BRAUER GROUPS AND DERIVED EQUIVALENCE FOR LAGRANGIAN FIBRATIONS

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To Suzan

Preface

Moduli spaces of sheaves on K3 surfaces give some of the most important examples of hyperkähler manifolds. We study their geometry using Hodge theory and derived categories. We are especially interested in Beauville–Mukai systems. This thesis is mostly based on [MS24; MM24].

We first investigate when a given moduli space of sheaves is fine, using a certain Brauer class called the *obstruction class.* These were defined and computed by Căldăraru in the case that the moduli space is itself a K3 surface. We extend his results to higher-dimensional moduli spaces using similar methods. An interesting new ingredient is the Căldăraru class. Căldăraru classes were used by Mukai [Muk87] and Căldăraru [Căl00], but were named in [MS24].

We apply Căldăraru classes to the study of derived equivalence for K3 surfaces. More precisely, we answer the question of whether every Fourier–Mukai partner of an elliptic K3 surface X is isomorphic to a Jacobian of X. This question was asked by Hassett and Tschinkel in [HT17]. The answer to the question is negative in general. The main ingredients for the proof are Căldăraru classes, Ogg–Shafarevich theory, and the Derived Torelli Theorem.

We use our explicit description of the obstruction class for a higher-dimensional moduli space to study birational and derived equivalence for Beauville–Mukai systems, and to generalise Ogg–Shafarevich theory to this setting.

For moduli spaces of sheaves on elliptic K3 surfaces, we provide a complete description of birational equivalence in terms of Căldăraru classes. We also use the results of Beckmann [Bec23] to show there exist moduli spaces M, M' such that there exists a Hodge isometry $T(M) \simeq T(M')$, but such that M and M' are not derived equivalent.

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Contents

Chapter 1

Introduction

In this thesis, we study the interplay between derived categories, Hodge theory, and birational geometry of hyperkähler manifolds, with emphasis on K3 surfaces and moduli spaces of sheaves on K3 surfaces.

The main characters

A K3 surface over $\mathbb{C}$ is a smooth, projective variety S of dimension 2 over $\mathbb{C}$, satisfying $\omega_S \simeq \mathcal{O}_S$, and $H^1(S, \mathcal{O}_S) = 0$. The second integral cohomology group $H^2(S, \mathbb{Z})$ is a torsion-free abelian group of rank 22 carrying a natural symmetric bilinear form given by the cup-product, turning $H^2(S, \mathbb{Z})$ into a lattice. Moreover, $H^2(S, \mathbb{Z})$ carries a natural Hodge structure of weight 2. One of the most fundamental results about K3 surfaces is the so-called *Torelli Theorem for K3 surfaces*, which asserts that two K3 surfaces S and S' are isomorphic if and only if there exists an isometry $H^2(S, \mathbb{Z}) \simeq H^2(S', \mathbb{Z})$ which is also an isomorphism of Hodge structures [PS71]. Such an isometry is called a *Hodge isometry.* This wonderful theorem allows us to study the geometry of K3 surfaces using lattice theory and Hodge theory.

The full cohomology group

$$H^*(S, \mathbb{Z}) = H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$$

also carries lattice structure and a Hodge structure, in which the integral $(1, 1)$-part is the *extended Néron–Severi lattice*

$$\mathrm{N}(S) \coloneqq H^0(S, \mathbb{Z}) \oplus \mathrm{NS}(S) \oplus H^4(S, \mathbb{Z}).$$

Given a (well-chosen) vector $v \in \mathrm{N}(S)$, we can form the moduli space $M(v)$ of (semi-stable) sheaves $\mathcal{F} \in \mathbf{Coh}(S)$ whose *Mukai vector*

$$v(\mathcal{F}) \coloneqq \mathrm{ch}(\mathcal{F})\sqrt{\mathrm{td}_X} \in \mathrm{N}(S)$$

is equal to v [GH96; OGr97; Huy06; BM14a; BM14b]. Such moduli spaces are examples of *hyperkähler manifolds*, which should be thought of as higher-dimensional generalisations of K3 surfaces. More precisely, the dimension of $M(v)$ is $v^2 + 2$ (we usually assume $v^2 \geq 0$). The geometry of $M(v)$ is a subject of great interest and importance. In this thesis, we study the derived categories and birational geometry of moduli spaces of sheaves on K3 surfaces, using lattice theory and Hodge theory as our main tools.

Let $H \subset S$ be a smooth and irreducible curve of genus g. It follows from the genus formula that $H^2 = 2g - 2$. Let $\omega \in H^4(S, \mathbb{Z})$ be the fundamental cocycle. Then, for any $d \in \mathbb{Z}$ the vector

$$v_d := H + (d + 1 - g)\omega \in \mathrm{N}(S)$$

gives rise to a $2g$-dimensional moduli space

$$\overline{\mathrm{Pic}}^d := M(v_d).$$

It admits a *Lagrangian fibration*

$$\overline{\mathrm{Pic}}^d \to |H| \simeq \mathbb{P}^g,$$

given by sending a sheaf to its support. Over a smooth, irreducible curve $C \in |H|$, the fibre of $\overline{\mathrm{Pic}}^d \to |H|$ is isomorphic to the Picard variety Pic^d_C, which explains the notation $\overline{\mathrm{Pic}}^d$ [Muk84; Bea91; Mar14; ADM16; HM23].

If $H \subset S$ is an elliptic curve, then we have an elliptic fibration $S \to |H|$. The dimension of $\overline{\mathrm{Pic}}^d$ is then 2, and $\overline{\mathrm{Pic}}^d$ is a K3 surface. In this case, we call $\overline{\mathrm{Pic}}^d$ the *d-th Jacobian* of S, and denote it by $\mathrm{J}^d(S) := \overline{\mathrm{Pic}}^d$.

Research question

For a smooth, projective variety X over a field k, we consider the abelian category $\mathbf{Coh}(X)$ of coherent sheaves on X. The associated bounded derived category

$$\mathcal{D}^b(X) := \mathcal{D}^b(\mathbf{Coh}(X))$$

is called the derived category of X. If Y is another smooth, projective variety over k for which there exists an equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$, then we say that X and Y are *derived equivalent* and that X and Y are *Fourier–Mukai partners*.

Question (Main Research Question)**.** *Given a K3 surface S with a smooth, irreducible curve $H \subset S$ of genus $g \geq 1$ and an integer $d \in \mathbb{Z}$, what are the Fourier–Mukai partners of $\overline{\mathrm{Pic}}^d$?*

In this thesis, we discuss the answer to this question. The cases $g = 1$ and $g > 1$ are considered separately. The reason for this is that derived equivalence for K3 surfaces is a much more highly-developed topic than derived equivalence for higher-dimensional hyperkähler manifolds.

Elliptic K3 surfaces: the case when $g = 1$

In Chapter 3, we study derived equivalence for elliptic K3 surfaces. The most fundamental result in this field is the so-called *Derived Torelli Theorem*, which asserts that two K3 surfaces X, Y are derived equivalent if and only if there is a Hodge isometry between the *transcendental lattices* $T(X)$ and $T(Y)$ [Muk87; Orl03]. The transcendental lattice is the orthogonal complement of the Néron–Severi lattice: $T(X) = \mathrm{NS}(X)^\perp \subset H^2(X, \mathbb{Z})$.

Let $X \to \mathbb{P}^1$ be an elliptic K3 surface. Since there is an isomorphism $X \simeq \mathrm{J}^1(X)$, the Main Research Question in this case is equivalent to asking what the Fourier–Mukai partners of X itself are. It turns out that X admits no non-trivial Fourier–Mukai partners as soon as the elliptic fibration $X \to \mathbb{P}^1$ admits a section. Therefore, we usually assume that the elliptic fibration $X \to \mathbb{P}^1$ has no sections. In this case, the lowest possible degree of a multisection is called the *multisection index* of the elliptic fibration, which we denote by t. Then $\mathrm{J}^k(X)$ is a Fourier–Mukai partner of X for any $k \in \mathbb{Z}$ such that $\gcd(k,t) = 1$ [Bri98, Theorem 1.2], [Căl00, Theorem 4.5.2]. These are called the *coprime Jacobians* of X. In Chapter 3, we answer the following question, which was asked by Hassett and Tschinkel in [HT17, Question 20]:

Question. *Let $X \to \mathbb{P}^1$ be an elliptic K3 surface, and let Y be a Fourier–Mukai partner of X. Does there exist an elliptic fibration $Y \to \mathbb{P}^1$ and an integer $k \in \mathbb{Z}$ such that Y is isomorphic to $\mathrm{J}^k(X)$ as an elliptic surface?*

We answer this question negatively for elliptic K3 surfaces of Picard rank 2. If X is an elliptic K3 surface of Picard rank 2, there exist integers $d \geq 0$, $t \geq 1$ and an isometry $\mathrm{NS}(X) \simeq \Lambda_{d,t}$, where $\Lambda_{d,t}$ is the rank two lattice with matrix

$$\begin{pmatrix} 2d & t \\ t & 0 \end{pmatrix}.$$

In this case, X admits a polarisation of degree $2d$, and the number t is the multisection index of any elliptic fibration $X \to \mathbb{P}^1$ [Ste04; SZ20]. The main result of Chapter 3 is the following theorem, which is also the main result of [MS24].

Theorem (See Theorem 3.1.2)**.** *[MS24, Theorem 1.2] Let X be an elliptic K3 surface of Picard rank 2. Let t be the multisection index of X and let $2d$ be the degree of a polarisation on X. Denote $m = \gcd(d,t)$.*

(i) If $m = 1$, then every Fourier–Mukai partner of X is isomorphic to a coprime Jacobian of a fixed elliptic fibration on X;

(ii) If $m = p^k$, for a prime p, then every Fourier–Mukai partner of X is isomorphic to a coprime Jacobian of one of the two elliptic fibrations on X;

(iii) If m is not a power of a prime, and X is very general with these properties, then X admits Fourier–Mukai partners which are not isomorphic to any Jacobian of any elliptic fibration on X.

The proof of this theorem relies on Ogg–Shafarevich Theory. This is the observation that, for an elliptic K3 surface with a section $S \to \mathbb{P}^1$, there exists an isomorphism $\mathrm{Br}(S) \simeq Ш(S)$ [Căl00], [Huy16, Corollary 11.5.5]. The group $\mathrm{Br}(S)$ is the *Brauer group* of S, which is isomorphic to $\mathrm{Hom}(T(S), \mathbb{Q}/\mathbb{Z})$. The group $Ш(S)$ is the *Tate–Shafarevich group* of S, which parametrises *torsors* of $S \to \mathbb{P}^1$. A torsor of $S \to \mathbb{P}^1$ is a pair $(X \to \mathbb{P}^1, \theta)$, consisting of an elliptic K3 surface $X \to \mathbb{P}^1$, together with an isomorphism $\theta\colon \mathrm{J}^0(X) \simeq S$ over $\mathbb{P}^1$ which preserves the distinguished sections of S and $\mathrm{J}^0(X)$.

Another ingredient of the proof is that of a *derived elliptic structure* of X. This is a pair (Y, g), where Y is a K3 surface which is derived equivalent to X and $g : Y \to \mathbb{P}^1$ is an elliptic fibration. We consider the set of isomorphism classes of derived elliptic structures, denoted $\mathrm{DE}(X)$, and relate it to Hodge-theoretical data using Ogg–Shafarevich theory.

More precisely, the Hodge-theoretical data is a Căldăraru class. Căldăraru classes are certain elements of a finite group associated to $\mathrm{N}(X)$. They were first studied by Căldăraru during his PhD, but they were given their name much later, in [MS24]. Căldăraru classes arise in the study of obstructions to the existence of universal sheaves.

Obstruction classes

Let S be a K3 surface with well-chosen Mukai vector $v \in \mathrm{N}(S)$. We consider the moduli space of sheaves $M(v)$. It is important to note that $M(v)$ is not necessarily a fine moduli space. This means that there is not always a universal sheaf $\mathcal{U}$ on $S \times M(v)$. However, a *twisted* universal sheaf always exists [Căl00, Proposition 3.3.2]. This is a $(1 \boxtimes \alpha)$-twisted sheaf $\mathcal{U}$ on $S \times M(v)$ with the property that $\mathcal{U}|_{S\times[\mathcal{F}]} \simeq \mathcal{F}$ for all $\mathbb{C}$-points $[\mathcal{F}] \in M(v)$. Here, α is a Brauer class on $M(v)$. Twisted universal sheaves are not unique, but the Brauer class α is uniquely determined by $M(v)$, i.e. α is the only Brauer class on $M(v)$ for which a $(1 \boxtimes \alpha)$-twisted universal sheaf exists. This is called the obstruction to the existence of a universal sheaf on $S \times M(v)$, or simply the *obstruction class.*

Căldăraru computed the obstruction class for a two-dimensional moduli space $M(v)$ explicitly in terms of cohomological data [Căl00]. His description uses what we now call Căldăraru classes. The order of the obstruction class is easy to compute using Căldăraru's results. To be precise, the order of the obstruction class of $M(v)$ is equal to the *divisibility* of v in $\mathrm{N}(S)$, denoted $\mathrm{div}(v)$. The divisibility of v is defined to be the smallest positive integer t such that there exists an element $w \in \mathrm{N}(S)$ with $v \cdot w = t$.

As a consequence, we obtain that a two-dimensional moduli space $M(v)$ is fine if and only if $\mathrm{div}(v) = 1$. This also follows from the Derived Torelli Theorem.

Beauville–Mukai systems: the case when $g > 1$

The three main tools used in Chapter 3 to study our Main Research Question for elliptic K3 surfaces are the Derived Torelli Theorem, Ogg–Shafarevich Theory, and Căldăraru's explicit computation of the obstruction class. All three of these tools are currently not fully understood in the higher-dimensional setting. In Chapter 2 and Chapter 4, we discuss recent progress on these three topics. We give a complete description of obstruction classes of higher-dimensional moduli spaces of sheaves in Chapter 2, and we give counterexamples to a higher-dimensional Derived Torelli statement and prove a higher-dimensional generalisation of Ogg–Shafarevich Theory in Chapter 4.

For the obstruction classes, we prove the following theorem in Chapter 2.

Theorem (See Theorem 2.4.15)**.** *[MM24, Theorem 4.5] Let S be a K3 surface, and let $v \in \mathrm{N}(S)$ be a primitive Mukai vector. There is a short exact sequence*

$$0 \to \langle \alpha \rangle \to \mathrm{Br}(M(v)) \to \mathrm{Br}(S) \to 0,$$

where $\alpha \in \mathrm{Br}(M(v))$ is the obstruction to the existence of a universal sheaf on $S \times M(v)$.

The order of the obstruction class can be easily computed, and this gives a useful method to check whether a given moduli space is fine. Similarly to the case of a two-dimensional moduli space, Căldăraru classes appear in the higher-dimensional generalisation as well, and again the order of the obstruction class equals the divisibility of v in $\mathrm{N}(S)$.

As an application of the explicit computation of the obstruction class, we obtain the following result, which is an application of a theorem by Beckmann [Bec23, Proposition 9.9].

Theorem (See Corollary 4.4.6)**.** *[MM24, Corollary 1.4] Let S be an elliptic K3 surface which admits a section. If M is a fine moduli space of sheaves on S, and M' is a non-fine moduli space on S, then M and M' are not derived equivalent, but there is a Hodge isometry $T(M) \simeq T(M')$.*

To prove a higher-dimensional generalisation of Ogg–Shafarevich Theory we rely on the new construction of the Tate–Shafarevich group of a pair (S, H) of [HM23]. In [HM23], the Tate–Shafarevich group $Ш(S, H)$ is defined to be a certain subquotient of the so-called special Brauer group $\mathrm{SBr}(S)$ of S. Instead, in this thesis we give an alternative definition of $Ш(S, H)$ in which an element of $Ш(S, H)$ is given by a $\overline{\mathrm{Pic}}^0$-torsor rather than by a special Brauer class on S. We prove that our definition of $Ш(S, H)$ is equivalent to the one given in [HM23].

Theorem (See Proposition 4.3.14)**.** *[MM24, Proposition 5.6] Let S be a K3 surface, and let $H \subset S$ be a smooth, irreducible curve of genus $g > 1$ whose class in $\mathrm{NS}(S)$ is primitive. For all $d \in \mathbb{Z}$, there is a short exact sequence*

$$0 \to \mathbb{Z}/n(d)\mathbb{Z} \to Ш(S, H) \to \mathrm{Br}(\overline{\mathrm{Pic}}^d) \to 0,$$

where

$$n(d) = \frac{\mathrm{div}(H)}{\gcd(\mathrm{div}(H), d + 1 - g)}.$$

This theorem generalises Ogg–Shafarevich theory to higher-dimensional moduli spaces of sheaves on K3 surfaces. This leaves the Derived Torelli Theorem as the last of the three main tools to be generalised to the higher-dimensional case. I expect that the Main Research Question can be resolved fully once all three of the main tools are extended to the setting of Beauville–Mukai systems.

Conventions and Notation

i) Unless stated otherwise, we work with varieties over the complex numbers. This is essential because many of our results rely on Hodge theory.

ii) In Chapter 3, we denote an elliptic K3 surface which does not admit a section by X, and an elliptic K3 surface which admits a section by S. In the other chapters, K3 surfaces are usually denoted by S.

iii) For a smooth, projective, complex variety X, and a cohomology class $x \in H^*(X, \mathbb{Z})$, we denote by $[x]_i$ the component of x that is contained in $H^i(X, \mathbb{Z})$.

Chapter 2

Brauer Groups and Hyperkähler Manifolds

2.1 Introduction

For a K3 surface S with primitive Mukai vector v and polarisation H, one can construct the moduli space $M_H(v)$ of H-Gieseker semistable sheaves with Mukai vector v. It is a hyperkähler manifold of dimension $v^2 + 2$, which is deformation equivalent to a Hilbert scheme of points on a K3 surface, provided H is generic [GH96; OGr97; Huy06; BM14a; BM14b]. These moduli spaces provide some of the most important examples of hyperkähler manifolds, and their geometry is a subject of great interest.

The moduli space $M_H(v)$ is generally a coarse moduli space. That is, there is not always a universal sheaf $\mathcal{U}$ on $S \times M_H(v)$ for which $\mathcal{U}|_{S\times[\mathcal{F}]} \simeq \mathcal{F}$ for all $[\mathcal{F}] \in M_H(v)$. The obstruction to the existence of a universal sheaf is a Brauer class on $M_H(v)$. More precisely, there is a unique Brauer class $\alpha \in \mathrm{Br}(M_H(v))$ for which there exists a $(1 \boxtimes \alpha)$-twisted universal sheaf on $S \times M_H(v)$. This Brauer class is called the *obstruction class* of $M_H(v)$. If we assume $v^2 = 0$, then $M_H(v)$ is a K3 surface, and in this setting, the obstruction class was computed explicitly in terms of cohomology by Căldăraru in his PhD thesis [Căl00]. His results showed that there exists a short exact sequence

$$0 \to \langle \alpha \rangle \to \mathrm{Br}(M_H(v)) \to \mathrm{Br}(S) \to 0, \tag{2.1.1}$$

and his very precise description of α also tells us the order of α in $\mathrm{Br}(M_H(v))$. The order of α equals the so-called *divisibility* of the Mukai vector v, which is a number that is easy to compute in practice.

Before we explain the main result of this chapter, let us briefly note some of the applications of Căldăraru's work. Firstly, $M_H(v)$ is a fine moduli space if and only if a universal sheaf on $S \times M_H(v)$ exists. This is equivalent to the obstruction class α being trivial. Therefore, Căldăraru's work shows that $M_H(v)$ is fine if and only if the divisibility of v is 1. This is important partially because of its implications for the derived categories of S and $M_H(v)$. Short exact sequence (2.1.1) is obtained from the well-known short exact sequence [Muk87, Proposition 6.4]

$$0 \to T(S) \to T(M_H(v)) \to \mathbb{Z}/\operatorname{div}(v)\mathbb{Z} \to 0, \tag{2.1.2}$$

which shows that $T(S)$ is Hodge isometric to $T(M_H(v))$ if and only if $M_H(v)$ is a fine moduli space. Recall the Derived Torelli Theorem for K3 surfaces:

Theorem 2.1.1 (Derived Torelli Theorem). *[Muk87; Orl03] Let X and Y be two K3 surfaces. The following are equivalent:*

i) There is an equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$;

ii) There is a Hodge isometry $T(X) \simeq T(Y)$;

iii) The K3 surface Y is isomorphic to a fine moduli space of sheaves on X.

The main goal of this chapter is to extend Căldăraru's results about Brauer groups to higher-dimensional moduli spaces of sheaves. Our main result is the following theorem. For a K3 surface S, we denote by $\mathrm{N}(S) := H^0(S,\mathbb{Z}) \oplus \mathrm{NS}(S) \oplus H^4(S,\mathbb{Z})$ the *extended Néron–Severi lattice.*

Theorem 2.1.2 (See Theorem 2.4.15). *Let S be a K3 surface, and let $v \in \mathrm{N}(S)$ be a Mukai vector of square $v^2 > 0$. Let $H \in \mathrm{NS}(S)$ be a generic polarisation on S, and write $M := M_H(v)$. Then there is a short exact sequence*

$$0 \to \langle \alpha \rangle \to \mathrm{Br}(M) \to \mathrm{Br}(S) \to 0,$$

where α is the obstruction class for M to be a fine moduli space of sheaves on S. Moreover, the order of α in $\mathrm{Br}(M)$ is equal to the divisibility of v in $\mathrm{N}(S)$.

Theorem 2.1.2 should be compared to [KK24, Proposition 6.6], where the obstruction class is computed for four-dimensional moduli spaces of twisted sheaves using a different method which only applies in dimension four.

Theorem 2.1.2 is completely analogous to (2.1.1). In particular, it also allows us to easily determine whether a given moduli space is fine or not.

Let us say a few words on the proof of Theorem 2.1.2. The proof is very closely related to Căldăraru's proof in the case $v^2 = 0$. Some modifications are needed in the higher-dimensional setting. Most notably, short exact sequence (2.1.2) has no analogue in the higher-dimensional setting, as in this case there is always a Hodge isometry $T(S) \simeq T(M)$. The way to derive (2.1.1) from (2.1.2) is by using the fact that $\mathrm{Br}(S) \simeq \mathrm{Hom}(T(S), \mathbb{Q}/\mathbb{Z}) \simeq T(S)^* \otimes \mathbb{Q}/\mathbb{Z}$ for a K3 surface S. However, this is not the case for higher-dimensional moduli spaces of sheaves on S. In this case, there is a certain subgroup $T'(M) \subset T(S)^*$, introduced in [HM23], for which there exists an isomorphism $\mathrm{Br}(M) \simeq T'(M) \otimes \mathbb{Q}/\mathbb{Z}$. The quotient $T(S)^*/T'(M)$ is cyclic of order $\mathrm{div}(v)$, thus there is a short exact sequence

$$0 \to T'(M) \to T(S)^* \to \mathbb{Z}/\operatorname{div}(v)\mathbb{Z} \to 0,$$

which gives rise to the short exact sequence of Theorem 2.1.2.

The proof that the kernel of the short exact sequence of Theorem 2.1.2 is generated by the obstruction class goes via a deformation argument very similar to Căldăraru's argument in the two-dimensional case. However, there are differences between our approach and his in the deformation argument as well. If M is a fine two-dimensional moduli space of sheaves on S, then Mukai's and Orlov's work shows that the Fourier–Mukai equivalence associated to the universal sheaf on $S \times M$ is a Hodge isometry $\tilde{H}(S,\mathbb{Z}) \simeq \tilde{H}(M,\mathbb{Z})$, where $\tilde{H}(S,\mathbb{Z})$ denotes the Mukai lattice. If, on the other hand, M has dimension $\dim M > 2$, then such a statement does not hold. Nevertheless, we could achieve the same result using the *Mukai pairing* on $H^*(M,\mathbb{Q})$, which was introduced by Căldăraru in [Căl03], which was later superseded by Căldăraru and Willerton in [CW10].

Theorem 2.1.2 has many applications. We discuss some of these applications in Chapter 4.

The structure of this chapter

In this chapter, we introduce many of the notions needed in the other chapters of this thesis.

In Section 2.2, we revise the necessary theory on lattices, hyperkähler manifolds, derived categories, and Hodge structures.

In Section 2.3, we discuss the relevant theory on moduli spaces. We study moduli spaces of sheaves on K3 surfaces, as well as moduli spaces of lattice polarised K3 surfaces. We prove a certain density lemma in Section 2.3.3, which was already noted in [Muk87; Căl00]. We prove that any (non-fine) moduli space of sheaves M on a K3 surface S can be deformed to a fine moduli space (for the exact statement, see Proposition 2.3.20).

In Section 2.4, we discuss obstruction classes of moduli spaces. Unlike in Section 2.2 and Section 2.3, many of the results and definitions in Section 2.4 are original. The most important definition in Section 2.4 is the definition of a Căldăraru class of a Mukai vector. The Căldăraru class is a certain element of the discriminant lattice of the extended Néron–Severi lattice, and this element appears in many of the main results of this thesis. Finally, we conclude this chapter with Theorem 2.4.15, which computes the obstruction to the existence of a universal sheaf for a moduli space of sheaves on a K3 surface.

The definition of the Căldăraru class comes from [MS24], and the rest of Section 2.3 and Section 2.4 is based on [MM24].

2.2 Preliminary Results

2.2.1 Lattices

Our main reference for lattice theory is [Nik80]. A lattice is a finitely generated free abelian group L together with a symmetric non-degenerate bilinear form $b\colon L\times L\to\mathbb{Z}$. We consider the quadratic form $q(x)=b(x,x)$ and we write $x\cdot y$ for $b(x,y)$ and x^2 for $q(x)$. A (metric) morphism of lattices between (L,b) and (L',b') is a group homomorphism $f\colon L\to L'$ which respects the bilinear forms, meaning $b(x,y)=b'(f(x),f(y))$ for all $x,y\in L$. An isomorphism of lattices is called an isometry. We write $O(L)$ for the group of isometries of L. The lattice L is called even if x^2 is even for all $x\in L$. All the lattices we consider will be assumed to be even.

Note that any metric morphism between non-degenerate lattices is injective. A metric morphism of lattices $N\hookrightarrow L$ is called a *primitive embedding* if the quotient L/N is torsion-free. A vector $v\in L$ is *primitive* if the morphism $\langle v\rangle\hookrightarrow L$ is a primitive embedding. The *divisibility* of a vector $v\in L$ is the positive integer

$$\operatorname{div}(v)\coloneqq\gcd_{u\in L}(u\cdot v)\,.$$

The dual of a lattice L is defined as $L^*\coloneqq\operatorname{Hom}(L,\mathbb{Z})$. It comes equipped with a natural bilinear form taking values in $\mathbb{Q}$. The bilinear form gives rise to a natural map $L\to L^*$ which is injective because we assume b to be non-degenerate; furthermore we have a canonical isomorphism

$$L^*\simeq\{x\in L\otimes\mathbb{Q}\mid\forall y\in L: x\cdot y\in\mathbb{Z}\}\subseteq L\otimes\mathbb{Q}. \tag{2.2.1}$$

The quotient $L^*/L = A_L$ is called the discriminant group of L. If the discriminant group is trivial, we call L unimodular. The discriminant group comes equipped with a quadratic form $\overline{q}\colon A_L \to \mathbb{Q}/2\mathbb{Z}$.

Any isometry $f\colon L \simeq L$ induces an isometry $\overline{f}\colon A_L \simeq A_L$. This defines a group homomorphism

$$O(L) \to O(A_L).$$

Note that there is a natural embedding

$$i_L\colon A_L \hookrightarrow L \otimes \mathbb{Q}/\mathbb{Z},$$

induced by (2.2.1). If we denote by $f_{\mathbb{Q}/\mathbb{Z}}\colon L \otimes \mathbb{Q}/\mathbb{Z} \simeq L \otimes \mathbb{Q}/\mathbb{Z}$ the isomorphism induced by f, then

$$i_L \circ \overline{f} = f_{\mathbb{Q}/\mathbb{Z}} \circ i_L.$$

For any primitive sublattice $N \hookrightarrow L$, we denote

$$i_{N,L}\colon A_N \hookrightarrow N \otimes \mathbb{Q}/\mathbb{Z} \hookrightarrow L \otimes \mathbb{Q}/\mathbb{Z}.$$

Lemma 2.2.1. *Let L be a unimodular lattice, and let $N \hookrightarrow L$ be a primitive sublattice. Write $T := N^\perp \subset L$ for the orthogonal complement. Then, in $L \otimes \mathbb{Q}/\mathbb{Z}$, we have*

$$(T \otimes \mathbb{Q}/\mathbb{Z}) \cap (N \otimes \mathbb{Q}/\mathbb{Z}) = i_{T,L}(A_T) = i_{N,L}(A_N).$$

Proof. Suppose $x \in (T \otimes \mathbb{Q}/\mathbb{Z}) \cap (N \otimes \mathbb{Q}/\mathbb{Z})$. Then there exist $\lambda \in T \otimes \mathbb{Q}$ and $v \in N \otimes \mathbb{Q}$ such that $x \equiv \lambda \pmod{L}$ and $x \equiv v \pmod{L}$. In particular, we have $\lambda - v \equiv 0 \pmod{L}$. For any integral vector $\zeta \in T$, we have $\lambda \cdot \zeta = (\lambda - v) \cdot \zeta \in \mathbb{Z}$, so that $\lambda \in T^*$. This means that $x \in i_{T,L}(A_T)$. By a similar argument, we have $x \in i_{N,L}(A_N)$. This shows that $(T \otimes \mathbb{Q}/\mathbb{Z}) \cap (N \otimes \mathbb{Q}/\mathbb{Z})$ is contained in $i_{T,L}(A_T)$ and $i_{N,L}(A_N)$.

The other two inclusions follow from a standard argument, c.f. [Muk87, Proposition 6.4]. We include it here for completeness. Let $\lambda \in i_{T,L}(A_T)$. Since it is clear that $\lambda \in T \otimes \mathbb{Q}/\mathbb{Z}$, we must show that $\lambda \in N \otimes \mathbb{Q}/\mathbb{Z}$. By slight abuse of notation, we also denote by λ any lift to T^*. Since T is a primitive sublattice of L and L is unimodular, there is a surjective map $L \to T^*$. That is, there exists a vector $x \in L$ such that $x \cdot \zeta = \lambda \cdot \zeta$ for every $\zeta \in T$. In particular, $x - \lambda$ is orthogonal to T, and thus $v := x - \lambda \in N \otimes \mathbb{Q}$. Therefore we have $\lambda + v \in L$, hence $\lambda \equiv -v \pmod{L}$, and we obtain $\lambda \in N \otimes \mathbb{Q}/\mathbb{Z}$. The final remaining inclusion is completely analogous. $\square$

Remark 2.2.2. Lemma 2.2.1 induces an isomorphism of groups $A_T \simeq i_{T,L}(A_T) = i_{N,L}(A_N) \simeq A_N$, which is an isomorphism of quadratic forms $A_T(-1) \simeq A_N$ [Nik80, Proposition 1.6.1].

Lemma 2.2.3. *[Nik80, Proposition 1.6.1] Let L be a unimodular lattice, let $N \hookrightarrow L$ be a primitive sublattice and write $T := N^\perp \subset L$. Suppose $g \in O(N)$ and $h \in O(T)$. Then $\overline{g}$ and $\overline{h}$ induce the same isometry on $i_{N,L}(A_N) = i_{T,L}(A_T)$ if and only if there is an isometry $f\colon L \simeq L$ preserving N and T, that satisfies $f|_N = g$ and $f|_T = h$.*

A primitive embedding of a lattice N into a unimodular lattice L determines a pair (T, ϕ) consisting of the orthogonal complement $T = N^\perp \subset L$, and an isometry $\phi\colon A_T \simeq A_N(-1)$. An isomorphism of two such pairs (T_1, ϕ_1) and (T_2, ϕ_2) is an isometry $g\colon T_1 \simeq T_2$ such that $\phi_1 = \phi_2 \circ \overline{g}$. Two primitive embeddings $i_1\colon N \hookrightarrow L$,

$i_2\colon N \hookrightarrow L$ are *in the same $O(L)$-orbit* if there exists an isometry $f\colon L \simeq L$ such that $i_1 = f \circ i_2$. In this case, the pairs (T_1, ϕ_1), (T_2, ϕ_2) determined by i_1 and i_2, respectively, are isomorphic. Indeed, the isomorphism $(T_1, \phi_1) \simeq (T_2, \phi_2)$ is given by $f|_{T_1}\colon T_1 \simeq T_2$. Conversely, if two primitive embeddings determine isomorphic pairs, then the embeddings are in the same $O(L)$-orbit. This is an important part of the following result by Nikulin.

Proposition 2.2.4. *[Nik80, Proposition 1.6.1] Let N be a lattice with signature (n_1, n_2), and let L be a unimodular lattice of signature (l_1, l_2). There is a bijection between $O(L)$-orbits of primitive embeddings $N \hookrightarrow L$ and isomorphism classes of pairs (T, ϕ), where T is a lattice with* $\operatorname{sign}(T) = (l_1 - n_1, l_2 - n_2)$ *and $\phi\colon A_T \simeq A_N(-1)$ is an isometry.*

Proposition 2.2.5. *Let N be a lattice of signature (n_1, n_2), and let L be a unimodular lattice of signature (l_1, l_2) such that $l_1 \geq n_1$ and $l_2 \geq n_2$. The number of $O(L)$-orbits of primitive embeddings $N \hookrightarrow L$ is equal to*

$$\sum_T |O(A_T)/O(T)|,$$

where the sum runs over all lattices T of signature $(l_1 - n_1, l_2 - n_2)$ which satisfy $A_T \simeq A_N(-1)$. Moreover, for a fixed lattice T, the order of the quotient $|O(A_T)/O(T)|$ is equal to the number of $O(L)$-orbits of primitive embeddings $N \hookrightarrow L$ with orthogonal complement $N^\perp \simeq T$.

Proof. For a fixed lattice T of signature $(l_1 - n_1, l_2 - n_2)$ which satisfies $A_T \simeq A_N(-1)$, the set of isometries $\phi\colon A_T \simeq A_N(-1)$, is a torsor under $O(A_T)$. Two such isometries $\phi\colon A_T \simeq A_N(-1)$ and $\psi\colon A_T \simeq A_N(-1)$ give rise to isomorphic pairs (T, ϕ), (T, ψ) if and only if there is an isometry $f\colon T \simeq T$ such that $\phi = \psi \circ \overline{f}$, that is, if and only if ϕ and ψ are in the same $O(T)$-orbit. Therefore, the number of isomorphism classes of pairs (T, ϕ) equals $|O(A_T)/O(T)|$, and the result follows from Proposition 2.2.4. □

We now use Proposition 2.2.5 to compute the number of primitive embeddings in the case $\operatorname{rk}(L) = \operatorname{rk}(N) + 1$. In this case, the orthogonal complement of N in L has rank 1.

Definition 2.2.6. We denote by $\langle 2n \rangle$ the lattice of rank 1 generated by an element x satisfying $x^2 = 2n$.

Corollary 2.2.7. *Let N be a lattice of signature (n_1, n_2) whose discriminant group A_N is cyclic of order $2n$ with $n > 1$. Suppose L is a unimodular lattice with* $\operatorname{sign}(L) = (n_1, n_2 + 1)$. *Then the number of $O(L)$-orbits of primitive embeddings of N in L is $2^{\omega(n)-1}$, where $\omega(n)$ is the number of distinct primes dividing n. If $n = 1$, the number of $O(L)$-orbits of primitive embeddings is 1.*

Proof. The negative-definite lattice T of rank 1 with discriminant group $A_T \simeq \mathbb{Z}/(2n)\mathbb{Z}$ is unique; it is the lattice $\langle -2n \rangle$.

Suppose first that $n > 1$. Then it is not hard to see that $O(A_{\langle -2n \rangle}) \simeq 2^{\omega(n)}$, see for example [Ste04, Corollary 2.2(i)]. Moreover, we clearly have $O(\langle -2n \rangle) = \{\pm \operatorname{id}\}$, which acts non-trivially on $O(A_{\langle -2n \rangle})$, so that $|O(A_T)/O(T)| = 2^{\omega(n)-1}$, as required.

On the other hand, if $n = 1$, then the group $O(A_{\langle -2n \rangle})$ is trivial, so that the number of $O(L)$-orbits of primitive embeddings is 1 by Proposition 2.2.5. □

A particularly important example of a lattice is the so-called *hyperbolic plane*, which is the lattice U of rank 2 whose quadratic form is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The hyperbolic plane is important because it is the unique unimodular (even, non-degenerate) lattice of the lowest possible rank. This makes it very versatile, as the following two lemmas show.

Lemma 2.2.8. *[Nik80, Proposition 1.14.1] Let L be a unimodular lattice, and let $N \hookrightarrow L$ be a primitive embedding. If there exists an embedding $U \subset N^\perp$, then for any $g \in O(N)$, there exists an isometry $f: L \simeq L$ such that $f|_N = g$.*

Proof. Write $T = N^\perp$. By assumption, there is an orthogonal decomposition $T = U \oplus T'$ for some lattice T'. We have $A_T \simeq A_U \oplus A_{T'} \simeq A_{T'}$, since U is unimodular. In particular, $\mathrm{rk}(T) = \mathrm{rk}(T') + 2 \geq \ell(A_{T'}) + 2$. Combined with [Nik80, Theorem 1.14.2], this implies that the homomorphism $O(T) \to O(A_T)$ is surjective. Choose $h \in O(T)$ such that $\overline{h}$ induces the same isomorphism on $i_{T,L}(A_T)$ as g, then by Lemma 2.2.3 there exists an isometry $L \simeq L$ which restricts to g on N. □

Lemma 2.2.9. *[GHS09, Proposition 3.3] Let $L = U \oplus U \oplus L'$ be a lattice containing two copies of the hyperbolic plane. Let $u, v \in L$ be two primitive vectors such that*

i) $\frac{u}{\mathrm{div}(u)} = \frac{v}{\mathrm{div}(v)} \in A_L$,

ii) $u^2 = v^2$.

Then there is an isometry $f: L \simeq L$ such that $\overline{f} = \mathrm{id}_{A_L}$ and such that $f(u) = v$.

There is an orthogonal direct sum decomposition

$$A_L = \bigoplus_p A_L^{(p)} \tag{2.2.2}$$

where $A_L^{(p)}$ consists of elements annihilated by a power of a prime p. The group $A_L^{(p)}$ coincides with the discriminant group of the p-adic lattice $L \otimes \mathbb{Z}_p$.

Definition 2.2.10. Two lattices L, L' are said to be in the same genus if they have the same signature and have isometric discriminant groups.

Lemma 2.2.11. *[Nik80, Corollary 1.13.4] Let L and L' be two lattices. Then L and L' are in the same genus if and only $L \oplus U \simeq L' \oplus U$.*

Lemma 2.2.12. *Let L, L' and T be lattices, and let H be a unimodular lattice. Suppose there are two primitive embeddings $i_1 : T \hookrightarrow H$, $i_2 : T \hookrightarrow H$ such that $i_1(T)^\perp \simeq L$ and $i_2(T)^\perp \simeq L'$. Then L and L' are in the same genus.*

Proof. It is easy to see that the signatures of L and L' are equal. Moreover, we have $A_L \simeq A_T(-1) \simeq A_{L'}$ by Remark 2.2.2, hence L and L' are in the same genus. □

An *overlattice* of a lattice T is a lattice L together with an embedding of lattices $T \hookrightarrow L$ of finite index. We say that two overlattices $T \hookrightarrow L$ and $T' \hookrightarrow L'$ are isomorphic if there exists a commutative diagram

$$\begin{tikzcd} T \arrow[r] \arrow[d, "f"'] & L \arrow[d, "\tilde{f}"] \\ T' \arrow[r] & L' \end{tikzcd}$$

where f and $\tilde{f}$ are isometries.

For any overlattice $T \hookrightarrow L$, there is a natural embedding of the cokernel $H_L := L/T$ in the discriminant group of T via the chain of embeddings

$$T \hookrightarrow L \hookrightarrow L^* \hookrightarrow T^*.$$

The subgroup H_L is isotropic for the quadratic form on A_T, and conversely any isotropic subgroup of A_T gives rise to an overlattice of T. The following result gives a complete classification of all overlattices of a given lattice T, up to isomorphism.

Lemma 2.2.13 ([Nik80, Proposition 1.4.2])**.** *Let T be a lattice, and let $T \hookrightarrow L$ and $T \hookrightarrow M$ be two overlattices of T. An isometry $f \in O(T)$ fits into a commutative diagram of the form*

$$\begin{tikzcd} T \arrow[r] \arrow[d, "f"] & L \arrow[d, "\simeq"] \\ T \arrow[r] & M \end{tikzcd} \tag{2.2.3}$$

if and only if the induced isometry $\overline{f} \in O(A_T)$ satisfies $\overline{f}(H_L) = H_M$. Moreover, the assignment $(T \hookrightarrow L) \mapsto H_L$ is a bijection between the set of isomorphism classes of overlattices of T and the set of $O(T)$-orbits of isotropic subgroups of A_T.

Note that (2.2.3) can be completed as follows:

$$\begin{tikzcd} T \arrow[r] \arrow[d, "f"] & L \arrow[r, two heads] \arrow[d, "\simeq"] & H_L \arrow[r, hook] \arrow[d, "\overline{f}|_{H_L}"] & A_T \arrow[d, "\overline{f}"] \\ T \arrow[r] & M \arrow[r, two heads] & H_M \arrow[r, hook] & A_T \end{tikzcd}$$

It will occasionally be useful to deal with quadratic spaces over $\mathbb{Q}$. These are much easier to work with, as explained by the following two results, both of which fail for lattices over $\mathbb{Z}$. We refer to [Ser73, §IV.1.5] for details on these results.

To avoid confusion with their integral counterparts, we refer to isometries of rational quadratic spaces as *rational isometries.*

Theorem 2.2.14 (Witt's Extension Theorem)**.** *Let V and W be quadratic spaces over $\mathbb{Q}$. If $V' \subset V$ and $W' \subset W$ are subspaces and $f \colon V' \simeq W'$ is a rational isometry, then there is a rational isometry $\tilde{f} \colon V \simeq W$ such that $\tilde{f}|_{V'} = f$.*

Theorem 2.2.15 (Witt's Cancellation Theorem)**.** *Let V_1, V_2 and V be quadratic spaces over $\mathbb{Q}$. If there is a rational isometry $V_1 \oplus V \simeq V_2 \oplus V$, then there is a rational isometry $V_1 \simeq V_2$.*

An immediate consequence of the Witt Cancellation Theorem is the following well-known result, see for example [HT17, Proof of Proposition 16]. We say that a lattice L *represents zero* if there is a non-zero element $v \in L$ such that $v^2 = 0$.

Lemma 2.2.16. *Let L and L' be two lattices in the same genus. Then L represents zero if and only if L' does.*

Proof. By Lemma 2.2.11, there is an isometry $L \oplus U \simeq L' \oplus U$. This induces a rational isometry $L_{\mathbb{Q}} \oplus U_{\mathbb{Q}} \simeq L'_{\mathbb{Q}} \oplus U_{\mathbb{Q}}$. By the Witt Cancellation Theorem, this means that there is a rational isometry $f : L_{\mathbb{Q}} \simeq L'_{\mathbb{Q}}$. Now suppose L represents zero, and let $v \in L$ be a non-zero element with $v^2 = 0$. Then $f(v) \in L'_{\mathbb{Q}}$ satisfies $f(v)^2 = 0$, and $f(v) \neq 0$. Let $n \in \mathbb{Z}$ be an integer such that $nf(v) \in L'$ is an integral vector. Then we have $(nf(v))^2 = n^2 f(v)^2 = 0$, so L' represents zero. The result follows from the symmetry of the situation. $\square$

2.2.2 Hyperkähler Manifolds

In this section, we study the lattices that arise from hyperkähler manifolds as cohomology groups. Let us first recall the definition of a hyperkähler manifold. We work over the complex numbers, unless explicitly stated otherwise. Our main references for hyperkähler manifolds are [Huy03; Huy99].

Definition 2.2.17. A hyperkähler manifold is a simply connected, compact Kähler manifold X which admits an everywhere non-degenerate holomorphic 2-form σ that generates $H^0(X, \Omega_X^2)$.

Note that the existence of a non-degenerate 2-form implies that X has even dimension $2n$. Moreover, σ induces an everywhere non-vanishing section of the canonical line bundle $\omega_X = \bigwedge^{2n} \Omega_X$, so that $\omega_X \simeq \mathcal{O}_X$. The fact that X is simply connected implies that $H^1(X, \mathbb{Z}) = 0$, and therefore, by Hodge theory, we have $H^1(X, \mathcal{O}_X) = 0$.

Definition 2.2.18. A K3 surface is a compact, smooth surface X with $\omega_X \simeq \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

It follows immediately from the discussion above that every hyperkähler manifold of dimension 2 is a K3 surface. The converse also holds, because every K3 surface is simply connected [Huy16, Corollary 1.1.4] and Kähler [Siu83; Tod80].

Lattices of K3 Surfaces

We first treat the 2-dimensional hyperkähler manifolds, that is, K3 surfaces. Our basic reference for K3 surfaces is [Huy16] and our basic references for Hodge theory are [Voi07a; Voi07b].

The Hodge diamond of a K3 surface S looks as follows:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

The singular cohomology group $H^2(S, \mathbb{Z})$ is a free abelian group of rank 22. Moreover, the cup-product is an integral symmetric bilinear form

$$H^2(S, \mathbb{Z}) \times H^2(S, \mathbb{Z}) \to H^4(S, \mathbb{Z}) \simeq \mathbb{Z}.$$

This bilinear form is even and non-degenerate, hence it turns $H^2(S,\mathbb{Z})$ into a lattice. As an abstract lattice, $H^2(S,\mathbb{Z})$ is isometric to the *K3 lattice*

$$\Lambda_{\mathrm{K3}} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}.$$

Here U is the hyperbolic plane and E_8 is the unique even, unimodular, positive-definite lattice of rank 8 (see [BPV12, §VIII.1] for details). It is important to note that $H^2(S,\mathbb{Z})$ is a unimodular lattice.

The cohomology group $H^2(S,\mathbb{Z})$ carries a natural Hodge structure of weight 2 given by

$$H^2(S,\mathbb{C}) \simeq H^2(S,\mathcal{O}_S) \oplus H^1(S,\Omega_S) \oplus H^0(S,\omega_S).$$

Since $\omega_S \simeq \mathcal{O}_S$, we have $H^{2,0}(S) \simeq H^0(S,\omega_S) = \mathbb{C}\cdot\sigma$, where σ is a non-vanishing section of ω_S.

Definition 2.2.19. We say that a Hodge structure of weight 2 on a free, finitely generated abelian group H is *of K3-type* if $H^{2,0} \simeq \mathbb{C}$, and $H^{p,q} = 0$ whenever $p < 0$ or $q < 0$.

The *Néron–Severi lattice* $\mathrm{NS}(S)$ is a sublattice of $H^2(S,\mathbb{Z})$, defined as the image of the first Chern class $c_1\colon \mathrm{Pic}(S) \hookrightarrow H^2(S,\mathbb{Z})$. Equivalently, the Néron–Severi lattice is the sublattice of $H^2(S,\mathbb{Z})$ consisting of all integral $(1,1)$-classes, i.e. $\mathrm{NS}(S) = H^{1,1}(S) \cap H^2(S,\mathbb{Z})$. We have $\mathrm{Pic}(S) \simeq \mathrm{NS}(S)$. The rank ρ of the Néron–Severi lattice is called the *Picard rank* of S.

The orthogonal complement $T(S) = \mathrm{NS}(S)^\perp \subseteq H^2(S,\mathbb{Z})$ is called the *transcendental lattice* of S. Since $\mathrm{NS}(S)$ is orthogonal to σ, it follows that we have $H^{2,0}(S) \subset T(S)\otimes\mathbb{C}$. In fact, $T(S)$ is the smallest primitive sublattice of $H^2(S,\mathbb{Z})$ such that $T(S)\otimes\mathbb{C}$ contains $H^{2,0}(S)$, and

$$\mathrm{NS}(S) = H^{2,0}(S)^\perp \cap H^2(S,\mathbb{Z}), \tag{2.2.4}$$

see [Huy16, Remark 15.1.2]. The transcendental lattice inherits a Hodge structure of K3-type from $H^2(S,\mathbb{Z})$ whose $(2,0)$-part is generated by σ.

The full integral cohomology group $H^*(S,\mathbb{Z}) = H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})$, admits a lattice structure given by

$$(r,l,s)\cdot(r',l',s') = l\cdot l' - rs' - sr'. \tag{2.2.5}$$

This quadratic form is called the *Mukai pairing*. With this lattice structure, $H^*(S,\mathbb{Z})$ is isometric to the *extended K3 lattice*

$$\tilde{\Lambda}_{\mathrm{K3}} \coloneqq U^{\oplus 3} \oplus E_8^{\oplus 2} \simeq \Lambda_{\mathrm{K3}} \oplus U. \tag{2.2.6}$$

It also inherits a Hodge structure of K3-type whose $(2,0)$-part is generated by the non-vanishing 2-form σ. We call the resulting Hodge structure the *Mukai lattice* and denote it by $\widetilde{H}(S,\mathbb{Z})$. More precisely, the Hodge structure of $\widetilde{H}(S,\mathbb{Z})$ is given by:

$$\begin{aligned}
\widetilde{H}^{0,2} &= H^{0,2}(S)\\
\widetilde{H}^{1,1} &= H^0(S,\mathbb{C}) \oplus H^{1,1}(S) \oplus H^4(S,\mathbb{C})\\
\widetilde{H}^{2,0} &= H^{2,0}(S).
\end{aligned}$$

We define the *extended Néron–Severi lattice* of S as the $(1,1)$-part of $\tilde{H}(S,\mathbb{Z})$:

$$\mathrm{N}(S) \coloneqq \tilde{H}^{1,1}(S,\mathbb{Z}) = H^0(S,\mathbb{Z}) \oplus \mathrm{NS}(S) \oplus H^4(S,\mathbb{Z}).$$

As a lattice, we have $\mathrm{N}(S) \simeq \mathrm{NS}(S) \oplus U$. The extended Néron–Severi lattice is also sometimes called the *numerical Grothendieck group* of S.

Definition 2.2.20. A lattice with a Hodge structure is called a Hodge lattice. Given two Hodge lattices H_1 and H_2, a group homomorphism $f\colon H_1 \to H_2$ is called a *Hodge metric morphism* if it has the following two properties:

i) It is a metric morphism, i.e. $h^2 = f(h)^2$ for all $h \in H_1$,

ii) It is a morphism of Hodge structures.

If f is additionally an isomorphism of groups, then we call f a *Hodge isometry.*

Example 2.2.21. Given a K3 surface S, the groups $T(S)$, $H^2(S,\mathbb{Z})$ and $\tilde{H}(S,\mathbb{Z})$ are all Hodge lattices. The two inclusions

$$T(S) \hookrightarrow H^2(S,\mathbb{Z}) \hookrightarrow \tilde{H}(S,\mathbb{Z})$$

are both Hodge metric morphisms.

Definition 2.2.22. For a Hodge lattice H of K3-type, the smallest primitive sublattice $T \subset H$ for which $T \otimes \mathbb{C}$ contains $H^{2,0}$ is called the *transcendental sublattice* of H.

For a K3 surface X, the transcendental sublattice of $H^2(X,\mathbb{Z})$ is by definition the transcendental lattice $T(X)$.

A *marked K3 surface* is a pair (S,ϕ), consisting of a K3 surface S and an isometry $\phi\colon H^2(S,\mathbb{Z}) \simeq \Lambda_{\mathrm{K3}}$, called a *marking* of S. The point $[\phi_{\mathbb{C}}(\sigma)] \in \mathbb{P}(\Lambda_{\mathrm{K3}} \otimes \mathbb{C})$ is called the *period* of (S,ϕ). Since $\sigma^2 = 0$ and $\sigma \cdot \overline{\sigma} > 0$, any period of S lies in the open subset

$$D \coloneqq \left\{\ell \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) \mid \ell^2 = 0 \text{ and } \ell \cdot \overline{\ell} > 0\right\}.$$

The following two results are among the most fundamental results about K3 surfaces.

Theorem 2.2.23 (Surjectivity of the Period Map)**.** *[Tod80] Any point in D is the period of a marked K3 surface, i.e. for any $\ell \in D$, there is a K3 surface S with an isometry $H^2(S,\mathbb{Z}) \to \Lambda_{K3}$ such that $H^2(S,\mathbb{C}) \to \Lambda_{K3} \otimes \mathbb{C}$ maps $H^{2,0}(S)$ to ℓ.*

Theorem 2.2.24 (Torelli Theorem for K3 Surfaces)**.** *[PS71] (see [Huy16, Theorem 5.5.3]) Let X and Y be K3 surfaces. Then X and Y are isomorphic if and only if there exists a Hodge isometry $H^2(X,\mathbb{Z}) \simeq H^2(Y,\mathbb{Z})$. Moreover, for any Hodge isometry $\psi\colon H^2(X,\mathbb{Z}) \simeq H^2(Y,\mathbb{Z})$ which preserves the ample cone, there is a unique isomorphism $f\colon X \simeq Y$ such that $\psi = f_*$.*

Note that the first part of Theorem 2.2.24 can be rephrased as follows: Two K3 surfaces X and Y are isomorphic if and only if there exist markings $\phi\colon H^2(X,\mathbb{Z}) \simeq \Lambda_{\mathrm{K3}}$, $\psi\colon H^2(Y,\mathbb{Z}) \simeq \Lambda_{\mathrm{K3}}$ such that

$$[\phi_{\mathbb{C}}(\sigma_X)] = [\psi_{\mathbb{C}}(\sigma_Y)],$$

where σ_X and σ_Y are non-vanishing 2-forms on X and Y, respectively. Indeed, in this case, the isometry $\psi^{-1} \circ \phi\colon H^2(X,\mathbb{Z}) \simeq H^2(Y,\mathbb{Z})$ is a Hodge isometry.

Lattices of hyperkähler manifolds

Let X be a hyperkähler manifold of dimension $2n > 2$. Similarly to K3 surfaces, the second cohomology group $H^2(X, \mathbb{Z})$ carries a Hodge structure of K3-type. Indeed, the $(2,0)$-part is $H^{2,0}(X) \simeq H^0(X, \Omega_X^2)$, which is 1-dimensional by assumption.

Recall that $H^2(X, \mathbb{Z})$ is a Hodge lattice when X is a K3 surface, with the bilinear form given by the cup-product. The cup-product in the higher-dimensional case is not an integral bilinear form, meaning that it does not take values in $\mathbb{Z}$, and therefore it looks like $H^2(X, \mathbb{Z})$ does not carry the structure of a Hodge lattice. However, one can define a quadratic form on $H^2(X, \mathbb{Z})$, called the *Beauville–Bogomolov–Fujiki* (*BBF*) form q [Bea83; Bog96; Fuj87]. The BBF form has the property that, for all $x \in H^2(X, \mathbb{Z})$, we have

$$\int_X x^{2n} = \lambda \cdot q(x)^n,$$

where $\lambda \in \mathbb{Q}$ is some fixed positive number that only depends on X.

The BBF form is non-degenerate, hence $H^2(X, \mathbb{Z})$ is a Hodge lattice, although it is not known whether $H^2(X, \mathbb{Z})$ is always even. It is important to note that, unlike in the case for K3 surfaces, $H^2(X, \mathbb{Z})$ is generally not unimodular if $\dim(X) > 2$ [Rap06].

Similar to the case of K3 surfaces, the *Néron–Severi lattice* $\mathrm{NS}(X)$ of X is the integral $(1,1)$-part of $H^2(X, \mathbb{Z})$, which is the image of $c_1 \colon \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$. The orthogonal complement $T(X) := \mathrm{NS}(X)^\perp$ is called the *transcendental lattice*, and it inherits a Hodge structure of K3-type from $H^2(X, \mathbb{Z})$.

We now restrict our attention to hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type, that is, hyperkähler manifolds that are deformation equivalent to the Hilbert scheme of n points on a K3 surface. We assume $n \geq 2$. Examples of hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type are smooth moduli spaces of (twisted) sheaves on a K3 surface, which are discussed in Section 2.3.1 below. For a hyperkähler manifold X of $\mathrm{K3}^{[n]}$-type, there exists an isometry of abstract lattices

$$H^2(X, \mathbb{Z}) \simeq U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2n-2 \rangle.$$

Note that $H^2(X, \mathbb{Z})$ is not unimodular in this case, as the discriminant group is isomorphic to $\mathbb{Z}/(2n-2)\mathbb{Z}$.

The Hodge lattice $H^2(X, \mathbb{Z})$ is a birational invariant of X. In other words, if X and Y are birational hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type, then there is a Hodge isometry $H^2(X, \mathbb{Z}) \simeq H^2(Y, \mathbb{Z})$ [OGr97, Proposition I.6.2]. It turns out that the converse does not hold, i.e. there exist non-birational hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type whose second cohomology groups are Hodge isometric [Yos01]. Despite this, Markman proved a wonderful theorem which is now known as the Birational Torelli Theorem for hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type.

For any hyperkähler manifold X of $\mathrm{K3}^{[n]}$-type, there is a natural $O(\tilde{\Lambda}_{\mathrm{K3}})$-orbit i_X of primitive embeddings $H^2(X, \mathbb{Z}) \hookrightarrow \tilde{\Lambda}_{\mathrm{K3}}$, where $\tilde{\Lambda}_{\mathrm{K3}}$ is the extended K3 lattice (2.2.6).

Theorem 2.2.25 (Birational Torelli Theorem). *[Mar11] Suppose X and Y are hyperkähler manifolds of* $\mathrm{K3}^{[n]}$*-type. The following are equivalent:*

i) X and Y are birational.

ii) There is a Hodge isometry $H^2(X,\mathbb{Z}) \simeq H^2(Y,\mathbb{Z})$ making the following diagram commute:

$$\begin{tikzcd} H^2(X,\mathbb{Z}) \arrow[r, "\simeq"] \arrow[d] & H^2(Y,\mathbb{Z}) \arrow[d] \\ \tilde{\Lambda}_{\mathrm{K3}} \arrow[r, "\simeq"] & \tilde{\Lambda}_{\mathrm{K3}} \end{tikzcd} \tag{2.2.7}$$

where the vertical arrows are embeddings contained in the orbits i_X and i_Y, respectively, and the isometry $\tilde{\Lambda}_{\mathrm{K3}} \simeq \tilde{\Lambda}_{\mathrm{K3}}$ is arbitrary.

Note that, in general, there exist multiple $O(\tilde{\Lambda}_{\mathrm{K3}})$-orbits of primitive embeddings $H^2(X,\mathbb{Z}) \hookrightarrow \tilde{\Lambda}_{\mathrm{K3}}$.

Lemma 2.2.26. *Let X be a hyperkähler manifold of $\mathrm{K3}^{[n]}$-type, where $n > 2$. The number of $O(\tilde{\Lambda}_{\mathrm{K3}})$-orbits of primitive embeddings $H^2(X,\mathbb{Z}) \hookrightarrow \tilde{\Lambda}_{\mathrm{K3}}$ is $2^{\omega(n-1)-1}$, where $\omega(n-1)$ is the number of distinct primes dividing $n-1$. If $n = 2$, there is exactly 1 orbit.*

Proof. This is a direct application of Corollary 2.2.7. □

A consequence of Theorem 2.2.25 and Lemma 2.2.26 is the following.

Corollary 2.2.27. *[Mar11] Let $n \geq 2$ such that $n-1$ is a power of a prime. If X and Y are hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type, then X and Y are birational if and only if there is a Hodge isometry $H^2(X,\mathbb{Z}) \simeq H^2(Y,\mathbb{Z})$.*

Proof. By Lemma 2.2.26, there is only one $O(\tilde{\Lambda}_{\mathrm{K3}})$-orbit of primitive embeddings $H^2(X,\mathbb{Z}) \hookrightarrow \tilde{\Lambda}_{\mathrm{K3}}$. Therefore, for any Hodge isometry $H^2(X,\mathbb{Z}) \simeq H^2(Y,\mathbb{Z})$ there exists an isometry of $\tilde{\Lambda}_{\mathrm{K3}}$ making (2.2.7) commute. In this case, X and Y are birational by the Birational Torelli Theorem 2.2.25. Conversely, if X and Y are birational, it follows from the Birational Torelli Theorem 2.2.25 that there is a Hodge isometry $H^2(X,\mathbb{Z}) \simeq H^2(Y,\mathbb{Z})$. □

2.2.3 Derived Categories

We recall some basic facts about derived categories of smooth, projective varieties, specifically K3 surfaces. We refer to [Huy06] for details. For a variety X over a field k, we denote by

$$\mathcal{D}^b(X) := \mathcal{D}^b(\mathbf{Coh}(X))$$

the bounded derived category of coherent sheaves on X. Two varieties X and Y are called *derived equivalent* if there is a linear, exact equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$. In this case, we say that X and Y are *Fourier–Mukai partners.* It is a well-known fact that for any linear, exact equivalence $F\colon \mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$, there exists an object $\mathcal{F}^\bullet \in \mathcal{D}^b(X \times Y)$ such that F is isomorphic to the functor

$$\begin{aligned} \Phi^{\mathcal{F}^\bullet}\colon \mathcal{D}^b(X) &\simeq \mathcal{D}^b(Y) \\ \mathcal{E} &\mapsto p_*\left(q^*(\mathcal{E}) \otimes \mathcal{F}^\bullet\right). \end{aligned}$$

Here, $q : X \times Y \to X$ and $p : X \times Y \to Y$ are the natural projections. We call functors of this form *Fourier–Mukai transforms,* and $\mathcal{F}^\bullet$ is called the *kernel* of $\Phi^{\mathcal{F}^\bullet}$.

Now, we let X and Y be smooth, projective varieties over $\mathbb{C}$. For an object $\mathcal{E}^\bullet \in \mathcal{D}^b(X)$, we define its *Mukai vector* to be

$$v(\mathcal{E}^\bullet) := \mathrm{ch}(\mathcal{E}^\bullet)\sqrt{\mathrm{td}_X} \in H^*(X, \mathbb{Q}).$$

Here, td_X denotes the *Todd class* of X. Recall that the full cohomology group $H^*(X, \mathbb{Q})$ has a ring structure. The class $\sqrt{\mathrm{td}_X}$ denotes a cohomology class whose square equals td_X. For example, for a K3 surface S, the Todd class of S is $\mathrm{td}_S = (1, 0, 2) \in \mathrm{N}(S)$, hence

$$\sqrt{\mathrm{td}_S} = (1, 0, 1) \in \mathrm{N}(S).$$

Therefore, the Mukai vector of an object $\mathcal{E}^\bullet \in \mathbf{Coh}(S)$ is

$$v(\mathcal{E}^\bullet) = (\mathrm{rk}(\mathcal{E}^\bullet), c_1(\mathcal{E}^\bullet), \mathrm{rk}(\mathcal{E}^\bullet) + \frac{1}{2}c_1(\mathcal{E}^\bullet)^2 - c_2(\mathcal{E}^\bullet)).$$

Given a cohomology class $\zeta \in H^*(X \times Y, \mathbb{Q})$, the *cohomological Fourier–Mukai transform with kernel* ζ is the group homomorphism

$$\begin{aligned} \Phi_H^\zeta : H^*(X, \mathbb{Q}) &\to H^*(Y, \mathbb{Q}) \\ x &\mapsto p_*(q^*(x) \wedge \zeta). \end{aligned}$$

We sometimes denote Φ_H^ζ by φ^ζ. By the Grothendieck–Riemann–Roch Theorem, for any object $\mathcal{E}^\bullet \in \mathcal{D}^b(X \times Y)$ and any $x \in H^*(X, \mathbb{Q})$, we have

$$\Phi_H^{v(\mathcal{E}^\bullet)}(x) = v\left(\Phi^{\mathcal{E}^\bullet}(x)\right).$$

Moreover, if $\Phi^{\mathcal{E}^\bullet}$ is an equivalence, then $\Phi_H^{v(\mathcal{E}^\bullet)}$ is a group isomorphism.

If X is a K3 surface, and the Fourier–Mukai transform $\Phi^{\mathcal{E}^\bullet} : \mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$ is an equivalence, then Y is a K3 surface and $\Phi_H^{v(\mathcal{E}^\bullet)}$ restricts to a Hodge isometry

$$\tilde{H}(X, \mathbb{Z}) \simeq \tilde{H}(Y, \mathbb{Z}),$$

where $\tilde{H}(X, \mathbb{Z})$ denotes the Mukai lattice as in Section 2.2.2 [Muk87; Orl03]. Therefore, it also restricts to a Hodge isometry between the transcendental lattices $T(X) \simeq T(Y)$. This observation is part of the Derived Torelli Theorem, due to Mukai and Orlov, see Section 2.3.2.

Now we let X and Y be arbitrary smooth, projective varieties. Let $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ be objects of $\mathcal{D}^b(X)$. Then for any equivalence $F : \mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$, we have $\mathrm{Ext}^i(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \mathrm{Ext}^i(F(\mathcal{F}^\bullet), F(\mathcal{G}^\bullet))$, hence

$$\chi(\mathcal{E}^\bullet, \mathcal{G}^\bullet) := \sum_i \dim\left(\mathrm{Ext}^i(\mathcal{F}^\bullet, \mathcal{G}^\bullet)\right) = \chi((F(\mathcal{E}^\bullet), F(\mathcal{G}^\bullet)). \tag{2.2.8}$$

If X is a K3 surface, then the Hirzebruch–Riemann–Roch Theorem implies that we have

$$\chi(\mathcal{F}^\bullet, \mathcal{E}^\bullet) = -v(\mathcal{F}^\bullet) \cdot v(\mathcal{E}^\bullet).$$

With this in mind, (2.2.8) implies that cohomological Fourier–Mukai equivalences between K3 surfaces restrict to isometries on the image of the Mukai vector. However, by the Derived Torelli Theorem, they even preserve the quadratic form on all of $\tilde{H}(X, \mathbb{Q})$. It turns out that cohomological Fourier–Mukai equivalences always preserve a certain quadratic form on $H^*(X, \mathbb{Q})$. In other words, K3 surfaces are not special with this property. Căldăraru was the first to realise this fact [Căl03].

Definition 2.2.28. Let X be any complex, smooth, projective variety. For $x \in H^*(X,\mathbb{C})$, we denote $x^\vee := \sum_i \sqrt{-1}^i[x]_i$, where $[x]_i$ is the component of x contained in $H^i(X,\mathbb{C})$. We define the *Mukai pairing* on $H^*(X,\mathbb{C})$ by

$$(x,y) := \int_X \exp(c_1(X)/2)x^\vee \wedge y.$$

Up to a sign, it coincides with our original definition of the Mukai pairing for the Mukai lattice of a K3 surface in (2.2.5). Indeed, for a K3 surface X, we have $c_1(X) = 0$, so that for $(r,l,s),(r',l',s') \in H^*(X,\mathbb{Z})$, we have

$$((r,l,s),(r',l',s')) = \int_X (r,l,s)^\vee \wedge (r',l',s') = -\left(l\cdot l' - rs' - r's\right),$$

which, up to a sign, equals (2.2.5).

Proposition 2.2.29. *[Căl03] (see also [Huy06, Proposition 5.44]) Let X and Y be complex, smooth, projective varieties. For any Fourier–Mukai equivalence $\Phi^{\mathcal{E}^\bullet} : \mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$, the corresponding cohomological Fourier–Mukai transform $\Phi_H^{\mathcal{E}^\bullet} : H^*(X,\mathbb{Q}) \simeq H^*(Y,\mathbb{Q})$ is an isometry with respect to the Mukai pairing.*

Proposition 2.2.29 is an immediate consequence of the following, more general statement.

Theorem 2.2.30. *[Căl03] Let X and Y be complex, smooth, projective varieties. Let $\Phi : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ be a functor which admits a left-adjoint $\Psi : \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$. Then, for all $x \in H^*(X,\mathbb{Q})$ and $y \in H^*(Y,\mathbb{Q})$, we have*

$$(\Psi(y),x) = (y,\Phi(x)).$$

Now we let X and Y be complex, smooth, projective varieties, and let $\dim(Y) = n$. If $\Phi = \Phi^{\mathcal{E}^\bullet} : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ is a Fourier–Mukai transform, then its left-adjoint is the Fourier–Mukai transform $\Psi = \Phi^{\mathcal{E}_L^\bullet} : \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$ with kernel

$$\mathcal{E}_L^\bullet := (\mathcal{E}^\bullet)^\vee \otimes p^*\omega_Y[n].$$

In this case, Theorem 2.2.30 follows from the computation in the proof of [Huy06, Proposition 5.44]. In that result, Φ is assumed to be an equivalence, but the computation does not rely on this assumption.

2.2.4 Brauer Groups and Twisted K3 Surfaces

We refer to [CS21] for basic facts on Brauer groups.

Definition 2.2.31. An Azumaya algebra $\mathcal{A}$ on a scheme X is a sheaf of $\mathcal{O}_X$-algebras, which is étale locally isomorphic to the sheaf of matrix algebras $\mathrm{M}_{n\times n}(\mathcal{O}_X)$.

We say that two Azumaya algebras $\mathcal{A}$, $\mathcal{B}$ are Br-*equivalent* if there exist locally free sheaves $\mathcal{E}$, $\mathcal{F}$ on X such that

$$\mathcal{A} \otimes \mathcal{E}nd(\mathcal{E}) \simeq \mathcal{B} \otimes \mathcal{E}nd(\mathcal{F}). \tag{2.2.9}$$

We say that $\mathcal{A}$ and $\mathcal{B}$ are SBr-*equivalent* if there exist locally free sheaves $\mathcal{E}$, $\mathcal{F}$ which, in addition to (2.2.9), satisfy

$$\det(\mathcal{E}) \simeq \det(\mathcal{F}) \simeq \mathcal{O}_X.$$

The set of isomorphism classes of Azumaya algebras carries a natural group structure given by the tensor product, and both Br-equivalence and SBr-equivalence are compatible with this group structure [Gro68a].

Definition 2.2.32. The Brauer group $\mathrm{Br}(X)$ of X is the group of Br-equivalence classes of Azumaya algebras of X. The special Brauer group $\mathrm{SBr}(X)$ is the group of SBr-equivalence classes of Azumaya algebras of X.

Now, let X be a complex smooth projective variety.

Definition 2.2.33. The *cohomological Brauer Group* of X is the group $H^2_{\text{ét}}(X, \mathbb{G}_m)$, or, equivalently, $H^2(X, \mathcal{O}_X^\times)_{\text{tors}}$.

The fact that $H^2_{\text{ét}}(X, \mathbb{G}_m)$ and $H^2(X, \mathcal{O}_X^\times)_{\text{tors}}$ are isomorphic is a consequence of the fact that $H^2(X, \mathbb{G}_m)$ is torsion, see [Huy16, Remark 18.1.4(ii)], combined with the Kummer sequence, see [Huy16, Remark 11.5.13].

It is a result by De Jong and Gabber that the Brauer group is naturally isomorphic to the cohomological Brauer group, see [CS21, §4.2], and we will frequently switch between the two viewpoints.

Under the additional assumption that $H^3(X, \mathbb{Z}) = 0$, we now derive a third version of the Brauer group from the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \overset{\exp}{\to} \mathcal{O}_X^\times \to 0.$$

The corresponding long exact sequence includes the following:

$$H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X^\times) \to H^3(X, \mathbb{Z}).$$

Since $H^1(X, \mathcal{O}_X^\times) \simeq \mathrm{Pic}(X)$, and the map $H^1(X, \mathcal{O}_X^\times) \to H^2(X, \mathbb{Z})$ is simply the first Chern class, we may rewrite the long exact sequence to:

$$0 \to \mathrm{NS}(X) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X^\times) \to H^3(X, \mathbb{Z}).$$

Assuming $H^3(X, \mathbb{Z}) = 0$, we obtain the short exact sequence

$$0 \to \frac{H^2(X, \mathbb{Z})}{\mathrm{NS}(X)} \to H^2(X, \mathcal{O}_X) \to H^2(X, \mathcal{O}_X^\times) \to 0.$$

The long exact sequence obtained by taking the tensor product with $\mathbb{Q}/\mathbb{Z}$ includes the following:

$$H^2(X, \mathcal{O}_X)_{\text{tors}} \to \mathrm{Br}(X) \to \frac{H^2(X, \mathbb{Z})}{\mathrm{NS}(X)} \otimes \mathbb{Q}/\mathbb{Z} \to H^2(X, \mathcal{O}_X) \otimes \mathbb{Q}/\mathbb{Z}. \tag{2.2.10}$$

Since $H^2(X, \mathcal{O}_X) \simeq \mathbb{C}$, the first and last terms of (2.2.10) vanish, and therefore we obtain an isomorphism

$$\mathrm{Br}(X) \simeq \frac{H^2(X, \mathbb{Z})}{\mathrm{NS}(X)} \otimes \mathbb{Q}/\mathbb{Z}. \tag{2.2.11}$$

For this reason, as in [HM23], we write

$$T'(X) := \frac{H^2(X,\mathbb{Z})}{\mathrm{NS}(X)}. \tag{2.2.12}$$

There is a well-defined injective group homomorphism $T'(X) \to T(X)^*$ given by

$$[\xi] \mapsto (\xi \cdot -)|_{T(X)}. \tag{2.2.13}$$

Since $T(X) \subset H^2(X,\mathbb{Z})$ is a primitive sublattice, the restriction map $H^2(X,\mathbb{Z})^* \to T(X)^*$ is surjective. This means that if $H^2(X,\mathbb{Z})$ is unimodular, i.e. if $H^2(X,\mathbb{Z}) \simeq H^2(X,\mathbb{Z})^*$, then for any element $f \in T(X)^*$, there exists a vector $\xi \in H^2(X,\mathbb{Z})$ such that $f(t) = \xi \cdot t$ for all $t \in T(X)$. In this case, the morphism (2.2.13) is an isomorphism. Recall that $H^2(X,\mathbb{Z})$ is unimodular if X is a K3 surface, hence the above discussion proves the following well-known lemma, see [Huy16, §18] and [Gee05].

Lemma 2.2.34. *Let X be a K3 surface. Then there is an isomorphism*

$$\mathrm{Br}(X) \simeq T(X)^* \otimes \mathbb{Q}/\mathbb{Z} \simeq \mathrm{Hom}(T(X), \mathbb{Q}/\mathbb{Z}).$$

In particular, $\mathrm{Br}(X)$ *is an infinite torsion group and for all integers $t \geq 1$ we have*

$$\mathrm{Br}(X)_{t-tors} \simeq \mathrm{Hom}(T(X), \mathbb{Z}/t\mathbb{Z}) \simeq (\mathbb{Z}/t\mathbb{Z})^{22-\rho}, \tag{2.2.14}$$

where ρ is the Picard number of X.

To summarise, for a K3 surface X, there are natural isomorphisms

$$\mathrm{Br}(X) \simeq H^2_{\text{ét}}(X,\mathbb{G}_m) \simeq H^2(X,\mathcal{O}_X^\times)_{\text{tors}} \simeq T'(X) \otimes \mathbb{Q}/\mathbb{Z} \simeq \mathrm{Hom}(T(X),\mathbb{Q}/\mathbb{Z}), \tag{2.2.15}$$

and we refer to each of these groups as the Brauer group of X.

On the other hand, if X is a higher-dimensional hyperkähler manifold of any of the currently known deformation types, then $H^2(X,\mathbb{Z})$ has a non-trivial discriminant [Rap06]. Thus, for a hyperkähler manifold, we generally have a strict inclusion $T'(X) \subset T(X)^*$.

Let us now consider the case when X is a higher-dimensional hyperkähler manifold of $\mathrm{K3}^{[n]}$-type. We note that the additional assumption that $H^3(X,\mathbb{Z}) = 0$ is satisfied in this situation. This follows from [Göt02, Equation (2.1)] and [Mar07, Theorem 1]. Now only the first four groups that appear in (2.2.15) are naturally isomorphic. We refer to each of the first four groups as the Brauer group of X.

Note that for any scheme X, there is a natural surjective group homomorphism $\mathrm{SBr}(X) \to \mathrm{Br}(X)$. It is a result by Grothendieck [Gro68b] that there is a short exact sequence

$$0 \to \mathrm{NS}(X) \otimes \mathbb{Q}/\mathbb{Z} \to \mathrm{SBr}(X) \to \mathrm{Br}(X) \to 0, \tag{2.2.16}$$

which, combined with (2.2.11), implies that we have

$$\mathrm{SBr}(X) \simeq H^2(X,\mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z},$$

whenever X is a K3 surface or, more generally, a hyperkähler manifold of $\mathrm{K3}^{[n]}$-type.

Definition 2.2.35. A *twisted K3 surface* is a pair (X, α) consisting of a K3 surface X and a Brauer class $\alpha \in \mathrm{Br}(X)$ (or a special Brauer class $\alpha \in \mathrm{SBr}(X)$).

For now, the notion of a twisted K3 surface may look rather pointless. However, we will soon be able to view them as geometric objects by defining twisted coherent sheaves, twisted derived categories, and twisted Hodge structures. In this sense, the tools used to study K3 surfaces can also be used to study twisted K3 surfaces. Moreover, twisted K3 surfaces naturally arise when one studies non-fine moduli spaces of sheaves on K3 surfaces. They arise via obstruction classes, which will be defined in Section 2.4.

Classically, the constructions of twisted sheaves, twisted derived categories, and twisted Hodge structures rely on extra choices. For the former two, this choice is a representative Čech cocycle for α or an Azumaya algebra representing α, and for the latter, it is the choice of a B-field lift. The equivalence classes of the twisted categories and the isomorphism classes of the twisted Hodge structures are independent of these choices, but the equivalences and Hodge isometries are not canonically defined, as we will see shortly. Recently, Huybrechts and Mattei reformulated these concepts in terms of special Brauer groups [HM23]. In their language, the ambiguity of choosing an equivalence or isometry disappears. We will now discuss both the classical constructions and the modern replacements. Twisted Hodge structures are discussed in the next section.

Let (X, α) be a twisted K3 surface. Choose an étale cover $(\mathcal{U}_i)_i$ of X and a Čech cocycle $\{\alpha_{ijk}\}$ representing $\alpha \in H^2_{\text{ét}}(X, \mathbb{G}_m)$. A *coherent* $\{\alpha_{ijk}\}$*-twisted sheaf* F is defined as an étale-local collection of coherent sheaves $F_i \in \mathbf{Coh}(\mathcal{U}_i)$ with identifications $\varphi_{ij} \colon F_i|_{\mathcal{U}_{ij}} \simeq F_j|_{\mathcal{U}_{ij}}$, satisfying the non-trivial cocycle condition

$$\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = \alpha_{ijk} \cdot \mathrm{id}\,.$$

We denote by $\mathbf{Coh}(X, \{\alpha_{ijk}\})$ the abelian category of $\{\alpha_{ijk}\}$-twisted sheaves. Note that this category depends on the choice of a cocycle representing α. However, for a different choice $\{\alpha'_{ijk}\}$, the resulting category $\mathbf{Coh}(X, \{\alpha'_{ijk}\})$ is non-canonically equivalent to $\mathbf{Coh}(X, \{\alpha_{ijk}\})$ by [Căl00, Lemma 1.2.8]. Because of this, we usually denote the category $\mathbf{Coh}(X, \{\alpha_{ijk}\})$ simply by $\mathbf{Coh}(X, \alpha)$.

A different, but equivalent, way to define the category $\mathbf{Coh}(X, \alpha)$ is via Azumaya algebras. Let $\mathcal{A}$ be an Azumaya algebra representing a Brauer class $\alpha \in \mathrm{Br}(X)$. We denote by $\mathbf{Coh}(X, \mathcal{A})$ the abelian category of coherent right $\mathcal{A}$-modules. Once again, choosing a different representative $\mathcal{A}'$ for α leads to a different category $\mathbf{Coh}(X, \mathcal{A}')$. However, if we have $\mathcal{A}' \simeq \mathcal{A} \otimes \mathcal{E}nd(\mathcal{E})$ for some locally free sheaf $\mathcal{E}$, there is an equivalence

$$\begin{aligned} \mathbf{Coh}(X, \mathcal{A}) &\simeq \mathbf{Coh}(X, \mathcal{A}') \\ \mathcal{F} &\mapsto \mathcal{F} \otimes \mathcal{E}. \end{aligned} \tag{2.2.17}$$

This implies that $\mathbf{Coh}(X, \mathcal{A}) \simeq \mathbf{Coh}(X, \mathcal{A}')$ for any two Br-equivalent $\mathcal{A}$ and $\mathcal{A}'$, since the Br-equivalence relation is generated by the relations of the form $\mathcal{A} \sim \mathcal{A} \otimes \mathcal{E}nd(\mathcal{E})$. The equivalence (2.2.17) depends on the choice of $\mathcal{E}$, which is not unique since $\mathcal{E}nd(\mathcal{E}) \simeq \mathcal{E}nd(\mathcal{E} \otimes \mathcal{L})$ for any line bundle $\mathcal{L}$.

Now, consider the *special* Brauer class $\alpha \in \mathrm{SBr}(X)$ represented by $\mathcal{A}$. If $\mathcal{A}'$ is another Azumaya algebra with the property that $\mathcal{A}' \simeq \mathcal{A} \otimes \mathcal{E}nd(\mathcal{E})$ for some locally free sheaf $\mathcal{E}$ with $\det(\mathcal{E}) \simeq \mathcal{O}_X$, then the difference with the classical case is that the sheaf $\mathcal{E}$ is uniquely determined. Indeed, suppose $\mathcal{E}$ and $\mathcal{E}'$ are locally free sheaves with $\det(\mathcal{E}) \simeq \det(\mathcal{E}') \simeq \mathcal{O}_X$ and such that $\mathcal{A} \otimes \mathcal{E}nd(\mathcal{E}) \simeq \mathcal{A} \otimes \mathcal{E}nd(\mathcal{E}')$. We claim that there exists a line bundle $\mathcal{L}$ such that $\mathcal{E} \simeq \mathcal{E}' \otimes \mathcal{L}$. To see this, note that we have an injective group homomorphism $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(k)$, where $k = k(X)$ is the function field of X. The group homomorphism is given by taking the stalk over the generic fibre. Since $\mathcal{A}_k \otimes_k \mathcal{E}nd(\mathcal{E})_k \simeq \mathcal{A}_k \otimes_k \mathcal{E}nd(\mathcal{E}')_k$, we have an isomorphism $\mathcal{E}nd(\mathcal{E})_k \simeq \mathcal{E}nd(\mathcal{E}')_k$ by [CS21, Theorem 1.2.4]. We can lift this to an isomorphism $\mathcal{E}nd(\mathcal{E}) \simeq \mathcal{E}nd(\mathcal{E}')$ by the injectivity of the map $\mathrm{Br}(X) \hookrightarrow \mathrm{Br}(k)$. Therefore, there exists a line bundle $\mathcal{L}$ such that $\mathcal{E} \simeq \mathcal{E}' \otimes \mathcal{L}$. Then

$$\mathcal{O}_X \simeq \det(\mathcal{E}') \simeq \det(\mathcal{E} \otimes \mathcal{L}) \simeq \det(\mathcal{E}) \otimes \mathcal{L}^{\otimes n},$$

where $n = \operatorname{rk} \mathcal{E}$. Therefore, $\mathcal{L}$ is a torsion element in $\mathrm{Pic}(X)$. Since X is a K3 surface, $\mathrm{Pic}(X)$ is torsion-free, hence $\mathcal{L} \simeq \mathcal{O}_X$. This shows that $\mathcal{E}$ is unique up to isomorphism. Therefore, for any two Azumaya algebras $\mathcal{A}$, $\mathcal{A}'$ representing the same special Brauer class, there is a distinguished equivalence $\mathbf{Coh}(X, \mathcal{A}) \simeq \mathbf{Coh}(X, \mathcal{A}')$. In the case $\mathcal{A}' \simeq \mathcal{A} \otimes \mathcal{E}nd(\mathcal{E})$ for some locally free sheaf $\mathcal{E}$ with $\det(\mathcal{E}) \simeq \mathcal{O}_X$, the equivalence is given by (2.2.17).

For a Brauer class $\alpha \in \mathrm{Br}(X)$, we now have the two associated categories $\mathbf{Coh}(X, \{\alpha_{ijk}\})$ and $\mathbf{Coh}(X, \mathcal{A})$, where $\{\alpha_{ijk}\}$ is a cocycle representing α, and $\mathcal{A}$ is an Azumaya algebra representing α. These two categories are equivalent by [Căl00, Theorem 1.3.7]. The idea of the proof is that for any Azumaya algebra $\mathcal{A}$ representing α, there exists a locally free $\{\alpha_{ijk}\}$-twisted sheaf $\mathcal{E}$ such that $\mathcal{A}$ is isomorphic to the sheaf $\mathcal{E}nd(\mathcal{E})$, and the equivalence is defined by

$$\begin{aligned} \mathbf{Coh}\,(X, \{\alpha_{ijk}\}) &\to \mathbf{Coh}\,(X, \mathcal{A}) \\ \mathcal{F} &\mapsto \mathcal{F} \otimes \mathcal{E}^\vee. \end{aligned}$$

For a (special) Brauer class α on X, we denote by

$$\mathcal{D}^b(X, \alpha) := \mathcal{D}^b(\mathbf{Coh}(X, \alpha))$$

the bounded derived category of the abelian category $\mathbf{Coh}(X, \alpha)$. If $\mathcal{A}$ is an Azumaya algebra representing α, we sometimes denote $\mathcal{D}^b(X, \alpha)$ by $\mathcal{D}^b(X, \mathcal{A})$. We may view elements of $\mathcal{D}^b(X, \alpha)$ as complexes of $\mathcal{A}$-modules, or as complexes of α-twisted sheaves.

For K3 surfaces X and Y, Brauer classes $\alpha, \alpha' \in \mathrm{Br}(X)$, and a morphism $f\colon Y \to X$, the usual derived functors

$$\begin{aligned} R\,\mathrm{Hom}\colon \quad & \mathcal{D}^b(X, \alpha) \times \mathcal{D}^b(X, \alpha') && \to \mathcal{D}^b(X, \alpha^{-1}\alpha') \\ \otimes^L\colon \quad & \mathcal{D}^b(X, \alpha) \times \mathcal{D}^b(X, \alpha') && \to \mathcal{D}^b(X, \alpha\alpha') \\ Lf^*\colon \quad & \mathcal{D}^b(X, \alpha) && \to \mathcal{D}^b(Y, f^*\alpha) \\ Rf_*\colon \quad & \mathcal{D}^b(Y, f^*\alpha) && \to \mathcal{D}^b(X, \alpha) \end{aligned}$$

are all well-defined, as was worked out in detail by Căldăraru [Căl00].

An important advantage of working with a special Brauer class is that there is a well-defined Chern character:

$$\mathrm{ch}_\alpha\colon\ \mathrm{D}^b(X, \alpha) \to \tilde{H}(X, \mathbb{Q}).$$

Explicitly, fix an Azumaya algebra $\mathcal{A}$ representing $\alpha \in \mathrm{SBr}(X)$. For an $\mathcal{A}$-module $\mathcal{F}$, we define

$$\mathrm{ch}_{\mathcal{A}}(\mathcal{F}) \coloneqq \mathrm{ch}(\mathcal{F})\sqrt{\mathrm{ch}(\mathcal{A})}^{-1}.$$

Proposition 2.2.36. *[HM23, Definition-Proposition 2.8] Let $\mathcal{A}$ and $\mathcal{A}'$ be two Azumaya algebras representing the same special Brauer class $\alpha \in \mathrm{SBr}(X)$. Then the diagram*

$$\begin{tikzcd} \mathcal{D}^b(X,\mathcal{A}) \arrow[rr, "\simeq"] \arrow[dr, "\mathrm{ch}_{\mathcal{A}}"'] & & \mathcal{D}^b(X,\mathcal{A}') \arrow[dl, "\mathrm{ch}_{\mathcal{A}'}"] \\ & \tilde{H}(X,\mathbb{Q}) & \end{tikzcd} \tag{2.2.18}$$

commutes, where the horizontal map is the canonical equivalence induced by (2.2.17).

Proof. Suppose first that $\mathcal{A}' \simeq \mathcal{A} \otimes \mathcal{E}nd(\mathcal{E})$ for some locally free sheaf $\mathcal{E}$ with $\det(\mathcal{E}) \simeq \mathcal{O}_X$. For any $\mathcal{A}$-module $\mathcal{F}$, we have

$$\begin{aligned} v_{\mathcal{A}'}(\mathcal{F}\otimes\mathcal{E}) &= \mathrm{ch}(\mathcal{F}\otimes\mathcal{E})\sqrt{\mathrm{ch}(\mathcal{A}\otimes\mathcal{E}nd(\mathcal{E}))}^{-1} \\ &= \mathrm{ch}(\mathcal{F})\,\mathrm{ch}(\mathcal{E})\sqrt{\mathrm{ch}(\mathcal{A})}^{-1}\sqrt{\mathrm{ch}(\mathcal{E}\otimes\mathcal{E}^\vee)}^{-1} \\ &= \mathrm{ch}(\mathcal{F})\,\mathrm{ch}(\mathcal{E})\sqrt{\mathrm{ch}(\mathcal{A})}^{-1}\sqrt{\mathrm{ch}(\mathcal{E})^2}^{-1} \\ &= \mathrm{ch}(\mathcal{F})\sqrt{\mathrm{ch}(\mathcal{A})}^{-1} \\ &= v_{\mathcal{A}}(\mathcal{F}), \end{aligned}$$

as required. The general case follows from the fact that the SBr-equivalence relation is generated by the relations of the form $\mathcal{A} \sim \mathcal{A}\otimes\mathcal{E}nd(\mathcal{E})$ for $\mathcal{E}$ a locally free sheaf with trivial determinant. □

Remark 2.2.37. Note that, if $\mathcal{E}$ is a locally free sheaf, it is not always the case that $\mathrm{ch}(\mathcal{E}\otimes\mathcal{E}^\vee) = \mathrm{ch}(\mathcal{E})^2$. For this equation to hold, we need both the assumption that X is a surface (or curve), and the assumption that $\det(\mathcal{E}) \simeq \mathcal{O}_X$. Indeed, writing $\mathrm{ch}(\mathcal{E}) = (\mathrm{rk}(\mathcal{E}), c_1, c_2)$, we have

$$\mathrm{ch}(\mathcal{E}\otimes\mathcal{E}^\vee) = (\mathrm{rk}(\mathcal{E})^2, 0, 2\,\mathrm{rk}(\mathcal{E})c_2 - c_1^2),$$

which only equals $\mathrm{ch}(\mathcal{E})^2$ if $c_1 = 0$. In particular, if $\mathcal{A}$ and $\mathcal{A}'$ represent the same (non-special) Brauer class, the diagram (2.2.18) does not necessarily commute.

2.2.5 Twisted Hodge Structures

We now move on to twisted Hodge structures. Our main references for twisted Hodge structures are [Huy05; HS05b; HS06]. The main goal of this section is to study twisted Hodge structures of twisted K3 surfaces. However, in Section 4.3.1, where we study derived equivalence for hyperkähler manifolds, it will be important to study twisted Hodge structures on non-unimodular Hodge lattices of K3-type. Therefore, we first study a slightly more general situation. Let H be a Hodge lattice of K3-type, with transcendental sublattice $T \subset H$, see Definition 2.2.22. We denote the integral $(1,1)$-part of H by $\mathrm{NS} \coloneqq T^\perp \subset H$. We write $\tilde{H} \coloneqq H \oplus U(-1)$, and denote $N \coloneqq T^\perp \subset \tilde{H}$. We consider $\tilde{H}$ with the Hodge structure of K3-type that it inherits from H, i.e. $N = \tilde{H}^{1,1}_{\mathbb{Z}}$. Similar to the Mukai lattice of a K3 surface, if $e, f \in U(-1)$ are the standard basis, we denote the element $re + \ell + sf \in \tilde{H}$

by (r, ℓ, s). In this set-up, $\tilde{H}$ carries a natural commutative ring structure with multiplication, denoted $\wedge$, given by:

$$(r, \ell, s) \wedge (r', \ell', s') = (rr', r\ell' + r'\ell, \ell \cdot \ell' + rs' + r's).$$

Note that the multiplicative identity element of $\tilde{H}$ is $(1, 0, 0)$.

Remark 2.2.38. In the following, the reader should keep in mind that we will apply the lemmas below in the setting where $H = H^2(X, \mathbb{Z})$ for some hyperkähler manifold X of $\mathrm{K3}^{[n]}$-type. If X is a K3 surface, $\tilde{H}$ is the Mukai lattice of X. In this case, NS is the Néron–Severi lattice of X, T is the transcendental lattice, and N is the extended Néron–Severi lattice. This is the reason for our suggestive notation.

There is an isometry $U \simeq U(-1)$, so that $\tilde{H}$ is Hodge isometric to $H \oplus U$. However, we write $U(-1)$ to make it clear that $(1, 0, 0) \cdot (0, 0, 1) = -1$. We do this so that the bilinear form on $\tilde{H}$ exactly matches that of the Mukai lattice of a K3 surface.

The following lemma will be used in Section 4.3.1, specifically in the proof of Proposition 4.3.7.

Lemma 2.2.39. *An isometry $f : \tilde{H} \simeq \tilde{H}$ with the property that $f(x) = x$ for all $x \in U(-1) \subset \tilde{H}$ is a ring isomorphism.*

Proof. Since f is an isometry, it suffices to show that it is compatible with the multiplication. Firstly, we have $f(1, 0, 0) = (1, 0, 0)$ by assumption. Moreover, since f preserves $U(-1)$, it also preserves $U(-1)^{\perp} = H$. For $(r, \ell, s), (r', \ell', s') \in \tilde{H}$, we have

$$\begin{aligned} f\left((r, \ell, s) \wedge (r', \ell', s')\right) &= f\left(rr', r\ell' + r'\ell, \ell \cdot \ell' + rs' + r's\right) \\ &= (rr', f(r\ell' + r'\ell), \ell \cdot \ell' + rs' + r's) \\ &= (rr', rf(\ell') + r'f(\ell), \ell \cdot \ell' + rs' + r's) \\ &= f\left(r, \ell, s)\right) \wedge f\left(r', \ell', s'\right). \end{aligned}$$

□

Given a rational element $B \in H \otimes \mathbb{Q}$, we define

$$\exp(B) \coloneqq \left(1, B, \frac{B^2}{2}\right)$$

and we obtain an isometry

$$\begin{aligned} \exp(B)\colon \quad \tilde{H} \otimes \mathbb{Q} &\longrightarrow \tilde{H} \otimes \mathbb{Q} \\ (r, \ell, s) &\longmapsto \exp(B) \wedge (r, \ell, s). \end{aligned} \tag{2.2.19}$$

The following result follows from a straightforward computation.

Lemma 2.2.40. *For $B, B' \in H \otimes \mathbb{Q}$, we have* $\exp(B) \circ \exp(B') = \exp(B + B')$. *Moreover, if $B \in H$, then* $\exp(B)$ *restricts to an isometry* $\exp(B) : \tilde{H} \simeq \tilde{H}$ *with inverse* $\exp(-B)$.

Definition 2.2.41. Given a rational element $B \in H \otimes \mathbb{Q}$, we define the B-twisted Hodge lattice $\tilde{H}(B)$ to be the Hodge lattice of K3-type whose underlying lattice is $\tilde{H}$ and whose $(2,0)$-part is generated by

$$\exp(B) \wedge \sigma = \sigma + B \wedge \sigma, \tag{2.2.20}$$

where $\sigma \in \tilde{H}_{\mathbb{C}}$ is a generator of $H^{2,0}$. We denote the integral $(1,1)$-part of $\tilde{H}(B)$ by $\mathrm{N}(B)$, and its orthogonal complement by $T(B) := N(B)^{\perp}$.

We now list a series of technical lemmas to help us study the twisted Hodge structure $\tilde{H}(B)$. This theory was set up in [Huy05].

Lemma 2.2.42. *The isometry* (2.2.19) *induces a Hodge isometry*

$$\exp(B)\colon \tilde{H}_{\mathbb{Q}} \simeq \tilde{H}(B)_{\mathbb{Q}}.$$

Proof. Since $\exp(B)$ is an isometry, it suffices to check that it preserves the Hodge structures. This follows immediately from (2.2.20). □

Lemma 2.2.43. *Consider the subgroup* $\exp(B) \wedge \tilde{H} \subset \tilde{H}_{\mathbb{Q}}$. *The bilinear form on* $\tilde{H}_{\mathbb{Q}}$ *restricts to an integral bilinear form on it, and it inherits a Hodge structure of K3-type from* $\tilde{H}_{\mathbb{Q}}$. *With this Hodge lattice structure, there is a Hodge isometry*

$$\exp(B) \wedge \tilde{H} \simeq \tilde{H}(-B).$$

Proof. This follows immediately from the following commutative diagram of Hodge metric morphisms:

$$\begin{tikzcd} \tilde{H}_{\mathbb{Q}} \arrow[r, "\exp(-B)"] & \tilde{H}(-B)_{\mathbb{Q}} \\ \exp(B) \wedge \tilde{H} \arrow[u, hook] \arrow[r, "\simeq"] & \tilde{H}(-B). \arrow[u, hook] \end{tikzcd}$$

□

Lemma 2.2.44. *Let* $B, B' \in H_{\mathbb{Q}}$. *If* $B' - B$ *is integral, then* $\exp(B' - B)$ *restricts to a Hodge isometry* $\tilde{H}(B) \simeq \tilde{H}(B')$.

Proof. Since $B' - B$ is integral, the Hodge isometry $\exp(B' - B) : \tilde{H}_{\mathbb{Q}} \simeq \tilde{H}_{\mathbb{Q}}$ preserves $\tilde{H}$. Moreover, since $\exp(B' - B) \wedge \exp(B) \wedge \sigma = \exp(B') \wedge \sigma$, $\exp(B' - B)$ restricts to a Hodge isometry $\tilde{H}(B) \simeq \tilde{H}(B')$. □

Lemma 2.2.45. *Let* $r \in \mathbb{Z}$ *be the smallest positive integer for which* $rB \in H$. *Then the element*

$$(r, rB, 0) \in \tilde{H}(B)$$

is contained in $\mathrm{N}(B)$. *Moreover, we have* $\mathrm{NS} \subset \mathrm{N}(B)$, *and* $(0,0,1) \in \mathrm{N}(B)$.

Proof. Let $\ell \in \mathrm{NS}$, and let $\sigma \in H_{\mathbb{C}}$ be a generator of $H^{2,0}$. We have the following equalities:

$$\begin{aligned} (0,\ell,0) \cdot (0,\sigma, B \cdot \sigma) &= 0 \\ (r, rB, 0) \cdot (0,\sigma, B \cdot \sigma) &= 0 \\ (0,0,1) \cdot (0,\sigma, B \cdot \sigma) &= 0. \end{aligned}$$

The claim follows from (2.2.4). □

We now consider the map

$$\begin{array}{rccc} (B\cdot -)\colon & T & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ & x & \longmapsto & B\cdot x, \end{array}$$

and we denote the kernel of this map by $\ker(B)$. In particular, $\ker(B)$ is a finite index sublattice of T which inherits a Hodge structure of K3-type from it.

Proposition 2.2.46. *The Hodge isometry* $\exp(B)\colon \tilde{H}_{\mathbb{Q}} \simeq \tilde{H}(B)_{\mathbb{Q}}$ *from* (2.2.19) *restricts to a Hodge isometry*

$$\ker(B) \simeq T(B).$$

Proof. Since $\exp(B)$ is a Hodge isometry, we have $\exp(B)(\ker(B)) \subset T(B)_{\mathbb{Q}}$. Moreover, for any element $x \in \ker(B)$, the image $\exp(B)(x) = (0, x, B\cdot x)$ is an integral vector by our assumption on x. This shows that we have

$$\exp(B)(\ker(B)) \subset T(B).$$

Conversely, suppose $x = (r, \ell, s) \in T(B)$. Since $(0,0,1) \in \mathrm{N}(B)$ by Lemma 2.2.45, we have $r = 0$ since x is orthogonal to $\mathrm{N}(B)$. Moreover, since $x \cdot (r, rB, 0) = 0$, again by Lemma 2.2.45, we have $rB\cdot \ell = rs$, hence $x = (0, \ell, B\cdot\ell) = \exp(B)\wedge \ell$. This shows that

$$T(B) \subset \exp(B)(\ker(B)),$$

hence $\exp(B)$ induces a Hodge isometry $\ker(B) \simeq T(B)$. □

Since $\ker(B)$ is a finite-index sublattice of T, we immediately obtain the following corollary.

Corollary 2.2.47. *There are rational Hodge isometries*

$$T_{\mathbb{Q}} \simeq T(B)_{\mathbb{Q}}, \quad \tilde{H}_{\mathbb{Q}} \simeq \tilde{H}(B)_{\mathbb{Q}}$$

and rational isometries

$$\mathrm{N}_{\mathbb{Q}} \simeq \mathrm{N}(B)_{\mathbb{Q}}, \quad \mathrm{NS}_{\mathbb{Q}} \simeq \mathrm{NS}(B)_{\mathbb{Q}}.$$

Proof. The existence of a rational Hodge isometry $T_{\mathbb{Q}} \simeq T(B)_{\mathbb{Q}}$ follows immediately from Proposition 2.2.46. The other isometries are obtained from it using the Witt Extension Theorem 2.2.14 and the Witt Cancellation Theorem 2.2.15. □

Twisted Hodge lattices for K3 surfaces

For the rest of this section, we specialise to K3 surfaces. Let S be a K3 surface with Brauer class $\alpha \in \mathrm{Br}(S)$.

Firstly, note that there is a natural surjection $H^2(S,\mathbb{Q}) \to \mathrm{Br}(S) \simeq T(S)^* \otimes \mathbb{Q}/\mathbb{Z}$. For a Brauer class $\alpha \in \mathrm{Br}(S)$, we call an element $B \in H^2(S,\mathbb{Q})$ whose image in $\mathrm{Br}(S)$ is α a *B-field lift* of α. For a B-field lift B of α, we define the twisted Hodge structure

$$\tilde{H}(S, B, \mathbb{Z})$$

to be the Hodge structure of K3-type whose underlying group structure is $\tilde{H}(S,\mathbb{Z})$, and whose $(2,0)$-part is generated by

$$\exp(B)\wedge\sigma = \sigma+\sigma\wedge B.$$

In the notation of Definition 2.2.41, this means that we have

$$\tilde{H}(S,B,\mathbb{Z}) \coloneqq \tilde{H}(S,\mathbb{Z})(B).$$

Similarly, we define

$$T(S,B) \coloneqq T(S)(B),$$

and

$$\mathrm{N}(S,B) \coloneqq \mathrm{N}(S)(B).$$

Note that, if B and B' are two distinct B-field lifts of α, then $B-B'$ maps to zero in

$$\mathrm{Br}(S)\simeq \frac{H^2(S,\mathbb{Z})}{\mathrm{NS}(S)}\otimes\mathbb{Q}/\mathbb{Z},$$

hence there is a rational vector $v\in\mathrm{NS}(S)_{\mathbb{Q}}$ such that $B-B'+v\in H^2(S,\mathbb{Z})$ is an integral vector. Now the map

$$\exp(B-B'+v)\colon \tilde{H}(S,B,\mathbb{Z})\simeq\tilde{H}(S,B',\mathbb{Z})$$

is a Hodge isometry. Note that the Hodge isometry depends on the choice of $v\in\mathrm{NS}(S)_{\mathbb{Q}}$, which is not unique. Therefore, the two twisted Hodge structures $\tilde{H}(S,B,\mathbb{Z})$ and $\tilde{H}(S,B',\mathbb{Z})$ are non-canonically Hodge isometric, and we usually denote them by $\tilde{H}(S,\alpha,\mathbb{Z})$. We denote by $\mathrm{N}(S,\alpha)$ the integral $(1,1)$-part of $\tilde{H}(S,\alpha,\mathbb{Z})$, which also depends on the choice of a B-field lift, up to a non-canonical isometry. The twisted transcendental lattice $T(S,\alpha)$ is defined to be the orthogonal complement $T(S,\alpha)=\mathrm{N}(S,\alpha)^{\perp}\subset\tilde{H}(S,\alpha,\mathbb{Z})$.

For a special Brauer class $\alpha\in\mathrm{SBr}(S)\simeq H^2(S,\mathbb{Z})\otimes\mathbb{Q}/\mathbb{Z}$, we define the twisted Hodge structure $\tilde{H}(S,\alpha,\mathbb{Z})$ completely analogously to the classical case; we choose a B-field lift $B\in H^2(S,\mathbb{Q})$ of α, and we define $\tilde{H}(S,B,\mathbb{Z})$ to be the Hodge lattice of K3-type with underlying lattice $\tilde{H}(S,\mathbb{Z})$ and whose $(2,0)$-part is generated by $\sigma+\sigma\wedge B$. If B and B' are two B-field lifts of the same special Brauer class $\alpha\in\mathrm{SBr}(S)$, then $B-B'\in H^2(S,\mathbb{Z})$ is an integral vector, hence there is a canonical Hodge isometry

$$\exp(B-B')\colon \tilde{H}(S,B',\mathbb{Z})\simeq\tilde{H}(S,B,\mathbb{Z})$$

by Lemma 2.2.44. If $\overline{\alpha}\in\mathrm{Br}(S)$ is the Brauer class induced by α via (2.2.16) we have a Hodge isometry

$$\tilde{H}(S,\alpha,\mathbb{Z})\simeq\tilde{H}(S,\overline{\alpha},\mathbb{Z}).$$

In other words, the Hodge isometry class $\tilde{H}(S,\alpha,\mathbb{Z})$ only depends on the image of α in $\mathrm{Br}(S)$.

2.3 Moduli Spaces

2.3.1 Moduli Spaces of Twisted Sheaves

We recall some facts about moduli spaces of (twisted) sheaves on K3 surfaces. For a K3 surface S, a Brauer class $\alpha\in\mathrm{Br}(S)$ and a Mukai vector $v\in\mathrm{N}(S,\alpha)$, the moduli space $M(v;\alpha)$ that we define in Theorem 2.3.2 depends on the choice of an Azumaya algebra, a cocycle, or a B-field. Different choices lead to different moduli spaces that need not even be birational as we discuss in Chapter 4, see Remark 4.2.15 for examples. For this reason, the moduli space $M(v;\alpha)$ is usually denoted by $M(v;\mathcal{A})$ or $M(v;B)$ in the literature, where $\mathcal{A}$ is an Azumaya algebra representing α and B is a B-field lift of α. Instead of this, we use the new theory of Huybrechts and Mattei [HM23], and take a *special* Brauer class $\alpha \in \mathrm{SBr}(S)$. In this case, the moduli space $M(v;\alpha)$ does not depend on any more choices, which is partially due to Proposition 2.2.36.

For the general theory, we refer to [HL10] for the untwisted case, to [HS05a] for the twisted case, and to [BM14a] for the generalisation to Bridgeland stability conditions.

Definition 2.3.1. Let S be a K3 surface, $\alpha \in \mathrm{SBr}(S)$. For $\mathcal{E}^\bullet \in \mathcal{D}^b(S,\alpha)$, we call

$$v_\alpha(\mathcal{E}^\bullet) := \mathrm{ch}_\alpha(\mathcal{E}^\bullet)\sqrt{\mathrm{td}_S} \in \mathrm{N}(S,\alpha) := \widetilde{H}^{1,1}(S,\alpha,\mathbb{Z})$$

the Mukai vector of $\mathcal{E}^\bullet$, where $\mathrm{ch}_\alpha(\mathcal{E}^\bullet)$ is the twisted Chern character of Proposition 2.2.36. A vector $v=(r,\ell,s) \in \mathrm{N}(S,\alpha)$ is called a *positive Mukai vector* if v has one of the following properties:

i) $r>0$,

ii) $r=0$ and ℓ is effective,

iii) $r=0$, $\ell=0$, $s>0$.

The following theorem, based on the pioneering work of Mukai [Muk87], is well known and due to many authors [GH96; Huy06; OGr97; BM14b; BM14a].

Theorem 2.3.2. *Let S be a K3 surface and let $\alpha \in \mathrm{SBr}(S)$ be a special Brauer class. Suppose $v \in \widetilde{H}(S,\alpha,\mathbb{Z})$ is a primitive Mukai vector. For H a generic polarisation, there exists a (possibly empty) coarse moduli space $M_H(v;\alpha)$ of H-Gieseker stable coherent α-twisted sheaves $\mathcal{E}$ with $v_\alpha(\mathcal{E})=v$. Moreover:*

i) $M_H(v;\alpha)$ is empty if $v^2<-2$,

ii) Provided v is positive, $M_H(v;\alpha)$ is a projective hyperkähler manifold of $K3^{[n]}$-type, where $2n=v^2+2$.

Example 2.3.3. i) The easiest example of a moduli space of sheaves on S is S itself. More precisely, the Mukai vector $v=(0,0,1)$ satisfies $M_H(v)\simeq S$.

ii) If $\alpha=0$ and $v=(1,0,1-n)$ for some $n>1$, then $M_H(v)$ is the Hilbert scheme $S^{[n]}$.

iii) If $H\subset S$ is a smooth, irreducible curve of genus g whose class in $\mathrm{NS}(S)$ is primitive, then the moduli space corresponding to the Mukai vector $v_d := (0,H,d+1-g)$ and special Brauer class $\alpha\in\mathrm{SBr}(S)$ is a (twisted) *Beauville–Mukai system* denoted $\overline{\mathrm{Pic}}^d_\alpha$. It has a Lagrangian fibration $\overline{\mathrm{Pic}}^d_\alpha \to |H|$ given by sending a sheaf to its Fitting support, and for a smooth curve $C\in|H|$, there is an isomorphism $(\overline{\mathrm{Pic}}^d_\alpha)_C \simeq \mathrm{Pic}^d_\alpha(C)$, where $\mathrm{Pic}^d_\alpha(C)$ is the twisted Picard variety of C. We refer to [Muk84; Bea91; HM23] for more details. We study Beauville–Mukai systems in Chapter 4.

iv) As a special case of iii), consider an elliptic curve $F \subset S$. Then S admits an elliptic fibration $S \to |F|$, and the moduli space $\mathrm{J}^k(S) \coloneqq M(0, F, k)$ is called the *k-th Jacobian of S*. It comes equipped with an elliptic fibration $\mathrm{J}^k(S) \to \mathbb{P}^1$. See [Huy16, §11] for details. We study Jacobians in Chapter 3.

2.3.2 Universal Sheaves

Let S be a K3 surface and fix a primitive Mukai vector $v \in \mathrm{N}(S)$. For a v-generic polarisation H, we write $M = M_H(v)$. Recall that M is a priori only a *coarse* moduli space, meaning that there is not necessarily a universal sheaf on $S \times M$. However, twisted universal sheaves and quasi-universal sheaves always exist, and we recall some basic facts about them now. Our basic reference for twisted sheaves is [Căl00].

Definition 2.3.4. Let $\alpha_M \in \mathrm{Br}(M)$ be a Brauer class. A $(1 \boxtimes \alpha_M)$-*twisted universal sheaf* is a $(1 \boxtimes \alpha_M)$-twisted sheaf $\mathcal{U}$ on $S \times M$ such that $\mathcal{U}|_{S \times [\mathcal{E}]} \simeq \mathcal{E}$ for all $[\mathcal{E}] \in M$. If α_M is trivial, we simply call $\mathcal{U}$ a *universal sheaf* on $S \times M$. If a $(1 \boxtimes \alpha_M)$-twisted universal sheaf exists, we call α_M the *obstruction to the existence of a universal sheaf* or simply the *obstruction class* of M.

We will see in Definition/Proposition 2.3.6 that, unlike universal sheaves, *twisted* universal sheaves always exist. Moreover, the Brauer class α_M is unique. We state this result in the relative setting, as we will need to be able to deal with families of moduli spaces in the next sections.

Proposition 2.3.5. *[HL10] Let $f \colon \mathcal{S} \to T$ be a projective morphism of schemes of finite type over $\mathbb{C}$ with connected fibres. Let $\mathcal{O}_{\mathcal{S}}(1)$ be a relatively ample line bundle on $\mathcal{S}$. Then for any polynomial P there exists a coarse relative moduli space $\mathcal{M} \coloneqq \mathcal{M}_{\mathcal{S}/T}(P) \to T$ for the functor*

$$\mathbf{M} \colon (\mathbf{Sch/T})^\circ \to \mathbf{Sets}$$

which associates to a T-scheme $X \to T$ the set of isomorphism classes of T-flat families of stable sheaves on the fibres of $\mathcal{S} \times_T X \to X$ with Hilbert polynomial P. In particular, for any $t \in T$, we have

$$\mathcal{M}_t \simeq M_{\mathcal{S}_t}(P).$$

Definition/Proposition 2.3.6. *[Căl00, Proposition 3.3.2] Keeping the notation of Proposition 2.3.5, there exists a unique Brauer class $\alpha_{\mathcal{M}} \in \mathrm{Br}(\mathcal{M})$ such that there is a $(1 \boxtimes \alpha_{\mathcal{M}})$-twisted universal sheaf on $\mathcal{S} \times_T \mathcal{M}$. This Brauer class is called the obstruction to the existence of a universal sheaf on $\mathcal{S} \times_T \mathcal{M}$. For any point $t \in T$, this twisted universal sheaf restricts to a $(1 \boxtimes \alpha_t)$-twisted universal sheaf on $\mathcal{S}_t \times \mathcal{M}_t$, hence α_t is the obstruction to the existence of a universal sheaf on $\mathcal{S}_t \times \mathcal{M}_t$.*

The main result of this chapter computes the obstruction Brauer class of a moduli space of sheaves on a K3 surface. The first result in this direction is the following.

Proposition 2.3.7. *[HL10, Theorem 4.6.5] Let S be a K3 surface with primitive Mukai vector $v \in \mathrm{N}(S)$ and v-generic polarisation H. If $\mathrm{div}(v) = 1$, then $M_H(v)$ is a fine moduli space.*

The main result of this chapter implies the converse of Proposition 2.3.7, c.f. Corollary 2.4.16.

Now, we fix S, v, H such that $M \coloneqq M_H(v)$ is non-empty and smooth. Moreover, we assume $v^2 \geq 2$ so that the dimension of M is at least 4. We denote by p and q the natural projections

$$\begin{tikzcd} & S \times M \arrow[dl, "q"'] \arrow[dr, "p"] & \\ S & & M. \end{tikzcd}$$

Let α_M be the obstruction to the existence of a universal sheaf on $S \times M$, and let $\mathcal{E}$ be a $(1 \boxtimes \alpha_M)$-twisted universal sheaf. There exists an α_M^{-1}-twisted locally free sheaf $\mathcal{F}$ of finite rank ρ on M by [Căl00, Theorem 1.3.5]. Now $\mathcal{U} \coloneqq \mathcal{E} \otimes p^*\mathcal{F}$ is a *quasi-universal sheaf of similitude* ρ on $S \times M$. This means that $\mathcal{E} \otimes p^*\mathcal{F}$ is an M-flat untwisted sheaf on $S \times M$ with the property that for any $[F] \in M$, we have $\mathcal{U}|_{S\times[F]} \simeq F^{\oplus\rho}$. We consider the Fourier–Mukai transform

$$\begin{aligned} \Phi^{\mathcal{U}^\vee}\colon \quad \mathcal{D}^b(S) &\longrightarrow \mathcal{D}^b(M) \\ F &\longmapsto Rp_*(q^*F \otimes \mathcal{U}^\vee). \end{aligned}$$

This Fourier–Mukai transform depends on the choices of $\mathcal{E}$ and $\mathcal{F}$, which determine $\mathcal{U}$. However, if $\mathcal{U}$ and $\mathcal{V}$ are two quasi-universal sheaves on $S \times M$, then there exist vector bundles E and F on M such that $\mathcal{U} \otimes p^*E \simeq \mathcal{V} \otimes p^*F$ by [Muk87, Appendix 2].

Definition 2.3.8. Let S be a K3 surface with primitive Mukai vector $v \in \mathrm{N}(S)$ for which $v^2 \geq 0$. Let H be a v-generic polarisation, and write $M \coloneqq M_H(v)$. Let $\mathcal{U}$ be a quasi-universal sheaf on $S \times M$ of similitude ρ. The normalised cohomological Fourier–Mukai transform

$$\begin{aligned} \varphi \coloneqq \tfrac{1}{\rho}\varphi^{v(\mathcal{U}^\vee)}\colon \quad H^*(S,\mathbb{Q}) &\longrightarrow H^*(M,\mathbb{Q}) \\ x &\longmapsto \tfrac{1}{\rho} p_*\left(q^*x \otimes v(\mathcal{U}^\vee)\right), \end{aligned}$$

is called the *Mukai morphism.*

Note that the Mukai morphism is also dependent on the choice of the quasi-universal sheaf $\mathcal{U}$. More precisely, let F be a vector bundle on M and write $\mathcal{U}' = \mathcal{U} \otimes p^*F$. Let φ' be the Mukai morphism corresponding to $\mathcal{U}'$, then

$$\varphi'(x) = \frac{\mathrm{ch}(F)}{\mathrm{rk}(F)}\varphi(x).$$

In particular, if we only consider the degree-2 part, we find

$$[\varphi'(x)]_2 = [\varphi(x)]_2 + \frac{c_1(F)}{\mathrm{rk}(F)}[\varphi(x)]_0. \tag{2.3.1}$$

Lemma 2.3.9. *[Muk87, Proof of Theorem 1.5], [OGr97] For any $x \in H^*(S,\mathbb{Q})$, we have*

$$[\varphi(x)]_0 = -x \cdot v. \tag{2.3.2}$$

In particular, whenever $x \in v^\perp \subset H^*(S,\mathbb{Q})$, $[\varphi(x)]_2$ is independent of the choice of quasi-universal sheaf.

Proof. Let $\omega \in H^{2n}(M, \mathbb{Z})$ be the fundamental cocycle. Let $\Phi^{\mathcal{U}^\vee} : \mathcal{D}^b(S) \to \mathcal{D}^b(M)$ be the Fourier–Mukai transform with kernel $\mathcal{U}^\vee$, and let

$$\varphi^{v(\mathcal{U}^\vee)} : H^*(S, \mathbb{Q}) \to H^*(M, \mathbb{Q})$$

be the corresponding cohomological Fourier–Mukai transform, i.e. $\varphi^{v(\mathcal{U}^\vee)} = \rho \cdot \varphi$. Recall the Mukai pairing on $H^*(M, \mathbb{Q})$ from Definition 2.2.28. We have

$$[\varphi^{v(\mathcal{U}^\vee)}(x)]_0 = (\varphi^{v(\mathcal{U}^\vee)}(x), \omega) = (x, \varphi_L^{v(\mathcal{U}^\vee)}(\omega)),$$

where $\varphi_L^{v(\mathcal{U}^\vee)} = \varphi^{v(\mathcal{U})}$ is the left-adjoint of $\varphi^{v(\mathcal{U}^\vee)}$. On the other hand, we have $\varphi_L^{v(\mathcal{U}^\vee)}(\omega) = v(\Phi^{\mathcal{U}}(\mathcal{O}_t)) = \rho v$ for any $t \in M$. Hence we obtain

$$[\varphi(x)]_0 = \frac{1}{\rho}[\varphi^{v(\mathcal{U}^\vee)}(x)]_0 = \frac{1}{\rho}(x, \rho v) = -x \cdot v,$$

as required. The final claim follows from (2.3.1) combined with (2.3.2). □

Mukai studied the Mukai morphism for an isotropic Mukai vector in his seminal paper [Muk87].

Theorem 2.3.10. *[Muk87, Theorem 1.5] Let S be a K3 surface, and let $v \in \mathrm{N}(S)$ be a primitive Mukai vector with $v^2 = 0$. Let H be a v-generic polarisation, and denote $M \coloneqq M_H(v)$. Then the Mukai morphism from Definition 2.3.8 induces a Hodge isometry*

$$v^\perp / \mathbb{Z}v \simeq H^2(M, \mathbb{Z}).$$

Here, $v^\perp$ denotes the orthogonal complement of v in $\tilde{H}(S, \mathbb{Z})$.

Theorem 2.3.10 was later generalised to the higher-dimensional case by O'Grady and Yoshioka.

Theorem 2.3.11. *[OGr97; Yos01] Let S be a K3 surface, and let $v \in \mathrm{N}(S)$ be a primitive Mukai vector with $v^2 > 0$. Let H be a v-generic polarisation, and denote $M \coloneqq M_H(v)$. Let $\varphi : H^*(S, \mathbb{Q}) \to H^*(M, \mathbb{Q})$ be the Mukai morphism from Definition 2.3.8. Then φ induces a Hodge isometry*

$$\begin{array}{rcl} \tilde{H}(S, \mathbb{Z}) \supset v^\perp & \xrightarrow{\sim} & H^2(M, \mathbb{Z}) \\ x & \longmapsto & [\varphi(x)]_2. \end{array}$$

Here, $[\varphi(x)]_2$ denotes the degree 2 part of $\varphi(x)$. Moreover, this Hodge isometry is independent of the choice of a quasi-universal sheaf on $S \times M$.

We are usually only interested in $M_H(v)$ up to birational equivalence. Since K3 surfaces are minimal, non-isomorphic K3 surfaces are never birational.

Assuming $v^2 > 0$, recall that the birational geometry of $M_H(v)$ is controlled by an $O(\tilde{\Lambda}_{\mathrm{K3}})$-orbit of lattice embedding $H^2(M_H(v), \mathbb{Z}) \hookrightarrow \tilde{\Lambda}_{\mathrm{K3}}$ by the Birational Torelli Theorem 2.2.25. The orbit of embeddings is given by

$$H^2(M_H(v), \mathbb{Z}) \simeq v^\perp \hookrightarrow \tilde{H}(S, \mathbb{Z}) \simeq \tilde{\Lambda}_{\mathrm{K3}},$$

where the first Hodge isometry is the inverse of the Mukai morphism from Theorem 2.3.11, and the second isometry is arbitrary, as the $O(\tilde{\Lambda}_{\mathrm{K3}})$-orbit of the embedding is independent of the choice of isometry $\tilde{H}(S, \mathbb{Z}) \simeq \tilde{\Lambda}_{\mathrm{K3}}$ [Mar11, §9.1].

In particular, the birational equivalence class of $M_H(v)$ is independent of the choice of H, and we sometimes suppress it from the notation for this reason.

We now turn to the most fundamental result about derived equivalence for K3 surfaces: the Derived Torelli Theorem.

Theorem 2.3.12 (The Derived Torelli Theorem)**.** *[Muk87; Orl03] Let X and Y be K3 surfaces. The following are equivalent:*

i) There is an equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$;

ii) There is a Hodge isometry $\tilde{H}(X,\mathbb{Z}) \simeq \tilde{H}(Y,\mathbb{Z})$;

iii) The K3 surface Y is isomorphic to a fine moduli space of sheaves on X. That is, there is a Mukai vector $v \in \mathrm{N}(X)$ and a v-generic polarisation H on X such that $\mathrm{div}(v) = 1$, and $Y \simeq M_H(v)$.

Proof sketch. We already saw in Section 2.2.3 that *i)* $\Longrightarrow$ *ii)*.

ii) $\Longrightarrow$ *iii)*. Assume there is a Hodge isometry $f : \tilde{H}(X,\mathbb{Z}) \simeq \tilde{H}(Y,\mathbb{Z})$. Let $v \in \tilde{H}(X,\mathbb{Z})$ be the vector such that $f(v) = (0,0,1)$. The Hodge isometry f induces a Hodge isometry between the lattices

$$v^{\perp}/\mathbb{Z}v \simeq (0,0,1)^{\perp}/\mathbb{Z}(0,0,1) \simeq H^2(Y,\mathbb{Z}).$$

By Theorem 2.3.10 and the Torelli Theorem, c.f. Theorem 2.2.24, this means $Y \simeq M_H(v)$ for a v-generic polarisation H. Moreover, since f is an isometry, we have $\mathrm{div}(v) = \mathrm{div}(f(v)) = \mathrm{div}(0,0,1) = 1$, as required.

iii) $\Longrightarrow$ *i)*. If Y is a fine moduli space of sheaves on X, then the Fourier–Mukai transform associated to universal family $\mathcal{U}$ on $X \times Y$ is an equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$ by [Orl03, Theorem 3.11]. □

Recall that for any K3 surface X, there is a chain of embeddings of Hodge lattices

$$T(X) \subset H^2(X,\mathbb{Z}) \subset \tilde{H}(X,\mathbb{Z}).$$

There is a wonderful interplay between the Torelli Theorem, the Derived Torelli Theorem, and this chain of embeddings.

Firstly, note that item *ii)* of Theorem 2.3.12 is equivalent to the existence of a Hodge isometry $T(X) \simeq T(Y)$. Indeed, every Hodge isometry $\tilde{H}(X,\mathbb{Z}) \simeq \tilde{H}(Y,\mathbb{Z})$ sends the transcendental lattice of X isometrically to the transcendental lattice of Y, and since there is a primitive embedding $U \subset \mathrm{N}(X)$, every Hodge isometry $T(X) \simeq T(Y)$ extends to a Hodge isometry $\tilde{H}(X,\mathbb{Z}) \simeq \tilde{H}(Y,\mathbb{Z})$ by Lemma 2.2.8.

Secondly, recall that the (non-derived) Torelli Theorem asserts that there is a Hodge isometry $H^2(X,\mathbb{Z}) \simeq H^2(Y,\mathbb{Z})$ if and only if there is an isomorphism $X \simeq Y$. Since any Hodge isometry $H^2(X,\mathbb{Z}) \simeq H^2(Y,\mathbb{Z})$ restricts to a Hodge isometry $T(X) \simeq T(Y)$ and extends to a Hodge isometry $\tilde{H}(X,\mathbb{Z}) \simeq \tilde{H}(Y,\mathbb{Z})$ (again due to Lemma 2.2.8), we obtain the obvious result that isomorphic K3 surfaces are derived equivalent. This should be seen as nothing more than a sanity check.

However, the existence of a Hodge isometry $H^2(X,\mathbb{Z}) \simeq H^2(Y,\mathbb{Z})$ is generally not equivalent to the existence of a Hodge isometry $T(X) \simeq T(Y)$, since $\mathrm{NS}(X) = T(X)^{\perp}$ does not necessarily contain a copy of the hyperbolic plane. In fact, there is an embedding $U \subset \mathrm{NS}(X)$ if and only if X admits an elliptic fibration with a section, as we will see in Chapter 3.

Unfortunately, it is currently not known whether a theorem such as Theorem 2.3.12 holds for hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type, as we discuss in Section 4.3.1. Note, however, that the Derived Torelli Theorem 2.3.12 implies that if X and Y are K3 surfaces with $T(X) \simeq T(Y)$, then Y is isomorphic to a moduli space of sheaves on X. This part, at least, also carries over to the higher-dimensional setting.

If $M = M_H(v)$ is a moduli space of sheaves of dimension $2n > 2$ on a K3 surface S, then we have $T(M) \simeq T(S)$. Indeed, this can be seen from the Hodge isometry $v^\perp \simeq H^2(M, \mathbb{Z})$ induced by the Mukai morphism, see Theorem 2.3.11, since the transcendental sublattice of $v^\perp$ is $T(S)$.

Proposition 2.3.13. *[Mar10] (see also [Add16, Proposition 4]) Let X be a hyperkähler manifold of* $\mathrm{K3}^{[n]}$*-type. Suppose there exists a K3 surface S and a Hodge isometry $T(X) \simeq T(S)$. Then X is birational to a moduli space of sheaves on S.*

Proof. Let $i_X \colon H^2(X, \mathbb{Z}) \hookrightarrow \tilde{\Lambda}_{\mathrm{K3}}$ be an embedding in the natural $O(\tilde{\Lambda}_{\mathrm{K3}})$-orbit (see Theorem 2.2.25). Fix any Hodge isometry $\psi \colon T(X) \simeq T(S)$. By Lemma 2.2.8, there exists an isometry $\tilde{\psi} \colon \tilde{\Lambda}_{\mathrm{K3}} \simeq \tilde{H}(S, \mathbb{Z})$ which makes the following diagram commute:

$$\begin{tikzcd} T(X) \arrow[r, "\psi"] \arrow[d, "i_X"'] & T(S) \arrow[d] \\ \tilde{\Lambda}_{\mathrm{K3}} \arrow[r, "\tilde{\psi}"] & \tilde{H}(S, \mathbb{Z}). \end{tikzcd} \tag{2.3.3}$$

It follows that $\tilde{\psi}$ is a Hodge isometry. Since $\tilde{\psi}(H^2(X, \mathbb{Z})) \subset \tilde{H}(S, \mathbb{Z})$ is a sublattice of rank 23, its orthogonal complement is a primitive sublattice of rank 1. By [Add16, Proof of Proposition 4], we may choose $\tilde{\psi}$ such that the orthogonal complement $\tilde{\psi}(H^2(X, \mathbb{Z}))$ is a positive Mukai vector (see Definition 2.3.1). Then we have a commutative diagram of Hodge metric morphisms:

$$\begin{tikzcd} H^2(X, \mathbb{Z}) \arrow[r, "\sim"] \arrow[d, "i_X"'] & v^\perp \arrow[d] \\ \tilde{\Lambda}_{\mathrm{K3}} \arrow[r, "\sim"] & \tilde{H}(S, \mathbb{Z}). \end{tikzcd} \tag{2.3.4}$$

Therefore, X is birational to $M(v)$ by the Birational Torelli Theorem 2.2.25. $\square$

In Appendix 4.A, we discuss a twisted version of Proposition 2.3.13.

2.3.3 Moduli Spaces of Lattice Polarised K3 Surfaces

In this section, we collect some basic facts about moduli spaces of lattice polarised K3 surfaces. Our main reference is [Dol95]. An important technical result in this section is Lemma 2.3.19, which is used to prove the main result of this chapter, namely Theorem 2.4.15.

Definition 2.3.14. For a lattice T of signature $(2, n)$, we define the period domain Ω_T of T to be one of the two connected components of

$$\left\{ \sigma \in \mathbb{P}(T \otimes \mathbb{C}) \mid \sigma^2 = 0 \text{ and } \sigma \cdot \overline{\sigma} > 0 \right\}. \tag{2.3.5}$$

The orthogonal group $O(T)$ acts naturally on the set (2.3.5), and we write

$$O^+(T) \coloneqq \{\sigma \in O(T) \mid \sigma(\Omega_T) = \Omega_T\}$$

for the subgroup of $O(T)$ consisting of isometries that preserve the connected component Ω_T. The index of $O^+(T) \subset O(T)$ is two.

Let N be an even lattice of signature $(1, \rho - 1)$ for some $1 \leq \rho \leq 20$. Suppose that there exists precisely one $O(\Lambda_{\mathrm{K3}})$-orbit of primitive embeddings $N \hookrightarrow \Lambda_{\mathrm{K3}}$. This is the case for example when $\rho \leq 10$ by [Nik80]. An *N-marked K3 surface* is a pair (S, j) consisting of a K3 surface S and a primitive embedding $j : N \hookrightarrow \mathrm{NS}(S)$. An isomorphism of N-marked K3 surfaces (X, i), (Y, j) is an isomorphism $f : X \simeq Y$ such that $i = f^* \circ j$.

Recall that a *marked K3 surface* is a pair (S, ϕ) consisting of a K3 surface S and an isometry $\phi : H^2(S, \mathbb{Z}) \simeq \Lambda_{\mathrm{K3}}$. Fix an embedding $N \subset \Lambda_{\mathrm{K3}}$. If (S, ϕ) is a marked K3 surface such that $N \subset \phi(\mathrm{NS}(S))$, then $(S, \phi^{-1}|_N)$ is an N-marked K3 surface.

Let $T = N^\perp \subset \Lambda_{\mathrm{K3}}$ be the orthogonal complement. We write

$$\tilde{O}^+(T) \coloneqq \ker\left(O^+(T) \to O(A_T)\right).$$

If (S, j) is an N-marked K3 surface, then there exists a marking $\phi : H^2(S, \mathbb{Z}) \simeq \Lambda_{\mathrm{K3}}$ such that the period of the marked K3 surface (S, ϕ), i.e. $[\phi(H^{2,0}(S, \mathbb{C}))]$, lies in Ω_T. Moreover, two isomorphic N-marked K3 surfaces give rise to periods which lie in the same $\tilde{O}^+(T)$-orbit.

Definition 2.3.15. We denote

$$\mathcal{F}_T \coloneqq \tilde{O}^+(T) \setminus \Omega_T.$$

Theorem 2.3.16. *[Dol95, §3] The quotient $\mathcal{F}_T$ is the coarse moduli space of N-marked K3 surfaces. The moduli space $\mathcal{F}_T$ is an equidimensional quasi-projective variety of dimension* $20 - \rho$.

Let S be a K3 surface, and let $v \in \mathrm{N}(S)$ be a primitive Mukai vector. Write $v = (r, E, s)$ for some $r, s \in \mathbb{Z}$ and $E \in \mathrm{NS}(S)$. Let $H \in \mathrm{NS}(S)$ be an ample divisor. Let M be the saturation of $\langle H, E\rangle$ in $\mathrm{NS}(S)$. That is, $M = (\langle H, E\rangle \otimes \mathbb{Q}) \cap \mathrm{NS}(S)$ is the smallest primitive sublattice of $\mathrm{NS}(S)$ which contains $\langle H, E\rangle$. Fix any marking $\phi : H^2(S, \mathbb{Z}) \simeq \Lambda_{\mathrm{K3}}$. Write $L \coloneqq \phi(M) \subset \Lambda_{\mathrm{K3}}$ and $e = \phi(E)$, $h = \phi(H)$. A point in the moduli space $\mathcal{F}_{L^\perp}$ corresponds to an L-marked K3 surface (X, i) for which the vector $(r, i(e), s) \in \mathrm{N}(X)$ is a Mukai vector, which we denote by $i(v)$. We wish to show that the locus in $\mathcal{F}_{L^\perp}$ of L-marked K3 surfaces (X, i) for which the Mukai vector $i(v)$ has divisibility 1 is dense.

From the definition of the period domain, it follows that we have

$$\Omega_{T'} = \Omega_T \cap \mathbb{P}(T' \otimes \mathbb{C}),$$

for any primitive sublattice $T' \subset T$. By a slight abuse of notation, we write $\mathcal{F}_{T'} \subset \mathcal{F}_T$ for the image of $\Omega_{T'}$ along the natural projection $\Omega_T \twoheadrightarrow \mathcal{F}_T$. If we have $\mathrm{rk}(T') + 1 = \mathrm{rk}(T)$, then $\mathcal{F}_{T'}$ is a divisor in $\mathcal{F}_T$.

Remark 2.3.17. Fix a primitive sublattice $N \subset \Lambda_{\mathrm{K3}}$ of signature $(1, \rho - 1)$ and write $T' = N^\perp$. For a point $[\ell] \in \Omega_{T'}$, we denote by NS_ℓ the integral $(1,1)$-part of the Hodge lattice of K3-type whose underlying lattice is Λ_{K3} and whose $(2,0)$-part is ℓ. It is a well-known fact that, for a very general point $[\ell]$, we have $\mathrm{NS}_\ell = N$. To see this, suppose that $[\ell] \in \Omega_N$ satisfies $N \not\subset \mathrm{NS}_\ell$. Then it follows from the definitions that $[\ell] \in \Omega_{\mathrm{NS}_\ell^\perp} \subset \Omega_{T'}$. Recall that we have $\dim(\Omega_{\mathrm{NS}_\ell^\perp}) = \mathrm{rk}(\mathrm{NS}_\ell^\perp) - 2 < \dim(\Omega_{T'}) = \mathrm{rk}(T') - 2$. Since there are countably many chains of primitive embeddings $N \hookrightarrow L \hookrightarrow \Lambda_{\mathrm{K3}}$, it follows that the points of $\Omega_{T'}$ for which N is a proper sublattice of NS_ℓ form a union of countably many lower-dimensional subvarieties of $\Omega_{T'}$. Thus, the very general point of $\Omega_{T'}$ satisfies $N = \mathrm{NS}_\ell$, as required.

Suppose now that we have another primitive sublattice $L \subset \Lambda_{\mathrm{K3}}$ of signature $(1, \rho-1)$ which is not equal to N as a sublattice of Λ_{K3}, and write $T = L^\perp$. By the above discussion, a very general point of Ω_T has $\mathrm{NS}_\ell = L$. In particular, we have $\Omega_T \neq \Omega_{T'}$. To rephrase this, if N and L are two primitive sublattices of Λ_{K3}, then we have $\Omega_{N^\perp} = \Omega_{L^\perp}$ if and only if $L = N$.

Proposition 2.3.18. *Let $L \subset \Lambda_{\mathrm{K3}}$ be a sublattice of signature $(1, \rho-1)$. Assume that $\rho \leq 9$. Suppose that $\{L_n\}_{n\in\mathbb{N}}$ is a set of pairwise non-isometric lattices of rank $(1,\rho)$, and for each $n \in \mathbb{N}$, we have a chain of primitive embeddings $L \hookrightarrow L_n \hookrightarrow \Lambda_{\mathrm{K3}}$. Then the set*

$$\bigcup_{n\in\mathbb{N}} \mathcal{F}_{L_n^\perp}$$

is dense in $\mathcal{F}_{L^\perp}$.

Proof. We first show that we have $\mathcal{F}_{L_n^\perp} \neq \mathcal{F}_{L_m^\perp}$ whenever $m \neq n$. We prove this by contradiction. Suppose that we have $\mathcal{F}_{L_m^\perp} = \mathcal{F}_{L_n^\perp}$ for some $m, n \in \mathbb{N}$ with $m \neq n$. Then we have

$$\bigcup_{g\in\tilde{O}^+(L^\perp)} g(\Omega_{L_n^\perp}) \cap \Omega_{L_m^\perp} = \Omega_{L_m^\perp}.$$

Note that for any $g \in \tilde{O}^+(L^\perp)$, the subspace $g(\Omega_{L_n^\perp}) \cap \Omega_{L_m^\perp}$ has codimension 1 or 0 in $\Omega_{L_m^\perp}$. Since $\tilde{O}^+(L^\perp)$ is countable, this means that there is a $g \in \tilde{O}^+(L^\perp)$ such that $g(\Omega_{L_n^\perp}) \cap \Omega_{L_m^\perp}$ has codimension 0, i.e. for which we have $g(\Omega_{L_n^\perp}) = \Omega_{L_m^\perp}$. Since $g(\Omega_{L_n^\perp}) = \Omega_{g(L_n^\perp)}$, it follows from Remark 2.3.17 that we have $g(L_n^\perp) = L_m^\perp$. Since we have $g \in \tilde{O}^+(L^\perp)$, we may extend g to an isometry $f : \Lambda_{\mathrm{K3}} \simeq \Lambda_{\mathrm{K3}}$ with $f|_L = \mathrm{id}_L$. Moreover, since $g(L_n^\perp) = L_m^\perp$, we have $f(L_n) = L_m$, i.e. f restricts to an isometry $L_n \simeq L_m$, which is a contradiction with the assumption that L_n and L_m are non-isometric.

To conclude, we use [MP23, Theorem 3.8 (and Remark 3.5)]: The Euclidian closure of the union

$$\bigcup_n \mathcal{F}_{L_n^\perp}$$

is contained in a Shimura subvariety of $\mathcal{F}_{L^\perp}$. Since all subspaces $\mathcal{F}_{L_n^\perp}$ have codimension 1, this Shimura subvariety has to be all of $\mathcal{F}_{L^\perp}$. $\square$

Lemma 2.3.19. *Let (S, ϕ) be a marked K3 surface. Let $v = (r, E, s) \in \mathrm{N}(S)$ be a primitive Mukai vector, and let $H \in \mathrm{NS}(S)$ be an ample divisor. Write $h = \phi(H)$ and $e = \phi(E)$. Let $L \subset \Lambda_{\mathrm{K3}}$ be the saturation of the sublattice $\langle h, e\rangle$. We consider the L-marked K3 surface $(S, \phi^{-1}|_L)$ as an element of the moduli space $\mathcal{F}_{L^\perp}$. Then the set of points $[(X, i)] \in \mathcal{F}_{L^\perp}$ for which*

$$\mathrm{div}_{\mathrm{N}(X)}(i(v)) = 1,$$

where $i(v) := (r, i(e), s)$*, is dense in* $\mathcal{F}_{L^\perp}$.

Proof. Let $e = me_0$ with e_0 primitive (in particular, $\gcd(r, m, s) = 1$ since v is primitive). We claim that there exists a vector $u \in \Lambda_{\mathrm{K3}}$ such that $u \cdot e_0 = 1$ and the lattice $\langle h, e, u\rangle$ has signature $(1, 2)$. Indeed, let $N = U \oplus E_8$. Fix any primitive embedding $L \hookrightarrow N$, which is possible by [Nik80, Corollary 1.12.3]. Since $\mathrm{rk}(L) \leq 10$, the primitive embedding $L \hookrightarrow \Lambda_{\mathrm{K3}}$ is unique up to the action of $O(\Lambda_{\mathrm{K3}})$. Therefore, we may assume that L is a sublattice of $N \subset \Lambda_{\mathrm{K3}}$. Since N is unimodular and e_0 is primitive, there exists a vector $u \in N$ such that $e_0 \cdot u = 1$. Now, for any $d \in E_8 \subset N^\perp \subset \Lambda_{\mathrm{K3}}$, write $u_d = u + d$, and let $L_d = \langle L, u_d\rangle \subset \Lambda_{\mathrm{K3}}$. If d is primitive, then $L_d \subset \Lambda_{\mathrm{K3}}$ is a primitive sublattice. Moreover, by computing the discriminant of L_d, we see that L_d is not isometric to $L_{d'}$ whenever $d^2 \neq (d')^2$. Now, for any $n \in \mathbb{N}$, let $d_n \in E_8$ be a primitive vector with $d_n^2 = 2n$. Then $\{L_n\}_{n\in\mathbb{N}}$ is a set of pairwise non-isometric lattices satisfying the assumptions of Proposition 2.3.18, and therefore $\cup_n \mathcal{F}_{L_n^\perp}$ is dense in $\mathcal{F}_{L^\perp}$. Note that for any $n \in \mathbb{N}$ and any point $[(X, i)] \in \mathcal{F}_{L_n^\perp}$, we have $\mathrm{div}_{\mathrm{N}(X)}(i(v)) = 1$, as required. □

We now provide a proof for the following proposition, previously noted by Mukai [Muk87] and Căldăraru [Căl00].

Proposition 2.3.20. *Let* S *be a K3 surface, let* $v = (r, E, s) \in \mathrm{N}(S)$ *be a primitive Mukai vector and let* H *be a* v*-generic polarisation. Then there exists a proper, smooth morphism* $\mathcal{S} \to T$ *of analytic spaces, such that for all* $i \in \mathbb{Z}$*, and all* $t \in T$*, we have an identification*

$$H^i(\mathcal{S}_t, \mathbb{Z}) \simeq H^i(\mathcal{S}, \mathbb{Z}). \tag{2.3.6}$$

and such that:

i) There exists a point $0 \in T$ *such that* $\mathcal{S}_0 \simeq S$.

ii) The vector $v_t \in \tilde{H}(\mathcal{S}_t, \mathbb{Z})$ *corresponding to* v *along* (2.3.6) *is contained in* $\mathrm{N}(\mathcal{S}_t)$ *for all* $t \in T$.

iii) For all $t \in T$*,* $H_t \in H^2(\mathcal{S}_t, \mathbb{Z})$ *is a* v*-generic polarisation.*

iv) There exists a point $1 \in T$ *such that* $\mathrm{div}_{\mathrm{N}(\mathcal{S}_1)}(v) = 1$.

Moreover, there is a smooth, proper family $\mathcal{M} \to T$ *with the following properties:*

v) For all $t \in T$*, there exists an isomorphism* $\mathcal{M}_t \simeq M_{H_t}(v_t)$.

vi) There is a Brauer class $\alpha \in \mathrm{Br}(\mathcal{M})$ *and a* $(1 \boxtimes \alpha)$*-twisted sheaf* $\mathcal{E}$ *on* $\mathcal{S} \times_T \mathcal{M}$ *which restricts to a twisted universal sheaf on each fibre* $\mathcal{S}_t \times \mathcal{M}_t$.

Proof. Choose any marking $\phi : H^2(S, \mathbb{Z}) \simeq \Lambda_{\mathrm{K3}}$ and let $h = \phi(H)$, $e = \phi(E)$, and let L be the saturation of $\langle h, e\rangle$ in Λ_{K3}. Then $(S, \phi^{-1}|_L)$ is an L-marked K3 surface, hence it represents a point $0 \in \mathcal{F}_{L^\perp}$. Let $0 \in T \subset \mathcal{F}_{L^\perp}$ be a small open neighbourhood, and let $\mathcal{S} \to T$ be a tautological family of K3 surfaces, which exists because the moduli space of L-marked K3 surfaces is a DM stack [Huy16, §5.4]. By shrinking T, for example until it is contractible, we may assume that we have the identifications (2.3.6). Moreover, by shrinking T further, we may assume that H_t is ample, and that H_t is v_t-generic for all $t \in T$ [HL10, Appendix 4.C]. Note that H_t and v_t are automatically algebraic, since we have $T \subset \mathcal{F}_{L^\perp}$. The existence of the relative moduli space $\mathcal{M} \to T$ and the relative twisted universal sheaf follows from [Căl00, Proposition 3.3.2]. Part *iv)* follows from Lemma 2.3.19. □

2.4 Obstruction Classes

Recall from Section 2.3.2 that for a non-fine moduli space M of sheaves on a K3 surface S the obstruction to the existence of a universal sheaf on $S \times M$ is a certain Brauer class $\alpha_M \in \mathrm{Br}(M)$. The main goal of this section is to prove Theorem 2.4.15, which computes the obstruction class α_M explicitly in cohomological terms. We discuss applications of this theorem in Chapter 4.

2.4.1 Căldăraru Classes

In Căldăraru's PhD thesis, he computed the obstruction class for a non-fine two-dimensional moduli space of sheaves on a K3 surface in purely cohomological terms. We will give a summary of this result in the next section. In this section, we discuss *Căldăraru classes*, which play a key role in Căldăraru's computation. Căldăraru classes were given their name in [MS24], on which Chapter 3 is based.

Definition 2.4.1. Let S be a K3 surface, and let $v \in \mathrm{N}(S)$ be a vector. We usually assume v is primitive, and that $v^2 \geq 0$. The *Căldăraru class* of v is the element

$$a_v \coloneqq -\frac{v}{\mathrm{div}(v)} \in A_{\mathrm{NS}(S)}.$$

The element $\omega_v \in A_{T(S)}$ corresponding to a_v via the natural isomorphism $A_{\mathrm{N}(S)} \simeq A_{T(S)}(-1)$ is called the *transcendental Căldăraru class* of v.

Remark 2.4.2. One should not pay too much attention to the sign that appears in the definition of the Căldăraru class. For all our applications of the Căldăraru class we only care about the orbit of the transcendental Căldăraru class under the natural action of the group of Hodge isometries of $T(S)$. Since $-\mathrm{id}_{T(S)} \in O(T(S))$ is a Hodge isometry, the transcendental Căldăraru class ω_v is always in the same orbit as $-\omega_v$, so that the sign in front of $-\frac{v}{\mathrm{div}(v)}$ can safely be ignored. The reason we include the sign will become apparent in the next section, see Remark 2.4.9.

For any lattice L and for any vector $v \in L$, the divisibility $\mathrm{div}(v)$ is the greatest integer m for which

$$\frac{v}{m} \in L^*.$$

Indeed, if $m > \mathrm{div}(v)$, and $u \in L$ is a vector with $u \cdot v < m$, then we have $\frac{v}{m} \cdot u \notin \mathbb{Z}$. Moreover, if v is a primitive vector, then by definition, we have $\frac{v}{m} \notin L$ for all $m > 1$. On the level of A_L, this means that the order of a_v is equal to $\mathrm{div}(v)$. This proves the following lemma.

Lemma 2.4.3. *Let $v \in \mathrm{N}(S)$ be a primitive Mukai vector. Then the order of the Căldăraru class $a_v \in A_{\mathrm{N}(S)}$ is $\mathrm{div}(v)$.*

As a trivial consequence of Lemma 2.4.3, we obtain the following.

Lemma 2.4.4. *For any primitive vector $v \in \mathrm{N}(S)$, the divisibility of v divides the order of $A_{\mathrm{N}(S)}$.*

Căldăraru classes will appear in Section 2.4.4 in our explicit computation of the obstruction class of a moduli space of sheaves on a K3 surface. They also appear in Chapter 3, where we study derived equivalence for elliptic K3 surfaces. Moreover, they make a final appearance in this dissertation in Chapter 4, where we use them to study birational equivalence for moduli spaces of sheaves on elliptic K3 surfaces.

The first time Căldăraru classes were important in the study of moduli spaces of sheaves on K3 surfaces was in Mukai's work [Muk87], namely in the following way. Let S be a K3 surface, and let $v \in \mathrm{N}(S)$ be a primitive Mukai vector with $v^2 = 0$. Then for a v-generic polarisation H, the moduli space $M := M_H(v)$ is a K3 surface. Recall from Theorem 2.3.10 that there is a Hodge isometry $H^2(M, \mathbb{Z}) \simeq v^\perp/\mathbb{Z}v$. Since v is contained in $\mathrm{N}(S) \subset \tilde{H}(S, \mathbb{Z})$, and $T(S) = \mathrm{N}(S)^\perp$, we have $T(S) \subset v^\perp \subset \tilde{H}(S, \mathbb{Z})$. This induces a Hodge metric morphism $T(S) \hookrightarrow v^\perp/\mathbb{Z}v \simeq H^2(M, \mathbb{Z})$. The image of $T(S)$ in $H^2(M, \mathbb{Q})$ is contained in $T(M)$. This inclusion exhibits $T(M)$ as an overlattice $T(S)$. Recall from Lemma 2.2.13 that overlattices of $T(S)$ correspond to isotropic subgroups of $A_{T(S)}$. The isotropic subgroup corresponding to the overlattice $T(S) \hookrightarrow T(M)$ is precisely the subgroup $\langle \omega_v \rangle \subset A_{T(S)}$, see [Muk87, Proposition 6.4].

While the Căldăraru class is an easy-to-understand element, its transcendental counterpart may seem a little more mysterious. Fortunately, there is an easy way to characterise the transcendental Căldăraru class in a way that makes it look more like the Căldăraru class itself.

Lemma 2.4.5. *Let $v \in \mathrm{N}(S)$ be a primitive Mukai vector of divisibility* $\mathrm{div}(v) = t$*. Then there exists a vector $x \in T(S)$ with the property that*

$$\frac{v + x}{t} \in \tilde{H}(S, \mathbb{Z}).$$

Then

$$\omega_v = \frac{x}{t} \in A_{T(S)}.$$

Proof. Since $\mathrm{N}(S)$ is a primitive sublattice of $\tilde{H}(S, \mathbb{Z})$, the natural map

$$\tilde{H}(S, \mathbb{Z}) \simeq \tilde{H}(S, \mathbb{Z})^* \to \mathrm{N}(S)^*$$

is surjective. Let $\zeta \in \tilde{H}(S, \mathbb{Z})$ be a preimage of $\frac{v}{t} \in \mathrm{N}(S)^*$. This means that for all $w \in \mathrm{N}(S)$, we have

$$\zeta \cdot w = \frac{v}{t} \cdot w.$$

This means that $x := t\zeta - v$ is an integral vector of $\tilde{H}(S, \mathbb{Z})$ contained in the orthogonal complement of $\mathrm{N}(S)$, which is $T(S)$. We have $v + x = t\zeta$, hence

$$\frac{v + x}{t} \in \tilde{H}(S, \mathbb{Z}).$$

Along the natural isometry $A_{\mathrm{N}(S)} \simeq A_{T(S)}(-1)$ of Lemma 2.2.1, the class $a_v = -\frac{v}{t} \in A_{\mathrm{N}(S)}$ corresponds to $\frac{x}{t} \in A_{T(S)}$, hence $\omega_v = \frac{x}{t}$. □

Since $\mathrm{N}(S) \simeq \mathrm{NS}(S) \oplus U$, there is a natural isomorphism $A_{\mathrm{N}(S)} \simeq A_{\mathrm{NS}(S)} \oplus A_U$, hence by unimodularity of U, we have

$$A_{\mathrm{N}(S)} \simeq A_{\mathrm{NS}(S)}. \tag{2.4.1}$$

The element of $A_{\mathrm{NS}(S)}$ corresponding to a_v via (2.4.1) is also called the Căldăraru class of v, with a slight abuse of language.

Lemma 2.4.6. *Let S be a K3 surface and let $v = (r, D, s) \in \mathrm{N}(S)$ be a primitive Mukai vector of divisibility $t = \mathrm{div}(v)$. The Căldăraru class of v in $A_{\mathrm{NS}(S)}$ is*

$$-\frac{1}{t}D.$$

Proof. Note that $t = \mathrm{div}(v) = \gcd(r, s, \mathrm{div}(D))$ divides both r and s. Therefore, we have

$$a_v = -\frac{(r, D, s)}{t} = -\frac{1}{t}(r, 0, s) - \frac{1}{t}D = -\frac{1}{t}D,$$

since $\frac{1}{t}(r, 0, s)$ is integral. □

2.4.2 The Obstruction Class of a Two-Dimensional Moduli Space

In this subsection, let S be a K3 surface with Mukai vector $v \in \mathrm{N}(S)$. Moreover, we assume that $v^2 = 0$. In this case, the moduli space $M := M_H(v)$ is a K3 surface, where H is a v-generic polarisation. In this setting, Căldăraru computed the obstruction class α_M explicitly as a class in $\mathrm{Hom}(T(M), \mathbb{Q}/\mathbb{Z}) \simeq \mathrm{Br}(M)$. We now give a brief summary of his results.

Remark 2.4.7. Note that, for any positive integer $t > 0$, the t-torsion subgroup of $\mathrm{Hom}(T(S), \mathbb{Q}/\mathbb{Z})$ is isomorphic to $\mathrm{Hom}(T(S), \mathbb{Z}/t\mathbb{Z})$, via the natural embedding $\mathbb{Z}/t\mathbb{Z} \hookrightarrow \mathbb{Q}/\mathbb{Z}$ given by

$$\bar{1} \mapsto \frac{1}{t} \pmod{\mathbb{Z}}.$$

With this in mind, we may consider a group homomorphism $T(S) \to \mathbb{Z}/t\mathbb{Z}$ as a Brauer class on S.

Recall that the Mukai morphism is a Hodge isometry $v^\perp/\mathbb{Z}v \simeq H^2(M, \mathbb{Z})$. Since we have $T(S) \subset v^\perp \subset \tilde{H}(S, \mathbb{Z})$, and $v \notin T(S)$, the Mukai morphism induces an embedding of Hodge lattices $T(S) \hookrightarrow T(M)$. Moreover, the quotient $T(M)/T(S)$ is finite and cyclic. Recall that there is a natural embedding $T(M)/T(S) \subset A_{T(S)}$. Via this embedding, $T(M)/T(S)$ is generated by the transcendental Căldăraru class of v, denoted ω_v by [Muk87, Proposition 6.4] (see Definition 2.4.1). This defines a Brauer class α_M on M as the composition

$$\alpha_M : T(M) \to T(M)/T(S) \simeq \mathbb{Z}/\operatorname{div}(v)\mathbb{Z}, \tag{2.4.2}$$

where the isomorphism $T(M)/T(S) \simeq \mathbb{Z}/\operatorname{div}(v)\mathbb{Z}$ is the one that maps ω_v to $\bar{1} \in \mathbb{Z}/\operatorname{div}(v)\mathbb{Z}$, and $\mathrm{div}(v)$ denotes the divisibility of v in $\mathrm{N}(S)$.

Theorem 2.4.8. *[Căl00, Theorem 5.4.3] The Brauer class α_M of equation (2.4.2) is the obstruction to the existence of a universal sheaf on $S \times M$.*

Remark 2.4.9. Theorem 2.4.8 is the reason for the choice of a sign in the definition of the Căldăraru class. The sign is there to ensure that α_M from (2.4.2) is the obstruction class, as opposed to $-\alpha_M$.

The description of the obstruction Brauer class α_M of Theorem 2.4.8 is useful because it is very explicit. For example, it is essential in Chapter 3 to study Fourier–Mukai partners of elliptic K3 surfaces. Unfortunately, the description uses the fact that $\mathrm{Br}(M) \simeq \mathrm{Hom}(T(M), \mathbb{Q}/\mathbb{Z})$, which is only the case when M is a K3 surface, see (2.2.11). Another, equivalent, way to view α_M is as follows.

Consider the short exact sequence

$$0 \to T(S) \to T(M) \to \langle \omega_v \rangle \to 0, \tag{2.4.3}$$

which induces the following short exact sequence by taking duals:

$$0 \to T(M)^* \to T(S)^* \to \mathbb{Z}/\operatorname{div}(v)\mathbb{Z} \to 0. \tag{2.4.4}$$

Since $v \in \tilde{H}(S,\mathbb{Z})$ is primitive, and $\tilde{H}(S,\mathbb{Z})$ is unimodular, it follows that the divisibility of v in $\tilde{H}(S,\mathbb{Z})$ is 1. That is, there exists a vector $u \in \tilde{H}(S,\mathbb{Z})$ with the property that $u \cdot v = 1$. We claim that

$$(u \cdot -)|_{T(S)} \in T(S)^*,$$

which is the image of u under the natural surjection $\tilde{H}(S,\mathbb{Z}) \to T(S)^*$, induces a generator of $T(S)^*/T(M)^*$. This is proven in Lemma 2.4.11 below in a more general setting. We abuse notation slightly and write u for $(u \cdot -)|_{T(S)}$. Let us write $w \in T(M)^*$ for the element which maps to $\operatorname{div}(v) \cdot u \in T(S)^*$.

Note that taking the tensor product of (2.4.4) with $\mathbb{Q}/\mathbb{Z}$ yields the exact sequence:

$$\mathrm{Tor}_1^{\mathbb{Z}}(T(S)^*, \mathbb{Q}/\mathbb{Z}) \to \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/\operatorname{div}(v)\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \to T(M)^* \otimes \mathbb{Q}/\mathbb{Z} \to T(S)^* \otimes \mathbb{Q}/\mathbb{Z} \to 0.$$

Since $T(S)^*$ is torsion-free, this is exactly the sequence:

$$0 \to \mathbb{Z}/\operatorname{div}(v)\mathbb{Z} \to \mathrm{Br}(M) \to \mathrm{Br}(S) \to 0. \tag{2.4.5}$$

The kernel of (2.4.5) is generated by

$$\frac{w}{\operatorname{div}(v)} \in \mathrm{Br}(M),$$

and this is precisely the obstruction class α_M of Theorem 2.4.8, see [Căl00, Theorem 5.3.1].

Note that the order of α_M in $\mathrm{Br}(M)$ is equal to the divisibility $\operatorname{div}(v)$ of v in $\mathrm{N}(S)$. We can see this from short exact sequence (2.4.5), since the kernel is generated by the obstruction class. We can also use (2.4.2) to find the order of α_M, since (2.4.2) exhibits α_M as a surjective group homomorphism $\alpha_M \colon T(M) \to \mathbb{Z}/\operatorname{div}(v)\mathbb{Z}$. This proves the following result, which also follows from the Derived Torelli Theorem 2.3.12.

Corollary 2.4.10. *[Căl00] Let S be a K3 surface with primitive Mukai vector $v \in \mathrm{N}(S)$ and v-generic polarisation H. The moduli space $M_H(v)$ is fine if and only if* $\operatorname{div}(v) = 1$.

In the remainder of this chapter, we show how to generalise Theorem 2.4.8 to higher-dimensional moduli spaces of sheaves. One of the main points of interest is that we compute the order of the obstruction class precisely.

2.4.3 Brauer Groups of Moduli Spaces

Let S be a K3 surface with primitive Mukai vector $v \in \mathrm{N}(S)$. Assume $v^2 > 0$. Let $H \in \mathrm{NS}(S)$ be a v-generic polarisation, and write $M := M_H(v)$. Then M is a hyperkähler manifold of $\mathrm{K3}^{[n]}$-type, where $2n = v^2 + 2$ by Theorem 2.3.2. The main goal of this subsection is to show that there is an exact sequence resembling (2.4.5) in this setting. Recall from Theorem 2.3.11 that the Mukai morphism induces a Hodge isometry $v^\perp \simeq H^2(M,\mathbb{Z})$. Therefore, we have a Hodge isometry $T(S) \simeq T(M)$, hence there is no short exact sequence of the form (2.4.3), which was necessary to derive (2.4.5). Instead, the correct first step is to recreate short exact sequence (2.4.4) using the group

$$T'(M) := \frac{H^2(M,\mathbb{Z})}{\mathrm{NS}(M)}$$

from (2.2.12).

Lemma 2.4.11. *Let S, v, H, and M be as above. Then there is a short exact sequence*

$$0 \to T'(M) \to T(S)^* \to \mathbb{Z}/\operatorname{div}(v)\mathbb{Z} \to 0.$$

Let $u \in \tilde{H}(S,\mathbb{Z})$ such that $u \cdot v \equiv 1 \pmod{\operatorname{div}(v)}$, then the image of u under the projection $\tilde{H}(S,\mathbb{Z}) \to T(S)^* \to T(S)^*/T'(M)$ is a generator for the cokernel.

Proof. The Mukai morphism is a Hodge isometry $H^2(M,\mathbb{Z}) \simeq v^\perp \subset \tilde{H}(S,\mathbb{Z})$ by Theorem 2.3.11. This induces an embedding of Hodge structures

$$T'(M) \simeq \frac{v^\perp}{(v^\perp)^{1,1}} \hookrightarrow \frac{\tilde{H}(S,\mathbb{Z})}{\mathrm{N}(S)} \simeq T(S)^*.$$

It follows from a straightforward computation that the cokernel of this embedding is generated by u: for any class $x \in \tilde{H}(S,\mathbb{Z})$ such that $x \cdot v = \lambda$, we have $(x - \lambda u)\cdot v \equiv 0 \pmod{\operatorname{div}(v)}$. Therefore, there exists an algebraic class $y \in \mathrm{N}(S)$ such that $y \cdot v = (x - \lambda u) \cdot v$. Therefore, we have $x - \lambda u - y \in v^\perp$, hence x and λu induce the same element in $T(S)^*/T'(M)$. Since $\operatorname{div}(v)u - z \in v^\perp$ for any $z \in \mathrm{N}(S)$ with $v \cdot z = \operatorname{div}(v)$, we obtain that u has order $\operatorname{div}(v)$ in $T(S)^*/T'(M)$. □

Let $w \in T'(M)$ be the vector which maps to $\operatorname{div}(v)\cdot u \in T(S)^*$. Using the description of $\mathrm{Br}(M)$ as $\mathrm{Br}(M) \simeq T'(M) \otimes \mathbb{Q}/\mathbb{Z}$ from (2.2.11), Lemma 2.4.11 immediately implies the following.

Proposition 2.4.12. *There is a short exact sequence*

$$0 \to \left\langle \frac{w}{\operatorname{div}(v)} \right\rangle \to \mathrm{Br}(M) \to \mathrm{Br}(S) \to 0.$$

Proposition 2.4.12 is derived from Lemma 2.4.11 in the same way that one derives (2.4.5) from (2.4.4).

We will see in the next section that $\frac{w}{\operatorname{div}(v)}$ is precisely the Brauer class of M that obstructs the existence of a universal sheaf on $S \times M$, see Theorem 2.4.15.

The following lemma gives us another way to view the obstruction class in terms of the Mukai morphism.

Lemma 2.4.13. *[Căl00, Proof of Theorem 5.3.1] Let $\varphi : \tilde{H}(S,\mathbb{Q}) \to H^*(M,\mathbb{Q})$ be the Mukai morphism from Definition 2.3.8. Consider the vector $[\varphi(u)]_2 \in H^2(M,\mathbb{Q})$, where $u \in \tilde{H}(S,\mathbb{Z})$ is as in Lemma 2.4.11. Then the Brauer class on M induced by $[\varphi(u)]_2$ along the composition*

$$H^2(M,\mathbb{Q}) \to T'(M)\otimes\mathbb{Q} \to T'(M)\otimes\mathbb{Q}/\mathbb{Z} \simeq \mathrm{Br}(M)$$

is precisely $\frac{w}{\mathrm{div}(v)}$. Moreover, this is independent of the choice of a quasi-universal sheaf on $S\times M$ and the choice of u.

Proof. The fact that $[\varphi(u)]_2$ induces the Brauer class $\frac{w}{\mathrm{div}(v)}$ is a straightforward chase through the identifications, see for example [Căl00, Theorem 5.3.1]. We check that the Brauer class $[\varphi(u)]_2$ is independent of our choice of quasi-universal sheaf $\mathcal{U}$. Recall that if $\mathcal{U}'$ is any other quasi-universal sheaf on $S\times M$, then there exist vector bundles $\mathcal{E}$ and $\mathcal{F}$ on M such that $\mathcal{U}\otimes p^*\mathcal{E} \simeq \mathcal{U}'\otimes p^*\mathcal{F}$ by [Muk87, Appendix 2]. For $\mathcal{F}\in\mathbf{Coh}(M)$ we have

$$[\varphi^{\mathcal{U}^\vee\otimes p^*\mathcal{F}}(u)]_2 = [\varphi^{\mathcal{U}^\vee}(u)]_2 + [\varphi^{\mathcal{U}^\vee}(u)]_0 \cdot \frac{c_1(\mathcal{F})}{\mathrm{rk}\,\mathcal{F}}.$$

Since

$$\frac{c_1(\mathcal{F})}{\mathrm{rk}\,\mathcal{F}} \in \mathrm{NS}(M)\otimes\mathbb{Q},$$

this shows that the class of $[\varphi(u)]_2$ in $\mathrm{Br}(M)\simeq T'(M)\otimes\mathbb{Q}/\mathbb{Z}$ does not depend on the choice of $\mathcal{U}$.

Now we check that $[\varphi(u)]_2$ is independent of the choice of u. Suppose that $u'\in\tilde{H}(S,\mathbb{Z})$ is another element such that $u'\cdot v\equiv 1 \pmod{\mathrm{div}(v)}$. Then $(u-u')\cdot v \equiv 0 \pmod{\mathrm{div}(v)}$. Therefore, there is an algebraic class $x\in\tilde{H}(S,\mathbb{Z})$ such that $x\cdot v = (u-u')\cdot v$. This means that $u-u'-x\in v^\perp$. This implies that $[\varphi(u-u'-x)]_2 \in H^2(M,\mathbb{Z})$ is integral, so in particular it vanishes in $\mathrm{Br}(M)$. Since φ is a morphism of Hodge structures, $\varphi(x)$ is algebraic, and we obtain $[\varphi(u)]_2 - [\varphi(u')]_2 = -[\varphi(x)]_2 = 0 \in \mathrm{Br}(M)$. □

2.4.4 The Obstruction Class of a Higher-Dimensional Moduli Space

We now prove Theorem 2.4.15, which computes the obstruction Brauer class explicitly when M is a higher-dimensional moduli space of sheaves. Our strategy is very similar to the proof of [Căl02, Theorem 1.1], but some modifications need to be made to the proof in our setting.

The main idea behind the proof is to deform M to a fine moduli space and then use the following result by Căldăraru.

Theorem 2.4.14. [Căl02, Theorem 4.1] Let $f : X \to T$ be a proper, smooth morphism of analytic spaces. Let $\mathcal{E}$ be a locally free α-twisted sheaf on X. Assume that $\alpha|_{X_0}$ is trivial, so that $\mathcal{E}_0 := \mathcal{E}|_{X_0}$ can be glued to an untwisted sheaf on X_0. Assume that for all $t\in T$, we have $H^i(X,\mathbb{Z})\simeq H^i(X_t,\mathbb{Z})$. Then

$$\alpha = -\frac{c_1(\mathcal{E}_0)}{\mathrm{rk}(\mathcal{E}_0)} \in \frac{H^2(X,\mathbb{Z})}{\mathrm{NS}(X)}\otimes\mathbb{Q}/\mathbb{Z}.$$

Theorem 2.4.15. *Keeping the notation as in Lemma 2.4.11 and Proposition 2.4.12, the Brauer class*

$$\frac{w}{\operatorname{div}(v)} \in \operatorname{Br}(M)$$

is the obstruction to the existence of a universal sheaf on $S \times M$ from Definition/Proposition 2.3.6.

Proof. We write $v = (r, E, s) \in \mathrm{N}(S)$. Let $\mathcal{S} \to T$ and $\mathcal{M} \to T$ be the families of K3 surfaces and moduli spaces of Proposition 2.3.20, and let $\alpha \in \operatorname{Br}(\mathcal{M})$ be the Brauer class, and $\mathcal{E}$ the $(1 \boxtimes \alpha)$-twisted universal sheaf on $\mathcal{S} \times_T \mathcal{M}$ from the same proposition. We identify $\mathcal{S}_0 \simeq S$ and $\mathcal{M}_0 \simeq M$.

Pick a locally free α^{-1}-twisted sheaf $\mathcal{F}$ on $\mathcal{M}$ of rank ρ. Fix $\mathcal{U} := \mathcal{E} \otimes p^*\mathcal{F}$, which is a relative quasi-universal sheaf on $\mathcal{S} \times_T \mathcal{M}$ of similitude ρ. For $t \in T$, we consider the induced Fourier–Mukai transforms

$$\Phi_t := \Phi^{\mathcal{U}_t^\vee} : \mathcal{D}^b(S_t) \to \mathcal{D}^b(M_t)$$

and morphisms of Hodge structures:

$$\begin{aligned} \varphi_t := \varphi^{(1/\rho)v(\mathcal{U}_t^\vee)} : H^*(S_t, \mathbb{Q}) &\to H^*(M_t, \mathbb{Q}) \\ x &\mapsto \tfrac{1}{\rho} \cdot p_* \left(v(\mathcal{U}_t^\vee) \cdot q^*(x)\right). \end{aligned}$$

satisfying $v\left(\Phi_t(-)\right) = \rho\varphi_t\left(v(-)\right)$. By shrinking T, we may assume that φ_t is a constant map on t.

Let $u \in \widetilde{H}(S, \mathbb{Z})$ be a vector such that $u \cdot v = 1 \pmod{\operatorname{div}(v)}$. Note that the class $[\varphi_0(u)]_2 \in H^2(M, \mathbb{Q})$ induces a Brauer class on M via the natural surjection

$$H^2(M, \mathbb{Q}) \to T'(M) \otimes \mathbb{Q} \to T'(M) \otimes \mathbb{Q}/\mathbb{Z} \simeq \operatorname{Br}(M).$$

By a slight abuse of notation, we also denote this Brauer class by $[\varphi_0(u)]_2$. By Lemma 2.4.13, we wish to prove that

$$\alpha|_M = [\varphi_0(u)]_2 \in \operatorname{Br}(M).$$

On the fibre over $1 \in T$, the Brauer class α_1 vanishes by Proposition 2.3.7, so that $\mathcal{E}_1$ is a universal sheaf. In particular, $\Phi' := \Phi^{\mathcal{E}_1^\vee}$ and $\varphi' := \varphi^{v(\mathcal{E}_1^\vee)}$ are well-defined. For any $\mathcal{G} \in \mathcal{D}^b(S)$ and $x \in \widetilde{H}(S, \mathbb{Q})$, we have $\Phi_1(\mathcal{G}) = \Phi'(\mathcal{G}) \otimes \mathcal{F}_1^\vee$ and

$$\varphi_1(x) = \varphi'(x)\frac{\operatorname{ch}(\mathcal{F}_1^\vee)}{\rho},$$

due to the projection formula. In particular, we have

$$[\varphi_1(u)]_2 = [\varphi'(u)]_2 + [\varphi'(u)]_0 \cdot \frac{-c_1(\mathcal{F}_1)}{\rho}.$$

Since this equation is purely topological, it also holds on the fibre at 0. We know by Theorem 2.4.14 that

$$\alpha|_M = \frac{c_1(\mathcal{F}_1)}{\rho}$$

(recall that $\mathcal{F}$ is α^{-1}-twisted).

Firstly, note that we have $[\varphi'(u)]_0 = -1$ as a consequence of Lemma 2.3.9, since we are assuming that $u \cdot v = 1$.

Secondly, we show that $[\varphi'(u)]_2$ is integral. For this, we first show that we can find a line bundle $\mathcal{L}$ on S_1 and choose $u = v(\mathcal{L})$.

Indeed, since the divisibility of v in $\mathrm{N}(S_1)$ is 1, we can find a divisor $D \in \mathrm{NS}(S_1)$ such that $\gcd(r, D \cdot E, s) = 1$. Let $\mathcal{L} = \mathcal{O}_{S_1}(D)$. Then $v(\mathcal{L}) = (1, D, \frac{D^2}{2} - 1)$, and we have $v(\mathcal{L}) \cdot v = -r \cdot (\frac{1}{2}D^2 - 1) - s + D \cdot E$. Note that r and s are divisible by $\mathrm{div}_{\mathrm{N}(S)}(v)$, hence by replacing D by mD for some $m \in \mathbb{Z}$ we obtain $v(\mathcal{L}) \cdot v \equiv 1 \pmod{\mathrm{div}_{\mathrm{N}(S)}(v)}$.

We now show that, by replacing $\mathcal{L}$ with $\mathcal{L}(k) \coloneqq \mathcal{L} \otimes \mathcal{O}_{S_1}(k)$ for $k \gg 0$, we may assume that $\mathcal{G} \coloneqq \Phi'(\mathcal{L})$ is locally free. It suffices to show that $R^j p_*(q^*\mathcal{L}(k) \otimes \mathcal{E}_1^\vee) = 0$ for all $j > 0$ and $k \gg 0$. For any point $[\mathcal{K}] \in M_1$, we have natural maps

$$R^j p_*(q^*\mathcal{L}(k) \otimes \mathcal{E}_1^\vee)_{[\mathcal{K}]} \to H^j(S_1, \mathcal{K} \otimes \mathcal{L}(k)).$$

By boundedness of M_1, all these cohomology groups vanish for all $[\mathcal{K}] \in M_1$, $j > 0$ and $k \gg 0$, which in turn implies that the $R^j p_*(q^*\mathcal{L}(k) \otimes \mathcal{E}_1^\vee)_{[\mathcal{K}]}$ vanish as well.

Note that this completes the proof, since we have

$$[\varphi'(u)]_2 = [v(\mathcal{G})]_2 = [c_1(\mathcal{G})] \in H^2(M_1, \mathbb{Z}).$$

□

Corollary 2.4.16. *Let S be a K3 surface and let $v \in \mathrm{N}(S)$ be a Mukai vector with $v^2 > 0$. Let H be a v-generic polarisation of S. Then the moduli space $M_H(v)$ is fine if and only if* $\mathrm{div}(v) = 1$.

For isotropic Mukai vectors, the conclusion of Corollary 2.4.16 follows immediately from the work of Căldăraru [Căl00], as well as from the Derived Torelli Theorem 2.3.12. In the higher-dimensional case, the fact that $M_H(v)$ is fine whenever $\mathrm{div}(v) = 1$ was already known, c.f. Proposition 2.3.7. The new part of Corollary 2.4.16 is that the divisibility of the Mukai vector is 1 for higher-dimensional fine moduli spaces.

Chapter 3

Elliptic K3 Surfaces

3.1 Introduction

Study of derived equivalence for complex K3 surfaces goes back to the work of Mukai. By the Derived Torelli Theorem [Muk87; Orl03], derived equivalence translates to a Hodge-theoretic concept. Building on the Derived Torelli theorem, and Nikulin's work on lattices [Nik80], one can deduce a formula for the number of Fourier–Mukai partners for a complex K3 surface [Ogu02; Hos+02].

Derived equivalences of elliptic K3 surfaces have been studied in [Ste04; Gee05]. One way to produce Fourier–Mukai partners of an elliptic surface $f : X \to \mathbb{P}^1$, is to take Jacobians $\mathrm{J}^k(X)$, which are moduli spaces parametrising stable torsion sheaves supported on a fibre of f and having degree $k \in \mathbb{Z}$. If k is coprime to the multisection index of f, then $\mathrm{J}^k(X)$ is derived equivalent to X and we refer to $\mathrm{J}^k(X)$ as a *coprime Jacobian* of X. This raises the question of whether the converse is also true:

Question 3.1.1. Is every Fourier–Mukai partner of an elliptic surface X a coprime Jacobian of X?

Question 3.1.1 was asked in 2014 by Hassett and Tschinkel in the case X is a K3 surface [HT17, Question 20]. In fact, since elliptic K3 surfaces can have several non-isomorphic elliptic fibrations, one can interpret this question differently depending on whether we fix a fibration on X in advance or not.

For elliptic surfaces of non-zero Kodaira dimension, as well as for bielliptic and Enriques surfaces, [BM01; BM19], Question 3.1.1 has an affirmative answer. We do not know the answer in the abelian case.

One of our main results is the following answer to Question 3.1.1 for K3 surfaces:

Theorem 3.1.2 (See Corollaries 3.5.14 and 3.5.15)**.** *Let X be an elliptic K3 surface of Picard rank 2. Let t be the multisection index of X and let $2d$ be the degree of a polarisation on X. Denote $m = \gcd(d, t)$.*

(i) If $m = 1$, then every Fourier–Mukai partner of X is isomorphic to a coprime Jacobian of a fixed elliptic fibration on X;

(ii) If $m = p^k$, for a prime p, then every Fourier–Mukai partner of X is isomorphic to a coprime Jacobian of one of the two elliptic fibrations on X;

(iii) *If m is not a power of a prime, and X is very general with these properties, then X admits Fourier–Mukai partners which are not isomorphic to any Jacobian of any elliptic fibration on X.*

Our method of proof of Theorem 3.1.2 relies on the Ogg–Shafarevich theory for elliptic surfaces, the Derived Torelli Theorem and lattice theory. In addition we introduce a new ingredient: a *derived elliptic structure*. The notion of the derived elliptic structure goes into the direction of describing an elliptic structure on X and its Fourier–Mukai partner in terms of the derived category $\mathcal{D}^b(X)$. We define a derived elliptic structure on a K3 surface X as a choice of an elliptic fibration on a Fourier–Mukai partner of X. Using this language, Question 3.1.1 translates to the question whether every derived elliptic structure on X is isomorphic to a coprime Jacobian of an actual elliptic structure on X.

We proceed to completely classify derived elliptic structures, for an elliptic K3 surface X of Picard rank two, in terms of certain *Lagrangian subgroups* of the discriminant lattice $A_{\mathrm{NS}(X)}$ of the Neron-Severi lattice of X. The final answer, at least when X is very general, is that the number of derived elliptic structures on X, up to coprime Jacobians, equals $2^{\omega(m)}$ where m is as in Theorem 3.1.2 and $\omega(m)$ is the number of distinct prime factors of m, that is $\omega(1) = 0$, $\omega(p^k) = 1$ and $\omega(m) > 1$ otherwise. This explains the condition on m appearing in Theorem 3.1.2.

Let us explain some difficulties that we encounter along the way. First of all, elliptic K3 surfaces of Picard rank two can have one or two elliptic fibrations, and in the latter case these elliptic fibrations are sometimes isomorphic. Thus, a direct comparison between the number of coprime Jacobians and Fourier–Mukai partners is complicated.

Secondly, many results that we state for arbitrary elliptic K3 surfaces X of Picard rank two simplify considerably when X is very general. Indeed in this case, the group G_X of Hodge isometries of the transcendental lattice $T(X)$ is trivial, that is $G_X = \{\pm \mathrm{id}\}$. In general this is a finite cyclic group of even order $|G_X| \leq 66$. This group appears in various bijections, similarly to how it appears in the counting formula of Fourier–Mukai partners [Hos+02]. The set of isomorphism classes of derived elliptic structures on X is in natural bijection with the set

$$\widetilde{\mathrm{L}}(A_{T(X)})/G_X,$$

see Theorem 3.5.11. Here $A_{T(X)}$ is the discriminant lattice of the transcendental lattice $T(X)$, and $\widetilde{\mathrm{L}}(A_{T(X)})$ denotes the set of Lagrangian elements (Definition 3.3.14). Taking a coprime Jacobian J^k of an elliptic structure translates into multiplying the corresponding Lagrangian element by k and changing elliptic fibrations on a given surface corresponds to an involution which can be described intrinsically in terms of $A_{T(X)}$. For very general X, $G_X = \{\pm \mathrm{id}\}$, and this group acts by multiplying Lagrangian elements by $-\mathrm{id}$. On the other hand, special X will have fewer Fourier–Mukai partners and fewer coprime Jacobians, however they will still match perfectly in cases (i) and (ii) of Theorem 3.1.2. See Example 3.4.24 for the most special (in terms of the size of G_X and $\mathrm{Aut}(X)$) elliptic K3 surface.

Similarly, when considering very general elliptic K3 surfaces, every isomorphism preserving the fibre class is necessarily an isomorphism over the base. This is false in general, and this is important, because the Ogg–Shafarevich theory works with elliptic surfaces over the base, whereas the natural equivalence relation is that of preserving the elliptic pencil. We provide a careful analysis of the difference between isomorphism over $\mathbb{P}^1$ and isomorphism as elliptic surfaces, which can be of independent interest. In particular, we are able to state which of the coprime Jacobians $\mathrm{J}^k(X)$ of an elliptic K3 surface X are isomorphic as elliptic surfaces (resp. over $\mathbb{P}^1$). Indeed, very general elliptic K3 surfaces with multisection index t have at most $\frac{\phi(t)}{2}$ coprime Jacobians, and the explicit number can be computed in all cases as follows:

Proposition 3.1.3. *(see Proposition 3.4.18) Let X be a complex elliptic K3 surface. There exist explicitly defined cyclic subgroups $B_X \subset \widetilde{B}_X$ of $(\mathbb{Z}/t\mathbb{Z})^*$, such that the number of isomorphism classes of coprime Jacobians $\mathrm{J}^k(X)$ considered up to isomorphism over the base (resp. preserving the elliptic pencil) equals $\phi(t)/|B_X|$ (resp. $\phi(t)/|\widetilde{B}_X|$).*

The group B_X can only be non-trivial if X is isotrivial with j-invariant 0 or 1728. We give examples when B_X and $\widetilde{B}_X$ are non-trivial, and when they are different.

Applications

We deduce from Theorem 3.1.2 that zeroth Jacobians of derived equivalent elliptic K3 surfaces are non-isomorphic in general (Corollary 3.5.17), that is passing to the Jacobian can not be defined solely in terms of the derived category (Remark 3.5.18).

Furthermore, Theorem 3.1.2 is relevant every time potential consequences of derived equivalence between K3 surfaces are considered. Let us explain two non-trivial situations when the explicit or geometric form of derived equivalence is desirable. The first is rational points over non-closed fields and the second is L-equivalence.

The motivation of Hassett–Tschinkel [HT17] was the question of existence of rational points on derived equivalent elliptic K3 surfaces over non-closed fields. Namely, since X and any of its coprime Jacobians $\mathrm{J}^k(X)$ are isogenous, it follows that X has a rational point if and only if $\mathrm{J}^k(X)$ has a rational point by the Lang-Nishimura theorem. Using Galois descent, as we know automorphism groups of elliptic K3 surfaces quite explicitly, we can partially extend Theorem 3.1.2 to subfields $k \subset \mathbb{C}$, and deduce the implication about rational points of Fourier–Mukai partners (see Corollary 3.5.22). We note that the question about the simultaneous existence of rational points on derived equivalent K3 surfaces still seems to be open.

Another application for Theorem 3.1.2 is to the question of L-equivalence of derived equivalent K3 surfaces X, Y [KS18]. For elliptic K3 surfaces the natural strategy is to prove L-equivalence for the generic fibres, which are genus one curves over the function field of the base, and then spread-out the L-equivalence over the total space. This strategy has been realised in [SZ20] for elliptic K3 surfaces of multisection index five. It follows from Theorem 3.1.2 that the same approach can work when the multisection index t is a power of a prime (and d is arbitrary).

Structure of this chapter

In Section 3.2, we discuss the basic theory of elliptic K3 surfaces and derived equivalence for K3 surfaces.

In Section 3.3, we study Néron–Severi lattices of elliptic K3 surfaces. We pay special attention to elliptic K3 surfaces of Picard rank 2, and study their Néron–Severi lattices in detail.

In Section 3.4, we revise Ogg–Shafarevich theory for elliptic K3 surfaces. We study functoriality of Ogg–Shafarevich theory and use this to study isomorphisms of Jacobians. Such isomorphisms are governed by the automorphisms of the zeroth Jacobian, and we describe this automorphism group. Finally, we study isotrivial elliptic K3 surfaces.

In Section 3.5, we study derived equivalence for elliptic K3 surfaces. The results of all the previous sections are combined to answer Question 3.1.1 for elliptic K3 surfaces of Picard rank 2. We also apply our results to study isomorphisms of zeroth Jacobians, and partially extend our results to non-closed fields of characteristic 0.

In the Appendix to this chapter, we show how to count Fourier–Mukai partners using the Counting Formula of [Hos+02], and we also give a relatively short proof of the Counting Formula. We then study maps between moduli spaces of lattice-polarised K3 surfaces. Finally, we study twisted derived equivalence for elliptic K3 surfaces.

This chapter is based on [MS24].

3.2 Preliminary Results

3.2.1 Derived Equivalence of K3 Surfaces

Our main references for derived equivalence for K3 surfaces are [Muk87; Orl03; Huy06].

The most fundamental result about derived equivalence for K3 surfaces is the Derived Torelli Theorem, which we already encountered in Section 2.3.2:

Theorem 3.2.1 (The Derived Torelli Theorem)**.** *[Muk87; Orl03] Let X and Y be K3 surfaces. The following are equivalent:*

i) There is an equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$;

ii) There is a Hodge isometry $\tilde{H}(X,\mathbb{Z}) \simeq \tilde{H}(Y,\mathbb{Z})$;

iii) The K3 surface Y is isomorphic to a fine moduli space of sheaves on X. That is, there is a Mukai vector $v \in \mathrm{N}(X)$ and a v-generic polarisation H on X such that $\mathrm{div}(v) = 1$, and $Y \simeq M_H(v)$.

If $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$, we say that X and Y are derived equivalent and that Y is a Fourier–Mukai partner of X.

Theorem 3.2.1 implies that in this case X and Y must have equal Picard numbers. Moreover, recall from Lemma 2.2.1 that there is an isomorphism $A_{\mathrm{NS}(X)} \simeq A_{T(X)}(-1)$. This shows that $A_{\mathrm{NS}(X)} \simeq A_{\mathrm{NS}(Y)}$, and since $\mathrm{NS}(X)$ and $\mathrm{NS}(Y)$ have equal ranks and signatures, it follows that they are in the same genus, see Definition 2.2.10 and Lemma 2.2.12.

Definition 3.2.2. For a K3 surface X we write $G_X \subset O(T(X))$ for the group of Hodge isometries of $T(X)$.

We have $G_X \simeq \mathbb{Z}/2g\mathbb{Z}$ for some integer $g \geq 1$ satisfying $\phi(2g) \mid \operatorname{rk} T(X)$ [Hos+02, Appendix B].

From the Derived Torelli Theorem one can deduce:

Theorem 3.2.3 (Counting Formula)**.** *[Hos+02] Let X be a K3 surface, and write* $\mathrm{FM}(X)$ *for the set of isomorphism classes of Fourier–Mukai partners of X. Then*

$$|\,\mathrm{FM}(X)| = \sum_{\Lambda} |O(\Lambda) \setminus O(A_\Lambda)/G_X|$$

where the sum runs over isomorphism classes of lattices Λ which are in the same genus as the Néron-Severi lattice $\mathrm{NS}(X)$*. Furthermore, each summand computes the number of isomorphism classes of Fourier–Mukai partners Y of X with* $\mathrm{NS}(Y) \simeq \Lambda$*.*

In Appendix 3.A, we use the Counting Formula to count Fourier–Mukai partners of certain elliptic K3 surfaces.

Definition 3.2.4. We say that a K3 surface X is *T-general* if $G_X = \{\pm\,\mathrm{id}\}$. A K3 surface that is not T-general is called *T-special*.

When X is T-general, the Counting Formula shows that the number of Fourier–Mukai partners is maximal (for a fixed $\mathrm{NS}(X)$) and only depends on $\mathrm{NS}(X)$. A similar effect holds for the invariants we study, see Theorem 3.5.11. Thus it is important to have explicit criteria for T-generality. If the Picard number ρ of X is odd, then $\phi(2g)$ must be odd, so $|G_X| = 2$ and X is T-general. Furthermore, we have the following result going back to Oguiso [Ogu02]:

Lemma 3.2.5 ([SZ20, Lemma 3.9])**.** *If X is a very general K3 surface in any moduli space of lattice polarised K3 surfaces, with Picard number $\rho < 20$, then X is T-general.*

See Example 3.4.24 for an explicit T-special K3 surface.

3.2.2 Elliptic K3 Surfaces

Recall that an elliptic surface is a surface X which admits a surjective morphism $f : X \to C$ where C is a smooth curve, such that the fibres of f are connected and the genus of the generic fibre is 1 [IS96, §10]. Our elliptic surfaces will be assumed to be relatively minimal, i.e. contain no (-1)-curves in the fibres of f; this is automatic for K3 surfaces. We say that an elliptic surface is isotrivial if all smooth fibres are isomorphic.

For an elliptic K3 surface we have the base $C \simeq \mathbb{P}^1$. There are two natural concepts of an isomorphism between elliptic K3 surfaces $f : X \to \mathbb{P}^1$ and $g : Y \to \mathbb{P}^1$.

Definition 3.2.6. (1) The surfaces X, Y are isomorphic as elliptic surfaces if there exists an isomorphism $X \simeq Y$ preserving the fibre classes, or equivalently there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\sim", "\beta"'] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ \mathbb{P}^1 \arrow[r, "\sim", "\overline{\beta}"'] & \mathbb{P}^1. \end{tikzcd}$$

In this case we say that the isomorphism $X \simeq Y$ twists the base by $\overline{\beta}$.

(2) The surfaces X and Y are isomorphic over $\mathbb{P}^1$ if there is an isomorphism $X \simeq Y$ twisting the base by the identity, or equivalently if there exists a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "\sim"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & \mathbb{P}^1. & \end{tikzcd}$$

Being isomorphic over $\mathbb{P}^1$ is more restrictive than being isomorphic as elliptic surfaces. For example, for every $\overline{\beta} \in \mathrm{Aut}(\mathbb{P}^1)$, $f : X \to \mathbb{P}^1$ and $\overline{\beta} f : X \to \mathbb{P}^1$ are isomorphic as elliptic surfaces, but usually not over $\mathbb{P}^1$.

Let $S \to \mathbb{P}^1$ be an elliptic K3 surface with a fixed section. We denote by $\mathrm{Aut}_{\mathbb{P}^1}(S)$ (resp. $\mathrm{Aut}(S, F)$) the group of automorphisms of S over $\mathbb{P}^1$ (resp. automorphisms of S preserving the fibre class). We have $\mathrm{Aut}_{\mathbb{P}^1}(S) \subset \mathrm{Aut}(S, F)$. We denote by $A_{\mathbb{P}^1}(S)$ (resp. $A(S, F)$) the group of automorphisms of S over $\mathbb{P}^1$ (resp. preserving the fibre class) which also preserve the zero-section. Such automorphisms will be called *group automorphisms* (see e.g. [DM22]).

Remark 3.2.7. The category of relatively minimal elliptic surfaces and their isomorphisms over $\mathbb{P}^1$ is equivalent to the category of genus one curves over $\mathbb{C}(t)$ and their isomorphisms. The functor is given by taking the generic fibre. This functor is an equivalence e.g. by [IS96, Theorem 7.3.3] or [DM22, Theorem 3.3].

3.3 Lattices of Elliptic K3 Surfaces

3.3.1 Isotropic Vectors

We collect some basic facts about lattices of elliptic K3 surfaces. Let $f : X \to \mathbb{P}^1$ be an elliptic fibration of a K3 surface X. Let $F \in \mathrm{NS}(X)$ be the class of a fibre. By the genus formula, we have $F^2 = 0$. Moreover, F is nef, and primitive, see [PS71].

Conversely, any primitive, nef, isotropic divisor $D \in \mathrm{NS}(X)$ is linearly equivalent to a smooth, irreducible genus 1 curve [Huy16, Proposition 2.3.10]. Therefore, we have $|D| \simeq \mathbb{P}^1$ and, the morphism $X \to |D| \simeq \mathbb{P}^1$ is an elliptic fibration.

Definition 3.3.1. Let L be a lattice. An *isotropic vector* of L is a non-zero vector $v \in L$ with $v^2 = 0$. An *isotropic element* of the discriminant group A_L is a non-zero element $a \in A_L$ such that $a^2 = 0$.

The following is a well-known result, see for example [Huy16, §2.3] and [PS71].

Lemma 3.3.2. *Let X be a K3 surface. The following is a one-to-one correspondence*

$$\left\{ \begin{array}{c} \mathrm{Aut}(X)\text{-orbits of non-zero, primitive,} \\ \text{nef, isotropic vectors in } \mathrm{NS}(X) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{elliptic fibrations of } X \end{array} \right\}$$
$$F \longmapsto (X \to |F|)$$

Proposition 3.3.3. *[Huy16, Proposition 11.1.3(i)] Let X be a K3 surface. Then X admits an elliptic fibration if and only if there is a non-zero class $D \in \mathrm{NS}(X)$ with $D^2 = 0$.*

Proposition 3.3.3 implies that a projective elliptic K3 surface has Picard rank $\rho \geq 2$. Namely, in this case, $\mathrm{NS}(X)$ contains a non-zero isotropic class F and an ample divisor H. These cannot be proportional because $H^2 > 0$ and $F^2 = 0$.

Definition 3.3.4. Let $f : X \to \mathbb{P}^1$ be an elliptic fibration of a K3 surface X, and let $F \in \mathrm{NS}(X)$ be the class of a fibre of f. A *multisection* of f is a smooth, irreducible curve $C \subset X$ such that $C \cdot F > 0$, and the number $C \cdot F$ is called the *degree* of the multisection C. The *multisection index* of f is the minimal degree of a multisection. We usually denote the multisection index by t.

The following lemma is a well-known result about the multisection index, c.f. [Huy16, §11.4], [Keu00].

Lemma 3.3.5. *Let $f : X \to \mathbb{P}^1$ be an elliptic fibration of a K3 surface X with multisection index t and fibre class $F \in \mathrm{NS}(X)$. We have*

$$t = \mathrm{div}(F) = \gcd_{D \in \mathrm{NS}(X)} (D \cdot F)\,.$$

Proof. It suffices to show that for any divisor $D \in \mathrm{NS}(X)$ for which $D \cdot F$ is positive, there exists an effective divisor E which satisfies $E \cdot F = D \cdot F$. Let $D \in \mathrm{NS}(X)$ be any divisor such that $D \cdot F > 0$. If $C \in \mathrm{NS}(X)$ is a component of a singular fibre, then $C \cdot F = 0$. Therefore, we may assume that C is not contained in the support of D. Then for $n \gg 0$,

$$(D + nF)^2 = D^2 + 2nD \cdot F > 0,$$

and for any (-2)-curve $C \in \mathrm{NS}(X)$,

$$(D + nF) \cdot C = D \cdot C + nC \cdot F. \tag{3.3.1}$$

If C is contained in a fibre, then $D \cdot C \geq 0$, since D is not supported on C, and $C \cdot F = 0$, hence (3.3.1) is non-negative for all $n \in \mathbb{N}$. On the other hand, if C is not contained in a fibre, then $C \cdot F > 0$, so that (3.3.1) is positive for $n \gg 0$. This shows that $D + nF$ is ample for $n \gg 0$. Therefore $h^0(D + nF) > 0$ for $n \gg 0$, so that $D + nF$ is linearly equivalent to an effective divisor E with $E \cdot F = (D + nF) \cdot F = D \cdot F + nF^2 = D \cdot F$, as required. □

Corollary 3.3.6. *Let X be a K3 surface. Then X admits an elliptic fibration with a section if and only if there is an embedding $U \hookrightarrow \mathrm{NS}(X)$.*

Proof. By Lemma 3.3.2 and Lemma 3.3.5, X admits an elliptic fibration with a section if and only if there exists a primitive, nef, isotropic vector $F \in \mathrm{NS}(X)$ of divisibility 1. Let $D \in \mathrm{NS}(X)$ be a divisor such that $D \cdot F = 1$. In this case, the sublattice $\langle D, F\rangle \subset \mathrm{NS}(X)$ is isometric to the hyperbolic plane U. Indeed, let $2m = D^2$, then $\langle D, F\rangle = \langle D - mF, F\rangle$ and we have $(D - mF)^2 = 0 = F^2$ and $(D - mF)F = 1$.

Conversely, suppose that there is an embedding $U \subset \mathrm{NS}(X)$. Let $F \in U$ be a primitive, isotropic vector with $\mathrm{div}(F) = 1$, and choose $D \in U$ such that $D \cdot F = 1$. By [Huy16, Remark 3.2.13], there exists an isometry $\sigma \in O(\mathrm{NS}(X))$ such that $\sigma(F)$ is nef. Since σ is an isometry, $\sigma(F)$ is primitive and we have $\sigma(F)^2 = 0$ and $\sigma(F) \cdot \sigma(D) = 1$. Therefore, $\sigma(F)$ is the fibre class of an elliptic fibration with a section by Lemma 3.3.2 and Lemma 3.3.5. □

Note that U is unimodular, so that any embedding $U \hookrightarrow \mathrm{NS}(X)$ is primitive and induces an orthogonal decomposition $\mathrm{NS}(X) \simeq U \oplus U^{\perp}$. The following result is a consequence of the Counting Formula, but we present a more straightforward proof.

Proposition 3.3.7. *[Hos+02, Proposition 2.7(3)] Let $S \to \mathbb{P}^1$ be an elliptic K3 surface which admits a section. Then S has no non-trivial Fourier–Mukai partners.*

Proof. Let S' be a Fourier–Mukai partner of S. We wish to prove that there is an isomorphism $S \simeq S'$. By the Derived Torelli Theorem 3.2.1, there exists a Hodge isometry $\tilde{\psi}\colon \tilde{H}(S,\mathbb{Z}) \simeq \tilde{H}(S',\mathbb{Z})$. Then $\tilde{\psi}$ restricts to a Hodge isometry $\psi\colon T(S) \simeq T(S')$. Since there exists an embedding $U \hookrightarrow \mathrm{NS}(S)$ by Corollary 3.3.6, the Hodge isometry ψ can be extended to a Hodge isometry $H^2(S,\mathbb{Z}) \simeq H^2(S',\mathbb{Z})$ by Lemma 2.2.8. By the Torelli Theorem 2.2.24, there is an isomorphism $S \simeq S'$, as required. □

3.3.2 Néron-Severi Lattices of Rank Two

We restrict our attention to elliptic K3 surfaces of Picard rank 2 following [Gee05; Ste04].

Proposition 3.3.8. *[Gee05, Remark 4.2], [SZ20, Lemma 3.3] Let X be an elliptic K3 surface of Picard rank 2, and let $F \in \mathrm{NS}(X)$ be the class of a fibre of an elliptic fibration of X of multisection index t. Then there exists a polarisation H on X such that H, F form a basis of $\mathrm{NS}(X)$ and $H \cdot F = t$. In particular, the Néron-Severi lattice of X is given by a matrix of the form*

$$\begin{pmatrix} 2d & t \\ t & 0 \end{pmatrix}. \tag{3.3.2}$$

We write $\Lambda_{d,t}$ for the lattice of rank 2 with matrix (3.3.2) with respect to some basis H, F. It is easy to see that the lattice $\Lambda_{d,t}$ has exactly two isotropic primitive vectors up to sign: one is F, and the other is

$$F' = \frac{1}{\gcd(d,t)}(tH - dF).$$

The following lemma describes when the class F' gives rise to another elliptic fibration on X.

Lemma 3.3.9. *[Gee05, §4.7] A K3 surface X with $\mathrm{NS}(X) \simeq \Lambda_{d,t}$ has two elliptic fibrations if and only if $d \not\equiv -1 \pmod{t}$. If $d \equiv -1 \pmod{t}$, X admits one elliptic fibration. If X is T-general, $t > 2$ and $d \not\equiv -1 \pmod{t}$, then the two fibrations are isomorphic (as elliptic surfaces) if and only if $d \equiv 1 \pmod{t}$.*

Corollary 3.3.10. *Let X be a K3 surface with $\mathrm{NS}(X) \simeq \Lambda_{d,t}$. Then every elliptic fibration of X has the same multisection index t.*

Proof. By Proposition 3.3.8, Lemma 3.3.9, and Lemma 3.3.2, X has either 1 or 2 elliptic fibrations up to isomorphism, with fibre classes given by F, and F' if F' is nef. Let $D = aH + bF \in \mathrm{NS}(X)$ for some $a, b \in \mathbb{Z}$. Then

$$D \cdot F' = \frac{1}{\gcd(d,t)}\left(adt + bt^2\right) = t \cdot \left(a \cdot \frac{d}{\gcd(d,t)} + b \cdot \frac{t}{\gcd(d,t)}\right)$$

which is a multiple of t. Moreover, if we choose a and b so that $ad + bt = \gcd(d, t)$, we obtain $D \cdot F' = t$. Therefore, we have $\operatorname{div}(F') = t = \operatorname{div}(F)$, and the result follows from Lemma 3.3.5. □

Remark 3.3.11. We note that Corollary 3.3.10 no longer holds if we drop the assumption that $\rho = 2$. For example, if X is a K3 surface with $\mathrm{NS}(X) = \Lambda_{d,t} \oplus \langle -2(d+t)\rangle$, with $1 < \gcd(d,t) < t$, the vector $F + H + D$ is primitive and isotropic, where D is a generator for $\langle -2(d+t)\rangle \subset \mathrm{NS}(X)$. Moreover, it is not hard to see that $\gcd(d,t) > 1$ implies that $\mathrm{NS}(X)$ contains no (-2)-curves, so that $F + H + D$ and F are nef. However, the elliptic fibrations corresponding to $F + H + D$ and F via Lemma 3.3.2 have multisection indices

$$\operatorname{div}(F + H + D) = \gcd(d,t) < t = \operatorname{div}(F).$$

The existence of a K3 surface with $\mathrm{NS}(X) \simeq \Lambda_{d,t} \oplus \langle -2(d+t)\rangle$ is guaranteed by the Surjectivity of the Period Map. Indeed, there exists a primitive embedding $\Lambda_{d,t} \oplus \langle -2(d+t)\rangle \hookrightarrow \Lambda_{\mathrm{K3}}$ by [Nik80, Theorem 1.12.4].

More generally, whenever $\mathrm{NS}(X)$ contains two primitive, isotropic classes whose divisibilities are not equal, X admits two elliptic fibrations with different multisection indices. Indeed let F_1 and F_2 be two such classes, then there are isometries $\sigma_1, \sigma_2 \in O(\mathrm{NS}(X))$ such that $\sigma_i(F_i)$ is nef for $i = 1, 2$ (c.f. [Huy16, Remark 3.2.13]), hence $\sigma_1(F_1)$ and $\sigma_2(F_2)$ correspond to elliptic fibrations with different multisection indices via Lemma 3.3.2.

We denote by $A_{d,t}$ the discriminant lattice of $\Lambda_{d,t}$ and we have

$$|A_{d,t}| = t^2.$$

It is easy to compute (see e.g. [Ste04, Proof of Lemma 3.2]) that the dual lattice $\Lambda_{d,t}^*$ is generated by

$$F^* = \frac{-2d}{t^2} F + \frac{1}{t} H, \quad H^* = \frac{1}{t} F \tag{3.3.3}$$

so that the images of (3.3.3) generate $A_{d,t}$.

The idea behind (3.3.3) is that F^* is the map $F^* \colon \mathrm{NS}(X) \to \mathbb{Z}$ satisfying $F^*(H) = 0$ and $F^*(F) = 1$, and H^* satisfies similar equations.

Furthermore for $a, b \in \mathbb{Z}$ we have

$$q(aF^* + bH^*) = \frac{2a(bt - ad)}{t^2}.$$

Lemma 3.3.12. *The discriminant group $A_{d,t}$ is isomorphic to $\mathbb{Z}/a\mathbb{Z} \oplus \mathbb{Z}/b\mathbb{Z}$ with $a = \gcd(2d, t)$ and $b = t^2/a$. In particular, $A_{d,t}$ is cyclic if and only if $\gcd(2d, t) = 1$.*

Furthermore, if Λ is a lattice in the same genus as $\Lambda_{d,t}$ then $\Lambda \simeq \Lambda_{e,t}$ with $\gcd(2e, t) = \gcd(2d, t)$.

Proof. The first claim follows by putting $\Lambda_{d,t}$ into Smith normal form.

Let Λ be a lattice in the same genus as $\Lambda_{d,t}$. Following the proof of [HT17, Proposition 16], Λ contains a primitive isotropic vector v. Hence, $\Lambda \simeq \Lambda_{e,s}$ for some $e, s \in \mathbb{Z}$, $s > 0$. Comparing discriminant groups of $\Lambda_{d,t}$ and $\Lambda_{e,s}$ we obtain $t = s$ and $\gcd(2d, t) = \gcd(2e, s)$. □

Example 3.3.13. Let $d = 0$, then by Lemma 3.3.12, $A_{0,t} \simeq \mathbb{Z}/t\mathbb{Z} \oplus \mathbb{Z}/t\mathbb{Z}$. Explicitly, generators (3.3.3) of the dual lattice $\Lambda_{0,t}^*$ are $F^* = \frac{1}{t}H$ and $H^* = \frac{1}{t}F$ and their images in $A_{0,t}$ are the two order t generators which are isotropic elements in $A_{0,t}$.

We introduce some properties of the discriminant groups which we will need to count Fourier–Mukai partners.

Definition 3.3.14. We call an isotropic element (see Definition 3.3.1) of order t in $A_{d,t}$ a *Lagrangian element*. We call a cyclic subgroup $H \subseteq A_{d,t}$ of order t a *Lagrangian subgroup* if every non-zero element of H is isotropic.

Definition 3.3.15. We denote by $\widetilde{\mathrm{L}}(A_{d,t})$ (resp. $\mathrm{L}(A_{d,t})$) the set of Lagrangian elements (resp. Lagrangian subgroups) of $A_{d,t}$.

The main reason we are interested in studying Lagrangians of $A_{d,t}$ is their correspondence with Fourier–Mukai partners which we establish in Section 3.5.

Proposition 3.3.16. *Let d, t be integers and let $m = \gcd(d, t)$. Then we have*

$$|\,\widetilde{\mathrm{L}}(A_{d,t})| = \phi(t) \cdot 2^{\omega(m)}, \quad |\,\mathrm{L}(A_{d,t})| = 2^{\omega(m)}. \tag{3.3.4}$$

Even though $\gcd(2d, t)$ is responsible for the structure of $A_{d,t}$, it is $\gcd(d, t)$ that appears in Proposition 3.3.16. For instance, if d and t are coprime and t is even, the discriminant group $A_{d,t}$ is not cyclic, but $|\,\mathrm{L}(A_{d,t})| = 1$.

Proof. Any cyclic subgroup $H \subset A_{d,t}$ of order t has $\phi(t)$ generators. H is a Lagrangian subgroup if and only if its generator is a Lagrangian element. Thus the two formulas in (3.3.4) are equivalent, and it suffices to prove the second one.

Let $t = \prod_p p^{k_p}$ be the prime factorisation of t. For any prime p, we have an isomorphism of p-adic lattices $\Lambda_{d,t} \otimes \mathbb{Z}_p \simeq \Lambda_{d,p^{k_p}} \otimes \mathbb{Z}_p$ (the isometry is given by $H \mapsto H$ and $F \mapsto \alpha F$, where α is the unit in $\mathbb{Z}_p$ given by $\alpha p^{k_p} = t$). By [Nik80, Proposition 1.7.1], $A_{d,t}$ is isometric to the orthogonal direct sum of $A_{d,p^{k_p}}$ over all primes p. Therefore we have

$$|\,\mathrm{L}(A_{d,t})| = \prod_p |\,\mathrm{L}(A_{d,p^{k_p}})|.$$

Therefore we need to prove that $|\,\mathrm{L}(A_{d,p^k})| = 1$ if d is coprime to p and $|\,\mathrm{L}(A_{d,p^k})| = 2$ otherwise. The result follows from Lemma 3.3.17 and Lemma 3.3.18 below. $\square$

Lemma 3.3.17. *The elements*

$$v = \frac{1}{t}F, \quad v' = \frac{1}{t}F' \tag{3.3.5}$$

are primitive isotropic vectors in $\Lambda_{d,t}^*$ and their images $\overline{v}$ and $\overline{v'}$ in $A_{d,t}$ generate Lagrangian subgroups in $A_{d,t}$. We have $\langle \overline{v} \rangle = \langle \overline{v'} \rangle$ if and only if $m := \gcd(d, t) = 1$, in which case

$$\overline{v'} = -d \cdot \overline{v}. \tag{3.3.6}$$

Proof. The first part is a simple computation. The corresponding Lagrangian subgroups are equal if and only if $v' = \frac{1}{tm}(tH - dF) = \frac{1}{m}H - \frac{d}{tm}F$ is a multiple of $v = \frac{1}{t}F$ modulo $\Lambda_{d,t}$. This is only the case when $m = 1$. $\square$

Lemma 3.3.18. *Let $t = p^k$ with p a prime number and $k \geq 1$. Then the subgroups $\langle \overline{v} \rangle, \langle \overline{v'} \rangle$ are the only Lagrangian subgroups of $A_{d,t}$.*

Proof. Write $d = \ell \cdot p^n$ for some $\ell \in \mathbb{Z}$ coprime to p and some $n \geq 0$. Note that whenever $n \geq k$, we have $d \equiv 0 \pmod{p^k}$, so that $\Lambda_{d,p^k} \simeq \Lambda_{0,p^k}$ and we can assume that $d = 0$. In this case we have $\overline{v'} = \overline{F^*}$ and it is easy to see that $\langle \overline{H^*} \rangle$ and $\langle \overline{F^*} \rangle$ are the only Lagrangian subgroups of A_{0,p^k} (see Example 3.3.13). Therefore we may assume $0 \leq n < k$.

In terms of generators (3.3.3) the quadratic form is given by

$$q(aF^* + bH^*) = \frac{2a}{p^{2k-n}} \left(bp^{k-n} - a\ell\right).$$

To find all Lagrangian subgroups, we start by describing the subgroup of elements in $A_{d,t}$ having order dividing $t = p^k$. We consider the vectors (3.3.5) which in our case are given by

$$\overline{v} = \frac{F}{p^k}, \quad \overline{v'} = \frac{H}{p^n} - \frac{\ell F}{p^k}.$$

Furthermore, the orders of $\overline{v}$ and $\overline{v'}$ are equal to p^k, and these elements satisfy a relation

$$p^n(\ell\overline{v} + \overline{v'}) = 0. \tag{3.3.7}$$

There are two cases to consider now. If $p > 2$, then

$$(A_{d,t})_{p^k-tors} = \left\langle \frac{F}{p^k}, \frac{H}{p^n} \right\rangle = \langle \overline{v}, \overline{v'} \rangle.$$

The vectors $\overline{v}$ and $\overline{v'}$ are isotropic and the discriminant form in terms of these elements equals

$$\overline{q}(a\overline{v} + b\overline{v'}) = \frac{2ab}{p^n}.$$

Hence an element $a\overline{v} + b\overline{v'}$ is isotropic if and only p^n divides ab. On the other hand, if $a\overline{v} + b\overline{v'}$ has order precisely p^k, then at least one of a or b is coprime to p. Hence isotropic elements of $A_{d,t}$ of order p^k are given by

$$a\overline{v} + bp^{n+j}\overline{v'}, \quad ap^{n+j}\overline{v} + b\overline{v'},$$

with both a and b coprime to p and $j \geq 0$. Using (3.3.7) we can rewrite these types of elements as

$$a'\overline{v}, \quad b'\overline{v'},$$

with a' and b' coprime to p. This finishes the proof in the $p > 2$ case.

If $p = 2$, then

$$\frac{1}{2^{n+1}} H \cdot F = \frac{2^k}{2^{n+1}} \quad \text{and} \quad \frac{1}{2^{n+1}} H^2 = 2\ell \cdot \frac{2^n}{2^{n+1}}$$

are both integers. This means that $\frac{1}{2^{n+1}}H$ is an element of $A_{d,2^k}$ by (2.2.1), and we have

$$(A_{d,t})_{2^k-tors} = \left\langle \frac{F}{2^k}, \frac{H}{2^{n+1}} \right\rangle \supsetneq \langle \overline{v}, \overline{v'} \rangle = \left\langle \frac{F}{2^k}, \frac{H}{2^n} \right\rangle.$$

However, a simple computation shows that all isotropic vectors are actually contained in $\langle \overline{v}, \overline{v'} \rangle$ and the proof works in the same way as in the $p > 2$ case. □

Lemma 3.3.18 allows us to define a canonical involution on the set of Lagrangian subgroups of $A_{d,t}$ as follows. For $H \subset A_{d,t}$ a Lagrangian, we take its primary decomposition with respect to (2.2.2)

$$H = \bigoplus_p H_p, \quad H_p \subset A_{d,t}^{(p)}$$

with each H_p a Lagrangian in $A_{d,t}^{(p)}$. We set $\iota_p(H_p)$ to denote the other Lagrangian subgroup as determined by Lemma 3.3.18; in the case p does not divide d, $\iota_p(H_p) = H_p$. We set

$$\iota(H) := \bigoplus_p \iota_p(H_p) \subset A_{d,t}. \tag{3.3.8}$$

The geometric significance of this involution is explained in Theorem 3.5.11. For now we note that

$$\iota(\langle \overline{v} \rangle) = \langle \overline{v'} \rangle \tag{3.3.9}$$

for $\overline{v}$, $\overline{v'}$ defined in Lemma 3.3.17.

3.4 Tate–Shafarevich Twists and Jacobians

Given an elliptic K3 surface $f : X \to \mathbb{P}^1$ and $k \in \mathbb{Z}$ we can define an elliptic K3 surface $\mathrm{J}^k(f) : \mathrm{J}^k(X) \to \mathbb{P}^1$, called the k-th Jacobian of X, as the moduli space of sheaves supported at the fibres of f and having degree k [Huy16, Chapter 11].

More precisely, let $F \in \mathrm{NS}(X)$ be the class of the fibre of the elliptic fibration f. Then $F^2 = 0$, and we define the k-th Jacobian of X to be the moduli space

$$\mathrm{J}^k(X) \coloneqq M(0, F, k).$$

Here, we suppress the choice of a generic polarisation in the notation, since $\mathrm{J}^k(X)$ does not depend on this choice. Since $F^2 = 0$, we have $(0, F, k)^2 = 0$, and therefore $\mathrm{J}^k(X)$ is a K3 surface. Moreover, $\mathrm{J}^k(X)$ admits a natural elliptic fibration $\mathrm{J}^k(X) \to |F|$ defined by sending a sheaf to its support. This is a two-dimensional example of a Beauville–Mukai system. We discuss Beauville–Mukai systems in more detail in Chapter 4.

The degree-0 Jacobian is the most important one, and we usually denote it by $S \coloneqq \mathrm{J}^0(X)$. The reason $\mathrm{J}^0(X)$ is so important is that the natural elliptic fibration on $\mathrm{J}^0(X)$ comes equipped with a distinguished section. This means that the generic fibre $\mathrm{J}^0(X)_\eta$ is an elliptic curve over $\mathrm{Spec}(\mathbb{C}(\eta))$, whereas the generic fibre of $X \to \mathbb{P}^1$ may not have a rational point over $\mathbb{C}(\eta)$. The generic fibre X_η is a so-called *torsor* of S_η. The set of isomorphism classes of S_η-torsors carries a natural group structure, and this group is called the *Weil–Châtelet group* of S_η, denoted $\mathrm{WC}(S_\eta)$. For any $k \in \mathbb{Z}$, $\mathrm{J}^k(X)_\eta$ is an S_η-torsor. Moreover, there exist infinitely many other elliptic K3 surfaces $Y \to \mathbb{P}^1$ for which Y_η is an S_η-torsor, and there exist infinitely many S_η-torsors that do not arise as the generic fibre of an elliptic K3 surface. Those that do, form a subgroup of $\mathrm{WC}(S_\eta)$ called the *Tate–Shafarevich group* of S. In this section, we study the Tate–Shafarevich group of an elliptic K3 surface with a section and compare it to its Brauer group.

3.4.1 Ogg–Shafarevich Theory for K3 Surfaces

Ogg–Shafarevich theory relates elements in the Brauer group $\mathrm{Br}(S)$ of an elliptic K3 surface S with a section, to S-torsors. For our purposes, the following definition of a torsor is convenient. Our main references for Tate–Shafarevich groups are [DG94; Fri98; Căl00; DP08; Huy16].

Definition 3.4.1. Let $f : S \to \mathbb{P}^1$ be an elliptic K3 surface with a section. An *f-torsor* is a pair $(g : X \to \mathbb{P}^1, \theta)$ where $g : X \to \mathbb{P}^1$ is an elliptic K3 surface and $\theta : \mathrm{J}^0(X) \to S$ is an isomorphism over $\mathbb{P}^1$ preserving the zero-sections, i.e. a group isomorphism over $\mathbb{P}^1$.

An isomorphism of f-torsors $(g : X \to \mathbb{P}^1, \theta)$ and $(h : Y \to \mathbb{P}^1, \eta)$ is an isomorphism $\gamma : X \to Y$ over $\mathbb{P}^1$ such that

$$\begin{tikzcd} \mathrm{J}^0(X) \arrow[rr, "\mathrm{J}^0(\gamma)"] \arrow[dr, "\theta"'] & & \mathrm{J}^0(Y) \arrow[dl, "\eta"] \\ & S & \end{tikzcd}$$

commutes.

Example 3.4.2. If X is an elliptic K3 surface, then X has a natural structure $(X, \mathrm{id}_{\mathrm{J}^0(X)})$ of a torsor over $\mathrm{J}^0(X)$. Since $\mathrm{J}^0(\mathrm{J}^k(X)) = \mathrm{J}^0(X)$ (this can be checked e.g. using Remark 3.2.7), all Jacobians $\mathrm{J}^k(X)$ also have a natural $\mathrm{J}^0(X)$-torsor structure.

The set of isomorphism classes of torsors of $f : S \to \mathbb{P}^1$ is in bijection with the Tate–Shafarevich group of $f : S \to \mathbb{P}^1$ [Huy16, 11.5.5(ii)], and we denote it $Ш(f : S \to \mathbb{P}^1)$ or just $Ш(S)$ if it can not lead to confusion.

A different, but equivalent, way to view the Tate–Shafarevich group is as follows. Let $X \to \mathbb{P}^1$ be an elliptic K3 surface with $S := \mathrm{J}^0(X)$. For any point $t \in \mathbb{P}^1$, there is an isomorphism of fibres $X_t \simeq S_t$. Moreover, there is an étale cover $\{U_i\}_i$ of $\mathbb{P}^1$, together with isomorphisms $S_{U_i} \simeq X_{U_i}$, hence X is obtained as a "regluing" of S. The details of this construction are given in for example [Căl00, §4.2] and [DP08, Chapter 2], and similar constructions for higher-dimensional fibrations can be found in [Mar14] and [AR23]. The upshot is that there is a natural isomorphism

$$Ш(S) \simeq H^1_{\text{ét}}(\mathbb{P}^1, \mathcal{X}),$$

where $\mathcal{X}$ is the sheaf of étale local sections of $S \to \mathbb{P}^1$ [Huy16, Proof of Proposition 5.6].

We now discuss a third way to view the Tate–Shafarevich group. Let $X \to \mathbb{P}^1$ be an elliptic K3 surface. If we write $S = \mathrm{J}^0(X)$, then the natural elliptic fibration $S \to \mathbb{P}^1$ has a section, turning the generic fibre $\mathrm{J}^0(S)_\eta$ into an elliptic curve over the function field $\mathbb{C}(\eta)$. Moreover, X_η is naturally a torsor over the abelian variety S_η. Recall that the group of torsors of a group scheme G is called the *Weil-Châtelet group* of G, denoted $\mathrm{WC}(G)$. By the discussion above, there is a group homomorphism

$$Ш(S) \to \mathrm{WC}(S_\eta) \tag{3.4.1}$$

defined by sending an S-torsor $X \to \mathbb{P}^1$ to its generic fibre X_η. Moreover, this morphism is injective. Indeed, suppose $X \to \mathbb{P}^1$ is an S-torsor such that X_η is the trivial S_η-torsor. Then X_η has a $\mathbb{C}(\eta)$-rational point whose closure is a section of $X \to \mathbb{P}^1$, hence $X \to \mathbb{P}^1$ is the trivial S-torsor by Lemma 3.4.5.

However, the group homomorphism (3.4.1) is not surjective. More precisely, there exist S_η-torsors whose unique relatively minimal models are not K3 surfaces. This is the content of the following proposition, see also [Huy16, Proposition 11.5.4, Corollary 11.5.5][1], [Fri98; DG94]. For a variety T over a field k, and $t \in T$ a (not necessarily closed) point, we denote by $k(\bar{t})$ the fraction field of the strict Henselisation of the stalk $\mathcal{O}_{T,t}$. Recall that the strict Henselisation of $\mathcal{O}_{T,t}$ is the stalk of $\mathcal{O}_T$ at t in the étale topology. For a T-scheme $X \to T$, we denote $X_{\bar{t}} := X \times_{\mathrm{Spec}(k(\eta))} \mathrm{Spec}(k(\bar{t}))$, where $k(\eta)$ is the fraction field of the generic point $\eta \in T$.

Proposition 3.4.3. *Let $S \to \mathbb{P}^1$ be an elliptic K3 surface with a section. Then there is an exact sequence*

$$0 \to Ш(S) \to \mathrm{WC}(S_\eta) \to \bigoplus_{t \in \mathbb{P}^1} \mathrm{WC}(S_{\bar{t}}) \to 0.$$

Another consequence of Proposition 3.4.3 is that the relatively minimal model $X \to \mathbb{P}^1$ of an S_η-torsor is a K3 surface if and only if for every point $t \in \mathbb{P}^1$, there is an étale open neighbourhood $t \in U \subset \mathbb{P}^1$ such that X_U has a $k(U)$-point. That is, if and only if étale locally, $X \to \mathbb{P}^1$ admits sections. This occurs precisely when $X \to \mathbb{P}^1$ has no multiple fibres. By *multiple fibres*, we mean fibres whose irreducible components are all non-reduced.

The following theorem is known as *Ogg–Shafarevich theory* for elliptic K3 surfaces, see [Huy16, Corollary 11.5.5].

Theorem 3.4.4 (Ogg–Shafarevich Theory). *If $S \to \mathbb{P}^1$ is an elliptic K3 surface with a section, then there is an isomorphism*

$$\mathrm{Br}(S) \simeq Ш(S). \tag{3.4.2}$$

The isomorphism (3.4.2) can be constructed in the following way. Consider an S-torsor $(X \to \mathbb{P}^1, \theta) \in Ш(S)$. The isomorphism $\theta : \mathrm{J}^0(X) \simeq S$ exhibits S as a moduli space of sheaves on X. In particular, the obstruction to the existence of a universal sheaf on $X \times S$ is a Brauer class $\alpha_X \in \mathrm{Br}(S)$, c.f. Section 2.3.2. The isomorphism (3.4.2) is the one that sends $(X \to \mathbb{P}^1, \theta)$ to α_X (see for example [Căl00, Theorem 4.4.1]). One of the main goals of this section is to understand how α_X changes when we change θ, so it would be more precise to include θ in the notation. However, to keep the notation light, we do not do that.

Recall that there is an isomorphism $\mathrm{Br}(S) \simeq \mathrm{Hom}(T(S), \mathbb{Q}/\mathbb{Z})$ (see Lemma 2.2.34), so that we may consider α_X as a group homomorphism $\alpha_X : T(S) \to \mathbb{Q}/\mathbb{Z}$.

Lemma 3.4.5. *Let (X, θ) be an S-torsor. Let t be the order of $\alpha_X \in \mathrm{Br}(S)$.*

i) X has a section if and only if $\alpha_X = 0$, in which case X is isomorphic to S as an S-torsor.

ii) For all $k \in \mathbb{Z}$ we have $\alpha_{\mathrm{J}^k(X)} = k \cdot \alpha_X$.

[1]See the erratum to [Huy16, Corollary 11.5.5] that was posted on https://www.math.uni-bonn.de/people/huybrech/ErratumK3.html

iii) The multisection index of X equals t.

iv) We have a Hodge isometry $T(X) \simeq \mathrm{Ker}(\alpha_X : T(S) \to \mathbb{Z}/t\mathbb{Z})$.

Proof. (i) It follows by construction that all S-torsor structures on S are isomorphic, and correspond to $0 \in \mathrm{Br}(S)$ under (3.4.2). Thus, if $\alpha_X = 0$, then X is isomorphic as S-torsor to S, in particular X and S are isomorphic as elliptic surfaces, hence X has a section. Conversely, if X has a section, then we have $S \simeq \mathrm{J}^0(X) \simeq X$ hence X is isomorphic as a torsor to some torsor structure on S, so that $\alpha_X = 0$ by the argument above.

Part (ii) is [Căl00, Theorem 4.5.2] and part (iv) is [Căl00, Theorem 5.4.3].

(iii) For a K3 surface X with a chosen elliptic fibration let us write $\mathrm{ind}(X)$ for the multisection index of the fibration. Since $\mathrm{J}^{\mathrm{ind}(X)}(X)$ admits a section, we have $\mathrm{J}^{\mathrm{ind}(X)}(X) \simeq S$ as torsors by (i). It follows using (ii) that $0 = \alpha_{\mathrm{J}^{\mathrm{ind}(X)}(X)} = \mathrm{ind}(X)\alpha_X$ hence $\mathrm{ord}(\alpha_X)$ divides $\mathrm{ind}(X)$. To prove their equality we use [Huy06, Ch. 4, (4.5), (4.6)] to deduce that for all $k \in \mathbb{Z}$

$$\mathrm{ind}(\mathrm{J}^k(X)) = \frac{\mathrm{ind}(X)}{\gcd(\mathrm{ind}(X), k)}.$$

In particular,

$$1 = \mathrm{ind}(\mathrm{J}^{\mathrm{ord}(\alpha_X)}(X)) = \frac{\mathrm{ind}(X)}{\gcd(\mathrm{ind}(X), \mathrm{ord}(\alpha_X))} = \frac{\mathrm{ind}(X)}{\mathrm{ord}(\alpha_X)}$$

so that $\mathrm{ind}(X) = \mathrm{ord}(\alpha_X)$, which proves part (ii). □

3.4.2 Functoriality of Ogg–Shafarevich Theory

In what follows, we sometimes write C, C' for bases of elliptic fibrations when they are not canonically isomorphic.

Lemma 3.4.6. *Let $X \to C$ and $X' \to C'$ be elliptic K3 surfaces with zeroth Jacobians $S \to C$ and $S' \to C'$, respectively. Then an isomorphism of elliptic surfaces $\gamma : X \simeq X'$ which twists the base by $\overline{\beta} : C \to C'$ (see Definition 3.2.6), induces a group isomorphism $\mathrm{J}^0(\gamma) : S \simeq S'$ twisting the base by $\overline{\beta}$.*

Proof. When $\overline{\beta}$ is the identity, this is a standard result which follows immediately from Remark 3.2.7. For the general case, see [DM22, §3, (3.3)]. □

Let $S \to \mathbb{P}^1$ be an elliptic K3 surface with a section. Recall that we denote by $A_{\mathbb{P}^1}(S)$ (resp. $A(\mathbb{P}^1, F)$) the group of group automorphisms of S over $\mathbb{P}^1$ (resp. group automorphisms of S preserving the fibre class $F \in \mathrm{NS}(S)$). We have $A_{\mathbb{P}^1}(S) \subset A(S, F)$, and we are interested in the orbits of these two groups acting on the Brauer group $\mathrm{Br}(S)$. We do this more generally, by explaining functoriality of $\text{Ш}(S)$ and $\mathrm{Br}(S)$ with respect to S.

Let $f : S \to C$ and $f' : S' \to C'$ be elliptic K3 surfaces with fixed sections. Assume that there exists a group isomorphism $\beta : S \simeq S'$ twisting the base by $\overline{\beta} : C \simeq C'$. We define a map $\beta_* : \text{Ш}(f : S \to C) \to \text{Ш}(f' : S' \to C')$ as follows:

$$\beta_*(g : X \to C, \theta) = (\overline{\beta} \circ g : X \to C', \beta \circ \theta).$$

Note that the element on the right-hand side belongs to $Ш(f')$ by Lemma 3.4.6.

Furthermore, in the same setting, we define

$$\beta_* : \mathrm{Hom}(T(S), \mathbb{Q}/\mathbb{Z}) \to \mathrm{Hom}(T(S'), \mathbb{Q}/\mathbb{Z})$$

by $\beta_*(\alpha) = \alpha \circ \beta^*$, where $\beta^* : T(S') \to T(S)$ is the Hodge isometry induced by β. It is important for applications that these two pushforwards are compatible with (3.4.2):

Lemma 3.4.7. *Let $f : S \to C$ and $f' : S' \to C'$ be elliptic K3 surfaces with fixed sections, and let $\beta : S \simeq S'$ be a group isomorphism twisting the base by $\overline{\beta}$. Then there is a commutative square of isomorphisms*

$$\begin{tikzcd} Ш(f : S \to C) \arrow[r, "\beta_*"] \arrow[d] & Ш(f' : S' \to C') \arrow[d] \\ \mathrm{Hom}(T(S), \mathbb{Q}/\mathbb{Z}) \arrow[r, "\beta_*"] & \mathrm{Hom}(T(S'), \mathbb{Q}/\mathbb{Z}), \end{tikzcd} \tag{3.4.3}$$

where the vertical arrows are induced by Lemma 2.2.34 and (3.4.2)*.*

Proof. The vertical arrows in (3.4.3) are the compositions of the vertical maps in the following diagram, with cohomology groups in étale and analytic topology respectively:

$$\begin{tikzcd} Ш(f : S \to C) \arrow[r, "\beta_*"] \arrow[d] & Ш(f' : S' \to C') \arrow[d] \\ H^1(C, \mathcal{X}_0) \arrow[r, "(1)"] \arrow[d] & H^1(C', \mathcal{X}'_0) \arrow[d] \\ H^2(S, \mathbb{G}_m) \arrow[r, "(2)"] \arrow[d] & H^2(S', \mathbb{G}_m) \arrow[d] \\ H^2_{an}(S, \mathcal{O}^*_S)_{tors} \arrow[r, "(3)"] \arrow[d, "(4)"'] & H^2_{an}(S', \mathcal{O}^*_{S'})_{tors} \arrow[d, "(4)"] \\ \mathrm{Hom}(T(S), \mathbb{Q}/\mathbb{Z}) \arrow[r, "\beta_*"] & \mathrm{Hom}(T(S'), \mathbb{Q}/\mathbb{Z}), \end{tikzcd} \tag{3.4.4}$$

c.f. [Huy16, Corollary 11.5.6]. Here $\mathcal{X}_0$ and $\mathcal{X}'_0$ are the sheaves of étale local sections of f and f', respectively. The horizontal arrows (1), (2), (3) are induced by $\overline{\beta}_*\mathcal{X}_0 \simeq \mathcal{X}'_0$ and $\beta_*\mathbb{G}_m \simeq \mathbb{G}_m$. Arrows (4) are induced by the exponential sequence. One can check commutativity for each square in (3.4.4), and this gives the desired result. □

Proposition 3.4.8. *Let $f : S \to C$, $f' : S' \to C'$ be elliptic K3 surfaces with sections. Let $(g : X \to C, \theta)$, $(g' : X' \to C', \theta')$ be torsors for f and f' respectively. Then there is a group isomorphism $\beta : S \simeq S'$, twisting the base by $\overline{\beta} : C \simeq C'$ and such that $\beta_*(g, \theta) \simeq (g', \theta')$ if and only if there is an elliptic surface isomorphism $X \simeq X'$ twisting the base by $\overline{\beta}$.*

Proof. Suppose there is a group isomorphism $\beta : S \simeq S'$ twisting the base by $\overline{\beta}$ and such that $\beta_*(g, \theta) = (g', \theta')$. Then it follows from the definition of β_* that there is an elliptic surface isomorphism $X \simeq X'$ twisting the base by $\overline{\beta}$. Conversely, suppose there is an elliptic surface isomorphism $\gamma : X \simeq X'$ twisting the base by $\overline{\beta}$. Consider the isomorphism $\beta := \theta' \circ \mathrm{J}^0(\gamma) \circ \theta^{-1} : S \to S'$. We can compute $\beta_*(g, \theta)$, decomposing β_* as a composition of isomorphisms

$$Ш(S) \xrightarrow{\theta_*^{-1}} Ш(\mathrm{J}^0(X)) \xrightarrow{\mathrm{J}^0(\gamma)_*} Ш(\mathrm{J}^0(X')) \xrightarrow{\theta'_*} Ш(S')$$

to see that $\beta_*(g, \theta) = (g', \theta')$. □

Remark 3.4.9. The proof of Proposition 3.4.8 in fact shows that given (g, θ), (g', θ') as in the statement, the set of isomorphisms between elliptic fibrations g and g' twisting the base by $\overline{\beta}$ (and ignoring the choice of θ, θ') is in natural bijection with the set of group isomorphisms β between S and S' twisting the base by $\overline{\beta}$ together with a chosen isomorphism γ between $\beta_*(g, \theta)$ and (g', θ').

It will be more convenient for us to work with the Brauer group instead of the Tate–Shafarevich group:

Proposition 3.4.10. *Using the same notation as in Proposition 3.4.8, there is a group isomorphism $\beta : S \simeq S'$, twisting the base by $\overline{\beta} : C \simeq C'$ and such that $\beta_*\alpha_X = \alpha_{X'}$ if and only if there is an elliptic surface isomorphism $X \simeq X'$ twisting the base by $\overline{\beta}$.*

Proof. This follows immediately from Proposition 3.4.8 and Lemma 3.4.7. □

Corollary 3.4.11. *Let $g : X \to C$, $g' : X' \to C'$ be elliptic K3 surfaces which are isomorphic via an isomorphism which twists the base by $\overline{\beta} : C \to C'$. Then for all $k \in \mathbb{Z}$, there exists an elliptic surface isomorphism $\mathrm{J}^k(X) \simeq \mathrm{J}^k(X')$ twisting the base by $\overline{\beta}$.*

Proof. Let $S \to C$ and $S' \to C'$ be the zeroth Jacobians of $X \to C$ and $X' \to C'$, respectively. By Proposition 3.4.10, there is a group isomorphism $\beta : S \to S'$ such that $\beta_*\alpha_X = \alpha_{X'}$. This means that $\beta_*(k \cdot \alpha_X) = k \cdot \beta_*\alpha_X = k \cdot \alpha_{X'}$ for all $k \in \mathbb{Z}$. Since the Brauer classes of $\mathrm{J}^k(X) \to C$ and $\mathrm{J}^k(X') \to C'$ are $k \cdot \alpha_X$ and $k \cdot \alpha_{X'}$, the result follows from Proposition 3.4.10. □

Corollary 3.4.12. *Let $S \to C$ be an elliptic K3 surface with a section. The set of $A(S, F)$-orbits (resp. $A_C(S)$-orbits) of $\mathrm{Br}(S)$ parametrises S-torsors up to isomorphism as elliptic surfaces (resp. up to isomorphism over C).*

Proof. We put $S = S'$ in Proposition 3.4.10, consider S-torsors (X, θ) and (X', θ') and write $\alpha_X, \alpha_{X'} \in \mathrm{Br}(S)$ for the corresponding Brauer classes. By Proposition 3.4.10 there is an isomorphism between elliptic surfaces X, X' twisting the base (resp. over the base) if and only if there exists $\beta \in A(S, F)$ (resp. $\beta \in A_C(S)$) such that $\beta_*(\alpha_X) = \alpha_{X'}$. Thus the resulting sets of orbits are as stated in the Corollary. □

Example 3.4.13. The automorphism $\beta = -1 \in A_C(S)$ acts on $\mathrm{Br}(S)$ as multiplication by -1. This way we always have (at least) two torsor structures on every elliptic K3 surface X. If X has no sections, these two torsor structures are isomorphic if and only if $\alpha_X \in \mathrm{Br}(X)$ has order two, which by Lemma 3.4.5 is equivalent to X having multisection index two.

We write EllK3 for the set of isomorphism classes of elliptic K3 surfaces (isomorphisms are allowed to twist the base). We can express Ogg–Shafarevich theory as a natural bijection between EllK3 and the set of isomorphism classes of twisted Jacobian K3 surfaces.

Definition 3.4.14. A *twisted Jacobian K3 surface* is a triple (S, f, α) where S is a K3 surface with elliptic fibration f together with a fixed section, and α is a Brauer class on S.

An isomorphism of two twisted Jacobian K3 surfaces $(S, f : S \to C, \alpha)$ and $(S', f' : S' \to C', \alpha')$ is a group isomorphism $\beta : S \simeq S'$ such that $\beta_* \alpha = \alpha'$. We write BrK3 for the set of isomorphism classes of twisted Jacobian K3 surfaces. The above results show the following.

Theorem 3.4.15. *The map* EllK3 $\to$ BrK3 *given by* $(X, g) \mapsto (\mathrm{J}^0(X), \mathrm{J}^0(g), \alpha_X)$ *is a bijection.*

Proof. From Proposition 3.4.10, it follows that the map EllK3 $\to$ BrK3 is well-defined and injective. For surjectivity, let $(S, f, \alpha) \in$ BrK3. Using the isomorphism (3.4.2), we obtain an S-torsor $(g : X \to \mathbb{P}^1, \theta : \mathrm{J}^0(X) \simeq S) \in Ш(f)$ corresponding to α. In particular, the map EllK3 $\to$ BrK3 assigns $(X, g) \mapsto (\mathrm{J}^0(X), \mathrm{J}^0(g), \theta_*^{-1}\alpha) \simeq (S, f, \alpha)$. □

3.4.3 Isomorphisms of Jacobians

Lemma 3.4.16. *[Căl00, Theorem 4.5.2] Let X be an elliptic K3 surface, and let $k, \ell \in \mathbb{Z}$. Then we have $\mathrm{J}^k(\mathrm{J}^\ell(X)) \simeq \mathrm{J}^{k\ell}(X)$ as torsors over $\mathrm{J}^0(X)$.*

Proof. By Lemma 3.4.5, we have $[\mathrm{J}^k(\mathrm{J}^\ell(X))] = k \cdot [\mathrm{J}^\ell(X)] = k\ell \cdot [X] = [\mathrm{J}^{k\ell}(X)]$ in the Tate–Shafarevich group of $\mathrm{J}^0(X)$. In particular, we have $\mathrm{J}^k(\mathrm{J}^\ell(X)) \simeq \mathrm{J}^{k\ell}(X)$ as torsors over $\mathrm{J}^0(X)$. □

Let t be the multisection index of X. We are especially interested in those Jacobians for which $\gcd(k, t) = 1$.

Definition 3.4.17. Let $X \to \mathbb{P}^1$ be an elliptic K3 surface with multisection index t. Let $k \in \mathbb{Z}$ be an integer such that $\gcd(k, t) = 1$. Then the Jacobian $\mathrm{J}^k(X)$ is called a *coprime Jacobian of X*.

By Theorem 3.5.1 below, every coprime Jacobian is a Fourier–Mukai partner of X. For all $k \in \mathbb{Z}$, we have well-known isomorphisms over $\mathbb{P}^1$:

$$\mathrm{J}^{k+t}(X) \simeq \mathrm{J}^k(X), \quad \mathrm{J}^{-k}(X) \simeq \mathrm{J}^k(X). \tag{3.4.5}$$

Here the first isomorphism follows by adding the multisection on the generic fibre, and then spreading out as in Remark 3.2.7, and the second isomorphism can be obtained, by the same token, from the dualisation of line bundles, or alternatively deduced from Proposition 3.4.10 with β acting by -1 on the fibres (see Example 3.4.13).

We see that there are at most $\phi(t)/2$ isomorphism classes of coprime Jacobians of X. The goal of the next result is to be able to compute this number precisely, see (3.4.11) for what this count will look like. Recall that a K3 surface X is T-general if $G_X = \left\{\pm \mathrm{id}_{T(X)}\right\}$, where G_X is the group of Hodge isometries of $T(X)$.

Proposition 3.4.18. *Let $X \to \mathbb{P}^1$ be an elliptic K3 surface with multisection index $t > 2$. Then $\mathrm{J}^k(X) \simeq \mathrm{J}^\ell(X)$ as $\mathrm{J}^0(X)$-torsors if and only if $k \equiv \ell \pmod t$. Furthermore there exist subgroups $B_X \subset \widetilde{B}_X \subset (\mathbb{Z}/t\mathbb{Z})^*$, such that for $k, \ell \in (\mathbb{Z}/t\mathbb{Z})^*$ we have*

$$\mathrm{J}^k(X) \simeq \mathrm{J}^\ell(X) \text{ over } \mathbb{P}^1 \iff k\ell^{-1} \in B_X,$$

and

$$\mathrm{J}^k(X) \simeq \mathrm{J}^\ell(X) \text{ as elliptic surfaces } \iff k\ell^{-1} \in \widetilde{B}_X.$$

Furthermore, B_X is a cyclic group of order 2, 4 or 6, containing $\{\pm 1\}$ and the case $B_X \simeq \mathbb{Z}/4\mathbb{Z}$ (resp. the case $B_X \simeq \mathbb{Z}/6\mathbb{Z}$) can occur only if X is an isotrivial elliptic fibration with j-invariant $j = 1728$ (resp. $j = 0$).

Finally, if X is T-general, then $B_X = \widetilde{B}_X = \{\pm 1\}$, that is in this case $\mathrm{J}^k(X)$ and $\mathrm{J}^\ell(X)$ are isomorphic over $\mathbb{P}^1$ if and only if they are isomorphic as elliptic surfaces if and only if $k \equiv \pm\ell \pmod t$.

In the statement we excluded the trivial cases $t = 1, 2$ because such elliptic K3 surfaces do not admit non-trivial coprime Jacobians.

Before we give the proof of the proposition, we need to set up some notation. Let S be an elliptic K3 with a section. For any subgroup $H \subset A(S, F)$ and any class $\alpha \in \mathrm{Br}(S)$ let H^α be the subgroup of H consisting of elements $\beta \in H$ with the property $\beta_*(\langle\alpha\rangle) \subset \langle\alpha\rangle$. Considering the action of H^α on $\langle\alpha\rangle = \mathbb{Z}/t\mathbb{Z}$ we get a natural homomorphism $H^\alpha \to (\mathbb{Z}/t\mathbb{Z})^*$ and we define

$$\overline{H}^\alpha \coloneqq \mathrm{Im}(H^\alpha \to (\mathbb{Z}/t\mathbb{Z})^*).$$

Proof of Proposition 3.4.18. Write $S = \mathrm{J}^0(X)$. We consider the following subgroups of $(\mathbb{Z}/t\mathbb{Z})^*$:

$$B_X \coloneqq \overline{A_{\mathbb{P}^1}(S)}^{\alpha_X} \tag{3.4.6}$$

$$\widetilde{B}_X \coloneqq \overline{A(S,F)}^{\alpha_X}. \tag{3.4.7}$$

We have $B_X \subset \widetilde{B}_X$, and $-1 \in A_{\mathbb{P}^1}(S)$ induces $-1 \in (\mathbb{Z}/t\mathbb{Z})^*$, in particular $\{\pm 1\} \subset B_X$. Note that we are assuming $t > 2$, hence $-1 \not\equiv 1 \pmod t$.

By Corollary 3.4.11, $\mathrm{J}^k(X)$ and $\mathrm{J}^\ell(X)$ are isomorphic over $\mathbb{P}^1$ if and only if $\mathrm{J}^{\ell^{-1}}(\mathrm{J}^k(X))$ and $\mathrm{J}^{\ell^{-1}}(\mathrm{J}^\ell(X))$ are isomorphic over $\mathbb{P}^1$. Here ℓ^{-1} is any integer such that $\ell\ell^{-1} \equiv 1 \pmod t$. By Lemma 3.4.16, we have $\mathrm{J}^{\ell^{-1}}(\mathrm{J}^k(X)) \simeq \mathrm{J}^{k\ell^{-1}}(X)$ and $\mathrm{J}^{\ell^{-1}}(\mathrm{J}^\ell(X)) \simeq \mathrm{J}^{\ell\ell^{-1}}(X) \simeq \mathrm{J}^1(X)$ over $\mathbb{P}^1$, where the last isomorphism follows from (3.4.5). By Corollary 3.4.12, this occurs if and only if $k\ell^{-1} \in B_X$. By the same argument, $\mathrm{J}^k(X)$ and $\mathrm{J}^\ell(X)$ are isomorphic as elliptic surfaces if and only if $\mathrm{J}^{k\ell^{-1}}(X)$ and X are isomorphic as elliptic surfaces if and only if $k\ell^{-1} \in \widetilde{B}_X$. The group B_X is a quotient of a subgroup of $A(S, F)$. The latter group, by Remark 3.2.7, is isomorphic to the group of elliptic curve automorphisms of the generic fibre of S. Thus, $A(S, F)$ (and hence B) is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, unless the j-invariant equals 1728 or 0 in which case $A(S, F)$ (and hence B_X) can be $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/6\mathbb{Z}$ respectively.

It remains to prove that $B_X = \widetilde{B}_X = \{\pm 1\}$ if X is T-general. By Proposition 3.4.10, an isomorphism $X \simeq \mathrm{J}^k(X)$ as elliptic surfaces would induce a group automorphism β of $S = \mathrm{J}^0(X)$ satisfying $\beta_*\alpha_X = k \cdot \alpha_X$. This means that $T(S)$ admits a Hodge isometry σ, which maps $T(X) = \mathrm{Ker}(\alpha_X)$ to itself. By T-generality, we get $\sigma = \pm\,\mathrm{id}$ so that $\beta_* = \pm 1$ and hence $k \equiv \pm 1 \pmod t$. □

Corollary 3.4.19. *If $A(\mathrm{J}^0(X), F) = A_{\mathbb{P}^1}(\mathrm{J}^0(X))$ then isomorphism classes of coprime Jacobians over $\mathbb{P}^1$ are the same as isomorphism classes of coprime Jacobians as elliptic surfaces.*

Proof. This follows from Proposition 3.4.18 as in this case $B_X = \widetilde{B}_X$ by construction. $\square$

Corollary 3.4.19 applies when singular fibres of $X \to \mathbb{P}^1$ lie over a non-symmetric set of points $Z \subset \mathbb{P}^1$, that is when $\overline{\beta} \in \mathrm{Aut}(\mathbb{P}^1)$ satisfies $\overline{\beta}(Z) = Z$ only for $\overline{\beta} = \mathrm{id}$. On the other hand, if Z is symmetric, and this symmetry can be lifted to an automorphism of $\mathrm{J}^0(X)$, we typically have $B_X \subsetneq \widetilde{B}_X$. For an explicit such surface, see Example 3.4.28.

3.4.4 Automorphisms and Hodge Isometries

Let $S \to \mathbb{P}^1$ be an elliptic K3 surface with a section. We saw in the previous section that the group $A(S, F)$, of automorphisms of S which preserve the elliptic fibration and the section, controls the isomorphisms of S-torsors. In this section, we investigate the group $A(S, F)$ in the case S has Picard rank 2. We also give a full description of the automorphism group $\mathrm{Aut}(S)$ in terms of the group of Hodge isometries of $T(S)$, denoted G_S (see Definition 3.2.2).

Lemma 3.4.20. *If X is a K3 surface of Picard rank 2, then G_X is a cyclic group of one of the following orders:*

$$2, 4, 6, 8, 10, 12, 22, 44, 50, 66.$$

Proof. The fact that G_X is a finite cyclic group of even order $2g$ where $\phi(2g) | \operatorname{rk} T(X)$ is proved in [Hos+02, Appendix B]. We solve the equation $\phi(2g) \mid 20$. Possible primes that can appear in the prime factorisation of $2g$ are 2, 3, 5, 11. Maximal powers of these primes such that $\phi(p^k) \mid 20$ are 2^3, 3, 5^2, 11 and the result follows by combining these or smaller prime powers. $\square$

Proposition 3.4.21. *Let X be an elliptic K3 surface of Picard rank 2. Then we have a canonical isomorphism*

$$\mathrm{Aut}(X) \simeq \mathrm{Ker}\left(G_X \to O\big(A_{T(X)}\big)/O^+\big(\,\mathrm{NS}(X)\big)\right),$$

where $O^+(\mathrm{NS}(X))$ is the group of isometries of $\mathrm{NS}(X)$ that preserve the ample cone. In particular, $\mathrm{Aut}(X)$ is a finite cyclic group and $|\,\mathrm{Aut}(X)| \leq 66$. Moreover, for any elliptic fibration $X \to \mathbb{P}^1$, the isomorphism above induces an isomorphism

$$\mathrm{Aut}(X, F) \simeq \mathrm{Ker}\left(G_X \to O(A_{T(X)})\right), \tag{3.4.8}$$

where $\mathrm{Aut}(X, F)$ is the group of automorphisms which fix the fibre class F of the elliptic fibration.

Proof. By the Torelli Theorem 2.2.24, there is a bijection between automorphisms of X and Hodge isometries of $H^2(X, \mathbb{Z})$ which preserve the ample cone. Using [Nik80, Corollary 1.5.2], we can write

$$\mathrm{Aut}(X) \simeq \left\{(\sigma, \tau) \in G_X \times O^+(\mathrm{NS}(X)) \;\middle|\; \overline{\sigma} = \overline{\tau} \in O(A_{T(X)})\right\}. \tag{3.4.9}$$

This isomorphism induces a surjective map $(\sigma, \tau) \mapsto \sigma$

$$\mathrm{Aut}(X) \to \mathrm{Ker}\left(G_X \to O(A_{T(X)})/O^+(\mathrm{NS}(X))\right). \tag{3.4.10}$$

The kernel of this map consists of the pairs $(\mathrm{id}_{T(X)}, \tau) \in G_X \times O^+(\mathrm{NS}(X))$ such that $\overline{\tau} = \mathrm{id}_{A_{T(X)}}$.

We claim that the homomorphism $O^+(\mathrm{NS}(X)) \to O(A_{T(X)})$ is injective. Since $\mathrm{NS}(X)$ contains four isotropic vectors $\pm F, \pm F'$, and $-\mathrm{id} \in O(\mathrm{NS}(X))$ never preserves the ample cone, we note that $O^+(\mathrm{NS}(X))$ must be either trivial, or isomorphic to $\mathbb{Z}/2\mathbb{Z}$ with non-trivial element swapping F with F'. The latter case is only possible when F' represents a class of an elliptic fibration on X, which by Lemma 3.3.9 corresponds to the case $d \not\equiv -1 \pmod t$. Then $\frac{1}{t}F$ and $\frac{1}{t}F'$ represent distinct classes in $A_{T(X)}$ (see (3.3.6)) and the element of $O^+(\mathrm{NS}(X))$ swapping F and F' has a non-trivial image in $O(A_{T(X)})$. Thus, since $O^+(\mathrm{NS}(X)) \to O(A_{T(X)})$ is injective, the map (3.4.10) is a bijection. The claim about isomorphism type of $|\mathrm{Aut}(X)|$ follows from Lemma 3.4.20. For the last statement, note that the only element of $O^+(\mathrm{NS}(X))$ which fixes F is the identity. Therefore, (3.4.8) also follows from (3.4.9). ☐

Example 3.4.22. Let X be an elliptic K3 surface with $\mathrm{NS}(X) \simeq \Lambda_{d,t}$ and assume that $\gcd(2d, t) = 1$. In this case $A_{d,t}$ is cyclic of order t^2 by Lemma 3.3.12. An isometry $\sigma \in O(A_{d,t})$ is given by multiplication by a unit $\alpha \in \mathbb{Z}/t^2\mathbb{Z}$ with $\alpha^2 \equiv 1 \pmod t$, so that the group $O(A_{d,t})$ is 2-torsion. Thus by Proposition 3.4.21, $\mathrm{Aut}(X) \subset G_X$ is a cyclic subgroup of index one or two.

Lemma 3.4.23. *Let S and S' be K3 surfaces of Picard rank 2 which admit elliptic fibrations with a section. Then every Hodge isometry between $T(S)$ and $T(S')$ lifts to a unique isomorphism between S and S'. In particular, we have* $\mathrm{Aut}(S) \simeq G_S$*. Finally, S admits a unique elliptic fibration with a unique section, hence every automorphism of S is a group automorphism.*

Proof. By Proposition 3.3.8 we have $\mathrm{NS}(S) \simeq \Lambda_{d,1}$, which is isomorphic to the hyperbolic lattice U, in particular $\mathrm{NS}(S)$ is unimodular and $A_{\mathrm{NS}(S)} = 0$. If there is a Hodge isometry between $T(S)$ and $T(S')$, extending it to a Hodge isometry between $H^2(S, \mathbb{Z})$ and $H^2(S', \mathbb{Z})$ preserving the ample cones, we obtain $S \simeq S'$, by the Torelli Theorem, as in the proof of Proposition 3.4.21. Thus we may assume that $S = S'$ in which case the result follows Proposition 3.4.21.

By Lemma 3.3.9, S admits a unique elliptic fibration. Since $\mathrm{NS}(S) = U$, there is a unique (-2)-curve which intersects the fibres of the elliptic fibration with multiplicity 1, i.e. a unique section. ☐

Example 3.4.24. Let $S \to \mathbb{P}^1$ be the elliptic K3 surface with a section given by the Weierstrass equation $y^2 = x^3 + t^{12} - t$. This surface is isotrivial with j-invariant 0. It was studied in [Keu16] and [Kon92]. We have $\mathrm{rk}\,\mathrm{NS}(S) = 2$, and S is T-special. In fact, the group G_S is cyclic of order 66, and S is unique with this property. Furthermore, $\mathrm{Aut}(S) \simeq \mathbb{Z}/66\mathbb{Z}$ by Lemma 3.4.23. The action of the subgroup $\mathbb{Z}/6\mathbb{Z} \subset \mathrm{Aut}(S)$ commutes with projection to $\mathbb{P}^1$ and rescales x and y coordinates, and the subgroup $\mathbb{Z}/11\mathbb{Z} \subset \mathrm{Aut}(S)$ preserves the fibre class $F \in \mathrm{NS}(S)$ and induces an order 11 automorphism $t \mapsto \zeta_{11} t$ on $\mathbb{P}^1$.

Corollary 3.4.25. *Let X be a T-general elliptic K3 surface of Picard rank 2 and multisection index $t > 2$, then* $\mathrm{Aut}(X, F) = \{\mathrm{id}\}$.

Proof. By Proposition 3.4.21, there is an isomorphism $\mathrm{Aut}(X, F) \simeq \mathrm{Ker}(G_X \to O(A_{T(X)}))$. We have $G_X = \{\pm \mathrm{id}\}$ by assumption. Since $t > 2$, and $A_{T(X)}$ has order t^2, we see that $-\mathrm{id}$ acts non-trivially on $A_{T(X)}$. Thus $\mathrm{Ker}(G_X \to O(A_{T(X)}))$ is trivial. □

3.4.5 Special Isotrivial Elliptic K3 Surfaces

By a *j-special isotrivial elliptic K3 surface* we mean an elliptic K3 surface with smooth fibres all having j-invariant 0 or 1728.

Remark 3.4.26. There exist Picard rank 2 isotrivial K3 surfaces with $j = 0$ (see Example 3.4.24), however for $j = 1728$ the minimal rank is 10 for the following reason. Let X be an isotrivial elliptic K3 surface X with $j = 1728$. The zeroth Jacobian S of X will have a Weierstrass equation $y^2 = x^3 + F(t)x$ with $F(t)$ a degree 8 polynomial in t. We have $\rho(S) = \rho(X)$. By semicontinuity of the Picard rank we may assume that $F(t)$ has distinct roots. In this case S has eight singular fibres, and the Weierstrass equation has ordinary double points at the singularities of the fibre, so S is the result of blowing up the Weierstrass model at these 8 points. Thus, in addition to the fibre class and the section class, S has 8 reducible fibres, so $\rho(S) \geq 10$. Isotrivial K3 surfaces with $j \neq 0, 1728$ are all Kummer and hence have Picard rank at least 17 [Saw14, Corollary 2].

We do not claim a direct relationship between the concepts of j-special and T-special, however both of these concepts require extra automorphisms.

Let $X \to \mathbb{P}^1$ be an elliptic K3 surface of multisection index $t > 2$, and let $S \to \mathbb{P}^1$ be its zeroth Jacobian. Let $H = A_{\mathbb{P}^1}(S)$; this group is $\mathbb{Z}/2\mathbb{Z}$ unless S is j-special, in which case it can be equal to $\mathbb{Z}/4\mathbb{Z}$ (resp. $\mathbb{Z}/6\mathbb{Z}$) when $j = 1728$ (resp. $j = 0$). By Proposition 3.4.18 the number of coprime Jacobians of X up to isomorphism over $\mathbb{P}^1$ equals $\phi(t)/|B_X|$, which is

$$\begin{cases} \phi(t)/2 & \text{if } X \to \mathbb{P}^1 \text{ is not isotrivial with } j = 0 \text{ or } j = 1728; \\ \phi(t)/4 & \text{for some isotrivial } X \text{ with } j = 1728, \text{ and } H = \mathbb{Z}/4\mathbb{Z}; \\ \phi(t)/6 & \text{for some isotrivial } X \text{ with } j = 0 \text{ and } H = \mathbb{Z}/6\mathbb{Z}. \end{cases} \tag{3.4.11}$$

We now show that the last two cases are indeed possible. For simplicity we assume that $t = p$, an odd prime.

Proposition 3.4.27. *Let $S \to \mathbb{P}^1$ be an elliptic K3 surface with a section. Assume S is isotrivial with $j = 1728$ (resp. $j = 0$) and $H = \mathbb{Z}/4\mathbb{Z}$ (resp. $H = \mathbb{Z}/6\mathbb{Z}$). Let $p > 2$ be a prime. Then S admits a torsor $X \to \mathbb{P}^1$ of multisection index p with exactly $\frac{\phi(p)}{4}$ (resp. $\frac{\phi(p)}{6}$) coprime Jacobians up to isomorphism over $\mathbb{P}^1$ if and only if $p \equiv 1 \pmod 4$ (resp. $p \equiv 1 \pmod 3$).*

Proof. Existence of such a torsor X implies the required numerical condition on p since 4 (resp. 6) divides $\phi(p) = p - 1$.

Conversely, assume that p satisfies the numerical condition. For every non-trivial element $\beta \in H$, the fixed subspace $(T(S) \otimes \mathbb{C})^{\langle \beta \rangle}$ is zero; this is because $S/\langle \beta \rangle$ admits a birational $\mathbb{P}^1$-fibration over $\mathbb{P}^1$, hence must be a rational surface. Thus $T(S) \otimes \mathbb{C}$, considered as a representation of a cyclic group H is a direct sum of one-dimensional representations corresponding to primitive roots of unity of order $|H|$.

This allows to describe $T(S) \otimes \mathbb{Q}$ as an H-representation, because irreducible $\mathbb{Q}$-representations of H are direct sums of Galois conjugate one-dimensional representations. Thus in both cases $T(S) \otimes \mathbb{Q} = V^{\oplus(\frac{22-\rho}{2})}$, where V is the 2-dimensional representation $\mathbb{Q}[i] = \mathbb{Q}[x]/(x^2+1)$ and $\mathbb{Q}[\omega] = \mathbb{Q}[x]/(x^2+x+1)$ respectively. At this point it follows that under our assumptions the Picard number $\rho = \rho(X)$ is even.

On the other hand, decomposition of the H-representation $T(S) \otimes \mathbb{Q}$ is induced from decomposition of $T(S) \otimes \mathbb{Z}[1/|H|]$, hence since $|H|$ is coprime to p, it induces a decomposition $T(X) \otimes \mathbb{F}_p \simeq V_p^{\oplus(\frac{22-\rho}{2})}$ with V_p defined by $\mathbb{F}_p[x]/(x^2+1)$ and $\mathbb{F}_p[x]/(x^2+x+1)$ respectively. Under the numerical condition on p, the corresponding polynomial has roots and the representation V_p is a direct sum of two one-dimensional representations $V_p = \chi \oplus \chi'$.

It follows that the dual representation $\mathrm{Br}(S)_{p-tors}$ induced by (2.2.14) splits into 1-dimensional representations χ, χ' as well. Take a generator $\alpha \in \mathrm{Br}(S)_{p-tors}$ for one of these representations, and let X be the corresponding torsor. The explicit description (3.4.6) shows that $B_X = H$. □

For explicit examples of surfaces satisfying conditions of Proposition 3.4.27, see Example 3.4.24 and Remark 3.4.26. Finally we illustrate the difference between isomorphism over $\mathbb{P}^1$ and isomorphism as elliptic surfaces.

Example 3.4.28. Consider the $j = 0$ isotrivial elliptic K3 surface $S \to \mathbb{P}^1$ of Example 3.4.24 above, and let $\beta \in A(S, F)$ be an automorphism of order 11. Note that $\beta \notin A_{\mathbb{P}^1}(S)$ so we may have $B_X \subsetneq \widetilde{B}_X$ in Proposition 3.4.18. By Lemma 3.4.23, β acts nontrivially on $T(S)$. As in the proof of Proposition 3.4.27, we deduce that for every prime $p \equiv 1 \pmod{11}$, the number of coprime Jacobians up to isomorphism as elliptic surfaces for an eigenvector torsor will be 11 times less than when they are considered up to isomorphism over $\mathbb{P}^1$.

3.5 Derived Equivalent K3 Surfaces and Jacobians

The following well-known result goes back to Mukai, see also [Căl00, Remark 5.4.6]. We provide the proof for completeness as it follows easily from what we have explained so far.

Theorem 3.5.1. *Let $S \to \mathbb{P}^1$ be an elliptic K3 surface with a section, and let $X \to \mathbb{P}^1$ be a torsor over $S \to \mathbb{P}^1$. Let $t \in \mathbb{Z}$ be the multisection index of $X \to \mathbb{P}^1$. Then $\mathrm{J}^k(X)$ is a Fourier–Mukai partner of X if and only if $\gcd(k,t) = 1$.*

Proof. Let $\alpha_X \in \mathrm{Br}(S)$ be the Brauer class of $X \to \mathbb{P}^1$. From Lemma 3.4.5 it is easy to deduce that

$$\det(T(X)) = t^2 \cdot \det(T(S)) \tag{3.5.1}$$

(cf [HS05b, Remark 3.1]).

Recall that $t = \mathrm{ord}(\alpha_X)$ by Lemma 3.4.5. We know $T(\mathrm{J}^k(X))$ is Hodge isometric to the kernel of $k \cdot \alpha_X : T(S) \to \mathbb{Z}/t\mathbb{Z}$, again by Lemma 3.4.5. If $\gcd(k,t) = 1$, then α_X and $k\alpha_X$ have the same kernel so that

$$T(\mathrm{J}^k(X)) \simeq \ker(k \cdot \alpha_X) = \ker(\alpha_X) \simeq T(X),$$

so $\mathrm{J}^k(X)$ is a Fourier–Mukai partner of X by the Derived Torelli Theorem.

Let us prove the converse implication. From (3.5.1), we get that for any $k \in \mathbb{Z}$, we have

$$\frac{\det(T(X))}{\det(T(\mathrm{J}^k(X)))} = \left(\frac{\mathrm{ord}(\alpha)}{\mathrm{ord}(k\alpha)}\right)^2 = \gcd(k, \mathrm{ord}(\alpha))^2.$$

Thus if X and $\mathrm{J}^k(X)$ are derived equivalent, then the left-hand side equals one by the Derived Torelli Theorem, hence k is coprime to $t = \mathrm{ord}(\alpha)$. □

The above proof nicely illustrates the methods we have discussed so far. However, there is an easier (but less conceptual) proof:

Alternative proof of Theorem 3.5.1. For $k \in \mathbb{Z}$, consider the Mukai vector $v_k = (0, F, k) \in \mathrm{N}(X)$, where $F \in \mathrm{NS}(X)$ is the fibre class of the elliptic fibration. This is the Mukai vector with the property that $M(v_k) \simeq \mathrm{J}^k(X)$. Recall from Lemma 3.3.5 that the divisibility of $F \in \mathrm{NS}(X)$ equals the multisection index t of $X \to \mathbb{P}^1$. On the other hand, we have $\mathrm{div}(v_k) = \gcd(k, \mathrm{div}(F)) = \gcd(k, t)$, hence

$$\mathrm{div}(v_k) = 1 \iff \gcd(k, t) = 1.$$

The result now follows from the Derived Torelli Theorem 3.2.1. □

3.5.1 Derived Elliptic Structures

In this subsection, we set up the theory of derived elliptic structures and Hodge elliptic structures.

Definition 3.5.2. Let X be a K3 surface. A *derived elliptic structure* on X is a pair (Y, ϕ), where Y is a K3 surface such that Y is derived equivalent to X and $\phi : Y \to \mathbb{P}^1$ is an elliptic fibration.

Definition 3.5.3. We say that two derived elliptic structures are isomorphic if they are isomorphic as elliptic surfaces. We denote by $\mathrm{DE}(X)$ (resp. $\mathrm{DE}_t(X)$) the set of isomorphism classes of derived elliptic structures on X (resp. derived elliptic structures on X of multisection index t).

Lemma 3.5.4. *Let X be a K3 surface. Then we have:*

(i) $\mathrm{DE}(X)$ *is a finite set;*

(ii) $\mathrm{DE}(X)$ *is nonempty if and only if X is elliptic;*

(iii) $\mathrm{DE}_t(X)$ *can be nonempty only for t such that t^2 divides the order of the discriminant group $A_{T(X)}$;*

(iv) If X is elliptic with $\rho(X) = 2$ and multisection index t, then every elliptic structure on every Fourier–Mukai partner of X also has multisection index t, that is $\mathrm{DE}(X) = \mathrm{DE}_t(X)$.

Proof. (i) The set of isomorphism classes of Fourier–Mukai partners of X is finite [BM01, Proposition 5.3], [Hos+02], and each of them has only finitely many elliptic structures up to isomorphism [Ste85]. It follows that $\mathrm{DE}(X)$ is a finite set.

(ii) If X elliptic, then X with its elliptic structure is an element of $\mathrm{DE}(X)$, hence it is nonempty. Conversely, if $\mathrm{DE}(X)$ is nonempty, then X admits a Fourier–Mukai partner which is an elliptic K3 surface. Then by the Derived Torelli Theorem NS(X) and NS(Y) are in the same genus, and since Y is elliptic, the intersection form NS(Y) represents zero by Lemma 2.2.16, hence a standard lattice theoretic argument shows that NS(X) also represents zero, and X is elliptic.

(iii) If (Y, ϕ) is a derived elliptic structure on X of multisection index t, then we have

$$|A_{T(X)}| = |A_{T(Y)}| = t^2 \cdot |A_{T(\mathrm{J}^0(Y))}|$$

where the first equality follows from the Derived Torelli Theorem and the second one can be deduced from (3.5.1) (cf [HS05b, Remark 3.1]). In particular, $\mathrm{DE}_t(X)$ is empty whenever t^2 does not divide the order of $A_{T(X)}$.

(iv) Every Fourier–Mukai partner Y of X also has Picard number $\rho(Y) = 2$. By Proposition 3.3.8, the multisection index of every elliptic fibration on Y equals the square root of $|A_{T(Y)}| = |A_{T(X)}| = t^2$. □

We can take coprime Jacobians of a derived elliptic structure (Y, ϕ), which we denote by $\mathrm{J}^k(Y, \phi)$. By Lemma 3.4.16 and Theorem 3.5.1 this defines a group action of $(\mathbb{Z}/t\mathbb{Z})^*$ on $\mathrm{DE}_t(X)$. The set of $(\mathbb{Z}/t\mathbb{Z})^*$-orbits on $\mathrm{DE}_t(X)$ parametrises derived elliptic structures up to taking coprime Jacobians, and it is sometimes a more natural set to work with.

We now explain Hodge-theoretic analogues of derived elliptic structures. The following definition is motivated by the Derived Torelli Theorem.

Definition 3.5.5. Let X be a K3 surface. A *Hodge elliptic structure* on X is a twisted Jacobian K3 surface (S, f, α) (see Definition 3.4.14) such that there exists a Hodge isometry $\mathrm{Ker}(\alpha) \simeq T(X)$.

The index of a Hodge elliptic structure is defined to be the order of its Brauer class α. An isomorphism of Hodge elliptic structures (S, f, α), (S', f', α') is an isomorphism $\gamma : S \to S'$ of elliptic surfaces such that $\gamma_*(\alpha) = \alpha'$. We denote by $\mathrm{HE}(X)$ the set of isomorphism classes of Hodge elliptic structures on X. We write $\mathrm{HE}_t(X)$ for the set of isomorphism classes of Hodge elliptic structures of index t. The operation $k * (S, f, \alpha) = (S, f, k\alpha)$ defines a group action of $(\mathbb{Z}/t\mathbb{Z})^*$ on $\mathrm{HE}_t(X)$.

Example 3.5.6. Let X be an elliptic K3 surface of Picard rank 2 and multisection index t. Let (S, f, α) be a Hodge elliptic structure on X. Since the discriminant of X equals t^2, from the sequence

$$0 \to T(X) \to T(S) \to \mathbb{Z}/t\mathbb{Z} \to 0,$$

we deduce that $T(S)$ is unimodular. Thus S is an elliptic K3 surface of Picard rank two, and it has a unique elliptic fibration, which has a unique section (see Lemma 3.3.9). We see that in the Picard rank two case f can be excluded from the data of a Hodge elliptic structure and we have a bijection

$$\mathrm{HE}_t(X) = \{(S, \alpha)\}/\simeq, \tag{3.5.2}$$

with isomorphisms understood as isomorphisms between K3 surfaces respecting the Brauer classes.

Proposition 3.5.7. *Let X be a K3 surface and let t be a positive integer. Then the bijection* EllK3 $\simeq$ BrK3 *of Theorem 3.4.15 induces a $(\mathbb{Z}/t\mathbb{Z})^*$-equivariant bijection* $\mathrm{DE}_t(X) \simeq \mathrm{HE}_t(X)$.

Proof. First of all note that by definition $\mathrm{DE}_t(X)$ is a subset of EllK3 consisting of isomorphism classes (Y,ϕ) with Y derived equivalent to X and ϕ having a multisection index t. Similarly, $\mathrm{HE}_t(X)$ is a subset of BrK3 consisting of (S,f,α) such that $\mathrm{ord}(\alpha)=t$ and $\mathrm{Ker}(\alpha)\simeq T(X)$. If $(Y,\phi)\in$ EllK3, then by Lemma 3.4.5, (Y,ϕ) belongs to $\mathrm{DE}_t(X)$ if and only if the corresponding triple $(\mathrm{J}^0(Y),\mathrm{J}^0(\phi),\alpha_Y)\in$ BrK3 belongs to $\mathrm{HE}_t(X)$.

The $(\mathbb{Z}/t\mathbb{Z})^*$-equivariance of the map is a direct consequence of the fact that $k\alpha_Y=\alpha_{\mathrm{J}^k(Y)}$, which holds again by Lemma 3.4.5. □

Definition 3.5.8. Let T be a lattice. For $t\in\mathbb{Z}$, we write $\mathrm{I}_t(A_T)$ for the set of cyclic, isotropic subgroups of order t in A_T, and we write $\widetilde{\mathrm{I}}_t(A_T)$ for the set of isotropic vectors of order t in A_T.

For a K3 surface X, there is a natural action of G_X, on $\mathrm{I}_t(A_{T(X)})$ and $\widetilde{\mathrm{I}}_t(A_{T(X)})$. Let (S,f,α) be a Hodge elliptic structure on X of index t. There is a unique isomorphism $r_\alpha:\mathbb{Z}/t\mathbb{Z}\simeq T(S)/\mathrm{Ker}(\alpha)$ such that the diagram

$$\begin{tikzcd} & T(S) \arrow[dl, "\alpha"'] \arrow[dr] & \\ \mathbb{Z}/t\mathbb{Z} \arrow[rr, "\sim", "r_\alpha"'] & & T(S)/\mathrm{Ker}(\alpha) \end{tikzcd} \tag{3.5.3}$$

commutes. Hence the Brauer class α singles out a generator $r_\alpha(\overline{1})$ of $T(S)/\mathrm{Ker}(\alpha)$. Fix any Hodge isometry $T(X)\simeq\mathrm{Ker}(\alpha)$. The natural inclusion $T(S)/T(X)\subset A_{T(X)}$ allows us to view $r_\alpha(\overline{1})$ as an element of $A_{T(X)}$, which we denote by w_α. We denote the subgroup of $A_{T(X)}$ generated by w_α by H_α. Note that w_α, and hence H_α, is only well-defined up to the G_X action on $A_{T(X)}$, since its construction depends on the original choice of Hodge isometry $T(X)\simeq\mathrm{Ker}(\alpha)$. On the other hand isomorphic Hodge elliptic structures on X give rise to isotropic vectors in the same G_X-orbit by Lemma 2.2.13. We define the map

$$w:\mathrm{HE}_t(X)\to\widetilde{\mathrm{I}}_t(A_{T(X)})/G_X,\quad w(S,f,\alpha)=w_\alpha. \tag{3.5.4}$$

The operation $k*w=k^{-1}w$, where k^{-1} is an inverse to k modulo t, defines a group action of $(\mathbb{Z}/t\mathbb{Z})^*$ on $\widetilde{\mathrm{I}}_t(A_T)/G_X$.

Lemma 3.5.9. *The map* (3.5.4) *is* $(\mathbb{Z}/t\mathbb{Z})^*$*-equivariant.*

Proof. Recall from Lemma 3.4.5(ii) that $\alpha_{\mathrm{J}^k(Y)}=k\cdot\alpha_Y$ in $\mathrm{Br}(\mathrm{J}^0(Y))$ for all $k\in\mathbb{Z}$. It follows from (3.5.3) that we have $r_{k\alpha}=k^{-1}r_\alpha$. Thus from the definitions we get

$$w_{k\alpha}=r_{k\alpha}(\overline{1})=k^{-1}r_\alpha(\overline{1})=k^{-1}w_\alpha=k*w_\alpha,$$

which means that the map w is equivariant. □

Proposition 3.5.7 and Lemma 3.5.9 give rise to the following commutative diagram with the vertical arrows being quotients by the corresponding $(\mathbb{Z}/t\mathbb{Z})^*$-actions:

$$\begin{tikzcd} \mathrm{DE}_t(X) \arrow[r, "\sim"] \arrow[d] & \mathrm{HE}_t(X) \arrow[r, "w"] \arrow[d] & \widetilde{\mathrm{I}}_t(A_{T(X)})/G_X \arrow[d] \\ \mathrm{DE}_t(X)/(\mathbb{Z}/t\mathbb{Z})^* \arrow[r, "\sim"] & \mathrm{HE}_t(X)/(\mathbb{Z}/t\mathbb{Z})^* \arrow[r] & \mathrm{I}_t(A_{T(X)})/G_X \end{tikzcd} \tag{3.5.5}$$

For (Y, ϕ) a derived elliptic structure of X, we consider $w_\phi := w_{\alpha_Y}$, the image of (Y, ϕ) under the composition of maps in the top row of (3.5.5). In particular, if $f : X \to \mathbb{P}^1$ is an elliptic fibration with fibre class $F \in \mathrm{NS}(X)$, then by construction, w_f is the Căldăraru class of the moduli space $\mathrm{J}^0(X)$ of sheaves with Mukai vector $(0, F, 0)$ on X, thus by Lemma 2.4.6 w_f corresponds to

$$\frac{1}{t}F \in \mathrm{I}_t(A_{\mathrm{NS}(X)})/G_X \tag{3.5.6}$$

(we can get rid of the minus sign in the formula at this point, as $-\mathrm{id} \in G_X$, see Remark 2.4.2).

3.5.2 Fourier–Mukai Partners in Rank 2

In this subsection, we work with an elliptic K3 surface of Picard rank 2, so that by Proposition 3.3.8 we have $\mathrm{NS}(X) \simeq \Lambda_{d,t}$ given by (3.3.2). The following result is one of the reasons why it is natural to concentrate on Picard rank two elliptic surfaces.

Lemma 3.5.10. *For an elliptic K3 surface X with $\mathrm{NS}(X) \simeq \Lambda_{d,t}$, all derived elliptic structures and all Hodge elliptic structures on X have the same index t.*

Proof. This follows from Lemma 3.5.4 and Proposition 3.5.7. □

For X as in Lemma 3.5.10, we have $\mathrm{DE}(X) = \mathrm{DE}_t(X)$. In particular, there is an action of $(\mathbb{Z}/t\mathbb{Z})^*$ on $\mathrm{DE}(X)$ by taking coprime Jacobians. Recall that for a K3 surface X with $\mathrm{NS}(X) \simeq \Lambda_{d,t}$, we have $A_{T(X)} \simeq A_{\mathrm{NS}(X)}(-1) \simeq A_{d,t}(-1)$, and it has order t^2 by Lemma 3.3.12. In this setting, isotropic elements (resp. cyclic isotropic subgroups) of order t are precisely Lagrangian elements (resp. Lagrangian subgroups), see Definition 3.3.14:

$$\mathrm{I}_t(A_{T(X)}) = \mathrm{L}(A_{T(X)}), \quad \widetilde{\mathrm{I}}_t(A_{T(X)}) = \widetilde{\mathrm{L}}(A_{T(X)}).$$

The following result is related to [Ma10, Proposition 3.3].

Theorem 3.5.11. *Let X be an elliptic K3 surface of Picard rank 2 and multisection index t. Then the map w (3.5.4) is a bijection. Furthermore, we have a bijection*

$$\mathrm{DE}(X)/(\mathbb{Z}/t\mathbb{Z})^* \simeq \mathrm{L}(A_{T(X)})/G_X. \tag{3.5.7}$$

Moreover, action (3.3.8) induces a $\mathbb{Z}/2\mathbb{Z}$-action on $\mathrm{L}(A_{T(X)})/G_X$ which under bijection (3.5.7) corresponds to the action on $\mathrm{DE}(X)$ swapping the two elliptic fibrations on Fourier–Mukai partners of X.

Proof. We first show that w is bijective. We start with bijection (3.5.2). For the injectivity of w, take (S, α) and (S', α') with $T(X) \simeq \mathrm{Ker}(\alpha) \simeq \mathrm{Ker}(\alpha')$. Assume that there exists a Hodge isometry $\sigma \in G_X$ with the property $\overline{\sigma}(w_\alpha) = w_{\alpha'}$. Then Lemma 2.2.13 implies that σ can be extended to a Hodge isometry $T(S) \to T(S')$. Since S and S' have Picard rank 2, Lemma 3.4.23 implies that this Hodge isometry is induced by a group isomorphism $\beta : S \simeq S'$. From $\overline{\sigma}(w_\alpha) = w_{\alpha'}$, it follows that (S, α) and (S', α') are isomorphic.

For the surjectivity of w, let $u \in A_{T(X)}$ be an isotropic vector of order t and $H = \langle u \rangle$. Via Lemma 2.2.13, H corresponds to an overlattice $i : T(X) \hookrightarrow T$ which inherits a Hodge structure from $T(X)$, i.e. $i : T(X) \hookrightarrow T$ is a Hodge overlattice. Note that T is unimodular, since the index of $T(X) \subset T$ is t and $A_{T(X)}$ has order t^2. Hence $T \oplus U$ is an even, unimodular lattice of rank 22 and signature $(3, 19)$. This means that it is isomorphic to the K3-lattice Λ_{K3}. By the surjectivity of the period map (Theorem 2.2.23), we obtain a K3 surface S with $T(S) \simeq T$ and $\mathrm{NS}(S) \simeq U$. Therefore the overlattice $i : T(X) \hookrightarrow T(S)$ is a Hodge overlattice with $T(S)/T(X) = H$. We define the Brauer class $\alpha : T(S) \to H \simeq \mathbb{Z}/t\mathbb{Z}$ where the second map is given by $u \mapsto \overline{1}$. Thus we have constructed a pair (S, α) with Căldăraru class u and $\mathrm{Ker}(\alpha) \simeq T(X)$.

Since w is bijective, the diagram (3.5.5) immediately implies (3.5.7). The action (3.3.8) induces the action on $\mathrm{L}(A_{T(X)})/G_X$ because ι commutes with G_X. Indeed this can be checked on each primary part (2.2.2), where there are at most two Lagrangian subgroups (see Lemma 3.3.18), hence the action of G_X factors through the action generated by ι_p. To show that ι corresponds to swapping the elliptic fibrations on Fourier–Mukai partners Y, we can use the identification $\mathrm{L}(A_{T(X)})/G_X = \mathrm{L}(A_{T(Y)})/G_Y$, and assume $Y = X$. The result follows from (3.3.9) because Lagrangian subgroups generated by $\overline{v}$ and $\overline{v'}$ correspond to the two elliptic fibrations on X via (3.5.7) by (3.5.6). □

Recall from Lemma 3.3.9 that a K3 surface X with $\mathrm{NS}(X) \simeq \Lambda_{d,t}$ admits two elliptic fibrations, except when $d \equiv -1 \pmod{t}$, in which case X admits only one elliptic fibration. Using Theorem 3.5.11 we can easily compare the coprime Jacobians of these two fibrations.

Example 3.5.12. Let X be an elliptic K3 surface of Picard rank two with $\mathrm{NS}(X) \simeq \Lambda_{d,t}$ such that $\gcd(d,t) = 1$ and $d \not\equiv -1 \pmod{t}$. Let (X, f) and (X, g) be two elliptic fibrations on X (see Lemma 3.3.9), and let w_f and w_g be their Căldăraru classes, which are Lagrangian elements in $A_{d,t}$. By Lemma 3.3.18, $A_{d,t}$ admits a unique Lagrangian subgroup, thus we have $\langle w_f \rangle = \langle w_g \rangle$. By Theorem 3.5.11 this implies that f and g are coprime Jacobians of each other. We can make this more precise as follows. Recall that by (3.5.6), w_f and w_g correspond to classes $\overline{v}$, $\overline{v'}$ (3.3.5) respectively. Using (3.3.6), we compute

$$w_g = \overline{v'} = -d\overline{v} = -dw_f = -d^{-1} * w_f.$$

Here d^{-1} is the inverse to d modulo t. Thus we have an isomorphism of elliptic surfaces

$$(X, g) \simeq \mathrm{J}^{-d^{-1}}(X, f) \simeq \mathrm{J}^{d^{-1}}(X, f)$$

and $(X, f) \simeq \mathrm{J}^{d}(X, g)$.

Corollary 3.5.13. *Let X be an elliptic K3 surface of Picard rank two. The set of Fourier–Mukai partners of X considered up to isomorphism as surfaces, and up to coprime Jacobians (on every derived elliptic structure of X) is in natural bijection with the double quotient*

$$\langle \iota \rangle \backslash \, \mathrm{L}(A_{T(X)})/G_X.$$

Proof. This is the consequence of the action of ι on $\mathrm{L}(A_{T(X)})/G_X$ by swapping the two elliptic fibrations as explained in Theorem 3.5.11. □

Corollary 3.5.14. *Let X be an elliptic K3 surface of Picard rank 2. Let $d, t \in \mathbb{Z}$ such that $\mathrm{NS}(X) \simeq \Lambda_{d,t}$, and write $m = \gcd(d,t)$.*

(i) If $m = 1$, then $\mathrm{DE}(X)$ *is a single* $(\mathbb{Z}/t\mathbb{Z})^*$*-orbit. Explicitly, every Fourier–Mukai partner of* X *will be found among the coprime Jacobians of a fixed elliptic fibration* (X, f).

(ii) If $m = p^k$, for a prime p and $k \geq 1$, then $\mathrm{DE}(X)$ *consists of at most two* $(\mathbb{Z}/t\mathbb{Z})^*$*-orbits, permuted by the involution* ι. *Explicitly every Fourier–Mukai partner of* X *will be found among the coprime Jacobians of one of the two elliptic fibrations on* X.

(iii) If m has at least 7 distinct prime factors then $\mathrm{DE}(X)$ *has at least three* $(\mathbb{Z}/t\mathbb{Z})^*$*-orbits. In particular, there exist Fourier–Mukai partners of* X *which are not isomorphic, as surfaces, to any of the Jacobians of elliptic structures on* X.

Proof. In each case we use Theorem 3.5.11 combined with the count of Lagrangians given in Proposition 3.3.16.

(i) Fix an elliptic fibration $f : X \to \mathbb{P}^1$ and let $H_f \subseteq A_{T(X)}$ be the corresponding Lagrangian subgroup. Since $m = 1$, Proposition 3.3.16 implies that $H_f \subseteq A_{T(X)}$ is the only Lagrangian subgroup. Therefore all derived elliptic structures are of the form $\mathrm{J}^k(X) \to \mathbb{P}^1$ for $k \in \mathbb{Z}$ coprime to t by Theorem 3.5.11.

(ii) By Proposition 3.3.16, $A_{T(X)}$ contains precisely two Lagrangian subgroups. The condition $m = p^k$ implies in particular that $d \not\equiv -1 \pmod{t}$, hence the surface X admits two elliptic fibrations $f : X \to \mathbb{P}^1$ and $g : X \to \mathbb{P}^1$. By Lemma 3.3.17, arguing like in Example 3.5.12, we see that the subgroups of $A_{T(X)}$ induced by the two elliptic fibrations are not equal. Hence H_f and H_g are the only two Lagrangians of $A_{T(X)}$, so every derived elliptic structure on X is either a coprime Jacobian of f or of g by Theorem 3.5.11.

(iii) Assume $\omega(m) \geq 7$. Since $-\mathrm{id} \in G_X$ acts trivially on $\mathrm{L}(A_{T(X)})$ and $|G_X| \leq 66$, by Proposition 3.3.16, the set $\mathrm{L}(A_{T(X)})/G_X$ has cardinality at least $2^{\omega(m)}/33 \geq 128/33$, that is there are at least three elements. The final statement follows from Corollary 3.5.13. □

Corollary 3.5.15. *Assume that* X *is a* T*-general elliptic K3 surface with* $\mathrm{NS}(X) = \Lambda_{d,t}$ *with* $t > 2$, *and let* $m = \gcd(d, t)$. *Then*

$$|\,\mathrm{DE}(X)| = 2^{\omega(m)-1} \cdot \phi(t), \quad |\,\mathrm{DE}(X)/(\mathbb{Z}/t\mathbb{Z})^*| = 2^{\omega(m)}. \tag{3.5.8}$$

In particular, if m is not a power of a prime, then X has Fourier–Mukai partners not isomorphic, as surfaces, to any Jacobian of an elliptic structure on X.

Proof. The second formula in (3.5.8) is an immediate consequence of Theorem 3.5.11, the fact that $G_X = \{\pm\,\mathrm{id}\}$ acts trivially on $\widetilde{\mathrm{L}}(A_{T(X)})$ and the Lagrangian count (3.3.4).

By Proposition 3.4.18, coprime Jacobians of a T-general elliptic K3 surface form $\phi(t)/2$ isomorphism classes. In other words, the orbits of the $(\mathbb{Z}/t\mathbb{Z})^*$-action on $\mathrm{DE}(X)$ are all of size $\phi(t)/2$ and the first formula in (3.5.8) follows from the second one.

The final statement follows from Corollary 3.5.13 because if m is not a power of a prime, $\mathrm{DE}(X)/(\mathbb{Z}/t\mathbb{Z})^*$ has at least four elements by (3.5.8) which thus can not form a single ι-orbit. □

3.5.3 The Zeroth Jacobian

In this subsection, we apply the results of Section 3.5.2 to investigate whether derived equivalent elliptic K3 surfaces have isomorphic zeroth Jacobians. A priori, this is a weaker question than Question 3.1.1. However, we now show that the two questions are equivalent in the very general case. In particular, the answer is negative.

Proposition 3.5.16. *Let $f : X \to \mathbb{P}^1$ be an elliptic K3 surface of Picard rank 2, and write $S := \mathrm{J}^0(X)$. Assume that $T(X)$ has no non-trivial rational Hodge isometries, that is*

$$O_{\mathrm{Hodge}}(T(X)_{\mathbb{Q}}) \simeq \mathbb{Z}/2\mathbb{Z}. \tag{3.5.9}$$

Let (Y, ϕ) be a derived elliptic structure on X such that $S' := \mathrm{J}^0(Y) \simeq S$. Then (Y, ϕ) is isomorphic to a coprime Jacobian of (X, f).

Proof. Fixing any Hodge isometry $T(X) \simeq T(Y)$ we view $T(X) \simeq T(Y) \hookrightarrow T(S')$ as an overlattice of $T(X)$. By assumption there exists a Hodge isometry $\beta^* : T(S') \simeq T(S)$ induced by an isomorphism $\beta : S \simeq S'$. Now β^* induces the rational Hodge isometry

$$T(X)_{\mathbb{Q}} \simeq T(S')_{\mathbb{Q}} \overset{\beta^*_{\mathbb{Q}}}{\simeq} T(S)_{\mathbb{Q}} \simeq T(X)_{\mathbb{Q}}$$

which by assumption equals $\pm\,\mathrm{id}$, hence β^* preserves $T(X)$ as a sublattice of $T(S)$ and $T(S')$. In particular, $\beta_*\alpha_X = k\alpha_Y$ for some $k \in \mathbb{Z}$, hence Y is a coprime Jacobian of X. □

It is well-known that if X is a very general $\Lambda_{d,t}$-polarised elliptic K3 surface then (3.5.9) is satisfied, see e.g. the argument of [SZ20, Lemma 3.9]. Thus, if X is a very general elliptic K3 surface of Picard rank two with two elliptic fibrations, Proposition 3.5.16 allows us to compare the corresponding zeroth Jacobians, which generalises [Gee05, Proposition 4.8].

Corollary 3.5.17. *Let X be an elliptic K3 surface of Picard rank two with $\mathrm{NS}(X) \simeq \Lambda_{d,t}$ and suppose $d \not\equiv \pm 1 \mod t$, so that X admits two non-isomorphic elliptic fibrations by Lemma 3.3.9. Assume (3.5.9) holds for X. Then the zeroth Jacobians of the two elliptic fibrations on X are isomorphic if and only if $\gcd(d, t) = 1$.*

Proof. If $\gcd(d, t) = 1$, the two fibrations on X are coprime Jacobians of each other by Corollary 3.5.14, hence the zeroth Jacobians are isomorphic. If $\gcd(d, t) \neq 1$, then by T-generality of X, the Căldăraru classes of the two fibrations on X are not proportional in $A_{T(X)}$, hence the two fibrations are not coprime Jacobians of each other and the result follows from Proposition 3.5.16. □

Remark 3.5.18. In the setting of Corollary 3.5.17, if zeroth Jacobians are not isomorphic, then they are also not derived equivalent. Indeed, elliptic K3 surfaces with a section do not admit nontrivial Fourier–Mukai partners [Hos+02, Proposition 2.7(3)].

3.5.4 Generalisation to Other Fields

In this subsection, we will use the theory of twisted forms to extend our results to a subfield $k \subset \mathbb{C}$. Let $f : X \to \mathbb{P}^1$ be a complex elliptic K3 surface with $\mathrm{NS}(X) \simeq \Lambda_{d,t}$. Recall that we denote by $\mathrm{Aut}(X, F)$ the group of automorphisms of X which fix the class of the fibre in $\mathrm{NS}(X)$. By Corollary 3.4.25, the group $\mathrm{Aut}(X, F)$ is trivial whenever $t > 2$ and X is T-general.

Let $k \subset L$ be a field extension. An *L-twisted form* of an elliptic K3 surface $(Y, \phi : Y \to C)$ over k is any elliptic K3 surface $(Y', \phi' : Y' \to C')$ over k such that (Y_L, ϕ_L) is isomorphic to (Y'_L, ϕ'_L) as elliptic surfaces.

Lemma 3.5.19. *Let (Y, ϕ) be an elliptic K3 surface over k such that $\mathrm{Aut}(Y_{\mathbb{C}}, F) = \{\mathrm{id}\}$. Then every $\mathbb{C}$-twisted form of (Y, ϕ) is isomorphic to Y as a surface.*

Proof. Any $\mathbb{C}$-twisted form (Y', ϕ') of (Y, ϕ) is also a $\overline{k}$-twisted form of (Y, ϕ) [Mil08, Lemma 16.27]. Thus it suffices to show that for any Galois extension L/k all L-twisted forms of (Y, ϕ) are isomorphic to Y. Let (Y', ϕ') be an L-twisted form of (Y, ϕ), and let $g : Y_L \simeq Y'_L$ be an isomorphism of elliptic surfaces, possibly twisting the base by an automorphism. Then for any $\sigma \in \mathrm{Gal}(L/k)$, the map $h := g \circ (\sigma g)^{-1}$ is an automorphism of Y_L as an elliptic surface.

Using injectivity of the map $\mathrm{Aut}(Y_L) \to \mathrm{Aut}(Y_{\mathbb{C}})$, c.f. [Stacks, Lemma 02VX], and the assumption about automorphisms of $Y_{\mathbb{C}}$, we see that h is the identity, that is g commutes with the Galois action. Therefore g descends to an isomorphism $Y \simeq Y'$ [Mil08, Proposition 16.9]. □

Lemma 3.5.20. *If (X, f) is an elliptic K3 surface over k such that $\rho(X_{\mathbb{C}}) = 2$, then all elliptic fibrations of $X_{\mathbb{C}}$ are induced by elliptic fibrations of X.*

Proof. By Lemma 3.3.9 $X_{\mathbb{C}}$ has one or two elliptic fibrations. If there is only fibration, it must come from the given elliptic fibration f. If there are two elliptic fibrations on $X_{\mathbb{C}}$, they are defined over some Galois extension L/k. Let F and F' be the corresponding divisor classes on X_L. These classes can not be permuted by the Galois group, because one of them corresponds to f, hence is fixed by the Galois group. Thus the other class is also fixed by the Galois group and the corresponding morphism $X \to C$ is defined over k, see e.g. [Lie17, Proposition 2.7, Theorem 3.4(2)]. □

Proposition 3.5.21. *Let X be an elliptic K3 surface over k with $\mathrm{NS}(X_{\mathbb{C}}) \simeq \Lambda_{d,t}$. Assume $\mathrm{Aut}(X_{\mathbb{C}}, F) = \{\mathrm{id}\}$. If d and t are coprime or have only one prime factor in common, then every Fourier–Mukai partner of X is isomorphic, as a surface, to a coprime Jacobian of one of the elliptic fibrations on X.*

Proof. Let Y be a Fourier–Mukai partner of X, and let $\phi : Y \to C$ be an elliptic fibration of Y, which exists by [HT17, Proposition 16]. By Corollary 3.5.14(i, ii), $\phi_{\mathbb{C}} : Y_{\mathbb{C}} \to C_{\mathbb{C}}$ is isomorphic to a coprime Jacobian $\mathrm{J}^k(X_{\mathbb{C}}, f_{\mathbb{C}})$ as elliptic surfaces, for some elliptic fibration $f_{\mathbb{C}}$ on $X_{\mathbb{C}}$. By Lemma 3.5.20, $f_{\mathbb{C}}$ comes from an elliptic fibration f on X, hence (Y, ϕ) is a $\mathbb{C}$-twisted form of $\mathrm{J}^k(X, f)$.

From the description of the automorphism groups given in Proposition 3.4.21 we deduce that

$$\mathrm{Aut}(\mathrm{J}^k(X_{\mathbb{C}}), F) \simeq \mathrm{Aut}(X_{\mathbb{C}}, F)$$

and by assumption this group is trivial. It follows from Lemma 3.5.19 that Y is isomorphic to $\mathrm{J}^k(X)$ as a surface. □

Proposition 3.5.21 implies the following:

Corollary 3.5.22. *Let X be as in Proposition 3.5.21. Let Y be any Fourier–Mukai partner of X. Then X has a k-rational point if and only if Y has a k-rational point.*

Proof. From Proposition 3.5.21, it follows that there is an elliptic fibration $f : X \to C'$ and an integer $\ell \in \mathbb{Z}$ such that $Y \simeq \mathrm{J}^\ell(X, f)$ as surfaces. There is a rational map $X \dashrightarrow \mathrm{J}^\ell(X) \simeq Y$ given by $P \mapsto \ell \cdot P$. By the Lang-Nishimura Theorem [Lan54], [Nis55], it follows that $X(k) \neq \emptyset$ implies $Y(k) \neq \emptyset$. Conversely, since X is also a coprime Jacobian of Y, the same argument shows that $Y(k) \neq \emptyset$ implies $X(k) \neq \emptyset$. $\square$

Appendix to Chapter 3

3.A Counting Elliptic Fourier–Mukai Partners

From the Derived Torelli Theorem 3.2.1, one can derive the following Counting Formula for Fourier–Mukai partners of K3 surfaces:

Theorem 3.A.1 (Counting Formula)**.** *[Hos+02] Let X be a K3 surface, and write* $\mathrm{FM}(X)$ *for the set of isomorphism classes of Fourier–Mukai partners of X. Then*

$$|\,\mathrm{FM}(X)| = \sum_{\Lambda} |O(\Lambda) \setminus O(A_\Lambda)/G_X|,$$

where the sum runs over isomorphism classes of lattices Λ which are in the same genus as the Néron-Severi lattice $\mathrm{NS}(X)$*. Furthermore, each summand computes the number of isomorphism classes of Fourier–Mukai partners Y of X with* $\mathrm{NS}(Y) \simeq \Lambda$.

We now give a relatively quick proof of the Counting Formula.

Proof of Theorem 3.A.1. To keep the notation light, we denote $T \coloneqq T(X)$, we write $\mathrm{Emb}(T)$ for the set of $O(\Lambda_{\mathrm{K3}})$-orbits of embeddings $T \hookrightarrow \Lambda_{\mathrm{K3}}$, and we denote the set of isomorphism classes of Fourier–Mukai partners of X by $\mathrm{FM}(X)$. Note that the Hodge isometries group G_X acts naturally on $\mathrm{Emb}(T)$.

Let $Y \in \mathrm{FM}(X)$ be a Fourier–Mukai partner. By the Derived Torelli Theorem 3.2.1, we may fix a Hodge isometry $f\colon T(Y) \simeq T$. We also fix an isometry $g\colon H^2(Y,\mathbb{Z}) \simeq \Lambda_{\mathrm{K3}}$. This determines the following primitive embedding:

$$i_Y\colon T \overset{f}{\simeq} T(Y) \hookrightarrow H^2(Y,\mathbb{Z}) \overset{g}{\simeq} \Lambda_{\mathrm{K3}},$$

and the element this defines in $\mathrm{Emb}(T)/G_X$ is independent of f and g. With this construction, we obtain a map

$$\begin{aligned} \mathrm{FM}(X) &\longrightarrow \mathrm{Emb}(T)/G_X \\ Y &\longmapsto i_Y. \end{aligned} \tag{3.A.1}$$

We now show that (3.A.1) is a bijection. Note that the surjectivity of (3.A.1) follows immediately from the surjectivity of the period map, see Theorem 2.2.23.

For injectivity, suppose $Y, Y' \in \mathrm{FM}(X)$ have the same image in $\mathrm{Emb}(T)/G_X$. By construction of $\mathrm{Emb}(T)/G_X$ and i_Y, there are a Hodge isometry $T \simeq T$, and an isometry $\Lambda_{\mathrm{K3}} \simeq \Lambda_{\mathrm{K3}}$ such that the following diagram commutes:

$$\begin{tikzcd} T(Y) \arrow[r, "\simeq"] \arrow[d, hook] & T \arrow[r, "\simeq"] \arrow[d, hook, "i_Y"] & T \arrow[d, hook, "i_{Y'}"] & T(Y') \arrow[l, "\simeq"'] \arrow[d, hook] \\ H^2(Y,\mathbb{Z}) \arrow[r, "\simeq"] & \Lambda_{\mathrm{K3}} \arrow[r, "\simeq"] & \Lambda_{\mathrm{K3}} & H^2(Y',\mathbb{Z}). \arrow[l, "\simeq"'] \end{tikzcd} \tag{3.A.2}$$

Since the top horizontal isometries of (3.A.2) are all Hodge isometries, the composition of the bottom horizontal isometries of (3.A.2) is a Hodge isometry $H^2(Y,\mathbb{Z}) \simeq H^2(Y',\mathbb{Z})$. By the Torelli Theorem 2.2.24, this means that there is an isomorphism $Y \simeq Y'$. Therefore, (3.A.1) is injective, hence it is a bijection.

To conclude, recall from Proposition 2.2.5 that we have

$$|\operatorname{Emb}(T)| = \sum_{\Lambda \in \mathcal{G}(\mathrm{NS}(X))} |O(\Lambda) \setminus O(A_\Lambda)|,$$

where for each $\Lambda \in \mathcal{G}(\mathrm{NS}(X))$ the number $|O(\Lambda) \setminus O(A_\Lambda)|$ equals the number of $O(\Lambda_{\mathrm{K3}})$-orbits of primitive embeddings $T \hookrightarrow \Lambda_{\mathrm{K3}}$ for which $T^\perp \simeq \Lambda$. From this, it follows that we have

$$|\operatorname{Emb}(T)/G_X| = \sum_{\Lambda \in \mathcal{G}(\mathrm{NS}(X))} |O(\Lambda) \setminus O(A_\Lambda)/G_X|,$$

as required. □

When we started the project that led to [MS24], which this chapter is based on, the original strategy for answering Question 3.1.1 was to use the Counting Formula to count Fourier–Mukai partners and to then compare that to the number of coprime Jacobians.

Eventually, with the use of Ogg–Shafarevich Theory, we arrived at the definition of a Hodge elliptic structure, and found a method to count those efficiently. Recall from Corollary 3.5.15 that we were able to count derived elliptic structures for T-general elliptic K3 surfaces of Picard rank 2. Using Corollary 3.5.15, we can also compute the number of Fourier–Mukai partners. This means that we computed the outcome of the Counting Formula, without actually computing the terms within the Counting Formula itself.

However, the first part of my PhD was spent computing the terms in the Counting Formula. At first, I computed the terms of the Counting Formula for $\Lambda_{d,t}$ with d and t coprime. The outcome was that every Fourier–Mukai partner was a Jacobian in this case. Of course, we would later use Hodge elliptic structures to show that the coprime case is rather unique with this property. In this section, I share my computations in the coprime case.

We now fix coprime integers $d \geq 0$ and $t > 2$, and consider the lattice $\Lambda_{d,t}$. Recall the following result by Van Geemen:

Proposition 3.A.2 ([Gee05, Remark 4.7])**.** *Let X be a K3 surface with $\mathrm{NS}(X) = \Lambda_{d,t}$. Then X admits two elliptic fibrations up to isomorphism if $d \not\equiv \pm 1 \mod t$. On the other hand, if $d \equiv \pm 1 \mod t$, X admits one fibration up to isomorphism.*

We combine Proposition 3.A.2 with the Counting Formula to count the number derived elliptic structures of X.

Corollary 3.A.3. *Let $X \to \mathbb{P}^1$ be an elliptic K3 surface of Picard rank 2. Then*

$$|\operatorname{DE}(X)| = \sum_{\Lambda \in \mathcal{G}(Y)} F_\Lambda \cdot |O(\Lambda) \setminus O(A_\Lambda)/G_X|,$$

where F_Λ is the number of elliptic fibrations on a K3 surface Y with $\mathrm{NS}(Y) \simeq \Lambda$ (this number depends only on Λ by Proposition 3.A.2).

Lemma 3.A.4. *[Gee05, Remark 4.7] We have*

$$F_{\Lambda_{d,t}} = \begin{cases} 1 & \text{if } d \equiv \pm 1 \pmod t \\ 2 & \text{if } d \not\equiv \pm 1 \pmod t. \end{cases}$$

To compute the sizes of the double quotients in the Counting Formula, we need to investigate the group of isometries of the discriminant lattices of $\Lambda_{d,t}$.

Lemma 3.A.5. *[Ste04] Let H, F denote the standard basis for the lattice $\Lambda_{d,t}$. Then the discriminant lattice $A_{d,t}$ of $\Lambda_{d,t}$ is cyclic of order t^2, and is generated by the element*

$$x = \frac{tH - 2dF}{t^2} \in A_{d,t},$$

which satisfies

$$x^2 = \frac{-2d}{t^2} \in \mathbb{Q}/\mathbb{Z}.$$

Lemma 3.A.6. *We have $O(A_{d,t}) \cong (\mathbb{Z}/2\mathbb{Z})^{\omega(t)}$, where $\omega(t)$ is the number of distinct primes dividing t.*

Proof. Since $A_{d,t}$ is cyclic of order t^2, an isometry $f : A_{d,t} \simeq A_{d,t}$ is given by multiplication with some integer $r \in \mathbb{Z}$ coprime to t^2. The fact that f is an isometry implies that

$$x^2 = f(x)^2 = r^2x^2 \pmod{\mathbb{Z}},$$

hence $r^2 \equiv 1 \pmod{t^2}$. Modulo t^2, there are exactly $2^{\omega(t)}$ choices for r, and f has order 2 in $O(A_{d,t})$, hence $O(A_{d,t}) \simeq (\mathbb{Z}/2\mathbb{Z})^{\omega(t)}$. □

Remark 3.A.7. Since $\Lambda_{d,t}$ contains at most two primitive isotropic vectors up to sign, it follows that it admits at most four isometries. Suppose $d^2 \equiv 1 \pmod t$. Then the matrix

$$M := \begin{pmatrix} -d & -t \\ \frac{d^2-1}{t} & d \end{pmatrix}$$

defines an isometry of $\Lambda_{d,t}$.

Lemma 3.A.8. *[Gee05, Lemma 4.6] For $d \in \mathbb{Z}$, $t > 2$ coprime integers, we have*

$$O(\Lambda_{d,t}) = \begin{cases} \{\pm \mathrm{id}\} & \text{if } d^2 \not\equiv 1 \pmod t \\ \{\pm \mathrm{id}, \pm M\} & \text{if } d^2 \equiv 1 \pmod t. \end{cases}$$

Lemma 3.A.9. *We have*

$$\operatorname{Ker}\left(O(\Lambda_{d,t}) \to O(A_{d,t})\right) = \begin{cases} 0 & \text{if } d \not\equiv \pm 1 \mod t \\ \mathbb{Z}/2\mathbb{Z} & \text{if } d \equiv \pm 1 \mod t. \end{cases}$$

Proof. By assumption, we have $t > 2$. This implies that $-\mathrm{id}_{A_{T(S)}} \neq \mathrm{id}_{A_{T(S)}}$, hence we have $-\mathrm{id}_{T(S)} \notin \operatorname{Ker}\left(O(\Lambda_{d,t}) \to O(A_{d,t})\right)$. Recall that $A_{d,t}$ is cyclic, and generated by

$$x = \frac{tH - 2dF}{t^2}.$$

We note that
$$(M-I)(x) = \frac{d-1}{t}H + \frac{(d-1)^2}{t^2}F,$$
which is contained in $\Lambda_{d,t}$ if and only if $d \equiv 1 \pmod{t}$. This shows that M maps to the identity on $A_{d,t}$ if and only if $d \equiv 1 \pmod{t}$. Similarly, $-M$ maps to the identity if and only if $d \equiv -1 \pmod{t}$. □

Proposition 3.A.10. *We have*
$$|O(\Lambda_{d,t}) \setminus O(A_{d,t})/G_X| = \begin{cases} 2^{\omega(t)-1} & \text{if } d^2 \not\equiv 1 \mod t \text{ or } d \equiv \pm 1 \mod t \\ 2^{\omega(t)-2} & \text{if } d^2 \equiv 1 \mod t \text{ and } d \not\equiv \pm 1 \mod t. \end{cases}$$
Moreover, we have
$$F_{\Lambda_{d,t}} \cdot |O(\Lambda_{d,t}) \setminus O(A_{d,t})/G_X| = \begin{cases} 2^{\omega(t)-1} & \text{if } d^2 \equiv 1 \pmod{t} \\ 2^{\omega(t)} & \text{if } d^2 \not\equiv 1 \pmod{t}. \end{cases}$$

Proof. By T-generality of X, we have $G_X = \{\pm \mathrm{id}\}$. From Lemma 3.A.9, it follows that the image of $O(\Lambda_{d,t})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if $e^2 \not\equiv 1 \mod t$ or $e \equiv \pm 1 \mod t$, and otherwise the image is $(\mathbb{Z}/2\mathbb{Z})^2$. Combining this with Lemma 3.A.6 and Lemma 3.A.4 gives the desired result. □

Lemma 3.A.11. *[Ste04, Lemma 3.2] If Λ is a lattice in the same genus as $\Lambda_{d,t}$, then there is a $k \in \mathbb{Z}$ coprime to t such that $\Lambda \simeq \Lambda_{dk^2,t}$. Moreover, for any $d, e \in \mathbb{Z}$, there is an isometry $\Lambda_{d,t} \simeq \Lambda_{e,t}$ if and only if $d \equiv e^{\pm 1} \pmod{t}$.*

Theorem 3.A.12. *Let X be a T-general K3 surface with $\mathrm{NS}(X) \simeq \Lambda_{d,t}$, where d, t are coprime integers, and $t > 2$. Then*
$$|\operatorname{DE}(X)| = \frac{\phi(t)}{2}.$$

Proof. We denote
$$\mathcal{G}' := \left\{\Lambda_{e,t} \in \mathcal{G}(\Lambda_{d,t}) \mid e^2 \equiv 1 \pmod{t}\right\} \subset \mathcal{G}(\Lambda_{d,t}).$$

Let N be the number of elements $e \in (\mathbb{Z}/t\mathbb{Z})^\times$ such that $\Lambda_{e,t} \in \mathcal{G}'$. Since we have $e \equiv e^{-1} \pmod{t}$ for such elements, we have $|\mathcal{G}'| = N$. We do not need to know the exact value of N for our computations, but it will appear in the formulas.

Note that
$$|\left\{dk^2 \mid k \in (\mathbb{Z}/t\mathbb{Z})^\times\right\}| = \frac{\phi(t)}{2^{\omega(t)}}.$$
Therefore, the number of elements $e \in (\mathbb{Z}/t\mathbb{Z})^\times$ such that $\Lambda_{e,t} \in \mathcal{G}(\Lambda_{d,t}) \setminus \mathcal{G}'$ equals $\frac{\phi(t)}{2^{\omega(t)}} - N$. However, since $e \not\equiv e^{-1} \pmod{t}$ for these elements, we have
$$|\mathcal{G}(\Lambda_{d,t}) \setminus \mathcal{G}'| = \frac{1}{2} \cdot \left(\frac{\phi(t)}{2^{\omega(t)}} - N\right)$$
We now combine Corollary 3.A.3, Proposition 3.A.10, and Lemma 3.A.11 to compute:
$$\begin{aligned} |\operatorname{DE}(X)| &= |\mathcal{G}(\Lambda_{d,t}) \setminus \mathcal{G}'| \cdot 2^{\omega(t)} + |\mathcal{G}'| \cdot 2^{\omega(t)-1} \\ &= \tfrac{1}{2} \cdot \left(\tfrac{\phi(t)}{2^{\omega(t)}} - N\right) \cdot 2^{\omega(t)} + N \cdot 2^{\omega(t)-1} = \tfrac{\phi(t)}{2}. \end{aligned}$$
□

Note that Theorem 3.A.12 gives the same number as Corollary 3.5.15. In particular, by computing the number of Jacobian derived elliptic structures of X as in Section 3.4, Theorem 3.A.12 also allows us to conclude that a K3 surface X satisfying the assumptions of Theorem 3.A.12 only has Jacobian derived elliptic structures.

There are a few drawbacks to this method of proving the result, though. Firstly, the computations get even more complicated when d and t are not coprime. moreover, this strategy is not capable of dealing with T-special K3 surfaces, whereas the action of G_X is nicely incorporated in the proof of Corollary 3.5.13. Lastly, this strategy is not sufficient to say anything about the zeroth Jacobian of X, which is something that we could accomplish in Section 3.5.3.

3.B Jacobian Moduli Maps

We set up the theory of Jacobian moduli maps. These are maps between moduli spaces of lattice polarised K3 surfaces that associate to an elliptic K3 surface its Jacobian. Recall that we call a K3 surface T-general if the only Hodge isometries of the transcendental lattice are $\pm\,\mathrm{id}$, see Definition 3.2.4. If $X \to \mathbb{P}^1$ is an elliptic K3 surface, and $S := \mathrm{J}^0(X) \to \mathbb{P}^1$ is its zeroth Jacobian, it is a natural question to ask whether T-generality of X implies T-generality of S. We use the theory of Jacobian moduli maps to show that for a very general choice of X, S will be T-general. It is currently unclear whether there exist T-general elliptic K3 surfaces for which the zeroth Jacobian is T-special.

Before we discuss Jacobian moduli maps, we consider a Hodge theoretic approach to the above question. Write G_X and G_S for the groups of Hodge isometries of $T(X)$ and $T(S)$, respectively. We consider the subgroup $H_X \subset G_X$ of Hodge isometries $f\colon T(X) \simeq T(X)$ whose action on $A_{T(X)}$ preserves the subgroup $T(S)/T(X) \subset A_{T(X)}$. Note that we have $\pm\,\mathrm{id} \in H_X$. We also define $H_S \subset G_S$ to be the group of Hodge isometries $g\colon T(S) \simeq T(S)$ satisfying $g(T(X)) = T(X)$. Again, we have $\pm\,\mathrm{id} \in H_S$.

Lemma 3.B.1. *We have an isomorphism*

$$\begin{array}{ccc} H_S & \xrightarrow{\sim} & H_X \\ g & \longmapsto & g|_{T(X)}. \end{array} \tag{3.B.1}$$

Proof. Note that (3.B.1) is well-defined, by Lemma 2.2.13. We first prove surjectivity. Suppose $f \in H_X$. By assumption, f preserves $T(S)/T(X) \subset A_{T(X)}$, therefore there exists a Hodge isometry $\tilde{f}\colon T(S) \simeq T(S)$ such that $\tilde{f}|_{T(X)} = f$ by Lemma 2.2.13. This means that $\tilde{f}$ is a preimage of f along (3.B.1). To prove the injectivity of (3.B.1), suppose $g \in H_S$ satisfies $g|_{T(X)} = \mathrm{id}_{T(X)}$. Then $g_{\mathbb{Q}} : T(S)_{\mathbb{Q}} \simeq T(S)_{\mathbb{Q}}$ is the identity, hence g is the identity. □

Lemma 3.B.1 is not enough to prove that T-generality for X is equivalent to T-generality for S, since H_X and H_S may be proper subgroups of G_X and G_S. However, it does give us a criterion for when T-specialness for X implies T-specialness for S.

3.B.1 Maps between Moduli Spaces of K3 Surfaces

We begin by recalling the constructions of moduli spaces of lattice-polarised K3 surfaces.

Let $\Lambda_{\mathrm{K3}} = U^3 \oplus E_8(-1)^2$ denote the K3 lattice. We fix an even, non-degenerate, primitive sublattice $\Lambda \subset \Lambda_{\mathrm{K3}}$ of signature $(1, \rho - 1)$, and denote its orthogonal complement by $T := \Lambda^{\perp}$. For simplicity, we assume that there is a unique $O(\Lambda_{\mathrm{K3}})$-orbit of primitive embeddings $\Lambda \hookrightarrow \Lambda_{\mathrm{K3}}$. This is true, for example, when $\rho \leq 10$ [Nik80, Theorem 1.14.4].

Definition 3.B.2. A Λ-*marked K3 surface* is a pair (X, ϕ) where X is a K3 surface and $\phi : H^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda_{\mathrm{K3}}$ is an isometry such that $\phi^{-1}(\Lambda) \subset \mathrm{NS}(X)$. We call (X, ϕ) an *ample* Λ-*marked K3 surface* if $\phi^{-1}(\Lambda) = \mathrm{NS}(X)$.

We write Ω_T for one of the two connected components of

$$\left\{\sigma \in \mathbb{P}(T \otimes \mathbb{C}) \mid \sigma^2 = 0,\ \sigma \cdot \overline{\sigma} > 0\right\}. \tag{3.B.2}$$

This is an open subset of a quadric in $\mathbb{P}(T \otimes \mathbb{C})$. We consider the subgroup $O^+(T)$ of isometries $f : T \simeq T$ which preserve the connected components of (3.B.2).

We denote

$$\Gamma_T := \tilde{O}^+(T) = \mathrm{Ker}\left(O^+(T) \to O(A_T)\right),$$

where A_T is the discriminant lattice of T. The quotient

$$\mathcal{F}_T := \Omega_T / \Gamma_T$$

is a quasi-projective variety. It is the coarse moduli space of Λ-marked K3 surfaces, that is, the moduli space of pairs (X, ϕ) where X is a K3 surface and $\phi : \Lambda \hookrightarrow \mathrm{NS}(X)$ is a primitive embedding.

We denote $\tilde{\Lambda}_{\mathrm{K3}} = U \oplus \Lambda_{\mathrm{K3}}$. We fix a primitive, isotropic vector $v \in \Lambda \oplus U$. Write $v = (r, \ell, s)$ for some $r, s \in \mathbb{Z}$ and $\ell \in \Lambda$. The vector $e = (0, 0, 1)$ is primitive and isotropic. We fix an isometry $f : \tilde{\Lambda}_{\mathrm{K3}} \to \tilde{\Lambda}_{\mathrm{K3}}$ which maps v to e. Such an isometry exists by Lemma 2.2.9. Then f induces an isometry

$$\overline{f} : v^{\perp}/\mathbb{Z}v \to e^{\perp}/\mathbb{Z} \cdot e \cong \Lambda_{\mathrm{K3}}.$$

We denote

$$\Lambda' := \left(v^{\perp} \cap (\Lambda \oplus U)\right) / \mathbb{Z}v$$

and consider it as a sublattice of Λ_{K3} via $\overline{f}$. We write $T' := \Lambda'^{\perp} \subset \Lambda_{\mathrm{K3}}$.

Our goal is to construct a holomorphic map $\mathcal{F}_T \to \mathcal{F}_{T'}$ using the vector v. Maps such as these have been studied already in [Ste08], and [Kon93].

Let (X, ϕ) be a Λ-marked K3 surface. This means that we have a commutative diagram:

$$\begin{tikzcd} H^2(X, \mathbb{Z}) \arrow[r, "\phi"] & \Lambda_{\mathrm{K3}} \\ \mathrm{NS}(X) \arrow[u, hook] & \Lambda \arrow[l, "\phi^{-1}"] \arrow[u, hook] \end{tikzcd} \tag{3.B.3}$$

Diagram (3.B.3) induces the diagram:

$$\begin{tikzcd} \tilde{H}(X,\mathbb{Z}) \arrow[r, "\tilde{\phi}"] & \tilde{\Lambda}_{\mathrm{K3}} \\ \mathrm{N}(X) \arrow[u, hook] & \Lambda \oplus U, \arrow[l, "\tilde{\phi}^{-1}"] \arrow[u, hook] \end{tikzcd} \tag{3.B.4}$$

where $\mathrm{N}(X) = \mathrm{NS}(X) \oplus U$ is the extended Néron-Severi lattice of X. The horizontal isometry $\tilde{\phi}$ of Diagram (3.B.4) is obtained by extending ϕ by the identity on U. Denote by $w \in \tilde{H}(X,\mathbb{Z})$ the vector which is mapped to v by $\tilde{\phi}$. Diagram (3.B.4) induces the diagram

$$\begin{tikzcd} w^{\perp}/\mathbb{Z}\cdot w \arrow[r, "\overline{\phi}"] & v^{\perp}/\mathbb{Z}v \arrow[r, "\overline{f}"] & \Lambda_{\mathrm{K3}} \\ w^{\perp}_{N(X)}/\mathbb{Z}\cdot w \arrow[u, hook] & \Lambda'. \arrow[l, "\overline{\phi}^{-1}"] \arrow[u, hook] & \end{tikzcd} \tag{3.B.5}$$

If we assume that v satisfies the conditions in [Muk87, Proposition 6.2], it follows that the moduli space $M(w)$ is again a K3 surface. Moreover, $M(w)$ comes equipped with a Hodge isometry $H^2(M(w),\mathbb{Z}) \to w^{\perp}/\mathbb{Z}\cdot w$. In particular, Diagram (3.B.5) turns $M(w)$ into a marked ample Λ'-marked K3 surface. Explicitly, the marking on $M(w)$ is given by $\overline{f} \circ \overline{\phi}$. It follows that the point in $\Omega_{T'}$ corresponding to $(M(w), \overline{f} \circ \overline{\phi})$ is $\overline{f}_{\mathbb{C}}(\omega)$, where ω is the point of Ω_T corresponding to (X,ϕ).

In light of the above discussion, we define the map $M_v : \Omega_T \to \Omega_{T'}$ by

$$M_v(\sigma) = \overline{f}_{\mathbb{C}}(\sigma).$$

Of course, M_v depends on the chosen isometry f. We need to check that M_v descends to a map $\overline{M}_v : \mathcal{F}_T \to \mathcal{F}_{T'}$. To check this, we construct a group homomorphism $\Gamma_T \to \Gamma_{T'}$, similarly to the strategy in [Kon93].

Recall that we may view T' as an overlattice of T via the isometry $v^{\perp}/\mathbb{Z}v \simeq \Lambda_{\mathrm{K3}}$. Now any isometry $\gamma \in \Gamma_T$ acts trivially on A_T. Therefore, γ preserves the subgroup $T'/T \subset A_T$, and can be extended to an isometry $\tilde{\gamma} : T' \simeq T'$. Moreover, $\tilde{\gamma}$ is the unique extension of γ to T' by an argument similar to the proof of injectivity of (3.B.1).

It is not hard to see that $\tilde{\gamma}$ acts as the identity on T'. To be more precise, write $H \coloneqq T'/T \subset A_T$, then there is an isometry $H^{\perp}/H \simeq A_{T'}$. Since $\tilde{\gamma}$ acts as the identity on A_T, it also acts as the identity on $H^{\perp} \subset A_T$, and therefore also on $H^{\perp}/H \simeq A_{T'}$. This shows that there is a group homomorphism $\Gamma_T \to \Gamma_{T'}$.

The final step in the construction of the map $\overline{M}_v$ is to show that the holomorphic map $M_v : \Omega_T \to \Omega_{T'}$ is equivariant with respect to our group homomorphism $\Gamma_T \to \Gamma_{T'}$. That is, we check that for any $\sigma \in \Omega_T$ and $\gamma \in \Gamma_T$, we have

$$M_v(\gamma(\sigma)) = \overline{\gamma}(M_v(\sigma)).$$

This follows from:

$$\begin{aligned} M_v(\gamma(\sigma)) &= \overline{f}_{\mathbb{C}}(\gamma(\sigma)) \\ &= \overline{\gamma}(\overline{f}_{\mathbb{C}}(\sigma)) \\ &= \overline{\gamma}(M_v(\sigma)). \end{aligned}$$

The above discussion proves:

Proposition 3.B.3. *Let Λ, Λ' and v be as above. Then the map $\overline{M}_v : \mathcal{F}_T \to \mathcal{F}_{T'}$ is well-defined and holomorphic.*

3.B.2 Jacobian Moduli Maps

We now apply the ideas of Section 3.B.1 to a specific choice for the Mukai vector v. We will obtain a holomorphic map which assigns to an elliptic K3 surface its zeroth Jacobian.

Let $\Lambda_{d,t}$ be the rank 2 lattice $\mathbb{Z}H \oplus \mathbb{Z}F$ with quadratic form

$$\begin{pmatrix} 2d & t \\ t & 0 \end{pmatrix}.$$

We fix an inclusion $\Lambda_{d,t} \subset \Lambda_{\mathrm{K3}}$, and write $T_{d,t} \coloneqq \Lambda_{d,t}^{\perp}$. We denote by $\mathcal{F}_{d,t} = \mathcal{F}_{T_{d,t}}$ the coarse moduli space of $\Lambda_{d,t}$-marked K3 surfaces. Let (X, ϕ) be an *ample* $\Lambda_{d,t}$-marked K3 surface. This means that X comes equipped with an isometry $\phi^{-1} : \mathrm{NS}(X) \simeq \Lambda_{d,t}$, and hence with a primitive isotropic vector $\phi(F)$. The class $\phi^{-1}(F)$ determines an elliptic fibration $f : X \to \mathbb{P}^1$. The multisection index of f is equal to t. The associated relative Jacobian fibration $\mathrm{J}^0(X) \to \mathbb{P}^1$ has a section. Therefore we have $\mathrm{NS}(\mathrm{J}^0(X)) \cong \Lambda_{0,1} = U$. Moreover, we have $\mathrm{J}^0(X) \cong M_v(X)$ where

$$v = (0, F, 0) \in \tilde{\Lambda}_{\mathrm{K3}}.$$

More generally, for any $k \in \mathbb{Z}$ we have $\mathrm{J}^k(X) \cong M_{v_k}(X)$, where $v_k = (0, F, k) \in \tilde{H}(X, \mathbb{Z})$.

It follows from Proposition 3.B.3 that there is a holomorphic map $\mathrm{J}^0 : \mathcal{F}_{d,t} \to \mathcal{F}_{0,1}$ which maps X to its Jacobian $\mathrm{J}^0(X)$.

Proposition 3.B.4. *The map $\mathrm{J}^0 : \mathcal{F}_{d,t} \to \mathcal{F}_{0,1}$ constructed above is dominant and has finite fibres.*

Proof. The fact that the map is dominant may be checked on the level of period domains. It is clear from the construction of M_v that it is dominant for any choice of v. Now fix any ample U-marked K3 surface (S, ϕ). There are only finitely many elliptic K3 surfaces $X \to \mathbb{P}^1$ for which $\mathrm{NS}(X) \simeq \Lambda_{d,t}$ and for which $\mathrm{J}^0(X) \simeq S$. Indeed any such elliptic K3 surface induces an element of $Ш(S)$ of order t, and since $Ш(S) \simeq \mathrm{Br}(S) \simeq (\mathbb{Q}/\mathbb{Z})^{20}$, there are only finitely many such $X \to \mathbb{P}^1$. For any such $X \to \mathbb{P}^1$, there are only finitely many ample $\Lambda_{d,t}$-markings $\Lambda_{d,t} \simeq \mathrm{NS}(X)$, since $O(\Lambda_{d,t})$ is finite. □

As a result of Proposition 3.B.4, we obtain the following corollary.

Corollary 3.B.5. *Let $f : X \to \mathbb{P}^1$ be a very general elliptic K3 surface with $\mathrm{NS}(X) \cong \Lambda_{d,t}$. Then $\mathrm{J}^0(X)$ is T-general.*

Proof. The very general elliptic K3 surface of Picard rank 2 with a section, i.e. a very general point of $\mathcal{F}_{0,1}$ is T-general. The result now follows from Proposition 3.B.4. □

3.C Twisted Elliptic K3 Surfaces

Recall from Section 2.2.4 that for a twisted K3 surface (X, α), we can define the derived category of α-twisted coherent sheaves $\mathcal{D}^b(X, \alpha) := \mathcal{D}^b(\mathbf{Coh}(X, \alpha)$.

Definition 3.C.1. We say that two K3 surfaces X, Y are *twisted derived equivalent* if there exist Brauer classes $\alpha \in \mathrm{Br}(X)$, $\beta \in \mathrm{Br}(Y)$ such that there exists an equivalence $\mathcal{D}^b(X, \alpha) \simeq \mathcal{D}^b(Y, \beta)$.

In this chapter, we saw that two derived equivalent elliptic K3 surfaces are not necessarily coprime Jacobians of one another. However, the following question is still open:

Question 3.C.2. Suppose X and Y are twisted derived equivalent K3 surfaces. Suppose X admits an elliptic fibration. Are there derived elliptic structures $(X', f) \in \mathrm{DE}(X)$ and $(Y', g) \in \mathrm{DE}(Y)$ such that $\mathrm{J}^e(Y', g) \simeq \mathrm{J}^d(X', f)$ for some $d, e \in \mathbb{Z}$?

We will see in Example 3.C.4 that the answer to this question is negative, but before we discuss this question let us note that it is a sensible one. Firstly, note that our negative answer to Question 3.1.1 is not enough to answer Question 3.C.2 negatively. Indeed, if X and Y are (non-twisted) derived equivalent, but Y is not isomorphic to a coprime Jacobian of X, then we may still choose $Y' = X$ and $X' = X$. In particular, if we remove the word *twisted* in Question 3.C.2, then the answer to the question is positive.

We now prove that the property of admitting an elliptic fibration is invariant under twisted derived equivalences.

Proposition 3.C.3. *Suppose X and Y are twisted derived equivalent K3 surfaces. Then X admits an elliptic fibration if and only if Y admits an elliptic fibration.*

Proof. Let $\alpha \in \mathrm{Br}(X)$ and $\beta \in \mathrm{Br}(Y)$ be Brauer classes for which there exists an equivalence $\mathcal{D}^b(X, \alpha) \simeq \mathcal{D}^b(Y, \alpha)$. Then, by [HS05b, Proposition 4.3], there is a Hodge isometry $T(X, \alpha) \simeq T(Y, \beta)$. By Proposition 2.2.46, this means that there is a rational Hodge isometry $T(X)_{\mathbb{Q}} \simeq T(Y)_{\mathbb{Q}}$. By the Witt Extension Theorem 2.2.14, there exists a rational isometry $\mathrm{NS}(X)_{\mathbb{Q}} \simeq \mathrm{NS}(Y)_{\mathbb{Q}}$, hence $\mathrm{NS}(X)$ represents zero if and only if $\mathrm{NS}(Y)$ represents zero. The result follows from Proposition 3.3.3. □

Example 3.C.4. Let X be a K3 surface with $\mathrm{NS}(X) \simeq \Lambda_{9,27}$. Since $\gcd(9, 27) = 9$, $A_{9,27}$ has 2 Lagrangian subgroups. They are generated by the Lagrangian elements

$$\overline{v} = \frac{1}{27}F, \qquad \overline{v'} = \frac{1}{9}H - \frac{1}{27}F,$$

respectively. The vector $\frac{1}{3}H \in A_{9,27}$ is isotropic, but not contained in either of the two Lagrangian subgroups. Denote by T the overlattice of $T(X)$ corresponding to $\frac{1}{3}H$. There is a K3 surface Y for which there exists a Hodge isometry $T(Y) \simeq T$ by Lemma 3.C.5 below. However, since $\frac{1}{3}H$ is not contained in any Lagrangian subgroup, it follows that Y is not isomorphic to a Jacobian of a derived elliptic structure on X.

Example 3.C.4 gives a counterexample to Question 3.C.2, as we explain now. Firstly, note that since the index of the overlattice $T(X) \hookrightarrow T(Y)$ is 3, the discriminant lattice $A_{T(Y)}$ has order $9 = 3^2$. Therefore, the multisection index of any elliptic fibration of Y is 3 by Corollary 3.3.10. Therefore Y does not have any non-trivial Fourier–Mukai partners. This follows from Corollary 3.5.15, as we now explain.

Note that $\mathrm{NS}(Y) \simeq \Lambda_{d,3}$, where $d = 0$ or $d = 1$. If $d = 0$, then $|\,\mathrm{DE}(Y)| = \phi(3) = 2$, but Y admits two non-isomorphic elliptic fibrations, hence Y has no Fourier–Mukai partners (if Y is T-special, it is possible that $|\,\mathrm{DE}(Y)| = 1$ and Y admits 1 elliptic fibration up to isomorphism). On the other hand, if $d = 1$, then $|\,\mathrm{DE}(Y)| = 1$, so that Y also has no Fourier–Mukai partners in this case.

Note that $\mathrm{J}^e(Y, g) \simeq (Y, g)$ for all elliptic fibrations $g : Y \to \mathbb{P}^1$ and all $e \in \mathbb{Z}$ not divisible by 3, since the multisection index of g is 3. Since (Y, g) is not a Jacobian of any Fourier–Mukai partner of X, this means that we have found a counterexample to Question 3.C.2.

Lemma 3.C.5. *[Ma10, Proof of Proposition 3.3] Let X be a K3 surface, and let $T(X) \hookrightarrow T$ be an overlattice. Then there exists a K3 surface Y for which there is a Hodge isometry $T(Y) \simeq T$.*

Proof. The isotropic subgroup $T/T(X) \subset A_{T(X)} \simeq A_{\mathrm{NS}(X)}(-1)$ induces an overlattice $\mathrm{NS}(S) \hookrightarrow N$. It is not hard to check that there is an isometry $A_N \simeq A_T(-1)$, and this induces an embedding $T \oplus N \hookrightarrow \Lambda_{\mathrm{K3}}$. The result follows from the surjectivity of the period map, see Theorem 2.2.23. $\square$

Chapter 4

Beauville–Mukai Systems

4.1 Introduction

We discuss applications of our computation of the obstruction class for a moduli space of sheaves on a K3 surface from Chapter 2. We mostly consider Beauville–Mukai systems, which are higher-dimensional analogues of elliptic K3 surfaces. We explore how some of the concepts from Chapter 3 translate to this higher-dimensional setting.

In our study of Fourier–Mukai partners of elliptic K3 surfaces in Chapter 3, the three main tools used in our strategy were:

i) The Derived Torelli Theorem for K3 surfaces (see Theorem 3.2.1).

ii) Căldăraru's computation of the obstruction class of a two-dimensional moduli space (see Theorem 2.4.8).

iii) Ogg–Shafarevich Theory (see Theorem 3.4.4).

To study Fourier–Mukai partners of Beauville–Mukai systems, it is important to generalise each of i), ii) and iii) to the higher-dimensional setting. In Chapter 2, we computed the obstruction class of a higher-dimensional moduli space of sheaves on a K3 surface. This dealt with ii). In this chapter, we make progress on i) and iii).

First we consider Ogg–Shafarevich theory for elliptic K3 surfaces. For an elliptic K3 surface with a section $S \to \mathbb{P}^1$, one can consider the Tate–Shafarevich group $Ш(S)$ consisting of elliptic K3 surfaces $X \to \mathbb{P}^1$ with a fixed isomorphism between the Jacobian fibration $\mathrm{J}^0(X) \to \mathbb{P}^1$ and $S \to \mathbb{P}^1$. Such elliptic K3 surfaces are called *torsors* of $S \to \mathbb{P}^1$. The Tate–Shafarevich group of an elliptic K3 surface with a section is naturally isomorphic to its Brauer group, with the isomorphism being given by sending a torsor $X \to \mathbb{P}^1$ to the obstruction to the existence of a universal sheaf on the product $X \times S$ by [Căl00, Theorem 4.4.1]. This description of the isomorphism

$$Ш(S) \simeq \mathrm{Br}(S) \tag{4.1.1}$$

was important in Chapter 3 in our study of Fourier–Mukai partners of elliptic K3 surfaces.

Tate–Shafarevich groups were recently generalised to the higher-dimensional case [Mar14; AR23; HM23]. For our applications, we rely on the new theory of Tate–Shafarevich groups introduced in [HM23].

For S a K3 surface, and $\mathcal{C} \to |H|$ a complete, generically smooth linear system with $H^2 = 2g - 2 \geq 0$, we may consider the relative Picard varieties $\overline{\mathrm{Pic}}^d(\mathcal{C}/|H|) \coloneqq M_L(0, H, d + 1 - g)$, where $d \in \mathbb{Z}$, and $L \in \mathrm{NS}(S)$ is a generic polarisation. These relative Picard varieties are called *Beauville–Mukai systems*, and we usually denote them by $\overline{\mathrm{Pic}}^d$ to keep notation light. For $d = 0$, the generic fibre $\overline{\mathrm{Pic}}^0_\eta$ is an abelian variety, and the Tate–Shafarevich group $\text{Ш}(S, H)$, introduced in [HM23], parametrises those torsors of $\overline{\mathrm{Pic}}^0_\eta$ which compactify to moduli spaces of (twisted) sheaves on S.

We give a different definition of the Tate–Shafarevich group than the one found in [HM23], and then show that it is equivalent to the definition given in [HM23]. The advantage of our definition is that it defines an element of the Tate–Shafarevich group to be a torsor of the generic fibre $\overline{\mathrm{Pic}}^0_\eta$ satisfying some conditions, which is more in line with the original construction of a Tate–Shafarevich group of an elliptic curve or an elliptic surface.

We prove that for any $d \in \mathbb{Z}$ there is a short exact sequence

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \text{Ш}(S, H) \to \mathrm{Br}(\overline{\mathrm{Pic}}^d) \to 0,$$

where

$$n = \frac{\mathrm{div}(H)}{\gcd(\mathrm{div}(H), d + 1 - g)}.$$

Somewhat surprisingly, the morphism $\text{Ш}(S, H) \to \mathrm{Br}(\overline{\mathrm{Pic}}^0)$ is generally not an isomorphism, but instead has a cyclic kernel of order 2 if the Picard rank of S is 1. This is surprising because it contrasts with the isomorphism (4.1.1) for elliptic K3 surfaces. On the other hand, we have an isomorphism $\text{Ш}(S, H) \simeq \mathrm{Br}(\overline{\mathrm{Pic}}^{g-1})$. This isomorphism may be seen as a higher-dimensional analogue of (4.1.1). Indeed, for an elliptic K3 surface $S \to \mathbb{P}^1$ with fibre class $F \in \mathrm{NS}(S)$, we have $\text{Ш}(S, F) \simeq \text{Ш}(S)$ by [HM23, §5.1]. In this case, we have $F^2 = 2g - 2 = 0$, hence $g - 1 = 0$. This means that we have an equality $\overline{\mathrm{Pic}}^{g-1} = \overline{\mathrm{Pic}}^0$, and $\overline{\mathrm{Pic}}^0 = \mathrm{J}^0(S)$ by definition. Moreover, since S is an elliptic K3 surface with a section, we have $\mathrm{J}^0(S) \simeq S$, see [MS24, §4] for details. Thus the isomorphism $\text{Ш}(S, F) \simeq \mathrm{Br}(\overline{\mathrm{Pic}}^{g-1})$ is another incarnation of (4.1.1).

Moreover, using our explicit computation of the obstruction class of Theorem 2.1.2, we show that for any $d \in \mathbb{Z}$, the morphism $\text{Ш}(S, H) \to \mathrm{Br}(\overline{\mathrm{Pic}}^d)$ maps the class $[\overline{\mathrm{Pic}}^1]$ to the obstruction $\alpha_d \in \mathrm{Br}(\overline{\mathrm{Pic}}^d)$ to the existence of a universal sheaf on $S \times \overline{\mathrm{Pic}}^d$. Using this fact, we explain that a result by Addington, Donovan and Meachan [ADM16] should be seen as an analogue to a result by Donagi and Pantev [DP08], see Theorem 4.3.12 and Theorem 4.3.13, respectively.

Lastly, we study birational equivalence of moduli spaces of sheaves on elliptic K3 surfaces. One of the main points of interest in this topic is the progress we make in item i) above. From Beckmann's work on derived equivalence for such moduli spaces [Bec23], we derive the following theorem.

Theorem 4.1.1 (See Theorem 4.4.5). *Let S be an elliptic K3 surface with a section, and let M be a moduli space of sheaves on S of dimension $2n$. Then the following are equivalent:*

i) M is a fine moduli space.

ii) *M is birational to the Hilbert scheme $S^{[n]}$.*

iii) *M is derived equivalent to the Hilbert scheme $S^{[n]}$.*

In particular, if M is a non-fine moduli space of sheaves on S, then M is not derived equivalent to $S^{[n]}$.

The equivalence *ii)* $\iff$ *iii)* was already noted by Beckmann in [Bec23, §9]. The inclusion of item *i)* is an application of Theorem 2.1.2. Theorem 4.1.1 rules out the possibility of generalising the Derived Torelli Theorem to higher-dimensional hyperkähler manifolds of K3$^{[n]}$-type using the transcendental lattice. More precisely, we obtain the following corollary, which gives counterexamples to a question raised in [KK24, Problem 1.1] and in [PR23, Question 2].

Corollary 4.1.2. *There exist moduli spaces of sheaves M and M' on the same K3 surface for which there is a Hodge isometry $T(M) \simeq T(M')$, but which are not derived equivalent.*

We also apply our results on birational equivalence for moduli spaces of sheaves to Beauville–Mukai systems. Our main result in this direction is the following theorem, which fully describes birational equivalences of Beauville–Mukai systems in this setting.

Theorem 4.1.3 (See Theorem 4.4.8)**.** *Let S be an elliptic K3 surface with a section. Let $\mathcal{C} \to |H|$ be a generically smooth complete linear system on S, and let $H^2 = 2g - 2 > 0$. The following are equivalent:*

i) *$\overline{\mathrm{Pic}}^d$ is birational to $\overline{\mathrm{Pic}}^e$.*

ii) *The obstruction classes of $\overline{\mathrm{Pic}}^d$ and $\overline{\mathrm{Pic}}^e$ have the same order.*

iii) *The Mukai vectors of $\overline{\mathrm{Pic}}^d$ and $\overline{\mathrm{Pic}}^e$ have the same divisibility. More precisely, we have $\gcd(\mathrm{div}(H), d + 1 - g) = \gcd(\mathrm{div}(H), e + 1 - g)$.*

The implication *i)* $\implies$ *ii)* is a general fact about birational moduli spaces [Bec23, Proof of Lemma 9.10]. The equivalence *ii)* $\iff$ *iii)* is an easy consequence of Theorem 2.1.2, and the implication *ii)* $\implies$ *i)* uses our explicit computation of the obstruction class.

The structure of this chapter

In Section 4.2, we recall the new construction of the Tate–Shafarevich group of a Beauville–Mukai system introduced in [HM23]. The main point of difference is that we define an element of the Tate–Shafarevich group to be a torsor of $\overline{\mathrm{Pic}}^0_\eta$. In [HM23], the Tate–Shafarevich group is defined to be a certain subquotient of the special Brauer group of the underlying K3 surface. We show that our definition is equivalent to the definition given in [HM23].

In Section 4.3, we study derived equivalence for hyperkähler manifolds, paying special attention to Beauville–Mukai systems. In Section 4.3.1, we show that the K3$^{[n]}$-*lattice* defined by Beckmann in [Bec23] is a birational invariant. In Section 4.3.2, we show how a theorem by Addington, Donovan and Meachan [ADM16] can be seen as a higher-dimensional generalisation of a theorem by Donagi and Pantev in [DP08]. To show this, we need our computation of the obstruction class from Chapter 2. We also prove that admitting a rational Lagrangian fibration is a derived invariant. That is, if X is a hyperkähler manifold of $\mathrm{K3}^{[n]}$-type which admits a rational Lagrangian fibration, we prove that all Fourier–Mukai partners of X also admit a rational Lagrangian fibration.

In Section 4.4, we study moduli spaces of sheaves on elliptic K3 surfaces. More precisely, we study birational equivalence for these moduli spaces. A particularly interesting result is that the natural generalisation of the Derived Torelli Theorem to higher-dimensional hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type is false. Moreover, we fully classify birational equivalence classes of moduli spaces of sheaves over elliptic K3 surfaces in terms of Căldăraru classes.

This chapter is based mostly on [MM24]. The parts that are not based on [MM24] are Section 4.3.1, Section 4.3.3, and Section 4.A.

4.2 Twisted Beauville–Mukai Systems

4.2.1 Lagrangian Fibrations

Definition 4.2.1. [Mat99; Mat05] Let X be a compact hyperkähler manifold. A Lagrangian fibration of X is a proper morphism $f : X \to B$ which satisfies $f_*\mathcal{O}_X \simeq \mathcal{O}_B$, where B is a normal variety with $0 < \dim(B) < \dim(X)$.

Theorem 4.2.2. *[Mat15] Let $f : X \to B$ be a Lagrangian fibration of a hyperkähler manifold of dimension $\dim(X) = 2n$. Then we have $\dim(B) = n$, and every smooth fibre X_b is an abelian subvariety of X.*

Example 4.2.3. The simplest example of a Lagrangian fibration is an elliptic fibration of a K3 surface $S \to \mathbb{P}^1$. Moreover, for any $n \in \mathbb{N}$, the induced morphism

$$S^{[n]} \to \mathrm{Sym}^n \mathbb{P}^1 \simeq \mathbb{P}^n$$

is a Lagrangian fibration.

In this section, we study (twisted) *Beauville–Mukai systems*, which are Lagrangian fibrations on moduli spaces of (twisted) sheaves on K3 surfaces. We recall the basic theory here. Our main references are [Muk84; Bea91].

Let S be a K3 surface, and let $H \subset S$ be a smooth, irreducible curve of genus $g > 0$ whose class in $\mathrm{NS}(S)$ is primitive. Let $\mathcal{C} \to |H|$ be the universal curve over the linear system $|H|$. This means that, for any $[C] = x \in |H|$, there is an isomorphism $\mathcal{C}_x \simeq C$. For $d \in \mathbb{Z}$, consider the Mukai vector

$$v_d \coloneqq (0, H, d + 1 - g). \tag{4.2.1}$$

For a v_d-generic polarisation $H' \in \mathrm{NS}(X)$, we write $M_d = M_{H'}(v_d)$. Note that a line bundle L of degree d on a smooth, irreducible curve $C \in |H|$ satisfies

$$v(i_*L) = (0, H, d + 1 - g),$$

where $i : C \hookrightarrow S$ is the inclusion. Therefore, we have an inclusion

$$\mathrm{Pic}^d(\mathcal{C}_{\mathrm{sm}}/|H|_{\mathrm{sm}}) \hookrightarrow M_d,$$

where $\mathcal{C}_{\mathrm{sm}} \to |H|_{\mathrm{sm}}$ is the restriction of the universal curve to the locus of smooth curves in $|H|$. For this reason, we usually denote

$$\overline{\mathrm{Pic}}^d(\mathcal{C}/|H|) := M_d,$$

and we denote

$$\mathrm{Pic}^d(\mathcal{C}/|H|) := \mathrm{Pic}^d(\mathcal{C}_{\mathrm{sm}}/|H|_{\mathrm{sm}}) \subset \overline{\mathrm{Pic}}^d(\mathcal{C}/|H|).$$

Moreover, to keep the notation light, we often omit $\mathcal{C}/|H|$ from the notation if this cannot lead to confusion. By Theorem 2.3.2, $\overline{\mathrm{Pic}}^d$ is a smooth hyperkähler manifold of $\mathrm{K3}^{[g]}$-type.

Note that there is a natural morphism

$$\begin{array}{rccc} f\colon & \overline{\mathrm{Pic}}^d & \longrightarrow & |H| \\ & [\mathcal{F}] & \longmapsto & \mathrm{Supp}(\mathcal{F}), \end{array}$$

which is a Lagrangian fibration by Theorem 4.2.2. Moreover, for a smooth curve $C \in |H|$, the fibre of f over C is isomorphic to $\mathrm{Pic}^d(C)$.

Remark 4.2.4. For L a line bundle on S with $L \cdot H = \mathrm{div}(H)$, we obtain standard birational maps between Beauville–Mukai systems by taking the tensor product with L :

$$\overline{\mathrm{Pic}}^d \overset{\sim}{\dashrightarrow} \overline{\mathrm{Pic}}^{d+\mathrm{div}(H)}.$$

We will see soon which Beauville–Mukai systems are isomorphic as $\overline{\mathrm{Pic}}^0_\eta$-torsors (Remark 4.2.16), and we will also see which ones are birational if S is an elliptic K3 surface (Theorem 4.4.8).

Now, fix a special Brauer class $\alpha \in \mathrm{SBr}(X)$ represented by an Azumaya algebra $\mathcal{A}$. Recall from Section 2.2.4 the definition of the twisted Mukai vector

$$\begin{array}{rccc} v_{\mathcal{A}}(-)\colon & \mathbf{Coh}(X, \mathcal{A}) & \longrightarrow & H^*(X, \mathbb{Q}) \\ & \mathcal{F} & \longmapsto & \mathrm{ch}(\mathcal{F})\sqrt{\mathrm{ch}(\mathcal{A})}^{-1}. \end{array}$$

For $d \in \frac{1}{d(\mathcal{A})} \cdot \mathbb{Z}$, we consider the Mukai vector

$$v_d = (0, H, d + 1 - g) \in H^*(X, \mathbb{Q}).$$

Then the discussion above also holds for the moduli space of stable $\mathcal{A}$-modules

$$\overline{\mathrm{Pic}}^d_\alpha := M_L(v_d; \alpha).$$

More precisely, $\overline{\mathrm{Pic}}^d_\alpha$ is a hyperkähler manifold of $\mathrm{K3}^{[g]}$-type, which admits a Lagrangian fibration $f : \overline{\mathrm{Pic}}^d_\alpha \to |H|$, defined by sending an α-twisted sheaf $[\mathcal{F}] \in \overline{\mathrm{Pic}}^d_\alpha$ to its support. In this case, the fibre of f over a smooth curve $C \in |H|$ is isomorphic to the moduli space of locally free $\mathcal{A}|_C$-modules of rank $d(\mathcal{A})$ and degree $d \cdot d(\mathcal{A})$, denoted

$$\mathrm{Pic}^d_{\alpha|_C}.$$

Note that we consider $\alpha|_C$ as a *special* Brauer class on C. In this case, if $\mathrm{Pic}^d_{\alpha|_C}$ is non-empty, then $\alpha|_C$ is the trivial Brauer class, by [HM23, Remark 3.3(ii)]. Just as in the untwisted case, we denote by $\mathrm{Pic}^d_\alpha := \mathrm{Pic}^d_\alpha(\mathcal{C}_{\mathrm{sm}}/|H|_{\mathrm{sm}}) \subset \overline{\mathrm{Pic}}^d_\alpha$ the open part of $\overline{\mathrm{Pic}}^d_\alpha$ living over the smooth curves in $|H|$.

Definition 4.2.5. A Lagrangian fibration of the form $f : \overline{\mathrm{Pic}}^d \to |H|$ is called a *Beauville–Mukai system,* and a Lagrangian fibration of the form $f : \overline{\mathrm{Pic}}^d_\alpha \to |H|$ is called a *twisted Beauville–Mukai system.*

Note that elliptic fibrations of K3 surfaces are Beauville–Mukai systems. The following theorem by Markman is a generalisation of this observation.

Theorem 4.2.6. *[Mar14, Theorem 7.13], [HM23, Theorem 1.2] Let $X \to \mathbb{P}^n$ be a non-special Lagrangian fibration of a hyperkähler manifold X of* $\mathrm{K3}^{[n]}$*-type. Then there exists a K3 surface S with a generically smooth linear system $|H|$ and a special Brauer class $\alpha \in \mathrm{SBr}(S)$ such that there exists a birational map $X \overset{\sim}{\dashrightarrow} \overline{\mathrm{Pic}}^d_\alpha$ making the following diagram commute:*

$$\begin{tikzcd} X \arrow[rr, dashed, "\sim"] \arrow[dr] & & \overline{\mathrm{Pic}}^d_\alpha \arrow[dl] \\ & \mathbb{P}^n. & \end{tikzcd}$$

Remark 4.2.7. A hyperkähler manifold of $\mathrm{K3}^{[n]}$-type X is *non-special* if there are no integral vectors in $H^{2,0}(X) \oplus H^{0,2}(X) \subset H^2(X, \mathbb{C})$ [Mar14]. If S is a K3 surface, $\alpha \in \mathrm{SBr}(S)$ is a special Brauer class, and $v \in \mathrm{N}(S, \alpha)$ is a primitive Mukai vector, then S is non-special if and only if $M(v; \alpha)$ is non-special. This follows from the fact that there is a rational Hodge isometry $T(S)_\mathbb{Q} \simeq T(M(v; \alpha))_\mathbb{Q}$, see Proposition 2.2.46, Corollary 2.2.47, and Theorem 2.3.11.

4.2.2 Tate–Shafarevich Groups

With this example in mind, we now turn our attention to Lagrangian fibrations of higher dimensions.

Definition 4.2.8. [HM23] For a K3 surface S with generically smooth linear system $|H|$, with $H^2 = 2g - 2 \geq 0$, we denote by

$$Ш(S, H)$$

the set of isomorphism classes of pairs $(X \to \mathbb{P}^g, \phi)$, where $X \to \mathbb{P}^g$ is a Lagrangian fibration of a hyperkähler manifold X which is birational to $\overline{\mathrm{Pic}}^d_\alpha$ for some $\alpha \in \mathrm{SBr}(S)$ and some $d \in \mathbb{Z}$, and $\phi : X_\eta \times \overline{\mathrm{Pic}}^0_{\mathcal{C}_\eta} \to X_\eta$ is a group action that turns X_η into a torsor over $\overline{\mathrm{Pic}}^0_{\mathcal{C}_\eta}$. Here, an isomorphism between such pairs $(X \to \mathbb{P}^g, \phi), (Y \to \mathbb{P}^g, \psi)$ is a birational map $X \overset{\sim}{\dashrightarrow} Y$ over $\mathbb{P}^g$ which induces an isomorphism of torsors $X_\eta \simeq Y_\eta$. We call $Ш(S, H)$ the *Tate–Shafarevich group of the pair* (S, H), and refer to elements of $Ш(S, H)$ as $\overline{\mathrm{Pic}}^0(\mathcal{C}/|H|)$*-torsors.*

Remark 4.2.9. Before we compute $Ш(S, H)$, let us briefly motivate why we call the set $Ш(S, H)$ the Tate–Shafarevich group of (S, H). The main motivation is that $Ш(S, H)$ parametrises the hyperkähler twists of $\overline{\mathrm{Pic}}^0$ (at least when S is non-special). This is completely analogous to how the Tate–Shafarevich group of an elliptic K3 surface with a section parametrises those torsors of the generic fibre that compactify to elliptic K3 surfaces (see Section 3.4.1).

We now compare $Ш(S,H)$ to the Tate–Shafarevich group $Ш^0$ defined by Markman in [Mar14]. We assume that the Picard rank of S is $\rho = 1$ and that H is an ample generator for $\mathrm{NS}(S)$. Then the group $Ш^0$ is defined as

$$Ш^0 \coloneqq H^1(|H|, \mathcal{A}^0),$$

where $\mathcal{A}^0$ is defined via the short exact sequence

$$0 \to \mathcal{A}^0 \to R^1p_*\mathcal{O}_{\mathcal{C}}^\times \stackrel{\deg}{\to} R^1p_*\mathbb{Z} \to 0,$$

where $p\colon \mathcal{C} \to |H|$ is the projection, and the map deg is obtained through the exponential sequence. An element $s \in Ш^0$ can be used to reglue $\overline{\mathrm{Pic}}^0$ to obtain a new hyperkähler manifold $\overline{\mathrm{Pic}}^0_s$ [Mar14, §7.2]. This regluing process is similar to how one reglues an elliptic K3 surface with a section to obtain Tate–Shafarevich twists, see [Căl00, §4.2] and [DP08, Chapter 2], which partially explains why Markman chose the notation $Ш^0$. The twist $\overline{\mathrm{Pic}}^d_s$ is projective if and only if $s \in Ш^0$ is a torsion element, as was proved in [AR23, Theorem 5.19]. Under our assumptions, there is an isomorphism [HM23, §4.4]

$$T(S) \otimes \mathbb{Q}/\mathbb{Z} \simeq Ш^0_{\mathrm{tors}},$$

and if $\alpha \in T(S) \otimes \mathbb{Q}/\mathbb{Z} \subset \mathrm{SBr}(S)$ is the special Brauer class corresponding to s along this isomorphism, then there is a birational equivalence $\overline{\mathrm{Pic}}^0_\alpha \overset{\sim}{\dashrightarrow} \overline{\mathrm{Pic}}^0_s$ [HM23, Proof of Theorem 1.2]. This fact, combined with Theorem 4.2.13 below, shows that there is a surjective group homomorphism $Ш^0 \twoheadrightarrow Ш(S,H)$, and two elements $s, s' \in Ш^0$ have the same image in $Ш(S,H)$ if and only if the generic fibres of $\overline{\mathrm{Pic}}^0_s$ and $\overline{\mathrm{Pic}}^0_{s'}$ are isomorphic as $\overline{\mathrm{Pic}}^0_\eta$-torsors.

Remark 4.2.9 explains why $Ш(S,H)$ bears its name, but it does not yet explain how it is a group. Note that there is a natural injective map $Ш(S,H) \hookrightarrow \mathrm{WC}(\overline{\mathrm{Pic}}^0_\eta)$, where $\mathrm{WC}(\overline{\mathrm{Pic}}^0_\eta)$ denotes the Weil-Châtelet group of the abelian variety $\overline{\mathrm{Pic}}^0_\eta$ (see Section 3.4.1). Therefore, to define a group structure on $Ш(S,H)$, it suffices to show that the image of $Ш(S,H)$ in $\mathrm{WC}(\overline{\mathrm{Pic}}^0_\eta)$ is closed under the group action.

Lemma 4.2.10. *[HM23, Remark 4.11] Let S be a K3 surface, and let $H \in \mathrm{NS}(S)$ be a smooth, irreducible curve. Let $\alpha, \beta \in \mathrm{SBr}(S)$ such that $\overline{\mathrm{Pic}}^0_\alpha$ and $\overline{\mathrm{Pic}}^0_\beta$ are non-empty. Then, in $\mathrm{WC}(\overline{\mathrm{Pic}}^0_\eta)$, we have*

$$[(\overline{\mathrm{Pic}}^0_\alpha)_\eta] \cdot [(\overline{\mathrm{Pic}}^0_\beta)_\eta] = [(\overline{\mathrm{Pic}}^0_{\alpha\beta})_\eta)].$$

In particular, $Ш(S,H)$ is naturally a subgroup of $\mathrm{WC}(\overline{\mathrm{Pic}}^0_\eta)$.

Proof. For any $\alpha \in \mathrm{SBr}(S)$, we have $[(\overline{\mathrm{Pic}}^0_\alpha)_\eta] = [\overline{\mathrm{Pic}}^0_{\alpha|_{\mathcal{C}_\eta}}]$ in $\mathrm{WC}(\overline{\mathrm{Pic}}^0_\eta)$. Therefore, the result follows from [HM23, Remark 4.11]. □

Recall that, for an elliptic K3 surface with a section $S \to \mathbb{P}^1$, there are isomorphisms

$$Ш(S) \simeq \mathrm{Br}(S) \simeq \mathrm{Hom}(T(S), \mathbb{Q}/\mathbb{Z}).$$

The rest of this section is devoted to proving Theorem 4.2.13 below, which is a similar structure result for the Tate–Shafarevich group of a pair (S,H) due to Huybrechts and Mattei [HM23].

Lemma 4.2.11. *[HM23, Proposition 4.13] Let $\alpha \in \mathrm{SBr}(S)$ be a special Brauer class such that $\overline{\mathrm{Pic}}^d_\alpha$ is non-empty. Then there exists a special Brauer class $\alpha_0 \in T(S) \otimes \mathbb{Q}/\mathbb{Z} \subset \mathrm{SBr}(S)$ with the property that*

$$[\overline{\mathrm{Pic}}^d_\alpha] = [\overline{\mathrm{Pic}}^0_{\alpha_0}]$$

in $Ш(S,H)$. Moreover $\overline{\mathrm{Pic}}^d_\alpha$ is non-empty for any $\alpha \in T(S) \otimes \mathbb{Q}/\mathbb{Z}$.

Combining Lemma 4.2.10 and Lemma 4.2.11, we immediately obtain the following proposition:

Proposition 4.2.12. *The map*

$$\begin{array}{rcl} T(S) \otimes \mathbb{Q}/\mathbb{Z} & \longrightarrow & Ш(S,H) \\ \alpha & \longmapsto & [\overline{\mathrm{Pic}}^0_\alpha] \end{array}$$

is a well-defined and surjective group homomorphism.

Recall that we have a short exact sequence

$$0 \to T(S) \to T(S)^* \to A_{T(S)} \to 0,$$

where $A_{T(S)}$ is the discriminant lattice of $T(S)$. Taking the tensor product of this short exact sequence by $\mathbb{Q}/\mathbb{Z}$, we obtain

$$0 \to A_{T(S)} \to T(S) \otimes \mathbb{Q}/\mathbb{Z} \to T(S)^* \otimes \mathbb{Q}/\mathbb{Z} \to 0, \tag{4.2.2}$$

and we view $A_{T(S)}$ as a subgroup of $T(S) \otimes \mathbb{Q}/\mathbb{Z}$ using this identification.

For a class $H \in \mathrm{NS}(S)$, we have

$$\frac{H}{\mathrm{div}(H)} \in A_{\mathrm{NS}(S)}.$$

Recall from Lemma 2.2.1 that there is a natural identification $A_{T(S)} \simeq A_{\mathrm{NS}(S)}(-1)$, so that $\frac{H}{\mathrm{div}(H)}$ corresponds to a unique element $\lambda \in A_{T(S)}$. In this notation, we write

$$\begin{array}{rrcl} \zeta_H\colon & A_{T(S)} & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ & x & \longmapsto & x \cdot \lambda. \end{array}$$

Theorem 4.2.13. *[HM23, Proposition 4.9] The natural morphism $T(S) \otimes \mathbb{Q}/\mathbb{Z} \to Ш(S,H)$ induces an isomorphism*

$$\frac{T(S) \otimes \mathbb{Q}/\mathbb{Z}}{\ker(\zeta_H)} \simeq Ш(S,H).$$

For any $d \in \mathbb{Z}$, the Beauville–Mukai system $\overline{\mathrm{Pic}}^d \to |H|$ defines an element of $Ш(S,H)$. For details, we refer to [HM23, Example 4.14].

The element of $(T(S) \otimes \mathbb{Q}/\mathbb{Z})/\ker(\zeta_H)$ corresponding to $[\overline{\mathrm{Pic}}^d]$ is the element induced by

$$\frac{-d}{\mathrm{div}(D)} \cdot D \in A_{\mathrm{NS}(S)} \subset T(S) \otimes \mathbb{Q}/\mathbb{Z}, \tag{4.2.3}$$

where $D \in \mathrm{NS}(S)$ is a divisor with $\operatorname{div}(D) = D \cdot H$. Such a divisor exists and can be constructed as follows. Let $u \in H^2(S, \mathbb{Z})$ be a class that satisfies $u \cdot H = 1$. This class exists since H is primitive and $H^2(S, \mathbb{Z})$ is unimodular. Also by unimodularity, there exists a divisor $D \in \mathrm{NS}(S)$, and an integer $m \in \mathbb{Z}$ such that $u \cdot E = \frac{D}{m} \cdot E$ for all $E \in \mathrm{NS}(S)$. Moreover, we have $\frac{D}{m} \in \mathrm{NS}(S)^*$, since u is an integral vector, hence $m \leq \operatorname{div}(D)$. On the other hand, from $\frac{D}{m} \cdot H = 1$, it follows that we have $m = D \cdot H$, hence $m \geq \operatorname{div}(D)$, and we find $m = \operatorname{div}(D)$.

Remark 4.2.14. Let us briefly explain our choice of D above. For simplicity, assume $d = 1$. In the notation of [HM23], for any vector bundle $\mathcal{E}$ on S of rank r and determinant L, there exists a birational map

$$\begin{array}{ccc} \overline{\mathrm{Pic}}^1 & \overset{\sim}{\dashrightarrow} & \overline{\mathrm{Pic}}^{d'}_{\mathcal{E}nd(\mathcal{E})} \\ F & \longmapsto & F \otimes \mathcal{E} \end{array}$$

where $d' = 1 + L \cdot H/r$. Moreover, the class $[\mathcal{E}nd(\mathcal{E})] \in \mathrm{SBr}(S)$ corresponds to the class $(1/r) \cdot L \in \mathrm{NS}(S) \otimes \mathbb{Q}/\mathbb{Z}$. Therefore, for any divisor $D \in \mathrm{NS}(S)$, one can pick $L = -D$ and $r = D \cdot H$ to get an isomorphism $\mathrm{Pic}^d \simeq \mathrm{Pic}^0_{\mathcal{E}nd(\mathcal{E})}$. However, $\mathcal{E}nd(\mathcal{E})$ only defines a class in $Ш(S, H)$ if $[\mathcal{E}nd(\mathcal{E})]$ lies in the image of $A_{\mathrm{NS}(S)} \subset \mathrm{NS}(S) \otimes \mathbb{Q}/\mathbb{Z}$, i.e. if $(-1/D \cdot H) \in \mathbb{Z}[1/\operatorname{div}(D)]$, equivalently $D \cdot H = \operatorname{div}(D)$.

Remark 4.2.15. The element (4.2.3) can be seen as a special Brauer class via the inclusion $T(S) \otimes \mathbb{Q}/\mathbb{Z} \hookrightarrow H^2(S, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} \simeq \mathrm{SBr}(S)$. The birational map

$$\overline{\mathrm{Pic}}^d \overset{\sim}{\dashrightarrow} \overline{\mathrm{Pic}}^0_{\mathcal{E}nd(\mathcal{E})}$$

shows that $\overline{\mathrm{Pic}}^d$ is birational to $\overline{\mathrm{Pic}}^0_{\alpha_d}$, where $\alpha_d \in \mathrm{SBr}(S)$ is a special Brauer class contained in $A_{\mathrm{NS}(S)} \subset \mathrm{SBr}(S)$. In particular, the image of α_d under the natural surjective map

$$\mathrm{SBr}(S) \simeq H^2(S, \mathbb{Z}) \otimes \mathbb{Q}/\mathbb{Z} \to T(S)^* \otimes \mathbb{Q}/\mathbb{Z} \simeq \mathrm{Br}(S)$$

vanishes. Now suppose we have $d, e \in \mathbb{Z}$ such that $\overline{\mathrm{Pic}}^d$ is not birational to $\overline{\mathrm{Pic}}^e$. Such integers exist for well-chosen S and H, see for example Theorem 4.4.8. Write $v_0 = (0, H, 1-g) \in \mathrm{N}(S)$. Then the twisted moduli spaces $M(v_0, \alpha_d)$ and $M(v_0, \alpha_e)$ are not birational, while α_d and α_e induce the same element of $\mathrm{Br}(S)$.

For now, let us note that $[\overline{\mathrm{Pic}}^1]$ corresponds to the image of $u \in H^2(S, \mathbb{Z})$ along the composition

$$H^2(S, \mathbb{Z}) \to T(S)^* \to T(S) \otimes \mathbb{Q}/\mathbb{Z}.$$

Remark 4.2.16. The element (4.2.3), corresponding to $[\overline{\mathrm{Pic}}^d] \in Ш(S, H)$, is trivial if and only if d is a multiple of $\operatorname{div}(H)$. Indeed, since $\operatorname{div}(D) = D \cdot H = k \operatorname{div}(H)$ for some k, we have

$$\zeta_H \left(\frac{-dD}{\operatorname{div}(D)} \right) = \frac{-d \operatorname{div}(D)}{\operatorname{div}(D)} = -\overline{d} \in \mathbb{Z}/\operatorname{div}(H)\mathbb{Z}.$$

This means that $\overline{\mathrm{Pic}}^d_\eta$ and $\overline{\mathrm{Pic}}^e_\eta$ are isomorphic $\overline{\mathrm{Pic}}^0_\eta$-torsors if and only if $d - e \equiv 0 \pmod{\operatorname{div}(H)}$. Notice that, in particular, $[\overline{\mathrm{Pic}}^1] = [-D/\operatorname{div}(D)]$ has order $\operatorname{div}(H)$.

4.3 Derived Equivalence for Beauville–Mukai Systems

4.3.1 Derived Equivalence for Hyperkähler Manifolds

Derived equivalence for K3 surfaces is a well-established subject. This is in no small part due to the Derived Torelli Theorem 2.3.12, which classifies Fourier–Mukai partners of K3 surfaces. If we turn our attention to derived equivalences of more general hyperkähler manifolds, the situation is much less clear. Even if we restrict our attention to hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type, no analogue of the Derived Torelli Theorem has been proved at the time of writing. This is not to say that there has been no progress in this subject at all. On the contrary, derived equivalence of hyperkähler manifolds is a very active area of research. In this section, we provide a summary of recent progress. We are especially interested in hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type, and even more so in moduli spaces of (twisted) sheaves on K3 surfaces.

Theorem 4.3.1. *[Hal21] Let S be a K3 surface, let $\alpha \in \mathrm{SBr}(S)$ be a special Brauer class, let $v \in \mathrm{N}(S, \alpha)$ be a primitive Mukai vector, $H \in \mathrm{NS}(S)$ a v-generic polarisation, and write $M = M_H(v; \alpha)$. If X is a complex, K-trivial, smooth, projective variety that is birational to M, then X is derived equivalent to M.*

As we mentioned above, there is currently no analogue for the Derived Torelli Theorem for hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type. Such a result would associate to any hyperkähler manifold X of $\mathrm{K3}^{[n]}$-type a Hodge lattice L_X, such that the following conjecture holds:

Conjecture 4.3.2. *Two hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type X, Y are derived equivalent if and only if there is a Hodge isometry $L_X \simeq L_Y$.*

There are two main contenders for a Hodge lattice that could play the role of L_X; they are the transcendental lattice $T(X)$ and the so-called $\mathrm{K3}^{[n]}$*-lattice* Λ_X introduced by Taelman and Beckmann, [Tae23; Bec23]. We discuss the $\mathrm{K3}^{[n]}$-lattice below. In Section 4.4, where we study birational equivalence for moduli spaces of sheaves on elliptic K3 surfaces, we prove that Conjecture 4.3.2 is false when we take $L_X = T(X)$. The question of whether Conjecture 4.3.2 holds for $L_X = \Lambda_X$ is still open.

If X and M are as in Theorem 4.3.1, then a birational map $X \overset{\sim}{\dashrightarrow} M$ induces a Hodge isometry $H^2(X, \mathbb{Z}) \simeq H^2(M, \mathbb{Z})$ by the Birational Torelli Theorem 2.2.25, hence we have a Hodge isometry $T(X) \simeq T(M)$. Similarly, in Proposition 4.3.7 below, we prove that a birational equivalence between X and M also gives rise to a Hodge isometry $\Lambda_X \simeq \Lambda_M$. This means that Theorem 4.3.1 is a requirement for Conjecture 4.3.2 to hold with $L_X = \Lambda_X$. Moreover, if such a Derived Torelli statement were true, Theorem 4.3.1 would have to hold for all hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type, not only for moduli spaces of sheaves. The question of whether birational equivalence implies derived equivalence for K-trivial varieties is part of an important conjecture by Bondal, Orlov, and Kawamata, called the *D-equivalence Conjecture* [BO02; Kaw02].

Corollary 4.3.3. *If Conjecture 4.3.2 holds with $L_X = \Lambda_X$, then the D-equivalence conjecture holds for hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type.*

Proof. This follows immediately from Proposition 4.3.7 below. □

We now turn our attention to the $\mathrm{K3}^{[n]}$-lattice Λ_X.

Definition 4.3.4. [Bec23] Let X be a hyperkähler manifold of $\mathrm{K3}^{[n]}$-type. Then the *Mukai lattice* of X is the Hodge lattice of K3-type:

$$\tilde{H}(X,\mathbb{Z}) \coloneqq U \oplus H^2(X,\mathbb{Z}),$$

whose $(2,0)$-part is $H^{2,0}(X)$. If $e, f \in U$ form the standard basis of U, the vector $re + x + sf \in \tilde{H}(X,\mathbb{Z})$, where $r, s \in \mathbb{Z}$ and $x \in H^2(X,\mathbb{Z})$, is denoted (r, x, s). Recall from Section 2.2.5 that $\tilde{H}(X,\mathbb{Z})$ admits a natural ring structure.

Recall that, if X and Y are K3 surfaces, then every Fourier–Mukai equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$ induces a Hodge isometry $\tilde{H}(X,\mathbb{Z}) \simeq \tilde{H}(Y,\mathbb{Z})$. This result holds because for any object $\mathcal{E}^\bullet \in \mathcal{D}^b(X \times Y)$, the Mukai vector $v(\mathcal{E}^\bullet) \in H^2(X \times Y, \mathbb{Q})$ is integral. This is not the case when X and Y are higher-dimensional moduli spaces of sheaves on K3 surfaces, and therefore cohomological Fourier–Mukai transforms do not necessarily induce Hodge isometries between the Mukai lattices.

As we discussed in Section 2.2.5, any rational element $B \in H^2(X,\mathbb{Z})$ induces a rational isometry $\exp(B) : \tilde{H}(X,\mathbb{Q}) \simeq \tilde{H}(X,\mathbb{Q})$. We will see in Theorem 4.3.9 that any Fourier–Mukai equivalence preserves $\exp(B) \wedge \tilde{H}(X,\mathbb{Z})$ for a well-chosen $B \in H^2(X,\mathbb{Q})$. We now explain what the correct choice for B is.

Definition 4.3.5. [Tae23; Bec23] Let X be a hyperkähler manifold of $\mathrm{K3}^{[n]}$-type. The $\mathrm{K3}^{[n]}$*-lattice* of X is the Hodge lattice

$$\Lambda_X \coloneqq \exp\left(\frac{\delta}{2}\right) \cdot \tilde{H}(X,\mathbb{Z}) \subset \tilde{H}(X,\mathbb{Q}).$$

Here, $\delta \in \tilde{H}(X,\mathbb{Z})$ is any primitive vector with $\delta^2 = 2 - 2n$ and $\mathrm{div}(\delta) = 2n - 2$. The Hodge structure on Λ_X is the one inherited from $\tilde{H}(X,\mathbb{Q})$, that is, $\Lambda_X^{2,0} = H^{2,0}(X)$.

Remark 4.3.6. If δ and δ' are two primitive vectors in $\tilde{H}(X,\mathbb{Q})$ with $\delta^2 = (\delta')^2 = 2 - 2n$ and $\mathrm{div}(\delta) = \mathrm{div}(\delta') = 2n - 2$, then we have $\frac{1}{2}(\delta' - \delta) \in \tilde{H}(X,\mathbb{Z})$. Indeed, the discriminant lattice $A = H^2(X,\mathbb{Z})^*/H^2(X,\mathbb{Z})$ is cyclic of order $2n-2$, and both $\frac{1}{2}\delta$ and $\frac{1}{2}\delta'$ are vectors of order 2 in A. Since A only contains one such vector, we have $\frac{1}{2}(\delta' - \delta) = 0 \in A$, hence $\frac{1}{2}(\delta' - \delta) \in H^2(X,\mathbb{Z})$. Recall from Lemma 2.2.40 that $\exp(\frac{1}{2}(\delta' - \delta))$ is an isometry which preserves $\tilde{H}(X,\mathbb{Z})$. This means that we have the following equalities of subsets of $\tilde{H}(X,\mathbb{Q})$:

$$\exp\left(\frac{\delta}{2}\right) \cdot \tilde{H}(X,\mathbb{Z}) = \exp\left(\frac{\delta}{2}\right) \cdot \exp\left(\frac{\delta' - \delta}{2}\right) \cdot \tilde{H}(X,\mathbb{Z}) = \exp\left(\frac{\delta'}{2}\right) \cdot \tilde{H}(X,\mathbb{Z}).$$

This shows that Λ_X is independent of the choice of δ.

Proposition 4.3.7. *Let X and Y be hyperkähler manifolds of* $\mathrm{K3}^{[n]}$*-type. If X and Y are birational, then Λ_X is Hodge isometric to Λ_Y.*

Proof. By [Huy03, Lemma 2.6], there exists a Hodge isometry

$$\psi \colon H^2(X,\mathbb{Z}) \simeq H^2(Y,\mathbb{Z}).$$

We obtain a Hodge isometry $\tilde{\psi} \colon \tilde{H}(X,\mathbb{Z}) \simeq \tilde{H}(Y,\mathbb{Z})$, extending by the identity on U. Lemma 2.2.39 shows that $\tilde{\psi}$ is a ring homomorphism. By taking the tensor product with $\mathbb{Q}$, we get a Hodge isometry $\tilde{\psi}_{\mathbb{Q}} : \tilde{H}(X,\mathbb{Q}) \simeq \tilde{H}(Y,\mathbb{Q})$ which restricts to $\tilde{\psi}$ on $\tilde{H}(X,\mathbb{Z})$. Let $\delta \in \tilde{H}(X,\mathbb{Z})$ be a primitive vector with $\delta^2 = 2-2n$ and $\operatorname{div}(\delta) = 2n-2$. Then $\tilde{\psi}(\delta) \in \tilde{H}(Y,\mathbb{Z})$ has the same square and divisibility as δ, because $\tilde{\psi}$ is an isometry. Using the fact that $\tilde{\psi}_{\mathbb{Q}}$ is a ring homomorphism, it follows that it restricts to a Hodge isometry

$$\Lambda_X = \exp\left(\frac{\delta}{2}\right) \cdot \tilde{H}(X,\mathbb{Z}) \simeq \exp\left(\frac{\tilde{\psi}(\delta)}{2}\right) \cdot \tilde{H}(Y,\mathbb{Z}) = \Lambda_Y,$$

where the final equality follows from Remark 4.3.6. □

Proposition 4.3.8. *[Bec23] For a hyperkähler manifold X of* $\mathrm{K3}^{[n]}$*-type, where $n \geq 2$, there is a Hodge isometry*

$$\Lambda_X \simeq \tilde{H}(X, \frac{\delta}{2}, \mathbb{Z}). \tag{4.3.1}$$

Moreover, we have

$$T(\Lambda_X) \simeq T(X), \tag{4.3.2}$$

where $T(\Lambda_X)$ is the transcendental sublattice of Λ_X, see Definition 2.2.22.

Proof. It follows immediately from Lemma 2.2.43 that there is a Hodge isometry $\Lambda_X \simeq \tilde{H}(X, -\frac{\delta}{2}, \mathbb{Z})$. Since $-\delta$ has the same square and divisibility as δ, (4.3.1) follows from Remark 4.3.6. For (4.3.2), we recall Proposition 2.2.46: we have a map

$$\begin{array}{rrcl} \left(\frac{\delta}{2}\cdot -\right): & T(X) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ & x & \longmapsto & \frac{\delta}{2}\cdot x, \end{array}$$

and there is a Hodge isometry

$$T(\Lambda_X) \simeq \ker\left(\frac{\delta}{2}\cdot -\right).$$

However, since δ has divisibility $2n-2$ in $\tilde{H}(X,\mathbb{Z})$, we have

$$\ker\left(\frac{\delta}{2}\cdot -\right) = T(X),$$

and the result follows. □

The following theorem, due to Taelman in the case $n = 2$, and Beckmann in the general case, provides one half of a Derived Torelli statement.

Theorem 4.3.9. *[Bec23; Tae23] Let X and Y be hyperkähler manifolds of* $\mathrm{K3}^{[n]}$*-type. Then any derived equivalence $\mathcal{D}^b(X) \simeq \mathcal{D}^b(Y)$ induces a Hodge isometry $\Lambda_X \simeq \Lambda_Y$.*

Corollary 4.3.10. *[Bec23, Corollary 9.3] Let X and Y be derived equivalent hyperkähler manifolds of* $\mathrm{K3}^{[n]}$*-type. Then there exists a Hodge isometry $T(X) \simeq T(Y)$.*

Proof. The transcendental sublattice of the Hodge lattice Λ_X is $T(X)$, by Proposition 4.3.8. Now the result follows from the fact that any Hodge isometry $\Lambda_X \simeq \Lambda_Y$ restricts to a Hodge isometry $T(\Lambda_X) \simeq T(\Lambda_Y)$ combined with Theorem 4.3.9. □

Theorem 4.3.11. *[Bec23, Corollary 9.6] Let S be a K3 surface, let $v \in \mathrm{N}(S)$ be a primitive Mukai vector, and let $H \in \mathrm{NS}(S)$ be a v-generic polarisation. Write $M \coloneqq M_H(v)$. If X is a hyperkähler manifold of $\mathrm{K3}^{[n]}$-type which is a Fourier–Mukai partner of M, then X is birational to a moduli space of sheaves on S.*

Proof. By Corollary 4.3.10, there is a Hodge isometry $T(X) \simeq T(M)$. By Theorem 2.3.11, the Mukai morphism induces a Hodge isometry $T(S) \simeq T(M)$, hence the Hodge lattices $T(X)$ and $T(S)$ are Hodge isometric. Therefore, X is birational to a moduli space of sheaves on S by Proposition 2.3.13. □

4.3.2 Ogg–Shafarevich Theory

Recall the following result by Addington, Donovan, and Meachan:

Theorem 4.3.12. *[ADM16] Let S be a K3 surface with $\mathrm{NS}(S) = \langle H \rangle$, where H is ample. Then for any $d, e \in \mathbb{Z}$, there is an equivalence*

$$\mathcal{D}^b(\overline{\mathrm{Pic}}^d, \alpha_d^e) \simeq \mathcal{D}^b(\overline{\mathrm{Pic}}^e, \alpha_e^{-d}).$$

The main goal of this subsection is to exhibit this result as a higher-dimensional generalisation of a result by Donagi and Pantev of [DP08]. This is done by Corollary 4.3.17 below. The statement and proof of Corollary 4.3.17 heavily relies on our explicit computation of the obstruction class in Theorem 2.4.15.

Twisted derived equivalences of elliptic K3 surfaces

First, we briefly recall the results of Donagi and Pantev. Their results hold more generally for elliptic surfaces $S \to \mathbb{P}^1$ whose fibres have at worst I_1-singularities, but we restrict our attention to elliptic K3 surfaces for now. Let $S \to \mathbb{P}^1$ be an elliptic K3 surface which admits a section. Recall that the Tate–Shafarevich group $Ш(S)$ of S parametrises pairs $(X \to \mathbb{P}^1, \phi)$, where $X \to \mathbb{P}^1$ is an elliptic K3 surface, and $\phi : \mathrm{J}^0(X) \simeq S$ is an isomorphism over $\mathbb{P}^1$ which preserves the chosen sections, see [Huy16, §11.5] for details. Elements of $Ш(S)$ are called S-torsors. Moreover, there is a canonical isomorphism

$$Ш(S) \simeq \mathrm{Br}(S). \tag{4.3.3}$$

This isomorphism can be understood as follows. Suppose $(X \to \mathbb{P}^1, \phi)$ is an S-torsor. Then ϕ exhibits S as a coarse moduli space of sheaves on X, hence there is a unique Brauer class $\alpha_X \in \mathrm{Br}(S)$ which is the obstruction to the existence of a universal sheaf on $X \times S$ by Definition/Proposition 2.3.6. The isomorphism (4.3.3) is given by

$$\begin{array}{rcl} Ш(S) & \longrightarrow & \mathrm{Br}(S) \\ (X \to \mathbb{P}^1, \phi) & \longmapsto & \alpha_X \end{array}.$$

Moreover, short exact sequence (2.4.5) takes the following form in this setting:

$$0 \to \langle \alpha_X \rangle \to \mathrm{Br}(S) \to \mathrm{Br}(X) \to 0,$$

and via (4.3.3), this sequence may be interpreted as the sequence

$$0 \to \langle [(X \to \mathbb{P}^1, \phi)] \rangle \to Ш(S) \to \mathrm{Br}(X) \to 0.$$

Let $(Y \to \mathbb{P}^1, \psi) \in Ш(S)$ be another S-torsor, and denote by $\overline{\alpha_Y} \in \mathrm{Br}(X)$ the image of $[(Y \to \mathbb{P}^1, \psi)] \in \mathrm{Br}(S)$ in $\mathrm{Br}(X)$. Similarly, we denote by $\overline{\alpha_X} \in \mathrm{Br}(Y)$ the image of α_X.

Theorem 4.3.13. *[DP08] Keeping the notation as above, there is an equivalence*

$$\mathcal{D}^b(X, \overline{\alpha_Y}) \simeq \mathcal{D}^b(Y, \overline{\alpha_X}^{-1}).$$

Tate–Shafarevich groups of Beauville–Mukai systems

We now turn our attention back to Beauville–Mukai systems. Let S be a K3 surface. For M a smooth moduli space of sheaves on S, recall that there is a natural chain of inclusions $T(S) \hookrightarrow T'(M) \hookrightarrow T(S)^*$. Here,

$$T'(M) := \frac{H^2(M, \mathbb{Z})}{\mathrm{NS}(M)}$$

satisfies $\mathrm{Br}(M) \simeq T'(M) \otimes \mathbb{Q}/\mathbb{Z}$, see Equation (2.2.12). We denote $A := T'(M)/T(S)$, and consider it as a subgroup of $A_{T(S)} \simeq T(S)^*/T(S)$. Using this, we obtain the short exact sequence

$$0 \to A \to T(S) \otimes \mathbb{Q}/\mathbb{Z} \to T'(M) \otimes \mathbb{Q}/\mathbb{Z} \to 0, \qquad (4.3.4)$$

where A is embedded in $T(S) \otimes \mathbb{Q}/\mathbb{Z}$ using the inclusion $A \subset A_{T(S)} \hookrightarrow T(S) \otimes \mathbb{Q}/\mathbb{Z}$.

Proposition 4.3.14. *Let S be a K3 surface, and let $H \subset S$ be a smooth, irreducible curve whose class in $\mathrm{NS}(S)$ is primitive, and such that $H^2 = 2g - 2 \geq 2$. If we take $M = \overline{\mathrm{Pic}}^d$ in short exact sequence (4.3.4), we obtain the short exact sequence:*

$$0 \to \frac{A}{\ker(\zeta_H)} \to Ш(S, H) \to \mathrm{Br}(\overline{\mathrm{Pic}}^d) \to 0. \qquad (4.3.5)$$

Moreover, the kernel is cyclic of order

$$\frac{\mathrm{div}(H)}{\mathrm{div}(v_d)} = \frac{\mathrm{div}(H)}{\gcd(\mathrm{div}(H), d + 1 - g)}.$$

Here, v_d is the Mukai vector of $\overline{\mathrm{Pic}}^d$, see (4.2.1).

Proof. Recall from the previous section that we have

$$\mathrm{Im}(T'(\overline{\mathrm{Pic}}^d) \hookrightarrow T(S)^*) = \{a \in T(S)^* \mid a \cdot v_d \in \mathrm{div}(v_d)\mathbb{Z}\}.$$

This implies that $\ker(\zeta_H) \subset A$, hence short exact sequence (4.3.5) is obtained from (4.3.4) by taking the quotient with $\ker(\zeta_H)$ and using the third isomorphism theorem. Choose any $a \in A$ such that $a \cdot v_d = \mathrm{div}(v_d)$. We claim that a generates $A/\ker(\zeta_H)$. Let $b \in A$ be any other element, then we can write $b \cdot H = \mathrm{div}(v_d) \cdot R$ for some $R \in \mathbb{Z}$. Since $Ra - b \in A$ satisfies $(Ra - b)(H) = 0$, it follows that $Ra = b$ in $A/\ker(\zeta_H)$. This shows that a generates $A/\ker(\zeta_H)$. Moreover, the order of a is precisely $\mathrm{div}(H)/\mathrm{div}(v_d)$. □

Corollary 4.3.15. *In the notation of Proposition 4.3.14, we have an isomorphism*

$$Ш(S, H) \simeq \mathrm{Br}(\overline{\mathrm{Pic}}^{g-1}).$$

Proof. Since $v_{g-1} = (0, H, 0)$ it follows that $\operatorname{div}(v_{g-1}) = \operatorname{div}(H)$, and now the statement follows from Proposition 4.3.14. $\square$

Corollary 4.3.15 is slightly surprising. One might have expected an isomorphism between $Ш(S,H)$ and $\mathrm{Br}(\overline{\mathrm{Pic}}^0)$, as is the case when $H^2 = 0$. Instead, if the Picard rank of S is 1, we have a short exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to Ш(S,H) \to \mathrm{Br}(\overline{\mathrm{Pic}}^0) \to 0$$

by Proposition 4.3.14. Note, however, that if $H^2 = 0$, we have $g = 1$, so that $\overline{\mathrm{Pic}}^0 = \overline{\mathrm{Pic}}^{g-1}$. Therefore, Corollary 4.3.15 is a generalisation of the well-known isomorphism $Ш(S) \simeq \mathrm{Br}(S)$ for an elliptic K3 surface S with a section.

Since we have, for any $d \in \mathbb{Z}$, a surjective group homomorphism $Ш(S,H) \to \mathrm{Br}(\overline{\mathrm{Pic}}^d)$, it is an important question what the preimages of the obstruction class are. We answer this question in the following proposition, using Theorem 2.4.15.

Proposition 4.3.16. *For any $d \in \mathbb{Z}$, the homomorphism $Ш(S,H) \to \mathrm{Br}(\overline{\mathrm{Pic}}^d)$ of Proposition 4.3.14 satisfies*

$$[\overline{\mathrm{Pic}}^1] \mapsto \alpha_d,$$

where $\alpha_d \in \mathrm{Br}(\overline{\mathrm{Pic}}^d)$ is the obstruction to the existence of a universal sheaf on $S \times \overline{\mathrm{Pic}}^d$.

Proof. Recall that the element $[\overline{\mathrm{Pic}}^1] \in Ш(S,H)$ is the image of the element

$$\overline{u} \in T(S) \otimes \mathbb{Q}/\mathbb{Z},$$

where $u \in H^2(S,\mathbb{Z})$ is a class with $u \cdot H = 1$. We show that the image of $\overline{u}$ in $\mathrm{Br}(\overline{\mathrm{Pic}}^d)$ is precisely the obstruction class. Consider the chain of surjections

$$\begin{array}{lll} T(S) \otimes \mathbb{Q}/\mathbb{Z} & \twoheadrightarrow T'(\overline{\mathrm{Pic}}^d) \otimes \mathbb{Q}/\mathbb{Z} & \twoheadrightarrow T(S)^* \otimes \mathbb{Q}/\mathbb{Z}. \\ & \quad \simeq \mathrm{Br}(\overline{\mathrm{Pic}}^d) & \quad \simeq \mathrm{Br}(S) \end{array}$$

Since $\overline{u} \in T(S)^*$, it follows that the image of $\overline{u}$ in $\mathrm{Br}(\overline{\mathrm{Pic}}^d)$ is in the kernel of the map $\mathrm{Br}(\overline{\mathrm{Pic}}^d) \to \mathrm{Br}(S)$, which is generated by α_d. From $u \cdot v_d = 1$, it follows that $\overline{u}$ maps to α_d by Theorem 2.4.15. $\square$

We can now see how Theorem 4.3.12 resembles Theorem 4.3.13.

Corollary 4.3.17. *Let S be a K3 surface of Picard rank 1. Let $H \in \mathrm{NS}(S)$ be an ample generator, and let $d, e \in \mathbb{Z}$. Denote by $\overline{\alpha_d} \in \mathrm{Br}(\overline{\mathrm{Pic}}^e)$, resp. $\overline{\alpha_e} \in \mathrm{Br}(\overline{\mathrm{Pic}}^d)$ the images of $[\overline{\mathrm{Pic}}^d]$, resp. $[\overline{\mathrm{Pic}}^e]$ in $\mathrm{Br}(\overline{\mathrm{Pic}}^e)$, resp. $\mathrm{Br}(\overline{\mathrm{Pic}}^d)$. Then there is an equivalence*

$$\mathcal{D}^b(\overline{\mathrm{Pic}}^d, \overline{\alpha_e}) \simeq \mathcal{D}^b(\overline{\mathrm{Pic}}^e, \overline{\alpha_d}^{-1}).$$

Proof. By Proposition 4.3.16, we have $\overline{\alpha_d} = \alpha_e^d \in \mathrm{Br}(\overline{\mathrm{Pic}}^e)$, and $\overline{\alpha_e} = \alpha_d^e \in \mathrm{Br}(\overline{\mathrm{Pic}}^d)$. Combining this with Theorem 4.3.12 proves the result. $\square$

It is an interesting open question whether Corollary 4.3.17 holds more generally. For example, when we consider the twisted Picard varieties of [HM23], or when we drop the assumption that $\rho = 1$.

4.3.3 Lagrangian Fibrations are a Derived Invariant

In this section, we discuss the following question.

Question 4.3.18. Let X and Y be derived equivalent hyperkähler manifolds. If X admits a rational Lagrangian fibration, does Y also admit a rational Lagrangian fibration?

If X and Y are K3 surfaces, the answer to Question 4.3.18 is affirmative by Lemma 2.2.16 and Lemma 3.3.2. Before we discuss the higher-dimensional analogue, let us define what a rational Lagrangian fibration is.

Definition 4.3.19. Let X be a hyperkähler manifold. A *rational Lagrangian fibration* on X is a composition

$$X \overset{\sim}{\dashrightarrow} Y \xrightarrow{f} B,$$

where Y is another hyperkähler manifold, and f is a Lagrangian fibration.

We note that the answer to Question 4.3.18 is negative if we drop the word *rational*, see for example [BM14a, §11]. We prove in this section that the answer to Question 4.3.18 is affirmative when X and Y are of any of the currently known deformation types. The currently known deformation types are: $\mathrm{K3}^{[n]}$, generalised Kummer, OG6, and OG10. Our strategy relies on the SYZ conjecture for hyperkähler manifolds:

Conjecture 4.3.20 (SYZ conjecture for hyperkähler manifolds)**.** *[Saw03] Let X be a hyperkähler manifold. Then X admits a rational Lagrangian fibration if and only if there is a non-zero isotropic class in* $\mathrm{NS}(X)$*.*

For all the currently known deformation types, Conjecture 4.3.20 has already been established [BM14a; Mar14; Yos16; MR21; MO22], but a general proof is unknown.

Lemma 4.3.21. *Let X and Y be derived equivalent hyperkähler manifolds of dimension $2n$. If either n is odd or $b_2(X)$ is odd, then there is a rational isometry* $\mathrm{NS}(X)_{\mathbb{Q}} \simeq \mathrm{NS}(Y)_{\mathbb{Q}}$.

Proof. We denote by $\tilde{H}(X,\mathbb{Q})$ the Hodge lattice of K3-type whose underlying lattice structure is $U \oplus H^2(X,\mathbb{Z})$ and whose Hodge structure is inherited from $H^2(X,\mathbb{Z})$. By our assumptions on n and $b_2(X)$, we may apply [Tae23, Theorem C]: There is a rational Hodge isometry $f : \tilde{H}(X,\mathbb{Q}) \simeq \tilde{H}(Y,\mathbb{Q})$. Since f is a Hodge isometry, it restricts to a rational Hodge isometry $T(X)_{\mathbb{Q}} \simeq T(Y)_{\mathbb{Q}}$. Therefore, f also restricts to a rational isometry $\mathrm{N}(X)_{\mathbb{Q}} \simeq \mathrm{N}(Y)_{\mathbb{Q}}$.

From the decompositions $\mathrm{N}(X)_{\mathbb{Q}} = \mathrm{NS}(X)_{\mathbb{Q}} \oplus U_{\mathbb{Q}}$ and $\mathrm{N}(Y)_{\mathbb{Q}} = \mathrm{NS}(Y)_{\mathbb{Q}} \oplus U_{\mathbb{Q}}$, it follows from the Witt Cancellation Theorem 2.2.15 that there is a rational isometry $\mathrm{NS}(X)_{\mathbb{Q}} \simeq \mathrm{NS}(Y)_{\mathbb{Q}}$, as required. □

Remark 4.3.22. The assumptions on n and $b_2(X)$ are satisfied for all the currently known deformation types. Namely, $b_2(X)$ is odd if X is of $\mathrm{K3}^{[n]}$-type ($n \geq 2$) or of generalised Kummer type, and n is odd if X is a K3 surface, or of OG6-type or OG10-type.

Proposition 4.3.23. *Let X be a hyperkähler manifold of any of the currently known deformation types. Let Y be a hyperkähler manifold which is derived equivalent to X. Then X admits a rational Lagrangian fibration if and only if Y does.*

Proof. By Lemma 4.3.21, there is a rational isometry $\mathrm{NS}(X)_{\mathbb{Q}} \simeq \mathrm{NS}(Y)_{\mathbb{Q}}$. Therefore, $\mathrm{NS}(X)$ represents zero if and only if $\mathrm{NS}(Y)$ does. The result follows from the proof of the SYZ conjecture for hyperkähler manifolds the currently known deformation types [BM14a; Mar14; Yos16; MR21; MO22]. □

4.4 Birational Equivalence for Moduli Spaces

In this section, we study birational equivalence for moduli spaces of sheaves on elliptic K3 surfaces with a section. More specifically, we study the birational geometry of Beauville–Mukai systems on such K3 surfaces. Theorem 4.4.8 below fully classifies which Beauville–Mukai systems over elliptic K3 surfaces are birational. This classification uses our explicit computation of the obstruction class.

It is a well-known fact that a K3 surface S admits an elliptic fibration with a section if and only if there is an isotropic class $F \in \mathrm{NS}(S)$ of divisibility 1, see for example [Huy16, Remark 3.2.13 and §11.4] and Lemma 3.3.5. This is equivalent to the existence of an embedding $U \subset \mathrm{NS}(S)$. In this case, the lattice $\mathrm{N}(S) \simeq \mathrm{NS}(S) \oplus U$ admits an embedding $U^{\oplus 2} \subset \mathrm{N}(S)$, so that we may apply Lemma 2.2.9 to $\mathrm{N}(S)$.

Recall that the main result about the birational geometry of $M_H(v)$ is Markman's *Birational Torelli Theorem*, c.f. Theorem 2.2.25. The Birational Torelli Theorem asserts that any hyperkähler manifold of $\mathrm{K3}^{[n]}$-type X comes equipped with a natural $O(\tilde{\Lambda}_{\mathrm{K3}})$-orbit of embeddings $H^2(X, \mathbb{Z}) \hookrightarrow \tilde{\Lambda}_{\mathrm{K3}}$. Here $\tilde{\Lambda}_{\mathrm{K3}} = \Lambda_{\mathrm{K3}} \oplus U = U^{\oplus 4} \oplus E_8^{\oplus 2}$ is the extended K3 lattice. Moreover, two hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type are birational if and only if there exists a commutative square of Hodge metric morphisms:

$$\begin{array}{ccc} H^2(X,\mathbb{Z}) & \longrightarrow & H^2(Y,\mathbb{Z}) \\ \downarrow & & \downarrow \\ \tilde{\Lambda}_{\mathrm{K3}} & \longrightarrow & \tilde{\Lambda}_{\mathrm{K3}}. \end{array}$$

If $M = M_H(v)$ is a moduli space of sheaves on a K3 surface S, then the orbit of embeddings $H^2(M, \mathbb{Z}) \hookrightarrow \tilde{\Lambda}_{\mathrm{K3}}$ is the one containing the following embedding:

$$H^2(M, \mathbb{Z}) \simeq v^{\perp} \hookrightarrow \tilde{H}(S, \mathbb{Z}) \simeq \tilde{\Lambda}_{\mathrm{K3}}. \tag{4.4.1}$$

Here, the first isometry is the Mukai morphism of Theorem 2.3.11, and the isometry $\tilde{H}(S, \mathbb{Z}) \simeq \tilde{\Lambda}_{\mathrm{K3}}$ is arbitrary. Note that the $O(\tilde{\Lambda}_{\mathrm{K3}})$-orbit of the embedding (4.4.1) is independent of the choice of isometry $\tilde{H}(S, \mathbb{Z}) \simeq \tilde{\Lambda}_{\mathrm{K3}}$.

Let S be a K3 surface. Recall from Definition 2.4.1 that for a vector $v \in \mathrm{N}(S)$, we call the element

$$a_v := -\frac{v}{\mathrm{div}(v)} \in A_{\mathrm{N}(S)}$$

the Căldăraru class of v, and that the element $\omega_v \in A_{T(S)}$ corresponding to a_v via the natural isomorphism $A_{\mathrm{N}(S)} \simeq A_{T(S)}(-1)$ of Remark 2.2.2 is called the transcendental Căldăraru class of v.

Proposition 4.4.1. *Let S be a K3 surface for which there exists an embedding $U \subset \mathrm{NS}(S)$. Let $u, v \in \mathrm{N}(S)$ be primitive Mukai vectors with $u^2 = v^2 = 2n-2$, $n > 1$. Then the moduli spaces $M(u)$ and $M(v)$ are birational if and only if there is a Hodge isometry $\psi : T(S) \simeq T(S)$ such that $\overline{\psi}(\omega_v) = \omega_u$.*

Proof. Suppose $M(u)$ and $M(v)$ are birational. Then by the Birational Torelli Theorem, there is a commutative diagram of Hodge isometries:

$$\begin{tikzcd} v^{\perp} \arrow[r, "\simeq"] \arrow[d] & u^{\perp} \arrow[d] \\ \tilde{H}(S,\mathbb{Z}) \arrow[r, "\simeq", "\Phi"'] & \tilde{H}(S,\mathbb{Z}). \end{tikzcd}$$

Denote $\phi \coloneqq \Phi|_{\mathrm{N}(S)} : \mathrm{N}(S) \simeq \mathrm{N}(S)$ and $\psi \coloneqq \Phi|_{T(S)} : T(S) \simeq T(S)$, and note that ψ is a Hodge isometry. Then $\phi(v) = \pm u$. Possibly replacing Φ by $-\Phi$, we may assume $\phi(v) = u$. We have $\overline{\phi}(a_v) = a_u$, hence $\overline{\psi}(\omega_v) = \omega_u$ by Lemma 2.2.3.

Conversely, suppose $\psi : T(S) \to T(S)$ is a Hodge isometry which satisfies $\overline{\psi}(\omega_v) = \omega_u$. Since $T(S)^{\perp} = \mathrm{N}(S)$, we may apply Lemma 2.2.8, and extend ψ to a Hodge isometry $\Psi : \tilde{H}(S,\mathbb{Z}) \to \tilde{H}(S,\mathbb{Z})$. Denote $\phi \coloneqq \Psi|_{\mathrm{N}(S)} : \mathrm{N}(S) \simeq \mathrm{N}(S)$. Since $\overline{\psi}(\omega_v) = \omega_u$, we have $a_{\phi(v)} = \overline{\phi}(a_v) = a_u$. By Lemma 2.2.9, there is an isometry $\iota : \mathrm{N}(S) \simeq \mathrm{N}(S)$ which maps $\phi(v)$ to u and which acts trivially on $A_{\mathrm{N}(S)}$. In particular, we can extend ι, by the identity on $T(S)$, to a Hodge isometry $\Gamma : \tilde{H}(S,\mathbb{Z}) \simeq \tilde{H}(S,\mathbb{Z})$. Now $\Gamma \circ \Psi$ is a Hodge isometry of $\tilde{H}(S,\mathbb{Z})$ which maps v to u, and this implies by the Birational Torelli Theorem that $M(v)$ and $M(u)$ are birational. $\square$

Remark 4.4.2. Since $-\mathrm{id}_{T(S)}$ is a Hodge isometry of $T(S)$, the transcendental Căldăraru class ω_v is always in the same orbit as $-\omega_v$. For this reason, we may drop the choice of a sign in front of the definition of the Căldăraru class, and we will also refer to the element

$$-a_v = \frac{v}{\mathrm{div}(v)} \in A_{\mathrm{N}(S)}$$

as the Căldăraru class of v.

Proposition 4.4.3. *Let S be a K3 surface for which there exists an embedding $U \subset \mathrm{NS}(S)$. If M is a fine(!) $2n$-dimensional moduli space of sheaves with Mukai vector $v \in \mathrm{N}(S)$, then M is birational to $S^{[n]}$.*

Proof. If $n = 1$, then M is a Fourier–Mukai partner of S by the Derived Torelli Theorem [Muk87; Orl03]. However, since $U \subset \mathrm{NS}(S)$, S does not have any non-trivial Fourier–Mukai partners by [Hos+02, Corollary 2.7(3)]. Therefore, we have $M \simeq S$, as required.

Now assume $n \geq 2$. By Theorem 2.4.15, we have $\mathrm{div}(v) = 1$. In particular, the Căldăraru class of v is

$$\frac{v}{\mathrm{div}(v)} = v = 0 \in A_{\mathrm{N}(S)}.$$

If $u \in \mathrm{N}(S)$ is any other Mukai vector with $u^2 = 2n-2$ and $\mathrm{div}(u) = 1$, then $a_u = 0$ as well, hence M is birational to $M(u)$ by Proposition 4.4.1. If we let $u = (1, 0, 1-n)$, then $M(u) \simeq S^{[n]}$, hence M is birational to $S^{[n]}$. $\square$

Corollary 4.4.4. *Let S be as in Proposition 4.4.3. Suppose moreover that* $\mathrm{NS}(S)$ *is unimodular. Then any moduli $2n$-dimensional space $M(v)$ of sheaves on S is birational to the Hilbert scheme $S^{[n]}$.*

Proof. Since $\mathrm{NS}(S)$ is unimodular, $\mathrm{N}(S)$ is also unimodular. Therefore, any primitive Mukai vector $v \in \mathrm{N}(S)$ has divisibility 1, hence the moduli space $M(v)$ is birational to a Hilbert scheme of points on S by Proposition 4.4.3. $\square$

Combining Proposition 4.4.3 with [Bec23, Corollary 9.9] yields the following interesting result, which provides examples of hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type with Hodge isometric transcendental lattices, but which are not derived equivalent. These are counterexamples to a question raised in [KK24, Problem 1.1], and a similar question raised in [PR23, Question 2].

Theorem 4.4.5. *[Bec23] Let S be a K3 surface for which there exists an embedding $U \subset \mathrm{NS}(S)$. Let $M = M_H(v)$ be a moduli space of sheaves on S of dimension $2n > 2$. Then the following are equivalent.*

i) M is a fine moduli space.

ii) M is birational to the Hilbert scheme $S^{[n]}$.

iii) M is derived equivalent to the Hilbert scheme $S^{[n]}$.

In particular, if M is a non-fine moduli space of sheaves, then M is not derived equivalent to $S^{[n]}$.

The equivalence *ii)* $\iff$ *iii)* in Theorem 4.4.5 was already noted by Beckmann in [Bec23]. The addition of item *i)* is a straightforward application of Theorem 2.4.15, combined with Proposition 4.4.3.

Proof. The implication *i)* $\implies$ *ii)* is Proposition 4.4.3. We now prove that *ii)* $\implies$ *i)*. Suppose M is birational to $S^{[n]}$. Let $u = (1, 0, 1-n) \in \mathrm{N}(S)$, so that $S^{[n]} \simeq M(u)$. Then the divisibility of v in $\mathrm{N}(S)$ is equal to the divisibility of $u = (1, 0, 1-n)$ in $\mathrm{N}(S)$ by Proposition 4.4.1, hence M is a fine moduli space by Corollary 2.4.16. The equality of the divisibilities of v and u is also an easy consequence of the Birational Torelli Theorem, and does not depend on the existence of a primitive embedding $U \subset \mathrm{NS}(S)$. Indeed, by Theorem 2.2.25, if M and $S^{[n]}$ are birational, there is a Hodge isometry $\phi : \tilde{H}(S, \mathbb{Z}) \simeq \tilde{H}(S, \mathbb{Z})$ which satisfies $\phi(u) = \pm v$. Since ϕ is an isometry, the divisibilities of u and v are equal.

The implication *iii)* $\implies$ *ii)* is [Bec23, Proposition 9.9]. The converse, *ii)* $\implies$ *iii)*, is [Hal21]. □

As an immediate consequence of Theorem 4.4.5, we obtain the following corollary.

Corollary 4.4.6. *There exist moduli spaces of sheaves M and M', on the same K3 surface, for which there is a Hodge isometry $T(M) \simeq T(M')$, but which are not derived equivalent.*

We now apply Proposition 4.4.1 to Beauville–Mukai systems. For $d \in \mathbb{Z}$, let $v_d = (0, H, d+1-g)$, where $H \subset S$ is a smooth, irreducible curve on a (not necessarily elliptic) K3 surface S, and assume H is primitive in $\mathrm{NS}(S)$, and $H^2 = 2g-2 \geq 2$. We write

$$a_d := \frac{v_d}{\mathrm{div}(v_d)}$$

for the Căldăraru class of v_d, see Definition 2.4.1 and Remark 4.4.2.

Lemma 4.4.7. *We have*

$$a_d = \frac{H}{\mathrm{div}(v_d)} \in A_{\mathrm{N}(S)}.$$

In particular, for $e \in \mathbb{Z}$, we have $a_d = a_e$ if and only if $\mathrm{div}(v_d) = \mathrm{div}(v_e)$.

Proof. By definition,

$$a_d = \frac{1}{\mathrm{div}(v_d)}(0, H, d+1-g) = \frac{H}{\mathrm{div}(v_d)} + (0, 0, \frac{d+1-g}{\mathrm{div}(v_d)}).$$

The result follows from the fact that $\mathrm{div}(v_d)$ divides $d+1-g$. $\square$

Theorem 4.4.8. *Let S be a K3 surface for which there exists an embedding $U \subset \mathrm{NS}(S)$. Let $H \subset S$ be a smooth irreducible curve with $H^2 = 2g-2$ and with primitive class in $\mathrm{NS}(S)$. Let $d, e \in \mathbb{Z}$. Then the following are equivalent:*

i) $\overline{\mathrm{Pic}}^d$ is birational to $\overline{\mathrm{Pic}}^e$.

ii) The obstruction classes of $\overline{\mathrm{Pic}}^d$ and $\overline{\mathrm{Pic}}^e$ have the same order.

iii) The Mukai vectors of $\overline{\mathrm{Pic}}^d$ and $\overline{\mathrm{Pic}}^e$ have the same divisibility. More precisely, we have $\gcd(\mathrm{div}(H), d+1-g) = \gcd(\mathrm{div}(H), e+1-g)$.

Proof. Note that $\gcd(\mathrm{div}(H), d+1-g) = \mathrm{div}(v_d)$ for all $d \in \mathbb{Z}$. The equivalence *ii)* $\iff$ *iii)* follows from the fact that the order of α_d is equal to $\mathrm{div}(v_d)$ for all $d \in \mathbb{Z}$ by Theorem 2.4.15.

By Proposition 4.4.1, $\overline{\mathrm{Pic}}^d$ and $\overline{\mathrm{Pic}}^e$ are birational if and only if there exists a Hodge isometry $\psi : T(S) \simeq T(S)$ such that $\overline{\psi}(\omega_d) = \omega_e$. In this case, the orders of ω_d and ω_e in $A_{T(S)}$ are equal. Since the order of ω_d equals the order of a_d, which is $\mathrm{div}(v_d)$, this means that $\mathrm{div}(v_d) = \mathrm{div}(v_e)$.

Conversely, if $\mathrm{div}(v_d) = \mathrm{div}(v_e)$, then $a_d = a_e$ by Lemma 4.4.7, and $\overline{\mathrm{Pic}}^d$ and $\overline{\mathrm{Pic}}^e$ are birational by Proposition 4.4.1. $\square$

To conclude, we note that Theorem 4.4.8 provides many examples of birational Beauville–Mukai systems $\overline{\mathrm{Pic}}^d$ and $\overline{\mathrm{Pic}}^e$ whose generic fibres are not isomorphic as $\overline{\mathrm{Pic}}^0_\eta$-torsors. Let S and H be as in Theorem 4.4.8. Recall from Remark 4.2.16 that the generic fibres of $\overline{\mathrm{Pic}}^e$ and $\overline{\mathrm{Pic}}^d$ are isomorphic $\overline{\mathrm{Pic}}^0_\eta$-torsors if and only if $d \equiv e \pmod{\mathrm{div}(H)}$. Therefore, if $\mathrm{div}(H)$ is big enough, we may find $d, e \in \mathbb{Z}$ such that $e \not\equiv d \pmod{\mathrm{div}(H)}$, but $\gcd(\mathrm{div}(H), d+1-g) = \gcd(\mathrm{div}(H), e+1-g)$. In this case, $\overline{\mathrm{Pic}}^e$ and $\overline{\mathrm{Pic}}^d$ are birational, but their generic fibres do not give isomorphic torsors.

It remains to be shown that such a smooth, irreducible curve H exists. Note that for $\rho = 2$, $\mathrm{NS}(S) \simeq U$ is unimodular. Then every primitive element of $\mathrm{NS}(S)$ has divisibility 1 (see Lemma 2.4.4), so Theorem 4.4.8 is trivial for $\rho = 2$. On the other hand, for $\rho > 2$, we can always find a smooth, irreducible curve $H \subset S$ whose class in $\mathrm{NS}(S)$ has higher divisibility. We now show how to construct these curves. Let $D \subset \mathrm{NS}(S)$ be a primitive divisor with $\mathrm{div}(D) > 1$, and let $L \in \mathrm{NS}(S)$ be any ample divisor. Let m be a large multiple of $\mathrm{div}(D)$. We claim that the general member H of the complete linear system $|mL + D|$ is a smooth and irreducible curve, and $\mathrm{div}(H) = k\,\mathrm{div}(D)$ for some $k \geq 1$.

Firstly, note that for any divisor $D' \in \mathrm{NS}(S)$, we have $(mL + D) \cdot D' = mL \cdot D' + D \cdot D'$, which is divisible by $\mathrm{div}(D)$, and therefore $\mathrm{div}(H) = \mathrm{div}(mL + D) = k \cdot \mathrm{div}(D)$ for some $k \geq 1$.

To check that the general member of $|mL + D|$ is smooth and irreducible, we show that the linear system is base-point free. Firstly, note that $mL + D$ is big and nef (see [Huy16, §2.3]), which follows from the openness of the ample cone. We may apply [Huy16, Corollary 2.3.15(ii)]: any big and nef complete linear system with a base locus is of the form $|nE + C|$, where $E \subset S$ is an elliptic curve, and $C \subset S$ is a rational curve with $C \cdot E = 1$. Note that the divisor $nE + C$ has divisibility 1, since $E \cdot (nE + C) = nE^2 + E \cdot C = 1$. Since the divisibility of $mL + D$ is $k\,\mathrm{div}(D) > 1$ for some $k \geq 1$, the complete linear system $mL + D$ has an empty base locus. It follows from [Huy16, Remark 2.3.7(ii)] that $|mL + D|$ contains an irreducible curve, and therefore by [Huy16, Corollary 2.3.6], the general member of $|mL + D|$ is a smooth, irreducible curve.

Suppose that H is not primitive. Then there is a primitive $H' \in \mathrm{NS}(S)$ and a $k' > 1$ such that $H = k'H'$. Write $\ell = \gcd(k', \mathrm{div}(D))$. We have

$$\frac{D}{\ell} = \frac{mL}{\ell} + \frac{H}{\ell}. \tag{4.4.2}$$

Since ℓ divides m, the vector on the right-hand-side of (4.4.2) is integral. However, since we chose D primitive, this means $\ell = 1$. Since k' divides $\mathrm{div}(H) = k \cdot \mathrm{div}(D)$, it follows that k' divides k, and

$$\mathrm{div}(H') = \frac{k}{k'}\,\mathrm{div}(D) \geq \mathrm{div}(D).$$

Since H' is still big and nef, the discussion above also applies to H', i.e. the general member of $|H'|$ is a smooth, irreducible curve in S, and this time it is also primitive in $\mathrm{NS}(S)$.

For any $d, e \in \mathbb{Z}$ such that $\gcd(\mathrm{div}(H'), d + 1 - g) = \gcd(\mathrm{div}(H'), e + 1 - g)$, but such that $d \not\equiv e \pmod{\mathrm{div}(H')}$, the Beauville–Mukai systems $\overline{\mathrm{Pic}}^d$ and $\overline{\mathrm{Pic}}^e$ are birational by Theorem 4.4.8, but their generic fibres are not isomorphic $\overline{\mathrm{Pic}}^0_\eta$-torsors by Remark 4.2.16.

Appendix to Chapter 4

4.A Transcendental Lattices of Moduli Spaces of Twisted Sheaves

The main inspiration for this section is the following result by Addington. We include a sketch of the proof for completeness.

Recall from the Birational Torelli Theorem 2.2.25 that for any hyperkähler manifold X of $\mathrm{K3}^{[n]}$-type, there is a natural orbit of embeddings $H^2(X,\mathbb{Z}) \hookrightarrow \tilde{\Lambda}_{\mathrm{K3}}$, where $\tilde{\Lambda}_{\mathrm{K3}} = U^{\oplus 4} \oplus E_8(-1)^{\oplus 2}$ is the extended K3 lattice. Such an embedding induces a Hodge structure of K3 type on $\tilde{\Lambda}_{\mathrm{K3}}$. In Proposition 4.A.1 below, $\tilde{\Lambda}^{1,1}_{\mathrm{K3}}$ denotes the integral $(1,1)$-part of $\tilde{\Lambda}_{\mathrm{K3}}$ with respect to this Hodge structure.

Proposition 4.A.1. *[Add16, Proposition 4] Let X be a hyperkähler manifold of $\mathrm{K3}^{[n]}$-type. The following are equivalent.*

i) There is a Hodge isometry $T(X) \simeq T(S)$ for some K3 surface S.

ii) X is birational to a moduli space of sheaves on some K3 surface S.

iii) There is a hyperbolic lattice $U \subset \tilde{\Lambda}^{1,1}_{\mathrm{K3}}$.

Moreover, in this case, we can take the same K3 surface in i) and ii).

Sketch of the proof. i) $\implies$ ii): Fix any Hodge isometry $T(X) \simeq T(S)$. Then there are two embeddings $T(S) \hookrightarrow \tilde{H}(S,\mathbb{Z})$ and $T(S) \simeq T(X) \hookrightarrow H^2(X,\mathbb{Z}) \hookrightarrow \tilde{\Lambda}_{\mathrm{K3}}$. Note that $\tilde{H}(S,\mathbb{Z})$ is isometric to $\tilde{\Lambda}_{\mathrm{K3}}$ as abstract lattices. Moreover, the embedding $T(S) \hookrightarrow \tilde{H}(S,\mathbb{Z})$ has $U \subset \mathrm{N}(S) = T(S)^{\perp}$, hence the two embeddings are isometric by Lemma 2.2.8. That is, there is an isometry $\tilde{H}(S,\mathbb{Z}) \simeq \tilde{\Lambda}_{\mathrm{K3}}$ which fixes $T(S)$. Such an isometry is automatically a Hodge isometry. Moreover, it exhibits $H^2(X,\mathbb{Z})$ as a corank 1 sublattice of $\tilde{H}(S,\mathbb{Z})$. Therefore we have $H^2(X,\mathbb{Z})^{\perp} = \langle v \rangle$ for some primitive $v \in \mathrm{N}(S)$. By setting up our isometries correctly, one may assume v is a Mukai vector so that $M(v)$ is a hyperkähler manifold of $\mathrm{K3}^{[n]}$-type (for a generic choice of polarisation). By the Birational Torelli Theorem 2.2.25, X and $M(v)$ are birational.

ii) $\implies$ iii): In this case there is a Hodge isometry $\tilde{\Lambda}_{\mathrm{K3}} \simeq \tilde{H}(S,\mathbb{Z})$, where $\tilde{\Lambda}_{\mathrm{K3}}$ is endowed with the Hodge structure induced by X, hence we have an isometry $\tilde{\Lambda}^{1,1}_{\mathrm{K3}} = \mathrm{N}(S)$, which contains a hyperbolic lattice.

iii) $\implies$ i): Decompose $\tilde{\Lambda}_{\mathrm{K3}} = \Lambda' \oplus U$, with $U \subset \tilde{\Lambda}^{1,1}_{\mathrm{K3}}$. Now Λ' is isometric to the K3 lattice, and it carries a natural Hodge structure of K3 type induced by $\tilde{\Lambda}_{\mathrm{K3}}$. By the surjectivity of the period map, there is a K3 surface S with $H^2(S,\mathbb{Z}) \simeq \Lambda'$. By construction, we have a Hodge isometry $T(S) \simeq T(X)$. □

There is a twisted version of Proposition 4.A.1, proved by Huybrechts in [Huy17, Lemma 2.6]. For a non-zero integer $k \in \mathbb{Z}$, we write $U(k)$ for the even lattice of rank 2 corresponding to the following matrix:

$$\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}.$$

Lemma 4.A.2. *Let X be a hyperkähler manifold of* $\mathrm{K3}^{[n]}$*-type. The following are equivalent.*

i) X is birational to a moduli space of twisted sheaves on a K3 surface.

ii) There is an embedding $U(k) \subset \tilde{\Lambda}^{1,1}_{\mathrm{K3}}$ for some $k \geq 1$.

Lemma 4.A.2, while applicable in a more general setting than Proposition 4.A.1, has a slightly less precise conclusion. This leads to the question of whether a hyperkähler manifold X for which there exists a Hodge isometry $T(X) \simeq T(S, \alpha)$ for some twisted K3 surface (S, α), with $\alpha \in \mathrm{SBr}(S)$, is birational to a moduli space of α-twisted sheaves on S. We now show that the answer to this question is negative.

Remark 4.A.3. Note that for an even non-degenerate lattice L, the existence of an integer k for which there is an embedding $U(k) \subset L$ is equivalent to the existence of a non-zero isotropic vector $v \in L$. Indeed, $U(k)$ certainly contains non-zero isotropic vectors. Conversely, let v be a non-zero isotropic vector. Let $w \in L$ be any vector such that $v \cdot w \neq 0$. Such a vector exists since L is non-degenerate. Write $w^2 = 2d$ and $v \cdot w = k$ for some $d, k \in \mathbb{Z}$. Then $kw - dv$ is isotropic and v and $kw - dv$ generate a lattice isometric to $U(m)$ for some $k \in \mathbb{Z}$ (more precisely, we have $m = kv \cdot w$).

Proposition 4.A.4. *Let X be a hyperkähler manifold of* $\mathrm{K3}^{[n]}$*-type. The following are equivalent.*

i) There is a Hodge isometry $T(X) \simeq T(S, \alpha)$ for some twisted K3 surface (S, α), where $\alpha \in \mathrm{SBr}(S)$.

ii) X is birational to a moduli space of twisted sheaves on some twisted K3 surface (S', α'), where $\alpha' \in \mathrm{SBr}(S')$.

iii) There is a (not necessarily primitive) embedding $U(k) \subset \tilde{\Lambda}^{1,1}_{\mathrm{K3}}$ for some $k \geq 1$.

However, in this case (S, α) is not necessarily isomorphic to (S', α'). In fact, (S, α) is not even necessarily derived equivalent to (S', α')

Proof. The equivalence $ii) \iff iii)$ is Lemma 4.A.2. The implication $ii) \implies i)$ is an easy consequence of the Birational Torelli Theorem 2.2.25. We now prove that $i) \implies iii)$. Since we have a rational isometry $T(X)_{\mathbb{Q}} \simeq T(S, \alpha)_{\mathbb{Q}}$ by Proposition 2.2.46, there is also a rational isometry $\tilde{\Lambda}^{1,1}_{\mathrm{K3},\mathbb{Q}} \simeq \mathrm{N}(S, \alpha)_{\mathbb{Q}}$. Since $\mathrm{N}(S, \alpha)$ represents zero, $\tilde{\Lambda}^{1,1}_{\mathrm{K3}}$ also represents zero, and the result follows from Remark 4.A.3.

For the final claim, choose any two twisted K3 surfaces (S, α), (S', α') such that there is a Hodge isometry $T(S, \alpha) \simeq T(S', \alpha')$, but for which there exists no Hodge isometry $\tilde{H}(S, \alpha, \mathbb{Z}) \simeq \tilde{H}(S', \alpha', \mathbb{Z})$. Such twisted K3 surfaces exist by [HS05b, Example 4.11]. Now any moduli space M of α-twisted sheaves on S has $T(M) \simeq T(S, \alpha) \simeq T(S', \alpha')$, but M is not birational to a moduli space of α'-twisted sheaves on S' by the Birational Torelli Theorem. □

Further Questions

The Derived Torelli Theorem

In my opinion, the most important question about derived equivalence for hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type is the following:

Question 1. Does Conjecture 4.3.2 hold for $L_X = \Lambda_X$?

As we discussed in Section 4.3.1, a positive answer to Question 1 would imply the D-equivalence conjecture for hyperkähler manifolds of $\mathrm{K3}^{[n]}$-type.

However, as we saw in Section 4.4, there exist moduli spaces of sheaves M and M' on a K3 surface S which are not derived equivalent (see Corollary 4.4.6). The transcendental lattices $T(M)$ and $T(M')$ are Hodge isometric, but we do not know whether the same holds for the $\mathrm{K3}^{[n]}$-lattices Λ_M and $\Lambda_{M'}$.

Question 2. Let M and M' be moduli spaces of sheaves on a K3 surface S. Is there a Hodge isometry $\Lambda_M \simeq \Lambda_{M'}$?

To answer Question 2, one needs to study the lattice $\Lambda_M^{1,1} = T(\Lambda_M)^\perp$. The most straightforward way to do this is by using the description $\Lambda_M \simeq \tilde{H}(M, \frac{\delta}{2}, \mathbb{Z})$, combined with Lemma 2.2.45. Note that a positive answer to Question 2 implies a negative answer to Question 1.

Twisted obstruction classes

Let (S, α) be a twisted K3 surface, with $\alpha \in \mathrm{SBr}(S)$. Let $v \in \mathrm{N}(S, \alpha)$ be a Mukai vector and let H be a v-generic polarisation. Write $M \coloneqq M_H(v; \alpha)$. The Mukai morphism is a Hodge isometry

$$\tilde{H}(S, \alpha, \mathbb{Z}) \supset v^\perp \simeq H^2(M, \mathbb{Z}),$$

hence we have a Hodge isometry $T(S, \alpha) \simeq T(M)$. Using the Mukai morphism, and writing

$$\mathrm{Br}(S, \alpha) \coloneqq \mathrm{Hom}(T(S, \alpha), \mathbb{Q}/\mathbb{Z})$$

we obtain a short exact sequence

$$0 \to \mathbb{Z}/\operatorname{div}(v)\mathbb{Z} \to \mathrm{Br}(M) \to \mathrm{Br}(S, \alpha) \to 0,$$

completely analogously to how we derived the short exact sequence of Proposition 2.4.12.

Question 3. Is the kernel the above short exact sequence generated by the obstruction class of M?

One step towards answering Question 3 is generalising Proposition 2.3.5 to moduli spaces of twisted sheaves.

Finally, it would be interesting to generalise Proposition 4.3.14 to the twisted case. This would lead to the following question about twisted Beauville–Mukai systems.

Question 4. Are there twisted derived equivalences as in Corollary 4.3.17 for twisted Beauville–Mukai systems $\overline{\mathrm{Pic}}^d_\alpha$ and $\overline{\mathrm{Pic}}^e_\alpha$? In this case, what are the Brauer classes $\overline{\alpha_d}$ and $\overline{\alpha_e}$?

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