Gromov–Witten and quasimap invariants of toric varieties – a geometric perspective

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Submitted for the degree of Doctor of Philosophy

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April 2024

Acknowledgements

Firstly, I would like to thank my supervisor Cristina Manolache for bringing to my attention such interesting questions, for all the mathematical discussions and for her support. Thank you also to my coauthors during this period, Etienne Mann, Renata Picciotto and Qaasim Shafi, for teaching me lots of mathematics, and some philosophy of life too.

I would also like to thank everyone in the AG&MP research group in Sheffield. From the very beginning of this journey, I have felt surrounded by amazing people, which also happen to be fantastic mathematicians (and/or physicists). The reading seminars and mathematical discussions have been very helpful, as have been all the hikes and climbing sessions.

Special thanks to Yannik, Jingxiang, Menelaos and Nadir, for all the beautiful moments that we have shared during these four years; to Yirui, who always had a few minutes to spare when I bumped into him; and to Maleeha, for sticking around (even when I confused cakes with pies).

Finalmente, agradezco a mi familia y amigos su apoyo incondicional a pesar de haber elegido una profesión difícil de compartir. En especial, gracias a Beatriz por acompañarme en este viaje, como en tantos otros, y seguir siendo una constante en mi vida.

ABSTRACT

We study two topics in enumerative geometry. The first one is the definition, in all genera, of reduced Gromov–Witten invariants of complete intersections in projective space. This definition is based on the blow-up of a stack along a coherent sheaf.

The second topic is the relation between Gromov–Witten and quasimap invariants of toric varieties. We construct a contraction morphism between stable maps and stable quasimaps to a toric variety. The main tool is the notion of degree of a quasimap at a basepoint, which we introduce and use to understand the functoriality of quasimaps along closed embeddings.

Contents

Introduction

Gromov–Witten theory

Enumerative algebraic geometry is the branch of algebraic geometry that deals with counting geometric objects. For example, there is a unique conic through 5 points in the plane and there are 2875 lines on a generic quintic threefold. We will focus on Gromov–Witten theory as a way to count curves inside a smooth variety X.

The main ingredient of Gromov–Witten theory is the moduli space of stable maps

$$\overline{\mathcal{M}}_{g,n}(X,\beta).$$

A point in this proper Deligne–Mumford stack corresponds to a morphism $f\colon C \to X$ from a nodal curve of genus g with n marked points to X of degree $f_*[C] = \beta \in H_2(X,\mathbb{Z})$, which is stable in the sense that (C,f) has finitely many automorphisms. The moduli space comes equipped with a universal curve $\pi\colon \mathcal{C} \to \overline{\mathcal{M}}_{g,n}(X,\beta)$, universal sections $s_i\colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to \mathcal{C}$ of π and an evaluation morphism $\mathrm{ev}\colon \mathcal{C} \to X$. This data recovers the curve C, the marked points and the map f on the fibres of π. Furthermore, the moduli space admits a virtual fundamental class

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}},$$

which is necessary for intersection-theoretic computations, given that $\overline{\mathcal{M}}_{g,n}(X,\beta)$ in general has several irreducible components of different dimensions.

The Gromov–Witten invariants of X are defined via intersection theory. Given cohomology classes $\gamma_1,\ldots,\gamma_n \in H^*(X,\mathbb{Z})$ such that $\sum_{i=1}^n \deg(\gamma_i)$ equals the virtual dimension of $\overline{\mathcal{M}}_{g,n}(X,\beta)$, the corresponding Gromov–Witten invariant is

$$\langle \gamma_1,\ldots,\gamma_n\rangle^X_{g,n,\beta} = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^n (\mathrm{ev}\circ s_i)^*\gamma_i.$$

This number can be interpreted as a *virtual* count of curves of genus-g and degree β in X meeting each class γ_i at the i-th marked point.

Despite their very interesting structure, Gromov–Witten invariants are not enumerative in general, meaning that they do not agree with the correct number of curves. The reason is that the moduli space of stable maps is a too large compactification of the moduli space of maps from smooth curves. This is our motivation for studying other compactifications and their relation to stable maps.

Reduced Gromov–Witten invariants

The Gromov–Witten invariants of $X = \mathbb{P}^r$ are not enumerative in positive genus. One of the reasons is geometric: the moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ is not irreducible in general. The different irreducible components appear because the source curve C of a stable map can become reducible too. As a consequence, the genus-g Gromov–Witten invariants of projective space have contributions from lower genus invariants. One would like to extract the "pure genus-g contribution", called *reduced* genus-g Gromov–Witten invariants, coming from the closure of the locus of stable maps with smooth source, called the *main component*.

For genus one, a birational model of the main component admitting a virtual fundamental class was built in [VZ08]. This model can be used to define reduced Gromov–Witten invariants. Moreover, a concrete relation between standard and reduced Gromov–Witten invariants in genus one was proven in [Zin08]. A similar conjecture was made in [Zin09a, HL11] for all-genus reduced Gromov–Witten invariants, see Conjecture 1.2.

In Chapter 1, we present a new definition of reduced Gromov–Witten invariants of projective spaces and complete intersections in any genus. Our approach is similar to that of [VZ08] in that we also build a birational model of the main component that admits a virtual fundamental class. However, while their construction was dictated by a direct study of the geometry of $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$, and is therefore specific to genus one, ours relies in the notion of blow-up of a stack along a sheaf and works in all genera. Proving the conjectural relation between standard and reduced Gromov–Witten invariants in higher genus is plausible with our definition. We give the first steps in that direction by studying the irreducible components of the abelian cone associated to a diagonal sheaf.

Stable toric quasimaps

Another prominent curve-counting theory for a toric variety X is *quasimap theory*. The starting point is the description of the functor of points in [Cox95] in terms of line bundle-section pairs depending on the fan of X. This is a generalization of well-known fact that a morphism $C \to \mathbb{P}^r$ is the same information as a line bundle on C with $r + 1$ sections which are not allowed to vanish simultaneously.

In [CFK10], the authors defined a proper Deligne–Mumford stack

$$\mathcal{Q}_{g,n}(X, \beta)$$

which parametrizes stable toric quasimaps to X. It admits a virtual fundamental class, which can be used to define quasimap invariants through intersection theory, as we did for Gromov–Witten invariants.

A point in the moduli space $\mathcal{Q}_{g,n}(X,\beta)$ corresponds to a prestable genus-g curve C with n marked points together with a collection of line bundle section-pairs satisfying all the conditions needed to define a morphism $C \to X$ except for the non-vanishing conditions on the sections, which can be violated at finitely many smooth unmarked points. As a result, a quasimap to X determines a rational morphism to X, which in general may not be regular. As for stable maps, a quasimap is stable if it admits only finitely many automorphisms. However, this condition is now stronger, in the sense that there exists stable maps which are not stable as quasimaps. This makes the task of finding a natural morphism between the moduli spaces $\overline{\mathcal{M}}_{g,n}(X,\beta)$ and $\mathcal{Q}_{g,n}(X,\beta)$ non-trivial.

Quasimap invariants, or their generating function called I-function, are typically easier to compute than Gromov–Witten invariants. They are also relevant for their presence in mirror symmetry. In fact, the the definition of the moduli space in [CFK10] is motivated by certain "toric compactifications" of the moduli space of maps $\mathbb{P}^1 \to X$ which appeared in the proof of the mirror theorem in [Giv98].

Our work in this area is motivated by the result in [CFK17] that Gromov–Witten and quasimap invariants of a smooth Fano toric variety are equivalent. Our purpose is to give a more conceptual proof of this result at the level of virtual fundamental classes, based on the geometry of the moduli spaces involved.

In Chapter 2, we construct a contraction morphism from a closed substack in $\overline{\mathcal{M}}_{g,n}(X,\beta)$ to $\mathcal{Q}_{g,n}(X,\beta)$ which extends the identity on the locus of stable maps which are stable as quasimaps. Furthermore, we show that the contraction morphism is surjective if X is Fano. This result is the first step towards a geometric comparison of Gromov–Witten and quasimap invariants. Along the way, we introduce the notion of *degree of a basepoint* and we use it to study the functoriality of quasimaps along closed embeddings.

Overview

We summarize the structure of the thesis and the main results.

In Chapter 1 we study reduced Gromov–Witten invariants of complete intersections in projective space. In Section 1.3 we review two equivalent constructions of the blow-up of a scheme along a sheaf. We also review the notion of diagonalization of a sheaf in

Section 1.4. We generalize both of these constructions to stacks in Section 1.4. Using our study of abelian cones in Section 1.6, we can use the blow-up of a stack along a sheaf to define reduced Gromov–Witten invariants in all genera in Section 1.7, see Definition 1.117. We expect our study of the irreducible components of an abelian cone associated to a diagonal sheaf in Section 1.6.3 to be useful in the future to prove Conjecture 1.2. Finally, in Section 1.8 we compare our definition of reduced invariants to the the previous construction in genus one in [VZ08], using the description in [HL10] in local coordinates.

In Chapter 2 we study the contraction morphism between stable maps and stable quasimaps to toric varieties. In Section 2.3 we introduce the notion of degree of a quasimap at a basepoint, see Definition 2.24. Inspired by this notion, in Section 2.4 we study the functoriality of quasimaps along a closed embedding ι of toric varieties, and we give a sufficient condition for the morphism $\mathcal{Q}(\iota)$ to be a closed embedding in Corollary 2.40. This criterion is used in Section 2.5 to define a contraction morphism from a closed substack of the moduli space of stable maps to the moduli space of stable quasimaps, see Construction 2.42. Finally, in Theorem 2.47 we show that the morphism is surjective if the target is Fano.

Chapter 1

Higher genus reduced Gromov-Witten invariants

The results in this chapter are joint work with Etienne Mann, Cristina Manolache and Renata Picciotto. It appeared originally as [CRMMP23].

Abstract. Given $\mathfrak{F}$ a coherent sheaf on a Noetherian integral algebraic stack $\mathfrak{P}$, we give two constructions of stacks $\widetilde{\mathfrak{P}}$, equipped with birational morphisms $p : \widetilde{\mathfrak{P}} \to \mathfrak{P}$ such that $p^*\mathfrak{F}$ is simpler: in the Rossi construction, the torsion-free part of $p^*\mathfrak{F}$ is locally free; in the Hu–Li diagonalization construction, the associated abelian cone $C(p^*\mathfrak{F})$ is a union of vector bundles. We use these constructions to define reduced Gromov–Witten invariants of complete intersections in all genera.

1.1 Introduction

Overview of the problem Let X be a smooth projective variety. We denote by $\overline{\mathcal{M}}_{g,n}(X,d)$ the moduli space of genus g, degree $d \in H_2(X;\mathbb{Z})$ stable maps to X (see [Kon95]). By [LT98, BF96], $\overline{\mathcal{M}}_{g,n}(X,d)$ has a virtual class $[\overline{\mathcal{M}}_{g,n}(X,d)]^{\mathrm{vir}} \in A_*(\overline{\mathcal{M}}_{g,n}(X,d))$. Gromov–Witten invariants of X are defined as intersection numbers against this virtual class. They are related to counts of curves in X of genus g and class d, but they often encode contributions from degenerate maps. These degenerate contributions can be explained by the geometry of the moduli space of stable maps.

We have little information about moduli spaces of stable maps to a variety X, even when X is complete intersection, but the moduli space of stable maps to projective spaces are better understood. The space of genus zero stable maps to a projective space $\overline{\mathcal{M}}_{0,n}(\mathbb{P}^r,d)$ is a smooth Deligne–Mumford stack and the resulting genus zero Gromov–Witten invariants are enumerative. For $g > 0$, the moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$ has several irreducible components and moreover, in genus one and two, we have explicit local equations for $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$: see [Zin09c, HL10, HLN12]. The existence of components consisting of maps with reducible domain is reflected by Gromov–Witten invariants: these components contribute in the form of lower genus stable maps.

In order to define invariants which do not have contributions from degenerate maps with reducible domains, we need to define a virtual class on the closure of the locus of maps with smooth domain. This is not possible directly, we need to replace this component with a birational one which admits a virtual class. In genus one and two, there are several such constructions [Zin09c, VZ08, HL10, RSPW19a, HLN12, BC23, HN19, HN20]. The resulting numbers are called *reduced* Gromov–Witten invariants.

Main result 1.1 (See Definition 1.117)**.** We define reduced Gromov–Witten invariants in any genus for any complete intersection in a projective space.

In genus one and two, our reduced Gromov–Witten invariants agree with the reduced invariants defined previously.

Genus one reduced invariants for varietes of any dimension are related to Gromov–Witten invariants [Zin08]. For threefolds the relation is much simpler. Let X be a threefold which is a complete intersection and let $\gamma \in H^*(X)^{\oplus n}$ be a collection of cohomology classes of X. Let $N^g_\beta(\gamma)$ be the genus g and degree β Gromov–Witten invariants of X with insertions given by γ, and let $r^g_\beta(\gamma)$ be the corresponding reduced invariants.

Conjecture 1.2. [Zin09a], [HL11, Conjecture 1.1] Let X be a Calabi–Yau threefold. Then, there are universal constants $C_h(g) \in \mathbb{Q}$, such that for $\deg(\beta) > 2g-2$, we have

$$N^g_\beta = \sum_{0 \le h \le g} C_h(g) r^h_\beta.$$

When X is the quintic threefold, the above formula in genus one is the formula in [Zin09a, LZ09]

$$N^1_d = \frac{1}{12} N^0_d + r^1_d. \tag{1.1}$$

If X is a Fano threefold, then reduced invariants are expected to be equal to Gopakumar–Vafa invariants. Indeed, the Gopakumar–Vafa invariants are by definition related to Gromov–Witten invariants by a recursive formula, which takes into account degenerate lower genus and lower degree boundary contributions. For Fano varieties, there are no lower degree contributions. Boundary contributions were computed by Pandharipande in [Pan99]. The conjectural equality between reduced Gromov–Witten invariants and Gopakumar–Vafa invariants (see [Pan99, Section 0.3]) for Fano threefolds gives the following.

Conjecture 1.3. [Zin09a, Zin11] Let X be a Fano threefold and let $C^X_{h,\beta}(g)$ be defined by the formula

$$\sum_{g \ge 0} C^X_{h,\beta}(g) t^{2g} = \left(\frac{\sin(t/2)}{t/2} \right)^{2h-2-K_X \cdot \beta}.$$

Then, we have the following

$$N^g_\beta(\gamma) = \sum_{h=0}^{g} C^X_{h,\beta}(g-h) r^h_\beta(\gamma).$$

The above should also hold in the Calabi–Yau case, where $C^X_{h,\beta}(g)$ do not depend on X and β.

The above conjectures have been proved in genus one and two. These are the only cases in which a definition of reduced invariants existed prior to this work.

Approach. In a first step, we use the geometry of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ in the following way. The moduli space of stable maps admits a map $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \to \mathfrak{Pic}$, where $\mathfrak{Pic}$ denotes the stack which parameterises genus g curves with n marked points together with a line bundle of degree d. One important observation is that $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ is an open substack in an abelian cone $\operatorname{Spec}\operatorname{Sym}\mathfrak{F}$, with $\mathfrak{F}$ a sheaf on $\mathfrak{Pic}$ (see (1.27) and (1.28)).

In a second step, we use the sheaf $\mathfrak{F}$ to construct $\widetilde{\mathfrak{Pic}}$ (or $\widetilde{\mathfrak{Pic}}^{HL}$) together with a birational morphism $p : \widetilde{\mathfrak{Pic}} \to \mathfrak{Pic}$. By base-change, this gives $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$, and $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ allows us to define reduced Gromov–Witten invariants.

The stack $\widetilde{\mathfrak{Pic}}$ can be constructed in two ways: using the Rossi construction (see Section 1.3) or using the Hu–Li diagonalization construction (see Section 1.4). In both approaches we start with an atlas $U_i \to \mathfrak{Pic}$, and then we consider $\widetilde{U}_i$ as defined by Rossi (see Section 1.3), or by Hu–Li (see Section 1.4). In the end, we glue $\widetilde{U}_i$ to a global object $\widetilde{\mathfrak{Pic}}$ (see Theorem 1.76). The resulting stack $\widetilde{\mathfrak{Pic}}$ is called the Rossi construction or the Hu–Li construction, depending on the definition of $\widetilde{U}_i$.

In general, the Rossi construction is different from the Hu–Li construction (see 1.72). By Example 1.133, the Rossi construction gives a new moduli space which is different from the Vakil–Zinger blow-up. However, by Proposition 1.116 this does not change the reduced invariants: they are the same for all birational models of $\mathfrak{Pic}$.

The Rossi construction for $\widetilde{\mathfrak{Pic}}$ is enough to prove the Main Result 1.1, but the Hu–Li construction is better behaved in relation to Conjecture 1.2 and Conjecture 1.3.

Relation to previous approaches The structure of this paper is different from the ones in [VZ08,HL10,HLN12,HN19,HN20]. In the mentioned papers, the authors have a three-step strategy to constructing the stack $\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d)$:

1 they find equations of local embeddings $U_i \subset \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ in smooth spaces V_i;

2 they blow up of U_i to obtain $\widetilde{U}_i$;

3 they show that $\widetilde{U}_i$ glue to a (smooth) stack $\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d)$.

The first step becomes involved in higher genus, due to the rather complicated geometry of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. Steps 2 and 3 are done by constructing an explicit blow of $\mathfrak{Pic}$ (or $\mathfrak{M}_{g,n}$). Finding a candidate for this blow-up is the hardest part of the construction.

We omit Step 1 completely. For us, Step 2 is minimal in a suitable sense – it is given by a universal property. The main ingredient in Step 3 is that the (local) constructions proposed in Section 1.3 and in Section 1.4 commute with flat pullbacks and this allows us to glue them. Explicit equations of the charts U_i are thus not necessary to construct $\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d)$. The advantage of this approach is that the gluing is conceptual and straightforward. This is similar to what Hu and Li do in [HL11] – our construction heavily relies on their ideas.

Main technical result. In Sections 1–6 we work in the following, completely general setup. Given $\mathfrak{P}$ a stack and $\mathfrak{F}$ a coherent sheaf on it, we want to construct $\widetilde{\mathfrak{P}}$ with a proper birational morphism $p : \widetilde{\mathfrak{P}} \to \mathfrak{P}$ such that $p^*\mathfrak{F}$ is better behaved. We present two general constructions:

1 the Rossi construction and

2 the Hu–Li diagonalization construction.

(1) Given an integral, Noetherian algebraic stack $\mathfrak{P}$, we construct $p : \widetilde{\mathfrak{P}} \to \mathfrak{P}$, such that the torsion-free part of $p^*\mathfrak{F}$ is locally free. We show that the Rossi construction has a universal property in Theorem 1.79. In particular, it is the minimal stack such that the torsion-free part of $p^*\mathfrak{F}$ is locally free. This construction does not change torsion sheaves.

(2) The Hu–Li diagonalization construction also produces a sheaf $p^*\mathfrak{F}$ whose torsion-free part is locally free. In addition to this, $p^*\mathfrak{F}$ also has a well-behaved torsion in the sense of Definition 1.46. For schemes, this is achieved by a construction of Hu and Li [HL11]. In Theorem 1.76 we produce a global object $\widetilde{\mathfrak{P}}^{HL}$. In Theorem 1.82 we show that $\widetilde{\mathfrak{P}}^{HL}$ satisfies a universal property, and in Theorem 1.97 we show that the irreducible components of $\operatorname{Spec}\operatorname{Sym}\ p^*\mathfrak{F}$ are vector bundles.

Applying this construction with $\mathfrak{P} = \mathfrak{Pic}$, we obtain a well-behaved $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ as follows. In the following we fix $d > 2g - 2$ (see Assumption 1.105 for details) and we consider X a hypersurface of degree k in $\mathbb{P}^r$. In Section 1.7 we define $\widetilde{\mathfrak{Pic}}$ and a proper and birational morphism $p_k : \widetilde{\mathfrak{Pic}} \to \mathfrak{Pic}$. We define

$$\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) := \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \times_{\mathfrak{Pic}} \widetilde{\mathfrak{Pic}},$$

which comes equipped with a morphism $\bar{p}_k : \widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. We denote by $\overline{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d)$ the closure in $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ of the locus of maps with smooth domain. The condition on d ensures that $\overline{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d)$ is generically smooth and unobstructed. As before, we consider the base-change

$$\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d) := \overline{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d) \times_{\mathfrak{Pic}} \widetilde{\mathfrak{Pic}}.$$

We have the following result in all genera.

Theorem 1.4 (See Theorem 1.124)**.** Let $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) = \cup_{\theta\in\Theta}\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{\theta}$, with $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{\theta}$ irreducible components of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. Then the following statements hold:

1 The stack $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ admits a virtual class.

2 The morphism $\bar{p}_k$ is proper, and we have $(\bar{p}_k)_*[\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)]^{\mathrm{vir}} = [\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)]^{\mathrm{vir}}$.

3 For any $\theta \in \Theta$, $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{\theta}$ is smooth over its image in $\widetilde{\mathfrak{Pic}}$; in particular, $\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d)$ is smooth over $\widetilde{\mathfrak{Pic}}$.

4 Let $\tilde{\pi}^{\theta} : \widetilde{\mathcal{C}}^{\theta} \to \widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{\theta}$ denote the universal curve. Then $\tilde{\pi}^{\theta}_* ev^*\mathcal{O}(k)$ is a locally free sheaf on $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{\theta}$; in particular, $\tilde{\pi}^{\circ}_* ev^*\mathcal{O}(k)$ is a locally free sheaf on $\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d)$.

History and related works Reduced genus 1 invariants are the output of a long and impressive project. Reduced invariants were defined using symplectic methods and compared to Gromov–Witten invariants by Zinger [Zin08, Zin07, Zin09b, Zin09a]. Li–Zinger showed [LZ07, LZ09] that reduced Gromov–Witten invariants are the integral of the top Chern class of a sheaf over the main component of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. This is an analogue, for reduced genus 1 invariants, of the quantum Lefschetz hyperplane property [LZ07, LZ09]. In view of [Zin09b], this also gives a proof of the formula (1.1). The algebraic definition requires a blow-up construction for the moduli space of stable maps to projective space, due to Vakil and Zinger [VZ08, VZ07]. See [Zin20] for a survey from the symplectic perspective.

Explicit local equations for the Vakil–Zinger blow-up in genus one are given in [Zin09c, HL10] and in genus two in [HLN12]. It is expected that the methods used in low genus could provide local equations for general moduli spaces of stable maps to projective spaces, but the combinatorics is likely to be tedious.

In [HL10, HLN12, HN19, HN20] the authors give a modular interpretation of reduced invariants in terms of graphs of degenerate maps. A modular interpretation via log maps has been given by Ranganathan, Santos-Parker and Wise [RSPW19a, RSPW19b].

Hu and Li introduce the diagonalization construction in [HL11]. They use this construction to define an Euler class on the moduli space of stable maps to projective spaces. This gives a non-intrinsic definition of reduced invariants of complete intersections. Conjecture 1.3 is hard to approach with this definition. In this paper, we rework their construction.

In a different direction, instead of replacing the moduli space of maps with a space which dominates $\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d)$, one can construct a space dominated by $\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d)$. This has been done by moduli spaces of maps from more singular curves, such as in [BCM20, BC23]. A modular interpretation comes for free with this approach, which makes these constructions particularly beautiful. A relationship between reduced invariants and invariants from maps with cusps was established in [BCM20]. Battistella and Carocci introduce a compactification of genus two maps to projective spaces [BC23]. An example of this compactification is given in [BC22].

More recently, reduced invariants for the quintic threefold have been compared to Gromov–Witten invariants using algebro-geometric methods by Chang and Li [CL15]. Chang–Li *define* reduced invariants as the integral against the top Chern class of a sheaf but, as discussed above, this gives the same reduced invariants as [Zin09b]. The algebraic comparison relies on the construction of maps with fields due to Chang and Li [CL12], and on Kiem–Li's cosection localised virtual class [KL13]. This method has been employed in [LO22, LO21] to extend the genus one relation between absolute and reduced Gromov–Witten invariants of complete intersections. In genus two, a similar work is done in [LLO22].

Zinger has computed reduced genus one invariants of projective hypersurfaces via localisation [Zin09a]. The computations in [Zin09c] and [Zin08] have been extended to complete intersections by Popa [Pop13].

Outline of the paper. In the following we give an outline of the paper and we highlight the main results.

In Section 1.2 we fix notation and briefly recall the background notions used, such as Fitting ideals, abelian cones, fractional ideals or the theorem of cohomology and base change.

In Section 1.3 we introduce the desingularization of a sheaf on a stack (Section 1.3.1) and we review the minimal desingularization, due to Rossi (Section 1.3.2). In Section 1.3.3 we give an algebraic desingularization in terms of Fitting ideals in the affine case, due to Oneto–Zatini and Villamayor. We show in Section 1.3.4 that the Rossi and Villamayor constructions agree. We show several properties of the minimal desingularization of a sheaf in Section 1.3.5.

In Section 1.4.1, we introduce the notion of diagonal sheaf (see Definition 1.49). In Section 1.4.2 we recall Construction 1.59, due to Hu and Li, which gives the minimal diagonalization of a sheaf. This is formalized in Section 1.4.3 via a universal property (Theorem 1.61). In Section 1.4.4, we collect properties of the Hu–Li blow-up, such as the existence of a morphism from the Hu–Li blow-up to the Rossi blow-up in Proposition 1.67. The two blow-ups are not isomorphic in general, as shown in Example 1.72. In Section 1.4.5 we construct a filtration of a diagonal sheaf under certain conditions (see Theorem 1.71). This filtration will then be used in Section 1.6 to describe the irreducible components of the abelian cone of a diagonal sheaf. Finally, in Section 1.4.6 we collect some remarks and examples regarding the minimality of the Hu-Li blow-up.

Section 1.6 is devoted to the study the irreducible components of the abelian cone $C(\mathcal{F})$ of a diagonal sheaf. Given a coherent sheaf $\mathcal{F}$ on an integral Noetherian scheme X, we introduce the main component (Definition 1.88) $C(\mathcal{F}^{\mathrm{tf}})$ of the abelian cone $C(\mathcal{F})$ in Section 1.6.1. General cones need not have a main component (Example 1.86). Furthermore, if $\mathcal{F}^{\mathrm{tf}}$ is locally free, then $C(\mathcal{F})$ is a pushout of its main component, which is a vector bundle (see Proposition 1.95 in Section 1.6.2). The remaining components are studied in Section 1.6.3. The best result is obtained for $\mathcal{F}$ a diagonal sheaf: each component of $C(\mathcal{F})$ is a vector bundle over its support (Theorem 1.97).

In Section 1.5 we generalize the Hu–Li and Rossi blow-ups to Artin stacks. These are constructed in Section 1.5.1, by first applying the Rossi and Hu–Li constructions for schemes to an atlas and then gluing. This works because the Hu–Li and Rossi constructions are local, they have a universal property, and they commute with flat base-change by Propositions 1.41 and 1.62. We extend to stacks the results on the schematic version of these blow-ups, such as the universal properties of desingularization in Theorem 1.79 and of diagonalization in Theorem 1.82.

In Section 1.7 we use the notion of desingularization of a sheaf on a stack to define reduced Gromov Witten invariants in all genera: see Definition 1.117. In Section 1.7.1 we recall how $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ can be naturally embedded as an open substack in an abelian cone over $\mathfrak{Pic}$ following [CL12]. In Definition 1.88 we introduce the main component of

$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ and show how it is compatible with the open embedding in an abelian cone. In Section 1.7.3 we consider a desingularization $\widetilde{\mathfrak{Pic}} \to \mathfrak{Pic}$ and use it to base change $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ to a new space $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. These space is used in Section 1.7.4 to define reduced Gromov-Witten invariants in any genus for a hypersurface in projective space (Definition 1.117). We also show independence of the chosen desingularization Proposition 1.116. Finally, in Section 1.7.5 we recall maps with fields and consider the analogue of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ for p-fields. These spaces can be used to compute reduced Gromov-Witten invariants (Proposition 1.127). In Theorem 1.124 we describe the irreducible components of the blown-up moduli spaces.

In Section 1.8 we compare the moduli spaces obtained from the Rossi desingularization and the Vakil–Zinger blow-up. While reduced invariants are independent of the birational model of $\mathfrak{Pic}$, the induced moduli spaces can be different. We study charts of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ and we show that the Rossi construction in genus one is different from the Vakil–Zinger blow-up.

How to read this paper Sections 1.2–1.6 are self-contained and of independent interest. The schematic version of the results in Section 1.5 are explained in Sections 1.3–1.4. The reader interested in reduced Gromov–Witten invariants can take the results in Section 1.5 for granted and read Section 1.7 and Section 1.8 directly.

Further work Our desingularizations do not come with a modular interpretation. It would be nice to have a modular interpretation of the resulting stack $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$, either in the spirit of [HL10, HLN12, HN19, HN20], or a log interpretation as in [RSPW19a]. It would be perhaps better to have a space of maps with more singular domains, as in [BCM20, BC23].

While a modular interpretation would be very interesting from a theoretical point of view, higher genus computations as done by Zinger in [Zin09a] are likely to be hard. The genus two blow-up $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ already involves several rounds of blow-ups, and a localisation computation would inherit the complexity of the blow-up. We hope that our construction sheds new light on this beautiful problem and will encourage more mathematicians to work on it.

On the positive side, we expect this construction to be enough for proving Conjecture 1.3. The main difference with [HL11] is that we blow up $\mathfrak{Pic}$, instead of blowing up $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. The advantage of blowing up $\mathfrak{Pic}$ is that now we have the ingredients used by

Chang–Li, Lee–Oh and Lee–Li–Oh to prove Conjecture 1.3 (and therefore Conjecture 1.2) in genus one and two. More precisely, we have fairly simple moduli spaces of maps with fields over $\widetilde{\mathfrak{Pic}}$, and these can be used to split the virtual class on $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. We hope to be able to prove Conjecture 1.3 without having explicit equations of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$, or a modular interpretation of $\widetilde{\mathfrak{Pic}}$. We will address this problem in future work.

Acknowledgements We are very grateful to Evgeny Shinder for useful discussions. The project started after Evgeny Shinder and Ananyo Dan pointed out to us the paper of Rossi [Ros68]. We thank Francesca Carocci for very useful discussions and Aleksey Zinger for very useful correspondence and for detailed explanations on [Zin11]. We thank Dhruv Ranganathan and Evgeny Shinder for their comments in a previous version of this draft.

CM was supported by a Dorothy Hodgkin fellowship. EM and RP benefit from the support of the French government "Investissements d'Avenir" program integrated to France 2030, bearing the following reference ANR-11-LABX-0020-01 and the PRC ANR-17-CE40-0014.

1.2 Background

In this section we recall several basic constructions and fix the notation used throughout the paper.

1.2.1 The relative Grassmannian

Let X be a scheme with a fixed quasi-coherent sheaf $\mathcal{E}$. The Grassmannian functor $\underline{\mathrm{Gr}}_X(\mathcal{E}, r) : ((\mathrm{Sch})/X)^{\mathrm{op}} \to (\mathrm{Set})$ is given on objects by

$$T \mapsto \{\mathcal{E}_T \twoheadrightarrow \mathcal{Q} \,|\, \mathcal{Q} \text{ is locally free of rank r } \} \tag{1.2}$$

with $\mathcal{E}_T \coloneqq \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_T$.

This functor is represented by a scheme $\mathrm{Gr}_X(\mathcal{E}, r)$ over X, which is projective if $\mathcal{E}$ is finitely generated. Moreover, the Grassmannian functor is compatible with base-change. In particular,

$$\mathrm{Gr}_X(\mathcal{O}_X^{\oplus n}, r) \cong \mathrm{Gr}(n, r) \times X$$

where $\mathrm{Gr}(n, r)$ is the usual Grassmannian of r-dimensional subspaces of $\mathbb{C}^n$ relative to a point. Since it represents a functor, the relative Grassmannian $\mathrm{Gr}_X(\mathcal{E}, r)$ comes with a universal sheaf and a universal quotient sheaf, which is locally free of rank r:

$$\mathcal{E}_{\mathrm{Gr}_X(\mathcal{E},r)} \twoheadrightarrow \mathcal{Q}_{\mathrm{Gr}_X(\mathcal{E},r)}.$$

As in the classical case, the relative Grassmannian admits the Plücker embedding:

$$\lambda_{n,r} : \mathrm{Gr}_X(\mathcal{O}^{\oplus n}, r) \to \mathrm{Gr}_X\left(\bigwedge^r \mathcal{O}^{\oplus n}, 1\right) \cong \mathbb{P}_X^{m-1},$$

with $m = \binom{n}{r}$. For the last isomorphism, consider an X-scheme T. A point of $\mathrm{Gr}_X(\mathcal{O}^{\oplus m}, 1)(T)$ is a surjection

$$\mathcal{O}_T^{\oplus m} \twoheadrightarrow \mathcal{L}$$

with $\mathcal{L}$ a line bundle on T. This is a pair of a line bundle and an m-tuple of generating sections, which is an object of $\mathbb{P}_X^{m-1}(T)$.

1.2.2 Fitting ideals

Definition 1.5. Let M be a finitely presented R-module. Let $F \xrightarrow{\varphi} G \to M \to 0$ be a presentation with F and G free modules and $\mathrm{rk}(G) = r$. Given $-1 \leq i < \infty$, the i-th Fitting ideal $F_i(M)$ of M is the ideal generated by all $(r-i) \times (r-i)$-minors of the matrix associated to φ after fixing basis of F and G. We use the convention that $F_i(M) = R$ if $r - i \leq 0$ and $F_{-1}(M) = 0$.

Intrinsically, $F_i(M)$ is the image of the map $\bigwedge^{r-i} F \otimes \bigwedge^{r-i} G^* \to R$ induced by

$$\bigwedge^{r-i} \varphi \colon \bigwedge^{r-i} F \to \bigwedge^{r-i} G.$$

The i-th Fitting ideal is well-defined in that it does not depend on the chosen presentation. Since determinants can be computed expanding by rows and columns, there are inclusions

$$0 = F_{-1}(M) \subset F_0(M) \subset F_1(M) \subset \ldots \subset F_k(M) \subset F_{k+1}(M) \subset \ldots$$

It follows from the definition and right-exactness of tensor product that Fitting ideals commute with base-change. That is, given a ring homomorphism $R \to S$ and a finitely presented R-module M, it holds that

$$F_i(M \otimes_R S) = F_i(M) \cdot S.$$

Similarly, for a scheme X and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ of finite presentation, we have ideal sheaves

$$0 = F_{-1}(\mathcal{F}) \subset F_0(\mathcal{F}) \subset \cdots \subset F_n(\mathcal{F}) \subset \cdots \subset \mathcal{O}_X,$$

which can be defined locally as described above. For $f : Y \to X$ a morphism of schemes, we have that

$$f^{-1}F_i(\mathcal{F}) \cdot \mathcal{O}_Y = F_i(f^*\mathcal{F}).$$

Fitting ideals describe the locus on X where the sheaf $\mathcal{F}$ is locally free of some rank. More precisely, we recall the following standard result.

Proposition 1.6. For any n, the sheaf $\mathcal{F}$ is locally free of rank n on the locally closed subscheme $V(F_{n-1}(\mathcal{F})) \setminus V(F_n(\mathcal{F}))$ of X.

Proof. See [Sta22, Tag 05P8]. □

1.2.3 Abelian cones

Let X be a Noetherian scheme. We recall the notions of cone and abelian cone over X and collect some basic properties.

Definition 1.7. Let $\mathcal{A} = \bigoplus_{d \geq 0} \mathcal{A}_d$ be a graded sheaf of $\mathcal{O}_X$-algebras such that the canonical map $\mathcal{O}_X \to \mathcal{A}_0$ is an isomorphism and such that $\mathcal{A}$ is locally generated by $\mathcal{A}_1$ as an $\mathcal{O}_X$-algebra. The *cone of* $\mathcal{A}$ is the scheme $\operatorname{Spec}_X(\mathcal{A})$ equipped with the natural projection

$$\operatorname{Spec}_X(\mathcal{A}) \to X.$$

The cone of $\mathcal{A}$ is *abelian* if the natural morphism $\operatorname{Sym}(\mathcal{A}_1) \to \mathcal{A}$ is an isomorphism of $\mathcal{O}_X$-algebras. A morphism of cones is a morphism over X induced by a graded morphism of sheaves of $\mathcal{O}_X$-algebras.

Definition 1.8. Let $\mathcal{F}$ be a coherent sheaf on X. The *abelian cone associated to* $\mathcal{F}$ is

$$C_X(\mathcal{F}) = \operatorname{Spec}_X(\operatorname{Sym}(\mathcal{F}))$$

equipped with the natural projection to X.

We will omit the subscript X in the formation of relative spectra and cones whenever it is possible to do so without introducing ambiguity.

Definition 1.8 is related to the total space of a locally free sheaf. If $\mathcal{E}$ is a locally free sheaf, then $C(\mathcal{E})$ is a vector bundle, but some authors (e.g. [Ful13, B.5.5]) prefer to define the total space of $\mathcal{E}$ as

$$\mathrm{Tot}(\mathcal{E}) \coloneqq C(\mathcal{E}^\vee) = \mathrm{Spec}\,(\mathrm{Sym}\,(\mathcal{E}^\vee)),$$

so that the sheaf of sections of $\mathrm{Tot}(\mathcal{E})$ over X is $(\mathcal{E}^\vee)^\vee \simeq \mathcal{E}$ by Lemma 1.9. When working with sheaves that may not be locally free, it is advisable to use $C(\mathcal{F})$ instead of $C(\mathcal{F}^\vee)$, see Lemma 1.9 and Example 1.10.

To an abelian cone $\pi\colon C(\mathcal{F}) \to X$ we can associate two natural sheaves of $\mathcal{O}_X$-modules. The sheaf of sections $\mathrm{Sect}(C(\mathcal{F}))$ of the projection π is given by

$$U \mapsto \mathrm{Hom}_U(U, C(\mathcal{F}) \mid_U).$$

The sheaf of functionals $\mathrm{Fun}(C(\mathcal{F}))$ is given by

$$U \mapsto \mathrm{Hom}_{\mathrm{Ab}}(C(\mathcal{F}) \mid_U, U \times \mathbb{A}^1),$$

where $\mathrm{Hom}_{\mathrm{Ab}}$ denotes morphisms of abelian cones.

Lemma 1.9. Given a coherent sheaf $\mathcal{F}$ in a Noetherian scheme X, there are natural isomorphisms of $\mathcal{O}_X$-modules

1 $\mathrm{Sect}(C(\mathcal{F})) \simeq \mathcal{F}^\vee$ and

2 $\mathrm{Fun}(C(\mathcal{F})) \simeq \mathcal{F}$.

Proof. It is enough to prove the statements in the affine case and for global sections. Let $X = \mathrm{Spec}\, R$ and $\mathcal{F} = \widetilde{M}$ for a Noetherian ring R and a coherent R-module M.

For 1, we see that

$$\mathrm{Hom}_X(X, C(\mathcal{F})) = \mathrm{Hom}_X(X, \mathrm{Spec}\,\mathrm{Sym}\,\mathcal{F}) \simeq \mathrm{Hom}_{R-\mathrm{alg}}(\mathrm{Sym}\, M, R) \simeq \mathrm{Hom}_{R-\mathrm{mod}}(M, R) \simeq M^\vee.$$

For 2 we have that

$$\mathrm{Hom}_{\mathrm{Ab}}(C(\mathcal{F}), X \times \mathbb{A}^1) = \mathrm{Hom}_{\mathrm{Gr}\, R-\mathrm{mod}}(\mathrm{Sym}\, R, \mathrm{Sym}\, M) \simeq \mathrm{Hom}_{R/\,\mathrm{mod}}(R, M) \simeq M,$$

where $\mathrm{Hom}_{\mathrm{Gr}\,R-\mathrm{mod}}$ denotes morphisms of graded R-modules and where we used that $X \times \mathbb{A}^1 \simeq C_X(\mathcal{O}_X)$. □

Example 1.10. Let $X = \mathrm{Spec}\,(R)$ with $R = \mathbb{C}[x]$ and let $I = (x)$ be the ideal of the origin 0 and let $M = R/I$ viewed as an R-module, that is, M is the skyscrapper sheaf supported at 0. Then

$$C(M) \simeq \mathrm{Spec}\,(\mathbb{C}[x,y]/(xy)) \subseteq X \times \mathbb{A}^1.$$

According to Lemma 1.9, $C(M)$ has no non-zero sections because $M^\vee = 0$, $C(M)$. This can be checked directly because any section must be 0 on $X \setminus 0$, thus everywhere. On the other hand, $C(M)$ has non-zero functionals

$$C(M) \to X \times \mathbb{A}^1 \colon (x,y) \mapsto (x, \lambda y)$$

for any $\lambda \in \mathbb{C}$. On the other hand, $M^\vee = 0$ so $C(M^\vee) = X$, which has no non-zero sections or functionals.

Lemma 1.11. For a cone $\pi \colon C = \mathrm{Spec}\,(\mathcal{A}) \to X$, the following are equivalent:

(1) C is a vector bundle over X,

(2) C is abelian and $\mathcal{A}_1$ is locally free over X,

(3) π is smooth.

Proof. The implication (1) ⇒ (3) is clear and the equivalence (1) $\iff$ (2) is standard. The implication (3) ⇒ (1) can be found in [BF96, Lemma 1.1]. □

1.2.4 Fractional ideals

Let R be an integral domain and let K be its field of fractions.

Definition 1.12. A *fractional ideal* over R is a finitely generated R-submodule of K.

We collect some useful properties and examples of fractional ideals.

Example 1.13. Every finitely generated ideal I of R is a fractional ideal over R.

Example 1.14. Let M be a finitely presented module over R such that $\dim_K M \otimes_R K = r$. Then there is an isomorphism

$$\bigwedge^r M \otimes_R K \simeq K \tag{1.3}$$

and the image of the composition

$$\bigwedge^r M \to \bigwedge^r M \otimes_R K \to K$$

is a fractional ideal over R.

Note that the isomorphism in Equation (1.3) is not canonical, but any other choice produces an isomorphic fractional ideal by Lemma 1.18 below. Therefore the isomorphism class of the fractional ideal constructed in Example 1.14 is an invariant of M, that we call the norm of M, following [Vil06].

Definition 1.15. Let M be a finitely presented module over R such that $\dim_K M \otimes_R K = r$. The *norm of* M is the isomorphism class $[\![M]\!]$ of the fractional ideal constructed in Example 1.14.

$$[\![M]\!] = \mathrm{Im}\left(\bigwedge^r M \to K \simeq \bigwedge^r M \otimes_R K\right). \tag{1.4}$$

Lemma 1.16. Let F be a fractional ideal over R. Then there exists $s \in R \setminus \{0\}$ such that $s \cdot F$ is an ideal of R. Conversely, if I is a finitely generated ideal in R and $s \in R \setminus \{0\}$, then $\frac{1}{s} I$ is a fractional ideal of R.

Proof. We have that $F = Rf_1 + \ldots + Rf_n$ for some $f_i \in F$. Write $f_i = r_i/s_i$ for some $r_i \in R$, $s_i \in R \setminus \{0\}$. Let $s = s_1 \cdot s_n$. Then $sr_i \in R$ for every i and $s \cdot F = Rsr_1 + \ldots Rsr_n$ is an ideal.

The converse is immediate. □

Corollary 1.17. Let F be a fractional ideal over R. Then there is an isomorphism of R-modules between F and an ideal of R.

Proof. Choose $s \in R \setminus \{0\}$ as in Lemma 1.16. Then, multiplication by s is an isomorphism between F and $s \cdot F$, which is an ideal of R. □

Lemma 1.18. For any two fractional ideals F_1 and F_2 over R,

$$\mathrm{Hom}_{R-\mathrm{mod}}(F_1, F_2) = \{k \in K \colon k \cdot F_1 \subseteq F_2\}$$

Proof. Given $k \in K$, multiplication by k is an endomoprhism of R-modules of K which restricts to a morphism $F_1 \to F_2$ if and only if $k \cdot F_1 \subseteq F_2$.

Conversely, let $\varphi\colon F_1 \to F_1$ be a homomorphism of R-modules. We need to find $k \in K$ such that $\varphi(f_1) = kf_1$ for every $f_1 \in F_1$. This condition implies that

$$k = \frac{\varphi(f_1)}{f_1} \in K.$$

To conclude, we need to show that k is independent of $f_1 \in F_1$. Indeed, choose $s \in R \setminus \{0\}$ such that $s \cdot F_1 \subseteq R$, which exists by Lemma 1.16. Then, given $f_1, f_1' \in F_1$ we have that

$$rf_1\varphi(f_1') = \varphi(rf_1f_1') = rf_1'\varphi(f_1) \Rightarrow \frac{\varphi(f_1')}{f_1'} = \frac{\varphi(f_1)}{f_1}. \qquad \square$$

Corollary 1.19. Let I and J be ideals in R which are isomorphic as fractional ideals and let $X = \mathrm{Spec}\,(R)$. Then

$$\mathrm{Bl}_I X = \mathrm{Bl}_J X$$

Proof. By Lemma 1.18, an isomorphism between I and J as fractional ideals is the same as an element $k \in K$ such that $k \cdot I = J$. Let $k = r/s$ with $r \in R$ and $s \in R \setminus \{0\}$. Then $(r)I = (s)J$. To conclude, observe that $\mathrm{Bl}_I X = \mathrm{Bl}_{(r)I} X$ because they satisfy the same universal property. $\square$

Definition 1.20. Let F be a fractional ideal over R and let $X = \mathrm{Spec}\,(R)$. The *blow-up of X along F* is

$$\mathrm{Bl}_F X := Bl_I X$$

for any ideal $I \subseteq R$ such that $I \cong F$ as fractional ideals.

In Section 1.3.3 we will study the particular case of Definition 1.20 where $F = [\![M]\!]$ is the norm of a moduule M and we will study its universal property Theorem 1.34.

1.2.5 Cohomology and base change

In Section 1.7 we will make extensive use of the so-called cohomology and basechange theorem. This is a result about the commutativity between cohomology and restriction to fibres. The two main references are [Har77, Theorem 12.11] and [Mum70, Section 5]. In this subsection, we recall the result following [Mum70].

The set-up is as follows. Let $f\colon X \to Y$ be a proper morphism between Noetherian schemes and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. For $y \in Y$, let $k(y)$ denote the residue field of y, let $X_y = X \times_Y \operatorname{Spec}(k(y))$, so that we have a Cartesian diagram

$$\begin{tikzcd} X_y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} k(y) \arrow[r] & Y, \end{tikzcd}$$

and let $\mathcal{F}_y = \mathcal{F} \otimes_{\mathcal{O}_Y} k(y)$.

With the previous notations, there is a natural morphism

$$\varphi^i(y)\colon R^i f_*(\mathcal{F}) \otimes k(y) \to H^i(X_y, \mathcal{F}_y).$$

We also consider the following functions $Y \to \mathbb{Z}$:

$$h^i(y, \mathcal{F}) = \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$$

and

$$\chi(\mathcal{F}_y) = \sum_{i \geq 0} (-1)^i \dim_{k(y)} H^i(X_y, \mathcal{F}_y).$$

Proposition 1.21. (Cohomology and Base Change [Mum70, Section 5]) Let $f\colon X \to Y$ be a proper morphism between Noetherian schemes and let $\mathcal{F}$ be a coherent sheaf on X which is flat over Y. The following holds

1 For each $i \geq 0$, the function $y \mapsto h^i(y, \mathcal{F})$ is upper semicontinuous on Y.

2 The function $y \mapsto \chi(\mathcal{F}_y)$ is locally constant on Y.

3 If, furthermore, Y is reduced and connected, the following are equivalent

   (a) the function $y \mapsto h^i(y, \mathcal{F})$ is constant,

   (b) the sheaf $R^i f_* \mathcal{F}$ is locally free on on Y and $\varphi^i(y)$ is an isomorphism for all $y \in Y$,

   and, in case that (a) and (b) hold, then $\varphi^{i-1}(y)$ is an isomoprhism for all i.

1.3 Desingularizations of coherent sheaves

In this section we introduce several constructions which "desingularize" a coherent sheaf $\mathcal{F}$ on a base scheme X.

1.3.1 Definition of desingularizations on stacks

We define our notion of desingularization and prove that it behaves well with composition. This part can be formulated directly for algebraic stacks instead of schemes, which will be useful later.

Definition 1.22. Let $\mathfrak{F}$ be a coherent sheaf on an integral algebraic stack $\mathfrak{X}$. A *desingularization* of $\mathfrak{F}$ is a morphism $p : \widetilde{\mathfrak{X}} \to \mathfrak{X}$ such that

1 $\widetilde{\mathfrak{X}}$ is integral,

2 p is birational and proper,

3 $(p^*\mathfrak{F})^{\mathrm{tf}}$ is a locally free sheaf.

Let us explain why a morphism as in Definition 1.22 deserves to be called a desingularization. The surjection $\mathfrak{F} \to \mathfrak{F}^{\mathrm{tf}}$ induces a closed embedding $C(\mathfrak{F}^{\mathrm{tf}}) \hookrightarrow C(\mathfrak{F})$ of abelian cones over $\mathfrak{X}$. By Lemma 1.11, the morphism $C(\mathfrak{F}^{\mathrm{tf}}) \to \mathfrak{X}$ is smooth if and only if $\mathfrak{F}^{\mathrm{tf}}$ is locally free. Therefore, Item 3 in Definition 1.22 es equivalent to saying that the morphism $C_{\widetilde{\mathfrak{X}}}((p^*\mathfrak{F})^{\mathrm{tf}}) \to \widetilde{\mathfrak{X}}$ is smooth.

Remark 1.23. If $\mathfrak{X}$ is a scheme and $\mathfrak{F}$ is a non-zero coherent ideal sheaf, the usual blow-up of $\mathfrak{X}$ at the closed subscheme defined by $\mathfrak{F}$ is a desingularization of $\mathfrak{F}$.

Lemma 1.24. Let $\mathfrak{X}$ be an integral algebraic stack and $\mathfrak{F}$ a coherent sheaf on X. Let $p : \widetilde{\mathfrak{X}} \to \mathfrak{X}$ be a desingularization of $\mathfrak{F}$ and let $q : \mathfrak{Y} \to \widetilde{\mathfrak{X}}$ be a proper birational morphism. Then, the composition $p \circ q : \mathfrak{Y} \to \mathfrak{X}$ is a desingularization of $\mathfrak{F}$.

Proof. The composition $r := p \circ q$ is birational and proper, so all we need to prove is that $(r^*\mathfrak{F})^{\mathrm{tf}}$ is locally free. In the following, we show that

$$(r^*\mathfrak{F})^{\mathrm{tf}} \simeq q^*((p^*\mathfrak{F})^{\mathrm{tf}}).$$

We have a commutative diagram of sheaves on $\mathfrak{Y}$

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{K}_1 \arrow[r] \arrow[d, dashed] & q^*p^*\mathfrak{F} \arrow[r] \arrow[d] & q^*((p^*\mathfrak{F})^{\mathrm{tf}}) \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & \mathfrak{K}_2 \arrow[r] & q^*p^*\mathfrak{F} \arrow[r] & (q^*p^*\mathfrak{F})^{\mathrm{tf}} \arrow[r] & 0
\end{tikzcd}$$

where the map $q^*p^*\mathfrak{F} \to q^*((p^*\mathfrak{F})^{\mathrm{tf}})$ is the pullback of the surjective map

$$p^*\mathfrak{F} \to (p^*\mathfrak{F})^{\mathrm{tf}},$$

$\mathfrak{K}_1$ and $\mathfrak{K}_2$ are the corresponding kernels and the solid vertical map is the identity. Since the image of the composition $\mathfrak{K}_1 \to (q^*p^*\mathfrak{F})^{\mathrm{tf}}$ is generically zero and $\mathfrak{X}$ is irreducible, we have that the morphism $\mathfrak{K}_1 \to (q^*p^*\mathfrak{F})^{\mathrm{tf}}$ is zero. This induces the left vertical map in the diagram, and therefore also the right vertical map. We have that $\mathfrak{K}_1$ and $\mathfrak{K}_2$ are torsion sheaves. Let $\mathfrak{K}_3$ be the cokernel of $\mathfrak{K}_1 \to \mathfrak{K}_2$. By the Snake Lemma, we have an exact sequence

$$0 \to \mathfrak{K}_3 \to q^*((p^*\mathfrak{F})^{\mathrm{tf}}) \to (r^*\mathfrak{F})^{\mathrm{tf}} \to 0.$$

By assumption $(p^*\mathfrak{F})^{\mathrm{tf}}$ is locally free, so $q^*((p^*\mathfrak{F})^{\mathrm{tf}})$ is locally free. Since $\mathfrak{X}$ is irreducible and $\mathfrak{K}_3$ is a torsion sheaf, we get that $\mathfrak{K}_3 = 0$. This proves the claim. □

1.3.2 Rossi's construction for affine schemes

In the following, we describe the desingularization construction proposed by Rossi in the analytic setup in [Ros68], which is very geometric in nature. The same construction was studied by Oneto and Zatini in the algebraic setup in [OZ91], under the name of Nash transformation. It gives a way of desingularizing coherent sheaves on integral Noetherian schemes by taking the closure of a graph into a Grassmannian. We present here Rossi's construction for affine schemes. A much more general gluing will be presented in Section 1.5. Our gluing recovers Rossi's original construction for schemes.

We have seen in Lemma 1.24 that desingularizations are not unique. The following construction is the minimal one in the sense of the universal property in Theorem 1.34.

Let $X = \mathrm{Spec}\,(R)$ be an integral affine Noetherian scheme and $\mathcal{F} = \widetilde{M}$ be the coherent sheaf associated to a coherent R-module M. This is a finitely generated module such that the kernel of any morphism $R^{\oplus n} \to M$ is finitely generated. Over the unique generic point $0 \in X$, the sheaf $\mathcal{F}|_0$ is locally-free of rank r. We define the *generic rank* of $\mathcal{F}$:

$$\mathrm{rk}(\mathcal{F}) := \mathrm{rk}(\mathcal{F}|_0) = r.$$

Note that if $\mathcal{F}$ is non-zero, there exists a non-empty open subscheme $U \subset X$ such that $\mathcal{F}\,|_U$ is locally free, for example by taking U to be the complement of the subscheme cut out by the smallest non-zero Fitting ideal of $\mathcal{F}$. For any such U, the rank of $\mathcal{F}\,|_U$ equals the generic rank of $\mathcal{F}$.

Since R is affine, we can find for some n a surjective morphism of sheaves

$$f : \mathcal{O}_X^{\oplus n} \twoheadrightarrow \mathcal{F}. \tag{1.5}$$

Restricted to a non-empty open subscheme U where $\mathcal{F}$ is locally free, f gives a U-point of $\mathrm{Gr}_U(\mathcal{O}_U^{\oplus n}, r)$, which is a morphism

$$\Gamma_f : U \to \mathrm{Gr}_U(\mathcal{O}_U^{\oplus n}, r)$$

such that the composition of Γ_f with the structure map $\mathrm{Gr}(\mathcal{O}_U^{\oplus n}, r) \to U$ is id_U. Moreover, under this isomorphism, the locally-free sheaf $\mathcal{F}|_U$ is identified with the restriction to $\Gamma_f(U)$ of the universal quotient sheaf $\mathcal{Q}_{\mathrm{Gr}_U(\mathcal{O}_U^{\oplus n}, r)}$. This is automatic from the description of the relative Grassmannian as the fine moduli scheme of the functor in (1.2).

By compatibility of the relative Grassmannian with base-change we have an open embedding $\mathrm{Gr}_U(\mathcal{O}_U^{\oplus n}, r) \subset \mathrm{Gr}_X(\mathcal{O}_X^{\oplus n}, r)$. We give the following definition.

Definition 1.25. The *blow-up of X at the coherent sheaf $\mathcal{F}$* is the schematic closure of $\Gamma_f(U)$ in $\mathrm{Gr}_X(\mathcal{O}_X^{\oplus n}, r)$, equipped with the morphism

$$p\colon \mathrm{Bl}_{\mathcal{F}}(X) = \overline{\Gamma_f(U)} \to X$$

obtained by restricting the natural projection $\mathrm{Gr}_X(\mathcal{O}_X^{\oplus n}, r) \to X$. Note that, since U is integral, $\mathrm{Bl}_{\mathcal{F}}X$ can also be described as the topological closure of $\Gamma_f(U)$ with the reduced induced structure (see [Sta22, Lemma 056B])

Remark 1.26. In the proof of Lemma 1.39, we will show that if $\mathcal{F}$ is a non-zero coherent ideal sheaf on X, then $\mathrm{Bl}_{\mathcal{F}}(X)$ is the usual blow-up and the above recovers its construction as the closure of a graph in the relative projective space.

Another classical example is the Nash blow-up of X, which is the particular case $\mathcal{F} = \Omega_X^1$. This case is related to resolution of singularities, see [Spi20]. The general case appeared in [OZ91] under the name of Nash transform.

Proposition 1.27. [cf. [Ros68, Theorem 3.5]] The blow-up $\mathrm{Bl}_{\mathcal{F}}(X)$ above is well-defined. Moreover,

$$p : \mathrm{Bl}_{\mathcal{F}}(X) \to X$$

is a projective birational morphism such that the torsion-free part of $p^*\mathcal{F}$,

$$(p^*\mathcal{F})^{\mathrm{tf}} = p^*\mathcal{F}/\mathrm{Tors}(p^*\mathcal{F})$$

is locally-free of rank r. That is, p is a desingularization of $\mathcal{F}$.

Proof. Observe that $p : \mathrm{Bl}_{\mathcal{F}}(X) \to X$ is a projective morphism, since $\mathrm{Gr}_X(\mathcal{O}_X^{\oplus n}, r)$ is a projective variety. It is also clear by construction that

$$\begin{tikzcd} \Gamma_f(U) \arrow[r, hook] \arrow[d, "\cong"] & \mathrm{Bl}_{\mathcal{F}}(X) \arrow[d, "p"] \\ U \arrow[r, hook] & X \end{tikzcd}$$

is Cartesian, in particular p is birational. To show that $(p^*\mathcal{F})^{\mathrm{tf}}$ is locally-free of rank r we show that it is the restriction of the universal quotient bundle on the Grassmannian.

We denote the restriction of the universal sub-bundle and quotient on $\mathrm{Gr}_X(\mathcal{O}_X^{\oplus n}, r)$ by S, respectively Q, and the kernel of the map $\mathcal{O}_{\mathrm{Bl}_{\mathcal{F}}(X)}^{\oplus n} \to (p^*\mathfrak{F})^{\mathrm{tf}}$ by K. We now consider the following diagram of sheaves on $\mathrm{Bl}_{\mathcal{F}}(X)$:

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r] \arrow[d, dashed] & \mathcal{O}_{\mathrm{Bl}_{\mathcal{F}}(X)}^{\oplus n} \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & K \arrow[r] & \mathcal{O}_{\mathrm{Bl}_{\mathcal{F}}(X)}^{\oplus n} \arrow[r] & (p^*\mathfrak{F})^{\mathrm{tf}} \arrow[r] & 0. \end{tikzcd}$$

Since the composition $S \to (p^*\mathfrak{F})^{\mathrm{tf}}$ is zero, we get a map $S \to K$ and thus a map $\theta : Q \to (p^*\mathfrak{F})^{\mathrm{tf}}$. By the construction of $\mathrm{Bl}_{\mathcal{F}}(X)$, θ is an isomorphism on $\Gamma_f(U)$. Since Q is locally free, the kernel of θ cannot be a torsion sheaf, and thus it has to be zero. This shows that θ is injective. On the other hand, the Snake Lemma shows that θ is surjective. This gives that

$$(p^*\mathcal{F})^{\mathrm{tf}} = \mathcal{Q}_{\mathrm{Gr}_X(\mathcal{O}_X^{\oplus n}, r)}|_{\mathrm{Bl}_{\mathcal{F}}(X)}.$$

To check that the above construction is well-defined, we need to check that it is independent of the choice of f and n in equation (1.5). Indeed, suppose we have another surjection

$$f' : \mathcal{O}_X^{\oplus n'} \twoheadrightarrow \mathcal{F}.$$

Since $\mathcal{O}_X^{\oplus n}$ is free, we can find a morphism γ such the following diagram commutes

$$\begin{tikzcd}
\mathcal{O}_X^{\oplus n'} \arrow[rr, "\gamma"] \arrow[dr, "f'"'] & & \mathcal{O}_X^{\oplus n} \arrow[dl, "f"] \\
& \mathcal{F}. &
\end{tikzcd}$$

Then γ induces an isomorphism

$$\begin{aligned}
\widetilde{\gamma}\colon \mathcal{O}_X^{\oplus n} \oplus \mathcal{O}_X^{\oplus n'} &\to \mathcal{O}_X^{\oplus n} \oplus \mathcal{O}_X^{\oplus n'} \\
(u, v) &\mapsto (u + \gamma(v), v).
\end{aligned}$$

For $h\colon \mathcal{O}_X^{\oplus(n+n')} \to \mathcal{O}_X^{\oplus n}$ the composition of $\widetilde{\gamma}$ with the projection in the first n factors and $g\colon \mathcal{O}_X^{\oplus(n+n')} \twoheadrightarrow \mathcal{F}$ the sum of f and f, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{O}_X^{\oplus(n+n')} \arrow[r, two heads, "g"] \arrow[d, "h"] & \mathcal{F} \\
\mathcal{O}_X^{\oplus n} \arrow[ur, "f"'] & .
\end{tikzcd}$$

The morphism h induces a morphism $\mathrm{Gr}(h)$ of Grassmannians such that the following diagram commutes

$$\begin{tikzcd}
U \arrow[r, "\Gamma_f"] \arrow[dr, "\Gamma_g"'] & \mathrm{Gr}_X(\mathcal{O}_X^{\oplus n}, r) \arrow[d, "\mathrm{Gr}(h)"] \\
& \mathrm{Gr}_X(\mathcal{O}_X^{\oplus(n+n')}, r).
\end{tikzcd}$$

Moreover, $\mathrm{Gr}(h)$ is an isomorphism onto its image. Then $\mathrm{Gr}(h)$ induces an isomorphism on the closures of $\Gamma_f(U)$ and $\Gamma_g(U)$. Similarly, we have an isomorphism between the closures of $\Gamma_{f'}(U)$ and $\Gamma_g(U)$. So we obtain an isomorphism $\overline{\Gamma_f(U)} \cong \overline{\Gamma_{f'}(U)}$ as required. $\square$

Proposition 1.28. Let $\mathcal{L}$ be a line bundle on X. Then we have a unique isomorphism

$$\begin{tikzcd}
\mathrm{Bl}_{\mathcal{F}}(X) \arrow[rr, "\exists!\widetilde{\phi}"] \arrow[dr, "p"'] & & \mathrm{Bl}_{\mathcal{F}\otimes\mathcal{L}}(X) \arrow[dl, "q"] \\
& X &
\end{tikzcd}$$

which makes the diagram commute.

Proof. Let $f : \mathcal{O}_X^{\oplus n} \to \mathcal{F}$ be a surjective morphism, let S denote the kernel of f and let U be an open subset of X where S is a vector bundle. We thus obtain a short exact sequence

$$0 \to S \otimes \mathcal{L}|_U \to \mathcal{O}_U^{\oplus n} \otimes \mathcal{L}|_U \overset{f\otimes id}{\to} \mathcal{F} \otimes \mathcal{L}|_U \to 0.$$

By possibly shrinking U we may assume we have an isomorphism $g : \mathcal{L}|_U \simeq \mathcal{O}_U$. We thus obtain a commutative diagram

$$\begin{tikzcd} \Gamma_f \arrow[r] \arrow[d] & \mathrm{Gr}_X(\mathcal{O}_X^{\oplus n}, r) \arrow[d] \\ \Gamma_{f\otimes id} \arrow[r] & \mathrm{Gr}_X(\mathcal{O}_X^{\oplus n}, r) \end{tikzcd}$$

where the right vertical arrow is induced by g. This gives a morphism $\widetilde{\phi} : \overline{\Gamma}_f \to \overline{\Gamma}_{f\otimes id}$ and proves the claim. $\square$

1.3.3 Villamayor's construction

We now present another construction of the blow-up $\mathrm{Bl}_{\mathcal{F}}X$ introduced in section 1.3.2 in terms of Fitting ideals of the sheaf $\mathcal{F}$. Most of these ideas first appeared in [OZ91]. We follow the exposition in [Vil06]. In Section 1.3.4 we show that this construction is equivalent to Rossi's construction.

Fitting ideals (see Section 1.2.2) are related to ranks of modules and flatness. Indeed, the local rank of M at a prime ideal P of R is k if and only if $F_{-1}(M) \subseteq F_0(M) \subseteq \ldots \subseteq F_{k-1}(M) \subseteq P$ but $F_k(M) \not\subset P$. As a corollary, if R is a domain then the generic rank of M is k if and only if $F_k(M)$ is the first non-zero Fitting ideal, and moreover M is flat if and only if it is free, if and only if $F_k(M) = R$.

The relationship between Fitting ideals and local freeness of the torsion-free part comes from Lipman's theorem.

Theorem 1.29 (cf. [Lip69, Lemma 1])**.** Let R be a local ring. Given a finitely presented module M and a non-negative integer r, the following are equivalent:

1 $F_0(M) = \ldots = F_{r-1}(M) = 0$ and $F_r(M)$ is invertible.

2 M has projective dimension at most one and $M/\mathrm{tor}(M)$ is free of rank r.

It follows from Theorem 1.29 that blowing up the first non-trivial Fitting ideal of M will make $(p^*M)^{\mathrm{tf}}$ locally free (with p the blow-up morphism). However, it is possible that M^{tf} is already locally free on $\operatorname{Spec} R$ even though its first non-trivial Fitting ideal is not principal, see Remark 1.35. In order to find a minimal transformation of $\operatorname{Spec}(R)$ on which M^{tf} is locally free, Villamayor proposes in [Vil06] the following construction.

Construction 1.30 ([Vil06, Remark 2.1]). Let R be a domain and let M be a finitely presented R-module or rank r. Choose generators $m_1, \dots, m_N$ for M. Then there is a short exact sequence

$$0 \to P \to R^N \to M \to 0.$$

Since M has rank r, there are elements $p_1, \dots, p_{N-r}$ in P which induce a morphism $R^{N-r} \to R^N$ of rank $N-r$. Let $P_1 \simeq R^{N-r}$ be the free module generated by $p_1, \dots, p_{N-r}$ and let $M_1 = R^N/P_1$, that is, the following is exact

$$0 \to P_1 \to R^N \to M_1 \to 0.$$

Then M_1 has projective dimension at most 1, $\operatorname{rk}(M_1) = \operatorname{rk}(M)$, there is a natural surjection $M_1 \to M$ and $M_1/\operatorname{tor}(M) = M/\operatorname{tor}(M)$.

Recall that we have introduced fractional ideals in Section 1.2.4. In particular, we have defined the notion of blow-up of an affine scheme along a fractional ideal in Definition 1.20.

Definition 1.31. Let R be a Noetherian integral ring, let $X = \operatorname{Spec} R$ and let M be a finitely presented R-module of generic rank r. The *blow-up of X along M* is

$$p : \mathrm{Bl}_M R := \mathrm{Bl}_{[\![M]\!]} R \to R,$$

where $[\![M]\!]$ is the norm of M from Definition 1.15.

Lemma 1.32. Under the assumptions of Definition 1.31, let M_1 be the R-module associated to M in Construction 1.30. Then $F_r(M_1)$ and $[\![M]\!]$ are isomorphic as fractional ideals over R. In particular, by Definition 1.20,

$$\mathrm{Bl}_{[\![M]\!]} X = \mathrm{Bl}_{F_r(M_1)} X.$$

Proof. See [Vil06, Proposition 2.5]. $\square$

Remark 1.33. As explained in [Vil06], the same ideas apply to any (Noetherian) ring R if we restrict to finitely presented R-modules M such that $M \otimes_R Q(R)$ is a free $Q(R)$-module, where $Q(R)$ is the total quotient ring of R.

Theorem 1.34. [Universal property of blow-up, affine case, [Vil06, Theorem 3.3]] Let R be an integral Noetherian ring, let M be a finitely presented R-module of generic rank r and let $X = \mathrm{Spec}\,(R)$. The blow-up $p\colon \mathrm{Bl}_M X \to X$ satisfies the following properties:

1 The sheaf $p^*M/\mathrm{tor}(p^*M)$ is locally free of rank r on $\mathrm{Bl}_M X$ and

2 for any morphism $q\colon Y \to X$ from a scheme Y for which $q^*M/\mathrm{tor}(q^*M)$ is locally free of rank r on Y, there is a unique morphism $q'\colon Y \to \mathrm{Bl}_M X$ such that $p \circ q' = q$.

Proof. The fractional ideal $[\![M]\!]$ is isomorphic to some ideal I of R. Let $\mathrm{Bl}_M X = \mathrm{Bl}_I X$ be the blow-up of X along I, which is independent of the choice of I. Since $q^*M/\mathrm{tor}(q^*M) = q^*M_1/\mathrm{tor}(q^*M_1)$ for any morphism $q\colon Y \to X$, we can replace M by M_1, which has projective dimension at most one. Conclude by Theorem 1.29 and the universal property of (the usual) blow-up. $\square$

Remark 1.35. In general, $\mathrm{Bl}_M X$ is not obtained by blowing up the first non-zero Fitting ideal of M. Indeed, let $I \subset R$ be an ideal and $M = R/I$. On the one hand, $M^{\mathrm{tf}} = 0$ is locally free of rank 0, which is the rank of M, so $\mathrm{Bl}_M X = X$. On the other hand, the first non-zero Fitting ideal of M is $F_0(M) = I$, so $\mathrm{Bl}_{F_0(M)} X = \mathrm{Bl}_M X$ if and only if I is principal.

However, if M has generic rank r and projective dimension ≤ 1, then $\mathrm{Bl}_M X = \mathrm{Bl}_{F_r(M)} X$.

1.3.4 Equivalence of the two constructions

We find it useful here to give a constructive comparison of the two approaches, which we find clarifies both.

Proposition 1.36 (Equivalence of Rossi constructions)**.** Let $X = \mathrm{Spec}\,(R)$ be an affine, integral Noetherian scheme and $\mathcal{F} = \widetilde{M}$ be a coherent sheaf. Then

$$\mathrm{Bl}_{\mathcal{F}} X = \mathrm{Bl}_M X,$$

where the first is defined in Definition 1.25 and the second in Definition 1.31.

The rest of Section 1.3.4 is devoted to the proof of Proposition 1.36, which is delayed until the end. The first observation we make is that both constructions only depend on $\bigwedge^r M$, where r is the generic rank. We suppose here that $r \neq 0$. Otherwise, the sheaf is already locally free up to torsion, so no blow-ups are necessary.

Lemma 1.37. Let X, M as above and $r = \mathrm{rk} M > 0$, we have canonical isomorphisms

$$\mathrm{Bl}_{\mathcal{F}} X = \mathrm{Bl}_{\bigwedge^r \mathcal{F}} X \tag{1.6}$$

and

$$\mathrm{Bl}_M X = \mathrm{Bl}_{\bigwedge^r M} X. \tag{1.7}$$

Proof. For (1.6), fix a surjection $f : \mathcal{O}_X^{\oplus n} \twoheadrightarrow \mathcal{F}$. Then $\wedge^r f : \bigwedge^r \mathcal{O}_X^{\oplus n} \twoheadrightarrow \bigwedge^r \mathcal{F}$ defines an embedding

$$\Gamma_{\wedge^r f} : U \to \mathbb{P}_U^{m-1} \subset \mathbb{P}_X^{m-1}$$

of the generic open, where $m = \binom{n}{r}$. With the definition of the Plücker embedding from Section 1.2.1, we have the following commutative diagram

$$\begin{tikzcd} U \arrow[r, "\Gamma_f"] \arrow[dr, "\Gamma_{\wedge^r f}"'] & \mathrm{Gr}_X(\mathcal{O}^{\oplus n}, r) \arrow[d, "\lambda_{n,r}"] \\ & \mathbb{P}_X^{m-1}. \end{tikzcd}$$

Since $\lambda_{n,r}$ is a closed embedding, it commutes with taking closures. Hence, it gives the required isomorphism.

For (1.7), recall the definition of the norm (1.4):

$$[\![M]\!] = \mathrm{Im}(\bigwedge^r M \to \bigwedge^r M \otimes_R K \cong K).$$

From this, it is clear that the Villamayor blow-up only depends on the top exterior power of M. □

For the Rossi construction, we reduce from rank 1 sheaves to ideal sheaves using the following lemma. Note that the Villamayor construction is already a usual blow-up at a (fractional) ideal $[\![M]\!]$.

Lemma 1.38. Let $\mathcal{F}$ and X as in Proposition 1.36. If $\mathcal{F}$ has rank 1, then there is an ideal sheaf $\mathcal{I}$ on X such that

$$\mathrm{Bl}_{\mathcal{F}} X = \mathrm{Bl}_{\mathcal{I}} X.$$

Moreover, $\mathcal{I}$ is the ideal sheaf associated to an ideal $I \cong [\![M]\!]$ as fractional ideals.

Proof. Let $\mathcal{K} = \widetilde{K(R)}$. Since $\mathrm{Bl}_{\mathcal{F}} X$ only depends on $\mathcal{F}\mid_U$ and since $\mathcal{F}$ and $\mathcal{F}^{\mathrm{tf}}$ are isomorphic over U, we can assume that $\mathcal{F}$ is torsion-free. In that case, we can replace $\mathcal{F}$ by its image in $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K} \simeq \mathcal{K}$ and we can find a section g of $\mathcal{K}$ such that $g \cdot \mathcal{F} \simeq \mathcal{I}$ is an ideal sheaf.

Remember that $\mathrm{Bl}_{\mathcal{F}} X = \overline{\Gamma_g(U)}$ where $\Gamma_g : U \to \mathbb{P}_U^{n-1}$ is induced by a surjection $f : \mathcal{O}_X^{\oplus n} \twoheadrightarrow \mathcal{F}$. Since g determines a section of $\mathcal{O}_X$ over U, we have another surjection

$$g|_U \cdot f|_U \colon \mathcal{O}_U^{\oplus n} \twoheadrightarrow \mathcal{I}.$$

By construction, $\Gamma_{g \cdot f} = \Gamma_f$, therefore $\mathrm{Bl}_{\mathcal{F}} X = \mathrm{Bl}_{\mathcal{I}} X$ follows. Let $I = \Gamma(\mathcal{I})$, then I is by construction equivalent to $[\![M]\!]$ as fractional ideals, since I is equivalent to the image of $M \to M \otimes_R K$. ☐

Lemma 1.39. Let X an affine integral and noetherian scheme and let $\mathcal{F} = \widetilde{I}$ be an ideal sheaf. Then

$$Bl_{\mathcal{F}} X = Bl_I X.$$

Proof. The right-hand side is the Villamayor blow-up of $[\![I]\!]$. As fractional ideals $[\![I]\!] = I$, so $Bl_I X$ is the usual blow-up of X along the ideal I. The left-hand side is the closure of the graph $\Gamma_f \colon U \to \mathbb{P}_U^{n-1}$ inside $\mathbb{P}_X^{n-1}$ where $f \colon \mathcal{O}_X^{\oplus n} \to \mathcal{F} \to 0$ is a choice of generators of the ideal sheaf $\mathcal{F}$ and U is the generic point of X. The lemma follows from the standard fact that the usual blow-up can be obtained as closure of a graph in projective space, see [EH99, Proposition IV-22]. ☐

Proof of Proposition 1.36. Combining Lemmas 1.37 and 1.38 we have that $\mathrm{Bl}_{\mathcal{F}} X$ is the usual blow-up $\mathrm{Bl}_{\mathcal{I}} X$ for an ideal sheaf $\mathcal{I} = \widetilde{I}$ such that $I \cong [\![\bigwedge^r M]\!]$ as fractional ideals. On the other hand,

$$\mathrm{Bl}_M X = \mathrm{Bl}_{\bigwedge^r M},$$

by Lemma 1.37, and this equals the usual blow-up of X along any ideal isomorphic as a fractional ideal to $[\![\bigwedge^r M]\!]$ by Lemma 1.32. ☐

1.3.5 Properties of the Rossi blow-up

In light of Proposition 1.36, from now on we will identify the Rossi and Villamayor blow-ups of a coherent sheaf $\mathcal{F}$ on an integral Noetherian scheme X, and we will denote the common scheme by $\mathrm{Bl}_{\mathcal{F}} X$. In this section, we collect some properties of the blow-up $\mathrm{Bl}_{\mathcal{F}} X$.

Due to the universal property Theorem 1.34 and compatibility with flat pullback (see Proposition 1.41) the blow-up $\mathrm{Bl}_{\mathcal{F}}X$ generalizes to integral Noetherian schemes X, and it inherits the universal property in Theorem 1.34. We will further generalize these results to Artin stacks in Section 1.5.

Proposition 1.40 (Universal property of blowing up)**.** Let X be a Noetherian integral scheme and $\mathcal{F}$ be a coherent sheaf of generic rank r. There exists a blow-up $p\colon \mathrm{Bl}_{\mathcal{F}}X \to X$ which satisfies

1 The sheaf $(p^*\mathcal{F})^{\mathrm{tf}} = p^*\mathcal{F}/\mathrm{tor}(p^*\mathcal{F})$ is locally free of rank r on $\mathrm{Bl}_{\mathcal{F}}X$ and

2 for any morphism $f\colon Y \to X$ from an scheme Y for which $f^*\mathcal{F}/\mathrm{tor}(f^*\mathcal{F})$ is locally free of rank r on Y, there is a unique morphism $f'\colon Y \to \mathrm{Bl}_{\mathcal{F}}X$ such that $p \circ f' = f$.

$$\begin{tikzcd} Y \arrow[r, dashed, "\exists! f'"] \arrow[dr, "f"'] & \mathrm{Bl}_{\mathcal{F}}(X) \arrow[d, "p"] \\ & X \end{tikzcd}$$

Proof. The scheme $\mathrm{Bl}_{\mathcal{F}}X$ can be constructed following the construction of the stack $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{X}$ in Section 1.5 and properties 1 and 2 can be deduced from Theorem 1.34 as we will do in Theorem 1.79 for $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{X}$. □

Proposition 1.41. Let $f : Y \to X$ be a morphism of Noetherian integral schemes and let $\mathcal{F}$ be a coherent sheaf on X of generic rank r. If $f^*\mathcal{F}$ has generic rank r then there is a unique morphism

$$\begin{tikzcd} \mathrm{Bl}_{f^*\mathcal{F}}(Y) \arrow[r, dashed, "\exists! \widetilde{f}"] \arrow[d] & \mathrm{Bl}_{\mathcal{F}}(X) \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

making the diagram commute. If, moreover, f is flat, then the square is Cartesian.

Proof. A unique morphism $\widetilde{f}$ making the diagram commute exists by the universal property of $\mathrm{Bl}_{\mathcal{F}}(X)$, which is Proposition 1.40. To show that the diagram is Cartesian, we use [Sta22, Lemma 0805], which is the analogous result for blow-ups along ideal sheaves. This requires checking that $f^{-1}[\![\mathcal{F}]\!] \cdot \mathcal{O}_Y = [\![f^*\mathcal{F}]\!]$, which holds since $\mathcal{F}$ and $f^*\mathcal{F}$ have the same rank and the norm $[\![\cdot]\!]$ is a determinantal ideal.

Indeed, we can work locally. Then we have $X = \mathrm{Spec}\,(A)$, $Y = \mathrm{Spec}\,(B)$, a ring homomorphism $f^{\#}\colon A \to B$ and $\mathcal{F} = \widetilde{M}$ for some finitely presented A-module M. To compute $[\![\mathcal{F}]\!]$, we take a presentation

$$A^m \xrightarrow{\Gamma} A^n \longrightarrow M \longrightarrow 0,$$

we choose a submatrix Γ' of Γ consisting of $n-r$ columns of Γ and then $[\![\mathcal{F}]\!]$ is represented by the ideal generated by all the minors $\Delta_i(\Gamma')$ of size $(n-r)\times(n-r)$ of Γ'. The choice of Γ' must be so that this ideal is non-zero and such a choice exists because $\mathrm{rk}(\mathcal{F}) = r$. Then $f^{-1}[\![\mathcal{F}]\!]\cdot B$ is the ideal in B generated by $f^{\#}(\Delta_i(\Gamma'))$ for all i. On the other hand, tensoring by $\otimes_A B$ we get a presentation

$$B^m \xrightarrow{f^*\Gamma} B^n \longrightarrow M\otimes_A B \longrightarrow 0.$$

Since $f^*\mathcal{F} = \widetilde{M\otimes_A B}$ and since $\mathrm{rk}(f^*\mathcal{F}) = r$, we can compute $[\![f^*\mathcal{F}]\!]$ in the same manner, i.e., taking all the minors of size $(n-r)\times(n-r)$ of a submatrix $(f^*\Gamma)''$ consisting of $n-r$ columns of $f^*\Gamma$. This means that $[\![f^*\mathcal{F}]\!]$ is generated by $\Delta_i((f^{\#}\Gamma)'')$. We can actually choose Γ' and $(f^*\Gamma)''$ so that they consist of the same columns, and in that case we are done because $f^{\#}$ is a ring homomorphism. □

Proposition 1.42. Let X be a Noetherian integral scheme and $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Assume that we have an exact sequence $0\to\mathcal{E}\to\mathcal{F}\to\mathcal{G}\to 0$.

1 If the sequence is locally split and $\mathcal{E}$ is locally free, then there is an isomorphism $\mathrm{Bl}_{\mathcal{F}}X \simeq \mathrm{Bl}_{\mathcal{G}}X$.

2 If $\mathcal{G}$ is locally free, then there is an isomorphism $\mathrm{Bl}_{\mathcal{F}}X \simeq \mathrm{Bl}_{\mathcal{E}}X$.

Proof. It is enough to prove the statements locally. Indeed, if $p_{\mathcal{F}}\colon \mathrm{Bl}_{\mathcal{F}}X\to X$ and $p_{\mathcal{G}}\colon \mathrm{Bl}_{\mathcal{G}}X\to X$ are the natural projections, then $\mathrm{Bl}_{\mathcal{F}}X\simeq\mathrm{Bl}_{\mathcal{G}}X$ if and only if $(p_{\mathcal{F}}^*\mathcal{G})^{\mathrm{tf}}$ and $(p_{\mathcal{G}}^*\mathcal{F})^{\mathrm{tf}}$ are locally free, and these are local statements.

Therefore, we may assume that we have $\mathcal{F}\simeq\mathcal{E}\oplus\mathcal{G}$. With this, we have that

$$\wedge^{\mathrm{top}}\mathcal{F} = \wedge^{\mathrm{top}}\mathcal{E}\otimes\wedge^{\mathrm{top}}\mathcal{G}. \tag{1.8}$$

By Lemma 1.37 we have that $\mathrm{Bl}_{\mathcal{G}}X\simeq\mathrm{Bl}_{\wedge^{\mathrm{top}}\mathcal{G}}X$ and $\mathrm{Bl}_{\mathcal{F}}X\simeq\mathrm{Bl}_{\wedge^{\mathrm{top}}\mathcal{F}}X\simeq\mathrm{Bl}_{\wedge^{\mathrm{top}}\mathcal{E}\otimes\wedge^{\mathrm{top}}\mathcal{G}}X$ using Equation (1.8). Suppose now that $\mathcal{E}$ is locally free, then $\wedge^{\mathrm{top}}\mathcal{E}$ is a line bundle and we conclude that $\mathrm{Bl}_{\mathcal{F}}X\simeq\mathrm{Bl}_{\mathcal{G}}X$ by Proposition 1.28.

If $\mathcal{G}$ is locally free, the sequence is locally split and a similar argument to the one above shows that $\mathrm{Bl}_{\mathcal{F}}X\simeq\mathrm{Bl}_{\mathcal{E}}X$. □

In the following we discuss a more general situation, when we have an open $U \subset X$ such that $0 \to \mathcal{E}_U \to \mathcal{F}_U \to \mathcal{G}_U \to 0$, with $\mathcal{E}_U$ locally free.

Proposition 1.43. Let $\mathcal{F}$ and $\mathcal{G}$ sheaves of ranks $r+a$ and r on an integral scheme X and $\mathcal{O}^{n+a} \to \mathcal{F}$ and $\mathcal{O}^n \to \mathcal{G}$ surjective morphisms. Suppose there exists $U \subset X$ an open subset and a commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & \\
\mathcal{O}_U^n \arrow[r] \arrow[d, shift left, "i"] & \mathcal{G}|_U \arrow[r] \arrow[d, shift left, "f"] & 0 \\
\mathcal{O}_U^{n+a} \arrow[r] \arrow[u, shift left] \arrow[d, shift left] & \mathcal{F}|_U \arrow[r] \arrow[u, shift left] \arrow[d, shift left] & 0 \\
\mathcal{O}_U^a \arrow[r, "h"] \arrow[u, shift left] \arrow[d] & \mathcal{E}_U \arrow[u, shift left] \arrow[d] & \\
0 & 0 &
\end{tikzcd}$$

with the columns split short exact sequences and h an isomorphism. Then, we get an induced morphism

$$\mathrm{Bl}_{\mathcal{G}} X \to \mathrm{Bl}_{\mathcal{F}} X.$$

In particular, $\mathrm{Bl}_{\mathcal{G}} X$ is a desingularisation of $\mathcal{F}$.

Proof. The diagram in the hypothesis gives a commutative diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & X \times \mathrm{Gr}(n, r) \arrow[d] \\
U \arrow[r] & X \times \mathrm{Gr}(n+a, r+a)
\end{tikzcd}$$

where the vertical map $\mathrm{Gr}(n, r) \to \mathrm{Gr}(n+a, r+a)$ is

$$(V \twoheadrightarrow Q) \mapsto (V \oplus W \twoheadrightarrow Q \oplus h(W)).$$

Here we denoted the fiber of the vector bundle $\mathcal{O}_U^a$ by W and (by a slight abuse of notation) the induced map by h. This gives a morphism between the closures $\mathrm{Bl}_{\mathcal{G}} X \to \mathrm{Bl}_{\mathcal{F}} X$.

$\square$

Corollary 1.44. Let $\mathcal{F}$ and $\mathcal{G}$ sheaves of ranks $r+a$ and r on an integral scheme X and $\mathcal{O}^n \to \mathcal{F}$ and $\mathcal{O}^{n+a} \to \mathcal{G}$ surjective morphisms. Suppose there exists $U \subset X$ an open subset and a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathcal{G}^\vee|_U \arrow[r] \arrow[d, shift left] & \mathcal{O}_U^n \arrow[d, shift left] \\
0 \arrow[r] & \mathcal{F}^\vee|_U \arrow[r] \arrow[u, shift left, "g"'] \arrow[d, shift left] & \mathcal{O}_U^{n+a} \arrow[u, shift left, "q"'] \arrow[d, shift left] \\
 & \mathcal{E}_U^\vee \arrow[r, "h"] \arrow[u, shift left] \arrow[d] & \mathcal{O}_U^a \arrow[u, shift left] \arrow[d] \\
 & 0 & 0
\end{tikzcd}$$

with the columns split short exact sequences and h an isomorphism. Then, we get an induced morphism

$$\mathrm{Bl}_{\mathcal{G}} X \to \mathrm{Bl}_{\mathcal{F}} X.$$

In particular, $\mathrm{Bl}_{\mathcal{G}} X$ is a desingularisation of $\mathcal{F}$.

Proof. This follows by dualising the statement in Proposition 1.43. Note that g is not the dual of f, but of its inverse.

Alternatively, one can copy the proof above. The only difference is that a map $\mathrm{Gr}(r,n) \to \mathrm{Gr}(r+a, n+a)$ is

$$(S \hookrightarrow V) \mapsto (g(S) \oplus W \hookrightarrow V \oplus h(W)). \quad \square$$

Remark 1.45. In general, $\mathrm{Bl}_{\mathcal{F}} X$ is not isomorphic to $\mathrm{Bl}_{\mathcal{F}^\vee} X$. For example, let X be a normal scheme and $\mathcal{F}$ an ideal sheaf. Then $\mathcal{F}^\vee$ is reflexive and it has rank one, so it is an invertible sheaf. This shows that $\mathrm{Bl}_{\mathcal{F}^\vee} X \simeq X$. If $\mathcal{F}$ is not locally free (see e.g. Example 1.99), then we have $\mathrm{Bl}_{\mathcal{F}} X \neq X$.

1.4 Diagonalization

The diagonalization process for certain coherent sheaves is introduced by Hu and Li in [HL11]. A coherent sheaf $\mathcal{F}$ on an integral Noetherian scheme X can locally be written as the cokernel of a morphism of locally free sheaves $\varphi\colon \mathcal{E}^{-1} \to \mathcal{E}^0$. Blowing up all the

Fitting ideals of $\mathcal{F}$ desingularizes both $\mathcal{F}$ and the kernel of φ, and makes the morphism φ diagonalizable (see Definition 1.46). We summarize the construction for schemes and its universal property and explore the possibility of finding a minimal blow-up which also desingularizes all the components of the abelian cone associated to $\mathcal{F}$. Applied to the moduli space of maps, this construction will be used in Section 1.7 to desingularize all the components of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. All schemes are assumed to be Noetherian and integral in this section.

1.4.1 Diagonalizable morphisms and diagonal sheaves

We recall the notion of diagonalizable morphism of locally free sheaves from [HL11] and introduce the notion of diagonal sheaf. We show that these two notions are equivalent in Proposition 1.51, in the sense that a morphism is diagonalizable if and only if its cokernel is diagonal.

Definition 1.46 (Diagonalizable morphism [HL11, Definition 3.2])**.** Let X be a scheme. A morphism $\varphi\colon \mathcal{O}_X^{\oplus p} \to \mathcal{O}_X^{\oplus q}$ is *diagonalizable* if there are direct sum decompositions by free sheaves

$$\mathcal{O}_X^{\oplus p} = G_0 \oplus \bigoplus_{i=1}^{\ell} G_i \quad \text{and} \quad \mathcal{O}_X^{\oplus q} = H_0 \oplus \bigoplus_{i=1}^{\ell} H_i \tag{1.9}$$

with $\varphi(G_i) \subseteq H_i$ for $0 \leq i \leq \ell$ such that

1 $\varphi \mid_{G_0} = 0$;

2 for every $1 \leq i \leq \ell$, there is an isomorphism $I_i : G_i \to H_i$;

3 the morphism $\varphi \mid_{G_i}\colon G_i \to H_i$ is given by $f_i I_i$ for some $0 \neq f_i \in \Gamma(\mathcal{O}_X)$;

4 $(f_{i+1}) \subsetneqq (f_i)$.

More generally, a morphism $\varphi\colon E^{-1} \to E^0$ of locally free sheaves on X is *locally diagonalizable* if X admits an open cover which trivializes E^{-1} and E^0 simultaneously and on which φ is diagonalizable.

Example 1.47. If $X = \operatorname{Spec}(R)$ for a principal ideal domain R, then every morphism $\varphi\colon R^p \to R^q$ is diagonalizable in the sense of Definition 1.46 and the diagonal form associated to φ is called the Smith normal form of φ.

We will be interested in the coherent sheaves arising as kernels and cokernels of such diagonalizable morphisms.

Proposition 1.48. Let X be a Noetherian integral scheme and let $\varphi\colon E^{-1} \to E^0$ be a locally diagonalizable morphism between locally free sheaves on X. Then $\ker(\varphi)$ is locally free.

Proof. The question is local, so we can assume that $E^{-1} = \mathcal{O}_X^{\oplus p}$ and $E^0 = \mathcal{O}_X^{\oplus q}$, and that they admit decompositions as in Equation (1.9). Then $\ker(\varphi) = G_0$ is free. $\square$

Definition 1.49. We say that a coherent sheaf $\mathcal{F}$ on a scheme X is *diagonal* if all the Fitting ideal sheaves $F_i(\mathcal{F})$ are locally principal.

Remark 1.50. By Theorem 1.29, if $\mathcal{F}$ is diagonal then $\mathcal{F}^{\mathrm{tf}} = \mathcal{F}/(\mathrm{tor}(\mathcal{F}))$ is locally free and $\mathcal{F}$ has tor-dimension at most 1.

Proposition 1.51. A morphism $\varphi : E^{-1} \to E^0$ of locally free sheaves is locally diagonalizable if and only if the coherent sheaf $\mathrm{Coker}(\varphi)$ is diagonal.

Proof. This result is contained in the proof of [HL11, Proposition 3.13]. Observe that the Fitting ideals $F_i(\mathcal{F})$ are just the determinantal ideals $\Delta_{(q-i)\times(q-i)}(\varphi)$, where $q = \mathrm{rk}(E^0)$. If φ is locally diagonalizable, take an open where it is of the form (1.9). Then the Fitting ideals of $\mathcal{F}$ are generated by products of the f_i's, so are principal in this open.

On the other hand, if $\mathcal{F}$ is diagonal, we can cover X by affine opens where all $F_i(\mathcal{F})$ are principal and where the E^i's are simultaneously trivialized. We quickly sketch how [HL11, Proposition 3.13] produces a decomposition as in (1.9), by possibly further restricting. The morphism φ is given by

$$\Gamma = (a_{i,j})$$

$i \in \{1, \dots, p\}$, $j \in \{1, \dots, q\}$. The Fitting ideal $F_{q-1}(\mathcal{F}) = \Delta_{1\times 1}(\Gamma)$ is principal if and only if, after further localization, there is an entry a_{i_0,j_0} which divides every other entry $a_{i,j}$. In that case, one can perform row and column operations to put Γ in the following form

$$\left(\begin{array}{c|ccc} a_{i_0,j_0} & 0 & \dots & 0 \\ \hline 0 & & & \\ \vdots & & \Gamma' & \\ 0 & & & \end{array}\right)$$

with Γ' a matrix of smaller size. The same argument works recursively since the remaining Fitting ideals of $\mathcal{F}$ and those of Γ' differ by the principal ideal (a_{i_0,j_0}). $\square$

Example 1.52. Any smooth curve X can be covered by affine open subschemes of the form $\mathrm{Spec}\,(R)$ with R a principal ideal domain. Therefore every coherent sheaf on X is locally diagonal and every morphism of locally free sheaves on X is locally diagonalizable.

We are interested in morphisms that transform a given coherent sheaf in a diagonal sheaf.

Definition 1.53. 1 Given a scheme X and a coherent sheaf $\mathcal{F}$, a *diagonalization* of $\mathcal{F}$ is a morphism $f : \widetilde{X} \to X$ such that $f^*\mathcal{F}$ is diagonal.

2 Given a scheme X and a morphism of locally-free sheaves $\varphi : E^{-1} \to E^0$, a *diagonalization* of φ is a morphism $f : \widetilde{X} \to X$ such that $f^*\varphi$ is locally diagonalizable and $\mathrm{rk}(\mathrm{Coker}(\varphi)) = \mathrm{rk}(f^*\mathrm{Coker}(\varphi))$.

Remark 1.54. From Proposition 1.51, we see that diagonalizing a coherent sheaf $\mathcal{F}$ is equivalent to diagonalizing any presentation $E^{-1} \to E^0 \twoheadrightarrow \mathcal{F}$ by locally free sheaves.

Remark 1.55. If $\varphi : E^{-1} \to E^0$ is a locally diagonalizable morphism on a scheme X, and $f : Y \to X$ is any morphism of Noetherian schemes, $f^*\varphi$ is locally diagonalizable. Similarly, if $\mathcal{F}$ is diagonal, $f^*\mathcal{F}$ is diagonal. Note that the generic ranks of $\mathcal{F}$ and $f^*\mathcal{F}$ will be different in general for non-dominant morphisms.

1.4.2 Construction of the Hu-Li blow-up

We recall the construction of the minimal diagonalization of a sheaf, introduced in [HL11], which we call the Hu-Li blow-up of a scheme along a sheaf.

Definition 1.56 (Maximal rank)**.** Let X be an integral Noetherian scheme and $\mathcal{F}$ a coherent sheaf of generic rank r. The maximal rank $\mathrm{mrk}(\mathcal{F})$ is

$$\mathrm{mrk}(\mathcal{F}) = \max_{p \in X}\{\mathrm{rk}(\mathcal{F}|_p)\} \geq r$$

which is the maximum rank of $\mathcal{F}$ when restricted to a closed point of $p \in X$. Equivalently, $\mathrm{mrk}(\mathcal{F})$ is such that the Fitting ideals $F_{\mathrm{mrk}(\mathcal{F})}(\mathcal{F})$ is $\mathcal{O}_X$ and $F_{\mathrm{mrk}(\mathcal{F})-1}(\mathcal{F}) \neq \mathcal{O}_X$, with the convention that $F_{-1}(\mathcal{F}) = 0$.

Remark 1.57. The above $\mathrm{mrk}(\mathcal{F})$ is finite. Indeed, the ascending chain condition on the Fitting ideals

$$F_{-1}(\mathcal{F}) \subset F_0(\mathcal{F}) \subset \cdots \subset F_n(\mathcal{F})$$

guarantees that there is some $\mathrm{mrk}(\mathcal{F})$ such that $F_{\mathrm{mrk}(\mathcal{F})}(\mathcal{F}) = F_{\mathrm{mrk}(\mathcal{F})+1}(\mathcal{F}) = \dots$.

Moreover, for any affine open $U \subset X$, the ascending chain of Fitting ideals stabilizes at $\mathcal{O}_U$, since $\mathcal{F}|_U = \widetilde{M}$ for a finitely generated module M. So the chain above must stabilize at $\mathcal{O}_X$.

Remark 1.58. There is a closed point $q \in X$ such that $\mathrm{rk}(\mathcal{F}|_q) = \mathrm{mrk}(\mathcal{F})$, and such that we have a resolution

$$\mathcal{O}_q^{\oplus p} \to \mathcal{O}_q^{\oplus \mathrm{mrk}(\mathcal{F})} \to \mathcal{F}|_q \to 0.$$

However, $\mathcal{F}$ may not be generated globally by $\mathrm{mrk}(\mathcal{F})$ sections. Indeed, it may not be globally generated at all!

Construction 1.59 (Hu–Li blow-up)**.** Let X be an integral Noetherian scheme and $\mathcal{F}$ a coherent sheaf of general rank r and maximal rank r_2. Recall from Section 1.3.3 that the Fitting ideals of $\mathcal{F}$ satisfy a chain of inclusions $F_{-1}(\mathcal{F}) \subseteq F_0(\mathcal{F}) \subseteq \dots$, that $F_{r_2}(\mathcal{F}) = \mathcal{O}_X$ and that $F_0(\mathcal{F}) = \dots = F_{r-1}(\mathcal{F}) = 0$ because $\mathcal{F}$ has rank r.

Let

$$\mathrm{Bl}^{HL}_{\mathcal{F}} X = \mathrm{Bl}_{F_r(\mathcal{F})\cdot\ldots\cdot F_{r_2-1}(\mathcal{F})} X,$$

with the convention that $\mathrm{Bl}^{HL}_{\mathcal{F}} X = X$ if $r = r_2$. By [Sta22, Lemma 080A], $\mathrm{Bl}^{HL}_{\mathcal{F}} X$ can also be constructed by successively blowing up X along (the total transforms of) the Fitting ideals of $\mathcal{F}$, that is

$$\mathrm{Bl}^{HL}_{\mathcal{F}} X = X_r \xrightarrow{p_r} \dots \longrightarrow X_{r_2-2} \xrightarrow{p_{r_2-2}} X_{r_2-1} \xrightarrow{p_{r_2-1}} X$$

where

- $X_{r_2-1} = \mathrm{Bl}_{F_{r_2-1}(\mathcal{F})} X$,
- $X_{r_2-2} = \mathrm{Bl}_{F_{r_2-2}(p^*_{r_2-1}\mathcal{F})} X_{r_2-1} = \mathrm{Bl}_{p^{-1}_{r_2-1} F_{r_2-2}(\mathcal{F}) \mathcal{O}_{X_{r_2-1}}} X_{r_2-1}$ and
- $X_i = \mathrm{Bl}_{F_i(p_i^* p_{i+1}^* \cdots p^*_{r_2-1}\mathcal{F})} X_{i+1}$ for all i with $r \leq i \leq r_2 - 2$.

Each p_i is the natural morphism coming from the blow-up construction and we denote by p the composition $p_{r_2-1} \circ \dots \circ p_r$.

1.4.3 The universal property of the Hu-Li blow-up

We are now ready to state the minimality properties for Construction 1.59. We can formulate a universal property for the morphism φ or, in light of Remark 1.54, we can formulate it to only depend on the cokernel sheaf $\mathcal{F}$.

Theorem 1.60 (Universal property of $\mathrm{Bl}^{HL}_{\mathcal{F}}X$ [HL11])**.** Let X be a Noetherian integral scheme and $\mathcal{F}$ a coherent sheaf on X of generic rank r. The natural projection $p\colon \mathrm{Bl}^{HL}_{\mathcal{F}}X \to X$ satisfies that

1 the sheaf $p^*\mathcal{F}$ has generic rank r and

2 the Fitting ideal $F_i(p^*\mathcal{F})$ is locally principal for all i.

Moreover, $p\colon \mathrm{Bl}^{HL}_{\mathcal{F}}X \to X$ satisfies the following universal property: for any morphism $f\colon Y \to X$ of Noetherian integral schemes such that

1 the sheaf $f^*\mathcal{F}$ has generic rank r and

2 the Fitting ideal $F_i(f^*\mathcal{F})$ is locally principal for all i,

there is a unique morphism $f'\colon Y \to \mathrm{Bl}^{HL}_{\mathcal{F}}X$ factoring f.

$$\begin{tikzcd} Y \arrow[r, dashed, "\exists! f'"] \arrow[dr, "f"'] & \mathrm{Bl}^{HL}_{\mathcal{F}}X \arrow[d, "p"] \\ & X \end{tikzcd}$$

Proof. Let r_2 denote the maximal rank of $\mathcal{F}$. Then $r = r_2$ if and only if $\mathcal{F}$ is locally free, in which case $\mathrm{Bl}^{HL}_{\mathcal{F}}X = X$ clearly has this property.

Otherwise, we must have $r_2 \geq r$, so $\mathrm{Bl}^{HL}_{\mathcal{F}}X$ is defined by the sequence of blow-ups in Construction 1.59. By construction, p is dominant and each Fitting ideal of $p^*\mathcal{F}$ is locally principal. The universality follows from the universal property of the usual blow-up as in [Har77, Proposition 7.14], using that $F_i(f^*\mathcal{F})$ is non-zero for all $r \leq i \leq r_2$, which holds by the assumption that $f^*\mathcal{F}$ and $\mathcal{F}$ have the same generic rank. □

Theorem 1.61 (Universal property of diagonalization [HL11])**.** Let $\varphi : E^{-1} \to E^0$ be a morphism of locally-free sheaves on a Noetherian integral scheme X. Let $\mathcal{F} = \mathrm{Coker}(\varphi)$ and let $\mathrm{Bl}^{HL}_{\mathcal{F}}X$ as in Construction 1.59. Then, the natural projection $p\colon \mathrm{Bl}^{HL}_{\mathcal{F}}X \to X$ is a diagonalization of φ. Moreover, p satisfies the following universal property: for any morphism $f : Y \to X$ such that $f^*\varphi$ is locally diagonalizable and $\mathrm{rk}(f^*\mathcal{F}) = \mathrm{rk}(\mathcal{F})$, there is a unique morphism $f' : Y \to \mathrm{Bl}^{HL}_{\mathcal{F}} X$ factoring f.

$$\begin{tikzcd} Y \arrow[r, dashed, "\exists ! f'"] \arrow[dr, "f"'] & \mathrm{Bl}^{HL}_{\mathcal{F}} X \arrow[d, "p"] \\ & X \end{tikzcd}$$

Proof. Follows immediately from Theorem 1.60 and Proposition 1.51. □

1.4.4 Properties of the Hu–Li blow up.

We collect properties of $\mathrm{Bl}^{HL}_{\mathcal{F}} X$.

Proposition 1.62. Let $f : Y \to X$ be a morphism of Noetherian integral schemes and let $\mathcal{F}$ be a coherent sheaf on X of generic rank r. If $f^*\mathcal{F}$ has generic rank r then there is a unique morphism

$$\begin{tikzcd} \mathrm{Bl}^{HL}_{f^*\mathcal{F}} Y \arrow[r, dashed, "\exists ! \widetilde{f}"] \arrow[d] & \mathrm{Bl}^{HL}_{\mathcal{F}} X \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

making the diagram commute. If, moreover, f is flat, then the square is Cartesian.

Proof. A unique morphism $\widetilde{f}$ making the diagram commute exists by the universal property, Theorem 1.61. To see that the diagram is Cartesian, we apply [Sta22, Lemma 0805] to each of the blow-ups defining $\mathrm{Bl}^{HL}_{\mathcal{F}} X$, using that the formation of Fitting ideals is compatible with pullbacks. □

Proposition 1.63. Let X be a Noetherian integral scheme, let $\mathcal{F}$ a coherent sheaf on X and let $\mathcal{L}$ be a line bundle on X. Then there is a unique isomorphism

$$\begin{tikzcd} \mathrm{Bl}^{HL}_{\mathcal{F}}(X) \arrow[rr, "\exists ! \widetilde{\phi}"] \arrow[dr, "p"'] & & \mathrm{Bl}^{HL}_{\mathcal{F}\otimes\mathcal{L}}(X) \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

which makes the diagram commute.

Proof. By Theorem 1.60, a unique factorization $\widetilde{\phi}$ of p through q exists if and only if $F_i(p^*(\mathcal{F} \otimes \mathcal{L}))$ is locally principal for all i, and this holds because $F_i(p^*\mathcal{F})$ is locally free for all i. Indeed, choose an open cover of X trivializing $\mathcal{L}$. The preimage by p of this cover induces a cover of $\mathrm{Bl}_{\mathcal{F}}(X)$ where $p^*(\mathcal{F}\otimes\mathcal{L}) \simeq p^*\mathcal{F}$. This shows that $F_i(p^*(\mathcal{F}\otimes\mathcal{L})) \simeq F_i(p^*\mathcal{F})$ locally, so $\widetilde{\phi}$ exists. The same argument shows there is a unique factorization of q through p, which must be the inverse of $\widetilde{\phi}$ by uniqueness. □

Proposition 1.64. Let X be a Noetherian integral scheme and $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ be coherent $\mathcal{O}_X$-modules. Assume that we have an exact sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$.

1 If the sequence is locally split and $\mathcal{E}$ is locally free, then there is an isomorphism $\mathrm{Bl}^{HL}_{\mathcal{F}} X \simeq \mathrm{Bl}^{HL}_{\mathcal{G}} X$.

2 If $\mathcal{G}$ is locally free, then there is an isomorphism $\mathrm{Bl}^{HL}_{\mathcal{F}} X \simeq \mathrm{Bl}^{HL}_{\mathcal{E}} X$.

Proof. It is enough to prove the statement locally, so we may assume that we have $\mathcal{F} \simeq \mathcal{E} \oplus \mathcal{G}$. With this, we have that

$$F_\ell(\mathcal{E} \oplus \mathcal{G}) = \sum_{k+k'=\ell} F_k(\mathcal{E})F_{k'}(\mathcal{G}) \tag{1.10}$$

by [Sta22, Lemma 07ZA]. If $\mathcal{G}$ is locally free, the sequence is locally split, therefore by symmetry it is enough to show one of the statements.

Without loss of generality, suppose that $\mathcal{G}$ is locally free, therefore

1 $F_{k'}(\mathcal{G}) = 0$ for all $k' < \mathrm{rk}(\mathcal{G})$ and

2 $F_{k'}(\mathcal{G}) = \mathcal{O}_X$ for all $k' \geq \mathrm{rk}(\mathcal{G})$

by [Sta22, Lemma 07ZD]. Combining Item 1 with Equation (1.10), if $\ell < \mathrm{rk}(\mathcal{G})$ then $F_\ell(\mathcal{F}) = 0$ because in the sum $k + k' = \ell$ we always have $0 \leq k' \leq \ell < \mathrm{rk}(\mathcal{G})$. On the other hand, for $\ell \geq \mathrm{rk}(\mathcal{G})$ we combine Equation (1.10), Items 1 and 2 and the chain of inclusions $F_0(\mathcal{E}) \subseteq F_1(\mathcal{E}) \subseteq \ldots$ to deduce that

$$F_\ell(\mathcal{F}) = F_\ell(\mathcal{E} \oplus \mathcal{G}) = \sum_{k+k'=\ell} F_k(\mathcal{E})F_{k'}(\mathcal{G}) = \sum_{0\leq k\leq \ell-\mathrm{rk}(\mathcal{G})} F_k(\mathcal{E})\,\mathcal{O}_X = F_{\ell-\mathrm{rk}(\mathcal{G})}(\mathcal{E}).$$

To sum up, we have shown that

$$F_\ell(\mathcal{F}) = F_\ell(\mathcal{E} \oplus \mathcal{G}) = \begin{cases} 0 & \text{if } \ell < \mathrm{rk}(\mathcal{G}) \\ F_{\ell-\mathrm{rk}(\mathcal{G})}(\mathcal{E}) & \text{if } \ell \geq \mathrm{rk}(\mathcal{G}). \end{cases}$$

This means that the collection of Fitting ideals of $\mathcal{F}$ and $\mathcal{E}$ agree, so $\mathrm{Bl}^{HL}_{\mathcal{F}} X \simeq \mathrm{Bl}^{HL}_{\mathcal{E}} X$. □

Proposition 1.65. Let X be a Noetherian integral scheme and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then for every positive integer n

$$\mathrm{Bl}^{HL}_{\mathcal{F}} X = \mathrm{Bl}^{HL}_{\mathcal{F}^{\oplus n}} X.$$

Proof. There is a natural morphism $\mathrm{Bl}^{HL}_{\mathcal{F}} X \to \mathrm{Bl}^{HL}_{\mathcal{F}^{\oplus n}} X$ over X. To see this, let $p\colon \mathrm{Bl}^{HL}_{\mathcal{F}} X \to X$ be the natural projection. Then $p^*\mathcal{F}$ is diagonal and it follows from Definition 1.46 that $p^*(\mathcal{F}^{\oplus n}) = (p^*\mathcal{F})^{\oplus n}$ is diagonalizable too. Then apply Theorem 1.61 to get the desired morphism.

Conversely, we show that there is a natural morphism $\mathrm{Bl}^{HL}_{\mathcal{F}^{\oplus n}} X \to \mathrm{Bl}^{HL}_{\mathcal{F}} X$ over X, which is enough to conclude the proof by the universal properties of both blow-ups. Remember that

$$\mathrm{Bl}^{HL}_{\mathcal{F}} X = \mathrm{Bl}_{\prod_\ell F_\ell(\mathcal{F})} X,$$

where the product is over all non-trivial Fitting ideals of $\mathcal{F}$, and similarly $\mathrm{Bl}^{HL}_{\mathcal{F}^{\oplus n}} X$ is the blow-up of X along $\prod_\ell F_\ell(\mathcal{F}^{\oplus n})$. By [Moo01], it suffices to show that $\prod_\ell F_\ell(\mathcal{F})$ divides a power of $\prod_\ell F_\ell(\mathcal{F}^{\oplus n})$ as fractional ideals. Actually, we show that every Fitting ideal $F_\ell(\mathcal{F}^{\oplus n})$ is a product of certain Fitting ideals $F_k(\mathcal{F})$, with each k appearing at least once as ℓ varies, and this is clearly enough.

By [BV88, Lemma 10.10], if A is any $\mathbb{Q}$-algebra, if $M = (a_{i,j})$ is any matrix with coefficients in A and if Δ_i denotes the ideal generated by all minors of M of size $i \times i$, then

$$\Delta_i \Delta_j \subseteq \Delta_{i+1}\Delta_{j-1}$$

whenever $i \leq j-2$. From this, we can conclude that if $\ell = ds + r$ with $r \in \{0, \ldots, s-1\}$

$$\sum_{j_1+\ldots+j_s=\ell} \prod_i \Delta_{j_i} = \Delta_d^{s-r}\Delta_{d+1}^r. \tag{1.11}$$

To conclude, remember that locally $\mathcal{F}$ is the cokernel of a morphism $\varphi\colon E^{-1} \to E^0$, that $F_i(\mathcal{F})$ is the ideal $\Delta_{r_2-i}(\varphi)$ of minors in φ of size $r_2 - i$, where $r_2 = \mathrm{rk}(E^0)$, and the expression for Fitting ideals of direct sums Equation (1.10). □

Example 1.66. Take $n = 2$ in Proposition 1.65. Then Equation (1.11) is equivalent to

$$F_\ell(\mathcal{F} \oplus \mathcal{F}) = \sum_{k+k'=\ell} F_k(\mathcal{F})F_{k'}(\mathcal{F}) = \begin{cases} F^2_{r_2-k} & \text{if } \ell = 2k \\ F_{r_2-k}F_{r_2-k-1} & \text{if } \ell = 2k+1 \end{cases}$$

where r_2 the maximal rank of $\mathcal{F}$ as in Construction 1.59.

Proposition 1.67. Let X be a Noetherian integral scheme and $\mathcal{F}$ a coherent sheaf on X. Then there is a natural morphism $\mathrm{Bl}^{HL}_{\mathcal{F}}X \to \mathrm{Bl}_{\mathcal{F}}X$.

Proof. Let $\pi\colon \mathrm{Bl}^{HL}_{\mathcal{F}}X \to X$ be the natural projection and let $r = \mathrm{rk}(\mathcal{F})$. By Theorem 1.34, it suffices to check that $(\pi^*\mathcal{F})^{\mathrm{tf}}$ is locally free of rank r. This can be checked locally. If $X = \mathrm{Spec}\,(R)$ for a local ring R, the result follows from Theorem 1.29. $\square$

1.4.5 The filtration of a diagonal sheaf

Given a diagonal sheaf $\mathcal{F}$ of generic rank 0, we construct a filtration $\mathcal{F}_\bullet$ such that $\mathcal{F}_i/\mathcal{F}_{i-1}$ is locally free on a Cartier divisor D_i (Construction 1.69). This filtration will be used in Section 1.6.3 to describe the irreducible components of the abelian cone of a diagonal sheaf (see Theorem 1.97).

Lemma 1.68. Let $\mathcal{F}$ be a diagonal coherent sheaf of generic rank r on a Noetherian integral scheme X. Then there is a short exact sequence

$$0 \to \mathcal{K} \to \mathcal{F} \to \mathcal{F}^{\mathrm{tf}} \to 0$$

with $\mathcal{F}^{\mathrm{tf}}$ locally free of rank r on X and $\mathcal{K}$ a diagonal coherent sheaf of generic rank 0.

Proof. Let $\mathcal{K} = \mathrm{tor}(\mathcal{F})$. Then $\mathcal{F}^{\mathrm{tf}}$ is locally free by Remark 1.50 and $\mathcal{K}$ is diagonal by the proof of Proposition 1.64. $\square$

Construction 1.69 (c.f. [Sta22, Tag 0ESU])**.** Let $\mathcal{F}$ be a diagonal coherent sheaf of generic rank zero and $\mathrm{mrk}(\mathcal{F}) = n$ on an integral scheme X. In the following, we construct an increasing filtration $\mathcal{F}_\bullet$:

$$\mathcal{F} = \mathcal{F}_n \supset \mathcal{F}_{n-1} \supset \cdots \supset \mathcal{F}_0 = 0$$

and effective Cartier divisors D_i such that for each i, the sheaf

$$\mathcal{F}_i/\mathcal{F}_{i-1}$$

is locally free of rank i on the closed locally principal subscheme defined by D_i.

Our formulation differs from the one in the reference, so we present the construction of the filtration in our context. We can work locally and assume that $\mathcal{F}$ has a presentation which is diagonalizable in the sense of Definition 1.46 that is $\varphi : \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus n}$ where φ is the diagonal matrix

$$\varphi = \operatorname{Diag}\left(\overbrace{f_1, \ldots, f_1}^{n_1}, \overbrace{f_2, \ldots, f_2}^{n_2}, \ldots, \overbrace{f_k, \ldots, f_k}^{n_k}\right)$$

with $n_1 + \cdots + n_k = n$ and non-zero f_i's satisfying $(f_{i+1}) \subsetneq (f_i)$. Note that locally $\mathcal{F}$ may not attain its maximal rank n, but we can always choose f_1 to be a unit to obtain a presentation of the correct rank.

Since $\mathcal{F}$ is diagonal, it has tor dimension at most 1 by Remark 1.50, therefore it admits a presentation by a *square* matrix φ.

Since we are working over a domain, $(f_{i+1}) \subseteq (f_i)$ is equivalent to $f_i | f_{i+1}$. We can define effective Cartier divisors $D_1, \ldots, D_n$ by taking ratios of successive entries of φ:

$$\begin{aligned} D_n &= F_{n-1} = (f_1) \\ D_i &= \left(\frac{\varphi_{n-i+1,n-i+1}}{\varphi_{n-i,n-i}}\right). \end{aligned}$$

In other words, D_i is the ideal generated by the ratio of the entries in position $n-i+1$ and $n-i$ in φ. Note that, while the generators of the ideals are only well-defined up to a unit, the ideals themselves are well-defined and do not depend on the chosen presentation of φ. In fact, they can be expressed as differences of the Fitting ideals of $\mathcal{F}$, which are independent of the chosen presentation.

The divisors D_i give closed locally principal subschemes of X, which are defined by (f_{k+1}/f_k) if $i = n - \sum_{j=1}^{k} n_k$ and are empty otherwise.

We define the increasing filtration of $\mathcal{F}_\bullet$ as follows. We set $\mathcal{F}_n := \mathcal{F}$, and define $\mathcal{F}_{n-1}$ as the cokernel of the morphism $\varphi' := \varphi/f_1$. That is,

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & (\mathcal{O}_X/(f_1))^{\oplus n} \arrow[r, "\sim"] & \mathcal{F}_n/\mathcal{F}_{n-1} \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{O}_X^{\oplus n} \arrow[r, "\varphi"] \arrow[u, two heads] & \mathcal{O}_X^{\oplus n} \arrow[r] \arrow[u, two heads] & \mathcal{F}_n \arrow[r] \arrow[u, two heads] & 0 \\ 0 \arrow[r] & \mathcal{O}_X^{\oplus n} \arrow[r, "\varphi'"] \arrow[u, hook, "\mathrm{id}"] & \mathcal{O}_X^{\oplus n} \arrow[r] \arrow[u, hook, "f_1 \cdot \mathrm{id}"] & \mathcal{F}_{n-1} \arrow[r] \arrow[u, hook] & 0. \end{tikzcd} \tag{1.12}$$

As $\mathcal{O}_{D_1} = \mathcal{O}_X/(f_1)$, the graded piece $\mathcal{F}_n/\mathcal{F}_{n-1}$ is locally free of rank n on D_n.

Now, φ' can be given by the diagonal matrix

$$\varphi' = \mathrm{Diag}\left(\overbrace{1,\ldots,1}^{n_1}, \overbrace{f_2/f_1,\ldots,f_2/f_1}^{n_2},\ldots,\overbrace{f_k/f_1,\ldots,f_k/f_1}^{n_k}\right).$$

We can pass to $\varphi'' : \mathcal{O}_X^{\oplus n-1} \to \mathcal{O}_X^{\oplus n-1}$ by removing the first entry. Clearly, $\mathcal{F}_{n-1} = \mathrm{Coker}\varphi''$. Then we can iterate the construction in (1.12), factoring our multiplication by the first entry φ''_1 of φ''

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] & (\mathcal{O}_X/(\varphi''_1))^{\oplus n-1} \arrow[r, "\sim"] & \mathcal{F}_{n-1}/\mathcal{F}_{n-2} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_X^{\oplus n-1} \arrow[r, "\varphi''"] \arrow[u, two heads] & \mathcal{O}_X^{\oplus n-1} \arrow[r] \arrow[u, two heads] & \mathcal{F}_{n-1} \arrow[r] \arrow[u, two heads] & 0 \\
0 \arrow[r] & \mathcal{O}_X^{\oplus n-1} \arrow[r, "\varphi'''"] \arrow[u, hook, "\mathrm{id}"] & \mathcal{O}_X^{\oplus n-1} \arrow[r] \arrow[u, hook, "\varphi''_1\cdot\mathrm{id}"] & \mathcal{F}_{n-2} \arrow[r] \arrow[u, hook] & 0.
\end{tikzcd}$$

This defines the next subsheaf $\mathcal{F}_{n-2}$ in the filtration and the new morphism φ'''. If $n_1 > 1$, $(\varphi''_1) = (1)$, so we will have $\mathcal{F}_{n-2} = \mathcal{F}_{n-1}$ and D_{n-1} defining the empty subscheme. Note that the sub-schemes defined by $D_{n-1},\ldots,D_{n-n_1+1}$ are all empty, and the filtration is constant until $\mathcal{F}_{n-n_1-1}$, which is the cokernel of

$$\mathcal{O}_X^{\oplus n-n_1} \xrightarrow{\psi} \mathcal{O}_X^{\oplus n-n_1}$$

with

$$\psi = \mathrm{Diag}\left(\overbrace{1,\ldots,1}^{n_2}, \overbrace{f_3/f_2,\ldots,f_3/f_2}^{n_3},\ldots,\overbrace{f_k/f_2,\ldots,f_k/f_2}^{n_k}\right)$$

and $D_{n-n_1} = (f_1/f_2)$. Iterating this construction clearly provides a filtration and a collection of effective divisors which satisfy the claims in the lemma.

The divisors D_i are defined globally in terms of Fitting ideals, and do not depend on the local expression of the matrix. In fact, unpacking the argument above, we can check that

$$\begin{pmatrix} F_0 \\ \vdots \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ & 1 & 2 & \cdots & n-1 \\ & & \ddots & \ddots & \vdots \\ & & & & \\ & & & 1 & 2 \\ & & & & 1 \end{pmatrix} \begin{pmatrix} D_1 \\ \vdots \\ D_n \end{pmatrix}. \tag{1.13}$$

Then,

$$\begin{pmatrix} D_1 \\ \vdots \\ D_n \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \\ & & & & 1 \end{pmatrix} \begin{pmatrix} F_0 \\ \vdots \\ F_{n-1} \end{pmatrix}. \tag{1.14}$$

With the notations of Construction 1.69, the associated graded sheaf to this filtration is

$$\mathcal{E} = \bigoplus_i \mathcal{E}_i,$$

where $\mathcal{E}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$.

We present an example to illustrate Construction 1.69.

Example 1.70. Take $R = \mathbb{C}[x,y,z]$, $X = \operatorname{Spec} R$. Let $\mathcal{F} = \widetilde{M}$ be the diagonal sheaf defined by

$$0 \to R^{\oplus 4} \xrightarrow{\varphi = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & xy & 0 \\ 0 & 0 & 0 & xyz \end{pmatrix}} R^{\oplus 4} \to M \to 0.$$

The divisors from the statement of Construction 1.69 are given by the ideals

$$D_4 = (x), D_3 = (1), D_2 = (y), D_1 = (z).$$

As a sanity check for Equation (1.13), we see that indeed

$$\begin{aligned}
F_0 &= D_1 + 2D_2 + 3D_3 + 4D_4 = (x^4y^2z),\\
F_1 &= D_2 + 2D_3 + 3D_4 = (x^3y),\\
F_2 &= D_3 + 2D_4 = (x^2y),\\
F_3 &= D_4 = (x).
\end{aligned}$$

Now, all the elements of φ are divisible by D_4, which is the ideal generated by the first entry. We set $\mathcal{F}_4 = \mathcal{F}$. To obtain the next step in the filtration, $\mathcal{F}_3$, we consider the decomposition $\varphi = x \cdot \varphi'$ below

$$\begin{tikzcd}
0 \arrow[r] & R^{\oplus 4} \arrow[r, "\varphi"] & R^{\oplus 4} \arrow[r] & M \arrow[r] & 0 \\
0 \arrow[r] & R^{\oplus 4} \arrow[r, "\varphi'"] \arrow[u, "\mathrm{id}"] & R^{\oplus 4} \arrow[r] \arrow[u, "x\cdot\mathrm{id}"] & M_3 \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

we set $\mathcal{F}_3 = \widetilde{M_3}$, the module defined by

$$\varphi' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & yz \end{pmatrix}$$

or equivalently as the cokernel of

$$R^{\oplus 3} \xrightarrow{\varphi' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & yz \end{pmatrix}} R^{\oplus 3}.$$

Similarly, $\mathcal{F}_2 = \widetilde{M_2}$ will be defined by

$$0 \to R^{\oplus 2} \xrightarrow{\begin{pmatrix} y & 0 \\ 0 & yz \end{pmatrix}} R^{\oplus 2} \to M_2 \to 0$$

and $\mathcal{F}_1 = \widetilde{M_1}$ by

$$0 \to R \xrightarrow{(z)} R \to M_1 \to 0.$$

Finally, $M_0 = 0$. In conclusion, we obtain the filtration

$$M = M_4 = (R/(x))^{\oplus 2} \oplus R/(xy) \oplus R/(xyz) \xhookleftarrow{\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ x & 0 \\ 0 & x \end{pmatrix}} M_3 = R/(y) \oplus R/(yz) \cong$$

$$\cong M_2 = R/(y) \oplus R/(yz) \xhookleftarrow{\begin{pmatrix} 0 & y \end{pmatrix}} M_1 = R/(z) \leftarrow 0 = M_0.$$

The graded pieces are

$$\begin{aligned} \mathcal{E}_4 &= \mathcal{F}_4/\mathcal{F}_3 = (\widetilde{R/(x)})^{\oplus 4} \\ \mathcal{E}_3 &= \mathcal{F}_3/\mathcal{F}_2 = 0 \\ \mathcal{E}_2 &= \mathcal{F}_2/\mathcal{F}_1 = (\widetilde{R/(y)})^{\oplus 2} \\ \mathcal{E}_1 &= \widetilde{R/(z)} \end{aligned}$$

and each of the sheaves $\mathcal{E}_i$ is locally free of rank i on the subscheme defined by D_i. Note that, for $i = 3$, such subscheme is empty.

Theorem 1.71. Let $\mathcal{F}$ be a diagonal coherent sheaf of generic rank r and maximal rank r_2 on a Noetherian integral scheme X. Then we have a filtration

$$\mathcal{F} \supset \mathcal{K} = \mathcal{K}_{r_2 - r} \supset \mathcal{K}_{r_2 - r - 1} \supset \cdots \supset \mathcal{K}_0 = 0$$

such that

$$\mathcal{F}^{\text{tf}} = \mathcal{F}/\mathcal{K}$$

is locally free of rank r and

$$\mathcal{E}_i = \mathcal{K}_i/\mathcal{K}_{i-1}$$

is locally free of rank i on the effective Cartier divisor D_i.

Proof. Immediate by Lemma 1.68 and Construction 1.69. $\square$

1.4.6 Remarks on minimality

We saw in Remark 1.35 that blowing up the first non-zero Fitting ideal of $\mathcal{F}$ is, in general, not the minimal way to make $\mathcal{F}^{\mathrm{tf}}$ locally free. Similarly, blowing up all the Fitting ideals of $\mathcal{F}$ is not the minimal way to turn $(\mathcal{F}\mid_{D_i})^{\mathrm{tf}}$ into locally free sheaves for all i. This is illustrated in the following examples.

Example 1.72. Take $X = \mathrm{Spec}\,(R)$ and $\mathcal{F} = \widetilde{M}$ for $M = R/I$ with $I \subset R$ a non-principal ideal. The only non-trivial Fitting ideal of $\mathcal{F}$ is $F_0(\mathcal{F}) = I$. Note that $M^{\mathrm{tf}} = 0$ is locally free and $M\mid_{V(I)}$ is locally free of rank 1. This means that $\mathcal{F}$ already has the desired property on X. However, blowing up all the Fitting ideals of $\mathcal{F}$ results in $\mathrm{Bl}_I(X)$, which is isomorphic to X if and only if I is invertible.

This example also shows that given a coherent sheaf $\mathcal{F}$ on an integral scheme X, the blow up $\mathrm{Bl}_{\mathcal{F}}(X)$ from Definition 1.25 and $\mathrm{Bl}^{HL}_{\mathcal{F}}X$ are different in general. Indeed, on the one hand, $\mathrm{Bl}_{\mathcal{F}}(X) = \mathrm{Bl}_M(X) = X$ because $M^{\mathrm{tf}} = 0$ is locally free. On the other hand, the only non-trivial Fitting ideal of M is $F_1(M) = I$, so $\mathrm{Bl}^{HL}_{\mathcal{F}}X = \mathrm{Bl}_I X$. Therefore, both blow ups agree if and only if I is invertible.

Example 1.73. Let P be the origin in $\mathbb{A}^2_{x,y}$ and consider the embedding $i\colon \mathbb{A}^2_{x,y} \to \mathbb{A}^3_{x,y,z}$ given by $i(x,y) = (x,y,0)$. The image of i is the plane $\Pi = \{z = 0\}$. Let $I = (x,y)$ be the ideal of P in $\mathbb{A}^2$ and consider the module $M = i_* I$ in $\mathbb{A}^3$.

Note that M has generic rank 0, $M^{\mathrm{tf}} = 0$ and that $M\mid_{\Pi}$ is the ideal I, which is torsion-free. This means that $M^{\mathrm{tf}} = 0$ is already locally free, but $(M\mid_{\Pi})^{\mathrm{tf}} = M\mid_{\Pi} = I$ is not locally free over Π.

To compute $\mathrm{Bl}^{HL}_M\mathbb{A}^3$, we start with the following resolution of M

$$R^3 \xrightarrow{\Gamma} R^2 \longrightarrow M \longrightarrow 0 \tag{1.15}$$

where $R = \mathbb{C}[x,y,z]$ and

$$\Gamma = \begin{pmatrix} y & z & 0 \\ -x & 0 & z \end{pmatrix}.$$

The Fitting ideals of M are

- $F_0(M) = z(x,y,z)$,
- $F_1(M) = (x,y,z)$,
- $F_n(M) = R$, for all $n \geq 3$

This reflects the fact that M has rank 0 on $\mathbb{A}^3 \setminus \Pi$, rank 1 on $\Pi \setminus P$ and rank 2 on P, as per Proposition 1.6. Then

$$\mathrm{Bl}^{HL}_M \mathbb{A}^3 = \mathrm{Bl}_{F_0(M)\cdot F_1(M)}\mathbb{A}^3 = \mathrm{Bl}_{z(x,y,z)^2}\mathbb{A}^3 = \mathrm{Bl}_{(x,y,z)}\mathbb{A}^3 = \mathrm{Bl}_P\mathbb{A}^3$$

is simply the blow-up of the origin in $\mathbb{A}^3$. Note that $\mathrm{Bl}^{HL}_M\mathbb{A}^3$ is distinct from $\mathrm{Bl}_M\mathbb{A}^3 = \mathbb{A}^3$ in this example, as the latter does not flatten $(M\mid_\Pi)^{\mathrm{tf}}$.

Remark 1.74. If the sheaf $\mathcal{F}$ has projective dimension ≤ 1, then the Rossi construction is equal to the blow-up of the first non-zero Fitting ideal. In this case, blowing up all the proper non-zero ideals as in the Hu–Li construction gives a minimal resolution with the property that $(\mathcal{F}\mid_{D_i})^{\mathrm{tf}}$ is locally free for all of the D_i's defined in terms of Fitting ideals by (1.14). For an ideal, having projective dimension 1 is equivalent to being principal.

Remark 1.75 (Extension of sheaves)**.** Let X be a scheme, let Y be a closed subscheme and let $\mathcal{F}_Y$ be a coherent sheaf of rank r on Y. In order to find a minimal resolution of this torsion sheaf, one may try to extend $\mathcal{F}_Y$ to X as a sheaf which not a torsion sheaf and perform a repeated Rossi construction. One can find an open cover of X and blow-ups of the charts such that on the blow-up the torsion-free part of the pullback of $\mathcal{F}$ is locally free on the support. However, the blown up charts may not glue to a global construction. Below, we explain that it is always possible to find *local* blow-ups.

Let X be an affine scheme and let Y be a closed subscheme. Let $\mathcal{F}_Y$ be a coherent sheaf of rank r on Y and assume that we have an exact sequence

$$\mathcal{O}_{U\cap Y}^{\oplus n-r} \xrightarrow{\widehat{M}} \mathcal{O}_{U\cap Y}^{\oplus n} \to \mathcal{F}_{U\cap Y} \to 0,$$

where $\widehat{M} \in M_{n,r-n}(\Gamma(O_Y))$. We may assume that $U = \operatorname{Spec} R$ and $Y \cap U = \operatorname{Spec} R/I$, where R is a ring and I is an ideal. Let $\widehat{M} = (\hat{f}_{ij})$, with $\hat{f}_{ij} \in R/I$. We now choose $f_{ij} \in R$ a lift of $\hat{f}_{ij}$ and we denote by M the matrix (f_{ij}). Then we have a morphism $\mathcal{O}_U^{\oplus n-r} \xrightarrow{M} \mathcal{O}_U^{\oplus n}$ and an exact sequence

$$\mathcal{O}_U^{\oplus n-r} \xrightarrow{M} \mathcal{O}_U^{\oplus n} \to \mathcal{F}_U \to 0,$$

where $\mathcal{F}_U$ denotes the cokernel of the map induced by M. Then, we have that $\mathcal{F}_U|_Y = \mathcal{F}_Y$ and the resolution above induces a morphism

$$U \dashrightarrow U \times \mathrm{Gr}(r,n).$$

Since there is no canonical choice for the lift M, the above morphisms do not glue in general.

1.5 Desingularization and diagonalization on stacks

In this section, we show that the constructions introduced so far in Section 1.3 and Section 1.4 make sense for algebraic stacks, since they are both local and they commute with flat base-change (see respectively Proposition 1.41 and Proposition 1.62). Thus, we define the desingularization and the diagonalization of a coherent sheaf on a Noetherian integral Artin stack, and we establish properties of both constructions.

1.5.1 Construction of $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{P}$ and $\mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P}$

So far, we have only constructed $\mathrm{Bl}_{\mathcal{F}}X$ and $\mathrm{Bl}^{HL}_{\mathcal{F}}X$ for an affine (Noetherian integral) scheme X. We need to generalize this construction and its universal properties to the cases where X is a scheme and, more generally, an algebraic space before we can proceed with Artin stacks. However, the arguments all run in the same way. So we assume below that we have taken a presentation of an algebraic space by affine schemes and have obtained the aforementioned results for X an algebraic space.

Denote by $\mathfrak{P}$ a Noetherian, integral Artin stack. Consider a smooth presentation of $\mathfrak{P}$, i.e. a groupoid in algebraic spaces (U_0, U_1, s, t, m) whose associated quotient stack $[U_1 \rightrightarrows U_0]$ is $\mathfrak{P}$. Here $[U_1 \rightrightarrows U_0]$ denotes the stackyfication of a category fibered in groupoids $[U_1 \rightrightarrows U_0]^{\mathrm{pre}}$. Recall that U_0, U_1 are algebraic spaces $m : U_1 \,{}_s\!\times_t U_1 \to U_1$ is the composition of arrows, $s, t : U_1 \to U_0$ are respectively source and target morphism and they are smooth morphisms. They satisfy some compatibility conditions that we will not use explicitly here (see [LMB00, §(4.3)] or [Sta22, Definition 0441]).

The reader can think of $\mathfrak{P}$ being the Picard stack $\mathfrak{Pic}_{g,n}$. Recall, that a S-point of $\mathfrak{Pic}_{g,n}$ is a couple (C, L) where C is a nodal curve of genus g with n distinct smooth marked points and L is a line bundle over it. It is well known that $\mathfrak{Pic}_{g,n}$ is a smooth Noetherian Artin stack over $\mathrm{Spec}(\mathbb{C})$ which is not of finite type as we do not fix the degree of the line bundle.

Let $\mathfrak{F}$ be a coherent sheaf on $\mathfrak{P}$, i.e. we have a coherent sheaf $\mathcal{F}_0$ on U_0 and also a coherent sheaf $\mathcal{F}_1$ on U_1 with two fixed isomorphisms

$$s^*\mathcal{F}_0 \simeq \mathcal{F}_1 \simeq t^*\mathcal{F}_0 \tag{1.16}$$

that satisfy the cocycle condition on $U_1 \,{}_s\!\times_t U_1$. We refer to the article of Olsson [Ols07, Proposition 6.12] for the equivalent definitions of coherent sheaves on an Artin stack.

We now proceed to use the smooth presentation of $\mathfrak{P}$ to define a stack $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{P}$ desingularizing the coherent sheaf $\mathfrak{F}$. All of the following discussion holds formally identical when we consider the procedure that diagonalises $\mathfrak{F}$ instead. The stack we obtain with the second procedure is denoted $\mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P}$.

Later, we prove that the blow-up stacks obtained in both cases are algebraic and come equipped with a representable (by a scheme), proper and birational morphism to $\mathfrak{P}$.

With the theory developed in §1.3, we can construct $\mathrm{Bl}_{\mathcal{F}_1}U_1$ and $\mathrm{Bl}_{\mathcal{F}_0}U_0$. Note that to apply the results in that section we require U_0, U_1 to be integral Noetherian schemes and not merely algebraic spaces. However, it is routine to extend the construction to algebraic spaces, and we omit that passage here.

Since the morphisms $s, t : U_1 \to U_0$ are smooth (hence flat), we apply flat base-change for blow-up of sheaves (see Proposition 1.41) to s, t and we get $\widetilde{s}, \widetilde{t} : \mathrm{Bl}_{\mathcal{F}_1}U_1 \to \mathrm{Bl}_{\mathcal{F}_0}U_0$. Using the fix isomorphisms (1.16), we obtain the following Cartesian diagrams

$$
\begin{tikzcd}
\mathrm{Bl}_{\mathcal{F}_1}U_1 \arrow[r, "\widetilde{s}"] \arrow[d, "q"] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Bl}_{\mathcal{F}_0}U_0 \arrow[d, "p"] \\
U_1 \arrow[r, "s"] & U_0
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Bl}_{\mathcal{F}_1}U_1 \arrow[r, "\widetilde{t}"] \arrow[d, "q"] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Bl}_{\mathcal{F}_0}U_0 \arrow[d, "p"] \\
U_1 \arrow[r, "t"] & U_0.
\end{tikzcd}
\tag{1.17}
$$

In addition, using Cartesian diagram on a cube, we construct a map

$$\widetilde{m} : \mathrm{Bl}_{\mathcal{F}_1}U_1 \ {}_{\widetilde{s}}\!\times_{\widetilde{t}} \mathrm{Bl}_{\mathcal{F}_1}U_1 \to \mathrm{Bl}_{\mathcal{F}_1}U_1.$$

More precisely, we have

$$
\begin{aligned}
\mathrm{Bl}_{\mathcal{F}_1}U_1 \ {}_{\widetilde{s}}\!\times_{\widetilde{t}} \mathrm{Bl}_{\mathcal{F}_1}U_1 &\simeq \left(\mathrm{Bl}_{\mathcal{F}_0}U_0 \ {}_p\!\times_s U_1\right) \ {}_{\widetilde{s}}\!\times_{\widetilde{t}} \left(\mathrm{Bl}_{\mathcal{F}_0}U_0 \ {}_p\!\times_t U_1\right) \\
&\simeq \mathrm{Bl}_{\mathcal{F}_0}U_0 \ {}_p\!\times_s U_1 \ {}_s\!\times_t U_1 \\
&\to \mathrm{Bl}_{\mathcal{F}_0}U_0 \ {}_p\!\times_s U_1 \text{ by applying } m : U_1 \ {}_s\!\times_t U_1 \to U_1 \\
&\simeq \mathrm{Bl}_{\mathcal{F}_1}U_1 \text{ by the Cartesian diagram (1.17).}
\end{aligned}
$$

We obtain a smooth groupoid in algebraic spaces

$$(\mathrm{Bl}_{\mathcal{F}_0}U_0, \mathrm{Bl}_{\mathcal{F}_1}U_1, \widetilde{s}, \widetilde{t}, \widetilde{m})$$

with a morphism of groupoids to (U_0, U_1, s, t, m). This defines a 1-morphism

$$p : [\mathrm{Bl}_{\mathcal{F}_0}U_0 \rightrightarrows \mathrm{Bl}_{\mathcal{F}_1}U_1]^{\mathrm{pre}} \to [U_0 \rightrightarrows U_1]^{\mathrm{pre}}.$$

Let $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{P}$ denote the stackyfication of $[\mathrm{Bl}_{\mathcal{F}_0}U_0 \rightrightarrows \mathrm{Bl}_{\mathcal{F}_1}U_1]^{\mathrm{pre}}$. By universal property, the morphism discussed above lifts to a morphism of stacks

$$\pi : \mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} \to \mathfrak{P}.$$

1.5.2 Properties of $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{P}$ and $\mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P}$

We collect some properties of $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{P}$ and $\mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P}$, including those of Sections 1.3.5 and 1.4.4, which naturally extend to stacks.

Theorem 1.76. Let $[U_1 \rightrightarrows U_0] \to \mathfrak{P}$ be an integral Noetherian Artin stack and $\mathfrak{F}$ be a coherent sheaf on it.

1 The stacks $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} = [\mathrm{Bl}_{\mathcal{F}_1}U_1 \rightrightarrows \mathrm{Bl}_{\mathcal{F}_0}U_0]$ and $\mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P} = [\mathrm{Bl}^{HL}_{\mathcal{F}_1}U_1 \rightrightarrows \mathrm{Bl}^{HL}_{\mathcal{F}_0}U_0]$ are integral Noetherian Artin stacks.

2 The morphisms $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} \to \mathfrak{P}$ and $\mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P} \to \mathfrak{P}$ are representable proper and birational.

Remark 1.77. Let S be a scheme and $f : S \to \mathfrak{P}$ be a flat morphism. Then we have

$$\begin{tikzcd} \mathrm{Bl}_{f^*\mathfrak{F}}S \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} \arrow[d] \\ S \arrow[r, "f"] & \mathfrak{P} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Bl}^{HL}_{f^*\mathfrak{F}}S \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P} \arrow[d] \\ S \arrow[r, "f"] & \mathfrak{P} \end{tikzcd} \tag{1.18}$$

In this section, we choose to use an atlas of $\mathfrak{P}$ to define the blow-ups on Artin stacks, but it is also possible to use the diagram above and fppf descent to define them. Hence, the definition above does not depend on the choice of the atlas.

Notice that as the Hu–Li diagonalization is a repeated blow-up of Fitting ideals on schemes, so the same holds for stacks.

Proof of Theorem 1.76. Once again, we will only discuss $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{P}$, as the argument for $\mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P}$ is identical, mutatis mutandis.

Part (2), together with the properties of the respective constructions on schemes, implies part (1) of the theorem. To establish part (2), it suffices to compute the fiber $U_0 \times_{\mathfrak{P}} \mathrm{Bl}_{\mathfrak{F}}\mathfrak{P}$ and show that it is $\mathrm{Bl}_{\mathcal{F}_0}U_0$. A priori, $\mathrm{Bl}_{\mathcal{F}_0}U_0 \to U_0$ is representable by an algebraic space, so the morphism $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} \to \mathfrak{P}$ will be representable by an algebraic space [Sta22, Tag 045G]. However, running the argument below for a covering of an algebraic space by schemes, one can show in two steps that the morphism is in fact representable by a scheme. Now consider the 2-Cartesian diagram of categories fibered in groupoids

$$\begin{tikzcd} \mathfrak{X} \arrow[rr] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & & U_0 \arrow[d] \\ {[\mathrm{Bl}_{\mathcal{F}_1}U_1 \rightrightarrows \mathrm{Bl}_{\mathcal{F}_0}U_0]^{\mathrm{pre}}} \arrow[rr, "p"] & & {[U_1 \rightrightarrows U_0]^{\mathrm{pre}}} \end{tikzcd} \tag{1.19}$$

where, by abuse of notation, U_0 is the category fibered in sets associated to this algebraic space. One computes (see the discussion around [Sta22, Tag 04Y4]) that the groupoid $\mathfrak{X}$ is given by (U_0', U_1', s', t', m') where

$$\begin{aligned} U_0' &= U_1 \times_{t,U_0,p} \mathrm{Bl}_{\mathcal{F}_0}U_0 \\ U_1' &= U_1 \times_{t,U_0,p\cdot s} \mathrm{Bl}_{\mathcal{F}_1}U_1 \\ s' &: (x,y) \mapsto (x, \widetilde{s}(y)) \\ t' &: (x,y) \mapsto (m(x,p(y)), \widetilde{t}(y)) \end{aligned}$$

By (1.17),

$$\begin{aligned} U_1' &= U_1 \times_{t,U_0,p\cdot s} (U_1 \times_{s,U_0,p} \mathrm{Bl}_{\mathcal{F}_0}U_0) \\ &= (U_1 \times_{t,U_0,s} U_1) \times_{t\cdot pr_2,U_0,p} \mathrm{Bl}_{\mathcal{F}_0}U_0 \\ s' &: ((x,y),z) \mapsto (x,z) \\ t' &: ((x,y),z) \mapsto (y,z) \end{aligned}$$

From this expression, $\mathfrak{X}$ is a banal groupoid whose stackyfication is equivalent to the scheme $\mathrm{Bl}_{\mathcal{F}_0}U_0$, as the relations s', t' identify all the points of the U_1 factor. Then the stackyfication of (1.19)gives us a 2-Cartesian diagram:

$$\begin{tikzcd} \mathrm{Bl}_{\mathcal{F}_0}U_0 \arrow[r, "\widetilde{\pi}"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & U_0 \arrow[d] \\ \mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} \arrow[r, "\pi"] & \mathfrak{P} \end{tikzcd} \tag{1.20}$$

This discussion proves that π is representable. Recall that a morphism of stacks is birational if there exists an isomorphism on dense open substacks on source and target (see [CMW12]). By Proposition 1.27 we deduce that π is proper and birational. $\square$

Proposition 1.78. Let $\mathfrak{F}$ be a coherent sheaf on $\mathfrak{P}$.

1 For any line bundle $\mathfrak{L}$ on $\mathfrak{P}$, we have $\mathrm{Bl}_{\mathfrak{F}\otimes\mathfrak{L}}\mathfrak{P} = \mathrm{Bl}_{\mathfrak{F}}\mathfrak{P}$ and also $\mathrm{Bl}^{HL}_{\mathfrak{F}\otimes\mathfrak{L}}\mathfrak{P} = \mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P}$.

2 Let $\mathfrak{E}$, $\mathfrak{F}, \mathfrak{G}$ be coherent $\mathcal{O}_{\mathfrak{P}}$-modules. Assume that we have an exact sequence $0 \to \mathfrak{E} \to \mathfrak{F} \to \mathfrak{G} \to 0$.

2.1 If the sequence is locally split and $\mathfrak{E}$ is locally free, then there are isomorphisms $\mathrm{Bl}_{\mathfrak{F}}X \simeq \mathrm{Bl}_{\mathfrak{G}}X$ and $\mathrm{Bl}^{HL}_{\mathfrak{F}}X \simeq \mathrm{Bl}^{HL}_{\mathfrak{G}}X$.

2.2 If $\mathfrak{G}$ is locally free, then there are isomorphisms $\mathrm{Bl}_{\mathfrak{F}}X \simeq \mathrm{Bl}_{\mathfrak{E}}X$ and $\mathrm{Bl}^{HL}_{\mathfrak{F}}X \simeq \mathrm{Bl}^{HL}_{\mathfrak{E}}X$.

Proof. For the first part of 1, let $p' \colon \mathrm{Bl}_{\mathfrak{F}\otimes\mathfrak{L}}\mathfrak{P} \to \mathfrak{P}$ and $p \colon \mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} \to \mathfrak{P}$ be the natural projections. By the Universal Property, Theorem 1.79, to show that $\mathrm{Bl}_{\mathfrak{F}\otimes\mathfrak{L}}\mathfrak{P} = \mathrm{Bl}_{\mathfrak{F}}\mathfrak{P}$ it suffices to show that $((p')^*\mathfrak{F})^{\mathrm{tf}}$ and $(p^*(\mathfrak{F}\otimes\mathfrak{L}))^{\mathrm{tf}}$ are locally free. These statements can be checked locally and they follow from Proposition 1.28.

Similarly, all the other statements are local, so they follow from the same statements on schemes with the two different blow-ups. For the Rossi blow-up $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{P}$, the schematic statements are Propositions 1.28 and 1.42 and for the Hu–Li blow-up $\mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P}$, it follows from Propositions 1.63 and 1.64. □

In Definition 1.46, we define the notion of diagonalizable morphism of sheaves on a scheme, we can extend it directly to algebraic stacks.

Theorem 1.79 (Universal property of the Rossi desingularization)**.** Let $\mathfrak{P}$ be an integral Noetherian Artin stack and $\mathfrak{F}$ be a coherent sheaf on $\mathfrak{P}$ of generic rank r. Let $\pi : \mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} \to \mathfrak{P}$ be the morphism constructed in Section 1.5.1. Then

1 The sheaf $(\pi^*\mathfrak{F})^{\mathrm{tf}}$ is locally free of generic rank r.

2 The morphism $\pi : \mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} \to \mathfrak{P}$ satisfies the following universal property: For any morphism of algebraic stacks $p : \mathfrak{Y} \to \mathfrak{P}$ such that $(p^*\mathfrak{F})^{\mathrm{tf}}$ is locally-free of generic rank r, there is a unique[1] morphism p', which makes the following diagram 2-commutative

$$\begin{tikzcd} \mathfrak{Y} \arrow[r, dashed, "\exists! p'"] \arrow[dr, "p"'] & \mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} \arrow[d, "\pi"] \\ & \mathfrak{P} \end{tikzcd}$$

[1]To be precise, there exists a morphism p', unique up to a unique 2-morphism.

Proof. The statement follows from the universal property of the Rossi blow-up for a scheme (Proposition 1.40) and the compatibility of the Rossi blow-up with flat pullback (Proposition 1.41). By invoking fppf descent and because all our stacks are quasi-separated, it suffices to prove this for any flat morphisms from affine Noetherian integral schemes $f : S \to \mathfrak{P}$ and $T \to \mathfrak{Y}$.

Let $T \to \mathfrak{Y}$ be a flat morphism from an affine Noetherian integral scheme. We construct a morphism

$$\begin{tikzcd} & & \mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} \arrow[d, "p"] \\ T \arrow[r] \arrow[urr, dashed, "a"] \arrow[rr, bend right, "g"'] & \mathfrak{Y} \arrow[r] & \mathfrak{P}. \end{tikzcd}$$

Let S be an affine integral Noetherian scheme and let $f\colon S \to \mathfrak{P}$ be a flat representable morphism. Recall that we have a Cartesian diagram

$$\begin{tikzcd} \mathrm{Bl}_{f^*\mathfrak{F}}S \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} \arrow[d] \\ S \arrow[r, "f"] & \mathfrak{P} \end{tikzcd}$$

as in (1.18), coming from Proposition 1.41. Since $T \to \mathfrak{Y}$ is flat, we have that $(g^*\mathfrak{F})^{\mathrm{tf}}$ is locally free of rank r, we define T_S by the Cartesian diagram

$$\begin{tikzcd} T_S \arrow[r, "g_S"] \arrow[d, "b"] & S \arrow[d, "f"] \\ T \arrow[r, "g"] & \mathfrak{P}. \end{tikzcd}$$

Then

$$(g_S^* f^* \mathfrak{F})^{\mathrm{tf}} = (b^* g^* \mathfrak{F})^{\mathrm{tf}} = b^*((g^*\mathfrak{F})^{\mathrm{tf}})$$

because b is flat. So $(a^* f^* \mathfrak{F})^{\mathrm{tf}}$ is locally free of rank r. By Proposition 1.40, there is a unique morphism $a_S\colon T_S \to \mathrm{Bl}_{f^*\mathfrak{F}}S$ over g_S.

$$\begin{tikzcd}
 & \mathrm{Bl}_{f^*\mathfrak{F}}S \arrow[d] \arrow[dr] & \\
T_S \arrow[ur, dashed, "a_S"] \arrow[r, "g_S"] \arrow[d, "b"] & S \arrow[dr, "f"] & \mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} \arrow[d] \\
T \arrow[rr, "g"] & & \mathfrak{P}.
\end{tikzcd}$$

The result follows from fppf descent, using that the lift of a_S of g_S is unique. □

From the beginning of Section 1.4 (Definitions 1.46 and 1.49 and Proposition 1.51), we can define the notion of diagonal sheaves or locally diagonalizable morphism of sheaves on Artin stacks as follows.

Definition 1.80.
- A coherent sheaf $\mathfrak{F}$ on $\mathfrak{P}$ is *diagonal* if for any scheme S and morphism $f : S \to \mathfrak{P}$, the sheaf $f^*\mathfrak{F}$ is diagonal, that is, its Fitting ideals $F_i(f^*\mathfrak{F})$ are locally principal.
- A *diagonalization* of a coherent sheaf $\mathfrak{F}$ is a morphism $\pi : \widetilde{\mathfrak{P}} \to \mathfrak{P}$ such that $\pi^*\mathfrak{F}$ is diagonal.

Remark 1.81. Using the presentation of $\mathfrak{P}$, we could also define that $\mathfrak{F}$ is diagonal if $\mathcal{F}_0$ is.

Theorem 1.82 (Universal property of the diagonalization)**.** Let $\pi : \mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P} \to \mathfrak{P}$ be as above. Then

1 The sheaf $\pi^*\mathfrak{F}$ is diagonal of the same generic rank as $\mathfrak{F}$.

2 The blow-up $\mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P}$ satisfies the universal property: For any morphism of algebraic stacks $f : \mathfrak{Y} \to \mathfrak{P}$ such that $f^*\mathfrak{F}$ is diagonal of the same generic rank as $\mathfrak{F}$, there is a unique morphism f' which makes the following diagram 2-commutative:

$$\begin{tikzcd}
\mathfrak{Y} \arrow[r, dashed, "\exists! f'"] \arrow[dr, "f"'] & \mathrm{Bl}^{HL}_{\mathfrak{F}}\mathfrak{P} \arrow[d, "\pi"] \\
 & \mathfrak{P}
\end{tikzcd}$$

Proof. The statement follows from the universal property of the Hu-Li blow-up for schemes, Theorem 1.60, and the compatibility of the Hu-Li blow-up with flat pullback, Proposition 1.62. The argument is the same as the proof of Theorem 1.79. □

Remark 1.83. If $\mathfrak{P}$ has the resolution property in the sense of [Tot04], then we have that $\pi : \mathrm{Bl}_{\mathfrak{F}}\mathfrak{P} \to \mathfrak{P}$ is projective. Indeed, if $\mathfrak{P}$ has the resolution property, then we have a global locally free sheaf $\mathfrak{E}$ with a surjective morphism

$$\mathfrak{E} \to \mathfrak{F} \to 0.$$

This allows us to define $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{P}$ via the graph construction and thus the resulting stack is projective over $\mathfrak{P}$. Note that projectivity is not local on the target, and thus, even though the local construction is projective, π may not be projective. By [Tot04] stacks which are not global quotient stacks do not have the resolution property. Many of the stacks that we work with are not global quotients. For examples of such stacks see [Kre13].

1.6 Components of abelian cones

Let $\mathcal{F}$ be a diagonal sheaf on an integral Noetherian scheme X, we study the irreducible components of the abelian cone $C(\mathcal{F}) = \mathrm{Spec}\,(\mathrm{Sym}\,\mathcal{F})$ associated to $\mathcal{F}$. We show that $C(\mathcal{F})$ has finitely many irreducible components, which we consider with their natural reduced structure. Each irreducible component is a vector bundle supported on a closed integral subscheme. All the cones in this section are taken over X, unless otherwise specified by the notation $C_{\mathrm{base}}(\mathrm{sheaf})$.

1.6.1 The main component of an abelian cone

We start our study of components of cones with the main component of an abelian cone. Our study is motivated by [AM98, Proposition 2.5], which we recall below as Proposition 1.85. It states that if $\pi\colon C = \mathrm{Spec}\,(\mathcal{A}) \to X$ is a cone with X integral and with $\mathcal{A}$ torsion-free outside of a closed $Z \subseteq X$, then the closure of $C \setminus \pi^{-1}(Z)$ inside C is equal to $\mathrm{Spec}\,(\mathcal{A}^{\mathrm{tf}})$.

In general, $\mathrm{Spec}\,(\mathcal{A}^{\mathrm{tf}})$ need not be irreducible, see Example 1.86. However, if the cone is abelian, that is, if $\mathcal{A} = \mathrm{Sym}\,\mathcal{F}$ for a coherent sheaf $\mathcal{F}$, then $\mathrm{Spec}\,(\mathcal{A}^{\mathrm{tf}})$ is an irreducible component that we call the main component of $C(\mathcal{F}) = \mathrm{Spec}\,\mathrm{Sym}\,\mathcal{F}$. Note that $(\mathrm{Sym}\,\mathcal{F})^{\mathrm{tf}}$ and $\mathrm{Sym}\,(\mathcal{F}^{\mathrm{tf}})$ need not agree in general (see Remark 1.89), but they do if $\mathcal{F}^{\mathrm{tf}}$ is locally free by Lemma 1.91. In particular, they agree for diagonal sheaves by Remark 1.50.

Let X be an integral Noetherian scheme, let $\mathcal{A}$ be an $\mathcal{O}_X$-algebra with the assumptions of Definition 1.7 and let

$$\pi \colon C = \mathrm{Spec}_X(\mathcal{A}) \to X$$

be the cone associated to $\mathcal{A}$. The natural surjection $\mathcal{A} \to \mathcal{A}^{\mathrm{tf}}$ induces a closed embedding

$$\mathrm{Spec}\,(\mathcal{A}^{\mathrm{tf}}) \hookrightarrow \mathrm{Spec}\,(\mathcal{A}),$$

which we want to understand geometrically.

Notation 1.84. Let X be a scheme and let $U \subseteq X$ be an open subscheme. We denote by $\mathrm{cl}_X U$ the closure of U in X, with its reduced induced structure, and by $\mathrm{cl}_X^{\mathrm{sch}} U$ the schematic closure of U in X. If U is reduced, then $\mathrm{cl}_X^{\mathrm{sch}} U = \mathrm{cl}_X U$ by [Sta22, Lemma 056B]

The following result is proven in [AM98] in the analytic category, but the proof copies.

Proposition 1.85. ([AM98, Proposition 2.5]) Let $U \subseteq X$ be a non-empty open such that $\mathcal{A}\,|_U$ is torsion free. Then

$$\mathrm{Spec}_X(\mathcal{A}^{\mathrm{tf}}) = \mathrm{cl}_{\mathrm{Spec}_X(\mathcal{A})}(\pi^{-1}(U)).$$

Furthermore, if $\pi^{-1}(U)$ is reduced, then

$$\mathrm{Spec}_X(\mathcal{A}^{\mathrm{tf}}) = \mathrm{cl}^{\mathrm{sch}}_{\mathrm{Spec}_X(\mathcal{A})}(\pi^{-1}(U)).$$

In general, $\mathrm{Spec}_X(\mathcal{A}^{\mathrm{tf}})$ may not be irreducible, see Example 1.86.

Example 1.86. The abelian cone $\mathrm{Spec}_X(\mathcal{A}^{\mathrm{tf}})$ may not be irreducible. For example, let $R = \mathbb{C}[x]$ and let $A = R[Y, Z]/(YZ)$ viewed as a graded R-algebra with Y, Z in degree 1. This is a cone over $\mathbb{A}^1 = \mathrm{Spec}\,(R)$. It is clear that A has no torsion as an R-module but $\mathrm{Spec}\,(R)$ has two irreducible components. By Proposition 1.85, we see that

$$\mathrm{Spec}\,(\mathrm{Sym}\,\mathcal{F})^{\mathrm{tf}} = \mathrm{cl}^{\mathrm{sch}}_{C(\mathcal{F})}(\pi^{-1}(U))$$

is the closure of an irreducible subset, therefore it is irreducible.

Now we focus on abelian cones. Firsly, we show in Proposition 1.87 that if $\mathcal{A} = \mathrm{Sym}\,\mathcal{F}$, then $\mathrm{Spec}_X(\mathcal{A}^{\mathrm{tf}})$ is an irreducible component of $\mathrm{Spec}_X(\mathcal{A})$.

Proposition 1.87. Let X be an integral Noetherian scheme and let $\mathcal{F}$ be a coherent sheaf on X. Then $\mathrm{Spec}\,(\mathrm{Sym}\,\mathcal{F})^{\mathrm{tf}}$ is an irreducible component of $C(\mathcal{F}) = \mathrm{Spec}\,\mathrm{Sym}\,\mathcal{F}$.

Proof. Let $U \subseteq X$ be a non-empty open such that $\mathcal{F}$ is locally free on U. Then, for $\pi\colon C(\mathcal{F}) \to X$ the projection, we have that $\pi^{-1}(U)$ is a vector bundle over U, thus it is integral. Let Z be the unique irreducible component of $C(\mathcal{F})$ containing $\pi^{-1}(U)$. Then

$$\mathrm{cl}^{\mathrm{sch}}_{C(\mathcal{F})}(\pi^{-1}(U)) = \mathrm{cl}^{\mathrm{sch}}_{Z}(\pi^{-1}(U)) = Z,$$

where the first equality follows from Lemma 1.108 and the second one is a basic property of the Zariski topology that the closure of an irreducible open in an irreducible space is the whole space. $\square$

Definition 1.88. Let X be an integral Noetherian scheme and let $\mathcal{F}$ be a coherent sheaf on X. We say that $\operatorname{Spec}(\operatorname{Sym}\mathcal{F})^{\mathrm{tf}}$ is the *main component* of the abelian cone $C(\mathcal{F}) = \operatorname{Spec}\operatorname{Sym}\mathcal{F}$.

Remark 1.89. With he assumptions of Definition 1.88, it is not true in general that the main component of $C(\mathcal{F}) = \operatorname{Spec}\operatorname{Sym}\mathcal{F}$ is equal to $C(\mathcal{F}^{\mathrm{tf}}) = \operatorname{Spec}\operatorname{Sym}(\mathcal{F}^{\mathrm{tf}})$. In fact, $C(\mathcal{F}^{\mathrm{tf}})$ need not be irreducible (see Example 1.90). The underlying reason for this discrepancy is that Sym and torsion-free part do not commute in general (see Remark 1.92). A particular case where $C(\mathcal{F}^{\mathrm{tf}})$ is clearly irreducible is if $\mathcal{F}^{\mathrm{tf}}$ is locally free. In that case,

$$(\operatorname{Sym}\mathcal{F})^{\mathrm{tf}} = \operatorname{Sym}(\mathcal{F}^{\mathrm{tf}}).$$

by Lemma 1.91 and so

$$\operatorname{Spec}(\operatorname{Sym}\mathcal{F})^{\mathrm{tf}} = \operatorname{Spec}\operatorname{Sym}(\mathcal{F}^{\mathrm{tf}}). \tag{1.21}$$

In particular, the equality (1.21) is true for a diagonal sheaf $\mathcal{F}$ by Remark 1.50.

Example 1.90. This is an example of a torsion free sheaf $\mathcal{G}$ on an integral Noetherian scheme X such that $\operatorname{Spec}\operatorname{Sym}\mathcal{G}$ is not irreducible. Let $X = \operatorname{Spec}(\mathbb{C}[x,y])$ be the affine plane, let $I = (x,y)$ be the ideal of the origin 0 and let $M = I \oplus I$. Then $\operatorname{Sym}(M) \simeq \mathbb{C}[x,y,A_1,A_2,B_1,B_2]/(yA_1 - xA_2, yB_1 - xB_2)$, so $C(M) = \operatorname{Spec}\operatorname{Sym}(M)$ has two irreducible components: one of them is $V(A_2B_1 - A_1B_2, yB_1 - xB_2, yA_1 - xA_2)$, which is the closure of the restriction of $C(M)$ to $X \setminus 0$; and the other one is $V(x,y)$, the fibre of $C(M)$ at 0. Both components have dimension 4.

1.6.2 Abelian cones as a pushout of their main component

We particularize our study of components of cones to an abelian cone $C(\mathcal{F})$ with $\mathcal{F}^{\mathrm{tf}}$ locally free. We first show that Sym and torsion-free part commute in that case

(Lemma 1.91), therefore the main component is $C(\mathcal{F}^{\mathrm{tf}})$, which is also abelian. We show that $C(\mathcal{F})$ admits a description as a pushout with $C(\mathcal{F}^{\mathrm{tf}})$ as one of the factors (Proposition 1.95).

Lemma 1.91. Let X be a Noetherian scheme and let $\mathcal{F}$ be a coherent sheaf on X. If $\mathcal{F}^{\mathrm{tf}}$ is locally free then

$$(\mathrm{Sym}\,\mathcal{F})^{\mathrm{tf}} = \mathrm{Sym}\,(\mathcal{F}^{\mathrm{tf}}).$$

Proof. We have the following commutative diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathrm{tor}(\mathrm{Sym}\,\mathcal{F}) \arrow[r, "e"] \arrow[d, "i'"] & \ker(p) \arrow[r, "e'"] \arrow[d, "i"] & \ker(p'') \arrow[r] \arrow[d, "i''"] & 0 \\
0 \arrow[r] & \mathrm{tor}(\mathrm{Sym}\,\mathcal{F}) \arrow[r, "f"] \arrow[d, "p'"] & \mathrm{Sym}\,\mathcal{F} \arrow[r, "f'"] \arrow[d, "p"] & (\mathrm{Sym}\,\mathcal{F})^{\mathrm{tf}} \arrow[r] \arrow[d, "p''"] & 0 \\
0 \arrow[r] & \mathrm{tor}(\mathrm{Sym}\,(\mathcal{F}^{\mathrm{tf}})) \arrow[r, "g"] \arrow[d] & \mathrm{Sym}\,(\mathcal{F}^{\mathrm{tf}}) \arrow[r, "g'"] \arrow[d] & (\mathrm{Sym}\,(\mathcal{F}^{\mathrm{tf}}))^{\mathrm{tf}} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The last two rows are clearly exact. Moreover, since $\mathcal{F}^{\mathrm{tf}}$ is locally free, we have that $\mathrm{tor}(\mathrm{Sym}\,(\mathcal{F}^{\mathrm{tf}})) = 0$ and g' is an isomorphism. The morphism i' is the identity on $\mathrm{tor}(\mathrm{Sym}\,\mathcal{F})$. The surjective morphism p comes from applying Sym to the surjection $\mathcal{F} \to \mathcal{F}^{\mathrm{tf}}$, because Sym preserves surjections. The morphism p'' is induced by p using that $\mathrm{tor}(\mathrm{Sym}\,(\mathcal{F}^{\mathrm{tf}})) = 0$. The first row is exact by the Snake Lemma. We want to show that $\ker(p'') = 0$ or, equivalently, that e is an isomorphism.

It follows from the above that we have

$$0 \longrightarrow \mathrm{tor}(\mathrm{Sym}\,\mathcal{F}) \xrightarrow{f \circ i'} \mathrm{Sym}\,\mathcal{F} \xrightarrow{p} \mathrm{Sym}\,(\mathcal{F}^{\mathrm{tf}}) \longrightarrow 0,$$

which is exact except possibly at $\mathrm{Sym}\,\mathcal{F}$. We conclude if we show exactness there. The inclusion $\mathrm{Im}(f \circ i') \subseteq \ker(p)$ is clear because $p \circ f = g \circ p' = 0$.

To show that $\ker(p) \subseteq \mathrm{Im}(f \circ i')$, we know that

$$\mathrm{tor}(\mathcal{F}) \otimes \mathrm{Sym}^{\,n-1}(\mathcal{F}) \to \mathrm{Sym}^{\,n}(\mathcal{F}) \to \mathrm{Sym}^{\,n}(\mathcal{F}^{\mathrm{tf}}) \to 0$$

is exact for all $n \geq 1$ by [Sta22, Lemma 01CJ]. Note that p is a morphism of graded algebras, therefore

$$\ker(p) = \bigoplus_n \ker(\mathrm{Sym}^n(\mathcal{F}) \to \mathrm{Sym}^n(\mathcal{F}^{\mathrm{tf}})).$$

It suffices to show that for each n, the morphism $\mathrm{tor}(\mathcal{F}) \otimes \mathrm{Sym}^{n-1}(\mathcal{F}) \to \mathrm{Sym}^n(\mathcal{F})$ factors through $\mathrm{tor}(\mathrm{Sym}(\mathcal{F}))$. Locally, $X = \mathrm{Spec}(R)$ and $\mathcal{F} = \widetilde{M}$ for some R-module M. Given $\lambda = \sum_j m_1^j \otimes \ldots \otimes m_n^j \in \mathrm{tor}(M) \otimes \mathrm{Sym}^{n-1}(M)$, we can choose for each j a non-zero divisor $r_j \in R$ such that $r_j m_1^j = 0$. Then $r = r_1 \cdots r_j$ is a non-zero divisor and $r\lambda = 0$, so $\lambda \in \mathrm{tor}(\mathrm{Sym}(M))$. □

Remark 1.92. Note that Lemma 1.91 does not holds in general if we do not assume that $\mathcal{F}^{\mathrm{tf}}$ is locally free. For example, let $\mathcal{F} = \mathcal{I}$ be the ideal sheaf of a closed point P on X. Then $(\mathrm{Sym}\,\mathcal{I})^{\mathrm{tf}} = \mathrm{Sym}\,\mathcal{I} = \bigoplus_{n\geq 0} \mathcal{I}^n$ if and only if P is regular. Another example is $R = \mathbb{C}[x,y]$ and $M = I \oplus I$ for $I = (x,y)$. Indeed, M is torsion-free but $\mathrm{Sym}\,M$ has torsion because $x(x \otimes y - y \otimes x) = x \otimes (xy) - (xy) \otimes x = y(x \otimes x - x \otimes x) = 0$.

Lemma 1.93. Let R be a ring, I be an ideal in R and M be an R-module. If M^{tf} is locally free and $I \cdot \mathrm{tor}(M) = 0$ then $I \cdot \mathrm{tor}(\mathrm{Sym}\,M) = 0$.

Proof. Note that $\mathrm{tor}(\mathrm{Sym}\,M) = \bigoplus_{n\geq 0} \mathrm{tor}(\mathrm{Sym}^n M)$. In the proof of Lemma 1.91 we show that $\mathrm{Sym}^n(M^{\mathrm{tf}}) \simeq (\mathrm{Sym}^n M)^{\mathrm{tf}}$. The following commutative diagram is exact by [Sta22, Lemma 01CJ].

$$\begin{tikzcd}
 & \mathrm{tor}(M) \otimes \mathrm{Sym}^{n-1}(M) \arrow[r] \arrow[d] & \mathrm{Sym}^n M \arrow[r] \arrow[d] & \mathrm{Sym}^n(M^{\mathrm{tf}}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{tor}(\mathrm{Sym}^n M) \arrow[r] & \mathrm{Sym}^n M \arrow[r] & \mathrm{Sym}^n(M)^{\mathrm{tf}} \arrow[r] & 0
\end{tikzcd}$$

The first row is exact by [Sta22, Lemma 01CJ], and the second row is also exact. By the Snake Lemma, $\mathrm{tor}(M) \otimes \mathrm{Sym}^{n-1}(M)$ surjects onto $\mathrm{tor}(\mathrm{Sym}^n M)$ and the claim follows. □

Lemma 1.94. Let R be a commutative ring, A be an R-algebra and I be an ideal of R. If $I \cdot \mathrm{tor}(A) = 0$ and A^{tf} is locally free, then the following square is Cartesian in the category of R-algebras

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A^{\mathrm{tf}} \arrow[d] \\
A \otimes R/I \arrow[r] & A^{\mathrm{tf}} \otimes R/I.
\end{tikzcd}$$

Proof. We have the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & IA \arrow[r, "e'"] \arrow[d, "i"] & IA^{\mathrm{tf}} \arrow[d, "i''"] & \\
0 \arrow[r] & \mathrm{tor}(A) \arrow[r, "f"] \arrow[d, "p'"] & A \arrow[r, "f'"] \arrow[d, "p"] & A^{\mathrm{tf}} \arrow[r] \arrow[d, "p''"] & 0 \\
 & \mathrm{tor}(A) \otimes R/I \arrow[r, "g"] \arrow[d] & A \otimes R/I \arrow[r, "g'"] \arrow[d] & A^{\mathrm{tf}} \otimes R/I \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The three columns are exact because $N \otimes R/I \simeq N/IN$ for any R-module N and because $I \cdot \mathrm{tor}(A) = 0$.

Observe that g is injective. This is equivalent to $\mathrm{Tor}_1(R/I, A^{\mathrm{tf}}) = 0$, which holds because A^{tf} is locally free. By the Snake Lemma, the natural morphism $e' \colon IA \to IA^{\mathrm{tf}}$ induced by f' is an isomorphism.

In order to prove the lemma, one can show that the square in question is a Cartesian square of R-modules and then check that it is also a Cartesian diagram of R-algebras. Both can be achieved by routine diagram chasing using the fact that e' is an isomoprhism. $\square$

Proposition 1.95. Let X be a Noetherian scheme, $\mathcal{F}$ a coherent sheaf on X and let $\pi \colon C(\mathcal{F}) = \mathrm{Spec}\,(\mathrm{Sym}\,\mathcal{F}) \to X$ be the corresponding abelian cone. Let $i \colon Z \hookrightarrow X$ be a closed subscheme in X with ideal sheaf $\mathcal{I}_Z$ such that $\mathcal{I}_Z \subseteq \mathrm{Ann}(\mathrm{tor}(\mathcal{F}))$. If $\mathcal{F}^{\mathrm{tf}}$ is locally free, then the following is a push-out of schemes

$$\mathrm{Spec}\,\mathrm{Sym}\,\mathcal{F} = \mathrm{Spec}\,(\mathrm{Sym}\,\mathcal{F}^{\mathrm{tf}}) \bigsqcup_{\mathrm{Spec}\, i_*(\mathrm{Sym}\,(\mathcal{F}^{\mathrm{tf}})|_Z)} \mathrm{Spec}\, i_*(\mathrm{Sym}\,(\mathcal{F}) \mid_Z)$$

If, moreover, X is integral, then $\mathrm{Spec}\,(\mathrm{Sym}\,\mathcal{F}^{\mathrm{tf}})$ is an irreducible component of the abelian cone $\mathrm{Spec}\,\mathrm{Sym}\,\mathcal{F}$.

Proof. Locally, $X = \mathrm{Spec}\, R$ is affine, $\mathcal{F} = \widetilde{M}$ for some finitely presented module M over R such that M^{tf} is locally free and $\mathcal{I}_Z = I$ is an ideal with $I \subseteq \mathrm{Ann}(\mathrm{tor}(M))$. Let $A = \mathrm{Sym}\, M$. Then $I \cdot \mathrm{tor}(A) = 0$ by Lemma 1.93, and Lemma 1.91 ensures that $A^{\mathrm{tf}} = \mathrm{Sym}\,(M^{\mathrm{tf}})$ is locally free. The result follows from Lemma 1.94.

The claim about $\mathrm{Spec}\,(\mathrm{Sym}\,\mathcal{F}^{\mathrm{tf}})$ being irreducible follows from Proposition 1.87 and Lemma 1.91. □

Remark 1.96. Remember that the support $\mathrm{supp}\,(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ can be defined set-theoretically by locally looking at the prime ideals where the stalk of $\mathcal{F}$ is non-zero. A scheme structure on $\mathrm{supp}\,(\mathcal{F})$ is given by the sheaf $\mathrm{Ann}(\mathcal{F})$. Therefore, the condition $\mathcal{I}_Z \subseteq \mathrm{Ann}(\mathrm{tor}(\mathcal{A}))$ in Proposition 1.95 implies that the closed Z must contain $\mathrm{supp}\,(\mathrm{tor}(\mathcal{F}))$.

Another natural scheme structure in $\mathrm{supp}\,(\mathcal{F})$ is given by $F_0(\mathcal{F})$, the 0-th Fitting ideal of $\mathcal{F}$. There is an inclusion $F_0(\mathcal{F}) \subseteq \mathrm{Ann}(\mathcal{F})$ by [Sta22, Lemma 07ZA], thus in Proposition 1.95 we can also take the particular case where $\mathcal{I}_Z = F_0(\mathrm{tor}(\mathcal{F}))$.

1.6.3 A decomposition of the abelian cone of a diagonal sheaf

We continue our study of components of cones by further specializing to the abelian cone of a diagonal sheaf $\mathcal{F}$. The pushout description of $C(\mathcal{F})$ in Proposition 1.95 is improved in Theorem 1.97: $C(\mathcal{F})$ is topologically a union of vector bundles.

Let $\mathcal{F}$ be a diagonal sheaf on an integral Noetherian scheme X. Remember that $\mathcal{F}^{\mathrm{tf}}$ is locally free by Proposition 1.67.

First we reduce from rank r to rank 0. By Proposition 1.95 and Lemma 1.91, we have a decomposition of $C(\mathcal{F})$ as a pushout

$$C(\mathcal{F}) = C(\mathcal{F}^{\mathrm{tf}}) \coprod_{C(\iota_*\mathcal{F}^{\mathrm{tf}}|_{\mathrm{Supp}(\mathrm{tor}(\mathcal{F}))})} C(i_*\mathcal{F}|_{\mathrm{Supp}(\mathrm{tor}(\mathcal{F}))})$$

Here all cones are taken over X and $C(\mathcal{F}^{\mathrm{tf}})$ is an irreducible component by Proposition 1.87. Replacing $\mathcal{F}$ by $i_*\mathcal{F}|_{\mathrm{Supp}(\mathrm{tor}(\mathcal{F}))}$, we may assyme that $\mathcal{F}$ has rank 0.

Let $\mathcal{F}$ be a rank 0 diagonal sheaf. Recall that, by Construction 1.69, $\mathcal{F}$ has a filtration with quotients supported on some effective Cartier divisors D_i for $i = 1, \dots, n$. Consider the finite collection of closed integral subschemes $\{Z_i^j\}_j$, which are the irreducible components of D_i taken with reduced structure. These are in the support of $\mathcal{F}$ and we will see in Lemma 1.98 that $(\mathcal{F}|_{Z_i^j})^{\mathrm{tf}}$ is locally free. Note that these collections are not necessarily disjoint for different i's. We denote the inclusion of Z_i^j in X simply by ι, without keeping track of the indices when it is not necessary.

Theorem 1.97. Let $\mathcal{F}$ be a diagonal sheaf of rank 0 on an integral Noetherian scheme X.

The cone of $\mathcal{F}$ is topologically a union of finitely many irreducible components

$$C(\mathcal{F}) = \bigcup_{i,j} C\left((\mathcal{F}\mid_{Z_i^j})^{\mathrm{tf}}\right) \cup X$$

where each $C\left((\mathcal{F}\mid_{Z_i^j})^{\mathrm{tf}}\right)$ a vector bundle supported on the integral subscheme Z_i^j.

Lemma 1.98. With the previous notations and assumptions, the cone $C_{Z_i^j}\left((\mathcal{F}\mid_{Z_i^j})^{\mathrm{tf}}\right) \to Z_i^j$ is a vector bundle of rank r_i^j, where

$$r_i^j = \max_k \{Z_i^j \subset D_k\}.$$

Proof. Since $\mathrm{tor}(\mathcal{F}|_U) = \mathrm{tor}(\mathcal{F})|_U$ for $U \subset X$ open, it is enough to prove it locally. We assume that $\mathcal{F}$ is the cokernel of a diagonal matrix

$$\mathrm{Diag}(f_1, \ldots, f_1, f_2, \ldots, f_2, \ldots, f_s),$$

where f_k divides f_{k+1}. Observe that, if $f_k \mid_{Z_i^j} = 0$, then $f_\ell \mid_{Z_i^j}$ also vanishes for all $\ell > k$. Take r_i^j as in the statement of the theorem: Z_i^j is a component of $D_{r_i^j}$ and the latter divides the last r_i^j entries.

Then the matrix presentation of $\mathcal{F}$ on Z_i^j looks like $\mathrm{Diag}(f_1 \mid_{Z_i^j}, \ldots, f_t \mid_{Z_i^j}, 0, \ldots, 0)$ where $f_t \mid_{Z_i^j} \neq 0$. Since Z_i^j is not, by assumption, a component of $Z(f_t)$ we see that the cokernel of $\mathrm{Diag}(f_1 \mid_{Z_i^j}, \ldots, f_t \mid_{Z_i^j})$ is a torsion sheaf and the torsion-free part of $\mathcal{F} \mid_{Z_i^j}$ is locally free of rank r_i^j. □

Proof of Theorem 1.97. To check the claim set-theoretically, it suffices to argue that any closed point of $C(\mathcal{F})$ is contained in at least one of the cones. Let $v \in C(\mathcal{F})$, the projection to X is $x \in X$. Then v is specified by some section $x \to \mathcal{F}|_x$. If $x \notin \bigcup_{i,j} Z_i^j$, $\mathcal{F}|_x = 0$, so we are done. Otherwise, we need to argue that $\mathcal{F}|_x \cong ((\mathcal{F}|_{Z_i^j})^{\mathrm{tf}})|_x$ for some i, j.

Let i be such that $x \in Z_i^j$ for some j but $x \notin Z_k^\ell$ for all $k > i$ and all ℓ. Then $x \in D_i$ but $x \notin D_k$ for any $k > i$.

By the construction of the D_i's, we know that $\mathrm{supp}\,(\mathrm{tor}(\mathcal{F}|_{Z_i^j})) \subset \bigcup_{k>i,\ell} Z_k^\ell$. Then $(\mathcal{F}_{Z_i^j}^{\mathrm{tf}})|_x = \mathcal{F}|_x$, and we are done.

The morphism $\bigcup_{i,j} C((\mathcal{F}|_{Z_i^j})^{\mathrm{tf}}) \to C(\mathcal{F})$ of topological spaces, given by the universal property of push-outs, is continuous and closed for the Zariski topology. Since we have just checked that it is also bijective, it is a homeomorphism. □

Example 1.99. The following example of blowing up the origin in $\mathbb{A}^2$ is simple, but it captures much of the essence of the decomposition in Proposition 1.95 (see (1.22) and (1.23)). We present it with full detail.

Let $R = k[x, y]$ and let $I = (x, y)$ be the ideal of the origin. The sheaf $\mathcal{F} = \widetilde{I}$ on $\mathbb{A}^2 = \mathrm{Spec}\,(R)$ is torsion-free but not locally free. Since $\mathcal{F}$ is an ideal sheaf, $\mathrm{Bl}_{\mathcal{F}}\mathbb{A}^2 = \mathrm{Bl}_0\mathbb{A}^2$ is just the usual blow up of $\mathbb{A}^2$ along the origin. Let $p\colon \mathrm{Bl}_0\mathbb{A}^2 \to \mathbb{A}^2$ be the natural projection. Then $p^*\mathcal{F}$ is not torsion-free, but $(p^*\mathcal{F})^{\mathrm{tf}}$ is a line bundle. To see that, we start with the following resolution of $\mathcal{F}$.

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} -y \\ x \end{pmatrix}} R \oplus R \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} I \longrightarrow 0.$$

Pulling back along p, we obtain a presentation of $p^*\mathcal{F}$:

$$\mathcal{O}_{\mathrm{Bl}_0\mathbb{A}^2} \xrightarrow{\begin{pmatrix} -ey' \\ ex' \end{pmatrix}} \mathcal{O}_{\mathrm{Bl}_0\mathbb{A}^2} \oplus \mathcal{O}_{\mathrm{Bl}_0\mathbb{A}^2} \xrightarrow{\begin{pmatrix} ex' & ey' \end{pmatrix}} p^*\mathcal{F} \longrightarrow 0.$$

Here e is a local coordinate for the exceptional divisor $E \subseteq \mathrm{Bl}_0\mathbb{A}^2$ and x' and y' correspond to the strict transforms of x and y. This induces a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[r] \arrow[d] & \mathrm{Coker}(e) \arrow[d, "id"] & \\
0 \arrow[r] & \mathcal{O}_{\mathrm{Bl}_0\mathbb{A}^2} \arrow[r, "\cdot e"] \arrow[d, "\begin{pmatrix} -ey' \\ ex' \end{pmatrix}"] & \mathcal{O}_{\mathrm{Bl}_0\mathbb{A}^2}(E) \arrow[r] \arrow[d, "\begin{pmatrix} y' \\ -x' \end{pmatrix}"] & \mathrm{Coker}(e) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{\mathrm{Bl}_0\mathbb{A}^2} \oplus \mathcal{O}_{\mathrm{Bl}_0\mathbb{A}^2} \arrow[r, "\mathrm{id}"] \arrow[d, "\begin{pmatrix} x'e & y'e \end{pmatrix}"] & \mathcal{O}_{\mathrm{Bl}_0\mathbb{A}^2} \oplus \mathcal{O}_{\mathrm{Bl}_0\mathbb{A}^2} \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & p^*\mathcal{F} \arrow[r] & \mathrm{Coker}(y', -x')^t \arrow[r] & 0 &
\end{tikzcd}$$

Applying the Snake Lemma and using that $\mathrm{Coker}(e) \simeq \mathcal{O}_E(E)$ and that $\mathrm{Coker}(y', -x')$ is the ideal sheaf generated by x' and y', we get a short exact sequence

$$0 \to \mathcal{O}_E(E) \to p^*\mathcal{F} \to (x', y') \to 0.$$

It follows that $\mathrm{tor}(p^*\mathcal{F}) \simeq \mathcal{O}_E(E)$ and $(p^*\mathcal{F})^{\mathrm{tf}} \simeq (x', y')$. In particular, $p^*\mathcal{F}$ is not torsion-free.

We can also describe the geometry of the abelian cones $\operatorname{Spec}\operatorname{Sym}\mathcal{F}$ and $\operatorname{Spec}\operatorname{Sym}(p^*\mathcal{F})$. We have

$$\pi_{\mathcal{F}}\colon \operatorname{Spec}\operatorname{Sym}\mathcal{F} = \operatorname{Spec}(R[X,Y]/(xY-yX)) \to \mathbb{A}^2,$$

which is irreducible and singular.

Next we describe $\operatorname{Spec}\operatorname{Sym}(p^*\mathcal{F})$. Let $S = k[x,y,x',y']/(xy'-yx')$ where the variables x', y' have degree 1 and x, y have degree 0. Then

$$\mathrm{Bl}_0\mathbb{A}^2 = \mathrm{Proj}(k[x,y,x',y']/(xy'-yx')).$$

A local equation for E is given by $e = x/x'$ or $e = y/y'$, depending on the chosen chart. Then

$$\pi_{p^*\mathcal{F}}\colon \operatorname{Spec}\operatorname{Sym}(p^*\mathcal{F}) = \operatorname{Spec} S[X,Y]/(e(x'Y-y'X)) \to \mathrm{Bl}_0\mathbb{A}^2$$

is reducible. It has two components:

$$\pi_{p^*\mathcal{F},\mathrm{main}}\colon C_{\mathrm{main}} = V(x'Y-y'X) \to \mathrm{Bl}_0\mathbb{A}^2, \tag{1.22}$$
$$\pi_{p^*\mathcal{F},\mathrm{tor}}\colon C_{\mathrm{tor}} = V(e) \to \mathrm{Bl}_0\mathbb{A}^2. \tag{1.23}$$

The main component C_{main} equals $\operatorname{Spec}\operatorname{Sym}(p^*\mathcal{F})^{\mathrm{tf}}$ and it is a vector bundle of rank 1. Meanwhile, C_{tor} corresponds to $\mathrm{tor}(\mathcal{F})$, it is supported over E and it is a vector bundle of rank 2 over its support.

1.7 Application to stable maps

In this section we apply the results in Section 1.5 to construct reduced Gromov–Witten invariants.

Given X a smooth subvariety in a projective space $\mathbb{P}^r$, there is an embedding of the moduli space of stable maps to X in the moduli space of stable maps to $\mathbb{P}^r$. The moduli space of genus *zero* stable maps to a projective space $\mathbb{P}^r$ is a smooth irreducible DM stack. If X is a hypersurface of degree k (or more generally a complete intersection) in $\mathbb{P}^r$, there is a locally free sheaf $\mathcal{E}_k$ on the moduli space of stable maps to $\mathbb{P}^r$, such that the moduli space of maps to X is cut out by the zero locus of a section of this sheaf. These statements are not true in higher genus. In general, the moduli space of stable maps to $\mathbb{P}^r$ has several irreducible components of different dimensions. We still have a natural sheaf $\mathcal{E}_k$ equipped with a section, but $\mathcal{E}_k$ is not locally free: its rank is different on different irreducible components.

There are several ways to use Section 1.5 to fix the above problem (see Remark 1.123). In this section we are concerned with finding and comparing various blow-ups the Picard stack along certain sheaves, which fix the above problem. More precisely, we consider $\widetilde{\mathfrak{Pic}} \to \mathfrak{Pic}$, such that $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) := \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \times_{\mathfrak{Pic}} \widetilde{\mathfrak{Pic}}$ desingularizes $\mathcal{E}_k$.

Under the assumption $d > 2g-2$ (see Assumption 1.105), we define $\widetilde{\mathcal{M}}^{\circ}_{g,n}(X, d)$ via the following Cartesian diagram

$$\begin{tikzcd}
\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r,d) \arrow[d] \\
\overline{\mathcal{M}}_{g,n}(X,d) \arrow[r] & \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d),
\end{tikzcd}$$

where $\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d)$ is the main component of the cone $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ (see Definition 1.110). We then define reduced invariants (see Definition 1.117) via an obstruction theory on $\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d)$ relative to $\widetilde{\mathfrak{Pic}}_k$ (see Theorem 1.124).

We also recall maps with fields [CL12] and then we construct a blow-up of it which makes the resulting stack as simple as possible. The resulting stack gives an alternative definition of reduced invariants, which is not intrinsic; the relation between these two invariants is similar in spirit to a Quantum Lefschetz theorem. The definition we give is more intrinsic, but working with maps with fields instead of maps is more suited to approaching Conjecture 1.2 and Conjecture 1.3. See [CL15, LO22, LO21] for the proof of Conjecture 1.3 in genus one and two.

1.7.1 Stable maps as open in an abelian cone

We recall how the moduli space of stable maps to projective space can be seen as an open substack of an abelian cone, following [CL12]. This observation motivates our study of components of cones in Section 1.6, as components of the ambient abelian cone are related to components of stable maps.

Let $\mathfrak{M}_{g,n}$ denote the stack of genus g pre-stable curves with n marked points, that is, $\mathfrak{M}_{g,n}$ parametrizes connected projective at-worst-nodal curves of arithmetic genus g with n distinct smooth marked points. Let $\underline{\mathfrak{C}}_{g,n}$ denote universal curve over $\mathfrak{M}_{g,n}$. Let $\mathfrak{Pic}_{g,n,d}$

denote the Artin stack which parameterises genus g pre-stable curves, with n marked points, together with a line bundle of degree d. Let $\mathfrak{Pic}^{\mathrm{st}}_{g,n,d}$ denote the open subset of $\mathfrak{Pic}_{g,n,d}$ consisting of $(C, p_1 \dots p_n, L)$ which satisfy the stability condition

$$L^{\otimes 3} \otimes \omega_C \left(\sum_{i=1}^{n} p_i \right) \text{ is ample.} \tag{1.24}$$

Notice that $\mathfrak{M}_{g,n}$ and $\mathfrak{Pic}_{g,n,d}$ are not separated, but they are smooth (see [Sta22, Lemma 0E6W] and [CFKM14, Proposition 2.11]) and irreducible. The stack $\mathfrak{Pic}_{g,n,d}$ is locally Noetherian and the stack $\mathfrak{Pic}^{\mathrm{st}}_{g,n,d}$ is Noetherian.

Notation 1.100. From now on, we fix g, n, d and the stability condition and we drop all the indices.

We define $\mathfrak{C}$ the universal curve over $\mathfrak{Pic}$ by the Cartesian diagram (1.25). Notice that we also have a universal line bundle $\mathfrak{L}$ over $\mathfrak{C}$.

$$\begin{tikzcd}
\mathfrak{L} \arrow[d] & \\
\mathfrak{C} \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\ulcorner"] & \underline{\mathfrak{C}} \arrow[d] \\
\mathfrak{Pic} \arrow[r] & \mathfrak{M}.
\end{tikzcd} \tag{1.25}$$

We form the cone of sections of $\mathfrak{L}$ as in Chang-Li ([CL15], Section 2)

$$S(\pi_* \mathfrak{L}) := \operatorname{Spec} \operatorname{Sym} \left(R^1 \pi_* (\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}) \right) \to \mathfrak{Pic}. \tag{1.26}$$

In the following we collect a list of remarks on the cone of sections defined above.

1 In [CL12, Proposition 2.2], the authors show that $S(\pi_*\mathfrak{L})$ is the moduli stack parameterizing (C, L, s) with $(C, L) \in \mathfrak{Pic}$ and $s \in H^0(C, L)$. Be aware that our $S(\pi_*\mathfrak{L})$ is denoted by $C(\pi_*\mathfrak{L})$ in [CL12].

2 This situation is similar to the discussion in Section 1.2.3 about the total space of a locally free sheaf. If $\mathfrak{E}$ is a locally free sheaf over $\mathfrak{Pic}$, then sections of $\mathfrak{E}$ correspond to sections of the vector bundle $\operatorname{Tot}(\mathfrak{E}) = \operatorname{Spec} \operatorname{Sym} (\mathfrak{E}^\vee)$ over $\mathfrak{Pic}$, but the same is not true if $\mathfrak{E}$ is not locally free.

3 In our set-up, the sheaf $R^0\pi_*\mathfrak{L}$ is not locally free. However, since we work with the universal family of curves $\mathfrak{C} \to \mathfrak{Pic}$, sections of the sheaf $R^0\pi_*\mathfrak{L}$ correspond

to sections of the abelian cone of its *Serre dual* $R^1\pi_*(\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}})$. This is proven in [CL12, Proposition 2.2].

4 Note that $R^0\pi_*\mathfrak{L}$ does not commute with base change but $R^1\pi_*\mathfrak{L}$ does by cohomology and base change.

For the rest of the section, let

$$\mathfrak{F} := R^1\pi_*(\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}). \tag{1.27}$$

Note that since π is proper, we have that $\mathfrak{F}$ is a coherent sheaf on $\mathfrak{Pic}$. As defined in Section 1.2.3, we consider the stack $\operatorname{Spec}\operatorname{Sym}\mathfrak{F}$, which is an abelian cone stack over $\mathfrak{Pic}$.

Let $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ be the moduli space of genus g, degree d stable maps, with n marked points.

Proposition 1.101. ([CL12, Proposition 2.7], [CFK10, Theorem 3.2.1]) The moduli space $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ is an open substack, cut out by the basepoint-free condition, of the stack

$$S(\pi_*\mathfrak{L}^{\oplus r+1}) = \operatorname{Spec}\operatorname{Sym}(\oplus_{i=0}^r \mathfrak{F}) \to \mathfrak{Pic}. \tag{1.28}$$

As before, a point of this cone over $(C, L) \in \mathfrak{Pic}$ is $(C, L, \underline{s})$ with $\underline{s} \in H^0(C, L)^{\oplus r+1}$. Note that

$$S(\pi_*\mathfrak{L}^{\oplus r+1}) = \overbrace{S(\pi_*\mathfrak{L}) \times_{\mathfrak{Pic}} \cdots \times_{\mathfrak{Pic}} S(\pi_*\mathfrak{L})}^{r+1 \text{ times}} \to \mathfrak{Pic}.$$

We define $\mathcal{L}, \mathcal{C}$ by the following Cartesian diagram

$$\begin{tikzcd}
\mathcal{L} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathfrak{L} \arrow[d] \\
\mathcal{C} \arrow[r] \arrow[d, "\overline{\pi}"] \arrow[dr, phantom, "\ulcorner", very near start] & \mathfrak{C} \arrow[d, "\pi"] \\
S(\pi_*\mathfrak{L}^{\oplus r+1}) \arrow[r, "\mu"] & \mathfrak{Pic}.
\end{tikzcd}$$

By [CL12], the complex

$$\oplus_{i=0}^r R^\bullet \overline{\pi}_* \mathcal{L}$$

is a dual obstruction theory for the natural projection

$$\mu : S(\pi_*\mathfrak{L}^{\oplus r+1}) = \operatorname{Spec}\operatorname{Sym}(\oplus_{i=0}^r \mathfrak{F}) \to \mathfrak{Pic}.$$

This perfect obstruction theory induces a virtual class

$$[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)]^{\mathrm{vir}} := \mu^![\mathfrak{Pic}] \in A_*(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)).$$

Proposition 1.102. We use Notation 1.100. For $\mathfrak{F}$ defined in eq. (1.27), we have an isomorphism of sheaves

$$\mathfrak{F}^\vee := \left(R^1\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}\right)^\vee \simeq \pi_*\mathfrak{L}$$

over $\mathfrak{Pic}$.

Proof. Using Grothendieck duality we have

$$\begin{aligned}
\left(R^\bullet\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}\right)^\vee &= R\mathcal{H}om_{\mathfrak{Pic}}(R^\bullet\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}, \mathcal{O}_{\mathfrak{Pic}}) \\
&= R^\bullet\pi_* R\mathcal{H}om_{\mathfrak{Pic}}(\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}, \omega_{\mathfrak{C}/\mathfrak{Pic}}[1]) \\
&= R^\bullet\pi_* R\mathcal{H}om_{\mathfrak{C}}(\mathcal{O}_{\mathfrak{C}}, \mathfrak{L} \otimes \omega^\vee_{\mathfrak{C}/\mathfrak{Pic}} \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}[1]) \\
&= R^\bullet\pi_* R\mathcal{H}om_{\mathfrak{C}}(\mathcal{O}_{\mathfrak{C}}, \mathfrak{L}[1]) \\
&= R^\bullet\pi_*\mathfrak{L}[1].
\end{aligned} \tag{1.29}$$

On the one hand, we have that

$$h^{-1}(R^\bullet\pi_*\mathfrak{L}[1]) = \pi_*\mathfrak{L}. \tag{1.30}$$

In the following we look at an explicit resolution of $\left(R^\bullet\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}\right)^\vee$ and compute its h^{-1}. This is similar to the discussion in [CFK20], Section 3.2. By the stability condition on $\mathfrak{Pic}$ (see Equation (1.24)), the universal curve over $\mathfrak{Pic}$ is projective. This ensures that we have an ample section on $\mathfrak{C}$ and we take A sufficiently large so that $R^0\pi_*\mathfrak{L}^\vee(-A) \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}} = 0$.

We now consider the exact sequence of sheaves on the universal curve over $\mathfrak{Pic}$

$$0 \to \mathfrak{L}^\vee(-A) \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}} \to \mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}} \to \mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}|_A \to 0.$$

Pushing forward the above to $\mathfrak{Pic}$, we get a long exact sequence

$$\begin{aligned}
0 \to R^0\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}} \to & R^0\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}|_A \to \\
& \to R^1\pi_*\mathfrak{L}^\vee(-A) \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}} \to R^1\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}} \to 0.
\end{aligned} \tag{1.31}$$

This gives

$$R^\bullet\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}} \simeq [R^0\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}|_A \to R^1\pi_*\mathfrak{L}^\vee(-A) \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}], \tag{1.32}$$

with the complex on the right, being a complex of vector bundles supported in $[0,1]$. This gives

$$(R^\bullet\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}})^\vee \simeq [(R^1\pi_*\mathfrak{L}^\vee(-A) \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}})^\vee \to (R^0\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}|_A)^\vee],$$

with the complex on the right being supported in $[-1,0]$.

Applying the functor $\mathcal{H}om(-,\mathcal{O})$ to (1.31) and using that it is left-exact, we get

$$0 \to \left(R^1\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}\right)^\vee \to (R^1\pi_*\mathfrak{L}^\vee(-A) \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}})^\vee \to (R^0\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}|_A)^\vee \tag{1.33}$$

This together with eq. (1.32) shows that

$$h^{-1}\left(\left(R^\bullet\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}\right)^\vee\right) = \left(R^1\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}\right)^\vee. \tag{1.34}$$

Equations (1.29), (1.30) and (1.34) imply that

$$\left(R^1\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}\right)^\vee = R^0\pi_*\mathfrak{L}. \qquad \square$$

Remark 1.103. As in the proof of Proposition 1.102, we have an explicit resolution of $\mathfrak{F}$ (see [CFK20], Section 3.2). Let A be a sufficiently high power of a very ample section of the morphism $\mathfrak{C} \to \mathfrak{Pic}$ such that $R^1\pi_*\mathfrak{L}(A) = 0$. Then, we have a morphism

$$0 \to \mathfrak{L} \to \mathfrak{L}(A) \to \mathfrak{L}(A)|_A \to 0. \tag{1.35}$$

This induces a long exact sequence

$$0 \to R^0\pi_*\mathfrak{L} \to R^0\pi_*\mathfrak{L}(A) \to R^0\pi_*\mathfrak{L}(A)|_A \to R^1\pi_*\mathfrak{L} \to 0, \tag{1.36}$$

which shows that $[R^0\pi_*\mathfrak{L}(A) \to R^0\pi_*\mathfrak{L}(A)|_A]$ is quasi-isomorphic to $R^\bullet\pi_*\mathfrak{L}$.

With our choice of A we have that $R^0\pi_*\mathfrak{L}(A)$ and $R^0\pi_*\mathfrak{L}(A)|_A$ are locally free sheaves over $\mathfrak{Pic}$. Sequence (1.36) together with the fact that $R^0\pi_*\mathfrak{L}(A)$ is a locally free sheaf over $\mathfrak{Pic}$ implies that $R^0\pi_*\mathfrak{L}$ is a torsion free sheaf on $\mathfrak{Pic}$.

Remark 1.104. The proof of Proposition 1.102 shows that we have a resolution of $\mathfrak{F} := R^1\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}$ to the left given by (1.31). The isomorphism in Equation (1.29) shows that the complex

$$[R^0\pi_*\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}|_A \to R^1\pi_*\mathfrak{L}^\vee(-A) \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}]$$

is dual to

$$[R^0\pi_*\mathfrak{L}(A) \to R^0\pi_*\mathfrak{L}(A)|_A].$$

We will use this duality in Section 1.8. Hu and Li work with the resolution to the right we have in (1.36). In the previous sections we used the resolution to the left Equation (1.36). Since dual morphisms have the same Fitting ideals, both morphisms give the same Hu–Li blow-up.

1.7.2 The main component of stable maps to $\mathbb{P}^r$

In the following we look at stable maps with a lower bound on the degree (see Assumption 1.105). In this situation the moduli space of stable maps has a *main (irreducible) component*. We discuss this main component and its relation to the main component of abelian cones.

We fix the following assumption from now on.

Assumption 1.105. In the following we fix $d > 2g-2$. For C a smooth genus g curve, L a line bundle of degree d, and $d > 2g-2$, we have that $H^1(C, L) = 0$. This shows that for $d > 2g-2$, the locus of stable maps with smooth domain is smooth and irreducible, so its closure is an irreducible component of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$.

Definition 1.106 (Main Component)**.** Consider the Zariski closure in $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ of the locus where the curve is smooth of genus g. We call this component the *main component* and we denote it by $\overline{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d)$.

We introduced the main component of an abelian cone in Definition 1.88. In our next result, Proposition 1.107, we show that the main component of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ and the main component of $\operatorname{Spec}\operatorname{Sym}(\oplus_{i=0}^r \mathfrak{F})$ are compatible along the open embedding Proposition 1.101. By the proof of proposition 1.107, on $\overline{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d)$ the universal curve is generically smooth and $\pi_*\mathcal{L}$ is generically a vector bundle.

Proposition 1.107. We have that $\overline{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d)$ is an open substack of $\operatorname{Spec}(\operatorname{Sym} \oplus_{i=0}^r \mathfrak{F})^{\mathrm{tf}}$.

Proof. Let $\mathcal{M}_{g,n}(\mathbb{P}^r, d)$ and $\mathfrak{Pic}^{\mathrm{sm}}$ denote the open substacks of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ and $\mathfrak{Pic}$ where the curve is smooth. The first step is to show that the sheaf $\mathfrak{F} = R^1\pi_*(\mathfrak{L}^\vee \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}})$ is locally free over $\mathfrak{Pic}^{\mathrm{sm}}$.

Let $\pi^{\mathrm{sm}}\colon \mathfrak{C}^{\mathrm{sm}} \to \mathfrak{Pic}^{\mathrm{sm}}$ denote the universal curve and $\mathfrak{L}^{\mathrm{sm}}$ the universal line bundle on $\mathfrak{C}^{\mathrm{sm}}$. Assumption 1.105 and cohomology and base change ensure that $R^1\pi^{\mathrm{sm}}_*\mathfrak{L}^{\mathrm{sm}} = 0$. Therefore, $R^0\pi^{\mathrm{sm}}_*\mathfrak{L}^{\mathrm{sm}}$ has constant rank, so it locally free. Using Serre Duality and local freeness of $\omega_{\pi^{\mathrm{sm}}}$, we see that

$$R^1\pi^{\mathrm{sm}}_*(\mathfrak{L}^{\mathrm{sm}\vee} \otimes \omega_{\pi^{\mathrm{sm}}}) \simeq (R^0\pi^{\mathrm{sm}}_*((\mathfrak{L}^{\mathrm{sm}\vee} \otimes \omega_{\pi^{\mathrm{sm}}})^\vee \otimes \omega_{\pi^{\mathrm{sm}}}))^\vee \simeq (R^0\pi^{\mathrm{sm}}_*\mathfrak{L}^{\mathrm{sm}})^\vee$$

is locally free. In particular, the $\mathcal{O}_{\mathfrak{Pic}}$-algebra $(\mathrm{Sym}\ \oplus_{i=0}^r \mathfrak{F})^{\mathrm{tf}}$ is locally free over $\mathfrak{Pic}^{\mathrm{sm}}$.

To simplify the notation, let $C = \mathrm{Spec}\,\mathrm{Sym}\ \oplus_{i=0}^r \mathfrak{F}$ and $C^{\mathrm{tf}} = \mathrm{Spec}\,(\mathrm{Sym}\ \oplus_{i=0}^r \mathfrak{F})^{\mathrm{tf}}$. By Proposition 1.85 and using that $\mathfrak{Pic}^{\mathrm{sm}}$ is generically reduced, we have that

$$\mathrm{cl}^{\mathrm{sch}}_C(\pi^{-1}(\mathfrak{Pic}^{\mathrm{sm}})) = C^{\mathrm{tf}}, \tag{1.37}$$

where $\pi^{-1}(\mathfrak{Pic}^{\mathrm{sm}})$ denotes the fibre product $\mathfrak{Pic}^{\mathrm{sm}} \times_{\mathfrak{Pic}} C$, which is open in C.

We conclude by the following chain of equalities

$$\begin{aligned}\overline{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d) &= \mathrm{cl}^{\mathrm{sch}}_{\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)}(\pi^{-1}(\mathfrak{Pic}^{\mathrm{sm}}) \times_C \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)) = \\ &= \mathrm{cl}^{\mathrm{sch}}_C(\pi^{-1}(\mathfrak{Pic}^{\mathrm{sm}})) \times_C \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) = \\ &= C^{\mathrm{tf}} \times_C \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d).\end{aligned}$$

The first equality is the definition of $\overline{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d)$, the second one is Lemma 1.108 (applied to $Y = C$ and W, V reduced open subschemes of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ and $\pi^{-1}(\mathfrak{Pic}^{\mathrm{sm}})$ respectively), and the last one is Equation (1.37). ☐

Lemma 1.108. Let Y be a scheme, let V, W be reduced open subschemes of Y. Then

$$\mathrm{cl}^{\mathrm{sch}}_W(V \times_Y W) = \mathrm{cl}^{\mathrm{sch}}_Y(V) \times_Y W$$

Proof. Since V and W are reduced, so is $V \times_Y W) V \cap W$. This means that the schematic closure is just the topological closure with the reduced induced structure by [Sta22, Lemma 056B]. Therefore the question is purely topological, and it is straightforward using that $\mathrm{cl}^{\mathrm{sch}}_Y W = \mathrm{cl}_Y W$ is the intersection of all the closed subsets C of Y that contain W. ☐

1.7.3 Blow-ups of the moduli space of stable maps to projective spaces

In this section we consider a desingularization $p\colon \widetilde{\mathfrak{Pic}} \to \mathfrak{Pic}$ of $\mathfrak{F}$ and the base change $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. By compatibility of abelian cones with pullback, $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ is an open substack of $C(\oplus_{i=0}^r p^*\mathfrak{F})$. We define the main component $\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r,d)$ of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$ to be the closure of the smooth locus (see Definition 1.110). This definition ensures that $\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r,d)$ is open in the main component of the ambient abelian cone $C(\oplus_{i=0}^r p^*\mathfrak{F})$ (Proposition 1.107), thus it is irreducible. In general, $\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r,d)$ does not agree with the pullback of $\overline{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r,d)$, which might be reducible (see Remark 1.112). Finally, we induce a virtual fundamental class on $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$.

We define

$$p\colon \widetilde{\mathfrak{Pic}} \to \mathfrak{Pic}$$

to be any desingularization (as in Definition 1.22) of the sheaf $\mathfrak{F}$ (defined in Equation (1.27)). By theorem 1.79, we have a proper birational map

$$\widetilde{\mathfrak{Pic}} \to \mathrm{Bl}_{\mathfrak{F}}\mathfrak{Pic} \tag{1.38}$$

where $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{Pic} \to \mathfrak{Pic}$ is the Rossi blow-up. We are mainly interested in $\widetilde{\mathfrak{Pic}}$ being the Rossi or the Hu-Li blow-up (see Section 1.5).

We define

$$\begin{tikzcd} \widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d) \arrow[r, "\overline{p}"] \arrow[d, "\widetilde{\mu}"] \arrow[dr, phantom, "\ulcorner"] & \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d) \arrow[d, "\mu"] \\ \widetilde{\mathfrak{Pic}} \arrow[r, "p"] & \mathfrak{Pic}. \end{tikzcd} \tag{1.39}$$

Note that $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$ is proper since p and $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$ are proper. Let $p:\widetilde{\mathfrak{Pic}}\to\mathfrak{Pic}$ be the natural projection. Consider the Cartesian diagram

$$\begin{tikzcd} \widetilde{\mathfrak{L}} \arrow[r] \arrow[d] & \mathfrak{L} \arrow[d] \\ \widetilde{\mathfrak{C}} \arrow[r, "q"] \arrow[d, "\widetilde{\pi}"] & \mathfrak{C} \arrow[d, "\pi"] \\ \widetilde{\mathfrak{Pic}} \arrow[r, "p"] & \mathfrak{Pic} \end{tikzcd}$$

where $\mathfrak{C}$ is the universal curve over $\mathfrak{Pic}$ and $\mathfrak{L}$ the universal line bundle. Recall that

$$\mathfrak{F} = R^1\pi_*(\mathfrak{L}^\vee \otimes \omega_\pi).$$

Lemma 1.109. In notation as before, we have an open embedding

$$\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \hookrightarrow \operatorname{Spec}\operatorname{Sym}\left(\oplus_{i=0}^r p^*\mathfrak{F}\right) \cong \operatorname{Spec}\operatorname{Sym}\left(\oplus_{i=0}^r R^1\widetilde{\pi}_*(\widetilde{\mathfrak{L}}^\vee \otimes \omega_{\widetilde{\pi}})\right).$$

Proof. By Proposition 1.101, we have an open embedding $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \subset \operatorname{Spec}_{\mathfrak{Pic}}\operatorname{Sym} \oplus_{i=0}^r \mathfrak{F}$. Thus, $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \subset \operatorname{Spec}_{\widetilde{\mathfrak{Pic}}}\operatorname{Sym} p^*\mathfrak{F}^{\oplus r+1}$ For the isomorphism, we see that cohomology commutes with base change since there are no higher derived pushforwards. This gives

$$p^*\mathfrak{F} \cong R^1\widetilde{\pi}_*(\widetilde{\mathfrak{L}}^\vee \otimes \omega_{\widetilde{\pi}}).$$

□

Definition 1.110. Consider the Zariski closure in $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ of the locus where the curve is smooth of genus g. We call this component the *main component* and we denote it by $\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d)$.

Proposition 1.111. The following hold:

1 We have an open embedding

$$\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d) \subset \operatorname{Spec}\left(\operatorname{Sym} \oplus_{i=0}^r p^*\mathfrak{F}\right)^{\mathrm{tf}} \simeq \operatorname{Spec}\operatorname{Sym}\left(\oplus_{i=0}^r (p^*\mathfrak{F})^{\mathrm{tf}}\right).$$

2 $\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d)$ is proper and smooth over $\widetilde{\mathfrak{Pic}}$.

Proof. By Lemma 1.109 we get $\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d) \hookrightarrow \operatorname{Spec}\left(\operatorname{Sym} \oplus_{i=0}^r p^*\mathfrak{F}\right)^{\mathrm{tf}}$ is an open embedding. By cohomology and base change we have $p^*\mathfrak{F} \simeq R^1\widetilde{\pi}_*(\widetilde{\mathfrak{L}}^\vee \otimes \omega_{\widetilde{\pi}})$. The argument in Proposition 1.107 applies to $R^1\widetilde{\pi}_*(\widetilde{\mathfrak{L}}^\vee \otimes \omega_{\widetilde{\pi}})$ and we obtain an open embedding

$$\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d) \hookrightarrow \operatorname{Spec}\left(\operatorname{Sym} \oplus_{i=0}^r R^1\widetilde{\pi}_*(\widetilde{\mathfrak{L}}^\vee \otimes \omega_{\widetilde{\pi}})\right)^{\mathrm{tf}}.$$

By construction $(p^*\mathfrak{F})^{\mathrm{tf}}$ is locally-free and, since the torsion-free part commutes with direct sums, the same is true for $(p^*\mathfrak{F}^{\oplus r+1})^{\mathrm{tf}}$. This shows that $\operatorname{Sym} \oplus_{i=0}^r (p_k^*\mathfrak{F})^{\mathrm{tf}}$ is locally free therefore we have an isomorphism

$$\operatorname{Spec}\left(\operatorname{Sym} \oplus_{i=0}^r p^*\mathfrak{F}\right)^{\mathrm{tf}} \simeq \operatorname{Spec}\operatorname{Sym}\left(\oplus_{i=0}^r (p^*\mathfrak{F})^{\mathrm{tf}}\right)$$

by Lemma 1.91.

The first part implies that $\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d)$ is smooth over $\widetilde{\mathfrak{Pic}}$ since it is open in a vector bundle. The properness of $\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d)$ follows from the properness of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. □

Remark 1.112. We consider the following Cartesian diagram:

$$
\begin{tikzcd}
\widetilde{\mathcal{M}}^\circ(\mathbb{P}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \overline{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d) \arrow[d, "\mu"] \\
\widetilde{\mathfrak{Pic}} \arrow[r, "p"] & \mathfrak{Pic}.
\end{tikzcd}
\tag{1.40}
$$

In general, $\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d) \hookrightarrow \widetilde{\mathcal{M}}^\circ(\mathbb{P})$ is not an isomorphism and thus the diagram below is only commutative

$$
\begin{tikzcd}
\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d) \arrow[r, "\overline{p}"] \arrow[d, "\widetilde{\mu}"] & \overline{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d) \arrow[d, "\mu"] \\
\widetilde{\mathfrak{Pic}} \arrow[r, "p"] & \mathfrak{Pic}.
\end{tikzcd}
\tag{1.41}
$$

This observation is a reflection of the fact that torsion-free part does not commute with pullback, see Remark 1.92. By pullback, Proposition 1.107 induces an open embedding

$$\widetilde{\mathcal{M}}^\circ(\mathbb{P}) \subset \operatorname{Spec} p^* \left(\operatorname{Sym}\left(\oplus_{i=0}^r \mathfrak{F}\right)\right)^{\mathrm{tf}},$$

which in general need not factor through $\operatorname{Spec}\ \left(\operatorname{Sym}\left(\oplus_{i=0}^r p^*\mathfrak{F}\right)\right)^{\mathrm{tf}} \not\simeq \operatorname{Spec} p^* \left(\operatorname{Sym}\left(\oplus_{i=0}^r \mathfrak{F}\right)\right)^{\mathrm{tf}}$. Meanwhile, as in the proof of Proposition 1.107, we have that

$$\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d) = \operatorname{Spec}\ \left(\operatorname{Sym}\left(\oplus_{i=0}^r p^*\mathfrak{F}\right)\right)^{\mathrm{tf}} \cap \widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d).$$

An intuitive way to think about the discrepancy between $\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d)$ and $\widetilde{\mathcal{M}}^\circ(\mathbb{P})$ is that the latter is total transform of $\overline{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d)$, while the former is the strict transform. In particular, $\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d)$ is always irreducible, while $\widetilde{\mathcal{M}}^\circ(\mathbb{P})$, in general, is not.

Let $\widehat{\pi} : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ be the universal curve and let $\widehat{q} : \widetilde{\mathcal{C}} \to \mathcal{C}$ the morphism induced by q. The morphism $\widetilde{\mu}$ has a dual perfect obstruction theory equal to

$$\oplus_{i=0}^r R^\bullet \widehat{\pi}_* \widehat{q}^* \mathcal{L}.$$

This perfect obstruction theory induces a virtual class

$$[\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)]^{\mathrm{vir}} := \widetilde{\mu}^! [\widetilde{\mathfrak{Pic}}] \in A_*(\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)),$$

where $\widetilde{\mu}^!$ is defined as in [Man11].

Remark 1.113. While $\mathfrak{Pic}$ is smooth, $\widetilde{\mathfrak{Pic}}$ does not need to be smooth. This is not a problem, all we need for a well-defined virtual class is that $\widetilde{\mathfrak{Pic}}$ has pure dimension. This is true, since $\mathfrak{Pic}$ has pure dimension and p is birational.

Proposition 1.114. We have the following equality

$$(\overline{p})_*[\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)]^{\mathrm{vir}} = [\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)]^{\mathrm{vir}}$$

Proof. Note that by cohomology and base-change we have that $R^\bullet \widetilde{\pi}_* \overline{p}^* \mathcal{L} = \overline{p}^* R^\bullet \overline{\pi}_* \mathcal{L}$. As p is birational and proper, we have $p_*[\widetilde{\mathfrak{Pic}}_k] = [\mathfrak{Pic}]$. We now apply Costello's Pushforward theorem ([HW22]) to Equation (1.39) and we get

$$\overline{p}_*[\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)]^{\mathrm{vir}} = [\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)]^{\mathrm{vir}}. \qquad \square$$

1.7.4 Definition of reduced GW invariants in all genera

In this section we define reduced Gromov–Witten invariants for hypersurfaces in projective spaces under Assumption 1.105. This is less straight-forward than for projective spaces, since we have no understanding of the geometry of moduli spaces of maps to hypersurfaces.

Let X be a smooth hypersurface on $\mathbb{P}^r$ defined by the vanishing of a regular section of s of $\mathcal{O}(k)$. We have that $\overline{\mathcal{M}}_{g,n}(X, d)$ is cut out in $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ by the vanishing of the section $\pi_* s$ of $\pi_* \mathcal{L}^{\otimes k}$ on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$. If $\pi_* \mathcal{L}^{\otimes k}$ is a vector bundle, we can use this to define the virtual class of $\overline{\mathcal{M}}_{g,n}(X, d)$ by virtual pullback. This generally fails for $g \geq 1$, so we will use the blow-ups we developed to ensure that the restriction of $\pi_* \mathcal{L}^{\otimes k}$ to the main component $\overline{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d)$ is locally free.

Construction 1.115. Let $X \subset \mathbb{P}^r$ be a smooth hypersurface as above. Let $p_k \colon \widetilde{\mathfrak{Pic}}_k \to \mathfrak{Pic}$ be any desingularization of $\mathfrak{F}$ and $R^0 \pi_* \mathfrak{L}^{\otimes k}$. We define

$$\begin{tikzcd} \overline{\mathcal{M}}^{\circ}_{g,n}(X, d) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \overline{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d) \arrow[d] \\ \overline{\mathcal{M}}_{g,n}(X, d) \arrow[r] & \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d). \end{tikzcd} \tag{1.42}$$

The main component $\overline{\mathcal{M}}^{\circ}_{g,n}(X, d)$ does not have a perfect obstruction theory. In order to define a perfect obstruction theory on it, we define

$$\begin{tikzcd}
\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r,d) \arrow[d] \\
\widetilde{\mathcal{M}}_{g,n}(X,d) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d) \arrow[d] \\
\overline{\mathcal{M}}_{g,n}(X,d) \arrow[r, "i"] & \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d),
\end{tikzcd} \tag{1.43}$$

where $\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r,d)$ and $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$ are defined in Section 1.7.3. Proposition 1.111 implies $\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d)$ is proper.

By construction we get that $(\tilde{\pi}_*\tilde{\mathcal{L}}^{\otimes k})^{\mathrm{tf}}$ is locally free on $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$.

We define

$$[\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d)]^{\mathrm{vir}} = i^![\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r,d)]. \tag{1.44}$$

Note that for any $1 \leq i \leq n$ we have morphisms

$$\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d) \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d) \xrightarrow{ev_i} \mathbb{P}^r$$

and

$$\widetilde{\mathcal{M}}_{g,n}(X,d) \to \overline{\mathcal{M}}_{g,n}(X,d) \xrightarrow{ev_i} X.$$

By abuse of notation we denote both of these compositions by ev_i.

Notice that the definition of the reduced virtual class in (1.44) does a priori depend on the choice of a desingularization of $\mathfrak{Pic}$. The following proposition shows that integration against this class does not depend on the desingularization. The proof of Proposition 1.116 is delayed, since we first proof Lemma 1.118.

Proposition 1.116. Under Assumption 1.105, let $p' : \widetilde{\mathfrak{Pic}}' \to \mathfrak{Pic}$ and $p'' : \widetilde{\mathfrak{Pic}}'' \to \mathfrak{Pic}$ be birational proper maps such that $((p')^*\mathfrak{F})^{\mathrm{tf}}$, $((p'')^*\mathfrak{F})^{\mathrm{tf}}$, $((p')^*(R^0\pi_*\mathfrak{L}^{\otimes k}))^{\mathrm{tf}}$ and $((p'')^*(R^0\pi_*\mathfrak{L}^{\otimes k}))^{\mathrm{tf}}$ are locally free. Consider $\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d)'$ and $\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d)''$ defined analogously to $\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d)$ above. Then we have

$$\int_{[\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d)']^{\mathrm{vir}}} \prod ev^*\gamma_i = \int_{[\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d)'']^{\mathrm{vir}}} \prod ev^*\gamma_i$$

Proposition 1.116 permits us to define the reduced Gromov–Witten invariants as they are independent of the blowing-up of $\mathfrak{Pic}$.

Definition 1.117. For $d > 2g-2$, we call *reduced* Gromov–Witten invariants of X, the following numbers

$$\int_{[\widetilde{\mathcal{M}}^{\circ}_{g,n}(X,d)]^{\mathrm{vir}}} \prod ev^*\gamma_i.$$

In order to prove Proposition 1.116, we first prove the following.

Lemma 1.118. Consider a commutative diagram of Artin stacks

$$\begin{tikzcd} \widehat{\mathfrak{Pic}} \arrow[rr] \arrow[dr] & & \widetilde{\mathfrak{Pic}} \arrow[dl] \\ & \mathfrak{Pic} & \end{tikzcd}$$

and let $\widehat{\mathcal{M}}(\mathbb{P})$ and $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$ be the corresponding fiber products $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d) \times_{\mathfrak{Pic}} \widehat{\mathfrak{Pic}}$, respectively $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d) \times_{\mathfrak{Pic}} \widetilde{\mathfrak{Pic}}$.

1 We have a diagram with Cartesian squares

$$\begin{tikzcd} \widehat{\mathcal{M}}(\mathbb{P}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d) \arrow[d] \\ \widehat{\mathfrak{Pic}} \arrow[r] & \widetilde{\mathfrak{Pic}} \arrow[r] & \mathfrak{Pic}. \end{tikzcd}$$

2 Suppose that $\widehat{\mathfrak{Pic}}$ and $\widetilde{\mathfrak{Pic}}$ are desingularizations of $\mathfrak{F}$. Let $\widehat{\mathcal{M}}^{\circ}(\mathbb{P})$ be the main component of $\widehat{\mathcal{M}}(\mathbb{P})$ in the sense of Definition 1.110. Under Assumption 1.105, the we have a commutative diagram

$$\begin{tikzcd} \widehat{\mathcal{M}}^{\circ}(\mathbb{P}) \arrow[r] \arrow[d] & \widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r,d) \arrow[r] \arrow[d] & \overline{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r,d) \arrow[d] \\ \widehat{\mathfrak{Pic}} \arrow[r] & \widetilde{\mathfrak{Pic}}' \arrow[r] & \mathfrak{Pic}. \end{tikzcd}$$

Proof. The first statement follows from the fact that the square on the right and the big square are Cartesian. This shows that the square on the left is Cartesian.

For the second statement, we apply point 1 and we consider the following extended diagram in which all squares are Cartesian

$$\begin{tikzcd}
\widehat{\mathcal{M}}^\circ \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \widetilde{\mathcal{M}}^\circ \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \overline{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d) \arrow[d] \\
\widehat{\mathcal{M}}(\mathbb{P}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \arrow[d] \\
\widehat{\mathfrak{Pic}} \arrow[r] & \widetilde{\mathfrak{Pic}} \arrow[r] & \mathfrak{Pic}.
\end{tikzcd}$$

By the definition of the main component of the moduli space of stable maps in Definition 1.106, we have solid maps in the diagram

$$\begin{tikzcd}
\widehat{\mathcal{M}}^\circ(\mathbb{P}) \arrow[r, dashed] \arrow[d] \arrow[rr, bend left] & \widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d) \arrow[r] \arrow[d] & \overline{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d) \arrow[d] \\
\widehat{\mathcal{M}}^\circ \arrow[r] & \widetilde{\mathcal{M}}^\circ \arrow[r] & \overline{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d).
\end{tikzcd}$$

The dashed arrow is the identity on maps with smooth domain. Since the map $\widehat{\mathcal{M}}^\circ \to \widetilde{\mathcal{M}}^\circ$ is proper it maps closed substacks to closed substacks, and thus the identity map extends to a map

$$\widehat{\mathcal{M}}^\circ(\mathbb{P}) \to \widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d). \quad \square$$

Proof of Proposition 1.116. Let $\widehat{\mathfrak{Pic}}$ denote the closure inside the fiber product $\widetilde{\mathfrak{Pic}}' \times_{\mathfrak{Pic}} \widetilde{\mathfrak{Pic}}''$ of locus of smooth curves. We then have a commutative diagram

$$\begin{tikzcd}
\widehat{\mathfrak{Pic}} \arrow[r] \arrow[d] & \widetilde{\mathfrak{Pic}}' \arrow[d, "p'"] \\
\widetilde{\mathfrak{Pic}}'' \arrow[r, "p''"] & \mathfrak{Pic}.
\end{tikzcd}$$

We define $\widehat{\mathcal{M}}(X)$ by the following Cartesian diagram

$$\begin{tikzcd}
\widehat{\mathcal{M}}(X) \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{g,n}(X, d) \arrow[d] \\
\widehat{\mathfrak{Pic}} \arrow[r] & \mathfrak{Pic}
\end{tikzcd}$$

and similarly we define

$$
\begin{tikzcd}
\widetilde{\mathcal{M}}_{g,n}(X,d)' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \overline{\mathcal{M}}_{g,n}(X,d) \arrow[d] \\
\widetilde{\mathfrak{Pic}}' \arrow[r, "p'"] & \mathfrak{Pic}
\end{tikzcd}
\quad
\begin{tikzcd}
\widetilde{\mathcal{M}}_{g,n}(X,d)'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \overline{\mathcal{M}}_{g,n}(X,d) \arrow[d] \\
\widetilde{\mathfrak{Pic}}'' \arrow[r, "p''"] & \mathfrak{Pic}
\end{tikzcd}
$$

Using the notation in (1.43) and Lemma 1.118, part 2, we obtain a commutative diagram

$$
\begin{tikzcd}
 & \widehat{\mathcal{M}}^\circ(X) \arrow[dl, "\widehat{p}'"'] \arrow[dr, "\widehat{p}''"] & \\
\widetilde{\mathcal{M}}^\circ_{g,n}(X,d)' \arrow[dr] & & \widetilde{\mathcal{M}}^\circ_{g,n}(X,d)'' \arrow[dl] \\
 & \overline{\mathcal{M}}^\circ_{g,n}(X,d) &
\end{tikzcd}
\tag{1.45}
$$

By Proposition 1.111 we have that $\widehat{p}'$ and $\widehat{p}''$ are proper. In the following we show that they are virtually birational.

Recall from Construction 1.115 that we have diagrams with Cartesian squares on the left

$$
\begin{tikzcd}
\widehat{\mathcal{M}}(X)^\circ \arrow[r] \arrow[d, "\widehat{p}'"] \arrow[dr, phantom, "\ulcorner"] & \widehat{\mathcal{M}}(\mathbb{P}^r)^\circ \arrow[r, "\widehat{\mu}'"] \arrow[d, "r'"] & \widehat{\mathfrak{Pic}} \arrow[d] \\
\widetilde{\mathcal{M}}^\circ_{g,n}(X,d)' \arrow[r, "i'"] & \widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r,d)' \arrow[r, "\mu'"] & \widetilde{\mathfrak{Pic}}'
\end{tikzcd}
\tag{1.46}
$$

and

$$
\begin{tikzcd}
\widehat{\mathcal{M}}(X)^\circ \arrow[r] \arrow[d, "\widehat{p}''"] \arrow[dr, phantom, "\ulcorner"] & \widehat{\mathcal{M}}(\mathbb{P}^r)^\circ \arrow[r, "\widehat{\mu}'"] \arrow[d, "r''"] & \widehat{\mathfrak{Pic}} \arrow[d] \\
\widetilde{\mathcal{M}}^\circ_{g,n}(X,d)'' \arrow[r, "i''"] & \widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r,d)'' \arrow[r, "\mu''"] & \widetilde{\mathfrak{Pic}}''
\end{tikzcd}
\tag{1.47}
$$

which give

$$[\widehat{\mathcal{M}}(X)^\circ]^{\mathrm{vir}} = (i')^![\widehat{\mathcal{M}}(\mathbb{P}^r)^\circ] \quad \text{and} \quad [\widehat{\mathcal{M}}(X)^\circ]^{\mathrm{vir}} = (i'')^![\widehat{\mathcal{M}}(\mathbb{P}^r)^\circ]. \tag{1.48}$$

Since r' and r'' are birational and proper we have

$$r'_*[\widehat{\mathcal{M}}(\mathbb{P}^r)^\circ] = [\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r,d)'] \quad \text{and} \quad r''_*[\widehat{\mathcal{M}}(\mathbb{P}^r)^\circ] = [\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r,d)'']. \tag{1.49}$$

Using (1.48), (1.49) and commutativity of pullbacks with push-forwards in eq. (1.46) and eq. (1.47), we get

$$\begin{aligned} \widetilde{p}'_*[\widehat{\mathcal{M}}(X)^\circ]^{\mathrm{vir}} &= [\widetilde{\mathcal{M}}^\circ_{g,n}(X,d)']^{\mathrm{vir}} \quad \text{and} \\ \widetilde{p}''_*[\widehat{\mathcal{M}}(X)^\circ]^{\mathrm{vir}} &= [\widetilde{\mathcal{M}}^\circ_{g,n}(X,d)'']^{\mathrm{vir}}. \end{aligned} \tag{1.50}$$

Intersecting both equations above with $\prod ev^*\gamma_i$ we get the conclusion. □

In the following we discuss several ways to blow up $\mathfrak{Pic}$, to desingularize $R^0\pi_*\mathfrak{L}^{\otimes k}$.

Lemma 1.119. Given $\mathcal{F}$, $\mathcal{G}$ sheaves on an integral scheme X, with $\mathcal{G}$ torsion free and $f : \mathcal{F} \to \mathcal{G}$ a morphism, we have that f factors through

$$\mathcal{F} \to \mathcal{F}^{\mathrm{tf}} \to \mathcal{G}.$$

Proof. Since X is integral and $\mathcal{G}$ is torsion free, we have that the composition $\mathrm{Tor}(\mathcal{F}) \to \mathcal{F} \to \mathcal{G}$ is zero. The claim now follows from the universal property of quotients. □

Proposition 1.120. In notation of Section 1.7.3, we have the following:

1 An isomorphism

$$\psi : (p^*R^0\pi_*\mathfrak{L})^{\mathrm{tf}} \to R^0\widetilde{\pi}_*\widetilde{\mathfrak{L}}.$$

2 $\mathrm{Bl}_{R^1\pi_*\mathfrak{L}^{\otimes k}}\mathfrak{Pic}$ is a desingularization of $R^0\pi_*\mathfrak{L}^{\otimes k}$, for any integer $k > 0$.

Proof. 1. As in Remark 1.103, let A a section of $\mathfrak{C} \to \mathfrak{Pic}$ such that $R^1\pi_*(\mathfrak{L}(A)) = 0$. By abuse of notation we denote by A the pull back of A to $\widetilde{\mathfrak{C}}$. We have exact sequences on $\widetilde{\mathfrak{Pic}}$ which fit into a commutative diagram

$$\begin{tikzcd} & p^*R^0\pi_*\mathfrak{L} \arrow[r] \arrow[d, dashed] & p^*R^0\pi_*(\mathfrak{L}(A)) \arrow[r] \arrow[d] & p^*R^0\pi_*(\mathfrak{L}(A)|_A) \arrow[r] \arrow[d] & p^*R^1\pi_*\mathfrak{L} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & R^0\widetilde{\pi}_*\widetilde{\mathfrak{L}} \arrow[r] & R^0\widetilde{\pi}_*(\widetilde{\mathfrak{L}}(A)) \arrow[r] & R^0\widetilde{\pi}_*(\widetilde{\mathfrak{L}}(A)|_A) \arrow[r] & R^1\widetilde{\pi}_*\widetilde{\mathfrak{L}} \arrow[r] & 0, \end{tikzcd}$$

where the vertical arrows are obtained by cohomology and base change and the solid arrows are isomorphisms.

By Lemma 1.119, we have a morphism

$$\psi : (p^*R^0\pi_*\mathfrak{L})^{\mathrm{tf}} \to R^0\widetilde{\pi}_*\widetilde{\mathfrak{L}}$$

which sits in a commutative diagram

$$\begin{tikzcd} p^*R^0\pi_*\mathfrak{L} \arrow[r] & (p^*R^0\pi_*\mathfrak{L})^{\mathrm{tf}} \arrow[r] \arrow[d] & p^*R^0\pi_*(\mathfrak{L}(A)) \arrow[d] \\ 0 \arrow[r] & R^0\widetilde{\pi}_*\widetilde{\mathfrak{L}} \arrow[r] & R^0\widetilde{\pi}_*(\widetilde{\mathfrak{L}}(A)). \end{tikzcd} \tag{1.51}$$

Since the Image of the map $(p^*R^0\pi_*\mathfrak{L})^{\mathrm{tf}} \to p^*R^0\pi_*(\mathfrak{L}(A))$ is equal to the kernel of $p^*R^0\pi_*(\mathfrak{L}(A)) \to p^*R^0\pi_*(\mathfrak{L}(A)|_A)$, and the Image of $R^0\widetilde{\pi}_*\widetilde{\mathfrak{L}} \to R^0\widetilde{\pi}_*\widetilde{\mathfrak{L}}(A)$, is equal to the kernel of $R^0\widetilde{\pi}_*(\widetilde{\mathfrak{L}}(A)) \to R^0\widetilde{\pi}_*(\widetilde{\mathfrak{L}}(A)|_A)$, the isomorphisms in the diagram show that

$$\operatorname{Im}\left((p^*R^0\pi_*\mathfrak{L})^{\mathrm{tf}} \to p^*R^0\pi_*(\mathfrak{L}(A))\right) \simeq \operatorname{Im}\left(R^0\widetilde{\pi}_*\widetilde{\mathfrak{L}} \to R^0\widetilde{\pi}_*(\widetilde{\mathfrak{L}}(A))\right). \tag{1.52}$$

Since $p^*R^0\pi_*\mathfrak{L} \to p^*R^0\pi_*(\mathfrak{L}(A))$ and $p^*R^0\pi_*\mathfrak{L} \to R^0\widetilde{\pi}_*\widetilde{\mathfrak{L}}$ are generically injective we have that $(p^*R^0\pi_*\mathfrak{L})^{\mathrm{tf}} \to p^*R^0\pi_*(\mathfrak{L}(A))$ and $(p^*R^0\pi_*\mathfrak{L})^{\mathrm{tf}} \to R^0\widetilde{\pi}_*\widetilde{\mathfrak{L}}$ are injective. This together with 1.52 shows that ψ is an isomorphism.

2. Without loss of generality we assume that $k = 1$. Let $p : \mathrm{Bl}_{R^1\pi_*\mathfrak{L}}\mathfrak{Pic} \to \mathfrak{Pic}$ denote the projection. By cohomology and base change we have $p^*(R^1\pi_*\mathfrak{L}) \simeq R^1\widetilde{\pi}_*\widetilde{\mathfrak{L}}$. With this, we have that $(R^1\widetilde{\pi}_*\widetilde{\mathfrak{L}})^{\mathrm{tf}}$ is locally free.

Since

$$\left((R^1\widetilde{\pi}_*\widetilde{\mathfrak{L}})^{\mathrm{tf}}\right)^{\vee} \simeq (R^1\widetilde{\pi}_*\widetilde{\mathfrak{L}})^{\vee}$$

and $(R^1\widetilde{\pi}_*\widetilde{\mathfrak{L}})^{\mathrm{tf}}$ is locally free, we get that $(R^1\widetilde{\pi}_*\widetilde{\mathfrak{L}})^{\vee}$ is locally free.

By Proposition 1.102 we have that $R^0\widetilde{\pi}_*\widetilde{\mathfrak{L}} \simeq (R^1\widetilde{\pi}_*\widetilde{\mathfrak{L}})^{\vee}$. This together with the above shows that $R^0\widetilde{\pi}_*\widetilde{\mathfrak{L}}$ is locally free. The claim now follows from the first part of the proposition. □

Proposition 1.121. In notation of Section 1.7.3, we have that $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{Pic}$ is a desingularization of $R^0\pi_*\mathfrak{L}^{\otimes k}$.

Proof. We prove the statement for $k = 1$. By cohomology and base change we have $p^*R^1\pi_*\mathfrak{L}^{\vee} \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}} \simeq R^1\widetilde{\pi}_*\widetilde{\mathfrak{L}}^{\vee} \otimes \omega_{\widetilde{\mathfrak{C}}/\widetilde{\mathfrak{Pic}}}$. By Proposition 1.102 we have $(R^1\widetilde{\pi}_*\widetilde{\mathfrak{L}}^{\vee} \otimes \omega_{\widetilde{\mathfrak{C}}/\widetilde{\mathfrak{Pic}}})^{\vee} \simeq R^0\widetilde{\pi}_*\widetilde{\mathfrak{L}}$. By Proposition 1.120 we have $R^0\widetilde{\pi}_*\widetilde{\mathfrak{L}} \simeq (p^*R^0\pi_*\mathfrak{L})^{\mathrm{tf}}$. With these we get

$$\left(\left(p^*R^1\pi_*\mathfrak{L}^{\vee} \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}\right)^{\mathrm{tf}}\right)^{\vee} \simeq \left(p^*R^1\pi_*\mathfrak{L}^{\vee} \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}\right)^{\vee} \simeq (p^*R^0\pi_*\mathfrak{L})^{\mathrm{tf}}.$$

This shows that $\left(p^*R^1\pi_*\mathfrak{L}^{\vee} \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}\right)^{\mathrm{tf}}$ is locally free implies $(p^*R^0\pi_*\mathfrak{L})^{\mathrm{tf}}$ is locally free.

Notice that in general the dual does not commute with pullback, so the above argument was needed.

In the following we show that $\mathrm{Bl}_{R^0\pi_*\mathfrak{L}}\mathfrak{Pic}$ is a desingularization of $R^0\pi_*\mathfrak{L}^{\otimes k}$, for any $k > 0$.

Locally we choose B a section such that $\mathfrak{L}^k \simeq \mathfrak{L}(B)$. Taking A such that $R^1\pi_*\mathfrak{L}(A) = 0$, we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & R^0\pi_*\mathfrak{L} \arrow[r, "\cdot A"] \arrow[d] & R^0\pi_*\mathfrak{L}(A) \arrow[d, "\cdot B"] \\
0 \arrow[r] & R^0\pi_*\mathfrak{L}(B) \arrow[r] & R^0\pi_*\mathfrak{L}(A+B).
\end{tikzcd}$$

Let U be the subset of $\mathfrak{Pic}$, where $R^1\pi_*\mathfrak{L} = 0$. Since we work under the assumption 1.105, U is a non-empty open subset. Then on U we have the following exact sequences

$$\begin{aligned}
0 \to R^0\pi_*\mathfrak{L} \to & R^0\pi_*\mathfrak{L}(B) \to R^0\pi_*\mathfrak{L}(B)|_B \to 0 \\
0 \to R^0\pi_*\mathfrak{L}(A) \to & R^0\pi_*\mathfrak{L}(A+B) \to R^0\pi_*\mathfrak{L}(A+B)|_B \to 0.
\end{aligned} \tag{1.53}$$

By possibly shrinking U, the section A may be chosen to avoid B. With this, we have that multiplication with A induces an isomorphism

$$R^0\pi_*\mathfrak{L}(B)|_B \simeq R^0\pi_*\mathfrak{L}(A+B)|_B.$$

By possibly shrinking U, we may assume that $R^0\pi_*\mathfrak{L}(A)$ and $R^0\pi_*\mathfrak{L}(A+B)$ are trivial, and that the sequences in 1.53 are split. The claim now follows from Corollary 1.44. □

In genus one, following [VZ08, HL10], one can define reduced Gromov-Witten invariants of degree-k hypersurfaces on $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{Pic}$. Below we give a direct proof of this fact. The proof below does not generalise to higher genus.

Lemma 1.122. Let $g = 1$. Then for every $k \geq 1$, the sheaf $\widetilde{\pi}_* \operatorname{ev}^* \mathcal{O}(k)$ is locally free on the main component $\widetilde{\mathcal{M}}^\circ_{1,n}(\mathbb{P}^r, d)$ of $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$.

Proof. Fix $k \geq 1$. By Equation (1.63), we need to show that $R^0\widetilde{\pi}_*(\mathfrak{L}^{\otimes k})$ is locally free over the image Z° of $\widetilde{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$ in $\widetilde{\mathfrak{Pic}}$ via the forgetful morphism.

In a neighbourhood of $(\widetilde{C}, \widetilde{L}) \in \widetilde{\mathfrak{Pic}}_1$ we can choose a section A of $\mathfrak{L}^{\otimes k-1}$. This gives an exact sequence

$$0 \to R^0\widetilde{\pi}_*\widetilde{\mathfrak{L}} \xrightarrow{\cdot A} R^0\widetilde{\pi}_*(\widetilde{\mathfrak{L}}^{\otimes k}) \to R^0\widetilde{\pi}_*(\widetilde{\mathfrak{L}}^{\otimes k}|_A) \to R^1\widetilde{\pi}_*\widetilde{\mathfrak{L}} \xrightarrow{\cdot A} R^1\widetilde{\pi}_*(\widetilde{\mathfrak{L}}^{\otimes k}) \to 0. \tag{1.54}$$

If $\tilde{L}$ is non-negative on each component of $\tilde{C}$, as is the case in Z°, we have the equality

$$h^1(\widetilde{C}, \widetilde{L}) = h^1(\widetilde{C}, \widetilde{L}^{\otimes k})$$

by a Riemann-Roch computation. Since cohomology in degree 1 commutes with base change and the map $H^1(\widetilde{C}, \widetilde{L}) \to H^1(\widetilde{C}, \widetilde{L}^{\otimes k})$ is an isomorphism, we have that the last arrow in sequence (1.54) is an isomorphism. This shows that we have a short exact sequence

$$0 \to R^0\widetilde{\pi}_*\widetilde{\mathfrak{L}} \to R^0\widetilde{\pi}_*(\widetilde{\mathfrak{L}}^{\otimes k}) \to R^0\widetilde{\pi}_*(\widetilde{\mathfrak{L}}^{\otimes k}|_A) \to 0.$$

Since $R^0\widetilde{\pi}_*(\mathfrak{L}^{\otimes k}|_A)$ is locally free and by Proposition 1.120 the sheaf $R^0\widetilde{\pi}_*\widetilde{\mathfrak{L}} \simeq (R^1\widetilde{\pi}_*\widetilde{\mathfrak{L}})^\vee$ is also locally on Z°, we get that $R^0\widetilde{\pi}_*(\widetilde{\mathfrak{L}}^{\otimes k})$ is locally free. □

Remark 1.123. Above we denoted by $\widetilde{\mathfrak{Pic}}$ any desingularization of $\mathfrak{F}$. We collect here various blow-ups of interest.

1 $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{Pic}$ in the sense of Rossi (see Section 1.5.1)

2 $\mathrm{Bl}_{\mathfrak{F}}^{HL}\mathfrak{Pic}$ in the sense of Hu–Li.

3 $\mathrm{Bl}_{R^1\pi_*\mathfrak{L}^{\otimes k}}\mathfrak{Pic}$

4 $\mathrm{Bl}_{R^0\pi_*\mathfrak{L}^{\otimes k}}\mathfrak{Pic}$

5 $\mathrm{Bl}^{HL}_{R^1\pi_*\mathfrak{L}^{\otimes k}}\mathrm{Bl}_{\mathfrak{F}}^{HL}\mathfrak{Pic}$

We have

$$\mathrm{Bl}_{\mathfrak{F}}^{HL}\mathfrak{Pic} \to \mathrm{Bl}_{\mathfrak{F}}\mathfrak{Pic} \quad \text{and} \quad \mathrm{Bl}^{HL}_{R^1\pi_*\mathfrak{L}^{\otimes k}}\mathrm{Bl}_{\mathfrak{F}}^{HL}\mathfrak{Pic} \to \mathrm{Bl}_{R^1\pi_*\mathfrak{L}^{\otimes k}}\mathrm{Bl}_{\mathfrak{F}}\mathfrak{Pic}.$$

By Proposition 1.121 and by proposition 1.120 we have

$$\mathrm{Bl}_{R^0\pi_*\mathfrak{L}^{\otimes k}}\mathfrak{Pic} \to Bl_{R^1\pi_*\mathfrak{L}^{\otimes k}}\mathfrak{Pic} \to \mathrm{Bl}_{\mathfrak{F}}\mathfrak{Pic}.$$

1.7.5 Reduced invariants from stable maps with fields

Reduced invariants are conjecturally related to Gromov–Witten invariants [Zin09a], [HL11, Conjecture 1.1]. One of the main difficulties in proving such conjectures is that one needs to understand how to split the virtual class of a moduli space of stable maps among its irreducible components. What makes this task particularly difficult is that almost nothing is known about the geometry of this moduli space of stable maps.

In genus one and two the existing algebraic proofs [CL15, LLO22, LO21, LO22] use an additional well-behaved moduli space of maps with fields [CL12]. In view of these conjectures, we discuss blow-ups of maps with fields and we show that the ingredients for the low genus proof are also available in higher genus.

Given X a hypersurface (or more generally a complete intersection) in $\mathbb{P}^r$, the associated moduli space of maps with fields has the following features:

1 in genus one and two it has well-understood geometry, such as local equations and irreducible components (see [HL10, HLN12]);

2 it has a virtual class and a localised virtual class (see [KL13], Definition 3.3); the latter is needed because the space of maps with fields is *not compact*;

3 the localised virtual class is supported on $\overline{\mathcal{M}}_{g,n}(X,d)$ and it coincides up to a sign to the virtual class of $\overline{\mathcal{M}}_{g,n}(X,d)$ (see [KL13], Theorem 1.1).

These properties allow us to work with the well-understood moduli space of stable maps with fields instead of $\overline{\mathcal{M}}_{g,n}(X,d)$, whose geometry is unavailable.

In the following we blow up the moduli space of maps with fields, to define reduced invariants in this context. The main theorem of this section, Theorem 1.124, shows that the blown-up moduli space of maps with fields also has the property 1 listed above. In future work we will also investigate the (intrinsic) normal cone of the space of maps with fields.

Review of maps with fields

We recall the construction and properties of the moduli space of maps with p-fields. This space was introduced in [CL12] to study higher genus Gromov-Witten invariants of the quintic threefold.

In the following we fix $k \in \mathbb{Z}$, $k > 1$, and we consider the sheaf

$$\pi_*(\mathfrak{L}^{\oplus r+1} \oplus (\mathfrak{L}^{\otimes -k} \otimes \omega_{\mathfrak{C}/\mathfrak{Pic}}))$$

on $\mathfrak{Pic}$ and its corresponding abelian cone

$$S\left(\pi_*(\mathfrak{L}^{\oplus r+1}\oplus(\mathfrak{L}^{\otimes -k}\otimes\omega_{\mathfrak{C}/\mathfrak{Pic}})\right)=\operatorname{Spec}\operatorname{Sym}R^1\pi_*\left(\left(\mathfrak{L}^\vee\otimes\omega_{\mathfrak{C}/\mathfrak{Pic}}\right)^{\oplus r+1}\oplus\mathfrak{L}^{\otimes k}\right)\xrightarrow{\mu^p}\mathfrak{Pic}.$$

Recall that, in our notation, we have already imposed on $\mathfrak{Pic}=\mathfrak{Pic}^{\mathrm{st}}_{g,n,d}$ the stability condition in Equation (1.24). Then, the *moduli space of maps with p-fields* is defined in [CL12, Section 3.1] as the abelian cone

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p=S\left(\pi_*(\mathfrak{L}^{\oplus r+1}\oplus(\mathfrak{L}^{\otimes -k}\otimes\omega_{\mathfrak{C}/\mathfrak{Pic}})\right)$$

Therefore, an element of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p$ over $(C,L,\underline{s})\in\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)$ is given by a choice of a section $p\in H^0(C,L^{\otimes -k}\otimes\omega_C)$.

Consider the Cartesian diagram

$$\begin{tikzcd}
\mathcal{C}^p \arrow[r, "\nu^p"] \arrow[d, "\overline{\pi}^p"] & \mathcal{C} \arrow[d, "\overline{\pi}"] \\
\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p \arrow[r] & \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d).
\end{tikzcd} \tag{1.55}$$

The complex

$$\mathbb{E}^\bullet:=R^\bullet\overline{\pi}^p_*(\oplus_{i=0}^r\mathcal{L}\oplus\mathcal{L}^{\otimes -k}\otimes\omega_{\overline{\pi}^p})$$

is a dual obstruction theory for the morphism μ^p. The stack $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p$ is not proper, but the perfect obstruction theory admits a cosection σ, that is, a morphism $\sigma\colon h^1(\mathbb{E}^\vee)\to\mathcal{O}_{\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p}$. This data gives a cosection localised virtual class $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p]^{\mathrm{vir}}$.

For X a smooth subvariety cut out by any regular section of $\mathcal{O}_{\mathbb{P}^r}(k)$, [CL20, Theorem 1.1] states that

$$[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p]^{\mathrm{vir}}=(-1)^{(r+1)d+1-g}[\overline{\mathcal{M}}_{g,n}(X,d)]^{\mathrm{vir}}\in A_{d^{\mathrm{vir}}(\overline{\mathcal{M}}_{g,n}(X,d))}(\overline{\mathcal{M}}_{g,n}(X,d)) \tag{1.56}$$

The particular case where $r=4$ and $k=5$ (therefore X is a quintic threefold) was the motivation for introducing p-fields in the first place. In [CL12, Theorem 1.1], the authors proved Equation (1.56) at the level of invariants before it was upgraded to classes in [CL20].

Blow-ups of maps with fields

In this section we discuss a non-minimal blow-up which has good properties.

In notation as before we consider

$$\mathfrak{E}_k := R^1\pi_*\mathfrak{L}^{\otimes k}. \tag{1.57}$$

We define

$$\widetilde{\mathfrak{Pic}}_k := \mathrm{Bl}^{HL}_{\mathfrak{E}_k}\mathrm{Bl}^{HL}_{\tilde{\mathfrak{F}}}\mathfrak{Pic}. \tag{1.58}$$

Similarly to Equation (1.39), we define $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p$ as the following Cartesian diagram:

$$\begin{tikzcd} \widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p \arrow[r] \arrow[d, "\widetilde{\mu}^p"] \arrow[dr, phantom, "\ulcorner", very near start] & \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p \arrow[d, "\mu^p"] \\ \widetilde{\mathfrak{Pic}}_k \arrow[r, "p_k"] & \mathfrak{Pic}. \end{tikzcd} \tag{1.59}$$

This gives a virtual class and a localised virtual class $[\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p]^{\mathrm{vir}}$. See [CL12, Section 3] for details.

Since $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d) \hookrightarrow \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p$, we have $\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d) \subset \widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p$. Under the assumption in 1.105, we have that $R^1\pi_*\mathcal{L}^{\otimes k} = 0$. In this case we define the main component of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p$ to be $\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d)$. From the obstruction theory, we see that under the assumption 1.105, $\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d)$ is indeed a component of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p$; in fact, it agrees with the main component of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p$ as an abelian cone.

Theorem 1.124. Denote by $(\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{p,\lambda})_{\lambda\in\Lambda}$ the irreducible components of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p$ and $(\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{\theta})_{\theta\in\Theta}$ the irreducible components of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$.

Let

$$\widehat{\pi}^{p,\lambda} : \widetilde{\mathcal{C}}^{\lambda} \to \widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{p,\lambda}$$
$$\tilde{\pi}^{\theta} : \widetilde{\mathcal{C}}^{\theta} \to \widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{\theta}$$

the associated universal curves. The following statements hold.

1 The morphism $\bar{p}_k$ is birational and proper.

2 The irreducible components $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{p,\lambda}$ and $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{\theta}$ are smooth over their image in $\widetilde{\mathfrak{Pic}}_k$. In particular, $\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d)$ is smooth over $\widetilde{\mathfrak{Pic}}_k$.

3 The sheaf $\widehat{\pi}^{p,\lambda}_* ev^*\mathcal{O}(k)$ is a locally free sheaf on $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{p,\lambda}$, the sheaf $\tilde{\pi}^{\theta}_* ev^*\mathcal{O}(k)$ is a locally free sheaf on $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{\theta}$. In particular, $\tilde{\pi}^{\circ}_* ev^*\mathcal{O}(k)$ is a locally free sheaf on $\widetilde{\mathcal{M}}^{\circ}_{g,n}(\mathbb{P}^r, d)$.

Proof. 1. We have that $\overline{p}_k$ is proper, as p_k is proper.

2. Consider the following diagram

$$\begin{tikzcd}
\operatorname{Spec}\operatorname{Sym} p_k^*(\mathfrak{F}\oplus\mathfrak{E}_k) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \operatorname{Spec}\operatorname{Sym}\mathfrak{F}\oplus\mathfrak{E}_k \arrow[d] \\
\widetilde{\mathfrak{Pic}}_k \arrow[r, "p_k"] & \mathfrak{Pic}.
\end{tikzcd} \tag{1.60}$$

We have that $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p$ is an open substack of $\operatorname{Spec}\operatorname{Sym} p_k^*(\mathfrak{E}_k \oplus \oplus_{i=0}^r \mathfrak{F})$ and by Theorem 1.97 we have that the irreducible components of the stacks

$$\operatorname{Spec}\operatorname{Sym} \ \oplus_{i=0}^r \mathcal{F} \oplus \mathfrak{E}_k \quad \text{and} \quad \operatorname{Spec}\operatorname{Sym} \ \oplus_{i=0}^r \mathfrak{F}$$

are smooth over their image in $\widetilde{\mathfrak{Pic}}_k$.

This shows that $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{p,\lambda}$ and $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^\theta$ are smooth over their image in $\widetilde{\mathfrak{Pic}}_k$. In particular

$$\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d) = \operatorname{Spec}\operatorname{Sym}(\oplus_{i=0}^r p_k^*\mathfrak{F})^{\mathrm{tf}} \text{ is smooth over } \widetilde{\mathfrak{Pic}}_k.$$

3. Let $\mu^\lambda : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^{p,\lambda} \to \mathfrak{Pic}$ be the restriction of μ^p. Let Z^λ be the image of μ^λ. Let $\pi^\lambda : \mathfrak{C}^\lambda \to Z^\lambda$ be the restriction of π. Let $\widetilde{Z}^\lambda$ be the fiber product

$$\begin{tikzcd}
\widetilde{Z}^\lambda \arrow[r, "p_k^\lambda"] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Z^\lambda \arrow[d] \\
\widetilde{\mathfrak{Pic}}_k \arrow[r, "p_k"] & \mathfrak{Pic}.
\end{tikzcd} \tag{1.61}$$

Let $\tilde{\pi}^\lambda : \widetilde{\mathfrak{C}}^\lambda \to Z^\lambda$ be the restriction of $\tilde{\pi}$ and let $q_k^\lambda : \widetilde{\mathfrak{C}}^\lambda \to \mathfrak{C}^\lambda$ be the restriction of q_k. By commutativity of proper push-forwards with base-change we have that

$$(p_k^\lambda)^* R^\bullet \pi_*^\lambda \mathfrak{L} \simeq R^\bullet \tilde{\pi}_*^\lambda (q_k^\lambda)^* \mathfrak{L}$$

Again, cohomology and base-change in the Cartesian diagram

$$\begin{tikzcd}
\widetilde{\mathcal{C}}^\lambda \arrow[r, "\nu^\lambda"] \arrow[d, "\tilde{\pi}^\lambda"] \arrow[dr, phantom, "\ulcorner"] & \widetilde{\mathfrak{C}}^\lambda \arrow[d, "\tilde{\pi}^\lambda"] \\
\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^\lambda \arrow[r, "\mu^\lambda"] & \widetilde{Z}^\lambda.
\end{tikzcd} \tag{1.62}$$

gives

$$R^\bullet\widehat{\pi}^\lambda_* ev^*\mathcal{O}(k) = (\mu^\lambda)^* R^\bullet\tilde{\pi}^\lambda_*\mathfrak{L}^{\otimes k}. \tag{1.63}$$

We have a short exact sequence

$$0 \to R^0\tilde{\pi}^\lambda_*\mathfrak{L}^k \to E^0 \xrightarrow{\phi} E^1 \to R^1\tilde{\pi}^\lambda_*\mathfrak{L}^{\otimes k} \to 0.$$

By construction we have that ϕ is locally diagonal. By Equation (1.63) we have that

$$R^\bullet\widehat{\pi}^\lambda_* ev^*\mathcal{O}(k) \simeq [(\mu^\lambda)^*E^0 \xrightarrow{(\mu^\lambda)^*\phi} (\mu^\lambda)^*E^1].$$

Since $(\mu^\lambda)^*\phi$ is locally diagonal, Proposition 1.48 implies that $R^0\widehat{\pi}^\lambda_* ev^*\mathcal{O}(k)$ is locally free.

A similar argument shows that $\tilde{\pi}^\theta_* ev^*\mathcal{O}(k)$ is a locally free sheaf on $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^\theta$ and in particular $\tilde{\pi}^\circ_* ev^*\mathcal{O}(k)$ is a locally free sheaf on $\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d)$. □

We also have the following.

Proposition 1.125. The localised invariants do not depend on the blow-up of $\mathfrak{Pic}$, more precisely,

$$\deg[\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p]^{\mathrm{vir}} = \deg[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p]^{\mathrm{vir}}.$$

Proof. It follows from Equation (1.56) and Proposition 1.116. □

Remark 1.126. The results of Proposition 1.116 and Proposition 1.125 can be stated at level of virtual classes. The statement of Proposition 1.116 with virtual classes is given by Equation (1.50).

Reduced invariants from maps with fields

Let $\mathfrak{C}_{\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p/\widetilde{\mathfrak{Pic}}} = \cup_i \mathfrak{C}_i$ where $\mathfrak{C}_i$ denotes an irreducible component and let $\mathfrak{C}_0$ denote the component supported on the main component of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p$. Let $[\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p]^{\mathrm{vir}}_i$ denote the class corresponding to the component $\mathfrak{C}_i$.

Proposition 1.127. Denote by $[\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p]^{\mathrm{vir}}_0$ the virtual fundamental class corresponding to the cone $\mathfrak{C}_0$. We have

$$\deg[\widetilde{\mathcal{M}}^\circ_{g,n}(X, d)]^{\mathrm{vir}} = (-1)^{kd-g+1}\deg[\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p]^{\mathrm{vir}}_0.$$

Proof. This follows the lines of proof of Corollary 4.4 in [CL15]. Let

$$\mathbb{E}^\bullet_1 := R^\bullet\widehat{\pi}^p_*(\oplus_{i=0}^r\mathcal{L}), \quad \mathbb{E}^\bullet_2 := R^\bullet\widehat{\pi}^p_*(\mathcal{L}^{\otimes -k}\otimes\omega_{\widehat{\pi}^p})$$

and let $\mathcal{E}_i = h^1/h^0(\mathbb{E}_i^\bullet)$ and $\mathcal{E} = h^1/h^0(\mathbb{E}^\bullet)$. We have that $\mathcal{E}_i$ is a vector bundle stack on $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p$ and $\mathcal{E} \simeq \mathcal{E}_1 \oplus \mathcal{E}_2$. Let $\mathcal{U}$ be the open subset of the main component of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p$, with consists of maps with fields with irreducible source. On $\mathcal{U}$ we have $R^1\widehat{\pi}_* f^*\mathcal{O}(k) = 0$, and thus $\mathcal{U}$ is also an open subset of $\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d)$. Using that $\mathcal{U}$ is smooth and unobstructed, we see that $\mathfrak{C}_{\mathcal{U}/\widetilde{\mathfrak{Pic}}}$ is isomorphic to the vector bundle stack $\mathcal{E}_1|_{\mathcal{U}}$. Since the embedding $\mathfrak{C}_{\mathcal{U}/\widetilde{\mathfrak{Pic}}} \hookrightarrow h^1/h^0(\mathcal{E}|_{\mathcal{U}})$ is

$$(\mathcal{E}_1 \oplus 0)|_{\mathcal{U}} \hookrightarrow (\mathcal{E}_1 \oplus \mathcal{E}_2)|_{\mathcal{U}}$$

and $\mathfrak{C}_{\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r,d)^p/\widetilde{\mathfrak{Pic}}} \hookrightarrow \mathcal{E}$ is a closed embedding, we get that $\mathfrak{C}_0 \simeq \mathcal{E}_1$. By the definition of the localised cosection virtual class, we get

$$[\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p]^{\mathrm{vir}}_0 = 0^!_{\sigma,loc}[\mathfrak{C}_0] = 0^![0_{\mathcal{E}_2}],$$

where $0_{\mathcal{E}_2}$ is the zero section of $\mathcal{E}_2|_{\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r,d)}$. By Lemma 4.3 in [CL15] with the complex $R := R^\bullet\widehat{\pi}^{p,\circ}_* \mathcal{L}^{\otimes k}$ and Theorem 1.124, part 3 we get

$$[\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p]^{\mathrm{vir}}_0 = c_{\mathrm{top}}(R^1\widehat{\pi}^{p,\circ}_*(\mathcal{L}^{\otimes -k} \otimes \omega_{\widehat{\pi}^{p,\circ}})) \cdot [\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d)].$$

By Serre duality we have that

$$c_{1-g+kd}(R^1\widehat{\pi}^{p,\circ}_*(\mathcal{L}^{\otimes -k} \otimes \omega_{\widehat{\pi}^p})) = (-1)^{1-g+kd} c_{1-g+kd}(R^0\widehat{\pi}^p_*\mathcal{L}^{\otimes k})$$

and thus

$$[\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p]^{\mathrm{vir}}_0 = (-1)^{1-g+kd} c_{1-g+kd}(R^0\widehat{\pi}^p_*\mathcal{L}^{\otimes k}) \cdot [\widetilde{\mathcal{M}}^\circ_{g,n}(\mathbb{P}^r, d)].$$

This proves the claim. □

Conjecture 1.128. Let X be a threefold which is a complete intersection in projective space. Then

$$\deg[\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)^p]^{\mathrm{vir}}_i = c_i \deg[\widetilde{\mathcal{M}}^\circ_{g_i,n}(X, d)]^{\mathrm{vir}},$$

for some $c_i \in \mathbb{Q}$ and $g_i < g$.

Remark 1.129. The conjecture has been proved for genus one [Zin09a, Zin09b, Zin08], [CL15], [LO21, LO22] and genus two [LLO22].

In genus $g = 1$ and X a Calabi–Yau threefold, the conjecture translates into

$$\deg[\overline{\mathcal{M}}_{1,n}(X, d)] = \deg[\widetilde{\mathcal{M}}^\circ_{1,n}(X, d)]^{\mathrm{vir}} + \frac{1}{12} \deg[\overline{\mathcal{M}}_{0,n}(X, d)]^{\mathrm{vir}}.$$

1.8 Desingularizations in genus one

In genus one, reduced Gromov–Witten invariants were originally defined using the desingularization constructed in [VZ08]. It consists of a sequence of blow-ups determined by the geometry of the moduli space $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d)$. In [HL10], local equations for the blow-up are determined. We aim to compare this desingularization with the one obtained using the Rossi blow-up $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{Pic}$, with $\mathfrak{F}$ as in Equation (1.27). In particular, we describe $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{Pic}$ locally in the spirit of [HL10].

Charts

In genus one, the original definition of refined Gromov–Witten invariants comes from [VZ08]. The main is idea is to apply a sequence of blow ups to $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ in order to desingularize the main component. Strictly speaking, the sequence of blow ups takes place in the stack $\mathfrak{M}_1^{wt}$ of genus-1 prestable curves endowed with a weight. Let Θ_k denote the closure of the loci in $\mathfrak{M}_1^{wt}$ of curves with k trees of rational curves attached to the core. Then one should blow up $\mathfrak{M}_1^{wt}$ along the loci Θ_1, Θ_2, Θ_3 and so on in order to produce a stack $\widetilde{\mathfrak{M}}_1^{wt}$. This process induces a blow-up $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ via fiber product.

Given a stratum $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)_\gamma$ corresponding to a weighted graph γ, local equations of $\widetilde{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ and the local description of Θ_k in that stratum are described explicitly in [HL10]. The purpose of this section is to summarize such local description, give coordinates for the new approach locally, and compare both.

It may be helpful to keep in mind the following diagram, described below.

$$
\begin{tikzcd}
\mathcal{E}_{\mathcal{V}} \arrow[rr, "\widetilde{\phi}"] & & E_\gamma \arrow[d] \\
(F=0) \arrow[u, hook] \arrow[r] & (\Phi_\gamma = 0) \arrow[ur, hook] & V_\gamma \\
\mathcal{U} \arrow[u] \arrow[r] \arrow[d] & \mathcal{V} \arrow[ur, "\phi"] \arrow[d] & \widetilde{\mathcal{V}} \arrow[l] \arrow[d] \arrow[dl, phantom, "\square"] \\
U \arrow[r] & \mathfrak{D}_1 \arrow[d] & \widetilde{\mathfrak{D}}_1 \arrow[l] \arrow[d] \arrow[dl, phantom, "\square"] \\
 & \mathfrak{M}_1^{wt} & \widetilde{\mathfrak{M}}_1^{wt} \arrow[l]
\end{tikzcd}
$$

Fix a weighted graph γ with root o. Let $\mathrm{Ver}(\gamma)$, $\mathrm{Ver}(\gamma)^t$ and $\mathrm{Ver}(\gamma)^*$ denote the vertices, the terminal vertices (or leaves) and the non-rooted vertices of γ, respectively. We take the natural ordering in $\mathrm{Ver}(\gamma)$ making the root o the minimal element. We assume that the weight in γ is non-negative on every vertex and that γ is terminally weighted, meaning that the vertices with non-zero weights are exactly those in $\mathrm{Ver}(\gamma)^t$. Let $\mathfrak{M}_1^{wt}$ is the stack of genus-1 prestable curves endowed with a weight. Remember that every element C parameterised by $\mathfrak{M}_1^{wt}$ has a dual (weighted) graph γ, which can be made terminally weighted and rooted by first declaring the root to be the (contraction of) the core of C and then pruning along all non-terminal positively weighted nodes. We will denote by o the root of any terminally weighted rooted tree, and by $a, b, \ldots$ the remaining vertices.

In the diagram, $\widetilde{\mathfrak{M}}_1^{wt}$ denotes the blow up of $\mathfrak{M}_1^{wt}$ described above and $\mathfrak{D}_1$ is the stack of stable pairs (C, D) with D an effective Cartier divisors supported in the smooth locus of C. Fix a point (C, D) in $\mathfrak{D}_1$ and a map in $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ with underlying curve C. Then U is a small open around the fixed map in $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$, $\mathcal{V}$ is a smooth chart around the point (C, D) in $\mathfrak{D}_1$ containing the image of U and $\mathcal{E}_{\mathcal{V}}$ is the total space of the sheaf $\rho_*\mathcal{L}(\mathcal{A})^{\oplus n}$ on $\mathcal{V}$.

Let $V_\gamma = \prod_{v\in \mathrm{Ver}(\gamma)^*} \mathbb{A}^1$ be an affine space that serves as model for local equations. We denote by $z_a, z_b, \ldots$ the natural coordinates in V_γ. Similarly, $E_\gamma = V_\gamma \times \prod_{v\in\mathrm{Ver}(\gamma)^t} (\mathbb{A}^1)^r$ and the coordinates on the affine space $(\mathbb{A}^1)^r$ corresponding to $a \in \mathrm{Ver}(\gamma)^t$ will be denoted by $w_{a,1}, \ldots, w_{a,r}$. The ideal $\Phi_\gamma = (\Phi_{\gamma,1}, \ldots, \Phi_{\gamma,r})$ will be described explicitly in Equation (1.64). The smooth morphism ϕ comes from the natural coordinates on $\mathcal{V}$, associated to the smoothing of each of the disconnecting nodes in C (which are in natural bijection with $\mathrm{Ver}(\gamma)^*$). The map $\mathcal{U} \to (F = 0)$ is an open embedding. Finally, $\widetilde{\phi}$ is induced by ϕ and $F = \widetilde{\phi}^*\Phi_\gamma$. It is in this sense that we can think of Φ_γ as the equations of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)_\gamma$.

Following [HL10], given a terminally weighted rooted graph γ, the ideal $\Phi_\gamma = (\Phi_{\gamma,1}, \ldots, \Phi_{\gamma,r})$ inside V_γ can be described as

$$\Phi_{\gamma,i} = \sum_{v\in\mathrm{Ver}(\gamma)^t} z_{[v,o]} w_{v,i} \quad 1 \le i \le r, \tag{1.64}$$

where

$$z_{[v,o]} = \prod_{o \prec a \preceq v} z_a.$$

Note that, for fixed i, the variables $w_{a,i}$ only appear in the i-th equation $\Phi_{\gamma,i}$. Due to the symmetry of the equations and the fact that all blow ups take place in V_γ, which has coordinates $\{z_a\}_{a\in\mathrm{Ver}(\gamma)^*}$ (but not the $w_{a,i}$), in the examples below we will not write down the index i in the equations $\Phi_{\gamma,i}$ nor in the variables $w_{a,i}$. For example, in the study of the equations $\Phi_{\gamma,i}$ after blowing up, it will be clear that the index i is irrelevant, in the sense that the way that $\Phi_{\gamma,i}$ changes is independent of i.

1.8.1 Local equations of desingularizations

The local equations of the loci that must be blown up are described, following [HL10].

Firstly, we describe how to assign an ideal I_γ to any semistable terminally weighted rooted tree γ. Here, *semistability* of γ means that every non-root vertex with weight zero has at least two edges.

The *trunk* of γ is the maximal chain $o = v_0 \prec v_1 \prec \ldots \prec v_r$ of vertices in γ such that each vertex v_i with $1 \leq i < r$ has exactly one immediate descendant and v_r is called a *branch vertex* if is it not terminal. Note that γ is a path tree if and only if it has no branch vertex.

Definition 1.130. Let γ be a semistable terminally weighted rooted tree with branch vertex v and let $a_1, \ldots, a_k$ be the immediate descendants of v. To γ we associate the ideal

$$I_\gamma = (z_{a_1}, \ldots, z_{a_k})$$

in V_γ.

First, we must blow up V_γ along the ideal I_γ. To describe the remaining steps we need to introduce the following operations.

Definition 1.131. Let γ be a terminally weighted semistable rooted tree.

- The *pruning* of γ along a vertex v is the new tree obtained by removing all the descendants of v (and the corresponding edges) and declaring the weight of v to be the sum of the original weight of v plus the weights of all removed vertices.

- The *advancing* of a vertex v with immediate ascendant $\overline{v}$ in γ is a new tree obtained by replacing every edge $(\overline{v}, v')$ with $v' \neq v$ by an edge $(\overline{v}, v)$ and pruning along all positively weighted non-terminal vertices as long as possible. In Section 1.8.1 we will denote by γ_v the advancing of v in γ and by γ'_v the same tree before pruning.

- Suppose γ has a branch vertex v. A *monoidal transform* of γ is a tree obtained by advancing one of the immediate descendants of v. The set of monoidal transforms of γ is $\mathrm{Mon}(\gamma)$.

It turns out that the ideal Φ_γ behaves nicely under monoidal transforms. Indeed, let γ be a semistable terminally weighted rooted tree with branch vertex v and let $a = a_1, a_2, \dots, a_k$ be the immediate descendants of v. Let γ_a be the tree advancing of a in γ. Let $\pi\colon \widetilde{V_\gamma} \to V_\gamma$ be the blow-up of V_γ along the ideal $I_\gamma = (z_{a_1}, \dots, z_{a_k})$. We view $\widetilde{V_\gamma}$ embedded inside $V_\gamma \times \mathbb{P}^{k-1}$. There is a natural way to associate to each generator z_{a_i} of I_γ one chart of $\mathbb{P}^{k-1}$, and thus also of $\widetilde{V_\gamma}$. We denote such chart by $\widetilde{V}_{\gamma,a_i}$. Let $\pi_a\colon \widetilde{V}_{\gamma,a} \to V_\gamma$ be the restriction of the natural projection, where $a = a_1$. Then, by the proof of [HL10, Lemma 5.14], one of the following must hold

- either γ_a is a path tree, and then the zero locus of $\pi_a^*(\Phi_\gamma)$ has smooth components;
- or γ_a is not a path tree and then

$$\pi_a^*(\Phi_\gamma) = \Phi_{\gamma_a}. \tag{1.65}$$

The whole blow-up process is summarized as follows. Fix γ. First blow up V_γ along I_γ. The pullback of Φ_γ is controlled by $\mathrm{Mon}(\gamma)$. If $\mathrm{Mon}(\gamma)$ consists only of path trees, we are done. Otherwise, for every element γ' in $\mathrm{Mon}(\gamma)$ which is not a path tree, blow up the chart of $\widetilde{V_\gamma}$ corresponding to γ' along $I_{\gamma'}$. Continue recursively. The process concludes by [HL10, Lemma 3.12].

Now we want to describe $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{Pic}$ locally. Namely, we want to describe which loci inside $\mathfrak{Pic}$ we are blowing up locally. We have an exact sequence

$$0 \to \rho_*\mathcal{L} \to \rho_*\mathcal{L}(\mathcal{A}) \to \rho_*\mathcal{L}(\mathcal{A})\mid_{\mathcal{A}}$$

by Equation (1.36). The change of notation is due to the fact that Equation (1.36) was global in $\mathfrak{Pic}$, but we now work locally. After a careful study of the second morphism, [HL10, Theorem 4.16] concludes that $\rho_*\mathcal{L}$ is the direct sum of a trivial bundle with the kernel of the morphism

$$\bigoplus_{v\in\mathrm{Ver}(\gamma)^t} \varphi_v\colon \mathcal{O}_{\mathcal{V}}^{\oplus\ell} \to \mathcal{O}_{\mathcal{V}}, \tag{1.66}$$

where ℓ is the cardinality of $\mathrm{Ver}(\gamma)^t$ and $\varphi_v = \prod_{o\prec v'\preceq v} \zeta_v$, with ζ_v the smoothing parameter of the disconnecting node corresponding to the vertex v.

By Proposition 1.102, the sheaf $\mathfrak{F}$ can be described locally as the dual of Equation (1.66). In particular, by Remark 1.134 we have that $\mathrm{Bl}_{\mathfrak{F}}\mathfrak{Pic}$ agrees, locally, with the blow-up along the ideal generated by the entries $(\varphi_v)_{v\in\mathrm{Ver}(\gamma)^t}$. In local coordinates, this ideal can be described as follows.

Definition 1.132. Let γ be a semistable terminally weighted rooted tree with $\mathrm{Ver}(\gamma)^t = \{v_1, \ldots, v_t\}$. To γ we associate the ideal

$$J_\gamma = (z_{[v_1,o]}, \ldots, z_{[v_t,o]}).$$

Similarly to Equation (1.65), in the same setup we have that

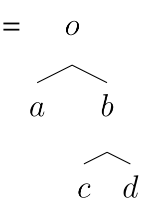
$$\pi_a^*(J_\gamma) = J_{\gamma_a},$$

independently of whether γ_a is has a branch vertex. This follows again from the proof of [HL10, Lemma 5.14].

Examples

For two concrete trees γ, we compute the equations Φ_γ as well as the ideals I_γ and J_γ. We describe the blow up process of Hu and Li and show that the result indeed desingularizes the main component of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ locally. Furthermore, we check that the ideal J_γ becomes locally principal in Hu–Li's blow-up.

Example 1.133. Consider the following labelled graph:

$\gamma =$ 

$$\Phi_\gamma = z_a w_a + z_b(z_c w_c + z_d w_d),$$
$$I_\gamma = (z_a, z_b),$$
$$J_\gamma = (z_a, z_b z_c, z_b z_d).$$

Let $\widetilde{V_\gamma}$ be the blow up along I_γ, that is the zero locus of $z_a z_b' - z_b z_a'$ inside $\mathbb{A}^4_{z_a,z_b,z_c,z_d} \times \mathbb{P}^1_{z_a',z_b'}$. The chart associated to a is that where $z_a' \neq 0$. Dehomogenizing amounts to the change of variables $z_b = z_b' z_a$. By doing so, we get that

$$\pi_a^*(\Phi_\gamma) = z_a(w_a + z_b'(z_c w_c + z_d w_d))$$

and that

$$\pi_a^*(J_\gamma) = (z_a, z_a z_b' z_c, z_a z_b' z_d) = (z_a).$$

This means that the zero locus of $\pi_a^*(\Phi_\gamma)$ already has smooth components, so no further blow-ups are needed on this chart, and that $\pi_a^*(J_\gamma)$ is principal on this chart too.

Below are the trees γ_a' obtained by advancing a without pruning, and γ_a obtained by advancing a. We know that $\pi_a^* J_\gamma = J_{\gamma_a}$, but we check it in this example.

$$\gamma'_a = \begin{array}{c} o \\ | \\ a \\ | \\ b \\ \diagup \ \diagdown \\ c \quad d \end{array} \qquad \gamma_a = \begin{array}{c} o \\ | \\ a \end{array} \qquad J_{\gamma_a} = (z_a).$$

Similarly, we now look at the chart associated to b, where $z'_b \neq 0$. The change of variables is now $z_a = z_b z'_a$. It follows that

$$\pi_b^*(\Phi_\gamma) = z_b(z'_a w_a + z_c w_c + z_d w_d)$$

and that

$$\pi_b^*(J_\gamma) = (z_b z'_a, z_b z_c, z_b z_d) = z_b(z'_a, z_c, z_d).$$

This means that we still need to blow up. This time the tree γ_b obtained by advancing b in γ (no pruning is needed) is not a path tree. We also check the identities $J_{\gamma_a} = \pi_a^* J_\gamma$ and $\pi_b^*(\Phi_\gamma) = \Phi_{\gamma_b}$.

$$\begin{array}{c} o \\ | \\ b \\ \diagup \ | \ \diagdown \\ a \quad c \quad d \end{array} \qquad \begin{aligned} \Phi_{\gamma_b} &= z_b(z_a w_a + z_c w_c + z_d w_d), \\ I_{\gamma_b} &= (z_a, z_c, z_d), \\ J_{\gamma_b} &= z_b(z_a, z_c, z_d). \end{aligned}$$

To conclude the example, we need to blow up along the ideal (z_a, z_c, z_d). We collect the result below.

Advancing a, or equivalently looking at the chart $z'_a \neq 0$, we have

$$\gamma'_{b,a} = \begin{array}{c} o \\ | \\ b \\ | \\ a \\ \diagup \ \diagdown \\ c \quad d \end{array} \qquad \gamma'_{b,a} = \begin{array}{c} o \\ | \\ b \\ | \\ a \end{array} \qquad \begin{gathered} \pi_a^*\pi_b^*(\Phi_\gamma) = \pi_a^*(\Phi_{\gamma_b}) = z_a z_b(w_a + z_c w_c + z_d w_d), \\ J_{\gamma_{b,a}} = (z_a z_b). \end{gathered}$$

Advancing c, or equivalently looking at the chart $z'_c \neq 0$, we have

$$\gamma'_{b,c} = \begin{array}{c} o \\ | \\ b \\ | \\ c \\ \frown \\ a \quad d \end{array} \qquad \gamma'_{b,c} = \begin{array}{c} o \\ | \\ b \\ | \\ c \end{array} \qquad \begin{gathered} \pi_c^*\pi_b^*(\Phi_\gamma) = \pi_c^*(\Phi_{\gamma_b}) = z_b z_c(z_a w_a + w_c + z_d w_d), \\ J_{\gamma_{b,c}} = (z_b z_c). \end{gathered}$$

And finally, advancing d, or equivalently looking at the chart $z'_d \neq 0$, we have

$$\gamma'_{b,d} = \begin{array}{c} o \\ | \\ b \\ | \\ d \\ \frown \\ a \quad c \end{array} \qquad \gamma'_{b,d} = \begin{array}{c} o \\ | \\ b \\ | \\ d \end{array} \qquad \begin{gathered} \pi_d^*\pi_b^*(\Phi_\gamma) = \pi_d^*(\Phi_{\gamma_b}) = z_b z_d(z_a w_a + z_c w_c + w_d), \\ J_{\gamma_{b,d}} = (z_b z_d). \end{gathered}$$

Remark 1.134. Example 1.133 shows that the Rossi blow-up process of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r, d)$ is not equal to the Vakil–Zinger blow-up. This is compatible with Remark 4.4. in [HN19]. Indeed, the Rossi blow-up around γ is given by $\mathrm{Bl}_{J_\gamma} V_\gamma$ and the Vakil–Zinger one is the iterated blow-up $\mathrm{Bl}_{I_{\gamma_b}} \mathrm{Bl}_{I_\gamma} V_\gamma$. We know there is a natural morphism

$$\mathrm{Bl}_{I_{\gamma_b}} \mathrm{Bl}_{I_\gamma} V_\gamma \to \mathrm{Bl}_{J_\gamma} V_\gamma$$

over V_γ, either by Proposition 1.67 or because we checked that J_γ pulls back to a principal ideal in $\mathrm{Bl}_{I_{\gamma_b}} \mathrm{Bl}_{I_\gamma} V_\gamma$. By contradiction, if there is a morphism

$$f\colon \mathrm{Bl}_{J_\gamma} V_\gamma \to \mathrm{Bl}_{I_{\gamma_b}} \mathrm{Bl}_{I_\gamma} V_\gamma$$

over V_γ, then we get a morphism

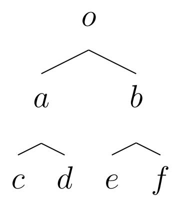
$$\widetilde{f}\colon \mathrm{Bl}_{J_\gamma} V_\gamma \to \mathrm{Bl}_{I_\gamma} V_\gamma$$

over V_γ. By [Moo01], there is a fractional ideal K in V_γ and a positive integer α such that

$$I_\gamma \cdot K = J_\gamma^\alpha.$$

This is not true for $I_\gamma = (z_a, z_b)$ and $J_\gamma = (z_a, z_b z_c, z_b z_d)$ in $V_\gamma = \mathbb{A}^4_{z_a, z_b, z_c, z_d}$.

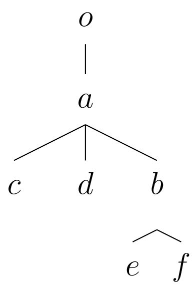
Example 1.135. We do a similar study for the following labelled graph γ:



$$\Phi_\gamma = z_a(z_c w_c + z_d w_d) + z_b(z_e w_e + z_f w_f),$$
$$I_\gamma = (z_a, z_b),$$
$$J_\gamma = (z_a z_c, z_a z_d, z_b z_e, z_b z_f).$$

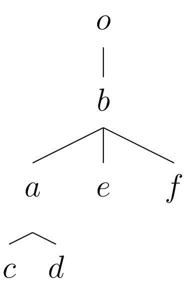
After blowing up, there are two charts, corresponding to the advacings of a and b respectively.

$\gamma_a =$



$$\Phi_{\gamma_a} = z_a(z_c w_c + z_d w_d + z_b(z_e w_e + z_f w_f)),$$
$$I_{\gamma_a} = (z_b, z_c, z_d),$$
$$J_{\gamma_a} = z_a(z_c, z_d, z_b z_e, z_b z_f).$$

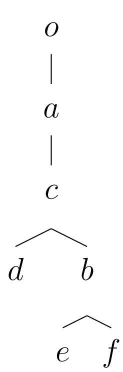
$\gamma_b =$



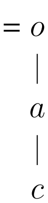
$$\Phi_{\gamma_b} = z_b(z_a(z_c w_c + z_d w_d) + z_e w_e + z_f w_f),$$
$$I_{\gamma_b} = (z_a, z_e, z_f),$$
$$J_{\gamma_b} = z_b(z_a z_c, z_a z_d, z_e, z_f).$$

By symmetry, it is enough to understand how to proceed in one of the charts. We choose the one corresponding to a. We get three new charts corresponding to the vertices c, d and b.

$\gamma'_{a,c} =$

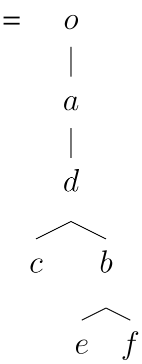


$\gamma_{a,c} =$

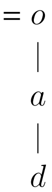


$$\pi_c^* \Phi_{\gamma_a} = z_a z_c(w_c + z_d w_d + z_b(z_e w_e + z_f w_f)),$$
$$J_{\gamma_{a,c}} = (z_a z_c).$$

$\gamma'_{a,d} =$

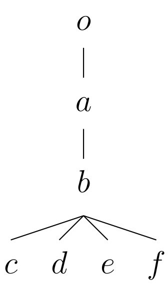


$\gamma_{a,d} =$



$$\pi_d^*\Phi_{\gamma_a} = z_a z_d(z_c w_c + w_d + z_b(z_e w_e + z_f w_f)),$$
$$J_{\gamma_{a,d}} = (z_a z_d).$$

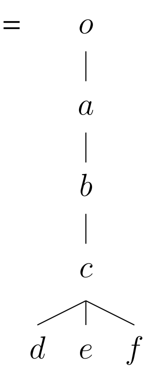
$\gamma_{a,b} =$



$$\Phi_{\gamma_{a,b}} = z_a z_b(z_c w_c + z_d w_d + z_e w_e + z_f w_f),$$
$$I_{\gamma_{a,b}} = (z_c, z_d, z_e, z_f),$$
$$J_{\gamma_{a,b}} = z_a z_b(z_c, z_d, z_e, z_f).$$

To conclude, we need to blow up the last chart along $I_{\gamma_{a,b}}$. This produces four new charts corresponding to c, d, e and f. We will only write down one of them since the rest are very similar.

$\gamma'_{a,b,c} =$



$\gamma_{a,b,c} =$

$$\pi_c^*\Phi_{\gamma_{a,b}} = z_a z_b z_c(w_c + z_d w_d + z_e w_e + z_f w_f),$$
$$J_{\gamma_{a,b,c}} = (z_a z_b z_c).$$

Smoothness

In genus one, $\overline{\mathcal{M}}_{1,n}(\mathbb{P}^r, d) \times_{\mathfrak{Pic}} \mathrm{Bl}_{\mathfrak{F}}\mathfrak{Pic}$ has simple normal crossings following the same argument as in [HL10, Theorem 5.24]. It is enough to show that the zero locus of the ideal Φ_γ becomes a simple normal crossing in the blow-up $\widetilde{V_\gamma}$ of V_γ along the ideal J_γ.

Remember that $\Phi_\gamma = (\Phi_{\gamma,1}, \ldots, \Phi_{\gamma,r})$ with

$$\Phi_{\gamma,i} = \sum_{v \in \mathrm{Ver}(\gamma)^t} z_{[v,o]} w_{b,i} \quad 1 \leq i \leq r,$$

where

$$z_{[v,o]} = \prod_{o \prec a \preceq v} z_a,$$

and that

$$J_\gamma = (z_{[v,o]})_{v \in \mathrm{Ver}(\gamma)^t}.$$

For given $v' \in \mathrm{Ver}(\gamma)^t$, the pullback of the equation $\Phi_{\gamma,i}$ on the chart corresponding to v' is equal to

$$z_{[v',o]} \left(w_{v',i} + \sum_{v' \neq v \in \mathrm{Ver}(\gamma)^t} z_{[v,o]} w_{b,i} \right)$$

by [HL10, Lemma 5.14]. This proves the claim.

Maps between blow-ups

By Proposition 1.67 there is a morphism from Vakil–Zinger's blow-up to Rossi's blow-up. In genus one, we can check it locally: it is equivalent to the fact that the pullback of the ideal J_γ to each chart $\widetilde{V_\gamma}$ of the Hu–Li blow-up of V_γ is principal. We have checked this in Example 1.133 and Example 1.135, More generally, we can give a proof for every γ as follows.

By Equation (1.65) if z_a is any of the generators of I_γ, then $\pi_a^*(J_\gamma) = J_{\gamma_a}$ where γ_a is the advancing of a in γ. In particular, it is enough to show that all the (natural) charts of $\widetilde{V_\gamma}$ correspond to path trees, which is proven in [HL10, Lemma 3.14].

Chapter 2

The contraction morphism between maps and quasimaps to toric varieties

This chapter represents work in progress, and will appear in the form of an article when completed.

Abstract. Given X a smooth projective toric variety, we construct a morphism from a closed substack of the moduli space of stable maps to X to the moduli space of quasimaps to X. If X is Fano, we show that this morphism is surjective. The construction relies on the notion of degree of a quasimap at a base-point, which we define. We show that a quasimap is determined by its regular extension and the degree of each of its basepoints.

2.1 Introduction

2.1.1 Results

To a smooth projective toric variety X we can associate two moduli spaces: $\overline{\mathcal{M}}_{g,n}(X,\beta)$, the moduli of stable maps, and $\mathcal{Q}_{g,n}(X,\beta)$, the moduli of stable toric quasimaps. These moduli spaces are two different compactifications of the space of maps from smooth curves to X. We study a geometric comparison of these compactifications. Our main result is the following.

Construction A (Construction 2.42)**.** Let X be a smooth projective toric variety. We construct a closed substack $\mathcal{V}$ of $\overline{\mathcal{M}}_{g,n}(X;\beta)$ and a morphism of stacks

$$c_X \colon \mathcal{V} \to \mathcal{Q}_{g,n}(X,\beta)$$

extending the identity on the locus of maps from a smooth source.

Our motivation comes from the comparison between Gromov–Witten and quasimap invariants of toric Fano varieties in enumerative geometry. For $X = \mathbb{P}^N$, the morphism $c_{\mathbb{P}^N}$, which is defined globally, agrees with the one used in [MOP11] to compare the virtual fundamental classes of both spaces. More generally, c_X is defined globally on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ if all toric divisors in X are nef, but not otherwise. Construction A thus opens the way to a geometric comparison in any genus. The following result makes the Fano case easier to treat.

Theorem B (Theorem 2.47)**.** Let X be a smooth Fano toric variety. The contraction morphism

$$c_X \colon \mathcal{V} \to \mathcal{Q}_{g,n}(X,\beta)$$

is surjective.

Construction A relies on the contraction morphism for (products of) projective spaces. One idea would be to use an embedding $\iota\colon X \hookrightarrow \mathbb{P}^N$ and the functoriality of maps and quasimaps. This fails because the morphism $\mathcal{Q}(\iota)\colon \mathcal{Q}_{g,n}(X,\beta) \to \mathcal{Q}_{g,n}(\mathbb{P}^N, \iota_*\beta)$ is not a closed embedding in general, see Example 2.30, which is due to Ciocan-Fontanine and which first appeared in [BN21, Remark 2.3.3]. This situation motivates the following result. Let $A_1(X)$ denote the group of 1-cycles on X modulo rational equivalence.

Theorem C (Corollary 2.40)**.** Let $\iota\colon X \to Y$ be a closed embedding between smooth projective toric varieties. If $\iota_*\colon A_1(X) \to A_1(Y)$ is injective, then the morphism

$$\mathcal{Q}(\iota)\colon \mathcal{Q}_{g,n}(X,\beta) \to \mathcal{Q}_{g,n}(Y, \iota_*\beta)$$

is a closed embedding.

Theorem C leads to Construction A following the strategy previously sketched. More precisely, we replace the embedding into projective space by an embedding $\iota\colon X \hookrightarrow \mathbb{P}^{N_1} \times \ldots \times \mathbb{P}^{N_s}$ such that ι_* is injective on curve classes (embeddings with this property are called *epic*, see Proposition 2.31). Such an embedding exists by Theorem 2.35. With this we obtain a diagram

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g,n}(X,\beta) \arrow[r, hook, "\overline{\mathcal{M}}(\iota)"] \arrow[d, dashed, "c_X"] & \overline{\mathcal{M}}_{g,n}(\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_k}, \iota_*\beta) \arrow[d, "c_{\mathbb{P}}"] \\
\mathcal{Q}_{g,n}(X,\beta) \arrow[r, hook, "\mathcal{Q}(\iota)"] & \mathcal{Q}_{g,n}(\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_k}, i_*\beta).
\end{tikzcd}$$

The dashed vertical arrow is defined on the locus $\mathcal{V}$ of maps in $\overline{\mathcal{M}}_{g,n}(X,\beta)$ for which $c_{\mathbb{P}} \circ \overline{\mathcal{M}}(\iota)$ factors through $\mathcal{Q}_{g,n}(X,\beta)$.

Most of the paper relies, directly or indirectly, on the notion of degree of a basepoint.

Definition D (Definition 2.24)**.** Let q be a quasimap to a smooth projective toric variety X and let x be a nonsingular point of the source. We construct a class $\beta_x \in A_1(X)$ called the degree of the quasimap q at the point x.

The degree β_x is constructed combinatorially in Proposition 2.21 and characterized in Proposition 2.23. We give a geometric interpretation, under certain assumptions, in Remark 2.45. It recovers (in the toric case) the notion of length of a basepoint in [CFKM14, Definition 7.1.1], see Lemma 2.29. We show in Corollary 2.27 that a toric quasimap is determined by its regular extension (Definition 2.13) and the degree of each of its basepoints. This notion is essential in Theorem C, and in the proof of Theorem B.

2.1.2 Context

The moduli space $\mathcal{Q}_{g,n}(X,\beta)$ of stable toric quasimaps was introduced in [CFK10], in relation to [Giv98, MOP11]. These constructions have been further generalized to include quasimaps to more general GIT quotients [CFKM14], stacks [CCFK15], and a stability parameter [CFK13, Tod11].

Quasimaps provide a compactification of the space of maps from smooth curves to X. The advantage over other compactifications relies in the fact that quasimap invariants, or more precisely the associated I-function, are often computable [CZ14, CFK16, KL18, CFK20, BN21]. Quasimaps have also been successfully used to prove results concerning Gromov–Witten invariants [LP18, FJR17, CRMS].

The contraction morphism c_X from Construction A appeared in [MOP11] in the case $X = \mathbb{P}^N$ and in [CFK10] for moduli spaces of maps and quasimaps with one parametrized component. A similar picture for Grassmannians appears in [PR03, Man14], where the authors show that the natural contraction morphism does not extend to the whole moduli space of stable maps.

The relation between Gromov–Witten invariants and quasimap invariants has been proved in many situations: via localization for projective spaces [MOP11] and for complete intersections in projective spaces [CJR17] and via wall-crossing [CFK17, CFK20]. In particular, Gromov-Witten and quasimap invariants of a smooth Fano toric variety agree [CFK17].

Our motivation is to use Construction A to prove this statement in a more geometric way as in [Man12, Man14] and at the level of derived structures and thus extend [KMMP22]. Besides of giving a better geometric understanding of the map quasimap wall-crossing, we hope such a result would contribute to developing techniques to relate virtual classes of moduli spaces which are "virtually birational", but not necessarily birational.

2.1.3 Outline of the paper

We summarize the content of each section, highlighting the main results.

In Section 2.2 we review basic content of toric quasimaps and introduce our notations.

The main purpose of Section 2.3 is to define the degree of a basepoint in Definition 2.24. Before that, we construct it combinatorially in Proposition 2.21. The importance of the definition is exhibited by its geometric characterization in Proposition 2.23. We collect basic properties of the degree of a basepoint in Proposition 2.25 and we show that it recovers the length of a point in Lemma 2.29.

We present the main problem of Section 2.4 in Example 2.30: given a closed embedding ι between smooth projective toric varieties, the induced morphism $\mathcal{Q}(\iota)$ between quasimap spaces may not be a closed embedding. We identify that the property of $\mathcal{Q}(\iota)$ being a monomorphism is related to the induced morphism ι_* on the Chow group A^1 being injective. This observation motivates the notion of *epic* morphism, which we discuss in Section 2.4.2. The main result of this subsection is Theorem 2.35: every smooth projective toric variety admits an epic closed embedding into a product of projective spaces. In Section 2.4.3 we show that if ι is epic, then $\mathcal{Q}(\iota)$ is a closed embedding (Corollary 2.40). The proof is as follows: first, we show (Corollary 2.27) that a quasimap is determined by its regular extension (introduced in Definition 2.13), and the degree of each of its basepoints; and this is used in Theorem 2.39 to characterize the fibres of $\mathcal{Q}(\iota)$, from which Corollary 2.40 follows. Finally, we give another proof of Corollary 2.40, directly at the level of families of quasimaps, in Section 2.4.4.

In Section 2.5, we construct a contraction morphism from a closed substack of the moduli space $\overline{\mathcal{M}}_{g,n}(X,\beta)$ to $\mathcal{Q}_{g,n}(X,\beta)$ (Construction 2.42). We describe explicitly the contraction morphism and the locus where it is defined in Proposition 2.43, at the level of closed points.

In Section 2.6, we use the notion of degree of a basepoint (Definition 2.24) to show that the contraction morphism is surjective for Fano targets (Theorem 2.47), and more generally for X satisfying the condition in Remark 2.57.

2.1.4 Acknowledgements

I would like to thank Cristina Manolache for posing this question and for the numerous discussions we have had about this topic. I would also like to thank Samuel Johnston, Jingxiang Ma, Etienne Mann, Navid Nabijou, Luis Manuel Navas Vicente and Menelaos Zikidis for helpful discussions and Navid Nabijou, Dhruv Ranganathan and Evgeny Shinder for their comments on previous drafts.

2.2 Background

2.2.1 Toric varieties

Let X be a toric variety, that is, a normal variety over $\mathbb{C}$ containing a torus T as a dense open subset and such that the natural group action of T extends to an action of T on X. We denote by N the character lattice of X and by $M = \mathrm{Hom}(N, \mathbb{Z})$ its co-character lattice. We will write N_X for N if there is any risk of confusion.

Every toric variety can be constructed from a fan Σ in $N \otimes_{\mathbb{Z}} \mathbb{R}$. We denote by $\Sigma(k)$ the set of k-dimensional cones in Σ and by $\sigma(k)$ the set of k-dimensional faces of a cone σ. Elements in $\Sigma(1)$ are called *rays of* Σ and will typically be denoted by the letter ρ. To each ray ρ we can associate a Cartier divisor D_ρ in X and an element $u_\rho \in N$, the unique generator of the semigroup $\rho \cap N$.

Recall that X is proper if and only if $\cup_{\sigma\in\Sigma}\sigma = N \otimes_{\mathbb{Z}} \mathbb{R}$, and that X is smooth if and only if for each cone $\sigma \in \Sigma$ it holds that the set $\{u_\rho \colon \rho \in \sigma(1)\}$ is a $\mathbb{Z}$-basis of N. The canonical divisor of X can be written as $K_X = -\sum_{\rho\in\Sigma(1)} D_\rho$. We say that a smooth toric variety X is Fano if the anticanonical divisor $-K_X$ ample. If X is smooth, Fano and proper, then $-K_X$ is very ample by [CLS11, Theorem 6.1.15]. In that case, we refer to the closed embedding $X \to \mathbb{P}(H^0(-K_X))$ as the *anticanonical embedding* of X.

For a (not necessarily toric) normal variety X, we denote by $A_k(X)$ the group of k-dimensional cycles on X modulo rational equivalence. If X is smooth of dimension n, intersection product induces a perfect pairing

$$A_k(X) \times A_{n-k}(X) \to \mathbb{Z}.$$

In that case, we denote $A^k(X) \coloneqq A_k(X)$. In particular, $A_1(X)$ is dual to $A^1(X)$, which is isomorphic to the group of Cartier divisors modulo principal divisors and also isomorphic to the Picard group $\mathrm{Pic}\,(X)$ of isomorphisms classes of invertible sheaves on X.

Any smooth proper toric variety X can be written as an almost geometric quotient

$$X \simeq (\mathbb{A}^{\Sigma(1)} \setminus Z(\Sigma)) /\!\!/ G,$$

where $\mathbb{A}^{\Sigma(1)} = \mathrm{Spec}\,(S)$ for $S = \mathbb{C}[z_\rho]_{\rho\in\Sigma(1)}$, where $G = \mathrm{Hom}(A_{\dim(X)-1}(X), \mathbb{C}^*)$ and where $Z(\Sigma)$ is the zero set of the ideal

$$B(\Sigma) = \left\langle z^{\hat{\sigma}} \colon \sigma \in \Sigma(\dim X) \right\rangle \subset S$$

where

$$z^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} z_\rho \in S$$

for each cone $\sigma \in \Sigma$. A subset $\mathcal{P}$ of $\Sigma(1)$ is a *primitive collection* if $\mathcal{P}$ is not contained in any cone of Σ, but each proper subset of $\mathcal{P}$ is contained in some cone of Σ. Then the irreducible decomposition of $Z(\Sigma)$ is

$$Z(\Sigma) = \bigcup_{\mathcal{P}} V(x_\rho \colon \rho \in \mathcal{P}), \tag{2.1}$$

where the union is over all the primitive collections $\mathcal{P}$ in $\Sigma(1)$.

2.2.2 Morphisms to a smooth toric variety

The functor of points of a smooth toric variety can be described in terms of line bundle-section pairs due to [Cox95].

Let X_Σ be a toric variety with fan Σ in a lattice N and let $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$.

Definition 2.1. A Σ-collection on a scheme S is a triplet

$$(L_\rho)_{\rho \in \Sigma(1)}, (s_\rho)_{\rho \in \Sigma(1)}, (c_m)_{m \in M}$$

of line bundles L_ρ on S, sections $s_\rho \in H^0(S, L_\rho)$ and isomorphisms

$$c_m \colon \otimes_{\rho \in \Sigma(1)} L_\rho^{\otimes \langle m, u_\rho \rangle} \simeq \mathcal{O}_S$$

satisfying the following conditions:

1 compatibility: $c_m \otimes c_{m'} = c_{m+m'}$ for all $m, m' \in M$,

2 non-degeneracy: for each $x \in S$ there is a maximal cone $\sigma \in \Sigma$ such that $s_\rho(x) \neq 0$ for all $\rho \not\subset \sigma$.

An equivalence between two Σ-collections on S is a collection of isomorphisms among the line bundles that preserve the sections and trivializations.

The compatible trivializations reflect the fact that the boundary divisors $(D_\rho)_{\rho \in \Sigma(1)}$ generate $\mathrm{Pic}\,(X_\Sigma)$, but usually not freely. The non-degeneracy condition can be thought geometrically as follows: X_Σ admits a GIT quotient presentation $\mathbb{A}^{\Sigma(1)} /\!\!/ (\mathbb{C}^*)^s$ by the Cox construction, in which non-degeneracy at $x \in S$ is equivalent to the image of x by the induced morphism to $\mathbb{A}^{\Sigma(1)}$ landing in the stable locus.

We can define a functor

$$\mathcal{C}_\Sigma \colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Set}$$

that associates to each scheme S the collection of Σ-collections on S up to equivalence, with morphisms naturally defined by pull-back. Then $\mathcal{C}_\Sigma$ is represented by X_Σ by [Cox95, Thm 1.1], the universal family being the Σ-collection on X_Σ that consists of the line bundles $\mathcal{O}_{X_\Sigma}(D_\rho)$ with their natural sections and trivializations c_m given by the characters χ^m of the dense torus in X_Σ. It is in this sense that morphisms to a smooth toric variety can be thought of as line bundle-section pairs.

2.2.3 Toric quasimaps

The previous interpretation of the functor of points of a smooth toric variety was used in [CFK10] to define the notion of quasimaps to a toric variety, by relaxing the non-degeneracy condition above.

Definition 2.2. Let X_Σ be a smooth projective toric variety. A *(prestable toric) quasimap* q to X_Σ consists of

1 a connected nodal projective curve C of genus g with n distinct non-singular markings,

2 line bundles L_ρ on C for $\rho \in \Sigma(1)$,

3 sections $s_\rho \in H^0(C, L_\rho)$ for $\rho \in \Sigma(1)$ and

4 trivializations $c_m \colon \bigotimes_{\rho\in\Sigma(1)} L_\rho^{\otimes\langle m,u_\rho\rangle} \simeq \mathcal{O}_C$ for $m \in M$

subject to

1 compatibility: $c_m \otimes c_{m'} = c_{m+m'}$ for all $m, m' \in M$,

2 quasimap non-degeneracy: there is a finite (possible empty) set $B \subseteq C$ of nonsingular points disjoint from the markings on C, such that for every $x \in C \setminus B$ there is a maximal cone $\sigma \in \Sigma$ such that $s_\rho(x) \neq 0$ for all $\rho \not\subset \sigma$.

Points in the set $B = B_q$ in Definition 2.2 are called *basepoints* of the quasimap. A toric quasimap defines a regular morphism $C \setminus B \to X_\Sigma$. If we view X_Σ as a GIT quotient

$\mathbb{A}^{\Sigma(1)}/\!\!/(\mathbb{C}^*)^s$ in the standard way, then a quasimap to X_Σ defines a morphism to the quotient stack $[\mathbb{A}^{\Sigma(1)}/(\mathbb{C}^*)^s]$, and the basepoints are precisely the points of C mapped into the unstable locus of the GIT stability.

Definition 2.3. The degree of a toric quasimap $q = (C, L_\rho, s_\rho, c_m)$ is the class $\beta = \beta_C \in A_1(X_\Sigma)$ determined by the conditions

$$\beta \cdot D_\rho = \deg(L_\rho) \tag{2.2}$$

for all $\rho \in \Sigma(1)$. Similarly, we associate a curve class $\beta_{C'} = \beta_{C',q} \in A_1(X_\Sigma)$ to every irreducible component C' of C by replacing $\deg(L_\rho)$ with $\deg(L_\rho \mid_{C'})$ in Equation (2.2). By [CFK10, Lemma 3.1.3], $\beta_{C'}$ is effective.

If q has no basepoints, this agrees with the usual notion of degree of a map. However, in the presence of basepoints, the degree of a quasimap does not agree with the degree of its associated regular map by Proposition 2.25. See Example 2.14.

2.2.4 Moduli space of stable toric quasimaps

The notion of stability for families of toric quasimaps was introduced in [CFK10].

Definition 2.4. Let X_Σ be a smooth projective toric variety and let $\{\alpha_\rho\}_{\rho\in\Sigma(1)}$ such that $L = \otimes_\rho \mathcal{O}_{X_\Sigma}(D_\rho)^{\otimes\alpha_\rho}$ is a very ample line bundle on X_Σ. Then a toric quasimap $q = (C, L_\rho, s_\rho, c_m)$ to X_Σ is *stable* if the line bundle

$$\omega_C(p_1 + \ldots + p_n) \otimes \mathcal{L}^\epsilon$$

is ample for all $\epsilon \in \mathbb{Q}_{>0}$, where $\mathcal{L} = \otimes_\rho L_\rho^{\otimes\alpha_\rho}$ and p_i denote the marked points in C.

Remark 2.5. The notion of a quasimap being stable is independent of the polarization chosen in Definition 2.4 by [CFK10, Lemma 3.1.3]. Also, stability imposes the inequality $2g - 2 + n \geq 0$, which we assume from now on.

Definition 2.6. A *family of genus g stable toric quasimaps to X_Σ of class β over S* is

1 a flat, projective morphism $\pi\colon \mathcal{C} \to S$ of relative dimension one,

2 sections $p_i\colon S \to \mathcal{C}$ of π for $1 \leq i \leq n$,

3 line bundles L_ρ on $\mathcal{C}$ for $\rho \in \Sigma(1)$,

4 sections $s_\rho \in H^0(\mathcal{C}, L_\rho)$ for $\rho \in \Sigma(1)$ and

5 compatible trivializations $c_m \colon \otimes_{\rho\in\Sigma(1)} L_\rho^{\otimes\langle m,u_\rho\rangle} \simeq \mathcal{O}_\mathcal{C}$ for $m \in M$

such that the restriction of the data to every geometric fibre of π is a stable toric quasimap of genus g and class β.

Theorem 2.7. ([CFK10, Theorems 3.2.1, 4.0.1]) The moduli space $\mathcal{Q}_{g,n}(X_\Sigma, \beta)$ of stable quasimaps of degree β to a smooth projective toric variety X_Σ from genus-g n-marked curves is a proper Deligne–Mumford stack, of finite type over $\mathbb{C}$.

The moduli space $\overline{\mathcal{M}}_{g,n}(X_\Sigma, \beta)$ of stable maps of degree β from genus-g n-marked curves to a smooth projective toric variety X_Σ can be recovered similarly. We define a *(prestable) map* to X_Σ to be a prestable toric quasimap (Definition 2.2) whose set of basepoints B is empty. Then, we impose the following stability condition for maps, which is different from Definition 2.4: we impose that

$$\omega_C(p_1 + \ldots + p_n) \otimes \mathcal{L}^2$$

is ample. As a result, every map is a quasimap, but stability need not be preserved. Note that both stability conditions are defined to ensure that the objects have finitely-may automorphisms, so that $\mathcal{Q}_{g,n}(X_\Sigma, \beta)$ and $\overline{\mathcal{M}}_{g,n}(X_\Sigma, \beta)$ are Deligne-Mumford stacks.

2.2.5 Functoriality of quasimaps

We fix a (not necessarily toric) closed embedding $\iota \colon X \to Y$ between smooth projective toric varieties for the rest of Section 2.2.5. The goal of this subsection is to describe the morphism

$$\mathcal{Q}(\iota) : \mathcal{Q}_{g,n}(X, \beta) \to \mathcal{Q}_{g,n}(Y, \iota_*\beta)$$

associated to ι. We follow [BN17, Appendix B].

Notation 2.8. We denote rays in the fan Σ_X of X by ρ and rays in the fan Σ_Y of Y by τ. Similarly, we denote the toric boundary divisor in X corresponding to ρ by D_ρ^X and the one in Y corresponding to τ by D_τ^Y. Let S^X denote the Cox ring on X, which is graded by $\mathrm{Pic}\,(X)$. Recall that X admits a quotient presentation $(\mathbb{A}^{\Sigma_X(1)} \setminus Z(\Sigma_X)) /\!\!/ \,\mathrm{Hom}(\mathrm{Pic}\,(X), \mathbb{G}_m)$.

By [Cox95, Theorem 3.2], the morphism ι corresponds to a collection of homogeneous polynomials $P_\tau \in S^X$ for $\tau \in \Sigma_Y(1)$ of degree $d_\tau := \iota^*\mathcal{O}_Y(D_\tau^Y) \in \mathrm{Pic}\,(X)$ satisfying the following conditions:

1 $\sum_{\tau\in\Sigma_Y(1)} \beta_\tau \otimes u_\tau = 0$ in $\mathrm{Pic}\,(Y)\otimes N$.

2 $(P_\tau(t_\rho)) \notin Z(\Sigma_Y)$ in $\mathbb{A}^{\Sigma_Y(1)}$ for every $(t_\rho)\notin Z(\Sigma_X)$ in $\mathbb{A}^{\Sigma_X(1)}$.

The relation between ι and (P_τ) is that the morphism

$$\tilde{\iota}\colon \mathbb{A}^{\Sigma_X(1)} \setminus Z(\Sigma_X) \to \mathbb{A}^{\Sigma_Y(1)} \setminus Z(\Sigma_Y)$$

defined by $\tilde{\iota}(t_\rho) \coloneqq (P_\tau(t_\rho))$ is a lift of ι.

For $\tau\in\Sigma_Y(1)$, we write the polynomial P_τ as follows:

$$P_\tau(t_\rho) = \sum_{\underline{a}} P_\tau^{\underline{a}}(t_\rho) = \sum_{\underline{a}} \mu_{\underline{a}} \prod_{\rho\in\Sigma_X(1)} t_\rho^{a_\rho}, \tag{2.3}$$

where the sum is over a finite number of indices $\underline{a} = (a_\rho) \in \mathbb{N}^{\Sigma_X(1)}$ and the coefficients $\mu_{\underline{a}}$ are non-zero. We choose a multi-index $\underline{a}$ appearing in Equation (2.3) and denote it by $\underline{a}^\tau$.

Construction 2.9. The morphism

$$\mathcal{Q}(\iota) : \mathcal{Q}_{g,n}(X,\beta) \to \mathcal{Q}_{g,n}(Y,\iota_*\beta)$$

associates to a family of quasimaps

$$(C, L_\rho, s_\rho, c_{m_X})$$

in $\mathcal{Q}_{g,n}(X,\beta)$ the quasimap

$$(C, L'_\tau, s'_\tau, c'_{m_Y})$$

described as follows:

$$L'_\tau = \bigotimes_{\rho\in\Sigma_X(1)} L_\rho^{\otimes a_\rho^\tau}, \quad s'_\tau = \mu_{\underline{a}^\tau} \prod_{\rho\in\Sigma_X(1)} s_\rho^{a_\rho^\tau}, \quad c'_{m_Y} = c_{m_X}, \tag{2.4}$$

where $m_X \in M_X$ is uniquely determined by the conditions

$$\langle m_X, u_\rho\rangle = \sum_\tau a_\rho^\tau \langle m_Y, u_\tau\rangle$$

for every $\rho\in\Sigma(1)$

Remark 2.10. In Construction 2.9, the underlying curve is the same because ι is a closed embedding, thus no stabilization is needed. The general case, where ι is not a closed embedding, is discussed in [BN17, Appendix B].

Remark 2.11. Let $\mathcal{Q}^{\mathrm{pre}}_{g,n}(X,\beta)$ denote the moduli stack of n-marked genus-g (prestable toric) quasimaps to X of class β. The description of $\mathcal{Q}(\iota)$ in Construction 2.9 in terms of line bundle-section pairs is the same as that of the morphisms

$$\mathcal{Q}(\iota)\colon \mathcal{Q}^{\mathrm{pre}}_{g,n}(X,\beta) \to \mathcal{Q}^{\mathrm{pre}}_{g,n}(Y,\iota_*\beta)$$

and

$$\overline{\mathcal{M}}(\iota)\colon \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(Y,\iota_*\beta).$$

Remark 2.12. The coefficients a^τ_ρ in Construction 2.9 satisfy the relations

$$\iota^*[D^Y_\tau] = \sum_{\rho\in\Sigma_X(1)} a^\tau_\rho [D^X_\rho]$$

for all $\tau \in \Sigma_Y(1)$.

2.2.6 Quasimaps as morphisms to an ambient stack

Let X be a smooth projective toric variety. Recall (Section 2.2.1) that X admits a quotient presentation

$$X \simeq (\mathbb{A}^{\Sigma(1)} \setminus Z(\Sigma)) /\!\!/ G$$

with $G = \mathrm{Hom}(\mathrm{Pic}\,(X), \mathbb{G}_m)$. Consider the quotient stack

$$\mathfrak{X} := [\mathbb{A}^{\Sigma(1)}/G],$$

which is a toric stack in the sense of [GS15], that is, it is a quotient of a toric variety by a subtorus of its torus. There is an open embedding $i_X\colon X \hookrightarrow \mathfrak{X}$, which is an isomorphism in codimension one. In particular, $\mathrm{Pic}\,(X) = \mathrm{Pic}\,(\mathfrak{X})$.

The functor of points of a toric stack is described in [GS15, Theorem 7.7]. For example, a morphism $S \to \mathfrak{X}$, with S a scheme, is equivalent to a triple (L_ρ, s_ρ, c_m) on S satisfying all the properties to be a Σ-collection (Definition 2.1) except possibly non-degeneracy. In fact, points of S where non-degeneracy holds are exactly those maps to $X \hookrightarrow \mathfrak{X}$. By his observation, we can view families of toric quasimaps to X as morphisms $\mathcal{C} \to \mathfrak{X}$, with quasimap non-degeneracy in Definition 2.2 reinterpreted as follows: each fibre of $\mathcal{C}$ must factor through $X \hookrightarrow \mathfrak{X}$ except at finitely many smooth unmarked points.

Furthermore, given a closed embedding $\iota\colon X \hookrightarrow Y$ of smooth projective toric varieties, there is an extension $\widetilde{\iota}\colon \mathfrak{X} \to \mathfrak{Y}$ making the following diagram commute

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[d, hook, "i_X"] & Y \arrow[d, hook, "i_Y"] \\ \mathfrak{X} \arrow[r, "\widetilde{\iota}"] & \mathfrak{Y}. \end{tikzcd}$$

To see this, note that ι is determined by the induced morphism

$$\iota^*\colon \mathrm{Pic}\,(Y) \to \mathrm{Pic}\,(X)$$

(see the argument in the proof of Corollary 2.40 in Section 2.4.4). But since $\mathrm{Pic}\,(\mathfrak{X}) \simeq \mathrm{Pic}\,(X)$ and $\mathrm{Pic}\,(\mathfrak{Y}) \simeq \mathrm{Pic}\,(Y)$, we have an induced morphism

$$\widetilde{\iota}^*\colon \mathrm{Pic}\,(Y) \to \mathrm{Pic}\,(X),$$

which is enough to determine a morphism $\widetilde{\iota}\colon \mathfrak{X} \to \mathfrak{Y}$ by the same reference. One can check that the functoriality of quasimaps described in Section 2.2.5 is simply composition by $\widetilde{\iota}$ when families of quasimaps are viewed as morphisms to the ambient stacks $\mathfrak{X}$ and $\mathfrak{Y}$.

2.2.7 Contraction morphism for projective space

Fix $N \geq 1$ and let $X_\Sigma = \mathbb{P}^N$. There is a natural morphism

$$c_{\mathbb{P}^N}\colon \overline{\mathcal{M}}_{g,n}(\mathbb{P}^N, \beta) \to \mathcal{Q}_{g,n}(\mathbb{P}^N, \beta).$$

The morphism $c_{\mathbb{P}^N}$ is called the contraction or comparison morphism for $\mathbb{P}^N$. It can be described as follows: a stable map is naturally a quasimap, but it is stable as a quasimap if and only if there are no rational tails. A rational tail T is a tree of rational components with no marked points and such that $T \cap \overline{(C \setminus T)}$ is a singleton (necessarily a node inside C). Let T be a rational tail of a stable map $f\colon C \to \mathbb{P}^N$, let $\hat{C} = \overline{C \setminus T}$ and let $p = T \cap \hat{C}$. One must contract T and define $\hat{L}_\rho = L_\rho \mid_{\hat{C}} \otimes \mathcal{O}_{\hat{C}}(d_{T,\rho}p)$ with $d_{T,\rho} = \deg(L_\rho \mid_T)$. Similarly, one must take $\hat{s}_\rho = s_\rho \otimes z_x^{d_{T,\rho}}$, with z_x a local parameter at x in $\hat{C}$. Doing this for every rational tail T of f produces the stable quasimap $c_{\mathbb{P}^N}(f)$. The description of $c_{\mathbb{P}^N}$ for families can be found in [PR03, Theorem 7.1].

2.3 The degree of a basepoint

We fix a smooth proper toric variety X with fan Σ for the rest of Section 2.3.

Every toric quasimap $q\colon C \dashrightarrow X$ has a degree β by Definition 2.3, which is an effective curve class in the target X. Moreover, q admits an associated regular morphism $q_{\mathrm{reg}}\colon C \to X_\Sigma$ (Definition 2.13), which also has a degree $\beta_{q_{\mathrm{reg}}}$. However, if q has basepoints, then $\beta \neq \beta_{q_{\mathrm{reg}}}$, see Example 2.14.

In Definition 2.24, we attach an effective non-zero curve class β_x to each basepoint x of q, called the degree of q at x. We show in Proposition 2.25 that the degree β_x is 0 unless x is a basepoint, and that, as x varies over all basepoints, the degrees β_x explain the discrepancy between β and $\beta_{q_{\mathrm{reg}}}$, since

$$\beta - \beta_{q_{\mathrm{reg}}} = \sum_{x \in B} \beta_x.$$

We show in Lemma 2.29 that the degree β_x recovers (in the toric case) the length of q at x, introduced in [CFKM14, Def. 7.1.1].

2.3.1 The regular extension of a quasimap

Definition 2.13. Let q be a quasimap to X. Then q defines a regular morphism

$$q\colon C \setminus B \to X.$$

Since X is proper and B consists of smooth points in C, there is a unique regular morphism

$$q_{\mathrm{reg}}\colon C \to X$$

extending q. We call q_{reg} the *(regular) extension* of q.

Example 2.14. In general, q and q_{reg} may have different degrees. Consider the quasimap

$$q\colon \mathbb{P}^1 \to \mathbb{P}^2\colon [x\colon y] \mapsto [0\colon 0\colon x]$$

which has a unique basepoint, at the point $[0\colon 1]$. Its extension is the constant map

$$q_{\mathrm{reg}}\colon \mathbb{P}^1 \to \mathbb{P}^2\colon [x\colon y] \mapsto [0\colon 0\colon 1].$$

Note that q has degree 1 but q_{reg} has degree 0. In particular, if we add two distinct marks to $\mathbb{P}^1$, distinct from $[0\colon 1]$, then q is a stable quasimap but q_{reg} is not a stable map.

2.3.2 Definition of the degree of a basepoint

As seen in Example 2.14, the degree of a quasimap may be different from the degree of its regular extension. We aim to assign a degree to each basepoint in order to explain this discrepancy. Such degree will be constructed using the vanishing order of the sections at a given basepoint.

Given a nodal curve C, a line bundle L on C, a non-zero section $s \in H^0(C, L)$ and a nonsingular point $x \in C$, we denote by $\operatorname{ord}_x(s)$ the vanishing order of s at x. Furthermore, we declare that $\operatorname{ord}_x(0) = \infty$ and we extend the natural operation and order on the monoid $\mathbb{Z}_{\geq 0}$ to $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ in the standard way. This means that we declare $a + \infty = \infty$ and $a < \infty$ for all $a \in \mathbb{Z}_{\geq 0}$. It follows from this definition that $\infty - a = \infty$ for every $a \in \mathbb{Z}_{\geq 0}$.

We will use the following basic result on toric geometry, whose proof we include for completeness.

Lemma 2.15. Let $\sigma \in \Sigma(\dim X)$. Then the classes $\{[D_\rho]\colon \rho \notin \sigma(1)\}$ form a basis of $\operatorname{Pic}(X)$.

Proof. The Picard group is generated by the classes $[D_\rho]$ for $\rho \in \Sigma(1)$, see [CLS11, Thm 4.2.1]. Let $n = \dim X$ and let $\sigma(1) = \{\rho_1, \ldots, \rho_n\}$. Then the ray generators $\{u_{\rho_1}, \ldots, u_{\rho_n}\}$ form a basis of the co-character lattice N by smoothness of Σ. Let $\{m_1, \ldots, m_n\}$ be its dual basis in the character lattice M. Then

$$\operatorname{div}(\chi^{m_i}) = \sum_{\rho \in \Sigma(1)} \langle m_i, u_\rho \rangle D_\rho = D_{\rho_i} + \sum_{\rho \notin \sigma(1)} \langle m_i, u_\rho \rangle D_\rho. \tag{2.5}$$

where χ^m is the character corresponding to $m \in M$. Taking linear equivalence, it follows that for $i = 1, \ldots, n$ the class $[D_{\rho_i}]$ lies in the subgroup generated by $\{[D_\rho]\colon \rho \notin \sigma(1)\}$. We conclude since $\operatorname{Pic}(X)$ is free of rank $|\Sigma(1)| - n$ by smoothness of X. □

Notation 2.16. Let q be a quasimap to X with underlying curve C and let $x \in C$ be a smooth point.

- We denote by C_x the irreducible component of C containing x and
- by V_x the set of rays $\rho \in \Sigma(1)$ such that s_ρ vanishes identically on C_x.

Construction 2.17. Let q be a quasimap to X with underlying curve C and let $x \in C$ be a smooth point. Given $\sigma \in \Sigma(\dim X)$ such that $V_x \subseteq \sigma(1)$, we define the curve class $\beta(x,\sigma)$ by the conditions

$$\beta(x,\sigma) \cdot [D_\rho] = \mathrm{ord}_x(s_\rho \mid_{C_x}) \; \forall \rho \notin \sigma(1).$$

The class $\beta(x,\sigma)$ is well-defined due to Lemma 2.15.

Similarly, to any cone $\sigma \in \Sigma(\dim X)$ and any $a = (a_\rho) \in \mathbb{R}^{\Sigma(1)}$ we can associate the unique curve class $\beta(a,\sigma) \in A_1(X)$ satisfying the conditions

$$\beta(a,\sigma) \cdot [D_\rho] = a_\rho \; \forall \rho \notin \sigma(1). \tag{2.6}$$

Before we use Construction 2.17, we need to prove an elementary lemma about the classes $\beta(a,\sigma)$.

Lemma 2.18. Let $n = \dim X$, let $\sigma \in \Sigma(\dim X)$ be a cone with rays $\sigma(1) = \{\rho_1, \ldots, \rho_n\}$ and let $\{m_1, \ldots, m_n\}$ be the dual basis to the ray generators $\{u_{\rho_1}, \ldots, u_{\rho_n}\}$. Let $a = (a_\rho) \in \mathbb{R}^{\Sigma(1)}$ and define $u = \sum_{\rho \in \Sigma(1)} a_\rho u_\rho$. Then for $i \in \{1, \ldots, n\}$ we have that

$$a_{\rho_i} \geq \beta(a,\sigma) \cdot [D_{\rho_i}] \iff \langle m_i, u \rangle \geq 0. \tag{2.7}$$

In particular, $a_{\rho_i} \geq \beta(a,\sigma) \cdot [D_{\rho_i}]$ for all $i \in \{1, \ldots, n\}$ if and only if $u \in \sigma$.

Proof. We show the equivalence in (2.7), the particular conclusion is immediate. Fix $i \in \{1, \ldots, n\}$. By Equation (2.5), we have that

$$\beta(a,\sigma) \cdot [D_{\rho_i}] = - \sum_{\rho \notin \sigma(1)} \langle m_i, u_\rho \rangle \beta(a,\sigma) \cdot [D_\rho].$$

On the other hand,

$$\langle m_i, u \rangle = \langle m_i, \sum_{\rho \in \Sigma(1)} a_\rho u_\rho \rangle = a_{\rho_i} + \sum_{\rho \notin \sigma(1)} a_\rho \langle m_i, u_\rho \rangle.$$

Using Equation (2.6), we conclude that indeed

$$a_{\rho_i} + \sum_{\rho \notin \sigma(1)} \langle m_i, u_\rho \rangle \beta(a,\sigma) \cdot [D_\rho] \geq 0 \iff a_{\rho_i} + \sum_{\rho \notin \sigma(1)} a_\rho \langle m_i, u_\rho \rangle \geq 0. \qquad \square$$

Proposition 2.19. Let q be a quasimap to X. Given a smooth point $x \in C$ such that $V_x = \emptyset$, there is a cone $\sigma \in \Sigma(\dim X)$ such that the curve class $\beta(x,\sigma)$ satisfies

$$\mathrm{ord}_x(s_\rho\ |_{C_x}) \geq \beta(x,\sigma)\cdot[D_\rho] \text{ for all } \rho \in \Sigma(1).$$

Proof. Let $u = \sum_{\rho\in\Sigma(1)} \mathrm{ord}_x(s_\rho\ |_{C_x})u_\rho$. Since Σ is complete, there is a cone $\sigma \in \Sigma(\dim X)$ such that $u \in \sigma$. We make one such choice. Then the inequality is true for $\rho \notin \sigma(1)$ by definition of $\beta(x,\sigma)$ and for $\rho \in \sigma(1)$ by Lemma 2.18 applied to $(a_\rho) = (\mathrm{ord}_x(s_\rho\ |_{C_x}))$. □

Next, we want the analogue of Proposition 2.19 without the assumption $V_x = \emptyset$.

Remark 2.20. Given a toric quasimap q with underlying curve C and a smooth point $x \in C$, there exists a maximal cone σ such that $V_x \subseteq \sigma(1)$ by generic non-degeneracy.

Proposition 2.21. Let q be a quasimap to X. Given a smooth point $x \in C$, there is a curve class β_x satisfying

$$\mathrm{ord}_x(s_\rho\ |_{C_x}) \geq \beta_x\cdot[D_\rho] \text{ for all } \rho \in \Sigma(1). \tag{2.8}$$

Moreover, we can choose β_x satisfying the following two extra conditions:

1 there is a cone $\sigma \in \Sigma(\dim X)$ with $V_x \subseteq \sigma(1)$ and

2 $\mathrm{ord}_x(s_\rho\ |_{C_x}) = \beta_x\cdot[D_\rho]$ for all $\rho \notin \sigma(1)$.

Proof. By Lemma 2.18, it is enough to take $\beta_x = \beta(a,\sigma)$ for certain integers $(a_\rho)_{\rho\in\Sigma(1)}$ satisfying

(i) there is $\sigma \in \Sigma(\dim X)$ with $V_x \subseteq \sigma(1)$ and such that $u = \sum_{\rho\in\Sigma(1)} a_\rho u_\rho \in \sigma$,

(ii) if $\rho \notin V_x$ then $a_\rho = \mathrm{ord}_x(s_\rho\ |_{C_x})$.

If $V_x = \emptyset$ it is enough to let $a_\rho = \mathrm{ord}_x(s_\rho\ |_{C_x})$ for all $\rho \in \Sigma(1)$ and proceed as in Proposition 2.19.

In general, let τ be the cone generated by the rays in V_x, which lies in Σ. Denote by N_τ the sublattice of N generated by $\tau \cap N$ and let $N(\tau) = N/N_\tau$. Let

$$u_0 = \sum_{\rho\notin V_x} \mathrm{ord}_x(s_\rho\ |_{C_x})u_\rho$$

and let $[u_0]$ denote its class in $N(\tau)$. Since X_Σ is complete, so is the closure $V(\tau)$ of the orbit corresponding to τ, which is a toric variety with fan the star of τ. It follows that there is a cone $\sigma \in \Sigma(\dim X)$ which contains τ as a face and such that $[u_0] \in [\sigma]$ in $N(\tau)$.

It is enough to choose $(a_\rho)_{\rho\in V_x}$ in such a way that $u \in N$ is a lift of $[u_0] \in N(\tau)$ satisfying $u \in \sigma$. □

Next we show that the class β_x from Proposition 2.21 is independent of the choice of $\sigma \in \Sigma$ used to define it.

Remark 2.22. Given a curve class $\beta \in A_1(X)$, we have that

$$\sum_{\rho\in\Sigma(1)} \beta \cdot [D_\rho]\, u_\rho = 0.$$

Therefore, for each smooth point $x \in C$ there is a natural isomorphism

$$\psi_{m,x,\beta} \colon \mathcal{O}_C\Big(\sum_{\rho\in\Sigma(1)} \beta \cdot [D_\rho]\, \langle m, u_\rho\rangle\, x\Big) = \mathcal{O}_C\Big(\langle m, \sum_{\rho\in\Sigma(1)} \beta \cdot [D_\rho]\, u_\rho\rangle\, x\Big) \to \mathcal{O}_C.$$

For fixed x and β, these isomorphisms satisfy the compatibility

$$\psi_{m+m',x,\beta} = \psi_{m,x,\beta} \otimes \psi_{m',x,\beta}.$$

Proposition 2.23. Let $q = (C, L_\rho, s_\rho, c_m)$ be a quasimap to X, let $x \in C$ be a smooth point and let z be a local coordinate at x. There is a unique curve class $\beta_x \in A_1(X)$ such that the following data defines a quasimap to X of which x is not a basepoint:

$$q' = (C, L'_\rho, s'_\rho, c'_m)$$

with

$$\begin{aligned} L'_\rho &:= L_\rho \otimes \mathcal{O}_C(-\beta_x \cdot [D_\rho]\, x), \\ s'_\rho &:= s_\rho \otimes z^{-\beta_x\cdot[D_\rho]}, \\ c'_m &= c_m \otimes \psi_{-m,x,\beta_x}, \end{aligned}$$

where ψ_{m,x,β_x} denotes the isomorphism constructed in Remark 2.22

Proof. Firstly, we have that

$$\bigotimes_\rho L'^{\otimes\langle m,u_\rho\rangle}_\rho = \left(\bigotimes_\rho L^{\otimes\langle m,u_\rho\rangle}_\rho\right) \otimes \mathcal{O}_C\Big(-\sum_\rho (\beta_x \cdot [D_\rho])\, \langle m, u_\rho\rangle\, x\Big).$$

The first term is isomorphic to $\mathcal{O}_C$ via c_m and the second one via ψ_{-m,x,β_x}.

Secondly, the function s'_ρ is a regular section of the line bundle L'_ρ if and only if

$$\operatorname{ord}_x(s_\rho\mid_{C_x}) \geq \beta_x \cdot [D_\rho]\ \forall \rho \in \Sigma(1). \tag{2.9}$$

Therefore, q' is a quasimap if and only if β_x satisfies Equation (2.8).

Similarly to Notation 2.16, let V'_x be the set of rays ρ such that s'_ρ vanishes identically on C_x. Note that $V_x = V'_x$

One can check that that x is not a basepoint of q' if and only if there exists a cone $\sigma \in \Sigma$ such that the collection of rays $\rho \in \Sigma(1)$ satisfying $s'_\rho(x) = 0$ is contained in σ. Therefore, x is not a basepoint of q' if and only if we take β_x with the extra conditions appearing in Proposition 2.21 that there is a cone $\sigma \in \Sigma(\dim X)$ with $V'_x \subseteq \sigma(1)$ and

$$\operatorname{ord}_x(s_\rho\mid_{C_x}) = \beta_x \cdot [D_\rho] \text{ for all } \rho \notin \sigma(1),$$

To conclude, we show that β_x is unique. By the previous discussion, it suffices to check that if we have two cones $\sigma, \sigma' \in \Sigma(\dim X)$ with $V_x \subseteq \sigma(1)$ and $V_x \subseteq \sigma'(1)$, then the curve classes $\beta \coloneqq \beta(x, \sigma)$ and $\beta' \coloneqq \beta(x, \sigma')$ defined in Construction 2.17 agree. By construction, we have that

$$\begin{aligned}
&\operatorname{ord}_x(s_\rho\mid_{C_x}) \geq \beta' \cdot [D_\rho] \text{ for all } \rho \in \Sigma(1) \text{ and}\\
&\operatorname{ord}_x(s_\rho\mid_{C_x}) = \beta \cdot [D_\rho] \text{ for all } \rho \notin \sigma(1).
\end{aligned}$$

It follows that

$$(\beta - \beta') \cdot D_\rho \geq \operatorname{ord}_x(s_\rho\mid_{C_x}) - \operatorname{ord}_x(s_\rho\mid_{C_x}) = 0$$

for all $\rho \notin \sigma(1)$. This uses that fact that, since $V_x \subseteq \sigma(1)$, we have that $\operatorname{ord}_x(s_\rho\mid_{C_x})$ is finite for all $\rho \notin \sigma(1)$.

Since every ample line bundle L on X_Σ can be written with non-negative coefficients in the basis $\{[D_\rho]\colon \rho \notin \sigma(1)\}$ of $\operatorname{Pic}(X_\Sigma)$, we see that $\beta - \beta'$ is non-negative on the ample cone, therefore also on the nef cone. This means that $\beta - \beta'$ lies in the dual of the nef cone, so it is effective. The same applies to $\beta' - \beta$, so we must have $\beta = \beta'$. □

Note that q' might not be stable quasimap even if q is, see Example 2.14.

Definition 2.24. In the setting of Proposition 2.23, we say that the class $\beta_x = \beta_{x,q} \in A_1(X)$ is the *degree of the quasimap q at the point x*. For a nodal point x, we define $\beta_x = 0$.

In Remark 2.45 we give a more geometric reformulation of β_x in terms of the contraction morphism under certain assumptions. The importance of this notion will become clear in Section 2.3.3. We strongly recommend Example 2.30 to understand how it can be useful.

2.3.3 Properties of the degree of a basepoint

Proposition 2.25. Let q be a quasimap to X of degree β with underlying curve C and basepoints B. For each smooth point $x \in C$, let β_x denote the degree of x. Let β_{reg} denote the degree of q_{reg}, the regular extension of q. The following holds

1 $\beta_x = 0$ if and only if $x \notin B$,

2 β_x is effective for all $x \in C$,

3 $\beta = \beta_{\text{reg}} + \sum_{x \in B} \beta_x$.

Proof. 1 Follows from Proposition 2.23.

2 The classes $\{[D_\rho] \colon \rho \in \Sigma(1)\}$ generate the effective cone by [CLS11, Lemma 15.1.8]. We know that $\beta_x = \beta(x, \sigma)$ for some cone σ. By construction, the classes $\beta(x, \sigma)$ lie in the dual of the effective cone, therefore also on the dual of the nef cone, so they must be effective.

3 Follows from applying Proposition 2.23 successively to each basepoint. The result is a quasimap without basepoints, which must then be the regular extension q_{reg}. □

The following property is an immediate corollary of Proposition 2.23.

Corollary 2.26. Let $q = (C, L_\rho, s_\rho, c_m)$ be a quasimap to X and let B denote the set of basepoints of q. For $x \in B$, let z_x denote a local coordinate at x. Then the regular extension q_{reg} of q is given by

$$(C, L'_\rho, s'_\rho, c'_m)$$

with

$$\begin{aligned} L'_\rho &:= L_\rho \otimes \mathcal{O}_C(-\sum_{x \in B} \beta_x \cdot [D_\rho]\, x) \\ s'_\rho &:= s_\rho \otimes \prod_{x \in B} z_x^{-\beta_x \cdot [D_\rho]}, \\ c'_m &= c_m \otimes \left(\bigotimes_{x \in B} \psi_{-m, x, \beta_x} \right). \end{aligned}$$

Corollary 2.27. Let q_1, q_2 be quasimaps to X with the same underlying curve. Then $q_1 = q_2$ if and only if the following conditions hold

1 $q_{1,\mathrm{reg}} = q_{2,\mathrm{reg}}$

2 $B_{q_1} = B_{q_2}$ and

3 $\beta_{p,q_1} = \beta_{p,q_2}$ for all $x \in B_{q_1} = B_{q_2}$.

Proof. It follows from Corollary 2.26. □

Using the description of $\mathcal{Q}(\iota)$ in Construction 2.9, we get the following compatibility with the degree of a basepoint.

Lemma 2.28. Let $\mathcal{Q}(\iota)\colon \mathcal{Q}^{\mathrm{pre}}_{g,n}(X,\beta) \to \mathcal{Q}^{\mathrm{pre}}_{g,n}(Y,\iota_*\beta)$, let q be a quasimap to X of class β with underlying curve C. Then for every point $x \in C$ we have the following equality in $A_1(Y)$:

$$\iota_*(\beta_{x,q}) = \beta_{x,\mathcal{Q}(\iota)(q)}.$$

Proof. If x is not smooth, both terms are 0 by definition. If x is smooth, one can check, using the explicit descriptions in Proposition 2.23, Remark 2.12 and Equation (2.4), that first twisting q by $\beta_{p,q}$ at x as in Proposition 2.23 and then applying $\mathcal{Q}(\iota)$ is the same as twisting $\mathcal{Q}(\iota)(q)$ by $\iota_*\beta_{x,q}$ at x. Therefore, both $\iota_*\beta_{x,q}$ and $\beta_{x,\mathcal{Q}(\iota)(q)}$ satisfy the uniqueness in Proposition 2.23 applied to $\mathcal{Q}(\iota)(q)$. □

2.3.4 The relation between degree and length

Given a prestable and not necessarily toric quasimap q with underlying curve C, in [CFKM14, Def. 7.1.1], the authors assign to every point $x \in C$ a number $\ell(x)$, called the *length of the quasimap q at the point x*, as follows.

Given $X = W/\!\!/G$ with $W = \mathrm{Spec}\,(A)$, a character θ on G and a quasimap $q = (C, P, u)$ with P a principal G-bundle on C and u a section of the fibre bundle $P \times_G W \to C$, let

$$\ell(x) := \min\left\{\frac{\mathrm{ord}_x(u^*s)}{m}\colon s \in H^0(W, L_{m\theta})^G, u^*s \neq 0, m > 0\right\}. \tag{2.10}$$

If the character θ is chosen such that $H^0(W, L_\theta)^G$ generated $\oplus_{m\geq 0} H^0(W, L_{m\theta})^G$ as an algebra over A^G, then we can replace $\mathrm{ord}_x(u^*s)/m$ by $\mathrm{ord}_x(u^*s)$. Note that

$$\{s\colon s \in H^0(W, L_{m\theta})^G, m > 0\}$$

is the ideal inside A cutting out the scheme theoretic unstable locus W^{us} inside W.

Toric varieties have a natural GIT presentation, whose unstable locus can be described combinatorially. We use this fact to show that, for toric quasimaps, the degree β_x recovers the length $\ell(x)$.

Lemma 2.29. Let q be a quasimap to X with underlying curve C. For every $x \in C$, the length $\ell(x)$ of q at x can be recovered combinatorially from the degree β_x of q at x as follows:

$$\ell(x) = \min \left\{ \sum_{\rho \notin \sigma(1)} \beta_x \cdot [D_\rho] \colon \sigma \in \Sigma_{\max}, V_x \subseteq \sigma(1) \right\}. \tag{2.11}$$

Proof. If x is singular, then $\beta_x = 0$ by definition and $\ell(x) = 0$ because x is not a basepoint. Therefore, we can assume that x is smooth, so that β_x is defined by the conditions in Proposition 2.23.

The toric variety X_Σ has a GIT presentation $\mathbb{A}^{\Sigma(1)} /\!\!/ G$. Let $\mathbb{A}^{\Sigma(1)} = \mathrm{Spec}\,(S)$, with $S = \mathbb{C}[z_\rho]_{\rho \in \Sigma(1)}$. For each cone $\sigma \in \Sigma$, let

$$z^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} z_\rho \in S.$$

If we choose a character θ on $G = \mathrm{Hom}(A_{\dim(X)-1}(X), \mathbb{C}^*)$ coming from an ample divisor class, then the unstable locus $Z(\Sigma)$ inside $\mathbb{A}^{\Sigma(1)}$ is cut out by the ideal

$$B(\Sigma) = \left\langle z^{\hat{\sigma}} \colon \sigma \in \Sigma_{\max} \right\rangle \subset S.$$

We can then rewrite (2.10) for a prestable quasimap (C, L_ρ, s_ρ, c_m) to a toric variety X_Σ as

$$\ell(x) = \min\{\mathrm{ord}_x(s^{\hat{\sigma}} \mid_{C_x}) \colon \sigma \in \Sigma_{\max}, s^{\hat{\sigma}} \mid_{C_x} \neq 0\} \tag{2.12}$$

where

$$s^{\hat{\sigma}} = \prod_{\rho \notin \sigma(1)} s_\rho \in H^0(C, \otimes_{\rho \notin \sigma(1)} L_\rho). \tag{2.13}$$

Note that the section $s^{\hat{\sigma}}$ vanishes identically on C_x if and only if for some $\rho \notin \sigma(1)$ the section s_ρ vanishes identically on C_x. It follows that

$$s^{\hat{\sigma}} \mid_{C_x} \neq 0 \iff V_x \subseteq \sigma(1). \tag{2.14}$$

Using Equations (2.12) to (2.14), we have that

$$\ell(x) = \min \left\{ \sum_{\rho \notin \sigma(1)} \mathrm{ord}_x(s_\rho) \colon \sigma \in \Sigma_{\max}, V_x \subseteq \sigma(1) \right\}. \tag{2.15}$$

To conclude, we need to show that Equation (2.15) equals the right hand side in Equation (2.11). One inequality follows immediately from Equation (2.9). On the other hand, in Equation (2.15) we can replace $\mathrm{ord}_x(s_\rho)$ by $\beta(x,\sigma) \cdot [D_\rho]$ by the definition of $\beta(x,\sigma)$ in Construction 2.17. This finishes the proof since we saw in the proof of Proposition 2.23 that the degree β_x equals $\beta(x,\sigma)$ for some $\sigma \in \Sigma_{\max}$ with $V_x \subseteq \sigma(1)$. □

2.4 Embeddings of toric varieties and quasimap spaces

A morphism $\iota\colon X \to Y$ between smooth projective varieties induces a morphism

$$\overline{\mathcal{M}}(\iota) : \overline{\mathcal{M}}_{g,n}(X,\beta) \to \overline{\mathcal{M}}_{g,n}(Y,\iota_*\beta),$$

via composition, which is a closed embedding when ι is. Similarly, if X and Y are toric, ι induces a morphism

$$\mathcal{Q}(\iota) : \mathcal{Q}_{g,n}(X,\beta) \to \mathcal{Q}_{g,n}(Y,\iota_*\beta).$$

See Construction 2.9. However, $\mathcal{Q}(\iota)$ may not be a closed embedding, even if ι is, see Example 2.30. Motivated by this example, we introduce a class of morphisms, that we call *epic*, in Section 2.4.2. We show that $\mathcal{Q}(\iota)$ is a closed embedding if ι is an epic closed embedding Corollary 2.40, and that every smooth projective toric variety admits an epic embedding into a product of projective spaces (Definition 2.34). Since our first proof of Corollary 2.40 is based on the degree of a basepoint, we work with quasimaps rather than families of them. Instead, we give another proof at the level of families in Section 2.4.4.

2.4.1 Quasimaps do not embed along closed embeddings

Let $\iota\colon X \hookrightarrow Y$ be a closed embedding between smooth projective toric varieties. The induced morphism

$$\mathcal{Q}(\iota)\colon \mathcal{Q}_{g,n}(X,\beta) \to \mathcal{Q}_{g,n}(Y,\iota_*\beta)$$

is not a closed embedding in general. Here is an example.

Example 2.30. Let $s\colon \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ be the Segre embedding, given in coordinates by

$$s([x\colon y],[z\colon w]) = [xz\colon xw\colon yz\colon yw].$$

Note that s is a toric closed embedding. Consider the following two quasimaps from $\mathbb{P}^1$, with homogeneous coordinates $[s\colon t]$, to $\mathbb{P}^1 \times \mathbb{P}^1$ of degree $(2,2)$

$$q_1([s\colon t]) = [s^2\colon st],[st\colon t^2], \qquad\qquad q_2([s\colon t]) = [st\colon t^2],[s^2\colon st].$$

Both of them have the same image under $\mathcal{Q}(s)$

$$\mathcal{Q}(s)(q_1)([s\colon t]) = \mathcal{Q}(s)(q_2)([s\colon t]) = [s^3t\colon s^2t^2\colon s^2t^2\colon st^3],$$

which is a quasimap to $\mathbb{P}^3$ of degree 4. This means that $\mathcal{Q}(s)$ is not a monomorphism, therefore it is not a closed embedding. Note that the quasimaps above are not stable, but the same conclusion holds if we add marked points. This example appeared in [BN21, Remark 2.3.3], where it is attributed to Ciocan-Fontanine.

Our contribution is the following: we can use the degree of a basepoint (see Definition 2.24) to explain why $\mathcal{Q}(s)$ is not a monomorphism. Both q_1 and q_2 have the same regular extension q_{reg}, namely, the diagonal embedding of $\mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$,

$$q_{\mathrm{reg}}([s\colon t]) = [s\colon t],[s\colon t].$$

Let $0 = [1\colon 0]$ and let $\infty = [0\colon 1]$. Then q_1 and q_2 both have the same basepoints $B_{q_1} = B_{q_2} = \{0,\infty\}$, but we can still distinguish q_1 from q_2 because the degrees at the basepoints differ:

$$\begin{aligned} \beta_{0,q_1} &= (0,1), & \qquad \beta_{\infty,q_1} &= (1,0),\\ \beta_{0,q_2} &= (1,0), & \qquad \beta_{\infty,q_2} &= (0,1). \end{aligned}$$

Here, we identify $A_1(\mathbb{P}^1 \times \mathbb{P}^1)$ with $\mathbb{Z}^2$ in the natural way. After applying $\mathcal{Q}(s)$, both quasimaps still have the same regular extension and the same set of basepoints. Moreover,

$$\beta_{0,\mathcal{Q}(s)(q_1)} = \beta_{\infty,\mathcal{Q}(s)(q_1)} = \beta_{0,\mathcal{Q}(s)(q_2)} = \beta_{\infty,\mathcal{Q}(s)(q_2)} = 1 \in A_1(\mathbb{P}^3). \tag{2.16}$$

Therefore, we can no longer distinguish them by Corollary 2.27. Note that, by Lemma 2.28, the degree of a basepoint is compatible with $\mathcal{Q}(s)$ and s_*, in the sense that $\beta_{x,\mathcal{Q}(s)(q_i)} = s_*\beta_{x,q_i}$; therefore, the equalities in Equation (2.16) come from the fact that $s_*(1,0) = s_*(0,1) =$

1.

Example 2.30 suggests that the reason why $\mathcal{Q}(s)$ is not a monomorphism (and therefore the reason why it is not a closed embedding, since it is proper) is that $s_*\colon A_1(\mathbb{P}^1 \times \mathbb{P}^1) \to A_1(\mathbb{P}^3)$ is not injective. This observation motivates Theorem 2.39 and Corollary 2.40.

2.4.2 Epic morphisms

Motivated by Example 2.30, we introduce the following class of morphisms

Proposition 2.31. Let X and Y be smooth projective varieties whose Picard group is free of finite rank and let $f\colon X \to Y$. The following are equivalent:

1 $f^*\colon \mathrm{Pic}\,(Y) \to \mathrm{Pic}\,(X)$ is surjective,

2 $f^*\colon A^1(Y) \to A^1(X)$ is surjective,

3 $f_*\colon A_1(X) \to A_1(Y)$ is injective.

If, furthermore, X and Y are toric and $\mathfrak{X}$, $\mathfrak{Y}$ denote their ambient toric stacks (as in Section 2.2.6), the above conditions are also equivalent to

4 $\widetilde{f}^*\colon \mathrm{Pic}\,(\mathfrak{Y}) \to \mathrm{Pic}\,(\mathfrak{X})$ is surjective,

Proof. The equivalence between Item 1 and Item 2 holds because for a smooth variety Pic and A^1 are isomorphic. Similarly, since the open embedding $X \hookrightarrow \mathfrak{X}$ is an isomorphism in codimension 1, we have that $\mathrm{Pic}\,(X) = \mathrm{Pic}\,(\mathfrak{X})$ and the equivalence between Item 1 and Item 4 follows. Finally, Item 2 and Item 3 follows from the assumption that the Picard groups are free of finite rank, because both maps are transpose to each other by the projection formula. $\square$

Definition 2.32. A morphism $f\colon X \to Y$ of smooth projective varieties whose Picard group is free of finite rank is *epic* if it satisfies any of the equivalent conditions in Proposition 2.31. The name is motivated by Item 1 in Proposition 2.31: epic morphisms induce an epimorphism on Picard groups.

We are particularly interested in varieties that admit an epic closed embedding in a product of projective spaces.

Proposition 2.33. Let X be a smooth projective variety such that $\mathrm{Pic}\,(X)$ is free of finite rank. The following are equivalent:

1 there exist $k, n_1, \ldots, n_k \geq 1$ and an epic closed embedding $X \hookrightarrow \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_k}$,

2 X admits a generating set of $\mathrm{Pic}\,(X)$ consisting of basepoint free divisors.

Proof. The implication Item 1 ⇒ Item 2 is clear.

Conversely, let $\mathcal{S}$ be a generating set $\mathcal{S}$ of $\mathrm{Pic}\,(X)$ whose elements are basepoint free, let $[A] \in \mathrm{Pic}\,(X)$ be a very ample class and let $\mathcal{S}' = \mathcal{S} \cup \{[A]\}$. Consider the induced morphism

$$\iota\colon X \to \mathbb{P} := \prod_{[D] \in \mathcal{S}'} \mathbb{P}(H^0(\mathcal{O}_X(D)))$$

associated to the complete linear systems $|D|$, for $[D] \in \mathcal{S}'$. Then ι^* is surjective on Picard groups because $\mathcal{S}'$ is a generating set. Finally, the diagonal arrow in the following diagram is a closed embedding, which implies that so is ι.

$$\begin{tikzcd}
X \arrow[r, "\iota"] \arrow[dr, "|A|"'] & \mathbb{P} \arrow[d, "\pi_A"] \\
& \mathbb{P}(H^0(\mathcal{O}_X(A)))
\end{tikzcd}$$

□

Definition 2.34. A smooth projective variety whose Picard group is free of finite rank is called *projectively epic* if it satisfies any of the equivalenc conditions in Proposition 2.33.

Theorem 2.35. Every smooth projective toric variety is projectively epic.

Proof. The nef and Mori cones of a smooth projective toric variety X are dual strongly convex rational polyhedral cones of full dimension by [CLS11, Theorem 6.3.12]. In particular, the semigroup of divisor classes which are nef admits a finite generating set $\mathcal{S}$ by Gordan's lemma, which also generated $\mathrm{Pic}\,(X)$. Furthermore, the elements of $\mathcal{S}$ are are basepoint free by [CLS11, Theorem 6.3.12], therefore X is projectively epic. □

Example 2.36. Let $X = \mathrm{Bl}_0\mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at the origin, which is a toric variety whose fan is pictured in Figure 2.1. Let $\pi\colon X \to \mathbb{P}^2$ be the natural projection and let H be the hyperplane class in $\mathrm{Pic}\,(\mathbb{P}^2)$. The group $\mathrm{Pic}\,(X)$ is generated by $L := \pi^*H$ and the class E of the exceptional divisor. Let $S = L - E$, which is the class of the strict transform of a line in $\mathbb{P}^2$. Then the nef cone of X is generated by S and L.

There are four toric boundary divisors in X, which we denote by $D_i = D_{\rho_i}$, with the conventions in Figure 2.1. They satisfy that

$$[D_0] = L, \qquad [D_1] = [D_2] = S, \qquad [D_3] = E.$$

Using this, together with the description of global sections of toric line bundles, one can easily describe in coordinates the following closed embeddings of X into products of projective spaces induced by complete linear systems. Such morphisms will be written using the ring $\mathbb{C}[x_0, x_1, x_2, x_3]$ of homogeneous coordinates on X, with x_i corresponding to ρ_i.

The set $\{S, L\}$ generates the nef cone of X. Although none of the two classes is ample, the morphism

$$\iota \colon X \hookrightarrow \mathbb{P}(H^0(\mathcal{O}_X(L)) \times \mathbb{P}(H^0(\mathcal{O}_X(S)) = \mathbb{P}^2 \times \mathbb{P}^1$$

is an epic closed embedding. One way to see that ι is indeed a closed embedding is to realize that this is the usual description of X as a closed subvariety of $\mathbb{P}^2 \times \mathbb{P}^1$. Its expression in coordinates is

$$\iota(x_0, x_1, x_2, x_3) = [x_0 \colon x_1x_3 \colon x_2x_3], [x_1 \colon x_2], \tag{2.17}$$

Note that none of S or L are ample.

The set $\{S, S + L\}$ also generates $\mathrm{Pic}\,(X)$, both classes are nef and $S + L$ is ample (thus very ample). This choice produces a closed embedding which could appear with the argument in the proof of Theorem 2.35

$$i \colon X \hookrightarrow \mathbb{P}(H^0(\mathcal{O}_X(S)) \times \mathbb{P}(H^0(\mathcal{O}_X(S + L)) = \mathbb{P}^1 \times \mathbb{P}^4$$

with

$$i(x_0, x_1, x_2, x_3) = [x_1 \colon x_2], [x_0x_2 \colon x_0x_1 \colon x_2^2x_3 \colon x_1x_2x_3 \colon x_1^2x_3].$$

Similarly, one could take the set $\{S + L, L\}$, which also generates $\mathrm{Pic}\,(X)$, with L nef and $S + L$ ample. The corresponding closed embedding is

$$j \colon X \hookrightarrow \mathbb{P}(H^0(\mathcal{O}_X(S + L)) \times \mathbb{P}(H^0(\mathcal{O}_X(L)) = \mathbb{P}^4 \times \mathbb{P}^2$$

with

$$j(x_0, x_1, x_2, x_3) = [x_0x_2 \colon x_0x_1 \colon x_2^2x_3 \colon x_1x_2x_3 \colon x_1^2x_3], [x_0, x_2x_3, x_1x_3].$$

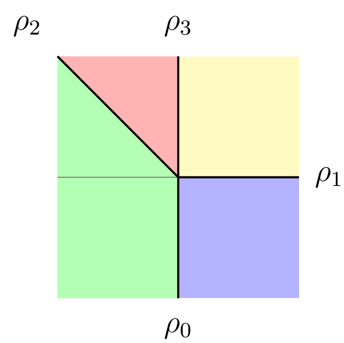


Figure 2.1: The fan of $\mathrm{Bl}_0\mathbb{P}^2$.

2.4.3 A criterion for quasimaps to embed along an embedding

We fix a closed embedding $\iota\colon X \hookrightarrow Y$ between smooth projective toric varieties for the rest of Section 2.4.3 and we study the morphism

$$\mathcal{Q}(\iota)\colon \mathcal{Q}_{g,n}(X,\beta) \to \mathcal{Q}_{g,n}(Y,\iota_*\beta)$$

on closed points. We firstly prove some lemmas which apply more generally to (prestable) toric quasimaps, see Remark 2.11.

Recall that we denote by B_q the set of basepoints of q (see Definition 2.2). In Definition 2.13, we introduced the notion of regular extension q_{reg} of a quasimap q to X. In this section, we denote q_{reg} by $r_X(q)$. Similarly, $r_Y(q')$ denotes the regular extension of a quasimap q' to Y.

Lemma 2.37. For every quasimap q to X, we have that

$$r_Y(\mathcal{Q}(\iota)q) = \overline{\mathcal{M}}(\iota)(r_X q).$$

Proof. The two maps have the same underlying curve C by Remark 2.10 and they are both extensions of the map $\iota \circ (q\,|_{C\setminus B})$ from the dense open $C \setminus B$ to C, so they must agree. □

Lemma 2.38. For every quasimap q to X, we have that

$$B_q = B_{\mathcal{Q}(\iota)(q)}.$$

Proof. Follows from Lemma 2.28 and proposition 2.25 because the pushforward along a closed embedding of an effective non-zero class is a non-zero class. □

Theorem 2.39. Let $\beta \in A_1(X)$ be an effective non-zero curve class and let q be an n-marked genus-g quasimap to Y of class $\iota_*\beta$ with underlying curve C. Consider the morphism induced by ι on prestable quasimaps

$$\mathcal{Q}(\iota)\colon \mathcal{Q}^{\mathrm{pre}}_{g,n}(X,\beta) \to \mathcal{Q}^{\mathrm{pre}}_{g,n}(Y,\iota_*\beta).$$

Then the fibre $\mathcal{Q}(\iota)^{-1}(q)$ of q along $\mathcal{Q}(\iota)$ is non-empty if and only if

1 there exists a morphism $f\colon C \to X$ such that $q_{\mathrm{reg}} = \iota \circ f$ and

2 there exist effective non-zero classes $(\beta^x)_{x\in B_q}$ in $A_1(X)$ such that

2.1 $\iota_*(\beta^x) = \beta_{x,q}$ for all $x \in B_q$,

2.2 $\beta = \beta_f + \sum_{x\in B_q} \beta^x$ and

2.3 $\mathrm{ord}_x(t_\rho) + \beta^x \cdot D_\rho \geq 0$ for every $\rho \in \Sigma_X(1)$ and every $x \in B_q$

with $\{t_\rho\}_{\rho\in\Sigma_X(1)}$ the underlying sections of f.

Furthermore, the fibre $\mathcal{Q}(\iota)^{-1}(q)$ is in one-to-one correspondence with the (finite) set of effective non-zero classes $(\beta^x)_{x\in B_q}$ in $A_1(X)$ satisfying Items 2.1 to 2.3.

If, moreover, q is stable, then so is every element in $\mathcal{Q}(\iota)^{-1}(q)$.

Proof. Firstly, let q_1 be a quasimap to X of class β such that $\mathcal{Q}(\iota)q_1 = q$. By Lemma 2.37,

$$\overline{\mathcal{M}}(\iota)(r_X q_1) = r_Y(\mathcal{Q}(\iota)q_1) = r_Y(q).$$

This means that $r_Y(q) = q_{\mathrm{reg}}$ factors through X and we can take $f := r_X(q_1) = q_{1,\mathrm{reg}}$. Using Lemma 2.38, the classes $\beta^x := \beta_{x,q_1}$, for $x \in B_q$, satisfy the desired conditions. Indeed, Item 2.1 follows from Lemma 2.28, Item 2.2 from Proposition 2.25 and Item 2.3 from Corollary 2.26, since

$$\mathrm{ord}_x(t_\rho) + \beta_{x,q_1} \cdot D_\rho = \mathrm{ord}_x(s^{(1)}_\rho) \geq 0,$$

where $s^{(1)}_\rho$ denote the sections of q_1.

Conversely, let q be a quasimap to Y of class $\iota_*\beta$ and let f and β^x, for $x \in B_q$, be as in the statement. Since $\overline{\mathcal{M}}(\iota)$ is a closed embedding, the morphism f is determined uniquely by the condition $q_{\mathrm{reg}} = \iota \circ f$. We show how to use the data in Item 2 to construct a unique quasimap q_1 to X of class β such that $\mathcal{Q}(\iota)q_1 = q$. We fix that $q_{1,\mathrm{reg}} = f$, and then we twist the line bundle-section pairs associated to f at each basepoint $x \in B_q$ as in Corollary 2.26, but changing the minus sign by a plus sign. This makes x a basepoint of $q_{1,\mathrm{reg}}$ with class β^x for every $x \in B_q$. By Corollary 2.27, the previous construction clearly determines a quasimap q_1 uniquely.

Finally, we show that if q is stable, then so is q_1. This follows from the fact that q and q_1 have the same curve (and marks) and the fact that components contracted by q_1 are also contracted by q. Indeed, if a component C' of the underlying curve of q and q_1 has degree $\beta_{C',q_1} = 0$ with respect to q_1, then by Proposition 2.25 and Lemma 2.28 we have that $\beta_{C',q} = i_*\beta_{C',q_1} = 0$. But then stability of q ensures that C' must have enough special points. □

Corollary 2.40. Let $\iota\colon X \hookrightarrow Y$ be an epic closed embedding. Then the morphism

$$\mathcal{Q}(\iota)\colon \mathcal{Q}_{g,n}(X,\beta) \to \mathcal{Q}_{g,n}(Y,\iota_*\beta)$$

induced by ι is a closed embedding over $\mathbb{C}$.

Proof. The morphism $\mathcal{Q}(\iota)$ is proper over $\mathrm{Spec}\,(\mathbb{C})$ by [Sta22, Lemma 01W6] because $\mathcal{Q}(X,\beta)$ is proper and $\mathcal{Q}(Y,\iota_*\beta)$ is separated, both over $\mathrm{Spec}\,(\mathbb{C})$. By [Sta22, Lemma 04XV], it is enough to show that $\mathcal{Q}(\iota)$ is also a monomorphism.

Since $\mathcal{Q}_{g,n}(X,\beta)$ is locally of finite type over $\mathrm{Spec}\,(\mathbb{C})$, so is the morphism $\mathcal{Q}(\iota)$ by [Sta22, Lemma 01T8]. Furthermore, the assumption that ι_* is injective on curve classes ensures, by Theorem 2.39, that for every (geometric) point s in $\mathcal{Q}_{g,n}(Y,\iota_*\beta)$, the natural morphism $\mathcal{Q}_{g,n}(X,\beta)_s \to s$ induced by $\mathcal{Q}(\iota)$ is an isomorphism. This ensures that $\mathcal{Q}(\iota)$ is a monomorphism by [Sta22, Lemma 05VH]. □

2.4.4 Another proof of Corollary 2.40

We give another, more direct proof of Corollary 2.40, which we find more elegant since it works for families of quasimaps directly.

Another proof of Corollary 2.40. All we need to show is that $\mathcal{Q}(\iota)$ is a monomorphism, since it is clearly proper.

Let $q = (\mathcal{C}, \mathcal{L}_\rho, s_\rho, c_m)$ and $q' = (\mathcal{C}', \mathcal{L}'_\rho, s'_\rho, c'_m)$ be two S-families of quasimaps to X such that $\mathcal{Q}(\iota)(q) \simeq \mathcal{Q}(\iota)(q')$. The first observation is that $\mathcal{C}$ and $\mathcal{C}'$ are isomorphic S-families. This holds because $\mathcal{Q}(\iota)$ preserves the underlying family of curves when ι is a closed embedding.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ denote the ambient stacks of X and Y as in Section 2.2.6. Since ι is epic, we have that $\widetilde{\iota}^*\colon \operatorname{Pic}(\mathfrak{Y}) \to \operatorname{Pic}(\mathfrak{X})$ is surjective by Proposition 2.31. With notations as in Section 2.2.6, we can view q and q' as morphisms

$$q, q' \colon \mathcal{C} \to \mathfrak{X}. \tag{2.18}$$

Composition with $\widetilde{\iota}$, q and q' give morphisms

$$\operatorname{Pic}(\mathfrak{Y}) \xrightarrow{\widetilde{\iota}^*} \operatorname{Pic}(\mathfrak{X}) \underset{(q')^*}{\overset{q^*}{\rightrightarrows}} \operatorname{Pic}(\mathcal{C})$$

The assumption that $\mathcal{Q}(\iota)(q) \simeq \mathcal{Q}(\iota)(q')$ implies that

$$q^* \circ \widetilde{\iota}^* = (\mathcal{Q}(\iota)(q))^* = (\mathcal{Q}(\iota)(q'))^* = (q')^* \circ \widetilde{\iota}^*.$$

Since $\widetilde{\iota}^*$ is surjective, we deduce that $q^* = (q')^*$. It follows that

$$(\mathcal{L}_\rho, s_\rho) \simeq q^*(\mathcal{O}_{\mathbb{P}}(D_\rho), t_\rho) = (q')^*(\mathcal{O}_{\mathbb{P}}(D_\rho), t_\rho) \simeq (\mathcal{L}'_\rho, s'_\rho).$$

So it remains to show that $c_m \simeq c'_m$ for all $m \in M$. Note that the isomorphism

$$c_m \colon \otimes_\rho \mathcal{L}_\rho^{\otimes \langle m, u_\rho \rangle} \to \mathcal{O}_{\mathcal{C}}$$

is determined by $g_m := c_m^{-1}(1) \in H^0(\mathcal{C}, \otimes_\rho \mathcal{L}_\rho^{\otimes\langle m, u_\rho\rangle})$. Similiarly, c'_m is determined by $g'_m := (c'_m)^{-1}(1)$ and the universal c_{χ^m} on X are determined by χ^m. As before, we get that

$$\begin{aligned}
(\otimes_\rho \mathcal{L}_\rho^{\otimes\langle m, u_\rho\rangle}, g_m) &= q^*(\otimes_\rho \mathcal{O}_{\mathbb{P}}(D_\rho)^{\otimes\langle m, u_\rho\rangle}, \chi^m),\\
(\otimes_\rho \mathcal{L}_\rho^{\otimes\langle m, u_\rho\rangle}, g'_m) &= (q')^*(\otimes_\rho \mathcal{O}_{\mathbb{P}}(D_\rho)^{\otimes\langle m, u_\rho\rangle}, \chi^m),
\end{aligned}$$

so they agree. $\square$

2.5 The contraction morphism for smooth projective toric varieties

We fix a smooth projective toric variety X for the rest of Section 2.5.

2.5.1 Construction of the contraction morphism

We construct a morphism of stacks

$$c = c_X\colon \mathcal{V} \to \mathcal{Q}_{g,n}(X, \beta),$$

with $\mathcal{V}$ a closed substack of $\overline{\mathcal{M}}_{g,n}(X, \beta)$. The morphism c_X, called the *contraction morphism* of X, will be a generalization of the contraction morphism $c_{\mathbb{P}^N}$ described in Section 2.2.7 and it extends the identity on the locus of stable maps which are stable as quasimaps. The construction of c_X relies on results from Section 2.4 and $c_{\mathbb{P}^N}$.

Remark 2.41. In Section 2.2.7 we have reviewed the morphism $c_{\mathbb{P}^N}$. One can check that a similar construction defines a (global) contraction morphism for products of projective spaces.

Construction 2.42. Let $\beta \in A_1(X)$ be an effective curve class. Choose an epic closed embedding

$$\iota\colon X \hookrightarrow \mathbb{P} := \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_k}.$$

Such an embedding always exists by Theorem 2.35. From Corollary 2.40 it follows that

$$\mathcal{Q}(\iota)\colon \mathcal{Q}_{g,n}(X, \beta) \hookrightarrow \mathcal{Q}_{g,n}(\mathbb{P}, \iota_*\beta)$$

is a closed embedding. Let

$$c_{\mathbb{P}}\colon \overline{\mathcal{M}}_{g,n}(\mathbb{P}, \iota_*\beta) \to \mathcal{Q}_{g,n}(\mathbb{P}, \iota_*\beta)$$

denote the contraction morphism for $\mathbb{P}$. Then $c_{\mathbb{P}}$ fits in the following diagram.

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n}(X, \beta) \arrow[r, hook, "\overline{\mathcal{M}}(\iota)"] & \overline{\mathcal{M}}_{g,n}(\mathbb{P}, \iota_*\beta) \arrow[d, "c_{\mathbb{P}}"] \\ \mathcal{Q}_{g,n}(X, \beta) \arrow[r, hook, "\mathcal{Q}(\iota)"] & \mathcal{Q}_{g,n}(\mathbb{P}, \iota_*\beta) \end{tikzcd} \tag{2.19}$$

We define the stack $\mathcal{V}$ by the following Cartesian diagram

$$\begin{tikzcd} \mathcal{V} \arrow[r, "j"] \arrow[d, "c_X"] & \overline{\mathcal{M}}_{g,n}(X, \beta) \arrow[d, "c_{\mathbb{P}} \circ \overline{\mathcal{M}}(\iota)"] \\ \mathcal{Q}_{g,n}(X, \beta) \arrow[r, "\mathcal{Q}(\iota)"] & \mathcal{Q}_{g,n}(\mathbb{P}, \iota_*\beta) \end{tikzcd}$$

and we call the induced morphism

$$c_X \colon \mathcal{V} \to \mathcal{Q}_{g,n}(X, \beta)$$

the *contraction morphism of* X.

Note that $j \colon \mathcal{V} \to \overline{\mathcal{M}}_{g,n}(X, \beta)$ is a closed embedding and $c_X \colon \mathcal{V} \to \mathcal{Q}_{g,n}(X, \beta)$ is locally of finite type.

We suspect that the morphism $c_X \colon \mathcal{V} \to \mathcal{Q}_{g,n}(X, \beta)$ from Construction 2.42 is independent of the chosen closed embedding $\iota \colon X \hookrightarrow \mathbb{P}$, as long as it is epic. For example, in Section 2.5.2 we describe the closed points of $\mathcal{V}$ (as a closed substack of $\overline{\mathcal{M}}_{g,n}(X, \beta)$) and show that these are independent of the chosen epic embedding ι (see Proposition 2.43). Independence of the chosen epic embedding reduces to showing that, given a family of quasimaps $q \in \mathcal{Q}_{g,n}(X, \beta)$, if there is an epic closed embedding ι into products of projective spaces $\mathbb{P}$ such that $c_{\mathbb{P}} \circ \overline{\mathcal{M}}(\iota)(q)$ factors through $\mathcal{Q}(\iota)$, then the same holds for any other epic embedding into a product of projective spaces. We will study this question in future work.

2.5.2 Description of contraction on points

We describe the closed substack $\mathcal{V}$ inside $\overline{\mathcal{M}}_{g,n}(X, \beta)$ and the morphism c_X from Construction 2.42 in terms of line bundle-section pairs. We use the notations introduced in Sections 2.2.5 and 2.5.1.

Proposition 2.43. Let (C, L_ρ, s_ρ, c_m) be the Σ-collection corresponding to an n-marked genus-g stable map $f \colon C \to X$. Let $T_1, \ldots, T_\ell$ be the rational tails of f, let $\tilde{C}$ be the curve obtained by contracting all the rational tails in C, viewed as a subcurve of C. For $1 \leq i \leq \ell$, let $x_i = T_i \cap \overline{(C \setminus T_i)}$ and let z_i denote a local coordinate at x_i inside $\tilde{C}$. Then

(i) The map $(C, L_\rho, s_\rho, c_{m_X})$ belongs to $\mathcal{V}$ if and only if

$$\operatorname{ord}_{x_i}(s_\rho \mid_{\tilde{C}}) + \beta_{T_i} \cdot D_\rho \geq 0$$

for every $\rho \in \Sigma_X(1)$ and every $1 \leq i \leq \ell$.

(ii) If $(C, L_\rho, s_\rho, c_{m_X})$ lies in $\mathcal{V}$ then its image under c_X is the quasimap $(\tilde{C}, \tilde{L}_\rho, \tilde{s}_\rho, \tilde{c}_{m_X})$

where

$$\begin{aligned}
\tilde{L}_\rho &= L_\rho \mid_{\tilde{C}} \otimes \mathcal{O}_{\tilde{C}}\left(\sum_{i=1}^{\ell} (\beta_{T_i} \cdot D_\rho) x_i\right), \\
\tilde{s}_\rho &= s_\rho \mid_{\tilde{C}} \otimes \prod_{i=1}^{\ell} z_i^{\beta_{T_i} \cdot D_\rho}, \\
\tilde{c}_{m_X} &= c_{m_X} \mid_{\tilde{C}} \otimes (\otimes_{i=1}^{\ell} \psi_{m_Y, x_i, \beta_{T_i}}),
\end{aligned}$$

where the isomorphisms $\psi_{m_X, x_i, \beta_{T_i}}$ are constructed in Remark 2.22.

Proof. The proof of Item (ii) follows from a careful study of the morphisms in Diagram (2.19). The morphism $c_{\mathbb{P}}$ is a generalization of the morphism $c_{\mathbb{P}^N}$, which is described in terms of line bundle-section pairs in Section 2.2.7. Similarly, $\overline{\mathcal{M}}(\iota)$ is described in Construction 2.9 (see Remark 2.11). These descriptions imply Item (ii).

For completeness, we spell out the details. Let us write

$$\begin{aligned}
(c_{\mathbb{P}} \circ \overline{\mathcal{M}}(\iota))(C, L_\rho, s_\rho, c_{m_X}) &= (\hat{C}, \hat{L}_\tau, \hat{s}_\tau, \hat{c}_{m_Y}), \\
\mathcal{Q}(\iota)(\tilde{C}, \tilde{L}_\rho, \tilde{s}_\rho, \tilde{c}_{m_X}) &= (\tilde{C}, \overline{L}_\tau, \overline{s}_\tau, \overline{c}_{m_Y}).
\end{aligned}$$

It suffices to show the equality

$$(\hat{C}, \hat{L}_\tau, \hat{s}_\tau, \hat{c}_{m_Y}) = (\tilde{C}, \overline{L}_\tau, \overline{s}_\tau, \overline{c}_{m_Y}).$$

Furthermore, in that case it follows that $(\tilde{C}, \tilde{L}_\rho, \tilde{s}_\rho, \tilde{c}_{m_X})$ is generically non-degenerate, because $(\tilde{C}, \overline{L}_\tau, \overline{s}_\tau, \overline{c}_{m_Y})$ is, by Lemma 2.38.

The equality $\hat{C} = \tilde{C}$ follows from the description of $c_{\mathbb{P}}$ and the fact that ι is a closed embedding, so $\overline{\mathcal{M}}(\iota)$ and $\mathcal{Q}(\iota)$ preserve the underlying curve.

For the line bundles, we have that

$$\begin{aligned}
\hat{L}_\tau &= \bigotimes_\rho L_\rho^{a_\rho^\tau} \mid_{\tilde{C}} \otimes \left(\bigotimes_{i=1}^{\ell} \mathcal{O}_{\tilde{C}}(\iota_*(\beta_{T_i}) \cdot D_\tau\, x_i)\right), \\
\overline{L}_\tau &= \bigotimes_\rho L_\rho^{a_\rho^\tau} \mid_{\tilde{C}} \otimes \left(\bigotimes_{i=1}^{\ell} \mathcal{O}_{\tilde{C}}(\beta_{T_i} \cdot (\sum_\rho a_\rho^\tau D_\rho)\, x_i)\right).
\end{aligned}$$

The desired equality follows from the projection formula and Remark 2.12, since

$$\iota_*(\beta_{T_i}) \cdot D_\tau = \beta_{T_i} \cdot \iota^*(D_\tau) = \beta_{T_i} \cdot (\sum_\tau a_\rho^\rho D_\rho).$$

For the sections, we have that

$$\hat{s}_\tau = \mu_{\underline{a}^\tau} \prod_\rho s_\rho^{a_\rho^\tau} \mid_{\tilde{C}} \prod_{i=1}^{\ell} z_i^{\iota_*(\beta_{T_i})\cdot D_\tau}$$
$$\overline{s}_\tau = \mu_{\underline{a}^\tau} \prod_\rho s_\rho^{a_\rho^\tau} \mid_{\tilde{C}} \prod_{i=1}^{\ell} z_i^{\beta_{T_i}\cdot(\sum_\rho a_\rho^\tau D_\tau)},$$

and we conclude by the same argument.

Finally, for the isomorphisms we have that

$$\hat{c}_{m_Y} = c_{m_X} \otimes \left(\bigotimes_i \psi_{m_Y, x_i, \iota_*(\beta_{T_i})} \right)$$
$$\overline{c}_{m_Y} = c_{m_X} \otimes \left(\bigotimes_i \psi_{m_X, x_i, \beta_{T_i}} \right)$$

where $m_X \in M_X$ is uniquely determined by the conditions

$$\langle m_X, u_\rho \rangle = \sum_\tau a_\rho^\tau \langle m_Y, u_\tau \rangle$$

for every $\rho \in \Sigma(1)$. One can check that this conditions implies the equality $\psi_{m_Y, x_i, \iota_*(\beta_{T_i})} = \psi_{m_X, x_i, \beta_{T_i}}$ for each i.

To conclude the proof, we show Item (i). By construction of $\mathcal{V}$, we need to find under what conditions the quasimap $q = (c_{\mathbb{P}} \circ \overline{\mathcal{M}}(\iota))(f)$ factors through the image of $\mathcal{Q}(\iota)$, for which we use Theorem 2.39. It suffices to rewrite Items 1 and 2 in Theorem 2.39 for q in terms of f itself.

Item 1 in Theorem 2.39 is automatic here. Indeed, $q_{\text{reg}} = (c_{\mathbb{P}}(\iota \circ f))_{\text{reg}}$ and $\iota \circ (f \mid_{\tilde{C}})$ are maps which agree on a dense open in $\tilde{C}$, therefore they must agree everywhere. This implies that q_{reg} factors through X.

Let us conclude with Item 2 in Theorem 2.39. By the proof of Theorem 2.39, if $q = \mathcal{Q}(\iota) q_1$ for some quasimap q_1 to X, then $\beta^{x_i} = \beta_{x_i, q_1}$. Combining this with Item 2.1 and the equality $\beta_{x_i,q} = \iota_*(\beta_{T_i,f})$, it follows $\iota_*(\beta^{x_i}) = \iota_*(\beta_{T_i,f})$, and therefore $\beta^{x_i} = \beta_{T_i,f}$ since ι_*

is injective. This is the only possible choice for the class β^{x_i}. Therefore, f lies in $\mathcal{V}$ if and only if Items 2.2 and 2.3 hold for this choice of β^{x_i}. Item 2.2 says

$$\beta_f = \beta_{f|_{\tilde{C}}} + \sum_{i=1}^{\ell} \beta_{T_i,f},$$

which is immediate. Thus, the only non-trivial condition for f to lie in $\mathcal{V}$ is Item 2.3, which says that

$$\mathrm{ord}_{x_i}(s_\rho \mid_{\tilde{C}}) + \beta_{T_i,f} \cdot D_\rho \geq 0$$

for every $\rho \in \Sigma(1)$ and every $1 \leq i \leq \ell$. □

Note that, if X has the property that all the toric boundary divisors D_ρ are nef, then c_X is defined on the whole of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ since the condition in Item (i) in Proposition 2.43 holds trivially.

Example 2.44. In order to illustrate the content of Proposition 2.43, we construct a family of stable maps whose special fibre lies in $\mathcal{V}$ but whose general fibre lies outside $\mathcal{V}$.

Let $X = \mathrm{Bl}_0\mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at the origin, whose fan is pictured in Figure 2.1. We will use the curve classes $L, S, E \in A_1(\mathrm{Bl}_0\mathbb{P}^2)$ introduced in Example 2.36. By [Cox95], a morphism to $\mathrm{Bl}_0\mathbb{P}^2$ is given by line bundle-section pairs (L_i, s_i) for $i = 0, \ldots, 3$ satisfying the following conditions

1 $L_0 \simeq L_1 \otimes L_3$ and $L_1 \simeq L_2$,

2 s_0 and s_3 do not vanish simultaneously and

3 s_1 and s_2 do not vanish simultaneously.

Such a map has degree β if $\beta \cdot D_i = \deg(L_i)$. For example, $(L \cdot D_i) = (1, 1, 1, 0)$, $(S \cdot D_i) = (1, 0, 0, 1)$ and $(E \cdot D_i) = (0, 1, 1, -1)$.

Let $C_1 = \mathbb{P}^1$ with homogeneous coordinates $[x_0 \colon x_1]$ and let $C_2 = \mathbb{P}^1$ with homogeneous coordinates $[y_0 \colon y_1]$. We follow the convention that $0 = [1 \colon 0]$. Let C be the rational nodal curve obtained by gluing of C_1 and C_2 along the origins. By abuse of notation, we denote the node by 0.

Consider the affine line $\mathbb{A}^1$ with coordinate t and let $W = C \times \mathbb{A}^1$, which we view as a trivial family of nodal curves over $\mathbb{A}^1$. We denote by W_t and 0_t the fibre and the node over $t \in \mathbb{A}^1$, respectively. Furthermore, we choose disjoint two sections of the projection

$W \to \mathbb{A}^1$ with the condition that they factor through $(C_1 \setminus 0) \times \mathbb{A}^1$. These will be our marked points.

Consider the following morphism $f \colon W \to \mathrm{Bl}_0\mathbb{P}^2$, whose components we denote by $s_0, \ldots, s_3$;

1 f restricted to $C_1 \times \mathbb{A}^1$ is given by

$$f([x_0 \colon x_1], t) = [x_0^2 \colon 0 \colon 2 \colon x_1^2 - tx_1x_0]$$

2 and f restricted to $C_2 \times \mathbb{A}^1$ is given by

$$f([y_0 \colon y_1], t) = [y_0^2 \colon y_1y_0 \colon y_1^2 - 3y_1y_0 + 2y_0^2 \colon 0]$$

One can check that the two descriptions glue along $0 \times \mathbb{A}^1$ and satisfy non-degeneracy, so f is a family in $\overline{\mathcal{M}}_{0,2}(\mathrm{Bl}_0\mathbb{P}^2, 2L)$ which restricts to a family of degree $2S$ on $C_1 \times \mathbb{A}^1$ and of degree $2E$ on $C_2 \times \mathbb{A}^1$.

For each t, Proposition 2.43 says that the morphism $f_t \colon W_t \to \mathrm{Bl}_0\mathbb{P}^2$ lies in the closed substack $\mathcal{V}$ of $\overline{\mathcal{M}}_{0,2}(\mathrm{Bl}_0\mathbb{P}^2, 2L)$ where $c_{\mathrm{Bl}_0\mathbb{P}^2}$ is defined if and only if

$$\mathrm{ord}_{0_t}(s_3 \mid_{C_1 \times \{t\}}) \geq -\beta_{C_1 \times \{t\}} \cdot D_3 = -2\, E \cdot E = 2.$$

Since $s_3 = x_1^2 - tx_1x_0$, the condition is satisfied if and only if $t = 0$. Therefore, f is indeed a family of stable maps whose general fibre does not lie in $\mathcal{V}$ but whose special fibre lies in $\mathcal{V}$.

As a sanity check for Proposition 2.43, we show that f_0 is indeed the only fibre that lies in $\mathcal{V}$ using Equation (2.19) directly. For that, we choose the closed embedding $\iota \colon \mathrm{Bl}_0\mathbb{P}^2 \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^1$ described in Example 2.36 and show that $c_{\mathbb{P}} \circ \overline{\mathcal{M}}(\iota)(f_t)$ lies in the image of $\mathcal{Q}_{0,2}(\mathrm{Bl}_0\mathbb{P}^2, 2L)$ if and only if $t = 0$.

Following the definition of ι in Equation (2.17), we have the following description of $\overline{\mathcal{M}}(\iota)(f)$:

1 on $C_1 \times \mathbb{A}^1$

$$\overline{\mathcal{M}}(\iota)(f)([x_0 \colon x_1], t) = [x_0^2 \colon 0 \colon 2(x_1^2 - tx_1x_0)], [0 \colon 2]$$

2 and, on $C_2 \times \mathbb{A}^1$,

$$\overline{\mathcal{M}}(\iota)(f)([y_0 \colon y_1], t) = [y_0^2 \colon 0 \colon 0], [y_1y_0 \colon y_1^2 - 3y_1y_0 + 2y_0^2]$$

Next, we apply $c_{\mathbb{P}}$, where $\mathbb{P} = \mathbb{P}^2 \times \mathbb{P}^1$. This means we contract $C_2 \times \mathbb{A}^1$ and twist by $\iota_*(2E) = (0, 2)$. The result is the morphism

$$g = c_{\mathbb{P}} \circ \overline{\mathcal{M}}(\iota)(f) \colon C_1 \times \mathbb{A}^1 \to \mathrm{Bl}_0\mathbb{P}^2$$

given by

$$g([x_0 \colon x_1], t) = [x_0^2 \colon 0 \colon 2(x_1^2 - t x_1 x_0)], [0 \colon 2x_1^2].$$

Observe that if we choose coordinates $[z_0 \colon z_1 \colon z_2]$ on $\mathbb{P}^2$ and $[w_0 \colon w_1]$ on $\mathbb{P}^1$, then the image of ι (see Equation (2.17)) is the closed with equation $z_1 w_1 - z_2 w_0$. The same equations determine if a stable map to $\mathbb{P}^2 \times \mathbb{P}^1$ factors through $\mathrm{Bl}_0\mathbb{P}^2$, but the situation is different for quasimaps. In fact, the sections defining g satisfy this equation, but we argue next that g_t lies in the image of $\mathcal{Q}_{0,2}(\mathrm{Bl}_0\mathbb{P}^2, 2L)$ if and only if $t = 0$.

The image of a family of quasimaps (C, L_i, s_i, c_m) along

$$\mathcal{Q}(\iota) \colon \mathcal{Q}_{0,2}(\mathrm{Bl}_0\mathbb{P}^2, 2L) \to \mathcal{Q}_{0,2}(\mathbb{P}^2 \times \mathbb{P}^1, (2, 2))$$

can be expressed in homogeneous coordinates as

$$[s_0 \colon s_1 s_3 \colon s_2 s_3], [s_1 \colon s_2].$$

In particular, for g_t to lie in the image of $\mathcal{Q}(\iota)$ we must have that the rational function

$$\frac{2(x_1^2 - t x_1 x_0)}{2x_1^2}$$

is regular (and, in fact, constant) on $C_1 \times \{t\}$. This happens if and only if $t = 0$.

Remark 2.45. The following observation can be deduced from Proposition 2.43, or directly from Construction 2.42. For every stable map $f \colon C \to X$ and every rational tail $T \subset C$ of f, the point $x = f(T)$ is a basepoint of the quasimap $c_X(f)$ of degree $\beta_{x, c_X(f)} = \deg(f \mid_T)$. Since the notion of degree of a basepoint is instrinsic, the same must hold for any other stable map f' with $c_X(f') = c_X(f)$. In other words, we can reinterpret the degree β_x of a basepoint x of a quasimap q as the degree of any rational tail T of a stable map $f \colon C \to X$ such that $c_X(f) = q$ and c_X contracts T to x, whenever one such map exists. For example, the existence of such f is guaranteed if X is Fano, since then c_X is surjective by Theorem 2.47.

Example 2.46. We highlight the relation, explained in Remark 2.45, between the degree of a basepoint and the degree of rational tails contracted to it.

Let $C = \mathbb{P}^1$ with homogeneous coordinates $[s_0 \colon s_1]$, consider $\mathbb{A}^1$ with coordinate t and let $W = C \times \mathbb{A}^1$. Consider the following morphism $f \colon W \to \mathbb{P}^2$

$$f([s_0 \colon s_1], t) = [s_0 \colon ts_0 \colon s_1].$$

Then f induces a morphism $\widetilde{f} \colon W \setminus W_0 \to \mathrm{Bl}_0\mathbb{P}^2$, which extends to a family of quasimaps q on W,

$$q([s_0 \colon s_1], t) = [s_0 \colon ts_0 \colon s_1 \colon 1]),$$

with a basepoint at the point $t = s_1 = 0$, the origin in W_0.

On the other hand, $\widetilde{f}$ can also be extended to a morphism

$$\widetilde{f} \colon \widetilde{W} = \mathrm{Bl}_{s=t=0}W \to \mathrm{Bl}_0\mathbb{P}^2,$$

which we view as a family of maps over $\mathbb{A}^1$. If we view $\widetilde{W}$ as the closed subvariety cut-out by $u_1t - s_1u_0$ inside $C \times \mathbb{A}^1 \times \mathbb{P}^1_{[u_0 \colon u_1]}$, then the expression of $\widetilde{f}$ is

$$\widetilde{f}([s_0 \colon s_1], t, [u_0 \colon u_1]) = [u_1s_0 \colon u_0s_0 \colon u_1 \colon u_1s_1].$$

The special fibre $\widetilde{W_0}$ has two components $T = V(s_1)$ and $\widetilde{C} = V(u_0)$. The restriction of $\widetilde{f}$ to each of them is

$$\widetilde{f}\mid_T ([u_0 \colon u_1]) = [u_1 \colon u_0 \colon u_1 \colon 0],$$

of degree E, and

$$\widetilde{f}\mid_{\widetilde{C}} ([s_0 \colon s_1]) = [s_0 \colon 0 \colon 1 \colon s_1],$$

of degree S.

Remark 2.45 tells us that we can read that $\deg(\widetilde{f}\mid_T) = E$ directly from q, by checking that $\beta_x = \beta_{x,q_0} = E$, where x is the point $s = t = 0$ in W. We check the claim in this particular example.

For that, we need to compute β_x following Proposition 2.21. Firstly,

$$q_0([s_0 \colon s_1]) = [s_0 \colon 0 \colon s_1 \colon 1])$$

has the following orders of vanishing at x: $(\mathrm{ord}_x(s_i)) = (0, \infty, 1, 0)$, and $V_x = \{\rho_1\}$. Let $\sigma_{i,j}$ denote the cone in Figure 2.1 spanned by ρ_i and ρ_j. The only cones $\sigma \in \Sigma(2)$ such that

$V_x \subseteq \sigma$ are $\sigma_{0,1}$ and $\sigma_{1,3}$. The former cone gives

$$\beta(x, \sigma_{0,1}) \cdot D_2 = 1,$$
$$\beta(x, \sigma_{0,1}) \cdot D_3 = 0.$$

Therefore $\beta(x, \sigma_{0,1}) = L$, but it does not satisfy the condition in Proposition 2.21 because

$$0 = \mathrm{ord}_x(s_0) \ngeq \beta(x, \sigma_{0,1}) \cdot D_0 = L \cdot L = 1.$$

Instead, we must take $\sigma_{1,3}$, which gives

$$\beta(x, \sigma_{1,3}) \cdot D_0 = 0,$$
$$\beta(x, \sigma_{1,3}) \cdot D_2 = 1.$$

Therefore $\beta(x, \sigma_{1,3}) = E$, and one easily checks that

$$\infty = \mathrm{ord}_x(s_1) \geq \beta(x, \sigma_{1,3}) \cdot D_1 = E \cdot S = 1,$$
$$0 = \mathrm{ord}_x(s_3) \geq \beta(x, \sigma_{1,3}) \cdot D_3 = E \cdot E = -1.$$

So $\beta_x = \beta(x, \sigma_{0,1}) = E$, as claimed.

2.6 Surjectivity for Fano targets

We have introduced contraction morphism c_X between stable maps and stable quasimaps for smooth projective toric varieties in Construction 2.42. In this section, we prove that c_X is surjective if the target X is Fano.

Theorem 2.47. Let X be a smooth Fano toric variety. The contraction morphism

$$c_X : \mathcal{V} \to \mathcal{Q}_{g,n}(X, \beta)$$

defined in Construction 2.42 is surjective.

The proof of Theorem 2.47 is delayed until Section 2.6.3. Before that, we introduce some terminology about the monoid of effective curve classes in Section 2.6.1 and an important construction in Section 2.6.2.

2.6.1 Factorizations of curve classes

We fix a smooth toric Fano variety X for the rest of Section 2.6.

Definition 2.48. Let β be a non-zero effective curve class in X. A *factorization* of β is an expression $\beta = \beta_1 + \ldots + \beta_k$ such that β_i is non-zero and effective for every i and $k \geq 2$. We say that β is *irreducible* if it does not admit any factorization.

Definition 2.49. The *length* of a curve class $\beta \in A_1(X)$ is

$$\lambda(\beta) = \deg(\iota_* \beta) = \beta \cdot (-K_X) = \sum_{\rho \in \Sigma(1)} \beta \cdot D_\rho,$$

with $\iota \colon X \hookrightarrow \mathbb{P}^N$ the anticanonical embedding.

Remark 2.50. The length λ defines a morphism of semigroups with unit $\lambda \colon A_1(X) \to \mathbb{Z}$. Moreover,

1 if β is effective then $\lambda(\beta) \geq 0$ with equality if and only if $\beta = 0$ and

2 if β is non-zero effective and $\beta = \beta_1 + \ldots + \beta_k$ is a factorization, then $\lambda(\beta) > \lambda(\beta_i)$ for all i.

2.6.2 Grafting trees on to quasimaps

We explain how to replace a basepoint of a quasimap with a rational curve and how to extend the quasimap there given certain extra data. This construction is fundamental for the proof of Theorem 2.47.

Construction 2.51. Let $q = (C, L_\rho, s_\rho, c_m)$ be a quasimap to X of degree β. Suppose for simplicity that q has a unique basepoint x and let β_x be the degree of q at x. Let $q_{\text{reg}} = (C, L'_\rho, s'_\rho, c'_m)$ denote the regular extension of q, constructed in Definition 2.13.

Given a rational irreducible curve $T \simeq \mathbb{P}^1$, a point $x_T \in T$ and sections $t_\rho \in H^0(T, \mathcal{O}_T(\beta_x \cdot [D_\rho]))$ such that $t_\rho(x_T) = s'_\rho(x)$ for all $\rho \in \Sigma(1)$, we can construct a new quasimap

$$q_{\text{gr}} = (\tilde{C}, \tilde{L}_\rho, \tilde{s}_\rho, \tilde{c}_m),$$

of class β by *grafting T on to q at x* as follows:

- The curve $\tilde{C}$ is obtained by gluing C and T along $x \in C$ and $x_T \in T$. By abuse of notation, we denote the resulting node in $\tilde{C}$ also by x.

- For each $\rho \in \Sigma_X(1)$, the line bundle $\tilde{L}_\rho$ on $\tilde{C}$ is obtained by gluing the line bundle L'_ρ on C and the line bundle $\mathcal{O}_T(\beta_x \cdot [D_\rho])$ on T.

- The sections $\tilde{s}_\rho$ of $\tilde{L}_\rho$ are obtained by gluing the sections t_ρ and s'_ρ.

- Finally, for each $m \in M$, the isomorphisms $\tilde{c}_m$ on $\tilde{C}$ are obtained by gluing the isomorphisms $c'_m \colon \otimes_{\rho\in\Sigma(1)} L'^{\otimes\langle m,u_\rho\rangle}_\rho \simeq \mathcal{O}_C$ and the isomorphisms $\psi_{m,\beta_x} \colon \mathcal{O}_T(\sum_{\rho\in\Sigma(1)} \beta_x \cdot [D_\rho]\,\langle m, u_\rho\rangle) \simeq \mathcal{O}_T$ analogous to those constructed in Remark 2.22.

The defining data of q_{gr} automatically satisfies generically non-degeneracy. This is clear on C, while on T it follows from the fact that q_{reg} is non-degenerate at x.

Remark 2.52. With the notations of Construction 2.51, the regular extension q_{reg} of a quasimap q has the following property: a section s_ρ is identically zero on an irreducible component C' of C if and only if s'_ρ is identically zero on C'.

It follows that, for each basepoint x of q, we can find sections t_ρ satisfying the assumptions of Construction 2.51. Indeed, we can choose t_ρ as follows

- if $\beta_x \cdot [D_\rho] < 0$, we must take $t_\rho = 0$ and the gluing condition holds because $s'_\rho(x) = 0$ by Corollary 2.26,

- if $\beta_x \cdot [D_\rho] = 0$ then we must take t_ρ constant with value $s'_\rho(x)$ and

- if $\beta_x \cdot [D_\rho] > 0$ then we can choose any section $t_\rho \in H^0(T, \mathcal{O}_T(\beta_x \cdot [D_\rho]))$ with $t_\rho(x_T) = s'_\rho(x)$.

Remark 2.53. Grafting can be undone. With the notations of Construction 2.51, we can recover q from q_{gr} by contracting T, restricting the rest of the data to C and twisting with the degree β_T of q_{gr} on T analogously to Proposition 2.43. In that case, we say that q is obtained by *pruning* q_{gr} *along* T. By Proposition 2.43, pruning all the rational tails of a stable map $f \colon C \to X$ we recover the quasimap $c_X(f)$.

Remark 2.54. Construction 2.51 can be extended to quasimaps q with more than one basepoints. The grafting of T on to q at a basepoint x, is obtained by replacing in Construction 2.51 the map q_{reg} by the quasimap obtained by extending q only at x. The description of such quasimap can be obtained using Corollary 2.26, replacing B in the formulas by the singleton $\{x\}$. Note that if we graft a quasimap q along all its basepoints $x_1, \dots, x_\ell$, the basepoints of the resulting quasimap q_{gr} must lie on the grafted rational curves $T_1, \dots, T_\ell$.

2.6.3 Proof of surjectivity

Proof of theorem 2.47. The morphism $c_X \colon \mathcal{V} \to \mathcal{Q}_{g,n}(X,\beta)$ is locally of finite type, therefore by [Sta22, Lemma 0487] it is enough to show that c_X is surjective on closed points.

Let $q = (C, L_\rho, s_\rho, c_m)$ be a stable quasimap to X in $\mathcal{Q}_{g,n}(X,\beta)$. We show by induction on $\lambda(\beta)$ (see Definition 2.49) that there exists a stable map f to X in $\overline{\mathcal{M}}_{g,n}(X,\beta)$ with $c_X(f) = q$.

If $\lambda(\beta) = 0$, then there are no basepoints by Proposition 2.25. This means that q is itself a stable map and there is nothing to show.

If C has basepoints $x_1, \dots, x_\ell$, grafting an irreducible rational curve T_i at each basepoint x_i induces an equality of curve classes

$$\beta = \sum_{i=1}^{\ell} \beta_{x_i} + (\beta - \sum_{i=1}^{\ell} \beta_{x_i}), \tag{2.20}$$

where $\beta_{x_i} \neq 0$ for all i by Proposition 2.25. If $\ell \geq 2$, Equation (2.20) is a factorization, and we can use Remark 2.50 to conclude by induction.

Therefore, we can restrict to the case that $\lambda(\beta) > 0$ and C has only one basepoint x. Furthermore, by restricting ourselves to the irreducible component containing the basepoint x, we can assume that C is irreducible. Let q_{gr} be obtained by grafting an irreducible rational curve T_1 to q at x. Then we get an expression

$$\beta = \beta_x + (\beta - \beta_x)$$

with $\beta_x \neq 0$. If $\beta - \beta_x$ is also non-zero, we conclude by induction on λ. Thus, we can assume that $\beta = \beta_x$.

In other words, we only need to deal with the particular case that on the new curve $C \cup T_1$ the degree is 0 on C and β on T_1. The strategy is to continue grafting irreducible rational curves to q_{gr} at the new basepoints, which necessarily lie on T_1 (see Remark 2.54). Again, the only problematic case is if there is a unique basepoint x_1 of q_{gr} on T_1 such that $\beta_{x_1} = \beta$, because we could enter a loop and end up with an infinite chain of $\mathbb{P}^1$'s. To rule this out, we use Lemma 2.55 as follows: when grafting T_1 to q, we are allowed to choose the sections t_ρ on T_1 (satisfying the conditions in Construction 2.51). If we show that we can choose the sections t_ρ with the property that

$$\text{there is } t_\rho \neq 0 \text{ such that } \mathrm{ord}_{x_1}(t_\rho) < \beta \cdot D_\rho, \tag{2.21}$$

then Lemma 2.55 (see also Remark 2.53), applied to $C_1 = C \cup T_1$ and $C_2 = T_2$, ensures that when we graft T_2 we cannot have $\beta_{T_1} = 0$ and $\beta_{T_2} = \beta$, and so we conclude by induction.

To get (2.21) for a specific class β, it suffices to show one of the following:

$$\text{there exists } \rho \in \Sigma(1) \text{ such that } s_\rho(x) = 0, s_\rho \neq 0 \text{ and } \beta \cdot D_\rho \geq 2, \tag{2.22}$$

or

$$\text{there exist } \rho \neq \rho' \in \Sigma(1) \text{ such that } \beta \cdot D_\rho = \beta \cdot D_{\rho'} = 1 \text{ and at most one is } 0 \text{ on } C_x. \tag{2.23}$$

In both cases, it is clear we can choose t_ρ on T_1 to have simple disjoint zeroes, which ensures (2.21).

To apply this strategy, we distinguish two cases, depending on whether β is irreducible or not. In the latter case we shall use that X is Fano, which is not needed in the former case.

Firstly, we assume that C is irreducible and has a unique basepoint x, that β is irreducible with $\lambda(\beta) > 0$ and that $\beta_x = \beta$. Since β is irreducible, it can be represented by the closure of the toric orbit associated to a wall $\tau \in \Sigma$; that is, $\beta = [V(\tau)]$. By [CLS11, Proposition 6.4.4], there exist two distinct rays $\rho_1, \rho_2 \in \Sigma(1)$ such that $\beta \cdot D_{\rho_i} = 1$ for $i = 1, 2$. By Lemma 2.56, there is $\rho \in \Sigma(1)$ such that $s_\rho(x) = 0$, $s_\rho \neq 0$ and $\beta \cdot D_\rho > 0$. If $\beta \cdot D_\rho \geq 2$, we are done by (2.22). Otherwise, either $\{\rho, \rho_1\}$ or $\{\rho, \rho_2\}$ satisfy (2.23).

Finally, we deal with the case where β is not irreducible. We also assume, as before, that $\lambda(\beta) > 0$, that C is irreducible, that there is a unique basepoint $x \in C$ and that $\beta = \beta_x$.

By Lemma 2.56, there is $\rho \in \Sigma(1)$ such that $s_\rho(x) = 0$, $s_\rho \neq 0$ and $\beta \cdot D_\rho > 0$.

We number the rays in $\Sigma(1)$ as $\rho = \rho_1, \rho_2, \ldots, \rho_r$. Since X is Fano, the anticanonical divisor $-K_X = \sum_{i=1}^r D_{\rho_i}$ is very ample (by [CLS11, Theorem 6.1.15]). Let $\iota\colon X_\Sigma \hookrightarrow \mathbb{P}^N$ be the corresponding closed embedding. Since β is not irreducible, let $\beta = \beta_1 + \beta_2$ be a factorization. Then

$$\sum_{i=1}^r \beta \cdot D_{\rho_i} = \deg(\iota_*\beta) = \deg(\iota_*\beta_1) + \deg(\iota_*\beta_2) \geq 2. \tag{2.24}$$

It follows that either $\beta \cdot D_\rho \geq 2$, and then we conclude by (2.22), or there is $i \neq 1$ such that $\beta \cdot D_{\rho_1} \geq 1$, and then $\{\rho, \rho_i\}$ satisfy (2.23).

As a result of the induction argument, we end up with a prestable map f to X whose contraction is the original quasimap q. If f is not stable, we can stabilize it (by contracting its irreducible components of degree 0 with no marks). The result is a stable map f which lies in $\mathcal{V}$ by construction. More precisely, one only needs to check the condition in Proposition 2.43, and it follows from the following observation: in Construction 2.51, we have that

$$\operatorname{ord}_x(\tilde{s}_\rho\mid_C) + \beta_{T_i}\cdot[D_\rho] = \operatorname{ord}_x(s'_\rho\mid_C) + \beta_x\cdot[D_\rho] = \operatorname{ord}_x(s_\rho) \geq 0.$$

It is also clear that f has degree β since the total curve class is preserved by grafting. $\square$

We conclude with two lemmas used in the proof of Theorem 2.47.

Lemma 2.55. Let q be a quasimap with underlying curve C consisting of two irreducible components C_1 and C_2 glued along a point x. Suppose C_2 has genus 0. Let q_1 and q_2 be the restrictions of q to C_1 and C_2 respectively. Assume $\beta_{C_1}=0$ and $\beta_{C_2}\neq 0$. Let q' be the quasimap obtained by pruning q along C_2. Then for every $\rho\in\Sigma(1)$ such that $s'_\rho\neq 0$, or equivalently such that $s_\rho\mid_{C_1}\neq 0$, the following holds

$$\operatorname{ord}_x(s'_\rho) = \beta_{C_2}\cdot D_\rho = \beta'\cdot D_\rho.$$

Proof. By definition,

$$s'_\rho = s_\rho\mid_{C_1}\cdot z^{\beta_{C_2}\cdot D_\rho},$$

with z a local coordinate at x inside C_1. Therefore, $s'_\rho = 0$ if and only if $s_\rho\mid_{C_1}=0$. Furthermore, if $s_\rho\mid_{C_1}\neq 0$ then $\operatorname{ord}_x(s_\rho\mid_{C_1})=0$ because $\beta_{C_1}=0$, and $\operatorname{ord}_x(s'_\rho)=\beta_{C_2}\cdot D_\rho = \beta'\cdot D_\rho$. $\square$

Lemma 2.56. Let $q=(C,L_\rho,s_\rho,c_m)$ be a stable quasimap to X of degree β with C irreducible and let x be a basepoint of q. Then there exists $\rho\in\Sigma(1)$ such that $s_\rho(x)=0$ but $s_\rho\neq 0$. In particular, $\beta\cdot D_\rho>0$.

Proof. Since x is a basepoint, by Equation (2.1) there is a primitive collection $\mathcal{P}\subseteq\Sigma(1)$ such that $s_\rho(x)=0$ for all $\rho\in\mathcal{P}$. By generic non-degeneracy, there exists $\rho\in\mathcal{P}$ such that s_ρ is not identically 0. The existence of s_ρ ensures that $\beta\cdot D_\rho=\deg(L_\rho)>0$. $\square$

Remark 2.57. In Theorem 2.47, we can replace the condition that X is Fano by the following property: for each effective non-irreducible non-zero curve class $\beta\in A_1(X)$ such that there exists $\rho\in\Sigma(1)$ with $\beta\cdot D_\rho=1$, there exists $\rho'\neq\rho$ such that $\beta\cdot D_{\rho'}>0$.

This condition is true if X is Fano by the argument involving Equation (2.24).

A way to control this condition is to look at

$$\sum_{\rho \in \Sigma(1)} \beta \dot{D}_\rho = \beta \cdot (-K_X).$$

For example, the above condition holds if $K_X \cdot \beta \geq 2$ for every effective non-zero curve class $\beta \in A_1(X)$. But this implies that $-K_X$ is very ample by [CLS11, Theorems 6.1.15 and 6.3.22], so X is Fano.

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