Model-Theoretic Considerations in Finite Combinatorics

Aristomenis-Dionysios (Aris) Papadopoulos

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- The results appearing in Chapter 3 originally appeared in [MP23b] and [MP23a]. Both of these works were written jointly with Nadav Meir.
- The results appearing in Chapter 4 originally appeared in [MPT23]. This work was written jointly with Nadav Meir and Pierre Touchard.
- The results in Chapter 5 are a joint work with Pantelis Eleftheriou. This work is currently being prepared for publication.
- Some of the results in Chapter 6 are based on joint work with Blaise Boissonneau and Pierre Touchard which originally appeared (amongst other results) in [MPT23].

In all cases, the contributions of all authors to the work presented here were equal. All results are reproduced in this thesis with the permission of all coauthors.

* * *

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IL FAUT CONFRONTER LES IDÉES VAGUES AVEC DES IMAGES CLAIRES

Jean-Luc Godard, *La Chinoise*

Global Acknowledgements[†]

First and foremost (and not just because it is traditional to do so) I'd like to thank all the people who, at times during the last three and a half years, were my supervisors (two at the beginning of my PhD, and two at the end, though never three at the same time). Starting a PhD in 2020 is definitely not something I would recommend to anyone, but **Dugald Macpherson** gave me immeasurable support and helped me get through it. I am so grateful and proud to call myself your student. **Isolde Adler** was always excited to work with me, and although our paths diverged early on, our discussions have heavily influenced my work. It's hard to summarise my relationship with **Pantelis Eleftheriou**, given the amount of hours we've spent working together and how generous he's been with me, so I won't really try to expand on this here. Pant, you should already know how grateful I am for everything.

I also greatly appreciate the efforts of my examiners **Vincenzo Mantova** and **Frank Wagner**, who read through this entire document and made (quite a fair amount of) suggestions which greatly improved its quality. All remaining mistakes in the pages that follow are entirely my fault!

The rest of these acknowledgments may appear somewhat sentimental. I apologise, if that's not your cup of tea, but that's what happens when one starts a PhD during a global pandemic...

* * *

Many more people have been very generous to me during this journey that started with no thesis and is hopefully coming to a close with this piece of writing (that would certainly like to call itself a thesis, and if you may permit its arrogance, henceforth will do so). They shared with me their mathematical ideas, their couches, and their beds. First and foremost, I am indebted to my collaborators **Blaise Boissonneau**, **Sam Braunfeld**, **Ioannis Eleftheriadis**, **Nadav Meir**, and **Pierre Touchard** for all the discussions, which (sometimes evolved into arguments, which, in turn,) evolved into theorems.

I consider myself extremely fortunate to belong to the broader **Model Theory community**. I will not attempt to write a long list of names here, but I am

[†]Local acknowledgements will be included at the end of some chapters. These should be understood as their union with the global ones, that is: the global acknowledgements apply to *all* chapters, not just the ones where no other thanks are given.

extremely grateful to have met each and every one of you. You are the reason I'm at a point where I can write all this down, but most importantly, the reason I had a lovely time getting here. The cliché that "mathematics does not happen in a vacuum" rang very true when after almost two years of lockdown I got to go to my first conference. That was the point I realised that the people are the reason we're all sticking this out. Well, at least, the reason why I am.

The **Leeds Logic group** needs a special mention. Since the doors of the SoM opened again and I got to exist in the Level 8 office, it really felt like a second home. This is not quite a comment on the hours I've spent there, but rather on the great people[1] who made it such a warm and lovely environment.

I was also lucky to belong to several awesome reading groups in the last few years. Their intersection, if I'm not mistaken, consists of **Paolo Marimon**.

* * *

I think this all began, a bit more than fourteen years ago (it was the 20th of February 2010, to be precise – take it from me, putting dates at the backs of books can be extremely helpful), when **my parents** brought home a strange little book called *Logicomix*. I should thank them for making the mistake of putting me down this path. But really, I should thank them for always being there for me, supporting me in every way ~~possible~~ imaginable. I may have learned no maths from you, but you taught me all the important things.

Perhaps I should now list some people who have not been mentioned yet and (mostly unknowingly) helped me end up here. **Ian Hodkinson**'s Logic course in my first year of undergrad was certainly pretty conclusive evidence that Logic was what I wanted to do. **David Evans**'s Mathematical Logic course in my third year of undergrad made the sale final. Writing a masters thesis under the kind supervision of **Zaniar Ghadernezhad** and **Lotte Kestner**, moved me a step closer to becoming a model theorist.[2] Following **Nick Ramsey**'s course on Model-Theoretic Tree Properties during my first year of PhD was the reason I kept thinking about model theory through lockdown. Of course, none of this would be possible without **Saharon Shelah**'s beautiful maths.

[1] Current: Tommy, Christian, **Asaf Karagila** (who insisted his name be in boldface), Pietro, Mervyn, Pablo, Matteo, Andrew, Calliope, Hope. Past: Francesco, Ibrahim, Richard, Carla, Gabriele, Giovanni, Rosario. WHO AM I FORGETTING? I'm so sorry!

[2] If **Andrew Brook-Taylor** had replied to that one e-mail from 2019 things may have turned out differently, but I'm glad he never did!

Finally, to all the non-model theory adjacent **friends** (people?) who were there along the way (I hope you know who you are, but if you think you should belong to this group of people, then you most certainly do): **I love all of you**. You should stop reading at the end of this page. (Okay, maybe you can check out the abstract, but no further!)

* * *

Luci and **Misirlou** you never made it easy, but you certainly made it good.

Abstract

This thesis contains a collection of results that lie primarily in the intersection of model theory and finite combinatorics. The two main areas of combinatorics that make an appearance in the pages to follow are *Structural Ramsey Theory* and *Extremal Graph Theory*.

Chapter 3 focusses on interactions between model theory and structural Ramsey theory. There, I essentially present two new results. The first is an extension of Scow's theorem connecting generalised indiscernibles and Ramsey classes, and the second is a construction of a strict hierarchy of "local" Ramsey classes.

In Chapter 4, the focus switches, momentarily, to pure model theory, and specifically *Classification Theory*, in the sense of Shelah. This chapter investigates dividing lines characterised by collapses of indiscernibles and transfer principles for products of structures for such dividing lines. Structural graph theory also shows up in the last section, dealing with bounds on the twin-width of classes of products of graphs.

In Chapter 5, I present results on extremal graph theory, in particular, Zarankiewicz's problem, restricted to model-theoretically tame contexts. These include semibounded o-minimal structures, models of Presburger arithmetic, and some Hrushovski constructions.

Finally, there is a chapter with two smaller results that both fall, at least in my opinion, within the broader context of intersections of model theory and combinatorics which closes the thesis.

Keywords: Model Theory, Classification Theory, Structural Ramsey Theory, Extremal Graph Theory.

Mathematics Subject Classification: Primary: 03C45, 05D10. Secondary: 03C52, 03C64, 05C35.

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Chapter 1

Introduction

> 'Logic issues in tautologies, mathematics in identities, philosophy in definitions; all trivial, but all part of the vital work of clarifying and organising our thought'
>
> Frank P. Ramsey, *The Foundations of Mathematics*

1.1 A Very Broad Introduction

This is not a thesis on the philosophy of mathematics. That being said, the results I will present in the following pages were often driven by a somewhat philosophical principle, which I want to talk about upfront.

Guiding Philosophy. Combinatorial truths, no matter how elementary they may sound, are usually true for "deep" reasons.[1] The language of model theory is particularly well-equipped to help us uncover these reasons, thus allowing us to:

(a) Gain a more fundamental understanding of combinatorics; and (b) Prove combinatorial results in the generality they deserve.

What follows is a brief discussion of this guiding principle with a non-mathematical audience in mind. For my more mathematically minded reader(s), especially those with time constraints, I suggest skipping right ahead to the next sections.

There are two questions that should spring to mind when reading the guiding

[1] The word *deep* is often used by mathematicians to describe a result whose proof requires a lot more mathematical machinery than its statement.

philosophy above: "What does he mean by a *combinatorial truth*?" and: "What is the *language of model theory*?" I will start by addressing the latter and then expand on the former through a couple of examples.

In the introduction to his excellent book [Hod93], Wilfrid Hodges attempts to define what model theory is. He gives the following definition:

"Model theory is the study of the construction and classification of structures within specified classes of structures".

The key tool in both the construction and classification aspects of model theory is the focus on *definable sets*, in the sense of *first-order logic*. Once we make the concession that the sets which can be described using formulas built with universal/existential quantifiers, function and relation symbols are the interesting ones, model theory offers us a unified way of viewing structures which at first sight may seem unrelated: By examining how similarly their definable sets behave. For example, in the eyes of a model theorist, linear independence in vector spaces and algebraic independence in (algebraically closed) fields look the same.

This is a rather brief (and maybe somewhat mysterious) introduction to model theory. Admittedly, model theory means many different things to many different people, and this was my attempt to say something that would be more-or-less universally accepted. This does have the unfortunate side-effect that what I said might be rather devoid of any actual content. Hopefully, though, if somebody who is not a model theorist is reading this, I managed to pique their interest. To that person, I would say that much more discussion can be found in [Poi00], and many more examples in [Hod93], so I will stop here.

As for *combinatorial truths*, this is an even broader umbrella. The aspects of combinatorics that will be of interest in this thesis are *(structural) Ramsey theory* and *Exrtemal Graph Theory*. In a sense, Ramsey theory is a formalisation of the idea that "absolute disorder is inevitable": Somehow, if we look at large enough combinatorial objects (say graphs), then they will necessarily contain some structured (e.g. homogeneous) subobject. On the other hand, it is slightly harder to give a very high-level description of extremal graph theory, but here is Bollobás's description, from [Bol78]:

"In extremal graph theory one is interested in the relations between various graph invariants [...], and also in the values of these invariants which ensure

that the graph has a given size."

A classical instance, and one which will be of interest later on, is the following: What can we say about the number of edges of a graph on n vertices if we know that this graph does not have a fixed graph as a subgraph? This is the problem of determining the *extremal number* of a given graph (with respect to the fixed omitted subgraph). Questions of this form have been puzzling graph theorists for at least 80 years. Hopefully, this will become clearer in the later chapters of this thesis.

Now that we are converging closer to the same page...

1.2 A Slightly Less Broad Introduction

One of the most successful themes of modern model theory, essentially starting with the work of Morley but later vastly generalised by Shelah, is on the classification side. Central here is the concept of a *dividing line*. Concisely put, a dividing line is a (local) combinatorial property of definable sets whose presence separates the class of all first-order theories into two sides, a *tame* side and a *wild* side. The theories on the tame side of a dividing line should all come with strong positive structural results, while those on the wild side should, in some concrete sense, be much more complicated. This idea of drawing dividing lines on the "map" of first-order theories is perhaps the reason why Hrushovski called model theory "the geography of tame mathematics".

As geographers, we really ought to have a map. Fortunately, G. Conant undertook this task and gave us one:

In Section 2.3, I will discuss some of the regions of the map above in more detail.

Apart from the plethora of beautiful structural results for tame theories that Shelah was aiming for, it has become evident that the dividing line methodology has rich applications in many fields of mathematics outside of mathematical logic, such as algebraic geometry (for instance, in [Hru96] Hrushovski proved the Mordell-Lang conjecture in any characteristic, using techniques originating from stability theory).

On the combinatorial side of things, Breuillard, Green, and Tao crucially used Hrushovski's *Lie model theorem* from [Hru12], a general theorem whose proof is

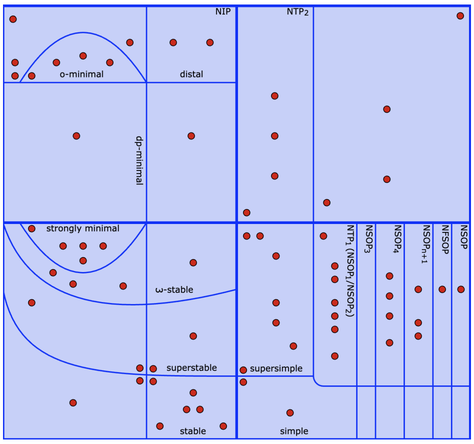


Figure 1.1: The Map of The Universe [Con].

based on techniques from local stability theory, to obtain a complete description of finite approximate groups in an arbitrary group G. In recent years, the dividing line methodology has been shown to play an important role in graph theory, pinpointing, for example, in very precise ways the various restricted contexts in which one can obtain strong forms of *Szemerédi's regularity lemma*, starting with the work of Malliaris and Shelah, in [MS14]. Of course, I have left out many important connections between model theory and combinatorics. For instance, as Laskowski observed in [Las92], model theorists had unknowingly been talking about theories with bounded VC-dimension from as early as 1971 when the notion of the *Independence Property* was introduced by Shelah.

Let me now briefly introduce the results that will occupy the remainder of this thesis.

1.3 An (incomplete) List of Results

It is standard to present the main results of a mathematical work in the first chapter, and this is what I will do in the next pages. Of course, the terminology involved has not yet been introduced, and I will not make any serious attempts to formally introduce it here. Thus, the results may appear somewhat apocryphal. In any case, I will do my best to present them with a brief introduction/discussion. Significantly more discussion will, of course, be given in the actual body of the thesis.

Chapter 3: Generalised Indiscernibles and Ramsey Theory

A very important tool that model theorists sometimes take for granted is *(order-)indiscernible sequences.* The key property behind their ubiquity is that given *any* sequence indexed by an infinite linear order, one can always realise its Ehrenfeucht-Mostowski type (EM-type) by an order indiscernible sequence. This is essentially an application of Ramsey's theorem and compactness.

It turns out that the connection between indiscernibles indexed by arbitrary structures, so-called *generalised indiscernibiles*, and *structural Ramsey theory* is very deep. Indeed, the main theorem of [Sco15] states that, under some technical assumptions, $\mathcal{N}$*-indexed indiscernibles* have the *modelling property* (the analogue of being able to find an indiscernible sequence realising a given EM-type) if, and only if, $\mathsf{Age}(\mathcal{N})$ is a Ramsey class (i.e. satisfies a structural analogue of Ramsey's theorem). In [MP23b], which is a joint work with N. Meir, we pass through "tame" expansions in infinitary logic to strengthen Scow's result, by removing all the technical assumptions. More precisely, we obtain the following:

Theorem (Theorem A)**.** *Let $\mathcal{N}$ an infinite, locally finite structure. Then $\mathsf{Age}(\mathcal{N})$ is a Ramsey class if, and only if, $\mathcal{N}$-indexed indiscernibles have the modelling property.*

We then use the theorem above to extend a known result about Ramsey classes to show that every Ramsey class consists of (essentially) *ordered* structures:

Theorem (Theorem B)**.** *Let $\mathcal{C}$ be a Ramsey class of finite $\mathcal{L}$-structures. Then there is an $\mathcal{L}_{\infty,0}$-formula (i.e., a possibly infinite Boolean combination of quantifier-free $\mathcal{L}$-formulas) that defines an order on all structures in $\mathcal{C}$.*

In [MP23a], which is also a joint work with N. Meir, a more local/combinatorial approach is taken, and we examine *finitary approximations* to the Ramsey property. By considering an interesting class of reducts of (generically ordered) random k-hypergraphs, the (generically ordered) *Kay-graphs*, which are higher-arity analogues of the well-studied class of *two-graphs*, we find the exact breaking point of the Ramsey property in these reducts, and show the following strict hierarchy of finitary approximations to the Ramsey property:

Theorem (Theorem D)**.** *For all $k \in \mathbb{N}_{\geq 1}$, there is a class $\mathcal{C}$ of finite ordered structures, which is Ramsey up to k, but not up to $k + 1$.*

Chapter 4: On Products of Structures

In [MPT23], which is a joint work with N. Meir and P. Touchard we take the rather novel approach of using generalised indiscernibles in structural Ramsey theory further. Using *collapsing indiscernibles* as a witness for the Ramsey property, we obtain a general "non-Ramsey" criterion, answering a question of P. Simon.[2] More precisely:

Theorem (Theorem G)**.** *Let $\mathcal{M}$ be a homogeneous n-ary $\aleph_0$-categorical structure in a finite relational language, and let $\mathcal{N}$ be a non-n-ary reduct of $\mathcal{M}$ in a finite relational language. Then $\mathsf{Age}(\mathcal{N})$ is not a Ramsey class.*

As a consequence of the above theorem, we can easily deduce the following general result about reducts of the generically ordered random k-hypergraph:

Theorem (Theorem H)**.** *Any proper reduct of the generically ordered random k-hypergraph which is not interdefinable with* DLO *is not Ramsey.*

In the same work, we also focus on *transfer principles* for products of structures. One of the goals of this project is to study which combinatorial configurations (viz. dividing lines) can be passed from two structures to their product. Our main tool is a complete characterisation of generalised indiscernible sequences in various kinds of products of structures, based on the behaviour of said sequences in the factors. For example, we obtain the following results for *lexicographic sums* of structures (a construction which generalises the *wreath product* of graphs).

Theorem (Theorem K)**.** *Let $\mathcal{M}$ be an $\mathcal{L}_{\mathcal{M}}$-structure and $\mathfrak{N} = \{\mathcal{N}_a\}_{a\in\mathcal{M}}$ be an $\mathcal{M}$-indexed collection $\mathfrak{N}$ of $\mathcal{L}_{\mathfrak{N}}$-structures. For*

$$P \in \{\textit{NIP}_n,\ \textit{indiscernible-triviality},\ \textit{monadic NIP},\ n\textit{-distality}\},$$

we have that both $\mathcal{M}$ and the common theory of $\{\mathcal{N}_a : a \in \mathcal{M}\}$ have P if, and only if, the lexicographic sum $\mathcal{M}[\mathfrak{N}]$ has P.

The key tool in the proof of the theorem above is, again, generalised indiscernible sequences. The general principle to keep in mind is that the *uncollapsed* indiscernibles that can exist in models of a theory give us a lot of information about "combinatorial traces" that the theory can see.

[2]Private correspondence.

Chapter 5: Around Zarankiewicz's Problem

Let me now discuss a very differently flavoured area of combinatorics, in which model-theoretic considerations have been shown to play an important role. To set the stage, recall the following well-known combinatorial question.

Question (Zarankiewicz's Problem, 1951). *What is the maximum number of edges (in terms of the number of vertices) a bipartite graph*[3] *can have, if it is* $K_{r,r}$-free *(i.e. it does not contain the complete bipartite graph* $K_{r,r}$ *as a subgraph) for some* $r \in \mathbb{N}$*?*

This question is known as *Zarankiewicz's Problem.* The upper bounds for arbitrary graphs and hypergraphs were given by Kővari-Sós-Turán and by Erdős, respectively. The picture, however, drastically changes if we restrict our attention to families of graphs that arise in algebrogeometric contexts. An important idea to keep in mind is that a lot of constructions used to prove sharp upper bounds in combinatorics are done in algebraic structures, and model theory can be used to show that the presence of algebra is, in a sense, inevitable. An example is given below.

One of the theorems in [Bas+21], essentially says that if $E \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ is a $K_{r,r}$-free *semilinear* relation (or, put in more model-theoretic terms, a $K_{r,r}$-free relation definable in $\langle \mathbb{R}, +, < \rangle$), then $E \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ has *linear* Zarankiewicz bounds. More precisely, there is some $\alpha \in \mathbb{R}_{>0}$ such that for every finite subset $B \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ we have that $|E \cap B| \leq \alpha |B|$. Moreover, using a deep result about o-minimal structures, the so-called *Trichotomy Theorem* of Peterzil and Starchenko [PS98], it is shown that if in some expansion of $\langle \mathbb{R}, +, < \rangle$ this is not the case, then that expansion *must* define a field, on some interval.

This leads to the following question: Is there a *"Zarankiewicz-like principle"* that guarantees the existence of a definable *full* field, i.e. a field that is not contained in any bounded interval? In joint work with P. Eleftheriou, we answer this question positively, by considering *eventually linear* relations (say, relations definable in $\mathbb{R}_{\mathsf{sbd}}$, the expansion of $\langle \mathbb{R}, +, < \rangle$ by all *bounded* semialgebraic subsets). In this case, we focus not on the definable relations that are $K_{r,r}$-free, but instead, on those that are $K_{r,r}$-free, when we restrict our attention to N*-distant* complete graphs, i.e. complete bipartite graphs

[3]Throughout this section I will restrict my attention to bipartite graphs, for the sake of notational simplicity, but all the results mentioned have natural "k-partite hypergraph" analogues.

whose vertices (on each part of the partition) are at distance at least N from each other. We dub such relations $K^N_{r,r}$-*free*, and prove that if we restrict our attention to eventually linear $K^N_{r,r}$-free graphs, then optimal bounds are achieved.

Theorem (Theorem O)**.** *Let $\mathcal{M} = \langle M, <, +, \dots \rangle$ be a saturated o-minimal expansion of an ordered group, and assume that $\mathcal{M}$ is semibounded. Then, for every (parameter-)definable $E \subseteq M^{d_1} \times M^{d_2}$, there is an $\alpha \in \mathbb{R}_{>0}$ such that if the infinite bipartite graph $(M^{d_1} \sqcup M^{d_2}, E)$ is $K^{\mathsf{tall}}_{\infty}$-free*[4]*, then for every tall finite grid $B = B_1 \times B_2 \subseteq M^{d_1} \times M^{d_2}$ we have:*

$$|E \cap B| \leq \alpha n,$$

where $n = \max\{|B_1|, |B_2|\}$.

Moreover, such bounds can be used to characterise the existence of *full fields* in a saturated o-minimal expansion of an ordered group. More precisely:

Theorem (Theorem P)**.** *Let $\mathcal{M}$ be a saturated o-minimal expansion of an ordered group. Then the following are equivalent:*

1. *$\mathcal{M}$ is semibounded.*

2. *("Linear" Zarankiewicz bounds) There is some $N \in M_{>0}$ such that: For all definable binary relations $E \subseteq \mathcal{M}^{d_1} \times \mathcal{M}^{d_2}$ there is $\alpha \in \mathbb{R}_{>0}$, such that if the infinite bipartite graph $(M^{d_1} \sqcup M^{d_2}, E)$, does not contain any infinite complete bipartite subgraphs, then for every finite N-distant grid $B = B_1 \times B_2 \subseteq M^{d_1} \times M^{d_2}$ we have:*

$$|E \cap B| \leq \alpha n,$$

   where $n = \max\{|B_1|, |B_2|\}$.

3. *(Full fields) There is no field definable on the whole of M.*

In the same work, we extend some of the more abstract results of [Bas+21] to obtain a general theorem about graphs omitting *(geometrically) independent* complete subgraphs in *(weakly) locally modular pregeometries*. As a corollary, we obtain an optimal result for graphs definable in Presburger arithmetic (see below) and in some Hrushovski constructions.

[4]Recall that a positive element t in a semibounded o-minimal expansion of an ordered group is *tall* if there is no field definable on the interval $(0, t)$.

Theorem (Theorem Q)**.** *Let $E \subseteq \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}$ be a binary relation definable in Presburger arithmetic. If E is $K_{r,r}$-free for some $r \in \mathbb{N}$ then there is a real number $\alpha \in \mathbb{R}_{>0}$ such that for all finite $B = B_1 \times B_2 \subseteq \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}$ we have*

$$|E \cap B| \leq \alpha n,$$

where $n = \max\{|B_1|, |B_2|\}$.

The Rest of the Thesis

The rest of this thesis has two more chapters:

- Chapter 2 contains background that is used everywhere in the thesis, sets up notation and is my attempt to put some of the later results in context.
- Chapter 6 contains a two smaller results, which I will discuss in more detail there.

Dependencies All (remaining) chapters of this thesis refer to results and terminology introduced in Chapter 2, but apart from that Chapters 3 to 6 are mostly independent of each other, except that some results of Chapter 3 are mentioned in Chapter 4.

Since the main chapters enjoy such a degree of independence, I have chosen to keep them as self-contained as possible (apart from global preliminaries), thus including a small introduction at the beginning of each of them, and briefly recalling facts/notation from previous chapters when necessary. Thus, it is possible (and perhaps even recommended) for the reader to start reading from the beginning of whichever of Chapters 3 to 6 seems more appealing to them at a given time and refer back to Chapter 2 if/when needed.

Chapter 2

Global Preliminaries[†]

> 'The sun shone, having no alternative, on the nothing new.'
>
> Samuel Beckett, *Murphy*

2.1 Introduction

When I started writing this document, I had the hope that it would be a piece of writing accessible to anyone, regardless of their background, provided they did not have any time constraints.[1] I must, however, admit that I have certainly failed in this regard, and I will indeed need to assume that the reader of this thesis is a mathematical logician familiar with model theory. That being said, in this chapter, I will try to summarise a lot of background material that will be useful when reading the rest of this thesis. This chapter is also here to set up a lot of the notation that will be persistent and (to the best of my abilities) will also be consistent throughout the thesis.

[†]This chapter, as hinted by the narrator of *Murphy*, contains no new results. It also does not contain all the background material required for reading the remainder of this thesis. It does, however, contain a superset of the intersection of preliminary material required for all the main chapters – I've opted to keep the more specialised background closer to where it will be used. To avoid making this chapter completely dry, I've included a couple of proofs I consider nice (which is not to say that I do not consider the proofs omitted equally nice, but choices had to be made). That being said, I recommend the reader familiar with the material indicated by the titles of the sections to just skip forward.

[1]To be completely honest, I had hoped that my recently/soon-to-be retired parents (who are not mathematicians and whose knowledge of English is somewhat limited) would be able to get something out of this piece of writing.

These “global preliminaries” are divided into six sections. More precisely:

1. **General Model Theory** – This little section is only here to fix the model-theoretic notation that will be used in the remainder of the thesis. I assume the reader is familiar with [TZ12, Chapters 1-5]. Much of the material regarding indiscernibles, from [TZ12, Chapter 5], will be recalled in the (Generalised) Indiscernibles subsection.

2. **Some Dividing Lines** – In Chapter 1 I included a picture of the *Map of the Universe*, and here I will quickly introduce some of the most classical (in my opinion) regions of that map (apologies *simple* theories, but you will not be included).

3. **Combinatorial Geometries** – An abstract overview of combinatorial geometries (matroids) will be given here, mainly to fix some notation and terminology.

4. **Structural Ramsey Theory** – This section gives a quick overview of structural Ramsey theory, starting from Ramsey’s theorem and ending with the Kechris-Pestov-Todorčević correspondence.

5. **(Generalised) Indiscernibles** – This section discusses various forms of indiscernibility. It ends rather abruptly (and somewhat surprisingly) without giving Scow’s theorem. This will be done later, I promise.

6. **A World of Structures** – This section, which occupies the most significant part of the preliminaries, contains what could reductively be described as “just a long list of structures”. But, if classification theory is interesting, it is because of the citizens of the various regions it maps out, and in this subsection, I want to introduce some of them.

With all this out of the way, let’s get going.

2.2 Some General Model Theory

Throughout this thesis, the letter $\mathcal{L}$ (and primed versions) will always refer to a first-order language.[2] (possibly multisorted). Languages will sometimes be identified with the collection of formulas or sentences built with their symbols,

[2] A first-order language is just a collection of symbols for *relations*, *functions*, and *constants* Symbols of the first two kinds come with an associated arity, and constant symbols can be viewed as 0-ary functions.

depending on the context. The letters $\mathcal{M}, \mathcal{N}, \ldots$ will typically denote infinite structures with domain $M, N, \ldots$, respectively, and sometimes the operator dom will be used to do this explicitly. If $\mathcal{M}$ is an $\mathcal{L}$-structure and $A \subseteq M$, then $\mathcal{L}(A)$ is the language $\mathcal{L}$ with fresh constant symbols for the elements of A. These are interpreted as the elements they are named after. Usually, the letters $\mathcal{A}, \mathcal{B}, \ldots$ will be reserved for finite(ly generated) structures with domain $A, B, \ldots$, and sometimes finite structures may be identified by their domain. If $A \subseteq \mathcal{M}$, then $\langle A \rangle_{\mathcal{M}}$ denotes the smallest substructure of $\mathcal{M}$ that contains A, that is, the substructure of $\mathcal{M}$ *generated by* A. When somewhat relevant, I will try to distinguish between elements and tuples, but this will not always be the case.[3]

In general, the rest of the model-theoretic notation throughout the thesis is either explained or standard. For example, given an $\mathcal{L}$-structure $\mathcal{M}$, a tuple $\bar{a} \in M^n$ and a subset $A \subseteq M$, I will denote by $\mathsf{tp}_{\mathcal{M}}(\bar{a}/A)$ the *type*[4] of $\bar{a}$ over A in $\mathcal{M}$, and if $\mathcal{M}$ is understood, then the subscript will be dropped. I will always write $\mathsf{tp}_{\mathcal{M}}(\bar{a})$ for $\mathsf{tp}_{\mathcal{M}}(\bar{a}/\emptyset)$. I will similarly write $\mathsf{qftp}_{\mathcal{M}}(\bar{a})$ for the *quantifier-free type* of $\bar{a}$ in $\mathcal{M}$, that is, the set of all quantifier-free $\mathcal{L}$-formulas $\phi(\bar{x})$ such that $\mathcal{M} \vDash \phi(\bar{a})$. If $\Delta \subseteq \mathcal{L}$ is a set of formulas in free variables from $\bar{x}$ I will write $\mathsf{tp}^{\Delta}_{\mathcal{M}}(\bar{a})$ for the restriction of $\mathsf{tp}_{\mathcal{M}}(\bar{a})$ to Δ-formulas, that is:

$$\mathsf{tp}^{\Delta}_{\mathcal{M}}(\bar{a}) = \{\phi(\bar{x}) \in \mathsf{tp}_{\mathcal{M}}(\bar{a}) : \phi(\bar{x}) \in \Delta \text{ or } \neg\phi(\bar{x}) \in \Delta\}.$$

Sometimes Δ will be all of $\mathcal{L}$, and in this case, the superscript will appear if multiple languages are present, to (hopefully) improve readability.

Sequences indexed by a structure $\mathcal{I}$ are just functions from the domain of $\mathcal{I}$. Sometimes I will write $(a_i : i \in \mathcal{I})$ to point out that we are working with an $\mathcal{I}$-indexed sequence, i.e. that the $\mathcal{I}$ structure on I matters, but sometimes, when there are two different relevant structures $\mathcal{I}, \mathcal{I}'$ on the same domain I, I may only write $(a_i : i \in I)$. Hopefully, this will not cause any confusion.

Often, *monster models* (see [TZ12, Chapter 6]) of a given $\mathcal{L}$-theory T will be employed. The letter $\mathbb{M}$ (or $\mathbb{I}, \mathbb{J}, \ldots$) will be reserved for monster models; these are $\kappa(\mathbb{M})$-saturated and $\kappa(\mathbb{M})$-homogeneous models of T, for some "very big" cardinal $\kappa(\mathbb{M})$ (where "very big" just means bigger than the cardinality of

[3] For instance, the distinction is rather irrelevant when working in an eq expansion (see [Pil96, Chapter 1]).

[4] In general, when I say type (resp. quantifier-free type), I mean a *complete* type (resp. quantifier-free type). Partial types will be called partial.

any other model of T that will be discussed). Any cardinal $\mu < \kappa(\mathbb{M})$ is then *small*. All models of T are thought of as small elementary substructures of the monster, and domainless elements are assumed to live in some monster, and the notation $\vDash \phi$, for an $\mathcal{L}$-sentence ϕ, is short for $\mathbb{M} \vDash \phi$.

Before closing this little section, here's the (Engeler,) Ryll-Nardzewski(, Svenonius) theorem:

Theorem 2.2.1 (Ryll-Nardzewski)**.** *Let $\mathcal{L}$ be a countable language, T a complete $\mathcal{L}$-theory with infinite models. Then, the following are equivalent:*

(1) Any two countable models of T are isomorphic (i.e. T is $\aleph_0$-categorical*).*

(2) T has a countable model $\mathcal{M}$ such that $\mathsf{Aut}(\mathcal{M})$ *is* oligomorphic.[5]

(3) For each $n \in \mathbb{N}$, the set S_n^T is finite.

(4) For each $n \in \mathbb{N}$, every $p \in \mathsf{S}_n^T$ is isolated.[6]

2.3 Some Dividing Lines

The story of classification theory starts in the mid-1960s with Morley's famous *categoricity* theorem, which confirmed a conjecture of Vaught:

Theorem 2.3.1 ([Mor65])**.** *Let T be a countable theory. If T is $\aleph_1$-categorical, then T is κ-categorical for all $\kappa \geq \aleph_1$.*

Recall that if T is a complete theory and κ a cardinal, then $I(T, \kappa)$ (the *spectrum function*) denotes the number of models of T of cardinality κ, up to isomorphism. In this notation, Morley's theorem says that:

$$I(T; \aleph_1) = 1 \implies \forall \kappa \geq \aleph_1 \;\; I(T; \kappa) = 1$$

Perhaps even more interesting than Morley's result is the beautiful theory that started being developed for its proof (and evolved into much more). Indeed, in the proof of Morely's theorem, one encounters things like *Morley rank*,[7] *strong minimality*,[8] abstract notions of independence, and more. See, for instance, [TZ12, Chapters 5 and 6].

[5] The action of $\mathsf{Aut}(\mathcal{M})$ on n-tuples has finitely many orbits for all $n \in \mathbb{N}$.

[6] There is a formula $\phi \in p$ which implies all other formulas in p, equivalently, p is an isolated point in the Stone space S_n^T.

[7] Cantor-Bendixon rank in the space of types.

[8] Recall: A structure $\mathcal{M}$ is *minimal* if every (parameter-)definable subset of M is either finite or cofinite. A theory T is *strongly minimal* if every model of T is minimal.

Morley's other important contribution to the development of classification theory was that he made the following two big conjectures:

(1) Let T be a countable theory, then $I(T;\kappa)$, as a function of κ is non-decreasing, when $\kappa \geq \aleph_1$.

(2) There is a version of the Categoricity Theorem for uncountable theories.

Shelah confirmed both of these conjectures [She90], and took an especially radical approach in proving (1): He decided to describe all possible (uncountable) spectra of (countable) first-order theories; a complete classification was actually achieved in [HHL00], using some tools from descriptive set theory. In this quest to describe the possible spectra of first-order theories, various important dividing lines naturally arose. Perhaps the most important one is *stability*.

Unless otherwise stated, throughout the rest of this subsection, T will always denote a complete $\mathcal{L}$-theory and $\mathbb{M} \models T$ a $\kappa(\mathbb{M})$-saturated and $\kappa(\mathbb{M})$-homogeneous (monster) model of T, for some very big cardinal $\kappa(\mathbb{M})$. Also, unless otherwise stated, all (tuples of) elements and subsets will come from this monster model and will be *small*, i.e. of size less than $\kappa(\mathbb{M})$.

Stability

Unfortunately, due to the lack of space and time, the discussion of stable theories will be rather short in this thesis. Fortunately, many authors have done an excellent job exposing it. I would certainly recommend [TZ12, Chapter 8], [Pil96], [Bal17], [Mak84]. Let me at least recall the definition:

Definition 2.3.2 (Order Property, Stability)**.** We say that a formula $\phi(\bar{x};\bar{y})$ has the *order property* in T if there exist $(\bar{a}_i : i \in \mathbb{N})$ and $(\bar{b}_i : i \in \mathbb{N})$ such that:

$$\models \phi(\bar{a}_i, \bar{b}_j) \text{ if, and only if, } i < j.$$

We say that T is *stable* if no formula has the order property in T.

After introducing indiscernibles, I will give another equivalent characterisation of stable theories. For now, let's briefly look at a generalisation of stable theories:

The Independence Property

The notion of NIP, originating in the work of Shelah, has been studied in the past decades, but over the past few years, applications of NIP in combinatorics have resulted in a "revived" focus on NIP theories, and the definition is standard. Again, I will restrict myself to simply giving the definition, and I will direct the reader to [Sim15], for a lot more information.

Definition 2.3.3 (Independence Property, NIP)**.** We say that a formula $\phi(\bar{x}; \bar{y})$ has the *independence property* in T if there exist $(\bar{a}_i)_{i \in \mathbb{N}}$ and $(\bar{b}_I)_{I \subseteq \mathbb{N}}$ such that:

$$\vDash \phi(\bar{b}_I, \bar{a}_i) \text{ if, and only if, } i \in I.$$

We say that T is *dependent* or *NIP* (No Independence Property) if no formula has the independence property in T.

The stable theories live within the NIP theories, but so do many other interesting classes of theories. I'll first discuss o-minimality.

o-Minimality

The class of *o-minimal* theories ("o" for *order*), was introduced by Pillay and Steinhorn in [PS86], as a way of generalising the work of van den Dries on $\mathbb{R}_{\mathsf{exp}}$, and at the same time as an "ordered" analogue of strong minimality. Since its introduction, o-minimality has been at the intersection of model theory with other areas of pure mathematics (ranging from point counting, through the Pila-Wilkie theorem, all the way to interactions with Hodge theory, say in the work Tsimerman). The standard reference for an introduction to o-minimality is [Dri98]. In the remainder of this section, I will recall a lot of the definitions and results that I will need later on.

Well, first of all, I suppose, I should define o-minimality:

Definition 2.3.4. Let $\mathcal{M} = \langle M, < \cdots \rangle$ be an ordered structure. We say that $\mathcal{M}$ is o-minimal if any definable (with parameters) subset of M is a finite union of points and intervals.

Unlike minimality (which is not a property of $\mathsf{Th}(\mathcal{M})$), o-minimality is an elementary property (so we don't need to talk of "strong o-minimality"). This is actually a consequence of the so-called *Cell Decomposition Theorem*, which is what I will discuss next.

The Cell Decomposition Theorem is one of the central structural results in o-minimal theories. I will give a statement of the theorem shortly, but first, I need to introduce cells.

Notation 2.3.5. The following notation will be persistent (unless otherwise stated) throughout this thesis. Let $\mathcal{M}$ be a structure $m, n \in \mathbb{N}$ with $m \leq n$. I will write $\hat{\pi}_m : M^n \to M^m$ for the natural projection map onto the first m coordinates and $\pi_m : M^n \to M$ for the projection onto the m-th coordinate. Sometimes I will view M^n as a product $\prod_{i=1}^{r} M^{d_i}$, for some $r \in \mathbb{N}_{\geq 2}$ and $d_1, \ldots, d_r \in \mathbb{N}_{\geq 1}$ with $\sum_{i=1}^{r} d_i = n$. In this case (and this will be made clear from the context), for $j \in [r]$ I will write $\hat{\pi}_j : M^n \to \prod_{i=1}^{j} M^{d_i}$ for the projection map onto the first $\sum_{i=1}^{j} d_i$ coordinates and $\pi_j : M^n \to M^{d_j}$ for the projection map onto the coordinates indexed by d_j.

Before introducing cells I will have to actually introduce some more notation. For a definable set $X \subseteq M^m$, I will write:

$$C(X) := \{f : X \to M : f \text{ is definable and continuous}\},$$

and

$$C_\infty(X) = C(X) \cup \{\pm\infty\},$$

where ∞ and $-\infty$ are viewed as constant functions on X. For $f, g \in C_\infty(X)$, the notation $f < g$ simply means that for all $x \in X$ we have that $f(x) < g(x)$, and in this case:

$$(f, g)_X := \{(x, y) \in X \times M : f(x) < y < g(x)\} \subseteq M^{m+1}.$$

In this notation:

Definition 2.3.6. Let $(i_1, \ldots, i_n) \in \{0, 1\}^n$ be a sequence of zeroes and ones. We define a $(i_1, \ldots, i_n)$-*cell* $C \subseteq M^n$, recursively, as follows:

Base case: If $C \subseteq M$ is a (i_1)-cell, then it has one of the of the following forms:

- A (0)-*cell* is just a singleton $\{a\} \subseteq M$.
- A (1)-*cell* is an open interval $(a, b) \subseteq M$, where $a, b \in M \cup \{\pm\infty\}$

Inductive part: Suppose that for $n \in \mathbb{N}$ and $(i_1, \ldots, i_n) \in \{0, 1\}^n$ we have defined $(i_1, \ldots, i_n)$-cells. Then:

- An $(i_1, \ldots, i_n, 0)$-*cell* is a set $C \subseteq \Gamma^{n+1}$ of the form:

$$\{(x, t) \in M^{n+1} : x \in D, \alpha(x) = t\},$$

where $D = \hat{\pi}_n(C)$ is a $(i_1, \ldots, i_n)$-cell and $\alpha \in C(D)$.

- An $(i_1, \ldots, i_n, 1)$-*cell* is a set $A \subseteq \Gamma^{m+1}$ of the form:

$$\{(x, y) \in M^{n+1} : x \in D, \alpha(x) < y < \beta(x)\},$$

where $D = \hat{\pi}_n(C)$ is a $(i_1, \ldots, i_n)$-cell, $\alpha < \beta \in C_\infty(X)$, i.e. a set of the from $(\alpha, \beta)_D$, for D an $(i_1, \ldots, i_n)$-cell and $\alpha < \beta \in C_\infty(D)$.

A *cell* in M^n is an $(i_1, \ldots, i_n)$-cell for some $(i_1, \ldots, i_n) \in \{0, 1\}^n$.

Now that we have cells, let's define decompositions:
Definition 2.3.7. A *cell decomposition* of M^n is a special kind of partition of M^n into finitely many cells, defined recursively, as follows:

Base case: A cell decomposition of M is a collection:

$$\{(-\infty, a_1), (a_1, a_2), \ldots, (a_k, +\infty)\} \cup \{\{a_1\}, \ldots, \{a_k\}\},$$

where $a_1 < \cdots < a_k$.

Inductive part: A cell decomposition of M^{n+1} is a finite partition $\mathcal{C}$ of M^{n+1} into cells, such that the set of projections:

$$\hat{\pi}_n(\mathcal{C}) := \{\hat{\pi}_n(C) : C \in \mathcal{C}\}$$

is a cell decomposition of M^n.

Okay, with all this covered, we have the following theorem:
Theorem 2.3.8 (Cell Decomposition Theorem)**.**

(1) Given any definable sets $X_1, \ldots, X_n \subseteq M^n$ there is a cell decomposition of M^n compatible[9] *with each of $X_1, \ldots, X_n$.*

(2) If $f : A \to M$ is a definable function, with $A \subseteq M^n$, then there is a cell decomposition $\mathcal{C}$ of M^n compatible with A such that whenever $C \in \mathcal{C}$ is a cell contained in A we have that $f|_C$ is continuous.

One can easily obtain a version of *uniform* Cell decomposition, for definable families. From this, we get the following:

[9]Recall: A cell decomposition $\mathcal{C}$ of M^n is *compatible* with a set X if each cell in $\mathcal{C}$ is either contained in X or in its complement.

Theorem 2.3.9. *Let $\{X_b : b \in I\}$ be a definable family. Then there is some $N \in \mathbb{N}$ such that for all $b \in I$ the set X_b has a decomposition into at most N parts.*

Another important consequence of o-minimality, in expansions of ordered groups is *definable choice*:

Theorem 2.3.10 (Definable Choice)**.** *Let $\mathcal{M}$ be an o-minimal expansion of an ordered (abelian) group and $\{X_b : b \in I\}$ be a definable family of subsets of $\mathcal{M}$. Then there is a unary definable function f such that for all $b \in I$, if X_b is non-empty then $f(b) \in X_b$.*

A recent generalisation of o-minimal thoeries is distality, which I shall now introduce:

Distality

Distality was introduced by P. Simon in [Sim13]. In a certain sense, the notion of distality was meant to capture the "purely unstable"[10] NIP structures. Apart from o-minimal theories (which are all distal), there are many more natural examples of distal structures (e.g. Presburger arithmetic, the $\mathbb{Q}_p$, and more). There is prolific literature on distality and its applications (e.g. in combinatorics), and many equivalent definitions of distality are used. Let me give one:

Definition 2.3.11. The theory T is called *distal* if for every indiscernible sequence $(a_i : i \in \mathbb{Q})$ in $\mathbb{M}$ and every tuple $b \in \mathbb{M}^{|b|}$ such that $(a_i : i \in \mathbb{Q} \setminus \{0\})$ is indiscernible over b we have that $(a_i : i \in \mathbb{Q})$ is indiscernible over b.

For more details on distal structures, excellent references are [Sim21, Chapter 9], as well as [Asc+22].

Remark 2.3.12. Let T be a distal theory and $\mathcal{M} \models T$. Then, every totally indiscernible sequence (see Definition 2.6.1) in $\mathcal{M}$ is constant.

Example 2.3.13. Here are some basic examples of distal and non-distal theories:

- Total linear orders are distal.
- Meet-trees $(T, \leq \wedge)$ are not distal, in general. For instance, the complete theory of a dense meet-tree is not distal: let $r \in T$ and $(a_i : i \in \mathbb{Q})$ are

[10] For example, a necessary condition for a theory to be distal is that it admits no infinite stable quotient.

distinct elements such that $a_i \wedge a_j = r$ for every $i \neq j$. Then $(a_i : i \in \mathbb{Q})$ is totally indiscernible (over r).

That's all for dividing lines, at least for now. More will be discussed in later chapters (especially in Chapter 4).

2.4 Combinatorial Geometries

This section is self-contained and does not involve much model theory. Here I'll try to summarise the basic terminology of combinatorial geometries (also known as *matroids*) that will be needed later on. Combinatorial geometries bring together many areas of pure mathematics and have a rich history; for more on this, I refer the reader to [CR74]. For a quick introduction to pregeometries with proofs I recommend [TZ12, Appendix C] and [Cas08]. Indeed, part of the material in this section, although standard, is based on the latter reference.

Definition 2.4.1 (Closure Operator, Pregeometry)**.** Let Ω be a (possibly infinite) set and $\mathsf{cl} : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ a function such that for all elements $a, b \in \Omega$ and all subsets $A, B \subseteq \Omega$ we have that:

(1) (Reflexivity). $A \subseteq \mathsf{cl}(A)$.

(2) (Transitivity). If $X \subseteq \mathsf{cl}(Y)$ then $\mathsf{cl}(X) \subseteq \mathsf{cl}(Y)$.[11]

(3) (Finite Character) $\mathsf{cl}(A) = \bigcup_{A'} \mathsf{cl}(A')$, where the union ranges over all finite subsets A' of A.

Then, we call cl a *closure operator* on Ω and refer to subsets of Ω which belong to the set $\{\mathsf{cl}(X) : X \subseteq \Omega\}$ as *closed*. If in addition to (1),(2), and (3), cl also satisfies:

(4) (Exchange) If $a \in \mathsf{cl}(Ab) \setminus \mathsf{cl}(A)$, then $b \in \mathsf{cl}(Aa)$.

for all elements $a, b \in \Omega$ and all subsets $A, B \subseteq \Omega$, then we say that (Ω, cl) is a *pregeometry*.

The prototypical example of a pregeometric closure operator is *linear span* in vector spaces.

Definition 2.4.2. Let (Ω, cl) be a pregeoemtry and $A \subseteq \Omega$. We say that A is:

(1) *(cl-)independent* if for all $a \in A$ we have that $a \notin \mathsf{cl}(A \setminus \{a\})$.

[11] Equivalently, $\mathsf{cl}(X) = \mathsf{cl}(\mathsf{cl}(X))$, assuming (Finite Character), which implies (Monotonicity).

(2) a *(*cl*-)generating set* if $\Omega = \mathsf{cl}(A)$.

(3) a *(*cl*-)basis* if it is a cl-independent cl-generating set.

By (Finite Character), a set $A \subseteq \Omega$ is cl-independent if, and only if, every finite subset of A is cl-independent.

Fact 2.4.3. *Every pregeometry has a basis. Moreover, all bases of a pregeometry have the same cardinality.*

Definition 2.4.4. Let (Ω, cl) be a pregeometry and fix a subset $C \subseteq \Omega$. We obtain two new pregeometries:

(1) The *localisation of* Ω *at* C (or *relativisation of* Ω *to* C): This is the pregeometry (Ω, cl_C) where $\mathsf{cl}_C : A \mapsto \mathsf{cl}(A \cup C)$, for all $A \subseteq \Omega$.

(2) The *restriction of* Ω *to* C: This is the pregeometry (C, cl^C), where $\mathsf{cl}^C : A \mapsto \mathsf{cl}(A) \cap C$, for all $A \subseteq \Omega$.

Definition 2.4.5. Let (Ω, cl) be a pregeometry. The cl-*rank* of (Ω, cl) is the cardinality of a basis of (Ω, cl). We denote this by $\mathsf{cl}\text{-}\mathsf{rk}(\Omega)$, in case we need to be more specific about the closure operator on Ω we may write $\mathsf{cl}\text{-}\mathsf{rk}((\Omega, \mathsf{cl}))$. Given a subset $C \subseteq \Omega$, we define $\mathsf{cl}\text{-}\mathsf{rk}(C) := \mathsf{cl}\text{-}\mathsf{rk}((C, \mathsf{cl}^C))$ and $\mathsf{cl}\text{-}\mathsf{rk}(\Omega/C) := \mathsf{cl}\text{-}\mathsf{rk}((\Omega, \mathsf{cl}_C))$.

Often, in literature, what was defined as the cl-rank of a pregeometry (Ω, cl) is referred to as the *dimension* of (Ω, cl). This choice of terminology is made to prevent confusion, since the term (cl-)dimension will be used later (see Definition 5.3.4).

Fact 2.4.6. *Let* (Ω, cl) *be a pregeometry and* $C \subseteq \Omega$. *If* A *is a basis of* (Ω, cl_C) *and* B *is a basis of* (C, cl^C)*, then* $A \cup B$ *is a basis for* (Ω, cl). *In this case, we say that* A *is a* basis over C *and* B *is a* basis of C. *In particular, this implies that:*

$$\mathsf{cl}\text{-}\mathsf{rk}(\Omega) = \mathsf{cl}\text{-}\mathsf{rk}(\Omega/C) + \mathsf{cl}\text{-}\mathsf{rk}(C).$$

Now for a localised version of Definition 2.4.2:

Definition 2.4.7. Let (Ω, cl) be a pregeoemtry and $A, C \subseteq \Omega$. We say that A is:

(1) *(*cl*-)independent over* C if for all $a \in A$ we have that $a \notin \mathsf{cl}(AC \setminus \{a\})$ (equivalently, if A is cl_C-independent in the localisation of Ω at C).

(2) a *(*cl*-)generating set over* C if $\Omega = \mathsf{cl}(A \cup C)$.

(3) a *(*cl*-)basis over* C if it is cl-independent over C and a cl-generating set over C.

One of the most important consequences of the axioms of a pregeometry is that they give rise to a notion of independence:

Definition 2.4.8. Let (Ω, cl) be a pregeometry and $A, B, C \subseteq \Omega$. We say that A *is independent from* B *over* C if for all finite $A_0 \subseteq A$ we have that:

$$\mathsf{cl}\text{-}\mathsf{rk}(A_0/C) = \mathsf{cl}\text{-}\mathsf{rk}(A_0/BC).$$

This gives rise to a ternary independence relation $\mathop{\smile\!\!\!\!\!\!\mid}$ on subsets of Ω as follows:

$$A \mathop{\smile\!\!\!\!\!\!\mid}_C B \text{ if, and only if, } A \text{ is } \mathsf{cl}\text{-independent from } B \text{ over } C,$$

for all subsets $A, B, C \subseteq \Omega$.

Remark 2.4.9. The following are equivalent:

(1) $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$.

(2) For all $A' \subseteq A$, if A' is independent over C then it is also independent over BC.

Fact 2.4.10. *Let* (Ω, cl) *be a pregeometry. Then* $\mathop{\smile\!\!\!\!\!\!\mid}$ *satisfies the following conditions, for all* $A, B, C, D \subseteq \Omega$*:*

(1) (cl-Independence)*.* $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ *if, and only if* $\mathsf{cl}(AC) \mathop{\smile\!\!\!\!\!\!\mid}_C \mathsf{cl}(BC)$

(2) (Monotonicity)*. If* $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ *then for all* $A_0 \subseteq A$ *and* $B_0 \subseteq B$ *we have that* $A_0 \mathop{\smile\!\!\!\!\!\!\mid}_C B_0$*.*

(3) (Symmetry)*.* $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ *if, and only if* $B \mathop{\smile\!\!\!\!\!\!\mid}_C A$*.*

(4) (Transitivity)*.* $A \mathop{\smile\!\!\!\!\!\!\mid}_C BD$ *if, and only if,* $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ *and* $A \mathop{\smile\!\!\!\!\!\!\mid}_{CB} D$*.*

(5) (Non-Degeneracy)*. If* $A \mathop{\smile\!\!\!\!\!\!\mid}_C B$ *and* $d \in \mathsf{cl}(AC) \cap \mathsf{cl}(BC)$ *then* $d \in \mathsf{cl}(C)$*.*

One of the most important examples of a closure operator often appearing naturally in model theory is the *algebraic closure operator*, acl, which recall is defined as follows, in a given $\mathcal{L}$-structure $\mathcal{M}$:

$$\mathsf{acl}(A) := \{b : \exists\phi \in \mathcal{L}(A) \text{ s.t. } \phi(M) \text{ is finite, and } b \in \phi(M)\}.$$

In many interesting cases, such as strongly minimal theories and o-minimal theories this is, in fact, a pregeometry.

There is more to be said about combinatorial geometries, and some of it will be said in Chapter 5.

2.5 (Structural) Ramsey Theory

In this section I will discuss the basics of structural Ramsey theory. This will be in no way a complete exposition of the theory, it's far from even being a complete introduction. An excellent introductory source, with a lot of examples, and many proofs (even of classical results) is Bodirsky's survey article, [Bod15].

Ramsey's Theorem

Before going deeper into the *structural* side of structural Ramsey theory, I find it helpful (psychologically, perhaps) to take a more historical approach and discuss a little bit of classical Ramsey theory.

In [Ram30] Frank P. Ramsey needed "certain theorems on combinations" to prove his *chief* theorem (a result on the decidability of certain fragments of first-order logic, which was certainly fashionable in the 1930s). According to him, the combinatorial results in his paper "have an independent interest". It turns out that from Ramsey's "theorems on combinations" sprung an entire area of mathematics... That's probably all I should say about the early days of Ramsey theory (apart from Ramsey's theorem, of course, which will be stated below). For a colourful and highly informative discussion of the beginnings of Ramsey theory I recommend [GS90].

Before stating Ramsey's theorem, I will need to introduce some notation. Let κ, λ be ordinals and ν, ρ be cardinals. The goal is to define the following:

$$\kappa \to (\lambda)^{\nu}_{\rho}, \tag{ER}$$

which is often referred to as the *Erdős-Rado partition arrow* (read: "kappa *arrows* lambda nu rho").

Let A be a set and ν, ρ be cardinals. Then:

- $\binom{A}{\nu} := \{B \subseteq A : |B| = \nu\}$.
- A *ρ-colouring* of a set A is simply a (surjective) function $\chi : A \to [\rho]$.

- Let $\chi : \binom{A}{\nu} \to [\rho]$ be a ρ-colouring of the ν-element subsets of A. A subset $X \subseteq A$ is called χ-*monochromatic* (or *monochromatic/homogeneous* when χ is understood) if:
$$\left|\mathsf{im}\left(\chi \restriction_{\binom{X}{\nu}}\right)\right| = 1,$$
that is, if there is some colour $i \in \rho$ such that for all $Y \in \binom{X}{\nu}$ we have that $\chi(Y) = i$.

Now, (ER) is short for the following statement: "For every ρ-colouring of the ν-element subsets of κ there is a homogeneous subset of order-type λ".

Theorem 2.5.1 (Ramsey's Theorem – Infinite Version)**.** *For all $k, r \in \mathbb{N}$:*

$$\omega \to (\omega)^k_r.$$

By an easy application of compactness, one obtains the following:

Corollary 2.5.2 (Finite Ramsey – Finite Version)**.** *For all $n, k, r \in \mathbb{N}$ there is some $N \in \mathbb{N}$ such that:*
$$N \to (n)^k_r$$

Structural Generalisations

In a (very formal) sense, which I hope becomes clear in the following, the finite version of Ramsey's theorem is really a theorem about the class of all finite linear orders. Intuitively, it says that given any finite linear order (of size n) and any linear order (of size k) inside the original, there is a big linear order (of size N) such that no matter how we r-colour its k-element (increasing) suborders, there will be a homogeneous suborder of size n.

Central to *structural* Ramsey theory is the idea that many classes of structures (not just linear orders) satisfy analogous versions of Ramsey's theorem. To make this precise, I will need to introduce some terminology.

Fraïssé Classes

Notation 2.5.3. Given $\mathcal{L}$-structures A, B, I will write $\binom{A}{B}$ for the set of all embeddings of B into A. Sometimes, this notation is used to denote the set of all isomorphic copies of B in A. Of course, in general, these two sets need not be equal. In the second part of Chapter 3, I will look at the relation between "isomorphic copies" and "embeddings" more closely. There, the notation will shift slightly and I will write $\mathsf{Emb}(B, A)$ for the set of all embeddings of B into A, and $\binom{A}{B}$ for the set of all substructures of A that are isomorphic to B.

Nevertheless, in the context of the Embedding Ramsey Property (see Definition 2.5.5(4)), all classes will necessarily consist of *rigid*[12] structures, and thus isomorphic copies and embeddings coincide, making the change in notation somewhat irrelevant and purely cosmetic.

Now, let's look at the structural analogue of the Erdős-Rado partition arrow:

Definition 2.5.4 (Structural Erdős-Rado Partition Arrow)**.** Let A, B, C be $\mathcal{L}$-structures (and without loss of generality assume that $A \subseteq B \subseteq C$) and let $r \in \mathbb{N}$. We write:

$$C \to (B)^A_r$$

if for each colouring $\chi : \binom{C}{A} \to r = 0, \ldots, r-1$ there exists some $\tilde{B} \in \binom{C}{B}$ such that $\chi \restriction \binom{\tilde{B}}{A}$ is constant.

Finally, the following definitions are all well-known and important:

Definition 2.5.5 (HP, JEP, AP, ERP)**.** Let $\mathcal{L}$ be a first-order language and $\mathcal{C}$ a class[13] of $\mathcal{L}$-structures. We say that $\mathcal{C}$ has the:

(1) *Hereditary Property* (HP) if whenever $A \in \mathcal{C}$ and $B \subseteq A$ we have that $B \in \mathcal{C}$.

(2) *Joint Embedding Property* (JEP) if whenever $A, B \in \mathcal{C}$ there is some $C \in \mathcal{C}$ such that both A and B are embeddable in C.

(3) *Amalgamation Property* (AP) if whenever $A, B, C \in \mathcal{C}$ are such that A embeds into B via $e : A \hookrightarrow B$ and into C via $f : A \hookrightarrow C$ there exist a $D \in \mathcal{C}$ and embeddings $g : B \hookrightarrow D$, $h : C \hookrightarrow D$ such that $g \circ e = h \circ f$.

(4) *Embedding Ramsey Property* (ERP) if whenever $A, B \in \mathcal{C}$ are such that $A \subseteq B$, then there is some $C \in \mathcal{C}$ such that $C \to (B)^A_2$.

Remark 2.5.6. One may wonder why in Definition 2.5.5(4), only 2-colourings are considered. It is fairly easy to see, by a standard induction argument, that if $\mathcal{C}$ has ERP (as defined above) then for all $r \in \mathbb{N}$ and all $A, B \in \mathcal{C}$ such that $A \subseteq B$, there is some $C \in \mathcal{C}$ such that $C \to (B)^A_r$.

Definition 2.5.7 (Fraïssé Class)**.** We say that a countable class $\mathcal{C}$ of (isomorphism types of) finite $\mathcal{L}$-structures is a *Fraïssé Class* if it has HP, JEP, and

[12]Recall: An $\mathcal{L}$-structure $\mathcal{M}$ is called *rigid* if it has no non-trivial automorphisms.

[13]Throughout this thesis, all classes of structures will be assumed to be closed under isomorphisms.

AP.

Recall that for a structure $\mathcal{M}$, the *age* of $\mathcal{M}$, denoted by $\mathsf{Age}(\mathcal{M})$, is the class of isomorphism types of finitely generated substructures of $\mathcal{M}$. In this thesis, a structure $\mathcal{M}$ will be called *homogeneous* if every isomorphism between finitely generated substructures of $\mathcal{M}$ extends to an automorphism of $\mathcal{M}$ (some authors refer to this property as *ultrahomogeneity*, and restrict their attention to countable structures, which is not an assumption I am making, in this thesis). An $\mathcal{L}$-structure $\mathcal{M}$ is called *locally finite* if for any finite $A \subseteq M$ we have that $\langle A \rangle_{\mathcal{M}}$ is finite.

The following celebrated theorem connects most of the notions defined above:

Fact 2.5.8 (Fraïssé's Theorem)**.** *Let $\mathcal{C}$ be a non-empty Fraïssé class. Then there exists a unique, up to isomorphism, countable homogeneous $\mathcal{L}$-structure $\mathcal{M}$ such that $\mathsf{Age}(\mathcal{M}) = \mathcal{C}$.*

Notation 2.5.9*.* In the notation of Fact 2.5.8, we will write $\mathsf{Flim}(\mathcal{C})$ to denote the unique, up to isomorphism, countable homogeneous $\mathcal{M}$ such that $\mathsf{Age}(\mathcal{M}) = \mathcal{C}$.

Definition 2.5.10 (Ramsey Class)**.** We say that an isomorphism-closed class of finite $\mathcal{L}$-structures, $\mathcal{C}$, is a *Ramsey Class* if it has HP, JEP, and ERP.

Example 2.5.11. In this updated terminology, Ramsey's theorem says that the class of all finite linear orders is a Ramsey class.

The following theorem of Nešetřil tells us that AP is a necessary condition for a class to be Ramsey. So countable Ramsey classes are necessarily Fraïssé classes.

Theorem 2.5.12 ([Neš05, Theorem 4.2(i)])**.** *Let $\mathcal{C}$ be a Ramsey class. Then $\mathcal{C}$ has AP.*

A natural generalisation of ERP comes from the notion of *Ramsey degrees*, which roughly says that even if we cannot always find monochromatic copies of our structures, we have some "global" control over the number of colours that are needed. More precisely:

Definition 2.5.13. Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures and $A \in \mathcal{C}$. Given $d \in \mathbb{N}$, we say that A has *Ramsey degree d (in $\mathcal{C}$)* if d is the least positive integer such that for any $B \in \mathcal{C}$ with $A \subseteq B$ there is some $C \in \mathcal{C}$ such that $B \subseteq C$ and for any $r \in \mathbb{N}$ and any colouring $\chi : \binom{C}{A} \to r$ there exists some $\tilde{B} \in \binom{C}{B}$ such that $|\mathsf{im}(\chi \restriction \binom{\tilde{B}}{A})| \leq d$.

Note here that often in the literature the notion just defined is referred to as a *small* Ramsey degree, to distinguish between it and the so-called *big* Ramsey degrees. I will not be discussing big Ramsey degrees here; see, for instance, [Maš21] for a relevant discussion.

The following Remark 2.5.14 is essentially a restatement of the definition:

Remark 2.5.14. In these terms, $\mathcal{C}$ has ERP if, and only if, for all $A \in \mathcal{C}$ we have that A has Ramsey degree 1.

Fact 2.5.15 ([Bod15, Lemma 2.9, and Corollary 2.10])**.** *Let $\mathcal{C}$ be a class of finite structures. Then, for all $A \in \mathcal{C}$, the Ramsey degree of A is at least $|\mathsf{Aut}(A)|$. In particular, if $\mathcal{C}$ has ERP, then all members of C are rigid (cf. Fact 3.8.2).*

The following fact is well known. It follows from a simple compactness argument.

Fact 2.5.16. *Let $\mathcal{M}$ be a locally finite structure. Then, the following are equivalent:*

(1) $\mathcal{M}$ has ERP.

(2) For all finitely generated substructures $A \subseteq B \subseteq \mathcal{M}$ we have that $\mathcal{M} \to (B)^A_2$.

Topological Dynamics

To close this section, let me give a quick overview of the central theorem of Kechris-Pestov-Todorčević, from [KPT05]. First, recall that a topological group is just an abstract group $(G, \cdot)$ together with a topology on G such that both the group operation operation and the inversion are continuous. An action of a topological group G on a topological space X is called continuous if it is a continuous function $G \times X \to X$. If X is a compact Hausdorff space then a continuous action of G on X is called an *G-flow*. A topological group G is *extremely amenable* if every G-flow has a fixed point.

Automorphism groups are closed subgroups of symmetric groups and thus inherit a natural topology (the *topology of pointwise convergence*).

Fact 2.5.17 ([KPT05, Theorem 4.8], [Bod21, Theorem 11.2.2])**.** *Let $\mathcal{C}$ be a Fraïssé class, $\mathcal{M} = \mathsf{Flim}(\mathcal{C})$, and $G = \mathsf{Aut}(\mathcal{M})$. Then, the following are equivalent:*

(1) The group G is extremely amenable.

(2) $\mathcal{C}$ consists of rigid structures and has ERP.

(3) *$\mathcal{C}$ is a Ramsey class and $\mathcal{M}$ admits an $\mathsf{Aut}(\mathcal{M})$-invariant linear order, that is, there exists a linear order $\preceq$ on M such that for all $i, j \in M$ and all $\sigma \in \mathsf{Aut}(\mathcal{M})$ we have that $i \preceq j$ if, and only if $\sigma(i) \preceq \sigma(j)$.*

2.6 (Generalised) Indiscernibles

As in the previous section, before reviewing generalised indiscernibles, it is sensible to discuss their more classical ancestors, *indiscernible sequences* and *indiscernible sets* (sometimes referred to as *totally indiscernible sequences*). Intuitively, an indiscernible set is a set of tuples from a structure all of whose subsets cannot be told apart using first-order logic, and an indiscernible sequence is a sequence of tuples where the same conclusion holds only of its increasing subsequences. More formally:

Definition 2.6.1. Let, $\mathcal{M}$ be a first-order structure, $A \subseteq M$ and $\mathcal{I}$ a linear order. We say that a sequence[14] $\mathfrak{I} = (\bar{a}_i : i \in I)$ of (same-length) tuples[15] from M is:

(1) An *order-indiscernible sequence over A* (or simply an *indiscernible sequence over A*) if for all $n \in \mathbb{N}$ and all increasing sequences $i_1 < \cdots < i_n, j_1 < \cdots < j_n$ from I we have that:

$$\mathsf{tp}(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}/A) = \mathsf{tp}(\bar{a}_{j_1}, \dots, \bar{a}_{j_n}/A).$$

(2) A *totally indiscernible sequence over A* (or, to the same extent, an *indiscernible set over A*) if for all $n \in \mathbb{N}$ and all $\{i_1, \dots, i_n\}, \{, j_1, \dots, j_n\} \in \binom{I}{n}$ we have that:

$$\mathsf{tp}(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}/A) = \mathsf{tp}(\bar{a}_{j_1}, \dots, \bar{a}_{j_n}/A).$$

Remark 2.6.2. When $A = \emptyset$, I will skip the "over $\emptyset$" part. Note that some authors (including, unfortunately, the younger self of the author of this thesis) sometimes call a sequence which is indiscernible over A an *A-indiscernible sequence*. For reasons that will become apparent when generalised indiscernibles show up, this will **not** be done in this thesis.

Here is the promised alternative characterisation of stability:

[14] Recall that formally a sequence is just a function from I to some Cartesian power of M.

[15] Throughout this thesis, unless otherwise stated, all sequences will consist of *same-length* tuples, and this is the last page on which the phrase "same-length" appears.

Theorem 2.6.3 ([She90, Theorem II.2.13]). *Let T be a first-order theory. Then, the following are equivalent:*

(1) T is stable.

(2) Every order-indiscernible sequence in a model of T is totally indiscernible.

Arguments using indiscernible sequences, instead of arbitrary ones, are usually much easier to carry out, since one may, at their leisure, exchange (finite) subsequences of elements with other ones in the sequence. Therefore, indiscernibles bring a very welcome form of uniformity to our arguments. That being said, if this were everything about indiscernibles, that probably wouldn't be enough to care too much about them. The amazing thing is that given *any* sequence $\mathfrak{I}$ in a first-order structure, one can always find (perhaps at the cost of moving to an elementary extension) an indiscernible sequence $\mathfrak{I}'$ which in a well defined way nicely approximates the increasing behaviour of $\mathfrak{I}$. To make this more precise, we need the concept of *Ehrenfeucht-Mostowski types*.

Definition 2.6.4. Let $\mathcal{M}$, A, and $\mathfrak{I} = (a_i : i \in I)$ be as in Definition 2.6.1. The *Ehrenfeucht-Mostowski type* of $\mathfrak{I}$ over A, denoted $\mathsf{EM\text{-}tp}(\mathfrak{I}/A)$ is the following partial type (in $\aleph_0$-many variables):

$$\mathsf{EM\text{-}tp}(\mathfrak{I}/A) := \{\phi(x_1, \dots, x_n) \in \mathcal{L}(A) : \forall i_1 < \cdots < i_n \in I, \\ \mathcal{M} \vDash \phi(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}),\ n \in \mathbb{N}\}.$$

Intuitively, $\mathsf{EM\text{-}tp}(\mathfrak{I})$ contains all the common behaviours of increasing finite subsequences of $\mathfrak{I}$. Let $\mathcal{I} = (I, <)$ and $\mathcal{J} = (J, <)$ be linear orders. We say that an I-indexed sequence $\mathfrak{I} = (a_i : i \in I)$ *realises* the EM-type over A of a $\mathcal{J}$-indexed sequence $\mathfrak{J} = (b_j : j \in J)$, denoted $\mathfrak{I} \vDash \mathsf{EM\text{-}tp}(\mathfrak{J}/A)$, if for all $n \in \mathbb{N}$ and all $i_1 < \cdots < i_n$ we have that $(a_{i_1}, \dots, a_{i_n}) \vDash \phi(x_1, \dots, x_n)$, for all $\phi \in \mathsf{EM\text{-}tp}(\mathfrak{J}/A)$.

Remark 2.6.5. It should be clear that $\mathfrak{I}$ is indiscernible over A if, and only if $\mathsf{EM\text{-}tp}(\mathfrak{I})$ is a maximally consistent set of $\mathcal{L}(A)$-formulas.

The following result is so folklore that it is sometimes (e.g. in [TZ12, Lemma 5.1.3]) referred to as the *standard lemma*. Many model theorists also refer to applications of the standard lemma by using the phrase "*by Ramsey and compactness*", a phrase that is ubiquitous in model theory papers. The reason why will, hopefully, be evident from the proof I will sketch below.

Theorem 2.6.6 (The Standard Lemma). *Let $\mathcal{I}$ and $\mathcal{J}$ be two infinite linear* *orders. For any parameter sets A, B and any sequence $\mathfrak{I} = (\bar{a}_i : i \in I)$ there is a sequence $\mathfrak{J} = (\bar{b}_j : j \in J)$ which is indiscernible over B and*

$$\mathfrak{J} \models \mathsf{EM\text{-}tp}(\mathfrak{I}/A).$$

Proof (sketch). By Ramsey's theorem and compactness. More precisely, observe that we can write down a set of $\mathcal{L}(AB)$-formulas $\Sigma(\bar{x}_j : j \in \mathcal{J})$ which says that any realisation of Σ is indiscernible over B and realises $\mathsf{EM\text{-}tp}(\mathfrak{I}/A)$ (this requires a bit of notational care, but can be done). It then suffices to show that such a set $\Sigma(x_j : j \in \mathcal{J})$ is satisfiable.

By *compactness*, it suffices to show that all finite subsets of Σ are satisfiable, and we can observe that this boils down to showing that for every finite set Δ of $\mathcal{L}(B)$-formulas, there is an infinite subset X of $\mathcal{I}$ such that every increasing subsequence of elements from X indexes elements which have the same Δ-type.

By *Ramsey's theorem*, such a subset X always exists. Indeed, let n be the maximal number of free variables appearing in formulas of Δ and colour the n-element subsets of $\mathcal{I}$ according to the Δ-type of the elements of $\mathfrak{I}$ that they index (in increasing order). Since Δ is finite, there are only finitely many Δ-types, and thus Ramsey's theorem applies. So, we can find an infinite monochromatic subset for this colouring, and the result follows. ☐

Now that we have the basics of indiscernible sequences down, we can move on up to *generalised indiscernibles*. These objects were first introduced by Shelah in [She90, Section VII.2], as a tool to study the tree property (which will not be discussed here). These notions have proven to be an important tool in classification theory, for instance in the study of other tree properties (see, [KK11] and [KKS13]). In Chapter 4 I will discuss how generalised indiscernibles can been used to define new dividing lines.

The main new idea here is that there is no need to index our sequences by linear orders (or unordered sets). Since sequences are just functions from the domain of the indexing structure to some Cartesian power of the structure we are working in, the indexing structure may be any first-order structure our hearts desire. Here's how to make this formal:

Definition 2.6.7 (Generalised Indiscernibles)**.** Let $\mathcal{L}'$ be a first-order language and $\mathcal{N}$ an $\mathcal{L}'$-structure. Given an $\mathcal{N}$-indexed sequence of tuples $\mathfrak{I} = (\bar{a}_i : i \in \mathcal{N})$ from $\mathbb{M}$, and a small subset $A \subseteq \mathbb{M}$, we say that $\mathfrak{I}$ is an *$\mathcal{N}$-indexed indiscernible* *sequence over* A or simply that $\mathfrak{I}$ is $\mathcal{N}$*-indiscernible over* A if for all $n \in \mathbb{N}$ and all sequences $i_1, \ldots, i_n, j_1, \ldots, j_n$ from N we have that:

$$\text{If } \mathsf{qftp}_{\mathcal{N}}(i_1, \ldots, i_n) = \mathsf{qftp}_{\mathcal{N}}(i_j, \ldots, j_n) \quad \text{then } \mathsf{tp}_{\mathbb{M}}(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n}/A) = \mathsf{tp}_{\mathbb{M}}(\bar{a}_{j_1}, \ldots, \bar{a}_{j_n}/A).$$

If $A = \emptyset$, we say that $\mathfrak{I}$ is simply an $\mathcal{N}$*-indiscernible sequence.*

Remark 2.6.8*.* Henceforth, as is usual, when discussing indiscernibles, I will not speak of (generalised) indiscernibility over a small set $A \subseteq \mathbb{M}$. This is, of course, since (when relevant) we can add constants naming A to $\mathcal{L}$ reducing this to indiscernibility over $\emptyset$ (in the $\mathcal{L}(A)$-structure $\mathbb{M}$, where the new constant symbols are naming the elements of A).

Example 2.6.9. It should, at this point, be clear that if $\mathcal{N} = \langle \mathbb{N}, < \rangle$ (or, more generally, $\mathcal{N}$ is any infinite linear order), then $\mathcal{N}$-indiscernible sequences are simply indiscernible sequences. If $\mathcal{N} = \langle \mathbb{N}, = \rangle$ (or, more generally, $\mathcal{N}$ is any infinite set in the language of pure equality), then $\mathcal{N}$-indiscernible sequences are simply totally indiscernible sequences.

The conclusion of the Standard Lemma is that for any sequence $\mathfrak{I}$ one can find an indiscernible sequence $\mathfrak{J}$ realising $\mathsf{EM\text{-}tp}(\mathfrak{I})$. Any good theorem serves better as a definition, so let's turn its conclusion into one. There are many ways of going about this, but the following terminology is essentially due to Scow [Sco12].

Definition 2.6.10 (The Modelling Property)**.** Let $\mathcal{L}'$ be a first-order language, $\mathcal{N}$ be an $\mathcal{L}'$-structure, and $\mathfrak{I} = (\bar{a}_i : i \in \mathcal{N})$ be an $\mathcal{N}$-indexed sequence of tuples from $\mathbb{M}$.

(1) Given an $\mathcal{N}$-indexed sequence of tuples $\mathfrak{J} = (\bar{b}_i : i \in \mathcal{N})$ from $\mathbb{M}$, we say that $\mathfrak{J}$ is *(locally) based on* $\mathfrak{I}$ if for all finite sets of $\mathcal{L}$-formulas $\Delta \subseteq \mathcal{L}$, all $n \in \mathbb{N}$ and all $i_1, \ldots, i_n$ from $\mathcal{N}$ there is some $j_1, \ldots, j_n$ from $\mathcal{N}$ such that:

   (a) $\mathsf{qftp}^{\mathcal{L}'}_{\mathcal{N}}(i_1, \ldots, i_n) = \mathsf{qftp}^{\mathcal{L}'}_{\mathcal{N}}(j_1, \ldots, j_n)$

   (b) $\mathsf{tp}^{\Delta}_{\mathbb{M}}(\bar{b}_{i_1}, \ldots, \bar{b}_{i_n}) = \mathsf{tp}^{\Delta}_{\mathbb{M}}(\bar{a}_{j_1}, \ldots, \bar{a}_{j_n})$

(2) We say that $\mathcal{N}$-indiscernible sequences have the *modelling property in* T if for each $\mathcal{N}$-indexed sequence $\mathfrak{I} = (\bar{a}_i : i \in \mathcal{N})$ of tuples from $\mathbb{M} \vDash T$ there exists an $\mathcal{N}$-indiscernible sequence $\mathfrak{J}$ (locally) based on $\mathfrak{I}$.

(3) We say that $\mathcal{N}$-indiscernible sequences have the *modelling property* if for

all first-order theories T, they have the modelling property in T.

The phrase "$\mathcal{N}$-indiscernible sequences have the modelling property" will usually be abbreviated by "$\mathcal{N}$ *has the Modelling Property*" (and often even by "$\mathcal{N}$ has *MP*.[16]")

Remark 2.6.11. It should be clear that when $\mathcal{N} \vDash \mathsf{LO}$, then an $\mathcal{N}$-indexed indiscernible sequence $\mathfrak{J}$ is locally based on an $\mathcal{N}$-indexed sequence $\mathfrak{I}$ if, and only if, $\mathfrak{I}$ has the same EM-type as $\mathfrak{J}$. So, in this case, the modelling property is precisely the conclusion of the Standard Lemma.

If this section feels like it's missing something, this is because it is. The connections between the Modelling Property and Structural Ramsey theory will be one of the central themes of Chapter 3.

2.7 A World of Structures[†]

In this section, my goal is to introduce most of the *dramatis personae* of the thesis, that is, the structures (or classes of structures) that will be showing up in the subsequent chapters. The material here is more-or-less well-known, but this is a good opportunity to fix notation.

2.7.1 Very Standard Classes of Structures

Pure sets.

The easiest (infinite) first-order structure one can imagine is a pure set, that is, an infinite set, in the empty language.[17] I will write $\mathcal{C}_{\mathsf{E}}$ for the class of all finite sets (in the language of pure equality). Of course, $\mathcal{C}_{\mathsf{E}}$ is a Fraïssé class, but since its members are far from rigid, it is not a Ramsey class. It is easy to see, by quantifier elimination, that $\mathsf{Flim}(\mathcal{C}_{\mathsf{E}})$ is strongly minimal.

Linear and Cyclic Orders

One step up the complexity hierarchy we get $\mathcal{C}_{\mathsf{LO}}$, the class of all finite linear orders. This one *is* a Ramsey class (by Ramsey's theorem), and $\mathsf{Flim}(\mathcal{C}_{\mathsf{LO}})$ is

[16] This should not be confused with the initials of the authors of [MP23b],[MP23a].

[†] Parts of this section appeared in [MPT23]. I am very grateful to Pierre Touchard both for making the lovely pictures that appear in [MPT23], and for letting me include them here.

[17] The equality symbol is always interpreted as *true equality* and is always assumed to be present.

the (unique up to isomorphism) countable dense linear order without endpoints, $(\mathbb{Q}, <)$. This is an o-minimal (NIP, distal, unstable) structure.

Recall that a (strict) *cyclic order* CO on a set X is a ternary relation such that, for all $x, y, z, w \in X$:

(1) $CO(x, y, z) \rightarrow CO(y, z, x)$;

(2) $CO(x, y, z) \rightarrow \neg CO(x, z, y)$

(3) $CO(x, y, z) \wedge CO(y, y, w) \rightarrow CO(x, y, w)$;

(4) if x, y, z are distinct, then $CO(x, y, w)$ or $CO(x, w, y)$.

I will write $\mathcal{C}_{\mathsf{CO}}$ for the class of all finite cyclic orders. This is a Fraïssé class whose Fraïssé limit is one of the three proper reducts.[18] of $(\mathbb{Q}, <)$)

2.7.2 (Ordered) Hypergraphs

One more step up the complexity hierarchy we have $\mathcal{C}_{\mathsf{H}_n}$ and $\mathcal{C}_{\mathsf{OH}_n}$, for $n \in \mathbb{N}$, the classes of all finite n-hypergraphs and all finite ordered n-hypergraphs, respectively. These will now be discusses in more detail.

Fix $m \in \mathbb{N}$. Let $\mathcal{L}_0 = \{R_i : i < m\}$ where each R_i is a relation symbol of arity r_i, for each $i < m$. A *hypergraph of type* $\mathcal{L}_0$ or $\mathcal{L}_0$*-hypergraph* is a structure $(A, (R_i)_{i<m})$ such that, for all $i < m$:

- (*Uniformity*): If $R_i(a_0, \ldots, a_{r_i-1})$ then all $a_0, \ldots, a_{r_i-1}$ are distinct.
- (*Symmetry*): If $R_i(a_0, \ldots, a_{r_i-1})$ then for any permutation $\sigma \in S_{r_i}$ we also have $R_i\left(a_{\sigma(0)}, \ldots, a_{\sigma(r_i-1)}\right)$.

The point is that each R_i is interpreted in A as an r_i-ary "hyperedge" relation, i.e. $R_i \subseteq [A]^{r_i}$.

Let $\mathcal{L}_0^+ = \mathcal{L}_0 \cup \{<\}$. If $\mathcal{M} = (A, (R_i)_{i<m}, <)$ is an $\mathcal{L}_0^+$-structure whose $\mathcal{L}_0$-reduct is an $\mathcal{L}_0$-hypergraph and $<$ is interpreted as a linear order in M, then we say that M is an *ordered* $\mathcal{L}_0^+$*-hypergraph*.

Let $\mathcal{C}$ be the class of all linearly ordered finite $\mathcal{L}_0$-hypergraphs. Then, $\mathcal{C}$ is a Fraïssé class and its Fraïssé limit is the *ordered random* $\mathcal{L}_0^+$*-hypergraph*, whose order is isomorphic to $(\mathbb{Q}, <)$.

[18] Discussed in more detail in Subsection 4.2.1

Fact 2.7.1 ([NR77])**.** *For any finite $\mathcal{L}_0^+$, let $\mathcal{C}$ be the class of all ordered $\mathcal{L}_0^+$-hypergraphs. Then $\mathcal{C}$ is a Ramsey class.*

In particular, if $\mathcal{L}_0^+ = \{<, R\}$, where R is a single relation symbol of arity n we call an ordered $\mathcal{L}_0^+$-hypergraph an *ordered n-uniform hypergraph*. The Fraïssé limit of the class of ordered n-uniform hypergraphs is the *ordered random n-uniform hypergraph*, which we denote by $\mathcal{OH}_n$.

Remark 2.7.2. A first-order $\mathcal{L}_0$-structure $\langle M, <, R\rangle$ is a model of $\mathsf{Th}(\mathsf{Flim}(\mathcal{C}_{\mathsf{OH}_n}))$ if, and only if:

- $\langle M, <, R\rangle$ is an ordered n-uniform hypergraph, in the sense above.
- $\langle M, <\rangle$ is a model of DLO.
- For all finite disjoint subsets $A_0, A_1 \subseteq M^{n-1}$ and any $b_0, b_1 \in M$ such that $b_0 < b_1$, there is some $b \in M$ such that:
  - $b_0 < b < b_1$.
  - For every $(a_{0,1}, \dots, a_{0,n-1}) \in A_0$, and $(a_{1,1}, \dots, a_{1,n-1}) \in A_1$, with that $a_{0,i}$ and the $a_{1,i}$ pairwise distinct, we have that:

$$R(b, a_{0,1}, \dots, a_{0,n-1}) \text{ and } \neg R(b, a_{1,1}, \dots, a_{1,n-1}).$$

In particular, an ordered random 1-hypergraph is a dense linear order with a dense co-dense subset. I will denote the ordered random 2-hypergraph $\mathcal{OH}_2$ by $\mathcal{OG}$ (since 2-hypergraphs are just graphs).

At times I will need to consider the structures obtained from $\mathcal{OH}_n$ by "forgetting" the order $<$ (in the terminology of Subsection 4.2.1, the *language reducts* to $\mathcal{L} = \{R\}$). These will be denoted by $\mathcal{H}_n$ (and for $n = 2$ by $\mathcal{G}$).

2.7.3 More Exotic Examples

I will now introduce two well-known homogeneous structures:

- The *C-relation*; and
- Its reduct, the *D-relation.*[19]

[19] We (meaning the authors of [MPT23]) are indebted to D. Bradley-Williams for his valuable comments on D-relations and D-related issues.

I will be rather succinct in the exposition here, but these are extremely interesting structures, and for more details, I recommend [Cam90, Section 5.1], [AM22], and [BJP16].

Generalised Chain Relations

Definition 2.7.3 ([BJP16, Paragraph 3.3])**.** A ternary relation $C(x; y, z)$ on a set X is called a *C-relation* if for all $a, b, c, d \in X$ we have that:

(1) $C(a; b, c) \to C(a; c, b)$;

(2) $C(a; b, c) \to \neg C(b; a, c)$;

(3) $C(a; b, c) \to (C(a; d, c) \vee C(d; b, c))$;

(4) $a \neq b \to C(a; b, b)$.

We say that a C-relation is *derived from a binary tree* or that it is *binary branching* if, in addition, for all distinct elements $a, b, c \in X$ we have that:

5. $C(a; b, c) \vee C(b; a, c) \vee C(c; a, b)$.

Let $\prec$ be a total order on X. We say that $\prec$ is *convex* for C if for all $a, b, c \in X$, if $C(a; b, c)$ and $a \prec c$, then either $a \prec b \prec c$ or $a \prec c \prec b$. We denote by $\mathcal{C}_{\mathsf{OC}}$ the class of all convexly ordered finite binary branching C-relations.

Fact 2.7.4 ([Bod15, Theorem 5.1])**.** *The class $\mathcal{C}_{\mathsf{OC}}$ is a Ramsey class.*

In particular, $\mathcal{C}_{\mathsf{OC}}$ is a Fraïssé class. I will write $\mathcal{OC}$ for $\mathsf{Flim}(\mathcal{C}_{\mathsf{OC}})$.

Generalised Direction Relations

Definition 2.7.5 ([BJP16, Paragraph 3.4])**.** A quaternary relation $D(x, y; z, w)$ on a set X is called a *D-relation* if for all $x, y, z, w, a \in X$ we have that:

1. $D(x, y; z, w) \to D(y, x; z, w) \wedge D(x, y; w, z) \wedge D(z, w; x, y)$;
2. $D(x, y; z, w) \to \neg D(x, z; y, w)$;
3. $D(x, y; z, w) \to (D(a, y; z, w) \vee D(x, y; z, a))$;
4. $(x \neq z \wedge y \neq z) \to D(x, y; z, z)$.

Similarly to Definition 2.7.3, we say that a D-relation is *derived from a binary tree* or that it is *binary branching* if, in addition for any four elements $x, y, z, w \in X$, if at least 3 of them are distinct, then we have:

(5) $D(x, y; z, w) \vee D(x, z; y, w) \vee D(x, w; y, z)$.

Remark 2.7.6. From a binary tree, we obtain such a D-relation on the set of leaves by setting, $D(a, b; c, d)$, for distinct a, b, c, d, if the paths between a and b and between c and d are disjoint. For instance, if a, b, c and d are arranged as follows:

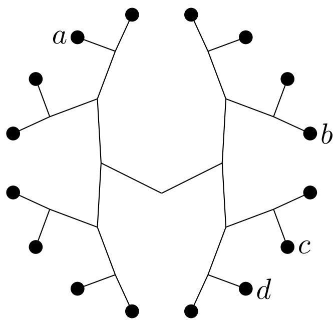
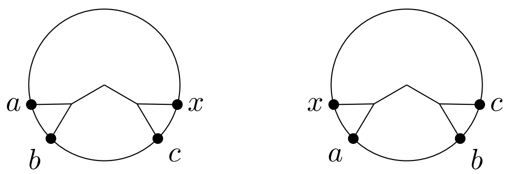


Figure 2.1: Example of Binary Branching D-Relation.

then $D(a, b; c, d)$ holds.

Definition 2.7.7. A cyclic order CO on a binary D-relation X is *convex* for D if for all distinct $x, a, b, c \in X$:

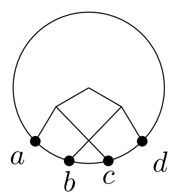
$$CO(a, b, c) \wedge CO(c, x, a) \rightarrow D(x, a; b, c) \vee D(a, b; c, x).$$



Equivalently, we have for all distinct $a, b, c, d \in X$,

$$CO(a, b, c) \wedge CO(b, c, d) \rightarrow \neg D(a, c; b, d).$$

Graphically, this means that the following configuration doesn't occur:



We shall denote by $\mathcal{C}_{\mathsf{COD}}$ the class of all convexly ordered finite binary branching D-relations.

Figure 2.2, below, has an abstract representation of a finite structure in $\mathcal{C}_{\mathsf{COD}}$ which will hopefully be helpful:

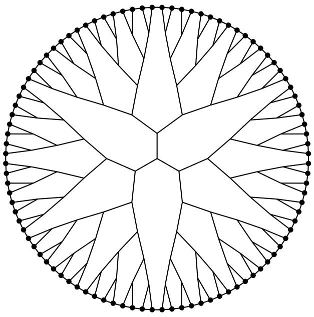


Figure 2.2: A large (abstract) cyclically ordered D-relation.

The following fact is probably well-known. A proof can be found in the preliminaries of [MPT23]

Fact 2.7.8. *The theory of dense cyclically ordered binary branching D-relations is complete, $\aleph_0$-categorical, and admits quantifier elimination in the language $\{D, CO\}$. It follows that $\mathcal{C}_{\mathsf{COD}} = \mathsf{Age}(\mathcal{M})$ is a Fraïssé class, where $\mathcal{M}$ is the unique homogeneous countable model of the theory of dense cyclically ordered binary branching D-relations.*

I will return to these classes in section 4.4 (Example 4.4.16), to discuss how, unlike $\mathcal{C}_{\mathsf{OC}}$, the class $\mathcal{C}_{\mathsf{COD}}$ is *not* a Ramsey class, even if one expands it by a generic order.

2.7.4 Hrushovski's *ab initio* Constructions

This subsection contains a quick review of the basic *ab initio* Hrushovski construction. The material here is classical, and the exposition is based on the sources from which I learnt it. More precisely, this section is based mostly on [Eva13] and [Zie13], and to a lesser extent [TZ12, Section 10.4]. I would recommend these sources (e.g., [Eva13, Section 3.4]) for historical remarks on the construction. Related to Hrushovski constructions (by work of Baldwin and Shelah [BS97]), and generic structures, but not related to the work in this thesis, is the excellent [Spe01].

Fix integers $r \geq 2$ and $m, n \geq 1$. Let $\mathcal{C}$ denote the class of all finite r-uniform hypergraphs and recall the definition of *Hrushovski's predimension* function,

$\delta : \mathcal{C} \to \mathbb{Z}$, on $\mathcal{C}$:

$$\delta(B) = n|B| - m|R^B|.$$

When $A \subseteq B$ are structures in $\mathcal{C}$, I shall, as is common, write $A \leq B$ if $\delta(A) \leq \delta(B')$ for all $A \subseteq B' \subseteq B$, and define

$$\mathcal{C}_0 = \{B \in \mathcal{C} : \emptyset \leq B\}.$$

By $\bar{\mathcal{C}}_0$ I will denote the class of structures all of whose finite substructures are in $\mathcal{C}_0$ and naturally extend $\leq$ on $\bar{\mathcal{C}}_0$, by setting $A \leq B$ if, and only if $A \cap X \leq X$ for all finite $X \subseteq B$.

Theorem 2.7.9 ([Hru93])**.** *There is a unique $\mathcal{M}_0 \in \bar{\mathcal{C}}_0$ such that:*

(1) $\mathcal{M}_0$ is the union of a $\leq$-chain of structures in $\mathcal{C}_0$.

(2) For all finite $X \leq \mathcal{M}_0$ and all finite $A \in \mathcal{C}_0$ such that $X \leq A$ there is an embedding $\alpha : A \to \mathcal{M}_0$ such that $\alpha \restriction_X = \mathsf{id}$ and $\alpha(A) \leq \mathcal{M}_0$.

The structure $\mathcal{M}_0$ is ω-stable and ω-saturated.

This $\mathcal{M}_0$ shall be referred to as the *generic structure* for the class $(\mathcal{C}_0, \leq)$.

2.7.5 o-Minimal Expansions of Ordered Groups

In this section, I will give a quick overview of o-minimal expansions of ordered groups. Recall that an *ordered group* is a group $(G, \cdot, 1)$ equipped with a binary operation $<$ such that for all $x, y, z \in G$ we have that:

$$x < y \implies x \cdot z < y \cdot z, \text{ and } z \cdot x < z \cdot y$$

It is well known that an o-minimal ordered group is abelian, divisible and torsion-free. Thus, I will write such groups with additive notation, and forego the word abelian.

A beautiful and important result about o-minimal expansions of ordered groups is the celebrated *Trichotomy theorem* of Peterzil and Starchenko [PS98]. First some terminology:

Definition 2.7.10. Let $\mathcal{M} = \langle M, <, +, \dots \rangle$ be an o-minimal structure. An element $a \in M$ is called *non-trivial* if there is an open interval I containing a and a parameter-definable function $F : I \times I \to M$ such that F is strictly monotone in each variable. A point is called *trivial* if it is not non-trivial. A *group interval* is a structure $\langle[-p,p],<,+,0\rangle$, where $p \in M$ belongs to a convex type-definable ordered group $(G,+,<)$, and $p>0$.

Intuitively, the trichotomy theorem tells us that every point in a saturated o-minimal structure is either trivial ("only sees the order structure"), belongs to a (closed) group interval, or belongs to an open interval on which there is a definable field. More formally:

Theorem 2.7.11 (Trichotomy Theorem)**.** *Let $\mathcal{M}$ be a saturated o-minimal structure. Given $a \in M$ one and only one of the following occurs:*

(1) a is trivial.

(2) The structure that $\mathcal{M}$ induces in some convex neighbourhood of a is an ordered vector space over a division ring. Furthermore, there is a closed interval containing a on which a group is definable.

(3) The structure that $\mathcal{M}$ induces on some open interval around a is an o-minimal expansion of a real closed field.

Thus, if $\mathcal{M} = \langle M,<,+,\dots\rangle$ is an expansion of an ordered group, there are naturally three mutually exclusive possibilities. I will first state them and then define them.

- $\mathcal{M}$ is either *linear*.
- $\mathcal{M}$ is *semibounded* (and non-linear).
- $\mathcal{M}$ *expands a real closed field.*

The third possibility is rather self-explanatory, but the first two certainly require a formal definition.

Definition 2.7.12. Let Λ be the set of all partial $\emptyset$-definable endomorphisms of $\langle M,<,+,0\rangle$,[20] and $\mathcal{B}$ the collection of all bounded parameter-definable sets of $\mathcal{M}$. Then, we say that:

(1) $\mathcal{M}$ is *linear* ([LP93]) if every definable subset of $\mathcal{M}$ is already definable in the structure $\langle M,<,+,0,\{\lambda\}_{\lambda\in\Lambda}\rangle$.

(2) $\mathcal{M}$ is *semibounded* ([Edm00; Pet92]) if every definable subset of $\mathcal{M}$ is already definable in the structure $\langle M,<,+,0,\{\lambda\}_{\lambda\in\Lambda},\{B\}_{B\in\mathcal{B}}\rangle$.

Obviously, if $\mathcal{M}$ is linear then it is semibounded. The following are known:

[20] That is, maps of the form $f:(a,b)\to M$ such that $f(x+t)-f(x)=f(y+t)-f(y)$, for all $x,y\in(a,b)$ and all $t\in M$ such that $x+t,y+t\in(a,b)$.

Fact 2.7.13 ([PS98],[Edm00]).

(1) $\mathcal{M}$ is linear if, and only if, there is no real closed field defined on an interval.

(2) $\mathcal{M}$ is semibounded if, and only if, $\mathcal{M}$ does not expand a real closed field whose universe is an unbounded interval of $\mathcal{M}$, if, and only if, $\mathcal{M}$ does not expand a real closed field.

Example 2.7.14. An important example of a semi-bounded nonlinear structure is the expansion of the ordered vector space $\langle \mathbb{R}, <, +, 0, \{x \mapsto \lambda x\}_{\lambda \in \mathbb{R}} \rangle$ by all bounded semialgebraic sets. Another important example is $\langle \mathbb{R}, <, +, 0, \cdot \restriction_{[0,1]^2} \rangle$, the expansion of the ordered vector space by the graph of multiplication, restricted on $[0,1]^2$. Here, it is clear that we can define multiplication on any bounded interval; however, this cannot be done uniformly, so if $\mathcal{R} \vDash \mathsf{Th}(\langle \mathbb{R}, <, +, 0, \cdot \restriction_{[0,1]^2} \rangle)$ is a saturated model we see that there is no field definable on all of $\mathcal{R}$.

2.7.6 Presburger Arithmetic

Presburger arithmetic is the theory of the structure $\langle \mathbb{Z}, <, +, 0 \rangle$, in the natural language $\mathcal{L}_{\mathsf{Pres}} = \{<, +, 0\}$. It will sometimes be denoted by T_{Pres}. A brief history of Presburger arithmetic can be found in [Haa18, Section 1]. It is clear that T_{Pres} is not o-minimal (it is distal) but it enjoys a lot of similarities with o-minimal theories, one of which is a cell decomposition theorem, which I will recall below.

Definition 2.7.15. Let $X \subseteq \Gamma^n$. A function $\alpha : X \to \Gamma$ is called *linear (on X)* if there exist $\gamma \in \Gamma$, integers $a_1, \dots, a_n \in \mathbb{Z}$, positive integers $s_1, \dots, s_n \in \mathbb{N}_{>0}$ and non-negative integers $e_1, \dots, e_n \in \mathbb{N}$ with $e_i \in \{0, \dots, s_i - 1\}$, for $i \in \{1, \dots, n\}$, such that:

- $X \subseteq \prod_{i=1}^n (e_i + s_i\mathbb{Z})$;
- For all $x \in X$ we have that:

$$\alpha(x) = \sum_{i=1}^{n} a_i \left(\frac{x_i - e_i}{s_i} \right) + \gamma.$$

The main structural result about sets definable in models of Presburger arithmetic is the *Cell Decomposition Theorem* (Theorem 2.7.17), due to Cluckers [Clu03]. For this, I will first need to introduce the notion of *Presburger cells.*

Definition 2.7.16. Let $(i_1, \ldots, i_n) \in \{0,1\}^n$ be a sequence of zeroes and ones. We define a $(i_1, \ldots, i_n)$-*Presburger cell* (or simply $(i_1, \ldots, i_n)$-*cell*) $C \subseteq \Gamma^n$, recursively, as follows:

Base case: If $C \subseteq \Gamma$ is a (i_1)-cell, then it has one of the of the following forms:

- A (0)-*cell* is just a singleton $\{a\} \subseteq \Gamma$.
- A (1)-*cell* is an infinite set C of the form

$$\{x \in \Gamma : \alpha \ \square_1 \ x \ \square_2 \ \beta, x \equiv c \pmod{n}\},$$

  where $\square_i$ are either $\leq$ or no condition, and n, $c \in [0, n)$ are fixed integers. We refer to (1)-cells as *Presburger intervals*.

Inductive part: Suppose that for $m \in \mathbb{N}$ and $(i_1, \ldots, i_m) \in \{0,1\}^m$ we have defined $(i_1, \ldots, i_m)$-cells. Then:

- A $(i_1, \ldots, i_m, 0)$-*cell* is a set $C \subseteq \Gamma^{m+1}$ of the form:

$$\{(x,t) \in \Gamma^{m+1} : x \in D, \alpha(x) = t\},$$

  where $D = \hat{\pi}_m(C)$ is a $(i_1, \ldots, i_m)$-cell and $\alpha : D \to \Gamma$ is a linear function of D.

- A $(i_1, \ldots, i_m, 1)$-*cell* is a set $A \subseteq \Gamma^{m+1}$ of the form:

$$\{(x,t) \in \Gamma^{m+1} : x \in D, \alpha(x) \ \square_1 \ t \ \square_2 \ \beta(x), t \equiv c \pmod{n}\},$$

  where $D = \hat{\pi}_m(C)$ is a $(i_1, \ldots, i_m)$-cell, $\alpha, \beta : D \to \Gamma$ are linear functions $\square_i$ either $\leq$ or no condition, and n and $c \in [0, n)$ are fixed integers, such that:

  > Unbounded Fibres: The cardinality of the fibres $C_x = \{t \in \Gamma : (x,t) \in C\}$ cannot be bounded uniformly in $x \in D$ by an integer.

To fix notation, given an $(i_1, \ldots, i_n)$-cell $C \subseteq \Gamma^n$ we write $\mathsf{Z}(C)$ for the set $\{k \in [n] : i_k = 0\}$ and $\mathsf{D}(C)$ for $[n] \setminus \mathsf{Z}(C)$.

Finally:

Theorem 2.7.17 (Cell Decomposition Theorem [Clu03, Theorem 1])**.** *Let $X \subseteq \Gamma^n$ be an $\mathcal{L}_{\mathsf{Pres}}$-definable set, possibly over parameters $A \subseteq \Gamma$. Then there is a finite partition $\mathcal{P}$ of X into Presberger cells, such that each $C \in \mathcal{P}$ is*

$\mathcal{L}_{\mathsf{Pres}}(S)$-*definable.*

Chapter 3

Generalised Indiscernibles and Ramsey Theory[†]

> 'Ravens, so we read, can only count up to seven. They can't tell the difference between two numbers greater than eight. First-order logic is much the same as ravens, except that the cutoff point is rather higher: it's ω rather than 8.'
>
> Wilfrid Hodges, *Model Theory*

Introduction

In Chapter 2, I gave a rather extensive introduction to two of the main themes of the first part of this chapter, namely: (a) *Structural Ramsey Theory*; and (b) *Generalised Indiscernibles.*

I tactfully avoided presenting how central the interaction between these two concepts (which at first sight may seem disconnected from each other) is, at least from the point of view of model theory. But fret not; this will be done in detail in the **first part** of this chapter. In the **second part** of this chapter,

[†]This chapter is largely based on [MP23b] and [MP23a]. Both these papers were written jointly with Nadav Meir, and at the time of writing, have been submitted for publication. The contributions of both authors to these papers are equal, and the results are reproduced here with the permission of Meir. The presentation I give follows the presentation in our papers, with minor differences, which I will not point out.

I will discuss a family of "local" versions of the Ramsey property, somewhat motivated by the discussion that takes place in the first part.

Part 1: Practical and Structural Infinitary Expansions

3.1 Introduction to Part 1

Recall from Section 2.5 that the original motivation behind what we now refer to as "Ramsey's theorem" was to show that certain fragments of first-order logic are decidable, essentially through a weak form of indiscernibility (though, of course, not under that name). On the other hand, as discussed in Chapter 2, the fact that given any sequence indexed by a linear order, one can always realise its EM-type by an indiscernible sequence is essentially an application of Ramsey's theorem and compactness.

Recently, Scow showed that the connection between *generalised* indiscernibility and *structural* Ramsey theory is even deeper than the previous paragraph may suggest. More precisely, the main theorem of [Sco15] states that under some technical hypotheses, $\mathcal{N}$-indexed indiscernibles have the modelling property if, and only if, $\mathsf{Age}(\mathcal{N})$ is a Ramsey class.

One of the main goals of Part 1 of this chapter is to strengthen Scow's theorem by showing that most of the technical assumptions made in [Sco15] are, in fact, not necessary. More precisely:

Theorem A (Theorem 3.5.12)**.** *Let $\mathcal{L}'$ be a first-order language and $\mathcal{N}$ an infinite, locally finite $\mathcal{L}'$-structure. Then, the following are equivalent:*

1. $\mathsf{Age}(\mathcal{N})$ *is a Ramsey class.*
2. *$\mathcal{N}$-indexed indiscernibles have the modelling property (Definition 2.6.10).*

As an almost immediate corollary of Theorem A, it follows that *every* Ramsey class admits an order, given by a union of quantifier-free types, extending a known result for *countable* Ramsey classes, from [Bod15]. More precisely:

Theorem B (Corollary 3.5.13)**.** *Let $\mathcal{N}$ be an $\mathcal{L}'$-structure such that $\mathsf{Age}(\mathcal{N})$ is a (not necessarily countable)*[1] *Ramsey class. Then there is an $\mathsf{Aut}(\mathcal{N})$-invariant linear order on $\mathcal{N}$, which is a union of quantifier-free types. More explicitly,*

[1]That is, $\mathcal{N}$ does not necessarily contain countably many isomorphism types of each finite structure.

there is a (possibly infinite) Boolean combination of atomic and negated atomic $\mathcal{L}'$*-formulas* $\Phi(x, y) := \bigvee_{i \in I} \bigwedge_{j \in J_i} \phi_{j_i}^{(-1)^{n_{j_i}}}(x, y)$*, such that* Φ *is a linear order for every structure in* $\mathsf{Age}(\mathcal{N})$*.*

Remark 3.1.1. This result will be generalised in the second part of this chapter, with a completely different proof, which is of independent interest.

Given an $\mathcal{L}'$-structure $\mathcal{N}$, in [MP23b] we introduce the following terminology: we say that $\mathcal{C} := \mathsf{Age}(\mathcal{N})$ is $\mathcal{L}'_{\infty,0}$*-orderable* (see Definition 3.4.1), essentially, if it satisfies the conclusion of Theorem B, that is, if there is a (possibly infinite) Boolean combination of atomic and negated atomic $\mathcal{L}'$-formulas which is a linear order on every structure in $\mathcal{C}$. We observe that this is the *minimal required condition* so that for every theory T, there is some $\mathcal{N}$-indexed indiscernible sequence in some sufficiently saturated model of T. For a more precise statement, see Proposition 3.4.6, which lists multiple equivalent conditions for the existence of $\mathcal{N}$-indexed indiscernibles.

Finally, in [MP23b], we also introduce and discuss a generalisation of the Ramsey Property, the *finitary Ramsey Property* (see Definition 3.6.1), which, we argue, is the natural generalisation of the Ramsey Property to classes of possibly infinite, not necessarily relational, structures. To this end, we show, first of all, that Theorem A holds without the assumption of local finiteness if one replaces "$\mathcal{C}$ is a Ramsey class" by "$\mathcal{C}$ is a finitary Ramsey class" (see Theorem 3.6.8, for a more precise statement). In this direction, we also obtain the following slight strengthening of the well-known correspondence of Kechris, Pestov, and Todorčević, which was discussed in Section 2.5:

Theorem C (Theorem 3.6.6)**.** *Let* $\mathcal{M}$ *be a homogeneous* $\mathcal{L}$*-structure. Then, the following are equivalent:*

1. $\mathsf{Aut}(\mathcal{M})$ *is extremely amenable.*
2. $\mathsf{Age}(\mathcal{M})$ *is a finitary Ramsey class.*

Structure of Part 1

In Section 3.2 I've included a useful result on Ramsey classes and the full statement of Scow's theorem since the latter will be used to derive Theorem A. Then, Section 3.3 introduces a set of techniques based around "conservative" expansions in infinitary logic, which are the main tools used in the proofs of Theorem A and Theorem B. Section 3.4 discusses $\mathcal{L}_{\infty,0}$-orderability as a necessary and sufficient condition for the existence of $\mathcal{N}$-indexed indiscernibles in (sufficiently saturated) models of arbitrary theories. Then, Section 3.5 combines all the previous results to prove Theorem A and Theorem B. In Section 3.6 the finitary-Ramsey property is introduced, and classes that no longer necessarily contain just finite structures come into the picture.

Local Notational Conventions

Unless otherwise stated, in this chapter, the word *definable* means definable *without parameters*. The same convention applies when (quantifier-free) types are discussed; that is, unless otherwise stated *quantifier-free type* means quantifier-free type over $\emptyset$. The reader may have noticed that in the statements of the results, I used the letter $\mathcal{L}'$ rather than the usual $\mathcal{L}$. Typically, $\mathcal{L}'$ will be the language used for *indexing structures*.

3.2 Local Preliminaries for Part 1

Let me start with a relatively elementary lemma, which is probably folklore, but I wasn't able to find a proof in the literature:

Lemma 3.2.1. *Let $\mathcal{C}$ be a class of finite structures. Let $A_1, \dots A_n \in \mathcal{C}$ be structures contained in some $B \in \mathcal{C}$, with A_i having (embedding) Ramsey degree d_i, for all $i \leq n$. Then, there is some $C \in \mathcal{C}$ such that $B \subseteq C$ and for all $r_i \in \mathbb{N}$ and all sets of colourings $\{\chi_i : \binom{C}{A_i} \to r_i : i \leq n\}$, there is some $\tilde{B} \in \binom{C}{B}$ such that $|\mathsf{im}(\chi_{i \restriction \binom{\tilde{B}}{A_i}})| \leq d_i$, for all $i \leq n$.*

Proof. The proof is by induction on n. The base case is precisely the definition of A_1 having Ramsey degree d_1 in $\mathcal{C}$. For the inductive step, assume that we are given $A_1, \dots, A_{n+1}, B \in \mathcal{C}$ such that $A_i \subseteq B$ and the Ramsey degree of A_i is d_i, for all $i \leq n+1$. By the inductive hypothesis, there is some $C_0 \in \mathcal{C}$ such that for all $r_i \in \mathbb{N}$, $i \leq n$, and all sets of colourings $\{\chi_i : \binom{C_0}{A_i} \to r_i : i \leq n\}$, there is some $\tilde{B} \in \binom{C_0}{B}$ such that $|\mathsf{im}(\chi_{i \restriction \binom{\tilde{B}}{A_i}})| \leq d_i$, for all $i \leq n$. Now, since A_{n+1} has Ramsey degree d_{n+1} in $\mathcal{C}$, there is some $C \in \mathcal{C}$ such that for all $r \in \mathbb{N}$ and all colourings $\chi : \binom{C}{C_0} \to r$ there exists some $\tilde{C}_0 \in \binom{C}{C_0}$ such that $|\mathsf{im}(\chi_{\restriction \binom{\tilde{C}_0}{A_{n+1}}})| \leq d_{n+1}$.

We claim that this C is the required structure. Indeed, suppose that we are given $r_i \in \mathbb{N}$, $i \leq n+1$, and a set of colourings $\{\chi_i : \binom{C}{A_i} \to r_i : i \leq n+1\}$. Let $\tilde{C}_0 \in \binom{C}{C_0}$ be such that $|\mathsf{im}(\chi_{\restriction \binom{\tilde{C}_0}{A_{n+1}}})| \leq d_{n+1}$. Then, by our choice of $\tilde{C}_0$, there is some $\tilde{B} \in \binom{\tilde{C}_0}{B}$ such that $|\mathsf{im}(\chi_{i \restriction \binom{\tilde{B}}{A_i}})| \leq d_i$, for all $i \leq n$. But since $\binom{\tilde{B}}{A_{n+1}} \subseteq \binom{\tilde{C}_0}{A_{n+1}}$, we must have that $\left|\mathsf{im}\left(\chi_{n+1 \restriction \binom{\tilde{B}}{A_{n+1}}}\right)\right| \leq \left|\mathsf{im}\left(\chi_{\restriction \binom{\tilde{C}_0}{A_{n+1}}}\right)\right| \leq d_{n+1}$, as required. □

The following useful little corollary is now immediate:

Corollary 3.2.2. *Let $\mathcal{C}$ be a Ramsey class of finite structures. Then for all $i \leq n$, all $A_1, \ldots A_n \in \mathcal{C}$ and all $B \in \mathcal{C}$ such that $A_i \subseteq B$ for each $i \leq n$, there is some $C \in \mathcal{C}$ such that for all sets of colourings $\{\chi_i : \binom{C}{A_i} \to [2] : i \leq n\}$, there is some $\tilde{B} \in \binom{C}{B}$ such that $\chi_{i \restriction \binom{\tilde{B}}{A_i}}$ is constant, for all $i \leq n$.*

The rest of this short section is dedicated to the statement of the main theorem of [Sco15]. First of all, recall (or quickly learn) the following definition, also from [Sco15]:

Definition 3.2.3 (qfi, [Sco15, Definition 2.3])**.** Let $\mathcal{N}$ be an $\mathcal{L}'$-structure. We say that $\mathcal{N}$ has qfi if for any quantifier-free type $q(x)$ realised in $\mathcal{N}$ there is a quantifier-free formula $\theta_q(x)$ such that $\mathsf{Th}_\forall(\mathcal{N}) \cup \theta_q(x)$ is consistent and:

$$\mathsf{Th}_\forall(\mathcal{N}) \cup \theta_q(x) \vdash q(x).$$

As commented in [Sco15], the name qfi refers to *quantifier-free types* (in particular, *realised* quantifier-free types) being *isolated*, in the universal part of the theory of $\mathcal{N}$ (and, in fact, by a quantifier-free formula). The main theorem of [Sco15], which draws the deep connection between generalised indiscernibles and Ramsey classes I promised in Section 2.6 and the introduction, is the following:

Theorem 3.2.4 ([Sco15, Theorem 3.12])**.** *Fix a first-order language $\mathcal{L}'$ and assume that $\mathcal{L}'$ contains a distinguished binary relation symbol $<$. Let $\mathcal{C}$ be a class of finite $\mathcal{L}'$-structures and $\mathcal{N}$ an infinite, locally finite $\mathcal{L}'$-structure with* qfi *such that $\mathsf{Age}(\mathcal{N}) = \mathcal{C}$ and $\langle \mathcal{N}, \leq \rangle \vDash \mathsf{LO}$ (where LO expresses that $\leq$ is a linear order). Then, the following are equivalent:*

1. *$\mathcal{C}$ is a Ramsey class.*
2. *$\mathcal{N}$-indiscernibles have the modelling property.*

In the following sections, I will explain how to eliminate both the order assumption and the qfi assumption from Theorem 3.2.4.

3.3 Expansions in Infinitary Logic

One central piece of machinery introduced in [MP23b] is the notion of *infinitary Morleyisations*. These are, essentially, conservative expansions in a fragment of the *infinitary logic* $\mathcal{L}_{\kappa,\lambda}$ (really, as it turns out, this fragment will be $\mathcal{L}_{\infty,0}$), which I feel obligated to briefly introduce.

Recall that for a language $\mathcal{L}$ and (not necessarily infinite) cardinals κ, λ, we denote by $\mathcal{L}^{\mathsf{pnf}}_{\kappa,\lambda}$ the collection of $\mathcal{L}$-formulas of the form:

$$\underbrace{(Q_1 y_1), \dots, (Q_\alpha y_\alpha)}_{\alpha<\lambda} \bigvee_{\beta<\xi_1} \left(\bigwedge_{\gamma<\xi_2} \phi_{\beta,\gamma}(\bar{x}, \bar{y}) \right), \tag{$\star$}$$

for cardinals $\xi_1, \xi_2 < \kappa$, and where each Q_i is a first-order quantifier (i.e. $\forall$ or $\exists$), and each $\phi_{\beta,\gamma}(\bar{x}, \bar{y})$ is an atomic or negated atomic $\mathcal{L}$-formula, in finitely many variables. When κ (resp. λ) is replaced by ∞, the intended meaning is that the number of conjunctions/disjunctions (resp. quantifiers) is unbounded. In this notation (by the *prenex normal form theorem*) traditional first-order logic is concerned only with $\mathcal{L}^{\mathsf{pnf}}_{\omega,\omega}$-formulas, and $\mathcal{L}^{\mathsf{pnf}}_{\infty,0}$ refers to arbitrarily large conjunctions/disjunctions of quantifier-free formulas.

Remark 3.3.1. In general, for infinite cardinals κ and λ, one needs to distinguish between:

- $\mathcal{L}_{\kappa,\lambda}$, the usual infinitary logic, whose formulas are built inductively by allowing for quantification over λ-tuples and conjunction/disjunction of κ-many formulas; and
- $\mathcal{L}^{\mathsf{pnf}}_{\kappa,\lambda}$, as defined above.

The reason is that for $\kappa \geq \aleph_1$ the prenex normal form theorem may fail. That being said, it follows immediately from the definitions that:

$$\mathcal{L}^{\mathsf{pnf}}_{\kappa,0} = \mathcal{L}_{\kappa,0},$$

for any infinite cardinal κ, or $\kappa = \infty$. So, in this case, the superscript pnf will be dropped.

Given an $\mathcal{L}$-structure $\mathcal{M}$, a set $X \subseteq M^n$ is $\mathcal{L}^{\mathsf{pnf}}_{\kappa,\lambda}$*-definable* if there is a formula $\Phi(\bar{x})$ as in ($\star$) such that $X = \{\bar{a} \in M : \mathcal{M} \vDash \Phi(\bar{a})\}$.

3.3.1 Infinitary Morleyisations

When working in model-theoretic classification theory, it is common to focus not on the syntactical properties of the definable sets in a given structure (e.g. if they are definable using quantifier-free formulas) but on the complexity of the lattice of its definable sets.

In particular, it is often easier to work in expansions of a given structure, in which definable sets are easier to describe. Still, the complexity of the lattice of definable sets and hence the notions of "tameness" and "wildness" (such as the ones discussed in Section 2.3) remain unchanged.

The prototypical example of such a process is what is known as the *Morleyisation*, named after M. Morley, of Morley's Theorem (Theorem 2.3.1). Given an $\mathcal{L}$-structure $\mathcal{M}$, the Morleyisation produces an expanded language $\widehat{\mathcal{L}} \supseteq \mathcal{L}$ and an $\widehat{\mathcal{L}}$-structure $\widehat{\mathcal{M}}$, on the same domain as $\mathcal{M}$, with the properties that:

1. The $\mathcal{L}$-definable sets in $\mathcal{M}$ are precisely the $\widehat{\mathcal{L}}$-definable sets in $\widehat{\mathcal{M}}$.
2. $\widehat{\mathcal{M}}$ has quantifier elimination.

To achieve this, we do the obvious thing: we name every definable subset of our structure by a new relation symbol! More precisely, $\widehat{\mathcal{L}} \supseteq \mathcal{L}$, contains for each $\mathcal{L}$-definable $X \subseteq M^n$ an n-ary relation symbol R_X. We interpret the new relation symbols in a natural way:

$$\widehat{\mathcal{M}} \vDash R_X(\bar{a}) \text{ if, and only if, } \bar{a} \in X,$$

for all $\mathcal{L}$-definable $X \subseteq M^n$ and all $\bar{a} \in M^n$.

It is now an easy task to show that:

1. In the resulting structure, $\widehat{\mathcal{M}}$, all definable sets can be defined by quantifier-free formulas.
2. The $\hat{\mathcal{L}}$-definable sets in $\widehat{\mathcal{M}}$ are precisely the $\mathcal{L}$-definable sets in $\mathcal{M}$.

Remark 3.3.2. Since many important model-theoretic properties of $\mathcal{M}$ (e.g. categoricity, stability, NIP, and more) are, in fact, properties of the lattice of the definable sets of $\mathcal{M}$ and not of the descriptive complexity of said definable sets, all of these are preserved by Morleyisations. It is thus usual practice, when working in classification theory, to assume that structures eliminate quantifiers since they can always be Morleyised.

In [MP23b], we introduce a more general form of this process. We then show that our generalisation is again a "conservative expansion" in a very well-defined sense, even though it may increase the complexity of the definable sets in the structure.

Definition 3.3.3. Let κ, λ be (not necessarily infinite) cardinals, or ∞. We define the $\mathcal{L}^{\mathsf{pnf}}_{\kappa,\lambda}$-*Morleyisation* of $\mathcal{M}$ to be the expansion $\widehat{\mathcal{M}}$ of $\mathcal{M}$ to the language:

$$\widehat{\mathcal{L}} := \mathcal{L} \cup \{R_X : X \text{ is an } \mathcal{L}^{\mathsf{pnf}}_{\kappa,\lambda}\text{-definable subset of } \mathcal{M}^n, \text{ for } n \in \mathbb{N}\},$$

where we naturally interpret the new relation symbols as follows:

$$\widehat{\mathcal{M}} \vDash R_X(\bar{a}) \text{ if, and only if } \bar{a} \in X,$$

for every $\mathcal{L}^{\mathsf{pnf}}_{\kappa,\lambda}$-definable subset X of $\mathcal{M}$ and every $\bar{a}$ in $\mathcal{M}$.

It should be immediately apparent that the $\mathcal{L}_{\omega,\omega}$-Morleyisation is just the usual Morleyisation we're all, by now, familiar with.

In [MP23b] we then focus our attention on a special kind of $\mathcal{L}^{\mathsf{pnf}}_{\kappa,\lambda}$-Morleyisation, namely the $\mathcal{L}_{\infty,0}$-*Morleyisation*. More precisely, we are concerned with two special reducts of the $\mathcal{L}_{\infty,0}$-Morleyisation. The following two subsections are dedicated to these reducts.

3.3.2 Quantifier-Free Type Morleyisation

It will turn out that adding predicates for all $\mathcal{L}_{\infty,0}$-definable sets is a somewhat complicated procedure, and to discuss generalised indiscernibles, one need no more than the preservation and reflection of equality of quantifier-free types (as stated in Lemma 3.3.5 and Lemma 3.3.7). This is achieved in the following way:

Definition 3.3.4. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. The *quantifier-free type Morleyisation* of $\mathcal{M}$ is the expansion $\widehat{\mathcal{M}}_{\mathsf{qfi}}$ of $\mathcal{M}$ to the language

$$\widehat{\mathcal{L}}_{\mathsf{qfi}} := \mathcal{L} \cup \{R_p : p \text{ is a quantifier-free type realised in } \mathcal{M}\},$$

where we interpret the new relation symbols in the natural way, that is:

$$\widehat{\mathcal{M}}_{\mathsf{qfi}} \vDash R_p(\bar{a}) \text{ if, and only if } \mathcal{M} \vDash p(\bar{a}),$$

for every $\bar{a} \in \mathcal{M}$ and every quantifier-free type p that is realised in $\mathcal{M}$.

An obvious remark about the quantifier-free type Morleyisation (and more generally about the $\mathcal{L}_{\infty,0}$-Morleyisation), which will be relevant later, is that equality of quantifier-free types does not change when one moves from $\mathcal{M}$ to $\widehat{\mathcal{M}}_{\mathsf{qfi}}$ (or $\widehat{\mathcal{M}}$). More precisely:

Lemma 3.3.5. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Then, for all tuples $\bar{a}, \bar{b}$ from $\mathcal{M}$ we have that:*

$$\mathsf{qftp}_{\widehat{\mathcal{M}}_{\mathsf{qfi}}}(\bar{a}) = \mathsf{qftp}_{\widehat{\mathcal{M}}_{\mathsf{qfi}}}(\bar{b}) \text{ if, and only if, } \mathsf{qftp}_{\mathcal{M}}(\bar{a}) = \mathsf{qftp}_{\mathcal{M}}(\bar{b}).$$

Proof. On the one hand, if $\mathsf{qftp}_{\mathcal{M}}(\bar{a}) \neq \mathsf{qftp}_{\mathcal{M}}(\bar{b})$, then, by definition there is a quantifier-free $\widehat{\mathcal{L}}_{\mathsf{qfi}}$-formula (in fact, an atomic one) which witnesses this, so $\mathsf{qftp}_{\widehat{\mathcal{M}}_{\mathsf{qfi}}}(\bar{a}) \neq \mathsf{qftp}_{\widehat{\mathcal{M}}_{\mathsf{qfi}}}(\bar{b})$. For the converse, suppose that $\mathsf{qftp}_{\widehat{\mathcal{M}}_{\mathsf{qfi}}}(\bar{a}) \neq \mathsf{qftp}_{\widehat{\mathcal{M}}_{\mathsf{qfi}}}(\bar{b})$. Then, there must be some $\widehat{\mathcal{L}}_{\mathsf{qfi}}$-quantifier-free formula ϕ which belongs to the first type but not to the second. Without loss of generality, we may assume that this formula is in $\widehat{\mathcal{L}}_{\mathsf{qfi}} \setminus \mathcal{L}$, for otherwise we would be done immediately. Of course, if $\bar{a}$ and $\bar{b}$ agree on all atomic $\widehat{\mathcal{L}}_{\mathsf{qfi}}$-formulas, then $\mathsf{qftp}_{\widehat{\mathcal{M}}_{\mathsf{qfi}}}(\bar{a}) = \mathsf{qftp}_{\widehat{\mathcal{M}}_{\mathsf{qfi}}}(\bar{b})$. So, in particular, there must be some $R_p \in \widehat{\mathcal{L}}_{\mathsf{qfi}}$ such that:

$$\widehat{\mathcal{M}}_{\mathsf{qfi}} \vDash R_p(\bar{a}) \wedge \neg R_p(\bar{b}).$$

But then, $\mathsf{qftp}(\bar{a}) = p$ and $\mathsf{qftp}(\bar{b}) \neq p$, so $\mathsf{qftp}_{\mathcal{M}}(\bar{a}) \neq \mathsf{qftp}_{\mathcal{M}}(\bar{b})$, as required. □

3.3.3 Quantifier-Free type Isolators

The conclusion of Lemma 3.3.5 is the crucial condition we need for our results. It is often, though, convenient to work with relational structures rather than arbitrary ones. To this end, let's consider the smallest relational reduct of the quantifier-free type Morleyisation of an $\mathcal{L}$-structure. More precisely:

Definition 3.3.6. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. The *quantifier-free type isolator*, or the $\mathcal{L}_{\infty,0}$*-isolator* of $\mathcal{M}$ is the reduct, $\mathcal{M}_{\mathsf{iso}}$, of $\widehat{\mathcal{M}}_{\mathsf{qfi}}$ to the language $\mathcal{L}_{\mathsf{iso}} = \widehat{\mathcal{L}}_{\mathsf{qfi}} \setminus \mathcal{L}$.

Explicitly, in the notation above, $\mathcal{L}_{\mathsf{iso}}$ is such that for every $\bar{a} \in \mathcal{M}$, if $p = \mathsf{qftp}_{\mathcal{M}}(\bar{a})$, then there is a unique relation symbol $R_p \in \mathcal{L}_{\mathsf{iso}}$ such that for every $\bar{b} \in \mathcal{M}$:

$$\mathcal{M}_{\mathsf{iso}} \vDash R_p(\bar{b}) \text{ if, and only if, } \mathsf{qftp}_{\mathcal{M}}(\bar{b}) = p.$$

Of course, as commented earlier, it is easy to see that the analogue of Lemma 3.3.5 still holds for quantifier-free type isolators. I'll write it down explicitly below for bookkeeping purposes:

Lemma 3.3.7. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Then, for all tuples $\bar{a}, \bar{b}$ from $\mathcal{M}$ we have that:*

$$\mathsf{qftp}_{\mathcal{M}_{\mathsf{iso}}}(\bar{a}) = \mathsf{qftp}_{\mathcal{M}_{\mathsf{iso}}}(\bar{b}) \text{ if, and only if, } \mathsf{qftp}_{\mathcal{M}}(\bar{a}) = \mathsf{qftp}_{\mathcal{M}}(\bar{b}).$$

Proof. This is essentially identical to the proof of Lemma 3.3.5. □

In the following sections, as discussed in Section 3.1, this machinery will be used to prove stronger versions of Scow's theorem. The quantifier-free type isolator construction, in particular, will be used to prove a variant of the Kechris-Pestov-Todorčević correspondence.

3.4 Existence of indiscernible sequences

In this section, I will discuss necessary and sufficient conditions on an $\mathcal{L}'$-structure $\mathcal{N}$ for $\mathcal{N}$-indiscernibles to exist in (sufficiently saturated) models of *arbitrary* theories. The key definition of this section is the following:

Definition 3.4.1. Let $\mathcal{C}$ be a class of $\mathcal{L}'$-structures. We say that $\mathcal{C}$ is $\mathcal{L}'_{\infty,0}$-*orderable* if there is a (possibly infinite) Boolean combination of atomic and negated atomic $\mathcal{L}'$-formulas $\phi_{j_i}^{(-1)^{n_{j_i}}}(x, y)$:

$$\Phi(x, y) := \bigvee_{i \in I} \bigwedge_{j \in J_i} \phi_{j_i}^{(-1)^{n_{j_i}}}(x, y),$$

such that Φ defines a linear order on every structure in $\mathcal{C}$, i.e. if there is an $\mathcal{L}'_{\infty,0}$-formula which is a linear order on all structures in $\mathcal{C}$. We say that an $\mathcal{L}'$-structure $\mathcal{N}$ is $\mathcal{L}'_{\infty,0}$-*orderable* if $\mathsf{Age}(\mathcal{N})$ is $\mathcal{L}'_{\infty,0}$-orderable.

I will start with the following easy lemma:

Lemma 3.4.2. *Let $\mathcal{M}_1, \mathcal{M}_2$ be structures in languages $\mathcal{L}_1, \mathcal{L}_2$, respectively, on the same set M, such that:*

$$\mathsf{qftp}_{\mathcal{M}_1}(\bar{a}) = \mathsf{qftp}_{\mathcal{M}_1}(\bar{b}) \iff \mathsf{qftp}_{\mathcal{M}_2}(\bar{a}) = \mathsf{qftp}_{\mathcal{M}_2}(\bar{b}), \qquad (\dagger)$$

for every $\bar{a}, \bar{b} \in M$. Then, the following are equivalent:

1. *$\mathcal{M}_1$ has MP.*

2. *$\mathcal{M}_2$ has MP.*

Proof. Assume that (†) holds for $\mathcal{M}_1$ and $\mathcal{M}_2$ as in the statement. It suffices to show that if $\mathcal{M}_1$ has MP, then so does $\mathcal{M}_2$, as the argument is symmetric. So let T be a first-order theory and $\mathbb{M} \vDash T$ a monster model of T which, without loss of generality, is $|M|^+$-saturated. Fix $\mathfrak{I} = (\bar{a}_m : m \in M)$. Since $\mathcal{M}_1$-indiscernibles have the modelling property, there is an $\mathcal{M}_1$-indiscernible $\mathfrak{J} = (\bar{b}_m : m \in M)$ which is based on $\mathfrak{I}$. By (†), $\mathfrak{J}$ is also $\mathcal{M}_2$-indiscernible, and the result follows. □

The next corollary is an almost immediate consequence of the above:

Corollary 3.4.3. *Let $\mathcal{N}$ be an $\mathcal{L}'$-structure and let $\preceq$ be a linear order on $\mathcal{N}$, given by a union of quantifier-free types. Let $\mathcal{N}_o = (\mathcal{N}, \preceq)$. If $\mathcal{N}_o$-indiscernibles have MP, then $\mathcal{N}$-indiscernibles have MP.*

Proof. Since $\preceq$ is a union of quantifier-free types of $\mathcal{N}$, we have that $\mathcal{N}$ and $\mathcal{N}_o$ satisfy (†), from Lemma 3.4.2, and thus the result follows. □

To see that, given an $\mathcal{L}'$-structure $\mathcal{N}$, the property that $\mathsf{Age}(\mathcal{N})$ is $\mathcal{L}'_{\infty,0}$-orderable is the minimal requirement for $\mathcal{N}$-indiscernibles to exist in every theory, one needs first to consider universal theories only:

Lemma 3.4.4. *Given an $\mathcal{L}'$-structure $\mathcal{N}$ and a universal*[2] *$\mathcal{L}$-theory T, the following are equivalent:*

1. *There is some $\mathcal{L}$-structure $\mathcal{M} \vDash T$ and a non-constant $\mathcal{N}$-indiscernible sequence of singletons in $\mathcal{M}$.*

2. *There is some $\mathcal{L}$-structure $\mathcal{M} \vDash T$ such that:*

   - *$N \subseteq M$ and $\langle N \rangle_{\mathcal{M}} = \mathcal{M}$, where $\langle N \rangle_{\mathcal{M}}$ denotes the $\mathcal{L}$-substructure of $\mathcal{M}$ generated*[3] *by N (see Section 2.2).*

   - *Every $\mathcal{L}$-definable subset of M restricted to N is equal to an $\mathcal{L}'_{\infty,0}$-definable subset of $\mathcal{N}$.*

Remark 3.4.5. When $\mathcal{L}$ is relational we have that $\langle N \rangle_{\mathcal{M}} = N$, so the lemma actually gives us an $\mathcal{L}'_{\infty,0}$-definable model of T *on* N, where, by an $\mathcal{L}'_{\infty,0}$-

[2] Recall a theory T is called *universal* if it admits an axiomatisation T_0 consisting of universal sentences.

[3] Equivalently, the smallest $\mathcal{L}$-substructure of $\mathcal{M}$ whose domain contains N.

definable structure on $\mathcal{N}$, I mean an $\mathcal{L}$-structure $\mathcal{M}$ with the same domain as $\mathcal{N}$, such that each relation in $\mathcal{L}$ on $\mathcal{M}$ is given by an $\mathcal{L}'_{\infty,0}$-formula in $\mathcal{N}$.

Proof. For the implication (2) $\implies$ (1), observe that if T is a universal theory and $\mathcal{M} \vDash T$ is an $\mathcal{L}'_{\infty,0}$-definable structure on $\mathcal{N}$, as in (2), then N is, by definition, an $\mathcal{N}$-indiscernible sequence of singletons in $\mathcal{M}$, where each element is indexed by itself.

For the implication (1) $\implies$ (2), let $\mathcal{M}' \vDash T$ and $(a_i : i \in \mathcal{N})$ be a non-constant $\mathcal{N}$-indiscernible sequence of singletons in $\mathcal{M}'$, as in (1).

Since T is universal and $\{a_i : i \in \mathcal{N}\} \subseteq \mathcal{M}' \vDash T$, it follows that

$$\langle \{a_i : i \in \mathcal{N}\} \rangle_{\mathcal{M}'} \vDash T.$$

Let $\mathcal{M} = \langle \{a_i : i \in \mathcal{N}\} \rangle_{\mathcal{M}'}$. This shows the first item of (2). To simplify notation, we identify a_i with i, for all $i \in N$.

To prove (2), it suffices to show that every $\mathcal{L}$-definable subset of $\mathcal{M}$ restricted to $N \subseteq \langle i : i \in \mathcal{N} \rangle_{\mathcal{M}'}$, is actually an $\mathcal{L}'_{\infty,0}$-definable subset of $\mathcal{N}$.

To this end, given an $\mathcal{L}$-formula $\phi(x_1, \dots, x_n)$, let

$$\Psi_\phi := \{\mathsf{qftp}_{\mathcal{N}}(i_1, \dots, i_n) : i_1, \dots, i_n \in N, \mathcal{M} \vDash \phi(i_1, \dots, i_n)\}.$$

By definition of Ψ, if $\mathcal{M} \vDash \phi(i_1, \dots, i_n)$, for some $i_1, \dots, i_n \in N$, then $i_1, \dots, i_n \vDash p$ for some $p \in \Psi_\phi$. Conversely, by $\mathcal{N}$-indiscernibility of $(i : i \in \mathcal{N})$, if $i_1, \dots, i_n \vDash p$, for some $p \in \Psi_\phi$ then $\mathcal{M} \vDash \phi(i_1, \dots, i_n)$. Thus,

$$\mathcal{M} \vDash \phi(a_{i_1}, \dots, a_{i_n}) \text{ if, and only if, } \mathcal{N} \vDash \bigvee_{p \in \Psi_\phi} \left(\bigwedge_{\psi \in p} \psi(i_1, \dots, i_n) \right),$$

which concludes the proof. □

Now for the main event of this section:

Proposition 3.4.6. *Given an* $\mathcal{L}'$*-structure* $\mathcal{N}$*, the following are equivalent:*

1. $\mathsf{Age}(\mathcal{N})$ *is* $\mathcal{L}'_{\infty,0}$*-orderable.*
2. *There is an infinite* $\{\leq\}$*-structure* $\mathcal{M}$ *such that* $(\mathcal{M}, \leq) \vDash \mathsf{LO}$ *and a non-constant* $\mathcal{N}$*-indiscernible sequence in* $\mathcal{M}$*.*

3. *There is an infinite $\{\leq\}$-structure $\mathcal{M}$ such that $(\mathcal{M}, \leq) \models \mathsf{LO}$ and a non-constant $\mathcal{N}$-indiscernible sequence* of singletons *in $\mathcal{M}$.*

4. *For every $\mathcal{L}$-theory T with infinite models and every sufficiently saturated model $\mathbb{M} \models T$ there is a non-constant $\mathcal{N}$-indiscernible sequence in $\mathbb{M}$.*

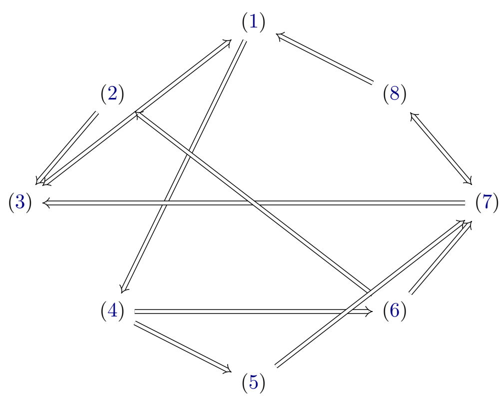
5. *For every $\mathcal{L}$-theory T with infinite models, and every sufficiently saturated model $\mathbb{M} \models T$ there is a non-constant $\mathcal{N}$-indiscernible sequence* of singletons *in $\mathbb{M}$.*

6. *For every* universal *$\mathcal{L}$-theory T with infinite models and every sufficiently saturated model $\mathbb{M} \models T$ there is a non-constant $\mathcal{N}$-indiscernible sequence in $\mathbb{M}$.*

7. *For every* universal *$\mathcal{L}$-theory T with infinite models, and every sufficiently saturated model $\mathbb{M} \models T$ there is a non-constant $\mathcal{N}$-indiscernible sequence* of singletons *in $\mathbb{M}$.*

8. *For every relational universal $\mathcal{L}$-theory T with infinite models, there is some $\mathcal{L}'_{\infty,0}$-definable model of T on $\mathcal{N}$.*

Proof. Notice that LO is a universal theory, so (8) $\implies$ (1), (6) $\implies$ (2), and (7) $\implies$ (3). Since order-indiscernible sequences exist in every theory with infinite models, and if $\mathcal{N}$ is an ordered structure, then $(\mathcal{N} \restriction \{<\})$-indiscernibility implies $\mathcal{N}$-indiscernibility, we have (1) $\implies$ (4). The implications (4) $\implies$ (5), (5) $\implies$ (7), (4) $\implies$ (6), (6) $\implies$ (7), and (2) $\implies$ (3) are trivial. The equivalences (1) $\iff$ (3) and (8) $\iff$ (7) are by Lemma 3.4.4. Thus, we have the following digraph of implications:



which is strongly connected. $\square$

Since if $\mathcal{N}$ has MP, non-constant $\mathcal{N}$-indiscernibles exist in every theory with infinite models (as we may always start with a non-constant $\mathcal{N}$-indexed sequence and find an $\mathcal{N}$-indiscernible sequence based on it) the proposition above gives the following corollary, which unlike Fact 2.5.17 does not assume that $\mathsf{Age}(\mathcal{N})$ is a countable class:

Corollary 3.4.7. *Let $\mathcal{N}$ be an $\mathcal{L}'$-structure with MP. Then $\mathcal{N}$ is $\mathcal{L}'_{\infty,0}$-orderable.*

This corollary will be used to show that the order assumption in Theorem 3.2.4 is not necessary. In particular, the key property from the corollary is that if $\mathcal{N}_o$ is the $\mathcal{L}'_{\infty,0}$-definable ordered expansion of $\mathcal{N}$ then, for all tuples $\bar{a}, \bar{b}$ from N we have that $\mathsf{qftp}_{\mathcal{N}}(\bar{a}) = \mathsf{qftp}_{\mathcal{N}}(\bar{b})$ if, and only if, $\mathsf{qftp}_{\mathcal{N}_o}(\bar{a}) = \mathsf{qftp}_{\mathcal{N}_o}(\bar{b})$.

3.5 Removing Assumptions From Scow's Theorem

The goal of this section is to show that in Theorem 3.2.4, the assumptions of: 1. $\mathcal{N}$ having qfi; and 2. $\mathcal{N}$ being linearly ordered, can be removed.

First, the qfi assumption is dealt with, in Subsection 3.5.1, by using *quantifier-free type Morleyisations*. Then, the order assumption is removed when $\mathsf{Age}(\mathcal{N})$ is countable, in Subsection 3.5.2, and finally, the countability assumption on $\mathsf{Age}(\mathcal{N})$ is lifted in Subsection 3.5.3. If the reader wants to skip ahead to the main result and work backwards, they are directed to Theorem 3.5.12.

3.5.1 The qfi assumption

First, I'll highlight some easy results regarding the quantifier-free type Morleyisation. The next lemma follows from the definition of the quantifier-free type Morleyisation:

Lemma 3.5.1. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Then $\widehat{\mathcal{M}}_{\mathsf{qfi}}$ has qfi.*

Proof. We need to show that every quantifier-free type which is realised in $\widehat{\mathcal{M}}_{\mathsf{qfi}}$ is isolated (in the universal part of the theory of $\widehat{\mathcal{M}}_{\mathsf{qfi}}$) by a quantifier-free formula. To this end, let $\bar{a}$ be a tuple from M, and write $\mathsf{q}(\bar{x}) = \mathsf{qftp}_{\widehat{\mathcal{M}}_{\mathsf{qfi}}}(\bar{a})$ for the quantifier-free type of $\bar{a}$ in $\widehat{\mathcal{M}}_{\mathsf{qfi}}$. By definition of $\widehat{\mathcal{M}}_{\mathsf{qfi}}$, the quantifier-free type $q(\bar{x})$ of $\bar{a}$ in $\mathcal{M}$ completely determines $\mathsf{q}(\bar{x})$. Thus, $\mathsf{q}(\bar{x})$ is isolated by the quantifier-free formula $R_q \in \widehat{\mathcal{L}}_{\mathsf{qfi}}$. □

The next lemma is the analogue of Lemma 3.4.2, for ERP:

Lemma 3.5.2. *Let $\mathcal{M}_1, \mathcal{M}_2$ be locally finite structures in languages $\mathcal{L}_1 \subseteq \mathcal{L}_2$, respectively, on the same domain M. Assume that for all tuples $\bar{a}, \bar{b}$ from M we have that:*

$$\mathsf{qftp}_{\mathcal{M}_1}(\bar{a}) = \mathsf{qftp}_{\mathcal{M}_1}(\bar{b}) \iff \mathsf{qftp}_{\mathcal{M}_2}(\bar{a}) = \mathsf{qftp}_{\mathcal{M}_2}(\bar{b}). \tag{†}$$

Moreover, assume that $\mathcal{L}_2 \setminus \mathcal{L}_1$ consists only of relation symbols. Then, the following are equivalent:

1. *$\mathcal{M}_1$ has ERP.*
2. *$\mathcal{M}_2$ has ERP.*

Proof. It suffices to show that if $\mathcal{M}_1$ has ERP, then so does $\mathcal{M}_2$, since the other implication can be proved in the same way. Let $A \subseteq B$ be finite subsets of M, with fixed enumerations, say $\bar{a} \subseteq \bar{b}$. Let $\mathcal{A}_i \subseteq \mathcal{B}_i$ be the finite substructures of $\mathcal{M}_i$ generated by A and B in M_i, for $i \in \{1, 2\}$. By assumption, for all $\bar{a}' \subseteq \bar{b}$ we have that:

$$\mathsf{qftp}_{\mathcal{M}_1}(\bar{a}') = \mathsf{qftp}_{\mathcal{M}_1}(\bar{a}) \iff \mathsf{qftp}_{\mathcal{M}_2}(\bar{a}') = \mathsf{qftp}_{\mathcal{M}_2}(\bar{a}).$$

First, notice that since $\mathcal{M}_1$ has ERP, $\mathsf{Age}(\mathcal{M}_1)$ consists of rigid structures and by (†) the same must be true for $\mathsf{Age}(\mathcal{M}_2)$. Observe now that since $\mathcal{L}_2 \setminus \mathcal{L}_1$ consists only of relation symbols we have that:

$$\binom{\mathcal{B}_1}{\mathcal{A}_1} = \left\{ \langle \bar{a}' \rangle_{\mathcal{M}_1} : \bar{a}' \subseteq \bar{b}, \text{ and } \mathsf{qftp}_{\mathcal{M}_1}(\bar{a}') = \mathsf{qftp}_{\mathcal{M}_1}(\bar{a}) \right\},$$

and:

$$\binom{\mathcal{B}_2}{\mathcal{A}_2} = \left\{ \langle \bar{a}' \rangle_{\mathcal{M}_2} : \bar{a}' \subseteq \bar{b}, \text{ and } \mathsf{qftp}_{\mathcal{M}_2}(\bar{a}') = \mathsf{qftp}_{\mathcal{M}_2,}(\bar{a}) \right\},$$

so, by (†) we have a one-to-one correspondence between these two sets.

Since $\mathcal{M}_1$ has ERP, there is some finite $C \subseteq M$ such that $\mathcal{C}_1 \to (\mathcal{B}_1)_2^{\mathcal{A}_1}$, where $\mathcal{C}_1$ is the substructure of $\mathcal{M}_1$ generated by C. But, arguing as above, we have a one-to-one correspondence between $\binom{\mathcal{C}_1}{\mathcal{A}_1}$ and $\binom{\mathcal{C}_2}{\mathcal{A}_2}$, where $\mathcal{C}_2$ is the substructure of $\mathcal{M}_2$ generated by C, and for every $\mathcal{B}_i' \subseteq \mathcal{C}_i$ such that $\mathcal{B}_i \simeq \mathcal{B}_i'$ we have a one-to-one correspondence between $\binom{\mathcal{B}_1'}{\mathcal{A}_1}$ and $\binom{\mathcal{B}_2'}{\mathcal{A}_2}$. Thus, by unpacking the definitions, it follows $\mathcal{C}_2 \to (\mathcal{B}_2)_2^{\mathcal{A}_2}$, as required. □

It is now almost immediate that the qfi assumption from Theorem 3.2.4 can be removed. More precisely:

Corollary 3.5.3. *Theorem 3.2.4 holds without assuming that $\mathcal{N}$ has* qfi.

Proof. By Lemma 3.3.5 we have that $\mathcal{N}$ and $\widehat{\mathcal{N}}_{\mathsf{qfi}}$ satisfy the assumptions of Lemmas 3.4.2 and 3.5.2. Thus, it suffices to show that the corollary holds for $\widehat{\mathcal{N}}_{\mathsf{qfi}}$, but this is immediate from Theorem 3.2.4 and Lemma 3.5.1. □

3.5.2 Removing the order assumption, in the countable case

Lemma 3.5.4. *Let $\mathcal{N}$ be an infinite locally finite $\mathcal{L}'$-structure with ERP and assume that $\mathsf{Age}(\mathcal{N})$ is countable. Then $\mathcal{N}$ is $\mathcal{L}_{\infty,0}$-orderable.*

Proof. First, by Theorem 2.5.12, it follows that $\mathsf{Age}(\mathcal{N})$ is a Fraïssé class, so we may consider $\mathcal{N}' := \mathsf{Flim}(\mathsf{Age}(\mathcal{N}))$. By [Bod15, Proposition 2.25], which states that homogeneous structures whose ages are Ramsey classes admit automorphism-invariant linear orders, we obtain an $\mathsf{Aut}(\mathcal{N}')$-invariant order $\preceq$ on $\mathcal{N}'$.

It is then clear, from the fact that $\mathcal{N}'$ is homogeneous that $\preceq$ is given by a (possibly infinite) Boolean combination of atomic or negated atomic $\mathcal{L}'$-formulas, thus $\mathsf{Age}(\mathcal{N}') = \mathsf{Age}(\mathcal{N})$ is $\mathcal{L}'_{\infty,0}$-orderable.

Let $\Phi(x,y)$ be the $\mathcal{L}_{\infty,0}$-formula which defines $\preceq$ on $\mathcal{N}'$. We claim that $\Phi(x,y)$ defines a linear order on $\mathcal{N}$, as well. By construction, $\mathsf{Age}(\mathcal{N}) = \mathsf{Age}(\mathcal{N}') = \mathcal{C}$, and $\Phi(x,y)$ defines a linear order on each $A \in \mathcal{C}$. Suppose that $\Phi(x,y)$ does not define a linear order on $\mathcal{N}$, then there is a finite substructure $A \subseteq \mathcal{N}$ on which $\Phi(x,y)$ is not a linear order, but $A \in \mathcal{C}$, contradicting the assumption that $\Phi(x,y)$ defines a linear order on all $A \in \mathcal{C}$. □

Combining the lemma above with Theorem 3.2.4 the following two corollaries can be deduced straightforwardly. I have opted to state them separately because the countability assumption appears only in the first.

Corollary 3.5.5. *Let $\mathcal{N}$ be an infinite locally finite $\mathcal{L}'$-structure such that $\mathsf{Age}(\mathcal{N})$ is countable. If $\mathcal{N}$ has ERP, then it has MP.*

Proof. Observe that if $\mathsf{Age}(\mathcal{N})$ is a countable Ramsey class, then Lemma 3.5.4 gives an $\mathcal{L}'_{\infty,0}$-definable linear order $\preceq$ on $\mathcal{N}$. Since $(\mathcal{N}, \preceq)$ is an expansion of $\mathcal{N}$ by a union of quantifier-free types, it follows that $\mathcal{N}$ and $(\mathcal{N}, \preceq)$ satisfy the assumptions of Lemmas 3.4.2 and 3.5.2.

Thus, by Lemma 3.5.2, we have that $(\mathcal{N}, \preceq)$ has ERP, since $\mathcal{N}$ has ERP. But then, by Corollary 3.5.3 we have that $(\mathcal{N}, \preceq)$ has MP. Finally, by Lemma 3.4.2, $\mathcal{N}$ has MP. □

Corollary 3.5.6. *Let $\mathcal{N}$ be an infinite locally finite $\mathcal{L}'$-structure. If $\mathcal{N}$ has MP, then $\mathcal{N}$ has the ERP.*

Proof. By Corollary 3.4.7 we may also assume that $\mathcal{N}$ is linearly ordered and still has MP, since the order is $\mathcal{L}'_{\infty,0}$-definable (again by Lemma 3.4.2, as in the previous proof). So, the result follows from Corollary 3.5.3. □

3.5.3 Reduction to the countable case

The main technical result of this subsection is the following proposition:

Proposition 3.5.7. *Let $\mathcal{N}$ be an $\mathcal{L}'$-structure, and let $\{\mathcal{N}_i : i \in I\}$ be $\mathcal{L}'$-structures, such that:*

1. *$\mathcal{N} = \bigcup_{i \in I} \mathcal{N}_i$.*
2. *For every finite $A \subseteq \mathcal{N}$, there is some $i \in I$, such that $A \subseteq \mathcal{N}_i$.*

If for all $i \in I$ we have that $\mathcal{N}_i$ has MP, then $\mathcal{N}$ has MP.

First, recall the following definition from [Sco15], which is a partial type, which, as its name suggests, enforces that an $\mathcal{N}$-indexed sequence is $\mathcal{N}$-indiscernible.

Definition 3.5.8 ([Sco15, Definition 2.10])**.** Let $\mathcal{L}$ and $\mathcal{L}'$ be first-order languages and fix an $\mathcal{L}'$-structure $\mathcal{N}$. We define $\mathsf{Ind}_{\mathcal{L}}^{\mathcal{N}}$ to be the following partial type in an $\mathcal{N}$-indexed sequence of variables $(x_i : i \in \mathcal{N})$:

$$\mathsf{Ind}_{\mathcal{L}}^{\mathcal{N}}(x_i : i \in \mathcal{N}) := \Big\{\phi(x_{i_1}, \dots, x_{i_n}) \to \phi(x_{j_1}, \dots, x_{j_n}) : n < \omega, \bar{i}, \bar{j} \in N^n, \\ \mathsf{qftp}_{\mathcal{N}}(\bar{i}) = \mathsf{qftp}_{\mathcal{N}}(\bar{j}),\ \phi(x_1, \dots, x_n) \in \mathcal{L}\Big\}.$$

The following is [Sco15, Definition 2.11] in disguise (see Remark 3.5.10). The idea behind this definition is a general notion of what it means for a partial type to be finitely satisfiable along a sequence indexed by an arbitrary structure.

Definition 3.5.9. Let $\mathcal{L}$ and $\mathcal{L}'$ be first-order languages, $\mathcal{M}$ an $\mathcal{L}$-structure, and $\mathcal{N}$ an $\mathcal{L}'$-structure. Let $\Sigma(\bar{x}_i : i \in N)$ be a (possibly partial) $\mathcal{L}$-type in $\mathcal{M}$, and let $\mathfrak{I} = (\bar{a}_i : i \in \mathcal{N})$ be an $\mathcal{N}$-indexed sequence of $|\bar{x}_i|$-tuples from $\mathcal{M}$. We say that Σ is *finitely satisfiable in* $\mathfrak{I}$ if for every finite $\Sigma_0 \subseteq \Sigma$ and every finite $A = \{i_1, \dots, i_n\} \subseteq N$, there is a finite $B = \{j_1, \dots, j_n\} \subseteq N$, such that:

(1) $\mathsf{qftp}_{\mathcal{N}}(i_1, \ldots, i_n) = \mathsf{qftp}_{\mathcal{N}}(j_1, \ldots, j_n)$, and

(2) $(\bar{a}_{j_1}, \ldots, \bar{a}_{j_n}) \vDash \Sigma_0 \restriction_{\{\bar{x}_{i_1}, \ldots, \bar{x}_{i_n}\}}$.

Remark 3.5.10. Let me comment here that the only difference between the definition above and [Sco15, Definition 2.11], is that in [Sco15], the analogous (2) asks that:

$$(\bar{a}_{j_1}, \ldots, \bar{a}_{j_n}) \vDash \Sigma \restriction_{\{\bar{x}_{i_1}, \ldots, \bar{x}_{i_n}\}},$$

that is, it does not range over the finite subsets $\Sigma_0 \subseteq \Sigma$. Of course, by compactness, the two definitions are equivalent in any $|\mathcal{N}|^+$-saturated structure.

Lemma 3.5.11. *Fix an $\mathcal{L}'$-structure $\mathcal{N}$, and a (not necessarily indiscernible) $\mathcal{N}$-indexed sequence $\mathfrak{I} = (\bar{a}_i : i \in \mathcal{N})$, in a sufficiently saturated structure. Then, the following are equivalent:*

(1) $\mathsf{Ind}_{\mathcal{L}}^{\mathcal{N}}$ *is finitely satisfiable in $\mathfrak{I}$.*

(2) There is an $\mathcal{N}$-indiscernible $\mathfrak{J} = (b_i : i \in \mathcal{N})$ that is (locally) based on $\mathfrak{I}$.

Proof. The implication (1) $\implies$ (2) is precisely [Sco15, Proposition 2(6)].

The implication (2) $\implies$ (1) follows almost immediately from local basedness and generalised indiscernibility. Indeed, pick an arbitrary finite subset $\Sigma_0 \subseteq \mathsf{Ind}_{\mathcal{L}}^{\mathcal{N}}$ and $(i_1, \ldots, i_n) \subseteq N$. Since $\mathfrak{J}$ is an $\mathcal{N}$-indiscernible based on $\mathfrak{I}$, there is some $(j_1, \ldots, j_n) \subseteq \mathcal{N}$, such that:

(a) $\mathsf{qftp}_{\mathcal{N}}(\bar{i}) = \mathsf{qftp}_{\mathcal{N}}(\bar{j})$;

(b) $\mathsf{tp}_{\mathbb{M}}^{\Sigma_0}(b_{i_1}, \ldots, b_{i_n}) = \mathsf{tp}_{\mathbb{M}}^{\Sigma_0}(a_{j_1}, \ldots a_{j_n})$.

Since $(b_{i_1}, \ldots, b_{i_n}) \vDash \Sigma_0 \restriction_{\{x_{i_1}, \ldots, x_{i_n}\}}$, by the (generalised) indiscernibility of $\mathfrak{J}$, it follows that $(a_{j_1}, \ldots, a_{j_n}) \vDash \Sigma_0 \restriction \{x_{i_1}, \ldots, x_{i_n}\}$. □

Now for the proof of Proposition 3.5.7.

Proof of Proposition 3.5.7. Let $\mathfrak{I} = (a_i : i \in \mathcal{N})$ be an $\mathcal{N}$-indexed sequence of tuples from $\mathbb{M}$. To show that $\mathcal{N}$ has MP, we need to produce an $\mathcal{N}$-indiscernible sequence $\mathfrak{J}$ which is locally based on $\mathfrak{I}$. By Lemma 3.5.11, it suffices to show that $\mathsf{Ind}_{\mathcal{L}}^{\mathcal{N}}$ is finitely satisfiable in $\mathfrak{I}$.

To this end, let $\Sigma_0 \subseteq \mathsf{Ind}_{\mathcal{L}}^{\mathcal{N}}$ be a finite set, and $(i_1, \ldots, i_n) \subseteq N$. We need to find some $(j_1, \ldots, j_n) \subseteq N$ such that:

$$\mathsf{qftp}_{\mathcal{N}}(i_1, \ldots, i_n) = \mathsf{qftp}_{\mathcal{N}}(j_1, \ldots, j_n) \text{ and } (a_{j_1}, \ldots, a_{j_n}) \vDash \Sigma_0 \restriction_{\{x_{i_1}, \ldots, x_{i_n}\}} .$$

By assumption, there is some substructure $\mathcal{N}_l \subseteq \mathcal{N}$ such that $\{i_1, \ldots, i_n\} \subseteq \mathcal{N}_l$, and $\mathcal{N}_l$-indiscernibles have MP.

Thus, we know that $\mathsf{Ind}_{\mathcal{L}}^{\mathcal{N}_l}$ is finitely satisfiable in the subsequence

$$\mathfrak{I} \restriction \mathcal{N}_l := (a_{i_k} : i_k \in \mathcal{N}_l),$$

which, by our assumption on $\mathcal{N}_l$, contains the part of $\mathfrak{I}$ indexed by $(i_1, \ldots, i_n)$. This, by definition, means that there is some $(j_1, \ldots, j_n) \subseteq N_l$ such that:

$$\mathsf{qftp}_{\mathcal{N}_l}(i_1, \ldots, i_n) = \mathsf{qftp}_{\mathcal{N}_l}(j_1, \ldots, j_n) \text{ and } (a_{j_1}, \ldots, a_{j_n}) \vDash \Sigma_0 \restriction_{\{x_{i_1}, \ldots, x_{i_n}\}}$$

However, since $\mathcal{N}_l$ is a substructure of $\mathcal{N}$, it follows that

$$\mathsf{qftp}_{\mathcal{N}_l}(\bar{i}) = \mathsf{qftp}_{\mathcal{N}}(\bar{i}),$$

for all tuples $\bar{i}$ from $\mathcal{N}_l$. Hence, the witnesses of the finite satisfiability of $\mathsf{Ind}_{\mathcal{L}}^{\mathcal{N}_l}$, witness the finite satisfiability of $\mathsf{Ind}_{\mathcal{L}}^{\mathcal{N}}$, and we are done. □

At this point, all the required tools for the proof of Theorem A have been developed, so let's put everything together:

Theorem 3.5.12. *Let $\mathcal{L}'$ be a first-order language and $\mathcal{N}$ an infinite, locally finite $\mathcal{L}'$-structure. Then, the following are equivalent:*

1. *$\mathcal{N}$ has ERP.*
2. *$\mathcal{N}$ has MP.*

Proof. The implication (2) $\implies$ (1) was already proved in Corollary 3.5.6. For the implication (1) $\implies$ (2), observe that since $\mathsf{Age}(\mathcal{N})$ is a Ramsey class, we can find countable $\mathcal{L}'$-structures $(\mathcal{N}_i : i \in I)$, for some indexing set I, satisfying the conditions of Proposition 3.5.7, i.e. such that:

1. $\mathcal{N} = \bigcup_{i \in I} \mathcal{N}_i$.
2. For every finite $A \subseteq \mathcal{N}$, there is some $i \in I$, such that $A \subseteq \mathcal{N}_i$.
3. For every $i \in I$ we have that $\mathsf{Age}(\mathcal{N}_i)$ is a Ramsey class.

To obtain the $\mathcal{N}_i$, we proceed as follows. Let $(A_i : i \in I)$ be some enumeration (not necessarily countable) of all finitely generated substructures of $\mathcal{N}$. For each finitely generated substructure A_i we construct $\mathcal{N}_i$ as follows. Let $B_0^i := A_i$, and then take $C_0^i \in \mathsf{Age}(\mathcal{N})$ to be an $\mathcal{L}$-structure as in the conclusion of Corollary 3.2.2, with respect to all finitely generated substructures $A_0^1, \dots, A_0^n$ of B_0^i. Let $B_1^i := C_0^i$ and repeat this process inductively, to obtain $(B_j^i : j \in \mathbb{N})$. Let $\mathcal{N}_i := \bigcup_{j\in\mathbb{N}} B_j^i$. By construction, $\mathsf{Age}(\mathcal{N}_i)$ is still a Ramsey class, and it is easy to see that $\mathcal{N}_i$ is countable, since it is the union of an increasing chain of finite structures. Then, by Corollary 3.5.5, for each $i \in I$, since $\mathsf{Age}(\mathcal{N}_i)$ is a countable Ramsey class, $\mathcal{N}_i$ has MP. From Proposition 3.5.7 we can conclude that $\mathcal{N}$ has MP. $\square$

As an immediate corollary of Theorem 3.5.12 and Corollary 3.4.7 we obtain the following:

Corollary 3.5.13. *Let $\mathcal{N}$ be an $\mathcal{L}'$-structure with the embedding Ramsey property. Then $\mathcal{N}$ is $\mathcal{L}'_{\infty,0}$-orderable.*

3.6 Around Local Finiteness

So far in this chapter, all classes considered have been classes of finite $\mathcal{L}'$-structures, or, to the same extent, classes of the form $\mathsf{Age}(\mathcal{N})$, where $\mathcal{N}$ is a *locally finite* $\mathcal{L}'$-structure. This umbrella assumption will be dropped in this section, to deal with the embedding Ramsey property in the wider context of structures that are not necessarily locally finite.

Let $\mathcal{N}$ be an (arbitrary) $\mathcal{L}'$-structure and let $\mathcal{C} = \mathsf{Age}(\mathcal{N})$. Given an $\mathcal{L}'$-structure $M \in \mathcal{C}$ and finite subsets $A \subseteq B$ of M, we overload the notation $\binom{B}{A}$ to mean:

$$\binom{B}{A} := \left\{ \tilde{A} \subseteq B : \mathsf{qftp}(\tilde{A}) = \mathsf{qftp}(A) \right\}.$$

Of course, in the definition above, different permutations of $\tilde{A}$ with the same quantifier-free type as A give rise to two different elements of $\binom{B}{A}$. Since, in the discussion that follows, I will be interested in classes consisting of rigid structures, this subtlety will not be important.

Definition 3.6.1 (f-ERP)**.** Let $\mathcal{C}$ be a class of (not necessarily finitely generated) $\mathcal{L}'$-structures. We say that $\mathcal{C}$ has the *finitary Embedding Ramsey Property* (f-ERP) if for any $M \in \mathcal{C}$ and any finite subsets $A, B \subseteq M$ there are an $N \in \mathcal{C}$ and a finite subset $C \subseteq N$ such that $C \to (B)_2^A$. We say that an $\mathcal{L}'$-structure $\mathcal{M}$ has the *finitary Embedding Ramsey Property* (f-ERP) if $\mathsf{Age}(\mathcal{M})$ the finitary Embedding Ramsey Property.

The analogue of Definition 2.5.10, for f-ERP is the following definition:

Definition 3.6.2. Let $\mathcal{C}$ be a class of $\mathcal{L}'$-structures. We say that $\mathcal{C}$ is a *finitary Ramsey class* if $\mathcal{C}$ has HP, JEP, and f-ERP.

Remark 3.6.3. If $\mathcal{C}$ is a class of finite structures, then $\mathcal{C}$ has f-ERP if, and only if, $\mathcal{C}$ has ERP. In particular, for any locally finite (e.g. relational) $\mathcal{L}'$-structure $\mathcal{N}$ we have that $\mathsf{Age}(\mathcal{N})$ is a Ramsey class if, and only if it is a finitary Ramsey class.

The following proposition is the natural generalisation of Fact 2.5.16. There are various folklore proofs of Fact 2.5.16, e.g. [Bod15, Proposition 2.18]. The proof given below is based on an argument I learned from [Ram20, Lemma 1.35]. Before proving it, I will need to introduce one more definition:

Definition 3.6.4 (Embedding Local Finiteness)**.** An $\mathcal{L}$-structure $\mathcal{M}$ has the *embedding local finiteness property* if for any finitely generated $N \subseteq \mathcal{M}$ and finite $A \subseteq N$ we have that $\binom{N}{A}$ is finite. More generally, an isomorphism-closed class $\mathcal{C}$ of $\mathcal{L}$-structures with HP has the *embedding local finiteness property* if for all $N \in \mathcal{C}$ and all finite $A \subseteq N$ we have that $\binom{N}{A}$ is finite.

Proposition 3.6.5. *Let $\mathcal{M}$ be a structure with the embedding local finiteness property. Then, the following are equivalent:*

1. *$\mathcal{M}$ has f-ERP.*
2. *For all finite subsets $A \subseteq B \subseteq \mathcal{M}$ we have that $\mathcal{M} \to (B)^A_2$.*

Proof. Suppose, first, that $\mathsf{Age}(\mathcal{M})$ has f-ERP, and let $A \subseteq B \subseteq M$, be finite sets. Let $N \in \mathsf{Age}(\mathcal{M})$ be such that $A \subseteq B \subseteq N$. Since $\mathsf{Age}(\mathcal{M})$ has f-ERP, there is some $N' \in \mathsf{Age}(\mathcal{M})$ and a finite subset $C \subseteq N'$ such that $C \to (B)^A_2$. Thus, it is clear, by the definitions, that $\mathcal{M} \to (B)^A_2$.[4]

Suppose that $\mathsf{Age}(\mathcal{M})$ does not have f-ERP. Then, by definition, there is some $N' \in \mathsf{Age}(\mathcal{M})$ and finite subsets $A \subseteq B \subseteq N'$ such that for all $N \in \mathsf{Age}(\mathcal{N})$ and all finite subsets $C \subseteq N$ we have that $C \not\to (B)^A_2$. That is, for all $N \in \mathsf{Age}(\mathcal{M})$ and all finite $C \subseteq N$ there is a colouring:

$$\chi_{N,C} : \binom{C}{A} \to 2$$

such that $\chi_{N,C}$ restricted to any $\tilde{B} \in \binom{C}{B}$ is non-constant.

[4]Observe that the embedding local finiteness property was not used in the proof of this implication.

We wish to show that for all finitely generated $N \subseteq \mathcal{M}$ and all finite $C \subseteq N$ there is a colouring $\chi^{\star}_{N,C} : \binom{C}{A} \to 2$ such that for all $\tilde{B} \in \binom{C}{B}$ we have that $\chi \restriction_{\binom{\tilde{B}}{A}}$ is non-constant. Since $A \subseteq B$ are finite, from this we can conclude that $\mathcal{M} \not\to (B)^A_2$.

Given a finitely generated $N \subseteq M$ define:

$$X_N := \{N' \in \mathsf{Age}(\mathcal{M}) : N \subseteq N'\}.$$

Of course, if $N_1, N_2 \subseteq M$ are finitely generated substructures then:

$$\begin{aligned} X_{N_1} \cap X_{N_2} &= \{N \in \mathsf{Age}(\mathcal{M}) : N_1 \subseteq N\} \cap \{N' \in \mathsf{Age}(\mathcal{M}) : N_2 \subseteq N'\} \\ &= X_{\langle N_1 \cup N_2 \rangle} \\ &\neq \emptyset. \end{aligned}$$

Thus $\{X_N : N \in \mathsf{Age}(\mathcal{M})\}$ generates a filter on $\mathsf{Age}(\mathcal{M})$, which we can extend to an ultrafilter $\mathcal{U}$ on $\mathsf{Age}(\mathcal{M})$. Now, given a finitely generated substructure $N \subseteq \mathcal{M}$ and a finite subset $C \subseteq N$ define a colouring:

$$\begin{aligned} \chi^{\star}_{N,C} : \binom{C}{A} &\to \{1, 2\} \\ \tilde{A} &\mapsto \begin{cases} 1 & \text{if } \{N' \in \mathsf{Age}(\mathcal{M}) : N \subseteq N',\, \chi_{N',C}(\tilde{A}) = 1\} \in \mathcal{U} \\ 2 & \text{otherwise.} \end{cases} \end{aligned}$$

Of course, since $\mathcal{U}$ is an ultrafilter this is well-defined, since the two colours partition X_N into two disjoint sets and thus only one of them can be in $\mathcal{U}$.

To finish the proof, take some finitely generated $N \subseteq \mathcal{M}$ and some finite $C \subseteq N$ by (2) we have that $\chi^{\star}_{N,C}$ is constant, say with value 1, on some $\tilde{B} \in \binom{C}{B}$. Then, by embedding local finiteness, and our assumption we have that:

$$\bigcap_{\tilde{A} \in \binom{N}{A}} \{N' \in \mathsf{Age}(\mathcal{M}) : N \subseteq N',\, \chi_{N',C}(\tilde{A}) = 1\} \in \mathcal{U}$$

and thus:

$$X_N \cap \bigcap_{\tilde{A} \in \binom{N}{A}} \{N' \in \mathsf{Age}(\mathcal{M}) : N \subseteq N',\, \chi_{N',C}(\tilde{A}) = 1\} \in \mathcal{U}.$$

Since $\mathcal{U}$ is an ultrafilter, it follows that:

$$X_N \cap \bigcap_{\tilde{A} \in \binom{N}{A}} \{N' \in \mathsf{Age}(\mathcal{M}) : N \subseteq N',\, \chi_{N',C}(\tilde{A}) = 1\} \neq \emptyset.$$

Any $N' \in \mathsf{Age}(\mathcal{N})$ in the intersection above is a finitely generated substructure of $\mathcal{M}$ containing $A \subseteq \tilde{B} \subseteq C$ such that $\chi_{N',C} \restriction_{\binom{\tilde{B}}{A}}$ is constant, which is a contradiction. □

The rest of this section, is an attempt to convenience the reader that f-ERP is the "correct" generalisation of ERP, to classes of not necessarily finite structures. The point is that the main results about ERP for classes of finite structures generalise, almost immediately, to arbitrary structures, if one replaces ERP with f-ERP. In particular, in this context, we have analogues of both the Kechris-Pestov-Todorčević correspondence and of Theorem 3.5.12 but we have now dropped all local finiteness/countability assumptions from $\mathcal{M}$.

Theorem 3.6.6. *Let $\mathcal{M}$ be an homogeneous $\mathcal{L}$-structure. Then, the following are equivalent:*

1. $\mathsf{Aut}(\mathcal{M})$ *is extremely amenable.*
2. $\mathcal{M}$ *has the finitary embedding Ramsey property.*

This theorem essentially follows from [KPT05], in the case where $\mathcal{M}$ is countable, and in the general case, it essentially appears[5] in [KP22]. It could also be proved by combining the results of [Bar13] with the proof of the main theorem of [KPT05]. Nonetheless, it is possible to give an alternative (completely rigorous proof) which showcases the power of the techniques introduced in Section 3.3.

The theorem will follow from the next remark:

Remark 3.6.7. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Recall that by $\mathcal{M}_{\mathsf{iso}}$ we denote its $\mathcal{L}_{\infty,0}$-isolator, introduced in Definition 3.3.6.

1. Clearly $\mathsf{Aut}(\mathcal{M}) = \mathsf{Aut}(\mathcal{M}_{\mathsf{iso}})$. In particular, $\mathsf{Aut}(\mathcal{M})$ is extremely amenable if, and only if, $\mathsf{Aut}(\mathcal{M}_{\mathsf{iso}})$ is extremely amenable.
2. $\mathcal{M}$ has the f-ERP if, and only if $\mathcal{M}_{\mathsf{iso}}$ has ERP. This follows easily from Lemma 3.3.7.

[5] As was pointed out in private correspondence by Krupiński where a definition *a posteriori* equivalent to f-ERP appears, see [KP22, Section 1.5]

Proof of Theorem 3.6.6. Given the two remarks above, the theorem follows from Fact 2.5.17, applied to $\mathcal{M}_{\mathsf{iso}}$, which is always relational, and is homogeneous, if, and only if, $\mathcal{M}$ is homogeneous. □

It is also possible to use the notion of f-ERP that was just introduced to remove *all* assumptions from Scow's theorem. This is the following theorem, which is, in a sense, the most general version of correspondence between generalised indiscernibles and Ramsey classes:

Theorem 3.6.8. *Let $\mathcal{L}'$ be a first-order language and $\mathcal{N}$ an $\mathcal{L}'$-structure. Then, the following are equivalent:*

1. *$\mathcal{N}$ has the finitary embedding Ramsey property.*
2. *$\mathcal{N}$ has the modelling property.*

Proof. Immediate. Simply apply Lemma 3.4.2, Lemma 3.3.7, and Theorem 3.5.12, to $\mathcal{N}_{\mathsf{iso}}$. □

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– End of Part 1 –

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Part 2: All these Approximate Ramsey Properties

3.7 Introduction to Part 2

The main novelty of [MP23a] is the definition of a *local Ramsey class*. This will be given formally in Section 3.10, but in order to summarise the main results of this part, I shall now describe it, somewhat informally. Fix $k \in \mathbb{N}$. We say that a class of finite structures, $\mathcal{C}$, is a *k-Ramsey class* (resp. embedding k-Ramsey class) or, to the same extent, that $\mathcal{C}$ is *Ramsey up to k* (resp. embedding Ramsey up to k) if the following assertion holds for all $A \in \mathcal{C}$ with $|A| \leq k$:

($\diamond_A$) For all $B \in \mathcal{C}$ with $A \subseteq B$, there is a $C \in \mathcal{C}$ such that for all n-colourings χ of the substructures of C isomorphic to A (resp. embeddings of A in C), there is a copy of B in C (resp. embedding of B in C) on which χ is constant.

Given this "definition", the following statements should be obvious:

(1) If $\mathcal{C}$ is a Ramsey class, then it is a k-Ramsey class, for all $k \in \mathbb{N}_{\geq 1}$.

(2) For all $k \in \mathbb{N}_{\geq 1}$, if $\mathcal{C}$ is Ramsey up to $k+1$, then it is Ramsey up to k.

So "Ramsey up to k" is a *local* version of the Ramsey property. My first goal in this part of the chapter is to prove that the implications in (2), above, are strict. More precisely, I will show the following:

Theorem D (Corollary 3.10.17)**.** *There is a class of finite ordered structures that is Ramsey up to k but not up to $k+1$, for all $k \in \mathbb{N}_{\geq 2}$.*

I have already pointed out in Section 2.5 that different authors use the term *Ramsey class* to refer to two non-equivalent notions: *structural* Ramsey classes and *embedding* Ramsey classes — the difference being whether the objects that are coloured are "isomorphic substructures" or "embeddings". However, these two notions coincide for classes of ordered structures (or more generally, for classes of rigid structures), so in the theorem above we capture both.

In the previous part of this chapter, I stated and proved a generalisation of a well-known fact that if $\mathcal{M}$ is a (countable) homogeneous structure whose age is a Ramsey class, then there exists a linear order on the domain of $\mathcal{M}$ which is preserved by all automorphisms of $\mathcal{M}$ (this was Theorem B, in Part 1). The second main result of [MP23a] is a generalisation this theorem, by showing that the conclusion follows only from assuming that $\mathsf{Age}(\mathcal{M})$ is embedding Ramsey up to 2. More precisely:

Theorem E (Theorem 3.11.1)**.** *Let $\mathcal{C}$ be a class of finite structures. If $\mathcal{C}$ is embedding Ramsey up to* 2 *(see Definition 3.8.4) then there is a possibly infinite Boolean combination of atomic formulas $\Phi(x, y)$ which defines a linear order on all structures in $\mathcal{C}$.*

As observed in the first part of this chapter, $\mathcal{L}_{\infty,0}$-orderability (the conclusion of Theorem E) is a naturally arising condition in various model-theoretic contexts (cf. Section 3.4). In [MP23a], we also observe that classes of finite structures which are $\mathcal{L}_{\infty,0}$-orderable must necessarily consist of rigid structures, but not all classes of finite rigid structures are $\mathcal{L}_{\infty,0}$-orderable (see Example 3.11.3).

Structure of Part 2

After some more local preliminaries in Section 3.8, the remainder of this chapter is organised as follows: In Section 3.9, Kay-*graphs* are introduced. These generalise *two*-graphs to higher arities and as shown in Section 3.10 serve as the constructions used to prove Theorem D. Finally, Section 3.11 has a proof of Theorem E.

Local notational conventions

Given finite structures A and B, I will momentarily deviate from the notation introduced in Section 2.5, and write $\binom{B}{A}$ the set of all substructures of B that are *isomorphic* to A and by $\mathsf{Emb}(A, B)$ the set of all *embeddings* of A into B. Hopefully this change in notation is not too disruptive.

Let $k \in \mathbb{N}_{\geq 2}$. Recall from Section 2.7 that a k-hypergraph is a structure $H = (V; R)$, where R is a k-ary relation symbol such that for all $x_1, \ldots, x_k \in V$ we have that:

- if $(x_1, \ldots, x_k) \in R$ then $|\{x_1, \ldots, x_k\}| = k$; and
- if $(x_1, \ldots, x_k) \in R$ then, for all $\sigma \in \mathsf{Sym}(k)$ we also have
$$\left(x_{\sigma(1)}, \ldots, x_{\sigma(k)}\right) \in R.$$

More concisely, if $H = (V, R)$ is a k-hypergraph, then $R \subseteq \binom{V}{k}$, where $\binom{V}{k}$

denotes the collection of all k-element subsets of V (this may also be denoted by $[V]^k$). I may write $\{v_1, \ldots, v_k\} \in R$ to mean $(v_1, \ldots, v_k) \in R$, and vice versa. In this part of the chapter only relational languages will be considered. For such a language $\mathcal{L}$, if H is an $\mathcal{L}$-structure and $R \in \mathcal{L}$ I will sometimes write $R(H)$ (or even just R if H is understood) for the set of realisations of R in H.

3.8 Local Preliminaries for Part 2

Let's start by formally defining some terminology mentioned in the previous section, and recalling some terminology from Section 2.5.

Definition 3.8.1. Let $\mathcal{C}$ be a class of finite structures, and $k \in \mathbb{N}_{\geq 1}$. We say that $A \in \mathcal{C}$ has *structural* (resp. *embedding*) *Ramsey degree at most* k if for every $B \in \mathcal{C}$ and every $n \in \mathbb{N}$ there is some $C \in \mathcal{C}$ such that for every colouring $\chi : \binom{C}{A} \to [n]$ (resp. $\chi : \mathsf{Emb}(A, C) \to [n]$) there is $B' \in \binom{C}{B}$ (resp. $B' \in \mathsf{Emb}(B, C)$) with $|\mathsf{Im}\left(\chi_{\restriction \binom{B'}{A}}\right)| \leq k$.

Recall the following well-known fact (a proof of which can be found in [Zuc15, Corollary 4.5]) connecting structural and embedding Ramsey degrees:

Fact 3.8.2. *Let $\mathcal{C}$ be a class of finite structures, and $A \in \mathcal{C}$. Then, A has* structural *Ramsey degree k if, and only if, A has* embedding *Ramsey degree $k|\mathsf{Aut}(A)|$.*

We call $A \in \mathcal{C}$ a *Ramsey object* (for $\mathcal{C}$) if it has embedding Ramsey degree at most 1. Thus, A is a Ramsey object for $\mathcal{C}$ if, and only if it is rigid and has structural Ramsey degree 1.

Remark 3.8.3. Connecting this new terminology with the terminology introduced in Chapter 2, a class $\mathcal{C}$ of finite structures has the Embedding Ramsey Property if every structure in $\mathcal{C}$ is a Ramsey object.

Now for the main definition:

Definition 3.8.4. For $k \in \mathbb{N}_{\geq 1}$ we say that a class $\mathcal{C}$ of finite structures has the *k-Embedding Ramsey Property* (*k-ERP*) if every structure of size k in $\mathcal{C}$ is a Ramsey object.

The following useful little corollary is a essentially a restatement of Corollary 3.2.2, for k-ERP:

Corollary 3.8.5. *Let $\mathcal{C}$ be a class of finite structures and assume that $\mathcal{C}$ has k-ERP. Then for all $i \leq n$, and $A_1, \ldots A_n \in \mathcal{C}$ with $|A_i| = k$ and all $B \in \mathcal{C}$ such that $A_i \subseteq B$ for each $i \leq n$, there is some $C \in \mathcal{C}$ such that for all sets of* *colourings* $\{\chi_i : \binom{C}{A_i} \to [2] : i \leq n\}$*, there is some* $\tilde{B} \in \binom{C}{B}$ *such that* $\chi_{i \restriction \binom{\tilde{B}}{A_i}}$ *is constant, for all* $i \leq n$*.*

3.9 *Kay*-graphs

Recall that given an ordered graph $G = (V; R, \leq)$, the *ordered two-graph* of G is defined to be the ordered 3-hypergraph $K(G) = \left(V; R^{(3)}, \leq\right)$, on the same ordered vertex set $(V; \leq)$, with hyperedge relation $R^{(3)}$ defined as follows:

$$R^{(3)}(x_1, x_2, x_3) \iff \left|\left\{\{i,j\} \in \binom{[3]}{2} : \{x_i, x_j\} \in R\right\}\right| \equiv 1 \pmod 2,$$

Two-graphs have been investigated deeply in the last 50 years. See, for instance, [Sei91], for an excellent survey. Here, a higher-arity generalisation of two-graphs will be given. This appears implicitly in [Tho96]. The generalisation to higher arities is probably what the reader should expect it to be:

Definition 3.9.1. Given a k-hypergraph $H = (V; R)$ we define the Kay-*graph* of H to be the $(k+1)$-hypergraph $K(H) = \left(V; R^{(k+1)}\right)$ on the same vertex set V as H with hyperedge relation $R^{(k+1)}$ defined as follows:

$$R^{(k+1)}(x_1, \ldots, x_{k+1}) \iff \left|\left\{\{i_1, \ldots, i_k\} \in \binom{[k+1]}{k} : \{x_{i_1}, \ldots, x_{i_k}\} \in R\right\}\right| \equiv k+1 \pmod 2.$$

We similarly define the *ordered* Kay-*graph* of an ordered hypergraph. If $\mathcal{H}$ is a class of (ordered) k-hypergraphs we will write $K(\mathcal{H})$ for the class of all (ordered) *Kay*-graphs of hypergraphs from $\mathcal{H}$, that is $K(\mathcal{H}) := \{K(H) : H \in \mathcal{H}\}$.

In the remainder of this section, only unordered *Kay*-graphs will be discussed, but all the arguments easily go through if one considers ordered *Kay*-graphs, instead.

Let $P^{(k)}$ be the class of all finite $(k+1)$-hypergraphs $(V; R)$ satisfying the following condition:

(†) For all $V_0 \in \binom{V}{k+2}$ the induced $(k+1)$-subhypergraph $(V_0, R_{\restriction V_0})$ has $|R_{\restriction V_0}| \equiv k \pmod 2$.

Recall from Section 2.7 that $\mathcal{C}_{\mathsf{H}_k}$ is the class of all finite k-hypergraphs. In this notation:

Proposition 3.9.2. *Let $k \in \mathbb{N}_{\geq 2}$. Then $P^{(k)} = K(\mathcal{C}_{\mathsf{H}_k})$.*

Before proving this proposition, I will prove a technical lemma, which will be useful later on. First, to fix some notation:

Given $H \in (V;S) \in P^{(k)}$ and any fixed vertex $\star \in V$ we define the *induced k-hypergraph* as the hypergraph on vertex set V with k-ary hyperedge relation $S_\star \subseteq \binom{V}{k}$, given by:

$$S_\star(x_1,\dots,x_k) \iff \begin{cases} S(x_1,\dots,x_k,\star), & \text{if } \star \notin \{x_1,\dots,x_k\} \\ \text{always}, & \text{if } \star \in \{x_1,\dots,x_k\}, \end{cases}$$

for all $\{x_1,\dots,x_k\} \in \binom{V}{k}$.

Lemma 3.9.3. *Let $(V;S) \in P^{(k)}$, and fix a vertex $\star \in V$. Then,*

$$K((V;S_\star)) = (V;S),$$

where $K(V,S_\star)$ is the induced Kay*-graph $\left(V;S_\star^{(k+1)}\right)$ of $(V;R_\star)$.*

Proof. Let $(V;S) \in P^{(k)}$ and fix $\star \in V$. To simplify notation, let $R = S_\star$. In this notation we wish to show that the induced *Kay*-graph $K((V;R)) = \left(V;R^{(k+1)}\right)$ is precisely $(V;S)$.

For bookkeeping purposes, observe that if $\{x_1,\dots,x_{k+1}\} \in \binom{V\setminus\{\star\}}{k+1}$ then we have:

$$\begin{aligned} &R^{(k+1)}(x_1,\dots,x_{k+1}) \\ \iff &\left|\left\{\{i_1,\dots,i_k\} \in \binom{[k+1]}{k} : \{x_{i_1},\dots,x_{i_k}\} \in R\right\}\right| \equiv k+1 \pmod 2 \\ \iff &\left|\left\{\{i_1,\dots,i_k\} \in \binom{[k+1]}{k} : \{x_{i_1},\dots,x_{i_k},\star\} \in S\right\}\right| \equiv k+1 \pmod 2 \\ \iff &\left|S_{\restriction\{x_1,\dots,x_{k+1},\star\}}) \setminus \{x_1,\dots,x_{k+1}\}\right| \equiv k+1 \pmod 2, \end{aligned} \tag{3.1}$$

essentially, by unfolding the definition of $R = S_\star$ and $R^{(k+1)}$.

With this out of the way, recall that our goal is to prove that

$$\left(V;R^{(k+1)}\right) = (V;S),$$

where $R = S_\star$. Explicitly, we need to show that for all $\{x_1,\dots,x_k\} \in \binom{V}{k+1}$ we have that:

$$(x_1,\ldots,x_{k+1}) \in R^{(k+1)} \text{ if, and only if,} (x_1,\ldots,x_{k+1}) \in S.$$

To this end, fix some arbitrary $\{x_1,\ldots,x_k\} \in \binom{V}{k}$. We consider two cases.

Case 1. $\{x_1,\ldots,x_{k+1}\} \in \binom{V\setminus\{\star\}}{k+1}$.

First, assume that $(x_1,\ldots,x_{k+1}) \in R^{(k+1)}$. Since $(V;S) \in P^{(k)}$, by (†), the induced subhypergraph $\left(\{x_1,\ldots,x_{k+1},\star\}; S_{\restriction\{x_1,\ldots,x_{k+1},\star\}}\right)$ of $(V;S)$ must have:

$$|S_{\restriction\{x_1,\ldots,x_{k+1},\star\}}| \equiv k \pmod 2$$

Since $(x_1,\ldots,x_{k+1}) \in R^{(k+1)}$, we know, by Equation (3.1) that:

$$|S_{\restriction\{x_1,\ldots,x_{k+1},\star\}}) \setminus \{x_1,\ldots,x_{k+1}\}| \equiv k+1 \pmod 2,$$

so we must have that $\{x_1,\ldots,x_{k+1}\} \in S$, as claimed.

Conversely, suppose that $(x_1,\ldots,x_{k+1}) \in S$. Then, by (†) the number of hyperedges in the induced subhyphergraph $\left(\{x_1,\ldots,x_{k+1},\star\}; S_{\restriction\{x_1,\ldots,x_{k+1},\star\}}\right)$ of $(V;S)$ must be congruent to k, modulo 2. In particular, if

$$\left|\left\{\{i_1,\ldots,i_k\} \in \binom{[k+1]}{k} : \{x_{i_1},\ldots,x_{i_k},\star\} \in S\right\}\right| \equiv k \pmod 2,$$

then we cannot have that $(x_1,\ldots,x_{k+1}) \in S$, contrary to our original assumption. Hence:

$$\left|\left\{\{i_1,\ldots,i_k\} \in \binom{[k+1]}{k} : \{x_{i_1},\ldots,x_{i_k},\star\} \in S\right\}\right| \equiv k+1 \pmod 2,$$

and thus, by Equation (3.1) we have that $(x_1,\ldots,x_{k+1}) \in R^{(k+1)}$, as required.

Case 2. $\{x_1,\ldots,x_{k+1}\} \in \binom{V}{k+1}$, and, without loss of generality, $x_{k+1} = \star$.

Suppose first that $(x_1,\ldots,x_k,\star) \in R^{(k+1)}$. By definition of R we have that

$(x_{i_1}, \ldots, x_{i_{k-1}}, \star) \in R$, for all $\{i_1, \ldots, i_{k-1}\} \in \binom{[k]}{k-1}$, and thus

$$\begin{aligned}\left|\left\{\{i_1, \ldots, i_k\} \in \binom{[k+1]}{k} : \{x_{i_1}, \ldots, x_{i_k}\} \in R\right\}\right| \\ &= k + \left|\left\{\{i_1, \ldots, i_k\} \in \binom{[k]}{k} : \{x_{i_1}, \ldots, x_{i_k}\} \in R\right\}\right| \\ &= \begin{cases} k+1 & \text{if } \{x_1, \ldots, x_k\} \in R \\ k & \text{otherwise.} \end{cases}\end{aligned}$$

But, since $(x_1, \ldots, x_k, \star) \in R^{(k+1)}$, the cardinality of this set is congruent to $k+1$ modulo 2, so we must have that $(x_1, \ldots, x_k) \in R$, and thus $(x_1, \ldots, x_k, \star) \in S$, as required.

Conversely, suppose that $(x_1, \ldots, x_k, \star) \in S$. If $(x_1, \ldots, x_k, \star) \notin R^{(k+1)}$, then, arguing as before we have that $(x_1, \ldots, x_k) \notin R$ and thus $(x_1, \ldots, x_k, \star) \notin S$, contradicting our assumption. Hence we must have that

$$(x_1, \ldots, x_k, \star) \in R^{(k+1)},$$

concluding the proof of the lemma. □

Proof. We first observe that the inclusion $P^{(k)} \subseteq K(\mathcal{C}_{\mathsf{H}_k})$ follows immediately from the previous lemma. Indeed let $(V; S)$ be a $(k+1)$-hypergraph and assume that $(V; S) \in P^{(k)}$. We need to show that there is a k-hypergraph $H = (V; R)$ such that $(V; S) = K(H)$. But, once we fix a vertex $\star \in V$, by the lemma above, we have that $(V; S) = K((V; R_\star))$.

We now move on to the inclusion $K(\mathcal{C}_{\mathsf{H}_k}) \subseteq P^{(k)}$. We will prove by induction on the cardinality of the hyperedge set of a finite k-hypergraph $H = (V; R) \in \mathcal{C}_{\mathsf{H}_k}$ that $K(H) \in P^{(k)}$.

Let $H = (V; R) \in \mathcal{C}_{\mathsf{H}_k}$. First, note that if $R = \emptyset$ then we can easily see that $K(H) \in P^{(k)}$. Indeed, if $k \equiv 0 \pmod 2$ then $R^{(k+1)} = \emptyset$ and hence $\left(V; R^{(k+1)}\right) \in P^{(k)}$, because (†) holds trivially for all $(k+2)$-element subsets of V. Similarly, if $k \equiv 1 \pmod 2$, then $R^{(k+1)} = \binom{V}{k+1}$, and hence $\left(V; R^{(k+1)}\right) \in P^{(k)}$, because for every $(k+2)$-element subset of V, the number of $(k+1)$-hyperedges is $\binom{k+2}{k+1} \equiv k+2 \equiv k \pmod 2$, and thus (†) holds.

Now, for the inductive step, assume that:

(IH) For any hyperedge $\{x_1, \ldots, x_k\} \in R$ we have that:

$$K((V; R')) \in P^{(k)},$$

where $R' = R \setminus \{(x_1, \ldots, x_k)\}$.

We need to show that $K((V; R)) \in P^{(k)}$.

Indeed, let $\{y_1, \ldots, y_{k+2}\} \subseteq V$. We need to show that in the induced subhypergraph:

$$\left(\{y_1, \ldots, y_{k+2}\}; R^{(k+1)}_{\restriction\{y_1,\ldots,y_{k+2}\}}\right)$$

we have that

$$\left|R^{(k+1)}_{\restriction\{y_1,\ldots,y_{k+2}\}}\right| \equiv k \pmod 2.$$

By (IH) we know that $K((V; R')) \in P^{(k)}$, so this is the case for

$$\left(\{y_1, \ldots, y_{k+2}\}; R'^{(k+1)}_{\restriction\{y_1,\ldots,y_{k+2}\}}\right),$$

where, as in (IH), $R' = R \setminus \{(x_1, \ldots, x_k)\}$. Of course, if $\{x_1, \ldots, x_k\} \not\subseteq \{y_1, \ldots, y_{k+2}\}$ then

$$\left(\{y_1, \ldots, y_{k+2}\}; R^{(k+1)}_{\restriction\{y_1,\ldots,y_{k+2}\}}\right) = \left(\{y_1, \ldots, y_{k+2}\}; R'^{(k+1)}_{\restriction\{y_1,\ldots,y_{k+2}\}}\right),$$

and we are done.

Thus, we may assume that $\{x_1, \ldots, x_k\} \subseteq \{y_1, \ldots, y_{k+2}\}$. To simplify notation, assume that $\{y_1, \ldots, y_{k+2}\} = \{x_1, \ldots, x_k, y_1, y_2\}$.

We consider the $(k+1)$-hyperedges in:

$$\left(\{x_1, \ldots, x_k, y_1, y_2\}; R'^{(k+1)}_{\restriction\{x_1,\ldots,x_k,y_1,y_2\}}\right).$$

These are associated with $(k+1)$-element subsets of $\{x_1, \ldots, x_k, y_1, y_2\}$, in the following way:

- Let n_1 be the number of k-hyperedges of R' on the set $\{x_1, \ldots, x_k, y_1\}$, without the hyperedge $\{x_1, \ldots, x_k\}$.
- Let n_2 be the number of k-hyperedges of R' on the set $\{x_1, \ldots, x_k, y_2\}$, without the hyperedge $\{x_1, \ldots, x_k\}$.
- Let n_3^I be the number of k-hyperedges of R' on the subsets $\{x_j : j \in$

$I\} \cup \{y_1, y_2\}$, for $I \in \binom{[k]}{k-2}$.

Then, since, by definition, to check if a $(k+1)$-subset forms a hyperedge in the *Kay*-graph, we simply need to check the parity of the number of its hyperedges mod 2, we have that the total number of $(k+1)$-hyperedges in:

$$\left(\{x_1, \dots, x_k, y_1, y_2\}; R'^{(k+1)}_{\restriction\{x_1,\dots,x_k,y_1,y_2\}}\right)$$

is precisely:

$$n_1 \pmod 2 + n_2 \pmod 2 + \sum_{I \in \binom{[k]}{k-2}} \left(n_3^I \pmod 2\right),$$

when k is even and:

$$(n_1+1) \pmod 2 + (n_2+1) \pmod 2 + \sum_{I \in \binom{[k]}{k-2}} \left((n_3^I+1) \pmod 2\right).$$

when k is odd.

In particular, by our inductive hypothesis, we have that:

$$n_1 + n_2 + \sum_{I \in \binom{[k]}{k-2}} n_3^I \equiv k \pmod 2,$$

when k is even and:

$$(n_1+1) + (n_2+1) + \sum_{I \in \binom{[k]}{k-2}} (n_3^I+1) \equiv k \pmod 2,$$

when k is odd.

Now, recall that $R = R' \cup \{x_1, \dots, x_k\}$, and after adding the hyperedge $\{x_1, \dots, x_k\}$ to R' the number of hyperedges in the induced graph on $\{x_1, \dots, x_k, y_1, y_2\}$ will be:

$$(n_1+1) + (n_2+1) + \sum_{I \in \binom{[k]}{k-2}} n_3^I \equiv k \pmod 2,$$

when k is even, and similarly:

$$((n_1+1)+1) + ((n_2+1)+1) + \sum_{I \in \binom{[k]}{k-2}} (n_3^I+1) \equiv k \pmod 2,$$

when k is odd, and thus the result follows. □

Proposition 3.9.4. *For all $k \in \mathbb{N}_{\geq 2}$ the class $K(\mathcal{C}_{\mathsf{H}_k})$ is a Fraïssé class.*

Proof. We start with the following easy claim:

Claim 1. Let $H_i = (V_i, R_i)$, for $i \in [2]$ be k-hypergraphs such that H_1 is an induced k-subhypergraph of H_2 then $K(H_1)$ is an induced $(k+1)$-subhypergraph of $K(H_2)$.

Proof of Claim 1. We need to show that for all $x_1, \ldots, x_{k+1} \in V_1$ we have that $K(H_1) \models R^{(k+1)}(x_1, \ldots, x_{k+1})$ if, and only if, $K(H_2) \models R^{(k+1)}(x_1, \ldots, x_{k+1})$. Since H_1 is an induced subhypergraph of H_2, by definition we have that:

$$H_1 \models R(x_{i_1}, \ldots, x_{i_k}) \text{ if, and only if } H_2 \models R(x_{i_1}, \ldots, x_{i_k}).$$

Hence:

$$\left\{ \{i_1, \ldots, i_k\} \in \binom{[k+1]}{k} : \{x_{i_1}, \ldots, x_{i_k}\} \in R(H_1) \right\} = \left\{ \{i_1, \ldots, i_k\} \in \binom{[k+1]}{k} : \{x_{i_1}, \ldots, x_{i_k}\} \in R(H_2) \right\},$$

and the result follows. ◀

By Proposition 3.9.2, it suffices to show that for all $k \geq 2$, the class $P^{(k)}$ is a Fraïssé class. We only show that $P^{(k)}$ has AP, since the argument for JEP is similar. Let $K_i = (V_i, S_i) \in P^{(k)}$ be finite *Kay*-graphs and suppose that we have embeddings $f_1 : K_0 \hookrightarrow K_1$ and $f_2 : K_0 \hookrightarrow K_2$. Fix a point $\star \in V_0$ (if $V_0 = \emptyset$ we can fix two points $\star_i \in V_i$, and run the argument given below, amalgamating over a single point) and let $H_i = (V_i; R_i)$, where $R_i = (S_i)_\star$, is the induced k-hypergraph, as defined in page 72.

Now, by Lemma 3.9.3, we have that $K(H_i) = K_i$, for $i = 1, 2$. Now, since the class of all k-hypergraphs has the free amalgamation property, let $H \in \mathcal{C}_{\mathsf{H}_k}$ be the free amalgam of H_1 and H_2 over H_0. By the the first claim, we have that $K_i = K(H_i)$ embed into $K(H)$, and these embeddings preserve the inclusions $K_0 \subseteq K_i$, for $i \in [2]$. Since $K(H)$ is a *Kay*-graph, we have that $K(H) \in P^{(k)}$, and the result follows. □

3.10 Local Ramsey properties for *Kay*-graphs

To prove Theorem D, I will first need to recall some machinery which *appears*[6] in [Zuc15]. To keep notation consistent, throughout this section, let $\mathcal{L} \subseteq \mathcal{L}^\star$ be relational languages and $\mathcal{C}$, $\mathcal{C}^\star$ classes of finite structures in $\mathcal{L}$ and $\mathcal{L}^\star$, respectively, such that $\mathcal{C}^\star$ is an expansion of $\mathcal{C}$ (i.e. for each $A^\star \in \mathcal{C}^\star$ there is some $A \in \mathcal{C}$ such that $A = A^\star_{\restriction \mathcal{L}}$). The main object of interest will be pairs of classes of structures of the form $(\mathcal{C}^\star, \mathcal{C})$.

Definition 3.10.1. We say that $\mathcal{C}^\star$ is a *reasonable expansion* of $\mathcal{C}$ if for any $A, B \in \mathcal{C}$, embedding $f : A \hookrightarrow B$, and expansion $A^\star \in \mathcal{C}^\star$ of A, then there is an expansion $B^\star \in \mathcal{C}^\star$ of B such that $f : A^\star \hookrightarrow B^\star$ is an embedding.

Fact 3.10.2 ([KPT05, Proposition 5.2])**.** *If $\mathcal{C}$ and $\mathcal{C}^\star$ are Fraïssé classes then $\mathcal{C}^\star$ is a reasonable expansion of $\mathcal{C}$ if, and only if* $\mathsf{Flim}(\mathcal{C}^\star)_{\restriction \mathcal{L}} = \mathsf{Flim}(\mathcal{C})$.

Definition 3.10.3. We say that $\mathcal{C}^\star$ is a *precompact expansion* of $\mathcal{C}$ if for all $A \in \mathcal{C}$ the set $\{A^\star \in \mathcal{C}^\star : A = A^\star_{\restriction \mathcal{L}}\}$ is finite.

Definition 3.10.4. We say that $\mathcal{C}^\star$ has the *expansion property* for $\mathcal{C}$ if for any $A^\star \in \mathcal{C}^\star$ there is some $B \in \mathcal{C}$ such that for any expansion $B^\star \in \mathcal{C}^\star$ of B, there is an embedding $f : A^\star \hookrightarrow B^\star$.

One more definition:

Definition 3.10.5. We call $(\mathcal{C}^\star, \mathcal{C})$ an *excellent pair* if $\mathcal{C}^\star$ is a reasonable precompact expansion of $\mathcal{C}$ with the expansion property (for $\mathcal{C}$) and $\mathcal{C}^\star$ has ERP.

The main result on excellent pairs that will be needed is the following:

Fact 3.10.6 ([Zuc15, Proposition 5.8])**.** *Let $(\mathcal{C}^\star, \mathcal{C})$ be an excellent pair. Then, every $A \in \mathcal{C}$ has finite Ramsey degree. Moreover, the Ramsey degree of A is equal to the number of expansions of A in $\mathcal{C}^\star$.*

As in Section 2.7, let $\mathcal{C}_{\mathsf{OH}_k}$ denote the class of all finite ordered k-hypergraphs. As discussed in Chapter 2, it is a well-known fact (due to [NR77], but also proved independently in [AH78]) that $\mathcal{C}_{\mathsf{OH}_k}$ has ERP. Given an ordered k-hypergraph $H = (V, R, \leq)$, I will, in the remainder of this section, write $\mathsf{E}(H) = \left(V; R, R^{(k+1)}, \leq\right)$, for the expansion of H by the *Kay*-graph it induces, and $\mathsf{E}(\mathcal{C}_{\mathsf{OH}_k})$ for the class $\{\mathsf{E}(H) : H \in \mathcal{C}_{\mathsf{OH}_k}\}$.

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[6] I was informed by S. Todorčević, that *really* the development of this machinery should be attributed to Nešetřil and Rödl, or Kechris, Pestov and Todorčević, but the exact statements (and terminology) I will recall in this section appear (with proofs) in Zucker's paper, which serves as an excellent entry-point to the theory, and has the appropriate attributions.

Let me start with the following straightforward remark:

Remark 3.10.7. Since, for all $H \in \mathcal{C}_{\mathsf{OH}_k}$ we have that $\mathsf{E}(H)$ is quantifier-freely definable from H, the fact that $\mathcal{C}_{\mathsf{OH}_k}$ has ERP implies that $\mathsf{E}(\mathcal{C}_{\mathsf{OH}_k})$ also has ERP.

In the sequel, $\mathcal{L} = \{S, \leq\}$ will be the language of $K(\mathcal{C}_{\mathsf{OH}_k})$, and $\mathcal{L}^\star = \{R, S, \leq\}$ the language of $\mathsf{E}(\mathcal{C}_{\mathsf{OH}_k})$.

The goal of the next few propositions is to show that $(\mathsf{E}(\mathcal{C}_{\mathsf{OH}_k}), K(\mathcal{C}_{\mathsf{OH}_k}))$ is an excellent pair, for $k \in \mathbb{N}_{\geq 2}$. I'll prove each of the individual requirements separately, and then collect them all in Corollary 3.10.16.

Let's start with the easiest one:

Proposition 3.10.8. *The expansion* $(\mathsf{E}(\mathcal{C}_{\mathsf{OH}_k}), K(\mathcal{C}_{\mathsf{OH}_k}))$ *is precompact.*

Proof. To see that the expansion is precompact observe that if $A \in K(\mathcal{C}_{\mathsf{OH}_k})$ the number of k-hyperedge relations that we could expand A by (not necessarily so that the resulting expansion is in $\mathsf{E}(\mathcal{C}_{\mathsf{OH}_k})$ is precisely $2^{|\binom{V(A)}{k}|}$, and hence $\{A^\star \in \mathsf{E}(\mathcal{C}_{\mathsf{OH}_k}) : A = A^\star_{\restriction \mathcal{L}}\}$ is finite. □

More generally, in the notation of Definition 3.10.5, it is easy to see, essentially from the proof above, that any expansion $\mathcal{C}^\star$ of $\mathcal{C}$ is precompact if $\mathcal{L}^\star \setminus \mathcal{L}$ is finite.

Before moving to the expansion property, I will need to introduce some terminology:

Definition 3.10.9. Let $A = (V; R, S, \leq) \in \mathsf{E}(\mathcal{C}_{\mathsf{OH}_k})$. We define the *complement* of A, denoted $\overline{A}$, to be

$$\overline{A} := (V; R', S', \leq) \in \mathsf{E}(\mathcal{C}_{\mathsf{OH}_k}),$$

where $R' := \binom{V}{k} \setminus R$ and $S' := \left(\binom{V}{k} \setminus R\right)^{(k+1)}$.

Lemma 3.10.10. *Let* $A = (V; R, S, \leq) \in \mathsf{E}(\mathcal{C}_{\mathsf{OH}_k})$ *and* $\overline{A} = ((V; \binom{V}{k} \setminus R, S', \leq))$ *the complement of* A*, as defined above. Then:*

1. $S' = \binom{V}{k+1} \setminus S$ *if, and only if,* $k \equiv 0 \pmod 2$.
2. $S = S'$ *if, and only if,* $k \equiv 1 \pmod 2$.

Proof. Let $A = (V; R, S, \leq) \in \mathsf{E}(\mathcal{C}_{\mathsf{OH}_k})$ and $V_0 \in \binom{V}{k+1}$ be arbitrary. We have that:

$$\left|\left(\binom{V_0}{k} \setminus R\right) \restriction_{V_0}\right| = \binom{k+1}{k} - |R \restriction_{V_0}| \equiv k+1-|R \restriction_{V_0}| \pmod 2.$$

Thus $\left|\left(\binom{V_0}{k} \setminus R\right) \restriction_{V_0}\right| \equiv 0 \pmod 2$ if, and only if, $V_0 \in S$. So, if $k \equiv 0 \pmod 2$ we have that $S' = \binom{V}{k+1} \setminus S$ and if $k \equiv 1 \pmod 2$ we have that $S' = S$, as claimed. □

Next, I will deal with the expansion property:

Proposition 3.10.11. *For all $k \in \mathbb{N}_{\geq 2}$ the pair $(\mathsf{E}(\mathcal{C}_{\mathsf{OH}_k}), K(\mathcal{C}_{\mathsf{OH}_k}))$ has the expansion property.*

Proof. Let $A^\star \in \mathsf{E}(\mathcal{C}_{\mathsf{OH}_k})$. We need to show that there is some $B \in K(\mathcal{C}_{\mathsf{OH}_k})$ such that for any expansion $B^\star \in \mathsf{E}(\mathcal{C}_{\mathsf{OH}_k})$ of B, there is an embedding $f : A^\star \hookrightarrow B^\star$. To this end, let $\tilde{A} := A^\star \sqcup \overline{A^\star}$, where $\overline{A^\star}$ is the complement of $A^\star$.

Now, since $\mathsf{E}(\mathcal{C}_{\mathsf{OH}_k})$ has ERP, let $D \in \mathsf{E}(\mathcal{C}_{\mathsf{OH}_k})$ be such that:

$$D \to \left(\tilde{A}\right)_2^{([k-1],\leq)}$$

and take $D^+ = D \sqcup \{\diamond\}$, where $\diamond$ is some new vertex, not in $V(D)$, with k-hyperedge relation given by:

$$R(D^+) := R(D) \sqcup \left\{\{\diamond\} \sqcup \{x_1, \dots, x_{k-1}\} : \{x_1, \dots, x_{k-1}\} \in \binom{V(D)}{k-1}\right\}.$$

Observe that by construction of D^+, for any $x_1, \dots, x_k \in V(D)$ we have that:

$$D^+ \vDash R^{(k+1)}(x_1, \dots, x_k, \diamond) \text{ if, and only if, } D^+ \vDash R(x_1, \dots, x_k). \tag{3.2}$$

To see this, note that by definition $D^+ \vDash R^{(k+1)}(x_1, \dots, x_k, \diamond)$ if, and only if:

$$\left|\left\{\{i_1, \dots, i_k\} \in \binom{[k+1]}{k} : \{x_{i_1}, \dots, x_{i_k}\} \in R\right\}\right| \equiv k+1 \pmod 2,$$

where $x_{k+1} = \diamond$. But:

$$\left|\left\{\{i_1,\dots,i_k\} \in \binom{[k+1]}{k} : \{x_{i_1},\dots,x_{i_k}\} \in R\right\}\right| \\ = \left|\left\{\{i_1,\dots,i_{k-1}\} \in \binom{[k]}{k-1} : \{x_{i_1},\dots,x_{i_{k-1}},\diamond\} \in R\right\}\right| + \mathsf{t}$$

where $\mathsf{t} = 1$ if, and only if, $D^+ \models R(x_1,\dots,x_k)$. Since in D^+, the vertex $\star$ forms a k-hyperedge with all $(k-1)$-element subsets, we have that

$$\left|\left\{\{i_1,\dots,i_{k-1}\} \in \binom{[k]}{k-1} : \{x_{i_1},\dots,x_{i_{k-1}},\diamond\} \in R\right\}\right| = \binom{k}{k-1} = k.$$

Thus $D^+ \models R^{(k+1)}(x_1,\dots,x_k,\star)$ if, and only if, $\mathsf{t} = 1$, as claimed.

We claim that $B = D^+ \restriction_{\mathcal{L}}$ witnesses the expansion property for A. Indeed, let us write $B = (V(D) \sqcup \{\diamond\}, S, \leq)$, and let $B^\star = (V(D) \sqcup \{\diamond\}; R^\star, S, \leq) \in \mathsf{E}(\mathcal{C}_{\mathsf{OH}_k})$ be some expansion of B.

Observe that, by definition, the $(k+1)$-hyperedge relation S of $B^\star$ is precisely the *Kay*-graph relation of D^+. In particular, for all $x_1,\dots,x_k \in V(D)$ we have that:

$$D^+ \models R^{(k+1)}(x_1,\dots,x_k,\diamond) \text{ if, and only if, } B^\star \models S(x_1,\dots,x_k,\diamond). \tag{3.3}$$

We need to show that there is an embedding $f : A^\star \hookrightarrow B^\star$. Consider the following colouring of the $(k-1)$-element substructures of $D \subseteq D^+$ (these are just linear orders), which depends on the expansion $B^\star$ of B:

$$c : \binom{D}{([k-1],\leq)} \to \{1,2\}$$
$$(d_1 \leq \cdots \leq d_{k-1}) \mapsto \begin{cases} 1 & \text{if } \{d_1,\dots,d_{k-1},\diamond\} \in R^\star \\ 2 & \text{otherwise.} \end{cases}$$

Since $D \to (\tilde{A})_2^{([k-1],\leq)}$, there is some $\tilde{A}' \in \binom{D^+ \setminus \{\diamond\}}{\tilde{A}}$ such that $c_{\restriction \tilde{A}'}$ is constant. By our choice of $\tilde{A}$ we have that $\tilde{A}' \simeq A^\star \sqcup \overline{A^\star}$. To simplify notation, let us write $A^\star \sqcup \overline{A^\star}$ for $\tilde{A}' \subseteq D^+$. We consider the two possible outcomes separately:

Case 1. $\mathsf{im}(c_{\restriction \tilde{A}'}) = \{1\}$. In this case we claim that for all $\{x_1,\dots,x_k\} \in \binom{A^\star}{k}$

we have that:

$$D^+ \vDash R(x_1, \dots, x_k) \text{ if, and only if, } B^\star \vDash R^\star(x_1, \dots, x_k),$$

in particular $A^\star \hookrightarrow B^\star$.

By Equations (3.2) and (3.3) we have that:

$$D^+ \vDash R(x_1, \dots, x_k) \text{ if, and only if, } B^\star \vDash S(x_1, \dots, x_k, \diamond),$$

thus, it suffices to show that:

$$B^\star \vDash S(x_1, \dots, x_k, \diamond) \text{ if, and only if } B^\star \vDash R^\star(x_1, \dots, x_k)$$

By definition, after putting $x_{k+1} = \diamond$ we have that $B^\star \vDash S(x_1, \dots, x_k, \diamond)$ if, and only if,

$$|\left\{\{i_1, \dots, i_k\} \in \binom{[k+1]}{k} : \{x_{i_1}, \dots, x_{i_k}\} \in R^\star\right\}| \equiv k+1 \pmod 2,$$

but:

$$\begin{aligned}\left|\left\{\{i_1, \dots, i_k\} \in \binom{[k+1]}{k} : \{x_{i_1}, \dots, x_{i_k}\} \in R^\star\right\}\right| \\ = \left|\left\{\{i_1, \dots, i_{k-1}\} \in \binom{[k]}{k-1} : \{x_{i_1}, \dots, x_{i_{k-1}}, \diamond\} \in R^\star\right\}\right| + \mathsf{t},\end{aligned}$$

where $\mathsf{t} = 1$ if, and only if, $B^\star \vDash R^\star(x_1, \dots, x_k)$. Since $\mathsf{im}(c_{\restriction \tilde{A}'}) = \{1\}$ we have that

$$\left|\left\{\{i_1, \dots, i_{k-1}\} \in \tbinom{[k]}{k-1} : \{x_{i_1}, \dots, x_{i_{k-1}}, \star\} \in R^\star\right\}\right| = \tbinom{k}{k-1} = k \tag{3.4}$$

So, we conclude that:

$$B^\star \vDash S(x_1, \dots, x_k, \diamond) \text{ if, and only if, } B^\star \vDash R^\star(x_1, \dots, x_k),$$

as claimed.

Case 2. $\mathsf{im}(c_{\restriction \tilde{A}'}) = \{2\}$. The argument will now vary slightly, depending on whether k is even or odd. We consider the two cases separately:

Assume k is even. In this case we again claim that for all $\{x_1, \dots, x_k\} \in \binom{A^\star}{k}$

we have that:

$$D^+ \vDash R(x_1, \ldots, x_k) \text{ if, and only if, } B^\star \vDash R^\star(x_1, \ldots, x_k),$$

which shows that $A^\star \hookrightarrow B^\star$.

The argument here is identical to the one given in Case 1, above, with the only difference that instead of Equation (3.4), we have that:

$$\left|\left\{\{i_1, \ldots, i_{k-1}\} \in \binom{[k]}{k-1} : \{x_{i_1}, \ldots, x_{i_{k-1}}, \diamond\} \in R^\star\right\}\right| = 0,$$

since $\mathsf{im}(c_{\restriction \tilde{A}'}) = \{2\}$. Again, this forces $\mathsf{t} = 1$, and the result follows as before.

Assume k is odd. In this case we claim that for all $\{x_1, \ldots, x_k\} \in \binom{\overline{A^\star}}{k}$ we have that:

$$D^+ \vDash \neg R(x_1, \ldots, x_k) \text{ if, and only if, } B^\star \vDash R^\star(x_1, \ldots, x_k).$$

Since k is odd, we know that the $(k+1)$-hyperedge relation on $\overline{A^\star}$ is the same as the $(k+1)$-hyperedge relation on $A^\star$, by Lemma 3.10.10. In particular, this means that $A^\star \hookrightarrow B^\star$.

Again, by Equations (3.2) and (3.3) it suffices to show that:

$$B^\star \vDash S(x_1, \ldots, x_k, \diamond) \text{ if, and only if } B^\star \vDash \neg R^\star(x_1, \ldots, x_k)$$

Arguing as in Case 1, $B^\star \vDash S(x_1, \ldots, x_k, \diamond)$ if, and only if,

$$\left|\left\{\{i_1, \ldots, i_{k-1}\} \in \binom{[k]}{k-1} : \{x_{i_1}, \ldots, x_{i_{k-1}}, \diamond\} \in R^\star\right\}\right| + \mathsf{t} \equiv 0 \pmod 2$$

where $\mathsf{t} = 1$ if, and only if $B^\star \vDash R^\star(x_1, \ldots, x_k)$. Since $\mathsf{im}(c_{\restriction \tilde{A}'}) = \{2\}$, just like the even case, we have that

$$\left|\left\{\{i_1, \ldots, i_{k-1}\} \in \binom{[k]}{k-1} : \{x_{i_1}, \ldots, x_{i_{k-1}}, \diamond\} \in R^\star\right\}\right| = 0$$

and hence $B^\star \vDash S(x_1, \ldots, x_k, \diamond)$ holds if, and only if, $\mathsf{t} = 0$. So, we conclude that:

$$B^\star \vDash S(x_1, \ldots, x_k, \diamond) \text{ if, and only if } B^\star \vDash \neg R^\star(x_1, \ldots, x_k),$$

as claimed. □

Finally, I will be dealing with reasonability.

Remark 3.10.12. It is well-known that for $k = 2$, the structure $\mathsf{Flim}(\mathsf{E}(\mathcal{C}_{\mathsf{OH}_2}))$ is precisely the expansion of the random graph by the *universal homogeneous* two-*graph*. In particular, in this case, it is immediate that:

$$\mathsf{Flim}(\mathsf{E}(\mathcal{C}_{\mathsf{OH}_2})) \restriction_S = \mathsf{Flim}(K(\mathcal{C}_{\mathsf{OH}_2})),$$

so the reasonability of $(\mathsf{E}(\mathcal{C}_{\mathsf{OH}_2}), K(\mathcal{C}_{\mathsf{OH}_2}))$ follows from Fact 3.10.2 and Proposition 3.9.4.

Proposition 3.10.13. *For all* $k \in \mathbb{N}_{\geq 2}$*, the expansion* $(\mathsf{E}(\mathcal{C}_{\mathsf{OH}_k}), K(\mathcal{C}_{\mathsf{OH}_k}))$ *is reasonable.*

Before going through the details of the proof, let me state a nice corollary:

Corollary 3.10.14. *For all* $k \in \mathbb{N}_{\geq 2}$ *we have that:*

$$\mathsf{Flim}(\mathsf{E}(\mathcal{C}_{\mathsf{OH}_k})) \restriction_S = \mathsf{Flim}(K(\mathcal{C}_{\mathsf{OH}_k})).$$

Proof. By Proposition 3.9.4 we know that $K(\mathcal{C}_{\mathsf{OH}_k}$ is a Fraïssé class. Thus, the corollary is immediate from Fact 3.10.2 and Proposition 3.10.13. □

The following useful little remark justifies some of the terminology that I shall be using in the proof.

Remark 3.10.15. Let $A = (V_A, R_A), B = (V_B, R_B)$ be k-hypergraphs. An injection $f : V_A \hookrightarrow V_B$ is an embedding of the *Kay*-graph $K(A) = (V_A, R_A^{(k+1)})$ into the *Kay*-graph $K(B) = (V_B, R_B^{(k+1)})$ if, and only if, f preserves the parity of k-hyperedges in every $(k+1)$-element subset of V_A, that is, if, and only if, for every $\{v_1, \ldots, v_{k+1}\} \in \binom{V_A}{k+1}$ we have that:

$$\left| R_A \restriction_{\{v_1,\ldots,v_{k+1}\}} \right| \equiv \left| R_B \restriction_{\{f(v_1),\ldots,f(v_{k+1})\}} \right| \pmod 2.$$

Proof of Proposition 3.10.13. The proof[7] is by induction on $k \in \mathbb{N}_{\geq 2}$. To fix notation, let $\mathcal{H}_k = \mathsf{Flim}(\mathcal{C}_{\mathsf{OH}_k}))$ and $\mathcal{K}_k = \mathsf{Flim}(\mathsf{E}(\mathcal{C}_{\mathsf{H}_k})) \restriction_S$. It suffices to show that $\mathcal{K}_k$ is homogeneous, for then, by Proposition 3.9.4 we have that $\mathcal{K}_k = \mathsf{Flim}(K(\mathcal{C}_{\mathsf{H}_k}))$ and the result follows by Fact 3.10.2.

We shall prove, by induction on $k \in \mathbb{N}_{\geq 2}$, the following statement:

$(\dagger)_k$ For all finite k-hypergraphs $A = (V_A, R_A) \subseteq \mathcal{H}_k = (V_k, R_k)$ and all

[7]For notational convenience, we shall ignore the order in this proof, and prove the result for *unordered* *Kay*-graphs. The proof with the order is essentially identical.

injections $f : V_A \hookrightarrow V_k$ which preserve the parity of k-hyperedges on every $(k+1)$-element subset of A there is some $\sigma \in \mathsf{Aut}(\mathcal{K}_k)$ such that $f = \sigma \restriction_{V_A}$.

The base case is, essentially, Remark 3.10.12. For the inductive step, we follow the proof of [Tho96, Theorem 2.6(2)]. In the notation of [Tho96, Theorem 2.6(2)], we consider only the case where $X = \{k-1\}$, and we shall prove that in this case $N = k-1$ and $\Phi^k_{\{k-1\}} = \{k+1\}$). In fact, the argument is a special case of Thomas's argument, and one need only note that in the case $X = \{k-1\}$ can avoid the use of Ramsey's theorem and start the inductive step at $N = k-1$.

So, let $n \in \mathbb{N}_{\geq 3}$, and assume that $(\dagger)_{k-1}$ holds. We shall prove $(\dagger)_k$, by induction on $|V_A|$. The result is clear when $|V_A| = k-1$, since in $\mathcal{H}_k$ all $(k-1)$-element subsets have the same quantifier-free types.

So, suppose that $(\dagger)_k$ is true whenever $|V_{A_0}| \leq n$ and let $(V_A, R_A) \subseteq \mathcal{H}_k$ be a finite k-hypergraph with $|V_A| = n+1$ and $f : V_A \hookrightarrow V_k$ be an injection that preserves the parity of k-hyperedges in every $(k+1)$-element subset of A. To fix notation, let $V_A = V_{A_0} \sqcup \{\star\}$. By induction we can find some $\sigma_0 \in \mathsf{Aut}(\mathcal{K}_k)$ such that $\sigma_0 \restriction_{V_{A_0}} = f$. Let $A_0^\star$ be the following $(k-1)$-hypergraph:

$$A_0^\star := (V_{A_0}, R^\star_{A_0}),$$

where:

$$\{v_1, \ldots, v_{k-1}\} \in R^\star_{A_0} :\Longleftrightarrow \{v_1, \ldots, v_{k-1}, \star\} \in R_A,$$

that is, in the terminology of [Tho96], $A_0^\star$ is the the $(k-1)$-hypergraph *induced on* A_0 *by* $\star$. Let $g := \sigma_0^{-1} \circ f : V_A \hookrightarrow V_k$ (so that $g \restriction_{V_{A_0}}$ is just the identity) and let $B_0^\star$ be the following $(k-1)$-hypergraph:

$$B_0^\star := (V_{A_0}, R^\star_{B_0}),$$

where:

$$\{v_1, \ldots, v_{k-1}\} \in R^\star_{B_0} :\Longleftrightarrow \{v_1, \ldots, v_{k-1}, g(\star)\} \in R_A,$$

which we can view as an arbitrarily chosen but fixed finite $(k-1)$-subhypergraph of $\mathcal{H}_{k-1}$. This induces an injection:

$$g_\star : A_0^\star \hookrightarrow \mathcal{H}_{k-1}.$$

We start with the following relatively easy claim:

Claim 1. The map $g_\star$ preserves the parity of $(k-1)$-hyperedges in every k-element subset of $A_0^\star$.

Proof of Claim 1. Let $\{v_1, \dots, v_k\} \in \binom{V_{A_0}}{k}$. By unfolding the definitions we see that:

$$\left| R_{k-1} \restriction_{\{g_\star(v_1), \dots, g_\star(v_k)\}} \right| = \left| R_k \restriction_{\{g(v_1), \dots, g(v_k), g(\star)\}} \right| - \mathsf{t},$$

where $\mathsf{t} = 1$ if $\{g(v_1), \dots, g(v_k)\} \in R_k$, and $\mathsf{t} = 0$, otherwise. Similarly that:

$$\left| R^\star_{A_0} \restriction_{\{v_1, \dots, v_k\}} \right| = \left| R_A \restriction_{\{v_1, \dots, v_k, \star\}} \right| + \mathsf{t}',$$

where $\mathsf{t}' = 1$ if $\{v_1, \dots, v_k\} \in R_k$, and $\mathsf{t}' = 0$, otherwise. Observe now that since on V_{A_0} we have that g is simply the identity, it follows that $\mathsf{t} = \mathsf{t}'$. Moreover, since $g = \sigma_0^{-1} \circ f$ and both σ_0^{-1} and f preserve the parity of k-hyperedges on $(k+1)$-element subsets of V_A, we have that:

$$\left| R_k \restriction_{\{g(v_1), \dots, g(v_k), g(\star)\}} \right| \equiv \left| R_A \restriction_{\{v_1, \dots, v_k, \star\}} \right| \pmod 2,$$

so the claim follows. ◀

By the claim above and $(\dagger)_{k-1}$ there is some $\tau_0 \in \mathsf{Aut}(\mathcal{K}_{k-1})$ such that $g^\star = \tau_0 \restriction_{V_{A_0}}$. By [Tho96, Theorem 2.5] we know that this map is induced by a *switch*[8] with respect to some $(k-2)$-element set X. In particular, there is some $\tau \in \mathsf{Aut}(\mathcal{K}_k)$ such that $g = \tau \restriction_{V_A}$, since we can consider the map induced by a switch with respect to $X \cup \{\star\}$ (see, for instance, the proof of [Tho96, Proposition 1.21]). Finally, taking $\sigma := \sigma_0 \circ \tau \in \mathsf{Aut}(\mathcal{K}_k)$ gives us the inductive step. □

Let's put everything together:

Corollary 3.10.16. *The pair* $(\mathsf{E}(\mathcal{C}_{\mathsf{OH}_k}), K(\mathcal{C}_{\mathsf{OH}_k}))$ *is excellent.*

Proof. We know, from Remark 3.10.7, that $\mathsf{E}(\mathcal{C}_{\mathsf{OH}_k})$ has ERP. In Proposition 3.10.8 we showed that it is precompact; in Proposition 3.10.11 we showed

[8] Recall: Let $A = (V_A, R_A), B = (V_B, R_B)$ be finite k-hypergraphs and $X \in \binom{A}{i}$, where $i \in \{0, \dots, k-1\}$. A *switch* with respect to X ([Tho96, Definition 1.9]) is a bijection $\pi : V_A \to V_B$ such that for all $Y \in \binom{A}{k}$ we have that $\pi \restriction_Y$ is an isomorphism if, and only if $X \not\subseteq Y$. Now, it is an almost immediate consequence of [Tho96, Theorem 2.5] that $\mathsf{Aut}(\mathcal{K}_k)$ is precisely the closed subgroup of $\mathsf{Sym}(V_k)$ generated by $\mathsf{Aut}(\mathcal{H}_k)$ and all $\pi \in \mathsf{Sym}(V_k)$ which are switches with respect to some $(k-1)$-subset of V_k.

that it has the expansion property; and in Proposition 3.10.13 we showed that it is reasonable. ☐

At this point, the main theorem of this part of Chapter 3 is an almost immediate corollary. Indeed:

Corollary 3.10.17. *Let $k \in \mathbb{N}_{\geq 2}$. Then $K(\mathcal{C}_{\mathsf{OH}_k})$ has k-ERP but not $(k+1)$-ERP.*

Proof. By Corollary 3.10.16, $(\mathsf{E}(\mathcal{C}_{\mathsf{OH}_k}), K(\mathcal{C}_{\mathsf{OH}_k}))$ is an excellent pair, so applying Fact 3.10.6, for all $A = (V, S, \leq) \in K(\mathcal{C}_{\mathsf{OH}_k})$ we have that the Ramsey degree of A is equal to the number of expansions of A in $\mathsf{E}(\mathcal{C}_{\mathsf{OH}_k})$. Observe that if $|V(A)| \leq k$ then $S(A) = \emptyset$, since hypergraphs are assumed to be uniform. Now, if $|V(A)| \leq k-1$, then A has a unique (trivial) expansion in $\mathsf{E}(\mathcal{C}_{\mathsf{OH}_k})$, namely $A^\star = (V; \emptyset, \emptyset, \leq)$. On the other hand, if $|V(A)| = k$, then A has two expansions in $\mathsf{E}(\mathcal{C}_{\mathsf{OH}_k})$, namely $A^\star_1 = (V; \{V\}, \emptyset, \leq)$ and $A^\star_2 = (V; \emptyset, \emptyset, \leq)$. ☐

Remark 3.10.18. Since the classes in Corollary 3.10.17, above, consist of ordered structures, the embedding Ramsey property and the structural Ramsey property coincide. Thus, if one considers the corresponding local versions to the *structural* Ramsey property, instead of the embedding Ramsey property, Corollary 3.10.17 is still applicable.

3.11 Orders in 2-Ramsey classes

This section is devoted to the proof of Theorem E.

Theorem 3.11.1. *Let $\mathcal{C}$ be a class of finite structures. If $\mathcal{C}$ has JEP and 2-ERP, then $\mathcal{C}$ is $\mathcal{L}_{\infty,0}$-orderable.*

Proof. By using the *quantifier-free type Morleyisation*, introduced in Section 3.3, we may assume that all quantifier-free types that are realised are isolated, in all structures in $\mathcal{C}$. Let $\mathcal{Q}$ be the set of quantifier-free $\mathcal{L}$-formulas in two variables isolating every quantifier-free type in two variables which is realised in some member of $\mathcal{C}$.

Abusing notation, let $\mathcal{C}/\cong$ be a set of representatives of structures in $\mathcal{C}$ under the equivalence relation of isomorphism. Let:

$$\mathcal{L}' := \mathcal{L} \cup \{<\} \cup \bigcup_{A \in \mathcal{C}/\cong} A,$$

where $<$ is a binary relation symbol and each $a \in \bigcup_{A \in \mathcal{C}/\cong} A$ is a constant symbol. Let T be the following $\mathcal{L}'$-theory:

$$\begin{aligned}&\mathsf{LO} \cup \bigcup_{A \in \mathcal{C}/\cong} \mathsf{Diag}(A)\\ &\quad \cup \{\forall x_1 \forall x_2 \forall y_1 \forall y_2 \, (Q(x_1, y_1) \wedge Q(x_2, y_2) \to (x_1 < y_1 \leftrightarrow x_2 < y_2)) : Q \in \mathcal{Q}\}\end{aligned}$$

where LO expresses that the new binary relation symbol $<$ is a linear order. To prove the theorem it suffices to show that T is (finitely) satisfiable since in this case we have that that some (not necessarily finite) intersection of quantifier-free types linearly orders all structures in $\mathcal{C}$. By JEP, it suffices to show that:

$$\begin{aligned}&\mathsf{LO} \cup \mathsf{Diag}(B)\\ &\quad \cup \{\forall x_1 \forall x_2 \forall y_1 \forall y_2 \, (Q(x_1, y_1) \wedge Q(x_2, y_2) \to (x_1 < y_1 \leftrightarrow x_2 < y_2)) : Q \in \mathcal{Q}\}\end{aligned}$$

is satisfiable for every $B \in \mathcal{C}$.

Given $B \in \mathcal{C}$, let $A_1, \ldots, A_n$ be the set of all substructures of B of size 2, and for each $i \leq n$ fix an enumeration $\{a_i^1, a_i^2\}$ of A_i.

Let $C \in \mathcal{C}$ be as promised from Corollary 3.8.5. Now let $\widehat{C}$ be an expansion of C to $\mathcal{L} \cup \{<\}$ such that $<$ is a linear order and for every $i \leq n$, define a colouring:

$$\begin{aligned}\chi_i : \binom{C}{A_i} &\to \{0, 1\}\\ \{a_i^1, a_i^2\} &\mapsto \begin{cases} 0 & \text{if } a_i^0 < a_i^1, \\ 1 & \text{if } a_i^1 < a_i^0, \end{cases}\end{aligned}$$

where the ordering is taken in $\widehat{C}$. Let $B' \in \binom{\widehat{C}}{B}$ be such that χ_i is constant on $\binom{B'}{A_i}$, for all $i \leq n$. The resulting B', with the ordering $<$ inherited from $\widehat{C}$ satisfies

$$\begin{aligned}&\mathsf{LO} \cup \mathsf{Diag}(B)\\ &\quad \cup \{\forall x_1 \forall x_2 \forall y_1 \forall y_2 \, (Q(x_1, y_1) \wedge Q(x_2, y_2) \to (x_1 < y_1 \leftrightarrow x_2 < y_2)) : Q \in \mathcal{Q}\}\end{aligned}$$

Indeed, suppose that we are given $b_1, b_2 \in B'$, then, for some i we have that $\{b_1, b_2\} = A_i$. Then for all $c_1, c_2 \in B'$, if we have that $B' \vDash Q(b_1, b_2) \wedge Q(c_1, c_2)$, for all $Q \in \mathcal{Q}$ then $\{c_1, c_2\} \simeq A_i$, hence $\chi_i(\{b_1, b_2\}) = \chi_i(\{c_1, c_2\})$. But since

$\chi_{i \restriction \binom{B'}{A_i}}$ is constant, we must have that b_1, b_2 and c_1, c_2 have the same order. □

Proposition 3.11.2. *Let $\mathcal{C}$ be a class of finite $\mathcal{L}$-structures. If $\mathcal{C}$ is $\mathcal{L}_{\infty,0}$-orderable, then every $A \in \mathcal{C}$ is rigid.*

Proof. Let $A \in \mathcal{C}$ and let $\widehat{A}$ be the expansion of A to $\mathcal{L} \cup \{\preceq\}$ where $\preceq$ is a binary relation symbol and for all $a, b \in A$ we have that $a \preceq b$ if, and only if $\Phi(a, b)$, where Φ is a Boolean combination of atomic and negated atomic formulas linearly ordering all structures in $\mathcal{C}$. Then $\mathsf{Aut}(A) = \mathsf{Aut}(\widehat{A})$, but since $\widehat{A}$ is a finite linearly ordered structure, it is rigid, and hence A must also be rigid. □

To conclude Chapter 3, I will give an example of a class of finite rigid structures which is not $\mathcal{L}_{\infty,0}$-orderable. This example is due to Cameron, and was communicated to Meir and me by Macpherson:

Example 3.11.3. Let $\mathcal{L} := \{R, C\}$, where R is a binary relation symbol and C is a ternary relation symbol. Let $\mathcal{C}$ be the class of finite $\mathcal{L}$ structures where R is a tournament and C is a C-relation derived from a binary tree (see Section 2.7 for the relevant definitions). It is an easy observation that all elements of $\mathcal{C}$ are rigid (since the automorphism groups of finite structures with C-relations derived from binary trees are 2-groups and the automorphism groups of tournaments have odd order). It is also fairly clear that $\mathcal{C}$ is an amalgamation class since $\mathcal{C} \restriction_R$ and $\mathcal{C} \restriction_C$ both have the strong amalgamation property. If $\mathcal{M} = \mathsf{Flim}(\mathcal{C})$ admitted an $\mathsf{Aut}(\mathcal{M})$-invariant linear order $\preceq$, then by homogeneity and antisymmetry, $\preceq$ would be a union of quantifier-free types in two variables x, y. Therefore $\preceq$ would be definable using only equality and R. So if $\mathcal{M}_2 := \mathcal{M}_{\restriction \{R\}}$, then $\preceq$ would be $\mathsf{Aut}(\mathcal{M}_2)$-invariant. Notice that $\mathcal{M}_2$ is the random tournament, which does not admit an invariant linear order (e.g. a 3-cycle is not rigid).

3.12 Local Acknowledgements†

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†Lifted almost directly from [MP23b] and [MP23a]. Here 'the authors' refers to Meir and me, the author of this thesis.

Chapter 4

On Products of Structures[†]

4.1 Introduction

I hope that in the last chapter I was able to convenience the reader that generalised indiscernibles highlight a strong link between model theory and structural Ramsey theory. This chapter, based on [MPT23], uses generalised indiscernibles as tools to prove results in both structural Ramsey theory and model theory.

All the way back in Chapter 2, I discussed some classical (i.e. relatively old) dividing lines. In a certain light, the various definitions of dividing lines given there (in Section 2.3) must seem rather ad hoc, especially for someone encountering them for the first time. Hence, a theme from a sort of "*meta-classification theoretic*" point of view is to systematically study dividing lines. Thus, rather than explicitly studying the tame and wild theories, according to some dividing line, it is conceivable to make the object of study the dividing lines themselves.

One of the starting points of the research presented in this chapter was to examine the behaviour of various dividing lines through the prism of *transfer principles* in some natural product constructions. I will discuss all of this in more detail in the remainder of this introduction. The dividing lines studied

[†]This chapter is based on [MPT23]. This manuscript was written jointly with Nadav Meir and Pierre Touchard, and at the time of writing is being prepared for submission. The contributions of all three authors to the results in this chapter are equal, and the results are reproduced here with the permission of Meir and Touchard. As in Chapter 3, the presentation I give here is largely new and differs from the arXiv version, and the differences will not be pointed out.

in this chapter admit definitions based on (generalised) indiscernible sequences. Some of them (e.g. NIP, distality) have already shown up and others will appear here for the first time (e.g. monadic NIP). A rather important way of defining dividing lines (at least as far as this chapter is concerned) will be introduced in the following paragraph.

Generalised Dividing Lines Various methods have been proposed as a uniform way of generalising and extending existing and well-studied dividing lines (see, for instance, [GHS17] and [GM22]). In this chapter, the focus will be on the approach taken in [GHS17] and developed further in [GH19], which provides a uniform scheme of dividing lines arising from *coding* classes of finite structures: Given a class of finite structures, $\mathbb{K}$, let $\mathsf{N}\mathscr{C}_{\mathbb{K}}$ be the class of all first-order theories that do not code all the members of $\mathbb{K}$ in a uniform manner (see Subsection 4.3.2 for the precise definition), and $\mathscr{C}_{\mathbb{K}}$, the complement of $\mathsf{N}\mathscr{C}_{\mathbb{K}}$, in the class of all first-order theories.

The study of dividing lines of the form $\mathsf{N}\mathscr{C}_{\mathbb{K}}$, where $\mathbb{K}$ is a Ramsey class, inevitably leads to the notion of $\mathsf{FLim}(\mathbb{K})$-indexed indiscernibles. This deep connection between generalised indiscernibles and structural Ramsey theory was the topic of the previous chapter. In this chapter, I will present connections between these notions and the study of dividing lines arising from coding Ramsey structures. Intuitively, the existence of "*uncollapsed*" $\mathsf{FLim}(\mathbb{K})$-indiscernible sequences (see Definition 4.3.3) for a Ramsey class $\mathbb{K}$ in some model of a first-order theory T indicates that T can somehow 'see a trace' of $\mathbb{K}$ and therefore T cannot be in $\mathsf{N}\mathscr{C}_{\mathbb{K}}$, and vice versa (see Theorem 4.4.3, which generalises [GH19, Theorem 3.14]). The main tool for analysing dividing lines of the form $\mathsf{N}\mathscr{C}_{\mathbb{K}}$ will be various notions of *indiscernible collapse*.

The starting point of understanding dividing lines in this way is Shelah's well-known theorem that in a stable theory every indiscernible sequence is totally indiscernible, and, in fact, this property characterises the class of stable theories (see Theorem 2.6.3). Results of this nature have been shown for various other dividing lines, and essentially amount to instances of Theorem 4.4.3 with the appropriate class $\mathbb{K}$ for each dividing line.

Transfer principles A *transfer principle* for a given theory T and a (model-theoretic) property P is a statement of the following form: A model $\mathcal{M}$ of T has property P if, and only if, some simpler structures related to $\mathcal{M}$ have property P. A great example comes from the celebrated *Ax-Kochen-Ershov theorem*, which states that Henselian valued fields of equicharacteristic zero are model complete relative to their residue fields and value groups[1]. There is an extensive literature in model theory that focuses on such transfer principles, for they are often an essential step toward characterising models with a given property.

At the same time, transfer principles give us information on how "well-behaved" a theory is. For instance, the theorem of Delon, which states that equicharacteristic zero Henselian valued fields transfer NIP from the residue field to the valued field itself [Del81], suggests that this class of valued fields is well-behaved for the model theorist. Analogous transfers for this class concerning many other dividing lines are known (or at least expected to be true).

The study of theories/classes of structures that are well-behaved with respect to a given dividing line is an active topic of research. In [MPT23], we take another approach and use transfer principles as a tool to reveal whether a dividing line itself is a well-behaved one. Our guiding principle is that a given dividing line is a "well-behaved" one if some natural transfer principles hold. More precisely, we are interested in the following general question:

Question. Let P be some notion of tameness (e.g. a dividing line). Is it true that two structures $\mathcal{M}$ and $\mathcal{N}$ are tame with respect to P if, and only if, their *full product*, $\mathcal{M} \boxtimes \mathcal{N}$ (see Definition 4.2.4), and their *lexicographic product*, $\mathcal{M}[\mathcal{N}]$ (see Definition 4.2.7), are tame with respect to P?

I'll now discuss this chapter's main results and leave it to the reader to see how they relate to the question above.

Main Results

As I have already mentioned, generalised indiscernibles, coding configurations, and Ramsey classes are closely interconnected notions. This connection was first established by Guingona and Hill for structures in finite relational language [GH19, Theorem 3.14]. The first main theorem of [MPT23] gives a more general version of this result:

[1] This thesis is not concerned with the model theory of valued fields, so I will let this statement remain somewhat mysterious. Nevertheless, if the reader is after sources demystifying the Ax-Kochen-Ershov theorem, there is a plethora of excellent introductory notes in the model theory of valued fields, such as [Mar08], and more specifically on the Ax-Kochen-Ershov theorem [dEl23]

Theorem F (Theorem 4.4.3)**.** *Let $\mathcal{I}$ be an $\aleph_0$-categorical Fraïssé limit of a Ramsey class. Then the following are equivalent for a theory T:*

1. *$T \in \mathsf{N}\mathscr{C}_{\mathsf{Age}(\mathcal{I})}$.*
2. *T collapses $\mathcal{I}$-indiscernibles.*

We then use generalised indiscernibles as a criterion to show that certain classes cannot be Ramsey. More precisely:

Theorem G (Theorem 4.4.10)**.** *Let $\mathcal{J}$ be an n-ary homogeneous $\aleph_0$-categorical structure, for some $n \in \mathbb{N}$. Let $\mathcal{I}$ be a non n-ary reduct of $\mathcal{J}$. Then, $\mathsf{Age}(\mathcal{I})$ is not a Ramsey class.*

From this we obtain the following:

Theorem H (Theorem 4.4.13)**.** *Any proper reduct of the generically ordered random k-hypergraph which is not interdefinable with DLO is not Ramsey.*

Next, we provide a negative answer to a question asked in [GP23, Section 7] and [GPS21, Question 4.7] about the linearity of the hierarchy given by coding $\mathbb{K}$-configurations. Let me explain: In [GPS21], the authors observe that one has the following strict inclusions:

$$\mathscr{C}_{\mathcal{C}_{\mathsf{E}}} \supset \mathscr{C}_{\mathcal{C}_{\mathsf{LO}}} \supset \mathscr{C}_{\mathcal{C}_{\mathsf{OG}}} = \mathscr{C}_{\mathcal{C}_{\mathsf{OH}_2}} \supset \mathscr{C}_{\mathcal{C}_{\mathsf{OH}_3}} \supset \cdots \supset \mathscr{C}_{\mathcal{C}_{\mathsf{OH}_n}} \supset \cdots$$

where, recall from Section 2.7, that:

- $\mathcal{C}_{\mathsf{E}}$ is the class of all finite sets (in the language of pure equality).
- $\mathcal{C}_{\mathsf{LO}}$ is the class of all finite linear orders.
- $\mathcal{C}_{\mathsf{OG}}$ the class of all finite ordered graphs.
- $\mathcal{C}_{\mathsf{OH}_n}$ the class of all finite ordered n-hypergraphs.

The question then becomes the following:

Question. Are there any other classes of the form $\mathscr{C}_{\mathbb{K}}$? If so, do these classes remain linearly ordered under inclusion?

We answer the first part of this question positively and the second part of this question negatively, as follows:

Theorem I (Example 4.5.4)**.** *Building on the notation above, let $\mathcal{C}_{\mathsf{OC}}$ be the class of finite convexly ordered binary branching C-relations (see Section 2.7),* *and let* $\mathcal{C}_{\mathsf{OG}} \boxtimes \mathcal{C}_{\mathsf{OC}}$ *be the full product the classes* $\mathcal{C}_{\mathsf{OG}}$ *and* $\mathcal{C}_{\mathsf{OC}}$ *(see Definition 4.2.4). Then:*

$$\begin{array}{ccccccc} & & & \mathscr{C}_{\mathcal{C}_{\mathsf{OG}}} & & & \\ & & \rotatebox{45}{$\supset$} & & \rotatebox{-45}{$\supset$} & & \\ \mathscr{C}_{\mathcal{C}_{\mathsf{E}}} & \supset & \mathscr{C}_{\mathcal{C}_{\mathsf{LO}}} & \begin{array}{c}\not\supset\\ \not\subset\end{array} & & \mathscr{C}_{\mathcal{C}_{\mathsf{OG}} \boxtimes \mathcal{C}_{\mathsf{OC}}} \supset & \cdots . \\ & & \rotatebox{-45}{$\supset$} & & \rotatebox{45}{$\supset$} & & \\ & & & \mathscr{C}_{\mathcal{C}_{\mathsf{OC}}} & & & \end{array}$$

Thus, the $\mathsf{N}\mathscr{C}_{\mathbb{K}}$ hierarchy is not linearly ordered.

We then turn our attention to transfer principles. One of our main tools is a characterisation of generalised indiscernibles in terms of generalised indiscernibles in the factors, in full products (Proposition 4.5.5) and lexicographic sums (Proposition 4.5.24).

We immediately obtain transfer theorems for dividing lines characterised by generalised indiscernibility. These results are summarised below:

Theorem J (Corollary 4.5.8, Corollary 4.5.10, Proposition 4.5.11)**.** *Let* $\mathcal{M}$, $\mathcal{N}$ *be structure in respective languages* $\mathcal{L}_{\mathcal{M}}$ *and* $\mathcal{L}_{\mathcal{N}}$. *For*

$$P \in \{\mathit{NIP}_n,\ n\text{-}\mathit{distal},\ \mathit{indiscernible\ triviality}\}^{2}$$

the following are equivalent:

1. $\mathcal{M}$ *and* $\mathcal{N}$ *have* P.
2. *The full product of* $\mathcal{M}$ *and* $\mathcal{N}$, $\mathcal{M} \boxtimes \mathcal{N}$, *has* P.

Similarly:

Theorem K (Corollary 4.5.23, Proposition 4.5.24, Corollary 4.5.26, Proposition 4.5.29)**.** *Let* $\mathcal{M}$ *be an* $\mathcal{L}_{\mathcal{M}}$*-structure and* $\mathfrak{N} = \{\mathcal{N}_a\}_{a \in \mathcal{M}}$ *be a collection* $\mathfrak{N}$ *of* $\mathcal{L}_{\mathfrak{N}}$*-structures indexed by* $\mathcal{M}$. *For:*

$$P \in \{\mathit{NIP}_n,\ n\text{-}\mathit{distal},\ \mathit{indiscernible\ triviality},\ \mathit{monadic\ NIP}\}^{3}$$

the following are equivalent:

1. $\mathcal{M}$ *and the* common theory[4] *of* $\{\mathcal{N}_a : a \in \mathcal{M}\}$ *have* P.
2. *The lexicographic sum of* $\mathcal{M}$ *and* $\mathfrak{N}$, $\mathcal{M}[\mathfrak{N}]$, *has* P.

[2] See Definition 4.2.17.
[3] See Definition 4.2.16.
[4] That is, the theory of any ultraproduct of the class.

Notice that in particular, monadic NIP transfers in lexicographic sums. We apply this result to generalise one of the results of [PS23], to lexicographic sums of ordered graphs of bounded twin-width. More precisely, once we obtain a result characterising ultraproducts of classes of lexicographic products, which is interesting in its own right (Proposition 4.6.1), we prove the following:

Theorem L (Corollary 4.6.5)**.** *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two hereditary classes of finite graphs with bounded twin-width. Then, the class of graphs consisting of lexicographic sums of graphs from $\mathcal{C}_1$ and $\mathcal{C}_2$ has bounded twin-width.*

Structure of the chapter

Sections 4.2 and 4.3 contain all the relevant terminology and background definitions needed for the remainder of this chapter. In Section 4.3, some preliminary results regarding coding configurations and collapsing indiscernibles are proved. Then, in Section 4.4 we prove Theorem F and Theorem G. We then devote Section 4.5 to proving Theorem J and Theorem K. In Section 4.6 we use the monadic NIP case in Theorem K to prove Theorem L. We conclude the paper with some open questions, in Section 4.7.

4.2 Local Preliminaries

4.2.1 Reducts

Often, when people refer to a reduct of an $\mathcal{L}$-structure $\mathcal{M}$, they mean a *language reduct*, that is, a structure $\mathcal{M}'$ on the same domain as $\mathcal{M}$, which is obtained from $\mathcal{M}$ by forgetting some of the symbols in $\mathcal{L}$ (this was, in fact, the way the word reduct was used in the previous chapter). The notion of reduct we work with in [MPT23] is a more general one, sometimes also referred to as a *definable* or *first-order reduct*. More precisely:

Definition 4.2.1. Let $\mathcal{M}$ and $\mathcal{M}'$ be structures on the same domain.

- We say that $\mathcal{M}'$ is a *first-order reduct* of $\mathcal{M}$ if every definable subset of $\mathcal{M}'$ is definable in $\mathcal{M}$, without parameters.

For simplicity, henceforth (unless otherwise stated), the word *reduct* should be understood to mean “first-order reduct”, in the sense just defined.[5] If $\mathcal{M}'$ is a reduct of $\mathcal{M}$ we also say that $\mathcal{M}$ is an *expansion* of $\mathcal{M}'$.

- We say that $\mathcal{M}$ and $\mathcal{M}'$ are *interdefinable*, if each is a reduct of the other.

[5]Some authors refer to this relation as $\emptyset$-*definable reduct*.

- We say that $\mathcal{M}'$ is a *quantifier-free reduct* of $\mathcal{M}$ if it is a reduct of $\mathcal{M}$ and for all tuples $\bar{a}, \bar{b}$ from M we have that $\mathsf{qftp}^{\mathcal{M}}(\bar{a}) = \mathsf{qftp}^{\mathcal{M}}(\bar{b})$ if, and only if, $\mathsf{qftp}^{\mathcal{M}}(\bar{a}) = \mathsf{qftp}^{\mathcal{M}}(\bar{b})$.
- We say that $\mathcal{M}_0$ and $\mathcal{M}_1$ are *quantifier-free interdefinable* if each is a quantifier-free reduct of the other.

Reducts of structures and, in particular, reducts of homogenous structures, play an important role in this chapter, as they will be used to construct dividing lines between tame and wild structures.

To give some additional context, recall Thomas's conjecture:
Conjecture (Thomas, [Tho96])**.** *If $\mathcal{M}$ is a countable homogeneous structure in a finite relational language, then $\mathcal{M}$ has finitely many reducts (up to interdefinability).*

This conjecture has been verified for many well-known examples. For instance, in [Cam76] it is shown that $(\mathbb{Q}, \leq)$ has 3 proper[6] reducts up to interdefinability (this was already mentioned in passing, in Section 2.7); in [Tho91] it is shown that the random graph has 3 proper reducts up to interdefinability; in [BPP15] it is shown that the ordered random graph has 42 proper reducts up to interdefinability. The general case remains open.

Let $\mathscr{R}(\mathcal{M})$ be the set of reducts of a structure $\mathcal{M}$ (up to interdefinability), and define a partial order $\sqsubset_{\mathsf{r}}$ on $\mathscr{R}(\mathcal{M})$ as follows:

$$\mathcal{R} \sqsubset_{\mathsf{r}} \mathcal{S} \text{ if, and only if, } \mathcal{R} \text{ is a reduct of } \mathcal{S}.$$

This makes $(\mathscr{R}(\mathcal{M}), \sqsubset_{\mathsf{r}})$ into a lattice.[7] Moreover, the lattice $(\mathscr{R}(\mathcal{M}), \sqsubset_{\mathsf{r}})$ is precisely the dual lattice of $(\{\mathsf{Aut}(\mathcal{R}) : \mathcal{R} \sqsubset_{\mathsf{r}} \mathcal{M}\}, \leq)$, the former being precisely the set of closed (with respect to the product topology) subgroups of $\mathsf{Sym}(\mathcal{M})$ which contain $\mathsf{Aut}(\mathcal{M})$ (as discussed in Section 2.5).

4.2.2 More Amalgamations

In Section 2.5, I defined the *Amalgamation Property*. For the purposes of this chapter, I need to define two well-known strengthenings:
Definition 4.2.2. A class of structures $\mathcal{C}_{\mathsf{C}}$ has the:

[6] Proper meaning different from the structure itself and an infinite set.
[7] Recall, a *lattice* is a partially ordered set in which any two elements have a meet and a join. The *dual* of a lattice is the lattice obtained by reversing the partial order.

1. The *Strong Amalgamation Property* (SAP) if whenever $A, B, C \in \mathcal{C}_\mathsf{C}$ are such that A embeds into B via $e : A \hookrightarrow B$ and into C via $f : A \hookrightarrow C$ there exists some $D \in \mathcal{C}_\mathsf{C}$ and embeddings $g : B \hookrightarrow D$, $h : C \hookrightarrow D$ such that $g \circ e = h \circ f$, and moreover for all $b \in B$ and $c \in C$, if $g(b) = h(c)$, there is some $a \in A$ such that $e(a) = b$ and $f(a) = c$.

2. The *Free Amalgamation Property* (FAP) if it consists of structures in a relational language $\mathcal{L}$ such that for all $A, B, C \in \mathcal{C}_\mathsf{C}$ with $C = A \cap B$ the *free amalgam* of A and B over C, denoted $A \otimes_C B$, is in $\mathcal{C}_\mathsf{C}$, where $A \otimes_C B$ is the $\mathcal{L}$-structure where for all $R \in \mathsf{Sig}(\mathcal{L})$ we have that $R^{A \otimes_C B} = R^A \cup R^B$.

Remark 4.2.3. For classes of relational structures, we have (FAP)$\Rightarrow$ (SAP) $\Rightarrow$ (AP), and both implications here are strict.

4.2.3 Product constructions

Let me now introduce the two product constructions[8] which will be central in this chapter:

1. The *full product* (Definition 4.2.4); and

2. The *lexicographic sum* (Definition 4.2.7).

Each of these constructions comes with a relative quantifier elimination result, which will be used later to describe generalised indiscernibles in full products and lexicographic sums, in terms of generalised indiscernibles in their factors. This description, as discussed in the introduction of this chapter, is a crucial tool we use in [MPT23], in order to prove transfer principles.

Full Product

Definition 4.2.4. For $i \in \{1, 2\}$, let $\mathcal{M}_i$ be an $\mathcal{L}_i$-structure (possibly multisorted) with main sort M_i. We define the *full product*[9] of $\mathcal{M}_1$ and $\mathcal{M}_2$, denoted $\mathcal{M}_1 \boxtimes \mathcal{M}_2$, to be the multisorted structure:

$$(M_1 \times M_2, M_1, M_2, \{\pi_{M_i} : M_1 \times M_2 \to M_i : i = 1, 2\})$$

where:

[8] When discussing products of structures, it will always be assumed that these structures are in *disjoint languages*.

[9] This notion of product should not be confused with the *Feferman product* (of two structures); full products are equipped with projection maps, which cannot always be recovered in the Feferman product.

- The sorts M_1 and M_2 are equipped with their respective $\mathcal{L}_i$-structure.
- For $i \in \{1,2\}$, $\pi_{M_i} : M_1 \times M_2 \to M_i$ is the natural projection.

We denote by $\mathcal{L}_{\mathcal{M}_1 \boxtimes \mathcal{M}_2}$ the corresponding language (which contains *disjoint* copies of $\mathcal{L}_1$ and $\mathcal{L}_2$).

The full product $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ is sometimes called the *direct product* (since $\mathsf{Aut}(\mathcal{M}_1 \boxtimes \mathcal{M}_2)$ is isomorphic to the direct product of $\mathsf{Aut}(\mathcal{M}_1)$ and $\mathsf{Aut}(\mathcal{M}_2)$) and can be described in a one-sorted language. However, the multisorted approach has the following immediate advantage:

Fact 4.2.5. *In the notation above $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ eliminates quantifiers relative to $\mathcal{M}_1$ and $\mathcal{M}_2$.*

Proof (Sketch). We may Morleyise $\mathcal{M}_1$ and $\mathcal{M}_2$, so, without loss, we may assume that they eliminate quantifiers in their respective languages $\mathcal{L}_1$ and $\mathcal{L}_2$, which we may assume are relational. We prove the fact by induction on the complexity $\mathcal{L}_{\mathcal{M}_1 \boxtimes \mathcal{M}_2}$-formulas. Let $\phi(\bar{x}) \in \mathcal{L}_{\mathcal{M} \boxtimes \mathcal{M}_2}$:

- If $\phi = R$ for some $R \in \mathcal{L}_i$, then, by definition:

$$\mathcal{M}_1 \boxtimes \mathcal{M}_2 \vDash \phi(\bar{x}) \iff \mathcal{M}_i \vDash \phi(\pi_i(\bar{x})).$$

- If $\phi(x, y)$ is $x = y$, then, again, by definition:

$$\mathcal{M}_1 \boxtimes \mathcal{M}_2 \vDash \phi(x, y) \iff \mathcal{M}_1 \vDash \phi(\pi_1(x, y)) \wedge \mathcal{M}_2 \vDash \phi(\pi_2(x, y)).$$

- If ϕ is of the form $\neg\psi$ or $\psi_1 \wedge \psi_2$, then the fact follows from the induction hypothesis.
- Finally, assume $\phi(\bar{x})$ is of the form $\exists y\, \psi(\bar{x}; y)$. We may assume ψ is in *disjunctive normal form*, and by induction hypothesis, we may actually assume that

  $$\psi(\bar{x}; y) = \bigvee_{i \in I} \bigwedge_{j \in J_i} \theta_{i,j}(\bar{x}; y),$$

  where $I, \{J_i\}_{i \in I}$ are finite and each $\theta_{i,j}$ is an atomic or negated atomic formula. As disjunctions commute with existential quantifiers, we may actually assume that

$$\psi(\bar{x}; y) = \bigwedge_{j \in J} \theta_j(\bar{x}; y).$$

Finally, since equalities in $M_1 \times M_2$ correspond to a conjunction of equalities in M_1 and M_2, we may assume ψ is

$$\psi(\bar{x}; y) = \psi_1(\pi_1(\bar{x}; y)) \wedge \psi_2(\pi_2(\bar{x}; y)).$$

But then, by definition, we have that:

$$\mathcal{M}_1 \boxtimes \mathcal{M}_2 \models \exists y \left(\psi_1(\pi_1(\bar{x}; y)) \wedge \psi_2(\pi_2(\bar{x}; y))\right) \iff \mathcal{M}_1 \models \exists y \psi_1(\pi_1(\bar{x}; y)) \text{ and } \mathcal{M}_2 \models \exists y \psi_2(\pi_2(\bar{x}; y)),$$

and the result follows.

□

We extend this definition to classes of structures:

Definition 4.2.6. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two classes of structures. We denote by $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ the smallest hereditary class of structures containing $C \boxtimes D$ for all $C \in \mathcal{C}_1$ and $D \in \mathcal{C}_2$.

Lexicographic Sum and Product

The *lexicographic sum* of relational structures was studied by Meir in [Mei16]. It is a method of constructing an $\mathcal{L}$-structure $\mathcal{M}[\mathcal{N}]$ from two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$, where $\mathcal{L}$ is a relational language, in a way that generalises the lexicographic (*wreath*[10]) product of graphs.

Definition 4.2.7. Let $\mathcal{M}$ be an $\mathcal{L}_{\mathcal{M}}$-structure and $\mathfrak{N} = \{\mathcal{N}_a : a \in M\}$ be a collection $\mathfrak{N}$ of $\mathcal{L}_{\mathfrak{N}}$-structures indexed by M. The *lexicographic sum of* $\mathfrak{N}$ *with respect to* $\mathcal{M}$, denoted by $\mathcal{M}[\mathfrak{N}]$, is the multisorted structure with:

- A main sort S, with base set $\bigcup_{a \in M} \{a\} \times N_a$.
- A sort for the structure $\mathcal{M}$,
- The natural projection map $\mathsf{v} : S \to M$.

[10] Consider automorphism groups, again!

We equip the main sort with an $\mathcal{L}_{\bullet,\mathfrak{N}}$-structure, where $\mathcal{L}_{\bullet,\mathfrak{N}}$ is just a copy of $\mathcal{L}_{\mathfrak{N}}$ (with an extra $\bullet$, in order to distinguish the symbols from those in the $\mathcal{N}_a$), as follows:

- For each n-ary relation symbol $P \in \mathcal{L}_{\mathfrak{N}}$ we set:

$$P_\bullet^{\mathcal{M}[\mathfrak{N}]} := \{((a, b_1), \ldots, (a, b_n)) : a \in M \text{ and } \mathcal{N}_a \vDash P(b_1, \ldots, b_n)\}.$$

- For each n-ary function symbol $f \in \mathcal{L}_{\mathfrak{N}}$ we set:

$$f_\bullet^{\mathcal{M}[\mathfrak{N}]}((a_1, b_1), \ldots, (a_n, b_n)) := \begin{cases} (a_1, f^{\mathcal{N}_a}(b_1, \cdots, b_n)) & \text{if } a_1 = \cdots = a_n, \\ \mathsf{u} & \text{otherwise.} \end{cases}.$$

where u is a fresh constant symbol (interpreted as a new element, outside of the above domains) which stands for *undefined*.

Finally, $\mathcal{M}[\mathfrak{N}]$, is the multisorted structure:

$$(S, M, \mathsf{v} : S \to M).$$

where S is taken with the $\mathcal{L}_{\bullet,\mathfrak{N}}$-structure defined above, and M with its $\mathcal{L}_{\mathcal{M}}$-structure. We write $\mathcal{L}_{\mathcal{M}[\mathfrak{N}]}$ for the multisorted language of the lexicographic sum.

If there is some $\mathcal{L}_{\mathfrak{N}}$-structure $\mathcal{N}$ such that for all $a \in M$ we have that $\mathcal{N}_a \cong \mathcal{N}$, we denote the lexicographic sum by $\mathcal{M}[\mathcal{N}]$, and we call it the *lexicographic product of* $\mathcal{M}$ *and* $\mathcal{N}$.[11]

Remark 4.2.8. Notice that the projection to the second coordinate is not in the language of the lexicographic product. But, to simplify notation, whenever $a \in \mathcal{M}$ we identify $\mathcal{N}_a$ with $\{a\} \times \mathcal{N}_a$ and write $\mathcal{N}_a \vDash \phi(c)$ for $c = (a, n) \in \mathcal{M}[\mathfrak{N}]$ and ϕ an $\mathcal{L}_{\mathfrak{N}}$-formula such that $\mathcal{N}_a \vDash \phi(n)$.

Notation 4.2.9. The sort $\mathcal{M}$ in $\mathcal{M}[\mathfrak{N}]$ can come with additional structure: we can define predicates

$$P_\phi^{\mathcal{M}} := \{a \in \mathcal{M} : \mathcal{N}_a \vDash \phi\} \qquad (\star)$$

for all $\mathcal{L}_{\mathfrak{N}}$-sentences ϕ. For technical reasons, these predicates will not generally be added in the language $\mathcal{L}_{\mathcal{M}[\mathfrak{N}]}$, but they will play an important role in the next sections.

[11] Observe that $\mathcal{M}[\mathcal{N}]$ is a reduct of $\mathcal{M} \boxtimes \mathcal{N}$.

Again, we extend the definition to classes of finite structures:

Definition 4.2.10. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two classes of finite structures. We denote by $\mathcal{C}_1[\mathcal{C}_2]$ the class of lexicographic sums $C[(D_c)_{c \in C}]$ where $C \in \mathcal{C}_1$ and $D_c \in \mathcal{C}_2$ for every $c \in C$.

Before giving the promised relative quantifier elimination result of lexicographic sums, I need to briefly discuss how elementary extensions of these objects look like.

Elementary extensions For the remainder of this paragraph, assume that $\mathcal{M}[\mathfrak{N}]$ is expanded as in Notation 4.2.9. Any complete 1-type $p \in \mathsf{S}_1^{\mathsf{Th}(\mathcal{M})}(\emptyset)$ is, of course, an ultrafilter on the set of $\emptyset$-definable subsets of M. By Łos's theorem, the theory of an ultraproduct $\prod_{\mathcal{U}} \mathcal{N}_a$ depends only on the ultrafilter $\mathcal{U}$, in particular, thinking of 1-types as ultrafilters we can view such ultraproducs as $\prod_p \mathcal{N}_a$, for $p \in \mathsf{S}_1^{\mathsf{Th}(\mathcal{M})}(\emptyset)$. Abusing notation, this may also be denoted by $\mathcal{N}_p$.

An elementary extension of $\mathcal{M}[\mathfrak{N}]$ will necessarily be of the form $\tilde{\mathcal{M}}[\tilde{\mathfrak{N}}]$ where $\tilde{\mathcal{M}}$ is an $\mathcal{L}_{\mathcal{M}}$-structure and $\tilde{\mathfrak{N}}$ is a collection of $\mathcal{L}_{\mathfrak{N}}$-structures $\{\tilde{\mathcal{N}}_{\tilde{a}} : \tilde{a} \in \tilde{M}\}$ such that:

- $\tilde{\mathcal{M}}$ is an elementary extension of $\mathcal{M}$ in $\mathcal{L}_{\mathcal{M}} \cup \{P_\phi\}_{\phi \in \mathcal{L}_{\mathfrak{N}}}$, where P_ϕ are the sentences from $(\star)$;
- For $a \in \mathcal{M}$, we have that $\tilde{\mathcal{N}}_a$ is an elementary extension of $\mathcal{N}_a$;
- For $\tilde{a} \in \tilde{\mathcal{M}} \setminus \mathcal{M}$, we have that $\tilde{\mathcal{N}}_{\tilde{a}}$ is elementary equivalent to the ultraproduct $\prod_p \mathcal{N}_a$ where $p = \mathsf{tp}(\tilde{a}) \in \mathsf{S}_1^{\mathsf{Th}(\mathcal{M})}(M)$.

Now for the relative quantifier elimination result:

Fact 4.2.11. *Consider the lexicographic sum $\mathcal{S} := \mathcal{M}[\mathfrak{N}]$ of a class of $\mathcal{L}_{\mathfrak{N}}$-structures $\mathfrak{N} := \{\mathcal{N}_a\}_{a \in \mathcal{M}}$ with respect to an $\mathcal{L}_{\mathcal{M}}$-structure $\mathcal{M}$. Assume that*

1. *For all sentences $\phi \in \mathcal{L}_{\mathfrak{N}}$, the set $\{a \in \mathcal{M} : \mathcal{N}_a \vDash \phi\}$ is $\emptyset$-definable in $\mathcal{M}$, as in $(\star)$;*
2. *For all $p \in \mathsf{S}_1^{\mathsf{Th}(\mathcal{M})}$, $\mathcal{N}_p = \prod_p \mathcal{N}_a$ admits quantifier elimination in $\mathcal{L}_{\mathfrak{N}}$.*

Then $\mathcal{M}[\mathfrak{N}]$ eliminates quantifiers relative to $\mathcal{M}$ in $\mathcal{L}_{\mathcal{M}[\mathfrak{N}]}$.

For a proof, the reader can refer to [Mei16, Theorem 2.7].

4.2.4 More Dividing Lines

This subsection can be seen as a continuation of Section 2.3. Unless otherwise stated, T will always denote a complete $\mathcal{L}$-theory and $\mathbb{M} \vDash T$ a $\kappa(\mathbb{M})$-saturated and $\kappa(\mathbb{M})$-homogeneous (monster) model of T, for some very big cardinal $\kappa(\mathbb{M})$, just like in Chapter 2. Again, unless otherwise stated all (tuples of) elements and subsets will come from this monster model and will be *small*, i.e. of size less than $\kappa(\mathbb{M})$.

Higher arity Generalisations of NIP

In Section 2.3, I introduced the *Independence Property*, and the relatively tame class of structures which do not have it, the *NIP* theories. Now, I will introduce a natural way of generalising NIP to structures of "higher arity". This notion (just like NIP) is due to Shelah, originally introduced in [She14]. As an abstract model-theoretic dividing line, it was later studied in more detail in [CPT19]. This is one of the prototypical examples of a dividing line arising from generalised indiscernibles.

Definition 4.2.12 (n-Independence Property, NIP_n)**.** We say that a formula $\phi(\bar{x}; \bar{y}_1, \ldots, \bar{y}_n)$ has the *n-Independence Property, IP_n* (in T) if there exist $(\bar{a}_i^1 {}^\frown \ldots {}^\frown \bar{a}_i^n : i \in \mathbb{N})$ and $(\bar{b}_I : I \subseteq \mathbb{N}^n)$ such that:

$$\vDash \phi(\bar{b}_I; \bar{a}_{i_1}^1, \ldots, \bar{a}_{i_n}^n) \text{ if, and only if, } (i_1, \ldots, i_n) \in I.$$

We say that T is *n-dependent* or *NIP_n* if no formula has IP_n in T.

Remark 4.2.13. It is easy to observe that NIP_1 corresponds precisely to NIP.

The following proposition is standard. I include a short proof in this thesis, because it will hopefully make some things clearer in Chapter 6.

Proposition 4.2.14. *Let T be a first-order theory. If T is* NIP_n *for some $n \in \mathbb{N}$ then T is* NIP_{n+1}.

Proof. Suppose that T has the $(n+1)$-independence property. Then, by definition, there is a formula $\phi(\bar{x}; \bar{y}_1, \ldots, \bar{y}_n, \bar{y}_{n+1})$ and sequences $(\bar{a}_i^1 {}^\frown \ldots {}^\frown \bar{a}_i^n {}^\frown \bar{a}_i^{n+1} : i \in \mathbb{N})$ and $(\bar{b}_I : I \subseteq \mathbb{N}^{n+1})$ such that:

$$\vDash \phi(\bar{b}_I; \bar{a}_{i_1}^1, \ldots, \bar{a}_{i_{n+1}}^{n+1}) \text{ if, and only if, } (i_1, \ldots, i_{n+1}) \in I.$$

For $i < n$ write $\bar{z}_i$ for $\bar{y}_i$ and write $\bar{z}_n$ for the concatenation $\bar{y}_n {}^\frown \bar{y}_{n+1}$. Let

$\psi(\bar{x}, \bar{z}_1, \ldots, \bar{z}_n)$ be $\phi(\bar{x}, \bar{y}_1, \ldots, \bar{y}_{n+1})$ and define a sequence of tuples $(\bar{c}_i^1 \frown \cdots \frown \bar{c}_i^n : i \in \mathbb{N})$ just as we did for the variables, by letting $\bar{c}_i^j$ be $\bar{a}_i^j$ for all $j < n$ and all $i \in \mathbb{N}$ and $\bar{c}_i^n$ be $\bar{a}_i^n \frown \bar{a}_0^{n+1}$ for all $i \in \mathbb{N}$. Define $(\bar{d}_I)_{I \subseteq \mathbb{N}^n}$ by $\bar{d}_I = \bar{b}_{I'}$, where

$$I' = \{(i_1, \ldots, i_n, 0) : (i_1, \ldots, i_n) \in I\},$$

for each $I \subseteq \mathbb{N}^n$. Then, by construction, we have that:

$$\begin{aligned} \vDash \psi(\bar{d}_I, \bar{c}_{i_1}^1, \ldots, \bar{c}_{i_n}^n) &\text{ if, and only if, } \vDash \phi(\bar{b}_{I'}, \bar{a}_{i_1}^1, \ldots, \bar{a}_{i_n}^n, \bar{a}_0^{n+1}) \\ &\text{ if, and only if, } (i_1, \ldots, i_n, 0) \in I' \\ &\text{ if, and only if, } (i_1, \ldots, i_n) \in I, \end{aligned}$$

as required. □

Moreover, for all $n \in \mathbb{N}$ the inclusion of NIP_n in NIP_{n+1} is strict, witnessed by the random $(n+1)$-hypergraph, which is NIP_{n+1}, but not NIP_n (e.g. the random graph has IP, but is NIP_2, etc). This is now well-known, and follows essentially from [CPT19, Proposition 6.5]. Later in this chapter, a different viewpoint on this fact will be given.

In the next subsections, I shall also recall the definitions of various classes of theories contained in NIP and NIP_n.

Higher arity Generalisations of Distality

Distality was briefly introduced in Section 2.3. Akin to NIP_n, there is a more recent higher arity generalisation of distality, called *n-distality*, introduced by Walker in [Wal23]. To give his definition, I will need to introduce some terminology first.

Let $\mathfrak{I}$ be an (order-)indiscernible sequence $(\bar{b}_i : i \in \mathcal{I})$ indexed by an infinite linear order $\mathcal{I} = (I, <)$. Suppose $\mathfrak{I}_0 + \cdots + \mathfrak{I}_n$ is a partition of $\mathfrak{I}$ corresponding to a Dedekind partition[12] $I_0 + \cdots + I_n$ of I. Let A be a sequence $(\bar{a}_0, \ldots, \bar{a}_{n-1})$, where $|\bar{b}_i| = |\bar{a}_j|$ for all $i \in I$ and $j < n$.

We say that A *inserts (indiscernibly) into* $\mathfrak{I}_0 + \cdots + \mathfrak{I}_n$ if the sequence remains

[12] That is, a partition into *Dedekind cuts*, meaning that each of the pieces of the partition is non-empty and has no maximal or minimal element.

indiscernible after inserting each $\bar{a}_i$ at the i^{th} cut, i.e. the sequence:

$$\mathfrak{I}_0{}^\frown(\bar{a}_0)^\frown\mathfrak{I}_1{}^\frown(\bar{a}_1)^\frown\cdots{}^\frown\mathfrak{I}_{n-1}{}^\frown(\bar{a}_{n-1})^\frown\mathfrak{I}_n$$

is indiscernible. Moreover, for any $A' \subseteq A$, we say that A' *inserts (indiscernibly) into* $\mathfrak{I}_0 + \cdots + \mathfrak{I}_n$ if the sequence remains indiscernible after inserting each $\bar{a}_i \in A'$ at the i^{th} cut. For simplicity, we may say that A *(or A') inserts into* $\mathfrak{I}$ when the partition of $\mathfrak{I}$ under consideration is clear.

Definition 4.2.15 ([Wal23, Definitions 3.5 and 3.13])**.** For $m > 0$ and an indiscernible sequence $\mathfrak{I}$, we say that the Dedekind partition $\mathfrak{I}_0 + \cdots + \mathfrak{I}_{m+1}$ is *m-distal* if every sequence $A = (\bar{a}_0, \ldots, \bar{a}_m)$ which does not insert into $\mathfrak{I}$ contains some m-element subsequence which does not insert into $\mathfrak{I}$. A theory is *m-distal* if all Dedekind partitions of indiscernible sequences in the monster model are m-distal.

As discussed in [Wal23], a theory is 1-distal if, and only if, it is distal, in fact, this is essentially the original definition of distality, from [Sim13, Section 2.1] and n-distality is a reasonable generalisation of distality (assuming NIP$_n$ is a reasonable generalisation of NIP), as every n-distal theory is NIP$_n$.

Monadic NIP

Another, differently flavoured, strengthening (i.e subclass) of NIP comes from the notion of *monadic NIP*. Monadic NIP was introduced by Baldwin and Shelah in [BS85], but the study of monadically NIP theories has seen a resurgence in recent years, both from the point of view of pure model theory (see, for instance, [BL21]), and from the point of view of theoretical computer science (see, for instance, [Bon+22a]).

Definition 4.2.16 (Monadic NIP)**.** We say that T is *monadically NIP* if, for every $\mathcal{M} \models T$ and every language expansion $\mathcal{M}'$ of $\mathcal{M}$ by unary predicates, we have that $\mathcal{M}'$ is NIP.

It is not immediately clear from the definition above that if $\mathcal{M}$ is such that every language expansion of $\mathcal{M}$ by unary predicates is NIP, then every model of $\mathsf{Th}(\mathcal{M})$ is monadically NIP. This follows from Fact 4.2.19, below. Before stating that theorem, some further model-theoretic machinery needs to be introduced. The following definition is due to Braunfeld and Laskowski, [BL21]:

Definition 4.2.17 (Indiscernible triviality)**.** We say that T has *indiscernible triviality* for any (order-)indiscernible sequence $(\bar{a}_i : i \in \mathbb{N})$, and every set B of parameters, if $(\bar{a}_i : i \in \mathbb{N})$ is indiscernible over each $b \in B$ then $(\bar{a}_i : i \in \mathbb{N})$ is indiscernible over B.

The following definition originating from [She14] is a strengthening of NIP:

Definition 4.2.18 (dp-rank, dp-minimality)**.** Fix $n \in \omega$. An *ICT-pattern of depth* n consists of a sequence of formulas $(\phi_i(\bar{x}, \bar{y}) : i \leq n)$ and an array of parameters $(\bar{a}_i^j : i \in \omega, j \leq n)$ such that for all $\eta : [n] \to \omega$ we have that:

$$\left\{\phi_j\left(\bar{x}, \bar{a}^j_{\eta(j)}\right) : j \leq n\right\} \cup \left\{\neg\phi_j\left(\bar{x}, \bar{a}^j_i\right) : j \leq n, i \neq \eta(j)\right\}$$

is consistent. We say that T has *dp-rank* n if there is an ICT-pattern of depth n, but no ICT pattern of depth $n+1$ (in the monster model of T). We say that T is *dp-minimal* if it has dp-rank 1.

These notions have been intensively studied see e.g. [CS19], [KOU13]. The reader will find in the literature various alternative ways of defining dp-minimality. For instance, in [OU11], it is shown that T is dp-minimal if and only if it is *inp-minimal* and NIP.

The following is one of the central theorems of [BL21], which includes a "Shelah-style forbidden-configuration" characterisation of monadic NIP.

Fact 4.2.19 ([BL21, Theorem 4.1])**.** *Let T be a first-order theory. Then, the following are equivalent:*

(1) T is monadically NIP.

(2) For all $\mathcal{M} \vDash T$, we cannot find an $\mathcal{L}$-formula $\phi(\bar{x}, \bar{y}, z)$, sequences of tuples $(\bar{a}_i : i \in \omega), (\bar{b}_j : j \in \omega)$, and a sequence of singletons $(c_{k,l} : k, l \in \omega)$, from $\mathcal{M}$ such that:

$$\mathcal{M} \vDash \phi(\bar{a}_i, \bar{b}_j, c_{k,l}) \text{ if, and only if, } (i,j) = (k,l).$$

(3) T is dp-minimal and has indiscernible triviality.

Here is a representation of the forbidden configuration from Fact 4.2.19(2):

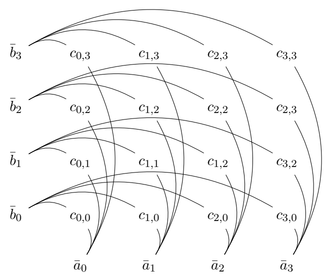


Figure 4.1: The Monadic NIP Forbidden Configuration

Suppose that $\mathcal{M}$ is a sufficiently saturated structure with IP. Then it is a well-known fact that we can find a formula $\phi(x, \bar{y})$, where $|x| = 1$, that witnesses IP in $\mathcal{M}$. If we are allowed to use unary predicates (or to add parameters [Sim21]), then the following result, from [BS85, Lemma 8.1.3] shows that we can achieve more:[13]

Fact 4.2.20. *If $\mathcal{M}$ is a sufficiently saturated structure with IP, then there is a monadic expansion $\mathcal{N}$ of $\mathcal{M}$ and a formula $\phi(x, y)$ with $|x| = |y| = 1$ which witnesses that $\mathcal{N}$ has IP.*

In particular, the following corollary follows immediately from the fact above and the fact that monadic expansions of monadic expansions are monadic expansions:

Corollary 4.2.21. *If T is not monadically NIP, then there is a monadic expansion of some $\mathcal{M} \vDash T$ which has IP witnessed by a formula $\phi(x, y)$ with $|x| = |y| = 1$.*

Example 4.2.22. I'll close this section by revisiting Example 2.3.13, to give some examples that one can keep in mind when discussing monadically NIP structures.

- Linear orders are monadically NIP.
- Meet-trees $(T, \leq \wedge)$ are monadically NIP. This follows from [Sim11, Proposition 4.7], where it is shown that coloured meet-trees (that is, monadic expansion of meet-trees) are dp-minimal, so in particular NIP.

The careful reader may wonder why at this point I'm not defining a higher arity generalisation of monadic NIP (say *monadic* NIP_n). This will be explained in Chapter 6.

[13] In Chapter 6 this result will be generalised to NIP_n theories.

4.3 Indiscernible Collapses and Coding Configurations

4.3.1 Indiscernibles Collapsing

In Section 2.6, I introduced generalised indiscernibles, and in the introduction to this chapter I promised to discuss how we can use them to construct new dividing lines.

I will follow the approach of Guingona, Hill, and Scow [GHS17]. Morally, the starting point is Theorem 2.6.3, which recall stated that if T is a stable theory, then a given sequence is order-indiscernible if, and only if, it is totally indiscernible, and this equivalence, in fact, characterises stability. As I will discuss in detail later in this chapter, this sort of phenomenon, which is made precise in the following definitions, can be used as an alternative definition for several dividing lines.

Lemma 4.3.1. *Let $\mathcal{I}$ and $\mathcal{I}'$ be two structures on the same domain, and assume that $\mathcal{I}'$ is a quantifier-free reduct of $\mathcal{I}$. Then, any $\mathcal{I}'$-indiscernible sequence is an $\mathcal{I}$-indiscernible sequence.*

Proof. Indeed, let I be the domain of $\mathcal{I}$ and $\mathcal{I}'$ and let $\mathfrak{I} = (\bar{a}_i : i \in I)$ be a $\mathcal{I}'$-indiscernible sequence. If $\mathsf{qftp}_{\mathcal{I}}(i_1, \ldots, i_n) = \mathsf{qftp}_{\mathcal{I}}(j_1, \ldots, j_n)$, then since $\mathcal{J}$ is a quantifier-free reduct of $\mathcal{I}$ we have that $\mathsf{qftp}_{\mathcal{J}}(i_1, \ldots, i_n) = \mathsf{qftp}_{\mathcal{J}}(j_1, \ldots, j_n)$, and since $\mathfrak{I}$ is $\mathcal{J}$-indiscernible it follows that $\mathsf{tp}(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n}) = \mathsf{tp}(\bar{a}_{j_1}, \ldots, \bar{a}_{j_n})$. Thus $\mathfrak{I}$ is $\mathcal{I}$-indiscernible. □

Of course, the converse need not always hold, and when it does hold, it is usually the conclusion of a strong structural result. Thus, in [MPT23] we turn it into a definition.

Definition 4.3.2. Let $\mathcal{I}$ and $\mathcal{I}'$ be two structures such that $\mathcal{I}'$ is a reduct of $\mathcal{I}$. We say that a theory T *collapses $\mathcal{I}$-indiscernibles to $\mathcal{I}'$-indiscernibles* if every $\mathcal{I}$-indiscernible sequence in the monster model is an $\mathcal{I}'$-indiscernible sequence.

We say that T *collapses $\mathcal{I}$-indiscernibles* if it collapses any $\mathcal{I}$-indiscernible sequence to an $\mathcal{I}'$-indiscernible sequence, where $\mathcal{I}'$ is a strict[14] quantifier-free reduct of $\mathcal{I}$.

[14]A *strict* quantifier-free reduct of $\mathcal{I}$ is any reduct of $\mathcal{I}$ which is not quantifier-free interdefinable with $\mathcal{I}$. It is allowed to be a pure set.

In [GH19], a slightly different definition trying to capture the same concept was given. More precisely:

Definition 4.3.3 (Non-collapsing indiscernibles [GH19, Definition 3.2])**.** Let $\mathcal{I}$ be a structure and let T be a theory. A sequence $(a_i : i \in \mathcal{I})$ of elements in a model $\mathcal{M}$ of T is a *non-collapsing indiscernible* if

$$\mathsf{qftp}_{\mathcal{I}}(i_1, \ldots, i_n) = \mathsf{qftp}_{\mathcal{I}}(j_1, \ldots, j_n) \text{ if, and only if, } \mathsf{tp}(a_{i_1}, \ldots, a_{i_n}) = \mathsf{tp}(a_{j_1}, \ldots, a_{j_n}),$$

for all $i_1, \ldots, i_n, j_1, \ldots, j_n \in \mathcal{I}$.

The definition above was given under the assumption $\mathcal{I}$ is a homogeneous structure in a finite relational language. The following lemma shows that, in this context (in particular, when $\mathcal{I}$ is $\aleph_0$-categorical), the two definitions coincide. Example 4.3.6 shows that Lemma 4.3.4 fails in general.

Lemma 4.3.4. *Let $\mathcal{I}$ be a homogeneous $\aleph_0$-categorical structure in a countable language, let $\mathcal{M}$ be some structure and $(\bar{a}_i : i \in \mathcal{I})$ be an $\mathcal{I}$-indiscernible sequence in $\mathcal{M}$. Then the following are equivalent:*

(1) $(\bar{a}_i : i \in \mathcal{I})$ is non-collapsing, according to Definition 4.3.3.

(2) For all reducts $\mathcal{I}'$ of $\mathcal{I}$ such that $(\bar{a}_i : i \in I)$ is $\mathcal{I}'$-indiscernible, we have that $\mathcal{I}$ and $\mathcal{I}'$ are quantifier-free interdefinable.

Proof. Let $\mathcal{L}_{\mathcal{I}}, \mathcal{L}_{\mathcal{M}}$ be the respective languages of $\mathcal{I}, \mathcal{M}$. Observe that since $\mathcal{I}$ is $\aleph_0$-categorical, every reduct of $\mathcal{I}$ must be $\aleph_0$-categorical too.

First, we prove the implication (1) $\implies$ (2): Let $\mathcal{I}'$ be a reduct of $\mathcal{I}$ and assume that $(a_i : i \in I)$ is $\mathcal{I}'$-indiscernible. Since $\mathcal{I}$ is $\aleph_0$-categorical and homogeneous, it has quantifier-elimination. Thus, it suffices to show that, for every $\mathcal{L}_{\mathcal{I}}$-formula $\phi(x_1, \ldots, x_n)$ there exists a quantifier-free $\mathcal{L}_{\mathcal{I}'}$-formula $\psi(x_1, \ldots, x_n)$ such that for all $i_1, \ldots, i_n \in \mathcal{I}$ we have that:

$$\mathcal{I} \vDash \phi(i_1, \ldots, i_n) \text{ if, and only if, } \mathcal{I}' \vDash \psi(i_1, \ldots, i_n).$$

By $\mathcal{I}'$-indiscernibility, Definition 4.3.3, and quantifier elimination in $\mathcal{I}$ we have that:

$$\begin{aligned} \mathsf{qftp}_{\mathcal{I}'}(i_1, \ldots, i_n) = \mathsf{qftp}_{\mathcal{I}'}(j_1, \ldots, j_n) & \\ \implies \mathsf{tp}(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n}) = \mathsf{tp}(\bar{a}_{j_1}, \ldots, \bar{a}_{j_n}) & \\ \implies \mathsf{qftp}_{\mathcal{I}}(i_1, \ldots, i_n) = \mathsf{qftp}_{\mathcal{I}}(j_1, \ldots, j_n) & \end{aligned} \tag{4.1}$$

for all $i_1, \ldots, i_n, j_1, \ldots, j_n \in I$. That is, equality of quantifier-free types in $\mathcal{I}'$ implies equality of (quantifier-free) types in $\mathcal{I}$.

Of course, since $\mathcal{I}$ has quantifier-elimination and $\mathcal{I}'$ is a (quantifier-free) reduct of $\mathcal{I}$ we can deduce that:

$$\begin{aligned} \mathsf{qftp}_{\mathcal{I}'}(i_1, \ldots, i_n) = \mathsf{qftp}_{\mathcal{I}'}(j_1, \ldots, j_n) \iff & \\ \mathsf{tp}_{\mathcal{I}}(i_1, \ldots, i_n) = \mathsf{tp}_{\mathcal{I}}(j_1, \ldots, j_n) & \end{aligned} \tag{4.2}$$

Given $\phi(x_1, \ldots, x_n)$, by the Ryll-Nardzewski theorem, there are finitely many complete n-types $p_1, \ldots, p_k \in S_n^{\mathcal{I}}(\emptyset)$ such that:

$$\mathcal{I} \vDash \phi(\bar{x}) \leftrightarrow \bigvee_{i=1}^{k} p_i(\bar{x}). \tag{4.3}$$

By $\aleph_0$-categoricity of $\mathcal{I}'$, there are only finitely many complete quantifier-free types. Thus, by Equation (4.1) for each p_i, there are complete quantifier-free types $q_{i,1}, \ldots, q_{i,l_i}$ in $\mathcal{I}'$ such that:

$$\mathcal{I}' \vDash \bigvee_{j=1}^{l_i} q_{i,j}(i_1, \ldots, i_n) \iff \mathcal{I} \vDash p_i(i_1, \ldots, i_n),$$

for all $i_1, \ldots, i_n \in \mathcal{I}$. In fact, by Equation (4.2) we can deduce that $l_i = 1$, for each $p \in \S_n^{\mathcal{I}}(\emptyset)$. At this point we are essentially done, since, by Equation (4.3) there are finitely many quantifier-free types $q_1, \ldots, q_m$ such that

$$\mathcal{I}' \vDash \bigvee_{j=1}^{m} q_j(i_1, \ldots, i_n) \iff \mathcal{I} \vDash \phi(i_1, \ldots, i_n) \tag{4.4}$$

for all $i_1, \ldots, i_n \in I$, and since $\mathcal{I}'$ is $\aleph_0$-categorical, we may replace each q_j in (4.4) by an isolating formula, so ϕ is $\emptyset$-quantifier-free-definable in $\mathcal{I}'$, and the result follows.

Now for the implication (2) $\implies$ (1): We need to show that given any $\mathcal{I}$-indiscernible sequence $(a_i : i \in \mathcal{I})$ we have that:

$$\mathsf{qftp}(i_1, \ldots, i_n) = \mathsf{qftp}(j_1, \ldots, j_n) \iff \mathsf{tp}(a_{i_1}, \ldots, a_{i_n}) = \mathsf{tp}(a_{j_1}, \ldots, a_{j_n}),$$

for all $i_1, \ldots, i_n, j_1, \ldots, j_n \in \mathcal{I}$.

To this end, we start by constructing a reduct $\mathcal{I}'$ of $\mathcal{I}$. Let $\mathbf{P}$ be the set of all complete n-types p in $\mathcal{M}$, realised by an n-tuple from $(a_i : i \in \mathcal{I})$ (for all $n \in \mathbb{N}$). Let $\mathcal{L}_{\mathcal{I}'}$ be the language consisting of an n-ary relation symbol R_p for every n-type $p \in \mathbf{P}$.

We interpret these relations in $\mathcal{I}'$ in the natural way, that is, for every n-type $p \in \mathbf{P}$ and every $i_1, \ldots, i_n \in \mathcal{I}$ we set

$$\mathcal{I}' \vDash R_p(i_1, \ldots, i_n) :\iff (a_{i_1}, \ldots, a_{i_n}) \vDash p$$

By construction, $(a_i : i \in I)$ is $\mathcal{I}'$-indiscernible, and in fact, by construction we have that:

$$\begin{aligned} \mathsf{qftp}_{\mathcal{I}'}(i_1, \ldots, i_n) = \mathsf{qftp}_{\mathcal{I}'}(j_1, \ldots, j_n) \iff \\ \mathsf{tp}(a_{i_1}, \ldots, a_{i_n}) = \mathsf{tp}(a_{j_1}, \ldots, a_{j_n}). \end{aligned} \tag{4.5}$$

Notice, now, that since $(a_i : i \in \mathcal{I})$ is $\mathcal{I}$-indiscernibile, by definition, (and quantifier elimination in $\mathcal{I}$) we have that equality of types in $\mathcal{I}$ implies equality of types in $\mathbf{P}$. In particular, since, by $\aleph_0$-categoricity of $\mathcal{I}$, there are only finitely many n-types in $\mathcal{I}$, for each $n \in \mathbb{N}$ (all of which are isolated), for every $p \in \mathbf{P}$ there are $\mathcal{L}_{\mathcal{I}}$-formulas $\psi_{p,1}, \ldots, \psi_{p,k(\phi)}$ such that:

$$\begin{aligned} \mathcal{I} \vDash \bigvee_{j=1}^{k(\phi)} \psi_{\phi,j}(i_1, \ldots, i_n) &\iff (a_{i_1}, \ldots, a_{i_n}) \vDash p \\ &\iff \mathcal{I}' \vDash R_p(i_1, \ldots, i_n). \end{aligned}$$

So $\mathcal{I}'$ is a reduct of $\mathcal{I}$. Therefore, by Item (2), $\mathcal{I}$ and $\mathcal{I}'$ are quantifier-free interdefinable.

Finally, let $i_1, \ldots, i_n, j_1, \ldots, j_n \in I$ be such that:

$$\mathsf{tp}(a_{i_1}, \ldots, a_{i_n}) = \mathsf{tp}(a_{j_1}, \ldots, a_{j_n}).$$

Then, by (4.5) we have that:

$$\mathsf{qftp}_{\mathcal{I}'}(i_1, \ldots, i_n) = \mathsf{qftp}_{\mathcal{I}'}(j_1, \ldots, j_n).$$

Since $\mathcal{I}$ and $\mathcal{I}'$ are quantifier-free interdefinable, this implies that:

$$\mathsf{qftp}_{\mathcal{I}}(i_1, \ldots, i_n) = \mathsf{qftp}_{\mathcal{I}}(j_1, \ldots, j_n),$$

as required. □

The following example shows that, in general, a *collapsing* $\mathcal{I}$-indiscernible sequence (in the sense of Definition 4.3.3) may collapse (in the sense of Definition 4.3.2) to an $\mathcal{I}'$-indiscernible sequence where $\mathcal{I}'$ is a strict quantifier-free reduct of $\mathcal{I}$.

Example 4.3.5. Let $\mathcal{B}^R = (V, E, R)$ be the countable random bipartite graph $\mathcal{B} = (V, E)$ equipped with a two-class equivalence relation R, for the partition. We may index $\mathcal{B}$ by itself, with the indexing structure viewed as a $\mathcal{B}^R$-structure, $(v : v \in \mathcal{B}^R)$. Clearly (since we are just indexing $\mathcal{B}$ by itself), this sequence is $\mathcal{B}^R$-indiscernible, but it is not uncollapsing, in the sense of Definition 4.3.3, since equality of (quantifier-free) types in $\mathcal{B}$ does not imply equality of (quantifier-free) types in $\mathcal{B}^R$ (e.g. any two vertices which are not joined have the same type in $\mathcal{B}$, but there are two different types of vertices which are not joined in $\mathcal{B}^R$).

Thus, in the sense of Definition 4.3.2, the sequence collapses to a $\mathcal{B}$-indiscernible sequence. One can then see that it cannot collapse any further, that is, $(g : g \in \mathcal{B})$ is an uncollapsing $\mathcal{B}$-indiscernible sequence.

The following example shows that Lemma 4.3.4 does not hold in general:

Example 4.3.6. Let $\mathcal{N}$ be the natural structure on $\mathbb{N}$ in the *full set-theoretic language* $\mathcal{L}_F$, i.e. the language which for every $A \subseteq \mathbb{N}^n$, contains an n-ary relation symbol $R_A \in \mathsf{Sig}(\mathcal{L}_F)$, and all $A \subseteq \mathbb{N}^n$ we naturally interpret:

$$R_A^{\mathcal{N}}(a_1, \ldots, a_n) \iff (a_1, \ldots, a_n) \in A.$$

Observe that $\mathcal{N}$ is $\aleph_0$-categorical[15] (though Theorem 2.2.1 cannot be applied here, since $\mathcal{L}_F$ is uncountable) and homogeneous. Let $\mathcal{N}_S$ be the (language) reduct of $\mathcal{N}$ to the language $\mathcal{L}_S \subset \mathcal{L}_F$ consisting only of unary predicates for the singleton sets, i.e. $\mathcal{L}_S := \{R_{\{a\}} : a \in \mathbb{N}\}$.

Then $\mathcal{N}$ codes any countable structure $\mathcal{M}$, and is, thus, in particular unstable. On the other hand, $\mathcal{N}_S$ is strongly minimal, and in particular, stable. Thus clearly, $\mathcal{N}$ is not a reduct of $\mathcal{N}_S$. To see that the equivalence in Lemma 4.3.4 fails, observe that:

1. The sequence $(i : i \in \mathcal{N})$ is a non-collapsing $\mathcal{N}$-indiscernible sequence in $\mathcal{N}$, in the sense of Definition 4.3.3.
2. $\mathcal{N}$ collapses $\mathcal{N}$-indiscernibles to $\mathcal{N}_S$ indiscernibles according to Definition 4.3.2. (In fact, every $\mathcal{N}$-indiscernible sequence in any model of any theory is also an $\mathcal{N}_S$-indiscernible sequence.)

Let's briefly remind ourselves why all of this is happening: An analogue of Shelah's theorem characterising stable theories as those that collapse order-indiscernible sequences to indiscernible sets was proved by Scow for NIP in [Sco12], and her result was generalised in [CPT19] to NIP$_n$, for $n > 1$.

Fact 4.3.7 ([CPT19, Theorem 5.4])**.** *Let $\mathcal{C}_{\mathsf{OH}_{n+1}}$ be the ordered random $(n+1)$-hypergraph. Then, the following are equivalent, for a first-order theory T:*

1. *T is NIP$_n$.*
2. *T collapses $\mathcal{C}_{\mathsf{OH}_{n+1}}$-indiscernibles into order-indiscernibles.*

4.3.2 $\mathbb{K}$-Configurations

One of the goals of this chapter is to develop the study of transfer principles for forbidding $\mathbb{K}$-*configurations*. Forbidding $\mathbb{K}$-configurations offers a unified way of describing tameness conditions that arise from coding combinatorial configurations, such as the *Order Property* and the *Independence Property*, and I hope I can convince the reader that this notion provides an interesting schema for defining new dividing lines.

[15]This is not immediately obvious, but the point is that in $\mathcal{N}$ we have uncountably (in fact, continuum) many infinite subsets X_i, any two of which have finite intersection. In particular, in any model with a non-standard element a each of these sets would have a non-standard element x_i greater than a (note that $<$ is definable in $\mathcal{N}$), and since the intersection of any two distinct X_i's is finite, all these new elements must be distinct.

The notion of $\mathbb{K}$-configurations originates from the work of Guingona and Hill [GH19], and was developed further in [GPS21]. In this section, fix a relational first-order language $\mathcal{L}_0$, and an arbitrary first-order language $\mathcal{L}$. Throughout, unless otherwise stated, $\mathbb{K}$ will denote an isomorphism-closed class of finite $\mathcal{L}_0$-structures.

Definition 4.3.8 ($\mathbb{K}$-configuration, [GPS21, Definition 5.1])**.** An $\mathcal{L}$-structure $\mathcal{M}$ *admits (or codes) a* $\mathbb{K}$*- configuration* if there exist an integer n, a function $I : \mathsf{Sig}(\mathcal{L}_0) \to \mathcal{L}$ and a sequence of functions $(f_A : A \in \mathbb{K})$ such that:

1. for all $A \in \mathbb{K}$, $f_A : A \to \mathcal{M}^n$,
2. for all $R \in \mathsf{Sig}(\mathcal{L}_0)$, for all $A \in \mathbb{K}$, for all $a \in A^{\mathsf{ar}(R)}$,

$$A \vDash R(a) \Leftrightarrow \mathcal{M} \vDash I(R)(f_A(a)).$$

A theory T admits a $\mathbb{K}$-configuration if some model $\mathcal{M} \vDash T$ admits a $\mathbb{K}$-configuration. We denote by $\mathscr{C}_{\mathbb{K}}$ the class of theories which admit a $\mathbb{K}$-configuration, and by $\mathsf{N}\mathscr{C}_{\mathbb{K}}$ the class of theories which *do not* admit a $\mathbb{K}$-configuration.

Remark 4.3.9. In the definition above, parameters do not appear at any point. This is for good reason. Recall the classical fact from stability theory that if a formula $\phi(\bar{x}, \bar{y}; \bar{c})$, for some constants $\bar{c}$, has the order property in $\mathbb{M} \vDash T$, say witnessed by $(\bar{a}_i : i \in \mathbb{N}), (\bar{b}_i : i \in \mathbb{N})$, then there is a parameter-free formula ψ which has the order property in T – of course ψ is just $\phi(\bar{x}, \bar{y}^\frown \bar{z})$ and the order property is witnessed by $(\bar{a}_i : i \in \mathbb{N}), (\bar{b}_i^\frown \bar{c} : i \in \mathbb{N})$.

This is a general feature of coding configurations, in particular, in particular if some $\mathcal{L}$-structure $\mathcal{M}$ admits a $\mathbb{K}$-configuration viewed as an $\mathcal{L}(M)$-structure (i.e. if $\mathcal{M}$ admits a $\mathbb{K}$-configuration *with parameters*) then, $\mathcal{M}$ admits $\mathbb{K}$-configuration as an $\mathcal{L}$-structure (i.e. without parameters, so, in the sense of the definition above).

Example 4.3.10. Recall that $\mathcal{C}_{\mathsf{LO}}$ and $\mathcal{C}_{\mathsf{CO}}$ denote respectively the class of finite linear orders and finite cyclic orders, $\mathcal{C}_{\mathsf{COD}}$ denotes the class of finite cyclically ordered D-relations and $\mathcal{C}_{\mathsf{OC}}$ the class of (convexly) ordered finite binary branching C-relations. One can observe that any structure admits a $\mathcal{C}_{\mathsf{CO}}$-configuration if, and only if, it admits a $\mathcal{C}_{\mathsf{LO}}$-configuration. In particular $\mathsf{N}\mathscr{C}_{\mathcal{C}_{\mathsf{CO}}} = \mathsf{N}\mathscr{C}_{\mathcal{C}_{\mathsf{LO}}}$. This is, of course, because after naming one constant a dense cyclic order is interdefinable with a standard linear order (see previous remark about naming constants). Similarly, any structure admits an $\mathcal{C}_{\mathsf{COD}}$-configuration if, and only if, it admits $\mathcal{C}_{\mathsf{OC}}$-configuration. Therefore $\mathsf{N}\mathscr{C}_{\mathcal{C}_{\mathsf{COD}}} = \mathsf{N}\mathscr{C}_{\mathcal{C}_{\mathsf{OC}}}$.

The following proposition is rather useful:

Proposition 4.3.11. *Let $\mathbb{K}$ be a class of (not necessarily finite) structures. Then*

$$\mathscr{C}_{\mathbb{K}} = \mathscr{C}_{\mathsf{Age}(\mathbb{K})} = \mathscr{C}_{\mathsf{HC}(\mathbb{K})},$$

where $\mathsf{Age}(\mathbb{K})$ denotes the class of all finitely generated substructures that embed in some structure in $\mathbb{K}$, and $\mathsf{HC}(\mathbb{K})$ denotes the hereditary closure *of $\mathbb{K}$.*

In particular, for a structure $\mathcal{M}$ we have that $\mathscr{C}_{\mathcal{M}} = \mathscr{C}_{\mathsf{Age}(\mathcal{M})}$, where $\mathscr{C}_{\mathcal{M}} := \mathscr{C}_{\{\mathcal{M}\}}$.

Proof. Since $\mathsf{Age}(\mathbb{K}) = \mathsf{Age}(\mathsf{HC}(\mathbb{K}))$, it suffices to show the first equality (as we may replace $\mathbb{K}$ with $\mathsf{HC}(\mathbb{K})$ to obtain the second).

For the inclusion $\mathscr{C}_{\mathbb{K}} \subseteq \mathscr{C}_{\mathsf{Age}(\mathbb{K})}$, suppose that $T \in \mathscr{C}_{\mathbb{K}}$, witnessed by I and $(f_{\mathcal{M}})_{\mathcal{M}\in\mathbb{K}}$. For every $A \in \mathsf{Age}(\mathbb{K})$ fix some $\mathcal{M}_A \in \mathbb{K}$ and embedding $e_A : A \hookrightarrow \mathcal{M}_A$; then letting $f_A := f_{\mathcal{M}} \circ e_A$, we get an $\mathsf{Age}(\mathcal{M})$-configuration.

For the inclusion $\mathscr{C}_{\mathsf{Age}(\mathbb{K})} \subseteq \mathscr{C}_{\mathbb{K}}$, suppose that $T \in \mathscr{C}_{\mathsf{Age}(\mathbb{K})}$. Fix I and $(f_A : A \in \mathsf{Age}(\mathbb{K}))$ as promised; then, for all $\mathcal{M} \in \mathbb{K}$, the type in $|\mathcal{M}|$-variables

$$\begin{aligned} T \cup \{ \ \ I(R)(x_{a_1}, \dots, x_{a_n}) : R \in \mathcal{L}(\mathbb{K}),\ \mathcal{M} \vDash \ \ R(a_1, \dots, a_n)\} \\ \cup \{\neg I(R)(x_{a_1}, \dots, x_{a_n}) : R \in \mathcal{L}(\mathbb{K}),\ \mathcal{M} \vDash \neg R(a_1, \dots, a_n)\} \end{aligned}$$

is finitely satisfiable, and therefore satisfiable by compactness. □

In light of Proposition 4.3.11 above, given a structure $\mathcal{M}$, we overload notation, and write $\mathscr{C}_{\mathcal{M}}$ for $\mathscr{C}_{\mathsf{Age}(\mathcal{M})} = \mathscr{C}_{\{\mathcal{M}\}}$, and similarly $\mathsf{N}\mathscr{C}_{\mathcal{M}}$ for $\mathsf{N}\mathscr{C}_{\mathsf{Age}(\mathcal{M})} = \mathsf{N}\mathscr{C}_{\{\mathcal{M}\}}$.

The following remark about coding quantifier-free formulas will be used in a couple of places later on in this chapter:

Remark 4.3.12. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and assume that $\mathcal{N}$ is an $\mathcal{L}'$-structure which admits an $\mathcal{M}$-configuration, witnessed by I, f. Let $\phi(\bar{x}) = \bigwedge_{i\in A} \bigvee_{j\in B} \psi_{i,j}(\bar{x})$ be a quantifier-free $\mathcal{L}$-formula, where each $\psi_{i,j}(\bar{x})$ is an atomic or negated atomic formula. In particular, for each $\psi_{i,j}(\bar{x})$, we can define $J(\psi_{i,j})$ to be $I(\psi_{i,j})$ if $\psi_{i,j}$ is a relation symbol, and $\neg I(\psi_{i,j})$ if $\psi_{i,j}$ is a negated relation symbol. Then, by definition:

$$\mathcal{M} \vDash \phi(a) \text{ if, and only if, } \mathcal{N} \vDash \bigwedge_{i \in A} \bigvee_{j \in B} J(\psi_{i,j})(f(a)),$$

for all $a \in \mathcal{M}$.

A somewhat different approach to coding $\mathbb{K}$-configurations is the focus of Walsberg in [Wal22]. There, the relevant notion is called *trace definability* and it is seen as a weak form of interpretability. I will not be discussing trace definability further in this thesis, except to say that from the point of view of trace definability, the following proposition is immediate, but if one works with (coding) $\mathbb{K}$-configurations, some proof is required.

Proposition 4.3.13. *Let $\mathcal{N}$ be an $\mathcal{L}_{\mathcal{N}}$-structure. If an $\mathcal{L}_{\mathcal{M}}$-structure $\mathcal{M}$ is interpretable in $\mathcal{N}$ and $\mathcal{M}$ admits a $\mathbb{K}$-configuration, then $\mathcal{N}$ also admits a $\mathbb{K}$-configuration. In particular, the class $\mathsf{N}\mathscr{C}_{\mathbb{K}}$ is closed under bi-interpretability.*

Proof. Assume that $\mathcal{N}$ interprets $\mathcal{M}$. There is a definable set $D \subseteq \mathcal{N}^m$, a definable equivalence relation $\sim$ on D and a bijection $\mathcal{M} \simeq D/\sim$ which interprets $\mathcal{M}$. This induces a map $* : \mathcal{L}_{\mathcal{M}} \to \mathcal{L}_{\mathcal{N}}$ which associates to any $\mathcal{L}_{\mathcal{M}}$-formula $\phi(x)$ an $\mathcal{L}_{\mathcal{N}}$-formula $\phi^*(x^*)$ interpreting it (where $|x^*| = m \times |x|$). Let $s : \mathcal{M} \to \mathcal{N}^m$ a section of the natural projection $\mathcal{D} \to \mathcal{D}/\sim$ (where we identify $\mathcal{M}$ with $D/\sim$).

If $\mathcal{M}$ admits a $\mathbb{K}$-configuration via:

$$\left(I : \mathsf{Sig}(\mathcal{L}_0) \to \mathcal{L}_{\mathcal{M}}, (f_A : A \to \mathcal{M}^n : A \in \mathbb{K})\right),$$

then $\mathcal{N}$ also admits a $\mathbb{K}$-configuration, via:

$$\left(* \circ I : \mathsf{Sig}(\mathcal{L}_0) \to \mathcal{L}_{\mathcal{N}}, (s \circ f_A : A \to \mathcal{M}^{nm} : A \in \mathbb{K})\right).$$

Indeed, by unravelling the definitions, we have that:

$$A \vDash R(a) \Leftrightarrow \mathcal{M} \vDash I(R)(f_A(a)) \Leftrightarrow \mathcal{N} \vDash I(R)^*(s(f_A(a))),$$

for all $R \in \mathsf{Sig}(\mathcal{L}_0)$ for all $A \in \mathbb{K}$ for all $a \in A^{\mathsf{arity}(R)}$. □

The following corollary is almost immediate:

Corollary 4.3.14. *The following classes cannot be of the form* $\mathsf{N}\mathscr{C}_{\mathbb{K}}$*, for any* $\mathbb{K}$*:*

1. *The class of dp-minimal structures.*
2. *The class of monadically NIP structures.*
3. *The class of distal structures.*

This follows from the fact the properties above are either not closed under bi-interpretability or under taking reducts.[16] For instance, an infinite set $\mathcal{S}$ in the language of pure equality is bi-interpretable with the full product $\mathcal{S} \boxtimes \mathcal{S}$ which has dp-rank 2. It is well-known that distality is not preserved under taking reducts. For example, $\langle \mathbb{Q}, < \rangle$ is distal, but its reduct to the language of pure equality is not.

A version of the following lemma appears in [GP23, Proposition 5.4(4)], under the assumption the classes are strong amalgamation classes, and $\mathcal{M}$ is a Fraïssé limit of such a class. In [MPT23], we show that these requirements are not needed and we provide a more elementary proof.

Lemma 4.3.15. *Let* $\mathcal{M}$ *be a structure with quantifier elimination and* $\mathbb{K}$ *a class of not necessarily finite structures. Then the following are equivalent:*

(1) $\mathcal{M} \in \mathscr{C}_{\mathbb{K}}$.

(2) $\mathscr{C}_{\mathcal{M}} \subseteq \mathscr{C}_{\mathbb{K}}$.

Proof. For (1) $\Rightarrow$ (2) suppose that $\mathcal{M} \in \mathscr{C}_{\mathbb{K}}$ and $\mathcal{N} \in \mathscr{C}_{\mathcal{M}}$. Let I, f witness that $\mathcal{M}$ admits a $\mathbb{K}$-configuration, and J, g witness that $\mathcal{N}$ admits an $\mathcal{M}$-configuration. By quantifier elimination, we may assume that $I(R)$ is quantifier-free for all $R \in \mathsf{Sig}(\mathbb{K})$. It is clear, using Remark 4.3.12, that $g \circ f$ and $J \circ I$ witness that $\mathcal{N}$ admits a $\mathbb{K}$-configuration. For (2) $\Rightarrow$ (1), observe that clearly $\mathcal{M} \in \mathscr{C}_{\mathcal{M}}$ and therefore, by assumption $\mathcal{M} \in \mathscr{C}_{\mathbb{K}}$. □

Fact 4.3.16 below was originally stated for classes of *finite* structures. However, notice that:

- $\mathsf{Age}(\mathbb{K}_1 \boxtimes \mathbb{K}_2) = \mathsf{Age}(\mathbb{K}_1) \boxtimes \mathsf{Age}(\mathbb{K}_2)$.
- $\mathsf{Age}(\mathbb{K}_1[\mathbb{K}_2]) = \mathsf{Age}(\mathbb{K}_1)[\mathsf{Age}(\mathbb{K}_2)]$.

[16] It is clear from the definition of $\mathsf{N}\mathscr{C}_{\mathbb{K}}$ that if $\mathcal{M} \in \mathsf{N}\mathscr{C}_{\mathbb{K}}$ and $\mathcal{N}$ is a reduct of $\mathcal{M}$ then $\mathcal{N} \in \mathsf{N}\mathscr{C}_{\mathbb{K}}$.

- $\mathsf{Age}(\mathbb{K}_1 \sqcup \mathbb{K}_2) = \mathsf{Age}(\mathbb{K}_1) \sqcup \mathsf{Age}(\mathbb{K}_2)$.

Therefore, by Proposition 4.3.11, we may phrase Fact 4.3.16 in the fullest generality:

Fact 4.3.16 ([GPS21, Theorem 4.27])**.** *Let $\mathbb{K}_1, \mathbb{K}_2$ be classes of structures in disjoint finite relational languages $\mathcal{L}_1, \mathcal{L}_2$, respectively. Then*

$$\mathscr{C}_{\mathbb{K}_1 \boxtimes \mathbb{K}_2} = \mathscr{C}_{\mathbb{K}_1[\mathbb{K}_2]} = \mathscr{C}_{\mathbb{K}_1 \sqcup \mathbb{K}_2} = \mathscr{C}_{\mathbb{K}_1} \cap \mathscr{C}_{\mathbb{K}_2}.$$

The fact above was originally phrased under a slightly weaker assumption that $\mathbb{K}_1$ and $\mathbb{K}_2$ both containing structures of size n, for every $n \in \mathbb{N}$. In [MPT23], we note that this requirement is only needed when considering free superposition (which I've tactfully avoided including here): For the full product and lexicographic product it is precisely [GPS21, Corollary 5.16] and [GPS21, Corollary 5.12], respectively. The fact above also did not include the disjoint union, however, this equality is almost trivial.

Theorem 4.3.17. *Let $\mathcal{N}$ be a structure and let $\mathcal{N}_1, \mathcal{N}_2$ be quantifier-free reducts of $\mathcal{N}$ such that $\mathcal{N} = \mathcal{N}_1 \vee \mathcal{N}_2$ in the lattice of quantifier-free reducts of $\mathcal{N}$, up to interdefinability (e.g. $\mathsf{Aut}(\mathcal{N}) = \mathsf{Aut}(\mathcal{N}_1) \cap \mathsf{Aut}(\mathcal{N}_2)$, when $\mathcal{N}$ is $\aleph_0$-categorical). Then $\mathscr{C}_{\mathcal{N}_1} \cap \mathscr{C}_{\mathcal{N}_2} = \mathscr{C}_{\mathcal{N}}$.*

Proof. The inclusion $\mathscr{C}_{\mathcal{N}} \subseteq \mathscr{C}_{\mathcal{N}_1} \cap \mathscr{C}_{\mathcal{N}_2}$ follows almost immediately, from the fact that $\mathcal{N}_i$ is a quantifier-free reduct of $\mathcal{N}$, for $i \in [2]$, and does not require the assumption that $\mathcal{N} = \mathcal{N}_1 \vee \mathcal{N}_2$. Suppose that $T \in \mathscr{C}_{\mathcal{N}}$ and let $\mathcal{M} \vDash T$ be a model admiting an $\mathcal{N}$-configuration, witnessed by I, f. Since $\mathcal{N}_1$ is a quantifier-free reduct of $\mathcal{N}$ for each $R \in \mathcal{L}_{\mathcal{N}_1}$, there is a quantifier-free $\mathcal{L}_{\mathcal{N}}$-formula ϕ_R such that $\mathcal{N} \vDash \phi_R(a)$ if, and only if, $\mathcal{N}_1 \vDash R(a)$, for all $a \in R$. Let $I(\phi_R)$ be as in Remark 4.3.12 and define

$$\begin{aligned} J : \mathsf{Sig}(\mathcal{L}_{\mathcal{N}_1}) &\to \mathcal{L}_{\mathcal{M}} \\ R &\mapsto I(\phi_R). \end{aligned}$$

It is clear that f, J witness that $\mathcal{M} \in \mathscr{C}_{\mathcal{N}_1}$. Similarly, $\mathcal{M} \in \mathscr{C}_{\mathcal{N}_2}$, and the result follows.

For the inclusion $\mathscr{C}_{\mathcal{N}_1} \cap \mathscr{C}_{\mathcal{N}_2} \subseteq \mathscr{C}_{\mathcal{N}}$, let $T \in \mathscr{C}_{\mathcal{N}_1} \cap \mathscr{C}_{\mathcal{N}_2}$. Observe that since $\mathcal{N} = \mathcal{N}_1 \vee \mathcal{N}_2$, there is a structure $\mathcal{N}'$ which is interdefinable with $\mathcal{N}$ such that $\mathcal{L}_{\mathcal{N}'} = \mathcal{L}_{\mathcal{N}_1} \cup \mathcal{L}_{\mathcal{N}_2}$. By Proposition 4.3.13 we may assume that $\mathcal{N} = \mathcal{N}'$. Let $\mathcal{M}_i \vDash T$ and let I_i, f_i be a coding of $\mathcal{N}_i$ in $\mathcal{M}_i$, for $i \in \{1, 2\}$. By taking a common elementary extension, we may assume $\mathcal{M}_1 = \mathcal{M}_2 = \mathcal{M}$. Then $f_i : N_i \to \mathcal{M}^{n_i}$ for some $n_1, n_2 \in \mathbb{N}$. Let $f : N \to \mathcal{M}^{n_1+n_2}$ be defined as $f_1 {}^\frown f_2$, i.e. $f(a) := f_1(a) {}^\frown f_2(a)$ for all $a \in \mathcal{N}$. Let π_i be the obvious projection of $\mathcal{M}^{n_1+n_2}$ onto $\mathcal{M}^{n_i}$, for $i \in \{1, 2\}$. Then, for every $i \in \{1, 2\}$ and every $R \in \mathcal{L}_i$, let $I(R) := I_i \circ \pi_i$. By construction, I, f is an $\mathcal{N}$-configuration in $\mathcal{M}$. □

Theorem 4.3.17 can be seen as a generalisation of Fact 4.3.16 in the context of the following remark:

Remark 4.3.18. Let $\mathbb{K}$ be any class of structures, and $\mathcal{C}_\mathsf{E}$ be the class of all finite sets. Then:

$$\mathscr{C}_\mathbb{K} = \mathscr{C}_{\mathbb{K} \boxtimes \mathcal{C}_\mathsf{E}} = \mathscr{C}_{\mathbb{K}[\mathcal{C}_\mathsf{E}]} = \mathscr{C}_{\mathcal{C}_\mathsf{E}[\mathbb{K}]} = \mathscr{C}_{\mathbb{K} \sqcup \mathcal{C}_\mathsf{E}}.$$

We can then deduce all the equalities in Fact 4.3.16 using that, up to quantifier-free interdefinability, we have the following, in the lattice of quantifier-free reducts:

- $\mathcal{M}[\mathcal{C}_\mathsf{E}] \vee \mathcal{C}_\mathsf{E}[\mathcal{N}] = \mathcal{M}[\mathcal{N}]$,
- $\mathcal{M} \boxtimes \mathcal{C}_\mathsf{E} \vee \mathcal{C}_\mathsf{E} \boxtimes \mathcal{N} = \mathcal{M} \boxtimes \mathcal{N}$,
- $\mathcal{M} \sqcup \mathcal{C}_\mathsf{E} \vee \mathcal{C}_\mathsf{E} \sqcup \mathcal{N} = \mathcal{M} \sqcup \mathcal{N}$.

4.4 Ramsey Theory Through Collapsing Indiscernibles

The main result of this section is Theorem 4.4.10, which roughly says that (ages of) "higher arity" reducts of $\aleph_0$-categorical homogeneous structures cannot have the Ramsey property. Surprisingly, the proof of the theorem is not combinatorial but uses exclusively model-theoretic tools, notably the notion of collapsing indiscernibles.

Let's start by recalling that, under some mild hypotheses, the notions of $\mathsf{N}\mathscr{C}_\mathbb{K}$ and theories collapsing generalised indiscernibles are closely related. More precisely:

Fact 4.4.1 ([GH19, Theorem 3.14, Corollary 3.15])**.** *Let $\mathcal{I}$ be the Fraïssé limit of a Ramsey class with the strong amalgamation property, in a finite relational language. Then the following are equivalent for a theory T:*

1. *$T \in \mathsf{N}\mathscr{C}_\mathcal{I}$.*

2. *T collapses $\mathcal{I}$-indiscernibles.*

Suppose moreover that $\mathcal{I}$ is unstable and $\mathcal{I}^<$ has the Ramsey property, where $\mathcal{I}^<$ is the free superposition[17] *of* $\mathsf{Age}(\mathcal{I})$ *and* $\mathcal{C}_{\mathsf{LO}}$. *Then the above are equivalent to the following:*

3. *T collapses $\mathcal{I}^<$-indiscernibles.*

The first equivalence above will be used later, in order to prove our transfer principles. For instance, as observed in [GH19, Example 3.16] the following is essentially a consequence of Fact 4.3.7 combined with Fact 4.4.1:

Fact 4.4.2. *Let $\mathcal{C}_{\mathsf{OH}_{n+1}}$ be the class of finite ordered $(n+1)$-hypergraphs. Then the following are equivalent for a first-order theory T:*

1. *T is NIP$_n$.*

2. *T is $\mathsf{N}\mathscr{C}_{\mathcal{C}_{\mathsf{OH}_{n+1}}}$.*

The path to Theorem 4.4.10 is through Fact 4.4.1, which will be used as a sort of criterion to witness the Ramsey property in a given class of structures. More precisely, if given a homogeneous Fraïssé limit $\mathcal{I}$, there is a theory T which codes $\mathcal{I}$ but collapses $\mathcal{I}$-indiscernibles, it must then be the case that $\mathcal{I}$ is not Ramsey.

First, I will give a slightly more general version of the equivalence between 1 and 2, from Fact 4.4.1. The underlying assumption in [GH19] is that all classes are in a *finite relational language*. In fact, the theorem may fail when infinite languages are involved. (See Example 4.4.4 below.)

Theorem 4.4.3. *Let $\mathcal{I}$ be an $\aleph_0$-categorical Fraïssé limit of a Ramsey class, in a finite relational language. Then the following are equivalent for a theory T:*

(1) $T \in \mathsf{N}\mathscr{C}_{\mathcal{I}}$.

(2) T collapses $\mathcal{I}$-indiscernibles.

Proof. To fix notation, say T is an $\mathcal{L}$-theory, and $\mathcal{I}$ is an $\mathcal{L}_{\mathcal{I}}$-structure. Observe that since $\mathcal{I}$ is, by assumption, an $\aleph_0$-categorical Fraïssé limit, by Lemma 4.3.4, the notion of collapsing $\mathcal{I}$-indiscernibles (in the sense of Definition 4.3.2) and the notion of not having non-collapsing $\mathcal{I}$-indiscernibles (in the sense of

[17]Recall: The *free superposition* of classes $\mathcal{C}_1$ and $\mathcal{C}_2$ in (disjoint) languages $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, is the class $\mathcal{C}_1 \star \mathcal{C}_2$ of all $\mathcal{L}_1 \sqcup \mathcal{L}_2$-structures whose language reduct to $\mathcal{L}_i$ is in $\mathcal{C}_i$, for $i = 1, 2$. When $\mathcal{C}_1$ and $\mathcal{C}_2$ have the strong amalgamation property their free superposition is again an amalgamation class (with SAP), see [Cam90, Example 3.9].

Definition 4.3.3) coincide, so they will be used interchangeably throughout the proof.

We start with the implication (1) $\implies$ (2): We argue by contraposition. Suppose that T does not collapse $\mathcal{I}$-indiscernibles. Then, by definition there is some $\mathcal{M} \vDash T$ and a non-collapsing $\mathcal{I}$-indiscernible $(\bar{a}_i : i \in \mathcal{I})$, in $\mathcal{M}$. Without loss of generality we may assume that $\mathcal{M}$ is Morleyised. We wish to show that $T \in \mathscr{C}_{\mathcal{I}}$, so we must find some $m \in \mathbb{N}$, a function $J : \mathsf{Sig}(\mathcal{L}_{\mathcal{I}}) \to \mathcal{L}$ and a function $f : I \to M^n$ such that:

$$\mathcal{I} \vDash R(a) \iff \mathcal{M} \vDash J(R)(f(a)),$$

for all $R \in \mathsf{Sig}(\mathcal{L}_{\mathcal{I}})$ and all $a \in I^{\mathsf{ar}(R)}$.

The idea is to code the configuration along the non-collapsing sequence $(\bar{a}_i : i \in \mathcal{I})$, that is, to use the function $J : i \mapsto \bar{a}_i$, for $i \in I$ (so $m = |\bar{a}_i|$).

To this end, let $R \in \mathcal{L}_{\mathcal{I}}$ be an n-ary relation symbol. Since $\mathcal{I}$ is $\aleph_0$-categorical and finitely homogeneous it has quantifier-elimination, so there are finitely many complete quantifier-free types $p_1, \dots, p_k \in S_n^{\mathcal{I}}(\emptyset)$, such that:

$$\mathcal{I} \vDash R(i_1, \dots, i_n) \iff \mathcal{I} \vDash \bigvee_{j=1}^{k} p_j(i_1, \dots, i_n) \tag{4.6}$$

for all $i_1, \dots, i_n \in \mathcal{I}$.

At this point, we have that:

$$\begin{aligned} \mathsf{qftp}(i_1, \dots, i_n) = \mathsf{qftp}(j_1, \dots, j_n) \iff \\ \mathsf{tp}(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}) = \mathsf{tp}(\bar{a}_{j_1}, \dots, \bar{a}_{j_n}) \end{aligned} \tag{4.7}$$

for all $i_1, \dots, i_n, j_1, \dots, j_n \in \mathcal{I}$.

The point is that for each quantifier-free type p_j in (4.6) there is a complete type $q_j \in S_n^{\mathcal{M}}(\emptyset)$ such that:

$$\mathcal{I} \vDash R(i_1, \dots, i_n) \iff (\bar{a}_{i_1}, \dots, \bar{a}_{i_n}) \vDash \bigvee_{j=1}^{k} \psi_{q_j}(\bar{x}_1, \dots, \bar{x}_n), \tag{4.8}$$

for all $i_1, \ldots, i_n \in \mathcal{I}$.

Let $\mathcal{M}_{\mathfrak{I}}$ be the structure $\mathcal{M}$ expanded by an m-ary relation symbol S, where $m = |\bar{a}_i|$, naming precisely the elements of the sequence $(\bar{a}_i : i \in \mathcal{I})$, and let $\mathcal{L}_{\mathfrak{I}} := \mathcal{L} \cup \{S\}$ We start with the following claim:

Claim 1. The sequence $(\bar{a}_i : i \in \mathcal{I})$ remains indiscernible in the structure $\mathcal{M}_{\mathfrak{I}}$. In particular, it is a non-collapsing indiscernible.

Proof of Claim 1. Suppose not. Then there are $i_1, \ldots, i_n, j_1, \ldots, j_n \in \mathcal{I}$ and an $\mathcal{L}_{\mathfrak{I}}$-formula $\phi(\bar{x}_1, \ldots, \bar{x}_n)$ such that:

$$\mathsf{qftp}(i_1, \ldots, i_n) = \mathsf{qftp}(j_1, \ldots, j_n)$$

and

$$\mathcal{M}_{\mathfrak{I}} \vDash \phi(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n}) \wedge \neg\phi(\bar{a}_{j_1}, \ldots, \bar{a}_{j_n}).$$

We prove that this is impossible by induction on the complexity of $\phi(\bar{x}_1, \ldots, \bar{x}_n)$. It is obviously impossible for quantifier-free formulas, and, in fact, it suffices to check that it is impossible when $\phi(\bar{x}_1, \ldots, \bar{x}_n)$ is of the form:

$$\exists \bar{y}(\bar{y} \in S \wedge \psi(\bar{x}_1, \ldots, \bar{x}_n, y)).$$

Of course, if S does not appear in a formula $\phi(\bar{x}_1, \ldots, \bar{x}_n)$, then since $(\bar{a}_i : i \in \mathcal{I})$ was $\mathcal{I}$-indiscernible in $\mathcal{M}$, $\phi(\bar{x}_1, \ldots, \bar{x}_n)$ will not be witnessing that it is not indiscernible in $\mathcal{M}_{\mathfrak{I}}$. Moreover, it is clear that we can move all quantifiers, to only need to consider formulas as above. Suppose that there is such a formula and $i_1, \ldots, i_n, j_1, \ldots, j_n \in \mathcal{I}$ such that:

$$\mathsf{qftp}(i_1, \ldots, i_n) = \mathsf{qftp}(j_1, \ldots, j_n)$$

and:

$$\mathcal{M}_{\mathfrak{I}} \vDash \exists \bar{y}(\bar{y} \in S \wedge \psi(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n}, y)) \wedge \neg \exists \bar{y}(\bar{y} \in S \wedge \psi(\bar{a}_{j_1}, \ldots, \bar{a}_{j_n}, y))$$

In particular, there is some $k \in \mathcal{I}$ such that:

$$\mathcal{M}_{\mathfrak{I}} \vDash \psi(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n}, \bar{a}_k).$$

Since $\mathcal{I}$ is homogeneous there is an automorphism σ of $\mathcal{I}$ sending i_l to j_l for all $l \in \{1, \ldots, n\}$. Then:

$$\mathsf{qftp}(i_1, \ldots, i_n, k) = \mathsf{qftp}(j_1, \ldots, j_n, \sigma(k)),$$

and by induction we have that:

$$\mathcal{M}_{\mathfrak{I}} \models \psi(\bar{a}_{j_1}, \ldots, \bar{a}_{j_n}, \bar{a}_{\sigma(k)})),$$

contradicting the fact that:

$$\mathcal{M}_{\mathfrak{I}} \models \neg\exists\bar{y}(\bar{y} \in S \wedge \psi(a_{j_1}, \ldots, a_{j_n}, y)).$$

The claim then follows. ◀

In particular, by the claim above, Equations (4.7) and (4.8) hold in $\mathcal{M}_{\mathfrak{I}}$.

Now, let $\mathfrak{I}$ be the structure induced by $\mathcal{M}_{\mathfrak{I}}$ on the set $\{\bar{a}_i : i \in \mathcal{I}\}$. It is easy to check that $\mathfrak{I}$ has quantifier-elimination.

Since $\mathcal{I}$ is $\aleph_0$-categorical, it has finitely many complete (quantifier-free) n-types, for all $n \in \mathbb{N}$, and thus by (4.7), and since $\mathfrak{I}$ has quantifier-elimination it follows that types in $\mathcal{M}_{\mathfrak{I}}$ determine (quantifier-free) types in $\mathfrak{I}$. So $\mathfrak{I}$ has finitely many complete n-types for all $n \in \mathbb{N}$, and thus $\mathfrak{I}$ is also $\aleph_0$-categorical. Therefore, by replacing the types q_j in (4.8) by complete types in the induced structure $\mathfrak{I}$, we can assume that each q_j is isolated (along the sequence), by some $\mathcal{L}$-formula in ψ_{q_j}. Replacing each q_j with ψ_{q_j} in (4.8) we get that

$$\mathcal{I} \models R(i_1, \ldots, i_n) \iff \mathfrak{I} \models \bigvee_{j=1}^{k} q_j(a_{i_1}, \ldots, a_{i_n}),$$

for all $i_1, \ldots, i_n \in \mathcal{I}$.

Since the domain of $\mathfrak{I}$ is precisely the range of the function $f : \mathcal{I} \to \mathcal{M}^{|\bar{a}_i|}$, it follows that:

$$\mathcal{I} \models R(i_1, \ldots, i_n) \iff \mathcal{M} \models \bigvee_{j=1}^{k} q_j(f(i_1), \ldots, f(i_n)),$$

Thus, setting $J(R) := \bigvee_{j=1}^{k} \psi_{q_j}$, gives us that $\mathcal{M} \in \mathscr{C}_{\mathcal{I}}$, as required.[18]

[18]Observe that the Ramsey assumption was not used in this implication. It will, however, be used in the converse.

Now for the implication (2) $\implies$ (1): Again we argue by contraposition. Suppose that $T \in \mathscr{C}_{\mathcal{I}}$. Then, we can find some $\mathcal{M} \vDash T$, some $m \in \mathbb{N}$, a function $J : \mathsf{Sig}(\mathcal{L}_{\mathcal{I}}) \to \mathcal{L}$ and a function $f : I \to M^n$, which we shall view as an I-indexed sequence $(f(i) : i \in I)$ such that:

$$\mathcal{I} \vDash R(i_1, \dots, i_m) \iff \mathcal{M} \vDash J(R)(f(i_1), \dots, f(i_m)),$$

for all m-ary $R \in \mathcal{L}_{\mathcal{I}}$ and all $i_1, \dots, i_m \in I$.

Since $\mathcal{I}$ is $\aleph_0$-categorical and homogeneous, all quantifier-free n-types $p \in S_n^{\mathcal{I}}(\emptyset)$ are isolated by a quantifier-free formula $\psi_p(x_1, \dots, x_n)$. Then, trivially, for all $p \in S_n^{\mathcal{I}}$ we have that:

$$\begin{aligned} &\mathsf{tp}^{\mathcal{M}}(f(i_1), \dots, f(i_n)) = \mathsf{tp}^{\mathcal{M}}(f(j_1), \dots, f(j_n)) \implies \\ &\mathcal{M} \vDash J(\psi_p)(f(i_1), \dots, f(i_n)) \leftrightarrow J(\psi_p)(f(j_1), \dots, f(j_n)), \end{aligned} \tag{4.9}$$

for all $i_1, \dots, i_n, j_1, \dots, j_n \in \mathcal{I}$, where $J(\psi_p)$ is constructed as in Remark 4.3.12.

In particular, it follows that:

$$\begin{aligned} \mathsf{tp}^{\mathcal{M}}(f(i_1), \dots, f(i_n)) = \mathsf{tp}^{\mathcal{M}}(f(j_1), \dots, f(j_n)) \implies& \\ \mathsf{qftp}^{\mathcal{I}}(i_1, \dots, i_n) = \mathsf{qftp}^{\mathcal{I}}(j_1, \dots, j_n),& \end{aligned} \tag{4.10}$$

for all $i_1, \dots, i_n, j_1, \dots, j_n \in \mathcal{I}$, and by (4.9), the implication in (4.10) is expressible in $\mathcal{L}$, that is, if $(\bar{a}_i : i \in \mathcal{I})$ is some $\mathcal{I}$-sequence (locally) based on $(f(i) : i \in \mathcal{I})$, then (4.10) holds along that sequence too.

Since $\mathcal{I}$ is Ramsey, we can find, in some elementary extension $\mathcal{N} \succcurlyeq \mathcal{M}$, an $\mathcal{I}$-indiscernible sequence $(\bar{a}_i : i \in \mathcal{I})$ which is based on $(f(i) : i \in I)$, by Theorem A. Since this sequence is $\mathcal{I}$-indiscernibile, we have that:

$$\begin{aligned} \mathsf{qftp}_{\mathcal{L}_{\mathcal{I}}}(i_1, \dots, i_n) = \mathsf{qftp}_{\mathcal{L}_{\mathcal{I}}}(j_1, \dots, j_n) \implies& \\ \mathsf{tp}_{\mathcal{L}}(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}) = \mathsf{tp}_{\mathcal{L}}(\bar{a}_{j_1}, \dots, \bar{a}_{j_n}),& \end{aligned} \tag{4.11}$$

for all $i_1, \dots, i_n, j_1, \dots, j_n \in \mathcal{I}$. Since $(\bar{a}_i : i \in \mathcal{I})$ is based on $(f(i) : i \in I)$, it follows from (4.9) and (4.10) that:

$$\mathsf{tp}^{\mathcal{M}}(\bar{a}_{i_1}, \ldots, \bar{a}_{i_n}) = \mathsf{tp}^{\mathcal{M}}(\bar{a}_{j_1}, \ldots, \bar{a}_{j_n}) \implies \mathsf{qftp}^{\mathcal{I}}(i_1, \ldots, i_n) = \mathsf{qftp}^{\mathcal{I}}(j_1, \ldots, j_n). \tag{4.12}$$

Thus, combining (4.11) and (4.12), we see that the sequence $(\bar{a}_i : i \in \mathcal{I})$ is non-collapsing, and hence, by Lemma 4.3.4, T does not collapse $\mathcal{I}$-indiscernibles. □

Without the assumption that the indexing structure is in a countable language,[19] the previous theorem can fail. In fact, this was already substantially discussed in Example 4.3.6, but it's worth revisiting, even briefly.

Example 4.4.4. Let $\mathcal{N}$ be the set $\mathbb{N}$ equipped with its full set-theoretic structure, and $\mathcal{N}_S$ be its reduct to unary predicates for singleton sets.

We can see that $\mathcal{N}_S$ doesn't collapse $\mathcal{N}$-indiscernibles (in the sense of Definition 4.3.3) but we clearly have that $\mathcal{N}_S \in \mathsf{N}\mathscr{C}_{\mathcal{N}}$.

Let's now move on to Theorem 4.4.10. I think it's useful to start with a motivating example.

Example 4.4.5. Let $\mathcal{OH}_3 = \mathsf{Flim}(\mathcal{C}_{\mathsf{OH}_3})$ be the random ordered 3-hypergraph, and let $\mathfrak{I} = (a_v : v \in \mathcal{OH}_3)$ be any $\mathcal{OH}_3$-indiscernible in $\langle \mathbb{Q}, < \rangle$ indexed by $\mathcal{OH}_3$. It's rather easy to see that $\mathfrak{I}$ collapses to an order-indiscernible sequence. On the one hand, this follows from the fact that $\langle \mathbb{Q}, < \rangle$ is NIP, but the most intuitive way of seeing this is that $\langle \mathbb{Q}, < \rangle$ is a *binary* structure, while $\mathcal{OH}_3$ is a *ternary* one, and an uncollapsed $\mathcal{OH}_3$-indiscernible would leave a trace of "arity 3" in $(\mathbb{Q}, <)$.

Soon, I will present the natural generalisation of the example above. More precisely: An n-ary structure must automatically collapse indiscernibles indexed by an m-ary structure, for any $m > n$.

First, recall the definitions:

Definition 4.4.6. Let $\mathcal{M}$ be a relational $\mathcal{L}$-structure. We say that $\mathcal{M}$ is:

1. *n-ary*, for $n \in \mathbb{N}$, if it admits quantifier elimination in a relational language that consists only of relation symbols of arity at most n.
2. *irreflexive*[20] if for all relation symbols R in $\mathcal{L}$ and tuples $\bar{i} \in \mathcal{M}^{\mathsf{ar}(R)}$, we have that $\mathcal{M} \vDash \neg R(\bar{i})$ if an element occurs twice in $\bar{i}$.

[19]Of course, the example is actually a non-Ramsey indexing structure. Since such structures don't have the modelling property it would be rather foolish to expect them to behave well with respect to indiscernibility.

[20]This was called *uniformity* in Section 2.7.

The following remark is clear.

Remark 4.4.7. Given a relational language $\mathcal{L}$ and an $\mathcal{L}$-structure $\mathcal{I}$, we may always find an $\mathcal{L}'$-structure $\mathcal{I}'$ such $\mathcal{I}$ and $\mathcal{I}$' are quantifier-free interdefinable, and $\mathcal{I}'$ is irreflexive. Moreover, this can be done in such a way that the notion of quantifier-free type remains the same: for all tuples $\bar{i}, \bar{j}$ we have $\mathsf{qftp}^{\mathcal{L}}(\bar{i}) = \mathsf{qftp}^{\mathcal{L}}(\bar{j})$ if, and only if, $\mathsf{qftp}^{\mathcal{L}'}(\bar{i}) = \mathsf{qftp}^{\mathcal{L}'}(\bar{j})$.

Intuitively, we can split up any non-irreflexive n-ary relation R into m-ary (for $m \leq n$) irreflexive ones, defined as the subsets of R on which the various combinations of coordinates of R are equal.

For instance, suppose that $R \in \mathcal{L}$ is a binary relation symbol. To obtain an irreflexive structure we introduce two new relation symbols, R_1 and R_2, where R_1 is unary and R_2 is binary, interpreted as:

$$R_1(x) \text{ if, and only if, } R(x, x),$$

and:

$$R_2(x, y) \text{ if, and only if, } R(x, y) \wedge x \neq y.$$

The same process (with significantly more notation) can be used for relations of any arity. Clearly, this produces a structure with the same quantifier-free types, i.e. a quantifier-free interdefinable structure.

Now, the promised generalisation of Example 4.4.5:

Proposition 4.4.8. *Fix $n \in \mathbb{N}$. Let $\mathcal{L}$ be a relational language and $\mathcal{L}_{\leq n}$ be the sublanguage consisting of symbols of arity at most n. Let $\mathcal{I}$ be an irreflexive $\mathcal{L}$-structure and $\mathcal{I}'$ the (language) reduct of $\mathcal{I}$ to $\mathcal{L}_{\leq n}$. Then any n-ary structure $\mathcal{M}$ collapses $\mathcal{I}$-indiscernibles to $\mathcal{I}'$-indiscernibles.*

Proof. Let $\mathcal{L}_{\mathcal{M}}$ be a relational language with symbols of arity at most n, in which $\mathcal{M}$ admits quantifier elimination. Let $(\bar{a}_i : i \in \mathcal{I})$ be an $\mathcal{I}$-indiscernible in $\mathcal{M}$. We need to show that $(\bar{a}_i : i \in I)$ is $\mathcal{I}'$-indiscernible. By quantifier elimination, it is enough to check that, for any relation $R \in \mathcal{L}_{\mathcal{M}}$, $(\bar{a}_i : i \in I)$ is $\mathcal{I}'$-indiscernible *with respect to the formula*[21]. $R(x_1, \ldots, x_{\mathsf{ar}(R)})$. Consider two $\mathsf{ar}(R)$-tuples $\bar{i} = (i_1, \ldots, i_{\mathsf{ar}(R)})$ and $\bar{j} = (j_1, \ldots, j_{\mathsf{ar}(R)})$ such that $\mathsf{qftp}_{\mathcal{L}_{\leq n}}(\bar{i}) = \mathsf{qftp}_{\mathcal{L}_{\leq n}}(\bar{j})$.

[21]This means the obvious thing: A sequence $\mathfrak{I} = (\bar{a}_i : i \in \mathcal{I})$ is $\mathcal{I}$-indiscernible *with respect to a formula* ϕ if for all tuples $\bar{i}, \bar{i}'$ from $\mathcal{I}$ with the same quantifier-free type we have that $\mathsf{tp}^{\phi}(\bar{a}_{\bar{i}}) = \mathsf{tp}^{\phi}(\bar{a}_{\bar{i}'})$, where $\mathsf{tp}^{\phi}(\bar{x})$ denotes the *ϕ-type* of $\bar{x}$

Since $\mathcal{I}$ is irreflexive and $\bar{i}$ has at most n elements, we have

$$\mathsf{qftp}^{\mathcal{L}_{\leq n}}(\bar{i}) \vdash \mathsf{qftp}^{\mathcal{L}}(\bar{i}),$$

and similarly for $\bar{j}$. In particular, it follows that $\mathsf{qftp}^{\mathcal{L}}(\bar{i}) = \mathsf{qftp}^{\mathcal{L}}(\bar{j})$, and thus, by $\mathcal{I}$-indiscernibility, for any two subtuples $\bar{a}_{\bar{i}}$ of $\bar{a}_{i_1}{}^\frown \cdots {}^\frown a_{i_{\mathsf{ar}(R)}}$ and $\bar{a}_{\bar{j}}$ of $\bar{a}_{j_1}{}^\frown \cdots {}^\frown a_{j_{\mathsf{ar}(R)}}$, such that $|\bar{a}_{\bar{i}}| = |\bar{a}_{\bar{j}}| = n$, consisting of corresponding elements we have that

$$\mathcal{M} \vDash R(\bar{a}_{\bar{i}}) \leftrightarrow R(\bar{a}_{\bar{j}}),$$

and the result follows. □

One example illustrating Proposition 4.4.8 has already been given, but here are a couple more, that will be useful later on:

Example 4.4.9. The examples below are concerned with (convexly ordered) binary branching C- and D-relations. Recall (or find out, in case you skipped it) that these classes of structures were introduced in Section 2.7.

1. Let $\mathcal{I} = \mathrm{Flim}(\mathcal{C}_{\mathsf{OC}})$ be the binary branching C-relation equipped with a convex order. Then any binary structure $\mathcal{M}$ collapses $\mathcal{I}$-indiscernibles to order-indiscernibles. Since $\mathcal{C}_{\mathsf{OC}}$ is Ramsey, by Theorem 4.4.3, $\mathcal{M} \in \mathsf{N}\mathscr{C}_{\mathcal{C}_{\mathsf{OC}}}$.
2. Let $\mathcal{I} = \mathrm{Flim}(\mathcal{C}_{\mathsf{COD}})$ be the binary branching D-relation equipped with a convex cyclic order. Then any ternary structure $\mathcal{M}$ collapses $\mathcal{I}$-indiscernible to cyclically ordered-indiscernibles.

Theorem 4.4.10. *Let $\mathcal{J}$ be an n-ary $\aleph_0$-categorical structure in a finite relational language $\mathcal{L}'$, and let $\mathcal{I}$ be a non-n-ary reduct of $\mathcal{J}$ which is homogeneous in a finite relational language $\mathcal{L}$. Then $\mathsf{Age}(\mathcal{I})$ is not a Ramsey class.*

Proof. Since $\mathcal{I}$ is a reduct of $\mathcal{J}$, it is also $\aleph_0$-categorical and by assumption it is homogeneous, so $\mathcal{I}$ is the Fraïssé limit of its age. Assume $\mathcal{I}$ is Ramsey. Since $\mathcal{I}$ is a reduct of $\mathcal{J}$, clearly $\mathcal{J} \in \mathscr{C}_{\mathcal{I}}$, therefore, by Theorem 4.4.3, $\mathcal{J}$ does not collapse $\mathcal{I}$-indiscernibles. On the other hand, since $\mathcal{J}$ is n-ary, by Proposition 4.4.8, it collapses $\mathcal{I}$-indiscernibles to $\mathcal{I}\restriction_{\mathcal{L}_{\leq n}}$-indiscernibles, where $\mathcal{L}_{\leq n}$ is the sublanguage of $\mathcal{L}$ consisting of symbols of arity at most n. Since, by assumption $\mathcal{I}$ is not n-ary, $\mathcal{I}\restriction_{\mathcal{L}_{\leq n}}$ is a strict reduct of $\mathcal{I}$, and thus we have a contradiction. □

Example 4.4.11. It follows immediately from the theorem above that $\mathcal{C}_C$ of all C-relations (without a linear order) is not Ramsey, since it is the age of a ternary reduct $(\mathbb{Q}, C)$ of the binary structure $(\mathbb{Q}, <)$. Of course, we already knew that this class was not Ramsey (as it is not ordered). Once I've developed a little more machinery, the theorem above will be put to good use in Theorem 4.4.13.

For the remainder of this section, for $n \in \mathbb{N}$, let $\mathcal{L}_{\mathsf{H}_n^\circ} = \{R, <\}$ be a language with a binary relation symbol $<$ and an n-ary relation symbol R (this was called $\mathcal{L}_0^+$ in Section 2.7, and I hope the change in notation is not too annoying). By an $\mathcal{L}_{\mathsf{H}_n^\circ}$-structure I shall always mean a structure in which $<$ is a *(total) linear order* and R is a *uniform symmetric n-ary relation* (i.e. $\mathcal{L}_{\mathsf{H}_n^\circ}$ structures are ordered n-uniform hypergraphs, in the sense of Section 2.7).

Proposition 4.4.12. *Let $n \geq 2$ and let $\mathcal{M}$ be an $\aleph_0$-categorical homogeneous $\mathcal{L}_{\mathsf{H}_n^\circ}$-structure and $\mathcal{N}$ be a reduct of $\mathcal{M}$. If $\mathcal{N}$ is a strict reduct of $\mathcal{M}$ and a proper expansion of $\mathcal{M} \restriction \{<\}$ then $\mathcal{N}$ is not n-ary.*

Proof. Since $\mathcal{N}$ is a proper reduct of $\mathcal{M}$ and $\mathsf{Aut}(\mathcal{N})$ preserves $<$, we must have that $\mathsf{Aut}(\mathcal{N})$ does not preserve R, otherwise, since $\mathcal{N}$ is $\aleph_0$-categorical, R would be definable in $\mathcal{N}$. Therefore, there are $a_1, \ldots, a_n \in \mathcal{M}$ and $g \in \mathsf{Aut}(\mathcal{N})$ such that:

$$\mathcal{M} \vDash R(a_1, \ldots, a_n) \wedge \neg R(g(a_1), \ldots, g(a_n)).$$

Since R is symmetric, after possibly permuting the a_i's, we may assume that $a_1 < \cdots < a_n$, and thus, since g preserves $<$ we have that $g(a_1) < \cdots < g(a_n)$.

Claim 1. For all $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ from $\mathcal{M}$ such that $x_1 < \cdots < x_n$ and $y_1 < \cdots < y_n$, there is some $f \in \mathsf{Aut}(\mathcal{N})$ such that $y_i = f(x_i)$ for all $1 \leq i \leq n$.

Proof of Claim 1. If $\mathcal{M} \vDash R(x_1, \ldots, x_n) \wedge R(y_1, \ldots, y_n)$, then, since $\mathcal{M}$ is homogeneous there is some $f \in \mathsf{Aut}(\mathcal{M}) \leq \mathsf{Aut}(\mathcal{N})$ such that $y_i = f(x_i)$, and the claim follows. Thus, without loss of generality, we may assume that:

$$\mathcal{M} \vDash R(x_1, \ldots, x_n) \wedge \neg R(y_1, \ldots, y_n).$$

In particular, there are $f_1, f_2 \in \mathsf{Aut}(\mathcal{M}) \leq \mathsf{Aut}(\mathcal{N})$ such that $f_1(x_i) = a_i$ and $f_2(g(a_i)) = y_i$ for all $1 \leq i \leq n$. By construction, $f = f_2 \circ g \circ f_1$ is as required. ◀

Thus, for all $m \leq n$, all $<$-increasing m-tuples in $\mathcal{N}$ satisfy the same relations in $\mathcal{N}$, so the only non-trivial definable relation in $\mathcal{N}$ of arity at most n is $<$. As $\mathcal{N}$ is a proper expansion of $\mathcal{M} \restriction \{<\}$, this implies $\mathcal{N}$ is not n-ary. $\square$

Theorem 4.4.13. *Let $n \in \mathbb{N}$ and $\mathcal{N}$ be a non-trivial reduct of $\mathcal{OH}_n$, other than* DLO. *Then $\mathcal{M}$ is not Ramsey.*

The proof of Theorem 4.4.13 will make use of Theorem B. I restate it below:

Theorem B. Let $\mathcal{C}$ be a Ramsey class of $\mathcal{L}'$-structures and $\mathcal{N}$ an $\mathcal{L}'$-structure such that $\mathsf{Age}(\mathcal{N}) = \mathcal{C}$. Then there is an $\mathsf{Aut}(\mathcal{N})$-invariant linear order on $\mathcal{N}$ which is the union of quantifier-free types.

Proof of Theorem 4.4.13. Assume towards a contradiction that $\mathcal{N}$ is Ramsey. Then, by Theorem B, there is a linear order $\lhd$ on $\mathcal{N}$ which is a union of quantifier-free types in $\mathcal{N}$. As $\mathcal{N}$ is a quantifier-free reduct of $\mathcal{OH}_n$, we see that $\lhd$ is a union of quantifier-free types in $\mathcal{OH}_n$.

Claim 1. The order $\lhd$ is either $<$ or $>$.

Proof of Claim 1. This is clear when $n \geq 3$, since R is uniform and thus the only quantifier-free types in two variables in $\mathcal{M}$ are $x < y$ and $x > y$. As $\lhd$ is antisymmetric, it cannot be equivalent to $x < y \vee x > y$.

We now discuss the case where $n = 2$. In this case, the only quantifier-free types in $\mathcal{M}$ are:

1. $x < y \wedge R(x, y)$.
2. $x > y \wedge R(x, y)$.
3. $x < y \wedge \neg R(x, y)$.
4. $x > y \wedge \neg R(x, y)$.

A straightforward check gives that the only unions of the types above that define a linear order in the random graph are the following:

$$(x < y \wedge R(x, y)) \vee (x < y \wedge \neg R(x, y))$$

and

$$(x > y \wedge R(x, y)) \vee (x > y \wedge \neg R(x, y)),$$

again, it follows that $\lhd$ is either $<$ or $>$. ◀

By Claim 1, we have that $\mathcal{N}$ is a proper expansion of $\mathcal{OH}_n \restriction \{<\}$ and a strict reduct of $\mathcal{OH}_n$. Therefore, by Proposition 4.4.12, $\mathcal{N}$ is not n-ary. By [Tho96, Theorem 2.7] we know that $\mathcal{N}$ is homogeneous in a finite relational language, therefore by Theorem 4.4.10, $\mathcal{N}$ is not Ramsey. □

Remark 4.4.14. In fact, a careful analysis of the proof of Theorem 4.4.13 gives a slightly more general result. This is stated below.

Corollary 4.4.15. *Let $n \geq 2$ and let $\mathcal{M}$ be an $\aleph_0$-categorical homogeneous $\mathcal{L}_{\mathsf{H}_n^o}$-structures. If $\mathcal{M}$ is not interdefinable with $(M, <_1, <_2)$, where $<_1$ and $<_2$ are two (independent) linear orders, then every proper quantifier-free reduct $\mathcal{N}$ of $\mathcal{M}$, which is homogeneous in a finite relational language and not interdefinable with $\mathcal{M} \restriction \{<\}$ is not Ramsey.*

Proof. In the case $n = 2$, in the proof of Claim 1 in Theorem 4.4.13, the only other unions of quantifier-free types that we would need to consider are:

$$(x < y \wedge R(x, y)) \vee (x > y \wedge \neg R(x, y)) \text{ and } (x > y \wedge R(x, y)) \vee (x < y \wedge \neg R(x, y)).$$

But, if either of the formulas above defines a linear order on $\mathcal{M}$, then $\mathcal{M}$ is interdefinable with a structure $(M, <_1, <_2)$, where $<_1$ and $<_2$ are independent linear orders, which is a contradiction. □

By [Bod15, Proposition 2.23], the class of finite cyclically ordered binary branching D-relations $\mathcal{C}_{\mathsf{COD}}$ is not Ramsey, since no total order is definable in $\mathsf{FLim}(\mathcal{C}_{\mathsf{COD}})$. We can see that, adding a generic order will not suffice to turn the class into a Ramsey class:

Example 4.4.16. The *free superposition*[22] $\mathcal{C}_{\mathsf{LO}} * \mathcal{C}_{\mathsf{COD}}$ is an amalgamation class, but is not a Ramsey class. Indeed, $\mathsf{FLim}(\mathcal{C}_{\mathsf{LO}} * \mathcal{C}_{\mathsf{COD}})$ is a reduct of $\mathsf{FLim}(\mathcal{C}_{\mathsf{LO}} * \mathcal{C}_{\mathsf{OC}})$. The latter is ternary, while the former is not. Therefore, by Theorem 4.4.10, the former is not Ramsey.

4.5 Transfer Principles

Let's now take a rather sharp turn, and use some of the machinery discussed earlier in this chapter to study transfer principles for products, full and lexicographic, with respect to the various dividing lines from Subsection 4.2.4. The common point of most of these properties is that they admit characterisations involving indiscernibility. Thus, the main tool for the transfer principles will

[22] See Fact 4.4.1.

be a description of indiscernible sequences, for full and lexicographic products, in terms of sequences in the factors.

4.5.1 Transfers in Full Products

First, *full products*. The upshot of this subsection is that the full product behaves extremely nicely with respect to the notion of collapsing indiscernibles to a specified reduct, but not necessarily for the dividing line arising from coding structures. It is fairly easy to describe indiscernible sequences in a full product $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ (this will be done in Proposition 4.5.5). Before that, I will give some negative results.

First, observe that the full product of structures $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ is almost never monadically NIP (or, for that matter, by the results of Chapter 6, monadically anything).

One can find a proof of the following proposition in [Tou23, Proposition 1.24].

Proposition 4.5.1. *Assume $\mathcal{M}_i$ is NIP for $i \in \{1, 2\}$. We have $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ is NIP of dp-rank $\mathsf{dp\text{-}rk}(\mathcal{M}_1) + \mathsf{dp\text{-}rk}(\mathcal{M}_2)$.*

From this, it is easy to deduce the following:

Corollary 4.5.2. *A full product $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ of infinite structures is never monadically NIP.*

Proof. From Proposition 4.5.1, it follows that if both $\mathcal{M}_1$ and $\mathcal{M}_2$ are infinite, then $\mathsf{dp\text{-}rk}(\mathcal{M}_1 \boxtimes \mathcal{M}_2) \geq 2$, and hence $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ is not dp-minimal, hence, by Fact 4.2.19 it is not monadically NIP. ☐

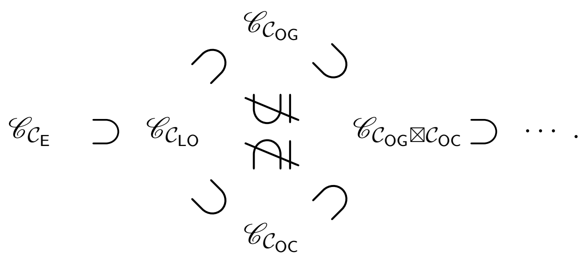
Remark 4.5.3. Indeed, $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ is monadically NIP precisely when one of the structures is finite and the other is monadically NIP.

The next example illustrates another negative result. More precisely, it illustrates that $\mathsf{N}\mathscr{C}_{\mathbb{K}}$ is not always closed under full products.

Example 4.5.4. Let $\mathcal{C}_{\mathsf{OG}}$ be the class of all finite ordered graphs, and $\mathcal{C}_{\mathsf{OC}}$ be the class of all finite convexly ordered binary branching C-relations. Consider the product $\mathcal{C}_{\mathsf{OG}} \boxtimes \mathcal{C}_{\mathsf{OC}}$ of the two classes, as in Definition 4.2.6. On the one hand, $\mathcal{OG} = \mathsf{Flim}(\mathcal{C}_{\mathsf{OG}})$ is in $\mathsf{N}\mathscr{C}_{\mathcal{C}_{\mathsf{OC}}}$ because the random ordered graph is binary, and therefore collapses $\mathsf{Flim}(\mathcal{C}_{\mathsf{OC}})$-indiscernibles to order-indiscernibles (see Example 4.4.9). To see why this is relevant, recall that by Fact 4.3.16 we have that $\mathscr{C}_{A\boxtimes B} = \mathscr{C}_A \cap \mathscr{C}_B$, so if $\mathcal{OG} \in \mathscr{C}_{\mathcal{C}_{\mathsf{OG}}\boxtimes\mathcal{C}_{\mathsf{OC}}}$ then $\mathcal{OG} \in \mathscr{C}_{\mathcal{C}_{\mathsf{OC}}}$, contradicting the previous statement. Thus $\mathcal{OG} \in \mathsf{N}\mathscr{C}_{\mathcal{C}_{\mathsf{OG}}\boxtimes\mathcal{C}_{\mathsf{OC}}}$.

The generic ordered binary branching C-relation, $\mathcal{OC} = \mathsf{Flim}(\mathcal{C}_{\mathsf{OC}})$, is known to be NIP, and therefore it is in $\mathsf{N}\mathscr{C}_{\mathcal{C}_{\mathsf{OG}}}$. So, again, arguing as in the previous paragraph, $\mathcal{OC} \in \mathsf{N}\mathscr{C}_{\mathcal{C}_{\mathsf{OG}}\boxtimes\mathcal{C}_{\mathsf{OC}}}$. However, since $\mathcal{OG} \boxtimes \mathcal{OC} = \mathsf{Flim}(\mathcal{C}_{\mathsf{OC}} \boxtimes \mathcal{C}_{\mathsf{OC}})$, it follows that $\mathcal{OG} \boxtimes \mathcal{OC} \in \mathscr{C}_{\mathcal{C}_{\mathsf{OG}}\boxtimes\mathcal{C}_{\mathsf{OC}}}$.

Example 4.5.4 above shows in particular that the hierarchy of $\mathscr{C}_{\mathbb{K}}$-configurations is not linearly ordered. Since $\mathcal{OG} \notin \mathscr{C}_{\mathcal{C}_{\mathsf{OC}}}$ and $\mathcal{OC} \notin \mathscr{C}_{\mathcal{C}_{\mathsf{OG}}}$, and both $\mathcal{OG}$ and $\mathcal{OC}$ have quantifier elimination, Lemma 4.3.15, gives the following:



Above, following the notation introduced in Section 2.7, $\mathcal{C}_{\mathsf{LO}}$ is the class of all finite linear orders, and $\mathcal{C}_{\mathsf{E}}$ is the class of all finite sets in the empty language. To summarise the discussion in Example 4.5.4:

- $\mathscr{C}_{\mathcal{C}_{\mathsf{OG}}} \not\subseteq \mathscr{C}_{\mathcal{C}_{\mathsf{OC}}}$, since $\mathcal{OG}$ is in $\mathscr{C}_{\mathcal{C}_{\mathsf{OG}}}$, but not in $\mathscr{C}_{\mathcal{C}_{\mathsf{OC}}}$.
- $\mathscr{C}_{\mathcal{C}_{\mathsf{OC}}} \not\subseteq \mathscr{C}_{\mathcal{C}_{\mathsf{OG}}}$, since $\mathcal{OC}$ is in $\mathscr{C}_{\mathcal{C}_{\mathsf{OC}}}$, but not in $\mathscr{C}_{\mathcal{C}_{\mathsf{OG}}}$.

Intuitively, $\mathsf{N}\mathscr{C}_{\mathcal{C}_{\mathsf{OG}}\boxtimes\mathcal{C}_{\mathsf{OC}}}$ is not closed under full products because $(\mathcal{OG} \boxtimes \mathcal{OC})$-indiscernibles can collapse to two "orthogonal" notions of indiscernibility, namely $\mathcal{OG}$-indiscernibles, and $\mathcal{OC}$-indiscernibles.

This provides a negative answer to [GPS21, Question 5.6], and also the question asked in [GP23, Section 7], as discussed in Section 4.1.

That's all for negative results regarding full products. In what follows, I will discuss how the situation from Example 4.5.4 (where we come across two "orthogonal" notions of indiscernibility) is, in some sense, the worst that can happen. The key tool is the following proposition:

Proposition 4.5.5. *Let $\mathcal{M}_i$ be an $\mathcal{L}_i$-structure, for $i \in \{1, 2\}$ and let $\mathfrak{I} = (\bar{c}_i : i \in I)$ be an $\mathcal{I}$-indexed sequence, for some structure $\mathcal{I}$, in $\mathcal{M}_1 \boxtimes \mathcal{M}_2$. Then, the following are equivalent for $B \subseteq \mathcal{M}_1 \boxtimes \mathcal{M}_2$*

1. *$\mathfrak{I}$ is $\mathcal{I}$-indiscernible over B in $\mathcal{M}_1 \boxtimes \mathcal{M}_2$.*

2. *$\pi_1(\mathfrak{I}) := (\pi_{M_1}(\bar{c}_i) : i \in I)$ is $\mathcal{I}$-indiscernible over $\pi_{M_1}(B)$ in $\mathcal{M}_1$ and $\pi_2(\mathfrak{I}) := (\pi_{M_2}(\bar{c}_i) : i \in I)$ is $\mathcal{I}$-indiscernible over $\pi_{M_2}(B)$ in $\mathcal{M}_2$.*

Since the reader may not quite be used to products of structures, before proceeding with the proof, I feel I should provide a bit of notational clarification:

Notation 4.5.6. Given a tuple $\bar{c} = (c_1{}^\frown \cdots {}^\frown c_n)$ in $\mathcal{M}_1 \boxtimes \mathcal{M}_2$, the components c_i may be in any of the three sorts of $\mathcal{M}_1 \boxtimes \mathcal{M}_2$. Then $\pi_{M_1}(\bar{c})$ denotes the subtuple of $\bar{c}$ (of size at most n) consisting of the projections $\pi_{M_1}(c_i)$ if $c_i \in \mathcal{M}_1 \boxtimes \mathcal{M}_2$, and of the elements c_j such that $c_j \in \mathcal{M}_1$ (in the correct order). In particular, to obtain $\pi_{M_1}(\bar{c})$ we remove the components of $\bar{c}$ that belong to the $\mathcal{M}_2$ sort, project the components of $\bar{c}$ that belong to the $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ sort to M_1, and keep the rest of the tuple unchanged. Of course, $\pi_2(\bar{c})$ is obtained analogously.

Proof. By relative quantifier elimination (Fact 4.2.5), any formula $\phi(\bar{x}, \bar{b})$ in $\mathcal{L}_{\mathcal{M}_1 \boxtimes \mathcal{M}_2}(B)$ is equivalent modulo the theory of $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ to a Boolean combination of formulas of the form:

$$\phi_1(\pi_{M_1}(\bar{x}), \pi_{M_1}(\bar{b})) \wedge \phi_2(\pi_{M_2}(\bar{x}, \pi_{M_1}(\bar{b})).$$

where $\phi_1(\bar{x}_1, \bar{y}_1)$ is an $\mathcal{L}_1$-formula and $\phi_2(\bar{x}_2, \bar{y}_2)$ is an $\mathcal{L}_2$-formula.

Given $\mathfrak{I} = (\bar{c}_i : i \in I)$ in $\mathcal{M}_1 \boxtimes \mathcal{M}_2$, checking whether $(\bar{c}_i : i \in I)$ is $\mathcal{I}$-indiscernible over B is equivalent to checking whether $(\bar{c}_i : i \in I)$ is $\mathcal{I}$-indiscernible with respect to the formulas $\phi_i(\pi_{M_i}(\bar{x}), \pi_{M_i}(\bar{b}))$, for $i \in \{1, 2\}$. But this is precisely if, and only if $\pi_i(\mathfrak{I})$ is $\mathcal{I}$-indiscernible over $\pi_{M_i}(B)$ in $\mathcal{M}_i$, and the result follows. □

So, Proposition 4.5.5 almost immediately gives transfer principles for full products and dividing lines which can be described by indiscernible collapses. This will come from the following corollary:

Corollary 4.5.7. *Let $\mathcal{I}$ be an $\mathcal{L}$-structure and $\mathcal{I}'$ a reduct of $\mathcal{I}$. Then, the following are equivalent, for any $\mathcal{L}_i$-structures $\mathcal{M}_i$, where $i \in \{1, 2\}$:*

1. *The full product $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ collapses $\mathcal{I}$-indiscernibles to $\mathcal{I}'$-indiscernibles.*

2. *Both $\mathcal{M}_1$ and $\mathcal{M}_2$ collapse $\mathcal{I}$-indiscernibles to $\mathcal{I}'$-indiscernibles.*

Proof. On the one hand, if both $\mathcal{M}_1$ and $\mathcal{M}_2$ collapse $\mathcal{I}$-indiscernibles to $\mathcal{I}'$-indiscernibles, then, given any $\mathcal{I}$-indiscernible $\mathfrak{I} = (\bar{c}_i : i \in \mathcal{I})$ in $\mathcal{M}_1 \boxtimes \mathcal{M}_2$, we have that both $\pi_1(\mathfrak{I})$ and $\pi_2(\mathfrak{I})$ are $\mathcal{I}$-indiscernible and hence $\mathcal{I}'$-indiscernible, in $\mathcal{M}_1$, and $\mathcal{M}_2$, respectively. Thus, by Proposition 4.5.5, $\mathfrak{I}$ is actually $\mathcal{I}'$-indiscernible.

On the other hand, suppose that $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ collapses $\mathcal{I}$-indiscernibles to $\mathcal{I}'$-indiscernibles. Given any $\mathcal{I}$-indiscernible sequence $\mathfrak{J} = (\bar{a}_i : i \in \mathcal{I})$ of n-tuples in $\mathcal{M}_1$, fix some $b \in M_2$ and let $\bar{c}_i = ((\bar{a}_i^1, b)^\frown \cdots {}^\frown (\bar{a}_i^n, b))$, for each $i \in I$. Clearly the constant sequence $(b : i \in \mathcal{I})$ is $\mathcal{I}$-indiscernible and thus, by Proposition 4.5.5 $\mathfrak{I} := (\bar{c}_i : i \in \mathcal{I})$ is $\mathcal{I}$-indiscernible in $\mathcal{M}_1 \boxtimes \mathcal{M}_2$. In particular, it is $\mathcal{I}'$-indiscernible, and hence, by Proposition 4.5.5, again, $\pi_1(\mathfrak{I}) = \mathfrak{J}$ is $\mathcal{I}'$-indiscernible in $\mathcal{M}_1$, so $\mathcal{M}_1$-collapses $\mathcal{I}$-indiscernibles to $\mathcal{I}'$-indiscernibles. An identical argument shows that this is also the case for $\mathcal{M}_2$, and this concludes the proof. □

The following corollary is now immediate from Corollary 4.5.7 combined with the characterisation of NIP$_n$ via an indiscernible collapse (Fact 4.3.7).

Corollary 4.5.8 (NIP$_n$ transfer for full products)**.** *Let $\mathcal{M}_i$ be an $\mathcal{L}_i$-structure, for $i \in \{1, 2\}$. For all $n \in \mathbb{N}$, the following are equivalent:*

(1) The full product $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ is NIP$_n$.

(2) Both $\mathcal{M}_1$ and $\mathcal{M}_2$ are NIP$_n$.

Somewhat similarly:

Proposition 4.5.9. *Let $\mathcal{M}_i$ be an $\mathcal{L}_i$-structure, for $i \in \{1, 2\}$. Then, the following are equivalent:*

(1) The full product $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ is distal.

(2) Both $\mathcal{M}_1$ and $\mathcal{M}_2$ are distal.

Proof. First, suppose that $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ is distal and let $(a_i : i \in \mathbb{Q})$ be an indiscernible sequence in $\mathcal{M}_1$ and fix some tuple $\bar{b}$ from $\mathcal{M}_1$ such that $(a_i : i \in \mathbb{Q} \setminus \{0\})$ is indiscernible over $\bar{b} = (b_1, \dots, b_k)$. Pick any element $c \in \mathcal{M}_2$. Then, by Proposition 4.5.5, the sequence $((a_i, c) : i \in \mathbb{Q})$ is indiscernible in $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ and $((a_i, c) : i \in \mathbb{Q} \setminus \{0\})$ is indiscernible over $(\bar{b}, c) = (b_1, c)^\frown \cdots {}^\frown (b_k, c)$. Since $\mathcal{M}_1 \times \mathcal{M}_2$ is distal, it follows that $((a_i, c) : i \in \mathbb{Q})$ is indiscernible over $(\bar{b}, c)$, and thus, again by Proposition 4.5.5, $(a_i : i \in \mathbb{Q})$ is indiscernible over $\bar{b}$, so $\mathcal{M}_1$ is distal. Similarly, $\mathcal{M}_2$ must also be distal.

To prove the other implication, assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ are both distal, and let $((a_i, b_i) : i \in \mathbb{Q})$ be a sequence in $\mathcal{M}_1 \boxtimes \mathcal{M}_2$, and B a set of parameters in $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ such that:

- $((a_i, b_i) : i \in \mathbb{Q})$ is indiscernible.
- $((a_i, b_i) : i \in \mathbb{Q} \setminus \{0\})$ is indiscernible over B,

By the characterisation in Proposition 4.5.5, we have:

1. $(a_i : i \in \mathbb{Q} \setminus \{0\})$ is indiscernible over $\pi_{\mathcal{M}_1}(B)$;
2. $(a_i : i \in \mathbb{Q})$ is indiscernible.

1'. $(b_i : i \in \mathbb{Q} \setminus \{0\})$ is indiscernible over $\pi_{\mathcal{M}_2}(B)$;

2'. $(b_i : i \in \mathbb{Q})$ is indiscernible.

Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are distal, we can deduce that:

- $(a_i : i \in \mathbb{Q})$ is indiscernible over $\pi_{\mathcal{M}_1}(B)$;
- $(b_i : i \in \mathbb{Q})$ is indiscernible over $\pi_{\mathcal{M}_2}(B)$.

Again by Proposition 4.5.5, $((a_i, b_i) : i \in \mathbb{Q})$ is indiscernible over B. This shows that $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ is distal and concludes the proof. □

Similarly, one can show more generally that m-distality also transfers to the full product.

Corollary 4.5.10. *Let $\mathcal{M}_i$ be an $\mathcal{L}_i$-structure, for $i \in \{1, 2\}$. For all $m \in \mathbb{N}$, the following are equivalent:*

(1) The full product $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ is m-distal.

(2) Both $\mathcal{M}_1$ and $\mathcal{M}_2$ are m-distal.

The proof follows essentially the same structure as that given for (1-)distality, just above it. The notion of m-distality will be discussed again in Proposition 4.5.29, where an analogous argument will be provided for lexicographic products.

Even though monadic NIP does not transfer to lexicographic products, indiscernible triviality does, and this is the content of the next proposition:

Proposition 4.5.11. *Let $\mathcal{M}_i$ be an $\mathcal{L}_i$-structure, for $i \in \{1, 2\}$. Then, the following are equivalent:*

(1) The full product $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ has indiscernible triviality.

(2) Both $\mathcal{M}_1$ and $\mathcal{M}_2$ have indiscernible triviality.

Proof. Observe first that any model of $\mathsf{Th}(\mathcal{M}_1 \boxtimes \mathcal{M}_2)$ is a full product of models of $\mathsf{Th}(\mathcal{M}_1)$ and $\mathsf{Th}(\mathcal{M}_2)$, so we may thus assume that $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ is a monster model. To show that a structure $\mathcal{M}$ has indiscernible triviality, it suffices to show that if a sequence $(\bar{a}_i : i \in \mathbb{N})$ from $\mathcal{M}$ is indiscernible over two tuples $\bar{b}$ and $\bar{b}'$, then it is indiscernible over $\{\bar{b}, \bar{b}'\}$. Of course, since formulas can only use finitely many variables, it follows then that if $(\bar{a}_i : i \in \mathcal{I})$ is indiscernible over each element of some set B, it has to be indiscernible over all of B. Observe also that $(\bar{a}_i : i \in \mathbb{N})$ is indiscernible over b if, and only if, the sequence $(\bar{a}_i {}^\frown (\bar{b}) : i \in \mathbb{N})$ is indiscernible.

With these technical remarks out of the way, suppose that $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ has indiscernible triviality, fix $\bar{b}, \bar{b}' \in M_1$, and let $(\bar{a}_i : i \in \mathbb{N})$ be an indiscernible sequence in $\mathcal{M}_1$ which is indiscernible over both $\bar{b}$ and $\bar{b}'$. Let $c \in M_2$ be arbitrary. By Proposition 4.5.5, the sequences:

$$((\bar{a}_i, c)^\frown(\bar{b}, c) : i \in \mathbb{N})) \text{ and } ((\bar{a}_i, c)^\frown(\bar{b}', c) : i \in \mathbb{N}),$$

where $(\bar{a}_i, c) := (a_i^1, c)^\frown \cdots ^\frown(a_i^{|\bar{a}_i|}, c)$ (and similarly for $\bar{b}, \bar{b}'$), are indiscernible. Since $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ has indiscernible triviality, it follows that

$$((\bar{a}_i, c)^\frown(\bar{b}, c)^\frown(\bar{b}', c) : i \in \mathbb{N})$$

is indiscernible, and thus, by Proposition 4.5.5 again, so is

$$(\bar{a}_i {}^\frown(\bar{b})^\frown(\bar{b}') : i \in \mathbb{N}).$$

Thus, $\mathcal{M}_1$ has indiscernible triviality. An analogous argument shows that the same must be true for $\mathcal{M}_2$.

Conversely, suppose that both $\mathcal{M}_1$ and $\mathcal{M}_2$ have indiscernible triviality, let $(\bar{a}_i : i \in \mathbb{N})$ be an indiscernible sequence of tuples in $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ and $\bar{b} = (\bar{b}_1, \bar{b}_2), \bar{b}' = (\bar{b}'_1, \bar{b}'_2)$, with $\bar{b}_i, \bar{b}'_i \in \mathcal{M}_i$, be tuples of parameters in $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ such that $(\bar{a}_i : i \in \mathbb{N})$ is indiscernible over both $\bar{b}$ and $\bar{b}'$. By the characterisation, $(\pi_{M_1}(\bar{a}_i)^\frown(\bar{b}_1) : i \in \mathbb{N})$ and $(\pi_{M_1}(\bar{a}_i)^\frown(\bar{b}'_1) : i \in \mathbb{N})$ are indiscernible in $\mathcal{M}_1$. By indiscernible triviality of $\mathcal{M}_1$:

$$(\pi_{M_1}(\bar{a}_i)^\frown(\bar{b}_1)^\frown(\bar{b}'_1) : i \in \mathbb{N})$$

is indiscernible in $\mathcal{M}_1$. Similarly:

$$(\pi_{M_2}(\bar{a}_i)^\frown(\bar{b}_2)^\frown(\bar{b}_2') : i \in \mathbb{N})$$

is indiscernible in $\mathcal{M}_2$, and by the characterisation again:

$$((\bar{a}_i)^\frown(\bar{b}_1, \bar{b}_2)^\frown(\bar{b}_1', \bar{b}_2') : i \in \mathbb{N})$$

is indiscernible in $\mathcal{M}_1 \boxtimes \mathcal{M}_2$, and this concludes the proof. □

4.5.2 Transfers in Lexicographic Sums

Now for *lexicographic sums*. To fix some notation, throughout this subsection (unless otherwise stated) $\mathcal{S}$ will be the lexicographic sum of an $\mathcal{L}_{\mathcal{M}}$-structure $\mathcal{M}$ and a class of $\mathcal{L}_{\mathfrak{N}}$-structures $\mathfrak{N} := \{\mathcal{N}_a\}_{a \in \mathcal{M}}$, as in Definition 4.2.7, under the hypothesis discussed in Notation 4.2.9:

$$\text{For } \phi \in \mathcal{L}_{\mathfrak{N}}, \text{ the set } \{a \in \mathcal{M} : \mathcal{N}_a \vDash \phi\} \text{ is } \emptyset\text{-definable in } \mathcal{M}. \qquad (\star)$$

This technical assumption ensures that the induced structure on the M-sort in $\mathcal{S}$ is exactly the $\mathcal{L}_{\mathcal{M}}$-structure $\mathcal{M}$ (this is a consequence of Theorem 4.2.11). For instance, $(\star)$ holds trivially in the case of a lexicographic product. Since most of the notions that will be considered are preserved under taking reducts, this is a mostly harmless assumption.

As in the case of full products, the main tool for transfer principles will be a description of the generalised indiscernible sequences of $\mathcal{M}[\mathfrak{N}]$ in terms of generalised indiscernibility in $\mathcal{M}$ and $\mathcal{N}_i$. But, unlike the full product, some assumptions will have to be made on the kinds of indexing structures that are considered. This will be illustrated in the following example:

Example 4.5.12. Let $\mathcal{M}$ be an infinite set, in the language of pure equality, say $\mathcal{M} = \omega$ and $\mathcal{N}$ be a countably infinite set equipped with an equivalence relation with two infinite equivalence classes, say $\mathcal{N} = R \sqcup B$, where R and B are disjoint copies of ω. The lexicographic product $\mathcal{M}[\mathcal{N}]$ is essentially an infinite set equipped with two equivalence relations E_1, E_2 such that all E_i-classes for $i = 1, 2$ are infinite, E_2 refines E_1 and each E_1-class is refined by exactly two E_2-classes. To fix notation, we may write

$$\mathcal{M}[\mathcal{N}] = \bigsqcup(\{i\} \times (\{r_i^j : j \in \omega\} \sqcup \{b_i^j : j \in \omega\})),$$

and for $(w,x),(y,z) \in \mathcal{M}[\mathcal{N}]$ we have:

- $(w,x)E_1(y,z)$ if, and only if, $w=y$.
- $(w,x)E_2(y,z)$ if, and only if $w=y$ and either $(x=r_w^j)\wedge(z=r_w^k)$ or $(x=b_w^j)\wedge(z=b_w^k)$, for some $j,k\in\omega$.

Let $\mathcal{J}=\mathcal{M}[\mathcal{N}]$ and $\mathcal{I}$ be the reduct of $\mathcal{M}[\mathcal{N}]$ to $\mathcal{L}_{\mathcal{J}}=\{E_1\}$, i.e. an infinite set equipped with an equivalence relation with infinitely many infinite equivalence classes.

Consider the sequence:

$$A=\left(\left((i,r_i^j):j\in\omega\right)^\frown\left((i,b_i^j):j\in\omega\right):i\in\omega\right).$$

We may view this as a $\mathcal{J}$-indexed sequence in $\mathcal{M}[\mathcal{N}]$, where $\mathcal{J}=\mathcal{M}[\mathcal{N}]$ is taken with the same enumeration as A. By quantifier elimination in $\mathcal{M}[\mathcal{N}]$, this sequence is $\mathcal{J}$-indiscernible. On the other hand, it is easy to see that if we view this sequence as an $\mathcal{I}$-indexed sequence, with the same enumeration, it is not $\mathcal{I}$-indiscernible.

Let us consider the "factors" of the sequence A, that is, the sequences:

- $\mathsf{v}(A)$, consisting of the first coordinates of the elements appearing in A, with the same enumeration, so:

$$\mathsf{v}(A)=((i:j\in\omega)^\frown(i:j\in\omega):i\in\omega).$$

- A_i, for $i\in\omega$, consisting of the second coordinates of elements in A whose first coordinate is i, so:

$$A_i=(r_i^j:j\in\omega)^\frown(b_i^j:j\in\omega)$$

We can view both sequences above as $\mathcal{I}$-indexed sequences in $\mathcal{M}$ and $\mathcal{N}$, respectively. In both cases, we can make sure that the sequences are $\mathcal{I}$-indiscernible (for $\mathsf{v}(A)$ we make sure that whenever two elements are equal they are in the same E_1-class, and for A_i we make sure that the E_1-classes refine the equivalence relation of $\mathcal{N}$).

Intuitively, the previous example shows that it is possible to have a $\mathcal{J}$-indexed sequence in a lexicographic product whose components are $\mathcal{J}$-indiscernible, but which is not $\mathcal{J}$-indiscernible. The first step towards the results of this subsection is to investigate conditions on $\mathcal{I}$ that allow for a relatively simple characterisation of $\mathcal{I}$-indiscernibles in a lexicographic sum in terms of $\mathcal{I}$-indiscernibles in the factors.

First, to fix some terminology, given a structure $\mathcal{I}$, define the following relations on $\{0,1\} \times I$:

$$E(C) := \{\{(\epsilon, i), (1-\epsilon, i)\} : i \in I, \epsilon \in \{0,1\}\},$$

and

$$E(D) := \{\{(\epsilon, i), (\epsilon, j)\} : i, j \in I, \epsilon \in \{0,1\}\}.$$

Let C and D be the graphs on $\{0,1\} \times \mathcal{I}$ with edge relation $E(C)$ and $E(D)$, respectively. These are shown below:

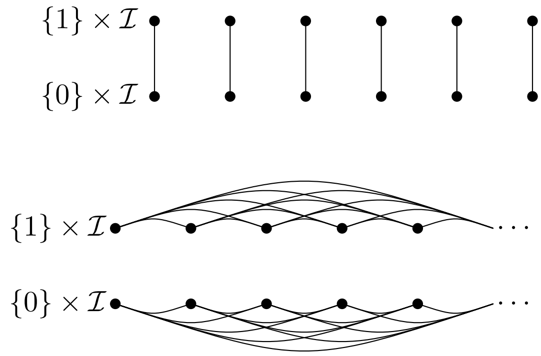


Figure 4.2: The graphs C (top) and D (bottom)

Definition 4.5.13. We call a structure $\mathcal{I}$ *a reasonable indexing structure* or simply *reasonable* if given any graph $G = (\{0,1\} \times I, R)$ whose vertex set is $\{0,1\} \times I$, such that $R((\epsilon_0, i), (\epsilon_1, j))$ depends only on $\mathsf{qftp}_{\mathcal{I}}(i,j)$ and on $\epsilon_0, \epsilon_1 \in \{0,1\}$, one of the following holds:

1. R is connected (i.e. there is a path joining any two vertices of $\{0,1\} \times I$),
2. R is contained in C (i.e. it is a subgraph of $(\{0,1\} \times I, C)$),
3. R is contained in D (i.e. it is a subgraph of $(\{0,1\} \times I, D)$).

It is clear that the structure $\mathcal{M}[\mathcal{N}]$ from Example 4.5.12 is not reasonable. Of course, that structure does not have the modelling property. The next example shows that, as a matter of fact, $\mathcal{I}$ having the modelling property does not imply that $\mathcal{I}$ is a reasonable indexing structure.

Example 4.5.14. An infinite linear order $(\mathcal{M}, <)$ equipped with an equivalence relation E with infinitely many, infinite, $<$-convex equivalence classes is not a reasonable indexing structure. Indeed, consider the graph $(\{0, 1\} \times M, R)$ where for $a, b \in M$ and $\epsilon_0, \epsilon_1 \in \{0, 1\}$, $R((\epsilon_0, a), (\epsilon_1, b))$ if, and only if, aEb and $\epsilon_0 \neq \epsilon_1$.

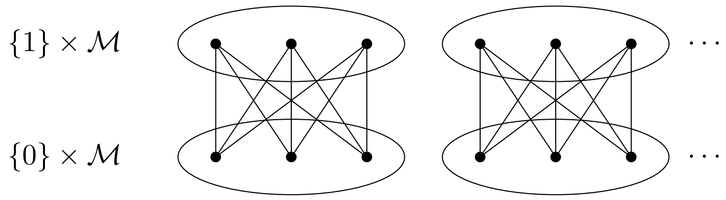


Figure 4.3: A non-reasonable Ramsey structure.

One sees that the edge relation of this graph depends only on the quantifier-free type of the nodes, but this graph does not satisfy any of the conditions 1,2 and 3 of Definition 4.5.13.

The definition of reasonability may, at first sight, seem rather ad hoc. It turns out that this is not quite the case, and modulo some mild assumptions, $\mathcal{I}$ is reasonable precisely when it is *primitive*. Recall that a permutation group G acting on a set Ω acts *primitively* if the only G-invariant equivalence relations on Ω are trivial (i.e. *equality* and *universality*). In the context of first-order structures:

Definition 4.5.15 (Primitivity)**.** We say that an $\mathcal{L}$-structure $\mathcal{M}$ is *primitive* if $\mathsf{Aut}(\mathcal{M})$ acts primitively on $\mathcal{M}$. Explicitly, $\mathcal{M}$ is primitive if, and only if, the only $\mathsf{Aut}(\mathcal{M})$-invariant equivalence relations on $\mathcal{M}$ are trivial.

Proposition 4.5.16. *Let $\mathcal{I}$ be an infinite homogeneous structure in a finite relational language. Then, the following are equivalent:*

(1) $\mathcal{I}$ is reasonable,

(2) $\mathcal{I}$ is primitive.

Proof. First for the implication (1) $\implies$ (2): Assume $\mathcal{I}$ is reasonable, and let E be a non-trivial $\mathsf{Aut}(\mathcal{I})$-invariant equivalence relation on $\mathcal{I}$. Since $\mathcal{I}$ is homogeneous, in a finite relational language, E must be quantifier-free definable. Now, we may define the graph R on $\{0, 1\} \times M$ as follows: For $a, b \in \mathcal{I}$ and

$\epsilon_0, \epsilon_1 \in \{0,1\}$,

$$R((\epsilon_0, a), (\epsilon_1, b)) \text{ if and only if } aEb.$$

By definition, the edges between the nodes $(\epsilon_0, a), (\epsilon_1, b)$ depend only on the quantifier-free type of a, b, and the ϵ_i. Since E is non-trivial there is a class with at least two elements; thus, the graph is not contained in the graphs C, D from Definition 4.5.13. It follows that the graph must be connected. But then, by transitivity of E, we must have that aEb for all $a, b \in I$. Thus, E is the universality equivalence relation, contradicting our initial assumption.

Now, for the implication (2) $\implies$ (1): Assume $\mathcal{I}$ is not reasonable. Then, by definition, we can find a graph R on $\{0,1\} \times \mathcal{I}$ such that $R((\epsilon_0, a), (\epsilon_1, b))$ depends only on ϵ_0, ϵ_1 and on the quantifier-free type of a, b, which is not connected, and not included in the graphs C and D from Definition 4.5.13. We can define two equivalence relations E_ϵ, for $\epsilon \in \{0,1\}$, by setting $aE_\epsilon b$ if, and only if, (ϵ, a) and (ϵ, b) are joined by an R-path. By assumption on R, it follows that E_ϵ is $\mathsf{Aut}(\mathcal{I})$-invariant. If either E_0 or E_1 is not trivial, then $\mathcal{I}$ is not primitive, so we are done. Thus, we may assume, without loss of generality, that they both are trivial.

Claim 1. It is not the case that both E_1 and E_2 have a single equivalence class.

Proof of Claim 1. Suppose not, then there are no edges of the form $R((0, a), (1, b))$, for otherwise the graph would be connected. But then, the graph would be included in D, which is absurd. ◀

Claim 2. If E_1 has one equivalence class and E_0 has only singleton classes, then $\mathcal{I}$ is not primitive.

Proof of Claim 2. First, since G is not included in the graph D from Definition 4.5.13, there must be at least one edge of the form

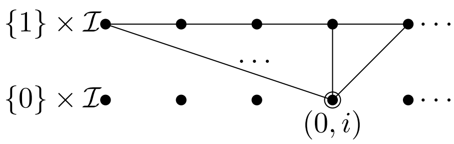
$$R((0, i), (1, j)).$$

Moreover, for all $i, i' \in \mathcal{I}$ such that:

$$R((0, i), (1, j)) \text{ and } R((0, i'), (1, j'))$$

then, since by assumption jE_1j', there is an R-path connecting $(1, j)$ and $(1, j')$, by definition of E_1, but then we have an R-path connecting $(0, i)$ and $(0, i')$, and thus iE_0i', which implies that $i = i'$, by assumption on E_0.

Thus there must be exactly one element $(0, i)$ which connects with some elements $(1, j)$ on the top row.



Then, we have a non trivial $\mathsf{Aut}(\mathcal{I})$-invariant equivalence relation E given by

$$E(j', j) \text{ if and only if } (j = i = j') \text{ or } (j \neq i \neq j'),$$

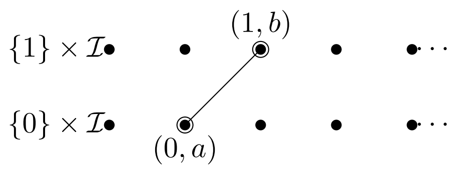
where $i \in \mathcal{I}$ is the unique element for which there exist $j \in \mathcal{I}$ such that $R((0, i), (1, j))$, thus $\mathcal{I}$ is not primitive. ◀

By symmetry, the result also follows if E_0 has one equivalence class and E_1 has only singleton classes.

Since we have assumed that both E_0 and E_1 are trivial and we have shown that they cannot both be universal, and that if one of the two is universal and the other is trivial with only singleton equivalence classes then $\mathcal{I}$ is not primitive, we may assume now that both E_0 and E_1 are trivial with singleton classes.

In particular, this means that R is the edge relation of a bipartite graph.

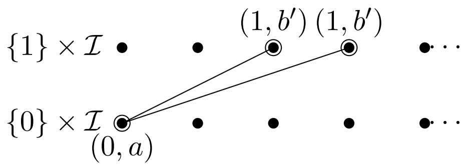
Let $a, b \in \mathcal{I}$. We say there is a *jump* from a to b if $R((0, a), (1, b))$, as shown in the figure below:



Observe that, by assumption on R, for any $\sigma \in \mathsf{Aut}(\mathcal{I})$ and any $a, b \in \mathcal{I}$, if there is a jump from a to b then there will also be a jump from $\sigma(a)$ to $\sigma(b)$.

We see that, given that E_1 is trivial with singleton classes, an element can jump to at most one element. Towards a contradiction, suppose not then there is some vertex with two different edges to $\{1\} \times \mathcal{I}$, say $R((0, a), (1, b))$ and

$R((0,a),(1,b'))$, but then $(1,b)$ and $(1,b')$ lie on the same path, which is a contradiction.



If an element does not jump to any other element, we can define a two-class $\mathsf{Aut}(\mathcal{I})$-invariant equivalence relation as follows: one class for jumping elements and one class for non-jumping elements. Similarly, if there is some $b \in \mathcal{I}$ such that no element jumps to b, we can again define an analogous two-class $\mathsf{Aut}(\mathcal{I})$-invariant equivalent relation.

Thus, without loss of generality, we may assume that we have a well-defined bijection $j : \mathcal{I} \to \mathcal{I}$ given by $a \mapsto b$, when a jumps to b. We have the following $\mathsf{Aut}(\mathcal{I})$-invariant equivalence relation $\sim_j$ given by:

$$a \sim_j b \text{ if, and only if, there is an integer } n \in \mathbb{Z} \text{ such that } j^n(a) = b.$$

Clearly, there must exist elements $a \neq b$ such that a jumps to b, otherwise R would be contained in the graph C from Definition 4.5.13. Thus, the equivalence relation $\sim_j$ has a non-singleton class. If $\sim_j$ is not universal, i.e. has at least two classes, then we are done. Otherwise, there is exactly one class, and we may consider the equivalence relation $\sim_{j^2}$ given by the squared function j^2, which has exactly 2 disjoint equivalence classes. We showed that $\mathcal{I}$ is not primitive. □

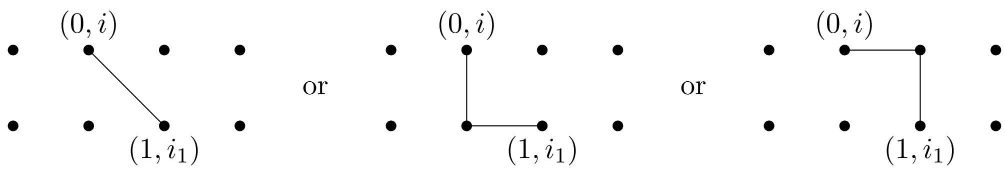
Lemma 4.5.17. *Let $\mathcal{I}$ be a transitive*[23] *Fraïssé limit of a free amalgamation class $\mathcal{C}$ of finite relational structures, possibly endowed with a generic order. Then $\mathcal{I}$ is a reasonable indexing structure.*

Proof. Let G be a graph on $\{0,1\} \times \mathcal{I}$ with edge relation $R((\epsilon_0, i), (\epsilon_1, j))$ that depends only on the quantifier-free type of (i,j) and $\epsilon_0, \epsilon_1 \in \{0,1\}$. Assume that G is not included in either C or D. We need to show that G is connected. Letting $i_0, i_0' \in \mathcal{I}$ be two distinct points, we will show that there is

[23] Recall that a structure $\mathcal{M}$ is called *transitive* if $\mathsf{Aut}(\mathcal{M})$ acts transitively on $\mathcal{M}$.

a path between $(0, i_0)$ and $(0, i'_0)$. The other paths from (ϵ_0, i_0) and (ϵ_1, i'_0) for $\epsilon_0, \epsilon_1 \in \{0, 1\}$ can be found similarly.

Claim 1. There exist $i, i_1 \in \mathcal{I}$ such that $i \neq i_1$ and there is a path of length at most 2 between $(0, i)$ and $(0, i_1)$ and such that the path has one of the following forms:



(Case 1) (Case 2) (Case 3)

Proof of Claim 1. The fact that there are distinct $i, i_1 \in \mathcal{I}$ for which there is a path of length at most 2 between $(0, i)$ and $(1, i_1)$ follows immediately from the fact that G is not included in C or D. So, we must argue that the path between $(0, i)$ and $(0, i_1)$ has one of the three forms from the statement of the claim.

To this end, suppose that we are not in the first case for any $i \neq i_1$. Then G does not contain edges of the form $R((0, i), (1, j))$, for $i \neq j$. Since G is not contained in either C or D it must contain at least one edge of the form $R((1, i), (1, j))$ and at least one edge of the form $R((0, i), (1, i))$ or at least one edge of the form $R((0, i), (0, j))$ and at least one edge of the form $R((0, i), (1, i))$. In the first case, it must contain all edges of the form $R((0, i), (1, i))$, since by transitivity $\mathsf{qftp}_{\mathcal{I}}(i)$ is constant, for all $i \in \mathcal{I}$, and thus it contains the second path, while in the latter case arguing similarly we see that G contains the third path. ◀

For the rest of this proof, we assume that we are in Case 1 of the claim, since the argument below will work identically for Cases 2 and 3.

By transitivity, we may assume that $i_0 = i$. By transitivity, again, there is some i'_1 with $\mathsf{qftp}_{\mathcal{I}}(i_0, i_1) = \mathsf{qftp}_{\mathcal{I}}(i'_0, i'_1)$. Without loss of generality, in what follows we assume that $i_1 < i_0$, in $\mathcal{I}$, and the case $i_0 < i_1$ follows by an almost identical argument.

By freely amalgamating $\{i_0, i'_0, i_1\}$ and $\{i_0, i'_0, i'_1\}$ over $\{i_0, i'_0\}$ we can find a structure $K = \{k_0, k'_0, k_1, k'_1\}$ in $\mathcal{C}$ such that:

1. $\mathsf{qftp}_{\mathcal{L}}(i_0, i'_0) = \mathsf{qftp}_{\mathcal{L}}(k_0, k'_0)$;
2. $\mathsf{qftp}_{\mathcal{L}}(i_0, i_1) = \mathsf{qftp}_{\mathcal{L}}(k_0, k_1) = \mathsf{qftp}_{\mathcal{L}}(k'_0, k'_1)$;
3. There are no relations between k_1 and k'_1 in K.

Now, since $\mathcal{I}$ is homogeneous, there is some $\sigma \in \mathsf{Aut}(\mathcal{I}|_{\mathcal{L}})$ such that $\sigma(k_0) = i_0$ and $\sigma(k'_0) = i'_0$. We let $i_2 := \sigma(k_1)$ and $i'_2 := \sigma(k'_1)$. It is now immediate, by construction, that

$$\mathsf{qftp}_{\mathcal{I}|_{\mathcal{L}}}(i_0, i_1) = \mathsf{qftp}_{\mathcal{I}|_{\mathcal{L}}}(i_0, i_2) = \mathsf{qftp}_{\mathcal{I}|_{\mathcal{L}}}(i'_0, i'_2),$$

and that there are no relations between i_2 and i'_2.

Since the order in $\mathcal{I}$ is generic, we can choose, in $\mathcal{I}$, i_2, i'_2 so that $i_2 < i_0$ and $i'_2 < i'_0$. In particular, this means that $\mathsf{qftp}_{\mathcal{I}}(i_0, i_1) = \mathsf{qftp}_{\mathcal{I}}(i_0, i_2) = \mathsf{qftp}_{\mathcal{I}}(i'_0, i'_2)$. So, by assumption on R, we have that:

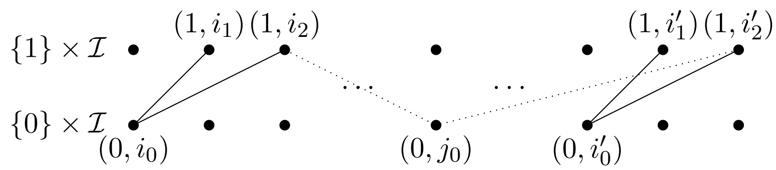
$$R((0, i_0), (1, i_2)) \text{ and that } R((0, i'_0), (1, i'_2))$$

We now use free amalgamation again, this time amalgamating $\{i_0, i_2\}$ and $\{i'_0, i'_2\}$ over i_0, which we identify with i'_0, since they have the same quantifier-free type, and we obtain a structure $P = \{p_0, p_2, p'_2\} \in \mathcal{C}$ where:

1. $\mathsf{qftp}_{\mathcal{L}}(p_0, p_2) = \mathsf{qftp}_{\mathcal{L}}(i_0, i_2)$;
2. $\mathsf{qftp}_{\mathcal{L}}(p_0, p'_2) = \mathsf{qftp}_{\mathcal{L}}(i_0, i'_2)$;
3. There are no relations between p_2 and p'_2 in P.

But observe that this means that $\mathsf{qftp}_{\mathcal{L}}(i_2, i'_2) = \mathsf{qftp}_{\mathcal{L}}(p_2, p'_2)$, so there is an automorphism $\sigma \in \mathsf{Aut}(\mathcal{I}|_{\mathcal{L}})$ such that $\sigma(p_2) = i_2$ and $\sigma(p'_2) = i'_2$.

Let $j_0 := \sigma(p_0)$:



Then we have that:

$$\mathsf{qftp}_{\mathcal{I}|_{\mathcal{L}}}(i_0, i_1) = \mathsf{qftp}_{\mathcal{I}|_{\mathcal{L}}}(j_0, i_2) = \mathsf{qftp}_{\mathcal{I}|_{\mathcal{L}}}(j_0, i'_2),$$

and, again, since the order is generic, we may assume that $j_0 > i_2, i'_2$. It follows that

$$\mathsf{qftp}_{\mathcal{I}}(i_0, i_1) = \mathsf{qftp}_{\mathcal{I}}(j_0, i_2) = \mathsf{qftp}_{\mathcal{I}}(j_0, i'_2).$$

Thus, G contains the path: $R((0, i_0), (1, i_2))$, $R((1, i_2), (0, j_0))$, $R((0, j_0), (1, i'_2))$, and $R((1, i'_2), (0, i'_0))$ as required. □

Remark 4.5.18. As an aside, combining Proposition 4.5.16 and Lemma 4.5.17 gives an extension of [MT11, Lemma 2.6] (which states that transitive Fraïssé limits of free amalgamation classes are primitive), to transitive Fraïssé limits of free amalgamation classes with a generic order. It was pointed out to me by Macpherson that an argument analogous to the one given in [MT11, Lemma 2.6] could be used to prove the lemma above. This would be done using D.G. Higman's criterion that a transitive permutation group H acting on a set X is primitive if and only if, for every orbit Ω of H on unordered pairs from X, the *orbital graph*[24] Γ_Ω with vertex set X and edge set Ω is connected. Such a proof works (and is rather simpler than the proof given above), but it is my feeling that the proof above illustrates better the definition of a reasonable structure.

Under this assumption of reasonable index sequence, it is now possible to describe the $\mathcal{I}$-indiscernible sequences in the lexicographic sum $\mathcal{S}$:

Proposition 4.5.19. *Let $\mathcal{M}$ be an $\mathcal{L}_{\mathcal{M}}$-structure, $\mathfrak{N} = \{\mathcal{N}_a\}_{a\in\mathcal{M}}$ be a collection of $\mathcal{L}_{\mathfrak{N}}$-structures indexed by $\mathcal{M}$ and let $\mathcal{S} = \mathcal{M}[\mathfrak{N}]$. Let $\mathcal{I}$ be a reasonable homogeneous indexing structure, and let $\mathfrak{I} = (c_i : i \in \mathcal{I})$ be a sequence of tuples of length λ (not necessarily finite) in $\mathcal{S}$. For $a \in \mathcal{M}$, denote by $\mathfrak{I}_a := (c^a_i : i \in I)$ the sequence of subtuples $c^a_i := (c_{\kappa_0,i}, c_{\kappa_1,i}, \cdots) \subseteq c_i$ consisting of all the elements $c_{\kappa_j,i}$ of c_i such that $\mathsf{v}(c_{\kappa_j,i})$ is equal to a. Then:*

I. If $(c_i : i \in \mathcal{I})$ is $\mathcal{I}$-indiscernible, then for all $\kappa < \lambda$ and all distinct $i, j \in \mathcal{I}$, if $\mathsf{v}(c_{\kappa,i}) = \mathsf{v}(c_{\kappa,j})$, then the sequence $(\mathsf{v}(c_{\kappa,i}) : i \in \mathcal{I})$ is constant.

We denote by $A \subseteq \mathcal{M}$ the subset of elements $a \in \mathcal{M}$ such that for some $\kappa < \lambda$ and all $i \in \mathcal{I}$, $\mathsf{v}(c_{\kappa,i}) = a$. In this notation we have that:

II. The following are equivalent, for any $B \subseteq \mathcal{S}$:

1. The sequence $\mathfrak{I}$ is $\mathcal{I}$-indiscernible over B.

2. The following conditions hold:

[24]The graph whose edge set is given by an orbit Ω of H on unordered pairs from X.

(a) *For all $a \in A$, the sequence $\mathfrak{I}_a$ is $\mathcal{I}$-indiscernible in $\mathcal{N}_a$ over $B \cap \mathcal{N}_a$.*

(b) *The sequence $\mathsf{v}(\mathfrak{I}) := (\mathsf{v}(c_i) : i \in \mathcal{I})$, where $\mathsf{v}(c_i)$ denotes the tuple $\mathsf{v}(c_{\kappa,i})_{\kappa<\lambda}$, is $\mathcal{I}$-indiscernible over $\mathsf{v}(B)$ in $\mathcal{M}$ and $\mathsf{tp}(c_i)$ is constant.*

Proof.

I. Fix $\kappa < \lambda$ and consider the graph G on $\{0,1\} \times \mathcal{I}$ whose edge relation is defined as follows:

$$R((\epsilon_0, i), (\epsilon_1, j)) \text{ if, and only if, } \mathsf{v}(c_{\kappa,i}) = \mathsf{v}(c_{\kappa,j}),$$

for any $\epsilon_0, \epsilon_1 \in \{0,1\}$.

Since $(c_i : i \in \mathcal{I})$ is $\mathcal{I}$-indiscernible, it follows that in G whether $R((\epsilon_0, i), (\epsilon_1, j))$ holds depends only on $\mathsf{qftp}(i, j)$. Suppose that for some distinct $i, j \in \mathcal{I}$ we have that $\mathsf{v}(c_{\kappa,i}) = \mathsf{v}(c_{\kappa,j})$. Thus, since $\mathcal{I}$ is reasonable, the graph G must be connected, since we have $R((0,k)(1,k))$, for all $k \in \mathcal{I}$, and by assumption $R((0,i),(0,j))$, and $R((0,i),(1,j))$. Thus, by construction, this means that $(\mathsf{v}(c_{\kappa,i}) : i \in \mathcal{I})$ is constant.

II. First, we deal with $1 \implies 2$ (the less interesting implication). Suppose that $\mathfrak{I} = (c_i : i \in \mathcal{I})$ is $\mathcal{I}$-indiscernible over B in $\mathcal{S}$.

(a) Let $a \in A$ (where a is as in the statement of the proposition). Given $i_1, \dots, i_n, j_1, \dots, j_n \in I$ such that $\mathsf{qftp}(i_1, \dots, i_n) = \mathsf{qftp}(j_1, \dots, j_n)$, we have that

$$\mathsf{tp}(c_{i_1}, \dots, c_{i_n}/B) = \mathsf{tp}(c_{j_1}, \dots, c_{j_n}/B),$$

and hence:

$$\mathsf{tp}(c^a_{\bar{i}}, /B \cap \mathcal{N}_a) = \mathsf{tp}(c^a_{\bar{j}}/B \cap \mathcal{N}_a),$$

where $c^a_{\bar{i}}$ is the concatenation of subtuples of c_{i_k} (for $k \leq n$) whose valuation is a.

(b) By Proposition 4.5.16, $\mathcal{I}$ is primitive, and therefore transitive, so, for all $i, j \in \mathcal{I}$ we have that $\mathsf{qftp}(i) = \mathsf{qftp}(j)$. We must thus have that $\mathsf{tp}(c_i) = \mathsf{tp}(c_j)$. The fact that $\mathsf{v}(\mathfrak{I})$ is indiscernible over $\mathsf{v}(B)$ is shown exactly as in (a) above.

Now for 2 $\implies$ 1. Let $(c_i : i \in I)$ be a sequence of tuples of size λ satisfying 2. Let $\phi(\mathbf{x}_0, \dots, \mathbf{x}_n)$ be an $\mathcal{L}_{\mathcal{M}[\mathfrak{N}]}$-formula, where $|\mathbf{x}_0| = \dots = |\mathbf{x}_{n-1}| = \lambda$. We denote by $\mathbf{x}$ the tuple $\mathbf{x}_0, \dots, \mathbf{x}_{n-1}$. Once we Morleyise the structures $\mathcal{M}$ and $\mathcal{N}_p$ for $p \in \mathsf{S}_1^{\mathsf{Th}(\mathcal{M})}$, we may assume, by relative quantifier elimination (Fact 4.2.11) that $\phi(\mathbf{x}_1, \dots, \mathbf{x}_n)$ is equivalent to a Boolean combination of formulas of the form:

- $\mathsf{v}(\mathbf{x}_{\kappa,l}) = \mathsf{v}(\mathbf{x}_{\kappa',m})$ for some $m < l < n, \kappa, \kappa' < \lambda$,
- $P_\bullet(\hat{\mathbf{x}})$, for $P \in \mathcal{L}_{\mathfrak{N}}$ and subtuples $\hat{\mathbf{x}}$ of $\mathbf{x}$,
- $P(\mathsf{v}(\hat{\mathbf{x}}))$, for $P \in \mathcal{L}_{\mathcal{M}}$ and subtuples $\hat{\mathbf{x}}$ of $\mathbf{x}$.

So, it suffices to check indiscernibility with respect to each formula $\psi(\mathbf{x}_1, \dots, \mathbf{x}_n)$ as above. More precisely, for all $i_0, \dots, i_n \in \mathcal{I}$ and all $i'_0, \dots, i'_n \in \mathcal{I}$ such that $\mathsf{qftp}(\bar{i}) = \mathsf{qftp}(\bar{i'})$ we have to show that

$$\mathcal{S} \models \psi(\bar{c}_{\bar{i}}) \leftrightarrow \psi(\bar{c}_{\bar{i}'}).$$

We start with the following, easy claim:
Claim 1. Fix $\kappa_0, \dots, \kappa_n < \lambda$. Let $i_0, \dots, i_n \in I$ not all equal. If

$$\mathsf{v}(c_{\kappa_0,i_0}) = \dots = \mathsf{v}(c_{\kappa_n,i_n}) = a,$$

for some $a \in \mathcal{M}$, then for all $i \in I$, we have that $\mathsf{v}(c_{\kappa_0,i}) = \dots = \mathsf{v}(c_{\kappa_n,i}) = a$.
Proof of Claim 1. For $n = 1$, the result follows immediately from Definition 4.5.13 applied to the graph G on $\{0,1\} \times I$, with edge relation given by:

$$R((\epsilon_0, i), (\epsilon_1, j)) \text{ if, and only if, } \mathsf{v}(c_{\kappa_{\epsilon_0},i}) = \mathsf{v}(c_{\kappa_{\epsilon_1},j}).$$

Clearly, in G whether $R((\epsilon_0, i), (\epsilon_1, j))$ holds depends only on $\mathsf{qftp}(i, j)$ and ϵ_0, ϵ_1, by indiscernibility of $\mathfrak{I}$. Since $i_0 \neq i_1$, the graph G is not contained in C, and since the graph has edges between $\{0\} \times \mathcal{I}$ and $\{1\} \times \mathcal{I}$, it is not contained in D. Thus, G is connected.

The statement for $n > 1$ follows by inductively repeating this argument, assuming, without loss of generality, that $i_0 \neq i_1$. ◀

If $i_0, \dots, i_n$ are all equal to some i, then, since $(\mathsf{tp}(c_i) : i \in \mathcal{I})$ is constant, for any formula $\psi(\mathbf{x}_0, \dots, \mathbf{x}_n)$, $\psi(c_{\kappa_0,i}, \dots, c_{\kappa_n,i})$ holds if, and only if,

$\psi(c_{\kappa_0,i'},\ldots,c_{\kappa_n,i'})$ holds for all $i' \in I$.

We may now assume that $i_0,\ldots,i_n$ are not all equal.

– Formulas of the form $\mathsf{v}(\mathbf{x}_{\kappa_0,i_0}) = \mathsf{v}(\mathbf{x}_{\kappa_1,i_1})$. We assumed that $i_0 \neq i_1$ and by the previous claim, for any $\kappa_0,\kappa_1 < \lambda$ we have that $\mathsf{v}(c_{\kappa_0,i_0}) = \mathsf{v}(c_{\kappa_1,i_1}) = a$ for some $a \in \mathcal{M}$ if, and only if, for all $i' \in I$, $\mathsf{v}(c_{\kappa_0,i'}) = \mathsf{v}(c_{\kappa_1,i'}) = a$. Thus, this case follows.

– Formulas of the form $P_\bullet(c_{\kappa_0,i_0},\ldots,c_{\kappa_n,i_n})$. If $\mathsf{v}(c_{\kappa_0,i_0}),\ldots,\mathsf{v}(c_{\kappa_n,i_n})$ are not all equal, then by the previous claim, for all $i'_0,\ldots,i'_n \in I$ with same quantifier-free type in $\mathcal{I}$, $\mathsf{v}(c_{\kappa_0,i'_0}),\ldots,\mathsf{v}(c_{\kappa_n,i'_n})$ are not all equal. In particular, for any $i'_0,\cdots,i'_n$ with the same quantifier-free type as $i_0,\cdots,i_n$, $P_\bullet(c_{\kappa_0,i'_0},\ldots,c_{\kappa_n,i'_n})$ doesn't hold.

  Thus, we may assume that there is $a \in \mathcal{M}$ such that $\mathsf{v}(c_{\kappa_1,i}) = \cdots = \mathsf{v}(c_{\kappa_n,i}) = a$ for all $i \in I$. This means that $c_{\kappa_1,i},\ldots,c_{\kappa_n,i} \in c_i^a$. Then, since, by assumption, (c_i^a) is $\mathcal{I}$-indiscernible in $\mathcal{N}_a$, we have that:

$$\mathcal{N}_a \vDash P(c_{\kappa_0,i_0},\ldots,c_{\kappa_n,i_n}) \text{ if, and only if, } \mathcal{N}_a \vDash P(c_{\kappa_0,i'_0},\ldots,c_{\kappa_n,i'_n}),$$

  for any $i'_0,i'_1,\ldots,i'_n$ with the same quantifier-free type in $\mathcal{I}$. This shows that $(c_i : i \in \mathcal{I})$ is $\mathcal{I}$-indiscernible with respect to the formula $P_\bullet(c_{\kappa_0,i_0},\ldots,c_{\kappa_n,i_n})$, as required.

– Formulas of the form $P(\mathsf{v}(c_{\kappa_0,i_0}),\cdots,\mathsf{v}(c_{\kappa_n,i_n}))$ where $P \in \mathcal{L}_\mathcal{M}$. This case is clear by the assumption that $(\mathsf{v}(c_{i_1}))_{i\in\mathcal{I}}$ is $\mathcal{I}$-indiscernible in $\mathcal{M}$.

This concludes the proof.

□

Remark 4.5.20. In the literature, it is sometimes the case that indiscernibles indexed by imprimitive (i.e. non-reasonable) structures are considered (see, for instance, [GHS17, Theorem 5.8] for a nice characterisation of NTP_2 theories). Partial results in this direction have already been obtained jointly with Meir and Touchard, and the full imprimitive case will be presented in future work.

Following the structure of the previous subsection on full products, the proposition above gives the following:

Corollary 4.5.21. *Let $\mathcal{M}$ be an $\mathcal{L}_\mathcal{M}$-structure, $\mathfrak{N} = \{\mathcal{N}_a\}_{a\in\mathcal{M}}$ be a collection*

$\mathfrak{N}$ of $\mathcal{L}_{\mathfrak{N}}$-structures indexed by $\mathcal{M}$ and let $\mathcal{S} = \mathcal{M}[\mathfrak{N}]$. Let $\mathcal{I}$ and $\mathcal{J}$ be two reasonable homogeneous indexing structures, such that $\mathcal{J}$ is a reduct of $\mathcal{I}$. Then, the following are equivalent:

(1) $\mathcal{S}$ collapses $\mathcal{I}$-indiscernibles to $\mathcal{J}$-indiscernibles.

(2) For all $p \in \mathsf{S}_1^{\mathsf{Th}(\mathcal{M})}$, $\mathcal{N}_p$ collapses $\mathcal{I}$-indiscernibles to $\mathcal{J}$-indiscernibles, and $\mathcal{M}$ collapses $\mathcal{I}$-indiscernibles to $\mathcal{J}$-indiscernibles.

Proof. As a monster model of $\mathsf{Th}(\mathcal{S})$ still is a lexicographic sum of a model of $\mathsf{Th}(\mathcal{M})$ with models $\mathsf{Th}(\mathcal{N}_p)$ where $p \in \mathsf{S}_1^{\mathsf{Th}(\mathcal{M})}$, so we may assume that $\mathcal{S} = \mathcal{M}[\mathfrak{N}]$ is a monster model. Let $(c_i : i \in \mathcal{I})$ be an $\mathcal{I}$-indiscernible sequence in $\mathcal{S}$. By Proposition 4.5.19, and using the same notation, we have that:

1. For all $a \in A$, the sequence $(c_i^a : i \in I)$ is $\mathcal{I}$-indiscernible in $\mathcal{N}_a$ over $B \cap \mathcal{N}_a$.
2. The sequence $(\mathsf{v}(c_i) : i \in I)$, where $\mathsf{v}(c_i)$ denotes the tuple $\mathsf{v}(c_{\kappa,i} : \kappa < \lambda)$, is $\mathcal{I}$-indiscernible over $\mathsf{v}(B)$ in $\mathcal{M}$ and $\mathtt{tp}(c_i)$ is constant.

By assumption, $\mathcal{M}$ and $\mathcal{N}_a$ for all $a \in \mathcal{M}$ collapse $\mathcal{I}$-indiscernible sequences to $\mathcal{J}$-indiscernible sequences. In particular, we have:

1. For all $a \in A$, $(c_i^a : i \in I)$ is $\mathcal{J}$-indiscernible in $\mathcal{N}_a$ over $B \cap \mathcal{N}_a$.
2. The sequence $(\mathsf{v}(c_i) : i \in I)$ is $\mathcal{J}$-indiscernible over $\mathsf{v}(B)$ in $\mathcal{M}$ and $\mathtt{tp}(c_i)$ is constant.

By Proposition 4.5.19, $(c_i : i \in I)$ is a $\mathcal{J}$-indiscernible sequence in $\mathcal{S}$. This concludes the proof. □

Observe that $\mathsf{N}\mathscr{C}_{\mathbb{K}}$ is stable under reducts, and thus Example 4.5.4 also applies to lexicographic products. More precisely:

Remark 4.5.22. Let $\mathcal{M}$ be the ordered random graph and $\mathcal{N}$ the convexly ordered C-relation, as in Example 4.5.4. Then $\mathcal{M}[\mathcal{N}]$ is in $\mathscr{C}_{\mathcal{C}_{\mathsf{OG}} \boxtimes \mathcal{C}_{\mathsf{OC}}}$ which by Fact 4.3.16 is actually equal to $\mathscr{C}_{\mathcal{C}_{\mathsf{OG}}[\mathcal{C}_{\mathsf{OC}}]}$. But, as we have already seen $\mathcal{M}$ is in $\mathsf{N}\mathscr{C}_{\mathcal{C}_{\mathsf{OC}}} \subset \mathsf{N}\mathscr{C}_{\mathcal{C}_{\mathsf{OG}} \boxtimes \mathcal{C}_{\mathsf{OC}}}$ and $\mathcal{N}$ is in $\mathsf{N}\mathscr{C}_{\mathcal{C}_{\mathsf{OG}}} \subset \mathsf{N}\mathscr{C}_{\mathcal{C}_{\mathsf{OG}} \boxtimes \mathcal{C}_{\mathsf{OC}}}$.

Applying Corollary 4.5.21 to the specific case of $\mathcal{C}_{\mathsf{H}_{n+1}}$-indiscernibles, where $\mathcal{C}_{\mathsf{H}_{n+1}}$ is the ordered random $(n+1)$-hypergraph, since the latter is primitive, gives:

Corollary 4.5.23 (NIP_n transfer for lexicographic sum)**.** *Let $\mathcal{M}$ be an $\mathcal{L}_{\mathcal{M}}$-structure, $\mathfrak{N} = \{\mathcal{N}_a\}_{a\in\mathcal{M}}$ be a collection of $\mathcal{L}_{\mathfrak{N}}$-structures indexed by $\mathcal{M}$ and let $\mathcal{S} = \mathcal{M}[\mathfrak{N}]$. For all $n \in \mathbb{N}$, the following are equivalent:*

(1) The lexicographic product $\mathcal{S}$ is NIP_n.

(2) Both $\mathcal{M}$ and $\mathcal{N}_p$, for all $p \in \mathsf{S}_1^{\mathsf{Th}(\mathcal{M})}$ are NIP_n.

Since for positive integers $m > n$, if $\mathcal{M}$ is NIP_n, then it is NIP_m, it is easy to see that a lexicographic product $\mathcal{M}[\mathcal{N}]$ of an NIP_m structure $\mathcal{M}$ with an NIP_n structure $\mathcal{N}$ is $\mathrm{NIP}_{\max(n,m)}$.

Next, I shall show that indiscernible triviality transfers to lexicographic sums. In the proof of the proposition below, I will need to make use of Assumption ($\star$), from page 137. Let me briefly recall it:

$$\text{For } \phi \in \mathcal{L}_{\mathfrak{N}}, \text{ the set } \{a \in \mathcal{M} : \mathcal{N}_a \vDash \phi\} \text{ is } \emptyset\text{-definable in } \mathcal{M}. \tag{$\star$}$$

The point is that if $\mathcal{S} = \mathcal{M}[\{\mathcal{N}_a\}_{a\in\mathcal{M}}]$ is a lexicographic sum satisfying ($\star$), then, for any $a, a' \in \mathcal{M}$, if $\mathsf{tp}(a) = \mathsf{tp}(a')$ then $\mathcal{N}_a \equiv \mathcal{N}_{a'}$. With this out of the way, let's get into the proof.

Proposition 4.5.24. *Let $\mathcal{M}$ be an $\mathcal{L}_{\mathcal{M}}$-structure, $\mathfrak{N} = \{\mathcal{N}_a\}_{a\in\mathcal{M}}$ be a collection of $\mathcal{L}_{\mathfrak{N}}$-structures indexed by $\mathcal{M}$ and let $\mathcal{S} = \mathcal{M}[\mathfrak{N}]$. Then, the following are equivalent:*

(1) The lexicographic product $\mathcal{S}$ has indiscernible triviality.

(2) Both $\mathcal{M}$ and $\mathcal{N}_p$, for all $p \in \mathsf{S}_1^{\mathsf{Th}(\mathcal{M})}$, have indiscernible triviality.

Proof. Again, we may assume that we are working in a monster model. Note, also, that may work with indiscernible sequences indexed by $(\mathbb{Q}, <)$, which is a homogeneous reasonable structure, rather than indiscernible sequences indexed by $(\mathbb{N}, <)$.

First, suppose that $\mathcal{S}$ has indiscernible triviality.

- $\mathcal{M}$ has indiscernible triviality: Let $(\bar{a}_i : i \in \mathbb{Q})$ be an (order)-indiscernible sequence of k-tuples in $\mathcal{M}$ and let $b, b' \in M$ such that $(\bar{a}_i : i \in \mathbb{Q})$ is indiscernible over b and over b'. We wish to show that it is indiscernible over $\{b, b'\}$. Without loss of generality, we may assume that all elements of the tuples in $(\bar{a}_i : i \in \mathbb{Q})$ are distinct, that is, for all $i, i' \in \mathbb{Q}$ and all $j, j' \leq k$ we have that $a_i^j \neq a_{i'}^{j'}$. Observe also that by ($\star$) and indiscernibility of $(\bar{a}_i : i \in \mathbb{Q})$ it follows that for all $i, i' \in \mathbb{Q}$ and all $j \leq k$ we have that $\mathcal{N}_{a_i^j} \equiv \mathcal{N}_{a_{i'}^j}$. Pick, for each $i \in \mathbb{Q}$ and $j \leq k$ an element $c_i^j \in \mathcal{N}_{a_i^j}$, so that when elements are chosen from elementarily equivalent structures, they can be chosen to satisfy the same arbitrarily chosen but fixed 1-type (which is possible since we are working in a monster model). For each $i \in \mathbb{Q}$ and each $j \leq k$ fix an element $c_i^j \in N_{a_i^j}$ realising our fixed type, and let $(\bar{a}_i, \bar{c}_i) = (a_i^1, c_i^1)^\frown \cdots ^\frown (a_i^k, c_i^k)$. By construction, the sequence $((\bar{a}_i, \bar{c}_i) : i \in \mathbb{Q})$ satisfies the two conditions of II.2 of Proposition 4.5.19, and is therefore indiscernible over (b, c) and over (b', c'), for any $c \in N_b$ and $c' \in N_{b'}$, in $\mathcal{S}$. Since $\mathcal{S}$ has indiscernible triviality, it follows that it is indiscernible over $\{(b, c), (b', c')\}$, and thus by Proposition 4.5.19 again, $(\bar{a}_i : i \in \mathbb{Q})$ is indiscernible over $\{b, b'\}$, as required.

- $\mathcal{N}_p$ has indiscernible triviality, for all $p \in \mathsf{S}_1^{\mathsf{Th}(\mathcal{M})}$: Fix some $p \in \mathsf{S}_1^{\mathsf{Th}(\mathcal{M})}$ and let $a \in \mathcal{M}$ realise p. Consider a sequence of k-tuples $(\bar{c}_i : i \in \mathbb{Q})$ from $\mathcal{N}_a$, which is indiscernible over $b \in N_a$ and over $b' \in N_a$. The sequence $((\bar{a}, \bar{c}_i) : i \in \mathbb{Q})$, where $(\bar{a}, \bar{c}_i) := (a, c_i^1)^\frown \cdots ^\frown (a, c_i^k)$ for each $i \in \mathbb{Q}$, satisfies the two conditions of II.2 of Proposition 4.5.19, and is therefore indiscernible over (a, b) and over (a, b'). Since $\mathcal{S}$ has indiscernible triviality, it follows that this sequence is indiscernible over $\{(a, b), (a, b')\}$, and using Proposition 4.5.19 again, it follows that $(\bar{c}_i : i \in \mathbb{Q})$ is indiscernible over $\{b, b'\}$, as required.

For the other implication, let $(\bar{c}_i : i \in \mathbb{Q})$ be a sequence of k-tuples in $\mathcal{S}$, which is indiscernible over $b \in \mathcal{S}$ and over $b' \in \mathcal{S}$. Then, of course, both $(\bar{c}_i{}^\frown b : i \in \mathbb{Q})$ and $(\bar{c}_i{}^\frown b' : i \in \mathbb{Q})$ are indiscernible in $\mathcal{S}$. For $a \in \mathcal{M}$ denote by b^a the second coordinate of b, if $\mathsf{v}(b) = a$, and the empty word $\emptyset$, otherwise (and similarly for b'^a). By Proposition 4.5.19, and using the same notation, we have:

1. For all $a \in A$ the sequences $(\bar{c}_i^a{}^\frown b^a : i \in \mathbb{Q})$ and $(\bar{c}_i^a{}^\frown b'^a : i \in \mathbb{Q})$ are indiscernible in $\mathcal{N}_a$.
2. $(\mathsf{v}(\bar{c}_i)^\frown \mathsf{v}(b) : i \in \mathbb{Q})$ and $(\mathsf{v}(\bar{c}_i)^\frown \mathsf{v}(b') : i \in \mathbb{Q})$ are indiscernible in $\mathcal{M}$ and $\mathsf{tp}(\bar{c}_i{}^\frown b)$ and $\mathsf{tp}(\bar{c}_i{}^\frown b')$ are constant.

Since, by assumption, the $\mathcal{N}_a$'s and $\mathcal{M}$ have indiscernible triviality we have that:

1. For all $a \in A$ the sequences $(\bar{c}_i^a {}^\frown b^a {}^\frown b'^a : i \in \mathbb{Q})$ is indiscernible in $\mathcal{N}_a$.
2. $(\mathsf{v}(c_i)^\frown \mathsf{v}(b)^\frown \mathsf{v}(b') : i \in \mathbb{Q})$ is indiscernible in $\mathcal{M}$.

Finally, by relative quantifier elimination and the above, $\mathsf{tp}(\bar{c}_i{}^\frown b^\frown b')$ is constant. So, by Proposition 4.5.19, we get that $(\bar{c}_i{}^\frown b^\frown b' : i \in \mathbb{Q})$ is indiscernible, and thus $\mathcal{S}$ has indiscernible triviality, as claimed. □

The next goal is to prove a transfer principle for monadic NIP. Indiscernible triviality has just been dealt with, so by Fact 4.2.19, it remains to show that dp-minimality transfers to lexicographic products. Thankfully, this was handled by Touchard:

Fact 4.5.25 ([Tou21, Theorem 2.16])**.** *Let $\mathcal{M}$ be an $\mathcal{L}_\mathcal{M}$-structure, $\mathfrak{N} = \{\mathcal{N}_a\}_{a\in\mathcal{M}}$ be a collection of $\mathcal{L}_\mathfrak{N}$-structures indexed by $\mathcal{M}$ and let $\mathcal{S} = \mathcal{M}[\mathfrak{N}]$. Then:*

$$\mathsf{bdn}(\mathcal{S}) = \sup\left(\{\mathsf{bdn}(\mathcal{M})\} \cup \left\{\mathsf{bdn}(\mathcal{N}_p) : p \in \mathsf{S}_1^{\mathsf{Th}(\mathcal{M})}\right\}\right).$$

In particular,[25] *the following are equivalent:*

(1) The lexicographic product $\mathcal{S}$ is inp-*minimal.*[26]

(2) Both $\mathcal{M}$ and $\mathcal{N}_p$, for all $p \in \mathsf{S}_1^{\mathsf{Th}(\mathcal{M})}$, are inp-*minimal.*

Putting Fact 4.2.19, Proposition 4.5.24, Corollary 4.5.23 (for $n = 1$) and Fact 4.5.25 together:

Corollary 4.5.26 (Transfer Principle For Monadic NIP Structures)**.** *Let $\mathcal{M}$ be an $\mathcal{L}_\mathcal{M}$-structure, $\mathfrak{N} = \{\mathcal{N}_a\}_{a\in\mathcal{M}}$ be a collection of $\mathcal{L}_\mathfrak{N}$-structures indexed by $\mathcal{M}$ and let $\mathcal{S} = \mathcal{M}[\mathfrak{N}]$. Then, the following are equivalent:*

(1) The lexicographic product $\mathcal{S}$ is monadically NIP.

(2) Both $\mathcal{M}$ and $\mathcal{N}_p$, for all $p \in \mathsf{S}_1^{\mathsf{Th}(\mathcal{M})}$, are monadically NIP.

Remark 4.5.27. Observe (it will be useful later) that Assumption ($\star$) is not actually required for Corollary 4.5.26, as if $\mathcal{M}$ is monadically NIP, then the expansion of $\mathcal{M}$ by unary predicates for the sets $\{a \in \mathcal{M} : \mathcal{N}_a \vDash \phi\}$, as ϕ ranges over $\mathcal{L}_\mathfrak{N}$-sentences will be automatically monadically NIP, by definition.

[25]The operator "bdn" above refers to *burden*, a notion introduced by Adler in [Adl07]. The precise definition is not relevant for the rest of the thesis, so I shall omit it and refer the reader to Adler's pre-print for more information. What matters for the purposes of this thesis is the rest of the statement.

[26]The notion of inp-*minimality* was introduced by Shelah in [She90, Chapter 3]. Again, the precise definition is not terribly important for our purposes. It suffices to recall the well-known fact that within NIP theories *inp*-minimality coincides with dp-minimality (see, for instance, [Adl07]).

Example 4.5.28. Given two meet-trees T_1 and T_2, the lexicographic product $T_1[T_2]$ is a meet-tree with an equivalence relation. By Corollary 4.5.26 and Example 4.2.22, it is monadically NIP. More generally, any meet-tree with finitely many equivalence relations $E_0, \dots, E_n$ such that E_{s+1} refines E_s for $s < n$ is monadically NIP.

The final result of this section is the promised transfer of m-distality to lexicographic products.

Proposition 4.5.29 (m-Distality transfer for lexicographic product)**.** *Let $\mathcal{M}$ be an $\mathcal{L}_{\mathcal{M}}$-structure, $\mathfrak{N} = \{\mathcal{N}_a\}_{a\in\mathcal{M}}$ be a collection of $\mathcal{L}_{\mathfrak{N}}$-structures indexed by $\mathcal{M}$ and let $\mathcal{S} = \mathcal{M}[\mathfrak{N}]$. For all $n \in \mathbb{N}$, the following are equivalent:*

(1) The lexicographic product $\mathcal{S}$ is m-distal.

(2) Both $\mathcal{M}$ and $\mathcal{N}_p$, for all $p \in \mathsf{S}_1^{\mathsf{Th}(\mathcal{M})}$, are m-distal.

Proof. Without loss, we may assume that $\mathcal{S} = \mathcal{M}[\mathfrak{N}]$ is a monster model. As in the proof for indiscernible triviality, we may assume that we are working with sequences indexed by $(\mathbb{Q}, <)$, rather than an arbitrary infinite linear order.

First, suppose that $\mathcal{S}$ is m-distal.

- $\mathcal{M}$ is m-distal: Let $\mathfrak{I} = \mathfrak{I}_0 + \cdots + \mathfrak{I}_{m+1}$ be an indiscernible sequence in $\mathcal{M}$, which is partitioned into $m+2$ subsequences (all of which are without loss indexed by $(\mathbb{Q}, <)$), and let $\bar{c}$ be an $(m+1)$-tuple. Following the analogous argument in Proposition 4.5.24, we can produce an indiscernible sequence $\mathfrak{I}' = \mathfrak{I}'_0 + \cdots + \mathfrak{I}'_{m+1}$ in $\mathcal{S}$ and lift $\bar{c}$ into an $(m+1)$-tuple $\bar{c}'$ in $\mathcal{S}$, so that $\mathsf{v}(\mathfrak{I}') = \mathfrak{I}$ and $\mathsf{v}(\bar{c}') = \bar{c}$. If $\bar{c}$ does not insert indiscernibly into $\mathfrak{I}$, then, by Proposition 4.5.19 $\bar{c}'$ will not insert indiscernibly into $\mathfrak{I}'$, and since $\mathcal{S}$ is m-distal this means that an m-subtuple of $\bar{c}'$ does not insert indiscernibly into $\mathfrak{I}'$. Taking valuations, this means that an m-subtuble of $\bar{c}$ does not insert indiscernibly into $\mathfrak{I}$.

- $\mathcal{N}_p$ is m-distal, for all $p \in \mathsf{S}_1^{\mathsf{Th}(\mathcal{M})}$: The argument here is identical (in structure) to the one given in Proposition 4.5.24. Given any sequences/elements in $\mathcal{N}_a$ (for $a \vDash p$), consider the corresponding elements in the lexicographic sum, with first coordinate a. By Proposition 4.5.19, all indiscernibility properties transfer, and this implication follows.

For the other direction, let $\mathfrak{I} = \mathfrak{I}_0 + \cdots + \mathfrak{I}_{m+1}$ be an indiscernible sequence partitioned into $m + 2$ subsequences (all of which are without loss indexed by

$(\mathbb{Q}, <))$. Let $\bar{c} = (c_0 ^\frown \cdots ^\frown c_{m+1})$ be an $(m+1)$-tuple from $\mathcal{S}$ that does not insert indiscernibly into $\mathfrak{I}$. So, by definition, the sequence:

$$\mathfrak{I}' = \mathfrak{I}_0 + c_0 + \cdots + c_m + \mathfrak{I}_{m+1}$$

is not indiscernible in $\mathcal{S}$. By Proposition 4.5.19, and using the same notation, we have the following cases to consider:

- *Case 1*. For some $a \in A$, $\mathfrak{I}'_a$ is not indiscernible in $\mathcal{N}_a$. Then, since $\mathcal{N}_a$ is m-distal, there is a m-subtuple of $\bar{c}$ which does not insert indiscernibly into $\mathfrak{I}_a$ (which is an indiscernible sequence in $\mathcal{N}_a$).
- *Case 2*. The sequence $\mathsf{v}(\mathfrak{I}')$ is not indiscernible in $\mathcal{M}$. Then, since $\mathcal{M}$ is m-distal, there is an m-subtuple of $\mathsf{v}(\bar{c})$ which does not insert indiscernibly into $\mathsf{v}(\mathfrak{I})$.
- *Case 3*. The sequence of types of elements in $\mathfrak{I}'$ is not constant. Since $\mathfrak{I}$ was indiscernible, this means that there is an element $c_i \in \bar{c}$, which does not have the same type as the elements in $\mathfrak{I}$.

In all cases, there is an m-subtuple of $\bar{c}$ which does not insert indiscernibly into $\mathfrak{I}$. Indeed, in the first two cases this is by Proposition 4.5.19, and in the third case, we find in fact a single c_i which does not insert indiscernibly into $\mathfrak{I}$. □

4.6 Ultraproducts and Twin-width

Finally, the transfer principles for monadic NIP discussed in the previous section will now be applied to construct new "algorithmically tame" hereditary[27] classes of graphs from given ones. In this section, algorithmically tame shall mean "of *bounded twin-width*" (a notion which should be taken as a black box).

The notions in this section are rather disconnected from the rest of this thesis, so the reader may take the view that they are indeed interesting in good faith. For a general introduction and precise definitions, a good reference is [Bon+21], where the notion of twin-width was introduced. For a basic background in parametrised complexity theory (which is not strictly necessary to understand the results in this section, but could aid in putting them in some context) one may consult [FG06].

[27] Recall: A class $\mathcal{C}$ of graphs is hereditary if it is closed under *induced* subgraphs.

The next proposition shows that an ultraproduct of the class of lexicographic sums of two classes is isomorphic to a lexicographic sum of ultraproducts of these classes. More precisely:

Proposition 4.6.1. *Fix a cardinal κ and $\mathcal{U}$ an ultrafilter on κ. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two classes of (not necessarily finite) structures in languages $\mathcal{L}_{\mathcal{C}_1}$ and $\mathcal{L}_{\mathcal{C}_2}$, respectively. Let $(G^i[H^i_g])_{i<\kappa}$ be a sequence of lexicographic sums in $\mathcal{C}_1[\mathcal{C}_2]$. Then we have:*

$$\prod_{\mathcal{U}} \left(G^i \left[H^i_g\right]\right) \simeq \left(\prod_{\mathcal{U}} G^i\right) \left[\mathcal{H}_g\right],$$

where, for each $g = [(g_i)]_{\mathcal{U}}$, $\mathcal{H}_g = \prod_{\mathcal{U}} H^i_{g_i}$.

Proof. First, notice that $\mathcal{H}_g$ doesn't depend on the choice of representative $(g_i)_i$ of $g = [(g_i)]_{\mathcal{U}}$.

Each $G^i[H^i_g]$ for $i \in \kappa$ carries a definable (through the projection v) equivalence relation $\sim$:

$$(g, h) \sim (g', h') \text{ if, and only if, } g = g'.$$

Of course, by Łos's theorem, $\sim$ is an equivalence relation on $\mathcal{M} = \prod_{\mathcal{U}} \left(G^i[H^i_g]\right)$, and for $a = [((g_i, h_i))_i]_{\mathcal{U}}$ and $b = [((g'_i, h'_i))_i]_{\mathcal{U}}$ in $\mathcal{M}$ we have:

$$\begin{aligned} \mathcal{M} \vDash a \sim b &\iff \left\{i : G^i[H^i_g] \vDash (g_i, h_i) \sim (g'_i, h'_i)\right\} \in \mathcal{U} \\ &\iff (g_i)_i = (g'_i)_i \mod \mathcal{U}. \end{aligned}$$

Similarly, if $P \in \mathcal{L}_{\mathcal{C}_1}$, for any tuple $a = [((g_i, h_i))_i]_{\mathcal{U}} \in \mathcal{M}$:

$$\begin{aligned} \mathcal{M} \vDash P(\mathsf{v}(a)) &\iff \left\{i : G^i[H^i_g] \vDash P(\mathsf{v}(g_i, h_i))\right\} \in \mathcal{U} \\ &\iff \left\{i : G^i \vDash P(g_i)\right\} \in \mathcal{U}. \end{aligned}$$

It follows that $\mathcal{M}/\sim$ is isomorphic to $\prod_{\mathcal{U}} G^i$.

It remains to show that each equivalence class is isomorphic to an ultraproduct of structures in $\mathcal{C}_2$. Fix $a = [(g_i, h_i)]_{\mathcal{U}} \in \mathcal{M}$. By the above, for any $b \in \mathcal{M}$ we have that $\mathcal{M} \vDash a \sim b$ if, and only if, b has a representative in $\prod_{i\in\kappa}\{g_i\} \times H^i_{g_i}$. Thus the class $[a]_\sim$ of a is an isomorphic copy of $\mathcal{H}_g = \prod_{\mathcal{U}} H^i_{g_i}$. □

Definition 4.6.2. A class $\mathcal{C}$ of structures is called *monadically NIP* if any ultraproduct of structures in $\mathcal{C}$ is monadically NIP.

Corollary 4.6.3. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two classes of structures that are monadically*

NIP. Then their lexicographic sum is monadically NIP.

Proof. This follows immediately Corollary 4.5.26, the fact that monadic NIP is closed under reducts and Proposition 4.6.1. □

It is conjectured (see, for instance, [Gaj+20, Conjecture 8.2]) that for hereditary classes of graphs, under a mild assumption from descriptive complexity theory (namely that $\mathsf{FPT} \neq \mathsf{AW}[\star]$, which, in particular, implies that first-order model checking for the class of all graphs is not tractable), the algorithmic tameness condition of having *fixed-parameter tractable model checking* coincides with the class being monadically NIP. There is strong evidence for this conjecture. In particular, in joint work with Braunfeld, Dawar, and Eleftheriadis [Bra+23], we show that the conjecture is, in fact, true for *monotone*[28] classes of relational structures, by using some nice (canonical) Ramsey arguments and Gaifman graphs, but this is perhaps a story for another day. More importantly, the conjecture is shown to be true for hereditary classes of ordered graphs [Bon+22a]. More precisely, the latter result is the following:

Fact 4.6.4 ([Bon+22a, Theorems 1 and 3])**.** *Let $\mathcal{C}$ be a hereditary class of finite, ordered binary structures. Then, the following are equivalent:*

1. *$\mathcal{C}$ is monadically NIP.*
2. *$\mathcal{C}$ has bounded twin-width.*
3. *(Assuming $\mathsf{FPT} \neq \mathsf{AW}[\star]$) Model-checking first-order logic is fixed-parameter tractable on $\mathcal{C}$.*

It is a fact that any finite graph G can be expanded by a total order resulting in an *ordered graph* of the same twin-width (see [Bon+21] or [ST21] for an explicit argument), but it is well-known that finding these orders is hard to do in a computationally efficient manner. In any case, given a class $\mathcal{C}$ of graphs with bounded twin-width we can assume that $\mathcal{C}$ consists of *ordered* graphs and still has bounded twin-width. Moreover, given any graph G, the twin-width of any (induced) subgraph of G is at most that of G (see [Bon+21]). In particular, given a class of graphs $\mathcal{C}$ with bounded twin-width we may assume that $\mathcal{C}$ is hereditary.

One of the results in [Bon+22b] is that the lexicographic product of two graphs

[28] Recall: A class of relational structures is called *monotone* if it is closed under *not necessarily induced substructures.*

of bounded twin-width has bounded twin-width. This was expanded for various other product notions (always involving two graphs) in [PS23]. Here the former result is generalised to lexicographic sums of graphs with bounded twin-width. More precisely we obtain the following:

Corollary 4.6.5. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two classes of finite graphs with bounded twin-width. Then, the reduct of $\mathcal{C}_1[\mathcal{C}_2]$ to the language of graphs has bounded twin-width.*

Proof. Without loss of generality, we may assume that $\mathcal{C}_1$ and $\mathcal{C}_2$ consist of ordered graphs and are hereditary. By Fact 4.6.4, it follows that $\mathcal{C}_1$ and $\mathcal{C}_2$ are monadically NIP. Now, by Corollary 4.6.3, it follows that the class $\mathcal{C}_1[\mathcal{C}_2]$, in which there is a natural total order coming from the orders from $\mathcal{C}_1$ and $\mathcal{C}_2$ is monadically NIP, and thus, its reduct to the language of ordered graphs is a monadically NIP hereditary class of ordered graphs. Thus, by Fact 4.6.4 it has bounded twin-width, as claimed. □

4.7 Open Questions

To finish off this chapter, I will collect here some open questions. Recall that a structure $\mathcal{M}$ *admits a distal expansion* if it is a reduct of a distal structure. With respect to some combinatorial properties, it is more relevant to ask if a structure has a distal expansion, rather than if the structure itself is distal. Clearly, given two structures each of which has a distal expansion, their full product will also have a distal expansion. However, the other direction remains unclear.

Remark 4.7.1. An expansion of $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ is not necessarily a reduct of a full product of expansions of $\mathcal{M}_1$ and $\mathcal{M}_2$. Assume $|\mathcal{M}_1| = |\mathcal{M}_2|$ and let $f : \mathcal{M}_1 \to \mathcal{M}_2$ be a bijection. Then $(\mathcal{M}_1 \boxtimes \mathcal{M}_2, f)$ cannot be such a reduct, as its theory implies $|\mathcal{M}_1| = |\mathcal{M}_2|$.

This motivates the following question:

Question 1. *Assume that a full product $\mathcal{M}_1 \boxtimes \mathcal{M}_2$ admits a distal expansion. Do both $\mathcal{M}_1$ and $\mathcal{M}_2$ admit a distal expansion?*

In Subsection 4.2.4, I recalled the fact that indiscernible triviality and dp-minimality characterise monadically NIP structures. As I mentioned there, dp-minimality cannot be characterised in terms of (forbidding) coding configurations, since it is not closed under bi-interpretations, but one can ask if there is a natural non-coding class strictly contained in NIP, that contains all monadically NIP structures. A natural candidate for such a class would be that of NIP structures with indiscernible triviality. This leads to the following question:

Question 2. *Is it possible to characterise indiscernible triviality in terms of collapsing indiscernibles or in terms of forbidden coding configurations?*

Finally, the following question naturally relates to the classification of homogeneous NIP_n structures:

Question 3. *Given an integer $n \geq 2$, is there a homogeneous countable structure with the Ramsey property, that is NIP_n, but not n-ary?*

Positive examples will give, by Proposition 4.4.8, some new instances of homogeneous countable structures $\mathcal{M}$ and $\mathcal{N}$ such that $\mathsf{N}\mathscr{C}_{\mathcal{N}}$ and $\mathsf{N}\mathscr{C}_{\mathcal{M}}$ are not included in each other.

More generally, one can ask about the structure of the $\mathsf{N}\mathscr{C}_{\mathbb{K}}$-hierarchy - say, when $\mathbb{K}$ ranges over the class of all homogeneous countable structures. By Fact 4.3.16 and Example 4.5.4, it is a rooted meet-tree, with a countable ascending chain and an antichain of size at least two. One can then reformulate [GPS21, Question 4.7]:

Question 4. *What is the* width[29] *of the $\mathsf{N}\mathscr{C}_{\mathsf{C}_\mathsf{K}}$-hierarchy?*

[29]Recall: The *width* of a poset is the maximal cardinality of any antichain.

4.8 Local Acknowledgements†

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† Lifted almost directly from [MPT23]. Here 'we' refers to Meir, Touchard, and me, the author of this thesis.

Chapter 5

Around Zarankiewicz's Problem[†]

5.1 Introduction

In this chapter, I will switch gears and discuss extremal graph theory in model-theoretically tame contexts. The combinatorial input in this chapter is the well-known *problem of Zarankiewicz*, which I will recall in the following.

Zarankiewicz's problem first appeared (at least in print) in the problems section of the 1951 Colloquium Mathematicum, [51]. It was **Problem 101**:

Soit R_n, où $n > 3$, un réseau plan formé de n^2 points rangés en n lignes[1] *et n colonnes.*

Trouver le plus petit nombre $k_2(n)$ tel que tout sous-ensembles de R_n formé de $k_2(n)$ points contienne 4 points situés simultanément dans 2 lignes et dans 2 colonnes de ce réseau.

D'une façon générale, trouver le plus petit nombre naturel $k_j(n)$ tel que tout sous-ensembles de R_n formé de $k_j(n)$ points contienne j^2 points situés simultanément

[†]This chapter is based on joint work with Pantelis Eleftheriou, which at the time of writing is being prepared for submission. The contributions of both authors to the results in this chapter are equal, and the results are reproduced here with the permission of Eleftheriou. Fact 5.4.4, Fact 5.4.5 and Fact 5.4.11, which appear in this chapter, are due to Eleftheriou and Mennuni (private correspondence, [EM24]), and I thank them for allowing me to include them here.

[1]Thanks to Christian d'Elbée for making sure lines and linen stay apart despite my mild dyslexia.

dans j lignes et dans j colonnes de ce réseau.

Okay, but this does not quite sound like a problem in graph theory (the first difficulty might be that it is in French), so let's cast it as one (and state it in English this time). To fix notation, I will write $K_{s,t}$ to denote the complete bipartite graph with s vertices on one part and t vertices on the other.[2]

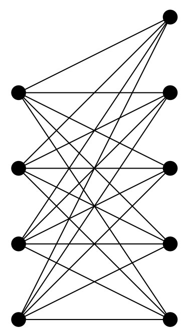

Figure 5.1: $K_{4,5}$

A graph G will be called $K_{s,t}$*-free* if it does not contain $K_{s,t}$ as a subgraph. Zarankiewicz's problem is the following:

Question (Zarankiewicz's Problem (for graphs[3]))**.** *For $n \in \mathbb{N}$, what is the maximum number of edges a $K_{s,t}$-free bipartite graph on n vertices can have (in terms of n, s and t)?*

A little bit of intuition behind this question may be helpful here. We should understand complete graphs as being *dense* and bounds on the number of edges of graphs as a *sparsity* condition. Thus, we are trying to find how *sparse* a given graph must be if it does not contain a fixed dense subgraph.

Zarankiewicz's problem is famously open in general, but much progress has been made in the last 70 years. To simplify the exposition, I will only focus on the case where $s = t = k$, for some $k \in \mathbb{N}$. The reader is referred to [Bol78, Section V.1.2] for a more detailed history, with proofs (at least for results published before 1978).

The first significant step was taken by Kővári–Sós–Turán in [KST54], where they proved the following:

---

[2]Formally (up to isomorphism) $K_{s,t} := ([s] \sqcup [t], \{(i,j),(j,i) : i \in [s], j \in [t]\})$

[3]Of course, it doesn't take a lot to see that this question has a natural hypergraph generalisation (as discussed in Chapter 2, hypergraphs are understood to be r-partite and r-uniform).

Theorem. *There is some $\alpha \in \mathbb{R}_{>0}$ such that for all bipartite graphs $G = (V, E)$ with $|V| = n$, we have that if G is $K_{2,2}$-free, then:*

$$|E| \leq \alpha n^{\frac{3}{2}}.$$

In [Erd64], Erdős showed that the proof of [KST54] works for arbitrary $k \in \mathbb{N}$, and generalised this result to r-hypergraphs (here the configuration that is omitted is the complete r-partite r-uniform hypergraph with k_i elements on the ith part, denoted $K_{k_1,\ldots,k_r}$).

Theorem. *There is some $\alpha \in \mathbb{R}_{>0}$ such that for all r-partite hypergraphs $H = (V, R)$ with $|V| = n$, we have that if H is $K_{k,\ldots,k}$-free, for some $k \in \mathbb{N}$, then:*

$$|R| \leq \alpha n^{r - \frac{1}{k^{r-1}}}.$$

Moreover, there exist r-partite hypergraphs on n vertices which are $K_{k,\ldots,k}$-free such that:

$$|R| \geq n^{r - \frac{C}{k^{r-1}}},$$

for sufficiently large C, independent of n, r, and k.

The "moreover" part was proved using Erdős's probabilistic method, and it shows that if one wants a result for the class of all graphs, then they can't expect the exponent in the theorem above to be "significantly" improved. So it's perhaps better to turn our attention to graphs that arise in more restricted contexts, particularly contexts that are in some sense *geometric*. One of the most influential results in this area is the celebrated Szemerédi-Trotter theorem, from [ST83]. I'll recall a slightly simplified version below.

Theorem. *Let P be a set of points in $\mathbb{R}^2$, and L a set of lines in $\mathbb{R}^2$. Write $I(P; L)$ for the* incidence relation *between P and L, that is:*

$$I(P; L) := \{(p, l) \subseteq P \times L : p \in l\},$$

and write $I(n; m)$ for:

$$\mathsf{max}\{I(P; L) : P \text{ is a set of } n \text{ points in } \mathbb{R}^2, \text{ and } L \text{ a set of } m \text{ lines in } \mathbb{R}^2\}.$$

Then:

$$|I(n; n)| = O\left(n^{\frac{4}{3}}\right),$$

and this bound is tight.

To see how this is relevant, note that $I(P;L)$ can be seen as the edge relation[4] of a bipartite graph with vertex set $P \sqcup L$. This graph is easily seen to be $K_{2,2}$-free, and the general Kővári–Sós–Turán bound would tell us that:

$$|I(n;n)| = O\left(n^{\frac{3}{2}}\right),$$

and since $\frac{4}{3} < \frac{3}{2}$, the Szemerédi-Trotter is a real improvement of the Kővari-Sós-Turán upper bound.

Bringing us closer to the historical present, a nice blend of recent model-theoretic and combinatorial techniques has yielded generalisations of the Szemerédi-Trotter theorem in a variety of geometrically tame settings, such as *semialgebraic* [Fox+17], o-minimal, and more generally distal [CGS20] graphs.

The starting point of the work in this chapter are the recent results of Basit, Chernikov, Starchenko, Tao, and Tran, who considered *semilinear hypergraphs* in [Bas+21]. In particular, they made a link between Zarankiewicz's problem and a classical theme in model theory, that of recognising the *existence of definable algebraic structure* from combinatorial data (e.g. the *Group Configuration Theorem*).

Motivated by some of the ideas in [Bas+21] and recent literature from the so-called *semibounded* o-minimal structures, in joint work with Eleftheriou, we introduce and prove an abstract version of Zarankiewicz's problem for hypergraphs whose vertices are "far apart" from each other in a certain sense made precise via a pregeometric closure independence (Theorem N). Applied to *short closure independence* from semibounded structures, our abstract result yields the same bound as in the case of semilinear hypergraphs in [Bas+21], but for uniformly distant hypergraphs (Theorem O). Furthermore, we obtain a combinatorial way to recognise the existence of an *unbounded* field definable in o-minimal structures (Theorem P). All these results (and more) will be stated more precisely in the following subsection.

[4] The observant reader will have noticed that this relation is not symmetric, but I still called it a graph. This will be a theme in this chapter, where any time a non-symmetric relation is referred to as an edge/hyperedge relation of a graph/hypergraph, one should *really* consider its symmetrisation. This process does not affect, of course, definability (when that is relevant), and it only increases the cardinality of a relation by a constant, which will be implicitly incorporated appropriately to the asymptotic bounds.

First, let me recall the setting of [Bas+21]. A set $X \subseteq \mathbb{R}^n$ is called *semilinear* if it is a finite union of sets defined using linear inequalities [Bas+21, Definition 1.4]. Equivalently, in model-theoretic terminology, X is semilinear if it is definable in the real ordered vector space $\mathbb{R}_{\mathsf{vec}} = \langle \mathbb{R}, <, +, \{x \mapsto rx\}_{r\in\mathbb{R}}\rangle$ (and again equivalently, if, and only if, X is definable in a *linear* o-minimal structure expanding $\langle \mathbb{R}, <, +\rangle$, as discussed in Chapter 2). In [Bas+21] they prove:

Theorem ([Bas+21, Corollary 5.12])**.** *Let $r \in \mathbb{N}_{\geq 2}$, and assume that $X \subseteq \mathbb{R}^d = \prod_{i\in[r]} \mathbb{R}^{d_i}$ is semilinear and is $K_{k,\ldots,k}$-free for some $k \in \mathbb{N}$. Then for any r-hypergraph H of the form $(V_1, \ldots, V_r; X \cap \prod_{i\in r} V_i)$, for finite $V_i \subseteq \mathbb{R}^{d_i}$ with $\sum_{i\in[r]} |V_i| = n$, we have that:*

$$\left|X \cap \prod_{i\in r} V_i\right| = O_{r,k}\left(n^{r-1}\right).$$

This statement explains the connection of definability with extremal combinatorics: We are interested in *finite* hypergraphs whose vertex sets need not be (parameter-free) definable, but their hyperedge relation is induced by a definable relation, on which we have imposed a global (in this case, "sparsity") assumption on that relation.

A converse of the above theorem is also proved in [Bas+21] and, together with the trichotomy theorem for o-minimal structures [PS98], establishes the following link to the theme of recognising definable fields (see [Bas+21, Definition 5.3, Corollary 5.11] for a precise statement): Suppose $\mathcal{R}$ is a saturated o-minimal structure. Then the following are equivalent:

- Every $K_{k,\ldots,k}$-free, for some $k \in \mathbb{N}$, hypergraph X definable in $\mathcal{R}$ satisfies the conclusion of [Bas+21, Corollary 5.12].
- $\mathcal{R}$ does not define a real closed field.

This is, in my opinion, sufficient context to understand the main results of this chapter, which involve semibounded o-minimal structures. The same joint work on which this chapter is based also contains results about hypergraphs definable in models of Presburger arithmetic, stable one-based theories, and *ab initio* Hrushovski constructions. All of these are summarised below.

Summary of Main Results

The first two actual results in this chapter concern Zarankiewicz's problem in a fairly abstract (pregeometric) setting. First, I need to make a little terminological remark:

Terminology. Throughout this section, an *r-grid* refers to a Cartesian product of r-many sets. Intuitively, restricting an infinite r-ary relation to a finite grid gives a finite r-partite hypergraph. It is to hypergraphs obtained in this manner that the results apply.

The first result, stated below, is a general counting theorem, which intuitively says that under some technical assumptions on a given relation E (cl-*tightness*, and (cl-Uniform Bounds), both explained later) one should expect to get *"linear"*[5] bounds on the hyperedges of finite hypergraphs whose edge relation is induced by E:

Theorem M (Theorem 5.3.5)**.** *Let $\mathcal{M}$ an $\mathcal{L}$-structure, $A \subseteq M$ be a subset, and cl a closure operator on $\mathcal{M}$. Let $\mathcal{E} := \{E_b : b \in I\}$ be an A-definable family of r-ary relations $E_b \subseteq \prod_{i\in[r]} M^{d_i}$, for fixed $d_1, \ldots, d_r \in \mathbb{N}$. Let $\mathcal{C}$ be a class of finite grids in $\prod_{i\in[r]} M^{d_i}$ such that cl and $\mathcal{C}$ satisfy a* uniform finiteness assumption *$\mathcal{C}$ of grids (see Assumption 2 on page 176, for the precise statement). Then, there is an $\alpha = \alpha(\mathcal{E}) \in \mathbb{R}_{>0}$ such that for every $b \in I$, if E_b is cl-tight (see Definition 5.3.1), then for every grid $B = B_1 \times \cdots \times B_r \in \mathcal{C}$, we have:*

$$|E_b \cap B| = \alpha n^{r-1},$$

where $n = \max\{|B_i| : i \in [r]\}$.

To apply this result, in practice, there should be some underlying geometry (in the sense of Section 2.4) which morally looks more like the geometry of a vector space rather than that of a field and ensures that hypergraphs omitting K_∞ (the infinitary analogue of $K_{k_1,\ldots,k_r}$) are cl-tight. More precisely:

Theorem N (Corollary 5.3.6)**.** *Let $\mathcal{M}$ an $\mathcal{L}$-structure, $A \subseteq M$ be a subset, and cl a closure operator on $\mathcal{M}$ which behaves similarly to acl in weakly locally modular pregeometric structures (see Assumption 1 on page 173 for more details). Let $\mathcal{E} := \{E_b : b \in I\}$ be a uniformly A-definable family of r-ary*

[5] The justification behind the term linear is that in the case of graphs, i.e. when $r = 2$, the bounds are of the form $O(n)$, so they *really* are linear.

relations $E_b \subseteq \prod_{i\in[r]} M^{d_i}$, for fixed $d_1, \dots, d_r \in \mathbb{N}$. Let $\mathcal{C}$ be a class of finite grids in $\prod_{i\in[r]} M^{d_i}$ such that cl *and $\mathcal{C}$ satisfy a* uniform finiteness assumption *$\mathcal{C}$ of grids (see Assumption 2 on page 176 for the precise statement). Then, there is an $\alpha = \alpha(\mathcal{E}) \in \mathbb{R}_{>0}$ such that for every $b \in I$, if E_b is K_∞^{cl}-free (see Section 5.3), then for every grid $B = B_1 \times \cdots \times B_r \in \mathcal{C}$, we have:*

$$|E_b \cap B| = \alpha n^{r-1},$$

where $n = \max\{|B_i| : i \in [r]\}$.

This abstract machinery will be applied in three different contexts:

1. Semibounded o-minimal expansions of ordered groups.
2. Presburger arithmetic.
3. Stable one-based theories and Hrushovski constructions.

I'll now state the rest of the results in this order.

First, for semibounded o-minimal expansions of ordered groups, the abstract machinery is applied to a closure operator called the *short closure* (Definition 5.4.3) to give the following result:

Theorem O (Corollary 5.4.17)**.** *Let $\mathcal{M} = \langle M, <, +, \dots \rangle$ be a saturated o-minimal expansion of a group, and assume that $\mathcal{M}$ is semibounded. Let $\mathcal{E} = \{E_b : b \in I\}$ be a parameter-definable family of partitioned r-ary relations:*

$$E_b \subseteq \prod_{i\in[r]} M^{d_i}.$$

Then, there is an $\alpha = \alpha(\mathcal{E}) \in \mathbb{R}_{>0}$ such that for every $b \in I$, if E_b is K_∞^{tall}-free (See Section 5.4), then for every tall finite grid $B = B_1 \times \cdots \times B_r \subseteq \prod_{i\in[r]} M^{d_i}$ (see Section 5.4, for the precise definitions) we have:

$$|E_b \cap B| = \alpha n^{r-1},$$

where $n = \max\{|B_i| : i \in [r]\}$.

In fact, the conclusion of Theorem O characterises semibounded o-minimal structures. This is the content of the following theorem:

Theorem P (Theorem 5.4.19)**.** *Let $\mathcal{M} = \langle M, <, +, \dots \rangle$ be a saturated o-minimal expansion of a group. Then, the following are equivalent:*

1. *$\mathcal{M}$ is semibounded.*
2. *There is some $N \in M_{>0}$ such that: For every parameter-definable family of $\mathcal{E} = \{E_b : b \in I\}$ of partitioned r-ary relations, where:*

$$E_b \subseteq \prod_{i\in[r]} M^{d_i},$$

there is an $\alpha = \alpha(\mathcal{E}) \in \mathbb{R}_{>0}$ such that for every $b \in I$, if E_b is K_∞^N-free (see Subsection 5.4.3), then for every N-distant finite grid $B = B_1 \times \cdots \times B_r \subseteq \prod_{i\in[r]} M^{d_i}$, we have

$$|E_b \cap B| = \alpha n^{r-1},$$

where $n = \max\{|B_i| : i \in [r]\}$.

3. *(Cf. the Szemerédi-Trotter theorem, from before) There is some $N \in M_{>0}$ such that: For every parameter-definable $E \subseteq M^{d_1} \times M^{d_2}$ there is some positive real number $\beta < \frac{4}{3}$, such that if E is K_∞^N-free, then for every N-distant finite grid $B = B_1 \times B_2 \subseteq M^{d_1} \times M^{d_2}$, we have*

$$|E \cap B| \leq \alpha n^{\beta},$$

where $n = \max\{|B_1|, |B_2|\}$.

4. *There is no parameter-definable field on the whole of M.*

Moving on to Presburger arithmetic:
Theorem Q (Theorem 5.5.38)**.** *Let $\Gamma \models \mathsf{Th}(\langle \mathbb{Z}, <, +, 0\rangle)$ be a model of Presburger arithmetic. Let $E \subseteq \prod_{i\in[r]} \Gamma^{d_i}$, for fixed $d_1, \ldots, d_r \in \mathbb{N}$ be a $\emptyset$-definable r-ary relation. Then, there is some $\alpha = \alpha(E) \in \mathbb{R}_{>0}$ such that if E is $K_{k,\ldots,k}$-free (for some $k \in \mathbb{N}$), then then for every finite grid $B \subseteq \Gamma^{d_1} \times \cdots \times \Gamma^{d_r}$, we have*

$$|E \cap B| \leq \alpha n^{r-1},$$

where $n = \max\{|B_i| : i \in [r]\}$.

In the context of n.f.c.p. stable one-based theories (the new terminology will be explained later on in this chapter), the following result is obtained:
Theorem R (Theorem 5.6.12)**.** *Let T be a stable one-based theory that does not have the finite cover property, and $\mathcal{M} \models T$ an $\aleph_1$-saturated model of T. For every partial type (possibly over parameters), $p(x_1, \ldots, x_r)$, consistent with*

T there is some $\alpha = \alpha(p) \in \mathbb{R}_{>0}$, such that if $p(M)$ does not contain the Cartesian product of r infinite sets (in the appropriate sorts – i.e. is K_∞-free), then for every finite $B = B_1 \times \cdots \times B_r \subseteq \prod_{i=1}^r M^{x_i}$ we have that:

$$|p(B_1, \ldots, B_r)| \leq \alpha n^{r-1},$$

where $n = \max\{|B_i| : i \in [r]\}$.

And finally, using a result of Evans [Eva05], the theorem above gives the following:

Theorem S (Corollary 5.6.13)**.** *Let $\mathcal{M}$ be an* ab initio *Hrushovski construction. For every partial type (possibly over parameters), $p(x_1, \ldots, x_r)$, consistent with T, there is some $\alpha = \alpha(p) \in \mathbb{R}_{>0}$ such that if there is some $k \in \mathbb{N}$ such that $p(M)$ does not contain the Cartesian product of r sets of size k (in the appropriate sorts – i.e. is $K_{k,\ldots,k}$-free), then for every finite $B = B_1 \times \cdots \times B_r \subseteq \prod_{i=1}^r M^{x_i}$ we have that:*

$$|p(B_1, \ldots, B_r)| \leq \alpha n^{r-1},$$

where $n = \max\{|B_i| : i \in [r]\}$.

Structure of this Chapter

This chapter is organised as follows. In Section 5.2, I will give some more background, which should ease the exposition of the rest of the chapter (the focus will be on weak local modularity and some general combinatorial machinery). Then, in Section 5.3, I will present the proofs of Theorem M and Theorem N, in Section 5.4 the proofs of Theorem O and Theorem P, in Section 5.5 the proof of Theorem Q, and finally in Section 5.6 the proofs of Theorem R and Theorem S.

5.2 Local Preliminaries

5.2.1 (Weak) Local Modularity

In this subsection, I will give some general background on *weak local modularity* and *linear*[6] o-minimal expansions of ordered groups. The material presented

[6] Somewhat unfortunately, the term linear has two different uses in this area of model-theoretic literature. In the context of (strongly) minimal theories, linearity is a condition on the rank of the canonical bases of plane curves (see[Pil96, Definition II.2.4]) and, in the context of o-minimal structures it refers to the trichotomy theorem [PS98], discussed in

here is not new and is largely based on [BV12] and [BV10].

The notion of weak local modularity was introduced in [BV10], in the context of lovely pairs of geometric structures, as a generalisation of one-basedness (see Subsection 5.6.1), which, in turn, essentially generalises the behaviour (i.e. *modularity*) of linear span in vector spaces. General discussion about the connections between weak local modularity, local modularity, one-basedness, and other related notions can be found in [BV10; BV12].

Recall that a theory T is *pregeometric* if for all $\mathcal{M} \vDash T$, the algebraic closure, acl, satisfies (EXCHANGE) in $\mathcal{M}$ (and thus is a pregeometry, in the sense of Definition 2.4.1).

Definition 5.2.1. Let T be a pregeometric theory. We say that T is *weakly locally modular* if for all saturated $\mathcal{M} \vDash T$ and all small $A, B \subseteq M$ there is some small $C \subseteq M$ such that:

$$AB \mathop{\downarrow}_{\emptyset} C \text{ and } A \mathop{\downarrow}_{\mathsf{acl}(AC)\cap\mathsf{acl}(BC)} B,$$

where $\downarrow$ denotes acl-independence.

If having many equivalent characterisations is evidence that a definition is natural, then weak local modularity certainly seems like a natural definition. Indeed, in [BV12, Theorem 2.20], the reader can find an extended list of equivalences of weak local modularity for (mostly geometric[7]) theories. The most important consequence of weak local modularity for our purposes is the following, which follows from [BV10, Proposition 6.8, Theorem 6.10(4)]:

Fact 5.2.2. *Let T be a complete o-minimal theory expanding a* DLO. *Then, the following are equivalent:*

1. *T is weakly locally modular.*
2. *T is linear in the sense of Definition 2.7.12 (i.e. if there is no field interpretable in any model of T, by Fact 2.7.13).*

Chapter 2. More discussion is provided in [BV10; BV12].

[7]See Section 5.3, for the definition of a *geometric theory.* While working on this project, it was observed that many of the equivalences in [BV12] hold with the weaker assumption that the theory is pregeometric, but expanding on this is beyond the scope of this chapter.

5.2.2 Combinatorial Setup

Before moving on to the main results, I will need to fix some general combinatorial notation. Let $A = \prod_{i=1}^{r} A_i$ be the Cartesian product of r-many finite sets $A_1, \ldots, A_r$. For $s \leq r$ we write $\delta_s^r(A)$ to denote the following quantity:

$$\delta_s^r(A) := \sum_{1 \leq i_1 < \cdots < i_s \leq r} \left(\prod_{j=1}^{s} |A_{i_j}| \right),$$

so that:

- $\delta_1^r(A) = \sum_{i=1}^{r} |A_i|$, and
- $\delta_r^r = \prod_{i=1}^{r} |A_i|$.

The most important quantity in this chapter will be $\delta_{r-1}^r(A)$:

$$\delta_{r-1}^r(A) = \sum_{1 \leq i \leq r} \left(\prod_{j \in [r] \setminus \{i\}} |A_j| \right).$$

It is clear that when $n = \max\{|A_i| : i \in [r]\}$ then $\delta_{r-1}^r(A) \leq rn^{r-1}$. The observant reader who likes to look ahead may have noticed that the statements of the main results as written in Section 5.1 differ slightly from the statements that will appear in the following sections. This inequality justifies why.

Let $\mathcal{M}$ be an $\mathcal{L}$-structure, and $r \geq 2$ and $d_1, \ldots, d_r$ be positive integers. An *r-ary definable relation of type $d_1, \ldots, d_r$* (possibly over parameters), is simply a subset $E \subseteq M^d$, where $d = \sum_{i=1}^{r} d_i$, given by partitioned $\mathcal{L}$-formula $\phi(x_1, \ldots, x_r)$ (again, possibly over parameters), where $|x_i| = d_i$ for $i = 1, \ldots, r$. I will refer to E as *$K_{\mathbf{k}}$-free* for $k \in \mathbb{N} \cup \{\infty\}$ (this is short for $K_{k,\ldots,k}$-free if for all $A_i \subseteq M^{d_i}$ with $|A_i| = k$ (or $|A_i| \geq \aleph_0$ if $k = \infty$), for $i \in [r]$, there is some $(a_1, \ldots, a_r) \in \prod_{i \in [r]} A_i$, such that $\mathcal{M} \models (a_1, \ldots, a_r) \notin E$, i.e. if $\prod_{i=1}^{r} A_i \not\subseteq E$.

Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Following the terminology of [Eva05], I shall call an r-ary relation E *sparse in $\mathcal{M}$* if there is a real number $\alpha \in \mathbb{R}_{>0}$ such that for all $A_i \subseteq M^{d_i}$, for $i \in [r]$, we have:

$$|E \cap \prod_{i=1}^{r} A_i| \leq \alpha \delta_{n-1}^n \left(\prod_{i=1}^{r} A_i \right).$$

Definition 5.2.3. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. We say that $\mathcal{M}$ has *linear Zarankiewicz bounds* (resp. *strong linear Zarankiewicz bounds*) in $\mathcal{M}$ if for every r-ary definable relation $E \subseteq \prod_{i=1}^{r} M^{|x_i|}$ (possibly over parameters) the condition (LZ) (resp. (SLZ)) holds, where:

(LZ) If E is $K_{\mathbf{k}}$-free for some $k \in \mathbb{N}$ then E is sparse in $\mathcal{M}$.

(SLZ) If E is K_∞-free then E is sparse in $\mathcal{M}$.

A theory T has *(strong) linear Zarankiewicz bounds* if every $\mathcal{M} \models T$ has (strong) linear Zarankiewicz bounds.

5.3 Abstract Zarankiewicz's Problem

In this section, I will give a proof of Theorem M, which is essentially one of the main tools that will be used in the remainder of this chapter. This section is structured as follows:

- First, I will attempt to draw a connection between abstract pregeometries (other than the usual acl) and model theory. This will be done by imposing additional (axiomatic) assumptions (Assumption 1) on the pregeometric closure operator and the independence relation it induces.
- Another ingredient needed for the proof of Theorem M, is a slight generalisation of a technical lemma from [Bas+21], which will be presented next.
- The final ingredient needed for the proof of Theorem M, which is an assumption that, in a sense, generalises uniform finiteness (elimination of the quantifier $\exists^\infty$) will then be given and discussed.
- Finally, I will put everything together, give a proof of the main theorem (Theorem 5.3.5), and deduce Theorem N.

Throughout this section, let $\mathcal{M}$ be an $\mathcal{L}$-structure and cl a pregeometric closure operator on M. Let $\mathop{\smile\!\!\!\!\!\mid} = \mathop{\smile\!\!\!\!\!\mid}^{\mathsf{cl}}$ be the induced notion of independence, as discussed in Chapter 2. Note that since $\mathop{\smile\!\!\!\!\!\mid}$ is a notion of independence that arises from a closure operator, it will automatically satisfy all the properties given in Fact 2.4.10.

The following assumptions on cl and $\mathop{\smile\!\!\!\!\!\mid}$ are meant to capture that cl behaves "rather well" with definable sets and that $\mathop{\smile\!\!\!\!\!\mid}$ behaves similarly to acl-independence in weakly locally modular theories.

Assumption 1. *We assume that* cl *satisfies the following properties, for all* $A, B, C, D \subseteq M$*:*

1. (cl-DEFINABILITY). *For all* $a, b \in M$*, if* $a \in \mathsf{cl}(Ab)$*, then there is an* $\mathcal{L}(A)$*-formula* $\phi(x, y)$ *such that* $\mathcal{M} \vDash \phi(a, b)$*, and whenever* $\mathcal{M} \vDash \phi(a', b')$*, we have that* $a' \in \mathsf{cl}(Ab')$*.*
2. (EXTENSION). *If* $A \mathop{\smile\!\!\!\!\!\!|}_C B$*, then for all* $d \in M$ *there is some* $A' \in M$ *such that* $A' \equiv_{BC} A$ *and* $A' \mathop{\smile\!\!\!\!\!\!|}_C Bd$*.*
3. (WEAK LOCAL MODULARITY, *under the assumption that* $\mathcal{M}$ *is saturated). For all small subsets*[8] $A, B \subseteq M$ *there is some small* $C \subseteq M$ *such that:*
$$C \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} AB \text{ and } A \mathop{\smile\!\!\!\!\!\!|}_{\mathsf{cl}(AC)\cap\mathsf{cl}(BC)} B$$

Here, perhaps, a little bit of elaboration may be needed. The (cl-DEFINABILITY) assumption does not say that the closure of a set is definable, but rather, that whenever $a \in \mathsf{cl}(Ab)$, there is an A-definable set X containing (a, b) and such that whenever $(a', b') \in X$ we have that $a' \in \mathsf{cl}(Ab')$. Observe that acl always satisfies this assumption. Indeed, if $a \in \mathsf{acl}(Ab)$, then there is a formula $\phi(x, b)$ with parameters from A such that:

$$\mathcal{M} \vDash \phi(a, b) \wedge |\phi(x, b)| = n,$$

for some $n \in \mathbb{N}$. Then, taking $\psi(x, y)$ to be the $\mathcal{L}(A)$-formula given by $\phi(x, y) \wedge (|\phi(M, y)| = n)$, we clearly have that $\mathcal{M} \vDash \psi(a, b)$, and whenever $\mathcal{M} \vDash \psi(a', b')$, it follows that $\phi(x, b')$ is an algebraic set containing a', i.e. $a' \in \mathsf{acl}(Ab')$. As for the (EXTENSION) assumption, this is needed to be able to define the cl-dimension of a parameter-definable set (see Definition 5.3.4) and to have that be independent of the choice of parameters. The (WEAK LOCAL MODULARITY) assumption is precisely what one would expect, given Definition 5.2.1, and it will be shown in action in the next proposition. First, I will need to introduce an additional definition:

Definition 5.3.1. Let $r \in \mathbb{N}_{\geq 2}$ and $d_1, \dots, d_r \in \mathbb{N}_{>0}$, and $A \subseteq M$. We say that $E \subseteq \prod_{i=1}^{r} M^{d_i}$ is cl*-tight over* A if whenever $\bar{a} = (a_1, \dots, a_r) \in E$ there is some $j \leq r$ such that $a_j \in \mathsf{cl}(A\bar{a}_{\neq j})$, where $\bar{a}_{\neq j}$ denotes the subtuple of $\bar{a}$ with a_j removed.

[8] A subset A of a κ-saturated structure $\mathcal{M}$ is called *small* if $|A| < \kappa$.

In the notation of Definition 5.3.1, above:

- An r-grid $B = B_1 \times \cdots \times B_r \subseteq \prod_{i=1}^r M^{d_i}$ is called cl-*independent over* A if B_i is cl-independent over A, for each $i \leq r$.
- A relation $E \subseteq \prod_{i=1}^r M^{d_i}$ is called K_∞^{cl}-*free* if it does not contain any infinite cl-independent grids.

The next proposition is a more abstract and slightly generalised version of [Bas+21, Lemma 5.5]. The main differences are that:

1. The proposition below works for an arbitrary closure operator satisfying Assumption 1, rather than acl in weakly locally modular structures (as given in [Bas+21]); and
2. The assumption is weakened from K_∞-free to K_∞^{cl}-free (of course every K_∞-free relation is automatically K_∞^{cl}-free, but the other implication need not hold).

The proof is a straightforward adaptation of the argument in [Bas+21], so only a sketch will be provided here.

Proposition 5.3.2. *Let $\mathcal{M}$ be a saturated $\mathcal{L}$-structure, cl a closure operator on $\mathcal{M}$ and assume that cl, and the independence relation $\mathop{\smile\!\!\!\!\!\mid}$ it induces satisfy Assumption 1. Let $r \in \mathbb{N}_{\geq 2}$ and fix positive integers $d_1, \ldots, d_r$, and an r-ary A-definable relation $E \subseteq \prod_{i \in [r]} M^{d_i}$. If E is K_∞^{cl}-free then E is cl-tight over A.*

Proof (Sketch). The proof is essentially the same as that of [Bas+21, Lemma 5.5]. We must only note that the proof given in [Bas+21] uses precisely the conditions from Assumption 1 to construct a copy of K_∞^{cl} inside E. I shall give a sketch of the proof for the sake of keeping the main chapters of this thesis somewhat self-contained.

- Step 1. Assume toward a contradiction that E is not cl-tight over A. This gives a tuple $\bar{a} = (a_1, \ldots, a_r) \in E$ such that $a_i \notin \mathsf{cl}(A\bar{a}_{\neq i})$ for all $i \in [r]$, where $\bar{a}_{\neq i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_r)$.
- Step 2. By (Weak Local Modularity) we obtain, for each $i \in [r]$ a set C_i such that:

$$C_i \mathop{\smile\!\!\!\!\!\mid}_{\emptyset} A\bar{a} \text{ and } a_i \mathop{\smile\!\!\!\!\!\mid}_{\mathsf{cl}(a_iC_i)\cap\mathsf{cl}(AC_i\bar{a}_{\neq i})} A\bar{a}_{\neq i}.$$

and by (Extension) we may assume that:

$$C_i \underset{\emptyset}{\downarrow} A\bar{a} \bigcup_{j<i} C_i,$$

for all $i \in [r]$. Thus, by (Transitivity) we have that:

$$\bigcup_{i\in[r]} C_i \underset{\emptyset}{\downarrow} A\bar{a}.$$

- Step 3. By Step 2, (Symmetry), and (Transitivity):

$$a_i \underset{\bigcap_{i\in[r]} \mathsf{cl}(AC\bar{a}_{\neq i})}{\downarrow} \bar{a}_{\neq i},$$

  for all $i \in [r]$. Moreover, by (Transitivity), we have that:

$$a_i \notin \mathsf{cl}\left(\bigcap_{i\in[r]} \mathsf{cl}(AC\bar{a}_{\neq i})\right),$$

  where $C = \bigcup_{i\in[r]} C_i$.

- Step 4. By induction, using the previous steps, (Symmetry), (Transitivity), and (Non-Degeneracy) build sequences $\mathcal{I}_i = (a^i_j : j \in \mathbb{N})$, for $i \in [r]$ such that:
  - For all $i \in [r]$ and all $j \in \mathbb{N}$ we have that $a^i_j \equiv_{A\bar{a}_{\neq i}} a_i$.
  - For all distinct $j, j' \in \mathbb{N}$ and all $i \in [r]$ we have that $a^i_j \neq a^i_{j'}$.
  - $a^i_j \notin \mathsf{cl}(Aa^i_{<j})$.

Putting everything together, we have constructed a Cartesian product of infinite cl-independent over A sets contained in E. This is the desired contradiction. □

Moving on. Recall that a theory T is called *geometric* if acl is a pregeometric closure operator in all models of T, and T eliminates the quantifier $\exists^\infty$. The latter condition means precisely that: For all $\mathcal{M} \vDash T$ and all formulas $\phi(x, y)$, where $|x| = 1$ the set:

$$\{b \in M^{|y|} : |\phi(M, b)| \geq \aleph_0\}$$

is definable. The following remark gives a more useful (at least for some combinatorial applications) characterisation of elimination of $\exists^\infty$:

Remark 5.3.3. T eliminates $\exists^\infty$ if, and only if, for all formulas $\phi(x, y)$ where $|x| = 1$, there is some $k_\phi \in \mathbb{N}$ such that for any $\mathcal{M} \models T$ and every $b \in M^{|y|}$:

$$\phi(M, b) \leq k_\phi \iff \phi(M, b) \text{ is finite.}$$

My next goal in this section is to give a condition which generalises elimination of $\exists^\infty$. To this end, I will first need to introduce the notion of cl-dimension. Recall that $\mathsf{cl\text{-}rk}(a/A)$ is the minimal cardinality of a subtuple a' of a such that $\mathsf{cl}(a \cup A) = \mathsf{cl}(a' \cup A)$.

Definition 5.3.4. Let $\mathcal{M}$ be an $\mathcal{L}$-structure and cl a pregeometric closure operator satisfying 1 and 2, from Assumption 1. Given a parameter-definable set $X \subseteq M^d$ we define its cl-*dimension*, denoted $\mathsf{cl\text{-}dim}$, as follows:

$$\mathsf{cl\text{-}dim}(X) := \max\{\mathsf{cl\text{-}rk}(a/A) : a \in X\},$$

where A is some set of parameters over which X is defined.

Note that, as observed previously, since cl satisfies (Extension) $\mathsf{cl\text{-}dim}(X)$ is well-defined and does not depend on the choice of parameters in the definition of X.

Assumption 2. *In addition to Assumption 1 we may also impose the following assumption on the closure operator* cl*, for a fixed class* $\mathcal{C}$ *of finite subsets of* $\prod_{i=1}^r M^{d_i}$.

4. ($\mathcal{C}$-Uniform Bounds) *Let* $\{X_b : b \in I\}$ *be a (parameter-)definable family of subsets* $X_b \subseteq \prod_{i=1}^r M^{d_i}$. *Then, there is* $N \in \mathbb{N}$ *such that for every* $b \in I$, *every* $i \in [r]$, *every* $Y = Y_1 \times \cdots \times Y_r \in \mathcal{C}$ *and every* $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_r) \in \pi_i(X_b)$, *if* $\mathsf{cl\text{-}dim}((X_b)_{\bar{a}_{\neq i}}) = 0$ *then* $|(X_b)_{\bar{a}_{\neq i}} \cap Y_i| \leq N$, *where* $(X_b)_{\bar{a}_{\neq i}}$ *denotes the set* $\{\bar{x} \in M^{d_i} : (\bar{a}_1, \ldots, \bar{a}_{i-1}, \bar{x}, \bar{a}_{i+1}, \ldots, \bar{a}_r) \in X_b\}$.

This was the last assumption needed for the proof of Theorem M:

Theorem 5.3.5. *Let* $\mathcal{M}$ *an* $\mathcal{L}$*-structure,* $A \subseteq M$ *be a subset, and* cl *a closure operator on* $\mathcal{M}$ *satisfying* (cl-Definability). *Let* $\{E_b : b \in I\}$ *be a uniformly* A*-definable family of* r*-ary relations* $E_b \subseteq \prod_{i\in[r]} M^{d_i}$, *for fixed* $d_1, \ldots, d_r \in \mathbb{N}$. *Let* $\mathcal{C}$ *be a class of finite grids in* $\prod_{i\in[r]} M^{d_i}$ *such that* cl *satisfies* ($\mathcal{C}$-Uniform Bounds). *Then there is* $\alpha \in \mathbb{R}_{>0}$, *such that for every* $b \in I$ *and* $k \in \mathbb{N}$, *if* E_b

is cl*-tight, then for every grid* $B \in \mathcal{C}$*, we have:*

$$|E_b \cap B| \leq \alpha\delta^r_{r-1}(|B|).$$

Proof. Let $\{X_b : b \in I\}$ be as above. By adding constants to the language of $\mathcal{M}$ for the elements of A, we may assume that $\{E_b : b \in I\}$ is $\emptyset$-definable (this is only to simplify the notation). Fix $N \in \mathbb{N}$ as guaranteed by ($\mathcal{C}$-Uniform Bounds), with respect to $\mathcal{C}$. Explicitly, N is such that for every $b \in I$, every $i \in [r]$, every grid $B = B_1 \times \cdots \times B_r \in \mathcal{C}$ and every $\bar{a} \in \pi_i(E_b)$, if $\mathsf{cl}\text{-}\mathsf{dim}((E_b)_{\bar{a}}) = 0$ then $|(E_b)_{\bar{a}} \cap B_i| \leq N$, where, again $\pi_i(E_b)$ denotes the projection of E_b to the coordinates indexed by d_i and $(E_b)_{\bar{a}}$ denotes the fibre of E_b above $\bar{a}$.

Let $b \in I$ be such that E_b is cl-tight. Then, for every $\bar{a} = (a_1, \ldots, a_r) \in E_b$, the set $\{a_1, \ldots, a_r\}$ is cl-dependent over b. That is, for all $\bar{a} = (a_1, \ldots, a_r) \in E_b$ there is some $i \in [r]$ such that $a_i \in \mathsf{cl}(\bar{a}_{\neq i}b)$.

By (cl-Definability), there is a b-definable set $X_{\bar{a}}$ such that $\bar{a} \in X_{\bar{a}}$, and for every $\bar{a}' = (a'_i, \ldots, a'_r) \in X_{\bar{a}}$, we have $a'_i \in \mathsf{cl}(\bar{a}'_{\neq i}b)$. Moreover, without loss of generality, we may assume that $X_{\bar{a}} \subseteq E_b$. Hence, we can write:

$$E_b = \bigcup_{\bar{a} \in E_b} X_{\bar{a}}.$$

By compactness, there is a finite subset of E_b, say $\{\bar{a}_1, \ldots, \bar{a}_n\}$ such that:

$$E_b = \bigcup_{j=1}^{n} X_{\bar{a}_j},$$

and by construction, for each $j \in [n]$ there is some $i_j \in [r]$ such that for every $\bar{a} = (a_1, \ldots, a_r) \in X_{\bar{a}_j}$ we have that $a_{i_j} \in \mathsf{cl}(\bar{a}_{\neq i_j}b)$. In particular, for every $\bar{a} \in \pi_{i_j}(X_{a_j})$ we have that:

$$\mathsf{cl}\text{-}\mathsf{dim}\left((X_{\bar{a}_j})_{\bar{a}_{\neq i_j}}\right) = 0,$$

and this holds for all $j \in [n]$.

Claim 1. We may assume that $n \leq r$.

Proof of Claim 1. Suppose that for $j, j' \in [n]$ we have that if $i_j = i_{j'} = i \in [r]$. Then, the b-definable set $X' := X_{a_j} \cup X_{a_{j'}} \subseteq E_b$ still witnesses that for every $\bar{a} = (a_1, \ldots, a_r) \in X'$, we have $a_i \in \mathsf{cl}(a_{\neq i}b)$, so we may replace $X_{\bar{a}_j}$ and $X_{\bar{a}_{j'}}$

with X'. ◀

More intuitively, our construction goes as follows: Since E_b is cl-tight, for each $\bar{a} \in E_b$ we must be able to find a fibre which is cl-dependent. Then, by (cl-Definability), we can find a b-definable set $X_{\bar{a}}$ witnessing this dependence. Then, using compactness on the union of all these sets, we obtain that E_b decomposes into finitely many of $X_{\bar{a}}$, and by the claim above, we may, in fact, assume that it decomposes to at most r many.

Now, to conclude the argument, let $B = B_1 \times \cdots \times B_r \in \mathcal{C}$. Then, since $X_{\bar{a}_j} \subseteq E_b$, for all $j \in [n]$, by our choice of N we have that:

$$\begin{aligned} |E_b \cap B| &\leq \sum_{j=1}^{n} |X_{\bar{a}_j} \cap (B_1 \times \cdots \times B_r)| \leq n \times N\delta^r_{r-1}(B) \\ &\leq (rN)\delta^r_{r-1}(B). \end{aligned}$$

Hence, for any $b \in I$, if E_b is cl-tight, then, for any $B \in \mathcal{C}$ we have that $|E_b \cap B| < \alpha\delta^r_{r-1}(B)$, where $\alpha = rN$, and this concludes the proof. □

From the theorem above and Proposition 5.3.2 the following corollary is immediate:

Corollary 5.3.6. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $A \subseteq M$ be a subset, and cl a closure operator on $\mathcal{M}$ satisfying Assumption 1. Let $\{E_b : b \in I\}$ be a uniformly A-definable family of r-ary relations $E_b \subseteq \prod_{i\in[r]} M^{d_i}$, for fixed $d_1, \ldots, d_r \in \mathbb{N}$. Let $\mathcal{C}$ be a class of finite grids in $\prod_{i\in[r]} M^{d_i}$ such that cl satisfies ($\mathcal{C}$-Uniform Bounds). Then there is $\alpha \in \mathbb{R}_{>0}$, such that for every $b \in I$ and $k \in \mathbb{N}$, if E_b is K^{cl}_∞-free, then for every grid $B \in \mathcal{C}$, we have:*

$$|E_b \cap B| \leq \alpha\delta^r_{r-1}(|B|).$$

5.4 Semibounded Zarankiewicz's Problem

In this section, I will return to semibounded o-minimal structures, which were briefly introduced in Chapter 2. This section culminates in Theorem P, which will be a corollary of Corollary 5.3.6, the Trichotomy for o-minimal expansions of ordered groups, and a semibounded version of (the lower bound of) the Szemerédi-Trotter theorem. First, I will have to discuss the *short closure* operator (which will be the closure operator of choice), and the *Cone*

Decomposition Theorem, for semibounded structures (which will be used to prove that the short closure is weakly locally modular). To get us started, though, let's fix some notation and terminology.

Throughout this section, let $\mathcal{M} = \langle M, <, +, 0, \dots \rangle$ denote a non-linear semibounded o-minimal expansion of a group, and fix an element $1 > 0$ such that some real closed field whose universe is $(0, 1)$ and whose order agrees with $<$ is definable in $\mathcal{M}$. This $\emptyset$-definable field will be denoted by R.

Notation 5.4.1. We denote by $\mathcal{M}_{\mathsf{lin}} = \langle M, <, +, 0, \{\lambda : \lambda \in \Lambda\} \rangle$, where Λ is the set of all partial $\emptyset$-definable endomorphisms of $\langle M, <, +, 0 \rangle$, the *linear reduct* of $\mathcal{M}$. A set definable in $\mathcal{M}_{\mathsf{lin}}$ is called *semilinear*.

5.4.1 Short Closure

Following [Pet09], an interval $I \subseteq M$ is called *short* if there is a parameter-definable bijection between I and $(0, 1)$; otherwise, it is called *long*. The point is that short intervals are precisely the ones on which there is a parameter-definable real closed field:

Fact 5.4.2 ([Pet09, Corollary 3.3])**.** *Suppose that $\mathcal{M}$ is saturated. An open interval $I \subseteq M$ is short if, and only if there is a parameter-definable real closed field with domain I.*

It is now easy to extend the definitions of *short* and *long*[9] to:

- Elements of $\mathcal{M}$: An element $a \in M$ is called *short* if either $a = 0$ or $(0, |a|)$ is a short interval; otherwise, it is called *tall*.
- Tuples from $\mathcal{M}$: A tuple $a \in M^n$ is called *short* if $|a| := |a_1| + \cdots + |a_n|$ is short, and *tall* otherwise.
- Definable subsets of $\mathcal{M}$: A definable set $X \subset M^n$ (or, to the same extent, its defining formula) is called *short*[10] if it is in definable bijection with a subset of $(0, 1)^n$; otherwise, it is called *long*.

Let me now introduce one of the main tools used in the proof of Theorem P, the *short closure*, which was originally introduced in [Ele12]:

Definition 5.4.3. For $A \subseteq M$, we define the *short closure of* A, denoted

[9]Which, as we shall see, will be replaced by the word *tall* when referring to elements or tuples.

[10]Of course, by Fact 5.4.2, for $n = 1$, the two definitions are equivalent.

$\mathsf{scl}(A)$, as follows:

$$\mathsf{scl}(A) = \{a \in M : \text{ there is an } A\text{-definable short interval containing } a\}.$$

Fact 5.4.4 ([EM24]). *Let $I \subseteq M$ be an A-definable short interval. Then, there is an A-definable bijection between I and R.*[11]

Proof (Sketch). By translating I, we may assume that $I = (0, a)$, where $a \in M_{>0}$ is a short element. By definition, there is a parameter-definable bijection $f : (0, a) \to R$. Say f is given by the formula $\phi(x, y, b_0)$, for some $b_0 \in M$, and let:

$$X = \{b \in M : \phi(x, y, b) \text{ defines a bijection between } (0, a) \text{ and } R\}.$$

Clearly X is A-definable. By definable choice there is some $b \in X \cap \mathsf{dcl}(A)$, and thus $\phi(x, y, b)$ gives an A-definable bijection between I and R. □

The first goal of this subsection is to prove that scl is a weakly locally modular pregeometric operator.

Fact 5.4.5 ([EM24]). *Let $A \subseteq M$. Then:*

$$\mathsf{scl}(A) = \mathsf{acl}(AR).$$

Proof (Sketch). On the one hand, suppose that $a \in \mathsf{dcl}(AR)$. By definition, there is an A-definable map f such that $a \in f(b)$, for some k-tuple b from R. Of course $f(R^k)$ is short for any $k \in \mathbb{N}$, and since R is $\emptyset$-definable, it follows that there is an A-definable short set containing a, so $a \in \mathsf{scl}(A)$.

For the converse, let X be an A-definable short interval containing a. By Fact 5.4.4 there is an A-definable bijection from X to R, and thus $a \in \mathsf{dcl}(AR)$, as required. □

In particular, since o-minimal expansions of ordered groups are geometric, it follows that scl defines a pregeometry on M. This was already observed in [Ele12, Lemma 5.5], but the characterisation offered by Fact 5.4.5 will be useful later on.

Now that we know that scl is a pregeometry, in order to deduce Theorem P,

[11] Recall that R is the $\emptyset$-definable real closed field on $(0, 1)$ whose order agrees with $<$.

it remains to show that it satisfies Assumption 1, and Assumption 2, for an appropriately chosen class of grids.[12]

The (cl-Definability) and (Extension) assumptions are easier to deal with:

Lemma 5.4.6. *Let $\mathcal{M}$ be a semibounded (nonlinear) expansion of a group. Then* scl *satisfies the (*cl*-Definability) assumption.*

Proof. This is [Ele12, Remark 5.3], which is essentially [Pet09, Corollary 3.7(3)]. The idea is, really, the same as for the proof that acl satisfies (cl-Definability). Still, intuitively, instead of forcing the number of realisations to be finite, we force the distance between realisations and parameters to be short. More precisely, if $a \in \mathsf{scl}(Ab)$, then there is a formula $\phi(x, y)$ such that $\mathcal{M} \vDash \phi(a, b)$ and $\phi(x, b)$ is short. We can find a short element $k \in M$ such that $\phi(x, b)$ has length at most k, and by [Pet09] such a k exists in $\mathsf{dcl}(\emptyset)$. Thus, (cl-Definability) is witnessed by

$$\phi(x, y) \wedge \text{“}\phi(M, y) \text{ has length at most } k\text{''}.$$

□

Let's also get (Extension) out of the way:

Lemma 5.4.7. *Let $\mathcal{M}$ be a semibounded (nonlinear) expansion of a group. Then* scl *satisfies the (Extension) assumption.*

Proof. This is essentially [Ele12, Lemma 5.7]. The argument is again a straightforward adaptation of the proof of the analogous result for acl. □

To prove that scl is weakly locally modular, the strategy that will be followed is to reduce the problem to $\mathcal{M}_{\mathsf{lin}}$. By Fact 5.2.2 we know that acl is weakly locally modular in $\mathcal{M}_{\mathsf{lin}}$, so (by the following lemma) it suffices to show that scl is a localisation (c.f. Definition 2.4.4(1)) of $\mathsf{acl}_{\mathsf{lin}}$, where $\mathsf{acl}_{\mathsf{lin}}$ denotes the algebraic closure in the linear reduct of $\mathcal{M}$.

Lemma 5.4.8. *Let* cl *be a pregeometric closure operator on a saturated structure $\mathcal{M}$ satisfying Assumption 1 and let $X \subseteq M$ be a small set.*

[12]Spoiler alert (although it was already spoiled in the introduction): It will be the *tall* grids.

Then, the localisation of cl *at* X*, that is:*

$$\mathsf{cl}_X : A \mapsto \mathsf{cl}(AX),$$

also satisfies Assumption 1.

Proof. The only non-trivial condition we need to check is (Weak Local Modularity). Let $A, B \subseteq M$ be small sets. Throughout the rest of the proof, we will write $\mathop{\smile\!\!\!\!\!\mid}^{\mathsf{cl}}$ for the independence relation induced by cl and $\mathop{\smile\!\!\!\!\!\mid}^{\mathsf{cl}_X}$ for the independence relation induced by cl_X, the localisation of cl at X. We wish to show that given small $A, B \subseteq M$ there is a set $C \subseteq M$ such that:

$$C \mathop{\smile\!\!\!\!\!\mid}_{\emptyset}^{\mathsf{cl}_X} AB \text{ and } A \mathop{\smile\!\!\!\!\!\mid}_{\mathsf{cl}(AC)\cap\mathsf{cl}(BC)}^{\mathsf{cl}_X} B.$$

By applying (Weak Local Modularity) to the small sets $AX = A \cup X$ and $BX = B \cup X$ there is a small set $C \subseteq M$ such that:

$$C \mathop{\smile\!\!\!\!\!\mid}_{\emptyset}^{\mathsf{cl}} (AX)(BX) \text{ and } AX \mathop{\smile\!\!\!\!\!\mid}_{\mathsf{cl}(AXC)\cap\mathsf{cl}(BXC)}^{\mathsf{cl}} BX,$$

We claim that this is the required set C witnessing that cl_X satisfies (Weak Local Modularity). Observe that:

$$A \mathop{\smile\!\!\!\!\!\mid}_{X}^{\mathsf{cl}} B \text{ if, and only if } A \mathop{\smile\!\!\!\!\!\mid}_{\emptyset}^{\mathsf{cl}_X} B.$$

Now, by (Monotonicity) and (Transitivity), since $C \mathop{\smile\!\!\!\!\!\mid}_{\emptyset}^{\mathsf{cl}} (AX)(BX)$, it follows that $C \mathop{\smile\!\!\!\!\!\mid}_{X}^{\mathsf{cl}} AB$, and thus

$$C \mathop{\smile\!\!\!\!\!\mid}^{\mathsf{cl}_X} AB.$$

Similarly, since $\mathsf{cl}(AXC) = \mathsf{cl}_X(AC)$, $\mathsf{cl}(BXC) = \mathsf{cl}_X(BC)$, and using (Monotonicity) again, since

$$AX \mathop{\smile\!\!\!\!\!\mid}_{\mathsf{cl}(AXC)\cap\mathsf{cl}(BXC)}^{\mathsf{cl}} BX$$

we have that

$$AX \mathop{\smile\!\!\!\!\!\mid}_{\mathsf{cl}_X(AC)\cap\mathsf{cl}_X(BC)}^{\mathsf{cl}} BX,$$

so in particular

$$A \underset{\mathsf{cl}_X(AC)\cap\mathsf{cl}_X(BC)}{\overset{\mathsf{cl}_X}{\downfree}} B,$$

and the result follows. □

Thus, it remains to show that scl is a localisation of $\mathsf{acl}_{\mathsf{lin}}$. Keeping in mind that this is the goal, let's take a deeper dive into semibounded o-minimal structures.

5.4.2 The Structure Theorem

First, I'll need to recall some definitions; these will start easy and culminate in the definitions of *(normalised) k-long cones* and *almost linear functions*. These notions will be the crucial building blocks in the Structure Theorem (Theorem 5.4.14). The material in this subsection can be found in [Ele12], which contains additional discussion.

Again, throughout this subsection, $\mathcal{M} = \langle M, <, +, 0, \dots \rangle$ will always be an o-minimal expansion of a group. Recall that a *partial endomorphism* of $\langle M, <, +, 0\rangle$ is a map $f : (a, b) \to M$ such that:

$$f(x + t) - f(x) = f(y + t) - f(y),$$

for all $x, y \in (a, b)$ and all $t \in M$ such that $x + t, y + t \in (a, b)$. Throughout this section, Λ will denote the set of all $\emptyset$-definable partial endomorphisms of $\langle M, <, +, 0\rangle$. If $\bar{v} = (v_1, \dots, v_n) \in \Lambda^n$ is an n-tuple of partial endomorphisms, then $\mathsf{dom}(v) := \cap_{i=1}^{n} \mathsf{dom}(v_i)$. Of course, if $t \in \mathsf{dom}(\bar{v})$ then we write $\bar{v}t$ for $(v_1t_1, \dots, v_nt_n) \in M^n$. I will usually drop the "overlining" for tuples of partial endomorphisms.

Definition 5.4.9. We say that $v_1, \dots, v_k \in \Lambda^n$ are *M-independent* if for all $t_1, \dots, t_k \in M$ with $t_i \in \mathsf{dom}(v_i)$ we have that:

$$\sum_{i=1}^{k} v_i t_i = 0 \implies t_1 = \dots = t_k = 0.$$

Now, a long cone is defined as follows:

Definition 5.4.10. Let $k \in \mathbb{N}$. A *k-long cone* $C \subseteq M^n$ is a definable set of the form:

$$C = \left\{ b + \sum_{i=1}^{k} v_i t_i : b \in B, t_i \in J_i \right\},$$

where:

- $B \subseteq M^n$ is a short cell.
- $v_1, \dots, v_k \in \Lambda^n$ are M-independent.
- $J_1, \dots, J_k$ are long intervals of the form $(0, a_i)$, for $a_i \in M_{>0} \cup \infty$, with $J_i \subseteq \mathsf{dom}(v_i)$.

A *long cone* is simply a k-long cone for some $k \in \mathbb{N}$.[13] We say that the long cone C is *normalised* if for all $x \in C$ there are unique $b \in B$ and $t_i \in J_i$ such that $x = b + \sum_{i=1}^{k} v_i t_i$. In this case, we write:

$$C = B + \sum_{i=1}^{k} v_i t_i | J_i.$$

A *long subcone* of a long cone C is a long cone contained in C.

In what follows, all "long cones" are assumed to be normalised, and all "cones" are assumed to be long. Thus, those words will often be dropped without further warning. The crucial thing to keep in mind is that the structure theorem (Theorem 5.4.14, below) gives us a decomposition into normalised cones.

Fact 5.4.11 ([EM24]). *Let $C = B + \sum_{i=1}^{k} v_i t_i | J_i \subseteq M^n$ be a normalised k-long cone. If C is A-definable, then so are all of B and $J_1, \dots, J_k$.*

Proof (Sketch). By definable choice, we can find an element $a \in C$ such that $a \in \mathsf{dcl}(A)$. To fix notation, say that $a = b + \sum_{i=1}^{k} v_i t_i$.

- Fix $i \leq k$ and let $a_i \in M$ be such that $J_i = (0, a_i)$. Then:

  $$a_i = \sup\{|s - t| : s, t \in M, a + sv_i, a + tv_i \in C\}.$$

  To see this, on the one hand, suppose that $s, t \in M$ are such that $a + sv_i, a + tv_i \in C$, then $a + sv_i = b + \sum_{j \in [k] \setminus \{i\}} v_j t_j + (t_i + s) v_i \in C$, and since C is normalised it follows that $t_i + s \in (0, a_i)$. Similarly $t_i + t \in (0, a_i)$. Thus, it follows that $|s - t| \leq a_i$. Conversely, if $a \geq |s - t|$ for all $s, t \in M$ such that $a + sv_i, a + tv_i \in (0, a_i)$, then arguing similarly we see that $a_i \leq a$.

[13] Observe that a short cell is just a 0-long cone.

- Similarly, since C is a normalised cone, we have that:

$$B = \{b \in M^n : \exists t_i \in J_i, b + v_i t_i \in C\},$$

  and since $J_1, \dots, J_k$ are A-definable, by the previous item, it follows that B is A-definable.

□

Definition 5.4.12. Let $C = B + \sum_{i=1}^{k} v_i t_i | J_i$ be a k-long cone, and $f : C \to M$ a definable continuous function. We say that f is *almost linear with respect to* C if there are:

- partial endomorphisms $\mu_1, \dots, \mu_k \in \Lambda$; and
- an extension $\tilde{f}$ of f to the set:

$$\left\{ b + \sum_{i=1}^{k} v_i t_i : b \in B, t_i \in J_i \cup \{0\} \right\},$$

such that for all $b \in B$ and all $t_i \in J_i \cup \{0\}$ we have that:

$$\tilde{f}\left(b + \sum_{i=1}^{k} v_i t_i\right) = \tilde{f}(b) + \sum_{i=1}^{k} \mu_i t_i.$$

Remark 5.4.13 ([Ele12, Remark 2.12]). In the notation of the definition above, since C is normalised, $\tilde{f}$ and $\mu_1, \dots, \mu_k$ are all uniquely determined by f and C. In particular, $\tilde{f}$ is continuous, so we may identify it with f.

And now, the promised Structure Theorem:

Theorem 5.4.14 (Structure Theorem [Ele12, Theorem 3.8])**.** *Let $X \subseteq M^n$ be an A-definable set. Then:*

1. *X is a finite union of A-definable long cones.*
2. *If X is the graph of an A-definable function $f : Y \to M$, for some $Y \subset M^{n-1}$, then there is a finite collection $\mathcal{C}$ of A-definable long cones, whose union is Y and such that f is almost linear with respect to each long cone in $\mathcal{C}$.*

Lemma 5.4.15. *Let $\mathcal{M}$ be a semibounded nonlinear expansion of a group. Then,* scl *is weakly locally modular.*

Proof. By Fact 5.4.5 and Lemma 5.4.8, it suffices to show, for every $A \subseteq M$ that

$$\mathsf{acl}(AR) = \mathsf{acl}_{\mathsf{lin}}(A\,\mathsf{acl}(R)),$$

where $R = (0, 1)$.

Let $A \subseteq M$. If $a \in \mathsf{acl}_{\mathsf{lin}}(A\,\mathsf{acl}(R))$, then $a \in \mathsf{acl}(A\mathsf{dcl}(R))$ which is, of course, equal to $\mathsf{acl}(AR)$.

Conversely, suppose that $a \in \mathsf{acl}(AR)$, since $\mathsf{acl} = \mathsf{dcl}$ in $\mathcal{M}$, we can, without loss of generality, find a tuple c from A and an R-definable function such that $a = f(c)$.

By the Structure Theorem, we may assume that there is an R-definable normalised cone $C = B + \sum_{i=1}^{k} v_i t_i | J_i$ containing c, and an almost linear, with respect to C, function $f : C \to M^n$. So:

$$c = b + \sum_{i=1}^{k} v_i t_i, \tag{5.1}$$

for some $b \in B$ and $t_i \in J_i$, and

$$a = f(b) + \sum_{i=1}^{k} \mu_i t_i, \tag{5.2}$$

for some $\mu_i \in \Lambda^n$.

By Fact 5.4.11, B is R-definable, hence $f(b) \in \mathsf{dcl}(R)$. Moreover, by Equation (5.1), and since cones are normalised, every $t_i \in \mathsf{dcl}_{\mathsf{lin}}(cb) \subseteq \mathsf{dcl}_{\mathsf{lin}}(A\mathsf{dcl}(R))$. Hence, by Equation (5.2) we have that $a \in \mathsf{dcl}_{\mathsf{lin}}(A\mathsf{dcl}(R))$, which is, of course, equal to $\mathsf{acl}_{\mathsf{lin}}(A\mathsf{dcl}(R))$, and we are done. □

Before moving on to the next lemma, I need to introduce two new terms:

- A *tall grid*: Fix positive integers $r \geq 2$ and $d_1, \ldots, d_r \in \mathbb{N}$. Let $B = B_1 \times \cdots \times B_r \subseteq \prod_{i=1}^{r} M^{d_i}$, where $B_i \subseteq M^{d_i}$, for $i \in [r]$, be an r-grid. We say that B is *tall* if for all $i \in [r]$ and all distinct points $x, y \in B_i$ we have that $|x - y|$ is tall.
- A K_∞^{tall}*-free relation*: We say that $E \subseteq \prod_{i=1}^{r} M^{d_i}$ is K_∞^{tall}*-free* if it does not contain any tall grid $B_1 \times \cdots \times B_r$, such that $|B_i| \geq \omega$, for all $i \in [r]$.

Lemma 5.4.16. *Let $\mathcal{M}$ be a semibounded expansion of a group, $A \subseteq M$,* *and $\{X_b : b \in I\}$ an A-definable family of subsets $X_b \subseteq \prod_{i=1}^r M^{d_i}$. Let $\mathcal{C}$ be the class of all finite tall grids in $\prod_{i=1}^r \mathcal{M}^{d_i}$. Then, there is $N \in \mathbb{N}$ such that for every $b \in I$, every $i \in [r]$, every $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_r) \in X_b$, and every $Y = Y_1 \times \cdots \times Y_r \in \mathcal{C}$, if $(X_b)_{\bar{a}_{\neq i}}$ is short then $|(X_b)_{\bar{a}_{\neq i}} \cap Y_i| \leq N$, where $(X_b)_{\bar{a}_{\neq i}}$ denotes the set $\{\bar{x} \in M^{d_i} : (\bar{a}_1, \ldots, \bar{a}_{i-1}, \bar{x}, \bar{a}_{i+1}, \ldots, \bar{a}_r) \in X_b\}$.*

In particular, scl *and $\mathcal{C}$ satisfy ($\mathcal{C}$-Uniform Bounds).*

Proof. By the uniform Cell Decomposition theorem (Theorem 2.3.9), for each $i \in [r]$ there is some N_i such that for all $b \in I$ and all $\bar{a} \in X_b$ we have that $(X_b)_{\bar{a}_{\neq i}}$ is a union of at most N_i open, connected cells. If $(X_b)_{\bar{a}_{\neq i}}$ is short for some $b \in I$, some $i \in [r]$, and some $\bar{a} \in X_b$ then all these cells are necessarily short.

We start with the following claim:

Claim 1. No open connected short set can contain two elements whose difference is tall.

Proof of Claim 1. Indeed, let C be an open connected short set and suppose that $a \in C$. We claim that:

$$C = \{a' \in C : |a - a'| \text{ is short}\},$$

from which, of course, the claim follows. Since C is an open connected set, it suffices to show that $\{a' \in C : |a - a'| \text{ is short}\}$ is clopen. Indeed, let $a' \in C$ and take a short open box B that contains a'. Then for every $c \in B \cap C$, we have $|a' - c|$ short, and hence:

$$|a - c| = |a - a' - (a' - c)| \text{ is short} \iff |a - a'| \text{ is short.}$$

Thus, both C and its complement are open, and the result follows. ◀

By the claim above, it follows that if $(X_b)_{\bar{a}_{\neq i}}$ is short for some $b \in I$, some $i \in [r]$, and some $\bar{a} \in X_b$ then $(X_b)_{\bar{a}_{\neq i}}$ contains at most N_i-many elements whose differences are tall (one for each of the N_i-many short cells in the decomposition of $(X_b)_{\bar{a}_{\neq i}}$). Thus, taking $N = \max\{N_i : i \in [r]\}$ gives the result. □

Putting everything together:

Corollary 5.4.17. *Let $\mathcal{M} = \langle M, <, +, \dots \rangle$ be a saturated o-minimal expansion of a group. If $\mathcal{M}$ is semibounded and nonlinear, then for every parameter-definable family $\{E_b : b \in I\}$ of partitioned r-ary relations, where:*

$$E_b \subseteq \prod_{i \in [r]} M^{d_i}$$

there is some $\alpha \in \mathbb{R}^{>0}$, such that: For every $b \in I$, if E_b is K_∞^{tall}-free, then for every tall finite grid $B \subseteq \prod_{i \in [r]} M^{d_i}$, we have

$$|E_b \cap B| \leq \alpha \delta_{r-1}^r(B).$$

Proof. We have spent the last few pages proving that:

1. scl satisfies Assumption 1 (this was Lemmas 5.4.6, 5.4.7 and 5.4.15). It is also clear by the definitions that if a relation E is K_∞^{tall}-free, then it is K_∞^{scl}-free (as the difference between any two scl-independent tuples must be tall).

2. scl together with the class $\mathcal{C}$ of all finite tall grids satisfy Assumption 2 (this was Lemma 5.4.16).

Thus, the result is an immediate application of Corollary 5.3.6. □

5.4.3 Semibounded Szemerédi-Trotter

I've already mentioned the Szemerédi-Trotter theorem in the introduction to this chapter. In this subsection, I will discuss the "easy" part of this theorem, namely the lower bound, which tells us that for arbitrarily large $n \in \mathbb{N}$ there is an arrangement of n points and n lines on the plane which have at least $n^{\frac{4}{3}}$ incidences. First, let me introduce some useful terminology:

Fix $N \in M_{>0}$. I will define the following terms (which will also appear in Section 5.5):

- N-distant grid: Fix positive integers $r \geq 2$ and $d_1, \dots, d_r \in \mathbb{N}$. Let $B = B_1 \times \cdots \times B_r \subseteq \prod_{i=1}^r M^{d_i}$, where $B_i \subseteq \Gamma^{d_i}$, for $i \in [r]$ be an r-grid. We say that B is *N-distant* if for all $i \in [r]$ and all distinct points $x, y \in B_i$ we have that $|x - y| > N$.

- K_∞^N-free relation: We say that E is *K_∞^N-free* if it does not contain any N-distant grid $B_1 \times \cdots \times B_r$ such that $|B_i| \geq \omega$, for all $i \in [r]$.

Proposition 5.4.18 (Semibounded Szemerédi-Trotter). *Let $\mathcal{M} = \langle M, <, +, \dots \rangle$ be an o-minimal expansion of an ordered group. Let $F \subseteq M$ be an open interval and suppose that there is a parameter-definable real closed field on F whose order agrees with $<$. Let $I \subseteq F^2 \times F^2$ be the bipartite relation given by:*

$$I(x_0, y_0; a, b) \text{ if, and only if "}(x_0, y_0) \text{ lies on the line } y = ax + b''.$$

For all $N \in F_{>0}$ there is an N-distant grid $B \subseteq F^2 \times F^2$ such that:

$$|I \cap B| = \Omega(|B|^{4/3}).$$

Proof. The proof is a minor adaptation of the usual proof of the lower bound from the Szemerédi-Trotter theorem. We follow the proof from [Mat13, Proposition 4.2.1]. The proof from [Mat13] lets us, for arbitrarily large n, construct n points and n lines on the real plane, with at least $\frac{1}{4}n^{4/3}$ incidences. In our context, this just says that there are arbitrarily large grids $B \subseteq F^2 \times F^2$, such that

$$|I \cap B| \geq \frac{1}{4}n^{4/3}.$$

We claim that with just a small tweak in the proof, we can always find such grids which are moreover N-distant, for any $N < |F|$. To this end, choose $r, s \in F^{>0}$ such that $r, s > N$.

The construction from [Mat13] is as follows:

$$P = \left\{(x, y) : x \in \{0, 1, \dots, k-1, \}, y \in \{0, 1, \dots, 4k^2 - 1\}\right\}$$

and

$$L = \left\{(a, b) : a \in \{0, 1, \dots, 2k-1\}, b \in \{0, 1, \dots, 2k^2 - 1\}\right\}.$$

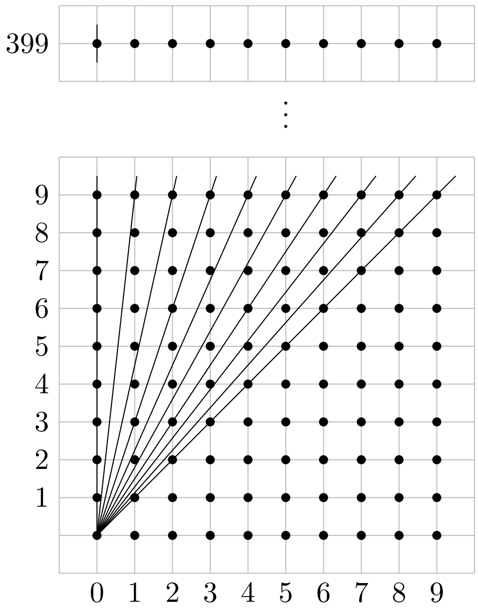

Figure 5.2: Szemerédi-Trotter lower bound configuration ($k = 10$).

We define:

$$P' = P = \left\{(x, y) : x \in \{0, r, \ldots, (k-1)r, \}, y \in \{0, rs, \ldots, (4k^2 - 1)rs\}\right\}$$

and

$$L' = \left\{(a, b) : a \in \{0, s, \ldots, (2k-1)s\}, b \in \{0, r, \ldots, (2k^2 - 1)rs\}\right\}.$$

Clearly, for any $x, y, a, b \in F$ we have that

$$y = ax + b \iff yrs = (ax + b)sr = (as)(xr) + brs$$

that is,

$$I(x, y, a, b) \iff I(xr, ysr, as, brs),$$

and since the map $(x, y, a, b) \mapsto (xr, ysr, as, brs)$ is a bijection, we get

$$|E \cap (P \times L)| = |E \cap (P' \times L')|.$$

It remains to check that P' and L' are N-distant.

To this end, let $(x, y), (x', y') \in P'$ be distinct points. If $x \neq x'$ then $x = k_1 r$,

$x' = k_2 r$ for some integers k_1, k_2, and thus $|x - x'| = |k_1 - k_2| r > N$. The same argument, with rs in place of r, works when $y \neq y'$, and the check is analogous for L'. □

5.4.4 The Main Semibounded Theorem

I have now gathered all the ingredients to deduce the main theorem of this section; all that remains is to put them together.

Theorem 5.4.19. *Let $\mathcal{M} = \langle M, <, +, \dots \rangle$ be a saturated o-minimal expansion of a group. Then, the following are equivalent:*

(1) $\mathcal{M}$ is semibounded.

(2) There is some $N \in M_{>0}$ such that for every parameter-definable family of $\mathcal{E} = \{E_b : b \in I\}$ of partitioned r-ary relations, where:

$$E_b \subseteq \prod_{i \in [r]} M^{d_i}$$

there is some $\alpha \in \mathbb{R}^{>0}$, such that: For every $b \in I$ and $k \in \mathbb{N}$, if E_b is K_∞^N-free, then for every N-distant finite grid $B \subseteq \prod_{i \in [r]} M^{d_i}$, we have

$$|E_b \cap B| \leq \alpha \delta_{r-1}^r(B).$$

(3) There is some $N \in M_{>0}$ such that for every parameter-definable $E \subseteq M^{d_1} \times M^{d_2}$ there is some positive real number $\beta < \frac{4}{3}$ such that if E is K_∞^N-free, then for every N-distant finite grid $B \subseteq M^{d_1} \times M^{d_2}$, we have

$$|E \cap B| \leq \alpha |B|^\beta.$$

(4) There is no field parameter-definable on the whole of M.

Proof.

- (1) $\implies$ (2): If $\mathcal{M}$ is non-linear then this is precisely Corollary 5.4.17, with $N \in M_{>0}$ any tall element. If $\mathcal{M}$ is linear then this follows (for any $N \in M_{>0}$) by [Bas+21, Corollary 5.11] (see Section 5.1).
- (2) $\implies$ (3): To see this, observe that when $r = 2$ we have that $r - 1 = 1$ and $1 < \frac{4}{3}$.

- (3) $\Longrightarrow$ (4): This follows from Proposition 5.4.18. Indeed, arguing by contraposition, if there were a parameter-definable field F on the whole of M, then using the point-line incidence relation on F we would be able to definably break the $\frac{4}{3}$ bounds on the class of all N-distant grids, for any $N \in M$, with a $K_{2,2}$-free (so, in particular, K_∞^N-free) relation.
- (4) $\Longrightarrow$ (1): This is precisely Fact 2.7.13(2).

□

5.5 Zarankiewicz's Problem in Presburger Arithmetic

I'll now change the setting and move from (semibounded) o-minimal structures to models of Presburger arithmetic. It would perhaps be useful for the reader to look back to Chapter 2 to remind themselves of the notation and main background results of Presburger arithmetic that will be used throughout this section.

For the remainder of this section, $\Gamma = \langle \Gamma, <, + \rangle$ will always denote a model of *Presburger arithmetic*, $T_{\mathsf{Pres}} = \mathsf{Th}(\langle \mathbb{Z}, <, + \rangle)$, in the natural Presburger language $\mathcal{L}_{\mathsf{Pres}} := \{<, +\}$.

The main result of this section is Theorem Q. In the terminology discussed in Subsection 5.2.2, I will actually present the proof of the following:

Theorem. *Let* $\Gamma \models T_{\mathsf{Pres}}$. *Then:*

(1) [Theorem 5.5.38] Γ *has linear Zarankiewicz bounds.*

(2) [Corollary 5.5.39] *If* Γ *is* $\aleph_1$*-saturated then* Γ *has strong linear Zarankiewicz bounds.*

This section is essentially entirely devoted to the proof of the theorem above.

This subsection is structured as follows:

(a) Since (2) follows easily from (1) and saturation, the main focus is on proving (1).

(b) In Subsection 5.5.1, I will present how to deduce from Corollary 5.3.6 a version of (2), for $\mathbb{Z}$-*distant* grids. This is Theorem 5.5.1.

(c) I will then present a reduction of (1) from Theorem 5.5.1, by first introducing a notion of N*-internal points*, for $N \in \mathbb{N}$, (in Subsection 5.5.2) and proving that for all $N \in \mathbb{N}$ every Presburger cell (see Definition 2.7.16) has an N-internal point. By considering *dilations* of boxes around N-internal points (for all $N \in \mathbb{N}$), one can then produce N-distant grids (as in Section 5.4) with the same *grid configuration* (see Definition 5.5.24) as any given grid. This is enough to build an *"increasing sequence of counterexamples"*, and deduce (1) from Theorem 5.5.1.

5.5.1 Saturated models of Presburger Arithmetic

I'll first state the main result of this subsection and explain the new terminology after the statement.

Theorem 5.5.1. *Let $\Gamma \models T_{\mathsf{Pres}}$ be $\aleph_1$-saturated. Let $\{E_b : b \in I\}$ be a definable family, where $E_b \subseteq \prod_{i=1}^{r} \Gamma^{d_i}$, for fixed positive integers $r \geq 2$ and $d_1, \dots, d_r \in \mathbb{N}$, for all $b \in I$. Then, there is some $\alpha \in \mathbb{R}_{>0}$ such that for all $b \in I$, if E_b is $K_\infty^{\mathbb{Z}}$-free, then for any finite $\mathbb{Z}$-distant grid $B \subseteq \prod_{i=1}^{r} \Gamma^{d_i}$ we have that:*

$$|E_b \cap B| \leq \alpha \delta_{r-1}^{r}(|B|).$$

The new terms that appeared in the statement above are, of course, the following:

- A $\mathbb{Z}$*-distant grid*: Fix positive integers $r \geq 2$ and $d_1, \dots, d_r \in \mathbb{N}$. Let $B = B_1 \times \cdots \times B_r \subseteq \prod_{i=1}^{r} \Gamma^{d_i}$, where $B_i \subseteq \Gamma^{d_i}$, for $i \in [r]$ be an r-grid. We say that B is $\mathbb{Z}$*-distant* if for all $i \in [r]$ and all distinct points $x, y \in B_i$ we have that $|x - y| > \mathbb{Z}$.
- A $K_\infty^{\mathbb{Z}}$*-free relation*: We say that E (as in the statement of Theorem 5.5.1) is $K_\infty^{\mathbb{Z}}$*-free* if it does not contain any $\mathbb{Z}$-distant grid $B_1 \times \cdots \times B_r$ such that $|B_i| \geq \omega$, for all $i \in [r]$.

Lemma 5.5.2. *Let $\{X_b : b \in I\}$ be an A-definable family of subsets $X_b \subseteq \prod_{i=1}^{r} \Gamma^{d_i}$, and $\mathcal{C}$ the class of all finite $\mathbb{Z}$-distant grids in $\prod_{i=1}^{r} \Gamma^{d_i}$. Then, there is $N \in \mathbb{N}$ such that for every $b \in I$, every $i \in [r]$, every $Y = Y_1 \times \cdots \times Y_r \in \mathcal{C}$ and every $\bar{a} = (\bar{a}_1, \dots, \bar{a}_r) \in \pi_i(X_b)$, if $(X_b)_{\bar{a}_{\neq i}}$ is finite then then $|(X_b)_{\bar{a}_{\neq i}} \cap Y_i| \leq N$, where, as before $(X_b)_{\bar{a}_{\neq i}}$ denotes the set $\{\bar{x} \in \Gamma^{d_i} : (\bar{a}_1, \dots, \bar{a}_{i-1}, \bar{x}, \bar{a}_{i+1}, \dots, \bar{a}_r) \in X_b\}$.*

In particular, acl *and $\mathcal{C}$ satisfy ($\mathcal{C}$-Uniform Bounds).*

To prove the lemma above, a version of Theorem 2.7.17 for definable families is needed (and given this, we can argue similarly to Lemma 5.4.16). This is not handled explicitly in [Clu03], and with the definition of cells considered in [Clu03] (i.e. Definition 2.7.16), it is not possible to obtain such a version.

To obtain a uniform statement a slightly more general version of Presburger cells needs to be introduced. This is done below. Intuitively, the new definition allows for $(i_1, \ldots, i_n, 1)$-cells to be finite (i.e. Presburger intervals are no longer necessarily infinite, and $(i_1, \ldots, i_n, 1)$-cells don't need to have arbitrarily large fibres). More precisely, Definition 2.7.16 is adapted as follows:

Definition 5.5.3 (Generalised Presburger Cells)**.** A *generalised Presburger cell* is defined exactly as in Definition 2.7.16, with the following adjustments:

- (1)-cells are allowed to be *finite* sets of the form:

$$\{x \in \Gamma : \alpha \,\square_1\, x \,\square_2\, \beta, x \equiv c \pmod{n}\},$$

  where $\square_i$ are either $\leq$ or no condition, and n, $c \in [0, n)$ are fixed integers.

- In the construction of $(i_1, \ldots, i_m, 1)$-cells, from Definition 2.7.16, the Unbounded Fibres condition is dropped.

Remark 5.5.4. In [Clu03], one of the uses of Presburger cells is as a characterisation of the dimension of definable sets (see [Clu03, Corollary 1]). This is no longer true if one considers generalised Presburger cells.

Since Presburger cells are, by definition, generalised Presburger cells, it is clear that Theorem 2.7.17 holds after replacing "Presburger cells" with "generalised Presburger cells". Intuitively, the new cell decomposition is obtained by grouping $(i_1, \ldots, i_n, 0)$-cells together, when appropriate.

The advantage of generalised Presburger cells is that fibres above elements (in their projections) remain generalised Presburger cells. This fact and an argument analogous to the situation in o-minimal structures (see, e.g. [Dri98, Section 3.3]) will provide a uniform cell-decomposition result for definable families.

First, some notation:

Notation 5.5.5. Given a set $A \subseteq \Gamma^{m+n}$, we denote by $\hat{\pi}_m(A)$ the projection of A onto the first m coordinates. For $b \in \hat{\pi}_m(A)$ we write A_b for the fibre of A above b, that is:

$$A_b := \{a \in \Gamma^n : (b, a) \in A\}.$$

In the notation just introduced, the following is a direct adaptation of [Dri98, Proposition III.3.5]:

Lemma 5.5.6.

(1) *Let $C \subseteq \Gamma^{m+n}$ be a generalised Presburger cell, and $a \in \hat{\pi}_m(C)$. Then C_a is a cell in Γ^n.*

(2) *Let $\mathcal{D}$ be a decomposition of Γ^{m+n} into generalised Presburger cells and $a \in \Gamma^m$. Then, the collection:*

$$\mathcal{D}_\alpha := \{C_a : C \in \mathcal{D} \text{ such that } a \in \hat{\pi}_m(C)\}$$

is a decomposition of Γ^n into generalised Presburger cells.

Proof. As noted in [Dri98], (2) follows from (1). For the proof of (1), we argue by induction on n. For $n = 1$, the result follows from the definition of generalised Presburger cells. Indeed, suppose that $C \subseteq \Gamma^{m+1}$ is a generalised Presburger cell and $a \in \hat{\pi}_m(C)$. We must consider two cases:

- *Case 1.* Suppose that C is a generalised $(i_1, \dots, i_m, 0)$-cell. By definition, it is of the form:
$$\{(x, \alpha(x)) : x \in D\},$$
for a linear function $\alpha : D \to \Gamma$ and a generalised Presburger cell $D \subseteq \Gamma^m$. If $a \in \hat{\pi}_m(C)$, then $C_a = (a, \alpha(a))$ and this is a (0)-cell.

- Suppose that C is a generalised $(i_1, \dots, i_m, 1)$-cell. By definition, it is of the form:
$$\{(x, y) : x \in D, \alpha(x) < y < \beta(x), y \equiv t \pmod r\},$$
for linear functions $\alpha, \beta : D \to \Gamma$, a generalised Presburger cell $D \subseteq \Gamma^m$, and positive integers $r \in \mathbb{Z}_{>0}$ and $t \in \{0, \dots, r-1\}$. If $a \in \hat{\pi}_m(C)$, then:
$$C_a = \{y : \alpha(a) < y < \beta(a), y \equiv t \pmod r\},$$
which is a generalised (1)-Presburger cell.

Now, for the inductive step. Suppose that the result holds for generalised Presburger cells $D \subseteq \Gamma^{m+n}$, and let $C \subseteq \Gamma^{m+(n+1)}$ be a generalised Presburger cell. Consider the projection map $\pi_1 : \Gamma^{m+(n+1)} \to \Gamma^{m+n}$. In this notation,

$\hat{\pi}_m \circ \hat{\pi}_1 : \Gamma^{m+(n+1)} \to \Gamma^m$ is precisely the projection onto the first m coordinates. Again, we consider two cases:

- *Case 1.* If C is a generalised $(i_1, \ldots, i_m, i_{m+1}, \ldots, i_{m+n}, 0)$-cell, say:

  $$C = \{(x, \alpha(x)) : x \in D\},$$

  for a linear function $\alpha : D \to \Gamma$ and a generalised $(i_1, \ldots, i_{m+n})$-cell $D \subseteq \Gamma^{m+n}$, for any $a \in \hat{\pi}_m(C)$ we have that:

  $$C_a = \{(x, \alpha_a(x)) : x \in D_a\},$$

  where D_a is a generalised Presburger cell, by induction and $\alpha_a(x) = \alpha(a, x)$.

- *Case 1.* If C is a generalised $(i_1, \ldots, i_m, i_{m+1}, \ldots, i_{m+n}, 0)$-cell, say of the form:

  $$\{(x, y) : x \in D, \alpha(x) < y < \beta(x), y \equiv t \pmod r\},$$

  for linear functions $\alpha, \beta : D \to \Gamma$, a generalised Presburger cell $D \subseteq \Gamma^{m+n}$, and positive integers $r \in \mathbb{Z}_{>0}$ and $t \in \{0, \ldots, r-1\}$. If $a \in \hat{\pi}_m(C)$, then:

  $$C_a = \{(x, y) : x \in D_a, \alpha_a(x) < y < \beta_a(x), y \equiv t \pmod r\},$$

  where D_a is a generalised Presburger cell, by induction and $\alpha_a(x) = \alpha(a, x)$, $\beta_a(x) = \beta(a, x)$.

□

The following corollary can now be deduced following word-for-word the proof of [Dri98, Corollary III.3.6]:

Corollary 5.5.7. *Let $\{E_b : b \in I\}$ be a definable family. Then, there is some $N_E \in \mathbb{N}$ such that for all $b \in I$ the set E_b has a partition into at most N_E generalised Presburger cells.*

Proof of Lemma 5.5.2. Let $\{X_b : b \in I\}$ be an A-definable family of subsets $X_b \subseteq \prod_{i=1}^r \Gamma^{d_i}$. By the previous corollary, for each $i \in [r]$ there is some $N_i \in \mathbb{N}$ such that for all $b \in I$ and all $\bar{a} = (\bar{a}_1, \ldots, \bar{a}_r) \in X_b$ the set $(X_b)_{\bar{a}_{\neq i}}$ has a partition into at most N_i generalised Presburger cells. Suppose that for some

$b \in I$ we know that $(X_b)_{\bar{a}_{\neq i}}$ is finite. Then, $(X_b)_{\bar{a}_{\neq i}}$ is contained in at most N_i copies of $\mathbb{Z}^{d_i}$, for otherwise it would necessarily be infinite, since if a generalised Presburger cell contains two points whose distance is greater than $\mathbb{Z}$, then that cell must be infinite. Thus for any $\mathbb{Z}$-distant set $Y \subseteq \Gamma^{d_i}$ we have that $|(X_b)_{\bar{a}_{\neq i}} \cap Y| \leq N_i$, and the result follows by taking $N := \max\{N_i : i \in [r]\}$. □

Proof of Theorem 5.5.1. To deduce Theorem 5.5.1 from Corollary 5.3.6 it suffices to show that:

(1) acl is *weakly locally modular* in T_{Pres}.

(2) Any relation E that is $K_\infty^{\mathbb{Z}}$-free is K_∞^{acl}-free.

(3) acl together with the class $\mathcal{C}$ of finite $\mathbb{Z}$-distant grids satisfy ($\mathcal{C}$-Uniform Bounds).

This is the last instance in this section where I will be discussing *generalised* Presburger cells. From now on, the word "cell" (or "Presburger cell") will refer to a cell in the sense of Definition 2.7.16.

For (1), we adapt the argument of [BV10, Proposition 6.9], where the authors show that linear o-minimal structures are weakly locally modular. Our argument is essentially the argument given there, with a few differences that will be pointed out. First, we need the following easy fact about definable functions in Presburger arithmetic:

Fact 5.5.8. *Let $a, b \in \Gamma$ and suppose that $b \in \mathsf{dcl}(a) \setminus \mathsf{dcl}(\emptyset)$. Then, there is an $\emptyset$-definable linear function f defined on a Presburger interval C around a such that $f(a) = b$, and f is strictly monotone on C.*

Proof of (1). First, we remark that in [BV10, Proposition 6.9], the authors prove the following:

(†) For any singletons a, b and tuple c from a saturated model $\mathcal{M} \vDash T$ if $a \in \mathsf{acl}(b, c)$ then there exists a tuple u in $\mathcal{M}$ and a singleton $d \in \mathsf{acl}(c, u)$ such that $u \mathop{\smile\!\!\!\!\!\mid} abc$ and $a \in \mathsf{acl}(b, d, u)$.

In [BV10, Theorem 4.3], the authors show that if T is a geometric theory, then 5.5.1 is equivalent to weak local modularity. It is an easy observation that the proof of [BV10, Theorem 4.3] does not, in fact, require the assumption that T is geometric, and it suffices to assume that T is pregeometric.[14] Thus, for

[14]We thank A. Berenstein and E. Vassiliev for confirming this.

Presburger arithmetic, (†) suffices to prove (1).

The proof of (†) is word-for-word the same as in [BV10, Proposition 6.9], after replacing the use of "continuity" in the o-minimal context, with "linearity" in models of Presburger arithmetic, and the group interval (b_1, b_2) with Γ, and using Fact 5.5.8 where appropriate. The use of [BV10, Proposition 6.8] is replaced by the fact that Γ is P-independent for any lovely pair[15] (Γ, P), which is trivial. □

For (2), one need only note that every acl-independent grid is $\mathbb{Z}$-distant. Indeed, let $B = B_1 \times \cdots \times B_r$ be an r-grid with each $B_i \subseteq \Gamma^{d_i}$ infinite and acl-independent. If $x = (x_1, \ldots, x_{d_i}), y = (y_1, \ldots, y_{d_i}) \in B_i$ are acl-independent, then they cannot be in the same copy of $\mathbb{Z}$, for otherwise there would be a 0-definable element $z = (z_1, \ldots, z_{d_i})$, such that $x + z = y$, and thus $y \in \mathsf{acl}(x)$, so B is $\mathbb{Z}$-distant.

Finally, (3) was proved in Lemma 5.5.2. □

The remainder of this section is devoted to deducing Theorem Q from Theorem 5.5.1.

5.5.2 The Standard Model

As I discussed previously the remainder of this section will be dedicated to deducing Theorem Q, which asserts that any model of T_{Pres} has linear Zarankiewicz bounds (meaning, for the class of all finite grids – not necessarily $\mathbb{Z}$-distant) from Theorem 5.5.1. To this end, I will start working in an arbitrary model $\Gamma \models T_{\mathsf{Pres}}$, and at some point (which I will make clear) I will switch to working in the *standard model*, that is $\langle \mathbb{Z}, <, +, 0 \rangle$. Since the statement of Theorem Q is first-order, it will suffice to develop some of the machinery only for $\langle \mathbb{Z}, <, + \rangle$.

Internal Points

Let $C \subseteq \Gamma^n$ be a Presburger cell, and $N \in \mathbb{N}$. In this subsection, I will first define what it means for a point $x \in C$ to be an *N-internal point of C* (Definition 5.5.12), for $N \in \mathbb{N}$. The main result of this subsection is that

[15] Lovely pairs of models of Presburger arithmetic are an interesting topic, but since their only relevance in this thesis is this one instance, I will omit the definition and direct the reader to [BV10] for all relevant discussion.

N-internal points exist, for every Presburger cell $C \subseteq \Gamma^n$, and every $N \in \mathbb{N}$ (Proposition 5.5.22). This is a key ingredient in the proof of the main result of the *Dilations* subsubsection.

To get started, I will discuss a notion of *cell description* for a Presburger cell $C \subseteq \Gamma^n$. To this end, I will need to fix some notation.

Congruences of cells Given an $(i_1, \ldots, i_n)$-cell $C \subseteq \Gamma^n$ it is straightforward to construct two sequences:

$$\mathbf{r}(C) = (r_1, \ldots, r_n),\ \mathbf{e}(C) = (e_1, \ldots, e_n),$$

where $r_i \in \mathbb{N}_{>0}$ and $e_i \in \{0, \ldots, r_i - 1\}$ for $i \in \{1, \ldots, n\}$, are such that:

($\star$) For all $i \in \{1, \ldots, n\}$ we have that $\pi_i(C) \subseteq e_i + r_i\mathbb{Z}$ and $\pi_i(C)$ contains only consecutive elements of $e_i + r_i\mathbb{Z}$. That is, for all $x \in \pi_i(C)$ there is some $y \in \pi_i(C)$ such that $|x - y| = r_i$.

The details of the construction of $\mathbf{r}(C)$ and $\mathbf{e}(C)$ are left to the reader. Given a cell $C \subseteq \Gamma^n$ with $\mathbf{r}(C) = (r_1, \ldots, r_n)$ we refer to r_i as the *i-th congruence of C*.

Cell descriptions Given an $(i_1, \ldots, i_n)$-cell $C \subseteq \Gamma^n$ a *cell description of C* is a triple $\mathsf{d} = (\mathbf{r}, \mathbf{e}, \bar{\alpha})$, where:

1. $\mathbf{r} = \mathbf{r}(C) = (r_1, \ldots, r_n)$.
2. $\mathbf{e} = \mathbf{e}(C) = (e_1, \ldots, e_n)$
3. $\alpha = (\alpha_{i_k} : k \in \mathsf{Z}(C), k \geq 2)$ consists of linear maps of the form:

$$\alpha_{i_k}(x_1, \ldots, x_{i_k - 1}) = \sum_{j=1}^{i_k - 1} a_j^{i_k} \left(\frac{x_j - e_j}{r_j} \right) + \gamma_{i_k},$$

such that:

$$\hat{\pi}_{i_k}(C) = \{(x, \alpha_{i_k}(x)) : x \in \hat{\pi}_{i_k - 1}(C)\}.$$

Given an $(i_1, \ldots, i_n)$-cell $C \subseteq \Gamma^n$, we write $\mathcal{D}(C)$ for the set of all cell descriptions of C. Observe that 1-dimensional cells have a unique cell description.

Given a cell description $\mathsf{d} = (\mathbf{r}, \mathbf{e}, \bar{\alpha})$ of a cell $C \subseteq \Gamma^n$ and $m \in \{1, \ldots, n\}$ we write d_m for the m-dimensional cell description of $\hat{\pi}_m(C)$ given by the restriction of d, that is:

$$\mathsf{d}_m = (\mathbf{r} \restriction_m, \mathbf{e} \restriction_m, \bar{\alpha} \restriction_{i_k \leq m}).$$

Let's now define *N-boxes* relative to a given Presburger cell C, around a point $x \in C$.

Boxes relative to cells around points Let $\mathsf{d} = (\mathbf{r}, \mathbf{e}, \bar{\alpha})$ be a cell description of an $(i_1, \ldots, i_n)$-cell $C \subseteq \Gamma^n$. We say that $x \in \Gamma^n$ *satisfies* d if:

- $x_i \equiv e_i \pmod{r_i}$ for all $i \in \{1, \ldots, n\}$.
- $\pi_{i_k}(x) = \alpha_{i_k}(\hat{\pi}_{i_k-1}(x))$ for all $k \in \mathsf{Z}(C)$ such that $k \geq 2$.

The next remark is clear:
Remark 5.5.9. Let $C \subseteq \Gamma^n$ be a cell. For any $\mathsf{d} \in \mathcal{D}(C)$ and any $x \in C$ we have that x satisfies d.
Definition 5.5.10. Let $C \subseteq \Gamma^n$ be an $(i_1, \ldots, i_n)$-cell and $\mathsf{d} = (\mathbf{r}, \mathbf{e}, \bar{\alpha}) \in \mathcal{D}(C)$ be a cell description of C. Given a point $x \in \Gamma^n$ which satisfies d, and a natural number $N \in \mathbb{N}$ we define the *N-box of type d around x*, denoted $\mathsf{B}_N(x, \mathsf{d})$, recursively, as follows:

Base case: If $C \subseteq \Gamma^n$ is an (i_1)-cell then $\mathsf{B}_N(x, \mathsf{d})$ is defined as follows:

- If $i_1 = 0$ then $\mathsf{B}_N(x, \mathsf{d}) := \{x\}$.
- If $i_1 = 1$ then $\mathsf{B}_N(x, \mathsf{d}) := x + r_1[-N, N]$.

Inductive part: Suppose we have defined N-boxes for all cells in Γ^{n-1}. Then $\mathsf{B}_N(x, \mathsf{d})$ is defined as follows:

- If C is an $(i_1, \ldots, i_{n-1}, 0)$-cell then:

$$\mathsf{B}_N(x, \mathsf{d}) := \{(x_0, \alpha_n(x_0)) : x_0 \in \mathsf{B}_N(\hat{\pi}_{n-1}(x), \mathsf{d}_{n-1})\}.$$

- If C is an $(i_1, \ldots, i_{n-1}, 1)$-cell then:

$$\mathsf{B}_N(x, \mathsf{d}) := \mathsf{B}_N(\hat{\pi}_{n-1}(x), \mathsf{d}_{n-1}) \times (\pi_n(x) + r_n[-N, N]).$$

The following is not too hard to verify:

Remark 5.5.11. Let $C \subseteq \Gamma^n$ be an $(i_1, \dots, i_n)$-cell, $\mathsf{d} \in \mathcal{D}(C)$, and $N \in \mathbb{N}$. If $x \in \Gamma^n$ satisfies d, then, for any $k \in \mathsf{Z}(C)$, $k > 1$, and any $x_0 \in \mathsf{B}_N(\hat{\pi}_{i_k-1}(x), \mathsf{d}_{i_k-1})$ we have that $\alpha_{i_k}(x_0)$ is a well-defined expression. In particular, the expression "$\alpha_n(x_0)$" in $(i_1, \dots, i_{n-1}, 0)$-case in the inductive part of Definition 5.5.10, above, is well-defined.

When the cell description of C we are working with is understood, I will simply write $\mathsf{B}_N(x, C)$ rather than $\mathsf{B}_N(x, \mathsf{d})$.

I shall now give the key definition of this section:

Definition 5.5.12. Let $C \subseteq \Gamma^n$ be a cell and $x \in C$. Given $N \in \mathbb{N}$, we say that $x \in C$ is an *N-internal point of C* (or simply that x is *N-internal in C*) if $\mathsf{B}_N(x, C) \subseteq C$.

Remark 5.5.13. An elementary inductive argument shows that the definition of N-internal points does not depend on the choice of cell description for C that we pick.

In light of the previous remark, and to simplify the exposition, I will adapt the following convention.

Convention. In the sequel, every Presburger cell $C \subseteq \Gamma^n$ will come with an arbitrarily chosen and fixed cell description $\mathsf{d} \in \mathcal{D}(C)$. Sometimes, abusing terminology, I will refer to this cell description as *the* cell description of C, even though it is not unique. The choice of cell description does not affect any parts of the arguments. When taking projections, the convention is that they will be the projection of the fixed cell description. It is also always implicitly assumed that whenever we write $\mathsf{B}_N(x, C)$, for a point $x \in \Gamma^n$, not necessarily in C, then x satisfies the cell description of C.

Definition 5.5.14. Let $C \subseteq \Gamma^n$ be a cell and $x \in C$. Given $y \in \Gamma^n$ and $N \in \mathbb{N}$ we say that y is *within N from x (relative to C)* if $y \in \mathsf{B}_N(x, C)$.

In both Definition 5.5.12 and Definition 5.5.14 I will omit reference to C when it is clear from the context.

Now that I have introduced all the required terminology, in the remainder of this subsection, I will build towards Proposition 5.5.22. This task is divided into three smaller subsubsections:

- *Preparatory Lemmas*, where I will present the proof of two small geometric results, used in the sequel.

- *Intersections of would-be fibres*, where I will present how to almost find N-internal points.
- *Existence of N-internal points*, where I will put everything together.

So let's get started.

Preparatory Lemmas

Lemma 5.5.15. *Let $C \subseteq \Gamma^n$ be an $(i_1, \ldots, i_n)$-cell. Fix $N \in \mathbb{N}$ and let $y \in \Gamma^n$ be a point satisfying the cell description of C. Then, there is some $M \in \mathbb{N}$ such that:*

$$\mathsf{B}_N(y, C) \subseteq y + \mathbf{r}(C)[-M, M]^n.$$

Proof. We prove this by induction on n. It is immediate for (0)-cells and (1)-cells, and in the inductive step it is by definition and inductive hypothesis when C is an $(i_1, \ldots, i_{n-1}, 1)$-cell.

In the case where C is an $(i_1, \ldots, i_{n-1}, 0)$-cell, by definition, we have that:

$$\mathsf{B}_N(y, C) = \{(x_0, \alpha(x_0)) : x_0 \in \mathsf{B}_N(\pi(y), \hat{\pi}_{n-1}(C))\}.$$

By inductive hypothesis there is some $M_0 \in \mathbb{N}$ such that:

$$\pi(\mathsf{B}_N(y, \mathsf{d})) = \mathsf{B}_N(\pi(y), \hat{\pi}_{n-1}(C)) \subseteq \pi(y) + \mathbf{r}(\hat{\pi}_{n-1}(C))[-M_0, M_0]^{n-1}.$$

Let $M = \max\{M_0, M_1\}$, where

$$M_1 = |\max(\pi_n(\mathsf{B}_N(\pi(y), \hat{\pi}_{n-1}(C))))| + |\min(\pi_n(\mathsf{B}_N(\pi(y), \hat{\pi}_{n-1}(C))))|$$

Then:

$$\mathsf{B}_N(x, \hat{\pi}_{n-1}(C)) \subseteq x + \mathbf{r}(C)[-M, M]^n,$$

as required. □

And for my next trick:

Lemma 5.5.16. *Let $C \subseteq \Gamma^n$ be a cell and $\alpha : C \to \Gamma$ be a linear function on C. For all $T \in \mathbb{N}$ there is a function $M : \mathbb{N} \to \mathbb{N}$ such that for all $x \in C$ we have that:*

$$\mathsf{max}\{\alpha(y) : y \in \mathsf{B}_T(x, C)\} = \alpha(x) + M(T).$$

Moreover, if α is non-constant on C, then M is strictly increasing in T.

Proof. We prove this by induction on n. The base case is immediate from the definitions of $\mathsf{B}_T(x, C) \subseteq \Gamma$ and the linearity of α.

For the inductive step, assume that:

(IH): For all cells $D \subseteq \Gamma^{n-1}$ and all linear functions $\beta : \Gamma^{n-1} \to \Gamma$ defined on D we have that: For all $T \in \mathbb{N}$ there is a function $M : \mathbb{N} \to \mathbb{N}$ such that for all $x_0 \in D$ we have that:

$$\mathsf{max}\{\beta(y) : y \in \mathsf{B}_T(x_0, D)\} = \beta(x_0) + M(T).$$

Moreover, if β is non-constant on D, then M is strictly increasing.

We now consider two cases. To simplify notation, given a cell $C \subseteq \Gamma^n$ we shall write $\pi(C)$ for the cell $\hat{\pi}_{n-1}(C) \subseteq \Gamma^{n-1}$.

Case 1. Suppose that C is an $(i_1, \ldots, i_{n-1}, 0)$-cell. Then C is of the form $\{(y, \gamma(y)) : y \in \pi(C)\}$, for some linear map $\gamma : \Gamma^{n-1} \to \Gamma$. By definition, $\mathsf{B}_T(x, C) = \{(y_0, \gamma(y_0)) : y_0 \in \mathsf{B}_T(\pi(x), \pi(C))\}$, so:

$$\mathsf{max}\{\alpha(y) : y \in \mathsf{B}_T(x, C)\} = \mathsf{max}\{\alpha((y_0, \gamma(y_0))) : y_0 \in \mathsf{B}_T(\pi(x), \pi(C))\}.$$

By applying (IH) to $\pi(C)$ and the linear function $\alpha' : \Gamma^{n-1} \to \Gamma$ given by $\alpha' : y_0 \mapsto \alpha((y_0, \gamma(y_0)))$, the existence of $M(T)$ follows. For the moreover part, observe that if α is non-constant on C, then α' is non-constant on $\pi(C)$, and we are done by (IH).

Case 2. Suppose that C is an $(i_1, \ldots, i_{n-2}, 1)$-cell. To fix notation, let $\alpha(x) = \sum_{i=1}^{n} a_i \left(\frac{x_i - e_i}{r_i}\right) + c$, and let $\alpha' : \Gamma^{n-1} \to \Gamma$ be the linear function obtained by restricting α to the first $(n-1)$-coordinates, that is $\alpha'(y) = \sum_{i=1}^{n-1} a_i \left(\frac{y_i - e_i}{r_i}\right) + c$. By (IH) there is some $M_1 \in \mathbb{N}$ such that:

$$\mathsf{max}\{\alpha'(y) : y \in \mathsf{B}_T(\pi(x), \pi(C))\} = \alpha'(\pi(x)) + M_1$$

By the base case, for (1)-cells, there is some $M_2 \in \mathbb{N}$ such that:

$$\mathsf{max}\{\alpha''(x) : x \in \pi_n(x) + r_n[-T, T]\} = \alpha''(x) + M_2,$$

where $\alpha'' : \Gamma \to \Gamma$ is the linear function $\alpha'' : x \mapsto a_n \left(\frac{x - e_n}{r_n}\right)$. Then, taking $M = M_1 + M_2$ gives the first part of the result. For the moreover part, if α is non-constant on C then at least one of α' and α'' must be non-constant on

C. If α' is non-constant then the fact that $M(T)$ is strictly increasing follows immediately from (IH), while if α' is constant but α'' is not, it follows from linearity of α''. □

Intersections of would-be fibres

For the remainder of this subsubsection fix an $(i_1, \ldots, i_{n-1}, 1)$-cell C of the form:

$$C = \{(x, x_n) : x \in \pi(C),\ \alpha(x) < x_n < \beta(x),\ x_n \equiv e_n \pmod{r_n}\},$$

where:

- $x = (x_1, \ldots, x_{n-1})$,
- $\alpha(x), \beta(x)$ are linear functions,
- $\pi(C) := \hat{\pi}_{n-1}(C) \subseteq \Gamma^{n-1}$ is an $(i_1, \ldots, i_{n-1})$-cell,
- r_n is a positive integer and $e_n \in \{0, \ldots, r_n - 1\}$.

Let's also fix here the notation for the linear functions $\alpha, \beta : \Gamma^{n-1} \to \Gamma$:

$$\alpha(x) = \sum_{i=1}^{n-1} a_i \left(\frac{x_i - e_i}{r_i}\right) + c_\alpha, \text{ and } \beta(x) = \sum_{i=1}^{n-1} b_i \left(\frac{x_i - e_i}{r_i}\right) + c_\beta.$$

For simplicity, throughout this subsection, I will work under the assumption that $a_i, b_i \neq 0$, and $a_i \neq b_i$ for all $i \in \{1, \ldots, n-1\}$. This assumption can be made without loss of generality (all the results still hold without this assumption, with proofs essentially identical to the ones we give when a_i or b_i are equal to 0 or when $a_i = b_i$, but we need to be careful in our calculations, to exclude terms that evaluate to 0 from denominators of fractions).

Let $\mathbf{r}(C) = (r_1, \ldots, r_n)$ and $\mathbf{e}(C) = (e_1, \ldots, e_n)$. Given $x \in \Gamma^{n-1}$ such that $x_i \equiv e_i \pmod{r_i}$ we write $\mathsf{w}C_x$ for the following set of points:

$$\mathsf{w}C_x := \{y \in \Gamma : \alpha(x) < y < \beta(x), x_n \equiv e_n \pmod{r_n}\}.$$

Remark 5.5.17. If $x \in \pi(C)$ then $\mathsf{w}C_x = C_x$, where C_x denotes the fibre of C above x.

By definition, $\mathsf{w}C_x$ is the fibre above x in the set:

$$\mathsf{w}C = \left\{(x, y) : x \in \prod_{i=1}^{n-1}(e_i + r_i\mathbb{Z}),\ \alpha(x) < y < \beta(x),\ y \equiv e_n \pmod{r_n}\right\}$$

If $x \notin \pi(C)$, but $x_i \equiv e_i \pmod{r_i}$ for all $i \in \{1, \ldots, n-1\}$, then we refer to $\mathsf{w}C_x$ as the *would-be fibre of* C above x.

Let $y \in \Gamma^{n-1}$ be a point satisfying the cell description of $\pi(C)$. We write $\mathsf{S}_N(y, C)$ for the following set of points:

$$\mathsf{S}_N(y, C) := \bigcap_{z \in \mathsf{B}_N(y, \pi(C))} \mathsf{w}C_z.$$

The following remark is immediate, from the definitions:
Remark 5.5.18. Fix $N \in \mathbb{N}$. Given $x = (x_1, \ldots, x_{n-1}) \in \Gamma^{n-1}$ such that $x_i \equiv d_i \pmod{r_i}$ for all $i \in \{1, \ldots, n-1\}$ then for all $y \in \mathbf{r}(\pi(C))[-M, M]^{n-1}$, where $M \in \mathbb{N}$, we have that $x_i \equiv e_i \pmod{r_i}$. Thus $\mathsf{w}C_y$ is well-defined, in the sense that the expressions '$\alpha(y)$' and '$\beta(y)$' are meaningful, for all $y \in \mathbf{r}(\pi(C))[-M, M]^{n-1}$, for $M \in \mathbb{N}$, and so, in particular, also, for all $y \in \mathsf{B}_N(x, \pi(C))$.

Intuitively, $\mathsf{S}_N(x, C)$ is the intersection of the would-be fibres of C, ranging over all points in the N-box of type $\pi(C)$ around x. The situation is clear when x is an N-internal point of $\pi(C)$. In particular, we have the following easy lemma:
Lemma 5.5.19. *Suppose that $x \in \pi(C)$ is N-internal and $|\mathsf{S}_N(x, C)| > 2N$. Then, there is an N-internal point in C.*

Proof. To see this, observe that when x is an N-internal point of $\pi(C)$, by definition $\mathsf{B}_N(x, \pi(C)) \subseteq \pi(C)$. Thus:

$$\mathsf{S}_N(x, C) = \bigcap_{y \in \mathsf{B}_N(x, \pi(C))} C_y.$$

Since $|\mathsf{S}_N(x, C)| > 2N$, we must have that there exists some $y_0 \in C_x$ such that $y_0 + r_n[-N, N] \subseteq \mathsf{S}_N(x, C)$. In particular, we get that:

$$\mathsf{B}_N(x, \pi(C)) \times (y_0 + r_n[-N, N]) = \mathsf{B}_N((x, y_0), C) \subseteq C,$$

that is, (x, y_0) is N-internal. □

Lemma 5.5.20. *Let $C \subseteq \Gamma^n$ be an $(i_1, \ldots, i_{n-1}, 1)$-cell, as above. For all $N \in \mathbb{N}$ there is some $N_C \in \mathbb{Z}$ such that for all $y \in \Gamma^{n-1}$ satisfying the cell description of $\pi(C)$ we have that:*

$$|\mathsf{S}_N(y, C)| \geq |\mathsf{w}C_y| + N_C$$

Proof. In the notation of the statement, above, let $M \in \mathbb{N}$ be such that $\mathsf{B}_N(y, \hat{\pi}_{n-1}(C)) \subseteq y + \mathbf{r}(\pi(C))[-M, M]^{n-1}$, as in Lemma 5.5.15. Then, clearly:

$$\mathsf{S}_N(y, C) \supseteq \bigcap_{u \in \mathbf{r}[-M,M]} \mathsf{w}C_{y+u},$$

where, to simplify notation, $\mathbf{r} = \mathbf{r}(C)$.

Now, by linearity, we have that the size of the intersection on the right-hand side, above, depends only on the would-be fibres in the "corners" of $y + \mathbf{r}[-M, M]^{n-1}$. More precisely:

$$\bigcap_{u \in \mathbf{r}[-M,M]} \mathsf{w}C_{y+u} = \bigcap_{u \in \mathbf{r}\{-M,M\}^{n-1}} \mathsf{w}C_{y+u}.$$

For any $u \in \mathbf{r}\{-M, M\}^{n-1}$, by definition, we have that $\mathsf{w}C_{y+u}$ is simply:

$$(\alpha\,(y + u)\,, \beta\,(y + u)) \cap (e_n + r_n\mathbb{Z}).$$

In particular:

$$\bigcap_{u \in \mathbf{r}\{-M,M\}^{n-1}} \mathsf{w}C_{y+u} = \{x \in \Gamma : \alpha_{\max} < x < \beta_{\min}, x \equiv e_n \pmod{r_n}\}$$

where

$$\alpha_{\max} = \max(\{\alpha(y + u) : u \in \mathbf{r}\{-M, M\}^{n-1}\})$$

and

$$\beta_{\min} = \min(\{\beta(y + u) : u \in \mathbf{r}\{-M, M\}^{n-1}\}).$$

Straightforward calculations show that:

$$\alpha_{\max} = \alpha(y + u_a), \text{ and } \beta_{\min} = \beta(y + u_b),$$

where for all $i \in \{1, \ldots, n-1\}$:

$$(u_a)_i = \begin{cases} M & \text{if } a_i > 0; \\ -M & \text{if } a_i < 0 \end{cases}, \text{ and } (u_b)_i = \begin{cases} M & \text{if } b_i < 0; \\ -M & \text{if } b_i > 0 \end{cases}.$$

Let N'_C denote the following quantity:

$$N'_C := M\left(\sum_{i\in N_\beta} b_i - \sum_{i\in P_\beta} b_i\right) + M\left(\sum_{i\in N_\alpha} a_i - \sum_{i\in P_\alpha} a_i\right),$$

where $P_\alpha = \{i \in [n-1] : a_i > 0\}$, $N_\alpha = \{i \in [n-1] : a_i < 0\}$, and P_β, N_β are defined similarly.

It is not hard to see, by unfolding the definitions, that:

$$\beta_{\min} - \alpha_{\max} = \left(\sum_{i=1}^{n-1}\left(\frac{y_i - e_i}{r_i}\right)(b_i - a_i) + c_\beta - c_\alpha\right) + N'_C.$$

Since, by definition:

$$\mathsf{w}C_y = \{y' \in e_n + r_n\mathbb{Z} : \alpha(\pi_{n-1}(y)) < y' < \beta(\pi_{n-1}(y))\}$$

it follows that:

$$|\mathsf{w}C_y| = \left|\left\{y' \in \mathbb{Z} : \sum_{i=1}^{n-1} a_i\left(\frac{y_i - e_i}{r_i}\right) + c_\alpha < y' < \sum_{i=1}^{n-1} b_i\left(\frac{y_i - e_i}{r_i}\right) + c_\beta\right\} \cap (e_n + r_n\mathbb{Z})\right|,$$

in particular:

$$|\mathsf{w}C_y| \geq \left\lfloor \frac{1}{r_n}\left(\sum_{i=1}^{n-1}\left(\frac{y_i - e_i}{r_i}\right)(b_i - a_i) + c_\beta - c_\alpha\right)\right\rfloor,$$

we have that:

$$\left|\bigcap_{u\in\mathbf{r}\{-M,M\}^{n-1}} \mathsf{w}C_{y+u}\right| > |C_y| + \left\lfloor \frac{N'_C}{r_n} \right\rfloor.$$

The lemma then follows from the equation above, by taking N_C to be $\left\lfloor \frac{N'_C}{r_n} \right\rfloor$.

□

The following is a corollary of the lemma above and Lemma 5.5.16:

Corollary 5.5.21. *Let $C \subseteq \Gamma^n$ be an $(i_1, \dots, i_{n-1}, 1)$-cell. For all $N \in \mathbb{N}$ there is some $T = T(N) \in \mathbb{N}$ such that for all $x \in \pi(C)$ there is a point $y \in \mathsf{B}_T(x, C)$ such that $|S_N(y, C)| > 2N$.*

Proof. Fix $N \in \mathbb{N}$ and define the linear function:

$$\gamma : \Gamma^{n-1} \to \Gamma$$
$$x \mapsto \beta(x) - \alpha(x).$$

There are two cases to consider here:

- If the function γ is constant, then, by definition of $(i_1, \dots, i_{n-1}, 1)$-cells, we must have that $\gamma(x)$ is infinite, for all $x \in \pi(C)$. In this case, for every $x \in \pi(C)$ we have that

$$\mathsf{S}_N(x, C) \geq |C_x| + N_C > 2N,$$

  where $N_C \in \mathbb{N}$ is as in Lemma 5.5.20, since $|C_x|$ infinite, and $N \in \mathbb{N}$.

- If the function γ is non-constant, then by Lemma 5.5.16, for all $T \in \mathbb{N}$ there is some $M = M(T) \in \mathbb{N}$ such that for all $x \in \pi(C)$:

$$\mathsf{max}\{\gamma(y) : y \in \mathsf{B}_T(x, \pi(C))\} = \gamma(x) + M,$$

  and, $M(T)$ is increasing in T. In particular, there is some $T \in \mathbb{N}$ such that $M(T) > r_n(2N - N_C)$, where N_C is as in Lemma 5.5.20. Then, for all $x \in C$ there is some $y \in \mathsf{B}_T(x, \pi(C))$ such that:

$$\gamma(y) = \gamma(x) + r_n(2N - N_C).$$

  Thus, it follows that:

$$|\mathsf{w}C_y| \geq \left\lfloor \frac{\beta(y) - \alpha(y)}{r_n} \right\rfloor = \left\lfloor \frac{\gamma(y)}{r_n} \right\rfloor > 2N - N_C,$$

  since $\gamma(x) > 0$, as $x \in \pi(C)$. Combining this with Lemma 5.5.20, it follows that $\mathsf{S}_N(y, C) > 2N$, as claimed.

□

Existence of Internal Points

Finally, it's time to prove that N-internal points exist.

Proposition 5.5.22 (Existence of Internal Points)**.** *Let $C \subseteq \Gamma^n$ be a cell. Then, for all $N \in \mathbb{N}$ there is an N-internal point $x \in C$.*

Proof. The proof is by induction on n. We first discuss the base case. Let $N \in \mathbb{N}$ be arbitrary. Then the result is clear if C is a (0)-cell, since it contains a single point, which is N-internal for all $N \in \mathbb{N}$. The result is also clear if C is a (1)-cell, which is unbounded above and below, since in this case every point in C is N-internal. Now, if C is bounded below (resp. above), then there is a least (resp. max) point $x \in C$ which is not N-internal, and since C is infinite, we are done.

Now, for the inductive step:

(IH): For all $N \in \mathbb{N}$ and for all $(i_1, \ldots, i_{n-1})$ cells $D \subseteq \Gamma^{n-1}$, D contains an N-internal point.

Fix $N \in \mathbb{N}$. We consider the cases of $(i_1, \ldots, i_{n-1}, 0)$- and $(i_1, \ldots, i_{n-1}, 1)$-cells separately.

- *Case 1.* Let $C \subseteq \Gamma^n$ be an $(i_1, \ldots, i_{n-1}, 0)$-cell, say $C = \{(x, \alpha(x)) : x \in D\}$, for some $(i_1, \ldots, i_{n-1})$-cell $D = \pi(C)$ and some linear function α. By inductive hypothesis, D contains an N-internal point, say y_0. But then, by unfolding the definition of an N-internal point for a $(i_1, \ldots, i_{n-1}, 0)$-cell, we see that the point $(y_0, \alpha(y_0))$ is N-internal.
- *Case 2.* Let $C \subseteq \Gamma^n$ be an $(i_1, \ldots, i_{n-1}, 1)$-cell. By Lemma 5.5.20, there is some $N_C \in \mathbb{N}$ such that:

  $$|\mathsf{S}_N(y, C)| \geq |\mathsf{w}C_y| + N_C$$

  for all $y \in \Gamma^{n-1}$ satisfying the cell description of $\pi(C)$. By Corollary 5.5.21 we can find some T such that for every $x \in \pi(C)$ there is some $y \in \mathsf{B}_T(x, \pi(C))$ such that $|\mathsf{w}C_y| > 2N - N_C$. Observe that for such y we have that:

  $$|\mathsf{S}_N(y, C)| > |\mathsf{w}C_y| + N_C > 2N.$$

  Now, by (IH) there is a $2K$-internal point $x_0 \in \pi(C)$, where $K = \max\{N, T\}$. In particular, by our choice of x and K, there is a K-

internal point $y \in \mathsf{B}_T(x_0, \pi(C))$ with $|\mathsf{S}_N(y, C)| \geq 2N$. Since we chose $K \geq N$, the result follows from Lemma 5.5.19.

□

Dilations

In this subsection I will present a proof of a "*grid dilation*" result (Proposition 5.5.37), which intuitively states that given any grid, inside a Presburger cell C, and any $N \in \mathbb{N}$, we can find, in C an N-distant grid with a "stronger" configuration (see Definition 5.5.34) than the original grid.

Notation 5.5.23. To facilitate exposition and avoid repetitions, throughout this subsection, we fix the following notation, which persists throughout this section:

- Positive integers $d_1, \ldots, d_r \in \mathbb{N}_{>0}$.
- For each $j \in \{1, \ldots, r\}$ we write $n_j = \sum_{i=1}^{j} d_i$, and we set $n_0 = 1$.
- For simplicity we write n for $n_r = \sum_{i=1}^{r} d_r$.
- For each $j \in \{1, \ldots, r\}$, by the *coordinates indexed by* d_j we mean the coordinates $n_{j-1}, \ldots, n_j$.
- For each $j \in \{1, \ldots, r\}$ we write $\hat{\pi}_{d_j}$ for the projection map to the coordinates indexed by d_j, so $\hat{\pi}_{d_j} : \Gamma^n \to \Gamma^{d_j}$.

Let's start by discussing in more detail some of the basic properties of grids.

About Grids Throughout this section an r-*grid* means, unless otherwise stated, an r-*grid* ***with respect to*** $d_1, \ldots, d_r$, i.e. a set $B \subseteq \Gamma^n$ of the form:

$$B = B_1 \times \cdots \times B_r,$$

where $B_j \subseteq \Gamma^{d_j}$, for all $j \in \{1, \ldots, r\}$.

Recall the following general definition:

Definition 5.5.24 (Grid configuration)**.** Let $B, B' \subseteq \Gamma^n$. We say that B and B' have the *same grid configuration with respect to* $d_1, \ldots, d_r$ if there exist enumerations $\{b_1, \ldots, b_m\}$ and $\{b'_1, \ldots, b'_m\}$ of B and B', respectively, such that for all $k, l \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, r\}$, we have

$$\hat{\pi}_{d_j}(b_k) = \hat{\pi}_{d_j}(b_l) \text{ if, and only if, } \hat{\pi}_{d_j}(b'_k) = \hat{\pi}_{d_j}(b'_l).$$

Example 5.5.25. It is easy to see that if $B = B_1 \times \cdots \times B_r$ is an $n_1 \times \cdots \times n_r$-grid and $B' = B'_1 \times \cdots \times B'_r$ is an $n'_1 \times \cdots \times n'_r$-grid then B and B' have the same grid configuration if, and only if $n_i = n'_i$, for all $i \in [r]$.

The following remark is immediate from the definition of having the same grid configuration:

Remark 5.5.26. Let $B, B' \subseteq \Gamma^n$ and suppose that B and B' have the same grid configuration with respect to $d'_1 = 1, \ldots, d'_n = 1$. Then they have the same grid configuration with respect to $d_1, \ldots, d_r$.

Given an arbitrary subset $B \subseteq \Gamma^n$ we have a natural way of building from it an r-grid, the *generated r-grid of B (with respect to $d_1, \ldots, d_r$)*, denoted $\mathsf{Gen}(B)$, namely:

$$\mathsf{Gen}(B) := \prod_{j=1}^{r} \hat{\pi}_{d_j}(B).$$

Lemma 5.5.27. *If $B, B' \subseteq \Gamma^n$ have the same grid configuration (with respect to $d_1, \ldots, d_r$) then $\mathsf{Gen}(B)$ and $\mathsf{Gen}(B')$ have the same grid-configuration (with respect to $d_1, \ldots, d_r$).*

Proof. To see this, given enumerations $\{b_1, \ldots, b_m\}$ and $\{b'_1, \ldots, b'_m\}$ of B and B' respectively, for each $j \in \{1, \ldots, r\}$ define $B_j := \{\hat{\pi}_{d_j}(b_1), \ldots, \hat{\pi}_{d_j}(b_m)\}$ and $B'_j := \{\hat{\pi}_{d_j}(b'_1), \ldots, \hat{\pi}_{d_j}(b'_m)\}$. Then, by definition $\mathsf{Gen}(B) = \prod_{j=1}^{r} B_j$ and $\mathsf{Gen}(B') = \prod_{j=1}^{r} B'_j$, and since B and B' have the same grid configuration, for all $j \in \{1, \ldots, r\}$ we have that $|B_j| = |B'_j|$.

Set $\beta_j = |B_j|$, and for each $(i_1, \ldots, i_r) \in \prod_{j=1}^{r}\{1, \ldots, \beta_j\}$ let $b_{i_1,\ldots,i_r} \in \Gamma^n$ be the element whose coordinates indexed by d_j are precisely b_{i_j} and define $b'_{i_1,\ldots,i_r}$ analogously. Consider the following enumerations $\{b_{i_1,\ldots,i_n} : (i_1, \ldots, i_r) \in \prod_{j=1}^{r}\{1, \ldots, \beta_j\}\}$ and $\{b'_{i_1,\ldots,i_n} : (i_1, \ldots, i_r) \in \prod_{j=1}^{r}\{1, \ldots, \beta_j\}\}$ of $\mathsf{Gen}(B)$ and $\mathsf{Gen}(B')$, respectively. By unfolding the definitions, and since B and B' have the same grid configuration we can see that these enumerations witness that $\mathsf{Gen}(B)$ and $\mathsf{Gen}(B')$ have the same grid configuration. □

Growing Fix a $(i_1, \ldots, i_n)$-cell $C \subseteq \Gamma^n$, a point $x \in C$ and positive integers $k, N \in \mathbb{N}$. We define the *k-fold dilation* of $\mathsf{B}_N(x, C)$, denoted $k\mathsf{B}_N(x, C)$ to be the subset of $\mathsf{B}_{kN}(x, C)$ consisting of *"every k-th point"* in each interval of $\mathsf{B}_{kN}(x, C)$. More precisely:

Definition 5.5.28 (k-fold Dilation)**.** In the notation above, we define $k\mathsf{B}_N(x, C)$ recursively as follows:

- If C is a (0)-cell, then $k\mathsf{B}_N(x,C) := \mathsf{B}_{kN}(x,C) = C$.
- If C is a (1)-cell, then $k\mathsf{B}_N(x,C) := x + kr_1[-N,N]$.

Once we have defined $k\mathsf{B}_N(x,C)$ for $(i_1,\dots,i_{n-1})$-cells, we define:

- If C is a $(i_1,\dots,i_{n-1},0)$-cell of the form $\{(x_0,\alpha(x_0)), x_0 \in \pi(C)\}$, then

$$k\mathsf{B}_N(x,C) := \{(x_0,\alpha(x_0) : x_0 \in k\mathsf{B}_N(\pi(x),\pi(C))\}$$

- If C is a $(i_1,\dots,i_{n-1})$-cell then:

$$k\mathsf{B}_N(x,C) := k\mathsf{B}_N(\pi(x),\pi(C)) \times (\pi_n(x) + kr_n[-N,N])$$

Observe that whenever $j \in \mathsf{D}(C)$, the distance between any two points in $\pi_j(k\mathsf{B}_N(x,C))$ is at least k. In particular, given $(\star)$ (from page 199) the following is easy to see:

Remark 5.5.29. Let $k,N \in \mathbb{N}$ and $x \in C$. Then $k\mathsf{B}_N(x,C)$ is k-distant, but observe that even if $\mathsf{B}_N(x,C)$ is a k-distant set (meaning that for all $i \in [r]$ and all distinct $a,b \in \pi_i(\mathsf{B}_N(x,C))$ we have that $|a-b| \geq k$), this does not mean that $\mathsf{Gen}(\mathsf{B}_N(x,C))$ is necessarily a k-distant grid (for a fixed choice of $d_1,\dots,d_r \in \mathbb{N}$). The point of the next lemma is that an appropriate dilation ensures that this will be the case.

Lemma 5.5.30. *Let $C \subseteq \Gamma^n$ be an $(i_1,\dots,i_n)$-cell and $x \in C$. For all $k \in \mathbb{N}$ there exists some $k' \in \mathbb{N}$ such that for all $N \in \mathbb{N}$*

$$\prod_{i=1}^{n} \pi_i(k'\mathsf{B}_N(x,C))$$

is k-distant.

The following lemma will be used in the proof:

Lemma 5.5.31. *Let $C \subseteq \Gamma^n$ be an $(i_1,\dots,i_n)$-cell and $x \in C$. Let $f : C \to \Gamma$ be a linear function on C. For all $k \in \mathbb{N}$ there is some $k' \in \mathbb{N}$ such that for $N \in \mathbb{N}$ and all distinct $z,z' \in k'\mathsf{B}_N(x,C)$ we have that if $f(z) \neq f(z')$ then $|f(z)-f(z')| \geq k$. Moreover, if $k'' \geq k'$, then for all $z,z' \in k''\mathsf{B}_N(x,C)$ we have that $|f(z)-f(z')| \geq km$.*

Proof. We argue by induction on n, and we will show the slightly stronger assertion that

($\star$) For all $k \in \mathbb{N}$ there is some $k' \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ and all distinct $z, z' \in k'\mathsf{B}_N(x, C)$ there is some $m \in \mathbb{N}$ such that

$$|f(z) - f(z')| = km.$$

Moreover, if $k'' \geq k'$, then for all $z, z' \in k''\mathsf{B}_N(x, C)$ we have that $|f(z) - f(z')| \geq km$.

The base case is trivial when C is a 0-cell and when C is a 1-cell, without loss of generality we may assume that $C = d + r\Gamma$, for some $r \in \mathbb{Z}_{>0}$ and some $r \in \{0, \ldots, r-1\}$, and that

$$\begin{aligned} f : C &\to \Gamma \\ x &\mapsto a\frac{x-d}{r} + \gamma, \end{aligned}$$

for some $a \in \mathbb{Z}$ and $\gamma \in \Gamma$. Let $k' = kr$. Then, for any $N \in \mathbb{N}$ and any two distinct $z, z' \in k'\mathsf{B}_N(x, C)$, by definition of $k'\mathsf{B}_N(x, C)$ we have that $|z - z'| = k'm_0 = krm_0$, for some $m_0 \in \mathbb{N}$. Thus:

$$\begin{aligned} |f(z) - f(z')| &= \left|\frac{a(z-d)}{r} - a\frac{a(z'-d)}{r}\right| = \left|a\left(\frac{z-z'}{r}\right)\right| \\ &= k|am_0|, \end{aligned}$$

as required (putting $m = |am_0|$). The moreover part is immediate.

For the inductive step, suppose that ($\star$) holds for all $(i_1, \ldots, i_{n-1})$ cells $C_0 \subseteq \Gamma^{n-1}$. More precisely:

(IH) For all cells $C_0 \subseteq \Gamma^{n-1}$, all $x_0 \in C$, all $k \in \mathbb{N}$ and all linear functions $f_0 : C_0 \to \Gamma$ there is some $k'_0 \in \Gamma$ such that for all $N \in \mathbb{N}$ and all distinct $z_0, z'_0 \in k'_0\mathsf{B}_N(x_0, C_0)$ we have that if $f_0(z_0) \neq f_0(z'_0)$ then there is some m_0 such that $|f_0(z_0) - f_0(z'_0)| = km_0$. Moreover, if $k''_0 \geq k'_0$, then for all $z_0, z'_0 \in k''\mathsf{B}_N(x, C)$ we have that $|f(z_0) - f(z'_0)| \geq km_0$.

Let $C \subseteq \Gamma^n$ be a cell, $x \in C$, and to fix notation, say that:

$$f(z) = \sum_{i=1}^{n} \frac{a_i(z_i - d_i)}{r_i} + \gamma,$$

for all $z \in C$, for some $a_i, r_i \in \mathbb{Z}$, $d_i \in \{0, \ldots, r_i - 1\}$, $\gamma \in \Gamma$. We need to consider two cases:

- *Case 1.* Suppose that C is an $(i_1, \dots, i_{n-1}, 0)$-cell and $f : C \to \Gamma$ a linear function. To fix notation, say:

$$C = \{(x, \alpha(x)) : x \in \pi(C)\}.$$

Observe that since $C \subseteq \mathsf{dom}(f)$, for all $x = (x_1, \dots, x_{n-1}) \in \pi(C)$ and all $i \in [n-1]$ we have that $x_i \equiv d_i \pmod{r_i}$. Thus, without loss of generality, we can write α in the following form:

$$\alpha(x) = \sum_{i=1}^{n-1} \frac{b_i(x_i - d_i)}{r_i} + \delta$$

Then, for any $z \in C$ we have that:

$$\begin{aligned} f(z) &= \sum_{i=1}^{n-1} \frac{a_i(z_i - d_i)}{r_i} + \left(\sum_{i=1}^{n-1} \frac{b_i(x_i - d_i)}{r_i} + \delta \right) + \gamma \\ &= \sum_{i=1}^{n-1} \frac{(a_i + b_i)(x_i - d_i)}{r_i} + (\gamma + \delta), \end{aligned}$$

and the result follows immediately from (IH).

- *Case 2.* Suppose that C is an $(i_1, \dots, i_{n-1}, 1)$-cell. Define two functions:

$$\begin{aligned} f_0 : \hat{\pi}_{n-1}(C) &\to \Gamma \\ z &\mapsto \sum_{i=1}^{n-1} \frac{a_i(z_i - d_i)}{r_i} + \gamma, \end{aligned}$$

and

$$\begin{aligned} f_1 : \pi_n(C) &\to \Gamma \\ z &\mapsto \frac{a_n(z - d_n)}{r_n}, \end{aligned}$$

so that for all $z \in C$ we have that:

$$f(z) = f_0(\hat{\pi}_{n-1}(z)) + f_1(\pi_n(z)).$$

By (IH), there is some k'_0 such that for $N \in \mathbb{N}$ and all distinct $z_0, z'_0 \in k'_0 \mathsf{B}_N(\hat{\pi}_{n-1}(x), \pi_{n-1}(C))$ then there is some $m_0 \in \mathbb{N}$ such that $|f_0(z_0) - f_0(z'_0)| = km_0$. We can argue as in the base case for (1)-cells to find some k'_1 such that for all $N \in \mathbb{N}$ and all distinct $z_1, z'_1 \in k'_1 \mathsf{B}_N(\pi_n(x), \pi_n(C))$ there is some $m_1 \in \mathbb{N}$ such that $|f_1(z_1) - f_1(z'_1)| = km_1$.

Let $k' = k'_0 k'_1$, and pick arbitrary $z, z' \in k'\mathsf{B}_N(x, C)$. Observe that by definition we have:

$$\begin{aligned}\hat{\pi}_{n-1}(z), \hat{\pi}_{n-1}(z') &\in k'\mathsf{B}_N(\hat{\pi}_{n-1}(x), C)\\ &\subseteq k'_0\mathsf{B}_{k'_1 N}(\hat{\pi}_{n-1}(x), C),\end{aligned}$$

and thus there is some m'_0 such that

$$|f_0(\hat{\pi}_{n-1}(z)) - f_0(\hat{\pi}_{n-1}(z'))| = km'_0.$$

Similarly:

$$\begin{aligned}\pi_n(z), \pi_n(z') &\in (\pi_n(x) + k_0 k'_1 r_n[-N, N])\\ &\subseteq (\pi_n(x) + k'_1 r_n[-k_0 N, k_0 N])\end{aligned}$$

and thus there is some m'_1 such that

$$|f_1(\pi_n(z)) - f_1(\pi_n(z'))| = km'_1.$$

Putting everything together:

$$\begin{aligned}|f(z) - f(z')| &= |[f_0(\hat{\pi}_{n-1}(z)) + f_1(\pi_n(z))] - [f_0(\hat{\pi}_{n-1}(z)) + f_1(\pi_n(z))]|\\ &= |f_0(\hat{\pi}_{n-1}(z)) - f_0(\hat{\pi}_{n-1}(z)) + f_1(\pi_n(z)) - f_1(\pi_n(z))|\\ &= |km'_0 + km'_1|\\ &= k|m'_0 + m'_1|,\end{aligned}$$

as required. The moreover part follows easily, from the analysis above and (IH), since, if $k'' \geq k'$ then for all $z, z' \in k''\mathsf{B}_N(x, C)$ we clearly (as in the (1)-cell case) have that:

$$|f_1(\hat{\pi}_{n-1}(z)) - f_0(\hat{\pi}_{n-1}(z'))| \geq km'_1,$$

and by the moreover part of (IH) we have that:

$$|f_0(\hat{\pi}_{n-1}(z)) - f_0(\hat{\pi}_{n-1}(z'))| \geq km'_0,$$

and thus $|f(z) - f(z') \geq k(m'_0 + m'_1)$.

This concludes the proof. □

Now the proof of Lemma 5.5.30 is almost immediate:

Proof of Lemma 5.5.30. Observe, first, that if the conclusion of the lemma holds for some $k' \in \mathbb{N}$, then, it holds for any $k'' > k'$. More precisely:

Claim 1. Suppose that for some $k' \in \mathbb{N}$ we have that:

$$\prod_{i=1}^{n} \pi_i(k' B_N(x, C))$$

is k-distant, and let $k'' > k'$. Then:

$$\prod_{i=1}^{n} \pi_i(k'' B_N(x, C))$$

is k-distant

Proof of Claim 1. This follows by easy induction and the moreover part of the previous lemma. ◀

Returning now to the proof of the lemma, we argue by induction on n and the structure of $\mathsf{B}_N(x, C)$. The base case is again trivial. Assume now that the result holds for $(i_1, \dots, i_{n-1})$-cells. More precisely:

(IH): For all $(i_1, \dots, i_{n-1})$-cells $C_0 \subseteq \Gamma^{n-1}$, all $k, N \in \mathbb{N}$ there is a $k'_0 \in \mathbb{N}$ such that:

$$\prod_{i=1}^{n-1} \pi_i(k'_0 \mathsf{B}_N(x, C))$$

is k-distant.

We have to consider two cases:

- *Case 1.* Suppose that C is an $(i_1, \dots, i_{n-1}, 0)$-cell. Then, by (IH) we get some k'_0 and by Lemma 5.5.31 we get some k'_1. Take $k' = \max\{k'_0, k'_1\}$, and the result follows.
- *Case 2.* Suppose that C is an $(i_1, \dots, i_{n-1}, 1)$-cell. By (IH) we get some k'_0. Then taking $k' = \max\{k'_0, k\}$ gives the result.

□

Lemma 5.5.32. *Let $x, y \in C$ and $k, N \in \mathbb{N}$. Then $\mathsf{B}_N(x, C)$ and $k\mathsf{B}_N(y, C)$ have the same grid configuration, with respect to $d'_1 = 1, \dots, d'_n = 1$.*

Proof. Fix $k, N \in \mathbb{N}$. We prove this by induction on n and the structure of $\mathsf{B}_N(x, C)$. Trivially, all singleton sets have the same grid configuration.

Moreover, it is clear (taking the increasing enumeration) that $x + r_1[-N, N]$ and $y + kr_1[-N, N]$ have the same grid configuration. Assume now that the result holds for $(i_1, \ldots, i_{n-1})$-cells. More precisely:

(IH): For all $(i_1, \ldots, i_{n-1})$-cells $C_0 \subseteq \Gamma^{n-1}$ and all $x_0, y_0 \in \Gamma^{n-1}$ there are enumerations $\{b_1, \ldots, b_m\}$ and $\{b'_1, \ldots, b'_m\}$ of $\mathsf{B}_N(x_0, C_0)$ and $k\mathsf{B}_N(y_0, C_0)$ such that for all $l, l' \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n-1\}$ we have that $\pi_j(b_l) = \pi_j(b_{l'})$ if, and only if, $\pi_j(b'_l) = \pi_j(b'_{l'})$.

If C is an $(i_1, \ldots, i_{n-1}, 0)$-cell, then we can enumerate $\mathsf{B}_N(x, C)$ and $k\mathsf{B}(y, C)$ with the enumerations induced by the enumerations we obtained from (IH). If, on the other hand, we have that C is an $(i_1, \ldots, i_{n-1}, 1)$-cell, then we can enumerate $\mathsf{B}_N(x, C)$ and $k\mathsf{B}(y, C)$ as lexicographic products of the enumerations from (IH) and the increasing enumerations of $\pi_n(x) + r_n[-N, N]$ and $\pi_n(y) + r_n k[-N, N]$, respectively. In each case we can easily see that the given enumerations witness that these sets have the same grid configuration. □

The following is an immediate consequence of Remark 5.5.26 and Lemma 5.5.32:

Corollary 5.5.33. *Let $x, y \in C$ and $k, N \in \mathbb{N}$. Then $\mathsf{B}_N(x, C)$ and $k\mathsf{B}_N(y, C)$ have the same grid configuration, with respect to $d_1, \ldots, d_r$.*

The next definition will be crucial in Proposition 5.5.37.

Definition 5.5.34 (Z-stronger grid)**.** Given an r-ary relation $E \subseteq \prod_{i=1}^{r} \Gamma^{d_i}$, and r-grids $B, B' \subseteq \prod_{i=1}^{r} \Gamma^{d_i}$ with the same grid configuration, we say that B' is *Z-stronger, for E,* than B if (in the notation of Definition 5.5.24) for all $k \in \{1, \ldots, l\}$, if $b_j \in E$ then $b'_j \in E$.

Lemma 5.5.35. *Let y be an N-internal point of C. Let $B_0 \subseteq C$ such that $B_0 = \mathsf{Gen}(B_0) \cap C$ and $B' \subseteq \mathsf{B}_N(y, C)$. If B_0 and B' have the same grid configuration, then $\mathsf{Gen}(B')$ is Z-stronger for C than $\mathsf{Gen}(B_0)$.*

Proof. By Lemma 5.5.27 we know that $\mathsf{Gen}(B')$ and $\mathsf{Gen}(B_0)$ have the same grid configuration. Since y is N-internal we know that $B' \subseteq \mathsf{B}_N(y, C) \subseteq C$ and thus, for any point $b \in \mathsf{Gen}(B_0)$, if $b \in C$ then $b \in B_0$ and hence the corresponding point in the enumeration of $\mathsf{Gen}(B')$ will belong to C. □

In order to prove the main result of this subsection I will need one final preparatory lemma.

Boxes and Dilations This is the point in the argument where I will have to switch from Γ being an arbitrary model of T_{Pres} to Γ being the standard model of T_{Pres}. For the remainder of this section fix an $(i_1, \ldots, i_n)$-cell $C \subseteq \mathbb{Z}^n$.

Lemma 5.5.36 (in the standard model)**.** *Let B be a finite r-grid. Then, there is some $x \in C$ and some $M \in \mathbb{N}$ such that:*

$$B \cap C \subseteq \mathsf{B}_M(x, C)$$

Proof. Let $B = B_1 \times \cdots \times B_r$ be an r-grid, where $B_j \subseteq \mathbb{Z}^{d_j}$, for each $j \in \{1, \ldots, r\}$. Let $M_0 \in \mathbb{N}$ be the maximum distance between any two distinct points in B; this exists since B is a finite subset of $\mathbb{Z}^n$. Let $b = \max\{|B_j| : j \in \{1, \ldots, r\}\}$. By Lemma 5.5.30 there exists some $M \in \mathbb{N}$ such that for all $x \in C$ we have that:

$$\prod_{i=1}^{n} \pi_i(M_1 \mathsf{B}_b(x, C))$$

is M_0-distant. Observe that $M_1 \mathsf{B}_b(x, C) \subseteq \mathsf{B}_{M_1 \times b}(x, C)$.

Let $x \in B \cap C$ be arbitrary (if the intersection is empty then the lemma follows immediately). We claim that $B \cap C \subseteq \mathsf{B}_{M_1 \times b}(x, C)$. Suppose that $y \in B \cap C$. Since $y \in C$, by definition, y satisfies d for any $\mathsf{d} \in \mathcal{D}(C)$. Thus, to show that $y \in \mathsf{B}_{M_1 \times b}(x, C)$ it suffices to show that for all $i \in \mathsf{D}(C)$ we have that $|x_i - y_i| < M_0 \times b$.

Since B is M_0-distant and not $(M_0 + 1)$-distant, for each $j \in \{1, \ldots, r\}$ we know that $\max\{\hat{\pi}_{d_j}(y)_i - x_i : i \in \{n_{j-1}, \ldots, n_j\}\} < M_0$, so we are done. □

Now, I can finally put everything together.

Proposition 5.5.37 (Grid Dilations, in the standard model)**.** *Let $B \subseteq \mathbb{Z}^n$ be a finite r-grid. For all $N \in \mathbb{N}$ there is an r-grid $B_N \subseteq \mathbb{Z}^n$ which is N-distant and Z-stronger for C than B.*

Proof. Let B be some r-grid. Observe that it suffices to show the proposition for grids of the form $B = \mathsf{Gen}(B_0)$, where $B_0 = B \cap C$. If B is such a grid, then since $B = \mathsf{Gen}(B \cap C)$, we have that:

$$B_0 = B \cap C = \mathsf{Gen}(B \cap C) \cap C = \mathsf{Gen}(B_0) \cap C.$$

Let $N \in \mathbb{N}$. By Lemma 5.5.36 we know that $B \cap C \subseteq \mathsf{B}_M(x, C)$ for some $x \in C$

and some $M \in \mathbb{N}$. Let $k \in \mathbb{N}$ be such that for all $L \in \mathbb{N}$ the set $\mathsf{Gen}(k\mathsf{B}_L(x, C))$ is N-distant; this exists, by Lemma 5.5.30. Let $y \in C$ be a kM-internal point of C, which exists by Proposition 5.5.22. By Corollary 5.5.33 we know that $\mathsf{B}_M(x, C)$ and $N\mathsf{B}_M(y, C)$ have the same grid configuration (with respect to $d_1 = \cdots = d_n = 1$, and thus, with any $d_1, \ldots, d_r$, by Remark 5.5.26). Let $B' \subseteq k\mathsf{B}_M(y, C)$ correspond (via the enumeration witnessing that $\mathsf{B}_M(x, C)$ and $N\mathsf{B}_M(y, C)$ have the same grid configuration) to the points in $B \cap C$. Then $\mathsf{Gen}(B')$ is N-distant, by our choice of k and has the same grid configuration as $\mathsf{Gen}(B \cap C)$. Finally, since y was kM-internal, by Lemma 5.5.35 we have that $\mathsf{Gen}(B')$ is Z-stronger than $\mathsf{Gen}(B \cap C)$. □

The Main Presburger Theorem

Theorem 5.5.38. *Let Γ be a model of Presburger arithmetic and $E \subseteq \prod_{i \in [r]} \Gamma^{d_i}$ a $\emptyset$-definable set, for fixed $d_1, \ldots, d_r \in \mathbb{N}$. Let $\mathcal{C}$ be the class of all finite r-grids in $\prod_{i \in [r]} \Gamma^{d_i}$. Then, there is $\alpha \in \mathbb{R}^{>0}$, such that for every $k \in \mathbb{N}$, if E is $K_{\mathbf{k}}$-free, then for every grid $B \in \mathcal{C}$, we have:*

$$|E \cap B| \leq \alpha \delta^r_{r-1}(B).$$

Proof. Observe that for all $k \in \mathbb{N}$ the statement "E is $K_{\mathbf{k}}$-free" as well as the statements "$|E \cap B| \leq \alpha \delta^r_{r-1}(B)$" for fixed $\alpha \in \mathbb{R}_{>0}$ and fixed sizes $B_1, \ldots, B_r$, where $B = B_1 \times \cdots \times B_r$, are expressible in first-order. Thus, it follows that if the conclusion of the theorem is true in $\mathbb{Z}$, it must be true in any $\Gamma \models T_{\mathsf{Pres}}$.

By the Cell Decomposition Theorem, it suffices to prove this when E is a cell, and by the previous observation we may assume that $E \subseteq \mathbb{Z}^n$. The proof is an easy compactness argument, based on grid dilations (Proposition 5.5.37). Assume toward a contradiction that for all $\alpha \in \mathbb{R}_{>0}$ there exists an r-grid B_α such that $|E \cap B_\alpha| > \alpha \delta^r_{r-1}(B_\alpha)$. For all $N \in \mathbb{N}$ and $\alpha \in \mathbb{R}_{>0}$ let $\phi_{N,\alpha}(x)$ be the formula expressing that x is an N-distant r-grid with the has the same configuration as B_α (for a fixed given enumeration of B_α), and also that x is Z-stronger for E than B_α, so $|E \cap B_\alpha| > \alpha \delta^r_{r-1}(B_\alpha)$. By compactness and Proposition 5.5.37 the set $\Sigma_\alpha(\bar{x}) = \{\phi_{N,\alpha}(x) : N \in \mathbb{N}\}$ is satisfiable. In particular, there is some saturated $\Gamma \succcurlyeq \mathbb{Z}$ a $\mathbb{Z}$-distant r-grid B such that $|E^\Gamma \cap B| > \alpha \delta^r_{r-1}(B)$. Since we can do this for all $\alpha \in \mathbb{R}_{>0}$, we have a contradiction to Theorem 5.5.1. □

From the theorem above, we can immediately deduce the following improved version of Theorem 5.5.1 for $\emptyset$-definable sets, by saturation:

Corollary 5.5.39. *In the context of Theorem 5.5.38, assume in addition that* Γ *is* $\aleph_1$*-saturated. Then, there is* $\alpha \in \mathbb{R}^{>0}$*, such that if* E *is* K_∞*-free, then for every grid* $B \in \mathcal{C}$*, we have:*

$$|E \cap B| \leq \alpha \delta_{r-1}^{r}(|B|).$$

5.6 Zarankiewicz's Problem in Hrushovski Constructions

In this section, I shall present a proof of Theorem R and Theorem S. The latter result is particularly interesting, because, in a sense, it gives a new class of examples of structures in which definable hypergraphs have linear Zarankiewicz bounds, structures whose geometry is *not* (weakly) locally modular. Of course, these structures still do not interpret infinite fields, so Theorem S hints at the possibility that the existence of (interpretable) fields is the *"essential"* obstruction to linear Zarankiewicz bounds.

This section is structured as follows:

(a) In Subsection 5.6.1 I give a very quick recount of the necessary background on stable one-baesed structures, and recall one of the main results of [Eva05].

(b) Then Subsection 5.6.2 has the proofs of Theorem R and Theorem S.

5.6.1 Local Preliminaries: One-Based Theories

Throughout, fix a complete stable L-theory T. Everything will take place inside a monster model $\mathbb{M} \vDash T^{\mathsf{eq}}$, and thus I will not distinguish between real and imaginary elements. Admittedly, the exposition in this subsection is rather terse, and I will only mention the definitions and facts that are needed for the proof of Theorem R. That being said, there is a rich theory lurking in the background, and a good starting point (with references and proofs) on stable one-based theories is [Pil96, Chapter 4].

Definition 5.6.1. We say that T is *one-based* if for all small algebraically closed subsets $A, B \subseteq \mathbb{M}^{\mathsf{eq}}$ we have that:

$$A \underset{A \cap B}{\downarrow} B.$$

And now for something completely different:

A *pseudoplane* is an incidence structure consisting of two sorts of objects, *points* and *lines*, such that:

1. Every point lies on infinitely many lines.
2. Every line contains infinitely many points.
3. No two distinct lines intersect at infinitely many points.
4. No two distinct points lie at the intersection of infinitely many lines.

In the language of geometric stability theory, this can be expressed as follows:
Definition 5.6.2. We say that the set of realisations of a complete type $r(x, y) = \mathsf{tp}(\mathsf{p}, \ell/\emptyset)$ is a *complete type-definable quasidesign* (in T) if:

1. $\mathsf{p} \notin \mathsf{acl}(\ell)$.
2. $\ell \notin \mathsf{acl}(\mathsf{p})$.
3. For any $\ell \neq \ell'$ the set $\{\mathsf{p} : \mathbb{M} \vDash r(\mathsf{p}, \ell) \wedge r(\mathsf{p}, \ell')\}$ is finite.

We say that the set of realisations of $r(x, y)$ is a *complete type-definable pseudoplane*, if, in addition, we also have that:

4. For any $\mathsf{p} \neq \mathsf{p}'$ the set $\{\ell : \mathbb{M} \vDash r(\mathsf{p}, \ell) \wedge r(\mathsf{p}', \ell)\}$ is finite.

This is the exact geometric configuration whose absence characterises one-based theories, amongst the stable ones. More precisely:
Theorem 5.6.3 ([Pil96, Proposition 4.1.7])**.** *The following are equivalent for a stable theory T:*

1. *T is one-based.*
2. *There is no complete type-definable quasidesign in T.*
3. *There is no complete type-definable pseudoplane in T.*

Remark 5.6.4. One-basedness is preserved after naming (small sets of) parameters.

The following result of Evans gives a connection between stable one-based theories and Hrushovski constructions (discussed in Chapter 2).

Theorem 5.6.5 ([Eva05, Theorem 1.9])**.** *Let $\mathcal{M}_0$ be an* ab initio *Hrushovski construction in the language $\mathcal{L} = \{R\}$. Then, there is a stable one-based trivial*[16] *$\mathcal{L}'$-structure $\mathcal{N}_0$, for a language*[17] *$\mathcal{L}' \supseteq \mathcal{L}$, whose language reduct to $\mathcal{L}$ is isomorphic to $\mathcal{M}_0$.*

5.6.2 Linear Zarankiewicz bounds and Gridless Quasidesigns

Let $\mathcal{M}$ be an $\mathcal{L}$-structure and fix positive integers $n \in \mathbb{N}$ and $d_1, \ldots, d_n$. By an *n-ary partial type* (possibly over parameters), I shall mean a consistent, with $\mathsf{Th}(\mathcal{M})$, set of partitioned formulas $\pi(x_1, \ldots, x_n)$ in free variables $x_1, \ldots, x_n$, where the x_i are allowed to be (finite) tuples. Generalising some of the machinery introduced in Section 5.2, a partial type will be called $K_{\mathbf{k}}$*-free* for $k \in \mathbb{N} \cup \{\infty\}$ (this is short for $K_{k,\ldots,k}$-free) if for all $A_i \subseteq M^{x_i}$ with $|A_i| = k$ (or $|A_i| \geq \aleph_0$ if $k = \infty$), for $i \in [n]$, there is some $(a_1, \ldots, a_n) \in \prod_{i\in[n]} A_i$, such that $(a_1, \ldots, a_n) \nvDash \pi(x_1, \ldots, x_n)$.

Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Following the terminology of [Eva05], an n-ary partial type $\pi(x_1, \ldots, x_n)$ will be called *sparse in $\mathcal{M}$* if there is a real number $\alpha \in \mathbb{R}_{>0}$ such that for all $A_i \subseteq M^{d_i}$, for $i \in [n]$, the following holds:

$$\left|\left\{(a_1, \ldots, a_n) \in \prod_{i\in[n]} A_i : (a_1, \ldots, a_n) \vDash \pi(x_1, \ldots, x_n)\right\}\right| \leq \alpha \delta^n_{n-1}(A).$$

We may recast the definition of *linear Zarankiewicz bounds* to partial types, rather than just definable sets:

Definition 5.6.6. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. We say that $\mathcal{M}$ has *strong linear Zarankiewicz bounds for partial types* in $\mathcal{M}$ if for every n-ary partial type $\pi(x_1, \ldots, x_n)$ (possibly over parameters) the condition (SLZ) holds, where:

(SLZ) If π is K_∞-free then π is sparse in $\mathcal{M}$.

A theory T has *(strong) linear Zarankiewicz bounds for partial types* if every $\mathcal{M} \vDash T$ has (strong) linear Zarankiewicz bounds.

---

[16] Recall: A stable theory T is called *trivial* if for any three tuples a, b, c of elements and any set A of parameters (from the monster model of T), if a, b, c are pairwise forking-independent over A, then a, b, are independent over Ac, see [Goo91].

[17] More precisely, $\mathcal{L}'$ is the language of *oriented hypergraphs*, described in detail in [Eva05, Definition 1.2, Remark 1.3].

The following remark follows immediately from the definitions:

Remark 5.6.7. If $\mathcal{M}$ has (strong) linear Zarankiewicz bounds for partial types and $\mathcal{N}$ is a reduct of $\mathcal{M}$ then $\mathcal{N}$ has (strong) linear Zarankiewicz bounds for partial types.

The remark above leads to the following natural question:

Question 5. *Suppose that $\mathcal{M}$ has (strong) linear Zarankiewicz bounds (for partial types) and $\mathcal{N}$ is interpretable in $\mathcal{M}$. Is it the case that $\mathcal{N}$ has (strong) linear Zarankiewicz bounds (for partial types)?*

Throughout the rest of this subsection, again, everything will be taking place inside a monster model $\mathbb{M}$ of a theory T, which will be assumed to have elimination of imaginaries (i.e. a monster model of T^{eq}). Thus, when a model is not mentioned, it should be understood that we are in $\mathbb{M}$.

I will start by giving the following variant of quasidesigns (which were briefly discussed in Subsection 5.6.1). Here, rather than enforcing that the intersection of two distinct "lines" contains finitely many "points" I will instead enforce that the intersection of any infinite set of pairwise distinct "lines" contains finitely many "points". More precisely:

Definition 5.6.8. We say that the set of realisations of partial type $r(x, y)$ defines a *partial gridless-quasidesign* if:

1. $\mathsf{p} \notin \mathsf{acl}(\ell)$, whenever $(\mathsf{p}, \ell) \vDash r(x, y)$.
2. $\ell \notin \mathsf{acl}(\mathsf{p})$, whenever $(\mathsf{p}, \ell) \vDash r(x, y)$.
3. For all sets $\{\ell_i : i \in \mathbb{N}\}$ consisting of pairwise distinct elements (in the y-sort) the set $\{\mathsf{p} : \mathbb{M} \vDash \bigwedge_{i\in\mathbb{N}} r(\mathsf{p}, \ell_i)\}$ is finite.

When $r(x, y) = \mathsf{tp}(a, b/\emptyset)$ is a complete type, we speak of *complete* gridless quasidesigns.

Remark 5.6.9. Let $r(x, y)$ be a partial gridless quasidesign. Then, by compactness there is some $N \in \mathbb{N}$ such that for all sets $\{\ell_i : i \in \mathbb{N}\}$ consisting of pairwise disinct elements in the y-sort the set $\{\mathsf{p} : \mathbb{M} \vDash \bigwedge_{i\leq N} r(\mathsf{p}, \ell_i)\}$ is finite.

To see this, suppose that this is not the case. Then, for all $N \in \mathbb{N}$ there is a set $\{\ell_i : i \in \mathbb{N}\}$ of pairwise distinct elements in the y-sort such that for all $M \in \mathbb{N}$ the set $\{\mathsf{p} : \mathbb{M} \vDash \bigwedge_{i<N} r(\mathsf{p}, \ell_i)\}$ has size at least M. Thus, the set:

$$\Sigma(y_i : i \in \mathbb{N}) = \{y_i \neq y_j : i \neq j \in \mathbb{N}\} \cup \left\{ |\bigwedge_{i<N} r(x, y_i)| \geq M : M, N \in \mathbb{N} \right\}$$

is finitely consistent. A realisation of Σ gives us an infinite set $\{\ell_i : i \in \mathbb{N}\}$, of pairwise distinct elements of the y-sort for which the set $\{\mathsf{p} : \mathbb{M} \models \bigwedge_{i\in\mathbb{N}} r(\mathsf{p}, \ell_i)\}$ is infinite.

On the way to the main theorem, I shall first show that stable one-based theories omit complete gridless quasidesigns.

Proposition 5.6.10. *If a theory T has a complete gridless quasidesign then T has a complete quasidesign. In particular, if T is a stable one-based theory, then T does not have a complete gridless quasidesign.*

Proof. The proposition follows from compactness and standard tricks. Suppose towards a contradiction that a stable one-baesed theory T admits a complete gridless quasidesign, say $r(x, y)$. Our goal is to construct a complete quasidesign in T, contradicting Theorem 5.6.3.

Observe that, since $r(x, y)$ is a gridless quasidesign, by definition, for all infinite sets $L = \{\mathsf{l}_i : i \in \mathbb{N}\}$ of elements in the y-sort $\mathsf{p} \in \mathsf{acl}(L)$. Thus, by compactness, there is some maximal $N \in \mathbb{N}$ and elements $\ell_1, \dots, \ell_N$ such that $\mathsf{p} \models \bigwedge_{i\leq N} r(x, \ell_i)$ and $\mathsf{p} \notin \mathsf{acl}(\ell_1, \dots, \ell_N)$. That is, for all $N' > N$ and $\{\ell_i : i \leq N'\}$, distinct elements, if $\mathsf{p} \models \bigwedge_{i\leq N'} r(x, \ell_i)$, then $\mathsf{p} \in \mathsf{acl}(\ell_1, \dots, \ell_{N'})$.

Let $\bar{\ell} =$ '$\{\ell_1, \dots, \ell_N\}$' $\in \mathbb{M}^{\mathsf{eq}}$ and $s(x, \bar{y}) = \mathsf{tp}(\mathsf{p}, \bar{\ell})$. Then, for any realisation $(b, \bar{c})$ of this type we have that $b \notin \mathsf{acl}(\bar{c})$ and $\bar{c} \notin \mathsf{acl}(b)$. Moreover, if $\bar{c}_1 =$ '$\{c_1^1, \dots, c_1^N\}$' $\neq \bar{c}_2 =$ '$\{c_2^1, \dots, c_2^N\}$', then, by our choice of N the set $\{\mathsf{p}' : \mathsf{p}' \models s(x, \bar{c}_1) \wedge s(x, \bar{c}_2)\}$, is finite. Thus, $s(x, \bar{y})$ is a complete quasidesign, contradicting the one-basedness of T.

□

Proposition 5.6.11. *Let T be a stable one-based theory without the finite cover property, and let $\mathcal{M} \models T^{\mathsf{eq}}$ be an $\aleph_1$-saturated model. Let $\pi(x_1, \dots, x_n) \subseteq M^{d_1} \times \cdots \times M^{d_n}$ be a partial type over a small set of parameters B. If $\pi(x_1, \dots, x_n)$ is K_∞-free, then for any $(a_1, \dots, a_n) \models \pi(x_1, \dots, x_n)$ there is some $i \in [n]$ such that $a_i \in \mathsf{acl}(\{a_j : j \in [n] \setminus \{i\}\}, B)$.*

In particular, in the terminology of Definition 5.3.1 p is acl*-tight.*

Proof. Without loss of generality, we may assume that $B = \emptyset$ (this is only to simplify notation). Suppose that $\pi(x_1, \dots, x_n)$ is as in the proposition, and assume, towards a contradiction, that the conclusion of the proposition is false. Then, there is some realisation $\bar{a} = (a_1, \dots, a_n) \vDash \pi(x_1, \dots, x_n)$ such that $a_i \notin \mathsf{acl}(\bar{a}_{\neq i})$ for all $i \in [n]$.

Since T is one-based, we know that:

$$\mathsf{acl}(a_i) \mathop{\perp}_{\mathsf{acl}(a_i) \cap \mathsf{acl}(\bar{a}_{\neq i})} \mathsf{acl}(\bar{a}_{\neq i}),$$

for all $i \in [r]$. Let $D = \bigcap_{i \in [r]} \mathsf{acl}(\bar{a}_{\neq i})$, and observe that for all $i \in [n]$ we have that $\mathsf{acl}(a_i) \cap \mathsf{acl}(\bar{a}_{\neq i}) \subseteq D \subseteq \mathsf{acl}(\bar{a}_{\neq i})$, so, by Monotonicity and Transitivity we have that

$$a_i \mathop{\perp}_{D} \bar{a}_{\neq i},$$

for all $i \in [n]$.

Let $s_i(x_i) = \mathsf{tp}(a_i/D)$, and observe that $D = \mathsf{acl}(D)$, since it is the intersection of algebraically closed sets. Since D is algebraically closed in $\mathbb{M}^{\mathsf{eq}}$ and T is stable, $s_i(x_i)$ is stationary for $i \in [r]$, and thus we can consider their Morley product.

Essentially by definition of the Morley product, since $(a_1, \dots, a_n)$ are a forking-independent over D tuple, we have that:

$$\mathsf{tp}(a_1, \dots, a_n/D) = s_1(x) \otimes \cdots \otimes s_n(x).$$

Now, take a Morley sequence in $s_1(x) \otimes \cdots \otimes s_n(x)$, say:

$$A := ((a_1^t, \dots, a_r^t) : t \in \omega) \vDash (s_1(x) \otimes \cdots s_n(x))^{\otimes \omega}$$

Each coordinate $A_i := (a_i^t : t \in \omega)$ is a Morley sequence in $s_i(x)$, since $\otimes$ is associative, and since by assumption these types are non-algebraic, each A_i is infinite. Finally, since T is stable, $\otimes$ is commutative, so for all $(a_1^{t_1}, \dots, a_n^{t_n}) \in \prod_{i \in [r]} A_i$ we have that:

$$(a_1^{t_1}, \dots, a_n^{t_n}) \vDash \mathsf{tp}(a_1, \dots, a_n/D).$$

In particular, $\prod_{i\in[n]} A_i \subseteq \pi(x_1, \ldots, x_n)$, which is a contradiction. □

Recall that a stable theory T *does not have the finite cover property* if T^{eq} eliminates the quantifier $\exists^{\infty}$ (see, for instance, *f.c.p. Theorem* [She90, Theorem II.4.4(8)]).

In [Eva05, Proposition 3.1], Evans proves that if T is a complete stable one-based theory that does not have the finite-cover property, then any type-definable pseudoplane $\pi(x, y)$ is sparse, that is, there is some $\alpha \in \mathbb{R}_{>0}$ such that for all finite $A \times B$ we have that:

$$|\{(a, b) \in A \times B : (a, b) \models \pi(x, y)\}| \leq \alpha\delta_1^2(A \times B).$$

First, it is easy to observe that the proof of [Eva05, Proposition 3.1] actually shows that if T is as above, then any type-definable *quasidesign* $\pi(x, y)$ is sparse. The proof of this is word-for-word the same as the proof that Evans gives.

Now, to relate this result with the combinatorial notion of linear Zarankiewicz bounds for partial types introduced at the beginning of this subsection, one need only make the following observation: Evans's proof yields that for any binary partial type $\pi(x, y)$ (in a stable one-based theory T which does not have the finite cover property) if $\pi(x, y)$ is $K_{2,\infty}$-free (or $K_{\infty,2}$-free) then $\pi(x, y)$ is sparse in $\mathbb{M} \models T$.

Proposition 5.6.10 gives the following generalisation of Evans's result. The argument below is similar to the proof of Theorem 5.3.5.

Theorem 5.6.12. *Let T be a stable one-based theory that does not have the finite cover property. Then:*

(1) If $\mathbb{M} \models T^{\mathrm{eq}}$ is an $\aleph_1$-saturated model of T then $\mathbb{M}$ has strong linear Zarankiewicz bounds for partial types.

(2) T has linear Zarankiewicz bounds (for definable sets).

Proof. Given (1), we can deduce (2) by an easy reduction. Indeed, suppose that $\mathcal{M} \models T$ and $\phi(x_1, \ldots, x_n)$ is $K_{\mathbf{k}}$-free. Let $\mathcal{N} \succcurlyeq \mathcal{M}$ be an $\aleph_1$-saturated. Clearly, the set of realisations of ϕ in $\mathcal{N}$ remains $K_{\mathbf{k}}$-free, and is thus K_{∞}-free. By (1), it has linear Zarankiewicz bounds.

So, it remains to prove (1). Our argument is standard, and analogous to both [Bas+21, Theorem 5.6] and [Eva05, Proposition 3.1]. Let $\pi(x_1,\dots,x_n)$ be K_∞-free. By Proposition 5.6.11, for all $(a_1,\dots,a_n)\models\pi(x_1,\dots,x_n)$ there is some $i\in[n]$ such that $a_i\in\mathsf{acl}(\{a_j : j\in[n]\setminus\{i\}\})$. By elimination of $\exists^\infty$, whenever $a_i\in\mathsf{acl}(\{a_j : j\in[n]\setminus\{i\}\})$ this is definable by a formula. More precisely, we can find formulas $\phi_{1,k}(x_1,\dots,x_n),\dots,\phi_{n,k}(x_1,\dots,x_n)$ and natural numbers $m_{i,k}$, for some index set K, such that:

$$\mathcal{M}\models\bigwedge_{i\in[n]}\forall x_1\cdots\forall x_{i-1}\forall x_{i+1}\cdots\forall x_n\exists^{<m_{i,k}}x_i\phi_{i,k}(x_1,\dots,x_n),$$

for all $k\in K$, and:

$$\mathcal{M}\models\pi(x_1,\dots,x_n)\to\bigvee_{k\in K}\left(\bigvee_{i\in[n]}\phi_{i,k}(x_1,\dots,x_n)\right).$$

So, by compactness, there is some formula $\rho(x_1,\dots,x_n)\in\pi(x_1,\dots,x_n)$, formulas $\phi_1(x_1,\dots,x_n),\dots,\phi_n(x_1,\dots,x_n)$ such that:

$$\mathcal{M}\models\rho(x_1,\dots,x_n)\to\bigvee_{i\in[n]}\phi_i(x_1,\dots,x_n).$$

and

$$\mathcal{M}\models\bigwedge_{i\in[n]}\forall x_1\cdots\forall x_{i-1}\forall x_{i+1}\cdots\forall x_n\exists^{<m}x_i\phi_i(x_1,\dots,x_n),$$

for some natural number $m\in\mathbb{N}$.

Claim 1. $\rho(x_1,\dots,x_n)$ is sparse.

Proof of Claim 1. To see this, let $A_i\subseteq\mathcal{M}^{x_i}$ be finite sets. Then, for every $(a_1,\dots,a_n)\in\prod_{i\in[m]}A_i$ such that $\mathcal{M}\models\rho(a_1,\dots,a_n)$ there is some $i\in[n]$ such that $\mathcal{M}\models\phi_i(a_1,\dots,a_n)$. Thus, we have that:

$$|\{(a_1,\dots,a_{i-1},a_{i+1},\dots,a_n) : \mathcal{M}\models\rho(a_1,\dots,a_n)\}|\le m.$$

Thus, $|\rho(A_1,\dots,A_n)|\le m\delta^n_{n-1}(\prod_{i\in[n]}A_i)$, as claimed. ◀

Since $\rho(x_1,\dots,x_n)\in\pi(x_1,\dots,x_n)$ it follows that $\pi(x_1,\dots,x_n)$ is also sparse. This concludes the proof. □

So, the following corollary is now immediate:

Corollary 5.6.13. *Let $\mathcal{M}$ be an* ab initio *Hrushovski construction (as briefly* *discussed in the previous section). Then, any partial type in* n *variables consistent with* $\mathsf{Th}(\mathcal{M})$ *has linear Zarankiewicz bounds in* $\mathcal{M}$*.*

Proof. By [Eva05, Subsection 3.2], the theory of the structures from Theorem 5.6.5 has nfcp. The result now immediate from Theorem 5.6.5, Remark 5.6.7, and Theorem 5.6.12. □

Chapter 6

Odds and Ends

> 'All this happened, more or less.'
>
> Kurt Vonnegut, *Slaughterhouse-Five*

Introduction

This chapter is divided into two independent parts. In the **first part**, I will discuss some results around NIP_n (see Definition 4.2.12) and monadic expansions, as hinted in Subsection 4.2.4. In the **second part**,[1] I will discuss the transfer of generalised indiscernibility (cf. Chapter 4) in *Mekler groups*. I have opted to keep the second part of this chapter relatively short and less self-contained than what (I hope) the rest of this thesis has been. That said, the reader can find all necessary terminology concerning Mekler groups in Appendix A.

[1] This is a relatively small part of the larger project [BPT24], which is co-authored with B. Boissonneau and P. Touchard

Part 1: Monadic NIP$_k$ and Coding Random Hypergraphs

6.1 Introduction to Part 1

In this section, I will present a proof of the following result:

Theorem T (Corollary 6.4.6)**.** *Let T be a complete first-order $\mathcal{L}$-theory. Then, the following are equivalent:*

(1) T is monadically NIP$_n$ *for some $n \in \mathbb{N}_{\geq 1}$.*[2]

(2) T is monadically NIP$_n$*, for all $n \in \mathbb{N}_{\geq 1}$.*

As a key step towards the theorem above, I will also give a detailed proof of the following, claimed in [Sim21], without proof:

Theorem U (Theorem 6.3.1)**.** *Let T be a complete first-order $\mathcal{L}$-theory and $\mathbb{M} \vDash T$ a monster model of T. Then, the following are equivalent for all $n \in \mathbb{N}_{\geq 1}$.*

(1) T is NIP$_n$*.*

(2) For any $A \subseteq \mathbb{M}$, all $\mathcal{L}(A)$-formulas $\phi(x, y_1, \ldots, y_n)$ where $|x| = |y_1| = \cdots = |y_n| = 1$ are NIP$_n$*.*

Also, a combinatorial construction of the 3-partite 3-uniform hypergraph inside a monadic expansion of the random graph will be given (Corollary 6.4.4).

For the remainder of this part, unless otherwise stated, T will be a fixed complete first-order theory and everything will be taking place inside a monster model $\mathbb{M} \vDash T$.

6.2 Local Preliminaries for Part 1

I've already discussed in Chapter 4 how (ordered) random hypergraphs relate to higher arity generalisations of NIP. To keep this chapter somewhat more self-contained, I shall recall some facts mentioned in Chapter 4, but perhaps in a bit more detail.

[2]Recall from Definition 4.2.16 that a theory T is monadically NIP if for every $\mathcal{M} \vDash T$ and every language expansion $\mathcal{M}'$ of $\mathcal{M}$ by unary relation symbols we have that $\mathcal{M}'$ is NIP. Analogously, T is *monadically* NIP$_n$ if in the previous sentence we replace "NIP" by "NIP$_n$". In light of Theorem T, monadic NIP$_n$ does not really deserve a proper definition.

In Definition 4.3.8, I defined what it means for an $\mathcal{L}$-structure $\mathcal{M}$ to *encode a* $\mathbb{K}$-*configuration*. It will be relevant, for the purposes of this part of the thesis, to "localise" this definition, and think about the formulas $I(R)$, from Definition 4.3.8 – that is, the formulas which "encode" the relations R of structures in $\mathbb{K}$. With this in mind, a formula $\phi(x_1, \ldots, x_n)$ *encodes* an n-partite n-uniform hypergraph $H = (V, R, P_1, \ldots, P_n)$ (where $P_1, \ldots, P_n$ are predicates for the parts of H) if there is a H-indexed set $(a_v : v \in V)$ such that:

$$\models \phi(a_{v_1}, \ldots, a_{v_n}) \text{ if, and only if, } \{v_1, \ldots, v_n\} \in R,$$

for all $v_i \in P_i$.

Recall from Section 2.7 that $\mathcal{H}_n$ is the random n-uniform hypergraph. I will write $\mathcal{H}_n^{\mathsf{p}}$ for the random n-partite n-uniform hypergraph, and $\mathcal{OH}_n^{\mathsf{p}}$ for the ordered random n-partite n-uniform hypergraph. I didn't explicitly define these structures in Section 2.7, but they are the natural objects their names suggest. Throughout this section $\mathcal{L}_{n,<}^{\mathsf{p}}$ will denote a language with n unary relation symbols $P_1, \ldots, P_n$ and a binary relation symbol $<$.

The first result I will need from [CPT19] is the following:

Fact 6.2.1 ([CPT19, Proposition 5.2])**.** *Let $\phi(x, y_1, \ldots, y_n)$ be an $\mathcal{L}$-formula. Then, the following are equivalent:*

(1) ϕ has IP_n.

(2) ϕ encodes every $(n+1)$-partite $(n+1)$-uniform hypergraph.

(3) ϕ encodes $\mathcal{H}_{n+1}^{\mathsf{p}}$ as a partite hypergraph.

(4) ϕ encodes $\mathcal{H}_{n+1}^{\mathsf{p}}$ as a partite hypergraph, by a $\mathcal{H}_{n+1}^{\mathsf{p}}$-indiscernible sequence.

The other result from [CPT19] that will be useful in the sequel is the following:

Fact 6.2.2 ([CPT19, Lemma 6.2])**.** *Let $\phi(x, y_1, \ldots, y_n)$ be a formula. Then, the following are equivalent:*

(1) ϕ has IP_n.

(2) There are b and $(a_g)_{g \in \mathcal{OH}_n^{\mathsf{p}}}$ such that:

(a) The sequence $(a_g)_{g \in \mathcal{OH}_n^{\mathsf{p}}}$ is $\mathcal{L}_{n,<}^{\mathsf{p}}$-indexed indiscernible, that is, when viewed as a sequence indexed by the language reduct of $\mathcal{OH}_n^{\mathsf{p}}$, where we "forget" the hyperedge relation, the sequence is a generalised indiscernible.

(b) *$\phi(b, y_0, \ldots, y_{n-1})$ encodes $\mathcal{OH}_n^{\mathsf{p}}$ on the a_g, respecting the partition in their indexing. That is:*

$$\models \phi(b, a_{g_{i_0}}, \ldots, a_{g_{i_{n-1}}}) \text{ if, and only if, } \mathcal{OH}_n^{\mathsf{p}} \models R(g_{i_0}, \ldots, g_{i_{n-1}}),$$

for all $g_i \in P_i$.

Finally, I will need to recall the following, from [CH21]:

Fact 6.2.3 ([CH21, Theorem 2.12(2)])**.** *The first order theory T has NIP$_n$ if, and only if, all $\mathcal{L}$-formulas $\phi(\bar{x}, y_1, \ldots, y_n)$ where $|y_1| = \cdots = |y_n| = 1$ are NIP$_n$.*

A slightly weaker version of this fact, where checking for IP$_n$ is reduced to checking the condition for formulas $\phi(x; \bar{y}_1, \ldots, \bar{y}_n)$ where $|x| = 1$ was implicit in Shelah's work, but given a detailed proof in [CPT19, Theorem 6.4].

6.3 Reducing the Number of Variables

The main theorem of this section is mentioned in [Sim21, Remark 4], but given there without proof. Recently,[3] Itay Kaplan asked if a formal proof has been given, and, to the best of my knowledge, the details have not been filled out before.

The proof I give below is essentially an adaptation of the argument of Simon, which mainly involves replacing the use of [Sim21, Lemma 1], used there for the NIP case, with Fact 6.2.1, which is needed for the argument to work in the general case. Moreover, a part of the argument of the NIP case, in [Sim21] uses a characterisation of the independence property in terms of picking out a dense co-dense subset of $\mathbb{Q}$ via an indiscernible sequence, a result which is not true in this form for higher arity independence properties. To patch this up, the proof below uses the natural analogue of this, which was stated in Fact 6.2.2 (i.e. [CPT19, Lemma 6.2]). With all that being said, let's get to the proof.

Theorem 6.3.1. *Let T be a first-order $\mathcal{L}$-theory and $\mathbb{M} \models T$ a monster model of T. For a positive integer $n \in \mathbb{N}_{\geq 1}$, the following are equivalent:*

(1) T is NIP$_n$.

(2) For any $A \subseteq \mathbb{M}$, all $\mathcal{L}(A)$-formulas $\phi(x, y_1, \ldots, y_n)$ where $|x| = |y_1| = \cdots = |y_n| = 1$ are NIP$_n$.

[3] In person, during the Final GeoMod Conference.

Proof. The direction (1) $\implies$ (2) is true by definition, so we only need to show that (2) $\implies$ (1). The proof is by (strong) induction on the length of the object variable, $\bar{x}$, and we argue essentially by minimal counterexamples.

Fix a positive integer $n \in \mathbb{N}_{\geq 1}$ and assume that T has IP_n, but for some $k \in \mathbb{N}_{>0}$ all formulas $\phi(\bar{x}, y_1, \dots, y_n)$ with $|\bar{x}| \leq k$ and $|y_1| = \cdots = |y_n| = 1$ are NIP_n. Note that, from Fact 6.2.3, we know that it suffices to check such formulas only. If we can show, given a formula (possibly over some parameters) $\psi(\bar{z}, w_1, \dots, w_n)$ with IP_n that there is a formula $\psi'(\bar{z}', w_1, \dots, w_n)$ (again, possibly over parameters, which need not be the same as the parameters in ψ) with $|\bar{z}'| < |\bar{z}|$ which also has IP_n, then, by induction, the result follows.

To this end, suppose that $\psi(\bar{z}, w_1, \dots, w_n)$ has IP_n, with $|\bar{z}| = m$. To simplify notation, assume that ψ is over $\emptyset$ (i.e. that we have added the parameters of ψ to the language). Then, by Fact 6.2.1 we have that ψ encodes $\mathcal{H}^{\mathsf{p}}_{n+1}$ as an $(n+1)$-partite hypergraph, by a $\mathcal{H}^{\mathsf{p}}_{n+1}$-indiscernible sequence, i.e. we can find a sequence $(\bar{b}_i, a_{1,i}, \dots, a_{n,i})_{i \in \mathbb{Q}}$ such that:

$$\vDash \psi(\bar{b}_i, a_{1,i_1}, \dots, a_{n,i_n}) \text{ if, and only if, } \mathcal{H}^{\mathsf{p}}_{n+1} \vDash R(i, i_1, \dots, i_n)$$

and such that $(\bar{b}_i, a_{1,i}, \dots, a_{n,i})_{i \in \mathbb{Q}}$ is $\mathcal{H}^{\mathsf{p}}_{n+1}$-indexed indiscernible. By the standard lemma, we may assume that this sequence is, in addition, order-indiscernible.

We split this sequence up into $(\bar{b}_i)_{i \in \mathbb{Q}}$ and $(a_{1,i}, \dots, a_{n,i})_{i \in \mathbb{Q}}$ and for each $i \in \mathbb{Q}$ we write b_i for the first element of the tuple $\bar{b}_i$, that is, $b_i := b_i^0$. We split the argument into two cases:

1. *Case 1.* The sequences $(b_i)_{i \in \mathbb{Q}}$ and $(a_{1,i}, \dots, a_{n,i})_{i \in \mathbb{Q}}$ are mutually indiscernible.[4]
2. *Case 2.* The sequences $(b_i)_{i \in \mathbb{Q}}$ and $(a_{1,i}, \dots, a_{n,i})_{i \in \mathbb{Q}}$ are not mutually indiscernible.

We will show that in each of the two cases above we can construct a formula $\psi'(\bar{z}', w_1, \dots, w_n)$ with $|\bar{z}'| < |\bar{z}|$ which also has IP_n.

Case 1. If the sequences $(b_i)_{i \in \mathbb{Q}}$ and $(a_{1,i}, \dots, a_{n,i})_{i \in \mathbb{Q}}$ are mutually indiscernible, then clearly the sequence $(a_{1,i}, \dots, a_{n,i})_{i \in \mathbb{Q}}$ is indiscernible over $\{b_0\}$. In this case, using the extension property of $\mathcal{H}^{\mathsf{p}}_{n+1}$ and indiscernibility it is not hard to

[4] By which I mean that each sequence is indiscernible over the other.

show that the $\mathcal{L}(b_0)$-formula $\psi'(\bar{z}', w_1, \ldots, w_n)$ given by fixing b_0 in ψ, that is:

$$\psi'(z_2, \ldots, z_m; w_1, \ldots, w_n) := \psi(b_0, z_2, \ldots, z_m; w_1, \ldots, w_n)$$

has IP_n.

Indeed, observe that the sequence $(a_{1,i}, \ldots, a_{i,n})_{i \in \mathbb{Q}}$ is, in fact, an $\mathcal{L}^{\mathsf{p}}_{n,<}$-indexed indiscernible (in the sense described in Fact 6.2.2) and by the extension property of the random $(n+1)$-ary ordered hypergraph the set

$$\{(i_1, \ldots, i_n) : \psi(\bar{b}_0, a_{1,i_1}, \ldots, a_{n,i_n})\}$$

is, essentially, the hyperedge relation of the (ordered) n-partite n-uniform hypergraph on $(a_{1,i}, \ldots, a_{i,n})_{i \in \mathbb{Q}}$. In particular, by Fact 6.2.2, it follows that $\psi'(\bar{z}'; w_1, \ldots, w_n)$, as defined above, has IP_n.

Case 2. If the sequences $(b_i)_{i \in \mathbb{Q}}$ and $(a_{0,i}, \ldots, a_{n-1,i})_{i \in \mathbb{Q}}$ are not mutually indiscernible, then, by definition this means that there must be some $m \in \omega$ and sequences $i_1 < \cdots < i_m$, $j_1 < \cdots < j_m$ and $i'_1 < \cdots < i'_m$, $j'_1 < \cdots < j'_m$ such that:

$$\mathsf{tp}(b_{i_1}, \ldots, b_{i_m}, \bar{a}_{j_1}, \ldots, \bar{a}_{j_m}) \neq \mathsf{tp}(b_{i'_1}, \ldots, b_{i'_m}, \bar{a}_{j'_1}, \ldots, \bar{a}_{j'_m}).$$

Since the sequence we started with was $\mathcal{H}^{\mathsf{p}}_{n+1}$-indiscernible, it follows that $(b_i, a_{1,i}, \ldots, a_{n,i})_{i \in \mathbb{Q}}$ is also an $\mathcal{H}^{\mathsf{p}}_{n+1}$-indiscernible sequence (it is simply a subtuple of the original sequence). This means that the indices above define two different finite $(n+1)$-uniform $(n+1)$-partite hypergraphs with m vertices in each part of the partition. To fix notation, let $H_\star = (V_\star, R_\star)$ where $V_\star = \{b_{i_1}, \ldots, b_{i_m}\} \cup \{\bar{a}_{j_1}, \ldots, \bar{a}_{j_m}\}$, and $R_\star$ is the induced hyperedge relation from the coding of $\mathcal{H}^{\mathsf{p}}_{n+1}$, and $H'_\star$ is defined analogously.

Since these two hypergraphs are different, they must differ in at least one hyperedge, say $\{k_1, k_2, \ldots, k_{n+1}\} \in R_\star$, but $\{k_1, k_2, \ldots, k_{n+1}\} \notin R'_\star$.

First, we consider $V_\star$ with the vertices which lie in the hyperedge distinguishing $H_\star$ and $H'_\star$ removed, that is:

$$U = \{b_{i_j} \in V_\star : i_j \neq k_1\} \cup \{\bar{a}_{i_j} \in V_\star : i_j \notin \{k_2, \ldots, k_{n+1}\}\}$$

Now, we define:

$$\begin{aligned} V_0 = \{b_j : j \in (k_1 - 1, < k_1 + 1), b_j \equiv_U b_{k_1}\} \\ \times \{a_{1,j} : j \in (k_2 - 1, k_2 + 1), a_{1,j} \equiv_U a_{1,k_2}\} \\ \times \dots \\ \times \{a_{n,j} : j \in (k_{n+1}, k_{n+1} + 1), a_{n,j} \equiv_U a_{n,k_{n+1}}\}. \end{aligned}$$

We can view V_0 as the vertex set of an infinite $(n+1)$-partite $(n+1)$-uniform hypergraph with hyperedge relation given by that of the $(n+1)$-partite $(n+1)$-uniform random hypergraph coded by (the indexing set of) $(\bar{b}_i, a_{1,i}, \dots, a_{n,i})_{i \in \mathbb{Q}}$ restricted to this subset. This is, of course, again a copy of the $(n+1)$-partite $(n+1)$-uniform random hypergraph.

By construction, for any tuple $(b_j, a_{1,l_1}, \dots, a_{n,l_n}) \in V_0$ we have that

$$\mathsf{tp}(b_j, a_{1,l_1}, \dots, a_{n,l_n}/U)$$

depends only on whether $(j, l_1, \dots, l_n)$ is a hyperedge in $\mathcal{H}^{\mathsf{p}}_{n+1}$, again since the sequence we started with was $\mathcal{H}^{\mathsf{p}}_{n+1}$-indiscernible. Let $\bar{e} := (b_j, a_{1,l_1}, \dots, a_{n,l_n}) \in V_0$ be such that $\{j, l_1, \dots l_n\}$ is a hyperedge and write $p(z, \bar{w})$ for $\mathsf{tp}(\bar{e}/U)$ and $\bar{e}' := (b_{j'}, a_{1,l'_1}, \dots, a_{n,l'_n}) \in V_0$ be such that $\{j', l'_1, \dots l'_n\}$ is not a hyperedge and write $q(z, \bar{w})$ for $\mathsf{tp}(\bar{e}'/U)$.

In this notation, since $p \neq q$, we can find a formula $\psi'(z, w_1, \dots, w_n) \in p(z, \bar{w})$ which does not belong to $q(z, \bar{w})$ and hence, for all $(j, l_1, \dots, l_n) \in \prod_{i=1}^{n}(k_i - 1, k_i + 1)$ we have that:

$$\models \psi'(b_j, a_{1,l_1}, \dots, a_{n,l_n}) \text{ if, and only if } R(j, l_1, \dots, l_n),$$

that is, ψ' encodes the hyperedge of a random $(n+1)$-partite $(n+1)$-uniform graph on V_0, since p was precisely the type of $(n+1)$-tuples which formed a hyperedge. This concludes the proof. □

6.4 Collapsing the Monadic NIP$_k$ Hierarchy

Let's take a closer look at the NIP$_k$ hierarchy. It's easy to see (Proposition 4.2.14):

$$\text{NIP} = \text{NIP}_1 \implies \text{NIP}_2 \implies \cdots \implies \text{NIP}_k \implies \text{NIP}_{k+1} \implies \dots$$

As I said near Proposition 4.2.14, that proof was only given so that it is clear how in Corollary 6.4.4 one may obtain, definably but with monadic parameters, a copy of the 3-partite 3-uniform hypergraph inside the random graph.

Moreover, as already discussed in Chapter 4 all of these implications are strict, at each point the random $(k+1)$-hypergraph witnesses this strictness, as it is NIP_{k+1} but has the k-independence property. This follows immediately from the fact that the random $(k+1)$-hypergraph has quantifier elimination and the following fact:

Fact 6.4.1 ([CPT19, Proposition 6.5])**.** *If T has elimination of quantifiers, then in order to check that T is NIP$_n$ it suffices to check that every atomic formula $\phi(x, \bar{y}_1, \dots, \bar{y}_n)$ with $|x| = 1$ is NIP$_n$.*

If one considers monadic expansions, this is no longer the case. The first strict implication above says precisely that the random graph has the Independence Property, i.e. it is not NIP_1, but is NIP_2. In the next lemma I will present a proof that the random graph is not monadically NIP_2. Indeed:

Lemma 6.4.2. *There exists a monadic expansion of the random graph which has the 2-Independence Property.*

Proof. Let $\mathcal{G}^+ = (V; E, P_0, P_1, Q)$ be an expansion of the random graph by three unary predicates, P_0, P_1, and Q. Pick P_0 and P_1 to name two disjoint and infinite subsets of V. To be more concrete, although this is not strictly necessary, we can assume that for $i \in \{0, 1\}$ we have that each P_i is countable, with fixed enumeration $(v_{j,i})_{j\in\omega}$. Moreover we can choose P_0 and P_1 to be independent sets with no edges between them. We will describe how Q is chosen later on in the proof.

Let $G_k = (V_{k,0} \sqcup V_{k,1}, E_k)$, for $k \in \mathcal{K}$, be a fixed enumeration of all finite bipartite graphs with no isolated vertices whose vertex set is a finite subset of $(\omega \times \{0\}) \sqcup (\omega \times \{1\})$. Note that this is not taken up to isomorphism. Observe that we have a one-to-one correspondence between such bipartite graphs and the finite subsets of $\omega \times \omega$. Indeed, each graph G_k corresponds to the subset $I_k \subseteq \omega \times \omega$, where:

$$(a, b) \in I_k \text{ if, and only if, } (a, 0) \in V_{k,0}, (b, 1) \in V_{k,1} \text{ and } \{(a, 0), (b, 1)\} \in E_k.$$

Our goal is to construct the following configuration in $\mathcal{G}^+$:

For each $k \in \mathcal{K}$ we want to choose a "fresh" clique K_{n_k} (i.e. different from all the cliques we have already chosen) where $n_k = |E_k| + 1$, none of whose vertices are in $P_0 \sqcup P_1$ and such that:

- For each $(a, b) \in I_k$ there is a unique vertex of K_{n_k} whose only neighbours in $P_0 \sqcup P_1$ are $v_{a,0}$ and $v_{b,1}$ (recall that $(v_{i,0})_{i\in\omega}$ and $(v_{i,1})_{i\in\omega}$ are our fixed enumerations of P_0 and P_1, respectively); and
- There is a vertex v_{I_k} in the clique which has no neighbours in $P_0 \sqcup P_1$, and is not connected to any of the previously chosen vertices.

Once we have chosen K_{n_k}, we add it to the predicate Q. At the end, we will have a collection of distinct cliques K_I, for each finite $I \subseteq \omega \times \omega$, such that each clique "codes" a bipartite graph on (a subset of) $P_0 \sqcup P_1$, and has an extra vertex v_I, which will be the "representative" of the clique. For example, the construction for the subset $I = \{(0, 2), (1, 2), (3, 1)\}$ would be:

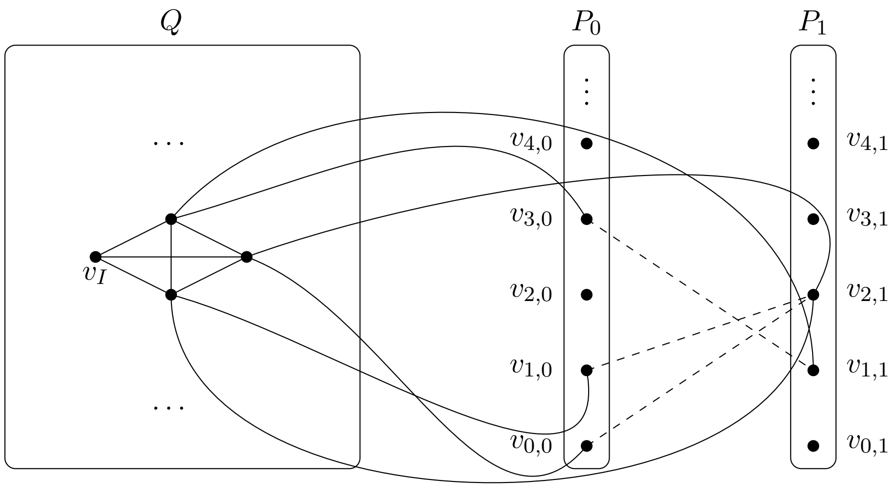

Figure 6.1: Coding the subset $I = \{(0, 2), (1, 2), (3, 1)\}$.

Now, assuming that we can construct the configuration described previously, we claim that $\mathcal{G}^+$ has IP$_2$. Indeed, let $\phi(x, y_0, y_1)$ be the following formula:

$$(P_0(y_0) \wedge P_1(y_1)) \wedge (\exists z)\,(Q(z) \wedge E(x, z) \wedge E(z, y_0) \wedge E(z, y_1))$$

Then, for each finite $I \in \mathcal{P}(\omega \times \omega)$, and any $(a, b) \in \omega \times \omega$ we then have that:

$$\mathcal{G}^+ \vDash \phi(v_I, v_{a,0}, v_{b,1}) \text{ if, and only if, } (a, b) \in I.$$

We need the two monadic predicates here to be able to code the existence of two coordinates – the same configuration could be defined in the normal random graph $\mathcal{G}$. In that case, we would have that the formula $\phi'(x, y_0, y_1)$, given by removing the unary predicates from $\phi(x, y_0, y_1)$, is such that $\mathcal{G} \models \phi(v_I, v_{a,0}, v_{b,0})$ if $(a, b) \in I$, but not the other way round.

Of course, if we can construct the configuration described above, then it follows, by compactness, that $\mathcal{G}^+$ is not monadically NIP$_2$.

Formally, we construct this configuration in some elementary extension of $\mathcal{G}$, by the extension property of the random graph and compactness. As above, let $G_k = (V_{k,0} \sqcup V_{k,1}, E_k)$ be our enumeration of all finite bipartite graphs with no isolated vertices with vertex a finite subset of $(\omega \times \{0\}) \sqcup (\omega \times \{1\})$.

We construct the cliques K_{n_k}, as described above for each $k \in \mathcal{K}$, by induction, as follows:

- *Step 1.* Given $k \in \mathcal{K}$, let:

$$\{\{a_{0,k}, b_{0,k}\}, \ldots, \{a_{e_k,k}, b_{e_k,k}\}\},$$

be an enumeration of the edges of G_k, where $e_k = |E_k|$. Then, consider the set of formulas $\Sigma_k\,(x_0, \ldots, x_{e_k})$ given by:

$$\left\{\bigwedge_{i<e_k}\left(\bigwedge_{k<e_k} E(x_i, v_{a_{0,k},0}) \wedge E(x_i, v_{b_{0,k},1})\right)^{(\text{if } i=k)}\right\} \cup \left\{\bigwedge_{0\leq i\neq j\leq e_k} E(x_i, x_j)\right\}$$
$$\cup \Bigg\{\left(\bigwedge_{i<e_k} (\exists! z_0)(P_0(z_0) \wedge E(x_i, z_0)) \wedge (\exists! z_1)(P_1(z_1) \wedge E(x_i, z_1))\right)$$
$$\wedge\, (\forall v)\,(P_0(v) \vee P_1(v) \to \neg E(x_{e_k}, v))\Bigg\}$$
$$\cup \left\{\left(\bigwedge_{i<e_k} (v_n \neq x_j)\right) \wedge Q(v_n) \to \neg E(v_n, x) : n \in \omega\right\},$$

where $(v_n : n \in \omega)$ is an enumeration of the vertex set of $\mathcal{G}$. Of course:

  - The first line in the definition of Σ_k says that each edge of G_k is coded by a vertex in $\{x_0, \ldots, x_{e_k-1}\}$ and that the vertices $\{x_0, \ldots, x_{e_k}\}$ form a clique.
  - The second and third lines say that each but one of the x_i is

connected to exactly one vertex in P_0 and exactly one vertex in P_1, and that x_{e_k} is connected to no vertices in $P_0 \sqcup P_1$.

– Finally, the fourth line says any vertex of $\mathcal{G}$, in particular, any vertex in $P_0 \sqcup P_1$, other than $\{x_0, \ldots, x_{e_k-1}\}$ connected to x_{e_k} is not in Q.

Clearly, the set $\Sigma_k(x_0, \ldots, x_{e_k})$ is finitely satisfiable, by the theory of the random graph. Hence by compactness there is an elementary extension $\mathcal{G}'$ of $\mathcal{G}$, in which we can find some realisations for this set of formulas, but of course this means that we can find a countable elementary extension $\mathcal{G}'$ in which $\Sigma_k(x_0, \ldots, x_{e_k})$ has a realisation, and thus, by ω-categoricity of $\mathcal{G}$, we can find a realisation in $\mathcal{G}$.

- *Step 2*. Once we have found the vertices of K_{n_k} in $\mathcal{G}$, we add all of them to the predicate Q and continue, by repeating the argument in Steps 1 and 2 for the next $k \in \mathcal{K}$.

The result then follows. □

Remark 6.4.3. We can carry out a similar argument to show that the random graph is not monadically NIP$_k$ for any $k > 1$. In this case we add new predicates $P_0, \ldots, P_{k-1}$ and Q and repeat the argument. For the higher-arity argument, we again use cliques in Q to "code" graphs on the subsets of $\prod_{i<k} \omega \times \{i\}$, where the i-th coordinate is coded by the predicate P_i.

Combining Lemma 6.4.2 with Fact 6.2.1 the following interesting corollary is immediate:

Corollary 6.4.4. *The* 3*-partite* 3*-uniform hypergraph is definable in a monadic expansion of the random graph.*

In particular:

Theorem 6.4.5. *Let T be a theory which has the k-independence property for some $k \geq 1$. Then some monadic expansion of T has the $(k+1)$-independence property.*

Proof. From Theorem 6.3.1 we can find, in a monadic expansion of T a formula $\phi(x, y_0, \ldots, y_{k-1})$ which has the k-independence property and such that x and $y_0, \ldots, y_{k-1}$ are singletons, rather than using parameters. We use monadic predicates which name singleton sets. Then, from Fact 6.2.1 it follows that $\phi(x, y_0, \ldots, y_{k-1})$ encodes $\mathcal{H}^{\mathsf{p}}_{k+1}$, as a partite hypergraph. We now work inside the $\mathcal{H}^{\mathsf{p}}_{k+1}$ encoded by ϕ. Since this has the k-Independence Property, from

Proposition 4.2.14 it follows that it has the Independence Property. Hence, using [BS85, Lemma 8.1.3] in some monadic expansion, we can find a formula $\psi(x, y)$ with x, y both singletons which has the independence property. Again using Fact 6.2.1 this means that $\psi(x, y)$ encodes the random bipartite graph $\mathcal{G}^{\mathsf{p}}$ (the random bipartite graph). Once more, we work inside the random bipartite graph $\mathcal{G}^{\mathsf{p}}$ encoded by $\psi(x, y)$. Our goal is to find a monadic expansion of this $\mathcal{G}^{\mathsf{p}}$ and a formula in this expansion which encodes the random graph. To fix notation, write $\mathcal{G}^{\mathsf{p}} = (V_0 \sqcup V_1, E)$.

By the extension property of $\mathcal{G}^{\mathsf{p}}$, using a similar argument as the one in Step 2 of Lemma 6.4.2, we can find, inside $\mathcal{G}^{\mathsf{p}}$ an infinite subset of vertices of V_1 such that any two of them belong to exactly one path of length 2 (i.e. a graph of the form $P_2 = (\{v_1, v_2, v_3\}, \{\{v_1, v_2\}, \{v_2, v_3\}\}))$ and no two of these paths pass through the same vertex of V_2. Let P be a unary predicate naming this subset of V_1 and let Q be a unary predicate naming the vertices in V_2 adjacent to vertices in P. The upshot is that for each pair of vertices $u, v \in P$ there is a unique vertex $w_{\{u,v\}}$ in Q such that u, v and $w_{u,v}$ form a path of length 2. Independently for each pair of vertices $u, v \in P$ remove $w_{\{u,v\}}$ from Q with some fixed positive probability, say $\frac{1}{2}$. Then, the formula $\chi(x, y)$ given by:

$$(\exists w)(Q(w) \wedge E(x, w) \wedge E(w, y))$$

defines, with probability 1 the edge relation of a random graph on P, hence the graph given by $(P, \chi(x, y))$ is a random graph inside $\mathcal{G}_{2,p}$. Using Remark 6.4.3 this is not monadically NIP_l for any $l \in \mathbb{N}$, so in particular it is not monadically NIP_{k+1}. Hence T has the $(k+1)$-independence property, as claimed. □

Combining the theorem above with Proposition 4.2.14, it follows that in monadic expansions the NIP_k hierarchy collapses, more precisely:

Corollary 6.4.6. *A theory T is monadically NIP_k for some $k \in \mathbb{N}$ if, and only if, it is monadically NIP_l, for all $l \in \mathbb{N}$.*

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– End of Part 1 –

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Part 2: Murphy's Law for 2-Nilpotent Groups[†]

6.5 Introduction to Part 2

An important aspect of Shelah's classification theory is, very roughly speaking, based on the idea that the presence/absence of simple combinatorial data (e.g. linear orders, random graphs, and more) can give us a lot of information on the "complexity" of a given theory (e.g. stability, NIP, and more).

The dichotomies given by the presence/absence of combinatorial configurations allow model theorists to divide (hence the name *"dividing lines"*) the class of all first-order theories into smaller regions, which can then be studied in more detail. Since the various regions of the model-theoretic universe are defined using combinatorial data, it is often the case that building combinatorial examples inhabiting each region is a much easier task than building *purely* algebraic ones.

Mekler's construction is now a classical technique for building a purely algebraic structure (a 2-nilpotent group of exponent p, for a fixed odd prime $p \in \mathbb{N}$) from a purely combinatorial one (a "nice" graph).[5]

The goal of this little section is to push this idea to its logical extreme, generalising the aforementioned results, and showing that Mekler's construction really does preserve the presence/absence of *all* interesting combinatorial data. So, in a sense, *Murphy's law* does indeed hold for the class of 2-nilpotent groups of exponent p – at least from the point of view of dividing lines that are characterised through generalised indiscernibles.

The main result is the following:

Theorem V. *Let $\mathcal{I}$ be an $\aleph_0$-categorial Fraïssé limit of a Ramsey class in a finite relational language, and $\mathcal{J}$ a proper reduct of $\mathcal{I}$, also in a finite relational language. For every nice graph C:*

[†]The main result in this section is part of larger joint work with Blaise Boissonneau and Pierre Touchard [BPT24].

[5]All the material around Mekler's construction is now standard, and a classical reference is [Hod93, Appendix A.3], all notation used in this section that is not introduced will be generally consistent with the notation from Hodges, and to avoid including a rather long preliminary section, which contains precisely the material from that book (and definitely does not live up to the standards of Hodges's exposition), I direct the reader there for the relevant material.

If:

- *$\mathcal{I}$ has specific collapsing (see Definition 6.6.2) to $\mathcal{J}$*
- *$\mathsf{Th}(\mathsf{C})$ collapses $\mathcal{I}$-indiscernibles to $\mathcal{J}$-indiscernibles*

Then:

- *$\mathsf{Th}(\mathsf{M}(\mathsf{C}))$ collapses $\mathcal{I}$-indiscernibles to $\mathcal{J}$-indiscernibles.*

Combining this with [ACT23, Lemma 4.14, Theorem 4.15], one can construct the first known examples of NFOP_k pure groups, for all $k \in \mathbb{N}_{>2}$, confirming an expectation of Abd Aldaim, Conant, and Terry [ACT23, Remark 2.13]. This is handled with more care in [BPT24].

6.6 Local Preliminaries for Part 2

Before reading this section, I suggest the reader takes a brief look at Appendix A to remind themself of the definitions and familiarise themself with the notation. Through this section, C will be a nice graph, and $\mathsf{M}(\mathsf{C})$ its Mekler group.

Recall the following key lemma from [CH19]:

Fact 6.6.1 ([CH19, Lemma 2.14])**.** *Let $\mathcal{M}$ a saturated model of $\mathsf{Th}(\mathsf{M}(\mathsf{C}))$. Let X be a transversal and $K_X \leq \mathsf{Z}(\mathcal{M})$ be such that $\mathcal{M} = \langle X \rangle \times K_X$. Let Y and Z be two small subsets of X and $\bar{h}_1, \bar{h}_2$ two tuples in K_X.*

If:

- *There is a bijection $f : Y \to Z$ which respects the 1^ν-, p-, 1^ι-parts , the handles, and $\mathsf{tp}_\Gamma(Y^\nu) = \mathsf{tp}_\Gamma(f(Y)^\nu)$.*
- *$\mathsf{tp}_{K_X}(\bar{h}_1) = \mathsf{tp}_{K_X}(\bar{h}_2)$.*

Then:

- *There is an automorphism $\sigma \in \mathsf{Aut}(\mathcal{M})$ extending f such that $\sigma(\bar{h}_1) = \bar{h}_2$.*

In [CH19], amongst other results, the authors essentially show the following ([CH19, Theorem 4.7]): If, for some nice graph $\mathcal{G}$, every model of $\mathsf{Th}(\mathcal{G})$ collapses H_k-indiscernibles (where H_k is the random k-partite hypergraph) to P_k-indiscernibles (where P_k is the random k-partite set), then so must every model of $\mathsf{Th}(\mathsf{M}(\mathcal{G}))$. Below, their argument is adapted for the general case of collapsing $\mathcal{I}$-indiscernibles (in place of H_k-indiscernibles) to $\mathcal{J}$-indiscernibles (in place of P_k-indiscernibles).

Definition 6.6.2 (Specific collapsing)**.** Let $\mathcal{I}$ be an $\aleph_0$-categorical Fraïssé limit of a Ramsey class and $\mathcal{J}$ a proper reduct of $\mathcal{I}$. We say that $\mathcal{I}$ *has specific collapsing to* $\mathcal{J}$ if for all theories T, and all $\mathcal{M} \vDash T$, every collapsing $\mathcal{I}$-indiscernible sequence in $\mathcal{M}$, is (a possibly collapsing) $\mathcal{J}$-indiscernible sequence.

Let's start with two easy lemmas:

Lemma 6.6.3. *Let T be a stable theory, $\mathcal{I}$ an $\aleph_0$-categorial Fraïssé limit of a Ramsey class in a finite relational language, and $\mathcal{J}$ a proper reduct of $\mathcal{I}$. If $\mathcal{I}$ has specific collapsing to $\mathcal{J}$ then every model of T collapses $\mathcal{I}$-indiscernibles to $\mathcal{J}$-indiscernibles.*

Proof. Since $\mathcal{I}$ is Ramsey, it expands a linear order, and since T is stable, we must have that, in models of T, $\mathcal{I}$-indiscernibles collapse to the reduct of $\mathcal{I}$ where we forget the order. By specific collapsing, this means that in models of T, $\mathcal{I}$-indiscernibles collapse to $\mathcal{J}$-indiscernibles. □

Lemma 6.6.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be structures such that $\mathcal{N}$ is interpretable in $\mathcal{M}$. Let $\phi(\bar{x})$ define the domain of $\mathcal{N}$ in $\mathcal{M}^{\mathsf{eq}}$ (modulo the definable equivalence relation $\sim$). Let $\mathcal{I}$ be an indexing structure and $(\bar{a}_i : i \in \mathsf{dom}(\mathcal{I}))$ an $\mathcal{I}$-indiscernible sequence such that $\mathcal{M} \vDash \phi(\bar{a}_i)$ for all $i \in \mathcal{I}$. Then $([\bar{a}_i]_\sim : i \in \mathsf{dom}(I))$ is an $\mathcal{I}$-indiscernible sequence in $\mathcal{N}$.*

Proof. Let $i_1, \dots, i_n, j_1, \dots, j_n \in \mathcal{I}$ be such that:

$$\mathsf{qftp}_{\mathcal{I}}(i_1, \dots, i_n) = \mathsf{qftp}_{\mathcal{I}}(j_1, \dots, j_n).$$

Since the sequence $(\bar{a}_i : i \in \mathsf{dom}(\mathcal{I}))$ is $\mathcal{I}$-indiscernible, it follows that

$$\mathsf{tp}(\bar{a}_{i_1}, \dots, \bar{a}_{i_n}) = \mathsf{tp}(\bar{a}_{j_1}, \dots, \bar{a}_{j_n})$$

and hence, in $\mathcal{M}^{\mathsf{eq}}$, we have that $\mathsf{tp}([\bar{a}_{i_1}]_\sim, \dots, [\bar{a}_{i_n}]_\sim) = \mathsf{tp}([\bar{a}_{j_1}]_\sim, \dots, [\bar{a}_{j_n}]_\sim)$, and the result follows. □

6.7 The Main Theorem

Theorem 6.7.1. *Let $\mathcal{I}$ be an $\aleph_0$-categorial Fraïssé limit of a Ramsey class in a finite relational language, and $\mathcal{J}$ a proper reduct of $\mathcal{I}$, also homogeneous in a finite relational language. For every nice graph* C*:*

If:

- *$\mathcal{I}$ has specific collapsing to $\mathcal{J}$*
- $\mathsf{Th}(\mathsf{C})$ *collapses $\mathcal{I}$-indiscernibles to $\mathcal{J}$-indiscernibles*

Then:

- $\mathsf{Th}(\mathsf{M}(\mathsf{C}))$ *collapses $\mathcal{I}$-indiscernibles to $\mathcal{J}$-indiscernibles.*

Proof. Assume toward a contradiction that $\mathsf{Th}(\mathsf{C})$ collapses $\mathcal{I}$-indiscernibles to $\mathcal{J}$-indiscernibles but $\mathsf{Th}(\mathsf{M}(\mathsf{C}))$ does not. Since both $\mathcal{I}$ and $\mathcal{J}$ are finitely homogeneous, without loss of generality they are both k-ary, for some $k \in \mathbb{N}$. By appropriately padding the symbols in $\mathcal{L}_\mathcal{I}$ and $\mathcal{L}_\mathcal{J}$, the respective languages of $\mathcal{I}$ and $\mathcal{J}$, we may assume that they all have arity exactly k. Moreover, since $\mathcal{I}$ has quantifier elimination we can also assume that $\mathcal{L}_\mathcal{J} \subseteq \mathcal{L}_\mathcal{I}$.

By assumption, we can find a model $\mathcal{M} \vDash \mathsf{Th}(\mathsf{M}(\mathsf{C}))$ and an $\mathcal{I}$-indiscernible sequence $B = (b_i : i \in \mathsf{dom}(\mathcal{I}))$ in $\mathcal{M}$ which is not $\mathcal{J}$-indiscernible. Since $\mathcal{I}$ has specific collapsing to $\mathcal{J}$, this must mean that B is non-collapsing. In particular, for all $(i_1, \ldots, i_k), (j_1, \ldots, j_k) \in \mathsf{dom}(\mathcal{I})^k$ we have that

$$\mathsf{qftp}_\mathcal{I}(i_1, \ldots, i_k) = \mathsf{qftp}_\mathcal{I}(j_1, \ldots, j_k) \iff \mathsf{tp}(b_{i_1}, \ldots, b_{i_k}) = \mathsf{tp}(b_{j_1}, \ldots, b_{j_k}).$$

Claim 1. For every k-ary relation symbol $R \in \mathcal{L}_\mathcal{I}$ we can find an $\mathcal{L}_{\mathsf{grp}}$-formula ϕ_R such that for all $i_1, \ldots, i_k \in \mathcal{I}$ we have that:

$$\mathcal{M} \vDash \phi_R(b_{i_1}, \ldots, b_{i_k}) \text{ if, and only if, } \mathcal{I} \vDash R(i_1, \ldots, i_k).$$

Proof of Claim 1. First, since $\mathcal{I}$ is $\aleph_0$-categorical there are finitely many quantifier-free k-types in $\mathsf{Th}(\mathcal{I})$, say $p_1, \ldots, p_n$. For every k-ary relation symbol $R \in \mathcal{L}_\mathcal{I}$ there is a subset $\mathbf{P}_R \subseteq \{p_1, \ldots, p_n\}$ such that R belongs to every type in $\mathbf{P}$ and to no types in $\{p_1, \ldots, p_n\} \setminus \mathbf{P}$. For each $p \in \{p_1, \ldots, p_n\}$ let ψ_p be its (quantifier-free) isolating formula, and let ψ_R be the formula $\bigvee_{p \in \mathbf{P}_R} \psi_p \wedge \bigwedge_{q \notin \mathbf{P}_R} \neg \psi_q$. Then:

$$\mathcal{I} \vDash \psi_R(i_1, \ldots, i_k) \text{ if, and only if, } \mathcal{I} \vDash R(i_1, \ldots, i_k).$$

Now, since B is an uncollapsed $\mathcal{I}$-indiscernible, for each quantifier-free k-type p of $\mathcal{I}$ there is a unique k-type q of $\mathcal{M}$ in B, corresponding to the k-tuples from B indexed by k-tuples from $\mathcal{I}$ whose quantifier-free type is p. For each such type we can find an $\mathcal{L}_{\mathsf{grp}}$-formula ϕ_p separating it from the rest.

Given a relation symbol $R \in \mathcal{L}_{\mathcal{I}}$, which belongs to a (unique) Boolean combination of isolating formulas ψ_R, as discussed above, take ϕ_R to be the formula $\bigwedge \bigvee \phi_i^{\epsilon_i}$, i.e. the corresponding Boolean combination of $\mathcal{L}_{\mathsf{grp}}$-formulas ϕ_p. ◀

Fix κ to be $|T|^+$ and $\mathbb{I} \succcurlyeq \mathcal{I}$ be an elementary extension of $\mathcal{I}$ with $|\mathbb{I}| = \kappa$. Without loss of generality, we may assume that $\mathcal{M}$ is saturated. Since $\mathcal{I}$ is Ramsey, $\mathcal{I}$-indiscernibles have the modelling property (by the generalised standard lemma, Theorem 3.5.12), so by compactness/saturation we can actually find, in $\mathcal{M}$, an $\mathbb{I}$-indexed indiscernible sequence $A = (a_i : i \in \mathsf{dom}(\mathbb{I}))$ which is based on B. Let $\mathbb{J} \succcurlyeq \mathcal{J}$ be the elementary extension of $\mathcal{J}$ given by taking the $\mathcal{L}_{\mathcal{J}}$-reduct of $\mathbb{I}$.

Claim 2. The sequence $(a_i : i \in \mathsf{dom}(\mathbb{I}))$ is not $\mathbb{J}$-indiscernible.

Proof of Claim 2. Since $(a_i : i \in \mathsf{dom}(\mathcal{I}))$ is not $\mathcal{J}$-indiscernible, we can find $i_1, \ldots, i_k, j_1, \ldots, j_k \in \mathsf{dom}(\mathcal{I})$ such that $\mathsf{qftp}_{\mathcal{J}}(i_1, \ldots, i_k) = \mathsf{qftp}_{\mathcal{J}}(j_1, \ldots, j_k)$ and $\mathsf{tp}(b_{i_1}, \ldots, b_{i_k}) \neq \mathsf{tp}(b_{j_1}, \ldots, b_{j_k})$. Let $\Delta_1 \subseteq \mathsf{tp}(b_{i_1}, \ldots, b_{i_k})$, and $\Delta_2 \subseteq \mathsf{tp}(b_{j_1}, \ldots, b_{j_k})$ be two finite sets of $\mathcal{L}_{\mathsf{grp}}$-formulas such that $\Delta_1 \cup \Delta_2$ is inconsistent.

Since $(a_i : i \in \mathsf{dom}(\mathbb{I}))$ is based on $(b_i : i \in \mathsf{dom}(\mathcal{I}))$ and $\mathbb{I} \succcurlyeq \mathcal{I}$, given $l_1, \ldots, l_k \in \mathbb{I}$ such that $\mathsf{qftp}_{\mathbb{I}}(l_1, \ldots, l_k) = \mathsf{qftp}_{\mathcal{I}}(i_1, \ldots, i_k)$, we can find a tuple $i'_1 \ldots, i'_k \in \mathcal{I}$ such that $\mathsf{qftp}_{\mathbb{I}}(l_1, \ldots, l_k) = \mathsf{qftp}_{\mathcal{I}}(i'_1, \ldots, i'_k)$ and thus

$$\mathsf{tp}^{\Delta_1}(a_{l_1}, \ldots, a_{l_k}) = \mathsf{tp}^{\Delta_1}(b_{i'_1}, \ldots, b_{i'_k}) = \mathsf{tp}^{\Delta_1}(b_{i_1}, \ldots, b_{i_k}).$$

Similarly, we find $m_1, \ldots, m_k$ such that $\mathsf{qftp}_{\mathbb{I}}(m_1, \ldots, m_k) = \mathsf{qftp}_{\mathcal{I}}(j_1, \ldots, j_k)$ and

$$\mathsf{tp}^{\Delta_2}(a_{m_1}, \ldots, a_{m_k}) = \mathsf{tp}^{\Delta_2}(b_{j_1}, \ldots, b_{j_k}).$$

But, by construction $\mathsf{qftp}_{\mathbb{J}}(l_1, \ldots, l_k) = \mathsf{qftp}_{\mathbb{J}}(m_1, \ldots, m_k)$ and thus $(a_i : i \in \mathsf{dom}(\mathbb{I}))$ is not $\mathbb{J}$-indiscernible. ◀

Using essentially [CH19, Proposition 2.18], we may write $\mathcal{M}$ in the form $\langle X \rangle \times \langle H \rangle$, where X is a transversal and $H \subseteq \mathcal{M}$ a set which is linearly independent over $[\mathcal{M}, \mathcal{M}]$.

After fixing an enumeration of $\mathsf{dom}(\mathbb{I})$, and rearranging A to be in the form $(a_i : i < \kappa)$, we express the elements in A as $\mathcal{L}_{\mathsf{grp}}$-terms built up from elements of X and H, as follows: For each $\lambda < \kappa$ let t_λ be an $\mathcal{L}_{\mathsf{grp}}$-term, and $\bar{x}_\lambda$, $\bar{h}_\lambda$ be finite tuples from X and H, respectively, such that $a_\lambda = t_\lambda(\bar{x}_\lambda, \bar{h}_\lambda)$.

By passing to a cofinal subsequence of A of cardinality κ in the fixed enumeration (recall that we set $\kappa = |T|^+$) we can find a term $t \in \mathcal{L}_{\mathsf{grp}}$ such that for all $\lambda < \kappa$ we have $t_\lambda = t$. Since, each $\bar{x}_\lambda$ is a tuple from X, we can assume that it is of the form $\bar{x}^\nu_\lambda{}^\frown\bar{x}^p_\lambda{}^\frown\bar{x}^\iota_\lambda$ (as discussed in Appendix A), where we simply list all elements of $\bar{x}_\lambda$ of the corresponding types. We may also append the handles of the elements in the tuple $\bar{x}^p_\lambda$ to the beginning of $\bar{x}^\nu_\lambda$ (so that the handle of the j-th element of $\bar{x}^p_\lambda$ is the j-th element of $\bar{x}^\nu_\lambda$, and we allow for repetition of elements).

Thus, at this point, after rearranging, we have an $\mathbb{I}$-indiscernible sequence $(t(\bar{x}_i, \bar{h}_i) : i \in \mathsf{dom}(\mathbb{I}))$. Of course, since $\mathcal{I} \preccurlyeq \mathbb{I}$, we may actually work with the subsequence $(t(\bar{x}_i, \bar{h}_i) : i \in \mathsf{dom}(\mathcal{I}))$. By construction, this sequence is $\mathcal{I}$-indiscernible, and arguing as in Claim 2, $(\bar{x}_i{}^\frown\bar{h}_i : i \in \mathsf{dom}(\mathcal{I}))$ is not $\mathcal{J}$-indiscernible.

By Claim 1 and the fact that A is based on B, for any $R \in \Sigma = \mathcal{L}_\mathcal{I} \setminus \mathcal{L}_\mathcal{J}$, which is non-empty since $\mathcal{J}$ is a strict reduct of $\mathcal{I}$, we can find an $\mathcal{L}_{\mathsf{grp}}$-formula ϕ_R such that:

$$\mathcal{M} \vDash \phi_R(a_{i_1}, \ldots, a_{i_k}) \text{ if, and only if, } \mathcal{I} \vDash R(i_1, \ldots, i_k).$$

Now, let $\Gamma(\mathcal{M})$ be a saturated model of $\mathsf{Th}(C)$, containing all the elements $(\bar{x}^\nu_i : i \in \mathsf{dom}(\mathcal{I}))$, which we now view as graph vertices. Since $\Gamma(\mathcal{M})$ is interpretable in $\mathcal{M}$, this sequence remains $\mathcal{I}$-indiscernible, by Lemma 6.6.4, and since $\mathsf{Th}(G)$ collapses $\mathcal{I}$-indiscernibles to $\mathcal{J}$-indiscernibles this is actually a $\mathcal{J}$-indiscernible sequence in Γ. Similarly, let $\mathcal{H}$ be a saturated model of $\mathsf{Th}(\langle H\rangle)$, in the language of groups. By Fact A.2.2(2) we know that $\mathsf{Th}(\langle H\rangle)$ is stable and has quantifier elimination. From the quantifier elimination, it is easy to see that $(\bar{h}_i : i \in \mathsf{dom}(\mathcal{I}))$ remains $\mathcal{I}$-indiscernible in $\mathcal{H}$, and since $\mathsf{Th}(\langle H\rangle)$ is stable, from Lemma 6.6.3, we have that $(\bar{h}_i : i \in \mathsf{dom}(\mathcal{I}))$ is $\mathcal{J}$-indiscernible in $\mathcal{H}$.

Since the sequence $(\bar{x}_i{}^\frown\bar{h}_i : i \in \mathsf{dom}(\mathcal{I}))$ is not $\mathcal{J}$-indiscernible, there are $i_1, \ldots, i_k, j_1, \ldots, j_k \in \mathsf{dom}(J)$ such that $\mathsf{qftp}_\mathcal{J}(i_1, \ldots, i_k) = \mathsf{qftp}_\mathcal{J}(j_1, \ldots, j_k)$ and $\mathsf{tp}((\bar{x}_{i_1}{}^\frown\bar{h}_{i_1}), \ldots, (\bar{x}_{i_k}{}^\frown\bar{h}_{i_k})) \neq \mathsf{tp}((\bar{x}_{j_1}{}^\frown\bar{h}_{j_1}), \ldots, (\bar{x}_{j_1}{}^\frown\bar{h}_{j_1}))$. Of course, by $\mathcal{I}$-indiscernibility of the sequence we must have that $\mathsf{qftp}_\mathcal{I}(i_1, \ldots, i_k) \neq$

$\mathsf{qftp}_{\mathcal{I}}(j_1, \ldots, j_k)$, so there exists some relation symbol $R \in \mathcal{L}_{\mathcal{I}}$ such that:

$$\mathcal{I} \models R(i_1, \ldots, i_k) \wedge \neg R(j_1, \ldots, j_k).$$

In particular, by Claim 1, we can find an $\mathcal{L}_{\mathsf{grp}}$-formula ϕ_R such that:

$$\mathcal{M} \models \phi_R((\bar{x}_{i_1}{}^\frown \bar{h}_{i_1}), \ldots, (\bar{x}_{i_k}{}^\frown \bar{h}_{i_k})) \wedge \neg\phi_R((\bar{x}_{j_1}{}^\frown \bar{h}_{j_1}), \ldots, (\bar{x}_{j_1}{}^\frown \bar{h}_{j_1}))$$

But, observe that at this point we are in the following situation:

- Since $\mathsf{qftp}_{\mathcal{J}}(i_1, \ldots, i_k) = \mathsf{qftp}_{\mathcal{J}}(j_1, \ldots, j_k)$ and the sequences $(\bar{x}_i^\nu : i \in \mathsf{dom}(\mathcal{I}))$ and $(\bar{h}_i : i \in \mathsf{dom}(\mathcal{I}))$ are $\mathcal{J}$-indiscernible we have that:

  $$\mathsf{tp}_\Gamma(\bar{x}_{i_1}^\nu, \ldots, \bar{x}_{i_k}^\nu) = \mathsf{tp}_\Gamma(\bar{x}_{j_1}^\nu, \ldots, \bar{x}_{j_k}^\nu)$$

  and

  $$\mathsf{tp}_{\langle H\rangle}(\bar{h}_{i_1}, \ldots, \bar{h}_{i_k}) = \mathsf{tp}_{\langle H\rangle}(\bar{h}_{j_1}, \ldots, \bar{h}_{j_k})$$

- The map sending $\bar{x}_{i_l}{}^\frown \bar{h}_{i_l} \mapsto \bar{x}_{j_l}{}^\frown \bar{h}_{j_l}$ respects the 1^ν-, p-, 1^ι-parts, and the handles.

Thus, by applying Fact 6.6.1, we can extend this map to an automorphism $\sigma \in \mathsf{Aut}(\mathcal{M})$ sending $(\bar{x}_{i_l}{}^\frown \bar{h}_{i_l})$ to $(\bar{x}_{j_l}{}^\frown \bar{h}_{j_l})$ for all $l \in \{1, \ldots, k\}$, which is a contradiction. $\square$

Appendix A

Mekler's construction

To keep this thesis a bit more self-contained, I have opted to include a brief exposition of Mekler's construction below. This construction gives us a way of building a group from any structure in a finite signature (it is well-known that it suffices to do this for graphs) in a way that preserves many important model-theoretic properties (see Fact A.3.1 and corollary A.3.4) originates from [Mek81]. The discussion here is largely lifted from [BPT24], but contains additional details which can be found in [Hod93, Appendix A.3] and [CH19]. Nothing in this section is original, and the reader can find everything presented here (probably written more nicely) in the aforementioned sources.

A.1 The Very Basics

Recall from Section 2.7 that a *graph* is just a structure C (to avoid confusion, I will try to keep graphs denoted by C and groups denoted by G, when discussing Mekler's construction) in a relational language with a single binary relation, which in this section will be denoted by E, such that E is irreflexive and symmetric.

Definition A.1.1. A graph $\mathsf{C} = (V, E)$ is called *nice* if it satisfies the following properties:

1. $|V| \geq 2$;
2. For any two distinct vertices $v_1, v_2 \in V$ there is some vertex $u \in V \setminus \{v_1, v_2\}$ such that $\{v_1, u\} \in E$ and $\{v_2, u\} \notin E$;
3. There are no triangles or squares in the graph.

For any graph C and (a fixed) odd prime p, let $\mathsf{M}(\mathsf{C})$ be the 2-nilpotent group of exponent p which is generated freely in the variety of 2-nilpotent groups of exponent p by the vertices of C, with only relations those imposing that two generators commute if and only if they are connected by an edge in C.

If C is a nice graph, we call $\mathsf{M}(\mathsf{C})$ *the Mekler group of* C. More generally, we say that a group G is *a Mekler group* if there is a nice graph C such that $G \equiv \mathsf{M}(\mathsf{C})$. An axiomatisation of the theory of Mekler groups can be found in [Hod93, Appendix A.3].

Let me now recall some (standard) terminology.

Fix a nice graph C with Mekler group $\mathsf{M}(\mathsf{C})$ and let G be a 2-nilpotent exponent p group such that $G \equiv \mathsf{M}(\mathsf{C})$. To fix some notation, I will write $\mathsf{Z} := \mathsf{Z}(G)$ for the centre of $\mathsf{M}(\mathsf{C})$, and for $g \in G$, I will write $\mathsf{C}_G(g)$ for the centraliser of g in G.

Definition A.1.2 (Equivalence relations $\sim$ and $\approx$)**.** We define the following two ($\emptyset$-definable) equivalence relations on G:

1. *Centraliser*: For all $g, h \in G$ we define $g \sim h$ if, and only if, $\mathsf{C}_G(g) = \mathsf{C}_G(h)$.
2. *Powers modulo centre*: For all $g, h \in G$ we define $g \approx h$ if, and only if, there is some $c \in \mathsf{Z}$ and some $\alpha \in \{0, \ldots, p-1\}$ such that $h = g^{\alpha}p$.

Remark A.1.3 ([Hod93, Lemma A.3.3])*.* The equivalence relation $\approx$ refines $\sim$, that is, for all $g, h \in G$, if $g \approx h$ then $g \sim h$.

Definition A.1.4. Let $g \in G$. We say that:

1. g is of *type* q, for $q \in \mathbb{N}$ if $[g]_\sim$ splits into exactly q-many $\approx$-classes.
2. g is *isolated* if every non-central element that commutes with g is $\approx$-equivalent to g.

An element $g \in G$ of type $q \in \mathbb{N}$ is said to have *type* q^ι if it is isolated and *type* q^ν otherwise.

Remark A.1.5. For all $q \in \mathbb{N}$, each of the sets of elements of type q, q^ι and q^ν in $\mathsf{M}(\mathsf{C})$ are $\emptyset$-definable, since $\sim$ and $\approx$ are $\emptyset$-definable.

The next fact follows from [Hod93, Lemmas A.3.6-A.3.10]:

Fact A.1.6. *All non-central elements of G can be partitioned into four different ($\emptyset$-definable, by the previous remark) classes:*

1. *Elements of type* 1^ν.
2. *Elements of type* 1^ι.
3. *Elements of type* p^ν.
4. *Elements of type* $(p-1)^\nu$.

Furthermore, all elements of type $p-1$ *can be written as the product of two* $\sim$*-inequivalent elements of type* 1^ν.

Definition A.1.7 (Handle)**.** For every element $g \in G$ of type p we call an element b of type 1^ν which commutes with g a *handle* of g. If $g, g' \in G$ are elements of type p with handles b, b', respectively, we say that they have the *same handle* if $b \sim b'$.

Remark A.1.8 ([CH19, Fact 2.5]). The handle of $g \in G$ is definable from g, up to $\sim$-equivalence.

Notation A.1.9. We denote by $\mathfrak{E}^\nu$, $\mathfrak{E}^p$ and $\mathfrak{E}^\iota$ the sets of elements of type, 1^ν, p and 1^ι, respectively.

Remark A.1.10. Both the centre of a Mekler group G as well as the quotient $G/\mathsf{Z}(G)$ are essentially $\mathbb{F}_p$-vector spaces. So, as is usual when discussing Mekler groups, *independence* over some supergroup of $\mathsf{Z}(G)$ will refer to linear independence (in the corresponding $\mathbb{F}_p$-vector space).

Definition A.1.11 (Transversal)**.** Let G be a Mekler group. A *transversal of* G is a set X which can be written as the union of three disjoint sets X^ν, X^p, and X^ι where:

- X^ν is a subset of $\mathfrak{E}^\nu$ linearly independent over Z and maximal for this property.
- X^p is a subset of $\mathfrak{E}^p$ linearly independent over $\langle \mathsf{Z}, \mathfrak{E}^\nu \rangle$ and maximal for this property.
- X^ι is a subset of $\mathfrak{E}^\iota$ linearly independent over $\langle \mathsf{Z}, \mathfrak{E}^\nu, \mathfrak{E}^p \rangle$ and maximal for this property.

Remark A.1.12. Every Mekler group G admits a transversal X, and all elements of G can be written as a finite product:

$$a_1^{r_1} \cdots a_n^{r_n} g_1^{s_1} \cdots g_m^{s_m} w_1^{t_1} \cdots w_k^{t_k} z,$$

where $a_i \in X^\nu$, $g_i \in X^p$, $w_i \in X^\iota, z \in \mathsf{Z}$ and r_i, s_i, t_i are in $\{0, \ldots, p-1\}$.

Fact A.1.13 ([CH19, Lemma 2.7])**.** *For every small set of tuples x, there is a* *partial type* $\phi(x)$ *such that* $G \models \phi(x)$ *if, and only if,* x *can be extended to a transversal of* M.

A.2 The Important Facts

Again, in this section, C will be a fixed nice graph, and unless otherwise stated, G will be a model of $\mathsf{Th}(\mathsf{M}(\mathsf{C}))$.

Here's how to interpret the graph from the group:

Fact A.2.1 ([CH19, Fact 2.8], essentially [Hod93, Corollary A.3.11])**.** *There is an interpretation* Γ *such that for any* $G \vDash \mathsf{Th}(\mathsf{M}(\mathsf{C}))$ *we have that* $\Gamma \vDash \mathsf{Th}(\mathsf{C})$*. More specifically, the graph* $\Gamma(G) = (V, R)$ *is given by:*

- $V := \{g \in G \setminus \mathsf{Z} : g \text{ is of type } 1^\nu\}$.
- $R := \{\{[g]_\sim, [h]_\sim\} : [g, h] = 1\}$.

Fact A.2.2 ([Hod93, Theorem A.3.14, Corollary A.3.15])**.** *Let* X *be a transversal in* G*. Then:*

(1) There is a subgroup of Z*, denoted* K_X *such that:*

$$G = \langle X \rangle \times K_X.$$

(2) The group K_X *is an elementary abelian* p*-group. In particular,* $\mathsf{Th}(K_X)$ *is stable and has quantifier-elimination.*

(3) If G *is saturated and uncountable then both the graph* $\Gamma(G)$ *and the group* K_X *are also saturated.*

(4) If G *is saturated then every automorphism of* $\Gamma(G)$ *can be lifted to an automorphism of* G

Fact A.2.3 ([CH19, Proposition 2.18])**.** *There is a partial type* $\pi(\bar{x}; \bar{y})$ *in* $\mathsf{Th}(\mathsf{M}(\mathsf{C}))$ *such that the following are equivalent for any* $\bar{a} \in G^{|\bar{x}|}$ *and any* $\bar{b} \in G^{|\bar{y}|}$*:*

(1) $G \vDash \pi(\bar{a}, \bar{b})$.

(2) $\bar{a}$ *can be extended to a transversal* X *and there is a subset* $H \subseteq \mathsf{Z}$ *containing* $\bar{b}$ *which is linearly independent over* $[G, G]$ *such that* $G = \langle X \rangle \times \langle H \rangle$.

A.3 Some Final Corollaries

For historical purposes, let me now summarise some of the known model-theoretic properties that are preserved by Mekler's construction:

Fact A.3.1. *A Mekler group* M *has the property* P *if and only if its associated graph* C *has the property* P*, where* P *is one of the following properties:*

- λ*-stability for every cardinal* λ*, [Mek81; Hod93];*
- *CM-triviality [Bau02];*
- *The* NIP_n*, for every* $n \in \mathbb{N}$*, [CH19];*
- *The tree property of the second kind, [CH19];*
- *The first and second strict order properties, [Ahn20];*
- *The anti-chain property, [AKL22].*

Moreover, in [BPT24] we develop a "relative quantifier elimination" from Mekler groups down to their nice graphs, and using this we are able to prove the following two theorems:

Theorem A.3.2 ([BPT24, Theorem B])**.** *Let* $\mathsf{M} \preccurlyeq \mathsf{M}'$ *be an elementary pair of Mekler groups of respective graphs* C *and* C'*.*

1. M *is stably embedded*[1] *in* M' *if and only if* C *is stably embedded in* C'*.*
2. M *is uniformly stably embedded in* M' *if and only if* C *is uniformly stably embedded in* C'*.*

and:

Theorem A.3.3 ([BPT24, Theorem A])**.** *Let* $\mathcal{I}$ *be a Ramsey structure, and* $\mathcal{J}$ *a reduct of* $\mathcal{I}$*. Let* M *be a Mekler group and* C *its associated nice graph. Then, the following are equivalent:*

1. C *collapses* $\mathcal{I}$*-indiscernibles (resp. to* $\mathcal{J}$*-indiscernibles).*
2. $\mathsf{M}(\mathsf{C})$ *collapses* $\mathcal{I}$*-indiscernibles (resp. to* $\mathcal{J}$*-indiscernibles).*

I have chosen not to present the main methods we developed in [BPT24] in this thesis as they involve long and delicate relative quantifier-elimination

[1] Recall: Let T be a complete theory. An elementary pair $\mathcal{M} \preceq \mathcal{N}$ is called *stably embedded* if for every formula $\phi(x, b)$ with b from N, there is a formula $\psi(x, c)$ with c from M such that $\phi(\mathcal{N}, b) = \psi(\mathcal{M}, c)$. The pair is called *uniformly stably embedded* if, in addition, the choice of the formula $\psi(x, z)$ depends only on $\phi(x, y)$, and is independent of the parameters b.

arguments which do not, in my opinion, fit in well with the rest of the themes of my thesis. I have, however, included Theorem 6.7.1, a slightly weaker version of Theorem A.3.3, which also appears in [BPT24].

As a final aside, from Theorem A.3.3 we can easily derive with the methods developed in this thesis (Theorem 4.4.3, to be precise) the following corollary, which fits rather neatly with some of the themes presented in Chapter 4:

Corollary A.3.4. *Let* $\mathbb{K}$ *be a Ramsey class in a countable language with an* $\aleph_0$*-categorical Fraïssé limit, and let* C *be a nice graph. Then, the following are equivalent:*

(1) $\mathsf{C} \in \mathsf{N}\mathscr{C}_{\mathbb{K}}$.

(2) $\mathsf{M}(\mathsf{C}) \in \mathsf{N}\mathscr{C}_{\mathbb{K}}$.

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