Combinatorial aspects of finite topological spaces via the representation theory of the symmetric group

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Abstract

We investigate the representation of the symmetric group S_n derived from linearizing the action of S_n on the power set, $\mathcal{P}(X_n)$, of $X_n := \{1, \ldots, n\}$, on the power set of the power set, $\mathcal{PP}(X_n)$, of X_n, and, finally, on the set of all topologies on X_n, $\mathsf{Top}(X_n)$. Moreover, we prove that the latter two cases give an algebra faithful representation of the symmetric group S_n.

We decompose the representation of the symmetric group S_n on $\mathbb{C}Y$, into irreducibles, for some particular invariant subsets, Y, of $\mathcal{P}(X_n)$ and $\mathcal{PP}(X_n)$, in the case $n = 2, 3, 4$. In the general case, we show that for some typical invariant subsets, $Y \subset \mathcal{PP}(X_n)$ the representation on $\mathbb{C}Y$ is explicitly a tensor product of representations that already have an explicit decomposition into irreducibles.

We reduce the action of S_n on $\mathsf{Top}(X_n)$ to an action on the set of reflexive, transitive relation on X_n. We use this presentation to find orbits, $\mathcal{O} \subset \mathsf{Top}(X_n)$, such that $\mathbb{C}\mathcal{O}$ is an algebra faithful representations, and orbits that are in bijection with orbits of the action of S_n on the set of Young tabloids.

Contents

List of Symbols

$\mathbb{C}$ complex number field

λ integer partition (defined in 2.4.1)

X_n the set $\{1, \ldots, n\}$

S_n the symmetric group on $\{1,2,\ldots,n\}$

D Young diagram (defined in 2.4.3)

$\mathcal{T}(D)$ the set of Young tableax with shape D (defined in 2.4.5)

t Young tableau (defined in 2.4.5)

$[t]$ Young tabloid (defined in 2.4.17)

Ω^λ the set of $\lambda - tabloids$ (defined in 2.4.19)

e_t a $\lambda-$polytabloid (defined in 2.4.22)

M^λ the permutation module corresponding to λ (defined in 2.4.3)

R_t the row stabilizer of Young tableau t (defined in 2.4.14)

C_t the column stabilizer of Young tableau t (defined in 2.4.14)

N_t the Young symmetriser associated to Young tableau t (defined in 2.4.28)

S^λ Specht module (defined in 2.4.25)

S_λ Young subgroup of S_n (defined in 2.4.12)

$\mathcal{P}(S)$ the power set of a set S

$\mathcal{P}_i(S)$ the set of subset of $\mathcal{P}(S)$ that all of which have the cardinality i (defined in 3.1.4)

$\mathcal{PP}(S)$ the power set of the power set of a set S

$\mathcal{P}_i\mathcal{P}(S)$ the set of subset of $\mathcal{P}\mathcal{P}(S)$ that all of which have the cardinality i (defined in 4.1.4)

$\mathsf{Top}(X)$ the set of all finite topological spaces on a set X

$\mathcal{P}^{top}\mathcal{P}(X_n)$ the set of all topologies on X_n

$\mathcal{P}^{top}_i\mathcal{P}(X_n)$ the set of subset of $\mathcal{P}^{top}\mathcal{P}(X_n)$ that all of which have the cardinality i

$\leq$ reflexive and transitive relation on a set X

$\leq_{(k:A_1,\ldots,A_k)}$ levelled and unrelated reflexive transitive relation, (defined in 5.4.3)

$\mathsf{Rel}(X)$ the set of all reflexive and transitive relation on a set X

$\mathrm{Orb}_{S_n}(\leq)$ the orbit of reflexive and transitive relation on a set X

$\triangleright$ the group action of S_n on $\mathsf{Rel}(X)$ (defined in 5.3.23)

$\blacktriangleright$ the group action of S_n on the power set (defined in 3.2.1)

$\triangleright$ the group action of S_n on the power set of the power set (defined in 4.1.6)

$\mathbb{C}(X)$ the free $\mathbb{C}$-vector space over X

$\mathbb{C}\mathcal{P}(S)$ the free $\mathbb{C}$-vector space over $\mathcal{P}(S)$

$\mathbb{C}\mathcal{P}\mathcal{P}(S)$ the free $\mathbb{C}$-vector space over $\mathcal{P}\mathcal{P}(S)$

$\mathbb{C}\mathsf{Top}(X)$ the free $\mathbb{C}$-vector space over $\mathsf{Top}(X)$

$\mathbb{C}S_n$ might denote the free $\mathbb{C}$-vector space over S_n or might denote group algebra which one is which should be clear from context.

Chapter 1

Introduction and Overview

1.1 Introduction

In the representation theory of finite groups there are special groups that, beyond some very general level of theory, have their own special representation theory. And then there are collections of groups that it is natural to try to study together.[1] For example, an important part of the representation theory of the symmetric groups S_n can be studied taking them together, with n as a variable. Eventually this unification breaks down. But in principle one can still ask questions about constructions in symmetric group representation theory that make sense for all n. Before setting out the problems we consider here, let us start by giving some more standard examples.

Let G be a finite group and $\mathbb{C}G$ the corresponding complex group algebra. For each $x \in \mathbb{C}G$ there is an orbit under the left action of $\mathbb{C}G$ on itself, denoted $\mathbb{C}Gx$. For example putting $x = \mathbf{e}_G := \sum_{g\in G} g$ we have $\mathbb{C}G\mathbf{e}_G = \mathbb{C}\mathbf{e}_G$. In other words element $\mathbf{e}_G$ induces a 1-dimensional left ideal and hence left module. Of course every group has a trivial representation, and this module $\mathbb{C}\mathbf{e}_G$ induces the trivial representation. In general there is a 1-dimensional space of such elements for each 1-dimensional left ideal. The number of such ideals depends on G but for example for the symmetric group S_n there is exactly one more, corresponding to the alternating representation: we write $\mathbf{f}_n = \sum_{g\in S_n} \mathrm{sgn}(g)\ g$.

[1] A note on references: This introduction aims for a very basic exposition, so for the most part hopefully references will not be needed until later. The first sentences are already an example of this. Possible general references on the collection of finite groups themselves would include [Asc04, CCea85], among very many others; but then informally (and loosely) the general representation theory could simply be expected to follow the 'pattern' of families of the groups themselves.

Given two symmetric groups S_n and S_m we write $S_n \boxtimes S_m$ for the obvious Young subgroup of S_{n+m}. Note that this induces an image of $\mathbf{e}_n := \mathbf{e}_{S_n}$ in $\mathbb{C}S_{n+m}$. Let us write $\mathbf{e}_n$ also for the image, when no ambiguity arises. This abuse of notation is relatively safe in that $\mathbf{e}_n\mathbf{e}_n = n!\mathbf{e}_n$ in both $\mathbb{C}S_n$ and $\mathbb{C}S_{n+m}$. We will call an element x a pre-idempotent if $xx = c_x x$ for some non-zero $c_x \in \mathbb{C}$. (It follows that $(1/c_x)x$ is idempotent.)

Given $x \in \mathbb{C}S_n$ and $y \in \mathbb{C}S_m$ we write $x \boxtimes y$ for the corresponding element in $\mathbb{C}S_{n+m}$. This gives a safer notation $\mathbf{e}_n \boxtimes 1$ (or even $\mathbf{e}_n \boxtimes 1_m$) for the image of $\mathbf{e}_n$ in $\mathbb{C}S_{n+m}$. Similarly $\mathbf{e}_2 \boxtimes \mathbf{e}_2 \in \mathbb{C}S_4$ and so on.

The $\boxtimes$ notation generalises in the obvious way: given a composition of n: $n = n_1 + n_2 + ... + n_l$ we have a subgroup $\boxtimes_{i=1}^l S_{n_i}$ of S_n. For example $\boxtimes_{i=1}^l \mathbf{e}_2$ is an element of $\mathbb{C}S_{2l}$. We can ask questions like: can we give closed form expression for the irreducible content of $\mathbb{C}S_{2l} \boxtimes_{i=1}^l \mathbf{e}_2$ for all l together?

Notation aside: Above we expressed n as a composition. But it will be clear that up to isomorphism the representation $\mathbb{C}S_n \boxtimes_{i=1}^l S_{n_i}$ does not depend on the order of terms in the composition. Thus here compositions have representative integer partitions — compositions of n such that $n_1 \geq n_2 \geq ... \geq n_l$. We will write Λ_n for the set of integer partitions of n (thus $\Lambda_3 = \{3, 2+1, 1+1+1\}$ and so on); and write Λ for the set of all integer partitions. Of course Λ_n indexes the cycle structures of elements of S_n; and hence can also be used to index complex irreducible representations (see later). We can note already that, denoting our partition of n by λ, and setting $\mathbf{e}_\lambda = \boxtimes_{i=1}^l \mathbf{e}_{n_i}$, then $\mathbb{C}S_n\mathbf{e}_\lambda$ is sometimes called the Young module Y_λ. We have, for example, $\mathbf{f}_{l+1}\mathbb{C}S_n \boxtimes_{i=1}^l \mathbf{e}_{n_i} = 0$ where n is at least $l+1$, simply by the pigeonhole principle and the identity $\mathbf{f}_2\mathbf{e}_2 = 0$. Similarly $dim(\mathbf{f}_{\lambda'}\mathbb{C}S_n\mathbf{e}_\lambda) = 1$ — the key fact for properties of Specht modules (see also later; here λ' is transpose λ). For example, let $\lambda = (2,1)$. Then we have

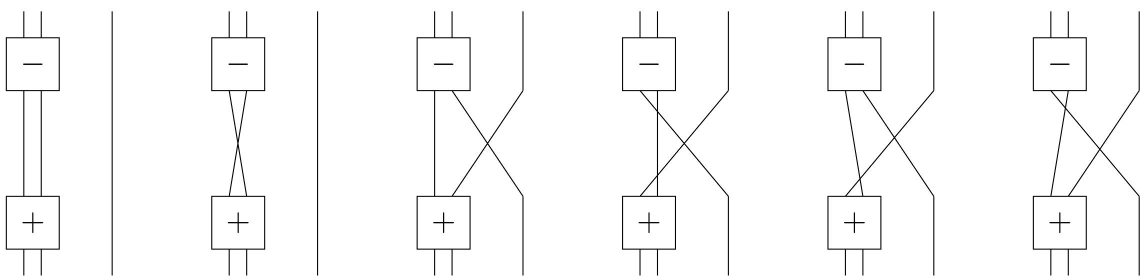


Figure 1.1: Spanning set in $\mathbf{f}_{(2,1)}\mathbb{C}S_3\mathbf{e}_{(2,1)}$.

Note that the first two elements in Figure1.1 are zero and the final four elements are nonzero dependent.

For another example of all-n S_n representation theory let us use the category $\mathcal{SET}$. Consider the set $\underline{n} := \{1, 2, ..., n\}$. The morphism sets of form $\mathcal{SET}(\underline{n}, \underline{n})$ are monoids, and indeed S_n is exactly the group of bijections in $\mathcal{SET}(\underline{n}, \underline{n})$. This means in particular that for each m the set $\mathcal{SET}(\underline{n}, \underline{m})$ is an S_n-set via the category composition, and the space $\mathbb{C}\mathcal{SET}(\underline{n}, \underline{m})$ is a complex representation, for every n. Observe for example that

$$\mathbf{f}_3 \mathbb{C}\mathcal{SET}(\underline{n}, \underline{2}) = 0 \tag{1.1}$$

for all $n \geq 3$. (A convenient way to represent elements f of $\mathcal{SET}(\underline{n}, \underline{m})$ is by $f \mapsto f(1)f(2)...f(n)$, i.e. as words of length n in $\underline{m}$ such as $11122, 12122 \in \mathcal{SET}(\underline{5}, \underline{2})$ — see e.g. [Gre80, §2.1]. If $m = 2$ then at least two of the first three 'letters' must be the same, and the $\mathbf{f}_3$ property follows.) We can deduce from this that the irreducible decomposition of the representation $\mathbb{C}\mathcal{SET}(\underline{n}, \underline{2})$ contains only irreducible representations with integer-partition labels containing at most two components (since these are the representations for which $\rho(\mathbf{f}_3) = 0$). Indeed $\mathbf{f}_{m+1}\mathbb{C}\mathcal{SET}(\underline{n}, \underline{m}) = 0$ for all n, leading to corresponding decomposition result for all n for every fixed m.

Recall that every function $F : S \to T$ defines an equivalence relation on set S by $s \sim^F s'$ if $F(s) = F(s')$. For $T = \underline{m}$ for some m, the 'shape' of such an equivalence relation on finite set S is the composition of $|S|$ giving the list of sizes of classes: $n_i = |F^{-1}(i)|$. Let us write $\mathcal{SET}_\lambda(S, \underline{m})$ for the subset of $\mathcal{SET}(S, \underline{m})$ of functions of shape λ (define $|\lambda|$ to be $\sum_i \lambda_i$, so $|\lambda| = |S|$). From our proof of (1.1) above we see that these subsets are fixed under the S_n action in case $S = \underline{n}$. Formally we have

$$\mathcal{SET}(S, \underline{m}) = \bigsqcup_{\lambda \in \mathbb{N}_0^m} \mathcal{SET}_\lambda(S, \underline{m}) \tag{1.2}$$

(with many of these subsets empty in general, unless $|\lambda| = |S|$). One can check that $\mathbb{C}\mathcal{SET}_\lambda(\underline{n}, \underline{m}) \cong \mathbb{C}S_n\mathbf{e}_\lambda$. Note that the irreducible content of the latter outer product is accessible by relatively straightforward counting rules (entirely straightforward in case $m = 2$; see later).

Much less straightforward is to analyse a further iteration of the $\mathcal{SET}(-, \underline{m})$ operation, say, $\mathcal{SET}(\mathcal{SET}(\underline{n}, \underline{2}), \underline{2})$, as an S_n-set. This time the decomposition as in (1.2) to $\mathcal{SET}_\lambda(\mathcal{SET}(\underline{n}, \underline{2}), \underline{2})$ has λ a composition of 2^n into two parts. Hence it is given in effect by $\lambda_1 \in \{0, 1, ..., 2^n\}$. This λ_1 organises the elements into groupings according to Pascal's triangle. Thus for example when $n = 3$ and $\lambda_1 = 2$ there are 28 elements (the third entry in the 2^3 row of the triangle). Returning to our word-representation we have $\mathcal{SET}(\underline{3}, \underline{2}) = \{111, 112, 121, 122, 211, 212, 221, 222\}$. Using this order then for example $21212222 \in \mathcal{SET}_{(2,6)}(\mathcal{SET}(\underline{3}, \underline{2}), \underline{2})$. The S_3 action is via the place-permutations on the labeling words 111,112, and so on. One sees almost immediately that 21212222 generates a subset of order 6 with no fixed points, thus a copy of the regular S_3-set. More generally the S_n action preserves the shape of the labelling words. And again each such shape μ, say, is determined by μ_1 (for 112 we have $\mu_1 = 2$ and so on). So for given λ_1 we can index subsets by partitions $\boldsymbol{\mu}$ lying in $\{0, 1, ..., n\}^{\lambda_1}$. (The subset of 6 above is part of the (2,1) subset, which altogether has order 9. The (1,0), (2,0), (1,1), (2,2), (3,1) (3,2) subsets all have order 3 and (3,0) has order 1, making 28.) But the general structure is quite rich, beyond such low rank examples.

Now we turn to consider constructions of form $\mathcal{P}(X)$ where $\mathcal{P}$ is the power-set function (compare for example with Manes [Man03, Ex.2.12]) and X is an S_n-set. In fact consider the map $\Theta : \mathcal{P}(X) \to \mathcal{SET}(X, \underline{2})$ given by $\Theta(a)(x) = 1$ if $x \in a$ and $= 2$ otherwise. The map $\bar{\Theta} : \mathcal{SET}(X, \underline{2}) \to \mathcal{P}(X)$ given by $x \in \bar{\Theta}(f) \iff f(x) = 1$ is inverse to Θ (see later), so they are isomorphisms. And so we can immediately apply decomposition results for $\mathcal{SET}(\underline{n}, \underline{2})$ and $\mathbb{C}\mathcal{SET}(\underline{n}, \underline{2})$ — such as those above — to $\mathcal{P}(X)$ and $\mathbb{C}\mathcal{P}(X)$ in case $X = \underline{n}$ (an S_n-set in the obvious way). What is less clear is the decomposition of $\mathcal{P}(X)$ for more general S_n-sets. A very natural case is to iterate the $\mathcal{P}$ function starting with $X = \underline{n}$. This gives us a heirarchy of representation theory questions: to determine the irreducible decomposition of $\mathcal{P}^d(\underline{n})$ for all n for different values of d. Of particular interest is the case $d = 2$. One of our original motivating questions (nominally unrelated to stability properties in S_n representation theory) is to try to find models of low-dimensional topology (i.e. Euclidean metric topology) in finite set topology. (A second motivating question on the topology side was to answer, in greater generality, a for-all-n finite set topology exam question from the Topology course [Mar21] — see later.) And of course the set of topologies on X, denoted by $\mathsf{Top}(X)$, is contained in $\mathcal{P}^2(X)$.

It is well known that a decomposition of a complex problem is breaking it down into simple distinct parts. Each part can be represented as subsets of the original problem to manage and better understand it. Therefore, this hierarchical partition help us to understand the original problem that we want to solve.

For example, in this work, we study $\mathcal{P}^2(X)$ under the action of symmetric group, first by breaking it down into closed subsets. Note that $\mathcal{P}(X)$ is a finite set of finite sets. Thus, there is an integer partition that describes the list of cardinalities of elements in an element of $\mathcal{P}\mathcal{P}(X)$. We can decompose such a set of sets according to fixing this integer partition. Let us write $\mathcal{P}_\lambda\mathcal{P}(X)$ for the subset of $\mathcal{P}^2(X)$ with this integer partition λ (for example $\{\{1\},\{1,2\}\}$ and $\{\{2\},\{1,2\}\}$ belong in $\mathcal{P}_{(2,1)}\mathcal{P}(X)$). The symmetric group action respects this decomposition. (It is the image under isomorphism Θ of the $\boldsymbol{\mu}$ decomposition mentioned above.) So, after understanding this decomposition, one possible next step is to see if the action decomposes further. Alternatively, we can pass at this stage to a linear representation from each component of the decomposition, and attempt to decompose linearly - for each into irreducible representations.

1.2 Overview of results

Let $n \in \mathbb{N}$ and $X_n = \{1, \ldots, n\}$, a set of cardinality n. Let $\mathcal{P}(X_n)$ be the power set of X_n. The action of the symmetric group S_n on X_n, where $f.x = f(x)$, gives an action of S_n on $\mathcal{P}(X)$, where $f \blacktriangleright A = f[A]$, the image of A under f. This gives a representation of S_n on $\mathbb{C}\mathcal{P}(X_n)$, by linearising. In Section 3.2.2 we will show that this is the same action as discussed in the Introduction.

As shown in the Introduction, if $n \geq 2$, the representation of S_n on $\mathbb{C}\mathcal{P}(X_n)$ does not contain all irreducible representations of S_n, and hence the representation is not algebra faithful. Furthermore, each irreducible representation of the decomposition of $\mathbb{C}\mathcal{P}(X_n)$ into irreducible arises from Young diagram of depth ≤ 2. In order to prove this in a different way, we will consider the following partition of $\mathcal{P}(X_n)$,

$$\mathcal{P}(X_n) = \bigsqcup_{i=0}^{n} \mathcal{P}_i(X_n).$$

where

$$\mathcal{P}_i(X_n) := \{A \subseteq X_n : |A| = i\}.$$

Because each $\mathcal{P}_i(X_n)$ is invariant under the action of symmetric group S_n, then we have, as representations of S_n,

$$\mathbb{C}\mathcal{P}(X_n) \cong \bigoplus_{i=0}^{n} \mathbb{C}\mathcal{P}_i(X_n).$$

This will enable us to apply the results in [Jam06, Theorem 3.2.10], as we now explain. Let $i \in \{0, 1, \ldots, n\}$. Let $\Omega^{(n-i,i)}$ be the set of young tabloids of shape $(n-i, i)$. Then

$$\mathcal{P}_i(X_n) \cong \Omega^{(n-i,i)}$$

as S_n-sets (See Proposition 3.2.13). A theorem in James [Jam06, Theorem 3.2.15] then says that

$$\mathbb{C}\mathcal{P}_i(X_n) \cong S^{(n)} \oplus S^{(n-1,1)} \oplus S^{(n-2,2)} \oplus \cdots \oplus S^{(n-i,i)}$$

when $i \leq n - i$.

A prime motivation for this thesis was to investigate the representation of S_n on $\mathcal{P}\mathcal{P}(X_n)$, the power set of the power set of X_n, and on the set $\mathsf{Top}(X_n)$ of topologies on X_n. A main conclusion is that the representations are considerably richer.

Let us look at the action of S_n on $\mathbb{C}\mathcal{P}\mathcal{P}(X_n)$. A general result is that all irreducible representation of S_n occur in $\mathbb{C}\mathcal{P}\mathcal{P}(X_n)$, and hence, we have that the representation of S_n is algebra faithful. This is done in §4.4. In order to have finer results, we consider the following partition of $\mathcal{P}\mathcal{P}(X_n)$,

$$\mathcal{P}\mathcal{P}(X_n) = \bigsqcup_{i=0}^{2^n} \mathcal{P}_i\mathcal{P}(X_n),$$

where all terms are invariant under the action of S_n. Furthermore, we can partition each $\mathcal{P}_i\mathcal{P}(X_n)$ into the subsets, $\mathcal{P}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n)$, where $a_1, \ldots, a_i \in \{0, \ldots, n\}$ satisfy $a_1 \leq a_2 \leq \cdots \leq a_i$, as defined below

$$\begin{aligned}\mathcal{P}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n) = &\{A \subseteq \mathcal{P}(X_n) : |A| = i; \text{ and, picking any order on the elements of } A, \text{ so} \\ &A = \{M_1, \ldots, M_i\}, \text{ then } |M_1| = a_{f(1)}, \ldots, |M_i| = a_{f(i)}, \text{for some} \\ &\text{permutation } f \text{ of } \{1, \ldots, i\}\}.\end{aligned}$$

We then have, see Equation (4.4):

$$\mathcal{P}_i\mathcal{P}(X_n) \;=\; \bigsqcup_{\substack{[a_1,\dots,a_i]\in\{0,1,\dots,n\}^i\\ 0\le a_1\le a_2\le\cdots\le a_i\le n}} \mathcal{P}_{i,[a_1,a_2,\dots,a_i]}\mathcal{P}(X_n)$$

Some of the terms in the partition may be empty sets, for instance, $\mathcal{P}_{n,[n,n,\dots,n]}\mathcal{P}(X_n)$, if $n \geq 2$.

Some sample results that we have are written below, and can be found in Theorem 4.3.24 and Section 4.4 are:

1. Let $0 \leq a_1 < a_2 < \cdots < a_i \leq n$, then we have:

$$\begin{aligned}&\mathbb{C}\mathcal{P}_{i,[a_1,\dots,a_i]}\mathcal{P}(X_n)\\ &\cong [\bigoplus_{j_1=0}^{min(a_1,n-a_1)} S^{(n-j_1,j_1)}]\otimes[\bigoplus_{j_2=0}^{min(a_2,n-a_2)} S^{(n-j_2,j_2)}]\otimes\cdots\otimes[\bigoplus_{j_i=0}^{min(a_i,n-a_i)} S^{(n-j_i,j_i)}]\end{aligned}$$

2.

$$\mathbb{C}\mathcal{P}_{n+1,[0,1,\dots,n]}\mathcal{P}(X_n) \cong \mathbb{C}S_n,$$

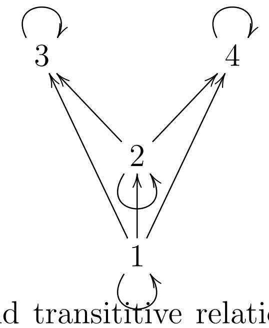
the regular representation of S_n, and so it is algebra faithful representation of S_n.

Now, let $\mathsf{Top}(X_n)$ be the set of all topologies on X_n. We know that $\mathsf{Top}(X_n) \subset \mathcal{P}\mathcal{P}(X_n)$, and moreover $\mathsf{Top}(X_n)$ is invariant under the action of symmetric group S_n (see §5.1). We also have that all irreducible representation of S_n occur in $\mathbb{C}\mathsf{Top}(X_n)$ (see §5.2), and as we will see in §5.4.5 in several different ways, and hence the representation of S_n on $\mathbb{C}\mathsf{Top}(X_n)$ is algebra faithful, (see §5.1.2).

We investigate the partition of $\mathsf{Top}(X_n)$ into orbits. Let $\mathsf{Rel}(X_n)$ be the set of all relations in X_n that are reflexive and transitive. We recall [May03, Proposition 1.16.] that we have a bijection

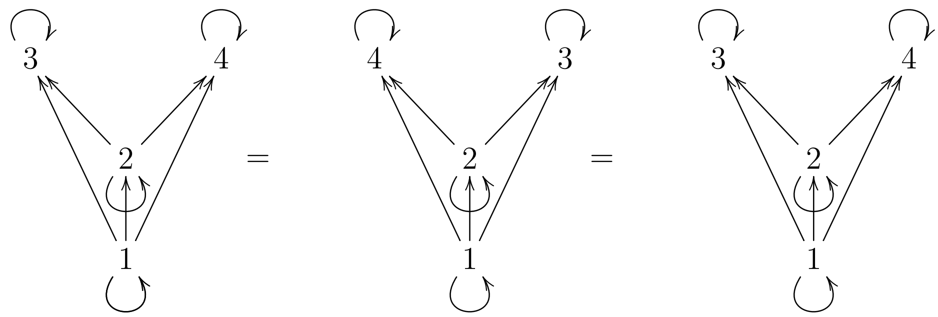
$$\mathsf{Rel}(X_n) \cong \mathsf{Top}(X_n),$$

which we prove in this thesis preserves the action of S_n; see Proposition 5.3.25. This allows us to understand diagrammatically the action of S_n on the set of topologies in X_n, and in which orbits the restriction of the action of S_n is free. For instance, the diagram



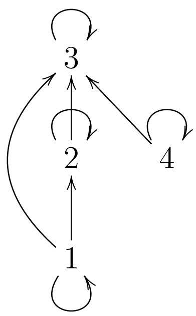
is identified with the, reflexive and transitive relation, $\leq$, on X_4, where $1 \leq 1$, $2 \leq 2$, $3 \leq 3$, $1 \leq 2$, $1 \leq 3$, $1 \leq 4$, $2 \leq 3$, $2 \leq 4$. We can see that the action of S_4 on $\mathrm{Orb}_{S_4}(\leq)$ is not free, because: e.g let $(34) \in S_4$. Then

$(34) \triangleright$



Whereas the action of S_4 of the orbit of the relation $\leq'$, on X_4, represented below:

$\leq' =$



is free, and in particular $\mathbb{C}(\mathrm{Orb}_{S_4}(\leq'))$ is algebra faithful.

Readers may have noticed some resemblance between these diagrams of relations and Young tabloids. We investigate this relation further in §5.4.6. In particular we prove that the set of Young tabloids can be embedded in the set of topologies in at least two ways, using leveled topologies.

Chapter 2

Preliminaries

In this chapter, we recall some fundamental concepts and some theoretical basis that be useful in our research.

2.1 A resumé of elementary linear algebra

Definition 2.1.1. Suppose X is a finite, non-empty set. Then the *free* $\mathbb{C}$*-vector space over* X, denoted $\mathbb{C}(X)$, has as underlying set the set of formal linear combinations, of elements of X, so

$$\mathbb{C}(X) := \left\{ \sum_{x \in X} \lambda_x x | \lambda_x \in \mathbb{C}, \text{for all } x \in X \right\}$$

The addition of elements of $\mathbb{C}(X)$, and their multiplication by scalars in $\mathbb{C}$, is defined below:

1.

$$\left(\sum_{x \in X} \lambda_x x \right) + \left(\sum_{x \in X} \alpha_x x \right) := \sum_{x \in X} (\lambda_x + \alpha_x) x,$$

2.

$$\alpha \left(\sum_{x \in X} \lambda_x x \right) = \sum_{x \in X} (\alpha \lambda_x) x.$$

Clearly X is a basis of $\mathbb{C}(X)$. By abuse of language we identify $x \in X$ with

$\sum_{y \in X} \delta(y, x) \cdot x \in \mathbb{C}(X)$, where

$$\delta(y, x) = \begin{cases} 1, y = x \\ 0, y \neq x. \end{cases}$$

.

Definition 2.1.2. Let X and Y be finite sets. Given a map $f \colon X \to Y$, the *linearisation* of f is the unique linear map $f^{\#} \colon \mathbb{C}(X) \to \mathbb{C}(Y)$ such that $x \mapsto f(x)$, on the basis X of $\mathbb{C}(X)$ and Y of $\mathbb{C}(Y)$. So

$$f^{\#}\left(\sum_{x \in X} \lambda_x x\right) = \sum_{x \in X} \lambda_x f(x).$$

Definition 2.1.3. Suppose X is a finite, non-empty, set. Let $f \colon X \to X$ be a map. Then the trace of a linear map $f^{\#} \colon \mathbb{C}(X) \to \mathbb{C}(X)$ is defined by :

$$\text{Tra}(f^{\#}) = |\{x \in X | f(x) = x\}|.$$

Here, $|S|$ denotes the cardinality of a finite set S.

2.2 A resumé of group representation theory

We first introduce the symmetric group. Then we show some general concepts about the representation of group.

Definition 2.2.1. (see for example [Fra97, Definition 2.4] and [Sag13, Section 1.1].) Let n be a positive integer and $X_n := \{1, \ldots, n\}$. The symmetric group, S_n, consists of the set of all bijective functions from X_n to itself, with group-law $fg = f \circ g$.

Notation 2.2.2. (See for example[Jam06, pp.5].) The elements f of S_n is called a permutation.

The order of the symmetric group is given by $|S_n| = n!$.

We will frequently write an element of S_n as an array of two lines:

$$f = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \\ f(1) & f(2) & f(3) & \dots & f(n) \end{pmatrix}.$$

Let $f \in S_n$. Then the sign of f will be defined as :

$$\operatorname{sgn}(f) = \begin{cases} +1 \text{ if } f & \text{is even permutation} \\ -1 \text{ if } f & \text{is odd permutation.} \end{cases}$$

Given a set X, we will also consider the group $\operatorname{Sym}(X)$, whose underlying set is the set of all bijections $f\colon X \to X$, and with group-law $f \cdot g = f \circ g$. Clearly $\operatorname{Sym}(X_n) = S_n$.

2.2.1 Generalities about group actions

Definition 2.2.3. (See for example [Fra97, Definition 3.17].) Let X be a set. Consider $(G, \cdot)$ a group, whose identity is denoted e. A (left) group action α of G on X is a function,

$$\begin{aligned} \alpha\colon G \times X &\to X \\ (g, x) &\mapsto g \cdot x. \end{aligned}$$

that satisfies the following two axioms:

1. $\forall g, h \in G \,\text{and}\, x \in X,$ then $g \cdot (h \cdot x) = (gh) \cdot x;$
2. $\forall x \in X,$ then $e \cdot x = x.$

A set X is equipped with an action of G is said to be a G-set.

Lemma 2.2.4. *Let X be a set. Consider $\operatorname{Sym}(X)$ is the symmetric group on X. Then we have an action of $\operatorname{Sym}(X)$ on X given by the mapping*

$$\begin{aligned} \alpha\colon \operatorname{Sym}(X) \times X &\to X \\ (f, x) &\mapsto f \cdot x \end{aligned}$$

where $f \cdot x := f(x)$.

Proof. We need to prove that α is a group action. This means we need to show the two axioms in Definition 2.2.3.

1. Let $f, g \in \mathrm{Sym}(X)$ and $y \in X$. Then

$$\begin{aligned} g.(f.y) &= g.(f(y)) \\ &= g(f(y)) \\ &= (g \circ f)(y) \\ &= (g \circ f).(y). \end{aligned}$$

   So the axiom (1) holds;

2. Let $id \in \mathrm{Sym}(X)$ is the identity in $\mathrm{Sym}(X)$, $y \in X$. Then:

$$\begin{aligned} id.y &= id(y) \\ &= y. \end{aligned}$$

   So the axiom (2) holds.

Therefore, α is a group action. □

Definition 2.2.5. [Lan12, pp 27.] Let G be a group. Let X and Y be G-sets. A *G-set map*, also known as a *morphism of G-sets*, or as a *map preserving the G-actions*, $f\colon X \to Y$ is a function $X \to Y$ such that

$$g.f(x) = f(g.x),$$

for all $g \in G, x \in X$.

Definition 2.2.6. Let G be a group. Let X and Y be G-sets. A *G-set map* $f\colon X \to Y$ is said to be isomorphism if it is bijection such that

$$g.f(x) = f(g.x),$$

for all $g \in G, x \in X$.

2.2.2 Generalities about group representations

Definition 2.2.7. (See for example [Lan12, Example in pp 8].) Let V is a finite dimensional complex vector space. We let $\mathrm{GL}(V)$ denote the group of all bijective linear map $T\colon V \to V$. The group-law is given by the usual composition: $TT' = T \circ T'$.

Definition 2.2.8. (See for example [FH13, Definition 1.1], [CSST10] and [Ste12, Definition3.1.1] .) Let G be a finite group. *A linear representation of* G is a pair (V, ρ) of a finite -dimensional complex vector space V and a group homomorphism

$$\rho\colon G \to \mathrm{GL}(V)$$
$$g \mapsto \rho(g).$$

This means that:

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2), \text{ for every } g_1, g_2 \in G.$$

The dimension or degree of ρ is, by definition $\dim_{\mathbb{C}}(V)$. If the representation ρ has dimension 0, then ρ is called a zero representation.

Sometimes we call either V or ρ is a representation and we often write gv for $\rho(g)(v)$.

Example 2.2.9. If V is any vector space, we have a representation of G on V where $\rho(g) = \mathrm{id}_V$, for all $g \in G$. If $V = \mathbb{C}$, we call this representation the trivial representation of G.

Example 2.2.10. Let $X_3 = \{1, 2, 3\}$. The underlying set of the group $S_3 = \mathrm{Sym}(X_3)$, as defined in (2.2.1), is

$$S_3 = \{id, (12), (23), (13), (132), (123)\}$$

Let $V = \mathbb{C}^3$. We use the basis $\{e_1, e_2, e_3\}$ of $\mathbb{C}^3$. Any linear map $T\colon \mathbb{C}^3 \to \mathbb{C}^3$ can be represented by 3×3 matrix. An example of a representation of S_3 in V is given by the group homomorphism

$$\rho\colon S_3 \to \mathrm{GL}(V)$$
$$\sigma \to (r_{ij})_{3\times 3},$$

where

$$r_{ij} = \begin{cases} 1, & \text{if } \sigma(j) = i \\ 0, & \text{otherwise.} \end{cases}$$

Explicitly

$$\rho(id) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho((12)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho((23)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\rho((13)) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \rho((132)) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho((123)) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note: This representation is called the defining representation.

2.2.3 The linearisation of a group action

Definition 2.2.11. Suppose that G has a (left) group action on a finite set X, denoted by:

$$G \times X \to X$$
$$(g, x) \mapsto g \cdot x.$$

The *linearised representation of G* associated to G-set X, has underlying vector space $\mathbb{C}(X)$, and

$$g \cdot \sum_{x \in X} \sigma_x x = \sum_{x \in X} \sigma_x (g \cdot x),$$

for all $g \in G$ and

$$\sum_{x \in X} \sigma_x x \in \mathbb{C}(X).$$

Definition 2.2.12. (See for example [Ste12, Definition 4.1.1.].) Let (V, ρ) and $(W, \rho^{'})$ be two linear representation of a finite group G over $\mathbb{C}$. A linear map $f \colon V \to W$ is called a morphism or intertwiner (homomorphism) between two representation of G if

$$f\rho(g) = \rho^{'}(g) f, \text{ for all } g \in G.$$

Lemma 2.2.13. *Let X and Y be sets and G has a (left) group action on X and Y. Suppose $f \colon X \to Y$ is a map preserving action of G as defined in Definition 2.2.5, I.e:*

$$\forall x \in X, g \in G\,, f(g \cdot x) = g(f(x)).$$

There is an intertwiner

$$\begin{aligned} f^{\#} \colon \mathbb{C}(X) &\to \mathbb{C}(Y) \\ \sum_{x \in X} \sigma_x\, x &\mapsto \sum_{x \in X} \sigma_x f(x) \end{aligned}$$

Definition 2.2.14. Let (V, ρ) and $(W, \rho^{'})$ be two linear representation of a finite group G over $\mathbb{C}$. We say they are isomorphic representation if there is a linear isomorphism $\phi \colon V \to W$ such that

$$\rho^{'}(g) = \phi \circ \rho(g) \circ \phi^{-1}$$

for all $g \in G$.

2.2.4 Subrepresentations of representations

Definition 2.2.15. (See for example [Ste12, Definition 3.1.10].) Let $\rho \colon G \to \mathrm{GL}(V)$ be a representation of a group G. A subspace $W \leq V$ is said to be $G-$*invariant* if one has $\rho(g)(w) \in W$, for all $g \in G, w \in W$.

Definition 2.2.16. (See for example [Ste12, page 16].) Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of a group G and $W \leq V$ be G-invariant subspace. The representation

$$\rho_{|W} : G \to \mathrm{GL}(W)$$
$$g \mapsto \rho(g)$$

is called *the subrepresentation* of ρ on W.

Definition 2.2.17. (See for example [Ste12] Definition 3.1.15.) A non-zero representation $\rho\colon G \to \mathrm{GL}(V)$ of a group G is called *irreducible or simple* if the only $G-$invariant subspaces of V are $\{0\}$ and V.

Example 2.2.18. The trivial representation of a group G is irreducible, since it has no proper non-zero subspace.

Definition 2.2.19. (See for example[Ste12, Definition 3.1.11, pp15.].) Let(V, ρ) and $(V^{'}, \rho^{'})$ be two representations of a group G over a field $\mathbb{C}$. *The (external) direct sum of these two representations* is the representation $(V \oplus V^{'}, \rho \oplus \rho^{'})$, denoted by:

$$\rho \oplus \rho^{'} : G \to \mathrm{GL}(V \oplus V^{'}),$$

where

$$(\rho \oplus \rho^{'})(g)(v, v^{'}) = (\rho(g)(v), \rho^{'}(g)(v^{'})),$$

for each $v \in V$ and $v^{'} \in V^{'}$.

Definition 2.2.20. (See for example [Ste12, Definition 3.1.21.]) A representation of a group G, $\rho\colon G \to \mathrm{GL}(V)$, is called *completely reducible or semisimple* if it can be written as the direct sum

$$V \cong V_1 \oplus V_2 \oplus \cdots \oplus V_n,$$

where V_i are $G-$invariant subspace and $\rho_{|V_i}$ is irreducible for all $i = 1, \ldots, n$.

Proposition 2.2.21. *(See for example [FH13, proposition 1.8.]) For any represen-*

tation V of a finite group G, there is a decomposition

$$V = a_1 V_1 \oplus a_2 V_2 \oplus \cdots \oplus a_k V_k,$$

where the V_i are distinct irreducible representations. The decomposition of V into a direct sum of the k factors is unique, as are the V_i that occur and their multiplicities a_i, where $i \in \{1, \ldots, k\}$.

2.2.5 Characters of group representation

Definition 2.2.22. (See for example [FH13, Definition 2.1, pp 13.]) If (V, ρ) is a linear representation of a group G, its character $\chi_V : G \to \mathbb{C}$ is the complex-valued function on the group defined by

$$\chi_V(g) = \text{Tra}(\rho(g)),$$

the trace of $\rho(g) : V \to V$, on V.

In particular, we have

$$\chi_V(hgh^{-l}) = \chi_V(g),$$

so that χ_V is constant on the conjugacy classes of G, such a function is called a class function. Note that $\chi_V(id) = dim(V)$.

Example 2.2.23. Let $X = \{1, 2, 3\}$ and suppose S_3 is the symmetric group. Then the character the defining representation of S_3, shown in Example 2.2.10, denoted by χ_d, is :

$$\chi_d(id) = \text{Tra}\Big(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Big) = 3,$$

$$\chi_d(12) = \text{Tra}\Big(\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Big) = 1,$$

$$\chi_d(123) = \text{Tra}\Big(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\Big) = 0,$$

Proposition 2.2.24. *(See for example [FH13, Proposition 2.1].) Let V be a representation of G written as*

$$V = V_1 \oplus \cdots \oplus V_k$$

where $V_1, \ldots, V_k$ are irreducible representations. Then

$$\chi_V = \chi_{V_1} + \cdots + \chi_{V_k}$$

where χ_V is the character of V and χ_{V_i} is the character of V_i.

Definition 2.2.25. (See for example[FH13, pp. 14.]) *The character table of a group G* is a square matrix whose rows are labelled by the irreducible representations of G, and columns by the conjugacy classes in G, where the entry in row and column is the character of the representation ρ on the conjugacy class .

2.3 Algebra-faithful and group-faithful representations: generalities

Definition 2.3.1. (See for example [EH18, Section 1.1.2].) Let G be a finite group. The group algebra $\mathbb{C}G$ is $\mathbb{C}$-algebra whose underlying vector space is the free vector space $\mathbb{C}G$ on the underlying set of G,

$$\mathbb{C}G = \left\{ \sum_{g \in G} \alpha_g g \quad \text{where} \quad \forall g \in G, \alpha_g \in \mathbb{C} \right\}$$

Multiplication: let

$$\sum_{g \in G} \alpha_g g \quad \text{and} \quad \sum_{h \in G} \lambda_h h$$

be two elements of $\mathbb{C}G$. Then we have

$$(\sum_{g \in G} \alpha_g g)(\sum_{h \in G} \lambda_h h) = \sum_{gh \in G} \alpha_g \lambda_h (gh)$$

The next proposition shows that a representation of a group is essentially the same as a representation of the corresponding group algebra.

Proposition 2.3.2. *(See for example [EH18, Proposition 2.41.].) A representation* $\rho\colon G \to \mathrm{GL}(V)$ *of a group* G *on* V*, gives rise to an algebra representation* $\hat{\rho}\colon \mathbb{C}G \to \mathrm{End}(V)$*, the algebra of linear maps* $V \to V$*, given by*

$$\sum_{g\in G} \alpha_g g \mapsto \sum_{g\in G} \alpha_g \rho(g).$$

Definition 2.3.3. (See for example [Lan12, ChapterXVIII, §1].) Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation of a group G, in a vector space V. Then ρ is called a *group-faithful representation* of G if ρ is injective, i.e.

$$g \neq h \Rightarrow \rho(g) \neq \rho(h).$$

Example 2.3.4. Let $S_2 = \{\mathrm{id}, (12)\}$ be the symmetric group of order 2. Suppose that the representation of symmetric group S_2 on $\mathbb{R}^2$ is given by the map

$$\rho\colon S_2 \times \mathbb{R}^2 \to \mathbb{R}^2$$

where, for all $(x, y) \in \mathbb{R}^2$, we have

$$\begin{aligned} \rho(id)(x, y) &= (x, y) \\ \rho((12))(x, y) &= (y, x). \end{aligned}$$

So, $\rho(id) \neq \rho((12))$. Thus, ρ is a group faithful.

Definition 2.3.5. A representation $\rho\colon G \to \mathrm{GL}(V)$ of a group G is called *algebra faithful* if the algebra map $\hat{\rho}\colon \mathbb{C}G \to \mathrm{End}(V)$ is injective.

So $\rho\colon G \to \mathrm{GL}(V)$ is algebra faithful if, and only if,

$$\sum_{g\in G} \lambda_g g \neq \sum_{h\in G} \lambda_h h \Rightarrow \hat{\rho}\Big(\sum_{g\in G} \lambda_g g\Big) \neq \hat{\rho}\Big(\sum_{h\in G} \lambda_h h\Big).$$

Lemma 2.3.6. *If a representation* ρ *of a group* G *is algebra faithful then* ρ *is group faithful.*

Proof. Let $\rho\colon G \to \mathrm{GL}(V)$ be a group representation. Suppose that $\hat{\rho}\colon \mathbb{C}G \to \mathrm{End}(V)$ is an algebra-faithful representation. This means $\hat{\rho}$ is injective. We want to prove ρ is injective. Let $g \neq h$, where $g, h \in G$. We know that if

$$\sum_{g\in G} \lambda_g g \neq \sum_{h\in G} \lambda_h h,$$

then

$$\hat{\rho}\big(\sum_{g\in G} \lambda_g g\big) \neq \hat{\rho}\big(\sum_{h\in G} \lambda_h h\big).$$

If $g \neq h$, then $g \neq h$ also as elements of $\mathbb{C}G$. So, $\hat{\rho}(g) \neq \hat{\rho}(h)$. Therefore, $\rho(g) \neq \rho(h)$. □

Caveat: It may happen that a representation is group-faithful without it being algebra faithful.

Example 2.3.7. Let $S_2 = \{\mathrm{id}, (12)\}$ be the symmetric group of order 2. Suppose that $V = \{(x, y) \in \mathbb{R}^2 : x + y = 0\}$. A representation of symmetric group S_2 on V is given by the map

$$\begin{aligned} \rho\colon S_2 \times V &\to V, \\ (g, v) &\mapsto g.v \end{aligned}$$

where for all $(x, y) \in V$, we have

$$\begin{aligned} \rho(id)(x, y) &= (x, y) \\ \rho((12))(x, y) &= (y, x). \end{aligned}$$

It is clear that ρ is group faithful. We now want to show that ρ is not algebra faithful. Consider the algebra map

$$\begin{aligned} \hat{\rho}\colon \mathbb{C}S_2 &\to \mathrm{End}(V) \\ \sum_{g\in S_2} \alpha_g g &\mapsto \sum_{g\in S_2} \alpha_g \rho(g). \end{aligned}$$

Let $id + (12) \in \mathbb{C}S_2$ and $0 \in \mathbb{C}S_2$. We do this by nothing that:

$$id + (12) \neq 0$$

but

$$\hat{\rho}\big((id + (12)\big) = \hat{\rho}\big(0\big)$$

Let us start for all $(x, y) \in \mathbb{R}^2$, we have

$$\begin{aligned}(id + (12))(x, y) &= id(x, y) + (12)(x, y)\\ &= (x, y) + (y, x)\\ &= (x + y, y + x)\\ &= (0, 0)\end{aligned}$$

and

$$0.(x, y) = (0, 0).$$

Lemma 2.3.8. *Let G be a group.*

1. *Suppose that V and W are isomorphic representations of G. If V is group faithful (resp. algebra faithful), then W is group faithful (resp. algebra faithful).*
2. *Suppose that W is a subrepresentation of V. If W is group faithful (resp. algebra faithful), then V is group faithful (resp. algebra faithful).*

Proof. 1. Let $\rho\colon G \to \mathrm{GL}(V)$ be a faithful group representation and $\rho^{'}\colon G \to \mathrm{GL}(W)$ be two group representations. Since V and W are isomorphic representation of G, then there is an isomorphism map $\phi\colon V \to W$ such that $\rho^{'}(g) = \phi \circ \rho(g) \circ \phi^{-1}$.

   We need to prove that $\rho^{'}$ is group faithful. This means we need to show that for all $g_1, g_2 \in G$, if $g_1 \neq g_2$, then $\rho^{'}(g_1) \neq \rho^{'}(g_2)$. By assumptions that ρ is faithful, then $\rho(g_1) \neq \rho(g_2)$. This implies that $\phi \circ \rho(g_1) \circ \phi^{-1} \neq \phi \circ \rho(g_2) \circ \phi^{-1}$. Hence, $\rho^{'}(g_1) \neq \rho^{'}(g_2)$. Therefore, $\rho^{'}$ is faithful.

   It is a similar with regarding to faithful algebra.

2. Let $\rho_{|W}\colon G \to GL(W)$ be a subrepresentation of $\rho\colon G \to GL(V)$. This means for $g \in G, w \in W$ we have $\rho_{|W}(g)(w) = \rho(g)(w)$. Suppose that W is a group faithful. We need to prove that V is also group faithful.

Assume that $g_1 \neq g_2$, where $g_1, g_2 \in G$. Since W is group faithful, then $\rho_{|W}(g_1) \neq \rho_{|W}(g_2)$. Hence, $\rho(g_1) \neq \rho(g_2)$. Therefore, V is group faithful.

It is a similar with regarding to faithful algebra.

□

Now, consider the left action of G on itself, by left multiplication: $g.x = gx$. The linearised representation of G associated to G-set G, has underlying vector space $\mathbb{C}(G)$. Moreover:

$$g \cdot \sum_{x \in G} \sigma_x x = \sum_{x \in G} \sigma_x (g \cdot x),$$

for all $g \in G$ and $\sum_{x \in G} \sigma_x x \in \mathbb{C}(G)$. This gives exactly the regular representation of G, below:

Definition 2.3.9. (See for example [S+16, Definition 4.4.1].) The left-regular representation of G is defined as being the representation of G on $\mathbb{C}G$, seen as a vector space, obtained from the left-action of G on G by left multiplication.

Using another notation, the left-regular representation of G is the homomorphism $L\colon G \to \mathrm{GL}(\mathbb{C}G)$, where $g \mapsto L_g$ defined as

$$L_g \sum_{h \in G} \alpha_h h = \sum_{h \in G} \alpha_h (g.h),$$

for $g, h \in G$. Notice that on a basis element $h \in G$, we have $L_g h = gh$. Noting that this representation has the well-known decomposition

$$D_1^{dim(D_1)} \oplus \cdots \oplus D_s^{dim(D_s)}$$

where $\{D_1, \ldots, D_s\}$ is a complete set of pairwise non-isomorphic irreducible $\mathbb{C}G-$modules.

Lemma 2.3.10. *The left-regular representation of G is algebra faithful.*

Proof. Let

$$L\colon G \to \mathrm{GL}(\mathbb{C}G)$$

be the left regular representation of the group G. We need to prove that L is algebra faithful by showing that the algebra map $\hat{L}\colon \mathbb{C}G \to \mathrm{End}(\mathbb{C}G)$ is injective. In other words, we need to show that if

$$\sum_{g\in G}\lambda_g g \neq \sum_{g\in G}\mu_g g,$$

then

$$\hat{L}\Big(\sum_{g\in G}\lambda_g g\Big) \neq \hat{L}\Big(\sum_{g\in G}\mu_g g\Big).$$

Now, suppose $h \in G$ and

$$\sum_{g\in G}\lambda_g g \neq \sum_{g\in G}\mu_g g,$$

then we have

$$\hat{L}\Big(\sum_{g\in G}\lambda_g g\Big)(h) = \sum_{g\in G}\lambda_g(gh)$$

and

$$\hat{L}\Big(\sum_{g\in G}\mu_g g\Big)(h) = \sum_{g\in G}\mu_g(gh).$$

Take $h = id_G$. Since

$$\sum_{g\in G}\lambda_g g \neq \sum_{g\in G}\mu_g g,$$

then this means there is at least one of elements of group g such that:

$$\lambda_g \neq \mu_g.$$

Therefore,

$$\hat{L}\Big(\sum_{g\in G}\lambda_g g\Big) \neq \hat{L}\Big(\sum_{g\in G}\mu_g g\Big).$$

□

2.4 A resumé of the representation theory of symmetric group S_n

Recall we considered the symmetric group and its actions in Definition 2.2.1 and Lemma 2.2.4.

We introduce some the representation theory of symmetric group S_n concepts and some examples. In addition, we recall the construction of all isomorphism classes of irreducible representation of the symmetric group S_n, which we realise as Specht modules.

2.4.1 Young Diagrams and Young Tableaux

Definition 2.4.1. (See for example [Jam06, Definition 2.2] and [Ful97, pp.1].) A *partition* of a positive integer n is given by an integer $m \in \{1, 2, \ldots, n\}$ and an m-tuple of positive integers, $(\lambda_1, \lambda_2, \ldots, \lambda_m)$, such that:

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m,$$

and

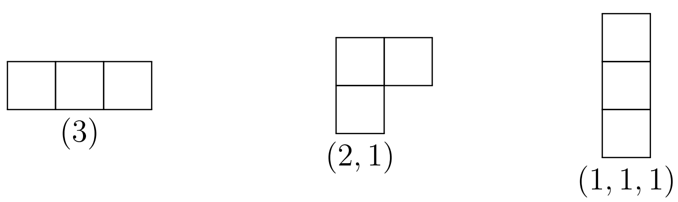
$$\lambda_1 + \lambda_2 + \cdots + \lambda_m = n.$$

We write $\lambda \vdash n$ to denote that λ is a partition of n. The set of partitions of n is denoted by P_n.

Example 2.4.2. The number 3 has three partitions, namely $(3), (2, 1), (1, 1, 1)$.

Definition 2.4.3. (See for example[FH13, pp.45] and [Ste12, Definition 10.1.4].) A *Young diagram*, D, is a finite collection of square boxes arranged in left-justified rows, with the row sizes weakly decreasing. The Young diagram associated to the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ is the one that has m rows, and λ_i boxes on the i^{th} row, for $i = 1, \ldots, m$.

Example 2.4.4. The Young diagrams corresponding to the partitions of 3, namely (3), $(2, 1)$, $(1, 1, 1)$, are:



Definition 2.4.5. (See for example [FH13, pp.45] and [Jam06, Definiton 3.6].) A *Young tableau*, t, is a Young diagram D, with n boxes, in which the n boxes have been filled with the numbers $1, \ldots, n$, with each number used exactly once. We say that the Young tableau t has the shape D. The set of Young tableax with shape D is denoted by $\mathcal{T}(D)$.

Example 2.4.6. These are all the tableaux corresponding to the partition $(2, 1)$, of $n = 3$:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$$

Given a Young diagram D, with n boxes, clearly has $n!$ Young tableaux with shape D.

Definition 2.4.7. (See for example [Ful97, pp.2].) A *standard Young tableau* is a Young tableaux whose entries are increasing across each row and each column.

Example 2.4.8. The only standard tableaux with shape the Young diagram given by the partition $(2, 1)$, of 3, are:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$$

Definition 2.4.9. A canonical Young tableau is the standard Young tableau that fills boxes of the given Young diagram with positive integers $1, 2, \dots, n$ in a lexicographical ordered way.

E.g.

$$t = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 \\ \cline{1-2} 9 \\ \cline{1-1}\end{array}$$

2.4.2 An action of S_n on the set of Young tableaux

Lemma 2.4.10. *(See for example [Jam06, pp.9] and [Ful97, pp. 83-84].) Let n be a positive integer. Let D be a Young diagram with n boxes. Let $\mathcal{T}(D)$ be the set of Young tableaux with shape D. We have an action, of S_n on $\mathcal{T}(D)$, denoted*

$$\begin{aligned} S_n \times \mathcal{T}(D) &\to \mathcal{T}(D) \\ (f, t) &\mapsto f.t, \end{aligned}$$

such that:

> *If $f \in S_n$ and $t \in \mathcal{T}(D)$, then if a certain box of t is filled in by i then the corresponding box of $f.t$ is filled in by $f(i)$.*

Example 2.4.11. Let $id, (12), (123) \in S_3$ and

$$t = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}.$$

Then the action of S_3 on t is such that:

$$\begin{aligned}
id.\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} &= \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \\
(12).\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} &= \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} \\
(123).\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array} &= \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \multicolumn{1}{|c}{} \\ \cline{1-1} \end{array}
\end{aligned}$$

Definition 2.4.12. (See for example [Sag13, Definition 2.1.2].) Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$. Then the corresponding Young subgroup of S_n is

$$S_\lambda = S_{\{1,2,\ldots,\lambda_1\}} \times S_{\{\lambda_1+1,\lambda_1+2,\ldots,\lambda_1+\lambda_2\}} \times \cdots \times S_{\{n-\lambda_m+1,n-\lambda_m+2,\ldots,n\}}$$

These subgroups are named in honor of the Reverend Alfred Young who was among the first to construct the irreducible representation of S_n[You28] and [You29].

Example 2.4.13.

$$\begin{aligned}
S_{(3,2,2,1)} &= S_{\{1,2,3\}} \times S_{\{4,5\}} \times S_{\{6,7\}} \times S_{\{8\}} \\
&\cong S_3 \times S_2 \times S_2 \times S_1.
\end{aligned}$$

We have that $S_{(\lambda_1,\lambda_2,\ldots\lambda_m)}$ and $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_m}$ are isomorphic as groups [Sag13, pp.54].

Definition 2.4.14. (See for example [Sag13, Definition 2.3.1].) Let $\lambda \in P_n$, and let t be Young tableau of λ has rows $R_1, R_2, \ldots, R_l$ and columns $C_1, C_2, \ldots, C_k$. The row stabilizer of t is

$$R_t = S_{R_1} \times S_{R_2} \times \cdots \times S_{R_l}.$$

The column stabilizer of t is

$$C_t = S_{C_1} \times S_{C_2} \times \cdots \times S_{C_k}.$$

Noting that R_t and C_t are subgroups of S_n. The labels in the i^{th} row of t define a subset R_i of $\{1, 2, 3, ..., n\}$ and the labels in the i^{th} column of t define a subset C_i of $\{1, 2, 3, ..., n\}$.

Given a set X, non-empty, then a partition $\mathbf{Q}$ of X is a set of non-empty, pairwise disjoint subsets of X, whose union is X. (This must not be confused with the notion of a partition of an integer.)

Given a partition $\mathbf{Q}$ of X we put:

$$S_{\mathbf{Q}} = \{f \in \mathrm{Sym}(X) \mid \forall A \in \mathbf{Q}, f(A) = A\}.$$

So, we will use $S_{\mathbf{Q}}$ instead of $S_{R_1} \times S_{R_2} \times \cdots \times S_{R_l}$ and $S_{C_1} \times S_{C_2} \times \cdots \times S_{C_k}$ as shown in the example below 2.4.15.

Now, we will explain the condition for permutation $\sigma \in S_n$ how to preserve every row or column as follows:

The labels in the i^{th} column of t define a subset C_i of $\{1, 2, 3, ..., n\}$. A permutation $\sigma \in S_n$ belongs to C_t if, and only if:

$$\sigma(C_i) = C_i,$$

for each $i \in \{1, \ldots, m\}$. Similarly, for the row stabiliser.

Example 2.4.15. Let $\lambda = (2, 1)$ and $t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$. Then we have:

$$\begin{aligned} R_t &= S_{(\{1,2\})} \times S_{(\{3\})} \cong S_{(\{1,2\},\{3\})} \\ &= \{id, (12)\} \end{aligned}$$

and

$$\begin{aligned} C_t &= S_{(\{1,3\})} \times S_{(\{2\})} \cong S_{(\{1,3\},\{2\})} \\ &= \{id, (13)\}. \end{aligned}$$

2.4.3 Young Tabloids and Specht module

Recall that the action of S_n on the set $X_n = \{1, 2, \ldots, n\}$, where $f.x = f(x)$. By linearising, this gives the defining representation of S_n denoted by $\mathbb{C}(X_n)$. The latter representation is not irreducible.

In this section, we will recall the construction of other representations of S_n, which are irreducible. This is done by using certain equivalence classes of tableaux, that are called tabloids.

Definition 2.4.16. (See for example [Sag13, Definition 2.1.4].) Two tableaux t_1 and t_2 are row equivalent, $t_1 \sim t_2$, if corresponding rows of the two tableaux contain the same elements.

Definition 2.4.17. (See for example [Jam06, Definition 3.9.].) A *Young tabloid* is an equivalence class of Young tableaux under the row equivalent relation $\sim$.

We write $[t]$, the equivalence class of t, as t without vertical lines between rows, as shown in the example below.

Example 2.4.18. Consider $\lambda = (2, 1) \in P_3$ and t as defined in Example 2.4.15 Then a Young tabloid will be

$$[t] = \begin{array}{l} \hline 1 \quad 2 \\ \hline 3 \\ \hline \end{array} = \left\{ \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 3 \\ \cline{1-1} \end{array} \right\}$$

Notation 2.4.19. Suppose $\lambda \vdash n$. We let

- Ω^λ denote the set of $\lambda - tabloids$.
- M^λ denote the free vector space on Ω^λ, so a basis of M^λ is the set of λ-tabloids.

Let n be a positive integer. Let $\lambda \in P_n$. For $\sigma \in S_n$ and $[t] \in \Omega^\lambda$, there is the action of S_n on $[t]$ as putting :

$$\sigma.[t] := [\sigma.t]. \tag{2.1}$$

The next lemma shows that this operation is well-defined, in other words it is independent of the chosen equivalence class representative.

Lemma 2.4.20. *(See for example[Jam06, pp. 13.].) Let $\lambda \in P_n$, let t_1 and t_2 be $\lambda-$tableaux, $\sigma \in S_n$. If $[t_1] = [t_2]$ then $[\sigma.t_1] = [\sigma.t_2]$.*

As a consequence we have a representation of S_n on M^λ, see Definition 2.2.11. [Jam06, p.p 13.]

Proposition 2.4.21. *(See for example [Jam06] 4.2.) Suppose $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, we have*

$$\dim\left(M^\lambda\right) = \frac{n!}{\lambda_1!\lambda_2!\ldots\lambda_m!}$$

Definition 2.4.22. (See for example [Jam06, Definition 4.3].) Let $\lambda \in P_n$, and let t be a λ-tableau with column stabilizer C_t. Set

$$k_t := \sum_{\sigma \in C_t} \operatorname{sgn}(\sigma)\sigma \in \mathbb{C}(S_n)$$

and define

$$\begin{aligned} e_t &:= k_t.[t] \\ &= \sum_{\sigma \in C_t} \operatorname{sgn}(\sigma)[\sigma.t] \in M^\lambda. \end{aligned}$$

So, e_t is called a $\lambda-$polytabloid. If t is a standard tableau, then e_t is called a standard $\lambda-$ polytabloid.

Example 2.4.23. Let $\lambda = (2, 1) \vdash 3$ and t as given in Example 2.4.15. Then the column stabilizer will be $C_t = \{id, (13)\}$ and $\lambda-$polytabloid will be

$$e_t = \begin{array}{cc} \hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array} - \begin{array}{cc} \hline 3 & 2 \\ \hline 1 & \\ \cline{1-1} \end{array}$$

Lemma 2.4.24. *(See for example [Jam06, pp.14].) Let t be a λ-tableau and $\sigma \in S_n$ be a permutation. Then*

$$\sigma.e_t = e_{\sigma.t}.$$

Proof. It can be found in [Sag13, Lemma2.2.3, pp.61] □

Definition 2.4.25. (See for example [Jam06, Definition4.3] and [Sag13, Definition 2.3.4].) The *Specht module* S^λ for the partition λ is the submodule of M^λ spanned by λ polytabloids, e_t, where t is λ-tableaux

Theorem 2.4.26. *(See for example [FH13, Lemma 4.25] .)*

1. *Given $\lambda \vdash n$, S^λ is an irreducible representation of S_n,*
2. *If $\lambda \neq \lambda'$, then the irreducible representations $S^\lambda$and $S^{\lambda'}$ are not isomorphic.*
3. *Given any representation, irreducible, of S_n, there exists a unique λ such that the irreducible representation is isomorphic to Specht module corresponding to λ.*

In particular, the Specht modules S^λ over $\mathbb{C}$, are a complete list of irreducibl representations of symmetric group S_n over $\mathbb{C}$, up to isomorphism.

Lemma 2.4.27. *(See for example [Jam06] Example 14.4) Let n, m be positive integers with $n - m \geq m$ and $\lambda = (n - m, m)$. Then*

$$\dim\ S^{(n-m,m)} = \binom{n}{m} - \binom{n}{m-1}$$

2.4.4 Young symmetrizers

In this subsection we follow for example[MG18]. Let n be a positive integer. Let us recall $\lambda = (\lambda_1, \ldots, \lambda_m) \in P_n$, be a partition of n and D be the corresponding Young diagram. let t be a tableau has the shape D. From Definition 2.4.14 we have two subgroups of the symmetric group, the column and row stabilisers, C_t, and R_t.

Note that C_t and R_t depend not only on the diagram D, supporting t, but explicitly in the labels in t. If $t \sim t'$ then $C_t = C_{t'}$, but we do not necessarily have $R_t = R_{t'}$.

Notation 2.4.28. Given a partition, we put $C_\lambda := C_t$ and $R_\lambda := R_t$ where t is the canonical Young tableaux has the shape D, associated to λ.

Let λ be a partition of n. Now refer to [FH13, Eq (4.1) and (4.2) in pp 46], we define two elements in the group algebra $\mathbb{C}S_n$ corresponding to an arbitrary λ-Young tableau t, namely:

$$A_t = \sum_{g \in R_t} g \in \mathbb{C}S_n$$

and

$$B_t = \sum_{g \in C_t} \operatorname{sgn}(g) g \in \mathbb{C}S_n.$$

Define the Young symmetriser, associated to t, as follows :

$$N_t = A_t.B_t \in \mathbb{C}S_n$$

Example 2.4.29. Let $\lambda = (2, 1)$ and $t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$. Consider R_t and C_t as defined in Example 2.4.15. Then

$$A_t = \sum_{g \in S_{\{\{1,2\},\{3\}\}}} g \in \mathbb{C}S_3$$

and

$$B_t = \sum_{h \in S_{\{\{1,3\},\{2\}\}}} \operatorname{sgn}(h) h \in \mathbb{C}S_3.$$

Now we have the Young symmetrizer as follows:

$$\begin{aligned} N_t &= A_t \ . \ B_t \in \mathbb{C}S_3 \\ &= \sum_{g \in S_{\{\{1,2\},\{3\}\}}} g \ . \sum_{h \in S_{\{\{1,3\},\{2\}\}}} \operatorname{sgn}(h) h \\ &= (id + (12))(id - (13)) \\ &= id + (12) - (13) - (132). \end{aligned}$$

Proposition 2.4.30. *(See for example [FH13, Theorem 4.3] and [JK81, Theorem 3.1.10, Theorem 2.3.21].) Let t be a Young tableau, with support D, a Young diagram corresponding to a partition $(\lambda_1, \ldots, \lambda_m)$ of n, a positive integer.*

Then, N_t is an preidempotent, more precisely:

$$N_t^2 = \alpha_t N_t,$$

where

$$\alpha_t = \frac{n!}{\dim(S^\lambda)}$$

We denoted $\frac{N_t}{\alpha_t}$ *by* E_t *that is idempotent associated to Young tableau* t*.*

Proof. See e.g. [FH13, Theorem 4.3, on pp 46]. □

Let we have a representation of group algebra $\mathbb{C}G$ as defined in Proposition 2.3.2. So we get the following theorem :

Theorem 2.4.31. *(See for example[FH13, Theorem 4.3] and [JK81].) Let,* λ *be partition of* n *and let* t *be a tableaux supported on* λ*. Let* $\mathbb{C}S_n$ *be, as before the group algebra, of* S_n*, and* $N_t \in \mathbb{C}S_n$ *be Young symmetrizer.*

1. *The image of* N_t*, under left-multiplication by* $\mathbb{C}S_n$*, namely:*

$$V^t := \{mN_t \mid m \in \mathbb{C}S_n\}$$

   is an irreducible representation of S_n*.*

2. *Moreover,* V^t *is isomorphic to* S^λ *defined in 2.4.25*

Proof. The first bit is in [FH13, Theorem 4.3]. The second bit follows by [JK81, 3.1.10 Theorem and 7.14 Lemma] □

The previous theorem hence gives an alternative construction of the Specht modules.

Definition 2.4.32. Let E_t and $E_{t'}$ two idempotents associated to Young tableau t and t' respectively. They are called orthogonal if their product is zero.

Definition 2.4.33. (See for example [Coh75] .) Let $\mathbb{C}G$ be semisimple. An idempotent is a primitive if it cannot be written as a sum of nonzero orthogonal idempotents.

Theorem 2.4.34. *(See for example [CR81, Proposition 9.14,(iii)].) If we have a primitive idempotent* E *in a semisimple algebra, its image* $\rho(E)$ *in one irreducible* ρ *has trace one and the image is zero in all other irreducible.*

Theorem 2.4.35. *Let n be a positive integer. Let $\lambda \vdash n$ be a partition of n. Let t be a canonical Young tableau, with support D, a Young diagram corresponding to a partition λ. Let (V, ρ) be any finite dimensional representation of S_n. Then the number of times that S^λ appears in the decomposition of V in irreducibles is the trace of the map*

$$\begin{aligned} L \colon V &\longrightarrow V \\ v &\longmapsto E_t.v, \end{aligned}$$

where E_t is defined in 2.4.30.

Proof. Let

$$V \cong \bigoplus_{\mu \vdash n} m_\mu S^\mu$$

be the decomposition of V into a direct sum of irreducible representation of S_n, where S^μ is the irreducible representation corresponding partition μ as defined in 2.4.26 and m_μ is a multiplicity of S^μ on V. We need to prove that the trace of L, denoted by $\operatorname{Tra}(L)$, is equal m_μ.

Since $\mathbb{C}S_nE_t.V$ is a left module of the group algebra $\mathbb{C}S_n$, then

$$\mathbb{C}S_nE_t.V \cong \bigoplus_{\mu} m_\mu \mathbb{C}S_nE_tS^\mu$$

is an isomorphism of modules and

$$\begin{aligned} E_t.V &\cong E_t \bigoplus_{\mu} m_\mu S^\mu \\ &\cong \bigoplus_{\mu} m_\mu E_tS^\mu \end{aligned}$$

is an isomorphism of vector spaces. Since the trace of E_t on S^μ from 2.4.34 is equal 1 if $\mu = \lambda$ and 0 otherwise (E_t is primitive by the property of $\mathbf{f}_{\lambda'}\mathbb{C}S_n\mathbf{e}_\lambda$ noted in the

Introduction — see e.g. [Mar08]),therefore,

$$\begin{aligned}\mathrm{Tra}(L) &= \sum_{\mu} \delta_{\lambda,\mu} m_{\mu} \\ &= m_{\lambda}\end{aligned}$$

□

2.4.5 Young's Rule

Let λ be a partition of n, a positive integer.

Definition 2.4.36. ([Sag13], Definition 2.9.1) A *generalized Young tableau* of shape λ, is an array t obtained by replacing the nodes of λ with positive integers, repetitions allowed. The type or content of T is the composition $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, where μ_i equals the number of $i's$ in t. Let

$$T_{\lambda\mu} = \{t : t \text{ has shape } \lambda \text{ and content } \mu\}$$

For example

$$\begin{array}{|c|c|c|}\hline 3 & 1 & 3 \\ \hline 1 & 4 \\ \cline{1-2}\end{array}$$

is an element of $T_{(3,2)(2,0,2,1)}$.

Definition 2.4.37. ([Sag13], Definition 9.2.5) A generalized tableau is *semistandard* if its rows weakly increase and its columns strictly increase. We let $\mathcal{T}^*_{\lambda\mu}$ denote the set of semistandard λ-tableaux of type μ.

For example,

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline 2 & 4 \\ \cline{1-2}\end{array} \in T_{(3,2)(2,1,0,2)}.$$

is semistandard, however

$$\begin{array}{|c|c|c|}\hline 4 & 1 & 1 \\ \hline 2 & 4 \\ \cline{1-2}\end{array} \in T_{(3,2)(2,1,0,2)}.$$

is not semistandard.

Definition 2.4.38. ([Sag13], Definition 2.11.1) The Kostka numbers are

$$\mathcal{K}_{\lambda\mu} = |\mathcal{T}^*_{\lambda\mu}|.$$

Theorem 2.4.39. *[Sag13, Theorem 2.11.5.]. (**Young's Rule**)The multiplicity of S^λ in M^μ is equal to the number of semistandard tableaux of shape λ and content μ, i.e.,*

$$M^\mu \cong \oplus_\lambda \mathcal{K}_{\lambda\mu} S^\lambda.$$

Example 2.4.40. ([Jam06, Example 14.2.]) Let $n = 7$ and $\mu = (3, 2, 2)$. Then

$$M^{(3,2,2)} \cong S^{(7)} \oplus 2S^{(6,1)} \oplus 3S^{(5,2)} \oplus 2S^{(4,3)} \oplus S^{(5,1,1)} \oplus 2S^{(4,2,1)} \oplus S^{(3,3,1)} \oplus S^{(3,2,2)}$$

Example 2.4.41. Let $n = 5$ and $\mu = (2, 2, 1)$. Then

$$M^{(2,2,1)} \cong S^{(5)} \oplus 2S^{(4,1)} \oplus S^{(3,2)} \oplus S^{(2,2,1)}$$

Example 2.4.42. Let $m \geq 1$ and $\mu = (m, 1)$. Then we have

$$M^{(m,1)} \cong S^{(m+1)} \oplus S^{(m,1)}$$

Example 2.4.43. Let $m \geq n$ and $\mu = (m, n)$. Then we have

$$M^{(m,n)} \cong S^{(m+n)} \oplus S^{(m+n-1,1)} \oplus S^{(m+n-2,2)} \oplus \cdots \oplus S^{(m,n)}$$

2.5 Warm up: the decomposition of the action of S_n on $\mathbb{C}(X_n)$ with $|X_n| = n$ into irreducibles

Let $n \in \mathbb{N}$ and $X_n := \{1, \ldots, n\}$. As a preparation for similar calculations that we will do later, let us use the result in Theorem 2.4.35 to prove the following well known result.

Proposition 2.5.1. *Let n be a positive integer and put $X_n := \{1, 2, \ldots, n\}$. Let*

$V = \mathbb{C}(X_n)$, and consider the representation ρ of S_n on V derived from linearising the action, as given in (2.2.4), of S_n on X_n, such that $f.a = f(a)$, if $f \in S_n$ and $a \in X_n$.

Consider

$$(n) = \underbrace{\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{\dots}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}}_{n}$$

The Specht module $S^{(n)}$ occurs exactly once in the decompositions of ρ into irreducibles.

Note that $S^{(n)}$ is the 1-dimensional trivial module.

Proof. Let

$$t = \boxed{1}\boxed{2}\boxed{3}\boxed{\dots}\boxed{n}$$

be the canonical Young tableau, supported in the Young diagram given by the partition $(n) \vdash n$.

Then:

$$\begin{aligned} E_t =& \frac{\dim(S^{(n)})}{n!} \sum_{g \in R_t} g \sum_{h \in C_t} \operatorname{sgn}(h)h \\ =& \frac{1}{n!} \sum_{g \in S_n} g \in \mathbb{C}S_n. \end{aligned}$$

Now we want to compute the trace of the linear map

$$\begin{aligned} L \colon V &\longrightarrow V \\ v &\longmapsto E_t.v \end{aligned}$$

and prove it is equal 1.

We use the fact that if $f, f' \colon V \to V$ are linear maps, and $a, a' \in \mathbb{C}$ then $\operatorname{Tra}(af+$

$a'f') = a\,\mathrm{Tra}(f) + a'\,\mathrm{Tra}(f')$. We have:

$$\begin{aligned}\mathrm{Tra}(L) &= \mathrm{Tra}\Big(\rho(E_t)\Big)\\ &= \mathrm{Tra}\Big(\frac{1}{n!}\sum_{g\in S_n}\rho(g)\Big)\\ &= \frac{1}{n!}\sum_{g\in S_n}\mathrm{Tra}\big(\rho(g)\big).\end{aligned}$$

So we need now to compute the value of $\mathrm{Tra}\big(\rho(g)\big)$. We do it as follows:

Given $g \in S_n$, then, by using proposition 2.1.3

$$\mathrm{Tra}\big(\rho(g)\big) = |\{x \in X_n | g.x = x\}|$$

Thus we have the following:

$$\begin{aligned}\mathrm{Tra}(L) &= \frac{1}{n!}\sum_{g\in S_n}\mathrm{Tra}\big(\rho(g)\big)\\ &= \frac{1}{n!}\sum_{g\in S_n}|\{x \in X_n | g.x = x\}|\\ &= |X_n/S_n|,\end{aligned}$$

by using Burnside's Lemma (see for example [Ste12], Chapter7, Corollary 7,2,9).

So, $\mathrm{Tra}(L)$ is the number of orbits of the action S_n on X_n. The value of X_n/S_n is equal to 1, because, given any $x \in X_n$:

$$\begin{aligned}\mathrm{Orb}_{S_n}(x) &= \{f(x) | f \in S_n\}\\ &= X_n,\end{aligned}$$

because given any $x, y \in X_n$, there exists a bijecton $f\colon X_n \to X_n$ such that $f(x) = y$. □

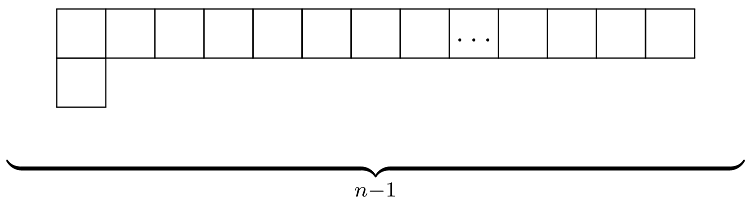
Notation 2.5.2. $\iota\colon S_{n-1} \to S_n$ is the inclusion homomorphism that means:

Let $f \in S_{n-1}$ so $f\colon X_{n-1} \to X_{n-1}$ is a bijection. Then $\iota(f) \in S_n$ is the bijection $\iota(f)\colon X_n \to X_n$, such that:

$$\iota(f)(a) = \begin{cases} f(a), & a \in X_{n-1} \\ a, & a = n. \end{cases}$$

Proposition 2.5.3. *Let n be a positive integer and $X_n = \{1, 2, \ldots, n\}$. Let $V = \mathbb{C}(X_n)$, and consider the representation ρ of S_n on V derived from the action of S_n on X_n, such that $f.a = f(a)$, if $f \in S_n$ and $a \in X_n$.*

Consider

$(n-1, n) =$ 

The Specht module $S^{(n-1,1)}$ occurs exactly once in the decompositions of ρ into irreducibles.

In the following proof, we will denote each element of X_n as $[a]$, where $a \in \{1, \ldots, n\}$.

Proof. Let

$$t = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & \ldots & n{-}1 \\ \hline n \\ \cline{1-1}\end{array}$$

be the canonical Young tableau, supported in the Young diagram given by the partition $(n-1, 1) \vdash n$. Then:

$$\begin{aligned} E_t &= \frac{\dim\left(S^{(n-1,1)}\right)}{n!} \sum_{g \in R_t} g \sum_{h \in C_t} \operatorname{sgn}(h) h \\ &= \frac{n-1}{n!} \sum_{g \in S_{n-1}} \iota(g).(id - (1n)), \text{by Lemma 2.4.27,} \end{aligned}$$

where, $\iota(g)$ defined as in Notation 2.5.2.

Now we want to compute the trace of linear map

$$\begin{aligned} L\colon V &\longrightarrow V \\ v &\longmapsto E_t\,.v, \end{aligned}$$

and prove it is equal 1.

Let us chose the basis X_n of V. Let us determine the value of

$$\underbrace{\left(\sum_{g \in S_{n-1}} \iota(g).(id - (1n)\,) \right)}_{K}.[a]$$

for $a \in X_n$. We consider several different cases.

- Suppose $[a] \in \{2, \ldots, n-1\}$, then $K.[a] = 0$, because

$$\begin{aligned} \big(\mathrm{id} - (1\ n)\big).[a] &= id.[a] - (1\,n).[a] \\ &= [a] - [a] \\ &= 0. \end{aligned}$$

- Suppose $a = [1]$. Then

$$\big(\mathrm{id} - (1\ n)\big).[1] = [1] - [n]$$

and

$$\begin{aligned} \sum_{g \in S_{n-1}} \iota(g).([1] - [n]) &= \sum_{g \in S_{n-1}} \iota(g).[1] &&- \sum_{g \in S_{n-1}} \iota(g).[n] \\ &= \qquad A &&- \qquad B. \end{aligned}$$

We compute each term separately, noting that we just need to find out the coefficient of the basis element [1], for it is only contribution for $\mathrm{Tra}(L)$.

  - The coefficient of [1] in A is the number of permutations in S_{n-1} that fix 1. These are exactly $(n-2)!$ of them.
  - Now, if $g \in S_{n-1}$, then $\iota(g).[n] \neq [1]$. Thus, the coefficient of [1], in B, is zero.

- Suppose $a = [n]$, then

$$\big(id - (1\ n)\big).\, [n] = [n] - [1]$$

and

$$\begin{aligned} \sum_{g \in S_{n-1}} \iota(g).([n] - [1]) &= \sum_{g \in S_{n-1}} \iota(g).[n] &&- \sum_{g \in S_{n-1}} \iota(g).[1] \\ &= \quad A &&- \quad B. \end{aligned}$$

We compute each term separately, noting that we just need to find out the coefficient of the basis element $[n]$, for it is only contribution for $\mathrm{Tra}(L)$.

  - If $g \in S_{n-1}$, then $\iota(g).[n] = [n]$. So $A = |S_{n-1}|.[n]$. Thus the coefficient of $[n]$ is $(n-1)!$.
  - If $g \in S_{n-1}$, then $\iota(g).[1] \neq [n]$. Thus, the coefficient of $[n]$ is zero.

So we have the following table to show the contributions for the trace of L:

$v \in X_n$	Coefficient of basis element $[v]$ in $K.[v] \in \mathbb{C}(X_n)$
$[a], \quad a \in \{2, \ldots, n-1\}$	0
$[1]$	$(n-2)!$
$[n]$	$(n-1)!$

Therefore, we have the following :

$$
\begin{aligned}
Tra(L) &= \frac{n-1}{n!}\Big((n-2)! + (n-1)!\Big) \\
&= \frac{n-1}{n!}\Big((n-2)!\big(1+n-1\big)\Big) \\
&= \frac{n-1}{n!}\Big(n(n-2)!\Big) \\
&= \frac{n(n-1)((n-2)!}{n!} \\
&= \frac{n!}{n!} \\
&= 1.
\end{aligned}
$$

□

Proposition 2.5.4. *(See for example [Jam06, Example5.1, pp.18].) Let n be a positive integer and $X_n = \{1, 2, \ldots, n\}$. Let $V = \mathbb{C}(X_n)$, and consider the representation ρ of S_n on V derived from the action of S_n on X_n, such that $f.a = f(a)$, if $f \in S_n$ and $a \in X_n$. Then*

$$V \cong S^{(n)} \oplus S^{(n-1,1)}$$

Proof. Since V is semisimple, then the decomposition of V into irreducible representation is unique form Proposition2.2.21. We know from Propositions 2.5.1 and 2.5.3 that both $S^{(n)}$ and $S^{(n-1,1)}$ appear exactly once in the decomposition into irreducible representation. Therefore, there is no space for more irreducible representation to appear. Let us now check the dimensions of them. So,

$$\dim(V) = n$$

and

$$
\begin{aligned}
\dim\left(S^{(n)} \oplus S^{(n-1,n)}\right) =& \dim\left(S^{(n)}\right) + \dim\left(S^{(n-1,1)}\right) \\
=& 1 + n - 1 \\
=& n.
\end{aligned}
$$

□

Chapter 3

Group G actions on power sets of G-sets

3.1 Notation for subsets $\mathcal{P}_i(S)$ of power sets

Definition 3.1.1. Let S be a set. The power set, $\mathcal{P}(S)$, of S is the set of all subsets of S, in other words

$$\mathcal{P}(S) := \{A : A \subseteq S\}.$$

Remark 3.1.2. *Let S be a finite set. The number of subsets of S is $|\mathcal{P}(S)| = 2^{|S|}$.*

Example 3.1.3. Recall $X_n = \{1, 2, ..., n\}$. Then

$$\mathcal{P}(X_0) = \{\emptyset\}$$

$$\mathcal{P}(X_1) = \{\emptyset, \qquad \{1\}\}$$

$$\mathcal{P}(X_2) = \{\emptyset, \quad \{1\}, \{2\}, \quad \{1, 2\}\}.$$

Notice the similarity of this sequence with Pascal's triangle.

Notation 3.1.4. Let S be a finite set. Given a non-negative integer i, we will use the following notation

$$\mathcal{P}_i(S) := \{A \subseteq S : |A| = i\} \subseteq \mathcal{P}(S).$$

Observe that

$$\mathcal{P}(S) = \bigsqcup_{i=0}^{|S|} \mathcal{P}_i(S). \tag{3.1}$$

Where the notation,

$$X = \bigsqcup_{i=1}^{|X|} A_i,$$

means that

$$X = \bigcup_{i=1}^{|X|} A_i,$$

and that given i, j, with $i \neq j$, then $A_i \cap A_j = \emptyset$.

Remark 3.1.5. *The number of subsets of S with cardinality i is $\mid \mathcal{P}_i(S) \mid = \binom{|S|}{i}$*

Example 3.1.6. Let $S = X_2 = \{1, 2\}$. Then we have

$$\begin{aligned}
\mathcal{P}_0(X_2) &= \{\emptyset\} \\
\mathcal{P}_1(X_2) &= \{\{1\}, \{2\}\} \\
\mathcal{P}_2(X_2) &= \{\{1, 2\}\}
\end{aligned}$$

Note that for any set S we have

$$\mathcal{P}_0(S) = \{\emptyset\} \tag{3.2}$$

So $\mathcal{P}_0(S)$ is never empty. And for each set S there is another set whose elements are the singleton sets each containing one element of S. This is $\mathcal{P}_1(S)$. This is empty if S is empty.

Let S be a set and let $f\colon S \to S$ be a function. For $A \subseteq S$ we will use the notation

$$f[A] = \{f(a) \ : \ a \in A\}. \tag{3.3}$$

3.2 Group actions on power sets of G-sets

Lemma 3.2.1. *Let G be a group and S be a G-set, as defined in 2.2.3 above, with action denoted:*

$$\alpha\colon G \times S \to S$$
$$(g, s) \mapsto g.s$$

(I) We have an action of G on $\mathcal{P}(S)$, given by the mapping

$$\blacktriangleright\colon G \times \mathcal{P}(S) \to \mathcal{P}(S)$$

defined as:

$$\textit{If } g \in G, \textit{then given } A \subseteq S, \textit{ then } g \blacktriangleright A = \{g.a|a \in A\}.$$

(II) For any $g \in G$, then $|g \blacktriangleright A| = |A|$.

Proof. (I) We need to prove that $\blacktriangleright$ is a group action. So, we will show the two axioms in Definition 2.2.3.

1. Let $g, h \in G$ and $A \subseteq S$. Then

$$\begin{aligned} g \blacktriangleright (h \blacktriangleright A) &= g \blacktriangleright \{h.a|a \in A\} \\ &= \{g.(h.a) : a \in A\} \\ &= \{(gh).a : a \in A\} \\ &= (gh) \blacktriangleright A. \end{aligned}$$

   So the axiom (1) holds;

2. Let Id_G is the identity in G, $A \subseteq S$. Then

$$\begin{aligned} Id_G \blacktriangleright A &= \{Id_G.a|a \in A\} \\ &= \{a : a \in A\} \\ &= A. \end{aligned}$$

   So the axiom (2) holds.

Therefore, $\blacktriangleright$ is a group action.

(II) Given that we have an action, fixing $g \in G$, the map $S \to S$ sending s to $g.s$ is a bijection. $\square$

Example 3.2.2. Let $X_n = \{1, \ldots, n\}$. We have an action of S_n on X_n, where $f.x = f(x)$. Applying the previous result this gives an action of S_n on $\mathcal{P}(X_n)$. Explicitly:

$$\blacktriangleright\colon S_n \times \mathcal{P}(X_n) \to \mathcal{P}(X_n),$$

where if $(f\colon X_n \to X_n) \in S_n$, then given $A \subseteq X_n$:

$$\begin{aligned} f \blacktriangleright A &= \{f.a \mid a \in A\} \\ &= \{f(a) \mid a \in A\} \\ &= f[A]. \end{aligned}$$

In the next lemma, we restrict the group action on $\mathcal{P}(S)$ and the group action closes on $\mathcal{P}_i(S)$.

Lemma 3.2.3. *Let G be a group and S be a G-set, as defined in 2.2.3 above, with*

$$\begin{aligned} \alpha\colon G \times S &\to S \\ (g, s) &\mapsto g.s \end{aligned}$$

Suppose $\mathcal{P}_i(S)$ is the set of subsets with cardinality i as defined in 3.1.4. We have an action of G on $\mathcal{P}_i(S)$, given by the mapping

$$\blacktriangleright\colon G \times \mathcal{P}_i(S) \to \mathcal{P}_i(S)$$

defined as:

$$\textit{If } g \in G, \textit{then given } A \in \mathcal{P}_i(S) : g \blacktriangleright A = \{g.a | a \in A\}.$$

Proof. The result follows from Lemma 3.2.1(II). $\square$

In particular, we have an action of S_n on $\mathcal{P}_i(X)$ where $f \blacktriangleright A = f[A]$.

3.2.1 Warm up: the decomposition of the action of S_n on $\mathbb{C}(\mathcal{P}_0(X_n))$ and $\mathbb{C}(\mathcal{P}_1(X_n))$ into irreducibles

In this subsection, we let $n \in \mathbb{N}$ and $X_n = \{1, \ldots, n\}$.

Lemma 3.2.4. *Let $i \in \{0, \ldots, n\}$. Consider the action $\blacktriangleright$, of S_n on $\mathcal{P}_i(X_n)$ as defined in 3.2.3. Let ρ_i be the linearised representation of S_n on $\mathbb{C}\mathcal{P}_i(X_n)$. Consider*

$$(n) = \underbrace{\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline \; & \; & \; & \; & \; & \ldots & \; & \; & \; & \; & \; \\ \hline \end{array}}_{n}$$

The Specht module $S^{(n)}$ occurs exactly once in the decompositions of ρ_i into irreducibles.

Proof. Let

$$t = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & \ldots & n \\ \hline \end{array}$$

be the canonical Young tableau, supported in the Young diagram given by the partition $(n) \vdash n$. Then:

$$\begin{aligned} E_t =& \frac{\dim S^{(n)}}{n!} \sum_{g \in R_t} g \sum_{h \in C_t} \operatorname{sgn}(h) h \\ =& \frac{1}{n!} \sum_{g \in S_n} g \in \mathbb{C}S_n. \end{aligned}$$

Using again theorem, 2.4.35 we want to compute the trace of linear map

$$\begin{aligned} F \colon \mathbb{C}\mathcal{P}_i(X_n) &\longrightarrow \mathbb{C}\mathcal{P}_i(X_n) \\ v &\longmapsto E_t \blacktriangleright v, \end{aligned}$$

and prove it is equal to 1.

By using the fact that if $f, f' \colon V \to V$ are linear maps, and $a, a' \in \mathbb{C}$ then

$\operatorname{Tra}(af + a'f') = a\operatorname{Tra}(f) + a'\operatorname{Tra}(f')$, we have

$$\begin{aligned}\operatorname{Tra}(F) &= \operatorname{Tra}\big(\rho_i(E_t)\big)\\ &= \operatorname{Tra}\Bigg(\frac{1}{n!}\sum_{g\in S_n}\rho_i(g)\Bigg)\\ &= \frac{1}{n!}\sum_{g\in S_n}\operatorname{Tra}\big(\rho_i(g)\big).\end{aligned}$$

We need now to compute the value of $\operatorname{Tra}\big(\rho_i(g)\big)$. We do it as follows, using again Proposition 2.1.3:

Given $g \in S_n$, then :

$$\begin{aligned}\operatorname{Tra}\big(\rho_i(g)\big) &= |\{A \in \mathcal{P}_i(X_n) \mid g \blacktriangleright A = A\}|\\ &= |\ \mathcal{P}_i(X_n)^g|.\end{aligned}$$

Here recall $\mathcal{P}_i(X_n)^g$ is the set of the elements of $\mathcal{P}_i(X_n)$ that are fixed by g.

Thus we have the following :

$$\begin{aligned}\operatorname{Tra}(F) &= \frac{1}{n!}\sum_{g\in S_n}\operatorname{Tra}\big(\rho_i(g)\big)\\ &= |\mathcal{P}_i(X_n)/S_n| \quad \text{by using Burnside's Lemma.}\end{aligned}$$

So, $\operatorname{Tra}(F)$ is the number of orbits of the action S_n on $\mathcal{P}_i(X_n)$. The value of $\mathcal{P}_i(X_n)$ is therefore equal to 1, because, given any $A \in \mathcal{P}_i(X_n)$

$$\begin{aligned}\operatorname{Orb}_{S_n}(A) &= \{f \blacktriangleright A | f \in S_n\}\\ &= \mathcal{P}_i(X_n),\end{aligned}$$

because any pair $A, B \in \mathcal{P}_i(X_n)$, there exists a bijecton $f\colon X_n \to X_n$ such that $f[A] = B$. □

Proposition 3.2.5. *Let $n \geq 1$. Suppose that*

$$\mathcal{P}_0(X_n) = \{A \subseteq X_n : |A| = 0\}.$$

Let $V = \mathbb{C}\mathcal{P}_0(X_n)$, and consider the representation ρ of S_n on V derived from the *action of S_n on $\mathcal{P}_0(X_n)$, such that: $f \blacktriangleright A = f[A]$ as defined in lemma 3.2.3 , if $f \in S_n$ and $A \in \mathcal{P}_0(X_n)$. Then:*

$$V \cong S^{(n)}.$$

Proposition 3.2.6. *Consider the action $\blacktriangleright$, of S_n on $\mathcal{P}_1(X_n)$ as defined in 3.2.3. Let ρ_1 be the linearised representation of S_n on $\mathbb{C}\mathcal{P}_1(X_n)$. Consider*

$$(n-1,n) = \underbrace{\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline \ & & & & & & & & \dots & & & & \\ \hline \multicolumn{1}{|c|}{\ } \\ \cline{1-1}\end{array}}_{n-1}$$

The Specht module $S^{(n-1,1)}$ occurs exactly once in the decompositions of ρ_1 into irreducibles and $n \geq 2$.

Proof. Let

$$t = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & \dots & n-1 \\ \hline \multicolumn{1}{|c|}{n} \\ \cline{1-1}\end{array}$$

be the canonical young tableau, supported in the Young diagram given by the partition $(n-1,1) \vdash n$. Then:

$$\begin{aligned} E_t &= \frac{\dim\left(S^{(n-1,1)}\right)}{n!} \sum_{g \in R_t} g \sum_{h \in C_t} sgn(h)h \\ &= \frac{n-1}{n!} \sum_{g \in S_{n-1}} \iota(g).(id - (1n)). \end{aligned}$$

where, $\iota(g)$ defined as in Notation 2.5.2.

Now we want to compute the trace of linear map

$$\begin{aligned} L \colon \mathbb{C}\mathcal{P}_1(X_n) &\longrightarrow \mathbb{C}\mathcal{P}_1(X_n) \\ v &\longmapsto E_t\,.v, \end{aligned}$$

and prove it is equal 1.

Let us chose the basis $\mathcal{P}_1(X_n)$ of $\mathbb{C}\mathcal{P}_1(X_n)$. Let us determine the value of

$$\underbrace{\left(\sum_{g\in S_{n-1}} \iota(g).(id-(1n)\,)\right)}_{K} \blacktriangleright v$$

for $v \in \mathcal{P}_1(X_n)$. We consider several different cases.

- Suppose $v=\{a\}, 1<a<n$, then $K \blacktriangleright v = 0$, because

$$\begin{aligned}\big(\mathrm{id}-(1\ n)\big) \blacktriangleright v &= id \blacktriangleright \{a\} - (1\,n).\{a\}\\ &= \{a\}-\{a\}\\ &= 0.\end{aligned}$$

- Suppose $v=\{1\}$. Then

$$\big(\mathrm{id}-(1\ n)\big) \blacktriangleright \{1\} = \{1\}-\{n\}$$

  and

$$\begin{aligned}\sum_{g\in S_{n-1}} \iota(g) \blacktriangleright (\{1\}-\{n\}) &= \sum_{g\in S_{n-1}} \iota(g) \blacktriangleright \{1\} &&- \sum_{g\in S_{n-1}} \iota(g) \blacktriangleright \{n\}\\ &= \qquad A &&- \qquad B.\end{aligned}$$

  We compute each term separately, noting that we just need to find out the coefficient of the basis element [1], for it is only contribution for $\mathrm{Tra}(L)$.

  - The coefficient of $\{1\}$ in A is the number of permutations in S_{n-1} that fix $\{1\}$. These are exactly $(n-2)!$ of them.
  - Now, if $g \in S_{n-1}$, then $\iota(g) \blacktriangleright \{n\} \neq \{1\}$. Thus, the coefficient of $\{1\}$, in B, is zero.

- Suppose $v=\{n\}$, then

$$\big(id-(1\ n)\big) \blacktriangleright \{n\} = \{n\}-\{1\}$$

  and

$$\begin{aligned}\sum_{g\in S_{n-1}} \iota(g) \blacktriangleright (\{n\}-\{1\}) &= \sum_{g\in S_{n-1}} \iota(g) \blacktriangleright \{n\} \quad - \sum_{g\in S_{n-1}} \iota(g) \blacktriangleright \{1\} \\ &= \quad A \quad - \quad B.\end{aligned}$$

We compute each term separately, noting that we just need to find out the coefficient of the basis element $\{n\}$, for it is only contribution for $\text{Tra}(L)$.

- If $g \in S_{n-1}$, then $\iota(g) \blacktriangleright \{n\} = \{n\}$. So, $A = |S_{n-1}|.\{n\}$. Thus the coefficient of $\{n\}$ is $(n-1)!$.
- If $g \in S_{n-1}$, then $\iota(g) \blacktriangleright \{1\} \neq \{n\}$. Thus, the coefficient of $\{n\}$ is zero.

Thus, we have the following table to show the contributions for the trace of L:

$v \in \mathcal{P}_1(X_n)$	Coefficient of basis element v in $K \blacktriangleright \{v\} \in \mathbb{C}\mathcal{P}_1(X_n)$
$\{a\}, \quad 1 < a < n$	0
$\{1\}$	$(n-2)!$
$\{n\}$	$(n-1)!$

Therefore we have the following :

$$\begin{aligned}Tra(L) &= \frac{n-1}{n!}\Big((n-2)! + (n-1)!\Big) \\ &= \frac{n-1}{n!}\Big((n-2)!\big(1+n-1\big)\Big) \\ &= \frac{n-1}{n!}\Big(n(n-2)!\Big) \\ &= \frac{n(n-1)((n-2)!}{n!} \\ &= \frac{n!}{n!} \\ &= 1.\end{aligned}$$

□

Proposition 3.2.7. *The map*

$$\begin{aligned} h\colon X_n &\to \mathcal{P}_1(X_n) \\ x &\mapsto \{x\} \end{aligned}$$

is bijective and it preserves the action of symmetric group, S_n.

Proof. It is obvious that h is bijective. We need to show that if $f \in S_n$, we have

$$h(f.x) = f \blacktriangleright h(x)$$

Let us start:

$$\begin{aligned} h(f.x) &= h(f(x)) \\ &= \{f(x)\}. \end{aligned}$$

Now,

$$\begin{aligned} f \blacktriangleright (h(x)) &= f \blacktriangleright \{x\} \\ &= \{f.x : x \in \{x\}\} \\ &= \{f(x)\} \end{aligned}$$

Therefore, h preserves the action of S_n. □

Recall that $f^{\#}$ is defined in Lemma 2.2.13

Proposition 3.2.8. *The map*

$$h^{\#}\colon \mathbb{C}(X_n) \to \mathbb{C}\mathcal{P}_1(X_n)$$

is an isomorphism of representations of S_n.

Proof. From Proposition 3.2.7, we have a bijective map preserves the action of S_n from X_n to $\mathcal{P}_1(X_n)$. It gives an isomorphic representation of S_n that means that $\mathbb{C}(X_n) \cong \mathbb{C}\mathcal{P}_1(X_n)$. □

Proposition 3.2.9. *Let $n \geq 2$. Suppose that*

$$\mathcal{P}_1(X_n) = \{A \subseteq X_n : |A| = 1\}.$$

Let $V = \mathbb{C}\mathcal{P}_1(X_n)$, and consider the representation ρ of S_n on V derived from the action of S_n on $\mathcal{P}_1(X_n)$, such that: $f \blacktriangleright A = f[A]$ as defined in lemma 3.2.3 , if $f \in S_n$ and $A \in \mathcal{P}_1(X_n)$. Then:

$$V \cong S^{(n)} \oplus S^{(n-1,1)}$$

Proof. We note that $\mathbb{C}(X_n)$ is isomorphic representation of S_n to $\mathbb{C}\mathcal{P}_1(X_n)$ by Proposition 3.2.8. Also, from proposition 2.5.4, we have $\mathbb{C}(X_n) \cong S^{(n)} \oplus S^{(n-1,1)}$. Therefore,

$$\mathbb{C}\mathcal{P}_1(X_n) \cong S^{(n)} \oplus S^{(n-1,1)}$$

□

We will use these type of techniques a lot in this thesis.

3.2.2 The decomposition of the action of S_n on $\mathbb{C}\mathcal{P}_i(X_n)$ into irreducibles, where $0 \leq i \leq n$.

In the previous subsection we gave elementary proofs of the decomposition of the linearisation of the action of S_n in $\mathcal{P}_i(X_n)$ into irreducibles, for $i = 0, 1$. We now deal with the general i case, by using a classical result appearing e.g. in [Jam06] and [Wil14].

Let $n \in \mathbb{N}$ and $X_n = \{1, \ldots, n\}$. Let $i \in \{0, \ldots, n\}$, with $i \leq n/2$.

Recall that $\Omega^{(n-i,i)}$ from 2.4.19 is the set of young tabloid of shape $(n-i, i)$. This has an action of S_n defined in (2.1). We recall that $M^{(n-i,i)}$ is the free vector space on $\Omega^{(n-i,i)}$ so it has linearised representation of S_n as in 2.4.3.

Theorem 3.2.10. *([Jam06, Example 17.17, pp 71].) Let $i \in \{0, \ldots, n\}$, with $i \leq n/2$. The representation $M^{(n-i,i)}$ of S_n decomposes as below, into irreducibles:*

$$M^{(n-i,i)} \cong S^{(n)} \oplus S^{(n-1,1)} \oplus S^{(n-2,2)} \cdots \oplus S^{(n-i,i)}.$$

□

Note that the computation in this theorem follows the Young Rule in Theorem 2.4.39.

Recall that

$$\mathcal{P}_i(X_n) = \{A \subseteq X_n : |A| = i\} \subseteq \mathcal{P}(X_n).$$

We will now show the decomposition of the representation of the action of S_n on $\mathbb{C}\mathcal{P}_i(X_n)$. Let us first to define two maps as follows:

Definition 3.2.11. Define the map

$$\alpha\colon \mathcal{P}_i(X_n) \to \Omega^{(n-i,i)}$$

such that given $\{x_1, \dots, x_i\} \in \mathcal{P}_i(X_n)$, then $\alpha(\{x_1, \dots, x_i\})$ is the unique Young tabloid of shape $(n-i, i)$ which contains $x_1, \dots, x_i$ in the second row.

Definition 3.2.12. let $\Omega^{(n-i,i)}$ be the set of Young tabloids of shape $(n-i, i)$. Define a map

$$\gamma\colon \Omega^{(n-i,i)} \to \mathcal{P}_i(X_n)$$

such that given $[t] \in \Omega^{(n-i,i)}$, then $\gamma([t])$ is the subset of X_n of cardinality i whose elements are the labels of the second row of $[t]$.

Proposition 3.2.13. *The maps*

$$\alpha\colon \mathcal{P}_i(X_n) \to \Omega^{(n-i,i)}$$

and

$$\gamma\colon \Omega^{(n-i,i)} \to \mathcal{P}_i(X_n)$$

are mutually inverse to each other and preserving the action of S_n.

Proof. **Firstly**, we need to prove that $\gamma\colon \Omega^{(n-i,i)} \to \mathcal{P}_i(X_n)$ which is defined in 3.2.12 is the inverse of the map $\alpha\colon \mathcal{P}_i(X_n) \to \Omega^{(n-i,i)}$ that is defined in 3.2.11 and α preserves the action of S_n.

1. We want to show that

$$\alpha \circ \gamma = Id_{\Omega^{(n-i,i)}}.$$

Let $[t] \in \Omega^{(n-i,i)}$ be an unique young tabloid of shape $(n-i, i)$ which contains

$x_1, \dots, x_i$ in the second row.

We now need to show that $\alpha(\gamma([t])) = [t]$. By definition 3.2.12 we have:

$$\gamma([t]) = \{x_1, x_2 \dots, x_i\}.$$

Since $\gamma([t]) = \{x_1, x_2 \dots, x_i\} \in \mathcal{P}_i(X_n)$, then from Definition 3.2.11 we have $\alpha(\gamma([t])) = \alpha(\{x_1, x_2 \dots, x_i\})$ is the unique young tabloid of shape $(n-i, i)$ which contains $x_1, \dots, x_i$ in the second row. Thus, $\alpha(\gamma([t])) = [t]$. Therefore, γ is the inverse of α.

2. We need to show that if $f \in S_n$ and $\{x_1, \dots, x_i\} \in \mathcal{P}_i(X_n)$, then

   $\alpha(f \blacktriangleright \{x_1, \dots, x_i\}) = f.(\alpha(\{x_1, \dots, x_i\})$. Let us start:

$$\begin{aligned} L.H.S &= \alpha(f \blacktriangleright \{x_1, \dots, x_i\}) \\ &= \alpha(\{f(x_1), \dots, f(x_i)\}) \end{aligned}$$

   This means $\alpha(\{f(x_1), \dots, f(x_i)\})$ is the unique young tabloid of shape $(n-i, i)$ which contains $f(x_1), \dots, f(x_i)$ in the second row from definition 3.2.11.

   R.H.S $= f.(\alpha(\{x_1, \dots, x_i\})$, by definition 3.2.11 we have that $\alpha(\{x_1, \dots, x_i\})$ is the unique young tabloid of shape $(n-i, i)$ which contains $x_1, \dots, x_i$ in the second row. Then when we act f on young tabloid, we have that $f.(\alpha(\{x_1, \dots, x_i\})$ is the unique young tabloid of shape $(n-i, i)$ which contains $f(x_1), \dots, f(x_i)$ in the second row. Therefore, α preserves the action of S_n.

Secondly, we need to prove that $\alpha \colon \mathcal{P}_i(X_n) \to \Omega^{(n-i,i)}$ which is defined in Definition 3.2.11 is the inverse of the map $\gamma \colon \Omega^{(n-i,i)} \to \mathcal{P}_i(X_n)$ that is defined in 3.2.12. Also, we need to show that γ is preserving the action of S_n.

We will show that

$$\gamma \circ \alpha = Id_{\mathcal{P}_i(X_n)}.$$

Let $A = \{x_1, \dots, x_i\} \in \mathcal{P}_i(X_n)$. Then by definition 3.2.11 $\alpha(A)$ is the unique young tabloid of shape $(n-i, i)$ which contains $x_1, \dots, x_i$ in the second row. Hence, by using definition 3.2.12

$$\begin{aligned}\gamma(\alpha(A)) &= \{x_1, \ldots, x_i\}\\ &= A.\end{aligned}$$

Thus, α is the inverse of γ.

□

Recall that $f^{\#}$ is defined in Lemma 2.2.13:

Proposition 3.2.14. *The map*

$$\alpha^{\#} \colon \mathbb{C}\mathcal{P}_i(X_n) \to \mathbb{C}\Omega^{(n-i,i)}$$

is an isomorphism representation of S_n.

Proof. Since there is a bijection preserving the action of S_n by Proposition 3.2.13, then we gives an isomorphism of representation of S_n that is

$$\mathbb{C}\mathcal{P}_i(X_n) \cong \mathbb{C}\Omega^{(n-i,i)}.$$

□

This means we have :

$$\mathbb{C}\mathcal{P}_i(X_n) \cong M^{(n-i,i)}.$$

Theorem 3.2.15. *Let $X_n = \{1, 2, \ldots, n\}$ and $i \in \{0, 1, 2, ..., n\}$. Then when $i \leq n - i$,*

$$\mathbb{C}\mathcal{P}_i(X_n) \cong S^{(n)} \oplus S^{(n-1,1)} \oplus S^{(n-2,2)} \cdots \oplus S^{(n-i,i)}.$$

Proof. We know that $\mathbb{C}\mathcal{P}_i(X_n) \cong \mathbb{C}\Omega^{(n-i,i)}$ from the previous proposition 3.2.14 and therefore from James's Theorem 3.2.10 we get the required. □

Recall that if $f \colon X \to X$ is a bijection, and $A \subset X$, then $f(X \setminus A) = X \setminus f(A)$.

Proposition 3.2.16. *Let n be non-negative integer. Let $i \in \{0, \ldots, n\}$. Then the bijection*

$$\mathcal{P}_i(X_n) \xrightarrow{g} \mathcal{P}_{n-i}(X_n),$$

such that

$$A \overset{g}{\mapsto} \hat{A},$$

where $\hat{A} = X_n \setminus A$, *preserves the action of* S_n. *So we have an isomorphism of representations of* S_n *if* $i > n - i$. *I.e.*

$$\mathbb{C}\mathcal{P}_i(X_n) \cong \mathbb{C}\mathcal{P}_{n-i}(X_n).$$

Proof. It is clear that g is bijective. We first need to show that if $f \in S_n$, then we have $g(f \blacktriangleright A) = f \blacktriangleright g(A)$. Let us start

$$\begin{aligned} L.H.S &= g(f \blacktriangleright A) \\ &= g(f[A]) \\ &= X_n \setminus f[A] \\ &= f[X_n \setminus A] \\ &= f[\hat{A}] \end{aligned}$$

$$\begin{aligned} R.H.S &= f \blacktriangleright g(A) \\ &= f \blacktriangleright \hat{A} \\ &= f[\hat{A}] \end{aligned}$$

Hence, g is preserving the action of S_n and therefore it gives an isomorphism of representation of S_n. □

So we can apply the previous result, which gives.

Lemma 3.2.17. *Let* $X_n = \{1, 2, \ldots, n\}$ *and* $i \in \{0, 1, 2, ..., n\}$. *Then when* $i > n-i$,

$$\mathbb{C}\mathcal{P}_i(X_n) \cong S^{(n)} \oplus S^{(n-1,1)} \oplus S^{(n-2,2)} \cdots \oplus S^{(i,n-i)}.$$

Hence, combining with Theorem 3.2.15 we have:

Lemma 3.2.18. *For all* $i \in \{0, \ldots, n\}$, *we have:*

$$\mathbb{C}\mathcal{P}_i(X_n) \cong \bigoplus_{a=0}^{\min(i,n-i)} S^{(n-a,a)}.$$

Corollary 3.2.18.1. *The decomposition of the action of S_n on $\mathbb{C}\mathcal{P}_i(X_n)$ into irreducibles has only tableaux with two rows. That is, the irreducibles are indexed by Specht modules labelled by partitions with at most two rows.*

Chapter 4

The irreducible content of the representation of S_n on some invariant subsets of $\mathcal{P}_i\mathcal{P}(X_n)$

4.1 The action of G on $\mathcal{P}(\mathcal{P}(S))$, where S is a G-set

In this section , we describe some notations we will use them later.

4.1.1 Notation for subsets $\mathcal{P}_i\mathcal{P}(S)$ of $\mathcal{P}\mathcal{P}(S)$

Notation 4.1.1. For S a set, we use the notation $\mathcal{P}\mathcal{P}(S) = \mathcal{P}(\mathcal{P}(S))$ for the power set of the power set of S. In other words,

$$\mathcal{P}\mathcal{P}(S) := \{A : A \subseteq \mathcal{P}(S)\}.$$

Remark 4.1.2. *The number of subsets of* $\mathcal{P}(S)$ *is* $|\mathcal{P}\mathcal{P}(S)| = 2^{(2^{|S|})}$.

Example 4.1.3. Let $X_2 = \{1, 2\}$. We know

$$\mathcal{P}(X_2) = \big\{\emptyset, \quad \{1\}, \quad \{2\}, \quad \{1, 2\}\big\}.$$

Therefore,

$$\begin{aligned}\mathcal{PP}(X_2) = \{&\emptyset,\\ &\{\emptyset\},\ \{\{1\}\},\ \{\{2\}\},\ \{\{1,2\}\},\\ &\{\emptyset,\{1\}\},\{\emptyset,\{2\}\},\{\emptyset,\{1,2\}\},\{\{1\},\{2\}\},\{\{1\},\{1,2\}\},\{\{2\},\{1,2\}\},\\ &\{\emptyset,\{1\},\{2\}\},\{\emptyset,\{1\},\{1,2\}\},\{\emptyset,\{2\},\{1,2\}\},\{\{1\},\{2\},\{1,2\}\},\\ &\{\emptyset,\{1\},\{2\},\{1,2\}\}\}.\end{aligned}$$

Notation 4.1.4. Let S be a set. Let i be a non-negative integer. We let $\mathcal{P}_i\mathcal{P}(S)$ be the set:

$$\mathcal{P}_i\mathcal{P}(S) := \{A \subseteq \mathcal{P}(S) : |A| = i\} \subseteq \mathcal{PP}(S).$$

Observe that

$$\mathcal{PP}(S) = \bigsqcup_{i=0}^{2^{|S|}} \mathcal{P}_i\mathcal{P}(S). \tag{4.1}$$

For example let $X_2 = \{1,2\}$. Then we have:

- $\mathcal{P}_0\mathcal{P}(X_2) = \{\emptyset\}$.
- $\mathcal{P}_1\mathcal{P}(X_2) = \{\{\emptyset\},\{\{1\}\},\{\{2\}\},\{\{1,2\}\}\}$.
- $\mathcal{P}_2\mathcal{P}(X_2) = \{\{\emptyset,\{1\}\},\{\emptyset,\{2\}\},\{\emptyset,\{1,2\}\},\{\{1\},\{2\}\},\{\{1\},\{1,2\}\},\{\{2\},\{1,2\}\}$.
- $\mathcal{P}_3\mathcal{P}(X_2) = \{\{\emptyset,\{1\},\{2\}\},\{\emptyset,\{1\},\{1,2\}\},\{\emptyset,\{2\},\{1,2\}\},\{\{1\},\{2\},\{1,2\}\}\}$.
- $\mathcal{P}_4\mathcal{P}(X_2) = \{\{\emptyset,\{1\},\{2\},\{1,2\}\}\}$

Therefore, we have $\mathcal{PP}(X_2) = \mathcal{P}_0\mathcal{P}(X_2) \sqcup \mathcal{P}_1\mathcal{P}(X_2) \sqcup \mathcal{P}_2\mathcal{P}(X_2) \sqcup \mathcal{P}_3\mathcal{P}(X_2) \sqcup \mathcal{P}_4\mathcal{P}(X_2)$.

For $\{1,2,3\}$ we have for example

$$\begin{aligned}&\mathcal{P}_2\mathcal{P}(\{1,2,3\}) = \{\{\emptyset,\{1\}\},\{\emptyset,\{2\}\},\{\emptyset,\{3\}\},\ \ \{\emptyset,\{1,2\}\},\{\emptyset,\{1,3\}\},\{\emptyset,\{2,3\}\},\ \ \{\emptyset,\{1,2,3\}\},\\ &\{\{1\},\{2\}\},\{\{1\},\{3\}\},\{\{3\},\{2\}\},\ \ \{\{1\},\{1,2\}\},\{\{2\},\{1,2\}\},\{\{3\},\{1,2\}\},\\ &\{\{1\},\{1,3\}\},\{\{2\},\{1,3\}\},\{\{3\},\{1,3\}\},\{\{1\},\{2,3\}\},\{\{2\},\{2,3\}\},\{\{3\},\{2,3\}\},...,\\ &\{\{1\},\{1,2,3\}\},...,\{\{1,2\},\{1,3\}\},\{\{1,2\},\{2,3\}\},\{\{1,3\},\{2,3\}\},...\{\{2,3\},\{1,2,3\}\}\}.\end{aligned} \tag{4.2}$$

It will be clear from this example that these S_n-invariant subsets themselves have invariant subsets. In §4.3 we determine these.

4.1.2 Notation for group actions on the power set of the power sets of a G-set

Lemma 4.1.5. *Let G be a group and S be a G-set, as defined in 2.2.3 above, denoted*

$$\alpha\colon G \times S \to S$$
$$(g, s) \mapsto g.s$$

(I) We have an action of G on $\mathcal{PP}(S)$, given by the mapping

$$\triangleright\colon G \times \mathcal{PP}(S) \to \mathcal{PP}(S)$$

defined as:

$$g \triangleright A = \{g \blacktriangleright a | a \in A\}$$

where $g \in G$, $\{a : a \in A\} := A \in \mathcal{PP}(S)$ and where $\blacktriangleright$ is defined in 3.2.1.

(II) For any $g \in G$and $A \in \mathcal{PP}(S)$, we have $|g \triangleright A| = |A|$.

Proof. (I) We need to prove that $\triangleright$ is a group action. This means we need to show the two axioms in Definition 2.2.3.

1. Let $f, g \in G$ and $A \in \mathcal{PP}(S)$. Then

$$\begin{aligned} g \triangleright (f \triangleright A) &= \{g \blacktriangleright (f \blacktriangleright a) : a \in A\}) \\ &= \{(gf) \blacktriangleright a : a \in A\}, \text{ by Lemma 3.2.1} \\ &= (g \circ f) \triangleright (A). \end{aligned}$$

So the axiom (1) holds;

2. Let I the identity of $G, A \in \mathcal{PP}(S)$. Then

$$\begin{aligned} I \rhd A &= \{I \blacktriangleright a : a \in A\} \\ &= \{a : a \in A\}, \text{ by Lemma 3.2.1} \\ &= A. \end{aligned}$$

So the axiom (2) holds.

Therefore, $\rhd$ is a group action.

(II) Given that we have an action, fixing $g \in G$, the map $S \to S$ sending s to $g.s$ is a bijection.

□

Example 4.1.6. Let $X_n = \{1, 2, \ldots, n\}$. We have an action of S_n on X_n, where $f.a = f(a)$ Applying the previous result this given an action of S_n on $\mathcal{PP}(X_n)$. Explicitly :

$$\rhd : S_n \times \mathcal{PP}(X_n) \to \mathcal{PP}(X_n)$$

defined as

$$f \rhd A = \{f \blacktriangleright a : a \in A\} = \{f[a] : a \in A\},$$

where $f : X_n \to X_n \in S_n$ *and* $A \in \mathcal{PP}(X_n)$, noting $A = \{a : a \in A\}$.

Lemma 4.1.7. *Let G be a group and S be a G-set, as defined in 2.2.3 above, with*

$$\begin{aligned} \alpha : G \times S &\to S \\ (g, s) &\mapsto g.s \end{aligned}$$

Suppose $\mathcal{P}_i\mathcal{P}(S)$ is the set of subsets of subsets with cardinality i as defined in 4.1.4. We have an action of G on $\mathcal{P}_i\mathcal{P}(S)$, given by the mapping

$$\rhd : G \times \mathcal{P}_i\mathcal{P}(S) \to \mathcal{P}_i\mathcal{P}(S)$$

defined as:

$$\textit{If } g \in G, \textit{then given } A \in \mathcal{P}_i\mathcal{P}(S) : g \triangleright A = \{g \blacktriangleright a | a \in A\}.$$

Proof. The result follows from Lemma 4.1.5 (II). □

In particular, we have an action of S_n on $\mathcal{P}_i\mathcal{P}(X)$ where $f \triangleright A = \{f \blacktriangleright a : a \in A\}$.

4.2 The decomposition of the representation of S_n on $\mathbb{C}\mathcal{P}_i\mathcal{P}(X_n)$ into irreducible for $2 \leq n \leq 4$.

The tables 4.1, 4.2, and 4.3 , below, show the decomposition of the representation of S_n on $\mathbb{C}\mathcal{P}_i\mathcal{P}(\{1, 2, 3, ..., n\})$ into irreducible for $2 \leq n \leq 4$. The full calculation were done by using GAP, Python, and the character table for Specht module of S_n (see[Gib21]) and the code can be found in the Appendix A.

The method that we used to decompose the representation of the action of S_n on $\mathbb{C}\mathcal{P}_i\mathcal{P}(X_n)$ into irreducible has many steps as follows:

1. Using GAP to calculate the points which are moved by a permutation in the action of S_n on $\mathbb{C}\mathcal{P}_i\mathcal{P}(X_n)$ that can be found in 2.4.3.

   For example, Let $n = 2$ and $i = 2$. Then ,by using GAP, we have the number of the points are moved by permutations id,(12) $\in S_2$ in the action of S_2 on $\mathbb{C}\mathcal{P}_2\mathcal{P}(X_2)$ which are 0 and 4 respectively.

2. Calculating the number of fixed points by the following equation:

   $$\binom{2^n}{i} - \text{ the number of the points are moved that be calculated in Step 1.}$$

   For example, we need to calculate the number of fixed points by permutation id and (12) in the action of S_2 on $\mathbb{C}\mathcal{P}_2\mathcal{P}(X_2)$, so we have

   $$\binom{4}{2} - 0 = 6$$

   and

   $$\binom{4}{2} - 4 = 2$$

respectively.

3. Constructing the system of equations as a matrix where the rows correspond to cycle types and the column correspond to irreducible characters for the Specht module S^λ can be founded in [Gib21] and the last column will be the number of fixed points that are calculated in Step 2.

   For example, we use the character table of S_2 and the number of fixed points to construct the matrix where the rows correspond to cycle types (the first row correspond to id and the second row (12)) and the column correspond to irreducible characters for the Specht module S^λ (the first column correspond to irreducible character for $S^{(2)}$, the second column correspond to irreducible character for $S^{(1^2)}$) and the last column will be the number of fixed points that be got in step 2). The matrix is :

$$\begin{bmatrix} 1 & 1 & \vdots & 6 \\ 1 & -1 & \vdots & 2 \end{bmatrix}$$

4. Using proposition 2.2.24 and Python to solve this system that can be found the code in A.2 and we get

$$\begin{bmatrix} 4 & 2 \end{bmatrix}$$

   Therefore, the representation of the action of group S_2 on $\mathbb{C}\mathcal{P}_2\mathcal{P}(X_2)$ has 4 copies of trivial representation $S^{(2)}$ and two copies of $S^{(1,1)}$. I.e.

$$\mathbb{C}\mathcal{P}_2\mathcal{P}(X_2) \cong 4S^{(2)} \oplus 2S^{(1,1)}$$

We will see later how this can be obtained by using more general techniques in the next sections.

Table 4.1: The decomposition of the representation of S_2 in $\mathbb{C}\mathcal{P}_i\mathcal{P}(X_2)$ into irreducibles.

$\mathbb{C}\mathcal{P}_i\mathcal{P}(X_2)$	Specht Module S^λ
$\mathbb{C}\mathcal{P}_0\mathcal{P}(X_2)$	$S^{(2)}$
$\mathbb{C}\mathcal{P}_1\mathcal{P}(X_2)$	$3S^{(2)}, S^{(1,1)}$
$\mathbb{C}\mathcal{P}_2\mathcal{P}(X_2)$	$4S^{(2)}, 2S^{(1,1)}$
$\mathbb{C}\mathcal{P}_3\mathcal{P}(X_2)$	$3S^{(2)}, S^{(1,1)}$
$\mathbb{C}\mathcal{P}_4\mathcal{P}(X_2)$	$S^{(2)}$

Table 4.2: The decomposition of the representation of S_3 in $\mathbb{C}\mathcal{P}_i\mathcal{P}(X_3)$ into irreducibles.

$\mathbb{C}\mathcal{P}_i\mathcal{P}(X_3)$	Specht Module S^λ
$\mathbb{C}\mathcal{P}_0\mathcal{P}(X_3)$	$S^{(3)}$
$\mathbb{C}\mathcal{P}_1\mathcal{P}(X_3)$	$4S^{(3)}, 2S^{(2,1)}$
$\mathbb{C}\mathcal{P}_2\mathcal{P}(X_3)$	$9S^{(3)}, 9S^{(2,1)}, S^{(1)^3}$
$\mathbb{C}\mathcal{P}_3\mathcal{P}(X_3)$	$16S^{(3)}, 18S^{(2,1)}, 4S^{(1)^3}$
$\mathbb{C}\mathcal{P}_4\mathcal{P}(X_3)$	$20S^{(3)}, 22S^{(2,1)}, 6S^{(1)^3}$
$\mathbb{C}\mathcal{P}_5\mathcal{P}(X_3)$	$16S^{(3)}, 18S^{(2,1)}, 4S^{(1)^3}$
$\mathbb{C}\mathcal{P}_6\mathcal{P}(X_3)$	$9S^{(3)}, 9S^{(2,1)}, S^{(1)^3}$
$\mathbb{C}\mathcal{P}_7\mathcal{P}(X_3)$	$4S^{(3)}, 2S^{(2,1)}$
$\mathbb{C}\mathcal{P}_8\mathcal{P}(X_3)$	$S^{(3)}$

Table 4.3: The decomposition of the representation of S_4 in $\mathbb{C}\mathcal{P}_i\mathcal{P}(X_4)$ into irreducibles.

$\mathbb{C}\mathcal{P}_i\mathcal{P}(X_4)$	Specht Module S^{λ}
$\mathbb{C}\mathcal{P}_0\mathcal{P}(X_4)$	$S^{(4)}$
$\mathbb{C}\mathcal{P}_1\mathcal{P}(X_4)$	$5S^{(4)}, 3S^{(3,1)}, S^{(2,2)}$
$\mathbb{C}\mathcal{P}_2\mathcal{P}(X_4)$	$17S^{(4)}, 21S^{(3,1)}, 11S^{(2,2)}, 6S^{(2,1,1)}$
$\mathbb{C}\mathcal{P}_3\mathcal{P}(X_4)$	$52S^{(4)}, 88S^{(3,1)}, 51S^{(2,2)}, 45S^(2,1,1), 7S^{(1)^4}$
$\mathbb{C}\mathcal{P}_4\mathcal{P}(X_4)$	$136S^{(4)}, 267S^{(3,1)}, 159S^{(2,2)}, 175S^{(2,1,1)}, 40S^{(1)^4}$
$\mathbb{C}\mathcal{P}_5\mathcal{P}(X_4)$	$284S^{(4)}, 616S^{(3,1)}, 377S^{(2,2)}, 455S^{(2,1,1)}, 117S^{(1)^4}$
$\mathbb{C}\mathcal{P}_6\mathcal{P}(X_4)$	$477S^{(4)}, 1105S^{(3,1)}, 689S^{(2,2)}, 868S^{(2,1,1)}, 234S^{(1)^4}$
$\mathbb{C}\mathcal{P}_7\mathcal{P}(X_4)$	$655S^{(4)}, 1561S^{(3,1)}, 979S^{(2,2)}, 1264S^{(2,1,1)}, 352S^{(1)^4}$
$\mathbb{C}\mathcal{P}_8\mathcal{P}(X_4)$	$730S^{(4)}, 1750S^{(3,1)}, 1098S^{(2,2)}, 1430S^{(2,1,1)}, 404S^{(1)^4}$
$\mathbb{C}\mathcal{P}_9\mathcal{P}(X_4)$	$655S^{(4)}, 1561S^{(3,1)}, 979S^{(2,2)}, 1264S^{(2,1,1)}, 352S^{(1)^4}$
$\mathbb{C}\mathcal{P}_{10}\mathcal{P}(X_4)$	$477S^{(4)}, 1105S^{(3,1)}, 689S^{(2,2)}, 868S^{(2,1,1)}, 234S^{(1)^4}$
$\mathbb{C}\mathcal{P}_{11}\mathcal{P}(X_4)$	$284S^{(4)}, 616S^{(3,1)}, 377S^{(2,2)}, 455S^{(2,1,1)}, 117S^{(1)^4}$
$\mathbb{C}\mathcal{P}_{12}\mathcal{P}(X_4)$	$136S^{(4)}, 267S^{(3,1)}, 159S^{(2,2)}, 175S^{(2,1,1)}, 40S^{(1)^4}$
$\mathbb{C}\mathcal{P}_{13}\mathcal{P}(X_4)$	$52S^{(4)}, 88S^{(3,1)}, 51S^{(2,2)}, 45S^(2,1,1), 7S^{(1)^4}$
$\mathbb{C}\mathcal{P}_{14}\mathcal{P}(X_4)$	$17S^{(4)}, 21S^{(3,1)}, 11S^{(2,2)}, 6S^{(2,1,1)}$
$\mathbb{C}\mathcal{P}_{15}\mathcal{P}(X_4)$	$5S^{(4)}, 3S^{(3,1)}, S^{(2,2)}$
$\mathbb{C}\mathcal{P}_{16}\mathcal{P}(X_4)$	$S^{(4)}$

We have seen from the previuos tables there is a duality as follows: Let $i \in \{0, 1, \ldots, n\}$. Suppose that $X_n = \{1, \ldots, n\}$. Then the bijection

$$\mathcal{P}_i\mathcal{P}(X_n) \xrightarrow{f} \mathcal{P}_{2^n-i}\mathcal{P}(X_n),$$

such that

$$M \overset{f}{\mapsto} \hat{M},$$

where $\hat{M} = \mathcal{P}(X_n) \setminus M$ preserves the action of S_n. So we have an isomorphism of representations of S_n. I.e.

$$\mathbb{C}\mathcal{P}_i\mathcal{P}(X_n) \cong \mathbb{C}\mathcal{P}_{2^n-i}\mathcal{P}(X_n).$$

4.3 Some subsets $\mathcal{P}_{i,[a_1,\ldots,a_i]}\ \mathcal{P}(X_n)$ of $\mathcal{P}_i\mathcal{P}(X_n)$

Let us now prepare to explain the low-rank decomposition tables 4.1, 4.2, 4.3 using explicit calculations; and to go to the general case. A crucial step is a decomposition of $\mathcal{P}_i\mathcal{P}(X_n)$ into subsets closed under the action of S_n.

Recall that an integer partition is a multiset of natural numbers, usually written in non-increasing order. Example $(7,3,3)$. The set Λ' of 'special' integer partitions is two disjoint copies of the set Λ of integer partitions, with elements of the second copy distinguished by ending in 0, thus $(7,3,3,0)$ and so on. That is, special integer partitions are multisets of natural numbers allowing also 0, but the multiplicity of 0 is at most 1.

(**4.3.1**) The 'shape' of $A \in \mathcal{PP}(X_n)$ is the multiset of orders of its elements as sets, denoted $sh(A)$. Example $sh(\{\emptyset,\{1\},\{2\},\{1,2\}\}) = (2,1,1,0)$. Observe that for $A \in \mathcal{P}_i\mathcal{P}(X_n)$ then $sh(A)$ is given by a sequence of length i (ending with at most one 0).

Observe that a function $f : S \to T$ on a set S induces a partition of the set into the inverse images of elements of the codomain T. Thus in particular $\mathcal{PP}(X_n)$ is partitioned by $sh : \mathcal{PP}(X_n) \to \Lambda'$; as is $\mathcal{P}_i\mathcal{P}(X_n)$ for each i. Let us write $\mathcal{P}_\lambda\mathcal{P}(X_n)$ for the $\lambda \in \Lambda'$ part of this partition. That is, we have

$$\mathcal{PP}(X_n) = \bigsqcup_{\lambda\in\Lambda'} \mathcal{P}_\lambda\mathcal{P}(X_n) \tag{4.3}$$

(Note that for fixed n most of these parts are empty, so the union is finite.)

Note from 3.2.1 that $\mathcal{P}(X_n)$ and hence $\mathcal{PP}(X_n)$ (and so on) inherit the property of S_n-set from X_n. Note also from 3.2.1 that each $\mathcal{P}_\lambda\mathcal{P}(X_n)$ is a sub S_n-set. Thus (4.3) is an S_n-set decomposition of $\mathcal{PP}(X_n)$. Thus we would solve the problem of determining the irreducible decomposition of $\mathcal{PP}(X_n)$ if we solve the corresponding problem for each $\mathcal{P}_\lambda\mathcal{P}(X_n)$.

Definition 4.3.2. Let $n \in \mathbb{N}$ and $X_n = \{1,\ldots,n\}$. Given $i \in \{0,\ldots,n\}$, and $a_1,\ldots,a_i \in \{0,\ldots,n\}$ such that $a_1 \le a_2 \le \cdots \le a_i$, define

$$\begin{aligned}\mathcal{P}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n) = \{ & A \subseteq \mathcal{P}(X_n) : |A| = i; \text{ and, picking any order on the elements of } A, \text{ so} \\ & A = \{M_1, \ldots, M_i\}, \text{ then } |M_1| = a_{f(1)}, \ \ldots, |M_i| = a_{f(i)}, \text{for some} \\ & \text{permutation } f \text{ of } \{1, \ldots, i\}\}.\end{aligned}$$

We will usually write the subsets $\{M_1, \ldots, M_i\}$ of $\mathcal{P}(X_n)$, with cardinality i, choosing an order so that $|M_1| \leq |M_2| \leq \cdots \leq |M_i|$, so usually f will not be needed.

For the case $i = 0$, we have:

$$\mathcal{P}_{0,[-]}\mathcal{P}(X_n) = \{\emptyset\}.$$

Here $[-]$ is the empty sequence.

Example 4.3.3. Let $X_2 = \{1, 2\}$, we have when $i = 0, 1$:

$$\begin{aligned}\mathcal{P}_{0,[-]}\mathcal{P}(X_2) &= \{\emptyset\}, \\ \mathcal{P}_{1,[0]}\mathcal{P}(X_2) &= \{\{\emptyset\}\}, \\ \mathcal{P}_{1,[1]}\mathcal{P}(X_2) &= \Big\{\{\{1\}\}, \{\{2\}\}\Big\}. \\ \mathcal{P}_{1,[2]}\mathcal{P}(X_2) &= \Big\{\{\{1, 2\}\}\Big\}.\end{aligned}$$

Therefore, we can see that

$$\mathcal{P}_1\mathcal{P}(X_2) = \mathcal{P}_{1,[0]}\mathcal{P}(X_2) \sqcup \mathcal{P}_{1,[1]}\mathcal{P}(X_2) \sqcup \mathcal{P}_{1,[2]}\mathcal{P}(X_2).$$

Example 4.3.4. Let $X_2 = \{1, 2\}$. For case $i = 2$, we have:

$$\begin{aligned}\mathcal{P}_{2,[0,1]}\mathcal{P}(X_2) &= \{\{\emptyset, \{1\}\}, \{\emptyset, \{2\}\}\}. \\ \mathcal{P}_{2,[0,2]}\mathcal{P}(X_2) &= \{\{\emptyset, \{1, 2\}\}\}. \\ \mathcal{P}_{2,[1,1]}\mathcal{P}(X_2) &= \{\{\{1\}, \{2\}\}\}. \\ \mathcal{P}_{2,[1,2]}\mathcal{P}(X_2) &= \{\{\{1\}, \{1, 2\}\}, \{\{2\}, \{1, 2\}\}\}.\end{aligned}$$

Therefore,

$$\mathcal{P}_2\mathcal{P}(X_2) = \mathcal{P}_{2,[0,1]}\mathcal{P}(X_2) \sqcup \mathcal{P}_{2,[0,2]}\mathcal{P}(X_2) \sqcup \mathcal{P}_{2,[1,1]}\mathcal{P}(X_2) \sqcup \mathcal{P}_{2,[1,2]}\mathcal{P}(X_2).$$

Example 4.3.5. Let $X_2 = \{1, 2\}$ and $i = 3$, we get:

$$\mathcal{P}_{3,[0,1,1]}\mathcal{P}(X_2) = \{\{\emptyset, \{1\}, \{2\}\}\}.$$
$$\mathcal{P}_{3,[0,1,2]}\mathcal{P}(X_2) = \{\{\emptyset, \{1\}, \{1, 2\}\}, \{\emptyset, \{2\}, \{1, 2\}\}\}.$$
$$\mathcal{P}_{3,[1,1,2]}\mathcal{P}(X_2) = \{\{\{1\}, \{2\}, \{1, 2\}\}\}.$$

Therefore,

$$\mathcal{P}_3\mathcal{P}(X_2) = \mathcal{P}_{3,[0,1,1]}\mathcal{P}(X_2) \sqcup \mathcal{P}_{3,[0,1,2]}\mathcal{P}(X_2) \sqcup \mathcal{P}_{3,[1,1,2]}\mathcal{P}(X_2).$$

Example 4.3.6. Let $X_2 = \{1, 2\}$. Then we have:

$$\mathcal{P}_{4,[0,1,1,2]}\mathcal{P}(X_2) = \{\{\varnothing, \{1\}, \{2\}, \{1, 2\}\}\}.$$

Therefore,

$$\mathcal{P}_4\mathcal{P}(X_2) = \mathcal{P}_{4,[0,1,1,2]}\mathcal{P}(X_2).$$

From all the above examples we also conclude that :

$$\mathcal{PP}(X_2) = \mathcal{P}_1\mathcal{P}(X_2) \sqcup \mathcal{P}_2\mathcal{P}(X_2) \sqcup \mathcal{P}_3\mathcal{P}(X_2) \sqcup \mathcal{P}_4\mathcal{P}(X_2).$$

Proposition 4.3.7. *In general we have*

$$\mathcal{P}_i\mathcal{P}(X_n) \;=\; \bigsqcup_{\substack{[a_1,\ldots,a_i]\in\{0,1,\ldots,n\}^i \\ 0\le a_1\le a_2\le\cdots\le a_i\le n}} \mathcal{P}_{i,[a_1,a_2,\ldots,a_i]}\mathcal{P}(X_n) \tag{4.4}$$

Proof. The disjoint union is clear from above Equation (4.3). We stop at n for the size of entries $\{0, 1, \ldots, n\}^i$ because the subset of X_n has at most n elements. □

Proposition 4.3.8. *$\mathcal{P}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n)$ is invariant under the action of S_n. I.e. If $f \in S_n, A \in \mathcal{P}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n)$, then $f \rhd A \in \mathcal{P}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n)$.*

Proof. The result follows from Lemma 4.1.5(II). □

Note that $\mathcal{P}_{i,[a_1,a_2,\ldots,a_i]}\mathcal{P}(X_n)$ is the empty set if more the one of the a_i is zero. Also, similarly, if more than n of the a_i is 1, then $\mathcal{P}_{i,[a_1,a_2,\ldots,a_i]}\mathcal{P}(X_n) = \emptyset$. More generally, in order that

$$\mathcal{P}_{i,[a_1,a_2,\ldots,a_i]}\mathcal{P}(X_n) \neq \emptyset,$$

then if $k \in \{0,\ldots,n\}$,

$$\Big|\{\{j \in \{0,\ldots,i\}|a_j = k\}\Big| \leq \binom{n}{k}.$$

4.3.1 The irreducible content of the action of S_n on $\mathbb{C}\mathcal{P}_{1,[a_1]}\mathcal{P}(X_n)$ and on $\mathbb{C}\mathcal{P}_{2,[a_1,a_2]}\mathcal{P}(X_n)$, when $2 \leq n \leq 5$. (tables)

In this subsection, we will show the computation of the irreducible content of the action of S_n on $\mathbb{C}\mathcal{P}_{0,[-]}\mathcal{P}(X_n)$, on $\mathbb{C}\mathcal{P}_{1,[a_1]}\mathcal{P}(X_n)$ and on $\mathbb{C}\mathcal{P}_{2,[a_1,a_2]}\mathcal{P}(X_n)$, when $2 \leq n \leq 5$ in Tables 4.4, 4.5, 4.6 and 4.7. We use the same method which in Section 4.2.

Note that the decomposition of $\mathbb{C}\mathcal{P}_{0,[-]}\mathcal{P}(X_n)$ is given by

$$\mathbb{C}\mathcal{P}_{0,[-]}\mathcal{P}(X_n) \cong S^{(n)},$$

for all n so we start with $i = 1$ in Table 4.4.

Table 4.4: The calculations of the irreducible content of the action of S_n on $\mathbb{C}\mathcal{P}_{1,[a_1]}\mathcal{P}(X_n)$, when $n = 2, 3, 4, 5$.

$\mathbb{C}\mathcal{P}_{1,[a_1]}\mathcal{P}(X_2)$	Specht Module S^λ
$\mathbb{C}\mathcal{P}_{1,[0]}\mathcal{P}(X_2)$	$S^{(2)}$
$\mathbb{C}\mathcal{P}_{1,[1]}\mathcal{P}(X_2)$	$S^{(2)}$, $S^{(1^2)}$
$\mathbb{C}\mathcal{P}_{1,[2]}\mathcal{P}(X_2)$	$S^{(2)}$

$\mathbb{C}\mathcal{P}_{1,[a_1]}\mathcal{P}(X_3)$	Specht Module S^λ
$\mathbb{C}\mathcal{P}_{1,[0]}\mathcal{P}(X_3)$	$S^{(3)}$
$\mathbb{C}\mathcal{P}_{1,[1]}\mathcal{P}(X_3)$	$S^{(3)}, S^{(2,1)}$
$\mathbb{C}\mathcal{P}_{1,[2]}\mathcal{P}(X_3)$	$S^{(3)}, S^{(2,1)}$
$\mathbb{C}\mathcal{P}_{1,[3]}\mathcal{P}(X_3)$	$S^{(3)}$

$\mathbb{C}\mathcal{P}_{1,[a_1]}\mathcal{P}(X_4)$	Specht Module S^λ
$\mathbb{C}\mathcal{P}_{1,[0]}\mathcal{P}(X_4)$	$S^{(4)}$
$\mathbb{C}\mathcal{P}_{1,[1]}\mathcal{P}(X_4)$	$S^{(4)}, S^{(3,1)}$
$\mathbb{C}\mathcal{P}_{1,[2]}\mathcal{P}(X_4)$	$S^{(4)}, S^{(3,1)}, S^{(2,2)}$
$\mathbb{C}\mathcal{P}_{1,[3]}\mathcal{P}(X_4)$	$S^{(4)}, S^{(3,1)}$
$\mathbb{C}\mathcal{P}_{1,[4]}\mathcal{P}(X_4)$	$S^{(4)}$

$\mathbb{C}\mathcal{P}_{1,[a_1]}\mathcal{P}(X_5)$	Specht Module S^λ
$\mathbb{C}\mathcal{P}_{1,[0]}\mathcal{P}(X_5)$	$S^{(5)}$
$\mathbb{C}\mathcal{P}_{1,[1]}\mathcal{P}(X_5)$	$S^{(5)}, S^{(4,1)}$
$\mathbb{C}\mathcal{P}_{1,[2]}\mathcal{P}(X_5)$	$S^{(5)}, S^{(4,1)}, S^{(3,2)}$
$\mathbb{C}\mathcal{P}_{1,[3]}\mathcal{P}(X_5)$	$S^{(5)}, S^{(4,1)}, S^{(3,2)}$
$\mathbb{C}\mathcal{P}_{1,[4]}\mathcal{P}(X_5)$	$S^{(5)}, S^{(4,1)}$
$\mathbb{C}\mathcal{P}_{1,[5]}\mathcal{P}(X_5)$	$S^{(5)}$

Table 4.5: The calculations of the irreducible content of the action of S_n on $\mathcal{P}_{2,[1,a_2]}\mathcal{P}(X_n)$, when $n = 2, 3, 4, 5$ and $1 < a_2$.

$\mathbb{C}\mathcal{P}_{2,[1,a_2]}\mathcal{P}(X_2)$	Specht Module S^λ
$\mathbb{C}\mathcal{P}_{2,[1,2]}\mathcal{P}(X_2)$	$S^{(2)}, S^{(1,1)}$

$\mathbb{C}\mathcal{P}_{2,[1,a_2]}\mathcal{P}(X_3)$	Specht Module S^λ
$\mathbb{C}\mathcal{P}_{2,[1,2]}\mathcal{P}(X_3)$	$2S^{(3)}, 3S^{(2,1)}, S^{(1^3)}$
$\mathbb{C}\mathcal{P}_{2,[1,3]}\mathcal{P}(X_3)$	$S^{(3)}, S^{(2,1)}$

$\mathbb{C}\mathcal{P}_{2,[1,a_2]}\mathcal{P}(X_4)$	Specht Module S^λ
$\mathbb{C}\mathcal{P}_{2,[1,2]}\mathcal{P}(X_4)$	$2S^{(4)}, 4S^{(3,1)}, 2S^{(2,2)}, 2S^{(2,1^2)}$
$\mathbb{C}\mathcal{P}_{2,[1,3]}\mathcal{P}(X_4)$	$2S^{(4)}, 3S^{(3,1)}, S^{(2,2)}, S^{(2,1^2)}$
$\mathbb{C}\mathcal{P}_{2,[1,4]}\mathcal{P}(X_4)$	$S^{(4)}, S^{(3,1)}$

$\mathbb{C}\mathcal{P}_{2,[1,a_2]}\mathcal{P}(X_5)$	Specht Module S^λ
$\mathbb{C}\mathcal{P}_{2,[1,2]}\mathcal{P}(X_5)$	$2S^{(5)}, 4S^{(4,1)}, 3S^{(3,2)}, 2S^{(3,1^2)}, S^{(2^2,1)}$
$\mathbb{C}\mathcal{P}_{2,[1,3]}\mathcal{P}(X_5)$	$2S^{(5)}, 4S^{(4,1)}, 3S^{(3,2)}, 2S^{(3,1^2)}, S^{(2^2,1)}$
$\mathbb{C}\mathcal{P}_{2,[1,4]}\mathcal{P}(X_5)$	$2S^{(5)}, 3S^{(4,1)}, S^{(3,2)}, S^{(3,1^2)}$
$\mathbb{C}\mathcal{P}_{2,[1,5]}\mathcal{P}(X_5)$	$S^{(5)}, S^{(4,1)}$

$\mathbb{C}\mathcal{P}_{2,[1,a_2]}\mathcal{P}(X_6)$	Specht Module S^λ
$\mathbb{C}\mathcal{P}_{2,[1,2]}\mathcal{P}(X_6)$	$2S^{(6)}, 4S^{(5,1)}, 3S^{(4,2)}, 2S^{(3,3)}, S^{(3,2,1)}, S^{(3,1^3)}$
$\mathbb{C}\mathcal{P}_{2,[1,3]}\mathcal{P}(X_6)$	$2S^{(6)}, 4S^{(5,1)}, 4S^{(4,2)}, 2S^{(3,3)}, 2S^{(3,2,1)}, 2S^{(3,1^3)}$
$\mathbb{C}\mathcal{P}_{2,[1,4]}\mathcal{P}(X_6)$	$2S^{(6)}, 4S^{(5,1)}, 3S^{(4,2)}, 2S^{(3,3)}, S^{(3,2,1)}, S^{(3,1^3)}$
$\mathbb{C}\mathcal{P}_{2,[1,5]}\mathcal{P}(X_6)$	$2S^{(6)}, 3S^{(5,1)}, S^{(4,2)}, S^{(3,3)}$
$\mathbb{C}\mathcal{P}_{2,[1,6]}\mathcal{P}(X_6)$	$S^{(6)}, S^{(5,1)}$

Table 4.6: The calculations of the irreducible content of the action of S_n on $\mathcal{P}_{2,[2,a_2]}\mathcal{P}(X_n)$, when $n = 2, 3, 4, 5$ and $2 < a_2$.

$\mathbb{C}\mathcal{P}_{2,[2,a_2]}\mathcal{P}(X_3)$	Specht Module S^λ
$\mathbb{C}\mathcal{P}_{2,[2,3]}\mathcal{P}(X_3)$	$S^{(3)}, S^{(2,1)}$

$\mathbb{C}\mathcal{P}_{2,[2,a_2]}\mathcal{P}(X_4)$	Specht Module S^λ
$\mathbb{C}\mathcal{P}_{2,[2,3]}\mathcal{P}(X_4)$	$2S^{(4)}, 4S^{(3,1)}, 2S^{(2,2)}, S^{(2,1^2)}$
$\mathbb{C}\mathcal{P}_{2,[2,4]}\mathcal{P}(X_4)$	$S^{(4)}, S^{(3,1)}, S^{(2,2)}$

$\mathbb{C}\mathcal{P}_{2,[2,a_2]}\mathcal{P}(X_5)$	Specht Module S^λ
$\mathbb{C}\mathcal{P}_{2,[2,3]}\mathcal{P}(X_5)$	$3S^{(5)}, 6S^{(4,1)}, 6S^{(3,2)}, 4S^{(3,1^2)}, 3S^{(2^2,1)}, S^{(2,1^3)}$
$\mathbb{C}\mathcal{P}_{2,[2,4]}\mathcal{P}(X_5)$	$2S^{(5)}, 4S^{(4,1)}, 3S^{(3,2)}, 2S^{(3,1^2)}, S^{(2^2,1)}$
$\mathbb{C}\mathcal{P}_{2,[2,5]}\mathcal{P}(X_5)$	$S^{(5)}, S^{(4,1)}, S^{(3,2)}$

Table 4.7: The calculations of the irreducible content of the action of S_n on $\mathcal{P}_{2,[3,a_2]}\mathcal{P}(X_n)$, when $n = 2, 3, 4, 5$ and $3 < a_2$.

$\mathbb{C}\mathcal{P}_{2,[3,a_2]}\mathcal{P}(X_4)$	Specht Module S^λ
$\mathbb{C}\mathcal{P}_{2,[3,4]}\mathcal{P}(X_4)$	$S^{(4)}, S^{(3,1)}$

$\mathbb{C}\mathcal{P}_{2[3,a_2]}\mathcal{P}(X_5)$	Specht Module S^λ
$\mathbb{C}\mathcal{P}_{2,[3,4]}\mathcal{P}(X_5)$	$2S^{(5)}, 4S^{(4,1)}, 3S^{(3,2)}, 2S^{(3,1^2)}, S^{(2^2,1)}$
$\mathbb{C}\mathcal{P}_{2,[3,5]}\mathcal{P}(X_5)$	$S^{(5)}, S^{(4,1)}, S^{(3,2)}$

$\mathbb{C}\mathcal{P}\mathcal{P}_{2,[4,a_2]}(X_5)$	Specht Module S^λ
$\mathbb{C}\mathcal{P}_{2,[4,5]}\mathcal{P}(X_5)$	$S^{(5)}, S^{(4,1)}$

4.3.2 General computation of the decomposition of $\mathbb{C}\mathcal{P}_{1,[a_1]}\mathcal{P}(X_n)$ into irreducibles.

In this subsection, we will see how the calculations in Table 4.4 can be explained in theoretical manner.

Let n be a positive integer, and $a_1 \in \{0, \ldots, n\}$. Recall that

$$\mathcal{P}_{1,[a_1]}\mathcal{P}(X_n) = \{A \subseteq \mathcal{P}(X_n) : |A| = 1, \text{ and } A = \{M\}, \text{ with } |M| = a_1\}.$$

Lemma 4.3.9. *There is a bijective map*

$$\begin{aligned} F^{(n)}_{(a_1)} \colon \mathcal{P}_{1,[a_1]}\mathcal{P}(X_n) &\to \mathcal{P}_{a_1}(X_n) \\ \{M\} &\mapsto M, \end{aligned}$$

preserving the action of S_n.

Proof. It is clear $F^{(n)}_{(a_1)}$ is bijective. We need to show $F^{(n)}_{(a_1)}$ preserves the action of S_n. Let $f \in S_n$. Then

$$\begin{aligned} F^{(n)}_{(a_1)}(f \triangleright \{M\}) &= F^{(n)}_{(a_1)}\{f \blacktriangleright M\} \\ &= f \blacktriangleright M \\ &= f \blacktriangleright F^{(n)}_{(a_1)}(\{M\}). \end{aligned}$$

□

Recall that $f^{\#}$ is defined in Lemma 2.2.13.

Lemma 4.3.10. *The map*

$$(F^{(n)}_{(a_1)})^{\#} \colon \mathbb{C}\mathcal{P}_{1,[a_1]}\mathcal{P}(X_n) \to \mathbb{C}\mathcal{P}_{a_1}(X_n)$$

is an isomorphism of representations of S_n.

Proof. By lemma 4.3.9, there is a bijective map $\mathcal{P}_{1,[a_1]}\mathcal{P}(X_n) \to \mathcal{P}_{a_1}(X_n)$ preserving the action of S_n. Then, from lemma 2.2.13 we get :

$$\mathbb{C}\mathcal{P}_{1,[a_1]}\mathcal{P}(X_n) \cong \mathbb{C}\mathcal{P}_{a_1}(X_n)$$

□

Lemma 4.3.11. *Let n be a positive integer and $a_1 \in \{0, \ldots, n\}$. Then we have:*

$$\mathbb{C}(\mathcal{P}_{1,[a_1]}\mathcal{P}(X_n)) \cong \bigoplus_{j=0}^{\min(a_1, n-a_1)} S^{(n-j,j)}$$

Proof. We have. $\mathbb{C}\mathcal{P}_{1,[a_1]}\mathcal{P}(X_n) \cong \mathbb{C}\mathcal{P}_{a_1}(X_n)$ from Lemma 4.3.10. Applying Lemma 3.2.18, then we have

$$\mathbb{C}(\mathcal{P}_{1,[a_1]}\mathcal{P}(X_n)) \cong \bigoplus_{j=0}^{min(a_1, n-a_1)} S^{(n-j,j)}$$

□

So, this lemma explained the calculation we have shown in the tables 4.4.

Lemma 4.3.12. *Consider the action $\triangleright$, of S_n on $\mathcal{P}_1\mathcal{P}(X_n)$ as defined in Lemma 4.1.6. Then:*

$$\mathbb{C}\mathcal{P}_1\mathcal{P}(X_n) \cong \mathbb{C}\mathcal{P}(X_n).$$

Proof. Obviously, there is a bijective map

$$\mathcal{P}_1\mathcal{P}(X_n) \xrightarrow{F} \mathcal{P}(X_n)$$
$$\{M\} \mapsto M.$$

Also, it is preserving the action of S_n because

$$\begin{aligned} F(f \triangleright \{M\}) &= F(\{f \blacktriangleright M\} \\ &= f \blacktriangleright M, \end{aligned}$$

for all $f \in S_n$. Therefore, $\mathbb{C}\mathcal{P}_1\mathcal{P}(X_n)$ is isomorphism to $\mathbb{C}\mathcal{P}(X_n)$. □

Corollary 4.3.12.1. *Each irreducible component of the representation of S_n on $\mathbb{C}\mathcal{P}_1\mathcal{P}(X_n)$ is isomorphic to a Spech module indexed by a partition with at most two rows.*

4.3.3 Towards the computation of the irreducible content of $\mathbb{C}\mathcal{P}_{2,[a_1,a_2]}\mathcal{P}(X_n)$, with $0 \leq a_1 < a_2 \leq n$

In this subsection, we will see how the calculations in Tables 4.5, 4.6, 4.7 can be explained in theoretical manner.

Proposition 4.3.13. *Let V and W be representation of a group G over $\mathbb{C}$. Then $V \otimes W$ is a representation of G by*

$$g(v \otimes w) = gv \otimes gw.$$

Proof. We want to show that:

$$g(g'(v \otimes w)) = (gg')(v \otimes w)$$

where $g, g' \in G$. Let us start:

$$\begin{aligned} g(g'(v \otimes w)) &= g(g'v \otimes g'w) \\ &= \big(g(g'v) \otimes g(g'w)\big) \\ &= \big((gg')v \otimes (gg')w\big) \text{ since } V \text{ and } W \text{ are representation of } G. \\ &= (gg')(v \otimes w) \end{aligned}$$

□

Lemma 4.3.14. *Let a group G acts on a set X and acts on a set Y. Then we have a 'product' action of G on $X \times Y$ as follows:*

$$\begin{aligned} G \times (X \times Y) &\overset{\alpha}{\mapsto} X \times Y \\ (g, (x, y)) &\mapsto g.(x, y) \end{aligned}$$

where

$$g.(x, y) = (g.x, g.y).$$

And then, as representation of G we have :

$$\mathbb{C}(X \times Y) \cong \mathbb{C}(X) \otimes \mathbb{C}(Y).$$

where $\otimes$ is defined in Proposition 4.3.13.

Proof. We need to prove that α is group action as follows:

1. Let $g, h \in G$ and $(x, y) \in X \times Y$. Then:

$$\begin{aligned} g.(h.(x,y)) &= g.\big((h.x, h.y)\big) \\ &= \big(g.(h.x), (g.(h.x)\big) \\ &= \big((gh(x), gh(y)\big) \\ &= (gh)(x,y). \end{aligned}$$

2. Let $g \in G$ and id_G is the identity. Then

$$\begin{aligned} id_G.(x,y) =& (id_G.x, id_G.y) \\ =& (x,y). \end{aligned}$$

□

The same idea allows us to make a representation of a group algebra from any two representations.

Lemma 4.3.15. *Let n be a positive integer. Let a_1, a_2 be integers with $0 \leq a_1 < a_2 \leq n$. Recall*

$$\mathcal{P}_{2,[a_1,a_2]}\mathcal{P}(X_n) = \{A \subseteq \mathcal{P}(X_n) \mid |A| = 2, \text{ and } A = \{M_1, M_2\}, |M_1| = a_1 \text{ and } |M_2| = a_2\}.$$

There is a bijective map

$$\begin{aligned} F^{(n)}_{(a_1,a_2)} \colon \mathcal{P}_{2,[a_1,a_2]}\mathcal{P}(X_n) &\to \mathcal{P}_{a_1}(X_n) \times \mathcal{P}_{a_2}(X_n) \\ \{M_1, M_2\} &\mapsto (M_1, M_2), \end{aligned}$$

preserving the action of S_n (taking our action to the product action).

Proof. It is obvious that $F^{(n)}_{(a_1,a_2)}$ is bijective. We need to show $F^{(n)}_{(a_1,a_2)}$ preserves the action of S_n.

$$\begin{aligned}
F^{(n)}_{(a_1,a_2)}(f \triangleright \{M_1, M_2\}) &= F^{(n)}_{(a_1,a_2)}(\{f \blacktriangleright M_1, f \blacktriangleright M_2\}) \\
&= (f \blacktriangleright M_1, f \blacktriangleright M_2) \\
&= f \blacktriangleright (M_1, M_2) \\
&= f \blacktriangleright F^{(n)}_{(a_1,a_2)}(\{M_1, M_2\}).
\end{aligned}$$

□

Recall that $f^{\#}$ is defined in Lemma 2.2.13:

Lemma 4.3.16. *The map*

$$(F^{(n)}_{(a_1,a_2)})^{\#} \colon \mathbb{C}\mathcal{P}_{2,[a_1,a_2]}\mathcal{P}(X_n) \to \mathbb{C}(\mathcal{P}_{a_1}(X_n) \times \mathcal{P}_{a_2}(X_n))$$

is an isomorphism of representations of S_n.

Proof. Let the symmetric group S_n acts on $\mathcal{P}_{a_1}(X_n)$ and on $\mathcal{P}_{a_2}(X_n)$. Then from Lemma 4.3.14 we have an action of S_n on $\mathcal{P}_{a_1}(X_n) \times \mathcal{P}_{a_2}(X_n)$ as:

$$g \blacktriangleright (M_1, M_2) = (g \blacktriangleright M_1, g \blacktriangleright M_2).$$

where $g \in S_n$, $M_1 \in \mathcal{P}_{a_1}(X_n)$ and $M_2 \in \mathcal{P}_{a_2}(X_n)$.

Since there is a bijective map

$$\begin{aligned}
F^{(n)}_{(a_1,a_2)} \colon \mathcal{P}_{2,[a_1,a_2]}\mathcal{P}(X_n) &\to \mathcal{P}_{a_1}(X_n) \times \mathcal{P}_{a_2}(X_n) \\
\{M_1, M_2\} &\mapsto (M_1, M_2)
\end{aligned}$$

preserving the action of S_n from Lemma 4.3.15, then we obtain that

$$(F^{(n)}_{(a_1,a_2)})^{\#} \colon \mathbb{C}\mathcal{P}_{2,[a_1,a_2]}\mathcal{P}(X_n) \to \mathbb{C}(\mathcal{P}_{a_1}(X_n) \times \mathcal{P}_{a_2}(X_n))$$

is an isomorphism of representations of S_n. □

Theorem 4.3.17. *Let n be a positive integer. Suppose $0 \leq a_1 < a_2 \leq n$. Then we* *have:*

$$\mathbb{C}(\mathcal{P}_{2,[a_1,a_2]}\mathcal{P}(X_n)) \cong [\bigoplus_{j=0}^{min(a_1,n-a_1,)} S^{(n-j,j)}] \otimes [\bigoplus_{k=0}^{min(a_2,n-a_2)} S^{(n-k,k)}]$$

And so:

$$\mathbb{C}(\mathcal{P}_{2,[a_1,a_2]}\mathcal{P}(X_n)) \cong \bigoplus_{j=0}^{\min(a_1,n-a_1)} \bigoplus_{k=0}^{\min(a_2,n-a_2)} S^{(n-j,j)} \otimes S^{(n-k,k)}$$

Proof. From Lemma 4.3.16, we have an isomorphism of representation

$$\mathbb{C}\mathcal{P}_{2,[a_1,a_2]}\mathcal{P}(X_n) \cong \mathbb{C}(\mathcal{P}_{a_1}(X_n) \times \mathcal{P}_{a_2}(X_n))$$

Since the symmetric group S_n acts on $\mathcal{P}_{a_1}(X_n)$ and on $\mathcal{P}_{a_2}(X_n)$, then, from Lemma 4.3.14, we have an action of S_n on $\mathcal{P}_{a_1}(X_n) \times \mathcal{P}_{a_2}(X_n)$ as:

$$g \blacktriangleright (M_1, M_2) = (g \blacktriangleright M_1, g \blacktriangleright M_2),$$

where $g \in S_n$, $M_1 \in \mathcal{P}_{a_1}(X_n)$ and $M_2 \in \mathcal{P}_{a_2}(X_n)$. Then, as representation of S_n, we have:

$$\mathbb{C}(\mathcal{P}_{a_1}(X_n) \times \mathcal{P}_{a_2}(X_n)) \cong \mathbb{C}\mathcal{P}_{a_1}(X_n) \otimes \mathbb{C}\mathcal{P}_{a_2}(X_n).$$

Thus, we get the following:

$$\mathbb{C}\mathcal{P}_{2,[a_1,a_2]}\mathcal{P}(X_n) \cong \mathbb{C}\mathcal{P}_{a_1}(X_n) \otimes \mathbb{C}\mathcal{P}_{a_2}(X_n)$$

Applying Lemma 3.2.18, we have

$$\mathbb{C}\mathcal{P}_{2,[a_1,a_2]}\mathcal{P}(X_n) \cong [\bigoplus_{j=0}^{min(a_1,n-a_1,)} S^{(n-j,j)}] \otimes [\bigoplus_{k=0}^{min(a_2,n-a_2)} S^{(n-k,k)}].$$

The final identity follows from the fact that the tensor product is distributive with respect to direct sums. □

This theorem explained the calculation we have done in tables above in 4.5, 4.6 and 4.7.

The general problem of computing tensor products of irreducible representations has been studied by many authors, but it is hard. A considerable amount of research has been done specifically on products of representations with two-row labels (as above), and here things are a bit easier than the general case. See for example [Ros00] and the references cited there. We will look at applying such methods in future work (and also look at applying our construction to address this classical problem). But for now, in this thesis, we restrict to considering some low-rank examples, as follows.

In the following examples, we are using GAP to compute the tensor product of specht modules of symmetric group and the code can be found in Appendix A.3.

Example 4.3.18. Let $X_2 = \{1, 2\}$ and S_2 be the symmetric group. Then the decomposition of the action of S_2 on $\mathbb{C}\mathcal{P}_{2,[a_1,a_2]}\mathcal{P}(X_2)$ into irreducibles will be as follows:

$$\begin{aligned}\mathbb{C}\mathcal{P}_{2,[1,2]}\mathcal{P}(X_2) &\cong [\bigoplus_{j=0}^{min(1,1)} S^{(n-j,j)}] \otimes [\bigoplus_{k=0}^{min(2,0)} S^{(n-k,k)}] \\ &\cong (S^{(2)} \oplus S^{(1,1)}) \otimes S^{(2)} \\ &\cong S^{(2)} \oplus S^{(1,1)},\end{aligned}$$

as expected since $S^{(2)}$ is the trivial (one dimensional) representation.

Example 4.3.19. Let $X_3 = \{1, 2, 3\}$ and S_3 be the symmetric group. Then we have the following decomposition of the action of S_3 on $\mathbb{C}\mathcal{P}_{2,[a_1,a_2]}\mathcal{P}(X_3)$ into irreducible :

- When $a_1 = 1$ and $a_2 = 2$, we have:

$$\begin{aligned}\mathbb{C}\mathcal{P}_{2,[1,2]}\mathcal{P}(X_3) &\cong [\bigoplus_{j=0}^{min(1,2)} S^{(n-j,j)}] \otimes [\bigoplus_{k=0}^{min(2,1)} S^{(n-k,k)}] \\ &\cong (S^{(3)} \oplus S^{(2,1)}) \otimes (S^{(3)} \oplus S^{(2,1)}) \\ &\cong (S^{(3)} \otimes S^{(3)}) \oplus (S^{(3)} \otimes S^{(2,1)}) \oplus (S^{(2,1)} \otimes S^{(3)}) \oplus (S^{(2,1)} \otimes S^{(2,1)}) \\ &\cong S^{(3)} \oplus S^{(2,1)} \oplus S^{(2,1)} \oplus S^{(3)} \oplus S^{(2,1)} \oplus S^{(1^3)} \\ &\cong 2S^{(3)} \oplus 3S^{(2,1)} \oplus S^{(1^3)}.\end{aligned}$$

  Hence all irreducible representation of S_3 occur in $\mathbb{C}\mathcal{P}_{2,[1,2]}\mathcal{P}(X_3)$, and hence in $\mathbb{C}\mathcal{PP}(X_3)$.

- When $a_1 = 1$ and $a_2 = 3$, we have:

$$\begin{aligned}\mathbb{C}\mathcal{P}_{2,[1,3]}\mathcal{P}(X_3) &\cong [\bigoplus_{j=0}^{min(1,2)} S^{(n-j,j)}] \otimes [\bigoplus_{k=0}^{min(3,0)} S^{(n-k,k)}] \\ &\cong (S^{(3)} \oplus S^{(2,1)}) \otimes S^{(3)} \\ &\cong (S^{(3)} \otimes S^{(3)}) \oplus (S^{(2,1)} \otimes S^{(3)}) \\ &\cong S^{(3)} \oplus S^{(2,1)}.\end{aligned}$$

as expected since $S^{(3)}$ is the trivial (one dimensional) representation.

- When $a_1 = 2$ and $a_2 = 3$, we have:

$$\begin{aligned}\mathbb{C}\mathcal{P}_{2,[2,3]}\mathcal{P}(X_3) &\cong [\bigoplus_{j=0}^{min(2,1)} S^{(n-j,j)}] \otimes [\bigoplus_{k=0}^{min(3,0)} S^{(n-k,k)}] \\ &\cong (S^{(3)} \oplus S^{(2,1)}) \otimes S^{(3)} \\ &\cong (S^{(3)} \otimes S^{(3)}) \oplus (S^{(3)} \otimes S^{(2,1)}) \\ &\cong S^{(3)} \oplus S^{(2,1)}.\end{aligned}$$

Example 4.3.20. Let $X_4 = \{1, 2, 3, 4\}$ and S_4 is the symmetric group. We need to calculate the irreducible of the action of S_4 on $\mathbb{C}\mathcal{P}_{2,[a_1,a_2]}\mathcal{P}(X_4)$ as the flowing:

- When $a_1 = 1$ and $a_2 = 2$, we have:

$$\begin{aligned}\mathbb{C}\mathcal{P}_{2,[1,2]}\mathcal{P}(X_4) &\cong [\bigoplus_{j=0}^{min(1,3)} S^{(n-j,j)}] \otimes [\bigoplus_{k=0}^{min(2,2)} S^{(n-k,k)}] \\ &\cong (S^{(4)} \oplus S^{(3,1)}) \otimes (S^{(4)} \oplus S^{(3,1)} \oplus S^{(2,2)}) \\ &\cong (S^{(4)} \otimes S^{(4)}) \oplus (S^{(4)} \otimes S^{(3,1)}) \oplus (S^{(4)} \otimes S^{(2,2)}) \oplus (S^{(3,1)} \otimes S^{(4)}) \oplus \\ &(S^{(3,1)} \otimes S^{(3,1)}) \oplus (S^{(3,1)} \otimes S^{(2,2)}) \\ &\cong S^{(4)} \oplus S^{(3,1)} \oplus S^{(2,2)} \oplus S^{(3,1)} \oplus S^{(4)} \oplus S^{(3,1)} \oplus S^{(2,2)} \oplus S^{(3,1)} \oplus S^{(2,1^2)} \oplus \\ &S^{(2,1^2)} \\ &\cong 2S^{(4)} \oplus 4S^{(3,1)} \oplus 2S^{(2,2)} \oplus S^{(2,1^2)}.\end{aligned}$$

- When $a_1 = 1$ and $a_2 = 3$, we have:

$$\begin{aligned}
\mathbb{C}\mathcal{P}_{2,[1,3]}\mathcal{P}(X_4) &\cong [\bigoplus_{j=0}^{min(1,3)} S^{(n-j,j)}] \otimes [\bigoplus_{k=0}^{min(3,1)} S^{(n-k,k)}] \\
&\cong (S^{(4)} \oplus S^{(3,1)}) \otimes (S^{(4)} \oplus S^{(3,1)}) \\
&\cong (S^{(4)} \otimes S^{(4)}) \oplus (S^{(4)} \otimes S^{(3,1)}) \oplus (S^{(3,1)} \otimes S^{(4)}) \oplus (S^{(3,1)} \otimes S^{(3,1)}) \\
&\cong S^{(4)} \oplus S^{(3,1)} \oplus S^{(3,1)} \oplus S^{(4)} \oplus S^{(3,1)} \oplus S^{(2,2)} \oplus S^{(2,1^2)} \\
&\cong 2S^{(4)} \oplus 3S^{(3,1)} \oplus S^{(2,2)} \oplus S^{(2,1^2)}.
\end{aligned}$$

- When $a_1 = 1$ and $a_2 = 4$, we have:

$$\begin{aligned}
\mathbb{C}\mathcal{P}_{2,[1,4]}\mathcal{P}(X_4) &\cong [\bigoplus_{j=0}^{min(1,3)} S^{(n-j,j)}] \otimes [\bigoplus_{k=0}^{min(4,0)} S^{(n-k,k)}] \\
&\cong (S^{(4)} \oplus S^{(3,1)}) \otimes S^{(4)} \\
&\cong (S^{(4)} \otimes S^{(4)}) \oplus (S^{(3,1)} \otimes S^{(4)}) \\
&\cong S^{(4)} \oplus S^{(3,1)}.
\end{aligned}$$

- When $a_1 = 2$ and $a_2 = 3$, we have:

$$\begin{aligned}
\mathbb{C}\mathcal{P}_{2,[2,3]}\mathcal{P}(X_4) &\cong [\bigoplus_{j=0}^{min(2,2)} S^{(n-j,j)}] \otimes [\bigoplus_{k=0}^{min(3,1)} S^{(n-k,k)}] \\
&\cong (S^{(4)} \oplus S^{(3,1)} \oplus S^{(2,2)}) \otimes (S^{(4)} \oplus S^{(3,1)}) \\
&\cong (S^{(4)} \otimes S^{(4)}) \oplus (S^{(4)} \otimes S^{(3,1)}) \oplus (S^{(3,1)} \otimes S^{(4)}) \oplus (S^{(3,1)} \otimes S^{(3,1)}) \oplus \\
&(S^{(2,2)} \otimes S^{(4)}) \oplus (S^{(2,2)} \otimes S^{(3,1)}) \\
&\cong S^{(4)} \oplus S^{(3,1)} \oplus S^{(3,1)} \oplus S^{(4)} \oplus S^{(3,1)} \oplus S^{(2,2)} \oplus S^{(2,1^2)} \oplus S^{(2,2)} \oplus S^{(3,1)} \oplus \\
&S^{(2,1^2)} \\
&\cong 2S^{(4)} \oplus 4S^{(3,1)} \oplus 2S^{(2,2)} \oplus S^{(2,1^2)}.
\end{aligned}$$

- When $a_1 = 2$ and $a_2 = 4$, we have:

$$\begin{aligned}
\mathbb{C}\mathcal{P}_{2,[2,4]}\mathcal{P}(X_4) &\cong [\bigoplus_{j=0}^{min(2,2)} S^{(n-j,j)}] \otimes [\bigoplus_{k=0}^{min(4,0)} S^{(n-k,k)}] \\
&\cong (S^{(4)} \oplus S^{(3,1)} \oplus S^{(2,2)}) \otimes S^{(4)} \\
&\cong (S^{(4)} \otimes S^{(4)}) \oplus (S^{(3,1)} \otimes S^{(4)}) \oplus (S^{(2,2)} \otimes S^{(4)}) \\
&\cong S^{(4)} \oplus S^{(3,1)} \oplus S^{(2,2)}.
\end{aligned}$$

- When $a_1 = 3$ and $a_2 = 4$, we have:

$$\begin{aligned}\mathbb{C}\mathcal{P}_{2,[3,4]}\mathcal{P}(X_4) &\cong [\bigoplus_{j=0}^{min(3,1)} S^{(n-j,j)}] \otimes [\bigoplus_{k=0}^{min(4,0)} S^{(n-k,k)}] \\ &\cong (S^{(4)} \oplus S^{(3,1)}) \otimes S^{(4)} \\ &\cong (S^{(4)} \otimes S^{(4)}) \oplus (S^{(3,1)} \otimes S^{(4)}) \\ &\cong S^{(4)} \oplus S^{(3,1)}.\end{aligned}$$

4.3.4 Towards the computation of the irreducible content of $\mathbb{C}\mathcal{P}_{i,[a_1,a_2,\ldots,a_i]}\mathcal{P}(X_n)$.

In this subsection, we show that the linearisation of the representation of S_n on $\mathbb{C}\mathcal{P}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n)$, when $0 \leq a_1 < a_2 \cdots < a_i \leq n$ reduces to a tensor product of representations.

Lemma 4.3.21. *Let $n \in \mathbb{N}$ and $X_n = \{1, \ldots, n\}$. Let $i \in \{1, \ldots, n\}$ and $0 \leq a_1 \leq a_2 \leq \cdots \leq a_i \leq n$. We have a bijection*

$$\begin{aligned} T\colon \mathcal{P}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n) &\to \mathcal{P}_{i,[n-a_1,\ldots,n-a_i]}\mathcal{P}(X_n) \\ \{M_1, \ldots, M_i\} &\mapsto \{X_n \setminus M_1, \ldots, X_n \setminus M_i\}\end{aligned}$$

preserving the action of S_n.

Proof. It is clear that T is bijective. Now, we need to show that T preserves the action of S_n. For all $f \in S_n$, we get:

$$\begin{aligned} f \triangleright \big(T(\{M_1, \ldots, M_i\})\big) &= f \triangleright (\{X_n \setminus M_1, \ldots, X_n \setminus M_i\}) \\ &= \{f[X_n \setminus M_1], \ldots, f[X_n \setminus M_i]\} \\ &= \{X_n \setminus f[M_1], \ldots, X_n \setminus f[M_i]\} \\ &= T\big(\{f[M_1], \ldots, f[M_i]\}\big) \\ &= T\big(f \triangleright \{M_1, \ldots, M_i\}\big).\end{aligned}$$

Where we used the fact that if $f\colon X \to Y$ is a bijective, and $A \subseteq X$, then $f(X \setminus A) = Y \setminus f(A)$. □

Proposition 4.3.22. *Let $i \in \{1, \ldots, n\}$ and $0 \leq a_1 < a_2 < \cdots < a_i \leq n$. There is* *a bijective map*

$$\begin{aligned} F^{(n)}_{(a_1,\ldots,a_i)} \colon \mathcal{P}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n) &\to (\mathcal{P}_{a_1}(X_n) \times \cdots \times \mathcal{P}_{a_i}(X_n)) \\ \{M_1, M_2, \ldots, M_i\} &\mapsto (M_1, M_2, \ldots, M_i), \end{aligned}$$

preserving the action of S_n. □

Proof. Clearly, $F^{(n)}_{(a_1,\ldots,a_i)}$ is bijective. We need to show the action of S_n is preserved.

$$\begin{aligned} F^{(n)}_{(a_1,\ldots,a_i)}(f \rhd (\{M_1, M_2, \ldots, M_i\})) &= F^{(n)}_{(a_1\ldots,a_i)}(\{f \blacktriangleright M_1, f \blacktriangleright M_2, \ldots, f \blacktriangleright M_i\}) \\ &= (f \blacktriangleright M_1, f \blacktriangleright M_2, \ldots, f \blacktriangleright M_i) \\ &= f \blacktriangleright (M_1, M_2, \ldots, M_i) \\ &= f \blacktriangleright F^{(n)}_{(a_1,\ldots,a_i)}(\{M_1, M_2, \ldots, M_i\}). \end{aligned}$$

□

Recall that $f^{\#}$ is defined in Lemma 2.2.13:

Proposition 4.3.23. *Let $i \in \{1, \ldots, n\}$ and $0 \le a_1 < a_2 < \cdots < a_i \le n$. The map*

$$(F^{(n)}_{(a_1,\ldots,a_i)})^{\#} \colon \mathbb{C}\mathcal{P}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n) \to \mathbb{C}(\mathcal{P}_{a_1}(X_n) \times \cdots \times \mathcal{P}_{a_i}(X_n))$$

is the isomorphic representation of S_n. □

Proof. From the previous proposition 4.3.22, there is a bijective map preserving the action of S_n from $\mathcal{P}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n)$ to $(\mathcal{P}_{a_1}(X_n) \times \cdots \times \mathcal{P}_{a_i}(X_n))$. Therefore,

$$\mathbb{C}\mathcal{P}_{i,[a_1,\ldots,a_l]}\mathcal{P}(X_n) \cong \mathbb{C}(\mathcal{P}_{a_1}(X_n) \times \cdots \times \mathcal{P}_{a_i}(X_n)).$$

□

Theorem 4.3.24. *Let n be a positive integer. Let $i \in \{1, \ldots, n\}$ and $0 \le a_1 < a_2 < \cdots < a_i \le n$. We have the following decomposition of the representation of S_n on $\mathbb{C}\mathcal{P}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n)$:*

$$\mathbb{C}\mathcal{P}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n) \cong [\bigoplus_{j_1=0}^{min(a_1,n-a_1)} S^{(n-j_1,j_1)}] \otimes [\bigoplus_{j_2=0}^{min(a_2,n-a_2)} S^{(n-j_2,j_2)}] \otimes \cdots \otimes [\bigoplus_{j_i=0}^{min(a_i,n-a_i)} S^{(n-j_i,j_i)}]$$

Proof. By Proposition 4.3.23, we have an isomorphism of representation of S_n

$$\mathbb{C}\mathcal{P}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n) \cong \mathbb{C}(\mathcal{P}_{a_1}(X_n) \times \cdots \times \mathcal{P}_{a_i}(X_n))$$

Let S_n acts on $\mathcal{P}_{a_1}(X_n)$, $\mathcal{P}_{a_2}(X_n)$, ..., $\mathcal{P}_{a_i}(X_n)$. Then by Lemma 4.3.14 we have an action of S_n on $\mathcal{P}_{a_1}(X_n) \times \mathcal{P}_{a_2}(X_n) \cdots \times \mathcal{P}_{a_i}(X_n)$ as:

$$g \blacktriangleright (M_1, \ldots, M_i) = (g \blacktriangleright M_1, g \blacktriangleright M_2, \ldots, g \blacktriangleright M_i),$$

where $g \in S_n$, $M_1 \in \mathcal{P}_{a_1}(X_n)$, $M_2 \in \mathcal{P}_{a_2}(X_n) \ldots$ and $M_i \in \mathcal{P}_{a_i}(X_n)$.

Then, as representation of S_n, we have:

$$\mathbb{C}(\mathcal{P}_{a_1}(X_n) \times \mathbb{C}(\mathcal{P}_{a_2}(X_n) \times \cdots \times \mathcal{P}_{a_i}(X_n)) \cong \mathbb{C}\mathcal{P}_{a_1}(X_n) \otimes \mathbb{C}(\mathcal{P}_{a_2}(X_n) \otimes \cdots \otimes \mathbb{C}\mathcal{P}_{a_i}(X_n).$$

Thus, we get the following:

$$\mathbb{C}\mathcal{P}_{i,[a_1,a_2,\ldots,a_i]}\mathcal{P}(X_n) \cong \mathbb{C}\mathcal{P}_{a_1}(X_n) \otimes \mathbb{C}\mathcal{P}_{a_2}(X_n) \otimes \ldots \mathbb{C}\mathcal{P}_{a_i}(X_n)$$

Applying Lemma 3.2.18, we have:

$$\mathbb{C}\mathcal{P}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n) \cong [\bigoplus_{j_1=0}^{min(a_1,n-a_1)} S^{(n-j_1,j_1)}] \otimes [\bigoplus_{j_2=0}^{min(a_2,n-a_2)} S^{(n-j_2,j_2)}] \otimes \cdots \otimes [\bigoplus_{j_i=0}^{min(a_i,n-a_i)} S^{(n-j_i,j_i)}]$$

□

In the next examples, we are using GAP to compute the tensor product of Specht modules of symmetric group and the code can be found in Appendix A.3.

Example 4.3.25. Let $X_3 = \{1, 2, 3\}$ and S_3 is the symmetric group. The decomposition of the action of S_3 on $\mathbb{C}\mathcal{P}_{4,[0,1,2,3]}\mathcal{P}(X_3)$ into irreducibles will be

$$\begin{aligned}\mathbb{C}\mathcal{P}_{4,[0,1,2,3]}\mathcal{P}(X_3) &\cong [\bigoplus_{j_1=0}^{min(0,3)} S^{(3-j_1,j_1)}] \otimes [\bigoplus_{j_2=0}^{min(1,2)} S^{(3-j_2,j_2)}] \otimes [\bigoplus_{j_3=0}^{min(2,1)} S^{(3-j_3,j_3)}] \otimes [\bigoplus_{j_4=0}^{min(3,0)} S^{(3-j_4,j_4)}] \\ &\cong S^{(3)} \otimes \left(S^{(3)} \oplus S^{(2,1)}\right) \otimes \left(S^{(3)} \oplus S^{(2,1)}\right) \otimes S^{(3)}. \\ &\cong 2S^{(3)} \oplus 3S^{(2,1)} \oplus S^{(1^3)}\end{aligned}$$

Hence, all irreducible representation of S_3 occur in $\mathbb{C}\mathcal{P}_{3,[0,1,2,3]}\mathcal{P}(X_4)$, and hence in $\mathbb{C}\mathcal{PP}(X_3)$.

Example 4.3.26. Let $X_4 = \{1, 2, 3, 4\}$ and S_4 is the symmetric group. The decomposition of the action of S_4 on $\mathbb{C}\mathcal{P}_{4,[0,2,3,4]}\mathcal{P}(X_4)$ into irreducibles will be

$$\begin{aligned}\mathbb{C}\mathcal{P}_{4,[0,2,3,4]}\mathcal{P}(X_4) &\cong [\bigoplus_{j_1=0}^{min(0,4)} S^{(4-j_1,j_1)}] \otimes [\bigoplus_{j_2=0}^{min(2,2)} S^{(4-j_2,j_2)}] \otimes [\bigoplus_{j_3=0}^{min(3,1)} S^{(4-j_3,j_3)}] \otimes [\bigoplus_{j_4=0}^{min(4,0)} S^{(4-j_4,j_4)}] \\ &\cong S^{(4)} \otimes \left(S^{(4)} \oplus S^{(3,1)} \oplus S^{(2,2)}\right) \otimes \left(S^{(4)} \oplus S^{(3,1)}\right) \otimes S^{(4)}. \\ &\cong S^{(4)} \oplus 2S^{(3,1)} \oplus S^{(2,2)} \oplus S^{(2,1^2)}\end{aligned}$$

Proposition 4.3.27. *Let $i \in \{1, \ldots, n\}$ and $0 \leq a_1 < a_2 < \cdots < a_i \leq n$. The dimension of $\mathbb{C}\mathcal{P}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n)$ is given by:*

$$\dim \mathbb{C}\mathcal{P}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n) := \sum_{j_1=0}^{min(a_1,n-a_1)} \dim S^{(n-j_1,j_1)} \times \cdots \times \sum_{j_i=0}^{min(a_i,n-a_i)} \dim S^{(n-j_i,j_i)}$$

Example 4.3.28. Let $X_3 = \{1, 2, 3\}$. Let i={0,1,2,3,4} and $0 < 1 < 2 < 3 < 4$. Then the dimension of $\mathbb{C}\mathcal{P}_{4,[0,1,2,3]}\mathcal{P}(X_3)$ is given by :

$$\begin{aligned}\dim \mathbb{C}\mathcal{P}_{4,[0,1,2,3]}\mathcal{P}(X_3) := &\sum_{j_1=0}^{min(0,3)} \dim S^{(3-j_1,j_1)} \times \sum_{j_2=0}^{min(1,2)} \dim S^{(3-j_2,j_2)} \times \\ &\sum_{j_3=0}^{min(1,2)} \dim S^{(4-j_3,j_3)} \times \sum_{j_4=0}^{min(0,3)} \dim S^{(4-j_4,j_4)}\end{aligned}$$

Therefore, by using Lemma 2.4.27 we get:

$$\begin{aligned}\dim \mathbb{C}\mathcal{P}_{4,[0,1,2,3]}\mathcal{P}(X_3) :=& \dim S^{(3)} \times (\dim S^{(3)} + \dim S^{(2,1)}) \times (\dim S^{(3)} + \dim S^{(2,1)}) \times \dim S^{(3)} \\ =& 9.\end{aligned}$$

4.4 The representation of S_n on $\mathbb{C}\mathcal{PP}(X_n)$ is algebra faithful

Fix an integer $n \geq 2$, and as usual set that $X_n = \{1, 2, \ldots, n\}$, so

$$(X_n)^n = \underbrace{X_n \times \cdots \times X_n}_{n \text{ times}}.$$

Let:

$$\mathfrak{X}_n := \{(a_1, a_2, \dots, a_n) \in (X_n)^n \mid \forall i, j : i \neq j \implies a_i \neq a_j\} \subseteq X_n^{(n)}.$$

Then S_n acts on $\mathfrak{X}_n$ as follows:

$$f.(a_1, a_2, \dots, a_n) = \big(f(a_1), f(a_2), \dots f(a_n)\big).$$

Below we let $U(S_n) = \{f \colon X_n \to X_n \mid f \text{ is a bijection}\}$ be the underlying set of the group S_n. So we have an action of S_n on $U(S_n)$, defined by $f.g = f \circ g$. The left-regular representation of S_n is therefore obtained as $\mathbb{C}U(S_n) = \mathbb{C}S_n$.

Lemma 4.4.1. *We have a bijection $g \colon \mathfrak{X}_n \to U(S_n)$, preserving the action of S_n.*

Proof. Define a map

$$\begin{aligned} g \colon \mathfrak{X}_n &\to U(S_n) \\ (a_1, a_2, \dots, a_n) &\mapsto \begin{pmatrix} 1 & 2 & \dots & n \\ \\ a_1 & a_2 & \dots & a_n \end{pmatrix}. \end{aligned}$$

Clearly, g is a bijection. And then, note that, given $(a_1, a_2, \dots, a_n) \in \mathfrak{X}_n$, and $f \in S_n$, then:

$$\begin{aligned} f.\big(g(a_1, a_2, \dots, a_n)\big) &= f.\left(\begin{pmatrix} 1 & 2 & \dots & n \\ \\ a_1 & a_2 & \dots & a_n \end{pmatrix} \right) \\ &= \begin{pmatrix} 1 & 2 & \dots & n \\ \\ f(a_1) & f(a_2) & \dots & f(a_n) \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} g\big(f.(a_1, a_2, \dots, a_n)\big) &= g.\big(f(a_1), f(a_2), \dots f(a_n)\big) \\ &= \begin{pmatrix} 1 & 2 & \dots & n \\ f(a_1) & f(a_2) & \dots & f(a_n) \end{pmatrix} \end{aligned}$$

□

Proposition 4.4.2. *We have that*

$$\mathbb{C}\mathfrak{X}_n \cong \mathbb{C}U(S_n),$$

as representations of S_n.

Proof. Since there is a bijective map $g\colon \mathfrak{X}_n \to U(S_n)$ preserving the action of S_n from above lemma 4.4.1, then

$$\mathbb{C}\mathfrak{X}_n \cong \mathbb{C}U(S_n),$$

as representations of S_n. □

Proposition 4.4.3. $\mathbb{C}\mathfrak{X}_n$ *is algebra faithful representation.*

Proof. Since $\mathbb{C}U(S_n)$ is the left regular representation, then from Lemma 2.3.10 we have $\mathbb{C}U(S_n)$ is algebra faithful. By Proposition 4.4.2 we have that $\mathbb{C}U(S_n)$ and $\mathbb{C}\mathfrak{X}_n$ are isomorphic representation of S_n, then we use lemma 2.3.8 to get that $\mathbb{C}\mathfrak{X}_n$ is algebra faithful. □

Lemma 4.4.4. *We have an injective map, preserving the action of S_n, defined as:*

$$\begin{aligned} \mathfrak{F}\colon \mathfrak{X}_n &\to \mathcal{PP}(X_n) \\ (a_1, \dots, a_n) &\mapsto \big\{\emptyset, \{a_1\}, \{a_1, a_2\}, \{a_1, a_2, a_3\}, \dots, X_n\big\}. \end{aligned}$$

Moreover, $\mathfrak{F}$ takes values in $\mathcal{P}_{n+1,[0,1,2,\dots,n]}\mathcal{P}(X_n) \subseteq \mathcal{PP}(X_n)$, which recall is invariant under the action of S_n on $\mathcal{PP}(X_n)$.

Proof. It is clear that $\mathfrak{F}$ is an injective. We now need to show that it preservs the action of S_n.

Let $f \in S_n$ and $(a_1, a_2, \dots, a_n) \in \mathfrak{X}_n$. we need to show that

$$\mathfrak{F}(f.(a_1, a_2, \dots, a_n)) = f \rhd \mathfrak{F}((a_1, a_2, \dots, a_n)).$$

Let us start as follows:

$$\begin{aligned}\mathfrak{F}\Big(f.(a_1, a_2, \dots, a_n)\Big) =& \mathfrak{F}\Big((f(a_1), f(a_2), \dots f(a_n))\Big)\\ =& \{\emptyset, \{f(a_1)\}, \{f(a_1), f(a_2)\}, \dots, \{f(a_1), f(a_2), \dots, f(a_n)\}\}\end{aligned}$$

and

$$\begin{aligned}f \rhd \mathfrak{F}\big((a_1, a_2, \dots, a_n)\big) =& f \rhd \big\{\emptyset, \{a_1\}, \{a_1, a_2\}, \dots, \{a_1, a_2 \dots, a_n\}\big\}\\ =& \big\{f \blacktriangleright \emptyset, f \blacktriangleright \{a_1\}, f \blacktriangleright \{a_1, a_2\}, \dots, f \blacktriangleright \{a_1, a_2, \dots, a_n\}\big\}\\ =& \big\{\emptyset, \{f(a_1)\}, \{f(a_1), f(a_2)\}, \dots, \{f(a_1), \dots, f(a_n)\}\big\}\end{aligned}$$

Therefore,

$$\mathfrak{F}(f.(a_1, a_2, \dots, a_n)) = f \rhd \mathfrak{F}((a_1, a_2, \dots, a_n)).$$

□

Proposition 4.4.5. *$\mathbb{C}\mathfrak{X}_n$ is isomorphic to subrepresentation of $\mathbb{C}\mathcal{PP}(X_n)$.*

Proof. Consider the map

$$\varphi\colon \mathfrak{X}_n \to \mathfrak{F}(\mathfrak{X}_n) \subseteq \mathcal{PP}(X_n)$$

defined in Lemma 4.4.4. We already know that φ preserves the action of S_n.

Since $\mathfrak{F}$ is injective, from previous lemma 4.4.4, then $\mathfrak{F}(\mathfrak{X}_n)$ is a subrepresentation of $\mathcal{PP}(X_n)$ and $\mathfrak{F}(\mathfrak{X}_n) \cong \mathfrak{X}_n$. Hence, φ is bijective and $\mathbb{C}\mathfrak{X}_n$ is isomorphic to subrepresentation of $\mathbb{C}\mathcal{PP}(X_n)$.

Corollary 4.4.5.1. *Every irreducible representation is a subrepresentation of $\mathbb{C}\mathcal{PP}(X_n)$.*

□

Theorem 4.4.6. *Let $X_n = \{1, 2, \ldots, n\}$. Then $\mathbb{C}\mathcal{PP}(X_n)$ is algebra faithful representation.*

Proof. Since $\mathbb{C}\mathfrak{X}_\mathfrak{n}$ is isomorphism to subrepresentation of $\mathbb{C}\mathcal{PP}(X_n)$ from the previous Proposition 4.4.5 and it is algebra faithful from Proposition 4.4.3, then by lemma 2.3.8 we have that $\mathbb{C}\mathcal{PP}(X_n)$ is algebra faithful representation as well. $\square$

Chapter 5

The action of S_n on the set of topologies on X_n

Let X be a set. Noting that any topology on a set X is an element of $\mathcal{PP}(X)$, we will study the induced action of $\mathrm{Sym}(X)$ on the set of topologies on X, which we will prove is invariant under the action of $\mathrm{Sym}(X)$ on $\mathcal{PP}(X)$. Let us recall the definition of topology as follows:

Definition 5.0.1. (See for example [Mun75, Section 12] .) Let X be a non-empty set. A set τ of subsets of X is said to be a *topology* on X if:

1. X and the empty set $\emptyset$ belong to τ;
2. The union of any (finite or infinite) collection of set in τ belongs to τ;
3. The intersection of any two finite sets in τ belongs to τ.

The pair (X, τ) is called a *topological space*.

Example 5.0.2. (See for example [Mun75, Section 12, Example 2].) If X is non-empty set, the set of all subsets of X is a topology on X; it is called the discrete topology. The set consisting of X and $\emptyset$ only is also a topology on X; we call it the indiscrete topology,

5.1 Explanation of the action of $\mathrm{Sym}(X)$ on the set of topologies on X

We have an action of S_n on $\mathcal{PP}X_n$ in section 4.1. More generally, we have an action of $\mathrm{Sym}(X)$ on $\mathcal{PP}X$, for any set X, possibly infinite. So, in this section we prove that this action is such that $f \rhd \tau$ is a topology on X if $f \in \mathrm{Sym}(X)$ and τ is a topology on X. We denote the set of all topologies on X by $\mathsf{Top}(X)$. Let us start by recalling some elementary lemmas.

Lemma 5.1.1. *Let X be a set. Consider $f : X \to X$ is injective and $A, B \subseteq X$, then:*

$$f(A \cap B) = f(A) \cap f(B)$$

Lemma 5.1.2. *Let X be a set. Consider $f : X \to X$ is a map and $A, B \subseteq X$, then:*

$$f(A \cup B) = f(A) \cup f(B)$$

Lemma 5.1.3. *Let X be a set. Let $f : X \to X \in \mathrm{Sym}(X)$ and $M \in \mathcal{PP}(X)$. Then:*

$$f \blacktriangleright \bigcup_{A \in M} A = \bigcup_{B \in f \rhd M} B$$

and

$$f \blacktriangleright \bigcap_{A \in M} A = \bigcap_{B \in f \rhd M} B.$$

Proof. We need to show that

$$f \blacktriangleright \bigcup_{A \in M} A = \bigcup_{B \in f \rhd M} B.$$

Let us to prove that in two steps:

Step 1: We need to show that

$$f \blacktriangleright \bigcup_{A \in M} A = \bigcup_{A \in M} (f \blacktriangleright A)$$

Firstly, Suppose $y \in f \blacktriangleright \bigcup_{A \in M} A$. Then there exists an $x \in \bigcup_{A \in M} A$ such that $f(x) = y$. Since $x \in \bigcup_{A \in M} A$, then there exists $B \in M$ such that $x \in B$. So, since $y \in f \blacktriangleright \bigcup_{A \in M} A$, then there exists $B \in M$ and $x \in B$ such that $f(x) = y$. This means if $x \in \bigcup_{A \in M} A$, then there exists $B \in M$ such that $y \in f(B)$. Hence, $y \in \bigcup_{A \in M} f \blacktriangleright A$. Therefore,

$$f \blacktriangleright \bigcup_{A \in M} A \subseteq \bigcup_{A \in M} (f \blacktriangleright A) \tag{5.1}$$

Secondly, Suppose $y \in \bigcup_{A \in M}(f \blacktriangleright A)$. Then there exists $B \in M$ such that $y \in f(B)$. Since, $B \subseteq \bigcup_{A \in M} A$, then $f(B) \subseteq f(\bigcup_{A \in M} A)$. So, $y \in f(\bigcup_{A \in M} A)$. Hence, $y \in f \blacktriangleright \bigcup_{A \in M} A$. Therefore,

$$\bigcup_{A \in M} (f \blacktriangleright A) \subseteq f \blacktriangleright \bigcup_{A \in M} A \tag{5.2}$$

So, from (1) and (2), we have:

$$f \blacktriangleright \bigcup_{A \in M} A = \bigcup_{A \in M} (f \blacktriangleright A) \tag{$*$}$$

Step 2 : We need to show that:

$$\bigcup_{A \in M} (f \blacktriangleright A) = \bigcup_{B \in f \triangleright M} B$$

Firstly, suppose $y \in \bigcup_{A \in M}(f \blacktriangleright A)$. Then there exists $C \in M$ such that $y \in f \blacktriangleright C$. Since $f \blacktriangleright C \in f \triangleright M$, then $y \in \bigcup_{B \in f \triangleright M} B$. Therefore,

$$\bigcup_{A \in M} (f \blacktriangleright A) \subseteq \bigcup_{B \in f \triangleright M} B \tag{5.3}$$

Secondly, suppose

$$y \in \bigcup_{B \in f \triangleright M} B.$$

Then there exists $A \in M$ such that $f \blacktriangleright A \in f \triangleright M$. So, $y \in f \blacktriangleright A$. Hence, $y \in \bigcup_{A \in M}(f \blacktriangleright A)$. Therefore,

$$\bigcup_{B \in f \triangleright M} B \subseteq \bigcup_{A \in M} (f \blacktriangleright A) \tag{5.4}$$

So, from (3) , (4), we have:

$$\bigcup_{A\in M}(f \blacktriangleright A) = \bigcup_{B\in f\triangleright M} B \tag{$**$}$$

Therefore, from (∗) and (∗∗), we have :

$$f \blacktriangleright \bigcup_{A\in M} A = \bigcup_{B\in f\triangleright M} B$$

Now, we want to prove that

$$f \blacktriangleright \bigcap_{A\in M} A = \bigcap_{B\in f\triangleright M} B$$

Let us to prove that in two steps:

Step 1: We need to show that

$$f \blacktriangleright \bigcap_{A\in M} A = \bigcap_{A\in M}(f \blacktriangleright A)$$

Firstly, suppose $y \in f \blacktriangleright \bigcap_{A\in M} A$. There there exists an $x \in \bigcap_{A\in M} A$ such that $f(x) = y$. Since $x \in \bigcap_{A\in M} A$, then for each $A \in M$ we have $x \in A$.

So, given a $y \in f \blacktriangleright \bigcap_{A\in M} A$, there exists x such that $x \in A$ for all $A \in M$ and also $y = f(x)$. So $y \in f(A)$, for all $A \in M$. So $y \in \bigcap_{A\in M}(f \blacktriangleright A)$. Therefore,

$$f \blacktriangleright \bigcap_{A\in M} A \subseteq \bigcap_{A\in M}(f \blacktriangleright A) \tag{5.5}$$

Secondly, Suppose $y \in \bigcap_{A\in M}(f \blacktriangleright A)$. Then for each $A \in M$, we have $y \in f(A)$. Since $A \in M$, then thre exists $a_m \in A$ such that $f(a_m) = y$. Since f is injective, then the elements a_m are all the same,let us call $a = a_m$. Thus, $a \in \bigcap_{A\in M} A$. So,

$y \in f(\bigcap_{A\in M} A)$. Hence, $y \in f \blacktriangleright \bigcap_{A\in M} A$. Therefore,

$$\bigcap_{A\in M} (f \blacktriangleright A) \subseteq f \blacktriangleright \bigcap_{A\in M} A \tag{5.6}$$

From (5) and (6), we have :

$$f \blacktriangleright \bigcap_{A\in M} A = \bigcap_{A\in M} (f \blacktriangleright A) \tag{\#}$$

Step 2 : We need to show that:

$$\bigcap_{A\in M} (f \blacktriangleright A) = \bigcap_{B\in f\triangleright M} B$$

Firstly, let $y \in \bigcap_{A\in M}(f \blacktriangleright A)$. Then for each $C \in M$ we have $y \in f \blacktriangleright C$. We want to prove that if $B \in f \triangleright M$ then $y \in B$. So let $B \in f \triangleright M$ then there exists a $A \in M$ such that $B = f(A)$.

Since $f \blacktriangleright C \in f \triangleright M$. Hence, $y \in \bigcap_{B\in f\triangleright M} B$. Therefore,

$$\bigcap_{A\in M} (f \blacktriangleright A) \subseteq \bigcap_{B\in f\triangleright M} B \tag{5.7}$$

Secondly, let

$$y \in \bigcap_{B\in f\triangleright M} B.$$

Then for each $A \in M$ such that $f \blacktriangleright A \in f \triangleright M$. So, $y \in f \blacktriangleright A$. Hence, $y \in \bigcap_{A\in M}(f \blacktriangleright A)$. Therefore,

$$\bigcap_{B\in f\triangleright M} B \subseteq \bigcap_{A\in M} (f \blacktriangleright A) \tag{5.8}$$

From (7), (8) we have:

$$\bigcap_{A\in M} (f \blacktriangleright A) = \bigcap_{B\in f\triangleright M} B \tag{\# \#}$$

Therefore, from (#) and (##) we have:

$$f \blacktriangleright \bigcap_{A\in M} A = \bigcap_{B\in f\triangleright M} B$$

$\square$

Proposition 5.1.4. *Let X be a set and recall that $\mathcal{P}\mathcal{P}X$ is the power of the power set of X. Then the set of all topologies on X, $\mathsf{Top}(X) \subset \mathcal{P}\mathcal{P}X$, is closed under the action $\triangleright$ of* $\mathrm{Sym}(X)$ *on $\mathcal{P}\mathcal{P}X$.*

Proof. Let X be a set and $f\colon X \to X \in \mathrm{Sym}(X)$. Fix $\tau \in \mathsf{Top}(X)$. Note that

$$f \triangleright \tau = \{f \blacktriangleright d : d \in \tau\}.$$

Then we need to prove that $f \triangleright \tau \in \mathsf{Top}(X)$.

Now, we need to show the three axioms in Definition 5.0.1.

1. We need to show that $X \in f \triangleright \tau$, *and* $\varnothing \in f \triangleright \tau$. By definition

   $$f \blacktriangleright X = f[X] = \{f(x) : x \in X\}.$$

   Since $f \in \mathrm{Sym}(X)$, then $f \blacktriangleright X = X$.

   Now since $X \in \tau$, then:
   $$X \in \{f \blacktriangleright d : d \in \tau\}.$$

   Hence, $X \in f \triangleright \tau$.

   Similarly, by definition
   $$f \blacktriangleright \varnothing = f[\varnothing] = \varnothing.$$

   Since $f \in \mathrm{Sym}(X)$, then $f \blacktriangleright \varnothing = \varnothing$.
   Now since $\varnothing \in \tau$, then:
   $$\varnothing \in \{f \blacktriangleright d : d \in \tau\}.$$

   Thus, $\varnothing \in f \triangleright \tau$.

2. We need to show closure under finite intersection. Let $m, n \in f \triangleright \tau$. Then there exist
   $$A, B \in \tau \ \textit{such that}\ m = f[A] \ \textit{and}\ n = f[B].$$

So, we need to show that : $m \cap n \in f \rhd \tau$. Now,

$$\begin{aligned} m \cap n &= f[A] \cap f[B] \\ &= f[A \cap B]. \end{aligned}$$

because f is injective.

Therefore,

$$m \cap n \in f \rhd \tau.$$

3. We need to show closure of $f \rhd \tau$ under arbitrary union. Consider $M \subseteq f \rhd \tau$ (M is a family of open sets). Then

$$M = \{f \blacktriangleright A : A \in N\}, where N \subseteq \tau.$$

Now, we need to show that

$$\bigcup_{m \in M} m \in f \rhd \tau$$

From Lemma 5.1.3, we have the following:

$$\bigcup_{m \in M \subseteq f \rhd \tau} m = f \blacktriangleright \bigcup_{A \in \tau} A$$

So, Since $A \in \tau$, then $\bigcup A \in \tau$. Hence,

$$f \blacktriangleright \bigcup_{A \in \tau} A \in f \rhd \tau$$

Therefore,

$$\bigcup_{m \in M \subseteq f \rhd \tau} m \in f \rhd \tau$$

So, $\mathsf{Top}(X)$ is closed under $\mathrm{Sym}(X)$ action.

□

Example 5.1.5. Let $X = X_2 = \{1, 2\}$, so $\mathrm{Sym}(X) = S_2 = \{id, (12)\}$. Then

$$\mathsf{Top}(X_2) = \big\{\{X_2, \varnothing\}, \{X_2, \varnothing, \{1\}\}, \{X_2, \varnothing, \{2\}\}, \{X_2, \varnothing, \{1\}, \{2\}\}\big\}.$$

The action of S_2 on $\mathsf{Top}(X_2)$ takes the form:

$id \triangleright \{X_2, \varnothing\} = \{X_2, \varnothing\}$

$id \triangleright \{X_2, \varnothing, \{1\}\} = \{X_2, \varnothing, \{1\}\}$

$id \triangleright \{X_2, \varnothing, \{2\}\} = \{X_2, \varnothing, \{2\}\}$

$id \triangleright \{X_2, \varnothing, \{1\}, \{2\}\} = \{X_2, \varnothing, \{1\}, \{2\}\}$

$(12) \triangleright \{X_2, \varnothing\} = \{X_2, \varnothing\}$

$(12) \triangleright \{X_2, \varnothing, \{1\}\} = \{X_2, \varnothing, \{2\}\}$

$(12) \triangleright \{X_2, \varnothing, \{2\}\} = \{X_2, \varnothing, \{1\}\}$

$(12) \triangleright \{X_2, \varnothing, \{1\}, \{2\}\} = \{X_2, \varnothing, \{1\}, \{2\}\}$

5.1.1 Some special subsets of $\mathsf{Top}(X_n)$

Definition 5.1.6. Let $n \in \mathbb{N}$ and $X_n = \{1, \ldots, n\}$. Given $i \in \{0, \ldots, n\}$, and $0 \leq a_1 \leq a_2 \leq \cdots \leq a_i \leq n$, define :

$$\mathcal{P}^{top}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n) = \mathsf{Top}(X_n) \cap \mathcal{P}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n).$$

Observe that by applying the previous result 4.3.7:

$$\mathsf{Top}(X_n) = \bigsqcup_{\substack{[a_1,\ldots,a_i] \in \{0,1,\ldots,n\}^i \\ 0 \leq a_1 \leq a_2 \leq \cdots \leq a_i \leq n}} \mathcal{P}^{top}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n)$$

Lemma 5.1.7. *Let $n \in \mathbb{N}$ and $X_n = \{1, \ldots, n\}$. Given $i \in \{0, \ldots, n\}$, and $0 \leq a_1 \leq a_2 \leq \cdots \leq a_i \leq n$. $\mathcal{P}^{top}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n)$ is invariant under the action of S_n.*

Proof. From Propositions 4.3.8 and 5.1.4, we note that both terms in the intersection in Definition 5.1.6 are invariant under the action of S_n, so then $\mathcal{P}^{top}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_n)$ is also invariant under the action of S_n. □

In the following example, we took advantage of the list of finite topologies on this wikipedia page [Wik24], see https://en.wikipedia.org/wiki/Finite_topological_space.

Example 5.1.8. Let $X_3 = \{1, 2, 3\}$ and $\mathsf{Top}(X_3)$ be the set of all topologies on X_3. Then we have 29 topologies on X_3 that naturally grouped together as follows:

- $\mathcal{P}^{top}_{2,[0,3]}\mathcal{P}(X_3) = \big\{\{\emptyset, X_3\}\big\}$. This gives one topology.
- $\mathcal{P}^{top}_{3,[0,1,3]}\mathcal{P}(X_3) = \big\{\{\emptyset, \{a\}, X_3\} \mid a \in \{1, 2, 3\}\big\}$. This gives three topologies.
- $\mathcal{P}^{top}_{3,[0,2,3]}\mathcal{P}(X_3) = \big\{\{\emptyset, \{a, b\}, X_3 \mid a \neq b \text{ and } a, b \in \{1, 2, 3\}\big\}$. This gives three topologies.
- $$\mathcal{P}^{top}_{4,[0,1,2,3]}\mathcal{P}(X_3) = \left\{ \{\emptyset, \{a\}, \{a, b\}, X_3\}, \{\emptyset, \{a\}, \{b, c\}, X_3\} \,\middle|\, \begin{matrix} a \neq b, b \neq c, a \neq c, \\ a, b, c \in \{1, 2, 3\} \end{matrix} \right\}.$$
  This gives nine topologies.
- $$\mathcal{P}^{top}_{5,[0,1,1,2,3]}\mathcal{P}(X_3) = \left\{ \{\emptyset, \{a\}, \{b\}, \{a, b\}, X_3\} \,\middle|\, \begin{matrix} a \neq b, b \neq c, a \neq b \\ \text{and} \\ a, b, c \in \{1, 2, 3\} \end{matrix} \right\}.$$
  This gives three topologies.
- $$\mathcal{P}^{top}_{5,[0,1,2,2,3]}\mathcal{P}(X_3) = \left\{ \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X_3\} \,\middle|\, \begin{matrix} a \neq b, b \neq c, a \neq b \\ \text{and} \\ a, b, c \in \{1, 2, 3\} \end{matrix} \right\}.$$
  This gives three topologies.
- $$\mathcal{P}^{top}_{6,[0,1,1,2,2,3]}\mathcal{P}(X_3) = \left\{ \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X_3\} \,\middle|\, \begin{matrix} a \neq b, b \neq c, a \neq c \\ \text{and } a, b, c \in \{1, 2, 3\}. \end{matrix} \right\}.$$
  This gives six topologies.
- $\mathcal{P}^{top}_{8,[0,1,1,1,2,2,2,3]}\mathcal{P}(X_3) = \mathcal{P}(X_3)$, this gives one topology.

Therefore, we the following partition of the set of all topologies in X_3:

$$\begin{aligned}\mathsf{Top}(X_3) =& \mathcal{P}^{top}_{2,[0,3]}\mathcal{P}(X_3) \sqcup \mathcal{P}^{top}_{3,[0,1,3]}\mathcal{P}(X_3) \sqcup \mathcal{P}^{top}_{3,[0,2,3]}\mathcal{P}(X_3) \sqcup \mathcal{P}^{top}_{4,[0,1,2,3]}\mathcal{P}(X_3) \\ &\sqcup \mathcal{P}^{top}_{5,[0,1,1,2,3]}\mathcal{P}(X_3) \sqcup \mathcal{P}^{top}_{5,[0,1,2,2,3]}\mathcal{P}(X_3) \sqcup \mathcal{P}^{top}_{6,[0,1,1,2,2,3]}\mathcal{P}(X_3) \sqcup \mathcal{P}^{top}_{8,[0,1,1,1,2,2,2,3]}\mathcal{P}(X_3).\end{aligned}$$

We will late on show a finer partition of $\mathsf{Top}(X_3)$.

5.1.2 The action of S_n on $\mathbb{C}\mathsf{Top}(X_n)$ is algebra faithful.

In this section, we will let $X_n = \{1, \dots, n\}$ and $\mathsf{Top}(X_n) \subset \mathcal{PP}(X_n)$ be the set of all topologies on X. Also, recall that the free vector space over the set of all topologies on X_n is denoted by $\mathbb{C}\mathsf{Top}(X_n)$. We will consider it with the representation of S_n obtained by linearising the action on $\mathsf{Top}(X_n)$.

Proposition 5.1.9. *Recall that from Section 4.4*

$$\mathfrak{X}_n := \{(a_1, a_2, \dots, a_n) \in (X_n)^n \mid \forall i, j : i \neq j \implies a_i \neq a_j\}.$$

Then $\mathbb{C}\mathfrak{X}_n$ *is isomorphic to a subrepresentation of* $\mathbb{C}\mathsf{Top}(X_n)$*.*

Proof. Recall the injective map

$$\begin{aligned} \mathfrak{F}\colon \mathfrak{X}_n &\to \mathcal{PP}(X_n) \\ (a_1, \dots, a_n) &\mapsto \big\{\emptyset, \{a_1\}, \{a_1, a_2\}, \{a_1, a_2, a_3\}, \dots, X_n\big\} \end{aligned}$$

that was defined as in Lemma 4.4.4. Then clearly $\mathfrak{F}(\mathfrak{X}_n) \subset \mathsf{Top}(X_n)$, which is invariant under the action of S_n. This means given $(a_1, \dots, a_n) \in \mathfrak{X}_\mathfrak{n}$, then $\mathfrak{F}\big((a_1, \dots, a_n)\big)$ is actually a topology. Thus, $\mathbb{C}\mathfrak{X}_\mathfrak{n}$ is isomorphic to subrepresentation of $\mathbb{C}\mathsf{Top}(X_n)$. □

Theorem 5.1.10. *The action of S_n on $\mathbb{C}\mathsf{Top}(X_n)$ is algebra faithful.*

Proof. By Proposition 4.4.3 and Proposition 5.1.9, we have that $\mathbb{C}\mathfrak{X}_\mathfrak{n}$ is algebra faithful representation that is isomorphic to subrepresentation of $\mathbb{C}\mathsf{Top}(X_n)$. Then from lemma 2.3.8 we obtain that $\mathbb{C}\mathsf{Top}(X_n)$ is algebra faithful. □

The proof of the theorem encodes a more general technique that we now explain.

Theorem 5.1.11. *Suppose that S_n acts on Y, a set, so that:*

- *action is transitive, namely, we have only one orbit, equivalently $\forall y \in Y$: $\mathrm{Orb}_{S_n}(y) = Y$;*
- *action is free, namely all stabiliser subgroups are trivial.*

Then:

1. *Given any $y \in Y$ we have a bijection,*

$$F_y \colon S_n \to Y$$
$$g \mapsto g.y,$$

that preserves the action of S_n, where S_n acts on itself by left multiplication.

2. *We have an isomorphism $F_y^{\#} \colon \mathbb{C}S_n \to \mathbb{C}Y$. So $\mathbb{C}Y$ is algebra faithful.*

Proof. 1. • We need first to prove that F_y is a bijective as follows:

(a) **Injective:** Let $F_y(g_1) = F_y(g_2)$. we need to show that $g_1 = g_2$. Since $F_y(g_1) = F_y(g_2)$, then $g_1.y = g_2.y$.

Since the action is free, then the only element that fixes y will be the identity id_G. This means $g_1^{-1}g_2 = id_G$ and this implies $g_1 = g_2$.

(b) **Surjective:** We need to show for every $z \in Y$, there is $g \in S_n$ such that $F_y(g) = z$. Since the action is transitive, for any $z \in Y$, there exists $g \in S_n$ such that $z = g.x = F_y(g)$.

• We need to show that $f.(F_y(g)) = F_y(fg)$.

$$\begin{aligned} f.(F_y(g)) &= f.(F_y(g)) \\ &= f.(g.y) \\ &= (fg)(y) \\ &= F_y(fg). \end{aligned}$$

2. From part1, $\mathbb{C}S_n \cong \mathbb{C}Y$, as representations of S_n. Since $\mathbb{C}S_n$ is the regular representation which is algebra faithful, then from Lemma 2.3.8, we have $\mathbb{C}Y$ is algebra faithful.

□

Clearly

$$\begin{aligned}\mathcal{P}^{top}_{n+1,[0,1,\dots,n]}\mathcal{P}(X_n) =& \Big\{\{\emptyset, \{a_1\}, \{a_1, a_2\}, \dots, X_n\} \mid i, j : a_i \neq a_j\Big\} \\ =& \,\mathrm{Orb}_{S_n}(\{\emptyset, \{a_1\}, \{a_1, a_2\}, \dots, X_n\})\end{aligned} \tag{5.9}$$

The action of S_n on $\mathcal{P}^{top}_{n+1,[0,1,\dots,n]}\mathcal{P}(X_n)$ is transitive and free because the only permutation that fix any element $A \in \mathcal{P}^{top}_{n+1,[0,1,\dots,n]}\mathcal{P}(X_n)$ is the identity permutation. From previous Theorem 5.1.11, we have that $\mathbb{C}\mathcal{P}^{top}_{n+1,[0,1,\dots,n]}\mathcal{P}(X_n)$ is algebra faithful.

We observe that any time we find an orbit of the action of S_n on $\mathsf{Top}(X_n)$ that is free then we found one more algebra faithful subrepresentation of S_n on $\mathbb{C}\mathsf{Top}(X_n)$. We will do this in subsection 5.4 by using the idea of passing to the action on the set of relations.

5.2 The irreducible content of the action of S_n on the free vector space over the set of all topologies on X_n, when $2 \leq n \leq 4$.

In this section, we calculate the irreducible content of the action of S_n on the free vector space over the set of all topologies on X_n when $2 \leq n \leq 4$ by using the same method in Section 4.2. Recall the set of all topologies on X_n denoted by $\mathsf{Top}(X_n)$ and by $\mathcal{P}^{top}\mathcal{P}(X_n)$. The latter notation is because as mentioned before any topology is an element of the power set of power set of X_n, and moreover the action we consider on $\mathsf{Top}(X_n)$ and by $\mathcal{P}^{top}\mathcal{P}(X_n)$ is the restriction of that action.

5.2.1 Computation the decomposition of the action of S_2 on $\mathbb{C}\mathsf{Top}(X_2)$

Let $X_2 = \{1, 2\}$ and $\mathsf{Top}(X_2)$ be the set of all topologies on X_2 which is:

$$\mathsf{Top}_{X_2} = \Big\{\{\varnothing, \{1,2\}\}, \{\varnothing, \{1\}, \{1,2\}\}, \{\varnothing, \{2\}, \{1,2\}\}, \{\varnothing, \{1\}, \{2\}, \{1,2\}\}\Big\}.$$

Then we observe that

$$\mathsf{Top}(X_2) = \mathcal{P}^{top}_{2,[0,2]}\mathcal{P}(X_2) \sqcup \mathcal{P}^{top}_{3,[0,1,2]}\mathcal{P}(X_2) \sqcup \mathcal{P}^{top}_{4,[0,1,1,2]}\mathcal{P}(X_2)$$

Now, by using the same method in Section 4.2 (using GAP and Python) to decompose the action of symmetric group S_n on $\mathbb{C}\mathcal{P}^{top}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_2)$ into irreducible, where $0 \leq a_1 \leq \cdots \leq a_i \leq 2$ and $2 \leq i \leq 4$, we have the following:

$$\begin{aligned}
\mathbb{C}\mathcal{P}^{top}_{2,[0,2]}\mathcal{P}(X_2) &\cong S^{(2)} \\
\mathbb{C}\mathcal{P}^{top}_{3,[0,1,2]}\mathcal{P}(X_2) &\cong S^{(2)} \oplus S^{(1^2)} \\
\mathbb{C}\mathcal{P}^{top}_{4,[0,1,1,2]}\mathcal{P}(X_2) &\cong S^{(2)}
\end{aligned}$$

Therefore, the irreducible content of the action of S_2 on the free vector space over $\mathbb{C}\mathsf{Top}(X_2)$ is

$$\mathbb{C}\mathsf{Top}(X_2) \cong 3S^{(2)} \oplus S^{(1^2)}.$$

5.2.2 Computation the decomposition of the action of S_3 on $\mathbb{C}\mathsf{Top}(X_3)$

Let $X_3 = \{1, 2, 3\}$ and $\mathsf{Top}(X_3)$ be the set of all topologies on X_3 that has 29 distinct topologies as be shown in Example 5.1.8.

We implement the same technique in Section 4.2 to decompose the action of symmetric group S_3 on $\mathbb{C}\mathcal{P}^{top}_{i,[a_1,\ldots,a_i]}\mathcal{P}(X_3)$ into irreducible, where $0 \leq a_1 \leq \cdots \leq a_i \leq 3$ and $2 \leq i \leq 8$, so we get the following (using GAP and Python):

$$\begin{aligned}
\mathbb{C}\mathcal{P}^{top}_{2,[0,3]}\mathcal{P}(X_3) &\cong S^{(3)} \\
\mathbb{C}\mathcal{P}^{top}_{3,[0,1,3]}\mathcal{P}(X_3) &\cong S^{(3)} \oplus S^{(2,1)} \\
\mathbb{C}\mathcal{P}^{top}_{3,[0,2,3]}\mathcal{P}(X_3) &\cong S^{(3)} \oplus S^{(2,1)} \\
\mathbb{C}\mathcal{P}^{top}_{4,[0,1,2,3]}\mathcal{P}(X_3) &\cong 2S^{(3)} \oplus 3S^{(2,1)} \oplus S^{(1^3)} \\
\mathbb{C}\mathcal{P}^{top}_{5,[0,1,1,2,3]}\mathcal{P}(X_3) &\cong S^{(3)} \oplus S^{(2,1)} \\
\mathbb{C}\mathcal{P}^{top}_{5,[0,1,2,2,3]}\mathcal{P}(X_3) &\cong S^{(3)} \oplus S^{(2,1)} \\
\mathbb{C}\mathcal{P}^{top}_{6,[0,1,1,2,2,3]}\mathcal{P}(X_3) &\cong S^{(3)} \oplus 2S^{(2,1)} \oplus S^{(1^3)} \\
\mathbb{C}\mathcal{P}^{top}_{8,[0,1,1,1,2,2,2,3]}\mathcal{P}(X_3) &\cong S^{(3)}
\end{aligned}$$

We notice that all irreducible representation of S_3 occurs in $\mathbb{C}\mathcal{P}^{top}_{4,[0,1,2,3]}\mathcal{P}(X_3)$ and in $\mathbb{C}\mathcal{P}^{top}_{6,[0,1,1,2,2,3]}\mathcal{P}(X_3)$ and hence in $\mathbb{C}\mathsf{Top}_{X_3}$.

As a result, we have the irreducible content of the action of S_3 on the free vector space over the set of all topologies on X_3 as:

$$\mathbb{C}\mathsf{Top}(X_3) \cong 9S^{(3)} \oplus 9S^{(2,1)} \oplus 2S^{(1^3)}.$$

We shall see later on a more conceptual way to obtain this.

5.2.3 Computation the decomposition of the action of S_4 on $\mathbb{C}\mathsf{Top}(X_4)$

In the following example, we took advantage of the list of finite topologies on this wikipedia page [Wik24], see https://en.wikipedia.org/wiki/Finite_topological_space.

Let $X_4 = \{1, 2, 3, 4\}$ and $\mathsf{Top}(X_4)$ be the set of all topologies on X_4. Then we have 355 different topologies on X_4 that are combined as follows:

Let $a, b, c, d \in \{1, 2, 3, 4\}$, where all the elements a, b, c, d are distinct. Then:

- $\mathcal{P}^{top}_{2,[0,4]}\mathcal{P}(X_4) = \{\emptyset, X_4\}$, this gives one topology.
- $\mathcal{P}^{top}_{3,[0,1,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a\}, X_4\}\Big\}$. This gives four topologies.
- $\mathcal{P}^{top}_{3,[0,2,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a, b\}, X_4\}\Big\}$. This gives six topologies.
- $\mathcal{P}^{top}_{3,[0,3,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a, b, c\}, X_4\}\Big\}$. This gives four topologies.
- $\mathcal{P}^{top}_{4,[0,1,2,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a\}, \{a, b\}, X_4\}\Big\}$.This gives twelve topologies.
- $\mathcal{P}^{top}_{4,[0,1,3,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a\}, \{a, b, c\}, X_4\}\Big\}$. This gives twelve topologies.
- $\mathcal{P}^{top}_{4,[0,2,2,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a, b\}, \{c, d\}, X_4\}\Big\}$.This gives three topologies.
- $\mathcal{P}^{top}_{4,[0,2,3,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a, b\}, \{a, b, c\}, X_4\}\Big\}$. This gives twelve topologies.
- $\mathcal{P}^{top}_{5,[0,1,1,2,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a\}, \{b\}, \{a, b\}X_4\}\Big\}$. This gives six topologies.
- $\mathcal{P}^{top}_{5,[0,2,3,3,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a, b\}, \{a, b, c\}, \{a, b, d\}, X_4\}\Big\}$. This gives six topologies.

- $$\mathcal{P}^{top}_{5,[0,1,2,3,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}, X_4\}\Big\} \cup \Big\{\{\emptyset, \{a\}, \{a,b\}, \{a,c,d\}, X_4\}\Big\} \cup \Big\{\{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}, X_4\}\Big\}.$$

  This gives forty-eight topologies.

- $$\mathcal{P}^{top}_{6,[0,1,1,2,3,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,d\}, X_4\}\Big\} \cup \Big\{\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, X_4\Big\}.,$$

  This gives twenty-four topologies.

- $$\mathcal{P}^{top}_{6,[0,1,2,2,3,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a\}, \{a,b\}, \{c,d\}, \{a,c,d\}, X_4\}\Big\} \cup \Big\{\{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}, X_4\}\Big\}.$$

  This gives twenty-four topologies.

- $$\mathcal{P}^{top}_{6,[0,1,2,3,3,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}, \{b,c,d\}, X_4\}\Big\} \cup \Big\{\{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, X_4\}\Big\}$$

  This gives twenty-four topologies.

- $\mathcal{P}^{top}_{7,[0,1,1,2,2,3,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}, X_4\}\Big\}$. This gives twenty-four topologies.
- $\mathcal{P}^{top}_{7,[0,1,2,2,3,3,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,b,d\}, X_4\}\Big\}$. This gives twenty-four topologies.
- $\mathcal{P}^{top}_{7,[0,1,1,2,3,3,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, X_4\}\Big\}$. This gives six topologies.

- $$\begin{aligned}\mathcal{P}^{top}_{8,[0,1,1,2,2,3,3,4]}\mathcal{P}(X_4) = &\Big\{\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,b,d\}, X_4\}\Big\} \\ &\cup \Big\{\{\{\emptyset, \{a\}, \{c\}, \{a,c\}, \{b,d\}, \{a,b,d\}\{b,c,d\}, X_4\}\Big\} \\ &\cup \Big\{\{\emptyset, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,b,d\}, X_4\}\Big\}\end{aligned}$$

  This gives fifty-four topologies.

- $$\mathcal{P}^{top}_{9,[0,1,1,2,2,2,3,3,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,d\}, \{b,c\}, \{a,b,c\}, \{a,b,d\}, X_4\}\Big\}.$$

  This gives twelve topologies.

- $$\mathcal{P}^{top}_{9,[0,1,1,1,2,2,2,3,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a\}, \{a,b\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,b,c\}, \{a,c,d\}, \{a,b,d\}X_4\}\Big\}.$$

  This gives four topologies.

- $$\mathcal{P}^{top}_{9,[0,1,2,2,2,3,3,3,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, X_4\}\Big\}$$

  This gives four topologies.

- $$\mathcal{P}^{top}_{10,[0,1,1,2,2,2,3,3,3,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, X_4\}\Big\}$$

  this gives twelve topologies.

- $$\mathcal{P}^{top}_{10,[0,1,1,1,2,2,2,3,3,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{a,b,d\}, X_4\}\Big\}.$$

This gives twelve topologies.

-

$$\mathcal{P}^{top}_{12,[0,1,1,1,2,2,2,2,3,3,3,4]}\mathcal{P}(X_4) = \Big\{\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, X_4\}\Big\}.$$

This gives twelve topologies.

- $\mathcal{P}^{top}_{16,[0,1,1,1,1,2,2,2,2,2,2,3,3,3,3,4]}\mathcal{P}(X_4) = \mathcal{P}(X_4)$, this gives one topology.

Then we are observing that:

$$\begin{aligned}\mathsf{Top}_{X_4} =& \mathcal{P}^{top}_{2,[0,4]}\mathcal{P}(X_4) \sqcup \mathcal{P}^{top}_{3,[0,1,4]}\mathcal{P}(X_4) \sqcup \mathcal{P}^{top}_{3,[0,2,4]}\mathcal{P}(X_4) \sqcup \mathcal{P}^{top}_{3,[0,3,4]}\mathcal{P}(X_4)\\ &\sqcup \mathcal{P}^{top}_{4,[0,1,2,4]}\mathcal{P}(X_4) \sqcup \mathcal{P}^{top}_{4,[0,1,3,4]}\mathcal{P}(X_4) \sqcup \mathcal{P}^{top}_{4,[0,2,2,4]}\mathcal{P}(X_4) \sqcup \mathcal{P}^{top}_{4,[0,2,3,4]}\mathcal{P}(X_4)\\ &\sqcup \mathcal{P}^{top}_{5,[0,1,1,2,4]}\mathcal{P}(X_4) \sqcup \mathcal{P}^{top}_{5,[0,2,3,3,4]}\mathcal{P}(X_4) \sqcup \mathcal{P}^{top}_{5,[0,1,2,3,4]}\mathcal{P}(X_4)\\ &\sqcup \mathcal{P}^{top}_{6,[0,1,1,2,3,4]}\mathcal{P}(X_4) \sqcup \mathcal{P}^{top}_{6,[0,1,2,3,3,4]}\mathcal{P}(X_4) \sqcup \mathcal{P}^{top}_{6,[0,1,2,2,3,4]}\mathcal{P}(X_4)\\ &\sqcup \mathcal{P}^{top}_{7,[0,1,1,2,2,3,4]}\mathcal{P}(X_4) \sqcup \mathcal{P}^{top}_{7,[0,1,2,2,3,3,4]}\mathcal{P}(X_4) \sqcup \mathcal{P}^{top}_{7,[0,1,1,2,3,3,4]}\mathcal{P}(X_4)\\ &\sqcup \mathcal{P}^{top}_{8,[0,1,1,2,2,3,3,4]}\mathcal{P}(X_4)\\ &\sqcup \mathcal{P}^{top}_{9,[0,1,1,2,2,2,3,3,4]}\mathcal{P}(X_4) \sqcup \mathcal{P}^{top}_{9,[0,1,1,1,2,2,2,3,4]}\mathcal{P}(X_4) \sqcup \mathcal{P}^{top}_{9,[0,1,2,2,2,3,3,3,4]}\mathcal{P}(X_4)\\ &\sqcup \mathcal{P}^{top}_{10,[0,1,1,2,2,2,3,3,3,4]}\mathcal{P}(X_4) \sqcup \mathcal{P}^{top}_{10,[0,1,1,1,2,2,2,3,3,4]}\mathcal{P}(X_4)\\ &\sqcup \mathcal{P}^{top}_{12,[0,1,1,1,2,2,2,2,3,3,3,4]}\mathcal{P}(X_4)\\ &\sqcup \mathcal{P}^{top}_{16,[0,1,1,1,1,2,2,2,2,2,2,3,3,3,3,4]}\mathcal{P}(X_4).\end{aligned}$$

Now, we utilize the same method in Section 4.2 (using GAP and Python) to show the irreducible content of the action of S_4 on the free vector space over the set of all topologies on X_4 as follows:

$$\begin{aligned}\mathbb{C}\mathcal{P}^{top}_{2,[0,4]}\mathcal{P}(X_4) &\cong S^{(4)}\\ \mathbb{C}\mathcal{P}^{top}_{3,[0,1,4]}\mathcal{P}(X_4) &\cong S^{(4)} \oplus S^{(3,1)}\\ \mathbb{C}\mathcal{P}^{top}_{3,[0,2,4]}\mathcal{P}(X_4) &\cong S^{(4)} \oplus S^{(3,1)} \oplus S^{(2,2)}\\ \mathbb{C}\mathcal{P}^{top}_{3,[0,3,4]}\mathcal{P}(X_4) &\cong S^{(4)} \oplus S^{(3,1)}\end{aligned}$$

$$\mathbb{C}\mathcal{P}^{top}_{4,[0,1,2,4]}\mathcal{P}(X_4) \cong S^{(4)} \oplus 2S^{(3,1)} \oplus S^{(2,2)} \oplus S^{(2,1^2)}$$
$$\mathbb{C}\mathcal{P}^{top}_{4,[0,1,3,4]}\mathcal{P}(X_4) \cong 2S^{(4)} \oplus 3S^{(3,1)} \oplus S^{(2,2)} \oplus S^{(2,1^2)}$$
$$\mathbb{C}\mathcal{P}^{top}_{4,[0,2,2,4]}\mathcal{P}(X_4) \cong S^{(4)} \oplus S^{(2,2)}$$
$$\mathbb{C}\mathcal{P}^{top}_{4,[0,2,3,4]}\mathcal{P}(X_4) \cong S^{(4)} \oplus 2S^{(3,1)} \oplus S^{(2,2)} \oplus S^{(2,1^2)}$$
$$\mathbb{C}\mathcal{P}^{top}_{5,[0,1,1,2,4]}\mathcal{P}(X_4) \cong S^{(4)} \oplus S^{(3,1)} \oplus S^{(2,2)}$$
$$\mathbb{C}\mathcal{P}^{top}_{5,[0,2,3,3,4]}\mathcal{P}(X_4) \cong S^{(4)} \oplus S^{(3,1)} \oplus S^{(2,2)}$$
$$\mathbb{C}\mathcal{P}^{top}_{5,[0,1,2,3,4]}\mathcal{P}(X_4) \cong 3S^{(4)} \oplus 7S^{(3,1)} \oplus 4S^{(2,2)} \oplus 5S^{(2,1^2)} \oplus S^{(1^4)}$$

$$\mathbb{C}\mathcal{P}^{top}_{6,[0,1,1,2,3,4]}\mathcal{P}(X_4) \cong 2S^{(4)} \oplus 4S^{(3,1)} \oplus 2S^{(2,2)} \oplus 2S^{(2,1^2)}$$
$$\mathbb{C}\mathcal{P}^{top}_{6,[0,1,2,2,3,4]}\mathcal{P}(X_4) \cong 2S^{(4)} \oplus 4S^{(3,1)} \oplus 2S^{(2,2)} \oplus 2S^{(2,1^2)}$$
$$\mathbb{C}\mathcal{P}^{top}_{6,[0,1,2,3,3,4]}\mathcal{P}(X_4) \cong 2S^{(4)} \oplus 4S^{(3,1)} \oplus 2S^{(2,2)} \oplus 2S^{(2,1^2)}$$
$$\mathbb{C}\mathcal{P}^{top}_{7,[0,1,1,2,2,3,4]}\mathcal{P}(X_4) \cong S^{(4)} \oplus 3S^{(3,1)} \oplus 2S^{(2,2)} \oplus 3S^{(2,1^2)} \oplus S^{(1^4)}$$
$$\mathbb{C}\mathcal{P}^{top}_{7,[0,1,2,2,3,3,4]}\mathcal{P}(X_4) \cong S^{(4)} \oplus 3S^{(3,1)} \oplus 2S^{(2,2)} \oplus 3S^{(2,1^2)} \oplus S^{(1^4)}$$
$$\mathbb{C}\mathcal{P}^{top}_{7,[0,1,1,2,3,3,4]}\mathcal{P}(X_4) \cong S^{(4)} \oplus S^{(3,1)} \oplus S^{(2,2)}$$

$$\mathbb{C}\mathcal{P}^{top}_{8,[0,1,1,2,2,3,3,4]}\mathcal{P}(X_4) \cong 3S^{(4)} \oplus 7S^{(3,1)} \oplus 5S^{(2,2)} \oplus 6S^{(2,1^2)} \oplus 2S^{(1^4)}$$
$$\mathbb{C}\mathcal{P}^{top}_{9,[0,1,1,2,2,2,3,3,4]}\mathcal{P}(X_4) \cong S^{(4)} \oplus 2S^{(3,1)} \oplus S^{(2,2)} \oplus S^{(2,1^2)}$$
$$\mathbb{C}\mathcal{P}^{top}_{9,[0,1,1,1,2,2,2,3,4]}\mathcal{P}(X_4) \cong S^{(4)} \oplus S^{(3,1)}$$
$$\mathbb{C}\mathcal{P}^{top}_{9,[0,1,2,2,2,3,3,3,4]}\mathcal{P}(X_4) \cong S^{(4)} \oplus S^{(3,1)}$$
$$\mathbb{C}\mathcal{P}^{top}_{10,[0,1,1,2,2,2,3,3,3,4]}\mathcal{P}(X_4) \cong S^{(4)} \oplus 2S^{(3,1)} \oplus S^{(2,2)} \oplus S^{(2,1^2)}$$
$$\mathbb{C}\mathcal{P}^{top}_{10,[0,1,1,1,2,2,2,3,3,4]}\mathcal{P}(X_4) \cong S^{(4)} \oplus 2S^{(3,1)} \oplus S^{(2,2)} \oplus S^{(2,1^2)}$$
$$\mathbb{C}\mathcal{P}^{top}_{12,[0,1,1,1,2,2,2,2,3,3,3,4]}\mathcal{P}(X_4) \cong S^{(4)} \oplus 2S^{(3,1)} \oplus S^{(2,2)} \oplus S^{(2,1^2)}$$
$$\mathbb{C}\mathcal{P}^{top}_{16,[0,1,1,1,1,2,2,2,2,2,2,3,3,3,3,4]}\mathcal{P}(X_4) \cong S^{(4)}$$

Consequently, we have the decomposition of the action of symmetric group S_4 on

$\mathbb{C}\mathsf{Top}(X_4)$ into irreducible as:

$$\mathbb{C}\mathsf{Top}(X_4) \cong 33S^{(4)} \oplus 54S^{(3,1)} \oplus 32S^{(2,2)} \oplus 30S^{(2,1^2)} \oplus 6S^{(1^4)}.$$

5.3 Topologies on finite sets and reflexive and transitive relations

We start by recalling some well-known facts about bases of topological spaces.

5.3.1 Basis of topology

Definition 5.3.1. (See for example [Mun75, Section 13].) Let X be a set and $\beta \subseteq \mathcal{P}(X)$. Then β is said to be a *basis in* X if the following holds:

1. for each $x \in X$, there is at least one $B \in \beta$ containing x;
2. Let $B_1, B_2 \in \beta$. If $x \in X$ belongs to the intersection of B_1 and B_2, then there is a $B_3 \in \beta$, containing x, such that $B_3 \subset B_1 \cap B_2$.

To the elements of β we call *basis elements*.

Example 5.3.2. Let X be a set. The set $\{\{x\} : x \in X\}$ is a basis in X.

Example 5.3.3. Let X be a set. The set $\beta = \{X\}$ is a basis in X.

Example 5.3.4. If τ is a topology on a set X, then it is a basis in X.

Lemma 5.3.5. *(See for example [Mun75, Section 13].) Let X be a set. Suppose that β is a basis in X and $\mathcal{P}(X)$ is the power set of X. Then:*

$$\tau_\beta := \{A \in \mathcal{P}(X) \mid \forall x \in A, \exists B \in \beta \colon x \in B \subset A\}$$

is a topology on X.

Example 5.3.6. Let $X = \{1, 2, 3\}$ and $\beta = \{\{1\}, \{2\}, \{3\}, \{1, 2\}\}$ be a basis in X. Then:

$$\tau_\beta = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, X\}$$

The basis elements in β are always opens sets in τ_β.

The three different bases in the above examples generate the discrete topology, indiscrete topology and the topology τ itself respectively.

Definition 5.3.7. [Mun75, Section 13]Let X be a set. Let β be a basis in X. To τ_β we call the *topology generated by* β. We may also say that β *generates* τ.

Lemma 5.3.8. *(See for example [Mun75, Lemma 13.1].) Let X be a set and β is a basis for a topology τ on X. Then τ_β is the set of all subsets of X that can be written as the union (arbitrary union) of elements of β.*

Different bases in X may generate the same topology. In the case when X is finite, however, there is a 'canonical' basis for any topology on X. We now follow Mays' note [May03]

Definition 5.3.9. (See for example [May03].) Let (X, τ) be a finite topological space. For $x \in X$, consider U_x^τ to be the intersection of the open sets that contain x. (Note that this is a finite intersection since we are working with a finite topological space, so U_x^τ is open.) Also put:

$$\mathrm{Min}(\tau) := \{U_x^\tau : x \in X\}.$$

Lemma 5.3.10. *(See for example [May03, Lemma 1.13].) Continuing the previous definition, we have:*

1. $\mathrm{Min}(\tau)$ *is a basis in* X.
2. *The topology generated by* $\mathrm{Min}(\tau)$ *is* τ, *itself; so*

$$\tau_{\mathrm{Min}(\tau)} = \tau.$$

Proof. 1. We need to prove that $\mathrm{Min}(\tau)$ is a basis as follows:

(a) For all $z \in X$. It is clear from Definition 5.3.9 that picking $U_z^\tau \in \mathrm{Min}(\tau)$, then $z \in U_z^\tau$. Hence, there is $U = U_z^\tau$ such that $z \in U$.

(b) Let $U_z^\tau, V_z^\tau \in \mathrm{Min}(\tau)$ and let $z \in U_z^\tau \cap V_z^\tau$. Since U_z^τ and V_z^τ are open sets containing z, then $W_z^\tau \subseteq U_z^\tau \cap V_z^\tau$. Now note that $z \in W_z^\tau$ and $W_z^\tau \in \mathrm{Min}(\tau)$.

2. We need to prove that $\tau_{\mathrm{Min}(\tau)} = \tau$ as the following:

(a) Firstly, we want to show if $U \in \tau$, then $U \in \tau_{\mathrm{Min}(\tau)}$. We need to prove given $x \in U$, there exists $U_x^\tau \in \mathrm{Min}(\tau)$ such that $x \in U_x^\tau \subseteq U$.

Now, let $x \in U$. Then $x \in U_x^\tau \subseteq U$ because U_x^τ is the intersection of all open set containing x and $x \in U$. Therefore, $U \in \tau_{\mathrm{Min}\,(\tau)}$.

(b) Secondly, we need to show if U is open in $\tau_{\mathrm{Min}(\tau)}$, then U is open in τ. Let $U \in \tau_{Min(\tau)}$. Then there exists an indexed family $\{U_{x_i}^\tau\}_{i \in I}$ with $U_{x_i}^\tau \in \mathrm{Min}(\tau)$ and $U = \bigcup_{i \in I} U_{x_i}^\tau$ which is open in τ since each $U_{x_i}^\tau$ is open in τ.

□

Notation 5.3.11. Let X be a set. We denoted the set of all bases in X by $B(X)$. Note $B(X) \subset \mathcal{PP}(X)$.

Lemma 5.3.12. *The action of* $\mathrm{Sym}(X)$ *on* $\mathcal{PP}(X)$ *restricts to an action of* $\mathrm{Sym}(X)$ *on* $B(X)$.

Proof. Suppose $f \in \mathrm{Sym}(X)$ and $\beta \in B(X)$. Define

$$\triangleright : \mathrm{Sym}(X) \times B(X) \to B(X)$$

as:

$$f \triangleright \beta = \{f \blacktriangleright b : b \in \beta\} = \{f[b] : b \in \beta\}.$$

We need to prove that if $\beta \in B(X)$ and $f \in \mathrm{Sym}(X)$, then $f \triangleright \beta \in B(X)$.

1. Firstly, we need to prove for all $x \in X$, there is $B \in f \triangleright \beta$ such that $x \in B$. Since $\beta \in B(X)$, then there exists a $B' \in \beta$ such that $f^{-1}(x) \in B'$. Thus, we have $x \in f[B']$ with $f[B'] \in f \triangleright \beta$.

2. Given $B_1, B_2 \in f \triangleright \beta$, for all $x \in B_1 \cap B_2$, we need to prove there exist $B_3 \in f \triangleright \beta$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

   Since $x \in B_1 \cap B_2$ with $B_1 = f(B_1')$ and $B_2 = f(B_2')$, where $B_1', B_2' \in \beta$, then $f^{-1}(x) \in B_1' \cap B_2'$. So, there exists $B_3' \in \beta$ such that $f^{-1}(x) \in B_3' \subseteq B_1' \cap B_2'$. Since, f is bijection, then $x \in f[B_3'] \subseteq f[B_1'] \cap f[B_2']$. Therefore, there exist $f[B_3'] \in f \triangleright \beta$ such that $x \in f[B_3'] \subseteq f[B_1'] \cap f[B_2']$.

□

Lemma 5.3.13. *Consider τ_β is the topology generated by a basis β and $f \in \mathrm{Sym}(X)$. Then we have:*

$$f \triangleright \tau_\beta = \tau_{f \triangleright \beta}$$

Proof.

1. Firstly, we need to prove that if $A \in f \triangleright \tau_\beta$, then $A \in \tau_{f \triangleright \beta}$.

   Let $A \in f \triangleright \tau_\beta$. Then for some $B \in \tau_\beta$, we can put: $A = f \triangleright B$. Since $B \in \tau_\beta$, then from Lemma 5.3.8, we get $B = \bigcup_{i \in I} B_i$, where $B_i \in \beta$. Thus, $A = f \triangleright (\bigcup_{i \in I} B_i)$. Since $f \in \mathrm{Sym}(X)$, then

   $$f \triangleright (\bigcup_{i \in I} B_i) = \bigcup_{i \in I} (f \triangleright B_i).$$

   Since $B_i \in \beta$, then $f \triangleright B_i \in f \triangleright \beta$. So, $A = \bigcup_{i \in I} (f \triangleright B_i)$ with $f \triangleright B_i \in f \triangleright \beta$. Therefore, $A \in \tau_{f \triangleright \beta}$.

2. Now, we need to prove that if $A \in \tau_{f \triangleright \beta}$, then $A \in f \triangleright \tau_\beta$.

   Let $A \in \tau_{f \triangleright \beta}$. Then

   $$A = \bigcup_{i \in I} (f \triangleright U_i),$$

   where $U_i \in \beta$. So,

   $$A = f \triangleright (\bigcup_{i \in I} U_i).$$

   But

   $$\bigcup_{i \in I} (U_i) \in \tau_\beta.$$

   Therefore, $A \in f \triangleright \tau_\beta$.

□

Lemma 5.3.14. *Let (X, τ) be topological space. Let $\mathrm{Min}(\tau)$ be defined as in Definition 5.3.9. Let $f \in \mathrm{Sym}(X)$. Then we have:*

$$f \triangleright \mathrm{Min}(\tau) = \mathrm{Min}(f \triangleright \tau).$$

Proof. By definition

$$\begin{aligned}\mathrm{Min}(\tau) =& \{U_x^\tau : x \in X\} \\ =& \Big\{ \bigcap_{\substack{V\in\tau \\ x\in V}} V : x \in X \Big\}.\end{aligned}$$

Then:

$$\begin{aligned} f \rhd \mathrm{Min}(\tau) &= f \rhd \Big\{ \bigcap_{\substack{V\in\tau \\ x\in V}} V : x \in X \Big\} \\ &= \Big\{ f[\bigcap_{\substack{V\in\tau \\ x\in V}} V] : x \in X \Big\} \\ &= \Big\{ \bigcap_{\substack{V\in\tau \\ x\in V}} f[V] : x \in X \Big\}, \qquad \text{since } f \text{ is bijective.} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Min}(f \rhd \tau) &= \Big\{ \bigcap_{\substack{U\in f\rhd\tau \\ y\in U}} U : y \in X \Big\} \\ &= \Big\{ \bigcap_{\substack{V\in\tau \\ y\in f[V]}} f[V] : y \in X \Big\} \\ &= \Big\{ \bigcap_{\substack{V\in\tau \\ x\in V}} f[V] : x \in X \Big\}, \qquad \text{since } f \text{ is bijective.} \end{aligned}$$

Therefore,

$$f \rhd \mathrm{Min}(\tau) = \mathrm{Min}(f \rhd \tau).$$

□

Lemma 5.3.15. *Let X be a finite set. Let $B(X) \subseteq \mathcal{P}\mathcal{P}(X)$ be the set of all bases in X and $\mathsf{Top}(X)$ be the set of all topologies on X. Then there are:*

1. *A surjective map*

$$\begin{aligned} \mathfrak{L} \colon B(X) &\twoheadrightarrow \mathsf{Top}(X) \\ \beta &\mapsto \tau_\beta \end{aligned}$$

that, moreover, preserves the action of $\mathrm{Sym}(X)$.

2. An injective map

$$\begin{aligned} \mathfrak{M} \colon \mathsf{Top}(X) &\hookrightarrow B(X) \\ \tau &\mapsto \mathrm{Min}(\tau). \end{aligned}$$

that, moreover, preserves the action of $\mathrm{Sym}(X)$*.*

Proof. 1. (a) We need to prove that $\mathfrak{L}$ is surjective. so we need to show is that if τ in Top_X, there is a basis β such that $\tau = \tau_\beta$. This clear from Lemma 5.3.10.

(b) We need to show that

$$f \rhd \mathfrak{L}(\beta) = \mathfrak{L}(f \rhd \beta)$$

Let us start

$$\begin{aligned} f \rhd \mathfrak{L}(\beta) &= f \rhd \tau_\beta \\ &= \tau_{f \rhd \beta} \qquad \text{from Lemma 5.3.13} \\ &= \mathfrak{L}(f \rhd \beta) \end{aligned}$$

2. (a) We need to prove that $\mathfrak{M}$ is injective. We need to show that

$$\tau \neq \tau^{'} \Longrightarrow \mathfrak{M}(\tau) \neq \mathfrak{M}(\tau^{'}).$$

We show the proof by using contradiction. Suppose $\mathfrak{M}(\tau) = \mathfrak{M}(\tau^{'})$. This means $\mathrm{Min}(\tau) = \mathrm{Min}(\tau^{'})$. However, we know if two topologies generated by the same basis that means the two topologies are equal. Thus, $\tau = \tau^{'}$. This is contradiction with our assumption. Therefore, $\mathfrak{M}(\tau) \neq \mathfrak{M}(\tau^{'})$.

3. We want to show that

$$f \rhd \mathfrak{M}(\tau) = \mathfrak{M}(f \rhd \tau).$$

Let us proceed as follows:

$$\begin{aligned} f \triangleright \mathfrak{M}(\tau) &= f \triangleright \operatorname{Min}(\tau) \\ &= \operatorname{Min}(f \triangleright \tau) \qquad \text{from Lemma 5.3.14} \\ &= \mathfrak{M}(f \triangleright \tau). \end{aligned}$$

□

5.3.2 Finite topologies and relations

Recall that a relation (or binary relation) on a set X is a subset of $X \times X$. (See[Hal60].)

A binary relation $\leq$ on a set X is reflexive if each element of X relates to itself: i.e. $\forall x \in X : x \leq x$. Also, a relation $\leq$, is called a transitive if given $x \leq y$ and $y \leq z$, then $x \leq z$. (See [Hal60].)

We now go back to the case when all sets are finite.

Lemma 5.3.16 ((See for example [May03, Lemma 1.13.].)). *Let X be a finite set with a reflexive and transitive relation $\leq$. The set of all sets of the form:*

$$B_x^{\leq} := \{y : y \leq x\},$$

where $x \in X$, is a basis in X (in the sense of (5.3.1)), denoted, $\mathcal{B}_{\leq}$.

Proof. We show $\mathcal{B}_{\leq}$ is a basis as follows:

1. Let $x \in X$. Since $\leq$ is reflexive, then it is clear that

$$x \in B_x^{\leq} = \{y : y \leq x\} \in \mathcal{B}_{\leq}$$

   because $x \leq x$. Therefore, the 1st condition of basis holds.

2. Let $x_1, x_2 \in X$. Let $x \in B_{x_1}^{\leq} \cap B_{x_2}^{\leq}$ we want to prove there exist $x_3 \in X$ such that $x \in B_{x_3^{\leq}} \subseteq B_{x_1}^{\leq} \cap B_{x_2}^{\leq}$. Define $x_3 = x$. We already know that $x \in B_x^{\leq}$.

   Claim. $B_x^{\leq} \subseteq B_{x_1}^{\leq} \cap B_{x_2}^{\leq}$.

   Proof. We want to prove if $z \in B_x^{\leq}$, then $z \in B_{x_1}^{\leq}$ and $z \in B_{x_2}^{\leq}$. Let $z \in B_x^{\leq}$.

Then $z \leq x$. Moreover, $x \leq x_1$ and $x \leq x_2$ because $x \in B_{x_1}^{\leq}$ and $x \in B_{x_2}^{\leq}$. By transitivity, we have $z \leq x_1$ and $z \leq x_2$. Therefore, $z \in B_{x_1}^{\leq} \cap B_{x_2}^{\leq}$. So, this proves the second condition of basis. □

□

Definition 5.3.17. If X is a set and $\leq$ is a reflexive and transitive relation, in X, we put

$$\tau_{\leq} := \tau_{\mathcal{B}_{\leq}}$$

Lemma 5.3.18. *Let τ be a topology on X, finite. If $x, y \in X$ then:*

$$x \in U_y^\tau \Longleftrightarrow U_x^\tau \subseteq U_y^\tau.$$

Proof. We prove each implication separately.

- $\Leftarrow$ Suppose $U_x^\tau \subseteq U_y^\tau$. We want to prove $x \in U_y^\tau$. Let $x \in U_x^\tau$. Hence, it is clear $x \in U_y^\tau$.
- $\Rightarrow$ Suppose $x \in U_y^\tau$, we want to prove that $U_x^\tau \subseteq U_y^\tau$. Since U_y^τ is open and U_x^τ is the smallest open set containing x, then $U_x^\tau \subseteq U_y^\tau$.

□

Lemma 5.3.19. *Let X be a finite set, with a topology τ. We have a reflexive transitive relation, on X, where $x \leq_\tau y$, if, and only if, $x \in U_y^\tau$.*

Proof. We need to prove that $\leq_\tau$ is reflexive and transitive relation as follows:

- **Reflexivity:** It is clear that $x \leq_\tau x$, since $x \in U_x^\tau$.
- **Transitivity:** Let $x \leq_\tau y$ and $y \leq_\tau z$. This means $x \in U_y^\tau$ and $y \in U_z^\tau$. So, $x \in U_y^\tau \subseteq U_z^\tau$. Therefore, $x \leq_\tau y$.

□

So, as in [May03, Proposition 1.16], we have a function

$$\begin{aligned} F : \mathsf{Top}(X) &\to \mathsf{Rel}(X) \\ \tau &\mapsto \leq_\tau . \end{aligned}$$

Also put:

$$\begin{aligned} \overline{F} : \mathsf{Rel}(X) &\to \mathsf{Top}(X) \\ \leq \quad &\mapsto \quad \tau_{\leq}. \end{aligned}$$

Our aim is to prove that they are mutually inverse, and that both preserve the action of $\mathrm{Sym}(X)$.

Lemma 5.3.20. *Let $\tau \in \mathsf{Top}(X)$. Then, if $y \in X$, $U_y^\tau = B_y^{\leq_\tau}$*

Proof.

$$\begin{aligned} z \in U_y^\tau &\iff z \leq_\tau y \quad \text{by Lemma 5.3.19} \\ &\iff z \in B_y^{\leq_\tau} = \{m : m \leq_\tau y\}. \end{aligned}$$

□

Lemma 5.3.21. *Let X be a finite set. Let $\leq$ be a reflexive and transitive relation in X. Then*

1. *$x \leq y$ happens if, and only if, $B_x^{\leq} \subseteq B_y^{\leq}$;*
2. *Given $x \in X$. $B_x^{\leq} = U_x^{\tau_{\leq}}$.*
3. *$\mathcal{B}_{\leq} = \mathrm{Min}(\tau_{\leq})$.*

Proof. 1. We prove each implication separately.

- $\Leftarrow$ Suppose $B_x^{\leq} \subseteq B_y^{\leq}$. We want to prove $x \leq y$. Since $x \in B_x^{\leq}$, then $x \in B_y^{\leq}$. Therefore, $x \leq y$.
- $\Rightarrow$ Suppose $x \leq y$, we want to prove that $B_x^{\leq} \subseteq B_y^{\leq}$. Let $z \in B_x^{\leq}$. Then $z \leq x$. Since, $x \leq y$, then by transitivity $z \leq y$. Thus, $z \in B_y^{\leq}$.

2. Given $x \in X$.

- We want to prove that $B_x^{\leq} \subseteq U_x^{\tau_{\leq}}$. Since $x \in U_x^{\tau_{\leq}}$ and $U_x^{\tau_{\leq}}$ is open in $\tau_{\leq}$, there exists a basis element $x \in B_z^{\leq} \subseteq U_x^{\tau_{\leq}}$. So, $x \leq z$. Hence, by transitivity we have $B_x^{\leq} \subseteq B_z^{\leq}$. Therefore, $B_x^{\leq} \subseteq U_x^{\tau_{\leq}}$.
- We need to prove that $U_x^{\tau_{\leq}} \subseteq B_x^{\leq}$. We know $x \in U_x^{\tau_{\leq}}$. Since $\{B_y^{\leq} : y \in X\}$ is a basis for $\tau_{\leq}$, then $B_x^{\leq}$ is open in $\tau_{\leq}$. We know $U_x^{\tau_{\leq}}$ is smallest open

set in $\tau_{\leq}$ containing x. Therefore, $U_x^{\tau_{\leq}} \subseteq B_x^{\leq}$.

3. It is clear that $\mathrm{Min}(\tau_{\leq}) = B_{\leq}$ from the previous statement (2).

□

Theorem 5.3.22 ([May03], Proposition 1.16)**.** *Let (X, τ) be a finite topological space. Let $\mathsf{Rel}(X)$ be the set of transitive, reflexive relations on X and $\mathsf{Top}(X)$ is the set of finite topologies spaces.*

The functions

$$F\colon \mathsf{Top}(X) \to \mathsf{Rel}(X)$$

and

$$\overline{F}\colon \mathsf{Rel}(X) \to \mathsf{Top}(X)$$

are mutually inverse to each other. I.e. $F \circ \overline{F} = id_{\mathsf{Rel}(X)}$ and $\overline{F} \circ F = id_{\mathsf{Top}(X)}$.

Proof. 1. We need to prove $F \circ \overline{F} = id_{\mathsf{Rel}(X)}$ .I.e. $\leq = \leq_{(\tau_{\leq})}$.

$$\begin{aligned} x \leq_{\tau_{\leq}} y &\iff x \in U_y^{\tau_{\leq}} \quad \text{by Lemma 5.3.19} \\ &\iff x \in B_y^{\leq} \quad \text{by Lemma 5.3.21 part 2} \\ &\iff x \in \{z : z \leq y\} \\ &\iff x \leq y \end{aligned}$$

2. We need to prove $\overline{F} \circ F = id_{\mathsf{Top}(X)}$. I.e. $\tau = \tau_{(\leq_\tau)}$. It is suffices to prove that

$$\mathrm{Min}(\tau) = \mathrm{Min}(\tau_{\leq_\tau}).$$

So, it is suffices to show that

$$\forall x \in X : U_x^{\tau_{\leq_\tau}} = U_x^{\tau}$$

Let us start as follows:

$$\begin{aligned} U_x^{\tau_{\leq_\tau}} &= B_x^{\leq_\tau} \quad \text{by Lemma 5.3.21 part 2} \\ &= U_x^\tau \quad \text{by Lemma 5.3.20.} \end{aligned}$$

□

We now discuss the action of Sym(X) on the set of transitive reflexive relations on X.

Proposition 5.3.23. *Let* $\mathsf{Rel}(X)$ *be the set of transitive, reflexive relations on* X*. Let* $\leq \in \mathsf{Rel}(X)$*. Then we have an action of* $\mathrm{Sym}(X)$ *on* $\mathsf{Rel}(X)$*, denoted by* $f \triangleright (\leq)$*, defined as:*

$$x\, f(\leq)\, y \iff f^{-1}(x) \leq f^{-1}(y).$$

Proof.

- **Claim 1:** $f(\leq)$ is reflexive.

  Proof. It is clear. □

- **Claim 2:** $f(\leq)$ is transitive.

  Proof. Let $x\, f(\leq)\, y$ and $y\, f(\leq)\, z$. Then

$$\begin{aligned} x\, f(\leq)\, y \,\text{and}\; y\, f(\leq)\, z &\iff f^{-1}(x) \leq f^{-1}(y) \,\text{and}\, f^{-1}(y) \leq f^{-1}(z) \\ &\iff f^{-1}(x) \leq f^{-1}(z) \\ &\iff x\, f(\leq)\, z. \end{aligned}$$

□

- **Claim 3** :

$$(fg)^{-1} \triangleright (\leq) = f^{-1}(g^{-1} \triangleright \leq)$$

Proof.

$$\begin{aligned} L.H.S. &= x\,(fg)(\leq) y \\ &= (f \circ g)^{-1}(x) \leq (f \circ g)^{-1}(y) \\ &= g^{-1} \circ f^{-1}(x) \leq g^{-1} \circ f^{-1}(y) \end{aligned}$$

$$\begin{aligned} R.H.S. &= x\, f(g(\leq))y \\ &= f^{-1}(x) \leq (g(\leq))f^{-1}(y) \\ &= g^{-1}(f^{-1}(x)) \leq g^{-1}(f^{-1}(y)) \\ &= g^{-1} \circ f^{-1}(x) \leq g^{-1} \circ f^{-1}(y) \end{aligned}$$

Therefore, L.H.S. =R.H.S. □

Lemma 5.3.24. *Let $f \in \mathrm{Sym}(X)$. Then*

$$f(U_x^\tau) = U_{f(x)}^{f(\tau)}.$$

Proof.

$$\begin{aligned} f(U_x^\tau) &= f(\bigcap_{\substack{x \in A \\ A \in \tau}} A) \\ &= \bigcap_{\substack{f(x) \in f(A) \\ f(A) \in f(\tau)}} f(A) \\ &= U_{f(x)}^{f(\tau)}. \end{aligned}$$

□

□

Proposition 5.3.25. *The bijections*

$$F\colon \mathsf{Top}(X) \to \mathsf{Rel}(X)$$

and

$$\overline{F}\colon \mathsf{Rel}(X) \to \mathsf{Top}(X)$$

preserve the action of symmetric group S_n.

Proof. $\forall f \in S_n$, $\tau \in \mathsf{Top}(X)$. Then we need to prove that

$$F(f \rhd \tau) = f \rhd F(\tau).$$

This means we need to show that

$$x \leq_{f(\tau)} y \iff x f(\leq_\tau) y.$$

Let us start

$$\begin{aligned} x \leq_{f(\tau)} y &\iff x \in U_y^{f(\tau)} \quad \text{by Lemma 5.3.19} \\ &\iff x \in f(U_x^\tau) \quad \text{by Lemma 5.3.24} \\ &\iff x f(\leq_\tau) y. \end{aligned}$$

□

Theorem 5.3.26. *Let X be a finite set. There is a bijection between the set of reflexive and transitive relations, $\leq$, on X and the set of all topologies in X. Moreover, the bijection preserves the action of* $\mathrm{Sym}(x)$.

The first statement is in [May03, Proposition 1.16].

Proof. The first statement from Theorem 5.3.22, and the second statement from Proposition 5.3.25. □

5.4 One more partition of set $\mathsf{Top}(X_n)$ via $\mathsf{Rel}(X_n)$

We now go back to considering $X = X_n$, where n is a positive integer. So $\mathrm{Sym}(X) = S_n$.

Recall that there is a bijection $\mathsf{Top}(X_n) \to \mathsf{Rel}(X_n)$ preserving the action of S_n from Theorem 5.3.26. Then the the S_n-sets $\mathsf{Top}(X_n)$ and $\mathsf{Rel}(X_n)$ are isomorphic . Therefore, as representations of S_n, we have $\mathbb{C}\mathsf{Rel}(X_n) \cong \mathbb{C}\mathsf{Top}(X_n)$.

Now, we have

$$\mathsf{Rel}(X_n) = \bigsqcup_{\mathcal{O} \in \mathsf{Rel}(X_n)/S_n} \mathcal{O},$$

a partition of a set into orbits. So, as representation of S_n, we obtain

$$\mathbb{C}\mathsf{Rel}(X_n) \cong \bigoplus_{\mathcal{O} \in \mathsf{Rel}(X_n)/S_n} \mathbb{C}\mathcal{O}.$$

Also, we have

$$\mathsf{Top}(X_n) = \bigsqcup_{\substack{[a_1,\dots,a_i]\in\{0,1,\dots,n\}^i \\ 0\le a_1\le a_2\le\cdots\le a_i\le n}} \mathcal{P}^{top}_{i,[a_1,\dots,a_i]}\mathcal{P}(X_n)$$

where each $\mathcal{P}^{top}_{i,[a_1,\dots,a_i]}\mathcal{P}(X_n)$ is invariant under the action of S_n. However they may contain more than one orbit. So, the partition

$$\mathsf{Rel}(X_n) = \bigsqcup_{\mathcal{O}\in\mathsf{Rel}(X)/S_n} \mathcal{O},$$

of $\mathsf{Rel}(X_n)$, is in theory finer that the partition of $\mathsf{Top}(X_n)$ is as below

$$\mathsf{Top}(X_n) = \bigsqcup_{\substack{[a_1,\dots,a_i] \\ 0\le a_1\le a_2\le\cdots\le a_i\le n}} \mathcal{P}^{top}_{i,[a_1,\dots,a_i]}\mathcal{P}(X_n).$$

So we have

$$\mathcal{P}^{top}_{i,[a_1,\dots,a_i]}\mathcal{P}(X_n) \cong \bigsqcup_{\mathcal{O}\in\mathcal{P}^{top}_{i,[a_1,\dots,a_i]}\mathcal{P}(X_n)/S_n} \mathcal{O}.$$

5.4.1 The full decomposition of $\mathsf{Top}(X_3)$ into orbits

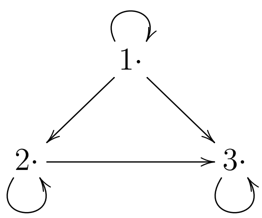
Now, we repeat the procedure that used in example 5.1.8, using this finer decomposition of $\mathsf{Top}(X_3)$.

1. We have:

$$\mathcal{P}^{top}_{4,[0,1,2,3]}\mathcal{P}(X_3) \cong \text{Orb}_{S_3}(\le) \sqcup \text{Orb}_{S_3}(\le'),$$

here we have two types of topologies, arising from two orbits of the action of S_3 on $\mathsf{Top}(X_3)$. The relations $\le$ and $\le'$ are :

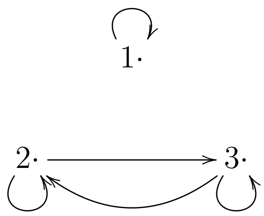
$\le=$



As the diagram indicates, this is the reflexive and transitive relation such that: $1\le 1$, $2\le 2$, $3\le 3$, $1\le 2$, $2\le 3$, $1\le 3$.

And

$\leq' =$



As shown in the diagram, this is the relation (reflexive and transitive) such that: $1 \leq' 1$, $2 \leq' 2$, $3 \leq' 3$, $1 \leq' 3$, $2 \leq' 3$, $3 \leq' 2$.

Let us see why this is the case. Suppose we have a topology in $\mathcal{P}^{top}_{4,[0,1,2,3]}\mathcal{P}(X_3)$, then

(a) we have one set of cardinality 1, say $\{a\}$, for some $a \in \{1, 2, 3\}$.

(b) We have one set with cardinality 2. There are two possible cases now, for the set of cardinality two.

   i. $\{a, b\}$ and the topology will then have the form $\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}$, where $a, b, c \in \{1, 2, 3\}$ are all different.

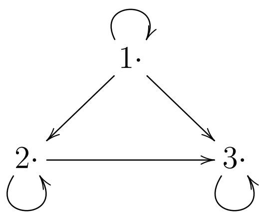
   ii. $\{b, c\}$ and the topology then has the form $\{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$, again for different $a, b, c \in \{1, 2, 3\}$.

So, we have

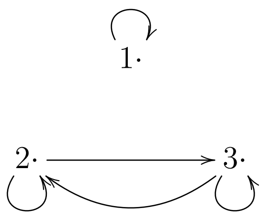
$$\mathcal{P}^{top}_{4,[0,1,2,3]}\mathcal{P}(X_3) = \mathrm{Orb}_{S_3}\big(\{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}\}\big) \sqcup \mathrm{Orb}_{S_3}\big(\{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}\}\big),$$

Therefore, we have an isomorphism of S_3-sets

$$\mathcal{P}^{top}_{4,[0,1,2,3]}\mathcal{P}(X_3) \cong \mathrm{Orb}_{S_3}\left(\quad \right)$$



$$\sqcup\ \mathrm{Orb}_{S_3}\left(\quad \right)$$



So we can see that the partition of $\mathsf{Top}(X_n)$ using the orbits on relations is finer in comparison with the one that merely use the number of subsets, and their cardinalities.

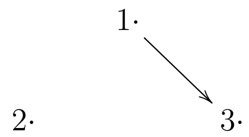
From now on, we will keep on describing a transitive and reflexive relation by a diagram. However we will omit the loops, even though we will always be describing reflexive relations.

2. We have :

$$\mathcal{P}^{top}_{6,[0,1,1,2,2,3]}\mathcal{P}(X_3) \cong \text{Orb}_{S_3}(\leq),$$

where

$\leq=$



As the diagram illustrates, this is the relation (reflexive and transitive) such that: $1 \leq 1$, $2 \leq 2$, $3 \leq 3$,$1 \leq 3$.

Let us see why this is the case. Assume we have a topology in $\mathcal{P}^{top}_{6,[0,1,1,2,2,3]}\mathcal{P}(X_3)$, then

(a) We have two singletons sets, let them be $\{a\}$ and $\{b\}$ with $a \neq b$, where $a, b \in \{1, 2, 3\}$.

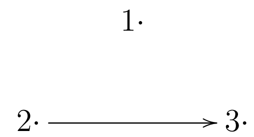
(b) We have two sets of cardinality 2 in the topology. One must be $\{a, b\}$ The other subset of cardinality 2 must be either $\{a, c\}$ or $\{b, c\}$, where $a, b, c \in \{1, 2, 3\}$ are all different, because we have only 3 elements in X_3.

So, the topology will have the form $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X_3\}$ and we get:

$$\mathcal{P}^{top}_{6,[0,1,1,2,2,3]}\mathcal{P}(X_3) = \text{Orb}_{S_3}\big(\{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}, X_3\}\big)$$

Therefore, we have an isomorphism of S_3-sets

$\mathcal{P}^{top}_{6,[0,1,1,2,2,3]}\mathcal{P}(X_3) \cong \text{Orb}_{S_3}$ (



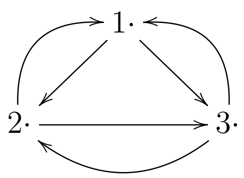
).

Now, we complete showing the partition of $\mathsf{Top}(X_3)$ in orbits of S_3. So we have the following:

3. We have:

$$\mathcal{P}^{top}_{2,[0,3]}\mathcal{P}(X_3) \cong \text{Orb}_{S_3}(\leq),$$

where the relation $\leq$ is as below:

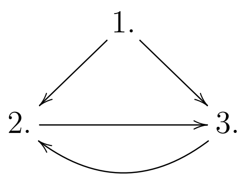


This relation gives the indiscrete topology and the action of S_3 is the trivial representation on this relation and on the topology.

4. We have:

$$\mathcal{P}^{top}_{3,[0,1,3]}\mathcal{P}(X_3) \cong \text{Orb}_{S_3}(\leq),$$

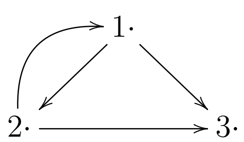
where the relation $\leq$ is as below:



5. We have :

$$\mathcal{P}^{top}_{3,[0,2,3]}\mathcal{P}(X_3) \cong \text{Orb}_{S_3}(\leq),$$

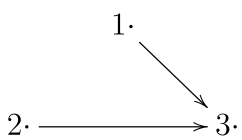
where the relation $\leq$ is as below:



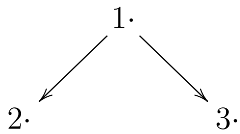
6. We have:

$$\mathcal{P}^{top}_{5,[0,1,1,2,3]}\mathcal{P}(X_3) \cong \text{Orb}_{S_3}(\leq),$$

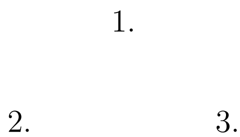
where the relation $\leq$ is as below:



7. We have:
$$\mathcal{P}^{top}_{5,[0,1,2,2,3]}\mathcal{P}(X_3) \cong \mathrm{Orb}_{S_3}(\leq),$$
where the relation $\leq$ is as below:



8. We have:
$$\mathcal{P}^{top}_{8,[0,1,1,1,2,2,2,3]}\mathcal{P}(X_3) \cong \mathrm{Orb}_{S_3}(\leq),$$
where the relation $\leq$ is as below:



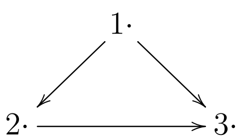
Here $x \leq y$, if, and only if, $x = y$. This relation gives the discrete topology and the action of S_3 is the trivial representation on this relation and on the topology.

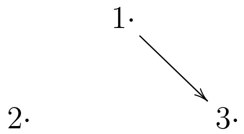
We have found the number of relations in the table in [May, pp.10].

5.4.2 Summary: the full decomposition of $\mathsf{Rel}(X_3)$ into S_3-orbits

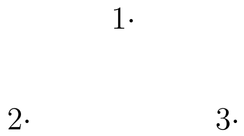
In this subsection, we will use Theorem 5.1.11. Any time we have an orbit $\mathcal{O}$ of S_n on $\mathsf{Rel}(X_n)$ that is free (i.e. all stabilisers are trivial) then $\mathbb{C}\mathcal{O}$ is isomorphic to the regular representation of S_n, and hence $\mathbb{C}\mathcal{O}$ is in particular algebra faithful. The formulation of the action of S_n on Top_{X_n} in terms of the action on $\mathsf{Rel}(X_n)$ makes it quite transparent to identify when an orbit is free. We now illustrate this for $n = 3$.

Summarising the previous subsection §5.4.1, $\mathsf{Rel}(X_3)$ is partitioned into the orbits of the relations below, and we have shown which of them are free, and their cardinality:

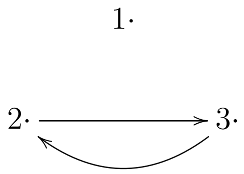
1. $\leq=$  . Free orbit. $|\,\mathrm{Orb}_{S_3}(\leq)| = 6$.

2. $\leq=$ 

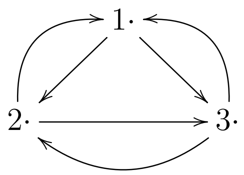
. Free orbit. $|\,\mathrm{Orb}_{S_3}(\leq)| = 6$.

3. $\leq=$ 

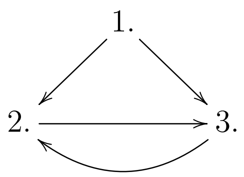
. Non-free orbit. $|\,\mathrm{Orb}_{S_3}(\leq)| = 1$.

4. $\leq=$ 

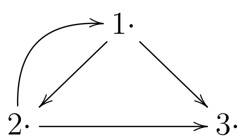
. Non-free orbit. $|\,\mathrm{Orb}_{S_3}(\leq)| = 3$.

5. $\leq=$ 

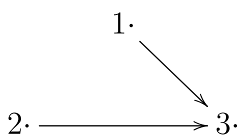
. Non-free orbit. $|\,\mathrm{Orb}_{S_3}(\leq)| = 1$.

6. $\leq=$ 

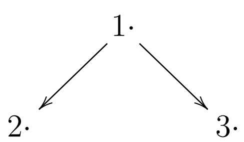
. Non-free orbit. $|\,\mathrm{Orb}_{S_3}(\leq)| = 3$.

7. $\leq=$ 

.Non-free orbit. $|\,\mathrm{Orb}_{S_3}(\leq)| = 3$.

8. $\leq=$ 

. Non-free orbit.$|\,\mathrm{Orb}_{S_3}(\leq)| = 3$.

9. $\leq=$ 

. Non-free orbit. $|\,\mathrm{Orb}_{S_3}(\leq)| = 3$.

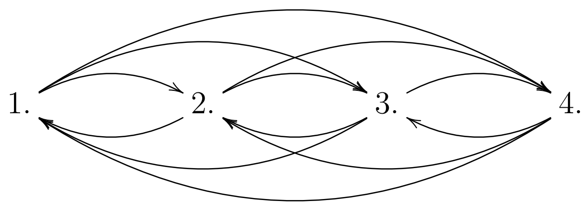
5.4.3 The full decomposition of $\mathsf{Top}(X_4)$ into orbits

In this subsection, we repeat the way that used in §5.2.3, using this finer decomposition of $\mathsf{Top}(X_4)$.

1. We have:

$$\mathcal{P}^{top}_{2,[0,4]}\mathcal{P}(X_4) \cong \mathrm{Orb}_{S_4}(\leq),$$

   where the relation $\leq$ is as below:



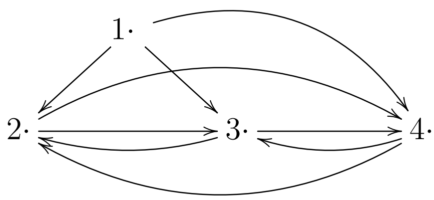
   As the diagram illustrates, this is the relation (reflexive and transitive) such that: $1 \leq 1$, $2 \leq 2$, $3 \leq 3$, $4 \leq 4$, $1 \leq 2$, $1 \leq 3$, $1 \leq 4$, $2 \leq 3$, $2 \leq 4$, $3 \leq 4$, $2 \leq 1$, $3 \leq 1$, $4 \leq 1$, $3 \leq 2$, $3 \leq 4$, $4 \leq 2$, $4 \leq 3$. This relation gives the indiscrete topology and the action of S_4 is the trivial representation on this relation and on the topology.

2. We have:

$$\mathcal{P}^{top}_{3,[0,1,4]}\mathcal{P}(X_4) \cong \mathrm{Orb}_{S_4}(\leq),$$

   where the relation $\leq$ is as below:

$\leq=$



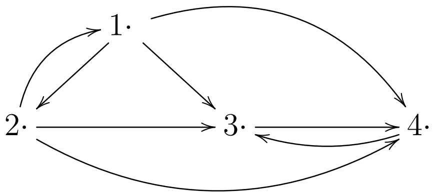
.

3. We have:

$$\mathcal{P}^{top}_{3,[0,2,4]}\mathcal{P}(X_4) \cong \mathrm{Orb}_{S_4}(\leq),$$

where the relation $\leq$ is as below:

$\leq=$



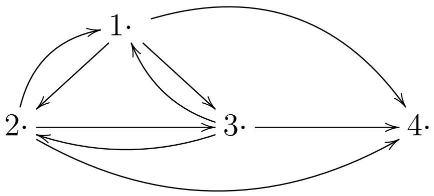
.

4. We have:

$$\mathcal{P}^{top}_{3,[0,3,4]}\mathcal{P}(X_4) \cong \text{Orb}_{S_4}(\leq),$$

where the relation $\leq$ is as below:

$\leq=$



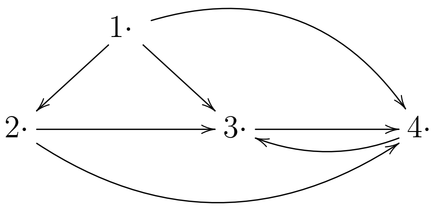
.

5. We have:

$$\mathcal{P}^{top}_{4,[0,1,2,4]}\mathcal{P}(X_4) \cong \text{Orb}_{S_4}(\leq),$$

where the relation $\leq$ is as below:

$\leq=$



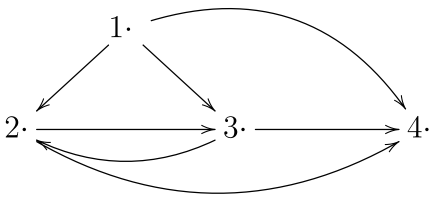
.

6. We have:

$$\mathcal{P}^{top}_{4,[0,1,3,4]}\mathcal{P}(X_4) \cong \text{Orb}_{S_4}(\leq) \sqcup \text{Orb}_{S_4}(\leq'),$$

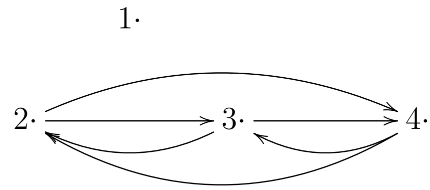
where the relation $\leq$ is as below:

$\leq=$



.

and

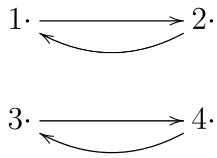
$\leq' =$ 

.

7. We have:

$$\mathcal{P}^{top}_{4,[0,2,2,4]}\mathcal{P}(X_4) \cong \text{Orb}_{S_4}(\leq),$$

where the relation $\leq$ is as below:

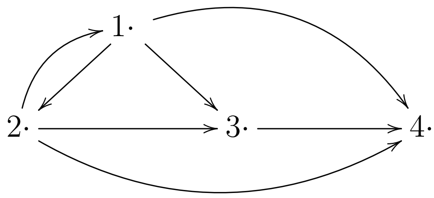
$\leq =$ 

.

8. We have:

$$\mathcal{P}^{top}_{4,[0,2,3,4]}\mathcal{P}(X_4) \cong \text{Orb}_{S_4}(\leq),$$

where the relation $\leq$ is as below:

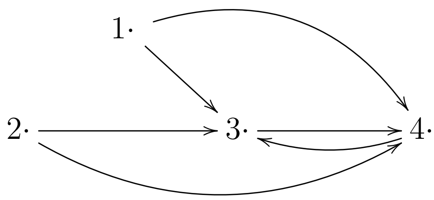
$\leq =$ 

.

9. We have:

$$\mathcal{P}^{top}_{5,[0,1,1,2,4]}\mathcal{P}(X_4) \cong \text{Orb}_{S_4}(\leq),$$

where the relation $\leq$ as below:

$\leq =$ 

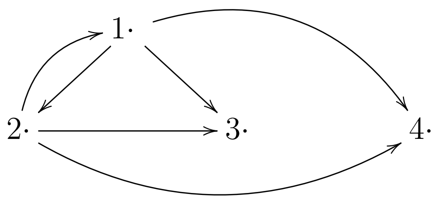
.

10. We have:

$$\mathcal{P}^{top}_{5,[0,2,3,3,4]}\mathcal{P}(X_4) \cong \operatorname{Orb}_{S_4}(\leq),$$

where the relation $\leq$ as below:

$\leq=$



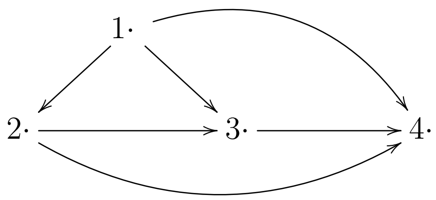
.

11. We have:

$$\mathcal{P}^{top}_{5,[0,1,2,3,4]}\mathcal{P}(X_4) = \operatorname{Orb}_{S_4}(\leq) \sqcup \operatorname{Orb}_{S_4)}(\leq') \sqcup \operatorname{Orb}_{S_4)}(\leq'').$$

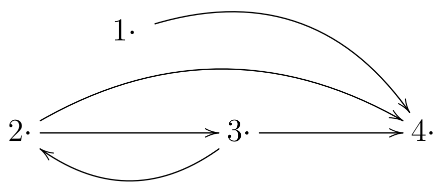
Where

$\leq=$



,

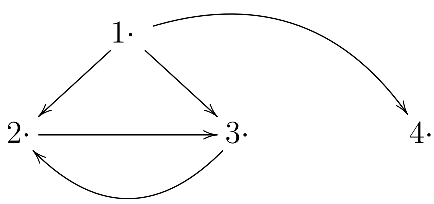
$\leq'=$



.

and

$\leq''=$

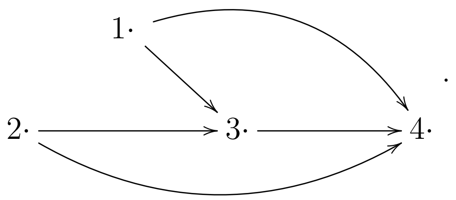


,

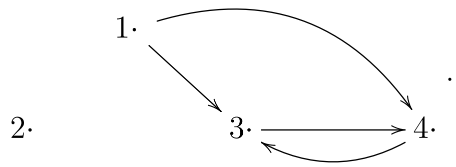
12. We have:

$$\mathcal{P}^{top}_{6,[0,1,1,2,3,4]}\mathcal{P}(X_4) \cong \operatorname{Orb}_{S_4}(\leq) \sqcup \operatorname{Orb}_{S_4)}(\leq'),$$

where the relation $\leq$ as below:

$\leq=$ 

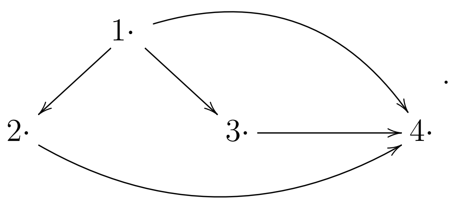
and

$\leq'=$ 

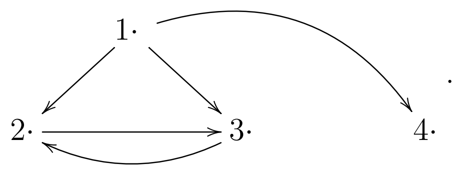
13. We have:

$$\mathcal{P}^{top}_{6,[0,1,2,2,3,4]}\mathcal{P}(X_4) \cong \text{Orb}_{S_4}(\leq) \sqcup \text{Orb}_{S_4}(\leq'),$$

where the relation $\leq$ as below:

$\leq=$ 

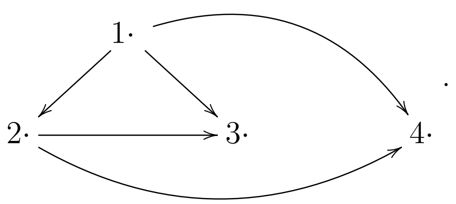
and

$\leq'=$ 

14. We have:

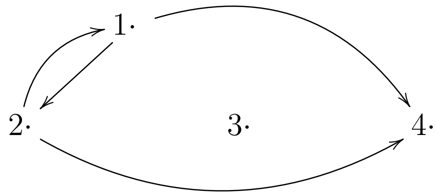
$$\mathcal{P}^{top}_{6,[0,1,2,3,3,4]}\mathcal{P}(X_4) \cong \text{Orb}_{S_4}(\leq) \sqcup \text{Orb}_{S_4}(\leq'),$$

where the relation $\leq$ as below:

$\leq=$ 

and

$\leq' =$



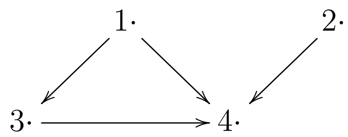
.

15. We have

$$\mathcal{P}^{top}_{7,[0,1,1,2,2,3,4]}\mathcal{P}(X_4) \cong \text{Orb}(\leq).$$

where the relation $\leq$ as below:

$\leq =$

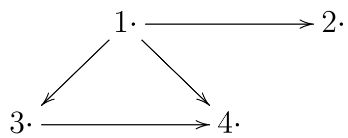


16. we have

$$\mathbb{C}\mathcal{P}^{top}_{7,[0,1,2,2,3,3,4]}\mathcal{P}(X_4) \cong \text{Orb}_{S_4}(\leq),$$

where the relation $\leq$ as below:

$\leq =$

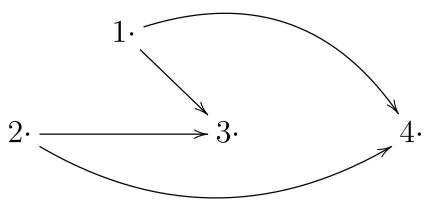


17. We have

$$\mathbb{C}\mathcal{P}^{top}_{7,[0,1,1,2,3,3,4]}\mathcal{P}(X_4) \cong \text{Orb}_{S_4}(\leq),$$

where the relation $\leq$ as below:

$\leq =$

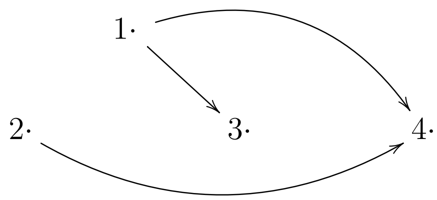


18. We have:

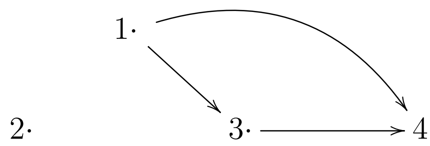
$$\mathcal{P}^{top}_{8,[0,1,1,2,2,3,3,4]}\mathcal{P}(X_4) = \text{Orb}_{S_4}(\leq) \sqcup \text{Orb}_{S_4}(\leq') \sqcup \text{Orb}_{S_4}(\leq'').$$

Where

$\leq =$

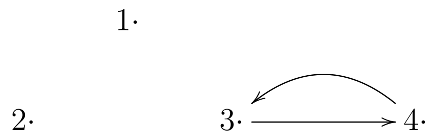


$\leq' =$



and

$\leq'' =$



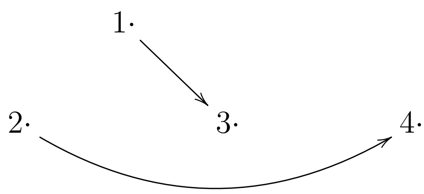
.

19. We have:

$$\mathbb{C}\mathcal{P}^{top}_{9,[0,1,1,2,2,2,3,3,4]}\mathcal{P}(X_4) \cong \text{Orb}_{S_4}(\leq),$$

where the relation $\leq$ as below:

$\leq =$

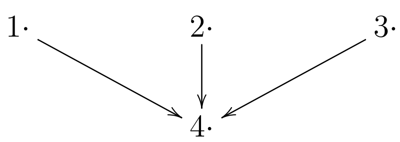


20. We have:

$$\mathbb{C}\mathcal{P}^{top}_{9,[0,1,1,1,2,2,2,3,4]}\mathcal{P}(X_4) \cong \text{Orb}_{S_4}(\leq),$$

where the relation $\leq$ as below:

$\leq =$

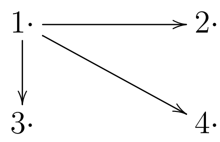


21. We have:

$$\mathbb{C}\mathcal{P}^{top}_{9,[0,1,2,2,2,3,3,3,4]}\mathcal{P}(X_4) \cong \text{Orb}_{S_4}(\leq),$$

where the relation $\leq$ as below:

$\leq=$

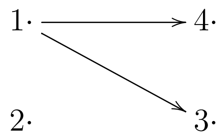


22. We have:

$$\mathbb{C}\mathcal{P}^{top}_{10,[0,1,1,2,2,2,3,3,3,4]}\mathcal{P}(X_4) \cong \text{Orb}_{S_4}(\leq),$$

where the relation $\leq$ as below:

$\leq=$

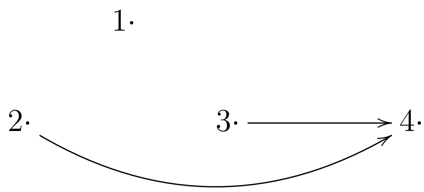


23. We have:

$$\mathbb{C}\mathcal{P}^{top}_{10,[0,1,1,1,2,2,2,3,3,4]}\mathcal{P}(X_4) \cong \text{Orb}_{S_4}(\leq),$$

where the relation $\leq$ as below:

$\leq=$

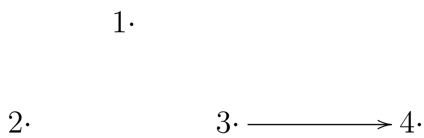


24. We have:

$$\mathbb{C}\mathcal{P}^{top}_{12,[0,1,1,1,2,2,2,2,3,3,3,4]}\mathcal{P}(X_4) \cong \text{Orb}_{S_4}(\leq),$$

where the relation $\leq$ as below:

$\leq=$



25. We have:

$$\mathcal{P}^{top}_{16,[0,1,1,1,1,2,2,2,2,2,2,3,3,3,3,4]}\mathcal{P}(X_4) \cong \text{Orb}_{S_s}(\leq),$$

where the relation $\leq$ as below:

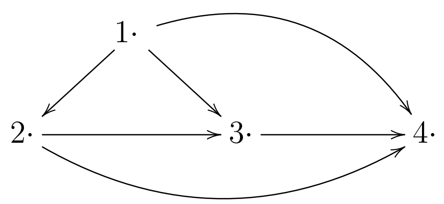
1. 2·

2. 3.

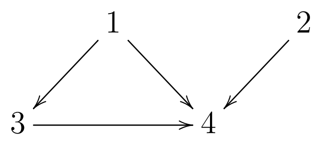
Here $x \leq y$, if, and only if, $x = y$. This relation gives the discrete topology and the action of S_4 is the trivial representation on this relation and on the topology.

5.4.4 Summary: the full decomposition of $\mathsf{Rel}(X_4)$ into S_4-orbits

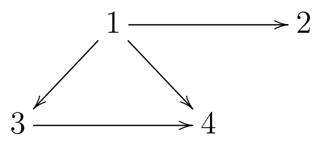
In this subsection, we summarise the previous subsection §5.4.3. Set $\mathsf{Rel}(X_4)$ is partitioned into the orbits of the relations below, and we should which orbits are free (and hence give algebra faithful representations of S_n), and the cardinality of each orbit.

1. $\leq=$  . Free orbit. $|\,\text{Orb}_{S_4}(\leq)| = 24$.

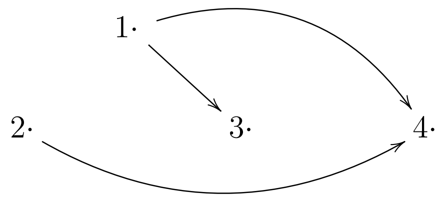
---

2. $\leq=$  . Free orbit. $|\,\text{Orb}_{S_4}(\leq)| = 24$.

---

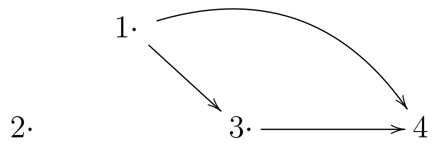
3. $\leq=$  . Free orbit. $|\,\text{Orb}_{S_4}(\leq)| = 24$.

---

4. $\leq=$ 

. Free orbit. $|\operatorname{Orb}_{S_4}(\leq)| = 24$.

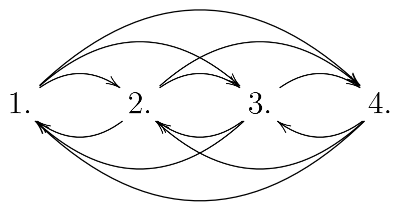
---

5. $\leq=$ 

. Free orbit. $|\operatorname{Orb}_{S_4}(\leq)| = 24$.

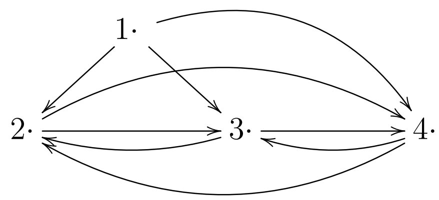
So, all above relations have free orbit and therefore they give algebra faithful representations, moreover isomorphic to $\mathbb{C}S_4$. Note that $\mathbb{C}S_4 \cong S^{(4)} \oplus 3S^{(3,1)} \oplus 2S^{(2,2)} \oplus 3S^{(2,1^2)} \oplus S^{(1^4)}$.

---

1. $\leq=$ 

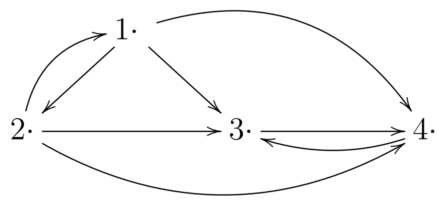
. Non free orbit. $|\operatorname{Orb}_{S_4}(\leq)| = 1$.

---

2. $\leq=$ 

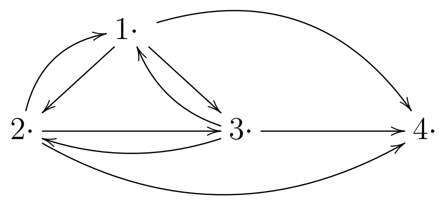
. Non free orbit. $|\operatorname{Orb}_{S_4}(\leq)| = 4$.

---

3. $\leq=$ 

.Non free orbit. $|\operatorname{Orb}_{S_4}(\leq)| = 6$.

---

4. $\leq=$ 

.Non free orbit. $|\operatorname{Orb}_{S_4}(\leq)| = 4$.

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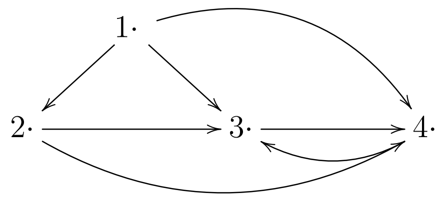
5. $\leq=$  . Non free orbit. $|\operatorname{Orb}_{S_4}(\leq)| = 12$.

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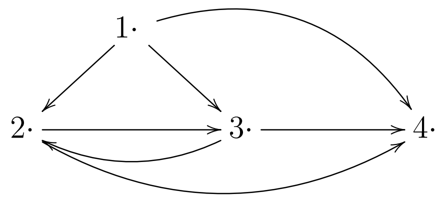
6. $\leq=$  . Non free orbit. $|\operatorname{Orb}_{S_4}(\leq)| = 12$.

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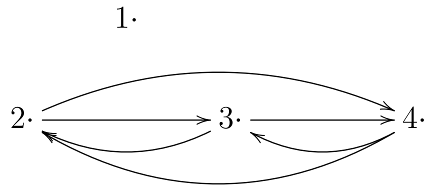
7. $\leq=$  .Non free orbit. $|\operatorname{Orb}_{S_4}(\leq)| = 4$.

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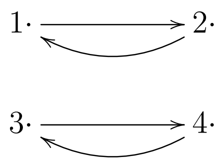
8. $\leq=$  . Non free orbit. $|\operatorname{Orb}_{S_4}(\leq)| = 4$.

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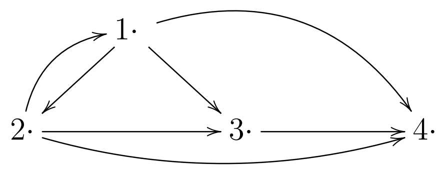
9. $\leq=$  . Non free orbit. $|\operatorname{Orb}_{S_4}(\leq)| = 12$.

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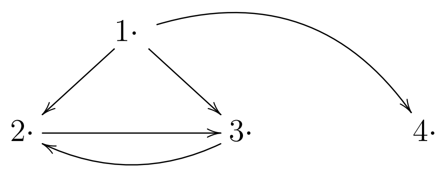
10. $\leq=$  .Non free orbit. $|\operatorname{Orb}_{S_4}(\leq)| = 12$.

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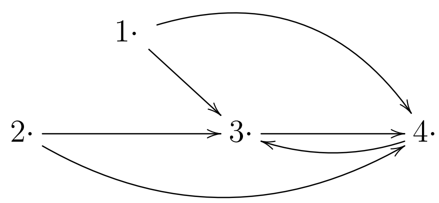
11. $\leq=$  . Non free orbit. $|\operatorname{Orb}_{S_4}(\leq)| = 12$.

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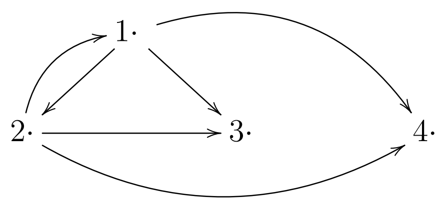
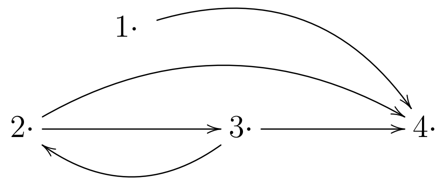
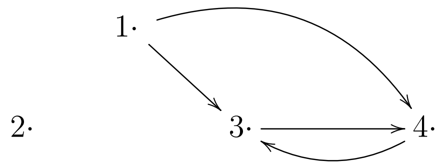
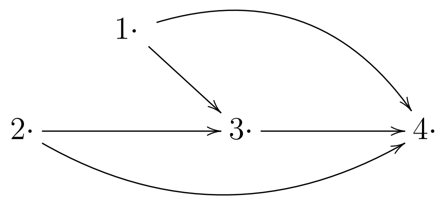
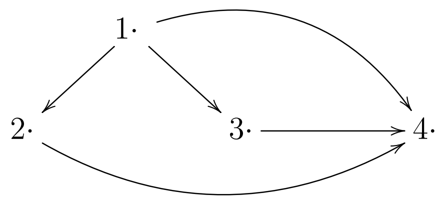
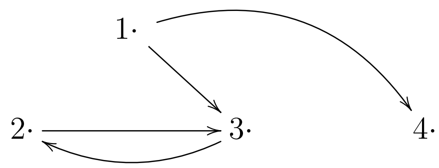
12. $\leq=$  . Non free orbit. $|\operatorname{Orb}_{S_4}(\leq)| = 12$

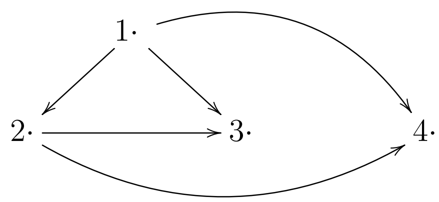
13. $\leq=$  .Non free orbit. $|\,\mathrm{Orb}_{S_4}(\leq)| = 12$.

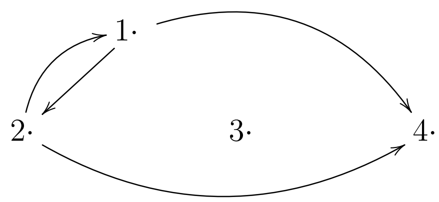
14. $\leq=$  . Non free orbit. $|\,\mathrm{Orb}_{S_4}(\leq)| = 12$.

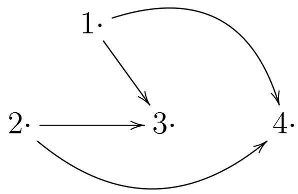
15. $\leq=$  . Non free orbit. $|\,\mathrm{Orb}_{S_4}(\leq)| = 12$.

16. $\leq=$  . Non free orbit. $|\,\mathrm{Orb}_{S_4}(\leq)| = 12$.

17. $\leq=$  . Non free orbit. $|\,\mathrm{Orb}_{S_4}(\leq)| = 12$.

18. $\leq=$  . Non free orbit. $|\,\mathrm{Orb}_{S_4}(\leq)| = 12$.

19. $\leq=$  . Non free orbit. $|\,\mathrm{Orb}_{S_4}(\leq)| = 12$.

20. $\leq=$ 

. Non free orbit. $|\,\mathrm{Orb}_{S_4}(\leq)| = 12$.

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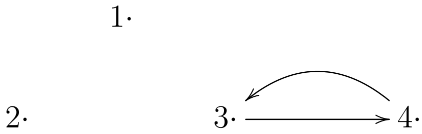
21. $\leq=$ 

. Non free orbit. $|\,\mathrm{Orb}_{S_4}(\leq)| = 6$.

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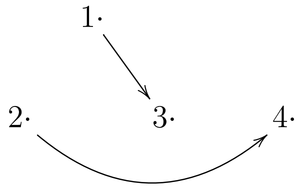
22. $\leq=$ 

. Non free orbit. $|\,\mathrm{Orb}_{S_4}(\leq)| = 12$.

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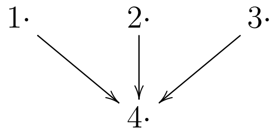
23. $\leq=$ 

. Non free orbit. $|\,\mathrm{Orb}_{S_4}(\leq)| = 4$.

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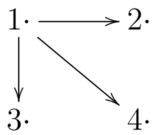
24. $\leq=$ 

. Non free orbit. $|\,\mathrm{Orb}_{S_4}(\leq)| = 4$.

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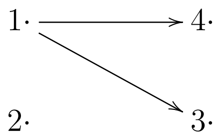
25. $\leq=$ 

. Non free orbit. $|\,\mathrm{Orb}_{S_4}(\leq)| = 12$.

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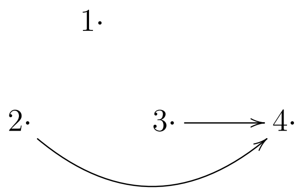
26. $\leq=$ 

. Non free orbit. $|\,\mathrm{Orb}_{S_4}(\leq)| = 12$.

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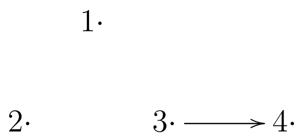
27. $\leq=$ 

. Non free orbit. $|\,\mathrm{Orb}_{S_4}(\leq)| = 12$.

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28. $\leq = \begin{array}{cc} 1. & 2\cdot \\ 3. & 4. \end{array}$. Non free orbit. $|\operatorname{Orb}_{S_4}(\leq)| = 1$.

5.4.5 Some free orbits for the general n case

Proposition 5.4.1. *Let $X_n = \{1, 2, \dots, n\}$. Consider the reflexive and transitive relation sketched below*

$$\leq = \quad 1\cdot \longrightarrow 2\cdot \longrightarrow 3\cdot \longrightarrow 4\cdot \longrightarrow 5\cdot \quad \dots\dots \quad n-2\cdot \longrightarrow n-1\cdot \longrightarrow n\cdot$$

Here the remaining arrows are either loops on an object, or a composition of arrows, in order to ensure transitivity. So, $x, y \in \{1, \dots, n\}$, $x \leq y$ has the usual meaning, in $\mathbb{Z}$. The action of S_n on $\operatorname{Orb}_{S_n}(\leq)$ is free and therefore it gives an algebra faithful representation of S_n.

Proof. Since

$$\operatorname{Orb}_{S_n}(\leq) = \mathcal{P}^{top}_{n+1,[0,1,2,\dots,n]}\mathcal{P}(X_n),$$

then from (5.9) we have $\mathbb{C}\operatorname{Orb}_{S_n}(\leq)$ is algebra faithful representation of S_n. It can also be seen, by inspection, that $\operatorname{Stab}_{S_n}(\leq)$ is trivial. So its orbit is free. $\square$

It is very easy to find more transitive and reflexive relations on X_n whose orbit is free as follows:

Proposition 5.4.2. *Let $X_n = \{1, 2, \dots, n\}$. Consider the reflexive and transitive relation sketched below (in all cases, the remaining arrows are either loops on an object, or a composition of arrows, in order to ensure transitivity):*

- *For $n \geq 4$, we have*

$$\begin{array}{l} \leq = \quad 1\cdot \longrightarrow 2\cdot \longrightarrow 3\cdot \longrightarrow 4\cdot \longrightarrow 5\cdot \dots\dots n-3\cdot \longrightarrow n-2\cdot \longrightarrow n-1\cdot \\ \hspace{22em} \uparrow \\ \hspace{22em} n\cdot \end{array}$$

So, $x, y \in \{1, \dots, n-1\}$, $x \leq y$ has the usual meaning, in $\mathbb{Z}$. We also have

$n \leq n-1$ and $n \leq n$.

- *For all n, we have*

$$\leq = \quad 1\cdot \longrightarrow 2\cdot \longrightarrow 3\cdot \longrightarrow 4\cdot \longrightarrow 5\cdot \ldots\ldots n-2\cdot \longrightarrow n-1\cdot$$

$$n\cdot$$

So, $x, y \in \{1, \ldots, n-1\}$, $x \leq y$ has the usual meaning, in $\mathbb{Z}$. We also have $n \leq n$.

- *For $n \geq 4$, we have*

$$\begin{array}{cccccccccccc} \leq = & 1\cdot & \longrightarrow & 2\cdot & \longrightarrow & 3\cdot & \longrightarrow & 4\cdot & \longrightarrow & 5\cdot \ldots\ldots n-2\cdot & \longrightarrow & n-1\cdot \\ & \downarrow & & & & & & & & & & \\ & n\cdot & & & & & & & & & & \end{array}$$

So, $x, y \in \{1, \ldots, n-1\}$, $x \leq y$ has the usual meaning, in $\mathbb{Z}$. We also have $1 \leq n$ and $n \leq n$.

- *For $n \geq 6$, we have*

$$\begin{array}{cccccccccccc} \leq = & 1\cdot & \longrightarrow & 2\cdot & \longrightarrow & 3\cdot & \longrightarrow & 4\cdot & \longrightarrow & 5\cdot \ldots & \longrightarrow & n-2\cdot \\ & & & & & & & & & & & \uparrow \\ & & & & & & & & & & & n\cdot \\ & & & & & & & & & & & \uparrow \\ & & & & & & & & & & & n-1\cdot \end{array}$$

So, $x, y \in \{1, \ldots, n-2\}$, $x \leq y$ has the usual meaning, in $\mathbb{Z}$. we also have $n-1 \leq n$ and $n \leq n-2$.

The action of S_n on the the orbits $\mathrm{Orb}_{S_n}(\leq)$ is free and therefore, linearising, it gives an algebra faithful representation of S_n.

Proof. Since the action of S_n on $\mathrm{Orb}_{S_n}(\leq)$ is transitive and free because we have only one orbit and all stabilisers are trivial, then it is isomorphic representation to $\mathbb{C}S_n$ from Theorem 5.1.11. Hence, it is algebra faithful representation of S_n. $\square$

5.4.6 Levelled topologies and Young tabloids

Definition 5.4.3. Let X be a non-empty set and let $k \in \mathbb{N}$. A *levelled and unrelated* reflexive transitive relation, *with k levels*, is one defined by:

1. Subsets $A_1, A_2, \dots A_k$ of X, that are non-empty and pairwise disjoint, and such that $A_1 \cup \dots \cup A_k = X$
2. $x \leq y \Leftrightarrow (x = y)$ or ($x \in A_i$ and $y \in A_j$, with $i < j$), where $<$ is usual sign in integer numbers.

We put $\leq = \leq_{(k:A_1,\dots,A_k)}$. The *shape* of such a relation is the pair $(k : |A_1|, |A_2|, \dots, |A_k|)$

Note: If

$$\leq_{(k:A_1,\dots,A_k)} = \leq_{(k':A'_1,\dots,A'_{k'})}$$

then $k = k'$, and $A_1 = A'_1$, $A_2 = A'_2$, $\dots$, $A_k = A'_k$.

Lemma 5.4.4. *A relation defined as in Definition 5.4.3 indeed is a transitive and reflexive relation.*

Proof. We need to prove that a relation as in Definition 5.4.3 is indeed, reflexive and transitive.

1. **Reflexive:** we want to show $x \leq x$.

   Since for any $x \in X$ we have $x = x$, then by definition 5.4.3 we have clearly that $x \leq x$.

2. **Transitive:** we need to show that if $x \leq y$ and $y \leq z$, then $x \leq z$.

   Suppose that $x \leq y$ and $y \leq z$. Then by definition 5.4.3 we have the following:

   (a) $(x = y)$ or ($x \in A_i$ and $y \in A_j$, with $i < j$)

   (b) $(y = z)$ or ($y \in A_j$ and $z \in A_l$, with $j < l$)

   Now, we need to show that $x \leq z$. This means we need to prove $(x = z)$ or ($x \in A_i$ and $z \in A_l$, with $i < l$). We have some cases to prove that $x \leq z$ as follows:

   Case 1: Suppose that $x = y$ and $y = z$. Then we have $x = z$. Thus, $x \leq z$.

Case 2: Suppose that $x = y$ and $y \in A_j$ and $z \in A_l$, with $j < l$. Then it is clear that $x \in A_j$. Therefore, we have $x \leq z$.

Case 3: Suppose that $y = z$ and $x \in A_i$ and $y \in A_j$, with $i < j$. Then it is clear that $z \in A_j$. Therefore, we have $x \leq z$.

Case 4: Suppose that $x \in A_i$, $y \in A_j$ and $z \in A_l$, with $i < j$ and $j < l$. Since we have $x \in A_i$ and $z \in A_l$ with $i < j$ and $j < l$, then $i < l$ from transitive inequality (in $\mathbb{Z}$). Therefore, $x \leq z$.

□

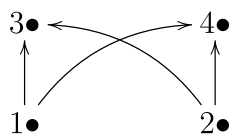
Example 5.4.5. Let $X = \{1, 2, 3, 4\}$. Put $k = 1$ and $A_1 = \{1, 2, 3, 4\}$. Then the corresponding levelled and unrelated reflexive transitive relation will be the one, $\leq$, defined by the diagram:

$\leq =$

Therefore $x \leq y$ if, and only if $x = y$. Also, we notice that $\overline{F}(\leq) = \tau_{\leq}$ is the discrete topology on X_4; see Section 5.3.2.

Example 5.4.6. Let $X = \{1, 2, 3, 4\}$. Put $k = 2$ and let $A_1 = \{1, 2\}, A_2 = \{3, 4\}$. Then the corresponding levelled and unrelated reflexive transitive relation will be defined by the diagram:

$\leq =$



Therefore, given any $x \in A_1$ and $y \in A_2$, $x \leq y$. The topology, $\overline{F}(\leq)$ has basis $\{\{1\}, \{2\}, \{1, 2, 3\}, \{1, 2, 4\}\}$ and explicitly

$$\overline{F}(\leq) = \tau_{\leq} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, X\}.$$

Definition 5.4.7. Let X be a non-empty set and let $k \in \mathbb{N}$. A *levelled and fully related* reflexive transitive relation, *with k levels*, is one defined by:

1. Subsets $A_1, \dots, A_k$ of X, that are pairwise disjoint and $A_1 \cup \dots \cup A_k = X$.
2. $x \leq y \Leftrightarrow$ (there exists $i \in \{1, 2, \dots, k\}$ such that $x, y \in A_i$) or ($x \in A_i$ and $y \in A_j$, with $i < j$), where $<$ is usual sign in integer numbers.

Proof. We need to prove that a levelled and related reflexive and transitive relation is, indeed, reflexive and transitive.

1. **Reflexive:** we want to show $x \leq x$.

   Since for any $x \in X$ we have $x \in A_i$, then by definition 5.4.7 we have clearly that $x \leq x$.

2. **Transitive:** we need to show that if $x \leq y$ and $y \leq z$, then $x \leq z$.

   Now, we need to show that $x \leq z$. This means ($x, z \in A_i$) or ($x \in A_i$ and $z \in A_l$, with $i < l$.) We have some cases to prove that $x \leq z$ as follows:

   Case 1: Suppose that $x, y \in A_i$ and $y, z \in A_i$. Then we have $x, z \in A_i$. Thus, $x \leq z$.

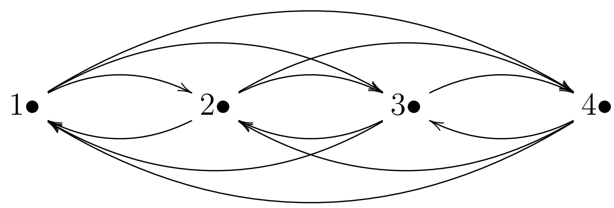
   Case 2: Suppose that $x, y \in A_i$ and $y \in A_i, z \in A_l$ with $i < l$. Then we have $x \in A_i$ and $z \in A_l$, with $i < l$. Therefore, we have $x \leq z$.

   Case 3: Suppose that $x \in A_i$, $y \in A_j$ and $z \in A_l$, with $i < j$ and $j < l$. Then we obtain that $x \in A_i$ and $z \in A_l$, with $i < l$. Thus, we have $x \leq z$.

□

Example 5.4.8. Let $X = \{1, 2, 3, 4\}$. Put $k = 1$. Let $A_1 = \{1, 2, 3, 4\}$. Then the corresponding levelled and fully related reflexive transitive relation will be defined by the diagram:

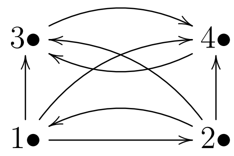
$\leq =$



Thus, given any $x, y \in A_1$, $x \leq y$. Also, we notice $\overline{F}(\leq) = \tau_{\leq}$ is the indiscrete topology on X_4; see Section 5.3.2.

Example 5.4.9. Let $X = \{1, 2, 3, 4\}$. Put $k = 2$. Let $A_1 = \{1, 2\}, A_2 = \{3, 4\}$. Then the corresponding levelled and fully related reflexive transitive relation is defined by the diagram:

$\leq=$



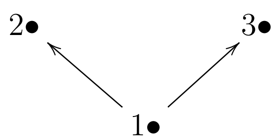
Therefore:

1. Given any $x, y \in A_1$, $x \leq y$;
2. Given any $x, y \in A_2$, $x \leq y$;
3. Given any $x \in A_1$ and $y \in A_2$, $x \leq y$.

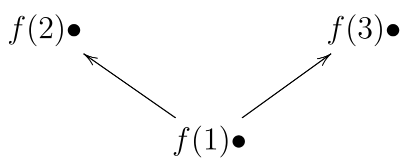
The topology has basis $\{\{1, 2\}, \{3, 4\}\}$ and expliclty

$$\overline{F}(\leq) = \tau_{\leq} = \{\emptyset, \{1, 2\}, \{3, 4\}, X\}.$$

Example 5.4.10. Let $X_3 = \{1, 2, 3\}$ and $k = 2$ with $A_1 = \{1\}, A_2 = \{2, 3\}$. Then we have the corresponding levelled and unrelated reflexive transitive relation, which can be visualised by the diagram:



Now, let $f \in S_3$, then the action of S_3 on $\leq$ is defined by diagram



We notice that the action S_3 on a such a relation returns a relation with the same shape.

Lemma 5.4.11. *Consider $X_n = \{1, 2, \ldots, n\}$, $k \in \mathbb{N}$. Let $A_1, \ldots, A_k$ be subsets of X_n, that are non-empty and pairwise disjoint, and such that $A_1 \cup \cdots \cup A_k = X_n$.*

Then:

- *If $f \in S_n$, then*

$$f. \leq_{(k:A_1,\ldots,A_k)} = \leq_{(k:f(A_1),\ldots,f(A_k))} .$$

- *The orbit of $\leq_{(k:A_1,\ldots,A_k)}$ consists of all relations with the same shape, $(k : |A_1|, \ldots, |A_k|)$.*

Proof. • 1. Since $f\colon X_n \to X_n$ is a bijection, then subsets $f(A_1), f(A_2), \ldots, f(A_K)$ of X_n, that are pairwise disjoint because from Lemma 5.1.1

$$f(A_1) \cap f(A_2) \cap \cdots \cap f(A_K) = f(A_1 \cap A_2 \cap \cdots \cap A_k) = \emptyset$$

and we have by Lemma 5.1.2.

$$f(A_1) \cup f(A_2) \cup \cdots \cup f(A_k) = f(A_1 \cup A_2 \cup \cdots \cup A_k) = X_n$$

2. We need to show that

$$x \quad f(\leq_{(k:A_1,\ldots,A_k)}) \quad y \Leftrightarrow x \quad \leq_{(k:f(A_1),\ldots,f(A_n))} \quad y$$

Let us denote the relations $f(\leq_{(k:A_1,\ldots,A_k)})$ and $\leq_{(k:f(A_1),\ldots,f(A_n))}$ as $\leq$, $\leq^{'}$ respectively.

We prove each implication separately.

(a) $\Rightarrow$: Since

$$\begin{aligned} xf(\leq)y &\Leftrightarrow f^{-1}(x) \leq f^{-1}(y), \text{ by proposition 5.3.23} \\ &\Leftrightarrow \left(f^{-1}(x) = f^{-1}(y)\right) \text{ or } \left(f^{-1}(x) \in A_i \text{ and } f^{-1}(y) \in A_j, i < j\right) \end{aligned}$$

Now, if we are applying f, then we have:

$$(x = y) \text{ or } (x \in f(A_i) \text{ and } y \in f(A_j),\ i < j) \Rightarrow x \leq^{'} y.$$

(b) $\Leftarrow$: Since from definition 5.4.3

$$x \leq^{'} y \Leftrightarrow (x = y) \text{ or } (x \in f(A_i) \text{ and } y \in f(A_j) \text{ with } i < j)$$

If $x = y$, then we get $f^{-1}(x) = f^{-1}(y)$. Thus, $f^{-1}(x) \leq f^{-1}(y)$ and hence $x \leq y$.

If $x \in f(A_i)$ and $y \in f(A_j)$ with $i < j$, then there is $x^{'} \in A_i$ and $y^{'} \in A_j$ such that $f(x^{'}) = x$ and $f(y^{'}) = y$.

Now, apply f^{-1}, we have: $f^{-1}(x) = x^{'} \in A_i$ and $f^{-1}(y) = y^{'} \in A_j$ with $i < j$. Hence, $f^{-1}(x) \leq f^{-1}(y)$ and therefore we have $x \leq y$.

- Let $\leq_{(k:A_1,A_2,\ldots,A_k)}$ be reflexive and transitive relations with shape

$$(k : |A_1|, |A_2|, \ldots, |A_k|).$$

Then we have pairwise disjoint subsets $A_1, A_2, \ldots A_k$ of X_n such that $A_1 \cup A_2 \cup \cdots \cup A_k = X_n$.

Suppose $\leq'_{(k':A'_1,A'_2,\ldots,A'_{k'})}$ be another reflexive and transitive relations with shape

$$(k : |A_1|, |A_2|, \ldots, |A_k|).$$

So in particular $k = k'$.

Then we have pairwise disjoint subsets $A^{'}_1, A^{'}_2, \ldots A^{'}_{k'}$ of X_n such that

$$A^{'}_1 \cup A^{'}_2 \cup \cdots \cup A^{'}_{k'} = X_n$$

and

$$|A_1| = |A'_1|, |A_2| = |A'_2|, \ldots, |A_k| = |A'_{k'}|.$$

Now, we need to show that the orbit of $\leq_{(k:A_1,\ldots,A_k)}$ contains this relation $\leq'_{(k':A'_1,A'_2,\ldots,A'_{k'})}$. Choose bijections:

$$g_1\colon A_1 \to A'_1$$
$$g_2\colon A_2 \to A'_2$$
$$\vdots$$
$$g_k\colon A_k \to A'_{k'}$$

Define a map:

$$g\colon X_n \to X_n,$$

where

$$g(x) = \begin{cases} g_1(x), x \in A_1 \\ g_2(x), x \in A_2 \\ \vdots \\ g_k(x), x \in A_k \end{cases}$$

Then, g is a bijection, and

$$x \leq_{(k:A_1,\ldots,A_k)} y \Leftrightarrow g(x) \leq'_{(k':A'_1,A'_2,\ldots,A'_{k'})} g(y).$$

So, $g \quad \triangleright \leq_{(k:A_1,\ldots,A_k)} = \leq'_{(k':A'_1,A'_2,\ldots,A'_{k'})}$. Hence $\leq'_{(k':A'_1,A'_2,\ldots,A'_{k'})}$ is in the orbit of $\leq_{(k:A_1,\ldots,A_k)}$.

□

Proposition 5.4.12. *Suppose that $\leq_{(k:A_1,A_2,\ldots,A_k)}$ is a levelled and unrelated reflexive transitive relation on X_n. Suppose (to simplify) that $|A_1| \geq |A_2| \geq \cdots \geq |A_k|$. Let D be the corresponding young diagram of shape $\mu = (|A_1|, |A_2|, \ldots, |A_k|)$. We have a bijection preserving the action of S_n,*

$$\mathrm{Orb}_{S_n}(\leq_{(k:A_1,A_2,\ldots,A_k)}) \longrightarrow \{ \textit{ Young tabloids with shape } D \quad \}$$

$$\leq_{(k:B_1,B_2,\ldots,B_k)} \longmapsto \begin{array}{l} \hline \cdot \;\; \cdots\;\cdots\;\cdots\;\cdots\;\cdots\;\cdots \;\; \cdot \\ \hline \cdot \;\; \cdots\;\cdots\;\cdots\;\cdots\;\cdots \;\; \cdot \\ \hline \cdot \;\; \cdots\;\cdots\;\cdots\;\cdots \;\; \cdot \\ \hline \cdot \;\; \cdots\;\cdots\;\cdots \;\; \cdot \\ \hline \cdot \;\; \cdots\;\cdots \;\; \cdot \\ \hline \cdot \;\; \cdots \;\; \cdot \\ \hline \cdots \\ \hline \end{array}$$

where the boxes in the first row have been filled with the elements of B_1, the boxes in the second row have been filled with the elements of B_2, and so on.

Lemma 5.4.13. *Suppose that $\leq_{(k:A_1,A_2,\ldots,A_k)}$ is a levelled and unrelated reflexive transitive relation on X_n. Let $\mu = (|A_1|, |A_2|, \ldots, |A_k|)$ and M^μ be defined in 2.4.3.*

Then we have:

$$\mathbb{C}\,\mathrm{Orb}_{S_n}(\leq_{(k:A_1,A_2,\ldots,A_k)}) \cong M^{\mu}$$

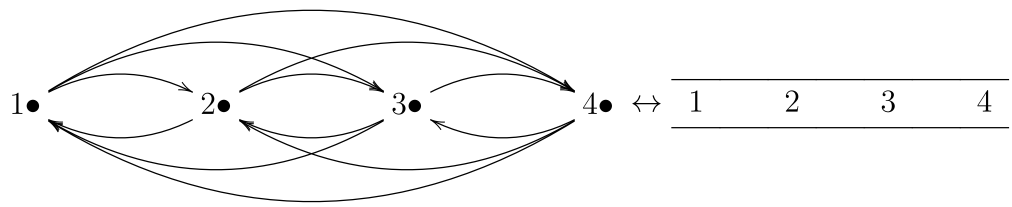
as representation of S_n.

Proof. It is clear from proposition 5.4.12. □

Example 5.4.14. Let $X_4 = \{1,2,3,4\}$ and S_4 the symmetric group. Suppose that $\leq$ as defined in Example 5.4.8. Then we have a bijection

$$\mathrm{Orb}_{S_4}(\leq_{(1:\{1,2,3,4\})}) \to \{\text{Young tabloids with shape}(4)\},$$

where all sets have cardinality 1, which is given by:



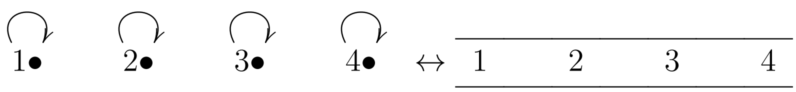
Thus, the decomposition of the action of S_4 on $\mathrm{Orb}_{S_4}(\leq)$ into irreducible is

$$M^{(4)} \cong S^{(4)}$$

Example 5.4.15. Let $X_4 = \{1,2,3,4\}$. Suppose that $\leq = \leq_{(1:\{1,2,3,4\})}$ as defined in Example 5.4.5. Then we have a bijective map

$$\mathrm{Orb}_{S_4}(\leq_{(1:1,2,3,4\})}) \to \{\text{Young tabloids with shape}(4)\}$$

is given by:



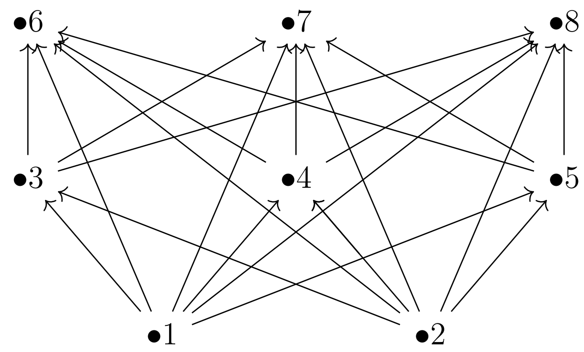
Then the decomposition of the action of S_4 on $\mathrm{Orb}_{S_4}(\leq)$ into irreducibles will be

$$M^{(4)} \cong S^{(4)}$$

We notice that the previous examples have different topologies, while the action of S_n on both topologies is trivial.

Example 5.4.16. Let $X_8 = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $k = 3$ with $A_1 = \{6, 7, 8\}$, $A_2 = \{3, 4, 5\}$ and $A_3 = \{1, 2\}$. The corresponding, levelled and unrelated, reflexive and transitive relation is defined by diagram

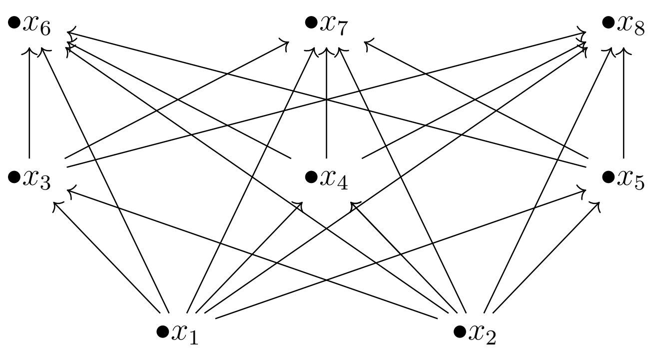
$\leq_{(3:A_1,A_2,A_3)}=$



Then we have a bijective map

$$\mathrm{Orb}_{S_8}(\leq_{(3:A_1,A_2,A_3)}) \leftrightarrow \{\text{Young tabloids of shape}(3,3,2)\}$$

given by:



$\leftrightarrow$

x_6	x_7	x_8
x_3	x_4	x_5
x_1	x_2	

Therefore, we have the representation of the action of S_n as follows:

$$\mathbb{C}\,\mathrm{Orb}_{S_n}(\leq_{(3:A_1,A_2,A_3)}) \cong M^{(3,3,2)}$$

Then by using Young rules we have the decomposition of the action of S_8 into irreducible :

$$M^{(3,3,2)} \cong S^{(8)} \oplus 2S^{(7,1)} \oplus 3S^{(6,2)} \oplus 2S^{(5,3)} \oplus S^{(4,3,3)} \oplus S^{(3,3,2)}$$

Appendix A

GAP4 and Python code

A.1 GAP Code (1)

In this section, we show GAP Code that we use to calculate how many points are moved by a permutation in the action of S_n on $\mathcal{P}_i\mathcal{P}(x_n)$, $i \in \{1, 2, \ldots, n\}$, n=2,3,4:

```
gap> S:=[1..2];
[ 1, 2 ]
gap> T:=Combinations(Combinations(S),1);;
gap> G:=SymmetricGroup(2);;
gap> phi:=ActionHomomorphism(G,T,OnSetsSets);
<action homomorphism>
gap> g:=(1,2);
(1,2)
gap> h:=g^phi;;
gap> NrMovedPoints(h);
2
gap> T:=Combinations(Combinations(S),2);;
gap> G:=SymmetricGroup(2);;
gap> phi:=ActionHomomorphism(G,T,OnSetsSets);
<action homomorphism>
gap> g:=(1,2);
(1,2)
gap> h:=g^phi;;
```

```
gap> NrMovedPoints(h);
4
gap> T:=Combinations(Combinations(S),3);;
gap> G:=SymmetricGroup(2);;
gap> phi:=ActionHomomorphism(G,T,OnSetsSets);
<action homomorphism>
gap> g:=(1,2);
(1,2)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
2
gap> T:=Combinations(Combinations(S),4);;
gap> G:=SymmetricGroup(2);;
gap> phi:=ActionHomomorphism(G,T,OnSetsSets);
<action homomorphism>
gap> g:=(1,2);
(1,2)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
0

gap > S:=[1..3];
[ 1 .. 3 ]
gap >  T:=Combinations(Combinations(S),1);;
gap > G:=SymmetricGroup(3);;
gap > phi:=ActionHomomorphism(G,T,OnSetsSets);
<action homomorphism>
 gap > g:=(1,2);
(1,2)
gap>  h:=g^{phi};;
gap> NrMovedPoints(h);
4
\item  gap >g:=(1,2,3);;
(1,2,3)
gap> h:=$$g^{phi};;
```

```
gap> NrMovedPoints(h);
6

gap>  T:=Combinations(Combinations(S),2);;
gap> G:=SymmetricGroup(3);;
gap> phi:=ActionHomomorphism(G,T,OnSetsSets);
<action homomorphism>
gap> g:=(1,2);
(1,2)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
20
gap> g:=(1,2,3);
(1,2,3)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
27

gap> T:=Combinations(Combinations(S),3);;
gap> G:=SymmetricGroup(3);;
gap> phi:=ActionHomomorphism(G,T,OnSetsSets);
<action homomorphism>
gap> g:=(1,2);
(1,2)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
44
gap> g:=(1,2,3);
(1,2,3)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
54

gap> T:=Combinations(Combinations(S),4);;
gap> G:=SymmetricGroup(3);;
gap> phi:=ActionHomomorphism(G,T,OnSetsSets);
```

```
<action homomorphism>
gap> g:=(1,2);
(1,2)
gap> h:=g^phi;;
gap> NrMovedPoints(h);
56
gap> g:=(1,2,3);
(1,2,3)
gap> h:=g^phi;;
gap> NrMovedPoints(h);
66

gap> T:=Combinations(Combinations(S),5);;
gap> G:=SymmetricGroup(3);;
gap> phi:=ActionHomomorphism(G,T,OnSetsSets);
<action homomorphism>
gap> g:=(1,2);
(1,2)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
44
gap> g:=(1,2,3);
(1,2,3)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
54

gap>  T:=Combinations(Combinations(S),6);;
gap> G:=SymmetricGroup(3);;
gap> phi:=ActionHomomorphism(G,T,OnSetsSets);
<action homomorphism>
gap> g:=(1,2);
(1,2)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
20
```

```
gap> g:=(1,2,3);
(1,2,3)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
27

gap >  T:=Combinations(Combinations(S),7);;
gap > G:=SymmetricGroup(3);;
gap > phi:=ActionHomomorphism(G,T,OnSetsSets);
<action homomorphism>
 gap > g:=(1,2);
(1,2)
gap>  h:=g^{phi};;
gap> NrMovedPoints(h);
4
\item   gap >g:=(1,2,3);;
(1,2,3)
gap> h:=g^{phi};;
gap> NrMovedPoints(h);
6

     gap> S:=[1..4];
[ 1 .. 4 ]
gap>  T:=Combinations(Combinations(S),1);;
gap>  G:=SymmetricGroup(4);;
gap>  phi:=ActionHomomorphism(G,T,OnSetsSets);
<action homomorphism>
gap> g:=(1,2);
(1,2)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
8
gap> g:=(1,2)(3,4);
(1,2)(3,4)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
```

```
12
gap> g:=(1,2,3);
(1,2,3)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
12
gap> g:=(1,2,3,4);
(1,2,3,4)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
14


gap>  T:=Combinations(Combinations(S),2);;
gap> G:=SymmetricGroup(4);;
gap> phi:=ActionHomomorphism(G,T,OnSetsSets);
<action homomorphism>
gap> g:=(1,2);
(1,2)
gap> h:=g^phi;;
gap> NrMovedPoints(h);
88
gap> g:=(1,2)(3,4);
(1,2)(3,4)
gap> h:=g^phi;;
gap>  NrMovedPoints(h);
108
gap> g:=(1,2,3);
(1,2,3)
gap> h:=g^phi;;
gap> NrMovedPoints(h);
114
gap> g:=(1,2,3,4);
(1,2,3,4)
gap> h:=g^phi;;
```

```
gap> NrMovedPoints(h);
118

gap> S:=[1..4];
[ 1 .. 4 ]
gap> T:=Combinations(Combinations(S),3);;
gap>  G:=SymmetricGroup(4);;
gap>  phi:=ActionHomomorphism(G,T,OnSetsSets);
<action homomorphism>
gap> g:=(1,2);
(1,2)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
472
gap> g:=(1,2)(3,4);
(1,2)(3,4)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
532
gap> g:=(1,2,3);
(1,2,3)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
552
gap> g:=(1,2,3,4);
(1,2,3,4)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
558

gap>  T:=Combinations(Combinations(S),4);;
gap> G:=SymmetricGroup(4);;
gap>  phi:=ActionHomomorphism(G,T,OnSetsSets);
<action homomorphism>
gap> g:=(1,2);
(1,2)
```

```
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
1632
gap>  g:=(1,2)(3,4);
(1,2)(3,4)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
1768
gap>  g:=(1,2,3);
(1,2,3)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
1803
gap>  g:=(1,2,3,4);
(1,2,3,4)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
1816
```

```
gap>  T:=Combinations(Combinations(S),5);;
gap> G:=SymmetricGroup(4);;
gap>  phi:=ActionHomomorphism(G,T,OnSetsSets);
<action homomorphism>
gap>  g:=(1,2);
(1,2)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
4040
gap>   g:=(1,2)(3,4);
(1,2)(3,4)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
4284
gap> g:=(1,2,3);
(1,2,3)
```

```
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
4344
gap>  g:=(1,2,3,4);
(1,2,3,4)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
4362


gap>  T:=Combinations(Combinations(S),6);;
gap>  G:=SymmetricGroup(4);;
gap>  phi:=ActionHomomorphism(G,T,OnSetsSets);
<action homomorphism>
gap>  g:=(1,2);
(1,2)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
7528
gap> g:=(1,2)(3,4);
(1,2)(3,4)
gap> h:=g^phi;;
gap> NrMovedPoints(h);
7892
gap>  g:=(1,2,3);
(1,2,3)
gap> h:=g^phi;;
gap> NrMovedPoints(h);
7986
gap>  g:=(1,2,3,4);
(1,2,3,4)
gap> h:=g^phi;;
gap> NrMovedPoints(h);
8002

gap> T:=Combinations(Combinations(S),7);;
```

```
gap> G:=SymmetricGroup(4);;
gap>  phi:=ActionHomomorphism(G,T,OnSetsSets);
<action homomorphism>
gap> g:=(1,2);
(1,2)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
10840
gap>  g:=(1,2)(3,4);
(1,2)(3,4)
gap> h:=g^phi;;
gap> NrMovedPoints(h);
11300
gap> g:=(1,2,3);
(1,2,3)
gap> h:=g^phi;;
gap> NrMovedPoints(h);
11412
gap>  g:=(1,2,3,4);
(1,2,3,4)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
11434

gap>  T:=Combinations(Combinations(S),8);;
gap> G:=SymmetricGroup(4);;
gap>  phi:=ActionHomomorphism(G,T,OnSetsSets);
<action homomorphism>
gap> g:=(1,2);
(1,2)
gap> h:=g^phi;;
gap> NrMovedPoints(h);
12224
gap> g:=(1,2)(3,4);
(1,2)(3,4)
```

```
gap> h:=g^phi;;
gap> NrMovedPoints(h);
12720
gap> g:=(1,2,3);
(1,2,3)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
12834
gap> g:=(1,2,3,4);
(1,2,3,4)
gap>  h:=g^phi;;
gap> NrMovedPoints(h);
12864
```

A.2 Python Code

In this section, we show Python code that we use to solve some equations by using the character table of specht module of S_n to find the multiplicity of irreducible representation of action S_n on $\mathcal{P}_i\mathcal{P}(x_n)$, $i \in \{1, 2, \ldots, n\}, 2 \leq n \leq 4$. At n= 3, we have :

- ```
  import numpy
  import scipy.linalg

  m = numpy.matrix([
        [1, 2, 1 ],
        [1, 0, -1],
        [1, -1, 1]
  ])

  res = numpy.matrix([[8], [4], [2]])

  print (numpy.linalg.solve(m, res))

  [[ 4.]
  ```
  ```

```
  [  2.]
  [-0.]]
```

- ```
  import numpy
  import scipy.linalg

  m = numpy.matrix([
       [1, 2, 1 ],
       [1, 0, -1],
       [1, -1, 1]
  ])

  res = numpy.matrix([[28], [8], [1]])

  print (numpy.linalg.solve(m, res))
  [[9.]
   [9.]
   [1.]]
  ```

- ```
  import numpy
  import scipy.linalg

  m = numpy.matrix([
       [1, 2, 1 ],
       [1, 0, -1],
       [1, -1, 1]
  ])

  res = numpy.matrix([[56], [12], [2]])

  print (numpy.linalg.solve(m, res))
  [[16.]
   [18.]
   [  4.]]
  ```

- ```
  import numpy
  ```
  ```

```
import scipy.linalg

m = numpy.matrix([
      [1, 2, 1 ],
      [1, 0, -1],
      [1, -1, 1]
])

res = numpy.matrix([[70], [14], [4]])

print (numpy.linalg.solve(m, res))
[[20.]
 [22.]
 [ 6.]]
```

At n=4, we have:

- ```
  import numpy
  import scipy.linalg

  m = numpy.matrix([
        [1, 3, 2, 3, 1],
        [1, 1, 0, -1, -1],
        [1, -1, 2, -1, 1],
        [1, 0, -1, 0, 1],
        [1, -1, 0, 1, -1]
  ])

  res = numpy.matrix([[16],[8],[4],[4],[2]])

  print (numpy.linalg.solve(m, res))
  [[ 5.]
   [ 3.]
   [ 1.]
   [-0.]
   [-0.]]
  ```
  ```

- ```
  import numpy
  import scipy.linalg

  m = numpy.matrix([
        [1, 3, 2, 3, 1],
        [1, 1, 0, -1, -1],
        [1, -1, 2, -1, 1],
        [1, 0, -1, 0, 1],
        [1, -1, 0, 1, -1]
  ])

  res = numpy.matrix([[120],[32],[12],[6],[2]])

  print (numpy.linalg.solve(m, res))
  [[17.]
   [21.]
   [11.]
   [ 6.]
   [-0.]]
  ```

- ```
  import numpy
  import scipy.linalg

  m = numpy.matrix([
        [1, 3, 2, 3, 1],
        [1, 1, 0, -1, -1],
        [1, -1, 2, -1, 1],
        [1, 0, -1, 0, 1],
        [1, -1, 0, 1, -1]
  ])

  res = numpy.matrix([[1820],[188],[52],[17],[4]])

  print (numpy.linalg.solve(m, res))
  [[52.]
  ```

```
 [88.]
 [51.]
 [45.]
 [ 7.]]
```

- ```
  import numpy
  import scipy.linalg

  m = numpy.matrix([
      [1, 3, 2, 3, 1],
      [1, 1, 0, -1, -1],
      [1, -1, 2, -1, 1],
      [1, 0, -1, 0, 1],
      [1, -1, 0, 1, -1]
  ])

  res = numpy.matrix([[1820],[188],[52],[17],[4]])

  print (numpy.linalg.solve(m, res))
  [[136.]
   [267.]
   [159.]
   [175.]
   [ 40.]]
  ```

- ```
  import numpy
  import scipy.linalg

  m = numpy.matrix([
      [1, 3, 2, 3, 1],
      [1, 1, 0, -1, -1],
      [1, -1, 2, -1, 1],
      [1, 0, -1, 0, 1],
      [1, -1, 0, 1, -1]
  ])
  ```

```
res = numpy.matrix([[4368],[328],[84],[24],[6]])

print (numpy.linalg.solve(m, res))
[[284.]
 [616.]
 [377.]
 [455.]
 [117.]]
```

- ```
  import numpy
  import scipy.linalg

  m = numpy.matrix([
      [1, 3, 2, 3, 1],
      [1, 1, 0, -1, -1],
      [1, -1, 2, -1, 1],
      [1, 0, -1, 0, 1],
      [1, -1, 0, 1, -1]
  ])

  res = numpy.matrix([[8008],[480],[116],[22],[6]])

  print (numpy.linalg.solve(m, res))
  ```
- ```
  import numpy
  import scipy.linalg

  m = numpy.matrix([
      [1, 3, 2, 3, 1],
      [1, 1, 0, -1, -1],
      [1, -1, 2, -1, 1],
      [1, 0, -1, 0, 1],
      [1, -1, 0, 1, -1]
  ])

  res = numpy.matrix([[11440],[600],[140],[28],[6]])
  ```

```
print (numpy.linalg.solve(m, res))
[[ 655.]
 [1561.]
 [ 979.]
 [1264.]
 [ 352.]]
```

- ```
  import numpy
  import scipy.linalg

  m = numpy.matrix([
      [1, 3, 2, 3, 1],
      [1, 1, 0, -1, -1],
      [1, -1, 2, -1, 1],
      [1, 0, -1, 0, 1],
      [1, -1, 0, 1, -1]
  ])

  res = numpy.matrix([[12870],[646],[150],[36],[6]])

  print (numpy.linalg.solve(m, res))
  [[ 730.]
   [1750.]
   [1098.]
   [1430.]
   [ 404.]]
  ```

A.3 GAP Code(2)

In this section, we show the Gap code that we use to compute the tensor product of specht modules of symmetric group

```
gap> c:=CharacterTable("symmetric",3);
CharacterTable( "Sym(3)" )
gap> irr:=Irr(c);
```
```

```
[ Character( CharacterTable( "Sym(3)" ), [ 1, -1, 1 ] ),
Character( CharacterTable( "Sym(3)" ), [ 2, 0, -1 ] ),
Character( CharacterTable( "Sym(3)" ), [ 1, 1, 1 ] ) ]
gap> CharacterParameters(c);
[ [ 1, [ 1, 1, 1 ] ], [ 1, [ 2, 1 ] ], [ 1, [ 3 ] ] ]
gap>  ten:=Tensored([irr[2]],[irr[2]])[1];
Character( CharacterTable( "Sym(3)" ), [ 4, 0, 1 ] )
gap>  SolutionMat(irr,ten);
[ 1, 1, 1 ]

gap> c:=CharacterTable("symmetric",4);
CharacterTable("Sym(4)" )
gap> irr:=Irr(c);
[ Character( CharacterTable( "Sym(4)" ), [ 1, -1, 1, 1, -1 ] ),
Character( CharacterTable( "Sym(4)" ), [ 3, -1, -1, 0, 1 ] ),
Character( CharacterTable( "Sym(4)" ), [ 2, 0, 2, -1, 0 ] ),
Character( CharacterTable( "Sym(4)" ), [ 3, 1, -1, 0, -1 ] ),
Character( CharacterTable( "Sym(4)" ), [ 1, 1, 1, 1, 1 ] ) ]
gap> CharacterParameters(c);
[ [ 1, [ 1, 1, 1, 1 ] ], [ 1, [ 2, 1, 1 ] ], [ 1, [ 2, 2 ] ], [ 1, [ 3, 1
[ 1, [ 4 ] ] ]
gap> ten:=Tensored([irr[4]],[irr[5]])[1];
Character( CharacterTable( "Sym(4)" ), [ 3, 1, -1, 0, -1 ] )
gap> SolutionMat(irr,ten);
[ 0, 0, 0, 1, 0 ]

gap> ten:=Tensored([irr[3]],[irr[4]])[1];
Character( CharacterTable( "Sym(4)" ), [ 6, 0, -2, 0, 0 ] )
gap> SolutionMat(irr,ten);
[ 0, 1, 0, 1, 0 ]

gap> ten:=Tensored([irr[4]],[irr[4]])[1];
Character( CharacterTable( "Sym(4)" ), [ 9, 1, 1, 0, 1 ] )
gap> SolutionMat(irr,ten);
[ 0, 1, 1, 1, 1 ]
```

```
    gap>  c:=CharacterTable("symmetric",6);
CharacterTable( "Sym(6)" )
gap> irr:=Irr(c);
[ Character( CharacterTable( "Sym(6)" ),
[ 1, -1, 1, -1, 1, -1, 1, -1, 1, 1, -1 ] ),
  Character( CharacterTable( "Sym(6)" ),
  [ 5, -3, 1, 1, 2, 0, -1, -1, -1, 0, 1 ] ),
  Character( CharacterTable( "Sym(6)" ),
  [ 9, -3, 1, -3, 0, 0, 0, 1, 1, -1, 0 ] ),
  Character( CharacterTable( "Sym(6)" ),
  [ 5, -1, 1, 3, -1, -1, 2, 1, -1, 0, 0 ] ),
  Character( CharacterTable( "Sym(6)" ),
  [ 10, -2, -2, 2, 1, 1, 1, 0, 0, 0, -1 ] ),
  Character( CharacterTable( "Sym(6)" ),
  [ 16, 0, 0, 0, -2, 0, -2, 0, 0, 1, 0 ] ),
  Character( CharacterTable( "Sym(6)" ),
  [ 5, 1, 1, -3, -1, 1, 2, -1, -1, 0, 0 ] ),
  Character( CharacterTable( "Sym(6)" ),
  [ 10, 2, -2, -2, 1, -1, 1, 0, 0, 0, 1 ] ),
  Character( CharacterTable( "Sym(6)" ),
  [ 9, 3, 1, 3, 0, 0, 0, -1, 1, -1, 0 ] ),
  Character( CharacterTable( "Sym(6)" ),
  [ 5, 3, 1, -1, 2, 0, -1, 1, -1, 0, -1 ] ),
  Character( CharacterTable( "Sym(6)" ),
  [ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 ] ) ]
gap>  CharacterParameters(c);
[ [ 1, [ 1, 1, 1, 1, 1, 1 ] ], [ 1, [ 2, 1, 1, 1, 1 ] ],
[ 1, [ 2, 2, 1, 1 ] ], [ 1, [ 2, 2, 2 ] ],

  [ 1, [ 3, 1, 1, 1 ] ], [ 1, [ 3, 2, 1 ] ], [ 1, [ 3, 3 ] ],
  [ 1, [ 4, 1, 1 ] ], [ 1, [ 4, 2 ] ], [ 1, [ 5, 1 ] ],
  [ 1, [ 6 ] ] ]
gap> ten:=Tensored([irr[11]],[irr[11]])[1];
Character( CharacterTable( "Sym(6)" ),
```

```
[ 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 ] )
gap> SolutionMat(irr,ten);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1 ]
gap> ten:=Tensored([irr[10]],[irr[10]])[1];
Character( CharacterTable( "Sym(6)" ),
[ 25, 9, 1, 1, 4, 0, 1, 1, 1, 0, 1 ] )
gap> SolutionMat(irr,ten);
[ 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1 ]
gap> ten:=Tensored([irr[9]],[irr[10]])[1];
Character( CharacterTable( "Sym(6)" ),
[ 45, 9, 1, -3, 0, 0, 0, -1, -1, 0, 0 ] )
gap> SolutionMat(irr,ten);
[ 0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 0 ]
```

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