Commutativity of Relative Pseudomonads

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I confirm that the work submitted is my own and that appropriate credit has been given where reference has been made to the work of others.

To Mum and Dad, for asking what I wanted to become and doing all they could to get me there.

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Abstract

Relative pseudomonads simultaneously generalise two important notions of category theory; that is, they generalise pseudomonads to non-endofunctors, and relative monads to bicategories. In this thesis, we study two aspects of the theory of relative pseudomonads: relative pseudomonads on 2-multicategories and pseudoalgebras for relative pseudomonads.

We develop the theory of relative monads on multicategories, expositing an analogue of the work of Kock on monads on monoidal categories. We define notions of strength, commutativity and idempotency for a relative monad T, as well as the notion of a relative multimonad. We go on to prove that idempotency implies commutativity, that a commutative relative monad is a relative multimonad, and that commutativity of T implies a multicategory structure on the Kleisli category $\mathrm{Kl}(T)$. Later, we extend this to the setting of relative pseudomonads on 2-multicategories, defining the corresponding two-dimensional notions and proving the corresponding implications.

We also develop the theory of pseudoalgebras for relative pseudomonads, constructing for a given relative pseudomonad T its Eilenberg-Moore bicategory of pseudoalgebras T-Alg. We then use pseudoalgebras to introduce the notion of 'algebraic lax idempotency' and characterise algebraically lax-idempotent relative pseudomonads; this is the counterpart in the relative setting to Kelly and Lack's characterisation of lax-idempotent pseudomonads as 'fully property-like'.

We apply our results in both cases to the presheaf relative pseudomonad $P : \mathrm{Cat} \to \mathrm{CAT}$, proving that its Eilenberg-Moore 2-category is biequivalent to the 2-category of locally-small cocomplete

categories and cocontinuous functors, and leading to a proof that the bicategory of profunctors Prof has a bimulticategorical structure.

Contents

Chapter 1

Introduction

Context and Motivation

The classical theory of monads provides a framework with which to study algebraic structures on objects of a category. Monads unify the study of categories of sets equipped with algebraic structures such as groups, monoids or lattices. A landmark in this field is Kock's theory of commutative monads [Koc70], developed in the setting of symmetric monoidal categories (such as Set, k-Vect or Rel). The basic notion in this theory is that of a *strong monad*, which comprises a monad T on a symmetric monoidal category equipped with a natural transformation with components

$$t_{X,Y} : X \otimes TY \to T(X \otimes Y),$$

called the *(left)-strength*. For example, every monad on Set with its Cartesian product is canonically strong; for example, if $T : \mathrm{Set} \to \mathrm{Set}$ is the list monad for which TX is the set of lists of elements in X, then the strength is given by

$$\begin{aligned} t_{X,Y} : \quad & X \times TY \to T(X \times Y) \\ & (x, (y_1, ..., y_n)) \mapsto ((x, y_1), (x, y_2), ..., (x, y_n)). \end{aligned}$$

Kock proves that the underlying endofunctor of a strong monad is a lax monoidal functor, and the monad unit is a monoidal natural transformation. Furthermore, Kock showed that the monad is *commutative*—a property of strength equivalent to asking that the operations of the monad's (possibly non-finitary) algebraic theory all commute—exactly when the monad is a monoidal monad, which is to say that the monad multiplication is monoidal.

Some nice properties follow when this happens. For example, if a symmetric monoidal category $\mathbb{C}$ has a closed structure and T is a commutative monad on $\mathbb{C}$, then the closed structure gives rise to one on the Eilenberg-Moore category of T-algebras.

Returning to our example, the list monad on Set is not commutative (note its algebras, monoids, are not described by commutative operations), and indeed we cannot put a closed structure on the category of monoids. However, the finite multiset monad on Set, whose algebras are *commutative* monoids, is commutative. Indeed, the set of monoid morphisms between commutative monoids can be given the structure of a commutative monoid.

Other examples of commutative monads on Set include the free abelian group monad and the powerset monad (whose algebras are sup-lattices).

We can extend the theory of monads by increasing the dimension of the monad's underlying category. Two-dimensional monad theory [BMP89] has traditionally studied the strict notion of a 2-monad, along with their algebras and lax, pseudo-, and strict algebra morphisms. In this setting Kelly [Kel74a] and Hyland & Power [HP02] extended Kock's theory to 2-monads, defining *pseudocommutative 2-monads*. Some aspects of the theory become more subtle; for example, one must distinguish between braiding and symmetry, and between closed structures and pseudo-closed structures.

For some applications, it is useful to consider instead the more general notion of a pseudomonad [Bun74, GL21, Lac00, Mar99], in which the axioms for a 2-monad hold only up to coherent isomorphisms. In this setting, an important notion is that of 'lax idempotency' [Koc95, Zö76], which is a property of a pseudomonad which entails that its pseudoalgebras are defined by a 'property-like structure' [KL97]. This is to say that pseudoalgebras for a lax-idempotent pseudomonad have an algebraic structure defined up to unique isomorphism. For example, consider the 2-monad on Cat which takes a category to its free cocompletion under finite coproducts. Its pseudoalgebras are the categories with chosen finite coproducts, and the finite coproduct is indeed defined up to unique isomorphism. A result which links these pseudomonads to the above discussion on commutativity is López Franco's [LF11] Theorem 7.3 that every lax-idempotent pseudomonad on a monoidal 2-category is pseudocommutative (extending work of Power, Cattani and Winskel in [CPW00]).

The move from monads to pseudomonads is not the only direction in which we can generalise the theory. Monads have been described (first in [Man76]; see also [Wal70, Har06, Mog91, MW10]) in a so-called 'no-iteration' form (also called a 'Kleisli triple' or an 'extension system') comprising an endofunctor $T : \mathbb{C} \to \mathbb{C}$, a transformation $1_{\mathbb{C}} \implies T$ and an extension operator $(-)^* : \mathbb{C}(X, TY) \to \mathbb{C}(TX, TY)$ satisfying three equations. Since this definition does not mention iteration of the underlying endofunctor, it is possible to adapt this notion for non-endofunctors.

Indeed, this is done by Altenkirch, Chapman and Uustalu in [ACU15], defining the notion of a 'relative monad' T along a base functor $J : \mathbb{C} \to \mathbb{D}$. It comprises a functor $T : \mathbb{C} \to \mathbb{D}$, a transformation $i : J \implies T$ and an extension operator $(-)^* : \mathbb{C}(JX, TY) \to \mathbb{C}(TX, TY)$ satisfying three equations. A relative monad structure can be given for example to the map $\mathrm{Set} \to \mathrm{Vect}_k$ sending a set to the k-vector space spanned by its elements.

The theory of relative monads has been explored further in [AM23, Lob23]. Just as monads can be generalised to pseudomonads, [FGHW18] generalises relative monads to relative pseudomonads between bicategories, exchanging the three equations for three families of invertible 2-cells and requiring these to satisfy two coherence equations. The sterling example of a relative pseudomonad is the presheaf construction $P : \mathrm{Cat} \to \mathrm{CAT}$ (where here Cat and CAT denote small categories and locally small categories respectively) taking a small category to its locally-small presheaf category; no unrestricted presheaf functor can be made into an ordinary pseudomonad due to size issues. However, one may avoid the size issues in other ways, such as restricting to small presheaves $P_s : \mathrm{CAT} \to \mathrm{CAT}$ [DL07], which is to say presheaves which are small colimits of representables.

Other examples of relative pseudomonads along the inclusion $\mathrm{Cat} \to \mathrm{CAT}$ are general cocompletions which take a small category to its free cocompletion under a certain class of colimit (for example, cocompletion under all coproducts, or under all filtered colimits).

Returning to the original motivation for the work in this thesis, Kock [Koc70] defines strong and commutative monads in the setting of symmetric monoidal closed categories. The work of Hyland and Power [HP02] on strong and pseudo-commutative 2-monads also operates in the symmetric monoidal closed context, although they pass through 2-multicategories in the course of proving that the 2-category of algebras for a pseudocommutative pseudomonad is closed. We follow the example of this paper throughout the thesis and work in multicategorical settings rather than monoidal ones. This allows us in general to obviate associativity and unitor coherences, at the expense of having to work in an unbiased way on general n-ary morphisms, instead of being able to consider only binary and nullary morphisms.

An assumption we drop is the existence of a closed structure. Although the notion of a closed multicategory is well-studied [BLM12], our through-line example of the presheaf relative pseudomonad has codomain CAT, and the 2-category of locally-small categories is for size reasons not closed. In particular, this means that though the López Franco result that every lax-idempotent pseudomonad is pseudocommutative makes essential use of a closed structure, our analogue (Theorem 5.18) explicitly does not require the existence of a closed structure.

Main results

Chapter 2 aims to extend the classical work of Kock [Koc70] from monads on monoidal categories to relative monads between multicategories in the one-dimensional setting. As such, we prove a chain of implications for strong relative monads.

Theorem 2.12. *If T is an idempotent strong relative monad between multicategories, then T is commutative.*

Theorem 2.14. *If T is a commutative relative monad between (symmetric) multicategories, then T is a (symmetric) relative multimonad.*

Theorem 2.16. *If T is a symmetric relative multimonad between symmetric multicategories, then T lifts to a relative monad between categories of commutative monoids.*

There are two main results in Chapter 3, which introduces the Eilenberg-Moore construction for relative pseudomonads. First, we generalise the work of [MW13] on no-iteration pseudoalgebras by constructing the Eilenberg-Moore bicategory for a relative pseudomonad.

Theorem 3.8. *Let T be a relative pseudomonad between bicategories. The T-pseudoalgebras, pseudomorphisms, and algebra 2-cells form a bicategory T-Alg, called the* Eilenberg-Moore bicategory.

We then apply this construction to the case of the presheaf relative pseudomonad, characterising its Eilenberg-Moore 2-category.

Theorem 3.23. *The Eilenberg-Moore 2-category P-Alg for the presheaf relative pseudomonad is biequivalent to the 2-category* COC *of cocomplete categories, cocontinuous functors and natural transformations.*

This gives another conceptual justification for referring to the presheaf construction as the 'free cocompletion' of a small category, mirroring the phenomenon whereby for example algebras for the free group monad are groups.

The main result of Chapter 4 is a counterpart in the relative setting to Kelly and Lack's Theorem 6.2 ($(v) \iff (i)$)in [KL97] on lax-idempotent pseudomonads.

Theorem 4.6. *Let $T : \mathbb{C} \to \mathbb{D}$ be a J-relative pseudomonad between bicategories. Then the following are equivalent:*

1. *T is algebraically lax-idempotent,*

2. *The forgetful 2-functor $U : T\text{-}Alg_l \to \mathbb{D}$ is locally fully faithful: that is, for every pair of T-pseudoalgebras $(A, {}^a)$ and $(B, {}^b)$, every map $h : A \to B$ has*

a unique lax morphism structure $(f, \bar{f}) : (A, {}^{a}) \to (B, {}^{b})$*, and furthermore every 2-cell* $\gamma : h \implies k : A \to B$ *is an algebra 2-cell.*

This theorem demonstrates the necessity in the relative setting of the notion of 'algebraic lax idempotency' as introduced in this chapter (which is shown in Proposition 4.5 not to be equivalent to the notion of 'lax idempotency' defined in [FGHW18] Section 5).

The main results of Chapter 5 parallel some of those in Chapter 2, as we generalise them to two dimensions and consider relative pseudomonads between 2-multicategories. We prove the following chain of implications for a strong relative pseudomonad between 2-multicategories.

Theorem 5.18. *If T is a lax-idempotent strong relative pseudomonad between 2-multicategories, then T is pseudocommutative.*

Theorem 5.16. *If T is a pseudocommutative relative pseudomonad between 2-multicategories, then T is a relative pseudomultimonad.*

We immediately make use of these results, proving that the presheaf relative pseudomonad is a lax-idempotent strong relative pseudomonad, and thus also pseudocommutative and a relative pseudomultimonad.

Finally, in Chapter 6 the main result is analogous to the result in [Gui80] Corollary 7 that the Kleisli category for a commutative monad has a monoidal structure.

Theorem 6.8. *The Kleisli bicategory for a pseudocommutative relative pseudomonad can be given the structure of a bimulticategory.*

This result gives the bicategory of profunctors Prof a bimulticategorical structure, since from [FGHW18] Example 4.2, Prof is biequivalent to the Kleisli bicategory for the presheaf relative pseudomonad.

Outline

In Chapter 2 we begin by extending the classical work of Kock [Koc70] from monads on monoidal categories to relative monads on multicategories. We assume familiarity with basic properties of multicategories, for which see [Lam69, Lei04]. In Section 2.1 we define the notions of a strong relative monad, and show that under suitable assumptions this is equivalent to the existing notion of a strong Kleisli triple. We then show that every strong relative monad extends to a multifunctor. We also define the notion of a commutative relative monad. In Section 2.2 we define idempotent strong relative monads, and prove that these are always commutative. In Section 2.3 we define symmetric relative multimonads and prove that every commutative relative monad is a symmetric relative multimonad. In Section 2.4 we define the category of commutative monoids in a symmetric multicategory $\mathbb{C}$, and prove that every symmetric relative multimonad $T : \mathbb{C} \to \mathbb{D}$ lifts to a relative monad $\tilde{T} : \mathrm{CMon}(\mathbb{C}) \to \mathrm{CMon}(\mathbb{D})$ between categories of commutative monoids.

Next, in Chapter 3, we introduce and explore the Eilenberg-Moore construction for relative pseudomonads. In Section 3.1 we define pseudoalgebras, pseudoalgebra morphisms and pseudoalgebra 2-cells over a relative pseudomonad, and prove that these form a bicategory T-Alg. In Section 3.2 we consider in particular the pseudoalgebras for the presheaf relative pseudomonad, proving that its Eilenberg-Moore bicategory is biequivalent to the 2-category of (locally small) cocomplete categories and cocontinuous functors between them.

Chapter 4 zeroes in on the notion of lax idempotency for relative pseudomonads. In Section 4.1 we recall the existing notion of lax idempotency as defined in [FGHW18, Sla23]. In Section 4.2 we define the more general notion of algebraic lax idempotency, and prove that it is equivalent to the forgetful 2-functor $U : T\text{-Alg} \to \mathbb{D}$ being locally fully faithful. We conclude the chapter by proving that the presheaf relative pseudomonad is algebraically lax idempotent.

We return to strength and commutativity in Chapter 5, which generalises many of the results of Chapter 2 to the two-dimensional setting of relative pseudomonads on 2-multicategories. In Section 5.1 we define strong relative pseudomonads and note that the presheaf construction can be given the structure of a strong relative pseudomonad. We also define pseudomultifunctors and prove that every strong relative pseudomonad is a pseudomultifunctor. In Section 5.2 we define pseudocommutative relative pseudomonads and relative pseudomultimonads, and prove that every pseudocommutative relative pseudomonad is a relative pseudomultimonad. In Section 5.3 we define lax-idempotent strong relative pseudomonads and prove that these are always pseudocommutative. We conclude the chapter in Section 5.4 by applying these results to the presheaf construction, proving that it is lax-idempotent and therefore both pseudocommutative and a relative pseudomultimonad.

Finally, in Chapter 6 we study the Kleisli construction for a relative pseudomonad. In Section 6.1 we prove the one-dimensional result that the Kleisli category for a commutative relative monad has the structure of a multicategory. In Section 6.2 we generalise this to two dimensions, proving that the Kleisli bicategory for a pseudocommutative relative pseudomonad has the structure of a bimulticategory. We close by remarking on how this result applies to the presheaf relative pseudomonad, whose Kleisli bicategory is biequivalent to the bicategory of profunctors.

Chapter 2

Commutative relative monads

Introduction

In this chapter we work entirely in a one-dimensional setting, both because the definitions and results are already novel here and because we can use the one-dimensional definitions to guide the more general definitions in two dimensions (which we explore in Chapter 5).

We define the notion of a strong relative monad on a multicategory (generalising the work of [Uus10]), and show that this recovers the existing definition of a strong monad on a monoidal category when the relative monad is along the identity and the multicategory is representable (in the sense of Hermida [Her00] 8.1). We go on to define notions of:

- idempotent strong relative monad (Definition 2.2),
- commutative relative monad (Definition 2.8), and
- symmetric relative multimonad (Definition 2.13),

showing that these recover the existing definitions of idempotent strong monad, commutative monad and symmetric monoidal monad respectively in the representable, $J = 1_{\mathbb{C}}$ case. We prove the following chain of implications for a strong relative monad T on a symmetric multicategory $\mathbb{C}$:

- every idempotent strong relative monad is commutative (Theorem 2.12),
- every commutative relative monad is a symmetric relative multimonad (Theorem 2.14), and
- every symmetric relative multimonad lifts to a relative monad between the multicategories of commutative monoids in $\mathbb{C}$ (Theorem 2.17).

2.1 Strong and commutative relative monads

Recall the definition of a relative monad [ACU15] Definition 2.1, which is patterned after the presentation of a monad in 'extension form' as described in [Man76] (Section 3, Exercise 12).

Definition 2.1. A relative monad $(T, i, {}^*)$ along a functor $J : \mathbb{C} \to \mathbb{D}$ comprises

- for each $X \in \mathrm{ob}\,\mathbb{C}$ an object $TX \in \mathrm{ob}\,\mathbb{D}$,
- for each $X \in \mathrm{ob}\,\mathbb{C}$ a *unit* map $i_X : JX \to TX$, and
- for each $X, Y \in \mathrm{ob}\,\mathbb{C}$ an *extension* map

  $$(-)^* : \mathbb{D}(JX, TY) \to \mathbb{D}(TX, TY)$$

  natural in both arguments, such that we have the following three equations:

  $$f = f^* i_Y, \qquad (f^* g)^* = f^* g^*, \qquad i_X^* = 1_{TX}$$

  for all $X, Y, Z \in \mathrm{ob}\,\mathbb{C}$ and $g : JX \to TY$, $f : JY \to TZ$.

In this section we seek to generalise Kock's notion of a strong monad on a monoidal category [Koc70]. A strong monad structure on a monoidal category is given by a map

$$t_{X,Y} : X \otimes TY \to T(X \otimes Y)$$

satisfying some axioms. As a motivation for our definition of strong relative monad in Definition 2.2, we generalise a relative monad's extension maps

$$\mathbb{C}(JX, TY) \xrightarrow{(-)^*} \mathbb{C}(TX, TY)$$

to n-ary hom-categories

$$\mathbb{C}(B_1, ..., JX, ..., B_n; TY) \xrightarrow{(-)^j} \mathbb{C}(B_1, ..., TX, ..., B_n; TY),$$

which we call *strengths*. To use this to construct the map t in the ordinary and representable case, we begin with the unit $i : X \otimes Y \to T(X \otimes Y)$. Passing to the underlying multicategory, this corresponds to a map $i : X, Y \to T(X \otimes Y)$. We can strengthen this map in the second argument to obtain

$$i^2 : X, TY \to T(X \otimes Y).$$

Now passing back to the original monoidal category we have found a strength map $X \otimes TY \to T(X \otimes Y)$, and one can check that this satisfies the strength axioms. This derivation, as well as Proposition 2.5 later on, motivates the use of the terminology 'strength' to refer to the maps

$$\mathbb{C}(B_1, ..., JX, ..., B_n; TY) \xrightarrow{(-)^j} \mathbb{C}(B_1, ..., TX, ..., B_n; TY)$$

below.

Throughout this section and later in the thesis we will employ the following notational shorthand.

- When the arity of a multimorphism is unimportant or can be easily inferred, we write $f : \bar{X} \to Y$ instead of $f : X_1, ..., X_n \to Y$.
- Given $f : \bar{X} \to Y$ and $g : \bar{W} \to X_j$ we abbreviate composites of the form $f \circ (1, ..., 1, g, 1, ..., 1)$ to $f \circ_j g$.
- We write $\bar{X} \cdot Z_j$ to abbreviate slightly more complicated domains of the form
$$X_1, ..., X_{j-1}, Z, X_{j+1}, ..., X_n.$$

Definition 2.2. A *strong relative monad* $(T, i, {}^\bullet)$ along a map of multicategories $J : \mathbb{C} \to \mathbb{D}$ comprises

- for each $X \in \operatorname{ob} \mathbb{C}$ an object $TX \in \operatorname{ob} \mathbb{D}$,
- for each $X \in \operatorname{ob} \mathbb{C}$ a map $i_X : JX \to TX$, and
- a family of maps $(-)^\bullet_{\bar{A},X,Y}$ which comprises for each arity n, index $1 \leq j \leq n$ and objects $\bar{A} \in \operatorname{ob} \mathbb{D}$, $X, Y \in \operatorname{ob} \mathbb{C}$ a map

  $$\begin{aligned} \mathbb{D}(\bar{A} \cdot JX_j; TY) &\xrightarrow{(-)^j} \mathbb{D}(\bar{A} \cdot TX_j; TY) \\ f &\longmapsto f^j \end{aligned}$$

  natural in all arguments,

such that we have

$$f = f^j \circ_j i, \qquad (f^j \circ_j g)^{j+k-1} = f^j \circ_j g^k, \qquad i^1 = 1$$

for all $\bar{A}, \bar{B} \in \operatorname{ob} \mathbb{D}$, $X, Y, Z \in \operatorname{ob} \mathbb{C}$ and $g : \bar{A} \cdot JX_k \to TY$, $f : \bar{B} \cdot JY_j \to TZ$.

Remark 2.3. On the full subcategories of unary maps of $\mathbb{C}$ and $\mathbb{D}$, this reduces to the definition of a relative monad. Hence on unary maps T has a functor structure given by $Tf := (i \circ Jf)^1$. We will see in Proposition 2.7 that furthermore T has the structure of a multifunctor.

Note that naturality of the maps $(-)^j$ stipulates in particular that, for n-ary f and m-ary g,

$$(f \circ_k g)^j = f^j \circ_k g$$

when $1 \leq j < k \leq n$, and

$$(f \circ_k g)^{j+m-1} = f^j \circ_k g$$

when $1 \leq k < j \leq n$.

We now justify this definition by showing that it recovers the existing definition of a strong monad on a monoidal category when the multicategory is representable and the relative monad is along the identity. To this end, we use the notion of a strong Kleisli triple from [Har06] Section 3.1, which is equivalent to the ordinary notion of strong monad on a monoidal category.

Definition 2.4. A *(right-)strong Kleisli triple* $(T, i, {}^r)$ on a monoidal category $\mathbb{C}$ comprises

- an endofunctor $T : \mathbb{C} \to \mathbb{C}$,
- a map $i_A : A \to TA$ for each object $A \in \operatorname{ob}\mathbb{C}$, and
- an extension map $\mathbb{C}(A \otimes B; TC) \xrightarrow{(-)^r} \mathbb{C}(A \otimes TB; TC)$ natural in all arguments

such that for all $A, B, C, D \in \operatorname{ob}\mathbb{C}$, $f : A \otimes B \to TC$, $g : B \otimes C \to TE$ and $h : A \otimes E \to TD$ we have

$$(i_A \circ \lambda)^r = \lambda : I \otimes TA \to TA,$$
$$f^r \circ (1 \otimes i_B) = f : A \otimes B \to TC,$$
$$h^r \circ (1 \otimes g^r) \circ \alpha = (h^r \circ (1 \otimes g) \circ \alpha)^r : (A \otimes B) \otimes TC \to TD.$$

Note that the fourth condition listed at [Har06] Section 3.1 corresponds to the stipulation here that $(-)^r$ be natural in the argument A. We will also need the notion of a *left-strong Kleisli triple*, whose extension map is instead of the form

$$\mathbb{C}(A \otimes B; TC) \xrightarrow{(-)^l} \mathbb{C}(TA \otimes B; TC),$$

and we call a Kleisli triple which is both right-strong and left-strong *bistrong*. We now prove the desired correspondence in one direction.

Proposition 2.5. *Let $\mathbb{C}$ be a monoidal category. A strong relative monad along the identity on the underlying multicategory of $\mathbb{C}$ has the structure of a bistrong Kleisli triple on $\mathbb{C}$.*

Proof. Let $(T, i, {}^\bullet)$ be a strong relative monad along the identity $1 : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, where $\bar{\mathbb{C}}$ is the underlying multicategory of a monoidal category $\mathbb{C}$. By Remark 2.3 T is an endofunctor, and we have the maps $i_A : A \to TA$ directly.

Recall that in any representable multicategory we have maps $u : () \to I$ and $v_{A,B} : A, B \to A \otimes B$ inducing bijections

$$\bar{\mathbb{C}}(..., A_{j-1}, I, A_{j+1}, ...; B) \xrightarrow{-\circ_j u} \bar{\mathbb{C}}(..., A_{j-1}, A_{j+1}, ...; B),$$
$$\bar{\mathbb{C}}(..., A_{j-1}, A_j \otimes A_{j+1}, A_{j+2}, ...; B) \xrightarrow{-\circ_j v} \bar{\mathbb{C}}(..., A_{j-1}, A_j, A_{j+1}, A_{j+2}, ...; B),$$

and that for $A, B, C, D \in \operatorname{ob} \mathbb{C}$, $f : A \to C$ and $g : B \to D$ we have identities

$$(f \otimes g) \circ v = v \circ (f, g) : A, B \to C \otimes D,$$
$$\lambda_A \circ v \circ_1 u = 1 : A \to A,$$
$$\rho_A \circ v \circ_2 u = 1 : A \to A,$$
$$\alpha_{A,B,C} \circ v \circ_1 v = v \circ_2 v : A, B, C \to A \otimes (B \otimes C).$$

Now we can define the left and right strengths respectively by

$$(-)^l \circ v = (- \circ v)^1, \quad (-)^r \circ v = (- \circ v)^2.$$

We now verify the three conditions for T to be right-strong. Firstly, to show that $(i \circ \lambda)^r = \lambda$ it suffices to show that $(i \circ \lambda)^r \circ v \circ_1 u = \lambda \circ v \circ_1 u$, and indeed

$$(i \circ \lambda)^r \circ v \circ_1 u = (i \circ \lambda \circ v)^2 \circ_1 u = (i \circ \lambda \circ v \circ_1 u)^2$$
$$= i^1 = 1 = \lambda \circ v \circ_1 u.$$

Secondly, to show that $f^r \circ (1 \otimes i) = f$ it suffices to show that $f^r \circ (1 \otimes i) \circ v = f \circ v$, and indeed

$$f^r \circ (1 \otimes i) \circ v = f^r \circ v \circ_2 i = (f \circ v)^2 \circ_2 i$$
$$= f \circ v.$$

Finally, to show that $g^r \circ (1 \otimes f^r) \circ \alpha = (g^r \circ (1 \otimes f) \circ \alpha)^r$ it suffices to show that

$$g^r \circ (1 \otimes f^r) \circ \alpha \circ v \circ_1 v = (g^r \circ (1 \otimes f) \circ \alpha)^r \circ v \circ_1 v,$$

and indeed

$$\begin{aligned}
g^r \circ (1 \otimes f^r) \circ \alpha \circ v \circ_1 v &= g^r \circ (1 \otimes f^r) \circ v \circ_2 v \\
&= g^r \circ v \circ_2 f^r \circ_2 v \\
&= (g \circ v)^2 \circ_2 (f \circ v)^2 \\
&= ((g \circ v)^2 \circ_2 f \circ_2 v)^3 \\
&= (g^r \circ v \circ_2 f \circ_2 v)^3 \\
&= (g^r \circ (1 \otimes f) \circ v \circ_2 v)^3 \\
&= (g^r \circ (1 \otimes f) \circ \alpha \circ v \circ_1 v)^3 \\
&= (g^r \circ (1 \otimes f) \circ \alpha \circ v)^2 \circ_1 v \\
&= (g^r \circ (1 \otimes f) \circ \alpha)^r \circ v \circ_1 v.
\end{aligned}$$

So T is right-strong, and a symmetric argument shows that T is also left-strong. Hence T is a bistrong Kleisli triple, as required. □

One may also show the desired correspondence in the other direction—that every bistrong Kleisli triple on a monoidal category has the structure of a strong relative monad along the identity on the category's underlying multicategory. We omit this result here for length, as defining the required $(-)^j$ extensions in full generality is notationally cumbersome.

Now we prove a generalisation of Theorem 2.1 in [Koc70] that every strong monad is a lax monoidal functor. To do this, we will extend our notation. Consider maps of the form $JX_1, ..., JX_n \to TY$, i.e. maps which can be strengthened in any index. In this case we can generalise from strengthenings in only one argument $f \mapsto f^j$ to strengthenings in any subset of the domain $f \mapsto f^S$ for $S \subseteq [n] := \{1, 2, ..., n\}$. Here we introduce the notation $- \circ_S g_j$ to mean 'compose with the map g_j at index j for all $j \in S$'.

Proposition 2.6. *Let T be a strong relative monad. Then for each n, subset $S \subseteq [n]$ and $JX_1, ..., JX_n \to TY$ we have a map $f^S : Z_1, ..., Z_n \to TY$, where*

$$Z_j = \begin{cases} TX_j & j \in S \\ JX_j & j \notin S \end{cases}$$

such that $(-)^S$ is natural in all arguments, and such that we have

$$f = f^S \circ_S i,$$
$$(f^{S_1} \circ_j g)^{S_2+j-1} = f^{S_1} \circ_j g^{S_2},$$

for all $j \in S_1$, $g : JX_1, ..., JX_m \to TY_j$, $f : JY_1, ..., JY_n \to TZ$, $S_1 \subseteq [n]$ and $S_2 \subseteq [m]$.

Proof. The action $(-)^S$ is defined by applying the strengths $(-)^j$ for $j \in S$ from left to right. We must now prove the two equalities. To show that $f = f^S \circ_S i$, we apply the equality $f^j \circ_j i = f$ in turn for each of the elements of S. To show that $(f^{S_1} \circ_j g)^{S_2+j-1} = f^{S_1} \circ_j g^{S_2}$, let $S_1 = U \sqcup U'$ where $U = S_1 \cap [j]$ (and $[j] := \{1, 2, ..., j\}$). Then

$$\begin{aligned}(f^{S_1} \circ_j g)^{S_2+j-1} &= (f^U \circ_j g)^{(S_2+j-1)\sqcup(U'+m-1)} \\ &= (f^U \circ_j g^{S_2})^{(U'+m-1)} \\ &= f^{S_1} \circ_j g^{S_2},\end{aligned}$$

as required. □

With this notation we can now prove the result, which is a generalisation of Remark 2.3.

Proposition 2.7. *Let T be a strong relative monad along a multifunctor $J : \mathbb{C} \to \mathbb{D}$. Then T is a multifunctor whose action on a multimorphism $f : X_1, ..., X_n \to Y$ is given by*

$$Tf := (i \circ Jf)^{[n]} : TX_1, ..., TX_n \to TY$$

where $[n] := \{1, 2, ..., n\}$.

Proof. We have two equations to check. To show that $T1_X = 1_{TX}$, we have

$$T1_X := (i_X \circ J1_X)^1 = (i_X \circ 1_{JX})^1 = i^1_X = 1_{TX}.$$

To show that $T(f \circ_j g) = Tf \circ_j Tg$, we have

$$\begin{aligned} T(f \circ_j g) &:= (i \circ J(f \circ_j g))^{[n+m-1]} = (i \circ Jf \circ_j Jg)^{[n+m-1]} \\ &= ((i \circ Jf)^{[j-1]} \circ_j Jg)^{j...(n+m-1)} \\ &= ((i \circ Jf)^{[j]} \circ_j (i \circ Jg))^{j...(n+m-1)} \\ &= ((i \circ Jf)^{[j]} \circ_j (i \circ Jg)^{[m]})^{(j+m-1)...(n+m-1)} \\ &= (i \circ Jf)^{[n]} \circ_j (i \circ Jg)^{[m]} \\ &= Tf \circ_j Tg. \end{aligned}$$

Hence a strong relative monad is a multifunctor. □

In the classical situation described in [Koc70], a strong monad with left-strength s and right-strength t can be given the structure of lax monoidal functor in two ways:

$$TX \otimes TY \xrightarrow{t} T(TX \otimes Y) \xrightarrow{Ts} TT(X \otimes Y) \xrightarrow{\mu} T(X \otimes Y)$$
$$TX \otimes TY \xrightarrow{s} T(X \otimes TY) \xrightarrow{Tt} TT(X \otimes Y) \xrightarrow{\mu} T(X \otimes Y)$$

It is then natural to ask about those strong monads for which these two composites are equal, which Kock called *commutative monads*.

Analogously, when we defined the subset strengths $(-)^S$, we had to choose an order (there, left to right) in which to apply the individual strengths. Commutativity, introduced in Definition 2.8 below, says that any choice of order gives the same result.

Definition 2.8. Let T be a strong relative monad. We say T is a *commutative relative monad* if for all $f : \bar{A} \cdot JX_J, JX_k \to TZ$ and $1 \leq j < k \leq n$ we have

$$f^{kj} = f^{jk} : \bar{A} \cdot TX_J, TX_k \to TZ.$$

Remark 2.9. A commutative relative monad along the identity on a representable multicategory is a commutative Kleisli triple; i.e. one for which

$$f^{rl} = f^{lr} : TA \otimes TB \to TC.$$

Indeed, it suffices to show that $f^{rl} \circ v = f^{lr} \circ v$, and we have

$$\begin{aligned} f^{rl} \circ v &= (f^r \circ v)^1 = (f \circ v)^{21} \\ &= (f \circ v)^{12} = (f^l \circ v)^2 \\ &= f^{lr} \circ v \end{aligned}$$

as required.

Note that being able to commute any two strengths lets us reorder the application of n strengths in any way we choose. This lets us manipulate the subset strengths more freely, as the following proposition shows. Recall that the notation $f^S \circ_s g_j$ for n-ary f and $S \subseteq [n]$ means 'strengthen f at the indices in S, and compose with g_j at index j for all $j \in S$'.

Proposition 2.10. *Let T be a commutative relative monad, let $f : JX_1, ..., JX_n \to TY$ be a map, let $S \subseteq [n]$, let $g_j : JZ_{j1}, ..., JZ_{jm_j} \to TX_j$ for $j \in S$, and let $S_j \subseteq [m_j]$. Then we have*

$$(f^S \circ_S g_j)^{\bigcup(S_j + k_j)} = f^S \circ_S g_j^{S_j},$$

where for all j, we have $k_j := \sum_{1 \leq k < j}(m_k - 1)$.

Proof. Since T is commutative, we can rearrange the indices of S so that any of them is rightmost. Thus if we start from $(f^S \circ_S g_j)^{\bigcup(S_j+k_j)}$, for each $j \in S$ in turn, we can

- shuffle S so that j is rightmost, then
- apply the axioms of a strength to bring the indices of S_j inside the parentheses.

Having done this for each $j \in S$, we obtain $f^S \circ_S g_j^{S_j}$ as required. $\square$

2.2 Idempotent strong relative monads

In the ordinary setting, a monad is idempotent if the multiplication $m : TTA \to TA$ and the unit $i : TA \to TTA$ are inverse to each other ([AT69] Section 6). We adapt this to our setting to define a notion of idempotent strong relative monad.

Definition 2.11. Let $T : \mathbb{C} \to \mathbb{D}$ be a strong relative monad. We say T is *idempotent* if the strengths are inverse to precomposition with the unit; that is, if the maps

$$\begin{tikzcd} \mathbb{D}(\bar{A} \cdot JX_j; TB) \arrow[r, shift left, "(-)^j"] & \mathbb{D}(\bar{A} \cdot TX_j; TB) \arrow[l, shift left, "- \circ_j i"] \end{tikzcd}$$

are inverses for all n, all $1 \leq j \leq n$ and all objects $A_1, ..., A_{j-1}, A_{j+1}, ..., A_n$ in $\mathbb{D}$ and X, Y in $\mathbb{C}$.

That is, as well as the equality $f^j \circ_j i = f : JX \to TY$ (which holds for all strong relative monads), we also have $(g \circ_j i)^j = g : TX \to TY$.

Every idempotent strong monad is commutative (this appears to be folklore, but see the two-dimensional generalisation at [LF11] Theorem 7.3); the analogous result holds in our setting.

Theorem 2.12. *If T is an idempotent strong relative monad, then T is commutative.*

Proof. Suppose T is idempotent and let $f : \bar{A} \cdot JX_j, JY_k \to TZ$ be an n-ary map with $1 \leq j < k \leq n$. Then

$$f^{kj} = (f^j \circ_j i)^{kj} = (f^{jk} \circ_j i)^j = f^{jk},$$

and so T is commutative. $\square$

2.3 Relative multimonads

In this section we generalise the notion of a symmetric monoidal monad on a symmetric monoidal category ([Koc70] Section 3) to our setting. Throughout this section, every multicategory will be symmetric; for all n the n-ary morphisms will be equipped with an action of S_n that respects composition. We denote the action of $\sigma \in S_n$ on $f : A_1, ..., A_n \to B$ by

$$f_\sigma : A_{\sigma(1)}, ..., A_{\sigma(n)} \to B.$$

We further suppose throughout that the strengths of a strong relative monad are compatible with the symmetry; that is, we have

$$(f^j)_\sigma = (f_\sigma)^{\sigma(j)}$$

for all n-ary f, $1 \leq j \leq n$ and $\sigma \in S_n$.

Definition 2.13. Let T be a relative monad. We say T is a *relative multimonad* if

- T is a multifunctor, and
- the multifunctoriality of T is compatible with the monad structure, which is to say that we have
  - $i \circ Jf = Tf \circ (i, ..., i)$ for all $f : X_1, ..., X_n \to Y$, and
  - if $h \circ Jf = Tf' \circ (g_1, ..., g_n)$ then also $h^* \circ Tf = Tf' \circ (g_1^*, ..., g_n^*)$:

$$\begin{tikzcd}
TX_1, ..., TX_n \arrow[r, "g_1^*, ..., g_n^*"] \arrow[d, "Tf"'] & TX_1', ..., TX_n' \arrow[d, "Tf'"] \\
TY \arrow[r, "h^*"'] & TY'
\end{tikzcd}$$

We further say that T is a *symmetric relative multimonad* if we have $(Tf)_\sigma = T(f_\sigma)$ for all n-ary f and $\sigma \in S_n$.

Now we have an analogue of the classical result ([Koc70] Theorem 3.2) that every commutative monad on a symmetric monoidal category is a symmetric monoidal monad.

Theorem 2.14. *Let T be a commutative relative monad along a symmetric multifunctor $J : \mathbb{C} \to \mathbb{D}$. Then T can be given the structure of a symmetric relative multimonad.*

Proof. Suppose T is commutative. Since T is strong, by Proposition 2.7 T has a multifunctor structure. We have two conditions to check to show that this can be extended to a relative multimonad structure on T. For the first, we simply have

$$i \circ Jf = (i \circ Jf)^{[n]} \circ (i, ..., i) = Tf \circ (i, ..., i).$$

Note that this holds for any strong relative monad, not necessarily commutative. For the second condition, suppose $h \circ Jf = Tf' \circ (g_1, ..., g_n)$. Then

$$\begin{aligned} h^* \circ Tf = h^* \circ (i \circ Jf)^{[n]} &= (h^* \circ i \circ Jf)^{[n]} \\ &= (h \circ Jf)^{[n]} = (Tf' \circ (g_1, ..., g_n))^{[n]} \\ &= ((i \circ Jf')^{[n]} \circ (g_1, ..., g_n))^{[n]} \\ &\overset{\dagger}{=} (i \circ Jf')^{[n]} \circ (g_1^*, ..., g_n^*) \\ &= Tf' \circ (g_1^*, ..., g_n^*), \end{aligned}$$

where the step marked † holds by Proposition 2.10 and the commutativity of T. To show that T is furthermore symmetric, we have

$$\begin{aligned} (Tf)_\sigma := ((i \circ Jf)^{[n]})_\sigma &= ((i \circ Jf)_\sigma)^{\sigma(1)...\sigma(n)} \\ &= ((i \circ Jf)_\sigma)^{[n]} = (i \circ Jf_\sigma)^{[n]} \\ &= T(f_\sigma). \end{aligned}$$

Hence indeed T is a symmetric relative multimonad. ☐

2.4 Commutative monoids in a multicategory

In this section, we generalise the result of Kock ([Koc70] Theorem 4.1) that every symmetric monoidal monad on a symmetric monoidal category $\mathbb{C}$ lifts to a monad on the category of commutative monoids in $\mathbb{C}$. To this end, we define the category of commutative monoids in a multicategory.

Definition 2.15. Let $\mathbb{C}$ be a symmetric multicategory. The category $\mathrm{CMon}(\mathbb{C})$ of (unbiased) commutative monoids in $\mathbb{C}$ comprises

- commutative monoid objects (M, m) consisting of an object $M \in \mathbb{C}$ and n-ary maps
$$m_n : M, ..., M \to M$$
for each n, such that
  - $m_1 = 1 : M \to M$,
  - $m_n \circ_k m_p = m_{n+p-1}$ for all $1 \leq k \leq n$ and $p \geq 0$, and
  - $(m_n)_\sigma = m_n$ for all $\sigma \in S_n$.
- monoid morphisms $f : (M, m) \to (M', m')$ comprising a map $f : M \to M'$ such that
$$\begin{tikzcd} M, ..., M \arrow[r, "m_n"] \arrow[d, "f,...,f"'] & M \arrow[d, "f"] \\ M', ..., M' \arrow[r, "m'_n"'] & M' \end{tikzcd}$$
commutes for all n.

We have a forgetful functor $U : \mathrm{CMon}(\mathbb{C}) \to \mathbb{C}$ with $U(M, m) = M$ and $Uf = f$.

In order for T along $J : \mathbb{C} \to \mathbb{D}$ to lift to a monad $\mathrm{CMon}(\mathbb{C}) \to \mathrm{CMon}(\mathbb{D})$, we need at the very least for J to lift to a functor $\tilde{J} : \mathrm{CMon}(\mathbb{D}) \to \mathrm{CMon}(\mathbb{C})$. The following proposition gives a sufficient condition for this to hold.

Proposition 2.16. *If $J : \mathbb{D} \to \mathbb{C}$ is a symmetric multifunctor between symmetric multicategories, then J lifts to a functor $\tilde{J} : \mathrm{CMon}(\mathbb{D}) \to \mathrm{CMon}(\mathbb{C})$.*

Proof. The map $\tilde{J}$ sends an object (M, m) to (JM, Jm); we see that this is a commutative monoid object since

$$Jm_n \circ_k Jm_p = J(m_n \circ_k m_p) = Jm_{n+p-1}$$
$$(Jm_n)_\sigma = J(m_n)_\sigma = Jm_n$$

by the symmetric multifunctoriality of J. On morphisms we have $\tilde{J}f = Jf$; we need to check that if $f : (M, m) \to (M', m')$ is a monoid morphism, then so is Jf. Indeed, we have

$$Jm'_n \circ (Jf, ..., Jf) = J(m'_n \circ (f, ..., f)) = J(f \circ m_n) = Jf \circ Jm_n,$$

as required. Functoriality follows from the functor structure of J. So indeed if J is a symmetric multifunctor then it lifts to $\tilde{J} : \mathrm{CMon}(\mathbb{D}) \to \mathrm{CMon}(\mathbb{C})$. □

Now we can prove the lifting result for T.

Theorem 2.17. *Let $(T, i, ^*)$ be a symmetric relative multimonad along the symmetric multifunctor $J : \mathbb{D} \to \mathbb{C}$. Then T lifts to a monad $(\tilde{T}, i, ^*)$ along $\tilde{J}$: $\mathrm{CMon}(\mathbb{D}) \to \mathrm{CMon}(\mathbb{C})$ such that*

$$U\tilde{T} = TU,$$
$$U(i) = i,$$
$$U(f^*) = f^*.$$

Proof. Suppose T is a symmetric multimonad along $J : \mathbb{D} \to \mathbb{C}$. Let $\tilde{T}(M, m) = (TM, Tm)$; this is a commutative monoid object due to the symmetric multifunctor structure on T, as above in Proposition 2.16.

The map $i : JM \to TM$ lifts to a monoid morphism $i : (JM, Jm) \to (TM, Tm)$ because the diagram

$$\begin{tikzcd} JM, ..., JM \arrow[r, "Jm_n"] \arrow[d, "{i,...,i}"'] & JM \arrow[d, "i"] \\ TM, ..., TM \arrow[r, "Tm_n"'] & TM \end{tikzcd}$$

commutes for all n, being one of the axioms of a multimonad.

Given a monoid morphism $f : (JM, Jm) \to (TM', Tm')$ we have that

$$\begin{tikzcd} JM, ..., JM \arrow[r, "Jm_n"] \arrow[d, "{f,...,f}"'] & JM \arrow[d, "f"] \\ TM', ..., TM \arrow[r, "Tm'_n"'] & TM' \end{tikzcd}$$

commutes for all n. Since T is a multimonad, we therefore also have that

$$\begin{tikzcd} TM, ..., TM \arrow[r, "Tm_n"] \arrow[d, "{f^*,...,f^*}"'] & TM \arrow[d, "f^*"] \\ TM', ..., TM \arrow[r, "Tm'_n"'] & TM' \end{tikzcd}$$

commutes for all n, and so f^* is also a monoid morphism. Hence T indeed lifts to the required monad $(\tilde{T}, i, {}^*)$ on CMon($\mathbb{C}$). $\square$

Chapter 3

The Eilenberg-Moore bicategory for a relative pseudomonad

Introduction

The Kleisli and Eilenberg-Moore categories for a relative monad have been defined in [ACU15] Section 2.3 and the Kleisli bicategory for a relative pseudomonad has been defined in [FGHW18] Section 4. However, the Eilenberg-Moore bicategory for a relative pseudomonad has not yet been defined. In this chapter we define pseudoalgebras for a relative pseudomonad, along with algebra morphisms and algebra 2-cells, and then prove that these form a bicategory T-Alg.

In ongoing work with Nathanael Arkor and Philip Saville, we aim to prove that this construction satisfies a universal property analogous to the Eilenberg-Moore category for an ordinary monad; namely that we have a J-relative pseudoadjunction (in the sense of [FGHW18] Definition 3.6)

$$\begin{tikzcd} & T\text{-Alg} \arrow[d, "U_T"] \\ \mathbb{C} \arrow[ur, "F_T"] \arrow[r, "J"'] & \mathbb{D} \end{tikzcd}$$

with $T = U_T F_T$ (a *resolution* of T) and that this one is biterminal among resolutions of T.

The second part of this chapter considers the example of the presheaf relative pseudomonad, and show that its Eilenberg-Moore 2-category of pseudoalgebras and pseudomorphisms is equivalent to the 2-category of cocomplete categories and cocontinuous functors.

3.1 Pseudoalgebras over a relative pseudomonad

Recall the definition of a relative pseudomonad ([FGHW18] Definition 2.1), which generalises Definition 2.1 to two dimensions.

Definition 3.1. Let $\mathbb{C}$ and $\mathbb{D}$ be bicategories and let $J : \mathbb{C} \to \mathbb{D}$ be a pseudofunctor between them. A *relative pseudomonad* $(T, i, {}^*; \eta, \mu, \theta)$ along J comprises

- for every $X \in \operatorname{ob}\mathbb{C}$ an object $TX \in \operatorname{ob}\mathbb{D}$ and a *unit map* $i_X : JX \to TX$,
- for every $X, Y \in \operatorname{ob}\mathbb{C}$ an *extension* functor $(-)^* : \mathbb{D}(JX, TY) \to \mathbb{D}(TX, TY)$ natural in both arguments.

Along with these we have three natural families of invertible 2-cells:

- $\eta_f : f \to f^*i$,
- $\mu_{f,g} : (f^*g)^* \to f^*g^*$, and
- $\theta_X : i_X^* \to 1_{TX}$ for $f : JX \to TY$ and $g : JW \to TX$,

satisfying the following two coherence conditions:

1. For $f : JX \to TY$, $g : JW \to TX$ and $h : JV \to TW$, the following diagram commutes:

$$\begin{tikzcd}
((f^*g)^*h)^* \arrow[rr, "\mu_{f^*g,h}"] \arrow[d, "(\mu_{f,g}h)^*"'] & & (f^*g)^*h^* \arrow[d, "\mu_{f,g}h^*"] \\
((f^*g^*)h)^* \arrow[d, "\cong"'] & & (f^*g^*)h^* \arrow[d, "\cong"] \\
(f^*(g^*h))^* \arrow[r, "\mu_{f,g^*h}"'] & f^*(g^*h)^* \arrow[r, "f^*\mu_{g,h}"'] & f^*(g^*h^*)
\end{tikzcd}$$

2. For $f : JX \to TY$, the following diagram commutes:

$$\begin{tikzcd} f^* \arrow[r, "(\eta_f)^*"] \arrow[drr, "\cong"'] & (f^*i)^* \arrow[r, "\mu_{f,i}"] & f^*i^* \arrow[d, "f^*\theta_X"] \\ & & f^*1_{TX} \end{tikzcd}$$

In defining pseudoalgebras over a relative pseudomonad, we simultaneously generalise algebras over relative monads ([ACU15] Definition 2.11) from one to two dimensions, and pseudoalgebras over no-iteration pseudomonads [MW13]. The 2-categorical (as opposed to bicategorical) case was treated in [Lew20], coinciding with the following definition in that case.

Definition 3.2. Let $J : \mathbb{C} \to \mathbb{D}$ be a pseudofunctor between bicategories. A *pseudoalgebra* for a J-relative pseudomonad $T : \mathbb{C} \to \mathbb{D}$ (or simply T*-pseudoalgebra*) comprises

- an object $A \in \mathbb{D}$;
- a natural family of functors $(-)^a : \mathbb{D}(JX, A) \to \mathbb{D}(TX, A)$ for each $X \in \mathbb{C}$, the *extension operator*;
- a natural family of invertible 2-cells $\tilde{a}_f : f \to f^a i_X$ for each $f : JX \to A$ in $\mathbb{D}$,

$$\begin{tikzcd} JX \arrow[rr, "f"] \arrow[dr, "i_X"'] & \arrow[d, phantom, "\Downarrow \tilde{a}_f"] & A \\ & TX \arrow[ur, "f^a"'] & \end{tikzcd}$$

- a natural family of invertible 2-cells $\hat{a}_{g,f} : (g^a f)^a \to g^a f^*$ for each $f : JX \to TY$ and $g : JY \to A$ in $\mathbb{D}$;

$$\begin{tikzcd} TX \arrow[rr, "(g^a f)^a"] \arrow[dr, "f^*"'] & \arrow[d, phantom, "\Downarrow \hat{a}_{g,f}"] & A \\ & TY \arrow[ur, "g^a"'] & \end{tikzcd}$$

such that

- for each $f : JX \to TY$, $g : JY \to TZ$, and $h : JZ \to A$ in $\mathbb{D}$, the following diagram commutes:

$$\begin{tikzcd}
((h^a g)^a f)^a \arrow[rr, "\hat{a}_{h^a g, f}"] \arrow[d, "(\hat{a}_{h,g} f)^a"'] & & (h^a g)^a f^* \arrow[d, "\hat{a}_{h,g} f^*"] \\
((h^a g^*) f)^a \arrow[d, "\cong"'] & & (h^a g^*) f^* \arrow[d, "\cong"] \\
(h^a (g^* f))^a \arrow[r, "\hat{a}_{h, g^* f}"'] & h^a (g^* f)^* \arrow[r, "h^a \mu_{g,f}"'] & h^a (g^* f^*)
\end{tikzcd} \tag{3.1}$$

- for each $f : JX \to A$ in $\mathbb{D}$, the following diagram commutes:

$$\begin{tikzcd}
f^a \arrow[r, "(\tilde{a}_f)^a"] \arrow[drr, bend right, "\cong"'] & (f^a i_X)^a \arrow[r, "\hat{a}_{f,i}"] & f^a {i_X}^* \arrow[d, "f^a \theta"] \\
& & f^a 1_{TX}
\end{tikzcd} \tag{3.2}$$

A pseudoalgebra is *strict* (or is a *strict algebra*) if each $\tilde{a}_f$ and each $\hat{a}_{f,g}$ is an identity 2-cell.

If we don't require the families of 2-cells $\tilde{a}$, $\hat{a}$ to be invertible, we almost obtain the notion of *lax algebra*; however, in this case we are required to add more coherence conditions (in a manner similar to coherence in monoidal versus lax monoidal categories), so we do not treat them here. As in the ordinary setting, we have a notion of free pseudoalgebra.

Proposition 3.3. *Let T be a relative pseudomonad along $J : \mathbb{C} \to \mathbb{D}$, and let Y be in $\mathbb{C}$. Then TY admits the structure of a T-pseudoalgebra, called the* free T-pseudoalgebra on Y.

Proof. The extension operator is given by the maps $(-)^* : \mathbb{D}(JX, TY) \to \mathbb{D}(TX, TY)$ for $X \in \mathbb{C}$; the natural families are given by μ and η respectively. The pseudoalgebra axioms are then exactly the axioms of a relative pseudomonad. $\square$

The following lemma will be useful in later proofs.

Lemma 3.4. *Let A be a pseudoalgebra for a relative pseudomonad T. For each $f : JX \to TY$ and $g : JY \to A$, the diagram*

$$\begin{tikzcd}
g^a f \arrow[r, "\tilde{a}_{g^a f}"] \arrow[drr, "g^a \eta_f"'] & (g^a f)^a i \arrow[r, "\hat{a}_{g,f} i"] & (g^a f^*) i \arrow[d, "\cong"] \\
& & g^a (f^* i)
\end{tikzcd} \tag{3.3}$$

also commutes.

Proof. Since $\tilde{a}$ is invertible, it suffices to prove that the diagram

$$\begin{tikzcd}
(g^a f)^a i \arrow[r, "\tilde{a}"] \arrow[drr, "\eta_f"'] & ((g^a f)^a i)^a i \arrow[r, "\hat{a}"] & ((g^a f^*) i)^a i \arrow[d, "\cong"] \\
& & (g^a (f^* i))^a i
\end{tikzcd}$$

commutes. By naturality and a pseudomonad coherence, we have

$$\begin{tikzcd}
(g^a f)^a i \arrow[r, "\eta"] \arrow[d, "\hat{a}"'] & (g^a (f^* i))^a i \arrow[d, "\hat{a}"] & \\
(g^a f^*) i \arrow[r, "\eta"] \arrow[drr, "\cong"'] & (g^a (f^* i)^*) i \arrow[r, "\mu"] & (g^a (f^* i^*)) i \arrow[d, "\theta"] \\
& & (g^a (f^* 1)) i
\end{tikzcd}$$

and so we may equivalently show that

$$\begin{tikzcd}
(g^a f)^a i \arrow[r, "\tilde{a}"] \arrow[dddd, "\hat{a}"'] & ((g^a f)^a i)^a i \arrow[r, "\hat{a}"] & ((g^a f^*) i)^a i \arrow[r, "\cong"] & (g^a (f^* i))^a i \arrow[d, "\hat{a}"] \\
& & & (g^a (f^* i)^*) i \arrow[d, "\mu"] \\
& & & (g^a (f^* i^*)) i \arrow[d, "\theta"] \\
& & & \\
(g^a f^*) i \arrow[rrr, "\cong"'] & & & (g^a (f^* 1)) i
\end{tikzcd}$$

commutes. Now by naturality and Equation 3.1, we have

$$\begin{tikzcd}
((g^a f)^a i)^a i \arrow[r, "\hat{a}"] \arrow[dd, "\hat{a}"] & ((g^a f^*) i)^a i \arrow[r, "\cong"] & (g^a (f^* i))^a i \arrow[d, "\hat{a}"] \\
 & & (g^a (f^* i)^*) i \arrow[d, "\mu"] \\
((g^a f)^a i^*) i \arrow[r, "\hat{a}"] & ((g^a f^*) i^*) i \arrow[r, "\cong"] \arrow[d, "\theta"] & (g^a (f^* i^*)) i \arrow[d, "\theta"] \\
 & (g^a f^*) i \arrow[r, "\cong"'] & (g^a (f^* 1)) i
\end{tikzcd}$$

and so we can reduce the problem to showing that

$$\begin{tikzcd}
(g^a f)^a i \arrow[r, "\tilde{a}"] \arrow[drrr, "\hat{a}"'] & ((g^a f)^a i)^a i \arrow[r, "\hat{a}"] & ((g^a f)^a i^*) i \arrow[r, "\hat{a}"] & ((g^a f^*) i^*) i \arrow[d, "\theta"] \\
 & & & (g^a f^*) i
\end{tikzcd}$$

commutes. But this may be filled as follows, using naturality and Equation 3.2:

$$\begin{tikzcd}
(g^a f)^a i \arrow[r, "\tilde{a}"] \arrow[drr, "1"'] & ((g^a f)^a i)^a i \arrow[r, "\hat{a}"] & ((g^a f)^a i^*) i \arrow[r, "\hat{a}"] \arrow[d, "\theta"] & ((g^a f^*) i^*) i \arrow[d, "\theta"] \\
 & & (g^a f)^a i \arrow[r, "\hat{a}"'] & (g^a f^*) i
\end{tikzcd}$$

and so indeed the original diagram commutes, as required. □

We now introduce the appropriate notion of morphism for pseudoalgebras. In fact, there are four such notions, depending on the 2-dimensional structure.

Definition 3.5. A *lax morphism* from a T-pseudoalgebra $(A, {}^a, \tilde{a}, \hat{a})$ to a T-pseudoalgebra $(B, {}^b, \tilde{b}, \hat{b})$ comprises

1. a morphism $h : A \to B$ in $\mathbb{D}$;

2. a natural family of 2-cells $\bar{h}_f : (hf)^b \to hf^a$ for each $f : JX \to A$ in $\mathbb{D}$,

such that

3. for each $f : JX \to A$ in $\mathbb{D}$, the following diagram commutes.

$$\begin{tikzcd} hf \arrow[r, "\tilde{b}_{hf}"] \arrow[drr, "h\tilde{a}_f"'] & (hf)^b i_X \arrow[r, "\bar{h}_f i_X"] & (hf^a) i_X \arrow[d, "\cong"] \\ & & h(f^a i_X) \end{tikzcd} \tag{3.4}$$

4. for each $f : JX \to TY$ and $g : JY \to A$ in $\mathbb{D}$, the following diagram commutes;

$$\begin{tikzcd} ((hg)^b f)^b \arrow[rr, "\hat{b}_{hg,f}"] \arrow[d, "(\bar{h}_g f)^b"'] & & (hg)^b f^* \arrow[d, "\bar{h}_g f^*"] \\ ((hg^a) f)^b \arrow[d, "\cong"'] & & (hg^a) f^* \arrow[d, "\cong"] \\ (h(g^a f))^b \arrow[r, "\bar{h}_{g^a f}"'] & h(g^a f)^a \arrow[r, "h\hat{a}_{g,f}"'] & h(g^a f^*) \end{tikzcd} \tag{3.5}$$

We say that $(h, \bar{h})$ is a *pseudomorphism* if each $\bar{h}_f$ is invertible; we say it is a *strict morphism* if each $\bar{h}_f$ is an identity 2-cell. A *colax morphism* is defined analogously, except that the direction of $\bar{h}$ is reversed.

The following lemma gives an example of a pseudomorphism out of a free pseudoalgebra.

Lemma 3.6. *Let T be a J-relative pseudomonad and let $(A, {}^a, \tilde{a}, \hat{a})$ be a T-pseudoalgebra. For each object $X \in \mathbb{C}$ and 1-cell $f : JX \to A$, the 1-cell $f^a : TX \to A$ admits the structure of a pseudomorphism from the free T-pseudoalgebra on X.*

Proof. The structure is defined by

$$\overline{f^a}_g := \hat{a}_{f,g} : (f^a g)^a \to f^a g^*.$$

With this structure, the coherence conditions 3.4 and 3.5 correspond exactly to the equation in Lemma 3.3 and Equation 3.1, respectively. □

Next, we introduce the appropriate notion of 2-cell for morphisms of pseudoalgebras.

Definition 3.7. Let $(h, \bar{h})$ and $(h', \overline{h'})$ be lax morphisms $(A, {}^a, \tilde{a}, \hat{a}) \to (B, {}^b, \tilde{b}, \hat{b})$. A *transformation* α from $(h, \bar{h})$ to $(h', \overline{h'})$ is a 2-cell $\alpha : h \implies h'$ such that the following diagram commutes for every $f : JX \to A$.

$$\begin{tikzcd} (hf)^b \arrow[r, "(\alpha f)^b"] \arrow[d, "\bar{h}_f"'] & (h'f)^b \arrow[d, "\overline{h'}_f"] \\ hf^a \arrow[r, "\alpha f^a"'] & h'f^a \end{tikzcd} \tag{3.6}$$

Transformations between colax morphisms are defined dually.

With all these notions defined, we can define the Eilenberg-Moore bicategory for a relative pseudomonad T.

Theorem 3.8. *Let T be a J-relative pseudomonad between bicategories. The T-pseudoalgebras, pseudomorphisms, and algebra 2-cells form a bicategory T-Alg, called the* Eilenberg-Moore bicategory.

Proof. We first show that for any pseudoalgebras $(A, {}^a)$ and $(B, {}^b)$, we have a hom-category of pseudomorphisms $T\text{-Alg}[(A, {}^a), (B, {}^b)]$. Since algebra 2-cells are just 2-cells in the underlying bicategory satisfying a property, it suffices to show that this property is preserved by identity and composition. That is, we need:

- $1_h : h \to h$ is an algebra 2-cell:

$$\begin{tikzcd} (hf)^b \arrow[r, "(1_h f)^b"] \arrow[d, "\bar{h}_f"'] & (hf)^b \arrow[d, "\bar{h}_f"] \\ hf^a \arrow[r, "1_h f^a"'] & hf^a \end{tikzcd}$$

- if $\alpha : h \to h'$ and $\beta : h' \to h''$ are algebra 2-cells, then so is $\beta\alpha : h \to h''$:

$$\begin{tikzcd} (hf)^b \arrow[r, "(\alpha f)^b"] \arrow[d, "\bar{h}_f"'] & (h'f)^b \arrow[r, "(\beta f)^b"] \arrow[d, "\bar{h}'_f"] & (h''f)^b \arrow[d, "\bar{h}''_f"] \\ hf^a \arrow[r, "\alpha f^a"'] & h'f^a \arrow[r, "\beta f^a"'] & h''f^a \end{tikzcd}$$

Hence indeed we have the required hom-categories.

Next, we want identity functors $1_{(A,{}^a)} : 1 \to T\text{-Alg}[(A, {}^a), (A, {}^a)]$. We need to show that 1_A can be given the structure of a pseudomorphism of algebras. We equip it with the 2-cell

$$(\overline{1_A})_g : (1_A g)^a \xrightarrow{\sim} g^a \xrightarrow{\sim} 1_A g^a.$$

We have two coherence conditions to check; condition (3.5) becomes:

$$\begin{tikzcd}
((1g)^a f)^a \arrow[rrr, "\hat{a}_{1g,f}"] \arrow[d, "\sim"'] & & & (1g)^a f^* \arrow[d, "\sim"] \\
(g^a f)^a \arrow[d, "\sim"'] & & & g^a f^* \arrow[d, "\sim"] \\
((1g^a) f)^a \arrow[d, "\sim"'] & & & (1g^a) f^* \arrow[d, "\sim"] \\
(1(g^a f))^a \arrow[r, "\sim"'] & (g^a f)^a \arrow[r, "\sim"'] & 1(g^a f)^a \arrow[r, "1\hat{a}_{g,h}"'] & 1(g^a f^*)
\end{tikzcd}$$

and using two naturality squares and two bicategory coherences, we can fill this in as follows:

$$\begin{tikzcd}
((1g)^a f)^a \arrow[rrr, "\hat{a}_{1g,f}"] \arrow[d, "\sim"'] & & & (1g)^a f^* \arrow[d, "\sim"] & \\
(g^a f)^a \arrow[d, "\sim"'] \arrow[ddr, equals] & & & g^a f^* \arrow[dd, "\sim"] \arrow[dr, "\sim"] & \\
((1g^a) f)^a \arrow[d, "\sim"'] & & & & (1g^a) f^* \arrow[dl, "\sim"] \\
(1(g^a f))^a \arrow[r, "\sim"'] & (g^a f)^a \arrow[r, "\sim"'] \arrow[uurr, "\hat{a}_{g,f}"] & 1(g^a f)^a \arrow[r, "1\hat{a}_{g,h}"'] & 1(g^a f^*) &
\end{tikzcd}$$

Condition (3.4) becomes:

$$\begin{tikzcd}
1f \arrow[r, "\tilde{a}_{1f}"] \arrow[ddrr, "1\tilde{a}_f"'] & (1f)^a i \arrow[r, "\sim"] & f^a i \arrow[d, "\sim"] \\
& & (1f^a)i \arrow[d, "\sim"] \\
& & 1(f^a i)
\end{tikzcd}$$

and using two naturality squares and one bicategory coherence, we can fill this in as follows:

$$\begin{tikzcd}
& (1f)^a i \arrow[dr, "\sim"] & & \\
1f \arrow[ur, "\tilde{a}_{1f}"] \arrow[r, "\sim"] \arrow[ddrr, "1\tilde{a}_f"'] & f \arrow[r, "\tilde{a}_f"] & f^a i \arrow[dd, "\sim"] \arrow[dr, "\sim"] & \\
& & & (1f^a)i \arrow[dl, "\sim"] \\
& & 1(f^a i) &
\end{tikzcd}$$

Thus we can define our identity functor $1_{(A,{}^a)} : 1 \to T\text{-Alg}[(A^a), (A, {}^a)]$ as picking out the pseudomorphism $(1_A, \overline{1_A})$.

We also need to define horizontal composition: for every triple of objects $(A, {}^a)$, $(B, {}^b)$ and $(C, {}^c)$ a functor

$$\circ : T\text{-Alg}[(B, {}^b), (C, {}^c)] \times T\text{-Alg}[(A, {}^a), (B, {}^b)] \to T\text{-Alg}[(A, {}^a), (C, {}^c)].$$

Given $(h, \bar{h}) : (B, {}^b) \to (C, {}^c)$ and $(h', \bar{h}') : (A, {}^a) \to (B, {}^b)$ we define the horizontal composite $(h, \bar{h}) \circ (h', \bar{h}')$ to be $(hh', \overline{hh'})$, where $\overline{hh'}_f$ for $f : JX \to A$ is defined as the composite

$$\overline{hh'}_f : ((hh')f)^c \xrightarrow{\sim} (h(h'f))^c \xrightarrow{\bar{h}_{h'f}} h(h'f)^b \xrightarrow{h\bar{h}'_f} h(h'f^a) \xrightarrow{\sim} (hh')g^a.$$

We must check that this composite actually gives hh' the structure of a pseudomorphism. For $f : JX \to A$, condition 3.4 becomes:

$$\begin{tikzcd}
(hh')f \arrow[r, "\tilde{c}"] \arrow[ddrrrr, "(hh')\tilde{a}"'] & ((hh')f)^c i \arrow[r, "\cong"] & (h(h'f))^c i \arrow[r, "\bar{h}"] & (h(h'f)^b)i \arrow[r, "\bar{h}'"] & (h(h'f^a))i \arrow[d, "\cong"] \\
&&&& ((hh')f^a)i \arrow[d, "\cong"] \\
&&&& (hh')(f^a i)
\end{tikzcd}$$

which we need to fill in. First, by naturality and bicategorical coherence we have

$$\begin{tikzcd}
(h(h'f))^c i \arrow[d, "\cong"'] & h(h'f) \arrow[l, "\tilde{c}"'] \arrow[r, "\tilde{a}"] \arrow[d, "\cong"'] & h(h'(f^a i)) \arrow[r, "\cong"] \arrow[d, "\cong"] & h((h'f^a)i) \arrow[r, "\cong"] & (h(h'f^a))i \arrow[d, "\cong"] \\
((hh')f)^c i & (hh')f \arrow[l, "\tilde{c}"] \arrow[r, "\tilde{a}"'] & (hh')(f^a i) & & ((hh')f^a)i \arrow[ll, "\cong"]
\end{tikzcd}$$

and so we can reduce the problem to showing

$$\begin{tikzcd}
h(h'f) \arrow[r, "\tilde{c}"] \arrow[ddrrr, "\tilde{a}"'] & (h(h'f))^c i \arrow[r, "\bar{h}"] & (h(h'f)^b)i \arrow[r, "\bar{h}'"] & (h(h'f^a))i \arrow[d, "\cong"] \\
&&& h((h'f^a)i) \arrow[d, "\cong"] \\
&&& h(h'(f^a i))
\end{tikzcd}$$

commutes. Using naturality and two instances of Equation 3.4 we can fill it in as follows:

$$\begin{tikzcd}
(h(h'f))^c i \arrow[r, "\bar{h}"] & (h(h'f)^b)i \arrow[r, "\bar{h}'"] \arrow[d, "\cong"] & (h(h'f^a))i \arrow[d, "\cong"] \\
h(h'f) \arrow[u, "\tilde{c}"] \arrow[r, "\tilde{b}"] \arrow[drr, "\tilde{a}"'] & h((h'f)^b i) \arrow[r, "\bar{h}'"] & h((h'f^a)i) \arrow[d, "\cong"] \\
&& h(h'(f^a i))
\end{tikzcd}$$

and hence the original diagram commutes, as required.

For the other condition, let $f : JX \to TY$ and $g : JY \to A$. Condition (3.5) becomes:

$$\begin{tikzcd}
(((hh')g)^c f)^c \arrow[rr, "\hat{c}_{(hh')g,f}"] \arrow[d, "\sim"'] & & ((hh')g)^c f^* \arrow[d, "\sim"] \\
((h(h'g))^c f)^c \arrow[d, "(\bar{h}_{h'g}f)^c"'] & & (h(h'g))^c f^* \arrow[d, "\bar{h}_{h'g}f^*"] \\
((h(h'g)^b)f)^c \arrow[d, "((h\bar{h}'_g)f)^c"'] & & (h(h'g)^b)f^* \arrow[d, "(h\bar{h}'_g)f^*"] \\
((h(h'g^a))f)^c \arrow[d, "\sim"'] & & (h(h'g^a))f^* \arrow[d, "\sim"] \\
(((hh')g^a)f)^c \arrow[d, "\sim"'] & & ((hh')g^a)f^* \arrow[dddd, "\sim"] \\
((hh')(g^a f))^c \arrow[d, "\sim"'] & & \\
(h(h'(g^a f)))^c \arrow[d, "\bar{h}_{h'(g^a f)}"'] & & \\
h(h'(g^a f))^b \arrow[d, "h\bar{h}_{g^a h}"'] & & \\
h(h'(g^a f)^a) \arrow[r, "\sim"'] & (hh')(g^a f)^a \arrow[r, "(hh')\hat{a}_{g,f}"'] & (hh')(g^a f^*)
\end{tikzcd}$$

which we fill in in stages. We first employ three naturality squares and one bicategory coherence to show that the diagram

$$\begin{tikzcd}
((h(h'g)^b)f)^c \arrow[r, "\sim"] \arrow[d, "((h\bar{h}'_g)f)^c"'] & (h((h'g)^b f))^c \arrow[r, "\bar{h}_{(h'g)^b f}"] \arrow[d, "(h(\bar{h}'_g f))^c"'] & h((h'g)^b f)^b \arrow[d, "h(\bar{h}'_g f)^b"] \\
((h(h'g^a))f)^c \arrow[r, "\sim"'] \arrow[d, "\sim"'] & (h((h'g^a)f))^c \arrow[r, "\bar{h}_{(h'g^a)f}"] \arrow[dd, "\sim"'] & h((h'g^a)f)^b \arrow[dd, "\sim"] \\
(((hh')g^a)f)^c \arrow[d, "\sim"'] & & \\
((h')(g^a f))^c \arrow[r, "\sim"'] & (h(h'(g^a f))^c \arrow[r, "\bar{h}_{h'(g^a f)}"'] & h(h'(g^a f))^b
\end{tikzcd}$$

commutes. We then employ two naturality squares, one bicategory coherence and

Equation (3.5) for h' to show that the diagram

$$\begin{tikzcd}
h((h'g)^b f)^b \arrow[rr, "h\hat{b}_{h'g,f}"] \arrow[d, "h(\bar{h}'_g f)^b"'] & & h((h'g)^b f^*) \arrow[r, "\sim"] \arrow[d, "h(\bar{h}'_g f^*)"'] & (h(h'g)^b) f^* \arrow[d, "(h\bar{h}'_g) f^*"] \\
h((h'g^a) f)^b \arrow[d, "\sim"'] & & h((h'g^a) f^*) \arrow[r, "\sim"] \arrow[d, "\sim"] & (h(h'g^a)) f^* \arrow[d, "\sim"] \\
h(h'(g^a f))^b \arrow[r, "h\bar{h}_{g^a h}"'] & h(h'(g^a f)^a) \arrow[r, "h(h'\hat{a}_{g,f})"] \arrow[d, "\sim"'] & h(h'(g^a f^*)) \arrow[d, "\sim"'] & ((hh')g^a) f^* \arrow[dl, "\sim"] \\
 & (hh')(g^a f)^a \arrow[r, "(hh')\hat{a}_{g,f}"'] & (hh')(g^a f^*) &
\end{tikzcd}$$

also commutes. Together, these reduce the problem to showing the diagram

$$\begin{tikzcd}
(((hh')g)^c f)^c \arrow[rr, "\hat{c}_{(hh')g,f}"] \arrow[d, "\sim"'] & & ((hh')g)^c f^* \arrow[d, "\sim"] \\
((h(h'g))^c f)^c \arrow[d, "(\bar{h}_{h'g} f)^c"'] & & (h(h'g))^c f^* \arrow[d, "\bar{h}_{h'g} f^*"] \\
((h(h'g)^b) f)^c \arrow[d, "\sim"'] & & (h(h'g)^b) f^* \arrow[d, "\sim"] \\
(h((h'g)^b f))^c \arrow[r, "\bar{h}_{(h'g)^b f}"'] & h((h'g)^b f)^b \arrow[r, "h\hat{b}_{h'g,f}"'] & h((h'g)^b f^*)
\end{tikzcd}$$

commutes. But we can employ a naturality square and Equation (3.5) for h to fill it in as follows:

$$\begin{tikzcd}
(((hh')g)^c f)^c \arrow[rr, "\hat{c}_{(hh')g,f}"] \arrow[d, "\sim"'] & & ((hh')g)^c f^* \arrow[d, "\sim"] \\
((h(h'g))^c f)^c \arrow[rr, "\hat{c}_{h(h'g),f}"] \arrow[d, "(\bar{h}_{h'g} f)^c"'] & & (h(h'g))^c f^* \arrow[d, "\bar{h}_{h'g} f^*"] \\
((h(h'g)^b) f)^c \arrow[d, "\sim"'] & & (h(h'g)^b) f^* \arrow[d, "\sim"] \\
(h((h'g)^b f))^c \arrow[r, "\bar{h}_{(h'g)^b f}"'] & h((h'g)^b f)^b \arrow[r, "h\hat{b}_{h'g,f}"'] & h((h'g)^b f^*)
\end{tikzcd}$$

Hence $(hh', \overline{hh'})$ is a pseudomorphism and our horizontal composition functor is defined on 1-cells.

On 2-cells, since algebra 2-cells are just 2-cells satisfying a property, it suffices to check that the horizontal composite of algebra 2-cells is again an algebra 2-cell. So let $\alpha : (h, \bar{h}) \implies (k, \bar{k})$ and $\alpha' : (h', \bar{h}') \implies (k', \bar{k}')$ be algebra 2-cells. We need to verify the commutativity of

$$\begin{tikzcd}
((hh')f)^c \arrow[r, "((\alpha\cdot\alpha')f)^c"] \arrow[d, "\overline{hh'}_f"'] & ((kk')f)^c \arrow[d, "\overline{kk'}_f"] \\
(kk')f^a \arrow[r, "(\alpha\cdot\alpha')f^a"'] & (kk')f^a
\end{tikzcd}$$

Using the definition of the lax morphism structures on dd' and ff', we can employ six naturality squares and Equation (3.6) for each of α and α' to fill this diagram in as follows:

$$\begin{tikzcd}
((hh')f)^c \arrow[r, "((\alpha h')f)^c"] \arrow[d, "\sim"'] & ((kh')f)^c \arrow[r, "((k\alpha')f)^c"] \arrow[d, "\sim"'] & ((kk')f)^c \arrow[d, "\sim"] \\
(h(h'f))^c \arrow[r, "(\alpha(h'f))^c"] \arrow[d, "\bar{h}_{h'f}"'] & (k(h'f))^c \arrow[r, "(k(\alpha'f))^c"] \arrow[d, "\bar{k}_{h'f}"] & (k(k'f))^c \arrow[d, "\bar{k}_{k'f}"] \\
h(h'f)^b \arrow[r, "\alpha(h'f)^b"'] \arrow[d, "h\bar{h}'_f"'] & k(h'f)^b \arrow[r, "k(\alpha'f)^b"] \arrow[d, "k\bar{h}'_f"'] & k(k'f)^b \arrow[d, "k\bar{k}'_f"] \\
h(h'f^a) \arrow[r, "\alpha(h'f^a)"'] \arrow[d, "\sim"'] & k(h'f^a) \arrow[r, "k(\alpha'f^a)"'] \arrow[d, "\sim"'] & k(k'f^a) \arrow[d, "\sim"] \\
(hh')f^a \arrow[r, "(\alpha h')f^a"'] & (kh')f^a \arrow[r, "(k\alpha')f^a"'] & (kk')f^a
\end{tikzcd}$$

Thus horizontal composition is also defined on 2-cells. Furthermore, horizontal composition inherits functoriality from the underlying bicategory, since being an algebra 2-cell is a property.

We now define associator and unitor 2-cells to be the same 2-cells as in the underlying bicategory. We must check that these are algebra 2-cells. For the left unitor, we need

$$\begin{tikzcd}
((1h)f)^b \arrow[r, "\sim"] \arrow[d, "\overline{1h}_f"'] & (hf)^b \arrow[d, "\bar{h}_f"] \\
(1h)f^a \arrow[r, "\sim"'] & hf^a
\end{tikzcd}$$

to commute. We employ a naturality square and two bicategory coherences to fill this in as follows:

$$\begin{tikzcd}
((1h)f)^b \arrow[r, "\sim"] \arrow[d, "\sim"'] & (hf)^b \arrow[ddl, equal] \arrow[ddddd, "\bar{h}_f"] \\
(1(hf))^b \arrow[d, "\sim"'] & \\
(hf)^b \arrow[d, "\sim"'] & \\
1(hf)^b \arrow[d, "1\bar{h}_f"'] & \\
1(hf^a) \arrow[d, "\sim"'] \arrow[dr, "\sim"] & \\
(1h)f^a \arrow[r, "\sim"'] & hf^a
\end{tikzcd}$$

So the left unitor is an algebra 2-cell. For the right unitor, we need

$$\begin{tikzcd}
((h1)f)^b \arrow[r, "\sim"] \arrow[d, "\overline{h1}_f"'] & (hf)^b \arrow[d, "\bar{h}_f"] \\
(h1)f^a \arrow[r, "\sim"'] & hf^a
\end{tikzcd}$$

to commute. We again employ a naturality square and two bicategory coherences to fill this in as follows:

$$\begin{tikzcd}
((h1)f)^b \arrow[r, "\sim"] \arrow[d, "\sim"'] & (hf)^b \arrow[ddddd, "\bar{h}_f"] \\
(h(1f))^b \arrow[ur, "\sim"'] \arrow[d, "\bar{h}_{1f}"'] & \\
h(1f)^a \arrow[d, "\sim"'] & \\
hf^a \arrow[d, "\sim"'] \arrow[ddr, equal] & \\
h(1f^a) \arrow[d, "\sim"'] & \\
(h1)f^a \arrow[r, "\sim"'] & hf^a
\end{tikzcd}$$

So the right unitor is also an algebra 2-cell. Finally, for the associator we need

$$\begin{tikzcd}
(((lk)h)f)^d \arrow[r, "\sim"] \arrow[d, "\overline{(lk)h}_f"'] & ((l(kh))f)^d \arrow[d, "\overline{l(kh)}_f"] \\
((lk)h)f^a \arrow[r, "\sim"'] & (l(kh))g^a
\end{tikzcd}$$

to commute. We employ two naturality squares and two bicategory axioms to fill the diagram in as follows:

$$\begin{tikzcd}
 & (((fg)h)k)^d \arrow[r, "\sim"] \arrow[d, "\sim"'] & ((f(gh))k)^d \arrow[dd, "\sim"] \\
 & ((fg)(hk))^d \arrow[d, "\sim"'] & \\
 & (f(g(hk)))^d \arrow[r, "\sim"] \arrow[d, "\bar{f}_{g(hk)}"'] & (f((gh)k))^d \arrow[d, "\bar{f}_{(gh)k}"] \\
 & f(g(hk))^c \arrow[d, "f\bar{g}_{hk}"'] & f((gh)k)^c \arrow[l, "\sim"] \\
(fg)(hk)^b \arrow[d, "(fg)\bar{h}_k"'] & f(g(hk)^b) \arrow[l, "\sim"'] \arrow[d, "f(g\bar{h}_k)"] & \\
(fg)(hk^a) \arrow[r, "\sim"] \arrow[dd, "\sim"'] & f(g(hk^a)) \arrow[d, "\sim"] & \\
 & f((gh)k^a) \arrow[d, "\sim"] & \\
((fg)h)k^a \arrow[r, "\sim"'] & (f(gh))k^a &
\end{tikzcd}$$

So the associator is an algebra 2-cell. The pentagon and triangle axioms hold since they hold in the underlying bicategory. Hence T-Alg is a bicategory. □

Remark 3.9. When $\mathbb{C}$ and $\mathbb{D}$ are (strict) 2-categories, T-Alg is also a 2-category.

Remark 3.10. We can specialise this result. If all 2-cells are identities we recover the one-dimensional Eilenberg-Moore category for a relative monad as in [ACU15] Definition 2.11, and if $J = 1_{\mathbb{C}}$ then we recover the 'no-iteration pseudoalgebras' of [MW10] Section 4.

Remark 3.11. We have a forgetful pseudofunctor $U : T\text{-Alg} \to \mathbb{D}$ sending a pseudoalgebra to its underlying object. This has a relative left biadjoint $F : \mathbb{C} \to T\text{-Alg}$ (in the sense of [FGHW18] Definition 3.6), which sends $X \in \mathbb{C}$ to the free pseudoalgebra $(TX, ^*)$.

Remark 3.12. By modifying the above proof, we obtain bicategories $TAlg_l$, $TAlg_c$ and (when $\mathbb{C}$ and $\mathbb{D}$ are 2-categories) $TAlg_s$, distinguished by containing the lax, colax and strict morphisms respectively.

The following proposition generalises an aspect of the phenomenon of doctrinal adjunction (treated in [Kel74b] for 2-monads and in [Nun18] for pseudomonads) to relative pseudomonads between bicategories.

Proposition 3.13. *Let $T : \mathbb{C} \to \mathbb{D}$ be a J-relative pseudomonad between bicategories, and let $(A, ^a)$ and $(B, ^b)$ be T-pseudoalgebras. Then for any adjunction*

$$\begin{tikzcd} A \arrow[r, bend left, "l"] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, "r"] \end{tikzcd}$$

we have a bijection between colax morphism structures $(l, \breve{l})$ on l and lax morphism structures $(r, \bar{r})$ on r.

Proof. (We omit bicategory-structure isomorphisms throughout for clarity.)

Denote by $\nu : 1 \to rl$, $\varepsilon : lr \to 1$ the unit and counit respectively of the adjunction $l \dashv r$. Let $(r, \bar{r})$ be a lax morphism structure on r with components $\bar{r}_g : (rg)^a \to rg^b$. We define a putative colax morphism structure on l by setting $\breve{l}_h$ to be

$$\breve{l}_h : lh^a \xrightarrow{l(\nu h)^a} l(rlh)^a \xrightarrow{l\bar{r}_{lh}} lr(lh)^b \xrightarrow{\varepsilon(lh)^b} (lh)^b.$$

We have two coherence equations to check. For the first:

$$\begin{tikzcd} lh \arrow[d, "l\tilde{a}_h"'] \arrow[rrd, "\tilde{b}_{lh}"] & & \\ l(h^a i) \arrow[r, "\sim"'] & (lh^a)i \arrow[r, "\breve{l}_h i"'] & (lh)^b i \end{tikzcd}$$

we write out our definition of $\breve{l}$ and fill in the resulting diagram with a triangle identity, two naturality squares and Equation 3.4 for the lax morphism structure on r:

$$\begin{tikzcd}
lh \arrow[rr, equal] \arrow[d, "\tilde{a}"'] \arrow[dr, "\nu"] & & lh \arrow[d, "\tilde{b}"] \\
lh^a i \arrow[d, "\nu"'] & lrlh \arrow[ur, "\varepsilon"] \arrow[dl, "\tilde{a}"'] \arrow[dr, "\tilde{b}"] & (lh)^b i \\
l(rlh)^a i \arrow[rr, "\bar{r}"'] & & lr(lh)^b i \arrow[u, "\varepsilon"']
\end{tikzcd}$$

For the second coherence condition:

$$\begin{tikzcd}
l(g^a h)^a \arrow[r, "\breve{l}_{g^a h}"] \arrow[d, "l\hat{a}_{g,h}"'] & (lg^a h)^b \arrow[r, "(\breve{l}_g h)^b"] & ((lg)^b h)^b \arrow[d, "\hat{b}_{lg,h}"] \\
lg^a h^* \arrow[rr, "\breve{l}_g h^*"'] & & (lg)^b h^*
\end{tikzcd}$$

we expand each instance of $\breve{f}$ and fill in the resulting diagram with ten naturality squares, a triangle equation and one instance of Equation 3.5 for the lax morphism structure on r:

$$\begin{tikzcd}
 & l(g^a h)^a \arrow[r, "\nu"] \arrow[dl, "\hat{a}"'] \arrow[d, "\nu"] & l(rlg^a h)^a \arrow[r, "\bar{r}"] \arrow[d, "\nu"] & lr(lg^a h)^b \arrow[r, "\varepsilon"] \arrow[d, "\nu"] & (lg^a h)^b \arrow[d, "\nu"] \\
lg^a h^* \arrow[d, "\nu"'] & l((rlg)^a h)^a \arrow[r, "\nu"] \arrow[dl, "\hat{a}"'] \arrow[d, "\bar{r}"] & l(rl(rlg)^a h)^a \arrow[r, "\bar{r}"] \arrow[d, "\bar{r}"] & lr(l(rlg)^a h)^b \arrow[r, "\varepsilon"] \arrow[d, "\bar{r}"] & (l(rlg)^a h)^b \arrow[d, "\bar{r}"] \\
l(rlg)^a h^* \arrow[ddrr, "\bar{r}"'] & l(r(lg)^b h)^a \arrow[r, "\nu"] \arrow[dr, equal] & l(rlr(lg)^b h)^a \arrow[r, "\bar{r}"] \arrow[d, "\varepsilon"] & lr(lr(lg)^b h)^b \arrow[r, "\varepsilon"] \arrow[d, "\varepsilon"] & (lr(lg)^b h)^b \arrow[d, "\varepsilon"] \\
 & & l(r(lg)^b h)^a \arrow[r, "\bar{r}"] & lr((lg)^b h)^b \arrow[r, "\varepsilon"] \arrow[dl, "\hat{b}"] & ((lg)^b h)^b \arrow[dl, "\hat{b}"] \\
 & & lr(lg)^b h^* \arrow[r, "\varepsilon"'] & (lg)^b h^* &
\end{tikzcd}$$

This demonstrates that $(l, \breve{l})$ is a colax morphism as required. A dual argument shows that every colax morphism structure on l induces a lax morphism structure on r, by defining $\bar{r}$ as the composite

$$\bar{r}_g : (rg)^a \xrightarrow{\nu(rg)^a} rl(rg)^a \xrightarrow{r\breve{l}_{rg}} r(lrg)^b \xrightarrow{r(\varepsilon g)^b} rg^b.$$

Finally, we need to show that these assignments are inverse to each other. In one direction, this amounts to showing that the following diagram commutes:

$$\begin{tikzcd}
(rg)^a \arrow[r, "\bar{r}"] \arrow[d, "\nu"'] & rg^b \\
rl(rg)^a \arrow[d, "\nu"'] & r(lrg)^b \arrow[u, "\varepsilon"'] \\
rl(rlrg)^a \arrow[r, "\bar{r}"'] & rlr(lrg)^b \arrow[u, "\varepsilon"']
\end{tikzcd}$$

And indeed we can fill this diagram in with three naturality squares and two triangle identities:

$$\begin{tikzcd}
& (rg)^a \arrow[r, "\bar{r}"] \arrow[dl, "\nu"'] \arrow[d, "\nu"] & rg^b \arrow[r, equal] \arrow[d, "\nu"] & rg^b \\
rl(rg)^a \arrow[dr, "\nu"'] & (rlrg)^a \arrow[r, "\bar{r}"] \arrow[d, "\nu"] & r(lrg)^b \arrow[ur, "\varepsilon"'] \arrow[d, "\nu"] \arrow[dr, equal] & \\
& rl(rlrg)^a \arrow[r, "\bar{r}"'] & rlr(lrg)^b \arrow[r, "\varepsilon"'] & r(lrg)^b
\end{tikzcd}$$

and this demonstrates one direction of the desired inverse. The other direction may be verified with the dual argument. Hence indeed colax morphism structures on l are in bijection with lax morphism structures on r. $\square$

3.2 Pseudoalgebras for the presheaf relative pseudomonad

In [FGHW18] Section 4 it was shown that the Kleisli bicategory for the presheaf relative pseudomonad P is equivalent to Prof, the bicategory of small categories and profunctors. The aim of this section is to characterise the Eilenberg-Moore 2-category for the presheaf relative pseudomonad as biequivalent to COC, the 2-category of cocomplete categories and cocontinuous functors (compare [GL12] Proposition 2.2 for the non-relative case, where one must take the small presheaf pseudofunctor $P_s : \mathrm{CAT} \to \mathrm{CAT}$).

In ongoing work with Nathanael Arkor and Phillip Saville, we hope to prove an analogue of the theorem that the Kleisli category for a monad embeds canonically into the Eilenberg-Moore category as its subcategory of free algebras ([Rie17], Lemma 5.2.13), which would specialise in the case of the presheaf relative pseudomonad P to the known result (see [CW97], Proposition 4.2.4) that Prof is biequivalent to the 2-category of presheaf categories PX for small X and cocontinuous functors between them.

We first show that every P-pseudoalgebra is equipped with a choice of colimit for every small diagram $\mathbb{D}$.

Definition 3.14. Denote by $i_{\mathbb{D}} : \mathbb{D} \to \mathbb{D}_\top$ the inclusion of $\mathbb{D}$ into the free category with a terminal object on $\mathbb{D}$. Explicitly, for $d, d' \in \mathbb{D}$ we have

$$\begin{aligned}
\operatorname{ob} \mathbb{D}_\top &:= \operatorname{ob} \mathbb{D} \sqcup \{\top\}, \\
\mathbb{D}_\top(\top, \top) &:= \{!_\top = 1_\top\}, \\
\mathbb{D}_\top(d, \top) &:= \{!_d\}, \\
\mathbb{D}_\top(d, d') &:= \mathbb{D}(d, d'), \\
\mathbb{D}_\top(\top, d) &:= \varnothing.
\end{aligned}$$

We can use this construction to characterise cocones under $\mathbb{D}$. Note that throughout we will use lowercase letters to denote functors so that our notation aligns with the rest of this section.

Lemma 3.15. *Let $f : \mathbb{D} \to \mathbb{A}$ be a diagram of shape $\mathbb{D}$ in $\mathbb{A}$. Then cocones under f are in bijection with functors $g : \mathbb{D}_\top \to \mathbb{A}$ such that $gi = f$.*

Proof. Given a cocone under f with nadir c, define $g : \mathbb{D}_\top \to \mathbb{A}$ such that $gi = f$, $gt = c$ and $g!_d$ is the leg of the cocone from fd.

Conversely, given such a functor g, the corresponding cocone under f has nadir gt and legs $g!_d$ for all d. □

Using the presheaf relative pseudomonad construction, we can find a distinguished cocone under any small diagram in a P-pseudoalgebra.

Lemma 3.16. *Let $(\mathbb{A}, {}^a)$ be a P-pseudoalgebra and let $f : \mathbb{D} \to \mathbb{A}$ be a small diagram in $\mathbb{A}$. Then there is a cocone $c : \mathbb{D}_\top \to \mathbb{A}$ under f with nadir $f^a \operatorname{colim} y_{\mathbb{D}}$.*

Proof. Consider the presheaf $s := \operatorname{colim} y_{\mathbb{D}} \in P\mathbb{D}$; it is the terminal presheaf sending every object of $\mathbb{D}$ to a singleton, and it has inclusion maps $\upsilon_d : yd \hookrightarrow s$ for all d. We have composites

$$fd \xrightarrow{(\tilde{a}_f)_d} f^a yd \xrightarrow{f^a \upsilon_d} f^a s$$

for all d, and these will form the legs of the required cocone. Indeed, for any morphism $h : d \to d'$ we can use a naturality square and the $\operatorname{colim} y$ cocone to show that the diagram

$$\begin{tikzcd} fd \arrow[r, "fh"] \arrow[d, "(\tilde{a}_f)_d"'] & fd' \arrow[d, "(\tilde{a}_f)_{d'}"] \\ f^a yd \arrow[r, "f^a yh"] \arrow[dr, "f^a \upsilon_d"'] & f^a yd' \arrow[d, "f^a \upsilon_{d'}"] \\ & f^a s \end{tikzcd}$$

commutes. □

We aim to prove that this distinguished cocone is in fact the colimit cocone. First, we prove something slightly weaker.

Lemma 3.17. *Let $(\mathbb{A}, {}^a)$ be a P-pseudoalgebra and let $f : \mathbb{D} \to \mathbb{A}$ be a small diagram in $\mathbb{A}$. Then the cocone c defined in Lemma 3.16 is weakly initial among all cocones under f. That is, for any cocone $g : \mathbb{D}_\top \to \mathbb{A}$ under f, there is a map of cocones $z_g : f^a s \to gt$.*

Proof. We first construct the required map z_g, and then show that it is a map of cocones. In the diagram below

$$\begin{tikzcd} \mathbb{D} \arrow[r, "y"] \arrow[d, "i"'] \arrow[dr, phantom, "\xRightarrow{\eta}"] & P\mathbb{D} \arrow[d, "(yi)^*"] \\ \mathbb{D}_\top \arrow[r, "y"] \arrow[dr, "g"'] & P\mathbb{D}_\top \arrow[d, "g^a"] \arrow[l, phantom, shift right=6, "\xRightarrow{\tilde{a}}"] \\ & \mathbb{A} \end{tikzcd}$$

consider the objects $s \in P\mathbb{D}$ and $yt \in P\mathbb{D}_\top$. Since t is terminal in $\mathbb{D}_\top$, yt is terminal in $P\mathbb{D}_\top$. The presheaf $(yi)^*s$ is in $P\mathbb{D}_\top$ and by terminality we have a unique map $(yi)^*s \xrightarrow{!} yt$ in $P\mathbb{D}_\top$. Applying the functor g^a to this map, we obtain $g^a(yi)^*s \xrightarrow{g^a!} g^a yt$, with which we can form the composite

$$f^a s = (gi)^a s \xrightarrow{\tilde{a}} (g^a yi)^a s \xrightarrow{\hat{a}} g^a (yi)^* s \xrightarrow{g^a !} g^a yt \xrightarrow{\tilde{a}^{-1}} gt.$$

This gives us a map $f^a s \to gt$, and this is how we define the desired z_g.

To show that this z_g is a map of cocones, we need to show that the diagram

$$\begin{tikzcd}
fd \arrow[r, equal] \arrow[d, "(\tilde{a}_f)_d"'] & gid \arrow[dd, "g!"] \\
f^a yd \arrow[d, "f^a \upsilon_d"'] & \\
f^a s \arrow[r, "z_g"'] & gt
\end{tikzcd}$$

commutes for all $d \in \operatorname{ob}\mathbb{D}$. Writing out our definition of z_g, we can fill this diagram with two equalities, four naturality squares, pseudoalgebra coherence 3.3 and the image under g^a of a unique map into the terminal object yt:

$$\begin{tikzcd}
fd \arrow[r, equal] \arrow[d, "\tilde{a}"'] & gid \arrow[r, "\tilde{a}"] \arrow[d, "\tilde{a}"'] & g^a yid \arrow[d, "\tilde{a}"'] \arrow[rrd, equal] & & & \\
f^a yd \arrow[r, equal] \arrow[d, "\upsilon_d"'] & (gi)^a yd \arrow[r, "\tilde{a}"] \arrow[d, "\upsilon_d"'] & (g^a yi)^a yd \arrow[r, "\hat{a}"'] \arrow[d, "\upsilon_d"'] & g^a (yi)^* yd \arrow[r, "\eta^{-1}"'] \arrow[d, "\upsilon_d"'] & g^a yid \arrow[r, "\tilde{a}^{-1}"] \arrow[d, "g^a y!"] & gid \arrow[d, "g!"] \\
f^a s \arrow[r, equal] & (gi)^a s \arrow[r, "\tilde{a}"'] & (g^a yi)^a s \arrow[r, "\hat{a}"'] & g^a (yi)^* s \arrow[r, "g^a !"'] & g^a yt \arrow[r, "\tilde{a}^{-1}"'] & gt
\end{tikzcd}$$

Hence $z_g : f^a s \to gt$ is indeed a map of cocones. □

We have found a weakly initial cocone under f; it remains only to show that it is actually initial; that is, that z_g is unique among maps of cocones from c to g.

Proposition 3.18. *Let $(\mathbb{A}, {}^a)$ be a P-pseudoalgebra. Then $\mathbb{A}$ has all small colimits.*

Proof. By Lemma 3.17, it suffices to show that z_g is unique among maps of cocones from c to g. Define a functor $h : \mathbb{D}_\top \to P\mathbb{D}$ by

$$hi := y, \ \ h(d \xrightarrow{!} t) := yd \xrightarrow{v_d} s.$$

Take a cocone g and consider natural transformations $\beta : f^a h \to g$ for which

$$(\beta \cdot i : f^a hi \to gi) = (\tilde{a}_f^{-1} : f^a y \to f).$$

These are determined by their component β_t, which only has to satisfy the naturality condition

$$\begin{tikzcd}
f^a yd \arrow[r, "\tilde{a}^{-1}"] \arrow[d, "f^a v_d"'] & fd \arrow[r, equal] & gid \arrow[d, "g!"] \\
f^a s \arrow[rr, "\beta_t"'] & & gt
\end{tikzcd}$$

stating precisely that β_t is a map of cocones from c to g. Hence we have a correspondence between

- natural transformations $\beta : f^a h \to g$ such that $\beta \cdot i = \tilde{a}^{-1}$, and
- maps of cocones from c to g.

Now let $\beta : f^a h \to g$ be such a natural transformation. Using six naturality squares and the pseudoalgebra coherences 3.1, 3.2 and 3.3 twice again, we can construct the following diagram:

$$
\begin{tikzcd}
 & & & f^a s \arrow[dlll, "\theta^{-1}"'] \arrow[d, equal] \\
f^a(hi)^* s \arrow[rr, "\hat{a}^{-1}"'] \arrow[d, "\eta"'] & & (f^a hi)^a s \arrow[r, "\beta_i = \tilde{a}^{-1}"] \arrow[dl, "\eta"'] \arrow[d, "\tilde{a}"] & (gi)^a s \arrow[d, "\tilde{a}"] \\
f^a(h^* yi)^* s \arrow[r, "\hat{a}^{-1}"] \arrow[dr, "\mu"'] & (f^a h^* yi)^a s \arrow[r, "\hat{a}^{-1}"'] & ((f^a h)^a yi)^a s \arrow[r, "\beta"] \arrow[d, "\hat{a}"] & (g^a yi)^a s \arrow[d, "\hat{a}"] \\
 & f^a h^*(yi)^* s \arrow[r, "\hat{a}^{-1}"] \arrow[d, "f^a h^* !"'] & (f^a h)^a (yi)^* s \arrow[r, "\beta"] \arrow[d, "(f^a h)^a !"] & g^a (yi)^* s \arrow[d, "g^a !"] \\
 & f^a h^* yt \arrow[r, "\hat{a}^{-1}"] \arrow[dr, "\eta^{-1}"'] & (f^a h)^a yt \arrow[r, "\beta"] \arrow[d, "\tilde{a}^{-1}"] & g^a yt \arrow[d, "\tilde{a}^{-1}"] \\
 & & f^a ht \arrow[r, "\beta_t"'] & gt
\end{tikzcd}
$$

where the clockwise composite is z_g. The anticlockwise composite is

$$
\begin{aligned}
f^a s &\xrightarrow{f^a\theta^{-1}} f^a y^* s = f^a(hi)^* s \xrightarrow{f^a\eta} f^a(h^* yi)^* s \xrightarrow{f^a\mu} f^a h^*(yi)^* s \\
&\xrightarrow{f^a h^* !} f^a h^* yt \xrightarrow{f^a\eta^{-1}} f^a ht \xrightarrow{\beta_t} gt,
\end{aligned}
$$

and by functoriality this composite is of the form

$$
f^a s \xrightarrow{f^a(\ldots)} f^a s \xrightarrow{\beta_t} gt
$$

for some map $s \to s$. But since s is terminal in $P\mathbb{D}$, the only such map is 1_s. Hence again by functoriality we have

$$
z_g = \beta_t f^a(1_s) = \beta_t 1_{f^a s} = \beta_t.
$$

So indeed the map of cocones $z_g : f^a s \to gt$ is unique, which implies

$$
f^a s \cong \operatorname{colim} f.
$$

Hence every presheaf pseudoalgebra $(A, {}^a; \tilde{a}, \hat{a})$ is cocomplete. $\square$

Corollary 3.19. *Let $(f, \bar{f}) : (\mathbb{A}, {}^a; \tilde{a}, \hat{a}) \to (\mathbb{B}, {}^b; \tilde{b}, \hat{b})$ be a pseudomorphism of P-pseudoalgebras. Then f preserves all small colimits, in that, for every functor $g : \mathbb{D} \to \mathbb{A}$, the canonical natural transformation*

$$\operatorname{colim} fg \implies f \operatorname{colim} g$$

is invertible. In particular, for any diagram $f : \mathbb{D} \to \mathbb{A}$ with small domain, the functor $f^a : P\mathbb{D} \to \mathbb{A}$ preserves small colimits.

Proof. We have the following chain of natural isomorphisms:

$$\operatorname{colim} fg \cong (fg)^b(\operatorname{colim} y_{\mathbb{D}}) \xrightarrow{\bar{f}_g} fg^a(\operatorname{colim} y_{\mathbb{D}}) \cong f \operatorname{colim} g$$

where the unnamed isomorphisms follow from Proposition 3.18. Precomposing with an inclusion $fgd \to \operatorname{colim} fg$ yields the commutative diagram

$$\begin{tikzcd}
fgd \arrow[r, "\tilde{b}"] \arrow[d, hook] & (fg)^b yd \arrow[r, "\bar{f}_g"] \arrow[d, hook] & fg^a yd \arrow[r, "\tilde{a}^{-1}"] \arrow[d, hook] & fgd \arrow[d, hook] \\
\operatorname{colim} fg \arrow[r, "\cong"] & (fg)^b(\operatorname{colim} y_{\mathbb{D}}) \arrow[r, "\bar{f}_g"] & fg^a(\operatorname{colim} y_{\mathbb{D}}) \arrow[r, "\cong"] & f \operatorname{colim} g
\end{tikzcd}$$

where the top row is the identity on fgd by Equation 3.4. Hence this is the canonical natural transformation out of $\operatorname{colim} fg$, and so f is cocontinuous as required.

That f^a for $f : \mathbb{D} \to \mathbb{A}$ then preserves small colimits follows from Lemma 3.6. □

We now establish the converse of Proposition 3.18.

Lemma 3.20. *Let $\mathbb{A}$ be a small-cocomplete category, let $f : \mathbb{D} \to \mathbb{A}$ and $g : P\mathbb{D} \to \mathbb{A}$ be functors with $\mathbb{D}$ small and g small-cocontinuous, and let $\alpha : f \to gy_{\mathbb{D}}$ be an invertible natural transformation (writing $y_{\mathbb{D}}$ for the Yoneda embedding). Then its transpose*

$$\alpha^\sharp : \operatorname{Lan}_y f \to g$$

is also invertible, where $\operatorname{Lan}_y f$ denotes the left Kan extension of f along $y_{\mathbb{D}}$.

Proof. Write $\eta_f : f \to (\mathrm{Lan}_y f)y$ for the 2-cell component of the left extension; since y is fully faithful η is invertible. Since $\mathrm{Lan}_y f$ and g are both cocontinuous, the composite

$$\mathrm{Lan}_y f \operatorname{colim} yh \simeq \operatorname{colim}(\mathrm{Lan}_y f)yh \xrightarrow{\eta^{-1}} \operatorname{colim} fh \xrightarrow{\alpha} \operatorname{colim} gyh \simeq g \operatorname{colim} yh$$

is invertible. Since every presheaf on $\mathbb{D}$ is the colimit of representables, these define an invertible transformation $\beta : \mathrm{Lan}_y f \to g$. Applied to the colimit of a single representable, this composite becomes

$$(\mathrm{Lan}_y f)yd = (\mathrm{Lan}_y f)yd \xrightarrow{\eta^{-1}} fd \xrightarrow{\alpha} gyd = gyd.$$

Hence

$$(\beta \cdot y)\eta_f = \alpha,$$

and so $\beta = \alpha^\sharp$. So indeed $\alpha^\sharp$ is invertible. □

Proposition 3.21. *Let $\mathbb{A}$ be a locally small category admitting small colimits. Then $\mathbb{A}$ may be equipped with the structure of a P-pseudoalgebra.*

Proof. For $f : \mathbb{D} \to \mathbb{A}$ define $f^a : P\mathbb{D} \to \mathbb{A}$ to be the left extension of f along y, which exists because $\mathbb{A}$ is cocomplete. Then we have a family of isomorphisms $\tilde{a}_f : f \to f^a y$ given by the 2-cell part of the left extension.

Define the other part of the pseudoalgebra structure $\hat{a}_{f,g} : (f^a g)^a \to f^a g^*$ to be the transpose of the invertible transformation

$$f^a g \xrightarrow{f^a \eta_g} f^a g^* i,$$

noting by Lemma 3.20 that $\hat{a}_{f,g}$ is invertible. By definition, then, the diagram

$$\begin{tikzcd} f^a g \arrow[r, "\tilde{a}"] \arrow[dr, "\eta"'] & (f^a g)^a y \arrow[d, "\hat{a}"] \\ & f^a g^* y \end{tikzcd}$$

commutes (this is Equation 3.3).

We now have two coherence conditions to check; we do so by considering their transposes. For condition 3.1, we need:

$$
\begin{tikzcd}
(f^a g)^a h \arrow[r, "\tilde{a}"] \arrow[d, "\tilde{a}"'] & ((f^a g)^a h)^a y \arrow[r, "\hat{a}"] & (f^a g)^a h^* y \arrow[dd, "\hat{a}"] \\
((f^a g)^a h)^a y \arrow[d, "\hat{a}"'] & & \\
(f^a g^* h)^a y \arrow[r, "\hat{a}"'] & f^a (g^* h)^* y \arrow[r, "\mu"'] & f^a g^* h^* y
\end{tikzcd}
$$

which using two naturality squares, Equation 3.3 twice and a pseudomonad coherence we can fill in as follows:

$$
\begin{tikzcd}
 & (f^a g)^a h \arrow[r, "\tilde{a}"] \arrow[dl, "\tilde{a}"'] \arrow[d, "\hat{a}"] \arrow[dr, "\eta"] & ((f^a g)^a h)^a y \arrow[d, "\hat{a}"] \\
((f^a g)^a h)^a y \arrow[d, "\hat{a}"'] & f^a g^* h \arrow[dl, "\tilde{a}"'] \arrow[d, "\eta"] \arrow[dr, "\eta"] & (f^a g)^a h^* y \arrow[d, "\hat{a}"] \\
(f^a g^* h)^a y \arrow[r, "\hat{a}"'] & f^a (g^* h)^* y \arrow[r, "\mu"'] & f^a g^* h^* y
\end{tikzcd}
$$

For condition 3.2, we need:

$$
\begin{tikzcd}
f \arrow[r, "\tilde{a}"] \arrow[ddrr, "\tilde{a}"'] & f^a y \arrow[r, "\tilde{a}"] & (f^a y)^a y \arrow[d, "\hat{a}"] \\
 & & f^a y^* y \arrow[d, "\theta"] \\
 & & f^a y
\end{tikzcd}
$$

which using a naturality square, Equation 3.3 and a pseudomonad coherence we can fill in as follows:

$$
\begin{tikzcd}
f \arrow[r, "\tilde{a}"] \arrow[dr, "\tilde{a}"'] & f^a y \arrow[r, "\tilde{a}"] & (f^a y)^a y \arrow[d, "\hat{a}"] \\
 & f^a y \arrow[ur, "\tilde{a}"] \arrow[r, "\eta"] \arrow[dr, equal] & f^a y^* y \arrow[d, "\theta"] \\
 & & f^a y
\end{tikzcd}
$$

Hence $(\mathbb{A}, {}^a; \tilde{a}, \hat{a})$ is a P-pseudoalgebra. □

Proposition 3.22. *Let $\mathbb{A}$ and $\mathbb{B}$ be cocomplete categories, and let $f : \mathbb{A} \to \mathbb{B}$ be a cocontinuous functor. Then f has a pseudomorphism structure as a map between P-pseudoalgebras.*

Proof. We need for $h : \mathbb{D} \to \mathbb{A}$ to define an invertible natural transformation

$$\bar{f}_h : (fh)^b \to fh^a : P\mathbb{D} \to \mathbb{B}.$$

We can use Lemma 3.20 to define it as the transpose of $f(\tilde{a}_h)^{-1} : fh \to fh^a y$; this immediately forces condition 3.4 to hold. We take transposes for the other condition 3.5, and need to show that

$$\begin{tikzcd}
(hg)^b f \arrow[d, "\tilde{b}"'] \arrow[dr, "\tilde{b}"] & & \\
((hg)^b f)^b i \arrow[d, "\bar{h}_g"'] & ((hg)^b f)^b i \arrow[r, "\hat{b}"] & (hg)^b f^* i \arrow[d, "\bar{h}"] \\
(hg^a f)^b i \arrow[r, "\bar{h}"'] & h(g^a f)^a i \arrow[r, "\hat{a}"'] & hg^a f^* i
\end{tikzcd}$$

commutes. But we can fill this in:

$$\begin{tikzcd}
 & & ((hg)^b f)^b i \arrow[d, "\hat{b}"] \\
((hg)^b f)^b i \arrow[d, "\bar{h}"'] & (hg)^b f \arrow[l, "\tilde{b}"'] \arrow[r, "\eta"'] \arrow[ur, "\tilde{b}"] \arrow[d, "\bar{h}"] & (hg)^b f^* i \arrow[d, "\bar{h}"] \\
(hg^a f)^b i \arrow[dr, "\bar{h}"'] & hg^a f \arrow[l, "\tilde{b}"'] \arrow[r, "\eta"] \arrow[d, "\tilde{a}"] & hg^a f^* i \\
 & h(g^a f)^a i \arrow[ur, "\hat{a}"'] &
\end{tikzcd}$$

with two naturality squares, Equation 3.3 twice, and Equation 3.4. Hence indeed f has the specified pseudomorphism structure. □

Now we combine the last few results to state the main theorem of this section.

Theorem 3.23. *The Eilenberg-Moore 2-category P-Alg for the presheaf relative pseudomonad is biequivalent to the 2-category* COC *of cocomplete categories, cocontinuous functors and natural transformations.*

Proof. By Proposition 3.18, Corollary 3.19, Proposition 3.21 and Proposition 3.22, we have functors $F : P\text{-Alg} \to \mathrm{COC}$ and $G : \mathrm{COC} \to P\text{-Alg}$. To extend these to 2-functors, we need to show that every natural transformation between cocontinuous functors is an algebra 2-cell. Taking transposes in Equation 3.6, we need to show that

$$\begin{tikzcd}
& hf \arrow[r, "\tilde{b}"] \arrow[dl, "\tilde{b}"'] & (hf)^b i \arrow[d, "\alpha"] \\
(hf)^b i \arrow[d, "\bar{h}"'] & & (h'f)^b i \arrow[d, "\bar{h}'"] \\
hf^a i \arrow[rr, "\alpha"'] & & h'f^a i
\end{tikzcd}$$

commutes. But we can fill this diagram

$$\begin{tikzcd}
& hf \arrow[r, "\tilde{b}"] \arrow[dl, "\tilde{b}"'] \arrow[d, "\tilde{a}"] \arrow[dr, "\alpha"] & (hf)^b i \arrow[dr, "\alpha"] & \\
(hf)^b i \arrow[r, "\bar{h}"'] & hf^a i \arrow[dr, "\alpha"'] & h'f \arrow[r, "\tilde{b}"] \arrow[d, "\tilde{a}"] & (h'f)^b i \arrow[dl, "\bar{h}'"] \\
& & h'f^a i &
\end{tikzcd}$$

with two naturality squares and two instances of Equation 3.3. So indeed every natural transformation is an algebra 2-cell, and F and G extend to 2-functors.

Now, $FG = 1_{\mathrm{COC}}$ strictly, and for a P-pseudoalgebra $(\mathbb{A}, {}^a)$, $GF(\mathbb{A}, {}^a)$ has the same underlying category with a possibly different choice of colimits, so $GF \simeq 1_{P\text{-Alg}}$.

Thus indeed P-Alg and COC are biequivalent. $\square$

Chapter 4

Lax idempotency for relative pseudomonads

The results in this chapter were developed in collaboration with Nathanael Arkor and Philip Saville.

Introduction

The property of a pseudomonad being lax-idempotent encodes the notion of an object equipped with a 'property-like structure' [Koc95, KL97]. The archetypal example of a lax-idempotent pseudomonads T takes a category $\mathbb{C}$ to its free cocompletion under a class of colimits; the T-pseudoalgebras are then categories with such colimits [KL00]. For relative pseudomonads, the notion of lax idempotency was first introduced in Section 5 of [FGHW18]. However, in this section we argue that this notion is insufficient in the setting of relative pseudomonads: the appropriate analogue of lax idempotency here is what we call *algebraic lax idempotency* (Definition 4.1). To justify this, we show that some known properties of lax-idempotent pseudomonads ([Koc95, KL97]) generalise to algebraically lax-idempotent relative pseudomonads and not, in general, to lax-idempotent relative pseudomonads (that algebraic lax idempotency implies lax idempotency is shown below in Proposition 4.4).

We close this section by showing that the presheaf relative pseudomonad is algebraically lax idempotent.

Parts of this chapter prefigure material in Chapter 5, in which we will define a notion of lax idempotency appropriate for 2-multicategories.

4.1 Lax idempotency

Recall the definition of a relative pseudomonad $(T, i, {}^*; \eta, \mu, \theta) : \mathbb{C} \to \mathbb{D}$ in Definition 3.1, including the unit maps $i_X : JX \to TX$, the extension functors $(-)^* : \mathbb{D}(JX, TY) \to \mathbb{D}(TX, TY)$ and the 2-cells $\eta_f : f \to f^* i$.

In this section we recall the definition of lax idempotency given in [Sla23] Definition 5.1, which we then show is equivalent to the definition from [FGHW18] in Proposition 4.2.

Definition 4.1. A J-relative pseudomonad $T : \mathbb{C} \to \mathbb{D}$ is *lax-idempotent* if the 2-cell $\eta_- : - \implies (-)^* i_X$ is the unit of an adjunction

$$\begin{tikzcd} \mathbb{D}(JX, TY) \arrow[r, bend left, "(-)^*"] \arrow[r, phantom, "\perp"] & \mathbb{D}(TX, TY) \arrow[l, bend left, "-\circ i"] \end{tikzcd}$$

for each $X, Y \in \mathbb{C}$.

Proposition 4.2. *Let $T : \mathbb{C} \to \mathbb{D}$ be a J-relative pseudomonad. Then the following are equivalent:*

1. *T is lax-idempotent in the sense of Definition 4.1,*

2. *$\eta_f : f \to f^* i$ exhibits f^* as the left Kan extension of f along i for each $f : JX \to TY$ in $\mathbb{D}$, and*

3. *T is lax-idempotent in the sense of [FGHW18] (Definition 5.1).*

Proof. The equivalence (1) $\iff$ (2) follows from the characterisation of left Kan extension along a map as a left adjoint to precomposition by that map (see for example [Mac71] Theorem 4.2). The equivalence (2) $\iff$ (3) holds as the second and third compatibility conditions listed in [FGHW18] Definition 5.1 are redundant ([FGHW18] Lemma 3.2 proves that they hold for any relative pseudomonad). ☐

4.2 Algebraic lax idempotency

Considering $(-)^*$ as the free pseudoalgebra structure on TY for $Y \in \mathbb{C}$ motivates the following definition.

Definition 4.3. A J-relative pseudomonad $T : \mathbb{C} \to \mathbb{D}$ is *algebraically lax-idempotent* if for every pseudoalgebra $(A, {}^a; \tilde{a}, \hat{a})$, the 2-cell $\tilde{a}$ is the unit of an adjunction

$$\begin{tikzcd} \mathbb{D}(JX, A) \arrow[r, bend left, "(-)^a"] \arrow[r, phantom, "\perp"] & \mathbb{D}(TX, A) \arrow[l, bend left, "-\circ i"] \end{tikzcd}$$

for each $X \in \mathbb{C}$.

Proposition 4.4. *Algebraic lax idempotency implies lax idempotency.*

Proof. Restricting Definition 4.3 to free pseudoalgebras gives the definition of a lax-idempotent relative pseudomonad. ☐

In the ordinary setting (when $J = 1$), the reverse implication also holds (as an analogue of [Koc95] Proposition 3.3). Thus, in the ordinary setting, algebraic lax idempotency is not a new notion. However, in the relative setting the reverse implication no longer holds.

Proposition 4.5. *Not every lax-idempotent relative pseudomonad is algebraically lax-idempotent.*

Proof. We give a counterexample to the one-dimensional version of the above statement: not every idempotent relative monad is algebraically idempotent.

That is, we give a relative monad T for which $(-)^*$ and $- \circ i$ are mutually inverse, but which has an algebra A such that $(-)^a$ and $- \circ i$ are not. Since every idempotent monad is lax-idempotent (for a locally discrete 2-category), this will prove the desired result.

Consider the category $\mathbb{D}$ generated by the diagram

$$\begin{tikzcd} J \arrow[r, "i"] & T \arrow[d, bend right, "g"'] \arrow[d, bend left, "h"] \\ & A \end{tikzcd}$$

subject to the constraint that $gi = hi =: f$. We have a relative monad $T : 1 \to \mathbb{D}$ along $J : 1 \to \mathbb{D}$ with unit i and extension $(-)^* : \mathbb{D}(J, T) \to \mathbb{D}(T, T)$ determined by $i^* = 1_T$.

Then $|\mathbb{D}(J, T)| = |\mathbb{D}(T, T)| = 1$ and $(-)^*$ is inverse to $- \circ i : \mathbb{D}(T, T) \to \mathbb{D}(J, T)$, so T is idempotent. However $|\mathbb{D}(J, A)| = 1 \neq 2 = |\mathbb{D}(T, A)|$ and so $- \circ i : \mathbb{D}(T, A) \to \mathbb{D}(J, A)$ is not invertible. Hence T is not algebraically lax idempotent. $\square$

In light of Proposition 4.5, we define algebraic lax idempotency as a separate notion. To justify its usefulness, we show an analogue of Theorem 6.2 $(v) \iff (i)$ in [KL97] (where (v) says 'algebra is adjoint to unit' and (i) is says 'the forgetful functor $U : T\text{-Alg}_l \to \mathbb{D}$ is locally fully faithful') to prove that lax-idempotent relative pseudomonads are *fully property-like*.

Theorem 4.6. *Let $T : \mathbb{C} \to \mathbb{D}$ be a J-relative pseudomonad between bicategories. Then the following are equivalent:*

1. *T is algebraically lax-idempotent,*

2. *The forgetful 2-functor $U : T\text{-}Alg_l \to \mathbb{D}$ is locally fully faithful: that is, for every pair of T-pseudoalgebras $(A, {}^a)$ and $(B, {}^b)$, every map $h : A \to B$ has a unique lax morphism structure $(f, \bar{f}) : (A, {}^a) \to (B, {}^b)$, and furthermore every 2-cell $\gamma : h \implies k : A \to B$ is an algebra 2-cell.*

Proof. (We omit bicategory-structure isomorphisms throughout for clarity.)

We first show that (1) implies (2). Suppose T is algebraically lax-idempotent. Then we have adjunctions $(\tilde{a}, \alpha) : (-)^a \dashv - \circ i$ and $(\tilde{b}, \beta) : (-)^b \dashv - \circ i$; this is to say that the unit and counit of the adjunction $(-)^a \dashv - \circ i$ are $\tilde{a}$ and α respectively, and likewise for the adjunction $(-)^b \dashv - \circ i$.

Let $h : A \to B$ be a map, and suppose it has a lax morphism structure. Then for all $g : JX \to A$, with respect to the adjunction

$$\begin{tikzcd} \mathbb{D}(JX, B) \arrow[r, bend left, "(-)^b"] \arrow[r, phantom, "\perp"] & \mathbb{D}(TX, B), \arrow[l, bend left, "-\circ i"] \end{tikzcd}$$

the component $\bar{h}_g : (hg)^b \to hg^a : TX \to B$ has transpose

$$hg \xrightarrow{\tilde{b}_{hg}} (hg)^b i \xrightarrow{\bar{h}_g i} hg^a i : JX \to B,$$

which by Equation 3.4 is equal to $h\tilde{a}_g : hg \to hg^a i$. Hence if the lax morphism structure exists, it is unique and equal to the transpose of $h\tilde{a}_g$.

It remains to show that the transpose of $h\tilde{a}_g$ indeed gives h the structure of a lax morphism. Equation 3.4 holds by definition; we need to show that Equation 3.5

$$\begin{tikzcd} ((hg)^b f)^b \arrow[rr, "\hat{b}_{hg,f}"] \arrow[d, "(\bar{h}_g f)^b"'] & & (hg)^b f^* \arrow[d, "\bar{h}_g f^*"] \\ (hg^a f)^b \arrow[r, "\bar{h}_{g^a f}"'] & h(g^a f)^a \arrow[r, "h\hat{a}_{g,f}"'] & hg^a f^* \end{tikzcd}$$

also holds. We do this by taking transposes, then filling in the resulting diagram with two naturality squares, two instances of Equation 3.3 and one instance of

Equation 3.4:

$$
\begin{tikzcd}
 & (hg)^b f \arrow[r, "\tilde{b}"] \arrow[dl, "\tilde{b}"'] \arrow[d, "\bar{h}"'] \arrow[dr, "\eta"] & ((hg)^b f)^b i \arrow[d, "\hat{b}"] \\
((hg)^b f)^b i \arrow[d, "\bar{h}"'] & hg^a f \arrow[dl, "\tilde{b}"'] \arrow[d, "\tilde{a}"] \arrow[dr, "\eta"] & (hg)^b f^* i \arrow[d, "\bar{h}"] \\
(hg^a f)^b i \arrow[r, "\bar{h}"'] & h(g^a f)^a i \arrow[r, "\hat{a}"'] & hg^a f^* i
\end{tikzcd}
$$

Thus indeed setting $\bar{h}_g$ to be the transpose of $h\tilde{a}_g$ gives h the structure of a lax morphism.

Now let $\gamma : h \to k$ be a 2-cell: to show that the algebra 2-cell condition (Equation 3.6) holds:

$$
\begin{tikzcd}
(hf)^b \arrow[r, "(\gamma f)^b"] \arrow[d, "\bar{h}_f"'] & (kf)^b \arrow[d, "\bar{k}_f"] \\
hf^a \arrow[r, "\gamma f^a"'] & kf^a
\end{tikzcd}
$$

we take transposes and fill the resulting diagram in with two naturality squares and two instances of Equation 3.4:

$$
\begin{tikzcd}
 & & (hf)^b i \arrow[dr, "\gamma"] & \\
 & hf \arrow[ur, "\tilde{b}"] \arrow[r, "\gamma"] \arrow[dl, "\tilde{b}"'] \arrow[d, "\tilde{a}"] & kf \arrow[r, "\tilde{b}"] \arrow[d, "\tilde{a}"'] & (kf)^b i \arrow[dl, "\bar{k}"] \\
(hf)^b i \arrow[r, "\bar{h}"'] & hf^a i \arrow[r, "\gamma"'] & kf^a i &
\end{tikzcd}
$$

Hence every 2-cell is an algebra 2-cell and so indeed $U : T\text{-Alg}_l \to \mathbb{D}$ is locally fully faithful.

We now show that (2) implies (1). Suppose $U : T\text{-Alg} \to \mathbb{D}$ is locally fully faithful. Then in particular every $g : TX \to A$ has a lax morphism structure, and we can use this to define the composite

$$\alpha_g : (gi)^a \xrightarrow{\bar{g}_i} gi^* \xrightarrow{g\theta} g.$$

These assemble into a natural transformation $\alpha : (-i)^a \implies 1$, as the naturality condition is formed from the fact that every 2-cell δ is an algebra 2-cell, as well as naturality of θ:

$$\begin{tikzcd}
(gi)^a \arrow[r, "\bar{g}_i"] \arrow[d, "(\delta i)^a"'] & gi^* \arrow[r, "g\theta"] \arrow[d, "\delta i^*"'] & g \arrow[d, "\delta"] \\
(hi)^a \arrow[r, "\bar{h}_i"'] & hi^* \arrow[r, "h\theta"] & h
\end{tikzcd}$$

It remains to show that $(\tilde{a}, \alpha)$ are the unit and counit of the desired adjunction $(-)^a \dashv - \circ i$. We fill in the triangle equation diagrams as follows:

$$\begin{tikzcd}
hi \arrow[r, "\tilde{a}"] \arrow[dr, "\eta"] \arrow[ddr, equal] & (hi)^a i \arrow[d, "\bar{h}"] \\
& hi^* i \arrow[d, "\theta"] \\
& hi_X
\end{tikzcd}
\qquad
\begin{tikzcd}
g^a \arrow[r, "\tilde{a}"] \arrow[ddr, equal, "1"'] & (g^a i)^a \arrow[d, "\overline{g^a} = \hat{a}"] \\
& g^a i^* \arrow[d, "g^a \theta_X"] \\
& g^a
\end{tikzcd}$$

so that the first commutes by Equation 3.4 and a relative pseudomonad coherence, while the second commutes as an instance of Equation 3.2. Hence we have the desired adjunction and so T is lax-idempotent. $\square$

We close this section by showing that our running example of the presheaf relative pseudomonad is algebraically lax idempotent.

Proposition 4.7. *The presheaf relative pseudomonad* $P : \mathrm{Cat} \to \mathrm{CAT}$ *is algebraically lax-idempotent.*

Proof. Let $(\mathbb{A}, {}^a; \tilde{a}, \hat{a})$ be a P-pseudoalgebra. Let $f : \mathbb{D} \to \mathbb{A}$ be a small diagram. By 3.19, $f^a : P\mathbb{D} \to \mathbb{A}$ preserves small colimits, and $f^a y$ is isomorphic to f. Since $P\mathbb{D}$ is the free cocompletion of $\mathbb{D}$, f^a is therefore the left Kan extension of f along $y_{\mathbb{D}}$ (see Theorem 4.51 in [Kel82]).

Hence we have an adjunction $(-)^a \dashv - \circ i$ and so P is algebraically lax-idempotent. $\square$

This allows us to immediately characterise the lax morphisms of P-pseudoalgebras.

Corollary 4.8. *Given two P-pseudoalgebras, every functor between their underlying cocomplete categories is (in a unique way) a lax morphism.*

Proof. By Theorem 4.6 and Proposition 4.7, the forgetful functor $P\text{-Alg} \to \mathrm{CAT}$ is locally fully faithful, and so every functor $f : A \to B$ between P-pseudoalgebras $(A, {}^{a})$ and $(B, {}^{b})$ has a unique lax morphism structure. $\square$

Chapter 5

Pseudocommutativity for relative pseudomonads

Introduction

This section generalises the work in Chapter 2, from relative monads on multicategories to relative pseudomonads on 2-multicategories. In doing so we provide an analogue of the theory of pseudocommutativity for pseudomonads ([HP02, LF11]) in the relative setting.

We define the notion of strong relative pseudomonad on a 2-multicategory (Definition 5.3; we expect this to be closely related to the notion of strong relative pseudomonad on a monoidal bicategory defined in [PS23] Section 4.2), and prove that for a strong relative pseudomonad, the underlying pseudofunctor becomes a pseudomultifunctor (Proposition 5.9) and the unit becomes multicategorical (part of Theorem 5.16). We also define the notion of pseudocommutative relative pseudomonad, which will amount to asking for, given

$$f : B_1, ..., B_{j-1}, JX, B_{j+1}, ..., B_{k-1}, JY, B_{k+1}, ..., B_n \to TZ,$$

an isomorphism

$$f^{kj} \cong f^{jk} : B_1, ..., B_{j-1}, TX, B_{j+1}, ..., B_{k-1}, TY, B_{k+1}, ..., B_n \to TZ$$

where $(-)^j$ and $(-)^k$ denote strengthenings in the jth and kth indices respectively (Definition 5.3). We then prove that every pseudocommutative relative pseudomonad is a relative pseudomultimonad (Theorem 5.16).

We define a notion of lax idempotency suitable for strong relative pseudomonads (Definition 5.17), extending earlier definitions in [DLLS23] and [FGHW18], and prove that every lax-idempotent strong relative pseudomonad is pseudocommutative (Theorem 5.18). This generalisation is unrelated to the definition of algebraic lax idempotency in Chapter 4, as the axes of generalisation are orthogonal: here we generalise from unary 2-categories to 2-multicategories, while in Chapter 4 we generalised from free pseudoalgebras $(TX, ^*)$ to general pseudoalgebras $(A, ^a)$.

To close, we apply these definitions and results to the example of presheaves (Theorem 5.20).

5.1 Strong relative pseudomonads

The 2-categories Cat and CAT possess more structure than simply being 2-categories; they are in particular cartesian 2-categories. Thus we will seek to develop Kock's theory of monads on symmetric monoidal closed categories [Koc70] for relative pseudomonads. To avoid some of the coherence inherent to working with monoidal 2-categories, we will work in the related setting of 2-multicategories (see Definition 5.1).

We seek to consider the notion of a relative pseudomonad along $J : \mathbb{C} \to \mathbb{D}$ when $\mathbb{C}$ and $\mathbb{D}$ are 2-multicategories. We will define a 'strong relative pseudomonad' from scratch to take this role, and note that a every strong relative pseudomonad induces a canonical relative pseudomonad structure. In order to do this, let us recall the definition of a 2-multicategory as in [Her00] (taking $V = \mathrm{Cat}$ to specialise the V-enriched theory).

Definition 5.1. (2-multicategory) A 2-multicategory $\mathbb{C}$ is a multicategory enriched in Cat. Unwrapping this statement a little, a 2-multicategory $\mathbb{C}$ is given by

1. a collection of objects $X \in \mathrm{ob}\,\mathbb{C}$, together with

2. a category of multimorphisms $\mathbb{C}(X_1, ..., X_n; Y)$ for all $n \geq 0$ and objects $X_1, ..., X_n, Y$ which we call a *hom-category*; an object of the hom-category $\mathbb{C}(X_1, ..., X_n; Y)$ is denoted by $f : X_1, ..., X_n \to Y$,

3. an identity multimorphism functor $\mathbf{1}_X : \mathbb{1} \to \mathbb{C}(X; X) : * \mapsto 1_X$ for all $X \in \mathrm{ob}\,\mathbb{C}$, and

4. composition functors

$$\mathbb{C}(X_1, ..., X_n; Y) \times \mathbb{C}(W_{1,1}, ..., W_{1,m_1}; X_1) \times ... \times \mathbb{C}(W_{n,1}, ..., W_{n,m_n}; X_n)$$

$$\to \mathbb{C}(W_{1,1}, ..., W_{n,m_n}; Y)$$

$$(f, g_1, ..., g_n) \mapsto f \circ (g_1, ..., g_n)$$

   for all arities $n, m_1, ..., m_n$ and objects $Y, X_1, ..., X_n, W_{1,1}, ..., W_{n,m_n}$ in $\mathbb{C}$.

where the identity and composition functors satisfy the usual associativity and identity axioms for an enrichment.

Recall the following notational shorthand first introduced in Chapter 2, which will again be used throughout this section.

- When arity is unimportant or can be easily inferred, we write $f : \bar{X} \to Y$ instead of $f : X_1, ..., X_n \to Y$.

- Given $f : \bar{X} \to Y$ and $g : \bar{W} \to X_j$ we abbreviate composites of the form $f \circ (1, ..., 1, g, 1, ..., 1)$ to $f \circ_j g$.

- We write $\bar{X} \cdot Y_j$ to abbreviate slightly more complicated domains of the form

$$X_1, ..., X_{j-1}, Y, X_{j+1}, ..., X_n.$$

  Similarly, we write $\bar{X} \cdot Y_j, Z_k$ to abbreviate

$$X_1, ..., X_{j-1}, Y, X_{j+1}, ..., X_{k-1}, Z, X_{k+1}, ..., X_n.$$

Remark 5.2. We can relate 2-multicategories to more familiar structures.

- Every 2-multicategory $\mathbb{C}$ restricts to a 2-category by considering only the unary hom-categories $\mathbb{C}(X;Y)$.
- (Strict) monoidal 2-categories (defined in for example [DS97]) have underlying 2-multicategories, where hom-categories $\mathbb{C}(X_1, ..., X_n; Y)$ are given by $\mathbb{C}(X_1 \otimes ... \otimes X_n, Y)$ (choosing the leftmost bracketing of the tensor product); this is shown in [Her00] Proposition 7.1 (2). For example, both Cat and CAT can be given 2-multicategorical structures.

We now define a notion of strong relative pseudomonad, generalising Definition 2.2 in Chapter 2 from one to two dimensions.

Definition 5.3. (Strong relative pseudomonad) Let $\mathbb{C}$ and $\mathbb{D}$ be 2-multicategories and let $J : \mathbb{C} \to \mathbb{D}$ be a (unary) 2-functor between them. A *strong relative pseudomonad* $(T, i, ^\bullet; \tilde{t}, \hat{t}, \theta)$ along J comprises:

- for every object X in $\mathbb{C}$ an object TX in $\mathbb{D}$ and unit map $i_X : JX \to TX$,
- for every n, index $1 \leq j \leq n$, objects $B_1, ..., B_{j-1}, B_{j+1}, ..., B_n$ in $\mathbb{D}$ and objects X, Y in $\mathbb{C}$ a functor

  $$\mathbb{D}(\bar{B} \cdot JX_j; TY) \xrightarrow{(-)^j} \mathbb{D}(\bar{B} \cdot TX_j; TY)$$

  called the *strength* (in the jth argument) and which is pseudonatural in all arguments.

Along with these we have three natural families of invertible 2-cells:

- $\tilde{t}_f : f \to f^j \circ_j i$,
- $\hat{t}_{f,g} : (f^j \circ_j g)^{j+k-1} \to f^j \circ_j g^k$, and
- $\theta_X : (i_X)^1 \to 1_{TX}$

for $f : \bar{B} \cdot JX_j \to TY$ and $g : \bar{C} \cdot JW_k \to TX$, satisfying the coherence conditions (1) and (2) shown below.

As a notational shorthand (in order not to have to keep track of index shifting due to composing multimorphims), when a map $f : \bar{B} \cdot JX_j \to TY$ has only one argument in the domain of the form JX for $X \in \operatorname{ob} \mathbb{D}$, we will denote its strengthening as f^t, rather than f^j. We will furthermore write $f^t \circ_t g$ to denote the composite of f^t with g in this strengthened argument. In this notation, the families of invertible 2-cells above are:

$$\begin{aligned} \tilde{t}_f &: f \to f^t \circ_t i, \\ \hat{t}_{f,g} &: (f^t \circ_t g)^t \to f^t \circ_t g^t, \\ \theta &: i^t \to 1. \end{aligned}$$

(We also omit subscripts from unit maps and from θ when unambiguous.) With this notation in hand, the two coherence conditions for these 2-cells are:

(1) for every $f : \bar{B} \cdot JX_j \to TY$, $g : \bar{C} \cdot JW_k \to TX$ and $h : \bar{D} \cdot JV_l \to TW$ the diagram

$$\begin{tikzcd} ((f^t \circ_t g)^t \circ_t h)^t \arrow[rr, "\hat{t}_{f^t \circ_t g, h}"] \arrow[d, "(\hat{t}_{f,g} \circ_t h)^t"'] & & (f^t \circ_t g)^t \circ_t h^t \arrow[d, "\hat{t}_{f,g} \circ_t h^t"] \\ (f^t \circ_t g^t \circ_t h)^t \arrow[r, "\hat{t}_{f, g^t \circ_t h}"'] & f^t \circ_t (g^t \circ_t h)^t \arrow[r, "f^t \circ_t \hat{t}_{g,h}"'] & (f^t \circ_t g^t) \circ_t h^t \end{tikzcd} \tag{5.1}$$

commutes, and

(2) for every $f : \bar{B} \cdot JX_j \to TY$ the diagram

$$\begin{tikzcd} f^t \arrow[r, "(\tilde{t}_f)^t"] \arrow[drr, equal] & (f^t \circ_t i)^t \arrow[r, "\hat{t}_{f,i}"] & f^t \circ_t i^t \arrow[d, "f^t \circ_t \theta"] \\ & & f^t \end{tikzcd} \tag{5.2}$$

commutes.

Remark 5.4. The stipulation that the maps

$$\mathbb{D}(\bar{B} \cdot JX_j; TY) \xrightarrow{(-)^j} \mathbb{D}(\bar{B} \cdot JX_j; TY)$$

be pseudonatural in all arguments asks in particular for invertible 2-cells of the form

$$(f \circ_k g)^j \cong f^j \circ_k g$$

for $g : \bar{C} \to B_k$ and $k \neq j$. Wherever such pseudonaturality isomorphisms arise in diagrams we will leave them anonymous, as they can be inferred from the source and target.

Remark 5.5. The data for a strong relative pseudomonad resembles that for a relative pseudomonad as in [FGHW18] Definition 2.1 very closely. Indeed, restricting $\mathbb{C}$ and $\mathbb{D}$ to their 2-categories of unary maps, $(T, i, ^\bullet)$ is exactly a relative pseudomonad as in Definition 3.1, with

$$(-)^* := (-)^1,$$
$$\eta := \tilde{t},$$
$$\mu := \hat{t},$$
$$\theta := \theta.$$

As with relative pseudomonads, we can derive more equalities of 2-cells for a strong relative pseudomonads. The proof of the following Lemma 5.6 is formally identical to the proof of Lemma 3.2 in [FGHW18].

Lemma 5.6. *Let T be a strong relative pseudomonad along $J : \mathbb{C} \to \mathbb{D}$. Then we have that:*

(1) for every $f : \bar{B} \cdot JX_j \to TY$ and $g : \bar{C} \cdot JW_k \to TX$, the diagram

$$\begin{tikzcd} f^t \circ_t g \arrow[r, "\tilde{t}_{f^t \circ_t g}"] \arrow[dr, "f^t \circ_t \tilde{t}_g"'] & (f^t \circ_t g)^t \circ_t i \arrow[d, "\hat{t}_{f,g} \circ_t i"] \\ & f^t \circ_t g^t \circ_t i \end{tikzcd} \tag{5.3}$$

commutes.

(2) for every $f : \bar{B} \cdot JX_j \to TY$*, the diagram*

$$\begin{tikzcd}
(i^t \circ f)^t \arrow[r, "\hat{t}_{i,f}"] \arrow[dr, "(\theta \circ f)^t"'] & i^t \circ f^t \arrow[d, "\theta \circ f^t"] \\
& f^t
\end{tikzcd} \tag{5.4}$$

commutes, and

(3) for every object $X \in \operatorname{ob} \mathbb{D}$*, the diagram*

$$\begin{tikzcd}
i \arrow[r, "\tilde{t}_i"] \arrow[dr, equal] & i^t \circ i \arrow[d, "\theta \circ i"] \\
& i
\end{tikzcd} \tag{5.5}$$

commutes.

Example 5.7. The presheaf relative pseudomonad can be given the structure of a strong relative pseudomonad. Given a multimorphism

$$f : B_1 \times \cdots \times B_{j-1} \times X \times B_{j+1} \times \cdots \times B_n \to PY$$

with $X, Y \in \mathrm{Cat}$ and $B_k \in \mathrm{CAT}$, which in line with our notation we shall write as $f : \bar{B} \cdot X_j \to PY$, its strengthening f^t is defined to be the left Kan extension

$$\begin{tikzcd}
\bar{B} \cdot X_j \arrow[r, "\bar{1} \cdot y_X"] \arrow[dr, "f"'] & \bar{B} \cdot PX_j \arrow[d, "f^t := \mathrm{Lan}_{\bar{1} \cdot y} f"] \arrow[dl, phantom, "\overset{\tilde{t}_f}{\Longrightarrow}"] \\
& PY
\end{tikzcd}$$

which also defines the 2-cells $\tilde{t}_f : f \to f^t \circ_t y$ (the desired left Kan extension exists as its colimit formula is isomorphic to a small colimit). As when giving P a relative pseudomonad structure, the 2-cells $\hat{t}_{f,g}$, θ are defined via the universal property of the left Kan extension. For details and a proof that this indeed endows P with a strong relative pseudomonad structure, see Proposition 5.19 in Section 5.4.

To generalise Proposition 2.7 (strong relative monads are multifunctors) to the two-dimensional setting, we need a notion of pseudomultifunctor.

Definition 5.8. (Pseudomultifunctor) Given 2-multicategories $\mathbb{C}, \mathbb{D}$, a *pseudo-multifunctor* $F : \mathbb{C} \to \mathbb{D}$ consists of:

- a function $\operatorname{ob}\mathbb{C} \xrightarrow{F} \operatorname{ob}\mathbb{D} : X \mapsto FX$,
- for each hom-category $\mathbb{C}(\bar{X}; Y)$ in $\mathbb{C}$ a functor
$$\mathbb{C}(\bar{X}; Y) \to \mathbb{D}(F\bar{X}; FY) : f \mapsto Ff,$$
- for each $X \in \operatorname{ob}\mathbb{C}$ an invertible 2-cell
$$\tilde{F}_X : F1_X \implies 1_{FX},$$
- for each $f : \bar{X} \to Y$, $1 \leq j \leq n$ and $g : \bar{W} \to X_i$ an invertible 2-cell
$$\hat{F}_{f,g} : F(f \circ_j g) \implies Ff \circ_j Fg$$

satisfying the following three coherence conditions which parallel the unit and associativity diagrams for a lax monoidal functor:

(1),(2) two unit axioms: for each $f : \bar{X} \to Y$ and $1 \leq j \leq n$ the diagrams

$$\begin{tikzcd} F(1_Y \circ f) \arrow[r, "\hat{F}_{1,f}"] \arrow[d, equal] & F1_Y \circ Ff \arrow[d, "\tilde{F}_Y \circ Ff"] \\ Ff \arrow[r, equal] & 1_{FY} \circ Ff \end{tikzcd} \qquad \begin{tikzcd} F(f \circ_j 1_{X_j}) \arrow[r, "\hat{F}_{f,1}"] \arrow[d, equal] & Ff \circ_i F1_{X_j} \arrow[d, "Ff \circ_j \tilde{F}_{X_j}"] \\ Ff \arrow[r, equal] & Ff \circ_j 1_{FX_j} \end{tikzcd}$$

commute, and

(3) one associativity axiom: for each $f : \bar{X} \to Y$, $1 \leq j \leq n$, $g : \bar{W} \to X_j$, $1 \leq k \leq m$ and $h : \bar{V} \to W_k$ the diagram

$$\begin{tikzcd}
F(f \circ_j (g \circ_k h)) \arrow[r, equal] \arrow[d, "\hat{F}_{f, g \circ_k h}"'] & F((f \circ_j g) \circ_{j+k-1} h) \arrow[d, "\hat{F}_{f \circ_j g, h}"] \\
Ff \circ_j F(g \circ_k h) \arrow[d, "Ff \circ_j \hat{F}_{g,h}"'] & F(f \circ_j g) \circ_{j+k-1} Fh \arrow[d, "\hat{F}_{f,g} \circ_{j+k-1} Fh"] \\
Ff \circ_j (Fg \circ_k Fh) \arrow[r, equal] & (Ff \circ_j Fg) \circ_{j+k-1} Fh
\end{tikzcd}$$

commutes.

If the 2-cells $\tilde{F}$, $\hat{F}$ are all identities we call F a *(strict) 2-multifunctor.*

Proposition 5.9. *Let T be a strong relative pseudomonad along a 2-multifunctor $J : \mathbb{C} \to \mathbb{D}$. Then T can be given the structure of a pseudomultifunctor $T : \mathbb{C} \to \mathbb{D}$.*

Proof. Suppose T is a strong relative pseudomonad. Given a map $f : \bar{X} \to Y$ let us define

$$\bar{f} := i_Y \circ Jf : J\bar{X} \to TY.$$

Now to show the T is a pseudo-multifunctor, we begin by defining the action of T on 1-cells by the functors

$$\mathbb{C}(\bar{X}; Y) \xrightarrow{(i \circ J-)^{[n]}} \mathbb{D}(T\bar{X}; TY),$$

(recalling that $(-)^{[n]}$ is defined in Chapter 2 as shorthand for $(-)^{123\ldots n}$) so that for $f : \bar{X} \to Y$ we have

$$Tf := (i \circ Jf)^{[n]} = \bar{f}^{[n]} : T\bar{X} \to TY.$$

We need to construct 2-cells $\tilde{T}_X : T1_X \implies 1_{TX}$ and $\hat{T}_{f,g} : T(f \circ_j g) \implies Tf \circ_j Tg$. We define the former by the map

$$T1_X = (i_X \circ J1_X)^1 = (i_X)^1 \xrightarrow{\theta} 1$$

and the latter by the composite

$$
\begin{aligned}
T(f \circ_j g) = (i \circ (Jf \circ_j Jg))^{[n+m-1]} &= (\bar{f} \circ_j Jg)^{[n+m-1]} \\
&\xrightarrow{\sim} (\bar{f}^{[j-1]} \circ_j Jg)^{j\ldots n+m-1} \\
&\xrightarrow{\tilde{t}} (\bar{f}^{[j]} \circ_j \bar{g})^{j\ldots n+m-1} \\
&\xrightarrow{\hat{t}\ldots\hat{t}} (\bar{f}^{[j]} \circ_j \bar{g}^{[m]})^{j+m\ldots n+m-1} \\
&\xrightarrow{\sim} \bar{f}^{[n]} \circ_i \bar{g}^{[m]} = Tf \circ_j Tg.
\end{aligned}
$$

It remains to show that the three coherence conditions hold. For the first, we write out our definitions and fill in the resulting diagram with two equalities and Equations 5.4 and 5.5:

$$
\begin{tikzcd}
(\bar{1} \circ Jf)^{[n]} \arrow[r, "\tilde{t}"] \arrow[dd, equal] & (\bar{1}^1 \circ \bar{f})^{[n]} \arrow[r, "\hat{t}\ldots\hat{t}"] \arrow[d, equal] & \bar{1}^1 \circ \bar{f}^{[n]} \arrow[d, equal] \\
& (i^1 \circ \bar{f})^{[n]} \arrow[r, "\hat{t}\ldots\hat{t}"] \arrow[dr, "\theta"] & i^1 \circ \bar{f}^{[n]} \arrow[d, "\theta"] \\
(i \circ Jf)^{[n]} \arrow[ur, "\tilde{t}"] \arrow[rr, equal] & & \bar{f}^{[n]}
\end{tikzcd}
$$

For the second condition we do the same, filling the resulting diagram in with equalities, a naturality square, and an instance of Equation 5.2:

$$
\begin{tikzcd}
(\bar{f} \circ_j J1)^{[n]} \arrow[r, "\sim"] \arrow[d, equal] & (\bar{f}^{[j-1]} \circ_j J1)^{j\ldots n} \arrow[r, "\tilde{t}"] \arrow[d, equal] & (\bar{f}^{[j]} \circ_j \bar{1})^{j\ldots n} \arrow[dl, equal] \arrow[d, "\hat{t}"] \\
\bar{f}^{[n]} \arrow[r, equal] \arrow[dddd, equal] & \bar{f}^{[n]} \arrow[d, "\tilde{t}"] & \\
& (\bar{f}^{[j]} \circ_j i)^{j\ldots n} \arrow[d, "\hat{t}"] & (\bar{f}^{[j]} \circ_j \bar{1}^1)^{j+1\ldots n} \arrow[dl, equal] \arrow[d, "\sim"] \\
& (\bar{f}^{[j]} \circ_j i^1)^{j+1\ldots n} \arrow[d, "\theta"] \arrow[dr, "\sim"] & \bar{f}^{[n]} \circ_j \bar{1}^1 \arrow[d, equal] \\
& (\bar{f}^{[j]} \circ_j 1)^{j+1\ldots n} \arrow[dl, equal] \arrow[dr, "\sim"] & \bar{f}^{[n]} \circ_j i^1 \arrow[d, "\theta"] \\
\bar{f}^{[n]} \arrow[rr, equal] & & \bar{f}^{[n]} \circ_j 1
\end{tikzcd}
$$

For the final coherence condition, in the interest of space we merely note here that verification only involves (aside from naturality squares) several uses of Equation 5.3. Thus every strong relative pseudomonad is indeed a pseudomultifunctor. □

Example 5.10. Proposition 5.9 will imply that the presheaf relative pseudomonad is a pseudomultifunctor. Using the coend formula for the left Kan extension we find for example that, given a functor $F : A \times B \times C \to D$ in Cat, the multicategorical action of P on F has the form

$$
\begin{aligned}
PF : PA \times PB \times PC &\to PD \\
(p, q, r) &\mapsto \int^{c \in C} \int^{b \in B} \int^{a \in A} p(a) \times q(b) \times r(c) \times y_{F(a,b,c)}.
\end{aligned}
$$

5.2 Pseudocommutative relative pseudomonads

Building on the work of Kelly [Kel74b], Hyland and Power [HP02] extend the notion of commutative monad [Koc70], which asks that the two composites

$$TA \otimes TB \xrightarrow{s} T(A \otimes TB) \xrightarrow{Tt} TT(A \otimes B) \xrightarrow{\mu} T(A \otimes B),$$
$$TA \otimes TB \xrightarrow{t} T(TA \otimes B) \xrightarrow{Ts} TT(A \otimes B) \xrightarrow{\mu} T(A \otimes B)$$

be equal, to the 2-categorical setting, defining *pseudocommutativity* by asking only for an invertible 2-cell between these two composites.

Analogously, there is some freedom in the pseudo-multifunctorial structure we place on a given strong relative pseudomonad T; we defined the action of T on a morphism $f : \bar{X} \to Y$ by

$$Tf := \bar{f}^{1\ldots n} : T\bar{X} \to TY,$$

but we could equally well have chosen

$$Tf := \bar{f}^{n\ldots 1} : T\bar{X} \to TY$$

with the strengthenings applied in the reverse order. We define pseudocommutativity in our more general setting so as to imply that the two choices of definition of Tf are coherently isomorphic.

Definition 5.11. (Pseudocommutative monad) Let T be a strong relative pseudomonad. We say that T is *pseudocommutative* if for every pair of indices $1 \leq j < k \leq n$ and map

$$f : \bar{B} \cdot JX_j, JY_k \to TZ$$

we have an invertible 2-cell

$$\gamma_f : f^{kj} \to f^{jk}$$

where f^{kj}, f^{jk} are maps $\bar{B} \cdot TX_j, TY_k \to TZ$, which is pseudonatural in all arguments and which satisfies five coherence conditions (two for $\tilde{t}$, two for $\hat{t}$, and a braiding condition) given below.

We will extend our notation in the following way. When there are two objects in the domain of a map

$$f : \bar{B} \cdot JX_j, JY_k \to TZ$$

in the image of J, let strengthening f at the leftmost of these be denoted by $f^s : \bar{B} \cdot TX_j, JY_k \to TZ$, with 2-cells $\tilde{s} : f \to f^s \circ_s i$ and $\hat{s} : (f^s \circ_s g)^s \to f^s \circ g^t$, and let the rightmost be denoted by $f^t : \bar{B} \cdot JX_j, TY_k \to TZ$ with 2-cells $\tilde{t}$, $\hat{t}$. When there are three such objects in the domain of f, we denote the corresponding three strengthenings from left to right by f^s, f^t and f^u (with corresponding 2-cell notation). This use of s, t, u allows us to write equations without keeping track of shifting indices caused by composing multimorphisms.

The coherence conditions γ must satisfy are as follows:

(1), (2) Precomposing γ_f in the jth or kth argument with a unit map i: the diagrams

$$\begin{tikzcd} f^t \arrow[r, "\tilde{s}_{f^t}"] \arrow[ddr, "(\tilde{s}_f)^t"'] & f^{ts} \circ_s i \arrow[d, "\gamma_f \circ_s i"] \\ & f^{st} \circ_s i \arrow[d, "\sim"] \\ & (f^s \circ_s i)^t \end{tikzcd} \qquad \begin{tikzcd} f^s \arrow[r, "(\tilde{t}_f)^s"] \arrow[ddr, "\tilde{t}_{f^s}"'] & (f^t \circ_t i)^s \arrow[d, "\sim"] \\ & f^{ts} \circ_t i \arrow[d, "\gamma_f \circ_t i"] \\ & f^{st} \circ_t i \end{tikzcd}$$

commute for $f : \bar{B} \cdot JX_j, JY_k \to TZ$.

(3), (4) Precomposing γ_f in the jth or kth argument with the strengthening of a map g in its lth argument: the diagrams

$$\begin{tikzcd}
(f^{ts} \circ_s g)^s \arrow[r, "\hat{s}_{f^t,g}"] \arrow[d, "(\gamma_f \circ_s g)^s"'] & f^{ts} \circ_s g^t \arrow[dd, "\gamma_f \circ_s g^t"] \\
(f^{st} \circ_s g)^s \arrow[d, "\sim"'] & \\
(f^s \circ_s g)^{ts} \arrow[d, "\gamma_{f^s \circ_s g}"'] & f^{st} \circ_s g^t \arrow[d, "\sim"] \\
(f^s \circ_s g)^{st} \arrow[r, "(\hat{s}_{f,g})^t"'] & (f^s \circ_s g^t)^t
\end{tikzcd}
\qquad
\begin{tikzcd}
(f^t \circ_t h)^{ts} \arrow[r, "(\hat{t}_{f,h})^s"] \arrow[d, "\gamma_{f^t \circ_t h}"'] & (f^t \circ_t h^t)^s \arrow[d, "\sim"] \\
(f^t \circ_t h)^{st} \arrow[d, "\sim"'] & f^{ts} \circ_t h^t \arrow[dd, "\gamma_f \circ_t h^t"] \\
(f^{ts} \circ_t h)^t \arrow[d, "(\gamma_f \circ_t h)^t"'] & \\
(f^{st} \circ_t h)^t \arrow[r, "\hat{t}_{f^s,h}"'] & f^{st} \circ_t h^t
\end{tikzcd}$$

commute for $f : \bar{B} \cdot JX_j, JY_k \to TZ$, $g : \bar{C} \cdot JW_l \to TX$ and $h : \bar{D} \cdot JV_m \to TY$.

(5) Braiding axiom relating the six ways to strengthen a map

$$f : \bar{B} \cdot JW_j, JX_k, JY_l \to TZ$$

in all three arguments: the diagram

$$\begin{tikzcd}
f^{uts} \arrow[r, "(\gamma_f)^s"] \arrow[d, "\gamma_{f^u}"'] & f^{tus} \arrow[r, "\gamma_{f^t}"] & f^{tsu} \arrow[d, "(\gamma_f)^u"] \\
f^{ust} \arrow[r, "(\gamma_f)^t"'] & f^{sut} \arrow[r, "\gamma_{f^s}"'] & f^{stu}
\end{tikzcd}$$

commutes for all $f : \bar{B} \cdot JW_j, JX_k, JY_l \to TZ$.

Remark 5.12. The braiding axiom (5) allows us to extend our notation. Given a map $f : J\bar{X} \to TY$ and a permutation $\sigma \in S_n$, we can construct maps

$$f^{1\ldots n} \to f^{\sigma(1)\ldots\sigma(n)}$$

as a composite of γ maps and their inverses. The braiding axiom (5) tells us that any two such composites of γ and γ^{-1} maps are equal; we will denote this map by

$$\gamma_{\sigma;f} : f^{1\ldots n} \to f^{\sigma(1)\ldots\sigma(n)}.$$

Remark 5.13. When J is the identity, Definition 5.11 reduces to the definition of pseudocommutativity found in [HP02] Definition 5. The correspondence between the coherence conditions given here and their conditions is enumerated in the following table:

Relative setting	Hyland & Power
(1), (2)	4., 5.
(3), (4)	6., 7.
(5)	1., 2., 3.

Example 5.14. The presheaf relative pseudomonad will be shown to be pseudocommutative in this sense in Theorem 5.20; recalling the formula for the multicategorical action of P on 1-cells in Example 5.10, one should be able to permute the order of strengthenings by means of Fubini isomorphisms for coends. However, proving that P is pseudocommutative directly in this way is challenging; in Section 5.4 we will discuss a property that implies pseudocommutativity and which is much easier to verify.

In Kock [Koc70] it is shown that a strong monad is lax-monoidal as a functor, and even that the monad unit for a strong monad is a monoidal transformation, but that in order for the monad multiplication (and thus the monad as a whole) to be monoidal, the monad must be commutative.

In our setting, every strong relative pseudomonad T has the structure of a pseudomultifunctor, and we are now interested in the question of when T further has the structure of a relative pseudomultimonad (defined below); that is, when the pseudomonadic structure of T is compatible with the ambient multicategorical structure. We will show in this section that every pseudocommutative relative pseudomonad is a multicategorical relative pseudomonad.

Definition 5.15. (Relative pseudomultimonad) Let $\mathbb{C}, \mathbb{D}$ be a pair of 2-multicategories and let T be a relative pseudomonad along $J : \mathbb{C} \to \mathbb{D}$. We say T is a *relative pseudomultimonad* if

- T is a pseudomultifunctor, and
- The unit and extension of T are compatible with the multicategorical structure, which is to say explicitly that
  - the monad unit i is multicategorical: for each $f : \bar{X} \to Y$ we have an invertible 2-cell

$$\iota_f : i_Y \circ Jf \to Tf \circ (i_{X_1}, ..., i_{X_n}),$$

$$\begin{tikzcd} J\bar{X} \arrow[r, "\bar{i}"] \arrow[d, "Jf"'] & T\bar{X} \arrow[d, "Tf"] \\ JY \arrow[r, "i"'] \arrow[ur, Rightarrow, "\iota_f"] & TY \end{tikzcd}$$

  - the monad extension $(-)^*$ is multicategorical: for each 2-cell of the form $\alpha : h \circ Jf \to Tf' \circ \bar{g}$ we have an extended 2-cell $\alpha^* : h^* \circ Tf \to Tf' \circ (g_1^*, ..., g_n^*)$:

$$\begin{tikzcd} J\bar{X} \arrow[r, "\bar{g}"] \arrow[d, "Jf"'] & T\bar{X}' \arrow[d, "Tf'"] \\ JY \arrow[r, "h"'] \arrow[ur, Rightarrow, "\alpha"] & TY' \end{tikzcd} \qquad \begin{tikzcd} T\bar{X} \arrow[r, "\bar{g}^*"] \arrow[d, "Tf"'] & T\bar{X}' \arrow[d, "Tf'"] \\ TY \arrow[r, "h^*"'] \arrow[ur, Rightarrow, "\alpha^*"] & TY' \end{tikzcd}$$

These must satisfy three coherence conditions (one for each of the families of 2-cells making T a relative pseudomonad).

(1) Compatibility with η: given a 2-cell $\alpha : h \circ Jf \to Tf' \circ \bar{g}$, the following two

composites are equal:

$$
\begin{tikzcd}
J\bar{X} \arrow[r, "\bar{i}"] \arrow[d, "Jf"'] & T\bar{X} \arrow[r, "\bar{g}^*"] \arrow[d, "Tf"] & T\bar{X}' \arrow[d, "Tf'"] \\
JY \arrow[r, "i"'] \arrow[ur, phantom, "\Nearrow \iota_f"] \arrow[rr, bend right, "h"'] & TY \arrow[r, "h^*"'] \arrow[ur, phantom, "\Nearrow \alpha^*"] \arrow[d, phantom, "\eta_h \Uparrow"] & TY' \\
& {} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& T\bar{X} \arrow[dr, "\bar{g}^*"] \arrow[d, phantom, "\bar{\eta}_{\bar{g}} \Uparrow"] & \\
J\bar{X} \arrow[ur, "\bar{i}"] \arrow[rr, "\bar{g}"] & {} & T\bar{X}' \arrow[d, "Tf'"] \\
JY \arrow[u, "Jf"] \arrow[rr, "h"'] \arrow[urr, phantom, "\Nearrow \alpha"] & & TY'
\end{tikzcd}
$$

(2) Compatibility with μ: given 2-cells $\alpha : Tf' \circ \bar{g}^* \to h \circ Jf$ and $\beta : Tf'' \circ \bar{g}'^* \to h' \circ Jf'$, the following two composites are equal:

$$
\begin{tikzcd}
T\bar{X} \arrow[r, "\bar{g}^*"] \arrow[d, "Tf"'] & T\bar{X}' \arrow[r, "\bar{g}'^*"] \arrow[d, "Tf'"] & T\bar{X}'' \arrow[d, "Tf''"] \\
TY \arrow[r, "h^*"'] \arrow[ur, phantom, "\Nearrow \alpha^*"] \arrow[rr, bend right, "(h'^*h)^*"'] & TY' \arrow[r, "h'^*"'] \arrow[ur, phantom, "\Nearrow \beta^*"] \arrow[d, phantom, "\mu_{h',h} \Uparrow"] & TY'' \\
& {} &
\end{tikzcd}
\qquad
\begin{tikzcd}
& T\bar{X}' \arrow[dr, "\bar{g}'^*"] \arrow[d, phantom, "\bar{\mu}_{\bar{g}',\bar{g}} \Uparrow"] & \\
T\bar{X} \arrow[ur, "\bar{g}^*"] \arrow[rr, "(\bar{g}'^*\bar{g})^*"] \arrow[d, "Tf"'] & {} & T\bar{X}'' \arrow[d, "Tf''"] \\
TY \arrow[rr, "(h'^*h)^*"'] \arrow[urr, phantom, "\Nearrow (\beta^*\alpha)^*"] & & TY''
\end{tikzcd}
$$

(where $(\beta^*\alpha)^*$ omits whiskerings), and

(3) Compatibility with θ: given $f : \bar{X} \to Y$, the following two composites are equal:

$$
\begin{tikzcd}
T\bar{X} \arrow[r, "\bar{1}"] \arrow[d, "Tf"'] & T\bar{X} \arrow[d, "Tf"] \\
TY \arrow[r, "1"] \arrow[r, bend right=60, "i^*"'] \arrow[r, phantom, shift right=3ex, "\theta \Uparrow"] & TY
\end{tikzcd}
\qquad
\begin{tikzcd}
T\bar{X} \arrow[r, dashed, "\bar{i}^*"] \arrow[r, bend left=60, "\bar{1}"] \arrow[r, phantom, shift left=3ex, "\bar{\theta} \Uparrow"] \arrow[d, "Tf"'] & T\bar{X} \arrow[d, "Tf"] \\
TY \arrow[r, "i^*"'] \arrow[ur, phantom, "\Nearrow \iota_f^*"] & TY
\end{tikzcd}
$$

In [Koc70] it is noted that the monad unit of a strong monad is always a monoidal transformation, but the monad multiplication is only a monoidal transformation if the monad is commutative. We shall see in the following proposition an analogous result: that for every strong relative pseudomonad, the monad unit is multicategorical (we can define the invertible 2-cells ι_f), but in order to make the monad extension multicategorical we require the relative pseudomonad to be pseudocommutative.

Theorem 5.16. *Let T be a strong relative pseudomonad along a 2-multifunctor $J : \mathbb{D} \to \mathbb{C}$. Suppose T is pseudocommutative. Then T is a relative pseudomulti-monad.*

Proof. By Proposition 5.9 we know that T is a pseudomultifunctor. We must check that the monad unit and extension are compatible with the multicategorical structure. For the unit, we need to find invertible 2-cells ι_f of shape

$$i \circ Jf \to Tf \circ \bar{i}$$

for $f : \bar{X} \to Y$. Since $Tf := (i \circ Jf)^{[n]} = \bar{f}^{[n]}$, we construct ι_f as the composite

$$\begin{aligned} i \circ Jf = \bar{f} &\xrightarrow{\tilde{t}} \bar{f}^1 \circ_1 i \\ &\xrightarrow{\tilde{t}} \bar{f}^{[2]} \circ_{[2]} \bar{i} \\ &\vdots \\ &\xrightarrow{\tilde{t}} \bar{f}^{[n]} \circ \bar{i} = Tf \circ \bar{i}. \end{aligned}$$

Note that we do not need the pseudocommutativity to construct the ι_f 2-cells. The construction of α^* given $\alpha : h \circ Jf \to Tf' \circ \bar{g}$ is more involved. We require a 2-cell of shape

$$h^* \circ Tf \to Tf' \circ \bar{g}^*.$$

We begin with the composite

$$\begin{aligned} h^* \circ Tf := h^1 \circ \bar{f}^{[n]} &\xrightarrow{\hat{t}^{-1}} (h^1 \circ \bar{f}^{[n-1]})^n \\ &\vdots \\ &\xrightarrow{\hat{t}^{-1}} (h^1 \circ \bar{f})^{[n]} \\ &\xrightarrow{\tilde{t}^{-1}} (h \circ Jf)^{[n]}, \end{aligned}$$

at which point we can compose with $\alpha^{[n]}$ to arrive at

$$(Tf' \circ \bar{g})^{[n]} := (\bar{f}'^{[n]} \circ \bar{g})^{[n]}.$$

From here we start needing the pseudocommutativity of T. Let $\sigma \in S_n$ be the cyclic permutation $(12...n)$. Now we compose as follows:

$$\begin{aligned}
&(\bar{f}'^{[n]} \circ \bar{g})^{[n]} \\
&\xrightarrow{\gamma_\sigma} (\bar{f}'^{2...1} \circ \bar{g})^{[n]} \xrightarrow{\hat{t}} (\bar{f}'^{2...1} \circ (g_1^1, g_2, ..., g_n))^{2...n} \\
&\xrightarrow{\gamma_\sigma} (\bar{f}'^{3...2} \circ (g_1^t, g_2, ..., g_n))^{2...n} \xrightarrow{\hat{t}} (\bar{f}'^{3...2} \circ (g_1^1, g_2^1, g_3, ..., g_n))^{3...n} \\
&\vdots \\
&\xrightarrow{\gamma_\sigma} (\bar{f}'^{1...n} \circ (g_1^1, ..., g_{n-1}^1, g_t))^n \xrightarrow{\hat{t}} \bar{f}'^{[n]} \circ \bar{g}^1 \\
&= Tf' \circ \bar{g}^*.
\end{aligned}$$

For example, the full composite in the case where f is a binary map is given by the diagram below:

$$\begin{tikzcd}
h^1 \circ \bar{f}^{12} \arrow[rr, dashed, "\alpha^*"] \arrow[dd, "\hat{t}^{-1}"'] & & f'^{12} \circ (g_1^*, g_2^*) \\
& & (f'^{12} \circ (g_1^*, g_2))^2 \arrow[u, "\hat{t}"'] \\
(h^1 \circ \bar{f}^1)^2 \arrow[dd, "\hat{t}^{-1}"'] & & (f'^{21} \circ (g_1^*, g_2))^2 \arrow[u, "\gamma"'] \\
& & (f'^{21} \circ (g_1, g_2))^{12} \arrow[u, "\hat{t}"'] \\
(h^1 \circ \bar{f})^{12} \arrow[r, "\tilde{t}^{-1}"'] & (h \circ Jf)^{12} \arrow[r, "\alpha"'] & (f'^{12} \circ (g_1, g_2))^{12} \arrow[u, "\gamma^{-1}"']
\end{tikzcd}$$

It now remains to verify that the three coherence conditions for a multicategorical relative pseudomonad. Here we shall only do this for binary maps, and we shall abbreviate the diagram chasing.

For the first condition, rewriting ι_f and α^* in terms of our constructions, we fill in the resulting diagram, aside from naturality squares, with four instances of Equation 5.3.

For the second coherence condition, unwrapping our definitions, we fill in the resulting diagram with naturality squares and:

- five instances of Equation 5.1, and
- axioms (3) and (4) from Definition 5.11.

For the third and final coherence condition, we rewriting everything in our terms and fill the resulting diagram with naturality squares and:

- instances of Equations 5.4 and 5.5,
- two uses of Equation 5.2, and
- axioms (1) and (2) from Definition 5.11.

Hence all three coherence conditions are satisfied, and thus we have shown that every pseudocommutative relative pseudomonad is a relative pseudomultimonad.

□

As the above proof demonstrates, working directly with the definitions of pseudocommutative relative pseudomonad and relative pseudomultimonad can be tedious. In the next section we will examine a condition on a relative pseudomonad which both implies pseudocommutativity and which is much easier to verify, being characterised by a universal property.

5.3 Lax-idempotent strong relative pseudomonads

In Chapter 4 extended lax idempotency for relative pseudomonads to general (not necessarily free) pseudoalgebras. To extend lax idempotency instead to strong relative pseudomonads on 2-multicategories, we generalise Definition 2.11 from Chapter 2 to the two-dimensional setting.

Definition 5.17. (Lax-idempotent strong relative pseudomonad) Let $J : \mathbb{C} \to \mathbb{D}$ be a pseudomultifunctor and let T be a strong relative pseudomonad along J. We say T is a *lax-idempotent strong relative pseudomonad* if the strengths are left adjoint to precomposition with the unit. That is, we have adjunctions

$$\begin{tikzcd} \mathbb{D}(\bar{B} \cdot JX_j; TY) \arrow[r, bend left, "(-)^j"] \arrow[r, phantom, "\perp"] & \mathbb{D}(\bar{B} \cdot JX_j; TY) \arrow[l, bend left, "- \circ_j i_X"] \end{tikzcd}$$

for every $1 \leq j \leq n$ and objects $\bar{B} \cdot JX_j; TY$ whose unit $- \implies (-)^j \circ_j i$ has components

$$\tilde{t}_f : f \to f^j \circ_j i_X$$

obtained from the strong structure (note that the unit is invertible).

We can equivalently state this condition in terms of left Kan extensions: T is lax-idempotent strong if for every map $f : \bar{B} \cdot JX_j \to TY$ the diagram

$$\begin{tikzcd} \bar{B} \cdot JX_j \arrow[r, "\bar{1} \cdot i"] \arrow[dr, "f"'] & \bar{B} \cdot TX_j \arrow[d, "f^t"] \arrow[dl, phantom, "\overset{\tilde{t}_f}{\Longrightarrow}"] \\ & TY \end{tikzcd}$$

exhibits f^t as the left Kan extension of f along $\bar{1} \cdot i$.

As a point of notation, we will use Greek letters to denote the counit of the lax idempotency adjunction; where the strengthening map is called $(-)^t$ and the unit $\tilde{t}$, the counit will be called

$$\tau_f : (f \circ_t i)^t \to f,$$

and where the strengthening is called $(-)^s$ and the unit $\tilde{s}$, the counit will be called

$$\sigma_f : (f \circ_s i)^s \to f$$

(and similarly for $(-)^u$ etc.).

Note that there is much less data to check in the course of showing that a relative pseudomonad is lax-idempotent compared with showing that it is pseudo-commutative. The following result generalises [LF11] Theorem 7.3, but the proof strategy is very different (not relying on the existence of a closed structure). The result gives us a shortcut for showing relative pseudomonads are pseudocommutative (and hence by Theorem 5.16 a relative pseudomultimonad).

Theorem 5.18. *Let $T : \mathbb{C} \to \mathbb{D}$ be a lax-idempotent strong relative pseudomonad. Then T is pseudocommutative, with a pseudocommutativity whose components $\gamma_g : f^{ts} \to f^{st}$ are given by the composite*

$$f^{ts} \xrightarrow{(\tilde{s}_f)^{ts}} (f^s \circ_s i)^{ts} \xrightarrow{\sim} (f^{st} \circ_s i)^s \xrightarrow{\sigma_{f^{st}}} g^{st}.$$

Proof. To begin, we first show that that putative γ_g is invertible. We will show that the composite

$$g^{st} \xrightarrow{(\tilde{t}_g)^{st}} (g^t \circ_t i)^{st} \xrightarrow{\sim} (g^{ts} \circ_t i)^t \xrightarrow{\tau_{g^{ts}}} g^{ts}$$

is its inverse. We have the commuting diagram

$$\begin{tikzcd}
g^{ts} \arrow[r, "\tilde{s}"] \arrow[d, "\tilde{t}"'] & (g^s \circ_s i)^{ts} \arrow[r, "\sim"] \arrow[d, "\tilde{t}"] & (g^{st} \circ_s i)^s \arrow[r, "\sigma"] \arrow[d, "\tilde{t}"] & g^{st} \arrow[d, "\tilde{t}"] \\
(g^t \circ_t i)^{ts} \arrow[r, "\tilde{s}"] \arrow[dd, "\tilde{s}"'] & ((g^t \circ_t i)^s \circ_s i)^{ts} \arrow[r, "\sim"] \arrow[d, "\sim"'] & ((g^t \circ_t i)^{st} \circ_s i)^s \arrow[r, "\sigma"] \arrow[d, "\sim"] & (g^t \circ_t i)^{st} \arrow[d, "\sim"] \\
& ((g^{ts} \circ_t i) \circ_s i)^{ts} \arrow[r, "\sim"'] \arrow[dl, equal] & ((g^{ts} \circ_t i)^t \circ_s i)^s \arrow[r, "\sigma"] \arrow[d, "\tau"] & (g^{ts} \circ_t i)^t \arrow[d, "\tau"] \\
((g^{ts} \circ_s i) \circ_t i)^{ts} \arrow[rr, "\tau"'] & & (g^{ts} \circ_s i)^s \arrow[r, "\sigma"'] & g^{ts}
\end{tikzcd}$$

whose clockwise composite is the composite $(\gamma_g)^{-1} \circ \gamma_g$, entirely composed of naturality squares. Then by the following diagram

$$\begin{tikzcd}
g^{ts} \arrow[d, "\tilde t"'] \arrow[dr, equal] & & \\
(g^t \circ_t i)^{ts} \arrow[r, "\tau"'] \arrow[d, "\tilde s"'] & g^{ts} \arrow[d, "\tilde s"'] \arrow[dr, equal] & \\
((g^{ts} \circ_s i) \circ_t i)^{ts} \arrow[r, "\tau"'] & (g^{ts} \circ_s i)^s \arrow[r, "\sigma"'] & g^{ts}
\end{tikzcd}$$

composed of a naturality square and two triangle identities, the anticlockwise composite of the first diagram is equal to the identity on g^{ts}, as required. The same argument (swapping the roles of s and t) demonstrates that the other composite $\gamma_g \circ (\gamma_g)^{-1}$ is also the identity, and so our γ_g is indeed invertible.

We now must show that our γ_g satisfies the coherence conditions for a pseudocommutativity. For the unit condition

$$\begin{tikzcd}
g^s \arrow[r, "(\tilde t_g)^s"] \arrow[drr, "\tilde t_{g^s}"'] & (g^t \circ_t i)^s \arrow[r, "\sim"] & f^{ts} \circ_t i \arrow[d, "\gamma_g \circ_t i"] \\
& & f^{st} \circ_t i
\end{tikzcd}$$

we write out $\gamma_g \circ_t i$ in terms of our composite and construct the commuting diagram

$$\begin{tikzcd}
g^s \arrow[r, "\tilde t"] \arrow[d, "\tilde s"'] & (g^t \circ_t i)^s \arrow[r, "\sim"] \arrow[d, "\tilde s"] & g^{ts} \circ_t i \arrow[d, "\tilde s"] \\
(g^s \circ_s i)^s \arrow[r, "\tilde t"] \arrow[dd, "\tilde t"'] & ((g^s \circ_s i)^t \circ_t i)^s \arrow[r, "\sim"] \arrow[d, "\sim"] & (g^s \circ_s i)^{ts} \circ_t i \arrow[d, "\sim"] \\
& ((g^{st} \circ_s i) \circ_t i)^s \arrow[r, "\sim"] \arrow[dl, equal] & (g^{st} \circ_s i)^s \circ_t i \arrow[d, "\sigma"] \\
((g^{st} \circ_t i) \circ_s i)^s \arrow[rr, "\sigma"'] & & g^{st} \circ_t i
\end{tikzcd}$$

comprising five naturality squares. Then the anticlockwise composite is, by the following commuting diagram

$$\begin{tikzcd}
g^s \arrow[d, "\tilde{s}"'] \arrow[dr, equal] & \\
(g^s \circ_s i)^s \arrow[r, "\sigma"'] \arrow[d, "\tilde{t}"'] & g^s \arrow[d, "\tilde{t}"] \\
((g^{st} \circ_t i) \circ_s i)^s \arrow[r, "\sigma"'] & g^{st} \circ_t i
\end{tikzcd}$$

of a naturality square and a triangle identity, equal to $\tilde{t}_{g^s}$, as required. The other unit condition is shown by the same argument, swapping the roles of s and t.

Now, for the strengthening condition

$$\begin{tikzcd}
(f^t \circ_t g)^{ts} \arrow[r, "(\hat{t}_{f,g})^s"] \arrow[d, "\gamma_{f^t \circ_t g}"'] & (f^t \circ_t g^t)^s \arrow[r, "\sim"] & f^{ts} \circ_t g^t \arrow[dd, "\gamma_f \circ_t g^t"] \\
(f^t \circ_t g)^{st} \arrow[d, "\sim"'] & & \\
(f^{ts} \circ_t g)^t \arrow[r, "(\gamma_f \circ_t g)^t"'] & (f^{st} \circ_t g)^t \arrow[r, "\hat{t}_{f^s,g}"'] & f^{st} \circ_t g^t
\end{tikzcd}$$

we can write out the anticlockwise composite in terms of our γ and construct a large commuting diagram filled in entirely with naturality squares and one triangle identity. The other strengthening condition is shown by the same argument, swapping the roles of s and t.

Finally, for the braiding coherence condition

$$\begin{tikzcd}
f^{uts} \arrow[r, "(\gamma_f)^s"] \arrow[d, "\gamma_{f^u}"'] & f^{tus} \arrow[r, "\gamma_{f^t}"] & f^{tsu} \arrow[d, "(\gamma_f)^u"] \\
f^{ust} \arrow[r, "(\gamma_f)^t"'] & f^{sut} \arrow[r, "\gamma_{f^s}"'] & f^{stu}
\end{tikzcd}$$

after writing each composite in terms of our γ we obtain a large diagram that may be filled in entirely with naturality squares. So all five coherence conditions are satisfied and hence indeed our γ is a pseudocommutativity for T. □

In summary, the previous sections have proved the following implications for T a relative pseudomonad along $J : \mathbb{C} \to \mathbb{D}$ between 2-multicategories:

- Every strong relative pseudomonad T is a pseudo-multifunctor (Proposition 5.9).
- Every pseudocommutative relative pseudomonad T is a multicategorical relative pseudomonad (Theorem 5.16).
- Every lax-idempotent strong relative pseudomonad T is pseudocommutative (Theorem 5.18).

5.4 The presheaf relative pseudomonad

We apply our results to the presheaf construction. As shown in [FGHW18] Example 3.9, the presheaf construction $P : \mathrm{Cat} \to \mathrm{CAT} : X \mapsto PX := [X^{op}, \mathrm{Set}]$ can be given the structure of a relative pseudomonad, where the units are given by the Yoneda embedding $y_X : X \to PX$ and the extension of a functor $f : X \to PY$ for small categories X, Y is given by the left Kan extension

$$\begin{tikzcd} X \arrow[r, "y"] \arrow[dr, "f"'] & PX \arrow[d, "f^* := \mathrm{Lan}_y f"] \arrow[dl, phantom, "\overset{\eta_f}{\Longrightarrow}"] \\ & PY \end{tikzcd}$$

along the Yoneda embedding, and this diagram also defines the map $\eta_f : f \to f^* y$.

In order to make use of the our results, we need to further show that the presheaf relative pseudomonad is strong.

Proposition 5.19. *The presheaf relative pseudomonad P along $J : \mathrm{Cat} \to \mathrm{CAT}$ is strong, with the strengthening of a functor*

$$f : \bar{B} \cdot X_j \to PY$$

defined as the left Kan extension

$$\begin{tikzcd} \bar{B} \cdot X_j \arrow[r, "\bar{1} \cdot y"] \arrow[dr, "f"'] & \bar{B} \cdot PX_j \arrow[d, "f^t := \mathrm{Lan}_{\bar{1} \cdot y} f"] \arrow[dl, phantom, "\overset{\tilde{t}_f}{\Longrightarrow}"] \\ & PY \end{tikzcd}$$

along $\bar{1} \cdot y$, and the 2-cell in the above diagram defines the map $\tilde{t}_f$.

Proof. We begin by constructing the rest of the data for a strong relative pseudomonad; namely, the invertible families of 2-cells

$$\hat{t}_{f,g} : (f^t \circ_t g)^t \to f^t \circ g^t,\ \theta : y^t \to 1.$$

Using the universal property of the left Kan extension, we define $\hat{t}_{f,g}$ and θ to be the unique 2-cells such that

$$\begin{tikzcd} f^t \circ_t g \arrow[r, "\tilde{t}_{f^t \circ_t g}"] \arrow[dr, "f^t \circ_t \tilde{t}_g"'] & (f^t \circ_t g)^t \circ_t y \arrow[d, "\hat{t}_{f,g} \circ_t y"] \\ & (f^t \circ_t g)^t \circ_t y \end{tikzcd} \qquad \begin{tikzcd} y \arrow[r, "\tilde{t}_y"] \arrow[dr, equal] & y^t \circ y \arrow[d, "\theta \circ y"] \\ & y \end{tikzcd}$$

commute, respectively. It remains to check the two coherence conditions of Definition 5.3. For the first:

$$\begin{tikzcd} ((f^t \circ_t g)^t \circ_t h)^t \arrow[rr, "\hat{t}_{f^t \circ_t g,h}"] \arrow[d, "(\hat{t}_{f,g} \circ_t h)^t"'] & & (f^t \circ_t g)^t \circ_t h^t \arrow[d, "\hat{t}_{f,g} \circ_t h^t"] \\ (f^t \circ_t g^t \circ_t h)^t \arrow[r, "\hat{t}_{f,g^t \circ_t h}"'] & f^t \circ_t (g^t \circ_t h)^t \arrow[r, "f^t \circ_t \hat{t}_{g,h}"'] & (f^t \circ_t g^t) \circ_t h^t \end{tikzcd}$$

by the universal property of the left Kan extension it suffices to show that the diagram

$$\begin{tikzcd} ((f^t \circ_t g)^t \circ_t h)^t \circ_t y \arrow[rr, "\hat{t}_{f^t \circ_t g,h} \circ_t y"] \arrow[d, "(\hat{t}_{f,g} \circ_t h)^t \circ_t y"'] & & (f^t \circ_t g)^t \circ_t h^t \circ_t y \arrow[d, "\hat{t}_{f,g} \circ_t h^t \circ_t y"] \\ (f^t \circ_t g^t \circ_t h)^t \circ_t y \arrow[r, "\hat{t}_{f,g^t \circ_t h} \circ_t y"'] & f^t \circ_t (g^t \circ_t h)^t \circ_t y \arrow[r, "f^t \circ_t \hat{t}_{g,h} \circ_t y"'] & (f^t \circ_t g^t) \circ_t h^t \circ_t y \end{tikzcd}$$

commutes. Rewriting terms we obtain the diagram

$$\begin{tikzcd} ((f^t \circ_t g)^t \circ_t h)^t \circ_t y \arrow[r, "\tilde{t}^{-1}"] \arrow[d, "\hat{t}"'] & (f^t \circ_t g)^t \circ_t h \arrow[r, "\tilde{t}"] & (f^t \circ_t g)^t \circ_t h^t \circ_t y \arrow[d, "\hat{t}"] \\ (f^t \circ_t g^t \circ_t h)^t \circ_t y \arrow[d, "\tilde{t}^{-1}"'] & & (f^t \circ_t g^t) \circ_t h^t \circ_t y \\ f^t \circ_t g^t \circ_t h \arrow[r, "\tilde{t}"'] & f^t \circ_t (g^t \circ_t h)^t \circ_t y \arrow[r, "\tilde{t}^{-1}"'] & f^t \circ_t g^t \circ_t h \arrow[u, "\tilde{t}"'] \end{tikzcd}$$

which we can fill in

$$\begin{tikzcd}
((f^t \circ_t g)^t \circ_t h)^t \circ_t y \arrow[r, "\tilde{t}^{-1}"] \arrow[d, "\hat{t}"'] & (f^t \circ_t g)^t \circ_t h \arrow[r, "\tilde{t}"] \arrow[d, "\hat{t}"'] & (f^t \circ_t g)^t \circ_t h^t \circ_t y \arrow[d, "\hat{t}"] \\
(f^t \circ_t g^t \circ_t h)^t \circ_t y \arrow[r, "\tilde{t}^{-1}"'] & f^t \circ_t g^t \circ_t h \arrow[r, "\tilde{t}"'] \arrow[d, bend right, "\tilde{t}"'] & (f^t \circ_t g^t) \circ_t h^t \circ_t y \\
& f^t \circ_t (g^t \circ_t h)^t \circ_t y \arrow[u, bend right, "\tilde{t}^{-1}"'] &
\end{tikzcd}$$

with two naturality squares. For the second:

$$\begin{tikzcd}
f^t \arrow[r, "(\tilde{t}_f)^t"] \arrow[drr, equal] & (f^t \circ_t y)^t \arrow[r, "\hat{t}_{f,y}"] & f^t \circ_t y^t \arrow[d, "f^t \circ_t \theta"] \\
& & f^t
\end{tikzcd}$$

again by the universal property of the left Kan extension we can equivalently show the diagram

$$\begin{tikzcd}
f^t \circ_t y \arrow[r, "(\tilde{t}_f)^t \circ_t y"] \arrow[drr, equal] & (f^t \circ_t y)^t \circ_t y \arrow[r, "\hat{t}_{f,y} \circ_t y"] & f^t \circ_t y^t \circ_t y \arrow[d, "f^t \circ_t \theta \circ_t y"] \\
& & f^t \circ_t y
\end{tikzcd}$$

commutes. Rewriting terms we obtain

$$\begin{tikzcd}
f^t \circ_t y \arrow[r, "\tilde{t}"] \arrow[drrr, equal] & (f^t \circ_t y)^t \circ_t y \arrow[r, "\tilde{t}^{-1}"] & f^t \circ_t y \arrow[r, "\tilde{t}"] & f^t \circ_t y^t \circ_t y \arrow[d, "\tilde{t}^{-1}"] \\
& & & f^t \circ_t y
\end{tikzcd}$$

which immediately commutes. Hence indeed the structure as defined makes P a strong relative pseudomonad. □

Having shown this, we can now apply the results of this chapter to the presheaf relative pseudomonad.

Theorem 5.20. *The presheaf relative pseudomonad is:*

(1) a lax-idempotent strong relative pseudomonad,

(2) a pseudocommutative relative pseudomonad, and

(3) a relative pseudomultimonad.

Proof. By Theorem 5.18 we know (1) $\implies$ (2), and by Theorem 5.16 we know (2) $\implies$ (3). So it suffices to check that P is a lax-idempotent strong relative pseudomonad. By Proposition 5.19 P is strong, and we have diagrams

$$\begin{tikzcd} \bar{B} \cdot X_j \arrow[r, "\bar{1} \cdot y"] \arrow[dr, "f"'] & \bar{B} \cdot PX_j \arrow[d, "f^t := \mathrm{Lan}_{\bar{1} \cdot y} f"] \arrow[dl, phantom, "\overset{\tilde{t}_f}{\Longrightarrow}"] \\ & PY \end{tikzcd}$$

exhibiting f^t as the left Kan extension of f along $\bar{1} \cdot y$. But this means precisely that we have an adjunction

$$(-)^t \dashv - \circ_t y$$

whose unit is $\tilde{t}$, as required. So indeed P is a lax-idempotent strong relative pseudomonad, and hence is also pseudocommutative and a multicategorical relative pseudomonad. $\square$

Chapter 6

Commutativity and the Kleisli construction

Introduction

In [FGHW18] Section 4 the Kleisli bicategory $\mathrm{Kl}(T)$ for a relative pseudomonad T is constructed, and it is shown that for the example of the presheaf relative pseudomonad P, its Kleisli bicategory $\mathrm{Kl}(P)$ is biequivalent to the bicategory Prof of profunctors.

Corollary 7 in the article [Gui80] proves that the Kleisli category for a commutative monad has a monoidal structure (one uses the map $TX \otimes TY \to T(X \otimes Y)$ to define the tensor product of morphisms); in this section we aim to prove an analogous result for relative pseudomonads: that the Kleisli bicategory for a pseudocommutative relative pseudomonad has the structure of a bimulticategory.

We apply our result to the case of the presheaf relative pseudomonad, constructing a bimulticategory structure on $\mathrm{Kl}(P)$ which corresponds via the biequivalence $\mathrm{Kl}(P) \cong \mathrm{Prof}$ to that induced by the pseudomonoidal structure on Prof where the product of two profunctors $f : Y^{op} \times X \to \mathrm{Set}$ and $g : B^{op} \times A \to \mathrm{Set}$ is given by the composite

$$(Y \times B)^{op} \times (X \times A) \xrightarrow{\cong} (Y^{op} \times X) \times (B^{op} \times A) \xrightarrow{f \times g} \mathrm{Set} \times \mathrm{Set} \xrightarrow{\times} \mathrm{Set}\,.$$

The question of whether the Eilenberg-Moore category of all pseudoalgebras can be given a bimulticategory structure too is interesting. In the ordinary pseudomonad setting, Theorem 14 of [HP02] shows that for a finitary pseudocommutative pseudomonad on Cat, the Eilenberg-Moore 2-category has a pseudomonoidal structure, induced by a pseudo-closed structure. I am yet to study closed structures in the relative setting and have not thus far attempted to prove an analogous result of Hyland and Power.

6.1 The Kleisli multicategory for a commutative relative monad

We first prove the one-dimensional result that the Kleisli category for a commutative relative monad can be given the structure of a multicategory. Recall from [ACU15] Section 2.3 the definition of the Kleisli category for a relative monad:

Definition 6.1. Let T be a relative monad along $J : \mathbb{C} \to \mathbb{D}$. The *Kleisli category* $\mathrm{Kl}(T)$ over T has the same objects as $\mathbb{C}$, but arrows in $\mathrm{Kl}(T)$ from X to Y are given by arrows $JX \to TY$ in $\mathbb{D}$. The composite of $f : JY \to TZ$ and $g : JX \to TY$ is defined to be f^*g, and the identity morphism on X is defined to be $i_X : JX \to TX$.

That this composition is associative and unital follows from the relative monad laws:

$$(f^*g)^*h = f^*(g^*h),$$
$$f^*i = f,$$
$$i^*f = f.$$

Now suppose $\mathbb{C}$ and $\mathbb{D}$ are multicategories and T is a commutative relative monad. Recall the notation f^S for the strengthening of an n-ary map f in the indices of a subset $S \subseteq [n]$ (from lowest to highest), and recall Lemma 2.10

for commutative relative monads, which states for $f : JX_1, ..., JX_n \to TY$ and $g_j : JZ_{j1}, ..., JZ_{jm_j} \to TX_j$ that

$$(f^S \circ_S g_j)^{\bigcup(S_j+k_j)} = f^S \circ_S g_j^{S_j},$$

where for all j, the index shift k_j is equal to the sum of $(m_k - 1)$ for $k < j$.

The following special case will be key to constructing the desired multicategory structure.

Corollary 6.2. *Let T be a commutative relative monad, let $f : J\bar{X} \to TY$ be an n-ary map and let $g_j : J\bar{Z}_j \to TX_j$ be m_j-ary maps for $1 \leq j \leq n$. Then we have*

$$(f^{[n]} \circ_{[n]} g_j)^{[\sum m_j]} = f^{[n]} \circ_{[n]} g_j^{[m_j]}.$$

Proof. This result is a special case of Lemma 2.10 when $S = [n]$ and $S_j = m_j$ for each j. □

Theorem 6.3. *Let T be a commutative relative monad along a multifunctor $J : \mathbb{C} \to \mathbb{D}$ between multicategories. Then the Kleisli category can be given the structure of a multicategory, where morphisms from $\bar{X}$ to Y are defined to be arrows $J\bar{X} \to TY$ in $\mathbb{D}$.*

Proof. We define the composite of an n-ary map $f : J\bar{X} \to TY$ and m_j-ary maps $g_j : J\bar{Z}_j \to TX_j$ for $1 \leq j \leq n$ to be equal to

$$f^{[n]} \circ_{[n]} g_j,$$

and we take the same identities as in the unary category $\mathrm{Kl}(T)$; that is, $i_X : JX \to TX$. Composition is unital:

$$\begin{aligned} i^* \circ f &= f, \\ f^{[n]} \circ_{[n]} i &= f^{[n-1]} \circ_{[n-1]} i \\ &= \dots \\ &= f. \end{aligned}$$

Note that this does not yet require T to be commutative. However, to show that composition is associative, we need to use Corollary 6.2. Let $f : J\bar{X} \to TY$ be an n-ary map, $g_j : J\bar{W}_j \to TX_j$ be an m_j-ary map for $1 \le j \le n$, and $h_{jk} : J\bar{V}_{jk} \to TW_{jk}$ be maps for $1 \le j \le n$, $1 \le k \le m_j$. Then Corollary 6.2 gives us the equality

$$(f^{[n]} \circ_{[n]} g_j)^{[\sum m_j]} \circ_{[\sum m_j]} h_{jk} = f^{[n]} \circ_{[n]} (g_j^{[m_j]} \circ_{[m_j]} h_{jk}),$$

and so composition is indeed associative. So $\mathrm{Kl}(T)$ can be given the required multicategory structure. □

Remark 6.4. For an n-ary map $f : J\bar{X} \to TY$, write f^T to denote $f^{[n]}$ (this allows us to unify notation for maps of different arities). Then these two equations, which together suffice to prove Theorem 6.3:

$$f = f^T \circ_T i,$$
$$(f^T \circ_T g_j)^T = f^T \circ_T g_j^T,$$

are in a sense (which will be exploited in the next section) multiary analogues of the following two conditions on a relative monad:

$$f = f^* \circ i,$$
$$(f^* \circ g)^* = f^* \circ g^*$$

6.2 The Kleisli bimulticategory for a pseudocommutative relative pseudomonad

To generalise Theorem 6.3 to the two-dimensional setting, we need to define the structure we want to construct on the Kleisli bicategory for a relative pseudomonad. We proceed by weakening the definition of a 2-multicategory (Definition 5.1) by replacing the associativity and identity axioms with invertible 2-cells.

Definition 6.5. A *bimulticategory* $\mathbb{C}$ comprises

- A collection of objects $X \in \operatorname{ob}\mathbb{C}$,
- for all objects Y and lists of objects $\bar{X}$, a category of multimorphisms $\mathbb{C}(\bar{X}; Y)$ called a *hom-category*; its objects are denoted by $f : \bar{X} \to Y$,
- for each $X \in \operatorname{ob}\mathbb{C}$ a functor $* \to \mathbb{C}(X; X)$ picking out 1_X, the *identity multimorphism*, and
- for each object Y, list of n objects $\bar{X}$ and n lists of objects $\bar{W}_j$ for $1 \le j \le n$, a composition functor

$$\begin{aligned} \mathbb{C}(\bar{X}; Y) \times \mathbb{C}(\bar{W}_1; X_1) \times \ldots \times \mathbb{C}(\bar{W}_n; X_n) &\to \mathbb{C}(\bar{\bar{W}}; Y) \\ (f, \bar{g}) &\mapsto f \circ \bar{g} \end{aligned}$$

along with the following families of invertible 2-cells:

- for each $f : \bar{X} \to Y$, *unitor* 2-cells

$$\begin{aligned} \lambda_f &: 1 \circ f \to f, \\ \rho_f &: f \to f \circ \bar{1}, \end{aligned}$$

- for each n-ary map $f : \bar{X} \to Y$, m_j-ary maps $g_j : \bar{W}_j \to X_j$ for $1 \le j \le n$ and maps $h_{jk} : \bar{V}_{jk} \to W_{jk}$ for $1 \le j \le n$, $1 \le k \le m_j$ an *associator* 2-cell

$$\alpha_{f,g,h} : (f \circ \bar{g}) \circ \bar{\bar{h}} \to f \circ \overline{(g_j \circ \bar{h}_j)}.$$

We finally require that the associators satisfy the pentagon identity, and that the unitors satisfy the triangle identity.

As examples illustrating the brevity of this 'vector' notation, in the case where all maps are binary,

$$\begin{tikzcd}
(((f(g_1,g_2))(h_1,h_2,h_3,h_4))(k_1,k_2,k_3,k_4,k_5,k_6,k_7,k_8) \arrow[r, "\alpha_{fg,h,k}"] \arrow[d, "\alpha_{f,g,h}k"'] & (f(g_1,g_2))(h_1(k_1,k_2),h_2(k_3,k_4),h_3(k_5,k_6),h_4(k_7,k_8)) \arrow[dd, "\alpha_{f,g,hk}"] \\
(f(g_1(h_1,h_2),g_2(h_3,h_4)))(k_1,k_2,k_3,k_4,k_5,k_6,k_7,k_8) \arrow[d, "\alpha_{f,gh,k}"'] & \\
f((g_1(h_1,h_2))(k_1,k_2,k_3,k_4),(g_2(h_3,h_4))(k_5,k_6,k_7,k_8)) \arrow[r, "f\alpha_{g,h,k}"'] & f(g_1(h_1(k_1,k_2),h_2(k_3,k_4)),g_2(h_3(k_5,k_6),h_4(k_7,k_8)))
\end{tikzcd}$$

is the pentagon axiom, and

$$\begin{tikzcd}
& f(g_1,g_2) \arrow[dl, "\rho_f"'] & \\
(f(1,1))(g_1,g_2) \arrow[rr, "\alpha_{f,1,g}"'] & & f(1g_1,1g_2) \arrow[ul, "\lambda_{\bar{g}}"']
\end{tikzcd}$$

is the triangle axiom.

If the unitor and associator 2-cells are identities, we recover 2-multicategories as defined in Definition 5.1.

Remark 6.6. Every bimulticategory is canonically a bicategory whose hom-categories are the unary hom-categories and whose composition is the restriction of the bimulticategory's composition.

We recall the construction of the Kleisli bicategory for a relative pseudomonad from [FGHW18] Section 4. Let T be a relative pseudomonad along $J : \mathbb{C} \to \mathbb{D}$. Then the Kleisli bicategory $\mathrm{Kl}(T)$ comprises the objects of $\mathbb{C}$, the hom-category $\mathrm{Kl}(T)(X,Y)$ is defined to be $\mathbb{D}(JX,TY)$, and the composition of $f : JX \to TY$ and $g : JW \to TX$ is defined by $f^*g : JW \to TX \to TY$. The unitor and associator 2-cells are defined in terms of the relative pseudomonad 2-cells.

Our proof strategy for showing the two-dimensional analogue of Theorem 6.3 will follow Remark 6.4. We aim to construct multiary analogues to the η and μ families of 2-cells for a relative pseudomonad, satisfying two corresponding coherence equations.

First, we show that such families of 2-cells suffice to give the Kleisli bicategory the structure of a bimulticategory.

Lemma 6.7. *Let T be a strong relative pseudomonad. Suppose we have the following two families of 2-cells:*

- *for* $f : J\bar{X} \to TY$*, an invertible 2-cell* $\tilde{T}_f : f \to f^T \circ_T i$*, and*
- *for* n*-ary* $f : J\bar{X} \to TY$ *and* $g_j : J\bar{W} \to TX_j$ *for* $1 \le j \le n$*, an invertible 2-cell* $\hat{T}_{f,g} : (f^T \circ_T g_j)^T \to f^T \circ_T g_j^T$*,*

and suppose further that these satisfy the following two coherence equations.

- *for an* n*-ary map* $f : J\bar{X} \to TY$*,* m_j*-ary maps* $g_j : J\bar{W}_j \to TX_j$ *for* $1 \le j \le n$ *and maps* $h_{jk} : J\bar{V}_{jk} \to TW_{jk}$ *for* $1 \le j \le n$*,* $1 \le k \le m_j$*, the diagram*

$$\begin{tikzcd}[column sep=small] ((f^T \circ_T g_j)^T \circ_T h_{jk})^T \arrow[rrrr, "\hat{T}_{fg,h}"] \arrow[d, "(\hat{T}_{f,g}h)^T"'] & & & & (f^T \circ_T g_j)^T \circ_T h_{jk}^T \arrow[d, "\hat{T}_{f,g}h^T"] \\ (f^T \circ_T g_j^T \circ_T h_{jk})^T \arrow[rr, "\hat{T}_{f,gh}"'] & & f^T \circ_T (g_j^T \circ_T h_{jk})^T \arrow[rr, "f^T\hat{T}_{g,h}"'] & & f^T \circ_T g_j^T \circ_T h_{jk}^T \end{tikzcd} \tag{6.1}$$

commutes, and

- *for an* n*-ary map* $f : J\bar{X} \to TY$*, the diagram*

$$\begin{tikzcd} f^T \arrow[r, "(\tilde{T}_f)^T"] \arrow[drr, equal] & (f^T \circ_T i)^T \arrow[r, "\hat{T}_{f,i}"] & f^T \circ_T i^* \arrow[d, "f^T \circ_T \theta"] \\ & & f^T \end{tikzcd} \tag{6.2}$$

commutes.

Then the Kleisli bicategory for T *can be given the structure of a Kleisli multibicategory, whose hom-categories* $\mathrm{Kl}(T)(\bar{X};Y)$ *are defined to be the categories* $\mathbb{D}(J\bar{X};TY)$ *and the identities are given by* $i_X : JX \to TX$*.*

Proof. We define the composition of n-ary $f : J\bar{X} \to TY$ and maps $g_j : JW_j \to TX_j$ for $1 \le j \le n$ by

$$f^{[n]} \circ \bar{g}.$$

We define the following unitor 2-cells for $f : J\bar{X} \to TY$ as follows:

$$\lambda_f := i^* \circ f \xrightarrow{\theta \circ f} f,$$
$$\rho_f := f \xrightarrow{\tilde{T}_f} f^T \circ_T i.$$

We define the associator 2-cell for an n-ary map $f : J\bar{X} \to TY$, m_j-ary maps $g_j : J\bar{W}_j \to TX_j$ for $1 \le j \le n$ and maps $h_{jk} : J\bar{V}_{jk} \to TW_{jk}$ for $1 \le j \le n$, $1 \le k \le m_j$ to be

$$\alpha_{f,g,h} := (f^T \circ_T g_j)^T \circ_T h_{jk} \xrightarrow{\hat{T}_{f,g}h} f^T \circ_T (g_j^T \circ_T h_{jk}).$$

It remains only to show that these 2-cells as defined satisfy the pentagon and triangle identities. In the following diagrams, we omit the composition operator $\circ_T$, as every composition is of this form. Thus, for example, the notation $f^T g_j$ is shorthand for $f^T \circ (g_1, ..., g_n)$.

To show the pentagon identity (for an n-ary map $f : J\bar{X} \to TY$, m_j-ary maps $g_j : J\bar{W}_j \to TX_j$ for $1 \le j \le n$, r_{jk}-ary maps $h_{jk} : J\bar{V}_{jk} \to TW_{jk}$ for $1 \le j \le n$, $1 \le k \le m_j$ and maps $p_{jkl} : J\bar{U}_{jkl} \to TV_{jkl}$ for $1 \le j \le n$, $1 \le k \le m_j$ and $1 \le l \le r_{jk}$)

$$\begin{tikzcd}
((f^T g_j)^T h_{jk})^T p_{jkl} \arrow[rr, "\alpha_{fg,h,p}"] \arrow[d, "\alpha_{f,g,h}p"'] & & (f^T g_j)^T (h_{jk}^T p_{jkl}) \arrow[d, "\alpha_{f,g,hp}"] \\
(f^T (g_j^T h_{jk}))^T p_{jkl} \arrow[r, "\alpha_{f,gh,p}"'] & f^T ((g_j^T h_{jk})^T p_{jkl}) \arrow[r, "f^T \alpha_{g,h,p}"'] & f^T (g_j^T (h_{jk}^T p_{jkl}))
\end{tikzcd}$$

we express α in terms of $\hat{T}$ and obtain the diagram

$$\begin{tikzcd}
((f^T g_j)^T h_{jk})^T p_{jkl} \arrow[rr, "\hat{T}_{fg,h}p"] \arrow[d, "(\hat{T}_{f,g}h)^T p"'] & & (f^T g_j)^T h_{jk}^T p_{jkl} \arrow[d, "\hat{T}_{f,g}h^T p"] \\
(f^T g_j^T h_{jk})^T p_{jkl} \arrow[r, "\hat{T}_{f,gh}p"'] & f^T (g_j^T h_{jk})^T p_{jkl} \arrow[r, "f^T \hat{T}_{g,h}p"'] & f^T g_j^T h_{jk}^T p_{jkl}
\end{tikzcd}$$

which is simply the whiskering of Equation 6.1 with the p_{jkl} maps. Hence the pentagon axiom holds.

To show the triangle identity (for n-ary map $f : J\bar{X} \to TY$ and maps $g_j : J\bar{W}_j \to TX_j$ for $1 \leq j \leq n$)

$$\begin{tikzcd} (f^T i)^T g_j \arrow[r, "\alpha_{f,i,g}"] \arrow[drr, equal] & f^T(i^* g) \arrow[r, "f^T \lambda_g"] & f^T g_j \arrow[d, "(\rho_f)^T g_j"] \\ & & (f^T i)^T g_j \end{tikzcd}$$

we express α, λ and ρ in terms of $\tilde{T}$ and $\hat{T}$ and obtain the diagram

$$\begin{tikzcd} (f^T i)^T g_j \arrow[r, "\hat{T}_{f,ig}"] \arrow[drr, equal] & f^T i^* g \arrow[r, "f^T \theta_g"] & f^T g_j \arrow[d, "(\tilde{T}_f)^T g_j"] \\ & & (f^T i)^T g_j \end{tikzcd}$$

Since all the involved 2-cells are invertible, we can express this diagram equivalently as

$$\begin{tikzcd} f^T g_j \arrow[r, "(\tilde{T}_f)^T g_j"] \arrow[drr, equal] & (f^T i)^T g_j \arrow[r, "\hat{T}_{f,ig}"] & f^T i^* g \arrow[d, "f^T \theta_g"] \\ & & f^T g_j \end{tikzcd}$$

and this is precisely Equation 6.2 whiskered with the g_j maps. Hence the triangle axiom holds, and thus we have a bimulticategory structure on the Kleisli bicategory for T. $\square$

To apply Lemma 6.7, we need to show that we can actually construct the families of 2-cells with the required properties; we do so below.

For a map $f : J\bar{X} \to TY$ we define $\tilde{T}_f : f \to f^T \circ_T i$ to be the composite

$$\begin{aligned} f &\xrightarrow{\tilde{t}} f^1 \circ (i, 1, ..., 1) \\ &\xrightarrow{\tilde{t}} f^{12} \circ_{12} (i, i, 1, ..., 1) \\ &\xrightarrow{\tilde{t}} \dots \\ &\xrightarrow{\tilde{t}} f^{[n]} \circ (i, i, ..., i) = f^T \circ_T i. \end{aligned}$$

Note that this does not require T to be pseudocommutative, only strong. Constructing the $\hat{T}$ maps will however require pseudocommutativity.

Recall the notation $\gamma_{\sigma;f}$ for σ a permutation in S_n denoting a composite of γ and γ^{-1} maps from $f^{12\ldots n}$ to $f^{\sigma(1)\sigma(2)\ldots\sigma(n)}$ (any choice giving the same result by coherence condition (5) for pseudocommutativity). To construct the desired map $\hat{T}_{f,g} : (f^T \circ_T g_j)^T \to f^T \circ_T g_j^T$ for n-ary map $f : J\bar{X} \to TY$ and m_j-ary maps $g_j : J\bar{Z}_j \to TX_j$ for $1 \leq j \leq n$, we start from

$$(f^T \circ_T g_j)^T = (f^{[n]} \circ_{[n]} g_j)^{\sum m_j}.$$

Then for each $j \in [n]$ in turn we can

- apply $\gamma_{\sigma;f}$ so that j is rightmost, then
- apply the $\hat{t}$ axioms of a strength to bring m_j indices inside the parentheses.

Having done this for each $j \in [n]$, we obtain

$$f^{[n]} \circ_{[n]} g_j^{m_j} = f^T \circ_T g_j^T$$

as required.

Thus, when T is a pseudocommutative relative pseudomonad, we can indeed construct the maps $\tilde{T}$ and $\hat{T}$.

To give an explicit example of a $\hat{T}$ map, consider the case where f, g_1 and g_2 are all binary maps. The desired $\hat{T}_{f,g}$ is then a map from $(f^{12} \circ (g_1, g_2))^{1234}$ to $f^{12} \circ (g_1^{12}, g_2^{12})$, and is the following composite:

$$\begin{tikzcd}
(f^{12} \circ (g_1, g_2))^{1234} \arrow[rr, dashed, "\hat{T}_{f,g}"] \arrow[d, "\gamma^{-1}"'] & & f^{12} \circ (g_1^{12}, g_2^{12}) \\
(f^{21} \circ (g_1, g_2))^{1234} \arrow[d, "\hat{t}"'] & & (f^{12} \circ (g_1^{12}, g_2^{1}))^{4} \arrow[u, "\hat{t}"'] \\
(f^{21} \circ (g_1^{1}, g_2))^{234} \arrow[r, "\hat{t}"'] & (f^{21} \circ (g_1^{12}, g_2))^{34} \arrow[r, "\gamma"'] & (f^{12} \circ (g_1^{12}, g_2))^{34} \arrow[u, "\hat{t}"']
\end{tikzcd}$$

The previous lemmas constitute most of the proof of our main result.

Theorem 6.8. *Let T be a pseudocommutative relative pseudomonad along a 2-multifunctor $J : \mathbb{C} \to \mathbb{D}$ between 2-multicategories. Then the Kleisli bicategory can be given the structure of a bimulticategory, whose hom-categories* $\mathrm{Kl}(T)(\bar{X}; Y)$

are defined to be the categories $\mathbb{D}(J\bar{X};TY)$ *and the identities are given by* $i_X : JX \to TX$*.*

Proof. By Lemma 6.7, it suffices to show that $\tilde{T}$ and $\hat{T}$ satisfy Equations 6.1 and 6.2 (the two coherence conditions enumerated in Lemma 6.7).

To begin with, Equation 6.2 is

$$\begin{tikzcd} f^T \arrow[r, "(\tilde{T}_f)^T"] \arrow[drr, equal] & (f^T \circ_T i)^T \arrow[r, "\hat{T}_{f,i}"] & f^T \circ_T i^* \arrow[d, "f^T \circ_T \theta"] \\ & & f^T \end{tikzcd}$$

and we prove this explicitly in the case where f is a binary map. Writing out $(\tilde{T}_f)^T$ as

$$f^{12} \xrightarrow{\tilde{t}} (f^1 \circ_1 i)^{12} \xrightarrow{\tilde{t}} (f^{12} \circ (i,i))^{12},$$

and writing out $\hat{T}_{f,i}$ as

$$\begin{aligned}(f^{12} \circ (i,i))^{12} &\xrightarrow{\gamma^{-1}} (f^{21} \circ (i,i))^{12} \xrightarrow{\hat{t}} (f^{21} \circ (i^*,i))^2 \\ &\xrightarrow{\gamma} (f^{12} \circ (i^*,i))^2 \xrightarrow{\hat{t}} f^{12} \circ (i^*,i^*),\end{aligned}$$

the resulting diagram can be filled in with the following two diagrams:

- four naturality squares, coherence equation (2) from the definition of pseudocommutativity (Definition 5.11) and strong relative pseudomonad coherence condition (Equation 5.2):

$$\begin{tikzcd}
& (f^{12} \circ (i,i))^{12} \arrow[r, "\gamma^{-1}"] & (f^{21} \circ (i,i))^{12} \arrow[d, "\sim"] & \\
f^{12} \arrow[r, "\tilde{t}"] \arrow[ddr, equal] & (f^1 \circ_1 i)^{12} \arrow[u, "\tilde{t}"] \arrow[r, "\tilde{t}"] \arrow[d, "\hat{t}"] & ((f^2 \circ_2 i)^1 \circ_1 i)^{12} \arrow[d, "\hat{t}"] & \\
& (f^1 \circ_1 i^*)^2 \arrow[r, "\tilde{t}"] \arrow[d, "\theta"] & ((f^2 \circ_2 i)^1 \circ_1 i^*)^2 \arrow[d, "\theta"'] \arrow[ddr, "\sim"] & \\
& f^{12} \arrow[r, "\tilde{t}"] & (f^2 \circ_2 i)^{12} \arrow[d, "\sim"] & \\
& & (f^{21} \circ_2 i)^2 \arrow[d, "\gamma"] & (f^{21} \circ (i^*,i))^2 \arrow[l, "\theta"'] \arrow[d, "\gamma"] \\
& & (f^{12} \circ_2 i)^2 & (f^{12} \circ (i^*,i))^2 \arrow[l, "\theta"']
\end{tikzcd}$$

and

- one naturality square, two multibicategory coherences, coherence equation (1) from the definition of pseudocommutativity (Definition 5.11) and the same strong relative pseudomonad coherence condition (Equation 5.2):

$$\begin{tikzcd}
(f^2 \circ_2 i)^{12} \arrow[r, "\sim"] & (f^{21} \circ_2 i)^2 \arrow[d, "\gamma"] & (f^{12} \circ (i^*,i))^2 \arrow[dl, "\theta"'] \arrow[d, "\sim"] \\
f^{12} \arrow[u, "\tilde{t}"] \arrow[r, "\tilde{t}"'] \arrow[ddr, equal] & (f^{12} \circ_2 i)^2 \arrow[d, "\hat{t}"] & ((f^1 \circ_1 i^*)^2 \circ_2 i)^2 \arrow[l, "\theta"] \arrow[d, "\hat{t}"] \\
& f^{12} \circ_2 i^* \arrow[d, "\theta"] & (f^1 \circ_1 i^*)^2 \circ_2 i^* \arrow[l, "\theta"'] \arrow[d, "\sim"] \\
& f^{12} & f^{12} \circ (i^*, i^*) \arrow[ul, "\theta"]
\end{tikzcd}$$

In the general case, the proof for n-ary f requires (aside from naturality) n invocations of conditions (1)/(2) from Definition 5.11 and n invocations of Equation 5.2.

Second, Equation 6.1 is

$$\begin{tikzcd}
((f^T \circ_T g_j)^T \circ_T h_{jk})^T \arrow[rr, "\hat{T}_{fg,h}"] \arrow[d, "(\hat{T}_{f,g}h)^T"'] & & (f^T \circ_T g_j)^T \circ_T h_{jk}^T \arrow[d, "\hat{T}_{f,g}h^T"] \\
(f^T \circ_T g_j^T \circ_T h_{jk})^T \arrow[r, "\hat{T}_{f,gh}"'] & f^T \circ_T (g_j^T \circ_T h_{jk})^T \arrow[r, "f^T\hat{T}_{g,h}"'] & f^T \circ_T g_j^T \circ_T h_{jk}^T
\end{tikzcd}$$

Writing out each $\hat{T}$ as constructed for a pseudocommutative T, we obtain a large diagram which may be filled in entirely using instances of coherence conditions (3), (4) and (5) from the definition of a pseudocommutative relative pseudomonad (Definition 5.11).

Hence indeed we have constructed a bimulticategory structure on the Kleisli category $\mathrm{Kl}(T)$. □

We apply Theorem 6.8 to the case of the presheaf relative pseudomonad. From [FGHW18] Example 4.3, the Kleisli bicategory $\mathrm{Kl}(P)$ is biequivalent to Prof, the bicategory of profunctors.

Corollary 6.9. *The bicategory* Prof *of profunctors has the structure of a bimulticategory.*

Proof. The presheaf relative pseudomonad P is pseudocommutative by Theorem 5.20. Hence by Theorem 6.8, $\mathrm{Kl}(P)$—and thus Prof—has the structure of a bimulticategory. □

What is this bimulticategory structure? Unravelling the proof of Theorem 6.8, we find that the hom-category $\mathrm{Prof}(X_1, ..., X_n; Y)$ is given by $\mathrm{CAT}(\prod X, PY)$. The composite of an n-ary profunctor $f : \prod X \to PY$ with an m-ary profunctor $g : \prod W \to PX_j$ (where $1 \leq j \leq n$) is given by $f^j \circ_j g$, where

$$f^j : X_1 \times \cdots \times X_{j-1} \times PX_j \times X_{j+1} \times \cdots X_n \to PY$$

is the left Kan extension of f along $1_{X_1} \times \cdots \times 1_{X_{j-1}} \times y_{X_j} \times 1_{X_{j+1}} \times \cdots \times 1_{X_n}$.

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