A Study of the Local Cohomology of Binomial Edge Ideals

DAVID GEORGE WILLIAMS

Supervised by Dr Mordechai Katzman

A Thesis Submitted for the Degree of Doctor of Philosophy of Mathematics

September 2024

The University of Sheffield

SCHOOL OF MATHEMATICS AND STATISTICS

Abstract

In this thesis, we apply local cohomology to the study of binomial edge ideals in order to establish three main results:

1. A combinatorial characterisation of graphs whose binomial edge ideals are of König type.
2. The calculation of the local cohomology modules of the binomial edge ideals of the complements of connected graphs of girth at least 5.
3. The calculation of the minimal attached primes of the local cohomology modules of the binomial edge ideals of block graphs.

These first appeared in [Wil23a], [Wil23b], and [Wil24] respectively.

We employ several prime characteristic tools, such as the Frobenius functor and Lyubeznik's $\mathscr{H}$-functor, in the proofs of some of these results, and prove several results of independent interest concerning these tools.

Specifically, we use Lyubeznik's $\mathscr{H}$-functor to establish a correspondence between F-stable secondary representations and primary decompositions in the category of F-finite F-modules, and construct a concrete example to demonstrate that there are rings for which this category is not closed under taking primary decompositions.

We also include a chapter compiling many graded analogues of local results, including some which do not seem to have appeared previously in the literature, which we hope will be of value to other researchers.

Acknowledgements

Firstly, I would like to express my deepest gratitude to my supervisor Dr Moty Katzman. This thesis would not have been possible without his endless guidance, patience and advice. His generosity with his time and encouragement have been invaluable. I am lucky to have been his student.

My family have been a constant source of love and support, for which I cannot thank them enough. Knowing that they have all been behind me at every step has meant so much on this journey. I would also like to congratulate them for putting up with me talking to them about algebra for the past four years, and to warn them that this condition does not necessarily resolve with the completion of a thesis.

My heartfelt thanks to all of my friends, both at Sheffield and further afield, who have supported me when I needed supporting, distracted me when I needed distracting, and allowed me to do the same (the latter, frequently). Special thanks to Brad for helping me settle into the city, David for our chats about commutative algebra, and Ahmed for our chats about everything other than maths.

I wish to thank Professor Charudatta Hajarnavis of the University of Warwick, whose commutative algebra course sparked my interest in the subject, as well as Dr Mark Cummings and Dr Adam Epstein for all of their help and advice during my time there as an undergraduate.

`Macaulay2` has been an indispensable tool throughout my PhD, I would like to thank the developers and package authors for all of their work on the system. I would also like to apologise to any computers I have blamed for my code not running; it *was*, in fact, something I had written.

Finally, I am grateful to the Engineering and Physical Sciences Research Council for the studentship which supported my work.

Contents

Introduction

Background

The principal objects of study in this thesis are *binomial edge ideals*. These associate to a graph G a homogeneous ideal $\mathcal{J}(G)$ in a polynomial ring over a field. Introduced by Herzog, Hibi, Hreinsdóttir, Kahle, and Rauh in [Her+10], and independently by Ohtani in [Oht11], they found immediate application to the subject of *conditional independence statements* in statistics, and have since become widely studied objects in commutative algebra in their own right. There are often correspondences between algebraic properties of $\mathcal{J}(G)$ and combinatorial properties of G, and much work has been done investigating the binomial edge ideals of various classes of graphs. See [HHO18, Chapter 7] for an overview of many of their properties, and some examples of these correspondences.

A key tool we will make use of in their study is *local cohomology*. This was developed by Grothendieck in the 1960s (see [GH67]) as a local analogue of the usual sheaf cohomology in algebraic geometry, and was used to prove (amongst other things) various connectedness theorems (see, for example, [Gro68]). In its algebraic form, it has found extensive use across commutative algebra, and has been applied to combinatorial, geometric, and many other settings. See [BS13] and [Iye+07] for some examples of such applications, as well as detailed developments of the theory.

One drawback of local cohomology is that, in general, local cohomology modules are not necessarily finitely generated, and so can be somewhat complex to work with. However it transpires that, in certain cases, they *are* Artinian, and so we may apply the theory of *Matlis Duality*, which was introduced by Matlis in [Mat58], over complete local rings. For such a ring R, the *Matlis Duality functor* establishes an anti-equivalence of categories between the categories of Artinian and Noetherian R-modules, and so, when applied to many local cohomology modules we obtain a Noetherian R-module, which is often much easier to understand. We will see that, in some cases, these are even simply Ext modules.

If we aim to study local cohomology modules via their Matlis duals, it is important to understand how applying the Matlis Duality functor affects the various properties of a module. For example, the annihilators of a module and its Matlis dual are equal. One powerful technique for investigating Noetherian modules is to look at their primary decompositions and associated primes. We will see that, particularly for Artinian modules, Matlis Duality provides us with a dual theory: that of *secondary representations* and *attached primes*. These notions were introduced independently by Kirby in [Kir73], Macdonald in [Mac73], and Moore in [Moo73], with Kirby's being the first paper received. In the case of local cohomology modules, there are situations where we can say a surprising amount about their attached primes.

We will also make use of various prime characteristic techniques, which exploit the fact that a ring of prime characteristic $p > 0$ comes equipped with the *Frobenius endomorphism* $r \mapsto r^p$. The first of these is the notion of *Frobenius actions*, which generalise the idea of the Frobenius endomorphism to modules. Crucially, we will see that many local cohomology modules possess a natural Frobenius action.

The central prime characterstic tool we will use is the *Frobenius functor* introduced by Peskine and Szpiro in [PS73]. In a sense, this allows a ring R of prime characteristic $p > 0$ to act on R-modules via "pth roots" of their usual action, producing a new module $F_R(M)$ with this modified action for any R-module M. A celebrated theorem of Kunz given in [Kun69] can be restated in terms of this functor: it characterises Noetherian regular rings of prime characteristic as precisely the Noetherian reduced rings for which this functor is exact. In many cases, this functor commutes with the Matlis Duality functor, and possesses a wide range of other desirable properties.

Building on this functor, Lyubeznik introduced the notion of *F-modules* in [Lyu97], which have since become objects of extensive study. These are the R-modules $\mathscr{M}$ for which $F_R(\mathscr{M}) \cong \mathscr{M}$, which includes many local cohomology modules. Moreover, such local cohomology modules satisfy an even stronger property: they are also *F-finite F-modules*. Again introduced by Lyubeznik in [Lyu97], these are the R-modules $\mathscr{M}$ which are isomorphic to the direct limit of the system induced by repeatedly applying the Frobenius functor to some R-homomorphism $\beta : M \to F_R(M)$, with M a finitely generated R-module. When this β is injective, we say that M is a *root* of $\mathscr{M}$. When R is regular, every F-finite F-module has a root, and many important properties of F-finite F-modules coincide with those of their roots. For example, their associated primes agree. Then, in a similar way to Matlis Duality, they allow us to deduce properties of potentially large modules via finitely generated ones.

Lyubeznik goes on to construct a functor in [Lyu97] which sends A-modules with a Frobenius action to F-finite F-modules over R for certain classes of rings A and R, denoted $\mathscr{H}_{R,A}$. This can be used to relate certain local cohomology modules of various rings. Again, certain properties of A-modules M with Frobenius actions can be inferred from $\mathscr{H}_{R,A}(M)$, and vice versa.

The final set of techniques we apply concern graded modules. Many of the results stated so far, especially those relating to Matlis Duality and local cohomology, are stated for local rings, often assuming completeness. However binomial edge ideals are defined over polynomial rings over a field. Fortunately, there is a general principle in commutative algebra that "most" results concerning modules over local rings have analogues which hold for graded modules over graded rings which have a unique proper homogeneous ideal which is maximal amongst the homogeneous ideals of the ring. There is also an analogue of completeness over such rings.

In this way, we will see that all of the results which will be necessary for our study of binomial edge ideals can be applied in the graded case also.

Outline of Thesis

Chapter 1

We begin with an overview of four of the main concepts in this thesis:

1. Matlis Duality: injective hulls, the Matlis Duality functor, and the Duality Theorem itself.
2. Local Cohomology: key properties and vanishing/non-vanishing theorems, Local Duality, and the Nagel-Schenzel Isomorphism.
3. Secondary Representations & Attached Primes: their existence for Artinian modules, some of their interactions with local cohomology, and their duality with associated primes.
4. Binomial Edge Ideals: their primary decompositions and a formula for their heights.

Chapter 2

We first introduce Frobenius actions and the Frobenius functor, then proceed to discuss F-modules, F-finite F-modules, and some of their important properties.

Next we describe Lyubeznik's $\mathscr{H}$-functor, as well as an original result providing a correspondence between secondary representations stable under Frobenius actions and primary decompositions in the category of F-finite F-modules.

Finally, we use this correspondence to deduce another result that we have not found in the literature: that the category of F-finite F-modules over a ring R is not, in general, closed under taking primary decompositions.

Chapter 3

In this chapter, we develop graded analogues of local theorems in order to apply our work in Chapter 1 to binomial edge ideals. Much of this chapter consists of a synthesis of results from disparate sources to provide a comprehensive account of these local analogues. Some of this work is done in slightly greater generality than is strictly necessary for the remainder of the thesis, but we hope that providing a consolidated reference for such results will be of value to other researchers.

There are however some results which do not seem to have appeared previously in the literature, such as a partial generalisation to the graded case of a theorem of Macdonald and Sharp.

Chapter 4

The majority of this chapter first appeared in [Wil23a].

Our main aim in this chapter is to give a combinatorial characterisation of binomial edge ideals which are of *König type*. These were introduced by Herzog, Hibi, and Moradi in [HHM21], as a generalisation of König graphs, in which the matching number is equal to the vertex cover number (see [HHM21, Section 1] for the details). A similar characterisation appeared in [LaC23, Lemma 5.3], however the techniques used in our proofs are very different, and we also obtain a certain independence result regarding our characterisation.

We then apply this characterisation to prove that certain classes of graphs have binomial edge ideals of König type, and conjecture that this holds for some other classes also.

We conclude this chapter by calculating an explicit root of the local cohomology module $H^{n-1}_{\mathcal{J}(G)}(R)$ as an F-finite F-module, where G is a Hamiltonian graph on n vertices (that is, it contains a cycle of length n as a subgraph).

Chapter 5

The majority of this chapter first appeared in [Wil23b].

In this chapter, we make use of a Hochster-type formula introduced by Àlvarez Montaner in [Àlv20] to calculate the local cohomology modules of the complements of connected graphs of girth at least 5 (that is, they contain no cycles of length 3 or 4 as subgraphs). We then use this calculation to deduce some properties of the binomial edge ideals of such graphs, and apply prime characteristic techniques to say even more in that setting.

Chapter 6

The majority of this chapter first appeared in [Wil24].

In our final chapter, we use an exact sequence obtained by applying [Oht11, Lemma 4.8] to vertices of block graphs with certain properties to calculate the minimal attached primes of the local cohomology modules of the binomial edge ideals of block graphs. This yields a combinatorial characterisation of which such local cohomology modules are non-vanishing.

During the writing of this thesis, it was shown by Lax, Rinaldo, and Romeo in [LRR24, Theorem 3.2] that the binomial edge ideals of block graphs are sequentially Cohen-Macaulay (see Definition 6.4.1). We will show that the main theorem of this chapter follows from their result, although the techniques used in our original proof of our theorem are very different to those of [LRR24].

We also provide an example of a graph whose binomial edge ideal has a local cohomology module with an embedded attached prime.

Conventions

Throughout this thesis, unless specified otherwise:

- All rings will be unital, commutative, and **Noetherian**.
- All modules will be unital.
- All graphs will be finite, simple, and undirected.
- By dimension, we mean Krull dimension, unless our ring is a field k, in which case, for any k-module M, $\dim_k(M)$ will denote the dimension of M as a k-vector space.
- By completion, we mean the $\mathfrak{m}$-adic completion of a local ring $(R, \mathfrak{m})$, or of a module over such a ring.
- By graded, we mean $\mathbb{Z}$-graded.

General Notation

- R will always denote a ring.
- For any ring R:
  - We denote by ***R*-Mod** the category of R-modules and R-homomorphisms.
  - We denote by ***R*-FinMod** the subcategory of ***R*-Mod** consisting of finitely generated R-modules and R-homomorphisms.
  - We denote by ***R*-ArtMod** the subcategory of ***R*-Mod** consisting of Artinian R-modules and R-homomorphisms.
  - For any R-modules M and N, we may denote the fact that an R-homomorphism $\varphi : M \to N$ is an R-isomorphism by writing $\varphi : M \xrightarrow{\sim} N$.
  - For any R-module M, we denote by Id_M the identity R-homomorphism on M.
  - For any ideal $\mathfrak{a}$ of R, we denote $\mathrm{grade}_R(\mathfrak{a}, R)$ by $\mathrm{grade}_R(\mathfrak{a})$.
  - For any ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R, we denote by $(\mathfrak{a} : \mathfrak{b})$ the colon ideal

    $$\{r \in R : r\mathfrak{b} \subseteq \mathfrak{a}\} \subseteq R$$

    and by $(\mathfrak{a} : \mathfrak{b}^\infty)$ the saturation

    $$\bigcup_{i=0}^{\infty} (\mathfrak{a} : \mathfrak{b}^i) \subseteq R$$

    We may denote these without brackets when there is no confusion.

    For any element $r \in R$, we set $(\mathfrak{a} : r) := (\mathfrak{a} : (r))$ and $(\mathfrak{a} : r^\infty) := (\mathfrak{a} : (r)^\infty)$.

Furthermore, for any R-module M and R-submodule N of M, we set

$$(N :_M \mathfrak{a}) := \{m \in M : \mathfrak{a}M \subseteq N\} \subseteq M$$

and

$$(N :_R M) := \{r \in R : rM \subseteq N\} \subseteq R$$

– For any ideal $\mathfrak{a}$ of R, we set

$$V(\mathfrak{a}) := \{\mathfrak{p} \in \operatorname{Spec}(R) : \mathfrak{a} \subseteq \mathfrak{p}\}$$

– For any R-module M and element $r \in R$, we denote by M_r the localisation $S^{-1}M$ of M at the multiplicatively closed subset

$$S = \{r^i : i \geq 0\} \subseteq R$$

– For any elements $r, s \in R$, we may write $r \mid s$ to denote that r divides s (that is, there exists some $a \in R$ such that $s = ar$).

- For each prime $p > 0$, we denote by $\mathbb{F}_p$ the finite field $\mathbb{Z}/p\mathbb{Z}$.
- For a matrix A, we denote by A^T the transpose of A.
- For any set S, we denote by $\mathcal{P}(S)$ the power set of S.
- For any finite set S, we denote by $|S|$ the number of elements of S.

Graph Notation & Terminology

- For any graph G, we denote by $V(G)$ the set of vertices of G, which will always be strictly positive integers, and by $E(G)$ the set of edges $\{i, j\}$ of G.
- By K_0 we denote the **null graph** with $V(K_0) = \varnothing$ and $E(K_0) = \varnothing$.
- By K_m for $m \geq 1$, we denote the complete graph with $V(K_m) = \{1, \ldots, m\}$
- By K_S, for a non-empty set of strictly positive integers S, we denote the complete graph with $V(K_S) = S$.
- We say that a graph G is a **clique** (on a non-empty set of strictly positive integers S) if it is a complete graph (with $V(G) = S$).
- By $K_{a,b}$, for $a, b \geq 1$, we denote the complete bipartite graph with $V(K_{a,b}) = \{1, \ldots, a+b\}$ and

$$E(G) = \{\{v, w\} : 1 \leq v \leq a \text{ and } a + 1 \leq w \leq a + b\}$$

We call the graph $K_{1,m}$, for $m \geq 1$, the **star with m edges**.

- By P_m, for $m \geq 1$, we denote the path graph with $V(P_m) = \{1, \ldots, m\}$ and

$$E(P_m) = \{\{i, i+1\} : 1 \leq i \leq m-1\}$$

- By C_m, for $m \geq 2$, we denote the cycle graph with $V(C_m) = \{1, \ldots, m\}$ and

$$E(C_m) = E(P_m) \cup \{\{1, m\}\}$$

  For $m = 1$, we set $C_m = P_m$.

- For any graph G:
  - We say that a graph H is **in** G, or **contained in** G, if it is a not-necessarily induced subgraph of G.
  - We say that a vertex v of G is a **cut vertex** if the induced subgraph of G obtained by removing v has a greater number of connected components than G.
  - We say that a vertex v of G is **universal** if it is adjacent to every other vertex of G, or **isolated** if it is adjacent to no other vertex of G.
  - We say that a vertex v of G is a **leaf** of G if it is adjacent to exactly one other vertex w of G, in which case we say that the edge $\{v, w\} \in E(G)$ is a **leaf edge** of G.
  - If G is a forest (that is, a disjoint union of trees), we say that any vertex v of G which is not isolated or a leaf is a **branch** of G.
  - For any vertex v of G, we denote the **(open) neighbourhood of v in G** by

$$N_G(v) = \{w \in V(G) : \{v, w\} \in E(G)\}$$

    and the **closed neighbourhood of v in G** by $N_G[v] = N_G(v) \cup \{v\}$.
  - As a slight abuse of notation, for any set of vertices $S \subseteq V(G)$, we denote by $G \setminus S$ the induced subgraph of G obtained by removing all vertices in S. For example, $P_3 \setminus \{3\} = P_2$.
  - As another abuse of notation, for any set $S \subseteq \mathbb{N}$ of strictly positive integers, and a set

$$E = \{\{i, j\} : i, j \in S \text{ with } i \neq j\}$$

    we denote be $G \cup E$ the graph with vertices $V(G) \cup S$ and edges $E(G) \cup E$. For example, $P_2 \cup \{\{2, 3\}\} = P_3$.

Chapter 1

Preliminaries

1.1 Matlis Duality

Introduced by Matlis in [Mat58], Matlis Duality gives, over a complete local ring, a duality between Noetherian and Artinian modules. Useful partial results also hold in the not-necessarily complete case.

We begin with an overview of certain types of injective modules:

1.1.1 Injective Hulls & Cogenerators

We first introduce a special type of R-module extension:

Definition 1.1.1. *Let M be an R-module and N an R-submodule of M. We say that M is an **essential extension** of N if, for any non-zero R-submodule L of M, we have $L \cap N \neq 0$.*

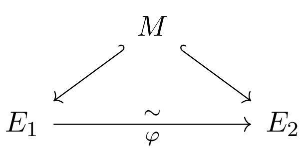
Definition 1.1.2. *Let M be an R-module. We say that an injective R-module E is an **injective hull** of M if M is an R-submodule of E and E is an essential extension of M.*

Note. *Some sources refer to injective hulls as **injective envelopes**.*

Proposition 1.1.3. [BH05, Proposition 3.2.4] *Let M be an R-module. Then:*

i) M admits an injective hull.

ii) If E_1 and E_2 are both injective hulls of M, then there exists an R-isomorphism $\varphi : E_1 \xrightarrow{\sim} E_2$ such that the diagram



commutes.

Notation 1.1.4. *Let M be an R-module. By Proposition 1.1.3, it makes sense to talk about **the** injective hull of M, which we will denote $E_R(M)$.*

Injective hulls behave well with respect to localisation:

Proposition 1.1.5. [BH05, Lemma 3.2.5] *Let M be an R-module, and S a multiplicatively closed subset of R. Then*

$$S^{-1}E_R(M) \cong E_{S^{-1}R}(S^{-1}M)$$

We next introduce another class of injective modules, and an important example of such a module:

Definition 1.1.6. *Let E be an injective R-module. We say that E is an **injective cogenerator** of R if, for any non-zero R-module M, $\mathrm{Hom}_R(M, E)$ is also non-zero.*

Proposition 1.1.7. [Mat86, Theorem 18.6 (i)] *Suppose that $(R, \mathfrak{m})$ is local. Then $E_R(R/\mathfrak{m})$ is an injective cogenerator of R.*

1.1.2 The Matlis Module & The Duality Theorem

Throughout this subsection, we assume that $(R, \mathfrak{m})$ is local.

With these concepts in hand, we now present the main result of this section:

Definition 1.1.8. *We define the **Matlis Duality functor** to be*

$$-^{\vee} := \mathrm{Hom}_R(-, E_R(R/\mathfrak{m})) : \boldsymbol{R}\textbf{-Mod} \to \boldsymbol{R}\textbf{-Mod}$$

*$E_R(R/\mathfrak{m})$ is sometimes called the **Matlis module** of R.*

Note that this functor is exact since $E_R(R/\mathfrak{m})$ is injective.

*For any R-module M, we say that $M^{\vee}$ is the **Matlis dual** of M.*

Note. *Some sources denote $-^{\vee}$ as $D(-)$ or $-^{*}$.*

Theorem 1.1.9 (Matlis Duality). *Let M be a Noetherian R-module, N an Artinian R-module, and L any R-module. Then:*

i) $L \hookrightarrow L^{\vee\vee}$.

ii) $M^{\vee}$ *is Artinian.*

Furthermore, if R is also complete, then

iii) $N^{\vee}$ *is Noetherian.*

iv) $M^{\vee\vee} \cong M$ *and* $N^{\vee\vee} \cong N$.

Proof.

i) See [BS13, Remarks 10.2.2].

ii) See [BS13, Theorem 10.2.19 (ii)].

iii) See [BS13, Theorem 10.2.12 (iii)].

iv) See [BS13, Theorem 10.2.12 (ii) & (iii)]. □

Another useful property of the Matlis Duality functor is the following:

Proposition 1.1.10. [BS13, Remarks 10.2.2 (ii)] *Let M be an R-module. Then*

$$\mathrm{Ann}_R(M) = \mathrm{Ann}_R(M^\vee)$$

It is also worth noting a few further properties of the Matlis Module:

Proposition 1.1.11. [BS13, Corollary 10.2.8] *Suppose that $(R, \mathfrak{m})$ is local, and let M be an R-module. Then M is Artinian if and only if $M \hookrightarrow E_R(R/\mathfrak{m})^t$ for some $t \geq 0$.*

This is similar to the fact that an R-module M is Noetherian if and only if $R^t \twoheadrightarrow M$ for some $t \geq 0$ (since R is Noetherian).

Proposition 1.1.12. [BS13, Remark 10.2.9 & Theorem 10.2.11] *$E_R(R/\mathfrak{m})$ has a natural $\widehat{R}$-action. Furthermore, for every $\varphi \in E_R(R/\mathfrak{m})^\vee$, there exists some $\widehat{r}_\varphi \in \widehat{R}$ such that $\varphi : e \mapsto \widehat{r}_\varphi e$ for all $e \in E_R(R/\mathfrak{m})$, and*

$$E_R(R/\mathfrak{m})^\vee \cong \widehat{R}$$

as R-modules via $\varphi \mapsto \widehat{r}_\varphi$.

Remark 1.1.13. When R is complete, Proposition 1.1.12 tells us that every R-homomorphism from $E_R(R/\mathfrak{m})$ to itself is given by multiplication by some element of R, and R-homomorphisms from $E_R(R/\mathfrak{m})^t$ to $E_R(R/\mathfrak{m})$ for $t \geq 1$ are given by diagonal $t \times t$ matrices with entries in R.

For example, take any R-homomorphism $\varphi : E_R(R/\mathfrak{m})^2 \to E_R(R/\mathfrak{m})$. φ can be viewed as the sum of two R-homomorphisms from $E_R(R/\mathfrak{m})$ to itself, and so will act on each component by multiplication. Then

$$\varphi : \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \mapsto r_\varphi e_1 + s_\varphi e_2 = \begin{bmatrix} r_\varphi & 0 \\ 0 & s_\varphi \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

for some $r_\varphi, s_\varphi \in R$.

We will make use of this observation in the proof of Proposition 2.2.15.

1.2 Local Cohomology

1.2.1 Definitions & First Properties

Introduced in its geometric form by Grothendieck in the 1960s (see [GH67]) to prove various connectedness theorems in algebraic geometry (see, for example, [Gro68]), local cohomology has since found widespread use in its algebraic form across commutative algebra.

There are several equivalent ways to define local cohomology algebraically, each with their own advantages. We will discuss several here, beginning with a purely functorial description:

Definition 1.2.1. *Let $\mathfrak{a}$ be an ideal of R, and M and N R-modules. We define the* ***$\mathfrak{a}$-torsion functor***

$$\Gamma_{\mathfrak{a}} : \boldsymbol{R}\text{-}\mathbf{Mod} \to \boldsymbol{R}\text{-}\mathbf{Mod}$$

on M by setting

$$\Gamma_{\mathfrak{a}}(M) \coloneqq \{m \in M : \mathfrak{a}^i m = 0 \textit{ for some } i \geq 0\}$$

and on an R-homomorphism $\varphi : M \to N$ by setting $\Gamma_{\mathfrak{a}}(\varphi) \coloneqq \varphi|_{\Gamma_{\mathfrak{a}}(M)}$.

Proposition 1.2.2. [BS13, Lemma 1.1.6] *Let $\mathfrak{a}$ be an ideal of R. Then $\Gamma_{\mathfrak{a}}$ is left-exact.*

Definition 1.2.3. *Let $\mathfrak{a}$ be an ideal of R, and $i \geq 0$. Since $\Gamma_{\mathfrak{a}}$ is left-exact by Proposition 1.2.2, we may define the* ***ith local cohomology functor with support at $\mathfrak{a}$*** *to be the ith right derived functor of $\Gamma_{\mathfrak{a}}$, which we will denote $H^i_{\mathfrak{a}}$.*

That is, given an R-module M, we take an injective resolution

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

of M (which we can do since $\boldsymbol{R}$-**Mod** *has enough injectives, see, for example, [BH05, Theorem 3.1.8]) and consider the cochain complex*

$$0 \xrightarrow{\delta^{-1}} \Gamma_{\mathfrak{a}}(I^0) \xrightarrow{\delta^0} \Gamma_{\mathfrak{a}}(I^1) \xrightarrow{\delta^1} \Gamma_{\mathfrak{a}}(I^2) \xrightarrow{\delta^2} \cdots$$

Then $H^i_{\mathfrak{a}}(M) \coloneqq \ker(\delta^i)/\operatorname{Im}(\delta^{i-1})$. By general homological arguments, this construction does not depend on the choice of injective resolution. Given an R-module N and R-homomorphism $M \to N$, we naturally obtain an R-homomorphism $H^i_{\mathfrak{a}}(M) \to H^i_{\mathfrak{a}}(N)$, and given an R-module L and short exact sequence

$$0 \longrightarrow N \hookrightarrow M \twoheadrightarrow L \longrightarrow 0$$

of R-modules, we naturally obtain a long exact sequence

$$\cdots \longrightarrow H^i_{\mathfrak{a}}(N) \longrightarrow H^i_{\mathfrak{a}}(M) \longrightarrow H^i_{\mathfrak{a}}(L) \longrightarrow H^{i+1}_{\mathfrak{a}}(N) \longrightarrow H^{i+1}_{\mathfrak{a}}(M) \longrightarrow H^{i+1}_{\mathfrak{a}}(L) \longrightarrow \cdots$$

of R-modules. Furthermore, $H^0_{\mathfrak{a}} = \Gamma_{\mathfrak{a}}$. See, for example, [Rot09, Section 6.2.3] for the details.

Note. *It can be checked (and will be particularly clear from Definition / Theorem 1.2.7) that if a map between two R-modules is given by $r\cdot$ for some $r \in R$, then the induced map on each of their local cohomology modules is also given by this same multiplication.*

This definition immediately allows us to deduce several useful properties of local cohomology:

Proposition 1.2.4. [Iye+07, Proposition 7.3 (2)] *Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of R such that $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$. Then we have $H^i_{\mathfrak{a}} = H^i_{\mathfrak{b}}$ as functors.*

Proof. Since R is Noetherian, we have $(\sqrt{\mathfrak{a}})^s \subseteq \mathfrak{a}$ and $(\sqrt{\mathfrak{b}})^t \subseteq \mathfrak{b}$ for some $s, t \geq 0$, so

$$\mathfrak{a}^t \subseteq (\sqrt{\mathfrak{a}})^t = (\sqrt{\mathfrak{b}})^t \subseteq \mathfrak{b}$$

and

$$\mathfrak{b}^s \subseteq (\sqrt{\mathfrak{b}})^s = (\sqrt{\mathfrak{a}})^s \subseteq \mathfrak{a}$$

Then if an element is killed by some power of $\mathfrak{b}$, it must also be killed by some power of $\mathfrak{a}$, and vice versa. This shows that $\Gamma_{\mathfrak{a}} = \Gamma_{\mathfrak{b}}$ as functors, and so their derived functors $H^i_{\mathfrak{a}}$ and $H^i_{\mathfrak{b}}$ agree also. □

Proposition 1.2.5. *Let $\mathfrak{a}$ be an ideal of R, M an R-module, and $i \geq 0$. Then for any $\mathfrak{p} \in \operatorname{Ass}_R(H^i_{\mathfrak{a}}(M))$, we have $\mathfrak{a} \subseteq \mathfrak{p}$.*

Proof. Since $\mathfrak{p} \in \operatorname{Ass}_R(H^i_{\mathfrak{a}}(M))$, we have $\mathfrak{p} = \operatorname{Ann}_R(h)$ for some $h \in H^i_{\mathfrak{a}}(M)$. Take any $a \in \mathfrak{a}$. With notation as in Definition 1.2.3, h is the equivalence class of an element of $\ker(\delta^i)$ under $\operatorname{Im}(\delta^{i+1})$. In particular, $a^j h = 0$ for some $j \geq 0$. Then $a^j \in \mathfrak{p}$, so since $\mathfrak{p}$ is prime we have $a \in \mathfrak{p}$, and we are done. □

Proposition 1.2.4 and Proposition 1.2.5 justify the geometric interpretation of local cohomology: by Hilbert's Nullstellensatz, in a polynomial ring over an algebraically closed field, ideals with the same radical give rise to the same affine variety, so one would hope that their local cohomology modules agree, and the fact that $\mathfrak{a}$ is contained in any associated prime of $H^i_{\mathfrak{a}}(M)$ justifies the use of the terminology "local cohomology with support at $\mathfrak{a}$".

Our second definition (intuitively) follows quickly from the first:

Definition / Theorem 1.2.6. *Let $\mathfrak{a}$ be an ideal of R. Then there is a natural isomorphism of functors*

$$H^i_{\mathfrak{a}}(-) \cong \varinjlim \operatorname{Ext}^i_R(R/\mathfrak{a}^j, -)$$

*from **R-Mod** to **R-Mod** for all $i \geq 0$.*

Sketch of Proof. Note that there is a natural isomorphism of functors

$$\Gamma_{\mathfrak{a}}(-) \cong \varinjlim \operatorname{Hom}_R(R/\mathfrak{a}^j, -)$$

from ***R*-Mod** to ***R*-Mod** since, for any R-module M, we have

$$\operatorname{Hom}_R(R/\mathfrak{a}^j, M) \cong (0 :_M \mathfrak{a}^j)$$

naturally via $\varphi \mapsto \varphi(1)$.

The direct system on the $\operatorname{Hom}_R(R/\mathfrak{a}^j, -)$ induced by the natural projection $R/\mathfrak{a}^{j+1} \twoheadrightarrow R/\mathfrak{a}^j$ is an example of a **filtered colimit** (see [Eis95, p. 709]), and so taking right derived functors here commutes with the direct limit (this follows, for example, from [Eis95, Proposition A6.4]), which yields the result. For a careful proof of this, see [BS13, Theorem 1.3.8]. □

The utility of this definition will become apparent later in this thesis (for example, in Proposition 2.1.23).

The last definition we will consider here comes from the cohomology of a certain Čech complex:

Definition / Theorem 1.2.7. [BH05, Theorem 3.5.6] *Let $\mathfrak{a} = (a_1 \ldots, a_t)$ be an ideal of R.*

Set

$$\Lambda = \mathcal{P}(\{1, \ldots, t\})$$

For any $S \in \Lambda$, let

$$a(S) = \prod_{i \in S} a_i \in R$$

with $a(\varnothing) = 1$.

Furthermore, for any $S \in \Lambda$ and $1 \leq i \leq t$, set

$$\sigma_{S,i} = |\{j \in S : j < i\}|$$

We denote by $C^\bullet$ the cochain complex

$$0 \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{t-2}} C^{t-1} \xrightarrow{\delta^{t-1}} C^t \xrightarrow{\delta^t} 0$$

where

$$C^m = \bigoplus_{\substack{S\in\Lambda \\ |S|=m}} R_{a(S)}$$

(recall that $R_{a(S)}$ denotes the localisation of R at the powers of $a(S)$), and $\delta^m : C^m \to C^{m+1}$ (for $0 \le m \le t-1$) is defined componentwise on each summand, with

$$\frac{r}{a(S)^l} \mapsto (-1)^{\sigma_{S,i}} \frac{a_i^l r}{a(T)^l}$$

when $T = S \sqcup \{i\}$ for some $1 \le i \le t$, and the zero map otherwise.

Then there is a natural isomorphism of functors

$$H^i_{\mathfrak{a}}(-) \cong H^i(C^\bullet \otimes_R -)$$

from $\boldsymbol{R}$-**Mod** *to* $\boldsymbol{R}$-**Mod** *for all $i \ge 0$.*

This definition immediately allows us to deduce some other important properties of local cohomology:

Theorem 1.2.8. [BS13, Theorem 3.3.1] *Let $\mathfrak{a} = (a_1, \dots, a_t)$ be an ideal of R, and M an R-module. Then $H^i_{\mathfrak{a}}(M) = 0$ for all $i > t$.*

Note. *By Proposition 1.2.4, we can use Theorem 1.2.8 to bound the arithmetic rank of an ideal (that is, its minimum number of generators up to radical).*

Theorem 1.2.9 (The Independence Theorem)**.** [BS13, Theorem 4.2.1] *Let $\mathfrak{a}$ be an ideal of R, S a ring, and $\varphi : R \to S$ a ring homomorphism. We can then view any S-module as an R-module, with the R-action induced by φ. Then there is a natural isomorphism of functors*

$$H^i_{\mathfrak{a}S}(-) \cong H^i_{\mathfrak{a}}(-)$$

from $\boldsymbol{S}$-**Mod** *to* $\boldsymbol{R}$-**Mod** *for all $i \ge 0$.*

Theorem 1.2.10 (The Flat Base Change Theorem)**.** [BS13, Theorem 4.3.2] *Let $\mathfrak{a}$ be an ideal of R, S a ring, and $\varphi : R \to S$ a ring homomorphism. We can then view any S-module as an R-module, with the R-action induced by φ. Then if φ is flat, there is a natural isomorphism of functors*

$$S \otimes_R H^i_{\mathfrak{a}}(-) \cong H^i_{\mathfrak{a}S}(S \otimes_R -)$$

from $\boldsymbol{R}$-**Mod** *to* $\boldsymbol{S}$-**Mod** *for all $i \ge 0$.*

The following two corollaries of the Flat Base Change Theorem are of particular importance:

Corollary 1.2.11. *Let $\mathfrak{a}$ be an ideal of R, and T a multiplicatively closed subset of R. Then there is a* *natural isomorphism of functors*

$$T^{-1}H^i_{\mathfrak{a}}(-) \cong H^i_{\mathfrak{a}T^{-1}R}(T^{-1}-)$$

from $\boldsymbol{R}$-**Mod** *to* $\boldsymbol{T^{-1}R}$-**Mod** *for all* $i \geq 0$.

Proof. This follows immediately from the Flat Base Change Theorem, since localisation is flat (see, for example, [Eis95, Theorem 7.2]). □

Corollary 1.2.12. *Suppose that* $(R, \mathfrak{m})$ *is local, and let* $\mathfrak{a}$ *be an ideal of* R. *Then there is a natural isomorphism of functors*

$$\widehat{R} \otimes_R H^i_{\mathfrak{a}}(-) \cong H^i_{\mathfrak{a}\widehat{R}}(\widehat{R} \otimes_R -)$$

from $\boldsymbol{R}$-**Mod** *to* $\widehat{\boldsymbol{R}}$-**Mod** *for all* $i \geq 0$.

Proof. This follows immediately from the Flat Base Change Theorem, since completion is flat (see, for example, [Mat86, Theorem 8.8]). □

When R is local, since completion is faithfully flat (see, for example, [Mat86, Theorem 8.1 (3)]), Corollary 1.2.12 often allows us to assume that our ring is complete when we are investigating the vanishing of local cohomology modules. By Cohen's Structure Theorem (see [Coh46]), in many cases (for example, in prime characteristic), we may assume that our ring is the quotient of the power series ring over a field. Coupled with applying the Independence Theorem to the natural projection from this power series ring onto its quotient, we can sometimes simply work over the power series ring itself, the properties of which are well understood.

Note. *There is another commonly used definition of local cohomology based on a direct limit of the cohomology of certain Koszul complexes, however we will not need this for our purposes. See [BS13, Section 5.2] for the details.*

One drawback of local cohomology modules is that they are not, in general, finitely generated. In fact, when $(R, \mathfrak{m})$ is local and M a finitely generated R-module of dimension $d > 0$, $H^d_{\mathfrak{m}}(M)$ *cannot* be finitely generated (see [BS13, Corollary 7.3.3]). However, the following theorem allows us in some sense to compensate for this:

Theorem 1.2.13. [BS13, Theorem 7.1.3] *Suppose that* $(R, \mathfrak{m})$ *is local, and let* M *be a finitely generated* R*-module. Then* $H^i_{\mathfrak{m}}(M)$ *is Artinian for all* $i \geq 0$.

Note. *When* $(R, \mathfrak{m})$ *is a complete local ring, by Theorem 1.2.13 and Matlis Duality (iii) we then have that* $H^i_{\mathfrak{m}}(M)^\vee$ *is finitely generated, and so is often much easier to work with than* $H^i_{\mathfrak{m}}(M)$ *itself. We will see in Subsection 1.2.3 that, in certain cases, we can explicitly describe these duals.*

We can also describe (Castelnuovo-Mumford) regularity using local cohomology. For an introduction to regularity, and its significance, see, for example, [Eis05, Section 4A]. We begin with a definition:

Definition 1.2.14. *Suppose that R is a polynomial ring over a field with the standard grading, and denote by $\mathfrak{m}$ the irrelevant ideal of R. Let M be a graded R-module. Then, denoting by M_d the dth graded component of M, we set*

$$\operatorname{end}_R(M) := \max\{d \in \mathbb{Z} : M_d \neq 0\}$$

with $\operatorname{end}_R(M) = \infty$ if no such d exists, or $\operatorname{end}_R(M) = -\infty$ if $M = 0$.

Theorem 1.2.15. [Eis05, Corollary 4.5] *Suppose that R is a polynomial ring over a field with the standard grading, and let $\mathfrak{m}$ denote the irrelevant ideal of R. For any finitely generated graded R-module M, we will see in Lemma 3.2.4 that $H^i_{\mathfrak{m}}(M)$ is also graded for each $i \geq 0$. Then we have*

$$\operatorname{reg}_R(M) = \max\{\operatorname{end}_R(H^i_{\mathfrak{m}}(M)) + i : i \geq 0\}$$

With these properties established, we will next explicitly compute an example of an important local cohomology module:

Example 1.2.16. [Iye+07, Example 7.16] Let $R = k[x_1, \ldots, x_n]$ for some field k and $n \geq 1$, and let $\mathfrak{m}$ be the irrelevant ideal of R. We will compute $H^n_{\mathfrak{m}}(R)$.

Note. *The same result, with minor modifications to this calculation, also holds for $R = k[[x_1, \ldots, x_n]]$.*

Set $y = x_1 \cdots x_n$, and

$$z_i = \prod_{\substack{j=1 \\ j \neq i}}^{n} x_j$$

for $1 \leq i \leq n$.

Then by Definition / Theorem 1.2.7, computing $H^n_{\mathfrak{m}}(R)$ amounts to computing

$$\operatorname{coker}\left(\bigoplus_{i=1}^{n} R_{z_i} \xrightarrow{\delta^{n-1}} R_y\right) = R_y \,/ \left(\sum_{i=1}^{n} \delta^{n-1}(R_{z_i})\right)$$

where δ^{n-1} is given by the natural maps

$$\frac{r}{z_i^t} \mapsto \frac{x_i^t r}{y^t}$$

up to some sign, but since the sign does not affect the image we can ignore it here.

Note that $R_y \cong k(x_1, \ldots, x_n)$, the ring of Laurent polynomials in n variables over k, as R-modules.

Furthermore, viewing each element of element of $\delta^{n-1}(R_{z_i})$ as a Laurent polynomial, the lowest exponent of x_i which can appear in each term is 0. Then quotienting out the sum of these R-modules in R_y leaves us with the R-submodule of $k[x_1^{-1}, \ldots, x_n^{-1}]$ in which, in each term of every element, every x_i has strictly negative degree. R acts on $k[x_1^{-1}, \dots, x_n^{-1}]$ as one would expect, but if under multiplication any element of a term would be raised to an exponent greater than 0, that term is killed.

In other words, $H^n_{\mathfrak{m}}(R)$ is generated over k by the set of monomials

$$\{x_1^{-a_1} \cdots x_n^{-a_n} : a_i \geq 1 \text{ for all } 1 \leq i \leq n\}$$

with the expected R-action, subject to the condition that

$$x_i(x_1^{-1} \cdots x_n^{-1}) = 0$$

for all $1 \leq i \leq n$. Notice that this is not a finitely generated R-module.

As a simple application of Example 1.2.16, take $n = 1$, and consider $H^1_{(x)}(k[x])$. From a geometric point of view, the fact that the equivalence class of $\frac{1}{x}$ in this module is non-vanishing corresponds to the fact that we cannot extend the domain of $\frac{1}{x}$ to the whole of $\mathbb{A}^1_k$ (see [BS13, Remark 2.3.3]).

When k is algebraically closed, by Serre's Affineness Criterion (see, for example, [BS13, Theorem 6.4.4]), the non-vanishing of $H^n_{\mathfrak{m}}(R)$ tells us that $\mathbb{A}^n_k \setminus \{\mathbf{0}\}$ is not affine for any $n \geq 2$.

For an ideal $\mathfrak{a}$ of R with t generators, we can also compute a useful description of $H^t_{\mathfrak{a}}(R)$, after first introducing some notation:

Notation 1.2.17. *For an ideal $\mathfrak{a} = (a_1, \dots, \mathfrak{a}_t)$ of R, and $l \geq 1$, we set*

$$\mathfrak{a}^{[l]} := (a_1^l, \dots, a_t^l)$$

Proposition 1.2.18. [BS13, Exercise 5.3.7] *Let $\mathfrak{a} = (a_1, \dots, a_t)$ be an ideal of R, and set $a = a_1 \cdots a_t$. Then*

$$H^t_{\mathfrak{a}}(R) \cong \varinjlim \left[R/\mathfrak{a} \xrightarrow{a\cdot} R/\mathfrak{a}^{[2]} \xrightarrow{a\cdot} R/\mathfrak{a}^{[3]} \xrightarrow{a\cdot} \cdots \right]$$

(where $a\cdot$ denotes multiplication by a).

Proof. Set

$$\overline{a}_i = a_1 \cdots a_{i-1} a_{i+1} \cdots a_t$$

for $1 \leq i \leq t$.

Then we have

$$H^t_{\mathfrak{a}}(R) \cong \operatorname{coker}(\delta^{t-1})$$

where

$$\delta^{t-1} : \bigoplus_{i=1}^{t} R_{\overline{a}_i} \to R_a$$

is the map detailed in Definition / Theorem 1.2.7.

This means that a typical element of $H^t_{\mathfrak{a}}(R)$ looks like $[\frac{r}{a^i}]$ for some $r \in R$ and $i \geq 0$, with the brackets denoting the fact that this is really an equivalence class of elements.

Furthermore, a typical element of the direct limit system in the statement of the proposition looks like $[r + \mathfrak{a}^{[i]}]$ for some $r \in R$ and $i \geq 0$, where again the brackets denote the fact that this is really an equivalence class of elements.

By [BS13, Lemma 5.3.2 (ii)], we have that $[\frac{r}{a^i}] = 0$ if and only if there exists some $l \geq i$ such that $a^{l-i}r \in \mathfrak{a}^{[l]}$. It is then easily checked that the map $[r + \mathfrak{a}^{[i]}] \mapsto [\frac{r}{a^i}]$ is well-defined and yields an isomorphism between the direct limit system and $H^t_{\mathfrak{a}}(R)$. □

1.2.2 Vanishing & Non-Vanishing Theorems

Much of the utility of local cohomology comes from its ability to compute or bound many important invariants of R-modules. We have already seen that local cohomology can be used to bound the arithmetic rank of an ideal, and to characterise regularity. We will see next that it also gives information about dimension, grade, depth, and (in certain cases) height.

We begin with two general results:

Theorem 1.2.19 (Grothendieck's Vanishing Theorem)**.** [BS13, Theorem 6.1.2] *Let $\mathfrak{a}$ be an ideal of R, and M an R-module. Then $H^i_{\mathfrak{a}}(M) = 0$ for all $i > \dim_R(M)$.*

Theorem 1.2.20. [BS13, Theorem 6.2.7] *Let $\mathfrak{a}$ be an ideal of R, and let M be a finitely generated R-module such that $\mathfrak{a}M \neq M$. Then $\operatorname{grade}_R(\mathfrak{a}, M)$ is the least integer i such that $H^i_{\mathfrak{a}}(M) \neq 0$.*

When R is local, we can precisely characterise dimension using local cohomology:

Theorem 1.2.21 (The Non-Vanishing Theorem)**.** [BS13, Theorem 6.1.4] *Suppose that $(R, \mathfrak{m})$ is local, and let M be a non-zero finitely generated R-module of dimension d. Then $H^d_{\mathfrak{m}}(M) \neq 0$.*

Note. *We will see a refined version of the Non-Vanishing Theorem in Theorem 1.3.8, which gives us further information about these local cohomology modules.*

Corollary 1.2.22. *Suppose that $(R, \mathfrak{m})$ is local, and let M be a non-zero finitely generated R-module. Then $\dim_R(M)$ is the greatest integer i such that $H^i_{\mathfrak{m}}(M) \neq 0$.*

Proof. The result follows immediately from Grothendieck's Vanishing Theorem and the Non-Vanishing Theorem. □

Again when R is local, we can also use local cohomology to calculate depth:

Corollary 1.2.23. *Suppose that $(R, \mathfrak{m})$ is local, and let M be a non-zero finitely generated R-module. Then $\operatorname{depth}_R(M)$ is the least integer i such that $H^i_{\mathfrak{m}}(M) \neq 0$.*

Proof. Nakayama's Lemma tells us that $\mathfrak{m}M \neq M$ since M is non-zero and finitely generated, and so applying Theorem 1.2.20 with $\mathfrak{a} = \mathfrak{m}$ yields the desired result. □

Note. *We will refer to the non-vanishing local cohomology modules at the highest and lowest indices as the **top** and **bottom** local cohomology modules respectively.*

We can then use local cohomology to determine when a finitely generated R-module over a local ring $(R, \mathfrak{m})$ is Cohen-Macaulay:

Corollary 1.2.24. *Suppose that $(R, \mathfrak{m})$ is local, and let M be a non-zero finitely generated R-module. Then M is Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(M) \neq 0$ for $i = \dim_R(M)$ and vanishes otherwise.*

Proof. This is immediate from Corollary 1.2.22 and Corollary 1.2.23. □

Local cohomology has many useful properties in prime characteristic, which we will see more of in Chapter 2. We state one here:

Theorem 1.2.25. [Iye+07, Theorem 21.29] *Suppose that R is a regular domain of prime characteristic $p > 0$, and let $\mathfrak{a}$ be an ideal of R. Then if $R/\mathfrak{a}$ is Cohen-Macaulay, we have that $H^i_{\mathfrak{a}}(R) = 0$ for $i \neq \operatorname{height}_R(\mathfrak{a})$.*

1.2.3 Canonical Modules & Local Duality

Throughout this subsection, we assume that $(R, \mathfrak{m})$ is local.

Canonical modules were, again, introduced by Grothendieck (see [GH67, Section 5]), mainly to prove results such as Local Duality. We begin with the definition of [BS13, Definition 12.1.2]:

Definition 1.2.26. *Let $n = \dim(R)$. We say that a finitely generated R-module C is a **canonical module** of R if*

$$C^\vee \cong H^n_{\mathfrak{m}}(R)$$

Note. *In some sources (for example, [GH67]), canonical modules are instead called **dualising modules**.*

Importantly, when such modules exist, they are unique:

Proposition 1.2.27. [BS13, Theorem 12.1.6] *Suppose that C_1 and C_2 are both canonical modules of R. Then $C_1 \cong C_2$.*

Notation 1.2.28. *By Proposition 1.2.27, it makes sense to talk about **the** canonical module of R (when one exists), which we will denote as ω_R.*

Not every local ring has a canonical module. In fact, not even every Cohen-Macaulay local ring has a canonical module (see, for example, [Nis12, Example 6.1]). However, we can describe exactly which Cohen-Macaulay local rings have canonical modules:

Proposition 1.2.29. [BS13, Remark 12.1.26 & Remarks 12.1.3 (v)] *Suppose that R is Cohen-Macaulay. Then R has a canonical module if and only if it is the homomorphic image of a Gorenstein local ring.*

In particular, when R is also complete, it has a canonical module by Cohen's Structure Theorem ([Coh46]).

Furthermore, we can characterise Gorenstein local rings using canonical modules:

Proposition 1.2.30. [BS13, Corollary 12.1.22] *Suppose that R is Cohen-Macaulay. Then R is Gorenstein if and only if $\omega_R \cong R$.*

They also behave well with respect to localisation and completion:

Proposition 1.2.31. [BS13, Theorem 12.1.18 (ii)] *Suppose that R is Cohen-Macaulay, and furthermore that it has a canonical module ω_R. Then, for any $\mathfrak{p} \in \operatorname{Spec}(R)$, $(\omega_R)_{\mathfrak{p}}$ is the canonical module of $R_{\mathfrak{p}}$ (and so we may denote it as $\omega_{R_{\mathfrak{p}}}$).*

Proposition 1.2.32. [BS13, Remarks 12.1.3 (ii)] *Suppose that R has a canonical module ω_R. Then $\widehat{\omega_R}$ is the canonical module of $\widehat{R}$ (and so we may denote it as $\omega_{\widehat{R}}$).*

It was mentioned in the note following Theorem 1.2.13 that, in some cases, we can explicitly describe the Matlis duals of local cohomology modules. We do this via Local Duality, which was first proved by Grothendieck (in much greater generality than we will need here, see [Har66, Chapter V]):

Theorem 1.2.33 (Local Duality)**.** [BS13, Theorem 12.1.20 (ii)] *Suppose that $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension n, and furthermore that it has a canonical module ω_R (in particular, when R is also complete by Proposition 1.2.29). Then there is a natural isomorphism of functors*

$$H^i_{\mathfrak{m}}(-) \cong \operatorname{Ext}^{n-i}_R(-, \omega_R)^{\vee}$$

*from **R-FinMod** to **R-Mod** for all $0 \leq i \leq n$ (that is, the isomorphism holds when the argument of $H^i_{\mathfrak{m}}$ is finitely generated).*

This is particularly useful when performing explicit calculations. For example, say we were interested in the vanishing of local cohomology modules over $R = k[x_1, \ldots, x_n]$ for some finite field k and $n \geq 0$. Then, for any finitely generated R-module M, we can explicitly compute $\operatorname{Ext}^i_R(M, R)$ for any $i \geq 0$ using computer algebra systems such as `Macaulay2`. Since $\widehat{R}$ is Gorenstein local, by Proposition 1.2.30 we have $\omega_{\widehat{R}} \cong \widehat{R}$. Then by Corollary 1.2.12 and Local Duality, we can compute all i for which $H^i_{\mathfrak{m}}(M) \neq 0$ (where $\mathfrak{m}$ is the irrelevant ideal of R), since we know that $H^i_{\mathfrak{m}}(M) = 0$ for all $i \geq n \geq \dim_R(M)$ by Grothendieck's Vanishing Theorem and we can compute $\dim_R(M)$.

1.2.4 The Nagel-Schenzel Isomorphism

Parts of this subsection first appeared in [Wil23a].

We conclude this section on local cohomology with a result of Nagel and Schenzel from [NS94]. Before giving its statement, we must introduce a generalisation of regular sequences. We first recall the definition of regular sequences themselves:

Definition 1.2.34. *Let M be an R-module. A sequence of elements $r_1, \ldots, r_t \in R$ is said to be an **M-sequence** if, for each $1 \leq i \leq t$, r_i is a non-zerodivisor on*

$$R/(r_1, \ldots, r_{i-1})$$

*When $M = R$, we simply call such a sequence a **regular sequence**.*

*Unless specified otherwise, we will assume that an M-sequence $r_1, \ldots, r_t$ is **proper**, by which we mean that*

$$R/(r_1, \ldots, r_t) \neq 0$$

We now define a generalisation:

Definition 1.2.35. [HQ19, Lemma 2.4] *Let I be an ideal of R, and M a finitely generated R-module. A sequence of elements $r_1, \ldots, r_t \in I$ is called **I-filter regular on M** if, for all $\mathfrak{p} \in \operatorname{Supp}_R(M) \setminus V(I)$ and $i \leq t$ such that $r_1, \ldots, r_i \in \mathfrak{p}$, we have that $\frac{r_1}{1}, \ldots, \frac{r_i}{1}$ is a (possibly improper) $M_{\mathfrak{p}}$-sequence.*

*When $M = R$, we simply say that such a sequence is **I-filter regular**.*

In a local ring, or in certain graded contexts, any permutation of a regular sequence is again a regular sequence (see, for example, [BS13, Proposition 1.1.6] and [Mat86, Theorem 16.3]). This is not the case with I-filter regular sequences. For example, take a field k, and set

$$R = k[[x, y, z]]/(xy, xz)$$

Then it is noted in [MQS20, p. 246] that $x + y, z$ is an (x, y, z)-filter regular sequence, but $z, x + y$ is not (in fact, even just z is not).

However, unlike regular sequences, there always exist I-filter regular sequences of arbitrary length:

Proposition 1.2.36. [AS03, Proposition 2.2] *Let I be an ideal of R, and M and R-module. Then, for any $t \geq 1$, there exists a sequence of elements $r_1, \ldots, r_t \in I$ which is I-filter regular on M.*

We are now ready to state the main theorem of this subsection:

Theorem 1.2.37 (The Nagel-Schenzel Isomorphism)**.** [NS94, Lemma 3.4] *Let I be an ideal of R, and M a finitely generated R-module. Furthermore, let $r_1, \dots, r_t$ be an I-filter regular sequence on M, and set $\mathfrak{a} = (r_1, \dots, r_t)$. Then*

$$H_I^i(M) \cong \begin{cases} H_{\mathfrak{a}}^i(M) & \text{if } 0 \le i < t \\ H_I^{i-t}(H_{\mathfrak{a}}^t(M)) & \text{if } i \ge t \end{cases}$$

In particular, for $i = t$, we have

$$H_I^t(M) \cong H_I^0(H_{\mathfrak{a}}^t(M)) = \Gamma_I(H_{\mathfrak{a}}^t(M))$$

Note. *The Nagel-Schenzel Isomorphism is stated in* [NS94, Lemma 3.4] *only for local rings $(R, \mathfrak{m})$ and $\mathfrak{m}$-filter regular sequences on finitely generated R-modules. [AS03, Proposition 2.3] naturally generalises this to I-filter regular sequences on finitely generated R-modules. In both of these papers, the proof of the isomorphism makes use of spectral sequences. For an elementary proof, and of a slightly more general result, see [HQ19, Theorem 2.7].*

1.3 Secondary Representations & Attached Primes

Parts of this section first appeared in [Wil24].

1.3.1 Definitions & First Properties

Note. *Whilst we assume that R is Noetherian, many of the results in this section hold without this assumption. See, for example, [Mac73] for statements of some such results with more precise conditions.*

Secondary representations were introduced independently by Kirby in [Kir73] (there called "coprimary decompositions"), Macdonald in [Mac73], and Moore in [Moo73] (again as "coprimary decompositions"), with Kirby's being the first paper received.

They represent a dual notion to the theory of primary decompositions, in a sense we will make precise.

We begin with their definitions:

Definition 1.3.1. *We say that a non-zero R-module S is **secondary** if, for each $r \in R$, either $rS = S$ or $r^m S = 0$ for some $m \ge 1$.*

*In this case, $\mathfrak{p} = \sqrt{\operatorname{Ann}_R(S)}$ is prime, and we say that S is **$\mathfrak{p}$-secondary**.*

Definition 1.3.2. *Let M be an R-module. If there exist $\mathfrak{p}_i$-secondary R-submodules S_i of M such that*

$$M = S_1 + \cdots + S_t$$

*for some $t \geq 1$, then we call this a **secondary representation** of M, and M is said to be **representable**.*

*Such a representation is said to be **minimal** when*

i) The $\mathfrak{p}_i$ are all distinct.

ii) For every $1 \leq i \leq t$, we have

$$S_i \nsubseteq \sum_{\substack{j=1 \\ j \neq i}}^{t} S_j$$

Note. *It is easily shown that the sum of any two $\mathfrak{p}$-secondary modules is $\mathfrak{p}$-secondary, and so any secondary representation can be refined to be minimal.*

We can already see some duality here, for an R-module M, an R-submodule N of M is $\mathfrak{p}$-primary for some $\mathfrak{p} \in \operatorname{Spec}(R)$ if and only if multiplication by each element of R is either injective or nilpotent on M/N, whereas it is $\mathfrak{p}$-secondary if multiplication by each element of R is either surjective or nilpotent on N.

There is a useful alternative characterisation of secondary modules (when R is Noetherian), dual to the fact that an R-module M has $\mathfrak{p} \in \operatorname{Spec}(R)$ as an associated prime if and only if we can embed $R/\mathfrak{p}$ into M:

Proposition 1.3.3. [Mac73, (2.5)] *Let M be an R-module, and $\mathfrak{p} \in \operatorname{Spec}(R)$. Then M is $\mathfrak{p}$-secondary if and only if there exists some R-submodule N of M such that $\operatorname{Ann}_R(M/N) = \mathfrak{p}$.*

There are several uniqueness theorems concerning secondary representations. The first allows us to define a notion dual to associated primes:

Theorem 1.3.4. [Mac73, (2.2)] *Let M be a representable R-module, and S_i the $\mathfrak{p}_i$-secondary summands in a minimal secondary representation of M for some $t \geq 1$. Then both t and the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ are independent of the choice of minimal secondary representation.*

Definition 1.3.5. *Let M be a representable R-module, and S_i the $\mathfrak{p}_i$-secondary summands in a minimal secondary representation of M for some $t \geq 1$. Then we set*

$$\operatorname{Att}_R(M) := \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$$

*and these are said to be the **attached primes** of M.*

*If an attached prime is minimal (with respect to inclusion) in this set, we say that it is **isolated**, otherwise we say that it is **embedded**.*

The second shows that the secondary modules in a secondary representation corresponding to the isolated attached primes are unique:

Theorem 1.3.6. *Let M be a representable R-module, and S the component in a minimal secondary representation of M corresponding to an isolated attached prime $\mathfrak{p}$. Then*

$$S = \bigcap_{r \in R \setminus \mathfrak{p}} rM$$

Furthermore, if M is Artinian, then $S = aM$ for some $a \in R \setminus \mathfrak{p}$.

Proof. For the first claim, apply [Mac73, (3.1)] to $R \setminus \mathfrak{p}$.

For the second, note that any chain

$$M \supseteq r_1 M \supseteq (r_1 M) \cap (r_2 M) \supseteq (r_1 M) \cap (r_2 M) \cap (r_3 M) \supseteq \cdots$$

with $r_i \in R$ must stabilise since M is Artinian. Then

$$S = \bigcap_{r \in R \setminus \mathfrak{p}} rM = \bigcap_{i=1}^{l} a_i M$$

for some $a_i \in R \setminus \mathfrak{p}$ and $l \geq 1$. Let $a = a_1 \cdots a_l$. This also belongs to $R \setminus \mathfrak{p}$ since $\mathfrak{p}$ is prime, so we have

$$S = \bigcap_{i=1}^{l} a_i M \supseteq aM \supseteq \bigcap_{r \in R \setminus \mathfrak{p}} rM = S$$

and the result follows. □

In the same way that any Noetherian module has a primary decomposition, the following is true of Artinian modules:

Theorem 1.3.7. [Mac73, (5.2)] *Let M be an Artinian R-module. Then M is representable.*

In particular, when $(R, \mathfrak{m})$ is local, $H^i_{\mathfrak{m}}(M)$ is representable for any finitely generated R-module M and $i \geq 0$ by Theorem 1.2.13 and Theorem 1.3.7. This leads us to the alternative version of the Non-Vanishing Theorem which was mentioned earlier, proved by Macdonald and Sharp in [MS72]:

Theorem 1.3.8. [MS72, Theorem 2.2] *Suppose that $(R, \mathfrak{m})$ is local, and let M be a finitely generated R-module of dimension d. Then*

$$\mathrm{Att}_R(H^d_{\mathfrak{m}}(M)) = \{\mathfrak{p} \in \mathrm{Ass}_R(M) : \dim(R/\mathfrak{p}) = d\}$$

1.3.2 Duality with Primary Decompositions & Associated Primes

Throughout this subsection, we assume that $(R, \mathfrak{m})$ is local.

As mentioned at the start of this section, secondary representations are in a certain sense dual to primary decompositions. This duality can be made explicit over local rings using Matlis Duality. We begin by "converting" primary decompositions into secondary representations:

Lemma 1.3.9. [Sha76, Section 3.2] *Let M be a Noetherian R-module, and N a $\mathfrak{p}$-primary R-submodule of M for some $\mathfrak{p} \in \operatorname{Ass}_R(M)$. Then $(M/N)^\vee$ is a $\mathfrak{p}$-secondary R-submodule of $M^\vee$.*

Lemma 1.3.10. [Sha76, Sections 3.3 – 3.5] *Let M be a Noetherian R-module with minimal primary decomposition*

$$Q_1 \cap \cdots \cap Q_t = 0 \subseteq M$$

with each Q_i being $\mathfrak{p}_i$-primary for some $\mathfrak{p}_i \in \operatorname{Ass}_R(M)$.

Then

$$M^\vee = (M/Q_1)^\vee + \cdots + (M/Q_t)^\vee$$

is a minimal secondary representation of $M^\vee$, with each $(M/Q_i)^\vee$ being $\mathfrak{p}_i$-secondary. In particular

$$\operatorname{Att}_R(M^\vee) = \operatorname{Ass}_R(M)$$

The dual results are also well known, we present proofs here for convenience:

Lemma 1.3.11. *Let M be an R-module, and N a $\mathfrak{p}$-secondary R-submodule of M for some $\mathfrak{p} \in \operatorname{Att}_R(M)$. Then $(M/N)^\vee$ is a $\mathfrak{p}$-primary R-submodule of $M^\vee$.*

Proof. Note that the elements of $(M/N)^\vee$ are maps from M/N to $E_R(R/\mathfrak{m})$, which we may view as the maps from M to $E_R(R/\mathfrak{m})$ such that N lies in their kernel.

Let

$$L = M^\vee/(M/N)^\vee$$

We want to show that multiplication by each element of $\mathfrak{p}$ is nilpotent on L, and that multiplication by each element of $R \setminus \mathfrak{p}$ is injective on L.

First, take any $r \in \mathfrak{p}$, so $r^m N = 0$ for some $m \geq 1$ since N is $\mathfrak{p}$-secondary. Then for any $\varphi \in M^\vee$, we have that

$$r^m \varphi(N) = \varphi(r^m N) = \varphi(0) = 0$$

so $r^m\varphi \in (M/N)^\vee$, and therefore multiplication by r is nilpotent on L.

Next, take any $r \in R \setminus \mathfrak{p}$ and suppose that, for some $\varphi \in M^\vee$, we have $r\varphi \in (M/N)^\vee$. That is, $r\varphi(N) = 0$. Since N is $\mathfrak{p}$-secondary, we have $rN = N$, and so

$$\varphi(N) = \varphi(rN) = r\varphi(N) = 0$$

Then $\varphi \in (M/N)^\vee$, so multiplication by r is injective on L, and we are done. ☐

Lemma 1.3.12. *Let M be a representable R-module with minimal secondary representation*

$$M = S_1 + \cdots + S_t$$

with each S_i being $\mathfrak{p}_i$-secondary for some $\mathfrak{p}_i \in \mathrm{Att}_R(M)$.

Then

$$(M/S_1)^\vee \cap \cdots \cap (M/S_t)^\vee = 0 \subseteq M^\vee$$

is a minimal primary decomposition of $M^\vee$, with each $(M/S_i)^\vee$ being $\mathfrak{p}_i$-primary. In particular

$$\mathrm{Ass}_R(M^\vee) = \mathrm{Att}_R(M)$$

Proof. Take any $A \subseteq \{1, \ldots, t\}$, and let

$$S_A = \sum_{i \in A} S_i$$

Furthermore, let

$$U_i = \{\varphi \in M^\vee : S_i \subseteq \ker(\varphi)\}$$

and

$$U_A = \{\varphi \in M^\vee : S_A \subseteq \ker(\varphi)\}$$

Then $(M/S_i)^\vee \cong U_i$ and $(M/S_A)^\vee \cong U_A$ as R-modules. Each U_i is $\mathfrak{p}_i$-primary by Lemma 1.3.11.

We have the exact sequence

$$0 \longrightarrow S_A \hookrightarrow M \twoheadrightarrow M/S_A \longrightarrow 0$$

so taking Matlis duals gives us the exact sequence

$$0 \longrightarrow U_A \hookrightarrow M^\vee \twoheadrightarrow S_A^\vee \longrightarrow 0$$

Now, $E_R(R/\mathfrak{m})$ is an injective cogenerator for R by Proposition 1.1.7, and so by exactness we have that $U_A = 0$ if and only if $S_A = M$. Set

$$I_A = \bigcap_{i \in A} U_i$$

Let $\varphi \in M^\vee$. If $S_i \subseteq \ker(\varphi)$ for each $i \in A$, then certainly $S_A \subseteq \ker(\varphi)$, so $I_A \subseteq U_A$. Conversely, if $S_A \subseteq \ker(\varphi)$ then $S_i \subseteq S_A \subseteq \ker(\varphi)$ for each $i \in A$, so $U_A \subseteq I_A$. Then $I_A = U_A$, and so the result follows by the minimality of the secondary representation of M. □

1.4 Binomial Edge Ideals

Throughout this section, we set

$$R = k[x_1, \dots, x_n, y_1, \dots, y_n]$$

for some field k and $n \geq 1$, and $\delta_{i,j} := x_i y_j - x_j y_i$.

Parts of this section first appeared in [Wil23a].

1.4.1 Definition & First Properties

Binomial edge ideals were introduced in [Her+10], and independently in [Oht11]. There is a rich interplay between the algebraic properties of these ideals and the combinatorial properties of the corresponding graph. For an overview of some of these interactions, see [HHO18, Chapter 7].

We begin with their definition:

Definition 1.4.1. *Let G be a graph. We define the **binomial edge ideal** of G as*

$$\mathcal{J}(G) := (\delta_{i,j} : \{i, j\} \in E(G))$$

We may also denote this ideal as $\mathcal{J}_G$.

A key fact about binomial edge ideals is the following:

Theorem 1.4.2. [Her+10, Corollary 2.2] *Binomial edge ideals are radical.*

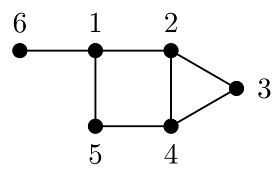
In fact, an explicit description of the primary decomposition can be given by purely combinatorial means, but we must first introduce some notation:

Notation 1.4.3. *Let G be a graph, and $S \subseteq V(G)$. Denote by $c_G(S)$ the number of connected components of $G \setminus S$ (we will just write $c(S)$ when there is no confusion). Then we set*

$$\mathcal{C}(G) := \{S \subseteq V(G) : S = \varnothing \text{ or } c(S \setminus \{v\}) < c(S) \text{ for all } v \in S\}$$

That is, when we "add back" any vertex in $S \in \mathcal{C}(G)$ to $G \setminus S$, it must reconnect at least two separate connected components of $G \setminus S$.

Example 1.4.4. Let

$G =$ 

Then

$$\mathcal{C}(G) = \{\varnothing, \{1\}, \{1,4\}, \{2,4\}, \{2,5\}\}$$

Notation 1.4.5. *Let G be a graph, and $S \subseteq V(G)$. Denote by $G_1, \dots, G_{c(S)}$ the connected components of $G \setminus S$, and let $\tilde{G}_i$ be the complete graph with vertex set $V(G_i)$. Then we set*

$$P_S(G) \coloneqq (x_i, y_i : i \in S) + \mathcal{J}(\tilde{G}_1) + \cdots + \mathcal{J}(\tilde{G}_{c(S)})$$

Proposition 1.4.6. *$\mathcal{J}(K_S)$ is prime for any $S \subseteq \{1, \dots, n\}$, and therefore $P_T(G)$ is prime for any graph G and $T \subseteq V(G)$ also.*

Proof. This follows from [BV88, Theorem 2.10]. □

Theorem 1.4.7. [Her+10, Corollary 3.9] *Let G be a graph. Then*

$$\mathcal{J}(G) = \bigcap_{S \in \mathcal{C}(G)} P_S(G)$$

is the primary decomposition of $\mathcal{J}(G)$.

We will also make use of the following:

Lemma 1.4.8. [Her+10, Lemma 3.1] *Let G be a graph, and $S \in \mathcal{C}(G)$. Then*

$$\operatorname{height}_R(P_S(G)) = |S| + n - c(S)$$

In particular, the height of a binomial edge ideal does not depend on the underlying field k.

Chapter 2

Prime Characteristic Tools

Throughout this chapter, we assume that R is of prime characteristic $p > 0$.

Parts of this chapter first appeared in [Wil23a].

2.1 The Frobenius Endomorphism

The **Frobenius endomorphism** equips any ring R of prime characteristic $p > 0$ with a non-trivial ring endomorphism sending $r \mapsto r^p$, since $(r + s)^p = r^p + s^p$ for any $r, s \in R$. This gives rise to many additional tools in prime characteristic, several of which we will make use of here.

2.1.1 Frobenius Actions

Notation 2.1.1. *We denote by f_R^e the* ***iterated Frobenius endomorphism*** *of R sending $r \mapsto r^{p^e}$. We will omit e when $e = 1$, and omit R when there is no confusion.*

There are several ways to define Frobenius maps. We begin with perhaps the most natural:

Definition 2.1.2. *Let M be an R-module. For any $e \geq 1$, we say that an additive map $\varphi : M \to M$ is an* ***eth Frobenius map*** *if*

$$\varphi(rm) = r^{p^e}\varphi(m)$$

for all $r \in R$ and $m \in M$. When $e = 1$, we simply say that φ is a ***Frobenius map****.*

Note. *Frobenius maps are <u>not usually R-homomorphisms</u>.*

Another makes use of a particular skew-polynomial ring:

Definition 2.1.3. *Let T be an indeterminate over R. For any $e \geq 1$, we define the* ***eth Frobenius skew-polynomial ring*** *to be the R-module*

$$R[T; f^e] = R \oplus RT \oplus RT^2 \oplus \cdots$$

which we turn into a usually noncommutative ring by setting $r \cdot T = rT$ and $T \cdot r = r^{p^e} T$ for all $r \in R$. As before, when $e = 1$, we omit e and simply say that $R[T; f]$ is the ***Frobenius skew-polynomial ring****.*

Note. *When discussing $R[T; f^e]$-modules and actions, we will always mean left $R[T; f]$-modules and actions.*

Specifying an e^{th} Frobenius map $\varphi : M \to M$ for some R-module M is the same as equipping M with an $R[T; f^e]$-module structure:

- Given an R-module M and e^{th} Frobenius map $\varphi : M \to M$, we can define an $R[T; f^e]$-action on M by extending the R-action additively, setting $(rT^l) \cdot m = r\varphi^l(m)$ for all $r \in R$ and $m \in M$.
- Given an $R[T; f^e]$-module M, we can view M as an R-module in the natural way, and define an e^{th} Frobenius map $\varphi : M \to M$ by setting $\varphi(m) = Tm$ for all $m \in M$.

With this equivalence in mind, we define the following:

Definition 2.1.4. *Suppose that M is an R-module with a Frobenius map $\varphi : M \to M$, or, equivalently, that M is an $R[T; f]$-module. When the Frobenius map or $R[T; f]$-action is clear, we say M has a* ***Frobenius action****.*

Many R-modules have Frobenius actions. For example, the Frobenius endomorphism is clearly a Frobenius action on R. The most important class of such modules for our purposes is the following:

Proposition 2.1.5. [Iye+07, Definition 21.14] *Let $\mathfrak{a}$ be an ideal of R. Then $H^i_{\mathfrak{a}}(R)$ has a Frobenius action for any $i \geq 0$, induced by the Frobenius endomorphism on R in the Čech complex definition of $H^i_{\mathfrak{a}}(R)$.*

In some cases, it is easy to describe this action explicitly:

Example 2.1.6. [Iye+07, Example 21.15] Let $R = k[x_1, \ldots, x_n]$, or $k[[x_1, \ldots, x_n]]$, for some field k and $n \geq 0$, and let $\mathfrak{m}$ be the irrelevant ideal of R. We saw in Example 1.2.16 that $H^n_{\mathfrak{m}}(R)$ is generated over k by the monomials

$$\{x_1^{-a_1} \cdots x_n^{-a_n} : a_i \geq 1 \text{ for all } 1 \leq i \leq n\}$$

Then the Frobenius action is given by the map

$$\alpha x_1^{-a_1} \cdots x_n^{-a_n} \mapsto \alpha^p x_1^{-pa_1} \cdots x_n^{-pa_n}$$

for all $\alpha \in k$.

Proposition 2.1.5 allows us to make the following definition:

Definition 2.1.7. *Suppose that $(R, \mathfrak{m})$ is local. By Proposition 2.1.5, $H^i_{\mathfrak{m}}(R)$ has a Frobenius action for any $i \geq 0$. If this action is injective for all $i \geq 0$, we say that R is **F-injective**.*

Note. *If $(R, \mathfrak{m})$ is local and $\mathfrak{a}$ an ideal of R, then by Proposition 2.1.5 and the Independence Theorem, $H^i_{\mathfrak{m}}(R/\mathfrak{a})$ has a Frobenius action as an R-module for any $i \geq 0$, and $R/\mathfrak{a}$ is F-injective if and only if this action is injective on $H^i_{\mathfrak{m}}(R/\mathfrak{a})$.*

F-injectivity is the subject of much research, and many classes of rings have been shown to be F-injective. Most importantly for our purposes:

Theorem 2.1.8. [DMN22, Corollary 3.6] *Let k be a field of prime characteristic $p > 0$, G a graph on n vertices for some $n \geq 1$, and*

$$R = k[x_1, \dots, x_n, y_1, \dots, y_n]$$

Then $R/\mathcal{J}_G$ is F-injective.

The effect of a Frobenius action on R-submodules is also a topic of particular importance.

Definition 2.1.9. *Let M be an R-module with a Frobenius action $\varphi : M \to M$, and N an R-submodule of M. If $\varphi(N) \subseteq N$, we say that N is **F-stable**.*

*Moreover, when M is representable, if M has a secondary representation in which every component is F-stable, we say that M has an **F-stable secondary representation**.*

Note. *Some sources use the terminology "F-stable" in a different sense, for example [Ene09], where F-stability in our sense is instead called **F-invariance**.*

As an illustration of the importance of this property, De Stefani and Ma showed in [DM21, Theorem 3.4] that, for a Noetherian local ring $(R, \mathfrak{m})$ of prime characteristic, if $H^i_{\mathfrak{m}}(R)$ has an F-stable secondary representation for all $i \geq 0$, then F-injectivity deforms (meaning that, if $R/(a)$ is injective for all non-zerodivisors $a \in R$, then R is F-injective). At the time of writing, both the question of the deformation of F-injectivity in general, and even the question of whether such local cohomology modules always have F-stable secondary representations, are open.

We do however know the following:

Proposition 2.1.10. [DM21, Lemma 3.2] *Let M be a representable R-module, and S a $\mathfrak{p}$-secondary submodule of M for some $\mathfrak{p} \in \operatorname{Att}_R(M)$. Then if $\mathfrak{p} \in \operatorname{MinAtt}_R(M)$, S is F-stable.*

In particular, when $(R, \mathfrak{m})$ is local of dimension n, $H^n_{\mathfrak{m}}(R)$ has an F-stable secondary representation by Theorem 1.3.8.

2.1.2 The Frobenius Functor

Introduced by Peskine and Szpiro in [PS73], the Frobenius Functor has found a wide range of applications across commutative algebra and algebraic geometry, and will be crucial for later results in this thesis.

It is a somewhat unusual construction, which we will give in two parts:

Definition 2.1.11. *For any $e \geq 1$, we denote by $F^e_* R$ the ring*

$$\{F^e_* r : r \in R\}$$

with

$$F^e_* r + F^e_* s = F^e_*(r+s)$$

and

$$(F^e_* r)(F^e_* s) = F^e_*(rs)$$

In other words, we take R, and simply "decorate" its elements with F^e_*.

We turn this into an R-<u>bimodule</u> by setting $r \cdot (F^e_* s) = F^e_*(rs)$ and $(F^e_* s) \cdot r = F^e_*(sr^{p^e})$ for all $r, s \in R$.

Definition 2.1.12. *For any $e \geq 1$, we define the* ***e*****th** ***Frobenius Functor*** *as the functor*

$$F^e_R(-) \coloneqq F^e_* R \otimes_R - : \boldsymbol{R}\textbf{-Mod} \to \boldsymbol{F_* R}\textbf{-Mod}$$

and identify $F^e_* R$ with R via $F^e_* r \mapsto r$ to view F^e_R as a functor from ***R*-Mod** *to* ***R*-Mod**. *Again, when $e = 1$, we omit e and simply say that F_R is the* ***Frobenius Functor***.

Note. *Some sources call F_R the* ***Peskine-Szpiro Functor***.

The Frobenius Functor behaves in the following way (here we are taking $e = 1$):

Let M be an R-module. Take any $m \in M$ and $r, s \in R$. Then

$$r \cdot (s \otimes_R m) = (rs) \otimes_R m$$

and

$$r \otimes_R (sm) = (s^p r) \otimes_R m$$

In a sense, the Frobenius Functor makes R act on M via "pth roots". If we were to denote this new action by $\circ$, we would have $r \circ m = r^{\frac{1}{p}} m$ for $r \in R$ and $m \in M$. Of course, not every element of R necessarily has a pth root, so they are "kept" on the left hand side of the tensor product, until we have at least p terms of "$r^{\frac{1}{p}}$" in a product, after which we can take "a whole r" across the tensor product to act on m in the usual sense.

Note. *It is routine to check that*

$$F^{e+1}_R(-) \cong F_R(F^e_R(-))$$

naturally as functors for every $e \geq 1$, so there is no ambiguity in our notation.

One of the key properties of this functor is a restatement of a theorem of Kunz:

Theorem 2.1.13 (Kunz's Theorem)**.** [Kun69, Corollary 2.7] *R is regular if and only if it is reduced and F_R is exact.*

With Kunz's Theorem in mind, we assume that R is regular for the remainder of this chapter.

Several other important properties follow from this:

Lemma 2.1.14. *We have that:*

i) *F_R commutes with arbitrary direct sums and finite intersections.* [Lyu97, Remarks 1.0 (b)]

ii) *F_R commutes with direct limits.* [Lyu97, Remarks 1.0 (g)]

iii) *F_R commutes with localisation.* [Lyu97, Remarks 1.0 (i)]

iv) *For any finitely generated R-module M, R-module N, and $i \geq 0$, we have*

$$F_R(\mathrm{Ext}^i_R(M, N)) \cong \mathrm{Ext}^i_R(F_R(M), F_R(N))$$

functorially in both M and N. [Lyu97, Remarks 1.0 (f)]

Furthermore, it also behaves well with respect to Matlis duality over complete local rings:

Lemma 2.1.15. [Lyu97, Lemma 4.1] *Suppose that $(R, \mathfrak{m})$ is complete local. Then there exists a natural isomorphism of functors*

$$\tau_- : F_R(-)^\vee \xrightarrow{\sim} F_R(-^\vee)$$

*from **R-ArtMod** to **R-ArtMod** (that is, the isomorphism holds when the argument of $F_R(-)^\vee$ is Artinian).*

The simplest application of the Frobenius Functor is to R itself:

Proposition 2.1.16. *We have $F_R(R) \cong R$.*

Proof. We can write any element of $F_R(R)$ in the form $r \otimes_R 1$ for some $r \in R$, since

$$\sum_{i=1}^{t} (r_i \otimes_R s_i) = \left(\sum_{i=1}^{t} r_i s_i^p \right) \otimes_R 1$$

Then the map $F_R(R) \to R$ sending $r \otimes_R 1 \mapsto r$ clearly provides the desired R-isomorphism. □

With these properties in hand, we can compute a presentation of $F_R(M)$ for any finitely generated R-module M. Note that, since R is Noetherian, finitely generated R-modules are also finitely presented. Then when we take a presentation

$$R^n \xrightarrow{A} R^m \twoheadrightarrow M$$

of M, we may view A as an $m \times n$ matrix.

Applying F_R, which is exact by Kunz's Theorem, we then have

$$F_R(R^n) \xrightarrow{F_R(A)} F_R(R^m) \twoheadrightarrow F_R(M)$$

which, by Proposition 2.1.16 and Lemma 2.1.14 (i), may be viewed as

$$R^n \xrightarrow{F_R(A)} R^m \twoheadrightarrow F_R(M)$$

and so to determine $F_R(M)$, it suffices to understand $F_R(A)$. We will do this carefully:

Proposition 2.1.17. *Let $A : R^n \to R^m$ be given by an $m \times n$ matrix with entries $(a_{i,j})$ for some $m, n \geq 1$ and $a_{i,j} \in R$. Then $F_R(A) : R^n \to R^m$ is given by the $m \times n$ matrix $A^{[p]} := (a_{i,j}^p)$.*

Proof. Note that by, for example [Rot09, Theorem 2.65], we have that $F_R(R)^l \cong F_R(R^l)$ via

$$\varphi : (a_1 \otimes_R r_1, \ldots, a_l \otimes_R r_l) \mapsto \sum_{i=1}^{l} (a_i \otimes_R \underbrace{(0, \ldots, 0, r_i, 0, \ldots, 0)}_{\text{0 except the } i^{\text{th}} \text{ position}})$$

and as shown in Proposition 2.1.16 we have $F_R(R) \cong R$ via $\psi : a \otimes_R b \mapsto b^p a$. Then the source and target of $F_R(A)$ are as claimed.

We have

$$A \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n} a_{1,i} r_i \\ \vdots \\ \sum_{i=1}^{n} a_{m,i} r_i \end{bmatrix}$$

Next:

- Let e_i denote the vector in R^n with 0 in every position except the i^{th}, where there is 1.
- Let z_i denote the vector in R^m with 0 in every position except the i^{th}, where there is
$$\sum_{j=1}^{n} a_{i,j}$$
- Let α_i denote the vector in $F_R(R)^m$ with 0 in every position except the i^{th}, where there is
$$r_i \otimes_R \left(\sum_{j=1}^{n} a_{i,j} \right)$$

- Let β_i denote the vector in R^m with 0 in every position except the i^{th}, where there is

$$\left(\sum_{j=1}^{n} a_{i,j}\right)^p r_i = \left(\sum_{j=1}^{n} a_{i,j}^p\right) r_i = \sum_{j=1}^{n} a_{i,j}^p r_i$$

Now, $F_R(A) = \mathrm{Id}_R \otimes_R A$, and so

$$F_R(A)\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \overset{\psi^{-1}}{=} F_R(A)\begin{bmatrix} r_1 \otimes_R 1 \\ \vdots \\ r_n \otimes_R 1 \end{bmatrix} \overset{\varphi}{=} F_R(A)\left(\sum_{i=1}^{n} r_i \otimes_R e_i\right) = \sum_{i=1}^{n}(F_R(A)(r_i \otimes_R e_i))$$
$$= \sum_{i=1}^{n}(r_i \otimes_R A(e_i)) = \sum_{i=1}^{n}(r_i \otimes_R z_i) \overset{\varphi^{-1}}{=} \sum_{i=1}^{n} \alpha_i$$
$$\overset{\psi}{=} \sum_{i=1}^{n} \beta_i = \begin{bmatrix} \sum_{i=1}^{n} a_{1,i}^p r_i \\ \vdots \\ \sum_{i=1}^{n} a_{m,i}^p r_i \end{bmatrix} = A^{[p]}\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

as desired. □

Corollary 2.1.18. *For any ideal $\mathfrak{a}$ of R, we have*

$$F_R(R/\mathfrak{a}) \cong R/\mathfrak{a}^{[p]}$$

Proof. Since R is Noetherian, we may write $\mathfrak{a} = (a_1, \ldots, a_t)$ for some $a_i \in \mathfrak{a}$ and $t \geq 0$. Then apply Proposition 2.1.17 to

$$R^t \xrightarrow{\begin{bmatrix} a_1 \\ \vdots \\ a_t \end{bmatrix}} R \twoheadrightarrow R/\mathfrak{a}$$

□

Note. *For any ideal $\mathfrak{a}$ of R and $e \geq 1$, we say that $\mathfrak{a}^{[p^e]}$ is the **e^{th} Frobenius power of $\mathfrak{a}$**.*

Another important property of F_R is the following:

Lemma 2.1.19. [HS93, Proposition 1.5] *For any injective R-module I, we have $F_R(I) \cong I$.*

In particular, when $(R, \mathfrak{m})$ is local, we have

$$F_R(E_R(R/\mathfrak{m})) \cong E_R(R/\mathfrak{m})$$

Proposition 2.1.17 gives us an outline of a quick proof for Lemma 2.1.15 (which can be made rigorous, as is done in [Lyu97, Lemma 4.1]):

Sketch of Proof. Suppose that $(R, \mathfrak{m})$ is complete local. By Proposition 1.1.11, for an Artinian R-module M we can obtain an exact sequence

$$M \hookrightarrow E_R(R/\mathfrak{m})^m \xrightarrow{A} E_R(R/\mathfrak{m})^n$$

for some $m, n \geq 1$. Note that

$$\operatorname{Hom}_R(E_R(R/\mathfrak{m}), E_R(R/\mathfrak{m})) = E_R(R/\mathfrak{m})^\vee \cong R$$

by Proposition 1.1.12 since R is complete, and so we can view A as an $n \times m$ matrix with entries in R. By Proposition 2.1.17, Proposition 2.1.16, and Lemma 2.1.14 (i), applying the Matlis Duality functor followed by F_R then yields

$$R^n \xrightarrow{(A^T)^{[p]}} R^m \twoheadrightarrow F_R(M^\vee)$$

since trivially $R^\vee \cong E_R(R/\mathfrak{m})$.

Using the same results, along with Lemma 2.1.19, applying F_R followed by the Matlis Duality functor yields

$$R^n \xrightarrow{(A^{[p]})^T} R^m \twoheadrightarrow F_R(M)^\vee$$

Then Lemma 2.1.15 simply amounts to noticing that taking the transpose of a matrix commutes with raising all entries to the pth power. ☐

Another important property of F_R is the following straightforward generalisation of [HS93, Corollary 1.6] from regular local rings to regular rings:

Lemma 2.1.20. *Let M be an R-module. Then*

$$\operatorname{Ass}_R(F_R(M)) = \operatorname{Ass}_R(M)$$

Proof. Take any $\mathfrak{p} \in \operatorname{Spec}(R)$. By Lemma 2.1.14 (iii), we have $F_R(M)_\mathfrak{p} \cong F_{R_\mathfrak{p}}(M_\mathfrak{p})$ as $R_\mathfrak{p}$-modules.

Furthermore, by [HS93, Corollary 1.6], we have

$$\operatorname{Ass}_{R_\mathfrak{p}}(F_{R_\mathfrak{p}}(M_\mathfrak{p})) = \operatorname{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p})$$

Then

$$\begin{aligned}\mathfrak{p} \in \operatorname{Ass}_R(F_R(M)) &\iff \mathfrak{p}R_\mathfrak{p} \in \operatorname{Ass}_{R_\mathfrak{p}}(F_R(M)_\mathfrak{p}) \\ &\iff \mathfrak{p}R_\mathfrak{p} \in \operatorname{Ass}_{R_\mathfrak{p}}(F_{R_\mathfrak{p}}(M_\mathfrak{p})) \\ &\iff \mathfrak{p}R_\mathfrak{p} \in \operatorname{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p}) \\ &\iff \mathfrak{p} \in \operatorname{Ass}_R(M)\end{aligned}$$

and we are done. □

2.1.3 *F*-Modules

We have seen in Proposition 2.1.16 that $F_R(R) \cong R$. There is a term for such modules, introduced by Lyubeznik in [Lyu97]:

Definition 2.1.21. *Let $\mathscr{M}$ be an R-module. If there exists an R-isomorphism $\theta : \mathscr{M} \xrightarrow{\sim} F_R(\mathscr{M})$, we say that $(\mathscr{M}, \theta)$ is an **F-module** (more properly, an **F_R-module**, but we will omit R when it is clear from context). When there is no confusion, we will just write $\mathscr{M}$, and say that θ is the **structure isomorphism** of $\mathscr{M}$.*

*Given two F-modules $(\mathscr{M}_1, \theta_1)$ and $(\mathscr{M}_2, \theta_2)$, we say than an R-homomorphism $\varphi : \mathscr{M}_1 \to \mathscr{M}_2$ is an **F-module homomorphism** if the diagram*

$$\begin{array}{ccc} \mathscr{M}_1 & \xrightarrow{\varphi} & \mathscr{M}_2 \\ {\scriptstyle \theta_1}\downarrow\wr & & \wr\downarrow{\scriptstyle \theta_2} \\ F_R(\mathscr{M}_1) & \xrightarrow[F_R(\varphi)]{} & F_R(\mathscr{M}_2) \end{array}$$

commutes.

*We say that an R-submodule $\mathscr{N}$ of $\mathscr{M}$ is an **F-submodule** of $\mathscr{M}$ if $\theta(\mathscr{N}) = F_R(\mathscr{N})$, viewing $F_R(\mathscr{N})$ as a subset of $F_R(\mathscr{M})$ by applying F_R to the inclusion $\mathscr{N} \hookrightarrow \mathscr{M}$. This inclusion is then an F-module homomorphism from $(\mathscr{N}, \theta|_\mathscr{N})$ to $(\mathscr{M}, \theta)$.*

With the work we have done so far, we can show that the local cohomology modules of F-modules are also F-modules ([Lyu97, Example 1.2(b)]). We first state a preliminary lemma:

Lemma 2.1.22. *Let $\mathfrak{a}$ be an ideal of R. Then there is a natural isomorphism of functors*

$$H^i_\mathfrak{a}(-) \cong \varinjlim \operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^j]}, -)$$

*from **R-Mod** to **R-Mod** for all $i \geq 0$.*

Proof. This follows from the direct limit definition of $H^i_\mathfrak{a}(-)$, since taking successive Frobenius powers of an ideal is cofinal to taking successive usual powers. □

Proposition 2.1.23. *Let $\mathscr{M}$ be an F-module. Then $H^i_{\mathfrak{a}}(\mathscr{M})$ is also an F-module for any ideal $\mathfrak{a}$ of R and $i \geq 0$.*

Proof. We have

$$\begin{aligned} F_R(H^i_{\mathfrak{a}}(\mathscr{M})) &\cong F_R(\varinjlim \operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^j]}, \mathscr{M})) \cong \varinjlim F_R(\operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^j]}, \mathscr{M})) \\ &\cong \varinjlim \operatorname{Ext}^i_R(F_R(R/\mathfrak{a}^{[p^j]}), F_R(\mathscr{M})) \cong \varinjlim \operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^{j+1}]}, \mathscr{M}) \\ &\cong \varinjlim \operatorname{Ext}^i_R(R/\mathfrak{a}^{[p^j]}, \mathscr{M}) \cong H^i_{\mathfrak{a}}(\mathscr{M}) \end{aligned}$$

by Lemma 2.1.14 (ii) and Lemma 2.1.14 (iv). □

2.1.4 *F*-Finite *F*-Modules

A particularly special class of F-modules can be determined entirely from a finitely generated R-module, even if they are not finitely generated themselves. Again, these were introduced by Lyubeznik in [Lyu97]:

Definition 2.1.24. *Let $\mathscr{M}$ be an F-module. We say that $\mathscr{M}$ is an **F-finite F-module** (more properly, an **F_R-finite F_R-module**, but we will again omit R when it is clear from context) if*

$$\mathscr{M} \cong \varinjlim \left[M \xrightarrow{\beta} F_R(M) \xrightarrow{F_R(\beta)} F^2_R(M) \xrightarrow{F^2_R(\beta)} \cdots \right]$$

for some finitely generated R-module M and R-homomorphism $\beta : M \to F_R(M)$.

*In this case, we say that β is a **generating morphism** of $\mathscr{M}$, and that M **generates** $\mathscr{M}$. If no morphism is specified, we say that a finitely generated R-module M generates $\mathscr{M}$ is there exists a generating morphism of $\mathscr{M}$ with source M.*

*If β is also injective, we say that it is a **root morphism** of $\mathscr{M}$, and that M is a **root** of $\mathscr{M}$. We can then view M as an R-submodule of $\mathscr{M}$ via β. Again, if no morphism is specified, we say that a finitely generated R-module M is a root of $\mathscr{M}$ is there exists a root morphism of $\mathscr{M}$ with source M.*

*If no other root of $\mathscr{M}$ is contained in M, we say that it is a **minimal root**.*

*We denote by **F_R-FinMod** the category of F-finite F-modules and F-module homomorphisms between them.*

Note. *To see that an F-finite F-module is an F-module, simply note that the diagram*

$$\begin{array}{ccccccc} M & \xrightarrow{\beta} & F_R(M) & \xrightarrow{F_R(\beta)} & F^2_R(M) & \xrightarrow{F^2_R(\beta)} & \cdots \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle F_R(\beta)} & & \downarrow{\scriptstyle F^2_R(\beta)} & & \\ F_R(M) & \xrightarrow[F_R(\beta)]{} & F^2_R(M) & \xrightarrow[F^2_R(\beta)]{} & F^3_R(M) & \xrightarrow[F^3_R(\beta)]{} & \cdots \end{array}$$

commutes, and the direct limits of the top and bottom rows are clearly isomorphic, so the vertical maps *induce a structure isomorphism* $\theta : \mathscr{M} \xrightarrow{\sim} F_R(\mathscr{M})$.

Crucially, every F-finite F-module has a root:

Proposition 2.1.25. [Lyu97, Proposition 2.3 (c) & Theorem 3.5] *Let $\mathscr{M}$ be an F-finite F-module. Then $\mathscr{M}$ has a root, and, furthermore, if $(R, \mathfrak{m})$ is also complete local, then $\mathscr{M}$ has a unique minimal root, which is contained in every other root of $\mathscr{M}$.*

$\boldsymbol{F_R}$**-FinMod** is closed under several common operations. We first state a definition:

Definition 2.1.26. *Let $\mathscr{M}$ be an F-finite F-module with root morphism $\beta : M \hookrightarrow F_R(M)$ for some finitely generated R-module M. We say that an R-submodule N of M is* ***β-compatible*** *if $\beta(N) \subseteq F_R(N)$.*

We will also require the following proposition:

Proposition 2.1.27. [Lyu97, Corollary 2.6] *Let $\mathscr{M}$ be an F-finite F-module with root morphism $\beta : M \hookrightarrow F_R(M)$ for some finitely generated R-module M. Then there is a one-to-one correspondence between β-compatible R-submodules of M and F-finite F-submodules of $\mathscr{M}$.*

Namely, a β-compatible R-submodule N of M gives rise to an F-finite F-submodule of $\mathscr{M}$ with root morphism $\beta|_N$, and an F-finite F-submodule $\mathscr{N}$ of $\mathscr{M}$ has root $\mathscr{N} \cap M$ which is β-compatible.

Proposition 2.1.28. *Let $\mathscr{M}$ be an F-finite F-module with root morphism $\beta : M \hookrightarrow F_R(M)$ for some finitely generated R-module M. Furthermore, let $\mathscr{N}_1$ and $\mathscr{N}_2$ be F-finite F-submodules of $\mathscr{M}$ with roots $N_1 \subseteq M$ and $N_2 \subseteq M$ respectively. Then $\mathscr{N}_1 \cap \mathscr{N}_2$ is an F-finite F-submodule of $\mathscr{M}$, with root $N_1 \cap N_2$.*

Proof. We have

$$\beta|_{N_1 \cap N_2} : N_1 \cap N_2 \hookrightarrow F_R(N_1) \cap F_R(N_2) \cong F_R(N_1 \cap N_2)$$

by Lemma 2.1.14 (i), and so $N_1 \cap N_2$ generates an F-finite F-submodule $\mathscr{L}$ of $\mathscr{M}$. The inclusion $\mathscr{L} \subseteq \mathscr{N}_1 \cap \mathscr{N}_2$ is easily checked, and the converse follows since any element of $\mathscr{N}_1 \cap \mathscr{N}_2$ has a representative in some $F_R^{i_1}(N_1)$ and $F_R^{i_2}(N_2)$, for some $i_1, i_2 \geq 0$, in their respective direct limit systems, and so we can find a representative for both in $F_R^{i}(N_1 \cap N_2)$, where $i = \max\{i_1, i_2\}$, by taking further applications of $F_R^{j}(\beta|_{N_1 \cap N_2})$ for increasing j to the representative in the earlier term in the limit if necessary. □

Proposition 2.1.29. *Let $\mathscr{M}$ be an F-finite F-module with root morphism $\beta : M \hookrightarrow F_R(M)$ for some finitely generated R-module M. Furthermore, let $\mathscr{N}$ be an F-finite F-submodule of $\mathscr{M}$ with root $N \subseteq M$. Then $\mathscr{M}/\mathscr{N}$ is an F-finite F-module, with root M/N.*

Proof. This follows from the proof of [Lyu97, Theorem 2.8]. □

One of the advantages of F-finite F-modules is that many of their properties can be deduced from those of a root, which, being finitely generated, is often easier to understand. For example, their associated primes agree (this is well known, we include a proof for convenience):

Proposition 2.1.30. *Let $\mathscr{M}$ be an F-finite F-module with generating morphism $\beta : M \to F_R(M)$ for some finitely generated R-module M. Then*

$$\mathrm{Ass}_R(\mathscr{M}) \subseteq \mathrm{Ass}_R(M)$$

Proof. Take any $\mathfrak{p} \in \mathrm{Ass}_R(\mathscr{M})$, so there exists some $\overline{x} \in \mathscr{M}$ such that $\mathrm{Ann}_R(\overline{x}) = \mathfrak{p}$. Now, $\overline{x}$ corresponds to the image in $\mathscr{M}$ of some $x \in F_R^e(M)$ for some $e \geq 0$, and we may take e to be minimal.

Let

$$\psi_i = F_R^{i-1}(\beta) \circ \cdots \circ F_R^e(\beta) : F_R^e(M) \to F_R^i(M)$$

for each $i > e$, and set $\psi_e = \mathrm{Id}_{F_R^e(M)}$. We then have

$$\mathrm{Ann}_R(\overline{x}) = \{r \in R : \psi_i(rx) = 0 \text{ for some } i \geq e\}$$

Since R is Noetherian, we can write $\mathfrak{p} = (a_1, \ldots, a_t)$ for some $a_i \in R$ and $t \geq 1$. Then, for each $1 \leq i \leq t$, there exists some $l_i \geq e$ such that $\psi_{l_i}(a_i x) = 0$.

Now, let

$$l = \max\{l_i : 1 \leq i \leq t\}$$

Then

$$a_i \psi_l(x) = \psi_l(a_i x) = 0$$

for each $1 \leq i \leq t$, so setting $y = \psi_l(x) \in F_R^l(M)$ we have $\mathfrak{p} \subseteq \mathrm{Ann}_R(y)$.

Conversely, if $r \in \mathrm{Ann}_R(y)$, then

$$\psi_l(rx) = r\psi_l(x) = ry = 0$$

and so $r \in \mathrm{Ann}_R(\overline{x}) = \mathfrak{p}$. Then $\mathrm{Ann}_R(y) \subseteq \mathfrak{p}$, so $\mathrm{Ann}_R(y) = \mathfrak{p}$, and therefore $\mathfrak{p} \in \mathrm{Ass}_R(F_R^l(M))$.

Then repeatedly applying Lemma 2.1.20 yields $\mathfrak{p} \in \mathrm{Ass}_R(M)$, and the result follows. □

Lemma 2.1.31. *Let $\mathscr{M}$ be an F-finite F-module with root morphism $\beta : M \hookrightarrow F_R(M)$ for some finitely generated R-module M. Then*

$$\mathrm{Ass}_R(\mathscr{M}) = \mathrm{Ass}_R(M)$$

Proof. Since M is a root of $\mathscr{M}$, the canonical inclusion is an injection, so $M \subseteq \mathscr{M}$. Then we have

$$\mathrm{Ass}_R(M) \subseteq \mathrm{Ass}_R(\mathscr{M})$$

and so we are done by Proposition 2.1.30. □

There is also an F-finite version of Proposition 2.1.23:

Proposition 2.1.32. [Lyu97, Proposition 2.10] *Let $\mathscr{M}$ be an F-finite F-module. Then $H^i_{\mathfrak{a}}(\mathscr{M})$ is also an F-finite F-module for any ideal $\mathfrak{a}$ of R and $i \geq 0$.*

In some cases, we can easily calculate an explicit generating morphism of the F-finite F-modules of Proposition 2.1.32:

Proposition 2.1.33. *Let $\mathfrak{a} = (a_1, \ldots, a_t)$ be an ideal of R. Then, setting $a = (a_1 \cdots a_t)^{p-1}$, we have that $a\cdot : R/\mathfrak{a} \to R/\mathfrak{a}^{[p]}$ is a generating morphism of $H^t_{\mathfrak{a}}(R)$ (where $a\cdot$ denotes multiplication by a).*

Proof. We have

$$\begin{aligned} H^t_{\mathfrak{a}}(R) &\cong \varinjlim \left[R/\mathfrak{a} \xrightarrow{a} R/\mathfrak{a}^{[2]} \xrightarrow{a^2\cdot} R/\mathfrak{a}^{[3]} \xrightarrow{a^3\cdot} \cdots \right] \\ &\cong \varinjlim \left[R/\mathfrak{a} \xrightarrow{a\cdot} R/\mathfrak{a}^{[p]} \xrightarrow{a^p\cdot} R/\mathfrak{a}^{[p^2]} \xrightarrow{a^{p^2}\cdot} \cdots \right] \\ &\cong \varinjlim \left[R/\mathfrak{a} \xrightarrow{a\cdot} F_R(R/\mathfrak{a}) \xrightarrow{F_R(a\cdot)} F^2_R(R/\mathfrak{a}) \xrightarrow{F^2_R(a\cdot)} \cdots \right] \end{aligned}$$

with the first isomorphism following from Proposition 1.2.18, and the result follows. ☐

In many cases, when the a_i form a regular sequence, Proposition 2.1.33 in fact gives us a root of $H^t_{\mathfrak{a}}(R)$:

Lemma 2.1.34. *Suppose that $a_1, \ldots, a_t \in R$ is a regular sequence, and that either*

1. *R is local.*
2. *R is non-negatively ($\mathbb{Z}$-)graded, and the a_i are hommogeneous of strictly positive degree.*

Let $\mathfrak{a} = (a_1, \ldots, a_t)$, take any $p \geq 2$, and set $a = (a_1 \cdots a_t)^{p-1}$. Then $(\mathfrak{a}^{[p]} : a) = \mathfrak{a}$.

Proof. Note that our conditions on R and the a_i ensure that $a_1^{\alpha_1}, \ldots, a_t^{\alpha_t}$ is also a regular sequence for all $\alpha_i \geq 1$, and that any permutation of this sequence is again a regular sequence (see [Mat86, Theorem 16.1] and the corollary to [Mat86, Theorem 16.3]).

Set

$$b_i = (a_1 \cdots a_i)^{p-1}$$

Now, clearly $\mathfrak{a} \subseteq (\mathfrak{a}^{[p]} : a)$, and so suppose that $r \in (\mathfrak{a}^{[p]} : a)$, so $ar \in \mathfrak{a}^{[p]}$. Then

$$b_t r = r_1^{(1)} a_1^p + \cdots + r_t^{(1)} a_t^p$$

for some $r_1^{(1)}, \ldots, r_t^{(1)} \in R$, so

$$a_t^{p-1}(b_{t-1} r - r_t^{(1)} a_t) = r_1^{(1)} a_1^p + \cdots + r_{t-1}^{(1)} a_{t-1}^p \in (a_1^p, \ldots, a_{t-1}^p)$$

and so, since $a_1^p, \dots, a_{t-1}^p, a_t^{p-1}$ is a also regular sequence, we have

$$b_{t-1}r - r_t^{(1)}a_t \in (a_1^p, \dots, a_{t-1}^p)$$

Then

$$b_{t-1}r = r_1^{(2)}a_1^p + \cdots + r_{t-1}^{(2)}a_{t-1}^p + r_t^{(1)}a_t$$

for some $r_1^{(2)}, \dots, r_{t-1}^{(2)} \in R$, so

$$a_{t-1}^{p-1}(b_{t-2}r - r_{t-1}^{(2)}a_{t-1}) = r_1^{(2)}a_1^p + \cdots + r_{t-2}^{(2)}a_{t-2}^p + r_t^{(1)}a_t \in (a_1^p, \dots, a_{t-2}^p, a_t)$$

and so, since $a_1^p, \dots, a_{t-2}^p, a_t, a_{t-1}^{p-1}$ is also a regular sequence, we have

$$b_{t-2}r - r_{t-1}^{(2)}a_{t-1} \in (a_1^p, \dots, a_{t-2}^p, a_t)$$

Then

$$b_{t-2}r = r_1^{(3)}a_1^p + \cdots + r_{t-2}^{(3)}a_{t-2}^p + r_{t-1}^{(2)}a_{t-1} + r_t^{(3)}a_t$$

for some $r_1^{(3)}, \dots, r_{t-2}^{(3)}, r_t^{(3)} \in R$, so

$$a_{t-2}^{p-1}(b_{t-3}r - r_{t-2}^{(3)}a_{t-2}) = r_1^{(3)}a_1^p + \cdots + r_{t-3}^{(3)}a_{t-3}^p + r_{t-1}^{(2)}a_{t-1} + r_t^{(3)}a_t \in (a_1^p, \dots, a_{t-3}^p, a_{t-1}, a_t)$$

and so, since $a_1^p, \dots, a_{t-3}^p, a_{t-1}, a_t, a_{t-2}^{p-1}$ is also a regular sequence, we have

$$b_{t-3}r - r_{t-2}^{(3)}a_{t-2} \in (a_1^p, \dots, a_{t-3}^p, a_{t-1}, a_t)$$

Then

$$b_{t-3}r = r_1^{(4)}a_1^p + \cdots + a_{t-3}^{(4)}a_{t-3}^p + r_{t-2}^{(3)}a_{t-2} + r_{t-1}^{(4)}a_{t-1} + r_t^{(4)}a_t$$

for some $r_1^{(4)}, \dots, r_{t-3}^{(4)}, r_{t-1}^{(4)}, r_t^{(4)} \in R$.

Continuing in this way, we eventually arrive at

$$b_0r = r_1^{(t)}a_1 + r_2^{(t+1)}a_2 + \cdots + r_t^{(t+1)}a_t \in \mathfrak{a}$$

for some $r_1^{(t)}, r_2^{(t+1)}, \dots, r_t^{(t+1)} \in R$, and so, since $b_0r = r$, we are done. □

2.2 Lyubeznik's $\mathscr{H}$-Functor

Throughout this section, we assume that $(R, \mathfrak{m})$ is complete regular local of dimension n, and that $(A, \mathfrak{n})$ is a local ring, with a surjective ring homomorphism $R \twoheadrightarrow A$ so that every A-module can be given the structure of an R-module. We denote the kernel of this homomorphism by $\mathfrak{a}$ (and so $A \cong R/\mathfrak{a}$).

2.2.1 Definition & First Properties

In [Lyu97, Theorem 4.2], Lyubeznik introduces a functor sending Artinian modules with a Frobenius action to F-finite F-modules:

Definition / Theorem 2.2.1. [Lyu97, Theorem 4.2] *For any Artinian $A[T;f]$-module M, we define an R-homomorphism $\gamma_M : F_R(M) \to M$ via*

$$r \otimes_R m \mapsto rTm$$

By Lemma 2.1.15, there is a canonical R-isomorphism $\tau_M : F_R(M)^\vee \xrightarrow{\sim} F_R(M^\vee)$, and so we can construct an R-homomorphism

$$\beta_M := \tau_M \circ \gamma_M^\vee : M^\vee \to F_R(M^\vee)$$

Now, by Matlis Duality (iii), $M^\vee$ is Noetherian since M is Artinian and R is complete, and so we may construct an F-finite F-module

$$\mathscr{H}_{R,A}(M) := \varinjlim \left[M^\vee \xrightarrow{\beta_M} F_R(M^\vee) \xrightarrow{F_R(\beta_M)} F_R^2(M^\vee) \xrightarrow{F_R^2(\beta_M)} \cdots \right]$$

Given another Artinian $A[T;f]$-module N and $A[T;f]$-homomorphism $\varphi : M \to N$, it can be checked that the diagram

$$\begin{array}{ccccccc} N^\vee & \xrightarrow{\beta_N} & F_R(N^\vee) & \xrightarrow{F_R(\beta_N)} & F_R^2(N^\vee) & \xrightarrow{F_R^2(\beta_N)} & \cdots \\ \downarrow{\scriptstyle \varphi^\vee} & & \downarrow{\scriptstyle F_R(\varphi^\vee)} & & \downarrow{\scriptstyle F_R^2(\varphi^\vee)} & & \\ M^\vee & \xrightarrow[\beta_M]{} & F_R(M^\vee) & \xrightarrow[F_R(\beta_M)]{} & F_R^2(M^\vee) & \xrightarrow[F_R^2(\beta_M)]{} & \cdots \end{array}$$

commutes, and so we have an induced F_R-module homomorphism

$$\mathscr{H}_{R,A}(\varphi) : \mathscr{H}_{R,A}(N) \to \mathscr{H}_{R,A}(M)$$

*Then $\mathscr{H}_{R,A}$ defines a contravariant, additive functor from **$A[T;f]$-ArtMod** to **F_R-FinMod**.*

When $A = R$ and there is no confusion, we will simply write $\mathscr{H}$.

Before examining the properties of this functor, we must introduce some notation:

Notation 2.2.2. *Let S be a ring of prime characteristic $p > 0$, and M an $S[T;f]$ module. Then we set*

$$M^* := \bigcap_{i=0}^{\infty} (ST^i M)$$

When M is Artinian, the descending chain

$$M \supseteq TM \supseteq T^2M \supseteq \cdots$$

must stabilise, and so $M^* = T^iM$ for some $i \geq 0$.

*If $M^* = 0$, we say that M is **nilpotent**.*

Furthermore, we set

$$M_{\text{nil}} := \{m \in M : T^im = 0 \textit{ for some } i \geq 0\}$$

and

$$M_{\text{red}} := M/M_{\text{nil}}$$

These operations commute:

Lemma 2.2.3. *Let S be a ring of prime characteristic $p > 0$, and M an $S[T;f]$-module. Then*

$$(M_{\text{red}})^* \cong (M^*)_{\text{red}}$$

Proof. We have

$$\begin{aligned}
(M_{\text{red}})^* &= \bigcap_{i=0}^{\infty}(ST^iM_{\text{red}}) \\
&= \bigcap_{i=0}^{\infty}(ST^i(M/M_{\text{nil}})) \\
&= \bigcap_{i=0}^{\infty}((ST^iM + M_{\text{nil}})/M_{\text{nil}}) \\
&= \left(\bigcap_{i=0}^{\infty}(ST^iM + M_{\text{nil}})\right)/M_{\text{nil}} \\
&= \left(\bigcap_{i=0}^{\infty}(ST^iM) + M_{\text{nil}}\right)/M_{\text{nil}} \\
&= (M^* + M_{\text{nil}})/M_{\text{nil}} \\
&\cong M^*/(M^* \cap M_{\text{nil}})
\end{aligned}$$

It is clear that

$$M^* \cap M_{\text{nil}} = (M^*)_{\text{nil}}$$

and so

$$(M_{\text{red}})^* \cong M^*/(M^* \cap M_{\text{nil}}) = M^*/(M^*)_{\text{nil}} = (M^*)_{\text{red}}$$

as desired. □

There is then no ambiguity in writing M^*_{red}.

Theorem 2.2.4. [Lyu97, Theorem 4.2 & Lemma 4.3] *We have that:*

i) $\mathscr{H}_{R,A}$ *is exact.*

ii) $\mathscr{H}_{R,A}(M) = 0$ *if and only if* M *is nilpotent.*

iii) $M^\vee$ *is a root of* $\mathscr{H}_{R,A}(M)$ *if and only if* $M = M^*$*, and* $(M^*_{\text{red}})^\vee$ *is the minimal root of* $\mathscr{H}_{R,A}(M)$*.*

iv) $\mathscr{H}_{R,A}(M) \cong \mathscr{H}_{R,A}(N)$ *in* $\boldsymbol{F_R}$**-FinMod** *if and only if* $M^*_{\text{red}} \cong N^*_{\text{red}}$ *in* $\boldsymbol{A[T;f]}$**-Mod***.*

The most important application of this functor for our purposes is the following:

Proposition 2.2.5. *By Proposition 2.1.5,* $H^i_{\mathfrak{n}}(A)$ *has a natural* $A[T;f]$*-module structure induced by the Frobenius endomorphism on* A*. It is easily checked that this gives rise to an* $R[T;f]$*-module structure on* $H^i_{\mathfrak{n}}(A)$*. By The Independence Theorem, we have* $H^i_{\mathfrak{n}}(A) \cong H^i_{\mathfrak{m}}(R/\mathfrak{a})$ *as* R*-modules, and so we can equip* $H^i_{\mathfrak{m}}(R/\mathfrak{a})$ *with an* $R[T;f]$*-module structure. Then, under this structure, we have*

$$\mathscr{H}(H^i_{\mathfrak{m}}(R/\mathfrak{a})) \cong H^{n-i}_{\mathfrak{a}}(R)$$

in $\boldsymbol{F_R}$**-FinMod** *for all* $0 \le i \le n$*, with the* F_R*-module structure on* $H^{n-i}_{\mathfrak{a}}(R)$ *being as in Proposition 2.1.23.*

Proof. The proof is essentially the same as that of [Lyu97, Example 4.8]. □

2.2.2 A Correspondence Theorem

Throughout this subsection, we assume that M is an Artinian $R[T;f]$-module. Furthermore, for simplicity, we also suppose that $M = M^*$ (so $M^\vee$ is a root of $\mathscr{H}(M)$ by Theorem 2.2.4 (iii)), and adopt the notation of Definition / Theorem 2.2.1.

In this subsection, we establish a correspondence between F-stable secondary representations of $R[T;f]$-modules and primary decompositions in $\boldsymbol{F_R}$**-FinMod**.

Lemma 2.2.6. *For any* $R[T;f]$*-submodule* N *of* M*, we have that* $(M/N)^\vee$ *is a* β_M*-compatible* R*-submodule of* $M^\vee$*.*

Proof. Since F_R is exact by Kunz's Theorem, we have

$$F_R(M)/F_R(N) \cong F_R(M/N)$$

via

$$\psi : r \otimes_R m + F_R(N) \mapsto r \otimes_R (m + N)$$

Letting $\pi : M \twoheadrightarrow M/N$ and $\rho : F_R(M) \twoheadrightarrow F_R(M)/F_R(N)$ be the canonical projections, we then have

$$\begin{array}{ccc} F_R(M/N)^\vee & \xhookrightarrow{F_R(\pi)^\vee} & F_R(M)^\vee \\ \psi^\vee \big\downarrow & \nearrow_{\rho^\vee} & \\ (F_R(M)/F_R(N))^\vee & & \end{array}$$

Take any $\varphi \in F_R(M/N)^\vee$ and $r \otimes_R m \in F_R(M)$. Then

$$F_R(\pi)^\vee(\varphi)(r \otimes_R m) = \varphi(F_R(\pi)(r \otimes_R m)) = \varphi(r \otimes_R (m + N))$$

and

$$\rho^\vee \circ \psi^\vee(\varphi)(r \otimes_R m) = \varphi(\psi(\rho(r \otimes_R m))) = \varphi(\psi(r \otimes_R m + F_R(N))) = \varphi(r \otimes_R (m + N))$$

so the diagram commutes and we may identify

$$F_R(M/N)^\vee \cong (F_R(M)/F_R(N))^\vee \cong \{\phi \in F_R(M)^\vee : F_R(N) \subseteq \ker(\phi)\} \subseteq F_R(M)^\vee$$

Now, again take any $\varphi \in (M/N)^\vee$, and any $r \otimes_R n \in F_R(N) \subseteq F_R(M)$. Then

$$\gamma_M^\vee(\varphi)(r \otimes_R n) = \varphi(\gamma_M(r \otimes_R n)) = \varphi(rTn) = r\varphi(Tn) = r \cdot 0 = 0$$

because $Tn \in N$ since N is an $R[T; f]$-module and $N \subseteq \ker(\varphi)$.

We then have that $F_R(N) \subseteq \ker(\gamma_M^\vee(\varphi))$, and so $\gamma_M^\vee(\varphi) \in F_R(M/N)^\vee$. Then

$$\gamma_M^\vee((M/N)^\vee) \subseteq F_R(M/N)^\vee$$

and applying $\tau_{M/N}$ from Lemma 2.1.15 completes the proof. □

Theorem 2.2.7. *If M has an F-stable secondary representation, then $\mathscr{H}(M)$ has a primary decomposition in* $\boldsymbol{F_R}$**-FinMod**.

Proof. Let

$$M = S_1 + \cdots + S_t$$

be an F-stable secondary representation of M, with $\operatorname{Att}_R(S_i) = \{\mathfrak{p}_i\}$ for some $\mathfrak{p}_i \in \operatorname{Spec}(R)$. Then, by Lemma 1.3.12, we have that

$$(M/S_1)^\vee \cap \cdots \cap (M/S_t)^\vee = 0 \subseteq M^\vee$$

is an irredundant primary decomposition of $M^\vee$, with each $(M/S_i)^\vee$ being $\mathfrak{p}_i$-primary.

Furthermore, each M/S_i is β_M-compatible by Lemma 2.2.6. Then, by Proposition 2.1.27, each $(M/S_i)^\vee$ generates an F-finite F-submodule of $\mathscr{H}(M)$, which we will denote $\mathscr{N}_i$, and

$$\mathscr{N}_1 \cap \cdots \cap \mathscr{N}_t = 0 \subseteq \mathscr{H}(M)$$

by Proposition 2.1.28, since the intersection of the generating R-modules is 0.

We know that $M^\vee/(M/S_i)^\vee$ generates $\mathscr{H}(M)/\mathscr{N}_i$ by Proposition 2.1.29, and so

$$\varnothing \neq \mathrm{Ass}_R(\mathscr{H}(M)/\mathscr{N}_i) \subseteq \mathrm{Ass}_R(M^\vee/(M/S_i)^\vee) = \{\mathfrak{p}_i\}$$

by Proposition 2.1.30. Then each $\mathscr{N}_i$ is $\mathfrak{p}_i$-primary, and we are done. □

We will now prove the reverse direction:

Proposition 2.2.8. *Let L be an R-module. Then*

$$\bigcap_{\phi \in L^\vee} \ker(\phi) = 0$$

Proof. Suppose that $l \in L$ belongs to this intersection, and consider $Rl \subseteq L$. Since $E_R(R/\mathfrak{m})$ is injective, we can extend any R-homomorphism from Rl to one on the whole of L. Then $\phi(l) = 0$ for any $\phi \in (Rl)^\vee$, so $(Rl)^\vee = 0$, and so since $E_R(R/\mathfrak{m})$ is an injective cogenerator by Proposition 1.1.7, we must have $Rl = 0$. Then $l = 0$, and we are done. □

Lemma 2.2.9. *If N is a β_M-compatible R-submodule of $M^\vee$, then $(M^\vee/N)^\vee$ is an $R[T;f]$-submodule of $M^{\vee\vee} \cong M$.*

Proof. We will first determine the $R[T;f]$-action on $M^{\vee\vee}$.

For any $m \in M$, let

$$\psi_m : M^\vee = \mathrm{Hom}_R(M, E_R(R/\mathfrak{m})) \to E_R(R/\mathfrak{m})$$

be given by $\varphi \mapsto \varphi(m)$. It is shown in the course of the proof of [BS13, Theorem 10.2.12 (ii)] that the map

$$\psi : M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, E_R(R/\mathfrak{m})), E_R(R/\mathfrak{m})) = M^{\vee\vee}$$

given by $m \mapsto \psi_m$ is an R-isomorphism. Then for any $z \in M^{\vee\vee}$, we have $z = \psi_{m_z}$ for some $m_z \in M$, and the $R[T;f]$-action on $M^{\vee\vee}$ is given by $Tz = \psi_{Tm_z}$.

Now, we may identify

$$(M^\vee/N)^\vee \cong \{z \in M^{\vee\vee} : N \subseteq \ker(z)\}$$

Furthermore, by Proposition 2.1.27 and [Lyu97, Lemma 4.3 (a)], we have $N = (M/L)^\vee$ for some $R[T;F]$-submodule $L \subseteq M$, and so we may identify

$$N = (M/L)^\vee \cong \{\varphi \in M^\vee : L \subseteq \ker(\varphi)\}$$

We can now show that $(M^\vee/N)^\vee$ is an $R[T;F]$-submodule of $M^{\vee\vee}$.

Take any $z \in (M^\vee/N)^\vee$, so $z \in M^{\vee\vee}$ is such that $N \subseteq \ker(z)$, and recall that we can write $z = \psi_{m_z}$ for some $m_z \in M$. Then we want to show that $\psi_{Tm_z}(N) = 0$ also.

By the definition of ψ_{m_z}, we have that $\varphi(m_z) = 0$ for all $\varphi \in N \cong (M/L)^\vee$, and so by Proposition 2.2.8 we have $m_z \in L$. Then, for any $\varphi \in N$, we have

$$\psi_{Tm_z}(\varphi) = \varphi(Tm_z) = 0$$

since $Tz_m \in L$, because L is an $R[T; f]$-submodule of M, and $L \subseteq \ker(\varphi)$ since $\varphi \in N$.

Then we have $\psi_{Tz_m}(M) = 0$, so $N \subseteq \ker(Tz)$, and so $Tz \in (M^\vee/N)^\vee$. We have then shown that $(M^\vee/N)^\vee$ is an $R[T; F]$-submodule of $M^{\vee\vee} \cong M$ as desired. □

Theorem 2.2.10. *If $\mathscr{H}(M)$ has an irredundant primary decomposition*

$$\mathscr{N}_1 \cap \cdots \cap \mathscr{N}_s = 0 \subseteq \mathscr{H}(M)$$

*in $\boldsymbol{F_R}$-**FinMod**, then M has an F-stable secondary representation.*

Proof. By Proposition 2.1.27, each $\mathscr{N}_i$ corresponds to a β_M-compatible R-submodule N_i of $M^\vee$, with $N_i = \mathscr{N}_i \cap M^\vee$. Then

$$N_1 \cap \cdots \cap N_s = (\mathscr{N}_1 \cap M^\vee) \cap \cdots \cap (\mathscr{N}_s \cap M^\vee) = (\mathscr{N}_1 \cap \cdots \cap \mathscr{N}_s) \cap M^\vee = 0 \cap M^\vee = 0$$

Say that $\mathscr{N}_i$ is $\mathfrak{p}_i$-primary for some $\mathfrak{p}_i \in \operatorname{Spec}(R)$. By Proposition 2.1.29, we have that $\mathscr{H}(M)/\mathscr{N}_i$ has root $M^\vee/N_i$, and so

$$\operatorname{Ass}_R(M^\vee/N_i) = \operatorname{Ass}_R(\mathscr{H}(M)/\mathscr{N}_i) = \{\mathfrak{p}_i\}$$

by Lemma 2.1.31. Then N_i is $\mathfrak{p}_i$-primary, and

$$N_1 \cap \cdots \cap N_s = 0 \subseteq M^\vee$$

is an irredundant primary decomposition of $M^\vee$.

We can now apply Lemma 1.3.10 and Lemma 2.2.9 to complete the proof. □

To summarise:

Theorem 2.2.11. *M has an F-stable secondary representation if and only if $\mathscr{H}(M)$ has a primary decomposition in $\boldsymbol{F_R}$-**FinMod**.*

For an example application of Theorem 2.2.11, we consider the following:

As noted previously in Subsection 2.1.1, [DM21, Theorem 3.4] gives us considerable interest in understanding for which local rings $(R, \mathfrak{m})$ we have that $H^i_{\mathfrak{m}}(R)$ has an F-stable secondary representation for all $i \geq 0$.

Let $R = k[[x_1, \ldots, x_n]]$, and take a homogeneous ideal $\mathfrak{a}$ of R. We denote by d the dimension of $R/\mathfrak{a}$.

By [DM21, Theorem 3.4], F-injectivity deforms for $R/\mathfrak{a}$ if and only if $H^i_{\mathfrak{m}/\mathfrak{a}}(R/\mathfrak{a})$ has an F-stable secondary representation over $R/\mathfrak{a}$ for all $i \geq 0$. It is routine to check that such a representation gives rise to an F-stable secondary representation over R, and we have

$$H^i_{\mathfrak{m}/\mathfrak{a}}(R/\mathfrak{a}) \cong H^i_{\mathfrak{m}}(R/\mathfrak{a})$$

as R-modules by The Independence Theorem.

We are only interested in $i \leq d$, since otherwise $H^i_{\mathfrak{m}}(R/\mathfrak{a})$ vanishes by Corollary 1.2.22. When $i = d$, every attached prime of $H^i_{\mathfrak{m}}(R/\mathfrak{a})$ is minimal by Theorem 1.3.8, and so $H^i_{\mathfrak{m}}(R/\mathfrak{a})$ has an F-stable secondary representation by Proposition 2.1.10. Then suppose that $0 \leq i < d$.

Now, we have

$$\mathscr{H}(H^i_{\mathfrak{m}}(R/\mathfrak{a})^*) = \mathscr{H}(H^i_{\mathfrak{m}}(R/\mathfrak{a})) \cong H^{n-i}_{\mathfrak{a}}(R)$$

by Theorem 2.2.4 (iv) and Proposition 2.2.5. Then, by Theorem 2.2.11, if $H^i_{\mathfrak{m}}(R/\mathfrak{a})^*$ is non-zero, it has an F-stable secondary representation if and only if $H^{n-i}_{\mathfrak{a}}(R)$ has a primary decomposition in $\boldsymbol{F_R}$**-FinMod**.

Take an $\mathfrak{a}$-filter regular sequence $a_1, \ldots, a_{n-i}$, and set $\mathfrak{a}_{n-i} = (a_1, \ldots, a_{n-i})$. Then

$$H^{n-i}_{\mathfrak{a}}(R) \cong \Gamma_{\mathfrak{a}}(H^{n-i}_{\mathfrak{a}_{n-i}}(R))$$

by the Nagel-Schenzel Isomorphism.

Let $a = (a_1 \cdots a_{n-i})^{p-1}$. By [Lyu97, Proposition 2.3], we have that

$$R/(\mathfrak{a}_{n-i} : a^\infty) \xhookrightarrow{a\cdot} R/(\mathfrak{a}_{n-i} : a^\infty)^{[p]}$$

is a root of the F-finite F-module with generating morphism

$$R/\mathfrak{a}_{n-i} \xrightarrow{a\cdot} R/\mathfrak{a}_{n-i}^{[p]}$$

and so, setting $\mathfrak{b} = (\mathfrak{a}_{n-i} : a^\infty)$, we have

$$\begin{aligned} H^{d-i}_{\mathfrak{a}}(R) &\cong \Gamma_{\mathfrak{a}}(H^{n-i}_{\mathfrak{a}_{n-i}}(R)) \\ &\cong \Gamma_{\mathfrak{a}}\left(\varinjlim \left[R/\mathfrak{a}_{n-i} \xrightarrow{a\cdot} R/\mathfrak{a}_{n-i}^{[p]} \xrightarrow{a^p\cdot} R/\mathfrak{a}_{n-i}^{[p^2]} \xrightarrow{a^{p^2}\cdot} \cdots\right]\right) \\ &\cong \Gamma_{\mathfrak{a}}\left(\varinjlim \left[R/\mathfrak{b} \xhookrightarrow{a\cdot} R/\mathfrak{b}^{[p]} \xhookrightarrow{a^p\cdot} R/\mathfrak{b}^{[p^2]} \xhookrightarrow{a^{p^2}\cdot} \cdots\right]\right) \\ &\cong \varinjlim \left[\Gamma_{\mathfrak{a}}(R/\mathfrak{b}) \xhookrightarrow{a\cdot} \Gamma_{\mathfrak{a}}(R/\mathfrak{b}^{[p]}) \xhookrightarrow{a^p\cdot} \Gamma_{\mathfrak{a}}(R/\mathfrak{b}^{[p^2]}) \xhookrightarrow{a^{p^2}\cdot} \cdots\right] \\ &\cong \varinjlim \left[(\mathfrak{b} : \mathfrak{a}^\infty)/\mathfrak{b} \xhookrightarrow{a\cdot} (\mathfrak{b}^{[p]} : \mathfrak{a}^\infty)/\mathfrak{b}^{[p]} \xhookrightarrow{a^p\cdot} (\mathfrak{b}^{[p^2]} : \mathfrak{a}^\infty)/\mathfrak{b}^{[p^2]} \xhookrightarrow{a^{p^2}\cdot} \cdots\right] \end{aligned}$$

by Proposition 2.1.33 (we will apply a technique similar to this in the proof of Lemma 4.1.11).

Now, we can compute $\mathfrak{a}$-filter regular sequences, as well as saturations, quotients, and $\dim(R/\mathfrak{a})$, using `Macaulay2` when k is a finite field. We can also compute a primary decomposition

$$(\mathfrak{c}_1/\mathfrak{b}) \cap \cdots \cap (\mathfrak{c}_t/\mathfrak{b}) = 0/\mathfrak{b} \subseteq (\mathfrak{b} : \mathfrak{a}^\infty)/\mathfrak{b}$$

of $(\mathfrak{b} : \mathfrak{a}^\infty)/\mathfrak{b}$ for some ideals $\mathfrak{b} \subseteq \mathfrak{c}_j \subseteq (\mathfrak{b} : \mathfrak{a}^\infty)$ of R and $t \geq 1$.

By our previous work in this subsection, if

$$a\mathfrak{c}_j/\mathfrak{b} \subseteq \mathfrak{c}_j^{[p]}/\mathfrak{b}^{[p]}$$

for all $1 \leq j \leq t$ (that is, each $\mathfrak{c}_j/\mathfrak{b}$ is $a\cdot$-compatible), then $H^{n-i}_{\mathfrak{a}}(R)$ has a primary decomposition in $\boldsymbol{F_R}$**-FinMod**.

Since we can compute whether or not these containments hold in `Macaulay2`, we are able to, starting only with $\mathfrak{a}$, perform a series of calculations which could establish that $H^i_{\mathfrak{m}}(R/\mathfrak{a})^*$ has an F-stable secondary representation when k is a finite field.

There are two major caveats to this procedure however. Firstly, $H^i_{\mathfrak{m}}(R/\mathfrak{a})^*$ may not necessarily equal $H^i_{\mathfrak{m}}(R/\mathfrak{a})$, and the existence of an F-stable secondary representation of $H^i_{\mathfrak{m}}(R/\mathfrak{a})^*$ may not necessarily imply the existence of such a representation for $H^i_{\mathfrak{m}}(R/\mathfrak{a})$. Secondly, the failure of this procedure does not necessarily establish that $H^i_{\mathfrak{m}}(R/\mathfrak{a})^*$ has no F-stable secondary representation, only that our particular choice of $\mathfrak{a}$-filter regular sequence and primary decomposition of $(\mathfrak{b} : \mathfrak{a}^\infty)/\mathfrak{b}$ does not give rise to one.

Note. *Being completely rigorous, we cannot apply* `Macaulay2` *exactly as described here, since it performs its calculations over a polynomial ring rather than a power series ring. However, in a similar fashion to Chapter 3, it is possible to develop the theory of this procedure in the graded case, though we will not do so here since it is not necessary for the remainder of this thesis. Many results concerning the theory of graded F-modules can be found, for example, in [LSW16, Section 2].*

2.2.3 Primary Decompositions in the Category of F-Finite F-Modules

Throughout this subsection, we let $R = \mathbb{F}_p[[x, y]]$ for some prime $p > 0$, and set $\mathfrak{m} = (x, y)$.

This subsection constitutes our main application of Theorem 2.2.11: to explicitly construct an F-finite F-module without a primary decomposition in the category of F-finite F-modules. Using our work in Subsection 2.2.2, we do this by applying Lyubeznik's $\mathscr{H}$-functor to an example given by De Stefani and Ma ([DM21, Example 3.3]) of an Artinian $R[T; f]$-module with no F-stable secondary representation.

We begin with the definition of their example:

Definition / Theorem 2.2.12. [DM21, Example 3.3] *Let*

$$W := R/\mathfrak{m} \oplus H^2_{\mathfrak{m}}(R)$$

Viewing $H^2_{\mathfrak{m}}(R)$ as in Example 1.2.16, we define a map $f : W \to W$ on W via

$$f : \begin{bmatrix} a + \mathfrak{m} \\ h \end{bmatrix} \mapsto \begin{bmatrix} a + \mathfrak{m} \\ ax^{-p}y^{-1} + h^p \end{bmatrix}$$

Then f is an injective Frobenius map on W, and W has no F-stable secondary representation.

The fact that f is injective tells us that $W_{\text{red}} = W$. We will now calculate W^*:

Proposition 2.2.13. *We have $W^* = W$.*

Proof. We want to compute

$$W^* = \bigcap_{i=1}^{\infty} (RT^i W) = \bigcap_{i=1}^{\infty} W^{f^i}$$

where W^{f^i} denotes the R-submodule of W generated by all elements of the form $f^i(w)$ for $w \in W$ (so we are using the notation of our first definition of Frobenius actions).

Since W is Artinian, we have $W^* = W^{f^e}$ for some $e \geq 1$, as noted when defining W^*.

Now, we have

$$f^e\left(\begin{bmatrix} 1 + \mathfrak{m} \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 + \mathfrak{m} \\ 0 \end{bmatrix} + \sum_{i=1}^{e} \begin{bmatrix} 0 + \mathfrak{m} \\ x^{-p^i}y^{-p^{i-1}} \end{bmatrix}$$

For each $1 \leq i \leq e$, we have

$$\begin{bmatrix} 0 + \mathfrak{m} \\ x^{-p^i}y^{-p^{i-1}} \end{bmatrix} = x^{p^e - p^i} y^{p^e - p^{i-1}} \cdot f^e\left(\begin{bmatrix} 0 + \mathfrak{m} \\ x^{-1}y^{-1} \end{bmatrix}\right) \in W^*$$

and so

$$\sum_{i=1}^{e} \begin{bmatrix} 0 + \mathfrak{m} \\ x^{-p^i}y^{-p^{i-1}} \end{bmatrix} \in W^*$$

which means that

$$\begin{bmatrix} 1 + \mathfrak{m} \\ 0 \end{bmatrix} = f^e\left(\begin{bmatrix} 1 + \mathfrak{m} \\ 0 \end{bmatrix}\right) - \sum_{i=1}^{e} \begin{bmatrix} 0 + \mathfrak{m} \\ x^{-p^i}y^{-p^{i-1}} \end{bmatrix} \in W^*$$

Then $\mathbb{F}_p \oplus 0 \subseteq W^*$.

Furthermore, for any $i, j \geq 1$, there exists some $l \geq e$ such that $p^l \geq \max\{i, j\}$, so

$$\begin{bmatrix} 0 + \mathfrak{m} \\ x^{-i}y^{-j} \end{bmatrix} = x^{p^l - i}y^{p^l - j} \cdot f^l \left(\begin{bmatrix} 0 + \mathfrak{m} \\ x^{-1}y^{-1} \end{bmatrix} \right) \in W^*$$

Then $0 \oplus H^2_{\mathfrak{m}}(R) \subseteq W^*$, and so $W^* = W$. □

We know that $H^2_{\mathfrak{m}}(R)$ is Artinian by Theorem 1.2.13, and clearly $R/\mathfrak{m}$ is Artinian, so W is an Artinian $R[T; f]$-module. Furthermore, by Proposition 2.2.13 and Theorem 2.2.4 (iii), we have shown that $W^\vee$ is the minimal root of $\mathscr{H}(W)$, which we know has no primary decomposition in $\boldsymbol{F_R}$**-FinMod** by Theorem 2.2.11.

We will now give an explicit description of $\mathscr{H}(W)$. Before doing so, we require a preparatory lemma. Whilst not difficult, we include a proof for completeness:

Lemma 2.2.14. *Let S be any ring, and suppose that we have exact sequences of S-modules*

$$A \xrightarrow{\ \alpha\ } B \xrightarrow{\ \beta\ } C \longrightarrow 0$$

and

$$A \xrightarrow{\ \alpha\ } B \xrightarrow{\ \gamma\ } D \longrightarrow 0$$

Then, slightly abusing the notation of preimages, the map $\delta : C \to D$ given by

$$c \mapsto \gamma(\beta^{-1}(c))$$

is a well-defined S-isomorphism.

Proof. We will first show that δ is well-defined. That is, we must show that, if $\beta(b_1) = \beta(b_2)$, then $\gamma(b_1) = \gamma(b_2)$.

We have

$$\beta(b_1 - b_2) = \beta(b_1) - \beta(b_2) = 0$$

and so

$$b_1 - b_2 \in \ker(\beta) = \operatorname{Im}(\alpha) = \ker(\gamma)$$

by exactness. Then

$$\gamma(b_1) - \gamma(b_2) = \gamma(b_1 - b_2) = 0$$

so $\gamma(b_1) = \gamma(b_2)$ as desired.

In order to show that δ is an S-isomorphism, we must first show that it is an S-homomorphism.

Take any $c_1, c_2 \in C$. Since β is surjective, there exist some $b_1, b_2 \in B$ such that $\beta(b_1) = c_1$ and $\beta(b_2) = c_2$. Then

$$\beta(b_1 + b_2) = \beta(b_1) + \beta(b_2) = c_1 + c_2$$

so

$$\delta(c_1 + c_2) = \gamma(b_1 + b_2) = \gamma(b_1) + \gamma(b_2) = \delta(c_1) + \delta(c_2)$$

Now take any $r \in R$ and $c \in C$. Again since β is surjective, there exists some $b \in B$ such that $\beta(b) = c$. Then

$$\beta(rb) = r\beta(b) = rc$$

so

$$\delta(rc) = \gamma(rb) = r\gamma(b) = r\delta(c)$$

and so δ is an S-homomorphism as desired.

We will next show that δ is injective. Take any $c \in C$ such that $\delta(c) = 0$. Since β is surjective, there exists some $b \in B$ such that $c = \beta(b)$. We then have

$$\gamma(b) = \delta(c) = 0$$

so

$$b \in \ker(\gamma) = \operatorname{Im}(\alpha) = \ker(\beta)$$

by exactness. Then

$$c = \beta(b) = 0$$

and so δ is injective.

Finally, we will show that δ is surjective. Since γ is surjective, for any $d \in D$ there exists some $b \in B$ such that $\gamma(b) = d$. Then we have

$$\delta(\beta(b)) = \gamma(b) = d$$

so we are done. □

Proposition 2.2.15. *$\mathscr{H}(W)$ has root morphism*

$$R/\mathfrak{m} \oplus R \xhookrightarrow{\begin{bmatrix} x^{p-1}y^{p-1} & y^{p-1} \\ 0 & 1 \end{bmatrix}} R/\mathfrak{m}^{[p]} \oplus R \cong F_R(R/\mathfrak{m} \oplus R)$$

Proof. Note that, since R is Gorenstein, it is a canonical module for itself by Proposition 1.2.30, and so we may also view $H^2_{\mathfrak{m}}(R)$ as $E := E_R(R/\mathfrak{m})$ by Local Duality. Then, throughout this proposition, we will view W as $R/\mathfrak{m} \oplus E$.

Furthermore, note that, for any $a \in \mathbb{F}_p$, we have $a^p = a$ by Fermat's Little Theorem. When referring to elements $a + \mathfrak{m}$ of $R/\mathfrak{m}$, we will always assume that $a \in \mathbb{F}_p$.

The goal of this proposition is construct R-isomorphisms μ and λ such that the diagram

$$\begin{array}{ccccc} R/\mathfrak{m}\oplus R & \xhookrightarrow{\begin{bmatrix} x^{p-1}y^{p-1} & y^{p-1}\\ 0 & 1\end{bmatrix}} & R/\mathfrak{m}^{[p]}\oplus R & \xrightarrow[\sim]{\lambda} & F_R(R/\mathfrak{m}\oplus R) \\ \mu\downarrow\wr & & & & \wr\downarrow F_R(\mu) \\ W^\vee & \xhookrightarrow[\gamma_W^\vee]{} & F_R(W)^\vee & \xrightarrow[\tau_W]{\sim} & F_R(W^\vee)\end{array} \qquad (\bigstar)$$

commutes, where $\gamma_W^\vee$ is as in Definition / Theorem 2.2.1 and τ_W is as in Lemma 2.1.15.

Our first aim is to calculate τ_W. We begin by constructing the start of an injective resolution of W:

$$0 \longrightarrow W \xhookrightarrow{\begin{bmatrix} a+\mathfrak{m} \\ e\end{bmatrix} \mapsto \begin{bmatrix} ax^{-1}y^{-1} \\ e\end{bmatrix}} E^2 \xrightarrow{\begin{bmatrix} x & 0 \\ y & 0\end{bmatrix}} E^2 \qquad (*)$$

Now, F_R is exact by Kunz's Theorem since R is regular, it is additive by Lemma 2.1.14 (i), and $F_R(E) \cong E$ canonically by Proposition 2.1.23 (recall that we are identifying E and $H^2_{\mathfrak{m}}(R)$), and so by Proposition 2.1.17, applying F_R to $(*)$ we obtain

$$0 \longrightarrow F_R(W) \xhookrightarrow{1\otimes_R \begin{bmatrix} a+\mathfrak{m} \\ e\end{bmatrix} \mapsto \begin{bmatrix} ax^{-p}y^{-p} \\ e^p\end{bmatrix}} E^2 \xrightarrow{\begin{bmatrix} x^p & 0 \\ y^p & 0\end{bmatrix}} E^2$$

Then, since $-^\vee$ is exact because E is injective by definition, clearly additive, and $E^\vee \cong R$ canonically by Proposition 1.1.12 since R is complete, applying $-^\vee$ (noting Remark 1.1.13) we obtain

$$R^2 \xrightarrow{\begin{bmatrix} x^p & y^p \\ 0 & 0\end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} r \\ s\end{bmatrix} \mapsto \left(1\otimes_R \begin{bmatrix} a+\mathfrak{m} \\ e\end{bmatrix} \mapsto rax^{-p}y^{-p}+se^p\right)} F_R(W)^\vee \longrightarrow 0$$

Since this last map is surjective, for any $\psi \in F_R(W)^\vee$ there exist some $r_\psi, s_\psi \in R$ such that ψ is given by

$$1\otimes_R \begin{bmatrix} a+\mathfrak{m} \\ e\end{bmatrix} \mapsto r_\psi ax^{-p}y^{-p} + s_\psi e^p$$

Conversely, if we first apply $-^\vee$ to $(*)$, by the same properties as before we obtain

$$R^2 \xrightarrow{\begin{bmatrix} x & y \\ 0 & 0\end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} r \\ s\end{bmatrix} \mapsto \left(\begin{bmatrix} a+\mathfrak{m} \\ e\end{bmatrix} \mapsto rax^{-1}y^{-1}+se\right)} W^\vee \longrightarrow 0$$

to which we now apply F_R, noting this time that $F_R(R) \cong R$ canonically by Proposition 2.1.16, to obtain

$$R^2 \xrightarrow{\begin{bmatrix} x^p & y^p \\ 0 & 0 \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} r \\ s \end{bmatrix} \mapsto r \otimes_R \left(\begin{bmatrix} a+\mathfrak{m} \\ e \end{bmatrix} \mapsto ax^{-1}y^{-1} \right) + s \otimes_R \left(\begin{bmatrix} a+\mathfrak{m} \\ e \end{bmatrix} \mapsto e \right)} F_R(W^\vee) \longrightarrow 0$$

Lemma 2.2.14 then tells us that τ_W is given by

$$(\psi : F_R(W) \to E) \mapsto r_\psi \otimes_R \left(\begin{bmatrix} a+\mathfrak{m} \\ e \end{bmatrix} \mapsto ax^{-1}y^{-1} \right) + s_\psi \otimes_R \left(\begin{bmatrix} a+\mathfrak{m} \\ e \end{bmatrix} \mapsto e \right)$$

We will next calculate $\gamma_W^\vee$. Recall that the Frobenius map on W is given in Definition / Theorem 2.2.12 as

$$f : \begin{bmatrix} a+\mathfrak{m} \\ e \end{bmatrix} \mapsto \begin{bmatrix} a+\mathfrak{m} \\ ax^{-p}y^{-1} + e^p \end{bmatrix}$$

which yields

$$\gamma_W : 1 \otimes_R \begin{bmatrix} a+\mathfrak{m} \\ e \end{bmatrix} \mapsto \begin{bmatrix} a+\mathfrak{m} \\ ax^{-p}y^{-1} + e^p \end{bmatrix}$$

and so

$$\gamma_W^\vee : (\psi : W \to E) \mapsto \left(1 \otimes_R \begin{bmatrix} a+\mathfrak{m} \\ e \end{bmatrix} \mapsto \psi \left(\begin{bmatrix} a+\mathfrak{m} \\ ax^{-p}y^{-1} + e^p \end{bmatrix} \right) \right)$$

Next, we will construct

$$\mu : R/\mathfrak{m} \oplus R \xrightarrow{\sim} W^\vee = (R/\mathfrak{m} \oplus E)^\vee$$

By Proposition 1.1.12, we have

$$R/\mathfrak{m} \oplus R \xrightarrow{\sim} (R/\mathfrak{m})^\vee \oplus E^\vee$$

via

$$\begin{bmatrix} a+\mathfrak{m} \\ r \end{bmatrix} \mapsto \begin{bmatrix} (1+\mathfrak{m}) \mapsto ax^{-1}y^{-1} \\ e \mapsto re \end{bmatrix}$$

and so

$$\mu : \begin{bmatrix} a+\mathfrak{m} \\ r \end{bmatrix} \mapsto \left(\begin{bmatrix} b+\mathfrak{m} \\ e \end{bmatrix} \mapsto abx^{-1}y^{-1} + re \right)$$

Finally, we will construct

$$\lambda : R/\mathfrak{m}^{[p]} \oplus R \xrightarrow{\sim} F_R(R/\mathfrak{m} \oplus R)$$

We have

$$R/\mathfrak{m}^{[p]} \oplus R \xrightarrow{\sim} F_R(R/\mathfrak{m}) \oplus F_R(R)$$

via

$$\begin{bmatrix} r+\mathfrak{m}^{[p]} \\ s \end{bmatrix} \mapsto \begin{bmatrix} r\otimes_R (1+\mathfrak{m}) \\ s\otimes_R 1 \end{bmatrix}$$

and so

$$\lambda : \begin{bmatrix} r+\mathfrak{m}^{[p]} \\ s \end{bmatrix} \mapsto r\otimes_R \begin{bmatrix} 1+\mathfrak{m} \\ 0 \end{bmatrix} + s\otimes_R \begin{bmatrix} 0+\mathfrak{m} \\ 1 \end{bmatrix}$$

We now claim that

$$\tau_W \circ \gamma_W^\vee \circ \mu\left(\begin{bmatrix} a+\mathfrak{m} \\ r \end{bmatrix}\right) = F_R(\mu)\circ\lambda\left(\begin{bmatrix} x^{p-1}y^{p-1} & y^{p-1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a+\mathfrak{m} \\ r \end{bmatrix}\right)$$

By our previous work, we have

$$\begin{aligned}
&\tau_W \circ \gamma_W^\vee \circ \mu\left(\begin{bmatrix} a+\mathfrak{m} \\ r \end{bmatrix}\right) \\
&= \tau_W \circ \gamma_W^\vee\left(\begin{bmatrix} b+\mathfrak{m} \\ e \end{bmatrix} \mapsto abx^{-1}y^{-1} + re\right) \\
&= \tau_W\left(1\otimes_R \begin{bmatrix} b+\mathfrak{m} \\ e \end{bmatrix} \mapsto abx^{-1}y^{-1} + rbx^{-p}y^{-1} + re^p\right) \\
&= (ax^{p-1}y^{p-1} + ry^{p-1})\otimes_R\left(\begin{bmatrix} b+\mathfrak{m} \\ e \end{bmatrix} \mapsto bx^{-1}y^{-1}\right) + r\otimes_R\left(\begin{bmatrix} b+\mathfrak{m} \\ e \end{bmatrix} \mapsto e\right)
\end{aligned}$$

and

$$\begin{aligned}
&F_R(\mu)\circ\lambda\left(\begin{bmatrix} x^{p-1}y^{p-1} & y^{p-1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a+\mathfrak{m} \\ r \end{bmatrix}\right) \\
&= F_R(\mu)\circ\lambda\left(\begin{bmatrix} ax^{p-1}y^{p-1} + ry^{p-1} + \mathfrak{m} \\ r \end{bmatrix}\right) \\
&= F_R(\mu)\left((ax^{p-1}y^{p-1} + ry^{p-1})\otimes_R \begin{bmatrix} 1+\mathfrak{m} \\ 0 \end{bmatrix} + r\otimes_R \begin{bmatrix} 0+\mathfrak{m} \\ 1 \end{bmatrix}\right) \\
&= (ax^{p-1}y^{p-1} + ry^{p-1})\otimes_R\left(\begin{bmatrix} a+\mathfrak{m} \\ e \end{bmatrix} \mapsto ax^{-1}y^{-1}\right) + r\otimes_R\left(\begin{bmatrix} a+\mathfrak{m} \\ e \end{bmatrix} \mapsto e\right)
\end{aligned}$$

so (★) commutes and the result follows. □

To summarise this subsection:

Theorem 2.2.16. *Let $R = \mathbb{F}_p[[x,y]]$ for some prime $p > 0$, and set $\mathfrak{m} = (x,y)$. Then the F-finite F-module with root morphism*

$$R/\mathfrak{m} \oplus R \xhookrightarrow{\begin{bmatrix} x^{p-1}y^{p-1} & y^{p-1} \\ 0 & 1 \end{bmatrix}} R/\mathfrak{m}^{[p]} \oplus R \cong F_R(R/\mathfrak{m} \oplus R)$$

*has no primary decomposition in $\boldsymbol{F_R}$-**FinMod**. Furthermore, $R/\mathfrak{m} \oplus R$ is the minimal root of this F-finite F-module.* □

Chapter 3

Graded Analogues of Local Theorems

Throughout this chapter, we assume that R is ($\mathbb{Z}$-)graded.

Many results, especially those relating to Matlis Duality and local cohomology, are stated for local rings, often assuming completeness. We will show that a broad range of such results will hold in the case that $R = k[x_1, \ldots, x_n]$ for some field k and $n \geq 0$, as well as in more general cases, when working with homogeneous ideals and graded modules. Most of these results are well known, but we collect them together and provide proofs (or references) for convenience.

3.1 Basic Definitions & Results

Definition 3.1.1. *We denote by* ****R*-Mod** *the category of graded R-modules and homogeneous R-homomorphisms of degree 0.*

Definition 3.1.2. *We denote by $^{*}\mathrm{Spec}(R)$ the set of homogeneous prime ideals of R.*

Definition 3.1.3. *Let $\mathfrak{a}$ be an ideal of R. We denote by $\mathfrak{a}^{*}$ the largest homogeneous ideal contained in $\mathfrak{a}$. Equivalently, it is the ideal generated by all the homogeneous elements of $\mathfrak{a}$.*

A key property of this operation is the following:

Proposition 3.1.4. [BH05, Lemma 1.5.6 (a)] *Let $\mathfrak{p} \in \mathrm{Spec}(R)$. Then we have $\mathfrak{p}^{*} \in {}^{*}\mathrm{Spec}(R)$.*

Note. *Proposition 3.1.4 tells us that when R is non-zero, we have $^{*}\mathrm{Spec}(R) \neq \varnothing$, since R must contain a maximal ideal $\mathfrak{m}$, and so $\mathfrak{m}^{*} \in {}^{*}\mathrm{Spec}(R)$.*

We also have the following relationship between $\mathfrak{p}$ and $\mathfrak{p}^{*}$:

Proposition 3.1.5. *Let M be a non-zero finitely generated graded R-module, and $\mathfrak{p}$ a prime ideal of R*

which is not homogeneous. Then

$$\operatorname{grade}_R(\mathfrak{p}, M) = \operatorname{grade}_R(\mathfrak{p}^*, M) + 1$$

Proof. Note that $\mathfrak{p}^*$ is prime by Proposition 3.1.4.

We see in the proof of [BH05, Theorem 1.5.9] that

$$\operatorname{rank}_{R/\mathfrak{p}}(\operatorname{Ext}^{i+1}_R(R/\mathfrak{p}, M)) = \operatorname{rank}_{R/\mathfrak{p}^*}(\operatorname{Ext}^i_R(R/\mathfrak{p}^*, M))$$

and so since

$$\operatorname{grade}_R(\mathfrak{a}, M) = \min\{0 \leq i \leq \dim_R(M) : \operatorname{Ext}(R/\mathfrak{a}, M) \neq 0\}$$

for any ideal $\mathfrak{a}$ of R by [BH05, Theorem 1.2.5 & Proposition 1.2.14], we are done if we can show that $\operatorname{grade}_R(\mathfrak{p}, M) \neq 0$. But if this were the case then we would have $\mathfrak{p} \in \operatorname{Ass}_R(M)$, since this would mean that

$$\operatorname{Hom}_R(R/\mathfrak{p}, M) = 0$$

We will see in Proposition 3.10.1 that all associated primes of M are homogeneous, so this cannot happen, which concludes the proof. □

Furthermore, ***R*-Mod** is closed under several common operations:

Lemma 3.1.6. *Let $\mathfrak{a}$ be a homogeneous ideal of R, M and L graded R-modules, N a graded R-submodule of M, and $\varphi : M \to L$ a homogeneous R-homomorphism. Then:*

i) Arbitrary sums and intersections of graded modules are graded. [Nor68, Section 2.13, Proposition 29]

ii) M/N is graded via $(M/N)_i = \pi(M_i)$, where $\pi : M \twoheadrightarrow M/N$ is the natural projection. [Nor68, pp. 116–117]

iii) $\sqrt{\mathfrak{a}}$ is homogeneous. [Nor68, Section 2.13, Proposition 32]

iv) $\mathfrak{a}M$ and $(N :_M \mathfrak{a})$ are graded R-submodules of M. [Nor68, Section 2.11, Proposition 31]

v) $(N :_R M)$ is homogeneous. In particular, $\operatorname{Ann}_R(M) = (0 :_R M)$ is homogeneous. [Nor68, Section 2.11, Proposition 30]

vi) $\operatorname{Im}(\varphi)$ and $\ker(\varphi)$ are graded R-submodules of L. [Nor68, Section 2.11, Lemma 11]

3.2 Graded Local Cohomology

We first define graded versions of Hom and Ext:

Definition 3.2.1. *Let M and N be graded R-modules. Then we denote by ${}^*\operatorname{Hom}_R(M, N)_i$ the R-*

submodule of $\mathrm{Hom}_R(M,N)$ *consisting of the homogeneous* R*-homomorphisms of degree* i*, and set*

$$^*\mathrm{Hom}_R(M,N) := \bigoplus_{i\in\mathbb{Z}} {}^*\mathrm{Hom}_R(M,N)_i$$

This is a graded R*-module, and is an* R*-submodule of* $\mathrm{Hom}_R(M,N)$*.*

It is well known that every graded R*-module has a graded free resolution (see, for example, [Eis95, p. 474]), and so we can define* $^*\mathrm{Ext}_R(-,N)$ *as the left derived functor of* $^*\mathrm{Hom}_R(-,N)$ *in* $^*\boldsymbol{R}$**-Mod***.*

Note. *Some sources denote* $^*\mathrm{Hom}_R$ *as* $\underline{\mathrm{Hom}}_R$ *or* HOM_R*, and* $^*\mathrm{Ext}^i_R$ *as* $\underline{\mathrm{Ext}}^i_R$ *or* EXT^i_R*.*

Forgetting the grading, these are in many cases equivalent to their non-graded counterparts. For example:

Lemma 3.2.2. *Let* M *be a finitely generated graded* R*-module, and* N *any graded* R*-module. Then*

$$^*\mathrm{Hom}_R(M,N) = \mathrm{Hom}_R(M,N)$$

and

$$^*\mathrm{Ext}^i_R(M,N) = \mathrm{Ext}^i_R(M,N)$$

for all $i \geq 0$*.*

Proof. The first claim is shown in [HOI88, Lemma 33.1 (2)], the second is then immediate. □

We can now define a graded version of local cohomology:

Definition 3.2.3. *Let* $\mathfrak{a}$ *be a homogeneous ideal of* R*,* M *a graded* R*-module, and* $i \geq 0$*. Then, analogously to the non-graded case, we define the* ***ith graded local cohomology functor with support at*** $\mathfrak{a}$ *to be*

$$^*H^i_{\mathfrak{a}}(-) := \varinjlim {}^*\mathrm{Ext}^i_R(R/\mathfrak{a}^j, -) : {}^*\boldsymbol{R}\textbf{-Mod} \to {}^*\boldsymbol{R}\textbf{-Mod}$$

This functor sends graded R*-modules to graded* R*-modules, and homogeneous* R*-homomorphisms to homogeneous* R*-homomorphisms, since the transition maps in the direct limit system are also graded.*

Note. *Some sources denote* $^*H^i_{\mathfrak{a}}$ *as* $\underline{H}^i_{\mathfrak{a}}$*.*

Forgetting the grading, this is also often equivalent to its non-graded counterpart. For example:

Lemma 3.2.4. [BS13, Remarks 13.4.6 (v)] *Let* M *be a finitely generated graded* R*-module, and* $\mathfrak{a}$ *a homogeneous ideal of* R*. Then*

$$^*H^i_{\mathfrak{a}}(-) \cong H^i_{\mathfrak{a}}(-)$$

naturally as functors from $^*\boldsymbol{R}$**-FinMod** *to* $\boldsymbol{R}$**-Mod** *for all* $i \geq 0$ *(that is, the isomorphism holds when the argument of* $^*H^i_{\mathfrak{a}}$ *is finitely generated).*

For a very careful treatment of graded local cohomology, see [BS13, Chapters 13 & 14].

3.3 *Local & *Complete Rings

Definition 3.3.1. *We say that a proper homogeneous ideal* $\mathfrak{m}$ *of* R *is* ***maximal** *if the only homogeneous ideal containing* $\mathfrak{m}$ *is* R*. If* R *has a unique *maximal ideal* $\mathfrak{m}$*, we say that* $(R, \mathfrak{m})$ *is* ***local**.

If, in addition to this, the ring R_0 *generated by the degree* 0 *elements of* R *is complete with respect to* $\mathfrak{m}_0 = R_0 \cap \mathfrak{m}$*, we say that* $(R, \mathfrak{m})$ *is* ***complete**.

Whilst *maximal ideals need not be maximal in the usual sense, they are prime:

Proposition 3.3.2. *Let* $\mathfrak{m}$ *a *maximal ideal of* R*. Then* $\mathfrak{m}$ *is prime.*

Proof. Since $\mathfrak{m}$ is *maximal, every non-zero homogeneous element in $R/\mathfrak{m}$ is invertible, and so we may apply [BH05, Lemma 1.5.7]. This tells us that either $R/\mathfrak{m} \cong k$, or $R/\mathfrak{m} \cong k[t, t^{-1}]$ for some field k and homogeneous indeterminate t of positive degree, as graded rings. In either case, we see that $R/\mathfrak{m}$ is an integral domain, and the result follows. □

There are, however, many situations where *maximal ideals are maximal in the usual sense. For example, as noted in [BH05, p. 35]:

Proposition 3.3.3. *Suppose that* R *is non-negatively graded, and let* $\mathfrak{m}$ *be a *maximal ideal of* R*. Then* $\mathfrak{m}$ *is maximal.*

Proof. As in the proof of Proposition 3.3.2, we see that either $R/\mathfrak{m} \cong k$, or $R/\mathfrak{m} \cong k[t, t^{-1}]$ for some field k and homogeneous indeterminate t of positive degree, as graded rings. But $R/\mathfrak{m}$ is non-negatively graded since R is, whereas $\deg(t^{-1}) < 0$. Then we must have $R/\mathfrak{m} \cong k$, and so $\mathfrak{m}$ is maximal as claimed. □

Just as $R_\mathfrak{m}$ is a faithfully flat R-module over a local ring $(R, \mathfrak{m})$, we have the following:

Lemma 3.3.4. [BH05, Proposition 1.5.15 (c)] *Suppose that* $(R, \mathfrak{m})$ *is *local. Then*

$$R_\mathfrak{m} \otimes_R - : {}^*\boldsymbol{R}\textbf{-Mod} \to \boldsymbol{R_\mathfrak{m}}\textbf{-Mod}$$

is faithfully flat.

We also have a graded version of Nakayama's Lemma:

Lemma 3.3.5 (Graded Nakayama's Lemma)**.** *Suppose that* $(R, \mathfrak{m})$ *is *local, and let* M *be a finitely generated graded* R*-module. Then* $\mathfrak{m}M = M$ *if and only if* $M = 0$*.*

Proof. This follows from [BH05, Exercise 1.5.24 (a)], taking $N = 0$. □

3.4 *Injective Hulls

Definition / Theorem 3.4.1. [BS13, Theorem 13.2.4 (i) & (ii)] *Injective hulls exist in* ****R*-Mod**, *and are unique up to graded R-isomorphism. We denote by ${}^*E_R(M)$ the* ****injective hull*** *of a graded R-module M in* ****R*-Mod**.

*Injective hulls are always contained in the usual, non-graded, injective hull (see [BS13, Theorem 13.2.4 (i)]). However, there are cases where the two coincide. For example:

Proposition 3.4.2. [BH05, Corollary 3.6.7] *Let $\mathfrak{m}$ be a *maximal ideal of R. Then there exists an R-isomorphism*

$$ {}^*E_R(R/\mathfrak{m}) \cong E_R(R/\mathfrak{m}) $$

We next introduce the graded analogue of Definition 1.1.1, and prove several graded versions of well-known results concerning essential extensions:

Definition 3.4.3. *Let M be a graded R-module and N a graded R-submodule of M. We say that M is a* ****essential extension*** *of N if, for any non-zero graded R-submodule L of M, we have $L \cap N \neq 0$.*

Proposition 3.4.4. *Let $N \subseteq M \subseteq L$ be graded R-modules. Then $N \subseteq L$ is *essential if and only if $N \subseteq M$ and $M \subseteq L$ are *essential.*

Proof. First suppose that $N \subseteq M$ and $M \subseteq L$ are *essential, and take any non-zero graded R-submodule A of L. Then $A \cap M \neq 0$ since $M \subseteq L$ is *essential. Now, $A \cap M$ is a graded R-submodule of M, and so

$$ A \cap N \supseteq (A \cap M) \cap N \neq 0 $$

since $N \subseteq M$ is *essential. Then $N \subseteq L$ is *essential as desired.

Now suppose that $N \subseteq L$ is *essential. Take any graded R-submodule B of M. Then B is also a graded R-submodule of L, so $B \cap N \neq 0$ since $N \subseteq L$ is *essential, and so $N \subseteq M$ is *essential. Next take any graded R-submodule C of L. Again since $N \subseteq L$ is *essential, we have $C \cap N \neq 0$, and so

$$ C \cap M \supseteq C \cap N \neq 0 $$

because $N \subseteq M$. Then $M \subseteq L$ is *essential, and we are done □

Lemma 3.4.5. **Injective hulls are maximal *essential extensions. That is, let M be a graded R-module. Then $M \subseteq {}^*E_R(M)$ is *essential, and for any graded R-module N with $M \subseteq N$ *essential, there exists a homogeneous R-isomorphism $\varphi : N \xrightarrow{\sim} L$ of degree 0 for some graded R-submodule L of ${}^*E_R(M)$, with $L \subseteq {}^*E_R(M)$ *essential, such that the following diagram commutes:*

$$ \begin{array}{ccc} M & \hookrightarrow & N \\ \downarrow & & \downarrow \wr \\ E_R(M) & \hookleftarrow & L \end{array} $$

Proof. This follows from the proof of [BH05, Theorem 3.6.2]. □

Lemma 3.4.6. *Let M be a graded R-module, and N a graded R-submodule of M such that $N \subseteq M$ is *essential. Then*

$$^*E_R(N) \cong {}^*E_R(M)$$

as graded R-modules.

Proof. Since $N \subseteq M$ is *essential by assumption, and $M \subseteq {}^*E_R(M)$ is *essential by Lemma 3.4.5, we have that $N \subseteq {}^*E_R(M)$ is *essential by Proposition 3.4.4. Then $^*E_R(M) \cong L$ for some graded R-submodule L of $^*E_R(N)$, with $L \subseteq {}^*E_R(N)$ *essential, by Lemma 3.4.5.

Since $^*E_R(N)$ is *injective, we then have $^*E_R(N) = A \oplus L$ for some graded R-submodule A of $^*E_R(N)$. Because this sum is direct, we have $A \cap L = 0$, but $L \subseteq {}^*E_R(N)$ is *essential, so $A = 0$ and therefore

$$^*E_R(N) = L \cong {}^*E_R(M)$$

This concludes the proof. □

3.5 Graded Chain Conditions

Next, we define versions of the familiar chain conditions in ****R*-Mod**:

Definition 3.5.1. *We say that a graded R-module M is ***Noetherian** (resp. ***Artinian**) if it satisfies the ascending (resp. descending) chain condition in ***R-Mod**.*

We have the following relationships between R-modules being *Noetherian/*Artinian and Noetherian/Artinian:

Lemma 3.5.2. [NO04, Theorem 5.4.7] *Let M be a *Noetherian R-module. Then M is Noetherian.*

Proposition 3.5.3. *Suppose that R is non-negatively graded, and let M be a *Artinian R-module. Then M is Artinian.*

Proof. This follows from [NO04, Proposition 5.4.5 (1)]. □

We can say even more in the *local case, and so we assume that $(R, \mathfrak{m})$ is *local for the remainder of this section.

Lemma 3.5.4. *Let M be a graded R-module. Then if $M_\mathfrak{m}$ is an Artinian $R_\mathfrak{m}$-module, we have that M is *Artinian.*

Proof. Take a descending chain

$$M \supseteq N_1 \supseteq N_2 \supseteq \cdots$$

of graded R-submodules of M. This gives us the descending chain

$$M_{\mathfrak{m}} \supseteq (N_1)_{\mathfrak{m}} \supseteq (N_2)_{\mathfrak{m}} \supseteq \cdots$$

of $R_{\mathfrak{m}}$-submodules of $M_{\mathfrak{m}}$.

Since $M_{\mathfrak{m}}$ is an Artinian $R_{\mathfrak{m}}$-module, there exists some $i \geq 1$ such that, for all $j \geq 0$, we have $(N_i)_{\mathfrak{m}} = (N_{i+j})_{\mathfrak{m}}$, and so $(N_i/N_{i+j})_{\mathfrak{m}} = 0$. Since the N_i/N_{i+j} are graded, we are then done by Lemma 3.3.4. □

This allows us to deduce that several important graded R-modules are *Artinian:

Corollary 3.5.5. *$^*E_R(R/\mathfrak{m})$ is *Artinian.*

Proof. We have

$$E_R(R/\mathfrak{m})_{\mathfrak{m}} \cong E_{R_{\mathfrak{m}}}((R/\mathfrak{m})_{\mathfrak{m}}) \cong E_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}})$$

by Proposition 1.1.5, which is Artinian as an $R_{\mathfrak{m}}$-module by Proposition 1.1.11, and so we are done by Lemma 3.5.4. □

Corollary 3.5.6. *Let M be a finitely generated graded R-module. Then $^*H^i_{\mathfrak{m}}(M)$ is *Artinian for all $i \geq 0$.*

Proof. We have

$$^*H^i_{\mathfrak{m}}(M)_{\mathfrak{m}} \cong H^i_{\mathfrak{m}}(M)_{\mathfrak{m}} \cong H^i_{\mathfrak{m}R_{\mathfrak{m}}}(M_{\mathfrak{m}})$$

as $R_{\mathfrak{m}}$-modules by Lemma 3.2.4 and Corollary 1.2.11, so we are done by Theorem 1.2.13 and Lemma 3.5.4. □

Analogously to Proposition 1.1.11 (as noted in [BH05, p. 142], since their proof of [BH05, Theorem 3.2.13 (4)] makes no use of completeness), we have the following:

Proposition 3.5.7. *Let M be a graded R-module. Then M is *Artinian if and only if*

$$M \subseteq \bigoplus_{i=1}^{t} {}^*E_R(R/\mathfrak{m})(a_i)$$

for some $t \geq 0$ and degree shifts $a_i \in \mathbb{Z}$.

Proof. We essentially follow the proof of [BH05, Theorem 3.2.13], but take gradings into consideration.

$^*E_R(R/\mathfrak{m})$ is *Artinian by Corollary 3.5.5, and so the reverse direction is immediate.

For the forward direction, we first define the ***socle** of M to be

$$^*\mathrm{Soc}_R(M) \coloneqq (0 :_M \mathfrak{m})$$

which is graded by Lemma 3.1.6 (v). We will show that ${}^*\mathrm{Soc}_R(M)$ is *essential in M:

Take any non-zero homogeneous $m \in M$, and consider the descending chain

$$Rm \supseteq \mathfrak{m}(Rm) \supseteq \mathfrak{m}^2(Rm) \supseteq \cdots$$

of graded R-submodules of Rm. Since Rm is Artinian, this chain must stabilise, say at some minimal integer i. Note that we must have $i > 0$ by Graded Nakayama's Lemma, since $Rm \neq 0$ and so we cannot have $Rm = \mathfrak{m}(Rm)$. Then $\mathfrak{m}(\mathfrak{m}^{i-1}(Rm)) = 0$ (again by Graded Nakayama's Lemma), and so

$$0 \neq \mathfrak{m}^{i-1}(Rm) \subseteq {}^*\mathrm{Soc}_R(M)$$

which means that

$$(Rm) \cap {}^*\mathrm{Soc}_R(M) \neq 0$$

We have then shown that ${}^*\mathrm{Soc}_R(M)$ is *essential in M, and so

$${}^*E_R(M) \cong {}^*E_R({}^*\mathrm{Soc}_R(M))$$

by Lemma 3.4.6.

Now, by [BH05, Exercise 1.5.20], and since M (and therefore ${}^*\mathrm{Soc}_R(M)$) is Artinian, we have

$${}^*\mathrm{Soc}_R(M) \cong \bigoplus_{i=1}^{t} (R/\mathfrak{m})(a_i)$$

for some $t \geq 0$ and degree shifts $a_i \in \mathbb{Z}$.

Taking *injective hulls commutes with taking finite direct sums (this can be shown using essentially the same argument as in the case of injective hulls. See, for example, [AF12, Proposition 18.12 (4)]), and so

$$\begin{aligned} M \subseteq {}^*E_R(M) \cong {}^*E_R({}^*\mathrm{Soc}_R(M)) &\cong {}^*E_R\left(\bigoplus_{i=1}^{t} (R/\mathfrak{m})(a_i)\right) \\ &\cong \bigoplus_{i=1}^{t} {}^*E_R((R/\mathfrak{m})(a_i)) = \bigoplus_{i=1}^{t} {}^*E_R(R/\mathfrak{m})(a_i) \end{aligned}$$

with the the last equality following directly from the construction of *injective hulls (again, see [BH05, Theorem 3.6.2]). □

This allows us to deduce the following:

Proposition 3.5.8. *Suppose that $\mathfrak{m}$ is maximal, and let M be a *Artinian R-module. Then M is Artinian.*

Proof. By Proposition 3.5.7, it suffices to show that ${}^*E_R(R/\mathfrak{m})$ is Artinian.

Note that ${}^*E_R(R/\mathfrak{m}) \cong E_R(R/\mathfrak{m})$ by Proposition 3.4.2, and take a descending chain

$$E_R(R/\mathfrak{m}) \supseteq N_1 \supseteq N_2 \supseteq \cdots$$

of R-submodules of $E_R(R/\mathfrak{m})$. This gives us the descending chain

$$E_R(R/\mathfrak{m})_\mathfrak{m} \supseteq (N_1)_\mathfrak{m} \supseteq (N_2)_\mathfrak{m} \supseteq \cdots$$

of $R_\mathfrak{m}$-submodules of $E_R(R/\mathfrak{m})_\mathfrak{m}$.

We have

$$E_R(R/\mathfrak{m})_\mathfrak{m} \cong E_{R_\mathfrak{m}}((R/\mathfrak{m})_\mathfrak{m}) \cong E_{R_\mathfrak{m}}(R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m})$$

by Proposition 1.1.5. This is Artinian as an $R_\mathfrak{m}$-module by Proposition 1.1.11, so there exists some $i \geq 1$ such that, for all $j \geq 0$, we have $(N_i)_\mathfrak{m} = (N_{i+j})_\mathfrak{m}$, and so $(N_i/N_{i+j})_\mathfrak{m} = 0$.

Now, $\operatorname{Ass}_R(E_R(R/\mathfrak{m})) = \{\mathfrak{m}\}$ by [BH05, Lemma 3.2.7], and so $\operatorname{Ass}_R(N_i) \subseteq \{\mathfrak{m}\}$ since $N_i \subseteq E_R(R/\mathfrak{m})$. In particular, $\operatorname{Supp}_R(N_i) = \{\mathfrak{m}\}$ since $\operatorname{MinSupp}_R(N_i) \subseteq \operatorname{Ass}_R(N_i)$. This means that $(N_i/N_{i+j})_\mathfrak{n} = 0$ for any $\mathfrak{n} \in \operatorname{MaxSpec}(R)$, so $N_i/N_{i+j} = 0$. Then $N_i = N_{i+j}$ for all $j \geq 0$, and we are done. □

Corollary 3.5.9. *Suppose that $\mathfrak{m}$ is maximal. Then ${}^*E_R(R/\mathfrak{m})$ is Artinian.*

Proof. This is immediate from Corollary 3.5.5 and Proposition 3.5.8. □

Corollary 3.5.10. *Suppose that $\mathfrak{m}$ is maximal, and let M be a finitely generated graded R-module. Then ${}^*H^i_\mathfrak{m}(M)$ is Artinian for all $i \geq 0$.*

Proof. This is immediate from Corollary 3.5.6 and Proposition 3.5.8. □

3.6 Graded Matlis Duality

Throughout this section, we assume that $(R, \mathfrak{m})$ is *local.

In this section, we will introduce a graded version of Matlis Duality.

Definition 3.6.1. *We define the **graded Matlis Duality functor** to be*

$$-^\vee := {}^*\operatorname{Hom}_R(-, {}^*E_R(R/\mathfrak{m})) : {}^*\boldsymbol{R}\textbf{-Mod} \to {}^*\boldsymbol{R}\textbf{-Mod}$$

Note that this functor is exact in ***R*-Mod** *since ${}^*E_R(R/\mathfrak{m})$ is *injective, and that it coincides with the usual Matlis Duality functor when its argument is finitely generated by Lemma 3.2.2.*

*For any graded R-module M, we say that $M^\vee$ is the **graded Matlis dual** of M.*

Note. *This definition is slightly different to that given in [BH05, p. 141], however they are shown to be equivalent in [BH05, Proposition 3.6.16 (b) & (c)].*

This functor differs from the usual Matlis Duality functor for many non-finitely generated modules, even in the *complete case. For example:

Proposition 3.6.2. [HOI88, Remark 34.9] *Suppose that R is *complete. Then*

$$^*\mathrm{Hom}_R(^*E_R(R/\mathfrak{m}), {}^*E_R(R/\mathfrak{m})) \cong R$$

but

$$\mathrm{Hom}_R(^*E_R(R/\mathfrak{m}), {}^*E_R(R/\mathfrak{m})) \cong \widehat{R_\mathfrak{m}}$$

However, this functor still shares many useful properties with the usual Matlis Duality functor:

Theorem 3.6.3 (Graded Matlis Duality)**.** *Let M be a *Noetherian R-module, N a *Artinian R-module, and L any graded R-module. Then*

i) $L \hookrightarrow L^{\vee\vee}$

ii) $M^\vee$ *is *Artinian.*

Furthermore, if R is also *complete, then

iii) $N^\vee$ *is *Noetherian.*

iv) $M^{\vee\vee} \cong M$ *and* $N^{\vee\vee} \cong N$ *as graded R-modules.*

Proof.

i) See the proof of Proposition 3.6.4.

ii) See [BS13, Exercise 14.4.2 (i)].

iii) See [BH05, Theorem 3.6.17 (a)].

iv) See [BH05, Theorem 3.6.17 (b)]. □

A graded analogue of Proposition 1.1.10 also holds:

Proposition 3.6.4. *Let M be a graded R-module. Then*

$$\mathrm{Ann}_R(M) = \mathrm{Ann}_R(M^\vee)$$

Proof. We adapt [BS13, Remarks 10.2.2] to the graded case, and so will first show that $M \hookrightarrow M^{\vee\vee}$.

For any non-zero homogeneous $m \in M$, note that Rm is a graded R-submodule of M by Lemma 3.1.6 (iv), and define

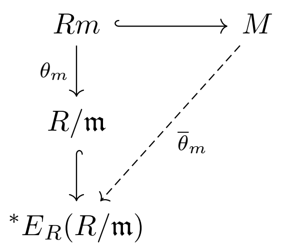
$$\begin{aligned} \theta_m : Rm &\to R/\mathfrak{m} \\ rm &\mapsto r + \mathfrak{m} \end{aligned}$$

We will first show that this is well-defined. Suppose that $rm = sm$ for some $r, s \in R$, and let r_i and s_i denote the i^{th} graded components of r and s respectively. Then, since m is homogeneous, we must have $r_i m = s_i m$ for all $i \geq 0$, and so $(r_i - s_i)m = 0$. Since $m \neq 0$, $r_i - s_i$ cannot be a unit, and so we must have $r_i - s_i \in \mathfrak{m}$ since it is homogeneous. Then $r - s \in \mathfrak{m}$, so θ_m is well-defined as claimed.

Since θ_m has degree $-\deg(m)$, and the natural homogeneous inclusion

$$R/\mathfrak{m} \hookrightarrow {}^*E_R(R/\mathfrak{m})$$

has degree 0, composing them we obtain a homogeneous R-homomorphism from Rm to ${}^*E_R(R/\mathfrak{m})$ of degree $-\deg(m)$. Because ${}^*E_R(R/\mathfrak{m})$ is *injective, we can extend this to a homogeneous R-homomorphism $\overline{\theta}_m$ of degree $-\deg(m)$ from M as below:



Then $\overline{\theta}_m \in M^\vee$, and $\overline{\theta}_m(m) \neq 0$.

Next, define

$$\begin{aligned} \varphi : M &\to M^{\vee\vee} \\ m &\mapsto (\psi \mapsto \psi(m)) \end{aligned}$$

It is routine to check that this is homogeneous of degree 0. Since $\overline{\theta}_m(m) \neq 0$ for any non-zero homogeneous $m \in M$, any homogeneous element of $\ker(\varphi)$ must be 0. But $\ker(\varphi)$ is a graded R-submodule of M by Lemma 3.1.6 (vi) since φ is homogeneous, and so $\ker(\varphi) = 0$ as desired.

Then

$$\operatorname{Ann}_R(M) \subseteq \operatorname{Ann}_R(M^\vee) \subseteq \operatorname{Ann}_R(M^{\vee\vee}) \subseteq \operatorname{Ann}_R(M)$$

and we are done. □

3.7 *Dimension

Next, we define a graded version of (Krull) dimension:

Definition 3.7.1. *We define the ***dimension** of R as*

$$^*\dim(R) := \max\{\operatorname{height}_R(\mathfrak{m}) : \mathfrak{m} \in {}^*\operatorname{MaxSpec}(R)\}$$

or ∞ if R contains homogeneous primes of arbitrarily large height.

For a graded R-module M, we set

$$^*\dim_R(M) := {}^*\dim(R/\operatorname{Ann}_R(M))$$

We do not need to introduce a notion of "*height", and be concerned with whether or not the primes in our chains when calculating these heights are homogeneous, due to the following result:

Proposition 3.7.2. *Let $\mathfrak{p} \in {}^*\operatorname{Spec}(R)$, and set $d = \operatorname{height}_R(\mathfrak{p})$. Then there exists a chain of homogeneous primes*

$$\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_d = \mathfrak{p}$$

Proof. This follows from [BH05, Theorem 1.5.8 (a)]. □

When $(R, \mathfrak{m})$ is *local, the *dimension of finitely generated graded R-modules has a particularly simple expression:

Lemma 3.7.3. *Suppose that $(R, \mathfrak{m})$ is *local, and let M be a finitely generated graded R-module. Then*

$$^*\dim_R(M) = \dim_{R_\mathfrak{m}}(M_\mathfrak{m})$$

In particular, for any homogeneous ideal $\mathfrak{a}$ of R, we have

$$^*\dim(R/\mathfrak{a}) = \dim(R_\mathfrak{m}/\mathfrak{a}R_\mathfrak{m})$$

Proof. Since M is finitely generated, we have

$$\operatorname{Ann}_R(M)_\mathfrak{m} = \operatorname{Ann}_{R_\mathfrak{m}}(M_\mathfrak{m})$$

Let $A = R/\operatorname{Ann}_R(M)$. Then

$$\begin{aligned} ^*\dim_R(M) &= {}^*\dim(A) = \operatorname{height}_A(\mathfrak{m}A) = \dim(R_\mathfrak{m}/\operatorname{Ann}_R(M)_\mathfrak{m}) = \dim(R_\mathfrak{m}/\operatorname{Ann}_{R_\mathfrak{m}}(M_\mathfrak{m})) \\ &= \dim_{R_\mathfrak{m}}(M_\mathfrak{m}) \end{aligned}$$

and the result follows. □

There are many cases where the usual dimension and the *dimension coincide. For example:

Lemma 3.7.4. *Suppose that R is non-negatively graded. Then*

$$\dim(R) = {}^*\dim(R)$$

Proof. This follows from [NO04, Theorem 5.5.1 (1)]. □

Note. *The definition of Krull dimension given in [NO04, Appendix B] is a generalisation of the usual definition in terms of chains of prime ideals to the possibly non-commutative case, however this definition agrees with the usual notion for commutative rings (see [GR73, Corollary 8.14]).*

Even without assuming that the grading is non-negative, these dimensions still cannot differ by much:

Lemma 3.7.5. *Suppose that $^*\dim(R) = d < \infty$. Then*

$$d \leq \dim(R) \leq d + 1$$

Proof. This follows from [NO04, Theorem 5.5.6 (2)]. □

In the *local case, we can say precisely when these dimensions coincide:

Proposition 3.7.6. *Suppose that $(R, \mathfrak{m})$ is *local. Then $^*\dim(R)$ and $\dim(R)$ are finite, with $\dim(R) = {}^*\dim(R)$ if and only if $\mathfrak{m}$ is maximal.*

Proof. Since R is the unique *maximal ideal of R, we have $^*\dim(R) = \dim(R_\mathfrak{m})$ by Lemma 3.7.3. This is finite because $R_\mathfrak{m}$ is local and Noetherian (since R is), so $\dim(R)$ is finite by Lemma 3.7.5, which proves the first claim.

If $\dim(R) = \dim(R_\mathfrak{m})$ then clearly $\mathfrak{m}$ must be maximal, and so it remains only to show the converse.

Then suppose that $\mathfrak{m}$ is maximal. We know that $\dim(R) = \dim(R_\mathfrak{n})$ for some $\mathfrak{n} \in \mathrm{MaxSpec}(R)$. If $\mathfrak{n}$ is homogeneous then $\mathfrak{n} = \mathfrak{m}$ and we are done, so suppose not. We then have

$$\dim(R_\mathfrak{n}) = \dim(R_{\mathfrak{n}^*}) + 1$$

by [BH05, Theorem 1.5.8 (b)].

Now, $\mathfrak{n}^* \subseteq \mathfrak{m}$, because $\mathfrak{n}^*$ is homogeneous and $(R, \mathfrak{m})$ is *local, and $\mathfrak{n}^* \subsetneq \mathfrak{n}$ since $\mathfrak{n}$ is not homogeneous, so we must have $\mathfrak{n}^* \subsetneq \mathfrak{m}$ because $\mathfrak{m}$ is maximal. Then

$$\dim(R) = \dim(R_\mathfrak{n}) = \dim(R_{\mathfrak{n}^*}) + 1 \leq \dim(R_\mathfrak{m}) \leq \dim(R)$$

and we are done. □

In the *local case, we can also characterise *dimension using local cohomology, similarly to Corollary 1.2.22:

Theorem 3.7.7. *Suppose that $(R, \mathfrak{m})$ is *local, and let M be a non-zero finitely generated graded R-module. Then ${}^*\dim_R(M)$ is the greatest integer i such that ${}^*H^i_{\mathfrak{m}}(M) \neq 0$.*

Proof. We have
$$ {}^*H^i_{\mathfrak{m}}(M)_{\mathfrak{m}} \cong H^i_{\mathfrak{m}}(M)_{\mathfrak{m}} \cong H^i_{\mathfrak{m}R_{\mathfrak{m}}}(M_{\mathfrak{m}}) $$
as $R_{\mathfrak{m}}$-modules by Lemma 3.2.4 and Corollary 1.2.11, and so we are done by Corollary 1.2.22, Lemma 3.3.4, and Lemma 3.7.3. □

3.8 *Depth

Throughout this section, we assume that $(R, \mathfrak{m})$ is *local.

Analogously to the local case, we make the following definition:

Definition 3.8.1. *Let M be a graded R-module. Then we define the ***depth** of M as*

$$ {}^*\mathrm{depth}_R(M) := \mathrm{grade}_R(\mathfrak{m}, M) $$

or ∞ if $\mathfrak{m}M = M$ (which, when M is finitely generated, can only happen if $M = 0$ by Graded Nakayama's Lemma).

Note. *Whilst we use the notation ${}^*\mathrm{depth}$ in this thesis for consistency with ${}^*\dim$, many sources simply use $\mathrm{depth}_R(M)$ to denote $\mathrm{grade}_R(\mathfrak{m}, M)$ in the *local case.*

When M is finitely generated, we have the following alternate characterisation of ${}^*\mathrm{depth}$:

Proposition 3.8.2. [BS13, Proposition 1.5.15 (e)] *Let M be a finitely generated graded R-module. Then*
$$ {}^*\mathrm{depth}_R(M) = \mathrm{depth}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}) $$

We cannot always find an M-sequence of length ${}^*\mathrm{depth}_R(M)$ consisting solely of homogeneous elements, even when $M = R$ and R is Cohen-Macaulay (see [HOI88, Remark 35.8] for an example of such a ring). However in many common situations, we can find such a sequence. For example:

Proposition 3.8.3. *Let M be a finitely generated graded R-module, and suppose that $\mathfrak{m}$ can be generated by elements of strictly positive degree. Then there exists an M-sequence of length ${}^*\mathrm{depth}_R(M)$ consisting of homogeneous elements.*

Proof. This follows from [BH05, Proposition 1.5.11]. □

In the general case, $\mathfrak{m}$ does not necessarily contain a regular sequence of maximal length even allowing non-homogeneous elements. For example, take $R = k[x, y]/(x^2, xy)$ for some field k. Then $\mathfrak{m} = \mathrm{Ann}_R(x)$, and so $\mathrm{grade}_R(\mathfrak{m}) = 0$, but $y + 1$ is a regular element of R. However, when $(R, \mathfrak{m})$ is Cohen-Macaulay and $\mathfrak{m}$ is maximal, such a situation cannot arise:

Proposition 3.8.4. *Suppose that R is Cohen-Macaulay, and that $\mathfrak{m}$ is maximal. Then $\mathfrak{m}$ contains a regular sequence of maximal length.*

Proof. Suppose that $\mathfrak{n}$ is a maximal ideal of R which is not homogeneous. Note that $\mathfrak{n}^*$ is prime by Proposition 3.3.2. Furthermore, we have $\mathfrak{n}^* \subseteq \mathfrak{m}$ since $\mathfrak{n}^*$ is homogeneous and $\mathfrak{m}$ is *maximal, and $\mathfrak{n}^* \subseteq \mathfrak{n}$ by definition, so we must have $\mathfrak{n}^* \subsetneq \mathfrak{m}$, since $\mathfrak{m}$ is maximal and $\mathfrak{m} \neq \mathfrak{n}$ since $\mathfrak{m}$ is homogeneous whereas $\mathfrak{n}$ is not.

Since R is Cohen-Macaulay, by [BH05, Corollary 2.1.4] we have

$$\mathrm{grade}_R(\mathfrak{a}) = \mathrm{height}_R(\mathfrak{a})$$

for any ideal $\mathfrak{a}$ of R, and so by Proposition 3.1.5 we have

$$\mathrm{grade}_R(\mathfrak{n}) = \mathrm{grade}_R(\mathfrak{n}^*) + 1 = \mathrm{height}_R(\mathfrak{n}^*) + 1 \leq \mathrm{height}_R(\mathfrak{m}) = \mathrm{grade}_R(\mathfrak{m})$$

which concludes the proof. □

We can also characterise *depth using local cohomology, similarly to Corollary 1.2.23:

Theorem 3.8.5. *Let M be a non-zero finitely generated graded R-module. Then $^*\mathrm{depth}_R(M)$ is the least integer i such that $H^i_{\mathfrak{m}}(M) \neq 0$.*

Proof. We have $\mathfrak{m}M \neq M$ by Graded Nakayama's Lemma, and so the result is immediate from Theorem 1.2.20. □

3.9 *Canonical Modules & Graded Local Duality

> Throughout this section, we assume that $(R, \mathfrak{m})$ is Cohen-Macaulay *local.

The theory of canonical modules also carries over to the graded case:

Definition 3.9.1. *Let $n = {}^*\mathrm{dim}(R)$. We say that a finitely generated graded R-module C is a ***canonical module** of R if there exists a homogeneous R-isomorphism*

$$C^\vee \cong {}^*H^n_{\mathfrak{m}}(R)$$

Note. *This definition differs from that of [BH05, Definition 3.6.8], however the two are reconciled in [BS13, Corollary 14.5.12].*

Similarly to the case of local rings, there are classes of *local rings where *canonical modules are known to exist:

Proposition 3.9.2. *Suppose that R is either *complete, or the graded homomorphic image of a Gorenstein *local ring. Then R has a *canonical module.*

Proof. For the first case, see [BH05, Theorem 3.6.19]. For the second, see [BS13, Example 14.5.2]. □

As might be expected given Lemma 3.2.4, we have the following result:

Proposition 3.9.3. [BH05, Proposition 3.6.9 (a)] *If C is a *canonical module of R, then C is also a canonical module of R.*

In some cases, we also have uniqueness:

Proposition 3.9.4. [BH05, Proposition 3.6.9 (b)] *Suppose that $\mathfrak{m}$ is maximal, and that R has a *canonical module C. Then C is unique up to homogeneous R-isomorphism.*

These modules also characterise *local Gorenstein rings as in the usual local case:

Proposition 3.9.5. *R is Gorenstein if and only if $R(a)$ is a *canonical module of R for some degree shift $a \in \mathbb{Z}$.*

Proof. This follows from [BH05, Proposition 3.6.11] and [BS13, Corollary 14.5.16]. □

Furthermore, they behave well with respect to localisation:

Proposition 3.9.6. [BS13, Proposition 14.5.3] *If C is a *canonical module of R, then $C_{\mathfrak{m}}$ is a canonical module of $R_{\mathfrak{m}}$.*

We can now state a graded version of Local Duality:

Theorem 3.9.7 (Graded Local Duality)**.** [BS13, Theorem 14.5.10] *Suppose that R has a *canonical module C (in particular, when R is also *complete by Proposition 3.9.2), and let $n = {}^*\dim(R)$. Then there is a natural isomorphism of functors*

$$^*H^i_{\mathfrak{m}}(-) \cong {}^*\mathrm{Ext}^{n-i}_R(-, C)^\vee$$

from ***R*-FinMod** *to* ***R*-Mod** *for all $0 \leq i \leq n$ (that is, the isomorphism holds when the argument of $^*H^i_{\mathfrak{m}}$ is finitely generated).*

3.10 *Secondary Representations & *Attached Primes

The theory of primary decompositions of graded modules is well known. For example:

Proposition 3.10.1. [BH05, Lemma 1.5.6 (b)] *Let M be a graded R-module. Then each $\mathfrak{p} \in \mathrm{Ass}_R(M)$ is homogeneous, and is the annihilator of a homogeneous element of M.*

There is an analogous theory of secondary representations of graded modules:

Definition / Proposition 3.10.2. [Sha86, Proposition 2.2] *Let S be a non-zero graded R-module. We say that S is* ****secondary*** *if, for each homogeneous $r \in R$, either $rS = S$ or $r^m S = 0$ for some $m \geq 1$. In this case, $\mathfrak{p} = \sqrt{\mathrm{Ann}_R(S)}$ is a homogeneous prime ideal of R, and we say that S is* ***$\mathfrak{p}$-*secondary****.*

Definition 3.10.3. *Let M be a graded R-module. If there exist $\mathfrak{p}_i$-*secondary submodules S_i in M such that*

$$M = S_1 + \cdots + S_t$$

for some $t \geq 1$, then we call this a ****secondary representation*** *of M, and M is said to be* ****representable****.*

Such a representation is said to be ***minimal*** *when*

i) The $\mathfrak{p}_i$ are all distinct.

ii) For every $1 \leq i \leq t$, we have

$$S_i \nsubseteq \sum_{\substack{j=1 \\ j \neq i}}^{t} S_j$$

Note. *It is easily shown that the sum of any two $\mathfrak{p}$-*secondary modules is $\mathfrak{p}$-*secondary, and so any *secondary representation can be refined to be minimal.*

The following graded version of [Mac73, (1.1)] is easily checked:

Proposition 3.10.4. *Let M be a $\mathfrak{p}$-*secondary R-module for some $\mathfrak{p} \in {}^*\mathrm{Spec}(R)$, and N a proper graded R-submodule of M. Then M/N is also $\mathfrak{p}$-*secondary.*

We next want to define a graded version of attached primes, but this requires showing the uniqueness of the primes associated with a minimal *secondary representation, as is done for standard secondary representations in [Mac73, (2.2)]. To do this, we will prove the following graded analogue of part of this theorem:

Lemma 3.10.5. *Let M be a *representable R-module with minimal *secondary representation*

$$M = S_1 + \cdots + S_t$$

for some *secondary R-submodules S_i of M, and $t \geq 1$.

Then, for any $\mathfrak{p} \in {}^*\mathrm{Spec}(R)$, we have that $\mathfrak{p} = \sqrt{\mathrm{Ann}_R(S_i)}$ for some $1 \leq i \leq t$ if and only if there exists a graded R-module N with $\sqrt{\mathrm{Ann}_R(N)} = \mathfrak{p}$ and a homogeneous R-epimorphism $M \twoheadrightarrow N$.

Proof. We essentially follow the proof of [Mac73, (2.2)], but take gradings into consideration.

First, assume without loss of generality that S_1 is $\mathfrak{p}$-*secondary, and set

$$Q = \sum_{i=2}^{t} S_i$$

This is graded, and it is easily checked that the maps in the various isomorphism theorems are homogeneous of degree 0, so

$$M/Q = (S_1 + Q)/Q \cong S_1/(S_1 \cap Q)$$

as graded R-modules. M/Q is then $\mathfrak{p}$-*secondary by Proposition 3.10.4, and so $\sqrt{\mathrm{Ann}_R(M/Q)} = \mathfrak{p}$. It is easily checked that the natural projection $M \twoheadrightarrow M/Q$ is homogeneous, which completes the proof of the forward direction.

For the converse, suppose that we have a graded R-module N with $\mathrm{Ann}_R(N) = \mathfrak{p}$, and a homogeneous R-epimorphism $\varphi : M \twoheadrightarrow N$. Set $Q = \ker(\varphi)$, and renumber the S_i so that $S_i \nsubseteq Q$ for $1 \leq i \leq l$ and $S_i \subseteq Q$ for $l+1 \leq i \leq t$ (for some integer l) if necessary. Note that $l \geq 1$ since if $S_i \subseteq Q$ for all $1 \leq i \leq t$ then $M = Q$, so $N = 0$ which would contradict that $\sqrt{\mathrm{Ann}_R(N)}$ is prime.

Then we have

$$N \cong M/Q = \sum_{i=1}^{t} (S_i + Q)/Q = \sum_{i=1}^{l} (S_i + Q)/Q$$

and so

$$\mathrm{Ann}_R(N) = \mathrm{Ann}_R\left(\sum_{i=1}^{l} (S_i + Q)/Q\right) = \bigcap_{i=1}^{l} \mathrm{Ann}_R((S_i + Q)/Q)$$

As before, we have

$$(S_i + Q)/Q \cong S_i/(S_i \cap Q)$$

and each $S_i/(S_i \cap Q)$ is $\mathfrak{p}_i$-*secondary by Proposition 3.10.4, so

$$\mathfrak{p} = \sqrt{\mathrm{Ann}_R(N)} = \bigcap_{i=1}^{l} \sqrt{\mathrm{Ann}_R(S_i/(S_i \cap Q))} = \bigcap_{i=1}^{l} \mathfrak{p}_i$$

Then $\mathfrak{p} = \mathfrak{p}_i$ for some $1 \leq i \leq l$ since it is prime, and we are done. □

Lemma 3.10.5 allows us to define the following:

Definition / Theorem 3.10.6. *Let M be a *representable R-module, and S_i the $\mathfrak{p}_i$-*secondary summands in a minimal *secondary representation of M for some $t \geq 1$. Then both t and the set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ are independent of the choice of minimal *secondary representation by Lemma 3.10.5. We call these primes the* ****attached primes*** *of M, denoted ${}^*\mathrm{Att}_R(M)$.*

Since R is Noetherian, we can refine Lemma 3.10.5 in a similar manner to [Mac73, (2.5)]:

Proposition 3.10.7. *Let M be a *representable R-module. Then, for any $\mathfrak{p} \in {}^*\mathrm{Spec}(R)$, we have $\mathfrak{p} \in {}^*\mathrm{Att}_R(M)$ if and only if there exists a graded R-module N with $\mathrm{Ann}_R(N) = \mathfrak{p}$ and a homogeneous R-epimorphism $M \twoheadrightarrow N$.*

Proof. We essentially follow the proof of [Mac73, (2.5)], but take gradings into consideration.

Since $\mathfrak{p}$ is prime we have $\sqrt{\mathfrak{p}} = \mathfrak{p}$, and so the reverse direction follows immediately from Lemma 3.10.5. We proceed then with notation as in the forward direction of Lemma 3.10.5.

If $\mathfrak{p}(M/Q) = 0$ then we are done, since we would then have

$$\mathfrak{p} \subseteq \mathrm{Ann}_R(M/Q) \subseteq \sqrt{\mathrm{Ann}_R(M/Q)} = \mathfrak{p}$$

and the natural projection $M \twoheadrightarrow M/Q$ is homogeneous.

Otherwise, since R is Noetherian, we have $\mathfrak{p}^a \subseteq \mathrm{Ann}_R(M/Q)$ for some $a \geq 1$, and so $\mathfrak{p}^a(M/Q) = 0$. Because we now assume that $M/Q \neq 0$, we must have $M/Q \neq \mathfrak{p}(M/Q)$, so

$$N = (M/Q)/\mathfrak{p}(M/Q)$$

is non-zero and therefore $\mathfrak{p}$-*secondary, again by Proposition 3.10.4. Then $\mathrm{Ann}_R(N) = \mathfrak{p}$ since

$$\mathfrak{p} \subseteq \mathrm{Ann}_R(N) \subseteq \sqrt{\mathrm{Ann}_R(N)} = \mathfrak{p}$$

as in the previous case.

Since $\mathfrak{p}$ is homogeneous and M/Q is graded, $\mathfrak{p}(M/Q)$ is graded also, so N is graded. It is easily checked that the natural projections

$$M \twoheadrightarrow M/Q \twoheadrightarrow N$$

are then homogeneous, which completes the proof. □

The standard existence theorem for secondary representations of Artinian modules also has a graded counterpart:

Theorem 3.10.8. [Sha86, Proposition 2.4] *Let M be a *Artinian R-module. Then M is *representable.*

In some cases (but not all, see Example 3.13.1), the notions of secondary and *secondary coincide. For example:

Lemma 3.10.9. [Ric03, Theorem 1.5] *Suppose that R is non-negatively graded, and let S be an Artinian $\mathfrak{p}$-*secondary R-module. Then S is $\mathfrak{p}$-secondary.*

We then have the following:

Theorem 3.10.10. *Suppose that R is non-negatively graded, and let M be an Artinian graded R-module. Then M has a minimal secondary representation in which each component is graded and each attached prime is homogeneous.*

Proof. This follows from Theorem 3.10.8 and Lemma 3.10.9. □

It is well known that, for a Noetherian R-module M, we have $\mathfrak{p} \in \mathrm{Ass}_R(M)$ if and only if there exists an R-submodule N of M such that $\mathrm{Ann}_R(N) = \mathfrak{p}$. We will next prove a graded analogue of this result:

Proposition 3.10.11. *Let M be a *Noetherian R-module. Then $\mathfrak{p} \in \mathrm{Ass}_R(M)$ if and only if there exists some graded R-submodule N of M such that $\mathrm{Ann}_R(N) = \mathfrak{p}$.*

Proof. If $\mathfrak{p} \in \mathrm{Ass}_R(M)$ then $\mathfrak{p} = \mathrm{Ann}_R(m)$ for some homogeneous $m \in M$ by Proposition 3.10.1, and $\mathfrak{p}$ is homogeneous. Then $R/\mathfrak{p}$ is graded, and the R-monomorphism $R/\mathfrak{p} \hookrightarrow M$ given by $1 + \mathfrak{p} \mapsto m$ is homogeneous (possibly after some degree shift), so the forward direction follows.

For the converse, note that since M is *Noetherian, it is also Noetherian by Lemma 3.5.2. Then N is also Noetherian, and so we can write

$$N = a_1 R + \cdots + a_t R$$

for some $a_1, \ldots, a_t \in N$. We then have

$$\mathfrak{p} = \mathrm{Ann}_R(N) = \mathrm{Ann}_R(a_1 R + \cdots + a_t R) = \bigcap_{i=1}^{t} \mathrm{Ann}_R(a_i) \supseteq \prod_{i=1}^{t} \mathrm{Ann}_R(a_i)$$

so $\mathrm{Ann}_R(a_i) \subseteq \mathfrak{p}$ for some $1 \leq i \leq t$ since $\mathfrak{p}$ is prime. Then

$$\mathfrak{p} = \mathrm{Ann}_R(N) \subseteq \mathrm{Ann}_R(a_i) \subseteq \mathfrak{p}$$

and so $\mathrm{Ann}_R(a_i) = \mathfrak{p}$. We then have $R/\mathfrak{p} \hookrightarrow M$ via $1 + \mathfrak{p} \mapsto a_i$, so $\mathfrak{p} \in \mathrm{Ass}_R(M)$ as desired. □

With these results in hand, we can now describe the effect of Graded Matlis Duality on attached and associated primes in a sufficiently nice setting, in a similar way to Lemma 1.3.10 and Lemma 1.3.12 (although here we assume *completeness for convenience):

Theorem 3.10.12. *Suppose that $(R, \mathfrak{m})$ is *complete *local, and let M be a *Noetherian R-module. Then*

$$^*\mathrm{Att}_R(M^\vee) = \mathrm{Ass}_R(M)$$

Proof. We essentially follow the argument of [BS13, Remark 10.2.14], but take gradings into consideration.

First, suppose that $\mathfrak{p} \in \mathrm{Ass}_R(M)$. This is homogeneous by Proposition 3.10.1, so we have an inclusion of graded R-modules $R/\mathfrak{p} \hookrightarrow M$. Then applying $-^\vee$, we obtain a homogeneous

R-epimorphism $M^\vee \twoheadrightarrow (R/\mathfrak{p})^\vee$. By Proposition 3.6.4, we have

$$\operatorname{Ann}_R((R/\mathfrak{p})^\vee) = \operatorname{Ann}_R(R/\mathfrak{p}) = \mathfrak{p}$$

and so $\mathfrak{p} \in {}^*\operatorname{Att}_R(M^\vee)$ by Proposition 3.10.7.

Conversely, if $\mathfrak{p} \in {}^*\operatorname{Att}_R(M^\vee)$, then by Proposition 3.10.7 there exists a graded R-module N with $\operatorname{Ann}_R(N) = \mathfrak{p}$ and a homogeneous R-epimorphism $M^\vee \twoheadrightarrow N$. Applying $-^\vee$ yields an inclusion

$$N^\vee \hookrightarrow M^{\vee\vee} \cong M$$

of graded R-modules by Graded Matlis Duality (iv). Again,

$$\operatorname{Ann}_R(N^\vee) = \operatorname{Ann}_R(N) = \mathfrak{p}$$

by Proposition 3.6.4. Since M is *Noetherian, $N^\vee$ is also, so $\mathfrak{p} \in \operatorname{Ass}_R(M)$ by Proposition 3.10.11 and we are done. □

Corollary 3.10.13. *Suppose that $(R, \mathfrak{m})$ is *complete *local, and let M be a *Artinian R-module. Then*

$$\operatorname{Ass}_R(M^\vee) = {}^*\operatorname{Att}_R(M)$$

Proof. This follows immediately from applying Theorem 3.10.12 to $M^\vee$, noting that $M^{\vee\vee} \cong M$ by Graded Matlis Duality (iv). □

We can use this to prove a graded analogue of Theorem 1.3.8. Note that Macdonald and Sharp do not require $(R, \mathfrak{m})$ to be Cohen-Macaulay or complete in [MS72, Theorem 2.2], however we suppose that $(R, \mathfrak{m})$ is Cohen-Macaulay and *complete for convenience, since this will suffice for the purposes of this thesis.

Theorem 3.10.14. *Suppose that $(R, \mathfrak{m})$ is Cohen-Macaulay *complete *local of *dimension n, and let M be a non-zero finitely generated graded R-module of *dimension d. Then*

$${}^*\operatorname{Att}_R(H^d_{\mathfrak{m}}(M)) = \{\mathfrak{p} \in \operatorname{Ass}_R(M) : {}^*\dim(R/\mathfrak{p}) = d\}$$

Proof. By Proposition 3.9.2, there exists a *canonical module C of R, and by Proposition 3.9.6 we have that $C_{\mathfrak{m}}$ is a canonical module of $R_{\mathfrak{m}}$.

Now, by the usual local version of Local Duality, we have

$$H^d_{\mathfrak{m}R_{\mathfrak{m}}}(M_{\mathfrak{m}}) \cong \operatorname{Ext}^{n-d}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, C_{\mathfrak{m}})^\vee$$

as $R_{\mathfrak{m}}$-modules, and so

$$\operatorname{Att}_{R_{\mathfrak{m}}}(H^d_{\mathfrak{m}R_{\mathfrak{m}}}(M_{\mathfrak{m}})) = \operatorname{Ass}_{R_{\mathfrak{m}}}(\operatorname{Ext}^{n-d}_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, C_{\mathfrak{m}}))$$

by [BS13, Corollary 10.2.20].

Since M is finitely generated, $\mathrm{Ext}^i_R(M,-)$ commutes with localisation for all $i \geq 0$. Then by Lemma 3.2.2, Lemma 3.2.4, Graded Local Duality, Theorem 3.10.12, and Lemma 3.7.3 we have

$$\begin{aligned}
\mathrm{Att}_{R_\mathfrak{m}}(H^d_{\mathfrak{m}R_\mathfrak{m}}(M_\mathfrak{m})) &= \mathrm{Ass}_{R_\mathfrak{m}}(\mathrm{Ext}^{n-d}_{R_\mathfrak{m}}(M_\mathfrak{m}, C_\mathfrak{m})) \\
&= \mathrm{Ass}_{R_\mathfrak{m}}(\mathrm{Ext}^{n-d}_R(M, C)_\mathfrak{m}) \\
&= \{\mathfrak{p}R_\mathfrak{m} : \mathfrak{p} \in \mathrm{Ass}_R(\mathrm{Ext}^{n-d}_R(M, C)) \text{ with } \mathfrak{p} \subseteq \mathfrak{m}\} \\
&= \{\mathfrak{p}R_\mathfrak{m} : \mathfrak{p} \in \mathrm{Ass}_R(\mathrm{Ext}^{n-d}_R(M, C))\} \\
&= \{\mathfrak{p}R_\mathfrak{m} : \mathfrak{p} \in {}^*\mathrm{Att}_R(H^d_\mathfrak{m}(M))\}
\end{aligned}$$

with the penultimate equality following since every associated prime of $\mathrm{Ext}^{n-d}_R(M, C)$ is homogeneous by Proposition 3.10.1.

We can also use the usual local version of [MS72, Theorem 2.2], along with Lemma 3.7.3, to obtain

$$\begin{aligned}
\mathrm{Att}_{R_\mathfrak{m}}(H^d_{\mathfrak{m}R_\mathfrak{m}}(M_\mathfrak{m})) &= \{\mathfrak{p}R_\mathfrak{m} \in \mathrm{Ass}_{R_\mathfrak{m}}(M_\mathfrak{m}) : \dim(R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m}) = d\} \\
&= \{\mathfrak{p}R_\mathfrak{m} \in \mathrm{Ass}_{R_\mathfrak{m}}(M_\mathfrak{m}) : {}^*\dim(R/\mathfrak{p}) = d\} \\
&= \{\mathfrak{p}R_\mathfrak{m} : \mathfrak{p} \in \mathrm{Ass}_R(M) \text{ with } \mathfrak{p} \subseteq \mathfrak{m} \text{ and } {}^*\dim(R/\mathfrak{p}) = d\} \\
&= \{\mathfrak{p}R_\mathfrak{m} : \mathfrak{p} \in \mathrm{Ass}_R(M) \text{ with } {}^*\dim(R/\mathfrak{p}) = d\}
\end{aligned}$$

where again we have applied Proposition 3.10.1 since M is graded, and so we are done. □

3.11 Some Properties of Cohen-Macaulay Graded Rings & Modules

Throughout this section, we assume that $(R, \mathfrak{m})$ is *local.

It is well known (for example, see [BH05, Theorem 2.1.2 (b)]) that, for a Cohen-Macaulay local ring $(R, \mathfrak{m})$ and ideal $\mathfrak{a}$ of R, we have

$$\mathrm{grade}_R(\mathfrak{a}) = \dim(R) - \dim(R/\mathfrak{a})$$

We will next prove a graded version of this result:

Lemma 3.11.1. *Suppose that R is Cohen-Macaulay and non-negatively graded. Then, for any homogeneous ideal $\mathfrak{a}$ of R, we have*

$$\mathrm{grade}_R(\mathfrak{a}) = {}^*\dim(R) - {}^*\dim(R/\mathfrak{a})$$

In particular, when R is non-negatively graded, we have

$$\mathrm{grade}_R(\mathfrak{a}) = \dim(R) - \dim(R/\mathfrak{a})$$

Regardless of the grading, we also have

$$\text{grade}_R(\mathfrak{a}) = \text{height}_R(\mathfrak{a})$$

Proof. We have

$$\text{grade}_R(\mathfrak{a}) \overset{(1)}{=} \text{grade}_{R_\mathfrak{m}}(\mathfrak{a}R_\mathfrak{m}) \overset{(2)}{=} \dim(R_\mathfrak{m}) - \dim(R_\mathfrak{m}/\mathfrak{a}R_\mathfrak{m}) \overset{(3)}{=} {}^*\dim(R) - {}^*\dim(R/\mathfrak{a})$$

where (1) follows from [BH05, Proposition 1.5.15 (e)], (2) follows from [BH05, Theorem 2.1.2] since R, and therefore $R_\mathfrak{m}$, is Cohen-Macaulay, and (3) follows from Lemma 3.7.3. This proves the first claim.

When R is non-negatively graded, $R/\mathfrak{a}$ is also, and so by Lemma 3.7.4 we have $\dim(R) = {}^*\dim(R)$ and $\dim(R/\mathfrak{a}) = {}^*\dim(R/\mathfrak{a})$, which proves the second claim.

The last claim follows from [BH05, Corollary 2.1.4]. □

It is also well known (for example, see [Mat86, Theorem 17.3 (i)] and [BH05, Corollary 2.14]) that, over a Cohen-Macaulay local ring, all associated primes of a finitely generated Cohen-Macaulay module over that ring are of the same height. This is also true in the *local case:

Lemma 3.11.2. *Let M be a finitely generated Cohen-Macaulay graded R-module, and take any $\mathfrak{p} \in \text{Ass}_R(M)$. Then ${}^*\dim(R/\mathfrak{p}) = {}^*\dim_R(M)$. Furthermore, if R is also Cohen-Macaulay, then we have*

$$\text{height}_R(\mathfrak{p}) = {}^*\dim(R) - {}^*\dim_R(M)$$

In particular, when R is non-negatively graded, we have

$$\text{height}_R(\mathfrak{p}) = \dim(R) - \dim_R(M)$$

Proof. Every associated prime of M is homogeneous by Proposition 3.10.1, and so must be contained in $\mathfrak{m}$. Then

$$\text{Ass}_{R_\mathfrak{m}}(M_\mathfrak{m}) = \{\mathfrak{p}R_\mathfrak{m} : \mathfrak{p} \in \text{Ass}_R(M) \text{ with } \mathfrak{p} \subseteq \mathfrak{m}\} = \{\mathfrak{p}R_\mathfrak{m} : \mathfrak{p} \in \text{Ass}_R(M)\}$$

Since M is a Cohen Macaulay R-module, $M_\mathfrak{m}$ is a Cohen-Macaulay $R_\mathfrak{m}$-module, and so $\dim(R_\mathfrak{m}/\mathfrak{p}R_\mathfrak{m}) = \dim_{R_\mathfrak{m}}(M_\mathfrak{m})$ by [Mat86, Theorem 17.3 (i)]. Then ${}^*\dim(R/\mathfrak{p}) = {}^*\dim_R(M)$ by Lemma 3.7.3. The last two claims then follow from Lemma 3.11.1 and Lemma 3.7.4. □

Before proving the final result of this chapter, we state an important lemma characterising when a *local ring is Cohen-Macaulay:

Lemma 3.11.3. [BH05, Exercise 2.1.27 (c)] *Let M be a finitely generated graded R-module. Then M is Cohen-Macaulay as an R-module if and only if $M_\mathfrak{m}$ is Cohen-Macaulay as an $R_\mathfrak{m}$-module.*

We can now prove a graded version of Corollary 1.2.24:

Lemma 3.11.4. *Let M be a non-zero finitely generated graded R-module. Then M is Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(M) \neq 0$ for $i = {}^*\dim_R(M)$ and vanishes otherwise.*

Proof. By Lemma 3.11.3, M is Cohen-Macaulay if and only if $M_{\mathfrak{m}}$ is Cohen-Macaulay as an $R_{\mathfrak{m}}$-module, which by the usual result on local rings holds if and only if $H^i_{\mathfrak{m}R_{\mathfrak{m}}}(M_{\mathfrak{m}}) \neq 0$ for $i = \dim_{R_{\mathfrak{m}}}(M_{\mathfrak{m}})$ and vanishes otherwise. Then since $\dim_{R_{\mathfrak{m}}}(M_{\mathfrak{m}}) = {}^*\dim_R(M)$ by Lemma 3.7.3, we are done by Lemma 3.3.4, since taking local cohomology commutes with localisation by Corollary 1.2.11. □

3.12 Summary of Graded Results

We now summarise the results we have collected in this chapter. We will assume that $(R, \mathfrak{m})$ is Gorenstein *complete *local, non-negatively graded of dimension n, and so all of the conditions on R for every result in this chapter are satisfied.

In particular, this is the case when $R = k[x_1, \dots, x_n]$ for some field k.

Throughout, M will denote a non-zero Noetherian graded R-module of dimension d and *depth t, N a non-zero Artinian graded R-module, and $\mathfrak{a}$ a homogeneous ideal of R.

Then:

1. d is the greatest integer i such that $H^i_{\mathfrak{m}}(M) \neq 0$.
   (Lemma 3.2.4, Theorem 3.7.7, and Lemma 3.7.4)
2. t is the least integer i such that $H^i_{\mathfrak{m}}(M) \neq 0$.
   (Lemma 3.2.4 and Theorem 3.8.5)
3. M is Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(M) \neq 0$ for $i = d$ and vanishes otherwise, and in this case all associated primes of M are of height $n - d$.
   (Lemma 3.11.4 and Lemma 3.11.2)
4. $H^i_{\mathfrak{m}}(M)$ is an Artinian graded R-module for all $i \geq 0$.
   (Lemma 3.2.4 and Corollary 3.5.10)
5. $M^\vee$ is Artinian and $N^\vee$ is Noetherian.
   (Graded Matlis Duality (ii), Graded Matlis Duality (iii), Proposition 3.5.8, and Lemma 3.5.2)
6. $M^{\vee\vee} \cong M$ and $N^{\vee\vee} \cong N$ as graded R-modules.
   (Graded Matlis Duality (iv))
7. N has a minimal secondary representation in which every component is graded and every attached prime is homogeneous.
   (Theorem 3.10.10)
8. We have
   $$H^i_{\mathfrak{m}}(M) \cong \operatorname{Ext}^{n-i}_R(M, R)^\vee$$
   as graded R-modules for all $0 \leq i \leq n$.
   (Lemma 3.2.4, Graded Local Duality, and Proposition 3.9.5)
9. We have
   $$\operatorname{Att}_R(N) = \operatorname{Ass}_R(N^\vee)$$

and

$$\mathrm{Ass}_R(M) = \mathrm{Att}_R(M^\vee)$$

(Theorem 3.10.12 and Corollary 3.10.13)

10. We have

$$\mathrm{Att}_R(H^d_{\mathfrak{m}}(M)) = \{\mathfrak{p} \in \mathrm{Ass}_R(M) : \dim(R/\mathfrak{p}) = d\}$$

(Lemma 3.2.4, Corollary 3.5.10, Lemma 3.10.9, Theorem 3.10.14, and Lemma 3.7.4)

11. We have that

$$\mathrm{height}_R(\mathfrak{a}) = \mathrm{grade}_R(\mathfrak{a}) = n - \dim(R/\mathfrak{a})$$

is the length of the longest homogeneous regular sequence contained in $\mathfrak{a}$.
(Lemma 3.11.1 and Proposition 3.8.3)

With these results established, we can apply many of the tools established in Chapter 1 to our coming work on binomial edge ideals.

3.13 A Counterexample & Further References

To see that the conditions we impose on R are, in many cases, necessary, we consider the following example:

Example 3.13.1. Let $R = k[t, t^{-1}]$ for some field k (the ring of Laurent polynomials over k). This is not non-negatively graded, but ${}^*\mathrm{Spec}(R) = \{0\}$, so it is *local, however clearly $0 \notin \mathrm{MaxSpec}(R)$. This causes many of the preceding results to fail. For example:

1. R is *Artinian as a graded R-module, but it is clearly not Artinian, consider for example

$$(1+t) \supsetneq (1+t)^2 \supsetneq \cdots$$

2. Viewing $(R, \mathfrak{m})$ as a *local ring with $\mathfrak{m} = (0)$, we have that $H^0_{\mathfrak{m}}(M) = M$ for any $M \in$ ***R*-Mod**, and so $H^0_{\mathfrak{m}}(M)$ is not *Artinian unless M itself is.

3. As noted in [Sha86, p. 215], R is *secondary as a graded R-module since homogeneous elements of R are of the form at^l for some $a \in k$ and $l \in \mathbb{Z}$, but not secondary since multiplication by $1 + t$ is neither surjective nor nilpotent.

However, some of the results we have presented here are far from the most general. Many more results concerning $\mathbb{Z}$-graded rings and modules, as well as $\mathbb{Z}$-graded local cohomology and Local Duality, can be found in [BH05, Sections 1.5 & 3.5]. [BS13, Chapters 13 & 14] extends many of these results to the case when R is graded by $\mathbb{Z}^t$ for $t \geq 1$. [NO04] contains many general results on rings and modules graded by any group, and [Nor68, Sections 2.11, 2.12 and 2.13] presents results on rings and modules graded by certain monoids.

Chapter 4

LF-Covers and Binomial Edge Ideals of König Type

Throughout this chapter, we set

$$R = k[x_1, \dots, x_n, y_1, \dots, y_n]$$

for some field k and $n \geq 1$, and $\delta_{i,j} := x_i y_j - x_j y_i$.

In this chapter, we give a combinatorial characterisation of connected graphs whose binomial edge ideals are of König type, developed independently to the similar characterisation given in [LaC23, Lemma 5.3], and exhibit some classes of graphs satisfying our criteria.

For any connected Hamiltonian graph G on n vertices, we also compute an explicit root of $H^{n-1}_{\mathcal{J}(G)}(R)$ as an F-finite F-module.

The majority of this chapter first appeared in [Wil23a].

4.1 The Main Theorem

4.1.1 Statement & Preliminaries

The notion of ideals of König type was introduced by Herzog, Hibi, and Moradi in [HHM21] as a generalisation of König graphs, for which the matching number is equal to the vertex cover number (see [HHM21, Section 1] for the details). There are several ways to define these ideals, but for binomial edge ideals there is a particularly elegant characterisation. We first introduce some terminology and notation:

Notation 4.1.1. *We say that a graph is a* ***linear forest*** *if every connected component is a path. For a graph G, we say that a linear forest in G is* ***maximal*** *if no other linear forest in G has a greater number of edges. We denote by $\mathsf{LF}(G)$ the number of edges of such a linear forest.*

Note. *In [HHM21] and [LaC23], the term **semi-path** is used in place of linear forest, however linear forest appears the more commonly used graph theoretic term, and so we adopt it here.*

Definition 4.1.2. [HHM21, Theorem 3.5] *Let G be a graph on n vertices. We say that $\mathcal{J}(G)$ is of **König type** if and only if*

$$\mathrm{grade}_R(\mathcal{J}(G)) = \mathsf{LF}(G)$$

The left-hand side of this equality is stated in [HHM21] as $2n - \dim(R/\mathcal{J}(G))$, but this is the same as our statement by Lemma 3.11.1, since we then have

$$2n - \dim(R/\mathcal{J}(G)) = \dim(R) - \dim(R/\mathcal{J}(G)) = \mathrm{grade}_R(\mathcal{J}(G))$$

We now introduce the notion of LF-covers:

Definition 4.1.3. *Let G be a graph. If there exists a maximal linear forest F in G, and a set $S \subseteq V(F)$ such that:*

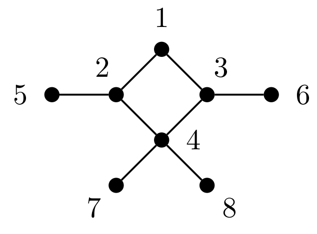
1. *No vertex in S is a leaf of F.*
2. *No two vertices in S are adjacent in F.*
3. *For every edge $\{i,j\} \in E(G)$, one of the following holds:*
   i) *At least one of i or j belongs to S. In this case, we say that this vertex **covers** $\{i,j\}$.*
   ii) *Both i and j belong to the same connected component of $F \setminus S$.*

*then we say that (F,S) is an **LF-cover** of G, and that G is **LF-coverable**.*

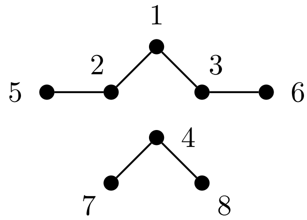
This notion is similar to that introduced in [LaC23, Lemma 5.3], with our final criterion expressed differently, however the techniques we use are very different to those of [LaC23].

To illustrate this concept, we consider several examples:

Example 4.1.4. Let

$G =$ 

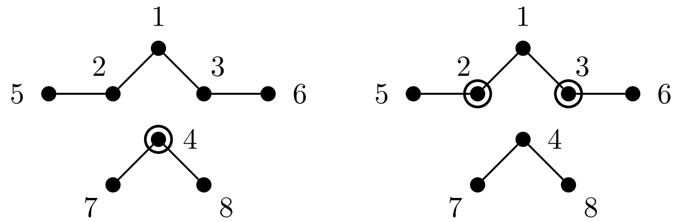
Then G has a unique maximal linear forest

$F =$ 

and LF-covers (F, S) for

$$S \in \{\{4\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}\}$$

In particular, the minimal S with respect to containment (here circled) are

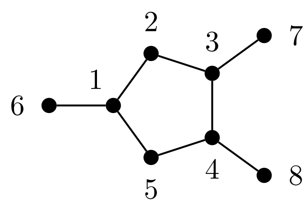


and so such minimal S need not contain the same number of vertices.

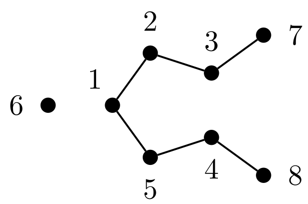
To see that these are LF-covers, note that $\{2, 4\}$ and $\{3, 4\}$ are the only edges of $G \setminus F$. In the first example, both $\{2, 4\}$ and $\{3, 4\}$ are covered by 4, whilst in the second example, $\{2, 4\}$ is covered by 2, and $\{3, 4\}$ is covered by 3. In both examples, no two vertices in S are adjacent in F, and no vertex in S is a leaf of F.

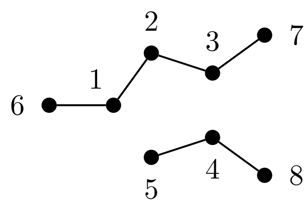
Example 4.1.5. $(P_n, \varnothing)$ is an LF-cover of C_n for any $n \geq 3$, because both endpoints of the only remaining edge $\{1, n\}$ in $C_n \setminus P_n$ belong to the same connected component of $P_n \setminus \varnothing$ (this example is also trivially true in the cases that $n = 1$ or $n = 2$, since then $C_n = P_n$).

Example 4.1.6. Let

$G =$ 

It is easily checked that, up to isomorphism, the maximal linear forests in G are

$F_1 =$ 

$F_2 =$ 

Note that (F_1, S_1) and (F_2, S_2) are LF covers of G if and only if

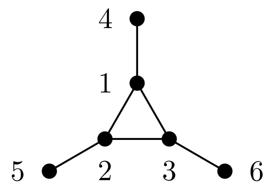
$$S_1, S_2 \in \{\{1,3\}, \{1,4\}, \{1,3,4\}\}$$

It is not a coincidence that these collections agree: we will see in Theorem 4.1.9 that, for any graph G, (F, S) being an LF-cover of G for some $S \subseteq V(G)$ is independent of the choice of maximal linear forest F.

Unfortunately, not all graphs are LF-coverable (we will see more such examples in Section 4.3):

Example 4.1.7. Let

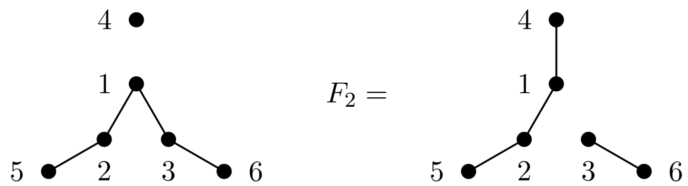
$G =$



This is sometimes called the **net** or **3-sunlet**.

Up to isomorphism, the maximal linear forests in G are

$F_1 =$ $F_2 =$



Suppose that we had an LF-cover (F_1, S_1) of G. The only way to cover $\{1,4\}$ would be to include 1 in S_1, since we cannot include 4 in S_1 because it is a leaf of F_1. However 2 and 3 would then belong to separate connected components of $F_1 \setminus S_1$, and so we would need to include either 2 or 3 in S_1 to cover $\{2,3\}$. But each of these are adjacent to 1 in F_1, and so cannot be added to S_1.

Now suppose that we had an LF-cover (F_2, S_2) of G. In order to cover $\{1,3\}$ and $\{2,3\}$, we would need to include both 1 and 2 in S_2, since we cannot add 3 to S_2 because it is a leaf of F_2. However 1 and 2 are adjacent in F_2, and so cannot both belong to S_2.

Note. *By the second claim of Theorem 4.1.9, to show that a graph G is not* LF*-coverable, it suffices to check a single maximal linear forest in G.*

We can characterise LF-covers algebraically:

Lemma 4.1.8. *Let G be a graph on n vertices, F a maximal linear forest in G, and $S \subseteq V(F)$. Then (F, S) is an* LF*-cover of G if and only if $S \in \mathcal{C}(F)$ and $\mathcal{J}(G) \subseteq P_S(F)$.*

Proof. Criterion 1 and Criterion 2 of Definition 4.1.3 are equivalent to saying that $S \in \mathcal{C}(F)$ since F is a linear forest. We have that $\mathcal{J}(G) \subseteq P_S(F)$ if and only if $\delta_{i,j} \in P_S(F)$ for every edge $\{i, j\} \in E(G)$, which is clearly equivalent to Criterion 3 in Definition 4.1.3 by the definition of $P_S(F)$, and the result follows. □

Our aim is to prove the following:

Theorem 4.1.9. *Let G be a graph on n vertices. Then $\mathcal{J}(G)$ is of König type if and only if G is* LF*-coverable.*

Furthermore, (F, S) being an LF*-cover of G for some $S \subseteq V(G)$ is independent of the choice of maximal linear forest F.*

The proof will be given in Subsection 4.1.3.

We begin by showing part of what is proved in [HHM21, Lemma 3.3], although we do so avoiding the use of initial ideals:

Proposition 4.1.10. *Let G be a graph on n vertices, and let $e_1, \ldots, e_t$ denote the elements of $\mathcal{J}(G)$ associated to the edges of G. Then the e_i form a regular sequence if and only if G is a linear forest.*

Proof. Note first that being a linear forest is equivalent to not containing any cycle C_m for $3 \leq m \leq n$ or $K_{1,3}$ (the star with 3 edges, sometimes called the claw). Since R is graded and the e_i are homogeneous of positive degree, if they form a regular sequence then any permutation of this sequence will remain a regular sequence (see, for example, [Mat86, Theorem 16.3]), and so to show necessity it is enough to show that $\delta_{1,2}, \delta_{2,3}, \ldots, \delta_{m-1,m}, \delta_{1,m}$ and $\delta_{1,2}, \delta_{1,3}, \delta_{1,4}$ do not form regular sequences (the second condition need not be checked when $n \leq 3$).

Firstly, we have

$$\begin{aligned}(\mathcal{J}(P_m) : \delta_{1,m}) &= \left(\bigcap_{S \in \mathcal{C}(P_m)} P_S(P_m) \right) : \delta_{1,m} = \bigcap_{S \in \mathcal{C}(P_m)} (P_S(P_m) : \delta_{1,m}) = \bigcap_{\substack{S \in \mathcal{C}(P_m) \\ \delta_{1,m} \notin P_S(P_m)}} P_S(P_m) \\ &= \bigcap_{\substack{S \in \mathcal{C}(P_m) \\ S \neq \varnothing}} P_S(P_m) \supsetneq \mathcal{J}(P_m)\end{aligned}$$

since the $P_S(P_m)$ are prime, and so $\delta_{1,2}, \delta_{2,3}, \ldots, \delta_{m-1,m}, \delta_{1,m}$ cannot be a regular sequence.

We similarly have that

$$(\mathcal{J}(K_{1,3}) : \delta_{1,4}) = ((x_1, y_1) \cap \mathcal{J}(K_3)) : \delta_{1,4} = ((x_1, y_1) : \delta_{1,4}) \cap (\mathcal{J}(K_3) : \delta_{1,4}) = \mathcal{J}(K_3) \supsetneq \mathcal{J}(K_{1,3})$$

and so $\delta_{1,2}, \delta_{1,3}, \delta_{1,4}$ cannot be a regular sequence either. This proves necessity.

For sufficiency, we will first prove by induction that $\delta_{1,2}, \delta_{2,3}, \ldots, \delta_{m-1,m}$ is a regular sequence for $m \geq 2$.

The case $m = 2$ is trivial since R is a domain.

We now proceed by induction, and so assume that $\delta_{1,2}, \delta_{2,3}, \ldots, \delta_{m-2,m-1}$ is a regular sequence.

As before, we have that

$$(\mathcal{J}(P_{m-1}) : \delta_{m-1,m}) = \bigcap_{\substack{S \in \mathcal{C}(P_{m-1}) \\ \delta_{m-1,m} \notin P_S(P_{m-1})}} P_S(P_{m-1}) = \bigcap_{S \in \mathcal{C}(P_{m-1})} P_S(P_{m-1}) = \mathcal{J}(P_{m-1})$$

since clearly $\delta_{m-1,m} \notin P_S(P_{m-1})$ for any $S \in \mathcal{C}(P_{m-1})$, and so $\delta_{1,2}, \delta_{2,3}, \ldots, \delta_{m-1,m}$ forms a regular sequence as desired.

Joining regular sequences from multiple disjoint paths will still result in a regular sequence, since they have no variables in common. Then we have shown sufficiency, and so we are done. □

4.1.2 LF-Covers & Local Cohomology

Throughout this subsection, we assume that k is of prime characteristic $p > 0$.

The next stage of our proof of Theorem 4.1.9 will make use of prime characteristic techniques. Our aim is to relate LF-covers to the associated primes of a certain local cohomology module.

The crux of our proof of the main theorem of this subsection is as follows:

Lemma 4.1.11. *Let G be a graph on n vertices, and set $\mathfrak{g} = \mathcal{J}(G)$. Furthermore, let F be a maximal linear forest in G, and set $\mathfrak{f} = \mathcal{J}(F)$. Then $(\mathfrak{f} : \mathfrak{g})/\mathfrak{f}$ is a root of $H^d_{\mathfrak{g}}(R)$, where $d = |E(F)|$.*

Proof. Let $e_1, \ldots, e_d$ be the elements of $\mathfrak{f}$ corresponding to the edges of F, which generate $\mathfrak{f}$. By Proposition 4.1.10, the e_i form a regular sequence. In particular, they form a $\mathfrak{g}$-filter regular sequence since $\mathfrak{f} \subseteq \mathfrak{g}$. Then, by the Nagel-Schenzel Isomorphism and Proposition 2.1.33, we have

$$\begin{aligned} H^d_{\mathfrak{g}}(R) \cong \Gamma_{\mathfrak{g}}(H^d_{\mathfrak{f}}(R)) &\cong \Gamma_{\mathfrak{g}}\left(\varinjlim \left[R/\mathfrak{f} \xhookrightarrow{u\cdot} R/\mathfrak{f}^{[p]} \xhookrightarrow{u^p\cdot} R/\mathfrak{f}^{[p^2]} \xhookrightarrow{u^{p^2}\cdot} \cdots\right]\right) \\ &\cong \varinjlim \left[\Gamma_{\mathfrak{g}}(R/\mathfrak{f}) \xhookrightarrow{u\cdot} \Gamma_{\mathfrak{g}}(R/\mathfrak{f}^{[p]}) \xhookrightarrow{u^p\cdot} \Gamma_{\mathfrak{g}}(R/\mathfrak{f}^{[p^2]}) \xhookrightarrow{u^{p^2}\cdot} \cdots\right] \\ &\cong \varinjlim \left[(\mathfrak{f} : \mathfrak{g}^\infty)/\mathfrak{f} \xhookrightarrow{u\cdot} (\mathfrak{f}^{[p]} : \mathfrak{g}^\infty)/\mathfrak{f}^{[p]} \xhookrightarrow{u^p\cdot} (\mathfrak{f}^{[p^2]} : \mathfrak{g}^\infty)/\mathfrak{f}^{[p^2]} \xhookrightarrow{u^{p^2}\cdot} \cdots\right] \end{aligned}$$

where $u = (e_1 \cdots e_d)^{p-1}$.

The injectivity of the first map follows from Lemma 2.1.34 since the e_i form a regular sequence, and the injectivity of the subsequent maps is a result of the exactness of the Frobenius functor (which is due to Kunz's Theorem since R is regular) and the left exactness of $\Gamma_{\mathfrak{g}}$.

Since $\mathfrak{f} = \mathcal{J}(F)$ is radical by Theorem 1.4.2, we have $(\mathfrak{f} : \mathfrak{g}^\infty) = (\mathfrak{f} : \mathfrak{g})$, and so we are done. □

The following lemma is not difficult (and holds over any ring), but we include a proof for completeness:

Lemma 4.1.12. *Let $\mathfrak{a}$ be a radical ideal of R with minimal primary decomposition*

$$\mathfrak{a} = \bigcap_{i=1}^{t} \mathfrak{p}_i$$

for some $\mathfrak{p}_i \in \operatorname{Spec}(R)$ and $t \geq 1$. Furthermore, let

$$\mathfrak{b} = \bigcap_{i=1}^{m} \mathfrak{p}_i$$

for some $1 \leq m < t$. Then

$$\operatorname{Ass}_R(\mathfrak{b}/\mathfrak{a}) = \{\mathfrak{p}_i : m+1 \leq i \leq t\}$$

Proof. Let $b \in \mathfrak{b}$, and suppose that $rb \in \mathfrak{a}$ for some $r \in R$. Then $rb \in \mathfrak{p}_i$ for all $1 \leq i \leq t$. Since the $\mathfrak{p}_i$ are prime, we must have that $r \in \mathfrak{p}_i$ for each $m+1 \leq i \leq t$ such that $b \notin \mathfrak{p}_i$. Then

$$\operatorname{Ann}_R(b+\mathfrak{a}) \subseteq \bigcap_{\substack{i=m+1 \\ b \notin \mathfrak{p}_i}}^{t} \mathfrak{p}_i$$

The converse clearly holds, and this ideal is only prime when $b \notin \mathfrak{p}_i$ for exactly one $m+1 \leq i \leq t$, so

$$\operatorname{Ass}_R(\mathfrak{b}/\mathfrak{a}) \subseteq \{\mathfrak{p}_i : m+1 \leq i \leq t\}$$

Since the primary decomposition of $\mathfrak{a}$ is minimal, no $\mathfrak{p}_i$ is redundant. Then we can find

$$b_j \in \bigcap_{\substack{i=1 \\ i \neq j}}^{t} \mathfrak{p}_i \setminus \mathfrak{p}_j$$

for each $m+1 \leq j \leq t$, and the result follows. □

We can now prove the main result of this subsection:

Theorem 4.1.13. *Let G be a graph on n vertices, and set $\mathfrak{g} = \mathcal{J}(G)$. Furthermore, let F be a maximal linear forest in G with $|E(F)| = d$, and set $\mathfrak{f} = \mathcal{J}(F)$. Then*

$$\operatorname{Ass}_R(H^d_{\mathfrak{g}}(R)) = \{P_S(F) : S \in \mathcal{C}(G) \textit{ such that } (F,S) \textit{ is an } \mathsf{LF}\textit{-cover of } G\}$$

Proof. We have that

$$(\mathfrak{f} : \mathfrak{g}) = \left(\bigcap_{S \in \mathcal{C}(F)} P_S(F) \right) : \mathfrak{g} = \bigcap_{S \in \mathcal{C}(F)} (P_S(F) : \mathfrak{g}) = \bigcap_{\substack{S \in \mathcal{C}(F) \\ \mathfrak{g} \not\subseteq P_S(F)}} P_S(F)$$

since each $P_S(F)$ is prime. Since $(\mathfrak{f} : \mathfrak{g})/\mathfrak{f}$ is a root of $H^d_{\mathfrak{g}}(R)$ by Lemma 4.1.11, their associated primes are equal by Lemma 2.1.31. By Lemma 4.1.12 and Lemma 4.1.8, these primes are exactly as claimed. □

Note. *At the time of writing, we do not know if Theorem 4.1.13 holds in characteristic* 0*.*

4.1.3 Proof of the Main Theorem

We now allow k to be of any characteristic again.

Whilst we have only proven Theorem 4.1.13 in prime characteristic, this will allow us to prove Theorem 4.1.9 in arbitrary characteristic:

Proposition 4.1.14. *Let G be a graph on n vertices, set $\mathfrak{g} = \mathcal{J}(G)$, and let F be a maximal linear forest in G. Then if G has an* LF*-cover (F, S) for some $S \subseteq V(G)$, $\mathfrak{g}$ is of König type.*

Proof. Let $d = |E(F)|$. Suppose first that k has prime characteristic $p > 0$. Since (F, S) is an LF-cover of G, then by Theorem 4.1.13 we know that $\operatorname{Ass}_R(H^d_{\mathfrak{g}}(R))$ is non-empty. In particular, we have that $H^d_{\mathfrak{g}}(R) \neq 0$, and so $\operatorname{grade}_R(\mathfrak{g}) \leq d$ by Theorem 1.2.20. But, by Lemma 1.4.8 and Lemma 3.11.1, $\operatorname{grade}_R(\mathfrak{g})$ is independent of the characteristic of k, and so this inequality holds in any case.

Conversely, the elements of $\mathfrak{g}$ corresponding to the edges of F form a regular sequence by Proposition 4.1.10, so $\operatorname{grade}_R(\mathfrak{g}) \geq d$ and we are done. □

Proposition 4.1.15. *Let G be a graph on n vertices, set $\mathfrak{g} = \mathcal{J}(G)$, and suppose that $\mathfrak{g}$ is of König type. Then G is* LF*-coverable.*

Proof. Let F be a maximal linear forest in G, and set $\mathfrak{f} = \mathcal{J}(F)$. Since $\mathfrak{g}$ is of König type, and $\mathfrak{f}$ is trivially of König type, we have

$$\operatorname{grade}_R(\mathfrak{g}) = \mathsf{LF}(G) = \mathsf{LF}(F) = \operatorname{grade}_R(\mathfrak{f})$$

and so

$$\dim(R/\mathfrak{g}) = \dim(R) - \operatorname{grade}_R(\mathfrak{g}) = \dim(R) - \operatorname{grade}_R(\mathfrak{f}) = \dim(R/\mathfrak{f})$$

by Lemma 3.11.1.

Let $d = \dim(R/\mathfrak{g})$. Then we can find $\mathfrak{p}_0, \ldots, \mathfrak{p}_d \in \operatorname{Spec}(R)$ such that

$$\mathfrak{f} \subseteq \mathfrak{g} \subseteq \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_d$$

Now, $\mathfrak{p}_0$ must be minimal over $\mathfrak{f}$, since otherwise we would have $\dim(R/\mathfrak{f}) > d$, and so $\mathfrak{p}_0 \in \operatorname{Ass}_R(\mathfrak{f})$. Then $\mathfrak{p}_0 = P_S(F)$ for some $S \subseteq V(F)$, and we are done by Lemma 4.1.8. □

Proof of Theorem 4.1.9. The first claim follows immediately from Proposition 4.1.14 and Proposition 4.1.15.

We will now prove the second claim, so suppose that (F, S) is an LF-cover of G, and take another maximal linear forest F' in G. When k has prime characteristic $p > 0$, we have

$$P_S(F) \in \mathrm{Ass}_R(H^d_{\mathfrak{g}}(R))$$

by Theorem 4.1.13, where $d = \mathsf{LF}(G)$ and $\mathfrak{g} = \mathcal{J}(G)$. This local cohomology module does not depend on the choice of maximal linear forest, and so, again by Theorem 4.1.13, we must have $P_S(F) = P_{S'}(F')$ for some $S' \subseteq V(F')$ such that (F', S') is an LF-cover of G. By definition, $x_i, y_i \in P_S(F)$ if and only if $i \in S$, and similarly $x_i, y_i \in P_{S'}(F')$ if and only if $i \in S'$. Then since $P_S(F) = P_{S'}(F')$, we must have $S = S'$, and so (F', S) is an LF-cover of G. Since LF-covers can be characterised purely combinatorially, and are therefore independent of characteristic, the result follows. □

4.2 Some Classes of LF-Coverable Graphs

We will next show that traceable, complete bipartite and trivially perfect graphs are LF-coverable. We will show the same for trees, by way of constructing an algorithm for calculating an explicit LF-cover.

We also conjecture that several well-known classes of graphs are LF-coverable, with computational evidence to support this.

Definition 4.2.1. *If a graph G on n vertices contains P_n, we say that G is **traceable**. If it contains C_n, we say it that is **Hamiltonian**.*

We recover, phrased differently, a similar result to one direction of [Oht11, Proposition 5.2]:

Proposition 4.2.2. *Traceable graphs are* LF*-coverable.*

Proof. Take F to be a path connecting all the vertices of the graph, and set $S = \varnothing$. □

Note. *It is shown in [Pós76] that almost all graphs are Hamiltonian. In particular, they are traceable, and so almost all binomial edge ideals have grade $n - 1$.*

Proposition 4.2.3. *Complete bipartite graphs $K_{a,b}$ are* LF*-coverable for all $a, b \geq 1$.*

Proof. We may assume without loss of generality that $a \geq b$. Let $v_1, \ldots, v_a$ and $w_1, \ldots, w_b$ be the vertices in each partition.

If $a = b$ then take

$$F = \{\{v_1, w_1\}, \{w_1, v_2\}, \{v_2, w_2\}, \ldots, \{v_a, w_b\}\}$$

and $S = \varnothing$.

If $a > b$ then take

$$F = \{\{v_1, w_1\}, \{w_1, v_2\}, \{v_2, w_2\}, \ldots, \{v_a, w_b\}, \{w_b, v_{a+1}\}\}$$

and $S = \{w_1, \ldots, w_b\}$. □

Definition 4.2.4. *We say that a graph G is **trivially perfect** if it can be constructed inductively, starting from a single vertex K_1, via either the disjoint union of smaller trivially perfect graphs, or the addition of a vertex adjacent to all other vertices of a smaller trivially perfect graph (that is, taking the join $*$ of a smaller trivially perfect graph with K_1).*

Proposition 4.2.5. *Trivially perfect graphs are* LF*-coverable.*

Proof. If a graph G_1 has an LF-cover (F_1, S_1), and a graph G_2 has an LF-cover (F_2, S_2), then $G_1 \sqcup G_2$ clearly has an LF-cover given by $F = F_1 \sqcup F_2$ and $S = S_1 \cup S_2$. Then we may assume that G is connected.

The graph K_1 trivially has an LF-cover, and so we may proceed by induction on $|V(G)|$.

We can obtain any connected trivially perfect graph G with more than one vertex by starting with a smaller trivially perfect graph G', then taking $G = G' * K_1$. We have an LF-cover (F', S') of G' by the inductive hypothesis.

If F' consists of a single connected component then it must be a path, so G' is traceable. Clearly the join of any traceable graph with a single point remains traceable, and so we are done by Proposition 4.2.2.

If F' consists of more than one connected component, then certainly two leaves from separate components, say l_1 and l_2, become joined in G. If we say $V(K_1) = \{v\}$, then we may extend F' by joining these two connected components along $\{l_1, v\}$ and $\{v, l_2\}$ to create a new linear forest F. Since l_1 and l_2 are leaves in F, we may set $S = S' \cup \{v\}$. This will cover all the new edges added by v in G, and so (F, S) is an LF-cover of G. □

There are several related classes of graphs which we conjecture are also LF-coverable:

Definition 4.2.6. *We say that a graph G is a **cograph** if it contains no induced P_4.*

Conjecture 4.2.7. *Cographs are* LF*-coverable.*

We have verified this conjecture using `Macaulay2` for all cographs with up to 13 vertices.

Trivially perfect graphs can be alternatively characterised as those with no induced P_4 or C_4 (see [JJC96, Theorem 3]), and so form a subclass of cographs. Then Conjecture 4.2.7 implies Proposition 4.2.5.

Definition 4.2.8. *Let $\rho = (\sigma_1, \ldots, \sigma_m)$ be a permutation of the integers $1, \ldots, m$. Then we define the* ***permutation graph*** *G associated to ρ by setting $V(G) = \{1, \ldots, m\}$ and*

$$E(G) = \{\{v, w\} : \sigma_w \text{ appears before } \sigma_v \text{ in } \rho, 1 \leq v < w \leq m\}$$

Conjecture 4.2.9. *Permutation graphs are* LF*-coverable.*

We have verified this conjecture using `Macaulay2` for all permutation graphs with up to 11 vertices.

Since permutation graphs are a superclass of cographs (see [BBL98, pp.280–281]), Conjecture 4.2.9 implies Conjecture 4.2.7.

Definition 4.2.10. *Let $I_1, \ldots, I_m$ be a collection of intervals on the real line. Then we define the* ***interval graph*** *G associated to this collection by setting $V(G) = \{1, \ldots, m\}$ and*

$$E(G) = \{\{v, w\} : I_v \cap I_w \neq \varnothing, 1 \leq v < w \leq m\}$$

Conjecture 4.2.11. *Interval graphs are* LF*-coverable.*

We have verified this conjecture using `Macaulay2` for all interval graphs with up to 10 vertices.

Interestingly, trivially perfect graphs are exactly the graphs which are both cographs and interval graphs (again, see [JJC96, Theorem 3]), and so Conjecture 4.2.11 implies Proposition 4.2.5 also.

Before proving that trees are LF-coverable, we first introduce a definition and some notation:

Definition 4.2.12. *Let T be a tree and b and branch point of T. We say that b is* ***extremal*** *if it is adjacent to at most one other branch point.*

Note. *Definition 4.2.12 is not established terminology.*

Notation 4.2.13. *For a tree T and vertex $v \in V(T)$, we denote by $L_T(v)$ the set of vertices of T which are leaves adjacent to v, and set $L_T[v] \coloneqq L_T(v) \cup \{v\}$.*

Proposition 4.2.14. *Let T be a tree on n vertices, where $n \geq 3$. Then T has an extremal branch point.*

Proof. The proof will be by induction on n. The case $n = 3$ is clear, so suppose that $n > 3$. Remove a leaf l and apply the inductive hypothesis to $T \setminus \{l\}$, so $T \setminus \{l\}$ has an extremal branch point b. When l is reconnected, if it is adjacent to a leaf l' of b then l' is now an extremal branch point. Otherwise b will remain an extremal branch point, and so we are done. □

We can now prove that trees are LF-coverable, and exhibit an algorithm for efficiently computing an LF-cover of any tree T. A distinct method is given in [LaC23, Algorithm 5.8] for computing a linear forest and set satisfying their criteria.

These algorithms are of independent interest, since they effectively calculate the dimension of $\mathcal{J}(T)$ by Lemma 3.11.1. For example, an algorithm to compute this dimension is given as a special case of [MR20, Theorem 2.2].

Theorem 4.2.15. *Let T be a tree on n vertices, where $n \geq 2$. Then T is* LF*-coverable, and we can inductively construct a maximal linear forest F in T and a set of vertices $S \subseteq V(F)$ such that (F, S) is an* LF*-cover of T.*

Proof. The proof will be by induction on n. The case $n = 2$ is trivial: take $F = T$ and $S = \varnothing$.

Then suppose that $n > 2$, and that the statement of the theorem holds for all trees with fewer than n vertices.

We may choose an extremal branch point b of T by Proposition 4.2.14, and consider two cases:

1. If b has a single leaf l, then b must be adjacent to another branch point b', since T cannot be a star because $n > 2$. Let $T' = T \setminus \{l\}$. By the inductive hypothesis, T' has an LF-cover (F', S').

   Let $F = F' \cup \{\{b, l\}\}$, and set $S = S'$. We claim that (F, S) is an LF-cover of T. The three conditions of Definition 4.1.3 are clearly satisfied, so it remains only to show that F is a maximal linear forest in G. Since $\{b, b'\}$ must satisfy the conditions of Definition 4.1.3 for (F', S'), we must either have that $\{b, b'\} \in E(F')$, in which case F is a linear forest since we will just be extending the path containing this edge, or $\{b, b'\} \notin E(F')$ and $b' \in S'$, in which case $\{b, l\}$ will be its own connected component in F, and so F is a linear forest in this case also. Maximality is clear, since T has one more edge than T', and F has one more edge than F' (which is maximal in T').

2. Now suppose that b has at least two leaves l_1 and l_2, and set $T' = T \setminus L_T[b]$. Again by the inductive hypothesis, T' has an LF-cover (F', S'). Note that if T is a star, T' will be the null graph K_0, which trivially has LF-cover $(\varnothing, \varnothing)$.

   Let
   $$F = F' \cup \{\{b, l_1\}, \{b, l_2\}\}$$
   and set $S = S' \cup \{b\}$. In this case, $\{\{b, l_1\}, \{b, l_2\}\}$ will be its own connected component in F, since we removed $L_T[b]$ from T when obtaining F', and so F is a linear forest. It is maximal in T since we are adding a star (centred at v) to T', and so we can add at most two edges to F' before introducing a claw (that is, $K_{1,3}$). Furthermore, every other leaf edge of b, and the edge $\{b, b'\}$ for some branch point $b' \in E(T)$ which will exist if T is not a star, will be covered by $b \in S$, and so (F, S) is an LF-cover of T. This concludes the proof. □

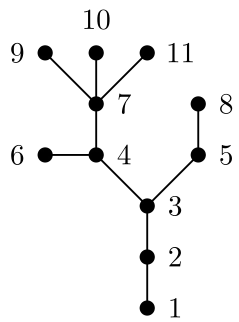
The resulting algorithm is very simple:

Algorithm 4.2.16.
Input: A tree T.
Output: An LF-cover (F, S) of T.

$F \leftarrow K_0$
$S \leftarrow \varnothing$
while $T \neq K_0$ **do**
  if T has a single edge e **then**
    $F \leftarrow F \cup \{e\}$
    $T \leftarrow K_0$
  else
    $b \leftarrow$ some extremal branch point of T
    if b has a single leaf l **then**
      $F \leftarrow F \cup \{\{b, l\}\}$
      $T \leftarrow T \setminus \{l\}$
    else
      $l_1 \leftarrow$ some leaf of b
      $l_2 \leftarrow$ some leaf of b other than l_1
      $F \leftarrow F \cup \{\{b, l_1\}, \{b, l_2\}\}$
      $S \leftarrow S \cup \{b\}$
      $T \leftarrow T \setminus L_T[b]$
    end
  end
end
return (F, S)

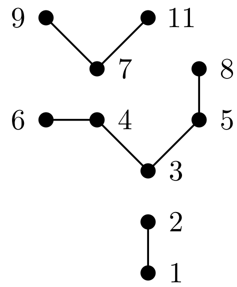
We will now illustrate the operation of Algorithm 4.2.16:

Example 4.2.17. Let

$T =$ 

We will compute F and S inductively, showing that we can obtain

$F =$



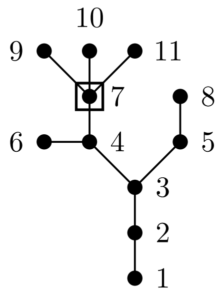
and $S = \{3, 7\}$ by following the proof of Theorem 4.2.15.

Note. *F and S may depend on choices of extremal branch points and leaves in Algorithm 4.2.16. In this example there are many alternative choices, as a somewhat trivially different* LF*-cover we could remove say* $\{7, 9\}$ *and add* $\{7, 10\}$ *in* $E(F)$ *whilst keeping S the same.*

We will explain in detail the first two steps of Algorithm 4.2.16 in this example, before presenting every step in a table.

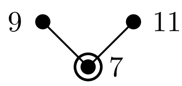
We begin with the full tree T, and select an extremal branch point. Here we choose $b = 7$ (as indicated by a square):

$T =$



This puts us in Case 2 of the proof of Theorem 4.2.15. With notation as in that case, we choose $l_1 = 9$ and $l_2 = 11$. Then the edges $\{7, 9\}$ and $\{7, 11\}$ will be added to the linear forest F, and 7 will be added to S (as indicated by a circle):

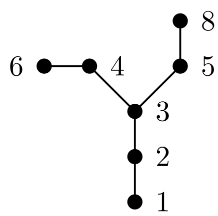
$F =$



Note that 7 then covers $\{4, 7\}$ and $\{7, 10\}$.

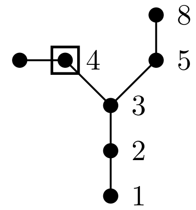
We then remove 7 and its leaves from T. This leaves us with

$T =$

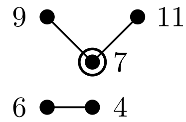


for which we must continue calculating an LF-cover.

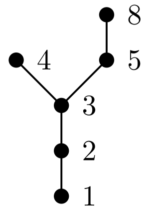
Then we next select the extremal branch point $b = 4$:

$T =$ 

We are then in Case 1 of the proof of Theorem 4.2.15. With notation as in that case, we then have $l = 6$, so $\{4, 6\}$ will be added to the linear forest F:

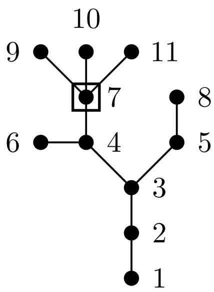
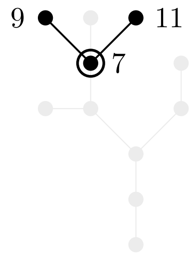
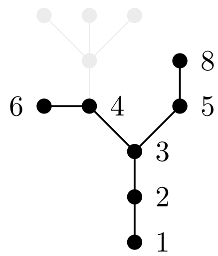
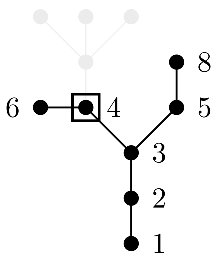
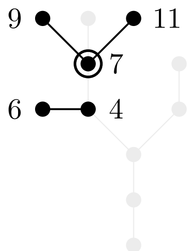
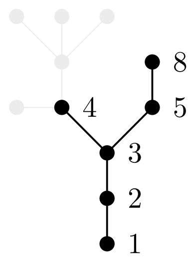
$F =$ 

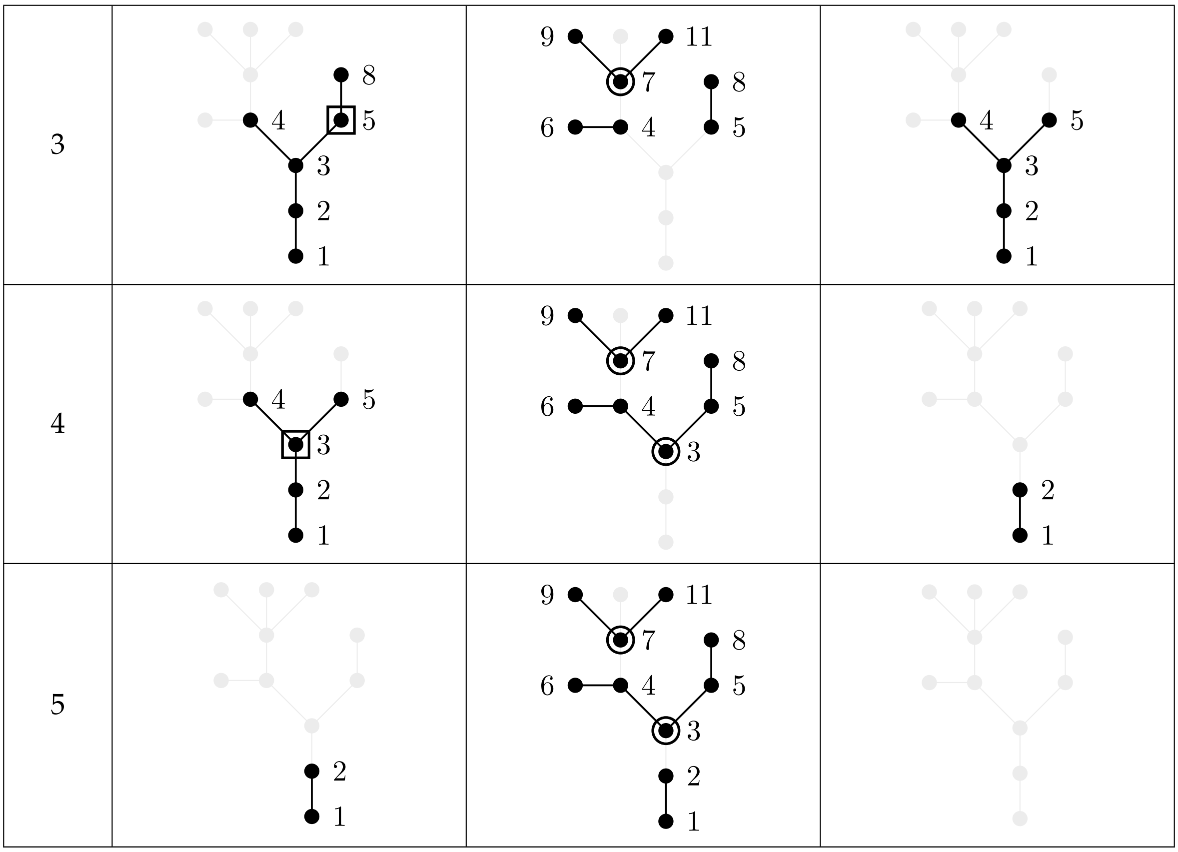
We then remove 6 from T. This leaves us with

$T =$ 

for which we must continue calculating an LF-cover.

The full execution of Algorithm 4.2.16 for this example is as follows (the light grey vertices and edges are simply placeholders to allow us to visualise each step in the context of the original tree):

Step	**Current T**	**Resulting F**	**Remaining T**
1			
2			

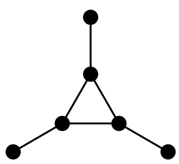


Then we have calculated F and S as claimed, and so by Theorem 4.1.9 we have

$$\text{grade}_R(\mathcal{J}(T)) = |E(F)| = 7$$

4.3 Some Graphs Without LF-Covers

Example 4.3.1. For $n = 6$, we saw directly in Example 4.1.7 that

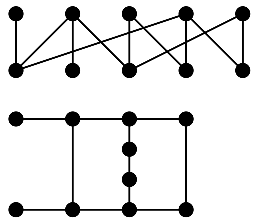
$G =$ 

is not LF-coverable. It can be checked using `Macaulay2` that $\text{grade}_R(\mathcal{J}(G)) = 5$, but $\mathsf{LF}(G) = 4$. This example is minimal in the sense that all graphs with 6 vertices or fewer are LF-coverable other than this one.

Being bipartite is also not enough to ensure LF-coverability:

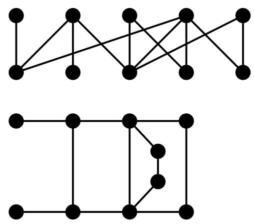
Example 4.3.2. For $n = 10$ and

$G =$ / $=$



it can be checked using `Macaulay2` that $\mathrm{grade}_R(\mathcal{J}(G)) = 9$, but $\mathsf{LF}(G) = 8$. This example is minimal in the sense that all bipartite graphs with fewer than 10 vertices are LF-coverable, and there is only one other bipartite graph on 10 vertices which is not LF-coverable:

$G =$ / $=$

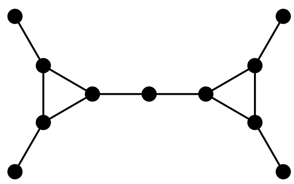


which also has $\mathrm{grade}_R(\mathcal{J}(G)) = 9$ and $\mathsf{LF}(G) = 8$, however this contains an additional edge, and there are no other bipartite examples with 11 edges or fewer.

In all of the examples we have seen so far, $\mathrm{grade}_R(\mathcal{J}(G))$ and $\mathsf{LF}(G)$ differ by at most 1. However this is not necessarily the case:

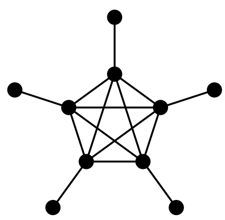
Example 4.3.3. For $n = 11$ and

$G =$



it can be checked using `Macaulay2` that $\mathrm{grade}_R(\mathcal{J}(G)) = 10$, but $\mathsf{LF}(G) = 8$. This example is minimal in the sense that there are no other such examples (where $\mathrm{grade}_R(\mathcal{J}(G))$ and $\mathsf{LF}(G)$ differ by more than 1) with 12 edges or fewer. There is however a single such example with 10 vertices or fewer, given by

$G =$



which has $\mathrm{grade}_R(\mathcal{J}(G)) = 9$ and $\mathsf{LF}(G) = 7$ for $n = 10$.

4.4 Calculating Explicit Roots

Lemma 4.1.11 suggests that we may be able to explicitly calculate roots of $H^d_{\mathcal{J}(G)}(R)$ when a graph G on n vertices has an LF-cover with d edges. In fact, using the Nagel-Schenzel Isomorphism, we could calculate roots of any local cohomology module of the binomial edge ideal of any graph if we could find a $\mathcal{J}(G)$-filter regular sequence of sufficient length.

However, in practice, proving that a sequence of elements is $\mathcal{J}(G)$-filter regular turns out to be quite difficult. Even in the case of LF-coverable graphs with covers with d edges, explicitly calculating roots of $H^d_{\mathcal{J}(G)}(R)$ is not easy.

We perform this calculation here for Hamiltonian graphs. We will first do this for the complete graph K_n, and then extend the result to all Hamiltonian graphs.

We now again assume that k is of prime characteristic $p > 0$, and that $n \geq 3$.

Furthermore, we define an $\mathbb{N}^n$-grading on R by setting

$$\deg(x_i) = \deg(y_i) = e_i$$

where e_i is the n-tuple with 0 everywhere except for 1 at the ith position.

Let $G = K_n$, and set $\mathfrak{g} = \mathcal{J}(G)$. We have that $(P_n, \varnothing)$ is an LF-cover of K_n by Proposition 4.2.2 since K_n is traceable, and we set $\mathfrak{a} = \mathcal{J}(P_n)$. We may assume that

$$E(P_n) = \{\{1,2\}, \ldots, \{n-1, n\}\}$$

By Lemma 4.1.11, we can obtain an explicit root of $H^{n-1}_{\mathfrak{g}}(R)$ by calculating $(\mathfrak{a} : \mathfrak{g})$.

Notation 4.4.1. *For $0 \leq l \leq n-2$, set*

$$z_l := x_2 \cdots x_{n-1-l} y_{n-l} \cdots y_{n-1}$$

Here l should be thought of as the number of ys in this expression.

For example, when $n = 5$ we have

$$\begin{aligned} z_0 &= x_2x_3x_4 \\ z_1 &= x_2x_3y_4 \\ z_2 &= x_2y_3y_4 \\ z_3 &= y_2y_3y_4 \end{aligned}$$

We aim to prove the following:

Theorem 4.4.2. *Let*

$$\mathfrak{b} = \mathfrak{a} + (z_0, \ldots, z_{n-2})$$

Then $(\mathfrak{a} : \mathfrak{g}) = \mathfrak{b}$*, and so* $\mathfrak{b}/\mathfrak{a}$ *is a root of* $H^{n-1}_{\mathfrak{g}}(R)$*.*

To show that $\mathfrak{b} \subseteq (\mathfrak{a} : \mathfrak{g})$, we must prove that, for any $0 \leq l \leq n-2$ and $1 \leq i < j \leq n$, we have $z_l \delta_{i,j} \in \mathfrak{a}$.

We will do this by proving a slightly stronger statement.

Notation 4.4.3. *For any* $1 \leq i < j \leq n$ *and* $0 \leq l \leq j-i-1$*, set*

$$z_l^{i,j} := x_{i+1} \cdots x_{j-1-l} y_{j-l} \cdots y_{j-1}$$

with $z_l^{i,i+1} = 1$*.*

These should be thought of as truncated versions of the z_l, with all indices lying strictly between i and j, but still l terms involving y appearing, so $z_l = z_l^{1,n}$.

For example, keeping $n = 5$ we have

$$\begin{aligned} z_0^{1,4} &= x_2 x_3 \\ z_1^{1,4} &= x_2 y_3 \\ z_2^{1,4} &= y_2 y_3 \end{aligned}$$

Proposition 4.4.4. *For any* $1 \leq i < j \leq n$ *and* $0 \leq l \leq j-i-1$*, we have* $z_l^{i,j}\delta_{i,j} \in \mathfrak{a}$*.*

Proof. The proof will be by induction on $j - i$.

If $j - i = 1$, then $\delta_{i,j} = \delta_{i,i+1} \in \mathfrak{a}$ since $\mathfrak{a} = \mathcal{J}(P_n)$, and so $z\delta_{i,j}$ belongs to $\mathfrak{a}$ for any $z \in R$.

Now suppose that $j - i \geq 2$ and that the result holds for all smaller such differences.

If $l = 0$ then $x_{i+1} \mid z_l^{i,j}$, and so

$$\begin{aligned} z_l^{i,j}\delta_{i,j} &= z_l^{i,j}(x_i y_j - x_j y_i) \\ &= z_l^{i,j} x_i y_j - z_l^{i+1,j} x_i x_j y_{i+1} + z_l^{i+1,j} x_i x_j y_{i+1} - z_l^{i,j} x_j y_i \\ &= x_i z_l^{i+1,j}(x_{i+1} y_j - x_j y_{i+1}) + x_j z_l^{i+1,j}(x_i y_{i+1} - x_{i+1} y_i) \\ &= x_i (z_l^{i+1,j}\delta_{i+1,j}) + (x_j z_l^{i+1,j})\delta_{i,i+1} \end{aligned}$$

We have $z_l^{i+1,j}\delta_{i+1,j} \in \mathfrak{a}$ by the inductive hypothesis, and $\delta_{i,i+1} \in \mathfrak{a}$ since $\mathfrak{a} = \mathcal{J}(P_n)$, so $z_l^{i,j}\delta_{i,j} \in \mathfrak{a}$.

Similarly, if $l \neq 0$ then $y_{j-1} \mid z_l^{i,j}$, and so

$$\begin{aligned} z_l^{i,j}\delta_{i,j} &= z_l^{i,j}(x_i y_j - x_j y_i) \\ &= z_l^{i,j} x_i y_j - z_{l-1}^{i,j-1} x_{j-1} y_i y_j + z_{l-1}^{i,j-1} x_{j-1} y_i y_j - z_l^{i,j} x_j y_i \\ &= y_j z_{l-1}^{i,j-1}(x_i y_{j-1} - x_{j-1} y_i) + y_i z_{l-1}^{i,j-1}(x_{j-1} y_j - x_j y_{j-1}) \\ &= y_j (z_{l-1}^{i,j-1}\delta_{i,j-1}) + (y_i z_{l-1}^{i,j-1})\delta_{j-1,j} \end{aligned}$$

We have $z_{l-1}^{i,j-1}\delta_{i,j-1} \in \mathfrak{a}$ by the inductive hypothesis, and $\delta_{j-1,j} \in \mathfrak{a}$ since $\mathfrak{a} = \mathcal{J}(P_n)$, so $z_l^{i,j}\delta_{i,j} \in \mathfrak{a}$.

Then, in either case, we have $z_l^{i,j}\delta_{i,j} \in \mathfrak{a}$ as desired. □

Corollary 4.4.5. *We have* $\mathfrak{b} \subseteq (\mathfrak{a} : \mathfrak{g})$.

Proof. This follows immediately from Proposition 4.4.4, since for any $1 \leq i < j \leq n$ and $0 \leq l \leq n-2$, we have that $z_{l'}^{i,j} \mid z_l$ for some $0 \leq l' \leq n-2$, l' is simply the number of occurrences of "y"s in z_l with indices strictly between i and j. □

We will now show the reverse inclusion, for which we need a few preparatory results:

Lemma 4.4.6. *For any* $2 \leq i \leq n-1$*, we have*

$$(\mathfrak{a} : \delta_{i-1,i+1}) = (x_i, y_i) + \mathfrak{a}$$

Proof. We have

$$\begin{aligned} (\mathfrak{a} : \delta_{i-1,i+1}) &= \bigcap_{S \in \mathcal{C}(P_n)} (P_S(P_n) : \delta_{i-1,i+1}) = \bigcap_{\substack{S \in \mathcal{C}(P_n) \\ \delta_{i-1,i+1} \notin P_S(P_n)}} P_S(P_n) = \bigcap_{\substack{S \in \mathcal{C}(P_n) \\ i \in S \\ i-1, i+1 \notin S}} P_S(P_n) \\ &= (x_i, y_i) + \mathcal{J}(P_{\{1,\ldots,i-1\}}) + \mathcal{J}(P_{\{i+1,\ldots,n\}}) = (x_i, y_i) + \mathfrak{a} \end{aligned}$$

since the $P_S(P_n)$ are prime.

Here $P_{\{v_1,\ldots,v_t\}}$ denotes the graph on $v_1, \ldots, v_t$ with edges $\{v_1, v_2\}, \ldots, \{v_{t-1}, v_t\}$, that is, the path on $v_1, \ldots, v_t$. □

Proposition 4.4.7. *Every homogeneous element of* R *with degree* $(0, 1, \ldots, 1, 0)$ *belongs to* $\mathfrak{b}$.

Proof. To see this, simply note that $x_i y_{i+1} = x_{i+1} y_i$ modulo $\mathfrak{a}$ for every $1 \leq i \leq n-1$, and so we may "swap" adjacent terms of xs and ys in our z_l whilst staying within $\mathfrak{b}$. □

Corollary 4.4.8. *We have* $(\mathfrak{a} : \mathfrak{g}) \subseteq \mathfrak{b}$.

Proof. Take any $c \in (\mathfrak{a} : \mathfrak{g})$. Since $\mathfrak{a}$ and $\mathfrak{g}$ are homogeneous we have that $(\mathfrak{a} : \mathfrak{g})$ is homogeneous also, and so we may assume that c is homogeneous. Because $c \in (\mathfrak{a} : \mathfrak{g})$, we have in particular that $c\delta_{i-1,i+1} \in \mathfrak{a}$ for every $2 \leq i \leq n-1$ since $\delta_{i-1,i+1} \in \mathfrak{g}$.

In Lemma 4.4.6 we saw that

$$(\mathfrak{a} : \delta_{i-1,i+1}) = (x_i, y_i) + \mathfrak{a}$$

and so we can write $c = f_2 + a_2$ for some $f_2 \in (x_2, y_2)$ and $a_2 \in \mathfrak{a}$. Note that $f_2 = c - a_2 \in (\mathfrak{a} : \mathfrak{g})$ also, so we can write $f_2 = f_3 + a_3$ for some $f_3 \in (x_3, y_3)$ and $a_3 \in \mathfrak{a}$. Continuing in this way, we arrive at

$$c = f_{n-1} + (a_2 + \cdots + a_{n-1})$$

where either $x_i \mid f_{n-1}$ or $y_i \mid f_{n-1}$ for each $2 \leq i \leq n-1$. This means that each term in f_{n-1} is divisible by a homogeneous element of R of degree $(0, 1, \ldots, 1, 0)$, so $f_{n-1} \in \mathfrak{b}$ by Proposition 4.4.7. Then $c \in \mathfrak{b}$ also, and we are done. □

This concludes the proof of Theorem 4.4.2 for $G = K_n$. We will now extend this result.

We now allow G to be any Hamiltonian graph on n vertices, otherwise our notation is as before. Since G is Hamiltonian it contains C_n, and we may assume that

$$E(C_n) = \{\{1,2\}, \ldots, \{n-1, n\}, \{1, n\}\}$$

Since K_n is Hamiltonian, to prove Theorem 4.4.2 for all Hamiltonian graphs, it will suffice to show that $(\mathfrak{a} : \mathfrak{g})$ is independent of the choice of Hamiltonian graph:

Proposition 4.4.9. *We have* $(\mathfrak{a} : \mathfrak{g}) = (\mathfrak{a} : \delta_{1,n})$.

Proof. We will first show that $(\mathfrak{a} : \delta_{1,n}) \subseteq (\mathfrak{a} : \delta_{i,j})$ for any $1 \leq i < j \leq n$.

We know

$$(\mathfrak{a} : \delta_{i,j}) = \bigcap_{S \in \mathcal{C}(P_n)} (P_S(P_n) : \delta_{i,j}) = \bigcap_{\substack{S \in \mathcal{C}(P_n) \\ \delta_{i,j} \notin P_S(P_n)}} P_S(P_n)$$

since the $P_S(P_n)$ are prime, and $\delta_{1,n} \in P_S(P_n)$ if and only if $S = \varnothing$. Clearly $\delta_{i,j} \in P_\varnothing(P_n)$ for any $1 \leq i < j \leq n$, and so the desired containment holds.

Then we have

$$(\mathfrak{a} : \mathfrak{g}) = \mathfrak{a} : \left(\sum_{\{i,j\} \in E(G)} (\delta_{i,j}) \right) = \bigcap_{\{i,j\} \in E(G)} (\mathfrak{a} : \delta_{i,j}) = (\mathfrak{a} : \delta_{1,n})$$

with the last equality following since G contains C_n because it is Hamiltonian and so $\{1, n\} \in E(G)$. □

This concludes the proof of Theorem 4.4.2. □

As an example application of Theorem 4.4.2, we show the following:

Proposition 4.4.10. *We have*

$$\mathrm{Ass}_R(H_{\mathfrak{g}}^{n-1}(R)) = \{\mathcal{J}(K_n)\}$$

Proof. By Theorem 4.4.2 and Proposition 4.4.9, we have that $\mathfrak{b}/\mathfrak{a}$ is a root of $H_{\mathfrak{g}}^{n-1}(R)$, and so

$$\mathrm{Ass}_R(H_{\mathfrak{g}}^{n-1}(R)) = \mathrm{Ass}_R(\mathfrak{b}/\mathfrak{a})$$

by Lemma 2.1.31.

Let $\mathfrak{p}$ be any associated prime of $\mathfrak{b}/\mathfrak{a}$, so $\mathfrak{p} = \mathrm{Ann}_R(b + \mathfrak{a})$ for some non-zero $b + \mathfrak{a} \in \mathfrak{b}/\mathfrak{a}$. This is equivalent to saying that $\mathfrak{p} = (\mathfrak{a} : b)$.

Since $b \in \mathfrak{b} = (\mathfrak{a} : \mathcal{J}(K_n))$ we have $b\mathcal{J}(K_n) \subseteq \mathfrak{a}$, so certainly $\mathcal{J}(K_n) \subseteq \mathfrak{p}$. We will now show the converse.

By definition, we have $\mathfrak{p}b \subseteq \mathfrak{a} \subseteq \mathcal{J}(K_n)$. Now, $\mathcal{J}(K_n) = P_{\varnothing}(K_n)$ is prime, and so either $\mathfrak{p} \subseteq \mathcal{J}(K_n)$ as desired, or $b \in \mathcal{J}(K_n)$. We know that $b\mathcal{J}(K_n) \subseteq \mathfrak{a}$, and so in this case we have $b^2 \in \mathfrak{a}$. But $\mathfrak{a} = \mathcal{J}(P_n)$ is radical, so $b \in \mathfrak{a}$. Then $b + \mathfrak{a} = 0 + \mathfrak{a}$, which contradicts that $b + \mathfrak{a}$ is non-zero, and so we are done. □

Chapter 5

Local Cohomology Modules of Binomial Edge Ideals of Complements of Graphs of Girth ≥ 5

Throughout this chapter, unless specified otherwise, we set

$$R = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

for some field k and $n \geq 1$, and $\delta_{i,j} := x_i y_j - x_j y_i$.

In this chapter, we calculate the local cohomology modules of the binomial edge ideals of the complements of connected graphs of girth at least 5 using the tools introduced in [Àlv20, Section 3]. We then use this calculation to compute or bound various invariants of these binomial edge ideals.

The majority of this chapter first appeared in [Wil23b].

5.1 Àlvarez Montaner's Hochster-Type Formula

The key tool we will make use of is a Hochster-type decomposition of the local cohomology modules of binomial edge ideals given by the (reduced) cohomology of intervals in the order complex of a certain poset associated to the binomial edge ideal, which was introduced by Àlvarez Montaner in [Àlv20, Section 3].

We begin by defining the poset itself:

Definition 5.1.1. *For a graph G, we construct the poset $\mathcal{Q}_{\mathcal{J}(G)}$ as follows:*

1. *We start with the associated primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ of $\mathcal{J}(G)$.*

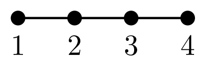
2. *Next, we add all sums of these primes to our poset.*
3. *If any of these sums is not prime, replace it with the primes in its primary decomposition.*
4. *Now add to the poset all sums of the previous elements and these new primes.*
5. *Repeat this process until all ideals in the poset are prime and all sums are included.*

This process terminates after a finite number of steps, as explained in [Àlv20, Definition 3.3].

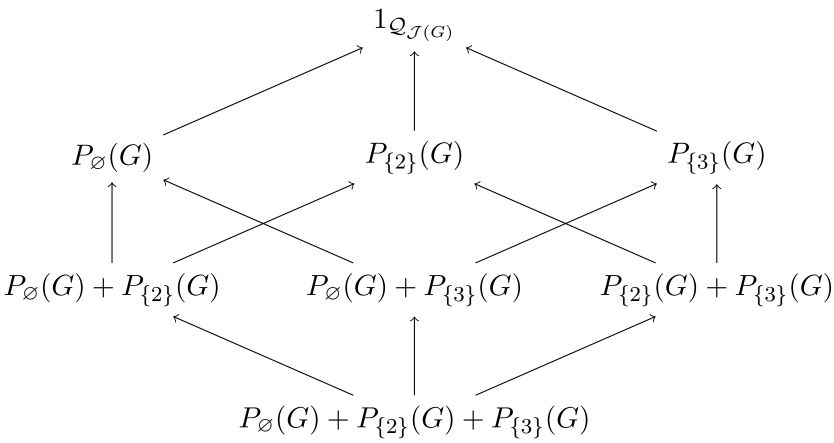
We then order these ideals by reverse inclusion, and adjoin a maximal element $1_{\mathcal{Q}_{\mathcal{J}(G)}}$.

Note. *These ideals are all Cohen-Macaulay, as explained in the introduction of [Àlv20, Section 3.2].*

Example 5.1.2. If

$G =$ 

then $1_{\mathcal{Q}_{\mathcal{J}(G)}}$ is given by



We now describe a simplicial complex associated to a poset:

Definition 5.1.3. *Let* $(\mathcal{P}, \leq)$ *be a poset. Then we define the* ***order complex*** *associated to* $(\mathcal{P}, \leq)$ *as the simplicial complex whose facets are the maximal chains in* $\mathcal{P}$.

Given $S_1, S_2 \in \mathcal{P}$ *with* $S_1 \leq S_2$*, we denote by* (S_1, S_2) *the order complex on that open interval in the poset, that is, the complex with facets given by the maximal chains strictly between* S_1 *and* S_2.

The key theorem is then as follows:

Theorem 5.1.4. [Àlv20, Theorem 3.9] *We have isomorphisms*

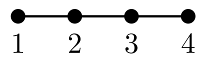
$$H^r_{\mathfrak{m}}(R/\mathcal{J}(G)) \cong \bigoplus_{\mathfrak{q} \in \mathcal{Q}_{\mathcal{J}(G)}} H^{d_{\mathfrak{q}}}_{\mathfrak{m}}(R/\mathfrak{q})^{M_{r,\mathfrak{q}}}$$

of graded k-vector spaces for all $r \geq 0$, where $d_{\mathfrak{q}} = \dim(R/\mathfrak{q})$ and

$$M_{r,\mathfrak{q}} = \dim_k(\widetilde{H}^{r-d_{\mathfrak{q}}-1}((\mathfrak{q}, 1_{\mathcal{Q}_{\mathcal{J}(G)}}); k))$$

Note. *We will compute reduced homology groups rather than reduced cohomology groups, which we can do since these have the same dimension as k-vector spaces when finite dimensional (see, for example, [MS05, p.11]).*

Example 5.1.5. Again, let

$G =$ 

and set

$$\mathfrak{p}_{S_1,S_2} = P_{S_1}(G) + P_{S_2}(G)$$
$$\mathfrak{q} = P_{\varnothing}(G) + P_{\{2\}}(G) + P_{\{3\}}(G)$$

for $S_1, S_2 \in \mathcal{C}(G)$.

Then we have

I	$\dim(R/I)$	$(I, 1_{\mathcal{Q}_{\mathcal{J}(G)}})$	$\dim_k(\widetilde{H}^{-1})$	$\dim_k(\widetilde{H}^{0})$	$\dim_k(\widetilde{H}^{1})$
$P_S(G)$ $S \in \mathcal{C}(G)$	5	$\varnothing$	1	0	0
$\mathfrak{p}_{S_1,S_2}$ $S_1, S_2 \in \mathcal{C}(G)$	4	$P_{S_1}(G)$ $P_{S_2}(G)$	0	1	0
$\mathfrak{q}$	3	$P_{\varnothing}(G)$ $P_{\{2\}}(G)$ $P_{\{3\}}(G)$ $\mathfrak{p}_{\varnothing,\{2\}}$ $\mathfrak{p}_{\varnothing,\{3\}}$ $\mathfrak{p}_{\{2\},\{3\}}$	0	0	1

and so

$$H^5_{\mathfrak{m}}(R/\mathcal{J}(G)) \cong \left[\bigoplus_{S \in \mathcal{C}(G)} H^5_{\mathfrak{m}}(R/P_S(G))\right] \oplus \left[\bigoplus_{S_1, S_2 \in \mathcal{C}(G)} H^4_{\mathfrak{m}}(R/\mathfrak{p}_{S_1,S_2})\right] \oplus H^3_{\mathfrak{m}}(R/\mathfrak{q})$$

as graded k-vector spaces, with all other local cohomology modules vanishing.

This (along with Lemma 3.11.4) tells us that $R/\mathcal{J}(G)$ is Cohen-Macaulay of dimension 5.

5.2 Some Properties of Complements of Graphs of Girth at Least 5

We begin with a definition:

Definition 5.2.1. *Let G be a graph. We define the **girth** of G to be the number of vertices in the smallest cycle contained in G, or ∞ if G is acyclic (or, equivalently, if G is a forest).*

For the remainder of this chapter, unless specified otherwise, G will denote a connected graph of girth at least 5.

We also suppose that G has no universal vertex, since then that vertex will be isolated in $\overline{G}$, and so will not affect $\mathcal{Q}_{\mathcal{J}(\overline{G})}$ (other than raising each $d_{\mathfrak{q}}$ by 2). For clarity, we will denote by $\overline{v}$ the vertex in $\overline{G}$ corresponding to the vertex v in G, although the two are really the same.

Furthermore, we may assume that $n \geq 5$: Any graph with fewer than 5 vertices but girth greater than 5 must be a tree, and it is easily checked that all trees on up to 4 vertices contain a universal vertex other than $P_4 = \overline{P_4}$, which we have already dealt with in Example 5.1.5.

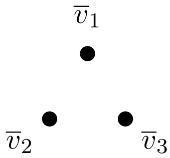
Finally, we denote by H the subgraph of G obtained by removing all leaves of G. Note that $V(H) \neq \varnothing$ since G has no universal vertex.

We first describe $\mathcal{C}(\overline{G})$:

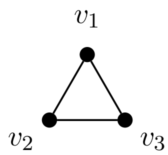
Lemma 5.2.2. *We have*

$$\mathcal{C}(\overline{G}) = \{\varnothing\} \cup \{N_{\overline{G}}(\overline{v}) : v \in V(H)\}$$

Proof. Trivially we have $\varnothing \in \mathcal{C}(\overline{G})$. Take any other $S \in \mathcal{C}(\overline{G})$. We will first show that $\overline{G} \setminus S$ consists of exactly two connected components. We know that it must have more than one. Say we had the induced subgraph

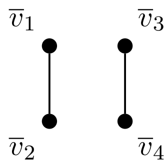


in $\overline{G} \setminus S$. Then we would have the 3-cycle

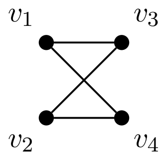


in G, a contradiction.

Next we will show that one of these components must be an isolated vertex. Suppose that we had the induced subgraph



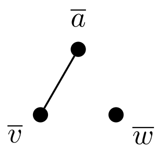
in $\overline{G} \setminus S$. Then we would have the 4-cycle



in G, again a contradiction, and so one component must be an isolated vertex as claimed.

Let $\overline{v}$ denote this isolated vertex. We now claim that $v \in V(H)$. Otherwise, suppose that v is adjacent to a single vertex w in G, so $\overline{v}$ is adjacent to every vertex in $\overline{G}$ other than $\overline{w}$. Then for $\overline{v}$ to be isolated with $\overline{G} \setminus S$ consisting of more than one connected component, we must have $S = V(\overline{G}) \setminus \{\overline{v}, \overline{w}\}$. This would mean that $\overline{G} \setminus S$ is just

Since $G \neq K_2$ and is connected, we know that w has at least one other neighbour, say a, and we must have $\overline{a} \in S$. But adding $\overline{a}$ back to $\overline{G} \setminus S$ gives



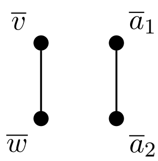
which does not decrease the number of connected components, contradicting that $\overline{a} \in S$.

This shows that $\overline{v}$ is as claimed. Since $\overline{v}$ is isolated in $\overline{G} \setminus S$, we must have $N_{\overline{G}}(\overline{v}) \subseteq S$.

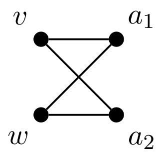
Now take any $\overline{a} \in S$, so adding $\overline{a}$ back to $\overline{G} \setminus S$ must decrease the number of connected components. We have shown there are only two connected components in $\overline{G} \setminus S$, one of which is $\{\overline{v}\}$, and so $\overline{a}$ must be adjacent to $\overline{v}$ in $\overline{G}$. This means that $S \subseteq N_{\overline{G}}(\overline{v})$, so $S = N_{\overline{G}}(\overline{v})$ as desired.

Finally, we will show that for any $v \in V(H)$, we have $N_{\overline{G}}(\overline{v}) \in \mathcal{C}(\overline{G})$. Take any $\overline{w} \in N_{\overline{G}}(\overline{v})$. This means that w is not adjacent to v in G.

We know that v has at least two neighbours, say a_1 and a_2, in G, since $v \in V(H)$. If we had the induced subgraph



in $\overline{G}$ then we would have the 4-cycle

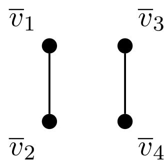


in G, a contradiction. Then $\overline{w}$ must be adjacent to both $\overline{v}$ and one of its neighbours in $\overline{G}$, so reintroducing it will reconnect the two disjoint components of $\overline{G} \setminus S$, and we are done. □

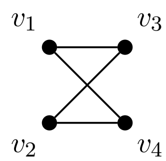
We will also make use of the following:

Proposition 5.2.3. *$\overline{G}$ is connected.*

Proof. If we had the induced subgraph



in $\overline{G}$, then we would have the 4-cycle



in G, a contradiction.

This means that either $\overline{G}$ is connected, or it contains an isolated vertex $\overline{v}$. But for $\overline{v}$ to be isolated in $\overline{G}$, it must be universal in G, contradicting our assumptions on G. □

5.3 An Equivalence of Posets

We next define a poset which we will show is equivalent to that of Definition 5.1.1 for $\overline{G}$.

Note. *As vertex sets, we have*

$$N_{\overline{G}}(\overline{v}) = V(G) \setminus N_G[v]$$

for any $v \in V(H)$.

Definition 5.3.1. *We first define a set $\mathcal{P}_{\overline{G}}$ consisting of the following ideals:*

$$\begin{aligned}
\mathfrak{j} &:= \mathcal{J}(K_n) & & \\
\mathfrak{a}_v &:= (x_i, y_i : i \in V(G) \setminus N_G[v]) + \mathcal{J}(K_{N_G(v)}) & & \text{for } v \in V(H) \\
\mathfrak{b}_v &:= (x_i, y_i : i \in V(G) \setminus N_G[v]) + \mathcal{J}(K_{N_G[v]}) & & \text{for } v \in V(H) \\
\mathfrak{c}_{\{v,w\}} &:= (x_i, y_i : i \in V(G) \setminus \{v, w\}) & & \text{for } \{v,w\} \in E(H) \\
\mathfrak{d}_{\{v,w\}} &:= (x_i, y_i : i \in V(G) \setminus \{v, w\}) + (\delta_{v,w}) & & \text{for } \{v,w\} \in E(H) \\
\mathfrak{e}_v &:= (x_i, y_i : i \in V(G) \setminus \{v\}) & & \text{for } v \in V(H) \text{ with } |N_H(v)| > 1
\end{aligned}$$

If there exist $v, w \in V(H)$ with $|N_H(w)| > 1$ and $w \notin N_G[v]$, then we also add $\mathfrak{m}$ to this set.

We next adjoin an element $1_{\mathcal{P}_{\overline{G}}}$, which will be maximal in the poset, and then turn $\mathcal{P}_{\overline{G}}$ into a poset by taking the transitive and reflexive closure of the following relations (the relation being reverse inclusion):

$$\begin{aligned}
\mathfrak{m} &\leq \mathfrak{e}_v & & \\
\mathfrak{e}_v &\leq \mathfrak{d}_{\{v,w\}} & & \text{for } w \in N_H(v) \\
\mathfrak{d}_{\{v,w\}} &\leq \mathfrak{b}_v, \mathfrak{b}_w, \mathfrak{c}_{\{v,w\}} & & \\
\mathfrak{b}_v &\leq \mathfrak{j}, \mathfrak{a}_v & & \\
\mathfrak{c}_{\{v,w\}} &\leq \mathfrak{a}_v, \mathfrak{a}_w & & \\
\mathfrak{j}, \mathfrak{a}_v &\leq 1_{\mathcal{Q}_{\mathcal{J}(\overline{G})}} & &
\end{aligned}$$

Note. *It is easily seen that all of these ideals are prime.*

Lemma 5.3.2. *We have $\mathcal{Q}_{\mathcal{J}(\overline{G})} = \mathcal{P}_{\overline{G}}$ as posets.*

Proof. We will first show that $\mathcal{Q}_{\mathcal{J}(\overline{G})} \subseteq \mathcal{P}_{\overline{G}}$.

By Lemma 5.2.2, $\mathfrak{j}$ and the $\mathfrak{a}_i$ are precisely the associated primes of $\mathcal{J}(\overline{G})$. Take any non-empty $S \subseteq V(H)$, and set

$$\sigma = \sum_{v \in S} \mathfrak{a}_v$$

If we have some v_1 and v_2 in S which are not adjacent and share no neighbours, then

$$N_G[v_1] \cap N_G[v_2] = \varnothing$$

and so $\sigma = \mathfrak{m}$. Note that if this is the case, since H is connected (because G is and we are only removing leaves), we must have a path P in H on at least 4 vertices with endpoints v_1 and v_2. If we consider the subpath P' starting at v_1 consisting of exactly 4 vertices, this must be an induced subgraph of G, since the girth of G is at least 5. Denote by w the neighbour of v_1 in P', and by v the neighbour of v_1 not adjacent to w in P'. Then $|N_H(w)| > 1$ and $w \notin N_G[v]$, so the criterion for the inclusion of $\mathfrak{m}$ in $\mathcal{P}_{\overline{G}}$ is satisfied.

We may now assume that each vertex in S is either adjacent to, or shares a neighbour with, every other.

If $S = \{v_1, v_2\}$ with v_1 and v_2 adjacent, then $\sigma = \mathfrak{c}_{\{v_1,v_2\}}$, since they cannot have a common neighbour as this would mean that G contains a 3-cycle.

Otherwise, we may assume that S consists of at least 3 vertices. At least 2 of these vertices, say v_1 and v_2, cannot be adjacent, since otherwise G would contain a 3-cycle. We next assume that they have a common neighbour, say w, since otherwise we are in the case $\sigma = \mathfrak{m}$. Note that $w \in V(H)$ also, since we know it has at least two neighbours. If v_1 and v_2 had another common neighbour then G would contain a 4-cycle, so this cannot be the case, and therefore

$$N_G[v_1] \cap N_G[v_2] = \{w\}$$

We then have

$$\mathfrak{a}_{v_1} + \mathfrak{a}_{v_2} = \mathfrak{e}_w$$

Since

$$\mathfrak{e}_w + \mathfrak{a}_v = \begin{cases} \mathfrak{e}_w & \text{if } w \in N_G[v] \\ \mathfrak{m} & \text{otherwise} \end{cases}$$

we now know that σ must be either $\mathfrak{a}_v$, $\mathfrak{c}_{\{v_1,v_2\}}$, $\mathfrak{e}_w$ or $\mathfrak{m}$ for some $v, v_1, v_2, w \in V(H)$.

Furthermore, we have

$$\begin{aligned} \mathfrak{a}_v + \mathfrak{j} &= \mathfrak{b}_v \\ \mathfrak{c}_{\{v,w\}} + \mathfrak{j} &= \mathfrak{d}_{\{v,w\}} \\ \mathfrak{e}_w + \mathfrak{j} &= \mathfrak{e}_w \\ \mathfrak{m} + \mathfrak{j} &= \mathfrak{m} \end{aligned}$$

and so $\mathcal{Q}_{\mathcal{J}(\overline{G})}$ is contained in $\mathcal{P}_{\overline{G}}$.

It is easily seen that we can achieve each $\mathfrak{b}_v$, $\mathfrak{c}_{\{v,w\}}$ and $\mathfrak{d}_{\{v,w\}}$ through sums of $\mathfrak{j}$ and the $\mathfrak{a}_i$. If we have any $\mathfrak{e}_w$ in $\mathcal{P}_{\overline{G}}$, then by definition we have some $w \in V(H)$ and $v_1, v_2 \in N_H(w)$, so

$$\mathfrak{a}_{v_1} + \mathfrak{a}_{v_2} = \mathfrak{e}_w$$

as shown above. Finally, if $\mathfrak{m}$ is in $\mathcal{P}_{\overline{G}}$ then by definition we have such a w, along with some $v \in V(H)$ with $w \notin N_G[v]$, so

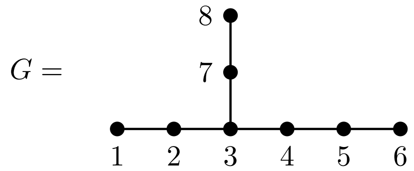
$$\mathfrak{e}_w + \mathfrak{a}_v = \mathfrak{m}$$

again as shown above. This shows that we can achieve every element of $\mathcal{P}_{\overline{G}}$ through sums of $\mathfrak{j}$ and the $\mathfrak{a}_i$, so $\mathcal{P}_{\overline{G}} \subseteq \mathcal{Q}_{\mathcal{J}(\overline{G})}$.

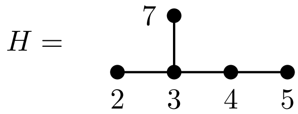
Clearly the orders on $\mathcal{Q}_{\mathcal{J}(\overline{G})}$ and $\mathcal{P}_{\overline{G}}$ agree, and so $\mathcal{Q}_{\mathcal{J}(\overline{G})} = \mathcal{P}_{\overline{G}}$ as posets as claimed. □

Note. *$\mathcal{P}_{\overline{G}}$ is also equal to the poset $\mathcal{P}_{\mathcal{J}(\overline{G})}$ of [Àlv20, Section 3.1], since each sum of the associated primes of $\mathcal{J}(\overline{G})$ is itself prime.*

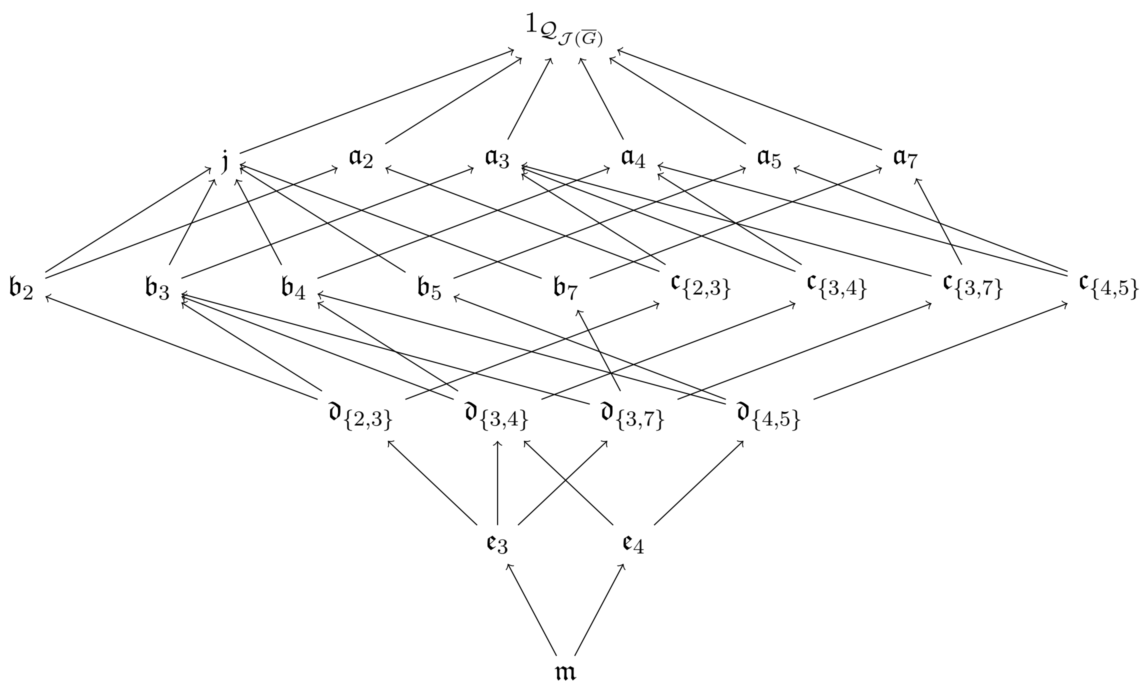
Example 5.3.3. Let



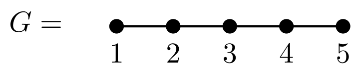
so



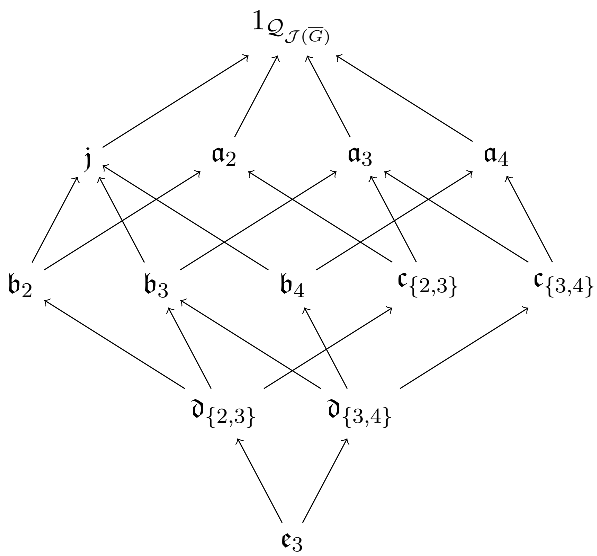
This yields the poset



Note. *We may not always get the $\mathfrak{e}_v$ and $\mathfrak{m}$ layers of the poset. For example, if*

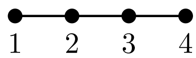


then we obtain the poset

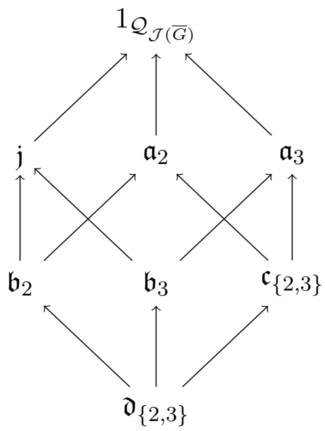


If

$G =$



then we obtain



However, since G has no universal vertex it cannot be a star, so H has at least one edge and therefore some $\mathfrak{d}_{\{v,w\}}$ will always be present.

5.4 The Main Theorem & Some Corollaries

In order to apply Theorem 5.1.4, we must calculate the dimensions of the ideals in our poset:

Proposition 5.4.1. *We have*

$$\begin{aligned}\dim(R/\mathfrak{j}) &= n+1\\ \dim(R/\mathfrak{a}_v) &= |N_G[v]|+2\\ \dim(R/\mathfrak{b}_v) &= |N_G[v]|+1\\ \dim(R/\mathfrak{c}_{\{v,w\}}) &= 4\\ \dim(R/\mathfrak{d}_{\{v,w\}}) &= 3\\ \dim(R/\mathfrak{e}_v) &= 1\end{aligned}$$

Proof. This follows immediately from the fact that the dimension of the binomial edge ideal of a complete graph on m vertices is $m+1$ (see for example [Her+10, Example 1.7 (a) & Corollary 3.4]). □

We must also calculate the necessary values of $M_{r,\mathfrak{q}}$. The most involved $\mathfrak{q} \in \mathcal{Q}_{\mathcal{J}(\overline{G})}$ for which to calculate $M_{r,\mathfrak{q}}$ is $\mathfrak{q} = \mathfrak{e}_v$. We will show that, in this case, $M_{r,\mathfrak{q}} = 0$ for all j.

Proposition 5.4.2. *For any $\mathfrak{e}_v \in \mathcal{Q}_{\mathcal{J}(\overline{G})}$, all reduced homology of the order complex Δ of the open interval $(\mathfrak{e}_v, 1_{\mathcal{Q}_{\mathcal{J}(\overline{G})}})$ vanishes.*

Proof. The facets of Δ are

$$\begin{gathered}\{\mathfrak{d}_{\{v,w\}}, \mathfrak{b}_v, \mathfrak{j}\}\\ \{\mathfrak{d}_{\{v,w\}}, \mathfrak{b}_v, \mathfrak{a}_v\}\\ \{\mathfrak{d}_{\{v,w\}}, \mathfrak{b}_w, \mathfrak{j}\}\\ \{\mathfrak{d}_{\{v,w\}}, \mathfrak{b}_w, \mathfrak{a}_w\}\\ \{\mathfrak{d}_{\{v,w\}}, \mathfrak{c}_{\{v,w\}}, \mathfrak{a}_v\}\\ \{\mathfrak{d}_{\{v,w\}}, \mathfrak{c}_{\{v,w\}}, \mathfrak{a}_w\}\end{gathered}$$

for $w \in N_H(v)$.

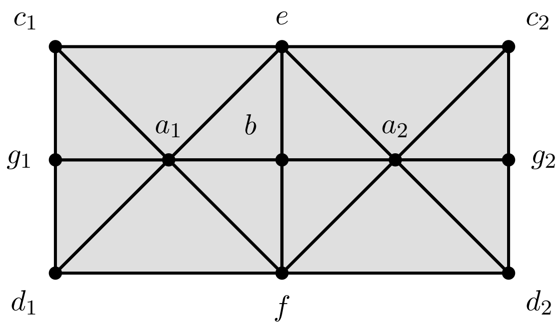
Let $w_1, \ldots, w_m$ denote the vertices in $N_H(v)$. To simplify notation, we make the relabellings

$$\begin{aligned}\mathfrak{d}_{\{v,w_i\}} &\mapsto a_i\\ \mathfrak{b}_v &\mapsto b\\ \mathfrak{b}_{w_i} &\mapsto c_i\\ \mathfrak{c}_{\{v,w_i\}} &\mapsto d_i\\ \mathfrak{j} &\mapsto e\\ \mathfrak{a}_v &\mapsto f\\ \mathfrak{a}_{w_i} &\mapsto g_i\end{aligned}$$

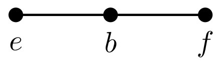
so our facets become

$$\bigcup_{i=1}^{m} \{\{a_i, b, e\}, \{a_i, b, f\}, \{a_i, c_i, e\}, \{a_i, c_i, g_i\}, \{a_i, d_i, f\}, \{a_i, d_i, g_i\}\}$$

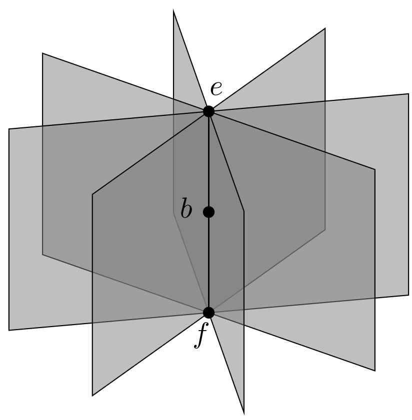
When $m = 2$, our complex is



Increasing m will simply add more squares joined along



We can take a geometric realisation (here pictured for $m = 8$) of Δ which looks like:



(e, b and f have been added to this illustration to highlight how the squares are joined in Δ, they are not actually points in $\mathbb{R}^3$).

This is clearly contractible, and so all reduced homology of Δ vanishes as desired. □

We are now ready to prove our main theorem:

Theorem 5.4.3. *We have the following isomorphisms*

$$H^5_{\mathfrak{m}}(R/\mathcal{J}(\overline{G})) \cong \left[\bigoplus_{\substack{v \in V(H) \\ |N_G[v]|=3}} [H^5_{\mathfrak{m}}(R/\mathfrak{a}_v) \oplus H^4_{\mathfrak{m}}(R/\mathfrak{b}_v)] \right] \oplus \left[\bigoplus_{\{v,w\} \in E(H)} [H^4_{\mathfrak{m}}(R/\mathfrak{c}_{\{v,w\}}) \oplus H^3_{\mathfrak{m}}(R/\mathfrak{d}_{\{v,w\}})] \right]$$

$$H^i_{\mathfrak{m}}(R/\mathcal{J}(\overline{G})) \cong \left[\bigoplus_{\substack{v \in V(H) \\ |N_G[v]|=i-2}} [H^i_{\mathfrak{m}}(R/\mathfrak{a}_v) \oplus H^{i-1}_{\mathfrak{m}}(R/\mathfrak{b}_v)] \right]$$

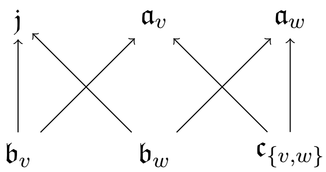
$$H^{n+1}_{\mathfrak{m}}(R/\mathcal{J}(\overline{G})) \cong H^{n+1}_{\mathfrak{m}}(R/\mathfrak{j}) \oplus \left[\bigoplus_{\substack{v \in V(H) \\ |N_G[v]|=n-1}} [H^{n+1}_{\mathfrak{m}}(R/\mathfrak{a}_v) \oplus H^{n}_{\mathfrak{m}}(R/\mathfrak{b}_v)] \right]$$

of graded k-vector spaces (for $5 < i < n+1$), with all other local cohomology modules vanishing.

Proof. We aim to apply Theorem 5.1.4 to $\mathcal{Q}_{\mathcal{J}(\overline{G})}$.

Note that, since G has no universal vertex by assumption, we cannot have any $v \in V(H)$ such that $|N_G[v]| > n-1$, and since every vertex in H has at least two neighbours in G, we cannot have $|N_G[v]| < 3$.

Taking into account Proposition 5.4.1, the following values are easily computed:

I	$\dim(R/I)$	$(I, 1_{\mathcal{Q}_{\mathcal{J}(\overline{G})}})$	$\dim_k(\widetilde{H}^{-1})$	$\dim_k(\widetilde{H}^{0})$	$\dim_k(\widetilde{H}^{1})$
$\mathfrak{j}$	$n+1$	$\varnothing$	1	0	0
$\mathfrak{a}_v$	$\lvert N_G[v]\rvert + 2$	$\varnothing$	1	0	0
$\mathfrak{b}_v$	$\lvert N_G[v]\rvert + 1$	$\mathfrak{j}$ $\mathfrak{a}_v$	0	1	0
$\mathfrak{c}_{\{v,w\}}$	4	$\mathfrak{a}_v$ $\mathfrak{a}_w$	0	1	0
$\mathfrak{d}_{\{v,w\}}$	3		0	0	1

Then, in light of Proposition 5.4.2, we are done if we can show that all reduced homology of the order complex Δ of the open interval $(\mathfrak{m}, 1_{\mathcal{Q}_{\mathcal{J}(\overline{G})}})$ vanishes.

We have that the length of a maximal chain in $(\mathfrak{m}, 1_{\mathcal{Q}_{\mathcal{J}(\overline{G})}})$ is 4, and so Δ has non-vanishing reduced homology at index 3 at the highest. From the definition of the $M_{r,\mathfrak{q}}$, we see that the smallest i for which the terms in the table can contribute to $H^i_{\mathfrak{m}}(R/\mathcal{J}(\overline{G}))$ is 5, and since $\dim_R(R/\mathfrak{m}) = 0$, the largest i for which $\mathfrak{m}$ can contribute to $H^i_{\mathfrak{m}}(R/\mathcal{J}(\overline{G}))$ is 4. Then, by Theorem 3.8.5, we are done if we can show that ${}^*\mathrm{depth}_R(R/\mathcal{J}(\overline{G})) > 4$.

By [RSK21, Theorem 5.2 & Theorem 5.3], we have that ${}^*\mathrm{depth}_R(R/\mathcal{J}(\overline{G})) \geq 4$ since $\overline{G}$ is connected by Proposition 5.2.3, with equality if and only if $\overline{G}$ can be written as a join

$$\overline{G} = (K_1 \sqcup K_1) * G'$$

for some graph G'. But if this were the case, then we would have $G = K_2 \sqcup \overline{G'}$, contradicting that G is connected. □

We can use this theorem to calculate the dimension, depth, and regularity of $R/\mathcal{J}(\overline{G})$. In the case where k is of prime characteristic $p > 0$, we can also calculate the cohomological dimension and bound the arithmetic rank.

Corollary 5.4.4. *We have*

$$^{*}\mathrm{depth}_R(R/\mathcal{J}(\overline{G})) = 5$$
$$\dim(R/\mathcal{J}(\overline{G})) = n+1$$

and $R/\mathcal{J}(\overline{G})$ *is Cohen-Macaulay if and only if* $G = P_4$.

Proof. We see from Theorem 5.4.3 that $H^5_{\mathfrak{m}}(R/\mathcal{J}(\overline{G}))$ and $H^{n+1}_{\mathfrak{m}}(R/\mathcal{J}(\overline{G}))$ always contain non-vanishing summands, and so are the bottom and top local cohomology modules respectively. The first claims then follow from Theorem 3.8.5 and Theorem 3.7.7.

We then have that $R/\mathcal{J}(\overline{G})$ is Cohen-Macaulay if and only if $n+1 = 5$ by Lemma 3.11.4 and Lemma 3.7.4. The only such graph satisfying our criteria on G is P_4, and so we are done. □

Corollary 5.4.5. *Suppose that* k *has prime characteristic* $p > 0$. *Then* $H^i_{\mathcal{J}(\overline{G})}(R)$ *is non-vanishing if and only if*

$$i \in \{n-1, 2n-5\} \cup \{2n - (N_G[v] + 2) : v \in V(H)\}$$

Consequently, we can compute the cohomological degree

$$\mathrm{cd}_R(\mathcal{J}(\overline{G})) = 2n - 5$$

and bound the arithmetic rank

$$2n - 5 \leq \mathrm{ara}_R(\mathcal{J}(\overline{G})) \leq 2n$$

with the lower bound being sharp.

Proof. Note that

$$\mathrm{cd}_R(\mathcal{J}(\overline{G})) = \max\{1 \leq i \leq 2n : H^i_{\mathcal{J}(\overline{G})}(R) \neq 0\}$$

by [Iye+07, Theorem 9.6].

We will make use of the functor $^{*}\mathscr{H}$ defined in [LSW16, Theorem 4.2]. This is the graded counterpart to the functor defined in Section 2.2.

By [LSW16, Proposition 2.8], we have that

$$^{*}\mathscr{H}(H^i_{\mathfrak{m}}(R/\mathcal{J}(\overline{G}))) \cong H^{2n-i}_{\mathcal{J}(\overline{G})}(R)$$

Now, by Theorem 2.1.8 we have that Frobenius is injective on $H^i_{\mathfrak{m}}(R/\mathcal{J}(\overline{G}))$, and so the first claim follows from [LSW16, Theorem 2.5 (2)] and Theorem 5.4.3.

We have $\mathrm{ara}_R(\mathcal{J}(\overline{G})) \leq 2n$ by [Iye+07, Theorem 9.13 & Remark 9.14], and $2n - 5 \leq \mathrm{ara}_R(\mathcal{J}(\overline{G}))$

by [Iye+07, Proposition 9.12], since we have shown that the cohomological dimension is $2n-5$. Furthermore, $\mathcal{J}(\overline{P_4})$ is generated by 3 elements, so the lower bound is sharp as claimed. □

Note. *At the time of writing, we do not know if Corollary 5.4.5 holds in characteristic* 0*.*

We need a few results about regularity before we compute it, these are not difficult but we include proofs for completeness:

Proposition 5.4.6. *Let* R *be a polynomial ring over a field with the standard grading,* I *a homogeneous ideal of* R*, and* z *an indeterminate. Then*

$$\operatorname{reg}_{R[z]}(R[z]/(IR[z]+(z))) = \operatorname{reg}_R(R/I)$$

Proof. Note that

$$R[z]/(IR[z]+(z)) \cong R/I$$

as $R[z]$-modules if we let z act on R/I by killing any element. Then the result follows immediately from applying [Eis05, Corollary 4.6] to R/I and $R \hookrightarrow R[z]$. □

Proposition 5.4.7. *Let* R *be a polynomial ring over a field with the standard grading,* I *a homogeneous ideal of* R*, and* z *an indeterminate. Then*

$$\operatorname{reg}_{R[z]}(R[z]/IR[z]) = \operatorname{reg}_R(R/I)$$

Proof. A minimal graded free resolution of R/I over R remains so over $R[z]$, since $R[z]$ is free (and therefore flat) over R, and its grading is compatible with that of R. □

We will make use of the following notation:

Notation 5.4.8. *Let* $1 \le i \le n$*, and let* G *be a graph with vertices* $1, \dots, j$ *for some* $j \le i$*. Then we set*

$$R'_i := k[x_1, \dots, x_i, y_1, \dots, y_i]$$

and denote by $\mathcal{J}_i(G)$ *the binomial edge ideal of* G *in* R'_i*.*

Proposition 5.4.9. *For any* $m \ge 2$*, we have*

$$\operatorname{reg}_{R'_m}(R'_m/\mathcal{J}_m(K_m)) = 1$$

Proof. Since

$$\operatorname{reg}_{R'_m}(\mathcal{J}_m(K_m)) = 2$$

(see for example [KS12, Remark 3.3]), the result is immediate from the fact that

$$\operatorname{reg}_{R'_m}(R/I) = \operatorname{reg}_{R'_m}(I) - 1$$

for any (homogeneous) ideal I of R. □

We can now compute the regularity of $R/\mathcal{J}(\overline{G})$:

Corollary 5.4.10. *We have*

$$\operatorname{reg}_R(R/\mathcal{J}(\overline{G})) = 3$$

Proof. Fix vertices v and w of H, and set $m = |N_G(v)|$. We may relabel the vertices of G so that $N_G(v) = \{1, \dots, m\}$ and $v = m + 1$.

By Theorem 1.2.15, for any finitely generated graded R-module M, we have

$$\operatorname{reg}_R(M) = \max\{\operatorname{end}_R(H^i_{\mathfrak{m}}(M)) + i : i \geq 0\}$$

Each ideal in $\mathcal{Q}_{\mathcal{J}(\overline{G})}$ is Cohen-Macaulay, and so only has a single non-vanishing local cohomology module. Furthermore, the isomorphisms in Theorem 5.4.3 are isomorphisms of graded k-vector spaces, and so to calculate $\operatorname{reg}_R(R/\mathcal{J}(\overline{G}))$ we need only calculate the regularities of $R/\mathfrak{j}$, $R/\mathfrak{a}_v$, $R/\mathfrak{b}_v$, $R/\mathfrak{c}_{\{v,w\}}$, and $R/\mathfrak{d}_{\{v,w\}}$, which we do as follows:

$\mathfrak{j}$: We have $\operatorname{reg}_R(R/\mathfrak{j}) = 1$ by Proposition 5.4.9.

$\mathfrak{a}_v$: Note that

$$R/\mathfrak{a}_v \cong R'_{m+1}/\mathcal{J}_{m+1}(K_m) \cong (R'_m/\mathcal{J}_m(K_m))[x_{m+1}, y_{m+1}]$$

We have

$$\operatorname{reg}_{R'_m}(R'_m/\mathcal{J}_m(K_m)) = 1$$

by Proposition 5.4.9. Next, we have

$$\operatorname{reg}_{R'_{m+1}}(R'_{m+1}/\mathcal{J}_{m+1}(K_m)) = \operatorname{reg}_{R'_{m+1}}((R'_m/\mathcal{J}_m(K_m))[x_{m+1}, y_{m+1}]) = 1$$

with the second equality following by Proposition 5.4.7, and so applying Proposition 5.4.6 we obtain $\operatorname{reg}_R(R/\mathfrak{a}_v) = 1$.

$\mathfrak{b}_v$: In this case, we have

$$R/\mathfrak{b}_v \cong R'_{m+1}/\mathcal{J}_{m+1}(K_{m+1})$$

and

$$\operatorname{reg}_{R'_{m+1}}(R'_{m+1}/\mathcal{J}_{m+1}(K_{m+1})) = 1$$

by Proposition 5.4.9, and so, again by applying Proposition 5.4.6, we have $\operatorname{reg}_R(R/\mathfrak{b}_v) = 1$.

$\mathfrak{c}_{\{v,w\}}$: Relabel the vertices again so that $v = 1$ and $w = 2$. Then $R/\mathfrak{c}_{\{v,w\}} \cong R'_2$. Now, R'_2 clearly has regularity 0 over itself, and so by Proposition 5.4.6 we have $\operatorname{reg}_R(R/\mathfrak{c}_{\{v,w\}}) = 0$.

$\mathfrak{d}_{\{v,w\}}$: Again, relabel the vertices so that $v = 1$ and $w = 2$. Then

$$R/\mathfrak{d}_{\{v,w\}} \cong R'_2/\mathcal{J}_2(K_2)$$

This has regularity 1 over R_2' by Proposition 5.4.9, and so, as before by Proposition 5.4.6, we have $\operatorname{reg}_R(R/\mathfrak{d}_{\{v,w\}}) = 1$.

This then yields

$$\begin{aligned}
\operatorname{end}_R(H^5_{\mathfrak{m}}(R/\mathcal{J}(\overline{G}))) &= \max\{\operatorname{end}_R(H^5_{\mathfrak{m}}(R/\mathfrak{a}_v)), \operatorname{end}_R(H^4_{\mathfrak{m}}(R/\mathfrak{b}_v)),\\
&\qquad \operatorname{end}_R(H^4_{\mathfrak{m}}(R/\mathfrak{c}_{\{v,w\}})), \operatorname{end}_R(H^3_{\mathfrak{m}}(R/\mathfrak{d}_{\{v,w\}}))\}\\
&= \max\{1-5, 1-4, 0-4, 1-3\}\\
&= \max\{-4, -3, -4, -2\}\\
&= -2
\end{aligned}$$

and for $5 < i < n+1$

$$\begin{aligned}
\operatorname{end}_R(H^i_{\mathfrak{m}}(R/\mathcal{J}(\overline{G}))) &= \max\{\operatorname{end}_R(H^i_{\mathfrak{m}}(R/\mathfrak{a}_v)), \operatorname{end}_R(H^{i-1}_{\mathfrak{m}}(R/\mathfrak{b}_v))\}\\
&= \max\{1-i, 1-(i-1)\}\\
&= \max\{1-i, 2-i\}\\
&= 2-i
\end{aligned}$$

and finally

$$\begin{aligned}
\operatorname{end}_R(H^{n+1}_{\mathfrak{m}}(R/\mathcal{J}(\overline{G}))) &= \max\{\operatorname{end}_R(H^{n+1}_{\mathfrak{m}}(R/\mathfrak{j})), \operatorname{end}_R(H^{n+1}_{\mathfrak{m}}(R/\mathfrak{a}_v)), \operatorname{end}_R(H^{n}_{\mathfrak{m}}(R/\mathfrak{b}_v))\}\\
&= \max\{1-(n+1), 1-(n+1), 1-n\}\\
&= \max\{-n, -n, 1-n\}\\
&= 1-n
\end{aligned}$$

Then

$$\begin{aligned}
\operatorname{reg}_R(R/\mathcal{J}(\overline{G})) &= \max\{-2+5, (2-i)+i, (1-n)+(n+1)\}\\
&= \max\{3, 2, 2\}\\
&= 3
\end{aligned}$$

and we are done. □

Chapter 6

Attached Primes of Local Cohomology Modules of Binomial Edge Ideals of Block Graphs

Throughout this chapter, we set

$$R = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

for some field k and $n \geq 1$, and $\delta_{i,j} := x_i y_j - x_j y_i$.

In this chapter, we calculate the minimal attached primes of the local cohomology modules of the binomial edge ideals of block graphs. In particular, we obtain a combinatorial characterisation of which of these modules are non-vanishing.

During the writing of this thesis, it was shown by Lax, Rinaldo, and Romeo in [LRR24, Theorem 3.2] that the binomial edge ideals of block graphs are sequentially Cohen-Macaulay (see Definition 6.4.1). We will show that this result implies the main theorem of this chapter, although the techniques used in our original proof of our main theorem are very different to those of [LRR24].

The majority of this chapter first appeared in [Wil24].

6.1 A Short Exact Sequence for Block Graphs

We begin with some definitions:

Definition 6.1.1. *A graph is said to be* ***biconnected*** *if it is connected, and remains connected if any one of its vertices is removed. Maximal biconnected subgraphs of a graph G are called* ***biconnected components*** *of G.*

Definition 6.1.2. *We say that a graph G is a **block graph** if every biconnected component of G is a clique.*

Note. *Block graphs are sometimes also called **clique trees**, since they roughly resemble trees in which the edges have been replaced by cliques.*

Definition 6.1.3. *We say that a maximal clique of a block graph G is a **leaf clique** of G if it contains at most one vertex which intersects with another maximal clique, or a **branch clique** of G otherwise.*

Note. *Definition 6.1.3 is not established terminology.*

Next, we introduce some notation:

Notation 6.1.4. *For any cut vertex v of G, we denote by G_v the graph obtained by adding the edges of the complete graph with vertex set $N_G(v)$ to G (that is, we complete the neighbourhood of v in G).*

Suppose that G has a cut vertex. By [Oht11, Lemma 4.8], we have

$$\mathcal{J}_G = \mathcal{J}_{G_v} \cap (\mathcal{J}_G + (x_v, y_v))$$

Note that

$$\mathcal{J}_G + (x_v, y_v) = \mathcal{J}_{G \setminus \{v\}} + (x_v, y_v)$$

and

$$\mathcal{J}_{G_v} + (\mathcal{J}_G + (x_v, y_v)) = \mathcal{J}_{G_v \setminus \{v\}} + (x_v, y_v)$$

For brevity, we set

$$\begin{aligned} G' &= G_v \\ G'' &= G \setminus \{v\} \\ H &= G_v \setminus \{v\} \end{aligned}$$

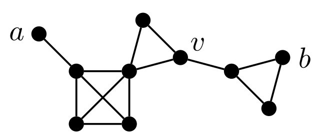
and

$$\begin{aligned} Q_1 &= \mathcal{J}_{G'} \\ Q_2 &= \mathcal{J}_{G''} + (x_v, y_v) \\ Q_3 &= \mathcal{J}_H + (x_v, y_v) \end{aligned}$$

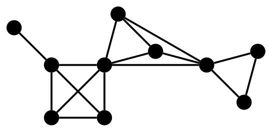
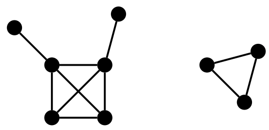
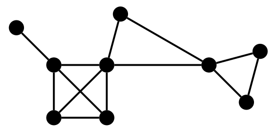
Then we obtain the short exact sequence

$$0 \longrightarrow R/\mathcal{J}_G \hookrightarrow R/Q_1 \oplus R/Q_2 \twoheadrightarrow R/Q_3 \longrightarrow 0$$

Example 6.1.5. If

$G =$ 

then the cliques containing the vertices a and b are leaf cliques, will all other cliques being branch cliques, and

$G' =$  $G'' =$  $H =$ 

6.2 Some Properties of Cut Vertices of Block Graphs

Throughout this section, we suppose that G is a block graph on n vertices which is not a disjoint union of cliques, and so it has at least one cut vertex.

v will always denote a cut vertex of G, and we adopt the notation of Section 6.1.

We will next compute $\mathcal{C}(G')$, as well as $\mathcal{C}(G'')$ and $\mathcal{C}(H)$ for certain v, relative to $\mathcal{C}(G)$.

We begin with a preliminary lemma:

Lemma 6.2.1. *For any $S \subseteq V(G)$ with $v \notin S$, and any vertices a and b of $G \setminus S$ other than v, the following are equivalent:*

1. *a and b are connected by a path in $G \setminus S$.*
2. *a and b are connected by a path in $G' \setminus S$.*
3. *a and b are connected by a path in $H \setminus S$.*

Proof.

(1) ⇒ (2): This follows immediately from the fact that $E(G') \supseteq E(G)$.

(2) ⇒ (3): Let P be a path connecting a and b in $G' \setminus S$. Since $H = G' \setminus \{v\}$, the only issue that may arise is if P passes through v. Then suppose that this is the case, so P contains edges $\{w_1, v\}$ and $\{v, w_2\}$ for some $w_1, w_2 \in N_G(v) \setminus S$. Since $N_G(v)$ has been completed in H, we may replace $\{w_1, v\}$ and $\{v, w_2\}$ in P with $\{w_1, w_2\}$. This new path then connects a and b in $H \setminus S$.

(3) ⇒ (1): Now let P be a path connecting a and b in $H \setminus S$. Since H is obtained by completing $N_G(v)$ in G and then removing v, the only issue that may arise here is if P includes any edges of the form $\{w_1, w_2\}$ for some $w_1, w_2 \in N_G(v) \setminus S$ with $\{w_1, w_2\} \notin E(G)$, so suppose that this is the case. Since $v \notin S$, we know that the edges $\{w_1, v\}$ and $\{v, w_2\}$ belong to $G \setminus S$, and so we may use these edges to replace $\{w_1, w_2\}$ in P. If P contains several such "bad" edges, the graph obtained by making these replacements in P, say Q, will no longer be a path. However, it will still be connected, and so we can find a subgraph of Q which is a path connecting a and b in $G \setminus S$. □

We can now compute $\mathcal{C}(G')$:

Proposition 6.2.2. *We have*

$$\mathcal{C}(G') = \{S \in \mathcal{C}(G) : v \notin S\}$$

Proof. We will first show that

$$\mathcal{C}(G') \supseteq \{S \in \mathcal{C}(G) : v \notin S\}$$

Take any $S \in \mathcal{C}(G)$ such that $v \notin S$, and any $w \in S$. To show that $S \in \mathcal{C}(G')$, we must show that adding w back to $G' \setminus S$ reconnects (at least) two vertices lying in separate connected components of $G' \setminus S$. We know that adding w back to $G \setminus S$ reconnects at least two vertices $a, b \in N_G(w) \setminus S$ lying in separate connected components of $G \setminus S$ since $S \in \mathcal{C}(G)$. By Lemma 6.2.1, a and b will lie in separate connected components of $G' \setminus S$ also, and will be reconnected in $G' \setminus S$ by adding w back to $G' \setminus S$. Then $S \in \mathcal{C}(G')$, and the first inclusion follows.

We will now show that

$$\mathcal{C}(G') \subseteq \{S \in \mathcal{C}(G) : v \notin S\}$$

Take any $S \in \mathcal{C}(G')$, and any $w \in S$. Note that any vertices in $N_G(v) \setminus S$ will be connected in $G' \setminus S$ (since we obtained G' from G by completing $N_G(v)$), and so we cannot have $v \in S$, since adding it back to $G' \setminus S$ cannot then reconnect any separate connected components of $G' \setminus S$, which would contradict that $S \in \mathcal{C}(G')$.

To show that $S \in \mathcal{C}(G)$, we must show that adding w back to $G \setminus S$ reconnects (at least) two vertices lying in separate connected components of $G \setminus S$. We know that adding w back to $G' \setminus S$ reconnects at least two vertices $a, b \in N_{G'}(w) \setminus S$ lying in separate connected components of $G' \setminus S$ since $S \in \mathcal{C}(G')$. By Lemma 6.2.1, a and b will lie in separate connected components of $G \setminus S$ also, and will be reconnected in $G \setminus S$ by adding w back to $G \setminus S$. Then $S \in \mathcal{C}(G)$, and the result follows. □

We will make use of the following two lemmas:

Lemma 6.2.3. *Suppose that G has at least two cut vertices. Then there exists a leaf clique of G which intersects with exactly one branch clique of G.*

Proof. The proof will be by induction on the number of cut vertices of G. If G has a exactly two cut vertices then the result is obvious, so suppose that G has more than two cut vertices.

Since G has a cut vertex, we may choose a leaf clique L of G with a cut vertex $w \in V(L)$, and set $A = G \setminus (C \setminus \{w\})$. That is, we obtain A by removing L from G, but keeping w. A has fewer cut vertices than G, and so we can find a leaf clique C of A which intersects with exactly one branch clique of A by induction.

If C remains a leaf clique in G, then it must intersect with exactly one branch clique of G, since we only removed a leaf clique from G to obtain A. The only way for C to become a branch clique in G is if L is a leaf clique intersecting with it. In this case, since C is a leaf clique in A, the only branch clique that L can intersect with in G is C. Then, in either case, we can find a leaf clique of G which intersects with exactly one branch clique of G, so we are done. □

Lemma 6.2.4. *Suppose that $S \in \mathcal{C}(H)$, and that $N_G(v) \subseteq S$. Then $S \in \mathcal{C}(G'')$.*

Proof. Since $N_G(v) \subseteq S$, we have $H \setminus S = G'' \setminus S$. The only edges in H that are not in G'' are between neighbours of v, but these are all removed in $H \setminus S$, so adding any single vertex back to either $H \setminus S$ or $G'' \setminus S$ will have the same effect. □

For the purposes of Subsection 6.3.2, we will need to choose v with a particular property:

Proposition 6.2.5. *There exists a cut vertex v of G such that*

$$\mathcal{C}(G'') = \{S \setminus \{v\} : S \in \mathcal{C}(G) \text{ with } v \in S\} \tag{†}$$

Proof. The inclusion

$$\mathcal{C}(G'') \supseteq \{S \setminus \{v\} : S \in \mathcal{C}(G) \text{ with } v \in S\}$$

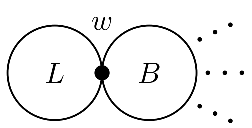
clearly holds for any cut vertex v of G, and so we will now find a cut vertex of G satisfying the reverse inclusion.

We proceed by induction on the number of cut vertices of G. If G has a single cut vertex then the result is obvious, so suppose that G has more than one cut vertex.

By Lemma 6.2.3, we can choose a leaf clique L of G, with cut vertex $w \in V(L)$, which intersects with exactly one branch clique B of G.

If this w belongs to at least two leaf cliques, then, for any $S \in \mathcal{C}(G'')$, we have $S \cup \{w\} \in \mathcal{C}(G)$, since adding w back to $G \setminus S$ will reconnect these cliques, and so the desired inclusion would be satisfied by taking $v = w$.

Otherwise, w belongs to a single leaf clique, and so we are in the situation

$G =$ 

(where circles denote cliques).

As in Lemma 6.2.3, set $A = G \setminus (L \setminus \{w\})$. A has fewer cut vertices than G, and so we can find some cut vertex v of A satisfying (†) for A by induction. We claim that this v also satisfies (†) for G itself.

Take any $S \in \mathcal{C}(G'')$. We aim to show that $S \cup \{v\} \in \mathcal{C}(G)$.

Now, clearly $S \setminus \{w\} \in \mathcal{C}(A'')$ (note that we do not necessarily have $w \in S$), and so by the inductive hypothesis we have

$$(S \setminus \{w\}) \cup \{v\} \in \mathcal{C}(A) \subseteq \mathcal{C}(G) \quad (\diamond)$$

Then if $w \notin S$ we are done, so suppose that $w \in S$.

For any $u \in S$, adding u back to $G'' \setminus S$ reconnects at least two vertices $a_u, b_u \in N_{G''}(u) \setminus S$ lying in separate connected connected components of $G'' \setminus S$ since $S \in \mathcal{C}(G'')$. Neither a_u nor b_u can be v since $v \notin V(G'')$, and so

$$a_u, b_u \in N_G(u) \setminus (S \cup \{v\})$$

Furthermore, we have

$$G'' \setminus S = G \setminus (S \cup \{v\})$$

so a_u and b_u will also lie in separate connected components of $G \setminus (S \cup \{v\})$, and will still be reconnected by adding u back to $G \setminus (S \cup \{v\})$.

Note in particular that, since $w \in S$, we have at least two vertices

$$a_w, b_w \in N_G(w) \setminus (S \cup \{v\})$$

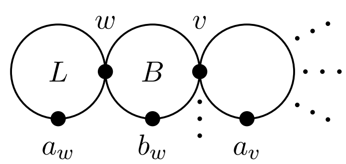
and since w belongs to only two cliques, L and B, in G, we may assume (without loss of generality) that $a_w \in V(L)$ and $b_w \in V(B)$, as a_w and b_w must lie in separate connected components of $G'' \setminus S$.

To conclude the proof, we wish to show that adding v back to $G \setminus (S \cup \{v\})$ reconnects (at least) two vertices lying in separate connected components of $G \setminus (S \cup \{v\})$.

For brevity, let $U = (S \setminus \{w\}) \cup \{v\}$. We saw in ($\diamond$) that $U \in \mathcal{C}(G)$, and so adding v back to $G \setminus U$ reconnects at least two vertices $a_v, b_v \in N_G(v) \setminus U$ lying in separate connected connected components of $G \setminus U$. Note that neither a_v nor b_v can belong to L, since $v \neq w$ and w is the only cut vertex belonging to L.

Removing w from $G \setminus U$ simply disconnects B and L, so, if neither a_v nor b_v is w, a_v and b_v will trivially lie in separate connected components of $G \setminus (S \cup \{v\})$, and will still be reconnected by adding v back to $G \setminus (S \cup \{v\})$.

However, if we have say $b_v = w$, then $b_v \notin G \setminus (S \cup \{v\})$, and so the result does not immediately follow. In this case, we have $v \in V(B)$, since the only branch clique that w belongs to is B, and so every cut vertex that w is adjacent to must belong to B also. Then we are in the following situation:

$G =$ 

We may then instead consider a_v and $b_w \in V(B) \setminus (S \cup \{v\})$. We have $b_w \in N_G(v) \setminus (S \cup \{v\})$, a_v and b_w will clearly lie in separate connected components of $G \setminus (S \cup \{v\})$, and adding v back to $G \setminus (S \cup \{v\})$ reconnects a_v and b_w. Then $S \cup \{v\} \in \mathcal{C}(G)$, and we are done. □

We can compute $\mathcal{C}(H)$ for such a v:

Proposition 6.2.6. *Let v be as in Proposition 6.2.5. Then we have*

$$\mathcal{C}(H) = \{S \in \mathcal{C}(G) : v \notin S \text{ and } N_G(v) \not\subseteq S\}$$

Proof. We will first show that

$$\mathcal{C}(H) \supseteq \{S \in \mathcal{C}(G) : v \notin S \text{ and } N_G(v) \not\subseteq S\}$$

Take any $S \in \mathcal{C}(G)$ such that $v \notin S$ and $N_G(v) \not\subseteq S$, and any $w \in S$. To show that $S \in \mathcal{C}(H)$, we must show that adding w back to $H \setminus S$ reconnects (at least) two vertices lying in separate connected components of $H \setminus S$. We know that adding w back to $G \setminus S$ reconnects at least two vertices $a, b \in N_G(w) \setminus S$ lying in separate connected components of $G \setminus S$ since $S \in \mathcal{C}(G)$. We may assume that neither a nor b is v, since $N_G(v) \not\subseteq S$ and so we may replace v with a neighbour if necessary. By Lemma 6.2.1, a and b will lie in separate connected components of $H \setminus S$ also, and will be reconnected in $H \setminus S$ by adding w back to $H \setminus S$. Then $S \in \mathcal{C}(H)$, and the first inclusion follows.

We will now show that

$$\mathcal{C}(H) \subseteq \{S \in \mathcal{C}(G) : v \notin S \text{ and } N_G(v) \not\subseteq S\}$$

Take any $S \in \mathcal{C}(H)$. Trivially, $v \notin S$. If $N_G(v) \subseteq S$, then, by Lemma 6.2.4, we have $S \in \mathcal{C}(G'')$. Since v is as in Proposition 6.2.5, we then have $S \cup \{v\} \in \mathcal{C}(G)$. But $N_G[v] \subseteq S \cup \{v\}$, so this clearly cannot belong to $\mathcal{C}(G)$, since adding v back to $G \setminus (S \cup \{v\})$ would not reconnect any separate connected components of $G \setminus (S \cup \{v\})$. Then we must have $N_G(v) \not\subseteq S$.

Now take any $w \in S$. To show that $S \in \mathcal{C}(G)$, we must show that adding w back to $G \setminus S$ reconnects (at least) two vertices lying in separate connected components of $G \setminus S$. We know that adding w back to $H \setminus S$ reconnects at least two vertices $a, b \in N_H(w) \setminus S$ lying in separate connected components of $H \setminus S$ since $S \in \mathcal{C}(H)$. By Lemma 6.2.1, a and b will lie in separate connected components of $G \setminus S$ also, and will be reconnected in $G \setminus S$ by adding w back to $G \setminus S$. Then $S \in \mathcal{C}(G)$, and the result follows. □

To summarise this section, when v is as in Proposition 6.2.5, we have

- $\mathcal{C}(G') = \{S \in \mathcal{C}(G) : v \notin S\}$ by Proposition 6.2.2.
- $\mathcal{C}(G'') = \{S \setminus \{v\} : S \in \mathcal{C}(G) \text{ with } v \in S\}$ by Proposition 6.2.5.
- $\mathcal{C}(H) = \{S \in \mathcal{C}(G) : v \notin S \text{ and } N_G(v) \not\subseteq S\}$ by Proposition 6.2.6.

6.3 The Main Theorem

Our goal is to show that, for any block graph G, we have

$$\mathrm{MinAtt}_R(H^i_{\mathfrak{m}}(R/\mathcal{J}_G)) = \{\mathfrak{p} \in \mathrm{Ass}_R(R/\mathcal{J}_G) : \dim(R/\mathfrak{p}) = i\}$$

We must prove a few results beforehand.

6.3.1 A Preliminary Lemma

Proposition 6.3.1. *Let R be a ring and z an indeterminate. Furthermore, set $S = R[z]$, and let M be an S-module such that $zM = 0$, viewed also as an R-module in the natural way. Then*

$$\mathrm{Ass}_S(M) = \{\mathfrak{q}S + (z) : \mathfrak{q} \in \mathrm{Ass}_R(M)\}$$

Proof. First, note that

$$\mathrm{Ann}_S(N) = \mathrm{Ann}_R(N)S + (z)$$

for any S-submodule N of M (with N also viewed as an R-module in the natural way).

If $\mathfrak{p} \in \mathrm{Ass}_S(M)$ then $\mathfrak{p} = \mathrm{Ann}_S(m)$ for some $m \in M$. We have that

$$R/\mathrm{Ann}_R(m) \cong S/(\mathrm{Ann}_R(m)S + (z)) = S/\mathfrak{p}$$

is an integral domain since $\mathfrak{p} \in \mathrm{Spec}(S)$, so $\mathrm{Ann}_R(m) \in \mathrm{Ass}_R(M)$ and therefore

$$\mathfrak{p} = \mathrm{Ann}_R(M)S + (z)$$

is of the desired form.

Conversely, if $\mathfrak{q} \in \mathrm{Ass}_R(M)$ then $\mathfrak{q} = \mathrm{Ann}_R(m)$ for some $m \in M$. We have that

$$S/\mathrm{Ann}_S(m) = S/(\mathrm{Ann}_R(m)S + (z)) \cong R/\mathrm{Ann}_R(M) = R/\mathfrak{q}$$

is an integral domain since $\mathfrak{q} \in \mathrm{Spec}(R)$, so

$$\mathfrak{q}S + (z) = \mathrm{Ann}_S(m) \in \mathrm{Ass}_S(M)$$

and we are done. □

Lemma 6.3.2. *Let R be a ring, $\mathfrak{a}$ an ideal of R, and z an indeterminate. Set $S = R[z]$, and let $\mathfrak{b} = \mathfrak{a}S + (z)$. Then*

$$\mathrm{Ass}_S(\mathrm{Ext}^i_S(S/\mathfrak{b}, S)) = \{\mathfrak{q}S + (z) : \mathfrak{q} \in \mathrm{Ass}_R(\mathrm{Ext}^{i-1}_R(R/\mathfrak{a}, R))\}$$

Proof. By [Ree56, Theorem 2.1] we have that

$$\operatorname{Ext}^i_S(S/\mathfrak{b}, S) \cong \operatorname{Ext}^{i-1}_R(R/\mathfrak{a}, R)$$

as S-modules by setting $A = S$, $\mathfrak{g} = (z)$, $M = S/\mathfrak{b}$ and $N = S$ in their notation, since $S/\mathfrak{b} \cong R/\mathfrak{a}$ as R-modules, and $z \operatorname{Ext}^i_S(S/\mathfrak{b}, S) = 0$, so we are done by Proposition 6.3.1. □

6.3.2 Proof of the Main Theorem

By Graded Local Duality and Theorem 3.10.12, we have

$$\operatorname{Att}_R(H^i_{\mathfrak{m}}(R/\mathcal{J}_G)) = \operatorname{Ass}_R(\operatorname{Ext}^{2n-i}_R(R/\mathcal{J}_G, R))$$

and so our main theorem amounts to showing that

$$\operatorname{MinAss}_R(\operatorname{Ext}^i_R(R/\mathcal{J}_G, R)) = \{\mathfrak{p} \in \operatorname{Ass}_R(R/\mathcal{J}_G) : \operatorname{height}_R(\mathfrak{p}) = i\}$$

since

$$\operatorname{height}_R(\mathfrak{p}) = 2n - \dim(R/\mathfrak{p})$$

by Lemma 3.11.1 (which we may apply since every associated prime of $\mathcal{J}_G$ is homogeneous by Proposition 3.10.1).

We first introduce some notation:

Notation 6.3.3. *For any R-module M and $i \geq 0$, we set*

$$\operatorname{Ass}^i_R(M) := \{\mathfrak{p} \in \operatorname{Ass}_R(M) : \operatorname{height}_R(\mathfrak{p}) = i\}$$

Notation 6.3.4. *For any ideal $\mathfrak{a}$ of R and $i \geq 0$, we set*

$$E^i_R(\mathfrak{a}) := \operatorname{Ext}^i_R(R/\mathfrak{a}, R)$$

Notation 6.3.5. *For any $1 \leq v \leq n$, we set*

$$R_v := k[x_i, y_i : 1 \leq i \leq n \textit{ with } i \neq v]$$

Our key proposition is as follows:

Proposition 6.3.6. *Let G be a block graph. Then any associated prime of $E^i_R(\mathcal{J}_G)$ contains an associated prime of $R/\mathcal{J}_G$ of height i.*

Proof. When G is a disjoint union of cliques, $R/\mathcal{J}_G$ is Cohen-Macaulay (this follows, for example, from [BV88, Corollary 2.8] and [BK02, Theorem 2.1]). By Theorem 3.10.14 and Lemma 3.7.4, the attached primes of $H^d_{\mathfrak{m}}(R/\mathcal{J}_G)$ are exactly the associated primes of $R/\mathcal{J}_G$ of height $\dim(R/\mathcal{J}_G)$.

Since $R/\mathcal{J}_G$ is Cohen-Macaulay in this case, it has no other such local cohomology modules by Lemma 3.11.4, and so the result is immediate.

We will induct first on n, the number vertices of G. When $n = 2$, we must have $G = K_2$, so G is a clique. We have already dealt with this case, and so the base case is established.

We now induct on the number of cut vertices of G. When G has no cut vertices, it must be a disjoint union of cliques. Again, we have already dealt with this case, and so the base case is established for this induction also.

We may assume then that G contains a cut vertex, and can choose such a cut vertex v of G as in Proposition 6.2.5. We adopt the notation of Section 6.1.

The short exact sequence

$$0 \longrightarrow R/\mathcal{J}_G \hookrightarrow R/Q_1 \oplus R/Q_2 \twoheadrightarrow R/Q_3 \longrightarrow 0$$

gives rise to the long exact sequence

$$\cdots \xrightarrow{\alpha_i} E^i_R(Q_1) \oplus E^i_R(Q_2) \longrightarrow E^i_R(\mathcal{J}_G) \longrightarrow E^{i+1}_R(Q_3) \xrightarrow{\alpha_{i+1}} \cdots$$

Setting

$$A^i := (E^i_R(Q_1) \oplus E^i_R(Q_2))/\operatorname{Im}(\alpha_i)$$

we then have the short exact sequence

$$0 \longrightarrow A^i \hookrightarrow E^i_R(\mathcal{J}_G) \twoheadrightarrow \ker(\alpha_{i+1}) \longrightarrow 0$$

Suppose that $E^i_R(\mathcal{J}_G) \neq 0$, and take any $\mathfrak{p} \in \operatorname{Ass}_R(E^i_R(\mathcal{J}_G))$. Localising at $\mathfrak{p}$ gives us the short exact sequence

$$0 \longrightarrow A^i_{\mathfrak{p}} \hookrightarrow E^i_R(\mathcal{J}_G)_{\mathfrak{p}} \twoheadrightarrow \ker(\alpha_{i+1})_{\mathfrak{p}} \longrightarrow 0$$

If $A^i_{\mathfrak{p}} = 0$, then

$$E^i_R(\mathcal{J}_G)_{\mathfrak{p}} = \ker(\alpha_{i+1})_{\mathfrak{p}} \hookrightarrow E^{i+1}_R(Q_3)_{\mathfrak{p}}$$

so $\mathfrak{p} \in \operatorname{Supp}_R(E^{i+1}_R(Q_3))$. Then $\mathfrak{p}$ contains some associated prime of $E^{i+1}_R(Q_3)$, and

$$\operatorname{Ass}_R(E^{i+1}_R(Q_3)) = \{\mathfrak{q}R + (x_v, y_v) : \mathfrak{q} \in \operatorname{Ass}_{R_v}(E^{i-1}_{R_v}(\mathcal{J}_H))\}$$

by Lemma 6.3.2 (where $\mathcal{J}_H$ is viewed here as an ideal in R_v).

By the (first) inductive hypothesis, we have

$$\operatorname{MinAss}_{R_v}(E^{i-1}_{R_v}(\mathcal{J}_H)) = \{\mathfrak{q} \in \operatorname{Ass}_{R_v}(R_v/\mathcal{J}_H) : \operatorname{height}_{R_v}(\mathfrak{q}) = i - 1\}$$

We know then that $\mathfrak{p}$ contains $\mathfrak{q}R + (x_v, y_v)$ for some $\mathfrak{q} \in \mathrm{Ass}_{R_v}(R_v/\mathcal{J}_H)$ with $\mathrm{height}_{R_v}(\mathfrak{q}) = i-1$, and we can write $\mathfrak{q} = P_S(H)$ for some $S \in \mathcal{C}(H)$. We claim that $P_S(G) \subseteq \mathfrak{p}$, and that $\mathrm{height}_R(P_S(G)) = i$.

This first claim follows since

$$P_S(G) \subseteq P_S(H)R + (x_v, y_v)$$

because the only edges of $G \setminus S$ which may not belong to $H \setminus S$ are of the form $\{v, w\}$ for some $w \in N_G(v)$, and so any elements of $P_S(G)$ introduced by these edges will be included in (x_v, y_v). We will now prove the second claim.

By Proposition 6.2.6, we have

$$\mathcal{C}(H) = \{S \in \mathcal{C}(G) : v \notin S \text{ and } N_G(v) \nsubseteq S\}$$

Then $c_G(S) = c_H(S)$, since v is the only vertex in $G \setminus S$ which is not in $H \setminus S$, and $N_G(v) \nsubseteq S$, so v will belong to the same connected component as at least one of its neighbours in $G \setminus S$.

By Lemma 1.4.8, we then have

$$\mathrm{height}_R(P_S(G)) = |S| + n - c_G(S) = (|S| + (n-1) - c_H(S)) + 1 = \mathrm{height}_{R_v}(P_S(H)) + 1 = i$$

as desired, and so the result holds in the case that $A^i_{\mathfrak{p}} = 0$.

If $A^i_{\mathfrak{p}} \neq 0$, then we must have either $\mathfrak{p} \in \mathrm{Supp}_R(E^i_R(Q_1))$, or $\mathfrak{p} \in \mathrm{Supp}_R(E^i_R(Q_2))$.

In the case that $\mathfrak{p} \in \mathrm{Supp}_R(E^i_R(Q_1))$, we have that $\mathfrak{p}$ contains some associated prime of $E^i_R(Q_1)$. Note that G' has fewer cut vertices than G (since we have completed the neighbourhood of v, and so it cannot be a cut vertex). Then we can apply the (second) inductive hypothesis to obtain

$$\mathrm{MinAss}_R(E^i_R(Q_3)) = \{\mathfrak{q} \in \mathrm{Ass}_R(R/\mathcal{J}_{G'}) : \mathrm{height}_R(\mathfrak{q}) = i\}$$

Then $\mathfrak{p}$ contains some $\mathfrak{q} \in \mathrm{Ass}_R(R/\mathcal{J}_{G'})$ with $\mathrm{height}_R(\mathfrak{q}) = i$, and we can write $\mathfrak{q} = P_S(G')$ for some $S \in \mathcal{C}(G')$. We also have $S \in \mathcal{C}(G)$ by Proposition 6.2.2, and $P_S(G) \subseteq P_S(G')$ (since we have only added edges to G to obtain G'), so we are done with this case if we can show that $\mathrm{height}_R(P_S(G)) = i$. By Lemma 1.4.8, this amounts to showing that $c_G(S) = c_{G'}(S)$. This follows from Lemma 6.2.1, which we may apply since $S \in \mathcal{C}(G')$ and so $v \notin S$ by Proposition 6.2.2. Then the result holds in this case also.

Finally, in the case that $\mathfrak{p} \in \mathrm{Supp}_R(E^i_R(Q_2))$, we have that $\mathfrak{p}$ contains some associated prime of $E^i_R(Q_2)$, and

$$\mathrm{Ass}_R(E^i_R(Q_2)) = \{\mathfrak{q}R + (x_v, y_v) : \mathfrak{q} \in \mathrm{Ass}_{R_v}(E^{i-2}(\mathcal{J}_{G''}))\}$$

by Lemma 6.3.2 (where $\mathcal{J}_{G''}$ is viewed here as an ideal in R_v).

By the (first) inductive hypothesis, we have

$$\mathrm{MinAss}_{R_v}(E^{i-2}_{R_v}(\mathcal{J}_{G''})) = \{\mathfrak{q} \in \mathrm{Ass}_{R_v}(R_v/\mathcal{J}_{G''}) : \mathrm{height}_{R_v}(\mathfrak{q}) = i-2\}$$

We know then that $\mathfrak{p}$ contains $\mathfrak{q}R+(x_v, y_v)$ for some $\mathfrak{q} \in \operatorname{Ass}_{R_v}(R_v/\mathcal{J}_{G''})$ with $\operatorname{height}_{R_v}(\mathfrak{q}) = i-2$, and we can write $\mathfrak{q} = P_S(G'')$ for some $S \in \mathcal{C}(G'')$.

Now, we have $S \cup \{v\} \in \mathcal{C}(G)$ (since we have chosen v to be as in Proposition 6.2.5), and

$$P_{S\cup\{v\}}(G) = P_S(G'')R + (x_v, y_v)$$

is clearly of the correct height (to see this using Lemma 1.4.8, note that $G \setminus \{S \cup \{v\}\} = G'' \setminus S$), so we are done. □

We can now prove our main theorem:

Theorem 6.3.7. *For any block graph G, we have*

$$\operatorname{MinAtt}_R(H^i_{\mathfrak{m}}(R/\mathcal{J}_G)) = \{\mathfrak{p} \in \operatorname{Ass}_R(R/\mathcal{J}_G) : \dim(R/\mathfrak{p}) = i\}$$

Proof. By [EHV92, Algorithm 1.5], for any finitely generated R-module M, we have

$$\operatorname{Ass}^i_R(M) = \operatorname{Ass}^i_R(\operatorname{Ext}^i_R(M, R))$$

Combining this with Proposition 6.3.6 completes the proof. □

Corollary 6.3.8. *Let G be a block graph. Then we have that $H^i_{\mathfrak{m}}(R/\mathcal{J}_G) \neq 0$ if and only if $R/\mathcal{J}_G$ has an associated prime of dimension i. This can be checked combinatorially via Lemma 1.4.8 (noting Lemma 3.11.1).*

Note. *Corollary 6.3.8 can be seen as a generalisation of the main result of [EHH11, Theorem 1.1] which, by Theorem 3.8.5, is equivalent to stating that, for a block graph G on n vertices, we have*

$$\min\{i \geq 0 : H^i_{\mathfrak{m}}(R/\mathcal{J}_G) \neq 0\} = n + c$$

where c is the number of connected components of G. It is easily seen that $n + c$ is the height of $P_\varnothing(G)$, and that this height is minimal amongst the heights of the associated primes of $R/\mathcal{J}_G$, so applying Corollary 6.3.8 yields this result.

Whilst block graphs are not explicitly given as the family of graphs in the statement of [EHH11, Theorem 1.1], it can been seen that these families are the same via, for example, [How79, Theorem 2.6].

6.4 An Alternative Proof of the Main Theorem

We first describe a generalisation of Cohen-Macaulay R-modules first introduced by Stanley:

Definition 6.4.1. [Sta96, p. 87, Definition 2.9] *Let k be a field, and R a non-negatively $\mathbb{Z}$-graded k-algebra of finite type such that the homogeneous elements of R of degree 0 are precisely k. We say that a finitely generated $\mathbb{Z}$-graded R-module M is **sequentially Cohen-Macaulay** if there exists a filtration* *of graded R-modules*

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{t-1} \subseteq M_t = M$$

for some $t \geq 0$ such that:

1. *For each $1 \leq i \leq t$, we have that M_i/M_{i-1} is Cohen-Macaulay.*
2. *We have*

$$\dim_R(M_j/M_{j-1}) < \dim_R(M_{j+1}/M_j)$$

for each $1 \leq j \leq t-1$.

An important alternative characterisation of sequentially Cohen-Macaulay R-modules was given by Peskine. We state it here in the specific case of a polynomial ring over a field:

Theorem 6.4.2. *Let $R = k[x_1, \dots, x_n]$ for some field k and $n \geq 0$ with the standard $\mathbb{Z}$-grading, and let M be a finitely generated $\mathbb{Z}$-graded R-module of dimension d. Then M is sequentially Cohen-Macaulay if and only if, for all $0 \leq i \leq d$, we have that $\operatorname{Ext}_R^{n-i}(M, R)$ is either 0 or Cohen-Macaulay of dimension i.*

Proof. This follows from [HS02, Theorem 1.4] and Proposition 3.9.5 since R is Gorenstein. □

Characterising families of graphs which give rise to sequentially Cohen-Macaulay binomial edge ideals is an active area of study. During the writing of this thesis, Lax, Rinaldo, and Romeo showed in [LRR24] that several well-known classes of graphs have sequentially Cohen-Macaulay binomial edge ideals. In particular:

Theorem 6.4.3. [LRR24, Theorem 3.2] *For any block graph G, $R/\mathcal{J}_G$ is sequentially Cohen-Macaulay.*

As was mentioned at the start of this chapter, this theorem implies Theorem 6.3.7. In fact, it implies the following stronger result:

Theorem 6.4.4. *For any block graph G, we have*

$$\operatorname{Att}_R(H^i_{\mathfrak{m}}(R/\mathcal{J}_G)) = \{\mathfrak{p} \in \operatorname{Ass}_R(R/\mathcal{J}_G) : \dim(R/\mathfrak{p}) = i\}$$

We will prove Theorem 6.4.4 using Theorem 6.4.3 and the following proposition:

Proposition 6.4.5. *Let $R = k[x_1, \dots, x_n]$ for some field k and $n \geq 0$ with the standard $\mathbb{Z}$-grading, and let M be a finitely generated $\mathbb{Z}$-graded R-module. Then, if M is sequentially Cohen-Macaulay, we have*

$$\operatorname{Ass}_R(\operatorname{Ext}^i_R(M, R)) = \operatorname{Ass}^i_R(M)$$

Proof. As noted in the proof of Theorem 6.3.7, we have

$$\operatorname{Ass}^i_R(\operatorname{Ext}^i_R(M, R)) = \operatorname{Ass}^i_R(M)$$

by [EHV92, Algorithm 1.5]. Then if $\mathrm{Ext}^i_R(M,R)=0$, we must have $\mathrm{Ass}^i_R(M)=\varnothing$, so the result is trivially true in this case.

Otherwise, by Theorem 6.4.2 we have that $\mathrm{Ext}^i_R(M,R)$ is Cohen-Macaulay of dimension $n-i$, so every associated prime of $\mathrm{Ext}^i_R(M,R)$ is of height $n-(n-i)=i$ by Lemma 3.11.2, and we are done. □

Proof of Theorem 6.4.4. By Theorem 6.4.3, we may apply Proposition 6.4.5 to $R/\mathcal{J}_G$. As we noted earlier, we have that

$$\mathrm{Att}_R(H^i_{\mathfrak{m}}(R/\mathcal{J}_G))=\mathrm{Ass}_R(\mathrm{Ext}^{2n-i}_R(R/\mathcal{J}_G,R))$$

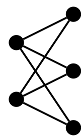
and so the result follows. □

6.5 Some Counterexamples

Theorem 6.3.7 is very far from true in general, as computations in `Macaulay2` (taking $k=\mathbb{Z}/2\mathbb{Z}$) show:

When $n=5$ and

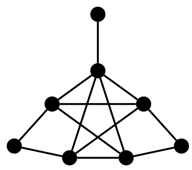
$G=$



we have $H^5_{\mathfrak{m}}(R/\mathcal{J}_G)\neq 0$, but $R/\mathcal{J}_G$ has no associated prime of dimension 5.

We can also have embedded attached primes. For example, let $n=8$ and

$G=$



Then `Macaulay2` tells us that there are $\mathfrak{p},\mathfrak{q}\in\mathrm{Spec}(R)$ such that

$$\mathrm{Att}_R(H^7_{\mathfrak{m}}(R/\mathcal{J}_G))=\{\mathfrak{p},\mathfrak{q}\}$$

with $\mathfrak{q}\subsetneq\mathfrak{p}$ and $\dim(R/\mathfrak{p})=7$, so $H^7_{\mathfrak{m}}(R/\mathcal{J}_G)$ has an embedded attached prime, and the minimal attached prime of $H^7_{\mathfrak{m}}(R/\mathcal{J}_G)$ is not of dimension 7.

Bibliography

[Àlv20] J. ÀLVAREZ MONTANER. *Local Cohomology of Binomial Edge Ideals and Their Generic Initial Ideals*. In: Collectanea Mathematica 71.2 (May 2020), pp. 331–348.

[AF12] F. W. ANDERSON and K. R. FULLER. *Rings and Categories of Modules*. Second Edition. Vol. 13. Graduate Texts in Mathematics. Springer, 2012.

[AS03] J. ASADOLLAHI and P. SCHENZEL. *Some Results on Associated Primes of Local Cohomology Modules*. In: Japanese Journal of Mathematics. New Series 29.2 (2003), pp. 285–296.

[BBL98] P. BOSE, J. F. BUSS, and A. LUBIW. *Pattern Matching for Permutations*. In: Information Processing Letters 65.5 (Mar. 1998), pp. 277–283.

[BK02] S. BOUCHIBA and S. KABBAJ. *Tensor Products of Cohen-Macaulay Rings: Solution to a Problem of Grothendieck*. In: Journal of Algebra 252.1 (June 2002), pp. 65–73.

[BS13] M. P. BRODMANN and R. Y. SHARP. *Local Cohomology: An Algebraic Introduction with Geometric Applications*. Second Edition. Vol. 136. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2013.

[BH05] W. BRUNS and J. HERZOG. *Cohen-Macaulay Rings*. Revised Edition. Vol. 39. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2005.

[BV88] W. BRUNS and U. VETTER. *Determinantal Rings*. Lecture Notes in Mathematics 1327. Springer, 1988.

[Coh46] I. S. COHEN. *On the Structure and Ideal Theory of Complete Local Rings*. In: Transactions of the American Mathematical Society 59.1 (Jan. 1946).

[DM21] A. DE STEFANI and L. MA. *F-Stable Secondary Representations and Deformation of F-Injectivity*. In: Acta Mathematica Vietnamica (Mar. 2021).

[DMN22] A. DE STEFANI, J. MONTAÑO, and L. NÚÑEZ-BETANCOURT. *Frobenius Methods in Combinatorics*. In: São Paulo Journal of Mathematical Sciences (Sept. 2022).

[Eis95] D. EISENBUD. *Commutative Algebra: With a View Toward Algebraic Geometry*. Vol. 150. Graduate Texts in Mathematics. Springer, 1995.

[Eis05] D. EISENBUD. *The Geometry of Syzygies: A Second Course in Commutative Algebra and Algebraic Geometry*. Vol. 229. Graduate Texts in Mathematics. Springer, 2005.

[EHV92] D. EISENBUD, C. L. HUNEKE, and W. VASCONCELOS. *Direct Methods for Primary Decomposition*. In: Inventiones Mathematicae 110.1 (Dec. 1992), pp. 207–235.

[EHH11] V. ENE, J. HERZOG, and T. HIBI. *Cohen-Macaulay Binomial Edge Ideals*. In: Nagoya Mathematical Journal 204 (Dec. 2011), pp. 57–68.

[Ene09] F. ENESCU. *Local Cohomology and F-stability*. In: Journal of Algebra 322.9 (Nov. 2009), pp. 3063–3077.

[GR73] R. GORDON and J. C. ROBSON. *Krull Dimension*. Memoirs of the American Mathematical Society 133. American Mathematical Society, 1973.

[GS] D. R. GRAYSON and M. E. STILLMAN. `Macaulay2`*, a software system for research in algebraic geometry*. Available at `http://www2.macaulay2.com`.

[Gro68] A. GROTHENDIECK. *Cohomologie Locale des Faisceaux Cohérents et Théorèmes de Lefschetz Locaux et Globaux: Séminaire de Géométrie Algébrique du Bois Marie 1962 (SGA 2)*. Vol. 2. Advanced Studies in Pure Mathematics. North-Holland Publishing Company, 1968.

[GH67] A. GROTHENDIECK and R. HARTSHORNE. *Local Cohomology: A Seminar Given by A. Grothendieck*. Lecture Notes in Mathematics 41. Springer, 1967.

[Har66] R. HARTSHORNE. *Residues and Duality*. Lecture Notes in Mathematics 20. Springer, 1966.

[HOI88] M. HERRMANN, U. ORBANZ, and S. IKEDA. *Equimultiplicity and Blowing Up: An Algebraic Study*. Springer, 1988.

[Her+10] J. HERZOG, T. HIBI, F. HREINSDÓTTIR, T. KAHLE, and J. RAUH. *Binomial Edge Ideals and Conditional Independence Statements*. In: Advances in Applied Mathematics 45.3 (Sept. 2010), pp. 317–333.

[HHM21] J. HERZOG, T. HIBI, and S. MORADI. *Graded Ideals of König Type*. In: Transactions of the American Mathematical Society 375.1 (Oct. 2021), pp. 301–323.

[HHO18] J. HERZOG, T. HIBI, and H. OHSUGI. *Binomial Ideals*. Vol. 279. Graduate Texts in Mathematics. Springer, 2018.

[HS02] J. HERZOG and E. SBARRA. *Sequentially Cohen-Macaulay Modules and Local Cohomology*. In: Proceedings of the International Colloquium on Algebra, Arithmetic and Geometry. Vol. 2. Tata Institute of Fundamental Research Studies in Mathematics. Narosa Publ. House, 2002, pp. 327–340.

[How79] E. HOWORKA. *On Metric Properties of Certain Clique Graphs*. In: Journal of Combinatorial Theory, Series B 27.1 (Aug. 1979), pp. 67–74.

[HS93] C. L. HUNEKE and R. Y. SHARP. *Bass Numbers of Local Cohomology Modules*. In: Transactions of the American Mathematical Society 339.2 (Feb. 1993), pp. 765–779.

[HQ19] D. T. HUONG and P. H. QUY. *Notes on the Frobenius Test Exponents*. In: Communications in Algebra 47.7 (July 2019), pp. 2702–2710.

[Iye+07] S. B. Iyengar, G. J. Leuschke, A. Leykin, C. Miller, E. Miller, A. K. Singh, and W. Uli. *Twenty-Four Hours of Local Cohomology*. Vol. 87. Graduate Studies in Mathematics. American Mathematical Society, 2007.

[JJC96] Y. Jing-Ho, C. Jer-Jeong, and G. J. Chang. *Quasi-Threshold Graphs*. In: Discrete Applied Mathematics 69.3 (Aug. 1996), pp. 247–255.

[KS12] D. Kiani and S. Saeedi Madani. *Binomial Edge Ideals of Graphs*. In: The Electronic Journal of Combinatorics 19.2 (June 2012), P44.

[Kir73] D. Kirby. *Coprimary Decomposition of Artinian Modules*. In: Journal of the London Mathematical Society s2-6.3 (May 1973), pp. 571–576.

[Kun69] E. Kunz. *Characterizations of Regular Local Rings of Characteristic p*. In: American Journal of Mathematics 91.3 (July 1969), pp. 772–784.

[LaC23] A. LaClair. *Invariants of Binomial Edge Ideals via Linear Programs*. June 2023. arXiv: `2304.13299[math]`.

[LRR24] E. Lax, G. Rinaldo, and F. Romeo. *Sequentially Cohen-Macaulay Binomial Edge Ideals*. May 2024. arXiv: `2405.08671[math]`.

[Lyu97] G. Lyubeznik. *F-Modules: Applications to Local Cohomology and D-Modules in Characteristic $p > 0$*. In: Journal für die reine und angewandte Mathematik (Crelles Journal) 491 (Oct. 1997), pp. 65–130.

[LSW16] G. Lyubeznik, A. Singh, and U. Walther. *Local Cohomology Modules Supported at Determinantal Ideals*. In: Journal of the European Mathematical Society 18.11 (2016), pp. 2545–2578.

[MQS20] L. Ma, P. H. Quy, and I. Smirnov. *Filter Regular Sequence Under Small Perturbations*. In: Mathematische Annalen 378.1-2 (Oct. 2020), pp. 243–254.

[MS72] I. G. Macdonald and R. Y. Sharp. *An Elementary Proof of the Non-Vanishing of Certain Local Cohomology Modules*. In: The Quarterly Journal of Mathematics 23.2 (1972), pp. 197–204.

[Mac73] I. G. Macdonald. *Secondary Representation of Modules Over a Commutative Ring*. In: Symposia Mathematica. Vol. XI. 1973, pp. 23–43.

[MR20] C. Mascia and G. Rinaldo. *Krull Dimension and Regularity of Binomial Edge Ideals of Block Graphs*. In: Journal of Algebra and Its Applications 19.07 (July 2020), p. 2050133.

[Mat58] E. Matlis. *Injective Modules Over Noetherian Rings*. In: Pacific Journal of Mathematics 8.3 (Sept. 1958), pp. 511–528.

[Mat86] H. Matsumura. *Commutative Ring Theory*. Trans. by M. Reid. Vol. 8. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1986.

[MS05] E. Miller and B. Sturmfels. *Combinatorial Commutative Algebra*. Vol. 227. Graduate Texts in Mathematics. Springer, 2005.

[Moo73] D. J. Moore. *Primary and Coprimary Decompositions*. In: Proceedings of the Edinburgh Mathematical Society 18.4 (Dec. 1973), pp. 251–264.

[NS94] U. NAGEL and P. SCHENZEL. *Cohomological Annihilators and Castelnuovo-Mumford Regularity*. In: Commutative Algebra: Syzygies, Multiplicities, and Birational Algebra. Ed. by W. J. HEINZER, C. L. HUNEKE, and J. D. SALLY. Vol. 159. Contemporary Mathematics. American Mathematical Society, 1994, pp. 307–328.

[NO04] C. NĂSTĂSESCU and F. VAN OYSTAEYEN. *Methods of Graded Rings*. Lecture Notes in Mathematics 1836. Springer, 2004.

[Nis12] J. NISHIMURA. *A Few Examples of Local Rings, I*. In: Kyoto Journal of Mathematics 52.1 (Jan. 2012).

[Nor68] D. G. NORTHCOTT. *Lessons on Rings, Modules and Multiplicities*. Cambridge University Press, 1968.

[Oht11] M. OHTANI. *Graphs and Ideals Generated by Some 2-Minors*. In: Communications in Algebra 39.3 (Mar. 2011), pp. 905–917.

[PS73] C. PESKINE and L. SZPIRO. *Dimension Projective Finie et Cohomologie Locale*. In: Publications Mathématiques de l'IHÉS 42 (1973), pp. 47–119.

[Pós76] L. PÓSA. *Hamiltonian Circuits in Random Graphs*. In: Discrete Mathematics 14.4 (1976), pp. 359–364.

[Ree56] D. REES. *A Theorem of Homological Algebra*. In: Mathematical Proceedings of the Cambridge Philosophical Society 52.4 (Oct. 1956), pp. 605–610.

[Ric03] A. S. RICHARDSON. *Attached Primes of Graded Modules*. In: Communications in Algebra 31.6 (Jan. 2003), pp. 2603–2613.

[Rot09] J. J. ROTMAN. *An Introduction to Homological Algebra*. Second Edition. Universitext. Springer, 2009.

[RSK21] M. ROUZBAHANI MALAYERI, S. SAEEDI MADANI, and D. KIANI. *Binomial Edge Ideals of Small Depth*. In: Journal of Algebra 572 (Apr. 2021), pp. 231–244.

[Sha76] R. Y. SHARP. *Secondary Representations for Injective Modules Over Commutative Noetherian Rings*. In: Proceedings of the Edinburgh Mathematical Society 20.2 (Sept. 1976), pp. 143–151.

[Sha86] R. Y. SHARP. *Asymptotic Behaviour of Certain Sets of Attached Prime Ideals*. In: Journal of the London Mathematical Society s2-34.2 (Oct. 1986), pp. 212–218.

[Sta96] R. P. STANLEY. *Combinatorics and Commutative Algebra*. Second Edition. Vol. 41. Progress in Mathematics. Birkhäuser, 1996.

[Wil23a] D. WILLIAMS. LF-*Covers and Binomial Edge Ideals of König Type*. Oct. 2023. arXiv: `2310.14410[math]`.

[Wil23b] D. WILLIAMS. *The Local Cohomology Modules of the Binomial Edge Ideals of the Complements of Connected Graphs of Girth at Least 5*. Dec. 2023. arXiv: `2312.01613[math]`.

[Wil24] D. WILLIAMS. *Minimal Attached Primes of Local Cohomology Modules of Binomial Edge Ideals of Block Graphs*. June 2024. arXiv: `2406.14368[math]`.