On Discrete Lagrangian Multiform Structures and their Reductions

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I confirm that the work submitted is my own, except where work which has formed part of jointly authored publications has been included. My contribution and the other authors to this work has been explicitly indicated below. I confirm that appropriate credit has been given within the thesis where reference has been made to the work of others.

Chapter 3, Chapter 4 and Appendix A are based on the work in [44], *Discrete Lagrangian multiforms for quad equations, tetrahedron equations, and octahedron equations*, J. J. Richardson and M. Vermeeren, arXiv preprint arXiv:2403.16845 (2024). JJR and MV jointly developed the ideas and the presentation of each Section. JJR provided the initial idea. MV contributed more towards the writing of the Introduction, Section 2, the Conclusion and the Appendix. JJR contributed more towards the writing and the calculations of Section 5.

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Abstract

This thesis develops the theory of Lagrangian multiforms. For integrable equations which belong to compatible systems of equations, this theory encapsulates the entire system in a single variational principle.

We introduce definitions to distinguish between Lagrangian 2-forms and weak Lagrangian 2-forms in the discrete setting of quadrilateral stencils. We present three novel types of discrete Lagrangian 2-form for the integrable quad equations of the ABS list. Two of our new Lagrangian 2-forms have the quad equations, or a system equivalent to the quad equations, as their Euler-Lagrange equations, whereas the third produces the tetrahedron equations. This is in contrast to the well-established Lagrangian 2-form for these equations, which produces equations that are weaker than the quad equations (they are equivalent to two octahedron equations). We use relations between the Lagrangian 2-forms to prove that the system of quad equations is equivalent to the combined system of tetrahedron and octahedron equations.

We formulate the double zero property of Lagrangian multiforms in the discrete setting. For each of the discrete Lagrangian 2-forms associated with quad equations, we show that double zero expansions can be derived in terms of their respective corner equations.

We develop a framework to periodically reduce discrete Lagrangian 2-forms into weak discrete Lagrangian 1-forms. This framework elevates periodic reductions to the Lagrangian multiform level.

We link trigonometric functions to discrete and continuous Lagrangian 1-forms, making use of their addition formula. We derive discrete commuting flows for the McMillan equation (discrete autonomous Painlevé II), derive Jacobi elliptic solutions and develop a discrete Lagrangian 1-form.

Contents

Chapter 1

Introduction

In this work, we develop the theory of Lagrangian multiforms, which began in the pioneering paper [28]. Specifically, we focus on discrete Lagrangian 2-forms and their reductions, such that compatible systems of discrete integrable equations arise from a single variational principle. These examples are interesting in their own right and our contributions also help unify the entire theory of continuous, discrete and semi-discrete Lagrangian multiforms into one cohesive framework.

The development of this mathematical theory is partly driven by the significance of Lagrangian variational principles throughout physics. Each fundamental equation in physics can be constructed from a Lagrangian variational principle (a stationary-action principle) [20]. It is anticipated that Lagrangian multiforms will contribute to the advancement of quantum path integrals [26, 51, 27], wherein the Lagrangian assumes a central role.

We note that when Lagrangians are an alternative to Hamiltonian constructions, they more easily handle: constraints, symmetries, generalised coordinates, covariance and Lorentz invariance [20]. The theory of Lagrangian multiforms is strengthening the Lagrangian approach by providing analogous features previously exclusive to Hamiltonians [16].

Integrable equations are non-linear equations, which are richer and more interesting than linear equations but can still be treated with exact methods [23]. We consider the notion of an integrable equation, as one which belongs to a compatible set of equations (multidimensionally consistent difference equations or a hierarchy of differential equations). Lagrangian multiforms provide a single variational principle [28], involving multiple independent variables, such that the compatible set of integrable equations arise variationally.

In recent years, the continuous version of the theory, describing integrable differential equations, has received a lot of attention [19, 16, 39, 41, 50, 55] and a semi-discrete version was formulated [49]. However, the origins of the Lagrangian multiform theory lie in the fully discrete setting [28] and this is what we focus on [31, 29, 62].

Lagrangian multiforms have now been connected with many features of integrability: Lax pairs [47], classical r-matrices [18], hamiltonian structures [52, 59, 17], as well as variational symmetries from Noether's theorem [41, 48, 59]. There are also connections to: pluri-harmonic functions in complex function theory and Baxter's notion of Z-invariance in the statistical mechanics of exactly solvable models [54]. After the first proposal of Lagrangian multiforms in [28], similar constructions were considered under the name of pluri-Lagrangian systems [54]. These constructions have lead to some useful developments however they drop an important postulate: the closure relation.

Discrete Lagrangian multiforms are well suited to be applied to discrete quantum theories of gravity, where discrete Lagrangians and symmetries play a key role [4, 5, 45, 46]. We also note that a connection has been made between Lagrangian multiforms and the Chern-Simons theory [33].

We specifically consider discrete integrable systems, which are receiving growing interest and are no longer seen as approximations to continuous integrable systems, but in fact richer and more general than their continuous counterparts [23]. One can take continuum limits of many discrete structures, including the discrete equations or the discrete Lagrangian multiforms and recover the continuous analogues [58].

We focus on discrete Lagrangian multiforms [28, 10, 61] for quad equations of the ABS list [1], which are integrable partial difference equations classified using multidimensional consistency. Soliton solutions have been derived for quad equations using this integrable property [14, 35]. These quad equations have many useful (integrable) features: they admit Lax representations, are related to Bäcklund transformations and have vanishing algebraic entropy [9, 38, 60, 1]. One important application of quad equations is that they arise when studying solutions to the quantum Yang-Baxter equation in star triangle form, as shown in [7, 6]. In the latter work, the authors utilise the fact that Q_4 is the most general quad equation and that Lagrangian structures exist for all quad equations.

In this thesis we will develop discrete Lagrangian 2-forms of quad equations, and their periodic reductions. We discover many interesting developments and also help unify the entire theory of Lagrangian multiforms, with the potential for future applications in quantum path integrals. In the next section we provide an overview of the chapters.

1.1 Thesis Overview

Chapter 2 is a pedagogical chapter, which introduces the following: integrability in the sense of compatible equations, the conventional discrete Lagrangian variational principle and the discrete Lagrangian multiform variational principle. We introduce these concepts by focusing on the H1 quad equation (from the ABS list) as a basic example of a discrete integrable partial difference equation. We review the pioneering work of [28] which started the theory of Lagrangian multiforms, which applies to the H1 equation. Our contribution in this chapter is to introduce a definition of a weak discrete Lagrangian 2-form, and contrast it to the full definition of a discrete Lagrangian 2-form. We discuss the discrete Lagrangian 2-form for H1, and how the compatible set of H1 equations are sufficient but not necessary for the Lagrangian multiform variational principle. We finish the chapter by discussing important features of Lagrangian multiforms.

In Chapter 3, we present two novel discrete Lagrangian 2-forms for any quad equation of the ABS list (including H1). We resolve the insufficiency discussed in Chapter 2, and show that the integrable quad equations are variational and follow from multiform Euler-Lagrange equations. We review how the well-established discrete Lagrangian 2-form encapsulates weak combinations of quad equations, which are E-equations or equivalently octahedron equations. We also introduce a third novel discrete Lagrangian 2-form encapsulating the tetrahedron equations, which are also weak combinations of quad equations. We use relations between these Lagrangian 2-forms to connect quad equations to tetrahedron and octahedron equations.

In Chapter 4 we reinforce that double zero expansions are a universal phenomenon of Lagrangian multiforms, by considering them in the discrete setting. We show that each of the discrete Lagrangian 2-forms considered in Chapter 3 admit a double zero expansion in terms of their polynomial corner equations (quad, tetrahedron, octahedron or E-polynomials).

In Chapter 5 we reinforce that integrable properties, including the Lagrangian multiform structure, are commonly preserved under reductions. We elevate periodic reductions, from integrable partial difference equations to integrable ordinary difference equations, to the Lagrangian (multiform) level. We present a novel framework to carry out periodic reductions of discrete Lagrangian 2-forms into discrete Lagrangian 1-forms. The framework is promising but leads to weak Lagrangian 1-forms. However, we resolve this for one example considered and this approach has much potential for future research.

In Chapter 6, we investigate Lagrangian 1-forms. We relate discrete and continuous

Lagrangian 1-forms to addition formulae for trigonometric and elliptic functions. Furthermore, we present a promising discrete Lagrangian 1-form construction for the McMillan equation (the discrete autonomous Painlevé II equation).

Chapter 2

H1 Quad Equation and the Development of Lagrangian Multiforms

In this Chapter we introduce the following: integrability in the sense of compatible equations, the conventional discrete Lagrangian variational principle and the discrete Lagrangian multiform variational principle. We introduce these concepts by focusing on the H1 quad equation as an ideal example of a discrete integrable partial difference equation.

2.1 Integrable Potential Korteweg-de Vries Equation

We consider the H1 quad equation [1], which is also known as the lattice potential Korteweg-de Vries (lpKdV) equation [36], as an archetypal example of a discrete integrable equation [23], given by

$$(u - u_{ij})(u_i - u_j) - \alpha_i + \alpha_j = 0\,. \tag{2.1}$$

We consider a field $u(\mathbf{n})$ over the lattice $\mathbf{n} = (n_i, n_j) \in \mathbb{Z}^2$, and lattice vectors $\mathbf{e}_i$ and $\mathbf{e}_j$ in the ith and jth direction, respectively. Equation (2.1) is a partial difference equation, where $u = u(\mathbf{n})$, $u_i = u(\mathbf{n} + \mathbf{e}_i)$, $u_{ij} = u(\mathbf{n} + \mathbf{e}_i + \mathbf{e}_j)$ and $u_j = u(\mathbf{n} + \mathbf{e}_j)$ are shifted field variables, and $\alpha_i, \alpha_j \in \mathbb{C}$ are constants. In the next section we study this equation as a discrete integrable equation. First of all, in this section, we will consider continuum limits and see the connection to the Korteweg-de Vries partial differential equation.

We will present some of the technical details of the continuum limit, although they are not important for our purposes. Please refer to [58] for further explanations or to [36, 23] for similar approaches. First we carry out the following non-autonomous transformation in preparation for the continuum limit:

$$v(\mathbf{n}) = u(\mathbf{n}) + \frac{n_i}{\lambda_i} + \frac{n_j}{\lambda_j}\,, \qquad \alpha_i = -\frac{1}{\lambda_i^2}\,, \qquad \alpha_j = -\frac{1}{\lambda_j^2}\,, \tag{2.2}$$

which leads to the equation

$$\left(-\frac{1}{\lambda_i} - \frac{1}{\lambda_j} + v - v_{ij}\right)\left(\frac{1}{\lambda_j} - \frac{1}{\lambda_i} + v_i - v_j\right) + \frac{1}{\lambda_i^2} - \frac{1}{\lambda_j^2} = 0\,. \tag{2.3}$$

Now we introduce independent time variables $(t_1, t_2, t_3, \cdots)$ as skew combinations of the discrete variables in the following way:

$$v = v(\mathbf{n},\, t_1 - 2n_i\lambda_i - 2n_j\lambda_j,\, t_2 + \frac{2n_i\lambda_i^2}{2} + \frac{2n_j\lambda_j^2}{2},\, t_3 - \frac{2n_i\lambda_i^3}{3} - \frac{2n_j\lambda_j^3}{3}, \cdots)\,. \tag{2.4}$$

If we Taylor expand in terms of the constants λ_i and λ_j, we find the following set of equations:

$$\frac{\partial v}{\partial t_2} = 0\,, \tag{2.5a}$$

$$\frac{\partial v}{\partial t_3} = 3\left(\frac{\partial v}{\partial t_1}\right)^2 + \frac{\partial^3 v}{\partial t_1^3}\,. \tag{2.5b}$$

The latter equation is the continuous potential Korteweg-de Vries equation [36]. Thus, we have derived the corresponding partial differential equation from a continuum limit of the partial difference equation. As is often the case for discrete integrable equations, the discrete equation should not be seen as an approximation of a continuous equation, but rich in itself and more general [23].

In fact we can go even further and consider more independent time variables $(t_1, t_2, t_3, t_4, t_5, \cdots)$ to derive an infinite hierarchy of compatible partial differential equations, starting with

$$\frac{\partial v}{\partial t_2} = 0\,, \tag{2.6a}$$

$$\frac{\partial v}{\partial t_3} = 3\left(\frac{\partial v}{\partial t_1}\right)^2 + \frac{\partial^3 v}{\partial t_1^3}\,, \tag{2.6b}$$

$$\frac{\partial v}{\partial t_4} = 0\,, \tag{2.6c}$$

$$\frac{\partial v}{\partial t_5} = 10\left(\frac{\partial v}{\partial t_1}\right)^3 + 5\frac{\partial^2 v}{\partial t_1^2} + 10\frac{\partial v}{\partial t_1}\frac{\partial^3 v}{\partial t_1^3} + \frac{\partial^5 v}{\partial t_1^5}\,, \tag{2.6d}$$

$$\dots$$

Note that for all even time variables t_{2m}, where $m \in \mathbb{N}$, the derivative vanishes trivially $\frac{\partial v}{\partial t_{2m}} = 0$. The derivatives in the odd time variables can all be written as combinations of t_1 derivatives (often distinguished with $x := t_1$). It then follows that the infinitely many odd time flows t_{2m+1} commute, in the following way:

$$\frac{\partial^2 v}{\partial t_{2m+1}\partial t_{2m'+1}} = \frac{\partial^2 v}{\partial t_{2m'+1}\partial t_{2m+1}} \qquad m, m' \in \mathbb{N}\,. \tag{2.7}$$

This gives us an infinite set of conservation laws for the pKdV partial differential equation, or equivalently an infinite hierarchy of compatible partial differential equations.

This hierarchy is a powerful notion of integrability for continuous equations, particularly relevant for Lagrangian multiforms. The lattice pKdV equation encapsulates the continuous pKdV equation and its entire integrable hierarchy. In the next section, we will focus on the lattice pKdV equation and the discrete analogue of an integrable hierarchy of equations: multidimensional consistency.

2.2 The Multidimensionally-Consistent H1 Quad Equation

We return to the lpKdV equation (2.1), which we will henceforth refer to as the H1 quad equation, written $Q(u, u_i, u_{ij}, u_j;\, \alpha_i, \alpha_j) = 0$ with [1]

$$Q(u, u_i, u_{ij}, u_j;\, \alpha_i, \alpha_j) = (u - u_{ij})(u_i - u_j) - \alpha_i + \alpha_j\,. \tag{2.8}$$

We fix $\alpha_i, \alpha_j \in \mathbb{C}$ as lattice parameters associated with the n_i and n_j directions, respectively.

First of all, let us discuss the properties of this discrete integrable equation, which characterise the more general quad equations of the ABS list that we will consider later [1]. This partial difference equation involves the values of the field u, u_i, u_{ij}, u_j, which lie on the square stencil. This is a polynomial equation and at most degree one in any one field variable. Hence, it is a *multiaffine quad equation*, meaning we can solve for any one field variable in terms of the other three.

Now let us consider its discrete symmetries. The variables u, u_i, u_{ij}, u_j are the values at the four corners of the square in rotational order. By observation, the H1 equation can be seen to be symmetric under rotation around the square in the following way:

$$Q(u_i, u_{ij}, u_j, u;\ \alpha_j, \alpha_i) = -Q(u, u_i, u_{ij}, u_j;\ \alpha_i, \alpha_j)\,, \tag{2.9a}$$

$$Q(u_{ij}, u_j, u, u_i;\ \alpha_i, \alpha_j) = Q(u, u_i, u_{ij}, u_j;\ \alpha_i, \alpha_j)\,, \tag{2.9b}$$

$$Q(u_j, u, u_i, u_{ij};\ \alpha_j, \alpha_i) = -Q(u, u_i, u_{ij}, u_j;\ \alpha_i, \alpha_j)\,. \tag{2.9c}$$

We can also see that the H1 equation is symmetric under reflection about the diagonal:

$$Q(u, u_j, u_{ij}, u_i;\ \alpha_j, \alpha_i) = -Q(u, u_i, u_{ij}, u_j;\ \alpha_i, \alpha_j)\,. \tag{2.10a}$$

Altogether, the H1 equation is symmetric under the symmetries of the square (or the dihedral group D_4).

Next we consider the defining integrability criterion of quad equations. Integrable equations are often part of a set of compatible equations, such as a hierarchy of differential equations, or in this case multidimensionally-consistent partial difference equations. In order to consider this property, we will attempt to impose H1 equations on the six faces of a cube. To do this, we can extend our field to $u(\mathbf{n})$ to be over a three-dimensional lattice with $\mathbf{n} = (n_i, n_j, n_k) \in \mathbb{Z}^3$. For example, we can consider the value of the field shifted one step in three different directions: $u_i = u(\mathbf{n} + \mathbf{e}_i)$, $u_j = u(\mathbf{n} + \mathbf{e}_j)$ and $u_k = u(\mathbf{n} + \mathbf{e}_k)$.

At this point, we introduce point inversion of the field variable on the cube, denoted by $\overleftrightarrow{\ }$, as follows:

$$\overleftrightarrow{u} = u_{ijk}\,, \qquad \overleftrightarrow{u_{ijk}} = u\,, \tag{2.11a}$$

$$\overleftrightarrow{u_i} = u_{jk}\,, \qquad \overleftrightarrow{u_{jk}} = u_i\,, \tag{2.11b}$$

$$\overleftrightarrow{u_j} = u_{ki}\,, \qquad \overleftrightarrow{u_{ki}} = u_j\,, \tag{2.11c}$$

$$\overleftrightarrow{u_k} = u_{ij}\,, \qquad \overleftrightarrow{u_{ij}} = u_k\,. \tag{2.11d}$$

Later we will permute indices ($i \to j$, $j \to k$, $k \to i$), such that u_{ij} permutes to u_{jk}, u_{jk} permutes to u_{ki} and u_{ki} permutes to u_{ij}. This is our reasoning for choosing the index ordering u_{ki} above and henceforth, which we are free to do as $u_{ki} = u_{ik}$. Due to the H1 equation satisfying the symmetries of the square, we can consider the equations at opposite faces of the cube using point inversion:

$$Q_{ij} = (u_i - u_j)(u - u_{ij}) - \alpha_i + \alpha_j = 0\,, \tag{2.12a}$$

$$Q_{jk} = (u_j - u_k)(u - u_{jk}) - \alpha_j + \alpha_k = 0\,, \tag{2.12b}$$

$$Q_{ki} = (u_k - u_i)(u - u_{ki}) - \alpha_k + \alpha_i = 0\,, \tag{2.12c}$$

$$\overleftrightarrow{Q}_{ij} = (u_{jk} - u_{ki})(u_{ijk} - u_k) - \alpha_i + \alpha_j = 0\,, \tag{2.12d}$$

$$\overleftrightarrow{Q}_{jk} = (u_{ki} - u_{ij})(u_{ijk} - u_i) - \alpha_j + \alpha_k = 0\,, \tag{2.12e}$$

$$\overleftrightarrow{Q}_{ki} = (u_{ij} - u_{jk})(u_{ijk} - u_j) - \alpha_k + \alpha_i = 0\,. \tag{2.12f}$$

Thus, the six quad equations on the faces of the cube that we consider are those that follow from permutation of indices and point inversion of the original equation $Q_{ij} = 0$.

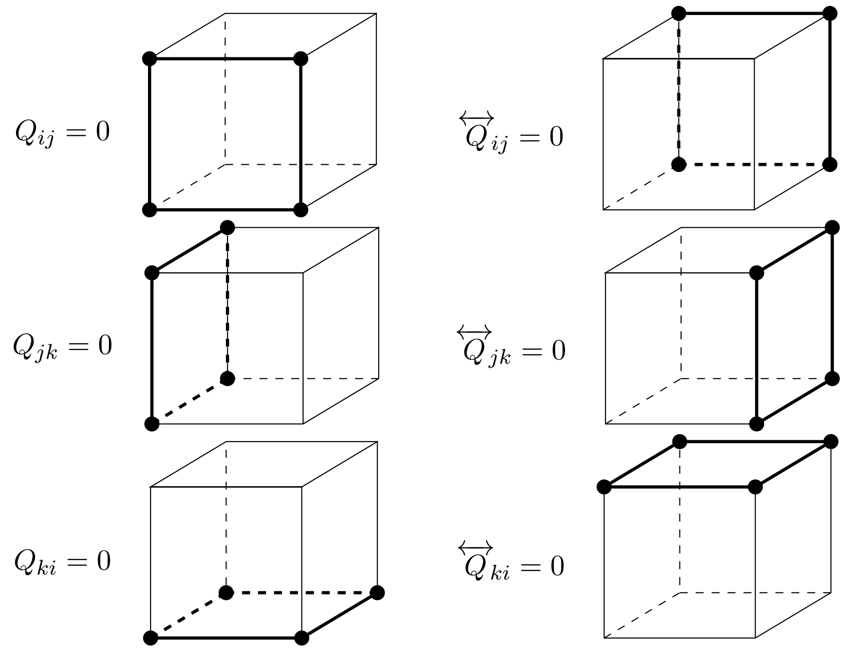


Figure 2.1: Depicting the compatible set of six quad equations on a cube

We consider an initial value problem of this set of equations around a cube. We start with values (u, u_i, u_j, u_k) and use $Q_{ij} = 0$ to calculate u_{ij}, $Q_{jk} = 0$ to calculate u_{jk}, and $Q_{ki} = 0$ to calculate u_{ki}. Ignoring potential singularities we can write this as

$$Q_{ij} = 0 \qquad \Longrightarrow \qquad u_{ij} = u - \frac{\alpha_i - \alpha_j}{u_i - u_j}\,, \tag{2.13a}$$

$$Q_{jk} = 0 \qquad \Longrightarrow \qquad u_{jk} = u - \frac{\alpha_j - \alpha_k}{u_j - u_k}\,, \tag{2.13b}$$

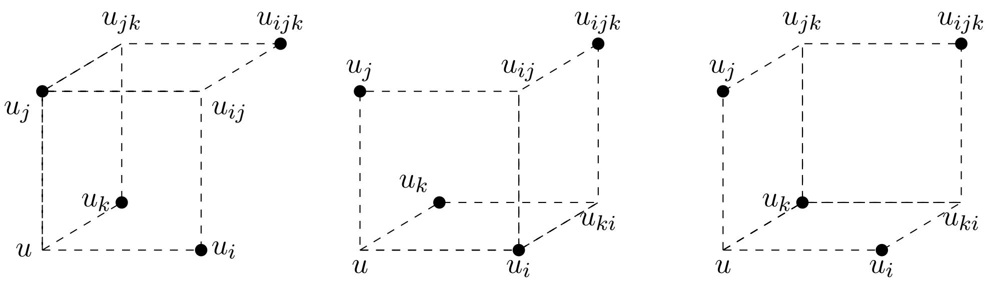
$$Q_{ki} = 0 \qquad \Longrightarrow \qquad u_{ki} = u - \frac{\alpha_k - \alpha_i}{u_k - u_i}\,. \tag{2.13c}$$

Then we can use any of the three equations $\overleftrightarrow{Q}_{ij} = 0$ and $\overleftrightarrow{Q}_{ij} = 0$ and $\overleftrightarrow{Q}_{ij} = 0$ to calculate u_{ijk} in terms of (u, u_i, u_j, u_k):

$$Q_{jk}, Q_{ki}, \overleftrightarrow{Q}_{ij} = 0 \qquad \Longrightarrow \qquad u_{ijk} = -\frac{\alpha_i - \alpha_j}{\left(u - \frac{\alpha_k - \alpha_i}{(u_k - u_i)}\right) - \left(u - \frac{\alpha_j - \alpha_k}{(u_j - u_k)}\right)} + u_k\,, \tag{2.14a}$$

$$Q_{ij}, Q_{ki}, \overleftrightarrow{Q}_{jk} = 0 \quad \Longrightarrow \quad u_{ijk} = -\frac{\alpha_j - \alpha_k}{\left(u - \frac{\alpha_i - \alpha_j}{(u_i - u_j)}\right) - \left(u - \frac{\alpha_k - \alpha_i}{(u_k - u_i)}\right)} + u_i \,, \tag{2.14b}$$

$$Q_{ij}, Q_{jk}, \overleftrightarrow{Q}_{ki} = 0 \quad \Longrightarrow \quad u_{ijk} = -\frac{\alpha_k - \alpha_i}{\left(u - \frac{\alpha_j - \alpha_k}{(u_j - u_k)}\right) - \left(u - \frac{\alpha_i - \alpha_j}{(u_i - u_j)}\right)} + u_j \,. \tag{2.14c}$$



(a) (b) (c)

Figure 2.2: Alternative multidimensionally consistent routes to calculate u_{ijk} from u, u_i, u_j, u_k using three quad equations: (a) using $Q_{ij}, Q_{jk}, \overleftrightarrow{Q}_{ki} = 0$, (b) using $Q_{ij}, Q_{ki}, \overleftrightarrow{Q}_{jk} = 0$, (c) using $Q_{jk}, Q_{ki}, \overleftrightarrow{Q}_{ij} = 0$.

These all lead to same expression for u_{ijk} in terms of (u_i, u_j, u_k):

$$\begin{aligned} Q_{jk}, Q_{ki}, \overleftrightarrow{Q}_{ij} = 0 \quad \text{or} \quad Q_{ij}, Q_{ki}, \overleftrightarrow{Q}_{jk} = 0 \quad \text{or} \quad Q_{ij}, Q_{jk}, \overleftrightarrow{Q}_{ki} = 0 \\ \Longrightarrow u_{ijk} = \frac{\alpha_i u_i (u_j - u_k) + \alpha_j u_j (u_k - u_i) + \alpha_k u_k (u_i - u_j)}{\alpha_i (u_j - u_k) + \alpha_j (u_k - u_i) + \alpha_k (u_i - u_j)} \\ = \frac{\alpha_i u_i (u_j - u_k) + \circlearrowleft_{ijk}}{\alpha_i (u_j - u_k) + \circlearrowleft_{ijk}} \,. \end{aligned} \tag{2.15}$$

Here we have abbreviated adding two more copies of an expression with indices permuted once and then twice, by $\circlearrowleft_{ijk}$. The fact that we can consistently calculate the value of u_{ijk} via different routes is *multidimensional consistency*. Furthermore, the expression we find can be rewritten as a multiaffine equation $T(u_{ijk}, u_k, u_i, u_j;\, \alpha_i, \alpha_j, \alpha_k) = 0$ on the tetrahedron stencil with

$$\begin{aligned} &T(u_{ijk}, u_k, u_i, u_j;\, \alpha_i, \alpha_j, \alpha_k) \\ &\quad = u_{ijk} \left(\alpha_i (u_j - u_k) + \circlearrowleft_{ijk}\right) - \left(\alpha_i u_i (u_j - u_k) + \circlearrowleft_{ijk}\right) \,. \end{aligned} \tag{2.16}$$

This is the *tetrahedron property*.

We can also consider point inverting this and find an analogous tetrahedron equation in terms of the other four variables $T(u, u_{ij}, u_{jk}, u_{ki};\, \alpha_i, \alpha_j, \alpha_k)$. With a slight abuse of notation we denote the two tetrahedron equations for H1 as

$$T = T(u, u_{ij}, u_{jk}, u_{ki};\, \alpha_i, \alpha_j, \alpha_k) \,,$$

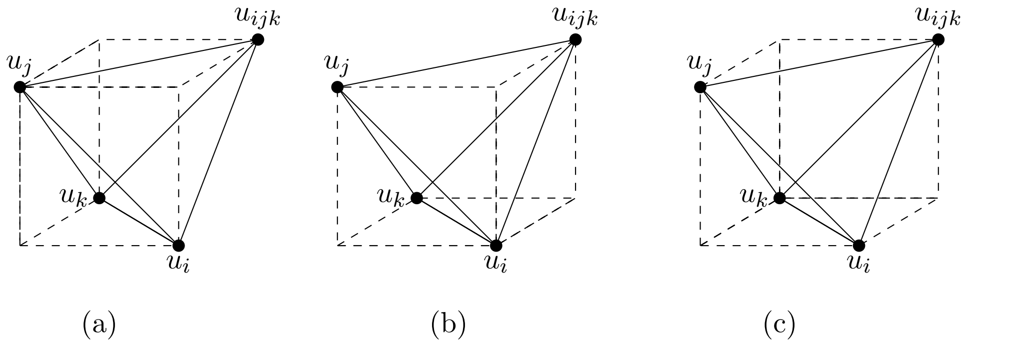


Figure 2.3: Alternative multidimensionally consistent routes giving rise the same tetrahedron equation $T(u_{ijk}, u_k, u_i, u_j;) = 0$: (a) using $Q_{ij}, Q_{jk}, \overleftrightarrow{Q}_{ki} = 0$, (b) using $Q_{ij}, Q_{ki}, \overleftrightarrow{Q}_{jk} = 0$, (c) using $Q_{jk}, Q_{ki}, \overleftrightarrow{Q}_{ij} = 0$.

$$= u\left(\alpha_i(u_{ki} - u_{ij}) + \circlearrowleft_{ijk}\right) - \left(\alpha_i u_{jk}(u_{ki} - u_{ij}) + \circlearrowleft_{ijk}\right) , \tag{2.17a}$$

$$\overleftrightarrow{T} = T(u_{ijk}, u_k, u_i, u_j;\, \alpha_i, \alpha_j, \alpha_k) ,$$

$$= u_{ijk}\left(\alpha_i(u_j - u_k) + \circlearrowleft_{ijk}\right) - \left(\alpha_i u_i(u_j - u_k) + \circlearrowleft_{ijk}\right) . \tag{2.17b}$$

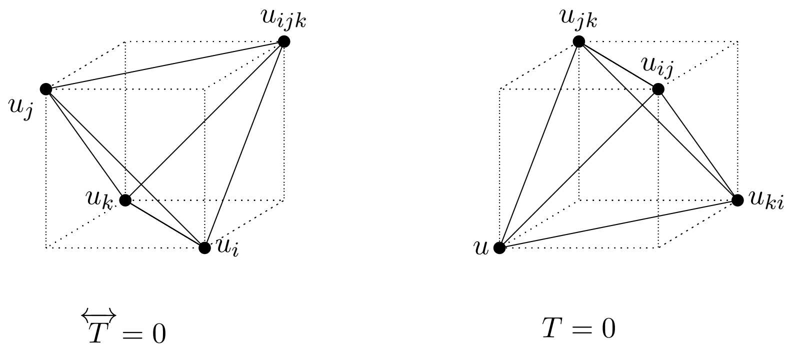


Figure 2.4: Two tetrahedron equations which arise from H1 equations around a cube

In the next section, we consider a conventional discrete Lagrangian structure which can be related to the H1 equation.

2.3 Discrete Lagrangians on Quadrilateral Stencils

In this section we review the discrete analogue of the conventional variational principle, specifically for discrete Lagrangians on quadrilateral stencils. We focus on the action and its criticality (with respect to local variations of the field). Then in the next section we consider Lagrangians associated with combinations of the H1 equation.

With calculus of variations in mind, we define an *action* over the lattice $\mathbb{Z}^2$ involving autonomous Lagrangians on quadrilateral stencils $\mathcal{L}_{ij} : (u, u_i, u_{ij}, u_j; \alpha_i, \alpha_j) \mapsto \mathbb{C}$, that is

$$S[u] = \sum_{\mathbf{n}\in\mathbb{Z}^2} \mathcal{L}_{ij}(u(\mathbf{n}), u(\mathbf{n}+\mathbf{e}_i), u(\mathbf{n}+\mathbf{e}_i+\mathbf{e}_j), u(\mathbf{n}+\mathbf{e}_j); \alpha_i, \alpha_j)\,,$$
$$= \sum_{\mathbf{n}\in\mathbb{Z}^2} \mathcal{L}_{ij}\,. \tag{2.18}$$

We could consider boundaries and boundary value problems in this action functional but it is not necessary for our purposes.

Now we consider the conventional variational principle. We denote criticality of the action with respect to local variations of the field by $\frac{\delta S[u]}{\delta u} = 0$, and it is defined in the following way. The field $u : \mathbb{Z}^2 \mapsto \mathbb{C}$ is a critical point of the action $S[u]$ if and only if the partial derivatives vanish at every point in the lattice:

$$\frac{\delta S[u]}{\delta u} = 0 \qquad \Longleftrightarrow \qquad \frac{\partial S[u]}{\partial u(\mathbf{n})} = 0 \quad \forall\, \mathbf{n} \in \mathbb{Z}^2\,. \tag{2.19}$$

In order to rewrite these partial derivatives in terms of Lagrangians, we introduce shift operators, T_i and T_j, which act on functions on the lattice, such that $T_i f(\mathbf{n}) = f(\mathbf{n}+\mathbf{e}_i)$ and $T_i^{-1} f(\mathbf{n}) = f(\mathbf{n}-\mathbf{e}_i)$. We can rewrite the partial derivatives of the action in (2.19) as a sum of four shifted Lagrangians, which we call the discrete Euler-Lagrange equation:

$$\frac{\delta S[u]}{\delta u} = 0 \iff \frac{\partial}{\partial u}\Big(\mathcal{L}_{ij} + T_j^{-1}\mathcal{L}_{ij} + T_i^{-1}\mathcal{L}_{ij} + T_j^{-1}T_i^{-1}\mathcal{L}_{ij}\Big) = 0 \quad \forall\, \mathbf{n} \in \mathbb{Z}^2\,. \tag{2.20}$$

In other words, criticality of the action is equivalent to the discrete Euler-Lagrange equation vanishing at every point in the lattice.

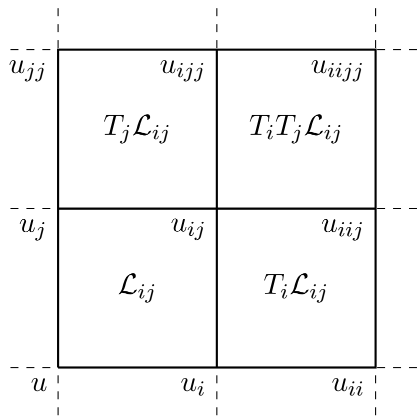


Figure 2.5: The four translated Lagrangians which contribute to the conventional Euler-Lagrange equation at u_{ij}

We restrict ourselves to autonomous discrete Lagrangians and thus the discrete Euler-Lagrange equations at every point in the lattice has the same form. For example, we could instead consider the discrete Euler-Lagrange equation centred at u_{ij}, which is

$$\frac{\partial}{\partial u_{ij}}\left(T_iT_j\mathcal{L}_{ij} + T_i\mathcal{L}_{ij} + T_j\mathcal{L}_{ij} + \mathcal{L}_{ij}\right) = 0\,. \tag{2.21}$$

Figure 2.5 depicts the four shifted Lagrangians from the action which lead to this equation. Furthermore, for a given Lagrangian, the corresponding variational equation is the one which arises from the discrete Euler-Lagrange equation (2.21).

Next we review a discrete Lagrangian for a combination of the H1 quad equation, and apply this conventional discrete variational principle.

2.4 Discrete Lagrangian for double H1

In this section we relate the conventional Lagrangian variational principle to the H1 quad equation. We review a Lagrangian where the Euler-Lagrange equation is a combination of two shifted H1 quad equations (double H1). We could consider the original 4-point Lagrangian for double H1 [36, 1] (which in Chapter 3 we refer to as the trident Lagrangian), but instead we present the Lagrangian introduced in [28]. Regardless of our choice of Lagrangian, we see that being on solutions to the H1 quad equation is sufficient but not necessary for the conventional action to be critical with respect to local variations of the field (for the Euler-Lagrange equations to be satisfied). Also note that here we only consider a single H1 quad equation in the two dimensional lattice $\mathbb{Z}^2$ without multidimensional consistency.

In [28], the following expression was introduced, which we refer to as the *triangle Lagrangian* for H1:

$$\mathcal{L}^{\triangleright}_{ij} = uu_i - uu_j - (\alpha_i - \alpha_j)\log(u_i - u_j)\,. \tag{2.22}$$

This defines an action (2.18) on the 2-dimensional lattice $\mathbb{Z}^2$.

The action is critical with respect to local variations of the field when the conventional discrete Euler-Lagrange equation (2.21) is satisfied. Calculating this explicitly using (2.22), we can write the Euler-Lagrange equation as a combination of two shifted H1 quad equations (2.12a):

$$\begin{aligned}&\frac{\partial}{\partial u_{ij}}\left(u_{ij}u_{iij} - u_{ij}u_{ijj} + u_ju_{ij} - (\alpha_i-\alpha_j)\log(u_{ij}-u_{jj}) - u_iu_{ij} - (\alpha_i-\alpha_j)\log(u_{ii}-u_{ij})\right)\\ &= u_{iij} - u_{ijj} + u_j - \frac{\alpha_i-\alpha_j}{u_{ij}-u_{jj}} - u_i + \frac{\alpha_i-\alpha_j}{u_{ii}-u_{ij}}\,,\end{aligned}$$

$$= -\frac{(u_i - u_{iij})(u_{ii} - u_{ij}) - \alpha_i + \alpha_j}{u_{ii} - u_{ij}} + \frac{(u_j - u_{ijj})(u_{ij} - u_{jj}) - \alpha_i + \alpha_j}{u_{ij} - u_{jj}},$$
$$= -\frac{T_i Q_{ij}}{u_{ii} - u_{ij}} + \frac{T_j Q_{ij}}{u_{ij} - u_{jj}}. \tag{2.23}$$

Thus, the equation which follows from the conventional Lagrangian variational principle, is not the H1 quad equation but a combination of two shifted copies of it, which we call double H1. In [24], it is shown precisely how the solution of double H1 is more general than H1. It is possible to obtain any solution of double H1 from a solution of H1 together with the solution of a linear partial difference equation.

We emphasise that imposing the H1 quad equation $Q_{ij} = 0$ throughout the plane is sufficient but not necessary for the criticality of the conventional action with respect to variations of the field:

$$Q_{ij} = 0 \quad \Longrightarrow \quad \frac{\delta S[u]}{\delta u} = 0\,. \tag{2.24}$$

The H1 quad equation implies criticality but criticality does not imply the H1 quad equation.

Now we note that the conventional variational principle considered so far only relates to a single partial difference equation on the 2-dimensional lattice $\mathbb{Z}^2$. The desire to relate multidimensionally-consistent sets of partial difference equations to a variational principle lead to the invention of Lagrangian multiforms, which we consider next.

2.5 Weak Discrete Lagrangian Multiforms on Quadrilaterals

The theory of Lagrangian multiforms can provide a single variational principle to encapsulate compatible sets of integrable equations (in the continuous, discrete and semi-discrete settings). This theory began in the pioneering paper [28], with constructions which we will define as *weak discrete Lagrangian 2-forms* on quadrilateral stencils. In Section 2.6, we discuss an example associated with combinations of the H1 quad equation. Then in Section 2.7 we discuss the additional property required for the full definition of a Lagrangian multiform.

With multidimensional consistency in mind, the authors of [28] considered an action over 2-dimensional quad surfaces surfaces made of elementary squares in a higher dimensional lattice. These quad surfaces are depicted in Figure 2.6. We simply consider the three dimensional square lattice $\mathbf{n} \in \mathbb{Z}^3$. However, this theory generalises to N-dimensional square lattices $\mathbb{Z}^N$ [11] and possibly also to the N-dimensional quadrilateral lattices

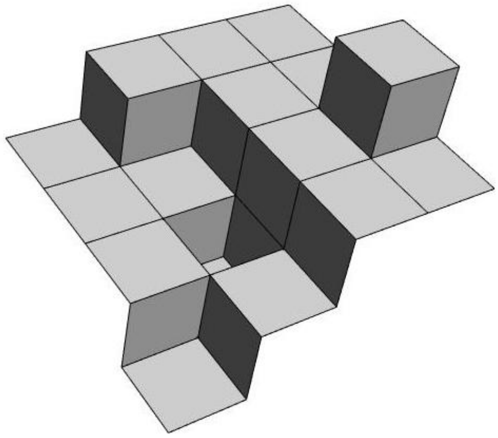


Figure 2.6: A quad surface σ in $\mathbb{Z}^3$ made from elementary squares.

considered in [1]. We consider oriented elementary squares at points $\mathbf{n} \in \mathbb{Z}^3$ defined by

$$\sigma_{ij}(\mathbf{n}) := (\mathbf{n}, \mathbf{n}+\mathbf{e}_i, \mathbf{n}+\mathbf{e}_i+\mathbf{e}_j, \mathbf{n}+\mathbf{e}_j)\,. \tag{2.25}$$

A quad surface σ is a connected set of these elementary squares, such that $\sigma_{ij}(\mathbf{n}) \in \sigma$. In the 3-dimensional lattice $\mathbb{Z}^3$, there are three pairs of directions and consequently a total of six oriented elementary squares: σ_{ij}, σ_{jk}, σ_{ki}, σ_{ji}, σ_{kj}, and σ_{ik}. Now consider three Lagrangians $\mathcal{L}_{ij}$, $\mathcal{L}_{jk}$ and $\mathcal{L}_{ki}$, which lie on quadrilateral stencils $\mathcal{L}_{ij} : (u, u_i, u_{ij}, u_j; \alpha_i, \alpha_j) \mapsto \mathbb{C}$, and are associated with the squares σ_{ij}, σ_{jk}, and σ_{ki}, respectively. Furthermore, consider Lagrangians which are anti-symmetric under the swap of directions: $\mathcal{L}_{ji} = -\mathcal{L}_{ij}$, such that σ_{ji}, σ_{kj}, and σ_{ik} corresponds to $-\mathcal{L}_{ij}$, $-\mathcal{L}_{jk}$, and $-\mathcal{L}_{ki}$, respectively.

Now we extend the conventional discrete action (2.18) and define an action over quad surfaces σ in $\mathbb{Z}^3$:

$$\begin{aligned} S[u,\sigma] &= \sum_{\sigma_{ij}(\mathbf{n})\in\sigma} \mathcal{L}_{ij}(u(\mathbf{n}), u(\mathbf{n}+\mathbf{e}_i), u(\mathbf{n}+\mathbf{e}_i+\mathbf{e}_j), u(\mathbf{n}+\mathbf{e}_j); \alpha_i, \alpha_j)\,, \\ &= \sum_{\sigma_{ij}(\mathbf{n})\in\sigma} \mathcal{L}_{ij}\,. \end{aligned} \tag{2.26}$$

Here, $\sigma_{ij}(\mathbf{n}) \in \sigma$ can be one of six oriented elementary squares, and then we add the corresponding signed Lagrangian ($\pm\mathcal{L}_{ij}$, $\pm\mathcal{L}_{jk}$, and $\pm\mathcal{L}_{ki}$).

We are interested in a variational principle, such that criticality of the action relates to multidimensionally-consistent sets of equations. We consider the effect that changing the quad surface has on the action. In both Figure 2.7 and Figure 2.8, we consider a quad surface with some fixed boundary and simply varying the surface locally by popping up a cube. We say that the action (2.26) is critical with respect to local variations of the

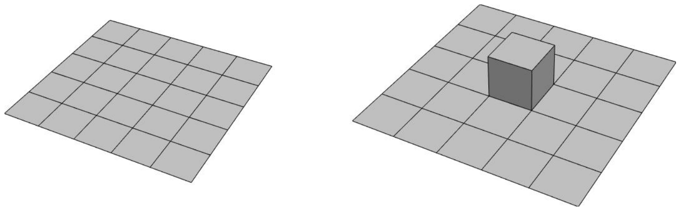


Figure 2.7: Two quad surfaces σ and $\sigma + \text{cube}$ (with the same boundary) which differ by the local variation of popping up a cube.

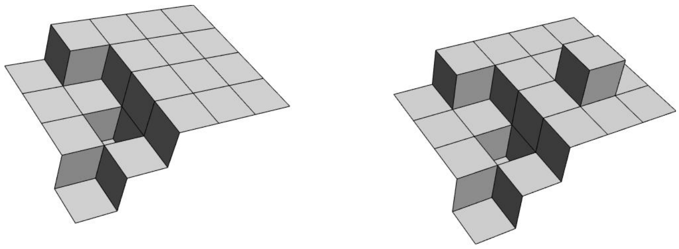


Figure 2.8: Two quad surfaces σ and $\sigma + \text{cube}$ (with the same boundary) which differ by the local variation of popping up a cube.

surface, if and only if the action does not change by the addition or subtraction of an elementary cube:

$$\frac{\delta S[u,\sigma]}{\delta\sigma} = 0 \qquad \Longleftrightarrow \qquad S[u,\sigma + \text{cube}] = S[u,\sigma]\,. \tag{2.27}$$

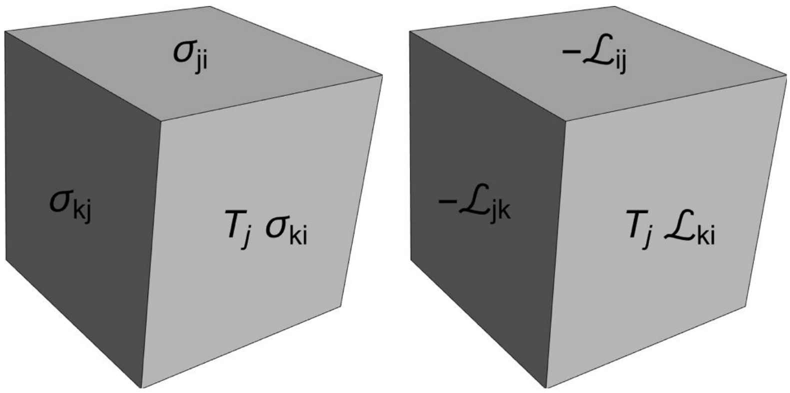


Figure 2.9: The left depicts three of the elementary oriented squares which make up the cube quad surface. The right depicts three Lagrangian contributions to the action $S[u, \text{cube}]$ over the cube, where each face of the cube contributes one Lagrangian term.

To understand this precisely, the quad surface of an oriented elementary cube (based at u), depicted in the left of Figure 2.9, is simply the collection of six oriented squares:

$$\text{cube} = \{T_k\sigma_{ij}, \sigma_{ji}, T_i\sigma_{jk}, \sigma_{kj}, T_j\sigma_{ki}, \sigma_{ik}\}\,. \tag{2.28}$$

The action on the elementary cube is the following, where we introduce the abbreviating notation:

$$S_{ijk} := S[u, \text{cube}] = (T_k\mathcal{L}_{ij} - \mathcal{L}_{ij}) + (T_i\mathcal{L}_{jk} - \mathcal{L}_{jk}) + (T_j\mathcal{L}_{ki} - \mathcal{L}_{ki})\,. \tag{2.29}$$

The six Lagrangians which contribute to this action are depicted in the right of Figure 2.9. Consequently, the criticality of the action with respect to local variations of the surface is equivalent to the vanishing of the action on an elementary cube:

$$\frac{\delta S[u,\sigma]}{\delta \sigma} = 0 \qquad \Longleftrightarrow \qquad S[u, \text{cube}] = S_{ijk} = 0\,. \tag{2.30}$$

We will often refer to the final equality, $S_{ijk} = 0$, as the *closure relation.* For given Lagrangians $\mathcal{L}_{ij}$, $\mathcal{L}_{jk}$ and $\mathcal{L}_{ki}$, later we will ask whether a field $u : \mathbb{Z}^3 \mapsto \mathbb{C}$ exists, such that the closure relation holds.

Now we consider criticality of the action with respect to variations of the field [11], generalising equation (2.19) to involve an action over quad surfaces. We consider a field $u : \mathbb{Z}^3 \mapsto \mathbb{C}$ and a quad surface σ with some boundary. We denote the internal lattice points of the quad surface as $V(\sigma) \subset \mathbb{Z}^3$. Now we say that the action is critical with respect to local variations of the field if and only if for generic quad surfaces σ the partial derivatives vanish at every internal point:

$$\frac{\delta S[u,\sigma]}{\delta u} = 0 \qquad \Longleftrightarrow \qquad \frac{\partial S[u,\sigma]}{\partial u(\mathbf{n})} = 0 \quad \forall\, \mathbf{n} \in V(\sigma)\,. \tag{2.31}$$

For criticality in this sense, we will see that instead of considering generic quad surfaces, we can simply consider the elementary cube. Since the cube is itself a quad surface, criticality for generic quad surfaces implies that the derivatives must vanish at the internal points of the action around the cube:

$$\frac{\delta S[u,\sigma]}{\delta u} = 0 \quad \Longrightarrow \quad \frac{\delta S[u,\text{cube}]}{\delta u} = 0 \quad \Longrightarrow \quad \nabla S_{ijk} = 0 \tag{2.32}$$

Note that this corresponds to eight *corner equations* of the cube:

$$\nabla S_{ijk} = \left(\frac{\partial S_{ijk}}{\partial u}, \frac{\partial S_{ijk}}{\partial u_i}, \frac{\partial S_{ijk}}{\partial u_j}, \frac{\partial S_{ijk}}{\partial u_k}, \frac{\partial S_{ijk}}{\partial u_{jk}}, \frac{\partial S_{ijk}}{\partial u_{ki}}, \frac{\partial S_{ijk}}{\partial u_{ij}}, \frac{\partial S_{ijk}}{\partial u_{ijk}} \right)\,. \tag{2.33}$$

Now making use of orientation, the derivative at any internal point of a generic quad surface can be made from derivatives of the corners of the cube. One example of this is depicted in Figure 2.10. The conventional discrete Euler-Lagrange expression (the

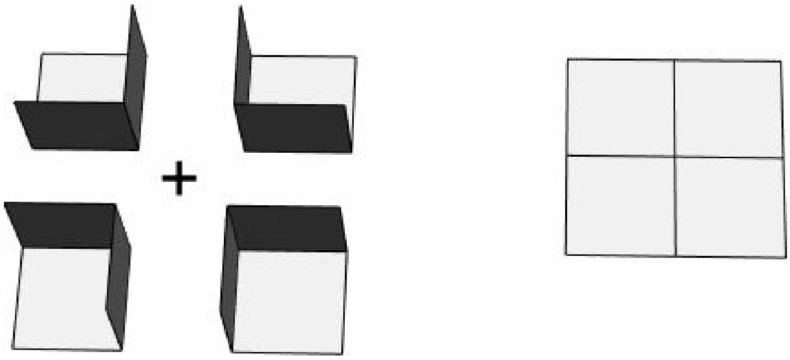


Figure 2.10: The derivative at the internal point of a simple plane can be made from four derivatives of corners of the cube, due to cancellations from orientation.

derivative at an internal point of a simple plane) is a combination of four translated corner derivatives of the cube:

$$\begin{aligned}&\frac{\partial}{\partial u_{ij}}\left(T_iT_j\mathcal{L}_{ij}+T_i\mathcal{L}_{ij}+T_j\mathcal{L}_{ij}+\mathcal{L}_{ij}\right)\\&\quad=\frac{\partial T_iT_jS_{ijk}}{\partial u_{ij}}+\frac{\partial T_iS_{ijk}}{\partial u_{ij}}+\frac{\partial T_jS_{ijk}}{\partial u_{ij}}+\frac{\partial S_{ijk}}{\partial u_{ij}}\,,\\&\quad=T_iT_j\frac{\partial S_{ijk}}{\partial u}+T_i\frac{\partial S_{ijk}}{\partial u_j}+T_j\frac{\partial S_{ijk}}{\partial u_i}+\frac{\partial S_{ijk}}{\partial u_{ij}}\,.\end{aligned} \tag{2.34}$$

For any internal point of a generic quad surface we find something similar. Then restricting ourselves to autonomous Lagrangians, it follows that criticality of the action with respect to local variations of the field on a generic quad surface is in fact equivalent to the criticality of the action on the elementary cube:

$$\frac{\delta S[u,\sigma]}{\delta u}=0 \quad\Longleftrightarrow\quad \frac{\delta S[u,\text{cube}]}{\delta u}=0 \quad\Longleftrightarrow\quad \nabla S_{ijk}=0\,. \tag{2.35}$$

With the right hand side equation, the eight corner equations of the cube are the *multiform Euler-Lagrange equations*. Here, we consider generic quad surfaces which: are closed surfaces, have a fixed boundary or have an open boundary. We demand criticality of the action in terms of local and internal variations of the field, such that we can ignore effects of the boundary on the multiform Euler-Lagrange equations.

Now we combine both types of criticality. An action over quad surfaces (2.26) and partial difference equations $K_1(u),\cdots,K_M(u)=0$ together define a weak discrete Lagrangian 2-form, if on solutions to those equations, the action is critical with respect to local variations of the surface and of the field:

$$K_1(u),\,\cdots,\,K_M(u)=0 \quad\Longrightarrow\quad \begin{cases}\dfrac{\delta S[u,\sigma]}{\delta\sigma}=0\\[2mm]\dfrac{\delta S[u,\sigma]}{\delta u}=0\,.\end{cases} \tag{2.36}$$

Using the results above (2.30) and (2.32), we define this type of weak Lagrangian multiform in terms of the action on an elementary cube S_{ijk}.

Definition 1. *An action over quad surfaces* (2.26) *and partial difference equations* $K_1(u), \cdots, K_M(u) = 0$ *together define a* ***weak discrete Lagrangian 2-form****, if the following property is satisfied. When the field* $u : \mathbb{Z}^3 \mapsto \mathbb{C}$ *satisfies the given equations, both the closure relation and the multiform Euler-Lagrange equations hold:*

$$K_1(u),\, \cdots,\, K_M(u) = 0 \quad \implies \quad \begin{cases} S_{ijk} = 0 \\ \nabla S_{ijk} = 0\,. \end{cases} \tag{2.37}$$

Here the given equations $K_1(u), \cdots, K_M(u) = 0$ implicitly correspond to a particular solution of the multiform Euler-Lagrange equations, but not necessarily the most general solution. In the example below, instead of considering the multiform Euler-Lagrange equations (a set of combinations of the H1 quad equation), the stronger set of each individual H1 equation is given. Closure is then proved on these given equations. The point is that the action alone does not define what equations we associate with a weak discrete Lagrangian 1-form. We can recognise a set of equations which make the multiform Euler-Lagrange equations vanish and then prove closure on that set. This is a weak definition because it is easier to verify than the definition we give later. In Chapter 5 we discover actions which immediately satisfy the conditions of a weak Lagrangian multiform. It is difficult to then verify the full definition of a Lagrangian multiform, which we present later.

Note that we can interchangeably talk about an action over quad surfaces or equivalently a set of Lagrangians (which define an action) satisfying the definition of a weak Lagrangian multiform. Moreover, a definition is not given in the original constructions of [28]. We have introduced Definition 1 which applies to all of their explicit constructions. This weak definition gives us something to contrast to when discussing recent advancements, including our results in Chapter 3. Next we present an example of a weak discrete Lagrangian 2-form from [28], where the given equations are the multidimensionally-consistent set of H1 quad equations.

2.6 Weak Discrete Lagrangian Multiform for H1

In [28], the triangle Lagrangian $\mathcal{L}_{ij}^{\triangle}$ for H1, given by equation (2.22), was introduced and considered in a higher dimensional lattice. Then we can define the action (2.26) over quad surfaces σ in terms of the three triangle Lagrangians in each plane: $\mathcal{L}^{\triangle}_{ij}$, $\mathcal{L}^{\triangle}_{jk}$ and $\mathcal{L}^{\triangle}_{ki}$. We take the given equations to be the multidimensionally-consistent set of quad equations (around the cube): $Q_{ij}, Q_{jk}, Q_{ki}, \overleftrightarrow{Q}_{ij}, \overleftrightarrow{Q}_{jk}, \overleftrightarrow{Q}_{ki} = 0$.

We can confirm criticality of the action with respect to local variations of the field and the surface, simply by considering the action around the elementary cube. Explicitly, the action around the elementary cube is

$$
\begin{aligned}
S^{\triangle}_{ijk} &= (T_k\mathcal{L}^{\triangle}_{ij} - \mathcal{L}^{\triangle}_{ij}) + (T_i\mathcal{L}^{\triangle}_{jk} - \mathcal{L}^{\triangle}_{jk}) + (T_j\mathcal{L}^{\triangle}_{ki} - \mathcal{L}^{\triangle}_{ki}) \\
&= u_k u_{ki} - u_k u_{jk} - (\alpha_i - \alpha_j)\log(u_{ki} - u_{jk}) \\
&\quad - u u_i + u u_j + (\alpha_i - \alpha_j)\log(u_i - u_j) \\
&\quad + u_i u_{ij} - u_i u_{ki} - (\alpha_j - \alpha_k)\log(u_{ij} - u_{ki}) \\
&\quad - u u_j + u u_k + (\alpha_j - \alpha_k)\log(u_j - u_k) \\
&\quad + u_j u_{jk} - u_j u_{ij} - (\alpha_k - \alpha_i)\log(u_{jk} - u_{ij}) \\
&\quad - u u_k + u u_i + (\alpha_k - \alpha_i)\log(u_k - u_i)\,.
\end{aligned}
\tag{2.38}
$$

In order to show that the closure relation holds on the H1 quad equations in [28], the following was observed:

$$
\begin{aligned}
&Q_{ij}, Q_{ki} = 0 \\
&\implies (u_{ij} - u_{ki}) = -\frac{(\alpha_j - \alpha_k)u_i + (\alpha_k - \alpha_i)u_j + (\alpha_i - \alpha_j)u_k}{(u_i - u_j)(u_j - u_k)(u_k - u_i)}(u_j - u_k) \\
&\qquad\qquad = A_{ijk}(u_j - u_k)\,.
\end{aligned}
\tag{2.39}
$$

Importantly, A_{ijk} is cyclically-symmetric (symmetric under permutations of i, j and k). Now we can take cyclic permutations and apply this to this action around the elementary cube:

$$
\begin{aligned}
&Q_{ij}, Q_{jk}, Q_{ki} = 0 \\
&\qquad \implies S^{\triangle}_{ijk} = A_{ijk}u_k(u_i - u_j) - (\alpha_i - \alpha_j)\log(A_{ijk}(u_i - u_j)) \\
&\qquad\qquad\qquad + (\alpha_i - \alpha_j)\log(u_i - u_j) + \circlearrowleft_{ijk}\,, \\
&\qquad\qquad = 0\,.
\end{aligned}
\tag{2.40}
$$

We find that on solutions to the multidimensionally-consistent H1 quad equations, the closure relation holds:

$$
Q_{ij}, Q_{jk}, Q_{ki} = 0 \quad \implies \quad S^{\triangle}_{ijk} = 0\,. \tag{2.41}
$$

Next we consider the corner equations of this action, starting at the u_i corner:

$$\begin{aligned}
\frac{\partial S_{ijk}^{\triangle}}{\partial u_i} &= -u + \frac{\alpha_i - \alpha_j}{u_i - u_j} + u_{ij} - u_{ki} + u - \frac{\alpha_k - \alpha_i}{u_k - u_i}\,, \\
&= -\frac{(u - u_{ij})(u_i - u_j) - \alpha_i + \alpha_j}{u_i - u_j} + \frac{(u - u_{ki})(u_k - u_i) - \alpha_k + \alpha_i}{u_k - u_i} \\
&= \frac{Q_{ij}}{u_j - u_i} - \frac{Q_{ki}}{u_i - u_k}\,.
\end{aligned} \tag{2.42}$$

We can also find analogous expressions at u_j and u_k by permuting indices. At the opposite corner of the cube u_{jk}, we find

$$\begin{aligned}
\frac{\partial S_{ijk}^{\triangle}}{\partial u_{jk}} &= -u_k + \frac{\alpha_i - \alpha_j}{u_{ki} - u_{jk}} - u_j - \frac{\alpha_k - \alpha_i}{u_{jk} - u_{ij}}\,, \\
&= \frac{\overleftrightarrow{Q}_{ij}}{u_{ki} - u_{jk}} - \frac{\overleftrightarrow{Q}_{ki}}{u_{jk} - u_{ij}}\,.
\end{aligned} \tag{2.43}$$

We find analogous expressions at u_{ki} and u_{ij} by permuting indices. We also find identically zero expressions at u and u_{ijk}. Thus, the eight corner derivatives on the cube are

$$\frac{\partial S_{ijk}^{\triangle}}{\partial u} = 0\,, \qquad \frac{\partial S_{ijk}^{\triangle}}{\partial u_{ijk}} = 0\,, \tag{2.44a}$$

$$\frac{\partial S_{ijk}^{\triangle}}{\partial u_i} = \frac{Q_{ij}}{u_j - u_i} - \frac{Q_{ki}}{u_i - u_k}\,, \qquad \frac{\partial S_{ijk}^{\triangle}}{\partial u_{jk}} = \frac{\overleftrightarrow{Q}_{ij}}{u_{ki} - u_{jk}} - \frac{\overleftrightarrow{Q}_{ki}}{u_{jk} - u_{ij}}\,, \tag{2.44b}$$

$$\frac{\partial S_{ijk}^{\triangle}}{\partial u_j} = \frac{Q_{jk}}{u_k - u_j} - \frac{Q_{ij}}{u_j - u_i}\,, \qquad \frac{\partial S_{ijk}^{\triangle}}{\partial u_{ki}} = \frac{\overleftrightarrow{Q}_{jk}}{u_{ij} - u_{ki}} - \frac{\overleftrightarrow{Q}_{ij}}{u_{ki} - u_{jk}}\,, \tag{2.44c}$$

$$\frac{\partial S_{ijk}^{\triangle}}{\partial u_k} = \frac{Q_{ki}}{u_i - u_k} - \frac{Q_{jk}}{u_k - u_j}\,, \qquad \frac{\partial S_{ijk}^{\triangle}}{\partial u_{ij}} = \frac{\overleftrightarrow{Q}_{ki}}{u_{jk} - u_{ij}} - \frac{\overleftrightarrow{Q}_{jk}}{u_{ij} - u_{ki}}\,. \tag{2.44d}$$

Thus, we have that the H1 quad equations around the cube imply the multiform Euler-Lagrange equations:

$$Q_{ij}, Q_{jk}, Q_{ki}, \overleftrightarrow{Q}_{ij}, \overleftrightarrow{Q}_{jk}, \overleftrightarrow{Q}_{ki} = 0 \quad \Longrightarrow \quad \nabla S_{ijk}^{\triangle} = 0 \tag{2.45}$$

From (2.45) and (2.41), we indeed find a weak discrete Lagrangian 2-form on quadrilateral stencils satisfying Definition 1. On solutions to the multidimensionally-consistent H1 quad equations, the closure relation holds and the multiform Euler-Lagrange equations vanish:

$$Q_{ij}, Q_{jk}, Q_{ki}, \overleftrightarrow{Q}_{ij}, \overleftrightarrow{Q}_{jk}, \overleftrightarrow{Q}_{ki} = 0 \quad \Longrightarrow \quad \begin{cases} S_{ijk}^{\triangle} = 0 \\ \nabla S_{ijk}^{\triangle} = 0\,. \end{cases} \tag{2.46}$$

Note, that like for the conventional discrete Euler-Lagrange equations (2.23), the multiform Euler-Lagrange equations (2.44) are combinations of H1 quad equations. Thus, the constructions, which started the theory of Lagrangian multiforms and we call weak Lagrangian multiforms, successfully relates the multidimensionally-consistent H1 equations to a single variational principle. However, the H1 equations are sufficient but not necessary for the criticality of the action. We would like the equations of interest to directly follow from the variational principle. With this in mind, next we discuss developments in theory of Lagrangian multiforms since its conception.

2.7 Lagrangian Multiforms

A key feature of a variational principle is that it produces the equations of interest. For the conventional variational principle, the Euler-Lagrange equations are the variational equations (2.20). It is important that for a given Lagrangian multiform the variational principle produces the integrable equations of interest. This is developed in the discrete quadrilateral setting in [11] in the context of so-called pluri-Lagrangian systems where the closure relation is an optional property and not part of the definition. We reconcile their developments while still retaining the original notion of a Lagrangian multiform, where the closure relation plays a central role. For a weak Lagrangian multiform (Definition 1) one does not need to prove the closure relation on the full multiform Euler-Lagrange equations, but can instead supply simpler equations which imply both the multiform Euler-Lagrange equations and the closure relation. Whereas a Lagrangian multiform is when the closure relation holds on the full set of multiform Euler-Lagrange equations. Thus, we introduce the following stronger and full definition of a discrete Lagrangian multiform

Definition 2. *An action over quad surfaces* (2.26) *defines a* ***discrete Lagrangian 2-form*** *if it satisfies the following property. When the field* $u : \mathbb{Z}^3 \mapsto \mathbb{C}$ *satisfies the multiform Euler-Lagrange equations, the closure relation holds:*

$$\nabla S_{ijk} = 0 \quad \Longrightarrow \quad S_{ijk} = 0\,. \tag{2.47}$$

Recall that the multiform Euler-Lagrange equations here are simply the eight corner equations of the cube. Given an action over quad surfaces (2.26) which satisfies this definition, the associated *variational equations* are exactly the multiform Euler-Lagrange equations. Furthermore, Definition 2 satisfies Definition 1 if the equations $K_1(u), \cdots, K_M(u) = 0$ are taken to be full set of multiform Euler-Lagrange equations. We can also say that: criticality with respect to local variations of the field produces multiform Euler-Lagrange equations, then on solutions to those equations, the action is critical with respect to local variations of the surface:

$$\frac{\delta S[u,\sigma]}{\delta u} = 0 \quad \implies \quad \frac{\delta S[u,\sigma]}{\delta \sigma} = 0\,. \tag{2.48}$$

Note that we will interchangeably talk about an action over quad surfaces or equivalently a set of Lagrangians (defining an action), which satisfy the definition of a Lagrangian multiform. The reason it is natural to refer to the functions in the action over a quad surface as Lagrangians is that multiform Euler-Lagrange equations imply each of the conventional Euler-Lagrange equation via (2.34). However, one cannot consider any set of discrete Lagrangians in three planes ($\mathcal{L}_{ij}$, $\mathcal{L}_{jk}$ and $\mathcal{L}_{ki}$) and their combined set of conventional Euler-Lagrange equations. The multiform Euler-Lagrange equations for this set of Lagrangians imply but are not necessarily equivalent to the combined set of conventional Euler-Lagrange equations. The multiform Euler-Lagrange equations and the closure relations involve combinations of Lagrangians in different planes. Thus, a Lagrangian multiform defines a compatible set of Lagrangians.

The distinction between Definition 1 and Definition 2 is key to our major result in Chapter 3. The results of [11] imply that the weak discrete Lagrangian 2-form we outlined above for H1, with corner equations (2.44), actually satisfies the stronger Definition 2. However the variational equations are not the H1 equations, but weak combinations of them. Our major result in Chapter 3 is that we present novel Lagrangian multiforms (satisfying Definition 2), such that the H1 equation (and other quad equations) are variational equations, and arise directly from the multiform Euler-Lagrange equations. Before we delve into this construction, with our clearer picture of what a Lagrangian multiform is, we discuss some important features in the next two sections.

2.8 The Form Structure of Lagrangian Multiforms

We have emphasised the action of a Lagrangian multiform and under what conditions the action is critical. However, the name Lagrangian multiform, comes from an equivalent formulation as a form over a manifold satisfying certain properties. We can define a discrete 2-form as [49, 50, 32]

$$L[u] = \mathcal{L}_{ij}\,\mathrm{dn}_i \wedge \mathrm{dn}_j + \mathcal{L}_{jk}\,\mathrm{dn}_j \wedge \mathrm{dn}_k + \mathcal{L}_{ki}\,\mathrm{dn}_k \wedge \mathrm{dn}_i\,. \tag{2.49}$$

The action (2.26) over quad surfaces can be written in terms of this form:

$$S[u,\sigma] = \sum_{\sigma} L[u]\,. \tag{2.50}$$

By taking the exterior derivative of this form [49], the coefficient which arises is exactly the action around the elementary cube (2.29):

$$\mathrm{d}L\,[u] = S_{ijk}\,\mathrm{dn}_i \wedge \mathrm{dn}_j \wedge \mathrm{dn}_k\,. \tag{2.51}$$

Then we could take the following to be the definition of a Lagrangian multiform L (equivalent to Definition 2):

$$\delta\,\mathrm{d}L\,[u] = 0 \qquad \Longrightarrow \qquad \mathrm{d}L\,[u] = 0\,. \tag{2.52}$$

A Lagrangian multiform is *closed* on its multiform Euler-Lagrange equations. Alternatively, the exterior derivative vanishes on its own variational derivatives. There are some technicalities and subtleties when it comes to considering discrete and semi-discrete forms with the discrete differentials dn and discrete analogues of Stokes' theorem [49]. Thus, for this work, we continue to emphasise the action of Lagrangian multiforms and under what conditions they are critical.

2.9 The Closure Relation and Conservation Laws

At this point, we would like to say a bit more about the significance of the closure relation. The connection between conservation laws and the closure relation for discrete Lagrangian 2-forms is still quite unexplored. In the case of discrete Lagrangian 1-forms, the closure relation vanishing is equivalent to the existence of common integrals of motion [52].

We mention the case for continuous Lagrangian 1-forms, where the closure relation is said to be "the direct counterpart of the Poisson involutivity of Hamiltonians, the Liouville criterion for integrability" [16]. We can define a continuous Lagrangian 1-form action over a continuous path Γ, with Lagrangians $\mathcal{L}_{t_1}, \cdots, \mathcal{L}_{t_N}$,

$$S[u,\Gamma] = \int_{\Gamma} \left(\mathcal{L}_{t_1}\,\mathrm{d}t_1 + \cdots \mathcal{L}_{t_N}\,\mathrm{d}t_N\right)\,. \tag{2.53}$$

Suppose we have Hamiltonians $H_1, \cdots, H_N$ defined via Legendre transformations. In [16], the following formula is derived, where the first term is the continuous closure relation of the Lagrangian multiform, the second term involves products of multiform Euler-Lagrange equations and the third term is a Poisson bracket of the corresponding Hamiltonians:

$$\left(\frac{\partial \mathcal{L}_k}{\partial t_j} - \frac{\partial \mathcal{L}_j}{\partial t_k}\right) + \Upsilon_k^m P_{mn} \Upsilon_j^n = \{H_k,\, H_l\}_R\,. \tag{2.54}$$

This formula is off solutions, and in fact an example of a double zero expansion, which we discuss in Chapter 4. The important thing to take away is that on solutions to the multiform Euler-Lagrange equations, the closure relation holding is equivalent to the Poisson brackets vanishing.

In [48], continuous Lagrangian 2-forms and 3-forms are considered. The closure relation itself is considered as a conservation law, and Noether's theorem is then used to show that every variational symmetry of a Lagrangian leads to a Lagrangian multiform. Furthermore, in [26] a quantum path integral analogue of the closure relation was formulated.

The closure relation and its connection to conservation laws is a powerful and important feature of Lagrangian multiforms, which we will touch upon in later Chapters.

Chapter 3

Discrete Lagrangian 2-Forms associated with Quad Equations

This chapter is based on sections 1-4 of [44] by J. J. Richardson and M. Vermeeren.

3.1 Chapter Introduction

The theory of Lagrangian multiforms describes integrable systems through a variational principle. In recent years, the continuous version of the theory, describing integrable differential equations, has received a lot of attention [19, 16, 39, 41, 50, 55] and a semi-discrete version was formulated, extending the theory of differential-difference equations [49]. However, the origins of Lagrangian multiform theory lie in the fully discrete setting and concern integrable *quad equations*, in particular those of the Adler-Bobenko-Suris (ABS) list [1]. This is a classification of scalar, multiaffine, multidimensionally consistent difference equations on a quadrilateral stencil (under the additional assumption that there exists a compatible "tetrahedron equation"). The observation that all equations on this list have a variational structure led to the introduction of Lagrangian multiforms in [28].

Lagrangian multiform theory describes a set of compatible equations (multidimensionally consistent difference equations or a hierarchy of differential equations) through a single variational principle involving a difference or differential form. This d-form is defined on the space of all independent variables of the set of equations. For any d-dimensional surface within the space of independent variables, we can consider an action, defined by the integral/sum of the Lagrangian d-form over the surface. In Lagrangian multiform theory (and in the closely related theory of *pluri-Lagrangian systems*), the variational principle requires these integrals to be critical no matter which surface is chosen.

The original Lagrangians of the ABS list [1] can be interpreted as defined on quadrilateral stencils. Whereas in [28], Lagrangians on triangle stencils were introduced and generalised to discrete Lagrangian 2-forms. These have been studied in [10, 11, 13, 28, 61] and produce a slightly weaker set of equations: the quad equations are sufficient conditions for the variational principle, but not necessary conditions. It is emphasised in [11] that: *"quad-equations are not variational; rather, discrete Euler-Lagrange equations for the two-forms given in [10, 28] are consequences of quad-equations"*. In [30], attempts were made to resolve this. However, in this chapter the literal interpretation of the first part of the statement is shown to be not true: quad equations *are* variational. We generalise the original Lagrangians on quadrilateral stencils to Lagrangian 2-forms and show that the generalised Euler-Lagrange equations produce the quad equations exactly.

The Lagrangian 2-form considered in the existing literature on quad equations produces Euler-Lagrange equations that are equivalent to a system of two octahedron equations [13]. This cannot be equivalent to the set of six quad equations on a cube, any four of which are independent. Furthermore, the two octahedron equations are independent of the two tetrahedron equations. In fact, the combination of these two sets, two octahedron equations and two tetrahedron equations, is equivalent to the full set of quad equations.

The structure of this chapter is as follows. In Section 3.2 we review the structure of the ABS equations and the assumptions behind their classification. In Sections 3.3.1–3.3.3 we present two new Lagrangian multiforms for which the variational principle produces exactly the set of quad equations, as well as a third multiform that produces only the tetrahedron equations (the latter has appeared in a slightly different context in [10]). In Section 3.3.4 we contrast this with the well-known Lagrangian multiform. In Section 3.3.5 we discuss the closure property of each of these 2-forms. In Section 3.4, we show that the relations between quad equations, tetrahedron equations, and octahedron equations follow from relations between the different Lagrangian 2-forms.

3.2 Quad Equations

3.2.1 The ABS Classification

The ABS list [1] is a classification of integrable difference equations that satisfy the following conditions:

- They are quad equations: they depend on a square stencil and on two parameters associated to the two lattice directions:

$$Q(u, u_i, u_j, u_{ij}, \alpha_i, \alpha_j) = 0\,. \tag{3.1}$$

- They are symmetric: Equation (3.1) is invariant under the symmetries of the square,

$$Q(u, u_i, u_j, u_{ij}, \alpha_i, \alpha_j) = \pm Q(u, u_j, u_i, u_{ij}, \alpha_j, \alpha_i) = \pm Q(u_i, u, u_{ij}, u_j, \alpha_i, \alpha_j)\,. \tag{3.2}$$

  Note that these two transformations of the tuple (u, u_i, u_j, u_{ij}) generate the symmetry group D_4, and that the α_i, α_j are interchanged accordingly.

- They are multiaffine: the function Q is a polynomial of degree one in each of the u, u_i, u_j, u_{ij} (but can be of higher total degree). This guarantees that we can solve the equations for any of the four variables, given the other three.

- They are three-dimensionally consistent: given initial values u, u_i, u_j, u_k, use the equations

$$Q_{ij} := Q(u, u_i, u_j, u_{ij}, \alpha_i, \alpha_j) = 0 \tag{3.3a}$$

$$Q_{jk} := Q(u, u_j, u_k, u_{jk}, \alpha_j, \alpha_k) = 0 \tag{3.3b}$$

$$Q_{ki} := Q(u, u_k, u_i, u_{ki}, \alpha_k, \alpha_i) = 0 \tag{3.3c}$$

  to determine u_{ij}, u_{jk}, u_{ki}; then u_{ijk} should be uniquely determined by

$$\overleftrightarrow{Q}_{ij} := Q(u_{ijk}, u_{jk}, u_{ki}, u_k, \alpha_i, \alpha_j) = 0 \tag{3.3d}$$

$$\overleftrightarrow{Q}_{jk} := Q(u_{ijk}, u_{ki}, u_{ij}, u_i, \alpha_j, \alpha_k) = 0 \tag{3.3e}$$

$$\overleftrightarrow{Q}_{ki} := Q(u_{ijk}, u_{ij}, u_{jk}, u_j, \alpha_k, \alpha_i) = 0\,. \tag{3.3f}$$

  This is illustrated in Figure 3.1.

- They satisfy the tetrahedron property: the value obtained for u_{ijk} in the computation above is independent of u, depending only on u_i, u_j, u_k.

Remark 3. *Note that, regardless of which signs occur in Equation* (3.2)*, we have that*

$$\overleftrightarrow{Q}_{ij} := Q(u_{ijk}, u_{jk}, u_{ki}, u_k, \alpha_i, \alpha_j) = Q(u_k, u_{ki}, u_{jk}, u_{ijk}, \alpha_i, \alpha_j),$$

because the last equality is obtained by applying both symmetries of the square twice. We will use the notation $\overleftrightarrow{(\cdots)}$ *more generally to denote the* point inversion $u \leftrightarrow u_{ijk}$, $u_i \leftrightarrow u_{jk}$,

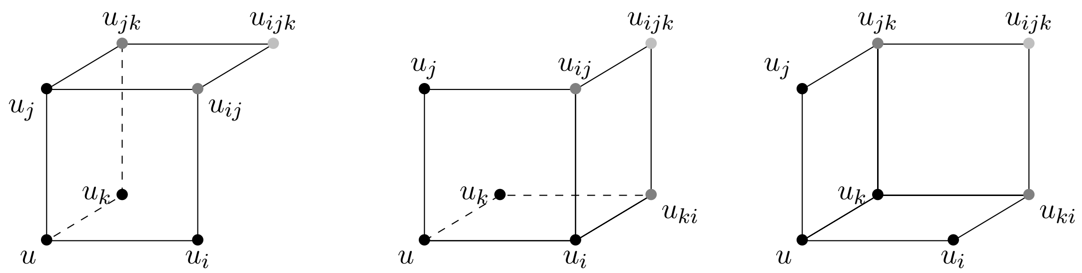


Figure 3.1: Multidimensional consistency demands that all three of these routes to calculate u_{ijk} from initial values u, u_i, u_j, u_k produce the same value

etc.

It turns out that each equation $Q(u, u_i, u_j, u_{ij}, \alpha_i, \alpha_j) = 0$ of the ABS list is equivalent to an equation in *three-leg form*:

$$\mathcal{Q}_{ij}^{(u)} := \psi(u, u_i, \alpha_i) - \psi(u, u_j, \alpha_j) - \phi(u, u_{ij}, \alpha_i - \alpha_j) = 0\,. \tag{3.4}$$

Here $\psi : (u, u_i, \alpha_i) \mapsto \mathbb{C}$ and $\phi : (u, u_{ij}, \alpha_i - \alpha_j) \mapsto \mathbb{C}$. This gives us a second, complementary, perspective on the ABS equations: we can either study the multiaffine expressions Q_{ij} or the three-leg forms $\mathcal{Q}_{ij}^{(u)}$. We use calligraphic letters to denote leg forms. The D_4-symmetry imposed on the quad equations has significant implications for the three-leg form. From the reflection symmetry it follows that $\phi(u, u_{ij}, \alpha_i - \alpha_j) = -\phi(u, u_{ij}, \alpha_j - \alpha_i)$. Applying rotational symmetry to Equation (3.4) we get equivalent three-leg forms based at the other three vertices of the square:

$$\mathcal{Q}_{ij}^{(u_i)} := \psi(u_i, u_{ij}, \alpha_j) - \psi(u_i, u, \alpha_i) - \phi(u_i, u_j, \alpha_j - \alpha_i) = 0\,, \tag{3.5a}$$

$$\mathcal{Q}_{ij}^{(u_{ij})} := \psi(u_{ij}, u_j, \alpha_i) - \psi(u_{ij}, u_i, \alpha_j) - \phi(u_{ij}, u, \alpha_i - \alpha_j) = 0\,, \tag{3.5b}$$

$$\mathcal{Q}_{ij}^{(u_j)} := \psi(u_j, u, \alpha_j) - \psi(u_j, u_{ij}, \alpha_i) - \phi(u_j, u_i, \alpha_j - \alpha_i) = 0\,. \tag{3.5c}$$

The three-leg forms based at different vertices are illustrated in Figure 3.2. We emphasise that

$$Q_{ij} = 0 \iff \mathcal{Q}_{ij}^{(u)} = 0 \iff \mathcal{Q}_{ij}^{(u_i)} = 0 \iff \mathcal{Q}_{ij}^{(u_j)} = 0 \iff \mathcal{Q}_{ij}^{(u_{ij})} = 0\,. \tag{3.6}$$

On the faces of a cube adjacent to a fixed vertex, the three-leg forms based at that vertex naturally combine to form an equation on a tetrahedral stencil, illustrated in Figure 3.3, for example:

$$\mathcal{T}^{(u)} := \phi(u, u_{ij}, \alpha_i - \alpha_j) + \phi(u, u_{jk}, \alpha_j - \alpha_k) + \phi(u, u_{ki}, \alpha_k - \alpha_i) \tag{3.7a}$$

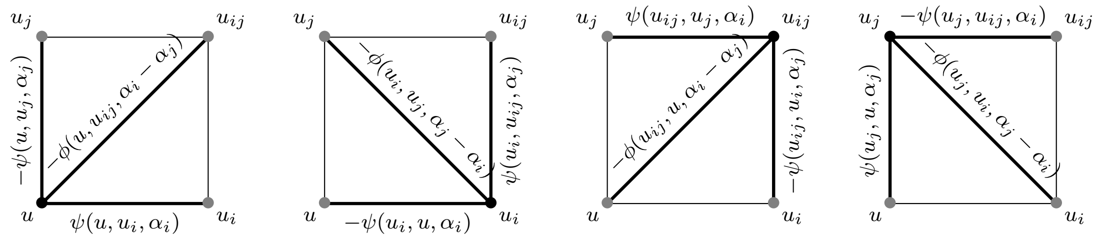


Figure 3.2: Graphical representation of the four orientations of a three-leg form.

$$= -\mathcal{Q}_{ij}^{(u)} - \mathcal{Q}_{jk}^{(u)} - \mathcal{Q}_{ki}^{(u)} = 0\,. \tag{3.7b}$$

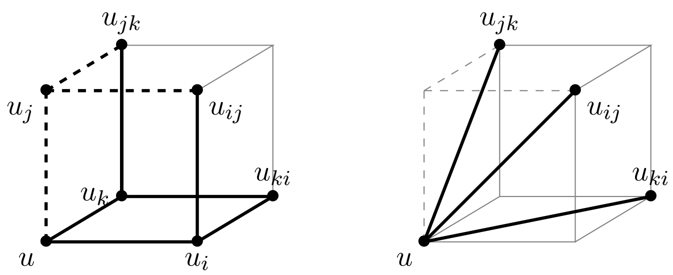


Figure 3.3: Tetrahedron equation from three quad equations

To relate the three-leg tetrahedron equation (3.7a) with a multiaffine equation, we will rely on some relations between the equations of the ABS list (which is given in the Appendix A). The ABS list contains three types of equation, labeled Q, H, and A. Those of type Q have the property that their short and long leg functions are the same, $\phi = \psi$. Each equation of type H and A shares its long leg function ϕ with an equation of type Q, but has a different short leg function ψ. Thus the quad equations of type H and type A have the same the tetrahedron equation as an equation of type Q. In fact, a multiaffine tetrahedron equation $T = 0$, equivalent to $\mathcal{T}^{(u)} = 0$, can be defined in terms of a quad polynomial of type Q, evaluated on a tetrahedron stencil:

$$T(u, u_{ij}, u_{jk}, u_{ki}, \alpha_i, \alpha_j, \alpha_k) = Q^{(\text{type Q})}(u, u_{ij}, u_{jk}, u_{ki}, \alpha_i - \alpha_j, -\alpha_j + \alpha_k). \tag{3.8}$$

Here $Q^{(\text{type Q})}$ represents the multiaffine polynomial corresponding to a quad equation of type Q. Consistent with the three-leg form (3.7a), this multiaffine polynomial is symmetric under permutation of indices.

Now we can apply the rotational symmetry of the quad equation of type Q and obtain equivalent three-leg forms of tetrahedron equation based at the other vertices:

$$\mathcal{T}^{(u_{ij})} := \phi(u_{ij}, u_{ki}, \alpha_j - \alpha_k) + \phi(u_{ij}, u, \alpha_i - \alpha_j) + \phi(u_{ij}, u_{jk}, \alpha_i - \alpha_k) = 0\,, \tag{3.9a}$$

$$\mathcal{T}^{(u_{ki})} := \phi(u_{ki}, u_{jk}, \alpha_i - \alpha_j) + \phi(u_{ki}, u_{ij}, \alpha_j - \alpha_k) + \phi(u_{ki}, u, \alpha_k - \alpha_i) = 0\,, \qquad (3.9b)$$

$$\mathcal{T}^{(u_{jk})} := \phi(u_{jk}, u, \alpha_j - \alpha_k) + \phi(u_{jk}, u_{ki}, \alpha_i - \alpha_j) + \phi(u_{jk}, u_{ij}, \alpha_i - \alpha_k) = 0\,. \qquad (3.9c)$$

Altogether we have that

$$T = 0 \iff \mathcal{T}^{(u)} = 0 \iff \mathcal{T}^{(u_{ij})} = 0 \iff \mathcal{T}^{(u_{jk})} = 0 \iff \mathcal{T}^{(u_{ki})} = 0\,. \qquad (3.10)$$

The same argument applies to the other four variables, so we can derive a second tetrahedron equation in three-leg form, which is related to the first by point inversion:

$$\overset{\leftrightarrow}{\mathcal{T}}{}^{(u_{ijk})} := \phi(u_{ijk}, u_k, \alpha_i - \alpha_j) + \phi(u_{ijk}, u_i, \alpha_j - \alpha_k) + \phi(u_{ijk}, u_j, \alpha_k - \alpha_i) = 0\,. \qquad (3.11)$$

Similarly, we define the three-leg expressions $\overset{\leftrightarrow}{\mathcal{T}}{}^{(u_i)}$, $\overset{\leftrightarrow}{\mathcal{T}}{}^{(u_j)}$, $\overset{\leftrightarrow}{\mathcal{T}}{}^{(u_k)}$, based at the remaining three vertices. Considering also the polynomial $\overset{\leftrightarrow}{T} = T(u_{ijk}, u_k, u_i, u_j, \alpha_i, \alpha_j, \alpha_k)$, we have the following equivalent equations:

$$\overset{\leftrightarrow}{T} = 0 \iff \overset{\leftrightarrow}{\mathcal{T}}{}^{(u_{ijk})} = 0 \iff \overset{\leftrightarrow}{\mathcal{T}}{}^{(u_k)} = 0 \iff \overset{\leftrightarrow}{\mathcal{T}}{}^{(u_i)} = 0 \iff \overset{\leftrightarrow}{\mathcal{T}}{}^{(u_j)} = 0\,. \qquad (3.12)$$

Example (H1), part 1. *One of the simplest equations on the ABS list is the lattice potential KdV equation, labeled* H1, *for which*

$$Q_{ij} = (u_i - u_j)(u - u_{ij}) - \alpha_i + \alpha_j\,. \qquad (3.13)$$

In this case, the leg functions are given by

$$\psi(u, u_i, \alpha_i) = u_i \qquad \text{and} \qquad \phi(u, u_{ij}, \alpha_i - \alpha_j) = \frac{\alpha_i - \alpha_j}{u - u_{ij}}\,. \qquad (3.14)$$

The quad equation $Q_{ij} = 0$ is equivalent to the three-leg equation $\mathcal{Q}^{(u)}_{ij} = 0$, where

$$\begin{aligned}\mathcal{Q}^{(u)}_{ij} &= \psi(u, u_i, \alpha_i) - \psi(u, u_j, \alpha_j) - \phi(u, u_{ij}, \alpha_i - \alpha_j)\\ &= u_i - u_j - \frac{\alpha_i - \alpha_j}{u - u_{ij}} = \frac{Q_{ij}}{u - u_{ij}}\,. \end{aligned} \qquad (3.15)$$

The three-leg form $\mathcal{T}^{(u)} = 0$ of the tetrahedron equation $T = 0$ is

$$\begin{aligned}\mathcal{T}^{(u)} &= \phi(u, u_{ij}, \alpha_i - \alpha_j) + \phi(u, u_{jk}, \alpha_j - \alpha_k) + \phi(u, u_{ki}, \alpha_k - \alpha_i)\\ &= \frac{\alpha_i - \alpha_j}{u - u_{ij}} + \frac{\alpha_j - \alpha_k}{u - u_{jk}} + \frac{\alpha_k - \alpha_i}{u - u_{ki}}\end{aligned}$$

$$= \frac{T}{(u-u_{ij})(u-u_{jk})(u-u_{jk})} \,. \tag{3.16}$$

The tetrahedron equation $T = 0$ can also be expressed in multiaffine form as

$$\begin{aligned} &T(u, u_{ij}, u_{jk}, u_{ki}, \alpha_i, \alpha_j, \alpha_k) \\ &\quad = (\alpha_i - \alpha_j)(uu_{ij} + u_{jk}u_{ki}) + (\alpha_j - \alpha_k)(uu_{jk} + u_{ij}u_{ki}) + (\alpha_k - \alpha_i)(uu_{ki} + u_{ij}u_{jk}) \,, \end{aligned} \tag{3.17}$$

which is exactly the multiaffine polynomial of ABS equation $Q1_{\delta=0}$ (see Appendix A).

3.2.2 Discrete Lagrangians on Quadrilateral Stencils

Already in the original ABS paper [1], Lagrangian functions (in the traditional sense considered in Section 2.3) were constructed for all equations from the ABS list. This construction is based on the observation that, after a suitable transformation of the variable u, there exist functions L and Λ such that the leg functions ψ and ϕ can be expressed as

$$\psi(x, y, \alpha) = \frac{\partial}{\partial x} L(x, y, \alpha) \,, \tag{3.18a}$$

$$\phi(x, y, \alpha - \beta) = \frac{\partial}{\partial x} \Lambda(x, y, \alpha - \beta) \,, \tag{3.18b}$$

where L and Λ have the following symmetries:

$$L(x, y, \alpha) = L(y, x, \alpha) \,, \tag{3.19a}$$

$$\Lambda(x, y, \alpha - \beta) = \Lambda(y, x, \alpha - \beta) \,, \tag{3.19b}$$

$$\Lambda(x, y, \alpha - \beta) = -\Lambda(x, y, \beta - \alpha) \,. \tag{3.19c}$$

Note that this implies that

$$\frac{\partial}{\partial y} L(x, y, \alpha) = \psi(y, x, \alpha) \,, \tag{3.20a}$$

$$\frac{\partial}{\partial y} \Lambda(x, y, \alpha - \beta) = \phi(y, x, \alpha - \beta) \,. \tag{3.20b}$$

The functions L and Λ can be combined into a 4-point Lagrangian [1]

$$\mathcal{L}^{\swarrow}_{ij}(u, u_i, u_j, u_{ij}, \alpha_i, \alpha_j) := L(u, u_i, \alpha_i) - L(u, u_j, \alpha_j) - \Lambda(u, u_{ij}, \alpha_i - \alpha_j) \,. \tag{3.21}$$

which we will refer to as the *trident Lagrangian*, inspired by the three-leg structure

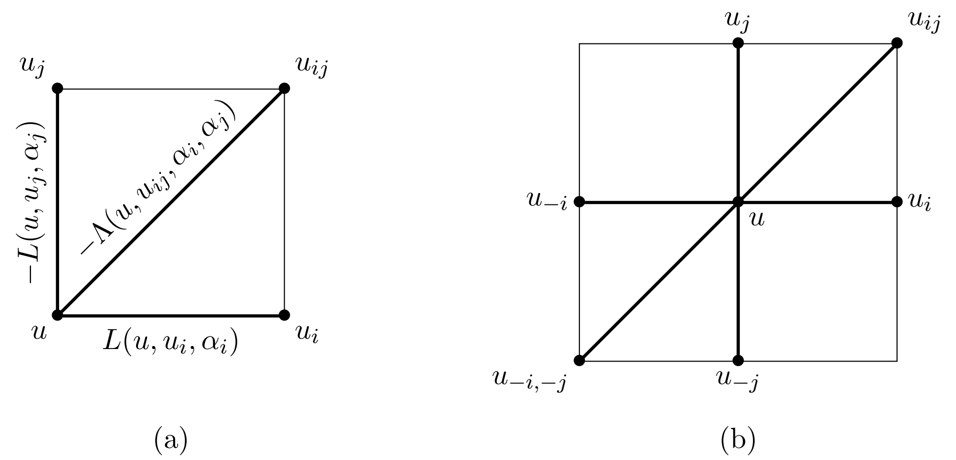


Figure 3.4: (a) The stencil of the trident Lagrangian. (b) The discrete Euler-Lagrange equation involves three-leg structures on two squares.

depicted Figure 3.4(a). The traditional Euler-Lagrange equation of $\mathcal{L}_\swarrow$, depicted in Figure 3.4(b), is

$$\begin{aligned}
&\frac{\partial}{\partial u}\Big(\mathcal{L}^{\swarrow}_{ij}(u, u_i, u_j, u_{ij}, \alpha_i, \alpha_j) + \mathcal{L}^{\swarrow}_{ij}(u_{-i}, u, u_{-i,j}, u_j, \alpha_i, \alpha_j) + \mathcal{L}^{\swarrow}_{ij}(u_{-j}, u_{i,-j}, u, u_{i,-j}, \alpha_i, \alpha_j) \\
&\qquad + \mathcal{L}^{\swarrow}_{ij}(u_{-i,-j}, u_{-j}, u_{-i}, u, \alpha_i, \alpha_j)\Big) \\
&= \psi(u, u_i, \alpha_i) - \psi(u, u_j, \alpha_j) - \phi(u, u_{ij}, \alpha_i - \alpha_j) \\
&\qquad + \psi(u, u_{-i}, \alpha_i) - \psi(u, u_{-j}, \alpha_j) - \phi(u, u_{-i-j}, \alpha_i - \alpha_j) = 0\,,
\end{aligned} \tag{3.22}$$

where the subscript $-i$ denotes a shift in the negative direction along the i-th coordinate axis. This Euler-Lagrange equation is the sum of two shifted instances of the quad equation (in the form (3.4) and (3.5b) respectively).

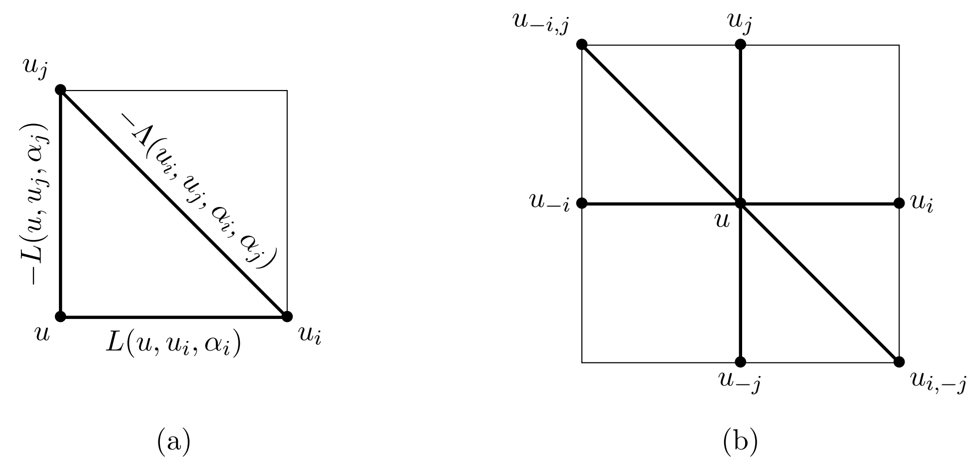


Figure 3.5: (a) The stencil of the triangle Lagrangian. (b) The discrete Euler-Lagrange equation involves three-leg structures on two squares.

The functions L and Λ can be combined into a 3-point Lagrangian [28]

$$\mathcal{L}_{ij}^{\triangle}(u,u_i,u_j,\alpha_i,\alpha_j) := L(u,u_i,\alpha_i) - L(u,u_j,\alpha_j) - \Lambda(u_i,u_j,\alpha_i-\alpha_j)\,, \tag{3.23}$$

which we will refer to as the *triangle Lagrangian*, inspired by the three-leg structure depicted Figure 3.5(a). The traditional Euler-Lagrange equation of $\mathcal{L}_{\triangle}$, depicted in Figure 3.5(b), is

$$\begin{aligned}
&\frac{\partial}{\partial u}\left(\mathcal{L}_{ij}^{\triangle}(u,u_i,u_j,\alpha_i,\alpha_j) + \mathcal{L}_{ij}^{\triangle}(u_{-i},u,u_{-i,j},\alpha_i,\alpha_j) + \mathcal{L}_{ij}^{\triangle}(u_{-j},u_{i,-j},u,\alpha_i,\alpha_j)\right)\\
&= \psi(u,u_i,\alpha_i) - \psi(u,u_{-j},\alpha_j) - \phi(u,u_{i,-j},\alpha_i-\alpha_j)\\
&\qquad + \psi(u,u_{-i},\alpha_i) - \psi(u,u_j,\alpha_j) - \phi(u,u_{-i,j},\alpha_i-\alpha_j) = 0\,,
\end{aligned} \tag{3.24}$$

where the subscript $-i$ denotes a shift in the negative direction along the i-th coordinate axis. This Euler-Lagrange equation is the sum of two shifted instances of the quad equation (in the form (3.5a) and (3.5c) respectively).

Both of these Lagrangians, for the entire ABS list, relate to the conventional action (2.18) over $\mathbb{Z}^2$ discussed in Section 2.3. In both cases, if the field u satisfies the corresponding quad equation in the plane, this implies criticality of the action with respect to local variations of field:

$$Q_{ij} = 0 \implies \frac{\partial}{\partial u}\left(\mathcal{L}_{ij} + T_j^{-1}\mathcal{L}_{ij} + T_i^{-1}\mathcal{L}_{ij} + T_j^{-1}T_i^{-1}\mathcal{L}_{ij}\right) = 0 \implies \frac{\delta S[u]}{\delta u} = 0\,. \tag{3.25}$$

In the upcoming sections we will generalise from conventional Lagrangians to Lagrangian multiforms. We will present a Lagrangian multiform variational principle which necessarily implies the quad equation, and furthermore necessarily implies the entire multidimensionally-consistent set of quad equations.

3.3 Lagrangian Multiforms

In Section 2.6 we discussed the earliest construction of Lagrangian multiforms from [28], and defined them as weak discrete Lagrangian 2-forms (Definition 1). Our major contribution is to construct discrete Lagrangian 2-forms satisfying Definition 2 (2.47) for the quad equations of the ABS list.

Similar to [28], the key idea is to interpret a Lagrangian $\mathcal{L}(u,u_i,u_j,u_{ij},\alpha_i,\alpha_j)$, which is skew-symmetric under the swap of indices $i \leftrightarrow j$, as a discrete 2-form on a higher-dimensional quadrilateral lattice. In terms of Definition 2, we have an action over oriented quad surfaces σ, where $\mathcal{L}_{ij}$, $\mathcal{L}_{jk}$ and $\mathcal{L}_{ki}$ are all of the same form, namely

$$S[u,\sigma] = \sum_{\sigma_{ij}(\mathbf{n})\in\sigma} \mathcal{L}_{ij}\,. \tag{3.26}$$

For a given Lagrangian function $\mathcal{L}_{ij}(u,u_i,u_j,u_{ij},\alpha_i,\alpha_j)$, under what conditions is this 2-form action critical?

The Lagrangian multiform variational principle is the following: criticality of the action with respect to local variations of the field produces multiform Euler-Lagrange equations; then on solutions to those equations, the action is critical with respect to local variations of the surface. It is equivalent to consider the criticality on the closed quad surface of the elementary cube

$$S_{ijk} := S[u,\text{cube}] = (T_k\mathcal{L}_{ij} - \mathcal{L}_{ij}) + (T_i\mathcal{L}_{jk} - \mathcal{L}_{jk}) + (T_j\mathcal{L}_{ki} - \mathcal{L}_{ki})\,. \tag{3.27}$$

Thus, for a given Lagrangian function $\mathcal{L}(u,u_i,u_j,u_{ij},\alpha_i,\alpha_j)$, we have a discrete Lagrangian 2-form satisfying Definition 2 if we show that

$$\left.\begin{array}{ll} \dfrac{\partial S_{ijk}}{\partial u} = 0 & \dfrac{\partial S_{ijk}}{\partial u_{ijk}} = 0 \\ \dfrac{\partial S_{ijk}}{\partial u_i} = 0 & \dfrac{\partial S_{ijk}}{\partial u_{jk}} = 0 \\ \dfrac{\partial S_{ijk}}{\partial u_j} = 0 & \dfrac{\partial S_{ijk}}{\partial u_{ki}} = 0 \\ \dfrac{\partial S_{ijk}}{\partial u_k} = 0 & \dfrac{\partial S_{ijk}}{\partial u_{ij}} = 0 \end{array}\right\} \quad \Longrightarrow \quad S_{ijk} = 0\,. \tag{3.28}$$

We require that on solutions to the corner equations, the closure relation holds.

In the next four subsections we will introduce three additional Lagrangian 2-forms and contrast them to the known Lagrangian 2-form. Initially, we will only study their corner equations. Then, in Subsection 3.3.5, we will prove that each of the Lagrangian 2-forms satisfies the definition, by showing that the closure relation holds on the respective set of corner equations.

3.3.1 Trident 2-Form: Quad Equations are Variational

One of our main results is that the quad equations of the ABS list are variational. We will show that the original 4-point Lagrangian construction in [1] generalises to a Lagrangian 2-form with corner equations that produce the quad equations directly. We recall the

Lagrangian (3.21) and interpret it as the trident 2-form

$$\mathcal{L}^{\not\swarrow}_{ij}(u,u_i,u_j,u_{ij},\alpha_i,\alpha_j) := L(u,u_i,\alpha_i) - L(u,u_j,\alpha_j) - \Lambda(u,u_{ij},\alpha_i-\alpha_j). \tag{3.29}$$

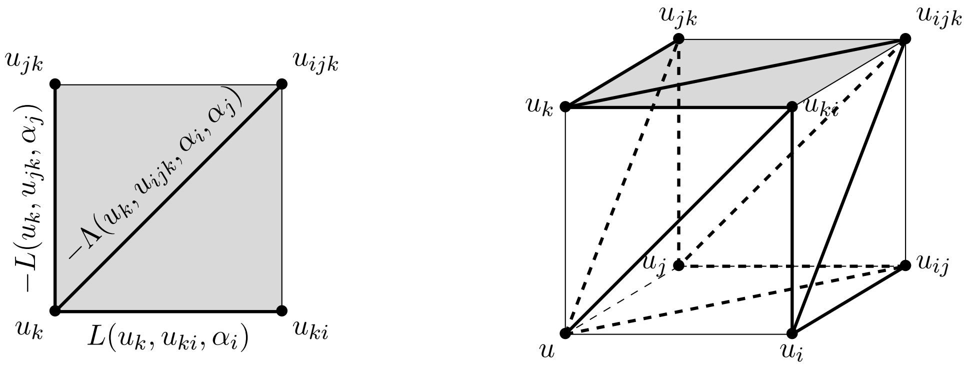


Figure 3.6: (a) The leg structure of a single Lagrangian $\mathcal{L}^{\not\swarrow}_{ij}(u_k,u_{ki},u_{jk},u_{ijk},\alpha_i,\alpha_j)$. (b) The leg structure for the action on an elementary cube of the trident 2-form $\mathcal{L}_{\not\swarrow}$.

The action over an elementary cube of the trident 2-form can be written as (see Figure 3.6(b))

$$\begin{aligned} S^{\not\swarrow}_{ijk} &= \mathcal{L}^{\not\swarrow}_{ij}(u_k,u_{ki},u_{jk},u_{ijk},\alpha_i,\alpha_j) - \mathcal{L}^{\not\swarrow}_{ij}(u,u_i,u_j,u_{ij},\alpha_i,\alpha_j) + \circlearrowleft_{ijk} \\ &= L(u_k,u_{ki},\alpha_i) - L(u_k,u_{jk},\alpha_j) - \Lambda(u_k,u_{ijk},\alpha_i-\alpha_j) \\ &\qquad - L(u,u_i,\alpha_i) + L(u,u_j,\alpha_j) + \Lambda(u,u_{ij},\alpha_i-\alpha_j) + \circlearrowleft_{ijk} \\ &= \left(L(u_i,u_{ij},\alpha_j) + \Lambda(u,u_{ij},\alpha_i-\alpha_j)\right) - \left(L(u_{jk},u_k,\alpha_j) + \Lambda(u_k,u_{ijk},\alpha_i-\alpha_j)\right) + \circlearrowleft_{ijk} \\ &= L(u_i,u_{ij},\alpha_j) + \Lambda(u,u_{ij},\alpha_i-\alpha_j) - \overline{(\cdots)} + \circlearrowleft_{ijk}\,, \end{aligned} \tag{3.30}$$

where the notation $-\overline{(\cdots)}$ represents subtracting an inverted copy of the preceding expression, and $+\circlearrowleft_{ijk}$ indicates the addition of terms obtained by cyclic permutation of (i,j,k). Thus, Equation (3.30) is manifestly symmetric under cyclic permutations and skew-symmetric under point inversion.

Now we consider the corner expression at u_{ij}

$$\begin{aligned} \frac{\partial S^{\not\swarrow}_{ijk}}{\partial u_{ij}} &= \frac{\partial}{\partial u_{ij}}\left(L(u_i,u_{ij},\alpha_j) - L(u_{ij},u_j,\alpha_i) + \Lambda(u,u_{ij},\alpha_i-\alpha_j)\right) \\ &= \psi(u_{ij},u_i,\alpha_j) - \psi(u_{ij},u_j,\alpha_i) + \phi(u_{ij},u,\alpha_i-\alpha_j) \\ &= -\mathcal{Q}^{(u_{ij})}_{ij}\,. \end{aligned} \tag{3.31}$$

This gives us exactly the original quad equation in three-leg form (3.5b). The expressions at u_{jk} and u_{ki} are analogous and follow from permutation of indices. At the u_k corner we find a similar expression, $\overleftrightarrow{\mathcal{Q}}^{(u_k)}_{ij}$, which is what we would expect from the skew-symmetry under point inversion. The expressions at u_i and u_j are analogous and follow from permutation of indices.

Let us consider the u corner expression produced by this 2-form

$$\begin{aligned}\frac{\partial S^{\swarrow}_{ijk}}{\partial u} &= \frac{\partial}{\partial u}\left(\Lambda(u, u_{ij}, \alpha_i - \alpha_j) + \Lambda(u, u_{jk}, \alpha_i - \alpha_j) + \Lambda(u, u_{ki}, \alpha_i - \alpha_j)\right)\\ &= \phi(u, u_{ij}, \alpha_i - \alpha_j) + \phi(u, u_{jk}, \alpha_i - \alpha_j) + \phi(u, u_{ki}, \alpha_i - \alpha_j)\\ &= \mathcal{T}^{(u)}\,. \end{aligned} \tag{3.32}$$

We find exactly the tetrahedron equation (3.7a) in three-leg form. At the u_{ijk} corner we find the same but negated and inverted $-\overleftrightarrow{\mathcal{T}}^{(u_{ijk})}$. Altogether we conclude the following:

Theorem 4. *The corner equations of the trident 2-form* (3.21) *are the quad equations and tetrahedron equations in three-leg form:*

$$\begin{aligned}
&\frac{\partial S^{\swarrow}_{ijk}}{\partial u} = \mathcal{T}^{(u)} = 0\,, &&\frac{\partial S^{\swarrow}_{ijk}}{\partial u_{ijk}} = -\overleftrightarrow{\mathcal{T}}^{(u_{ijk})} = 0\,,\\
&\frac{\partial S^{\swarrow}_{ijk}}{\partial u_i} = \overleftrightarrow{\mathcal{Q}}^{(u_i)}_{jk} = 0\,, &&\frac{\partial S^{\swarrow}_{ijk}}{\partial u_{jk}} = -\mathcal{Q}^{(u_{jk})}_{jk} = 0\,,\\
&\frac{\partial S^{\swarrow}_{ijk}}{\partial u_j} = \overleftrightarrow{\mathcal{Q}}^{(u_j)}_{ki} = 0\,, &&\frac{\partial S^{\swarrow}_{ijk}}{\partial u_{ki}} = -\mathcal{Q}^{(u_{ki})}_{ki} = 0\,,\\
&\frac{\partial S^{\swarrow}_{ijk}}{\partial u_k} = \overleftrightarrow{\mathcal{Q}}^{(u_k)}_{ij} = 0\,, &&\frac{\partial S^{\swarrow}_{ijk}}{\partial u_{ij}} = -\mathcal{Q}^{(u_{ij})}_{ij} = 0\,.
\end{aligned} \tag{3.33}$$

Proof. Take derivatives of (3.30) and recognise the three-leg terms (3.5b) (and their point inversions), as well as the three-leg terms (3.7a) (and their point inversions). □

Example (H1), part 2. *For H1 we have*

$$L(u, u_i, \alpha_i) = u u_i\,, \tag{3.34a}$$

$$\Lambda(u_i, u_j, \alpha_i - \alpha_j) = (\alpha_i - \alpha_j)\log(u_i - u_j)\,. \tag{3.34b}$$

Note in this form the symmetry $\Lambda(u_j, u_i, \alpha_i - \alpha_j) = \Lambda(u_i, u_j, \alpha_i - \alpha_j)$ is up to an addition of a complex constant. We comment on options to make this strictly symmetric in Appendix A.

The trident Lagrangian is

$$\mathcal{L}^{\swarrow}_{ij}(u, u_i, u_j, u_{ij}, \alpha_i - \alpha_j) = u u_i - u u_j - (\alpha_i - \alpha_j)\log(u - u_{ij}). \tag{3.35}$$

Its action around the elementary cube of the 2-form is

$$\begin{aligned} S^{\swarrow}_{ijk} &= u_k u_{ki} - u_k u_{jk} - (\alpha_i - \alpha_j)\log(u_k - u_{ijk}) - u u_i + u u_j + (\alpha_i - \alpha_j)\log(u - u_{ij}) \\ &\qquad + \circlearrowleft_{ijk} \\ &= (u_i u_{ij} + (\alpha_i - \alpha_j)\log(u - u_{ij})) - (u_{jk} u_k + (\alpha_i - \alpha_j)\log(u_{ijk} - u_k)) + \circlearrowleft_{ijk} \\ &= u_i u_{ij} + (\alpha_i - \alpha_j)\log(u - u_{ij}) - \overline{(\cdots)} + \circlearrowleft_{ijk}\,. \end{aligned} \tag{3.36}$$

This leads to 8 corner expressions. For example, we have

$$\begin{aligned} \frac{\partial S^{\swarrow}_{ijk}}{\partial u} &= \frac{\alpha_i - \alpha_j}{u - u_{ij}} + \frac{\alpha_j - \alpha_k}{u - u_{jk}} + \frac{\alpha_k - \alpha_i}{u - u_{ki}} \\ &= \mathcal{T}^{(u)} = \frac{T}{(u - u_{ij})(u - u_{jk})(u - u_{ki})}\,. \end{aligned} \tag{3.37}$$

In addition, we have

$$\begin{aligned} \frac{\partial S^{\swarrow}_{ijk}}{\partial u_{ij}} &= u_i - u_j - \frac{\alpha_i - \alpha_j}{u - u_{ij}} \\ &= -\mathcal{Q}^{(u_{ij})}_{ij} = \frac{Q_{ij}}{u - u_{ij}}. \end{aligned} \tag{3.38}$$

Similarly, we find

$$\frac{\partial S^{\swarrow}_{ijk}}{\partial u_i} = \overleftrightarrow{\mathcal{Q}}^{(u_{jk})}_{jk} = \frac{\overleftrightarrow{Q}_{jk}}{(u_i - u_{ijk})}\,, \tag{3.39a}$$

$$\frac{\partial S^{\swarrow}_{ijk}}{\partial u_{ijk}} = -\overleftrightarrow{\mathcal{T}}^{(u_{ijk})} = \frac{\overleftrightarrow{T}}{(u_i - u_{ijk})(u_j - u_{ijk})(u_k - u_{ijk})}\,. \tag{3.39b}$$

3.3.2 Cross 2-Form: Tetrahedron Equations are Variational

We will show that the tetrahedron equations arise as corner equations of the following 2-form, which we call the *cross Lagrangian*,

$$\mathcal{L}^{\times}_{ij}(u, u_i, u_j, u_{ij}, \alpha_i - \alpha_j) = \Lambda(u_i, u_j, \alpha_i - \alpha_j) - \Lambda(u, u_{ij}, \alpha_i - \alpha_j)\,. \tag{3.40}$$

The name for this Lagrangian is inspired on its leg structure, which is shown in Figure 3.7(a). This Lagrangian was studied in [10] in the more general context of Laplace type equations on a bipartite quad graph. In our present context, it is particularly relevant because it links our new trident Lagrangian to the known triangle Lagrangian.

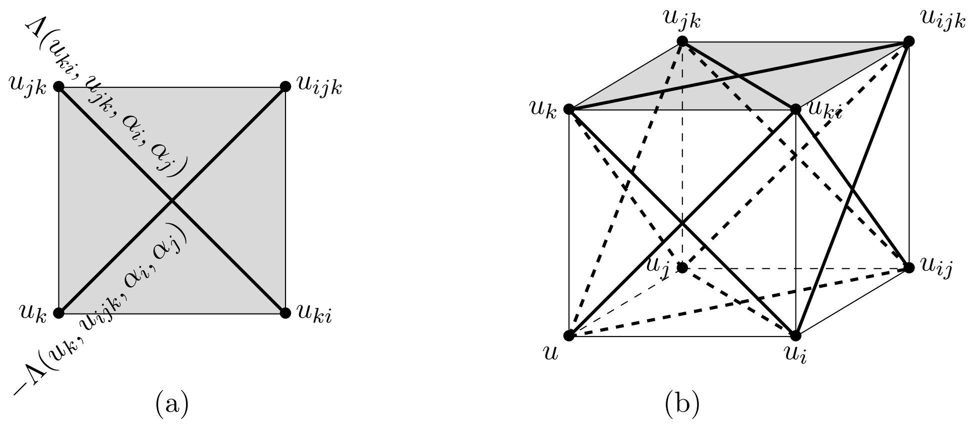


Figure 3.7: (a) The leg structure of a single Lagrangian $\mathcal{L}^{\times}_{ij}(u_k, u_{ki}, u_{jk}, u_{ijk}, \alpha_i, \alpha_j)$. (b) The leg structure for the action on an elementary cube of the cross 2-form $\mathcal{L}_{\times}$.

Consider the action over an elementary cube and note that

$$\begin{aligned} S^{\times}_{ij} &= \Lambda(u_{ki}, u_{jk}, \alpha_i - \alpha_j) - \Lambda(u_k, u_{ijk}, \alpha_i - \alpha_j) \\ &\qquad - \Lambda(u_i, u_j, \alpha_i - \alpha_j) + \Lambda(u, u_{ij}, \alpha_i - \alpha_j) + \circlearrowleft_{ijk} \\ &= \Lambda(u, u_{ij}, \alpha_i - \alpha_j) - \Lambda(u_i, u_j, \alpha_i - \alpha_j) - \overline{(\cdots)} + \circlearrowleft_{ijk}\,. \end{aligned} \tag{3.41}$$

By taking partial derivatives of this action we can conclude the following:

Theorem 5. *The corner equations of the cross 2-form* (3.40) *are the two tetrahedron equations, in different three-leg forms, that is*

$$\begin{aligned} \frac{\partial S^{\times}_{ijk}}{\partial u} &= \mathcal{T}^{(u)} = 0\,, & \frac{\partial S^{\times}_{ijk}}{\partial u_{ijk}} &= -\overleftrightarrow{\mathcal{T}}^{(u_{ijk})} = 0\,, \\ \frac{\partial S^{\times}_{ijk}}{\partial u_i} &= -\overleftrightarrow{\mathcal{T}}^{(u_i)} = 0\,, & \frac{\partial S^{\times}_{ijk}}{\partial u_{jk}} &= \mathcal{T}^{(u_{jk})} = 0\,, \\ \frac{\partial S^{\times}_{ijk}}{\partial u_j} &= -\overleftrightarrow{\mathcal{T}}^{(u_j)} = 0\,, & \frac{\partial S^{\times}_{ijk}}{\partial u_{ki}} &= \mathcal{T}^{(u_{ki})} = 0\,, \\ \frac{\partial S^{\times}_{ijk}}{\partial u_k} &= -\overleftrightarrow{\mathcal{T}}^{(u_k)} = 0\,, & \frac{\partial S^{\times}_{ijk}}{\partial u_{ij}} &= \mathcal{T}^{(u_{ij})} = 0\,. \end{aligned} \tag{3.42}$$

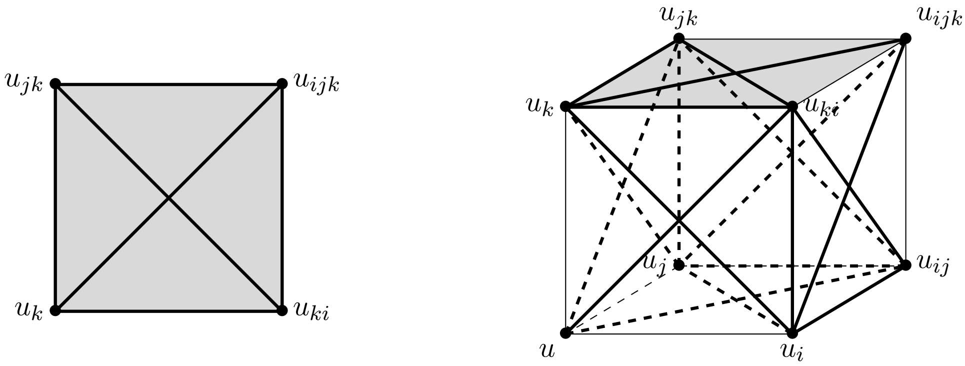
Proof. In the partial derivatives of (3.41), we recognise three-leg forms of the tetrahedron equations: Equations (3.7a)–(3.9c) and their counterparts obtained by point inversion. □

3.3.3 Cross-Square 2-Form

Here we study a second 2-form that produces a system of corner equations equivalent to the quad equations. It will be particularly useful in later sections when we consider the double zero property of the exterior derivative of a 2-form. The function defining this 2-form was introduced in [10], where it was studied on a single quad. To our knowledge, it was never before considered as a Lagrangian multiform. It is given by

$$
\begin{aligned}
\mathcal{L}^{\boxtimes}_{ij} := L(u, u_i, \alpha_i) + L(u_{ij}, u_j, \alpha_i) - L(u, u_j, \alpha_j) - L(u_{ij}, u_i, \alpha_j) \\
- \Lambda(u, u_{ij}, \alpha_i, \alpha_j) - \Lambda(u_i, u_j, \alpha_i, \alpha_j)
\end{aligned}
\tag{3.43}
$$

Inspired by its leg structure, illustrated in Figure 3.8, we call it the *cross-square Lagrangian.*



(a) (b)

Figure 3.8: (a) The leg structure of a single Lagrangian $\mathcal{L}^{\boxtimes}_{ij}(u_k, u_{ki}, u_{jk}, u_{ijk}, \alpha_i, \alpha_j)$. (b) The leg structure for the action on an elementary cube of the cross-square 2-form $\mathcal{L}_{\boxtimes}$.

The action over an elementary cube of the cross-square Lagrangian is

$$
\begin{aligned}
S^{\boxtimes}_{ijk} &= 2L(u_i, u_{ij}, \alpha_j) + \Lambda(u, u_{ij}, \alpha_i - \alpha_j) + \Lambda(u_i, u_j, \alpha_i - \alpha_j) - \overline{(\cdots)} + \circlearrowleft_{ijk}\,, \\
&= 2S^{\swarrow}_{ijk} - S^{\times}_{ijk}\,.
\end{aligned}
\tag{3.44}
$$

so its corner equations consist of linear combinations of the corner equations of the trident and cross Lagrangians:

Theorem 6. *The corner equations of the cross-square 2-form* (3.43) *are combinations of quad equations and tetrahedron equations in three-leg form, namely*

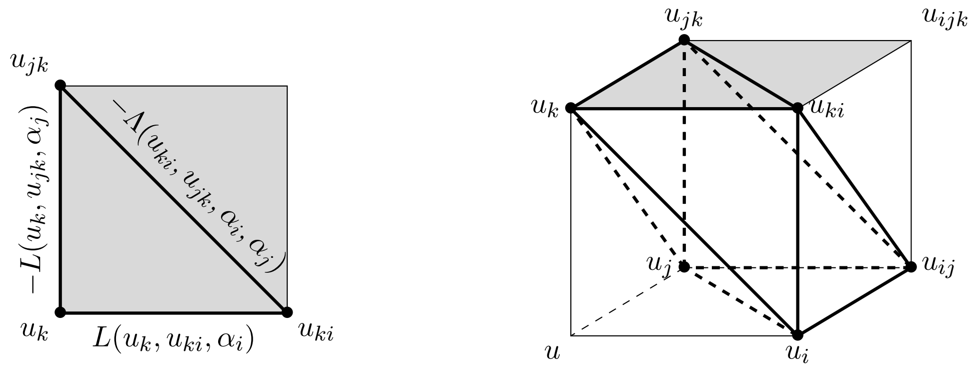
$$
\begin{aligned}
&\frac{\partial S^{\boxtimes}_{ijk}}{\partial u} = -\mathcal{T}^{(u)} = 0\,, && \frac{\partial S^{\boxtimes}_{ijk}}{\partial u_{ijk}} = \overleftrightarrow{\mathcal{T}}^{(u_{ijk})} = 0\,, \\
&\frac{\partial S^{\boxtimes}_{ijk}}{\partial u_i} = 2\overleftrightarrow{\mathcal{Q}}^{(u_i)}_{jk} + \overleftrightarrow{\mathcal{T}}^{(u_i)} = 0\,, && \frac{\partial S^{\boxtimes}_{ijk}}{\partial u_{jk}} = -2\mathcal{Q}^{(u_{jk})}_{jk} - \mathcal{T}^{(u_{jk})} = 0\,, \\
&\frac{\partial S^{\boxtimes}_{ijk}}{\partial u_j} = 2\overleftrightarrow{\mathcal{Q}}^{(u_j)}_{ki} + \overleftrightarrow{\mathcal{T}}^{(u_j)} = 0\,, && \frac{\partial S^{\boxtimes}_{ijk}}{\partial u_{ki}} = -2\mathcal{Q}^{(u_{ki})}_{ki} - \mathcal{T}^{(u_{ki})} = 0\,, \\
&\frac{\partial S^{\boxtimes}_{ijk}}{\partial u_k} = 2\overleftrightarrow{\mathcal{Q}}^{(u_k)}_{ij} + \overleftrightarrow{\mathcal{T}}^{(u_k)} = 0\,, && \frac{\partial S^{\boxtimes}_{ijk}}{\partial u_{ij}} = -2\mathcal{Q}^{(u_{ij})}_{ij} - \mathcal{T}^{(u_{ij})} = 0\,.
\end{aligned}
\tag{3.45}
$$

These are equivalent to the set of multiaffine quad equations around the cube (3.3)*.*

Proof. From (3.44) we deduce that the corner equations are the corresponding linear combination of the corner equations (3.33) and (3.42) of the trident and cross Lagrangian 2-forms. If multiaffine quad equations hold, then all of the three-leg forms involved in the system (3.45) vanish, so the corner equations are satisfied. Conversely, if all equations of the system (3.45) hold, then in particular the tetrahedron equations in three-leg form, centred at u and u_{ijk}, are satisfied. Then by the symmetries (3.10) and (3.12), the tetrahedron three-leg forms centred at $u_i, u_j, u_k, u_{ij}, u_{jk}, u_{ki}$ also vanish. The remaining corner equations reduce to individual quad equations in three-leg form. These are equivalent to the multiaffine quad equations. □

3.3.4 Comparison with the Triangle 2-Form

Now we compare the above constructions to the known 2-form, which started the theory of Lagrangian multiforms [28]. We review the results of [11], and in what way this original example is a Lagrangian 2-form in the sense of Definition 2. We recall the Lagrangian (3.23) and now interpret it as the triangle 2-form. We review how this 2-form is critical on a set of equations weaker than the quad equations and we will relate it to our new 2-forms via $\mathcal{L}^{\triangleright}_{ij} = \mathcal{L}^{\swarrow}_{ij} - \mathcal{L}^{\times}_{ij}$.



(a) (b)

Figure 3.9: (a) The leg structure of a single Lagrangian $\mathcal{L}^{\triangleright}_{ij}(u_k, u_{ki}, u_{jk}, \alpha_i, \alpha_j)$. (b) The leg structure for the action on an elementary cube of the triangle 2-form $\mathcal{L}^{\triangleright}_{ij}$ sits on an octahedral stencil.

The action around the elementary cube of the triangle Lagrangian depends on an octahedral stencil depicted in Figure 3.9:

$$S^{\triangleright}_{ijk} = L(u_i, u_{ij}, \alpha_j) + \Lambda(u_i, u_j, \alpha_i - \alpha_j) - \overline{(\cdots)} + \circlearrowleft_{ijk}. \tag{3.46}$$

Since u and u_{ijk} does not appear in this action, there are no corner equations at these points. The corner equation at u_{ij} in terms of leg functions is

$$
\begin{aligned}
0 &= \frac{\partial S^{\triangle}_{ijk}}{\partial u_{ij}} \\
&= \frac{\partial}{\partial u_{ij}} \left(L(u_i, u_{ij}, \alpha_j) - L(u_j, u_{ij}, \alpha_i) - \Lambda(u_{ij}, u_{ki}, \alpha_j - \alpha_k) - \Lambda(u_{jk}, u_{ij}, \alpha_k - \alpha_i) \right) \\
&= \psi(u_{ij}, u_i, \alpha_j) - \psi(u_{ij}, u_j, \alpha_i) - \phi(u_{ij}, u_{ki}, \alpha_j - \alpha_k) - \phi(u_{ij}, u_{jk}, \alpha_k - \alpha_i) \qquad (3.47) \\
&=: \mathcal{E}_{ij} \,.
\end{aligned}
$$

This can be written as a combination of quad equations in three-leg form:

$$
\mathcal{E}_{ij} = \overleftrightarrow{\mathcal{Q}}^{(u_{ij})}_{jk} + \overleftrightarrow{\mathcal{Q}}^{(u_{ij})}_{ki} \,. \qquad (3.48)
$$

In other words, the four-leg equations $\mathcal{E}_{ij} = 0$ can be obtained by eliminating the variable u_{ijk} from two quad equations in three-leg form. Similarly, at the corner u_i, we find the corner equation $-\overleftrightarrow{\mathcal{E}}_{ij} = 0$, where $\overleftrightarrow{\mathcal{E}}_{ij} = \mathcal{Q}^{(u_k)}_{jk} + \mathcal{Q}^{(u_k)}_{ki}$.

Theorem 7. *The corner equations of the triangle 2-form are*

$$
\begin{aligned}
&\frac{\partial S^{\triangle}_{ijk}}{\partial u} = 0 \,, && \frac{\partial S^{\triangle}_{ijk}}{\partial u_{ijk}} = 0 \,, \\
&\frac{\partial S^{\triangle}_{ijk}}{\partial u_i} = -\overleftrightarrow{\mathcal{E}}_{jk} = 0 \,, && \frac{\partial S^{\triangle}_{ijk}}{\partial u_{jk}} = \mathcal{E}_{jk} = 0 \,, \\
&\frac{\partial S^{\triangle}_{ijk}}{\partial u_j} = -\overleftrightarrow{\mathcal{E}}_{ki} = 0 \,, && \frac{\partial S^{\triangle}_{ijk}}{\partial u_{ki}} = \mathcal{E}_{ki} = 0 \,, \\
&\frac{\partial S^{\triangle}_{ijk}}{\partial u_k} = -\overleftrightarrow{\mathcal{E}}_{ij} = 0 \,, && \frac{\partial S^{\triangle}_{ijk}}{\partial u_{ij}} = \mathcal{E}_{ij} = 0 \,.
\end{aligned} \qquad (3.49)
$$

Proof. Using the action (3.46) and the definition (3.47) alongside cyclicity and point inversion gives us these six expressions. □

This 2-form produces a set of equations which vanish on the quad equations but are not equivalent to them because they lack the variables u and u_{ijk}. Furthermore, we can relate each of the four-leg equations (3.49) to a quad equation and a tetrahedron equation. From $\mathcal{L}^{\triangle}_{ij} = \mathcal{L}^{\swarrow}_{ij} - \mathcal{L}^{\times}_{ij}$, we have

$$
S^{\triangle}_{ijk} = S^{\swarrow}_{ijk} - S^{\times}_{ijk} \qquad (3.50)
$$

and thus

$$
\mathcal{E}_{ij} = -\mathcal{Q}^{(u_{ij})}_{ij} - \mathcal{T}^{(u_{ij})} \,. \qquad (3.51)
$$

Example (H1), part 3. *For H1 we have*

$$\mathcal{L}^{\triangle}_{ij} = uu_i - uu_j - (\alpha_i - \alpha_j)\log(u_i - u_j)\,, \tag{3.52}$$

so

$$\begin{aligned} S^{\triangle}_{ijk} &= u_k u_{ki} - u_k u_{jk} - (\alpha_i - \alpha_j)\log(u_{ki} - u_{jk}) + (\alpha_i - \alpha_j)\log(u_i - u_j) + \circlearrowleft_{ijk}\,, \\ &= u_i u_{ij} + (\alpha_i - \alpha_j)\log(u_i - u_j) - \overline{(\cdots)} + \circlearrowleft_{ijk} \end{aligned} \tag{3.53}$$

Two corner equations are identically zero and the other six have a four-leg form. The corner equations at u_{ij}, u_{jk} and u_{ki} are all of the form

$$0 = \frac{\partial S^{\triangle}_{ijk}}{\partial u_{ij}} = u_i - u_j - \frac{\alpha_j - \alpha_k}{u_{ij} - u_{ki}} + \frac{\alpha_j - \alpha_k}{u_{jk} - u_{ij}} = \overleftrightarrow{\mathcal{Q}}^{(u_{ij})}_{jk} + \overleftrightarrow{\mathcal{Q}}^{(u_{ij})}_{ki} = \mathcal{E}_{ij}\,. \tag{3.54}$$

Clearly these equations vanish on the quad equations. Similarly, the other three corner equations are of the form

$$0 = \frac{\partial S^{\triangle}_{ijk}}{\partial u_k} = u_{ki} - u_{jk} + \frac{\alpha_k - \alpha_i}{u_k - u_i} - \frac{\alpha_j - \alpha_k}{u_j - u_k} = -\mathcal{Q}^{(u_{ij})}_{jk} - \mathcal{Q}^{(u_{ij})}_{ki} = -\overleftrightarrow{\mathcal{E}}_{ij}\,. \tag{3.55}$$

3.3.5 Closure Relations

In this section we will show that each of the 2-forms satisfies the closure relation on their respective corner equations, which we stated in the theorems above and recall here:

1. The corner equations of the trident Lagrangian $\mathcal{L}^{\swarrow}_{ij}$ are equivalent to the system of quad equations.
2. The corner equations of the cross Lagrangian $\mathcal{L}^{\times}_{ij}$ are the tetrahedron equations.
3. The corner equations of the cross-square Lagrangian $\mathcal{L}^{\boxtimes}_{ij}$ are equivalent to the system of quad equations.
4. The corner equations of the triangle Lagrangian $\mathcal{L}^{\triangle}_{ij}$ are equivalent to the system of four-leg $\mathcal{E}$ equations.

In [28] it was verified (for most of the ABS equations) by direct computation that the action $S^{\triangle}_{ijk}$ of the triangle Lagrangian vanishes on solutions of the quad equations. In [10] a conceptual proof of this property was given, which applies to all ABS equations. In [11] it was observed that the same technique can be used to show that the closure property for the triangle Lagrangian holds on the corner equations (i.e. $\mathcal{E}$-equations). Below we will adapt this proof to our new Lagrangian multiforms.

Theorem 8. *On solutions of the quad equations,* $S^{\swarrow}_{ijk} = 0$.

On solutions of the tetrahedron equations, $S^{\times}_{ijk} = 0$.

On solutions of the quad equations, $S^{\boxtimes}_{ijk} = 0$.

Note that the closure relation $S^{\times}_{ijk} = 0$ can be seen as a consequence of the star-triangle relation of [10, Theorem 2].

An essential fact required to prove Theorem 8 is that the space of solutions is connected:

Lemma 9. *The following sets are connected:*

1. *The space of complex-valued solutions to the tetrahedron equations,*

$$\{(u, u_i, u_j, u_k, u_{ij}, u_{jk}, u_{ki}, u_{ijk}) \in \mathbb{C}^8 \mid T(u, u_{ij}, u_{jk}, u_{ki}, \alpha_i, \alpha_j, \alpha_k) = 0 \\ \textit{and } T(u_{ijk}, u_k, u_i, u_j, \alpha_i, \alpha_j, \alpha_k) = 0\}.$$

2. *The space of complex-valued solutions to the quad equations,*

$$\{(u, u_i, u_j, u_k, u_{ij}, u_{jk}, u_{ki}, u_{ijk}) \in \mathbb{C}^8 \mid \textit{Equations } (3.3) \textit{ hold}\}.$$

Proof. We give the proof of case 2. We parametrise the space of solutions by initial conditions $(u, u_i, u_j, u_k) \in \mathbb{C}^4$. Clearly, every solution corresponds to a unique element of $\mathbb{C}^4$, but not every such 4-tuple generates a solution. For example, if $\frac{\partial Q_{ij}}{\partial u_{ij}} = 0$, we cannot solve $Q_{ij} = 0$ for Q_{ij}. A solution exists with given initial values, if no such singulatity occurs on any of the faces. Each of the six singularity sets has complex codimension one, so real codimension two. Removing sets of codimension two from $\mathbb{C}^4$ does not disconnect it, so the space of initial conditions avoiding singularities is connected. □

Proof of Theorem. Let $S_{ijk} = S^{\times}_{ijk}$, $S_{ijk} = S^{\swarrow}_{ijk}$, or $S = S^{\boxtimes}_{ijk}$. We consider the action S_{ijk} as a function of $(u, u_i, u_j, u_k, u_{ij}, u_{jk}, u_{ki}, u_{ijk})$. The corner equations express that the gradient of S_{ijk} with respect to these variables vanishes. Together with Lemma 9, this shows that the action is constant on the set of solutions.

Consider any solution $v = (u, u_i, u_j, u_k, u_{ij}, u_{jk}, u_{ki}, u_{ijk})$ to the corner equations. Then also $\overleftrightarrow{v} := (u_{ijk}, u_{jk}, u_{ki}, u_{ij}, u_k, u_i, u_j, u)$ is a solution. By the symmetry of L and Λ we have

$$S_{ijk}(\overleftrightarrow{v}) = -S_{ijk}(v). \tag{3.56}$$

Since S_{ijk} is constant on the set of solutions, this implies that $S_{ijk} = 0$. □

The cross square 2-form satisfies an even stronger property than the closure relation. Up to an additive term depending only on the lattice parameters, the Lagrangian vanishes on a single quad as a consequence of the quad equation:

Proposition 10 ([10, Theorem 1])**.** *On solutions of the quad equation, $\mathcal{L}^{\boxtimes}_{ij} = g(\alpha_i, \alpha_j)$, for a function $g(\alpha_i, \alpha_j)$ which does not depend on the fields (u, u_i, u_j, u_{ij}).*

Proof. We start by considering the derivatives of an individual cross Lagrangian

$$\frac{\partial}{\partial u}\mathcal{L}^{\boxtimes}_{ij} = \mathcal{Q}^{(u)}_{ij} \qquad \frac{\partial}{\partial u_i}\mathcal{L}^{\boxtimes}_{ij} = \mathcal{Q}^{(u_i)}_{ij} \qquad \frac{\partial}{\partial u_j}\mathcal{L}^{\boxtimes}_{ij} = \mathcal{Q}^{(u_j)}_{ij} \qquad \frac{\partial}{\partial u_{ij}}\mathcal{L}^{\boxtimes}_{ij} = \mathcal{Q}^{(u_{ij})}_{ij}\,. \tag{3.57}$$

These are exactly the quad equations in three-leg form. Hence, on solutions of $Q_{ij} = 0$ we have $\nabla \mathcal{L}^{\boxtimes}_{ij} = 0$. Similar to Lemma 9 we have that the space of solutions of a single quad equation is connected. It follows that the value of $\mathcal{L}^{\boxtimes}_{ij}$ (on solutions) does not depend on (u, u_i, u_j, u_{ij}). □

3.4 Quad Tetrahedron and Octahedron Equations

Above we discussed the three-leg forms equivalent to the multiaffine quad and tetrahedron equations. In this section, we discuss the polynomial equations associated with the four-leg corner equations (3.49) of the triangle 2-form. We review the related multiaffine *octahedron equations* (also known as *octahedron relations*) and their relationship to quad equations. Most of this section is based on [13], but in Proposition 12 we provide an additional variational interpretation of the relation between the different types of polynomial equations.

The four-leg equations $\mathcal{E}_{ij} = 0$, where the left-hand side is defined in Equation (3.48), can be obtained by eliminating the variable u_{ijk} from two quad equations in three-leg form. An equivalent polynomial equation is obtained by eliminating u_{ijk} from the corresponding two multiaffine quad equations. This yields the equation $E_{ij} = 0$, where the left-hand side is the polynomial

$$E_{ij} = \frac{\partial \overleftrightarrow{Q}_{jk}}{\partial u_{ijk}}\overleftrightarrow{Q}_{ki} - \frac{\partial \overleftrightarrow{Q}_{ki}}{\partial u_{ijk}}\overleftrightarrow{Q}_{jk}\,. \tag{3.58}$$

We have two more polynomials by permuting indices and three more from point inversion:

$$\overleftrightarrow{E}_{ij} = \frac{\partial Q_{jk}}{\partial u}Q_{ki} - \frac{\partial Q_{ki}}{\partial u}Q_{jk}\,. \tag{3.59}$$

These polynomials lead to equations that are equivalent to those involving the four-leg expressions $\mathcal{E}_{ij}$:

$$E_{ij}(u_i, u_j, u_{ij}, u_{jk}, u_{ki}, \alpha_i, \alpha_j, \alpha_k) = 0 \iff \mathcal{E}_{ij}(u_i, u_j, u_{ij}, u_{jk}, u_{ki}, \alpha_i, \alpha_j, \alpha_k) = 0\,. \tag{3.60}$$

From their definition as combinations of quad equations, it is clear that $E_{ij} = 0$ and $\overleftrightarrow{E}_{ij} = 0$ are consequences of the quad equations. However, they are not equivalent to the quad equations! Only two of these six equations are independent (see Proposition 11 below).

The relation (3.51) suggests that the polynomial E_{ij} is related to the polynomials Q_{ij} and T. Indeed, eliminating the variable u from the system $T = 0$, $Q_{ij} = 0$ must lead to an equation equivalent to $E_{ij} = 0$, so we find

$$\frac{\partial Q_{ij}}{\partial u} T - \frac{\partial T}{\partial u} Q_{ij} = \gamma(\alpha_i, \alpha_j, \alpha_k) E_{ij}\,. \tag{3.61}$$

Here, $\gamma(\alpha_i, \alpha_j, \alpha_k)$ does not contain field variables, because the left hand side depends linearly on $(u_i, u_j, u_{jk}, u_{ki})$ and quadratically on u_{ij}, as does E_{ij}. Under point inversion we find the analogous relation

$$\frac{\partial \overleftrightarrow{Q}_{ij}}{\partial u_{ijk}} \overleftrightarrow{T} - \frac{\partial \overleftrightarrow{T}}{\partial u_{ijk}} \overleftrightarrow{Q}_{ij} = \gamma(\alpha_i, \alpha_j, \alpha_k) \overleftrightarrow{E}_{ij}\,. \tag{3.62}$$

The main result of [13] is that for every member of the ABS list there exist two octahedron equations which are equivalent to the set of E-equations. These are of the form

$$\Omega_1(u_i, u_j, u_k, u_{ij}, u_{jk}, u_{ki}, \alpha_i, \alpha_j, \alpha_k) = 0\,, \tag{3.63a}$$

$$\Omega_2(u_i, u_j, u_k, u_{ij}, u_{jk}, u_{ki}, \alpha_i, \alpha_j, \alpha_k) = 0\,, \tag{3.63b}$$

where Ω_1, Ω_2 are multiaffine polynomials in the field variables $u_i, u_j, u_k, u_{ij}, u_{jk}, u_{ki}$, which form an octahedral stencil.

Note for Q4 and A2, we define Ω_1 and Ω_2 as two linearly independent cyclic combinations of the R_i, R_j, R_k from [13, Theorem 4.1]. For Q4 and A2 we have point inversion symmetry such that $\overleftrightarrow{\Omega}_1 = \Omega_1$ and $\overleftrightarrow{\Omega}_2 = \Omega_2$. For the rest of ABS list there holds $\overleftrightarrow{\Omega}_1 = -\Omega_1$ and $\overleftrightarrow{\Omega}_2 = \Omega_2$.

In all cases we can get E-polynomials by eliminating a variable form the system of octahedron equations. For Q4 and A2, eliminating u_k and u_{ij} leads to, respectively,

$$\frac{\partial\Omega_1}{\partial u_k}\Omega_2 - \frac{\partial\Omega_2}{\partial u_k}\Omega_1 = \mu(u_i, u_j, u_{jk}, u_{ki}, \alpha_i, \alpha_j, \alpha_k) E_{ij}\,, \tag{3.64a}$$

$$\frac{\partial\Omega_1}{\partial u_{ij}}\Omega_2 - \frac{\partial\Omega_2}{\partial u_{ij}}\Omega_1 = \mu(u_{jk}, u_{ki}, u_i, u_j, \alpha_i, \alpha_j, \alpha_k) \overleftrightarrow{E}_{ij}\,, \tag{3.64b}$$

where the factor $\mu(u_i, u_j, u_{jk}, u_{ki}, \alpha_i, \alpha_j, \alpha_k)$ is polynomial in u_i, u_j, u_{jk}, u_{ki}. For the rest of ABS list, we have the explicit expressions

$$\frac{\partial\Omega_1}{\partial u_k}\Omega_2 - \frac{\partial\Omega_2}{\partial u_k}\Omega_1 = g_1\left(\frac{\partial\Omega_1}{\partial u_{ij}} - \frac{\partial\Omega_1}{\partial u_k} \right) E_{ij}\,, \tag{3.65a}$$

$$-\frac{\partial\Omega_1}{\partial u_{ij}}\Omega_2 + \frac{\partial\Omega_2}{\partial u_{ij}}\Omega_1 = g_1\left(-\frac{\partial\Omega_1}{\partial u_k} + \frac{\partial\Omega_1}{\partial u_{ij}} \right) \overleftrightarrow{E}_{ij}\,, \tag{3.65b}$$

where $g_1 = g_1(\alpha_i, \alpha_j, \alpha_k)$ is defined explicitly on a case by case basis in [13, Proposition 5.3].

Example (H1), part 4. *For H1 the four-leg equations $\mathcal{E}_{ij} = 0$ and $\overleftrightarrow{\mathcal{E}}_{ij} = 0$ are equivalent to the following polynomials, respectively,*

$$E_{ij} = (u_i - u_j)(u_{ij} - u_{jk})(u_{ij} - u_{ki}) + \alpha_i(u_{ki} - u_{ij}) + \alpha_j(u_{ij} - u_{jk}) + \alpha_k(u_{jk} - u_{ki})\,, \tag{3.66a}$$

$$\overleftrightarrow{E}_{ij} = (u_{jk} - u_{ki})(u_k - u_i)(u_k - u_j) + \alpha_i(u_j - u_k) + \alpha_j(u_k - u_i) + \alpha_k(u_i - u_j)\,. \tag{3.66b}$$

These can written in terms of the following multiaffine octahedron polynomials

$$\Omega_1 = u_i(u_{ki} - u_{ij}) + \circlearrowleft_{ijk}\,, \tag{3.67a}$$

$$\Omega_2 = \alpha_i(u_k - u_j + u_{ij} - u_{ki}) + u_i u_{jk}(u_j - u_k - u_{ij}u_{ki}) + \circlearrowleft_{ijk}\,, \tag{3.67b}$$

The equivalence between the corner equations of $\mathcal{L}_{\triangleright}$ (i.e. the E-equations obtained by setting the expressions (3.58) and (3.59) to zero) and octahedron equations can be seen as a particular case of the following statement. This holds in the sense of fractional ideals, as explained in [13].

Proposition 11. *Out of the set of equations consisting of the six E-equations and two octahedron equations, any 2 equations imply the other six.*

Further to this, a dimension-counting argument suggests that if we add the tetrahedron equations to the set of octahedron equations (or E-equations), we should obtain a system equivalent to the quad equations. The Lagrangian multiform structure provides an explicit proof of this fact:

Proposition 12. *The tetrahedron equations and octahedron equations together are equivalent to the quad equations.*

Proof. From Equation (3.7a) (or from tetrahedron property as assumed in the ABS classification [1]) it follows that the quad equations imply the tetrahedron equations. From (3.64) or (3.65) it follows that the E-equations, and hence the quad equations, imply the octahedron equations (in the sense of fraction ideals, as in [13]).

To prove the other implication, assume that the octahedron and tetrahedron equations are satisfied. Then the actions of $\mathcal{L}_{ij}^{\triangleright}$ and $\mathcal{L}_{ij}^{\times}$ are critical. Since $\mathcal{L}_{ij}^{\angle} = \mathcal{L}_{ij}^{\triangleright} + \mathcal{L}_{ij}^{\times}$, it follows that the action of $\mathcal{L}_{ij}^{\angle}$ is critical, hence the quad equations hold. □

2-Form	Corner polynomials	Equivalent system
$\mathcal{L}_{ij}^{\angle}$	$(T, \overleftrightarrow{Q}_{jk}, \overleftrightarrow{Q}_{ki}, \overleftrightarrow{Q}_{ij}, Q_{ij}, Q_{jk}, Q_{ki}, \overleftrightarrow{T})$	$\Omega_1, \Omega_2, T, \overleftrightarrow{T} = 0$
$\mathcal{L}_{ij}^{\triangleright}$	$(0, \overleftrightarrow{E}_{jk}, \overleftrightarrow{E}_{ki}, \overleftrightarrow{E}_{ij}, E_{ij}, E_{jk}, E_{ki}, 0)$	$\Omega_1, \Omega_2 = 0$
$\mathcal{L}_{ij}^{\times}$	$(T, \overleftrightarrow{T}, \overleftrightarrow{T}, \overleftrightarrow{T}, T, T, T, \overleftrightarrow{T})$	$T, \overleftrightarrow{T} = 0$
$\mathcal{L}_{ij}^{\boxtimes}$	$(T, \sim \overleftrightarrow{Q}_{jk}, \sim \overleftrightarrow{Q}_{ki}, \sim \overleftrightarrow{Q}_{ij}, \sim Q_{ij}, \sim Q_{jk}, \sim Q_{ki}, \overleftrightarrow{T})$	$\Omega_1, \Omega_2, T, \overleftrightarrow{T} = 0$

Table 3.1: Overview of the four types of Lagrangian multiform, with their corner equations (in polynomial form) and a symmetric set of equations forming an equivalent system. Note in the final row "$\sim \overleftrightarrow{Q}_{jk}$" represents an expression such that, after the elimination of a tetrahedron polynomial $\overleftrightarrow{T}$, a quad polynomial $\overleftrightarrow{Q}_{jk}$ remains.

Proposition 12 relates the three sets of equations we are dealing with: those produced by $\mathcal{L}_{ij}^{\angle}$ (or $\mathcal{L}_{ij}^{\boxtimes}$), $\mathcal{L}_{ij}^{\times}$, and $\mathcal{L}_{ij}^{\triangleright}$. (See Table 3.1 for an overview.) The corner equations of $\mathcal{L}_{ij}^{\times}$ can be immediately identified with the tetrahedron equations. The equations produced by $\mathcal{L}_{ij}^{\triangleright}$ can be understood from two points of view. On the one hand they are generated by the 5-point E-equations, which have an obvious variational interpretation, but are far less symmetric than the other equations considered. On the other hand they are generated by the octahedron equations, which have cyclic and point inversion symmetry, but have no previously known variational interpretation. The equations produced by $\mathcal{L}_{ij}^{\angle}$ (or $\mathcal{L}_{ij}^{\boxtimes}$) are the six quad equations around the cube (together with the two tetrahedron equations), which are equivalent to the combined set of two tetrahedron equations and two octahedron equations.

3.5 Chapter Conclusions

The major result in this chapter is the elevation of the Lagrangians from [1] and [10] to full-fledged Lagrangian multiforms related to the ABS quad equations: $\mathcal{L}_{ij}^{\angle}$, $\mathcal{L}_{ij}^{\times}$ and $\mathcal{L}_{ij}^{\boxtimes}$.

We showed that, contrary to common belief, the ABS quad equations are variational: they are the corner equations of $\mathcal{L}_{ij}^{\swarrow}$ (and equivalent to the corner equations of $\mathcal{L}_{ij}^{\boxtimes}$).

Using the relations between these new Lagrangian 2-forms and the well-established Lagrangian 2-form $\mathcal{L}_{ij}^{\triangle}$, we showed that the quad equations are equivalent to the combined system of tetrahedron and octahedron equations.

Chapter 4

Discrete Lagrangian 2-Forms and Double Zero Expansions

This chapter is based on section 5 of [44] by J. J. Richardson and M. Vermeeren.

4.1 Chapter Introduction

An important property of Lagrangian multiform theory is that, on solutions to the variational principle, the Lagrangian d-form is closed. This means in particular that the action over a closed surface vanishes on solutions. Recently, an algebraic interpretation of this property has been given more emphasis: we can write the exterior derivative of the Lagrangian form as (a sum of) product(s) of Euler-Lagrange expressions. In other words, the exterior derivative is not just zero on solutions of the Euler-Lagrange equations, but attains a *double zero* on this set of equations. In the continuous (and semi-discrete) case, the double zero property has been used implicitly in [48, 54] and discussed explicitly in [16, 42, 49]. In the discrete setting, a double zero expansion for the lattice Boussinesq equation was recently obtained in [40]. In this Chapter, we present double zero expansions for the Lagrangian multiforms of the ABS list. With future work, double zero expansions may be used to investigate conservation laws [43, 53] and variational symmetries [48, 54] in this discrete quadrilateral setting.

We show that the exterior derivatives of all four 2-forms admit a double zero expansion (in terms of quad, tetrahedron, and octahedron polynomials, respectively). We give a general construction of these double zero expansions, as well as explicit expressions for the quad equation H1.

To motivate the formulation of the double zero property, we first present some observations regarding a linear quad equation. Then we formalise the definition of a double zero expansion and provide constructions of double zero expansions for each of the 2-forms for any member of ABS list. Afterwards we investigate explicit expansions associated with each of the 2-forms for H1.

4.2 Double Zero Expansion Associated with a Linear Quad Equation

We consider the linear quad equation $Q_{ij} = 0$, with

$$Q_{ij} = (\alpha_i + \alpha_j)(u_i - u_j) - (\alpha_i - \alpha_j)(u - u_{ij}). \tag{4.1}$$

This equation can be considered as a linearisation of H1 and is associated to the following triangle Lagrangian

$$\mathcal{L}^{\triangleright}_{ij} = u(u_i - u_j) - \frac{\alpha_i + \alpha_j}{2(\alpha_i - \alpha_j)}(u - u_{ij})^2 \,. \tag{4.2}$$

Its action over an elementary cube can be written as

$$\begin{aligned} S^{\triangleright}_{ijk} &= \mathcal{L}^{\triangleright}_{ij}(u_k, u_{ki}, u_{jk}, \alpha_i - \alpha_j) + \mathcal{L}^{\triangleright}_{jk}(u_i, u_{ij}, u_{ki}, \alpha_j - \alpha_k) + \mathcal{L}^{\triangleright}_{ki}(u_j, u_{jk}, u_{ij}, \alpha_k - \alpha_i) \\ &\quad - \mathcal{L}^{\triangleright}_{ij}(u, u_i, u_j, \alpha_i - \alpha_j) - \mathcal{L}^{\triangleright}_{jk}(u, u_j, u_k, \alpha_j - \alpha_k) - \mathcal{L}^{\triangleright}_{ki}(u, u_k, u_i, \alpha_k - \alpha_i) \\ &= \frac{O_1\, O_2}{(\alpha_i - \alpha_j)(\alpha_j - \alpha_k)(\alpha_k - \alpha_i)}, \end{aligned} \tag{4.3}$$

where

$$O_1 := (\alpha_j - \alpha_k)u_i + (\alpha_j - \alpha_k)u_{jk} \; + \circlearrowleft_{ijk} , \tag{4.4a}$$

$$O_2 := \alpha_i(\alpha_j - \alpha_k)u_i - \alpha_i(\alpha_j - \alpha_k)u_{jk} \; + \circlearrowleft_{ijk} . \tag{4.4b}$$

Hence, the action over an elementary cube can be explicitly written as a product of two expressions O_1 and O_2. In keeping with the terminology of [13], we refer to these as octahedron expressions since they lie on the stencil $(u_i, u_j, u_k, u_{ij}, u_{jk}, u_{ki})$. They are symmetric under cyclic permutation of the indices and under point inversion, in the sense that $\overleftrightarrow{O}_1 = O_1$ and $\overleftrightarrow{O}_2 = -O_2$.

In analogy to multiple zeros of polynomials, we say that $S_{\triangleright}$ has a *double zero* on the equations $O_1 = 0$ and $O_2 = 0$. The factorisation (4.3) implies that for any

$v \in \{u, u_i, u_j, u_k, u_{ij}, u_{jk}, u_{ki}, u_{ijk}\}$ there holds

$$\frac{\partial S^{\triangle}_{ijk}}{\partial v} = \frac{1}{(\alpha_i - \alpha_j)(\alpha_j - \alpha_k)(\alpha_k - \alpha_i)} \left(\frac{\partial O_1}{\partial v} O_2 + O_1 \frac{\partial O_2}{\partial v} \right), \tag{4.5}$$

which is zero if both $O_1 = 0$ and $O_2 = 0$. Hence, the double zero property tells us that the action is critical if the two equations $O_1 = 0$ and $O_2 = 0$ hold. This implies that all corner equations are consequences of these two equations!

Note that both O_1 and O_2 vanish on solutions to the linear quad equations. Indeed, we have

$$O_1 = Q_{ij} + Q_{jk} + Q_{ki} = \overleftrightarrow{Q}_{ij} + \overleftrightarrow{Q}_{jk} + \overleftrightarrow{Q}_{ki}, \tag{4.6a}$$

$$O_2 = -\alpha_k Q_{ij} - \alpha_i Q_{jk} - \alpha_j Q_{ki} = \alpha_k \overleftrightarrow{Q}_{ij} + \alpha_i \overleftrightarrow{Q}_{jk} + \alpha_j \overleftrightarrow{Q}_{ki}. \tag{4.6b}$$

This linear example shows the power of the double zero property: if S can be written as a product of two expressions, then the fact that both these expressions vanish is a sufficient condition for criticality. Hence, in this situation, two equations together are equivalent to the full system of corner equations. In some examples, the double zero expansion of S_{ijk} involves more than two expressions and is a sum of products, rather than a single product. In the next section we give a suitably general definition of a double zero expansion.

4.3 Definition of a Double Zero Expansion

Now we give a suitably general definition of a double zero expansion in this discrete setting.

Definition 13. *We say that* $S_{ijk}(u, u_i, u_j, u_k, u_{ij}, u_{jk}, u_{ki}, u_{ijk})$ *has a* double zero *on a set of equations* $\{K_m(u, u_i, u_j, u_k, u_{ij}, u_{jk}, u_{ki}, u_{ijk}) = 0 \mid m = 1, \ldots M\}$ *if it can be written as*

$$S_{ijk} = \sum_{1 \le m \le m' \le M} d_{m,m'} \, K_m \, K_{m'} \, . \tag{4.7}$$

Here $d_{m,m'}$ *are coefficients which can depend upon any variable or parameter, but must be nonsingular on generic points of* $\{K_m(u, u_i, u_j, u_k, u_{ij}, u_{jk}, u_{ki}, u_{ijk}) = 0 \mid m = 1, \ldots M\}$. *We call the right hand side of Equation* (4.7) *a* double zero expansion *of* S_{ijk}.

In other words, we are describing a function S_{ijk} which can be written as a sum of products of pairs of the expressions $\{K_m\}_{m=1,\cdots,M}$. When we impose the set of equations

$\{K_m = 0\}_{m=1,\cdots,M}$ on the function S_{ijk}, it "doubly" vanishes. Below we discuss how this is useful for Lagrangian multiforms. When we impose $\{K_m = 0\}_{m=1,\cdots,M}$, we simply need that $d_{m,m'}$ is well-defined; it is not a problem if $d_{m,m'} = 0$.

This definition applies to the example above, because Equation (4.3) is of the form (4.7) with $M = 2$, $K_1 = O_1$, $K_2 = O_2$, $d_{1,1} = d_{2,2} = 0$, and

$$d_{1,2} = \frac{1}{(\alpha_i - \alpha_j)(\alpha_j - \alpha_k)(\alpha_k - \alpha_i)} \,. \tag{4.8}$$

Proposition 14. *If the action over an elementary cube S_{ijk} of a Lagrangian 2-form L_{ij} has a double zero on a set of equations $\{K_m = 0 \mid m = 1, \ldots M\}$, then this set implies the multiform Euler-Lagrange equations of L_{ij}.*

Proof. For any $v \in \{u, u_i, u_j, u_k, u_{ij}, u_{jk}, u_{ki}, u_{ijk}\}$ we have

$$\frac{\partial S_{ijk}}{\partial v} = \sum_{1 \le m \le m' \le M} \left(\frac{\partial K_m}{\partial v} K_{m'} + K_m \frac{\partial K_{m'}}{\partial v} \right) , \tag{4.9}$$

which vanishes on $\{K_m = 0 \mid m = 1, \ldots M\}$. □

Below we will use Taylor expansions to derive double zero expansions. For example, for the action around the cube $S^{\times}_{ijk}$ of the cross Lagrangian 2-form we will later see that it can be Taylor expanded in terms of T and $\overleftrightarrow{T}$ in the following way

$$S^{\times}_{ijk} = \sum_{n=0}^{\infty} d_n \, T^n - \sum_{n=0}^{\infty} \overleftrightarrow{d}_n \, \overleftrightarrow{T}^n \,. \tag{4.10}$$

Since this is the cross Lagrangian 2-form, we have that $\nabla S^{\times}_{ijk} = 0 \iff T, \overleftrightarrow{T} = 0$ and $T, \overleftrightarrow{T} = 0 \implies S^{\times}_{ijk} = 0$. Those defining Lagrangian multiform properties give us that the zeroth and first order terms in the above Taylor expansion must vanish and thus

$$S^{\times}_{ijk} = \sum_{n=2}^{\infty} d_n \, T^n - \sum_{n=2}^{\infty} \overleftrightarrow{d}_n \, \overleftrightarrow{T}^n \,. \tag{4.11}$$

This is a double zero expansion satisfying Definition 13.

4.4 Double Zero Expansions for ABS 2-Forms

In this subsection we derive double zero expansions for the action over an elementary cube for each of the discrete 2-forms for arbitrary members of the ABS list. The double zero expansions are constructed from Taylor expansions in one variable, where the variable represents either a quad polynomial Q, a tetrahedron polynomial T or an E polynomial. In this Taylor expansion, the zeroth order term vanishes due to the closure relation and the first order term vanishes as a consequence of the corner equations. Hence, each discrete 2-form has a double zero expansion in terms of the polynomials associated with its corner equations. These double zero expansions are manifestly symmetric under cyclic permutation of the indices and under point inversion.

Cross-square 2-form. Recall that $Q_{ij} = 0$ implies that $\mathcal{L}_{ij}^{\boxtimes} = g(\alpha_i, \alpha_j)$ and that $\nabla \mathcal{L}_{ij}^{\boxtimes} = 0$ (see Proposition 10 and its proof). With that in mind, we consider a change of variables from (u, u_i, u_j, u_{ij}) to $(Q_{ij}, u_i, u_j, u_{ij})$, so that we can easily perform a Taylor expansions "about solutions", i.e. about $Q_{ij} = 0$.

Since the quad polynomial Q_{ij} is multiaffine, we can write it as

$$Q_{ij}(u, u_i, u_j, u_{ij}, \alpha_i, \alpha_j) = r(u_i, u_j, u_{ij}, \alpha_i, \alpha_j)u + s(u_i, u_j, u_{ij}, \alpha_i, \alpha_j), \tag{4.12}$$

so it is possible to write u as a rational function of Q_{ij}, u_i, u_j, u_{ij}:

$$u = V(Q_{ij}, u_i, u_j, u_{ij}, \alpha_i, \alpha_j) := \frac{Q_{ij} - s(u_i, u_j, u_{ij}, \alpha_i, \alpha_j)}{r(u_i, u_j, u_{ij}, \alpha_i, \alpha_j)}\,. \tag{4.13}$$

In the case of the cross square 2-form, we can consider the Lagrangian on a single square, apply this variable transformation, and then Taylor expand about $Q_{ij} = 0$

$$\begin{aligned} \mathcal{L}_{ij}^{\boxtimes} &= \mathcal{L}_{ij}^{\boxtimes}\left(V(Q_{ij}, u_i, u_j, u_{ij}, \alpha_i, \alpha_j), u_i, u_j, u_{ij}, \alpha_i, \alpha_j\right) \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{n!} \frac{\partial^n}{\partial {Q_{ij}}^n} \mathcal{L}_{ij}^{\boxtimes}(V(Q_{ij}, u_i, u_j, u_{ij}, \alpha_i, \alpha_j), u_i, u_j, u_{ij}, \alpha_i, \alpha_j)\Big|_{Q_{ij}=0} \right) Q_{ij}^n\,. \end{aligned} \tag{4.14}$$

Now, the fact that for $Q_{ij} = 0$ there holds $\mathcal{L}_{ij}^{\boxtimes} = g(\alpha_i, \alpha_j)$ and $\nabla \mathcal{L}_{ij}^{\boxtimes} = 0$ implies that the zeroth order term is the constant $g(\alpha_i, \alpha_j)$ and that the first order term vanishes. Hence we have:

$$\mathcal{L}_{ij}^{\boxtimes} = g(\alpha_i, \alpha_j) + \sum_{n=2}^{\infty} \left(\frac{1}{n!} \frac{\partial^n}{\partial {Q_{ij}}^n} \mathcal{L}_{ij}^{\boxtimes}(V(Q_{ij}, u_i, u_j, u_{ij}, \alpha_i, \alpha_j), u_i, u_j, u_{ij}, \alpha_i, \alpha_j)\Big|_{Q_{ij}=0} \right) Q_{ij}^n. \tag{4.15}$$

This has a significant consequence for the action of $\mathcal{L}_{ij}^{\boxtimes}$ over an elementary cube,

$$S_{ijk}^{\boxtimes} = -\mathcal{L}_{ij}^{\boxtimes} + \overleftrightarrow{\mathcal{L}_{ij}^{\boxtimes}} + \circlearrowleft_{ijk}\,. \tag{4.16}$$

Indeed, we can substitute in the expansion (4.15), using the fact that $g(\alpha_i, \alpha_j) - \overline{(\cdots)} = 0$, to conclude the following:

Proposition 15. *The action of the Lagrangian 2-form $\mathcal{L}^{\boxtimes}_{ij}$ over an elementary cube has a double zero expansion in terms of Q_{ij}, Q_{jk}, Q_{ki}, $\overleftrightarrow{Q}_{ij}$, $\overleftrightarrow{Q}_{jk}$ $\overleftrightarrow{Q}_{ki}$:*

$$S^{\boxtimes}_{ijk} = -\sum_{n=2}^{\infty}\left(\frac{1}{n!}\frac{\partial^n}{\partial {Q_{ij}}^n}\mathcal{L}^{\boxtimes}_{ij}(V(Q_{ij},u_i,u_j,u_{ij},\alpha_i,\alpha_j),u_i,u_j,u_{ij},\alpha_i,\alpha_j)\Big|_{Q_{ij}=0}\right)Q^n_{ij} - \overline{(\cdots)} + \circlearrowleft_{ijk}\,. \tag{4.17}$$

Cross 2-form. Recall that $T,\overleftrightarrow{T}=0$ implies that $S^{\times}_{ijk}=0$ (see Theorem 8) and that $\nabla S^{\times}_{ijk}=0$ (these are the corner equations). With that in mind, and with the intention of finding a series expansion about $T,\overleftrightarrow{T}=0$, we consider a change of variables form $(u,u_i,u_j,u_k,u_{ij},u_{jk},u_{ki},u_{ijk})$ to $(T,u_i,u_j,u_k,u_{ij},u_{jk},u_{ki},\overleftrightarrow{T})$. Since the tetrahedron polynomial T is multiaffine, it is possible to rewrite u in terms of T (and u_{ijk} in terms of $\overleftrightarrow{T}$) in the following way for the entire ABS list:

$$u = W(T,u_{ij},u_{jk},u_{ki},\alpha_i,\alpha_j,\alpha_k)\,, \tag{4.18a}$$

$$u_{ijk} = W(\overleftrightarrow{T},u_k,u_i,u_j,\alpha_i,\alpha_j,\alpha_k)\,. \tag{4.18b}$$

Here, W is a fraction with a numerator which is multiaffine in (T,u_{ij},u_{jk},u_{ki}) and a denominator which is multiaffine in (u_{ij},u_{jk},u_{ki}). Now we consider the action on an elementary cube of the cross 2-form, apply the variable transformation, and Taylor expand about $T=0$ and $\overleftrightarrow{T}=0$:

$$\begin{aligned} S^{\times}_{ijk} &= -\mathcal{L}_{\times}(u,u_i,u_j,u_{ij},\alpha_i,\alpha_j) - \overline{(\cdots)} + \circlearrowleft_{ijk} \\ &= -\Lambda(u_i,u_j,\alpha_i-\alpha_j) + \Lambda(W(T,u_{ij},u_{jk},u_{ki},\alpha_i,\alpha_j),u_{ij},\alpha_i-\alpha_j) - \overline{(\cdots)} + \circlearrowleft_{ijk} \\ &= -\Lambda(u_i,u_j,\alpha_i-\alpha_j) + \sum_{n=0}^{\infty}\left(\frac{1}{n!}\frac{\partial^n}{\partial T^n}\Lambda(W(T,u_{ij},u_{jk},u_{ki},\alpha_i,\alpha_j),u_{ij},\alpha_i-\alpha_j)\Big|_{T=0}\right)T^n \\ &\quad - \overline{(\cdots)} + \circlearrowleft_{ijk}\,. \end{aligned} \tag{4.19}$$

Now, the fact that $T,\overleftrightarrow{T}=0 \implies S^{\times}_{ijk}, \nabla S^{\times}_{ijk}=0$ means that the zeroth and first order terms must vanish (i.e. the sum starts at $n=2$), so we have:

Proposition 16. *The action of the Lagrangian 2-form $\mathcal{L}^{\times}_{ij}$ over an elementary cube has a double zero expansion in terms of $T,\overleftrightarrow{T}$*

$$S^{\times}_{ijk} = \sum_{n=2}^{\infty}\left(\frac{1}{n!}\frac{\partial^n}{\partial T^n}\Lambda(W,u_{ij},\alpha_i-\alpha_j)\Big|_{T=0}\right)T^n - \overline{(\cdots)} + \circlearrowleft_{ijk}\,, \tag{4.20}$$

where $W = W(T, u_{ij}, u_{jk}, u_{ki}, \alpha_i, \alpha_j)$.

Triangle 2-form. We note that $S^{\boxtimes}_{ijk} = 2S^{\triangle}_{ijk} + S^{\times}_{ijk}$ and that $S^{\triangle}_{ijk}(u_i, u_j, u_k, u_{ij}, u_{jk}, u_{ki})$ does not depend on u or u_{ijk}. Thus, imposing the equations $T, \overleftrightarrow{T} = 0$ has no effect on $S^{\triangle}_{ijk}$: we can take $T = 0$ and $\overleftrightarrow{T} = 0$ as the definition of u and u_{ijk} in terms of the variables $(u_i, u_j, u_k, u_{ij}, u_{jk}, u_{ki})$ that occur in $S^{\triangle}_{ijk}$. Since $T, \overleftrightarrow{T} = 0$ implies $S^{\times}_{ijk} = 0$, we have that

$$T, \overleftrightarrow{T} = 0 \qquad \Longrightarrow \qquad S^{\boxtimes}_{ijk} = 2S^{\triangle}_{ijk}. \tag{4.21}$$

Now recall that $E, \overleftrightarrow{E} = 0$ implies $S^{\triangle}_{ijk} = 0$ (see Theorem 8) and $\nabla S^{\triangle}_{ijk} = 0$ (these are the corner equations). With all of this in mind, we note that the polynomial identity (3.61) between T, Q_{ij} and E_{ij} implies that

$$T, \overleftrightarrow{T} = 0 \qquad \Longrightarrow \qquad Q_{ij} = \frac{\gamma(\alpha_i, \alpha_j, \alpha_k)}{\frac{\partial T}{\partial u}} E_{ij}\,. \tag{4.22}$$

Applying these observations to the double zero expansion of the action of the cross square Lagrangian (4.17) we find:

Proposition 17. *The action of the Lagrangian 2-form* $\mathcal{L}^{\triangle}_{ijk}$ *over an elementary cube has a double zero expansion in terms of* E_{ij}, E_{jk}, E_{ki}, $\overleftrightarrow{E}_{ij}$, $\overleftrightarrow{E}_{jk}$, $\overleftrightarrow{E}_{ki}$

$$S^{\triangle}_{ijk} = -\frac{1}{2} \sum_{n=2}^{\infty} \frac{\frac{\partial^n}{\partial {Q_{ij}}^n} \mathcal{L}^{\boxtimes}_{ij}(V, u_i, u_j, u_{ij}, \alpha_i, \alpha_j)\Big|_{Q_{ij}=0}}{n! \left(\frac{\partial T}{\partial u} \frac{1}{\gamma(\alpha_i,\alpha_j,\alpha_k)}\right)^n} E^n_{ij} - \overline{(\cdots)} + \circlearrowleft_{ijk}\,, \tag{4.23}$$

where $V(Q_{ij}, u_i, u_j, u_{ij}, \alpha_i, \alpha_j)$.

Trident 2-form. Recall that $Q, \overleftrightarrow{Q} = 0$ implies $S^{\curlyvee}_{ijk} = 0$ (see Theorem 8) and $\nabla S^{\curlyvee}_{ijk} = 0$, and that

$$S^{\curlyvee}_{ijk} = \frac{1}{2} S^{\times}_{ijk} + \frac{1}{2} S^{\boxtimes}_{ijk}\,. \tag{4.24}$$

From the three-leg equation (3.7a) and its point inversion, we can derive the following relations between the multiaffine polynomials T in terms of Q_{ij}, Q_{jk} and Q_{ki}

$$c\,T = d_{ij}\,Q_{ij} + d_{jk}\,Q_{jk} + d_{ki}\,Q_{ki}\,, \tag{4.25a}$$

$$\overleftrightarrow{c}\,\overleftrightarrow{T} = \overleftrightarrow{d}_{ij}\,\overleftrightarrow{Q}_{ij} + \overleftrightarrow{d}_{jk}\,\overleftrightarrow{Q}_{jk} + \overleftrightarrow{d}_{ki}\,\overleftrightarrow{Q}_{ki}\,, \tag{4.25b}$$

where d and c_{ij} are polynomials. We can apply these observations to the double zero expansion of the cross 2-form (4.20) and cross-square 2-form (4.17) and conclude the following:

Proposition 18. *The action of Lagrangian 2-form $\mathcal{L}_{ij}^{\swarrow}$ over a an elementary cube has a double zero expansion in terms of Q_{ij}, Q_{jk}, Q_{ki}, $\overleftrightarrow{Q}_{ij}$, $\overleftrightarrow{Q}_{jk}$ $\overleftrightarrow{Q}_{ki}$*

$$
\begin{aligned}
S_{ijk}^{\swarrow} = \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{1}{n!}\frac{\partial^n}{\partial T^n}\Lambda(W, u_{ij}, \alpha_i - \alpha_j)\Big|_{T=0}\right)\frac{1}{c^n}\left(d_{ij}\, Q_{ij} + d_{jk}\, Q_{jk} + d_{ki}\, Q_{ki}\right)^n \\
- \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{1}{n!}\frac{\partial^n}{\partial {Q_{ij}}^n}\mathcal{L}_{ij}^{\boxtimes}(V, u_i, u_j, u_{ij}, \alpha_i, \alpha_j)\Big|_{Q_{ij}=0}\right) Q_{ij}^n - \overline{(\cdots)} + \circlearrowleft_{ijk}\,,
\end{aligned}
\tag{4.26}
$$

where $W = W(T, u_{ij}, u_{jk}, u_{ki}, \alpha_i, \alpha_j)$ and $V = V(Q_{ij}, u_i, u_j, u_{ij}, \alpha_i, \alpha_j)$.

4.4.1 Double Zero Expansions in terms of Octahedron Equations

Some of the double zero expansions above can be written in terms of octahedron polynomials. For Q4 and A2 we can rewrite the expansion (4.23) using Equation (3.64). With the cyclic symmetry and point inversion symmetry of Ω_1 and Ω_2, we can conclude that:

Proposition 19. *For Q4 and A2, a double zero expansion for the cube action of the triangle 2-form in terms of Ω_1, Ω_2 is given by*

$$
S_{ijk}^{\triangle} = -\frac{1}{2}\sum_{n=2}^{\infty}\frac{\frac{\partial^n}{\partial {Q_{ij}}^n}\mathcal{L}_{ij}^{\boxtimes}(V, u_i, u_j, u_{ij}, \alpha_i, \alpha_j)\Big|_{Q_{ij}=0}}{n!\left(\frac{\partial T}{\partial u}\frac{\mu(u_i, u_j, u_{jk}, u_{ki}, \alpha_i, \alpha_j, \alpha_k)}{\gamma(\alpha_i, \alpha_j, \alpha_k)}\right)^n}\left(\frac{\partial\Omega_1}{\partial u_k}\Omega_2 - \frac{\partial\Omega_2}{\partial u_k}\Omega_1\right)^n - \overline{(\cdots)} + \circlearrowleft_{ijk}\,. \tag{4.27}
$$

where $V = V(Q_{ij}, u_i, u_j, u_{ij}, \alpha_i, \alpha_j)$.

For the rest of the ABS list we can rewrite the expansion (4.23) using Equation (3.65) and can conclude that:

Proposition 20. *For Q1, Q2, Q3, H1, H2, H3 and A1, a double zero expansion for the cube action of the triangle 2-form in terms of Ω_1, Ω_2 is given by*

$$
S_{ijk}^{\triangle} = -\frac{1}{2}\sum_{n=2}^{\infty}\frac{\frac{\partial^n}{\partial {Q_{ij}}^n}\mathcal{L}_{ij}^{\boxtimes}(V, u_i, u_j, u_{ij}, \alpha_i, \alpha_j)\Big|_{Q_{ij}=0}}{n!\left(\frac{\partial T}{\partial u}\frac{1}{\gamma(\alpha_i, \alpha_j, \alpha_k)}g_1\left(\frac{\partial\Omega_1}{\partial u_{ij}} - \frac{\partial\Omega_1}{\partial u_k}\right)\right)^n}\left(\frac{\partial\Omega_1}{\partial u_k}\Omega_2 - \frac{\partial\Omega_2}{\partial u_k}\Omega_1\right)^n - \overline{(\cdots)} + \circlearrowleft_{ijk}\,, \tag{4.28}
$$

where $V = V(Q_{ij}, u_i, u_j, u_{ij}, \alpha_i, \alpha_j)$.

These propositions give a possible variational interpretation to the octahedron equations. The double zero expansions imply that the equations $\Omega_1 = 0$ and $\Omega_2 = 0$ are sufficient conditions for criticality, hence they imply the corner equations.

4.5 Example: Double Zero Expansions for H1

In this subsection we show that the general construction described above leads to succinct double zero expansions for the actions over the cube of each of the discrete 2-forms associated with H1. For this example, we will show by explicit computation that the zeroth and first order terms vanish in the Taylor expansions.

Cross-square Lagrangian. Solving the quad equation $Q_{ij} = 0$ for u, with Q_{ij} given by Equation (3.13), we find a change of variables expressing u in terms of the quad polynomial Q_{ij}:

$$u = \frac{\alpha_i - \alpha_j + Q_{ij}}{u_i - u_j} + u_{ij} \,. \tag{4.29}$$

Now we apply this to a single cross-square Lagrangian and Taylor expand in Q_{ij}

$$\begin{aligned} \mathcal{L}^{\boxtimes}_{ij} &= uu_i + u_j u_{ij} - uu_j - u_i u_{ij} - (\alpha_i - \alpha_j)\log(u - u_{ij}) - (\alpha_i - \alpha_j)\log(u_i - u_j)\,, \\ &= (u - u_{ij})(u_i - u_j) - (\alpha_i - \alpha_j)\log\left((u - u_{ij})(u_i - u_j)\right)\,, \\ &= \alpha_i - \alpha_j + Q_{ij} - (\alpha_i - \alpha_j)\log(\alpha_i - \alpha_j + Q_{ij}) \\ &= (\alpha_i - \alpha_j)(1 - \log(\alpha_i - \alpha_j)) - \sum_{n=2}^{\infty} \frac{Q_{ij}^n}{n(\alpha_j - \alpha_i)^{n-1}}\,. \end{aligned} \tag{4.30}$$

We find that the first order term vanishes as expected, but the zeroth order term does not. It depends only on the lattice parameters, in line with Proposition 10. (These lattice parameter terms could be removed by redefining the leg terms of Lagrangians.) When we consider the action around the cube, the zeroth order contributions cancel and we find

$$\begin{aligned} S^{\boxtimes}_{ij} &= \overleftrightarrow{\mathcal{L}^{\boxtimes}_{ij}} - \mathcal{L}^{\boxtimes}_{ij} + \circlearrowleft_{ijk} \\ &= \sum_{n=2}^{\infty} \frac{Q_{ij}^n}{n(\alpha_j - \alpha_i)^{n-1}} - \overline{(\cdots)} + \circlearrowleft_{ijk}\,. \end{aligned} \tag{4.31}$$

Equation (4.31) is a double zero expansion for the action around the cube of the cross square 2-form associated with H1 in terms of Q_{ij},Q_{jk},Q_{ki},$\overleftrightarrow{Q}_{ij}$, $\overleftrightarrow{Q}_{jk}$,$\overleftrightarrow{Q}_{ki}$.

Cross Lagrangian. In order to derive a double zero expansion for the cross 2-form associated with H1, we consider a variable transformation for u in terms of T, obtained from Equation (3.17),

$$u = \frac{T - ((\alpha_i - \alpha_j)u_{jk}u_{ki} + \circlearrowleft_{ijk})}{((\alpha_i - \alpha_j)u_{ij} + \circlearrowleft_{ijk})}\,. \tag{4.32}$$

From this we obtain

$$u-u_{ij}=\frac{T+(\alpha_i-\alpha_j)(u_{ij}-u_{jk})(u_{ki}-u_{ij})}{((\alpha_i-\alpha_j)u_{ij}+\circlearrowleft_{ijk})}\,. \tag{4.33}$$

We have an analogous transformation for u_{ijk} in terms of $\overleftrightarrow{T}$. Now we apply this to the action around the cube of the cross 2-form and Taylor expand in T and $\overleftrightarrow{T}$. Using the point inversion symmetry we find

$$\begin{aligned} S^{\times}_{ijk} &= -\mathcal{L}^{\times}_{ij}+\overleftrightarrow{\mathcal{L}}^{\times}_{ij}+\circlearrowleft_{ijk} \\ &= -(\alpha_i-\alpha_j)\log(u_i-u_j)+(\alpha_i-\alpha_j)\log(u-u_{ij})-\overline{(\cdots)}+\circlearrowleft_{ijk} \\ &= (\alpha_i-\alpha_j)\log(u_{jk}-u_{ki})+(\alpha_i-\alpha_j)\log(u-u_{ij})-\overline{(\cdots)}+\circlearrowleft_{ijk} \\ &= (\alpha_i-\alpha_j)\log((u-u_{ij})(u_{jk}-u_{ki}))-\overline{(\cdots)}+\circlearrowleft_{ijk}\,. \end{aligned} \tag{4.34}$$

Now we substitute u using Equation (4.33):

$$S^{\times}_{ijk}=(\alpha_i-\alpha_j)\log\left(\frac{T(u_{jk}-u_{ki})+(\alpha_i-\alpha_j)(u_{ij}-u_{jk})(u_{ki}-u_{ij})(u_{jk}-u_{ki})}{((\alpha_i-\alpha_j)u_{ij}+\circlearrowleft_{ijk})}\right) -\overline{(\cdots)}+\circlearrowleft_{ijk}\,. \tag{4.35}$$

Then we observe that

$$\begin{aligned} (\alpha_i-\alpha_j)\log((u_{ij}-u_{jk})(u_{ki}-u_{ij})(u_{jk}-u_{ki}))+\circlearrowleft_{ijk}=0\,, \\ (\alpha_i-\alpha_j)\log((\alpha_i-\alpha_j)u_{ij}+\circlearrowleft_{ijk})+\circlearrowleft_{ijk}=0\,, \end{aligned} \tag{4.36}$$

because the logarithms are invariant under cyclic permutations of i,j,k, so we find

$$\begin{aligned} S^{\times}_{ijk} &= (\alpha_i-\alpha_j)\log\left(\frac{T}{(u_{ij}-u_{jk})(u_{ki}-u_{ij})}+(\alpha_i-\alpha_j)\right)-\overline{(\cdots)}+\circlearrowleft_{ijk} \\ &= -\sum_{n=2}^{\infty}\frac{T^n}{n(\alpha_i-\alpha_j)^{n-1}(u_{ij}-u_{jk})^n(u_{ij}-u_{ki})^n}-\overline{(\cdots)}+\circlearrowleft_{ijk}\,. \end{aligned} \tag{4.37}$$

Note that the zeroth and first order terms vanish. Hence, Equation (4.37) is a double zero expansion in terms of T and $\overleftrightarrow{T}$.

Triangle Lagrangian. In order to derive a double zero expansion for the action on the elementary cube of the triangle 2-form, we note the following relation for H1:

$$T=0 \qquad \Longrightarrow \qquad Q_{ij}=\frac{(\alpha_i-\alpha_j)E_{ij}}{(\alpha_i(u_{ij}-u_{ki})+\circlearrowleft_{ijk})}\,. \tag{4.38}$$

$T, \overleftrightarrow{T} = 0 \implies S^{\times}_{ijk} = 0$, so we deduce form Equations (3.44) and (3.50) that

$$T, \overleftrightarrow{T} = 0 \qquad \implies \qquad S^{\triangle}_{ijk} = \frac{1}{2} S^{\boxtimes}_{ijk} - \frac{1}{2} S^{\times}_{ijk} = \frac{1}{2} S^{\boxtimes}_{ijk} \,. \tag{4.39}$$

Since $S^{\triangle}_{ijk}(u_i, u_j, u_k, u_{ij}, u_{ji}, u_{ki})$ does not depend on u or u_{ijk}, we can assume that $T = 0$ and $\overleftrightarrow{T} = 0$, without affecting the action. Thus, we can write the cube action for the triangle 2-form as

$$\begin{aligned} S^{\triangle}_{ijk} &= \frac{1}{2} \sum_{n=2}^{\infty} \frac{1}{n(\alpha_j - \alpha_i)^{n-1}} \left(\frac{(\alpha_i - \alpha_j) E_{ij}}{(\alpha_i (u_{ij} - u_{ki}) + \circlearrowleft_{ijk})} \right)^n - \overline{(\cdots)} + \circlearrowleft_{ijk} \\ &= \frac{1}{2} \sum_{n=2}^{\infty} \frac{(\alpha_j - \alpha_i)(-1)^n}{n \left(\alpha_i (u_{ij} - u_{ki}) + \circlearrowleft_{ijk}\right)^n} E_{ij}^n - \overline{(\cdots)} + \circlearrowleft_{ijk} \,. \end{aligned} \tag{4.40}$$

Equation (4.40) is a double zero expansion for the triangle 2-form in terms of E_i, E_j, E_k, $\overleftrightarrow{E}_i$, $\overleftrightarrow{E}_j$, $\overleftrightarrow{E}_k$.

We can use the identities (3.65), where $g_1 = 1$ for H1, to rewrite this in terms of Ω_1 and Ω_2:

$$S^{\triangle}_{ijk} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{(\alpha_j - \alpha_i)(-1)^n}{n \left(\alpha_i (u_{ij} - u_{ki}) + \circlearrowleft_{ijk}\right)^n} \left(\frac{\frac{\partial \Omega_1}{\partial u_k} \Omega_2 - \frac{\partial \Omega_2}{\partial u_k} \Omega_1}{\frac{\partial \Omega_1}{\partial u_{ij}} - \frac{\partial \Omega_1}{\partial u_k}} \right)^n - \overline{(\cdots)} + \circlearrowleft_{ijk} \,. \tag{4.41}$$

Equation (4.41) is a double zero expansion for the triangle 2-form in terms of the octahedron polynomials.

Trident Lagrangian. To derive the double zero expansion for action of the trident 2-form, we consider equation (4.25), which shows that cyclic combinations of quad equations lead to the tetrahedron equation. For H1, this can be written explicitly as

$$T = -(u - u_{jk})(u - u_{ki}) Q_{ij} - (u - u_{ki})(u - u_{ij}) Q_{jk} - (u - u_{ij})(u - u_{jk}) Q_{ki}. \tag{4.42}$$

Using Equation (3.44), we can write the action on an elementary cube of the trident 2-form as

$$\begin{aligned} S^{\swarrow}_{ijk} &= \frac{1}{2} S^{\times}_{ijk} + \frac{1}{2} S^{\boxtimes}_{ijk} \\ &= \frac{1}{2} \sum_{n=2}^{\infty} \left(\frac{Q_{ij}^n}{n(\alpha_j - \alpha_i)^{n-1}} - \frac{T^n}{n(\alpha_i - \alpha_j)^{n-1} (u_{ij} - u_{jk})^n (u_{ij} - u_{ki})^n} \right) - \overline{(\cdots)} + \circlearrowleft_{ijk} \\ &= \frac{1}{2} \sum_{n=2}^{\infty} \left(\frac{Q_{ij}^n}{n(\alpha_j - \alpha_i)^{n-1}} - \frac{(-Q_{ij}(u - u_{jk})(u - u_{ki}) + \circlearrowleft_{ijk})^n}{n(\alpha_i - \alpha_j)^{n-1} (u_{ij} - u_{jk})^n (u_{ij} - u_{ki})^n} \right) - \overline{(\cdots)} + \circlearrowleft_{ijk} \,. \end{aligned} \tag{4.43}$$

Equation (4.43) a double zero expansion for the cube action of the trident 2-form in terms of the quad polynomials Q_{ij},Q_{jk},Q_{ki},$\overleftrightarrow{Q}_{ij}$,$\overleftrightarrow{Q}_{jk}$,$\overleftrightarrow{Q}_{ki}$.

4.6 Example: Double Zero Expansions for $Q1_{\delta=0}$

In this subsection, we apply the construction above to $Q1_{\delta=0}$ and note down explicit double zero expansions for three of the 2-forms in terms of their multiaffine corner polynomials.

The cube action of the $Q1_{\delta=0}$ cross square 2-form can be written as a double zero expansion in terms of Q_{ij},Q_{jk},Q_{ki},$\overline{Q}_{ij}$, $\overline{Q}_{jk}$,$\overline{Q}_{ki}$

$$S^{\boxtimes}_{ijk} = \sum_{n=2}^{\infty} \left(\frac{1}{n\alpha_i^{n-1}(u_i - u_{ij})^n (u_i - u_j)^n} - \frac{1}{n(\alpha_i - \alpha_j)^{n-1}(u_i - u_{ij})^n(-u_j + u_{ij})^n} - \frac{1}{n\alpha_j^{n-1}(u_i - u_j)^n(u_j - u_{ij})^n} \right) Q^n_{ij} - \overline{(\cdots)} + \circlearrowleft_{ijk} . \tag{4.44}$$

On the level of polynomials, if we can impose $T, \overline{T} = 0$ then we can rewrite Q_{ij} in terms of E_{ij} in the following way

$$T, \overline{T} = 0 \qquad \Longrightarrow \qquad Q_{ij} = \frac{(\alpha_j - \alpha_i)E_{ij}}{\alpha_k(\alpha_i(u_{ki} - u_{ij}) + \circlearrowleft_{ijk})} . \tag{4.45}$$

We can use $S^{\triangle}_{ijk} = \frac{1}{2}S^{\boxtimes}_{ijk} + \frac{1}{2}S^{\times}_{ijk}$ and use the fact that $S^{\triangle}_{ijk}$ is independent of u to safely apply $T, \overline{T} = 0$. This leads to a double zero expansion of the cube action of the triangle 2-form in terms of E_{ij},E_{jk},E_{ki},$\overline{E}_{ij}$, $\overline{E}_{jk}$,$\overline{E}_{ki}$

$$S^{\triangle}_{ijk} = \frac{1}{2}\sum_{n=2}^{\infty} \left(\frac{1}{n\alpha_i^{n-1}(u_i - u_{ij})^n (u_i - u_j)^n} - \frac{1}{n(\alpha_i - \alpha_j)^{n-1}(u_i - u_{ij})^n(-u_j + u_{ij})^n} - \frac{1}{n\alpha_j^{n-1}(u_i - u_j)^n(u_j - u_{ij})^n} \right) \frac{(\alpha_j - \alpha_i)^n E^n_{ij}}{\alpha_k^n(\alpha_i(u_{ki} - u_{ij}) + \circlearrowleft_{ijk})^n} - \overline{(\cdots)} + \circlearrowleft_{ijk} . \tag{4.46}$$

Recall that all tetrahedron equations associated with ABS quad equations are of type Q. The tetrahedron equation for this example of $Q1_{\delta=0}$ is the same as it is for H1. Identical to H1, the cube action of the $Q1_{\delta=0}$ cross 2-form can be written as a double zero expansion in terms of T, $\overline{T}$

$$S^{\times}_{ijk} = -\sum_{n=2}^{\infty} \frac{T^n}{n(\alpha_i - \alpha_j)^{n-1}(u_{ij} - u_{jk})^n(u_{ij} - u_{ki})^n} - \overline{(\cdots)} + \circlearrowleft_{ijk} . \tag{4.47}$$

Next we consider some of these double zero constructions for another member of the

ABS list.

4.7 Example: Double Zero Expansions for H2

In this subsection, we apply the construction to the H2 quad equation. We simply consider the H2 cross square 2-form and H2 cross 2-form, for brevity.

This is a double zero expansion for the cube action of the H2 cross square 2-form in terms of Q_{ij},Q_{jk},Q_{ki},$\overline{Q}_{ij}$, $\overline{Q}_{jk}$,$\overline{Q}_{ki}$

$$\begin{aligned} S^{\boxtimes}_{ijk} = \sum_{n=2}^{\infty} \frac{1}{\alpha_i - \alpha_j - u_i + u_j} \Bigg(& \frac{1}{n!\, n(\alpha_j - \alpha_i)^n(\alpha_j + u_i + u_{ij})^n} - \frac{1}{n!\, n(\alpha_j - \alpha_i)^n(\alpha_i + u_j + u_{ij})^n} \\ & + \frac{1}{n!\,(-\alpha_i + \alpha_j - u_i + u_j)^n(\alpha_i + u_j + u_{ij})^n} \\ & - \frac{1}{n!\,(-\alpha_i + \alpha_j - u_i + u_j)^n(\alpha_j + u_i + u_{ij})^n} \Bigg) Q^n_{ij} - \overline{(\cdots)} + \circlearrowleft_{ijk} . \end{aligned} \tag{4.48}$$

This is a double zero expansion for the cube action of the H2 cross 2-form in terms of $T,\overline{T}$

$$\begin{aligned} S^{\times}_{ijk} = \sum_{n=2}^{\infty} \frac{1}{(\alpha_i(u_{ki} - u_{ij}) + \circlearrowleft_{ijk})} \Bigg(& \frac{1}{n!\,(\alpha_i - \alpha_j)^n(\alpha_j - \alpha_k + u_{ij} - u_{ki})^n(\alpha_i - \alpha_k - u_{ij} + u_{jk})^n} \\ & - \frac{1}{n!\,(\alpha_i - \alpha_j)^n(\alpha_j - \alpha_k - u_{ij} + u_{ki})^n(\alpha_i - \alpha_k + u_{ij} - u_{jk})^n} \Bigg) T^n - \overline{(\cdots)} + \circlearrowleft_{ijk} . \end{aligned} \tag{4.49}$$

4.8 Chapter Conclusions

We formulated the double-zero property, which has recently seen a lot of emphasis in the continuous and semi-discrete settings, on the discrete level. We showed that the exterior derivative of the discrete 2-forms (the action around the cube) for the ABS quad equations can be written as double zero expansions of their multiaffine corner equations. We emphasised that the double zero expansions follows from the property $\nabla S_{ijk} = 0 \implies S_{ijk} = 0$, which defines a discrete Lagrangian 2-form.

The Trident and Cross square Lagrangian 2-forms admit double zero expansions in terms of quad equations, emphasising that they encapsulate the quad equations variationally. The Cross Lagrangian 2-form also admits double zero expansions in terms of tetrahedron equations, emphasising that it encapsulates the tetrahedron equations variationally. The octahedron equations had previously only been considered on the level of the equations. Now we have written double zero expansions of discrete 2-forms (particularly the triangle 2-form) in terms of the octahedron equations, emphasising how they are a minimal and symmetric set of multiaffine equations.

This reinforces the universality of the double zero property for all Lagrangian multiforms, beyond the examples considered here. These double zero expansions set us up for investigation of conservation laws [43, 53] and variational symmetries [48, 54] in this discrete quadrilateral setting.

Chapter 5

Periodically Reducing Quad 2-Forms to Discrete 1-Forms

In this Chapter, we develop a novel framework to carry out periodic reductions of discrete Lagrangian 2-forms to discrete Lagrangian 1-forms. We reinforce the phenomenon that integrable properties, including the Lagrangian multiform structure, are preserved under periodic reductions [23].

This framework applies to the discrete Lagrangian 2-forms outlined in Chapter 3, such that we elevate periodic reductions of quad equations to the Lagrangian multiform level. Integrable ordinary difference equations have been derived from periodic reductions of the lattice potential Korteweg-de Vries (H1) equation in [15, 24]. Furthermore in [26], periodic reductions were applied to the multidimensionally-consistent linear quad equation (which admits a discrete Lagrangian 2-form). This periodic reduction on the level of the equations produced a system of multicomponent ordinary difference equations. Then reduced variables were applied to derive a commuting system of discrete harmonic oscillators. A discrete Lagrangian 1-form was derived and then quantised in the sense of path integrals. The goal of our framework is to derive more general discrete Lagrangian 1-forms, with multiform Euler-Lagrange equations that are non-linear and richer than a system of harmonic oscillator equations. Furthermore, with future work, we would like these more general Lagrangian 1-forms to be quantisable in the sense of path integrals, in order to generalise the work of [26].

We note that there is a previous example of a discrete Lagrangian 2-form being periodically reduced to a discrete Lagrangian 1-form [12], however its periodicity is in one direction and not on a staircase. Furthermore, our framework offers a unique picture of the periodic reduction on the action involving a four dimensional periodic quad surface, which is a novel and promising perspective.

Our starting points are the discrete Lagrangian 2-forms discussed so far, but generalised to act on quad surfaces in a four dimensional lattice $\mathbf{n} = (n_i, n_j, n_k, n_l) \in \mathbb{Z}^4$. Four dimensions will allow us to consider a periodic staircase in the n_i and n_j directions and evolution perpendicular to this plane, in the n_k or n_l directions. Even though this higher dimensional picture is more abstract, we will see that isolating the periodic reduction from the evolution leads to a useful geometric picture. In the next section we outline how discrete Lagrangian 2-forms can be generalised to act on quad surfaces in a four dimensional lattice.

5.1 Discrete Lagrangian 2-forms in $\mathbb{Z}^4$

A useful feature of discrete Lagrangian 2-forms (Definition 2) which encapsulate multidimensionally-consistent equations, is that the formulation immediately generalises from 3-dimensional lattices to higher-dimensional lattices [28]. For the framework in this Chapter, it will be useful to consider a field $u(n_i, n_j, n_k, n_l)$ over the four dimensional lattice $\mathbb{Z}^4$. We have (2-dimensional) quad surfaces σ made from oriented elementary squares (σ_{ij}, σ_{ik}, σ_{il}, σ_{jk}, σ_{jl}, σ_{kl}) in the four dimensional lattice. The action over quad surfaces is defined analogously to (2.26)

$$S_2[u, \sigma] = \sum_{\sigma_{ij}(\mathbf{n}) \in \sigma} \mathcal{L}_{ij} \tag{5.1}$$

We introduce the subscript 2 to specify that this action corresponds to a discrete Lagrangian 2-form (and not a discrete Lagrangian 1-form). As we are now in $\mathbb{Z}^4$, there are four elementary cubes to consider (cube_{ijk}, cube_{jkl}, cube_{kli}, and cube_{lij}). The statements in Definition 2 apply to each of these cubes. We also introduce the following notation, such that on cube_{ijk} we have

$$\left.\begin{array}{ll}
\dfrac{\partial S_2[u, \text{cube}_{ijk}]}{\partial u} = 0 & \dfrac{\partial S_2[u, \text{cube}_{ijk}]}{\partial T_i T_j T_k u} = 0 \\
\dfrac{\partial S_2[u, \text{cube}_{ijk}]}{\partial T_i u} = 0 & \dfrac{\partial S_2[u, \text{cube}_{ijk}]}{\partial T_j T_k u} = 0 \\
\dfrac{\partial S_2[u, \text{cube}_{ijk}]}{\partial T_j u} = 0 & \dfrac{\partial S_2[u, \text{cube}_{ijk}]}{\partial T_k T_i u} = 0 \\
\dfrac{\partial S_2[u, \text{cube}_{ijk}]}{\partial T_k u} = 0 & \dfrac{\partial S_2[u, \text{cube}_{ijk}]}{\partial T_i T_j u} = 0
\end{array}\right\} \quad \Longrightarrow \quad S_2[u, \text{cube}_{ijk}] = 0\,. \tag{5.2}$$

We have the same statement for: cube_{jkl}, cube_{kli}, and cube_{lij}. On solutions to the multiform Euler-Lagrange equations, the closure relation holds. In other words, the corner equations of the four cubes (cube_{ijk}, cube_{jkl}, cube_{kli}, and cube_{lij}), define the multiform Euler-Lagrange equations and on solutions to those equations the action on each of the cubes vanish.

Our framework produces weak discrete Lagrangian 1-forms which act on discrete paths. Thus we outline what discrete Lagrangian 1-forms are in the next section, with two relevant definitions.

5.2 Discrete Lagrangian 1-Forms

Now we introduce an N-component field $w(\mathbf{n}) = (w_1(\mathbf{n}), \cdots, w_N(\mathbf{n}))$ on the 2-dimensional lattice $\mathbf{n} = (n_k, n_l) \in \mathbb{Z}^2$. It is sufficient to consider $\mathbb{Z}^2$, then it is straightforward to generalise to a higher dimensional quadrilateral lattice. Later when we are on solutions, this field will satisfy (integrable) ordinary difference equations separately in the n_k and n_l directions. For now, we impose nothing.

We consider directed discrete edges at points $\mathbf{n} \in \mathbb{Z}^2$ defined by

$$\Gamma_{\pm k}(\mathbf{n}) := (\mathbf{n}, \mathbf{n} \pm \mathbf{e}_k)\,, \tag{5.3a}$$

$$\Gamma_{\pm l}(\mathbf{n}) := (\mathbf{n}, \mathbf{n} \pm \mathbf{e}_l)\,, \tag{5.3b}$$

A discrete path Γ is a connected set of these discrete edges, depicted in Figure 5.1 and such that $\Gamma_{\pm k}(\mathbf{n}) \in \Gamma$. We define an action over discrete paths Γ by

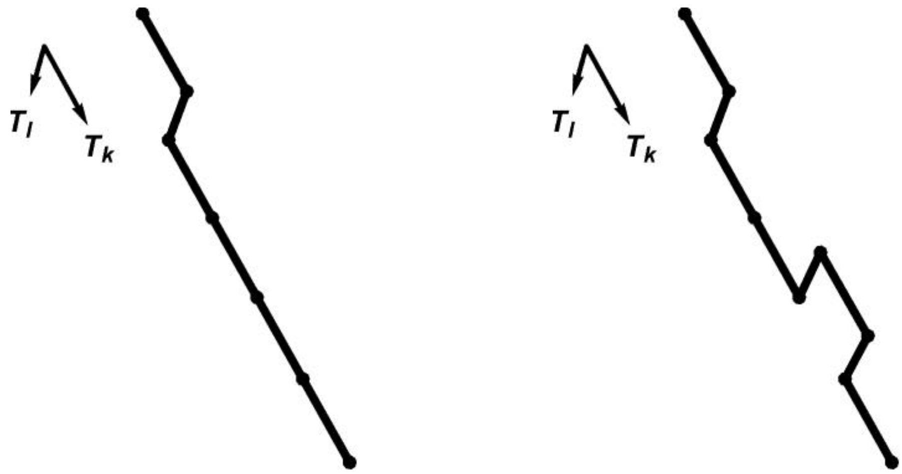
$$S_1[w, \Gamma] = \sum_{\Gamma_{\pm k}(\mathbf{n}) \in \Gamma} \pm \mathcal{L}_k\,. \tag{5.4}$$

Here, the sum $\sum_{\Gamma_{\pm k}(\mathbf{n}) \in \Gamma}$ implicitly sums over all discrete edges in the n_k and n_l directions in the discrete path Γ. Furthermore, each discrete edge is associated with a function at a point in the lattice, defined as:

$$\mathcal{L}_k : (w(\mathbf{n}),\, w(\mathbf{n} + \mathbf{e}_k)) \mapsto \mathbb{C}\,, \tag{5.5a}$$

$$\mathcal{L}_l : (w(\mathbf{n}),\, w(\mathbf{n} + \mathbf{e}_l)) \mapsto \mathbb{C}\,. \tag{5.5b}$$

Note that we do not a priori define these as Lagrangians, as that follows from the definition of a Lagrangian multiform. Also note that each directed edge in the discrete path is signed and is associated with a signed Lagrangian in (5.4).



(a) (b)

Figure 5.1: Two discrete paths Γ and $\Gamma + \text{square}_{kl}$ in the two dimensional lattice, which differ by a local variation of adding a square.

In this setting, let us consider the full definition of a Lagrangian multiform from Section 2.7. We would like this action over discrete paths to satisfy the following:

$$\frac{\delta S_1[w,\Gamma]}{\delta w} = 0 \quad \Longrightarrow \quad \frac{\delta S_1[w,\Gamma]}{\delta \Gamma} = 0\,. \tag{5.6}$$

This is that criticality of the action with respect to local variations of the field produces multiform Euler-Lagrange equations, then on solutions to those equations, the action is critical with respect to local variations of the path. Now we will define both types of criticality for discrete Lagrangian 1-forms.

Similar to discrete Lagrangian 2-form with cubes, it is sufficient to just consider the behaviour of the action on elementary squares, which are defined by

$$\text{square}_{kl} = \{\Gamma_k,\, T_k\Gamma_l,\, T_l\Gamma_{-k},\, \Gamma_{-l}\}\,. \tag{5.7}$$

To see this, let us first consider Figure 5.1. For each of these, consider a discrete path with some fixed end points and simply vary it locally by popping up a square. If the action (5.4) is critical with respect to local variations of the path, then the action should not change by the addition or subtraction of an elementary square. The action around the closed path of the elementary square is

$$S_1[w, \text{square}_{kl}] = \mathcal{L}_k + T_k\mathcal{L}_l - T_l\mathcal{L}_k - \mathcal{L}_l\,. \tag{5.8}$$

We define criticality of the action with respect to local variations of the path to be equivalent to the vanishing of the action on the elementary square

$$\frac{\delta S_1[w,\Gamma]}{\delta\Gamma}=0 \qquad \Longleftrightarrow \qquad S_1[w,\mathrm{square}_{kl}]=0\,. \tag{5.9}$$

Now we consider criticality of the action with respect to local variations of the field. We consider a field $w:\mathbb{Z}^2\mapsto\mathbb{C}^N$ and a discrete path Γ with some boundary. We denote the internal lattice points of the discrete path as $V(\Gamma)\subset\mathbb{Z}^2$. Now we say that the action is critical with respect to local variations of the field if and only if for generic discrete paths Γ the partial derivatives vanish at every internal point:

$$\frac{\delta S_1[w,\Gamma]}{\delta w}=0 \qquad \Longleftrightarrow \qquad \frac{\partial S_1[w,\Gamma]}{\partial u(\mathbf{n})}=0 \quad \forall\,\mathbf{n}\in V(\Gamma)\,. \tag{5.10}$$

Since the elementary square is a discrete path itself, the derivatives at every internal point (every corner) of the action around the square must vanish:

$$\frac{\delta S_1[w,\Gamma]}{\delta w}=0 \qquad \Longrightarrow \qquad \nabla S_1[w,\mathrm{square}_{kl}]=0\,. \tag{5.11}$$

Here we use a slight abuse of notation to denote the four corner derivatives of the action around the square as

$$\nabla S_1[w,\mathrm{square}_{kl}]=\left(\frac{\partial}{\partial w},\frac{\partial}{\partial T_k w},\frac{\partial}{\partial T_l w},\frac{\partial}{\partial T_kT_l w}\right)S_1[w,\mathrm{square}_{kl}]\,. \tag{5.12}$$

Every internal point of a path is either a corner (like the corner of a square) or a straight segment. We can write any straight segment as the combination of two corners of the square:

$$\begin{aligned}\frac{\partial}{\partial T_k w}\left(\mathcal{L}_k+T_k\mathcal{L}_k\right)&=\frac{\partial}{\partial T_k w}\left(\mathcal{L}_k+T_k\mathcal{L}_l-T_k\mathcal{L}_l+T_k\mathcal{L}_k\right)\\ &=\frac{\partial S_1[w,\mathrm{square}_{kl}]}{\partial T_k w}+\frac{\partial S_1[w,T_k\mathrm{square}_{kl}]}{\partial T_k w}\,.\end{aligned} \tag{5.13}$$

Now, since our Lagrangians are autonomous, it follows that criticality of the action with respect to local variations of the field for a generic discrete path is equivalent to criticality of the action on the elementary square:

$$\frac{\delta S_1[w,\Gamma]}{\delta w}=0 \qquad \Longleftrightarrow \qquad \nabla S_1[w,\Gamma]=0\,. \tag{5.14}$$

With (5.6), (5.9) and (5.14), we can define the following

Definition 21. *An action over discrete paths* (5.4) *defines a* ***discrete Lagrangian 1-form*** *if it satisfies the following property. When the field* $w:\mathbb{Z}^2\mapsto\mathbb{C}^N$ *satisfies the* *multiform Euler-Lagrange equations, the closure relation holds:*

$$\nabla S_1[w, square_{kl}] = 0 \quad \Longrightarrow \quad S_1[w, square_{kl}] = 0\,. \tag{5.15}$$

However, for our framework we also need to consider weak Lagrangian multiforms, defined analogously to Definition 1, as follows:

Definition 22. *An action over discrete paths* (5.4) *and ordinary difference equations* $K_1(w), \cdots, K_M(w) = 0$ *together define a* ***weak discrete Lagrangian 1-form****, if the following property is satisfied. When the field* $w : \mathbb{Z}^2 \mapsto \mathbb{C}^N$ *satisfies the given equations, both the closure relation and the multiform Euler-Lagrange equations hold:*

$$K_1(w), \cdots, K_M(w) = 0 \quad \Longrightarrow \quad \begin{cases} S_1[w, square_{kl}] = 0 \\ \nabla S_1[w, square_{kl}] = 0\,. \end{cases} \tag{5.16}$$

5.3 Example: Periodically Reducing Lagrangian 2-Form for H1 Quad Equation

5.3.1 The Staircase Z and Periodic Map θ_Z

In this section, we introduce our periodic reduction framework with an example, to avoid unnecessarily abstract notation. Later we will discuss how this generalises. We will consider a period $N = 4$ staircase and a discrete Lagrangian 2-form associated with the H1 equation.

First of all we introduce an example of a finite staircase $Z \subset \mathbb{Z}^2 \subset \mathbb{Z}^4$ made from $N = 4$ elementary steps in the n_i and n_j directions, depicted in Figure 5.2, that is

$$\begin{aligned} Z = \{&(n_i, n_j, n_k, n_l),\ (n_i+1, n_j, n_k, n_l),\ (n_i+1, n_j+1, n_k, n_l), \\ &(n_i+2, n_j+1, n_k, n_l),\ (n_i+2, n_j+2, n_k, n_l)\} \subset \mathbb{Z}^4\,. \end{aligned} \tag{5.17}$$

Figure 5.2: A finite staircase in Z in the four dimensional lattice, made from $N = 4$ steps in the n_i and n_j directions.

Now we consider our example staircase with $N = 4$, such that there are 4 components

$w = (w_1, w_2, w_3, w_4)$. With this we introduce a periodic map $\theta_Z : u \mapsto w$ which maps to a field where Z is periodic staircase. The field w allows for evolution of the periodic staircase in the n_k and n_l directions. The periodic map θ_Z is defined in the following way, simply as a relabelling:

$$\begin{aligned} w_1 &= \theta_Z(u) & w_2 &= \theta_Z(T_i\, u) & w_3 &= \theta_Z(T_jT_i\, u) \\ w_4 &= \theta_Z(T_jT_i^2\, u) & w_1 &= \theta_Z(T_j^2T_i^2\, u) & & \\ T_k\, w_1 &= \theta_Z(T_k\, u) & T_k\, w_2 &= \theta_Z(T_kT_i\, u) & T_k\, w_3 &= \theta_Z(T_kT_jT_i\, u) \\ T_k\, w_4 &= \theta_Z(T_kT_jT_i^2\, u) & T_k\, w_1 &= \theta_Z(T_kT_j^2T_i^2\, u) & & \end{aligned} \tag{5.18}$$

Figure 5.3 depicts the image of this mapping.

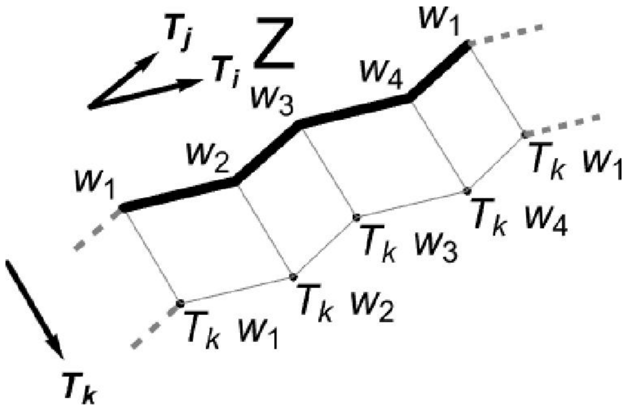


Figure 5.3: This shows the periodic variables $w = (w_1, w_2, w_3, w_4)$ and them shifted in the n_k direction, $T_kw = (T_kw_1, T_kw_2, T_kw_3, T_kw_4)$.

Our framework is fully on the level of the Lagrangians. However, it is useful to first understand the periodic reduction and the map θ_Z on the level of the equations. We apply this periodic reduction to the H1 quad equations. With multidimensional-consistency we can impose H1 quad equations throughout the $\mathbb{Z}^4$ lattice. There are six possible pairs of directions in $\mathbb{Z}^4$, thus the following six H1 quad equations define evolution of the field u in the n_i n_j n_k and n_l direction

$$Q_{ij} = (u - T_iT_ju)(T_iu - T_ju) - \alpha_i + \alpha_j \;= 0\,, \tag{5.19a}$$

$$Q_{jk} = (u - T_jT_ku)(T_ju - T_ku) - \alpha_j + \alpha_k \;= 0\,, \tag{5.19b}$$

$$Q_{ki} = (u - T_kT_iu)(T_ku - T_iu) - \alpha_k + \alpha_i \;= 0\,, \tag{5.19c}$$

$$Q_{kl} = (u - T_kT_lu)(T_ku - T_lu) - \alpha_k + \alpha_l = 0\,, \tag{5.19d}$$

$$Q_{li} = (u - T_lT_iu)(T_lu - T_iu) - \alpha_l + \alpha_i \;= 0\,, \tag{5.19e}$$

$$Q_{jl} = (u - T_jT_lu)(T_ju - T_lu) - \alpha_j + \alpha_l \;= 0\,. \tag{5.19f}$$

Now we consider periodicity along the staircase Z and evolution of this staircase. Now we consider an example of applying the periodic staircase and only consider shifted copies of two of these equations ((5.19b) or (5.19c)). We consider periodic initial conditions $w = (w_1, w_2, w_3, w_4)$ along Z and evolution in the n_k direction. This only involves applying periodicity to shifted copies of two H1 quad equations ((5.19b) or (5.19c)). The evolution $(w_1, w_2, w_3, w_4) \mapsto T_k(w_1, w_2, w_3, w_4)$ is defined by the following four equations

$$\theta_Z(Q_{ki}) = (w_1 - T_k w_2)(T_k w_1 - w_2) - \alpha_k + \alpha_i = 0\,, \tag{5.20a}$$

$$\theta_Z(T_i Q_{jk}) = (w_2 - T_k w_3)(w_3 - T_k w_2) - \alpha_j + \alpha_k = 0\,, \tag{5.20b}$$

$$\theta_Z(T_j T_i Q_{ki}) = (w_3 - T_k w_4)(T_k w_3 - w_4) - \alpha_k + \alpha_i = 0\,, \tag{5.20c}$$

$$\theta_Z(T_i T_j T_i Q_{jk}) = (w_4 - T_k w_1)(w_1 - T_k w_4) - \alpha_j + \alpha_k = 0\,. \tag{5.20d}$$

Let us show more explicitly how one of these works, emphasising that θ_Z is simply a relabelling map that commutes

$$\begin{aligned} \theta_Z(T_i(Q_{jk}(u,\, u_j,\, u_{jk},\, u_k))) &= Q_{jk}(\theta_Z(u_i),\, \theta_Z(u_{ij}),\, \theta_Z(u_{ijk}),\, \theta_Z(u_{ki})) \\ &= Q_{jk}(w_2,\, w_3,\, T_k w_3,\, T_k w_2) \\ &= (w_2 - T_k w_3)(w_3 - T_k w_2) - \alpha_j + \alpha_k\,. \end{aligned} \tag{5.21}$$

Similarly we can define evolution of the periodic staircase in the n_l direction, $(w_1, w_2, w_3, w_4) \mapsto T_l(w_1, w_2, w_3, w_4)$,

$$\theta_Z(Q_{li}) = (w_1 - T_l w_2)(T_l w_1 - w_2) - \alpha_l + \alpha_i = 0\,, \tag{5.22a}$$

$$\theta_Z(T_i Q_{jl}) = (w_2 - T_l w_3)(w_3 - T_l w_2) - \alpha_j + \alpha_l = 0\,, \tag{5.22b}$$

$$\theta_Z(T_j T_i Q_{li}) = (w_3 - T_l w_4)(T_l w_3 - w_4) - \alpha_l + \alpha_i = 0\,, \tag{5.22c}$$

$$\theta_Z(T_i T_j T_i Q_{jk}) = (w_4 - T_l w_1)(w_1 - T_l w_4) - \alpha_j + \alpha_l = 0\,. \tag{5.22d}$$

5.3.2 Periodic Quad Surfaces and a 1-Form Action

Now that we understand the effect of θ_Z on the level of the equations, we no longer impose any equations on our fields. We now consider things on the level of Lagrangians and actions.

We consider the discrete Lagrangian 2-form defined by the H1 trident Lagrangians

$$\mathcal{L}_{ij} = u u_i - u u_j - (\alpha_i - \alpha_j)\log(u - u_{ij}) \tag{5.23}$$

We have the action on quad surfaces σ

$$S_2[u,\sigma] = \sum_{\sigma_{ij}(\mathbf{n})\in\sigma} \mathcal{L}_{ij} \tag{5.24}$$

And for any three pairs of directions (any four of the possible elementary cubes): the corner equations of the action around the elementary cube are the quad equations

$$\left.\begin{array}{ll} \dfrac{\partial S_2[u,\text{cube}_{ijk}]}{\partial u}=0 & \dfrac{\partial S_2[u,\text{cube}_{ijk}]}{\partial T_iT_jT_ku}=0 \\ \dfrac{\partial S_2[u,\text{cube}_{ijk}]}{\partial T_iu}=0 & \dfrac{\partial S_2[u,\text{cube}_{ijk}]}{\partial T_jT_ku}=0 \\ \dfrac{\partial S_2[u,\text{cube}_{ijk}]}{\partial T_ju}=0 & \dfrac{\partial S_2[u,\text{cube}_{ijk}]}{\partial T_kT_iu}=0 \\ \dfrac{\partial S_2[u,\text{cube}_{ijk}]}{\partial T_ku}=0 & \dfrac{\partial S_2[u,\text{cube}_{ijk}]}{\partial T_iT_ju}=0 \end{array}\right\} \iff Q_{ij}, Q_{jk}, Q_{ki}, \overleftrightarrow{Q}_{ij}, \overleftrightarrow{Q}_{jk}, \overleftrightarrow{Q}_{ki} = 0\,. \tag{5.25}$$

Then on solutions to the quad equations, the corresponding closure relation holds

$$Q_{ij}, Q_{jk}, Q_{ki}, \overleftrightarrow{Q}_{ij}, \overleftrightarrow{Q}_{jk}, \overleftrightarrow{Q}_{ki} = 0 \quad \Longrightarrow \quad S_2[u,\text{cube}_{ijk}] = 0\,. \tag{5.26}$$

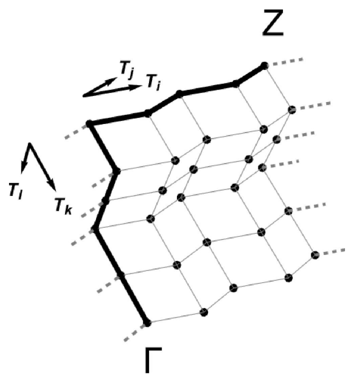


Figure 5.4: A 2-dimensional quad surface $\sigma = \Gamma \times Z$ in 4D, represented in 3D. It is made from a periodic staircase $Z \subset \mathbb{Z}^2$ (in the n_k and n_l direction) and an oriented discrete path Γ (in the n_i and n_j direction).

A key ingredient is to consider specific quad surfaces of the form $\sigma = Z \times \Gamma$ made from the finite staircase Z and a discrete path Γ. Note that Z and Γ live in perpendicular planes, such that $\sigma = Z \times \Gamma$ is a 2-dimensional quad surface in the four dimensional square lattice. For any discrete Lagrangian 2-form, the action

$$S_2[u, \Gamma \times Z] \tag{5.27}$$

is well defined and criticality with respect to variations of the field implies criticality with respect to variations of the surface.

Now we can define an action over discrete paths Γ (in the n_i and n_j direction), by applying the periodic map θ_Z to this action

$$S_1[w,\Gamma] = \theta_Z\left(S_2[u, Z\times\Gamma]\right) . \tag{5.28}$$

The map θ_Z deforms the finite quad surface $Z\times\Gamma$ into a periodic cylindrical surface. The action on left contains the field value along Z inside the components of w and thus it is a well-defined action on the discrete path Γ.

In order to consider the Lagrangian 1-form aspects of this action, we first consider the contributions from the quad surface $\sigma = Z\times\Gamma$. Visually we can see that is a collection of oriented squares. For this example, there are four oriented squares in each row and five rows, which we can explicitly write as

$$\begin{aligned} Z\times\Gamma = \{ \quad & \sigma_{ik}, & T_i\sigma_{jk}, \quad & T_jT_i\sigma_{ik}, & T_jT_i^2\sigma_{jk}, \\ & T_k\sigma_{il}, & T_kT_i\sigma_{jl}, \quad & T_kT_jT_i\sigma_{il}, & T_kT_jT_i^2\sigma_{jl}, \\ & T_lT_k\sigma_{il}, & T_lT_kT_i\sigma_{jl}, \quad & T_lT_kT_jT_i\sigma_{il}, & T_lT_kT_jT_i^2\sigma_{jl}, \\ & T_l^2T_k\sigma_{ik}, & T_l^2T_kT_i\sigma_{jk}, \quad & T_l^2T_kT_jT_i\sigma_{ik}, & T_l^2T_kT_jT_i^2\sigma_{jk}, \\ & T_l^2T_k^2\sigma_{ik}, & T_l^2T_k^2T_i\sigma_{jk}, \quad & T_l^2T_k^2T_jT_i\sigma_{ik}, & T_l^2T_k^2T_jT_i^2\sigma_{jk} \quad \} . \end{aligned} \tag{5.29}$$

We can write the action explicitly using the Lagrangian 2-form and the periodic map, which for this example is

$$\begin{aligned} S_1[w,\Gamma] = \theta_Z\Big(\quad & \mathcal{L}_{ik} + T_i\mathcal{L}_{jk} + T_jT_i\mathcal{L}_{ik} + T_jT_i^2\mathcal{L}_{jk} + \\ & T_k\mathcal{L}_{il} + T_kT_i\mathcal{L}_{jl} + T_kT_jT_i\mathcal{L}_{il} + T_kT_jT_i^2\mathcal{L}_{jl} + \\ & T_lT_k\mathcal{L}_{il} + T_lT_kT_i\mathcal{L}_{jl} + T_lT_kT_jT_i\mathcal{L}_{il} + T_lT_kT_jT_i^2\mathcal{L}_{jl} + \\ & T_l^2T_k\mathcal{L}_{ik} + T_l^2T_kT_i\mathcal{L}_{jk} + T_l^2T_kT_jT_i\mathcal{L}_{ik} + T_l^2T_kT_jT_i^2\mathcal{L}_{jk} + \\ & T_l^2T_k^2\mathcal{L}_{ik} + T_l^2T_k^2T_i\mathcal{L}_{jk} + T_l^2T_k^2T_jT_i\mathcal{L}_{ik} + T_l^2T_k^2T_jT_i^2\mathcal{L}_{jk} \Big) . \end{aligned} \tag{5.30}$$

Next we can identify that rows of this sum correspond to one step in the discrete

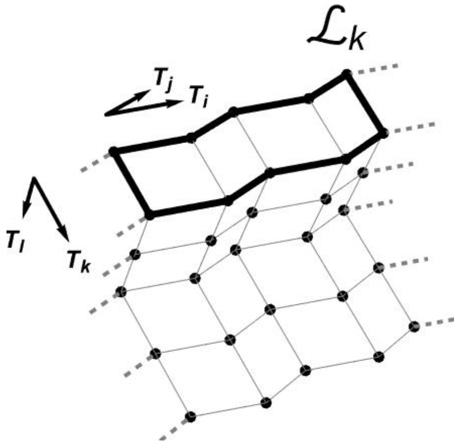


Figure 5.5: The function $\mathcal{L}_k$ made from summing periodically reduced 2-form Lagrangians along a row (parallel to the staircase Z). This function is associated to a single step in the n_k direction.

path. This function is depicted in Figure 5.5 and defined as

$$\begin{aligned}\mathcal{L}_k &= \mathcal{L}_k(w,\, T_k w)\\ &:= \theta_Z(\mathcal{L}_{ik} + T_i\mathcal{L}_{jk} + T_jT_i\mathcal{L}_{ik} + T_jT_i^2\mathcal{L}_{jk})\,,\\ &= w_1w_2 - w_1T_kw_1 - (\alpha_i-\alpha_k)\log(w_1 - T_kw_2)\\ &\quad + w_2w_3 - w_2T_kw_2 - (\alpha_j-\alpha_k)\log(w_2 - T_kw_3)\\ &\quad + w_3w_4 - w_3T_kw_3 - (\alpha_i-\alpha_k)\log(w_3 - T_kw_3)\\ &\quad + w_4w_1 - w_4T_kw_4 - (\alpha_i-\alpha_k)\log(w_4 - T_kw_1)\,,\end{aligned} \tag{5.31}$$

We can also identify rows in the n_l and define the following function

$$\begin{aligned}\mathcal{L}_l &= \mathcal{L}_l(w, T_l w)\\ &:= \theta_Z(\mathcal{L}_{il} + T_i\mathcal{L}_{jl} + T_jT_i\mathcal{L}_{il} + T_jT_i^2\mathcal{L}_{jl})\,.\\ &= w_1w_2 - w_1T_lw_1 - (\alpha_i-\alpha_l)\log(w_1 - T_lw_2)\\ &\quad + w_2w_3 - w_2T_lw_2 - (\alpha_j-\alpha_l)\log(w_2 - T_lw_3)\\ &\quad + w_3w_4 - w_3T_lw_3 - (\alpha_i-\alpha_l)\log(w_3 - T_lw_3)\\ &\quad + w_4w_1 - w_4T_lw_4 - (\alpha_i-\alpha_l)\log(w_4 - T_lw_1)\,.\end{aligned} \tag{5.32}$$

With this identification the action (5.30) over the discrete path Γ can be written in the form of a Lagrangian 1-form, that is

$$S_1[w,\Gamma] = \mathcal{L}_k + T_k\mathcal{L}_l + T_lT_k\mathcal{L}_l + T_l^2T_k\mathcal{L}_k + T_l^2T_k^2\mathcal{L}_k\,. \tag{5.33}$$

We can easily generalise this to a generic path Γ

$$S_1[w,\Gamma] = \sum_{\Gamma_{\pm k}(\mathbf{n})\in\Gamma} \pm\mathcal{L}_k \,. \tag{5.34}$$

Next we consider in what sense this action corresponds to the definition of a Lagrangian 1-form.

5.3.3 Deriving a Weak Lagrangian 1-Form

Let us consider the path around an elementary square depicted in Figure 5.6

$$S_1[w,\text{square}_{kl}] = \theta_Z\Big(S_2[\,u,\,\text{cube}_{ikl} + T_i\text{cube}_{jkl} + T_jT_i\text{cube}_{ikl} + T_jT_i^2\text{cube}_{jkl}\,]\Big)\,. \tag{5.35}$$

Let us also consider derivatives, for this example there are $16 = 4\times 4$ corner equations for the 4 corners and 4 components. They are all of the following form, linear combinations of Lagrangian 2-form corner equations with the periodic map applied,

$$\frac{\partial S_1[w,\text{square}_{kl}]}{\partial T_k w_1} = \theta_Z\left(\frac{\partial}{\partial T_k u}S_2[\,u,\,\text{cube}_{ikl}] + \frac{\partial}{\partial T_k u}S_2[\,u,\,T_jT_i^2\text{cube}_{ikl}]\right). \tag{5.36}$$

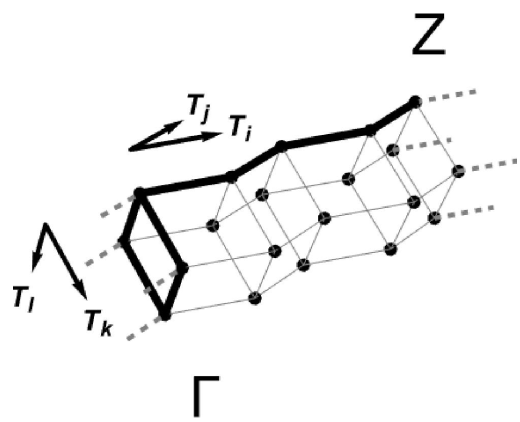


Figure 5.6: The path around a square corresponds to a periodic quad surface built out of cubes.

Recall that we have (5.25) for any cube, including the four involved here: cube_{ikl}, $T_i\text{cube}_{jkl}$, $T_jT_i\text{cube}_{ikl}$ and $T_jT_i^2\text{cube}_{jkl}$. Thus the periodic reductions of the quad equations throughout the lattice implies the vanishing of the $16 = 4\times 4$ corner equations of this 1-form action:

$$\theta_Z(Q_{ij}), \theta_Z(Q_{jk}), \theta_Z(Q_{ki}), \theta_Z(Q_{kl}), \theta_Z(Q_{il}), \theta_Z(Q_{jl}) = 0 \implies \nabla S_1[w,\text{square}_{kl}] = 0\,. \tag{5.37}$$

Similarly recall that we have (5.26) for any cube, including the four involved here:

cube_{ikl}, $T_i\text{cube}_{jkl}$, $T_jT_i\text{cube}_{ikl}$ and $T_jT_i^2\text{cube}_{jkl}$. Thus the periodic reductions of the quad equations throughout the lattice implies that the closure relation holds:

$$\theta_Z(Q_{ij}), \theta_Z(Q_{jk}), \theta_Z(Q_{ki}), \theta_Z(Q_{kl}), \theta_Z(Q_{il}), \theta_Z(Q_{jl}) = 0 \implies S_1[w, \text{square}_{kl}] = 0\,. \tag{5.38}$$

Lemma 23. *Consider an action* (5.34) *over discrete paths* Γ *with 1-form Lagrangians* (5.31) *and* (5.32). *This satisfies the definition of a weak discrete Lagrangian 1-form (Definition 22), where the given equations are the periodically reduced (θ_Z) H1 quad equations.*

Proof. With the staircase Z and map θ_Z, the periodic reduction of the multidimensionally-consistent H1 quad equations is well defined by (5.20) and (5.22). Combining together (5.37) and (5.38) gives us (5.16) on the periodically reduced H1 quad equations and Definition 22:

$$\theta_Z(Q_{ij}), \theta_Z(Q_{jk}), \theta_Z(Q_{ki}), \theta_Z(Q_{kl}), \theta_Z(Q_{il}), \theta_Z(Q_{jl}) = 0 \implies \begin{cases} S_1[w, \text{square}_{kl}] = 0\,, \\ \nabla S_1[w, \text{square}_{kl}] = 0\,. \end{cases} \tag{5.39}$$

□

5.4 Periodically Reducing Discrete Lagrangian 2-Forms on Quadrilateral Stencils

Lemma 23 straightforwardly generalises to generic periodic staircases and the trident discrete Lagrangian 2-form (3.21), for any quad equation of the ABS list. We define a finite staircase $Z \subset \mathbb{Z}^2$ as the $N+1$ vertices which arise from a sequence of N single positive steps in the n_i and n_j directions. Then we can define a periodic map $\theta_Z : u(n_i, n_j, n_k, n_l) \mapsto w(n_k, n_l)$, such that Z becomes a periodic staircase, with the field w having N components for each value along the staircase. Refer to [56] for a general discussion on well-posed periodic initial value problems for partial difference equations. One can straightforwardly apply the staircases Z and periodically reduce any quad equation $Q_{ij}(u, u_i, u_{ij}, u_j; \alpha_i, \alpha_j) = 0$ of the ABS list. With this, we conclude that

Theorem 24. *Consider a staircase* Z *and a trident discrete Lagrangian 2-form* (3.21), *whose multiform Euler-Lagrange equations are quad equations. Consider an action* (5.34) *over discrete paths* Γ *with Lagrangians analogous to* (5.31) *and* (5.32). *This satisfies Definition 22 of a weak discrete Lagrangian 1-form, where the given equations are the* *periodically reduced (θ_Z) quad equations.*

Proof. The proof of Lemma 23 immediately generalises to a generic staircase Z and general quad equations $Q_{ij}(u, u_i, u_{ij}, u_j; \alpha_i, \alpha_j) = 0$. For any multidimensionally-consistent set of multiaffine quad equations in four dimensions, it is well-posed to apply the staircase Z as a periodic initial value problem [56]. An analogues construction gives us that the periodic reduction of the quad equations throughout the lattice implies that the closure relation and the multiform Euler-Lagrange equations hold:

$$\theta_Z(Q_{ij}), \theta_Z(Q_{jk}), \theta_Z(Q_{ki}), \theta_Z(Q_{kl}), \theta_Z(Q_{il}), \theta_Z(Q_{jl}) = 0 \implies \begin{cases} S_1[w, \text{square}_{kl}] = 0\,, \\ \nabla S_1[w, \text{square}_{kl}] = 0\,. \end{cases} \tag{5.40}$$

□

Remark 25. *We consider generalising Theorem 24 to arbitrary single-component discrete Lagrangian 2-forms on quadrilateral stencils satisfying Definition 2, such as the triangle 2-form* (3.23) *and the cross 2-form* (3.40)*. The construction straightforwardly generalises such that instead of taking periodic reductions of quad equations, one takes periodic reductions of the single-component multiform Euler-Lagrange equations in the four dimensional lattice. We only need to check that this periodic initial value problem with staircase Z is well posed in the sense discussed in [56]. In other words we need to check that the following is a compatible set of ordinary difference equations: $\theta_Z(\nabla S_2[u, cube_{ijk}]) = 0$, $\theta_Z(\nabla S_2[u, cube_{jkl}]) = 0$, $\theta_Z(\nabla S_2[u, cube_{kli}]) = 0$, and $\theta_Z(\nabla S_2[u, cube_{lij}]) = 0$.*

Remark 26. *We consider generalising Theorem 24 to arbitrary multi-component discrete Lagrangian 2-forms on quadrilateral stencils satisfying Definition 2 (including the Boussinesq 2-form from [29]). The construction straightforwardly generalises such that instead of taking periodic reductions of quad equations, one takes periodic reductions of the multi-component multiform Euler-Lagrange equations in the four dimensional lattice. We only need to check that this periodic initial value problem with staircase Z is well posed in the sense discussed in [56].*

5.5 Example: Periodically Reducing 2-Form for Linear Quad Equation

In this section we pick a particularly simple example, so that we can go beyond our framework and in fact derive a (non-weak) discrete Lagrangian 1-form. Let us start with a field $u(n_i, n_j, n_k, n_l)$ in a four dimensional lattice. For each quad in this lattice, such as (u, u_i, u_j, u_{ij}), we can associate a Lagrangian coefficient from a discrete Lagrangian 2-form, which is quadratic in u variables

$$\mathcal{L}_{ij} := u\,(T_i u - T_j u) - \frac{1}{2}\frac{\alpha_i + \alpha_j}{\alpha_i - \alpha_j}\,(u - T_j T_i u)^2\,. \tag{5.41}$$

This is a full Lagrangian 2-form, because the multiform Euler-Lagrange equations are multidimensionally consistent and lead to the vanishing of the closure relation. Furthermore, the linear quad equations appear directly from the multiform Euler-Lagrange equations

$$\begin{aligned}
\frac{\partial S_2[u, \mathrm{cube}_{ijk}]}{\partial u_i} &= T_i Q_{jk} & \frac{\partial S_2[u, \mathrm{cube}_{ijk}]}{\partial u_{jk}} &= Q_{jk} \\
\frac{\partial S_2[u, \mathrm{cube}_{ijk}]}{\partial u_j} &= T_j Q_{ki} & \frac{\partial S_2[u, \mathrm{cube}_{ijk}]}{\partial u_{ki}} &= Q_{ki} \\
\frac{\partial S_2[u, \mathrm{cube}_{ijk}]}{\partial u_k} &= T_k Q_{ij} & \frac{\partial S_2[u, \mathrm{cube}_{ijk}]}{\partial u_{ij}} &= Q_{ij} \\
\frac{\partial S_2[u, \mathrm{cube}_{ijk}]}{\partial u} &= -Q_{ij} - Q_{jk} - Q_{jk} & \frac{\partial S_2[u, \mathrm{cube}_{ijk}]}{\partial u_{ijk}} &= -T_i Q_{jk} - T_j Q_{ki} - T_k Q_{ij}
\end{aligned}$$

Here, the Q's are linear quad expressions with

$$Q_{ij} = T_i u - T_j u - \frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j}(u - T_j T_i u)\,. \tag{5.42}$$

One can check that the cube action can be written in terms of the quad equations, but furthermore it can be written as double zero expansion in terms of them.

The construction (5.31) gives us a discrete Lagrangian 1-form in the n_k direction

$$\begin{aligned}
\mathcal{L}_k &= \mathcal{L}_{n_k}(w,\, T_{n_k} w) \\
&:= \theta_Z(\mathcal{L}_{ik} + T_i \mathcal{L}_{jk} + T_j T_i \mathcal{L}_{ik} + T_j T_i^2 \mathcal{L}_{jk}) \\
&= w_1\,(w_2 - T_k w_1) - \frac{1}{2}\frac{\alpha_i + \alpha_k}{\alpha_i - \alpha_k}\,(u_i - T_k w_2)^2 \\
&\quad + w_2\,(w_3 - T_k w_2) - \frac{1}{2}\frac{\alpha_j + \alpha_k}{\alpha_j - \alpha_k}\,(w_2 - T_k w_3)^2 \\
&\quad + w_3\,(w_4 - T_k w_3) - \frac{1}{2}\frac{\alpha_i + \alpha_k}{\alpha_i - \alpha_k}\,(w_3 - T_k w_4)^2 \\
&\quad + w_4\,(w_1 - T_k w_4) - \frac{1}{2}\frac{\alpha_j + \alpha_k}{\alpha_j - \alpha_k}\,(w_4 - T_k w_1)^2
\end{aligned} \tag{5.43}$$

The Lagrangian in the n_4 direction is analogous. This leads to 16 discrete multiform

Euler-Lagrange expressions of the form

$$\frac{\partial S_1[w,\mathrm{square}_{kl}]}{\partial w_2} = \theta_Z\left(-T_iQ_{jk} - T_iQ_{lj} \right) \tag{5.44}$$
$$\frac{\partial S_1[w,\mathrm{square}_{kl}]}{\partial T_k w_2} = \theta_Z\left(Q_{ik} + T_iT_kQ_{lj} \right) \tag{5.45}$$
$$\frac{\partial S_1[w,\mathrm{square}_{kl}]}{\partial T_l w_2} = \theta_Z\left(T_iT_lQ_{jk} + Q_{li} \right) \tag{5.46}$$
$$\frac{\partial S_1[w,\mathrm{square}_{kl}]}{\partial T_lT_k w_2} = \theta_Z\left(-T_lQ_{ik} - T_kQ_{li} \right) \tag{5.47}$$

From the construction outlined in this chapter we know that closure could also be written as a sum of periodically reduced linear quad equations $\theta_Z(Q)$. This gives us a weak discrete Lagrangian 1-form.

Instead, for this particular example, we would like to check if closure vanishes on the 1-form corner equations. One can check that 8 Euler-Lagrange expressions are independent and equivalent to the 16 given above. The 8 we consider are all of the form:

$$\frac{\partial S_1[w,\mathrm{square}_{kl}]}{\partial w_2} = \theta_Z\left(-T_iQ_{jk} - T_iQ_{lj} \right) \tag{5.48}$$
$$\frac{\partial S_1[w,\mathrm{square}_{kl}]}{\partial T_lT_k w_2} = \theta_Z\left(-T_lQ_{ik} - T_kQ_{li} \right) \tag{5.49}$$

It turns out that we can write the square closure relation in terms of these 8 expressions in the following way:

$$\begin{aligned}
&S_1[w,\mathrm{square}_{34}] \qquad (5.50)\\
&= \theta_Z\Big(S_2[\,u,\ \mathrm{cube}_{ikl} + T_i\mathrm{cube}_{jkl} + T_jT_i\mathrm{cube}_{ikl} + T_jT_i^2\mathrm{cube}_{jkl}\,]\Big)\\
&= \sum_Z \theta_Z\ (\,\mathrm{cube}_{ikl}\,)\\
&= \sum_Z \theta_Z\ (\,\mathcal{L}_{ik} - T_l\mathcal{L}_{ik} + \mathcal{L}_{il} - T_k\mathcal{L}_{il}\,)\\
&= \sum_Z \Bigg(\frac{(\alpha_i+\alpha_k)(\alpha_i+\alpha_l)}{4\alpha_i(\alpha_k-\alpha_l)} \left(\frac{\partial S_1[w,\mathrm{square}_{kl}]}{\partial T_lT_k w_2}^2 - \frac{\partial S_1[w,\mathrm{square}_{kl}]}{\partial w_1}^2 \right)\\
&\qquad + \frac{1}{2}\frac{\alpha_k+\alpha_l}{\alpha_k-\alpha_l}\left((T_kw_1 - T_lw_1)^2 - (T_kw_2 - T_lw_2)^2 \right)\\
&\qquad + T_k(w_1T_lw_1) - T_k(w_2T_lw_2) + T_l(w_1T_kw_1) - T_l(w_2T_kw_2) \Bigg)\\
&= \sum_Z \left(\frac{(\alpha_i+\alpha_k)(\alpha_i+\alpha_l)}{4\alpha_i(\alpha_k-\alpha_l)} \left(\frac{\partial S_1[w,\mathrm{square}_{kl}]}{\partial T_lT_k w_2}^2 - \frac{\partial S_1[w,\mathrm{square}_{kl}]}{\partial w_1}^2 \right) \right). \qquad (5.51)
\end{aligned}$$

In the third equality we utilised periodicity, orientation and the sum across the staircase to remove the Lagrangians in the n_k and n_l direction. The fourth equality was found by playing with the explicit terms of the Lagrangians, and grouping expressions in terms of the corner equations. The final terms in the fourth equality vanish by the sum over the staircase. What we are left with in the fifth and final equality is that the square closure relation can be written as a double zero expansion in terms of the eight corner equations. Thus, we find that closure vanishes on the discrete 1-form Euler-Lagrange equations. Hence, this is not weak and it satisfies the full definition of a discrete Lagrangian 1-form (Definition 21):

$$\left.\begin{array}{ll} \dfrac{\partial S_1[w,\text{square}_{kl}]}{\partial w}=0 & \dfrac{\partial S_1[w,\text{square}_{kl}]}{\partial T_iT_jw}=0 \\ \dfrac{\partial S_1[w,\text{square}_{kl}]}{\partial T_iw}=0 & \dfrac{\partial S_1[w,\text{square}_{kl}]}{\partial T_jw}=0 \end{array}\right\} \quad\Longrightarrow\quad S_1[w,\text{square}_{kl}]=0\,. \tag{5.52}$$

This example gives us hope, that with future research, this framework can be developed such that it gives full discrete Lagrangian 1-forms in a general case.

5.6 Chapter Conclusions

In this chapter, we outline a framework to periodically reduce discrete Lagrangian 2-forms on quadrilateral stencils to weak discrete Lagrangian 1-forms. This framework is a promising development in the theory of Lagrangians multiforms, elevating the periodic reductions of multidimensionally consistent quad equations to the Lagrangian level. Furthermore, we have a clear understanding of moving between the discrete 2-form action over quad surfaces and the discrete 1-form action over discrete paths. On an example, associated with the linear quad system, we derive a discrete Lagrangian 1-form, that is not weak. This gives us hope, that with future research, this framework can be developed such that it gives (full) discrete Lagrangians 1-forms in the general case.

Chapter 6

Deriving Lagrangian 1-Forms

In this chapter we are motivated to find Lagrangian 1-forms, which could in future work be quantised with path integrals. In [26], path integrals were calculated for a specific Lagrangian 1-form, with linear multiform Euler-Lagrange equations, and this started the theory of quantum multiforms. The next step is to quantise a Lagrangian 1-form with nonlinear multiform Euler-Lagrange equations. We expect the singularities from the nonlinearity to play an important role. A noteworthy nonlinear 1-form structure is that of the discrete-time Calogero-Moser system in [62]. This system has many nice integrability properties, but also some technicalities which make it not ideal as a nonlinear toy system to investigate quantum multiforms.

In the first half of this chapter, we would like to develop the understanding of Lagrangian 1-forms, even for quite simple integrable systems, so that we can discover useful nonlinear toy systems for the study of quantum multiforms. We link the addition formulae for trigonometric functions to a Lagrangian multiform structure. In Section 6.2 we reconsider the continuous and discrete Lagrangian 1-forms associated with linear equations and sinusoidal solutions. In Section 6.3 we discover a continuous and discrete Lagrangian 1-form associated with nonlinear equations and cosecant solutions.

In Section 6.4, we reconsider periodic reductions of discrete Lagrangian 2-forms for the H1 quad equation considered in Lemma 23. We will review how this periodic reduction of H1 (or lpKdV) leads to the McMillan equation (autonomous discrete Painlevé II equation) [34, 37, 22, 21]. In [25], the standard Lagrangian of this discrete ordinary difference equation was quantised in terms of path integrals, and a 2-step propagator was derived. We are motivated to generalise this to a quantum multiform structure, however developments in the classical setting are required. Our major contribution is the derivation of classical commuting equations, and finding Jacobi elliptic solutions involving non-trivial parameters. We then develop a discrete Lagrangian 1-form. We find that already on the classical level the equations and Lagrangian 1-form are non-trivial and involve subtle square-root terms.

6.1 Discrete and Continuous Lagrangian 1-Forms

In this section, we first note a useful and equivalent formulation of a discrete Lagrangian 1-form (Definition 21), inspired by [52, 62]. We now use the notation $x(\mathbf{n}) = x(n_k, n_l)$ for the field over the lattice $\mathbb{Z}^2$ (instead of $w(\mathbf{n})$), and as before denote discrete shifts with $x_k = x(\mathbf{n} + \mathbf{e}_k)$. Consider (5.13) and the fact that the corner equations are autonomous and hold anywhere in the lattice. One can check that the four multiform Euler-Lagrange equations are, in fact, equivalent to two conventional Euler-Lagrange equations and one corner equation:

$$\nabla S_1[\mathbf{x}, \mathrm{square}_{kl}] = 0 \qquad \Longleftrightarrow \qquad \begin{cases} \dfrac{\partial \mathcal{L}_k}{\partial x_k} + \dfrac{\partial T_k \mathcal{L}_k}{\partial x_k} = 0 \\ \dfrac{\partial \mathcal{L}_l}{\partial x_l} + \dfrac{\partial T_l\, \mathcal{L}_l}{\partial x_l} = 0 \\ \dfrac{\partial \mathcal{L}_k}{\partial x} = \dfrac{\partial \mathcal{L}_l}{\partial x} \end{cases} \tag{6.1}$$

Thus, we can equivalently say that a discrete Lagrangian 1-form is a pair of functions $\mathcal{L}_k : (x, x_k;\, \lambda_k) \mapsto \mathbb{C}$ and $\mathcal{L}_l : (x, x_l;\, \lambda_l) \mapsto \mathbb{C}$ where

$$\left.\begin{aligned} &\frac{\partial \mathcal{L}_k}{\partial x_k} + \frac{\partial T_k \mathcal{L}_k}{\partial x_k} = 0 \\ &\frac{\partial \mathcal{L}_l}{\partial x_l} + \frac{\partial T_l\, \mathcal{L}_l}{\partial x_l} = 0 \\ &\frac{\partial \mathcal{L}_k}{\partial x} = \frac{\partial \mathcal{L}_l}{\partial x} \end{aligned}\right\} \qquad \Longrightarrow \qquad T_l \mathcal{L}_k - \mathcal{L}_k = T_k \mathcal{L}_l - \mathcal{L}_l\,. \tag{6.2}$$

Here we have written the closure relation in a different form. We emphasise that the property (6.2) defines the Lagrangian multiform: on solutions to the multiform Euler-Lagrange equations, the closure relation holds.

Now we comment on how we have restricted the form of Lagrangians in this definition of a discrete Lagrangian 1-form. The Lagrangians $\mathcal{L}_k : (x, x_k;\, \lambda_k)$ and $\mathcal{L}_l : (x, x_l;\, \lambda_l)$ are first order and do not involve discrete shift in the other direction, x_l and x_k, respectively. This restriction means that the following equations vanish trivially:

$$\frac{\partial \mathcal{L}_k}{\partial x_l} = 0\,, \tag{6.3a}$$

$$\frac{\partial \mathcal{L}_l}{\partial x_k} = 0\,. \tag{6.3b}$$

Later we will define a continuous Lagrangian 1-form in an analogous way to a discrete Lagrangian 1-form, however the Lagrangians will not be restricted in this way. Continuous equations analogous to (6.3) will not vanish identically and will instead be considered as additional multiform Euler-Lagrange equations.

Now consider the corner equation in (6.2), and note that it corresponds to the existence of some shared conjugate momentum $y(n_k, n_l)$ on the lattice $\mathbb{Z}^2$, defined by

$$y = \frac{\partial \mathcal{L}_k}{\partial x} = \frac{\partial \mathcal{L}_l}{\partial x}\,. \tag{6.4}$$

This is key to keep in mind when attempting to derive Lagrangian 1-forms. One can start with any two conventional Lagrangians, but having this compatibility is a rare property.

Furthermore, we can also generalise a discrete Lagrangian 1-form from 2 to N compatible Lagrangians and from a field $x(\mathbf{n})$ over the 2-dimensional lattice $\mathbb{Z}^2$ to one over the N-dimensional lattice $\mathbb{Z}^N$. For example, consider $N = 4$, with four Lagrangians $\mathcal{L}_i(x, x_i;\, \lambda_i)$, $\mathcal{L}_j(x, x_j;\, \lambda_j)$, $\mathcal{L}_k(x, x_k;\, \lambda_k)$ and $\mathcal{L}_l(x, x_l;\, \lambda_l)$, as well as a field $x(n_i, n_j, n_k, n_l)$ over the lattice $\mathbb{Z}^4$. The four Lagrangians are compatible and a discrete Lagrangian 1-form, if for all six pairs of Lagrangians the property (6.2) holds.

The above is more visibly a discrete analogue of the definition of a continuous Lagrangian 1-form given in [52], which we now present. Let us now consider a field $x(t_\alpha, t_\beta)$ with two continuous independent variables $(t_\alpha, t_\beta) \in \mathbb{R}^2$. We denote continuous derivatives in the following way:

$$x_{t_\alpha} := \frac{\partial x}{\partial t_\alpha}\,, \qquad x_{t_\alpha t_\alpha} := \frac{\partial^2 x}{\partial t_\alpha^2}\,, \tag{6.5a}$$

$$x_{t_\beta} := \frac{\partial x}{\partial t_\beta}\,, \qquad x_{t_\beta t_\beta} := \frac{\partial^2 x}{\partial t_\beta^2}\,. \tag{6.5b}$$

This notation is not to be confused with the notation for discrete shifts (x_k, x_{kk}, x_l, x_{ll}, x_i, x_l) or the notation for constants: $\lambda_k, \lambda_l, \lambda_\alpha, \lambda_\beta \in \mathbb{C}$.

We only consider a continuous Lagrangian 1-form with two directions without loss of generality. We define an action over one-dimensional continuous paths $\Gamma \in \mathbb{R}^2$ as

$$S_1[x, \Gamma] = \int_\Gamma \left(\mathcal{L}_\alpha \,\mathrm{d}t_\alpha + \mathcal{L}_\beta \,\mathrm{d}t_\beta\right)\,, \tag{6.6}$$

with functions

$$\mathcal{L}_\alpha : (x, x_{t_\alpha}, x_{t_\beta};\, \lambda_\alpha) \mapsto \mathbb{C}\,, \tag{6.7a}$$

$$\mathcal{L}_\beta : (x, x_{t_\alpha}, x_{t_\beta};\, \lambda_\beta) \mapsto \mathbb{C}\,. \tag{6.7b}$$

Now we give the following definition similar to the constructions in [52, 62].

Definition 27. *A continuous Lagrangian 1-form is given by an action* (6.6) *over continuous paths, which satisfies the following property. On solutions to the multiform Euler-Lagrange equations (the variational equations), the closure relation holds:*

$$\left.\begin{aligned}
&\frac{\partial \mathcal{L}_\alpha}{\partial x_{t_\beta}} = 0\\
&\frac{\partial \mathcal{L}_\beta}{\partial x_{t_\alpha}} = 0\\
&\frac{\partial}{\partial t_\alpha}\frac{\partial \mathcal{L}_\alpha}{\partial x_{t_\alpha}} - \frac{\partial \mathcal{L}_\alpha}{\partial x} = 0\\
&\frac{\partial}{\partial t_\beta}\frac{\partial \mathcal{L}_\beta}{\partial x_{t_\beta}} - \frac{\partial \mathcal{L}_\beta}{\partial x} = 0\\
&\frac{\partial \mathcal{L}_\alpha}{\partial x_{t_\alpha}} = \frac{\partial \mathcal{L}_\beta}{\partial x_{t_\beta}}
\end{aligned}\right\} \quad \Longrightarrow \quad \frac{\partial \mathcal{L}_\alpha}{\partial t_\beta} = \frac{\partial \mathcal{L}_\beta}{\partial t_\alpha}\,. \tag{6.8}$$

The first two multiform Euler-Lagrange handle “alien derivatives” as discussed in [58]. The third and fourth equations are the conventional Euler-Lagrange equations. The fifth equation is the corner equation and implies that there exist some shared conjugate momentum $y(t_\alpha, t_\beta)$, defined by

$$y = \frac{\partial \mathcal{L}_\alpha}{\partial x_{t_\alpha}} = \frac{\partial \mathcal{L}_\beta}{\partial x_{t_\beta}}\,. \tag{6.9}$$

It is key to keep this in mind when attempting to derive Lagrangian 1-forms.

6.2 Lagrangian 1-Form for Common Sinusoidal Solutions

In this section we will derive a continuous Lagrangian 1-form, such that a general solution to the continuous multiform Euler-Lagrange equations is a common sine solution:

$$x(t_\alpha, t_\beta) = A\sin(\xi + \lambda_\alpha t_\alpha + \lambda_\beta t_\beta)\,. \tag{6.10}$$

To do that, we will find a system of equations involving the parameters, $\lambda_\alpha, \lambda_\beta$, such that two initial conditions determine A and ξ. Then we will derive a discrete Lagrangian 1-form, such that a general solution to the multiform Euler-Lagrange equations is a common sine solution:

$$x(n_k, n_l) = A\sin(\xi + \lambda_k n_k + \lambda_l n_l)\,. \tag{6.11}$$

To do that, we will find a system of equations involving the parameters, λ_k, λ_l, such that two initial conditions determine A and ξ. Hence, the perspective of this section is to derive Lagrangian multiforms from special functions. This will prepare us for later sections, where we will generalise this to non-trivial examples. Furthermore, we want a system of equations in terms of x which encapsulates the following key functional properties of sin

$$\cos(a) = \sin(a + \pi/2) \tag{6.12a}$$

$$\frac{\partial}{\partial a}\sin(a) = \cos(a) \tag{6.12b}$$

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b) \tag{6.12c}$$

$$\sin^2(a) + \cos^2(a) = 1 \tag{6.12d}$$

Here, a and b are arbitrary.

Let us consider continuous equations. For the time variable t_α, sinusoidal solutions of the form $x = A\sin(\xi + \lambda_\alpha t_\alpha)$ arise from the harmonic oscillator equation, which is a linear second order differential equation

$$x_{t_\alpha t_\alpha} + \lambda_\alpha^2\, x = 0\,. \tag{6.13}$$

The parameters A and ξ in the solution are determined by two initial conditions. Furthermore, solutions $x = A\sin(\xi + \lambda_\alpha t_\alpha)$ satisfy the following biquadratic first order differential equation, involving the initial condition parameter A:

$$\frac{1}{\lambda_\alpha^2}\, x_{t_\alpha}^2 + x^2 - A^2 = 0\,. \tag{6.14}$$

This equation reconciles the property (6.12d) of sine as well as the fact that the parameter A is invariant with time. If we also consider similar equations in the t_β direction, we have two separate solutions $x = A\sin(\xi + \lambda_\alpha t_\alpha)$ and $x = A'\sin(\xi' + \lambda_\beta t_\beta)$. How do we link together evolution in the t_α direction and t_β direction and make the parameter A shared? This can be done with the following linear differential equation

$$\frac{x_{t_\alpha}}{\lambda_\alpha} = \frac{x_{t_\beta}}{\lambda_\beta}\,. \tag{6.15}$$

Then, the general solution to this system of equations is $x = A\sin(\xi + \lambda_\alpha t_\alpha + \lambda_\beta t_\beta)$. Furthermore, this equation defines a shared momentum $y(t_\alpha, t_\beta)$ in the following way:

$$y = \frac{x_{t_\alpha}}{\lambda_\alpha} = \frac{x_{t_\beta}}{\lambda_\beta}\,. \tag{6.16}$$

In terms of the general solution, we have that $y(t_\alpha, t_\beta) = A\cos(\xi + \lambda_\alpha t_\alpha + \lambda_\beta t_\beta)$.

Now we consider discrete equations. Sinusoidal solutions of the form $x = A\sin(\xi + \lambda_k n_k)$ arise from the discrete harmonic oscillator equation, which is a linear second order difference equation:

$$x_{kk} - 2c_k\, x_k + x = 0\,. \tag{6.17}$$

Here, we introduce the constants $c_k = \cos(\lambda_k)$, $s_k = \sin(\lambda_k)$, $c_l = \cos(\lambda_l)$ and $s_l = \sin(\lambda_l)$. The subscript notation is subtle, and we emphasise that $\lambda_k, c_k, s_k, \lambda_l, c_l, s_l \in \mathbb{C}$ are constants, whereas $x_k = x(n_k + 1, n_l)$, $x_{kk} = x(n_k + 2, n_l)$, $x_l = x(n_k, n_l + 1)$ and $x_{ll} = x(n_k, n_l + 2)$ are discrete shifts of the field. Furthermore, solutions $x = A\sin(\xi + \lambda_k n_k)$ satisfy the following biquadratic first order differential equation, involving the initial condition parameter A:

$$\frac{1}{s_k^2}(x_k - c_k\, x)^2 + x^2 - A^2 = 0\,. \tag{6.18}$$

This equation expresses the property (6.12d) of sine as well as the fact that the parameter A is invariant with time. If we also consider similar equations in the n_l direction, we have two separate solutions $x = A\sin(\xi + \lambda_k n_k)$ and $x = A'\sin(\xi' + \lambda_l n_l)$. How do we link together evolution in the n_k direction and n_l direction and make the parameter A shared? This can be done with the following linear difference equation, which follows from the addition formulae (6.12c):

$$\frac{x_k - c_k\, x}{s_k} = \frac{x_l - c_l\, x}{s_l} \tag{6.19}$$

Then, the general solution to this system of equations is $x = A\sin(\xi + \lambda_k n_k + \lambda_l n_l)$. Furthermore, this equation defines a shared momentum $y(n_k, n_l)$ in the following way:

$$y = \frac{x_k - c_k\, x}{s_k} = \frac{x_l - c_l\, x}{s_l} \tag{6.20}$$

In terms of the general solution, we have that $y = A\cos(\xi + \lambda_k n_k + \lambda_l n_l)$.

Next we will construct a continuous Lagrangian 1-form for this system, and then a discrete Lagrangian 1-form.

6.2.1 Continuous Lagrangian 1-Form

Let us start with a Lagrangian with a general form and derive its terms by imposing that that the 1-form Euler-Lagrange equations have sinusoidal solutions. For this pedagogical example, let us consider Lagrangians $\mathcal{L}_\alpha$ quadratic in x_{t_α} with coefficients which are functions $f, g, h : (x;\, \lambda_\alpha) \mapsto \mathbb{C}$

$$\mathcal{L}_\alpha = f(x;\, \lambda_\alpha) x_{t_\alpha}^2 + g(x;\, \lambda_\alpha) x_{t_\alpha} + h(x;\, \lambda_\alpha)\,, \tag{6.21a}$$

$$\mathcal{L}_\beta = f(x;\, \lambda_\beta) x_{t_\beta}^2 + g(x;\, \lambda_\beta) x_{t_\beta} + h(x;\, \lambda_\beta)\,. \tag{6.21b}$$

Now we impose that the standard continuous Euler-Lagrange equation is proportional to the second order differential equation (6.13) with sine solutions

$$\begin{aligned} &\frac{\partial}{\partial t_\alpha}\frac{\partial \mathcal{L}_\alpha}{\partial x_{t_\alpha}} - \frac{\partial \mathcal{L}_\alpha}{\partial x} \\ &\quad = 2f'(x;\, \lambda_\alpha) x_{t_\alpha}^2 + 2f(x;\, \lambda_\alpha) x_{t_\alpha t_\alpha} - h'(x;\, \lambda_\alpha) \\ &\quad \propto x_{t_\alpha t_\alpha} + \lambda_\alpha^2 x \end{aligned} \tag{6.22}$$

This restricts f, g and h in the following way

$$f'(x;\, \lambda_\alpha) = 0 \qquad\qquad - h'(x;\, \lambda_\alpha) = 2f(x;\, \lambda_\alpha)\lambda_\alpha^2 x\,. \tag{6.23}$$

Thus, the Lagrangian of the harmonic oscillator has the following form

$$\mathcal{L}_\alpha = f(\lambda_\alpha)\left(x_{t_\alpha}^2 - \lambda_\alpha^2 x^2\right) + g(x;\, \lambda_\alpha) x_{t_\alpha}\,. \tag{6.24}$$

The final term is a total derivative and does not effect the standard Euler-Lagrange equation.

Now we impose the corner Euler-Lagrange equation by introducing a free function $\phi(y, x)$ and using the corner equations (6.16)

$$\frac{\partial \mathcal{L}_\alpha}{\partial x_{t_\alpha}} = 2f(\lambda_\alpha) x_{t_\alpha} + g(x;\, \lambda_\alpha) = \phi\left(\frac{x_{t_\alpha}}{\lambda_\alpha},\, x\right) = \phi(y,\, x)\,. \tag{6.25}$$

This forces $f(\lambda_\alpha) = \gamma/\lambda_\alpha$ with $\gamma \in \mathbb{C}$ and $g(x;\, \lambda_\alpha) = g(x)$. Now we have that the following Lagrangians satisfy the multiform Euler-Lagrange equations:

$$\mathcal{L}_\alpha = \gamma \left(\frac{1}{\lambda_\alpha}\, x_{t_\alpha}^2 - \lambda_\alpha\, x^2 \right) + g(x)\, x_{t_\alpha}\,, \tag{6.26a}$$

$$\mathcal{L}_\beta = \gamma \left(\frac{1}{\lambda_\alpha}\, x_{t_\beta}^2 - \lambda_\beta\, x^2 \right) + g(x)\, x_{t_\beta}\,. \tag{6.26b}$$

The final term is what we call an exact Lagrangian 1-form and it is a freedom which does not effect the Lagrangian 1-form structure.

Next we consider the continuous closure relation

$$\begin{aligned} &\frac{\partial \mathcal{L}_\alpha}{\partial t_\beta} - \frac{\partial \mathcal{L}_\beta}{\partial t_\alpha} \\ &\quad = 2\gamma \left(\frac{1}{\lambda_\alpha}\, x_{t_\alpha t_\beta} x_{t_\alpha} - \lambda_\alpha x_{t_\beta} x \right) - 2\gamma \left(\frac{1}{\lambda_\beta}\, x_{t_\alpha t_\beta} x_{t_\beta} - \lambda_\beta x_{t_\alpha} x \right) \\ &\quad = 2\gamma \left(\frac{x_{t_\alpha}}{\lambda_\alpha} - \frac{x_{t_\beta}}{\lambda_\beta} \right) \left(x_{t_\alpha t_\beta} + \lambda_\alpha \lambda_\beta x \right)\,. \end{aligned} \tag{6.27}$$

What we find is, in fact, a double zero expansion. This is stronger than having the closure relation on solutions to the multiform Euler-Lagrange equations. Altogether, we conclude the following.

Proposition 28. *The double zero expansion* (6.27) *gives us a continuous Lagrangian 1-form (Definition 27). The general solution to the multiform Euler-Lagrange equations can be written in terms of the sine function, as in* (6.10).

We can also Legendre transform the Lagrangian and recover the invariant equation (6.14), by setting $g(x) = 0$

$$I_\alpha = \gamma \left(\frac{1}{\lambda_\alpha}\, x_{t_\alpha}^2 + \lambda_\alpha x^2 \right)\,. \tag{6.28}$$

On the general solution, with initial conditions defining A, we have

$$(6.10) \quad \Longrightarrow \quad I_\alpha = \gamma \lambda_\alpha\, A^2\,. \tag{6.29}$$

Next we consider an analogous discrete Lagrangian 1-form construction.

6.2.2 Discrete Lagrangian 1-Form

Let us start with two discrete Lagrangians of the following form, whose terms will be derived by imposing a sinusoidal discrete Lagrangian 1-form structure:

$$\mathcal{L}_k = f(x_k x;\, \lambda_k) + g(x_k; \lambda_k) + h(x;\, \lambda_k)\,, \tag{6.30a}$$

$$\mathcal{L}_l = f(x_l x;\, \lambda_l) + g(x_l; \lambda_l) + h(x;\, \lambda_l)\,. \tag{6.30b}$$

Here we take $\mathcal{L}_k$ and $\mathcal{L}_l$ to have the same form. We allow for generic functions f, g and h, and note that $f(x_k x;\, \lambda_k)$ acts on the symmetric product $x_k x$. We could instead start with a Lagrangian quadratic in x and x_k. However, it is interesting to see how imposing the sinusoidal multiform structure takes us from function terms to polynomial terms.

Let us start with

$$\begin{aligned}
&\frac{\partial}{\partial x_k}\left(\mathcal{L}_k + T_k\mathcal{L}_k\right) \\
&\quad = f(x_k x;\, \lambda_k)x + f'(x_{kk}x_k;\, \lambda_k)x_{kk} + g'(x_k;\lambda_k) + h'(x_k;\, \lambda_k) \\
&\quad \propto x_{kk} - 2c_k\, x_k + x = 0\,.
\end{aligned} \tag{6.31}$$

This restricts f, g and h in the following way

$$f(x_k x;\, \lambda_k) = \zeta(\lambda_k)x_k x \qquad g'(x_k;\, \lambda_k) + h'(x_k;\, \lambda_k) = -2\zeta(\lambda_k)c_k x_k\,. \tag{6.32}$$

This leads to

$$f(x_k x;\, \lambda_k) = \zeta(\lambda_k)\, x_k x \qquad h(x;\, \lambda_k) = -\zeta(\lambda_k)c_k\, x^2 - g(x;\, \lambda_k)\,. \tag{6.33}$$

Thus, Lagrangians with discrete harmonic oscillators as standard Euler-Lagrange equations have the following form:

$$\mathcal{L}_k = \zeta(\lambda_k)\left(x_k x - c_k\, x^2\right) + g(x_k;\, \lambda_k) - g(x;\, \lambda_k)\,. \tag{6.34}$$

Now we impose that the corner Euler-Lagrange equation holds, utilising equation (6.20) and a free function $\phi(y, x)$:

$$\begin{aligned}
-\frac{\partial \mathcal{L}_k}{\partial x} &= -\zeta(\lambda_k)\left(x_k - 2c_k x\right) - g'(x;\, \lambda_k) \\
&= \phi\left(\frac{x_k - c_k\, x}{s_k},\, x\right) = \phi(y, x)
\end{aligned} \tag{6.35}$$

This leads to Lagrangians of the following form satisfying discrete multiform Euler-Lagrange equations for sine:

$$\mathcal{L}_k = \zeta\left(\frac{1}{s_k}x_k x - \frac{c_k}{2s_k}\left(x_k^2 + x^2\right)\right) + G(x_k) - G(x)\,, \tag{6.36}$$

Now let us consider the discrete closure relation for these Lagrangians

$$\begin{aligned}
&T_l\mathcal{L}_k - \mathcal{L}_k - T_k\mathcal{L}_l + \mathcal{L}_l \\
&= \frac{\zeta\, s_k s_l}{2(c_l s_k - c_k s_l)} \left(\left(\frac{\partial \mathcal{L}_k}{\partial x} - \frac{\partial \mathcal{L}_l}{\partial x} \right)^2 - \left(\frac{\partial T_k\mathcal{L}_l}{\partial x_{kl}} - \frac{\partial T_l\mathcal{L}_k}{\partial x_{kl}} \right)^2 \right) \\
&\quad + \frac{\zeta\, \left((s_l^2 + c_l^2 - 1)s_k^2 - (s_k^2 + c_k^2 - 1)s_l^2 \right) (x_k^2 - x_l^2)}{2 s_k s_l (c_l s_k - c_k s_l)} \,.
\end{aligned} \tag{6.37}$$

What we find is, in fact, a double zero expansion, which is stronger than the closure relation. Note that this double zero expansion only requires the following identity $s_k^2 + c_k^2 - 1 = 0$ (and $s_l^2 + c_l^2 - 1 = 0$) between the coefficients s_k and c_k. Altogether, we conclude the following.

Proposition 29. *The double zero expansion* (6.37) *gives us a discrete Lagrangian 1-form (Definition 21). The general solution to the multiform Euler-Lagrange equations can be written in terms of the sine function* (6.11)*.*

We can also take derivatives of the Lagrangian with respect to a lattice parameter to find the following integral of motion (6.18) on common sine solutions [52]

$$I_k = \frac{\partial}{\partial \lambda_k}\mathcal{L}_k = \zeta \left(-\frac{1}{s_k^2} x_k x - \frac{c_k}{2 s_k} \left(x_k^2 + x^2 \right) \right) \tag{6.38}$$

On the general sine solution, we recover find the constant parameter A defined by initial conditions

$$(6.11) \qquad \Longrightarrow \qquad I_k = -\frac{\zeta}{2} A^2 \,. \tag{6.39}$$

In the next section we similarly derive a continuous and discrete Lagrangian 1-form, but from common cosecant solutions.

6.3 Lagrangian 1-Form for Common Cosecant Solution

In this section we will derive a continuous Lagrangian 1-form, such that a general solution to the continuous multiform Euler-Lagrange equations is a common cosecant solution:

$$x(t_\alpha, t_\beta) = A \csc(\xi + \lambda_\alpha t_\alpha + \lambda_\beta t_\beta) \,. \tag{6.40}$$

To do that, we will find a system of equations involving the parameters, $\lambda_\alpha, \lambda_\beta$, such that two initial conditions determine A and ξ. Then we will derive a discrete Lagrangian 1-form, such that a general solution to the multiform Euler-Lagrange equations is a common cosecant solution:

$$x(n_k, n_l) = A\csc(\xi + \lambda_k n_k + \lambda_l n_l)\,. \tag{6.41}$$

To do that, we will find a system of equations involving the parameters, λ_k, λ_l, such that two initial conditions determine A and ξ.

Let us consider what form the multiform Euler-Lagrange equations should take, starting with the continuous equations. The following second order differential equation in the t_α direction has general solutions of the form $x = A\csc(\xi + \lambda_\alpha t_\alpha)$:

$$x\, x_{t_\alpha t_\alpha} - 2x_{t_\alpha}^2 - \lambda_\alpha^2 x^2 = 0 \tag{6.42}$$

These solutions satisfy the following continuous invariant equation

$$\frac{x^4\lambda_\alpha^2}{x_{t_\alpha}^2 + \lambda_\alpha^2 x^2} - A^2 = 0 \tag{6.43}$$

Alternatively, we can write this invariant equation in the following form

$$\frac{x_{t_\alpha}^2}{\lambda_\alpha^2 x^4} + \frac{1}{x^2} - \frac{1}{A^2} = 0 \tag{6.44}$$

To get corner equations we could consider the equality of two invariant equations in different directions. If we also consider similar equations in the t_β direction, we have two separate solutions $x = A\csc(\xi + \lambda_\alpha t_\alpha)$ and $x = A'\csc(\xi' + \lambda_\beta t_\beta)$. We can link together evolution in both directions with the following differential equation, which also defines a shared momentum $y(t_\alpha, t_\beta)$

$$y = -\frac{\lambda_\alpha x^2}{x_{t_\alpha}} = -\frac{\lambda_\beta x^2}{x_{t_\beta}} \tag{6.45}$$

For the discrete evolution, the following second order difference equation in terms of x, x_k and x_{kk} has general solutions in terms the cosecant function

$$x_k(x_{kk} + x) - 2c_k\, x_{kk} x = 0 \tag{6.46}$$

Here, we introduce the constants $c_k = \cos(\lambda_k)$, $s_k = \sin(\lambda_k)$, $c_l = \cos(\lambda_l)$ and $s_l = \sin(\lambda_l)$. The subscript notation is subtle, and we emphasise that $\lambda_k, c_k, s_k, \lambda_l, c_l, s_l \in \mathbb{C}$ are constants, whereas $x_k = x(n_k + 1, n_l)$, $x_{kk} = x(n_k + 2, n_l)$, $x_l = x(n_k, n_l + 1)$ and $x_{ll} = x(n_k, n_l + 2)$ are discrete shifts of the field. These solutions satisfy the following equation expressing the invariance of the initial condition parameter A:

$$\frac{(1-c_k^2)x_k^2x^2}{x_k^2-2c_kx_kx+x^2}-A^2=0 \tag{6.47}$$

Alternatively we can write this "invariant" equation as

$$-\frac{2c_k}{s_k^2}\frac{1}{x_kx}+\frac{1}{s_k^2}\left(\frac{1}{x_k^2}+\frac{1}{x^2}\right)-\frac{1}{A^2}=0\,. \tag{6.48}$$

If we also consider similar equations in the n_l direction, we have two separate solutions $x=A\csc(\xi+\lambda_kn_k)$ and $x=A'\csc(\xi'+\lambda_ln_l)$. We can link together evolution in both directions with the following differential equation, which also defines a shared momentum $y(n_k,n_l)$

$$y=\frac{s_k\,x_kx}{x-c_k\,x_k}=\frac{s_l\,x_lx}{x-c_l\,x_l}\,. \tag{6.49}$$

Then, the general solution to this system of equations is $x=A\csc(\xi+\lambda_kn_k+\lambda_ln_l)$, and we also have that $y=A\sec(\xi+\lambda_kn_k+\lambda_ln_l)$.

First we will construct a continuous Lagrangian 1-form, then a discrete Lagrangian 1-form.

6.3.1 Continuous Lagrangian 1-Form

We will start with a Lagrangian in a general form and then impose the continuous Lagrangian 1-form structure. Let us consider the conventional Euler-Lagrange equation, and assume that it gives us an equation which is proportional to (6.42), that is

$$\frac{\partial}{\partial t_\alpha}\frac{\partial\mathcal{L}_\alpha}{\partial x_{t_\alpha}}-\frac{\partial\mathcal{L}_\alpha}{\partial x}\propto\ x_{t_\alpha t_\alpha}-\frac{2x_{t_\alpha}^2}{x}-\lambda_\alpha^2x=0\,. \tag{6.50}$$

With this in mind, let us assume a Lagrangian of the following general form

$$\mathcal{L}_\alpha=f(x;\,\lambda_\alpha)x_{t_\alpha}^2+g(x;\,\lambda_\alpha)+h(x;\,\lambda_\alpha)x_{t_\alpha}\,. \tag{6.51}$$

This leads to

$$\begin{aligned}\frac{\partial}{\partial t_\alpha}\frac{\partial\mathcal{L}_\alpha}{\partial x_{t_\alpha}}-\frac{\partial\mathcal{L}_\alpha}{\partial x}&=2\frac{\partial f(x;\,\lambda_\alpha)}{\partial x}x_{t_\alpha}^2+2f(x;\,\lambda_\alpha)x_{t_\alpha t_\alpha}+\frac{\partial h(x;\,\lambda_\alpha)}{\partial x}x_{t_\alpha}\\&\quad-\frac{\partial f(x;\,\lambda_\alpha)}{\partial x}x_{t_\alpha}^2-\frac{\partial g(x;\,\lambda_\alpha)}{\partial x}-\frac{\partial h(x;\,\lambda_\alpha)}{\partial x}x_{t_\alpha}\,,\\&=2f(x;\,\lambda_\alpha)x_{t_\alpha t_\alpha}+\frac{\partial f(x;\,\lambda_\alpha)}{\partial x}x_{t_\alpha}^2-\frac{\partial g(x;\,\lambda_\alpha)}{\partial x}\end{aligned} \tag{6.52}$$

$$\propto x_{t_\alpha t_\alpha} - \frac{2x_{t_\alpha}^2}{x} - \lambda_\alpha^2 x$$

This results in two differential equations constraining f and g. Let us consider the f equation first

$$\frac{\partial f(x;\,\lambda_\alpha)}{\partial x} = -4\frac{f(x;\,\lambda_\alpha)}{x} \implies f(x;\,\lambda_\alpha) = \frac{\theta(\lambda_\alpha)}{x^4} \tag{6.53}$$

Integration gave rise to a free function $\theta(\lambda_\alpha)$. Next let us consider the constraint equation on g, which is

$$-\frac{\partial g(x;\,\lambda_\alpha)}{\partial x} = -2f(x;\,\lambda_\alpha)\lambda_\alpha^2 x = -\frac{2\theta(\lambda_\alpha)\lambda_\alpha^2}{x^3} \implies g(x;\,\lambda_\alpha) = -\frac{\theta(\lambda_\alpha)\lambda_\alpha^2}{x^2} \tag{6.54}$$

This leads to the following Lagrangian which satisfies the standard Euler-Lagrange equation:

$$\mathcal{L}_\alpha = \frac{\theta(\lambda_\alpha)}{x^4}x_{t_\alpha}^2 - \frac{\theta(\lambda_\alpha)\lambda_\alpha^2}{x^2} + h(x;\,\lambda_\alpha)x_{t_\alpha}\,. \tag{6.55}$$

We have a multiplicative freedom by a function of the lattice parameter and a total difference term.

Now we impose the multiform Euler-Lagrange equations. Let us assume that the "shared momentum" is a function of y and x. With this in mind let us write down the following equation

$$\frac{1}{yx^2} = -\frac{x_{t_\alpha}}{\lambda_\alpha x^4} = -\frac{x_{t_\beta}}{\lambda_\beta x^4}\,. \tag{6.56}$$

Let us assume that the corner equation is proportional to this in the following way, where we introduce a constant $\gamma \in \mathbb{C}$ and a function $F(x)$

$$\frac{\partial \mathcal{L}_\alpha}{\partial x_{t_\alpha}} = \frac{2\theta(\lambda_\alpha)}{x^4}x_{t_\alpha} + h(x;\,\lambda_\alpha) = -\frac{\gamma x_{t_\alpha}}{\lambda_\alpha x^4} + F(x) = \frac{\gamma}{yx^2} + F(x)\,. \tag{6.57}$$

This gives us the following Lagrangians which satisfy the multiform Euler-Lagrange equations:

$$\mathcal{L}_\alpha = \gamma\left(\frac{x_{t_\alpha}^2}{\lambda_\alpha x^4} - \frac{\lambda_\alpha}{x^2}\right) + F(x)x_{t_\alpha}\,, \tag{6.58a}$$

$$\mathcal{L}_\beta = \gamma\left(\frac{x_{t_\beta}^2}{\lambda_\beta x^4} - \frac{\lambda_\beta}{x^2}\right) + F(x)x_{t_\beta}\,. \tag{6.58b}$$

The last term is an exact 1-form, which makes no contributions to the continuous Lagrangian 1-form.

Now we consider the continuous closure relation, that is

$$
\begin{aligned}
\frac{\partial \mathcal{L}_{t_\alpha}}{\partial t_\beta} - \frac{\partial \mathcal{L}_{t_\beta}}{\partial t_\alpha} &= \gamma \left(\frac{2x_{t_\alpha} x_{t_\alpha t_\beta}}{\lambda_\alpha x^4} - \frac{4x_{t_\alpha}^2 x_{t_\beta}}{\lambda_\alpha x^5} + \frac{2\lambda_\alpha x_{t_\beta}}{x^3} \right) \\
&\quad - \gamma \left(\frac{2x_{t_\beta} x_{t_\alpha t_\beta}}{\lambda_\beta x^4} - \frac{4x_{t_\beta}^2 x_{t_\alpha}}{\lambda_\beta x^5} + \frac{2\lambda_\beta x_{t_\alpha}}{x^3} \right) \\
&= \gamma \left(\frac{x_{t_\alpha}}{\lambda_\alpha x^4} - \frac{x_{t_\beta}}{\lambda_\beta x^4} \right) \left(x_{t_\alpha t_\beta} - \frac{2x_{t_\alpha} x_{t_\beta}}{x} - \lambda_\alpha \lambda_\beta x \right) . \qquad (6.59)
\end{aligned}
$$

This is a double zero expansion in terms of the multiform Euler-Lagrange equations. This is stronger than having the closure relation on solutions to the multiform Euler-Lagrange equations.

Proposition 30. *The double zero expansion* (6.59) *defines a continuous Lagrangian 1-form (Definition 27). The general solution to the multiform Euler-Lagrange equations can be written in terms of the cosecant function* (6.40)*.*

We can also Legendre transform the Lagrangian and recover the invariant we considered in the common solutions (6.44), by setting $g(x) = 0$:

$$I_\alpha = \gamma \left(-\frac{x_{t_\alpha}^2}{\lambda_\alpha x^4} - \frac{\lambda_\alpha}{x^2} \right) . \qquad (6.60)$$

On the general solution, with initial conditions defining A, we have

$$(6.40) \quad \Longrightarrow \quad I_\alpha = -\gamma \lambda_\alpha \frac{1}{A^2} . \qquad (6.61)$$

At this point we realise that the transformation $x \mapsto \frac{1}{x}$ takes the continuous Lagrangian 1-form for sine (6.26a) to the continuous Lagrangian 1-form for cosecant (6.58a). Perhaps it is unsurprising that the Lagrangian multiform structure is preserved by this transformation. Thus we can transform between Lagrangian 1-forms with linear multiform Euler-Lagrange equations and Lagrangians 1-forms with non-linear multiform Euler-Lagrange equations. Next we consider an analogous discrete Lagrangian 1-form construction.

6.3.2 Discrete Lagrangian 1-Form

In this section we derive a discrete Lagrangian 1-form where the general solution of the multiform Euler-Lagrange equations is the cosecant function, as in (6.41). Now we apply the transformation $x \mapsto \frac{1}{x}$ and $x_k \mapsto \frac{1}{x_k}$ on the discrete 1-form Lagrangians (6.36) with sinusoidal solutions. This leads to the following discrete Lagrangians:

$$\mathcal{L}_k = \zeta \left(\frac{1}{s_k} \frac{1}{x_k x} - \frac{c_k}{2 s_k} \left(\frac{1}{x_k^2} + \frac{1}{x^2} \right) \right) + G\left(\frac{1}{x_k} \right) - G\left(\frac{1}{x} \right) . \tag{6.62}$$

One can check that the conventional Euler-Lagrange equation is proportional to equation (6.46), and the corner Euler-Lagrange equation is proportional to equation (6.49)

Now let us consider the discrete closure relation

$$\begin{aligned} &T_l \mathcal{L}_k - \mathcal{L}_k - T_k \mathcal{L}_l + \mathcal{L}_l \\ &= \frac{\zeta\, s_k s_l}{2(c_l s_k - c_k s_l)} \left(\left(\frac{\partial \mathcal{L}_k}{\partial x} - \frac{\partial \mathcal{L}_l}{\partial x} \right)^2 - \left(\frac{\partial T_k \mathcal{L}_l}{\partial x_{kl}} - \frac{\partial T_l \mathcal{L}_k}{\partial x_{kl}} \right)^2 \right) \\ &\quad + \frac{\zeta \left((s_l^2 + c_l^2 - 1) s_k^2 - (s_k^2 + c_k^2 - 1) s_l^2 \right) (x_l^2 - x_k^2)}{2 s_k s_l (c_l s_k - c_k s_l) x_k^2 x_l^2} . \end{aligned} \tag{6.63}$$

What we find is, in fact, a double zero expansion, which is stronger than the closure relation. Note that this double zero expansion only requires the identity $s_k^2 + c_k^2 - 1 = 0$ (and $s_l^2 + c_l^2 - 1 = 0$) between the coefficients s_k and c_k. Altogether, we conclude the following.

Proposition 31. *The double zero expansion* (6.63) *gives us a discrete Lagrangian 1-form (Definition 21). The general solution to the multiform Euler-Lagrange equations can be written in terms of the cosecant function, as in* (6.41).

We can also take derivatives of the Lagrangian with respect to a lattice parameter to find the following integral of motion (6.48):

$$I_k = \frac{\partial \mathcal{L}_k}{\partial \lambda_k} = \zeta \left(-\frac{c_k}{s_k^2} \frac{1}{x_k x} + \frac{1}{2 s_k^2} \left(\frac{1}{x_k^2} + \frac{1}{x^2} \right) \right) \tag{6.64}$$

On the general cosecant solution, we recover the constant parameter A determined by initial conditions

$$(6.41) \qquad \Longrightarrow \qquad I_k = \frac{\zeta}{2} \frac{1}{A^2} . \tag{6.65}$$

So far we have considered Lagrangian 1-forms associated with sine/cosine solutions and cosecant/secant solutions. In the next section we will see an example of how generalising to elliptic solutions is quite non-trivial. We consider periodic reductions of lpKdV (or the H1 quad equation) and develop a discrete Lagrangian 1-form.

6.4 Developing Lagrangian 1-Form for the Discrete McMillan Equation

In this section we consider the periodic reductions of the H1 quad equation (or lpKdV) on the $N = 4$ staircase considered in 5.3 and (5.22). In this section we will emphasise it as a periodic reduction of lpKdV and use different notation.

6.4.1 Periodic Reduction of Lattice Korteweg-de Vries Equation

In this subsection we derive novel integral ordinary difference equations which commute with the McMillan equation in the four-dimensional lattice $\mathbb{Z}^4$. In order to do this, we review a derivation of the McMillan equation from a periodic reduction of lattice potential Korteweg-de Vries (lpKdV) [3, 15, 25]. We note the work in [24], where higher dimensional commuting maps are not considered, but more general periodic reductions of lpKdV are derived. We start with a field $u(n_i, n_j, n_k, n_l)$, satisfying the lpKdV equation, written in the following way

$$(p - q - u_i + u_j)(p + q + u - u_{ij}) = (p^2 - q^2)\,. \tag{6.66}$$

Here, p and q are lattice parameters associated with n_i and n_j directions respectively. We impose a periodic initial staircase along the steps (T_i, T_j, T_i, T_j) such that there is periodicity $T_i^2 T_j^2 = \mathbb{1}$. The following periodic staircase is imposed: $w_1 \coloneqq u$, $w_2 \coloneqq u_i$, $w_3 \coloneqq u_{ij}$ and $w_4 \coloneqq u_{iij}$. If we consider evolution of staircase variables in the n_i direction we get the map $(w_1, w_2, w_3, w_4) \mapsto (T_i w_1, T_i w_2, T_i w_3, T_i w_4)$

$$T_i w_1 = w_2\,, \tag{6.67a}$$

$$(p - q - T_i w_2 + w_3)(p + q + w_2 - w_4) = (p^2 - q^2)\,, \tag{6.67b}$$

$$T_i w_3 = w_4\,, \tag{6.67c}$$

$$(p - q - T_i w_4 + w_1)(p + q + w_4 - w_2) = (p^2 - q^2)\,. \tag{6.67d}$$

Analogously, if we consider evolution of staircase variables in the n_j direction we get the implicit map $(w_1, w_2, w_3, w_4) \mapsto (T_j w_1, T_j w_2, T_j w_3, T_j w_4)$

$$(p - q - w_2 + T_j w_1)(p + q + w_1 - w_3) = (p^2 - q^2)\,, \tag{6.68a}$$

$$T_j w_2 = w_3\,, \tag{6.68b}$$

$$(p - q - w_4 + T_j w_3)(p + q + w_3 - w_1) = (p^2 - q^2)\,, \tag{6.68c}$$

$$T_j w_4 = w_1\,. \tag{6.68d}$$

The next step of this reduction involves reduction variables. To find these, we first define the lpKdV Lax matrices [23, 36]:

$$L_p = \begin{pmatrix} p - u_i & 1 \\ k^2 + p(u - u_i) - u u_i & p + u \end{pmatrix} \tag{6.69a}$$

$$L_q = \begin{pmatrix} q - u_j & 1 \\ k^2 + q(u - u_j) - u u_j & q + u \end{pmatrix} \tag{6.69b}$$

We can then take the trace of the monodromy matrix, which is a shifted product of Lax matrices along the staircase:

$$\begin{aligned} &\mathrm{Tr}(\, T_i T_j T_i L_p \cdot T_i T_j L_q \cdot T_i L_q \cdot L_p \,) \\ &\quad = (w_1 - w_3)^2 (w_2 - w_4)^2 - (p^2 + q^2)((w_1 - w_3) + (w_2 - w_4))^2 \\ &\qquad - 2pq \left(((w_1 - w_3) + (w_2 - w_4))^2 + 2(w_1 - w_3)(w_2 - w_4) \right) + 2p^2 q^2 + \mathcal{O}(k^2)\,. \end{aligned} \tag{6.70}$$

This leads to an invariant in terms of (w_1, w_2, w_3, w_4) on the staircase once we ignore higher order terms of k. In this expression we observe the pairings $\pm(w_1 - w_3)$ and $\pm(w_2 - w_4)$. Thus, we make the following natural choice for our definitions of x and y

$$x \coloneqq w_1 - w_3 \qquad y \coloneqq w_2 - w_4\,. \tag{6.71}$$

We also introduce the following parameter abbreviation, which we will henceforth use alongside p and q for brevity:

$$\epsilon \coloneqq p + q \qquad \gamma \coloneqq p - q\,. \tag{6.72}$$

Now we can rewrite the invariant above, where we utilise the freedom to subtract constants for brevity, in the form

$$J(x, y) \coloneqq 2\,\epsilon\,\gamma\,x\,y - \epsilon^2\,(x^2 + y^2) + x^2\,y^2\,. \tag{6.73}$$

With these reduction variables, we can derive the following equations for evolution in the n_i direction

$$y_i = -x + \frac{2\,\epsilon\,\gamma\,x_i}{\epsilon^2 - x_i^2}\,, \tag{6.74a}$$

$$y = x_i\,. \tag{6.74b}$$

For evolution in the n_j direction we get

$$y = x_j + \frac{2\,\epsilon\,\gamma\,x}{\epsilon^2 - x^2}\,, \tag{6.75a}$$

$$y_j = -x\,. \tag{6.75b}$$

This leads to the following two second order ordinary difference equations

$$x_{ii} - \frac{2\,\epsilon\,\gamma\,x_i}{\epsilon^2 - x_i^2} + x = 0\,, \tag{6.76a}$$

$$x_{jj} + \frac{2\,\epsilon\,\gamma\,x_j}{\epsilon^2 - x_j^2} + x = 0\,. \tag{6.76b}$$

Due to the symmetry of this periodic staircase, evolution of x in the n_i and n_j directions are the same up to opposite signs of the parameter. One can easily check that the invariants, $J(x, x_i)$ and $J(x_j, -x)$, are conserved under the respective second order difference equations.

Now we consider the implicit map $(w_1, w_2, w_3, w_4) \mapsto (T_k w_1, T_k w_2, T_k w_3, T_k w_4)$ which arises from considering evolution in the direction n_k, perpendicular to the plane of the periodic staircase:

$$(p - r - w_2 + T_k w_1)(p + r + w_1 - T_k w_2) = (p^2 - r^2)\,, \tag{6.77a}$$

$$(q - r - w_3 + T_k w_2)(q + r + w_2 - T_k w_3) = (q^2 - r^2)\,, \tag{6.77b}$$

$$(p - r - w_4 + T_k w_3)(p + r + w_3 - T_k w_4) = (p^2 - r^2)\,, \tag{6.77c}$$

$$(q - r - w_1 + T_k w_4)(q + r + w_4 - T_k w_1) = (q^2 - r^2)\,. \tag{6.77d}$$

As this map arises from a periodic reduction of the multidimensionally-consistent lpKdV equations, we expect it to commute with the maps (6.67) and (6.68a), in the plane of the staircase. In order to derive equations in terms of x and y, we choose to exchange w_1 and w_2 for x and y, using $w_1 \equiv w_3 + x$ and $w_2 \equiv w_4 + y$ to get

$$(p - r + (T_k w_3 - w_4) + x_k - y)(p + r - (T_k w_4 - w_3) + x - y_k) = p^2 - r^2\,, \tag{6.78a}$$

$$(q - r + (T_k w_4 - w_3) + y_k)(q + r - (T_k w_3 - w_4) + y) = q^2 - r^2\,, \tag{6.78b}$$

$$(p - r + (T_k w_3 - w_4))(p + r - (T_k w_4 - w_3)) = p^2 - r^2\,, \tag{6.78c}$$

$$(q - r + (T_k w_4 - w_3) - x)(q + r - (T_k w_3 - w_4) - x_k) = q^2 - r^2\,. \tag{6.78d}$$

It turns out that the remaining four variables $w_3, w_4, T_k w_3, T_k w_4$ appear only in two pairs $(T_k w_3 - w_4)$ and $(T_k w_4 - w_3)$. Now by eliminating the pairs and applying the quadratic formula, we derive the following equations for y and y_k

$$y = \frac{\epsilon\,\gamma\,x}{\epsilon^2 - x^2} \pm_3 \frac{\sqrt{\gamma^2\epsilon^2 + (4r^2 - \gamma^2 - \epsilon^2)(x-\epsilon)(x_k-\epsilon) + (x-\epsilon)^2(x_k-\epsilon)^2}}{2(x-\epsilon)} \pm_3 \frac{\sqrt{\gamma^2\epsilon^2 + (4r^2 - \gamma^2 - \epsilon^2)(x+\epsilon)(x_k+\epsilon) + (x+\epsilon)^2(x_k+\epsilon)^2}}{2(x+\epsilon)}\,, \tag{6.79a}$$

$$y_k = \frac{\epsilon\,\gamma\,x_k}{\epsilon^2 - x_k^2} \mp_3 \frac{\sqrt{\gamma^2\epsilon^2 + (4r^2 - \gamma^2 - \epsilon^2)(x-\epsilon)(x_k-\epsilon) + (x-\epsilon)^2(x_k-\epsilon)^2}}{2(x_k-\epsilon)} \mp_3 \frac{\sqrt{\gamma^2\epsilon^2 + (4r^2 - \gamma^2 - \epsilon^2)(x+\epsilon)(x_k+\epsilon) + (x+\epsilon)^2(x_k+\epsilon)^2}}{2(x_k+\epsilon)}\,. \tag{6.79b}$$

We denote the two possible choice of signs associated in the n_k direction with $\pm_k$. The two simplest options for the behaviour of the signs on solutions is that they remain fixed or that they flip once on every step. Such flipping leads to the trivial solution $y = y_{kk}$, whereas we use fixed signs from henceforth and derive consistent non-trivial equations. There is another possibility, which we do not consider, that the signs update in a non-trivial way upon being shifted. The square roots which appear here are symmetric in x and x_k and hence we define the following

$$\Gamma_r(v) \coloneqq \sqrt{\gamma^2\epsilon^2 + (4r^2 - \gamma^2 - \epsilon^2)v + v^2}\,. \tag{6.80}$$

We can now write down the set of commuting equations, in directions n_i, n_j and n_k, as well as a fourth direction n_l which is analogous to n_k but with lattice parameter s

$$y - \frac{\epsilon\,\gamma\,x}{\epsilon^2 - x^2} = -\frac{\epsilon\,\gamma\,x}{\epsilon^2 - x^2} + x_i\,, \tag{6.81a}$$

$$y_i - \frac{\epsilon\,\gamma\,x_i}{\epsilon^2 - x_i^2} = \frac{\epsilon\,\gamma\,x_i}{\epsilon^2 - x_i^2} - x\,, \tag{6.81b}$$

$$y - \frac{\epsilon\,\gamma\,x}{\epsilon^2 - x^2} = \frac{\epsilon\,\gamma\,x}{\epsilon^2 - x^2} + x_j\,, \tag{6.81c}$$

$$y_j - \frac{\epsilon\,\gamma\,x_j}{\epsilon^2 - x_j^2} = -\frac{\epsilon\,\gamma\,x_j}{\epsilon^2 - x_j^2} - x\,, \tag{6.81d}$$

$$y - \frac{\epsilon\,\gamma\,x}{\epsilon^2 - x^2} = \pm_k \frac{\Gamma_r((x-\epsilon)(x_k-\epsilon))}{2(x-\epsilon)} \pm_k \frac{\Gamma_r((x+\epsilon)(x_k+\epsilon))}{2(x+\epsilon)}\,, \tag{6.81e}$$

$$y_k - \frac{\epsilon\,\gamma\,x_k}{\epsilon^2 - x_k^2} = \mp_k \frac{\Gamma_r((x-\epsilon)(x_k-\epsilon))}{2(x_k-\epsilon)} \mp_k \frac{\Gamma_r((x+\epsilon)(x_k+\epsilon))}{2(x_k+\epsilon)}\,, \tag{6.81f}$$

$$y - \frac{\epsilon\,\gamma\,x}{\epsilon^2 - x^2} = \pm_l \frac{\Gamma_s((x-\epsilon)(x_l-\epsilon))}{2(x-\epsilon)} \pm_l \frac{\Gamma_s((x+\epsilon)(x_l+\epsilon))}{2(x+\epsilon)}\,, \tag{6.81g}$$

$$y_l - \frac{\epsilon\,\gamma\,x_l}{\epsilon^2 - x_l^2} = \mp_l \frac{\Gamma_s((x-\epsilon)(x_l-\epsilon))}{2(x_l-\epsilon)} \mp_l \frac{\Gamma_s((x+\epsilon)(x_l+\epsilon))}{2(x_l+\epsilon)}\,. \tag{6.81h}$$

Here, we consider four dimensions, so that we can consider two maps, T_k and T_l, which evolve perpendicular to the plane of the staircase. Unlike the T_i and T_j maps, the

T_k and T_l maps each have their own independent parameter, r and s, respectively. These perpendicular maps and their parameters involve lots of interesting non-triviality. Furthermore, we could use the fact that the parameters r and s are to free to generalise this to an integrable system of N commuting maps. We will also that we can use a limit to recover the simpler T_i and T_j maps from the non-trivial T_k and T_l maps. Note that this set of equations here suggests a Lagrangian 1-form structure is possible, which is discussed in the next section. For now we investigate these equations and the invariants further. We can make a link between the shared variable y and the invariant J

$$y - \frac{\epsilon\,\gamma\,x}{\epsilon^2 - x^2} = \pm\frac{\sqrt{\epsilon^2 x^4 + (J + (\gamma^2 - \epsilon^2)\epsilon^2)x^2 - J\epsilon^2}}{\epsilon^2 - x^2}\,, \tag{6.82a}$$

$$y_i - \frac{\epsilon\,\gamma\,x_i}{\epsilon^2 - x_i^2} = \pm\frac{\sqrt{\epsilon^2 x_i^4 + (J + (\gamma^2 - \epsilon^2)\epsilon^2)x_i^2 - J\epsilon^2}}{\epsilon^2 - x_i^2}\,. \tag{6.82b}$$

Now we write down second order difference corner equations in terms of $x(n_i, n_j, n_k, n_l)$. We can eliminate two equations involving y to derive a corner equation for (x, x_i, x_j)

$$x_i - \frac{2\epsilon\,\gamma\,x}{\epsilon^2 - x^2} - x_j = 0\,. \tag{6.83}$$

We can eliminate two equations involving y_j to derive the following trivial corner equation

$$x_{ij} = -x\,. \tag{6.84}$$

We can eliminate two equations involving y to derive a corner equation for (x, x_i, x_k)

$$x_i - \frac{\epsilon\,\gamma\,x}{\epsilon^2 - x^2} \mp_k \frac{\Gamma_r((x-\epsilon)(x_k-\epsilon))}{2(x-\epsilon)} \mp_k \frac{\Gamma_r((x+\epsilon)(x_k+\epsilon))}{2(x+\epsilon)} = 0\,. \tag{6.85}$$

We can eliminate two equations involving y to derive a corner equation for (x, x_k, x_l)

$$\begin{aligned}&\pm_k \frac{\Gamma_r((x-\epsilon)(x_k-\epsilon))}{2(x-\epsilon)} \pm_k \frac{\Gamma_r((x+\epsilon)(x_k+\epsilon))}{2(x+\epsilon)}\\ &\mp_l \frac{\Gamma_s((x-\epsilon)(x_l-\epsilon))}{2(x-\epsilon)} \mp_l \frac{\Gamma_s((x+\epsilon)(x_l+\epsilon))}{2(x+\epsilon)} = 0\,.\end{aligned} \tag{6.86}$$

Furthermore, we can eliminate two equations involving y_k to derive a corner equation for (x, x_k, x_{kl})

$$\begin{aligned}&\mp_k \frac{\Gamma_r((x-\epsilon)(x_k-\epsilon))}{2(x_k-\epsilon)} \mp_k \frac{\Gamma_r((x+\epsilon)(x_k+\epsilon))}{2(x_k+\epsilon)}\\ &\mp_l \frac{\Gamma_s((x_k-\epsilon)(x_{kl}-\epsilon))}{2(x_k-\epsilon)} \mp_l \frac{\Gamma_s((x_k+\epsilon)(x_{kl}+\epsilon))}{2(x_k+\epsilon)} = 0\,.\end{aligned} \tag{6.87}$$

We now eliminate two equations involving y_k to derive the second order difference equation for evolution in the n_k direction involving (x, x_k, x_{kk})

$$\begin{aligned} &\pm_k \frac{\Gamma_r((x-\epsilon)(x_k-\epsilon))}{2(x_k-\epsilon)} \pm_k \frac{\Gamma_r((x+\epsilon)(x_k+\epsilon))}{2(x_k+\epsilon)} \\ &\pm_k \frac{\Gamma_r((x_k-\epsilon)(x_{kk}-\epsilon))}{2(x_k-\epsilon)} \pm_k \frac{\Gamma_r((x_k+\epsilon)(x_{kk}+\epsilon))}{2(x_k+\epsilon)} = 0\,. \end{aligned} \tag{6.88}$$

This can be expanded out to give a polynomial, which is a polynomial and a second order difference equation in (x, x_k, x_{kk})

$$\begin{aligned} &4x_k^6(x+x_{kk})^2 + 8x_k^5(x+x_{kk})(4r^2-\gamma^2+x\,x_{kk}) \\ &+ 4x_k^4((4r^2-\gamma^2)^2-\epsilon^4-\epsilon^2(x^2+x_{kk}^2)+2(4r^2-\gamma^2-2\epsilon^2)x\,x_{kk}) \\ &- 4x_k^3(x+x_{kk})(\epsilon^2(12r^2-\gamma^2+\epsilon^2)+(4r^2-\gamma^2+3\epsilon^2)x\,x_{kk}) \\ &- 4x_k^2(\epsilon^2(16r^4-(\gamma^2-\epsilon^2)^2)+4r^2\epsilon^2(x+x_{kk})^2+(16r^4-8r^2\gamma^2+\gamma^4-\epsilon^4)x\,x_{kk}) \\ &+ 4x_k\epsilon^2(4r^2-\gamma^2+\epsilon^2)(x+x_{kk})(\epsilon^2-\gamma^2+x\,x_{kk}) + \epsilon^2(4r^2-\gamma^2+\epsilon^2)(x+x_{kk})^2 \\ &= 0 \end{aligned} \tag{6.89}$$

This polynomial is symmetric in x and x_{kk}, and it is of sixth degree in x_k. It is a generalisation of the McMillan equations (6.76). These commuting flows must be compatible with the McMillan equations (6.76), but also generalise them, which we will show in later sections.

Similarly, (6.86) can be squared to give a sixth degree equation in x and fourth degree in x_k and x_l,

$$\begin{aligned} 0 =\,& c_6(x_k, x_l, \epsilon, \gamma, r, s)x^6 + c_5(x_k, x_l, \epsilon, \gamma, r, s)x^5 + c_4(x_k, x_l, \epsilon, \gamma, r, s)x^4 \\ &+ c_3(x_k, x_l, \epsilon, \gamma, r, s)x^3 + c_2(x_k, x_l, \epsilon, \gamma, r, s)x^2 + c_1(x_k, x_l, \epsilon, \gamma, r, s)x \\ &+ c_0(x_k, x_l, \epsilon, \gamma, r, s) \end{aligned} \tag{6.90}$$

The coefficients c_i and the entire equation is symmetric under $(x_k, r) \leftrightarrow (x_l, s)$.

We now derive an equations defining the invariant J in terms of x and x_k. Let us start with

$$\pm\frac{\sqrt{\epsilon^2x^4+(J+(\gamma^2-\epsilon^2)\epsilon^2)x^2-J\epsilon^2}}{\epsilon^2-x^2} = \pm_k\frac{\Gamma_r((x-\epsilon)(x_k-\epsilon))}{2(x-\epsilon)} \pm_k \frac{\Gamma_r((x+\epsilon)(x_k+\epsilon))}{2(x+\epsilon)} \tag{6.91}$$

We can expand this out. Along the way this gives J in terms of x and x_k

$$\begin{aligned} J =& -2r^2\epsilon^2 - \frac{1}{2}\epsilon^2(x^2+x_k^2) + \frac{1}{2}(4r^2-\gamma^2-\epsilon^2)xx_k + \frac{1}{2}x^2x_k^2 \\ &+ \frac{1}{2}\Gamma_r((x-\epsilon)(x_k-\epsilon))\Gamma_r((x+\epsilon)(x_k+\epsilon)) \end{aligned} \tag{6.92}$$

Expanding out gives the following biquadratic polynomial for x and x_k involving the invariant J:

$$\begin{aligned} 0 = & -J + \frac{J(8r^2 - 2(\gamma^2+\epsilon^2)^2) + \epsilon^2(16r^4 - (\gamma^2-\epsilon^2)^2))}{2J + 8r^2\epsilon^2} x x_k \\ & - \frac{\epsilon^2(4J + (4r^2 - \gamma^2 + \epsilon^2)^2)}{4J + 16r^2\epsilon^2}(x^2 + x_k^2) + x^2 x_k^2 \,. \end{aligned} \tag{6.93}$$

This can also been seen as a quadratic equation in terms of the invariant J.

6.4.2 Jacobi Elliptic Solutions

In this subsection we derive Jacobi elliptic function solutions for the McMillan equation and its system of corner equations.

Theorem 32. *Given a value for the invariant $J(x, x_i)$ from two initial conditions x and x_i, as well lattice parameters p, q, r and s, we have a general Jacobi elliptic solution for the commuting McMillan equations*

$$x(n_i, n_j, n_k, n_l) = A\,\mathrm{sn}(\xi + \lambda_i n_i + \lambda_j n_j + \lambda_k n_k + \lambda_l n_l;\, k)\,. \tag{6.94}$$

Here, the equations (6.97)*,* (6.98)*,* (6.99) *and* (6.100) *define $A(J,p,q)$, $k(J,p,q)$, $\xi(J,p,q)$, $\lambda_i(J,p,q))$, $\lambda_j(J,p,q))$, $\lambda_k(J,p,q,r)$ and $\lambda_l(J,p,q,s)$.*

Two prove this theorem we first derive a first order difference equation for $x(n_i) = A\,\mathrm{sn}(\xi + \lambda_i n_i;\, k)$, with the parameters A, k and λ_i appearing explicitly in the equation

$$\frac{A^4}{k^2} + \frac{2A^2\mathrm{cn}(\lambda_i)\mathrm{dn}(\lambda_i)}{k^2\mathrm{sn}(\lambda_i)^2} x x_i - \frac{A^2}{k^2\mathrm{sn}(\lambda_i)^2}(x^2 + x_i^2) + x^2 x_i^2 = 0 \tag{6.95}$$

This can be derived from the addition formula for the Jacobi sn function. Now, let us recall the McMillan invariant (6.73) for x and x_i, namely

$$J(x,y) \coloneqq 2\,\epsilon\,\gamma\,x\,x_i - \epsilon^2\,(x^2 + x_i^2) + x^2\,x_i^2\,.$$

Equating coefficients of the two equations leads to

$$J = -\frac{A^4}{k^2}\,, \tag{6.96a}$$

$$2\,\epsilon\,\gamma = \frac{2A^2\mathrm{cn}(\lambda_i)\mathrm{dn}(\lambda_i)}{k^2\mathrm{sn}(\lambda_i)^2}\,, \tag{6.96b}$$

$$-\epsilon^2 = -\frac{A^2}{k^2\mathrm{sn}(\lambda_i)^2}\,. \tag{6.96c}$$

We can solve these three equations alongside $\mathrm{cn}(\lambda_i,k)^2\mathrm{dn}(\lambda_i,k)^2 \equiv (1-\mathrm{sn}(\lambda_i,k)^2)(1-$

$k^2\mathrm{sn}(\lambda_i, k)^2$), to get solution parameters in terms of $\epsilon(p,q), \gamma(p,q), J(x, x_i, p, q)$ and x

$$A^2 = -\frac{J + \gamma^2\epsilon^2 - \epsilon^4}{2\epsilon^2} \pm \frac{\sqrt{(J + \gamma^2\epsilon^2 + \epsilon^4)^2 - (2\gamma\epsilon^3)^2}}{2\epsilon^2}\,, \tag{6.97a}$$

$$k^2 = -\frac{J^2 + 2J\gamma^2\epsilon^2 + \epsilon^4(\gamma^2 - \epsilon^2)}{2J\epsilon^4} \pm \frac{(J + \gamma^2\epsilon^2 - \epsilon^4)\sqrt{(J + \gamma^2\epsilon^2 + \epsilon^4)^2 - (2\gamma\epsilon^3)^2}}{2J\epsilon^4}\,, \tag{6.97b}$$

$$\xi = \mathrm{sn}^{-1}(x/A) \tag{6.97c}$$

The following equations determine the parameters λ_i and λ_j, which define evolution in the n_i and n_j direction:

$$\lambda_i\,, \lambda_j \in \mathbb{C} \quad \text{s.t.} \tag{6.98a}$$

$$\mathrm{cn}(\lambda_i, k)\mathrm{dn}(\lambda_i, k) = \frac{\gamma}{\epsilon}\,, \tag{6.98b}$$

$$\mathrm{cn}(\lambda_j, k)\mathrm{dn}(\lambda_j, k) = -\frac{\gamma}{\epsilon}\,, \tag{6.98c}$$

$$\mathrm{sn}(\lambda_i, k)^2 = \mathrm{sn}(\lambda_j, k)^2 = \frac{A^2}{k^2\epsilon^2}\,. \tag{6.98d}$$

These parameters can be solved for by inverting the elliptic functions and make a choice with the freedom of the elliptic periods. Alternatively, one could solve for these parameters numerically.

Now we derive the solutions for a commuting flow equation in the n_k direction, recalling (6.93). This leads to (6.97) and the following equations, which determine the parameter λ_k, defining evolution in the n_k direction:

$$\lambda_k \in \mathbb{C} \quad \text{s.t.} \tag{6.99a}$$

$$\mathrm{cn}(\lambda_k, k)\mathrm{dn}(\lambda_k, k) = \frac{J(8r^2 - 2(\gamma^2 + \epsilon^2)^2) + \epsilon^2(16r^4 - (\gamma^2 - \epsilon^2)^2))}{\epsilon^2(4J + (4r^2 - \gamma^2 + \epsilon^2)^2)}\,, \tag{6.99b}$$

$$\mathrm{sn}(\lambda_k, k)^2 = \frac{4J + 16r^2\epsilon^2}{\epsilon^2(4J + (4r^2 - \gamma^2 + \epsilon^2)^2)}\frac{A^2}{k^2}\,. \tag{6.99c}$$

We can consider another analogous flow in the fourth direction with λ_l associated with s

$$\lambda_l \in \mathbb{C} \quad \text{s.t.} \tag{6.100a}$$

$$\mathrm{cn}(\lambda_l, k)\mathrm{dn}(\lambda_l, k) = \frac{J(8s^2 - 2(\gamma^2 + \epsilon^2)^2) + \epsilon^2(16s^4 - (\gamma^2 - \epsilon^2)^2))}{\epsilon^2(4J + (4s^2 - \gamma^2 + \epsilon^2)^2)}\,, \tag{6.100b}$$

$$\mathrm{sn}(\lambda_l, k)^2 = \frac{4J + 16s^2\epsilon^2}{\epsilon^2(4J + (4s^2 - \gamma^2 + \epsilon^2)^2)}\frac{A^2}{k^2}\,. \tag{6.100c}$$

This concludes this section, given us everything we need for the theorem above. We have commuting Jacobi elliptic solutions with non-trivial solution parameters.

6.4.3 Partial Discrete Lagrangian 1-Form

The corner equations (6.81) above can be integrated to give the following Lagrangians

$$\mathcal{L}_i \coloneqq -xx_i - \frac{\gamma\epsilon}{2}\log\big(\epsilon^2 - x^2\big) - \frac{\gamma\epsilon}{2}\log\big(\epsilon^2 - x_i^2\big)\,, \tag{6.101a}$$

$$\mathcal{L}_j \coloneqq -xx_j + \frac{\gamma\epsilon}{2}\log\big(\epsilon^2 - x^2\big) + \frac{\gamma\epsilon}{2}\log\big(\epsilon^2 - x_j^2\big)\,, \tag{6.101b}$$

$$\mathcal{L}_k \coloneqq \mp_k \int^{(x+\epsilon)(x_k+\epsilon)} \frac{\Gamma_r(v)}{2v}\,\mathrm{d}v \mp_k \int^{(x-\epsilon)(x_k-\epsilon)} \frac{\Gamma_r(v)}{2v}\,\mathrm{d}v\ , \tag{6.101c}$$

$$\mathcal{L}_l \coloneqq \mp_l \int^{(x+\epsilon)(x_l+\epsilon)} \frac{\Gamma_s(v)}{2v}\,\mathrm{d}v \mp_l \int^{(x-\epsilon)(x_l-\epsilon)} \frac{\Gamma_s(v)}{2v}\,\mathrm{d}v\ . \tag{6.101d}$$

The corner equations are recovered from the Euler-Lagrange equations in the following way:

$$\begin{aligned} -\frac{\partial}{\partial x}\mathcal{L}_k &= \pm_k \frac{\Gamma_r((x+\epsilon)(x_k+\epsilon))}{2(x+\epsilon)(x_k+\epsilon)}\frac{\partial((x+\epsilon)(x_k+\epsilon))}{\partial x} \\ &\quad \pm_k \frac{\Gamma_r((x-\epsilon)(x_k-\epsilon))}{2(x-\epsilon)(x_k-\epsilon)}\frac{\partial((x-\epsilon)(x_k-\epsilon))}{\partial x} \\ &= \pm_k \frac{\Gamma_r((x+\epsilon)(x_k+\epsilon))}{2(x+\epsilon)} \pm_k \frac{\Gamma_r((x-\epsilon)(x_k-\epsilon))}{2(x-\epsilon)}\,, \end{aligned} \tag{6.102a}$$

$$\begin{aligned} \frac{\partial}{\partial x_k}\mathcal{L}_k &= \mp_k \frac{\Gamma_r((x+\epsilon)(x_k+\epsilon))}{2(x+\epsilon)(x_k+\epsilon)}\frac{\partial((x+\epsilon)(x_k+\epsilon))}{\partial x_k} \\ &\quad \mp_k \frac{\Gamma_r((x-\epsilon)(x_k-\epsilon))}{2(x-\epsilon)(x_k-\epsilon)}\frac{\partial((x-\epsilon)(x_k-\epsilon))}{\partial x_k} \\ &= \mp_k \frac{\Gamma_r((x+\epsilon)(x_k+\epsilon))}{2(x_k+\epsilon)} \mp_k \frac{\Gamma_r((x-\epsilon)(x_k-\epsilon))}{2(x_k-\epsilon)}\,. \end{aligned} \tag{6.102b}$$

We note the integrals in the Lagrangians above have the following forms in terms of known functions

$$\begin{aligned} \int \frac{\Gamma_r(v)}{2v}\,\mathrm{d}v \equiv{}& \frac{1}{2}\Gamma_r(v) + \gamma\epsilon\ \mathrm{arctanh}\left(\frac{v-\Gamma_r(v)}{\gamma\epsilon}\right) \\ &- \frac{4r^2-\gamma^2-\epsilon^2}{4}\log\Big(-(4r^2-\gamma^2-\epsilon^2) - 2v + 2\Gamma_r(v)\Big) \end{aligned} \tag{6.103}$$

We note the following relevant identity:

$$\mathrm{arctanh}(z) \equiv \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\,. \tag{6.104}$$

We also note the following freedom. Each Lagrangian can be replaced with $\mathcal{L}_i \to \mathcal{L}_i'$, without breaking the multiform Euler-Lagrange structure:

$$\mathcal{L}_i' \coloneqq f(\epsilon,\gamma)\mathcal{L}_i + g(x,\epsilon,\gamma) - g(x_i,\epsilon,\gamma) + h_i(\epsilon,\gamma)\,, \tag{6.105a}$$

$$\mathcal{L}'_j := f(\epsilon,\gamma)\mathcal{L}_j + g(x,\epsilon,\gamma) - g(x_j,\epsilon,\gamma) + h_j(\epsilon,\gamma)\,, \tag{6.105b}$$

$$\mathcal{L}'_k := f(\epsilon,\gamma)\mathcal{L}_k + g(x,\epsilon,\gamma) - g(x_k,\epsilon,\gamma) + h_k(\epsilon,\gamma,r)\,, \tag{6.105c}$$

$$\mathcal{L}'_l := f(\epsilon,\gamma)\mathcal{L}_l + g(x,\epsilon,\gamma) - g(x_l,\epsilon,\gamma) + h_l(\epsilon,\gamma,s)\,. \tag{6.105d}$$

The multiform Euler-Lagrange should guarantee constant closure. However, further work is required to prove directly that these Lagrangians have constant closure, and then vanishing closure up to certain choices of functions f, g and h_i. We note that we can show vanishing closure between $\mathcal{L}_i$ and $\mathcal{L}_j$, but this is trivial

$$x_{ij} = -x \qquad \Longrightarrow \qquad \mathcal{L}_i - T_j\mathcal{L}_i - \mathcal{L}_j + T_i\mathcal{L}_j = 0\,. \tag{6.106}$$

Overall, we can only conclude the following.

Theorem 33. *The four Lagrangians* (6.101) *are a partial Lagrangian 1-form in the following sense. The closure relation has not been proved, but the multiform Euler-Lagrange equations correspond to a non-trivial integrable system of commuting maps (with general solutions in terms of Jacobi elliptic functions given in Theorem 32).*

6.4.4 Degenerations from Commuting McMillan to McMillan

In this subsection we describe how the commuting McMillan flows generalise the standard McMillan flows in the plane of the staircase. When considering periodic reductions, the flows perpendicular to the plane of the periodic staircase are fundamental. In this case the degenerations $r \mapsto p$ and $x_k \mapsto x_i$ takes the commuting flows in n_k to the McMillan equations in n_i. Analogously, one can consider $r \mapsto q$ and $x_k \mapsto x_j$. The square roots degenerate as follows:

$$\lim_{\substack{r\to\pm p\\ x_k\to x_i}} \Gamma_r((x-\epsilon)(x_k-\epsilon)) \equiv \gamma\epsilon + \epsilon^2 - \epsilon(x+x_i) + xx_i\,, \tag{6.107a}$$

$$\lim_{\substack{r\to\pm p\\ x_k\to x_i}} \Gamma_r((x+\epsilon)(x_k+\epsilon)) \equiv \gamma\epsilon + \epsilon^2 + \epsilon(x+x_i) + xx_i\,. \tag{6.107b}$$

$$\lim_{\substack{r\to\pm q\\ x_k\to x_j}} \Gamma_r((x-\epsilon)(x_k-\epsilon)) \equiv \gamma\epsilon - \epsilon^2 + \epsilon(x+x_j) - xx_j\,, \tag{6.108a}$$

$$\lim_{\substack{r\to\pm q\\ x_k\to x_j}} \Gamma_r((x+\epsilon)(x_k+\epsilon)) \equiv \gamma\epsilon - \epsilon^2 - \epsilon(x+x_j) - xx_j\,. \tag{6.108b}$$

All corner equations (6.81) degenerate accordingly, for example

$$\lim_{\substack{r\to\pm p\\ x_k\to x_i}} \pm_k \frac{\Gamma_r((x-\epsilon)(x_k-\epsilon))}{2(x-\epsilon)} \pm_k \frac{\Gamma_r((x+\epsilon)(x_k+\epsilon))}{2(x+\epsilon)} \equiv \pm_k\Big(-\frac{\epsilon\,\gamma\,x}{\epsilon^2-x^2} + x_i\Big)\,, \tag{6.109a}$$

$$\lim_{\substack{r\to\pm p\\ x_k\to x_i}} \mp_k \frac{\Gamma_r((x-\epsilon)(x_k-\epsilon))}{2(x_k-\epsilon)} \mp_k \frac{\Gamma_r((x+\epsilon)(x_k+\epsilon))}{2(x_k+\epsilon)} \equiv \mp_k(\frac{\epsilon\,\gamma\,x_i}{\epsilon^2-x_i^2}-x)\,. \tag{6.109b}$$

One can also apply these degenerations to the Jacobi elliptic solutions above and see that the dependence on $J(x, x_k)$ in the solution parameter λ_k disappears and simplifies.

6.4.5 Linearisation to Recover Discrete Harmonic Oscillator

In this section, we show that the discrete nonlinear 1-form above linearises to the discrete harmonic oscillator 1-form structure from [26]. In order to carry out a linearisation we first recognise a trivial solution to the set of equations, which is $x(n_i, n_j, n_k, n_l) = 0$. The square roots with this solution lead to

$$\pm_k \Gamma_r(\epsilon^2) \coloneqq \pm_k 2r\epsilon\,. \tag{6.110}$$

With this it can be easily checked that all of the corner equations (6.81) vanish under $x(n_i, n_j, n_k, n_l) = 0$ and $(y(n_i, n_j, n_k, n_l) = 0)$. Now we can linearise the Lagrangians about this solution to obtain

$$\lim_{x\to 0} \mathcal{L}_j \equiv -xx_j - \frac{\gamma}{2\epsilon}(x^2+x_j^2)\,, \tag{6.111a}$$

$$\lim_{x\to 0} \mathcal{L}_k \equiv \frac{\gamma^2-\epsilon^2-4r^2}{4r\epsilon}xx_k + \frac{\gamma^2-\epsilon^2+4r^2}{8r\epsilon}(x^2+x_k^2) - h_k(\epsilon,\gamma,r)\,. \tag{6.111b}$$

With $f(\epsilon,\gamma) = -\epsilon$, $P \coloneqq pq \equiv \frac{1}{4}(\epsilon^2-\gamma^2)$, $Q \coloneqq q^2$ and $R \coloneqq r^2$, we get

$$\lim_{x\to 0} \mathcal{L}'_j \equiv \frac{1}{q}\left((P+Q)xx_j + \frac{1}{2}(P-Q)(x^2+x_j^2)\right)\,. \tag{6.112a}$$

$$\lim_{x\to 0} \mathcal{L}'_k \equiv \frac{1}{r}\left((P+R)xx_k + \frac{1}{2}(P-R)(x^2+x_k^2)\right)\,. \tag{6.112b}$$

Hence linearisation recovers the discrete Lagrangian 1-forms of the harmonic oscillator, as in [26].

6.5 Chapter Conclusions

In this chapter we developed Lagrangian 1-forms associated with special functions, including the sine function, the cosecant function and Jacobi elliptic functions. In Section 6.3, we derived novel a discrete Lagrangian and a continuous Lagrangian 1-form, where the general solution to the multiform Euler-Lagrange equations are common cosecant functions. With future work, the discrete Lagrangian 1-form could be quantised in the sense of path integrals to generalise the work of [26]. In Section 6.4, we derived a partial discrete Lagrangian 1-form, associated with the McMillan equation, where the general solution to the multiform Euler-Lagrange equations is a common Jacobi elliptic function. Despite the McMillan equation being a simple integrable ordinary difference equation arising from a periodic reduction of lpKdV, the system of commuting maps and the corresponding Lagrangian 1-form structure is rich and non-trivial. Further work is required to prove that the discrete Lagrangian 1-form satisfies constant closure (and then vanishing closure) on the multiform Euler-Lagrange equations. We find that the non-trivial commuting flows generalise and degenerate to the McMillan equations. Furthermore, our partial discrete Lagrangian 1-form for the McMillan equation can be linearised to recover the discrete harmonic oscillator Lagrangian 1-form from [26].

Chapter 7

Conclusion

The theory of Lagrangian multiforms is continuing to develop at a rapid pace. In this thesis, we developed discrete Lagrangian 2-forms on quadrilateral stencils and their reductions. We have highlighted some new aspects of the theory and clarified a number of issues. We believe there are many opportunities for future developments and applications, particularly in quantum integrable systems and quantum path integrals.

7.1 Summary of Results

In Chapter 2 we partly reviewed the concepts of integrable equations and discrete Lagrangian multiforms, focussing on an example of the H1 quad equation. Importantly, we introduced a definition of a weak discrete Lagrangian 2-form to describe the earliest (pioneering) constructions of [28]. We then introduced a stronger definition of a discrete Lagrangian 2-form to reconcile recent developments in the discrete setting [11], as well as developments in the continuous [16] and semi-discrete setting [49].

Armed with this stronger definition, In Chapter 3 we present novel discrete Lagrangian 2-forms for the quad equations of the entire ABS list [1]. To achieve this, we reviewed the quad equations of the ABS list which are classified by multidimensional consistency. We emphasised the three leg form of the quad equations, as well as the related tetrahedron polynomials and octahedron polynomials. The major result in this chapter is the elevation of the Lagrangians from [1] and [10] to full-fledged Lagrangian multiforms related to the ABS quad equations: $\mathcal{L}_{ij}^{\llcorner\!\!\!\swarrow}$, $\mathcal{L}_{ij}^{\times}$ and $\mathcal{L}_{ij}^{\boxtimes}$. We showed that, contrary to common belief, the ABS quad equations are variational: they are the corner equations of $\mathcal{L}_{ij}^{\llcorner\!\!\!\swarrow}$ (and equivalent to the corner equations of $\mathcal{L}_{ij}^{\boxtimes}$). Using the relations between these new Lagrangian 2-forms and the well-established Lagrangian 2-form $\mathcal{L}_{ij}^{\triangle}$, we showed that the quad equations are equivalent to the combined system of tetrahedron and octahedron equations.

In Chapter 4 we reinforce that double zero expansions are a universal phenomenon of Lagrangian multiforms, by formulating them in the discrete setting. We show that each of the discrete Lagrangian 2-forms considered in Chapter 3 admit a double zero expansion in terms of their respective multiaffine corner equations.

In Chapter 5, we develop a framework to periodically reduce discrete Lagrangian 2-forms into (weak) discrete Lagrangian 1-forms. We show that periodic reductions from integrable partial difference equations to integrable ordinary difference equations, can be carried out on the Lagrangian level, such that the Lagrangian multiform structure is partially preserved. The key insight in this Chapter is to make use of multidimensional consistency and consider evolution perpendicular to the periodic staircase. This leads to an action over periodic quad surfaces in a four-dimensional lattice. This seems quite abstract at first, however the properties of a weak discrete Lagrangian 1-form follow from linearity. We consider this framework on a simple example and periodically reduce the discrete Lagrangian 2-form associated with the linear quad equation. For this example, we derive a discrete Lagrangian 1-form (satisfying the stronger definition), by explicitly deriving a double zero expansion for it. This framework is promising and it lays the groundwork to find a general framework to periodically reduce Lagrangian multiforms.

In Chapter 6, we investigate Lagrangian 1-forms. We relate discrete and continuous Lagrangian 1-forms to addition formulae for trigonometric and reciprocal trigonometric functions. We derive discrete commuting flows for the McMillan equation and develop a discrete Lagrangian 1-form.

7.2 Future Work

As we emphasise from our discussion in the Introduction, there are potential applications of discrete Lagrangian multiforms in quantum integrable systems, quantum path integrals and discrete quantum theories of gravity. For instance, in [26], the authors considered the quantum path integral of a discrete Lagrangian 1-form associated with the discrete harmonic oscillator. One powerful property of discrete integrable equations (and discrete Lagrangian multiforms) in this context is that they provide exact integrable discretisations of their continuous counterparts, which resolves many of the mathematical issues associated with the time-slicing limit. We believe discrete Lagrangian 1-forms related to work in Chapter 5 and Chapter 6 will lead to non-linear generalisations of the quantum multiform construction in [26]. Furthermore, our work in Chapter 3 provides an ideal discrete

Lagrangian 2-form which could be used to quantise the lpKdV equation and other quad equations.

Our definitions in Chapter 2 and novel constructions in Chapter 3 help unify discrete Lagrangian multiforms with the semi-discrete and continuous case. So far there are limited examples of semi-discrete Lagrangian multiforms [49], and we expect our work to help in the discovery of more.

The double zero expansions of the discrete Lagrangian 2-forms around the elementary cube in Chapter 4 are ripe for further exploration. We expect future investigations into 2-dimensional analogous of the discrete integrals of motion in [53]. Furthermore, it should be possible to find a discrete Lagrangian 2-form analogue of the expressions in [16] or the variational symmetries in [48, 54]. There is the question of what equivalent set of equations should the double zero expansion be in terms of? We explored expansions in terms of multiaffine equations in Chapter 4, but it is worth exploring expansions in terms of the non-linear three-leg forms. We also believe there is more to be understood from three leg forms as integrals of reciprocal biquadratics [61].

Appendix A

The ABS list

Here we present the equations of the ABS list. For each we give leg functions ϕ and ϕ making up a possible three-leg form. There is freedom both in how to present the equations (such as Möbius transformations of the fields and point transformations of the parameters). The forms below are chosen such that (except for Q4) they can be explicitly integrated to give the Lagrangian functions L and Λ.

For H1–H3, A1–A2, and Q1 these are taken from [28] (with some modifications in H3 and A2 to clarify the case $\delta = 0$ and to enforce the symmetry of Λ). The Lagrangians for the equations Q2–Q3 are inspired on the ideas of that paper and on the three-leg forms presented in [1, 10].

H1

$$
\begin{aligned}
&Q = (u - u_{ij})(u_i - u_j) - \alpha_i + \alpha_j \\
&\psi = u_i \\
&\phi = \frac{\alpha_i - \alpha_j}{u - u_{ij}} \\
&L = u u_i \\
&\Lambda = (\alpha_i - \alpha_j) \log(u - u_{ij})
\end{aligned}
$$

A comment on the symmetry of Λ is in order. If the fields are real-valued we can interpret the log-term as $\log(|u - u_i|)$, which makes the symmetry $u \leftrightarrow u_i$ manifest. If the fields are complex-valued, we have $\log(u - u_i) - \log(u_i - u) = \pi i$, so the symmetry holds up to a constant which does not affect the corner equations. Of course we could write an explicitly symmetric expression, $\Lambda = \frac{1}{2}(\alpha_i - \alpha_j)\log\big((u - u_i)^2\big)$, but for the sake of conciseness we choose not to write such symmetrised expressions.

H2

$$
\begin{aligned}
Q &= -\alpha_i^2 + \alpha_j^2 - (\alpha_i - \alpha_j)(u + u_i + u_{ij} + u_j) + (u - u_{ij})(u_i - u_j) \\
\psi &= \log(\alpha_i + u + u_i) + 1 \\
\phi &= \log(\alpha_i - \alpha_j + u - u_{ij}) - \log(-\alpha_i + \alpha_j + u - u_{ij}) \\
L &= (\alpha_i + u + u_i)\log(\alpha_i + u + u_i) \\
\Lambda &= (\alpha_i - \alpha_j + u - u_{ij})\log(\alpha_i - \alpha_j + u - u_{ij}) \\
&\quad + (\alpha_i - \alpha_j - u + u_{ij})\log(-\alpha_i + \alpha_j + u - u_{ij})
\end{aligned}
$$

H3 $_{\delta=0}$

$$
\begin{aligned}
Q &= (uu_i + u_{ij}u_j)e^{\alpha_i} - (u_iu_{ij} + uu_j)e^{\alpha_j} \\
\psi &= \frac{\log(uu_i)}{u} \\
\phi &= \frac{\alpha_i - \alpha_j}{u} - \frac{\log\left(-\frac{ue^{\alpha_i - \alpha_j}}{u_{ij}} + 1\right)}{u} + \frac{\log\left(-\frac{ue^{-\alpha_i + \alpha_j}}{u_{ij}} + 1\right)}{u} \\
L &= \frac{1}{2}\log(uu_i)^2 \\
\Lambda &= (\alpha_i - \alpha_j)\log\left(\frac{u}{u_{ij}}\right) + \mathrm{Li}_2\left(\frac{ue^{\alpha_i - \alpha_j}}{u_{ij}}\right) - \mathrm{Li}_2\left(\frac{ue^{-\alpha_i + \alpha_j}}{u_{ij}}\right)
\end{aligned}
$$

where Li_2 denotes the dilogarithm function. The symmetry of Λ follows from the dilogarithm identity

$$
\mathrm{Li}_2(z) + \mathrm{Li}_2\left(\frac{1}{z}\right) = -\frac{\pi^2}{6} - \frac{(\ln(-z))^2}{2}.
$$

H3 $_{\delta=1}$

$$
\begin{aligned}
Q &= (uu_i + u_{ij}u_j)e^{\alpha_i} - (u_iu_{ij} + uu_j)e^{\alpha_j} + e^{2\alpha_i} - e^{2\alpha_j} \\
\psi &= \frac{\alpha_i}{u} + \frac{\log(uu_ie^{-\alpha_i} + 1)}{u} \\
\phi &= \frac{\alpha_i - \alpha_j}{u} - \frac{\log\left(-\frac{ue^{\alpha_i - \alpha_j}}{u_{ij}} + 1\right)}{u} + \frac{\log\left(-\frac{ue^{-\alpha_i + \alpha_j}}{u_{ij}} + 1\right)}{u} \\
L &= \alpha_i\log(uu_i) - \mathrm{Li}_2(-uu_ie^{-\alpha_i}) \\
\Lambda &= (\alpha_i - \alpha_j)\log\left(\frac{u}{u_{ij}}\right) + \mathrm{Li}_2\left(\frac{ue^{\alpha_i - \alpha_j}}{u_{ij}}\right) - \mathrm{Li}_2\left(\frac{ue^{-\alpha_i + \alpha_j}}{u_{ij}}\right)
\end{aligned}
$$

A1 $_{\delta=0}$

$$
Q = (uu_{ij} + u_iu_j)(\alpha_i - \alpha_j) + (uu_i + u_{ij}u_j)\alpha_i - (u_iu_{ij} + uu_j)\alpha_j
$$

$$\psi = \frac{\alpha_i}{u+u_i}$$
$$\phi = \frac{\alpha_i - \alpha_j}{u-u_{ij}}$$
$$L = \alpha_i \log(u+u_i)$$
$$\Lambda = (\alpha_i - \alpha_j)\log(u-u_{ij})$$

A1$_{\delta=1}$

$$Q = -(\alpha_i-\alpha_j)\alpha_i\alpha_j + (uu_{ij}+u_iu_j)(\alpha_i-\alpha_j) + (uu_i+u_{ij}u_j)\alpha_i - (u_iu_{ij}+uu_j)\alpha_j$$
$$\psi = \log(\alpha_i+u+u_i) - \log(-\alpha_i+u+u_i)$$
$$\phi = -\log(\alpha_i-\alpha_j-u+u_{ij}) + \log(-\alpha_i+\alpha_j-u+u_{ij})$$
$$L = (\alpha_i+u+u_i)\log(\alpha_i+u+u_i) + (\alpha_i-u-u_i)\log(-\alpha_i+u+u_i)$$
$$\Lambda = (\alpha_i-\alpha_j-u+u_{ij})\log(\alpha_i-\alpha_j-u+u_{ij}) + (\alpha_i-\alpha_j+u-u_{ij})\log(-\alpha_i+\alpha_j-u+u_{ij})$$

A2

$$Q = -(uu_iu_{ij}u_j+1)\sin(\alpha_i-\alpha_j) + (u_iu_{ij}+uu_j)\sin(\alpha_i) - (uu_i+u_{ij}u_j)\sin(\alpha_j)$$
$$\psi = \frac{i\alpha_i}{u} - \frac{\log\left(-uu_ie^{i\alpha_i}+1\right)}{u} + \frac{\log\left(-uu_ie^{-i\alpha_i}+1\right)}{u}$$
$$\phi = \frac{i\alpha_i - i\alpha_j}{u} - \frac{\log\left(-\frac{ue^{i\alpha_i-i\alpha_j}}{u_{ij}}+1\right)}{u} + \frac{\log\left(-\frac{ue^{-i\alpha_i+i\alpha_j}}{u_{ij}}+1\right)}{u}$$
$$L = i\alpha_i\log(uu_i) + \mathrm{Li}_2(uu_ie^{i\alpha_i}) - \mathrm{Li}_2(uu_ie^{-i\alpha_i})$$
$$\Lambda = (i\alpha_i - i\alpha_j)\log\left(\frac{u}{u_{ij}}\right) + \mathrm{Li}_2\left(\frac{ue^{i\alpha_i-i\alpha_j}}{u_{ij}}\right) - \mathrm{Li}_2\left(\frac{ue^{-i\alpha_i+i\alpha_j}}{u_{ij}}\right)$$

Q1$_{\delta=0}$

$$Q = \alpha_i(u-u_j)(u_i-u_{ij}) + \alpha_j(u-u_i)(u_{ij}-u_j)$$
$$\psi = \frac{\alpha_i}{u-u_i}$$
$$L = \alpha_i\log(u-u_i)$$

For equations of type Q, the long-leg function ϕ is identical to the short leg function ψ, and correspondingly Λ is identical to L.

Q1$_{\delta=1}$

$$Q = (\alpha_i-\alpha_j)\alpha_i\alpha_j + \alpha_i(u-u_j)(u_i-u_{ij}) + \alpha_j(u-u_i)(u_{ij}-u_j)$$

$$\psi = \log(\alpha_i + u - u_i) - \log(-\alpha_i + u - u_i)$$

$$L = (\alpha_i + u - u_i)\log(\alpha_i + u - u_i) + (\alpha_i - u + u_i)\log(-\alpha_i + u - u_i)$$

Q2

$$\begin{aligned} Q &= -(\alpha_i^2 - \alpha_i\alpha_j + \alpha_j^2 - u - u_i - u_{ij} - u_j)(\alpha_i - \alpha_j)\alpha_i\alpha_j \\ &\quad + \alpha_i(u - u_j)(u_i - u_{ij}) + \alpha_j(u - u_i)(u_{ij} - u_j) \\ \psi &= \frac{\log\left(\alpha_i + \sqrt{u} + \sqrt{u_i}\right)}{2\sqrt{u}} + \frac{\log\left(\alpha_i + \sqrt{u} - \sqrt{u_i}\right)}{2\sqrt{u}} \\ &\quad - \frac{\log\left(-\alpha_i + \sqrt{u} + \sqrt{u_i}\right)}{2\sqrt{u}} - \frac{\log\left(-\alpha_i + \sqrt{u} - \sqrt{u_i}\right)}{2\sqrt{u}} \\ L &= (\alpha_i + \sqrt{u} + \sqrt{u_i})\log\left(\alpha_i + \sqrt{u} + \sqrt{u_i}\right) + (\alpha_i + \sqrt{u} - \sqrt{u_i})\log\left(\alpha_i + \sqrt{u} - \sqrt{u_i}\right) \\ &\quad + (\alpha_i - \sqrt{u} - \sqrt{u_i})\log\left(-\alpha_i + \sqrt{u} + \sqrt{u_i}\right) + (\alpha_i - \sqrt{u} + \sqrt{u_i})\log\left(-\alpha_i + \sqrt{u} - \sqrt{u_i}\right) \end{aligned}$$

Q3 $_{\delta=0}$

$$\begin{aligned} Q &= -(uu_{ij} + u_iu_j)\sin(\alpha_i - \alpha_j) + (uu_i + u_{ij}u_j)\sin(\alpha_i) - (u_iu_{ij} + uu_j)\sin(\alpha_j) \\ \psi &= \frac{i\alpha_i}{u} - \frac{\log\left(-\frac{ue^{(i\alpha_i)}}{u_i} + 1\right)}{u} + \frac{\log\left(-\frac{ue^{(-i\alpha_i)}}{u_i} + 1\right)}{u} \\ L &= i\alpha_i \log\left(\frac{u}{u_i}\right) + \mathrm{Li}_2\left(\frac{ue^{i\alpha_i}}{u_i}\right) - \mathrm{Li}_2\left(\frac{ue^{-i\alpha_i}}{u_i}\right) \end{aligned}$$

Q3 $_{\delta=1}$ An explicit function L is only known after a change of variable $u \to \cosh(u)$:

$$\begin{aligned} Q &= \sinh(\alpha_i)(\cosh(u)\cosh(u_i) + \cosh(u_{ij})\cosh(u_j)) \\ &\quad - \sinh(\alpha_j)(\cosh(u_i)\cosh(u_{ij}) + \cosh(u)\cosh(u_j)) \\ &\quad - \sinh(\alpha_i - \alpha_j)(\cosh(u)\cosh(u_{ij}) + \cosh(u_i)\cosh(u_j)) \\ &\quad - \sinh(\alpha_i - \alpha_j)\sinh(\alpha_i)\sinh(\alpha_j) \\ \psi &= \alpha_i - \log\left(-e^{(\alpha_i - u + u_i)} + 1\right) - \log\left(-e^{(\alpha_i - u - u_i)} + 1\right) \\ &\quad + \log\left(-e^{(-\alpha_i - u + u_i)} + 1\right) + \log\left(-e^{(-\alpha_i - u - u_i)} + 1\right) \\ L &= \alpha_i(u - u_i) - \mathrm{Li}_2(e^{(\alpha_i - u + u_i)}) - \mathrm{Li}_2(e^{(\alpha_i - u - u_i)}) + \mathrm{Li}_2(e^{(-\alpha_i - u + u_i)}) + \mathrm{Li}_2(e^{(-\alpha_i - u - u_i)}) \end{aligned}$$

Q4: We present Q4 in its elliptic parametrisation [38]. Consider the following parameter pairs related to α_i, α_j through the Weierstrass elliptic function $\wp$:

$$(a, A) = \left(\wp(\alpha_i), \wp'(\alpha_i)\right) ,$$

$$(b,B) = \big(\wp(\alpha_j), \wp'(\alpha_j)\big) \ ,$$
$$(c,C) = \big(\wp(\alpha_j - \alpha_i), \wp'(\alpha_j - \alpha_i)\big) \ .$$

Then, after a change of variable $u \to \wp(u)$, Q4 can be written as

$$\begin{aligned} Q = {} & A((\wp(u)-b)(\wp(u_j)-b)-(a-b)(c-b))((\wp(u_i)-b)(\wp(u_{ij})-b)-(a-b)(c-b)) \\ & + B((\wp(u)-a)(\wp(u_i)-a)-(b-a)(c-a))((\wp(u_j)-a)(\wp(u_{ij})-a)-(b-a)(c-a)) \\ & - ABC(a-b)\,. \end{aligned}$$

Its three-leg form is given by [2]:

$$\psi = \log\left(\frac{\sigma(u+u_i+\alpha_i)\sigma(u-u_i+\alpha_i)}{\sigma(u+u_i-\alpha_i)\sigma(u-u_i-\alpha_i)}\right) \ .$$

A corresponding function L was constructed in [57] by taking a power series expansion of ψ and integrating it term by term. No closed form formula for L is known. The analytic properties of L may be developed by considering elliptic dilogarithms, such as in the work of [8] on scattering amplitudes.

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