Towards Algebraic n-Categories of Manifolds and Cobordisms

Jack Romö

Submitted in accordance with the requirements for the degree of
Doctor of Philosophy

School of Mathematics
University of Leeds
United Kingdom

September 17, 2024

To George, whose endless love and toil
Raised that house by Lake Erie
In which much of this thesis was written.

Declaration

I confirm that the work submitted is my own and that appropriate credit has been given where reference has been made to the work of others.

This copy has been supplied on the understanding that it is copyright material and that no quotation from the thesis may be published without proper acknowledgement.

The right of Jack Tyler Romö to be identified as Author of this work has been asserted by Jack Tyler Romö in accordance with the Copyright, Designs and Patents Act 1988.

The entirety of this thesis is my own work. Some parts of this thesis are taken directly from the preprint [Rom23], in particular many parts of chapters 2, 3 and 5, albeit with various modifications and corrections. I am the sole author of this preprint.

Acknowledgements

I take this time to first and foremost thank my advisors, João Faria Martins and Nicola Gambino, for their extraordinary support throughout these last four years of my life. To begin a doctorate in the midst of a pandemic only compounds the isolation inherent to specialized research, yet I can say for certain it was the guidance of both João and Nicola that truly mitigated this feeling. I thank you both for attending to my countless random ideas and helping to cohere them into a substantial research project, for helping to dispel the imposter syndrome that plagues me as it does all mathematicians, and for providing me with the skills needed to engage with both the work and lifestyle of mathematical research. The devil is in the details, as I have come to learn, and I thank you both for making me a fine exorcist.

Over the course of my PhD, I have had the good fortune to meet a number of wonderfully talented and personable colleagues, to who I should express my gratitude. I thank Eugenia Cheng, for helpful discussions that elucidated to me the significance of Trimble n-categories as a line of inquiry for constructing homotopy n-categories of (∞, n)-categories. I thank Lukas Waas and Charles Rezk, for insights that eventually culminated in the construction of a Reedy fibrant replacement functor that forms the fourth chapter of this thesis. I thank Simona Paoli for our conversations in Skye on the relationships between complete 2-fold Segal spaces and the simplicial models of weak 2-category. I also thank Julie Bergner for enlightening discussions in Skye on complete n-fold Segal spaces. Finally, while also at Skye, I thank David Green and Mike Johnson for discussions on both this thesis and future explorations of homotopy n-category constructions. I thank Sarah Whitehouse for sharing her expertise in operads to better inform my forays into the 'operadic' models of algebraic n-category. I thank Nima Rasekh, Lyne Moser and Martina Rovelli for arranging and overseeing the fantastic "Online Workshop on $(\infty, 2)$-Categories," from which I learned of countless models for $(\infty, 2)$-categories and of the manifold threads of equivalence tethering them together. I thank Brandon Shapiro for his supervision of my own presentation for that workshop. I reiterate my thanks expressed in [Rom23] to Ross Street and again Julie Bergner for helpful email correspondences, and to Udit Mavinkurve and Tim Porter for suggesting useful references that contributed to the completion of that preprint;

the contents of said document now form a significant portion of this thesis. Moreover, I repeat my thanks to the developers of the q.uiver.app software, which I used to create many of the diagrams in this thesis and the original preprint. I thank Lukas Müller for suggesting useful references for this thesis. I also thank all the organizers and speakers at the many workshops, seminars, conferences and other events I have had the pleasure to attend during the course of my PhD.

Turning closer to home, I extend my thanks to all the PhD students and faculty at the University of Leeds, who have stood by my side on this pilgrimage into the world of mathematical research. I owe special thanks to Matteo Spadetto, Luca Mesiti, Fiona Torzewska, Gabriele Lobbia and Andrew Slattery; it was a great comfort and pleasure to meet you all at a time of isolation at the start of my doctorate. I wish the best for your future endeavors and sincerely hope our friendships continue well into our careers. I also thank Tathagata Ghosh, Federico Olimpieri, Benjamin Morris, Asaf Karagila, Andrew Brooke-Taylor, Paul Martin and countless others for adding joy and color to the department I have had the pleasure of joining for these past four years. Between internal seminars, reading groups and countless discussions in the tea room, Leeds has proven itself both a fine garden in which to cultivate myself as a mathematician and a spring of many valued friendships.

There are a vast number of people who have had their say in the course of my life up to this point and who have directly impacted my adventures for the past four years of my academic career. Many of these people are not to be found in the office, nor at conferences; they are at home, in pubs and on vacations, nextdoor and a thousand miles away. Of all these people I owe my greatest and deepest gratitude to my parents, who have shaped my identity, passions and purpose, and who have been there for me at the lowest points in my degree and beyond. I thank you both for everything that I am and everything I shall ever be; know that my every achievement is your own, my every joy thanks to your love.

I thank my eldest sister Nina, who has shared in my life since its beginning and been one of my closest friends and confidants ever since. Though I may not have mastered the mathematics of Pokémon EV training like you, I hope this thesis will serve as a step towards compensation. I also thank my youngest sister Mia, whose passions for justice and empathy in the world inspire those around her, myself included. The world is ill-prepared for the righteous fire in your heart; may it be cooled by naught but the Barrel at Frosty Treat. I thank my grandmother and late grandfather, who created together a haven in Hanford Bay I will cherish until my final days. I extend my gratitude to all of my family, who have molded my life both within that sacred grove and beyond.

I thank my brother-in-law Cameron for not only entering my life as a beloved friend and a new part of our family, but for the role you now play in the life of my sister Nina. It has brought me great joy to see the bond you two have forged; I look forward immensely to what the future holds for the two of you in marriage and the three of us in friendship. The sky is the limit, Cloud District notwithstanding. With you has come another cast of immensely close friends I owe a great deal of my wellbeing and happiness to. I thank Josh, Ella, Jackson, Alice, Kieran and Rin. Despite the great physical distance dividing us, I rest easy knowing of the strong bond holding us together from now and until the end of time, built of both love and tabletop games. Though she is certainly a part of this group too, I would take a moment to especially thank Thea for the many years of friendship we have shared since the beginnings of secondary school. May your future blessings be as uncountable as your niche speculative fiction reading recommendations. Please pet Bokke every time you read this sentence.

There are innumerable others still left to thank, so I conclude with but a few. I thank Aki, Graham, Tom and Agathe for your hands in steering my life at York and my first forays into computer science and mathematics. Though we have chosen different courses after our graduation, that our bonds have grown no weaker is the sign of a fine friendship. May our future together be as long as Avi is wide. I thank Mitya for his wonderful mentorship throughout my Bachelor's degree and beyond. I thank the friends I made at the Hyde Park Book Club in Leeds for shining an artistic spark upon many cold and cloudy days. Though there are many others to mention, I should like to finish with a thanks to my beloved cat Pooh, to whom I dedicated my Bachelor's thesis prior. As he cannot yet read, know well if you are reading this far that he shall receive a fine scratching behind the ears soon after I type these last few words.

This thesis was made possible by the Leeds Doctoral Scholarship, awarded by the Faculty of Engineering and Physical Sciences at the University of Leeds. I humbly thank the University for funding my research and education in this manner.

Abstract

Advancements in the study of extended topological quantum field theories have recently relied upon constructing suitable symmetric monoidal n-categories whose objects are smooth m-dimensional manifolds and whose k-morphisms are $(m+k)$-dimensional cobordisms, with composition given by gluing along boundaries and monoidal structure by disjoint union. A number of such constructions currently prevail in the literature, in particular those defined as symmetric monoidal bicategories and those obtained as symmetric monoidal (∞, n)-categories, usually using the model of complete n-fold Segal spaces. The former have proven suitable for explicit computations, often admitting finite presentations that quickly yield new topological quantum field theories. The latter instead revel in their generality; with them, it is possible to consider full extension to all $n > 0$ and to consider all degrees of extension, using m-dimensional manifolds for any $m \geq 0$ as the objects.

There is an unfortunate divide between these two approaches to TQFTs - until the work in this thesis was developed, there was no complete direct investigation into obtaining fully weak homotopy bicategories from 2-fold Segal spaces, which has left researchers without the technology needed to translate results about $(\infty, 2)$-categorical TQFTs into their bicategorical counterparts. Moreover, there has been no means to develop TQFTs using algebraic models of n-category for general n, such as those of Trimble or Batanin and Leinster. Such models may be more amenable to presentations by generators and relations, in a similar manner to bicategories.

In this thesis, we take steps towards developing homotopy n-categories of n-fold Segal spaces with the goal of application to higher categories of manifolds and cobordisms. In particular, we functorially obtain the unbiased homotopy bicategory of a Reedy fibrant 2-fold Segal space, constructed by choosing sections of the Segal maps and homotopies between sections. We compare this with a simpler approach obtained by formally inverting the homotopy 1-functors of the Segal maps; our belief is the former method will more easily extend to homotopy n-categories for $n > 2$. We then concretely establish a general Reedy fibrant replacement functor, which we specialize to projective fibrant 2-fold Segal spaces. We obtain and discuss the resulting homotopy bicategory of a projective fibrant 2-fold Segal space by this method. We then apply our constructions to obtain fundamental bigroupoids of topological spaces, which serves to illustrate our notion of homotopy bicategory in action. Finally, we consider some resulting characterizations of completeness and equivalences between complete 2-fold Segal spaces.

Contents

List of Figures

Chapter 1

Introduction

Over the past several decades, an extraordinary synergy of category theory, physics and topology has made itself apparent in the form of *topological quantum field theories*. The mathematical definition of a topological quantum field theory (TQFT) was designed to model those quantum field theories (QFTs) invariant under diffeomorphisms. These particular quantum field theories enjoy a formal mathematical description not easily extended to more general QFTs, which was first discovered by Segal in the case of *conformal* field theories [Seg88] and later extended to the topological case by Atiyah [Ati88]. This latter contribution has blossomed into the modern category-theoretic perspective on the subject of TQFTs.

Henceforth, we use the following notation:

Notation 1.0.1. *If M is an oriented smooth manifold, then write $\overline{M}$ to denote the same smooth manifold with opposite orientation.*

In general, a topological quantum field theory may be formalized as a symmetric monoidal functor

$$Z : \mathbf{Bord}_n \to \mathbf{B}$$

where the codomain $\mathbf{B}$ is some symmetric monoidal category, often with duals, and usually of a 'linear' nature, such as the category **Hilb** of Hilbert spaces or $\mathbf{Rep}_{\mathbb{C}}(G)$ of complex representations for some finite group G.

The domain $\mathbf{Bord}_n$, following [Lur09b, Def. 1.1.1] in the oriented case, has as its objects compact $(n-1)$-dimensional smooth manifolds M, possibly with framing, orientation or some other structure. We will only discuss the oriented and unoriented cases here. This category then has as its morphisms diffeomorphism classes of n-dimensional cobordisms

$$M \hookrightarrow \Sigma \hookleftarrow N$$

namely smooth (un)oriented n-dimensional compact smooth manifolds, where in the oriented case the maps above induce an orientation-preserving diffeomorphism

$$\overline{M} \sqcup N \cong \partial\Sigma.$$

In the unoriented case, the diffeomorphism is of the form $M \sqcup N \cong \partial\Sigma$.

The diffeomorphisms identifying two cobordisms as the same morphism in $\mathbf{Bord}_n$ must respect these boundaries, meaning such a diffeomorphism $f : \Sigma \cong \Sigma'$ between two such cobordisms must be such that the diagram

$$\begin{tikzcd} & \Sigma \arrow[dd, "f"] & \\ M \arrow[ur, hook] \arrow[dr, hook] & & N \arrow[ul, hook'] \arrow[dl, hook'] \\ & \Sigma' & \end{tikzcd}$$

commutes.

The source and target are given by the source and target boundaries M and N of the equivalence class of cobordisms $[\Sigma]$ in question. Composition of cobordisms $M \hookrightarrow \Sigma_1 \hookleftarrow N$ and $N \hookrightarrow \Sigma_2 \hookleftarrow P$ are given by pushouts in **Top**

$$\begin{tikzcd} & & \Sigma_1 \sqcup_N \Sigma_2 & & \\ & \Sigma_1 \arrow[ur] & & \Sigma_2 \arrow[ul] & \\ M \arrow[ur] & & N \arrow[ul] \arrow[ur] & & P \arrow[ul] \end{tikzcd}$$

where smooth structure is induced by collars, a nontrivial matter in general [Mil65, Thm. 1.4]. Identities are given by cylinders $M \hookrightarrow M \times [0,1] \hookleftarrow M$, while the symmetric monoidal structure is induced by disjoint union [Lur09b, Ex. 1.1.3].

There is interest in TQFTs not only from the side of physics, but also geometry; indeed, a TQFT Z immediately supplies an invariant of n-dimensional smooth manifolds [Lur09b, pg. 7]. Consider some such smooth manifold M, perhaps with orientation, depending on Z. This defines a cobordism $\emptyset \to M \leftarrow \emptyset$, where $\emptyset$ is the unit of the monoidal structure on $\mathbf{Bord}_n$. Then, if for instance $\mathbf{B} = \mathbf{Vect}_{\mathbb{C}}$ is the category of complex vector spaces, interpreting M as such a cobordism implies a linear map

$$Z(M) : \mathbb{C} \cong Z(\emptyset) \to Z(\emptyset) \cong \mathbb{C}$$

namely a complex number $Z(M) \in \mathbb{C}$ assigned to M. This number is multiplicative in connected components, as $Z(M \sqcup N) \cong Z(M) \otimes Z(N)$. More importantly, since Z is defined on diffeomorphism classes of cobordisms, we have that the complex number $Z(M)$ is a diffeomorphism invariant of M. The efficacy of TQFTs as an invariant of smooth manifolds has been an active topic of research [RS22] [Dav11].

A major advantage of TQFTs as an invariant of smooth manifolds, at least in low dimensions, is that they are highly computable. For instance, it is a classical result that, due to a finite presentation of $\mathbf{Bord}_2$ in the oriented case by generators and relations as a symmetric monoidal category, the 2-dimensional oriented TQFTs with target $\mathbf{Vect}_\mathbf{k}$ are classified by commutative Frobenius algebras [Koc03].

Unfortunately, this story of finite presentations cannot be expected to hold for the case of $\mathbf{Bord}_3$; the objects of this category are 2-dimensional compact smooth manifolds, of which there are an infinite number of diffeomorphism classes even in the path-connected case. This is in direct contrast to $\mathbf{Bord}_2$, where all objects are generated by S^1 under disjoint union. We cannot therefore expect any finite presentation of $\mathbf{Bord}_n$ for $n \geq 3$, which makes the classification of higher-dimensional TQFTs considerably more complex. This is somewhat troublesome, as invariants of smooth manifolds really only become interesting above dimension 2. After all, the finite presentation of oriented 2-dimensional TQFTs is largely due to Euler characteristic being a complete invariant of compact oriented 2-dimensional smooth manifolds already.

A solution to this issue is to be found in *higher category theory*. Instead of constructing a presentation for the category $\mathbf{Bord}_3$, we should instead consider a presentation for a *bicategory* $\mathbf{Bord}_{3,2}$, whose objects are 1-dimensional smooth manifolds, morphisms are 2-dimensional cobordisms and 2-morphisms are '2-cobordisms' between cobordisms. A more precise definition, found for instance in [Sch14b, Def. 3.9], establishes higher cobordisms between $(n-1)$-dimensional cobordisms $M \hookrightarrow \Sigma_1 \hookleftarrow N$ and $M \hookrightarrow \Sigma_2 \hookleftarrow N$ as diffeomorphism classes of diagrams

$$\begin{tikzcd}
M \arrow[r, hook] \arrow[d, hook] & \Sigma_1 \arrow[d] & N \arrow[l, hook'] \arrow[d, hook] \\
M \times [0,1] \arrow[r] & Q & N \times [0,1] \arrow[l] \\
M \arrow[r, hook] \arrow[u, hook] & \Sigma_2 \arrow[u] & N \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}$$

where Q is an n-dimensional compact smooth manifold with corners, whose boundary ∂Q is given by the four maps into Q in the above diagram. Composition is given by pushouts in the vertical or horizontal directions once again, while disjoint unions now induce a symmetric monoidal bicategory structure on $\mathbf{Bord}_{n,n-1}$ [Sch14b, pg. 186-188].

There is now a greater hope of constructing finite presentations once more: where 2-dimensional manifolds were once the objects in $\mathbf{Bord}_3$ and so could not be decomposed into cobordisms, they are now the 1-morphisms in $\mathbf{Bord}_{3,2}$ and such a decomposition is now possible. Success has been found in obtaining presentations of certain such bicategories, such as the seminal work of [Sch14b] on the unoriented and oriented cases of $\mathbf{Bord}_{2,1}$, the work of [Pst14] in the framed case of $\mathbf{Bord}_{2,1}$ and [Bar+14] in the case of $\mathbf{Bord}_{3,2}$ with orientation.

Of course, the problems we faced with $\mathbf{Bord}_3$ will inevitably rear their heads again with $\mathbf{Bord}_{4,3}$; the objects are once more compact 2-dimensional smooth manifolds, which cannot be decomposed into a finite number of isomorphism classes by disjoint union. We now require a *symmetric monoidal tricategory* $\mathbf{Bord}_{4,3,2}$, or perhaps a *symmetric monoidal tetracategory* $\mathbf{Bord}_{4,3,2,1}$. In general, for any n, we should expect a need for at least a *symmetric monoidal n-category* $\mathbf{Bord}_{n+1,\cdots,2}$ in order to obtain finite presentations and construct computable invariants of $(n+2)$-dimensional compact smooth manifolds. Many usually go further and demand a symmetric monoidal $(n+1)$-category $\mathbf{Bord}_{n+1,\cdots,1}$, yielding what is often referred to as a *fully extended* topological quantum field theory.

1.1 Higher Categories

It is now apparent that we require a general theory of higher categories to proceed further with extended TQFTs. The diverse needs of different applications for higher categories and the current lack of a single 'best' definition meeting all of these requirements at once have resulted in of countless models for higher category theory being introduced in the literature. One could argue that the simplest of these models is that of *strict n-categories*. The definition of a strict n-category is classical, to be found for instance in [Lei04, Def. 1.4.1]; one simply defines the category of (small) *strict 0-categories* $\mathbf{StrCat}_0$ to be $\mathbf{Set}$ and inductively defines the category of (small) *strict n-categories* $\mathbf{StrCat}_n$ to be the category of small categories enriched over $\mathbf{StrCat}_{n-1}$. One must show for the induction to proceed that each $\mathbf{StrCat}_n$ is itself monoidal, though the Cartesian product handles this issue easily.

Tragically, such a simple approach to higher category theory is doomed to fail at providing a description of $\mathbf{Bord}_{n,\cdots,2}$ for $n > 4$. As noted in [Lur09b, pg. 13], associativity of the composition operation can only be expected to hold up to diffeomorphism. Moreover, composition with units can only hold up to diffeomorphism, where the diffeomorphism in question must collapse the composed interval $M \times [0,1]$ down to M. We cannot truly expect to rephrase this definition such that we obtain a meaningful strict n-category, as not all weak n-categories are equivalent to strict ones for $n > 2$ [Sim98]. We must therefore prepare for the worst and assume the necessity of a *weak n*-category of manifolds and higher cobordisms.

There is regrettably an immediate issue with this situation, presented not by smooth manifolds but rather n-categories themselves: the theory of fully weak n-categories is woefully incomplete and terribly complicated. Even the mere definition of a weak n-category becomes entirely intractable beyond $n = 3$, as one can see in the complete definition of a weak tetracategory written up by Trimble in [Tri]. For higher n, we are thus forced to rely upon other technologies to bear the brunt of the complexity.

1.1.1 (∞, n)-Categories

There are several approaches to handling the immense amount of data inherent to a weak n-category. One of the more popular techniques is to assume the *homotopy hypothesis*, a desideratum for the theory of higher groupoids posed by Grothendieck in [Gro]. The hypothesis concerns so-called ∞-*groupoids*, a special kind of higher category with n-morphisms for all $n \geq 1$ such that all morphisms are equivalences, in some suitably weak sense. While our definition of ∞-groupoid thus far may seem appallingly vague, what Grothendieck noticed makes these spaces far more familiar. The homotopy hypothesis amounts to the following statement:

> *The higher category theory of ∞-groupoids is equivalent to the homotopy theory of CW complexes.*

One should expect to be able to interpret a CW complex X as an ∞-groupoid, though some extra implicit cells would need to be considered for the interpretation to be precise. The objects of the ∞-groupoid in question are the 0-cells $x \in X_0$ of X and the n-morphisms are the n-cells $\alpha \in X_n$. Composition is given by gluing n-cells along boundary k-cells for $k < n$, an act that is weakly associative by repraparameterization. For weak units, one considers the constant weak $(n+1)$-cells implicitly glued onto every n-cell for $n \geq 0$. For weak inverses, one notes that every n-cell for $n > 0$ implies another n-cell glued in the opposite orientation, the concatenation with which gives an n-cell that can be contracted down to an $(n-1)$-cell. Conversely, we should expect to be able to see an ∞-groupoid as a CW complex, by starting with the set of objects and attaching n-cells for each n-morphism inductively in n to its source and target cells.

The interpretation is not only intrinsic within a given ∞-groupoid, but extrinsic as well. Any map between CW complexes can classically be made cellular, which thus corresponds to a map between the induced ∞-groupoids. Homotopies can be similarly restructured, so they correspond to natural isomorphisms, while homotopy equivalences precisely correspond to equivalences of higher groupoids. The homotopy hypothesis claims this mapping precisely transmits all results about ∞-groupoids to CW complexes and vice versa.

Depending on one's perspective, the homotopy hypothesis can either be a definition or a theorem. The latter case can be found in models of n-category such as the Tamsamani and weakly globular models [Pao19], where homotopy theory is not immediately baked into higher groupoids and must be carefully extracted [BP15]. Constructions analogous to our description of ∞-groupoids realized from topological spaces can be found for the Trimble n-category model in [Che11, Sec. 1.1] and the Batanin-Leinster ω-category model in [Lei04, Ex. 9.2.7], though these are perhaps more analogous to singular homology than what we have described; they assume no particular CW structure on the topological space X in question and simply take all possible disks $D^k \to X$. In the former case however, where the homotopy hypothesis is assumed rather than proven, one simply takes CW complexes or some adjacent model of weak homotopy theory as the chosen definition of higher groupoids. In this approach, higher groupoid theory is merely a reinterpretation of homotopy theory.

This latter approach has garnered considerable success in higher category theory. With the starting point of a fully weak ∞-groupoid, new models of higher category can be obtained by appending extra structure onto homotopy types or by weakening the chosen model of weak homotopy theory. As noted in [Rom23], approaches in these directions include quasicategories [Joy02], complicial sets [Ver08], Segal n-categories [HS01], complete n-fold Segal spaces [Bar05], Θ_n-spaces [BR13a] and n-quasi-categories [Ara14]. We will refer to all such models as *homotopy-theoretic models* of higher category theory.

The model most commonly associated with TQFTs in this direction is that of *complete n-fold Segal spaces.* These are a form of so-called *(∞, n)-category*, which are higher categories where all morphisms of dimension $> n$ are weakly invertible. It should be noted that these are a natural generalization of ∞-groupoids, which are $(\infty, 0)$-categories themselves. The definition of a complete n-fold Segal space is inductive, starting from the $n = 0$ case with complete 0-fold Segal spaces being Kan complexes[1].

Note the choice of Kan complex as the model of ∞-groupoid, or equivalently of (weak) homotopy type. In the case of Kan complexes, one usually interprets weak homotopy theory for the purposes of higher groupoids in the form of *model category theory*, employing the classical Quillen model structure on the category **sSet** of simplicial sets. This approach underpins a more general tactic in homotopy-theoretic models, where one constructs a model structure on some category of presheaves whose fibrant objects are precisely the models of higher category in question. The model structure then describes the higher category

[1]It is not necessarily standard to consider complete 0-fold Segal spaces in the literature; rather, one usually starts with the case of $n = 1$. It is, however, perfectly consistent to start with $n = 0$.

theory, or at least its equivalences by the weak equivalences and higher natural isomorphisms by the left or right homotopies. After all, one cannot expect homotopy theory to capture the 'lax' or 'oplax' aspects of higher category theory, such as noninvertible natural transformations, since homotopies can always be inverted in a model category.

The inductive procedure for obtaining complete n-fold Segal spaces from complete $(n-1)$-fold Segal spaces induces a model structure at each step, starting with the Quillen model structure on **sSet**, a model structure that is assumed to be present throughout this thesis. Our inductive intuition here is a natural extension of the setup of Lurie in [Lur09b, pg. 25], in the case $n = 1$ and moreover modified for Reedy fibrancy. It is also similar to the approach of Haugseng in [Hau18] more generally, though differs to suit our purposes. Assume then that one has a category $\mathbf{CSSP}_{n-1}$ of complete $(n-1)$-fold Segal spaces, which is the full subcategory of fibrant objects in some model category. The basic datum to consider is then a simplicial object in this category, that is a functor

$$X : \Delta^{op} \to \mathbf{CSSP}_{n-1}$$

where Δ is the usual simplicial category. Unwinding this induction yields an equivalent functor

$$X : (\Delta^{op})^n \to \mathbf{sSet}.$$

We may interpret $X_0 \in \mathbf{CSSP}_{n-1}$ as the *underlying ∞-groupoid* of X; it will be an $(\infty, n-1)$-category that is in fact an $(\infty, 0)$-category. We may then interpret $X_1 \in \mathbf{CSSP}_{n-1}$ as the $(\infty, n-1)$-category of pseudofunctors $[1] \to X$, where $[1]$ is the category $\{0 \to 1\}$ with two objects and one morphism between them. The two maps $X_1 \to X_0$ then identify the source and target of the image of $[1]$ in this interpretation, via pullback along the two objects $[0] \to [1]$ where $[0] = \{0\}$ is the trivial one-object category.

One might note an oddity to this intuition - if X_1 is to be a higher category of pseudofunctors and higher natural transformations, then the two images of a 1-morphism in X_1 of the form $\alpha : f \Rightarrow g$ in X_0 should be maps $\mathbf{dom}(f) \to \mathbf{dom}(g)$ and $\mathbf{cod}(f) \to \mathbf{cod}(g)$ between objects, which might not in general be equivalences and thus invalidate that X_0 is an ∞-groupoid. We should therefore in our interpretation restrict our attention to natural transformations that are equivalences on objects, which gives our model of higher categories a markedly 'globular' flavor.

The interpretation continues to higher X_m. We have the categories

$$[m] := \{0 \to \cdots \to m\}$$

of totally ordered poset categories of $m + 1$ objects for all $m \geq 0$, with all functors between them, as a natural alternative to Δ. We might then interpret X_m as the $(\infty, n-1)$-category of pseudofunctors $[m] \to X$. Note that our interpretation necessitates *pseudofunctors* rather than *strict functors*, so that elements of X_m contain more data than a mere chain of m morphisms. For instance, in the case of $m = 2$, we should expect to find a diagram in X of the form

$$\begin{tikzcd} & y \arrow[dr, "g"] \arrow[d, phantom, "\alpha\Downarrow"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}$$

where x, y, z are objects, f, g, h are morphisms and α is some 2-equivalence from $g \circ f$ to h. In some sense, this diagram exhibits h as a *composite* of g and f.

This last point is of particular importance in homotopy-theoretic models of higher category. Consider a CW complex T. The composite of two 1-cells $f, g : [0,1] \to T$ is not necessarily uniquely defined, as the concatenation of f and g could be done with many different parameterizations. All of these are however equivalent by homotopies changing the parameterization. This equivalence goes further than mere existence of homotopies: the *space* of paths $[0,1] \to T$ of the form

$$[0,1] \xrightarrow{\phi} [0,2] \cong [0,1] \sqcup_* [0,1] \xrightarrow{f \sqcup_* g} T$$

for some endpoint-preserving map ϕ is contractible, as this is equivalently just the space of endpoint-preserving maps $\phi : [0,1] \to [0,2]$. Hence, any two points have a path in this space, meaning a homotopy, between them, while any two homotopies are themselves homotopic and so forth.

It is this general philosophy on composition that is taken by the homotopy-theoretic models. Rather than specifying the singular composite $g \circ f$ of two morphisms g and f, one instead presents a contractible space of possible composites h. Hence, we should simultaneously regard X_2 as the $(\infty, n-1)$-category of *composites* of chains of two 1-morphisms in X.

Proceeding up to $m = 3$, we have a natural interpretation of elements of X_3 as diagrams of the form

$$\begin{tikzcd} & x \arrow[r, "g"] \arrow[d, phantom, "\Downarrow\alpha"] & y \arrow[dr, "h"] \arrow[dl, phantom, "\Downarrow\beta"] & \\ w \arrow[ur, "f"] \arrow[urr] \arrow[rrr, "k"'] & & & z \end{tikzcd} \overset{\Omega}{\Longrightarrow} \begin{tikzcd} & x \arrow[r, "g"] \arrow[drr] \arrow[d, phantom, "\Downarrow\delta"] & y \arrow[dr, "h"] \arrow[d, phantom, "\gamma\Downarrow"] & \\ w \arrow[ur, "f"] \arrow[rrr, "k"'] & & & z \end{tikzcd}$$

where Ω is now an invertible 3-morphism. This therefore gives us a similar interpretation of X_3 as the $(\infty, n-1)$-category of composites of chains of three morphisms in X. In general, one may naturally interpret X_m as the $(\infty, n-1)$-category of composites of chains of m morphisms.

To perhaps make this discussion more precise, there is a natural map, called the *Segal map*, for each $m \geq 2$ of the form

$$\gamma_m : X_m \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

induced by considering the m maps $[1] \to [m]$ identifying the morphism $(i-1) \to i$ for all $1 \leq i \leq m$. A typical element of the codomain is an m-tuple of 1-morphisms $(f_1, \cdots, f_m)$ where $f_i \in X_1$ for all $0 \leq i \leq n$, such that the target object of f_{i-1} is moreover the source object of f_i for all $i > 0$. We thus might see the codomain as the $(\infty, n-1)$-category of *chains* of m morphisms in X. The domain, in turn, consists of all such chains together with composites, while the map in question forgets the composites of the chain.

We can therefore see the fiber

$$\begin{tikzcd} X(f_1, \cdots, f_m) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_m \arrow[d] \\ \{(f_1, \cdots, f_m)\} \arrow[r] & X_1 \times_{X_0} \cdots \times_{X_0} X_1 \end{tikzcd}$$

as the collection of all composites of the chain $f_1 \cdots f_m$. We will assert a few properties of the Segal map to obtain the contractibility property noted for CW complexes:

1. The Segal maps must be *fibrations* in the model structure for complete $(n-1)$-fold Segal spaces. This guarantees, since fibrations are always preserved under pullback, that $X(f_1, \cdots, f_m)$ is fibrant and thus a complete $(n-1)$-fold Segal space.

2. The Segal maps must be *weak equivalences* in the same model structure, so equivalences of $(\infty, n-1)$-categories. This implies that the Segal maps are in fact trivial fibrations, so that the terminal map

$$X(f_1, \cdots, f_m) \to \{(f_1, \cdots, f_m)\} \cong *$$

is a weak equivalence. Hence, the $(\infty, n-1)$-category of composites of a given chain is always contractible, as this is the more general meaning of contractibility in an arbitrary model category.

A few other properties are needed as well, though these are less significant than the one above, which we term the *Segal condition*. A condition called *Reedy fibrancy* generalizes the fact that the Segal maps are fibrations to all face maps $X_m \to X_{m-1}$, while *essential constancy* ensures for instance that X_0 is indeed merely an ∞-groupoid. Finally, one has *completeness*, which guarantees that the paths in X_0 and the 'invertible' morphisms in X_1 are in direct correspondence with one another, proven in the case $n = 1$ for instance in [Rez00, Prop. 6.4].

We then define a *complete n-fold Segal space* to be a functor X as above that satisfies the Segal condition, is Reedy fibrant, is complete and finally satisfies essential constancy. One then proceeds to define a natural model structure on the category $\mathbf{sSet}^{(\Delta^{op})^n}$ whose fibrant objects are the (Reedy fibrant) complete n-fold Segal spaces and whose weak equvialences are levelwise between the fibrant objects. We write $\mathbf{CSSP}_n$ for the full subcategory of $\mathbf{sSet}^{(\Delta^{op})^n}$ whose objects are the complete n-fold Segal spaces.

One can also drop the requirement of completeness at each level to inductively define *n-fold Segal spaces*, which also admit a sensible model structure with levelwise weak equivalences between fibrant objects. We call the category of these structures $\mathbf{SeSp}_n$, each object of which is a functor $\Delta^{op} \to \mathbf{SeSp}_{n-1}$ satisfying the Segal condition and essential constancy. Due to the lack of completeness, the different notions of equivalence given at each inductive level are not required to agree with one another.

1.1.2 Collars, Projective Fibrancy and Completeness

Complete n-fold Segal spaces are particularly amenable to constructing higher categories of TQFTs for a few reasons. First, as a homotopy-theoretic model, they make no demand for what would be an irritatingly technical composition operation on cobordisms, necessarily defined by gluing manifolds along boundaries with a smooth structure given by collars. Another reason is that the fundamental 'diagrams' in a complete n-fold Segal space are cube-like in nature. For instance, for a complete 2-fold Segal space X, a typical 0-simplex α in the Kan complex $X_{1,1}$ has *horizontal* faces $f, g \in X_{1,0}$, *vertical* faces $i, j \in X_{0,1}$ and *corners* $w, x, y, z \in X_{0,0}$. We can arrange this data diagrammatically as follows:

$$\begin{tikzcd} w \arrow[r, "f"] \arrow[d, "i"'] \arrow[dr, phantom, "\alpha\Downarrow"] & x \arrow[d, "j"] \\ y \arrow[r, "g"'] & z \end{tikzcd}$$

Essential constancy will make the 1-morphisms i and j equivalences. Thus, the 2-morphisms of X, namely the elements of $X_{1,1}$ (the 1-morphisms in the $(\infty, 1)$-category of 1-morphisms, according to our interpretation of X) should really be squares with essentially constant vertical sides. This is precisely how the 2-morphisms in the bicategory $\mathbf{Bord}_{n,n-1}$ were defined, where the vertical sides of a 2-cobordism were forced to be cylinders. In general, unwinding our induction, an element of $X_{n,m}$ should be thought to consist of a diagram of

$\alpha_{i,j} \in X_{1,1}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$ of the form

$$\begin{tikzcd}
x_{0,0} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{1,1}\Downarrow"] & x_{1,0} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{2,1}\Downarrow"] & \cdots \arrow[r] \arrow[dr, phantom, "\alpha_{n,1}\Downarrow"] & x_{n,0} \arrow[d] \\
x_{0,1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha_{1,2}\Downarrow"] & x_{1,1} \arrow[r] \arrow[d] & \cdots \arrow[r] & x_{n,1} \arrow[d] \\
\vdots \arrow[d] \arrow[dr, phantom, "\alpha_{1,m}\Downarrow"] & \vdots \arrow[d] & \ddots & \vdots \arrow[d] \\
x_{0,m} \arrow[r] & x_{1,m} \arrow[r] & \cdots \arrow[r] & x_{n,m}
\end{tikzcd}$$

which we may call an (n, m)-*chain*, together with all composites, both vertical and horizontal. This intuition should continue to complete n-fold Segal spaces, using n-dimensional hypercubes instead of squares. (We might then call a diagram analogous to the above one, with i_k hypercubes in direction k for $1 \leq k \leq n$, an $(i_1, \cdots, i_n)$-*chain*.)

A final reason is that, in defining each Kan complex $X_{i_1,\cdots,i_n}$ for a complete n-fold Segal space X, it suffices to find some *moduli space* of $(i_1, \cdots, i_n)$-chains in the (∞, n)-category X in question. One can then usually apply some 'singular set' functor **Sing** to obtain the appropriate Kan complex. For cobordisms, there are natural moduli spaces of manifolds one can construct, which when appended with additional data, such as collars, serve as natural candidates for the ∞-groupoids needed once converted into Kan complexes.

There have several success stories in obtaining complete n-fold Segal spaces of manifolds and higher cobordisms. The first of these was in the work of Lurie [Lur09b], a construction that was later modified by Calaque and Scheimbauer [CS19]. A different construction was produced more recently by Grady and Pavlov [GP22a], while another still was given by Schommer-Pries [Sch17]. Many of these models are largely similar in their approaches to defining the (∞, d)-category $\mathbf{Bord}_{n,d}$ of n-dimensional smooth manifolds and higher cobordisms. In general, the elements of $(\mathbf{Bord}_{n,d})_{i_1,\cdots,i_d}$ correspond to $(n + d)$-dimensional smooth manifolds. These smooth manifolds are then decorated with a series of 'collars', which divvy up the manifold into an $(i_1, \cdots, i_n)$-chain of d-cobordisms between n-dimensional smooth manifolds. In particular, should any $i_j = 0$, the singular collar in the j^{th} direction will usually in essence cover the entire manifold, rendering it a cylinder and thus effectively a manifold of one dimension lower. A typical element of $(\mathbf{Bord}_{0,2})_{2,3}$ in the definition posed in [CS19] is depicted in Figure 1.1, inspired by the diagrams in [CS19, pg. 34-35].

The approaches of Lurie, Calaque and Scheimbauer and of Grady and Pavlov are significantly different past these high-level similarities. For instance, Grady and Pavlov's construction is levelwise the nerve of a groupoid, rather than the singular set of a moduli space as in Calaque and Scheimbauer's construction or simply a topological space as per Lurie. Another difference is that, while Grady and Pavlov's approach seems to be Reedy fibrant, though this appears unproven, the Lurie-style higher categories of cobordisms are not Reedy fibrant if $d > 1$. For example, Reedy fibrancy would imply the maps

$$(\mathbf{Bord}_{0,2})_{1,1} \to (\mathbf{Bord}_{0,2})_{0,1}$$

are fibrations, so that solutions Q should always exist to lifting problems of the form

$$\begin{tikzcd} \Delta[0] \arrow[r, "x"] \arrow[d, hook] & (\mathbf{Bord}_{0,2})_{1,1} \arrow[d] \\ \Delta[1] \arrow[r, "p"'] \arrow[ur, dashed, "Q"] & (\mathbf{Bord}_{0,2})_{0,1} \end{tikzcd}$$

A counterexample is found if one assumes x to be a 2-cobordism as depicted in Figure 1.2 and p to be a path of cobordisms as visualized in Figure 1.3.

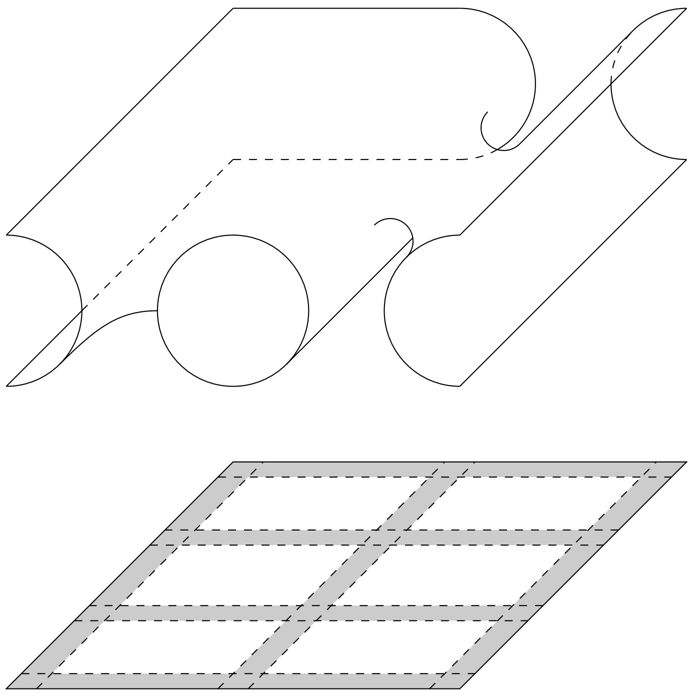


Figure 1.1: A typical element (M, I) of $(\mathbf{Bord}_{0,2})_{2,3}$, with M a manifold and $I = (I_1, I_2)$ a pair of lists of subintervals of $(0, 1)$, which identify the gray regions below. I_1 contains the three intervals in the horizontal direction, while I_2 contains the four in the vertical direction. The manifold M is embedded into $\mathbb{R}^k \times (0, 1)^2$ for some suitably high k; the 'collars' are then those submanifolds of M existing above each interval in I, on which there are submersivity conditions.

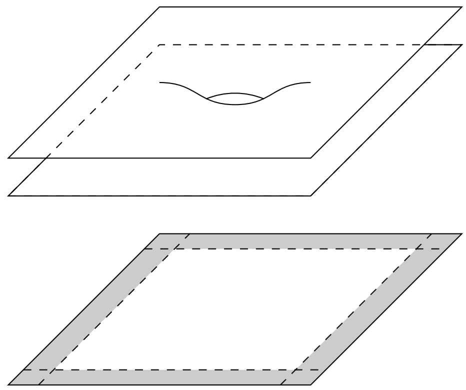


Figure 1.2: An element $x = (M, I)$ of $(\mathbf{Bord}_{0,2})_{1,1}$.

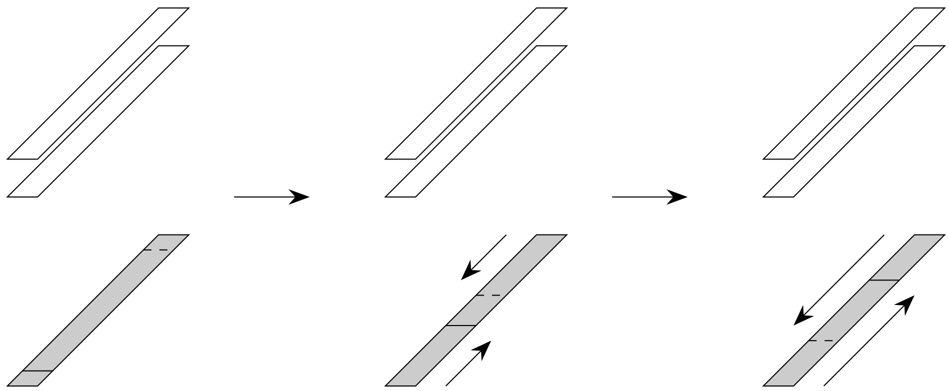


Figure 1.3: A path $p_t = (M_t, I_t)$ for $t \in [0,1]$ in $(\mathbf{Bord}_{0,2})_{0,1}$. To distinguish the vertical collars, one is depicted as a solid line and the other as a dashed line.

It is not possible for the collars to pass over the hole in x, so no path Q can possibly exist. Note that Grady and Pavlov's construction avoids this issue, as their collars are not rigid straight lines and can freely bend around the hole in x above. We are unaware at present if either a proof against Reedy fibrancy akin to the above or a proof for Reedy fibrancy of Grady and Pavlov's construction have been considered thus far in the literature.

This motivates the need for a version of complete n-fold Segal spaces that are *projective* fibrant, rather than *Reedy* fibrant, namely ones where the Segal maps may merely be weak equivalences rather than trivial fibrations. Indeed, it should be noted that Reedy fibrancy is never claimed, assumed or even needed by Lurie in [Lur09b], nor by Calaque and Scheimbauer in [CS19]; projective fibrancy is all that these works require. There is a perfectly reasonable notion of a projective fibrant complete n-fold Segal space, as explored in [Lur09b] [JS17] [CS19], though the space $X(f_1, \cdots, f_m)$ of composites for a chain of m

morphisms is no longer necessarily fibrant and thus not always an $(\infty, n-1)$-category. Moreover, the spaces $X_1 \times_{X_0} \cdots \times_{X_0} X_1$ are no longer guaranteed to be fibrant as the face maps $X_1 \to X_0$ are not fibrations, so a homotopy pullback $X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1$ is needed instead to obtain a higher category of chains per se. In this situation, the elements are no longer chains of morphisms, but rather chains

$$x_0 \xrightarrow{f_1} x_1 \overset{p_1}{\rightsquigarrow} x_1' \xrightarrow{f_2} x_2 \overset{p_2}{\rightsquigarrow} \cdots \overset{p_{n-2}}{\rightsquigarrow} x_{n-2}' \xrightarrow{f_{n-1}} x_{n-1} \overset{p_{n-1}}{\rightsquigarrow} x_{n-1}' \xrightarrow{f_n} x_n$$

where each $f_i \in X_1$ and each p_i is a path in X_0 from x_i to x_i'.

Another issue to note in the setup of [CS19] is that this model does not immediately seem to be *complete* either; this issue is discussed in [Lur09b, Warning 2.2.8]. This means that we cannot rely on, for instance, the paths in $(\mathbf{Bord}_{n,d})_{0,\cdots,0}$ having anything to do with the weakly invertible 1-morphisms in $(\mathbf{Bord}_{n,d})_{1,0,\cdots,0}$. One could perform a *completion* on this space to force it to be complete, using some generalization of the methods in [Rez00, pg. 32], as is done in [Lur09b] and [CS19], though it has been argued in [Lur09b, Warning 2.2.8] that choosing not to perform such a completion may allow one to obtain more refined information about the resulting TQFTs.

It is worth mentioning that one may extend these notions to obtain a form of *symmetric monoidal* complete n-fold Segal space as is done for the higher categories of cobordisms defined in [CS19] and [GP22a], which are both symmetric monoidal by the usual disjoint union structure. In essence, one may define a symmetric monoidal (∞, n)-category by considering *Segal's category* Γ of finite pointed sets $\langle n \rangle = \{0, \cdots, n\}$ with pointed maps $f : \langle n \rangle \to \langle m \rangle$, namely such that $f(0) = 0$, between them. One then takes a symmetric monoidal complete n-fold Segal space to be a functor

$$X : \Gamma \to \mathbf{CSSP}_n$$

such that the natural induced map

$$X\langle n \rangle \to \prod_{i=1}^{n} X\langle 1 \rangle$$

is a weak equivalence for all $n \geq 0$. The underlying complete n-fold Segal space is $X\langle 1 \rangle$, while $X\langle n \rangle$ is to be interpreted as the (∞, n)-category of tuples of n objects $(x_1, \cdots, x_n)$ together with all tensorings of the x_i, in analogy to X_n versus $X_1 \times_{X_0} \cdots \times_{X_0} X_1$. In the above models of higher category of cobordisms, $X\langle 1 \rangle$ is as usual, while $X\langle n \rangle$ simply consists of disjoint tuples of n such higher cobordisms.

1.2 Homotopy n-Categories

Given there has been success in obtaining both (∞, n)-categorical and bicategorical notions of TQFT, there should be some explicit means to translate between the two. Results about one should immediately obtain results about the other. Moreover, a suitable conversion may form a sanity check of sorts, ensuring various approaches to defining extended TQFTs agree with one another as we should expect them to.

One might wonder what the appropriate method to compare these two cases of TQFT might be. A possible answer is that of a *homotopy bicategory* functor. Given an (∞, n)-category, it is natural to expect an n-category to result from identifying any two n-morphisms that have an $(n+1)$-equivalence between them. This should amount to taking path components in the ∞-groupoids of n-morphisms $\alpha : f \Rightarrow g$ between two fixed $(n-1)$-morphisms f and g. In effect, this is the natural extension of the path components functor π_0 from the $(\infty, 0)$ case to general (∞, n).

The major complication of this construction arises from choosing the model of (∞, n)-category converted from and model of n-category converted to. For us, the clear contender for the former model is that of complete n-fold Segal spaces while the latter for $n = 2$ is bicategories. Note however the discrepancy between these two notions of higher category: the former is a homotopy-theoretic model, only guaranteeing contractible spaces of compositions for morphisms, while the latter is not, instead exhibiting precise choices of composites, together with higher coherence isomorphisms between them.

We are thus faced with the task of converting from a *homotopy-theoretic* model of higher category to what we might describe as an *algebraic* model. Algebraic models of higher category would include classical bicategories, classical tricategories, Trimble n-categories [Che11] and Batanin-Leinster style n-categories [Lei04]. In all of these notions of higher category, one may explicitly know the singular composition of some pasting diagram in the higher category in question. The challenge is then one of coherently choosing composition operations and coherence morphisms from the contractible spaces of compositions, such that the necessary coherence conditions are satisfied.

It is the objective of this thesis to establish a notion of homotopy bicategory that is directly applicable to comparing current models of $(\infty, 2)$-categories and bicategories of manifolds and cobordisms. While some work in the literature is available to help in this task, it seems at present that there is a need for a direct, fully concrete such construction tailored to send the $(\infty, 2)$-categories of cobordisms discussed thus far to bicategories of cobordisms explicitly similar to those established in the literature, such as those of Schommer-Pries in [Sch14b]. We expect such a construction to serve as an effective foundation for comparison of the work on TQFTs in each of these domains.

Our main results in this regard, stated somewhat informally, are the construction of functors

$$h_2 : \mathbf{SeSp}_2^{inj} \to \mathbf{Bicat}$$
$$h_2^{proj} : \mathbf{SeSp}_2^{proj} \to \mathbf{Bicat}$$

from categories $\mathbf{SeSp}_2^{inj}$ and $\mathbf{SeSp}_2^{proj}$ of Reedy fibrant and projective fibrant 2-fold Segal spaces respectively to a category **Bicat** of bicategories and pseudofunctors. We claim these to be suitable models of homotopy bicategory for 2-fold Segal spaces in the context of TQFTs.

1.2.1 Previous Work

There has been some effort directed to considering homotopy bicategories of 2-fold Segal spaces previously in the literature, which our work builds upon. In particular, one approach to obtaining homotopy bicategories of 2-fold Segal spaces is conjectured in [JS17, Def. 2.12] though originally appears in Scheimbauer's thesis [Sch14a], the original source for the higher category of cobordisms constructed in [CS19]. This first requires the definition of a *homotopy (1-)category* $h_1(X)$ of a projective fibrant (1-fold) Segal space X. Recall that a projective fibrant Segal space is a functor $X : \Delta^{op} \to \mathbf{sSet}$ that is levelwise fibrant and such that the Segal maps

$$X_n \to X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1$$

to *homotopy pullbacks* are weak equivalences. This characterization of projective fibrant Segal spaces is a result of Horel [Hor15]. Thus, one may consider the category $h_1(X)$ defined for instance in [JS17, Def. 2.2] as follows, using the notation

$$X^h(x, y) := X_1 \times^h_{(X_0)^2} \{(x, y)\}$$

for the *homotopy mapping space*, itself being a Kan complex, and $\pi_0 : \mathbf{sSet} \to \mathbf{Set}$ for the *path components* functor, sending a simplicial set K to $K_0/\cong_K$, where $\cong_K$ is the equivalence relation on K_0 generated by the 1-simplices:

1. The objects are given by the set $\mathbf{ob}(h_1(X)) := (X_0)_0$;

2. The hom-sets are of the form

$$\mathbf{Hom}_{h_1(X)}(x, y) := \pi_0(X^h(x, y));$$

3. Identities are given by applying π_0 to the degeneracy maps

$$* \cong X_0 \times_{(X_0)^2} \{(x, x)\} \to X_1 \times^h_{(X_0)^2} \{(x, x)\};$$

4. Composition is given by considering the diagram

$$\begin{aligned}\pi_0(X^h(x,y)) \times \pi_0(X^h(y,z)) &\to \pi_0(\{x\} \times^h_{X_0} (X_1 \times^h_{X_0} X_1) \times^h_{X_0} \{z\}) \\ &\xleftarrow{\sim} \pi_0(\{x\} \times^h_{X_0} X_2 \times^h_{X_0} \{z\})) \\ &\to \pi_0(X^h(x,z))\end{aligned}$$

and noting that the leftwards-facing map is a bijection, as it is the image under π_0 of a weak equivalence between Kan complexes. We will see later in this thesis that π_0 behaves on Kan complexes precisely like path components of topological spaces in this manner.

One may prove that this is a category; we will give an alternative means to prove this is the case later in this thesis. There is also a version of this construction if X is Reedy fibrant, found for instance in [Rez00, pg. 13], where all homotopy pullbacks are replaced with strict pullbacks. Hence, the hom-sets of a Reedy fibrant Segal space X's homotopy category are of the form $\pi_0(X(x,y)) := \pi_0(X_1 \times_{(X_0)^2} \{(x,y)\})$ rather than $\pi_0(X^h(x,y))$.

Armed with this construction, one might then naturally expect an immediate generalization to bicategories. This is precisely what is proposed in [JS17, Def. 2.12]. Given a projective fibrant 2-fold Segal space $X : (\Delta^{op})^2 \to \mathbf{sSet}$, the *homotopy bicategory* $h_2(X)$ should be obtained as follows, where we freely reuse the notation $X^h(x,y)$ in the same manner, now representing a projective fibrant Segal space rather than a Kan complex:

1. The objects should be $\mathbf{ob}(h_2(X)) := (X_{0,0})_0$;

2. The hom-categories should be of the form

$$\mathbf{Hom}_{h_2(X)}(x,y) := h_1(X^h(x,y));$$

3. Identities are given by applying h_1 to the degeneracy maps

$$* \cong X_0 \times_{(X_0)^2} \{(x,x)\} \to X_1 \times^h_{(X_0)^2} \{(x,x)\};$$

4. As the map $X_2 \to X_1 \times^h_{X_0} X_1$ is a levelwise weak equivalence, one may prove that the induced maps

$$\{x\} \times^h_{X_0} X_2 \times^h_{X_0} \{z\} \to \{x\} \times^h_{X_0} (X_1 \times^h_{X_0} X_1) \times^h_{X_0} \{z\}$$

are weak equivalences levelwise. Moreover, h_1 sends such weak equivalences to equivalences of categories. Thus, one may once again consider for composition the categorified diagram

$$\begin{aligned}h_1(X^h(x,y)) \times h_1(X^h(y,z)) &\to h_1(\{x\} \times^h_{X_0} (X_1 \times^h_{X_0} X_1) \times^h_{X_0} \{z\}) \\ &\xleftarrow{\sim} h_1(\{x\} \times^h_{X_0} X_2 \times^h_{X_0} \{z\}) \\ &\to h_1(X^h(x,z))\end{aligned}$$

as h_1, like π_0, commutes with products. One may then consider an inverse to the leftwards-facing functor to obtain a composition operation.

This conjectured bicategory, while quite natural in its definition, presents some technical obstacles in the pursuit of our intended applications. For instance, the chosen inverse to the functor

$$h_1(\{x\} \times^h_{X_0} X_2 \times^h_{X_0} \{z\}) \to h_1(\{x\} \times^h_{X_0} (X_1 \times^h_{X_0} X_1) \times^h_{X_0} \{z\})$$

may correspond to no possible (weak) inverse to the original map of $(\infty, 1)$-categories

$$\{x\} \times^h_{X_0} X_2 \times^h_{X_0} \{z\} \to \{x\} \times^h_{X_0} (X_1 \times^h_{X_0} X_1) \times^h_{X_0} \{z\}.$$

As an analogous example, consider the *fundamental groupoid* $\Pi_1(\mathbb{R})$ of the real line. Since $\mathbb{R}$ is simply connected and path-connected, the fundamental groupoid is contractible. Consider then the characteristic function of the rationals $\chi_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$, namely the function defined such that

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$$

This function defines a perfectly valid functor $\Pi_1(\mathbb{R}) \to \Pi_1(\mathbb{R})$, as indeed *any* map on objects between these two categories extends to a unique functor. However, it is not realized by any continuous map $\mathbb{R} \to \mathbb{R}$. Hence, functors between fundamental groupoids may have very little to do with the original topological spaces.

We will see later that Π_1 in fact factors through h_1 in the Reedy fibrant case by a chain of maps $\mathbf{Top} \to \mathbf{SeSp}_1 \to \mathbf{Cat}$. Hence, this is evidence of a more general phenomenon when inverting induced equivalences between homotopy categories: for two Segal spaces X and Y, functors $h_1(X) \to h_1(Y)$ do not necessarily originate from any map $X \to Y$. In short, h_1 is not a full functor.

While this issue is not necessarily severe if one is only interested in 'the big picture', as all such choices should result in canonically equivalent bicategories, it is somewhat obtrusive if one wants to work within the resulting bicategory itself. Highly abnormal though technically correct composition operations may result from such considerations, which may make constructions within the homotopy bicategory unnecessarily difficult to reason about. It would be preferable for the chosen inverse to result from some map at the level of the $(\infty, 1)$-categories, rather than just the 1-categorical truncation, as an assurance that the composition operations will be closer to what the original $(\infty, 2)$-category would suggest. This would for instance ensure that homotopy bicategories of $(\infty, 2)$-categories of cobordisms better resemble what one might obtain by constructing bicategories of higher cobordisms by hand, as the composition operations would have to be induced by valid simplicial maps between the singular sets of some moduli spaces of cobordisms. Similar concerns may be raised about associators and unitors; it would be preferable to have some direct interpretation of these natural isomorphisms in the original $(\infty, 2)$-category, so the resulting bicategory is clearer on a concrete level and thus easier to work within.

1.2.2 Homotopy Bicategories of Reedy Fibrant 2-fold Segal Spaces

Our first contribution, as first developed in the author's preprint [Rom23], is the construction of the homotopy bicategory $h_2(X)$ of a Reedy fibrant 2-fold Segal space X. As a part of this endeavor, we also develop a homotopy bicategory functor h_2^{tr} which can accept as its composition operations any chosen inverses to the functors

$$h_1\Big(X_n \times_{(X_0)^{n+1}} \{(x_0, \cdots, x_n)\}\Big) \to h_1\Big((X_1 \times_{X_0} \cdots \times_{X_0} X_1) \times_{(X_0)^{n+1}} \{(x_0, \cdots, x_n)\}\Big)$$

for any objects $x_0, \cdots, x_n \in (X_{0,0})_0$, similarly to the conjectured definition of Johnson-Freyd and Scheimbauer in [JS17], albeit modified to suit the Reedy fibrant case. To obtain coherence isomorphisms and coherence conditions, we apply modified techniques from Lack and Paoli's work on obtaining bicategories from Tamsamani 2-categories in [LP08]. This method, though correct, still faces the technical obstacles we have noted due to composition operations being chosen after taking homotopy categories.

Beyond the concerns raised thus far, at the level of pure higher category theory, one might simply wonder in a 2-fold Segal space X where composition operations, associators, unitors, pentagon conditions and so forth come from concretely at the $(\infty, 2)$-categorical level. Truncation to homotopy categories before obtaining this data perhaps obfuscates the source of this structure in X itself, so one may take an inherent interest in obtaining this data from X directly in the pursuit of better understanding 2-fold Segal spaces as a model of higher category.

Our panacea to the above issues amounts to an alternative construction of homotopy bicategory, involving choosing composition operations and identifying coherence isomorphisms and coherence conditions *before* truncation by solving natural families of lifting problems. In this way, there is a guarantee that the composition operations in the resulting bicategory will agree with the truncated ∞-categorical information, which should yield more natural composition maps. For instance, in the case of $\mathbf{Bord}_{n,2}$, the composition maps will be induced by simplicial maps between singular sets of the moduli spaces of cobordisms at each level.

Consider some Reedy fibrant 2-fold Segal space X. In the model structure defining Reedy fibrant 1-fold Segal spaces, all objects are cofibrant and the Segal maps are trivial fibrations. Hence, one has a natural lifting problem with solutions μ_n

$$\begin{tikzcd} & X_n \arrow[d] \\ X_1 \times_{X_0} \cdots \times_{X_0} X_1 \arrow[r, "id"'] \arrow[ur, dashed, "\mu_n"] & X_1 \times_{X_0} \cdots \times_{X_0} X_1 \end{tikzcd}$$

for all $n \geq 2$. We argue that such *horizontal compositions* μ_n are the natural notion of composition operations in a 2-fold Segal space; they are simply sections of the Segal maps, which within our interpretations of n-fold Segal spaces amounts to coherently choosing an element

$$\mu_n(f_1, \cdots, f_n) \in X(f_1, \cdots, f_n)$$

for each chain of 1-morphisms $(f_1, \cdots, f_n) \in X_1 \times_{X_0} \cdots \times_{X_0} X_1$ in X. We may identify the composition operation by postcomposing with a map $X_n \to X_1$ and taking a pullback over objects $x_0, \cdots, x_n \in (X_{0,0})_0$, thus obtaining an $(\infty, 1)$-functor $\circ^{x_0, \cdots, x_n}$ of the form

$$\begin{aligned} X(x_0, x_1) \times \cdots \times X(x_{n-1}, x_n) &\hookrightarrow (X_1 \times_{X_0} \cdots \times_{X_0} X_1) \times_{(X_0)^2} \{(x_0, x_n)\} \\ &\xrightarrow{\mu_n \times_{(X_0)^2} 1_{\{(x_0, x_n)\}}} X_n \times_{(X_0)^2} \{(x_0, x_n)\} \\ &\to X(x_0, x_n). \end{aligned}$$

Note that this is remarkably similar to the conjectured homotopy bicategory construction posed earlier, though this time constructed in the $(\infty, 1)$-categorical case. Moreover, there is a guarantee by setting μ_n to be a *section* of the Segal maps γ_n that, under our interpretation of X, the map μ_n chooses a composite that is indeed a composite of the source 1-morphisms. If μ_n was just a *homotopy inverse* instead, namely a map with homotopies $\gamma_n \mu_n \sim id$ and $\mu_n \gamma_n \sim id$, the choice $\mu_n(f_1, \cdots, f_n)$ may only be a composite of a chain of maps $(f_1', \cdots, f_n')$ that is *equivalent* to $(f_1, \cdots, f_n)$ in $X_1 \times_{X_0} \cdots \times_{X_0} X_1$.

From a *choice of horizontal compositions* $(\mu_n)_{n \geq 0}$, we would like to extract associators and unitors that satisfy the pentagon and triangle axioms. First, we must determine the appropriate maps to compare. If one considers for instance the composition operation $\bullet \bullet \bullet \mapsto \bullet(\bullet\bullet)$, the appropriate map is given by the two induced solutions from μ_2 of the lifting problems

$$\begin{tikzcd}
 & X_2 \arrow[r] \arrow[d] & X_1 \\
 & X_1 \times_{X_0} X_1 & \\
 & X_2 \times_{X_0} X_1 \arrow[u] \arrow[d] & \\
X_1 \times_{X_0} X_1 \times_{X_0} X_1 \arrow[r, "id"'] \arrow[uuur, dashed] \arrow[ur, dashed] & X_1 \times_{X_0} X_1 \times_{X_0} X_1 &
\end{tikzcd}$$

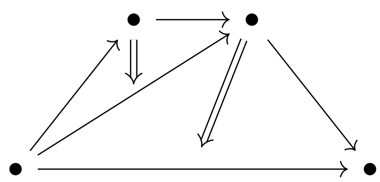
We can rearrange this into a single solution to a lifting problem

$$\begin{tikzcd}
 & (X_2 \times_{X_0} X_1) \times_{X_1 \times_{X_0} X_1} X_2 \arrow[r] \arrow[d] & X_1 \\
X_1 \times_{X_0} X_1 \times_{X_0} X_1 \arrow[r, "id"'] \arrow[ur, dashed] & X_1 \times_{X_0} X_1 \times_{X_0} X_1 &
\end{tikzcd}$$

If we were to extend $X : \Delta^{op} \to \mathbf{sSet}^{\Delta^{op}}$ to a functor $X : \mathbf{sSet}^{op} \to \mathbf{sSet}^{\Delta^{op}}$ by sending colimits of representables $\Delta[n] \in \mathbf{sSet}$ for $[n] \in \Delta$ to limits, then one would see that the pullback space above is simply X_K for

$$K = (\Delta[2] \sqcup_{\Delta[0]} \Delta[1]) \sqcup_{\Delta[1] \sqcup_{\Delta[0]} \Delta[1]} \Delta[2] \in \mathbf{sSet}.$$

Portraying K graphically suggests an immediate interpretation of X_K as the space of composites $h(gf)$ for chains of three morphisms f, g, h in X:



The composite of this form is obtained by applying μ_2 twice. Similarly, for the composition operation $\bullet\bullet\bullet \mapsto (\bullet\bullet)\bullet$, one obtains another natural lifting problem and solution from μ_2 of the form

$$\begin{tikzcd}
 & (X_1 \times_{X_0} X_2) \times_{X_1 \times_{X_0} X_1} X_2 \arrow[r] \arrow[d] & X_1 \\
X_1 \times_{X_0} X_1 \times_{X_0} X_1 \arrow[r, "id"'] \arrow[ur, dashed] & X_1 \times_{X_0} X_1 \times_{X_0} X_1 &
\end{tikzcd}$$

The uppermost object can be rewritten as X_H, where

$$H = (\Delta[1] \sqcup_{\Delta[0]} \Delta[2]) \sqcup_{\Delta[1] \sqcup_{\Delta[0]} \Delta[1]} \Delta[2] \in \mathbf{sSet}$$

is the diagram of the form

$$\begin{tikzcd}
& \bullet \arrow[r] \arrow[drr] \arrow[d, phantom, "\Downarrow"] & \bullet \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\
\bullet \arrow[ur] \arrow[rrr] & & & \bullet
\end{tikzcd}$$

giving composites $(hg)f$ of chains f, g, h in X.

We may then use the solutions to the lifting problems given above to obtain a commutative diagram

$$\begin{tikzcd}
& & X_3 \arrow[r] \arrow[dl] \arrow[dr] \arrow[dd] & X_1 \\
& X_K \arrow[urr, bend left] \arrow[dr] & & X_H \arrow[u] \arrow[dl] \\
X_1 \times_{X_0} X_1 \times_{X_0} X_1 \arrow[rr] \arrow[ur, dashed] \arrow[urrr, dashed] & & X_1 \times_{X_0} X_1 \times_{X_0} X_1 &
\end{tikzcd}$$

From this, we can proceed in one of two ways. One is to prove that the two maps $X_3 \to X_K$ and $X_3 \to X_H$ induced by the inclusions $K \hookrightarrow \Delta[3]$ and $H \hookrightarrow \Delta[3]$ are trivial fibrations, allowing us to solve the induced lifting problems to obtain two distinct solutions $\mu_{\bullet(\bullet\bullet)}$ and $\mu_{(\bullet\bullet)\bullet}$ to the lifting problem

$$\begin{tikzcd}
& X_3 \arrow[r] \arrow[d] & X_1 \\
X_1 \times_{X_0} X_1 \times_{X_0} X_1 \arrow[r] \arrow[ur, dashed, bend left, "\mu_{\bullet(\bullet\bullet)}"] \arrow[ur, dashed, bend right, "\mu_{(\bullet\bullet)\bullet}"'] & X_1 \times_{X_0} X_1 \times_{X_0} X_1 &
\end{tikzcd}$$

whose postcompositions with the map $X_3 \to X_1$ give precisely the two bracketed composition operations induced by μ_2 when fibered over objects w, x, y and z. It is then a classical result in model category theory, proven for instance in [Hir09, Prop. 7.6.13], that any two solutions to the same lifting problem necessarily are related by a homotopy. There is a reasonable amount of freedom in choosing what is meant by a 'homotopy'; we choose for a homotopy L between the two solutions above to take the form

$$L : (X_1 \times_{X_0} X_1 \times_{X_0} X_1) \times N(I[1]) \to X_3$$

where $N : \mathbf{Cat} \to \mathbf{CSSP}_1$ is the *classifying diagram* functor in [Rez00], while $I[1]$ is the standard *walking category*, consisting of two objects and a single isomorphism between them. The $(\infty, 1)$-category $N(I[1])$ can then be interpreted as a countable ∞-groupoid analogue to the closed interval $[0, 1]$.

The reason we make this choice is because there is a natural isomorphism $h_1 \circ N \cong id$. Hence, if we fiber the above homotopy over objects $w, x, y, z \in (X_{0,0})_0$ and postcompose with the map $X_3 \to X_1$, then one obtains a map

$$X(w,x) \times X(x,y) \times X(y,z) \times N(I[1]) \to X(w,z)$$

which when h_1 is applied becomes a functor

$$h_1(X(w,x)) \times h_1(X(x,y)) \times h_1(X(y,z)) \times I[1] \to h_1(X(x,z)).$$

A functor $C \times I[1] \to D$ is precisely a natural isomorphism, so we may take L to be our chosen associator in X.

In this thesis, we take a slightly different approach. Given we have already chosen μ_3, we instead obtain a homotopy between the two solutions

$$\begin{tikzcd}
 & X_3 \arrow[r] & X_K \arrow[r] \arrow[d] & X_1 \\
X_1 \times_{X_0} X_1 \times_{X_0} X_1 \arrow[ur, dashed, "\mu_3"] \arrow[urr, dashed] \arrow[rr] & & X_1 \times_{X_0} X_1 \times_{X_0} X_1 &
\end{tikzcd}$$

to obtain a homotopy from the composition operation $(f, g, h) \mapsto h \circ (g \circ f)$ to μ_3, which we interpret as *unbiased trinary composition*, sending $(f, g, h) \mapsto h \circ g \circ f$. A similar diagram to above with K replaced by H obtains a homotopy from $(f, g, h) \mapsto (h \circ g) \circ f$ to μ_3 as well. If one desires the original associator, one can invert the latter homotopy and concatenate the two. However, this approach more immediately suggests an interpretation as an *unbiased bicategory* as in [Lei00, Def. 1.2.1], with n-ary composition given by μ_n for all n. An unbiased bicategory is similar to a bicategory, though has unbiased n-ary composition operations

$$(f_1, \cdots, f_n) \mapsto \bigcirc_{i=1}^{n} f_i$$

together with coherence 2-isomorphisms adding a unary composition

$$f \cong (f)$$

and ones which remove a level of bracketing, such as

$$(f \circ g) \circ (h \circ k) \cong f \circ g \circ h \circ k.$$

These must fulfil a coherence condition; the more pressing case for us concerns the latter 2-isomorphism and amounts to saying that layers of brackets can be removed in any order. For instance, the diagram

$$\begin{tikzcd}
((f\circ g)\circ(h))\circ((k\circ l)\circ(m\circ n\circ p)) \arrow[d] \arrow[dr] & \\
(f\circ g\circ h)\circ(k\circ l\circ m\circ n\circ p) \arrow[d] & (f\circ g)\circ(h)\circ(k\circ l)\circ(m\circ n\circ p) \arrow[dl] \\
f\circ g\circ h\circ k\circ l\circ m\circ n\circ p &
\end{tikzcd}$$

is required to commute.

Obtaining such coherence homotopies is not a difficult generalization of our discussion thus far. We are led to consider more general simplicial sets than K and H, which we take to calling *simplicial composition diagrams*. Henceforth, for convenience, we will begin writing

$$Sp(n) := \Delta[1] \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} \Delta[1]$$

for the *n-spine*. A simplicial composition diagram of arity n is then a diagram in **sSet** $Sp(n) \hookrightarrow K \hookleftarrow \Delta[1]$, where K is inductively either $\Delta[n]$ or a diagram

$$(K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_r) \sqcup_{Sp(r)} \Delta[r]$$

such that K_i is a simplicial composition diagram of arity k_i for all i and $\sum_i k_i = n$. Clearly the examples of H and K above are simplicial composition diagrams, though more generally any nested composition operation can be represented by a unique diagram K.

There is a natural category $\mathbf{SCD}_n$ of such diagrams of arity n for all $n \geq 0$, with maps commuting with the two morphisms from $Sp(n)$ and $\Delta[1]$. For the above picture, we have a natural commuting square of such diagrams in Figure 1.4.

If we name these four simplicial composition diagrams (top to bottom, left to right) as Q_1, Q_2, Q_3 and Q_4, where evidently $Q_4 = \Delta[8]$, we obtain a new diagram of given solutions to lifting problems

$$\begin{tikzcd}
 & & X_8 = X_{Q_4} \arrow[rr] \arrow[dl] \arrow[dr] & & X_1 \\
 & X_{Q_2} \arrow[urrr] \arrow[dr] & & X_{Q_3} \arrow[ur] \arrow[dl] & \\
 & & X_{Q_1} \arrow[uurr, bend right] \arrow[d] & & \\
X_{Sp(8)} \arrow[rr, "id"'] \arrow[uuurr, dashed, bend left, "\mu_8"] \arrow[uur, dashed, "\mu_{Q_2}"] \arrow[uurrr, dashed, "\mu_{Q_3}"] \arrow[urr, dashed, "\mu_{Q_1}"] & & X_{Sp(8)} & &
\end{tikzcd}$$

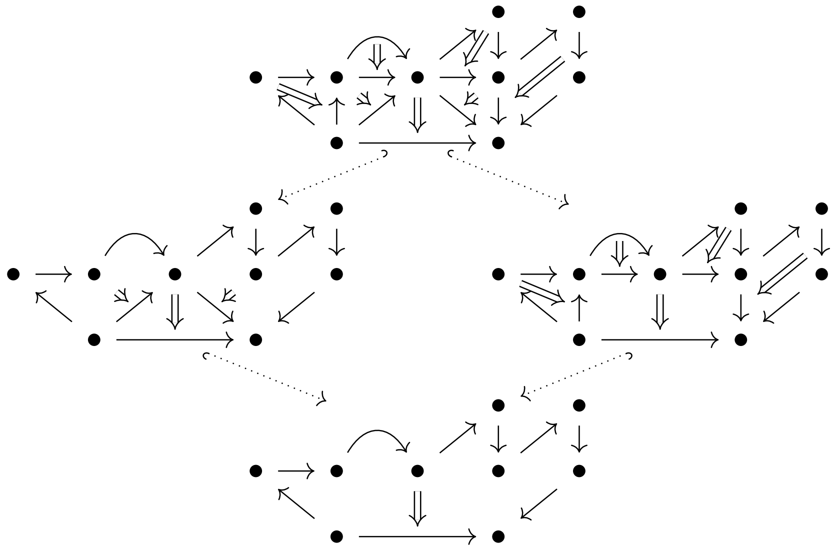


Figure 1.4: A commuting diagram in $\mathbf{SCD}_8$.

where we have begun writing $\mu_K : X_{Sp(n)} \to X_K$ for the natural lift induced by μ_r's. In the above diagram, there are two chains of left homotopies from μ_{Q_1} to μ_8, corresponding to the two paths in the associativity condition. We can concatenate these two paths to obtain two homotopies

$$V, W : X_{Sp(8)} \times N(I[1]) \to X_{Q_1}$$

from μ_{Q_1} to $\iota \circ \mu_8$, where $\iota : X_8 \to X_{Q_1}$ is the map induced by the inclusion $Q_1 \hookrightarrow \Delta[8]$. The homotopies V and W pass through μ_{Q_2} and μ_{Q_3}, respectively.

Note that these two homotopies are themselves solutions to the same lifting problem

$$\begin{tikzcd}
X_{Sp(8)} \sqcup X_{Sp(8)} \arrow[r, "\mu_{Q_1} \sqcup (\iota \circ \mu_8)"] \arrow[d, hook] & X_{Q_1} \arrow[d, two heads, "\sim"] \\
X_{Sp(8)} \times N(I[1]) \arrow[r] \arrow[ur, dashed, bend left, "V"] \arrow[ur, dashed, "W"'] & X_{Sp(8)}
\end{tikzcd}$$

Hence, one should expect a *homotopy between homotopies* comparing V and W, what we might call a *2-homotopy.*

A naïve construction of such a 2-homotopy might take the form of a solution

χ to the lifting problem

$$\begin{tikzcd}
\Big(X_{Sp(8)} \times N(I[1])\Big) \sqcup \Big(X_{Sp(8)} \times N(I[1])\Big) \arrow[r, "V \sqcup W"] \arrow[d, hook] & X_{Q_1} \arrow[d, two heads, "\sim"] \\
X_{Sp(8)} \times N(I[1] \times I[1]) \arrow[r, two heads] \arrow[ur, dashed, "\chi"] & X_{Sp(8)}
\end{tikzcd}$$

However, the resulting map χ, when postcomposed with the map $X_{Q_1} \to X_1$, fibered over objects $x_0, \cdots, x_8 \in (X_{0,0})_0$ and converted to a functor with h_1, results in a functor

$$\Big(\prod_{i=1}^{8} h_1(X(x_{i-1}, x_i))\Big) \times I[1] \times I[1] \to h_1(X(x_0, x_8)).$$

We would like this datum to witness that the natural isomorphisms ϕ and ψ induced by V and W respectively are *equal*, as this is precisely the content of the coherence condition we are tasked with checking. Unfortunately, the above functor merely provides us with a pair of natural automorphisms S on the source of ϕ and ψ, and T on the target of ϕ and ψ, such that $T\phi = \psi S$. We would like to ensure that T and S are identities. To do so, we enforce *globularity* on the 2-homotopy χ by constraining it to a more substantial lifting problem

$$\begin{tikzcd}
\Big(X_{Sp(8)} \times N(I[1])\Big) \sqcup_{X_{Sp(8)} \sqcup X_{Sp(8)}} \Big(X_{Sp(8)} \times N(I[1])\Big) \arrow[r, "V \sqcup_{\mu_{Q_1} \sqcup \iota \circ \mu_8} W"] \arrow[d, hook] & X_{Q_1} \arrow[d, two heads, "\sim"] \\
\Big(X_{Sp(8)} \times N(I[1] \times I[1])\Big) \sqcup_{\big(X_{Sp(8)} \times N(\{0,1\} \times I[1])\big)} (X_{Sp(8)} \sqcup X_{Sp(8)}) \arrow[r, two heads] \arrow[ur, dashed, "\chi"] & X_{Sp(8)}
\end{tikzcd}$$

which guarantees that T and S are identities as needed. Thus, χ is a *globular 2-homotopy*, which when h_1 is applied becomes an equality between coherence isomorphisms as needed.

Overall, we find that a somewhat inductive procedure for obtaining coherence conditions presents itself: composition operations are given by solutions to certain lifting problems, compositions of which induce multiple solutions to the same aggregate lifting problem. Such solutions must have a homotopy between them, which gives a higher coherence condition. These may themselves be composed, yielding solutions to the same lifting problem once again, giving higher homotopies that amount to even higher coherences. For the purposes of homotopy bicategories at least, it suffices to stop at 2-homotopies; any higher conditions would simply collapse to identities between identities.

In this way, we have therefore extracted all of the algebraic data of an unbiased bicategory from a homotopy-theoretic model of $(\infty, 2)$-category. We are currently unaware of other comparable projects tackling this problem in a similar manner, let alone ones amenable to constructing algebraic higher categories of cobordisms.

It should also be noted that our construction is capable of extracting, from a map $F : X \to Y$ between 2-fold Segal spaces each equipped with choices $(\mu_n)_{n\geq 2}$ and $(\nu_n)_{n\geq 2}$ of horizontal compositions respectively, an *unbiased pseudofunctor* $h_2(F) : h_2(X) \to h_2(Y)$ that is fully weak. The weakness stems from the map F not needing to commute with maps μ_n and ν_n for any n; the coherence isomorphisms such as

$$h_2(F)(f) \circ h_2(F)(g) \cong h_2(F)(f \circ g)$$

come from obtaining induced left homotopies between solutions

$$\begin{tikzcd}
 & X_2 \arrow[r, dashed, "F_2"] & Y_2 \arrow[d, two heads, "\sim"] \\
X_{Sp(2)} \arrow[ur, dashed, "\mu_2"] \arrow[urr, dashed, "\nu_2 \circ F_{Sp(2)}"'] \arrow[rr, "F_{Sp(2)}"'] & & Y_{Sp(2)}
\end{tikzcd}$$

to the same lifting problem. Our construction is functorial in this manner. Finally, for any two choices μ_n or ν_n of horizontal compositions for a given $(\infty, 2)$-category X, we have that the induced homotopy bicategories are canonically equivalent by an equivalence that acts as the identity on objects and hom-categories.

1.2.3 Reedy Fibrant Replacement

While our means of obtaining homotopy bicategories seems to be quite natural in the Reedy fibrant case, generalizing to the projective fibrant case is not so straightforward. Given some projective fibrant 2-fold Segal space X, there is no guarantee that the Segal maps

$$\gamma_n : X_n \to X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1$$

admit any section or even a homotopy inverse, namely a map $\iota_n : X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1 \to X_n$ equipped with homotopies $\gamma_n \iota_n \sim id$ and $\iota_n \gamma_n \sim id$. As is noted in [DS95, Lemma 4.24], a homotopy inverse can usually be expected for a weak equivalence when the domain and codomain are both fibrant and cofibrant. One can slim down their assumptions to the mere requirement that the domain is fibrant and the codomain is cofibrant. However, while X_n will be fibrant in a *projective* model structure, cofibrancy is not such a simple condition in this model structure and may not be satisfied by $X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1$. Thus, inverses to these maps at the level of homotopy categories may not be reflected by anything at the level of X itself.

In order to enforce a reasonable interpretation of composition at the level of $(\infty, 2)$-categories, a natural approach is to take a *fibrant replacement* $R(X)$ of X, where $R(X)$ is some Reedy fibrant 2-fold Segal space, together with a weak equivalence $X \to R(X)$. One might then take the homotopy bicategory $h_2(R(X))$ of $R(X)$, where the composition operations chosen may now be reflected as chosen sections of the maps $R(X)_n \to R(X)_{Sp(n)}$.

Often, Reedy fibrant replacement functors are simply assumed to exist; they always will, as the terminal map $X \to *$ may be factorized into a trivial cofibration followed by a fibration in the Reedy model structure. However, for the purposes of computing homotopy bicategories, it will be necessary for us to know precisely what the contents of a Reedy fibrant replacement $R(X)$ of X will be.

We propose in this thesis a general means to take a Reedy fibrant replacement $R(Y)$ of a functor $Y : \mathscr{C} \to \mathbf{sSet}$ that is levelwise fibrant, where $\mathscr{C}$ is a Reedy category. As far as we are aware, our approach is novel. In the simple case of $\mathscr{C} = \Delta^{op}$, the procedure inductively defines $R(Y)_n$ for increasing n, starting with setting

$$R(Y)_0 := Y_0.$$

There is then a natural map $Y_0 \to R(Y)_0$ given by the identity, so is a weak equivalence.

Proceeding to $R(Y)_1$, we require that the map $R(Y)_1 \to R(Y)_0^2$ intuitively identifying sources and targets of 1-morphisms, given by the maps $\Delta[0] \hookrightarrow \Delta[1]$, is a fibration. Writing **nerve** for the standard quasi-categorical nerve $\mathbf{Cat} \to \mathbf{sSet}$ and again $I[1]$ for the walking category, one obtains a natural factorization of the map $Y_1 \to R(Y)_0$ of the form

$$Y_1 \hookrightarrow Y_0^{\mathbf{nerve}(I[1])} \times_{Y_0} Y_1 \times_{Y_0} Y_0^{\mathbf{nerve}(I[1])} \to Y_0^2 = R(Y)_0^2$$

where the second map is a fibration and the first a trivial cofibration. We thus set $R(Y)_1$ to be this middle space. Intuitively, an element of $R(Y)_1$ is a 1-morphism f with two paths in Y_0, one to its source object x and one from its target object y. We might depict such an element diagrammatically as

$$x' \overset{p}{\rightsquigarrow} x \xrightarrow{f} y \overset{q}{\rightsquigarrow} y'$$

where $f \in Y_1$ and $p, q : \mathbf{nerve}(I[1]) \to Y_0$. The source and target maps in $R(Y)$ identify the objects x' and y' respectively, rather than x and y.

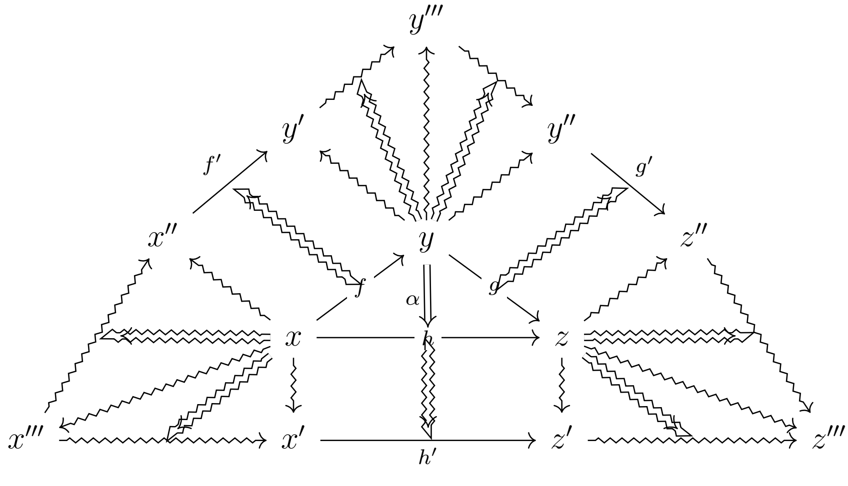
For the degeneracy map, there is an evident morphism $Y_0 \to R(Y)_1$ obtained by a factorization $Y_0 \to Y_1 \to R(Y)_1$. This sends an object to its identity 1-morphism, with constant paths on either end. One may check that the simplicial identities are satisfied.

Stepping up a dimension, writing $\partial\Delta[n]$ for the boundary of the n-simplex $\Delta[n]$, we now consider the induced map $Y_2 \to Y_{\partial\Delta[2]} \to R(Y)_{\partial\Delta[2]}$. Note that this is indeed a step up in induction, as $R(Y)_0^2 \cong R(Y)_{\partial\Delta[1]}$. We again factorize this map to obtain

$$Y_2 \to Y_2 \times_{R(Y)_{\partial\Delta[2]}} R(Y)_{\partial\Delta[2]}^{\mathbf{nerve}(I[1])} \to R(Y)_{\partial\Delta[2]}.$$

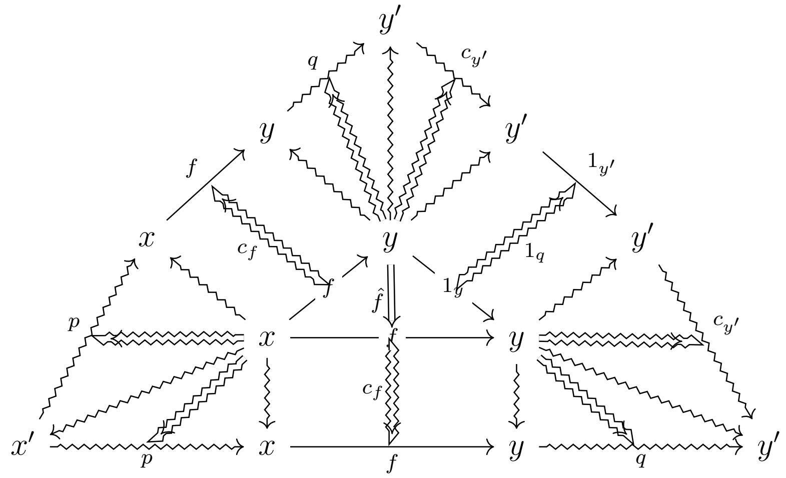
This latter map is a fibration by definition, which is the required property for Reedy fibrancy. Moreover, the former map is a trivial cofibration, so we may set $R(Y)_2$ to be this middle object.

An element in $R(Y)_{\partial\Delta[2]}$ is three elements of $R(Y)_1$ arranged into an oriented triangle. We may thus draw an element of $R(Y)_2$ as a diagram



The map $Y_2 \to R(Y)_2$ then identifies a diagram of this form where all paths and squares are constant.

The two degeneracy maps are now more complex; we cannot simply factor through $Y_2 \to R(Y)_2$, as $R(Y)_1$ now contains extra data that Y_1 does not. A typical degeneracy map will obtain from a triple $(p, f, q) \in R(Y)_1$ a 2-simplex of the form



where $c_{(-)}$ denotes constant paths. The triangular paths in the diagram are then homotopies that retract the paths down to identities along one coordinate.

As one proceeds to higher dimensions, it continues to be possible to inductively set

$$R(Y)_n := Y_n \times_{R(Y)_{\partial\Delta[n]}} R(Y)_{\partial\Delta[n]}^{\mathbf{nerve}(I[1])}.$$

The face maps are straightforward, as is the map $Y_n \to R(Y)_n$, which includes into a constant path of boundaries. The degeneracy maps are more nuanced and require special treatment, each obtained by an ensemble of degenerate homotopies. It is for this last reason that we suspect this construction has not been considered thus far in the literature.

We can generalize the above procedure to a replacement for a 2-fold Segal space X by setting $R(X)_{0,0} := X_{0,0}$ and inductively in n and m defining

$$R(X)_{n,m} := X_{n,m} \times_{R(X)_{\partial\Delta[m,n]}} R(X)_{\partial\Delta[m,n]}^{\mathbf{nerve}(I[1])}.$$

The reversal of m and n in the boundary bisimplex is a matter of convention in this thesis.

Thankfully, $R(X)$ will preserve the Segal maps as weak equivalences and the property of essential constancy, so will indeed be a Reedy fibrant 2-fold Segal space. However, this object is considerably more complicated than X was alone. Note that even each level $R(X)_n$ will have an infinte amount of data appended as one inductively proceeds up the spaces $R(X)_{n,m}$ for increasing m. Considering even low dimensions, such as $R(X)_{3,1}$, proves to be entirely intractable at the present moment. A more concise, perhaps non-inductive, formulation of $R(X)$ would be a useful endeavor in this regard, which we leave to future work.

For the time being, $R(X)$ still serves to have some utility. In the case of $(\infty, 1)$-categories, we demonstrate an alternative method to construct homotopy categories of projective fibrant Segal spaces by taking homotopy categories of their Reedy fibrant replacements. As it turns out, the two homotopy categories $h_1(X)$ and $h_1(R(X))$ are in natural bijection on objects and morphisms, so the complexity of $R(X)$ can be entirely forgotten and its use relegated to simpler proofs of coherence. A similar story thankfully holds for the $(\infty, 2)$ case: for a projective fibrant 2-fold Segal space X, there is a natural isomorphism of categories

$$h_1(\{x\} \times^h_{X_0} X_1 \times^h_{X_0} \{y\}) \cong h_1(R(X)(x, y)).$$

We should perhaps not be surprised, as the objects of both categories are the same, namely 1-morphisms augmented with paths of their source and target objects. With these isomorphisms, we are able to prove that for any choice of inverses to the functors

$$h_1(\{x\} \times^h_{X_0} X_2 \times^h_{X_0} \{z\}) \to h_1(\{x\} \times^h_{X_0} (X_1 \times^h_{X_0} X_1) \times^h_{X_0} \{z\})$$

there is a valid choice of coherence isomorphisms that obtains an unbiased bicategory structure. Though we suspect there is an elementary description of these coherence isomorphisms that does not involve $R(X)$ at all, we have not yet obtained it. We leave this as future work.

In total, we are able to apply our Reedy fibrant replacement functor R to demonstrate the correctness of the conjectured construction of homotopy bicategory for projective fibrant 2-fold Segal spaces of Johnson-Freyd and Scheimbauer in [JS17]. This demonstrates how our approach to homotopy bicategories not only unravels the algebraic data implicit in the $(\infty, 2)$-category in the Reedy fibrant case, but also generalizes to establish conjectured constructions in the projective fibrant case. Moreover, should one be interested, the projective fibrant case is also obtained by solving a natural series of lifting problems that can be concretely described by our model of Reedy fibrant replacement.

We believe there is further potential to this approach. For instance, one should be able to obtain *symmetric monoidal structures* on bicategories in this regard. Consider a symmetric monoidal 2-fold Segal space $X : \Gamma \to \mathbf{SeSp}_2$. The category Γ is certainly not Reedy, though it is a *generalized Reedy category* in the sense of [BM10], a notion developed by Berger and Moerdijk to extend Reedy categories to permit non-trivial automorphisms of objects. If one were to augment our construction of R to generalized Reedy categories, one could instead begin with a functor X that is *generalized Reedy fibrant*, where the required weak equivalences in the monoidal structure are instead trivial fibrations. One then obtains new lifting problems with solutions κ_n

$$\begin{tikzcd}
 & X\langle n\rangle \arrow[d, two heads, "\sim"] \\
\prod_{i=1}^n X\langle 1\rangle \arrow[r, "id"'] \arrow[ur, dashed, "\kappa_n"] & \prod_{i=1}^n X\langle 1\rangle
\end{tikzcd}$$

that serve as natural contenders for a choice of unbiased tensor product. The same calculus of higher homotopies should be possible once again, which should serve to obtain symmetric monoidal homotopy bicategories. Again using our Reedy fibrant replacement functor, these results can be transmitted to the projective fibrant case instantly, producing symmetric monoidal bicategories of cobordisms suitable for use in TQFTs.

1.2.4 To $(\infty, 3)$ and Beyond

We suspect that our means of obtaining unbiased homotopy bicategories of Reedy fibrant 2-fold Segal spaces should continue to extend to higher dimensions, or at least some modification thereof. The general calculus of obtaining higher homotopies as coherence conditions between existing data seems sensible as a path to comparing algebraic and homotopy-theoretic models of higher category. It certainly appears more plausible than attempting to extend those methods discussed thus far that obtain composition operations and coherences after truncation; extending these approaches would involve inverting certain $(n-1)$-equivalences after truncating hom-$(\infty, n-1)$-categories to their homotopy $(n-1)$-categories, which would likely demand explicit computation of all the combinatorial higher algebraic structure of a weak n-category at the level of $(n-1)$-categories. By working at the level of (∞, n)-categories instead, as our approach demonstrates, much of this challenge can be reduced to obtaining homotopies between solutions to lifting problems in a model category, a method not currently available to algebraic notions of n-categories which seem at present to have no model structures proven in the literature.

A particular model that we find promising is that of *Trimble n-categories*, as defined in [Che11]. To add a pinch of formality to our discussion, what we construct from a Reedy fibrant 2-fold Segal space X is an unbiased bicategory whose objects are the elements of the set $(X_{0,0})_0$ and whose hom-categories are of the form $h_1(X(x,y))$ for all objects x, y. Our means of obtaining composition operations immediately takes on an operadic flavor after this point. For a start, simplicial composition diagrams form an operad **SCD** valued in **Cat**, which at level n is the category $\mathbf{SCD}_n$ of diagrams of arity n. The operad map, for

$n \geq 1$ and $k_1, \cdots, k_n \geq 0$ of the form

$$\mathbf{SCD}_n \times \prod_{i=1}^{n} \mathbf{SCD}_{k_i} \to \mathbf{SCD}_{\sum_i k_i}$$

sends diagrams $Q, K_1, \cdots, K_n$ to the diagram

$$(K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_n) \sqcup_{Sp(n)} Q$$

which is easily checked to be a new simplicial composition diagram.

Our above arguments, for $n \geq 0$ and objects $x_0, \cdots, x_n$, amount to the construction of functors

$$\mathbf{SCD}_n \times \prod_{i=1}^{n} h_1(X(x_{i-1}, x_i)) \to h_1(X(x_0, x_n))$$

that abide by an operad law. This law states that, for a tuple of objects

$$Y = ((x_0^1, \cdots, x_{k_1}^1), \cdots, (x_0^n, \cdots, x_{k_n}^n))$$

where $x_{k_i}^i = x_0^{i+1}$ for all $i < n$ and such that flattening Y and removing the extraneous $x_{k_i}^i$'s for $i < n$ produces a tuple $(x_0, \cdots, x_r)$, we have that the diagram

$$\begin{array}{ccc} \mathbf{SCD}_n \times \prod_{i=1}^{n} \left(\mathbf{SCD}_{k_i} \times \prod_{j=1}^{k_i} h_1(X(x_{j-1}^i, x_j^i)) \right) & \longrightarrow & \mathbf{SCD}_r \times \prod_{i=1}^{r} h_1(X(x_{i-1}, x_i)) \\ \downarrow & & \downarrow \\ \mathbf{SCD}_n \times \prod_{i=1}^{n} h_1(X(x_0^i, x_{k_i}^i)) & \longrightarrow & h_1(X(x_0, x_r)) \end{array}$$

commutes. This is precisely the definition of a Trimble 2-category, though with a different choice of operad: in [Che11], the operad in question is the levelwise fundamental groupoid of the *little intervals operad* E, where for $k \geq 0$

$$E_k \subseteq [0, k]^{[0,1]}$$

is the subspace of continuous endpoint-preserving maps $[0, 1] \to [0, k]$, while the operad maps are given by concatenation. An important trait of this operad is that it is *contractible*, so that all coherence isomorphisms and higher conditions indeed form some contractible collection. It is notable then that **SCD** is not even a groupoid. However, the image of every map in $\mathbf{SCD}_n$ is a natural isomorphism, so we could simply consider the groupoid generated by $\mathbf{SCD}_n$ for each n. It remains to be proven that these groupoids, or at least some minor modifications of them, are contractible; we strongly suspect this to be the case.

In order to obtain higher Trimble n-categories, the aim would then be to inductively obtain each h_{n-1} and perform a construction similar to above with coherence conditions going up to dimension n. One would need a suitable 'n-categorification', or perhaps even '(∞, n)-categorification', of **SCD** for this to be possible. This problem is a topic of future work. We suspect it will pull upon some modification of higher *opetopes* [BD98], as simplicial composition diagrams seem to be equivalent to the lowest-dimensional nontrivial case of these. Other modifications to the above methods would also be necessary, such as either introducing a notion of weak n-functor between Trimble n-categories similar to our construction or considering only strict n-functors, that is, maps $X \to Y$ between n-fold Segal spaces commuting with the chosen horizontal compositions. If all of these issues were resolved, it would result in a completely general construction of algebraic homotopy n-categories of homotopy-theoretic (∞, n)-categories for all n.

There are also other models to consider, such as those of Batanin and Leinster [Lei04], though these models will require a substantially different perspective in order to generate suitable lifting problems. We leave this to future work.

1.2.5 Related Constructions

A few alternative constructions of the homotopy bicategory $h_2(X)$ of a complete 2-fold Segal space X are already available in the literature. One of these was obtained by Moser in [Mos21], where a Quillen pair between Lack's model structure on strict 2-categories and the standard model structure for Reedy fibrant complete 2-fold Segal spaces is obtained, the left adjoint of which is a homotopy 2-category functor

$$h_2^{Mos} : \mathbf{CSSP}_2 \to 2\mathbf{Cat}.$$

This approach, more concerned with model category theory than with constructing particular homotopy bicategories, elects to produce strict 2-categories instead of more general weak ones. While indeed every bicategory is equivalent to a strict 2-category, such a construction would not directly obtain an explicitly weak bicategory of manifolds and cobordisms akin to, for instance, the construction of Schommer-Pries in [Sch14b]. For this reason, we elect not to proceed with the functor h_2^{Mos} for our specific purposes. It should be noted that this construction is in fact well-defined on any functor $X : (\Delta^{op})^2 \to \mathbf{sSet}$ rather than just complete 2-fold Segal spaces, though still produces strict 2-categories in this general case.

Another approach is to be found via the work of Campbell [Cam20], which demonstrates a method to produce from a *2-quasicategory* C, a homotopy-theoretic model of $(\infty, 2)$-category, a corresponding *Tamsamani 2-category*

$h_2^{Camp}(C)$, a model of bicategory. In order to understand 2-quasicategories, we first need a quick definition:

Definition 1.2.1 ([Rez10, pg. 11]). *Suppose C is a category. Define ΘC to be the category whose objects are tuples $([m], c_1, \cdots, c_m)$ where $[m] \in \Delta$ and $c_i \in C$. Morphisms between $([m], c_1, \cdots, c_m)$ and $([n], d_1, \cdots, d_n)$ consist of*

1. *A morphism $f : [m] \to [n]$ in Δ and*
2. *For each $1 \leq i \leq m$ and $1 \leq j \leq n$ such that $f(i-1) \leq j \leq f(i)$, a map $f_{ij} : c_i \to d_j$ in C.*

We should think of $\Theta_2 := \Theta\Delta$ as the category of 2-*dimensional pasting diagrams*, which are a special class of 2-categories that represent 'arities' of 2-categorical composition operations. A typical element is of the form $([m], [n_1], \cdots, [n_m])$ for $m, n_1, \cdots, n_m \geq 0$. As an example, we should interpret $([4], [3], [1], [0], [2])$ as the 2-category

$$\begin{tikzcd}
\bullet \arrow[r, bend left=80] \arrow[r, bend left=30] \arrow[r, bend right=30] \arrow[r, bend right=80] \arrow[r, phantom, shift left=4ex, "\Downarrow"] \arrow[r, phantom, "\Downarrow"] \arrow[r, phantom, shift right=4ex, "\Downarrow"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \arrow[r] & \bullet \arrow[r, bend left] \arrow[r] \arrow[r, bend right] \arrow[r, phantom, shift left=2ex, "\Downarrow"] \arrow[r, phantom, shift right=2ex, "\Downarrow"] & \bullet
\end{tikzcd}$$

A 2-quasicategory is then a presheaf $\Theta_2^{op} \to \mathbf{Set}$ satisfying certain natural conditions we will not elaborate on here.

By work of Tamsamani [Tam99] that was later refined by Lack and Paoli in [LP08], one may then convert a Tamsamani 2-category Q into a fully weak bicategory $\tilde{Q}$, as is done in [Cam20]. Of course, one must somehow first convert a complete 2-fold Segal space X into an equivalent 2-quasicategory. A web of functors that have been already studied is possible to piece together to this effect[2] of the form

$$\begin{tikzcd}
\mathbf{sSet}^{(\Delta^{op})^2} \arrow[r, bend left=40, "d_*"] \arrow[r, bend right=40, "d_\#"'] \arrow[r, phantom, shift left=2ex, "\top"] \arrow[r, phantom, shift right=2ex, "\top"] & \mathbf{sSet}^{\Theta_2^{op}} \arrow[l, "d^*" description] \arrow[r, bend left=15, "\mathbf{Real}_D"] \arrow[r, phantom, "\perp"] & \mathbf{Set}^{\Theta_2^{op}} \arrow[l, bend left=15, "\mathbf{Sing}_D"] \arrow[r, bend right=15, "d^*"'] \arrow[r, phantom, "\top"] \arrow[rd, "h_2^{Camp}"'] & \mathbf{sSet}^{\Delta^{op}} \arrow[l, bend right=15, "d_*"'] \arrow[d, "\tau^{\Delta^{op}}"] \\
& & & \mathbf{Cat}^{\Delta^{op}} \arrow[d, "(\tilde{-})"] \\
& & & \mathbf{Bicat}
\end{tikzcd}$$

which draws upon the following functors:

[2] I learned about much of this chain of equivalences, along with citations that gave more of this chain, through the fantastic *Online Workshop on $(\infty, 2)$-Categories* run by Lyne Moser, Nima Rasekh and Martina Rovelli. I express my gratitude to all the organizers and participants for this event and the many things I learned from it.

- $d : \Delta \times \Delta \to \Theta_2$ is the functor sending $([n], [m])$ to the pasting diagram $([n], [m], \cdots, [m])$;
- $\mathbf{Real}_D$ and $\mathbf{Sing}_D$ are defined in [Ara14];
- $\tau : \mathbf{sSet} \to \mathbf{Cat}$ is the left adjoint to the nerve functor $\mathbf{nerve} : \mathbf{Cat} \to \mathbf{sSet}$. It restricts to the *homotopy (1-)category* functor for quasicategories.

The rest of the functors are induced; d^* is a pullback map, while d_* and $d_\#$ are induced right and left adjoints. Many of the adjoint pairs of functors depicted above have been proven to form Quillen pairs on reasonable model structures [BR20] [Ara14] [Cam20].

There are a few nontrivial subtleties to this picture if we wanted to compose some chain of these maps to obtain homotopy bicategories. The first is that there seems to be no sensible path of *left* adjoints from $\mathbf{sSet}^{(\Delta^2)^{op}}$ to $\mathbf{Set}^{\Theta_2^{op}}$. One might desire such a chain as, in general, homotopy bicategory functors should form the *left* adjoints to some 'nerve' functor in the opposite direction, as is the case for the constructions of Moser [Mos21] and Campbell [Cam20]. One might believe that the composite $\mathbf{Real}_D \circ d_\#$ might be a viable option. Unfortunately, the left adjoint $d_\#$ to d^* is Quillen on the wrong model structure. Indeed, it is left Quillen on the *projective* model structures rather than the *injective* ones [BR20, pg. 5-6], while $\mathbf{Real}_D$ is Quillen on the *injective* model structure on $\mathbf{sSet}^{\Theta_2^{op}}$ [Ara14] [BR13b]. One might remedy this with intermediate injective fibrant replacements, though this is a nontrivial step. Moreover, one should not expect fibrant replacement functors to be left Quillen, which means one functor in the chain may not be a left adjoint. A composite of such functors appears less likely to be a left adjoint itself, which may lead to challenges further down the line.

Another obstacle stems from the fact that $\mathbf{Bord}_{n,2}$ is not Reedy fibrant. A Reedy fibrant replacement is thus necessary at the start of the chain, as many of these constructions only work on Reedy model structures. We will address sensible Reedy fibrant replacement functors in time as a part of our own constructions of homotopy bicategories. Moreover, its lack of completeness means the above constructions must be expected to work without a completeness assumption or require the incorporation of a completion functor.

Despite the challenges thus far mentioned, it is perfectly plausible that a construction of this form could yield a useful notion of homotopy bicategory for 2-fold Segal spaces. We leave investigating such an approach and comparisons thereof with the results of this thesis to future work.

Finally, it should be mentioned that if a sensible chain of functors is found across the above diagram and a homotopy bicategory functor is obtained therein, the resulting construction will necessarily present similar issues to the approach conjectured in [JS17]. Indeed, the final functor from Tamsamani 2-categories to classical bicategories detailed in [LP08] chooses composition operations in essentially the same manner, so the final composition operations will be chosen after taking homotopy categories rather than before. Regardless, if successfully investigated, this approach may still serve as a useful model of homotopy bicategory for TQFTs. We leave exploration of possible applications for such an alternative approach to future work.

1.3 Outline

This thesis is organized into three major chapters, with one final short chapter on applications and examples. Most of chapters 2 and 3, along with the discussion of fundamental bigroupoids in chapter 5, are all taken directly from the author's preprint [Rom23], albeit with various modifications and corrections.

Chapter 2 covers all relevant background material; it assumes some aptitude with category theory, bicategories and model structures. Simplicial sets and higher simplicial spaces, Reedy, injective and projective model structures, left Bousfield localization, tensors and cotensors, homotopy limits and the model structures for both Reedy and projective fibrant (complete) n-fold Segal spaces for $n = 1, 2$ are covered in this chapter. Unbiased bicategories are then defined along with unbiased pseudofunctors, together with an established functor which sends an unbiased bicategory to an equivalent biased one. Finally, the notion of an equivalence of unbiased bicategories is presented.

Chapter 3 then focuses on the construction of unbiased homotopy bicategories. It begins with a construction of homotopy bicategories after truncating to homotopy categories of mapping spaces, using methods analogous to those used by Lack and Paoli in [LP08]. Issues are noted with this approach, motivating the search for the data of a homotopy bicategory within the original $(\infty, 2)$-category. To obtain this data, the theory of simplicial composition diagrams and choices of horizontal compositions is then developed, culminating in the construction of the operad **SCD**. A detailed treatment of left homotopies and globular left 2-homotopies for Reedy fibrant Segal spaces then follows, establishing how the homotopy category functor h_1 sends left homotopies to natural isomorphisms, how to concatenate certain left homotopies and identifying situations when globular 2-homotopies exist between left homotopies. The construction of h_2 is then described in full; a great deal of technical work is needed to show how to functorially concatenate and compose left homotopies in an operadic manner, such that the resulting left homotopy solves certain useful lifting problems. The case of unbiased pseudofunctors is then treated, resulting in a completely functorial construction that is invariant under choice of horizontal composition up to canonical isomorphism. To end the chapter, the two constructions of homotopy bicategory are shown to precisely agree, up to some potential discrepancy in certain coherence isomorphisms. We conjecture equality on these as well.

Chapter 4 presents our Reedy fibrant replacement functor. The construction is first given in full generality over any Reedy category, which is later specialized to n-fold Segal spaces for $n = 1, 2$. We analyze the behavior of our replacement functor on various example Reedy categories to better understand our construction. We then demonstrate how to apply our Reedy fibrant replacement functors to understand homotopy categories of projective fibrant Segal spaces, as an illustration of our construction's efficacy. Finally, we turn to the approach of [JS17, Def. 2.12] to defining homotopy bicategories of projective fibrant 2-fold Segal spaces and use our Reedy fibrant replacement functor to prove its consistency.

Chapter 5 finally demonstrates some applications of our homotopy bicategory constructions and examples of them in action. First, we demonstrate how our construction functorially obtains the fundamental bigroupoid of a topological space. Next, we consider a new characterization of equivalences between complete 2-fold Segal spaces in both the projective fibrant and Reedy fibrant cases, akin to Dwyer-Kan equivalences though now employing homotopy bicategories. Finally, we give a characterization of completeness for Reedy fibrant 2-fold Segal spaces using homotopy bicategories. The thesis then concludes with some remarks on potential avenues for future work.

1.4 Notation and Conventions

We adopt a few notational conventions throughout this thesis.

In general, we write $\mathscr{C}$ or $\mathscr{D}$ to refer to generic categories. We will write $\mathbf{Fun}(\mathscr{C}, \mathscr{D})$ or $\mathscr{D}^{\mathscr{C}}$ interchangeably to refer to the category of functors and natural transformations for two categories. In general, in a category with exponents, we write Y^X for the inner hom object.

Δ will denote the *simplicial category* of finite ordinals with weakly monotonic maps for morphisms. We use the notation

$$[n] := \{0 < \cdots < n\}$$

to refer to the object of size n in Δ. Maps in Δ are written in turn as

$$\langle i_1, \cdots, i_n \rangle_m : [n] \to [m]$$

for $0 \leq i_k \leq m$, which denotes the map sending $j \mapsto i_j$ for all $0 \leq j \leq n$. The only requirement on these maps is that $i_k \leq i_{k+1}$ for all $0 \leq k < n$. Sometimes, we will omit the subscript and simply write $\langle i_1, \cdots, i_n \rangle$.

We write **Cat** for the category of small categories and functors, and **Bicat** for the category of small bicategories and pseudofunctors. We write **Set** for the category of sets, **Top** for a 'convenient category of topological spaces' which we will define more precisely in due course and $\mathbf{sSet} := \mathbf{Set}^{\Delta^{op}}$ for the category of simplicial sets. For a simplicial set $X : \Delta^{op} \to \mathbf{Set}$, we write $X_n := X([n])$ and $X_f := X(f) : X_n \to X_m$ for a map $f : [m] \to [n]$ in Δ.

We will write $\mathbf{sSpace}_n := \mathbf{sSet}^{(\Delta^{op})^n}$ to refer to the category of *n-uple simplicial spaces*. In general, for an n-uple simplicial space $X : (\Delta^{op})^n \to \mathbf{sSet}$ we write

$$(X_{k_1,\cdots,k_n})_m := \Big(X([k_n], \cdots, [k_1])\Big)([m]).$$

Note the reversal of indices $k_1, \cdots, k_n$. This may, by hom-tensor adjunction, be alternatively seen as a functor $X : (\Delta^{op})^{n+1} \cong \Delta^{op} \times (\Delta^{op})^n \to \mathbf{Set}$. In this case, we write

$$(X_{k_1,\cdots,k_n})_m := X([m], [k_n], \cdots, [k_1]).$$

We will also write, for $r < n$, that

$$X_{k_1,\cdots,k_r} := X_{k_1,\cdots,k_r,\underbrace{\bullet,\cdots,\bullet}_{n-r}} : (\Delta^{op})^{n-r} \to \mathbf{sSet}.$$

In particular, we have that $X_k = X_{k,\bullet,\cdots,\bullet}$. Moreover, we have that

$$(X_{k_1,\cdots,k_r})_{k_{r+1},\cdots,k_n} = X_{k_1,\cdots,k_r,k_{r+1},\cdots,k_n}.$$

We write $I[n]$ to refer to the groupoid with $n+1$ objects of the form $\{0, \cdots, n\}$, which has precisely one morphism between any two objects. In general, we will call a category or groupoid *contractible* if every hom-set is a singleton. A contractible category $\mathscr{C}$ is always a groupoid and moreover is equivalent to a terminal category $*$ by the terminal map $\mathscr{C} \to *$. Evidently, a category is contractible if and only if the terminal map is an equivalence of categories. Note that we in general refer to *the* terminal object $*$ and *the* terminal map $X \to *$ from X in any category, even though terminal objects are in general only unique up to unique isomorphism.

At times, we will describe a property for either a functor $X : \mathscr{C} \to \mathscr{D}$ or a natural transformation $f : X \Rightarrow Y$ between such functors as being *levelwise*. By this, we mean for every $c \in \mathscr{C}$ the property holds either of $X(c)$ or $f_c : X(c) \to Y(c)$. For instance, one can compute the product $X \times Y$ of two simplicial sets $X, Y : \Delta^{op} \to \mathbf{Set}$ 'levelwise', meaning to compute $(X \times Y)(c) := X(c) \times Y(c)$ for each $c \in \Delta^{op}$.

We employ the framework of model categories developed by Hirschhorn in [Hir09] throughout this thesis. In particular, this comes with the assumption that all model categories, which we generally write as $\mathscr{M}$, are equipped with chosen functorial factorizations for all morphisms.

Chapter 2

Background

Some technical background is necessary before we can begin discussing the constructions of this thesis. In order to understand the model of (∞, n)-category we will be using repeatedly, namely complete n-fold Segal spaces, we will require some aptitude with simplicial sets and the construction of model structures. We will also need some understanding of lower dimensional category theory, in particular bicategories and unbiased bicategories.

For the rest of this thesis, we assume the following:

Definition 2.0.1. *Let **Top** be the category of k-spaces, as in [Ste67].*

Recall that a k-space is a Hausdorff space X such that, for every closed subset $K \subseteq X$, if $Q \subseteq K$ is such that $Q \cap C \neq \emptyset$ for all compact $C \subseteq K$, then Q is itself closed [Ste67, pg. 2].

Our restriction to k-spaces is motivated by a desire for Cartesian closedness. Indeed, our chosen definition of **Top** is Cartesian closed [Wyl73] and suitably rich to admit a model structure [Hir09, Thm. 7.10.10], making it an especially convenient workspace for our homotopy theoretic needs.

We should clarify now that there are no new results anywhere in this chapter. There are many ways to go about describing complete 2-fold Segal spaces; our choices are equivalent to the others in the literature, such as the Reedy fibrant definitions given in [BR20] and [Mos21] and the projective fibrant case studied in [JS17]. What is original here is merely a matter of notation, reaching the same constructions by analogous means.

2.1 Simplicial Sets and Spaces

Simplicial sets are the fundamental building blocks for many models of (∞, n)-category theory. The model we will use is no different; in the end, a complete n-fold Segal space is little more than a diagram of simplicial sets that one might interpret as representing some (∞, n)-category. The diagram is required to satisfy some properties that make the interpretation more believable, a feat accomplished with the technology of model structures. Before we explore such methods, we must understand how to construct and interpret the diagram itself. A solid background of simplicial sets is therefore our first priority.

A simplicial set, in essence, is a geometric object built by gluing together higher-dimensional tetrahedra. Simplicial sets share many similarities with CW-complexes, including in their homotopy theory. We will touch on this point more formally when we discuss model categories. For now, the basic shapes one should keep in mind are the following:

Definition 2.1.1 ([GJ09c, pg. 3])**.** *Let* $n \geq 0$*. The* topological n-simplex *is the space*

$$\Delta_t[n] := \left\{(x_0, \cdots, x_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} x_i = 1, \forall i\ x_i \geq 0\right\} \subseteq \mathbb{R}^{n+1}$$

equipped with the subspace topology.

A few illustrations of low-dimensional cases should elucidate why we should see these as analogous to tetrahedra in all dimensions:

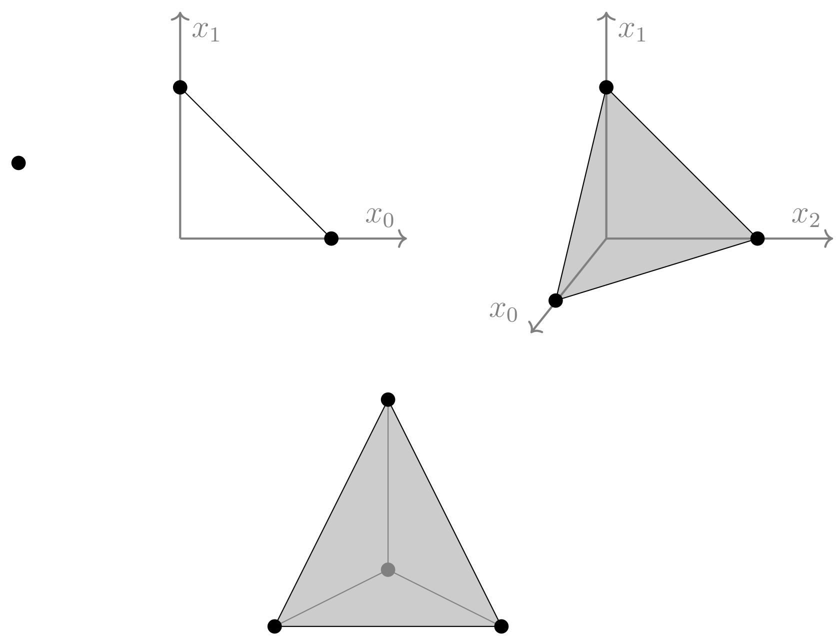


Figure 2.1: The spaces $\Delta_t[n]$ for $n = 0, 1, 2$ from left to right. $\Delta_t[3]$ is depicted below.

Of course, we cannot faithfully represent $\mathbb{R}^4$ in any straightforward way on two-dimensional paper, so visualizing $\Delta_t[3]$ as a tetrahedron will have to do.

One might note that the simplices depicted above fit into one another as vertices, edges and faces. For instance, there are three natural embeddings of $\Delta_t[1]$ into $\Delta_t[2]$ identifying the edges of the topological 2-simplex of the form $d_2^0, d_2^1, d_2^2 : \Delta_t[1] \to \Delta_t[2]$ defined by

$$d_2^0(x_0, x_1) := (0, x_0, x_1)$$
$$d_2^1(x_0, x_1) := (x_0, 0, x_1)$$
$$d_2^2(x_0, x_1) := (x_0, x_1, 0)$$

for all $(x_0, x_1) \in \Delta_t[1]$. These are really just special cases of the affine maps

$$\langle i_0, \cdots, i_n \rangle_m : \Delta_t[n] \to \Delta_t[m]$$

that send $e_n^j \mapsto e_m^{i_j}$, where $e_n^i \in \Delta_t[n]$ is the point such that $(e_n^i)_j = \delta_j^i$ is the Kronecker delta. For instance, $e_2^0 = (1, 0, 0)$ and $e_4^2 = (0, 0, 1, 0, 0)$. In particular, we have $d_2^0 = \langle 1, 2 \rangle_2$, $d_2^1 = \langle 0, 2 \rangle_2$ and $d_2^2 = \langle 0, 1 \rangle_2$.

We will only be interested in maps $f : \Delta_t[n] \to \Delta_t[m]$ that are affine, preserve vertices and preserve an *ordering* on vertices. More precisely, we only care for maps f such that if $i \leq j$, then $f(e_n^i) = e_m^{f_i}$ and $f(e_n^j) = e_m^{f_j}$ such that $f_i \leq f_j$. Hence, the full ranges of maps we will care for are ones of the form

$$f = \langle f_0, \cdots, f_n \rangle_m, \;\; i \leq j \Rightarrow f_i \leq f_j.$$

It should be imagined that our simplices have a particular orientation they prefer. This means we exclude any automorphisms of the n-simplex that permute the vertices, such as $\langle 1, 0 \rangle_1 : \Delta_t[1] \to \Delta_t[1]$ sending $(x_0, x_1) \mapsto (x_1, x_0)$. One might imagine depicting topological n-simplices with arrows on their edges:

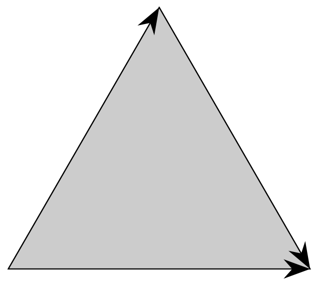


Figure 2.2: The topological 2-simplex.

We can safely abstract such simplices and maps between them by considering only the vertices and the behavior of the maps on these vertices. We obtain the following category:

Definition 2.1.2 ([GJ09c, pg. 3]). *The* simplicial category Δ *is the category whose objects are the ordered sets*

$$[n] := \{0 \leq 1 \leq \cdots \leq n-1 \leq n\}$$

for $n \in \mathbb{Z}_{\geq 0}$ and whose morphisms are the maps

$$\langle f_0, \cdots, f_n \rangle_m : [n] \to [m]$$

sending $i \mapsto f_i$, where $i \leq j \Rightarrow f_i \leq f_j$.

Proposition 2.1.3 ([GJ09c, pg. 3])**.** *There is a functor $\bullet_t : \Delta \to \textbf{\textit{Top}}$ sending $[n] \mapsto \Delta_t[n]$ and $\langle f_0, \cdots, f_n \rangle_m : [n] \to [m]$ to the analogous map $\langle f_0, \cdots, f_n \rangle_m : \Delta_t[n] \to \Delta_t[m]$.*

Proof. $\Delta_t[n]$ is clearly a k-space, so the functor is well-defined. Functoriality is then trivial. □

It should be noted that Goerss and Jardine do not make mention of using a convenient category of topological spaces such as k-spaces, as we do.

As promised, a simplicial set will then simply be an arbitrary gluing of objects in Δ, namely an object in Δ's cocompletion, which we may model with the category of presheaves $\mathbf{Set}^{\Delta^{op}}$:

Definition 2.1.4 ([GJ09c, pg. 3])**.** *A* simplicial set *is a presheaf $\Delta^{op} \to \textbf{\textit{Set}}$. The category of simplicial sets $\textbf{\textit{sSet}}$ is the presheaf category $\textbf{\textit{Set}}^{\Delta^{op}}$.*

Notation 2.1.5. *Let X be a simplicial set. For $[n] \in \Delta$, write $X_n := X([n]) \in \textbf{\textit{Set}}$ for the set of n-*simplices *in X. For $\langle i_0, \cdots, i_n \rangle_m : [n] \to [m]$, write $X_{\langle i_0, \cdots, i_n \rangle} : X_m \to X_n$.*

Note that m is omitted in the latter notation above. It will either be clear from context or stated explicitly henceforth.

There are a few particularly important maps in Δ one should always keep in mind. These include the maps d_2^i we explored earlier, identifying the faces of a 2-simplex, along with a totally new class of maps called the *degeneracies*, which identify two adjacent vertices and collapse an n-simplex into an $(n-1)$-simplex by doing so:

Definition 2.1.6 ([GJ09c, pg. 4])**.** *The* cofaces *$d_n^i : [n-1] \to [n]$ for $n \geq 1$ and $0 \leq i \leq n$ are the maps*

$$d_n^i := \langle 0, \cdots, i-1, \hat{i}, i+1, \cdots, n \rangle_n$$

sending j to j if $j < i$ or $j+1$ otherwise.

The codegeneracies *$s_n^i : [n+1] \to [n]$ for $n \geq 0$ and $0 \leq i \leq n$ are the maps*

$$s_n^i := \langle 0, \cdots, i-1, i, i, i+1, \cdots, n \rangle_n$$

sending j to j if $j \leq i$ or $j-1$ otherwise.

These satisfy a number of basic identities, called the *cosimplicial identities*:

Proposition 2.1.7 ([GJ09c, pg. 4]). *The following identities hold in Δ:*

$$d_n^j d_{n-1}^i = \begin{cases} d_n^i d_{n-1}^{j-1} & i < j \\ d_n^{i+1} d_{n-1}^j & i \geq j \end{cases}$$

$$s_n^j d_{n+1}^i = \begin{cases} d_n^i s_{n-1}^{j-1} & i < j \\ 1 & i = j, j+1 \\ d_n^{i-1} s_{n-1}^j & i > j+1 \end{cases}$$

$$s_n^j s_{n+1}^i = \begin{cases} s_n^i s_{n+1}^{j+1} & i \leq j \\ s_n^{i-1} s_{n+1}^j & i > j \end{cases}.$$

Proof. Exercise. □

These identities are actually sufficient to generate a simplicial set, so they are of particular importance.

Proposition 2.1.8 ([GJ09c, pg. 4]). *Let $X_n \in$ **Set** for all $n \geq 0$ and choose a collection of maps*

$$\delta_n^i : X_n \to X_{n-1}, \quad n \geq 1, 0 \leq i \leq n$$

and

$$\sigma_n^i : X_n \to X_{n+1}, \quad n \geq 0, 0 \leq i \leq n.$$

Suppose they satisfy the analogue in Δ^{op} of the identities in Proposition 2.1.7. Then there is a unique simplicial set X where each X_n is as the chosen sets above, $X_{d_n^i} = \delta_n^i$ and $X_{s_n^i} = \sigma_n^i$.

Notation 2.1.9. *For a simplicial set X, we call the maps $X_{d_n^i}$ the* face maps *and $X_{s_n^i}$ the* degeneracy maps.

It is high time we constructed a few example simplicial sets.

Notation 2.1.10. *Let $\Delta[n] \in$ **sSet** be the representable simplicial set corresponding to $[n]$. We call this simplicial set the n*-simplex.

A number of simplicial sets then become available to us.

Definition 2.1.11. *Define the n*-spine *$Sp(n) \in$ **sSet** to be the simplicial set*

$$\underbrace{\Delta[1] \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} \Delta[1]}_{n}$$

for $n \geq 1$. For $n = 0$, let $Sp(0) := \Delta[0]$.

We can imagine the n-spine as a chain of arrows:

$$0 \longrightarrow 1 \longrightarrow \quad \cdots \quad \longrightarrow n-1 \longrightarrow n$$

Note that there is a natural inclusion $Sp(n) \hookrightarrow \Delta[n]$, given by the colimit of the maps $\langle i-1, i\rangle_n : \Delta[1] \to \Delta[n]$ for $1 \le i \le n$.

Definition 2.1.12 ([HM22, pg. 60])**.** *For a simplicial set X, define the* n-skeleton *for $n \ge 0$ to be the minimal sub-simplicial set $\boldsymbol{sk}_n X \subseteq X$ which contains every $x \in X_k$ for $k \le n$.*

We can think of the n-skeleton of a simplicial set as consisting solely of the simplices up to dimension n and their degeneracies. For instance, $\mathbf{sk}_0 Sp(n)$ is just a disjoint union of $n+1$ instances of $\Delta[0]$, while $\mathbf{sk}_1\Delta[2]$ is a hollow triangle. We can generalize this latter example:

Definition 2.1.13. *Define the* boundary *of the n-simplex to be*

$$\partial\Delta[n] := \boldsymbol{sk}_{n-1}\Delta[n].$$

There is of course a natural inclusion $\partial\Delta[n] \hookrightarrow \Delta[n]$. Alternative descriptions are given of $\partial\Delta[n]$ in [HM22, pg. 161] and [GJ09c, pg. 6], though we find Definition 2.1.13 to be more succinct.

There is one last particular simplicial set we will have some use for:

Definition 2.1.14 ([HM22, pg. 161])**.** *Let $n \ge 0$ and $0 \le k \le n$. Define the* (n,k)-horn *or simply* horn Λ^n_k *to be the subobject of $\Delta[n]$ such that, for $p \ge 0$,*

$$(\Lambda^n_k)_p = \Big\{ \langle f_0, \cdots, f_p\rangle_n : [p] \to [n] \mid \exists i \neq k \ \big(\forall j \ i \neq f_j\big) \Big\}.$$

One can also define Λ^n_k to be the maximal sub-simplicial set of $\Delta[n]$ not containing the $(n-1)$-simplex

$$d^k_n : \Delta[n-1] \to \Delta[n].$$

The horns are also subobjects of the boundary $\partial\Delta[n]$. One should see Λ^n_k as consisting of all the faces of $\Delta[n]$ excluding the face d^k_n opposite to the vertex $\langle k\rangle_n$.

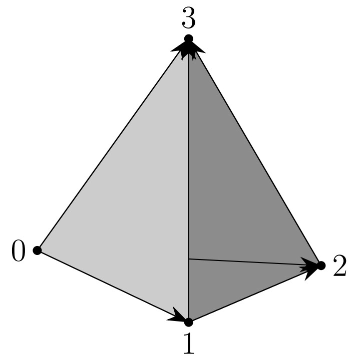


Figure 2.3: The horn Λ^3_0.

One might get a sense from the above image for why these might be named 'horns'; in Λ^3_0 above, the vertex $\langle 0 \rangle$ is the 'mouthpiece', while the 'rim' of the horn is the boundary $\partial\Delta[2] \hookrightarrow \Lambda^3_0$ that naturally fits into the commuting diagram

$$\begin{tikzcd}
\partial\Delta[2] \arrow[r, hook] \arrow[d, hook] & \Lambda^3_0 \arrow[d, hook] \\
\Delta[2] \arrow[r, hook, "d^0_3"'] & \Delta[3]
\end{tikzcd}$$

Note also that $\Lambda^2_1 = Sp(2)$.

With horns, we are now able to establish perhaps the most significant type of simplicial set for our purposes: the *Kan complexes*, which are those simplicial sets that perhaps admit the best interpretation as combinatorial models of topological spaces.

Definition 2.1.15 ([GJ09c, pg. 11])**.** *A* Kan complex *X is a simplicial set such that, for every $n > 0$, for every $0 \leq k \leq n$ and for every map $q : \Lambda^n_k \to X$, there exists a map $p : \Delta[n] \to X$ such that the diagram*

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "q"] \arrow[d, hook] & X \\
\Delta[n] \arrow[ur, "p"'] &
\end{tikzcd}$$

commutes.

There are a few important functors from and to **sSet** that deserve discussion. The first of these, a rich source of Kan complexes and a partial justification for interpreting Kan complexes as homotopy types, comes from the realization that any space $T \in \mathbf{Top}$ naturally induces a simplicial set called the *singular set*, whose n-simplices are the maps $\Delta_t[n] \to T$:

Definition 2.1.16 ([GJ09c, pg. 3])**.** *Define the* singular set *functor* $\boldsymbol{Sing} : \boldsymbol{Top} \to \boldsymbol{sSet}$ *such that* $T \mapsto \boldsymbol{Sing}(T)$ *where for* $n \geq 0$ *we have*

$$\boldsymbol{Sing}(T)_n := \boldsymbol{Hom}_{\boldsymbol{Top}}(\Delta_t[n], T)$$

with simplicial maps given by precomposition. For $f : T \to U$ *in* $\boldsymbol{Top}$*, define* $\boldsymbol{Sing}(f)$ *levelwise by sending* $\Delta_t[n] \to X$ *to* $\Delta_t[n] \to X \xrightarrow{f} Y$*.*

Proposition 2.1.17 ([GJ09c, Lemma 3.3])**.** *For every space* $T \in \boldsymbol{Top}$*, the simplicial set* $\boldsymbol{Sing}(T)$ *is a Kan complex.*

The functor **Sing** is in fact a special case of a general way to obtain certain functors, which always yield a left adjoint:

Proposition 2.1.18 ([MM94, Thm. 2, pg. 41])**.** *Suppose* $A : \mathscr{C} \to \mathscr{E}$ *is a functor from a small category* $\mathscr{C}$ *to a cocomplete category* $\mathscr{E}$*. Consider the Yoneda embedding* $\boldsymbol{y}_{\mathscr{E}} : \mathscr{E} \to \boldsymbol{Set}^{\mathscr{E}^{op}}$*.*

Let $R : \mathscr{E} \to \boldsymbol{Set}^{\mathscr{C}^{op}}$ *be the composite functor*

$$\mathscr{E} \xrightarrow{\boldsymbol{y}_{\mathscr{E}}} \boldsymbol{Set}^{\mathscr{E}^{op}} \xrightarrow{(-)\circ A} \boldsymbol{Set}^{\mathscr{C}^{op}}$$

sending $E \in \mathscr{E}$ *to the presheaf* $R(E)$*, where for* $c \in \mathscr{C}$

$$R(E)(c) := \boldsymbol{Hom}_{\mathscr{E}}(A(c), E).$$

Then R *admits a left adjoint* L*.*

The left adjoint in question is given by a coend

$$L(X) := \int^{c:\mathscr{C}} A(c) \cdot X(c).$$

The situation in Proposition 2.1.18 is in general referred to as a *nerve-realization paradigm* in [Lor21, Prop. 3.2.2]. The functor $A : \mathscr{C} \to \mathscr{E}$ is itself referred to as a *nerve-realization context*, or *NR context* for short [Lor21, Def. 3.2.1].

We can apply this immediately to obtain a left adjoint to **Sing**, by setting A to be the functor $\bullet_t : \Delta \to \mathbf{Top}$ sending $[n]$ to $\Delta_t[n]$ for all $n \geq 0$:

Definition 2.1.19 ([GJ09c, pg. 7])**.** *Define the* geometric realization *functor* $|\cdot| : \boldsymbol{sSet} \to \boldsymbol{Top}$ *to send* X *to the space*

$$|X| := \int^{n \geq 0} \Delta_t[n] \cdot X_n.$$

One should see this functor as forming a topological space from X with an instance of $\Delta_t[n]$ for each n-simplex in X, glued together according to the simplicial maps.

Proposition 2.1.20. *$\textbf{\textit{Sing}}$ is right adjoint to $|\cdot|$.*

Proof. This is immediate by Proposition 2.1.18. □

An alternative proof to the same effect is provided by Goerss and Jardine in [GJ09c, Prop. 2.2, pg. 7].

Another such adjunction is available to us via a totally different interpretation of Δ, one which perhaps loses some geometric color but is more attentive to the orderings of vertices:

Definition 2.1.21. *Define the functor $\bullet_c : \Delta \to \textbf{\textit{Cat}}$ to send $[n]$ to the poset category $[n] := \{0 \to 1 \to \cdots \to (n-1) \to n\}$, with maps sent to appropriate functors.*

One could identify Δ with its image in **Cat**, as this functor is clearly fully faithful. We obtain from $\bullet_c$ a natural functor from categories to simplicial sets:

Definition 2.1.22. *Define the* nerve *functor $\textbf{\textit{nerve}} : \textbf{\textit{Cat}} \to \textbf{\textit{sSet}}$ to be such that, for $\mathscr{C} \in \textbf{\textit{Cat}}$ and $n \geq 0$,*

$$\textbf{\textit{nerve}}(\mathscr{C})_n := \textbf{\textit{Hom}}_{\textbf{\textit{Cat}}}([n], \mathscr{C}).$$

This functor is clearly a right adjoint by Proposition 2.1.18. We will not make use of the left adjoint here, so omit a construction. A description of this left adjoint and an alternative direct definition of the nerve functor are given in [HM22, pg. 65]. Another definition is given in [MM94, pg. 453].

One final construction on simplicial sets sprouts from the apparent connection between simplicial sets and topological spaces. Given a simplicial set X, we should expect to be able to take its set of *path components*:

Definition 2.1.23 ([GJ09c, pg. 61])**.** *Let $X \in \textbf{\textit{sSet}}$. Define $\pi_0(X) \in \textbf{\textit{Set}}$ to be the set*

$$X_0/\cong_X$$

where $\cong_X$ is the equivalence relation generated by declaring $x \cong_X y$ if there is a 1-simplex $f \in X_1$ such that $X_{d_1^1}(f) = x$ and $X_{d_1^0}(f) = y$.

This evidently induces a functor $\pi_0 : \textbf{sSet} \to \textbf{Set}$. Kan complexes enjoy a simpler description of π_0:

Proposition 2.1.24 ([GJ09c, Lemma 6.1])**.** *Suppose X is a Kan complex. Then for $x \in X_0$, we have that $x \cong_X y$ if and only if there is a 1-simplex $f \in X_1$ such that $X_{d_1^1}(f) = x$ and $X_{d_1^0}(f) = y$.*

There is then an immediate classical equivalence with the standard path components functor:

Notation 2.1.25. *Write $\pi_0^t : \boldsymbol{Top} \to \boldsymbol{Set}$ for the topological path components functor, sending $T \in \boldsymbol{Top}$ to its set of path components $\pi_0^t(T)$.*

Proposition 2.1.26. *There is a natural isomorphism $\pi_0 \circ \boldsymbol{Sing} \cong \pi_0^t$.*

Proof. Note that $\pi_0(\mathbf{Sing}(T)) = \mathbf{Sing}(T)_0/\cong_{\mathbf{Sing}(T)}$ and $\pi_0^t(T) = T/\simeq$, where $\simeq$ is the equivalence relation induced on points by paths in T. Thus, these two sets are quotients of the same set under different equivalence relations. We must prove then that $x \cong_{\mathbf{Sing}(T)} y$ if and only if $x \simeq y$.

This is however immediate; $\mathbf{Sing}(T)$ is Kan, so $x \cong_{\mathbf{Sing}(T)} y$ if and only if there is a 1-simplex $f \in \mathbf{Sing}(T)_1$ whose faces are x and y. By definition, such a 1-simplex is a path $f : [0,1] \to T$ such that $f(0) = x$ and $f(1) = y$. □

This holds more generally for higher homotopy groups and so-called *simplicial homotopy groups*, as discussed in [GJ09c, pg. 60]; their methods are in turn more powerful and unneeded for our more simplified situation.

We should perhaps note that, due to the clear similarities between topological spaces and Kan complexes that the above result touches on, we will at times allow ourselves to refer to a 1-simplex in a Kan complex X as a *path* in X, whose source and target are given by the faces. By horn fillings, paths can be concatenated and inverted, though not uniquely. By degeneracies, constant paths always exist.

We will require one final important classical fact about path components:

Proposition 2.1.27. *π_0 preserves finite products.*

Proof. Suppose $X, Y \in \mathbf{sSet}$. We have that $\cong_{X\times Y}$ is generated by elements of $(X \times Y)_1 = X_1 \times Y_1$. Hence, we have that $(x, y) \cong_{X\times Y} (x', y')$ if and only if there is a chain of 0-simplices $(x, y) = (x_0, y_0), \cdots, (x_n, y_n) = (x', y')$ and a chain of 1-simplices (f_i, g_i) for $1 \leq i \leq n$ such that for each $i \geq 1$, either

$$(x_{i-1}, y_{i-1}) = (X_{d_1^1}(f_i), Y_{d_1^1}(g_i)) \qquad (x_i, y_i) = (X_{d_1^0}(f_i), Y_{d_1^0}(g_i))$$

or

$$(x_{i-1}, y_{i-1}) = (X_{d_1^0}(f_i), Y_{d_1^0}(g_i)) \qquad (x_i, y_i) = (X_{d_1^1}(f_i), Y_{d_1^1}(g_i)).$$

By removing those 1-simplices f_i or g_i which are degenerate, we may allow the two chains of maps $f_1, \cdots, f_n$ and $g_1, \cdots, g_m$ to be any two distinct lengths n and m. Thus, $\cong_{X\times Y}$ is precisely the equivalence relation $\cong_X \times \cong_Y$ on $X_0 \times Y_0$. Note finally then that

$$\pi_0(X) \times \pi_0(Y) = (X_0/\cong_X) \times (Y_0/\cong_Y) \cong (X_0 \times Y_0)/(\cong_X \times \cong_Y).$$

□

2.1.1 n-uple Simplicial Spaces

Our next task is to understand the kinds of diagrams in **sSet** we wish to interpret as (∞, n)-categories. For us, these will be *n-uple simplicial spaces*, whose definition is rather straightforward:

Definition 2.1.28 ([CS19, pg. 12])**.** *Let $n \geq 1$. An n-*uple simplicial space *is a functor $X : (\Delta^{op})^n \to$ **sSet**.*

Notation 2.1.29. *Write $\mathbf{sSpace}_n$ for the category of n-uple simplicial spaces with natural transformations between them. Write **sSpace** if $n = 1$ and **ssSpace** if $n = 2$. We will call a 1-uple simplicial space a* simplicial space *and a 2-uple simplicial space a* bisimplicial space.

Clearly $\mathbf{sSpace}_0 \cong \mathbf{sSet}$.

At times, we will want to decompose an n-uple simplicial space $X : (\Delta^{op})^n \to$ **sSet** by hom-tensor adjunction into, for $0 \leq k \leq n$, some functor

$$(\Delta^{op})^k \to \mathbf{sSet}^{(\Delta^{op})^{n-k}} = \mathbf{sSpace}_{n-k}.$$

However, there are many ways we could choose to perform such a decomposition, by applying the hom-tensor adjunction to different instances of Δ^{op} within $(\Delta^{op})^n$. Hence, we should establish some convention of what choices we will generally make.

In this thesis, we choose to prioritize the identification of categories

$$\begin{aligned}
\mathbf{sSpace}_n &= \mathbf{sSet}^{\overbrace{\Delta^{op} \times \cdots \times \Delta^{op}}_{n}} \\
&= \mathbf{sSet}^{\overbrace{\Delta^{op} \times \cdots \times \Delta^{op}}_{n-k} \times \overbrace{\Delta^{op} \times \cdots \times \Delta^{op}}_{k}} \\
&\cong \left(\mathbf{sSet}^{\overbrace{\Delta^{op} \times \cdots \times \Delta^{op}}_{n-k}}\right)^{\overbrace{\Delta^{op} \times \cdots \times \Delta^{op}}_{k}} = \mathbf{sSpace}_{n-k}^{(\Delta^{op})^k}.
\end{aligned}$$

Under this identification, a functor $X \in \mathbf{sSpace}_n$ of the form $(\Delta^{op})^n \to \mathbf{sSet}$ is reinterpreted as a functor $X' : (\Delta^{op})^k \to \mathbf{sSpace}_{n-k}$ such that, for all $i_1, \cdots, i_n \geq 0$, we have an equality

$$\Big(X'([i_k], \cdots, [i_1])\Big)([i_n], \cdots, [i_{k+1}]) = X([i_n], \cdots, [i_1]).$$

We could inductively unravel X in this manner into an object $\overline{X}$ in the category

$$\overline{X} \in (\cdots (\mathbf{sSet}^{\Delta^{op}})^{\Delta^{op}} \cdots)^{\Delta^{op}}$$

such that

$$\Big(\cdots \Big(\overline{X}([i_1])\Big) \cdots\Big)([i_n]) = X([i_n], \cdots, [i_1]).$$

We convert this decision into notation, which will henceforth have the consequence of different descriptions of indices being reversals of one another:

Notation 2.1.30. *Let $X \in \boldsymbol{sSpace}_n$. Write $X_{i_1,\cdots,i_k} := X([i_k],\cdots,[i_1]) \in \boldsymbol{sSpace}_{n-k}$, where X is now interpreted as a functor $(\Delta^{op})^k \to \boldsymbol{sSpace}_{n-k}$ as above.*

We thus have for $X \in \mathbf{sSpace}_n$, for $0 \leq k \leq n$ and for $i_1, \cdots, i_n \geq 0$ that

$$X_{i_1,\cdots,i_n} = (X_{i_1,\cdots,i_k})_{i_{k+1},\cdots,i_n}.$$

Notation 2.1.31. *Write $\Delta[j, i_n, \cdots, i_1] \in \boldsymbol{sSpace}_n$ for the representable preseheaf of $([j],[i_n],\cdots,[i_1]) \in \Delta^{op} \times (\Delta^{op})^n$.*

With this choice, the Yoneda lemma gives us for $X \in \mathbf{sSpace}_n$ that

$$\mathbf{Hom}_{\mathbf{sSpace}_n}(\Delta[j, i_n, \cdots, i_1], X) \cong (X_{i_1,\cdots,i_n})_j.$$

Proposition 2.1.32. *$\boldsymbol{sSpace}_n$ is Cartesian closed.*

This is due to $\mathbf{sSpace}_n$ being a presheaf category, which implies it to be Cartesian closed by [MM94, Prop. 1, pg. 46]. We will write the mapping space for $X, Y \in \mathbf{sSpace}_n$ as Y^X. Levelwise, we have again by [MM94, pg. 46] that

$$((Y^X)_{i_1,\cdots,i_n})_j = \mathbf{Hom}_{\mathbf{sSpace}_n}(X \times \Delta[j, i_n, \cdots, i_1], Y).$$

We will also need some modifications to these mapping spaces before we can proceed. Suppose X is an n-uple simplicial space, so that for all $k \geq 0$, we have $X_k \in \mathbf{sSpace}_{n-1}$. We could interpret X_k as the '$(n-1)$-uple simplicial space of k-simplices in X'. More generally, consider the space

$$X_{Sp(3)} := X_1 \times_{X_0} X_1 \times_{X_0} X_1.$$

This in turn could be seen as the '$(n-1)$-uple simplicial space of 3-spines in X'. In general, we may be interested in the space X_K where $K \in \mathbf{sSet}$. This should be obtained by taking a limit over all the simplices in K. We will have need for more general enrichments of $\mathbf{sSpace}_n$ in $\mathbf{sSpace}_k$ for $k \leq n$, where for instance K is a k-uple simplicial space, so we immediately extend the above situation to this more general case.

Before all else, we will need a few conversions between different levels of simplicial space. Three such conversions will be of particular importance to us for each $0 \leq k \leq n$, of the form

$$\begin{aligned} \rho_n^k &: \mathbf{sSpace}_n \to \mathbf{sSpace}_k \\ F_n^k &: \mathbf{sSpace}_k \to \mathbf{sSpace}_n \\ \iota_n^k &: \mathbf{sSpace}_k \to \mathbf{sSpace}_n. \end{aligned}$$

All of these functors will be identities if $k = n$. Intuitively, ρ_n^k will be a *truncation* functor that forgets everything above level 0 in the discarded dimensions.

F_n^k will produce a *levelwise discrete* higher simplicial space, making all the new dimensions degenerate, while ι_n^k will instead produce a *levelwise constant* higher simplicial space, adding identical copies of the old space in each level of the new dimensions.

More precisely, given an n-uple simplicial space $X \in \mathbf{sSpace}_n$, we may obtain a k-uple simplicial space $\rho_n^k(X) \in \mathbf{sSpace}_k$ from it where

$$(\rho_n^k(X)_{i_1,\cdots,i_k})_j = (X_{0,\cdots,0,i_1,\cdots,i_k})_j.$$

In the opposite direction, given a k-uple simplicial space $Y \in \mathbf{sSpace}_k$, we may obtain an n-uple simplicial space $F_n^k(Y) \in \mathbf{sSpace}_n$ where

$$(F_n^k(Y)_{i_1,\cdots,i_k,i_{k+1},\cdots,i_n})_j = (Y_{i_1,\cdots,i_k})_{i_{k+1}}.$$

This is to say, for fixed $i_1, \cdots, i_{k+1}$, we have that the induced functor

$$(F_n^k(Y))_{i_1,\cdots,i_k,i_{k+1},\bullet,\cdots,\bullet})_\bullet : (\Delta^{op})^{n-k} \times \Delta^{op} \to \mathbf{Set}$$

is constant. For instance, if $K \in \mathbf{sSet}$, note that $F_n^0(K)$ is levelwise a constant $(n-1)$-uple simplicial space whose image is a constant simplicial set; for all $m \geq 0$, we have that $F_n^0(K)_m$ is the constant space defined by the set K_m.

Finally, in the case of ι_n^k, if we have some $Z \in \mathbf{sSpace}_k$, we may obtain from it an n-uple simplicial space $\iota_n^k(Z) \in \mathbf{sSpace}_n$ such that

$$(\iota_n^K(Z)_{i_1,\cdots,i_n})_j = (Z_{i_{n-k+1},\cdots,i_n})_j.$$

We have then for all $i_1, \cdots, i_{n-k}$ and $j_1, \cdots, j_{n-k}$ that

$$\iota_n^K(Z)_{i_1,\cdots,i_{n-k},\underbrace{\bullet,\cdots,\bullet}_k} = \iota_n^K(Z)_{j_1,\cdots,j_{n-k},\underbrace{\bullet,\cdots,\bullet}_k} = Z_{\underbrace{\bullet,\cdots,\bullet}_k}.$$

Note this chain of equalities takes place in $\mathbf{sSet}$.

We now formalize all of these constructions:

Definition 2.1.33. *Let $n \geq 1$. Define the map*

$$\rho_n : \boldsymbol{sSpace}_n \to \boldsymbol{sSpace}_{n-1}$$

so that $\rho_n(X) = X_0$. More precisely, ρ_n is defined by the precomposition map on presheaf categories induced by the functor $\Delta \times \Delta^{n-1} \to \Delta \times \Delta^n$ sending

$$([j], [i_{n-1}], \cdots, [i_1]) \mapsto ([j], [i_{n-1}], \cdots, [i_1], [0]).$$

Notation 2.1.34. *Write*

$$\rho_n^k := \rho_{k+1} \circ \cdots \circ \rho_{n-1} \circ \rho_n.$$

Hence, $\rho_n = \rho_n^{n-1}$. Moreover, set $\rho_n^n = id$.

We see then that ρ_n^k is given by precomposition with the functor $\Delta \times \Delta^k \to \Delta \times \Delta^n$ sending

$$([j], [i_k], \cdots, [i_1]) \mapsto ([j], [i_k], \cdots, [i_1], \underbrace{[0], \cdots, [0]}_{n-k}).$$

Definition 2.1.35. *Let $0 \leq k \leq n$. Define the map*

$$F_n^k : \boldsymbol{sSpace}_k \to \boldsymbol{sSpace}_n$$

such that, for any $K \in \boldsymbol{sSpace}_k$,

$$(F_n^k(K)_{i_1,\cdots,i_n})_{i_{n+1}} = (K_{i_1,\cdots,i_k})_{i_{k+1}}.$$

More formally, this is given by the precomposition map on presheaf categories induced by the functor $\Delta \times \Delta^n \to \Delta \times \Delta^k$ such that

$$([i_{n+1}], [i_n], \cdots, [i_1]) \mapsto ([i_{k+1}], \cdots [i_1]).$$

Definition 2.1.36. *Let $0 \leq k \leq n$. Define the map*

$$\iota_n^k : \boldsymbol{sSpace}_k \to \boldsymbol{sSpace}_n$$

such that, for any $K \in \boldsymbol{sSpace}_k$ and $i_0, \cdots, i_n \geq 0$,

$$(\iota_n^k(K)_{i_1,\cdots,i_n})_j = (K_{i_{n-k+1},\cdots,i_n})_j.$$

More formally, this is given by the precomposition map on presheaf categories induced by the functor $\Delta \times \Delta^n \to \Delta \times \Delta^k$ sending

$$([j], [i_n], \cdots, [i_1]) \mapsto ([j], [i_n], \cdots [i_{n-k+1}]).$$

Proposition 2.1.37. *For any $0 \leq k \leq n$, the functors ρ_n^k, ι_n^k and F_n^k are symmetric monoidal with respect to the Cartesian closed structures.*

Proof. All three maps are pullback functors between categories of presheaves. $\square$

It should be noted that several of these functors appear in [JT07] in the context of simplicial spaces: the functor $\rho_1^0 : \mathbf{sSpace} \to \mathbf{sSet}$ is denoted by i_2^*, while $\iota_1^0 : \mathbf{sSet} \to \mathbf{sSpace}$ is denoted by p_2^*. The map $F_1^0 : \mathbf{sSet} \to \mathbf{sSpace}$ can be written in Joyal and Tierney's notation as $(-)\square 1$.

The significance of ρ_n^k lies in showing concretely how to enrich $\mathbf{sSpace}_n$ in $\mathbf{sSpace}_k$, simply by taking mapping spaces to be $\rho_n^k(Y^X)$:

Notation 2.1.38. *For any $0 \leq k \leq n$ and some $X, Y \in \mathbf{sSpace}_n$, write $\mathbf{Map}_n^k(X,Y) \in \mathbf{sSpace}_k$ for the induced k-uple mapping space between X and Y by ρ_n^k. More concretely, this is*

$$\mathbf{Map}_n^k(X,Y)_{i_1,\cdots,i_k} = (Y^X)_{\underbrace{0,\cdots,0}_{n-k},i_1,\cdots,i_k}.$$

If $n = k$, this is just the inner hom.

Proposition 2.1.39. *For any $0 \leq k \leq n$, the category $\mathbf{sSpace}_n$ is enriched in $\mathbf{sSpace}_k$ with mapping spaces given by $\mathbf{Map}_n^k(-,-)$.*

Proof. Apply the change of base results for enriched categories in [Rie14, Lemma 3.4.3] and the enrichment of $\mathbf{sSpace}_n$ in itself by the Cartesian structure. □

Note that $\mathbf{Map}_n^0$ defines an enrichment of $\mathbf{sSpace}_n$ in simplicial sets. In the case that $k = 0$ and $n = 1$, this is precisely the definition of the enrichment of **sSpace** by **sSet** given in [JT07, pg. 10]. We will see a comparison with another such mapping space in the literature later.

The significance of F_n^k will be its interaction with the representable objects $\Delta[j, i_k, \cdots, i_1]$ in $\mathbf{sSpace}_k$. For example, an enriched analogue of the Yoneda lemma will be possible for n-uple simplicial spaces using F_n^k and $\mathbf{Map}_n^k(-,-)$. We first consider a few simpler results, including some about boundaries of our representable n-uple simplicial spaces; these will be needed once we discuss model structures.

Notation 2.1.40. *Write $\partial\Delta[i_n, \cdots, i_0]$ for the maximal sub-presheaf of the representable presheaf $\Delta[i_n, \cdots, i_0]$ missing the element $(1_{[i_n]}, \cdots, 1_{[i_0]})$.*

Notation 2.1.41. *Write*

$$F_n^k(i_k, \cdots, i_0) = F_n^k(\Delta[i_k, \cdots, i_0])$$
$$\partial F_n^k(i_k, \cdots, i_0) = F_n^k(\partial\Delta[i_k, \cdots, i_0]).$$

Lemma 2.1.42. *There is an isomorphism*

$$F_n^k(i_k, \cdots, i_0) \cong \Delta[0, \cdots, 0, i_k, \cdots, i_0].$$

Proposition 2.1.43. *For all $n \geq 1$ and $i_0, \cdots, i_n \geq 0$, the space $\partial\Delta[i_n, \cdots, i_0]$ is isomorphic to*

$$\Big(\Delta[i_n, \cdots, i_1, 0] \times \partial F_n^0(i_0)\Big) \sqcup_{\partial\Delta[i_n,\cdots,i_1,0]\times\partial F_n^0(i_0)} \Big(\partial\Delta[i_n, \cdots, i_1, 0] \times F_n^0(i_0)\Big).$$

Proof. The pushout contains all maps in Δ^n of the form $([j_n], \cdots, [j_0]) \to ([i_n], \cdots, [i_0])$ such that either $[j_0] \to [i_0]$ is not a surjection or such that the maps $[j_h] \to [i_h]$ are not all surjections for $1 \le h \le n$. Thus, all maps which are not surjections on every coordinate are contained, which is an alternative characterization of $\partial\Delta[i_n, \cdots, i_0]$. $\square$

As promised, there is an interesting complementary interaction between F_n^k and the mapping spaces induced by ρ_n^{n-k-1}:

Proposition 2.1.44. *For any $0 \le k < n$ and $i_0, \cdots, i_k \ge 0$,*

$$\boldsymbol{Map}_n^{n-k-1}(F_n^k(i_k, \cdots, i_0), X) \cong X_{i_0, \cdots, i_k}.$$

Proof. We can compare the two sides levelwise. Note that this isomorphism takes place in $\mathbf{sSpace}_{n-k-1}$. Thus, for $j_1, \cdots, j_{n-k} \ge 0$:

$$\begin{aligned}
&(\mathbf{Map}_n^{n-k-1}(F_n^k(\Delta[i_k, \cdots, i_0]), X)_{j_1, \cdots, j_{n-k-1}})_{j_{n-k}} \\
&= ((X^{F_n^k(\Delta[i_k, \cdots, i_0])})_{0, \cdots, 0, j_1, \cdots, j_{n-k-1}})_{j_{n-k}} \\
&\cong \mathbf{Hom}_{\mathbf{sSpace}_n}(\Delta[j_{n-k}, \cdots, j_1, 0, \cdots, 0], X^{F_n^k(\Delta[i_k, \cdots, i_0])}) \\
&\cong \mathbf{Hom}_{\mathbf{sSpace}_n}(F_n^k(\Delta[i_k, \cdots, i_0]) \times \Delta[j_{n-k}, \cdots, j_1, 0, \cdots, 0], X) \\
&\cong \mathbf{Hom}_{\mathbf{sSpace}_n}(\Delta[0, \cdots, 0, i_k, \cdots, i_0] \times \Delta[j_{n-k}, \cdots, j_1, 0, \cdots, 0], X) \\
&\cong \mathbf{Hom}_{\mathbf{sSpace}_n}(\Delta[j_{n-k}, \cdots, j_1, i_k, \cdots, i_0], X) \\
&\cong ((X_{i_0, \cdots, i_k})_{j_1, \cdots, j_{n-k-1}})_{j_{n-k}}
\end{aligned}$$

which completes the proof. $\square$

The case of $k = 0$ and $n = 1$ is discussed in [Rez00, pg. 6]. More generally, the case $k = 0$ and $n \ge 1$ is discussed in [Rez10, pg. 6]. We will see a proof that these mapping spaces agree with the enrichment in **sSet** Rezk considers in the latter of these sources shortly; that they agree with the former is on the nose.

Finally, we will need a means to *tensor* by lower-dimensional simplicial spaces, wherein lies the substance of ι_n^k:

Notation 2.1.45. *For $0 \le k \le n$, write $-\Box_n^k- : \boldsymbol{sSpace}_n \times \boldsymbol{sSpace}_k \to \boldsymbol{sSpace}_n$ for the functor defined such that, for $X \in \boldsymbol{sSpace}_n$ and $K \in \boldsymbol{sSpace}_k$,*

$$X \Box_n^k K := X \times \iota_n^k(K).$$

For $k = 0$, this agrees with the $\Box$-product given in [GJ09b, pg. 370].

Proposition 2.1.46. *Let $0 \le k \le n$. Then $\iota_n^k : \boldsymbol{sSpace}_k \to \boldsymbol{sSpace}_n$ is a left adjoint to $\rho_n^k : \boldsymbol{sSpace}_n \to \boldsymbol{sSpace}_k$.*

Proof. We will construct a unit and counit for this pair of functors. Note first that

$$\rho_n^k \circ \iota_n^k = 1_{\mathbf{sSpace}_k}$$

so that the unit $\epsilon_K : K \to \rho_n^k(\iota_n^k(K)) = K$ can simply be set to be 1_K. This is clearly a natural transformation and satisfies the required universality property.

For the counit, we have for $N \in \mathbf{sSpace}_n$ that

$$\iota_n^k(\rho_n^k(N))_{i_1,\cdots,i_n} = ()_{\underbrace{0,\cdots,0}_{n-k},i_{n-k+1},\cdots,i_n}.$$

The counit $\mu_N : \iota_n^k(\rho_n^k(N)) \to N$ may be defined levelwise by the maps in $\mathbf{sSpace}_k$ of the form

$$(\mu_N)_{i_1,\cdots,i_{n-k}} := N_{\langle 0,\cdots,0\rangle_0,\cdots,\langle 0,\cdots,0\rangle_0} : N_{0,\cdots,0} \to N_{i_1,\cdots,i_{n-k}}.$$

Note that this is then the natural transformation of the form

$$\mu_{i_1,\cdots,i_{n-k}} := ()_{\langle 0,\cdots,0\rangle_0,\cdots,\langle 0,\cdots,0\rangle_0}$$

so naturality is given immediately in N levelwise. For naturality in $i_1,\cdots,i_{n-k}$, note that the map

$$\langle 0,\cdots,0\rangle_0,\cdots,\langle 0,\cdots,0\rangle_0 : ([i_{n-k}],\cdots,[i_1]) \to ([0],\cdots,[0])$$

is the terminal map from $([i_{n-k}],\cdots,[i_1])$ in Δ^{n-k} and so is the initial map from this object in $(\Delta^{op})^{n-k}$. Thus, for any map $f : ([i_{n-k}],\cdots,[i_1]) \to ([j_{n-k}],\cdots,[j_1])$ in Δ^{n-k}, we have a commuting diagram

$$\begin{tikzcd}
N_{0,\cdots,0} \arrow[r, "1_{N_{0,\cdots,0}}"] \arrow[d, "N_{\langle 0,\cdots,0\rangle,\cdots,\langle 0,\cdots,0\rangle}"'] & N_{0,\cdots,0} \arrow[d, "N_{\langle 0,\cdots,0\rangle,\cdots,\langle 0,\cdots,0\rangle}"] \\
N_{j_1,\cdots,j_{n-k}} \arrow[r, "N_f"'] & N_{i_1,\cdots,i_{n-k}}
\end{tikzcd}$$

as needed. Thus, $\mu : \iota_n^k \circ \rho_n^k \Rightarrow 1_{\mathbf{sSpace}_n}$ is a valid natural transformation.

To prove the required universal property, consider some $K \in \mathbf{sSpace}_k$ and a map $g : \iota_n^k(K) \to N$. We need to show existence of a unique morphism $h : \iota_n^k(K) \to \iota_n^k(\rho_n^k(N))$ which g factors through. We find that the only possible such map is $h := \iota_n^k(g_{0,\cdots,0})$, which factors due to the commutative diagram

$$\begin{tikzcd}
\iota_n^k(K)_{j_1,\cdots,j_{n-k}} = K \arrow[r, "g_{0,\cdots,0}"'] \arrow[rr, bend left, "g_{j_1,\cdots,j_{n-k}}"] \arrow[d, "\iota_n^k(K)_f = 1_K"'] & N_{0,\cdots,0} \arrow[r, "N_{\langle 0,\cdots,0\rangle,\cdots,\langle 0,\cdots,0\rangle}"'] \arrow[d, "1_{N_{0,\cdots,0}}"'] & N_{j_1,\cdots,j_{n-k}} \arrow[d, "N_f"] \\
\iota_n^k(K)_{i_1,\cdots,i_{n-k}} = K \arrow[r, "g_{0,\cdots,0}"] \arrow[rr, bend right, "g_{i_1,\cdots,i_{n-k}}"'] & N_{0,\cdots,0} \arrow[r, "N_{\langle 0,\cdots,0\rangle,\cdots,\langle 0,\cdots,0\rangle}"] & N_{i_1,\cdots,i_{n-k}}
\end{tikzcd}$$

so that this is the counit as needed. $\square$

This result implies that $\mathbf{Map}_n^0(-,-)$ agrees for instance with the simplicial enrichment of $\mathbf{sSpace}_n$ defined by Rezk in [Rez10, pg. 6], where Rezk also notes a more general version of the adjunction between ι_n^0 and ρ_n^0 must hold for *simplicial presheaves*, which generalize n-uple simplicial spaces.

Proposition 2.1.47. *Let $0 \leq k \leq n$. Suppose $W, X, Y \in \boldsymbol{sSpace}_n$. There is a natural isomorphism*

$$\boldsymbol{Map}_n^k(X \times W, Y) \cong \boldsymbol{Map}_n^k(X, Y^W).$$

Proof. This is just the natural isomorphism $Y^{X \times W} \cong (Y^W)^X$ postcomposed with ρ_n^k. □

Proposition 2.1.48. *Let $0 \leq k \leq n$. Suppose $K \in \boldsymbol{sSpace}_k$ and $M \in \boldsymbol{sSpace}_n$. Then there is a natural isomorphism*

$$\rho_n^k(M)^K \cong \rho_n^k(M^{\iota_n^k(K)}).$$

Proof. Consider some $X \in \mathbf{sSpace}_k$. We have that

$$\begin{aligned}\mathbf{Hom}_{\mathbf{sSpace}_k}(X, \rho_n^k(M)^K) &\cong \mathbf{Hom}_{\mathbf{sSpace}_n}(\iota_n^k(X \times K), M)\\ &\cong \mathbf{Hom}_{\mathbf{sSpace}_n}(\iota_n^k(X) \times \iota_n^k(K), M)\\ &\cong \mathbf{Hom}_{\mathbf{sSpace}_n}(\iota_n^k(X), M^{\iota_n^k(K)})\\ &\cong \mathbf{Hom}_{\mathbf{sSpace}_k}(X, \rho_n^k(M^{\iota_n^k(K)})).\end{aligned}$$

Varying over all X completes the proof. □

Proposition 2.1.49. *Let $0 \leq m \leq k \leq n$. Suppose $K \in \boldsymbol{sSpace}_k$ and $X, Y \in \boldsymbol{sSpace}_n$. Then there is a natural isomorphism*

$$\boldsymbol{Map}_n^m(X \square_n^k K, Y) \cong \boldsymbol{Map}_k^m(K, \boldsymbol{Map}_n^k(X, Y)).$$

Proof. We have that

$$\mathbf{Map}_n^m(X \square_n^k K, Y) \cong \rho_n^m(Y^{X \times \iota_n^k(K)}) \cong \rho_n^m((Y^X)^{\iota_n^k(K)}) = \rho_k^m(\rho_n^k((Y^X)^{\iota_n^k(K)}))$$

and

$$\mathbf{Map}_k^m(K, \mathbf{Map}_n^k(X, Y)) \cong \rho_k^m(\rho_n^k(Y^X)^K)$$

so the isomorphism holds by Proposition 2.1.48. □

2.2 Model Category Theory

Central to our chosen definition of $(\infty, 2)$-category is the construction of a suitable model structure. Our intention is to encode $(\infty, 2)$-categories as the fibrant objects in a model structure on **ssSpace**.

The first and most fundamental model structure we build all other results from in this thesis is the following classical model structure on **sSet**:

Theorem 2.2.1 ([GJ09c, Thm. 11.3])**.** *There exists a model structure on **sSet** whose weak equivalences are maps $f : X \to Y$ such that the geometric realization $|f| : |X| \to |Y|$ is a weak equivalence of topological spaces and whose cofibrations are the levelwise monomorphisms.*

We will always assume this model structure on **sSet**. The fibrant objects are the Kan complexes, while the fibrations are more generally the *Kan fibrations*, namely those maps with the right lifting property with respect to all horn inclusions $\Lambda^n_k \hookrightarrow \Delta[n]$ for $0 \leq k \leq n$ and $n \geq 1$ [GJ09c, pg. 10] [Hir09, Def. 7.10.8].

We will at times need the following classical facts:

Proposition 2.2.2 ([GJ09c, Prop. 11.1])**.** *Suppose X is a Kan complex. Then the unit $\eta_X : X \to \mathbf{Sing}(|X|)$ of the adjunction between $|\cdot|$ and **Sing** induces an isomorphism $\pi_0(X) \to \pi_0(\mathbf{Sing}(|X|))$.*

In fact, it is classical that $|\cdot|$ and **Sing** induce a Quillen equivalence between **sSet** and a model structure on **Top** whose weak equivalences are weak homotopy equivalences and whose fibrations are the Serre fibrations. Though we will at times use this more general fact, for the purposes of computations, the details of π_0 are somewhat more pressing.

Corollary 2.2.3. *Suppose X is a Kan complex. Then there is an isomorphism $\pi_0(X) \cong \pi_0^t(|X|)$.*

Proof. Simply consider the chain

$$\pi_0(X) \to \pi_0(\mathbf{Sing}(|X|)) \to \pi_0^t(|X|).$$

□

Corollary 2.2.4. *Suppose $f : X \to Y$ is a weak equivalence in **sSet** between Kan complexes. Then $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is an isomorphism.*

Proof. We have a natural diagram of maps

$$\begin{array}{ccccc}
\pi_0(X) & \longrightarrow & \pi_0(\mathbf{Sing}(|X|)) & \longrightarrow & \pi_0^t(|X|) \\
\downarrow & & \downarrow & & \downarrow \\
\pi_0(Y) & \longrightarrow & \pi_0(\mathbf{Sing}(|Y|)) & \longrightarrow & \pi_0^t(|Y|)
\end{array}$$

where all the horizontal maps are isomorphisms. The rightmost vertical map is an isomorphism by the definition of a weak equivalence between Kan complexes, so the other vertical maps are as well. ☐

Proposition 2.2.5. *Suppose $F : \mathscr{C} \to \mathscr{D}$ is an equivalence of categories. Then the induced map $\boldsymbol{nerve}(F) : \boldsymbol{nerve}(\mathscr{C}) \to \boldsymbol{nerve}(\mathscr{D})$ is a weak equivalence of simplicial sets.*

Proof. An equivalence of categories is sent by **nerve** to a *categorical equivalence* as defined in [Rez22, Sec. 24.1] by the discussion in [Rez22, pg. 58], which is always a weak equivalence as noted in [Rez22, pg. 131]. ☐

There are several insightful resources which explore the model category theory we will make use of in this thesis. The one we will employ the most is Hirschhorn [Hir09], with occasional additions from other sources, like May and Ponto [MP12], Goerss and Jardine [GJ09a], Johnson-Freyd and Scheimbauer [JS17], Hovey [Hov07], Rezk [Rez00] and Bergner [BR13b] [BR20], and several others. It should be noted that [JS17, App. A] explores a great deal of the model category theory relevant to constructing projective complete n-fold Segal spaces, so runs in parallel to our own discussions in many places.

2.2.1 Reedy Model Structures

A model structure on **ssSpace** for Reedy fibrant complete 2-fold Segal spaces is explicitly described by Bergner and Rezk in [BR20], while another for projective fibrant complete 2-fold Segal spaces is described by Johnson-Freyd and Scheimbauer in [JS17]. They are each natural extensions and modifications of the model structure Rezk defined in [Rez00] for Reedy fibrant complete Segal spaces. The general strategy both approaches use is to take a left Bousfield localization of a pre-existing model structure that automatically exists on **ssSpace**, whose weak equivalences are levelwise weak equivalences in **sSet**. Taking such a starting point sets the stage for a truly 'homotopy-theoretic' approach to higher categories, inheriting structure from **sSet** to describe ∞-categorical phenomena in a manner derivative from the homotopy hypothesis. We will discuss this more later.

This initial model structure on **ssSpace** we now consider is known more commonly as the *Reedy model structure*. The most general case is studied in-depth in [Hir09], but we will need only the specific cases of simplicial and bisimplicial spaces. This will not be the only such initial model structure we employ, as is done in [JS17].

Our final chosen approach to defining Reedy fibrant complete 2-fold Segal spaces will be a fusion of the methods in [BR20] and [JS17], where we will begin with the Reedy model structure as in the former but localize against the maps presented in the latter. This keeps us close to the literature on $(\infty, 2)$-categorical TQFTs while reaping the added benefits of Reedy fibrancy we will later need.

We begin with a few definitions with regards to Reedy categories in general, which will later be specialized to the cases of particular interest in this thesis:

Definition 2.2.6 ([Hir09, Def. 15.1.2])**.** *A* Reedy category $\mathscr{C}$ *is a small category equipped with two wide subcategories* $\mathscr{C}^+$ *and* $\mathscr{C}^-$ *called the* direct *and* inverse subcategories *and a function* $\boldsymbol{deg} : \boldsymbol{ob}(\mathscr{C}) \to \mathbb{Z}_{\geq 0}$ *called the* degree function[1]*, such that:*

1. *Every morphism f in $\mathscr{C}$ admits a unique factorization $f = \alpha^+ \circ \alpha^-$, where $\alpha^+ \in \mathscr{C}^+$ and $\alpha^- \in \mathscr{C}^-$;*
2. *For every $\alpha^+ : c \to d$ in $\mathscr{C}^+$ we have $\boldsymbol{deg}(c) \leq \boldsymbol{deg}(d)$ and for every $\alpha^- : a \to b$ in $\mathscr{C}^-$ we have $\boldsymbol{deg}(a) \geq \boldsymbol{deg}(b)$. Equality holds in either case if and only if the morphism in question is an identity.*

Proposition 2.2.7 ([Hir09, Example 15.1.12])**.** *Δ is a Reedy category, with degree map $\boldsymbol{deg}([i]) = i$, Δ^+ the wide subcategory of injections and Δ^- the wide subcategory of surjections.*

Proposition 2.2.8. *$(\Delta^{op})^n$ is a Reedy category, with degree map*

$$\boldsymbol{deg}(([i_1], \cdots, [i_n])) = \sum_j i_j$$

and with $((\Delta^{op})^n)^+ := ((\Delta^-)^{op})^n$ and $((\Delta^{op})^n)^- := ((\Delta^+)^{op})^n$.

Proof. Apply [Hir09, Prop. 15.1.5] and [Hir09, Prop. 15.1.6]. □

For any model category $\mathscr{M}$ and any Reedy category $\mathscr{C}$, the Reedy model structure is a particular model structure on the category $\mathscr{M}^{\mathscr{C}}$ whose weak equivalences are levelwise. The substance of Reedy categories lies in their inherently inductive nature. To see this in action, we need a few definitions:

Definition 2.2.9 ([Hir09, Def. 15.2.3])**.** *Suppose $\mathscr{C}$ is a Reedy category with an object $x \in \mathscr{C}$.*

1. *The* latching category $\partial(\mathscr{C}^+ \downarrow x)$ *is the full subcategory of $(\mathscr{C}^+ \downarrow x)$ containing all objects except for 1_x.*
2. *The* matching category $\partial(x \downarrow \mathscr{C}^-)$ *is the full subcategory of $(x \downarrow \mathscr{C}^-)$ containing all objects except for 1_x.*

[1] Bergner and Rezk in [BR13b] set the codomain of **deg** to be $\mathbb{N}$. This is a trivial difference that will not affect any results.

Definition 2.2.10 ([Hir09, Def. 15.2.5])**.** *Suppose $\mathscr{C}$ is a Reedy category with an object $x \in \mathscr{C}$. Let $\mathscr{M}$ be a model category, with a functor $X : \mathscr{C} \to \mathscr{M}$.*

1. *The* latching object of X at x $L_xX \in \mathscr{M}$ *is the colimit*

$$L_xX := colim_{\partial(\mathscr{C}^+\downarrow x)}X.$$

The latching map of X at x *is the natural map* $L_xX \to X(x)$*.*

2. *The* matching object of X at x $M_xX \in \mathscr{M}$ *is the limit*

$$M_xX := \lim_{\partial(x\downarrow\mathscr{C}^-)} X.$$

The matching map of X at x *is the natural map* $X(x) \to M_xX$*.*

Note that latching objects and matching objects are evidently functorial in X. We thus obtain functors

$$L_x, M_x : \mathscr{M}^{\mathscr{C}} \to \mathscr{M}.$$

The latching and matching maps are in turn natural transformations $L_x \Rightarrow (-)_x$ and $(-)_x \Rightarrow M_x$.

We will introduce the terminology of a *latching-matching map* to refer to the following morphisms:

Definition 2.2.11. *Suppose $\mathscr{C}$ is a Reedy category with an object $x \in \mathscr{C}$. Let $\mathscr{M}$ be a model category, with a functor $X : \mathscr{C} \to \mathscr{M}$.*

The latching-matching map of X at x *is the natural map* $L_xX \to M_xX$ *[Hir09, Rem. 15.2.10].*

This induces a natural transformation $L_x \Rightarrow M_x$. Then, for any $x \in \mathscr{C}$, as noted in [Hir09, Sec. 15.2] there must be a natural commuting diagram

$$\begin{tikzcd} & X_x \arrow[dr] & \\ L_xX \arrow[ur] \arrow[rr] & & M_xX \end{tikzcd}$$

that consists of the latching map, matching map and latching-matching map of X at x. We could of course phrase this as a commuting diagram of natural transformations if we wished.

What is interesting about Reedy categories is that, since there are no non-identity morphisms between objects of the same degree, *any* factorization of the latching-matching map at x is enough to define a functor $X : \mathscr{C} \to \mathscr{M}$ on x:

Proposition 2.2.12 ([Hir09, Rem. 15.2.10])**.** *Suppose $\mathscr{C}$ is a Reedy category and $\mathscr{M}$ is a model category. Write $\mathscr{C}^{<n}$ and $\mathscr{C}^{\leq n}$ for the Reedy subcategories of $\mathscr{C}$ consisting of objects of degree strictly below n or of degree up to n, respectively. Consider a functor $X : \mathscr{C}^{<n} \to \mathscr{M}$.*

For every $x \in \mathscr{C}$ such that $\boldsymbol{deg}(x) = n$, consider some factorization $L_xX \to X'_x \to M_xX$ of the latching-matching map[2] *for X at x. Then these factorizations uniquely determine a functor $X' : \mathscr{C}^{\leq n} \to \mathscr{M}$ such that $X'|_{\mathscr{C}^{<n}} = X$, with latching maps*

$$L_xX' = L_xX \to X'_x$$

and matching maps

$$X'_x \to M_xX = M_xX'$$

given by the factorizations in question.

We can thus inductively define a functor $X : \mathscr{C} \to \mathscr{M}$ by choosing any images for the objects of degree 0 [Hir09, pg. 281], then inductively factorizing the latching-matching maps.

There is an analogous result for inductively defining natural transformations between such functors:

Proposition 2.2.13 ([Hir09, Sec. 15.2.11])**.** *Suppose $\mathscr{C}$ is a Reedy category and $\mathscr{M}$ is a model category. Write $\mathscr{C}^{<n}$ and $\mathscr{C}^{\leq n}$ for the Reedy subcategories of $\mathscr{C}$ consisting of objects of degree strictly below n or of degree up to n, respectively. Consider two functors $X, Y : \mathscr{C} \to \mathscr{M}$ and a natural transformation $\kappa : X|_{\mathscr{C}^{<n}} \Rightarrow Y|_{\mathscr{C}^{<n}}$.*

For every $x \in \mathscr{C}$ such that $\boldsymbol{deg}(x) = n$, consider some map $\kappa'_x : X_x \to Y_x$ such that the diagram

$$\begin{tikzcd}
L_xX \arrow[r] \arrow[d, "L_x\kappa"'] & X_x \arrow[r] \arrow[d, "\kappa'_x"'] & M_xX \arrow[d, "M_x\kappa"] \\
L_xY \arrow[r] & Y_x \arrow[r] & M_xY
\end{tikzcd}$$

commutes[3]*. Then the maps κ'_x uniquely induce a natural transformation $\kappa' : X|_{\mathscr{C}^{\leq n}} \Rightarrow Y|_{\mathscr{C}^{\leq n}}$.*

We can moreover obtain from this an inductive technique to solve *lifting problems* in $\mathscr{M}^{\mathscr{C}}$. This procedure is what informs the definition of a Reedy model structure.

[2]Note that this is well-defined, even though x is not in the domain of X. A more careful treatment of this issue is given by Hirschhorn in [Hir09, Sec. 15.2].

[3]Note that $L_x\kappa$ and $M_x\kappa$ are both well-defined, even though κ is not defined on x yet. Again, a more precise treatment of this issue is given in [Hir09, Sec. 15.2.11].

Corollary 2.2.14 ([Hir09, Sec. 15.2.11])**.** *Suppose $\mathscr{C}$ is a Reedy category and $\mathscr{M}$ is a model category. Write $\mathscr{C}^{<n}$ and $\mathscr{C}^{\leq n}$ for the Reedy subcategories of $\mathscr{C}$ consisting of objects of degree strictly below n or of degree up to n, respectively. Consider four functors $W, X, Y, Z : \mathscr{C} \to \mathscr{M}$ and four natural transformations inducing a commutative diagram in $\mathscr{M}^{\mathscr{C}}$*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & W \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

Suppose moreover that we have a natural transformation $\omega : Y|_{\mathscr{C}^{<n}} \Rightarrow W|_{\mathscr{C}^{<n}}$ commuting with the above diagram when restricted to $\mathscr{C}^{<n}$.

Consider, for each $x \in \mathscr{C}$ with $\boldsymbol{deg}(x) = n$, some choice of map $\omega'_x : Y_x \to W_x$ making the diagram

$$\begin{tikzcd} X_x \sqcup_{L_x X} L_x Y \arrow[r] \arrow[d] & W_x \arrow[d] \\ Y_x \arrow[r] \arrow[ur, dashed, "\omega'_x"] & Z_x \times_{M_x Z} M_x W \end{tikzcd}$$

commute. Then the maps ω'_x uniquely determine an extension ω' of ω to $\mathscr{C}^{\leq n}$ commuting with the above diagram restricted to $\mathscr{C}^{\leq n}$.

It seems natural then to consider a cofibration to be a natural transformation $X \Rightarrow Y$ such that all the maps $X_x \sqcup_{L_x X} L_x Y \to Y_x$ are cofibrations in $\mathscr{M}$. Moreover, one should then take the fibrations to be those transformations such that the maps $X_x \to Y_x \times_{M_x Y} M_x X$ are fibrations in $\mathscr{M}$. As noted in [Hir09, pg. 288], the above corollary motivates the role of these maps in defining a model structure as follows:

Theorem 2.2.15 ([Hir09, Thm. 15.3.4])**.** *Suppose $\mathscr{C}$ is a Reedy category and $\mathscr{M}$ is a model category. Then the following data specifies a model structure on $\mathscr{M}^{\mathscr{C}}$, termed the* Reedy model structure*:*

1. *The weak equivalences $X \to Y$ are levelwise weak equivalences in $\mathscr{M}$;*

2. *The cofibrations $X \to Y$ are the maps such that for all $x \in \mathscr{C}$, the* relative latching maps

$$X_x \sqcup_{L_x X} L_x Y \to Y_x$$

are cofibrations in $\mathscr{M}$;

3. *The fibrations $X \to Y$ are the maps such that for all $x \in \mathscr{C}$, the* relative matching maps
$$X_x \to Y_x \times_{M_x Y} M_x X$$
are fibrations in $\mathscr{M}$.

For the purposes of (∞, n)-categories, we will only discuss the instances of this model structure we will make use of, namely the case where $\mathscr{C} = (\Delta^{op})^n$ for $n \leq 2$. In this situation, the matching objects are straightforward to construct:

Proposition 2.2.16. *Let $X : (\Delta^{op})^k \to \boldsymbol{sSpace}_{n-k}$ be an n-uple simplicial space. Then for $([i_k], \cdots, [i_1]) \in (\Delta^{op})^k$, we have*
$$M_{([i_k],\cdots,[i_1])}X \cong \boldsymbol{Map}_n^{n-k}(\partial F_n^{k-1}(i_k, \cdots, i_1), X).$$

Proof. Note by definition that
$$\partial F_n^{k-1}(i_k, \cdots, i_1) \cong \operatorname{colim}_{\partial\left(([i_k],\cdots,[i_1])\downarrow((\Delta^{op})^k)^-\right)} F_n^{k-1}.$$
Thus, since $\mathbf{Map}_n^{n-k}(-, X)$ sends colimits to limits, the result follows. $\square$

This is an extension of Rezk's description in [Rez00] of the Reedy model structure for simplicial spaces, where $\mathscr{M} = \mathbf{sSet}$ and $C = \Delta^{op}$:

Definition 2.2.17 ([Rez00, pg. 6])**.** *A map $f : X \to Y$ in $\boldsymbol{sSpace}$ is a weak equivalence in the Reedy model structure if and only if it is such levelwise in $\boldsymbol{sSet}$. It is a* Reedy fibration of simplicial spaces *if and only if, for every $n \geq 0$, the map*
$$X_n \to Y_n \times_{\boldsymbol{Map}_1^0(\partial F_1^0(n),Y)} \boldsymbol{Map}_1^0(\partial F_1^0(n), X)$$
is a fibration in $\boldsymbol{sSet}$, namely a Kan fibration.

Evidently, it is the case that this produces a model structure by taking cofibrations to be maps with the left lifting property with respect to Reedy fibrations. An elementary description of this specific model structure is detailed by Goerss and Jardine in [GJ09b].

We can now inductively make a similar statement about the Reedy model structure on n-uple simplicial spaces, in particular where $\mathscr{M} = \mathbf{sSpace}_{n-1}$ and $C = \Delta^{op}$:

Definition 2.2.18. *Let $f : X \to Y$ be a map in $\boldsymbol{sSpace}_n$. Then f is a weak equivalence in the Reedy model structure on $\boldsymbol{sSpace}_n$ if and only if it is levelwise one in $\boldsymbol{sSet}$ and a* Reedy fibration of n-uple simplicial spaces *if and only if, for every $k \geq 0$, the map*
$$X_k \to Y_k \times_{\boldsymbol{Map}_n^{n-1}(\partial F_n^0(k),Y)} \boldsymbol{Map}_n^{n-1}(\partial F_n^0(k), X)$$
is a fibration in the Reedy model structure on $\boldsymbol{sSpace}_{n-1}$.

A more elementary description not invoking Reedy fibrations of simplicial spaces is also possible, by instead taking $\mathscr{M} = \mathbf{sSet}$ and $C = (\Delta^{op})^n$. It is shown by [Hir09, Theorem 15.5.2] that these choices will yield identical model structures on $\mathbf{sSpace}_n$. We thus have:

Proposition 2.2.19. *Let $n \geq 1$. A map $f : X \to Y$ is a Reedy fibration in $\boldsymbol{sSpace}_n$ if and only if, for every $i_1, \cdots, i_n \geq 0$, the map*

$$X_{i_1,\cdots,i_n} \to Y_{i_1,\cdots,i_n} \times_{\boldsymbol{Map}^0_n(\partial F^{n-1}_n(i_n,\cdots,i_1),Y)} \boldsymbol{Map}^0_n(\partial F^{n-1}_n(i_n, \cdots, i_1), X)$$

is a Kan fibration.

Of particular interest to us are *Reedy fibrant* objects, as a special case of these will be our $(\infty,1)$-categories in the case of **sSpace** and $(\infty,2)$-categories for **ssSpace**. In the former case, Rezk notes in [Rez00, pg. 6] that the Reedy fibrant simplicial spaces are just those spaces X such that the map $X_n \to \mathbf{Map}^0_1(\partial F^0_1(n), X)$ is a Kan fibration for all $n \geq 0$. Similarly, we find that a Reedy fibrant bisimplicial space X is one such that the map $X_{n,m} \to \mathbf{Map}^0_2(\partial F^1_2(m,n), X)$ is a Kan fibration for all $n, m \geq 0$. In particular, X_0 and $X_{\bullet,0}$ are both Reedy fibrant simplicial spaces. We alternatively might say that the maps $X_n \to \mathbf{Map}^1_2(\partial F^0_2(n), X)$ are Reedy fibrations of simplicial spaces.

A useful fact about Reedy fibrations is that they are also levelwise Reedy fibrations:

Proposition 2.2.20 ([GJ09b, pg. 366, Cor. 2.6])**.** *Reedy fibrations and cofibrations in $\boldsymbol{sSpace}_n$ are also levelwise Reedy fibrations and cofibrations in $\boldsymbol{sSpace}_{n-1}$.*

In particular, every Reedy fibrant bisimplicial space is levelwise a Reedy fibrant simplicial space, which is then levelwise a Kan complex.

Bergner and Rezk in [BR13b] demonstrate that Δ is in fact an *elegant Reedy category*:

Definition 2.2.21 ([BR13b, Def. 2])**.** *Let $\mathscr{C}$ be a Reedy category and $X : \mathscr{C}^{op} \to \boldsymbol{Set}$ a presheaf. Let $c \in \mathscr{C}$ and $x \in X(c)$. Then x is* degenerate *if and only if there exists some non-identity map $\alpha : c \to d$ in $\mathscr{C}^-$ such that $x = X(\alpha)(y)$ for some $y \in X(d)$. x is likewise* nondegenerate *if it is not degenerate.*

Definition 2.2.22 ([BR13b, Def. 4])**.** *An* elegant Reedy category *$\mathscr{C}$ is a Reedy category such that, for every presheaf $X : \mathscr{C}^{op} \to \boldsymbol{Set}$, every $c \in C$ and $x \in X(c)$, there exists a unique map $\alpha : c \to d$ in $\mathscr{C}^-$ and unique nondegenerate $y \in X(d)$ such that $x = X(\alpha)(y)$.*

Proposition 2.2.23 ([BR13b, Cor. 4.4])**.** *Δ is an elegant Reedy category.*

A consequence of this fact, stated as [BR13b, Prop. 3.10], shows that the Reedy model structures on **ssSpace** and **sSpace** are both just the *injective model structures*, where weak equivalences and cofibrations are levelwise:

Proposition 2.2.24 ([BR13b, Prop. 3.10])**.** *Let $\mathscr{C}$ be an elegant Reedy category and $\mathscr{D}$ a category. Let $\mathscr{M} := \boldsymbol{sSet}^{\mathscr{D}^{op}}$ be the model category of simplicial presheaves endowed with an injective model structure, meaning weak equivalences and cofibrations are levelwise. Then the injective and Reedy model structures on $\mathscr{M}^{\mathscr{C}^{op}}$ coincide.*

The cases of this result we have need for have already been concretely defined, setting $\mathscr{D} = *$ or $\mathscr{D} = \Delta$. What is important for us is the following consequence:

Corollary 2.2.25. *A map $f : X \to Y$ in $\boldsymbol{sSpace}_n$ is a Reedy cofibration if and only $(f_{i_1,\cdots,i_n})_j : (X_{i_i,\cdots,i_n})_j \to (Y_{i_1,\cdots,i_n})_j$ is an inclusion of sets for all $j, i_1, \cdots, i_n \geq 0$.*

Proof. A cofibration in an injective model structure is a levelwise cofibration. Inductively set $\mathscr{M} = \mathbf{sSet}$, $\mathscr{D} = \Delta^k$ and $\mathscr{C} = \Delta$ for $0 \leq k < n$ to obtain that a Reedy cofibration in $\mathbf{sSpace}_n$ is levelwise one in **sSet**. Since cofibrations in **sSet** are levelwise inclusions, this completes the proof. □

Corollary 2.2.26. *Every object in $\boldsymbol{sSpace}_n$ is Reedy cofibrant.*

Another result we will need is with regards to F_n^k:

Proposition 2.2.27. *Let $f : U \to V$ be a Reedy cofibration in $\boldsymbol{sSpace}_k$ with $0 \leq k \leq n$. Then $F_n^k(f)$ is a Reedy cofibration in $\boldsymbol{sSpace}_n$.*

Proof. By Corollary 2.2.25, it will suffice to prove that $(F_n^k(f)_{i_0,\cdots,i_{n-1}})_{i_n}$ is an inclusion of sets for every $i_0, \cdots, i_n \geq 0$. This is just the map $f_{i_0,\cdots,i_k} : (U_{i_0,\cdots,i_{k-1}})_{i_k} \to (V_{i_0,\cdots,i_{k-1}})_{i_k}$. By Corollary 2.2.25, since f is a Reedy cofibration, the result is immediate. □

Note however that *trivial* cofibrations are not necessarily preserved by F_n^k. For instance, $g : \Lambda_1^2 \cong \Delta[1] \sqcup_{\Delta[0]} \Delta[1] \hookrightarrow \Delta[2]$ is a trivial cofibration in **sSet**, but $F_1^0(g)$ is not - indeed, we find that $F_1^0(g)_1$ is a map from a two-element discrete simplicial set, the 1-simplices of the domain, to a three-element discrete simplicial set. This cannot possibly be a weak equivalence, so $F_1^0(g)$ is not a levelwise weak equivalence.

One should note however that ι_n^k does preserve both cofibrations and trivial cofibrations, as it simply produces many copies of the same (trivial) cofibration levelwise. This was noted in another form by Joyal and Tierney in [JT07, Prop. 4.6] for the case of ι_1^0. Hence, ι_n^k and ρ_n^k induce a Quillen pair between the Reedy model structures on $\mathbf{sSpace}_n$ and $\mathbf{sSpace}_k$. We will not need this result, though it is a hint of the relevance of these functors to our Reedy model structures, which we will elaborate upon shortly.

Finally, we should note that one may consider our concrete explorations of Reedy model structures on $\mathbf{sSpace}_n$ in a more general context, if the category $\mathbf{sSpace}_k$ for $k < n$ already has a finer model structure imposed upon it:

Definition 2.2.28. *Let $\mathscr{M}$ be a model category whose underlying category is $\boldsymbol{sSpace}_k$. A* weak equivalence *in the Reedy model structure on $\mathscr{M}^{\Delta^{op}}$ is taken levelwise, while a* Reedy fibration *in the Reedy model structure on $\mathscr{M}^{\Delta^{op}}$ is a map $f : A \to B$ in $\mathscr{M}^{\Delta^{op}}$ such that for all $n \geq 0$, the induced map*

$$A_n \to B_n \times_{\boldsymbol{Map}_2^1(\partial F_2^0(n),B)} \boldsymbol{Map}_2^1(\partial F_2^0(n), A)$$

is a fibration in $\mathscr{M}$.

2.2.2 Projective and Injective Model Structures

The Reedy model structure on a category of functors $\mathscr{M}^{\mathscr{C}}$ is in general sandwiched between two other structures, known as the *projective* and *injective* model structures. We have already briefly interacted with the latter. Neither of these model structures are in general guaranteed to exist, though they will in all circumstances we care for.

Definition 2.2.29 ([Lur08, Def. A.2.8.1])**.** *Suppose $\mathscr{M}$ is a model category and $\mathscr{C}$ is a small category. Then a morphism $f : A \to B$ in $\mathscr{M}^{\mathscr{C}}$ is:*

1. *a* weak equivalence *if it is a weak equivalence levelwise in $\mathscr{M}$;*
2. *an* injective cofibration *if it is a levelwise cofibration in $\mathscr{M}$;*
3. *a* projective fibration *if it is a levelwise fibration in $\mathscr{M}$;*
4. *an* injective fibration *if it satisfies the right lifting property with respect to all injective cofibrations that are also weak equivalences;*
5. *a* projective cofibration *if it satisfies the left lifting property with respect to all projective fibrations that are also weak equivalences.*

It is not necessarily guaranteed that the weak equivalences and injective fibrations and cofibrations, nor the weak equivalences and projective fibrations and cofibrations, will form a model category. However, should $\mathscr{M}$ be a *combinatorial* model category, such model structures do exist:

Proposition 2.2.30 ([Lur08, Prop. A.2.8.2])**.** *Suppose $\mathscr{M}$ is a combinatorial model category [Lur08, Def. A.2.6.1] and $\mathscr{C}$ is a small category. Then there is a model structure on $\mathscr{M}^{\mathscr{C}}$, called the* injective model structure*, whose cofibrations, weak equivalences and fibrations are the injective cofibrations, weak equivalences and injective fibrations respectively.*

Proposition 2.2.31 ([Lur08, Prop. A.2.8.2])**.** *Suppose $\mathscr{M}$ is a combinatorial model category [Lur08, Def. A.2.6.1] and $\mathscr{C}$ is a small category. Then there is a model structure on $\mathscr{M}^{\mathscr{C}}$, called the* projective model structure*, whose cofibrations, weak equivalences and fibrations are the projective cofibrations, weak equivalences and projective fibrations respectively.*

Indeed, in all cases we care for this structure is satisfied. $(\Delta^{op})^n$ is of course a small category, while it is classical that **sSet** is combinatorial. We thus have two model structures immediately on $\mathbf{sSpace}_n$, the injective and projective model structures using $\mathscr{M} = \mathbf{sSet}$ and $\mathscr{C} = (\Delta^{op})^n$. We will distinguish these by the following:

Notation 2.2.32. *Write $\boldsymbol{sSpace}_n^{inj}$ for the model category of n-uple simplicial spaces with the injective model structure. Write $\boldsymbol{sSpace}_n^{proj}$ for the model category of n-uple simplicial spaces with the projective model structure.*

Note of course that $\mathbf{sSpace}_n^{inj}$ is just the Reedy model structure, owing to our discussion on elegant Reedy categories.

The following proposition may be proven by using [Lur08, Prop. A.2.8.7]:

Proposition 2.2.33. *Suppose $\mathscr{M}$ is a combinatorial model category and $\mathscr{C}$ is a small category. Then if f is an injective fibration, it is also a projective fibration. Moreover, if f is a projective cofibration then it is an injective cofibration.*

It will be relevant for us to identify a few useful projective cofibrations.

Proposition 2.2.34 ([Hir09, Def. 11.5.7] [Hir09, Thm. 11.6.1])**.** *Suppose $\mathscr{M}$ is a cofibrantly generated model category and $\mathscr{C}$ is a small category. Then the projective model structure on $\mathscr{M}^{\mathscr{C}}$ exists. Moreover, for every cofibrant object X in $\mathscr{M}$, the objects F_X^x, for $x \in \mathscr{C}$, are projective cofibrant, where for $y \in \mathscr{C}$ we have*

$$F_X^x(y) := \coprod_{\boldsymbol{Hom}_{\mathscr{C}}(x,y)} X$$

and maps induced by postcomposition and 1_X.

Proposition 2.2.35. *For any $0 \leq k < n$ and $i_0, \cdots, i_k \geq 0$, under the identification*

$$\boldsymbol{sSpace}_n \cong \boldsymbol{sSpace}_{n-k-1}^{(\Delta^{op})^{k+1}}$$

the n-uple simplicial space $F_n^k(i_k, \cdots, i_0)$ is isomorphic to $F_{\Delta[0,\cdots,0]}^{([i_k],\cdots,[i_0])}$.

Proof. By Lemma 2.1.42, we have that $F_n^k(i_k, \cdots, i_0)$ is just the representable presheaf $\Delta[0, \cdots, 0, i_k, \cdots, i_0]$. The result then holds by inspection. □

Proposition 2.2.36. *For any $0 \leq k < n$ and $i_0, \cdots, i_k \geq 0$, the n-uple simplicial space $F_n^k(i_k, \cdots, i_0)$ is cofibrant in $\mathbf{sSpace}_n^{proj}$.*

Proof. By Proposition 2.2.35, we have that $F_n^k(i_k, \cdots, i_0)$ is just the object $F_{\Delta[0,\cdots,0]}^{([i_k],\cdots,[i_0])}$. This will be cofibrant if $\Delta[0, \cdots, 0]$ is projective cofibrant in $\mathbf{sSpace}_{n-k-1}$. However, this space is itself just $F_n^{n-1}(0) \cong F_{\Delta[0]}^{([0],\cdots,[0])}$, so is projective cofibrant as $\Delta[0]$ is cofibrant in $\mathbf{sSet}$. □

One important application of projective and injective model structures is the study of *homotopy colimits* and *homotopy limits*, respectively. We will only have need for the latter in this thesis. The significance of the injective model structure in this regard amounts to the following:

Proposition 2.2.37 ([Lur08, Prop. A.2.8.7])**.** *Suppose $\mathscr{M}$ is a combinatorial model category. Let $\mathscr{C}$ be a small category. Then the adjunction*

$$\begin{tikzcd} \mathscr{M} \arrow[r, bend left, "d_{\mathscr{C}}"] \arrow[r, phantom, "\perp"] & \mathscr{M}^{\mathscr{C}} \arrow[l, bend left, "\lim_{\mathscr{C}}"] \end{tikzcd}$$

is a Quillen pair between $\mathscr{M}$ and the injective model structure on $\mathscr{M}^{\mathscr{C}}$, where $d_{\mathscr{C}}$ is the constant functor map.

We moreover have the following characterization of homotopy limits:

Proposition 2.2.38 ([Rie20, Thm. 5.2.6])**.** *Consider a model category $\mathscr{M}$ and a small category $\mathscr{C}$. Then should the injective model structure on $\mathscr{M}^{\mathscr{C}}$ exist, the* homotopy limit *functor*

$$\mathbb{R}\lim_{\mathscr{C}} : \mathscr{M}^{\mathscr{C}} \to \mathscr{M}$$

exists and may be computed as the limit of an injective fibrant replacement of the original diagram.

To make this truly functorial demands a 'functorial injective fibrant replacement,' as is implicit in the use of [Rie20, Cor. 4.2.4] to prove the above result. Such a notion is made precise for example as the *functorial (cofibrant) fibrant approximations* in [Hir09, Def. 8.1.15 (2)], similarly to cofibrant replacement functors. However, as is standard, we will permit 'case-by-case' calculation of such injective fibrant replacements. Indeed, $\lim_{\mathscr{C}}$ will preserve weak equivalences between injectively fibrant diagrams [Lur08, pg. 657]. Thus, any two injective fibrant replacements of a particular diagram $\mathscr{C} \to \mathscr{M}$ will induce weakly equivalent limits, so either will be admissible as a notion of 'homotopy limit.'

2.2.3 Mapping Spaces and (Co)Fibrations

The interactions between our Reedy model structures and the various mapping spaces we have constructed will be necessary for defining our models of higher category. More generally, our situation fits into the greater problem of how categorical enrichments interact with model structures. Our primary source for this topic is [MP12]. We will assume henceforth that all enrichments are over symmetric monoidal categories.

Definition 2.2.39 ([MP12, pg. 329])**.** *A* cosmos $\mathscr{V}$ *is a bicomplete closed symmetric monoidal category.*

Proposition 2.2.40. $\boldsymbol{sSpace}_k$ *is a cosmos for all* $k \geq 0$ *under the Cartesian monoidal structure.*

Recall by Proposition 2.1.39 that for all $n \geq k \geq 0$, the category $\mathbf{sSpace}_n$ is enriched over $\mathbf{sSpace}_k$ via $\mathbf{Map}_n^k(-,-)$. We wish to prove that this enrichment is somehow compatible with model structures on $\mathbf{sSpace}_n$ and $\mathbf{sSpace}_k$, in particular the injective, and thus Reedy, model structures $\mathbf{sSpace}_n^{inj}$ and $\mathbf{sSpace}_k^{inj}$. The appropriate notions which make this objective precise are *tensors* and *cotensors*:

Definition 2.2.41 ([MP12, pg. 328])**.** *Suppose $\mathscr{M}$ is a $\mathscr{V}$-enriched category, where $\mathscr{V}$ is symmetric monoidal. Then a* tensor *(or copower)* $\odot : \mathscr{M} \times \mathscr{V} \to \mathscr{M}$ *and* cotensor *(or power)* $\pitchfork : \mathscr{V}^{op} \times \mathscr{M} \to \mathscr{M}$ *are functors that induce a series of natural isomorphisms*

$$\boldsymbol{Hom}_{\mathscr{M}}(X \odot V, Y) \cong \boldsymbol{Hom}_{\mathscr{V}}(V, \mathscr{M}(X,Y)) \cong \boldsymbol{Hom}_{\mathscr{M}}(X, V \pitchfork Y).$$

Note that it is not always guaranteed that tensors and cotensors exist. We invoke special terminology for when they do:

Definition 2.2.42 ([MP12, pg. 329])**.** *A $\mathscr{V}$-enriched category $\mathscr{M}$ is $\mathscr{V}$-*bicomplete *if and only if it is bicomplete and has all tensors and cotensors.*

Proposition 2.2.43. $\boldsymbol{sSpace}_n$ *is* $\boldsymbol{sSpace}_k$*-bicomplete for all* $0 \leq k \leq n$.

Proof. That $\mathbf{sSpace}_n$ is bicomplete is immediate from it being a category of presheaves. For tensors and cotensors, consider setting $\odot := \square_n^k$ and $\pitchfork := (-)^{\iota_n^k(-)}$. Then we have the natural bijection induced by the adjunction between ι_n^k and ρ_n^k of the form

$$\begin{aligned}
\mathbf{Hom}_{\mathbf{sSpace}_n}(X \odot V, Y) &= \mathbf{Hom}_{\mathbf{sSpace}_n}(X \square_n^k V, Y) \\
&= \mathbf{Hom}_{\mathbf{sSpace}_n}(X \times \iota_n^k(V), Y) \\
&\cong \mathbf{Hom}_{\mathbf{sSpace}_n}(\iota_n^k(V), Y^X) \\
&\cong \mathbf{Hom}_{\mathbf{sSpace}_k}(V, \mathbf{Map}_n^k(X,Y)).
\end{aligned}$$

Moreover, we have a natural bijection induced by the same adjunction of the form

$$\begin{aligned}\mathbf{Hom}_{\mathbf{sSpace}_k}(V, \mathbf{Map}_n^k(X,Y)) &\cong \mathbf{Hom}_{\mathbf{sSpace}_n}(\iota_n^k(V), Y^X)\\ &\cong \mathbf{Hom}_{\mathbf{sSpace}_n}(\iota_n^k(V)\times X, Y)\\ &\cong \mathbf{Hom}_{\mathbf{sSpace}_n}(X, Y^{\iota_n^k(V)}).\end{aligned}$$

These bijections establish the conditions for tensors and cotensors as needed. □

In general, we will not use the notation of $\odot$ and $\pitchfork$ and simply write $\Box_n^k$ and $(-)^{(-)}$. Note in the latter case that we omit the ι_n^k implicit in the exponent.

It should be noted that these are the natural extension of the tensor and cotensor in the projective model structure $\mathscr{M}^{\mathscr{C}}$ for a simplicial cofibrantly generated model category $\mathscr{M}$ and small category $\mathscr{C}$ in [Hir09, Def. 11.7.1]. Indeed, $\mathbf{sSpace}_n^{proj}$ is cofibrantly generated and simplicial by inductively applying this definition and [Hir09, Thm. 11.7.3]; we have that the simplicial enrichment Hirschhorn describes is precisely $\mathbf{Map}_n^0$. We are unaware of a result in the literature establishing the above general case of $\mathbf{sSpace}_n$ and $\mathbf{sSpace}_k$ when $k \neq 0$.

We now wish to show that our tensor and cotensor are well-behaved with respect to model structures. The appropriate such interactions are as follows:

Lemma 2.2.44 ([MP12, Lemma 16.4.5])**.** *Suppose $\mathscr{M}$ and $\mathscr{V}$ are model categories such that $\mathscr{V}$ is a cosmos and $\mathscr{M}$ is $\mathscr{V}$-bicomplete. Let $\odot$ and $\pitchfork$ be a tensor and cotensor on $\mathscr{M}$. Then the following conditions are equivalent:*

1. *Given a cofibration $f : X \to Y$ in $\mathscr{M}$ and a cofibration $p : U \to V$ in $\mathscr{V}$, the induced map*

$$f\Box p : (Y \odot U) \sqcup_{X\odot U} (X \odot V) \to Y \odot V$$

is a cofibration in $\mathscr{M}$, which is trivial if either f or p is.

2. *Given a cofibration $f : W \to X$ and a fibration $g : Y \to Z$ in $\mathscr{M}$, the induced map*

$$\mathscr{M}[f,g] : \mathscr{M}(X,Y) \to \mathscr{M}(W,Y) \times_{\mathscr{M}(W,Z)} \mathscr{M}(X,Z)$$

is a fibration in $\mathscr{V}$, which is trivial if either f or g is.

3. *Given a cofibration $p : U \to V$ in $\mathscr{V}$ and a fibration $g : Y \to Z$ in $\mathscr{M}$, the induced map*

$$\pitchfork [p,g] : V \pitchfork Y \to U \pitchfork Y \times_{U\pitchfork Z} V \pitchfork Z$$

is a fibration in $\mathscr{M}$, which is trivial if either p or g is.

Proposition 2.2.45 ([Hov07, Prop. 4.2.8]). *The category* $\boldsymbol{sSet}$*, enriched over itself, with tensor given by Cartesian product and cotensor by the exponent, satisfies the equivalent conditions in Lemma 2.2.44.*

Proposition 2.2.46. *The tensor and cotensor in Proposition 2.2.43 satisfy all the equivalent conditions in Lemma 2.2.44.*

Proof. We will prove the first condition holds. Suppose $f : U \to V$ is a cofibration in $\mathbf{sSpace}_n$ and $g : W \to X$ is a cofibration in $\mathbf{sSpace}_k$. By Corollary 2.2.25, these are both simply levelwise cofibrations in $\mathbf{sSet}$, so checking that the pushout product $f \square g : (V \square_n^k W) \sqcup_{U \square_n^k W} (U \square_n^k X) \to (V \square_n^k X)$ is a cofibration amounts to checking this levelwise. This problem reduces to checking that for every $i_1, \cdots, i_n \geq 0$, the map

$$\begin{array}{c}(V_{i_1,\cdots,i_n} \times W_{i_{n-k+1},\cdots,i_n}) \sqcup_{U_{i,\cdots,i_n} \times W_{i_{n-k+1},\cdots,i_n}} (U_{i_1,\cdots,i_n} \times X_{i_{n-k+1},\cdots,i_n}) \\ \big\downarrow \\ V_{i_1,\cdots,i_n} \times X_{i_{n-k+1},\cdots,i_n}\end{array}$$

is a cofibration in $\mathbf{sSet}$. By Lemma 2.2.45, $f \square g$ is then immediately a cofibration as desired. The above reasoning extends to the case where f or g is a trivial cofibration, as these are again just levelwise trivial cofibrations in $\mathbf{sSet}$. $\square$

We thus have the following useful results that we will need many times over in this thesis, allowing us to manipulate and construct Reedy fibrations in $\mathbf{sSpace}_n^{inj}$.

Corollary 2.2.47. *Suppose $f : U \to V$ is a Reedy cofibration and $p : Y \to Z$ a Reedy fibration in* $\boldsymbol{sSpace}_n$*. Then the induced map*

$$\boldsymbol{Map}_n^k(f,p) : \boldsymbol{Map}_n^k(V,Y) \to \boldsymbol{Map}_n^k(U,Y) \times_{\boldsymbol{Map}_n^k(U,Z)} \boldsymbol{Map}_n^k(V,Z)$$

is a Reedy fibration in $\boldsymbol{sSpace}_k$*, which is trivial if either f or p is.*

Corollary 2.2.48. *If Y is Reedy fibrant in* $\boldsymbol{sSpace}_n$ *and $f : U \to V$ is a (trivial) Reedy cofibration in* $\boldsymbol{sSpace}_n$*, then the map*

$$\boldsymbol{Map}_n^k(f,Y) : \boldsymbol{Map}_n^k(V,Y) \to \boldsymbol{Map}_n^k(U,Y)$$

is a (trivial) Reedy fibration in $\boldsymbol{sSpace}_k$*.*

We note again that this fact is not necessarily new in all cases. For instance, the case $k = 0$ and $n = 1$ is established in [JT07, Prop. 2.4]. More generally, the case $k = 0$ for all n is already known:

Proposition 2.2.49. *The mapping spaces* $\boldsymbol{Map}_n^0$*, tensors* $\square_n^0$ *and cotensor* $(-)^{\iota_n^0(-)}$ *are precisely those obtained from the natural simplicial model structure on* $\boldsymbol{sSpace}_n$ *described in [GJ09b, pg. 370].*

Moreover, it is precisely the enrichment from the natural simplicial model structure on $\boldsymbol{sSpace}_n^{proj}$ *obtained inductively via [Hir09, Def. 11.7.2] and from the enrichment of* ***sSet*** *by itself.*

2.2.4 Localizations

A few words should be said as well about left Bousfield localizations and how they will interact with injective and projective model structures. We first need an auxiliary definition:

Definition 2.2.50 ([Hir09, Not. 16.1.1] [Hir09, Def. 16.1.2])**.** *Let* $\mathscr{M}$ *be a model category. Let* $X \in \mathscr{M}$*. Define* $\boldsymbol{cc}_*(X) \in \mathscr{M}^\Delta$ *to be the constant functor to* X*.*

A cosimplicial resolution *of* X *is a cofibrant approximation* $\tilde{X} \to \boldsymbol{cc}_*(X)$ *in the Reedy model category structure*[4] *on* $\mathscr{M}^\Delta$*.*

Within the scope of this thesis, we are not too interested in the precise behavior of such a Reedy model structure. For us, two facts are important. The first of these uses that every simplicial model category is naturally tensored and cotensored over **sSet** as in [Hir09, Def. 9.1.6], a tensor we write as $\otimes$:

Proposition 2.2.51 ([Hir09, Prop. 16.1.3])**.** *Let* $\mathscr{M}$ *be a simplicial model category. If* X *is an object of* $\mathscr{M}$ *and* $W \to X$ *is a cofibrant approximation to* X*, then the cosimplicial object* $\tilde{W}$*, where* $\tilde{W}^n := W \otimes \Delta[n]$*, is a cosimplicial resolution of* X*.*

The place where this structure matters to us is in consideration of *homotopy function complexes*:

Definition 2.2.52 ([Hir09, Def. 17.1.1 and Notation 16.4.1])**.** *Let* $\mathscr{M}$ *be a model category and* $X, Y \in \mathscr{M}$*. A* left homotopy function complex *from* X *to* Y *is a triple*

$$(\tilde{X}, \hat{Y}, \mathscr{M}(\tilde{X}, \hat{Y}))$$

where $\tilde{X}$ *is a cosimplicial resolution of* X*,* $\hat{Y}$ *is a fibrant approximation to* Y *and* $\mathscr{M}(\tilde{X}, \hat{Y})$ *is the simplicial set which at level* n *is the set* $\boldsymbol{Hom}_{\mathscr{M}}(\tilde{X}_n, \hat{Y})$*.*

For our purposes, left homotopy function complexes are sufficient. There are dual right homotopy function complexes [Hir09, Def. 17.2.1] and two-sided complexes [Hir09, Def. 17.3.1], which involve simplicial resolutions [Hir09, Def. 16.1.2]. We will refer to left homotopy function complexes as simply *homotopy*

[4] Note that the source Reedy category is now Δ rather than Δ^{op}.

function complexes as in [Hir09, Def. 17.4.1], as the left nature of them is unimportant.

Corollary 2.2.53. *Let $X, Y \in \boldsymbol{sSpace}_n$ such that Y is Reedy fibrant. Then, setting $\tilde{X}$ to be the cosimplicial object where $\tilde{X}^n = X\Box_n^0\Delta[n]$ as in Proposition 2.2.51, the tuple*

$$(\tilde{X}, Y, \boldsymbol{Map}_n^0(X, Y))$$

is a left homotopy function complex for X and Y in the Reedy model structure on $\boldsymbol{sSpace}_n$.

Proof. Apply the cosimplicial resolution in Proposition 2.2.51, using X as a cofibrant approximation for itself by Corollary 2.2.26. □

We will use the complex in Corollary 2.2.53 as our homotopy function complex and may write it as $\mathbf{map}(X, Y)$ when doing so. Note that our homotopy function complexes are in fact functorial in X and Y:

Definition 2.2.54 ([Hir09, Def. 17.5.2])**.** *Suppose $\mathscr{M}$ is a model category and $\mathscr{K}$ is a subcategory of $\mathscr{M}^{op} \times \mathscr{M}$. Then a* functorial left homotopy function complex *on $\mathscr{K}$ is a pair (F, ϕ), where*

$$F : \mathscr{K} \to (\mathscr{M}^{\Delta})^{op} \times \mathscr{M}^{\Delta^{op}}$$

and ϕ is a natural transformation $(\boldsymbol{cc}_*X, \boldsymbol{cs}_*Y) \to F(X, Y)$ where $\boldsymbol{cs}_*Y : \Delta^{op} \to \mathscr{M}$ is the constant functor on $Y \in \mathscr{M}$, such that

1. *$\phi(X, Y)$ defines a left homotopy function complex from X to Y for every object (X, Y) of $\mathscr{K}$ in that it identifies a pair of maps $\tilde{X} \to \boldsymbol{cc}_*X$ and $\boldsymbol{cs}_*Y \to \boldsymbol{cs}_*\hat{Y}$ giving a cosimplicial resolution and fibrant approximation respectively, and*

2. *F takes maps of $\mathscr{K}$ to compositions of maps of left homotopy function complexes.*

Proposition 2.2.55. *The left homotopy function complex in Corollary 2.2.53 defines a functorial left homotopy function complex on $\boldsymbol{sSpace}_n^{inj}$ for the full subcategory $\mathscr{K} \subseteq \boldsymbol{sSpace}_n^{op} \times \boldsymbol{sSpace}_n$ of pairs (X, Y) where Y is fibrant.*

Proof. It is evident that $F : (X, Y) \mapsto (\tilde{X}, Y)$ is a functor. The natural transformation $\phi : (cc_*X, cs_*Y) \to (\tilde{X}, Y)$ is defined by natural map $\tilde{X} \to cc_*X$ and $1_Y : Y \to Y$. This evidently defines the appropriate left homotopy function complex. The second point is immediate by construction. □

One should consider then how to proceed with the projective model structure. Unlike in the Reedy model structure, projective cofibrancy is not always immediate; we will in general be unable to prove it holds in our use cases. Thus, we have the following weaker results:

Proposition 2.2.56 ([Hir09, Prop. 8.1.17]). *There exists a* cofibrant replacement functor *on $\boldsymbol{sSpace}_n^{proj}$, namely a functor $Q : \boldsymbol{sSpace}_n \to \boldsymbol{sSpace}_n$ and natural transformation $\eta : Q \Rightarrow 1_{\boldsymbol{sSpace}_n}$ such that $Q(X)$ is projective cofibrant and $\eta_X : Q(X) \to X$ is a trivial projective fibration for all $X \in \boldsymbol{sSpace}_n$.*

Note that Hirschhorn in [Hir09, Def. 8.1.15] refers to Q and η together as a *functorial fibrant cofibrant approximation* on $\mathbf{sSpace}_n$.

Proposition 2.2.57. *Suppose $X, Y \in \boldsymbol{sSpace}_n$ are such that Y is projective fibrant. Then, setting $\tilde{Q(X)}$ to be as in Proposition 2.2.51, the tuple*

$$(\tilde{Q(X)}, Y, \boldsymbol{Map}_n^0(Q(X), Y))$$

is a left homotopy function complex for X and Y in the projective model structure on $\boldsymbol{sSpace}_n$. Moreover, if X is also projective cofibrant, then setting $\tilde{X}$ to again be as in Proposition 2.2.51, the tuple

$$(\tilde{X}, Y, \boldsymbol{Map}_n^0(X, Y))$$

is another such left homotopy function complex.

Proof. Apply the cosimplicial resolution in Proposition 2.2.51, using $Q(X)$ or X itself as a cofibrant approximation for X respectively. □

These can be made functorial in a straightforward manner, using the same methods as for the injective model structure on $\mathbf{sSpace}_n$:

Proposition 2.2.58. *The left homotopy function complexes on $\boldsymbol{sSpace}_n^{proj}$ in Proposition 2.2.57 are functorial on the full subcategory of pairs (X, Y) such that Y is projective fibrant or on the full subcategory of such pairs where Y is projective fibrant and X is projective cofibrant, respectively.*

Definition 2.2.59 ([Hir09, Def. 3.1.4]). *Let $\mathscr{M}$ be a model category and C be a class of maps in $\mathscr{M}$.*

*An object W of $\mathscr{M}$ is C-*local *if W is fibrant and for every element $f : A \to B$ of C, the induced map of functorial left homotopy function complexes*

$$f^* : \boldsymbol{map}(B, W) \to \boldsymbol{map}(A, W)$$

*is a weak equivalence. In turn, a map $g : X \to Y$ in $\mathscr{M}$ is a C-*local equivalence *if and only if for every C-local object W, the induced map*

$$g^* : \boldsymbol{map}(Y, W) \to \boldsymbol{map}(X, W)$$

is a weak equivalence.

Note that we have specified that the homotopy function complexes must be functorial, unlike Hirschhorn in [Hir09, Def. 3.1.4], to ensure functoriality of the maps f^* and g^* in a manner of our choosing.

Definition 2.2.60 ([Hir09, Def. 3.3.1]). *Let $\mathscr{M}$ be a model category and C be a class of maps in $\mathscr{M}$.*

The left Bousfield localization *of $\mathscr{M}$ with respect to C, if it exists, is a model category structure $L_C\mathscr{M}$ on the underlying category of $\mathscr{M}$ such that:*

1. *The weak equivalences of $L_C\mathscr{M}$ are the C-local equivalences of $\mathscr{M}$;*
2. *The cofibrations of $L_C\mathscr{M}$ are the cofibrations of $\mathscr{M}$.*

The fibrations are then induced.

Note then that a left Bousfield localization of a model structure will have the same cofibrations and trivial fibrations as before, while adding more weak equivalences and trivial cofibrations and removing some fibrations [Hir09, Prop. 3.3.3]. Hence, in our eventual model structures defining $(\infty, 2)$-categories and $(\infty, 1)$-categories, levelwise weak equivalences will still be weak equivalences. Note also that fibrations in the localization will be Reedy fibrations, but the other way around will no longer be the case in general.

The particular significance of C-local objects in left Bousfield localizations goes past merely characterizing the weak equivalences: they are precisely the fibrant objects in such model structures.

Proposition 2.2.61 ([Hir09, Prop. 3.4.1 (1)]). *Let $\mathscr{M}$ be a left proper model category and C be a class of maps in $\mathscr{M}$. Then the fibrant objects in $L_C\mathscr{M}$, if this model structure exists, are precisely the C-local objects in $\mathscr{M}$.*

Existence of left Bousfield localizations will be guaranteed inductively by the following results acting in tandem:

Proposition 2.2.62 ([Hir09, Theorem 13.1.13] [Hir09, Proposition 12.1.4]). ***sSet*** *is proper and cellular.*

Proposition 2.2.63. *The Reedy model structure on $\mathbf{sSpace}_n$ is proper and cellular.*

Proof. These model structures can be seen to be Reedy on the underlying model category **sSet**, which is proper. By [Hir09, Theorem 15.3.4(2)], this means the Reedy model structure on $\mathbf{sSpace}_n$ is proper. A similar story holds for cellularity using [Hir09, Theorem 15.7.6]. □

Proposition 2.2.64. *The projective model structure on $\mathbf{sSpace}_n$ is proper and cellular.*

Proof. Cellularity is given by [Hir09, Prop. 12.1.5] and Proposition 2.2.62. Properness is proven by [Hir09, Thm. 13.1.14]. □

Theorem 2.2.65 ([Hir09, Thm. 4.1.1])**.** *Let $\mathscr{M}$ be a left proper and cellular model category and let C be a class of maps in $\mathscr{M}$. Then $L_C\mathscr{M}$ exists and is moreover left proper and cellular.*

It will be useful to us to organize our sets of maps to localize by with the following data structure:

Definition 2.2.66 ([JS17, Def. A.3])**.** *A* presentation (C, M) *consists of a small category C and a set of maps M in $\boldsymbol{sSet}^C$.*

We will only care for presentations of the form (Δ^{op}, M) and $((\Delta^{op})^2, N)$. The upshot for us is the ability to construct instances of the latter from the former:

Definition 2.2.67 ([JS17, Prop. A.9])**.** *Let (C, M) and (D, N) be presentations. Define a new presentation $(C, M) \boxtimes (D, N)$ whose underlying category is the product of categories $C \times D$ and whose set of distinguished maps in $\boldsymbol{sSet}^{C\times D}$ is the set*

$$M \boxtimes N := \{m \boxtimes 1_{\boldsymbol{z}(d)}\}_{(m,d)\in M\times D} \cup \{1_{\boldsymbol{y}(c)} \boxtimes n\}_{(c,n)\in C\times N}$$

where $\boldsymbol{y} : C \to \boldsymbol{Set}^C$ and $\boldsymbol{z} : D \to \boldsymbol{Set}^D$ are the Yoneda embeddings and $\boxtimes : \boldsymbol{sSet}^C \times \boldsymbol{sSet}^D \to \boldsymbol{sSet}^{C\times D}$ denotes the functor sending a pair of presheaves (A, B) to the presheaf $(A \boxtimes B) : (x, y) \mapsto A(x) \times B(y)$.

Let C be a Reedy category. For us, to localize with respect to a presentation (C, M) simply means to localize the Reedy model structure on $\mathbf{sSet}^C$ with respect to the maps M, namely to take the model category $L_M\mathbf{sSet}^C$. We will also do this with the projective model structure on $\mathbf{sSet}^C$, as is done in [JS17, App. A] to obtain projective fibrant complete n-fold Segal spaces.

We will also have need for a few natural results about fibrations and weak equivalences involving localizations:

Proposition 2.2.68 ([Hir09, Thm. 3.2.13 (1)])**.** *Suppose $\mathscr{M}$ is a model category and $\mathscr{C}$ is a class of maps in $\mathscr{M}$. If X and Y are C-local and $f : X \to Y$ is a C-local equivalence, then f is a weak equivalence in $\mathscr{M}$.*

Proposition 2.2.69 ([Hir09, Prop. 3.3.16 (1)])**.** *Suppose $\mathscr{M}$ is a model category and C is a class of maps in $\mathscr{M}$. If X and Y are C-local objects in $\mathscr{M}$, then a map $f : X \to Y$ is a fibration in $L_C\mathscr{M}$ if and only if it is one in $\mathscr{M}$.*

2.2.5 Homotopy Pullbacks and Path Spaces

We will have need for a suitable notion of *homotopy pullback* for cospans of simplicial sets. We begin with a consideration of general path spaces in a model category:

Definition 2.2.70 ([DS95, pg. 22])**.** *Let $\mathscr{M}$ be a model category and $C \in \mathscr{M}$. A* path object *for C is a factorization*

$$C \xrightarrow{d} \mathcal{P}(C) \xrightarrow{p} C \times C$$

of the diagonal map $C \to C \times C$ such that d is a weak equivalence. It is a good path object *if p is a fibration and a* very good path object *if, moreover, d is a trivial cofibration.*

The purpose of path objects is often to obtain *right homotopies*, a notion of homotopy amenable to model categories:

Definition 2.2.71 ([DS95, pg. 22])**.** *A* right homotopy *H from f to g for $f, g : C \to D$ in a model category $\mathscr{M}$ is a map*

$$H : C \to \mathcal{P}(D)$$

for some path object $\mathcal{P}(D)$ of D such that the two compositions $C \to \mathcal{P}(D) \to D$ result in f and g respectively. It is called a good *or* very good *right homotopy if $\mathcal{P}(D)$ is a good or very good path object, respectively.*

We will encounter so-called *left homotopies* later in this thesis, which are somehow dual to right homotopies.

There are two path objects we will take an interest in for **sSet**, neither of which we claim originality for. The first of these appears for instance in [RV22, Def. 1.1.23]. Before we present it, we need a quick definition:

Definition 2.2.72. *Let $I[n]$ be the groupoid whose object set is $[n]$ and with all hom-sets singletons.*

For some topological connection to this object, consider the following standard construction:

Definition 2.2.73 ([May99, ch. 2])**.** *The* fundamental groupoid *functor Π_1 : **Top** $\to$ **Grpd** sends a topological space X to the groupoid $\Pi_1(X)$ whose objects are points in X and whose morphisms are homotopy classes of paths relative to start and end points.*

Then, in particular, one might notice that $I[n]$ is isomorphic to a full subcategory of the fundamental groupoid $\Pi_1(\Delta_t[n])$ of the n-simplex whose objects are the 'corners' of $\Delta_t[n]$, namely those points $(x_0, \cdots, x_n)$ where some $x_i = 1$.

This is equivalent to the entirety of $\Pi_1(\Delta_t[n])$ along the inclusion functor, a fact quickly proven by the n-simplex being contractible. We might therefore consider $I[1]$ as a finite model for a contractible homotopy type with two points and $I[n]$ similarly for $n+1$ points.

The extension of these ideas to simplicial sets for us will be the space $\mathbf{nerve}(I[1])$:

Definition 2.2.74. *Suppose $X \in \boldsymbol{sSet}$ is a Kan complex. Then define $I := \boldsymbol{nerve}(I[1])$ and in turn X^I to be the space*

$$X \xrightarrow{c_X} X^I := X^{\boldsymbol{nerve}(I[1])} \xrightarrow{\langle s_X, t_X \rangle} X \times X$$

where the maps $c_X : X \to X^I$ and $s_X, t_X : X^I \to X$ are given by applying $X^{\boldsymbol{nerve}(-)}$ to the natural maps

$$* \sqcup * \to I[1] \to *$$

in $\boldsymbol{Cat}$. We will refer to c_X as the constant path map*, s_X as the* source map *and t_X as the* target map.

Our second path object is built upon the pushout of the natural span $\mathbf{nerve}(I[1]) \leftarrow * \to \mathbf{nerve}(I[1])$, where the leftmost map identifies the object 1 while the rightmost map identifies 0:

Definition 2.2.75. *Suppose $X \in \boldsymbol{sSet}$ is a Kan complex. Then define $\Lambda := \boldsymbol{nerve}(I[1]) \sqcup_* \boldsymbol{nerve}(I[1])$ and in turn X^Λ to be the space*

$$X \xrightarrow{\lambda_X} X^\Lambda := X^{\boldsymbol{nerve}(I[1]) \sqcup_* \boldsymbol{nerve}(I[1])} \xrightarrow{\langle \sigma_X, \tau_X \rangle} X \times X$$

where the maps $\lambda_X : X \to X^\Lambda$ and $\sigma_X, \tau_X : X^\Lambda \to X$ are given by the natural maps

$$* \sqcup * \to \boldsymbol{nerve}(I[1]) \sqcup_* \boldsymbol{nerve}(I[1]) \to *$$

in $\boldsymbol{sSet}$. We will refer to λ_X as the constant path map*, σ_X as the* source map *and τ_X as the* target map.

Note that X^Λ does not result from some diagram in $\mathbf{Cat}$; indeed, appending two copies of $I[1]$ along $*$ in $\mathbf{Cat}$ would yield $I[2]$ instead, rather than the above simplicial set. We can see Λ as a 'horn' of sorts, though where each 1-simplex has an inverse.

Proposition 2.2.76. *If X is a Kan complex, then X^I is a very good path object for X.*

Proof. The map $X^I \to X \times X$ being a fibration follows from the inclusion of objects $* \sqcup * \hookrightarrow \mathbf{nerve}(I[1])$ being a cofibration and either Corollary 2.2.47 in the case $n = 0$ or [Hir09, Ex. 9.1.13], which states that **sSet** has a natural simplicial model structure.

The map $X \to X^I$ is clearly a cofibration. To show it is a weak equivalence, note that each map $* \to \mathbf{nerve}(I[1])$ is a trivial cofibration, so that the two maps $X^I \to X$ are trivial fibrations. The composites $X \to X^I \to X$ are the identity, so by 2-out-of-3 we have a trivial cofibration as needed. □

Proposition 2.2.77. *If X is a Kan complex, then X^Λ is a very good path object for X.*

Proof. The proof is similar to X^I. Again, $X^\Lambda \to X \times X$ is a fibration since $* \sqcup * \hookrightarrow \Lambda$ is a cofibration. Moreover, the constant path map $X \to X^\Lambda$ is immediately a cofibration. For it to be trivial, it again suffices to show the two maps $* \to \Lambda$ are trivial cofibrations. Considering the pushout diagram

$$\begin{tikzcd} & & * \arrow[d, "k"] \\ & * \arrow[r] \arrow[d, "i"'] & \mathbf{nerve}(I[1]) \arrow[d, "j"] \\ * \arrow[r] & \mathbf{nerve}(I[1]) \arrow[r] & \Lambda \end{tikzcd}$$

we see that since i is a trivial cofibration, j must be one too. Since k is a trivial cofibration, the composite jk is a trivial cofibration as needed. A similar story holds for the horizontal maps. □

We will perhaps take a slightly unusual definition of 'homotopy pullback' in **sSet**, electing to use Λ rather than I. We do not require much specific insight into homotopy limits; for those interested, one may consult [Hir09, ch. 18-19] or [Rie14, ch. 5].

Definition 2.2.78. *Suppose $f : X \to Y \leftarrow Z : g$ is a cospan of Kan complexes in **sSet**. Define the* homotopy pullback *of this cospan to be*

$$X \times_Y^h Z := X \times_Y Y^\Lambda \times_Y Z,$$

where the limit is taken over the diagram

$$X \xrightarrow{f} Y \xleftarrow{\sigma_Y} Y^\Lambda \xrightarrow{\tau_Y} Y \xleftarrow{g} Z.$$

A more standard definition would perhaps be $X \times_Y Y^I \times_Y Z$ or something analogous, such as $X \times_Y Y^{\mathbf{Sing}([0,1])} \times_Y Z$, which are both perfectly serviceable in the scenario of Kan complexes. All of these differ solely by the choice of path object used. We employ the above as it will be the easiest to cohere with the results of our upcoming computations.

As a show of good faith, we should prove this is indeed a reasonable notion of homotopy pullback. We will first need a result about factorizing maps between Kan complexes that will prove crucial in our work on Reedy fibrant replacement later. This result is entirely standard; for instance, both its statement and much of its proof are a special case of the *factorization lemma* in [Bro73]. Brown states in the proof thereof that this derives from standard methods in homotopy theory.

Lemma 2.2.79. *Suppose $f : X \to Y$ is a map in $\boldsymbol{sSet}$ between Kan complexes X and Y. Then the induced map*

$$X \times_Y Y^{\boldsymbol{nerve}(I[1])} \to Y$$

by the uppermost horizontal maps in the diagram

$$\begin{tikzcd}
X \times_Y Y^{\boldsymbol{nerve}(I[1])} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y^{\boldsymbol{nerve}(I[1])} \arrow[r, "t_Y"] \arrow[d, "s_Y"] & Y \\
X \arrow[r, "f"'] & Y &
\end{tikzcd}$$

is a fibration. Moreover, the map $X \to X \times_Y Y^{\boldsymbol{nerve}(I[1])}$, given by 1_X and $X \xrightarrow{f} Y \xrightarrow{c_Y} Y^{\boldsymbol{nerve}(I[1])}$, is a trivial cofibration.

Proof. Consider the diagram

$$\begin{tikzcd}
X \times_Y Y^{\mathbf{nerve}(I[1])} \arrow[r, "\cong"] \arrow[ddr, two heads, "t"'] & (X \times Y) \times_{Y \times Y} Y^{\mathbf{nerve}(I[1])} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & Y^{\mathbf{nerve}(I[1])} \arrow[d, two heads] \\
& X \times Y \arrow[r, "f \times 1_Y"'] \arrow[d, two heads] & Y \times Y \\
& Y &
\end{tikzcd}$$

By Proposition 2.2.76, the rightmost vertical map is a fibration, so the pullback map is a fibration. Projections from products of fibrant objects are also fibrations, meaning the map t is a fibration as needed.

Note then that the composite map $X \to X \times_Y Y^{\mathbf{nerve}(I[1])} \to X$ is 1_X. Because the map $X \times_Y Y^{\mathbf{nerve}(I[1])} \to X$ is a trivial fibration, by 2-out-of-3 the map $X \to X \times_Y Y^{\mathbf{nerve}(I[1])}$ is a weak equivalence. That it is a cofibration is immediate. □

Proposition 2.2.80. *Let $f : X \to Y \leftarrow Z : g$ be a cospan of Kan complexes in $\boldsymbol{sSet}$. Then $X \times_Y^h Z$ is a homotopy pullback of this cospan.*

Proof. It suffices to prove that the cospan

$$X \times_Y Y^I \to Y \leftarrow Y^I \times_Y Z$$

is injective fibrant as a diagram $\mathscr{K} \to \mathbf{sSet}$, where $\mathscr{K}$ is the natural cospan diagram.

Note that $\mathscr{K}$ may be interpreted as a Reedy category $a \to b \leftarrow c$, where $\mathbf{deg}(a) = \mathbf{deg}(c) = 1$ and $\mathbf{deg}(b) = 0$. As there are no maps of positive degree, the Reedy cofibrations in $\mathbf{sSet}^{\mathscr{K}}$ are injective cofibrations, so Reedy and injective fibrancy coincide. It thus suffices to prove that the maps $X \times_Y Y^I \to Y$ and $Y^I \times_Y Z \to Y$ are fibrations, which is immediate by Lemma 2.2.79. □

2.3 Complete n-fold Segal Spaces

We are now ready to begin discussing *complete n-fold Segal spaces*, our model of choice for (∞, n)-categories. We will first consider the Reedy fibrant case of $n = 1$, then move on to $n = 2$. Finally, we will turn to the projective fibrant case, as this demands special care.

2.3.1 Complete Segal Spaces

Complete Segal spaces were first developed by Rezk as a model for $(\infty, 1)$-categories in [Rez00]. Though his intention was moreso to use complete Segal spaces as models for homotopy theories akin to model categories, it soon became apparent that his construction could be iterated to obtain a definition of (∞, n)-category, called *complete n-fold Segal spaces*. These objects are studied for instance in [Lur09a], [BR20] and [JS17]. For us, the case $n = 2$ will suffice.

We draw from the intuitions given in [Lur09b]. Our aim is to construct a model for an $(\infty, 1)$-category X. A reasonable place to commence this endeavor is the *homotopy hypothesis*, as considered originally by Grothendieck in [Gro], which amounts to the declaration that the theory of ∞-groupoids is equivalent to the homotopy theory of CW complexes. From this perspective, we may infer that Kan complexes are a suitable model for ∞-groupoids, allowing us to at least concretely specify the 'underlying ∞-groupoid' $X_0 \in \mathbf{sSet}$ of X. Intuitively, X_0 should represent the result of stripping away from X all non-invertible higher morphisms.

Our next obstacle is then to obtain these non-invertible morphisms in X which X_0 omits. Consider that a morphism should be represented by an '∞-functor' $[1] \to X$, from the poset category $[1] = \{0 \to 1\}$ to X. We may then be inclined to imagine the '∞-groupoid of such functors' as another Kan complex $X_1 \in \mathbf{sSet}$. One might note that X_0 could similarly be interpreted as the ∞-groupoid of functors $[0] \to X$, where $[0]$ is the discrete category with one object.

We immediately consider there to be maps $s, t : X_1 \to X_0$ obtained by 'precomposing' with the two functors $[0] \to [1]$, each extracting the source and target of a 1-morphism respectively. Moreover, s and t should be accompanied by a map $i : X_0 \to X_1$ induced by the projection $[1] \to [0]$, which we interpret as supplying the identity 1-morphism of an object.

Given a sequence of 1-morphisms in X, we must now provide some means to identify their composite. This challenge may be addressed by considering a new '∞-groupoid' $X_2 \in \mathbf{sSet}$ of '∞-functors' $[2] \to X$, where in general $[n]$ is the poset category $\{0 < \cdots < n\}$ for $n \geq 0$. Such a functor identifies a chain of two morphisms $x \xrightarrow{f} y \xrightarrow{g} z$ and a third morphism $x \xrightarrow{g \circ f} z$ such that said maps commute up to some higher equivalence. Indeed, a weak functor between higher categories need not strictly respect composition of morphisms. There are clearly two maps $X_1 \to X_2$ given by inserting identities on the left or right of the chain, along with three maps $X_2 \to X_1$ given by extracting f, g or $g \circ f$.

Being able to take unique specified compositions of morphisms now reduces to demanding that the induced *Segal map*

$$\gamma_2 : X_2 \to X_1 \times_{t, X_0, s} X_1$$

is invertible. Then, we have a path $X_1 \times_{t, X_0, s} X_1 \to X_2 \to X_1$ sending (f, g) to $g \circ f$.

The next and final quantum of information necessary to fully model an ∞-category X becomes a specification of coherence conditions. For a hypothetical ∞-category, we should imagine an unending tower of higher and higher coherence morphisms, with associators and unitors between composites, pentagonators between diagrams of associators, *ad infinitum*. To surmount this combinatorial barrier to describing X, we take a similar approach to quasi-categories and abstain from choosing a particular composite for a sequence of morphisms. Instead, we will only ask that there is a 'contractible space of options' for composites of two maps. More formally, we will ask that γ_2 is a mere *weak equivalence* of simplicial sets rather than an isomorphism. In fact, we will go further and assert it is a *trivial fibration*; if everything is in fact a Kan complex as we have thus far assumed, one would find for each pair of composable morphisms $(f, g) \in X_1 \times_{X_0} X_1$ that the fiber $\gamma_2^{-1}((f, g)) \subseteq X_2$ of 'composites' of f and g is a contractible Kan complex, as desired.

To ensure this perspective does in fact guarantee coherence conditions such as weak associativity, we will include spaces X_n of '∞-functors' $[n] \to X$, representing chains of length n and all possible unbiased composites. Maps between these spaces are given by functors $[m] \to [n]$. We will then demand that all the remaining Segal maps

$$\gamma_n : X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

are trivial fibrations. The result of this discussion is a functor $X : \Delta^{op} \to \mathbf{sSet}$ known as a *Segal space.*

Definition 2.3.1 ([Rez00, pg. 11])**.** *A* Reedy fibrant Segal space *is a Reedy fibrant simplicial space $X : \Delta^{op} \to$ **sSet** such that the Segal maps,*

$$\gamma_n : X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1,$$

are weak equivalences for all $n \geq 2$.

We have importantly specified the prefix of 'Reedy fibrant' in the above definition. In due course, we will define *projective fibrant* Segal spaces. Once we have done so, when we write 'Segal space' without specifying Reedy fibrancy, we will mean projective fibrancy, as this will necessarily include both the projective and Reedy fibrant cases.

Definition 2.3.2. *Let $\mathbf{SeSp}^{inj}$ be the full subcategory of **sSpace** whose objects are the Reedy fibrant Segal spaces.*

Note that, given a Reedy fibrant Segal space X, the spaces X_0 and X_1 are Kan complexes and the maps $s := X_{\langle 0 \rangle} = X_{d_1^1}$ and $t := X_{\langle 1 \rangle} = X_{d_1^0}$ must both be Kan fibrations. Hence, as noted in [Rez00, pg. 11], the space $X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is actually a homotopy limit. In particular, this implies that the natural map

$$X_1 \times_{X_0} \cdots \times_{X_0} X_1 \to X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1$$

is a weak equivalence.

Note also that the maps γ_n are necessarily trivial fibrations [Rez00, pg. 11], a fact we must introduce more notation to explain. Recall the *n-spine* from Definition 2.1.11.

Definition 2.3.3. *For $n \geq 1$, let $g_n : Sp(n) \hookrightarrow \Delta[n]$ be the inclusion $\langle 0, 1 \rangle_n \sqcup_{\langle 1 \rangle_n} \cdots \sqcup_{\langle n-1 \rangle_n} \langle n-1, n \rangle_n$. For $n = 0$, let $g_0 = 1_{\Delta[0]}$.*

We can clearly see then that γ_n is just $\mathbf{Map}^0_1(F^0_1(g_n), X)$; indeed, recall that $\mathbf{Map}^0_1$ is the natural enrichment of **sSpace** by **sSet** and that for all $n \geq 0$, we have a natural isomorphism

$$\mathbf{Map}^0_1(F^0_1(n), X) \cong X_n.$$

Moreover, F^0_1 preserves colimits and $\mathbf{Map}^0_1(-, X)$ sends colimits to limits. In the case that $n < 2$, γ_n is just an identity. The map $F^0_1(g_n)$ is clearly a Reedy cofibration by Proposition 2.2.27, so γ_n is a Reedy fibration by Corollary 2.2.48 as needed.

An example is now in order. One of the classic litmus tests for any new definition of higher category is whether it supports a notion of *fundamental higher groupoid* of a topological space. This structure should encode all the weak homotopy theory of the space in question. The means we present to accomplish this in the $(\infty, 1)$ and later $(\infty, 2)$ cases are by no means novel, though an explicit reference in the literature remains elusive.

Definition 2.3.4. *Let $\boldsymbol{Sing_{sS}} : \boldsymbol{Top} \to \boldsymbol{sSpace}$ be the functor sending any $X \in \boldsymbol{Top}$ to the space $\boldsymbol{Sing_{sS}}(X)$, defined levelwise such that*

$$\boldsymbol{Sing_{sS}}(X)_n := \boldsymbol{Sing}(X^{\Delta_t[n]})$$

with simplicial maps induced by precomposition.

We find that

$$\mathbf{Sing_{sS}}(X)_{n,m} \cong \mathbf{Hom_{Top}}(\Delta_t[n] \times \Delta_t[m], X).$$

Again, by Proposition 2.1.18, this has a left adjoint $|\bullet|_{\mathbf{sS}}$ given by a certain coend. We do not take a great deal of interest in this adjoint, so we will not discuss it further.

It would be prudent for us to evaluate how the intuitions that led us to defining Segal spaces apply to this example. Given a space X, the Kan complex $\mathbf{Sing_{sS}}(X)_0$ is just $\mathbf{Sing}(X)$, which is indeed the prototypical example of an ∞-groupoid, namely the fundamental ∞-groupoid of a topological space. This will be the underlying ∞-groupoid of our $(\infty, 1)$-category.

The space $\mathbf{Sing_{sS}}(X)_1$ is then the fundamental ∞-groupoid of the space $X^{\Delta_t[1]}$ of paths in X. As we may hope, the 1-morphisms in our $(\infty, 1)$-category will therefore be paths in our topological space, with the source and target of a path $p : [0, 1] \to X$ given by $p(0)$ and $p(1)$, respectively. Looking at $\mathbf{Sing_{sS}}(X)_n$ reveals in general that the n-simplices will simply be the topological n-simplices in X.

Our attention now turns to the Segal maps γ_n. Starting with the simplest non-trivial case of $n = 2$, the Segal map γ_2 on $\mathbf{Sing_{sS}}(X)$ sends a 2-simplex $\phi : \Delta_t[2] \to X$ to the restriction $\phi|_{|Sp(2)|} : |Sp(2)| \to X$. That the Segal map is a weak equivalence is immediate: there is a deformation retract $\Delta_t[2] \to |Sp(2)|$ of the spine inclusion, which induces a deformation retract of the Segal map itself by precomposition. Thus, given a chain of two 1-morphisms f and g in $\mathbf{Sing_{sS}}(X)_1$, a composite $g \circ f$ may be obtained thereof by applying this deformation retract to obtain a new 1-morphism in $\mathbf{Sing_{sS}}(X)$. One may in general compose such a chain of length n by doing the same with $\Delta_t[n]$ and $|Sp(n)|$.

Proposition 2.3.5. *Suppose $X \in$ **Top**. Then $\textbf{Sing}_{\textbf{sS}}(X)$ is a Reedy fibrant Segal space.*

Proof. By the above reasoning, the Segal maps in $\mathbf{Sing_{sS}}(X)$ are weak equivalences. We are thus left with proving Reedy fibrancy. Consider the matching object

$$\begin{aligned} M_n\mathbf{Sing_{sS}}(X) \cong \lim_{[k]\in\partial(\Delta^{op}\downarrow[n])} \mathbf{Sing_{sS}}(X)_k &\cong \lim_{k\in\partial\Delta[n]} \mathbf{Sing}(X^{\Delta_t[k]}) \\ &\cong \mathbf{Sing}(X^{\operatorname{colim}_{k\in\partial\Delta[n]}\Delta_t[k]}) \\ &\cong \mathbf{Sing}(X^{\partial\Delta_t[n]}) \end{aligned}$$

where $\partial\Delta_t[n] := |\partial\Delta[n]|$. The map $\partial\Delta[n] \hookrightarrow \Delta[n]$ is a cofibration, so is sent to a cofibration in **Top** as $|\cdot|$ is left Quillen. Thus, the maps

$$X^{\Delta_t[n]} \to X^{\partial\Delta_t[n]}$$

are Serre fibrations. **Sing** is right Quillen, so then sends these maps to fibrations, implying the matching maps are fibrations as needed. □

One might expect some notion of a 'mapping space' in a definition of ∞-category, namely an ∞-groupoid of higher morphisms between two fixed objects. This is easily obtained with Reedy fibrant Segal spaces:

Definition 2.3.6 ([Ber18, pg. 10])**.** *Let X be a Reedy fibrant Segal space and $x, y \in X_{0,0}$. The* mapping space *between x and y, written here as $X(x,y)$, is defined to be the pullback*

$$\begin{tikzcd} X(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X_1 \arrow[d] \\ \{(x,y)\} \arrow[r] & X_0 \times X_0 \end{tikzcd}$$

in ***sSet***.

Note that since X is Reedy fibrant, the mapping spaces $X(x,y)$ are once again given by homotopy pullbacks [Ber18, pg. 16]. Moreover, each such mapping space is clearly fibrant, as fibrations are preserved under pullback and the discrete space $\{(x,y)\}$ is necessarily fibrant, as noted in [Ber18, pg. 16]. This will however not be guaranteed for projective fibrant Segal spaces, as the rightmost vertical map may then not be a fibration.

Henceforth, we will employ the following notation:

Notation 2.3.7. *Let $X \in \boldsymbol{sSpace}_k$. Let $x_1, \cdots, x_n \in (X_{0,\cdots,0})_0$. Then write $(-)^{x_1,\cdots,x_n} : (\boldsymbol{sSpace}_{k-1})_{/(X_0)^n} \to \boldsymbol{sSpace}_{k-1}$ be the functor sending $A \to (X_0)^n$ to the pullback*

$$A^{x_1,x_2,\cdots,x_n} := \{(x_1,\cdots,x_n)\} \times_{(X_0)^n} A.$$

For instance, $X(a, b) = X_1^{a,b}$, where we regard X_1 as an object in the category $\mathbf{sSpace}_0/_{(X_0)^2}$ via the natural map $X_1 \to X_0 \times X_0$. We will similarly start writing

$$X(a_0, \cdots, a_n) := \big(X_1 \times_{X_0} \cdots \times_{X_0} X_1\big)^{a_0,\cdots,a_n}.$$

We should interpret this as the space of chains of n morphisms between the objects $a_0, \cdots, a_n$. Moreover, we have Segal maps on fibers

$$\gamma_n^{a_0,\cdots,a_n} : X_n^{a_0,\cdots,a_n} \to X(a_0, \cdots, a_n).$$

In [Rez00, pg. 12], the domain is written as $\mathrm{map}_X(x_0, \cdots, x_n)$, with the codomain decomposed as $\mathrm{map}_X(x_0, x_1) \times \cdots \times \mathrm{map}_X(x_{n-1}, x_n)$.

That the map $\gamma_n^{a_0,\cdots,a_n}$ even exists is because γ_n commutes with the natural maps to $(X_0)^{n+1}$ from both its domain and its codomain. We would like a convenient term for when this condition holds more generally:

Definition 2.3.8. *Consider a cospan $A \to (X_0)^m \leftarrow B$ in $\boldsymbol{sSpace}_k$. A map $f : A \to B$ is* object-fibered *with respect to this cospan if f commutes with the cospan. If the cospan is evident, simply say f is object-fibered.*

Note from [Hir09, Thm. 7.6.5 (2)] that given a model category $\mathscr{M}$ and an object $A \in \mathscr{M}$, there is a natural model structure on $\mathscr{M}_{/A}$ whose cofibrations, fibrations and weak equivalences are those morphisms whose underlying maps in $\mathscr{M}$ are as such. Thus, $(\mathbf{sSpace}_k)_{/(X_0)^n}$ admits two natural model structures, from $\mathbf{sSpace}_k^{inj}$ and $\mathbf{sSpace}_k^{proj}$ respectively. We may use this to prove for instance that the maps $\gamma_n^{x_0,\cdots,x_n}$ are trivial fibrations. That these maps are indeed trivial fibrations is noted in [Rez00, pg. 12].

Proposition 2.3.9. *$(-)^{x_1,\cdots,x_n}$ preserves fibrations and trivial fibrations in $\boldsymbol{sSpace}_k^{inj}$ and $\boldsymbol{sSpace}_k^{proj}$.*

Proof. Consider a (trivial) fibration $f : A \to B$ in $(\mathbf{sSpace}_k)_{/(X_0)^n}$, given by the model structure induced either by the injective or projective model structures on $\mathbf{sSpace}_k$. We have a collection of pullback diagrams

$$\begin{tikzcd}
A^{x_1,\cdots,x_n} \arrow[r] \arrow[d, "f^{x_1,\cdots,x_n}"'] \arrow[dr, phantom, "\ulcorner"] & A \arrow[d, "f"] \\
B^{x_1,\cdots,x_n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d] \\
\{(x_1, \cdots, x_n)\} \arrow[r] & (X_0)^n
\end{tikzcd}$$

where the outermost and bottom squares are pullback diagrams. Thus, the uppermost square is also a pullback diagram, so the upper left vertical map is the pullback of a (trivial) fibration, as needed. ◻

The maps $\gamma_n^{a_0,\cdots,a_n}$ are then pullbacks of trivial fibrations of the form

$$\begin{tikzcd}
X_n^{a_0,\cdots,a_n} \arrow[rr] \arrow[d, "\gamma_n^{a_0,\cdots,a_n}"'] \arrow[drr, phantom, "\lrcorner", very near start] & & X_n \arrow[d, "\gamma_n"] \\
X(a_0,\cdots,a_n) \arrow[rr, hook] & & X_1 \times_{X_0} \cdots \times_{X_0} X_1
\end{tikzcd}$$

A useful operation on $(\infty,1)$-categories, and indeed one which underpins the central objective of this thesis, is the *homotopy category* construction, which 'collapses' such an $(\infty,1)$-category X down to a mere 1-category $h_1(X)$. The resulting category should have the same objects as X but now retain only the path components of the original mapping spaces as its morphisms.

In the case of Reedy fibrant Segal spaces, this procedure is not at all difficult to formalize. Note that the below definition, while generally following the approach of [JS17, Def. 2.2], is slightly modified to suit Reedy fibrancy: the mapping spaces are given by strict pullbacks rather than more general homotopy pullbacks, as is also the case with the codomain of the Segal map γ_2. We will return to the projective fibrant case studied by Johnson-Freyd and Scheimbauer, which their definition of homotopy category is catered to, in due course. Another definition is given by Rezk in [Rez00, pg. 12-13], which agrees precisely with the definition below. We choose to follow Johnson-Freyd and Scheimbauer instead, as it is their description of homotopy bicategories we wish to eventually approach.

Definition 2.3.10 ([JS17, Def. 2.2]). *Let X be a Reedy fibrant Segal space. The* homotopy category $h_1(X) \in \boldsymbol{Cat}$ *is the category whose objects are the elements of the set $(X_0)_0$ and whose hom-sets are of the form*

$$\boldsymbol{Hom}_{h_1(X)}(x,y) := \pi_0(X(x,y))$$

with identities given by the degeneracies and composition by applying π_0 to the zig-zag diagram

$$\begin{aligned} X(x,y) \times X(y,z) \cong X(x,y,z) \hookrightarrow (X_1 \times_{X_0} X_1)^{x,z} \\ \xleftarrow{\gamma_2^{x,z}} X_2^{x,z} \\ \xrightarrow{X_{d_2^1}^{x,z}} X(x,z). \end{aligned}$$

Because $\gamma_2^{x,z}$ is a trivial fibration and thus a weak equivalence of simplicial sets, the map $\pi_0(\gamma_2^{x,z})$ is a bijection. Therefore, after applying π_0, we will have a single function from the first object in the zig-zag to the last.

Note that since the Segal maps $\gamma_2^{x,z}$ are trivial fibrations, we have a lifting problem of the form

$$\begin{tikzcd}
& & X_2^{x,z} \arrow[r] \arrow[d, "\gamma_2^{x,z}"'] & X(x,z) \\
X(x,y) \times X(y,z) \arrow[r, hook] & (X_1 \times_{X_0} X_1)^{x,z} \arrow[r] \arrow[ur, dashed, "\mu_2^{x,z}"] & (X_1 \times_{X_0} X_1)^{x,z} &
\end{tikzcd}$$

which admits the solution $\mu_2^{x,z}$. Taking π_0 of this chain of morphisms induces the same composition map as above. Rezk and Rasekh make a similar observation in [Rez00] and [Ras18] respectively, though instead work pointwise; each notes that since $\gamma_2^{x,z}$ is a trivial fibration, any element of $(X_1 \times_{X_0} X_1)^{x,z}$ can be lifted to an element of $X_2^{x,z}$, with any two such liftings residing in the same path component. All of these approaches yield the same category. We will make explicit use of the approach of solving the entire lifting problem when we discuss homotopy bicategories.

Proposition 2.3.11 ([Rez00, Prop. 5.4])**.** *Let X be a Segal space. Then $h_1(X)$ is a category.*

One may also prove that any map $X \to Y$ between Segal spaces induces a functor $h_1(X) \to h_1(Y)$ in a functorial manner, due to commutativity with Segal maps and degeneracy maps. This results in a functor

$$h_1 : \mathbf{SeSp}^{inj} \to \mathbf{Cat}.$$

It is perhaps useful to cement our intuitions by studying how h_1 acts on an example Reedy fibrant Segal space. To this end, we consider $\mathbf{Sing_{sS}}(X)$ for $X \in \mathbf{Top}$.

Recall the *fundamental groupoid* functor $\Pi_1 : \mathbf{Top} \to \mathbf{Cat}$ from Definition 2.2.73.

Proposition 2.3.12. *There is a natural isomorphism*

$$h_1 \circ \boldsymbol{Sing_{sS}} \cong \Pi_1.$$

Proof. Consider a topological space X and take the category $C := h_1(\mathbf{Sing_{sS}}(X))$. The objects of this category are given by the underlying set of $\mathbf{Sing}_1(X)_0$, which is just the underlying set of X. Hence, there is a clear bijection from the objects of C to those of $\Pi_1(X)$.

Now, let $x, y \in X$. Recall that $\mathbf{Sing_{sS}}$ is right adjoint and thus commutes with limits. This observation, together with Proposition 2.1.26, implies that

$$\mathbf{Hom}_C(x,y) = \pi_0(\mathbf{Sing_{sS}}(X)(x,y)) \cong \pi_0^t(\{(x,y)\} \times_{X^2} X^{\Delta_t[1]}).$$

It therefore suffices to construct a natural bijection

$$\pi_0^t(\{(x,y)\} \times_{X^2} X^{\Delta_t[1]}) \to \mathbf{Hom}_{\Pi_1(X)}(x,y).$$

Note that two paths $\Delta_t[1] \cong [0,1] \to X$ from x to y are identified in $\Pi_1(X)$ if and only if there is a path $[0,1] \to X^{\Delta_t[1]}$ between these paths that is constant on source and target. This happens if and only if they are in the same path component of $\pi_0^t(\{(x,y)\} \times_{X^2} X^{\Delta_t[1]})$, so there is a natural bijection between these sets sending $[\gamma]$ to $[\gamma]$ for any path γ from x to y in X.

That these bijections respect composition and identities is evident. □

One may construct a model structure for Segal spaces, using the following presentation:

Definition 2.3.13. *The* Segal presentation (Δ^{op}, C) *is given by the set C of maps in* ***sSet***$^{\Delta^{op}}$ = ***sSpace*** *consisting of the* Segal maps

$$F_1^0(g_n) : F_1^0(Sp(n)) \hookrightarrow F_1^0(n)$$

for all $n \geq 2$.

Our name for this presentation is not standard in the literature, as far as we are aware.

Theorem 2.3.14 ([Rez00, Theorem 7.1])**.** *There is a model structure obtained as a left Bousfield localization of the Reedy model structure on* ***sSpace****, which we will write as $SeSp^{inj}$, whose fibrant objects are the Reedy fibrant Segal spaces.*

The above model structure is simply the localization of $\mathbf{sSpace}^{inj}$ by the Segal presentation. This is equivalent to the localization Rezk uses in [Rez00, Sec. 10] to obtain this model category; he localizes with respect to the single map obtained as the coproduct of all the Segal maps together.

Despite the evident intuitiveness and efficacy of Segal spaces as a model of $(\infty,1)$-categories, they remain somewhat incomplete. One final addition is necessary to ensure that the 'homotopical' data of a Reedy fibrant Segal space's underlying ∞-groupoid and the 'categorical' data, displayed for instance in the homotopy category, are in mutual agreement. Resolving this will allow for a more elegant description of the equivalences between such $(\infty,1)$-categories, reminiscent of how equivalences between categories are often defined.

Definition 2.3.15 ([Rez00, pg. 14])**.** *Let X be a Reedy fibrant Segal space. Then $X_{heq} \subseteq X_1 \in$* ***sSet****, the* space of homotopy equivalences of X*, is the subspace of path components of X_1 whose 0-simplices are mapped to isomorphisms in $h_1(X)$.*

It is clear that the degeneracy map $X_0 \to X_1$ factors through X_{heq}, as identity maps are indeed isomorphisms [Rez00, pg. 14].

Definition 2.3.16 ([Rez00, pg. 14])**.** *A* Reedy fibrant complete Segal space *X is a Reedy fibrant Segal space such that the map $X_0 \to X_{heq}$ is a weak equivalence.*

Notation 2.3.17. *Write $\boldsymbol{CSSP}^{inj}$ for the full subcategory of $\boldsymbol{sSpace}$ whose objects are the Reedy fibrant complete Segal spaces.*

Once again, it is worthwhile checking what happens to singular spaces.

Proposition 2.3.18. *Let $X \in \boldsymbol{Top}$. Then $\boldsymbol{Sing_{sS}}(X)$ is a Reedy fibrant complete Segal space.*

Proof. We know that $h_1(\mathbf{Sing_{sS}}(X)) \cong \Pi_1(X)$ is a groupoid. Hence, $\mathbf{Sing_{sS}}(X)_{heq} = \mathbf{Sing_{sS}}(X)_1$, so we need only show that the map

$$X^{\Delta_t[0]} \to X^{\Delta_t[1]}$$

induced by the degeneracy $\Delta_t[1] \to \Delta_t[0]$ is a weak equivalence. This is indeed the case, as the degeneracy map is a homotopy equivalence. □

One may tighten the model structure $SeSp^{inj}$ by adding further maps to localize against to produce a sensible homotopy theory for complete Segal spaces. In doing so, the weak equivalences between the fibrant objects, and in fact more generally between Segal spaces, will obtain an alternative description analogous to equivalences of categories:

Definition 2.3.19 ([Rez00, pg. 16])**.** *A map $f : X \to Y$ between Reedy fibrant Segal spaces is a* Dwyer-Kan equivalence *if and only if the following conditions hold:*

1. *The induced functor $h_1(f) : h_1(X) \to h_1(Y)$ is an equivalence of categories.*
2. *For every $x, y \in X_{0,0}$, the induced map $X(x, y) \to Y(f(x), f(y))$ is a weak equivalence.*

This is much more akin to the usual 'fully faithful and essentially surjective' definition of an equivalence of categories. The second condition is the extension of full faithfulness to what we might expect of an equivalence of $(\infty, 1)$-categories, inducing weak equivalences on mapping spaces rather than bijections on hom-sets. Without it, the first condition implies only bijections of path components.

Theorem 2.3.20 ([Rez00, Thm. 7.2] [Rez00, Thm. 7.7] [Rez00, pg. 28]). *There is a closed model structure obtained as a left Bousfield localization of the Reedy model structure on* ***sSpace****, which we will write as $CSSP^{inj}$, whose fibrant objects are the complete Segal spaces and whose weak equivalences between Segal spaces are the Dwyer-Kan equivalences.*

The left Bousfield localization on $\mathbf{sSpace}^{inj}$ in question may be done with respect to the following presentation, a superset of the Segal presentation, to obtain the desired model structure:

Definition 2.3.21 ([JS17, pg. 55-56]). *The* Rezk presentation (Δ^{op}, S) *is given by the set S of maps in $\boldsymbol{sSet}^{\Delta^{op}} = \boldsymbol{sSpace}$ consisting of:*

1. *The* ***Segal maps*** $F_1^0(g_n) : F_1^0(Sp(n)) \hookrightarrow F_1^0(n)$.
2. *The* ***completeness map***

$$F_1^0(0) \sqcup_{F_1^0(\langle 0,0\rangle_0), F_1^0(1), F_1^0(\langle 0,2\rangle_3)} F_1^0(3) \sqcup_{F_1^0(\langle 1,3\rangle_3), F_1^0(1), F_1^0(\langle 0,0\rangle_0)} F_1^0(0) \to F_1^0(0)$$

given by the maps $1_{F_1^0(0)}$ and $F_1^0(\langle 0,0,0,0\rangle_0)$.

We thus find that a weak equivalence between Reedy fibrant Segal spaces in $SeSp^{inj}$ is always a Dwyer-Kan equivalence [Rez00, Thm. 7.7].

Note that the above presentation is not precisely the same as the one given in [Rez00], though will induce the same model structure. Indeed, it is shown by Rezk in [Rez10, Sec. 10] that locality with respect to the completeness map above induces precisely the completeness condition on Segal spaces. This is due to the following result on the matter:

Proposition 2.3.22 ([Rez10, Prop. 10.1]). *For $X \in \boldsymbol{SeSp}^{inj}$, the natural map*

$$\boldsymbol{Map}_1^0\big(F_1^0(\Delta[0] \sqcup_{\langle 0,0\rangle, \Delta[1], \langle 0,2\rangle} \Delta[3] \sqcup_{\langle 1,3\rangle, \Delta[1], \langle 0,0\rangle} \Delta[0]), X\big) \to X_1$$

determined by $\langle 1,2\rangle_3 : \Delta[1] \to \Delta[3]$ factors through X_{heq}. Moreover, the corestriction to X_{heq} is a weak equivalence.

2.3.2 Complete 2-fold Segal Spaces

We are now ready to begin our foray into the world of $(\infty, 2)$-categories. Our model of choice will be *complete 2-fold Segal spaces*, which will be assembled from complete Segal spaces in a similar way to how complete Segal spaces were built out of Kan complexes.

In order to facilitate the structure of an $(\infty, 2)$-category X, we should consider both horizontal and vertical composition. Hence, for each $n, m \geq 0$, we will now have an ∞-groupoid $X_{n,m}$ of 'grids' of 2-morphisms of horizontal length n and vertical length m, which we will write as $X_{n,m}$. We could see this as the ∞-groupoid of ∞-functors $[n] \boxtimes [m] \to X$, where we might imagine $\boxtimes$ as a 'Gray tensor product' of ∞-categories.

This should, for a fixed $n \geq 0$, induce a complete Segal space $X_{n,\bullet}$ where composition is vertical. In the special case $n = 1$, we have what we might call the $(\infty, 1)$-category of 1-morphisms $X_{1,\bullet}$, whose objects $f \in (X_{1,0})_0$ are 1-morphisms in X, whose morphisms $\alpha \in (X_{1,1})_0$ are 2-morphisms in X and whose higher morphisms provide vertical composites of 2-morphisms in X. We then have $(\infty, 1)$-categories $X_{\bullet,n}$ whose compositions are instead horizontal in X.

A starting point to formalize this intuition is as follows:

Definition 2.3.23 ([Ber18, Def. 6.1])**.** *A* Reedy fibrant double Segal space *is a Reedy fibrant bisimplicial space such that the Segal maps*

$$\gamma_{n,\bullet} : X_{n,\bullet} \to X_{1,\bullet} \times_{X_{0,\bullet}} \cdots \times_{X_{0,\bullet}} X_{1,\bullet},$$
$$\gamma_{\bullet,n} : X_{\bullet,n} \to X_{\bullet,1} \times_{X_{\bullet,0}} \cdots \times_{X_{\bullet,0}} X_{\bullet,1},$$

are weak equivalences for all $n \geq 2$, so $X_{\bullet,k}$ and $X_{k,\bullet}$ are Reedy fibrant Segal spaces for all $k \geq 0$.

There is a model structure on $\mathbf{sSpace}_2$ such that these are the fibrant objects, obtained by localizing $\mathbf{sSpace}_2^{inj}$ with respect to the following presentation, a subset of one of the *Lurie presentations* in [JS17, pg. 56] that we will return to later:

Definition 2.3.24. *The* 2-uple Segal presentation *is the presentation* $(\Delta^{op}, C) \boxtimes (\Delta^{op}, C)$.

Definition 2.3.25. *Let SSp_2^{inj} be the model structure on $\boldsymbol{sSpace}_2$ obtained by localizing $\boldsymbol{sSpace}_2^{inj}$ with respect to the* 2*-uple Segal presentation.*

Definition 2.3.26. *Let $\boldsymbol{SSp}_2^{inj}$ be the full subcategory of $\boldsymbol{sSpace}_2$ whose objects are the Reedy fibrant double Segal spaces.*

It is clear that $\mathbf{SSp}_2^{inj}$ is the full subcategory of fibrant objects in the model structure SSp_2^{inj} on $\mathbf{sSpace}_2$.

One might begin to notice an analogy with how complete Segal spaces are constructed from an assortment of ∞-groupoids. At each level, a Reedy fibrant double Segal space X will be a Reedy fibrant Segal space. We could indeed choose to see X_0 as the 'underlying $(\infty, 1)$-category' of X, containing only the invertible 2-morphisms. More generally, X_n will be a Reedy fibrant Segal space, which we could choose to interpret as the $(\infty, 1)$-category of $(\infty, 2)$-functors $[n] \to X$. Precomposition yields a functor from Δ^{op} to $\mathbf{SeSp}^{inj}$, which is a Reedy fibrant bisimplicial space satisfying a Segal condition valued in $(\infty, 1)$-categories instead of ∞-groupoids. This can be made more precise:

Proposition 2.3.27 ([Ber18, Prop. 6.3]). *A bisimplicial space X is a Reedy fibrant double Segal space if and only if it is Reedy fibrant as a functor $\Delta^{op} \to SeSp^{inj}$ and is such that the Segal maps for all $n \geq 2$*

$$\gamma_n : X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

are weak equivalences in $SeSp^{inj}$.

Proof. Recall that (trivial) fibrations between local objects in a left Bousfield localization are simply (trivial) fibrations in the original model structure. Hence, by [Hir09, Cor. 15.3.12], X is Reedy fibrant in $SeSp^{inj}$ if and only if it is Reedy fibrant in $\mathbf{sSpace}^{inj}$ and is levelwise a Reedy fibrant Segal space.

The maps γ_n are then fibrations in $SeSp^{inj}$. Thus, since Reedy trivial fibrations and trivial fibrations in $SeSp^{inj}$ are the same [Hir09, Prop. 3.3.3], the maps γ_n are weak equivalences in $SeSp^{inj}$ if and only if they are weak equivalences levelwise. This means that for each $m \geq 0$, the map

$$\gamma_{n,m} : X_{n,m} \to X_{1,m} \times_{X_{0,m}} \cdots \times_{X_{0,m}} X_{1,m}$$

is a weak equivalence, which is precisely the first case of Segal maps for a double Segal space. The second case is then given if and only if each X_n is a Segal space, which is guaranteed by Reedy fibrancy in $SeSp^{inj}$. $\square$

One may note that this is precisely the notion of 2-fold Segal space in [Ber12, Def. 3.4] minus 'essential constancy', a property we will return to later.

A useful construction pertaining to Reedy fibrant double Segal spaces will be an analogue of the mapping spaces defined previously for Reedy fibrant Segal spaces. To this end, recall that for a bisimplicial space X, for $n \geq 0$ the notation X_n denotes the simplicial space $X_{n,\bullet}$.

Definition 2.3.28 ([Ber18, Def. 6.6]). *Let X be a Reedy fibrant double Segal space. Let $x, y \in (X_{0,0})_0$. Then the* mapping space *$X(x,y)$ is defined to be the pullback in **sSpace** of the form*

$$\begin{tikzcd} X(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_1 \arrow[d] \\ \{(x,y)\} \arrow[r] & X_0 \times X_0. \end{tikzcd}$$

In [Ber18, Def. 6.6], this construct is referred to as the *mapping object* of X and is denoted as $\underline{\mathrm{map}}_X(x,y)$. We could alternatively write it as $X_{1,\bullet}^{x,y}$ or $X_1^{x,y}$ using Notation 2.3.7.

One particular fact about Reedy fibrant double Segal spaces is important for us to establish presently:

Proposition 2.3.29. *Let X be a Reedy fibrant double Segal space. Then the Segal maps*

$$\gamma_n : X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

are trivial fibrations in $CSSP^{inj}$, $SeSp^{inj}$ and $\textbf{sSpace}^{inj}$.

Proof. That X is a Reedy fibrant double Segal space implies it is a Reedy fibrant functor $\Delta^{op} \to \mathbf{sSpace}^{inj}$. Thus, the Segal maps γ_n are Reedy fibrations of simplicial spaces. We also have by definition that the maps γ_n are levelwise weak equivalences in $\mathbf{sSet}$. This implies that each γ_n is a Reedy trivial fibration, meaning it is as such in $SeSp^{inj}$ and $CSSP^{inj}$; indeed, by [Hir09, Prop. 3.3.3], left Bousfield localizations of a given model structure have the same trivial fibrations as the model structure being localized. □

We henceforth employ the notation $A^{a_1,\cdots,a_n}$ and $X(a_1, \cdots, a_n)$ similarly to for simplicial spaces. Again, note that the fibers of the Segal maps $\gamma_n^{a_1,\cdots,a_n}$ for a Reedy fibrant double Segal space are trivial fibrations in $\mathbf{sSpace}^{inj}$, $SeSp^{inj}$ and $CSSP^{inj}$.

As detailed in [Ber18, pg. 15], Reedy fibrant double Segal spaces alone should not be expected to accurately model $(\infty, 2)$-category theory. If we choose to continue following our intuitions for Reedy fibrant double Segal spaces developed thus far, we should then consider the Kan complex $X_{1,1}$ to represent the ∞-groupoid of 2-morphisms in a Reedy fibrant double Segal space X. We might interpret the face maps $X_{1,1} \to X_{1,0}$ as retrieving the source and target 1-morphisms of a given 2-morphism. However, there are now four possible source and target objects, given by the four distinct maps $X_{1,1} \to X_{0,0}$. These do not have to equate to each other to give a 'globular picture' of 2-morphisms, as per the following illustration:

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

Instead, we must contend with the 'vertical' 1-morphisms in $X_{0,1}$ which we have thus far neglected. We discover a 'cubical' picture of 2-morphisms, which we may visualize as follows:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

More explicitly, given some $f \in (X_{1,1})_0$, we could identify the parts of this diagram as

$$\begin{tikzcd}
X_{(d_1^1,d_1^1)}(f) \in X_{0,0} \arrow[r, "X_{(d_1^1,id)}(f)\in X_{1,0}"] \arrow[d, "X_{(id,d_1^1)}(f)\in X_{0,1}"'] \arrow[dr, phantom, "\Downarrow f\in X_{1,1}"] & X_{(d_1^1,d_1^0)}(f) \in X_{0,0} \arrow[d, "X_{(id,d_1^0)}(f)\in X_{0,1}"] \\
X_{(d_1^0,d_1^1)}(f) \in X_{0,0} \arrow[r, "X_{(d_1^1,id)}(f)\in X_{1,0}"'] & X_{(d_1^0,d_1^0)}(f) \in X_{0,0}
\end{tikzcd}$$

In general, we consider the globular case to be closer to our expectations of higher category theory, better resembling both bicategories and the higher categories of manifolds and cobordisms explored in [Sch14b] and [MP23]. Hence, the $(\infty, 1)$-category of vertical 1-morphisms $X_{0,\bullet}$ will have to be brushed under the rug in some natural manner.

A suitable requirement would be for $X_{0,\bullet}$ to in fact represent an ∞-groupoid, so that all vertical morphisms are equivalences. We will realize this intuition by demanding $X_{0,\bullet}$ is *essentially constant*:

Definition 2.3.30 ([CS19, Def. 2.3] [JS17, Def. 2.7])**.** *A simplicial space X is* essentially constant *if and only if the natural map $q : \iota_1^0(X_0) \to X$, defined levelwise such that $q_n : X_0 \to X_n$ is induced by the terminal map $\langle 0, \cdots, 0\rangle_n : [n] \to [0]$ in Δ, is a levelwise weak equivalence.*

The map q is of course simply the map induced by $1_{X_0} : X_0 \to \rho_1^0(X) = X_0$ under the adjunction with ι_1^0.

It will be beneficial for us to establish a running example of an $(\infty, 2)$-category. This will be done in much the same way as with Reedy fibrant complete Segal spaces:

Definition 2.3.31. *Suppose $X \in$ **Top**. Then define $\boldsymbol{Sing_{ssS}}(X)$ to be the bisimplicial space such that, for all $n \geq 0$,*

$$\boldsymbol{Sing_{ssS}}(X)_n := \boldsymbol{Sing_{sS}}(X^{\Delta_t[n]}).$$

Note then that

$$(\mathbf{Sing_{ssS}}(X)_{a,b})_c \cong \mathbf{Hom_{Top}}(\Delta_t[a] \times \Delta_t[b] \times \Delta_t[c], X).$$

It is evident that $\mathbf{Sing_{ssS}}(X)$ is a Reedy fibrant double Segal space for all $X \in$ **Top**. Indeed, we have that $\mathbf{Sing_{ssS}}(X)_{\bullet,k} \cong \mathbf{Sing_{ssS}}(X)_{k,\bullet} = \mathbf{Sing_{sS}}(X^{\Delta_t[k]})$, which is in fact a Reedy fibrant complete Segal space. Reedy fibrancy of $\mathbf{Sing_{ssS}}(X)$ overall may be established by an extension of the proof for Proposition 2.3.5.

We should establish that $\mathbf{Sing_{ssS}}(X)$ aligns with our intuitions for Reedy fibrant double Segal spaces. The notion that $\mathbf{Sing_{ssS}}(X)_0$ should be seen as the underlying $(\infty, 1)$-category of $\mathbf{Sing}_{\mathbf{ssS}}(X)$ somewhat immediately applies by definition. Moreover, the interpretation of $\mathbf{Sing}_{\mathbf{ssS}}(X)_1$ as the $(\infty, 1)$-category of morphisms also applies rather directly. The intuition carries on for higher levels as needed.

Now, consider the cubical picture for 2-morphisms in a Reedy fibrant double Segal space we have previously identified. Whether by coincidence or not, the space $\mathbf{Sing}_{\mathbf{ssS}}(X)_{1,1}$ is quite literally $\mathbf{Sing}(X^{\Delta_t[1]\times\Delta_t[1]})$, the space of squares inside X. The natural simplicial maps to $\mathbf{Sing}_{\mathbf{ssS}}(X)_{1,0}$, $\mathbf{Sing}_{\mathbf{ssS}}(X)_{0,1}$ and $\mathbf{Sing}_{\mathbf{ssS}}(X)_{0,0}$ identify the edges and corners of this square. With the Segal maps, we can compose these squares either horizontally or vertically by aligning edges.

Note also that the vertical maps are in fact 'essentially constant'; they are paths, which can be contracted so the square is essentially 'pinched' into a globular picture, up to homotopy. Hence, the vertical data is rather negligible. Of course, in this example the horizontal data is also as such since $\mathbf{Sing}_{\mathbf{ssS}}(X)$ is really an ∞-groupoid in disguise, but more general examples of $(\infty, 2)$-categories are not so simple.

We will begin our strengthening of Reedy fibrant double Segal spaces by first introducing essential constancy to our definitions:

Definition 2.3.32 ([Ber12, Def. 3.4])**.** *A* Reedy fibrant 2-fold Segal space X *is a Reedy fibrant double Segal space such that X_0 is essentially constant.*

There is a model structure on $\mathbf{sSpace}_2$ for which the Reedy fibrant 2-fold Segal spaces are the fibrant objects. This is given by the following presentation, again a subset of one of the Lurie presentations in [JS17, pg. 56]:

Definition 2.3.33. *The* 2-fold Segal presentation *is the* 2*-uple Segal presentation $(\Delta^{op}, C) \boxtimes (\Delta^{op}, C)$ together with the following maps:*

1. *The **essential constancy maps**, for each $m \geq 0$, of the form*

$$F_2^0(\underbrace{\langle 0, \cdots, 0\rangle_0}_{m}, \langle 0\rangle_0) : F_2^0(m, 0) \to F_2^0(0, 0).$$

Definition 2.3.34. *Let $SeSp_2^{inj}$ be the model structure on $\boldsymbol{sSpace}_2$ obtained by localizing $\boldsymbol{sSpace}_2^{inj}$ with respect to the 2-fold Segal presentation.*

Definition 2.3.35. *Let $\boldsymbol{SeSp}_2^{inj}$ be the full subcategory of $\boldsymbol{sSpace}_2$ whose objects are the Reedy fibrant 2-fold Segal spaces.*

While Reedy fibrant 2-fold Segal spaces are an improvement on Reedy fibrant double Segal spaces, they are still incomplete as a model of $(\infty, 2)$-categories. In particular, we should expect such an $(\infty, 2)$-category X to yield genuine $(\infty, 1)$-categories $X_{n,\bullet}$ and $X_{\bullet,n}$ for all $n \geq 0$. Currently, we can only guarantee these to be Segal spaces rather than *complete* Segal spaces. It is this shortcoming we must somehow rectify.

For us, a complete 2-fold Segal space will be equivalent to a Reedy fibrant rendition of the definition given in [JS17, Def. 2.7], shifting explicitly from levelwise Segal spaces to levelwise complete Segal spaces in both coordinates. We make this decision to ensure our work remains as close as possible to efforts establishing (∞, n)-categories of cobordisms, such as in [JS17] and [CS19]. When we come to studying *projective fibrant* complete 2-fold Segal spaces, our definitions will then precisely align with these sources.

Definition 2.3.36. *A* Reedy fibrant complete 2-fold Segal space X *is a Reedy fibrant* 2*-fold Segal space such that* $X_{k,\bullet}$ *and* $X_{\bullet,k}$ *are both Reedy fibrant complete Segal spaces for all* $k \geq 0$.

Proposition 2.3.37. *There is a model category structure on* $\boldsymbol{sSpace}_2$*, which we will denote by* $CSSP_2^{inj}$*, obtained as a left Bousfield localization of* $\boldsymbol{sSpace}_2^{inj}$*, whose fibrant objects are precisely the Reedy fibrant complete* 2*-fold Segal spaces.*

Bergner and Rezk in [BR20, Theorem 5.6] make a similar statement, albeit with respect to a different definition of Reedy fibrant complete 2-fold Segal space.

Recall that the *Rezk presentation* (Δ^{op}, S) obtains the model structure $CSSP^{inj}$ whose fibrant objects are the complete Segal spaces. With this in mind, following [JS17, pg. 55-56] though building upon $\mathbf{sSpace}_2^{inj}$ rather than $\mathbf{sSpace}_2^{proj}$, our model structure is the left Bousfield localization of $\mathbf{sSpace}_2^{inj}$ with regards to the following presentation:

Definition 2.3.38. *The* 2-fold Lurie presentation *is the presentation* $(\Delta^{op}, S) \boxtimes (\Delta^{op}, S)$ *together with the essential constancy maps of the* 2*-fold Segal presentation.*

This can be generalized to all $\mathbf{sSpace}_n$ to obtain the *Lurie presentations* in [JS17, pg. 56].

There is a clear example of a Reedy fibrant complete 2-fold Segal space already available to us:

Proposition 2.3.39. *Let* $X \in \boldsymbol{Top}$*. Then* $\boldsymbol{Sing_{ssS}}(X)$ *is a Reedy fibrant complete* 2*-fold Segal space.*

2.3.3 Projective Fibrant Complete n-Fold Segal Spaces

The theories of Reedy fibrant complete Segal spaces and Reedy fibrant complete 2-fold Segal spaces seem reasonably straightforward to define in nature: one considers a model structure on $\mathbf{sSpace}_n$ for $n = 1, 2$ and localizes with respect to some relevant presentation. It seems then that there should be no obstacle to obtaining a *projective fibrant* analogue to these constructions, where one starts with $\mathbf{sSpace}_n^{proj}$ rather than $\mathbf{sSpace}_n^{inj}$. We begin investigating this insight in the case $n = 1$, reproducing the definition of projective fibrant Segal space found in [JS17, Def. 2.1] and the relevant model structure by the same localization:

Definition 2.3.40. *Let $SeSp^{proj}$ be the model structure on **sSpace** obtained by localizing $\boldsymbol{sSpace}^{proj}$ with respect to the Segal presentation.*

A projective fibrant Segal space X *is a fibrant object in the model structure $SeSp^{proj}$.*

*Let $\boldsymbol{SeSp}^{proj}$ be the full subcategory of **sSpace** whose objects are the projective fibrant Segal spaces.*

Due to the evident similarities between how projective and Reedy fibrant Segal spaces are defined, one might instinctively expect a projective fibrant Segal space to be a projective fibrant functor $X : \Delta^{op} \to \mathbf{sSet}$, meaning one where each X_n is a Kan complex, such that the Segal maps

$$X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

are weak equivalences in $\mathbf{sSet}$. However, technical issues begin to arise if we inspect this suggestion further. All we may guarantee of X is that it is local with respect to the Segal maps $F_1^0(Sp(n)) \to F_1^0(n)$, which only tells us that the maps of homotopy function complexes

$$\mathbf{map}(F_1^0(n), X) \to \mathbf{map}(F_1^0(Sp(n)), X)$$

are weak equivalences. Recall that $\mathbf{Map}_n^0(-,-)$ was proven to be a model for homotopy function complexes only in the Reedy fibrant case; this relied on the fact that all objects in $\mathbf{sSpace}_n^{inj}$ are cofibrant. We cannot depend upon such a crutch anymore in $\mathbf{sSpace}_n^{proj}$, where cofibrancy is not so straightforward to guarantee. Instead, we only have by Proposition 2.2.57 that the maps

$$\mathbf{Map}_1^0\big(Q(F_1^0(n)), X\big) \to \mathbf{Map}_1^0\big(Q(F_1^0(Sp(n))), X\big)$$

are weak equivalences, for some projective cofibrant replacement functor Q. Note that by Proposition 2.2.36, the spaces $F_1^0(n)$ are projective cofibrant. Hence, by [Hir09, Prop. 17.1.6], we have for each $n \geq 2$ that the map $\mathbf{Map}_1^0(F_1^0(n), X) \to \mathbf{Map}_1^0(Q(F_1^0(n)), X)$ induced by precomposition with the cofibrant replacement map $Q(F_1^0(n)) \to F_1^0(n)$ is a weak equivalence. Thus, for a projective fibrant Segal space X, the composite maps

$$X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1 \to \mathbf{Map}_1^0\big(Q(F_1^0(Sp(n))), X\big)$$

are weak equivalences. To proceed further would require us to identify a projective cofibrant replacement of $F_1^0(Sp(n))$.

Thankfully, we will not have to define such a replacement presently: the nuances of projective fibrant Segal spaces have been addressed by Horel in [Hor15], where a rather straightforward alternative condition is identified.

Proposition 2.3.41 ([Hor15, Prop. 2.3])**.** *A projective fibrant object $X : \Delta^{op} \to$ **sSet** in **sSpace**^{proj} is fibrant in $SeSp^{proj}$ if and only if, for all $m, n \geq 0$, the maps*

$$X_{m+n} \to X_m \times_{X_0} X_n \to X_m \times^h_{X_0} X_n$$

given by $\langle 0, \cdots, m\rangle_{m+n} : [m] \to [m+n]$ and $\langle m, \cdots, m+n\rangle_{m+n} : [n] \to [m+n]$ in Δ are weak equivalences.

Calaque and Scheimbauer note in [CS19, Rem. 1.5] that this holds if and only if the maps

$$X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1 \to X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1$$

are weak equivalences for all $n \geq 2$.

It should be noted that Johnson-Freyd and Scheimbauer in [JS17, Def. 2.1] use this latter property as their definition of a projective fibrant Segal space, though without enforcing any particular model of homotopy pullback. Horel also makes no assertion of which model of homotopy pullback is used; indeed, any model of homotopy pullbacks will yield the same result. In particular, if X is Reedy fibrant, we may use the strict pullback as a model of homotopy pullback to show directly that every Reedy fibrant Segal space is also a projective fibrant Segal space. One may alternatively use [Hor15, Prop. 2.3 (6)].

We now consider a similar definition for projective fibrant complete Segal spaces, whose model structure is obtained in precisely the same manner as in [JS17]. We will prove that the fibrant objects of this model structure are as described in [JS17, Def. 2.3] once we have studied Reedy fibrant replacement functors. It should be noted that our proof will be by no means novel; a similar proof can be obtained via [JS17, Lemma 2.5].

Definition 2.3.42. *Let $CSSP^{proj}$ be the model structure on* **sSpace** *obtained by localizing* **sSpace**^{proj} *with respect to the Rezk presentation.*

A projective fibrant complete Segal space *X is a simplicial space that is fibrant in the model structure $CSSP^{proj}$.*

Let ***CSSP***^{proj} *be the subcategory of* **sSpace** *whose objects are the projective fibrant complete Segal spaces.*

Note that the fibrant objects of this model structure will always be projective fibrant Segal spaces. It is moreover evident that every Reedy fibrant complete Segal space is also a projective fibrant such object; being local with respect to some class of maps is independent of the choice of homotopy function complex, as is noted by Hirschhorn in [Hir09, Def. 3.1.4 (1a)].

We can now establish the projective fibrant analogues of Reedy fibrant double Segal spaces, Reedy fibrant 2-fold Segal spaces and Reedy fibrant complete 2-fold Segal spaces. Once more, the model structures and fibrant objects presented here are identical to those considered in [JS17], with the model structures defined in the same manner.

Definition 2.3.43. *Let SSp_2^{proj} be the model structure on $\boldsymbol{sSpace}_2$ obtained by localizing $\boldsymbol{sSpace}_2^{proj}$ with respect to the 2-uple Segal presentation.*
A projective fibrant 2-uple Segal space *is a fibrant object in SSp_2^{proj}.*
Let $\boldsymbol{SSp}_2^{proj}$ be the category of projective fibrant 2-uple Segal spaces.

We may extract from this a characterization of projective fibrant 2-uple Segal spaces identical to the definition given by Johnson-Freyd and Scheimbauer in [JS17, Def. 2.6], though specifying the model of homotopy pullback:

Proposition 2.3.44. *Suppose $X : (\Delta^{op})^2 \to \boldsymbol{sSet}$ is a projective fibrant object in $\boldsymbol{sSpace}_2^{proj}$. Then X is fibrant in SSp_2^{proj} if and only if for all $n \geq 2$ and $k \geq 0$, the maps*

$$X_{n,k} \to X_{1,k} \times^h_{X_{0,k}} \cdots \times^h_{X_{0,k}} X_{1,k}$$
$$X_{k,n} \to X_{k,1} \times^h_{X_{k,0}} \cdots \times^h_{X_{k,0}} X_{k,1}$$

are weak equivalences. That is to say, $X_{k,\bullet}$ and $X_{\bullet,k}$ must both be projective fibrant Segal spaces for all $k \geq 0$.

Proof. We proceed by a modified version of the proof of [Hor15, Prop. 2.3]. Note that locality is invariant under levelwise weak equivalences of projective fibrant bisimplicial spaces, by [Hir09, Lemma 3.2.1]. To show the postulated condition is invariant under levelwise weak equivalences as well, we employ a modified version of the proof of [Hor15, Prop. 1.11]. Consider a levelwise weak equivalence $X \to Y$ of bisimplicial spaces. Let $n \geq 2$ and $k \geq 0$. Then we have a natural diagram in $\mathbf{sSet}$ of the form

$$\begin{tikzcd}
X_{n,k} \arrow[r] \arrow[d] & X_{1,k} \times_{X_{0,k}} \cdots \times_{X_{0,k}} X_{1,k} \arrow[r] \arrow[d] & X_{1,k} \times^h_{X_{0,k}} \cdots \times^h_{X_{0,k}} X_{1,k} \arrow[d] \\
Y_{n,k} \arrow[r] & Y_{1,k} \times_{Y_{0,k}} \cdots \times_{Y_{0,k}} Y_{1,k} \arrow[r] & Y_{1,k} \times^h_{Y_{0,k}} \cdots \times^h_{Y_{0,k}} Y_{1,k}
\end{tikzcd}$$

where the leftmost and rightmost vertical maps are weak equivalences. Thus, the uppermost composite of horizontal maps is a weak equivalence if and only if the lowermost composite is as such. A similar diagram handles the other maps in the condition. This completes the required modified version of the proof of [Hor15, Prop. 1.11].

Thus, it suffices to prove these conditions are equivalent in the case that X is in fact a Reedy fibrant bisimplicial space. If this is so, the homotopy pullbacks $X_{1,k} \times^h_{X_{0,k}} \cdots \times^h_{X_{0,k}} X_{1,k}$ and $X_{k,1} \times^h_{X_{k,0}} \cdots \times^h_{X_{k,0}} X_{k,1}$ may be modelled by strict pullbacks. Hence, X will be local with respect to the 2-uple Segal presentation if and only if it is a Reedy fibrant 2-uple Segal space, which completes the proof. $\square$

We now turn to the projective fibrant analogues of Reedy fibrant 2-fold Segal spaces and Reedy fibrant complete 2-fold Segal spaces:

Definition 2.3.45. *Let $SeSp_2^{proj}$ be the model structure on $\boldsymbol{sSpace}_2$ obtained by localizing $\boldsymbol{sSpace}_2^{proj}$ with respect to the 2-fold Segal presentation.*

A projective fibrant 2-fold Segal space *is a fibrant object in $SeSp_2^{proj}$.*

Let $\boldsymbol{SeSp}_2^{proj}$ be the category of projective fibrant 2-fold Segal spaces.

Proposition 2.3.46. *A projective fibrant object $X : (\Delta^{op})^2 \to \boldsymbol{sSet}$ in $\boldsymbol{sSpace}_2^{proj}$ is fibrant in $SeSp_2^{proj}$ if and only if it is a projective fibrant 2-uple Segal space and such that X_0 is essentially constant.*

Proof. The essential constancy maps' domains and codomains are all projective cofibrant by Proposition 2.2.36, so the concrete effect of locality with respect to these maps is the same as for the Reedy fibrant case, by using the homotopy function complex in Proposition 2.2.57. $\square$

Thus, the definition we have provided for projective fibrant 2-fold Segal spaces is the same as in [JS17, Def. 2.7], though again now specifying the model of homotopy pullback.

Definition 2.3.47. *Let $CSSP_2^{proj}$ be the model structure on $\boldsymbol{sSpace}_2$ obtained by localizing $\boldsymbol{sSpace}_2^{proj}$ with respect to the 2-fold Lurie presentation.*

A projective fibrant complete 2-fold Segal space *is a fibrant object in $CSSP_2^{proj}$.*

Let $\boldsymbol{CSSP}_2^{proj}$ be the category of projective fibrant complete 2-fold Segal spaces.

Proposition 2.3.48. *A projective fibrant object $X : (\Delta^{op})^2 \to \boldsymbol{sSet}$ in $\boldsymbol{sSpace}_2^{proj}$ is fibrant in $CSSP_2^{proj}$ if and only if it is a projective fibrant 2-fold Segal space and such that $X_{\bullet,k}$ and $X_{k,\bullet}$ are local with respect to the completeness map for all $k \geq 0$.*

Proof. A similar proof to Proposition 2.3.44 is in order. Again, locality is invariant under levelwise weak equivalence between projective fibrant bisimplicial spaces, so we may assume X is Reedy fibrant. Then locality holds if and only if X is a Reedy fibrant complete 2-fold Segal space, which is true if and only if it is a projective fibrant 2-fold Segal space such that $X_{\bullet,k}$ and $X_{k,\bullet}$ are local with respect to the completeness map. $\square$

This is precisely the definition of a projective fibrant complete 2-fold Segal space given in [JS17, Def. 2.7], though again now specifying models of homotopy pullback. We will not be so bold as to claim our proofs characterizing the fibrant objects in SSp_2^{proj}, $SeSp_2^{proj}$ or $CSSP_2^{proj}$ are novel; an alternative set of proofs is stated to be possible by Johson-Freyd and Scheimbauer in [JS17, pg. 55-56] by using [JS17, Prop. A.9], though we were unable to verify this directly ourselves. We have instead simply recycled the tactic Horel uses in [Hor15] to prove Proposition 2.3.41 in various localized model structures.

At this point, we have established that the above notions of projective fibrant Segal space, 2-uple Segal space, 2-fold Segal space and complete 2-fold Segal space all agree on the nose with the definitions given in [JS17, Sec. 2]. They also agree at the level of model structures with the model categories posed in [JS17, App. A], a fact that holds for $CSSP^{proj}$ as well; though Johnson-Freyd and Scheimbauer only explicitly declare presentations of model structures for projective fibrant complete Segal spaces, 'projective fibrant complete 2-uple Segal spaces' and projective fibrant complete 2-fold Segal spaces in this article, it is straightforward to obtain the other model structures we discuss from their work by simply removing the completeness maps.

Similarly to Reedy and projective fibrant Segal spaces, it is easily seen that a Reedy fibrant complete Segal space, 2-uple Segal space, 2-fold Segal space or complete 2-fold Segal space is necessarily a projective fibrant such object. Thus, henceforth, when we write 'Segal space', 'complete Segal space', '2-uple Segal space', '2-fold Segal space' or 'complete 2-fold Segal space', we will mean the projective fibrant cases unless stated otherwise.

It should be noted that Horel in [Hor15, Sec. 2.5] provides an explicit characterization of completeness for projective fibrant Segal spaces; we will not use this description here, opting to leave the issue until we can address it ourselves in a different manner. This will be done in tandem with obtaining *homotopy categories* of projective fibrant Segal spaces, which we will consider in depth after we begin studying Reedy fibrant replacement functors, resulting in a description identical to that of [JS17, Def. 2.3] obtained by related methods.

2.4 Biased and Unbiased Bicategories

Armed with a reasonable understanding of the domain of our homotopy bicategory construction, we are now ready to consider the codomain. Our first goal in this regard is to extend the principles of the homotopy category functor h_1 from Reedy fibrant Segal spaces to a homotopy bicategory functor for Reedy fibrant 2-fold Segal spaces. It would seem sensible then that the codomain of our functor should be a category of bicategories of some description.

We believe that a more natural target may be *unbiased bicategories*. Indeed, consider the nature of composition in a complete 2-fold Segal space X; the spaces X_n allow us to humor composites of chains of n morphisms rather than just binary composition. Such a situation would be most faithfully represented by an 'unbiased' construction, which for each n accommodates a distinct operation for composing n morphisms at once.

In the below definitions and throughout this thesis, we take the convention that for natural isomorphisms $\phi : F \Rightarrow G$, $\psi : G \Rightarrow H$ and $\theta : Q \Rightarrow R$, vertical composition is written $\psi\phi : F \Rightarrow H$ and horizontal is $\theta \circ \phi : Q \circ F \Rightarrow R \circ G$.

Definition 2.4.1 ([Lei00, Def. 1.2.1])**.** *An* unbiased bicategory $\mathscr{B}$ *consists of the following data:*

1. *A collection of objects* $\boldsymbol{ob}(\mathscr{B})$*;*

2. *For each pair of objects x and y, a category $\boldsymbol{Hom}_{\mathscr{B}}(x, y)$ of* 1*-morphisms $x \to y$ and* 2*-morphisms $f \Rightarrow g$ between them;*

3. *For each tuple of objects $X = (x_0, \cdots, x_n)$ for $n \in \mathbb{Z}_{>0}$, a functor*

$$\circ^X : \boldsymbol{Hom}_{\mathscr{B}}(x_0, x_1) \times \cdots \times \boldsymbol{Hom}_{\mathscr{B}}(x_{n-1}, x_n) \to \boldsymbol{Hom}_{\mathscr{B}}(x_0, x_n)$$

   which we alternatively write as

$$(f_1, \cdots, f_n) \mapsto (f_n \circ \cdots \circ f_1) = \bigcirc_{i=1}^n f_i$$

   for $f_i \in \boldsymbol{Hom}_{\mathscr{B}}(x_{i-1}, x_i)$ all 1*-morphisms or all* 2*-morphisms;*

4. *For each object x, a functor $\circ^x : * \to \boldsymbol{Hom}_{\mathscr{B}}(x, x)$ from the discrete singleton category, identifying a* 1*-morphism we denote as $() : x \to x$;*

5. *For each $n \in \mathbb{Z}_{>0}$ and $k_1, \cdots, k_n \in \mathbb{Z}_{\geq 0}$, for each sequence of tuples $X_i = (x_0^i, \cdots, x_{k_i}^i)$ such that $x_{k_i}^i = x_0^{i+1}$ for all $1 \leq i < n$, setting*

$$\begin{aligned} X &:= (X_1, \cdots, X_n) \\ Y &:= (x_0^1, \cdots, x_{k_1}^1, x_1^2, \cdots, x_{k_2}^2, \cdots, x_1^n, \cdots, x_{k_n}^n) \\ Z &:= (x_0^1, x_{k_1}^1, x_{k_2}^2, \cdots, x_{k_n}^n) \end{aligned}$$

   a natural isomorphism

$$\gamma_X : \circ^Z \circ (\circ^{X_1} \times \circ^{X_2} \times \cdots \times \circ^{X_n}) \Rightarrow \circ^Y$$

   between $((f_{k_n}^n \circ \cdots \circ f_1^n) \circ \cdots \circ (f_{k_1}^1 \circ \cdots \circ f_1^1))$ and $(f_{k_n}^n \circ \cdots \circ f_1^1)$, written levelwise as

$$\gamma_{((f_1^1, \cdots, f_{k_1}^1), \cdots, (f_1^n, \cdots, f_{k_n}^n))} : \bigcirc_{i=1}^n (\bigcirc_{j=1}^{k_i} f_j^i) \to \bigcirc_{i=1}^n \bigcirc_{j=1}^{k_i} f_j^i;$$

6. *For each pair of objects x and y, a natural isomorphism $\iota_{x,y} : 1_{\boldsymbol{Hom}_{\mathscr{B}}(x,y)} \Rightarrow \circ^{x,y}$ between f and (f), written levelwise as $\iota_f : f \to (f)$,*

such that:

1. *(associativity) for any $n, m_1, \cdots, m_n \in \mathbb{Z}_{>0}$, integers $k_1^1, \cdots, k_{m_n}^n \in \mathbb{Z}_{\geq 0}$ and thrice-nested sequence of objects $(((x_{p,q,r})_{r=0}^{k_q^p})_{q=1}^{m_p})_{p=1}^n$ such that $x_{p,q,k_q^p} = x_{p,q+1,0}$ for $q < m_p$ and $x_{p,m_p,k_{m_p}^p} = x_{p+1,1,0}$ for $p < n$, together with any sequence of 1-morphisms $f_{p,q,r} \in \boldsymbol{Hom}_{\mathscr{B}}(x_{p,q,r-1}, x_{p,q,r})$ for $p, q, r \geq 1$, the diagram*

$$\begin{tikzcd}[column sep=tiny]
& \bigcirc_{p=1}^n(\bigcirc_{q=1}^{m_p}(\bigcirc_{r=1}^{k_q^p} f_{p,q,r})) \arrow[dl, "\bigcirc_{p=1}^n \gamma_{D_p}"'] \arrow[dr, "\gamma_D"] & \\
\bigcirc_{p=1}^n(\bigcirc_{q=1}^{m_p} \bigcirc_{r=1}^{k_q^p} f_{p,q,r}) \arrow[dr, "\gamma_E"'] & & \bigcirc_{p=1}^n \bigcirc_{q=1}^{m_p} (\bigcirc_{r=1}^{k_q^p} f_{p,q,r}) \arrow[dl, "\gamma_F"] \\
& \bigcirc_{p=1}^n \bigcirc_{q=1}^{m_p} \bigcirc_{r=1}^{k_q^p} f_{p,q,r} &
\end{tikzcd}$$

commutes, where

(a) $D_p = ((f_{p,q,r})_{r=1}^{k_q^p})_{q=1}^{m_p}$;

(b) $D = ((\bigcirc_{r=1}^{k_q^p} f_{p,q,r})_{q=1}^{m_q})_{p=1}^n$;

(c) $E = ((f_{p,q,r})_{r=1,q=1}^{k_q^p,m_p})_{p=1}^n$;

(d) $F = ((f_{p,q,r})_{r=1}^{k_q^p})_{q=1,p=1}^{m_q,n}$;

2. *(unitality) for any sequence of objects $x_0, \cdots, x_n \in \boldsymbol{ob}(\mathscr{B})$ and 1-morphisms $f_i \in \boldsymbol{Hom}_{\mathscr{B}}(x_{i-1}, x_i)$, the diagram*

$$\begin{tikzcd}
& \bigcirc_{i=1}^n f_i \arrow[dl, "\iota_{\bigcirc_{i=1}^n f_i}"'] \arrow[dd, "1_{\bigcirc_{i=1}^n f_i}"] \arrow[dr, "\bigcirc_{i=1}^n \iota_{f_i}"] & \\
(\bigcirc_{i=1}^n f_i) \arrow[dr, "\gamma_{((f_1,\cdots,f_n))}"'] & & \bigcirc_{i=1}^n (f_i) \arrow[dl, "\gamma_{((f_1),\cdots,(f_n))}"] \\
& \bigcirc_{i=1}^n f_i &
\end{tikzcd}$$

commutes.

Note the existence of object identities given by the 1-morphisms $()$. The unitality condition on these is now implicit in what we will refer to as the *associators* γ_X, rather than what we refer to henceforth as the *unitors* $\iota_{x,y}$, which take a role closer to the unit of a monad rather than a monoid. Indeed, the analogous notion of a (small) *unbiased monoidal category* is explained in [Lei04, pg. 69-70] to be a weak algebra for a certain strict 2-monad. In [Lei00, Def. 1.2.1], Leinster absorbs the identities () into the collection of composition operations by allowing $n = 0$, so that $X = (x_0)$. It should also be noted that we have made the addition of indexing the objects in the associators γ_X and associativity conditions, along with declaring when indices are allowed to equal zero or not.

It will be necessary for us to possess some notion of pseudofunctor between these unbiased bicategories. We establish a suitable such definition presently:

Definition 2.4.2 ([Lei00, Def. 1.2.3])**.** *Let $\mathscr{B}$ and $\mathscr{C}$ be unbiased bicategories. Write the composition functors in $\mathscr{B}$ as $\circ_{\mathscr{B}}^{x_0,\cdots,x_n}$ and the same functors in $\mathscr{C}$ as $\circ_{\mathscr{C}}^{y_0,\cdots,y_n}$, for objects $x_0,\cdots,x_n$ in $\mathscr{B}$ and $y_0,\cdots,y_n$ in $\mathscr{C}$.*

An unbiased pseudofunctor *or* unbiased weak functor *$P : \mathscr{B} \to \mathscr{C}$ between $\mathscr{B}$ and $\mathscr{C}$ then consists of the following collection of data:*

1. *A mapping of objects $\boldsymbol{ob}(\mathscr{B}) \to \boldsymbol{ob}(\mathscr{C})$, mapping $x \mapsto P(x)$;*
2. *For each pair of objects x and y in $\mathscr{B}$, a functor*
$$P_{x,y} : \boldsymbol{Hom}_{\mathscr{B}}(x,y) \to \boldsymbol{Hom}_{\mathscr{C}}(P(x),P(y));$$
3. *For each $n \in \mathbb{Z}_{>0}$ and objects $x_0,\cdots,x_n$, a natural isomorphism*
$$\pi_{x_0,\cdots,x_n} : \circ_{\mathscr{C}}^{P(x_0),\cdots,P(x_n)} \circ (P_{x_0,x_1} \times \cdots \times P_{x_{n-1},x_n}) \Rightarrow P_{x_0,x_n} \circ \circ_{\mathscr{B}}^{x_0,\cdots,x_n}$$
which is levelwise a morphism $\pi_{(f_1,\cdots,f_n)} : \bigcirc_{i=1}^n P_{x_{i-1},x_i}(f_i) \to P_{x_0,x_n}(\bigcirc_{i=1}^n f_i)$ for $f_i \in \boldsymbol{Hom}_{\mathscr{B}}(x_{i-1},x_i)$;
4. *A natural isomorphism $\pi_x : \circ_{\mathscr{C}}^{P(x)} \Rightarrow P_{x,x} \circ \circ_{\mathscr{B}}^{x}$ of the form $() \to P(())$,*

such that:

1. *For any $n > 0$, integers $k_1,\cdots,k_n \in \mathbb{Z}_{\geq 0}$ and nested sequence of objects $((x_j^i)_{j=0}^{k_i})_{i=1}^n$ in $\mathscr{B}$ such that $x_{k_i}^i = x_0^{i+1}$ for $i < n$, along with morphisms $f_j^i \in \boldsymbol{Hom}_{\mathscr{B}}(x_{j-1}^i, x_j^i)$ for $1 \leq i \leq n$ and $1 \leq j \leq k_i$, the diagram*

$$\begin{tikzcd}
\bigcirc_{i=1}^n(\bigcirc_{j=1}^{k_i} P_{x_{j-1}^i,x_j^i}(f_j^i)) \arrow[rr, "\gamma_D"] \arrow[d, "\bigcirc_{i=1}^n \pi_{(f_1^i,\cdots,f_{k_i}^i)}"'] & & \bigcirc_{i=1}^n \bigcirc_{j=1}^{k_i} P_{x_{j-1}^i,x_j^i}(f_j^i) \arrow[dd, "\pi_{(f_1^1,\cdots,f_{k_n}^n)}"] \\
\bigcirc_{i=1}^n P_{x_0^i,x_{k_i}^i}(\bigcirc_{j=1}^{k_i} f_j^i) \arrow[d, "\pi_{(\bigcirc_{j=1}^{k_1} f_j^1,\cdots,\bigcirc_{j=1}^{k_n} f_j^n)}"'] & & \\
P_{x_0^1,x_{k_n}^n}(\bigcirc_{i=1}^n(\bigcirc_{j=1}^{k_i} f_j^i)) \arrow[rr, "P_{x_0^1,x_{k_n}^n}(\gamma_{D'})"'] & & P_{x_0^1,x_{k_n}^n}(\bigcirc_{i=1}^n \bigcirc_{j=1}^{k_i} f_j^i)
\end{tikzcd}$$

commutes, where:

(a) $D = ((P_{x^i_{j-1},x^i_j}(f^i_j))^{k_i}_{j=1})^n_{i=1}$;

(b) $D' = ((f^i_j)^{k_i}_{j=1})^n_{i=1}$;

2. *For each $x, y \in \boldsymbol{ob}(\mathscr{B})$ and $f \in \boldsymbol{Hom}_{\mathscr{B}}(x,y)$, the diagram*

$$\begin{tikzcd} P_{x,y}(f) \arrow[r, "\iota_{P_{x,y}(f)}"] \arrow[dr, "P_{x,y}(\iota_f)"'] & (P_{x,y}(f)) \arrow[d, "\pi_{(f)}"] \\ & P_{x,y}((f)) \end{tikzcd}$$

commutes.

Note again the subtle insertion of () here, with $\pi_{x_0} : () \to P_{x_0,x_0}(())$. This is absorbed in [Lei00, Def. 1.2.3] by allowing $n = 0$ in the definition of what we will refer to as the *compositors* $\pi_{x_0,\cdots,x_n}$. We also note that in [Lei00], Leinster only establishes the terminology of 'unbiased weak functor' to refer to instances of the above construction. We will use the term 'unbiased pseudofunctor' for them instead in this thesis.

The codomain of our homotopy bicategory functor should then be a category **UBicat** of small unbiased bicategories, namely those unbiased bicategories whose collections of objects are sets, with unbiased pseudofunctors between them. To construct this category, we must first define the composite of two unbiased pseudofunctors:

Definition 2.4.3 ([Lei00, pg. 7-8])**.** *Let $\mathscr{B}, \mathscr{C}$ and $\mathscr{D}$ be unbiased bicategories. Let $P : \mathscr{B} \to \mathscr{C}$ and $Q : \mathscr{C} \to \mathscr{D}$ be unbiased pseudofunctors, with compositors π and θ, respectively. Then define $Q \circ P : \mathscr{B} \to \mathscr{D}$ to be the unbiased pseudofunctor with:*

1. *Objects mapping as $x \mapsto Q(P(x))$;*

2. $(Q \circ P)_{x,y} := Q_{P(x),P(y)} \circ P_{x,y}$ *for $x, y \in \boldsymbol{ob}(\mathscr{B})$;*

3. *For $x_0, \cdots, x_n \in \boldsymbol{ob}(\mathscr{B})$, natural isomorphisms*

$$\psi_{x_0,\cdots,x_n} := \big(Q_{P(x_0),P(x_n)} \circ \pi_{x_0,\cdots,x_n}\big)\big(\theta_{P(x_0),\cdots,P(x_n)} \circ (P_{x_0,x_1} \times \cdots \times P_{x_{n-1},x_n})\big)$$

which are thus levelwise, for $f_i \in \boldsymbol{Hom}_{\mathscr{B}}(x_{i-1}, x_i)$, of the form

$$\bigcirc^n_{i=1} Q_{P(x_{i-1}),P(x_i)}(P_{x_{i-1},x_i}(f_i)) \to Q_{P(x_0),P(x_n)}(P_{x_0,x_n}(\bigcirc^n_{i=1} f_i));$$

4. *For $x \in \boldsymbol{ob}(\mathscr{B})$, natural isomorphisms $\psi_x := \big(Q_{x,x} \circ \pi_x\big)\theta_{P(x)}$.*

Again, in [Lei00, pg. 7-8], the natural isomorphisms ψ_x are handled as a special case of the more general $\psi_{x_0,\cdots,x_n}$.

That this procedure defines a genuine unbiased pseudofunctor is routine. One may also check that the induced composition operation for unbiased pseudofunctors is associative and unital, with identity unbiased pseudofunctors being identities on objects and hom-categories with trivial natural isomorphisms, as noted in [Lei00, pg. 8]. Thus, we have the following:

Proposition 2.4.4 ([Lei00, pg. 8]). *There is a category* ***UBicat****, whose objects are small unbiased bicategories and whose morphisms are unbiased pseudofunctors, with composition defined as in Definition 2.4.3.*

In [Lei00], Leinster denotes this category as $\mathbf{UBicat}_{\mathrm{wk}}$, though we have added the assumption of smallness.

2.4.1 Classical Versus Unbiased Bicategories

One might balk at the definition of an unbiased bicategory. Why make use of such a definition when classical bicategories are better understood and more widely used? They have existed since Bénabou's definition in [Bén67] and remain one of the oldest and best-understood instances of a higher category.

Such questions can be answered with the insight that nothing has truly been lost. Unbiased bicategories can be converted into bicategories easily enough:

Definition 2.4.5 ([Lei00, pg. 10]). *Let $\mathscr{B}$ be an unbiased bicategory. The* underlying bicategory $\tilde{\mathscr{B}}$ *is the classical bicategory given by the following data:*

1. *Objects and hom-categories are as in $\mathscr{B}$;*

2. *The composition functors $\circ^{x,y,z} : \boldsymbol{Hom}_{\tilde{\mathscr{B}}}(x,y)\times\boldsymbol{Hom}_{\tilde{\mathscr{B}}}(y,z) \to \boldsymbol{Hom}_{\tilde{\mathscr{B}}}(x,z)$ for $x,y,z \in \boldsymbol{ob}(\tilde{\mathscr{B}})$ are the binary composition functors in $\mathscr{B}$;*

3. *Identity functors $1_x : * \to \boldsymbol{Hom}_{\tilde{\mathscr{B}}}(x,x)$ for all $x \in \boldsymbol{ob}(\tilde{\mathscr{B}})$ are given by the functors $\circ^x$ in $\mathscr{B}$;*

4. *The associators $\gamma_{w,x,y,z}$ for $w,x,y,z \in \boldsymbol{ob}(\tilde{\mathscr{B}})$ are given by the tuples $X_1 := ((w,x),(x,y,z))$ and $X_2 := ((w,x,y),(y,z))$, along with the composites*

$$\alpha_{w,x,y,z} := \big(1_{\circ^{w,y,z}} \circ (1_{\circ^{x,y,z}} \times \iota_{y,z}^{-1})\big)\gamma_{X_2}^{-1}\gamma_{X_1}\big(1_{\circ^{w,x,z}} \circ (\iota_{w,x} \times 1_{\circ^{x,y,z}})\big)$$

of domain $\circ^{w,x,z} \circ (1_{\boldsymbol{Hom}_{\tilde{\mathscr{B}}}(w,x)} \times \circ^{x,y,z})$ and codomain $\circ^{w,y,z} \circ (\circ^{w,x,y} \times 1_{\boldsymbol{Hom}_{\tilde{\mathscr{B}}}(y,z)})$;

5. *The unitors* $\lambda_{x,y}$ *and* $\rho_{x,y}$ *for* $x, y \in \boldsymbol{ob}(\tilde{\mathscr{B}})$ *are given by the natural isomorphisms*

$$\rho_{x,y} := \iota_{x,y}^{-1}\gamma_{(x),(x,y)}\big(1_{\circ^{x,x,y}}\circ(1_{\circ^x}\times\iota_{x,y})\big) : \circ^{x,x,y}\circ(\circ^x\times 1_{\boldsymbol{Hom}_{\tilde{\mathscr{B}}}(x,y)}) \Rightarrow 1_{\boldsymbol{Hom}_{\tilde{\mathscr{B}}}(x,y)}$$

and

$$\lambda_{x,y} := \iota_{x,y}^{-1}\gamma_{(x,y),(y)}\big(1_{\circ^{x,y,y}}\circ(\iota_{x,y}\times 1_{\circ^y})\big) : \circ^{x,y,y}\circ(1_{\boldsymbol{Hom}_{\tilde{\mathscr{B}}}(x,y)}\times\circ^y) \Rightarrow 1_{\boldsymbol{Hom}_{\tilde{\mathscr{B}}}(x,y)}.$$

The pentagon and triangle axioms may be checked to hold from the axioms of an unbiased bicategory. Indeed, the pentagon axiom intuitively follows from pondering the following diagram:

$$\begin{tikzcd}
 & & ((fg)h)k \arrow[dll, dotted] \arrow[dl, dashed] \arrow[dr, dashed] \arrow[drr, dotted] & & \\
(f(gh))k \arrow[r, dashed] \arrow[dr, dashed] \arrow[ddr, dotted] & (fgh)k \arrow[r] & fghk & (fg)hk \arrow[l] & (fg)(hk) \arrow[l, dashed] \arrow[dl, dashed] \arrow[ddl, dotted] \\
 & f(gh)k \arrow[ur] & f(ghk) \arrow[u] & fg(hk) \arrow[ul] & \\
 & f((gh)k) \arrow[u, dashed] \arrow[ur, dashed] \arrow[rr, dotted] & & f(g(hk)) \arrow[ul, dashed] \arrow[u, dashed] &
\end{tikzcd}$$

The dotted arrows are the newly constructed associators in the supposed biased bicategory, while the dashed and solid arrows are associators in the original unbiased bicategory. Commutativity now follows from all the five triangles and quadrilaterals in the pentagon's interior commuting. The triangles commute by definition, while the quadrilaterals are each an instance of the associativity condition of an unbiased bicategory. We should thus not be surprised to obtain a bicategory from Definition 2.4.5. Note that this reasoning is not entirely rigorous, due to some subtle unitors lurking in the dashed and solid arrows. Moreover, this approach does not appear to be used by Leinster to prove that the above construction yields a bicategory in [Lei00, App. A]. We are at present unaware of its existence elsewhere in the literature.

One may also construct a pseudofunctor between classical bicategories from an unbiased pseudofunctor, by restricting to binary composition and inducing the natural isomorphisms in a similar manner [Lei00, pg. 11]. The laws of a pseudofunctor are then induced. This conversion induces a functor $\tilde{\bullet}$: **UBicat** $\to$ **Bicat**, to the category of bicategories and pseudofunctors:

Theorem 2.4.6 ([Lei00, Thm. 1.3.1, Cor. 1.3.2])**.** *For any bicategory* $\mathscr{B}$*,* $\tilde{\mathscr{B}}$ *is a bicategory. Moreover,* $\tilde{\bullet}$ *sends unbiased pseudofunctors to pseudofunctors and defines a genuine functor from* ***UBicat*** *to the category* ***Bicat*** *of bicategories and pseudofunctors. This functor is fully faithful and essentially surjective.*

It is clear however that there is no 'canonical' inverse to the functor $\tilde{\bullet}$. There are many ways to obtain an unbiased bicategory from a biased one, with an infinite number of choices resulting just from building the unbiased compositions out of different nestings of binary composition operations. We will not explore this issue further here; that an inverse exists will suffice for our needs.

We finish our exposition on unbiased bicategories with a consideration of equivalences. There is a natural notion of equivalence between unbiased bicategories; we are unaware if this notion has been considered before or defined in the literature, though it is hardly profound or novel enough for us to claim originality for rigorously defining it here. Indeed, it is simply a pseudofunctor whose image under $\tilde{\bullet}$ is an equivalence of bicategories:

Definition 2.4.7. *An* equivalence of unbiased bicategories *is an unbiased pseudofunctor $F : \mathscr{C} \to \mathscr{D}$ that satisfies the following two properties:*

1. Full faithfulness*: The functors on hom-categories*

$$\boldsymbol{Hom}_{\mathscr{C}}(x, y) \to \boldsymbol{Hom}_{\mathscr{D}}(F(x), F(y))$$

are all equivalences of categories;

2. Essential surjectivity*: For all $y \in \boldsymbol{ob}(\mathscr{D})$, there exists some $x \in \boldsymbol{ob}(\mathscr{C})$ and an internal equivalence $F(x) \simeq y$ in $\mathscr{D}$.*

We will make use of this notion when we come to characterizing weak equivalences between complete 2-fold Segal spaces using our homotopy bicategory constructions.

Chapter 3

Homotopy Bicategories of 2-fold Segal Spaces

We now turn to the problem of obtaining the *homotopy bicategory* $h_2(X)$ of a 2-fold Segal space X. We will first consider the case of Reedy fibrant 2-fold Segal spaces, as these admit a more immediate method to construct composition operations, associators and higher coherence conditions.

We begin by investigating a more standard approach to obtaining homotopy bicategories, built upon existing approaches to related problems in the literature. This will involve composition operations being chosen after truncation of hom-categories rather than before. We discuss our complaints with this strategy and proceed to constructing our proposed method, which obtains composition operations, coherence isomorphisms and coherence conditions in the original $(\infty, 2)$-category before truncation.

In short, our proposed homotopy bicategory for a Reedy fibrant 2-fold Segal space X will have as objects the elements of the set $(X_{0,0})_0$ of 0-simplices and as hom-categories the categories $h_1(X(x, y))$ for $x, y \in (X_{0,0})_0$, while composition operations will be induced by chosen solutions μ_n to lifting problems

$$\begin{tikzcd}
 & X_n \arrow[r, "X_{\langle 0,n\rangle}"] \arrow[d, "\gamma_n"] & X_1 \\
X_1 \times_{X_0} \cdots \times_{X_0} X_1 \arrow[ur, dashed, "\mu_n"] \arrow[r, "id"'] & X_1 \times_{X_0} \cdots \times_{X_0} X_1 &
\end{tikzcd}$$

then fibered over objects $x_0, \cdots, x_n$ and converted to functors by applying h_1. Associators will be given by naturally induced *homotopies* between nested composition operations, while coherence conditions will be in turn given by *2-homotopies* between composites of these homotopies. The precise definitions of all of these notions will be established in due course.

Our construction goes farther than just obtaining bicategories. Indeed, it functorially obtains weak pseudofunctors from maps $X \to Y$ in $\mathbf{SeSp}_2^{inj}$

between the corresponding homotopy bicategories by similar methods. We believe that this calculus of inductively obtaining higher homotopies between homotopies should extend to constructions of homotopy n-category functors for more general (∞, n)-categories, as well as to other higher structures, like homotopy symmetric monoidal bicategories of symmetric monoidal $(\infty, 2)$-categories. These topics are beyond the scope of this thesis and are left to future work.

For the remainder of this chapter, we assume 2-fold Segal spaces and Segal spaces are Reedy fibrant unless otherwise stated. Moreover, we introduce some new notation for mapping spaces defined by a k-uple simplicial space of dimension $k < 2$, in an effort to simplify our forthcoming notation:

Notation 3.0.1. *Suppose $X \in \boldsymbol{sSpace}_k$. Let $0 \leq m < k$. Write*

$$X_\bullet : (\boldsymbol{sSpace}_m)^{op} \to \boldsymbol{sSpace}_{k-m-1}$$

for the functor $\boldsymbol{Map}_k^{k-m-1}(F_k^m(\bullet), X)$, sending

$$K \mapsto X_K := \boldsymbol{Map}_k^{k-m-1}(F_k^m(K), X).$$

For instance, we have for $X \in \mathbf{sSpace}_k$ that

$$\begin{aligned}
X_{\Delta[n]} &\cong X_n = X_{n,\bullet,\cdots,\bullet} \\
X_{Sp(n)} &\cong X_1 \times_{X_0} \cdots \times_{X_0} X_1 \\
&= X_{1,\bullet,\cdots,\bullet} \times_{X_{0,\bullet,\cdots,\bullet}} \cdots \times_{X_{0,\bullet,\cdots,\bullet}} X_{1,\bullet,\cdots,\bullet} \\
X_{\partial\Delta[n]} &\cong M_n X \\
X_{\partial\Delta[i_k,\cdots,i_1]} &\cong M_{([i_k],\cdots,[i_1])} X.
\end{aligned}$$

It is clear that if $j : K \to L$ is a Reedy cofibration of m-uple simplicial spaces, the map $j^* = X_j : X_L \to X_K$ is a Reedy fibration of $(k - m - 1)$-uple simplicial spaces.

In particular, setting $k = 2$ and $m = 0$ will give us, for $X \in \mathbf{sSpace}_2$ and $K_1, K_2 \in \mathbf{sSet}$, a simplicial space X_{K_1} and a simplicial set $X_{K_1,K_2} := (X_{K_1})_{K_2}$. Moreover, if $Q \in \mathbf{sSpace}$, we get a simplicial set X_Q.

3.1 A 2-Truncated Construction

Before all else, we consider the approach outlined by Johson-Freyd and Scheimbauer in [JS17, Def. 2.12], modified to suit the case of Reedy fibrant 2-fold Segal spaces rather than projective fibrancy. We will cohere this conjectured construction by making use of the techniques presented by Lack and Paoli in [LP08, Sec. 7], which are applied to obtaining a concrete bicategory from a given Tamsamani 2-category. We will modify these methods to better accord with the task at hand.

Consider some 2-fold Segal space X. We wish for our homotopy bicategory $h_2(X)$ to have as its objects the elements of the set $(X_{0,0})_0$, while its morphisms should be obtained from the categories $h_1(X(x,y))$ for all $x, y \in (X_{0,0})_0$. To obtain horizontal composition operations in $h_2(X)$, recall first the notation $(-)^{x_0,\cdots,x_n}$ in Notation 2.3.7. Consider some $x_0, \cdots, x_n \in (X_{0,0})_0$. As h_1 commutes with products up to natural isomorphism, a fact we will prove in Lemma 3.3.1, we then note that the morphisms

$$h_1(\gamma_n^{x_0,\cdots,x_n}) : h_1(X_n^{x_0,\cdots,x_n}) \to h_1(X_{Sp(n)}^{x_0,\cdots,x_n}) \cong h_1(X(x_0,x_1))\times\cdots\times h_1(X(x_{n-1},x_n))$$

are the images of trivial fibrations $\gamma_n^{x_0,\cdots,x_n}$ and as such are equivalences of categories. Taking inspiration from Lack and Paoli in [LP08, pg. 16], composition operations may thus be obtained using what we term *fillers* for the Segal maps, which are simply a matter of notational convenience:

Definition 3.1.1. *Consider a span of categories $\mathscr{C} \xleftarrow{g} \mathscr{D} \xrightarrow{f} \mathscr{E}$ such that f is an equivalence. Then a* lax filler *for this span is a pair (h, η) of a functor $h : \mathscr{E} \to \mathscr{C}$ and natural transformation $\eta : g \Rightarrow h \circ f$ making the diagram*

$$\begin{tikzcd} \mathscr{D} \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\Searrow\eta"] & \mathscr{E} \arrow[dl, dashed, "h"] \\ \mathscr{C} & \end{tikzcd}$$

commute. It is called a filler *if η is a natural isomorphism.*

Proposition 3.1.2. *Let $\mathscr{C} \xleftarrow{g} \mathscr{D} \xrightarrow{f} \mathscr{E}$ be a span of categories such that f is an equivalence. Then there exists a filler for this span.*

Proof. Set $h := g \circ \overline{f}$, for some inverse $\overline{f} : \mathscr{E} \to \mathscr{D}$ of f. With this inverse is associated some natural isomorphism $\beta : 1_{\mathscr{D}} \Rightarrow \overline{f} \circ f$. Thus, the pair $(h, 1_g \circ \beta)$ constitutes a filler for the span as needed. $\square$

Consider then the span in $\mathbf{SeSp}^{inj}$ of the form

$$X(x_0,x_n) \xleftarrow{X_{\langle 0,n\rangle}^{x_0,x_n}} X_n^{x_0,x_n} \hookleftarrow X_n^{x_0,\cdots,x_n} \xrightarrow{\gamma_n^{x_0,\cdots,x_n}} X_{Sp(n)}^{x_0,\cdots,x_n} \cong \prod_{i=1}^{n} X(x_{i-1},x_i).$$

It is indeed the case that this is a span in $\mathbf{SeSp}^{inj}$, as $(-)^{a_0,\cdots,a_k}$ preserves fibrations and thus fibrancy in any model structure on **sSpace**.

As we have noted earlier, by Proposition 2.3.29, the right-facing map is a Reedy trivial fibration and thus weak equivalence of Segal spaces in $CSSP^{inj}$.

Therefore, it is a Dwyer-Kan equivalence and will be sent by h_1 to an equivalence of categories.

By Proposition 3.1.2, we may thus immediately obtain a filler for the image of the above span under h_1. This amounts to a functor $\bullet^{x_0,\cdots,x_n}$ and natural isomorphism $\eta^{x_0,\cdots,x_n}$ making the diagram

$$\begin{tikzcd}
h_1(X_n^{x_0,\cdots,x_n}) \arrow[r, "h_1(\gamma_n^{x_0,\cdots,x_n})"] \arrow[d, hook] \arrow[dr, phantom, "\Rightarrow \eta^{x_0,\cdots,x_n}"] & h_1(X(x_0,x_1)) \times \cdots \times h_1(X(x_{n-1},x_n)) \arrow[ddl, dashed, "\bullet^{x_0,\cdots,x_n}"] \\
h_1(X_n^{x_0,x_n}) \arrow[d, "h_1(X^{x_0,x_n}_{\langle 0,n\rangle})"'] & \\
h_1(X(x_0,x_n)) &
\end{tikzcd}$$

commute.

We immediately package the above choices into a single structure:

Definition 3.1.3. *Let $X \in \boldsymbol{SeSp}_2^{inj}$. A* choice of 2-truncated compositions *is a choice, for each $n \geq 0$ and $x_0, \cdots, x_n \in (X_{0,0})_0$, of one filler $(\bullet^{x_0,\cdots,x_n}, \eta^{x_0,\cdots,x_n})$ for each span*

$$h_1(X(x_0,x_n)) \leftarrow h_1(X_n^{x_0,\cdots,x_n}) \rightarrow \prod_{i=1}^{n} h_1(X(x_{i-1},x_i)).$$

We write these choices collectively as $(\bullet^{x_0,\cdots,x_n}, \eta^{x_0,\cdots,x_n})_{n\geq 0}^{x_0,\cdots,x_n \in (X_{0,0})_0}$.

Proposition 3.1.4. *Suppose $X \in \boldsymbol{SeSp}_2^{inj}$. Then there exists a choice of 2-truncated compositions for X.*

It should be noted that our construction now differs from the approach of Lack and Paoli for Tamsamani 2-categories in [LP08, Sec. 7], beyond just being tailored to 2-fold Segal spaces and unbiased bicategories: the identity maps $\bullet^x$ will not necessarily be given by degeneracies as Lack and Paoli specify for Tamsamani 2-categories, but rather by any possible filler for the appropriate span of categories. When we come to defining homotopy bicategories before truncation to hom-categories, we will turn to obtaining identities from degeneracy maps in this manner as well.

Choices of 2-truncated compositions paired with 2-fold Segal spaces can be neatly arranged to form a category:

Definition 3.1.5. *Define $\boldsymbol{SeSp}_2^{2comp}$ to be the category whose objects are pairs $(X, (\bullet^{x_0,\cdots,x_n}, \eta^{x_0,\cdots,x_n})_{n\geq 0}^{x_0,\cdots,x_n \in (X_{0,0})_0})$ where $X \in \boldsymbol{SeSp}_2^{inj}$ and*

$$(\bullet^{x_0,\cdots,x_n}, \eta^{x_0,\cdots,x_n})_{n\geq 0}^{x_0,\cdots,x_n \in (X_{0,0})_0}$$

is a choice of 2*-truncated compositions for* X*, and whose morphisms*

$$(X, (\bullet_X^{x_0,\cdots,x_n}, \eta_X^{x_0,\cdots,x_n})_{n\geq 0}^{x_0,\cdots,x_n\in(X_{0,0})_0}) \to (Y, (\bullet_Y^{y_0,\cdots,y_n}, \eta_Y^{y_0,\cdots,y_n})_{n\geq 0}^{y_0,\cdots,y_n\in(Y_{0,0})_0})$$

are maps $X \to Y$ *in* $\boldsymbol{SeSp}_2^{inj}$*.*

We seek a homotopy bicategory functor that yields bicategories determined by choices of 2-truncated compositions. Our discussion of unbiased bicategories leads us to more precisely seek a homotopy *unbiased* bicategory functor, of the form

$$h_2^{tr} : \mathbf{SeSp}_2^{2comp} \to \mathbf{UBicat}.$$

Our definition for choices of 2-truncated compositions was designed with the codomain **UBicat** in mind; each $\bullet^{x_0,\cdots,x_n}$ in such a choice determines the n-ary composition functor we will use between the relevant hom-categories.

We write h_2^{tr} for our desired functor rather than simply h_2 to emphasize the use of choices of 2-truncated compositions. We save the term h_2 for later, when we establish how to choose composition operations and coherence isomorphisms within the original 2-fold Segal space X directly.

Note that we can easily remove the decoration of 2-truncated compositions if we prefer:

Proposition 3.1.6. *The forgetful functor* $\boldsymbol{SeSp}_2^{2comp} \to \boldsymbol{SeSp}_2^{inj}$ *is an equivalence of categories.*

Proof. Essential surjectivity is immediate, while full faithfulness is by inspection of the definition of a morphism in $\mathbf{SeSp}_2^{2comp}$. □

Thus, there exists an inverse to this functor which we can precompose with to obtain an alternative functor

$$h_2' : \mathbf{SeSp}_2^{inj} \to \mathbf{UBicat}$$

where the choices of compositions are left up to mere existence.

Let us return to the construction of the homotopy bicategory of some 2-fold Segal space X equipped with a choice $(\bullet^{x_0,\cdots,x_n}, \eta^{x_0,\cdots,x_n})_{n\geq 0}^{x_0,\cdots,x_n\in(X_{0,0})_0}$ of 2-truncated compositions. The coherence isomorphisms and coherence conditions stem from some standard facts about categories, which we were inspired to consider by Lack and Paoli's construction of associators and unitors from Tamsamani 2-categories in [LP08, pg. 17-18]:

Definition 3.1.7. *Suppose* $\mathscr{C} \xleftarrow{g} \mathscr{D} \xrightarrow{f} \mathscr{E}$ *is a span of categories such that* f *is an equivalence. Let* (h, η) *and* (k, ϵ) *be lax fillers for this span.*

A morphism of lax fillers $\phi : (h, \eta) \to (k, \epsilon)$ *is a natural transformation* $\phi : h \Rightarrow k$ *such that*

$$\epsilon = (\phi \circ 1_f)\eta.$$

It is a morphism of fillers *if both lax fillers are in fact fillers.*

Proposition 3.1.8. *Let $\mathscr{C} \xleftarrow{g} \mathscr{D} \xrightarrow{f} \mathscr{E}$ be a span of categories such that f is an equivalence. Suppose (h, η) is a filler and (k, ϵ) a lax filler for this span. Then there exists a unique morphism of lax fillers from (h, η) to (k, ϵ).*

Proof. For existence, note that since f is an equivalence there is some functor $\overline{f} : \mathscr{E} \to \mathscr{D}$ and a natural isomorphism $\beta : f \circ \overline{f} \Rightarrow 1_{\mathscr{E}}$. We then obtain a natural isomorphism

$$h \xRightarrow{1_h \circ \beta^{-1}} h \circ f \circ \overline{f} \xRightarrow{(\epsilon\eta^{-1}) \circ 1_{\overline{f}}} k \circ f \circ \overline{f} \xRightarrow{1_k \circ \beta} k.$$

For uniqueness, consider two such morphisms $\phi, \psi : (h, \eta) \to (k, \epsilon)$. We have immediately that

$$\phi \circ 1_f = \epsilon\eta^{-1} = \psi \circ 1_f.$$

This in particular induces a natural isomorphism

$$\phi \circ \beta = (1_h \circ \beta)(\phi \circ 1_f \circ 1_{\overline{f}}) = (1_h \circ \beta)(\psi \circ 1_f \circ 1_{\overline{f}}) = \psi \circ \beta.$$

Thus, we have

$$\phi = (\phi \circ \beta)(1_h \circ \beta^{-1}) = (\psi \circ \beta)(1_h \circ \beta^{-1}) = \psi$$

as needed. □

This implies in particular that a filler (h, η) for the above span is precisely a left Kan extension of g along an equivalence of categories f, as defined for instance in [Rie14, Def. 1.1.1]. A similar proof shows such a filler must be a right Kan extension as well. Thus, a choice of 2-truncated composition operations is precisely a choice of Kan extensions of the 'target morphism' maps $h_1(X^{x_0,x_n}_{\langle 0,n\rangle})$ along the Segal maps $h_1(\gamma_n^{x_0,\cdots,x_n})$.

Proposition 3.1.8 relies heavily on the choice of natural isomorphism η in a filler (h, η). Without this choice being specified and the commutativity property being demanded of morphisms between fillers, there could exist many such natural isomorphisms. Consider for instance the category $\mathbf{Set}^{\mathbb{Z}/2\mathbb{Z}}$ of functors from the group $\mathbb{Z}/2\mathbb{Z}$ seen as a category into $\mathbf{Set}$. This is precisely the category of sets S equipped with an involution $p : S \to S$, meaning $p^2 = 1$, together with functions equivariant under these involutions. Then the span

$$\mathbf{Set}^{\mathbb{Z}/2\mathbb{Z}} \xleftarrow{id} \mathbf{Set}^{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{id} \mathbf{Set}^{\mathbb{Z}/2\mathbb{Z}}$$

is such that fillers (h, η) may be defined, as the right-facing map is an equivalence. Consider $h := id : \mathbf{Set}^{\mathbb{Z}/2\mathbb{Z}} \to \mathbf{Set}^{\mathbb{Z}/2\mathbb{Z}}$. The functor h then admits two natural automorphisms, given by $1_{id} : id \Rightarrow id$ and the automorphism $P : id \Rightarrow id$ given by the involutions. However, P is not a morphism of fillers from $(id, 1_{id})$ to $(id, 1_{id})$ as it fails the commutativity requirement with 1_{id}.

Likewise, 1_{id} fails the commutativity requirement to be a morphism of fillers from $(id, 1_{id})$ to (id, P). The unique morphism in the former case is 1_{id}, while it is P in the latter case.

We may now package fillers and morphisms thereof into a category, with compositions and identities being given in an evident manner:

Definition 3.1.9. *Let $\mathscr{C} \xleftarrow{g} \mathscr{D} \xrightarrow{f} \mathscr{E}$ be a span of categories such that f is an equivalence. Then define $\boldsymbol{Fill}(g, f)$ be the category whose objects are fillers (h, η) of this span and whose morphisms $(h, \eta) \to (h', \eta')$ are morphisms of fillers.*

Note that this is a subcategory of ${}_{g/}(\mathscr{C}^{\mathscr{D}})$, consisting solely of the morphisms $\eta : g \Rightarrow h \circ f$ which are moreover isomorphisms. By Proposition 3.1.8, this is a rather simple subcategory:

Corollary 3.1.10. *$\boldsymbol{Fill}(g, f)$ is a contractible groupoid.*

This implies in particular that every morphism of fillers is given by a natural isomorphism.

We now have sufficient infrastructure to construct associators and unitors for our homotopy bicategories $h_2^{tr}(X)$, following the approach of [LP08] though adapted instead to the unbiased setting. Consider some $n \in \mathbb{Z}_{>0}$ and $k_1, \cdots, k_n \in \mathbb{Z}_{\geq 0}$. Let each $T_i := (x_0^i, \cdots, x_{k_i}^i)$ be a tuple of elements of $(X_{0,0})_0$ such that for all $i < n$ we have $x_{k_i}^i = x_0^{i+1}$. Set $T := (T_1, \cdots, T_n)$ and define

$$S := (x_0^1, \cdots, x_{k_1}^1, x_1^2, \cdots, x_{k_2}^2, \cdots, x_1^n, \cdots, x_{k_n}^n).$$

Let the elements of S be enumerated as $x_0, \cdots, x_r$. Moreover, consider the tuple $R = (x_0^1, x_{k_i}^1, x_{k_2}^2, \cdots, x_{k_n}^n)$. We then have that the span

$$h_1(X(x_0, x_r)) \leftarrow h_1(X_r^{x_0, \cdots, x_r}) \to \prod_{i=1}^{r} h_1(X(x_{i-1}, x_i))$$

has two natural fillers, given by the diagrams

$$\begin{tikzcd}
h_1(X_r^S) \arrow[r, "\simeq"] \arrow[d] & \prod_{i=1}^n h_1(X_{k_i}^{T_i}) \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\Rightarrow \prod_{i=1}^n \eta^{T_i}"] & \prod_{i=1}^r h_1(X(x_{i-1}, x_i)) \arrow[dl, "\prod_{i=1}^n \bullet^{T_i}"] \\
h_1(X_n^R) \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\Rightarrow \eta^R"] & \prod_{i=1}^n h_1(X(x_0^i, x_{k_i}^i)) \arrow[dl, "\bullet^R"] & \\
h_1(X(x_0, x_r)) & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
h_1(X_r^S) \arrow[rr, "\simeq"] \arrow[d] \arrow[r, phantom, "\Rightarrow \eta^S"] & & h_1(X(x_0,x_1)) \times \cdots \times h_1(X(x_{r-1},x_r)) \arrow[dll, dashed, "\bullet^S"] \\
h_1(X(x_0,x_r)) & &
\end{tikzcd}$$

Thus, there must be a unique morphism of fillers

$$\gamma_T : \bullet^R \circ \prod_{i=1}^{n} \bullet^{T_i} \Rightarrow \bullet^S$$

between them. The resulting natural isomorphisms γ_T will be our associators.

For the unitors, note for any $x, y \in (X_{0,0})_0$ that we have a diagram

$$\begin{tikzcd}
h_1(X(x,y)) \arrow[r, "id"] \arrow[d, "id"'] \arrow[r, phantom, "\Rightarrow id"'] & h_1(X(x,y)) \arrow[dl, "id"] \\
h_1(X(x,y)) &
\end{tikzcd}$$

so that there is a unique induced morphism of fillers

$$\iota_{x,y} : id \Rightarrow \bullet^{x,y}.$$

To prove the unitality axiom, consider some $n \geq 0$ and sequence of objects $x_0, \cdots, x_n \in (X_{0,0})_0$. We have that the three relevant natural isomorphisms are all induced by morphisms of fillers between the filler

$$\begin{tikzcd}
h_1(X_n^{x_0,\cdots,x_n}) \arrow[rr] \arrow[d] \arrow[r, phantom, "\Rightarrow \eta^{x_0,\cdots,x_n}"] & & h_1(X(x_0,x_1)) \times \cdots \times h_1(X(x_{n-1},x_n)) \arrow[dll, "\bullet^{x_0,\cdots,x_n}"] \\
h_1(X(x_0,x_n)) & &
\end{tikzcd}$$

and itself again. Hence, all three composites must be equal by Corollary 3.1.10.

For associativity, consider as required some $n, m_1, \cdots, m_n > 0$ and some sequence $k_1^1, \cdots, k_{m_n}^n \geq 0$. Consider some sequence of tuples of elements of $(X_{0,0})_0$

$$R_{p,q} := (x_{p,q,r})_{r=0}^{k_q^p}$$

for $1 \leq p \leq n$ and $1 \leq q \leq m_p$; we demand that $x_{p,q,k_q^p} = x_{p,q+1,0}$ for $q < m_p$ and $x_{p,m_p,k_{m_p}^p} = x_{p+1,1,0}$ for $p < n$. Set each list

$$Q_p := (x_{p,1,0}, \cdots, x_{p,1,k_1^p}, x_{p,2,1}, \cdots, x_{p,m_p,k_{m_p}^p})$$

to be alternatively written as $(x_{p,0}, \cdots, x_{p,s_p})$; these are the flattened versions of the nested lists $(R_{p,1}, \cdots, R_{p,m_p})$ for each $1 \leq p \leq n$. Next, set

$$P := (x_{1,0}, \cdots, x_{1,s_1}, x_{2,1}, \cdots, x_{n,s_n})$$

to be alternatively written as $(x_0, \cdots, x_r)$, which is the flattened version of the list $(Q_1, \cdots, Q_n)$. Then, for each $1 \leq p \leq n$ write

$$M_p := (x_{p,1,0}, x_{p,1,k_1^p}, x_{p,2,k_2^p}, \cdots, x_{p,m_p,k_{m_p}^p})$$

and write

$$M := (x_{1,1,0}, x_{1,1,k_1^1}, \cdots, x_{1,m_1,k_{m_1}^1}, x_{2,1,k_1^2}, \cdots, x_{n,m_n,k_{m_n}^n}).$$

This is the flattened version of the nested list $(M_1, \cdots, M_n)$. Finally, write

$$N := (x_{1,0}, x_{1,s_1}, x_{2,s_2}, \cdots, x_{n,s_n}).$$

We have that the two paths in the desired commutative diagram of natural isomorphisms constitute two induced morphisms of fillers from the induced filler

$$\begin{tikzcd}
h_1(X_r^P) \arrow[r] \arrow[d] & \prod_{p=1}^n h_1(X_{s_p}^{Q_p}) \arrow[r] \arrow[d] & \prod_{p,q=1}^{n,m_p} h_1(X_{k_q^p}^{R_{p,q}}) \arrow[r] \arrow[d] & \prod_{i=1}^r h_1(X(x_{i-1}, x_i)) \arrow[dl, "\prod_{p,q=1}^{n,m_p} \bullet^{R_{p,q}}"] \\
h_1(X^M_{\sum_{p=1}^n m_p}) \arrow[r] \arrow[d] & \prod_{p=1}^n h_1(X_{m_p}^{M_p}) \arrow[r] \arrow[d] & \prod_{p,q=1}^{n,m_p} h_1(X(x_{p,q,0}, x_{p,q,k_q^p})) \arrow[dl, "\prod_{p=1}^n \bullet^{M_p}"] & \\
h_1(X_n^N) \arrow[r] \arrow[d] & \prod_{p=1}^n h_1(X(x_{p,0}, x_{p,s_p})) \arrow[dl, "\bullet^N"] & & \\
h_1(X(x_0, x_r)) & & &
\end{tikzcd}$$

to the filler

$$\begin{tikzcd}
h_1(X_r^P) \arrow[rr] \arrow[d] \arrow[r, phantom, "\Rrightarrow \eta^P"] & & h_1(X(x_0, x_1)) \times \cdots \times h_1(X(x_0, x_r)) \arrow[dll, "\bullet^P"] \\
h_1(X(x_0, x_r)) & &
\end{tikzcd}$$

so that they must be equal, by Corollary 3.1.10.

These insights culminate in the following result:

Theorem 3.1.11. *Suppose X is a Reedy fibrant 2-fold Segal space, with a choice of 2-truncated compositions $(\bullet^{x_0,\cdots,x_n}, \eta^{x_0,\cdots,x_n})_{n\geq 0}^{x_0,\cdots,x_n\in(X_{0,0})_0}$. Then the following data constitutes a bicategory*

$$h_2^{tr}\big(X, (\bullet^{x_0,\cdots,x_n}, \eta^{x_0,\cdots,x_n})_{n\geq 0}^{x_0,\cdots,x_n\in(X_{0,0})_0}\big)$$

alternatively shortened to $h_2^{tr}(X)$ when the choice of 2-truncated compositions is evident:

1. *The object set $(X_{0,0})_0$;*
2. *Hom-categories $h_1(X(x,y))$ for each $x, y \in (X_{0,0})_0$;*
3. *Composition operations given by the choices $\bullet^{x_0,\cdots,x_n}$ of 2-truncated compositions;*
4. *Functors $\bullet^x : * \cong h_1(X_0^x) \to h_1(X(x,x))$ induced by the choice of 2-truncated compositions;*
5. *Associators and unitors induced by the unique morphisms of fillers between the relevant fillers.*

A similar story to homotopy bicategories may be obtained for unbiased pseudofunctors. We believe our approach to this particular problem to be novel, as no analogous construction seems to have been explicitly established for Tamsamani 2-categories in for instance [LP08]. However, we suspect a similar such construction is implicit in the left biadjoint functor from Tamsamani 2-categories to bicategories Lack and Paoli study in this paper.

Consider a morphism in $\mathbf{SeSp}_2^{2comp}$

$$f : (X, (\bullet_X^{x_0,\cdots,x_n}, \eta_X^{x_0,\cdots,x_n})_{n\geq 0}^{x_0,\cdots,x_n\in(X_{0,0})_0}) \to (Y, (\bullet_Y^{y_0,\cdots,y_n}, \eta_Y^{y_0,\cdots,y_n})_{n\geq 0}^{y_0,\cdots,y_n\in(Y_{0,0})_0}).$$

We seek some induced pseudofunctor between the homotopy bicategories of the domain and codomain of f. To this end, we first identify a natural mapping of objects $(f_{0,0})_0 : (X_{0,0})_0 \to (Y_{0,0})_0$. As a matter of convenience, for each $x, y \in (X_{0,0})_0$ we introduce the notation

$$f_1^{x,y} : X(x,y) \hookrightarrow X_1 \times_{Y_0^2} \{(f(x), f(y))\} \xrightarrow{f_1^{f(x),f(y)}} Y(f(x), f(y))$$

so as to obtain natural functors $h_1(f_1^{x,y}) : h_1(X(x,y)) \to h_1(Y(f(x), f(y)))$ between hom-categories.

For compositors, let $n \geq 0$ and $x_0, \cdots, x_n \in (X_{0,0})_0$. We consider the natural span of categories

$$\begin{tikzcd}
h_1(X_n^{x_0,\cdots,x_n}) \arrow[r, "h_1(\gamma_n^{x_0,\cdots,x_n})"] \arrow[d, hook] & h_1(X_{Sp(n)}^{x_0,\cdots,x_n}) \cong \prod_{i=1}^n h_1(X(x_{i-1},x_i)) \\
h_1(X_n^{x_0,x_n}) \arrow[d, "h_1(f_n^{x_0,x_n} \circ X_{\langle 0,n\rangle}^{x_0,x_n})"'] & \\
h_1(Y(f(x_0),f(x_n))) &
\end{tikzcd}$$

which admits two fillers, given by the diagrams

$$\begin{tikzcd}
h_1(X_n^{x_0,\cdots,x_n}) \arrow[r, "\simeq"] \arrow[d] & \prod_{i=1}^n h_1(X(x_{i-1},x_i)) \arrow[d] \\
h_1(Y_n^{f(x_0),\cdots,f(x_n)}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow \eta_Y^{f(x_0),\cdots,f(x_n)}"] & \prod_{i=1}^n h_1(Y(f(x_{i-1}),f(x_i))) \arrow[dl, "\bullet_Y^{f(x_0),\cdots,f(x_n)}"] \\
h_1(Y(f(x_0),f(x_n))) &
\end{tikzcd}$$

and

$$\begin{tikzcd}
h_1(X_n^{x_0,\cdots,x_n}) \arrow[r, "\simeq"] \arrow[d] \arrow[dr, phantom, "\Downarrow \eta_X^{x_0,\cdots,x_n}"] & \prod_{i=1}^n h_1(X(x_{i-1},x_i)) \arrow[dl, "\bullet_X^{x_0,\cdots,x_n}"] \\
h_1(X(x_0,x_n)) \arrow[d] & \\
h_1(Y(f(x_0),f(x_n))) &
\end{tikzcd}$$

Our compositor for the objects $x_0, \cdots, x_n$ will then be the natural isomorphism

$$\pi_{x_0,\cdots,x_n} : \bullet_Y^{f(x_0),\cdots,f(x_n)} \circ \prod_{i=1}^n h_1(f_1^{x_{i-1},x_i}) \Rightarrow h_1(f_1^{x_0,x_n}) \circ \bullet_X^{x_0,\cdots,x_n}$$

induced by the unique morphism of fillers between the fillers above.

To verify the coherence conditions involved for pseudofunctors, consider some $n > 0$ and $k_1, \cdots, k_n \geq 0$. Consider some sequence of tuples $S_i = (x_0^i, \cdots, x_{k_i}^i)$ for $1 \leq i \leq n$ of objects $x_j^i \in (X_{0,0})_0$ for all i and j, such that $x_{k_i}^i = x_0^{i+1}$ for $i < n$. Set $T := (X_1, \cdots, X_n)$ and

$$Q := (x_0^1, \cdots, x_{k_1}^1, x_1^2, \cdots, x_{k_n}^n)$$

which we alternatively write as $(x_0, \cdots, x_r)$. This is the flattened version of the nested tuple $(S_1, \cdots, S_n)$.

Note that in the required commuting diagram, the two paths of natural isomorphisms are given by morphisms of fillers between the two fillers

$$
\begin{tikzcd}
h_1(X_r^{x_0,\cdots,x_r}) \arrow[r] \arrow[d] & \prod_{i=1}^n h_1(X_{k_i}^{S_i}) \arrow[r] \arrow[d] & \prod_{i=1}^r h_1(X(x_{i-1},x_i)) \arrow[d] \\
h_1(Y_r^{f(x_0),\cdots,f(x_r)}) \arrow[r] \arrow[d] & \prod_{i=1}^n h_1(Y_{k_i}^{f(S_i)}) \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4, "\Rightarrow"] & \prod_{i=1}^r h_1(Y(f(x_{i-1}),f(x_i))) \arrow[dl, "\prod_{i=1}^n \bullet_Y^{f(S_i)}"] \\
h_1(Y_n^{f(x_0^1),f(x_{k_1}^1),\cdots,f(x_{k_n}^n)}) \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4, "\Rightarrow"] & \prod_{i=1}^n h_1(Y(f(x_0^i),f(x_{k_i}^i))) \arrow[dl, "\bullet_Y^{f(x_0^1),f(x_{k_1}^1),\cdots,f(x_{k_n}^n)}"] & \\
h_1(Y(f(x_0),f(x_r))) & &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
h_1(X_r^{x_0,\cdots,x_r}) \arrow[r] \arrow[d] & \prod_{i=1}^n h_1(X_{k_i}^{S_i}) \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4, "\Rightarrow"] & \prod_{i=1}^r h_1(X(x_{i-1},x_i)) \arrow[dl, "\prod_{i=1}^n \bullet_X^{S_i}"] \\
h_1(X_n^{x_0^1,x_{k_1}^1,\cdots,x_{k_n}^n}) \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4, "\Rightarrow"] & \prod_{i=1}^n h_1(X(x_0^i,x_{k_i}^i)) \arrow[dl, "\bullet_X^{x_0^1,x_{k_1}^1,\cdots,x_{k_n}^n}"] & \\
h_1(X(x_0,x_r)) \arrow[d] & & \\
h_1(Y(f(x_0),f(x_r))) & &
\end{tikzcd}
$$

so is implied by Corollary 3.1.10. The second condition to be a pseudofunctor amounts to two sequences of natural isomorphisms between the two fillers

$$
\begin{tikzcd}
h_1(X(x,y)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow id"] & h_1(X(x,y)) \arrow[dl, "id"] \\
h_1(X(x,y)) \arrow[d] & \\
h_1(Y(f(x),f(y))) &
\end{tikzcd}
\qquad
\begin{tikzcd}
h_1(X(x,y)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow \eta_X^{x,y}"] & h_1(X(x,y)) \arrow[dl, "\bullet_X^{x,y}"] \\
h_1(X(x,y)) \arrow[d] & \\
h_1(Y(f(x),f(y))) &
\end{tikzcd}
$$

so is similarly induced. This discussion results in the construction of a valid unbiased pseudofunctor from a morphism f in $\mathbf{SeSp}_2^{2comp}$, mapping between the homotopy bicategories of its source and target. We formalize this observation with the following theorem:

Theorem 3.1.12. *Suppose*

$$f : (X, (\bullet_X^{x_0,\cdots,x_n}, \eta_X^{x_0,\cdots,x_n})_{n\geq 0}^{x_0,\cdots,x_n\in(X_{0,0})_0}) \to (Y, (\bullet_Y^{y_0,\cdots,y_n}, \eta_Y^{y_0,\cdots,y_n})_{n\geq 0}^{y_0,\cdots,y_n\in(Y_{0,0})_0})$$

is a morphism in $\boldsymbol{SeSp}_2^{2comp}$. Then the following data defines an unbiased pseudofunctor $h_2^{tr}(f) : h_2^{tr}(X) \to h_2^{tr}(Y)$:

1. *The mapping of objects $(f_{0,0})_0 : (X_{0,0})_0 \to (Y_{0,0})_0$;*
2. *The functors on hom-categories*
$$h_1(f_1^{x,y}) : h_1(X(x,y)) \to h_1(Y(f(x), f(y)));$$
3. *Natural isomorphisms $\pi_{x_0,\cdots,x_n}$ induced by the unique morphisms of fillers between the relevant fillers.*

We would like to establish functoriality for the unbiased homotopy bicategory and unbiased homotopy pseudofunctor constructions in Theorem 3.1.11 and Theorem 3.1.12, respectively. This is handled by the following results:

Proposition 3.1.13. *Suppose $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a chain in $\boldsymbol{SeSp}_2^{2comp}$. Then*

$$h_2^{tr}(g) \circ h_2^{tr}(f) = h_2^{tr}(g \circ f).$$

Proof. We will write $X, Y, Z \in \mathbf{SeSp}_2^{inj}$ for the underlying 2-fold Segal spaces of the corresponding objects in $\mathbf{SeSp}_2^{2comp}$. Moreover, we will write $\pi^h_{a_0,\cdots,a_n}$ for $h \in \{f, g, g \circ f\}$ to denote the compositors of $h_2^{tr}(h)$.

For objects, we have that the left hand side and right hand side are given by the mappings $(g_{0,0})_0 \circ (f_{0,0})_0$ and $((g \circ f)_{0,0})_0$, respectively, which are evidently equal. For hom-categories, it suffices to note that we have a commuting diagram for each $x, y \in (X_{0,0})_0$ of the form

$$\begin{tikzcd} X(x,y) \arrow[r, "f_1^{x,y}"] \arrow[dr, "(g\circ f)_1^{x,y}"'] & Y(f(x), f(y)) \arrow[d, "g_1^{f(x),f(y)}"] \\ & Z\big((g\circ f)(x), (g\circ f)(y)\big) \end{tikzcd}$$

which is quickly proven by inspection.

We must now prove that the coherence isomorphisms agree. In the case of $h_2^{tr}(g) \circ h_2^{tr}(f)$, all coherence isomorphisms are, for some $n \geq 0$ and $x_0, \cdots, x_n \in (X_{0,0})_0$, of the form

$$\psi_{x_0,\cdots,x_n} := \big(h_1(g_1^{f(x_0),f(x_n)}) \circ \pi^f_{x_0,\cdots,x_n}\big)\big(\pi^g_{f(x_0),\cdots,f(x_n)} \circ \prod_{i=1}^n h_1(f_1^{x_{i-1},x_i})\big)$$

where the product is set to the identity functor if $n = 0$. Note however that such natural isomorphisms induce morphisms of fillers between the fillers defining $\pi^{g\circ f}_{x_0,\cdots,x_n}$. Thus, by Corollary 3.1.10, these natural isomorphisms must be equal as needed. □

Proposition 3.1.14. *Suppose $X \in \boldsymbol{SeSp}_2^{2comp}$. Then $h_2^{tr}(1_X) = 1_{h_2^{tr}(X)}$.*

Proof. The behavior on objects and hom-categories is immediately as needed. The compositors are then forced to be identities as well, by inspection of the relevant fillers. □

This is enough to define a *homotopy bicategory* functor:

Definition 3.1.15. *Define $h_2^{tr} : \boldsymbol{SeSp}_2^{2comp} \to \boldsymbol{UBicat}$ to be the functor sending objects*

$$(X, (\bullet^{x_0,\cdots,x_n}, \eta^{x_0,\cdots,x_n})_{n\geq 0}^{x_0,\cdots,x_n\in(X_{0,0})_0})$$

to $h_2^{tr}(X)$ and morphisms f to $h_2^{tr}(f)$.

Though h_2^{tr} does depend upon choices of 2-truncated compositions, we have rather immediately that these choices are quite irrelevant up to isomorphism in **UBicat**:

Corollary 3.1.16. *Suppose $X \in \boldsymbol{SeSp}_2^{inj}$, with two choices of 2-truncated compositions of the form*

$$(\bullet^{x_0,\cdots,x_n}, \eta^{x_0,\cdots,x_n})_{n\geq 0}^{x_0,\cdots,x_n\in(X_{0,0})_0}$$

and

$$(\otimes^{x_0,\cdots,x_n}, \epsilon^{x_0,\cdots,x_n})_{n\geq 0}^{x_0,\cdots,x_n\in(X_{0,0})_0}.$$

*Then there is an isomorphism in **UBicat***

$$h_2^{tr}\big(X, (\bullet^{x_0,\cdots,x_n}, \eta^{x_0,\cdots,x_n})_{n\geq 0}^{x_0,\cdots,x_n\in(X_{0,0})_0}\big) \to h_2^{tr}\big(X, (\otimes^{x_0,\cdots,x_n}, \epsilon^{x_0,\cdots,x_n})_{n\geq 0}^{x_0,\cdots,x_n\in(X_{0,0})_0}\big)$$

that moreover acts as the identity on objects and hom-categories.

3.1.1 The Issue of Premature Truncation

It would appear at a first glance that h_2^{tr} is a perfectly admissible notion of bicategory, even without a complete comparison with known constructions like in [Mos21] and [Cam20]. Moreover, it seems perfectly usable for the purposes of obtaining specific homotopy bicategories of 2-fold Segal spaces.

Issues do however begin to arise as we commence the computation of example homotopy bicategories. For instance, consider some $T \in \mathbf{Top}$. We would imagine that, for some choice of 2-truncated compositions, the homotopy bicategory $h_2^{tr}(\mathbf{Sing_{ssS}}(T))$ should be a reasonable model for the *fundamental* *bigroupoid* of T, similar to the work of [HKK01]. It would seem that such a computation is a reasonable starting point on the path to homotopy bicategories of $(\infty, 2)$-categories of manifolds and cobordisms, as these are usually rather topological themselves; see [Lur09b] and [CS19] for examples of such constructions.

The objects and hom-categories of $\Pi_2^{tr}(T) := h_2^{tr}(\mathbf{Sing}_{\mathbf{ssS}}(T))$ seem reasonable; indeed, $\mathbf{ob}(\Pi_2^{tr}(T))$ is evidently the set of points in T, while, adopting the notation

$$T(x, y) := \{(x, y)\} \times_{T \times T} T^{[0,1]}$$

for convenience, we have by Proposition 2.3.12 that

$$\mathbf{Hom}_{\Pi_2^{tr}(T)}(x, y) \cong \Pi_1(T(x, y)).$$

We then turn to the issue of choices of horizontal compositions. We seek fillers for the cospans

$$\Pi_1(T(x_0, x_n)) \leftarrow \Pi_1(\{(x_0, \cdots, x_n)\} \times_{T^n} T^{\Delta_t[n]}) \rightarrow \prod_{i=1}^{n} \Pi_1(T(x_{i-1}, x_i)).$$

It is at this point that problems may arise in explicit constructions. We have the following result:

Proposition 3.1.17. *Suppose T is 2-connected. For any $x, y \in T$, the groupoid $\Pi_1(T(x, y))$ is contractible.*

Proof. Consider two objects in $\Pi_1(T(x, y))$, namely paths $f, g : [0, 1] \to T$ such that $f(0) = x = g(0)$ and $f(1) = y = g(1)$. These define a loop $S^1 \to T$, for which there must be a filler $D^2 \to T$ by T having trivial fundamental groups. Thus, every two objects in the above groupoid have at least one morphism between them.

Consider then two such morphisms between f and g, given by homotopies

$$H_1, H_2 : [0, 1] \times [0, 1] \to T$$

such that

$$\begin{aligned} H_1(-, 0) &= x = H_2(-, 0) \\ H_1(-, 1) &= y = H_2(-, 1) \\ H_1(0, -) &= f = H_2(0, -) \\ H_1(1, -) &= g = H_2(1, -). \end{aligned}$$

Then H_1 and H_2 together define a continuous map $S^2 \to T$. We have that T has trivial second homotopy groups, so that there must exist a filler $D^3 \to T$, which determines a homotopy between H_1 and H_2 suitably fixed on x and y. This proves that any two objects in the above groupoid have at most one morphism between them. $\square$

This implies that all three categories in the above span are contractible; for the middle category, the fact that the right-facing map is fully faithful into a contractible category implies contractibility. Thus, for example, given any $x, y, z \in S^2$, any choice of function

$$\mathbf{ob}(\Pi_1(S^2(x,y))) \times \mathbf{ob}(\Pi_1(S^2(y,z))) \to \mathbf{ob}(\Pi_1(S^2(x,z)))$$

will uniquely yield a filler that restricts to this choice of mapping on object sets. In general, given $n \geq 0$ and $x_0, \cdots, x_n \in T$, consider a path $f : [0,n] \to T$ with $f(i) = x_i$ for $0 \leq i \leq n$. We may choose to send f to quite literally *any* path $F : [0,1] \to T$ with $F(0) = x_0$ and $F(1) = x_n$ as its 'composite' path, should T have trivial fundamental and second homotopy groups. There is moreover no need for this choice to vary continuously with f.

One might claim these oddities to present no real issue; in the case of 2-connectedness, T is trivial in the dimensions relevant to fundamental bigroupoids. It is however somewhat unnatural to allow such strange choices of composition operations on $\Pi_2^{tr}(T)$. If we ever wished to work with this bigroupoid directly, we may find it difficult to perform calculations owing to the potentially highly unusual composition operations. More generally, it is possible that $\Pi_2^{tr}(U)$ for any general space U may admit strange composition operations that have no clear foundation in U, with their idiosyncrasies written off as being isomorphic to whatever more natural definition might exist.

The particularly cynical reader may even object to the bigroupoid $\Pi_2^{tr}(T)$ having anything to do with T at all. Indeed, the choices of composition operations, associators and unitors are made entirely at the level of categories rather than within the topological space T itself, so could be argued to be nothing more than artefacts of the truncation to fundamental groupoids and of abstract category theory. It would be enlightening to develop all of the relevant data at the level of T itself, rather than after truncation, to justify this construction of fundamental bigroupoid. More generally, obtaining composition operations, associators, unitors and coherence conditions for homotopy bicategories $h_2^{tr}(X)$ at the level of X may help justify our approach to homotopy bicategories, as well as other related such constructions.

One may further object to the dependence upon category-theoretic arguments by noting the difficulty of extending the techniques discussed thus far to the computation of 'homotopy n-groupoids' $\Pi_n(T)$ for higher n. In general, we should hope to extract from a Reedy fibrant n-fold Segal space X, as defined for instance in [CS19], its 'homotopy n-category' $h_n(X)$, an instance of some algebraic model of n-categories. Our over-reliance on properties of the 2-category $\mathbb{C}\mathbf{at}$ of categories, functors and natural transformations to obtain coherence conditions does not obviously extend to algebraic models of more general n-categories. This is in part due to the theory of n-categories for general n being somewhat more sparse than that of (∞, n)-categories, owing to models of the latter generally having model structures to work with, which models of the former presently lack. For this reason, it may be best to try and use model-category-theoretic methods to obtain as much algebraic structure on an (∞, n)-category as possible before passing to some chosen model of n-category. Doing so in the case $n = 2$ should serve as an enlightening first step in this direction.

3.2 Simplicial Composition Diagrams

It would seem that the natural next step in analyzing our homotopy bicategory functor h_2^{tr} is to adapt it such that composition operations, coherence isomorphisms and coherence conditions are identified within the original 2-fold Segal space X, rather than being obtained after collapsing to the level of categories. Henceforth, we will speak of the imagined homotopy bicategory constructed from X in this way as $h_2(X)$, to distinguish it from $h_2^{tr}(X)$.

Our first step to constructing $h_2(X)$ is lifting the choice of 2-truncated compositions from categories to Segal spaces. In the spirit of our intuition for Reedy fibrant 2-fold Segal spaces, constructing a binary composition functor should for instance proceed by solving a lifting problem in $SeSp^{inj}$ of the form

$$\begin{tikzcd} & X_2 \arrow[r, "X_{\langle 0,2\rangle}"] \arrow[d, "\gamma_2"] & X_1 \\ X_1 \times_{X_0} X_1 \arrow[ur, dashed, "\mu_2"] \arrow[r, "id"'] & X_1 \times_{X_0} X_1 & \end{tikzcd}$$

then fibering the map $X_{\langle 0,2\rangle} \circ \mu_2$ over some pair $\{(x, z)\} \in ((X_0 \times X_0)_0)_0$ and finally precomposing with the inclusion of $X(x, y) \times X(y, z)$ for some $y \in (X_{0,0})_0$ to obtain a map

$$\circ^{x,y,z} : X(x, y) \times X(y, z) \hookrightarrow (X_1 \times_{X_0} X_1)^{x,z} \xrightarrow{\mu_2^{x,z}} X_2^{x,z} \xrightarrow{X_{\langle 0,2\rangle}^{x,z}} X(x, z).$$

We should think of this as binary composition on the hom-spaces $X(x, y)$ and $X(y, z)$. In general, we can take any sequence of objects $x_0, \cdots, x_n \in (X_{0,0})_0$ and solve a lifting problem

$$\begin{tikzcd} & X_n \arrow[r, "X_{\langle 0,n\rangle}"] \arrow[d, "\gamma_n"] & X_1 \\ X_1 \times_{X_0} \cdots \times_{X_0} X_1 \arrow[ur, dashed, "\mu_n"] \arrow[r, "id"'] & X_1 \times_{X_0} \cdots \times_{X_0} X_1 & \end{tikzcd}$$

to obtain a composition map

$$\circ^{x_0,\cdots,x_n} : \prod_{i=1}^{n} X(x_{i-1}, x_i) \hookrightarrow X_{Sp(n)}^{x_0,x_n} \xrightarrow{\mu_n^{x_0,x_n}} X_n^{x_0,x_n} \xrightarrow{X_{\langle 0,n\rangle}^{x_0,x_n}} X(x_0, x_n).$$

Note that μ_n must be object-fibered with respect to the cospan

$$X_n \xrightarrow{X_{\langle 0\rangle}\times\cdots\times X_{\langle n\rangle}} (X_0)^{n+1} \xleftarrow{(X_{\langle 0\rangle}\times X_{\langle 1\rangle})\times_{1_{X_0}}\cdots\times_{1_{X_0}}(X_{\langle 0\rangle}\times X_{\langle 1\rangle})} X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

as defined in Definition 2.3.8 by the following result:

Proposition 3.2.1. *Let X be a 2-fold Segal space. Consider a solution g to a lifting problem in $SeSp^{inj}$*

$$\begin{tikzcd} A \arrow[r, "s"] \arrow[d, "i"'] & B \arrow[d, "p"] \\ C \arrow[r, "t"'] \arrow[ur, dashed, "g"] & D \end{tikzcd}$$

where every morphism besides g is object-fibered over some maps to $(X_0)^{n+1}$. Then g is object-fibered and for any $x_0, \cdots, x_n \in (X_{0,0})_0$, the diagram[1]

$$\begin{tikzcd} A^{x_0,\cdots,x_n} \arrow[r, "s^{x_0,\cdots,x_n}"] \arrow[d, "i^{x_0,\cdots,x_n}"'] & B^{x_0,\cdots,x_n} \arrow[d, "p^{x_0,\cdots,x_n}"] \\ C^{x_0,\cdots,x_n} \arrow[r, "t^{x_0,\cdots,x_n}"'] \arrow[ur, dashed, "g^{x_0,\cdots,x_n}"] & D^{x_0,\cdots,x_n} \end{tikzcd}$$

commutes.

Proof. Consider the chosen maps $f_Z : Z \to (X_0)^{n+1}$ for $Z \in \{A, B, C, D\}$. Then

$$f_C = f_D \circ t = f_D \circ (p \circ g) = (f_D \circ p) \circ g = f_B \circ g$$

and so g is object-fibered. Commutativity then follows from functoriality of $(-)^{x_0,\cdots,x_n}$. $\square$

The map $\circ^{x_0,\cdots,x_n}$ is then 'unbiased composition,' taking a sequence of 1-morphisms in X

$$x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} x_{n-1} \xrightarrow{f_n} x_n$$

and composing them into a single map $f_n \circ \cdots \circ f_1$ at once. These operations amount to horizontal composition in our $(\infty, 2)$-category, while the Segal maps in each Segal space X_n induce vertical composition. We package this into a definition, which will serve as our extension of 2-truncated compositions to $(\infty, 1)$-categories:

[1] Note that this may no longer be a lifting problem, as $i^{x_0,\cdots,x_n}$ may not be a (trivial) cofibration. For us, usually $A = \emptyset$ and all objects are cofibrant when we want a lifting problem, so this doesn't matter.

Definition 3.2.2. *Let X be a 2-fold Segal space. Then a* choice of horizontal compositions *is a sequence of maps $\mu_n : X_1 \times_{X_0} \cdots \times_{X_0} X_1 \to X_n$ for $n \geq 2$ solving the lifting problems in $SeSp^{inj}$ of the form*

$$\begin{tikzcd} & X_n \arrow[d, "\gamma_n"] \\ X_1 \times_{X_0} \cdots \times_{X_0} X_1 \arrow[r, "id"'] \arrow[ur, dashed, "\mu_n"] & X_1 \times_{X_0} \cdots \times_{X_0} X_1 \end{tikzcd}$$

Notation 3.2.3. *Given a choice of horizontal composition maps μ_n for $n \geq 2$, write $\mu_1 = 1_{X_1}$ and $\mu_0 = 1_{X_0}$.*

Recall that since $\gamma_1 : X_1 \to X_1$ and $\gamma_0 : X_0 \to X_0$ are just identities, the maps μ_1 and μ_0 are solutions of similar lifting problems. In fact, they are the only solutions of their respective lifting problems.

With horizontal compositions now available to us, we choose to construct a domain for our homotopy bicategory functor as follows:

Definition 3.2.4. *Let $\boldsymbol{SeSp}_2^{comp}$ be the category whose objects $(X, (\mu_n)_{n\geq 0})$ are pairs of Reedy fibrant 2-fold Segal spaces with a choice of horizontal composition and whose morphisms $(X, (\mu_n)_{n\geq 0}) \to (Y, (\nu_n)_{n\geq 0})$ are maps $X \to Y$ in $\boldsymbol{SeSp}_2^{inj}$.*

Note how the morphisms do not account for the choice of horizontal compositions in either the domain or codomain, much like for $\mathbf{SeSp}_2^{2comp}$.

We hence seek a functor

$$h_2 : \mathbf{SeSp}_2^{comp} \to \mathbf{UBicat}.$$

Obtaining composition operations will be accomplished by applying h_1 to the maps $\circ^{x_0,\cdots,x_n}$. which we will explore in greater depth when more machinery has been developed to this end. This homotopy bicategory construction will ideally factor through h_2^{tr} by a natural 'truncation' functor

$$Tr : \mathbf{SeSp}_2^{comp} \to \mathbf{SeSp}_2^{2comp}$$

though we will not be able to fully define Tr until we have explored how to obtain natural isomorphisms from applying h_1 to *homotopies*, which we will study in due course.

Notation 3.2.5. *For a 2-fold Segal space X, given $n > 0$ and $x_0, \cdots, x_n \in (X_{0,0})_0$, along with a choice of horizontal compositions $(\mu_n)_{n\geq 0}$, write $\circ^{x_0,\cdots,x_n} : \prod_{i=1}^{n} X(x_{i-1}, x_i) \to X(x_0, x_n)$ for the map described above. If $n = 0$, write $\circ^{x_0} = X_{s_0^0}^{x_0,x_0} : * \cong \{x_0\} \hookrightarrow X(x_0, x_0)$.*

Note we did not need to separately specify $\circ^{x_0}$, as if $n = 0$ then

$$\prod_{i=1}^{n} X(x_{i-1}, x_i) \cong *.$$

We do so simply for clarity; in future, the $n = 0$ case will be implicitly covered in this manner.

Note also that there is almost never a unique choice of horizontal compositions; in general, there are a multitude of possible solutions to the relevant lifting problems. We will see in time that these choices, while not necessarily equal, are in fact always equivalent in a way we will make precise later.

One might wonder how more complex 'nested' composition operations look, for instance composing a chain of 1-morphisms in X of the form $v \xrightarrow{f} w \xrightarrow{g} x \xrightarrow{h} y \xrightarrow{k} z$ into the composite $(k \circ h) \circ (g \circ f)$. This is of course merely a matter of applying our operations $\circ^{x_0, \cdots, x_n}$ in sequence, yet a more direct interpretation is also possible, one which will be of immense importance in developing our homotopy bicategories.

Note that everything we have written can be rephrased using Notation 3.0.1:

$$\begin{tikzcd}
 & X_n \arrow[r, "X_{\langle 0,n \rangle}"] \arrow[d, "\gamma_n"] & X_1 \\
X_{Sp(n)} \arrow[r, "id"'] \arrow[ur, dashed, "\mu_n"] & X_{Sp(n)} &
\end{tikzcd}$$

The composition operation $(- \circ -) \circ (- \circ -)$ can now be constructed from the solutions to the sequence of two lifting problems

$$\begin{tikzcd}
 & X_2 \arrow[r, "X_{\langle 0,2 \rangle}"] \arrow[d, "\gamma_2"] & X_1 \\
 & X_{Sp(2)} & \\
 & X_{\Delta[2] \sqcup_{\Delta[0]} \Delta[2]} \arrow[u, "X_{\langle 0,2 \rangle} \times_{1_{X_0}} X_{\langle 0,2 \rangle}"'] \arrow[d, "\gamma_2 \times_{1_{X_0}} \gamma_2"] & \\
X_{Sp(4)} \arrow[r, "id"'] \arrow[ur, dashed, "\mu_2 \times_{1_{X_0}} \mu_2"] \arrow[uuur, dashed, bend left, "\mu_2 \circ (X_{\langle 0,2 \rangle} \times_{1_{X_0}} X_{\langle 0,2 \rangle}) \circ (\mu_2 \times_{1_{X_0}} \mu_2)"] & X_{Sp(4)} &
\end{tikzcd}$$

The overall map is then, for $v, w, x, y, z \in (X_{0,0})_0$, the composite

$$\begin{tikzcd}
X(v,w) \times X(w,x) \times X(x,y) \times X(y,z) \arrow[r, hook] & X^{v,z}_{Sp(4)} \arrow[d, "X^{v,z}_{\langle 0,2 \rangle} \circ (\mu_2 \circ (X_{\langle 0,2 \rangle} \times_{1_{X_0}} X_{\langle 0,2 \rangle}) \circ (\mu_2 \times_{1_{X_0}} \mu_2))^{v,z}"'] \\
 & X(v,z)
\end{tikzcd}$$

We can reinterpret this algebraic monstrosity in a more geometric light. Consider how the cospan

$$X_2 \xrightarrow{\gamma_2} X_{Sp(2)} \xleftarrow{X_{\langle 0,2\rangle} \times_{1_{X_0}} X_{\langle 0,2\rangle}} X_{\Delta[2] \sqcup_{\Delta[0]} \Delta[2]}$$

appears in the above lifting problem. The two lifts can be combined into a single lift to the pullback of the cospan

$$X_{\Delta[2] \sqcup_{\Delta[0]} \Delta[2]} \times_{X_{Sp(2)}} X_2$$

which, since F_2^0 preserves colimits and $\mathbf{Map}_2^1(-, X)$ sends colimits to limits, we can rephrase as

$$X_{(\Delta[2] \sqcup_{\Delta[0]} \Delta[2]) \sqcup_{Sp(2)} \Delta[2]}$$

to obtain a new diagram

$$\begin{tikzcd}
 & X_{(\Delta[2] \sqcup_{\Delta[0]} \Delta[2]) \sqcup_{Sp(2)} \Delta[2]} \arrow[r, "\tau^*"'] \arrow[d, "\iota^*"] & X_1 \\
X_{Sp(4)} \arrow[ur, dashed, "\mu"] \arrow[r, "id"'] & X_{Sp(4)} &
\end{tikzcd}$$

where ι^* and τ^* are induced by the maps ι and τ in the cospan in **sSet**

$$Sp(4) \overset{\iota}{\hookrightarrow} (\Delta[2] \sqcup_{\Delta[0]} \Delta[2]) \sqcup_{Sp(2)} \Delta[2] \overset{\tau}{\hookleftarrow} \Delta[1].$$

The domain of the map μ is well-understood to us by this point: it is simply the ∞-category of chains of length four in X. The codomain, however, is new. We should interpret it as the ∞-category of diagrams in X of the form

$$\begin{tikzcd}
 & w \arrow[dr, "g"] \arrow[dd, Rightarrow, "\alpha"] & & y \arrow[dddr, "k"] \arrow[dd, Rightarrow, "\beta"'] & \\
 & & x \arrow[ur, "h"] \arrow[dr] \arrow[dd, Rightarrow, "\gamma"'] & & \\
 & q \arrow[ur] & & r \arrow[dr] & \\
v \arrow[uuur, "f"] \arrow[ur] \arrow[rrrr, "p"'] & & & & z
\end{tikzcd}$$

where $\alpha, \beta, \gamma \in (X_{2,0})_0$ are compositions, $f, g, h, k, q, r, p \in (X_{1,0})_0$ are 1-morphisms and $v, w, x, y, z \in (X_{0,0})_0$ are objects in our $(\infty, 2)$-category.

The morphism μ can now be seen as taking a chain $v \xrightarrow{f} w \xrightarrow{g} x \xrightarrow{h} y \xrightarrow{k} z$ and extending it to a diagram as above, choosing $\alpha, \beta, \gamma, q, r$ and p. The shape of this diagram can in fact tell us directly what the composition operation in question is, as p is a composite of q with r, which are in turn composites of f with g and h with k respectively.

The general shape of such ‘composition diagrams’ is evident by an induction. In the below definition, write

$$s, t : \Delta[0] \to Sp(n)$$

for the maps in **sSet** identifying the first and last objects.

Definition 3.2.6. *A* simplicial composition diagram *is a cospan* $Sp(n) \xrightarrow{\iota} K \xleftarrow{\tau} \Delta[1]$ *defined inductively such that either:*

1. $K = \Delta[n]$*,* $\iota = g_n : Sp(n) \hookrightarrow \Delta[n]$ *and* $\tau = \langle 0, n\rangle : \Delta[1] \hookrightarrow \Delta[n]$ *for some* $n \geq 0$*;*

2. $K = (K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_d) \sqcup_{Sp(d)} \Delta[d]$ *for some* $d > 1$*, where* $Sp(k_i) \xrightarrow{\iota_i} K_i \xleftarrow{\tau_i} \Delta[1]$ *are simplicial composition diagrams for* $1 \leq i \leq d$ *and the pushouts are taken over the diagrams*

$$K_1 \xleftarrow{\iota_1 \circ t} \Delta[0] \xrightarrow{\iota_2 \circ s} \cdots \xleftarrow{\iota_{d-1} \circ t} \Delta[0] \xrightarrow{\iota_d \circ s} K_d$$

and

$$(K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_d) \xleftarrow{\tau_1 \sqcup_{1_{\Delta[0]}} \cdots \sqcup_{1_{\Delta[0]}} \tau_d} Sp(d) \hookrightarrow \Delta[d].$$

The maps are then ι *as in the diagram*

$$\begin{tikzcd}
Sp(n) \arrow[r, phantom, "="] & Sp(k_1 + \cdots + k_d) \arrow[ddr, "\iota"'] \arrow[r, phantom, "\cong"] & Sp(k_1) \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} Sp(k_d) \arrow[d, "\iota_1 \sqcup_{1_{\Delta[0]}} \cdots \sqcup_{1_{\Delta[0]}} \iota_d"] \\
& & K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_d \arrow[d, hook] \\
& & K
\end{tikzcd}$$

and $\tau : \Delta[1] \xrightarrow{\langle 0, d\rangle} \Delta[d] \hookrightarrow K$*.*

We will say the arity *of the composition diagram* K *is* n*. Its* depth *is either* 1 *if* $K = \Delta[n]$ *or* $1 + \max_{1 \leq i \leq d} d_i$ *otherwise, where* d_i *is the depth of* K_i*.*

Note the inclusion of $\Delta[0] \xrightarrow{id} \Delta[0] \xleftarrow{X_{\langle 0,0\rangle}} \Delta[1]$, which amounts to the problem of taking identities. Note also that it is entirely unambiguous to refer to a simplicial composition diagram solely by the space K; there is only one possibility for the maps ι and τ.

A similar notion to these diagrams is to be found in the membranes obtained from polygonal decompositions of convex $(n+1)$-gons used to define 2-Segal spaces in [DK19]. They also strongly resemble the opetopes of [BD98]. Alternatively, one might note a simplicial composition diagram is simply a series of necklaces as in [DS11] stacked upon one another. Making these connections precise and exploring their full influence will be future work.

The space of composition diagrams in question will be X_K for a 2-fold Segal space X and a simplicial composition diagram K. It is the case that these are always Segal spaces:

Proposition 3.2.7. *Let X be a 2-fold Segal space. Suppose K is a simplicial composition diagram. Then X_K is a Segal space.*

Proof. It suffices to prove that X_K is fibrant in the model structure $SeSp^{inj}$. We proceed by induction on the depth of K. The result is clear if $K = \Delta[n]$, as $X_{\Delta[n]} \cong X_n$. Now, suppose the depth is greater than 1, so $K = (K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_n) \sqcup_{Sp(n)} \Delta[n]$. This implies that X_K is isomorphic to the pullback

$$\begin{tikzcd} X_K \arrow[r, "\cong"] & (X_{K_1} \times_{X_0} \cdots \times_{X_0} X_{K_n}) \times_{X_{Sp(n)}} X_n \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X_n \arrow[d, "\gamma_n"] \\ & X_{K_1} \times_{X_0} \cdots \times_{X_0} X_{K_n} \arrow[r] & X_{Sp(n)} \end{tikzcd}$$

Since the Segal map γ_n is a trivial fibration in $SeSp^{inj}$, the induced map by the pullback

$$X_K \to X_{K_1} \times_{X_0} \cdots \times_{X_0} X_{K_n}$$

is a trivial fibration in $SeSp^{inj}$. By induction, each X_{K_i} is fibrant in $SeSp^{inj}$, so the maps $X_{K_i} \to X_0$ are Reedy fibrations between fibrant objects in $SeSp^{inj}$ and thus fibrations in this model structure. Therefore, the codomain $X_{K_1} \times_{X_0} \cdots \times_{X_0} X_{K_n}$ is fibrant in $SeSp^{inj}$, implying the space X_K is also fibrant as desired. □

Moreover, the map ι always induces a trivial fibration:

Proposition 3.2.8. *Let X be a 2-fold Segal space. Suppose $Sp(n) \xrightarrow{\iota} K \xleftarrow{\tau} \Delta[1]$ is a simplicial composition diagram. Then the map $\iota^* : X_K \to X_{Sp(n)}$ induced by ι is a trivial fibration in $SeSp^{inj}$.*

Proof. If $K = \Delta[n]$, then $\iota^* = \gamma_n$ and the result is immediate. Suppose then that $K = (K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_r) \sqcup_{Sp(r)} \Delta[r]$. Then ι^* is given by the composite map

$$X_K \to X_{K_1} \times_{X_0} \cdots \times_{X_0} X_{K_r} \xrightarrow{\iota^*_{K_1} \times_{1_{X_0}} \cdots \times_{1_{X_0}} \iota^*_{K_r}} X_{Sp(n)}.$$

The first of these maps is a trivial fibration in $SeSp^{inj}$, as shown in the proof of Proposition 3.2.7. It thus suffices to prove the latter is as such.

Assume by induction on the depth of K that all the maps $\iota_{K_i}^* : X_{K_i} \to X_{Sp(k_i)}$ are trivial fibrations in $SeSp^{inj}$, where K_i has arity k_i for all i. Then we have a natural pullback diagram

$$\begin{tikzcd}
X_{K_1} \times_{X_0} \cdots \times_{X_0} X_{K_r} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \prod_{i=1}^r X_{k_i} \arrow[d, "\prod_{i=1}^r \iota_{K_i}^*"] \\
X_{Sp(n)} \arrow[r] & \prod_{i=1}^r X_{Sp(k_i)}
\end{tikzcd}$$

where the bottom horizontal map is induced by the natural map

$$Sp(k_1) \sqcup \cdots \sqcup Sp(k_r) \to Sp(k_1) \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} Sp(k_r) \cong Sp(k_1 + \cdots + k_r) = Sp(n).$$

Thus, since products of trivial fibrations are themselves trivial fibrations [Hir09, Prop. 7.2.5], the rightmost vertical map must be a trivial fibration as needed. □

Note also that the maps ι^* are tautologically object-fibered with respect to the maps induced by $\Delta[0] \hookrightarrow Sp(n) \xrightarrow{\iota} K$. The same can be said for $\tau^* : X_K \to X_1$ with respect to the maps $\Delta[0] \hookrightarrow \Delta[1] \xrightarrow{\tau} K$. These two maps also agree with the two endpoint inclusions of the spine composed with ι.

Simplicial composition diagrams can be used to nest existing composition operations in an evident way, a natural extension of our example:

Definition 3.2.9. *Let X be a 2-fold Segal space with a choice of horizontal compositions $(\mu_n)_{n\geq 0}$. Let $Sp(n) \xrightarrow{\iota} K \xleftarrow{\tau} \Delta[1]$ be a simplicial composition diagram. Then the* induced composition on K *is the map μ_K solving the lifting problem*

$$\begin{tikzcd}
 & X_K \arrow[r, "\tau^*"] \arrow[d, "\iota^*"] & X_1 \\
X_{Sp(n)} \arrow[ur, dashed, "\mu_K"] \arrow[r, "id"'] & X_{Sp(n)} &
\end{tikzcd}$$

where ι^* and τ^* are induced by ι and τ respectively, such that:

1. *If $K = \Delta[n]$ then $\mu_K = \mu_n$;*
2. *If $K = (K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_r) \sqcup_{Sp(r)} \Delta[r]$ where each K_i has arity k_i, then μ_K is the pullback map induced by $(\mu_{K_1} \times_{1_{X_0}} \cdots \times_{1_{X_0}} \mu_{K_r})$ and*

$$\mu_r \circ (\tau_1^* \times_{1_{X_0}} \cdots \times_{1_{X_0}} \tau_r^*) \circ (\mu_{K_1} \times_{1_{X_0}} \cdots \times_{1_{X_0}} \mu_{K_r})$$

where τ_i^* is induced by the map $\tau_i : \Delta[1] \to K_i$ for $1 \leq i \leq r$.

It is not hard to see how μ_K, by induction, defines a nesting of composition operations described in the structure of K. In particular, if $K = (K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_r) \sqcup_{Sp(r)} \Delta[r]$, then the composition operation in question composes the first k_1 morphisms by μ_{K_1}, the second k_2 morphisms by μ_{K_2} and so on up to μ_{K_r}, then composes the resulting chain of n morphisms by μ_r. This can be generalized:

Proposition 3.2.10. *Suppose $K = (K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_r) \sqcup_{Sp(r)} K_0$ is a simplicial composition diagram of arity n for diagrams $\Delta[k_i] \xrightarrow{\iota_i} K_i \xleftarrow{\tau_i} \Delta[1]$, where $k_0 = r$. Then μ_K is the pullback map induced by $(\mu_{K_1} \times_{1_{X_0}} \cdots \times_{1_{X_0}} \mu_{K_r})$ and*

$$Q := \mu_{K_0} \circ (\tau_1^* \times_{1_{X_0}} \cdots \times_{1_{X_0}} \tau_r^*) \circ (\mu_{K_1} \times_{1_{X_0}} \cdots \times_{1_{X_0}} \mu_{K_r}).$$

Proof. The proof is by induction on the depth of K_0. Either $K_0 = \Delta[n]$, for which the proof is trivial, or $K_0 = (K_0^1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_0^p) \sqcup_{Sp(p)} \Delta[p]$ for simplicial composition diagrams $\Delta[h_j] \xrightarrow{\iota_j^0} K_0^j \xleftarrow{\tau_j^0} \Delta[1]$. Suppose we are in the latter case. We have that

$$K = (K_1' \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_p') \sqcup_{Sp(p)} \Delta[p]$$

for simplicial composition diagrams $\Delta[q_j] \xrightarrow{\iota_j'} K_j' \xleftarrow{\tau_j'} \Delta[1]$. We have moreover for each $1 \leq j \leq p$ that $K_j' = (H_1^j \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} H_{h_j}^j) \sqcup_{Sp(h_j)} K_0^j$ for simplicial composition diagrams $\Delta[s_{j,r}] \xrightarrow{\iota_{j,r}} H_r^j \xleftarrow{\tau_{j,r}} \Delta[1]$. The spaces H_r^j are clearly just the spaces K_i.

We wish to show that the induced pullback map

$$\begin{tikzcd}
X_{Sp(n)} \arrow[dr, dashed] \arrow[drr, bend left, "\mu_{K_1}\times_{1_{X_0}}\cdots\times_{1_{X_0}}\mu_{K_r}"] \arrow[ddr, bend right, "\mu_{K_0}\circ(\tau_1^*\times_{1_{X_0}}\cdots\times_{1_{X_0}}\tau_r^*)\circ(\mu_{K_1}\times_{1_{X_0}}\cdots\times_{1_{X_0}}\mu_{K_r})"'] & & \\
& X_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X_{K_1\sqcup_{\Delta[0]}\cdots\sqcup_{\Delta[0]}K_r} \arrow[d, "\tau_1^*\times_{1_{X_0}}\cdots\times_{1_{X_0}}\tau_r^*"] \\
& X_{K_0} \arrow[r, "\iota_0^*"'] & X_{Sp(r)}
\end{tikzcd}$$

is μ_K. Our strategy is to consider the diagram

$$\begin{tikzcd}
X_{Sp(n)} \arrow[drrr, "\mu_{K_1}\times_{1_{X_0}}\cdots\times_{1_{X_0}}\mu_{K_r}"] \arrow[dr, dashed] \arrow[dddr, "Q"'] & & & \\
& X_K \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X_{K'_1\sqcup_{\Delta[0]}\cdots\sqcup_{\Delta[0]}K'_p} \arrow[d] & X_{K_1\sqcup_{\Delta[0]}\cdots\sqcup_{\Delta[0]}K_r} \arrow[l, "R"] \arrow[dd] \\
& X_p \arrow[r, "\gamma_p"'] & X_{Sp(p)} & \\
& X_{K_0} \arrow[u, "((\tau_1^0)^*\times_{1_{X_0}}\cdots\times_{1_{X_0}}(\tau_p^0)^*)\times_{1_{Sp(p)}}1_{X_p}"'] \arrow[rr, "\iota_0^*"'] & & X_{Sp(r)}
\end{tikzcd}$$

where

$$R := 1_{X_{K_1\sqcup_{\Delta[0]}\cdots\sqcup_{\Delta[0]}K_r}} \times_{1_{X_{Sp(r)}}} (\mu_{K_0^1} \times_{1_{X_0}} \cdots \times_{1_{X_0}} \mu_{K_0^p})$$

and show that the induced morphisms $X_{Sp(n)} \to X_{K'_1\sqcup_{\Delta[0]}\cdots\sqcup_{\Delta[0]}K'_p}$ and $X_{Sp(n)} \to X_p$ are $(\mu_{K'_1} \times_{1_{X_0}} \cdots \times_{1_{X_0}} \mu_{K'_p})$ and

$$\mu_p \circ ((\tau'_1)^* \times_{1_{X_0}} \cdots \times_{1_{X_0}} (\tau'_p)^*) \circ (\mu_{K'_1} \times_{1_{X_0}} \cdots \times_{1_{X_0}} \mu_{K'_p})$$

respectively. This will imply the induced pullback map is μ_K by the inner pullback square.

Note first that, for each $1 \leq j \leq p$, by induction on depth we have that the diagram

$$\begin{tikzcd}
X_{Sp(q_j)} \arrow[d, "\mu_{H_1^j}\times_{1_{X_0}}\cdots\times_{1_{X_0}}\mu_{H_1^j}"'] \arrow[dr, "\mu_{K'_j}"] & \\
X_{H_1^j\sqcup_{\Delta[0]}\cdots\sqcup_{\Delta[0]}H_{h_j}^j} \arrow[r, "1_{X_{H_1^j\sqcup_{\Delta[0]}\cdots\sqcup_{\Delta[0]}H_{h_j}^j}}\times_{1_{X_{Sp(h_j)}}}\mu_{K_0^j}"'] & X_{K'_j}
\end{tikzcd}$$

commutes. Thus, by taking a pullback of these maps, the first morphism is as required. For Q, it will suffice to show that the diagram

$$\begin{tikzcd}
X_{Sp(n)} \arrow[dr] \arrow[drr, "\mu_{K_1}\times_{1_{X_0}}\cdots\times_{1_{X_0}}\mu_{K_r}"] \arrow[dddd] & & \\
& X_{K_1'\sqcup_{\Delta[0]}\cdots\sqcup_{\Delta[0]}K_p'} \arrow[d] & X_{K_1\sqcup_{\Delta[0]}\cdots\sqcup_{\Delta[0]}K_r} \arrow[d, "\tau_1^*\times_{1_{X_0}}\cdots\times_{1_{X_0}}\tau_r^*"] \\
& X_{K_0^1\sqcup_{\Delta[0]}\cdots\sqcup_{\Delta[0]}K_0^p} \arrow[d, "(\tau_0^1)^*\times_{1_{X_0}}\cdots\times_{1_{X_0}}(\tau_0^p)^*"] & X_{Sp(r)} \arrow[l, "\mu_{K_0^1}\times_{1_{X_0}}\cdots\times_{1_{X_0}}\mu_{K_0^p}"'] \arrow[dd, "\mu_{K_0}"] \\
& X_{Sp(p)} \arrow[dl, "\mu_p"'] & \\
X_p & & X_{K_0} \arrow[ll, "((\tau_1^0)^*\times_{1_{X_0}}\cdots\times_{1_{X_0}}(\tau_p^0)^*)\times_{1_{Sp(p)}}1_{X_p}"]
\end{tikzcd}$$

commutes, where the map $X_{Sp(n)} \to X_p$ is induced by the other morphisms. The leftmost pentagon commutes by definition, while the bottom right pentagon commutes by the definition of μ_{K_0} and the fact that the lowest horizontal map is simply the projection map from the pullback defining X_{K_0} in terms of X_p and the spaces $X_{K_0^j}$.

For the uppermost pentagon, we consider each K_j' and repective K_j^0 in turn for varying j. The respective diagram will commute in each case by induction. Thus, the whole pentagon commutes and the leftmost vertical map is indeed the map we seek. $\square$

We can package the nested nature of μ_K into a genuine statement about composition operations analogous to the maps $\circ^{x_0,\cdots,x_n}$:

Notation 3.2.11. *Let X be a 2-fold Segal space and $Sp(n) \xrightarrow{\iota} K \xleftarrow{\tau} \Delta[1]$ a simplicial composition diagram. Let $x_0, \cdots, x_n \in (X_{0,0})_0$. Then define*

$$\begin{tikzcd}
\prod_{i=1}^n X(x_{i-1}, x_i) \arrow[r, hook] \arrow[drr, "\circ_K^{x_0,\cdots,x_n}"'] & X_{Sp(n)}^{x_0,x_n} \arrow[r, "\mu_K^{x_0,x_n}"] & X_K^{x_0,x_n} \arrow[d, "(\tau^*)^{x_0,x_n}"] \\
& & X(x_0, x_n)
\end{tikzcd}$$

Note again that if $n = 0$, the domain of $\circ_K^x$ is the terminal object $* \cong \{x\}$.

Theorem 3.2.12. *Suppose X is a 2-fold Segal space and $Sp(k_i) \xrightarrow{\iota_i} K_i \xleftarrow{\tau_i} \Delta[1]$ are simplicial composition diagrams for $0 \leq i \leq n$, with $k_0 = n \geq 1$. Let $r = \sum_i k_i$. Suppose*

$$Y = ((x_0^1, \cdots, x_{k_1}^1), \cdots, (x_0^n, \cdots, x_{k_n}^n))$$

is a nested list of elements of $(X_{0,0})_0$ such that $x^i_{k_i} = x^{i+1}_0$ for $i < n$. Let $(x_0, \cdots, x_r)$ be the flattened version of this list where all $x^i_{k_i}$ have been removed for $i < n$.

Then, setting $K := (K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_n) \sqcup_{Sp(n)} K_0$, we have that

$$\circ_K^{x_0,\cdots,x_r} = \circ_{K_0}^{x^1_0,x^2_0,\cdots,x^n_0,x^n_{k_n}} \circ (\circ_{K_1}^{x^1_0,\cdots,x^1_{k_1}} \times \cdots \times \circ_{K_n}^{x^n_0,\cdots,x^n_{k_n}}).$$

Before we can prove this theorem, we need some notation for compactness:

Notation 3.2.13. *For a series of spans $X_0 \leftarrow A_i \rightarrow X_0$ with $1 \leq i \leq n$ and $n > 0$, write*

$$\prod_{X_0}^{1\leq i\leq n} A_i := A_1 \times_{X_0} \cdots \times_{X_0} A_n.$$

If $n = 0$, set this to be the terminal object $*$.

Proof. By Proposition 3.2.10, it will suffice to prove that the diagram

$$\begin{tikzcd}
\prod_{i=1}^n \prod_{j=1}^{k_i} X(x^i_{j-1}, x^i_j) \arrow[r, hook, "\cong"] \arrow[d, hook] & \prod_{i=1}^r X(x_{i-1}, x_i) \arrow[d, hook] \\
\prod_{i=1}^n (\prod_{X_0}^{1\leq i\leq k_i} X_1)^{x^i_0,x^i_{k_i}} \arrow[r, hook] \arrow[d, "\prod_{i=1}^n \mu_{K_i}^{x^i_0,x^i_{k_i}}"'] & (\prod_{X_0}^{1\leq i\leq r} X_1)^{x_0,x_r} \arrow[d, "(\mu_{K_1}\times_{X_0}\cdots\times_{X_0}\mu_{K_n})^{x_0,x_r}"] \\
\prod_{i=1}^n X_{K_i}^{x^i_0,x^i_{k_i}} \arrow[r, hook] \arrow[d, "\prod_{i=1}^n (\tau_i^*)^{x^i_0,x^i_{k_i}}"'] & X^{x_0,x_r}_{K_1\sqcup_{\Delta[0]}\cdots\sqcup_{\Delta[0]}K_n} \arrow[d, "(\tau_1^*\times_{X_0}\cdots\times_{X_0}\tau_n^*)^{x_0,x_r}"] \\
\prod_{i=1}^n X(x^i_0, x^i_{k_i}) \arrow[r, hook] & X^{x_0,x_r}_{Sp(n)} \arrow[d, "\mu_{K_0}^{x_0,x_r}"'] \\
 & X^{x_0,x_r}_{K_0} \arrow[d, "(\tau_0^*)^{x_0,x_r}"'] \\
 & X(x_0, x_r)
\end{tikzcd}$$

commutes, where the inclusions are in general of the form $A \times_{X_0} \{x\} \times_{X_0} B \hookrightarrow A \times_{X_0} B$, where $A \rightarrow X_0 \leftarrow B$ is a cospan of simplicial spaces and $x \in (X_{0,0})_0$.

One can reduce the problem to checking each of the three squares in the diagram commutes. The first square is immediate, as the two paths are simply a matter of removing objects x_i in different orders. The second two squares commute trivially; it is a matter of applying the inclusion before or after the vertical morphisms, which has no effect. Hence, the result is immediate, as the two maps being equated are both paths in this diagram. □

When we come to define homotopy bicategories, our hom-categories will be $h_1(X(x,y))$ with composition functors $h_1(\circ^{x_0,\cdots,x_n})$. To obtain coherence isomorphisms like associators, we need an essential technical property of these maps with respect to *maps between simplicial composition diagrams*:

Definition 3.2.14. *A* map of simplicial composition diagrams *is a map $f : K_1 \to K_2$ in* **sSet** *for simplicial composition diagrams $\Delta[n] \xrightarrow{\iota_i} K_i \xleftarrow{\tau_i} \Delta[1]$ for $i = \{1, 2\}$, such that $f \circ \iota_1 = \iota_2$ and $f \circ \tau_1 = \tau_2$.*

Note that these maps do not necessarily respect some given inductive presentation of K_1 and K_2 as simplicial composition diagrams built out of smaller such diagrams. For the purposes of this thesis, it will suffice to not make such specifications. However, if we should for instance desire uniqueness of such morphisms, such considerations may then be of interest. We leave these quandaries to future work.

Definition 3.2.15. *Let $\boldsymbol{SCD}_n$ be the category whose objects are simplicial composition diagrams of arity n and whose morphisms are maps of simplicial composition diagrams.*

Proposition 3.2.16. *Let $Sp(n) \xrightarrow{\iota_i} K_i \xleftarrow{\tau_i} \Delta[1]$ for $i \in \{1, 2\}$ be two simplicial composition diagrams of arity $n \geq 0$ with $f : K_1 \to K_2$ a map in $\boldsymbol{SCD}_n$ between them. Let X be a 2-fold Segal space with some choice of horizontal compositions $(\mu_n)_{n\geq 0}$.*

Then the map

$$\mu'_{K_2} : X_{Sp(n)} \xrightarrow{\mu_{K_2}} X_{K_2} \xrightarrow{f^*} X_{K_1}$$

is naturally object-fibered and solves the same lifting problem as μ_{K_1}, namely giving a commutative diagram

$$\begin{tikzcd} & X_{K_1} \arrow[d, "\iota_1^*"] \\ X_{Sp(n)} \arrow[ur, dashed, "\mu'_{K_2}"] \arrow[r, "id"'] & X_{Sp(n)} \end{tikzcd}$$

such that for any $x_0, \cdots, x_n \in (X_{0,0})_0$, the diagram

$$\begin{tikzcd} \prod_{i=1}^n X(x_{i-1}, x_i) \arrow[r, hook] \arrow[rrd, "\circ_{K_2}^{x_0,\cdots,x_n}"'] & X^{x_0,x_n}_{Sp(n)} \arrow[r, "(\mu'_{K_2})^{x_0,x_n}"] & X^{x_0,x_n}_{K_1} \arrow[d, "(\tau_1^*)^{x_0,x_n}"] \\ & & X(x_0, x_n) \end{tikzcd}$$

commutes.

Proof. We have that μ'_{K_2} is object-fibered, as f^*'s commutativity with the maps ι_i^* implies it is object-fibered. It is then clear that this is a solution to the lifting problem, since $\iota_1^* \circ \mu'_{K_2} = (f \circ \iota_1)^* \circ \mu_{K_2} = \iota_2^* \circ \mu_{K_2} = id$. Moreover, the second diagram commutes, since $\tau_1^* \circ f^* = (f \circ \tau_1)^* = \tau_2^*$ and we obtain the original definition of $\circ_{K_2}^{x_0,\cdots,x_n}$. □

The consequence of this result we need is that, for any $f : K_1 \to K_2$ as above, the maps $\circ_{K_i}^{x_0,\cdots,x_n}$ are of the form $\beta \circ \mu^{(i)} \circ \alpha$, where the two $\mu^{(i)}$ for $i \in \{1, 2\}$ are solutions to the same lifting problem. We will find that this implies there is a homotopy between the two $\circ_{K_i}^{x_0,\cdots,x_n}$, which will descend under h_1 to our associator.

A final object we should define is the following functor:

Definition 3.2.17. *Let $n > 0$ and $k_1, \cdots, k_n \geq 0$. Define the functor*

$$\mathcal{G}^n_{k_1,\cdots,k_n} : \boldsymbol{SCD}_n \times \prod_{i=1}^{n} \boldsymbol{SCD}_{k_i} \to \boldsymbol{SCD}_{k_1+\cdots+k_n}$$

by sending

$$(K, K_1, \cdots, K_n) \mapsto (K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_n) \sqcup_{Sp(n)} K$$

with similar behavior for maps.

One might note that this creates a (nonsymmetric) operad $\mathbf{SCD}_\bullet$valued in **Cat**. We will not need this fact particularly so we omit a proof, but this same structure will reappear later.

3.3 Homotopies and Globular 2-Homotopies

It is now time for us to address the question of coherence isomorphisms. These will be achieved by a certain notion of *homotopy* between maps $f, g : X \to Y$ in $\mathbf{SeSp}^{inj}$. In general, it is our aim to show that, for a given 2-fold Segal space X with some choice of horizontal compositions $(\mu_n)_{n\geq 0}$, maps $f : K_1 \to K_2$ of simplicial composition diagrams induce homotopies from μ_{K_1} to $f^* \circ \mu_{K_2}$, which will in turn induce homotopies between $\circ_{K_1}^{x_0,\cdots,x_n}$ and $\circ_{K_2}^{x_0,\cdots,x_n}$ for all $x_0, \cdots, x_n \in (X_{0,0})_0$.

The payoff of this result is in the images of these homotopies under h_1. We will design our notion of homotopy to be such that the image of any homotopy under h_1 is a natural isomorphism between functors. Moreover, we will show that, under certain conditions, any two homotopies between the same two functors will be related by a higher 'homotopy between homotopies', which we will refer to as a *globular* 2*-homotopy.* The image of such a 2-homotopy under h_1 will be an equality between natural isomorphisms. Our coherence isomorphisms will then be the images of certain homotopies between composition operations $\circ_K^{x_0,\cdots,x_n}$ under h_1, which will satisfy the coherence conditions because of the existence of certain globular 2-homotopies.

The main concrete result of this section is an enrichment of h_1 by Kan complexes, whose domain $\mathbf{SeSp}^{inj}$ will be such that the hom-space from A to B will be the Kan complex of 'higher homotopies' between maps $A \to B$. The 1-simplices will be homotopies, while the 2-simplices will be some generalization of globular 2-homotopies to a simplicial context. Moreover, the codomain **Cat** will be enriched by natural isomorphisms between functors. We will also give a more abstract justification for globular 2-homotopies, proving they exist in a general model category and that their presence between certain homotopies is not specific to simplicial spaces or Segal spaces.

3.3.1 Homotopy Categories of Segal Spaces

Before we can proceed further, we will need a natural extension of the fact that π_0 preserves products to h_1. We do not claim originality for this elementary observation as it is noted in [Rez00, pg. 31]. However, we have not yet found a proof in the literature, so we provide one here:

Lemma 3.3.1. *h_1 preserves products up to natural isomorphism of categories.*

Proof. Let X and Y be Segal spaces. We seek to specify an isomorphism in **Cat** of the form $p : h_1(X \times Y) \to h_1(X) \times h_1(Y)$, which is natural in X and Y.

On objects, there is an evident natural bijection $\mathbf{ob}(h_1(X\times Y)) \to \mathbf{ob}(h_1(X)\times h_1(Y))$. For morphisms, note that

$$\begin{aligned}(X \times Y)((x_1, y_1), (x_2, y_2)) &:= \{(x_1, y_1)\} \times_{X_0\times Y_0} (X_1 \times Y_1) \times_{X_0\times Y_0} \{(x_2, y_2)\} \\ &\cong (\{x_1\} \times_{X_0} X_1 \times_{X_0} \{x_2\}) \times (\{y_1\} \times_{Y_0} Y_1 \times_{Y_1} \{y_2\}) \\ &= X(x_1, x_2) \times Y(x_2, y_2).\end{aligned}$$

Hence, there are natural bijections

$$\mathbf{Hom}_{h_1(X\times Y)}((x_1, y_1), (x_2, y_2)) \to \mathbf{Hom}_X(x_1, x_2) \times \mathbf{Hom}_Y(y_1, y_2)$$

induced by π_0 preserving products. That these respect identities and composition is easily verified. □

With this out of the way, the current aim for us is to show that h_1 sends some notion of homotopies to natural isomorphisms. To understand what this means in a precise sense, we will devise a suitable interpretation of the categories $\mathbf{SeSp}^{inj}$ and **Cat** as **sSet**-enriched categories that h_1 respects, where $\mathbf{SeSp}^{inj}$ is enriched via homotopies between maps and **Cat** is enriched via natural isomorphisms between functors. The latter is classical, though we will need a modification of it suitable for our methods. Recall the definition of $I[n]$ in Definition 2.2.72.

Definition 3.3.2. *Define* $\boldsymbol{Iso} : \boldsymbol{Cat} \to \boldsymbol{Grpd}$ *to be the functor sending* $\mathscr{C}$ *to the maximal subgroupoid* $\boldsymbol{Iso}(\mathscr{C})$ *of* $\mathscr{C}$.

Note that $\mathbf{Iso}$ is right adjoint to the natural inclusion $\iota : \mathbf{Grpd} \to \mathbf{Cat}$, which itself evidently preserves finite products and terminal objects. Thus, the functors $\mathbf{Iso}, \iota$ and the right adjoint functor $\mathbf{nerve} : \mathbf{Cat} \to \mathbf{sSet}$ are all monoidal under the monoidal structures on $\mathbf{Cat}, \mathbf{Grpd}$ and $\mathbf{sSet}$ given by Cartesian products. This implies the functor

$$\mathbf{nerve} \circ \iota \circ \mathbf{Iso} : \mathbf{Cat} \to \mathbf{sSet}$$

is monoidal under this structure on $\mathbf{Cat}$ and $\mathbf{sSet}$.

Note moreover that $\mathbf{Cat}$ is naturally enriched over itself by considering categories $\mathbf{Fun}(-,-)$ of functors and natural transformations. Thus, by [Rie14, Lemma 3.4.3], applying $\mathbf{nerve} \circ \iota \circ \mathbf{Iso}$ to this enrichment allows us to obtain the following enrichment of $\mathbf{Cat}$ by simplicial sets:

Definition 3.3.3. *Let* $\boldsymbol{Cat}_s$ *be the* $\boldsymbol{sSet}$*-enriched category of small categories with hom-spaces of the form*

$$\boldsymbol{Hom}_{\boldsymbol{Cat}_s}(\mathscr{C}, \mathscr{D}) := \boldsymbol{nerve}(\boldsymbol{Iso}(\boldsymbol{Fun}(\mathscr{C}, \mathscr{D}))).$$

The set of n-simplices for this simplicial set may be written as

$$\boldsymbol{Hom}_{\boldsymbol{Cat}_s}(\mathscr{C}, \mathscr{D})_n \cong \boldsymbol{Hom}_{\boldsymbol{Cat}}(\mathscr{C} \times I[n], \mathscr{D}).$$

There is a clear cosimplicial object $I[\bullet] : \Delta \to \mathbf{Grpd}$ sending each $[n]$ to $I[n]$, defining the simplicial maps here. To see how the above description of natural isomorphisms applies, note that a natural isomorphism $F \cong G$ of functors $F, G : \mathscr{C} \to \mathscr{D}$ can be alternatively be written as a functor $\mathscr{C} \times I[1] \to \mathscr{D}$.

3.3.2 Cosimplicial Resolutions

The story for Segal spaces demands some more advanced technology. We could certainly state that $SeSp^{inj}$ is enriched in $\mathbf{sSet}$ almost immediately, using $\mathbf{Map}_1^0(-,-)$. This is however not the structure we seek. Our wish is for the 1-simplices in our enrichment to represent homotopies of some description, which should be sent by h_1 to natural isomorphisms. More precisely, we wish to have a notion of homotopy between maps $f, g : X \to Y$ in $\mathbf{SeSp}^{inj}$ of the form $H : X \times M \to Y$ for some fixed $M \in \mathbf{SeSp}^{inj}$ such that $h_1(M) \cong I[1]$. If this condition is satisfied, then we may apply h_1 to obtain a natural isomorphism from H between $h_1(f)$ and $h_1(g)$ of the form

$$h_1(X) \times I[1] \cong h_1(X) \times h_1(M) \cong h_1(X \times M) \xrightarrow{h_1(H)} h_1(Y).$$

We would moreover like these maps $X \times M \to Y$ to be the 1-simplices in our enrichment of $\mathbf{SeSp}^{inj}$ and in turn h_1.

Let $X, Y \in \mathbf{SeSp}^{inj}$. A 1-simplex in $\mathbf{Map}_1^0(X, Y)$ is a morphism $X \square_1^0 \Delta[1] \to Y$. It is at this point we notice a fundamental obstruction: the constant bisimplicial set $\iota_1^0\Delta[1]$ is not Reedy fibrant, so is not an object in $\mathbf{SeSp}^{inj}$. For instance, the demand that the map $\mathbf{Map}_1^0(F_1^0(1), \iota_1^0\Delta[1]) \to \mathbf{Map}_1^0(\partial F_1^0(1), \iota_1^0\Delta[1])$ is a fibration in $\mathbf{sSet}$ is asking for the diagonal map $(id, id) : \Delta[1] \to \Delta[1] \times \Delta[1]$ to be a fibration in $\mathbf{sSet}$. This is clearly not the case, as we can set up a horn lifting problem

$$\begin{tikzcd}
\Delta[0] \arrow[r, hook, "\langle 0 \rangle"] \arrow[d, hook, "\langle 0 \rangle"'] & \Delta[1] \arrow[d, "(id,id)"] \\
\Delta[1] \arrow[r, "(id, \langle 0,0 \rangle)"'] & \Delta[1] \times \Delta[1]
\end{tikzcd}$$

which has no solution. Moreover, the simplicial set $\Delta[1]$ is itself not a Kan complex - the map

$$\langle 0,1 \rangle \sqcup_{\langle 0 \rangle} \langle 0,0 \rangle : \Delta[1] \sqcup_{\langle 0 \rangle, \Delta[0], \langle 0 \rangle} \Delta[1] \cong \Lambda_0^2 \to \Delta[1]$$

has no filler to $\Delta[2] \to \Delta[1]$, so the level 0 condition for Reedy fibrancy also fails. We might interpret this failure as the fact that the poset category of two objects and one morphism between them is not an ∞-groupoid, as this one morphism has no inverse. This observation also implies that $\iota_1^0\Delta[1]$ is not projective fibrant, so cannot possibly be a projective fibrant Segal space either.

There is another potential enrichment of $\mathbf{SeSp}^{inj}$ we could turn to, where the mapping space from X to Y is of the form

$$\mathbf{Map}_F(X, Y) := ((Y^X)_\bullet)_0.$$

The n-simplices are now of the form $f : X \times F_1^0(n) \to Y$. Notably, $F_1^0(n)$ is Reedy fibrant as it is levelwise discrete, as noted more generally in [Rez00, pg. 6]. Moreover, it is evidently a Segal space; indeed, the Segal maps in this case are bijections. Thus, we may apply h_1 to such a 1-simplex f and obtain a functor

$$h_1(f) : h_1(X) \times h_1(F_1^0(1)) \cong h_1(X) \times [1] \to h_1(Y).$$

This is precisely a natural transformation between functors. However, we are interested only in natural *isomorphisms*, as all of the coherence 2-morphisms we wish to construct for our homotopy bicategories must be as such. Hence, though one could likely proceed with the above enrichment, we it will be beneficial for us to instead pick an enrichment of $\mathbf{SeSp}^{inj}$ whose 1-simplices are guaranteed to be sent by h_1 to natural isomorphisms on the nose.

A sensible path forwards is to identify some deeper mechanism which gives rise to the appearance of $\Delta[1]$ in our theory and consider whether $\Delta[1]$ can be thus substituted for something more well-behaved, while continuing to reap all the benefits said mechanism provided. The correct such structure is that of a *homotopy function complex* and the underlying *cosimplicial resolution*, as defined in Definition 2.2.50. The reason to think about these is that cosimplicial resolutions are closely tied to notions of left homotopy, always exhibiting valid cylinder objects:

Definition 3.3.4 ([DS95, Def. 4.2]). *Let $\mathscr{M}$ be a model category and C an object in $\mathscr{M}$. A* cylinder object *of C is a factorization*

$$C \sqcup C \xrightarrow{i} C \wedge I \xrightarrow{p} C$$

of the map $C \sqcup C \to C$ such that p is a weak equivalence. The cylinder object is a good cylinder object *if i is a cofibration and a* very good cylinder object *if it is good and p is a trivial fibration.*

Definition 3.3.5 ([DS95, pg. 19]). *Let $\mathscr{M}$ be a model category and $f, g : C \to D$ be morphisms in $\mathscr{M}$. Suppose $C \sqcup C \xrightarrow{i} C \wedge I \xrightarrow{p} C$ is a cylinder object for C. A* left homotopy *$H : C \wedge I \to D$* from *f* to *g is a map H such that $H \circ i$ is the map $f + g : C \sqcup C \to D$.*

The map H is moreover a good left homotopy *if $C \wedge I$ is a good cylinder object and a* very good left homotopy *if $C \wedge I$ is a very good cylinder object.*

Proposition 3.3.6 ([Hir09, Prop. 16.1.6]). *Let $\mathscr{M}$ be a model category. If $\tilde{X}$ is a cosimplicial resolution of X in $\mathscr{M}$, then*

$$\tilde{X}_0 \sqcup \tilde{X}_0 \xrightarrow{\tilde{X}_{\langle 0\rangle} \sqcup \tilde{X}_{\langle 1\rangle}} \tilde{X}_1 \xrightarrow{\tilde{X}_{\langle 0,0\rangle}} \tilde{X}_0$$

is a good cylinder object of $\tilde{X}_0$.

Recall from Corollary 2.2.53 that $\mathbf{Map}_1^0(-,-)$ is a left homotopy function complex induced by a cosimplicial resolution whose cylinder objects are of the form $X \sqcup X \to X \square_1^0 \Delta[1] \to X$. It seems reasonable then that replacing this cosimplicial resolution is the correct way to replace $\Delta[1]$. We should choose the resolution so that the resulting cylinder objects are of the form $X \sqcup X \to X \times K \to X$ for some K such that $h_1(K) \cong I[1]$. This will imply we obtain an enrichment in homotopy function complexes whose 1-simplices are sent by h_1 to natural isomorphisms. We will find that this indeed leads to the enrichment we seek.

Another advantage to this approach to building an enrichment is that our mapping spaces will always be Kan complexes:

Proposition 3.3.7 ([Hir09, Prop. 17.1.3])**.** *Suppose $\mathscr{M}$ is a model category and X and Y are objects of $\mathscr{M}$. Then a left homotopy function complex is a Kan complex.*

This will later let us compose our left homotopies vertically.

Our chosen cosimplicial resolution involves the *classifying diagram* functor. Recall that $\mathbf{Iso}(\mathscr{C})$ for a category $\mathscr{C}$ is its maximal subgroupoid. Moreover, recall that $\mathbf{Iso} : \mathbf{Cat} \to \mathbf{Grpd}$ is right adjoint to the inclusion $\mathbf{Grpd} \to \mathbf{Cat}$.

Definition 3.3.8 ([Rez00, pg. 8])**.** *The* classifying diagram functor N : $\boldsymbol{Cat} \to \boldsymbol{sSpace}$ *is defined such that for any category $\mathscr{C}$ and $n \geq 0$,*

$$N\mathscr{C}_n := \boldsymbol{nerve}(\boldsymbol{Iso}(\mathscr{C}^{[n]}))$$

with the evident simplicial maps and behavior on functors.

Proposition 3.3.9 ([Rez00, Prop. 6.1])**.** *Let $\mathscr{C} \in \boldsymbol{Cat}$. Then $N\mathscr{C}$ is a complete Segal space.*

Proposition 3.3.10 ([Rez00, Thm. 3.7])**.** *Let $\mathscr{C}, \mathscr{D} \in \boldsymbol{Cat}$. Then $N(\mathscr{C} \times \mathscr{D}) \cong N(\mathscr{C}) \times N(\mathscr{D})$.*

Proposition 3.3.11 ([Rez00, pg. 13])**.** *There is a canonical natural isomorphism $h_1 \circ N \cong 1_{\boldsymbol{Cat}}$.*

Proof. The objects of $h_1(N\mathscr{C})$ are the elements of the set $(N\mathscr{C}_0)_0$, which are the 0-simplices of $\mathbf{nerve}(\mathbf{Iso}(\mathscr{C}))$, namely the objects of $\mathscr{C}$. Hence, the object sets are the same. For morphisms, we note both $\mathbf{nerve} : \mathbf{Cat} \to \mathbf{sSet}$ and $\mathbf{Iso} : \mathbf{Cat} \to \mathbf{Grpd}$ are right adjoints. Thus, they commute with pullbacks, meaning for any objects x and y of $\mathscr{C}$,

$$\mathbf{nerve}(\mathbf{Iso}(\mathscr{C}^{[1]})) \times_{(\mathbf{nerve}(\mathbf{Iso}(\mathscr{C})))^2} \{(x,y)\} \cong \mathbf{nerve}(\mathbf{Iso}(\mathscr{C}^{[1]} \times_{\mathscr{C}^2} \{(x,y)\}))$$

Note that $\mathscr{C}^{[1]} \times_{\mathscr{C}^2} \{(x,y)\}$ is always discrete as a category - indeed, all morphisms are squares whose objects are x and y and whose vertical maps are identities. Hence, the entire nerve is discrete, so the hom-sets of $h_1(N\mathscr{C})$ are path components of discrete spaces, and so are just the original hom-sets as needed.

Identities are also clearly the same, as is composition. Hence, $h_1(N\mathscr{C}) \cong \mathscr{C}$ as needed. Functors share a similar fate by inspection, so the proof is complete. $\square$

As noted in [Rez00, pg. 13], the identifications of objects and hom-sets of $h_1(N(\mathscr{C}))$ for some category $\mathscr{C}$ with those of $\mathscr{C}$ itself are already stated in [Rez00, Rem. 5.2], which are in many ways the core components of this proof. We claim no originality for the above proof, having simply added some intermediate working for the reader's benefit.

For a Segal space X, we take our cosimplicial resolution to be $\tilde{X} := X \times (N \circ I[\bullet])$, so that $\tilde{X}_n := X \times N(I[n])$.

Proposition 3.3.12. *Suppose $X \in \boldsymbol{SeSp}^{inj}$. Then the functor $X \times N(I[\bullet])$, sending $[n] \mapsto X \times N(I[n])$, defines a cosimplicial resolution of X in $SeSp^{inj}$.*

Proof. We need to prove that the natural map $X \times N(I[\bullet]) \to \mathbf{cc}_*(X)$ is a levelwise weak equivalence. This is in fact a trivial fibration levelwise; each map $X \times N(I[n]) \to X$ is a pullback map for the cospan $X \to * \leftarrow N(I[1])$. Note that since N has its image in $\mathbf{CSSP}^{inj}$ by Proposition 3.3.9, the map $N(I[1]) \to *$ is a fibration in $SeSp^{inj}$. It is moreover a weak equivalence since $I[1] \to *$ is an equivalence of categories and N sends equivalences to levelwise weak equivalences of simplicial spaces [Rez00, Thm. 3.7], which are thus weak equivalences in $SeSp^{inj}$.

It now suffices to prove that $X \times N(I[\bullet])$ is Reedy cofibrant. In particular, we must prove that the latching maps for all $n \geq 0$ of the form

$$\mathrm{colim}_{k \in \partial\Delta[n]}\big(X \times N(I[k])\big) \cong X \times \big(\mathrm{colim}_{k \in \partial\Delta[n]} N(I[k])\big) \to X \times N(I[n])$$

are cofibrations in $SeSp^{inj}$, namely monomorphisms, for which it is sufficient to prove that the maps

$$L(n) := \mathrm{colim}_{k \in \partial\Delta[n]} N(I[k]) \to N(I[n])$$

are monomorphisms. Thus, we are tasked with proving that the cosimplicial object $N(I[\bullet]) : \Delta \to SeSp^{inj}$ is Reedy cofibrant.

Let $m \geq 0$. Note that the above maps are monomorphisms if and only if they are levelwise, namely if and only if each map

$$L(n)_m \cong \mathrm{colim}_{k \in \partial\Delta[n]} N(I[k])_m = L_n(N(I[\bullet])_m) \to N(I[n])_m$$

is a cofibration in $\mathbf{sSet}$. Hence, it suffices to prove that the cosimplicial simplicial sets $N(I[\bullet])_m : \Delta \to \mathbf{sSet}$ are Reedy cofibrant.

These cosimplicial simplicial sets are, for $n \geq 0$, of the form

$$N(I[n])_m \cong \mathbf{nerve}(\mathbf{Iso}(I[n]^{[m]})) = \mathbf{nerve}(I[n]^{[m]}).$$

By [Hir09, Cor. 15.9.10], a cosimplicial simplicial set $X : \Delta \to \mathbf{sSet}$ is Reedy cofibrant if and only if the equalizer of the map

$$X(0) \sqcup X(0) \to X(1)$$

is empty. However, this is clearly the case above; the two maps $\mathbf{nerve}(I[0]^m) \to \mathbf{nerve}(I[1]^{[m]})$ send a k-simplex $[k] \times [m] \to I[0]$ to a map $[k] \times [m] \to I[1]$ with constant image in either the object 0 or 1, respectively. Hence, each $N(I[\bullet])_m$ is a Reedy cofibrant cosimplicial simplicial set, implying $N(I[\bullet])$ is Reedy cofibrant in $(SeSp^{inj})^{\Delta}$, as needed. □

Hence, since Segal spaces are fibrant in the model structure $SeSp^{inj}$, we can build a Kan complex $\mathbf{Map}^I_{\mathbf{sS}}(X, Y)$ for any Segal spaces X and Y, which at level k is the set of maps $X \times N(I[k]) \to Y$, given by the left homotopy function complex induced by our cosimplicial resolution.

Proposition 3.3.13. *$\boldsymbol{Map}^I_{\boldsymbol{sS}}(-,-) : (\boldsymbol{SeSp}^{inj})^{op} \times \boldsymbol{SeSp}^{inj} \to \boldsymbol{sSet}$ defines an enrichment of $\boldsymbol{SeSp}^{inj}$ in Kan complexes.*

Proof. Suppose X, Y and Z are Segal spaces and let $k \geq 0$. Let $f : X \times N(I[k]) \to Y$ and $g : Y \times N(I[k]) \to Z$. Then we have a natural map

$$X \times N(I[k]) \xrightarrow{1_X \times D_k} X \times N(I[k]) \times N(I[k]) \xrightarrow{f \times 1_{N(I[k])}} Y \times N(I[k]) \xrightarrow{g} Z$$

where the map $D_k : N(I[k]) \to N(I[k]) \times N(I[k])$ is the diagonal map.We will use this as our composite of g and f.

The resulting operations are compatible with the simplicial structure on $\mathbf{Map}^I_{\mathbf{sS}}(-,-)$, giving an operation

$$\circ : \mathbf{Map}^I_{\mathbf{sS}}(X, Y) \times \mathbf{Map}^I_{\mathbf{sS}}(Y, Z) \to \mathbf{Map}^I_{\mathbf{sS}}(X, Z).$$

Identities are given by $X \times N(I[1]) \to X \times N(I[0]) \cong X \xrightarrow{1_X} X$. Associativity and identity laws are routine. □

Note that there is a more conceptual reason for the existence of this enrichment. Consider the functor $N \circ I[\bullet] : \Delta \to \mathbf{sSpace}$. By Proposition 2.1.18, this induces a right adjoint functor

$$I' : \mathbf{sSpace} \to \mathbf{sSet}$$

sending X to the simplicial set $I'(X)$ such that

$$I'(X)_n := \mathbf{Hom}_{\mathbf{sSpace}}(N(I[n]), X).$$

Since I' is a right adjoint, it is monoidal under the monoidal structures on **sSpace** and **sSet** given by Cartesian products. Thus, by [Rie14, Lemma 3.4.3], this induces an **sSet**-enrichment on **sSpace**, by change-of-base on the natural **sSpace**-enrichment of **sSpace** given by internal hom's. This can then be restricted to the full subcategory $\mathbf{SeSp}^{inj}$.

Explicitly, we have that the induced mapping space from X to Y in $\mathbf{SeSp}^{inj}$ is levelwise of the form

$$I'(Y^X)_n := \mathbf{Hom}_{\mathbf{sSpace}}(N(I[n]), Y^X) \cong \mathbf{Hom}_{\mathbf{SeSp}^{inj}}(N(I[n]) \times X, Y)$$

as needed.

We thus obtain our $\mathbf{sSet}$-enrichment for $\mathbf{SeSp}^{inj}$. Moreover, the enrichment of h_1 is now immediate: there is an evident simplicial map for Segal spaces X, Y of the form

$$\mathbf{Map}^I_{\mathbf{sS}}(X, Y) \to \mathbf{nerve}(\mathbf{Iso}(\mathbf{Fun}(h_1(X), h_1(Y))))$$

sending $f : X \times N(I[k]) \to Y$ to

$$\begin{aligned} h_1(X) \times I[k] \cong h_1(X) \times h_1(N(I[k])) &\cong h_1(X \times N(I[k])) \\ &\xrightarrow{h_1(f)} h_1(Y). \end{aligned}$$

Proposition 3.3.14. *By the above mapping, h_1 extends to an* ***sSet****-enriched functor.*

Proof. It is clear that identities are preserved, so we are left to show the same for composition. However, one notes that if a natural isomorphism $\alpha : F \Rightarrow G$ between functors $F, G : \mathscr{C} \to \mathscr{D}$ is written in the form $\mathscr{C} \times I[1] \to \mathscr{D}$, then horizontal composition is exactly as stated for $\mathbf{Map}^I_{\mathbf{sS}}$ at all levels of $\mathbf{nerve}(\mathbf{Iso}(\mathbf{Fun}(\mathscr{C}, \mathscr{D})))$. Hence, composition is preserved as needed. □

We now have the desired simplicial set enrichment. We also have that the 1-simplices in $\mathbf{Map}^I_{\mathbf{sS}}(X, Y)$ are valid left homotopies, as $X \times N(I[1])$ is a cylinder object for X owing to it being part of a cosimplicial resolution. We thus have established how h_1 can transmit left homotopies to natural isomorphisms in an '∞-groupoidal' manner.

Notation 3.3.15. *Suppose $H : X \times N(I[1]) \to Y$ is a left homotopy from f to g, where $f, g : X \to Y$ are maps in $\mathbf{SeSp}^{inj}$. Then write $h_1(H) : h_1(X) \times I[1] \to h_1(Y)$ to refer to the natural isomorphism*

$$h_1(X) \times I[1] \cong h_1(X) \times h_1(N(I[1])) \cong h_1(X \times N(I[1])) \xrightarrow{h_1(H)} h_1(Y)$$

from $h_1(f)$ to $h_1(g)$.

A useful consequence of this particular choice of cylinder is that it is in fact a very good cylinder object:

Proposition 3.3.16. *Let X be a simplicial space. Then $X \times N(I[1])$ is a very good cylinder object for X in $SeSp^{inj}$.*

Proof. By Proposition 3.3.6 and Proposition 3.3.12, we have a good cylinder object. As stated in the proof of Proposition 3.3.12, the object $N(I[1])$ is fibrant in $SeSp^{inj}$, so that the map $X \times N(I[1]) \to X$ is moreover a fibration as needed. $\square$

One final fact we will need about left homotopies is their capacity to be composed. That left homotopies can be composed in general is classical; by [Hir09, Def. 7.4.3], given two good cylinder objects $X \wedge I$ and $X \wedge I'$ of a cofibrant object X in a model category $\mathscr{M}$, one may explicitly produce a 'composite' of any two good left homotopies $H : X \wedge I \to Y$ and $H' : X \wedge I' \to Y$ in $\mathscr{M}$ by using the pushout

$$(X \wedge I) \sqcup_X (X \wedge I')$$

as a new good cylinder object. We provide the details for our case, which concretely shows moreover how we may continue to use precisely the same kind of cylinder objects as we have before:

Definition 3.3.17. *Consider a lifting problem of Segal spaces*

$$\begin{tikzcd} & B \arrow[d, "p"] \\ C \arrow[r] & D \end{tikzcd}$$

where p is a trivial fibration. Suppose $f, g, h : C \to B$ are solutions, with left homotopies $H, K : C \times N(I[1]) \to B$ over D, from f to g and from g to h respectively. Then a composite *of H and K is a left homotopy $KH : C \times N(I[1]) \to B$ over D from f to h defined by the composition $Q \circ i$ in the diagram*

$$\begin{tikzcd} & C \sqcup C \sqcup C \arrow[d, hook] \arrow[dr, "f \sqcup g \sqcup h"] & \\ & C \times (N(I[1]) \sqcup_* N(I[1])) \arrow[r, dashed, "H \sqcup_g K"] \arrow[d, hook, "j"'] & B \arrow[d, "p"] \\ C \times N(I[1]) \arrow[r, hook, "i"] & C \times N(I[2]) \arrow[r] \arrow[ur, dashed, "Q"] & D \end{tikzcd}$$

where i is the image of $\langle 0, 2 \rangle : \Delta[1] \to \Delta[2]$ in the cosimplicial object $C \times N(I[\bullet])$, while j on each component $C \times N(I[1])$ in its domain is the image of the maps $\langle 0, 1 \rangle$ and $\langle 1, 2 \rangle$ respectively.

Note that Q need not be unique in the above definition; in general, there may be many candidate composites for two given homotopies.

A trivial result is then immediate:

Proposition 3.3.18. *In the situation of Definition 3.3.17, the map Q is a horn filler for the 2-horn $\Lambda_1^2 \to \boldsymbol{Map}^I_{\boldsymbol{sS}}(C,B)$ defined by H and K on each of its respective 1-simplices. The homotopy KH represents the edge $\langle 0,2\rangle$ of Q.*

Hence, it is reasonable to see KH as a valid composite of H and K in the Kan complex $\mathbf{Map}^I_{\mathbf{sS}}(C,B)$. Of course, KH is not at all unique. This will not matter when we come to need this result.

3.3.3 Globular 2-Homotopies

We are in need of some result that establishes when h_1 sends two left homotopies $K, H : X \times N(I[1]) \to Y$ between the same maps $f, g : X \to Y$ to the same natural isomorphism. A suitable such criterion is the existence of a *globular 2-homotopy* between K and H:

Definition 3.3.19. *Let $X, Y \in \boldsymbol{SeSp}^{inj}$. Let $f, g : X \to Y$ and let $K, H : X \times N(I[1]) \to Y$ be two left homotopies from f to g. Then a* globular 2-homotopy *$\alpha : K \Rightarrow H$ is a left homotopy from K to H, namely a functor $X \times N(I[1]) \times N(I[1]) \to Y$, such that the restrictions to $X \times \{0\} \times N(I[1])$ and $X \times \{1\} \times N(I[1])$ are constantly f and g respectively.*

Globular 2-homotopies are notably similar to the *correspondences* of [Qui67, pg. 23, Def. 2], for our particular model category and cylinder objects. The two are not precisely equal, though we believe they should be equivalent in some suitable sense. We find no need to explore such a comparison further here, as we are not concerned with manipulating globular 2-homotopies so much as we are with establishing their mere existence.

Globular 2-homotopies are, in spirit, homotopies between two homotopies of maps of Segal spaces. They moreover enjoy a place in our enrichment of $\mathbf{SeSp}^{inj}$ by supposed Kan complexes of 'higher homotopies', helping to justify this interpretation thereof. Consider the two maps

$$A, B : I[2] \to I[1] \times I[1]$$

sending $0 \mapsto (0,0)$ and $2 \mapsto (1,1)$, where A sends 1 to $(0,1)$ while B sends 1 to $(1,0)$. Then for a globular 2-homotopy $\alpha : X \times N(I[1]) \times N(I[1]) \to Y$, we obtain two maps

$$X \times N(I[2]) \to X \times N(I[1] \times I[1]) \cong X \times N(I[1]) \times N(I[1]) \to Y.$$

In our enrichment of $\mathbf{SeSp}^{inj}$ by $\mathbf{Map}^I_{\mathbf{sS}}(-,-)$, these correspond precisely to those 2-simplices $\alpha \in \mathbf{Map}^I_{\mathbf{sS}}(X,Y)_2$ such that either the 1-simplex of the form $\mathbf{Map}^I_{\mathbf{sS}}(X,Y)_{\langle 0,1\rangle}(\alpha)$ or the 1-simplex of the form $\mathbf{Map}^I_{\mathbf{sS}}(X,Y)_{\langle 1,2\rangle}(\alpha)$ is degenerate. We may thus interpret globular 2-homotopies as the 2-morphisms in each ∞-groupoid $\mathbf{Map}^I_{\mathbf{sS}}(X,Y)$.

A particularly appealing aspect of our specific definition for globular 2-homotopies is that, upon an application of h_1, a globular 2-homotopy between two homotopies H and K will explicitly identify the natural isomorphisms $h_1(H)$ and $h_1(K)$:

Proposition 3.3.20. *Let X, Y, f, g, K, H be as above. Suppose $\alpha : K \Rightarrow H$ is a globular 2-homotopy from K to H. Then the two natural isomorphisms, denoted by $h_1(H)$ and $h_1(K)$, of the form*

$$h_1(X) \times I[1] \cong h_1(X) \times h_1(N(I[1])) \cong h_1(X \times N(I[1])) \xrightarrow{h_1(K), h_1(H)} h_1(Y)$$

are equal.

Proof. This is a matter of unwinding definitions. We find that the map, which we denote by $h_1(\alpha)$, of the form

$$\begin{aligned} h_1(X) \times I[1] \times I[1] &\cong h_1(X) \times h_1(N(I[1])) \times h_1(N(I[1])) \\ &\cong h_1(X \times N(I[1]) \times N(I[1])) \\ &\xrightarrow{h_1(\alpha)} h_1(Y) \end{aligned}$$

reduces to $h_1(f)$ on $h_1(X) \times \{0\} \times I[1]$ and $h_1(g)$ on $h_1(X) \times \{1\} \times I[1]$. Moreover, it restricts to $h_1(K)$ and $h_1(H)$ on the other evident restrictions. It is then clear that $h_1(K) = h_1(H)$ as needed. $\square$

An important consequence to this proposition for us lies in comparing solutions of lifting problems. Consider a lifting problem with two possible solutions

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "p"] \\ C \arrow[r, "g"'] \arrow[ur, dashed, bend left, "\delta"] \arrow[ur, dashed, bend right, "\epsilon"'] & D \end{tikzcd}$$

where i is a cofibration and p a trivial fibration. We have the following useful result:

Proposition 3.3.21 ([Hir09, Prop. 7.6.13])**.** *Let $\mathscr{M}$ be a model category. Consider the above solid arrow diagram, where adding in either δ or ϵ alone still makes the diagram commute. Then there is a left homotopy from δ to ϵ in the model category $(A \downarrow \mathscr{M} \downarrow D)$.*

It will be important for us to understand how this is proven.

Proof. There is a model structure on $(A \downarrow \mathscr{M} \downarrow D)$ given in [Hir09, Thm. 7.6.5], where weak equivalences, fibrations and cofibrations are those which restrict to such maps in $\mathscr{M}$.

Now, factor the map $C \sqcup_A C \to C$ as $C \sqcup_A C \xrightarrow{j} C \wedge I \xrightarrow{r} C$ as a good cylinder object, so j is a cofibration and r a weak equivalence. Consider the solid arrow diagram

$$\begin{tikzcd}
C \sqcup_A C \arrow[rr, "\delta \sqcup_f \epsilon"] \arrow[d, "j"'] & & B \arrow[d, "p"] \\
C \wedge I \arrow[r, "r"'] \arrow[urr, dashed, "H"] & C \arrow[r, "g"'] & D
\end{tikzcd}$$

where j is a cofibration and p is a trivial fibration. Thus, there is a dotted arrow H making the diagram commute, which makes H a left homotopy from δ to ϵ in $(A \downarrow \mathscr{M} \downarrow D)$ as needed. □

A side result should be noted with regards to these homotopies and taking fibers:

Proposition 3.3.22. *Suppose $\mathscr{M} = SeSp^{inj}$ in the above lifting problem, X is a 2-fold Segal space and that f, i, p, g, δ and ϵ are all object-fibered over $(X_0)^{n+1}$. Then the induced homotopy H will also be object-fibered, with respect to the vertex maps $C \wedge I \to C \to (X_0)^{n+1}$. Moreover, for any $x_0, \cdots, x_n \in (X_{0,0})_0$, the map $H^{x_0,\cdots,x_n} : (C \wedge I)^{x_0,\cdots,x_n} \to B^{x_0,\cdots,x_n}$ is a left homotopy from $\delta^{x_0,\cdots,x_n}$ to $\epsilon^{x_0,\cdots,x_n}$.*

Proof. The map j is clearly object-fibered, by the naturally induced map from the pushout $C \sqcup_A C \to (X_0)^{n+1}$. The map $\delta \sqcup_f \epsilon$ must also be object-fibered, while p and g are by assumption. Thus, by Proposition 3.2.1, the solution H of the lifting problem must also be object-fibered such that commutativity of the diagram is preserved after taking a fiber.

Note that r is object-fibered. Moreover, $r^{x_0,\cdots,x_n}$ is a trivial fibration and thus a weak equivalence, so $(C \wedge I)^{x_0,\cdots,x_n}$ is a cylinder object, though perhaps not a good or very good one anymore. Hence, $H^{x_0,\cdots,x_n}$ is a valid left homotopy as claimed. □

Note that in the case $C \wedge I = C \times N(I[1])$, the homotopy will in fact stay a very good left homotopy.

If $A = \emptyset$ is the initial object, the cylinder object in question is also a cylinder object in $(\mathscr{M} \downarrow D)$, as well as in $\mathscr{M}$, yielding a lifting problem of the form

$$\begin{tikzcd} C \sqcup C \arrow[r, "\delta\sqcup\epsilon"] \arrow[d, "i"'] & B \arrow[d, "p"] \\ C \wedge I \arrow[r, "g"'] \arrow[ur, dashed, "H"] & D \end{tikzcd}$$

which we encapsulate in the following corollary:

Corollary 3.3.23. *Given a model category $\mathscr{M}$ and a diagram*

$$\begin{tikzcd} & B \arrow[d, "p"] \\ C \arrow[r, "g"'] \arrow[ur, dashed, bend left, "\delta"] \arrow[ur, dashed, bend right, "\epsilon"'] & D \end{tikzcd}$$

where C is cofibrant, $p : B \to D$ is a trivial fibration and δ and ϵ are each solutions to the lifting problem, then there is a left homotopy in $(\mathscr{M} \downarrow D)$ from δ to ϵ.

We can then compare two such left homotopies $H, K : C \wedge I \to B$. As per the proof of Proposition 3.3.21, these are two solutions to a new lifting problem, letting us inductively apply the proposition to obtain a left homotopy Γ from H to K in $(C \sqcup C \downarrow \mathscr{M} \downarrow D)$. Of course, the cylinder object this homotopy will be defined with respect to is not guaranteed to be such that, for instance, we precisely obtain a globular 2-homotopy in the case of Segal spaces. We will show how to resolve this situation in the most general context possible, as the structure of a globular 2-homotopy between homotopies of solutions to a lifting problem may be of interest in other model categories.

We should spend some time understanding precisely what such a left homotopy is in our use case. Consider, for $f : X \to Z$ a map of simplicial spaces and $p : Y \to Z$ a trivial fibration in **sSpace** with the model structure $SeSp^{inj}$, two solutions α and β of the lifting problem of the form

$$\begin{tikzcd} & Y \arrow[d, two heads, "p"] \\ X \arrow[r, "f"'] \arrow[ur, dashed, bend left, "\alpha"] \arrow[ur, dashed, bend right, "\beta"'] & Z \end{tikzcd}$$

Then consider two left homotopies $H, K : \alpha \Rightarrow \beta$ induced as solutions to the lifting problem

$$\begin{tikzcd}
X \sqcup X \arrow[r, "\alpha\sqcup\beta"] \arrow[d, hook] & Y \arrow[d, two heads, "p"] \\
X \times N(I[1]) \arrow[r, "f\circ(1_X\times!)"'] \arrow[ur, dashed, bend left=10, "H"] \arrow[ur, dashed, bend right=10, "K"'] & Z
\end{tikzcd}$$

Then there is an induced left homotopy Γ in $(X \sqcup X \downarrow \mathbf{sSpace} \downarrow Z)$ from H to K. We need a suitable cylinder object in this model category to be able to write down Γ directly. One may produce a general such cylinder object. Before we do so, a technical lemma is needed for the construction:

Lemma 3.3.24. *Suppose $C \sqcup C \xrightarrow{i} C \wedge I \xrightarrow{p} C$ is a very good cylinder object for C in a model category $\mathscr{M}$. Then there is a very good cylinder object*

$$(C \wedge I) \sqcup (C \wedge I) \xrightarrow{h} C \wedge I \wedge I \xrightarrow{r} C \wedge I$$

for $C \wedge I$ and a cofibration v that solves the lifting problem

$$\begin{tikzcd}
(C \sqcup C) \sqcup (C \sqcup C) \arrow[r, "i\sqcup i"] \arrow[d, "\tau"'] & (C \wedge I) \sqcup (C \wedge I) \arrow[r, "h"] & C \wedge I \wedge I \arrow[dd, "r"] \\
(C \sqcup C) \sqcup (C \sqcup C) \arrow[d, "i\sqcup i"'] & & \\
(C \wedge I) \sqcup (C \wedge I) \arrow[r, "p\sqcup p"'] \arrow[uurr, dashed, "v"] & C \sqcup C \arrow[r, "i"'] & C \wedge I
\end{tikzcd}$$

where $\tau : (C \sqcup C) \sqcup (C \sqcup C) \to (C \sqcup C) \sqcup (C \sqcup C)$ is the isomorphism permuting the C's by the permutation $(1324)^2$.

Proof. Consider the pushout diagram

$$\begin{tikzcd}
(C \sqcup C) \sqcup (C \sqcup C) \arrow[r, "i\sqcup i"] \arrow[d, "\tau"'] & (C \wedge I) \sqcup (C \wedge I) \arrow[dd, "h'"] \\
(C \sqcup C) \sqcup (C \sqcup C) \arrow[d, "i\sqcup i"'] & \\
(C \wedge I) \sqcup (C \wedge I) \arrow[r, "v'"'] \arrow[ur, phantom, "\lrcorner"] & C \wedge \square
\end{tikzcd}$$

We note that h' and v' are both cofibrations, by preservation along pushouts. Now, a universal map $f : C \wedge \square \to C \wedge I$ is obtained from the pushout by the

[2] Read this as the map $(a_1a_2a_3a_4)$ sending $i \mapsto a_i$, rather than the cyclic permutation.

diagram

$$\begin{tikzcd}
(C \sqcup C) \sqcup (C \sqcup C) \arrow[r, "i \sqcup i"] \arrow[d, "\tau"'] & (C \wedge I) \sqcup (C \wedge I) \arrow[dd, "h'"] \arrow[ddr, bend left, two heads] & \\
(C \sqcup C) \sqcup (C \sqcup C) \arrow[d, "i \sqcup i"'] & & \\
(C \wedge I) \sqcup (C \wedge I) \arrow[r, "v'"'] \arrow[d, "p \sqcup p"] \arrow[ur, phantom, "\lrcorner"] & C \wedge \Box \arrow[dr, dashed, "f"] & \\
C \sqcup C \arrow[rr, "i"] & & C \wedge I
\end{tikzcd}$$

Factorize this into a cofibration followed by a trivial fibration $C \wedge \Box \xrightarrow{\iota} C \wedge I \wedge I \xrightarrow{r} C \wedge I$. The map h is then defined to be $\iota \circ h'$, concluding the construction of the very good cylinder object. The map v is then set to be $\iota \circ v'$. This is clearly a cofibration and solves the required lifting problem by commutativity of the pushout diagram. $\Box$

Note that for us, given $\mathscr{M} = SeSp^{inj}$, some Segal space X and using the cylinder object $X \times N(I[1])$, Lemma 3.3.24's induced cylinder object can be set to be $X \times N(I[1]) \times N(I[1])$, while v is given by the evident inclusion of the 'vertical' isomorphisms.

Lemma 3.3.25. *Suppose $C \sqcup C \xrightarrow{i} C \wedge I \xrightarrow{p} C$ is a very good cylinder object for a cofibrant object C in a left proper model category $\mathscr{M}$. Let $h, r, C \wedge I \wedge I$ and v be as in Lemma 3.3.24.*

Then there is a cylinder object for $C \wedge I$ in $(C \sqcup C \downarrow \mathscr{M})$ of the form

$$(C \wedge I) \sqcup_{C \sqcup C} (C \wedge I) \xrightarrow{b} C \wedge B \xrightarrow{z} C \wedge I$$

where $C \wedge B$ is the pushout

$$\begin{tikzcd}
(C \wedge I) \sqcup (C \wedge I) \arrow[r, "p \sqcup p"] \arrow[d, "v"'] & C \sqcup C \arrow[d, "s"] \\
C \wedge I \wedge I \arrow[r, "k"'] \arrow[ur, phantom, "\lrcorner"] & C \wedge B.
\end{tikzcd}$$

Proof. We need to define the two maps b and z. It will then be sufficient to show that z is a weak equivalence and that zb is the appropriate projection map

$$(C \wedge I) \sqcup_{C \sqcup C} (C \wedge I) \to C \wedge I$$

induced by the pushout.

The map z is easiest, so we start here: it is simply given by the universal map from the pushout, induced by the maps $i : C \sqcup C \to C \wedge I$ and $r : C \wedge I \wedge I \to C \wedge I$. That these commute is due to the lifting problem v is a solution for.

As an intermediate step, note that $p \sqcup p$ is a weak equivalence. Indeed, the two maps $C \to C \sqcup C \xrightarrow{i} C \wedge I$ are weak equivalences by 2-out-of-3. As C is cofibrant, these are moreover trivial cofibrations. Then, note that the map

$$C \sqcup C \hookrightarrow (C \sqcup C) \sqcup (C \sqcup C) \xrightarrow{i \sqcup i} (C \wedge I) \sqcup (C \wedge I) \xrightarrow{p \sqcup p} C \sqcup C$$

is simply the identity. The composite of the first two maps is a coproduct of trivial cofibrations so is itself a trivial cofibration [Hir09, Prop. 7.2.5 (2)]. Hence, $p \sqcup p$ is a weak equivalence by 2-out-of-3.

Note then that k is a weak equivalence, as $\mathscr{M}$ is left proper, v is a cofibration and $p \sqcup p$ is a weak equivalence. This implies by 2-out-of-3 with r that z is a weak equivalence, as needed.

Now we turn to b. Consider the two morphisms $i_1, i_2 : C \sqcup C \to (C \wedge I) \sqcup (C \wedge I)$. It will suffice to show that $khi_1 = khi_2$, as this will induce a pushout map to $C \wedge B$. Extending the pushout diagram defining $C \wedge B$ reveals two larger commutative diagrams, for $j \in \{1, 2\}$, of the form

$$\begin{tikzcd}
C \sqcup C \arrow[r, "\alpha_j"] \arrow[d, "i_j"'] & (C \wedge I) \sqcup (C \wedge I) \arrow[r, "p \sqcup p"] \arrow[d, "v"'] & C \sqcup C \arrow[d, "s"] \\
(C \wedge I) \sqcup (C \wedge I) \arrow[r, "h"'] & C \wedge I \wedge I \arrow[r, "k"'] & C \wedge B
\end{tikzcd}$$

where α_1 and α_2 are induced by the lifting problem defining v. Commutativity is given by this same lifting problem. It thus suffices to show that $s(p \sqcup p)\alpha_1 = s(p \sqcup p)\alpha_2$. However, $(p \sqcup p)\alpha_1$ and $(p \sqcup p)\alpha_2$ give the identity on $C \sqcup C$, so they are equal as needed, in turn defining b.

We finally need to show that b and z satisfy the requirements of a cylinder object, namely that $z \circ b$ is the projection map from the pushout. This is a matter of a quick diagram chase and an application of the same property of h and r. Indeed, if $q : (C \wedge I) \sqcup (C \wedge I) \to (C \wedge I) \sqcup_{C \sqcup C} (C \wedge I)$ is the natural map, then $zbq = zkh = rh$ by inspection, which is the required projection. $\square$

Note that we did not prove this to be a very good cylinder object, as this is not needed for our purposes. We have not even proven it to be a good cylinder object. Regardless, though this cylinder object is a useful intermediate construction, we will never need to work with it directly:

Lemma 3.3.26. *Suppose, in the situation of Lemma 3.3.25, we have $D \in \mathscr{M}$ and two homotopies $H, K : C \wedge I \to D$, both between maps $f, g : C \to D$.*

Then there is a left homotopy from H to K in $(C \sqcup C \downarrow \mathscr{M})$ if and only if there is a left homotopy

$$\Gamma : C \wedge I \wedge I \to D$$

from H to K in $\mathscr{M}$ such that $\Gamma \circ v = (f \circ p) + (g \circ p)$.

Proof. Suppose the map Γ exists. The property it satisfies corresponds exactly to having a map to D from the span which $C \wedge B$ is the pushout of. Thus, a left homotopy $\Gamma' : C \wedge B \to D$ is induced by universality of the pushout.

Now, suppose $\kappa : C \wedge B' \to D$ is a left homotopy in $(C \sqcup C \downarrow \mathscr{M})$, for some good cylinder object

$$(C \wedge I) \sqcup_{C \sqcup C} (C \wedge I) \xrightarrow{b'} C \wedge B' \xrightarrow{z'} C \wedge I.$$

Some work is needed to obtain a left homotopy using the cylinder object $C \wedge B$ itself, as we have not proven it to be a good cylinder object. For now, take any factorization of b

$$(C \wedge I) \sqcup_{C \sqcup C} (C \wedge I) \xrightarrow{\gamma} C \wedge \beta \xrightarrow{\omega} C \wedge B$$

into a cofibration γ followed by a trivial fibration ω. $C \wedge \beta$ will serve just fine as a good cylinder object. Hence, we can assume $C \wedge B' = C \wedge \beta$.

Now, take a lift

$$\begin{tikzcd}
(C \wedge I) \sqcup (C \wedge I) \arrow[r, "p \sqcup p"] \arrow[d, "v"'] & C \sqcup C \arrow[r] & (C \wedge I) \sqcup_{C \sqcup C} (C \wedge I) \arrow[r, "\gamma"] & C \wedge \beta \arrow[d, "\omega"] \\
C \wedge I \wedge I \arrow[rrr, "k"'] \arrow[urrr, dashed, "\kappa"] & & & C \wedge B
\end{tikzcd}$$

The map κ, together with the natural map $C \sqcup C \to C \wedge \beta$ given by γ, induces a map by universality of the pushout $e : C \wedge B \to C \wedge \beta$. Since k and ω are weak equivalences, κ is a weak equivalence. Thus, since κ and k are weak equivalences, e must be a weak equivalence. Note that e commutes with the maps $C \sqcup C \to C \wedge \beta$ and $s : C \sqcup C \to C \wedge B$, since e is induced by the pushout's universal property. Precomposing with e thus gives the left homotopy $C \wedge B \to D$ in $(C \sqcup C \downarrow \mathscr{M})$ desired, which when precomposed with k gives Γ. □

In particular, we find the following:

Corollary 3.3.27. *Consider the lifting problem in Corollary 3.3.23, along with the data given in Lemma 3.3.25. Any two left homotopies $H, K : C \wedge I \to B$ over D between δ and ϵ induced as in the corollary's methods will be related by a left homotopy over D*

$$\Gamma : C \wedge I \wedge I \to B$$

such that $\Gamma \circ v = (\delta \circ p) + (\epsilon \circ p)$.

Proof. H and K are solutions of the same lifting problem. Thus, by Proposition 3.3.21, there is a left homotopy between them in $(C \sqcup C \downarrow \mathscr{M} \downarrow D) = (C \sqcup C \downarrow (\mathscr{M} \downarrow D))$, implying the result by Lemma 3.3.26. $\square$

Note then that Γ, in our use case, is exactly the definition of a globular 2-homotopy. Thus, we have the following result:

Theorem 3.3.28. *Consider a lifting problem*

$$\begin{tikzcd} & B \arrow[d, "p"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

in $SeSp^{inj}$, where C is cofibrant and p is a trivial fibration. Any two solutions will be related by a left homotopy in $(\boldsymbol{SeSp}^{inj} \downarrow D)$ using the cylinder object $C \times N(I[1])$, while any two such left homotopies will be related by a globular 2-homotopy.

Corollary 3.3.29. *Let $f, g : C \to B$ be two solutions to the lifting problem above. Let $H, K : C \times N(I[1]) \to B$ be two left homotopies over D between them. Then the enrichment of h_1 in Kan complexes sends H and K to the same natural isomorphism $h_1(C) \times I[1] \to h_1(D)$.*

As an aside, a consequence of our approach to left homotopies that now deserves mentioning is that h_1 will send a section of a trivial fibration to its inverse:

Proposition 3.3.30. *Consider a lifting problem of Segal spaces*

$$\begin{tikzcd} & A \arrow[d, "f"] \\ B \arrow[r, "id"'] \arrow[ur, dashed, "g"] & B \end{tikzcd}$$

with solution g and a trivial fibration f. Then $h_1(g)$ is a categorical inverse to $h_1(f)$.

Proof. We have that $h_1(f) \circ h_1(g) = 1_{h_1(B)}$. Moreover, by the 2-out-of-3 property of weak equivalences, we have that g is a Dwyer-Kan equivalence and thus $h_1(g)$ is an equivalence.

The proof will be complete if we produce a left homotopy $H : A \times N([I]) \to A$ from $g \circ f$ to 1_A, as h_1^c will send this to the necessary natural isomorphism.

Such a left homotopy can be produced by noting that $g \circ f$ and 1_A are both solutions to the lifting problem

$$\begin{tikzcd} & A \arrow[d, "f"] \\ A \arrow[ur, dashed, bend left, "g\circ f"] \arrow[ur, dashed, "id"'] \arrow[r, "f"'] & B \end{tikzcd}$$

completing the proof. □

Our results also make it possible to prove directly that composition of left homotopies and natural isomorphisms coincide, without needing the simplicial enrichment of h_1. Recall our convention of identifying $h_1(X \times N(I[1]))$ with $h_1(X) \times I[1]$ for some Segal space X.

Proposition 3.3.31. *In the situation of Definition 3.3.17, the vertical composite of natural isomorphisms $h_1(K)h_1(H) : h_1(C) \times I[1] \to h_1(B)$ is equal to $h_1(KH) : h_1(C) \times I[1] \to h_1(B)$ for any composite $KH : C \times N(I[1]) \to B$.*

Proof. Let Q be as in Proposition 3.3.17. The composite of $h_1(K)$ and $h_1(H)$ amounts to considering the induced functor

$$F : h_1(C) \times I[2] \to h_1(B)$$

which restricts to $h_1(H)$ and $h_1(K)$. There is precisely one such functor. However, $h_1(Q) : h_1(C) \times I[2] \cong h_1(C \times N(I[2])) \to h_1(B)$ also shares this property, so $h_1(Q) = F$. Hence, restriction yields $h_1(KH)$ for any KH as needed. □

3.4 Homotopy Unbiased Bicategories

Let $X \in \mathbf{SeSp}_2^{inj}$ be a 2-fold Segal space. We wish to define an unbiased bicategory $h_2(X)$. As we will soon find, it will not be reasonable to expect such a bicategory to exist without additional data. This will take the form of a choice of horizontal compositions.

We set $\mathbf{ob}(h_2(X)) := (X_{0,0})_0$, for consistency with h_1. We also define

$$\mathbf{Hom}_{h_2(X)}(x, y) := h_1(X(x, y)) = h_1((X_{1,\bullet})^{x,y}).$$

We then take a choice of horizontal compositions $(\mu_n)_{n\geq 0}$ for X as in Definition 3.2.2. Composition functors, for $n \geq 0$ and $x_0, \cdots, x_n \in \mathbf{ob}(h_2(X))$, will then be the maps

$$\bullet^{x_0,\cdots,x_n} : \prod_{i=1}^{n} \mathbf{Hom}_{h_2(X)}(x_{i-1}, x_i) \cong h_1(\prod_{i=1}^{n} X(x_{i-1}, x_i)) \xrightarrow{h_1(\circ^{x_0,\cdots,x_n})} \mathbf{Hom}_{h_2(X)}(x_0, x_n).$$

In general, for a simplicial composition diagram K of arity n, write

$$\bullet_K^{x_0,\cdots,x_n} : \prod_{i=1}^{n} \mathbf{Hom}_{h_2(X)}(x_{i-1}, x_i) \cong h_1(\prod_{i=1}^{n} X(x_{i-1}, x_i)) \xrightarrow{h_1(\circ_K^{x_0,\cdots,x_n})} \mathbf{Hom}_{h_2(X)}(x_0, x_n).$$

Note the domain is $* \cong \{x_0\}$ if $n = 0$, as with $\circ^{x_0}$.

3.4.1 Truncated Compositions

Our first attempt will be to induce h_2 from h_2^{tr}. We now have the technology to obtain choices of 2-truncated compositions from choices of horizontal compositions, as we can now construct the relevant natural isomorphisms. Suppose $(X, (\mu_n)_{n\geq 0}) \in \mathbf{SeSp}_2^{comp}$. We seek to construct fillers

$$\begin{tikzcd} h_1(X_n^{x_0,\cdots,x_n}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow \eta^{x_0,\cdots,x_n}"] & \prod_{i=1}^{n} h_1(X(x_{i-1}, x_i)) \arrow[dl, dashed, "\bullet^{x_0,\cdots,x_n}"] \\ h_1(X(x_0, x_n)) & \end{tikzcd}$$

for each $x_0, \cdots, x_n \in (X_{0,0})_0$.

Note first that the composition

$$X_n^{x_0,\cdots,x_n} \xrightarrow{\gamma_n^{x_0,\cdots,x_n}} X_{Sp(n)}^{x_0,\cdots,x_n} \xrightarrow{\mu_n^{x_0,\cdots,x_n}} X_n^{x_0,\cdots,x_n}$$

is a weak equivalence by 2-out-of-3. Moreover, since both the source and target of this map are both fibrant and cofibrant, we have that this map admits a *homotopy inverse*:

Proposition 3.4.1 ([DS95, Lemma 4.24]). *Suppose $f : X \to Y$ is a morphism in a model category $\mathscr{M}$ where X and Y are both fibrant and cofibrant. Then f is a weak equivalence if and only if it has a* homotopy inverse, *namely a map $g : Y \to X$ together with (left or right) homotopies $f \circ g \sim 1_Y$ and $g \circ f \sim 1_X$.*

We modify the definition of homotopy inverse to include the homotopies as data.

As a matter of fact, we do not even need this more general result: we have a natural lifting problem in $SeSp^{inj}$ with two distinct solutions

$$\begin{tikzcd} & X_n \arrow[d] \\ X_n \arrow[r] \arrow[ur, dashed, bend left, "1_{X_n}"] \arrow[ur, dashed, bend right, "\mu_n \circ \gamma_n"'] & X_{Sp(n)} \end{tikzcd}$$

so that there must exist a left homotopy between them. Moreover, this left homotopy will be fibered over objects. Thus, we have the following:

Proposition 3.4.2. *Suppose $X \in \boldsymbol{SeSp}_2^{inj}$ with a choice of horizontal compositions $(\mu_n)_{n\geq 0}$. Then for every $n \geq 0$ and $x_0, \cdots, x_n \in (X_{0,0})_0$, there exists a left homotopy $K_n : X_n \times N(I[1]) \to X_n$ from the composition $\mu_n \circ \gamma_n$ to 1_{X_n} that is fibered over objects in X_0 and is constant on postcomposition with γ_n. Moreover, setting*

$$H_n^{x_0,\cdots,x_n} : X_n^{x_0,\cdots,x_n} \times N(I[1]) \xrightarrow{K_n^{x_0,\cdots,x_n}} X_n^{x_0,\cdots,x_n} \hookrightarrow X_n^{x_0,x_n} \xrightarrow{X_{\langle 0,n\rangle}^{x_0,x_n}} X(x_0,x_n)$$

we have that all such left homotopies K_n induce the same filler

$$\begin{tikzcd} h_1(X_n^{x_0,\cdots,x_n}) \arrow[rr] \arrow[d] \arrow[dr, Rightarrow, "h_1(H_n^{x_0,\cdots,x_n})"] & & \prod_{i=1}^n h_1(X(x_{i-1},x_i)) \arrow[dll, dashed] \\ h_1(X(x_0,x_n)) & \bullet^{x_0,\cdots,x_n} & \end{tikzcd}$$

for the span $h_1(X(x_0,x_n)) \leftarrow h_1(X_n^{x_0,\cdots,x_n}) \to \prod_{i=1}^n h_1(X(x_{i-1},x_i))$.

Proof. The left homotopy K_n is induced between two solutions to the same lifting problem, so the required properties are satisfied on the nose. Moreover, any two K_n and K_n' will be related by a globular 2-homotopy, which will in turn be fibered over objects. Thus, any H_n and H_n' will be related by such a globular 2-homotopy, proving uniqueness of the filler. □

This gives us the required conversion from horizontal compositions to 2-truncated compositions:

Definition 3.4.3. *Define the functor*

$$Tr : \boldsymbol{SeSp}_2^{comp} \to \boldsymbol{SeSp}_2^{2comp}$$

to send $(X, (\mu_n)_{n\geq 0})$ to the pair

$$(X, (h_1(\circ^{x_0,\cdots,x_n}), h_1(H_n^{x_0,\cdots,x_n}))_{n\geq 0}^{x_0,\cdots,x_n\in(X_{0,0})_0})$$

and to send maps to the same underlying maps.

That this is functorial is trivial, due to morphisms ignoring the choices of horizontal or 2-truncated compositions.

Note that Tr only identifies 2-truncated compositions that are identities on unary composition and that are given by degeneracy maps on nullary composition. Thus, the composite

$$\mathbf{SeSp}_2^{comp} \to \mathbf{SeSp}_2^{2comp} \xrightarrow{h_2^{tr}} \mathbf{UBicat}$$

has in its image only those unbiased bicategories where the unitors $\iota_{x,y}$ are identities and identity 1-morphisms are given by degeneracy maps. Moreover, Tr identifies only those truncated composition operations that admit some description as compositions in X itself.

3.4.2 Coherence Isomorphisms

It is somewhat unenlightening to define h_2 by factoring through h_2^{tr}; there is no clear reason that associators, unitors or the coherence conditions should exist. We would like to demonstrate that all of this data exists intrinsically in X itself, as we expect such results to better lend themselves to computing homotopy n-categories for higher n in the future.

Our first order of business is to provide the associators and unitors. We will in fact provide much more general machinery here, which will be needed to work with functors as well. For associators, let $n \in \mathbb{Z}_{>0}$ and $k_1, \cdots, k_n \in \mathbb{Z}_{\geq 0}$, with $r = \sum_{i=1}^n k_i$. Consider any n-tuple of tuples of elements of $(X_{0,0})_0$

$$Y := ((x_0^1, x_1^1, \cdots, x_{k_1}^1), \cdots, (x_0^n, \cdots, x_{k_n}^n))$$

such that $x_{k_i}^i = x_0^{i+1}$ for every $i < n$. Let $(x_0, \cdots, x_r)$ be the flattened tuple after removing each $x_{k_i}^i$ for $i < n$.

Proposition 3.4.4. *Let Y be as above. Let $Sp(r) \xrightarrow{\iota} K \xleftarrow{\tau} \Delta[1]$ be a simplicial composition diagram where $K = (K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_n) \sqcup_{Sp(n)} K_0$, with each diagram K_i of arity k_i where $k_0 = n$. Then*

$$\bullet_K^{x_0,\cdots,x_r} = \bullet_{K_0}^{x_0^1,\cdots,x_0^n,x_{k_n}^n} \circ (\bullet_{K_1}^{x_0^1,\cdots,x_{k_1}^1} \times \cdots \times \bullet_{K_n}^{x_0^n,\cdots,x_{k_n}^n}).$$

Proof. We have by Theorem 3.2.12 that the diagram

$$\begin{tikzcd}
\prod_{i=1}^r \mathbf{Hom}_{h_2(X)}(x_{i-1}, x_i) \arrow[r, "\cong"] \arrow[d, "\cong"'] & h_1(\prod_{i=1}^r X(x_{i-1}, x_i)) \arrow[dl, "\cong"'] \arrow[dddl, "h_1(\circ_{K_1}^{x_0^1,\cdots,x_{k_1}^1} \times \cdots \times \circ_{K_n}^{x_0^n,\cdots,x_{k_n}^n})"'] \arrow[ddd, "h_1(\circ_K^{x_0,\cdots,x_r})"] \\
\prod_{i=1}^n h_1(\prod_{j=1}^{k_i} X(x_{j-1}^i, x_j^i)) \arrow[d, "\prod_{i=1}^n h_1(\circ_{K_i}^{x_0^i,\cdots,x_{k_i}^i})"'] & \\
\prod_{i=1}^n \mathbf{Hom}_{h_2(X)}(x_0^i, x_{k_i}^i) \arrow[d, "\cong"'] & \\
h_1(\prod_{i=1}^n X(x_0^i, x_{k_i}^i)) \arrow[r, "h_1(\circ_{K_0}^{x_0^1,\cdots,x_0^n,x_{k_n}^n})"'] & \mathbf{Hom}_{h_2(X)}(x_0, x_r)
\end{tikzcd}$$

commutes. The result follows. □

Let $Sp(n) \xrightarrow{\iota_{K_i}} K_i \xleftarrow{\tau_{K_i}} \Delta[1]$, for $i \in \{1,2\}$, be simplicial composition diagrams of arity n. Consider a map $f : K_1 \to K_2$ in $\mathbf{SCD}_n$, as in Definition 3.2.14. We have as in Proposition 3.2.16 the map $\mu'_{K_2} := f^* \circ \mu_{K_2}$. The two maps μ_{K_1} and μ'_{K_2} both solve the lifting problem

$$\begin{tikzcd}
 & X_{K_1} \arrow[r, "\tau^*_{K_1}"] \arrow[d] & X_1 \\
X_{Sp(n)} \arrow[r] \arrow[ur, dashed, bend left, "\mu'_{K_2}"] \arrow[ur, dashed, bend right, "\mu_{K_1}"'] & X_{Sp(n)} &
\end{tikzcd}$$

and thus admit an induced left homotopy $\beta_f : X_{Sp(n)} \times N(I[1]) \to X_{K_1}$ between them. This left homotopy is object-fibered and is constant on postcomposition with $\iota^*_{K_1} : X_{K_1} \to X_{Sp(n)}$. We will write β_f for a left homotopy induced by a map f in $\mathbf{SCD}_n$ in this manner. We will in general speak of an 'induced left homotopy' when a left homotopy is produced from two maps being solutions to the same lifting problem.

Recall that even though β_f is not in general unique, any two induced left homotopies β_f and β'_f by a map f will have a globular 2-homotopy $\alpha : X_{Sp(n)} \times N(I[1]) \times N(I[1]) \to X_{K_1}$ between them, by Theorem 3.3.28. It is this property that asserts functoriality of the following construction:

Definition 3.4.5. *Let X be a 2-fold Segal space with a choice of horizontal compositions $(\mu_n)_{n\geq 0}$. Let $x, y \in (X_{0,0})_0$ and set $n \geq 0$. Define the functor*

$$\omega^{x,y}_{(X,(\mu_n)_{n\geq 0})} : \boldsymbol{SCD}_n \to \boldsymbol{Fun}\Big(h_1(X^{x,y}_{Sp(n)}), h_1(X(x,y))\Big)$$

to send $K \mapsto h_1((\tau^*_K)^{x,y} \circ \mu^{x,y}_K)$ and $f : K_1 \to K_2$ to the natural isomorphism

$$h_1((\tau^*_{K_1})^{x,y} \circ \beta^{x,y}_f) : h_1(X^{x,y}_{Sp(n)}) \times I[1] \xrightarrow{h_1(\beta^{x,y}_f)} h_1(X^{x,y}_{K_1}) \xrightarrow{h_1((\tau^*_{K_1})^{x,y})} h_1(X(x,y))$$

for an induced left homotopy $\beta_f : X_{Sp(n)} \times N(I[1]) \to X_{K_1}$ from μ_{K_1} to μ'_{K_2}.

Proposition 3.4.6. *$\omega^{x,y}_{(X,(\mu_n)_{n\geq 0})}$ is a functor.*

Proof. By the above discussion, $\omega^{x,y}_{(X,(\mu_n)_{n\geq 0})}(f)$ is well-defined for any morphism f in $\mathbf{SCD}_n$.

Suppose $K_1 \xrightarrow{f} K_2 \xrightarrow{g} K_3$ is a diagram in $\mathbf{SCD}_n$ for simplicial composition diagrams $\Delta[n] \xrightarrow{\iota_i} K_i \xleftarrow{\tau_i} \Delta[1]$. We wish to show that

$$\omega^{x,y}_{(X,(\mu_n)_{n\geq 0})}(g)\omega^{x,y}_{(X,(\mu_n)_{n\geq 0})}(f) = \omega^{x,y}_{(X,(\mu_n)_{n\geq 0})}(g \circ f).$$

These three natural isomorphisms can be rewritten as

$$\begin{aligned}
\omega^{x,y}_{(X,(\mu_n)_{n\geq 0})}(f) &= (\tau_1^*)^{x,y} \circ \beta_f^{x,y} \\
\omega^{x,y}_{(X,(\mu_n)_{n\geq 0})}(g) &= (\tau_1^*)^{x,y} \circ (f^* \circ \beta_g)^{x,y} \\
\omega^{x,y}_{(X,(\mu_n)_{n\geq 0})}(g \circ f) &= (\tau_1^*)^{x,y} \circ \beta_{g\circ f}^{x,y}
\end{aligned}$$

for left homotopies $\beta_f^{x,y}$, $(f^* \circ \beta_g)^{x,y}$ and $\beta_{g\circ f}^{x,y}$ induced between $\mu_{K_1}^{x,y}$ and $(f^* \circ \mu_{K_2})^{x,y}$, $(f^* \circ \mu_{K_2})^{x,y}$ and $(f^* \circ g^* \circ \mu_{K_3})^{x,y}$ and $\mu_{K_1}^{x,y}$ and $(f^* \circ g^* \circ \mu_{K_3})^{x,y}$ respectively, all induced by the maps $\mu_{K_1}, (f^* \circ \mu_{K_2})$ and $(f^* \circ g^* \circ \mu_{K_3})$ being solutions to the lifting problem of the form

$$\begin{tikzcd}
 & X_{K_1} \arrow[d, "\iota^*_{K_1}"] \\
X_{Sp(n)} \arrow[ur, dashed, bend right, "\mu_{K_1}"'] \arrow[ur, dashed, "(f^*\circ\mu_{K_2})"] \arrow[ur, dashed, bend left, "(f^*\circ g^*\circ\mu_{K_3})"] \arrow[r] & X_{Sp(n)}
\end{tikzcd}$$

We may therefore take a composite $C = ((f^* \circ \beta_g)^{x,y})(\beta_f^{x,y})$ as in Definition 3.3.17, which is a left homotopy from $\mu_{K_1}^{x,y}$ to $(f^* \circ g^* \circ \mu_{K_3})^{x,y}$. By Proposition 3.3.31, we have that $h_1(C)$ is the composite of $h_1((f^* \circ \beta_g)^{x,y})$ and $h_1(\beta_f^{x,y})$.

Note however that C and $\beta_{g\circ f}^{x,y}$ are both left homotopies induced between the same solutions of a lifting problem. Hence, by Corollary 3.3.29, we have that $h_1(\beta_{g\circ f}^{x,y}) = h_1(C)$ is the composite of the natural isomorphisms $h_1((f^* \circ \beta_g)^{x,y})$ and $h_1(\beta_f^{x,y})$, as needed.

Now, suppose K is a simplicial composition diagram of arity n. We wish to show that

$$\omega^{x,y}_{(X,(\mu_n)_{n\geq 0})}(1_K) = 1_{h_1((\tau_K^*)^{x,y} \circ \mu_K^{x,y})}.$$

The left homotopy β_{1_K} can be set to be a constant left homotopy, as its source and target are equal. Since any choice of β_{1_K} will yield the same natural isomorphism, the result is evident. □

Definition 3.4.7. *Let X be a 2-fold Segal space with a choice of horizontal compositions $(\mu_n)_{n\geq 0}$. Let $x_0, \cdots, x_n \in (X_{0,0})_0$. Then define the map*

$$\phi^{x_0,\cdots,x_n}_{(X,(\mu_n)_{n\geq 0})} : \boldsymbol{Fun}\Big(h_1(X^{x_0,x_n}_{Sp(n)}), h_1(X(x_0,x_n))\Big) \to \boldsymbol{Fun}\Big(\prod_{i=1}^n h_1(X(x_{i-1},x_i)), h_1(X(x_0,x_n))\Big)$$

to be the functor given by precomposition with the inclusion

$$F_{x_0,\cdots,x_n} : \prod_{i=1}^n h_1(X(x_{i-1},x_i)) \cong h_1(\prod_{i=1}^n X(x_{i-1},x_i)) \hookrightarrow h_1(X^{x_0,x_n}_{Sp(n)}).$$

Write $\Phi^{x_0,\cdots,x_n}_{(X,(\mu_n)_{n\geq 0})} := \phi^{x_0,\cdots,x_n}_{(X,(\mu_n)_{n\geq 0})} \circ \omega^{x_0,x_n}_{(X,(\mu_n)_{n\geq 0})}$.

For our given Y, let $K = (\Delta[k_1] \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} \Delta[k_n]) \sqcup_{Sp(n)} \Delta[n]$ and take the map $f_{k_1,\cdots,k_n} : K \to \Delta[r]$ in $\mathbf{SCD}_r$ to be the map restricting to the morphisms

$$\langle \sum_{j=1}^{i-1} k_j, \cdots, \sum_{j=1}^{i-1} k_j + k_i \rangle : \Delta[k_i] \to \Delta[r]$$

$$\langle 0, k_1, k_1 + k_2, \cdots, \sum_{j=1}^{n} k_j \rangle : \Delta[n] \to \Delta[r].$$

Our associators for $h_2(X)$ will be the natural isomorphisms

$$\Phi^{x_0,\cdots,x_r}_{(X,(\mu_n)_{n\geq 0})}(f_{k_1,\cdots,k_n}) : \bullet^{x_0^1,\cdots,x_0^n,x_{k_n}^n} \circ (\bullet^{x_0^1,\cdots,x_{k_1}^1} \times \cdots \times \bullet^{x_0^n,\cdots,x_{k_n}^n}) \Rightarrow \bullet^{x_0,\cdots,x_r}.$$

Unitors for us will be trivial; we choose to set them to be identities. We are thus now ready to declare the definition of our unbiased bicategory in full:

Definition 3.4.8. *Let X be a 2-fold Segal space. Let $(\mu_n)_{n\geq 0}$ be a choice of horizontal compositions.*

Then the unbiased homotopy bicategory $h_2(X, (\mu_n)_{n\geq 0})$ of X, *written as $h_2(X)$ if the μ_n are known, is the unbiased bicategory defined such that:*

1. $\boldsymbol{ob}(h_2(X)) := (X_{0,0})_0$*;*
2. $\boldsymbol{Hom}_{h_2(X)}(x, y) := h_1(X(x, y))$ *for all $x, y \in (X_{0,0})_0$;*
3. *For each $n > 0$ and $x_0, \cdots, x_n \in (X_{0,0})_0$, composition is given by the functor $\bullet^{x_0,\cdots,x_n}$;*
4. *For each $x \in (X_{0,0})_0$, identities are given by the functor $\bullet^x$;*
5. *Let $n > 0$ and $k_1, \cdots, k_n \in \mathbb{Z}_{\geq 0}$ with $r = \sum_i k_i$. Consider any n-tuple of tuples of elements of $(X_{0,0})_0$*

   $$Y := ((x_0^1, x_1^1, \cdots, x_{k_1}^1), \cdots, (x_0^n, \cdots, x_{k_n}^n))$$

   such that $x_{k_i}^i = x_0^{i+1}$ for every $i < n$. Let $(x_0, \cdots, x_r)$ be the flattened tuple after removing each $x_{k_i}^i$ for $i < n$. Then the natural isomorphism γ_Y is defined to be

   $$\Phi^{x_0,\cdots,x_r}_{(X,(\mu_n)_{n\geq 0})}(f_{k_1,\cdots,k_n});$$
6. *For each $x, y \in (X_{0,0})_0$, the natural isomorphism*

   $$\iota_{x,y} : 1_{\boldsymbol{Hom}_{h_2(X)}(x,x)} \Rightarrow 1_{\boldsymbol{Hom}_{h_2(X)}(x,x)} = \bullet^{x,x}$$

   is an identity.

We can now prove that the data in Definition 3.4.8, excluding the associators, indeed agrees with our first attempt at homotopy bicategories:

Proposition 3.4.9. *The data for $h_2(X, (\mu_n)_{n\geq 0})$, excluding the associators, is equal to that specifying the unbiased bicategory $h_2^{tr}(Tr(X, (\mu_n)_{n\geq 0}))$.*

Proof. Equality on objects, hom-categories, identities and composition is evident. Moreover, equality on the natural isomorphisms $\iota_{x,y}$ is immediate, as these are identities in both cases. □

We leave the associators for future work, which we conjecture to be equal in both cases so that $h_2(X, (\mu_n)_{n\geq 0}) = h_2^{tr}(Tr(X, (\mu_n)_{n\geq 0}))$ for all $(X, (\mu_n)_{n\geq 0}) \in$ $\mathbf{SeSp}^{comp}$.

3.4.3 Coherence Conditions

We now come to proving the coherence conditions on our proposed data for an unbiased bicategory. Though a possible line of inquiry, it is unenlightening to reduce a proof of these coherence conditions on our homotopy bicategories to a comparison with h_2^{tr}. If our goal is indeed to understand how all the algebraic data of a bicategory appears in a 2-fold Segal space, we should seek to prove the coherence conditions by instead identifying globular 2-homotopies between the relevant left homotopies.

We will go further than this; our final proof of coherence will be to establish an 'operadic' action of $\mathbf{SCD}_\bullet$ via $\Phi^{(-)}_{(X,(\mu_n)_{n\geq 0})}$ on our hom-categories and functors between them. This construction will rely on identifying and composing left homotopies in a suitably 'operadic' manner. A technical lemma is first required:

Lemma 3.4.10. *Consider a diagram in a model category $\mathscr{M}$*

$$\begin{tikzcd} & & & E \arrow[r, "s"] \arrow[d, "q"'] & F \arrow[r, "w"] \arrow[d, "r"] & Z \\ A \arrow[r, "h"] \arrow[d, "f"'] & B \arrow[r, "u"] \arrow[d, "g"] & & G \arrow[r, "t"'] \arrow[ur, dashed, "\epsilon"] & H & \\ C \arrow[r, "k"'] \arrow[ur, dashed, "\delta"] & D & & & & \end{tikzcd}$$

where the two squares define lifting problems with solutions δ and ϵ, such that

g and r are trivial fibrations. Then the square in the diagram

$$\begin{tikzcd}
A \arrow[r, "h\times_{tuh}\epsilon uh"] \arrow[dd, "f"'] & B\times_H F \arrow[r, two heads] \arrow[d, two heads] & F \arrow[r, "w"] & Z \\
 & B \arrow[d, "g"] & & \\
C \arrow[r, "k"'] \arrow[uur, dashed, "\delta\times_{tu\delta}\epsilon u\delta"] & D & &
\end{tikzcd}$$

is also a lifting problem in $\mathscr{M}$ with the given solution. Moreover, the map $C \to B \times_H F \to F \to Z$ in this diagram is equal to $w\epsilon u\delta$.

Proof. It is clear that the rightmost vertical map will be a trivial fibration if g and r are as such. Thus, the diagram is a valid lifting problem. The last equality is easily checked by the definition of the chosen lift. $\square$

The following lemma is what allows us to compose left homotopies horizontally in an 'operadic' manner, which forms the foundations of our proof of coherence conditions:

Lemma 3.4.11. *Let X be a Segal space and set $n \geq 1$ and $k_1, \cdots, k_n \geq 0$. Take Segal spaces A_i^j for $1 \leq i \leq n$ and $1 \leq j \leq k_i$, Segal spaces B_i and B_i' for $1 \leq i \leq n$ and some Segal space C, all fibered over $X \times X$. Take between these a set of maps in $SeSp^{inj}$ for $1 \leq i \leq n$ of the form*

$$f_i, g_i : \prod_{X}^{1\leq j\leq k_i} A_i^j \to B_i$$
$$t_i : B_i \to B_i'$$
$$\phi, \psi : \prod_{X}^{1\leq i\leq n} B_i' \to C.$$

Consider a finite collection of left homotopies in $SeSp^{inj}$ emerging from lifting problems in commutative diagrams fibered over $X \times X$

$$\begin{tikzcd}
\prod_X^{1\leq j\leq k_i} A_i^j \sqcup \prod_X^{1\leq j\leq k_i} A_i^j \arrow[r, "f_i\sqcup g_i"] \arrow[d] & B_i \arrow[r, "t_i"] \arrow[d] & B_i' \\
\prod_X^{1\leq j\leq k_i} A_i^j \times N(I[1]) \arrow[r] \arrow[ur, dashed, "H_i"] & \prod_X^{1\leq j\leq k_i} A_i^j &
\end{tikzcd}$$

for $1 \leq i \leq n$ where the right vertical map is a trivial fibration and

$$\begin{tikzcd}
\prod_X^{1\leq i\leq n} B_i' \sqcup \prod_X^{1\leq i\leq n} B_i' \arrow[r, "\psi\sqcup\phi"] \arrow[d] & C \arrow[d] \\
\prod_X^{1\leq i\leq n} B_i' \times N(I[1]) \arrow[r] \arrow[ur, dashed, "K"] & \prod_X^{1\leq i\leq n} B_i'
\end{tikzcd}$$

where again the right vertical map is a trivial fibration. For all $k \geq 1$, set $D_k : \prod_X^{i,j} A_i^j \times N(I[1]) \to \prod_X^{i,j} A_i^j \times N(I[1])^k$ to be the diagonal map.

Then the left homotopy

$$Q := K \circ \Big(\Big((\prod_X^{1 \leq i \leq n} t_i \circ H_i) \circ D_n \Big) \times 1_{N(I[1])} \Big) \circ D_2$$

is equal to $p \circ F$, where p and F are fibered naturally over $X \times X$ and fit into a lifting problem in $SeSp^{inj}$

$$\begin{tikzcd}
\prod_X^{i,j} A_i^j \sqcup \prod_X^{i,j} A_i^j \arrow[r, "\iota"] \arrow[dd] & \prod_X^i B_i \times_{\prod_X^i B_i'} C \arrow[r, two heads, "p"] \arrow[d] & C \\
& \prod_X^i B_i \arrow[d] & \\
\prod_X^{i,j} A_i^j \times N(I[1]) \arrow[r] \arrow[uur, dashed, "F"] & \prod_X^{i,j} A_i^j &
\end{tikzcd}$$

such that p is the pullback projection and ι is induced by the maps

$$\prod_X^{i,j} A_i^j \sqcup \prod_X^{i,j} A_i^j \xrightarrow{\prod_X^i f_i \sqcup \prod_X^i g_i} \prod_X^i B_i$$

and

$$\prod_X^{i,j} A_i^j \sqcup \prod_X^{i,j} A_i^j \xrightarrow{\prod_X^i f_i + \prod_X^i g_i} \prod_X^i B_i \sqcup \prod_X^i B_i \xrightarrow{\prod_X^i t_i + \prod_X^i t_i} \prod_X^i B_i' \sqcup \prod_X^i B_i' \xrightarrow{\psi \sqcup \phi} C.$$

Proof. Define

$$q : \prod_X^{i,j} A_i^j \sqcup \prod_X^{i,j} A_i^j \hookrightarrow \prod_X^{1 \leq i \leq n} (\prod_X^j A_i^j \sqcup \prod_X^j A_i^j) \cong \bigsqcup_{k=1}^{2^n} \prod_X^{i,j} A_i^j$$

to be the pullback map induced by the n maps $\rho_i \sqcup \rho_i$ for $1 \leq i \leq n$, where $\rho_i : \prod_X^{i,j} A_i^j \to \prod_X^j A_i^j$ is the projection to the i^{th} coordinate. This is implicitly an inclusion.

We have that $(\prod_X^i H_i) \circ D_n$ is a solution of the lifting problem defined by the outermost square of the diagram

$$\begin{tikzcd}
\prod_X^{i,j} A_i^j \sqcup \prod_X^{i,j} A_i^j \arrow[d, hook, "q"'] \arrow[dr, bend left, "\prod_X^{1\leq i\leq n} f_i \sqcup \prod_X^{1\leq i\leq n} g_i"] & \\
\prod_X^{1\leq i\leq n}(\prod_X^j A_i^j \sqcup \prod_X^j A_i^j) \arrow[r, "\prod_X^{1\leq i\leq n}(f_i \sqcup g_i)"] \arrow[d, hook] & \prod_X^{1\leq i\leq n} B_i \arrow[d] \\
\prod_X^{1\leq i\leq n}(\prod_X^j A_i^j \times N(I[1])) \arrow[r] \arrow[d, two heads] \arrow[ur, dashed, "\prod_X^{1\leq i\leq n} H_i"] & \prod_X^{i,j} A_i^j \\
\prod_X^{i,j} A_i^j \times N(I[1]) \arrow[ur] &
\end{tikzcd}$$

which demonstrates $(\prod_X^i H_i) \circ D_n$ is an induced left homotopy from $\prod_X^i f_i$ to $\prod_X^i g_i$.

We now have a commutative diagram of the form

$$\begin{tikzcd}
& \prod_X^{1\leq i\leq n} B_i' \sqcup \prod_X^{1\leq i\leq n} B_i' \arrow[r, "\psi\sqcup\phi"] \arrow[d] & C \arrow[d] \\
& \prod_X^{1\leq i\leq n} B_i' \times N(I[1]) \arrow[r, two heads] \arrow[ur, dashed, "K"] & \prod_X^{1\leq i\leq n} B_i' \\
(\prod_X^{i,j} A_i^j \sqcup \prod_X^{i,j} A_i^j) \times N(I[1]) \arrow[r, "(\prod_X^i f_i \sqcup \prod_X^i g_i)\times 1_{N(I[1])}"] \arrow[d] & \prod_X^{1\leq i\leq n} B_i \times N(I[1]) \arrow[u, "\prod_X^i t_i \times 1_{N(I[1])}"'] \arrow[d] & \\
\prod_X^{i,j} A_i^j \times N(I[1])^{n+1} \arrow[r, two heads] \arrow[d, two heads] \arrow[ur, dashed, "(\prod_X^i H_i)\times 1_{N(I[1])}"] & \prod_X^{i,j} A_i^j \times N(I[1]) & \\
\prod_X^{i,j} A_i^j \times N(I[1])^2 \arrow[ur, two heads] & &
\end{tikzcd}$$

which, by Lemma 3.4.10, produces a single lifting problem with solution F'

within the diagram

$$\begin{tikzcd}
\prod^{i,j}_X A^j_i \sqcup \prod^{i,j}_X A^j_i \arrow[d, "(id,\langle 0\rangle)+(id,\langle 1\rangle)"'] \arrow[dr, "\iota'"] & & C \\
(\prod^{i,j}_X A^j_i \sqcup \prod^{i,j}_X A^j_i) \times N(I[1]) \arrow[r] \arrow[d] & \left(\prod^i_X B_i \times N(I[1])\right) \times_{\prod^i_X B'_i} C \arrow[r, two heads, "R"] \arrow[d] & \prod^i_X B_i \times_{\prod^i_X B'_i} C \arrow[u, "p"'] \arrow[dd] \\
\prod^{i,j}_X A^j_i \times N(I[1])^2 \arrow[r, two heads] \arrow[d, two heads] \arrow[ur, dashed, "F'"] & \prod^{i,j}_X A^j_i \times N(I[1]) \arrow[dr, two heads] & \\
\prod^{i,j}_X A^j_i \times N(I[1]) \arrow[rr, two heads] & & \prod^{i,j}_X A^j_i
\end{tikzcd}$$

where the maps $\langle 0\rangle, \langle 1\rangle : * \to N(I[1])$ identify the two objects of $I[1]$ and R is induced by the projection forgetting $N(I[1])$. The outermost square of this diagram is another lifting problem, solved by $F := R \circ F' \circ D_2$. It is clear then that $p \circ F = Q$ by design. Moreover, since the diagram

$$\begin{tikzcd}
\prod^{i,j}_X A^j_i \sqcup \prod^{i,j}_X A^j_i \arrow[rr, "\prod^i_X f_i + \prod^i_X g_i"] \arrow[d, "(id,\langle 0\rangle)+(id,\langle 1\rangle)"'] & & \prod^i_X B_i \sqcup \prod^i_X B_i \arrow[d, "\prod^i_X t_i + \prod^i_X t_i"] \\
\left(\prod^{i,j}_X A^j_i \sqcup \prod^{i,j}_X A^j_i\right) \times N(I[1]) \arrow[d, "(\prod^i_X f_i \sqcup \prod^i_X g_i)\times 1_{N(I[1])}"'] & & \prod^i_X B'_i \sqcup \prod^i_X B'_i \arrow[ddd, "\psi \sqcup \phi"] \\
\left(\prod^i_X B_i\right) \times N(I[1]) \arrow[d, "(\prod^i_X t_i)\times 1_{N(I[1])}"'] & & \\
\left(\prod^i_X B'_i\right) \times N(I[1]) \arrow[rr, "K"'] & & C
\end{tikzcd}$$

commutes, setting $\iota := R \circ \iota'$ completes the proof. □

A crucial and nontrivial lemma is now needed, which carries a remarkably operadic flavor to it. We will not explore any subtle appearances of operads in this thesis; such concerns will be left to future work. One could argue this lemma is the reason we have coherence:

Lemma 3.4.12. *Let X and the list Y be as above, with some choice of hori-*

zontal compositions $(\mu_n)_{n\geq 0}$*. Consider the functor* $(-)\cdot(-)$ *of the form*

$$\begin{array}{c}
\boldsymbol{Fun}\Big(\prod_{i=1}^{n} h_1(X(x_0^i, x_{k_i}^i)), h_1(X(x_0, x_r))\Big) \times \boldsymbol{Fun}\Big(\prod_{i=1}^{r} h_1(X(x_{i-1}, x_i)), \prod_{i=1}^{n} h_1(X(x_0^i, x_{k_i}^i))\Big) \\
\Big\downarrow {\scriptstyle (-)\cdot(-)} \\
\boldsymbol{Fun}\Big(\prod_{i=1}^{r} h_1(X(x_{i-1}, x_i)), h_1(X(x_0, x_r))\Big)
\end{array}$$

defined by horizontal composition. Then

$$\Phi^{x_0^1,\cdots,x_0^n,x_{k_n}^n}_{(X,(\mu_n)_{n\geq 0})} \cdot \Big(\prod_{i=1}^{n} \Phi^{x_0^i,\cdots,x_{k_i}^i}_{(X,(\mu_n)_{n\geq 0})}\Big) = \Phi^{x_0,\cdots,x_r}_{(X,(\mu_n)_{n\geq 0})} \circ \mathcal{G}^n_{k_1,\cdots,k_n}.$$

Proof. That this equivalence holds on objects is by construction and Proposition 3.4.4. For morphisms, consider for $0 \leq i \leq n$ a collection of maps $f_i : K_i \to K_i'$ in $\mathbf{SCD}_{k_i}$, where $Sp(k_i) \xrightarrow{\iota_i} K_i \xleftarrow{\tau_i} \Delta[1]$ and $Sp(k_i) \xrightarrow{\iota_i'} K_i' \xleftarrow{\tau_i'} \Delta[1]$ are simplicial composition diagrams and $k_0 = n$. Then we have an induced map $f : K \to K'$, where

$$\begin{aligned} K &:= (K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_n) \sqcup_{Sp(n)} K_0 \\ K' &:= (K_1' \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_n') \sqcup_{Sp(n)} K_0' \end{aligned}$$

with maps $Sp(r) \xrightarrow{\iota} K \xleftarrow{\tau} \Delta[1]$ and $Sp(r) \xrightarrow{\iota'} K' \xleftarrow{\tau'} \Delta[1]$. We wish to show that

$$\Phi^{x_0^1,\cdots,x_0^n,x_{k_n}^n}_{(X,(\mu_n)_{n\geq 0})}(f_0) \circ \prod_{i=1}^{n} \Phi^{x_0^i,\cdots,x_{k_i}^i}_{(X,(\mu_n)_{n\geq 0})}(f_i) = \Phi^{x_0,\cdots,x_r}_{(X,(\mu_n)_{n\geq 0})}(f).$$

The left hand side of this equality can be expanded out as

$$h_1((\tau_0^*)^{x_0^1,x_{k_n}^n}) \circ h_1(\beta_{f_0}^{x_0^1,x_{k_n}^n}) \circ F_{x_0^1,\cdots,x_0^n,x_{k_n}^n} \circ \Big(\prod_{i=1}^{n} h_1((\tau_i^*)^{x_0^i,x_{k_i}^i}) \circ h_1(\beta_{f_i}^{x_0^i,x_{k_i}^i}) \circ F_{x_0^i,\cdots,x_{k_i}^i}\Big)$$

where each β_{f_i} is a left homotopy induced between μ_{K_i} and $\mu'_{K_i'}$. The right hand side can then be expanded out as

$$h_1((\tau^*)^{x_0,x_r}) \circ h_1(\beta_f^{x_0,x_r}) \circ F_{x_0,\cdots,x_r}.$$

If we can show that the left hand side is also of the form $h_1((\tau^*)^{x_0^1,x_{k_n}^n}) \circ h_1(H^{x_0,x_r}) \circ F_{x_0,\cdots,x_r}$ for some left homotopy H induced between μ_K and $\mu'_{K'}$, the proof will be complete, as these will then have to be identified by Corollary 3.3.29.

We should note that, if we are to treat these natural isomorphisms as functors of the form $\mathscr{C} \times I[1] \to \mathscr{D}$, we will have to rewrite the left-hand side as

$$h_1((\tau_0^*)^{x_0^1,x_{k_n}^n} \circ \beta_{f_0}^{x_0^1,x_{k_n}^n}) \circ F'_{x_0^1,\cdots,x_0^n,x_{k_n}^n} \circ \left(\Big(\prod_{i=1}^n h_1((\tau_i^*)^{x_0^i,x_{k_i}^i} \circ \beta_{f_i}^{x_0^i,x_{k_i}^i}) \circ F'_{x_0^i,\cdots,x_{k_i}^i} \circ D'_n \Big) \times 1_{I[1]} \right) \circ D'_2$$

where we now have the inclusion

$$F'_{a_0,\cdots,a_k} : \prod_{i=1}^k h_1(X(a_{i-1},a_i)) \times I[1] \hookrightarrow h_1(X_{Sp(k)}^{a_0,a_k} \times N(I[1]))$$

and $D'_k : \prod_{i=1}^r \mathbf{Hom}_{h_2(X)}(x_{i-1},x_i) \times I[1] \to \prod_{i=1}^r \mathbf{Hom}_{h_2(X)}(x_{i-1},x_i) \times I[1]^k$ the diagonal map for all $k > 0$.

Using naturality of the isomorphism $h_1(-) \times h_1(-) \cong h_1(- \times -)$, we can convert this into a map $h_1((\tau_0^*)^{x_0^1,x_{k_n}^n} \circ P)$, where

$$P := \beta_{f_0}^{x_0^1,x_{k_n}^n} \circ \mathcal{F}_{x_0^1,\cdots,x_0^n,x_{k_n}^n} \circ \left(\Big(\prod_{i=1}^n (\tau_i^*)^{x_0^i,x_{k_i}^i} \circ \beta_{f_i}^{x_0^i,x_{k_i}^i} \circ \mathcal{F}_{x_0^i,\cdots,x_{k_i}^i} \circ D_n^{x_0,\cdots,x_r} \Big) \times 1_{N(I[1])} \right) \circ D_2^{x_0,\cdots,x_r}$$

with the inclusions

$$\mathcal{F}_{a_0,\cdots,a_k} : \prod_{i=1}^k X(a_{i-1},a_i) \to X_{Sp(k)}^{a_0,a_k}$$

and the diagonal maps $D_k : X_{Sp(r)} \times N(I[1]) \to X_{Sp(r)} \times N([1])^k$ on $N(I[1])$.

Note that the maps $\mathcal{F}_{a_0,\cdots,a_k}$ form part of a more general natural transformation $(-)^{a_0,\cdots,a_k} \Rightarrow (-)^{a_0,a_k}$. Using this naturality, we can prove that P is in fact equal to a map $Q^{x_0,x_r} \circ \mathcal{F}_{x_0,\cdots,x_r}$ such that

$$Q := \beta_{f_0} \circ \left(\Big((\prod_{X_0}^{1 \leq i \leq n} \tau_i^* \circ \beta_{f_i}) \circ D_n \Big) \times 1_{N(I[1])} \right) \circ D_2.$$

This precisely places us in the situation of Lemma 3.4.11. Note however that the resulting induced left homotopy is of the form

$$H : X_{Sp(r)} \times N(I[1]) \to X_K$$

and maps from μ_K to $\mu'_{K'}$ by Proposition 3.2.10. Moreover, given the inclusion $b : K_0 \hookrightarrow K$, it is such that $b^* \circ H = Q$. Since $\tau_0^* \circ b^* = \tau^*$, we have our left homotopy H such that our original natural isomorphism is of the form

$$h_1(\tau^{x_0,x_r}) \circ h_1(H^{x_0,x_r}) \circ F_{x_0,\cdots,x_r}.$$

This confirms by Corollary 3.3.29 that the equality holds. □

Theorem 3.4.13. *Let X be a 2-fold Segal space and $(\mu_n)_{n\geq 0}$ a choice of horizontal compositions. Then $h_2(X,(\mu_n)_{n\geq 0})$ is an unbiased bicategory.*

Proof. We begin by dealing with associativity conditions. Let $n, m_1, \cdots, m_n \in \mathbb{Z}_{>0}$ and $k_1^1, \cdots, k_n^{m_n} \in \mathbb{Z}_{\geq 0}$. Take a thrice-nested tuple of elements of $(X_{0,0})_0$ of the form

$$L = (L_p)_{p=1}^n = ((L_{p,q})_{q=1}^{m_p})_{p=1}^n = (((x_{p,q,s})_{s=0}^{k_p^q})_{q=1}^{m_p})_{p=1}^n$$

where $x_{p,q,k_p^q} = x_{p,q+1,0}$ if $q < m_p$ and $x_{p,m_p,k_p^{m_p}} = x_{p+1,1,0}$ if $p < n$. Let $t_p = \sum_{q=1}^{m_p} k_p^q$ and $r = \sum_{p=1}^n t_p$. Set $(x_0, \cdots, x_r)$ to be the flattened version of L after removing the elements x_{p,q,k_p^q}, not including $x_{n,m_n,k_n^{m_n}}$. Let $(x_0^p, \cdots, x_{t_p}^p)$ be the flattened version of L_p after removing each x_{p,q,k_p^q} not including $x_{p,m_p,k_p^{m_p}}$.

We take an interest in the four composition operations $\bullet_{Q_i}^{x_0,\cdots,x_r}$ for $i \in \{1,2,3,4\}$, where the four simplicial composition diagrams Q_i are as follows:

$$\begin{aligned}
Q_1 :=& \Big(Q_{11} \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} Q_{1n}\Big) \sqcup_{Sp(n)} \Delta[n] \\
Q_2 :=& \Big(\Delta[t_1] \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} \Delta[t_n]\Big) \sqcup_{Sp(n)} \Delta[n] \\
Q_3 :=& \Big(\Delta[k_1^1] \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} \Delta[k_n^{m_n}]\Big) \sqcup_{Sp(\sum_{i=1}^n m_i)} \Delta[\sum_{i=1}^n m_i] \\
Q_4 :=& \Delta[r]
\end{aligned}$$

where, for $1 \leq p \leq n$,

$$Q_{1p} := (\Delta[k_p^1] \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} \Delta[k_p^{m_p}]) \sqcup_{Sp(m_p)} \Delta[m_p].$$

There is a clear commutative diagram in $\mathbf{SCD}_r$

$$\begin{tikzcd}
& Q_1 \arrow[dl, "\iota_{12}"'] \arrow[dr, "\iota_{13}"] & \\
Q_2 \arrow[dr, "\iota_{24}"'] & & Q_3 \arrow[dl, "\iota_{34}"] \\
& Q_4 &
\end{tikzcd}$$

An example of such a diagram is found in Figure 1.4. We wish to show that this induces the diagram of natural isomorphisms

$$\begin{tikzcd}
& \bullet_{Q_1}^{x_0,\cdots,x_r} \arrow[dl, Rightarrow, "\alpha_{12}"'] \arrow[dr, Rightarrow, "\alpha_{13}"] & \\
\bullet_{Q_2}^{x_0,\cdots,x_r} \arrow[dr, Rightarrow, "\alpha_{24}"'] & & \bullet_{Q_3}^{x_0,\cdots,x_r} \arrow[dl, Rightarrow, "\alpha_{34}"] \\
& \bullet_{Q_4}^{x_0,\cdots,x_r} &
\end{tikzcd}$$

that we seek to prove commutes, where

$$\begin{aligned}
\alpha_{12} &:= 1_{\bullet^{x_{1,1,0},\cdots,x_{n,1,0},x_{n,m_n,k_n^{m_n}}}} \circ \prod_{p=1}^{n} \Phi^{x_0^p,\cdots,x_{t_p}^p}_{(X,(\mu_n)_{n\geq 0})}(f_{k_1^p,\cdots,k_{m_p}^p}) \\
\alpha_{13} &:= \Phi^{x_{1,1,0},\cdots,x_{n,1,0},x_{n,m_n,k_n^{m_n}}}_{(X,(\mu_n)_{n\geq 0})}(f_{m_1,\cdots,m_n}) \circ \prod_{p=1}^{n} 1_{\left(\bullet^{x_0^p,\cdots,x_{t_p}^p}_{Q_{1p}}\right)} \\
\alpha_{24} &:= \Phi^{x_0,\cdots,x_r}_{(X,(\mu_n)_{n\geq 0})}(f_{t_1,\cdots,t_n}) \\
\alpha_{34} &:= \Phi^{x_0,\cdots,x_r}_{(X,(\mu_n)_{n\geq 0})}(f_{k_1^1,\cdots,k_n^{m_n}}).
\end{aligned}$$

Using Lemma 3.4.12, it is straightforward to identify each α_{ij} with the corresponding $\Phi^{x_0,\cdots,x_r}_{(X,(\mu_n)_{n\geq 0})}(\iota_{ij})$. Thus, by functoriality, the diagram commutes as needed.

The conditions on unitors is then trivial, since all involved natural isomorphisms are identities. Indeed, the relevant associators are $\Phi^{x_0,\cdots,x_n}_{(X,(\mu_n)_{n\geq 0})}(f_{1,\cdots,1})$, which are constant as $f_{1,\cdots,1} = 1_{\Delta[n]}$. □

3.5 Homotopy Unbiased Pseudofunctors

In order for h_2 to be a genuine functor, we need to demonstrate how to convert a map $f : (X,(\mu_n)_{n\geq 0}) \to (Y,(\nu_n)_{n\geq 0})$ in $\mathbf{SeSp}_2^{comp}$ into a pseudofunctor

$$h_2(f) : h_2(X,(\mu_n)_{n\geq 0}) \to (Y,(\nu_n)_{n\geq 0}).$$

Before we can do so, a great deal of the technology we developed for h_2's behavior on objects now needs to be extended to handle maps $X \to Y$ of 2-fold Segal spaces.

We will first need to extend the notion of being object-fibered to a map between 2-fold Segal spaces:

Proposition 3.5.1. *Let $F : X \to Y$ be a map in $\boldsymbol{sSpace}_k$. Consider two commutative diagrams in $\boldsymbol{sSpace}_{k-1}$*

$$\begin{tikzcd}
A \arrow[rr, "F_A"] \arrow[dr] \arrow[dd, "f"'] & & K \arrow[d] \\
& (X_0)^n \arrow[r, "(F_0)^n"] & (Y_0)^n \\
B \arrow[rr, "F_B"'] \arrow[ur] & & L \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[rr, "F_A"] \arrow[d] & & K \arrow[dl] \arrow[dd, "g"] \\
(X_0)^n \arrow[r, "(F_0)^n"] & (Y_0)^n & \\
B \arrow[rr, "F_B"'] \arrow[u] & & L \arrow[ul]
\end{tikzcd}$$

for some chosen maps F_A and F_B, so that f and g are object-fibered. Let $x_0, \cdots, x_n \in (X_{0,\cdots,0})_0$.

Then, defining $F_B^{x_0,\cdots,x_n}$ *to be the map*

$$\begin{tikzcd} B^{x_0,\cdots,x_n} \arrow[r, "F_B^{x_0,\cdots,x_n}"] \arrow[d, hook] & L^{F(x_0),\cdots,F(x_n)} \\ B^{F(x_0),\cdots,F(x_n)} \arrow[ur, "F_B^{F(x_0),\cdots,F(x_n)}"'] & \end{tikzcd}$$

and similarly for $F_A^{x_0,\cdots,x_n}$*, we have that*

$$F_B^{x_0,\cdots,x_n} \circ f^{x_0,\cdots,x_n} = (F_B \circ f)^{f(x_0),\cdots,f(x_n)} \circ \mathcal{I}^{F,A}_{x_0,\cdots,x_n},$$

where $\mathcal{I}^{F,A}_{x_0,\cdots,x_n} : A^{x_0,\cdots,x_n} \hookrightarrow A^{f(x_0),\cdots,f(x_n)}$ *is the inclusion. Similarly, we have*

$$g^{f(x_0),\cdots,f(x_n)} \circ F_A^{x_0,\cdots,x_n} = (g \circ F_A)^{f(x_0),\cdots,f(x_n)} \circ \mathcal{I}^{F,A}_{x_0,\cdots,x_n}.$$

Proof. Diagram chases show both statements to be true. □

The central idea controlling our composition, coherence isomorphisms and coherence conditions was the collections of functors $\Phi^{x_0,\cdots,x_n}_{(X,(\mu_n)_{n\geq 0})}$ for all $x_0,\cdots,x_n \in (X_{0,0})_0$ and $n \geq 0$. The flavor of results we will need to prove for pseudofunctors are similar, so we seek an augmented version of this construct.

In essence, what $\Phi^{x_0,\cdots,x_n}_{(X,(\mu_n)_{n\geq 0})}$ accomplished was to convert diagrams in $\mathbf{SCD}_n$ into commutative diagrams of coherence isomorphisms. A reasonable extension of this category for the purposes of pseudofunctors is $\mathbf{SCD}_n \times [1]$, with the morphisms $(1_K, 0 < 1)$ representing a homotopy from composition in the codomain to the domain:

Definition 3.5.2. *Let* $f : (X, (\mu_n)_{n\geq 0}) \to (Y, (\nu_n)_{n\geq 0})$ *be a map in* $\boldsymbol{SeSp}_2^{comp}$*. Let* $x, y \in (X_{0,0})_0$*. Then define the functor*

$$\omega_f^{x,y} : \boldsymbol{SCD}_n \times [1] \to \boldsymbol{Fun}\Big(h_1(X^{x,y}_{Sp(n)}), h_1(Y(x,y))\Big)$$

to be the functor sending

$$(K,0) \mapsto h_1((\tau_K^*)^{f(x),f(y)} \circ \nu_K^{f(x),f(y)} \circ f^{x,y}_{Sp(n)})$$
$$(K,1) \mapsto h_1((\tau_K^*)^{f(x),f(y)} \circ f_K^{x,y} \circ \mu_K^{x,y})$$

and sending a morphism to the map generated by the following cases:

1. $(g, 1_0) \mapsto \omega^{f(x),f(y)}_{(Y,(\nu_n)_{n\geq 0})}(g) \circ h_1(f^{x,y}_{Sp(n)})$*, where* $g : K_1 \to K_2$*;*

2. $(g, 1_1) \mapsto h_1(f_1^{x,y}) \circ \omega^{x,y}_{(X,(\mu_n)_{n\geq 0})}(g)$*, where* g *is as above;*

3. $(1_K, 0 < 1) \mapsto h_1((\tau_K^*)^{f(x),f(y)}) \circ h_1(\eta_K^{f(x),f(y)}) \circ h_1(\mathcal{I}_{x,y}^{f,X_{Sp(n)}})$, *where* η_K *is an induced left homotopy from* $\nu_K \circ f_{Sp(n)}$ *to* $f_K \circ \mu_K$ *by the lifting problem*

$$\begin{tikzcd} & X_K \arrow[r, dashed, "f_K"] & Y_K \arrow[d] \\ X_{Sp(n)} \arrow[ur, dashed, "\mu_K"] \arrow[urr, dashed, "\nu_K \circ f_{Sp(n)}"'] \arrow[rr] & & Y_{Sp(n)} \end{tikzcd}$$

Proposition 3.5.3. $\omega_f^{x,y}$ *is a functor.*

Proof. Functoriality on the subcategory $\mathbf{SCD}_n \times \{0, 1\}$ is clear. We need only prove that, for a map $g : K_1 \to K_2$,

$$\omega_f^{x,y}(g, 1_1)\omega_f^{x,y}(1_{K_1}, 0 < 1) = \omega_f^{x,y}(1_{K_2}, 0 < 1)\omega_f^{x,y}(g, 1_0).$$

Expanding out terms, we have that

$$\begin{aligned} \omega_f^{x,y}(g, 1_0) &= h_1\Big((\tau_{K_1}^*)^{f(x),f(y)} \circ \alpha_g^{f(x),f(y)} \circ (f_{Sp(n)} \times 1_{N(I[1])})^{f(x),f(y)} \circ \mathcal{I}_{x,y}^{f,X_{Sp(n)} \times N(I[1])}\Big) \\ \omega_f^{x,y}(1_{K_1}, 0 < 1) &= h_1\Big((\tau_{K_1}^*)^{f(x),f(y)} \circ \eta_{K_1}^{f(x),f(y)} \circ \mathcal{I}_{x,y}^{f,X_{Sp(n)} \times N(I[1])}\Big) \\ \omega_f^{x,y}(1_{K_2}, 0 < 1) &= h_1\Big((\tau_{K_2}^*)^{f(x),f(y)} \circ \eta_{K_2}^{f(x),f(y)} \circ \mathcal{I}_{x,y}^{f,X_{Sp(n)} \times N(I[1])}\Big) \\ \omega_f^{x,y}(g, 1_1) &= h_1\Big(f_1^{f(x),f(y)} \circ (\tau_{K_1}^*)^{f(x),f(y)} \circ \beta_g^{f(x),f(y)} \circ \mathcal{I}_{x,y}^{f,X_{Sp(n)} \times N(I[1])}\Big) \end{aligned}$$

for induced left homotopies α_g from ν_{K_1} to ν'_{K_2} and β_g from μ_{K_1} to μ'_{K_2}.

We are able to produce the two compositions $(f_{K_1}^{f(x),f(y)} \circ \beta_g^{f(x),f(y)})(\eta_{K_1}^{f(x),f(y)})$ and $((g^*)^{f(x),f(y)} \circ \eta_{K_2}^{f(x),f(y)})\Big(\alpha_g^{f(x),f(y)} \circ (f_{Sp(n)}^{f(x),f(y)} \times 1_{N(I[1])})\Big)$, which yield the same natural isomorphisms as before. As these compositions are left homotopies induced between the same solutions to the same lifting problem, the equality holds. $\square$

Definition 3.5.4. *Let* $f : (X, (\mu_n)_{n \geq 0}) \to (Y, (\nu_n)_{n \geq 0})$ *be a map in* $\boldsymbol{SeSp}_2^{comp}$. *Let* $x_0, \cdots, x_n \in (X_{0,0})_0$. *Then define the map*

$$\phi_f^{x_0,\cdots,x_n} : \boldsymbol{Fun}\Big(h_1(X_{Sp(n)}^{x_0,x_n}), h_1(Y(x_0, x_n))\Big) \to \boldsymbol{Fun}\Big(\prod_{i=1}^n h_1(X(x_{i-1}, x_i)), h_1(Y(x_0, x_n))\Big)$$

to be precomposition with the inclusion

$$F_{x_0,\cdots,x_n} : \prod_{i=1}^n h_1(X(x_{i-1}, x_i)) \cong h_1(\prod_{i=1}^n X(x_{i-1}, x_i)) \hookrightarrow h_1(X_{Sp(n)}^{x_0,x_n}).$$

Write $\Phi_f^{x_0,\cdots,x_n} := \phi_f^{x_0,\cdots,x_n} \circ \omega_f^{x_0,x_n}$.

Henceforth, for a map $f : (X, (\mu_n)_{n\geq 0}) \to (Y, (\nu_n)_{n\geq 0})$, write $\bullet_A^{x_0,\cdots,x_n}$ or $\bullet_{K,A}^{x_0,\cdots,x_n}$ for $A \in \{X, Y\}$ to distinguish the composition operations in each of these spaces.

The 'operadic' flavor of $\Phi^{x_0,\cdots,x_n}_{(X,(\mu_n)_{n\geq 0})}$ is admittedly somewhat diluted in the following lemma concerning $\Phi_f^{x_0,\cdots,x_n}$. Future work will include seeking out a perhaps more natural category than $\mathbf{SCD}_n \times [1]$, which appears too 'coarse' in the following results to lend itself to such an interpretation. Nonetheless, some operadic color still shines through, at least enough for us to prove what we need:

Lemma 3.5.5. *Let $f : (X, (\mu_n)_{n\geq 0}) \to (Y, (\nu_n)_{n\geq 0})$ be a map in $\boldsymbol{SeSp}_2^{comp}$. Let $n > 0$ and choose integers $k_1, \cdots, k_n \in \mathbb{Z}_{\geq 0}$ and a nested sequence of elements $((x_j^i)_{j=0}^{k_i})_{i=1}^n$ of $(X_{0,0})_0$ such that $x_{k_i}^i = x_0^{i+1}$ for $i < n$. Let $(x_0, \cdots, x_r)$ be the flattened version of the sequence with all $x_{k_i}^i$ removed for $i < n$. Set K to be the simplicial composition diagram*

$$K = (K_1 \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} K_n) \sqcup_{Sp(n)} K_0$$

such that K_i has arity k_i and $k_0 = n$. Then the diagram

$$\begin{tikzcd}
\bullet^{f(x_0),\cdots,f(x_r)}_{K,Y} \circ h_1(f^{x_0,\cdots,x_r}_{Sp(r)}) \arrow[dd, "\Phi_f^{x_0,\cdots,x_r}(1_K,0<1)"'] \arrow[dr, "1_{\bullet^{f(x_0^1),\cdots,f(x_0^n),f(x_{k_n}^n)}_{K_0,Y}} \circ \prod_{i=1}^n \Phi_f^{x_0^i,\cdots,x_{k_i}^i}(1_{K_i},0<1)"] & \\
& \bullet^{f(x_0^1),\cdots,f(x_0^n),f(x_{k_n}^n)}_{K_0,Y} \circ h_1(f^{x_0^1,\cdots,x_0^n,x_{k_n}^n}_{Sp(n)}) \circ \prod_{i=1}^n \bullet^{x_0^i,\cdots,x_{k_i}^i}_{K_0,X} \arrow[dl, "\Phi_f^{x_0^1,\cdots,x_0^n,x_{k_n}^n}(1_{K_0},0<1) \circ \prod_{i=1}^n 1_{\bullet^{x_0^i,\cdots,x_{k_i}^i}_{K_i,X}}"] \\
h_1(f_1^{x_0,x_r}) \circ \bullet^{x_0,\cdots,x_r}_{K,X} &
\end{tikzcd}$$

commutes.

Proof. The proof is similar to that of Lemma 3.4.12. Expanding out all the terms, we obtain the expressions

$$R := h_1((\tau_K^*)^{f(x_0),f(x_r)}) \circ h_1(\eta_K^{f(x_0),f(x_r)}) \circ h_1(\mathcal{I}^{f,X}_{x_0,x_r}) \circ F_{x_0,\cdots,x_r}$$

$$S := \bullet^{f(x_0^1),\cdots,f(x_0^n),f(x_{k_n}^n)}_{K_0,Y} \circ \prod_{i=1}^n h_1((\tau_{K_i}^*)^{f(x_0^i),f(x_{k_i}^i)}) \circ h_1(\eta_{K_i}^{f(x_0^i),f(x_{k_i}^i)}) \circ h_1(\mathcal{I}^{f,X}_{x_0^i,x_{k_i}^i}) \circ F_{x_0^i,\cdots,x_{k_i}^i}$$

$$T := h_1((\tau_{K_0}^*)^{f(x_0),f(x_r)}) \circ h_1(\eta_K^{f(x_0),f(x_r)}) \circ h_1(\mathcal{I}^{f,X}_{x_0,x_r}) \circ F_{x_0^1,\cdots,x_0^n,x_{k_n}^n} \circ \prod_{i=1}^n \bullet^{f(x_0^i),\cdots,f(x_{k_i}^i)}_{K_0,Y}$$

If we can show both S and T to be of the form

$$h_1((\tau_K^*)^{f(x_0),f(x_r)}) \circ h_1(Q_A) \circ h_1(\mathcal{I}^{f,X}_{x_0,x_r}) \circ F_{x_0,\cdots,x_r}$$

for some suitable induced left homotopies Q_A where $A \in \{S, T\}$, then we will be done.

Set $D_k : X_{Sp(r)} \times N(I[1]) \to X_{Sp(r)} \times N(I[1])^k$ to the diagonal. For $A \in \{X, Y\}$, define the map

$$\mathcal{F}^A_{a_0,\cdots,a_k} : \prod_{i=1}^{k} A(a_{i-1}, a_i) \to A^{a_0,a_k}_{Sp(k)}$$

to be the evident inclusion. Define $I_{K,X}$ to be the identity homotopy from μ_K to itself and $I_{K,Y}$ to be as such for ν_K.

Similarly to the proof of Lemma 3.4.12, we have that $S = h_1((\tau_{K_0}^*)^{f(x_0),f(x_r)}) \circ h_1(\mathcal{S})$, where $\mathcal{S}$ is the map

$$(I^{f(x_0),f(x_r)}_{K_0,Y}) \circ \mathcal{F}^Y_{f(x_0^1),\cdots,f(x_0^n),f(x^n_{k_n})} \circ \left(\left(\prod_{i=1}^{n} Q_i \circ D_n^{x_0,\cdots,x_r}\right) \times 1_{N(I[1])}\right) \circ D_2^{x_0,\cdots,x_r}$$

where

$$Q_i := (\tau_{K_i}^* \circ \eta_{K_i})^{f(x_0^i),f(x^i_{k_i})} \circ \mathcal{I}^{f,X_{Sp(k_i)}\times N(I[1])}_{x_0^i,x^i_{k_i}} \circ \mathcal{F}^X_{x_0^i,\cdots,x^i_{k_i}}$$

and that $T = h_1((\tau_{K_0}^*)^{f(x_0),f(x_r)}) \circ h_1(\mathcal{T})$, where $\mathcal{T}$ is the map

$$\eta_K^{f(x_0),f(x_r)} \circ \mathcal{I}^{f,X_{Sp(n)}\times N(I[1])}_{x_0,x_r} \circ \mathcal{F}^X_{x_0^1,\cdots,x_0^n,x^n_{k_n}} \circ \left(\left(\prod_{i=1}^{n} W_i \circ D_n^{x_0,\cdots,x_r}\right) \times 1_{N(I[1])}\right) \circ D_2^{x_0,\cdots,x_r}$$

where

$$W_i := (\tau_{K_i}^*)^{x_0^i,x^i_{k_i}} \circ I^{x_0^i,x^i_{k_i}}_{K_i,X} \circ \mathcal{F}^X_{x_0^i,\cdots,x^i_{k_i}}.$$

By similar manipulations to those in Lemma 3.4.12, we can reduce each of these further such that $\mathcal{S} = Q_S^{f(x_0),f(x_r)} \circ \mathcal{I}^{f,X}_{x_0,x_r} \circ \mathcal{F}^X_{x_0,\cdots,x_r}$ where

$$Q_S := I_{K_0,Y} \circ \left(\left(\prod_{i=1}^{n} (\tau_{K_i}^* \circ \eta_{K_i}) \circ D_n\right) \times 1_{N(I[1])}\right) \circ D_2$$

and such that $\mathcal{T} = Q_T^{f(x_0),f(x_r)} \circ \mathcal{I}^{f,X}_{x_0,x_r} \circ \mathcal{F}^X_{x_0,\cdots,x_r}$ where

$$Q_T := \eta_K \circ \left(\left(\prod_{i=1}^{n} (\tau_{K_i}^* \circ I_{K_i,X}) \circ D_n\right) \times 1_{N(I[1])}\right) \circ D_2.$$

Both of these are of suitable form for application of Lemma 3.4.11, the resulting homotopies from which are both of the form $X_{Sp(r)} \times N(I[1]) \to Y_K$ and can be composed to complete the proof. $\square$

Armed with these results, we are able to finally declare our definition of $h_2(f)$ and prove its correctness.

Definition 3.5.6. *Let $f : (X, (\mu_n)_{n\geq 0}) \to (Y, (\nu_n)_{n\geq 0})$ be a morphism in $\boldsymbol{SeSp}_2^{comp}$. Then define*

$$h_2(f) : h_2(X, (\mu_n)_{n\geq 0}) \to h_2(Y, (\nu_n)_{n\geq 0})$$

to be the pseudofunctor defined such that:

1. *For every $x \in (X_{0,0})_0$, $h_2(f)(x) = (f_{0,0})_0(x)$;*
2. *For every $x, y \in (X_{0,0})_0$, $h_2(f)_{x,y} = h_1(f_1^{x,y}) : h_1(X(x, y)) \to h_1(Y(f(x), f(y)))$;*
3. *For every $n \in \mathbb{Z}_{>0}$ and $x_0, \cdots, x_n \in (X_{0,0})_0$, the natural isomorphism*
$$\pi_{x_0,\cdots,x_n} := \Phi_f^{x_0,\cdots,x_n}(1_{\Delta[n]}, 0 < 1);$$
4. *For every $x \in (X_{0,0})_0$, the natural isomorphism π_x is the identity.*

Theorem 3.5.7. *Let $f : (X, (\mu_n)_{n\geq 0}) \to (Y, (\nu_n)_{n\geq 0})$ be a morphism in $\boldsymbol{SeSp}_2^{comp}$. Then $h_2(f)$ is a pseudofunctor.*

Proof. Let $n \in \mathbb{Z}_{>0}$ and $k_1, \cdots, k_n \in \mathbb{Z}_{\geq 0}$, with $r = \sum_{i=1}^n k_i$. Consider any n-tuple of tuples of elements of $(X_{0,0})_0$

$$Y := ((x_0^1, x_1^1, \cdots, x_{k_1}^1), \cdots, (x_0^n, \cdots, x_{k_n}^n))$$

such that $x_{k_i}^i = x_0^{i+1}$ for every $i < n$. Let $(x_0, \cdots, x_r)$ be the flattened tuple after removing each $x_{k_i}^i$ for $i < n$.

Consider moreover the composition diagram

$$K := (\Delta[k_1] \sqcup_{\Delta[0]} \cdots \sqcup_{\Delta[0]} \Delta[k_n]) \sqcup_{Sp(n)} \Delta[n].$$

We are tasked with showing that the diagram

$$\begin{tikzcd}
\bullet_{K,Y}^{f(x_0),\cdots,f(x_r)} \circ h_1(f_{Sp(r)}^{x_0,\cdots,x_r}) \arrow[r, Rightarrow, "\Phi_f^{x_0,\cdots,x_r}(f_{k_1,\cdots,k_n},1_0)"] \arrow[d, Rightarrow, "1_{\bullet_Y^{f(x_0^1),\cdots,f(x_0^n),f(x_{k_n}^n)}} \circ \prod_{i=1}^n \Phi_f^{x_0^i,\cdots,x_{k_i}^i}(1_{\Delta[k_i]},0<1)"] & \bullet_Y^{x_0,\cdots,x_r} \circ h_1(f_{Sp(r)}^{x_0,\cdots,x_r}) \arrow[dd, Rightarrow, "\Phi_f^{x_0,\cdots,x_r}(1_{\Delta[r]},0<1)"] \\
\bullet_Y^{f(x_0^1),\cdots,f(x_0^n),f(x_{k_n}^n)} \circ h_1(f_{Sp(n)}^{x_0^1,\cdots,x_0^n,x_{k_n}^n}) \circ \prod_{i=1}^n \bullet_X^{x_0^i,\cdots,x_{k_i}^i} \arrow[d, Rightarrow, "\Phi_f^{x_0^1,\cdots,x_0^n,x_{k_n}^n}(1_{\Delta[n]},0<1) \circ \prod_{i=1}^n 1_{\bullet_X^{x_0^i,\cdots,x_{k_i}^i}}"] & \\
h_1(f_1^{x_0,x_r}) \circ \bullet_{K,X}^{x_0,\cdots,x_r} \arrow[r, Rightarrow, "\Phi_f^{x_0,\cdots,x_r}(f_{k_1,\cdots,k_n},1_1)"'] & h_1(f_1^{x_0,x_r}) \circ \bullet_X^{x_0,\cdots,x_r}
\end{tikzcd}$$

commutes. This, however, must hold, by Lemma 3.5.5 and functoriality of $\Phi_f^{x_0,\cdots,x_r}$. The other commuting diagram is trivial, as all involved natural isomorphisms are identities. $\square$

We now need to show that h_2, given the behavior on objects from Definition 3.4.8 and morphisms from Definition 3.5.6, defines a valid functor into **UBicat**:

Theorem 3.5.8. *Let*

$$(X,(\mu_n)_{n\geq 0}) \xrightarrow{f} (Y,(\nu_n)_{n\geq 0}) \xrightarrow{g} (Z,(\omega_n)_{n\geq 0})$$

be a chain of morphisms in $\boldsymbol{SeSp}_2^{comp}$. *Then* $h_2(g\circ f) = h_2(g)\circ h_2(f)$.

Proof. The equivalence is evident on the object mapping and behavior on hom-categories. For the compositors, let the isomorphisms for g be labelled as θ and for f be π. The ones ψ for $g\circ f$ must be proven to be such that

$$\psi_{x_0,\cdots,x_n} = \big(h_1(g_1^{f(x_0),f(x_n)})\circ(\pi_{x_0,\cdots,x_n})\big)\big(\theta_{f(x_0),\cdots,f(x_n)}\circ h_1(f_{Sp(n)}^{x_0,\cdots,x_n})\big).$$

Expanding out all of these terms gives us that

$$\begin{aligned}
\pi_{x_0,\cdots,x_n} &= h_1(Y_{\langle 0,n\rangle}^{f(x_0),f(x_n)})\circ h_1(\eta_n^{f(x_0),f(x_n)})\circ h_1(\mathcal{I}_{x_0,x_n}^{f,X_{Sp(n)}})\circ F_{x_0,\cdots,x_n}^X\\
\theta_{f(x_0),\cdots,f(x_n)} &= h_1(Z_{\langle 0,n\rangle}^{gf(x_0),gf(x_n)})\circ h_1(\kappa_n^{gf(x_0),gf(x_n)})\circ h_1(\mathcal{I}_{f(x_0),f(x_n)}^{g,Y_{Sp(n)}})\circ F_{f(x_0),\cdots,f(x_n)}^Y\\
\psi_{x_0,\cdots,x_n} &= h_1(Z_{\langle 0,n\rangle}^{gf(x_0),gf(x_n)})\circ h_1(\zeta_n^{gf(x_0),gf(x_n)})\circ h_1(\mathcal{I}_{x_0,x_n}^{gf,X_{Sp(n)}})\circ F_{x_0,\cdots,x_n}^X
\end{aligned}$$

where $F_{a_0,\cdots,a_n}^A$ is the map $F_{a_0,\cdots,a_n}$ adjusted in the obvious way for $A\in\{X,Y\}$ and the left homotopies η_n, κ_n and ζ_n are respectively from $\nu_n\circ f_{Sp(n)}$ to $f_n\circ\mu_n$, from $\omega_n\circ g_{Sp(n)}$ to $g_n\circ\nu_n$ and from $\omega_n\circ g_{Sp(n)}\circ f_{Sp(n)}$ to $g_n\circ f_n\circ\mu_n$.

Note that $\theta_{f(x_0),\cdots,f(x_n)}\circ h_1(f_{Sp(n)}^{x_0,\cdots,x_n})$ is equal to

$$h_1(Z_{\langle 0,n\rangle}^{gf(x_0),gf(x_n)})\circ h_1((\kappa_n\circ(f_{Sp(n)}\times 1_{N(I[1])}))^{gf(x_0),gf(x_n)})\circ h_1(\mathcal{I}_{x_0,x_n}^{gf,X_{Sp(n)}})\circ F_{x_0,\cdots,x_n}^X$$

while similarly $h_1(g_1^{f(x_0),f(x_n)})\circ(\pi_{x_0,\cdots,x_n})$ is equal to

$$h_1(g_1^{f(x_0),f(x_n)})\circ h_1(Y_{\langle 0,n\rangle}^{f(x_0),f(x_n)})\circ h_1(\eta_n^{f(x_0),f(x_n)})\circ h_1(\mathcal{I}_{x_0,x_n}^{f,X_{Sp(n)}})\circ F_{x_0,\cdots,x_n}^X$$

which is then equal to

$$h_1(Z_{\langle 0,n\rangle}^{gf(x_0),gf(x_n)})\circ h_1((g_n\circ\eta_n)^{f(x_0),f(x_n)})\circ h_1(\mathcal{I}_{x_0,x_n}^{f,X_{Sp(n)}})\circ F_{x_0,\cdots,x_n}^X.$$

All of these three cases are now of the form $h_1(Z_{\langle 0,n\rangle}^{gf(x_0),gf(x_n)})\circ h_1(\Gamma^{f(x_0),f(x_n)})\circ h_1(\mathcal{I}_{x_0,x_n}^{f,X_{Sp(n)}})\circ F_{x_0,\cdots,x_n}^X$ for induced left homotopies Γ. We find that the left homotopies $g_n\circ\eta_n$ and $\kappa_n\circ(f_{Sp(n)}\times 1_{N(I[1])})$ are now readily composable, so by Proposition 3.3.31 and Corollary 3.3.29 the equality holds. $\square$

Theorem 3.5.9. *Let $(X, (\mu_n)_{n\geq 0})$ be an object in $\boldsymbol{SeSp}_2^{comp}$. Then $h_2(1_X) = 1_{h_2(X,(\mu_n)_{n\geq 0})}$.*

Proof. The behavior on objects and hom-categories is self-evident. Moreover,

$$\Phi_{1_X}^{x_0,\cdots,x_n}(1_{\Delta[n]}, 0 < 1)$$

is built of a left homotopy with the same domain and codomain, so can be assumed to be constant, making the natural isomorphisms trivial. □

We are finally able to state what we consider to be the central definition of this thesis:

Definition 3.5.10. *Define*

$$h_2 : \boldsymbol{SeSp}_2^{comp} \to \boldsymbol{UBicat}$$

to be the functor that sends $(X, (\mu_n)_{n\geq 0})$ to $h_2(X, (\mu_n)_{n\geq 0})$ and a morphism f to $h_2(f)$.

A useful consequence of functoriality is that we may finally show how all choices of horizontal compositions made in a 2-fold Segal space X ultimately have no real consequence in the homotopy bicategory:

Corollary 3.5.11. *Let X be a 2-fold Segal space. Let $(\mu_n)_{n\geq 0}$ and $(\nu_n)_{n\geq 0}$ be two choices of horizontal compositions for X. Then there is a pseudofunctor $h_2(X, (\mu_n)_{n\geq 0}) \to h_2(X, (\nu_n)_{n\geq 0})$ which is an isomorphism in **UBicat** and acts as the identity on objects and hom-categories.*

3.5.1 Factoring Through h_2^{tr}

We have that h_2 and $h_2^{tr} \circ Tr$ agree precisely on objects, excluding perhaps associators. We now wish to show a similar result holds for morphisms, for which the proof is similiar to that of objects:

Proposition 3.5.12. *Suppose $f : (X, (\mu_n)_{n\geq 0}) \to (Y, (\nu_n)_{n\geq 0})$ is a morphism in $\boldsymbol{SeSp}_2^{comp}$. Then $h_2(f)$ and $h_2^{tr}(Tr(f))$ agree on all data excluding compositors.*

Proof. That equality holds on objects and hom-categories is immediate. Moreover, equality on the natural isomorphisms π_x is trivial, as these are identities. □

We leave a full comparison of compositors to future work. We believe an equality of compositors should also hold, so that h_2 is indeed equal to $h_2^{tr} \circ Tr$.

Note that h_2 and $h_2^{tr} \circ Tr$ are necessarily not full. In fact, the image under h_2 and thus under $h_2^{tr} \circ Tr$ of any map will be a *normal homomorphism*, as described in [LP08]:

Definition 3.5.13 ([LP08, pg. 1]). *A* normal homomorphism $F : \mathscr{C} \to \mathscr{D}$ *is a pseudofunctor between bicategories that strictly preserves identities.*

This is similar to the bifunctor from *Tamsamani* 2*-categories* to the 2-category **NHom** of bicategories, normal homomorphisms and certain pseudonatural natural transformations defined by Lack and Paoli in [LP08]. We suspect our definition of h_2, if extended to incorporate pseudonatural transformations, may factor through this construction at least up to equivalence. We leave such investigations to future work.

Chapter 4

Reedy Fibrant Replacement

In the realm of topological quantum field theories, Reedy fibrancy seems in general more of a suggestion than an assertion. Several constructions of (∞, n)-category of manifolds and cobordisms, such as those of [CS19] and [Lur09b], build upon a foundation of *projective fibrant* n-fold Segal spaces rather than *injective fibrant*, or equivalently Reedy fibrant. This apparent lack of Reedy fibrancy means that our construction of homotopy bicategory may not be immediately applicable to many known $(\infty, 2)$-categorical constructions of topological quantum field theories.

A few possible solutions present themselves. One is to completely revise our homotopy bicategory functor h_2 to accommodate projective fibrant 2-fold Segal spaces from the start. Obtaining such a construction lies beyond the scope of this thesis. Indeed, if $X \in \mathbf{SeSp}_2^{proj}$ then the projective fibrant analogue to our usual lifting problem

$$\begin{tikzcd} & X_{2,\bullet} \arrow[d, "\gamma_2"] \\ & X_{Sp(2),\bullet} \arrow[d] \\ X_{1,\bullet} \times^h_{X_{0,\bullet}} X_{1,\bullet} \arrow[r, "id"'] & X_{1,\bullet} \times^h_{X_{0,\bullet}} X_{1,\bullet} \end{tikzcd}$$

is no longer in general a lifting problem. It fails on both possible fronts: cofibrancy in the projective model structure cannot be so easily assumed, while the right vertical map is merely a weak equivalence. Thus, there is no guarantee of sections existing for the projective fibrant analogues of the Segal maps $X_{n,\bullet} \to X_{1,\bullet} \times^h_{X_{0,\bullet}} \cdots \times^h_{X_{0,\bullet}} X_{1,\bullet}$.

The situation is even worse than a lack of sections: it is not even guaranteed for the Segal maps to have *homotopy inverses*, as in Proposition 3.4.1. While fibrancy is assured, a lack of cofibrancy implies we may not even have this weaker alternative to a section of the Segal maps. However, upon applying h_1, weak equivalences are converted to equivalences of categories, so an inverse may be found in the homotopy categories at least. Such inverses potentially would not reflect any data originally in the overarching $(\infty, 2)$-category due to the lack of homotopy inverse, but this is perhaps still acceptable in many circumstances. Such lines of reasoning follows from the suggested approach of [JS17], which if employed to extend the definition of h_2^{tr} to the projective fibrant case may lead to a suitable construction. We leave this to future work.

Another solution is to try and handle the case of 2-fold Segal space where the Segal maps have a homotopy inverse of some sort. In the case of the (∞, n)-category of cobordisms $\mathbf{Bord}_n$ defined in [CS19], the Segal maps are proven to be weak equivalences by exhibiting levelwise deformation retracts thereof, which if proven to commute would identify such an inverse. One may then proceed with an extension of our construction under such assumptions. This is also left to future work.

Yet another approach to this issue would be to start with an (∞, n)-category of manifolds and cobordisms that is Reedy fibrant *a priori*. This is an ongoing project of the author that did not reach its completion within the time constraints of the thesis, so will be left to future work as well. One could also turn to the constructions of Grady and Pavlov [GP22a], which seem by nature to circumvent the pathological case outlined in the introduction violating Reedy fibrancy. These higher categories of cobordisms have thus far not yet been proven to be Reedy fibrant in the literature as far as we are aware, though it is indeed possible that they are suitable for our needs.

A final tactic, which we will investigate in this chapter, is to convert a general projective fibrant 2-fold Segal space X into an equivalent Reedy fibrant one. Such an object would be described as a *Reedy fibrant replacement* $R(X)$ of X. Reedy fibrant replacements have been computed in particular instances [DFT22] [Ste19, Lemma 6.4.1], though it appears that there is no fully explicit general construction of such an operation in the literature not based upon the small object argument at present, such as in [Rie14, ch. 12]. This is not in general an issue, as it often suffices to simply know a fibrant replacement must exist, with functoriality being an occasional required property. Indeed, one may simply perform a factorization

$$X \to R(X) \to *$$

of the terminal map $X \to *$ into a trivial cofibration and fibration in sequence. If one knows a functorial factorization system for the Reedy model structure in question, then this will induce a functorial Reedy fibrant replacement.

For us, the precise contents of $R(X)$ are important and will be scrutinized in detail when taking a homotopy bicategory, so we will need to introduce a precise definition of $R(X)$ that appends as little data to X as possible. For this reason, methods involving the small object argument are not suitable for our purposes. Of course, this is by no means a shortcoming specific to the small object argument; fibrant replacement should in general be expected to introduce an unmanageable amount of new data, much like an injective resolution of a chain complex. Regardless, such resolutions are often necessary to perform computations of algebraic structure. One might draw a rather direct analogy between X and an object in an abelian category, between $R(X)$ and an injective resolution thereof, and between $h_2(R(X))$ and homology.

Our construction applies generally for any Reedy category together with the target model category $\mathscr{M} = \mathbf{sSet}$. It is a direct application of the natural method to factorize maps in a Reedy model structure, as described in [Hir09, pg. 293]. We recall this method presently. Consider some Reedy category $\mathscr{C}$ and model category $\mathscr{M}$. Suppose $f : X \to Y$ is a morphism in $\mathscr{M}^{\mathscr{C}}$. We wish to obtain a factorization

$$X \xrightarrow{q} Z \xrightarrow{p} Y$$

where q is a Reedy trivial cofibration and p is a Reedy fibration. One first considers each $c \in \mathscr{C}$ with $\mathbf{deg}(c) = 0$ and chooses a factorization

$$X(c) \xrightarrow{q_c} Z(c) \xrightarrow{p_c} Y(c)$$

of the map $f_c : X(c) \to Y(c)$ into trivial cofibrations and fibrations. Starting with these choices, one then proceeds by induction in degree: if $c \in \mathscr{C}$ and all q_d and p_d have been defined for $\mathbf{deg}(d) < \mathbf{deg}(c)$, then this suffices to define the object

$$Z_{<\mathbf{deg}(c)} : \mathscr{C}^{<\mathbf{deg}(c)} \to \mathscr{M}$$

and factorizations

$$L_cX \xrightarrow{L_cq} L_cZ \xrightarrow{L_cp} L_cY \qquad\qquad M_cX \xrightarrow{M_cq} M_cZ \xrightarrow{M_cp} M_cY$$

of the maps $L_cf : L_cX \to L_cY$ and $M_cf : M_cX \to M_cY$.

As discussed in [Hir09, pg. 293], it then suffices to produce a factorization

$$X(c) \sqcup_{L_cX} L_cZ \to Z(c) \to Y(c) \times_{M_cY} M_cZ$$

of the map $X(c) \times_{L_cX} L_cZ \to Y(c) \times_{M_cY} M_cZ$ into a trivial cofibration and fibration in $\mathscr{M}$. Moreover, the fact that the former map is a trivial cofibration is enough by [Hir09, Thm. 15.3.15] to ensure that q will be a trivial cofibration, while p is a Reedy fibration by definition. In this manner, one may inductively construct Z, p and q levelwise.

Capitalizing upon the existence of this known algorithm, we may set $Y = *$ to obtain Reedy fibrant replacements of X. In particular, if we assume that X is already levelwise fibrant, we may set $Z(c) := X(c)$ and then factorize maps

$$X(c) \sqcup_{L_cX} L_cZ \to Z(c) \to M_cZ$$

inductively in the degree of c. We will present a particular choice of this factorization when $\mathscr{M} = \mathbf{sSet}$. The goal with this construction is to produce $R(X) = Z$ when $\mathscr{C} = (\Delta^{op})^2$ and $X \in \mathbf{SeSp}_2^{proj}$ such that $R(X) \in \mathbf{SeSp}_2^{inj}$ and the resulting homotopy bicategory $h_2(R(X))$ is tractable.

In total, what we seek is a functor, for $n \in \{1, 2\}$, of the form

$$R : \mathbf{SeSp}_n^{proj} \to \mathbf{SeSp}_n^{proj}$$

such that $R(X) \in \mathbf{SeSp}_n^{inj}$, together with a natural transformation $\kappa : id \Rightarrow R$ such that $\kappa_X : X \to R(X)$ is a trivial cofibration in $SeSp_n^{inj}$. Note once again that many potential such functors may exist; we have chosen one for our needs.

The functor that we will construct relies on Lemma 2.2.79, which asserts that, for a map $f : X \to Y$ between two Kan complexes X and Y, there is a natural factorization

$$X \xrightarrow{i} X \times_Y Y^{\mathbf{nerve}(I[1])} \xrightarrow{p} Y$$

where the middle object has as its 0-simplices pairs $(x \in X_0, q : \mathbf{nerve}(I[1]) \to Y)$ such that $q(0) = f(x)$. On 0-simplices, i sends $x \in X_0$ to the pair $(x, c_{f(x)})$ where $c_{f(x)} : \mathbf{nerve}(I[1]) \to Y$ is the constant path on $f(x)$, while p is the 'target map' sending $(x, p) \mapsto p(1)$. What is key is that i is always a trivial cofibration, while p is always a fibration.

Now, suppose $X : \mathscr{C} \to \mathbf{sSet}$ is some projective fibrant functor from a Reedy category $\mathscr{C}$. In order to obtain our Reedy fibrant replacement $R(X)$, we have established the need to define $R(X)(c)$ inductively as a factorization

$$X(c) \sqcup_{L_c X} L_c R(X) \to R(X)(c) \to M_c R(X).$$

Our proposal is to set $R(X)(c)$ to be the pullback

$$\begin{tikzcd}
R(X)(c) \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & (M_c R(X))^{\mathbf{nerve}(I[1])} \arrow[d] \\
X(c) \arrow[r] & M_c X \arrow[r] & M_c R(X)
\end{tikzcd}$$

which is the factorization of the map $X(c) \to M_c R(X)$, a map we assume to be between two Kan complexes by induction. The induced map $X(c) \to R(X)(c)$ by both $1_{X(c)}$ and the inductively defined map

$$X(c) \to M_c X \to M_c R(X) \to (M_c R(X))^{\mathbf{nerve}(I[1])}$$

forms the levels of the levelwise trival cofibration κ_X. It is then by design that the matching maps $R(X)(c) \to M_c R(X)$ are fibrations, so that we have Reedy fibrancy.

Of course, this is not sufficient to induce a valid Reedy fibrant replacement: we require a *latching map* $L_cR(X) \to R(X)(c)$ that commutes with $X(c) \to R(X)(c)$ on L_cX. It is this last step that underpins the majority of the infrastructure behind our construction. In essence, we produce an inductive series of degenerate homotopies that allow for a suitable latching map defined by setting the new paths in $R(X)(c)$ not present in $L_cR(X)$ to be constant.

As a simple example of R in action, consider a levelwise fibrant functor

$$A : \Delta^{op}_{\leq 1} \to \mathbf{sSet}$$

where $\Delta_{\leq 1}$ is the full subcategory of Δ consisting only of objects of degree ≤ 1. Thus, A may be interpreted as a 1-skeletal simplicial space.

The case of $R(A)(0)$ is straightforward: we simply set this to be $A(0)$. For the case of $R(A)(1)$, we have that $M_1R(A) \cong A(0)^2$, so that

$$R(A)(1) := A(1) \times_{A(0)^2} (A(0)^2)^{\mathbf{nerve}(I[1])}.$$

A typical 0-simplex in this space is a triple (p, f, q), where $f \in A(1)_0$ and $p, q : \mathbf{nerve}(I[1]) \to A(0)$ are such that $p(0) = A(\langle 0\rangle)(f)$ and $q(0) = A(\langle 1\rangle)(f)$. Considering A as the truncation of what may be a Segal space, we might interpret such a triple as a 1-morphism f with paths p and q out of its source and target objects. Should we have completeness, we could interpret these paths as equivalences appended onto f on either side.

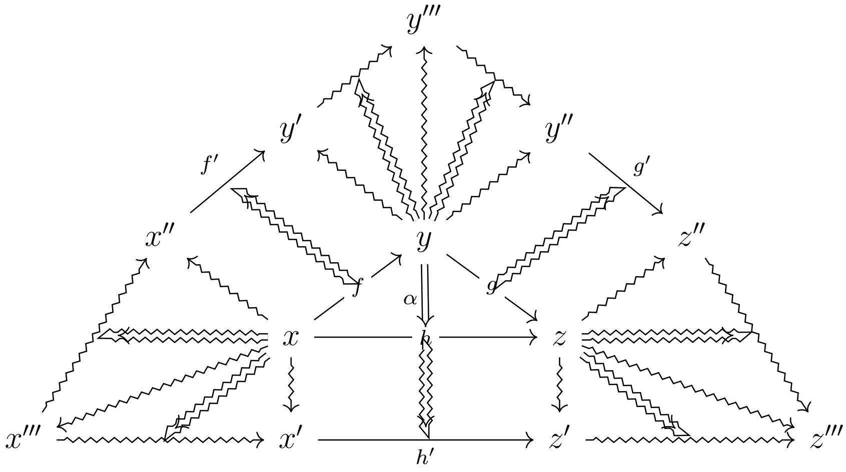
We could visualize such a triple (p, f, q) as follows, where the true direction of p has been inverted to induce a single direction of arrows:

$$x' \overset{p}{\rightsquigarrow} x \xrightarrow{f} y \overset{q}{\rightsquigarrow} y'$$

The two face maps $R(A)(1) \to R(A)(0)$ then send (p, f, q) to either $p(1)$ or $q(1)$ respectively. As for the degeneracy map $R(A)(0) \to R(A)(1)$, our approach will result in a map sending x to $(c_x, A_{\langle 0,0\rangle}(x), c_x)$, where c_x is the constant path on x. This could be visualized as simply an arrow of the form

$$x \xrightarrow{f} y$$

The data of this Reedy fibrant replacement functor becomes inductively more and more complicated in higher dimensions as more paths of paths are appended at each level, an issue we will explore in due course. For instance, if we were to consider extending A to $\Delta^{op}_{\leq 2}$, we would have that a typical 0-simplex of $R(A)(2)$ would consist of a pair (α, p) of a 0-simplex α of $A(2)$ and a path p in $M_2R(A)$ such that $p(0)$ agrees with the boundary of α. We can visualize such an object as follows:



If we should then extend A further to $\Delta^{op}_{\leq 3}$, a full description of $R(A)(3)$ will turn out to be somewhat intractable, requiring several pages to naïvely unravel the induction in full. We hope to eventually produce a simpler description of R that is not inductive at all; indeed, we believe for a Reedy category $\mathscr{C}$, object $c \in \mathscr{C}$ and $X : \mathscr{C} \to \mathbf{sSet}$ projective fibrant that $R(X)(c)$ is in fact a homotopy limit over the diagram $(c \downarrow \mathscr{C}^-)$, due to similarities with the construction of homotopy limits in [Hir09, Def. 18.1.8] and for reasons we will discuss in due course. We leave this analysis to future work.

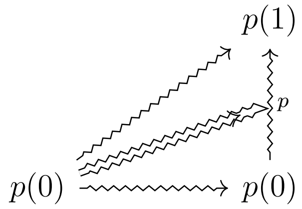
Thankfully, in many cases this complexity can be ignored; in the case of projective fibrant 2-fold Segal spaces, we will find that one may replace the resulting infinite number of higher paths with a much simpler structure, yielding an elegant description of the resulting homotopy bicategory of the Reedy fibrant replacement.

A necessary and appealing consequence of our construction R for obtaining homotopy bicategories is that R will always preserve locality. Thus, for instance, our functor will transmit a projective fibrant 2-fold Segal space to a Reedy fibrant 2-fold Segal space and will similarly preserve completeness. Its generality implies immediate extension to other models of (∞, n)-category, such as Θ_n-spaces [Rez10]. We do not explore these connections here. Moreover, we have that R preserves finite products and weak equivalences.

It should be noted finally that while it appears Reedy fibrant replacement functors have not been defined more generally in a fully explicit manner without the small object argument, a general Reedy *cofibrant* replacement functor where the underlying model category is **sSet** has been defined by Horel in [Hor15, Sec. 6.3]. The approach Horel uses seems rather different from our construction in nature and we do not know how the two relate to one another at present, or if there is a dual fibrant replacement functor one may obtain from Horel's aproach. Such considerations will be left as future work.

4.1 Paths in Path Spaces

For the purposes of this chapter, we will find utility in a few minor constructions worth establishing before we proceed further. In particular, we will need a standard homotopy of path spaces that retracts a path down to its source. We visualize this homotopy as, given a path $p : \mathbf{nerve}(I[1]) \to X$, producing a path of paths $D_X(p) : \mathbf{nerve}(I[1]) \to X^{\mathbf{nerve}(I[1])}$ of the form



More formally, this can be given by a map

$$\mathbf{nerve}(I[1]) \times \mathbf{nerve}(I[1]) \cong \mathbf{nerve}(I[1] \times I[1]) \xrightarrow{\mathbf{nerve}(Q)} \mathbf{nerve}(I[1]) \xrightarrow{p} X$$

where $Q : I[1] \times I[1] \to I[1]$ is the map sending $(i, j) \mapsto i \times j$ for $i, j \in \{0, 1\} = \mathbf{ob}(I[1])$. Visually, if we depict $I[1] \times I[1]$ as the category

$$\begin{tikzcd}
(0,0) \arrow[r, "\cong"] \arrow[d, "\cong"'] & (1,0) \arrow[d, "\cong"] \\
(0,1) \arrow[r, "\cong"'] & (1,1)
\end{tikzcd}$$

then its image under Q corresponds to the diagram in $I[1]$ of the form

$$\begin{tikzcd}
0 \arrow[r, "1_0"] \arrow[d, "1_0"'] & 0 \arrow[d, "\cong"] \\
0 \arrow[r, "\cong"'] & 1
\end{tikzcd}$$

This evidently induces the desired homotopy of paths; the image of the leftmost vertical map under postcomposition with p will be the constant path on $p(0)$, while the image of the rightmost will be p itself.

We formalize this discussion as follows:

Definition 4.1.1. *Suppose $B \in \boldsymbol{sSet}$. Then define*

$$D_B : B^{\boldsymbol{nerve}(I[1])} \to \left(B^{\boldsymbol{nerve}(I[1])}\right)^{\boldsymbol{nerve}(I[1])}$$

to be the map defined by precomposing the isomorphism

$$B^{\boldsymbol{nerve}(I[1]\times I[1])} \cong \left(B^{\boldsymbol{nerve}(I[1])}\right)^{\boldsymbol{nerve}(I[1])}$$

with $B^{\boldsymbol{nerve}(Q)}$*, where* $Q : I[1] \times I[1] \to I[1]$ *is the map defined such that*

$$Q(i,j) := i \times j$$

for $i, j \in \{0,1\} = \boldsymbol{ob}(I[1])$*.*

Proposition 4.1.2. D^B *defines a very good right homotopy to* $1_{B^{\boldsymbol{nerve}(I[1])}}$ *from the map*

$$B^{\boldsymbol{nerve}(I[1])} \xrightarrow{s_B} B \xrightarrow{c_B} B^{\boldsymbol{nerve}(I[1])}.$$

Proof. We consider the two projections

$$B^{\mathbf{nerve}(I[1])} \xrightarrow{D_B} \left(B^{\mathbf{nerve}(I[1])}\right)^{\mathbf{nerve}(I[1])} \to B^{\mathbf{nerve}(I[1])}$$

in turn, where the latter map is set either to $s_{B^{\mathbf{nerve}(I[1])}}$ or $t_{B^{\mathbf{nerve}(I[1])}}$. They are each given by precomposition with functors

$$I[1] \to I[1] \times I[1] \xrightarrow{Q} I[1]$$

where the first functor is either of the form $id \times \{0\}$ or $id \times \{1\}$. It is clear then that the first map sends $i \mapsto 0$, while the second sends $i \mapsto i$, as needed. □

We will make use of this homotopy repeatedly to prove that the constant maps $X \to R(X)$ given by constant paths are weak equivalences; in fact, we will construct a partial deformation retract thereof. Returning to the visual depiction of a 0-simplex (p, f, q) in $R(A)(1)$ for $A : \Delta^{op}_{\leq 1} \to \mathbf{sSet}$ projective fibrant of the form

$$x' \overset{p}{\rightsquigarrow} x \xrightarrow{f} y \overset{q}{\rightsquigarrow} y'$$

we will produce a map $R(A)(1) \to A(1)$ sending $(p, f, q) \mapsto f$ and a homotopy built from $D_\bullet$ that deforms $R(A)(1) \to A(1) \to R(A)(1)$ to $1_{R(A)(1)}$, by gradually extending the paths p and q to their full lengths.

Another fact about $D_\bullet$ is also necessary:

Proposition 4.1.3. *The map*

$$B \xrightarrow{c_B} B^{\boldsymbol{nerve}(I[1])} \xrightarrow{D_B} \left(B^{\boldsymbol{nerve}(I[1])}\right)^{\boldsymbol{nerve}(I[1])}$$

is a constant homotopy from c_B *to* c_B*.*

Proof. Unwinding the definitions of c_B and D_B reveals this map is induced by the terminal map

$$I[1] \times I[1] \to I[1] \to *.$$

□

4.2 Reedy Fibrant Replacement Functors

Our goal in this section is to construct the Reedy fibrant replacement of a general levelwise fibrant functor

$$X : \mathscr{C} \to \mathbf{sSet}$$

for some Reedy category $\mathscr{C}$. That is to say, we seek a new functor $R(X) : \mathscr{C} \to \mathbf{sSet}$ such that for all $c \in \mathscr{C}$, the map $R(X)(c) \to M_c R(X)$ is a fibration of simplicial sets, together with a levelwise weak equivalence $\kappa_X : X \to R(X)$.

In total, we will obtain a functor

$$R : (\mathbf{sSet}^{\mathscr{C}})_{proj} \to (\mathbf{sSet}^{\mathscr{C}})_{proj}$$

where the domain and codomain are the full subcategories of *projective*, namely *levelwise*, fibrant functors, such that $R(X)$ is Reedy fibrant for all X. We will also find a natural transformation $\kappa : id \Rightarrow R$ such that for each X, $\kappa_X : X \to R(X)$ is levelwise a weak equivalence.

Moreover, setting R^+ to be the postcomposition with the functor i^* defined by precomposition with $i : \mathscr{C}^+ \hookrightarrow \mathscr{C}$ of the form

$$\begin{tikzcd}
(\mathbf{sSet}^{\mathscr{C}})_{proj} \arrow[r, "R"] \arrow[dr, "R^+"'] & (\mathbf{sSet}^{\mathscr{C}})_{proj} \arrow[d, "i^*"] \\
 & (\mathbf{sSet}^{\mathscr{C}^+})_{proj}
\end{tikzcd}$$

and $\kappa^+ : i^* \Rightarrow R^+$ to be the whiskering

$$\begin{tikzcd}
(\mathbf{sSet}^{\mathscr{C}})_{proj} \arrow[r, bend left, "id"{name=U}] \arrow[r, bend right, "R"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\kappa"] & (\mathbf{sSet}^{\mathscr{C}})_{proj} \arrow[r, "i^*"] & (\mathbf{sSet}^{\mathscr{C}^+})_{proj}
\end{tikzcd}$$

we will then devise a natural transformation $\alpha : R^+ \Rightarrow i^*$ such that $\alpha\kappa^+ = 1_{i^*}$. For instance, if $\mathscr{C} = \Delta^{op}$, we have that the restriction of a simplicial space X to $\mathscr{C}^+$ forgets the face maps and leaves us with only degeneracies, so that R^+, α and κ^+ would act on this restricted object.

While we will not necessarily have conversely that $\kappa^+\alpha = 1_{R^+}$, we will construct a natural transformation

$$H : R^+ \Rightarrow (-)^{\mathbf{nerve}(I[1])} \circ R^+$$

which yields for each $X \in (\mathbf{sSet}^{\mathscr{C}})_{proj}$ and each $c \in \mathscr{C}$ a right homotopy

$$(H_X)(c) : (\kappa_X^+)(c) \circ (\alpha_X)(c) \sim 1_{R^+(X)(c)}.$$

We will later specialize our construction to the case $\mathscr{C} = (\Delta^{op})^n$ for $n \in \{1, 2\}$ in order to take Reedy fibrant replacements of projective fibrant n-fold Segal spaces.

The entirety of our construction largely hinges on Lemma 2.2.79. We will repeatedly apply this factorization in higher and higher degrees to force maps to matching objects to be fibrations. This alone does not explain how we will obtain maps from latching objects; to do so, we require α and H.

Notation 4.2.1. *Write $\mathscr{C}^{\leq n}$ for the full subcategory of a Reedy category $\mathscr{C}$ whose objects are those of degree less than or equal to n.*

In particular, note that $\mathscr{C}^{\leq 0}$ is a discrete category [Hir09, Ex. 15.1.23]. We will henceforth write as a minor abuse of notation $i^* : (\mathbf{sSet}^{\mathscr{C}^{\leq n}})_{proj} \to (\mathbf{sSet}^{(\mathscr{C}^{\leq n})^+})_{proj}$ for all n.

Notation 4.2.2. *Suppose $X : \mathscr{C} \to \boldsymbol{sSet}$ is functor from some category $\mathscr{C}$. Write $X^{\boldsymbol{nerve}(I[1])} : \mathscr{C} \to \boldsymbol{sSet}$ for the evident functor such that*

$$c \mapsto X(c)^{\boldsymbol{nerve}(I[1])}$$

with maps induced by postcomposition.

For the rest of this section, assume that $\mathscr{C}$ is some fixed Reedy category.

Notation 4.2.3. *Write X^+ for the restriction of $X : \mathscr{C} \to \boldsymbol{sSet}$ along the inclusion $\mathscr{C}^+ \subseteq \mathscr{C}$. Similarly, write $\gamma^+ : X^+ \Rightarrow Y^+$ for the whiskering of a natural transformation $\gamma : X \Rightarrow Y$ between such functors with this inclusion.*

We first construct R, κ, α and H for objects of degree 0:

Definition 4.2.4. *Suppose $X : \mathscr{C}^{\leq 0} \to \boldsymbol{sSet}$ is levelwise fibrant. Let $c \in \mathscr{C}^{\leq 0}$. Define:*

- *$R^{\leq 0}(X)(c) := X(c)$;*
- *$(\kappa_X^{\leq 0})(c) : X(c) \to R^{\leq 0}(X)(c)$ to be the identity;*
- *$(\alpha_X^{\leq 0})(c) : R^{\leq 0}(X)^+(c) \to X^+(c)$ to be the identity;*
- *$(H_X^{\leq 0})(c) : (\kappa_X^{\leq 0})^+(c) \circ (\alpha_X^{\leq 0})(c) \sim 1_{R^{\leq 0}(X)^+(c)}$ to be the trivial homotopy.*

These together define

$$R^{\leq 0} := 1_{(\boldsymbol{sSet}^{\mathscr{C}^{\leq 0}})_{proj}} : (\boldsymbol{sSet}^{\mathscr{C}^{\leq 0}})_{proj} \to (\boldsymbol{sSet}^{\mathscr{C}^{\leq 0}})_{proj}$$

and $\kappa_X^{\leq 0} : id \Rightarrow R^{\leq 0}$, $\alpha_X^{\leq 0} : (R^{\leq 0})^+ \Rightarrow i^*$ and $H_X^{\leq 0} : (\kappa_X^{\leq 0})^+ \alpha_X^{\leq 0} \sim 1_{R^{\leq 0}(X)^+}$.

It is clear that these are all valid functors and natural transformations, as $\mathscr{C}^{\leq 0}$ is discrete. Moreover, $\alpha_X^{\leq 0}(\kappa_X^{\leq 0})^+ = id$ by definition.

Before we can proceed to the inductive case, we need to establish a few constructions:

Notation 4.2.5. *Suppose $X : \mathscr{C}^{\leq n} \to \boldsymbol{sSet}$ is levelwise fibrant, where $n > 0$. Suppose that, writing X to also mean $X^{\leq n-1} : \mathscr{C}^{\leq n-1} \to \boldsymbol{sSet}$, the functor and maps*

$$\begin{aligned} R^{\leq n-1}(X) &: \mathscr{C}^{\leq n-1} \to \boldsymbol{sSet} \\ \kappa_X^{\leq n-1} &: X^{\leq n-1} \Rightarrow R^{\leq n-1}(X) \\ \alpha_X^{\leq n-1} &: R^{\leq n-1}(X)^+ \Rightarrow (X^+)^{\leq n-1} \\ H_X^{\leq n-1} &: (\kappa_X^{\leq n-1})^+ \alpha_X^{\leq n-1} \sim 1_{R^{\leq n-1}(X)^+} \end{aligned}$$

are all defined to be natural in X and such that $\alpha_X^{\leq n-1}(\kappa_X^{\leq n-1})^+ = 1_{(X^+)^{\leq n-1}}$. Suppose $c \in \mathscr{C}$ such that $\boldsymbol{deg}(c) = n$.

Then write

$$\begin{aligned} M_c\kappa_X^{\leq n-1} &: M_cX = M_cX^{\leq n-1} \to M_cR^{\leq n-1}(X) \\ L_c\kappa_X^{\leq n-1} &: L_cX = L_cX^{\leq n-1} \to L_cR^{\leq n-1}(X) \end{aligned}$$

for the natural maps defined via $\kappa_X^{\leq n-1}$.

Note that this notation is not just an application of the functors $M_c, L_c : \mathbf{sSet}^{\mathscr{C}^{\leq n}} \to \mathbf{sSet}$, as $R^{\leq n-1}(X)$ and $\kappa_X^{\leq n-1}$ are not defined on all of $\mathscr{C}^{\leq n}$. Indeed, we find that M_c and L_c can be instead defined on the domain category $\mathbf{sSet}^{\mathscr{C}^{\leq n-1}}$.

We will need to be more careful with inducing maps on latching objects via α and H, as these are moreover only defined on $(\mathscr{C}^+)^{\leq n-1}$. We will write $\mathscr{L}_c$ instead of L_c in these cases, to remind ourselves there is no map $R^{\leq n-1}(X) \to X^{\leq n-1}$ nor homotopy $R^{\leq n-1}(X) \to R^{\leq n-1}(X)^{\mathbf{nerve}(I[1])}$ we are starting with.

Definition 4.2.6. *Suppose $\mathscr{C}$ is a Reedy category with $c \in \mathscr{C}$ such that $\boldsymbol{deg}(c) = n$. Then define*

$$\mathscr{L}_c : \boldsymbol{sSet}^{(\mathscr{C}^+)^{\leq n-1}} \to \boldsymbol{sSet}$$

to send $X \mapsto colim_{\partial(\mathscr{C}^+ \downarrow c)} X$.

This is indeed well-defined, as the categories $\partial(\mathscr{C}^+ \downarrow c)$ are entirely contained within $(\mathscr{C}^+)^{\leq n-1}$.

Proposition 4.2.7. *Suppose $\mathscr{C}$ is a Reedy category with $c \in \mathscr{C}$ such that $\boldsymbol{deg}(c) = n$. Then the diagram*

$$\begin{tikzcd} \boldsymbol{sSet}^{\mathscr{C}^{\leq n}} \arrow[r, "L_c"] \arrow[d] & \boldsymbol{sSet} \\ \boldsymbol{sSet}^{(\mathscr{C}^{\leq n-1})^+} \arrow[ur, "\mathscr{L}_c"'] & \end{tikzcd}$$

commutes.

Proof. This is by definition. □

We immediately obtain the following:

Proposition 4.2.8. *Assume the conditions of Notation 4.2.5. Then there is a natural map*

$$\mathscr{L}_c\alpha_X^{\leq n-1} : L_c R^{\leq n-1}(X) \to L_c X^{\leq n-1} = L_c X$$

induced by $\alpha_X^{\leq n-1}$.

Proposition 4.2.9. *Assume the conditions of Notation 4.2.5. Then there is a natural map*

$$\mathscr{L}_c H_X^{\leq n-1} : L_c R^{\leq n-1}(X) \to L_c(R^{\leq n-1}(X)^{\boldsymbol{nerve}(I[1])})$$

induced by $H_X^{\leq n-1}$.

Proof. $H^{\leq n-1}$ is defined as a natural transformation of functors

$$\begin{tikzcd} (\mathbf{sSet}^{(\mathscr{C}^{\leq n-1})^+})_{proj} \arrow[r, bend left, "(R^{\leq n-1})^+"] \arrow[r, bend right, "((R^{\leq n-1})^+)^{\mathbf{nerve}(I[1])}"'] \arrow[r, phantom, "H^{\leq n-1}\Downarrow"] & (\mathbf{sSet}^{(\mathscr{C}^{\leq n-1})^+})_{proj} \end{tikzcd}$$

and so is sufficient to define a map on latching objects as stated. □

Proposition 4.2.10. *Assume the conditions of Notation 4.2.5. Then*

$$L_c\kappa_X^{\leq n-1} \circ \mathscr{L}_c\alpha_X^{\leq n-1} = \mathscr{L}_c\big((\kappa_X^{\leq n-1})^+\alpha_X^{\leq n-1}\big)$$
$$\mathscr{L}_c\alpha_X^{\leq n-1} \circ L_c\kappa_X^{\leq n-1} = \mathscr{L}_c\big(\alpha_X^{\leq n-1}(\kappa_X^{\leq n-1})^+\big).$$

Proof. We have that $L_c\kappa_X^{\leq n-1} = \mathscr{L}_c(\kappa_X^{\leq n-1})^+$. The result then follows. □

We will need a particular homotopy induced by these maps. Note importantly that since $M_c(R^{\leq n-1}(X)^{\mathbf{nerve}(I[1])})$ is defined as a limit, we have a natural isomorphism

$$M_c(R^{\leq n-1}(X)^{\mathbf{nerve}(I[1])}) \cong M_c(R^{\leq n-1}(X))^{\mathbf{nerve}(I[1])}.$$

Notation 4.2.11. *Assume the conditions of Notation 4.2.5. Then write*

$$
\begin{aligned}
L_c(s_{R^{\leq n-1}(X)}), L_c(t_{R^{\leq n-1}(X)}) &: L_c(R^{\leq n-1}(X)^{\boldsymbol{nerve}(I[1])}) \to L_c(R^{\leq n-1}(X)) \\
M_c(s_{R^{\leq n-1}(X)}), M_c(t_{R^{\leq n-1}(X)}) &: M_c(R^{\leq n-1}(X)^{\boldsymbol{nerve}(I[1])}) \to M_c(R^{\leq n-1}(X))
\end{aligned}
$$

for the maps induced by $s_{R^{\leq n-1}(X)}$ and $t_{R^{\leq n-1}(X)}$, respectively.

Proposition 4.2.12. *Assume the conditions of Notation 4.2.5. Then, letting g_0, g_1, g_2, g_3 be latching-matching maps for appropriate functors, the diagram*

$$
\begin{tikzcd}
 & L_cX \arrow[dl, "g_0"'] \arrow[d, "L_c\kappa_X^{\leq n-1}"] & L_cR^{\leq n-1}(X) \arrow[l, "\mathscr{L}\alpha_X^{\leq n-1}"'] \arrow[r, "id"] \arrow[d, "\mathscr{L}_cH_X^{\leq n-1}"'] & L_cR^{\leq n-1}(X) \arrow[d, "id"] \\
M_cX \arrow[dr, "M_c\kappa_X^{\leq n-1}"'] & L_cR^{\leq n-1}(X) \arrow[d, "g_1"] & L_c(R^{\leq n-1}(X)^{\boldsymbol{nerve}(I[1])}) \arrow[l, "L_c(s_{R^{\leq n-1}(X)})"'] \arrow[r, "L_c(t_{R^{\leq n-1}(X)})"] \arrow[d, "g_2"'] & L_cR^{\leq n-1}(X) \arrow[d, "g_3"] \\
 & M_c(R^{\leq n-1}(X)) \arrow[d, "id"'] & M_c(R^{\leq n-1}(X)^{\boldsymbol{nerve}(I[1])}) \arrow[l, "M_c(s_{R^{\leq n-1}(X)})"] \arrow[r, "M_c(t_{R^{\leq n-1}(X)})"'] \arrow[d, "\cong"'] & M_c(R^{\leq n-1}(X)) \arrow[d, "id"] \\
 & M_c(R^{\leq n-1}(X)) & (M_cR^{\leq n-1}(X))^{\boldsymbol{nerve}(I[1])} \arrow[l, "s_{M_cR^{\leq n-1}(X)}"] \arrow[r, "t_{M_cR^{\leq n-1}(X)}"'] & M_c(R^{\leq n-1}(X))
\end{tikzcd}
$$

commutes. Moreover, this diagram is natural in X.

Proof. This is a matter of checking definitions. □

We are now ready to commence the inductive step to constructing our Reedy fibrant replacement functor:

Definition 4.2.13. *Assume the conditions of Notation 4.2.5. Then define $R^{\leq n}|_{\mathscr{C}^{\leq n-1}} := R^{\leq n-1}$. This is thus sufficient to build $M_cR^{\leq n}(X)$ and a map $M_cX \to M_cR^{\leq n}(X)$ via $M_c\kappa_X^{\leq n-1}$. Hence, define $R^{\leq n}(X)(c)$ to be the pullback*

$$
\begin{tikzcd}
R^{\leq n}(X)(c) \arrow[r] \arrow[dd] \arrow[dr, phantom, very near start, "\ulcorner"] & X(c) \arrow[d] \\
 & M_cX \arrow[d, "M_c\kappa_X^{\leq n-1}"] \\
(M_cR^{\leq n}(X))^{\boldsymbol{nerve}(I[1])} \arrow[r, "s_{M_cR^{\leq n}(X)}"'] & M_cR^{\leq n}(X)
\end{tikzcd}
$$

Moreover, we have a natural map

$$
R^{\leq n}(X)(c) \to (M_cR^{\leq n}(X))^{\boldsymbol{nerve}(I[1])} \xrightarrow{t_{M_cR^{\leq n}(X)}} M_cR^{\leq n}(X)
$$

where the first map is pullback projection. Note moreover that $L_cR^{\leq n}(X)$ is already defined. Then, consider the diagram

$$\begin{tikzcd}
L_cR^{\leq n}(X) \arrow[rr, "\mathscr{L}_c\alpha^{\leq n-1}"] \arrow[dr, dashed] \arrow[dd, "\mathscr{L}_cH_X^{\leq n-1}"'] & & L_cX \arrow[d] \\
& X(c) \times_{M_cR^{\leq n}(X)} (M_cR^{\leq n}(X))^{\boldsymbol{nerve}(I[1])} \arrow[r] \arrow[dd] & X(c) \arrow[d] \\
L_c(R^{\leq n}(X)^{\boldsymbol{nerve}(I[1])}) \arrow[d] & & M_cX \arrow[d, "M_c\kappa_X^{\leq n-1}"] \\
M_c(R^{\leq n}(X)^{\boldsymbol{nerve}(I[1])}) \arrow[r, "\cong"] & (M_cR^{\leq n}(X))^{\boldsymbol{nerve}(I[1])} \arrow[r, "s_{M_cR^{\leq n}(X)}"] & M_cR^{\leq n}(X)
\end{tikzcd}$$

This commutes by Proposition 4.2.12, so yields a map from the latching object. As the map

$$\begin{aligned}
L_cR^{\leq n}(X) \xrightarrow{\mathscr{L}_cH_X^{\leq n-1}} L_c(R^{\leq n}(X)^{\boldsymbol{nerve}(I[1])}) &\to M_c(R^{\leq n}(X)^{\boldsymbol{nerve}(I[1])}) \\
&\to (M_cR^{\leq n}(X))^{\boldsymbol{nerve}(I[1])} \\
&\xrightarrow{t_{M_cR^{\leq n}(X)}} M_cR^{\leq n}(X)
\end{aligned}$$

is precisely the latching-matching map by Proposition 4.2.12, we have that the composition

$$L_cR^{\leq n}(X) \to R^{\leq n}(X)(c) \to M_cR^{\leq n}(X)$$

is precisely the latching-matching map. This is thus sufficient by Proposition 2.2.12 to define $R^{\leq n}(X)$.

A natural definition of $R^{\leq n}(f)$ for $f : X \to Y$ results in the functor

$$R^{\leq n} : (\boldsymbol{sSet}^{\mathscr{C}^{\leq n}})_{proj} \to (\boldsymbol{sSet}^{\mathscr{C}^{\leq n}})_{proj}.$$

The fact that $R^{\leq n}$ is functorial is due to Proposition 4.2.12, in particular the fact that the diagram in question is natural in X. This allows us to construct morphisms $R^{\leq n}(f)$ using Proposition 2.2.13.

Before we proceed further, note that we will not be able to prove our inductive definition of $\kappa_X^{\leq n}$ is a natural transformation until we have established the corresponding inductive definition of $H_X^{\leq n}$. However, we are able to define its levels without yet having naturality:

Definition 4.2.14. *Assume the conditions of Notation 4.2.5. Then define $(\kappa_X^{\leq n})(c) : X(c) \to R^{\leq n}(X)(c)$ by the identity on $X(c)$ together with the map*

$$X(c) \to M_cX \xrightarrow{M_c\kappa_X^{\leq n-1}} M_cR^{\leq n}(X) \xrightarrow{c_{M_cR^{\leq n}(X)}} (M_cR^{\leq n}(X))^{\boldsymbol{nerve}(I[1])}.$$

Moreover, define $(\kappa_X^{\leq n})(d) := (\kappa_X^{\leq n-1})(d)$ for $\boldsymbol{deg}(d) \leq n-1$.

Definition 4.2.15. *Assume the conditions of Notation 4.2.5. Then define* $(\alpha_X^{\leq n})(c) : R^{\leq n}(X)(c) \to X(c)$ *by the pullback projection to* $X(c)$*. Moreover, define* $(\alpha_X^{\leq n})(d) := (\alpha_X^{\leq n-1})(d)$ *for* $\boldsymbol{deg}(d) \leq n-1$*.*

Some explanation is likely needed for why α_X is only defined on $\mathscr{C}^+$ rather than all of $\mathscr{C}$. We should indeed not expect a retract $R(X) \to X$ of κ_X for all of X, as such a map would imply that the matching maps $X(c) \to M_c X$ would be retracts of the maps $R(X)(c) \to M_c R(X)$. Indeed, we would have diagrams of the form

$$\begin{tikzcd} X(c) \arrow[r, "\kappa_X(c)"'] \arrow[d] \arrow[rr, bend left, "id"] & R(X)(c) \arrow[r, "\alpha_X(c)"'] \arrow[d] & X(c) \arrow[d] \\ M_c X \arrow[r, "M_c \kappa_X"] \arrow[rr, bend right, "id"'] & M_c R(X) \arrow[r, "M_c \alpha_X"] & M_c X \end{tikzcd}$$

As fibrations are closed under retract, this would imply that X was already Reedy fibrant, which is not generally the case. Thus, it must not be possible in general to define $M_c \alpha_X$ and therefore to define α_X on $\mathscr{C}^-$. Regardless, it is perfectly reasonable to define α_X on $\mathscr{C}^+$ as we will prove later, which is more than sufficient for our purposes.

Note that in the case of $\mathscr{C} = (\Delta^{op})^n$, the category $\mathscr{C}^+$ consists of solely the degeneracy maps.

We now turn to defining our homotopies. Note that $(R(X)^{\leq n}(c))^{\mathbf{nerve}(I[1])}$ may alternatively be written as

$$X(c)^{\mathbf{nerve}(I[1])} \times_{(M_c R^{\leq n}(X))^{\mathbf{nerve}(I[1])}} \left((M_c R^{\leq n}(X))^{\mathbf{nerve}(I[1])}\right)^{\mathbf{nerve}(I[1])}.$$

Definition 4.2.16. *Assume the conditions of Notation 4.2.5. Then define* $(H_X^{\leq n})(c) : R^{\leq n}(X)(c) \to (R^{\leq n}(X)(c))^{\boldsymbol{nerve}(I[1])}$ *to be induced by the map of cospans*

$$\begin{tikzcd} (M_c R^{\leq n}(X))^{\boldsymbol{nerve}(I[1])} \arrow[r] \arrow[d, "D_{M_c R^{\leq n}(X)}"'] & M_c R^{\leq n}(X) \arrow[d, "c_{M_c R^{\leq n}(X)}"'] & X(c) \arrow[l] \arrow[d, "c_{X(c)}"] \\ \left((M_c R^{\leq n}(X))^{\boldsymbol{nerve}(I[1])}\right)^{\boldsymbol{nerve}(I[1])} \arrow[r] & (M_c R^{\leq n}(X))^{\boldsymbol{nerve}(I[1])} & X(c)^{\boldsymbol{nerve}(I[1])} \arrow[l] \end{tikzcd}$$

Moreover, define $(H_X^{\leq n})(d) := (H_X^{\leq n-1})(d)$ *for* $\boldsymbol{deg}(d) \leq n-1$*.*

We now must prove that $\kappa^{\leq n}, \alpha^{\leq n}$ and $H^{\leq n}$ are natural in $\mathscr{C}$ and in $(\mathbf{sSet}^{\mathscr{C}})_{proj}$. Before we do so, a result is needed:

Proposition 4.2.17. *Assume the conditions of Notation 4.2.5. Then the map $(H^{\leq n}_X)(c) \circ (\kappa^{\leq n}_X)^+(c)$ is the constant homotopy on $(\kappa^{\leq n}_X)^+(c)$.*

Proof. We have that this map is defined by the pullback of the two composite maps

$$X(c) \xrightarrow{(\kappa^{\leq n}_X)^+(c)} X(c) \times_{M_cR^{\leq n}(X)} (M_cR^{\leq n}(X))^{\mathbf{nerve}(I[1])} \to X(c) \xrightarrow{c_{X(c)}} X(c)^{\mathbf{nerve}(I[1])}$$

and

$$\begin{aligned} X(c) &\xrightarrow{(\kappa^{\leq n}_X)^+(c)} X(c) \times_{M_cR^{\leq n}(X)} (M_cR^{\leq n}(X))^{\mathbf{nerve}(I[1])} \\ &\to (M_cR^{\leq n}(X))^{\mathbf{nerve}(I[1])} \\ &\xrightarrow{D_{M_cR^{\leq n}(X)}} \big((M_cR^{\leq n}(X))^{\mathbf{nerve}(I[1])}\big)^{\mathbf{nerve}(I[1])}. \end{aligned}$$

The former of these maps is evidently just c_X. The latter is then the map

$$\begin{aligned} X(c) \to M_cX \xrightarrow{M_c\kappa^{\leq n-1}_X} M_cR^{\leq n}(X) &\xrightarrow{c_{M_cR^{\leq n}(X)}} (M_cR^{\leq n}(X))^{\mathbf{nerve}(I[1])} \\ &\xrightarrow{D_{M_cR^{\leq n}(X)}} \big((M_cR^{\leq n}(X))^{\mathbf{nerve}(I[1])}\big)^{\mathbf{nerve}(I[1])} \end{aligned}$$

which by Proposition 4.1.3 is the map

$$\begin{aligned} X(c) \to M_cX \to M_cR^{\leq n}(X) &\xrightarrow{c_{M_cR^{\leq n}(X)}} (M_cR^{\leq n}(X))^{\mathbf{nerve}(I[1])} \\ &\xrightarrow{c_{(M_cR^{\leq n}(X))^{\mathbf{nerve}(I[1])}}} \big((M_cR^{\leq n}(X))^{\mathbf{nerve}(I[1])}\big)^{\mathbf{nerve}(I[1])}. \end{aligned}$$

Thus, we have a commutative diagram

$$\begin{tikzcd}
X(c) \arrow[rr, "id"] \arrow[dr, "(\kappa^{\leq n}_X)^+(c)"'] \arrow[dd] & & X(c) \arrow[dd, "c_{X(c)}"] \\
& R^{\leq n}(X)(c) \arrow[d, dashed, "(H^{\leq n}_X)(c)"'] & \\
M_cX \arrow[d, "M_c\kappa^{\leq n-1}_X"'] & R^{\leq n}(X)(c)^{\mathbf{nerve}(I[1])} \arrow[r] \arrow[dd] \arrow[ddr, phantom, "\lrcorner", very near start] & X(c)^{\mathbf{nerve}(I[1])} \arrow[dd] \\
M_cR^{\leq n}(X) \arrow[d, "c_{M_cR^{\leq n}(X)}"'] & & \\
(M_cR^{\leq n}(X))^{\mathbf{nerve}(I[1])} \arrow[r, "c_{(M_cR^{\leq n}(X))^{\mathbf{nerve}(I[1])}}"'] & \big((M_cR^{\leq n}(X))^{\mathbf{nerve}(I[1])}\big)^{\mathbf{nerve}(I[1])} \arrow[r] & (M_cR^{\leq n}(X))^{\mathbf{nerve}(I[1])}
\end{tikzcd}$$

which means the composite $(H^{\leq n}_X)(c) \circ (\kappa^{\leq n}_X)^+(c)$ is the constant homotopy on $(\kappa^{\leq n}_X)^+(c)$, as needed. $\square$

Proposition 4.2.18. *Assume the conditions of Notation 4.2.5. Then* $\kappa_X^{\leq n} : X \to R^{\leq n}(X)$ *is a valid morphism.*

Proof. By Proposition 2.2.13, it suffices to show that the diagram

$$\begin{tikzcd}
L_c X^{\leq n} \arrow[r] \arrow[d, "L_c \kappa_X^{\leq n}"'] & X(c) \arrow[r] \arrow[d, "(\kappa_X^{\leq n})(c)"'] & M_c X \arrow[d, "M_c \kappa_X^{\leq n}"] \\
L_c R^{\leq n}(X) \arrow[r] & R^{\leq n}(X)(c) \arrow[r] & M_c R^{\leq n}(X)
\end{tikzcd}$$

commutes. It is immediate that the right-hand square commutes. For the left-hand side, it suffices to show that the two diagrams

$$\begin{tikzcd}
L_c X \arrow[r] \arrow[d, "L_c \kappa_X^{\leq n}"'] & X(c) \arrow[dd, "id"] \\
L_c R^{\leq n}(X) \arrow[d, "\mathscr{L}_c \alpha_X^{\leq n}"'] & \\
L_c X \arrow[r] & X(c)
\end{tikzcd}$$

and

$$\begin{tikzcd}
L_c X \arrow[rr] \arrow[d, "L_c \kappa_X^{\leq n}"'] & & X(c) \arrow[d] \\
L_c R^{\leq n}(X) \arrow[d, "\mathscr{L}_c H_X^{\leq n}"'] & & M_c X \arrow[d, "M_c \kappa_X^{\leq n}"] \\
L_c(R^{\leq n}(X)^{\mathbf{nerve}(I[1])}) \arrow[d] & & M_c R^{\leq n}(X) \arrow[d] \\
M_c(R^{\leq n}(X)^{\mathbf{nerve}(I[1])}) \arrow[rr, "\cong"] & & (M_c R^{\leq n}(X))^{\mathbf{nerve}(I[1])}
\end{tikzcd}$$

commute. The former commutes because $\alpha_X^{\leq n-1}(\kappa_X^{\leq n-1})^+ = 1_{(X^+)^{\leq n-1}}$ by induction, while the latter commutes because

$$H_X^{\leq n-1}(\kappa_X^{\leq n-1})^+ : (\kappa_X^{\leq n-1})^+ \sim (\kappa_X^{\leq n-1})^+$$

is levelwise the trivial homotopy by induction with Proposition 4.2.17. □

Proposition 4.2.19. *Assume the conditions of Notation 4.2.5. Then* $\alpha_X^{\leq n} : R^{\leq n}(X)^+ \to (X^{\leq n})^+$ *is a valid morphism.*

Proof. By Proposition 2.2.13 and the fact that $(\mathscr{C}^+)^{\leq n}$ exhibits trivial matching objects, defining such a functor requires only a commutative diagram

$$\begin{tikzcd}
L_c R(X)^{\leq n} \arrow[r] \arrow[d, "\mathscr{L}_c \alpha_X^{\leq n}"'] & R(X)^{\leq n}(c) \arrow[d, "\alpha_X^{\leq n}(c)"] \\
L_c X^{\leq n} \arrow[r] & X^{\leq n}(c)
\end{tikzcd}$$

Note however that by the definition of the uppermost horizontal map, this commutativity is trivial. ☐

To prove naturality of $H_X^{\leq n}$, we will need an intermediate result:

Proposition 4.2.20. *Assume the conditions of Notation 4.2.5. Then*

$$(H_X^{\leq n})(c)^{\textbf{\textit{nerve}}(I[1])} \circ (H_X^{\leq n})(c) = D_{R^{\leq n}(X)(c)} \circ (H_X^{\leq n})(c).$$

Proof. Unwinding definitions, we have that $(H_X^{\leq n})(c)^{\mathbf{nerve}(I[1])} \circ (H_X^{\leq n})(c)$ is given by the map of cospans

$$\begin{tikzcd}
(M_c R^{\leq n}(X))^{\mathbf{nerve}(I[1])} \arrow[r] \arrow[d, "D_{M_c R^{\leq n}(X)}"'] & M_c R^{\leq n}(X) \arrow[d, "c_{M_c R^{\leq n}(X)}"'] & X(c) \arrow[l] \arrow[d, "c_{X(c)}"] \\
(M_c R^{\leq n}(X))^{\mathbf{nerve}(I[1]\times I[1])} \arrow[r] \arrow[d, "D^{\mathbf{nerve}(I[1])}_{(M_c R^{\leq n}(X))}"'] & (M_c R^{\leq n}(X))^{\mathbf{nerve}(I[1])} \arrow[d, "c^{\mathbf{nerve}(I[1])}_{(M_c R^{\leq n}(X))}"'] & X(c)^{\mathbf{nerve}(I[1])} \arrow[l] \arrow[d, "c^{\mathbf{nerve}(I[1])}_{X(c)}"] \\
(M_c R^{\leq n}(X))^{\mathbf{nerve}(I[1]\times I[1]\times I[1])} \arrow[r] & (M_c R^{\leq n}(X))^{\mathbf{nerve}(I[1]\times I[1])} & X(c)^{\mathbf{nerve}(I[1]\times I[1])} \arrow[l]
\end{tikzcd}$$

so that the three vertical composites are given by the map $I[1]^3 \to I[1]$ sending $(i, j, k) \mapsto ijk$ and the map $I[1]^2 \to *$. We then see that the map $D_{R^{\leq n}(X)(c)} \circ (H_X^{\leq n})(c)$ is given in turn by the composite

$$\begin{tikzcd}
(M_c R^{\leq n}(X))^{\mathbf{nerve}(I[1])} \arrow[r] \arrow[d, "D_{M_c R^{\leq n}(X)}"'] & M_c R^{\leq n}(X) \arrow[d, "c_{M_c R^{\leq n}(X)}"'] & X(c) \arrow[l] \arrow[d, "c_{X(c)}"] \\
(M_c R^{\leq n}(X))^{\mathbf{nerve}(I[1]\times I[1])} \arrow[r] \arrow[d, "D_{(M_c R^{\leq n}(X))^{\mathbf{nerve}(I[1])}}"'] & (M_c R^{\leq n}(X))^{\mathbf{nerve}(I[1])} \arrow[d, "D_{M_c R^{\leq n}(X)}"'] & X(c)^{\mathbf{nerve}(I[1])} \arrow[l] \arrow[d, "D_{X(c)}"] \\
(M_c R^{\leq n}(X))^{\mathbf{nerve}(I[1]\times I[1]\times I[1])} \arrow[r] & (M_c R^{\leq n}(X))^{\mathbf{nerve}(I[1]\times I[1])} & X(c)^{\mathbf{nerve}(I[1]\times I[1])} \arrow[l]
\end{tikzcd}$$

which is given by precisely the same maps, as needed. ☐

Proposition 4.2.21. *Assume the conditions of Notation 4.2.5. Then* $H_X^{\leq n} : R^{\leq n}(X)^+ \to (R^{\leq n}(X)^+)^{\textbf{\textit{nerve}}(I[1])}$ *is a valid morphism.*

Proof. Again, it suffices to show that the diagram

$$\begin{tikzcd}
L_c R^{\leq n}(X) \arrow[r] \arrow[d, "\mathscr{L}_c H_X^{\leq n}"'] & R^{\leq n}(X)(c) \arrow[d, "H_X^{\leq n}(c)"] \\
L_c(R^{\leq n}(X)^{\mathbf{nerve}(I[1])}) \arrow[r] & R^{\leq n}(X)(c)^{\mathbf{nerve}(I[1])}
\end{tikzcd}$$

commutes.

We subdivide this question by splitting up $(R^{\leq n}(X)(c))^{\mathbf{nerve}(I[1])}$ into $X(c)^{\mathbf{nerve}(I[1])}$ and $\big((M_c R^{\leq n}(X))^{\mathbf{nerve}(I[1])}\big)^{\mathbf{nerve}(I[1])}$. It suffices then to check that the maps into these two objects induce commutative diagrams.

One may then first check that the diagram

$$\begin{tikzcd}
L_c R^{\leq n}(X) \arrow[rr] \arrow[d, "\mathscr{L}_c H_X^{\leq n}"'] & & R^{\leq n}(X)(c) \arrow[d, "\alpha_X^{\leq n}(c)"] \\
L_c(R^{\leq n}(X)^{\mathbf{nerve}(I[1])}) \arrow[r] \arrow[d, "\mathscr{L}_c((\alpha_X^{\leq n})^{\mathbf{nerve}(I[1])})"'] & R^{\leq n}(X)(c)^{\mathbf{nerve}(I[1])} \arrow[dr, "\alpha_X^{\leq n}(c)^{\mathbf{nerve}(I[1])}"'] & X(c) \arrow[d, "c_{X(c)}"] \\
L_c(X^{\mathbf{nerve}(I[1])}) \arrow[rr] & & X(c)^{\mathbf{nerve}(I[1])}
\end{tikzcd}$$

commutes, by unwinding the definition of $H_X^{\leq n}$ inductively and checking first the bottom-left square and then the upper-right commuting diagram.

It then suffices to prove that the diagram

$$\begin{tikzcd}
L_c R^{\leq n}(X) \arrow[rr] \arrow[d, "\mathscr{L}_c H_X^{\leq n}"'] \arrow[dr, "\mathscr{L}_c H_X^{\leq n}"] & & R^{\leq n}(X)(c) \arrow[d] \\
L_c(R^{\leq n}(X)^{\mathbf{nerve}(I[1])}) \arrow[dr, "\mathscr{L}_c((H_X^{\leq n})^{\mathbf{nerve}(I[1])})"'] & L_c(R^{\leq n}(X)^{\mathbf{nerve}(I[1])}) \arrow[r] \arrow[d, "L_c D_{R^{\leq n}(X)(\bullet)}"] & (M_c R^{\leq n}(X))^{\mathbf{nerve}(I[1])} \arrow[d, "D_{M_c R^{\leq n}(X)}"] \\
& L_c\big((R^{\leq n}(X)^{\mathbf{nerve}(I[1])})^{\mathbf{nerve}(I[1])}\big) \arrow[r] & \big((M_c R^{\leq n}(X))^{\mathbf{nerve}(I[1])}\big)^{\mathbf{nerve}(I[1])}
\end{tikzcd}$$

commutes. By Proposition 4.2.20, we have that the leftmost polygon commutes, while the remaining two are by inspection.

That this diagram suffices is because one may show that the diagram

$$\begin{tikzcd}
L_c(R^{\leq n}(X)^{\mathbf{nerve}(I[1])}) \arrow[r] \arrow[d, "\mathscr{L}_c H_X^{\leq n}"'] & R^{\leq n}(X)(c)^{\mathbf{nerve}(I[1])} \arrow[d, two heads] \\
L_c((R^{\leq n}(X)^{\mathbf{nerve}(I[1])})^{\mathbf{nerve}(I[1])}) \arrow[r] & M_c((R^{\leq n}(X)^{\mathbf{nerve}(I[1])})^{\mathbf{nerve}(I[1])})
\end{tikzcd}$$

commutes. $\square$

Proposition 4.2.22. *Assume the conditions of Notation 4.2.5. Then $H_X^{\leq n}, \kappa_X^{\leq n}$ and $\alpha_X^{\leq n}$ are all natural in X.*

Proof. All involved constructions to define these natural transformations are themselves natural in X, so the result inductively holds in n by inspection. □

We complete our induction by proving all the necessary results about $\kappa^{\leq n}, \alpha^{\leq n}$ and $H^{\leq n}$ for the next inductive step in n.

Proposition 4.2.23. *Assume the conditions of Notation 4.2.5. Then*

$$\alpha_X^{\leq n}(\kappa_X^{\leq n})^+ = 1_{(X^+)^{\leq n}}.$$

Proof. This is by definition. □

Proposition 4.2.24. *Assume the conditions of Notation 4.2.5. Then $(H_X^{\leq n})(c)$ is a right homotopy from $(\kappa_X^{\leq n})^+(c) \circ (\alpha_X^{\leq n})(c)$ to $1_{R^{\leq n}(X)^+(c)}$.*

Proof. Consider, for some $c \in \mathscr{C}$, the composite morphism

$$R^{\leq n}(X)(c) \xrightarrow{H_X^{\leq n}(c)} R^{\leq n}(X)(c)^{\mathbf{nerve}(I[1])} \to R^{\leq n}(X)(c)$$

where the final map is either the source or target map. Looking at the components of this map, we have the two maps of the form

$$X(c) \xrightarrow{c_{X(c)}} X(c)^{\mathbf{nerve}(I[1])} \to X(c)$$

which are both identities, along with the two maps of the form

$$(M_cR^{\leq n}(X))^{\mathbf{nerve}(I[1])} \xrightarrow{D_{M_cR^{\leq n}(X)}} ((M_cR^{\leq n}(X))^{\mathbf{nerve}(I[1])})^{\mathbf{nerve}(I[1])} \to (M_cR^{\leq n}(X))^{\mathbf{nerve}(I[1])}.$$

If the last map in this chain of morphisms is the source map, then the overall map $R^{\leq n}(X)(c) \to R^{\leq n}(X)(c)$ is given by the map of cospans

$$\begin{tikzcd}
X(c) \arrow[r] \arrow[d, "id"'] & M_cX \arrow[r, "M_c\kappa_X^{\leq n}"] \arrow[d, "id"'] & M_cR^{\leq n}(X) \arrow[d, "id"] & (M_cR^{\leq n}(X))^{\mathbf{nerve}(I[1])} \arrow[l, "s_{M_cR^{\leq n}(X)}"'] \arrow[d, "s_{M_cR^{\leq n}(X)}"] \\
X(c) \arrow[r] \arrow[d, "id"'] & M_cX \arrow[r, "M_c\kappa_X^{\leq n}"] \arrow[d, "id"'] & M_cR^{\leq n}(X) \arrow[d, "id"] & M_cR^{\leq n}(X) \arrow[l, "id"'] \arrow[d, "c_{M_cR^{\leq n}(X)}"] \\
X(c) \arrow[r] & M_cX \arrow[r, "M_c\kappa_X^{\leq n}"'] & M_cR^{\leq n}(X) & (M_cR^{\leq n}(X))^{\mathbf{nerve}(I[1])} \arrow[l, "s_{M_cR^{\leq n}(X)}"]
\end{tikzcd}$$

It is clear that the first of these maps is $(\alpha_X^{\leq n})(c)$ by the isomorphism $X(c) \cong X(c) \times_{M_cR^{\leq n}(X)} M_cR^{\leq n}(X)$, so that the induced map of pullbacks $R^{\leq n}(X)(c) \to$

$R^{\leq n}(X)(c)$ is precisely the result of applying $(\kappa_X^{\leq n})^+(c) \circ (\alpha_X^{\leq n})(c)$. If instead the final map in the composite is the target map, we have that the induced map $R^{\leq n}(X)(c) \to R^{\leq n}(X)(c)$ the identity. Thus, the homotopy $(H_X^{\leq n})(c)$ is between the correct morphisms as desired. □

With all of these inductive pieces in place, we are able to continue the induction to all n and obtain our construction in full:

Definition 4.2.25. *Suppose $X : \mathscr{C} \to$ **sSet** is levelwise fibrant. Define $R(X)$ such that $R(X)|_{\mathscr{C}^{\leq n}} = R^{\leq n}(X)$. Do similarly for κ_X, α_X and H_X. This defines R, κ, α and H in their entirety.*

This completes the definition of R. We must now prove that we have indeed obtained a valid Reedy fibrant replacement:

Proposition 4.2.26. *$\kappa_X : X \to R(X)$ is a levelwise trivial cofibration.*

Proof. Consider some $c \in \mathscr{C}$ with $\mathbf{deg}(c) = n$. It will suffice to prove that

$$\kappa_X^{\leq n}(c) : X(c) \to X(c) \times_{M_c R^{\leq n}(X)} (M_c R^{\leq n}(X))^{\mathbf{nerve}(I[1])}$$

is a trivial cofibration in **sSet**.

Note that the existence of $\alpha_X^{\leq n}(c)$ implies that this is a levelwise inclusion, hence a cofibration. Moreover, $H_X^{\leq n}(c)$ exhibits $\kappa_X^{\leq n}(c)$ as a weak equivalence in the model structure in **sSet**; indeed, applying geometric realization to the induced map $R^{\leq n}(X)(c) \times \mathbf{nerve}(I[1]) \to R^{\leq n}(X)(c)$ gives a homotopy

$$\begin{aligned} |R^{\leq n}(X)(c)| \times [0,1] &\cong |R^{\leq n}(X)(c)| \times |\Delta[1]| \\ &\to |R^{\leq n}(X)(c)| \times |\mathbf{nerve}(I[1])| \\ &\to |R^{\leq n}(X)(c)| \end{aligned}$$

as geometric realization commutes with products and by the map $\Delta[1] \to \mathbf{nerve}(I[1])$ identifying the morphism $0 \to 1$ in $I[1]$. This homotopy is from $|\kappa_X^{\leq n}(c)| \circ |\alpha_X^{\leq n}(c)|$ to the identity, which together with the fact that $|\alpha_X^{\leq n}(c)| \circ |\kappa_X^{\leq n}(c)| = id$ shows we have a weak equivalence of simplicial sets and therefore a trivial cofibration as needed. □

Note by 2-out-of-3 that α_X is also a levelwise weak equivalence. Moreover, note that we have not shown that κ_X is in general a Reedy trivial cofibration, only a weak equivalence and levelwise cofibration. However, in the case of *elegant* Reedy categories, we do in fact have a Reedy trivial cofibration. This will be the only case we have need for in this thesis.

In order to prove that $R(X)$ is Reedy fibrant, we will need an intermediate result on Reedy model categories:

Proposition 4.2.27 ([Hir09, Cor. 15.3.12])**.** *Let $\mathscr{C}$ be a Reedy category and $\mathscr{M}$ a model category. Suppose $X \in \mathscr{M}^{\mathscr{C}}$ is Reedy fibrant and $x \in \mathscr{C}$. Then $M_x X$ is fibrant in $\mathscr{M}$.*

Proposition 4.2.28. *Suppose X is projective fibrant. Then $R(X)$ is Reedy fibrant.*

Proof. We prove this by induction. Suppose $c \in \mathscr{C}$ has $\mathbf{deg}(c) = 0$. Then $R(X)(c) = X(c)$, so we have Reedy fibrancy at c so that $R^{\leq 0}(X)$ is Reedy fibrant.

For the inductive step, suppose $R^{\leq n-1}(X)$ is Reedy fibrant. Then, following the methods in [Hir09, pg. 293], for every $d \in \mathscr{C}$ of degree n we can factorize the map $X(d) \sqcup_{L_d X} L_d R^{\leq n-1}(X) \to M_d R^{\leq n-1}(X)$ into a trivial cofibration followed by a fibration. This yields a Reedy fibrant functor

$$Q(X) : \mathscr{C}^{\leq n} \to \mathbf{sSet}$$

where $Q(X)|_{\mathscr{C}^{\leq n-1}} = R^{\leq n-1}(X)$. Then we have that $M_c Q(X) = M_c R^{\leq n-1}(X)$ is a Kan complex by Proposition 4.2.27. Thus, we have a map $(\kappa_X)(c) : X(c) \to M_c R(X) = M_c R^{\leq n-1}(X)$ between Kan complexes, which induces the map

$$R(X)(c) = X(c) \times_{M_c R(X)} (M_c R(X))^{\mathbf{nerve}(I[1])} \to M_c R(X)$$

as a fibration by Lemma 2.2.79. □

One final result will be crucial for our construction, which we have not discussed yet: the preservation of locality with respect to some class of morphisms. This will in particular imply that projective fibrant n-fold Segal spaces will be sent to their Reedy fibrant counterparts.

Proposition 4.2.29. *Suppose X is levelwise fibrant and C-local with respect to some collection of morphisms C. Then $R(X)$ is also C-local.*

Proof. Note that κ_X is a levelwise weak equivalence between fibrant objects X and $R(X)$ in the projective model structure on $\mathbf{sSet}^{\mathscr{C}}$. Hence, the result holds by [Hir09, Lemma 3.2.1]. □

Armed with these results, we may now specialize our situation to n-fold Segal spaces, which for us is restricted to the cases $n \in \{1, 2\}$:

Definition 4.2.30. *Let $R : \boldsymbol{SeSp}_n^{proj} \to \boldsymbol{SeSp}_n^{inj}$ be the functor R applied to the case $\mathscr{C} = (\Delta^{op})^n$ and restricted in domain to n-fold Segal spaces for $n \in \{1, 2\}$. Define κ similarly.*

Corollary 4.2.31. *κ is a levelwise cofibration in the Reedy model structure.*

Proof. Cofibrations in the injective model structure are levelwise and $(\Delta^{op})^n$ is an elegant Reedy category. □

We know already that the matching objects for an n-uple simplicial space are of the form $M_{([i_n],\cdots,[i_1])}X \cong X_{\partial\Delta[i_n,\cdots,i_1]}$. Hence, we have that for a projective fibrant n-uple simplicial space X,

$$R(X)_{i_1,\cdots,i_n} \cong X_{i_1,\cdots,i_n} \times_{R(X)_{\partial\Delta[i_n,\cdots,i_1]}} R(X)^{\mathbf{nerve}(I[1])}_{\partial\Delta[i_n,\cdots,i_1]}.$$

We will also have use for a few particular properties of R:

Proposition 4.2.32. *$f : X \to Y$ is a levelwise weak equivalence in $(\boldsymbol{sSet}^{\mathscr{C}})_{proj}$ if and only if $R(f)$ is a levelwise weak equivalence.*

Proof. For any $c \in \mathscr{C}$, we have a natural diagram

$$\begin{tikzcd} X(c) \arrow[r, "f_c"] \arrow[d, "(\kappa_X)_c"'] & Y(c) \arrow[d, "(\kappa_Y)_c"] \\ X(c) \times_{M_cX} (M_cX)^{\mathbf{nerve}(I[1])} \arrow[r, "R(f)_c"'] & Y(c) \times_{M_cY} (M_cY)^{\mathbf{nerve}(I[1])} \end{tikzcd}$$

where all but the horizontal maps are weak equivalences. By 2-out-of-3, the result holds. □

Proposition 4.2.33. *R preserves finite products up to natural isomorphism.*

Proof. It is evident that there is a levelwise isomorphism

$$R(X \times Y)(c) \cong R(X)(c) \times R(Y)(c)$$

and that this commutes with matching object maps. For latching object maps, one may assume by induction that the latching objects of $R(X) \times R(Y)$ and $R(X \times Y)$ are isomorphic. Then one quickly checks that the same latching maps are induced in both instances. □

We suspect that R should have more powerful properties than these, but we will only need these results for the purposes of this chapter.

4.2.1 Taking Examples to the (Homotopy) Limit

The fact that R may be applied to Reedy categories other than $(\Delta^{op})^n$ suggests an opportunity to expand our intuitions for R's behavior by studying other diagram categories. A trivial example is given by $\mathscr{C}$ being discrete, on which R does nothing. More generally, any Reedy category $\mathscr{C}$ such that $\mathscr{C}^-$ is discrete will have trivial Reedy fibrant replacements. These are not particularly enlightening demonstrations of our constructions in action, so we ignore them henceforth.

The simplest non-trivial categories to examine are those Reedy categories $\mathscr{C}$ such that $\mathscr{C}^+$ is discrete. In this case, the latching objects and thus latching maps are all trivial, so we can entirely focus on the matching objects. An interesting consequence of this fact is that the Reedy cofibrations in $\mathbf{sSet}^{\mathscr{C}}$ are precisely the levelwise cofibrations, meaning the Reedy and injective model structures agree. Our construction therefore necessarily obtains a model of homotopy limit over such diagrams.

Consider, for instance, the cospan diagram $C := \{x \to y \leftarrow z\}$, where $\mathbf{deg}(x) = \mathbf{deg}(z) = 1$ and $\mathbf{deg}(y) = 0$. A projective fibrant functor $F : C \to \mathbf{sSet}$ then amounts to a cospan of simplicial sets

$$C_x \to C_y \leftarrow C_z$$

where C_x, C_y and C_z are all Kan complexes. The Reedy fibrant replacement $R(F)$ is then easily checked to be the cospan

$$C_x \times_{C_y} C_y^{\mathbf{nerve}(I[1])} \to C_y \leftarrow C_y^{\mathbf{nerve}(I[1])} \times_{C_y} C_z.$$

If one were to take a pullback of this cospan, one immediately obtains our chosen definition of homotopy pullback $C_x \times^h_{C_y} C_z$ in terms of Λ.

Another example comes from the diagram for equalizers, namely $E := \{x \rightrightarrows y\}$, with $\mathbf{deg}(x) = 1$ and $\mathbf{deg}(y) = 0$. Then we have, for a functor $G : E \to \mathbf{sSet}$, that $R(G)$ is the diagram

$$E_y \times_{E_x^2} (E_x^2)^{\mathbf{nerve}(I[1])} \rightrightarrows E_x.$$

The limit of this diagram is similarly a reasonable model of homotopy-coherent equalizer, whose 0-simplices are triples (p, e, q), where $e \in (E_y)_0$ and $p, q : \mathbf{nerve}(I[1]) \to E_y$ are paths such that $p(1) = q(0) = e$ and the image of $p(0)$ under the first map $E_x \to E_y$ is equal to the image of $q(1)$ under the second map.

For an infinite example, consider the poset category $M := \{\cdots \to 3 \to 2 \to 1 \to 0\}$, whose objects are the elements of $\mathbb{Z}_{\geq 0}$ with a morphism $a \to b$ if and only if $b \leq a$. This is a Reedy category where $\mathbf{deg}(x) = x$ for all objects $x \in M$. For a levelwise fibrant functor $J : M \to \mathbf{sSet}$, we then have that $R(J)$ is the functor

$$J_0 \leftarrow J_1 \times_{J_0} J_0^{\mathbf{nerve}(I[1])} \leftarrow J_2 \times_{J_1 \times_{J_0} J_0^{\mathbf{nerve}(I[1])}} (J_1 \times_{J_0} J_0^{\mathbf{nerve}(I[1])})^{\mathbf{nerve}(I[1])} \leftarrow \cdots$$

Writing down $R(J)_n$ without recursively referring to $R^{\leq n-1}(J)$ would appear to be difficult; naively 'flattening out' the definition produces an exponentially growing expression of nested pullbacks. There is however a simpler non-inductive way to write down this functor, taking inspiration from the construction of homotopy limits in [Hir09, Def. 18.1.8]; indeed, if we restrict J to $M^{\leq n}$, then we have that $\lim_{M^{\leq n}} R(J) \cong R(J)_n$, so each $R(J)_n$ should be an inductively growing model of homotopy limit:

Proposition 4.2.34. *Suppose $J : M \to$ **sSet** is as above. Then for any $n \geq 0$, we have*

$$R(J)_n \cong \boldsymbol{eq}\Big(\prod_{i=0}^{n} J_i^{\boldsymbol{nerve}(I[n-i])} \rightrightarrows \prod_{j=0}^{n-1} J_j^{\boldsymbol{nerve}(I[n-j-1])}\Big)$$

where the first map acts by precomposition with $\boldsymbol{nerve}(I[n-j-1]) \to \boldsymbol{nerve}(I[n-j])$ sending $i \mapsto i$ on objects, while the second map acts by postcomposition with $J_j \to J_{j-1}$ if $j > 0$ and the terminal map $J_0 \to *$.

Moreover, the maps $R(J)_n \to R(J)_{n-1}$ are given by commutative diagrams

$$\begin{array}{ccc}
\prod_{i=0}^{n} J_i^{\boldsymbol{nerve}(I[n-i])} & \rightrightarrows & \prod_{j=0}^{n-1} J_j^{\boldsymbol{nerve}(I[n-j-1])} \\
\downarrow & & \downarrow \\
\prod_{i=0}^{n-1} J_i^{\boldsymbol{nerve}(I[n-i-1])} & \rightrightarrows & \prod_{j=0}^{n-2} J_j^{\boldsymbol{nerve}(I[n-j-2])}
\end{array}$$

where the leftmost vertical map is given on $J_i^{\boldsymbol{nerve}(I[n-i])}$ for $i < n$ by precomposition with maps $I[n-i-1] \to I[n-i]$ sending $j \mapsto j+1$ for objects j and on J_n by the terminal map $J_n \to *$. The rightmost vertical map is similar.

Finally, the maps $(\kappa_J)(n) : J_n \to R(J)_n$ are given by the maps

$$J_n \to J_i \to J_i^{\boldsymbol{nerve}(I[n-i])}.$$

Proof. The case $n = 0$ is trivial. Suppose then that the result holds for $R(J)_{n-1}$. We have that $R(J)_n$ is isomorphic to the limit of the diagram

$$\begin{aligned}
&J_n \times_{\mathbf{eq}\left(\prod_{i=0}^{n-1} J_i^{\mathbf{nerve}(I[n-i-1])} \rightrightarrows \prod_{j=0}^{n-2} J_j^{\mathbf{nerve}(I[n-j-2])}\right)} \\
&\quad \mathbf{eq}\Big(\prod_{i=0}^{n-1} J_i^{\mathbf{nerve}(I[n-i-1])} \rightrightarrows \prod_{j=0}^{n-2} J_j^{\mathbf{nerve}(I[n-j-2])}\Big)^{\mathbf{nerve}(I[1])} \\
\cong &J_n \times_{\mathbf{eq}\left(\prod_{i=0}^{n-1} J_i^{\mathbf{nerve}(I[n-i-1])} \rightrightarrows \prod_{j=0}^{n-2} J_j^{\mathbf{nerve}(I[n-j-2])}\right)} \\
&\quad \mathbf{eq}\Big(\prod_{i=0}^{n-1} J_i^{\mathbf{nerve}(I[n-i-1]\times I[1])} \rightrightarrows \prod_{j=0}^{n-2} J_j^{\mathbf{nerve}(I[n-j-2]\times I[1])}\Big).
\end{aligned}$$

Note that

$$J_n \cong \mathbf{eq}\Big(\prod_{i=0}^{n} J_i \rightrightarrows \prod_{j=0}^{n-1} J_j\Big)$$

where the two maps are induced by the diagrams

$$\begin{array}{ccccc}
J_n & J_{n-1} & \cdots & J_1 & J_0 \\
\downarrow & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} & \downarrow{\scriptstyle id} \\
* & J_{n-1} & \cdots & J_1 & J_0
\end{array}$$

and

$$\begin{tikzcd}
J_n \arrow[dr] & J_{n-1} \arrow[dr] & \cdots \arrow[dr] & J_1 \arrow[dr] & J_0 \arrow[dr] & \\
& J_{n-1} & \cdots & J_1 & J_0 & *
\end{tikzcd}$$

Thus, since limits commute, we may reinterpret $R(J)_n$ as the equalizer

$$\begin{aligned}
\mathbf{eq}\Big(& J_n \times \prod_{i=0}^{n-1} J_i \times_{J_i^{\mathbf{nerve}(I[n-i-1])}} J_i^{\mathbf{nerve}(I[n-i-1]\times I[1])} \\
& \rightrightarrows J_{n-1} \times \prod_{i=0}^{n-2} J_i \times_{J_i^{\mathbf{nerve}(I[n-i-2])}} J_i^{\mathbf{nerve}(I[n-i-2]\times I[1])}\Big)
\end{aligned}$$

The induction now holds, by the isomorphisms

$$J_i^{\mathbf{nerve}(I[n-i])} \cong J_i \times_{J_i^{\mathbf{nerve}(I[n-i-1])}} J_i^{\mathbf{nerve}(I[n-i-1]\times I[1])}$$

given by the maps $I[n-i] \to *$ and by $I[n-i] \to I[n-i-1] \times I[1]$ sending $0 \mapsto (0,0)$ and $j \mapsto (j-1,1)$ for $0 < j \leq n-i$. The maps $R(J)_n \to R(J)_{n-1}$ are as needed by inspection, as are the maps $(\kappa_J)_n : J_n \to R(J)_n$; indeed, the matching object $M_n R^{\leq n-1}(J)$ is just $R(J)_{n-1}$. $\square$

We have that the 0-simplices of $R(J)_n$ correspond naturally to tuples

$$(p_n \in J_n, p_{n-1} : \mathbf{nerve}(I[1]) \to J_{n-1}, \cdots, p_0 : \mathbf{nerve}(I[n]) \to J_0)$$

of maps $p_i : \mathbf{nerve}(I[n-i]) \to J_i$, where $p_i|_{\mathbf{nerve}(I[n-i-1])}$, restricted under the map $\mathbf{nerve}(I[n-i-1]) \to \mathbf{nerve}(I[n-i])$ sending $j \mapsto j$ for all $j \in I[n-i-1]$, is equal to the composition

$$\mathbf{nerve}(I[n-i-1]) \xrightarrow{p_{i+1}} J_{i+1} \to J_i$$

for all $i < n$. That is to say, a 0-simplex corresponds to a commutative diagram

$$\begin{tikzcd}
* \arrow[r, "\cong"] & \mathbf{nerve}(I[0]) \arrow[r] \arrow[d, "p_n"'] & \mathbf{nerve}(I[1]) \arrow[r] \arrow[d, "p_{n-1}"'] & \cdots \arrow[r] & \mathbf{nerve}(I[n-1]) \arrow[r] \arrow[d, "p_1"'] & \mathbf{nerve}(I[n]) \arrow[d, "p_0"] \\
& J_n \arrow[r] & J_{n-1} \arrow[r] & \cdots \arrow[r] & J_1 \arrow[r] & J_0
\end{tikzcd}$$

Taking the limit $\lim_M R(J)$ in turn yields a Kan complex whose 0-simplices are infinite sequences of finite tuples

$$((p_0^0), (p_0^1, p_1^1), (p_0^2, p_1^2, p_2^2), (p_0^3, p_1^3, p_2^3, p_3^3), \cdots)$$

such that $p_i^i \in (J_i)_0$ for all $i \geq 0$, while the maps

$$p_j^i : \mathbf{nerve}(I[i-j]) \to J_j$$

restrict along the maps $\psi^i_j : I[i-j-1] \to I[i-j]$ sending $k \mapsto k$ for $k \in I[i-j-1]$ to p^i_{j+1} composed with the map $J_{j+1} \to J_j$. Moreover, considering the maps $\phi^i_j : I[i-j-1] \to I[i-j]$ sending $k \mapsto k+1$, we have that the diagrams

$$\begin{tikzcd} \mathbf{nerve}(I[i-j-1]) \arrow[rr, "\mathbf{nerve}(\phi^i_j)"] \arrow[dr, "p^{i-1}_j"'] & & \mathbf{nerve}(I[i-j]) \arrow[dl, "p^i_j"] \\ & J_j & \end{tikzcd}$$

commute.

What we have then is, for each $i \geq 0$, a commutative diagram of the form

$$\begin{tikzcd} \mathbf{nerve}(I[0]) \arrow[r, "\mathbf{nerve}(\phi^{i+1}_i)"] \arrow[d, "p^i_i"'] & \mathbf{nerve}(I[1]) \arrow[r, "\mathbf{nerve}(\phi^{i+2}_i)"] \arrow[dl, "p^{i+1}_i"'] & \mathbf{nerve}(I[2]) \arrow[r] \arrow[dll, "p^{i+2}_i"] & \cdots \\ J_i & & & \end{tikzcd}$$

Let $\mathbb{Z}_{\geq 0}$ be equipped with the natural poset category structure. Considering the functor $\phi^i : \mathbb{Z}_{\geq 0} \to \mathbf{Cat}$ sending $[n] \mapsto I[n]$ and the morphism $a \leq a+1$ for $a \in \mathbb{Z}_{\geq 0}$ to ϕ^{i+a}_i, we could alternatively represent this as a map

$$p^i : \mathrm{colim}_{\mathbb{Z}_{\geq 0}} \mathbf{nerve}(\phi^i) \to J_i.$$

The object $\mathrm{colim}_{\mathbb{Z}_{\geq 0}} \mathbf{nerve}(\phi^i)$ is naturally identified with the nerve $\mathbf{nerve}(I[\infty])$ of the contractible groupoid $I[\infty]$ whose objects are the elements of $\mathbb{Z}_{\geq 0}$. The maps $I[n] \to I[\infty]$ then send $k \mapsto n-k$ for all $n \geq 0$.

Consider then the map $\psi : \mathbf{nerve}(I[\infty]) \to \mathbf{nerve}(I[\infty])$ induced by the maps ψ^i_j. Concretely, this map sends $n \mapsto n+1$ for all $n \in \mathbb{Z}_{\geq 0}$. By the first commutativity condition, we have that a 0-simplex of $\lim_M R(J)$ precisely corresponds to a commutative diagram of the form

$$\begin{tikzcd} \mathbf{nerve}(I[\infty]) \arrow[d, "p^0"'] & \mathbf{nerve}(I[\infty]) \arrow[l, "\psi"'] \arrow[d, "p^1"'] & \mathbf{nerve}(I[\infty]) \arrow[l, "\psi"'] \arrow[d, "p^2"'] & \cdots \arrow[l] \\ J_0 & J_1 \arrow[l] & J_2 \arrow[l] & \cdots \arrow[l] \end{tikzcd}$$

This seems analogous to another description of homotopy limits along the same diagram M, though with target category **Top**, in [Rie14, Ex. 6.5.6]; here, Riehl explains that the homotopy limit of a functor $M \to \mathbf{Top}$ where the maps $J_{i+1} \to J_i$ are inclusions can be described by a map $f : [0, \infty) \to J_0$ such that $f([n, \infty)) \subseteq J_n$ for all $n \geq 0$.

The case of $\mathscr{C} = M$ seems in particular to suggest that, given a levelwise fibrant functor $X : \mathscr{C} \to \mathbf{sSet}$, the replacement $R(X)(x)$ for each $x \in \mathscr{C}$

should be at least strongly related to homotopy limits of X over each category $(x \downarrow \mathscr{C}^-)$. Indeed, note that for any $x \in \mathscr{C}$, the natural restriction $R(X)|_{(x\downarrow\mathscr{C}^-)}$ is in fact a Reedy fibrant replacement for $X|_{(x\downarrow\mathscr{C}^-)}$ and thus an injective fibrant replacement. Hence, $R(X)_x$ must be a homotopy limit of the diagram $X|_{(x\downarrow\mathscr{C}^-)}$. This suggests that what we have really constructed levelwise for $R(X)$ is a series of inductively defined homotopy limits over finite-degree Reedy categories.

It would be of interest to move past this inductive definition to see $R(X)$ explicitly at each level. We have not yet been able to establish such a construction in general, so we are left to handle the cases of Δ^{op} and $(\Delta^{op})^2$ by hand, as we shall do for 1-fold and 2-fold Segal spaces.

4.3 Homotopy Categories via Reedy Fibrant Replacement

As a demonstration of the utility of our Reedy fibrant replacement functor, we consider the challenge of obtaining the *homotopy category* $h_1^p(X)$ of a projective fibrant Segal space X. Our construction will in the end be an instance of the approach given by Johnson-Freyd and Scheimbauer in [JS17, Def. 2.2], in particular with a chosen model of homotopy pullback. This section serves to elucidate how our construction of Reedy fibrant replacement suffices to obtain other standard constructions from projective fibrant situations directly.

Definition 4.3.1. *Suppose $X \in$ **sSpace**. Let $x_1, \cdots, x_n \in (X_0)_0$. Define the* homotopy fiber *functor*

$$(-)_h^{x_1,\cdots,x_n} : \boldsymbol{sSet}_{/(X_0)^n} \to \boldsymbol{sSet}$$

to be the functor sending $Y \to (X_0)^n$ to the pullback

$$\begin{tikzcd} Y_h^{x_1,\cdots,x_n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \times_{(X_0)^n} (X_0^{\boldsymbol{nerve}(I[1])})^n \arrow[d] \\ \{(x_1,\cdots,x_n)\} \arrow[r] & (X_0)^n \end{tikzcd}$$

Recall that if $\mathscr{M}$ is a model category, then for any $x \in \mathscr{M}$ there is a natural model structure on $\mathscr{M}_{/x}$ where cofibrations, fibrations and weak equivalences are those maps whose underlying maps in $\mathscr{M}$ are cofibrations, fibrations or weak equivalences respectively. This is proven to give a model structure in [Hir09, Thm. 7.6.5].

Proposition 4.3.2. $(-)_h^{x_1,\cdots,x_n}$ *preserves weak equivalences and fibrancy.*

Proof. Consider a weak equivalence $Y \to Z$ commuting with maps $Y \to (X_0)^n$ and $Z \to (X_0)^n$. Note that the pullbacks $Y \times_{(X_0)^n} (X_0^{\mathbf{nerve}(I[1])})^n$ and $Z \times_{(X_0)^n} (X_0^{\mathbf{nerve}(I[1])})^n$ are homotopy pullbacks by Proposition [Lur08, Rem. A.2.4.5], as **sSet** is proper and the map $(X_0^{\mathbf{nerve}(I[1])})^n \to (X_0)^n$ is a fibration. Thus, the map

$$Y \times_{(X_0)^n} (X_0^{\mathbf{nerve}(I[1])})^n \to Z \times_{(X_0)^n} (X_0^{\mathbf{nerve}(I[1])})^n$$

is a weak equivalence.

We then have a natural induced diagram of weak equivalences levelwise between cospans

$$\begin{tikzcd}
Y \times_{(X_0)^n} (X_0^{\mathbf{nerve}(I[1])})^n \arrow[r] \arrow[d, "\sim"'] & (X_0)^n \arrow[d, "\sim"] & \{(x_1,\cdots,x_n)\} \arrow[l] \arrow[d, "\sim"] \\
Z \times_{(X_0)^n} (X_0^{\mathbf{nerve}(I[1])})^n \arrow[r] & (X_0)^n & \{(x_1,\cdots,x_n)\} \arrow[l]
\end{tikzcd}$$

By [Lur08, Rem. A.2.4.5], we have that the pullbacks of both the upper and lower cospans are in fact homotopy pullbacks, so the weak equivalences induce a weak equivalence of homotopy pullbacks as needed.

Fibrancy of $Y_h^{x_1,\cdots,x_n}$ for $Y \in \mathbf{sSet}$ a Kan complex is induced by the map $Y \times_{(X_0)^n} (X_0^{\mathbf{nerve}(I[1])})^n \to X_0^n$ being a fibration, implying the pullback projection $Y_h^{x_1,\cdots,x_n} \to \{(x_1,\cdots,x_n)\}$ is a fibration. □

Lemma 4.3.3. *Suppose* $Y \to (X_0)^n$ *is a fibration. Then there is a natural weak equivalence*

$$Y^{x_1,\cdots,x_n} \to Y_h^{x_1,\cdots,x_n}.$$

Proof. There is a natural weak equivalence

$$Y \to Y \times_{(X_0)^n} (X_0^{\mathbf{nerve}(I[1])})^n$$

given by constant path maps. Then consider the diagram

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "\sim"'] & (X_0)^n \arrow[d, "\sim"'] & \{(x_1,\cdots,x_n)\} \arrow[l] \arrow[d, "\sim"] \\
Y \times_{(X_0)^n} (X_0^{\mathbf{nerve}(I[1])})^n \arrow[r] & (X_0)^n & \{(x_1,\cdots,x_n)\} \arrow[l]
\end{tikzcd}$$

By [Lur08, Rem. A.2.4.5], we have that the pullbacks of both the upper and lower cospans are in fact homotopy pullbacks. Thus, we have an induced weak equivalence between pullbacks as needed. □

Notation 4.3.4. *For a projective fibrant Segal space X with $x, y \in (X_0)_0$, we will write*

$$X^h(x,y) := (X_1)_h^{x,y}$$

for the homotopy mapping space *of X from x to y.*

Note that if X is projective fibrant then $X^h(x,y)$ is in fact a Kan complex now. We may interpret this space as the ∞-groupoid whose objects are triples (p, f, q), where $f : x' \to y'$ is a 1-morphism and $p, q : \mathbf{nerve}(I[1]) \to X_0$ are paths in X_0 from x to x' and from y' to y, respectively.

Lemma 4.3.5. *Suppose X is a Reedy fibrant Segal space. Then there is a natural weak equivalence*

$$X(x,y) \to X^h(x,y)$$

for all $x, y \in (X_0)_0$.

Proof. The map $X_1 \to X_0 \times X_0$ is a fibration. $\square$

Using this fact, we can quickly prove that the following defines a category, using again a mild modification of [JS17, Def. 2.2] and [CS19, Def. 1.9]:

Definition 4.3.6 ([CS19, Def. 1.9])**.** *Let X be a projective fibrant Segal space. Then the* projective homotopy category *$h_1^p(X)$ is defined to be the category whose objects are the elements of the set $(X_0)_0$, whose hom-sets are of the form*

$$\boldsymbol{Hom}_{h_1^p(X)}(x,y) := \pi_0(X^h(x,y))$$

with identities given by the degeneracies and composition by applying π_0 to the zig-zag diagram

$$\begin{aligned} X^h(x,y) \times X^h(y,z) \to (X_1 \times^h_{X_0} X_1)_h^{x,z} &\leftarrow (X_1 \times_{X_0} X_1)_h^{x,z} \\ &\leftarrow (X_2)_h^{x,z} \\ &\to X^h(x,z) \end{aligned}$$

and simply inverting all resulting bijections.

Before we prove this is a category, we introduce a new useful simplicial set:

Definition 4.3.7. *The n-*wheel *$Wh(n) \in \boldsymbol{sSet}$ is the simplicial set*

$$Wh(n) := Sp(n) \sqcup_{\Delta[0] \sqcup \Delta[0]} \Delta[1]$$

where the map $\Delta[0] \sqcup \Delta[0] \to Sp(n)$ is such that composing with $Sp(n) \hookrightarrow \Delta[n]$ gives the 0-simplices $\langle 0 \rangle$ and $\langle n \rangle$.

We can visualize the n-wheel as a diagram (which perhaps more resembles a flat car tire)

$$\begin{tikzcd} & 1 \arrow[r] & \cdots \arrow[r] & n-1 \arrow[dr] & \\ 0 \arrow[ur, bend left] \arrow[rrrr, bend right] & & & & n \end{tikzcd}$$

There is an evident map $Wh(n) \hookrightarrow \Delta[n]$ that the inclusion of the n-spine factors through, which is itself an inclusion if $n \geq 2$. There is moreover an evident map $Wh(n) \hookrightarrow K$ for any simplicial composition diagram K of arity n.

Proposition 4.3.8. *Suppose X is a projective fibrant Segal space. Then $h_1^p(X)$ is a category.*

Proof. Note that the Reedy fibrant replacement $R(X)$ will have a homotopy category $h_1(R(X))$ isomorphic to $h_1^p(X)$ in objects and hom-sets, as the sets of objects are equal and by Proposition 4.3.5 we have a natural bijection for each $x, y \in \mathbf{ob}(h_1(R(X)))$

$$\pi_0(X(x,y)) \cong \pi_0(X^h(x,y)).$$

These bijections on hom-sets respect identities by inspection. Moreover, we may prove these bijections to naturally respect the composition operations on $h_1^p(X)$ and $h_1(R(X))$. To do so, note that there is an isomorphism

$$R(X)_{Wh(2)} \cong ((X_1 \times^h_{X_0} X_1) \times_{X_0^2} (X_0^2)^{\mathbf{nerve}(I[1])}) \times_{X_0^2} (X_1 \times_{X_0^2} (X_0^2)^{\mathbf{nerve}(I[1])})$$

given by unwinding definitions. Moreover, we have a natural map from $X_2 \times_{X_0^2} (X_0^2)^{\mathbf{nerve}(I[1])}$ into $R(X)_{Wh(2)}$ given by the composite map

$$X_2 \to X_1 \times_{X_0} X_1 \to X_1 \times^h_{X_0} X_1 \to (X_1 \times^h_{X_0} X_1) \times_{X_0^2} (X_0^2)^{\mathbf{nerve}(I[1])}$$

and by the map

$$X_2 \xrightarrow{X_{\langle 0,2 \rangle}} X_1 \to X_1 \times_{X_0^2} (X_0^2)^{\mathbf{nerve}(I[1])}.$$

Using this resultant map, upon solving the lifting problem

$$\begin{tikzcd} X_2 \arrow[d, hook, "\sim"'] \arrow[rr, "(\kappa_X)_2"] & & R(X)_2 \arrow[d, two heads] \\ X_2 \times_{X_0^2} (X_0^2)^{\mathbf{nerve}(I[1])} \arrow[d] \arrow[rr] \arrow[urr, dashed] & & R(X)_{Wh(2)} \\ ((X_1 \times^h_{X_0} X_1) \times_{X_0^2} (X_0^2)^{\mathbf{nerve}(I[1])}) \times_{X_0^2} (X_1 \times_{X_0^2} (X_0^2)^{\mathbf{nerve}(I[1])}) \arrow[urr] & & \end{tikzcd}$$

whose solution is naturally a weak equivalence by 2-out-of-3, there is a commutative diagram

$$\begin{tikzcd}
X^h(x,y) \times X^h(y,z) \arrow[r, "\cong"] \arrow[d] & R(X)(x,y) \times R(X)(y,z) \arrow[d] \\
(X_1 \times^h_{X_0} X_1)^{x,z}_h \arrow[r, "\cong"] & R(X)^{x,z}_{Sp(2)} \\
(X_1 \times_{X_0} X_1)^{x,z}_h \arrow[u] & \\
(X_2)^{x,z}_h \arrow[u] \arrow[r] \arrow[d] & R(X)^{x,z}_2 \arrow[uu] \arrow[d] \\
X^h(x,z) \arrow[r, "\cong"] & R(X)(x,y)
\end{tikzcd}$$

where all horizontal maps are weak equivalences. This implies that the horizontal maps upon applying π_0 are bijections, which means that composition operations are identified. Thus, associativity and identity laws are induced on $h_1^p(X)$, making it a category as needed. $\square$

We can similarly see that h_1^p and $h_1 \circ R$ will agree precisely on morphisms. Thus, we have that h_1^p defines a valid functor $\mathbf{SeSp}^{proj} \to \mathbf{Cat}$. This is not a new result at all; both [CS19] and [JS17] define this construction, without employing R or any other Reedy fibrant replacement functor. However, the method of using this particular Reedy fibrant replacement functor to prove coherence and functoriality seems to be novel in this context.

Note that we have an agreement between the notions of homotopy category for Reedy and projective fibrancy respectively:

Proposition 4.3.9. *Suppose $X \in \boldsymbol{SeSp}^{inj}$. Then $h_1^p(X) \cong h_1(X)$.*

Proof. There is an equality of objects and a natural bijection on hom-sets. Moreover, this defines a valid functor by an analogous proof to that of Proposition 4.3.8. $\square$

Some more results are in order, which are standard enough that we will not claim originality:

Proposition 4.3.10. *There is a natural isomorphism $h_1^p(X \times Y) \cong h_1^p(X) \times h_1^p(Y)$ for all $X, Y \in \boldsymbol{SeSp}^{proj}$.*

Proof. This is given by the fact that h_1 and R both preserve products. $\square$

Proposition 4.3.11. *h_1^p preserves weak equivalences between projective fibrant Segal spaces.*

Proof. This is given by the fact that R preserves weak equivalences. □

We expect this latter proposition could alternatively be proven by using [Hor15, Prop. 2.11].

We also finally have a more standard characterization of completeness using h_1^p:

Proposition 4.3.12. *Suppose X is a projective fibrant Segal space. Then X is complete if and only if the natural map*

$$X_0 \to R(X)_{heq} \subseteq X_1 \times_{X_0^2} (X_0^2)^I$$

is a weak equivalence.

Proof. X will be local with respect to the completeness map if and only if $R(X)$ is. □

Note that there is a natural factorization

$$X_0 \to X_{heq} \to R(X)_{heq}$$

where X_{heq} for a projective fibrant complete Segal space X is the subspace of X_1 of path components of those $f \in (X_1)_0$ whose image in $R(X)_1$ lies in $R(X)_{heq}$. That is, these are the morphisms $f : x \to y$ such that (c_x, f, c_y) is invertible in $h_1^p(X)$, where c_x and c_y are the constant paths on $x, y \in (X_0)_0$ respectively.

It is evident that the latter map above is a weak equivalence; indeed, $(H_X)_1 : R(X)_1 \to R(X)_1^{\mathbf{nerve}(I[1])}$ defines paths from each $(p, f, q) \in R(X)_{heq}$ to $(c_{X_{\langle 0 \rangle}(f)}, f, c_{X_{\langle 1 \rangle}(f)}) \in R(X)_{heq}$, implying that $f \in X_{heq}$. This means $(\alpha_X)_1 : R(X)_1 \to X_1$, sending $(p, f, q) \mapsto f$ on 0-simplices, restricts and corestricts on the path components defining X_{heq} and $R(X)_{heq}$ to a map $R(X)_{heq} \to X_{heq}$, which is a homotopy inverse of the inclusion $X_{heq} \to R(X)_{heq}$. Thus, we have the following:

Proposition 4.3.13. *Suppose X is a projective fibrant Segal space. Then X is complete if and only if the natural map*

$$X_0 \to X_{heq}$$

is a weak equivalence.

A similar characterization is now possible of projective fibrant complete 2-fold Segal spaces, giving us precisely the definition of a projective fibrant complete 2-fold Segal space in [JS17, Def. 2.7].

Note again that these characterizations are certainly not new; we believe the descriptions of completeness for projective fibrant Segal spaces by Horel in [Hor15, Prop. 2.6] should be equivalent to our own. Moreover, Proposition 4.3.13 is precisely the definition of completeness employed by Johson-Freyd and Scheimbauer in [JS17, Def. 2.3]. We suspect all of these approaches to be little more than different masks on the same beast, inevitably reducing to similar techniques to prove correctness. A complete comparison will be left to future work.

4.4 Homotopy Bicategories of Projective Fibrant 2-fold Segal Spaces

We now turn to the challenge of constructing the homotopy bicategory of a projective fibrant 2-fold Segal space X. It seems as though this task is somewhat complete already; we may compose for instance $h_2' : \mathbf{SeSp}_2^{inj} \to \mathbf{UBicat}$ with $R : \mathbf{SeSp}_2^{proj} \to \mathbf{SeSp}_2^{inj}$ to retrieve an unbiased bicategory $h_2'(R(X))$ functorially from any $X \in \mathbf{SeSp}_2^{proj}$.

While plausibly correct as a model of homotopy bicategory, this approach quickly presents challenges if we wish to study the contents of the resulting unbiased bicategory directly. Much of the problem is with Reedy fibrant replacement: while R does produce for us Reedy fibrant analogues of any projective fibrant 2-fold Segal space X, a price must be paid in complexity. Indeed, $R(X)_{n,m}$ inductively builds upon $R(X)_{a,b}$ for all $a \leq n$ and $b \leq m$. We may therefore find it rather difficult to work with $R(X)$ given only concrete knowledge of X. For example, the standard lifting problems we should be expected to solve to obtain a choice of horizontal compositions, such as

$$\begin{tikzcd} & R(X)_2 \arrow[d] \\ R(X)_{Sp(2)} \arrow[r, "id"'] & R(X)_{Sp(2)} \end{tikzcd}$$

rely on simplicial spaces such as $R(X)_2$, which implicitly contains an infinite amount of new structure in the form of each simplicial set $R(X)_{2,n}$.

It is the goal of this section to try and mitigate this complexity to only those parts that are absolutely necessary. First, we will investigate what explicitly are the objects, morphisms and 2-morphisms of such an unbiased bicategory. After this is done, we will consider methods to simplify the hom-categories of this bicategory and in doing so obtain a more straightforward description of potential composition operations. This discussion will result in what may be considered in many ways an analogue to h_2^{tr} in the projective fibrant case, whose consistency is proven by the Reedy fibrant case's correctness and which precisely agrees with the definition of homotopy bicategory described by Johnson-Freyd and Scheimbauer in [JS17, Def. 2.12], albeit with some chosen models of homotopy pullbacks.

For the rest of this chapter, we will resume the conventions of writing

$$\begin{aligned} I &:= \mathbf{nerve}(I[1]) \\ \Lambda &:= \mathbf{nerve}(I[1]) \sqcup_* \mathbf{nerve}(I[1]). \end{aligned}$$

Let $X \in \mathbf{SeSp}_2^{proj}$. Fix some choice of horizontal compositions $(\mu_n)_{n\geq 0}$ for $R(X) \in \mathbf{SeSp}_2^{inj}$. Our precise interest is in understanding the bicategory

$$\mathscr{B} := h_2(R(X), (\mu_n)_{n\geq 0})$$

along with potentially some other unbiased bicategories isomorphic in **UBicat** to $\mathscr{B}$ and obtained by tractable methods. We cannot expect to explicitly describe all $R(X)_{n,m}$ nor all $\mu_{n,m}$ in service of this goal at present; without a non-inductive description of R, the description for each of these will boundlessly grow in complexity as $n+m$ increases. Instead, we should try to obtain only the data necessary to construct $\mathscr{B}$. Our first port of call in this regard is to understand the objects and higher morphisms.

The collection of objects of $\mathscr{B}$ is straightforward to understand, as this is just

$$\mathbf{ob}(\mathscr{B}) = (R(X)_{0,0})_0 = (X_{0,0})_0.$$

The hom-categories for $x, y \in (X_{0,0})_0$ are then of the form

$$\mathbf{Hom}_{\mathscr{B}}(x,y) = h_1(R(X)(x,y)).$$

The 1-morphisms of $\mathscr{B}$ from x to y are thus elements of the set

$$(R(X)(x,y)_0)_0 \cong (\{x\} \times_{X_{0,0}} X_{0,0}^I \times_{X_{0,0}} X_{1,0} \times_{X_{0,0}} X_{0,0}^I \times_{X_{0,0}} \{y\})_0$$

meaning they are tuples (p, f, q), where $f \in (X_{1,0})_0$ and $p, q : I \to X_{0,0}$ are paths from $X_{\langle 0\rangle,0}(f)$ to x and from $X_{\langle 1\rangle,0}(f)$ to y, respectively. These tuples may be visualized as diagrams of the form

$$x \xleftarrow{p} X_{\langle 0\rangle,0}(f) \xrightarrow{f} X_{\langle 1\rangle,0}(f) \xrightarrow{q} y$$

We could then envision a choice of horizontal composition μ_2 as taking a chain of two such diagrams of the form

$$x \xleftarrow{p} X_{\langle 0\rangle,0}(f) \xrightarrow{f} X_{\langle 1\rangle,0}(f) \xrightarrow{q} y \xleftarrow{r} X_{\langle 0\rangle,0}(g) \xrightarrow{g} X_{\langle 1\rangle,0}(g) \xrightarrow{s} z$$

and producing a path of such diagrams to the spine of some 2-simplex $\gamma \in (X_{2,0})_0$, retracting the inner two paths q and r to constant paths in doing so, then yielding the remaining edge $X_{\langle 0,1\rangle,0}(\gamma)$ of γ of the form

$$x \overset{p'}{\leftsquigarrow} X_{\langle 0\rangle,0}(f) \xrightarrow{g\circ f} X_{\langle 1\rangle,0}(g) \overset{s'}{\rightsquigarrow} z$$

as the composite, for new paths p' and s'. Note that in general, while x and z are preserved in such a composite, the outermost paths p and s may not be.

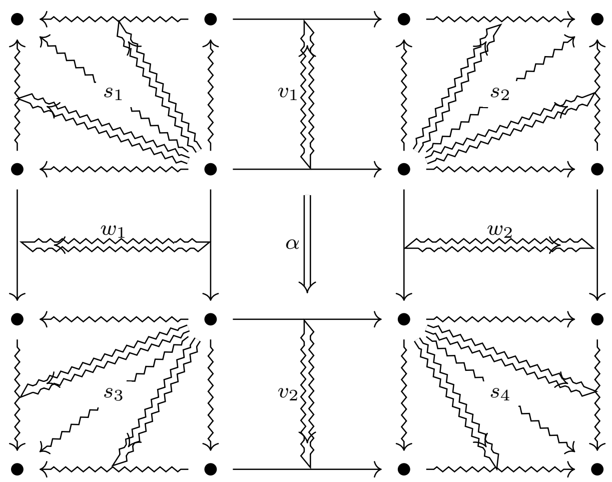
The 2-morphisms are slightly more complex, as these are now defined to be path components in certain subspaces of the simplicial set $R(X)(x,y)_1$. We must therefore now understand both 0-simplices and 1-simplices in this space. Building again upon the fact that

$$R(X)_{1,1} = X_{1,1} \times_{R(X)_{\partial\Delta[1,1]}} R(X)^I_{\partial\Delta[1,1]}$$

we find that $R(X)_{1,1}$ is precisely the limit of a diagram

$$\begin{tikzcd}
X_{0,0}^{\Lambda\times I} \arrow[r] \arrow[d] \arrow[dr] & X_{0,0}^{I} \arrow[dr] & & X_{1,0}^{I} \arrow[ll] \arrow[rr] \arrow[d] & & X_{0,0}^{I} \arrow[dl] & X_{0,0}^{\Lambda\times I} \arrow[l] \arrow[dl] \arrow[d] \\
X_{0,0}^{I} \arrow[dr] & X_{0,0}^{\Lambda} \arrow[r] \arrow[d] & X_{0,0} & X_{1,0} \arrow[l] \arrow[r] & X_{0,0} & X_{0,0}^{\Lambda} \arrow[l] \arrow[d] & X_{0,0}^{I} \arrow[dl] \\
& X_{0,0} & X_{0,0} \arrow[ul] & & X_{0,0} \arrow[ur] & X_{0,0} & \\
X_{0,1}^{I} \arrow[uu] \arrow[r] \arrow[dd] & X_{0,1} \arrow[u] \arrow[d] & & X_{1,1} \arrow[ul] \arrow[uu] \arrow[ur] \arrow[ll] \arrow[rr] \arrow[dl] \arrow[dd] \arrow[dr] & & X_{0,1} \arrow[u] \arrow[d] & X_{0,1}^{I} \arrow[l] \arrow[uu] \arrow[dd] \\
& X_{0,0} & X_{0,0} \arrow[dl] & & X_{0,0} \arrow[dr] & X_{0,0} & \\
X_{0,0}^{I} \arrow[ur] & X_{0,0}^{\Lambda} \arrow[u] \arrow[r] & X_{0,0} & X_{1,0} \arrow[l] \arrow[r] & X_{0,0} & X_{0,0}^{\Lambda} \arrow[l] \arrow[u] & X_{0,0}^{I} \arrow[ul] \\
X_{0,0}^{\Lambda\times I} \arrow[u] \arrow[ur] \arrow[r] & X_{0,0}^{I} \arrow[ur] & & X_{1,0}^{I} \arrow[ll] \arrow[rr] \arrow[u] & & X_{0,0}^{I} \arrow[ul] & X_{0,0}^{\Lambda\times I} \arrow[l] \arrow[ul] \arrow[u]
\end{tikzcd}$$

We can thus visualize a 0-simplex in this space, for some $\alpha \in (X_{1,1})_0$, as a diagram of the form



More precisely, a typical 0-simplex is then a pair (α, r) where $\alpha \in (X_{1,1})_0$ and $r : \mathbf{nerve}(I[1]) \to R(X)_{\partial\Delta[1,1]}$. This means r reduces to paths $w_1, w_2 : \mathbf{nerve}(I[1]) \to X_{0,1}$, maps

$$s_1, s_2, s_3, s_4 : \Lambda \times I \cong \mathbf{nerve}(I[1]^2) \sqcup_{\mathbf{nerve}(I[1])} \mathbf{nerve}(I[1]^2) \to X_{0,0}$$

and paths $v_1, v_2 : \mathbf{nerve}(I[1]) \to X_{1,0}$. These maps are all constrained to agree appropriately on endpoints as per the requirements of the pullbacks. Thus, a typical 0-simplex is the tuple $(\alpha, v_1, v_2, s_1, s_2, s_3, s_4, w_1, w_2)$.

A typical 1-simplex is, then, a tuple

$$(\beta, V_1, V_2, S_1, \ldots, S_4, W_1, W_2)$$

where $\beta \in (X_{1,1})_1$ and the remaining elements take the forms

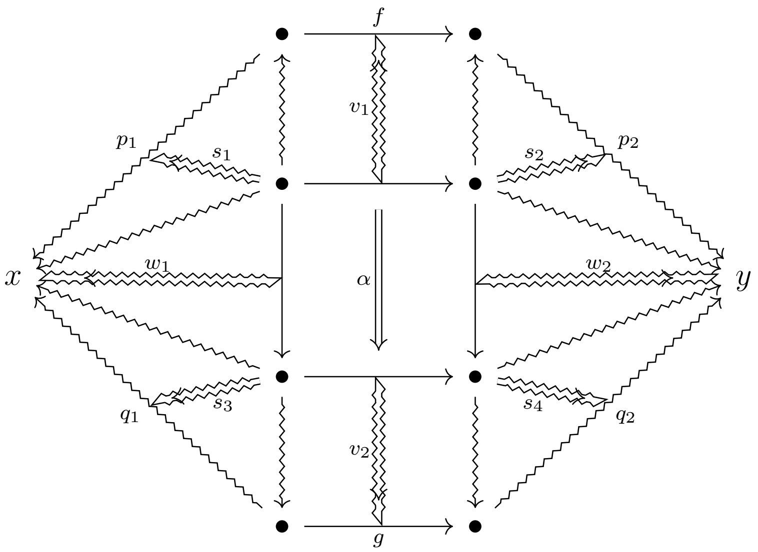
$$\begin{aligned} V_1, V_2 &: \mathbf{nerve}(I[1]) \times \Delta[1] \to X_{1,0} \\ S_1, \ldots, S_4 &: \Big(\mathbf{nerve}(I[1]^2) \times_{\mathbf{nerve}(I[1])} \mathbf{nerve}(I[1]^2)\Big) \times \Delta[1] \to X_{0,0} \\ W_1, W_2 &: \mathbf{nerve}(I[1]) \times \Delta[1] \to X_{0,1}. \end{aligned}$$

Again, these are constrained to agree on endpoints appropriately.

The elements of $\mathbf{Hom}_{\mathscr{B}}(x,y)((p_1, f, p_2), (q_1, g, q_2))$ are therefore equivalence classes of tuples

$$[(\alpha, v_1, v_2, s_1, s_2, s_3, s_4, w_1, w_2)]$$

where w_1 and w_2 are required to terminate at x and y respectively, s_1, v_1 and s_2 must together terminate at (p_1, f, p_2) and s_3, v_2 and s_4 must together terminate at (q_1, g, q_2). The equivalence relation is given by 1-simplices that similarly restrict constantly to $x, y, (p_1, f, p_2)$ and (q_1, g, q_2). A 2-morphism can thus be visualized as an equivalence class of far more ‘globular’ diagrams, of the form



The equivalence relation in question is then paths of such diagrams relative to the boundary. Note that no globularity is required of the internal 2-simplex α. However, essential constancy of X guarantees that every $\alpha \in (X_{1,1})_0$ may be realized in such a diagram, so appears as a 2-morphism in $\mathscr{B}$.

One could in turn imagine horizontal and vertical composition similarly to the discussion of composition for 1-morphisms, albeit by concatenation of such diagrams either vertically or horizontally. By the equivalence relation, such compositions will be uniquely defined relative to the boundary 1-morphisms.

4.4.1 Path Objects and Homotopy Pullbacks of Simplicial Spaces

In a bid to at least simplify the homotopy categories somewhat, we will introduce some methods to extend our approach to homotopy categories to more general simplicial spaces. We first require a suitable notion of path space and homotopy pullback in **sSpace**^{proj}. We will simply apply the $(-)^{\mathbf{nerve}(I[1])}$ operation levelwise to do so, using the cotensor construction we established for **sSpace** by **sSet**:

Definition 4.4.1. *Suppose $X \in$ **sSpace**. Then define*

$$X^I := X^{\iota_1^0(\boldsymbol{nerve}(I[1]))}.$$

We will sometimes omit ι_1^0 if it is clear from context.

Proposition 4.4.2. *Suppose $X \in$ **sSpace** is levelwise fibrant. Then*

$$X \to X^I := X^{\boldsymbol{nerve}(I[1])} \to X \times X$$

where the latter map is given by $\langle 0\rangle + \langle 1\rangle : * \sqcup * \to I[1]$ and the former by $I[1] \to *$, is a good path object for X in the projective model structure.

Proof. We need to prove that the latter map is a levelwise fibration and the former is a levelwise weak equivalence. Both of these are evident, as for each $n \geq 0$ the diagram

$$X_n \to (X_n)^I \to X_n \times X_n$$

defines a very good path object for X_n. □

We can then use this approach to obtain a reasonable notion of homotopy pullback, given levelwise by Λ:

Proposition 4.4.3. *For $X \in$ **sSpace** projective fibrant, the maps*

$$X \to X^\Lambda \to X \times X$$

given levelwise by Λ define a good path object in $\boldsymbol{sSpace}^{proj}$.

Proof. The proof is identical to that of X^I. □

Definition 4.4.4. *For a cospan $X \to Y \leftarrow Z$ of projective fibrant Segal spaces, define the* homotopy pullback *to be the space*

$$X \times^h_Y Z := X \times_Y Y^\Lambda \times_Y Z.$$

This is again a valid notion of homotopy pullback in $\mathbf{sSpace}^{proj}$, though we care little for the details of this fact here.

4.4.2 Simplifying the Hom-Categories

Our hope is to directly extend our approach to h_1^p to obtain a tractable description of a functor h_2^p from some category of projective fibrant 2-fold Segal spaces to **UBicat**. We begin with a simpler description of the mapping $(\infty, 1)$-categories in terms of our new path objects:

Definition 4.4.5. *Suppose $X \in \boldsymbol{SeSp}_2^{proj}$, with $x, y \in (X_{0,0})_0$. Then define*

$$X^h(x, y) := \{x\} \times_{X_0} X_0^I \times_{X_0} X_1 \times_{X_0} X_0^I \times_{X_0} \{y\}.$$

Proposition 4.4.6. *Suppose $X \in \boldsymbol{SeSp}_2^{proj}$. Then there is a natural levelwise weak equivalence*

$$q_X : X_1 \times_{X_0^2} (X_0^2)^I \to R(X)_1$$

that is fibered over X_0. Moreover, the diagram

$$\begin{tikzcd} X_1 \arrow[rr, "(\kappa_X)_1"] \arrow[dr] & & R(X)_1 \\ & X_1 \times_{X_0^2} (X_0^2)^I \arrow[ur, "q_X"'] & \end{tikzcd}$$

commutes, where the bottom left map is induced by the constant path map.

Proof. We seek a weak equivalence in **sSet** for each $n \geq 0$ of the form

$$X_{0,n}^I \times_{X_{0,n}} X_{1,n} \times_{X_{0,n}} X_{0,n}^I \to R(X)_{1,n} = X_{1,n} \times_{R(X)_{\partial\Delta[n,1]}} R(X)_{\partial\Delta[n,1]}^I.$$

We will obtain this map levelwise by induction on n, constructing for each $n \geq 0$ a pair of maps α_n and β_n making the diagram

$$\begin{tikzcd} X_{1,n} \times_{X_{0,n}^2} (X_{0,n}^I)^2 \arrow[dr, dashed] \arrow[drr, "\alpha_n"] \arrow[ddr, "\beta_n"'] & & \\ & R(X)_{1,n} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X_{1,n} \arrow[d] \\ & R(X)_{\partial\Delta[n,1]}^I \arrow[r] & R(X)_{\partial\Delta[n,1]} \end{tikzcd}$$

commute.

The map α_n is simply a pullback projection to $X_{1,n}$. The latter map is given by induction. For $n = 0$, we have that $\partial\Delta[0,1] \cong \Delta[0,0] \sqcup \Delta[0,0]$. Hence, β_0 may simply be given by the pullback projection

$$X_{1,0} \times_{X_{0,0}^2} (X_{0,0}^I)^2 \to (X_{0,0}^I)^2.$$

Now, assume we have defined α_{n-1} and β_{n-1}. To obtain β_n, we note first that

$$R(X)_{\partial\Delta[n,1]}^I \cong R(X)_{1,\partial\Delta[n]}^I \times_{(R(X)_{0,\partial\Delta[n]}^2)^I} (R(X)_{0,n}^2)^I$$

so obtaining a map into $R(X)_{\partial\Delta[n,1]}^I$ amounts to considering maps to the two components. Hence, we seek maps ϕ_n, ψ_n making the diagram

$$\begin{tikzcd}
X_{1,n} \times_{X^2_{0,n}} (X^I_{0,n})^2 \arrow[dr, dashed] \arrow[drr, "\phi_n"] \arrow[ddr, "\psi_n"'] & & \\
& R(X)^I_{\partial\Delta[n,1]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & R(X)^I_{1,\partial\Delta[n]} \arrow[d] \\
& (R(X)^2_{0,n})^I \arrow[r] & (R(X)^2_{0,\partial\Delta[n]})^I
\end{tikzcd}$$

commute.

The map ψ_n to $(R(X)^2_{0,n})^I$ is given by $((\kappa_X)^2_{0,n})^I$. For ϕ_n, we have inductively in n a map

$$X_{1,n} \times_{X^2_{0,n}} (X^2_{0,n})^I \to X_{1,\partial\Delta[n]} \times_{X^2_{0,\partial\Delta[n]}} (X^2_{0,\partial\Delta[n]})^I \to R(X)_{1,\partial\Delta[n]}.$$

We obtain from this a homotopy

$$X_{1,n} \times_{X^2_{0,n}} (X^2_{0,n})^I \to \left(X_{1,n} \times_{X^2_{0,n}} (X^2_{0,n})^I\right)^I \to R(X)^I_{1,\partial\Delta[n]}$$

where the first map is given by a constant homotopy in $X_{1,n}$ and by the homotopy $D_{X^2_{0,n}}$.

One may check that the resulting maps commute enough to establish a map to $R(X)_{1,n}$, while the resulting map is natural in n. The map then clearly factors through $(\kappa_X)_1$ as needed. That it is a weak equivalence is induced by 2-out-of-3. $\square$

Note moreover that the above map is natural in X itself.

Proposition 4.4.7. *Suppose $X \in \boldsymbol{SeSp}^{proj}_2$. Then for any $x, y \in (X_{0,0})_0$, $X^h(x,y) \in \boldsymbol{SeSp}^{proj}_1$. Moreover, there is a natural isomorphism of categories*

$$h^p_1((q_X)^{x,y}) : h^p_1(X^h(x,y)) \cong h_1(R(X)(x,y)).$$

Proof. Consider the map $(q_X)^{x,y} : X^h(x,y) \to R(X)(x,y)$. Levelwise, we have that this is induced by a diagram

$$\begin{tikzcd}
\{(x,y)\} \arrow[r] \arrow[d, "\simeq"'] & X^2_{0,m} \arrow[d, "\simeq"] & X_{1,m} \times_{X^2_{0,m}} (X^2_{0,m})^I \arrow[l] \arrow[d, "\simeq"] \\
\{(x,y)\} \arrow[r] & R(X)^2_{0,m} & R(X)_{1,m} \arrow[l]
\end{tikzcd}$$

for all $m \geq 0$. This is a diagram in **sSet**, where the two rightmost horizontal maps are both fibrations. By right properness of **sSet** and the characterization of homotopy pullbacks related by Lurie in [Lur08, Rem. A.2.4.5], we have that $X^h(x,y)_m$ and $R(X)(x,y)_m$ are homotopy pullbacks of simplicial sets for all $m \geq 0$. Thus, the above levelwise weak equivalence of cospans induces a weak equivalence $X^h(x,y)_m \to R(X)(x,y)_m$ and thus a levelwise weak equivalence $X^h(x,y) \to R(X)(x,y)$. It is evident that $X^h(x,y)$ is projective fibrant. Moreover, the levelwise weak equivalence $X^h(x,y) \to R(X)(x,y)$ implies $X^h(x,y) \in \mathbf{SeSp}_1^{proj}$ by [Hir09, Lemma 3.2.1 (1)], as $R(X)(x,y) \in \mathbf{SeSp}_1^{proj}$. Hence, this levelwise weak equivalence is transported by h_1^p to an equivalence of categories.

Note moreover that the map is an identity on objects. Indeed, at level 0 we have a map

$$X_{1,0} \times_{X^2_{0,0}} (X^2_{0,0})^I \to X_{1,0} \times_{X^2_{0,0}} (X^2_{0,0})^I$$

that acts as the identity map. Therefore, the above functor is an equivalence of categories that is bijective on objects, so it is in fact an isomorphism of categories. $\square$

One could argue that this immediately obtains a simpler description of the homotopy bicategory $\mathscr{B}$ of a projective fibrant 2-fold Segal space X, in particular simplifying the vertical composition. However, it does not yet elucidate the nature of horizontal composition, let alone the coherence isomorphisms. Though we do not yet have a complete description of these homotopy bicategories that makes no reference to the Reedy fibrant replacement, we provide some insights that may yield more detail in future work.

For a 2-fold Segal space X and each $n \geq 2$, we may consider a lifting problem in $\mathbf{sSpace}^{inj}$ of the form

$$\begin{tikzcd}
X_n \arrow[r, "(\kappa_X)_n"] \arrow[d, hook, "\sim"'] & R(X)_n \arrow[d, two heads] \\
X_n \times_{X_0^{n+1}} (X_0^{n+1})^I \arrow[r] \arrow[ur, dashed] \arrow[d] & R(X)_{Wh(n)} \\
((X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1) \times_{X_0^2} (X_0^2)^I) \times_{X_0^2} (X_1 \times_{X_0^2} (X_0^2)^I) \arrow[ur] &
\end{tikzcd}$$

where the bottom-left vertical map is given by the pullback of the n maps for $1 \leq i \leq n$ of the form

$$m_i : X_n \times_{X_0^{n+1}} (X_0^{n+1})^I \to X_1 \times_{X_0^2} (X_0^2)^I$$

given by the map $X_{\langle i-1,i\rangle} : X_n \to X_1$ and the projection $(\mathbf{pr}_{i-1}, \mathbf{pr}_i) : (X_0^{n+1})^I \to (X_0^2)^I$. Note that this induces a commutative diagram of the form

$$\begin{tikzcd}
X_n \times_{X_0^{n+1}} (X_0^{n+1})^I \arrow[d, "\gamma_n \times_{X_0^{n+1}} 1_{(X_0^{n+1})^I}"'] \arrow[dr] & \\
X_{Sp(n)} \times_{X_0^{n+1}} (X_0^{n+1})^I \arrow[r] & (X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1) \times_{X_0^2} (X_0^2)^I
\end{tikzcd}$$

The solutions to the above lifting problem then each amount to a levelwise weak equivalence

$$\omega_n : X_n \times_{X_0^{n+1}} (X_0^{n+1})^I \to R(X)_n$$

by 2-out-of-3. Moreover, each of these maps are fibered over X_0^{n+1}. Armed with this data, we may consider diagrams

$$\begin{tikzcd}
(X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1) \times_{X_0^2} (X_0^2)^I \arrow[r, "\sim"] & R(X)_{Sp(n)} \\
X_n \times_{X_0^{n+1}} (X_0^{n+1})^I \arrow[r, "\sim"] \arrow[u] \arrow[d] & R(X)_n \arrow[u] \arrow[d] \\
X_1 \times_{X_0^2} (X_0^2)^I \arrow[r, "\sim"] & R(X)_1
\end{tikzcd}$$

This picture can be fibered over objects $x_0, \cdots, x_n \in (X_{0,0})_0$ while preserving all weak equivalences, as all the resulting pullbacks will necessarily be homotopy pullbacks by an analysis similar to the proof of Proposition 4.4.7. Moreover, upon applying h_1^p everywhere, we obtain diagrams where the vertical upwards maps are equivalences of categories, as are all horizontal maps, while the uppermost and lowermost horizontal maps are isomorphisms.

We may thus obtain, given some choice of 2-truncated compositions for $R(X)$

$$(\bullet_R^{x_0,\cdots,x_n}, \eta_R^{x_0,\cdots,x_n})_{n\geq 0}^{x_0,\cdots,x_n\in(X_{0,0})_0}$$

a particular filler for the cospan

$$h_1^p(X^h(x_0,x_n)) \leftarrow h_1^p((X_n)_h^{x_0,x_n}) \leftarrow h_1^p((X_n)_h^{x_0,\cdots,x_n}) \xrightarrow{\simeq} \prod_{i=1}^n h_1^p(X^h(x_{i-1},x_i))$$

of the form

$$\begin{tikzcd}
h_1^p((X_n)_h^{x_0,\cdots,x_n}) \arrow[rr, "\simeq"] \arrow[ddd] \arrow[dr, "h_1^p(\omega_n^{x_0,\cdots,x_n})"'] & & \prod_{i=1}^n h_1^p(X^h(x_{i-1},x_i)) \arrow[d, "\cong"'] \\
& h_1(R(X)_n^{x_0,\cdots,x_n}) \arrow[r, "\simeq"] \arrow[dd] \arrow[dr, Rightarrow, "\eta_R^{x_0,\cdots,x_n}"] & \prod_{i=1}^n h_1(R(X)(x_{i-1},x_i)) \arrow[dddl, "\bullet_R^{x_0,\cdots,x_n}"] \\
& & \\
h_1^p((X_n)_h^{x_0,x_n}) \arrow[d] & h_1(R(X)_n^{x_0,x_n}) \arrow[d] & \\
h_1^p(X^h(x_0,x_n)) \arrow[r, "\cong"] & h_1(R(X)(x_0,x_n)) &
\end{tikzcd}$$

In particular, we may induce such 2-truncated compositions by applying the functor $Tr : \mathbf{SeSp}_2^{comp} \to \mathbf{SeSp}_2^{2comp}$ to the pair $(R(X),(\mu_n)_{n\geq 0})$.

Note moreover that any other filler of the above cospan $(C^{x_0,\cdots,x_n},\iota^{x_0,\cdots,x_n})$ will induce a unique morphism of fillers

$$\epsilon^{x_0,\cdots,x_n} : C^{x_0,\cdots,x_n} \Rightarrow \bullet_R^{x_0,\cdots,x_n}.$$

We may show this to be sufficient to establish an unbiased homotopy bicategory, with composition operations given by $C^{x_0,\cdots,x_n}$, by the following result:

Lemma 4.4.8. *Suppose $\mathscr{B}$ is an unbiased bicategory with composition operations $\circ^{x_0,\cdots,x_n}$, associators γ_X for nested tuples X and unitors $\iota_{x,y}$. Suppose, for every $n \geq 0$ and $x_0,\cdots,x_n \in \boldsymbol{ob}(\mathscr{B})$, we have some functor*

$$C^{x_0,\cdots,x_n} : \prod_{i=1}^n \boldsymbol{Hom}_{\mathscr{B}}(x_{i-1},x_i) \to \boldsymbol{Hom}_{\mathscr{B}}(x_0,x_n)$$

together with a natural isomorphism $\eta^{x_0,\cdots,x_n} : C^{x_0,\cdots,x_n} \Rightarrow \circ^{x_0,\cdots,x_n}$. Suppose moreover that η^x and $\eta^{x,y}$ are identities for all $x,y \in \boldsymbol{ob}(\mathscr{B})$.

Then there is an unbiased bicategory $\mathscr{B}_{C,\eta}$, whose objects, hom-categories, identities and unitors $\iota_{x,y}$ are the same as in $\mathscr{B}$, whose composition functors are the functors $C^{x_0,\cdots,x_n}$ and whose associators χ_Y, for nested tuples of objects in $\mathscr{B}$

$$Y = ((x_0^1,\cdots,x_{k_1}^1),\cdots,(x_0^n,\cdots,x_{k_n}^n))$$

with $x_{k_i}^i = x_0^{i+1}$ for $i<n$ and flattened versions $(x_0,\cdots,x_r)$ with $x_{k_i}^i$ removed for $i<n$, are given by

$$\chi_Y := (\eta^{x_0,\cdots,x_r})^{-1}\gamma_Y\big(\eta^{x_0^1,x_{k_1}^1,\cdots,x_{k_n}^n} \circ \prod_{i=1}^n \eta^{x_0^i,\cdots,x_{k_i}^i}\big).$$

Proof. For the associators, we consider some $n, m_1, \cdots, m_n \in \mathbb{Z}_{>0}$, integers $k_1^1, \cdots, k_{m_n}^n \in \mathbb{Z}_{\geq 0}$ and thrice-nested sequence of objects $(((x_{r,q,p})_{r=0}^{k_q^p})_{q=1}^{m_p})_{p=1}^n$ such that $x_{k_q^p,q,p} = x_{0,q+1,p}$ for $q < m_p$ and $x_{k_{m_p}^p,m_p,p} = x_{0,1,p+1}$ for $p < n$. Then define

1. $D_p = ((x_{r,q,p})_{r=0}^{k_q^p})_{q=1}^{m_p}$;

2. $D = ((x_{0,0,p}, x_{k_0^p,0,p}, x_{k_1^p,1,p}, \cdots, x_{k_{m_p}^p,m_p,p}))_{p=1}^n$;

3. $E = ((x_{r,q,p})_{r=0,q=1}^{k_q^p,m_p})_{p=1}^n$;

4. $F = ((x_{r,q,p})_{r=0}^{k_q^p})_{q=1,p=1}^{m_p,n}$;

We wish to show that the diagrams

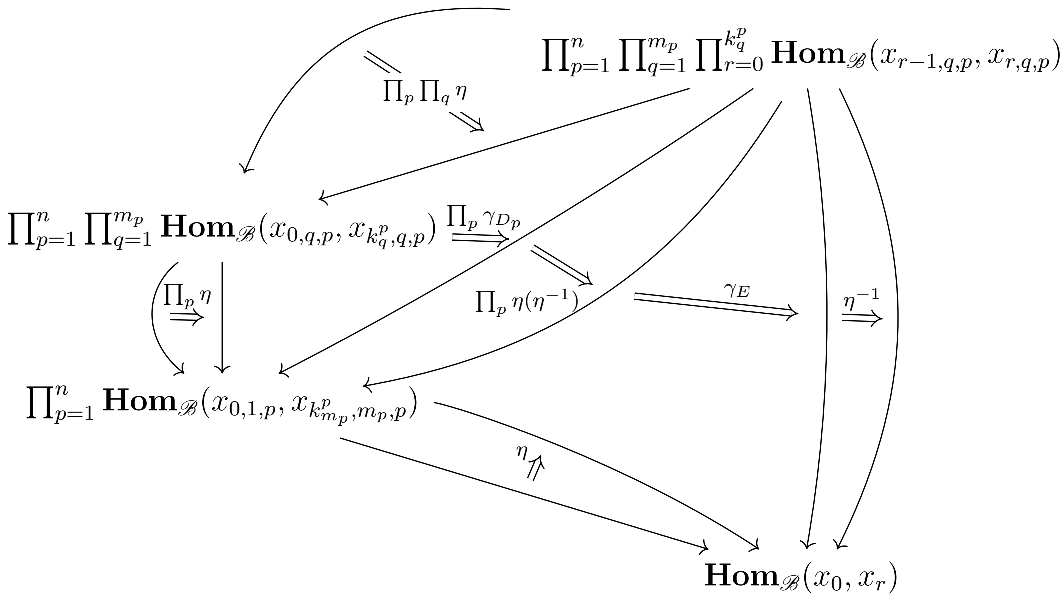
$$\begin{tikzcd}
& & \prod_{p=1}^n \prod_{q=1}^{m_p} \prod_{r=0}^{k_q^p} \mathbf{Hom}_{\mathscr{B}}(x_{r-1,q,p}, x_{r,q,p}) \arrow[dll] \arrow[ddd, bend left] \\
\prod_{p=1}^n \prod_{q=1}^{m_p} \mathbf{Hom}_{\mathscr{B}}(x_{0,q,p}, x_{k_q^p,q,p}) \arrow[d] \arrow[ddrr, bend left] & & \\
\prod_{p=1}^n \mathbf{Hom}_{\mathscr{B}}(x_{0,1,p}, x_{k_{m_p}^p,m_p,p}) \arrow[drr] \arrow[r, phantom, "\overset{\chi_D}{\Rightarrow}"] & {} \arrow[r, phantom, "\overset{\chi_F}{\Rightarrow}"] & {} \\
& & \mathbf{Hom}_{\mathscr{B}}(x_0, x_r)
\end{tikzcd}$$

and

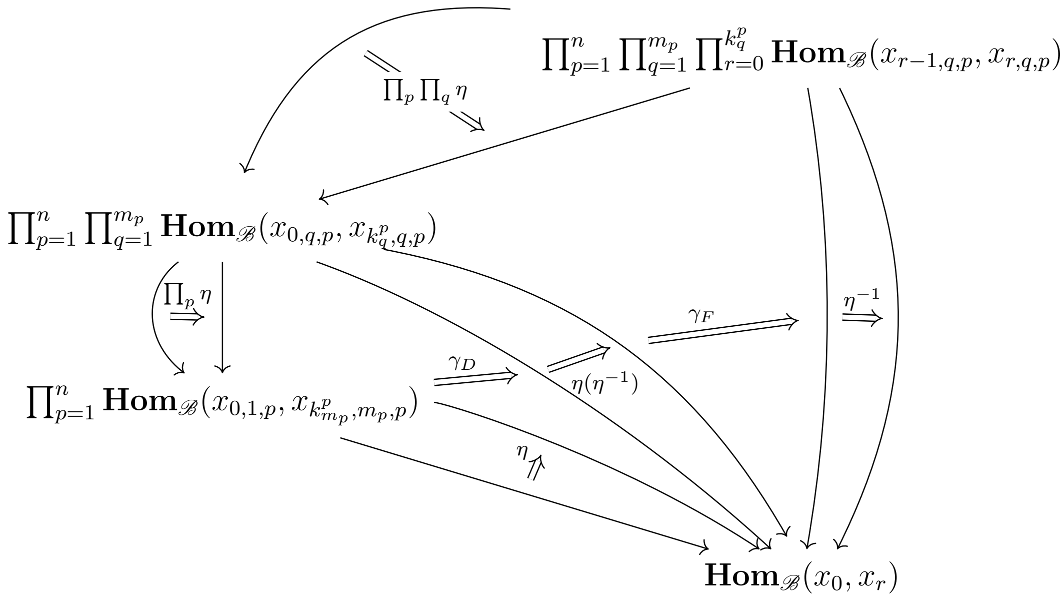
$$\begin{tikzcd}
& & \prod_{p=1}^n \prod_{q=1}^{m_p} \prod_{r=0}^{k_q^p} \mathbf{Hom}_{\mathscr{B}}(x_{r-1,q,p}, x_{r,q,p}) \arrow[dll] \arrow[ddll, bend left] \arrow[ddd, bend left] \\
\prod_{p=1}^n \prod_{q=1}^{m_p} \mathbf{Hom}_{\mathscr{B}}(x_{0,q,p}, x_{k_q^p,q,p}) \arrow[d] \arrow[r, phantom, "\overset{\prod_p \chi_{D_p}}{\Rightarrow}"] & {} \arrow[r, phantom, "\overset{\chi_E}{\Rightarrow}"] & {} \\
\prod_{p=1}^n \mathbf{Hom}_{\mathscr{B}}(x_{0,1,p}, x_{k_{m_p}^p,m_p,p}) \arrow[drr] & & \\
& & \mathbf{Hom}_{\mathscr{B}}(x_0, x_r)
\end{tikzcd}$$

commute with each other.

Note however that, omitting some indices, the latter diagram is equivalent to a diagram of the form



while the former is equivalent to one of the form



Removing the instances of $\eta\eta^{-1}=id$ shows that the two diagrams commute with one another, as $\mathscr{B}$ is a bicategory and the same condition holds of the associators γ.

For the unitors, consider some $n\geq 0$ and $x_0,\cdots,x_n\in\mathbf{ob}(\mathscr{B})$. Note that the required commuting diagram may be split into checking two given natural isomorphisms $C^{x_0,\cdots,x_n} \Rightarrow C^{x_0,\cdots,x_n}$ are equal to $1_{C^{x_0,\cdots,x_n}}$, one of the form

$$
\begin{tikzcd}
 & \prod_{i=1}^n \mathbf{Hom}_{\mathscr{B}}(x_{i-1},x_i) \arrow[dl, bend right=40, "id"'] \arrow[dl, bend left=20] \arrow[dd] \arrow[dd, bend left=50, "C^{x_0,\cdots,x_n}"] \arrow[dl, phantom, "\Downarrow \prod_{i=1}^n \eta^{x_{i-1},x_i}\iota_{x_{i-1},x_i}"] \\
\prod_{i=1}^n \mathbf{Hom}_{\mathscr{B}}(x_{i-1},x_i) \arrow[dr, bend right=30, "C^{x_0,\cdots,x_n}"'] \arrow[dr, bend left=10] \arrow[r, phantom, "\xRightarrow[\gamma_{((x_0,x_1),\cdots,(x_{n-1},x_n))}]{}"] \arrow[dr, phantom, "\Uparrow \eta^{x_0,\cdots,x_n}"] & \xRightarrow{(\eta^{x_0,\cdots,x_n})^{-1}} \\
 & \mathbf{Hom}_{\mathscr{B}}(x_0,x_n)
\end{tikzcd}
$$

which is as needed since $\eta^{x,y}$ is the identity and $\mathscr{B}$ satisfies the unitality condition, and another of the form

$$
\begin{tikzcd}
 & \prod_{i=1}^n \mathbf{Hom}_{\mathscr{B}}(x_{i-1},x_i) \arrow[dl, bend right=40, "C^{x_0,\cdots,x_n}"'] \arrow[dl, bend left=20] \arrow[dd] \arrow[dd, bend left=50, "C^{x_0,\cdots,x_n}"] \arrow[dl, phantom, "\Downarrow \eta^{x_0,\cdots,x_n}"] \\
\mathbf{Hom}_{\mathscr{B}}(x_0,x_n) \arrow[dr, bend right=30, "id"'] \arrow[dr, bend left=10] \arrow[r, phantom, "\xRightarrow[\gamma_{((x_0,\cdots,x_n))}]{}"] \arrow[dr, phantom, "\Uparrow \eta^{x_0,x_n}\iota_{x_0,x_n}"] & \xRightarrow{(\eta^{x_0,\cdots,x_n})^{-1}} \\
 & \mathbf{Hom}_{\mathscr{B}}(x_0,x_n)
\end{tikzcd}
$$

which is as needed for similar reasons. Thus, $\mathscr{B}_{C,\eta}$ is a bicategory. $\square$

We are unaware of a proof of the above lemma or its statement in the literature, either as written for unbiased bicategories or in some analogous form for bicategories. However, it seems elementary enough that it is likely well-known; all we have shown is that new unbiased bicategorical structures can be induced by conjugation of given coherence isomorphisms.

We claim in general that there is an isomorphism $F : \mathscr{B} \to \mathscr{B}_{C,\eta}$ in **UBicat** that is an identity on objects and hom-categories, with compositors given by $\eta^{x_0,\cdots,x_n}$. We leave this to future work.

This immediately suggests an unbiased bicategory structure, with natural coherence isomorphisms, for any choice of horizontal compositions on $R(X)$:

Theorem 4.4.9. *Suppose $X \in \boldsymbol{SeSp}_2^{proj}$ and consider the following data, excluding the associators γ_Y, for an unbiased bicategory:*

1. *A set of objects* $(X_{0,0})_0$*;*
2. *For each* $x, y \in (X_{0,0})_0$*, hom-categories* $h_1^p(X^h(x,y))$*;*
3. *For each* $n > 0$ *and* $x_0, \cdots, x_n \in (X_{0,0})_0$*, a composition functor* $\circ^{x_0,\cdots,x_n}$ *given by choosing a filler* $(\circ^{x_0,\cdots,x_n}, \psi^{x_0,\cdots,x_n})$ *for the cospan*

$$h_1^p(X^h(x_0,x_n)) \leftarrow h_1^p((X_n)_h^{x_0,x_n}) \leftarrow h_1^p((X_n)_h^{x_0,\cdots,x_n}) \xrightarrow{\simeq} \prod_{i=1}^{n} h_1^p(X^h(x_{i-1},x_i))$$

such that the choice for $n = 1$ *and any* $x_0, x_1 \in (X_{0,0})_0$ *is the identity* $\circ^{x_0,x_1} = 1_{h_1^p(X^h(x_0,x_1))}$ *coupled with the identity natural isomorphism;*

4. *For each* $x \in (X_{0,0})_0$*, a functor*

$$\circ^x : * \cong h_1^p(X_0^{x,x}) \to h_1^p((X_0)_h^{x,x}) \to h_1^p(X^h(x,x))$$

where the second map is given by constant paths;

5. *For each* $x, y \in (X_{0,0})_0$*, a natural isomorphism* $\iota_{x,y} : 1_{h_1^p(X^h(x,y))} \to \circ^{x,y}$ *given by the identity, as these two functors are equal.*

Then there exists some choice of associators γ_Y *making the above data an unbiased bicategory.*

We suspect there is a characterization of the natural isomorphisms γ_Y that is independent of $R(X)$ and is uniquely determined by the choices of fillers. Moreover, we claim this unbiased bicategory is isomorphic in **UBicat** to $h_2(R(X), (\mu_n)_{n\geq 0})$ for some choice of horizontal compositions $(\mu_n)_{n\geq 0}$ for $R(X)$. We leave such considerations to future work. We also leave the discussion of induced pseudofunctors making the above construction functorial to future work; it is of course functorial if one chooses composition functors to be given by the Reedy fibrant replacement, but more general choices demand further inspection.

Henceforth, we will write our homotopy bicategories of projective fibrant 2-fold Segal spaces as follows, keeping the above potential simpler description thereof in mind:

Definition 4.4.10. *Fix some inverse* λ *to the projection* $\boldsymbol{SeSp}_2^{comp} \to \boldsymbol{SeSp}_2^{inj}$*. Write*

$$h_2^{proj} : \boldsymbol{SeSp}_2^{proj} \to \boldsymbol{UBicat}$$

for the composite functor of the form

$$\boldsymbol{SeSp}_2^{proj} \xrightarrow{R} \boldsymbol{SeSp}_2^{inj} \xrightarrow{\lambda} \boldsymbol{SeSp}_2^{comp} \xrightarrow{h_2} \boldsymbol{UBicat}.$$

Note that h_2^{proj} is uniquely defined up to natural isomorphism, as any two inverses to an equivalence of categories have a natural isomorphism between them.

Chapter 5

Applications and Examples

We now have a means to obtain the homotopy bicategory of both a Reedy fibrant 2-fold Segal space and a projective fibrant 2-fold Segal space. Armed with these constructions, we now proceed to some applications and examples.

Our first order of business is to extend Π_1 to the *fundamental bigroupoid* of a topological space; the entirety of the section explaining this construction is copied from the author's preprint [Rom23] verbatim, though tweaked to accommodate that h_2 now supports Reedy fibrant 2-fold Segal spaces that are not complete, along with other minor modifications and corrections. Our construction has the advantage of providing complete control over the composition operations and giving explicit descriptions of the associators and (trivial) unitors. In doing so, we establish that our construction can be expected to have the correct behavior on the special case of ∞-groupoids.

A few simple applications are then explored, in particular those pertaining to characterizing and understanding completeness of 2-fold Segal spaces. As a part of this effort, we establish a description of equivalences between complete 2-fold Segal spaces that forms an $(\infty, 2)$-categorical analogue to Dwyer-Kan equivalences between complete Segal spaces, which holds in both the Reedy fibrant and projective fibrant cases. We finally conclude this thesis by considering a definition of completeness for Reedy fibrant 2-fold Segal spaces that relies upon h_2 more explicitly.

5.1 Fundamental Bigroupoids

To take stock of our progress in defining homotopy bicategories, we now consider the effect of h_2 on the case of $S := \mathbf{Sing}_{\mathbf{ssS}}(U)$ for some given $U \in \mathbf{Top}$. We will find this yields a sensible notion of *fundamental bigroupoid* of a topological space, similar in nature to [HKK01] though with differences in horizontal composition.

To begin, we have a set of objects $\mathbf{ob}(h_2(S))$ equal to the set of points in

U, along with hom-categories of the form

$$\mathbf{Hom}_{h_2(S)}(x,y) \cong \Pi_1(\{(x,y)\} \times_{U\times U} U^{\Delta_t[1]})$$

which is precisely what we should expect.

For composition, we observe that $\mathbf{Sing_{sS}}$ commutes with limits, since it is a right adjoint. Hence,

$$\prod_{S_0}^{1\le i\le n} S_1 \cong \mathbf{Sing_{sS}}(U^{Sp_t(n)})$$

where $Sp_t(n) := \Delta_t[1] \sqcup_{\Delta_t[0]} \cdots \sqcup_{\Delta_t[0]} \Delta_t[1]$. Thus, a choice of horizontal compositions for S actually reduces to solving lifting problems of the form

$$\begin{tikzcd} & \mathbf{Sing_{sS}}(U^{\Delta_t[n]}) \arrow[d, "\mathbf{Sing_{sS}}(U^{|g_n|})"] \\ \mathbf{Sing_{sS}}(U^{Sp_t(n)}) \arrow[r, "id"'] \arrow[ur, dashed, "\mu_n"] & \mathbf{Sing_{sS}}(U^{Sp_t(n)}) \end{tikzcd}$$

where $|g_n| : Sp_t(n) \hookrightarrow \Delta_t[n]$ is the topological spine inclusion.

Note that the map $|g_n|$ is a trivial Hurewicz cofibration; it is the inclusion of a subcomplex and is moreover a homotopy equivalence. This in turn implies that $U^{\Delta_t[n]} \to U^{Sp_t(n)}$ is a trivial Hurewicz fibration. Hence, solutions to this lifting problem can be obtained from any of the sections of $U^{|g_n|}$, which necessarily exist. We can obtain a simple example of such a section by finding a retract $R_n : \Delta_t[n] \twoheadrightarrow Sp_t[n]$ of $|g_n|$: if we understand that $Sp_t(n) \subseteq \Delta_t[n]$ and define $e_i \in \Delta_t[n]$ such that e_i's i^{th} entry is 1, we choose to map $e_i \mapsto (\frac{n-i}{n}, 0, \cdots, 0, \frac{i}{n})$, inducing linearly a retraction r_n to the edge from e_0 to e_n in $\Delta_t[n]$ that sends

$$(x_0, \cdots, x_n) \mapsto (\sum_{i=0}^{n} x_i \frac{n-i}{n}, 0, \cdots, 0, \sum_{i=0}^{n} x_i \frac{i}{n}).$$

Composing this with a map s_n to $Sp_t(n)$ that sends

$$(1-x_n, 0, \cdots, 0, x_n) \mapsto ((i+1) - nx_n)e_i + (nx_n - i)e_{i+1}, \quad \frac{i}{n} \le x_n \le \frac{i+1}{n}$$

gives us a final retraction $R_n = s_n \circ r_n : \Delta_t[n] \to Sp_t(n)$, sending

$$(x_0, \cdots, x_n) \mapsto \big((i+1) - \sum_{j=0}^{n} jx_j\big)e_i + \big(\sum_{j=0}^{n} jx_j - i\big)e_{i+1}, \quad i \le \sum_{j=0}^{n} jx_j \le i+1$$

We should show that this is the identity on $Sp_t(n)$. Indeed, consider some $(x_0, \cdots, x_n) = (1-t)e_i + te_{i+1} = (0, \cdots, 0, 1-t, t, 0, \cdots, 0)$ for $0 \leq t \leq 1$. Then

$$\sum_{j=0}^{n} jx_j = i(1-t) + (i+1)t = i - it + it + t = i + t$$

which ranges from i to $i+1$ with t.

Applying R_n then gives us

$$R_n(0, \cdots, 0, 1-t, t, 0, \cdots, 0) = ((i+1)-(i+t))e_i + (i+t-i)e_{i+1} = (1-t)e_i + te_{i+1}$$

which confirms that the inclusion $|g_n| : Sp_t(n) \subseteq \Delta_t[n]$ is a section of R_n, as needed.

We now have our composition map, which by using the identification

$$\mathbf{Hom}_{h_2(S)}(x_0, x_1) \times \cdots \times \mathbf{Hom}_{h_2(S)}(x_{n-1}, x_n) \cong \Pi_1(\{x_0, \cdots, x_n\} \times_{U^{n+1}} U^{Sp_t(n)})$$

can be phrased in the form

$$\begin{tikzcd}
\Pi_1(\{x_0, \cdots, x_n\} \times_{U^{n+1}} U^{Sp_t(n)}) \arrow[d] & \\
\Pi_1(\{x_0, x_n\} \times_{U \times U} U^{Sp_t(n)}) \arrow[r, "\Pi_1(1_{\{x_0,x_n\}} \times 1_{U^2} U^{R_n})"] & \Pi_1(\{x_0, x_n\} \times_{U \times U} U^{\Delta_t[n]}) \arrow[d] \\
& \Pi_1(\{x_0, x_n\} \times_{U \times U} U^{\Delta_t[1]})
\end{tikzcd}$$

which, in the end, amounts to composing a sequence of n paths in a topological space into one path. This composition, by the way R_n was defined, is 'unbiased' - the composite path $[0,1] \to U$ sends the interval $[\frac{i}{n}, \frac{i+1}{n}]$ to the i^{th} path in the chain by the map $x \mapsto nx - i$.

We should note that this is but one possible choice of composition operation. In our approach, we established that choice of composition could be built upon a choice of retract for the inclusion $Sp_t(n) \hookrightarrow \Delta_t[n]$. There are in fact an uncountably infinite number of such retracts. For instance, if $n = 2$, choosing any $t \in (0,1)$ induces a distinct retract $f_t : \Delta_t[2] \to Sp_t(2)$ sending

$$(a, b, c) \mapsto \begin{cases} (1 - \frac{bt+c}{t}, \frac{bt+c}{t}, 0) & bt + c < t \\ (0, 1 - \frac{bt+c-t}{1-t}, \frac{bt+c-t}{1-t}) & bt + c \geq t. \end{cases}$$

This results in a composition of paths which, as before, concatenates the paths in an affine manner, but differs in the parameterization of the concatenation. More precisely, the parameter t specifies the point in $[0,1]$ where the first path ends and the second begins in the composite. We can generalize this phenomenon to all $\Delta_t[n]$ in a straightforward way.

Of course, the space of all retracts $\Delta_t[n] \to Sp_t[n]$ of $|g_n|$ need not contain only piecewise linear elements or even only smooth ones. Moreover, our particular construction of the homotopy bicategory of S may have used these retracts, but there is certainly no need to rely on these to obtain the necessary lifts at all. Any section of $U^{|g_n|} : U^{\Delta_t[n]} \to U^{Sp_t(n)}$ will do. One could go further and say the section need not result from a map on these underlying topological spaces - it need only be defined on the resulting Segal spaces, so behavior levelwise could potentially vary drastically.

We now turn to associators in $h_2(S)$, as unitors are of course trivial. In order to work within **Top** efficiently, we need to show that one may convert from homotopies defined using the interval $[0, 1]$ to ones defined using $N(I[1])$.

Proposition 5.1.1. *There is a weak equivalence of Segal spaces* $\tau : N(I[1]) \to$ $\boldsymbol{Sing_{sS}}([0,1])$ *such that the diagram*

$$\begin{tikzcd} * \sqcup * \arrow[r] \arrow[d] & \boldsymbol{Sing_{sS}}([0,1]) \arrow[d] \\ N(I[1]) \arrow[r] \arrow[ur, dashed, "S"] & * \end{tikzcd}$$

commutes.

Proof. Note that this is a valid lifting problem in $SeSp^{inj}$, since $\mathbf{Sing_{sS}}(-)$ preserves weak equivalences levelwise and has its image in $\mathbf{SeSp}^{inj}$, so that the rightmost vertical map is a trivial fibration in $SeSp^{inj}$. Thus, the map S is simply a solution of a lifting problem, so will exist. Moreover, it must be a weak equivalence by 2-out-of-3.

One can concretely define S levelwise. Let $n, m \geq 0$. Recall that

$$N(I[1])_{n,m} \cong \mathbf{Hom_{Cat}}([n] \times [m], I[1]).$$

Note that $[n] \times [m] \cong \mathbf{Ho}(\Delta[n] \times \Delta[m])$, where $\mathbf{Ho} : \mathbf{sSet} \to \mathbf{Cat}$ is the left adjoint to $\mathbf{nerve} : \mathbf{Cat} \to \mathbf{sSet}$ [RV22, pg. 6-7]. Thus, we have that

$$N(I[1])_{n,m} \cong \mathbf{Hom_{sSet}}(\Delta[n] \times \Delta[m], \mathbf{nerve}(I[1])).$$

Applying $|\cdot| : \mathbf{sSet} \to \mathbf{Top}$ then gives a map

$$N(I[1])_{n,m} \to \mathbf{Hom_{Top}}(\Delta_t[n] \times \Delta_t[m], |\mathbf{nerve}(I[1])|).$$

This induces a map[1] $N(I[1]) \to \mathbf{Sing_{sS}}(|\mathbf{nerve}(I[1])|)$ in $\mathbf{SeSp}^{inj}$. Now, we seek a map $|\mathbf{nerve}(I[1])| \to [0, 1]$ in **Top**, as postcomposition with this will yield our desired morphism. By adjunction, it will suffice to find a map q : $\mathbf{nerve}(I[1]) \to \mathbf{Sing}([0, 1])$.

[1]Note that this is not levelwise a bijection.

This map can be defined levelwise. q_0 is a map from a two-element set; we send one element to 0 and the other to 1. The two nondegenerate 1-simplices are then sent by q_1 to the maps $\Delta_t[1] \to [0,1]$ sending $t \mapsto t$ and $t \mapsto 1-t$ respectively. This clearly respects degeneracy and face maps at level 0 and 1. Now, since $\mathbf{nerve}(I[1])$ is a nerve, the maps

$$\mathbf{nerve}(I[1])_n \to \mathbf{nerve}(I[1])_1 \times_{\mathbf{nerve}(I[1])_0} \cdots \times_{\mathbf{nerve}(I[1])_0} \mathbf{nerve}(I[1])_1$$

are bijections for all $n \geq 2$, so every such q_n is uniquely induced in a way commuting immediately with face and degeneracy maps. This completes the construction of q and therefore S. Since S solves the above lifting problem, by 2-out-of-3 S is again a weak equivalence. □

Another way to understand q is to notice, as noted by Rezk in [Rez00, pg. 24] analogously in the case of $F_1^0(\mathbf{nerve}(I[1]))$, that the elements of $\mathbf{nerve}(I[1])_n$ can be identified with sequences $(w_i)_{0 \leq i \leq n}$ where $w_i \in \{0,1\}$ for all i, with face and degeneracy maps given by copying or removing elements of the sequence. Indeed, an element of $\mathbf{nerve}(I[1])_n$ is a chain of morphisms of length n in $I[1]$, which is uniquely identified by the sequence of objects in the chain. The image of such a sequence is then the unique affine map $\Delta_t[n] \to [0,1]$ sending $e_i \mapsto w_i$. This is precisely the notion of a *thick* 1-*simplex* given by Getzler in [Get13], who notes this has previously been called $E(1)$ by Rezk in [Rez00] and $\Delta'[1]$ by Joyal and Tierney in [JT07].

This means, given we specify our homotopies classically in **Top**, we can transmit them by $\mathbf{Sing_{sS}}$ and precomposition with S to the format of left homotopy we have built h_2 upon.

Suppose then we have chosen sections $p_n : U^{Sp_t(n)} \to U^{\Delta_t[n]}$ of $U^{|g_n|}$. It will suffice, for each $n > 0$ and $k_1, \cdots, k_n \geq 0$ with $r := \sum_i k_i$, to find a homotopy

$$U^{Sp_t(r)} \times [0,1] \to U^{(\Delta_t[k_1] \sqcup_{\Delta_t[0]} \cdots \sqcup_{\Delta_t[0]} \Delta_t[k_n]) \sqcup_{Sp_t(n)} \Delta_t[n]}$$

from $U^{|f_{k_1,\cdots,k_n}|} \circ p_r$ to

$$\left((1_{U^{\Delta_t[k_1]}} \times_{1_U} \cdots \times_{1_U} 1_{U^{\Delta_t[k_n]}}) \times_{1_{U^{Sp_t(n)}}} p_n\right) \circ \left(p_{k_1} \times_{1_U} \cdots \times_{1_U} p_{k_n}\right)$$

that is constant on postcomposition with the natural map to $U^{Sp_t(r)}$. Both of these maps are again sections of a trivial Hurewicz fibration, so there will necessarily be a homotopy between them.

In the simple case that $p_n := U^{R_n}$, our challenge shrinks to the problem of finding a homotopy from $U^{|f_{k_1,\cdots,k_n}|} \circ U^{R_r}$ to $U^{(id \sqcup_{id} \cdots \sqcup_{id} id) \sqcup_{id} R_n} \circ U^{(R_{k_1} \sqcup_{id} \cdots \sqcup_{id} R_{k_n})}$. It then suffices to construct a homotopy of the form

$$\Big((\Delta_t[k_1] \sqcup_{\Delta_t[0]} \cdots \sqcup_{\Delta_t[0]} \Delta_t[k_n]) \sqcup_{Sp_t(n)} \Delta_t[n]\Big) \times [0,1] \to Sp_t(r)$$

from $R_r \circ |f_{k_1,\cdots,k_n}|$ to $(R_{k_1} \sqcup_{id} \cdots \sqcup_{id} R_{k_n}) \circ (id \sqcup_{id} R_n)$. We choose to identify $Sp_t(r)$ with $[0, r]$ in the evident way, sending $e_i \mapsto i$, and set our homotopy to be linear interpolation.

To understand more concretely what our homotopy does, consider the case $n = 2$ and $k_1 = 1, k_2 = 2$. The homotopy is from the composition map $((- \circ -) \circ -)$ to $(- \circ - \circ -)$. We have reduced this to a homotopy of retracts

$$\Big((\Delta_t[1] \sqcup_{\Delta_t[0]} \Delta_t[2]) \sqcup_{Sp_t(2)} \Delta_t[2]\Big) \times [0,1] \to [0,3] \cong Sp_t(3).$$

The important behavior of the homotopy is on the unit square $\Delta_t[1] \times [0, 1]$ in the domain, identified by the inclusion

$$\Delta_t[1] \xrightarrow{|\langle 0,2\rangle|} \Delta_t[2] \cong (\emptyset \sqcup_\emptyset \emptyset) \sqcup_\emptyset \Delta_t[2] \hookrightarrow \Big((\Delta_t[1] \sqcup_{\Delta_t[0]} \Delta_t[2]) \sqcup_{Sp_t(2)} \Delta_t[2]\Big)$$

as the image of this path is the end result of the composition operation. We find that $R_3 \circ |f_{1,2}|$ acts on this interval as the morphism $[0, 1] \to [0, 3]$ sending $x \mapsto 3x$, while $(R_1 \sqcup_{id} R_2) \circ (id \sqcup_{id} R_2)$ is the piecewise linear map sending $0 \mapsto 0$, $\frac{1}{2} \mapsto 1$, $\frac{3}{4} \mapsto 2$ and $1 \mapsto 3$. Our associator is a piecewise linear interpolation between these two maps.

As we might expect, there are many possible choices of homotopy to exhibit the associators in $h_2(S)$. Linear interpolation is but one option; a suitable homotopy may be only polynomial, smooth or merely continuous. However, all of these will produce identical associators, as the induced natural isomorphisms will levelwise be homotopic paths. In order to generate truly distinct associators, they must be given as natural isomorphisms that are levelwise paths which are not homotopic, which is not possible by the given approach nor indeed by any method within the confines of our constructions of homotopy bicategories.

Now we consider the production of pseudofunctors. We add the following definition:

Definition 5.1.2. *Let $\boldsymbol{Top}^{comp}$ be the category of pairs $(T, (p_n)_{n\geq 0})$ where $T \in \boldsymbol{Top}$ and $p_n : T^{Sp_t(n)} \to T^{\Delta_t[n]}$ is a section of the map $T^{|g_n|}$, together with maps $(T, p_n) \to (U, q_n)$ given by continuous maps $T \to U$ in $\boldsymbol{Top}$.*

We now have a functor

$$\mathbf{Sing}_{\mathbf{ssS}}^{comp} : \mathbf{Top}^{comp} \to \mathbf{SeSp}_2^{comp}$$

that extends $\mathbf{Sing}_{\mathbf{ssS}}$ by inducing horizontal compositions from the maps p_n.

Consider a continuous map $f : (T, p_n) \to (U, q_n)$ in $\mathbf{Top}$. Let $S_T := \mathbf{Sing}_{\mathbf{ssS}}(T)$ and $S_U := \mathbf{Sing}_{\mathbf{ssS}}(U)$. We already have specified horizontal compositions in S_U and S_T. Write μ_n for the horizontal compositions induced by p_n and likewise ν_n for those from q_n. We thus obtain a map

$$F := \mathbf{Sing}_{\mathbf{ssS}}(f) : (S_T, \mu_n) \to (S_U, \nu_n).$$

We will write F for the underlying map $S_T \to S_U$ as well.

The pseudofunctor $h_2(F) : h_2(S_T, (\mu_n)_{n\geq 0}) \to h_2(S_U, (\nu_n)_{n\geq 0})$ evidently sends $x \mapsto f(x)$ for objects $x \in T$. On hom-categories, for $x, y \in T$, we have the induced functor

$$\Pi_1(\{(x,y)\} \times f^{\Delta_t[1]}) : \Pi_1(\{(x,y)\} \times T^{\Delta_t[1]}) \to \Pi_1(\{(f(x), f(y))\} \times U^{\Delta_t[1]})$$

which sends a 1-morphism $p : [0,1] \to T$ from x to y to the path $f \circ p$. Moreover, it sends a homotopy class of paths $[H]$ for $H : [0,1] \times [0,1] \to T$ to $[f \circ H]$.

Vertical composition is respected on the nose. For horizontal composition, we must understand the compositors $\pi_{x_0,\cdots,x_n}$. It suffices to find a homotopy

$$T^{Sp_t(n)} \times [0,1] \to U^{\Delta_t[n]}$$

from $f^{\Delta_t[n]} \circ p_n$ to $q_n \circ f^{Sp_t(n)}$ that is constant on the vertices. Such a homotopy will necessarily exist, as both are solutions to the same lifting problem

$$\begin{tikzcd}
& U^{\Delta_t[n]} \arrow[d] \\
T^{Sp_t(n)} \arrow[ur, dashed, bend left, "f^{\Delta_t[n]}\circ p_n"] \arrow[ur, dashed, bend right, "q_n\circ f^{Sp_t(n)}"'] \arrow[r] & U^{Sp_t(n)}
\end{tikzcd}$$

in the *Hurewicz model structure* on **Top**, as defined for instance in [MP12, Sec. 17.1]. Any two such homotopies will by Corollary 3.5.11 induce the same natural isomorphism, so the pseudofunctor $h_2(F)$ is specified entirely.

In the special case that $p_n = T^{R_n}$ and $q_n = U^{R_n}$, this is much simpler. In fact, the two maps being homotoped between are equal, so the natural isomorphisms $\pi_{x_0,\cdots,x_n}$ are all identities.

Now, suppose one introduced a different set of deformation retracts $R'_n : \Delta_t[n] \to Sp_t(n)$. These induce a different choice of horizontal compositions $(\eta_n)_{n\geq 0}$ on S_T. We thus have by Corollary 3.5.11 that there is a canonical pseudofunctor $P : h_2(S_T, (\mu_n)_{n\geq 0}) \to h_2(S_T, (\eta_n)_{n\geq 0})$ that is the identity on objects and hom-categories. Indeed, P is induced by the identity morphism $1_T : T \to T$.

The only interesting aspect of this pseudofunctor's structure is in the natural isomorphisms $\pi_{x_0,\cdots,x_n}$. These will be induced by homotopies

$$T^{Sp_t(n)} \times [0,1] \to T^{\Delta_t[n]}$$

from U^{R_n} to $U^{R'_n}$. For such purposes, it suffices to find a left homotopy

$$\Delta_t[n] \times [0,1] \to Sp_t(n)$$

from R_n to R'_n that preserves $Sp_t(n)$. We may choose our left homotopy to be linear interpolation pointwise, choosing to interpret $Sp_t(n) \cong [0,n]$. We find thus that, for a sequence of paths $x_0 \xrightarrow{f_1} x_1 \cdots \xrightarrow{f_n} x_n$, the 2-morphism $\pi_{(f_1,\cdots,f_n)}$ is the homotopy class $[H]$ of the map $H : [0,1] \to U^{[0,1]}$ such that $0 \mapsto (f_0 \sqcup_* \cdots \sqcup_* f_n) \circ R_n$ and $1 \mapsto (f_0 \sqcup_* \cdots \sqcup_* f_n) \circ R'_n$, with all other points defined by the linear interpolation between R_n and R'_n. Any other homotopy will yield the same natural isomorphism, so this description covers all possibilities.

In the end, composing $\mathbf{Sing}^{comp}_{\mathbf{ssS}}$ and h_2 yields a 'fundamental bigroupoid' functor

$$\Pi_2^{comp} : \mathbf{Top}^{comp} \to \mathbf{UBicat}.$$

Using some coherent choice of maps p_n, for instance those induced by R_n, allows us to make the domain **Top**. This then yields a functor

$$\Pi_2 : \mathbf{Top} \to \mathbf{UBicat}.$$

5.2 Completeness and Equivalences

We have not yet compared our constructions of homotopy bicategories with other approaches in the literature explicitly; we leave this task to future work. For the time being, there are other means available to help convince ourselves that h_2^{tr}, h_2 and h_2^{proj} all deserve the title of 'homotopy bicategory'. For instance, we may establish a characterization of weak equivalences between Reedy fibrant complete 2-fold Segal spaces in $CSSP_2^{inj}$:

Definition 5.2.1. *A map $F : X \to Y$ in $\boldsymbol{CSSP}_2^{inj}$ is a* Dwyer-Kan biequivalence *if and only if the following properties hold:*

1. *For every $x, y \in (X_{0,0})_0$ and $f, g \in (X(x,y)_0)_0$, the map*

$$X(x,y)(f,g) \to Y(F(x), F(y))(F(f), F(g))$$

*is a weak equivalence in **sSet**;*

2. *The functor*

$$h_2(F) : h_2(X) \to h_2(Y)$$

is an equivalence of unbiased bicategories.

Note that we have omitted a choice of horizontal compositions for X and Y in taking their homotopy bicategories. Any choices will yield the same end result, as they will produce isomorphic unbiased bicategories.

Theorem 5.2.2. *Suppose $f : X \to Y$ is a map in $\boldsymbol{CSSP}_2^{inj}$. Then f is a weak equivalence in $CSSP_2^{inj}$ if and only if it is a Dwyer-Kan biequivalence.*

The reason we might have to believe such a fact is an alternative characterization of equivalences between Reedy fibrant complete 2-fold Segal spaces due to Bergner and Rezk:

Theorem 5.2.3 ([BR20, Thm. 8.18]). *A map $f : X \to Y$ in $\boldsymbol{CSSP}_2^{inj}$ is a weak equivalence in $CSSP_2^{inj}$ if and only if the following hold:*

1. *For every $x, y \in (X_{0,0})_0$, the maps*
$$f_1^{x,y} : X(x, y) \to Y(f(x), f(y))$$
are weak equivalences in $CSSP^{inj}$;

2. *The functor*
$$h_1(f_{\bullet,0}) : h_1(X_{\bullet,0}) \to h_1(Y_{\bullet,0})$$
is an equivalence of categories.

Note that Bergner and Rezk's statement of the above theorem applies to more general objects than Reedy fibrant complete 2-fold Segal spaces. In particular, they state it for *Segal objects in Θ_n-spaces*, as defined in [BR20, Def. 5.1]. We will only need the case of Reedy fibrant complete 2-fold Segal spaces presently.

Bergner and Rezk call a map satisfying the conditions above a *Dwyer-Kan equivalence*, so we avoid overloading this terminology by using the name *biequivalence* instead.

We will need a few intermediate results about equivalences in homotopy bicategories. First, we need the analogous notion of a *homotopy category* for unbiased bicategories:

Notation 5.2.4. *Let $[\bullet] : \boldsymbol{UBicat} \to \boldsymbol{Cat}$ be the functor sending an unbiased bicategory $\mathscr{B}$ to the category $[\mathscr{B}]$ whose objects are the elements of the set $\boldsymbol{ob}(\mathscr{B})$ and whose hom-sets $\boldsymbol{Hom}_{[\mathscr{B}]}(x, y)$ are the sets of isomorphism classes for each $\boldsymbol{Hom}_{\mathscr{B}}(x, y)$.*

It is indeed acceptable to refer to sets of isomorphism classes, rather than classes thereof or some other notion of large collection; much like **Cat**, we assert that all unbiased bicategories in **UBicat** are necessarily small.

Lemma 5.2.5. *Suppose $X \in \boldsymbol{CSSP}_2^{inj}$. Then there is an isomorphism in $\boldsymbol{Cat}$*
$$h_1(X_{\bullet,0}) \cong [h_2(X)].$$

Proof. We have for any $x, y \in (X_{0,0})_0$ by [Rez00, Cor. 6.5] that $\pi_0(X_{\bullet,0}(x, y)) = \pi_0(X(x, y)_0) \cong h_1(X(x, y))/\cong$, where $\cong$ is the isomorphism relation in $h_1(X(x, y))$.

Now, consider some $x, y, z \in (X_{0,0})_0$. We have a natural diagram

$$\begin{tikzcd}
\pi_0(X(x,y)_0) \times \pi_0(X(y,z)_0) \arrow[r, "\cong"] \arrow[d] & \left(h_1(X(x,y))/\cong\right) \times \left(h_1(X(y,z))/\cong\right) \arrow[d] \\
\pi_0(X^{x,z}_{Sp(2),0}) \arrow[r, "\cong"] & h_1(X^{x,z}_{Sp(2)})/\cong \\
\pi_0(X^{x,z}_{2,0}) \arrow[r, "\cong"] \arrow[u, "\cong"] \arrow[d] & h_1(X^{x,z}_2)/\cong \arrow[u, "\cong"'] \arrow[d] \\
\pi_0(X(x,z)_0) \arrow[r, "\cong"] & h_1(X(x,z))/\cong
\end{tikzcd}$$

so that the composition operations in $h_1(X_{\bullet,0})$ and $[h_2(X)]$ are the same. For identities, we have a natural diagram

$$\begin{tikzcd}
* \arrow[r] \arrow[d, "\cong"'] & * \arrow[d, "\cong"] \\
\pi_0(X^{x,x}_{0,0}) \arrow[r, "\cong"] \arrow[d] & h_1(X^{x,x}_0)/\cong \arrow[d] \\
\pi_0(X(x,x)_0) \arrow[r, "\cong"] & h_1(X(x,x))/\cong
\end{tikzcd}$$

These two diagrams establish functoriality of the map $h_1(X_{\bullet,0}) \to [h_2(X)]$ acting as the identity on objects and by the isomorphisms $\pi_0(X(x,y)_0) \cong h_1(X(x,y))/\cong$ on hom-sets. This map is both fully faithful and bijective on objects, so is invertible. $\square$

We now introduce the notation $\simeq$ to refer to internal equivalence in a given bicategory or unbiased bicategory.

Corollary 5.2.6. *Suppose $X \in \boldsymbol{CSSP}_2^{inj}$. If $x, y \in (X_{0,0})_0$, then $x \cong y$ in $h_1(X_{\bullet,0})$ if and only if $x \simeq y$ in $h_2(X)$.*

Proof of Theorem 5.2.2. First, suppose that f is an equivalence in $CSSP_2^{inj}$. We have for every $x, y \in (X_{0,0})_0$ that the map $X(x,y) \to Y(f(x), f(y))$ is a Dwyer-Kan equivalence, so that for every $h, k \in (X(x,y)_0)_0$ the map

$$(f_1^{x,y})_1^{h,k} : X(x,y)(h,k) \to Y(f(x), f(y))(f(h), f(k))$$

is a weak equivalence as needed. Moreover, the maps

$$h_1(X(x,y)) \to h_1(Y(f(x), f(y)))$$

are equivalences for all x, y, giving full faithfulness of $h_2(f)$. Note that the converses of all of these implications also hold, so we have that $h_2(f)$ is fully faithful and the maps $(f_1^{x,y})_1^{f,g}$ are weak equivalences of simplicial sets if and only if the maps $f_1^{x,y}$ are equivalences of Reedy fibrant complete Segal spaces. Note this implies that $h_1(f_{\bullet,0})$ is itself fully faithful, as it implies that the maps $\pi_0(f_{1,0}^{x,y})$ are bijections.

For essential surjectivity, consider that the functor $h_1(f_{\bullet,0})$ is essentially surjective. This implies that, for any $y \in (Y_{0,0})_0$, there exists some $x \in (X_{0,0})_0$ such that $f(x) \cong y$. This holds if and only if $f(x)$ is equivalent to y in $h_2(X)$, so the functor $h_1(f_{\bullet,0})$ is essentially surjective if and only if $h_2(f)$ is essentially surjective. □

This implies in particular that h_2 sends weak equivalences in $CSSP_2^{inj}$ between Reedy fibrant complete 2-fold Segal spaces to equivalences of unbiased bicategories. We leave the case of non-completeness to future work.

These results may be extended to the projective fibrant case:

Proposition 5.2.7. *Suppose $X \in \boldsymbol{CSSP}_2^{proj}$. Then there is an isomorphism in* ***Cat***

$$h_1^p(X_{\bullet,0}) \cong [h_2^{proj}(X)].$$

Proof. We have a chain of isomorphisms

$$[h_2^{proj}(X)] = [h_2(R(X))] \cong h_1(R(X)_{\bullet,0}) \cong h_1^p(X_{\bullet,0}).$$

□

Definition 5.2.8. *Suppose $F : X \to Y \in \boldsymbol{CSSP}_2^{proj}$. Then F is a* projective Dwyer-Kan biequivalence *if and only if the following hold:*

1. *For every $x, y \in (X_{0,0})_0$ and $f, g \in (X^h(x,y)_0)_0$, the map*

$$X^h(x,y)^h(f,g) \to Y^h(F(x),F(y))^h(F(f),F(g))$$

is a weak equivalence in ***sSet****;*

2. *The functor*

$$h_2^{proj}(F) : h_2^{proj}(X) \to h_2^{proj}(Y)$$

is an equivalence of unbiased bicategories.

Theorem 5.2.9. *Suppose $f : X \to Y$ is a map in $\boldsymbol{CSSP}_2^{proj}$. Then f is a weak equivalence in $CSSP_2^{proj}$ if and only if it is a projective Dwyer-Kan biequivalence.*

Proof. The result follows from the fact that R preserves and reflects weak equivalences.

Indeed, it suffices to observe that F will be a projective Dwyer-Kan biequivalence if and only if $R(F) : R(X) \to R(Y)$ is a Dwyer-Kan biequivalence. Then F will be a projective fibrant Dwyer-Kan biequivalence if and only if $R(F)$ is a levelwise weak equivalence, which holds if and only if F is a levelwise weak equivalence between projective fibrant complete 2-fold Segal spaces as needed. □

We also have an alternative characterization of completeness for 2-fold Segal spaces. This requires a formalization of a result in [JS17], which depended upon the conjectured existence of homotopy bicategories for projective fibrant 2-fold Segal spaces. As we now have such a construction that is isomorphic to theirs, we may now precisely state the following result:

Lemma 5.2.10 ([JS17, Lemma 2.8])**.** *Suppose $X \in \boldsymbol{SeSp}_2^{proj}$. Then $X \in \boldsymbol{CSSP}_2^{proj}$ if and only if $X_{\bullet,0}$ is complete and every $X_{k,\bullet}$ is complete for $k \geq 0$.*

We may specialize this result to the Reedy fibrant case very easily:

Corollary 5.2.11. *Suppose $X \in \boldsymbol{SeSp}_2^{inj}$. Then $X \in \boldsymbol{CSSP}_2^{inj}$ if and only if $X_{\bullet,0}$ is complete and every $X_{k,\bullet}$ is complete for $k \geq 0$.*

We were made aware of this characterization by the discussion in [Ber18, Rem. 6.10]. In fact, one does not even require that every $X_{k,\bullet}$ is complete; it is sufficient to demand that each $X(x, y)$ is complete instead, as is done in the definition for complete Segal objects given in [BR20, Def. 5.3]:

Proposition 5.2.12 ([BR20, Def. 5.3])**.** *Suppose $X \in \boldsymbol{SeSp}_2^{inj}$. Then $X \in \boldsymbol{CSSP}_2^{inj}$ if and only if $X_{\bullet,0}$ is complete and for all $x, y \in (X_{0,0})_0$, the Segal space $X(x, y)$ is complete.*

We do not explore the proof of this fact here, though we do take an interest in a resulting characterization of completeness:

Definition 5.2.13. *Suppose $X \in \boldsymbol{SeSp}_2^{inj}$. Define $X_{heq2} \subseteq X_{1,0}$ to be the subobject consisting of the union of the path components of 1-morphisms that are equivalences in $h_2(X)$.*

Note that $X_{heq2} = (X_{\bullet,0})_{heq}$. Thus, we immediately have the following:

Corollary 5.2.14. *Suppose $X \in \boldsymbol{SeSp}_2^{inj}$. Then $X \in \boldsymbol{CSSP}_2^{inj}$ if and only if:*

1. *The morphism $X_{0,0} \to X_{heq2}$ is a weak equivalence;*

2. *For all* $x, y \in (X_{0,0})_0$, *the morphisms*

$$X(x, y)_0 \to X(x, y)_{heq}$$

are weak equivalences.

Note then that if $X \in \mathbf{CSSP}_2^{inj}$, two objects $x, y \in h_2(X)$ will be equivalent if and only if there is a path in $X_{0,0}$ from x to y. This is a reasonable extension of the case of complete Segal spaces, where for $Y \in \mathbf{CSSP}^{inj}$ we had that Y_0 was a moduli space of objects whose paths corresponded to isomorphisms in $h_1(Y)$.

Chapter 6

Conclusion

In summary, we have constructed two functorial means to obtain the unbiased homotopy bicategory of a Reedy fibrant 2-fold Segal space X, one involving choices of 2-truncated compositions and another via some choice of horizontal compositions μ_n for $n \geq 2$. Moreover, in both cases, we have constructed homotopy pseudofunctors of maps between Reedy fibrant 2-fold Segal spaces decorated with such choices. We have demonstrated that these constructions largely agree with one another, in that the latter factors through the former up to potential disagreement on certain coherence isomorphisms, which we conjecture to be equal regardless. Furthermore, we have extended our work to the case of projective fibrancy by the construction of a general Reedy fibrant replacement functor for projective fibrant functors $\mathscr{C} \to \mathbf{sSet}$, where $\mathscr{C}$ is a Reedy category. Finally, we have considered the behavior of our constructions on singular bisimplicial spaces to obtain fundamental bigroupoids and proven the applicability of our homotopy bicategories to characterizing both completeness of 2-fold Segal spaces and equivalences between such complete 2-fold Segal spaces, in both the Reedy and projective fibrant scenarios.

A number of intermediate constructions have been introduced in this work that may be of independent interest, such as simplicial composition diagrams and their **Cat**-valued nonsymmetric operad, globular left 2-homotopies, the means to compose globular left homotopies of simplicial spaces vertically and horizontally and the usage of higher homotopies to establish coherence conditions. Moreover, our Reedy fibrant replacement functor may be of interest on its own, as it is perfectly applicable to other notions of (∞, n)-category, like Θ_n-spaces. Investigating this functor further and proving more properties of it would potentially be a useful future endeavor.

It is our belief that h_2 should not only provide a large number of bicategories of manifolds and cobordisms, by for instance considering $(\infty, 2)$-categories of n-manifolds and higher cobordisms for any $n \geq 0$ and considering various appended data like framings and orientations, but also eventually lead to a natural homotopy symmetric monoidal bicategory by similar methods, namely by solving some natural family of lifting problems. We also suspect that results such as the cobordism hypothesis and its proof sketch by Lurie in [Lur09b] or the geometric cobordism hypothesis of Grady and Pavlov in [GP22b] may be transmitted to these bicategories, extending the results of Schommer-Pries in [Sch14b] and Pstrągowski in [Pst14]. Moreover, the methodology used to obtain h_2 should provide useful insights in the task of constructing homotopy n-categories of more general (∞, n)-categories, using models such as those of Trimble or Batanin and Leinster. Such work may elucidate connections between 'homotopy-theoretic' and 'algebraic' models of higher category.

6.1 Future Work

A number of future projects naturally branch from the results in this thesis. The foremost is to apply h_2 to some suitable $(\infty, 2)$-category of manifolds and cobordisms. While it is possible to do so with projective fibrant constructions, we believe it would be more suitable to choose some 2-fold Segal space of cobordisms that is Reedy fibrant on the nose. It appears that the work of Grady and Pavlov in [GP22b] should perhaps fit this criterion, though a complete proof appears to be pending in the literature at present. Alternatively, a new construction of such an $(\infty, 2)$-category could prove useful in and of itself.

Another goal is to extend h_2 to accommodate symmetric monoidal structures. One approach to defining a symmetric monoidal complete 2-fold Segal space, given in [CS19], is to consider a functor

$$A : \Gamma \to \mathbf{CSSP}_2^{proj}$$

where Γ is the category of finite pointed sets

$$\langle n \rangle := \{0, \cdots, n\}$$

with pointed maps

$$f : \langle n \rangle \to \langle m \rangle, \quad f(0) = 0$$

between them, such that for all $n \geq 0$ the natural map

$$A\langle n \rangle \to (A\langle 1 \rangle)^n$$

is a weak equivalence. We could consider a version of this definition where instead we have a functor

$$A' : \Gamma \to \mathbf{CSSP}_2^{inj}$$

such that the above maps are trivial Reedy fibrations. Such a structure would perhaps introduce suitable lifting problems whose solutions would obtain naturally a symmetric monoidal bicategory structure on $h_2(A\langle 1 \rangle)$. For instance, one would obtain a natural product on objects and hom-categories by considering solutions to the lifting problem

$$\begin{tikzcd} & A\langle 2\rangle \arrow[r] \arrow[d, two heads, "\sim"] & A\langle 1\rangle \\ A\langle 1\rangle \times A\langle 1\rangle \arrow[r] \arrow[ur, dashed] & A\langle 1\rangle \times A\langle 1\rangle & \end{tikzcd}$$

and restricting to levels 0 and 1. It should be noted that Γ, while not a Reedy category per se, is in fact a *generalized Reedy category* as defined in [BM10], so our Reedy fibrant replacement functor could perhaps be extended to this situation to handle the general case of the above maps being mere weak equivalences. We leave this to future work.

Before one gets too eager however, it is prudent to check that h_2 does behave as we should expect. For instance, identifying that h_2 is equivalent to other known constructions of homotopy bicategory in the literature, such as those of [Mos21] and [Cam20], would be a useful verification that h_2 indeed implements the expected notion of a 'homotopy bicategory'. Moreover, finding a nerve functor that is (Quillen) right adjoint to h_2 would be a useful feat. Producing a Quillen adjunction would require a model structure on **UBicat** or perhaps **Bicat**, though no such model structure seems to have yet been constructed in the literature. Model structures exist instead on bicategories with *strict* 2-functors [Lac04] and on *bigroupoids* with weak 2-functors [Bes18], the former of which was sufficient to establish the Quillen adjunction in [Cam20], though it may instead be prudent to wait for such a model structure to be introduced.

There is reason to suspect that our notion of h_2 will not admit a right adjoint per se, but perhaps something weaker such as a right *biadjoint*. Indeed, the possibility that h_2 factors through h_2^{tr} suggests some agreement with the work of Lack and Paoli in [LP08] on Tamsamani 2-categories and bicategories. For instance, h_2 identifies only the *normal homomorphisms*, as is the case with Lack and Paoli's realization functor from Tamsamani 2-categories to bicategories. If this should be the case, then Lack and Paoli's construction of a biadjoint rather than a strict adjoint suggests we should seek such a structure instead, under some suitable extension of $\mathbf{SeSp}_2^{inj}$ to a bicategory or perhaps simplicial category.

Some other basic results would be of use barring the more powerful ones listed thus far, such as demonstrating how to obtain from h_2 pseudonatural transformations and modifications. This requires a suitable notion of pseudonatural transformation of 2-fold Segal spaces, perhaps using the work on such structures in [JS17].

Looking beyond bicategories, it would be of great interest to construct extensions of h_2 to more general *homotopy n-category* functors h_n from (∞, n)-

categories to some suitable algebraic notion of n-category, such as Trimble n-categories or Batanin-Leinster n-categories. Such constructions may yield greater advancements in the world of algebraic n-categories, allowing perhaps for the transmission of model structures from the (∞, n)-categorical world down to models of n-category, as well as establishing lifting problems whose solutions naturally obtain algebraic higher-categorical data from models such as those of complete n-fold Segal spaces. There may moreover be situations where it is advantageous to work with algebraic n-categories, such as using computads to present higher categories in these models.

Having such homotopy n-category functors may moreover allow one to instantiate topological quantum field theories in the algebraic models of n-category, where the algebraic structure may more immediately permit presentations of higher categories of cobordisms by generators and relations in a similar style to [Sch14b]. Such advantages depend on progress in algebraic models of higher category, a task that may in and of itself be expedited by constructions like h_n. Until further results are obtained in this direction, the complete impact algebraic n-categories shall have on TQFTs will likely remain unknown.

Bibliography

[Ara14] Dimitri Ara. "Higher Quasi-Categories vs Higher Rezk Spaces". In: *Journal of K-theory* 14.3 (Dec. 2014), pp. 701–749. ISSN: 1865-2433, 1865-5394. DOI: 10.1017/S1865243315000021. arXiv: 1206.4354 [math]. (Visited on 10/21/2023).

[Ati88] Michael Atiyah. "Topological quantum field theories". en. In: *Publications Mathématiques de l'Institut des Hautes Études Scientifiques* 68.1 (Jan. 1988), pp. 175–186. ISSN: 1618-1913. DOI: 10.1007/BF02698547. URL: https://doi.org/10.1007/BF02698547 (visited on 10/31/2020).

[Bar+14] Bruce Bartlett et al. *Extended 3-dimensional bordism as the theory of modular objects*. 2014. arXiv: 1411.0945 [math.GT]. URL: https://arxiv.org/abs/1411.0945.

[Bar05] Clark Barwick. "(∞, n)-Cat as a Closed Model Category". PhD thesis. University of Pennsylvania, 2005.

[BD98] John C. Baez and James Dolan. "Higher-Dimensional Algebra III.n-Categories and the Algebra of Opetopes". In: *Advances in Mathematics* 135.2 (May 1998), pp. 145–206. ISSN: 0001-8708. DOI: 10.1006/aima.1997.1695. (Visited on 06/20/2022).

[Bén67] Jean Bénabou. "Introduction to Bicategories". In: *Reports of the Midwest Category Seminar*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1967, pp. 1–77. ISBN: 978-3-540-35545-8.

[Ber12] Julia E. Bergner. "Models for (∞, n)-Categories and the Cobordism Hypothesis". In: *arXiv:1011.0110 [math]* (Dec. 2012). arXiv: 1011.0110 [math]. (Visited on 11/27/2020).

[Ber18] Julia E. Bergner. "A Survey of Models for (∞, n)-Categories". In: *arXiv:1810.10052 [math]* (Oct. 2018). arXiv: 1810.10052 [math]. (Visited on 03/29/2021).

[Bes18] Martijn den Besten. *A Quillen model structure for bigroupoids and pseudofunctors*. en. arXiv:1809.00963 [math]. Sept. 2018. URL: http://arxiv.org/abs/1809.00963 (visited on 07/11/2024).

[BM10] Clemens Berger and Ieke Moerdijk. "On an extension of the notion of Reedy category". In: *Mathematische Zeitschrift* 269.3–4 (Sept. 2010), pp. 977–1004. ISSN: 1432-1823. DOI: 10.1007/s00209-010-0770-x. URL: http://dx.doi.org/10.1007/s00209-010-0770-x.

[BP15] David Blanc and Simona Paoli. "Segal-type algebraic models of n-types". In: *Algebraic & Geometric Topology* 14.6 (Jan. 2015). arXiv:1204.5101 [math], pp. 3419–3491. ISSN: 1472-2739, 1472-2747. DOI: 10.2140/agt.2014.14.3419. URL: http://arxiv.org/abs/1204.5101 (visited on 03/23/2023).

[BR13a] Julia E. Bergner and Charles Rezk. "Comparison of Models for (∞,n)–Categories, I". In: *Geometry & Topology* 17.4 (July 2013), pp. 2163–2202. ISSN: 1364-0380. DOI: 10.2140/gt.2013.17.2163. (Visited on 03/22/2023).

[BR13b] Julia E. Bergner and Charles Rezk. "Reedy Categories and the \$\$\varTheta \$\$-Construction". In: *Mathematische Zeitschrift* 274.1 (June 2013), pp. 499–514. ISSN: 1432-1823. DOI: 10.1007/s00209-012-1082-0. (Visited on 05/22/2023).

[BR20] Julia E. Bergner and Charles Rezk. "Comparison of Models for \$(\infty, n)\$-Categories, II". In: *Journal of Topology* 13.4 (Dec. 2020), pp. 1554–1581. ISSN: 1753-8416, 1753-8424. DOI: 10.1112/topo.12167. arXiv: 1406.4182. (Visited on 03/29/2021).

[Bro73] Kenneth S. Brown. "Abstract Homotopy Theory and Generalized Sheaf Cohomology". In: *Transactions of the American Mathematical Society* 186 (1973), pp. 419–458. ISSN: 00029947. URL: http://www.jstor.org/stable/1996573 (visited on 11/13/2024).

[Cam20] Alexander Campbell. "A Homotopy Coherent Cellular Nerve for Bicategories". In: *Advances in Mathematics* 368 (July 2020), p. 107138. ISSN: 00018708. DOI: 10.1016/j.aim.2020.107138. arXiv: 1907.01999 [math]. (Visited on 03/09/2023).

[Che11] Eugenia Cheng. "Comparing operadic theories of \$n\$-category". en. In: *Homology, Homotopy and Applications* 13.2 (2011), pp. 217–249. ISSN: 15320073, 15320081. DOI: 10.4310/HHA.2011.v13.n2.a14. URL: http://www.intlpress.com/site/pub/pages/journals/items/hha/content/vols/0013/0002/a014/ (visited on 01/19/2024).

[CS19] Damien Calaque and Claudia Scheimbauer. "A Note on the \$(\infty,n)\$-Category of Cobordisms". In: *Algebraic & Geometric Topology* 19.2 (Mar. 2019), pp. 533–655. ISSN: 1472-2739, 1472-2747. DOI: 10.2140/agt.2019.19.533. arXiv: 1509.08906. (Visited on 11/25/2020).

[Dav11] Kelly J. Davis. “Axiomatic TQFT, Axiomatic DQFT, and Exotic 4-Manifolds”. In: *arXiv:1106.2358 [math]* (Oct. 2011). arXiv: 1106.2358. URL: `http://arxiv.org/abs/1106.2358` (visited on 10/31/2020).

[DFT22] Julien Ducoulombier, Benoit Fresse, and Victor Turchin. “Projective and Reedy Model Category Structures for (Infinitesimal) Bimodules over an Operad”. en. In: *Applied Categorical Structures* 30.5 (Oct. 2022), pp. 825–920. ISSN: 0927-2852, 1572-9095. DOI: `10.1007/s10485-022-09675-z`. URL: `https://link.springer.com/10.1007/s10485-022-09675-z` (visited on 07/07/2024).

[DK19] Tobias Dyckerhoff and Mikhail Kapranov. *Higher Segal Spaces I*. Vol. 2244. 2019. DOI: `10.1007/978-3-030-27124-4`. arXiv: `1212.3563 [math]`. (Visited on 10/22/2023).

[DS11] Daniel Dugger and David I. Spivak. “Rigidification of quasi-categories”. In: *Algebraic & Geometric Topology* 11.1 (Jan. 2011). Publisher: Mathematical Sciences Publishers, pp. 225–261. ISSN: 1472-2739. DOI: `10.2140/agt.2011.11.225`. URL: `https://msp.org/agt/2011/11-1/p07.xhtml` (visited on 07/09/2024).

[DS95] W.G. Dwyer and J. Spalinski. “Homotopy Theories and Model Categories”. In: *Handbook of Algebraic Topology*. Elsevier, 1995, pp. 73–126. ISBN: 978-0-444-81779-2. DOI: `10.1016/B978-044481779-2/50003-1`. (Visited on 02/02/2021).

[Get13] Ezra Getzler. *Thick Simplices and Quasi-Categories*. Oct. 2013. DOI: `10.48550/arXiv.1307.6924`. arXiv: `1307.6924 [math]`. (Visited on 11/20/2023).

[GJ09a] Paul G. Goerss and John F. Jardine. “Model Categories”. In: *Simplicial Homotopy Theory*. Modern Birkhäuser Classics. Basel: Birkhäuser, 2009, pp. 1–64. ISBN: 978-3-0346-0189-4. DOI: `10.1007/978-3-0346-0189-4_2`.

[GJ09b] Paul G. Goerss and John F. Jardine. “Reedy Model Categories”. In: *Simplicial Homotopy Theory*. Modern Birkhäuser Classics. Basel: Birkhäuser, 2009, pp. 353–387. ISBN: 978-3-0346-0189-4. DOI: `10.1007/978-3-0346-0189-4_7`.

[GJ09c] Paul G. Goerss and John F. Jardine. “Simplicial Sets”. In: *Simplicial Homotopy Theory*. Modern Birkhäuser Classics. Basel: Birkhäuser, 2009, pp. 1–64. ISBN: 978-3-0346-0189-4. DOI: `10.1007/978-3-0346-0189-4_1`.

[GP22a] Daniel Grady and Dmitri Pavlov. *Extended field theories are local and have classifying spaces.* en. arXiv:2011.01208 [math-ph]. Feb. 2022. URL: http://arxiv.org/abs/2011.01208 (visited on 06/11/2023).

[GP22b] Daniel Grady and Dmitri Pavlov. *The geometric cobordism hypothesis.* arXiv:2111.01095 [math-ph]. June 2022. DOI: 10.48550/arXiv.2111.01095. URL: http://arxiv.org/abs/2111.01095 (visited on 06/11/2023).

[Gro] Alexander Grothendieck. *Pursuing Stacks.*

[Hau18] Rune Haugseng. "Iterated spans and classical topological field theories". en. In: *Mathematische Zeitschrift* 289.3-4 (Aug. 2018), pp. 1427–1488. ISSN: 0025-5874, 1432-1823. DOI: 10.1007/s00209-017-2005-x. URL: http://link.springer.com/10.1007/s00209-017-2005-x (visited on 03/11/2022).

[Hir09] Philip Hirschhorn. *Model Categories and Their Localizations.* Vol. 99. Mathematical Surveys and Monographs. Providence, Rhode Island: American Mathematical Society, Aug. 2009. ISBN: 978-0-8218-4917-0. DOI: 10.1090/surv/099. (Visited on 10/16/2020).

[HKK01] K. A. Hardie, K. H. Kamps, and R. W. Kieboom. "A Homotopy Bigroupoid of a Topological Space". In: *Applied Categorical Structures* 9.3 (May 2001), pp. 311–327. ISSN: 1572-9095. DOI: 10.1023/A:1011270417127. (Visited on 04/19/2023).

[HM22] Gijs Heuts and Ieke Moerdijk. *Simplicial and Dendroidal Homotopy Theory.* Ergebnisse Der Mathematik Und Ihrer Grenzgebiete. 3. Folge / a Series of Modern Surveys in Mathematics 3. Folge, Volume 75. Cham: Springer Nature, 2022. ISBN: 978-3-031-10446-6. DOI: 10.1007/978-3-031-10447-3.

[Hor15] Geoffroy Horel. "A model structure on internal categories in simplicial sets". In: *arXiv:1403.6873 [math]* (May 2015). arXiv: 1403.6873. URL: http://arxiv.org/abs/1403.6873 (visited on 01/19/2021).

[Hov07] Mark Hovey. *Model Categories.* Nachdr. Mathematical Surveys and Monographs 63. Providence, RI: American Mathematical Society, 2007. ISBN: 978-0-8218-4361-1.

[HS01] André Hirschowitz and Carlos Simpson. "Descente Pour Les N-Champs (Descent for n-Stacks)". In: *arXiv:math/9807049* (Mar. 2001). arXiv: math/9807049. (Visited on 08/04/2021).

[Joy02] A. Joyal. "Quasi-Categories and Kan Complexes". In: *Journal of Pure and Applied Algebra*. Special Volume Celebrating the 70th Birthday of Professor Max Kelly 175.1 (Nov. 2002), pp. 207–222. ISSN: 0022-4049. DOI: 10.1016/S0022-4049(02)00135-4. (Visited on 10/21/2023).

[JS17] Theo Johnson-Freyd and Claudia Scheimbauer. "(Op)Lax Natural Transformations, Twisted Quantum Field Theories, and "Even Higher" Morita Categories". In: *Advances in Mathematics* 307 (Feb. 2017), pp. 147–223. ISSN: 00018708. DOI: 10.1016/j.aim.2016.11.014. arXiv: 1502.06526. (Visited on 01/24/2021).

[JT07] André Joyal and Myles Tierney. "Quasi-Categories vs Segal Spaces". In: *Contemporary Mathematics* 431 (2007). Ed. by Alexei Davydov et al., pp. 277–326. DOI: 10.1090/conm/431/08278. (Visited on 11/20/2023).

[Koc03] Joachim Kock. *Frobenius Algebras and 2-D Topological Quantum Field Theories*. London Mathematical Society Student Texts. Cambridge: Cambridge University Press, 2003. ISBN: 978-0-521-83267-0. DOI: 10.1017/CBO9780511615443.

[Lac04] Stephen Lack. "A Quillen Model Structure for Bicategories". en. In: *K-Theory* 33.3 (Nov. 2004), pp. 185–197. ISSN: 15730514, 09203036. DOI: 10.1007/s10977-004-6757-9.

[Lei00] Tom Leinster. *Operads in Higher-Dimensional Category Theory*. Nov. 2000. DOI: 10.48550/arXiv.math/0011106. arXiv: math/0011106. (Visited on 07/13/2023).

[Lei04] Tom Leinster. *Higher operads, higher categories*. en. London Mathematical Society lecture note series 298. Cambridge, UK ; New York: Cambridge University Press, 2004. ISBN: 978-0-521-53215-0.

[Lor21] Fosco Loregian. *Coend calculus*. en. arXiv:1501.02503 [math]. June 2021. DOI: 10.1017/9781108778657. URL: http://arxiv.org/abs/1501.02503 (visited on 09/15/2024).

[LP08] Stephen Lack and Simona Paoli. "2-Nerves for Bicategories". In: *K-Theory* 38.2 (Jan. 2008), pp. 153–175. ISSN: 15730514, 09203036. DOI: 10.1007/s10977-007-9013-2. arXiv: math/0607271. (Visited on 10/14/2021).

[Lur08] Jacob Lurie. "Higher Topos Theory". In: *arXiv:math/0608040* (July 2008). arXiv: math/0608040. (Visited on 10/27/2020).

[Lur09a] Jacob Lurie. "(Infinity,2)-Categories and the Goodwillie Calculus I". In: *arXiv:0905.0462 [math]* (May 2009). arXiv: 0905.0462 [math]. (Visited on 01/24/2021).

[Lur09b] Jacob Lurie. "On the Classification of Topological Field Theories". In: *arXiv:0905.0465 [math]* (May 2009). arXiv: `0905.0465 [math]`. (Visited on 10/16/2020).

[May99] J. Peter May. "A Concise Course in Algebraic Topology". In: University of Chicago Press, 1999. ISBN: 0226511839.

[Mil65] John Milnor. *Lectures on the h-Cobordism Theorem*. Princeton: Princeton University Press, 1965. ISBN: 9781400878055. DOI: `doi:10.1515/9781400878055`. URL: `https://doi.org/10.1515/9781400878055`.

[MM94] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. Universitext. New York, NY: Springer New York, 1994. ISBN: 978-0-387-97710-2. DOI: `10.1007/978-1-4612-0927-0`. (Visited on 10/16/2020).

[Mos21] Lyne Moser. *A double $(\infty,1)$-categorical nerve for double categories*. en. arXiv:2007.01848 [math]. May 2021. URL: `http://arxiv.org/abs/2007.01848` (visited on 02/15/2024).

[MP12] Jon Peter May and Kate Ponto. *More Concise Algebraic Topology: Localization, Completion, and Model Categories*. Chicago Lectures in Mathematics Series. Chicago, Ill.: University of Chicago Press, 2012. ISBN: 978-0-226-51178-8.

[MP23] João Faria Martins and Timothy Porter. *A categorification of Quinn's finite total homotopy TQFT with application to TQFTs and once-extended TQFTs derived from strict omega-groupoids*. en. arXiv:2301.02491 [math]. Jan. 2023. URL: `http://arxiv.org/abs/2301.02491` (visited on 03/09/2023).

[Pao19] Simona Paoli. *Simplicial Methods for Higher Categories: Segal-type Models of Weak n-Categories*. en. Vol. 26. Algebra and Applications. Cham: Springer International Publishing, 2019. ISBN: 978-3-030-05673-5 978-3-030-05674-2. DOI: `10.1007/978-3-030-05674-2`. URL: `http://link.springer.com/10.1007/978-3-030-05674-2` (visited on 07/17/2024).

[Pst14] Piotr Pstrągowski. "On Dualizable Objects in Monoidal Bicategories, Framed Surfaces and the Cobordism Hypothesis". In: *arXiv:1411.6691 [math]* (Nov. 2014). arXiv: `1411.6691 [math]`. (Visited on 10/16/2020).

[Qui67] Daniel G. Quillen. *Homotopical Algebra*. en. Lecture Notes in Mathematics. Springer Berlin, Heidelberg, 1967. ISBN: 978-3-540-03914-3.

[Ras18] Nima Rasekh. "Introduction to Complete Segal Spaces". In: *arXiv:1805.03131 [math]* (May 2018). arXiv: `1805.03131 [math]`. (Visited on 03/16/2021).

[Rez00] Charles Rezk. "A Model for the Homotopy Theory of Homotopy Theory". In: *arXiv:math/9811037* (Mar. 2000). arXiv: `math/9811037`. (Visited on 01/19/2021).

[Rez10] Charles Rezk. "A Cartesian Presentation of Weak N-Categories". In: *Geometry & Topology* 14.1 (Jan. 2010), pp. 521–571. ISSN: 1364-0380, 1465-3060. DOI: `10.2140/gt.2010.14.521`. arXiv: `0901.3602`. (Visited on 03/16/2021).

[Rez22] Charles Rezk. "Introduction to Quasicategories". en. In: (2022). URL: `https://people.math.rochester.edu/faculty/doug/otherpapers/Rezk-quasicats-intro3.pdf` (visited on 11/11/2024).

[Rie14] Emily Riehl. *Categorical Homotopy Theory*. 1st ed. Cambridge University Press, May 2014. ISBN: 978-1-107-26145-7. DOI: `10.1017/CBO9781107261457`. (Visited on 10/14/2023).

[Rie20] Emily Riehl. *Homotopical Categories: From Model Categories to $(\infty,1)$-Categories*. July 2020. arXiv: `1904.00886 [math]`. (Visited on 06/11/2023).

[Rom23] Jack Romo. *Homotopy Bicategories of Complete 2-fold Segal Spaces*. 2023. arXiv: `2311.11983 [math.CT]`. URL: `https://arxiv.org/abs/2311.11983`.

[RS22] David Reutter and Christopher Schommer-Pries. *Semisimple Field Theories Detect Stable Diffeomorphism*. en. arXiv:2206.10031 [math]. Dec. 2022. URL: `http://arxiv.org/abs/2206.10031` (visited on 07/15/2024).

[RV22] Emily Riehl and Dominic Verity. *Elements of ∞-Category Theory*. 1st ed. Cambridge University Press, Jan. 2022. ISBN: 978-1-108-93688-0 978-1-108-83798-9. DOI: `10.1017/9781108936880`. (Visited on 08/02/2024).

[Sch14a] Claudia Scheimbauer. "Factorization Homology as a Fully Extended Topological Field Theory". PhD thesis. ETH Zürich, 2014.

[Sch14b] Christopher J. Schommer-Pries. "The Classification of Two-Dimensional Extended Topological Field Theories". In: *arXiv:1112.1000 [math]* (Aug. 2014). arXiv: `1112.1000 [math]`. (Visited on 12/14/2020).

[Sch17] Christopher Schommer-Pries. *Invertible Topological Field Theories*. en. arXiv:1712.08029 [math]. Dec. 2017. URL: `http://arxiv.org/abs/1712.08029` (visited on 09/15/2024).

[Seg88] G. B. Segal. "The Definition of Conformal Field Theory". en. In: *Differential Geometrical Methods in Theoretical Physics*. Ed. by K. Bleuler and M. Werner. Dordrecht: Springer Netherlands, 1988, pp. 165–171. ISBN: 978-94-015-7809-7. DOI: 10.1007/978-94-015-7809-7_9. URL: https://doi.org/10.1007/978-94-015-7809-7_9 (visited on 07/15/2024).

[Sim98] Carlos Simpson. *Homotopy types of strict 3-groupoids*. arXiv:math/9810059. Oct. 1998. DOI: 10.48550/arXiv.math/9810059. URL: http://arxiv.org/abs/math/9810059 (visited on 03/10/2023).

[Ste19] Raffael Stenzel. "On univalence, Rezk Completeness and presentable quasi-categories". en. phd. University of Leeds, Mar. 2019. URL: https://etheses.whiterose.ac.uk/24342/ (visited on 08/02/2024).

[Ste67] N. E. Steenrod. "A convenient category of topological spaces." In: *Michigan Mathematical Journal* 14.2 (May 1967). Publisher: University of Michigan, Department of Mathematics, pp. 133–152. ISSN: 0026-2285, 1945-2365. DOI: 10.1307/mmj/1028999711. (Visited on 03/06/2024).

[Tam99] Zouhair Tamsamani. "Sur des notions de n—categorie et n—groupoide non strictes via des ensembles multi-simpliciaux (On the notions of a nonstrict n–category and n–groupoid via multisimplicial sets)". In: *K-Theory* 16.1 (1999), pp. 51–99. DOI: 10.1023/A:1007747915317.

[Tri] Todd Trimble. *Tetracategories*. URL: https://math.ucr.edu/home/baez/trimble/tetracategories.html (visited on 06/20/2022).

[Ver08] D. R. B. Verity. "Weak Complicial Sets I. Basic Homotopy Theory". In: *Advances in Mathematics* 219.4 (Nov. 2008), pp. 1081–1149. ISSN: 0001-8708. DOI: 10.1016/j.aim.2008.06.003. (Visited on 06/21/2022).

[Wyl73] Oswald Wyler. "Convenient categories for topology". In: *General Topology and its Applications* 3.3 (Sept. 1973), pp. 225–242. ISSN: 0016-660X. DOI: 10.1016/0016-660X(72)90014-1. URL: https://www.sciencedirect.com/science/article/pii/0016660X72900141 (visited on 03/06/2024).