Countless Contractions

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Abstract

We examine the model-theoretic properties of the induced structure on the archimedean classes of a non-archimedean expansion of $\mathbb{R}_{\exp}$ equipped with trans-exponential functions. This is mainly done by studying Contraction groups and Asymptotic triples.

Contraction groups are a type of model-theoretic structure introduced by F.V Kuhlmann consisting of an ordered abelian group G along with a unary function $\chi : G \to G$ which collapses entire archimedean classes to a single point. The canonical example being the action of the logarithm on the archimedean classes of a non-archimedean model of $\mathbb{R}_{\exp}$. In the papers F.-V. Kuhlmann, 1994 and F.-V. Kuhlmann, 1995, it was shown that the theory of centripetal contraction groups has quantifier elimination and is weakly o-minimal.

Similarly, Asymptotic triples consist of an ordered abelian group G and a unary function $\psi : G^{\neq 0} \to G$ collapsing entire archimedean classes to a point, with the canonical example being the action of the **logarithmic derivative** on the archimedean classes of the germs of functions on $\mathbb{R}_{\exp}$. In the works of Aschenbrenner and Van den Dries it was also shown that some formulation of these into first-order logic has quantifier elimination and is weakly o-minimal.

The works mentioned above only deal with regular logarithms and exponentials. In this thesis, we extend the works above, to so-called 'Hyper-logarithms' and 'Trans-exponentials', which can intuitively be thought of as the composition of log and exp infinitely many times. The main results for this thesis are quantifier elimination and weak o-minimality for n-contraction groups along with quantifier elimination for $\theta - L$-contraction groups.

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Chapter 1

Introduction

The purpose of this thesis is to examine the model theoretic properties of the induced structure on the archimedean classes of a non-archimedean expansion of $\mathbb{R}_{\exp}$ equipped with trans-exponential functions. This is mainly done by studying Contraction groups and Asymptotic triples.

Contraction groups are a type of model theoretic structure introduced in F.-V. Kuhlmann, 1994 consisting of an ordered abelian group G along with a unary function $\chi : G \to G$ which collapses entire archimedean classes (these will be defined later but for now think of them as the equivalence classes under big-O-notation) to a single point. The canonical example is the action of the logarithm on the archimedean classes of a non-archimedean model of $\mathbb{R}_{\exp}$. In that paper, it was shown that the theory of centripetal contraction groups has quantifier elimination and is weakly o-minimal.

Asymptotic triples were introduced in Aschenbrenner and van den Dries, 2000, similarly consisting of an ordered abelian group G and a unary function $\psi : G^{\neq 0} \to G$ collapsing entire archimedean classes to a point, with the canonical example being the action of the **logarithmic derivative** on the archimedean classes of the germs of functions on $\mathbb{R}_{\exp}$. In the initial paper, it was also shown that some formulation of these into first order logic has quantifier elimination and is weakly o-minimal.

Note that the works mentioned above only dealt with regular logarithms and exponentials. In this thesis, we extend the works above, to so called 'Hyper-logarithms' and 'Trans-exponentials', which can intuitively be thought of as the composition of log and exp infinitely many times. The main results for this thesis are quantifier elimination and weak o-minimality for n-contraction groups (Theorem 3.4.17 and Theorem 3.5.21) along with quantifier elimination for $\theta - L$-contraction groups (Theorem 4.4.13).

We have just introduced two potentially unfamiliar concepts, archimedean classes and trans-exponentials/hyper-logarithms. Let us expand on both starting with the latter.

1.1 Model theory of $\mathbb{R}$

The model theory of the real numbers began with Alfred Tarski (see Vaught, 1986), who showed that the theory of real closed fields in the language of ordered rings has quantifier elimination and is complete. A field F is real closed if it is orderable and for every odd degree polynomial along with every polynomial of the form $x^2 - a$, where $a \in F^{>0}$, has a root in F. We can express this theory in first order logic as:

Definition 1.1.1. Let $\mathcal{L}_{or} := \langle +. -, \cdot, 0, 1, < \rangle$ be the language of ordered rings. The theory of **ordered fields** states that:

1. $\langle +, -, \cdot, 0, 1 \rangle$ forms a field
2. $<$ is a linear ordering
3. For any $a, b > 0$ and $c, d < 0$:
   (a) $-a < 0$
   (b) $a + b$, $a \cdot b$, $c \cdot d$ are all greater than 0

The theory of **real closed fields** further asserts that:

4. Every positive element in F has a square root:

$$\forall y (y > 0 \rightarrow \exists x (x^2 = y))$$

5. Every odd degree polynomial with coefficients in F has a root in F, this can be expressed via the collection of formulas $(\phi_n)_{n<\omega}$ where ϕ_n is:

$$\forall a_0, a_1, \ldots, a_n \exists x \left(\sum_{i=0}^{n} a_i x^i = 0 \right)$$

Theorem 1.1.2 (Tarski)**.** *The theory of real closed fields has quantifier elimination, is complete and is decidable.*

An immediate consequence of this result is o-minimality, which is a model theoretic property of structures expanding dense linear orders proclaiming that every definable set in one dimension can be described just using the ordering.

Definition 1.1.3. Let $\mathcal{M} := (M, <, \ldots)$ expand a dense linear order. $\mathcal{M}$ is **o-minimal** if every definable set in M^1 is a finite union of points and intervals.

By quantifier elimination, every real closed field must be o-minimal. Note that o-minimality is preserved under elementary equivalence, unlike similar model theoretic properties like minimality or weak o-minimality. A proof can be found in van den Dries, 1998, but the main reason is that o-minimal structures eliminate $\exists^\infty$, i.e for any formula $\phi(x, y)$, if for all $m \in M$, the set $|\phi(M,m)|$ is finite, then there exists some $n < \omega$ such that for all $m \in M$, $|\phi(M,m)| < n$.

O-minimality is significant because it gives a complete description of the definable sets in all dimensions, as a finite union of 'cells'. We give a quick definition of o-minimal cells.

Lemma 1.1.4 (Monotonicity Theorem,van den Dries, 1998)**.** *Let $\mathcal{M} \coloneqq (M, <, +, 0, \ldots)$ be an o-minimal structure expanding an ordered abelian groups. For any definable function $f : M \to M$, there exists some collection of points $a_0 < a_1 < \ldots < a_n$ in M, such that on each C_i, where:*

$$C_0 \coloneqq (-\infty, a_0) < C_1 \coloneqq (a_0, a_1) < \ldots < C_n \coloneqq (a_{n-1}, a_n) < C_{n+1} \coloneqq (a_n, \infty)$$

the function $f \mid C_i$ is monotone and continuous.

This itself is a very strong statement, but we can use it to induct on the dimension of a set to describe definable sets in all dimensions:

Definition 1.1.5 (O-minimal Cells)**.** Let $\mathcal{M} \coloneqq (M, <, +, 0, \ldots)$ be an o-minimal structure expanding an ordered abelian groups. A 0-cell is any point in M. A 1-cell is any interval $(a, b) \subseteq M$. Given an n-cell $X \subseteq M^{m+n}$, and two definable functions $f, g : X \to M$ with $f(x) \leq g(x)$ for all $x \in X$,

- $\Gamma(f) \coloneqq \{(x, y) \in M^{m+n} \times M \mid f(x) = y\}$ is an n-cell in M^{m+n+1}
- $\Gamma(f, g) \coloneqq \{(x, y) \in M^{m+n} \times M \mid f(x) \leq y \leq g(y)\}$ is an $n+1$-cell in M^{m+n+1}.

Theorem 1.1.6 (Cell Decomposition, van den Dries, 1998)**.** *Let $\mathcal{M} \coloneqq (M, <, +, 0, \ldots)$ be an o-minimal structure expanding an ordered abelian groups. Any definable set in M^n is a finite union of cells.*

Hopefully the theorems above demonstrate why o-minimality is a desirable property to have in a structure.

1.2 Adding exponentials

By o-minimality for real closed fields, we can totally classify the definable sets in $\mathbb{R}$ as an ordered field and show that they are 'tame' in some sense. Naturally, we could then ask if the same is true the reals expanded with an exponential function. It was proved in Wilkie, 1996 that the theory of $(\mathbb{R}, \exp)$ in the language $\mathcal{L}_{or} \cup \langle \exp \rangle$ is model complete and o-minimal. Thus any set defined via polynomials involving exp must also be 'tame', in the sense that it must a boolean combination of o-minimal cells as defined in Definition 1.1.5.

Going even further, it could then be asked what how definable sets look like when we adjoin to $\mathbb{R}$ a function that grows faster than any iteration of exp. By this we mean some function $E : \mathbb{R} \to \mathbb{R}^{>0}$ such that for any $n < \omega$, we have:

$$E(x) > \exp_n(x) := \underbrace{\exp \circ \ldots \circ \exp}_{n\text{-times}}(x)$$

for all sufficiently large $x \in \mathbb{R}$. We call such a function a **trans-exponential**, and its inverse a **hyper-logarithm**.

Before it is attempted to prove model theoretic results for structures with such functions, it should probably be checked that such functions do exist. There are quite a few expositions trans-exponentials, and we highlight a few that will be particularly relevant in this thesis.

The first construction was given in Boshernitzan, 1986. They proved that there exists a 'Hardy field' which contains a function E satisfying the functional equation:

$$E(x+1) = E(\exp(x))$$

and that such a function is 'obviously trans-exponential'. So what is a Hardy field? They were initially introduced in Bourbaki, 1976, but we give a presentation from S. Kuhlmann, 2000.

Definition 1.2.1. Let $\mathcal{C}$ be a collection of functions from $\mathbb{R}^{>0}$ to $\mathbb{R}$. Define the equivalence relation $\approx$ on $\mathcal{C}$ as:

$$f \approx g \iff \exists N \in \mathbb{R}^{>0} \text{ such that } \forall x > N, f(x) = g(x)$$

Note that pointwise addition and multiplication of functions carry over to equivalence classes of $\approx$. If $\mathcal{K} := \mathcal{C}/\approx$ is a field under these operations and closed under derivation, we say $\mathcal{K}$ is a **Hardy field**. Note that any Hardy field is orderable, since for $[f]_\approx \in \mathcal{K}, [f]_\approx \neq [0]_\approx$, we must have $[\frac{1}{f}]_\approx \in \mathcal{K}$, thus f must be eventually positive or eventually negative, hence we can assert $[f]_\approx > [g]_\approx$ if and only if $f - g$ is eventually positive.

Examples of Hardy fields include $\mathbb{Q}, \mathbb{R}$ (consider each element as a constant function) and $\mathbb{R}(x)$, where each equivalence class consists solely of one polynomial. We write $[f]_\approx$ as just f when dealing with Hardy fields.

We say a Hardy field is **exponential** if it contains the function exp.

Note that the Hardy field given by Boshernitzan was not closed under composition. A trans-exponential Hardy field that is closed under composition was constructed in Padgett, 2022. The advantage of Hardy fields over other trans-exponential constructions is that they are in some sense explicit, rather than an arbitrary collection of formal series.

The last exposition we mention are the various forms of 'Hyperseries'. These were initially developed in the thesis of Schmeling, 2001, and were further developed in papers including 'Logarithmic Hyperseries' van den Dries, van der Hoeven, and Kaplan, 2019 and 'Hyperserial fields' Bagayoko, van der Hoeven, and Kaplan, 2021). In the 'Hyperserial fields' paper, the authors construct a formal series, which they call 'Hyperseries', that forms an ordered differential real closed field and is closed under composition, containing functions $E_0 \coloneqq \exp, E_1, E_2, \ldots, L_0 \coloneqq \log, L_1, L_2, \ldots$ satisfying the functional equations:

$$E_i(x+1) = E_{i-1}(E_i(x))$$
$$L_i(L_{i-1}(x)) = L_i(x) - 1$$

for all $i > 0$ and sufficient large x.

Going forward, we assume the existence of the following fields:

Theorem 1.2.2 (Bagayoko, van der Hoeven, and Kaplan, 2021)**.** *There exists a non-archimedean ordered differential real closed field K extending $\mathbb{R}$ with functions $(E_i : K^{>\mathbb{R}} \to K^{>\mathbb{R}}, L_i : K^{>\mathbb{R}} \to K^{\mathbb{R}})_{i<\omega}$ such that for all $i \geq 0$, L_i is the compositional inverse of E_i and:*

L1 $L_{i+1}(L_i(x)) = L_{i+1}(x) - 1$ *for any* $x > \mathbb{R}$.

L2 L_i *is strictly increasing.*

L3 *For all* $n \in \mathbb{N}$ *and* $x > \mathbb{R}$*:*

$$0 < L_{i+1}(x) < L_i^n(x)$$

L4 L_i *considered as a function from* $K^{>\mathbb{R}} \to K^{>\mathbb{R}}$ *is surjective.*

Theorem 1.2.3 (Padgett, 2022)**.** *There exists a real closed exponential Hardy field extending $\mathbb{R}(x)$ closed under composition, containing a trans-exponential function E. Moreover it's functional inverse L satisfies* (**L1**) - (**L4**)*, where $L_1 \coloneqq L$ and $L_0 \coloneqq \log$.*

1.2.1 O-minimal Transexponential exapansions of $\mathbb{R}$

Naturally, to extend the o-minimality of $\mathbb{R}_{\exp}$, we could ask if there exists an o-minimal expansion of $\mathbb{R}$ as an ordered field containing a trans-exponential function. In fact, this was posed as an open question in Speissegger, 2002. One possible avenue to tackle this question is to study the value group of a trans-exponential field K, prove the structure induced by the trans-exponential is tame in some sense, specifically is weakly o-minimal, and hope that this tameness lifts back into the underlying field. Thus we have some motivation to study the value group of a trans-exponential field. As stated in the abstract, this is done via contraction groups and asymptotic couples.

1.3 Archimedean Classes

Let $(K, +, -, \cdot, 0, 1, <_K)$ be an ordered field, then we can define the archimedean equivalence relation from section (2.3.1) on $K \setminus \{0\}$ via the ordered abelian group structure. As a reminder, for any pair $x, y \in K^{\neq 0}$:

$$x \asymp y \iff \exists n_1, n_2 \in \mathbb{N} \text{ such that } n_1|x| > |y| \text{ and } n_2|y| > |x|$$

The quotient $K^{\neq 0} \;\asymp$ is denoted as $[K]$. Moreover, ordering $<_K$ induces a linear order on $[K]$, denoted $<_{[K]}$, defined as:

$$[x] \geq_{[K]} [y] \iff \exists n \in \mathbb{N} \text{ such that } n|x| > |y| \tag{1.1}$$

Where x, y are elements of $K^{\neq 0}$. This was extracted solely from the ordered abelian group structure on K, but since we also have a field structure, we can define an ordered abelian group structure on $[K]$ itself as follows:

- The group operation $+$ on $[K]$ is defined as $[x] + [y] = [xy]$
- The identity on $[K]$ is $0 \coloneqq [1]$, where 1 is the multiplicative identity on K
- Additive inverses are given by $-[x] = [\frac{1}{x}]$

This makes $\mathcal{G} \coloneqq ([K], <, +, 0)$ an ordered abelian group. Furthermore, if K is real closed, $\mathcal{G}$ is also divisible, since $\frac{[x]}{n} = [\sqrt[n]{x}]$ for any $n \in \mathbb{N}$, $x > 0$.

Remark 1.3.1. Due to a peculiar convention in the literature, when referring to the archimedean classes of a field, most authors order the quotient via the reversal of the ordering in Equation (1.1). To avoid confusion, when we follow this convention, we use value group notation for the equivalence classes. So $v(x) = [x]$ denotes the archimedean class of x, and $v(x) + v(y) = v(x \cdot y)$, but:

$$[x] < [y] \iff v(x) > v(y)$$

We call v the **natural valuation** and $v(K)$ the **value group** of K.

Example 1.3.2. Let consider the field $\mathbb{R}(X)$ with the ordering:

$$f(X) \coloneqq a_n X^n + \ldots + a_m X^m > 0 \iff a_n > 0$$

where $m, n \in \mathbb{N}, n > m$, $a_i \in \mathbb{R}$ and $a_n, a_m \neq 0$. Then $(\mathbb{R}(X), <)$ is an ordered field. Write the polynomials f, g as:

$$f(X) \coloneqq \sum_{i=m}^{n} a_i X^i \qquad\qquad g(X) \coloneqq \sum_{i=m}^{n} b_i X^i$$

Then $f \succeq g$ if and only if $\deg(f) < \deg(g)$. Hence the archimedean class of a polynomial

f is all polynomials of the same degree, and $[f] > [g]$ if and only if $\deg(f) > \deg(g)$. Note that if $f, g \neq 0$, then $\deg(fg) = \deg(f) + \deg(g)$ hence via the identification with f to it's degree as an integer, we have an isomorphism from $[\mathbb{R}(X)]$ to $\mathbb{Z}$ as ordered abelian groups.

Example 1.3.3 (S. Kuhlmann, 2000, p. 93)**.** Let K be a Hardy field (Definition 1.2.1), then for any $f, g \in K, g \neq 0$, the function $\frac{f(x)}{g(x)}$ must eventually be monotone, thus:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \in \mathbb{R} \cup \{\pm\infty\}$$

Consider the equivalence relation defined by the valuation:

$$v(f) = v(g) \iff \lim_{x\to\infty} \frac{f(x)}{g(x)} \in \mathbb{R}^{\neq 0}$$

and the ordering on $v(K^{\neq 0})$:

$$v(f) > v(g) \iff \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$$

Then $([K], <_{[K]})$, the set archimedean classes of $[K]$, is isomorphic to $(v(K), <)$ as ordered sets. To see this, suppose $[f] = [g]$, then there exists some $n_1, n_2 \in \mathbb{Q}^{>0}$ such that for sufficiently large $x \in \mathbb{R}$,

$$n_1|g(x)| < |f(x)| < n_2|g(x)|$$

Since $[g] \neq 0$, we can assume g is eventually greater than zero or eventually less than zero, thus:

$$n_1\left|\frac{g(x)}{g(x)}\right| < \left|\frac{f(x)}{g(x)}\right| < n_2\left|\frac{g(x)}{g(x)}\right|$$

so for all sufficiently large x,

$$n_1 < \left|\frac{f(x)}{g(x)}\right| < n_2$$

since $\left|\frac{f(x)}{g(x)}\right|$ is an element of our Hardy field, we know it must be eventually monotone, hence it must have a limit, so

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} \in (-n_2, -n_1) \cup (n_1, n_2) \subseteq \mathbb{R}^{\neq 0}$$

Conversely, suppose $v(f) = v(g)$, i,e.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = r \in \mathbb{R}^{\neq 0}$$

Then there exists some $\epsilon \in (0, r)_{\mathbb{R}}$ such that for all sufficiently large x, we have:

$$\left|\frac{f(x)}{g(x)}\right| < r + \epsilon$$

thus $|f(x)| = (r + \epsilon)|g(x)|$, so $[f] = [g]$. The homomorphism of ordered sets follows from a similar calculation.

1.4 Contraction Groups

1.4.1 Logarithm on the Archimedean classes

Suppose further that K is a model of $Th(\mathbb{R}_{\exp})$ (but still non-archimedean i.e. $v(K)$ has more than one element), hence $\log : K^{>0} \to K$ is well-defined. We can define a function $\chi : v(K) \to v(K)$ as follows:

$$\chi(v(f)) \coloneqq \begin{cases} v(\log(f)) & \text{if } v(f) < 0 \text{ and } f > 0 \\ 0 & \text{if } v(x) = 0 \\ -\chi\left(v\left(\frac{1}{f}\right)\right) & \text{if } v(f) > 0 \text{ and } f > 0 \end{cases} \tag{1.2}$$

Since $v(-f) = v(f)$, χ is defined on all elements of $v(f)$, moreover, since for any $f, g \in K^{>0}$, $v(f) = v(g)$ implies $v(\log(x)) = v(\log(y))$, it is well defined on $v(K)$. We have the following easily verifiable characteristics of χ:

Fact 1.4.1. *The map $\chi : v(K) \to v(K)$ as defined in (1.2) is well defined, moreover for any $f, g \in K^{\neq 0}$, χ satisfies the following:*

- $\chi v(f) = 0 \iff v(f) = 0$ *(if and only if x is archimedean equivalent to 1_K in K)*
- *If $v(f) < v(g)$, then $\chi(v(f)) \leq \chi(v(g))$*
- $\chi(-v(f)) = -\chi(v(f))$
- *If $v(f)$ is archimedean equivalent to $v(g)$ as elements of vK, then $\chi(v(f)) = \chi(v(g))$.*
- *χ is surjective*
- *For any $v(f) > 0$, we have $0 < \chi(v(f)) < v(f)$*

This motivates the following definition:

Definition 1.4.2. F.-V. Kuhlmann, 1994, p. 223 Let $\mathcal{L}_1 = \langle +, -, 0, <, \chi \rangle$ be a first order language, where χ is a unary function. A $\mathcal{L}_1$-structure $(G, +, -, 0, <, \chi)$ is a **precontraction group** if and only if for all $a, b \in G$,

CG $(G, +, -, 0, <)$ is an ordered abelian group,

C0 $\chi a = 0 \iff a = 0$

C≤ χ respects $\leq$

C− $\chi(-a) = -\chi a$

CA If a is archimedean equivalent to b **purely within the ordered abelian group** G, and they both have the same sign, then $\chi a = \chi b$.

This can be expressed in first order logic by the set for formulas; for every $n \in \mathbb{N}$, we have:

$$0 < a < b \wedge na > b \to \chi a = \chi b$$

Furthermore, (G, χ) is a contraction group if the following two axioms hold:

CS χ is surjective.

CD $(G, +, <, 0)$ is a divisible ordered abelian group.

Finally, a precontraction group is **centripetal** if it satisfies:

CP $a > 0 \to \chi a < a$

and is **centrifugal** if:

CF $a > 0 \to \chi a > a$

In the two contraction group papers by F.V Kuhlmann, the following was proved:

Theorem 1.4.3 (F.-V. Kuhlmann, 1994 , F.-V. Kuhlmann, 1995)**.** *The theory of centripetal contraction groups has quantifier elimination, is complete and is weakly o-minimal.*

Since K is real closed as a field we know vK must be divisible as an ordered abelian group. Combined with fact 1.4.1, we see that the action of log on the value group of a non-archimedean model of $Th(\mathbb{R}_{\exp})$ is indeed a centripetal contraction group.

Corollary 1.4.4. *The archimedean classes of a non-archimedean model of* $Th(\mathbb{R}_{\exp})$ *along with the action of* log *is a weakly o-minimal structure.*

The entirety of this thesis only concerns contraction groups that are centripetal. **From now on, we simply refer to centripetal (pre)contraction groups as (pre)contraction groups!**

1.4.2 Hyper-logarithmic Contraction Groups

Suppose further that we can define a hyper-logarithm $L_1 : K^{>0} \to K$ as in Theorem 1.2.2. It turns out that L_1 gives a well defined function on vK and interacts with χ, the function induced by log in a particular way:

Proposition 1.4.5. *Define the function $\chi_2 : vK \to vK$ as:*

$$\chi_2 v(f) = \begin{cases} v\left(L_1(f)\right) & \text{if } v(f) > 0 \text{ and } f > 0 \\ 0 & \text{if } v(f) = 0 \\ -\chi_2\left(v\left(\frac{1}{f}\right)\right) & \text{if } v(f) < 0 \text{ and } x > 0 \end{cases}$$

Then χ_2 is well defined.

Proof. As $v(-f) = v(f)$, it is clear that χ_2 is indeed defined on all elements of vK. To prove χ_2 is well defined, it is sufficient to show that for any $f \in K^{>\mathbb{R}}$, $v(L_1(f)) = v(L_1(2f))$, since the positive part of an archimedean class is convex. Since L_1 is strictly increasing, we know that $v(L_1(f)) \leq v(L_1(2f))$. For the converse, note the following:

$$\begin{aligned} 2L_1(f) &= 2(L_1(e^f) - 1) && \text{Re-arrange (L1) with } x := e^f \\ &= L_1(e^f) + (L_1(e^f) - 2) \\ &\geq L_1(2f) + (L_1(e^f) - 2) && \text{Since } e^f > 2f \text{ and } L_1 \text{ is strictly increasing} \\ &\geq L_1(2f) && \text{Since } L_1(e^f) > \mathbb{R} \text{ hence } L_1(e^f) - 2 > \mathbb{R} \end{aligned}$$

Hence $2L_1 f > L_1(2f)$, so $[L_1(f)] \leq [L_1(2f)]$. ☐

Theorem 1.4.6. *The map $\chi_2 : vK \to vK$ is a centripetal contraction and if $\chi : vK \to vK$ is the contraction induced by* log *as defined in* (1.2)*, then for every $v(f) \in vK^{>0}$ and $n \in \mathbb{N}$, the pair χ and χ_2 satisfy:*

(**H**1) $0 < \chi_2 v(f) < \chi^n v(f)$

(**H**2) $\chi_2 v(f) = \chi_2(\chi v(f))$

Proof. Fix some $f, g \in K^{>0}$, we go through each of the contraction group axioms and verify that they hold.

- **C0** Suppose $v(f) > 0$, if $\chi_2 v(f) = 0$ then $v(L_1(f)) = 0$, so $L_1(f) \in \mathbb{R}$. But we assumed the codomain of a hyperlogarithm was $K^{\mathbb{R}}$, so this cannot happen. Since we defined χ_2 to be an odd function, we also have $\chi_2 v(f) \neq 0$ for all $v(f) < 0$.
- **C**$\leq$ Suppose $v(f) > v(g) > 0$. That means $f, g \in K^{>\mathbb{R}}$ and $f > g$, hence $L_1(f) > L_1(g) > \mathbb{R}$ (as L_1 is strictly increasing), thus $\chi_2 v(f) \geq \chi_2 v(g) < 0$. Combined with χ_2's oddity, $\chi_2 v(f) \leq \chi_2 v(g)$ when $v(f) < v(g) < 0$. Finally, since $\chi_2(vK^{>0}) > 0$, and (subsequently) $\chi_2(vK^{<0}) < 0$, for $v(f) < 0 < v(g)$, we have $\chi_2 v(f) < 0 < \chi_2 v(g)$ so $\chi_2 v(f) < \chi_2 v(g)$.
- **C**$-$ Immediate from the definition of χ_2.

CA It is sufficient to show that $\chi_2 v(f) = \chi_2(2v(f))$, for all $x \in K^{>\mathbb{R}}$. Since L_1 is increasing on $K^{>\mathbb{R}}$, we know that

$$L_1(x) < L_1(x^2) < L_1(e^x)$$

Hence by (C≤), we have:

$$\chi_2 v(f) \leq \chi_2 v(f^2) = \chi_2(2v(f)) \leq \chi_2 v(e^f)$$

By applying (**L1**), and since $v(L_1(e^f)) > v(1)$, we have $v(L_1(f)) = v(L_1(e^f) - 1) = v(L_1(e^f))$, hence $\chi_2 v(f) = \chi_2 v(e^f)$, hence $\chi_2 v(f) = \chi_2(2v(f))$.

CS This follows from the surjectivity of L_1.

CP We need to show that for any $f \in K^{>\mathbb{R}}$ that $L_1(f) \prec f$. But by (**L3**), we have:

$$0 < L_1(f) < \log(f) \prec f$$

Hence $L_1(f) \prec f$.

H1 The following is true for $f \in K^{>\mathbb{R}}$ and $n \in \mathbb{N}$:

$$v(f) > v(\log(f)) > v(\log^2(f)) > \ldots > v(\log^n(f))$$

But $L_1(f) < \log^n(f)$, hence $0 > v(L_1(f)) > v(\log^n(f))$, which tells us for all $a \in \mathcal{G}^{>0}$, $0 < \chi_2(a) < \chi^n(a)$.

H2 We know $L_1(\log(x)) = L_1(x) - 1$, so $v(L_1(\log(x)) = v(L_1(x))$, hence $\chi_2(\chi(a)) = \chi_2(a)$.

□

Similar to Definition 1.4.2, the previous lemma motivates the following:

Definition 1.4.7. Let $\mathcal{L}_2 = \langle +, -, 0, <, \chi_1, \chi_2 \rangle$ be a first order language. Define the theory T_2, the theory of 2 contraction groups as asserting:

$\langle +, -, 0, < \rangle$ is a divisible ordered abelian group,

$\langle +, -, 0, <, \chi_i \rangle$ forms a centripetal contraction group for $i = 1, 2$,

H1 $\forall a (\chi_2 a = \chi_2(\chi_1 a))$

H2 For every $n \in \mathbb{N}$, the formula $(\forall a > 0)(0 < \chi_2 a < \chi_1^n a)$

In Section 3.3, we prove the following result analogous to Theorem 1.4.3 (see Theorem 3.3.19).

Theorem 1.4.8. *The theory T_2 has quantifier elimination, is complete and is weakly o-minimal.*

1.4.3 Multiple contractions

By Theorem 1.4.8 know what L_1 does on vK and how it interacts with log. The straightforward progression would be to ask what each L_i does. It turns out $\log, L_1, L_2, L_3, \ldots$ all induce well defined functions $\chi_1, \chi_2, \chi_3, \chi_4, \ldots$ on the archimedean classes of K, and moreover each χ_i is a centripetal contraction. We also have the following equations, for any $i > j$, $m \in \mathbb{N}$:

$$\chi_i(\chi_j(a)) = \chi_i(a)$$
$$0 < \chi_i(a) < \chi_j^m(a)$$

This gives motivation to the following definition (which is identical to Definition 1.4.7 except we allow i to range up to any natural number):

Definition 1.4.9. Let $\mathcal{L}_n = \langle +, -, 0, <, \chi_1, \ldots, \chi_n \rangle$ be a first order language. Define theory of n-contraction groups T_n as:

- $\langle +, -, 0, < \rangle$ is a divisible ordered abelian group
- For each $i \leq n$, $\langle +, -, 0, <, \chi_i \rangle$ forms a centripetal contraction group
- For any $j < i \leq n$, the formula $\forall a(\chi_i(\chi_j(a)) = \chi_i(a))$
- For any $j < i \leq n$ and $m \in \mathbb{N}$, the formula $(\forall a > 0)(\chi_i(a) < \chi_j^m(a))$

Analogously to Theorem 1.4.3 and Theorem 1.4.8, we prove the following (Theorem 3.4.17 and Theorem 3.5.21):

Theorem 1.4.10. *The theory T_n has quantifier elimination, is complete, has a prime model and is weakly o-minimal.*

We should also remark that in proving weak o-minimality, we show that every definable function $f : G \to G$ for some n-contraction group $\mathcal{G}$ can be finitely decomposed into 'simple' functions called χ_n-polynomials (see Theorem 3.5.20).

1.4.4 Other works on Contraction groups

We briefly highlight some other results concerning contraction groups.

In Bautista, 2019, the author examines the structure of the value group of a logarithmic trans-series. Logarithmic trans-series can be loosely thought of as a formal series of infinite sums closed under log, but not exp. The lack of exponentiation means log will not be surjective on the series, thus χ will not be surjective on the value group. The author highlights that the image of χ will be a discrete set, and develops a model complete and complete theory characterising the behaviour of χ in such a setting.

Another result on contraction groups can be found in Krapp and S. Kuhlmann, 2023. There the authors give necessary and sufficient conditions to recover an exponential field from a countable contraction group. They also give a method to produce a countable centripetal contraction group, which we exploit to do the same for T_n for all $n \in \mathbb{N}$ (Section 3.4.5).

1.5 Derivatives on the Archimedean Classes

1.5.1 Logarithmic Derivatives

Now suppose K is an exponential Hardy field expanding $\mathbb{R}(x)$. It turns out that the derivative is a well defined function on vK, thus by composing the derivative with χ, we deduce that the logarithmic derivative is well defined on vK.

Proposition 1.5.1. *Let K be a Hardy field expanding $\mathbb{R}(x)$. Then for any $f, g \in K^{\neq 0}$, we have:*

$$v(f) = v(g) \implies v(f') = v(g')$$

Proof. Recall from Example 1.3.3, that:

$$v(f) = v(g) \iff \lim_{x\to\infty} \frac{f(x)}{g(x)} \in \mathbb{R}^{\neq 0}$$

Suppose $v(f) = v(g)$, so $\lim_{x\to\infty} \frac{f(x)}{g(x)} = r \in \mathbb{R}^{\neq 0}$. Then by L'Hôpital's rule, we have:

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = r$$

thus $v(f') = v(g')$. □

Suppose K is closed under log. Set $G := vK$, and consider the element $v(\log(f)')$ in G, where $|f|$ is not in the convex hull of $\mathbb{R}^{>0}$ in K (which is the same as saying $v(f) \neq 0$). We calculate it to be:

$$\begin{aligned} v(\log(f)') &= v\left(\frac{f'}{f}\right) && \text{By the chain rule} \\ &= v(f') - v(f) && \text{Since } v(f\cdot g) = v(f) + v(g) \text{ and } v(f^{-1}) = -v(f) \end{aligned}$$

By Proposition 1.5.1, we know that $v(f')$ is a well defined function on $G^{\neq 0}$, thus $\psi : G^{\neq 0} \to G : v(f) \mapsto v(f') - v(f)$ is also a well defined function. We call such a pair an

asymptotic couple. As stated earlier these were initially introduced in Aschenbrenner and van den Dries, 2000. This motivated the following definition:

Definition 1.5.2 (Aschenbrenner and van den Dries, 2000, p. 353)**.** Let $\mathcal{L}_\psi$ be the language $\langle +, -, <, 0, 1, \psi, P(x), (\delta_n)_{n>0}\rangle$, where ψ and all the δ_n are unary functions and P is a unary predicate. The theory of **asymptotic triples**, T_ψ^-, is the universal theory asserting the following:

- $+, -, 0, <$ form a divisible ordered abelian group
- δ_n is the function which divides something by n
- ψ is defined on all non-zero elements (so not defined on 0),
- ψ satisfies the following:

  A1 $\psi(1) = 1$

  A2 $\psi(nv) = \psi(v)$ for all $n \in \mathbb{Z}^{\neq 0}$

  A3 $\psi(v) < \psi(w) + |w|$ for all $v, w \neq 0$

  A4 ψ is **decreasing** on the **positive** part of the ordered abelian group. (So by (A2), it is increasing on the negative part).

- P is a downwards closed set containing the image of ψ but disjoint from the image of $|x| + \psi(x)$

The theory of **closed asymptotic triples**, T_ψ, asserts further that:

C1 $\{\psi(x) \mid x \neq 0\}$ has no maximal element

C2 $\{\psi(x) \mid x \neq 0\}$ is everything below the image of the function $(id + \psi)$ restricted to positive elements, i.e:

$$\forall y \Big(\exists x (y = \psi(x)) \leftrightarrow (\forall w > 0)(y < w + \psi(w)) \Big)$$

The theory of the value group of a Hardy field was initially developed by Rosenlicht (see Rosenlicht, 1981), but the model theoretic properties were developed by Aschenbrenner and van den Dries in Aschenbrenner and van den Dries, 2000. We follow the presentation given in the former.

Remark 1.5.3. Let $\mathcal{V} := (V, \psi, P_\mathcal{V})$ be an asymptotic triple, then note that the downward closure of $\psi(V^{\neq 0})$ is one possible choice for $P_\mathcal{V}$. But it may not be the only choice, e.g. if there exists some $a \in V$ with

$$\psi(V^{\neq 0}) < a < (id + \psi)(V^{>0})$$

We call such an element a **H-point** (Definition 4.2.10) and it can be shown that at most one can exist (Proposition 4.2.9). Note however that if $\mathcal{V}$ is closed, then no H-point can exist, and the only possible value for $P_{\mathcal{V}}$ is $\psi(V^{\neq 0})$.

Remark 1.5.4. Th action of the derivative on G can be expressed as $(id + \psi)(v(f)) = v(f) + \psi(v(f)) = v(f) + v(f') - v(f) = v(f')$.

Suppose K is any exponential real closed Hardy field expanding $\mathbb{R}(x)$. Let us verify that the structure introduced on $G = vK$ is indeed an asymptotic triple.

Proposition 1.5.5. *Let K be any exponential real closed Hardy field expanding $\mathbb{R}(x)$. Then $\mathcal{G} := (G, \psi, \psi(G^{\neq 0}))$, where $1_G := v\left(\frac{1}{x}\right)$ and $\psi(v(f)) := v(f') - v(f)$, is an asymptotic triple. Suppose further that K is closed under integration, i.e. for all $f \in K$ there exists some $g \in K$ with $g' = f$, then $\mathcal{G}$ is a closed asymptotic triple.*

Proof. Since K is real closed we know that G is a divisible ordered abelian group. Let us verify the remaining axioms:

(A1): Simply calculate the logarithmic derivative of $\frac{1}{x}$:

$$\begin{aligned} \psi\left(v\left(\frac{1}{x}\right)\right) &= v\left(\frac{-1}{x^2}\right) - v\left(\frac{1}{x}\right) \\ &= v\left(\left(\frac{1}{x^2}\right) \cdot \left(\frac{1}{x}\right)\right) \\ &= v\left(\frac{1}{x}\right) \end{aligned}$$

(A2): Pick some $v(f) \neq 0$ and some $n \in \mathbb{Z}^{\neq 0}$, so $nv(f) = v(f^n)$, thus:

$$\begin{aligned} \psi(nv(f)) &= \psi(v(f^n)) \\ &= v((f^n)') - v(f^n) \\ &= v(nf'f^{n-1}) - v(f^n) \\ &= v(f') + (n-1)v(f) - nv(f) \\ &= v(f') - v(f) \end{aligned}$$

(A3): Pick some $f, g \in K$ with $f > \mathbb{R}$ and $0 < g < \mathbb{R}$, so $v(f) < 0$ and $v(g) > 0$. Since log is strictly increasing on K and $\log(\mathbb{R}) = \mathbb{R}^{>0}$, we must have $\log(f) > \mathbb{R}$ hence $[\log(f)] < 0$ thus by Example 1.3.3:

$$\lim_{x \to \infty} \frac{\log f(x)}{g(x)} = \infty$$

So by L'Hôpital's rule, we have:

$$\lim_{x\to\infty} \frac{(\log f(x))'}{(g(x))'} = \infty$$

thus $[(\log f)'] < v(g')$. Since $[(\log f)'] = \psi v(f)$ and $v(g') = v(g) + \psi v(g)$, we are done.

(A4): We prove ψ is increasing on $G^{<0}$, which will be sufficient since (A2) implies ψ is symmetric. Pick $v(f), v(g) \in G^{<0}$, with $v(f) < v(g)$, so $f > g > \mathbb{R}$, then since log is strictly increasing and surjective on $K^{>\mathbb{R}}$, we must have $\log(f) > \log(g) > \mathbb{R}$, thus:

$$\lim_{x\to\infty} \frac{\log g(x)}{\log f(x)} \in \mathbb{R}$$

So by L'Hôpital's rule, we have:

$$\lim_{x\to\infty} \frac{(\log g(x))'}{(\log f(x))'} \in \mathbb{R}$$

thus $v((\log f)') < v((\log g)')$.

(C1): Pick some $v(f) \in G^{\neq 0}$, then note that $\psi v(\log(f)) > \psi v(f)$, thus $P := \psi(G^{\neq 0})$ has no max.

(C2): Suppose $v(f) < (id + \psi)(G^{>0})$. We know there exists some $g \in K$ with $g' = f$. Then $\psi v(e^g) = v(g'e^g) - v(e^g) = v(g')$, so $v(f) \in \psi(G^{\neq 0})$. □

Remark 1.5.6. While we have shown that the value group of a real closed Hardy field forms an asymptotic triple, it is possible to show that the value group of any H-field (an ordered differential field with a valuation that interacts nicely with the derivative) also forms an asymptotic couple. Thus the Hyperseries mentioned in Theorem 1.2.2 will also give an asymptotic couple when we take the action of the logarithmic derivative on the value group.

In the paper, the following was proved:

Theorem 1.5.7 (Aschenbrenner and van den Dries, 2000, p. 355)**.** *The theory T_ψ has quantifier elimination and is complete. It is also the model completion of T_ψ^-.*

1.5.2 Hyper-Logarithmic Derivatives

Now suppose K is a trans-exponential Hardy field as in Theorem 1.2.3. We know that the hyper-logarithm L satisfies the functional equation:

$$L(x) = L(\log(x)) + 1$$

Thus by differentiating both sides, we get:

$$L'(x) = (L(\log(x)) + 1)' \tag{1.3}$$
$$= (L(\log(x)))' \tag{1.4}$$
$$= \frac{1}{x} L'(\log(x)) \tag{1.5}$$

Thus for any non-zero $f \in K$, we have:

$$\begin{aligned} v\left((L(f))'\right) &= v\left(\frac{f'}{f} L'(\log(f))\right) \\ &= v\left(\frac{f'}{f}\right) + v\left(L'(\log(x))\right) \\ &= v(f') - v(f) + v(L'(\log(f))) \\ &= \psi(v(f)) + v\left(L'(\log(f))\right) \end{aligned}$$

Since $v(g') = (id + \psi)(v(g))$ for any $v(g) \neq 0$, we then have:

$$v(L'(\log(f))) = (id + \psi)(v(L(f))) - \psi v(f)$$

For $v(f) < 0$, set $\theta(v(f)) := -v(L'(\log(f)))$, then since the derivative and L are well defined on non-zero elements of $v(g)$, θ itself must also be well-defined, we can also characterise its behaviour with respect to χ:

Proposition 1.5.8. *Define the function $\theta : G \to G$ as:*

$$\theta(v(f)) := \begin{cases} \psi v(f) - (id + \psi)(\chi_2 v(f)) & \text{If } v(f) < 0 \\ 0 & \text{If } v(f) = 0 \\ -\theta\left(v\left(\frac{1}{f}\right)\right) & \text{If } v(f) > 0 \end{cases} \tag{1.6}$$

Then θ is a centripetal precontraction on G, and satisfies the functional equation:

$$\theta(x) = \chi(x) + \theta \circ \chi(x)$$

where χ is the action of log *from Fact 1.4.1, χ_2 is the action of L from Proposition 1.4.5, and ψ is the logarithmic derivative from Proposition 1.5.5.*

Proof. Axioms (C0), (C−) follow immediately from the definition. (CA) follow from (CA) for χ_2 and (A2) for ψ.

(CP)**:** Pick some $f \in K^{>\mathbb{R}}$, so $v(f) < 0$. Thus by (**L3**) and (**L2**), we have:

$$\log^n(f) \succ L(f) \succ \frac{1}{\log^n(f)}$$

By L'Hôpital's rule, the dominance relation is preserved between all three functions above after differentiating, thus:

$$\frac{f'}{f}\left(\prod_{i=1}^{n-1}\log^i(f)\right)^{-1} \succ \frac{f'}{f}L'(\log(f)) \succ \frac{f'}{f}\left(\prod_{i=1}^{n-1}\log^i(f)(\log^n(f))^2\right)^{-1}$$

Thus by quotienting to the value group, we have:

$$v\left(\frac{f'}{f}\left(\prod_{i=1}^{n-1}\log^i(f)\right)^{-1}\right) < v\left(\frac{f'}{f}L'(\log(f))\right) < v\left(\frac{f'}{f}\left(\prod_{i=1}^{n-1}\log^i(f)(\log^n(f))^2\right)^{-1}\right)$$

Since $v\left(\frac{f'}{f}\right) = \psi v(f)$ and $v\left(\frac{1}{\log(f)}\right) = -\chi v(f) > 0_G$, we then have,

$$0_G < -\sum_{i=1}^{n-1}\chi v(f) < -\theta v(f) < -\sum_{i=1}^{n-1}\chi^i v(f) - 2\chi^n v(f) \tag{1.7}$$

So,

$$0_G > \sum_{i=1}^{n-1}\chi v(f) > \theta v(f) > \sum_{i=1}^{n-1}\chi^i v(f) + 2\chi^n v(f) \tag{1.8}$$

Thus by (CP) for χ, we get (CP) for θ.

(C≤): Suppose $v(f)v(g) < 0$, so $f \succ g \succ \mathbb{R}$. Since log is increasing, we must also have $\log(f) \succ \log(g)$. Since $\log \succ L$, we must have $\log' \succ L'$, so $\frac{1}{x} \succ L'(x)$, hence:

$$L'(\log(f)) \prec L'(\log(g))$$

Thus $v(L'(\log(f))) = -\theta v(f) > v(L'(\log(g))) = -\theta v(g)$, so $\theta v(f) < \theta v(g)$, thus we have (C≤).

(CS): We know that χ is surjective on G, thus apply Lemma 4.3.5.

θ-equation: We know from the proof of (CP) that for all $a > 0$, $\chi(a) < \theta(a) < 2\chi(a)$, so by (A2) and (CA) for χ_2, we know that $\psi \circ \chi(a) = \psi \circ \theta(a)$ and $\chi_2 \circ \chi(a) = \chi_2 \circ \theta(a)$.

$$
\begin{aligned}
\theta(a) &= (id+\psi)\circ\chi_2(a) - \psi(a) && \text{Apply Equation (1.6)}\\
&= (id+\psi)\circ\chi_2\circ\theta(a) - \psi(a) && \text{Since } \chi_2\circ\theta = \chi_2\\
&= (id+\psi)\circ\chi_2\circ\theta(a) - (\psi\circ\chi(a) - \chi(a)) && \text{Since } \psi(x) = (id+\psi)\circ\chi(x) \text{ for } x>0\\
&= \chi(a) + (id+\psi)\circ\chi_2\circ\theta(a) - \psi\circ\chi(a) && \text{Rearrange the right hand side}\\
&= \chi(a) + (id+\psi)\circ\chi_2\circ\theta(a) - \psi\circ\theta(a) && \text{As } [\chi(a)] = [\theta(a)]\\
&= \chi(a) + \theta\circ\chi(a) && \text{Unapply Equation (1.6)}
\end{aligned}
$$

A similar calculation proves (Cθ) for $a < 0$. □

This motivates the following definition:

Definition 1.5.9. Let $\mathcal{L}_{\psi\text{-}\chi}$ be the language $\langle +, -, <, 0, 1, \psi, P(x), (\delta_n)_{n>0}, \chi, L, \theta\rangle$, where ψ and all the δ_n are unary functions and P is a unary predicate. The theory of **asymptotic contractions**, $T^{-}_{\psi\text{-}\chi}$, is the universal theory asserting the following:

- $+, -, 0, <$ form a divisible ordered abelian group
- δ_n is the function which divides something by n
- ψ is defined on all non-zero elements (so not defined on 0),
- ψ satisfies the following:

  A1 $\psi(1) = 1$

  A2 $\psi(nv) = \psi(v)$ for all $n \in \mathbb{Z}^{\neq 0}$

  A3 $\psi(v) < \psi(w) + |w|$ for all $v, w \neq 0$

  A4 $v \ll w$ implies $\psi(v) \geq \psi(w)$
- P is a downwards closed set containing the image of ψ but disjoint from the image of $|x| + \psi(x)$
- χ, L, θ are centripetal precontractions
- χ and L form a 2-precontraction group
- For all x, $\theta(x) = \chi(x) + \theta(\chi(x))$
- For all $x > 0$, $(id + \psi)(L(x)) = \psi(x) + \theta(x)$
- For all $x < 0$, $(id + \psi)(\chi(x)) = \psi(x)$

Moreover, the various functions interact in the following way:

CL χ and L form a 2-precontraction group with $\chi_1 := \chi$ and $\chi_2 := L$

Cθ For all x, $\theta(x) = \chi(x) + \theta(\chi(x))$

Cψ For all $x < 0$,

$$(id + \psi)(\chi(x)) = \psi(x)$$

hence for all $x > 0$, $(-id + \psi)(\chi(x)) = \psi(x)$

CI For all $x < 0$,

$$(id + \psi)(L(x)) = \psi(x) - \theta(x)$$

hence for all $x > 0$, $(-id + \psi)(L(x)) = \psi(x) + \theta(x)$

The theory of **closed asymptotic contractions**, $T_{\psi\text{-}\chi}$, asserts further that:

C1 P has no maximal element

C2 P is everything below the image of the function $(id + \psi)$ restricted to positive elements, i.e:

$$P(x) \iff (\forall w > 0)(x < w + \psi(w))$$

C3 All the contractions are surjective

Note that the value group of a trans-exponential Hardy field K as in Theorem 1.2.3 is only a model of the universal theory $T^{-}_{\psi-\chi}$, since we do not know if K is closed under integration hence (C2) may not be satisfied. For an example of a trans-exponential field that is closed under integration, and hence will produce a model of the theory $T_{\psi-\chi}$, see Theorem 6.23 in Bagayoko, 2022.

Analogous to Theorem 4.2.40 we would expect the following result to hold for $T_{\psi\text{-}\chi}$.

Conjecture 1.5.10. *The theory $T_{\psi\text{-}\chi}$ has quantifier elimination and is complete.*

We do not prove that in this thesis but we do have an intermediate result (Section 4.4). Let $\mathcal{L}_{\theta\text{-}L} \coloneqq \langle +, -, 0, <, \chi, L, \theta \rangle$, and $T_{\theta-L}$ be the theory asserting:

- $+, -, 0, <$ form a divisible ordered abelian group
- χ, L, θ are centripetal contractions
- χ and L for a 2-precontraction group
- For all x, $\theta(x) = \chi(x) + \theta(\chi(x))$

Then we have the following result (Theorem 4.4.13):

Theorem 1.5.11. *The theory $T_{\theta-L}$ has quantifier elimination, has a prime model and is complete.*

Chapter 2

Preliminaries

2.1 Basics of Model theory

Definition 2.1.1. A **first order language** $\mathcal{L} \coloneqq \langle (f_i)_i, \ldots, (R_i)_i, (c_i)_i \rangle$ is a formal collection of symbols where:

- $f_i : (\cdot)^{n_i} \to (\cdot)$ represent n_i-ary functions
- $R_i : (\cdot)^{n_i} \to \{0, 1\}$ represent n_i-ary relations
- c_i represent constants

An $\mathcal{L}$-structure is a collection of objects $\mathcal{M} \coloneqq (M, (f_i)_i, (R_i)_i, (c_i)_i)$ is a where:

- M is a set
- $f_i : M^{n_i} \to M$ is an n_i-ary function
- $R_i : M^{n_i} \to \{0, 1\}$ is an n_i-ary relation
- $c_i \in M$ is a constant in M

$|\mathcal{M}|$ denotes M, the domain of $\mathcal{M}$. We use calligraphic font to denote a structure and standard font to denote the domain of a structure, so $|\mathcal{M}_1| = M_1$, $|\overline{\mathcal{M}}| = \overline{M}$ and so on.

Example 2.1.2. Let $\mathcal{L} \coloneqq \langle Even(x), Odd(x) \rangle$ be the language containing two relation symbols. The collection

$$\mathcal{Z} \coloneqq \langle \mathbb{Z}, Even_{\mathcal{Z}}(x), Odd_{\mathcal{Z}}(x) \rangle$$

where for any $z \in \mathbb{Z}$,

$$Even_{\mathcal{Z}}(z) \iff z \text{ is } \mathbf{odd}$$
$$Odd_{\mathcal{Z}}(z) \iff z \text{ is } \mathbf{even}$$

Is an $\mathcal{L}$-structure. This example demonstrates that the symbols in $\mathcal{L}$ have no meaning by themselves aside from the number of arguments they take. Only when we interpret them in an $\mathcal{L}$-structure that the symbols in $\mathcal{L}$ have a semantic value.

Example 2.1.3. Let $\mathcal{L}_{og} \coloneqq \langle +, -, 0 < \rangle$ be the language of ordered groups, where $+$ is a function with two arguments, $-$ is a function of one argument, $<$ is a relation of two arguments and 0 is a constant.

- The structure $\mathcal{Z} \coloneqq (\mathbb{Z}, +, -, <, 0)$, where all the functions are interpreted as you would expect, is an $\mathcal{L}$-structure.
- Similarly the structures $\mathcal{Q} \coloneqq (\mathbb{Q}, +, -, <, 0)$, $\mathcal{R} \coloneqq (\mathbb{R}[x], +, -, <, 0)$ are $\mathcal{L}$-structures

Definition 2.1.4. Fix some language $\mathcal{L}$. An **$\mathcal{L}$-formula** is a formal symbol made up of finitely many applications of the symbols in the language $\mathcal{L}$, variables $x_1, x_2, x_3, \ldots$ and the logical symbols:

$$\underset{\text{And}}{\wedge} \qquad \underset{\text{And or}}{\vee} \qquad \underset{\text{Not}}{\neg} \qquad \underset{\text{Implies}}{\rightarrow} \qquad \underset{\text{For all}}{\forall} \qquad \underset{\text{Exists}}{\exists} \qquad \underset{\text{Equals}}{=}$$

For example, if we fix $\mathcal{L} \coloneqq \mathcal{L}_{og}$ from Example 2.1.3, the following are formulas:

- $(x_1 + x_2 = 0) \wedge \neg(x_1 = -x_3)$
- $\forall x_1(x_1 + x_2 = 0) \vee (0 = 0)$
- $(\forall x_1 \exists x_2(x_2 + x_2 = x_1)) \rightarrow x_4 = 0$

The following **is not** a formula:

$$x_1 = 0 \vee x_1 + x_1 = 0 \vee x_1 + x_1 + x_1 = 0 \vee \ldots$$

since it was constructed using infinitely many applications of $\vee$.

A formula is a **sentence** if all the variables are bound by quantifiers, so:

- $\exists x_1(x_1 + x_1 = x_2)$ is not a sentence, since x_2 is not bound
- $\forall x_2 \exists x_1(x_1 + x_1 = x_2)$ is a sentence.

Note that within a structure, a sentence can either be true or false. Given a sentence ϕ, and a structure $\mathcal{M}$, we say $\mathcal{M} \models \phi$ if ϕ is true in $\mathcal{M}$.

Definition 2.1.5. Fix some language $\mathcal{L}$, an $\mathcal{L}$-theory T is some collection of $\mathcal{L}$-sentences closed under implication.

Definition 2.1.6.
- A theory T has **quantifier elimination** if for all formulas $\phi(\overline{x})$, there exists some quantifier free formula $\psi(\overline{x})$ such that for any model $\mathcal{M}$ of T, we have:

$$\mathcal{M} \models \forall \overline{x}(\phi(\overline{x}) \leftrightarrow \psi(\overline{x}))$$

- Let $\mathcal{M} \subseteq \mathcal{N}$ be an extension of $\mathcal{L}$ structures. We say the extension is existentially closed if for all quantifier free $\mathcal{L}$-formulas $\phi(x, \overline{y})$ and $\overline{a} \in A^{|\overline{y}|}$, the following holds:

$$\mathcal{M} \models \exists x \phi(x, \overline{a}) \iff \mathcal{N} \models \exists x \phi(x, \overline{a})$$

- Given some $b \in B \setminus A$, we denote $\mathcal{A}\langle b \rangle_{\mathcal{B}}$ as the structure generated by A and b in $\mathcal{B}$, which formally is the intersection of all substructures of $\mathcal{B}$ containing A, b in their domains. When there is no ambiguity over $\mathcal{B}$, we denote it as $\mathcal{A}\langle b \rangle$.

Types

Definition 2.1.7. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, $\overline{x} = (x_1, \ldots, x_n)$ be a tuple of variables and $A \subseteq M$. An n-type $p(\overline{x})$ is some collection of $\mathcal{L}$-formulas with parameters in A with free variables $\overline{x}$, such that for any finite subset $\Delta(\overline{x}) \subseteq p(\overline{x})$, there exists some $\overline{a} \in M^n$ such that $\mathcal{M} \models \Delta(\overline{a})$, we say p is finitely satisfiable in this case.

If $\mathcal{M} \models p(\overline{a})$, then we say $\overline{a}$ is a realisation of p in $\mathcal{M}$ and if there exists such an $\overline{a}$, we say p is realised in $\mathcal{M}$.

Given some $b \in M$, define $\text{tp}(b, A)(x)$ and $\text{qftp}(b, A)(x)$ as:

- $\text{tp}_{\mathcal{M}}(b, A)$ is all the set of formulas with parameters in $A \cup \{b\}$ that $\mathcal{M}$ proves, with the occurrences of b replaced with the variable x:

$$\text{tp}_{\mathcal{M}}(b, A) := \{\phi(x, \overline{a}) \mid \overline{a} \in A^{|\overline{a}|}, \mathcal{M} \models \phi(b, \overline{a}), \phi(x, \overline{y}) \text{ is any } \mathcal{L}\text{-formula}\}$$

- $\text{qftp}_{\mathcal{M}}(b, A)$ is all the set of quantifier free formulas in $\text{tp}(b, A)(x)$:

$$\text{qftp}_{\mathcal{M}}(b, A) := \{\phi(x, \overline{a}) \mid \overline{a} \in A^{|\overline{a}|}, \mathcal{M} \models \phi(b, \overline{a}), \phi(x, \overline{y}) \text{ is any quantifier free } \mathcal{L}\text{-formula}\}$$

When the context (i.e. the underlying structure $\mathcal{M}$) is clear, we drop the subscript. We say a type $q(\overline{x})$ determines some other type $p(\overline{x})$, denoted $q \vdash p$, if for any $\psi(\overline{x}) \in p(\overline{x})$, there exists some finite $\Delta(\overline{x}) \subseteq q(\overline{x})$ such that $\mathcal{M} \models \Delta(\overline{x}) \implies \mathcal{M} \models \phi(\overline{x})$.

Definition 2.1.8. We say that a structure $\mathcal{M}$ is κ-saturated, for some cardinal κ, if for any $A \subseteq M$ with $|A| < \kappa$, and any type $p(\overline{x})$ of $\mathcal{M}$ over A, there exists a realisation of p in $\mathcal{M}$.

Factoring Property

Definition 2.1.9. Fix some language $\mathcal{L}$, let T be some $\mathcal{L}$-theory and let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures. We say $\mathcal{N}$ has **the factoring property** over $\mathcal{M}$ with respect to T if:

- There exists an embedding $f : \mathcal{M} \to \mathcal{N}$
- For any model $\mathcal{O} \models T$ and embedding $g : \mathcal{M} \to \mathcal{O}$, there exists an embedding $h : \mathcal{N} \to \mathcal{O}$ such that $g = h \circ f$

Note that a prime model $\mathcal{A}$ of some theory T has the factoring property over any model $\mathcal{B}$ of T.

Definition 2.1.10. Let T be any theory, and $T_\forall$ be its universal theory. For any $\mathcal{A} \models T_\forall$, we say some $\overline{\mathcal{A}} \models T$ is a T**-hull** of $\mathcal{A}$ if it has the factoring property over $\mathcal{A}$ with respect to T. We say T has the **closure property** if any model of $T_\forall$ has a T-hull. In general we denote some T-hull of $\mathcal{B} \models T_\forall$ as $\overline{\mathcal{B}}$.

Existential Closedness

Fact 2.1.11. *Let $\mathcal{A} \subseteq \mathcal{B}$ be structures. If $\mathcal{B}$ is embeddable over $\mathcal{A}$ in some elementary extension of $\mathcal{A}$, then $\mathcal{A}$ is existentially closed in $\mathcal{B}$.*

Conversely, if $\mathcal{A}$ is existentially closed in $\mathcal{B}$, then there is an embedding of $\mathcal{B}$ into every $|B|^+$-saturated elementary extension of $\mathcal{A}$.

Proof. Suppose $\mathcal{B} \preceq \mathcal{A}^*$ for some elementary extension $\mathcal{A}^*$ of $\mathcal{A}$, and pick some formula $\phi(x, \overline{y})$, and a tuple $\overline{a} \in A^{|\overline{y}|}$. Suppose $\mathcal{B} \models \exists x \phi(a, \overline{a})$, then since $\mathcal{B} \preceq \mathcal{A}^*$ and $\mathcal{A} \preceq \mathcal{A}^*$, we know $\mathcal{A}^* \models \exists x \phi(a, \overline{a})$, and hence $\mathcal{A} \models \exists x \phi(a, \overline{a})$. Thus $\mathcal{A}$ is existentially closed in $\mathcal{B}$.

For the second statement, fix some $|B|^+$ saturated elementary extension $\mathcal{A}^*$ of $\mathcal{B}$, and enumerate $B \setminus A$ as $(b_i)_{i<\gamma}$ for some ordinal $\gamma \leq |B|^+$. Assume for induction that for some $\alpha < \gamma$, $\mathcal{A}_\alpha := \mathcal{A}\langle\{b_i \mid i \leq \alpha\}\rangle$ embeds into $\mathcal{A}^*$, so by the previous statement we know $\mathcal{A}_\alpha \preceq \mathcal{A}^*$. Consider the type $\mathrm{tp}(b/A_\alpha)(x)$, by $|B|^+$ saturation it must have a realisation $\overline{b} \in A^*$. Embed $\mathcal{A}_\alpha\langle b\rangle$ into $\mathcal{A}^*$ via $b \mapsto \overline{b}$. The limit and base case are proved as expected, thus by transfinite induction we know $\mathcal{B}$ embeds into $\mathcal{A}^*$. □

Fact 2.1.12. *Fix some theory T such that $\mathcal{A}, \mathcal{B} \models T$. Let $b \in B \setminus A$, and suppose there is some collection of formulas $p_b(x)$ that is a type of $\mathcal{A}$ and determines the quantifier free type of b over A (meaning $p_b \vdash \mathrm{qftp}_{\mathcal{B}}(b, A)$). Then $\mathcal{A}$ is existentially closed in $\mathcal{A}\langle b\rangle_{\mathcal{B}}$, and hence existentially closed in any T-hull $\overline{\mathcal{A}\langle b\rangle_{\mathcal{B}}}$ of $\mathcal{A}\langle b\rangle_{\mathcal{B}}$.*

Proof. Let $\mathcal{A}^*$ be some $|A|^+$ saturated elementary extension of $\mathcal{A}$. Then since p_b is a type of $\mathcal{A}$, there exists some realisation $\overline{b}$ in $\mathcal{A}^*$. Since p_b determines the quantifier free type of b over A, we have an isomorphism over $\mathcal{A}$ of generated structures:

$$\mathcal{A}\langle b\rangle_{\mathcal{B}} \cong \mathcal{A}\langle \bar{b}\rangle_{\mathcal{A}^*} \tag{2.1}$$

Hence $\mathcal{A}\langle b\rangle_{\mathcal{B}}$ and $\overline{\mathcal{A}\langle b\rangle_{\mathcal{B}}}$ are embeddable into $\mathcal{A}^*$ over $\mathcal{A}$, so by fact (2.1.11), we see that $\mathcal{A}$ is existentially closed in both. □

Remark 2.1.13. Since $\bar{b} \models \text{qftp}(b, A)(x)$ is equivalent to the generated structures akin to (2.1) being isomorphic over $\mathcal{A}$, it is also sufficient to show that for any $\mathcal{C}$ with $\mathcal{A} \subseteq \mathcal{C}$ and $c \in C$ with $\mathcal{C} \models p_b(c)$, we have an isomorphism over $\mathcal{A}$:

$$\mathcal{A}\langle b\rangle_{\mathcal{B}} \cong \mathcal{A}\langle c\rangle_{\mathcal{C}}$$

and that p_b is a type of $\mathcal{A}$, in order to prove the conclusion of fact (2.1.12).

Criterion for Quantifier Elimination

Theorem 2.1.14. *Let T be a theory. If it satisfies the following:*

1. *T has the closure property.*
2. *For any $\mathcal{A} \subseteq \mathcal{B}$ which are models of T, and any $b \in B \setminus A$, $\mathcal{A}$ is existentially closed in $\mathcal{A}\langle b\rangle$ and the T-hull of $\mathcal{A}\langle b\rangle$.*
3. *There exists a model $\mathcal{I}$ of T that embeds into every other model of T.*

Then T has quantifier elimination, has a prime model and is complete.

Proof. First we show T is model complete. Pick some models $\mathcal{A}, \mathcal{B}$ of T, some quantifier free formula $\phi(x, \overline{y})$ and $a \in A^{|\overline{y}|}$ such that:

$$\mathcal{B} \models \exists x \phi(x, \overline{a})$$

Let $b \in B$ be a realisation of $\phi(x, \overline{a})$ in $\mathcal{B}$, then:

$$\mathcal{A}\langle b\rangle \models \exists x \phi(x, \overline{a})$$

But we assumed that $\mathcal{A}$ is existentially closed in $\mathcal{A}\langle b\rangle$, hence:

$$\mathcal{A} \models \exists x \phi(x, \overline{a})$$

Hence $\mathcal{A}$ is existentially closed in $\mathcal{B}$, so T is model complete. Let $\overline{\mathcal{I}}$ be a T-hull of $\mathcal{I}$, so for any models $\mathcal{M}, \mathcal{N}$ of T, $\overline{\mathcal{I}}$ embeds elementarily into both of them, by model completeness. Apply this to your favourite method to prove quantifier elimination.

Finally, notice that the T-hull of the initial structure is a prime model of T. □

Sometimes we may not be able to directly show $\mathcal{A}$ is existentially closed in $\mathcal{A}\langle b\rangle$ for all arbitrary B, so we might need to construct a chain of substructures of $\mathcal{B}$:

$$\mathcal{A}_0 \coloneqq \mathcal{A} \subseteq \mathcal{A}_1 \subseteq \ldots \subseteq \mathcal{A}_n \supseteq \mathcal{A}\langle b\rangle$$

such that for all $i \in [0, n-1]$, $\mathcal{A}_i$ is existentially closed in $\mathcal{A}_{i+1}$. So we also have the following criterion:

Theorem 2.1.15. *Let T be a theory. If it satisfies the following:*

1. *T has the closure property*
2. *For any $\mathcal{A} \subseteq \mathcal{B}$ which are models of T, we can enumerate $B \setminus A$ as $\{b_i\}_{i<\alpha}$ such that for all $\beta < \alpha$, if we define $\mathcal{A}_\beta$ as:*

$$\mathcal{A}_\beta \coloneqq \overline{\mathcal{A}\left\langle \{b_i \mid i < \beta\}\right\rangle}^T$$

then $\mathcal{A}_\beta$ is existentially closed in $\mathcal{A}_\beta\langle b_\beta\rangle$.

3. *There exists a model of $\mathcal{I}$ of T that embeds into every other model of T*

Then T has quantifier elimination, has a prime model and is complete.

Proof. Since existential closedness is a transitive property, (2) implies that $\mathcal{A}$ is existentially closed in $\mathcal{B}$, so we can repeat the proof of Theorem 2.1.14 to get quantifier elimination, a prime model and completeness. □

2.2 Weak o-minimality

Definition 2.2.1. A structure $\mathcal{M} \coloneqq (M, <, \ldots)$ expanding a dense linear order is **weakly o-minimal** if any definable set in one variable, $A \subseteq M$, is a finite union of convex sets. (Points are convex sets!)

Unlike o-minimality, weak o-minimality is not preserved under elementary equivalence, see Macpherson, Marker, and Steinhorn, 2000. Moreover, we may not have piecewise monotonicity as we do for o-minimal structures, but we do have local monotonicity.

2.3 Ordered Abelian Groups

2.3.1 Archimedean classes

Let $\mathcal{G} \coloneqq (G, +, <, 0)$ be an ordered abelian group, that is an abelian group with a linear order $<$ such that for all $x, y, z \in G$:

$$x \leq y \implies x + z \leq x + y$$

We can define the relation $\preceq$, of dominance on G, to say that one element is roughly as big, or much bigger in magnitude than another. Specifically for $x, y \in G$:

$$x \succeq y \iff \exists n \in \mathbb{N} \text{ such that } n|x| > |y|$$
$$x \succ y \iff \forall n \in \mathbb{N}, |x| > n|y|$$

Furthermore, we say $x \asymp y$, meaning x is **archimedean equivalent** to y, if and only if $x \preceq y$ and $y \preceq x$. Then $\asymp$ happens to be an equivalence relation on G, so we denote the quotient as $[G]$, and the equivalence class of x as $[x]$. Note that each equivalence class that is not $[0] = \{0\}$ is made up of two convex components, one in the positive cone of G and its reflection in the negative cone, hence we can define a linear ordering $<_{[G]}$ on $[G]$:

$$[x] >_{[G]} [y] \iff |x| \succ |y|$$

Where $|\Delta| \coloneqq \{|\delta| \mid \delta \in \Delta\}$. Note that we could have also defined $<_{[G]}$ as:

$$[x] \geq_{[G]} [y] \iff x \succeq y, [x] >_{[G]} [y] \iff x \succ y$$

So $[x] >_{[G]} [y]$ can be thought of as saying "$|x|$ is much bigger than $|y|$".

Example 2.3.1. Let $\mathcal{G} \coloneqq (\mathbb{R}[X], +, <_{lex}, 0)$ be the set of real polynomials with the lexicographical ordering, so:

$$\ldots <_{lex} -X^2 <_{lex} -X <_{lex} -1 <_{lex} 0 <_{lex} 1 <_{lex} \frac{1}{2}X <_{lex} X <_{lex} 2X <_{lex} X^2 <_{lex} \ldots$$
$$f(X) \coloneqq a_nX^n + a_{n-1}X^{n-1} + \ldots + a_1X + a_0 > 0 \iff a_n > 0$$

Where $a_i \in \mathbb{R}$ and $a_n \neq 0$. We denote $<_{lex}$ as $<$ from now on. Then for any polynomial $f(X) \in \mathbb{R}[X]$ the class of f is:

$$[f(X)] = \{h(X) \in \mathbb{R}[X] \mid \deg(f) = \deg(g)\}$$

and for any two $f(X), g(X) \in \mathbb{R}[X]$, we have

$$f(X) \prec g(X) \iff [f(X)] < [g(X)] \iff \deg(f) < \deg(g)$$

where $\deg(r) = 0$ for any $r \in \mathbb{R}^{\neq 0}$ and $\deg(0) = -1$.

Definition 2.3.2. Given two ordered abelian groups G, H, we say a group homomorphism $\phi : G \to H$ is an order homomorphism if for all $g_1, g_2 \in G$, we have:

$$g_1 <_G g_2 \iff \phi(g_1) <_H \phi(g_2)$$

Note that any order homomorphism must be injective, so we say G is a subgroup of H as an **ordered** abelian group if such a map exists.

Fix an extension of ordered abelian groups $\phi : G \to H$. Since the archimedean equivalence relation was defined purely through the ordered group structure, it induces a map on the quotients:

$$\overline{\phi} : [G] \to [H] : [g] \to [\phi(g)]$$

It can be shown that $\overline{\phi}$ is injective, hence we can think of $[G]$ as lying inside $[H]$. Furthermore if $\overline{\phi}$ is surjective then we say $[G] = [H]$, or that the extension is **value preserving**.

Fact 2.3.3. *Let $G \subseteq H$ be an extension of ordered abelian groups, and $b_i \in H \setminus G$ for all $i \in [0, n]$. Then*

$$\left[G + \sum_{i=0}^{n} \mathbb{Z} b_i\right] \setminus [G]$$

has at most n elements.

Fact 2.3.4. *The equivalence relation $[\cdot]$ satisfies:*

$$[x + y] \leq \max\{[x], [y]\}$$

We call the equation above the ***ultrametric inequality****.*

Hahn Products and sums

Hahn products and sums were originally defined by Hans Hahn in 1907, and their main purpose is to express the Hahn embedding theorem (Theorem 2.3.8) which we state without proof. It was originally derived in Hahn, 1995, however, that paper is in German, thus we refer the reader to Gravett, 1956 and S. Kuhlmann, 2000.

Definition 2.3.5 (Hahn Sums and Products)**.** Let Γ be an ordered set and $\{\mathcal{R}_\gamma\}_{\gamma \in \Gamma}$ be a collection of ordered abelian groups. The **Hahn sum**, denoted $\coprod_{i \in \Gamma} \mathcal{R}_\gamma$ is the set of all functions:

$$f : \Gamma \to \bigsqcup_{\gamma \in \Gamma} \mathcal{R}_\gamma$$

such that $f(\gamma) \in \mathcal{R}_\gamma$ for all $\gamma \in \Gamma$, and with finite support, that is the set:

$$\operatorname{supp}(f) \coloneqq \{\gamma \in \Gamma \mid f(\gamma) \neq 0_\gamma\} \tag{2.2}$$

is finite. This set can be made into an ordered abelian group via component wise addition, so for all $f, g \in \amalg_{i \in \Gamma}$ and $\delta \in \Gamma$:

(*i*) $f + g(\delta) \coloneqq f(\delta) + g(\delta)$

(*ii*) $(-f)(\delta) \coloneqq -r_\delta$ where $f(\delta) = r_\delta$

(*iii*) 0 is the function that sends every δ to 0_δ.

(*iv*) $f > 0$ if and only if $f(\delta) > 0_\delta$ where $\delta \coloneqq \max(\operatorname{supp}(f))$.

Similarly the **Hahn product** $\prod_{i \in \Gamma} \mathcal{R}_\gamma$ is the set of all functions of the form (2.2) with **well-based** (that is reverse well-ordered) support, equipped with an ordered abelian group structure defined in the same way as the Hahn sum.

Notation 2.3.6. We denote arbitrary elements of a Hahn sum/product indexed by Γ as $(r_\gamma)_{\gamma \in \Gamma}$, where $r_\gamma \in \mathcal{R}_\gamma$. Moreover we denote q_γ as the function f defined as:

$$f(\alpha) \coloneqq \begin{cases} 0 & \text{If } \alpha \neq \gamma \\ q & \text{If } \alpha = \gamma \end{cases}$$

Example 2.3.7. Let $\Gamma \coloneqq \{X^0, X^1, X^2, X^3, \ldots\}$ with $X^i < X^{i+1}$ be an ordered set. Then we can think of the Hahn sum $\amalg_\Gamma \mathbb{R}$ as the set of real polynomials $\mathbb{R}[X]$ with the ordering defined in Example 2.3.1, so:

$$f(X) \coloneqq a_n X^n + \ldots + a_0 X^0 > 0 \iff a_n > 0$$

where $a_n \neq 0$.

Theorem 2.3.8 (Hahn Embedding theorem, Gravett, 1956, p. 59)**.** *Let G be an ordered abelian group. Then there is an order homomorphism $\phi : G \hookrightarrow \prod_{[G]} \mathbb{R}$*

2.3.2 Linear independence

Definition 2.3.9. Let $G \subseteq H$ be an extension of ordered abelian groups with H divisible. We say some collection of elements of H, $B \coloneqq \{b_i\}_i$ is **linearly independent** over G if and only if for any finite subset $\{b_1, \ldots, b_n\} \subseteq B$, and $q_1, \ldots, q_n \in \mathbb{Q}$ not all zero, we have

$$q_1 b_1 + \dots q_n b_n \notin G$$

Example 2.3.10. We list some examples:

- Let $G := (\mathbb{R}[X, X^4], <_{lex})$ and $H := (\mathbb{R}[X, X^2, X^3, X^4], <_{lex})$, and let $i : G \hookrightarrow H$ be the embedding which sends 1_G to 1_H, X^i_G to X^i_H for $i = 1, 4$. Then the set $\{X^2, X^3\} \subseteq H$ is linearly independent over G.

- Let $G := 0$ and $H := \Pi_{-\mathbb{N}}\mathbb{R}$. Then the collection $\{1_{-n} \mid n \in \mathbb{N}\} \cup \left\{ \sum_{n\in\mathbb{N}} 1_{-n} \right\}$ is linearly independent over G, since we cannot do infinite sums between elements of H.

- Let $G := \amalg_{-\mathbb{N}}\mathbb{R}$ and $H := \Pi_{-\mathbb{N}}\mathbb{R}$. Then we can embed G into H via the embedding $1_n \mapsto 1_n$. The pair $\left\{ \sum_{n\in\mathbb{N}} 1_{-2n}, \sum_{n\in\mathbb{N}} 1_{-2n-1} \right\}$ is linearly independent over G, since there every element of G must have finite support.

- Let $G := \mathbb{Q}$ and $H := \mathbb{R}$, and embed G into H in the obvious way. Then the set $\{\sqrt[4]{2}, \sqrt[2]{2}\}$ is linearly independent over $\mathbb{Q}$. The set $\{2\sqrt{2}, \sqrt{2}\}$ is not linearly independent over $\mathbb{Q}$.

- Let $G := \mathbb{Z}$ and $H := \mathbb{Q}$. Then $\frac{1}{2}$ is not linearly independent over G.

Definition 2.3.11. Let $(C, <) \subseteq (D, <)$ be an extension of ordered sets. We say $d_1, d_2 \in D$ induce the same cut in C if and only if:

$$\{x \in C \mid x < d_1\} = \{x \in C \mid x < d_2\}$$
$$\{x \in C \mid x > d_1\} = \{x \in C \mid x > d_2\}$$

Lemma 2.3.12. *Let $G \subseteq H$ be an extension of **divisible** ordered abelian groups with $b \in H \setminus G$. Then:*

$$[G + \mathbb{Z}b] = [G] \iff \forall g \in G, z \in \mathbb{Z} \text{ with } g + zb > 0, \text{ there exists some } \overline{g} \in G \text{ such that } \overline{g} < g + zb < 2\overline{g} \tag{2.3}$$

Thus for any other divisible ordered abelian group $\overline{H}$ with $G \subseteq \overline{H}$ and $\overline{b} \in H \setminus G$, if b and $\overline{b}$ satisfy the same cut in G, then:

$$[G + \mathbb{Z}b] = [G] \iff [G + \mathbb{Z}\overline{b}] = [G]$$

The same statement holds when $\mathbb{Z}$ is replaced with $\mathbb{Q}$.

Proof. For the forward direction of (2.3), pick some $g+zb \in (G+\mathbb{Z}b)^{>0}$, then $[g+zb] \in [G]$. By definition, this means that there exists some $h \in G^{>0}$ such that $[h] = [g + zb]$ i.e there exists some $n_1, n_2 \in \mathbb{N}$ such that $n_1 g > g + zb$ and $n_2(g + zb) > h$. Some multiple of h involving n_1 and n_2 will give the appropriate $\overline{g}$.

Conversely, if for any arbitrary $g + zb$ we can find some $h \in G$ such that $h < g + zb < 2h$, this by definition implies that $g + zb$ and h are archimedean equivalent, thus $[g + zb] \in [G]$.

For the latter statement, suppose $\overline{b}$ satisfies the same cut in G as b. If $[G + \mathbb{Z}b] \neq [G]$, namely $[b - g] \notin [G]$ for some $g \in G$. If we assume $b - g > 0$ (hence $\overline{b} - g > 0$ as well!), then since $b - g$ is not archimedean equivalent to any element of G, we have for all $h \in G^{>0}$, either $nh < b - g$ for all $n \in \mathbb{N}^{>0}$, or $b - g < nh$ for all $n \in \mathbb{N}^{>0}$. Since $\overline{b}$ satisfies the same cut, then the same is true for $\overline{b} - g$, thus it also is not archimedean equivalent to anything in G.

Finally, if $[G + \mathbb{Z}b] = [G]$, then by (2.3), for any pair $g \in G$ and $z \in \mathbb{Z}$ with $g + zb > 0$, we can find some $h \in G$ such that $h < g + zb < 2h$. Since b and $\overline{b}$ satisfy the same cut, this means $h < g + z\overline{b} < 2h$, thus by (2.3) again, $[G + \mathbb{Z}\overline{b}] = [G]$.

The proof is identical when we replace $\mathbb{Z}$ with $\mathbb{Q}$ since G and H are divisible. □

Lemma 2.3.13. *Let $G, H, \overline{H}$ be divisible ordered abelian groups, with $G \subseteq H$ and $G \subseteq \overline{H}$. Let I be some indexing set, and suppose $B := \{b_i\}_{i \in I} \subseteq H \setminus G$, $\overline{B} := \{\overline{b}_i\}_{i \in I} \subseteq \overline{H} \setminus G$. If for any tuple $i_1, \ldots, i_n \in I$, $z_i \in \mathbb{Z}$ and $g \in G$, we have:*

$$\sum_{j=1}^{n} z_j b_{i_j} = g \iff \sum_{j=1}^{n} z_j \overline{b}_{i_j} = g \tag{2.4}$$

$$\sum_{j=1}^{n} z_j b_{i_j} > g \iff \sum_{j=1}^{n} z_j \overline{b}_{i_j} > g \tag{2.5}$$

Then there exists an isomorphism of ordered abelian groups:

$$\phi : G_b := G + \sum_{i \in I} \mathbb{Z} b_i \to G_{\overline{b}} := G + \sum_{i \in I} \mathbb{Z} \overline{b}_i : g + z_1 b_{i_1} + \ldots + z_n b_{i_n} \mapsto g + z_1 \overline{b}_{i_1} + \ldots + \overline{b}_{i_n}$$

Furthermore, if $[b_i] \neq [b_j]$ for all $i \neq j$ and for each $i \in I$ either:

(i) $[b_i] \notin [G]$

(ii) $[G + \mathbb{Z}b_i] = [G]$, *and* $[b_j] \notin [G]$ *for all* $j \neq i$

Then for any $i \in I$ and $x \in G_b$, we have $[x] \in [G] \cup \{[b_i] \mid i \in I\}$ and:

$$[x] = [b_i] \iff [\phi(x)] = [b_i]$$

The same holds when $\mathbb{Z}$ is replaced with $\mathbb{Q}$.

Proof. To show ϕ is well defined, pick two sums from G_b,

$$\beta := g_\beta + \sum_{j=0}^{n} z_i b_i \qquad\qquad \gamma := g_\gamma + \sum_{j=0}^{n} w_i b_i$$

Where $b_i \in B$ and $z_i, w_i \in \mathbb{Z}$. Suppose $\beta = \gamma$, then $\beta - \gamma = 0$, thus:

$$\sum_{j=0}^{n} (z_i - w_i) b_i = g_\gamma - g_\beta$$

So by (2.4), we have:

$$\sum_{j=0}^{n} (z_i - w_i) \bar{b}_i = g_\gamma - g_\beta$$

Thus:

$$\phi(\beta) = g_\beta \sum_{j=0}^{n} z_i \bar{b}_i = g_\gamma + \sum_{j=0}^{n} w_i \bar{b}_i = \phi(\gamma)$$

Hence ϕ is well defined. Bijectivity follows similarly from (2.4), and (2.5) tells us it is also an order isomorphism.

Suppose further that (*i*) and (*ii*) hold. Pick some $x \in G_b$, written as:

$$x := \sum_{i=0}^{n-1} +(g + z_i b_i) \tag{2.6}$$

where $g \in G$, $z_0, \ldots, z_{n-1} \in \mathbb{Z}^{\neq 0}, z_n \in \mathbb{Z}$, $b_i \in B$, and $[b_i] \notin [G]$ for all $i \in [0, n-1]$, and b_n satisfies (*ii*). Then since all the terms in Equation (2.6) are distinct, the ultrametric inequality, tells us:

$$[x] = \max\{[b_0], \ldots, [b_{n-1}], [g + z_n b_n]\}$$

Similarly, by applying Lemma 2.3.12, we deduce that the natural valuation of $\phi(x)$ is:

$$[\phi(x)] = \max\{[\bar{b}_0], \ldots, [\bar{b}_{n-1}], [g + z_n \bar{b}_n]\}$$

If $[x] = [b_i]$ for $i \leq n-1$, then $[b_i] > [b_j], [g + z_n b_n]$ for all $j \neq 1, j \in [1, n-1]$, thus by (2.3) and (2.5), we deduce that $[\phi(x)] = [\bar{b}_i]$. Similarly if $[x] = [g+z_n b_n]$ then $[\phi(x)] = [g+z_n\bar{b}_n]$. The same argument applies when there is so $i \in I$ satisfying (*ii*). □

2.3.3 Pure Valuational Extensions

Definition 2.3.14. We say an extension $G \subseteq H$ of divisible ordered abelian groups is **pure valuational** if for all $h \in H \setminus G$, there exists some $g \in G$ such that $[h - g] \notin [G]$

Proposition 2.3.15. *The property of being a pure valuational extension is transitive and hence closed under infinite chains.*

Proof. To prove transitivity, suppose $G \subseteq H$ and $H \subseteq K$ are pure extensions. Pick some $k \in K \setminus H$, then there exists some $h \in H$ such that $[k - h] \notin [H]$. If $h \in G$ then we are done. If $h \notin G$, then there exists some $g \in G$ such that $[h - g] \notin [G]$. Consider $k - g$:

$$\begin{aligned}
[k-g] &= [(k-h)+(h-g)] \\
&= \max\{[k-h],[h-g]\} && \text{Since } [k-h]\notin[H] \text{ and } [h-g]\in[H] \\
&\notin [G] && \text{Since both } [k-h],[h-g]\notin[G]
\end{aligned}$$

Thus for any infinite chain of pure extensions $G_0 \subseteq G_1 \subseteq G_2 \subseteq \ldots$, we deduce that $G_0 \subseteq G_\omega$ is pure, where:

$$G_\omega := \bigcup_{i\in\omega} G_i$$

□

2.4 Set theory

Definition 2.4.1. An ordinal α is a transitive set (so $x \in y \in \alpha \implies x \in \alpha$) whose elements are well ordered by the membership relation. We say an ordinal β is a **successor ordinal** if it is of the form $\alpha + 1 := \alpha \cup \{\alpha\}$ for some ordinal α. If no such α exists, we say β is a **limit ordinal**. We can define addition, multiplication and exponentiation on ordinals as follows:

Fix some ordinals α and β. Define $\alpha + \beta$ as follows:

- If $\beta = 0$, then $\alpha + \beta = \beta$
- If $\beta = \gamma + 1$, then $\alpha + \beta := (\alpha + \gamma) + 1$
- If β is a limit ordinal, then $\alpha + \beta := \bigcup_{\gamma<\beta} \alpha + \gamma$

Define $\alpha \cdot \beta$ as:

- If $\beta = 0$, then $\alpha \cdot \beta = 0$

- If $\beta = \gamma + 1$, then $\alpha \cdot \beta \coloneqq (\alpha \cdot \gamma) + \alpha$
- If β is a limit ordinal, then $\alpha \cdot \beta \coloneqq \bigcup_{\gamma<\beta} \alpha \cdot \gamma$

Define α^β as:

- If $\beta = 0$, then $\alpha^\beta = 1$
- If $\beta = \gamma + 1$, then $\alpha^\beta \coloneqq (\alpha^\gamma) \cdot \alpha$
- If β is a limit ordinal, then $\alpha^\beta \coloneqq \bigcup_{\gamma<\beta} \alpha^\gamma$

Note that for any distinct ordinals α and β, either $\alpha \in \beta$ or $\beta \in \alpha$. We say $\alpha < \beta$ iff $\alpha \in \beta$.

Example 2.4.2. $0 \coloneqq \varnothing$ is an ordinal, $1 \coloneqq 0 \cup \{0\} = \varnothing$ is also an ordinal. Thus every natural number can be written as $n \coloneqq (n-1) \cup \{n-1\} = \{0, 1, \ldots, n-1\}$, hence is a (successor) ordinal. Let $\omega \coloneqq \{0, 1, \ldots, n, \ldots\}$, then ω is a limit ordinal.

Addition, multiplication and exponentiation for natural numbers as ordinals work as you would expect. However, note that $1 + \omega = \omega \in \omega + 1$. Similarly $2\omega = \omega \in \omega 2$.

Definition 2.4.3. Given a cardinal κ, κ^+ denotes the next cardinal.

Fact 2.4.4. *Any well ordered set is order isomorphic to an ordinal. Thus assuming the axiom of choice, we can assume any set can be enumerated by an ordinal.*

Fact 2.4.5 (Cantor normal form)**.** *Let α be an ordinal less than ω^ω, then there exists some $n, c_i \in \omega$ such that:*

$$\alpha = \omega^n c_n + \ldots + \omega c_1 + c_0$$

Chapter 3

Just Contractions

3.1 Introduction

This section aims to study the structure induced on the value group of a trans-exponential field by **just** the various logarithms. Fix a trans-exponential field K as in Theorem 1.2.2, so a field K with functions $\log, L_1, L_2, \ldots : K^{>\mathbb{R}} \to K^{>\mathbb{R}}$ satisfying (**L1**)-(**L4**).

As shown in Fact 1.4.1 and Theorem 1.4.6, we can show that the actions of log and L_1 of $G \coloneqq vK$ form contraction groups $\chi_1 \coloneqq [\log]$ and $\chi_2 \coloneqq [L_1]$. A similar line of reasoning will also show that $L_2, L_3, \ldots$ also induce centripetal contractions $\chi_3, \chi_4, \ldots$, and the maps together form a model of the theory T_n.

The outline of this chapter is as follows. In Section 3.2, we present a slightly different proof of QE and completeness for 1-contraction groups to the original from F.-V. Kuhlmann, 1994. Then in Section 3.3 and Section 3.4, we show how to use that proof to prove QE and completeness for 2 and n-contraction groups. In Section 3.4.5 we then extend the result of Krapp and S. Kuhlmann, 2023 to construct a countable model of T_n, and in Section 3.5 we generalise the proof by F.-V. Kuhlmann, 1995 of weak o-minimality for 1-contraction groups to n-contraction groups.

3.2 One Contractions

We give a slightly different presentation of the proof of quantifier elimination for contraction groups given by Franz-Victor Kuhlmann in F.-V. Kuhlmann, 1994, mainly so that we can set up an inductive argument for proving quantifier elimination for groups with n-contractions. We want to satisfy the conditions of Theorem (2.1.14), which means showing the theory of contraction groups has the closure property, any contraction group is existentially closed in the structure generated by it and one extra element from a superstructure, and that there exists a contraction group that embeds into every other contraction group.

First, let us fix the definitions and some notation.

Definition 3.2.1. Let $\mathcal{L}_1$ be the language $\langle +, -, 0, <, \chi \rangle$. The theory T_1 asserts that a structure is a contraction group, and T_1^- asserts that a structure is a precontraction group (so we remove divisibility for the underlying group (CD) and surjectivity for χ (CS)). Thus T_1^- is the universal theory of T_1

We usually hide the ordered abelian group notation when writing down a contraction group, so (G, χ) denotes the structure $(G, +, -, 0. <, \chi)$.

3.2.1 Closure Property

Lemma 3.2.2. *Let $(G, +, <, 0) \subseteq (H, +, <, 0)$ be an extension of ordered abelian groups, with $[G] = [H]$. Suppose further we can endow G with a map χ such that $\mathcal{G} := (G, \chi) \models T_1^-$, then there is a unique extension of χ to H such that $\mathcal{H} := (H, \chi)$ is also a model of T_1^- extending $\mathcal{G}$ as $\mathcal{L}_1$-structures. In particular:*

(i) If (G, χ) is a precontraction group and $\overline{G}$ is the divisible hull of G, then there is a unique extension of χ to $\overline{G}$ such that $(\overline{G}, \chi)$ is a precontraction group extending (G, χ).

(ii) Similarly if $\overline{G}$ is the Hahn product $\prod_{[G]} \mathbb{R}$, then again there is a unique extension of χ to $\overline{G}$ such that $(\overline{G}, \chi) \models T_1^-$ and $(\overline{G}, \chi)$ contains (G, χ) as a substructure.

Proof. For some $h \in H$, there exists some $g \in G$ such that $[h] = [g]$ so set $\chi(h) := \chi(g)$, a routine calculation shows that $\mathcal{H}$ is indeed a precontraction group. Moreover, since $\chi(\{g \in G \mid [g] = [h]\})$ is a single point (because of axiom (CA)), the only possible thing we can map h to is $\chi(g)$, hence the extension of χ is unique. $\square$

Lemma 3.2.3 (F.-V. Kuhlmann, 1994, p. 227)**.** *Let $\mathcal{G} := (G, \chi)$ be a 1-precontraction group. There exists some 1-contraction group (H, χ) containing (G, χ) as a substructure. Moreover, for all $h \in H$, there exists some $n \in \mathbb{N}$ such that $\chi^n(h) \in G$.*

Proof. Let $\mathcal{G}_0 := \mathcal{G}$. Given some $\mathcal{G}_n$ with $n \in \mathbb{N}$, define $\mathcal{G}_{n+1} := (G_{n+1}, \chi)$ as follows. The ordered abelian group G_{n+1} is the Hahn **product**:

$$G_{n+1} := \prod_{\Gamma} \mathbb{R}$$

Where Γ is defined as:

$$\Gamma := [G_n] \cup (G_n^{>0} \setminus \chi(G_n))$$

The ordering on Γ is as follows. For $\gamma_1, \gamma_2 \in [G_n]$, we say $\gamma_1 <_\Gamma \gamma_2$ if and only if $\gamma_1 <_{[G_n]} \gamma_2$. Similarly for $a, b \in G_n^{>0} \setminus \chi(G_n)$, we say $a <_\Gamma b$ if and only if $a <_{G_n} b$. For some $\gamma \in [G_n]$

and $a \in G_n^{>0} \setminus \chi(G_n)$, note that since $a \notin \chi(G_n)$, either $|\chi(\gamma)| >_{G_n} a$ or $|\chi(\gamma)| <_{G_n} a$. We set $a <_\Gamma \gamma$ in the first case and $a >_\Gamma \gamma$ in the latter case.

To define a contraction χ on G_{n+1}, it is sufficient to fix the values of $\chi(1_\gamma)$ for all $\gamma \in \Gamma$. If $\gamma \in [G_n]$, then set $\chi(1_\gamma) := |\chi(\gamma)| \in G_n$, and when $\gamma \in G_n^{>0} \setminus \chi(G_n)$, set $\chi(1_\gamma) := a$, where a is the element of $G_n^{>0}$ we construed γ from. By the Hahn embedding theorem (2.3.8), we see that G_n embeds into G_{n+1} as an ordered abelian group, and by (CA), this embedding is an $\mathcal{L}_1$-homomorphism.

Define $\mathcal{G}_\omega$ as:

$$\mathcal{G}_\omega := \bigcup_{i \in \omega} \mathcal{G}_i$$

Then $\mathcal{G}_\omega$ is a model of T_1^-, and since for any $a \in G_\omega$, we can find some $n \in \omega$ such that G_n contains a preimage for a under χ, we see that $\mathcal{G}_\omega \models T_1$.

Finally, note that $\chi(G_{n+1}) \subseteq G_n$, so for any $h \in G_n$, we know that $\chi^n(h) \in G$. □

Lemma 3.2.4 (F.-V. Kuhlmann, 1994, p. 228)**.** *Every precontraction group $\mathcal{G}$ has a T_1-hull. Moreover for any T_1-hull $\overline{\mathcal{G}}$ and for any $\overline{g} \in \overline{G}$, there exists some $n \in \mathbb{N}$ such that $\chi^n(\overline{g}) \in G$.*

Proof. By Lemma 3.2.2 (i), we may assume G is divisible. Via Lemma 3.2.3, embed $\mathcal{G}$ into some model $\mathcal{H}$ of T_1. We keep filling in the gaps in the image of χ on G by adding elements of H, and show that the resulting structure has the factoring property.

Set $\mathcal{G}_0 := \mathcal{G}$, and assume we have defined some $\mathcal{G}_\alpha \models T_1^- \cup$(CD) with the factoring property over $\mathcal{G}$ with respect to T_1 **and hence** embeds into $\mathcal{H}$ over $\mathcal{G}$. If $\mathcal{G}_\alpha$ is a model of T_1, i.e. χ is surjective on G_α, then stop. Otherwise, define $\mathcal{G}_{\alpha+1}$ as follows; pick some $a \in G_\alpha^{>0} \setminus \chi(G_\alpha)$, we can find some $b \in H$ such that $\chi(b) = a$. Fix this particular b, and define $G_{\alpha+1}$ as:

$$G_{\alpha+1} := G_\alpha + \mathbb{Q}b$$

Note that $[b] \notin [G_\alpha]$, else by (CA), a would have some preimage under χ in G_α. Hence $[G_{\alpha+1}] = [G_\alpha] \cup [b]$. Once we set $\chi_{\alpha+1}(\gamma) := \chi_\alpha(\gamma)$ for $[\gamma] \in [G_\alpha]$, and $\chi_{\alpha+1}(\gamma) := a$ when $[\gamma] = [b]$, for any $\gamma \in G_{\alpha+1}$, the $\mathcal{G}_{\alpha+1}$ is a model of $T_1^- \cup$ (CD). It remains to show that $\mathcal{G}_{\alpha+1}$ has the factoring property over $\mathcal{G}$ with respect to T_1, for which it is sufficient to show that $\mathcal{G}_{\alpha+1}$ has the factoring property over $\mathcal{G}_\alpha$ over T_1.

Suppose $\mathcal{G}_\alpha$ embeds into some contraction group $\mathcal{K}$ via some function $i : \mathcal{G}_\alpha \hookrightarrow \mathcal{K}$. Then $a \in K$ thus has a preimage under χ in K, $k \in K$. Define the map $\bar{i} : \mathcal{G}_{\alpha+1} \to \mathcal{K}$ as $\bar{i}(g_\alpha + qb) = i(g_\alpha) + qk$, where $g_\alpha \in G_\alpha$ and $q \in \mathbb{Q}$. The axiom (C≤) implies that b and k induce the same cut in G_α, so by Lemma 2.3.13, the map $\bar{i}$ is an ordered group homomorphism. Since $\chi(b) = \chi(k) = a$ and $[k] \notin [G_\alpha]$, we see that it must also be an $\mathcal{L}_1$-homomorphism.

For limit ordinals β, if we have already defined $\mathcal{G}_\alpha$ for all $\alpha < \beta$, set:

$$\mathcal{G}_\beta := \bigcup_{\alpha<\beta} \mathcal{G}_\alpha$$

Then again $\mathcal{G}_\beta$ is a model of $T_1^- \cup$(CD), and has the factoring property over $\mathcal{G}$ with respect to T_1. Since $\mathcal{H}$ is a model of T_1, for some $\alpha \le |H|^+$, $\mathcal{G}_\alpha$ must be a model of T_1.

To get the final statement, note that any T_1-hull of $\mathcal{G}$ embeds into the structure (H, χ) from Lemma 3.2.3. □

Remark 3.2.5. If we examine the construction of a 1-contraction hull of some 1-precontraction group (G, χ) (Lemma 3.2.4), we see that the 1-contraction hull constructed $(\overline{G}, \chi)$ can be written as:

$$G + \sum_{i\in I} \mathbb{Q}\delta_i$$

where $(\delta_i)_{i\in I} > 0$ is a collection of indeterminates such that for all $i, j \in I$, $i \neq J$:

- $[\delta_i] \notin [G]$
- $[\delta_i] \neq [\delta_j]$
- $[\delta_i]$ is not less than $[G]$ i.e. there exists some $g \in G$ with $[g] < [\delta]$
- $\delta_i < \delta_j$ if and only if $\chi(\delta_i) < \chi(\delta_j)$ and $\delta_i < g$ if and only if $\chi^n(\delta_i) <_G \chi^n(g)$ for some $n \in \mathbb{N}$.

Thus $G \subseteq \overline{G}$ is a pure valuation extension of ordered abelian groups and $[G]$ is coinitial in $[\overline{G}]$.

3.2.2 Existential Closedness

Fix some models $\mathcal{G} \subseteq \mathcal{H}$ of T_1, for any $b \in H \setminus G$, we need to find some 'simple' (meaning easy to show is finitely realised in $\mathcal{G}$) set of formulas $p_b(x)$ such that $p_b \vdash \mathrm{qftp}_{\mathcal{H}}(b, G)$.

Definition 3.2.6 (F.-V. Kuhlmann, 1994, pp. 233–234)**.** For any $b \in H \setminus G$, a sequence $(b_i, g_i)_{i<\omega}$, where $b_i \in H$ and $g_i \in G$, is a **characteristic sequence** of b in $\mathcal{G}$ if it satisfies:

- If $[G] \subsetneq [G + \mathbb{Z}b]$ then $b_0 = b - g_0$ and $[b_0] \notin [G]$.
- If $[G] = [G + \mathbb{Z}b]$ then $b_0 := b$ and $g_0 := 0$.
- For any $i \geq 1$, if $b_i \notin G$ then:

  - $[b_0] > [b_1] > \ldots > [b_i]$
  - If $[G + \mathbb{Z}\chi(b_i)] = [G]$, then $g_{i+1} := 0$ and $b_{i+1} := \chi(b_i)$
  - If $[G + \mathbb{Z}\chi(b_i)] \neq [G]$ then $[\chi(b_i) - g_{i-1}] \notin [G]$ (g_{i+1} might still be zero!)
- For any $i \geq 1$, if $b_i \in G$ then $b_{j+1} = 0$ and $g_j = 0$ for all $j \geq i$

For the rest of this section fix some $b \in H \setminus G$.

Lemma 3.2.7 (F.-V. Kuhlmann, 1994, p. 233)**.** *A characteristic sequence of b in $\mathcal{G}$ exists.*

Proof. Consider the extension of ordered abelian groups $G \subseteq G + \mathbb{Z}b$. Suppose the extension is value preserving, so $[G + \mathbb{Z}b] = [G]$, then set $b_0 := b$, $b_1 := \chi(b_0)$, and $b_{i+2}, g_i := 0$ for all $i \in \omega$.

If the extension $G \subseteq G + \mathbb{Z}b$ is **not** value preserving, then there exists some $g \in G$ such that $[b - g] \notin [G]$. Set $g_0 := g$ and $b_0 := b - g_0$

Assume we have defined $(b_j, g_j)_{j \leq i}$ for some $i \in \omega$, where $b_{j+1} := \chi(b_j) - g_{j+1}$. If $b_i \in G$, then set $b_j, g_j := 0$ for all $j > i$. If not, consider $\chi(b_i)$, and the extension:

$$G + \sum_{j=0}^{i} \mathbb{Z}b_j \subseteq G + \sum_{j=0}^{i} \mathbb{Z}b_j + \mathbb{Z}\chi(b_i) \tag{3.1}$$

- If $\chi(b_i) \in G$, then set $b_{i+1} := \chi(b_i)$, $b_{j+1}, g_j := 0$ for all $j \geq i + 1$.
- If the extension (3.1) is value preserving, then set $b_{i+1} := \chi(b_i)$, $b_{i+2} := \chi(b_{i+1}) = \chi^2(b_i)$, and $g_j := 0$ for all $j \geq i + 1$.
- If the extension (3.1) is **not** value preserving, then there exists some $\gamma \in G + \sum_{j=0}^{i} \mathbb{Z}b_j$, such that:

  $$[\chi(b_i) - \gamma] \notin \left[G + \sum_{j=0}^{i} \mathbb{Z}b_j\right]$$

  Then since $[b_0] > [b_1] > \ldots > [b_i] > [\chi(b_i)]$, with the last inequality coming from axiom (CP), we see that $\gamma \in G$. Set $g_{i+1} := \gamma$ and $b_{i+1} := \chi(b_i) - g_{i+1}$

It can be verified easily that the sequence $(b_i, g_i)_{i<\omega}$ satisfies the conditions of Definition 3.2.6, hence we are done. ∎

Definition 3.2.8 (F.-V. Kuhlmann, 1994, p. 234)**.** Fix some $b \in H \backslash G$ and 1-characteristic sequence $(b_i, g_i)_{i<\omega}$. If $b_i \notin G$ for all $i < \omega$, we say b **is 1-transcendental** and say b has **1-characteritic length** ω. Conversely, we call b **1-algebraic** when $b_i = 0$ for some $i < \omega$, and say b has **1-characteristic length** α where α is the least such i (So $b_{\alpha-1} \in G^{\neq 0}$).

We extend Kuhlmann's definition by the following; when b is 1-algebraic with $[b_{\alpha-1}] \in [G]$ (so $[G+\mathbb{Z}b_{\alpha-1}] = [G]$), we say b is **1-archimedean-algebraic**, and when $[b_{\alpha-1}] \notin [G]$ we say it is **1-value-algebraic**.

Lemma 3.2.9. *The characteristic sequences of b in $\mathcal{G}$ have the following properties.*

(i) For any two 1-characteristic sequences $(b_i, g_i)_{i\in\omega}$ and $(\bar{b}_i, \bar{g}_i)_{i\in\omega}$, for any $\alpha < \omega$, we have:

$$[b_\alpha] = [\bar{b}_\alpha]$$

(ii) If b has a 1-characteristic sequence of length α, then all it's 1-characteristic sequences have length α, thus the 1-characteristic sequence length of b is well-defined, hence so are the definitions of 1-transcendental, 1-algebraic, 1-value-algebraic and 1-archimedean-algebraic.

(iii) If $(b_i, g_i)_{i<\omega}$ is a 1-characteristic sequence, then for all $\alpha < \omega$, the sequence $(b_{\alpha+i}, g_{\alpha+i})_{i<\omega}$ is an 1-characteristic sequence of b_α.

Proof. We prove (i) by induction. If $[G+\mathbb{Z}b]$ is value preserving, then both b_0 and $\bar{b}_0$ must be equal to b, thus $[b_0] = [\bar{b}_0]$. If the extension is not value preserving, then both $[b_0]$ and $[\bar{b}_0]$ **do not** lie in $[G]$. By Fact 2.3.3 we have

$$|[G+\mathbb{Z}b] \setminus [G]| = 1$$

Thus $[b_0] = [\bar{b}_0]$. Suppose $[b_l] = [\bar{b}_l]$ for all $l < k$, where $k < \omega$. Consider the possible configurations of b_{k-1} and $\bar{b}_{k-1}$. Either **both** $[b_{k-1}]$ and $[b_{k-1}]$ **are outside** $[G]$, or **both are inside** $[G]$. When they are inside $[G]$, we may have $b_{k-1} \in G$ or $b_{k-1} \notin G$, and the same for $\bar{b}_{k-1}$. We need to consider all these cases separately.

- If $[b_{k-1}] = [\bar{b}_{k-1}] \notin [G]$, then by (CA) we have $\chi(b_{k-1}) = \chi(\bar{b}_{k-1})$. Again from Fact 2.3.3, we know that:

  $$|[G+\mathbb{Z}\chi b_{n-1}]\ [G]| \leq 1$$

  Thus by the construction of 1-characteristic sequences, we have $[\chi(b_{k-1}) - g_k] = [\chi(\bar{b}_{k-1}) - \bar{g}_k]$, so $[b_k] = [\bar{b}_k]$.

- If $[b_{k-1}] = [\bar{b}_{k-1}] \in [G]$, but $b_{k-1} \notin G$, then by construction, $b_{k-1} = \chi(b_{k-2})$, but we assumed $[b_{k-2}] = [\bar{b}_{k-2}]$, thus $\bar{b}_{k-1} = \chi(\bar{b}_{k-2}) = b_{k-1}$, so $b_k = \bar{b}_k \in G$.

- Similarly if $[b_{k-1}] = [\bar{b}_{k-1}] \in [G]$, but $\bar{b}_{k-1} \notin G$, then we would again have $b_k = \bar{b}_k \in G$

- Finally if $b_{k-1} \in G$, then $b_k = \chi(b_{k-1})$, but we assumed $[b_{k-1}] = [\bar{b}_{k-1}]$, so $\bar{b}_k = \chi(\bar{b}_{k-1}) = b_k$.

Thus we have proved (i). (ii) is an easy consequence of (i). (iii) is verified just by checking the definition of 1-characteristic sequences. □

Thus we can now fix a 1-characteristic sequence $(b_i, g_i)_{i<\omega}$ of b in $\mathcal{G}$.

Lemma 3.2.10. *The structure $\mathcal{G}\langle b\rangle$ has domain:*

$$G_b \coloneqq G + \sum_{i\in\omega} \mathbb{Z}b_i \tag{3.2}$$

Proof. Since each b_i can be written as an $\mathcal{L}_1$-term with parameters in G, we have $G_b \subseteq |\mathcal{G}\langle b\rangle|$. To prove the converse it is sufficient to show that the G_b is closed under χ. Pick some element $x \in G_b$, we can write it as:

$$x \coloneqq g + z_0 b_0 + \ldots + z_n b_n$$

Where $g \in G, z_i \in \mathbb{Z}$ and $z_n \neq 0$. We can further assume (by replacing g with $g + z_n b_n$) that $b_n \notin G$.

Since $[b_0] > [b_1] > \ldots > [b_{n-1}] > [b_n]$ and $[b_0], \ldots, [b_{n-1}] \notin [G]$, we deduce that:

$$[x] = \max\{[z_0 b_0], \ldots, [z_{n-1} b_{n-1}], [g + z_n b_n]\}$$

But since $\chi(b_i)$ always lands in G_b for all $i \in [1, n]$, and $[g + z_n b_n] \in [G] \cup [b_n]$, we deduce that $\chi(x)$ is also in G_b, so we are done. □

Definition 3.2.11. For $n \in \omega$, let x_n be the term with the same form as b_n but with b replaced with the variable x, so:

$$x_0 \coloneqq x - g_0 \tag{3.3}$$

$$x_n \coloneqq \begin{cases} \chi(x_{n-1}) - g_n & \text{If } b_{n-1} \notin G \\ 0 & \text{Otherwise} \end{cases} \tag{3.4}$$

If we view each x_n as a function $G \to G$, then by (C≤) and (CS), x_n must be increasing and surjective.

Similarly, Define $y_{m,n}(x)$ to be the term b_n but with the occurrence b_m replaced with the variable x, so $y_{m,n}(b_m) = b_n$. Each $y_{m,n}$ is also increasing and surjective as a function to and from G.

Let p_b be the set of formulas:

$$p_b(x) := \bigcup_{n\in\omega} D_{b_n}(x_n) \tag{3.5}$$

Where for some fixed constant $h \in H$ and variable y, $D_h(y)$ is defined as:

$$D_h(y) := \begin{cases} \left\{ g < y < \overline{g} \mid g, \overline{g} \in G, g < h < \overline{g} \right\} & \text{If } h \notin G \\ \left\{ y = h \right\} & \text{If } h \in G \end{cases} \tag{3.6}$$

This must be a consistent set of formulas, since $b \models p_b(x)$.

Lemma 3.2.12. *For any $b \in H \setminus G$, the 'type' p_b is a type of $\mathcal{G}$.*

Proof. Suppose b is 1-algebraic with a characteristic sequence of length $n+1$. We can represent some finite subset $\Delta \subseteq p_b$ via the single formula:

$$\Delta(x) := \left(\bigwedge_{i=0}^{n-1} a_i < x_i < \overline{a}_i \right) \wedge x_n = a_n$$

Where $a_i, \overline{a_i} \in G$. Define $h, \overline{h}$ as:

$$h := \max\{y_{i,n-1}(a_i) \mid 0 \le i < n\} \in G$$
$$\overline{h} := \min\{y_{i,n-1}(\overline{a}_i) \mid 0 \le i < n\} \in G$$

Then since for each i, $y_{i,n-1}$ is increasing and $b_{n-1} \notin G$, we must have $h < b_{n-1} < \overline{h}$. Let $A := \chi^{-1}(b_n) \subseteq G$, then A must be a convex set. We claim that $(h, \overline{h})_G \cap A \neq \varnothing$. Suppose the intersection is empty, say because $A > \overline{h}$, then pick some $g \in G$ with

$$\overline{h} < g < A$$

We can do this because A must be a collection of archimedean classes, so if for example $\overline{h} > 0$, then $\overline{h} < 2\overline{h} < A$. Since χ is increasing and A is the preimage of b_n under χ, we have:

$$\chi(\overline{h}) \le \chi(g) < a_n$$

So $\chi(\overline{h}) < a_n = \chi(b_{n-1})$, but since χ is increasing, and $\overline{h} > b_{n-1}$, this is a contradiction. Similarly, we cannot have $A < h$, hence $A \cap (h, \overline{h}) \neq \varnothing$. Any element of $x_{n-1}^{-1}(A \cap (h, \overline{h}))$ will realise Δ.

If b is 1-transcendental, then we can write Δ as:

$$\Delta(x) := \left(\bigwedge_{i=0}^{n-1} a_i < x_i < \overline{a}_i \right)$$

Define $h, \overline{h}$ just as before, then any element of $x_{n-1}^{-1}(h, \overline{h})$ realises Δ.

□

Lemma 3.2.13. *Let $\overline{\mathcal{H}}$ be some other model of T_1 with $\mathcal{G}$ as a substructure, and $\overline{b} \in \overline{H} \setminus G$ with $\overline{b} \models p_b(x)$. Then there exists an isomorphism over $\mathcal{G}$:*

$$\phi : \mathcal{G}\langle b \rangle_{\mathcal{H}} \to \mathcal{G}\langle \overline{b} \rangle_{\overline{\mathcal{H}}}$$

Proof. Suppose b has characteristic sequence of length α, where $\alpha \leq \omega$, and set $\overline{b}_i := x_i(b)$. We see that $\{b_i\}_{i<\alpha}$ and $\{\overline{b}_i\}_{i<\alpha}$ satisfy the conditions of Lemma 2.3.13 (since they both realise p_b and are strictly decreasing in valuation), thus we have an isomorphism ϕ of ordered abelian groups:

$$\phi : G + \sum_{i<\alpha} \mathbb{Z} b_i \to G + \sum_{i<\alpha} \mathbb{Z} \overline{b}_i : b_i \mapsto \overline{b}_i \tag{3.7}$$

Note that the image of ϕ is definitely a subset of $\left|\mathcal{G}\langle \overline{b} \rangle_{\mathcal{H}}\right|$. If we can show that ϕ is an $\mathcal{L}_1$-homomorphism, then that would automatically imply that the image of ϕ is closed under χ, and hence we would have the equality:

$$\left|\mathcal{G}\langle \overline{b} \rangle_{\mathcal{H}}\right| = G_{\overline{b}} := G + \sum_{i<\alpha} \mathbb{Z} \overline{b}_i$$

Assume b is 1-transcendental, so $\alpha = \omega$ and $b_i \notin G$ for all $i \in \omega$, then by (CA) and (CS), $[b_i] \notin [G]$ for all i. Since $\overline{b}_i$ induces the same cut in G as b_i, we also have $[\overline{b}_i] \notin [G]$ for all i. Pick some element $x \in |\mathcal{G}\langle b \rangle|$, we can write it as:

$$x := g + z_0 b_0 + \ldots + z_n b_n$$

where $g \in G$, $z_i \in \mathbb{Z}$ and $z_n \neq 0$, hence $\phi(x)$ can be written as:

$$\begin{aligned} \phi(x) &= \phi(g + z_0 b_0 + \ldots + z_n b_n) \\ &= \phi(g) + \phi(z_0 b_0) + \ldots + \phi(z_n b_n) \\ &= g + z_0 \overline{b}_0 + \ldots z_n \overline{b}_n \end{aligned}$$

Note that $[x] \in \{[g], [z_0 b_0], \ldots, [z_n b_n]\}$ and $\phi(x) \in \{[g], [z_0 \overline{b}_0], \ldots, [z_n \overline{b}_n]\}$, moreover:

$$[x] = [b_i] \iff [\phi(x)] = [\bar{b}_i]$$
$$[x] = [g] \iff [\phi(x)] = [g]$$

Suppose $[x] = [b_i]$, then:

$$\begin{aligned}
\chi(\phi(x)) &= \chi(\bar{b}_i) && \text{As } [\phi(x)] = [\bar{b}_i] \text{ and axiom (CA)}\\
&= \bar{b}_{i+1} + g_{i+1} && \text{Since } \bar{b}_{i+1} := x_{n+1}(\bar{b})\\
&= \phi(b_{i+1}) + g_{i+1} && \text{Since } \phi(b_{i+1}) := \bar{b}_{i+1}\\
&= \phi(b_{i+1} + g_{i+1}) && \phi \text{ is a group homomorphism and is constant on } G\\
&= \phi(\chi(b_i)) && \text{Again by construction of characteristic sequences}\\
&= \phi(\chi(x)) && \text{As } [x] = [b_i] \text{ and axiom (CA)}
\end{aligned}$$

Similarly, if $[x] = [g]$, then:

$$\begin{aligned}
\chi(\phi(x)) &= \chi(g) && \text{As } [\phi(x)] = [g] \text{ and axiom (CA)}\\
&= \phi(\chi(g)) && \text{Since } \phi \text{ is constant on } G\\
&= \phi(\chi(x)) && \text{We assumed } [x] = [g] \text{ so by (CA), } \chi(x) = \chi(g)
\end{aligned}$$

Thus $\phi(\chi(x)) = \chi(\phi(x))$ for all x in $\mathcal{G}\langle b \rangle$. Similarly, when b is 1-value-algebraic (so $[b_{\alpha-1}] \notin [G]$), the exact same argument as above confirms that ϕ respects χ, since $[b_i], [\bar{b}_i] \notin [G]$ for all $i \in [1, \alpha - 1]$ and $b_\alpha = \bar{b}_\alpha \in G$.

The only bothersome case is when b is 1-archimedean-algebraic (so $[G + \mathbb{Z}b_{\alpha-1}] = [G]$). To simplify things set $\alpha = 1$ and $b_0 := b$ so $b \notin [G]$ but $[G + \mathbb{Z}b] = [G]$. By Lemma 2.3.12, we see that any $\bar{b}$ realising the same cut in G as b also satisfies $[G + \mathbb{Z}\bar{b}] = [G]$. Pick some $x \in G + \mathbb{Z}b$, written as $x = g + zb$, so $\phi(x) = g + z\bar{b}$. Then since b and $\bar{b}$ induce the same cut in G and $[x], [\phi(x)] \in [G]$, we see that $[x] = [\phi(x)]$. Thus $\chi(x) = \chi(\phi(x))$, and since ϕ is constant on G, we have $\phi(\chi(x)) = \chi(\phi(x))$. □

Thus an immediate application of Fact 2.1.12 yields us:

Proposition 3.2.14. *For any extension $\mathcal{G} \subseteq \mathcal{H}$ of contraction groups and $b \in H \setminus G$, $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b \rangle$.*

3.2.3 Initial structures

Lemma 3.2.15. *There exists a model of T_1^- that embeds into every other model of T_1^-*

Proof. Consider $\mathcal{Q} := (Q, \chi)$, where Q is the Hahn **sum**:

$$\coprod_{\mathbb{N}} \mathbb{Z}$$

With $\chi(1_{-n}) := 1_{-n-1}$ for any $n \in \omega$. Then for any $\mathcal{G} \models T_1^-$ and $g \in G$, we have an isomorphism:

$$\langle g \rangle \cong \mathcal{Q}$$

□

3.2.4 Final Result

Since all the assumptions of Theorem 2.1.14 have been satisfied through Lemma 3.2.4, Proposition 3.2.14 and Lemma 3.2.15, we have:

Theorem 3.2.16 (F.-V. Kuhlmann, 1994)**.** *The theory T_1 has quantifier elimination, has a prime model and is complete.*

3.3 Two Contractions

As we showed in Theorem 1.4.6, the action of a hyper-logarithm on an appropriate field K induces a centripetal contraction group which interacts with the contraction induced by log in a particular way, and in Definition 1.4.7 we defined a theory which characterises this action.

Definition 3.3.1. Let T_2 be the theory from Definition 1.4.7. Define T_2^- be T_2 but with the axioms requiring χ and L to be surjective, and divisibility of the underlying group erased. Say $\mathcal{G}$ is a **2-precontraction group** if $\mathcal{G} \models T_2^-$, and a **2-contraction group** if it models T_2.

The purpose of this section is to prove that T_2 has quantifier elimination in the language $\mathcal{L}_2$. The proof has the same structure as the proof for 1-contractions from Section 3.2, which means we need to satisfy the conditions of Theorem 2.1.14.

3.3.1 Closure operator

For some 2-precontraction group (G, χ_1, χ_2), we can consider the reduct to just the first contraction χ, and take the T_1 hull with respect to just that contraction.

Definition 3.3.2. Let $\mathcal{G} := (G, \chi_1, \chi_2)$ be a 2-precontraction group, and (G, χ_1) be its reduct to the language $\mathcal{L}_1$. Some $\mathcal{L}_1$-structure $\mathcal{K} := (K, \chi_1)$ is a **1-contraction hull of $\mathcal{G}$** if $\mathcal{K}$ is a T_1-hull of $\mathcal{G}$.

As a reminder, Lemma 3.2.4 tells us that for any 1-contraction hull $(\overline{G}, \chi)$ of (G, χ) and any $\overline{g} \in \overline{G}$, there exists some $n \in \mathbb{N}$ such that $\chi^n(\overline{g}) \in G$. We leverage this to show that any 2-precontraction group has a T_2-hull. First, we make an easy observation:

Lemma 3.3.3. *Let $\mathcal{G}$ be a 2-precontraction group and $\mathcal{K} := (K, \chi_1)$ be a 1-contraction hull of $\mathcal{G}$. Then there exists a unique extension of χ_2 to K such that (K, χ_1, χ_2) is a model of*

T_2^- extending $\mathcal{G}$. Moreover, the model (K, χ_1, χ_2) has the factoring property over $\mathcal{G}$ with respect to T_2.

Proof. For all $x \in K \setminus G$, there exists some $n \in \mathbb{N}$ such that $\chi_1^n(x) = g_x \in G$, hence by axiom (H1), if χ_2 is a 2-contraction on $\mathcal{K}$ over G, the value of $\chi_2(x)$ is fixed and must be $\chi_2(g_x)$, hence the extension of L to K must be unique. Moreover, it is easy to verify that L is a 1-precontraction on K and interacts with χ in the right way, hence (K, χ_1, χ_2) is a 2-precontraction group.

To show $\mathcal{K}$ has the factoring property, pick some model $\mathcal{H} := (H, \chi_1, \chi_2)$ of T_2 extending $\mathcal{G}$ as $\mathcal{L}_2$ structures. Since (K, χ_1) is a T_1-hull of (G, χ_1), it has the factoring property with respect to T_1, hence there exists an $\mathcal{L}_1$-embedding $\phi : (K, \chi_1) \to (H, \chi_1)$ that is constant on G, and by (H1), it must also be an $\mathcal{L}_2$-embedding. □

We would like to show that any 2-precontraction group is embedded in some (possibly very large) 2-contraction group.

Proposition 3.3.4. *Let (G, χ_1, χ_2) be a 2-precontraction group. Then there is some 2-contraction group (K, χ_1, χ_2) such that (G, χ_1, χ_2) is a substructure of (K, χ_1, χ_2).*

Proof. Let $\mathcal{G}_0 = \mathcal{G}$. We define a chain of 2-precontraction groups $\mathcal{G}_0 := \mathcal{G} \subseteq \mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \dots$ where $\mathcal{G}_n := (G_n, \chi_1, \chi_2)$, and our desired structure will be

$$\mathcal{K} := \bigcup_{n \in \mathbb{N}} \mathcal{G}_n \tag{3.8}$$

Given $\mathcal{G}_{2k}$, choose $\mathcal{G}_{2k+1}$ to be some 1-contraction hull of $(\mathcal{G}_{2k}, \chi_1)$, so by Lemma 3.3.3, there is a unique extension of χ_2 to G_{2k+1} such that $\mathcal{G}_{2k+1} := (G_{2k+1}, \chi_1, \chi_2)$ is a model of T_2^- containing $\mathcal{G}_{2k}$ as a substructure.

Given $\mathcal{G}_{2k-1}$, set $\mathcal{G}_{2k}$ to be the structure with domain $G_{2k} := \prod_{\Gamma} \mathbb{R}$, where

$$\Gamma = [G_{2k-1}] \cup \left(\omega \times \{G_{2k-1}^{>0} - \chi_2(G_{2k-1})\} \right)$$

is ordered as follows:

- For $(n, a) \in \omega \times \{G_{2k-1}^{>0} - \chi_2(G_{2k-1})\}$, $\beta \in [G_{2k-1}]$, we have:

$$(n, a) < \beta \iff a < \chi_2 \beta$$

- For $(n, a), (m, b) \in \omega \times \{G_{2k-1}^{>0} - \chi_2(G_{2k-1})\}$, $a \neq b$, we have:

$$(n, a) < (m, b) \iff a < b$$

- For $(n, a) \in \omega \times \{G_{2k-1}^{>0} - \chi_2(G_{2k-1})\}$, we have $(n+1, a) < (n, a)$, thus:

$$(n, a) <_\Gamma (m, a) \iff n < m$$

- For any two elements in $[G_{2k-1}]$, use the ordering from $[G_{2k-1}]$ itself.

Define the contractions on G_{2k} as follows. Let $(r)_\alpha \in \prod_\Gamma \mathbb{R}$ denote the element which has value r in the place of the ordinal α, and is 0 everywhere else. As contraction maps are constant on archimedean classes, we only need to define χ_1 and χ_2 on elements of the form $(1)_\alpha$.

(i) For $\alpha \in v\mathcal{G}_{2n-1}$, set $\chi_i(1)_\alpha = \chi_i(\alpha)$ for $i = 1, 2$.

(ii) For $\alpha = (n, a) \in \omega \times \{G_{2k-1}^{>0} - \chi_2(G_{2k-1})\}$, set $\chi_1(1)_\alpha = (1)_{(n+1,a)}$, and $\chi_2(1)_\alpha = a$.

By the Hahn embedding theorem ((2.3.8)) we see that G_{2k} embeds into G_{2k+1} as ordered abelian groups, and by (i), we see that it must be an $\mathcal{L}_2$-embedding. Hence define $\mathcal{K}$ as the union (3.8), we see that both χ_1 and χ_2 are surjective on $\mathcal{K}$ thus it is a model of T_2. $\square$

We can now show that T_2 has the closure property

Proposition 3.3.5. *Any 2-precontraction group $\mathcal{G}$ has a T_2-hull.*

Proof. Embed $\mathcal{G}$ into some model $\mathcal{K}$ of T_2 via Proposition 3.3.4. We alternate between taking the χ_1-hull and filling in the gaps in $\chi_2(G)$ with elements of K while ensuring the factoring property still holds, and eventually after at most $|K|^+$ steps, we end up with a model of T_2. Set $\mathcal{G}_0 := \mathcal{G}$. Given some successor ordinal $\alpha = \eta + n$, where η is a limit ordinal and $n \in \omega$, suppose we have defined some $\mathcal{G}_\alpha \models T_n^-$ with the factoring property over $\mathcal{G}$ with respect to T_2, thus embedding into $\mathcal{K}$ over $\mathcal{G}$. Define $\mathcal{G}_{\alpha+1}$ as follows:

n is even: Set $\mathcal{G}_{\alpha+1}$ to be some 1-contraction hull of $\mathcal{G}_\alpha$. By Lemma 3.3.3, $\mathcal{G}_{\alpha+1}$ has the factoring property over $\mathcal{G}$ with respect to T_1, and hence can be thought of as an $\mathcal{L}_2$-substructure of $\mathcal{K}$.

n is odd: Pick some $a \in G_\alpha^{>0} \setminus \chi_2(G_\alpha)$ (if the set is empty then set $\mathcal{G}_{\alpha+1} := \mathcal{G}_\alpha$). We know there exists some $b \in K$ such that $\chi_2(b) = a$, so define $G_{\alpha+1}$ as:

$$G_{\alpha+1} := G_\alpha + \sum_{j\in\omega} \mathbb{Q}\chi_1^j(b)$$

By (H1), we have $\chi_2(\chi_1^i(b)) = a$, thus by (CA) we see that $[\chi_1^i(b)] \notin [G_\alpha]$ for all $i \in \omega$, hence:

$$[G_{\alpha+1}] = [G_\alpha] \cup \bigcup_{j\in\omega} [\chi_1^j(b)]$$

From which we deduce that $G_{\alpha+1}$ is closed under χ_i for all i, hence $\mathcal{G}_{\alpha+1} := (G_{\alpha+1}, \chi_1, \chi_2)$ is a model of T_2^- contained in $\mathcal{K}$. It remains to show that it has the factoring property over $\mathcal{G}_\alpha$ with respect to T_2. Pick any model $\mathcal{H}$ of T_2 with $\mathcal{G}_\alpha$ as a substructure, and pick some $\bar{b} \in H$ such that $\chi_2(\bar{b}) = a$. Then by (C≤), the sets $\{\chi_1^i(b)\}_{i\in\omega}$ and $\{\chi_1^i(\bar{b})\}_{i\in\omega}$ satisfy the conditions of Lemma 2.3.13, thus we have an isomorphism of ordered abelian groups over G_α:

$$\phi : G_\alpha + \sum_{j\in\omega} \mathbb{Q}\chi_1^j(b) \to G_\alpha + \sum_{j\in\omega} \mathbb{Q}\chi_1^j(\bar{b}) : \chi_1^i(b) \mapsto \chi_1^i(\bar{b})$$

Since $\chi_1(b) = a$, we see that ϕ is also an $\mathcal{L}_2$-isomorphism, thus $\mathcal{G}_{\alpha+1}$ embeds into $\mathcal{H}$ over $\mathcal{G}_\alpha$, thus $\mathcal{G}_{\alpha+1}$ has the factoring property over $\mathcal{G}_\alpha$ with respect to T_2.

For a limit ordinal β set $\mathcal{G}_\beta$ to be the union of all the structures before it. Since $\mathcal{K}$ is a model of T_2, eventually $\mathcal{G}_\beta$ must also be a model of T_2, and this will be T_2-hull of $\mathcal{G}$. □

3.3.2 Existential closedness

Fix some extension of 2-contradictions groups $\mathcal{G} \subseteq \mathcal{H}$, and some $b \in H \setminus G$. We want to show that $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b\rangle$. To begin with, we find a generating set for $\mathcal{G}\langle b\rangle$ as an ordered abelian group.

Definition 3.3.6. Let $(G, \chi) \subseteq (H, \chi)$ be an extension of 1-contraction groups, and $b \in H \setminus G$ have 1-characteristic sequence $(b_i, g_i)_{i<\omega}$. We say b is 1-**super-transcendental** if b is 1-transcendental and the shift sequence $(g_i)_{i\in\omega}$ is eventually zero, and call $i \in \omega$ the 1-**null point** if $g_i \neq 0$ but $f_j = 0$ for all $j > i$. By Lemma 3.2.9, this definition is well-defined.

Definition 3.3.7. Let $\mathcal{G}, \mathcal{H}$ and b be as above. Some sequence $(b_i, g_i)_{i<\omega^2}$ is a 2-characteristic sequence of b in $\mathcal{G}$ if:

(a) For any $m \in \omega$, the sequence $(b_{\omega m+i}, g_{\omega m+i})_{i\in\omega}$ is a 1-characteristic sequence for $b_{\omega m}$ in (G, χ_1).

(b) Fix some $j \in \omega$

  (i) If $b_{\omega j+i}$ is 1-super-transcendental, then $b_{\omega(j+1)} = \chi_2(b_{\omega j+k})$, where k the 1-null point.

  (ii) Otherwise $b_{\omega(j+1)+\alpha}, g_{\omega(j+1)+\alpha}$ are all 0 for any $\alpha < \omega^2$.

We say b is 2-**algebraic** if $b_\alpha \in G$ for some $\alpha < \omega^2$, and we say a characteristic sequence has length α if $b_\alpha = 0$ and $b_\beta \neq 0$ for all $\beta < \alpha$.

For the rest of this section, fix some $b \in H \setminus G$.

Lemma 3.3.8. *A 2-characteristic sequence of b in $\mathcal{G}$ exists. Moreover for any 2-characteristic sequence $(b_i, g_i)_{i\in\omega^2}$, if $\alpha < \beta < \omega^2$ and $b_\beta \notin G$, then $[b_\alpha] \notin [G]$ and $[b_\alpha] > [b_\beta]$.*

Proof. The construction is evident from Definition 3.3.7. To show it is decreasing in valuation throughout its length, it is sufficient to show $[b_{\omega j}] < [b_{\omega(j-1)+i}]$ for all $i, j \in \omega$, $j > 0$ and $b_{\omega j} \notin G$. But this follows from axiom (H2). □

Lemma 3.3.9. *All 2-characteristic sequences of b in $\mathcal{G}$ satisfy the following.*

(i) For any two 2-characteristic sequences $(b_i, g_i)_{i\in\omega^2}$ and $(\bar{b}_i, \bar{g}_i)_{i\in\omega^2}$ and $\alpha < \omega^2$, we have:

$$[b_\alpha] = [\bar{b}_\alpha]$$

(ii) If b has a 2-characteristic sequence of length α, then all it's 2-characteristic sequences have length α, thus the 2-characteristic sequence length of b is well-defined, hence so are the definitions of 2-algebraic, 2-transcendental and 1-super-transcendental.

(iii) If $(b_i, g_i)_{i<\omega^2}$ is a 2-characteristic sequence, then for all $\alpha < \omega^2$, the sequence $(b_{\alpha+i}, g_{\alpha+i})_{i<\omega^2}$ is an 2-characteristic sequence of b_α.

Proof. Identical to Lemma 3.2.9. □

Now we can fix some 2-characteristic sequence $(b_i, g_i)_{i<\omega^2}$ of b in $\mathcal{G}$.

Lemma 3.3.10. *Suppose b is* ***not*** *1-super-transcendental in (G, χ_1). Then the structure generated by b in $\mathcal{G}$, $\mathcal{G}\langle b\rangle_{\mathcal{H}}$ has domain:*

$$G_b := G + \sum_{i\in\omega} \mathbb{Z}b_i \tag{3.9}$$

Proof. As a reminder, the 1-characteristic sequence has the following form:

$$b_0 := b - g_0$$
$$b_n := \chi(b_{n-1}) - g_n \text{ for } n > 0$$

Since each element of the 1-characteristic sequence of b is made up of finitely many compositions of elements of G and $\mathcal{L}_1$-functions, we definitely know that G_b is contained in $|\mathcal{G}\langle b\rangle|$. For the converse, assume b is 1-transcendental, and write an arbitrary element x of G_b as:

$$x = g + z_0 b_0 + \ldots + z_n b_n$$

Where $g \in G$ and $z_i \in \mathbb{Z}$. Then $[x] \in \{[g], [z_0 b_0], \ldots, [z_n b_n]\}$, hence to show (3.9) is closed under χ_1 and χ_2 it is sufficient to show $\chi_i(b_j)$ lie in G for $i = 1, 2$ and $j \in \omega$. For $i = 1$ this follows from the construction of 1-characteristic sequences, specifically,

$\chi_1(b_j) = b_{j+1} - g_{j+1}$. For $i = 2$, fix some b_j, and let $k > 0$ be the least natural number such that $g_{n+k} \neq 0$, which must exist since b is **not** 1-super-transcendental, hence:

$$b_{n+k} = \chi_1^k(b_n) - g_{n+k}$$

We know $[b_{n+k}] \notin [G]$, and $[g_{n+k}] \in [G]$. Moreover, since $g_{n+k} \neq 0$, we must have $[\chi_1^k(b_n)] \in [G]$. Hence by the ultrametric inequality we have $[\chi_1^k(b_n)] = [g_{n+k}]$, thus

$$\begin{aligned} \chi_2(b_n) &= \chi_2(\chi_1^k(b_n)) && \text{By axiom (H1)} \\ &= \chi_2(g_{n+k}) && \text{Since } [g_{n+k}] = [\chi_1^k(b_n)] \text{ and (CA)} \end{aligned}$$

Identical arguments apply when b is 1-algebraic, with the only modification being the valuation of x when b is 1-archimedean-algebraic. Suppose the 1-characteristic sequence of b has length $n + 1$, then:

$$[x] \in \{[g - z_n b_n], [b_0], \ldots, [b_{n-1}]\}$$

But $[g - z_n b_n] \in G$ since $[G + \mathbb{Z}b_n] = [G]$, and by the 1-transcendental argument we see that $\chi_2(b_i) \in [G]$ for $i \in [0, n-1]$. ☐

Lemma 3.3.11. *No matter what the status of the 2-characteristic sequence of b, the $\mathcal{G}\langle b\rangle$ has domain:*

$$G + \sum_{i<\omega^2} \mathbb{Z}b_i \tag{3.10}$$

Proof. Suppose the characteristic sequence has length α, then by Lemma 3.3.8, any element of (3.10) has valuation lying in

$$[G] \cup \{[b_\beta] \mid \beta < \alpha\}$$

for some $\beta < \alpha$, thus it is sufficient ot show $\chi_i(b_j)$ lies in (3.10) for all i, j where $i = 1, 2, j < \alpha$. Fix $j = \omega m + n$. If b_j is not 1-super-transcendental in (G, χ_1), then by Lemma 3.3.10, we see $\chi_i(b_j)$ is in (3.10), so suppose b is 1-super-transcendental. If there exists some least $k \in \omega$ such that $g_{j+k+1} \neq 0$, then $[\chi_1^k(b_j)] = [g_{j+k+1}]$, thus $\chi_2(b_j) = \chi_2(g_{j+k+1}) \in G$. If $g_{j+k+1} = 0$ for all $k \in \omega$, then by construction of 2-characteristic sequences, we have $b_{j+\omega} = \chi_2(b_j)$, which must be in (3.10). ☐

Let us extend Definition 3.2.11 to 2-contraction groups.

Definition 3.3.12. Let $b \in H \setminus G$ have 2-characteristic sequence $(b_i, g_i)_{i<\omega^2}$. Let x_α be the term b_α but with b replaced with the variable x, so:

$$x_\alpha := \begin{cases} 0 & \text{If } b_\alpha = 0 \\ x - g_\alpha & \text{If } \alpha = 0 \\ \chi_1(x_{\alpha-1}) - g_\alpha & \text{If } \alpha \text{ is a successor ordinal and } b_\alpha \neq 0 \\ \chi_2(x_\beta) - g_\alpha & \text{If } \alpha \text{ is a limit ordinal, } b_\alpha \neq 0 \text{ and } b_\alpha = \chi_2(b_\beta) \end{cases}$$

Given some $\gamma < \omega^2$ set $y_{\gamma,\alpha}(x)$ to be the term with variable to x which returns b_α when x is substituted with b_γ, so for $\alpha \geq \gamma$:

$$y_{\gamma,\alpha} := \begin{cases} 0 & \text{If } b_\alpha = 0 \\ x & \text{If } \alpha = \gamma \\ \chi_1(y_{\gamma,\alpha-1}(x)) - g_\alpha & \text{If } \alpha > \gamma \text{ is a successor ordinal and } b_\alpha \neq 0 \\ \chi_2(y_\beta(x)) - g_\alpha & \text{If } \alpha > \gamma \text{ is a limit ordinal, } b_\alpha \neq 0 \text{ and } b_\alpha = \chi_2(b_\beta) \end{cases}$$

As with 1-contraction groups, the functions x_α, $y_{\beta,\alpha}$ are all surjective and increasing.

Example 3.3.13. Let b have 2-characteristic sequence as follows:

Definition 3.3.14. Let $p_b(x)$ be just as we defined in Equation (3.5) from Definition 3.2.11, but with the index ranging though ω^2 instead of ω:

$$p_b(x) := \bigcup_{n \in \omega^2} D_{b_n}(x_n) \tag{3.11}$$

Lemma 3.3.15. *The type p_b is a type of $\mathcal{G}$.*

Proof. Suppose b is 2-algebraic, with a characteristic sequence of length $\alpha + 1 < \omega^2$, meaning b_α lies in G but is not equal to 0. Pick some finite sequence of ordinals $\alpha_1 < \ldots < \alpha_n < \alpha$, and write some finite subset of p_b, Δ as:

$$\Delta(x) := \Delta_1(x) \wedge \Delta_2(x)$$

Where the Δ_i's are defined as:

$$\Delta_1(x) := \bigwedge_{i=1}^{n} a_{\alpha_i} < x_{\alpha_i} < \overline{a}_{\alpha_i}$$
$$\Delta_2(x) := x_\alpha = b_\alpha$$

Define h and $\overline{h}$ as:

$$h := \max\{y_{\alpha_i,\alpha-1}(a_{\alpha_i}) \mid 1 \leq i < n\}$$
$$\overline{h} := \min\{y_{\alpha_i,\alpha-1}(\overline{a}_{\alpha_i}) \mid 1 \leq i < n\}$$

Since each term $y_{\alpha_i,\alpha}$ thought of as functions on H and G is increasing, we have:

$$h < b_{\alpha_n} < \overline{h}$$

By the same argument as in Lemma 3.2.12, we see that $(h, \overline{h})_G$ and $A := \chi^{-1}_{\alpha_n,\alpha}(b_\alpha)$ have non-empty intersection in G, thus any element of $y^{-1}_{\alpha_n}((h, \overline{h})_G) \cap A$ realises Δ.

If b is 2-transcendental, or 2-algebraic with a characteristic sequence of limit ordinal length, then we can write Δ as just Δ_1, thus if we define h and $\overline{h}$ as before, any element of $x^{-1}_{\alpha_n}(h, \overline{h})$ will realise Δ. □

Lemma 3.3.16. *For any other 2-contraction group $\overline{\mathcal{H}}$ with $\mathcal{G} \subseteq \overline{\mathcal{H}}$, and $\bar{b} \in \overline{H} \setminus G$ with $\bar{b} \models p_b(x)$, there exists an $\mathcal{L}_2$ isomorphism ϕ over $\mathcal{G}$:*

$$\phi : \mathcal{G}\langle b \rangle_{\mathcal{H}} \to \mathcal{G}\langle \bar{b} \rangle_{\overline{\mathcal{H}}} \tag{3.12}$$

Proof. By Lemma 3.3.11, we know $\mathcal{G}\langle b \rangle$ has domain:

$$G + \sum_{i \in \omega^2} \mathbb{Z} b_i$$

Set $\bar{b}_i := x_i(b_i)$ for all $i \in \omega^2$, then since $\bar{b}$ realises the type p_b, Lemma 2.3.13 establishes an isomorphism of ordered abelian groups:

$$\phi : G + \sum_{i \in \omega^2} \mathbb{Z} b_i \to G + \sum_{i \in \omega^2} \mathbb{Z} \bar{b}_i : b_i \mapsto \bar{b}_i \tag{3.13}$$

Just as in Lemma 3.2.13, it is sufficient to show ϕ is an $\mathcal{L}_2$-homomorphism, which will imply that the domain of $\mathcal{G}\langle \bar{b} \rangle$ is the image of ϕ. Pick some element $x \in |\mathcal{G}\langle b \rangle|$, then for any $i \in \omega^2$ and $g \in G$:

$$[x] = [b_i] \iff [\phi(x)] = [\bar{b}_i]$$
$$[x] = [g] \iff [\phi(x)] = [g]$$

If $[x] = b_i$, then the same argument as in Lemma 3.2.13, tells us that $\chi_1 \circ \phi(x) = \phi \circ \chi_1(x)$, thus it remains show $\chi_2 \circ \phi(x) = \phi \circ \chi_2(x)$. Suppose i is before the 1-null point of b, i.e. there exists some $n \in \omega, n > 0$ such that $g_{i+n} \neq 0$, then:

$$\begin{aligned}
\chi_2(\phi(x)) &= \chi_2(\bar{b}_i) && \text{We assumed } [\phi(x)] = [\bar{b}_i] \\
&= \chi_2(\chi_1^n(\bar{b}_i)) && \text{By (H1)} \\
&= \chi_2(g_{i+n}) && \text{Since } [\chi_1^n(\bar{b}_i)] = [g_{i+n}] \\
&= \phi(\chi_2(g_{i+n})) && \psi \text{ is constant on } G \\
&= \phi(\chi_2(\chi_1^n(b_i))) && \text{Since } [\chi_1^n(b_i)] = [g_{i+n}] \\
&= \phi(\chi_2(b_i)) && \text{By (H1)}
\end{aligned}$$

If i is at or past the 1-null point, i.e. for all $n \in \omega$ $n > 0$ we have $g_{i+n} = 0$, then:

$$\begin{aligned}
\chi_2(\phi(x)) &= \chi_2(\bar{b}_i) && \text{We assumed } [\phi(x)] = [\bar{b}_i] \\
&= \bar{b}_{i+\omega} && \text{Since } \chi_2(\bar{b}_i) = \bar{b}_{i+\omega} \\
&= \phi(b_{i+\omega}) && \text{By the definition of } \phi \\
&= \phi(\chi_2(b_i)) \\
&= \phi(\chi_2(x))
\end{aligned}$$

When $[x] = [g]$, then the same argument as in Lemma 3.2.13 shows $\chi_i \circ \phi(x) = \phi \circ \chi_i(x)$ for $i = 1, 2$. □

As with the 1-contraction group case, Fact 2.1.12 gives us:

Proposition 3.3.17. *$\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b\rangle$.*

Proof. Use Lemma 3.3.16 and Lemma 3.3.15. □

3.3.3 Initial structures

Lemma 3.3.18. *The theory T_2 has an initial model.*

Proof. Consider the Hahn sum $\coprod_{\omega^2} \mathbb{Z}$, and denote the element with value r in the place of the ordinal α but 0 everywhere else as $(r)_\alpha$. To give a 2-contraction group structure on this set, we only need to define $\chi(1)_\alpha$ and $L(1)_\alpha$ for every ordinal $\alpha < \omega^2$. We can write α as $\omega n + m$, where n, m are natural numbers. Hence let $\chi(1)_{\omega n+m} = (1)_{\omega n+m+1}$, and $L(1)_{\omega n+m} = (1)_{\omega(n+1)}$.

It is easy to see that this is a 2-precontraction group, it remains to show that this structure embeds into every 2-precontraction group. Let (G, χ_1, χ_2) be any precontraction group, and pick some $a \in \mathcal{G}$. We will show that $\langle a\rangle \cong (\coprod_{\omega^2} \mathbb{Z}, \chi, L)$. Using axioms (H1) and (CA) for both χ and L, we see that:

$$\begin{aligned}\langle a\rangle &= \mathbb{Z}a + \mathbb{Z}\chi a + \mathbb{Z}\chi^2 a + \dots \\ &\quad + \mathbb{Z}L(a) + \mathbb{Z}\chi L(a) + \dots \\ &\quad + \mathbb{Z}L^2(a) + \mathbb{Z}\chi L^2(a) + \dots \\ &\vdots \\ &= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\mathbb{Z}\chi^i L^j(a)\end{aligned}$$

Define a map $\phi : \langle a\rangle \to (\mathbb{H}_{\omega^2}\mathbb{Z}, \chi, L)$ as $\phi(\chi^i L^j(a)) = (1)_{\omega j+i}$. Then ϕ is an isomorphism, hence the structure $(\mathbb{H}_{\omega^2}\mathbb{Z}, \chi, L)$ embeds into (G, χ_1, χ_2). By the universal property of 2-contraction hulls, we know that any 2-contraction hull of $(\mathbb{H}_{\omega^2}\mathbb{Z}, \chi, L)$ embeds into any 2-contraction hull of (G, χ_1, χ_2), hence the result is proved. □

3.3.4 Final Result

By applying Proposition 3.3.5, Proposition 3.3.17 and Lemma 3.3.18 to Theorem 2.1.14, we have:

Theorem 3.3.19. *The theory of* 2*-contraction groups has quantifier elimination, is complete and has a prime model.*

3.4 Multiple Contractions

Definition 3.4.1. Let $\mathcal{L}_n$ be the language $\langle +, -, 0, <, \chi_1, \dots, \chi_n\rangle$ be the language of ordered abelian groups along with n unary functions. We will call a structure $(\mathcal{G}, +, -, 0, <, \chi_1, \dots, \chi_n)$, (or just $(G, \chi_1, \dots, \chi_n)$) an n**-precontraction group** if $(G, +, -, 0, <)$ is an ordered abelian group and:

C1$_n$ For all $i \leq n$, the structure (G, χ_i) is a precontraction group

C2$_n$ For all $i < n$ and $x \in \mathcal{G}$, we have $\chi_{i+1}(\chi_i(x)) = \chi_{i+1}(x)$, and hence for all $i < j \leq n$, $\chi_j(\chi_i(x)) = \chi_j(x)$

C3$_n$ For all $i < n$, $k \in \mathbb{N}$ and $x \in \mathcal{G}$, $x > 0$, we have $\chi_{i+1}(x) < \chi_i^k(x)$

Furthermore, if all the maps χ_i are surjective and G is divisible, we will call $(G, \chi_1, \dots, \chi_n)$ an n**-contraction group**. Let T_n be the theory of n-contraction groups, and T_n^- be the theory of n-precontraction groups (so T_n^- is the universal theory of T_n).

In this section we prove that for all $n < \omega$, the theory T_n has QE and is complete, using the criterion from Theorem 2.1.14.

3.4.1 Closure Operator

First, we prove that T_n has the closure property. We need to induct on the following statement:

$(\Phi)_n$ Any n-precontraction group $\mathcal{G}$ has a T_n-hull, and moreover, any T_n-hull $\overline{\mathcal{G}}$ and $\overline{g} \in \overline{G}$, there exists some $k \in \mathbb{N}$ such that $\chi_n^k(\overline{g}) \in G$.

Definition 3.4.2. Assume $(\Phi)_{n-1}$, and let $\mathcal{G} \coloneqq (G, \chi_1, \ldots, , \chi_n)$ be an n-precontraction group. Consider the reduct of $\mathcal{G}$ to $\mathcal{L}_{n-1}$, $(G, \chi_1, \ldots, \chi_{n-1})$. An $(n-1)$**-contraction hull** of $\mathcal{G}$ is some T_{n-1} hull $\mathcal{K} \coloneqq (K, \chi_1, \ldots, \chi_{n-1})$ of $(G, \chi_1, \ldots, \chi_{n-1})$.

Lemma 3.4.3. *Assume $(\Phi)_{n-1}$. Let $\mathcal{K}$ be a $(n-1)$-contraction hull of $\mathcal{G}$. There is a unique extension of χ_n to K such that $(K, \chi_1, \ldots, \chi_n)$ is a model of T_n^-. Moreover, $(K, \chi_1, \ldots, \chi_n)$ has the factoring property over $\mathcal{G}$ with respect to T_n.*

Proof. Identical to Lemma 3.3.3. □

Lemma 3.4.4. *Assume $(\Phi)_{n-1}$. Any n-precontraction group $\mathcal{G}$, embeds into some n-contraction group $\mathcal{K}$. Moreover, for all $g \in K$, there exists some $k \in \mathbb{N}$ such that $\chi_n^k(g) \in G$.*

Proof. Set $\mathcal{K} \coloneqq \bigcup_{i\in\omega} \mathcal{G}_i$. Define $\mathcal{G}_0 \coloneqq \mathcal{G}$, and for $i > 0$, set $\mathcal{G}_i$ as follows.

For odd i: Let $(G_i, \chi_1, \ldots, \chi_{n-1})$ be some $n-1$-contraction hull of $\mathcal{G}_{i-1}$. By Lemma 3.4.3, there exists a unique extension of χ_n to G_{i-1} such that $\mathcal{G}_{i-1} \coloneqq (G_{i-1}, \chi_1, \ldots, \chi_n)$ is an n-precontraction group extending $\mathcal{G}_{i-1}$, thus $\mathcal{G}_{i-1} \subseteq \mathcal{G}_i$. Moreover, we can show $\chi_n(G_i) \subseteq G_{i-1}$: pick some $g \in G_i$, so by $(\Phi)_{n-1}$ there exists some $k \in K$ such that $\chi_{n-1}^k(g) \in G$, thus by $(\mathbf{C2}_n)$, $\chi_n(g) = \chi_n(\chi_{n-1}^k(g)) \in G$.

For even i, set G_i to be the Hahn **product**

$$G_i \coloneqq \prod_{\Gamma} \mathbb{R} \tag{3.14}$$

Where Γ is an ordered set with domain:

$$[G_{i-1}] \cup \left(\omega^{n-1} \times (G_{i-1}^{>0} \setminus \chi_n(G_{i-1}))\right) \tag{3.15}$$

With $<_\Gamma$ defined as follows:

- $[G_{i-1}]$ keeps its usual ordering
- $\omega^{n-1} \times (G_{i-1}^{>0} \setminus \chi_n(G_{i-1}))$ is ordered anti-lexicographically, so for $(\alpha, a), (\beta, b)$, where $\alpha, \beta \in \omega^{n-1}$ and $a, b \in G_{i-1}^{>0} \setminus \chi_n(G_{i-1})$, if $a = b$ then:

  $$(\alpha, a) >_\Gamma (\beta, b) \iff \alpha < \beta$$

  Otherwise:

$$(\alpha, a) >_\Gamma (\beta, b) \iff a > b$$

- For $\gamma \in [G_{i-1}]$ and (α, a), note that $a \notin \chi_n(G_{i-1})$, hence set:

$$\gamma >_\Gamma (\alpha, a) \iff \chi_n(\gamma) > a$$

Define the contractions on G_{i-1} as follows:

(i) For $\gamma \in [G_{i-1}]$ and $j \in [1, n]$, set $\chi_j(1_\gamma) = \chi_j(\gamma)$ i.e the point which the archimedean class of γ maps to in $\mathcal{G}_{i-1}$.

(ii) For (α, a) and $j < n$, set:

$$\chi_j(1_{(\alpha,a)}) \coloneqq 1_{(\alpha+\omega^{j-1},a)}$$

For χ_n, set:

$$\chi_n(1_{(\alpha,a)}) \coloneqq a$$

It can be easily verified that $\mathcal{G}_{i-1} \coloneqq (G, \chi_1, \dots, \chi_n)$ is an n-precontraction group. Moreover, by the Hahn embedding theorem we know G_{i-1} embeds into G_i as ordered abelian groups, and from the way we defined the contractions on G_i, we see that this embedding is also an $\mathcal{L}_n$-homomorphism, thus $\mathcal{G}_{i-1} \subseteq \mathcal{G}_i$. Furthermore, note that $\chi_n(G_{i+1}) \subseteq G_i$

Given some i and $a \in G_i$, the preimages of a under χ_i are added in either $\mathcal{G}_{i+1}$ or $\mathcal{G}_{i+2}$, thus all the contractions must be surjective on $\mathcal{K}$. Finally, note that $\chi_n(G_{i+1}) \subseteq G_i$ for all $i < \omega$, thus for all $k \in G_i$, we have $\chi_n^i(k) \in G$, thus for all $k \in K$, there exists some $i \in K$ such that $\chi_n^i(k) \in G$. □

Lemma 3.4.5. $(\Phi)_{n-1} \implies (\Phi)_n$

Proof. Embed some n-precontraction group $\mathcal{G}$ into some n-contraction group $\mathcal{K}$. We define some strictly increasing chain of n-precontraction groups $\mathcal{G}_i$ all contained in $\mathcal{K}$. Since $\mathcal{K} \models T_n$, after atmost $|K|^+$ steps, the resulting structure must also be a model of T_n.

Set $\mathcal{G}_0 \coloneqq \mathcal{G}$. Assume we have some n-precontraction group $\mathcal{G}_\alpha$ with the factoring property over $\mathcal{G}_\beta$ for all $\beta < \alpha$, in particular $\mathcal{G}_\alpha$ embeds into $\mathcal{K}$ over $\mathcal{G}_\beta$.

When α is odd, set $\mathcal{G}_{\alpha+1}$ to be some $n-1$-contraction hull of $\mathcal{G}_\alpha$, by Lemma 3.4.3, $\mathcal{G}_\alpha$ is an n-precontraction group with the factoring property over $\mathcal{G}_\alpha$ with respect to T_n.

For even α, pick some $a \in G_\alpha^{>0} \setminus \chi_n(G_\alpha)$. There exists some $b \in K$ such that $\chi_n(b) = a$. Set the domain of $\mathcal{G}_{\alpha+1}$ to be:

$$G_{\alpha+1} \coloneqq G_\alpha + \sum_{\overline{r}\in\mathbb{N}^n} \mathbb{Z}\overline{\chi}^{\overline{r}}(b)$$

Where for some tuple $\overline{r} \coloneqq (r_1, \dots, r_n) \in \mathbb{N}^n$, we define $\overline{\chi}^{\overline{r}}(x)$ as:

$$\overline{\chi}^{\overline{r}}(x) \coloneqq \chi_1^{r_1} \circ \dots \circ \chi_n^{r_n}(x)$$

Note that by (**C3**$_n$), each element of $\{\overline{\chi}^{\overline{r}}(b) \mid \overline{r} \in \mathbb{N}^n\}$ has distinct valuations, and for any $i \in [1, n]$:

$$\begin{aligned}
\chi_i(\overline{\chi}^{\overline{r}})(b) &= \chi_i \circ \chi_1^{r_1} \circ \dots \circ \chi_i^{r_i} \circ \dots \circ \chi_1^{r_1} \circ (b) \\
&= \chi_i^{r_i+1} \circ \dots \circ \chi_1^{r_1}(b) \\
&= \overline{\chi}^{\overline{s}}(b) \in G_{\alpha+1}
\end{aligned}$$

Thus $G_{\alpha+1}$ is closed under all contractions, hence is an n-precontraction group. It remains to show $\mathcal{G}_{\alpha+1}$ has the factoring property over $\mathcal{G}_\alpha$. Let $\mathcal{H}$ be an arbitrary n-contraction group, so we have an $\mathcal{L}_n$-embedding $\mathcal{G}_\alpha \hookrightarrow \mathcal{H}$. Pick some $\overline{b} \in H$ with $\chi_n(b) = a$, then the sets:

$$B \coloneqq \{\overline{\chi}^{\overline{r}}(b) \mid \overline{r} \in \mathbb{N}^n\} \qquad\qquad \overline{B} \coloneqq \{\overline{\chi}^{\overline{r}}(\overline{b}) \mid \overline{r} \in \mathbb{N}^n\}$$

together satisfy the conditions of Lemma 2.3.13, thus we have an ordered abelian group isomorphism:

$$\phi : G_\alpha + \sum_{\overline{r}\in\mathbb{N}^n} \mathbb{Z}\overline{\chi}^{\overline{r}}(b) \to G_\alpha + \sum_{\overline{r}\in\mathbb{N}^n} \mathbb{Z}\overline{\chi}^{\overline{r}}(\overline{b}) : \overline{\chi}^{\overline{r}}(b) \mapsto \overline{\chi}^{\overline{r}}(\overline{b})$$

The usual calculations further show ϕ is an $\mathcal{L}_n$-homomorphism, thus $\mathcal{G}_{\alpha+1}$ embeds into $\mathcal{H}$ over $\mathcal{G}_\alpha$.

For limit ordinals α set $\mathcal{G}_\alpha$ to be:

$$\bigcup_{\beta<\alpha} \mathcal{G}_\beta$$

Then there must be some ordinal $\gamma < |K|^+$ such that $\mathcal{G}_\gamma$ is a T_n-hull of $\mathcal{G}$. Given an arbitrary T_n-hull $\overline{\mathcal{G}}$ of $\mathcal{G}$, we know $\overline{\mathcal{G}}$ embeds into $\mathcal{K}$, thus for all $\overline{g} \in G_\gamma$ there exists some $k \in \omega$ such that $\chi_n^k(\overline{g}) \in G$. □

3.4.2 Existential Closedness

Given two n-contraction groups $\mathcal{G} \subseteq \mathcal{H}$, and some $b \in H \setminus G$, we need to construct some sequence $(b_i)_i$ which gives a generating set for $\mathcal{G}\langle b\rangle$ as an abelian group. Again, we induct on the number of contractions, so the base case is just the regular characteristic sequence for 1-contraction groups, then given the $n-1$-sequence, so a generating set for $(G, \chi_1, \ldots, \chi_{n-1})\langle b\rangle$, we construct the n-characteristic sequence.

Definition 3.4.6. We define n-characteristic sequences as follows. For $n = 1$, an n-characteristic sequence of some b is any 1-characteristic sequence of b in (G, χ_1). For some $n > 1$, an n-characteristic sequence of b in $(G, \chi_1, \ldots, \chi_n)$ is some sequence $(b_i, g_i)_{i<\omega^n}$ such that:

(I)$_n$ For any ordinal α, the sequence $(b_{\alpha+i}, g_{\alpha+i})_{i<\omega^{n-1}}$ is an $(n-1)$-characteristic sequence of b_α in $(G, \chi_1, \ldots, \chi_{n-1})$, in particular this holds for $\alpha = \omega^{n-1}j$ where $j \in \omega$.

(II)$_n$ Fix some $j \in \omega$, if b_{ω^j} is $(n-1)$-super-transcendental then:

$$b_{\omega(j+1)} = \chi_n(b_{i+\gamma})$$

where γ is the point from which $(g_{\omega j+i})_{i<\omega^{n-1}}$ is null. We say γ is the $(n-1)$**-null point of** b.

(III)$_n$ Otherwise, if $b_{\omega j}$ is $(n-1)$-algebraic or $(n-1)$-transcendental but not $(n-1)$-super-transcendental, set $b_{\omega(j+1)} = 0$.

The existence of n-characteristic sequences is evident from the definition. If $b_\alpha = 0$ for some $\alpha < \omega^n$, we say b is n**-algebraic**, and has n**-characteristic sequence length** α, where α is the least such. Otherwise we say b is n**-transcendental**, and has n**-characteristic sequence length** ω^n. If b is n-transcendental but $(g_i)_{i<\omega^n}$ is eventually zero, we say it is n**-super-transcendental**. These are well defined because:

For the rest of this section fix some $b \in H \setminus G$.

Lemma 3.4.7. *The n-characteristic sequences of b in $\mathcal{G}$ satisfy the following:*

(i) For any two n-characteristic sequences $(b_i, g_i)_{i\in\omega^n}$ and $(\bar{b}_i, \bar{g}_i)_{i\in\omega^n}$ and $\alpha < \omega^n$, we have:

$$[b_\alpha] = [\bar{b}_\alpha]$$

(ii) If b has an n-characteristic sequence of length α, then all its n-characteristic sequences have length α (thus the n-characteristic length of b is well-defined, hence so are the definitions of n-algebraic, n-transcendental and n-super-transcendental).

(iii) If $(b_i, g_i)_{i<\omega^n}$ *is a* n*-characteristic sequence, then for all* $\alpha < \omega^n$*, the sequence* $(b_{\alpha+i}, g_{\alpha+i})_{i<\omega^n}$ *is an* n*-characteristic sequence of* b_α*.*

Proof. Identical to Lemma 3.2.9 and Lemma 3.3.9. ☐

Lemma 3.4.8. *The* n*-characteristic sequences of* b *in* $\mathcal{G}$ ***also*** *satisfy the following:*

(i) For all ordinals $\alpha < \beta$ *with* $b_\beta \neq 0$ *we have* $[b_\alpha] > [b_\beta]$

(ii) If b *is* n*-transcendental,* ***or*** n*-algebraic with an* n*-characteristic sequence of limit ordinal length, then for all* $\alpha < \omega^n$*,*

$$b_\alpha \neq 0 \implies [b_\alpha] \notin [G]$$

(iii) If b *is* n*-algebraic with an* n*-characteristic length* α*, where* α *is a successor ordinal, then for all* $\beta < \alpha$*,* $[b_\beta] \notin [G]$ *and*

$$[b_\alpha] \in [G] \iff [G + \mathbb{Z}b_\alpha] = [G]$$

Thus the ordered abelian group generated by G *and the elements of the* n*-characteristic sequence has natural valuation:*

$$\left[G + \sum_{i<\omega^n} \mathbb{Z}b_i\right] = G \cup \{[b_\beta] \mid \beta < \alpha\}$$

where α *is the length of any* n*-characteristic sequence.*

Proof. For the first, note that $[b_\alpha] > [b_{\alpha+1}]$ from the construction of 1-characteristic sequences, and $[b_\alpha] > [b_{\alpha+\omega^k}]$ for all $k < n$, since $b_{\alpha+\omega^k} = \chi_{k+1}(b_{\alpha+\gamma})$ for some $\gamma < \omega^k$. Points (ii) and (iii) follow from Lemma 3.4.7 and the construction of n-characteristic sequences. The statement about the natural valuation of the group generated by G and the n-characteristic sequence follows from the fact that the valuations of $[b_\alpha]$ are all distinct from each other and $[G]$ for α less than the n-characteristic length of b. ☐

Lemma 3.4.9. *Fix some* n*-characteristic sequence* $(b_i, g_i)_{i<\omega^n}$*. Let* $\alpha < \beta < \omega^n$ *be ordinals. Suppose* $g_\gamma = 0$ *for all* γ *satisfying* $\alpha < \gamma \leq \beta$*. If we write* β *as:*

$$\beta = \alpha + \omega^{n-1}c_{n-1} + \ldots + \omega c_1 + c_0$$

where $c_i \in \omega$*, then* b_β *can be written as:*

$$\chi_1^{c_0} \circ \ldots \circ \chi_n^{c_{n-1}}(b_\alpha)$$

Proof. For any k with $1 \leq k \leq n$, let Ψ_k be:

Ψ_k For $\overline{b} \in H \setminus G$ with k-characteristic sequence $(\overline{b}_i, g_i)_{i<\omega^k}$, and any ordinals $\gamma < \eta < \omega^k$ with η written as:

$$\gamma + \omega^{k-1} d_{k-1} + \ldots + \omega d_1 + d_0$$

where $d_0, \ldots, d_{k-1} \in \omega$, if $g_i = 0$ for all $i \in (\gamma, \eta]$, then we can write b_η as:

$$\chi_1^{d_0} \circ \ldots \chi_k^{d_{k-1}}(b_\gamma)$$

It is sufficient to prove Ψ_k for all $k \in [1, n]$. Ψ_1 follows from the definition of 1-characteristic sequences, so assume Ψ_{k-1}. Fix some $\alpha < \beta < \omega^k$ with:

$$\beta := \alpha + \omega^{k-1} c_{k-1} + \ldots + \omega c_1 + c_0$$

where $c_0, \ldots, c_k \in \omega$, and suppose $g_i = 0$ for all i with $\alpha < i \leq \beta$. Let $\overline{\beta} := \alpha + \omega^{k-1} c_{k-1}$, then we can write β as:

$$\beta = \overline{\beta} + \omega^{k-2} c_{k-2} + \ldots + \omega c_1 + c_0$$

then by Ψ_{k-1} applied to β and $\overline{\beta}$ we know:

$$b_\beta = \chi_1^{c_0} \circ \ldots \circ \chi_{k-1}^{c_{k-2}}(b_{\overline{\beta}})$$

But from the definition of n-characteristic sequences, we know $b_{\overline{\beta}} = \chi_k(b_\alpha)$, thus:

$$b_\beta = \chi_1^{c_0} \circ \ldots \circ \chi_k^{c_{k-1}}(b_{\overline{\alpha}})$$

□

Lemma 3.4.10. *For any $b \in H \setminus G$, the structure $\mathcal{G}\langle b \rangle$ has domain:*

$$G + \sum_{i<\omega^n} \mathbb{Z} b_i$$

where $(b_i, g_i)_{i<\omega^n}$ is any n-characteristic sequence of b.

Proof. For any $k \in [1, n]$, let Ψ_k be the statement:

For any $\bar{b} \in H \setminus G$ and k-characteristic sequence $(\bar{b}_i, g_i)_{i<\omega^k}$, the structure $(G, \chi_1, \ldots, \chi_k)\langle\bar{b}\rangle$ has domain:

$$G + \sum_{i<\omega^k} \mathbb{Z}\bar{b}_i$$

We know Ψ_1 is true from Lemma 3.2.10, so assume for induction that Ψ_{k-1} holds for some k. Fix some $b \in H \setminus G$ and some k-characteristic sequence $(b_i, g_i)_{i<\omega^k}$. By Lemma 3.4.8, to prove Ψ_k it is sufficient to show:

$$\chi_j(b_i) \in G + \sum_{i<\omega^k} \mathbb{Z}b_i$$

for all $i < \omega^k$ and $j \in [1, k]$. By Ψ_{k-1}, we know this is true for all $j < k$ and $i < \omega^k$. Fix some $\alpha < \omega^k$, then $(b_{\alpha+i}, g_{\alpha+i})_{i\in\omega^{k-1}}$ is an $(k-1)$-characteristic sequence of b_α.

1. Suppose b_α is not $(k-1)$-super-transcendental.

   (a) Suppose there exists some $\beta \in (\alpha, \alpha + \omega^{k-1})$, with $g_\beta \neq 0$, we can assume β is the least such. Then there exists some $\gamma \in [\alpha, \beta)$ and $j \in [1, k-1]$ such that

   $$b_\beta = \chi_j(b_\gamma) - g_\beta$$

   By the construction of characteristic sequences, we know $[\chi_j(b_\gamma)] = [g_\beta]$. Suppose we write γ as:

   $$\gamma = \alpha + \omega^{k-2}c_{k-2} + \ldots + \omega c_1 + c_0$$

   Then by Lemma 3.4.9, we can write b_γ as:

   $$b_\gamma = \chi_{k-1}^{c_{k-2}} \circ \ldots \circ \chi_1^{c_0}(b_\alpha)$$

   With this, we can now calculate $\chi_n(b_\gamma)$:

   $$\begin{aligned} \chi_k(b_\alpha) &= \chi_k \circ \chi_j \circ \chi_{k-1}^{c_{k-2}} \circ \ldots \circ \chi_1^{c_0}(b_\alpha) \\ &= \chi_k(\chi_j(b_\gamma)) \\ &= \chi_k(g_\beta) \in G \end{aligned}$$

   (b) However, if there is no $\beta \in (\alpha, \alpha + \omega^{k-1})$ with $g_\beta \neq 0$, then since b_α **is not** $(k-1)$-super-transcendental, it must be $(k-1)$-algebraic. By the same argument as the previous case, we see that $\chi_n(b_\alpha) \in G$.

2. If b is $(k-1)$-super-transcendental, then if there exists some $\beta \in (\alpha, \alpha + \omega^{k-1})$ with $g_\beta \neq 0$, then repeat (1a). Otherwise, by definition of k-characteristic sequences, we must have $\chi_k(b_\alpha) = b_{\alpha+\omega^{k-1}}$, thus $\chi_n(b_\alpha)$ is in the group generated by the characteristic sequence.

□

From now on fix some n-characteristic sequence $(b_i, g_i)_{i<\omega^n}$. Let us further extend Definition 3.2.11, and Definition 3.3.12 to n-contraction groups.

Definition 3.4.11. Let x_α be the term obtained by replacing the occurrence of b in b_α with the variable x, so:

$$x_\alpha := \begin{cases} 0 & \text{If } b_\alpha = 0 \\ x - g_\alpha & \text{If } \alpha = 0 \\ \chi_1(x_{\alpha-1}) - g_\alpha & \text{If } \alpha \text{ is a successor ordinal and } b_\alpha \neq 0 \\ \chi_k(x_\beta) - g_\alpha & \text{If } (*) \end{cases}$$

Where (*) asserts that α is a limit ordinal, $b_\alpha \neq 0$ and $\beta \in [\omega n, \omega(n+1))$ is past the null point of $b_{\omega n}$

Definition 3.4.12. Define the type p_b as:

$$p_b(x) := \bigcup_{\alpha \in \omega^n} D_{b_\alpha}(x_\alpha) \tag{3.16}$$

Where $D_h(y)$ is defined in Definition 3.2.11.

Lemma 3.4.13. *$p(x)$ is a type of $\mathcal{G}$.*

Proof. Let b have an n-characteristic sequence of length α. Pick some finite subset Δ of p_b. If α is a successor ordinal, then we can find some finite sequence of ordinals $\alpha_1, \ldots, \alpha_n, \alpha$, such that Δ can be written as:

$$\Delta(x) := \Delta_1(x) \wedge \Delta_2(x)$$

Where Δ_i is defined as:

$$\Delta_1(x) := \bigwedge_{i=1}^{n} a_{\alpha_i} < x_{\alpha_i} < \bar{a}_{\alpha_i}$$
$$\Delta_2(x) := x_\alpha = b_\alpha$$

By the same arguments as in Lemmas (3.2.12) and (3.3.15), we see that Δ is finitely realised in $\mathcal{G}$. Similarly, if α is a limit ordinal, then we can write Δ as just Δ_1, and again by the same arguments as the aforementioned lemmas, we see that Δ still must be realised in $\mathcal{G}$.

□

Lemma 3.4.14. *For any other n-contraction group $\overline{\mathcal{H}}$ with $\mathcal{G} \subseteq \overline{\mathcal{H}}$, and $\bar{b} \in \overline{H} \setminus G$ with $\bar{b} \models p_b(x)$, there exists an $\mathcal{L}_n$ isomorphism ϕ over $\mathcal{G}$:*

$$\phi : \mathcal{G}\langle b\rangle_{\mathcal{H}} \to \mathcal{G}\langle \bar{b}\rangle_{\overline{\mathcal{H}}} \tag{3.17}$$

Proof. Apply Lemma 2.3.13 and Lemma 3.4.10 in the same way as in the proofs of Lemma 3.2.13 and Lemma 3.3.16. □

Thus by Fact 2.1.12 applied to Lemma 3.4.13 and Lemma 3.4.14, we have:

Proposition 3.4.15. *$\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b\rangle$.*

3.4.3 Initial structures

Lemma 3.4.16. *There is some n-contraction group that embeds into all n-contraction groups.*

Proof. Consider the set $\coprod_{\omega^n} \mathbb{R}$, where

$$\chi_i(1[\omega^{n-1}c_n + \ldots + \omega^{i-1}c_i + \ldots c_1]) = 1[\omega^{n-1}c_n + \ldots + \omega^{i-1}(c_i + 1)]$$

For any n-contraction group $(G, \chi_1, \ldots, \chi_n)$ and $a \in \mathcal{G}$, we see that $(\coprod_{\omega^n} \mathbb{R}, \chi_1, \ldots, \chi_n) \cong (0, \chi_1, \ldots, \chi_n)\langle a\rangle$. So we just take the n-contraction hull, and the resulting structure embeds into all n-contraction groups. □

3.4.4 Final Result

By Theorem 2.1.14, combined with Lemma 3.4.5, Proposition 3.4.15 and Lemma 3.4.16, we get:

Theorem 3.4.17. *The theory of n-contraction groups has quantifier elimination, has a prime model and is complete.*

3.4.5 Countable Models

In the paper Krapp and S. Kuhlmann, 2023, the authors construct an explicit example of a centripetal contraction group.

Theorem 3.4.18 (Krapp and S. Kuhlmann, 2023, p. 21)**.** *There is an explicit construction of a countable centripetal contraction group.*

Proof. Let V be the Hahn **sum**:

$$\coprod_{\mathbb{Q}} \mathbb{Q}$$

Then V is countable and $[V]$ is isomorphic to $(\mathbb{Q}, <)$. Since any two linear orders are isomorphic, there exists an order isomorphism $\sigma : \mathbb{Q} \to V^{>0}$. The map

$$\chi : V^{>0} \to V^{>0} : 1_q \mapsto \sigma(q) : (r_i)_i \mapsto \sigma(\max \operatorname{supp}(r_i))$$

extended to V in the expected way, is a contraction on V, but not necessarily centripetal. To make it so, we need to choose σ in a particular way. For all $z \in \mathbb{Z}$ set $\sigma(z) := 1_{z-1}$, and extend σ from $(z, z+1)$ to $(1_{z-1}, 1_{z-2})$ via any order isomorphism. Then for $q \in [z, z+1]$, we have $\chi(1_q) \in [1_{z-1}, 1_{z-2}] < 1_{q-1}$, thus χ is centripetal. □

We can extend this idea very easily to n-contraction groups.

Theorem 3.4.19. *For all $n \in \omega$, there exists a countable n-contraction group.*

Proof. Let V be the Hahn **sum**:

$$\coprod_{(\mathbb{Q}^n, <_{lex})} \mathbb{Q}$$

For some $i \in [1, n]$, define the map $\sigma_i : \mathbb{Q}^{n-i} \times \mathbb{Z} \to V^{>0}$ as:

$$\sigma_i(q_n, \ldots, q_{i+1}, z) = 1_{q_n, \ldots, q_{i+1}, z-10}$$

for $q_j \in \mathbb{Q}$ and $z \in \mathbb{Z}$. Extend σ to $\mathbb{Q}^{n-i+1}$ in any order preserving way, then σ must be a surjection, since the image of σ can get arbitrarily big in $V^{>0}$. Define $\chi_i : V^{>0} \to V^{>0}$ as:

$$\chi_i(1_{q_n, \ldots, q_1}) = \sigma(q_n, \ldots, q_i)$$

Then each χ_i is a centripetal contraction, once we extend them to the entirety of V. It remains to verify ($\mathbf{C}2_n$) and ($\mathbf{C}3_n$).

($\mathbf{C}2_n$): Fix some $\overline{q} := q_n, \ldots, q_1 \in \mathbb{Q}^n$. Note that for all $k \in \mathbb{N}$, $\chi_{i-1}^k(1_{\overline{q}}) \in A = 1_{q_n, \ldots, q_i, \mathbb{Q}, 0, \ldots, 0}$, but $\chi_i(A) = \sigma_i(q_n, \ldots, q_i)$ is a single point, thus $\chi_i(\chi_{i-1}^k(1_{\overline{q}})) = \chi(1_{\overline{q}})$.

($C3_n$): Fix some $\overline{q} \in \mathbb{Q}^n$ as before, and remember that for all $k \in \mathbb{N}$, $\chi_{i-1}^k(1_{\overline{q}}) \in A$. But $\chi_i(1_{\overline{q}}) = \sigma(q_n, \ldots, q_i) < 1_{q_n,\ldots,q_i-5,0,\ldots,0} < A$. Thus $\chi_i(1_{\overline{q}}) < \chi_{i-1}^k(1_{\overline{q}})$. □

3.4.6 Infinitely many contractions

Definition 3.4.20. Let $\mathcal{L}_\alpha$ be the language $\langle +, -, 0, <, \{\chi_i\}_{i<\alpha} \rangle$ be the language of ordered abelian groups along with n unary functions, where α is an ordinal. We will call a structure $(\mathcal{G}, +, -, 0, <, \{\chi_i\}_{i<\alpha})$, (or just $(G, (\chi_i)_{i<\alpha})$) an **n-precontraction group** if:

$C1_\alpha$ For all $i \leq \alpha$, the structure $(\mathcal{G}, \chi_i)$ is a centripetal precontraction group

$C2_\alpha$ For all $j < i < \alpha$ and $x \in \mathcal{G}$, we have $\chi_i(\chi_j(x)) = \chi_i(x)$

$C3_\alpha$ For all $j < i < \alpha$, $k \in \mathbb{N}$ and $x \in \mathcal{G}$, $x > 0$, we have $\chi_i(x) < \chi_j^k(x)$

Furthermore, if all the maps χ_i are surjective, we will call $(G, (\chi_i)_{i<\alpha})$ an **α-contraction group**. Let T_α be the theory of α-contraction groups.

Since the reduct of an α-contraction group to finitely many contractions will be a model of T_n, we get:

Theorem 3.4.21. *For any ordinal α, T_α has quantifier elimination.*

We could also replace α with any linearly ordered set and get the same result.

3.5 Weak o-minimality

It was proved by F-V Kuhlmann that contraction groups are weakly-o-minimal:

Theorem 3.5.1 (F.-V. Kuhlmann, 1995)**.** *The theory of divisible centripetal contraction groups is weakly o-minimal.*

We should mention that the proof was originally published in F.-V. Kuhlmann, 1995, but a more complete version can be found in the appendix of S. Kuhlmann, 2000.

The goal of this section is to show that for any n, the theory of n-contraction groups is weakly o-minimal. **From now on we fix some $n \in \mathbb{N}$ and work within a model $\mathcal{G} := (G, \chi_1, \ldots, \chi_n)$ of T_n!**

Notation 3.5.2. For any two contractions $\eta_1, \eta_2 \in \{\chi_1, \ldots, \chi_n\}$, with $\eta_1 := \chi_i$ and $\eta_2 := \chi_j$, we say $\eta_1 > \eta_2$ if and only if $i > j$.

3.5.1 χ_n-monomials and polynomials

To examine definable sets in some contraction group, we need a concise way to describe unary functions in our theory. For 1-contraction groups, this was done using χ-polynomials, which essentially look like nested applications of χ and translations. A **χ-monomial** (F.-V. Kuhlmann, 1995, p. 8) is a function of the form:

$$\chi[a_1,\ldots,a_k](x) = \chi(\chi\ldots(\chi(x-a_1)-\ldots-a_{k-1})-a_k)$$

And a χ-**polynomial** is a term of the form:

$$\sum_{i=1}^{k} z_i\chi[a_1,\ldots,a_i](x) + z_0x + c$$

The proof of weak o-minimality for 1-contraction groups involved showing that every definable function is piecewise equal to a χ-polynomial, then showing that χ-polynomials were monotonic (on some sufficient region). For n-contraction groups, we will do the same.

Definition 3.5.3. For some $a \in G$, define $\mathcal{M}^a, \mathcal{O}^a$ as follows:

$$\mathcal{M}^a := \{x \in G \mid [x] < [a]\}$$
$$\mathcal{O}^a := \{x \in G \mid [x] \leq [a]\}$$

So $\mathcal{M}^a$ is the largest convex subgroup of G not containing a, and $\mathcal{O}^a$ is the smallest convex subgroup that does contain a. Given this, we can define sets of the form $d + \mathcal{O}^b, d + \mathcal{M}^b$ for some $d, b \in G$ as you would expect.

Fact 3.5.4. *For any non-zero $a, b \in G$, either $a + \mathcal{M}^a \cap b + \mathcal{M}^b = \varnothing$ or $a + \mathcal{M}^a = b + \mathcal{M}^b$*

Proof. If $a + \mathcal{M}^a \cap b + \mathcal{M}^b \neq \varnothing$, then we must have $[a] = [b]$ since for all $x \neq 0$, $x + \mathcal{M}^x$ is contained within $[x]$. Fix some $z \in a + \mathcal{M}^a \cap b + \mathcal{M}^b$, and let $a + \gamma \in a + \mathcal{M}^a$, so $[\gamma] < [a]$, then:

$$\begin{aligned}
[b-(a+\gamma)] &= [(b-z)+(z-a)+\gamma] \\
&\leq \max\{[b-z],[z-a],[\gamma]\} && \text{By the ultrametric inequality} \\
&< [b] && \text{Since } [b-z],[z-a] \text{ and } [\gamma] \text{ are less than } [b]
\end{aligned}$$

Hence $a + \gamma \in b + \mathcal{M}^b$, so $a + \mathcal{M}^a \subseteq b + \mathcal{M}^b$, and by a symmetrical argument we also have $b + \mathcal{M}^b \subseteq a + \mathcal{M}^a$. □

Definition 3.5.5. The χ_n-monomials are a collection of **terms** which we define inductively:

- x is a χ_n-monomial, with characteristic domain $C(x) := \mathcal{G}$ and length $l(x) = 0$
- If $t(x)$ is a χ_n-monomial with characteristic domain $C(t(x))$, then for any $i \leq n$, $\chi_i(t(x))$ is a χ_n-monomials with characteristic domain $C(t)$, and length $l(t) + 1$.
- If $t(x)$ is a χ_n-monomial with characteristic domain $C(t)$, then for any $i \leq n$, $\chi_i(t(x) - a)$ is a χ_n monomial with characteristic domain $C(t(x)) \cap t^{-1}(a + \mathcal{M}^a)$

(this can sometimes be empty!), and length $l(t)+1$. We denote the term $\chi_i(x-a)$ as $\chi_i[a](x)$.

Note that a χ_n-monomial is a term and not a function! We list some examples:

Example 3.5.6. Let us again stress that χ_n-monomials are defined **syntactically**, so two χ_n-monomials which have the same characteristic domain and are equal as functions on the characteristic domain are still different χ_n-monomials.

- $\chi_2(x-a)$ has characteristic domain $a+\mathcal{M}^a$ and length 1
- $\chi_2(\chi_1(x-a))$ has characteristic domain $a+\mathcal{M}^a$ and length 2. Note that this term considered as a function on $\mathcal{G}$ takes the fact same values as $\chi_2(x-a)$, but since χ_n-monomials are terms, both are distinct as χ_n-monomials (but have the same characteristic domains!).
- For some sequence of contractions $\delta_1,\ldots,\delta_m$, and constants $a_1,\ldots,a_m \in G^{\neq 0}$, the term:

$$\delta_m[a_m]\circ\ldots\circ\delta_1[a_1](x)$$

is a χ_n-monomial of length m, and characteristic domain:

$$(a_1+\mathcal{M}^{a_1})\cap\delta_1[a_1]^{-1}(a_2+\mathcal{M}^{a_2})\cap\ldots\cap\delta_{m-1}[a_{m-1}]^{-1}(a_m+\mathcal{M}^{a_m})$$

The motivation behind the definition of characteristic domains is that outside it, a χ_n-monomial is piecewise equal to some ‘simpler’ χ_n-monomial. We will make this idea more precise in Lemma 3.5.10, but for now consider the monomial $\chi_1(x-a)$, if $0<x<a+\mathcal{M}^a$, then $v(x-a)=v(a)$, so $\chi_1(x-a)=\chi_1(a)$, hence the monomial is constant. Conversely, if $x>a+\mathcal{M}^a>0$, then $v(x-a)=v(x)$, hence $\chi_1(x-a)=\chi_1(x)$.

Definition 3.5.7. Fix some sequence of contractions $\delta_1,\ldots\delta_k$, and constants $a_1,\ldots,a_m\in G$. A **successive sequence of χ_n-monomials** is some sequence $(f_i(x))_{i\leq k}$ defined as:

$$f_i(x) \coloneqq \delta_i[a_i]\circ f_{i-1}(x)=\delta_1[a_i]\circ\delta_{i-1}[a_{i-1}]\circ\ldots\circ\delta_1[a_1](x) \tag{3.18}$$

where $\delta_i[a_i] \coloneqq \delta_i(x-a_i)$. Given such a sequence, with $z_i\in\mathbb{Q}, z_k\neq 0$ and $c\in G$, a χ_n-**polynomial** is a **term** of the form:

$$f(x)=\sum_{i=0}^{k} z_i f_i(x)+c=\sum_{i=0}^{k} z_i\delta_i[a_i]\circ\ldots\circ\delta_1[a_1](x)+c$$

Observe that $C(f_k)\subseteq C(f_{k-1})\subseteq\ldots\subseteq C(f_0)$, so we set the characteristic domain and length of f to be the same as f_k. Unless otherwise stated, f and g will denote the χ_n

polynomials:

$$f(x) = \sum_{i=0}^{k} z_i f_i(x) \qquad\qquad g(x) = \sum_{i=0}^{l} \overline{z}_i g_i(x) \tag{3.19}$$

$$f_i(x) = \chi_{n_i}(f_{i-1}(x) - a_i) \qquad\qquad g_i(x) = \chi_{m_i}(g_{i-1}(x) - b_i)$$

where $z_i, \overline{z}_i \in \mathbb{Z}$ and $a_i, b_i \in G$. We call $(f_i), (g_i)$ the **terms** of a χ_n-polynomial, and we say the terms of f and g **coincide** up to the d-th term if f_i and g_i are the χ_n-monomial for all $i \leq d$.

Definition 3.5.8. We can consider any two $\mathcal{L}_n$-terms with only one free-variable $t_1(x)$ and $t_2(x)$, as functions on G. We say they are equal on G, denoted as $t_1 \equiv t_2$ if and only if they take the same values of each element of G. Given some $M \subseteq G$, we write $t_1 \equiv_M t_2$ if and only if they take the same values on each element of M. So to clarify:

$$t_1 \equiv t_2 \iff \text{For all } x \in G,\ t_1(x) = t_2(x)$$
$$t_1 \equiv_M t_2 \iff \text{For all } x \in M,\ t_1(x) = t_2(x)$$

Example 3.5.9. Assuming all the a_i are different constants and δ_i's are different contractions, the following **are not** χ_n polynomials:

$$x + \chi_1[a_1] \circ \chi_2[a_2](x) + \chi_2[a_2] \circ \chi_1[a_1](x)$$
$$x + \chi_2[a_1] \circ \chi_1(x) + \chi_2[a_1](x)$$

The following **is** a χ_n-polynomial:

$$x + \chi_1(x - a_1) + \chi_2(\chi_1(x - a_1) - a_2) - \chi_1(\chi_2(\chi_1(x - a_1) - a_2) - a_3)$$

As stated earlier, the reason for defining characteristic domains is that outside of its domain, a χ_n-polynomial has a different form. The following lemma makes this idea precise.

Lemma 3.5.10. *Let $f(x)$ be a χ_n-**polynomial** with characteristic domain C_f. There is a finite convex partition $\mathcal{P}$ of $G \setminus C_f$ such that for each $M \in \mathcal{P}$, there exists some χ_n-polynomial F_M with $M \subseteq C(F_M)$ and $f \equiv_M F_M$.*

Proof. Assume f has the same form as (3.19). We construct a sequence $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_k$ of finite convex partition of $G \setminus C(f)$ such that $\mathcal{P}_i$ is a refinement of $\mathcal{P}_{i+1}$, and $\mathcal{P}_k$ will be our desired partition. To simplify the proof, assume $a_i \neq 0$ for all $i \leq k$, otherwise $C(f_i) = C(f_{i-1})$, so we do not need to do any partitioning at the i-th step.

Define the following partition $\mathcal{P}_1$ of $M \coloneqq G \setminus C(f)$:

$$M_1 \coloneqq \{x \in M \mid |x| > |a_1| + \mathcal{M}^{a_1} \wedge \operatorname{sign}(x) = \operatorname{sign}(a_1)\}$$
$$M_2 \coloneqq \{x \in M \mid x \in a_1 + \mathcal{M}^{a_1}\}$$
$$M_3 \coloneqq \{x \in M \mid \ |x| < a_1 + \mathcal{M}^{a_1}\}$$
$$M_4 \coloneqq \{x \in M \mid |x| \geq |a_1| + \mathcal{M}^{a_1} \wedge \operatorname{sign}(x) = -\operatorname{sign}(a_1)\}$$

Note that each M_i is a convex set, except in the case where C_f bisects one of them. In that case, we split the bisected set, M_i into M_i^-, M_i^+, into the convex components above and below C_f (one or both may be empty!, e.g if $f(x) \coloneqq \chi_1(x - a_1)$). This can only happen at the first step, and we still end up with a finite convex partition, so we can safely ignore this case.

Fix some sequence $\delta = (\delta_1, \ldots, \delta_{i-1}) \in \{1, 2, 3, 4\}^{i-1}$, and suppose we have constructed some M_δ and some successive sequence of χ_n-monomials $(F_j^\delta)_{j<i}$ (whose elements will be denoted denote as F_j), such that $f_j \equiv_{M_\delta} F_j$ for all $j < i$ and F_i has characteristic domain containing M_δ. From this, we construct $\{M_{\delta,j}\}_{j=1,2,3,4}$ as follows:

$$M_{\delta,1} \coloneqq \{x \in M_\delta \mid |\chi[a_{i-1}](x)| > |a_i| + \mathcal{M}^{a_i} \wedge sign(\chi[a_{i-1}](x)) = sign(a_i)\}$$
$$M_{\delta,2} \coloneqq \{x \in M_\delta \mid \chi[a_{i-1}](x) \in a_i + \mathcal{M}^{a_i}\}$$
$$M_{\delta,3} \coloneqq \{x \in M_\delta \mid \ |\chi[a_{i-1}](x)| < a_i + \mathcal{M}^{a_i}\}$$
$$M_{\delta,4} \coloneqq \{x \in M_\delta \mid |\chi[a_{i-1}](x)| \geq |a_i| + \mathcal{M}^{a_i} \wedge sign(\chi[a_{i-1}](x)) = -sign(a_i)\}$$

We should examine how f_i behaves on each:

- $x \in M_{\delta,1}, M_{\delta,4}$: For any such x, we have $[F_{i-1}(x) - a_i] = [F_{i-1}(x)]$, hence $f_i(x) = \chi_{n_i}(F_{i-1}(x))$, and the characteristic domain of $\chi_{n_i}(F_{i-1}(x))$ is the same as the characteristic domain of $F_{i-1}(x)$,
- $x \in M_{\delta,2}$: Evidently $f_i(x) = \chi_{n_i}(F_{i-1}(x) - a_i)$. Since M_δ is within the characteristic domain of F_{i-1}, by definition $M_{\delta,2}$ is contained in the characteristic domain of $\chi_{n_i}(F_{i-1}(x) - a_i)$.
- $x \in M_{\delta,3}$: Note that $[F_{i-1}(x) - a_i] = [a_i]$, hence f_i is constant.

Using this inductive process, we can construct $\mathcal{P} = \{M_\delta \mid \delta \in \{1, 2, 3, 4\}^k\}$ which is a convex partition of $G \setminus C(f)$ (note that $M_{2,\ldots,2}$ is contained in $C(f)$!). It can be shown that on every non-empty $Q \in \mathcal{P}$, f can be written as χ_n-polynomial with characteristic domain containing Q. □

3.5.2 Strategy for proving Weak o-minimality

By quantifier elimination, to prove some structure is weakly o-minimal, it is sufficient to show that for any term $t(x)$, the sets:

$$\{x \in \mathcal{G} \mid t(x) = 0\}$$
$$\{x \in \mathcal{G} \mid t(x) > 0\}$$

can be written as a finite union of open convex sets. For that, it is sufficient to show that for any term $t(x)$, we can form a partition of G, say $\mathcal{P}$, of finitely many open convex sets such that on each element of $\mathcal{P}$, $t(x)$ is either constant or monotonic. And for that, it is sufficient to prove the following two statements:

- Every χ_n polynomial is monotonic on its characteristic domain.
- For any term $t(x)$, we can split up G into finitely many open convex sets $Q_1 \ldots Q_k$ such that for each i, there exists some χ_n-polynomial f_i with $Q_i \subseteq C(f_i)$ and $t \equiv_{Q_i} f_i$.

3.5.3 Monotonicity of χ_n polynomials

Fix some χ_n-polynomial f.

Lemma 3.5.11. *For all $x \in C(f)$, we have :*

$$[f_1(x)] > [f_2(x)] > \ldots > [f_k(x)]$$

Proof. Let $1 < i \leq k$, then:

$$\begin{aligned} [f_i(x)] &= [\chi_{n_i}(f_{i-1} - a_i)] \\ &< [f_{i-1}(x) - a_i] \\ &< [f_{i-1}(x)] \end{aligned}$$

The last inequality is true since $f_{i-1}(x) \in a_i + \mathcal{M}^{a_i}$. □

Lemma 3.5.12. *Every χ_n-monomial is monotonic on the entirety of G.*

Proof. Given some $t : G \to G$ which is monotonic on G, we know that both $\delta(t(x))$ and $t(x) - c$ are also monotonic on G, where $\delta \in \{\chi_1, \ldots, \chi_n\}$ and $c \in G$. So we can induct on the length of some χ_n-monomial to show it is monotonic. □

The following is very similar to the corresponding proof for χ-polynomials (Lemma A.45 in S. Kuhlmann, 2000, p. 146)

Lemma 3.5.13. *The χ_n-polynomial f is monotonic on its characteristic domain.*

Proof. Let $m \leq k$ be the least integer such that $z_m \neq 0$, and without loss of generality assume $z_m > 0$. Pick any $a, b \in C(f)$, we want to show that $a < b \implies f(a) \leq f(b)$. If $m = k$ then f is a χ_n-monomial so by Lemma 3.5.12 the statement is already true, so assume $0 \leq m < k$. There are three cases to consider:

1. $[f_m(a) - f_m(b)] \geq [f_{m+1}(a)]$
2. $[f_m(a) - f_m(b)] \geq [f_{m+1}(b)]$
3. $[f_m(a) - f_m(b)] < [f_{m+1}(a)], [f_{m+1}(b)]$

- Case 1: Since $f_{m+1}(a) = \delta_{m+1}(f_m(a) - a_{m+1})$, we deduce that $[f_{m+1}(a)] < [f_m(a) - a_{m+1}]$. By the ultrametric inequality, we know that any x, y and z satisfy:

$$[x - y] < [x - z] \implies [x - z] = [y - z]$$

hence by setting $x \coloneqq f_m(a), y \coloneqq f_m(b)$ and $z = a_{m+1}$ we get:

$$[f_m(a) - a_{m+1}] = [f_m(b) - a_{m+1}]$$

Which tells us $f_i(a) = f_i(b)$ for all $i > m$. Since $f_m(a) < f_m(b)$, we deduce that $f(a) \leq f(b)$.

- Case 2: This is symmetrical to case 1.
- Case 3: By Lemma 3.5.11, we have for all $i > m$:

$$[f_m(a) - f_m(b)] < [f_i(a)], [f_i(b)]$$

We know that for tuples $x_0, \ldots, x_l$ and $x'_0 \ldots, x'_l$, the valuation v satisfies:

$$x_0 < x'_0 \text{ and } [x_l - x'_l] < \max_{0<i\leq l}\{[x_i], [x'_i]\} \implies \sum_{0\leq i\leq l} x_i < \sum_{0\leq i\leq l} x'_i$$

From which the inequality $f(a) < f(b)$ follows easily.

□

Subsequently, Lemma 3.5.10 tells us that any χ_n-polynomial is piecewise monotonic on the entirety of G:

Lemma 3.5.14. *Any χ_n-polynomial is piecewise monotonic on G.*

Proof. By Lemma 3.5.10, we can partition $G \setminus C_f$ into finitely many convex sets $\mathcal{P}$ such that on each $M \in \mathcal{P}$, f is equal to some χ_n-polynomial f_M such that $M \subseteq C(f_M)$. But we know f_M is monotonic on its characteristic domain, hence monotonic on M, hence one each $M \in \mathcal{P}$, f is monotonic, hence f is piecewise monotonic on G. □

3.5.4 Every term is locally a χ_n polynomial

The goal for this section is to show that for any term $t(x)$, we can form a partition a finite open convex partition $\mathcal{P}$ of G, such that for each $M \in \mathcal{P}$, $t(x)$ is equal to some χ_n polynomial f_M, and M is a subset of the characteristic domain of f_M. We will call $t(x)$ **$\mathcal{P}$-nice** if $\mathcal{P}$ satisfies what was just stated, and call $t(x)$ **nice** if such a partition exists. We will call a χ_n-polynomial **nice on** P, for some open convex P, if we can find an open convex partition $\mathcal{P}$ of P such that f is $\mathcal{P}$-nice.

Remark 3.5.15. To show a term t is nice on some set $C \subseteq G$, it is enough to give a finite convex partition $\mathcal{Q}$ of C and show t is nice on every $Q \in \mathcal{Q}$. Also, if t is nice on C, then it is also nice on any $D \subseteq C$.

To show that every term is nice, we will induct on the terms of the language, hence it is sufficient to show for any two χ_n-polynomials $f(x), g(x)$ and contraction χ_i that both $f(x) + g(x)$ and $\chi_i(f(x))$ are nice. Let us begin with the former, with the proof building on the same proof for χ-polynomials (Lemma A.44 in S. Kuhlmann, 2000, p. 145).

Lemma 3.5.16. *Let $f(x), g(x)$ be χ_n-polynomials. Then $f(x) + g(x)$ is nice on $C := C_f \cap C_g$.*

Proof. Let d be the least integer such that f_d and g_d do not coincide (so are different χ_n-monomials). We give a finite convex partition $\mathcal{P}$ of C such that on each $M \in \mathcal{P}$, we can rewrite f, g as χ_n-polynomials which are equal up to the d-th terms. Then we partition M in the same way, so f and g coincide piecewise up to the $d+1$-th term, and repeat until $f + g$ eventually coincide piecewise up to the $\min(\text{len}(f), \text{len}(g))$-th term.

Stated precisely, suppose we have a convex subset Q of C such that there are two χ_n-polynomials F, G with terms F_i and G_i respectively, which satisfy:

- $F \equiv_Q f$, $G \equiv_Q g$ and $Q \subseteq C(F), C(G)$.
- $\text{len}(F) \geq \text{len}(f)$ and $\text{len}(G) \geq \text{len}(g)$.
- F and G have the same terms up to (and **not** including) the d-th element, so for all $i < d$ F_i is the same χ_n-monomial as G_i.

We give a finite convex partition $\mathcal{P}$ of Q such that on each $M \in \mathcal{P}$, either one of the following happens:

(I) One of F_d or G_d is constant, **or**

(II) We can find χ_n-polynomials F_M, G_M (with coefficients $z_{M,i}, \overline{z}_{M,i}$ and terms $F_{M,i}, G_{M,i}$) such that:

  (i) The terms of F_M, G_M coincide **up to and including** the d-th term

  (ii) Both the characteristic domain of F_M, G_M contain M

  (iii) The sum $(F + G)(x)$ is equal to $(F_M + G_M)(x)$ for all $x \in M$

(iv) One of the following occurs:

(II.a) $\operatorname{len}(F_M) = \operatorname{len}(F) + 1$ and $\operatorname{len}(G_M) = \operatorname{len}(G)$

(II.b) $\operatorname{len}(F_M) = \operatorname{len}(F)$ and $\operatorname{len}(G_M) = \operatorname{len}(G) + 1$

(II.c) $\operatorname{len}(F_M) = \operatorname{len}(F)$ and $\operatorname{len}(G_M) = \operatorname{len}(G)$

By repeating this process a finite number of times (consider the pair $(\operatorname{len}(F)-d, \operatorname{len}(G)-d)$, at least one will reach 0 as d increases), we eventually get a finite convex partition $\mathcal{P}$ of C such that on each $Q \in \mathcal{P}$, we set f, g equal to some F_Q, G_Q whose terms coincide up to the length of the smallest one (in other words F_Q, G_Q coincide up to the $\min(\operatorname{len}(F_Q), \operatorname{len}(G_Q))$-th term.)

So fix some χ_n-polynomials F, G (with terms $(F_i)_i, (G_i)_i$), who have the same terms up to and **not** including the d-th term, and some convex set $M \subseteq C$. Set $y(x) \coloneqq F_{d-1}(x) = G_{d-1}(x)$, and write F, G as follows:

$$F_i(x) \coloneqq \eta_i(F_{i-1}(x) - a_i) \qquad G_i(x) \coloneqq \overline{\eta}_i(G_{i-1}(x) - b_i)$$

$$F(x) \coloneqq \sum_{i=0}^{\operatorname{len}(F)} z_i F_i(x) \qquad G(x) \coloneqq \sum_{i=0}^{\operatorname{len}(G)} \overline{z}_i G_i(x)$$

For ease of notation write a_d and b_d as a and b. First, we deal with the cases when at least one of a, b is equal to zero. Note that in these cases, we do not need to partition M to get F_M, G_M equal up to and including the d-th element.

- $a = 0, b \neq 0$: We know that $y(x) \in b + \mathcal{M}^b$, hence $F_d(x) \equiv_M \eta_d(y(x)) \equiv_M \eta_d(b)$ which is constant, hence we are in scenario (I).
- $a \neq 0, b = 0$: Same as above, so we are in scenario (I)
- $a, b = 0$: Since F_d and G_d are different terms, we must have $\eta_d \neq \overline{\eta}_d$. Suppose $\eta_d > \overline{\eta}_d$, then by axiom (**C2**$_n$) we have $F_d(x) \equiv_M \eta_d\overline{\eta}_d(y(x))$. Set $G_M \coloneqq G$, and define F_M as:

$$F_{M,i}(x) \coloneqq \begin{cases} G_i(x) & \text{if } i \leq d \\ F_{i-1,d}(\overline{\eta}_d(y(x))) & \text{if } \operatorname{len}(G) + 1 \geq i > d \end{cases}$$

$$z_{M,i} \coloneqq \begin{cases} z_i & \text{if } i < d \\ 0 & \text{if } i = d \\ z_{i-1} & \text{if } \operatorname{len}(F) + 1 \geq i > d \end{cases}$$

Then we can write $F + G$ as:

$$
\begin{aligned}
(F+G)(x) &= \sum_{i=0}^{d-1}(z_i+\overline{z}_i)G_i(x) + \sum_{i=d}^{\text{len}(F)} z_iF_i(x) + \sum_{i=d}^{\text{len}(G)} \overline{z}_iG_i(x) \\
&= \sum_{i=0}^{d-1}(z_i+\overline{z}_i)G_i(x) + 0G_d + \overline{z}_dG_d + \sum_{i=d}^{\text{len}(F)} z_iF_i(x) + \sum_{i=d+1}^{\text{len}(G)} \overline{z}_iG_i(x) \\
&= \sum_{i=0}^{d}(z_{M,i}+\overline{z}_{M,i})G_{M,i}(x) + \sum_{i=d+1}^{\text{len}(F)+1} z_{M,i}F_{M,i}(x) + \sum_{i=d+1}^{\text{len}(G)} \overline{z}_{M,i}G_{M,i}(x)
\end{aligned}
$$

Hence $F+G = F_M + G_M$. Since $F_M(x) = F(x)$ for any $x \in \mathcal{G}$, F_M must have the same characteristic domain as F, and its length has only increased by one, hence we are in scenario (II.a). Similarly, if $\overline{\eta}_d > \eta_d$, then we are in case scenario (II.b).

- $a = b \neq 0$: This is identical to the previous case.

Now suppose both $a, b \neq 0$ and $a \neq b$. Since $y(x)$ lies in both $a+\mathcal{M}^a, b+\mathcal{M}^b$, Fact 3.5.4 tells us that $a+\mathcal{M}^a = b+\mathcal{M}^b$. Define M_1, M_2, M_3, M_4, M_5 as follows:

$$
\begin{aligned}
M_1 &:= M \cap y^{-1}(\{x \in a+\mathcal{M}^a \mid x < a+\mathcal{M}^{b-a}\}) \\
M_2 &:= M \cap y^{-1}(a+\mathcal{M}^{b-a}) \\
M_3 &:= M \cap y^{-1}(\{x \in a+\mathcal{M}^a \mid a+\mathcal{M}^{b-a} < x < b+\mathcal{M}^{b-a}\}) \\
M_4 &:= M \cap y^{-1}(b+\mathcal{M}^{b-a}) \\
M_5 &:= M \cap y^{-1}(\{x \in a+\mathcal{M}^a \mid x > b+\mathcal{M}^{b-a}\})
\end{aligned}
$$

Then $\{M_i \mid i = 1,2,3,4,5\}$ is a finite convex partition of M. We define new χ_n-polynomials on each M_i depending on what $y(x)$ does on each set.

- $x \in M_1$: We know $y(x) \in a+\mathcal{M}^a$, and $y(x) < a+\mathcal{M}^{b-a}$, hence we get $y(x) = a-\epsilon$, where $[a] > [\epsilon] \geq [b-a]$. This tells us that $y(x)-a = -\epsilon$ and $y(x)-b = (a-b)-\epsilon$. Using the ultrametric inequality, we can do the following:

$$
\begin{aligned}
[y(x)-a)] &= [-\epsilon] \\
&= [(a-b)-\epsilon] \\
&= [y(x)-b]
\end{aligned}
$$

  Hence $\chi(y(x)-a) = \chi(y(x)-b)$ for any contraction χ. We then split into the following cases:

  - $\eta_d = \overline{\eta}_d$: Define F_{M_1} as:

$$F_{M_1,i}(x) := \begin{cases} G_i(x) & \text{if } i \leq d \\ F_{i,d}(G_d(x)) & \text{if } \operatorname{len}(F) \geq i > d \end{cases}$$

$$z_{M_1,i} = z_i$$

and set $G_M := G$. Then F and F_{M_1} have the same characteristic domain and length, so we are in case scenario (II.c).

– $\eta_d \neq \overline{\eta}_d$: We apply the same argument as when a, b are both zero, so if $\eta_d > \overline{\eta}_d$ then we are in case scenario (II.a), and if $\eta_d < \overline{\eta}_d$ then we are in scenario (II.b).

- $x \in M_2$: Write $y(x)$ as $a+\epsilon$, where $[\epsilon] \leq [b-a]$. Then $[y(x)-b] = [(a-b)+\epsilon] = [a-b]$, hence $\overline{\eta}_d(y(x) - b)$ is constant, so we are in case scenario (I).
- $x \in M_3$: Write $y(x)$ as $a+\epsilon$, where $\mathcal{M}^{b-a} < \epsilon < b-a+\mathcal{M}^{b-a}$. Then $[y(x)-a] = [\epsilon]$, and $[y(x) - b] = [(a - b) + \epsilon] = [\epsilon]$, **but** $y(x) - a$ and $y(x) - b$ have opposite signs. Again we split into cases depending on whether η_d and $\overline{\eta}_d$ are equal.

– $\eta_d = \overline{\eta}_d$: Note that $\eta_d(y(x) - a) = -\overline{\eta}_d(y(x) - b)$. Set $G_{M_3,i} := G$ and define F_{M_3} as:

$$F_{M_3,i}(x) := \begin{cases} G_i(x) & \text{if } i \leq d \\ \eta_i(F_{M_3,i-1}(x) - (-a_i)) & \text{if } \operatorname{len}(F) \geq i > d \end{cases}$$

Since $y(x)-a$ and $y(x)-b$ have the same valuation but opposite signs, we have:

$$\begin{aligned} F_{M_3,d} &= \eta_d[b_d] \circ F_{d-1}(x) \\ &= \eta_d(F_{d-1}(x) - b_d) \\ &= -\eta_d(F_{d-1} - a_d) \\ &= -F_d(x) \end{aligned}$$

Hence by induction, we have for all $i \geq d$:

$$F_{M_3,i} = -F_i$$

So to ensure we do not change the value of the sum $(F+G)(x)$ when replacing F with F_{M_3}, we need to flip the sign of the z_i's from the d-th term onwards:

$$z_{M_3,i} := \begin{cases} z_i & \text{if } i < d \\ -z_i & \text{if } \operatorname{len}(F) > i \geq d \end{cases}$$

Now let us compute $(F + G)$:

$$
\begin{aligned}
(F+G)(x) &= \sum_{i=0}^{d-1}(z_i+\overline{z}_i)G_i(x) + z_d F_d(x) + \overline{z}_d G_d(x) + \sum_{i=d+1}^{\operatorname{len}(F)} z_i F_i(x) + \sum_{i=d+1}^{\operatorname{len}(G)} \overline{z}_i G_i(x) \\
&= \sum_{i=0}^{d-1}(z_i+\overline{z}_i)G_i(x) - z_d G_d(x) + \overline{z}_d G_d(x) + \sum_{i=d+1}^{\operatorname{len}(F)} z_i F_i(x) + \sum_{i=d+1}^{\operatorname{len}(G)} \overline{z}_i G_i(x) \\
&= \sum_{i=0}^{d}(z_{M_3,i}+\overline{z}_{M_3,i})G_{M_3,i}(x) + \sum_{i=d+1}^{\operatorname{len}(F)} z_{M_3,i} F_{M_3,i}(x) + \sum_{i=d+1}^{\operatorname{len}(G)} \overline{z}_{M_3,i} G_{M_3,i}(x) \\
&= (F_{M_3}+G_{M_3})(x)
\end{aligned}
$$

Let us verify that the characteristic domain of F_{M_3} contains M_3. Pick some $x \in M_3$. Note that:

$$
C(F_{M_3,d}) = C(F_{d-1}) \cap (\eta_d[b_d] \circ F_d)^{-1}(-a_{d+1} + \mathcal{M}^{a_{d+1}})
$$

Since $M \subseteq C(F)$, we have $x \in C(F)$, so

$$
\eta[a_d] \circ F_d(x) \in a + \mathcal{M}^a
$$

Since $F_{M_3,d}(x) = F_d(x)$, we also have

$$
\eta[b_d] \circ F_d(x) \in -a_{d+1} + \mathcal{M}^{a_{d+1}}
$$

Hence $x \in C(F_{M_3,d})$. By a similar calculation, we also have $x \in C(F_{M_3,i})$ for all $i \geq d$, hence $M_3 \subseteq C(F_{M_3})$. Subsequently, we are in scenario (II.c).

– $\eta_d \neq \overline{\eta}_d$: Say $\eta_d > \overline{\eta}_d$, so we have $\eta_d(y(x)-a) = -\eta_d(\overline{\eta}_d(y(x)-b)$. Set $G_{M_3} := G$ and define F_{M_3} as:

$$
F_{M_3,i}(x) := \begin{cases} G_i(x) & \text{if } i \leq d \\ \eta_d(G_d(x)) & \text{if } i = d+1 \\ \eta_{i-1}(F_{M_3,i-2}(x) - (-a_{i-1})) & \text{if } \operatorname{len}(F)+1 \geq i > d+1 \end{cases}
$$

So for all $i > d$ we have $F_{M_3,i}(x) = -F_{i-1}(x)$, hence we flip the sign of the relevant z_i's.

$$z_{M_3,i} \coloneqq \begin{cases} z_i & \text{if } i < d \\ 0 & \text{if } i = d \\ -z_i & \text{if } \operatorname{len}(F) + 1 \geq i > d \end{cases}$$

Let us again calculate $F + G$:

$$\begin{aligned}(F+G)(x) &= \sum_{i=0}^{d-1}(z_i + \overline{z}_i)G_i(x) + \sum_{i=d}^{\operatorname{len}(F)} z_i F_i(x) + \sum_{i=d}^{\operatorname{len}(G)} \overline{z}_i G_i(x) \\ &= \sum_{i=0}^{d-1}(z_i + \overline{z}_i)G_i(x) + 0G_d + \overline{z}_d G_d + \sum_{i=d}^{\operatorname{len}(F)} z_i F_i(x) + \sum_{i=d+1}^{\operatorname{len}(G)} \overline{z}_i G_i(x) \\ &= \sum_{i=0}^{d}(z_{M_3,i} + \overline{z}_{M_3,i})G_{M_3,i}(x) + \sum_{i=d+1}^{\operatorname{len}(F)+1} z_{M,i} F_{M,i}(x) \\ &+ \sum_{i=d+1}^{\operatorname{len}(G)} \overline{z}_{M,i} G_{M,i}(x)\end{aligned}$$

By a calculation similar to the one done in the previous case, we see that $M_3 \subseteq C(F_{M_3})$, hence we are in scenario (II.a). Similarly, if $\eta_d < \overline{\eta}_d$, then we are in scenario (II.b).

The calculations M_4, M_5 are symmetrical to M_2, M_1 respectively, hence omitted.

□

Just like with Lemma 3.5.13, we use Lemma 3.5.10 to show that $f + g$ is nice on the whole of G.

Lemma 3.5.17. *Let $f(x), g(x)$ be χ_n polynomials. Then $f(x) + g(x)$ is 'nice' on G.*

Proof. By Lemma 3.5.10, we a finite convex partition $\mathcal{P}_f$ of G such that for all $M \in \mathcal{P}_f$, there is some χ_n-polynomial f_m such that $f \equiv_M f_M$ and $M \subseteq C(f_M)$. Similarly, we have another finite convex partition $\mathcal{P}_g$ of G. Define $\mathcal{P}$ as:

$$\mathcal{P} \coloneqq \{M \cap N \mid M \in \mathcal{P}_f, N \in \mathcal{P}_g\}$$

Then $\mathcal{P}$ is also a finite convex partition of G. Pick some $M \cap N \in \mathcal{P}$, then $f + g$ is equal to $f_M + g_N$. Apply Lemma 3.5.16 to $f_M + g_M$, then $f + g$ is nice on $M \cap N$. Hence $f + g$ is nice on every element of $\mathcal{P}$, hence nice on G. □

The following proof is almost identical to lemma A.46 in S. Kuhlmann, 2000.

Lemma 3.5.18. *Let f be a χ_n polynomial of the form:*

$$f(x) = \sum_{i=0}^{k} z_i f_i(x) + c$$

where $c \in G$ is a constant. Then for all $i \leq n$, $\chi_i(f(x))$ is nice on $C \coloneqq C(f)$.

Proof. Fix some contraction χ_i, and to ease notation write it as χ. Let m be the least integer such that $z_m \neq 0$, and let $\tilde{c} = -\frac{c}{z_m}$. If $m = k$, then

$$\chi(f(x)) = \chi(f_m(x) - \tilde{c})$$

Consider the following partition of C:

$$\begin{aligned}
M_1 &\coloneqq \{x \in C \mid |f_m(x)| > \left|\tilde{c} + \mathcal{M}^{\tilde{c}}\right| \text{ and } sign(f_m(x)) = sign(\tilde{c})\} \\
M_2 &\coloneqq \{x \in C \mid f_m(x) \in \tilde{c} + \mathcal{M}^{\tilde{c}}\} \\
M_3 &\coloneqq \{x \in C \mid |f_m(x)| < \left|\tilde{c} + \mathcal{M}^{\tilde{c}}\right|\} \\
M_4 &\coloneqq \{x \in C \mid |f_m(x)| \geq \left|\tilde{c} + \mathcal{M}^{\tilde{c}}\right| \text{ and } sign(f_m(x)) = -sign(\tilde{c})\}
\end{aligned}$$

Exactly as done in the proof of Lemma 3.5.10, we see that $\{M_1, M_2, M_3, M_4\}$ is a finite convex partition of C and on each we can write $\chi(f(x))$ as an appropriate χ_n-polynomial.

So assume $m < k$, and set $y(x) \coloneqq f_m(x) - a_{m+1}$, hence we can write f as:

$$\begin{aligned}
f(x) = F(y) = &\sum_{i=m+2}^{k} z_i f_{i,m+2}(\delta(y)) \\
&+ z_{m+1}\delta(y) \\
&+ z_m y \\
&- z_m(-a_{m+1} + \tilde{c})
\end{aligned} \tag{3.20}$$

where $\eta = \eta_{m+1}$. Write $d \coloneqq \tilde{c} - a_{m+1}$. If $d = 0$ then we have:

$$\chi(f(x)) = \chi\left(\sum_{i=m+2}^{k} z_i f_{i,m+2}(\eta(y)) + z_{m+1}\eta(y) + z_m y\right)$$

Since we are assuming $x \in C(f)$, we can apply Lemma 3.5.11 to get:

$$\left[\sum_{i=m+2}^{k} z_i f_{i,m+2}(\eta(y))\right] < [\eta(y)] < [y]$$

and by applying axiom (**C2**$_n$) we get:

$$\chi(f(x)) = \chi(y(x)) = \chi(f_m(x) - a_{m+1})$$

which is a χ_n polynomial with characteristic domain containing in $C(f)$, so we are done. So now suppose $d \neq 0$, then we can partition $C(f)$ into 5 open convex sets as follows:

$$\begin{aligned}
M_1 &= C \cap y^{-1}(G^{<0} - \mathcal{O}^d)\\
M_2 &= C \cap y^{-1}(\{g \in \mathcal{O}^d \mid g < d + \mathcal{M}^d\})\\
M_3 &= C \cap y^{-1}(d + \mathcal{M}^d)\\
M_4 &= C \cap y^{-1}(\{g \in \mathcal{O}^d \mid g > d + \mathcal{M}^d\})\\
M_5 &= C \cap y^{-1}(G^{>0} - \mathcal{O}^d).
\end{aligned}$$

Note that $M_1 < M_2 < M_3 < M_4 < M_5$, assuming $y(x)$ is increasing (by Lemma 3.5.12, $y(x)$ must be monotonic on $\mathcal{G}$). We will show how $\chi f(x)$ behaves on each M_i.

- $x \in M_1$: Since $[y(x)] > [d]$, we know what $\chi(f(x)) = \chi(y(x))$, which is a χ_n-polynomial with characteristic domain equal to $C(f_{m+1})$, hence $M_1 \subseteq C(\chi(y(x)))$
- $x \in M_2$: We see that $[y - d] = [v]$, hence $\chi(y(x)) = -sign(z_m)\chi(d)$
- $x \in M_4$: Similar to the case for M_2, we have $\chi(y(x)) = sign(z_m)\chi(d)$
- $x \in M_5$: Similar to the case for M_1, we have $\chi f(x) = \chi(y(x))$.

It remains to analyse the behaviour of $\chi(f(x))$ on M_3. Since $y(x) \in d + \mathcal{M}^d$, we know that $\eta(y(x))$ is constant on M_3, hence the first two lines of equation (3.20) are constant. Set

$$\gamma \coloneqq \tilde{c} - z_m^{-1} \sum_{i=m+1}^{k} f_{i,m+1}(\tilde{c})$$

Then we have:

$$\begin{aligned}
\chi(f(x)) &= \chi\left(\sum_{i=m+2}^{k} z_i f_{i,m+2}(\eta(y)) + z_{m+1}\eta(y) + z_m y - z_m(-a_{m+1} + \tilde{c})\right)\\
&= \chi(z_m(y + a_{m+1} - \gamma))\\
&= sign(z_m)\chi(f_m(x) - \gamma)\\
&= sign(z_m)\chi[\gamma](f_m(x))
\end{aligned}$$

Set $g(x) \coloneqq \chi[\gamma](f_m(x))$, then it remains to show that $M_3 \subseteq C(g)$.

Claim 3.5.18.1. *If $M_3 \neq \varnothing$, then $(\tilde{c} + \mathcal{M}^d) \cap C(f_{k,m}) \neq \varnothing$.*

Proof. Since $y(x) \coloneqq f_m(x) - a_{m+1}$, we can write M_3 as:

$$M_3 = C(f_k) \cap f_m^{-1}(\tilde{c} + \mathcal{M}^d)$$

Pick some $h \in M_3$, then $f_m(h) \in \tilde{c} + \mathcal{M}^d$, as $h \in f_m^{-1}(\tilde{c} + \mathcal{M}^d)$, and $f_m(h) \in C(f_{k,m})$, as $h \in C(f_k)$.

■

Claim 3.5.18.2. *If $(\tilde{c} + \mathcal{M}^d) \cap C(f_{k,m}) \neq \varnothing$, then $[\gamma - \tilde{c}] < [d]$ and $[\tilde{c}] = [\gamma]$.*

Proof. Pick some $h \coloneqq \tilde{c} + \epsilon \in \tilde{c} + \mathcal{M}^d$, where $\epsilon \in \mathcal{M}^d$. Then we have:

$$\begin{aligned}[h - a_{m+1}] &= [\tilde{c} + \epsilon - a_{m+1}] \\ &= [d - \epsilon] \\ &= [d]\end{aligned}$$

Hence $f_{m+1,m}(x) = \eta_{m+1}(x - a_{m+1})$, and subsequently $f_{i,m}(x)$ for all $i > m$ are constant on $\tilde{c} + \mathcal{M}^d$.

This implies that $(\tilde{c} + \mathcal{M}^d) \subseteq C(f_{k,m})$, since for any $x \in \tilde{c} + \mathcal{M}^d$, we have $f_{m+1,m}(x) = f_{m+1,m}(\tilde{c})$, in particular $\tilde{c} \in C(f_{k,m})$. Since

$$\tilde{c} - \gamma = z_m^{-1} \sum_{i=m+1}^{k} f_{i,m+1}(\tilde{c})$$

Lemma 3.5.11 tells us that:

$$[\tilde{c} - \gamma] = [f_{m+1}(\tilde{c})]$$

But $f_{m+1}(\tilde{c}) = \eta(\tilde{c} - a_{m+1}) = \eta(d)$, hence

$$[\tilde{c} - \gamma] = [\eta(d)] < [d]$$

Similarly, since $\tilde{c} \in C(f_{k,m})$, we know that

$$\left[\sum_{i=m+1}^{k} f_{i,m}(\tilde{c})\right] < [\tilde{c}]$$

hence $[\gamma] = [\tilde{c}]$. ■

So by combining the previous two claims, we know that $M_3 \neq \varnothing$ implies $[\gamma - \tilde{c}] < [d]$ and $[\tilde{c}] = [\gamma]$. So let us assume $M_3 \neq \varnothing$, which means:

$$\begin{aligned} M_3 &= C \cap y^{-1}(d + \mathcal{M}^d) \\ &\subseteq y^{-1}(d + \mathcal{M}^d) \\ &\subseteq f_m^{-1}(\tilde{c} + \mathcal{M}^d) && \text{as } f_m(x) = y(x) - a_{m+1} \text{ and } d = \tilde{c} = a_{m+1} \\ &\subseteq f_m^{-1}(\gamma + \mathcal{M}^d) && \text{as } [\tilde{c} - \gamma] > [d] \text{ and } [\tilde{c}] = [\gamma] \\ &\subseteq f_m^{-1}(\gamma + \mathcal{M}^\gamma) && \text{as } \gamma + \mathcal{M}^d \subseteq \gamma + \mathcal{M}^\gamma \\ &\subseteq C(g) && \text{where } g(x) := \eta(f_m(x) - \gamma) \end{aligned}$$

Hence we are done. □

Again just like Lemma 3.5.13 and Lemma 3.5.16, we now shot $\chi(f)$ is nice on the entirety of G.

Lemma 3.5.19. *Let f be a χ_n polynomial. Then for all $i \leq n$, $\chi_i(f(x))$ is 'nice' on G.*

Proof. This is a direct application of Lemma 3.5.10. There is a finite convex partition $\mathcal{P}$ of G such that for each $M \in \mathcal{P}$, f is equal to some f_M with $M \subseteq C(f_M)$. Simply apply Lemma 3.5.18 to each f_M. □

By combining lemmas 3.5.18 and 3.5.16 (and inducting on complexity), we get the following:

Theorem 3.5.20. *Let $t(x)$ be any term. Then t is nice.*

This combined with quantifier elimination gives us the following:

Theorem 3.5.21. *For any $n \in \mathbb{N}$, the theory of n-contraction groups is weakly o-minimal.*

3.6 Remarks on Algebraicity

We conclude this chapter with some remarks on how the notion of n-algebraicity is different from other notions of algebraicity.

3.6.1 Model theoretic algebraic closure

It was proven by F.V-Kulnmann in F.-V. Kuhlmann, 1995, p. 13 that the definable closure of some subset A in some 1-contraction group $\mathcal{G}$ is the divisible hull of the precontraction group generated by A. However, we only consider the notion of 1-algebraicity when we have an extension of 1-contraction groups $\mathcal{G} \subseteq \mathcal{H}$ and some $b \in H \setminus G$. Thus comparing 1-algebraicty to model-theoretic algebraicity is somewhat meaningless since the definable closure $\mathcal{G}$ is just $\mathcal{G}$ itself. Suppose however we dropped the requirement that $\mathcal{G}$ must be a 1-contraction group and allow it to be a 1-precontraction group. Then model-theoretic algebraicity implies 1-algebraicity, but the converse does not hold.

Lemma 3.6.1. *Let $\mathcal{G} \subseteq \mathcal{H}$ be* 1*-precontraciton groups, with* H *a* 1*-contraction group, and pick some* $b \in H \setminus G$*. If* b *is in the (model-theoretic) algebraic closure of* G*, then* b *is* 1*-algebraic over* $\mathcal{G}$*, with characteristic sequence of length* 1*.*

Proof. By Theorem 1.3 from F.-V. Kuhlmann, 1995, we know that b is in the divisible hull of G, thus $[G + \mathbb{Q}b] = [G]$, so $\chi(G + \mathbb{Q}) \subseteq G$, thus b is 1-algerbraic, and has characteristic sequence $(b, \chi(b), 0, \ldots)$. □

To see why the converse does not hold, pick some $b \in H$ with $\chi(b) \in G$ but $[b] > [G]$. Then b is 1-algebraic, with 1-characteristic sequence of length 1, but of course, b is outside the divisible hull of G, so is not in the (model-theoretic) algebraic closure of G.

3.6.2 Transitivity of Algebraic Closure

Say an extension $\mathcal{G} \subseteq \mathcal{H}$ of n-contraction groups is algebraic if every element b of $H \setminus G$ is n-algebraic over G. This definition is not immediately useful, because if we pick some $b, b' \in H \setminus G$, it is not clear whether b' is n-algebraic over $\mathcal{G}_b := \overline{\mathcal{G}\langle b \rangle}$. Regardless, we can prove such a relation is transitive.

Lemma 3.6.2. *Let $\mathcal{G} \subseteq \mathcal{H} \subseteq \mathcal{K}$ be n-contraction groups. If the extensions $\mathcal{G} \subseteq \mathcal{H}$ and $\mathcal{H} \subseteq \mathcal{K}$ are* 1*-algerbaic, then the extension $\mathcal{G} \subseteq \mathcal{K}$ is* 1*-algebraic.*

Proof. Pick some $k \in K \setminus H$, we need to show k is n-algebraic over $\mathcal{G}$. We know k is n-algebraic over $\mathcal{H}$, so has an n-characteristic sequence $(k_{\mathcal{H},i}, h_i)_{i \leq \alpha}$, where $\alpha < \omega^n$ is written as:

$$\alpha = \omega^{n-1} c_{n-1} + \ldots + \omega^1 c_1 + c_0$$

where $c_i \in \omega$ for all $i < n$. Let $(k_{\mathcal{G},i}, g_i)_{i < \omega^n}$ be an n-characteristic sequence of k in $\mathcal{G}$. We need to show this sequence has finite length, for which it is sufficient to show $k_{\mathcal{G},i} \in H$ for some $i \in \omega^n$. Suppose we have shown $k_{\mathcal{G},i} = k_{\mathcal{H},i}$ and $g_i = h_i$ for all $i \leq \beta$, for some $\beta < \alpha$.

1. Suppose $h_i = 0$ for all $i \in [\beta, \alpha]$, then for all $i \in [\beta, \alpha]$, there is no $h \in H$, and thus no $g \in G$ such that:

   $$[(k_{\mathcal{G},i} + h_i) - h], [(k_{\mathcal{G},i} + h_i) - g] < [k_{\mathcal{G},i} + h_i]$$

   Thus $g_i = 0$ and hence $k_{\mathcal{G},i} = k_{\mathcal{H},i}$ for all $i \in [\beta, \alpha]$, so we are done.

2. Suppose there exists some least $\gamma \geq \beta$ with $h_\gamma \neq 0$. Then by the same argument as in the previous case, we see that $g_i = h_i = 0$ and $k_{\mathcal{G},i} = k_{\mathcal{H},i}$ for all $i \in [\beta, \gamma)$. Furthermore, since $h_\gamma \neq 0$, we know that $[k_{\mathcal{G},\gamma} + h_\gamma] \in [H]$.

(a) If is no $g \in G$ with $[(k_{\mathcal{G},\gamma} + h_\gamma) - g] < [k_{\mathcal{G},\gamma} + h_\gamma]$, then we must have $g_\gamma = 0$ and $k_{\mathcal{G},\gamma+1} = \chi(k_{\mathcal{G},\gamma}) \in H$ by (CA), so we are done.

(b) If there is some $g \in G$ with $[(k_{\mathcal{G},\gamma} + h_\gamma) - g] < [k_{\mathcal{G},\gamma} + h_\gamma]$, then we can assume $g_\gamma = h_\gamma = g$. Restart this process but replace β with γ.

Due to the construction of n-characteristic sequences, this process must eventually terminate, hence we see that k is n-algebraic over $\mathcal{G}$. □

Chapter 4

Contractions and Derivations

4.1 Introduction

The goal of this chapter is to examine the action of the logarithmic derivative along with various contractions on the value group of a trans-exponential expansion of $\mathbb{R}$.

Let K be a trans-exponential Hardy field as in Theorem 1.2.3. Then by Proposition 1.5.5, we know that $G \coloneqq v(K)$ along with the well defined map $\psi : vf \mapsto vf' - vf$ will give a model of T_ψ^-, and if K is closed under integration, then (G, ψ) is a model of T_ψ. Recall that the function $(id + \psi) : G^{\neq 0} \to G$ maps $v(f)$ to $v(f')$, which suggests that the action of log, on the value group G, must be definable in G via:

$$v(\log(f)) = (id + \psi)^{-1}(\psi(v(f)))$$

The following proposition shows how we can make this idea precise.

Proposition 4.1.1. *Let $\mathcal{G}$ be an asymptotic couple such that $\psi(G^{\neq 0})$ has no maximal element. If we define the map $\chi : G \to G$ as:*

$$\chi(x) \coloneqq \begin{cases} (id + \psi)^{-1}(\psi(x)) & \text{if } x < 0 \\ -\chi(-x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then χ is well defined and (G, χ) is a centripetal precontraction group. Moreover if $\psi(G^{\neq 0})$ is downwards closed, then χ is surjective, hence (G, χ) is a centripetal contraction group.

Furthermore, for all $x < 0$ and $n \in \mathbb{N}$, the functions χ and ψ satisfy the equation:

$$\psi(\chi^n(x)) = \psi(x) + \sum_{i=1}^{n} \chi^i(x) \tag{4.1}$$

Proof. To show χ is well defined, pick some $b \in \psi(G^{\neq 0})$. Since $\psi(G^{\neq 0})$ has no maximum element, we know there exists some $c \in \psi(G^{\neq 0})$ such that $b < c$. By Lemma 4.2.7 (3b), we know that b has a preimage under $(id + \psi)$, thus χ is well defined on $\psi^{-1}(b) \cap G^{<0}$, and hence well defined on the entirety of G.

To show χ is a centripetal precontraction, we need to verify the axioms in Definition 1.4.2. Axioms (C0), and (C−) are evident from the definition. (CA) follows from (A2) for ψ and (C≤) follows since both ψ and $(id + \psi)^{-1}$ are increasing on negative elements of $\mathcal{G}$. To prove (CP), choose some $x < 0$, and suppose for a contradiction that $\chi(x) \leq x$. Then:

$$(id + \psi)^{-1}(\psi(x)) \leq x$$

So by applying $(id + \psi)$ to both sides we have:

$$\psi(x) \leq (id + \psi)(x)$$

Then if we subtract $\psi(x)$ from both sides, we would have $x \geq 0$ which is a contradiction.

If $\psi(G^{\neq 0})$ is downwards closed, then again by Lemma 4.2.7, we have $\psi(G^{\neq 0}) = (id + \psi)(G^{<0})$ from which surjectivity of χ immediately follows.

The equation (4.1) follows since $(id + \psi)(\chi(x)) = \psi(x)$ for all $x < 0$. □

Note that we do not need (C2) to define χ or even make χ-surjective. Moreover, if the map χ is surjective, that does not imply that $\mathcal{G}$ satisfies (C2). **However**, if χ is surjective, then we can deduce that $\psi(G^{\neq 0})$ is downwards closed.

Proposition 4.1.2. *Let $\mathcal{G}$ be an asymptotic triple such that $\psi(G^{\neq 0})$ has no maximal element. Define χ as in Proposition 4.1.1. If χ is surjective, then $\psi(G^{\neq 0})$ is downwards closed.*

Proof. By Lemma 4.2.7 we know that $(id + \psi)(G^{<0})$ is a downwards closed set, and since χ is surjective, we also have $\chi(G^{<0}) = G^{<0}$. Since ψ satisfies:

$$\psi(x) = (id + \psi)(\chi(x))$$

For all $x < 0$, we deduce that $\psi(G^{<0}) = \psi(G^{\neq 0})$ is downwards closed. □

Hyper-logarithms

So far we know that the logarithm χ, logarithmic derivative ψ and hyper-logarithm $L := \chi_2$ are all well defined on vK. As shown in Proposition 1.5.8, we can define further structure on the value group, via the map $\theta : G \to G$. As a reminder, θ is a centripetal contraction satisfying the functional equations:

$$\theta(x) = \chi(x) + \theta(\chi(x))$$
$$(id + \psi)(L(x)) = \psi(x) - \theta(x)$$

for all $x < 0$. The intuition behind θ is that it is the infinite sum

$$\theta(x) := \sum_{i<\omega} \chi^i(x)$$

But obviously, we do not have an operation for infinite sums in the language of ordered abelian groups, thus it becomes necessary to add the function θ to our language, to describe the action of the derivative of the hyper-logarithm on vK.

To check that this collection of functions describes completely the structure induced by the hyper-logarithmic derivative on vK, we would need to show that the theory $T_{\psi-\chi}$ described in Definition 1.5.9 is complete. As stated in the introduction, we did not manage to do this, but we did manage to prove quantifier elimination and completeness for the structure with domain G and just the contractions χ, L and θ (Theorem 4.4.13).

The outline of this chapter is as follows. In Section 4.2 we present the proof of quantifier elimination and completeness for T_ψ, which was originally proved in Aschenbrenner and van den Dries, 2000. Our exposition will be identical to that in the original paper, except we go into more detail on a few details in certain proofs. We also highlight some potential gaps in the original paper and provide explanations on how to get around them (see the remarks after Lemma 4.2.19 and Theorem 4.2.40). In Section 4.3, we present a proof of quantifier elimination and completeness for the structure containing **just** χ and θ, then in Section 4.4 we do the same but with the three contractions χ, L and θ. Finally in Section 4.5 we give a partial proof of quantifier elimination and completeness for $T_{\psi-\chi}$, the theory with all the contractions and ψ.

4.2 Asymptotic Triples

In the original paper on asymptotic triples by Aschenbrenner and van den Dreis, quantifier elimination is proved for a 2-sorted theory involving an ordered abelian group and a real closed field, with the group being a vector space equipped with scalar multiplication over the field. They then assert that a proof of QE for the one sorted structure defined in Definition 1.5.2 follows easily. However, in the two sorted case, the axiom (A4) is replaced with:

A4' For all v, w, if $[v] < [w]$ then $\psi(v) > \psi(w)$

In the one sorted structure we do not have such a powerful axiom, the most we can express is the same statement but with non-strict inequalities. Thus some small difficulties arise when applying the proof of QE for two sorted triples to the one sorted case. We highlight them as they occur, and present solutions.

Definition 4.2.1. Let $\mathcal{L}_\psi$ be the language from Definition 1.5.2. Let $\mathcal{L}'_\psi$ be $\mathcal{L}_\psi$ with the predicate P removed. If $\mathcal{V}$ is a (closed) asymptotic triple, we call the reduct of $\mathcal{V}$ to $\mathcal{L}'_\psi$ a **(closed) asymptotic couple**. Note that T_ψ can be expressed completely in $\mathcal{L}'_\psi$, since in a model $\mathcal{V}$ of T_ψ, the only value for $P_\mathcal{V}$ is $\psi(V^{\neq 0})$, so we denote the reduct of T_ψ to $\mathcal{L}'_\psi$ as $T_\psi \mid \mathcal{L}_\psi$.

Notation 4.2.2. For an asymptotic triple $\mathcal{V}$, we denote $\Psi_\mathcal{V}$ to be $\psi(V^{\neq 0})$. When the context is clear we just denote it as Ψ.

4.2.1 Preliminaries on asymptotic triples

Definition 4.2.3. Let C, D be ordered sets and $f : C \to D$ be a function. We say f has the **intermediate value property** (IVP) if for any $x, y \in C$ with $f(x) < f(y)$ and $a \in (f(x), f(y))$, there exists some $z \in C$ between x and y such that $f(z) = a$.

The following lemma will be crucial in many proofs going forward.

Lemma 4.2.4 (Aschenbrenner and van den Dries, 2000, p. 319)**.** *Let $(G, +, <, 0)$ be an arbitrary ordered abelian group and $C \subseteq G$ be a convex subset. Suppose for all $a, x, y \in C$, $\eta : C \to G$ satisfies:*

1. *If $x \neq y$ then $[x - y] > [\eta x - \eta y]$*
2. *Both of the following hold*
   (a) *If $0 < a < x < y$ and $[y - x] < [y - a]$ then $\eta x = \eta y$.*
   (b) *If $0 > a > x > y$ and $[y - x] < [y - a]$ then $\eta x = \eta y$.*

Then $f : C \to \mathcal{G} : x \mapsto x + \eta x$ is strictly increasing and has the intermediate value property

Proof. Let $x, y \in C$ with $x > y$, then:

$$f(x) - f(y) = x - y + (\eta x - \eta y)$$

But (1) states that $[x - y] > [\eta x - \eta y]$, hence:

$$\operatorname{sign}(f(x) - f(y)) = \operatorname{sign}(x - y)$$

so $f(x) > f(y)$, hence f is strictly increasing. To prove IVP, we can assume without loss of generality that C is of the form $[0, c]$ or $[-c, 0]$ for some $c \in G^{>0}$ and that $\eta(0) = 0$. Thus by (1), we have for all $x \in C^{\neq 0}$, $[x] > [\eta(x)]$.

Suppose $C = [0, c]$, pick some $v \in \mathcal{G}$ such that $0 < v < c + \eta c$, then it is sufficient to prove that there is some $x \in [0, c]$ such that $v = x + \eta x$. Note that by (1), we have the following series of deductions:

$$[c + \eta c] = [c] \implies [v] \leq [c + \eta c] = [c] \implies [c - v] \leq [c]$$

This means we only have to consider the following two cases:

Case 1 $[c] > [c - v]$**:** Set $u := v - \eta c$, we claim that $0 < u < c$. Since $v < c + \eta c$, we know that $u < c$, so suppose $u \leq 0$. Then $0 < v < \eta c$, which means $[v] < [\eta c] < [c]$, which then means $[c - v] = [c]$ by the ultrametreic inequality, which contradicts the assumption $[c] > [c - v]$. Hence our goal is to show that $u + \eta u = v$, for which it is sufficient to show that $\eta u = \eta c$. But note that $[c - u] < [c]$, since:

$$[c - u] = [(c - v) + (v - u)] = [c - v + \eta c] < [c]$$

with the final inequality holding since both $[c - v], [\eta c] < [c]$. So we can apply (2.a) to deduce $\eta u = \eta c$ hence we are done.

Case 2 $[c] = [c - v]$**:** Set $u := v - \eta v$, like before we want to show that both $u \in (0, c)$ and $u + \eta u = v$. Since $v > 0$ and $v \succ \eta v$ (recall that $x \succ y$ mean $[x] > [y]$), u must be positive. Suppose $v \geq c$, then since $v < c + \eta c$, , we have $0 \leq v - c \leq \eta c \prec c$ which means $[c - v] < [c]$ which is a contradiction. Then $0 < u < v$ and $[u - v] = [\eta v] < [v]$, hence applying (2.a) yields us $\eta u = \eta v$, so we are done.

The same calculations work when $C = [-c, 0]$, except we apply (2.b) instead. □

Remark 4.2.5. If $\eta(x)$ is constant on archimedean classes then $\eta(x)$ and $\eta(x - a)$ both satisfy (2), for any constant $a \in C$ with $C - a \subseteq C$. Thus we know the function $x + \psi(x - a)$ is strictly increasing and has the intermediate value property on both convex components of $G^{\neq a}$.

Proposition 4.2.6 (Aschenbrenner and van den Dries, 2000, p. 320)**.** *Let $\mathcal{V}$ be an asymptotic triple and $x \neq y$ be non-zero elements of V, then ψ satisfies the following:*

1. $[x] = [y] \implies \psi(x) = \psi(y)$
2. *If $x, y \in P_{\mathcal{V}}$ then $\psi(x - y) > \min\{x, y\}$, so in particular, $\psi(\psi x - \psi y) > \min\{\psi x, \psi y\}$*
3. $[x - y] > [\psi x - \psi y]$
4. *The function $id + \psi$ is increasing on $V^{\neq 0}$, so $x > y \implies (id + \psi)(x) > (id + \psi)(y)$*

Proof. Axioms (A2) and (A4) of asymptotic triples imply that ψ is constant on archimedean classes. For (2), suppose $x < y \in P_{\mathcal{V}}$, then by A3, we have:

$$\psi(x - y) + (y - x) > y \implies \psi(x - y) > x = \min\{x, y\}$$

To prove (3), note that $[x-y] < [x],[y]$ implies $[x]=[y]$, which implies $\psi(x)=\psi(y)$ so the statement must be true. So assume $[x-y]=\max\{[x],[y]\}$ and $[x]\geq[y]$, so $\psi x\leq\psi y$. From (2) we know that

$$\psi(\psi x-\psi y) > \min\{\psi x,\psi y\} = \psi x$$

Hence the contrapositive of axiom A4 implies $[\psi x-\psi y] < [x] = [x-y]$. Finally, for (4), use (1) and (2) to satisfy the hypotheses of Lemma 4.2.4. □

Lemma 4.2.7 (Aschenbrenner and van den Dries, 2000, p. 320)**.** *Let $\mathcal{V}$ be an asymptotic triple.*

1. *The set $(id+\psi)(V^{>0})$ is closed upwards*
2. *The set $(-id+\psi)(V^{>0})$ is closed downwards*
3. *The following equalities hold:*
   (a) $(-id+\psi)(V^{>0}) = (id+\psi)(V^{<0})$
   (b) $(-id+\psi)(V^{>0}) = \{a\in V \mid a<b \text{ for some } b\in\Psi_{\mathcal{V}}\}$

Proof. For the first two, apply Lemma 4.2.4 to show that the functions $x+\psi(x)$ and $-x+\psi(-x)$ have IVP and are strictly increasing on $V^{>0}$, using Proposition 4.2.6 to satisfy the two hypotheses. The equality in (3a), follows since $\psi(x)=\psi(-x)$. Finally, for (3b), pick some $x\in(id+\psi)(V^{>0})$, which by (3a) we can write $x=y+\psi y$ for some $y<0$, hence $x<\psi(y)$. Conversely, pick some $x\in V$ such that $x<\psi b$ for some $b\in V$, $b>0$, and set:

$$y \coloneqq \min\{b,\psi b-x\} > 0$$

Then since $0<y\leq b$ and $\psi y\geq\psi b$ we have:

$$x \leq \psi b - y \leq \psi y - y$$

Hence by (2), we see that $x\in(id+\psi)(V^{>0})$. □

Definition 4.2.8. Given some asymptotic triple $\mathcal{V}$, an **H-cut** is some downwards closed set containing $\Psi_{\mathcal{V}}$ but disjoint from $(id+\psi)(V^{>0})$. So $P_{\mathcal{V}}$ must be an H-cut and moreover

Proposition 4.2.9 (Aschenbrenner and van den Dries, 2000, p. 321)**.** *Let $\mathcal{V}$ be an asymptotic triple.*

1. *There is at most one $v\in V$ such that*

$$\Psi < v < (id + \psi)(V^{>0}) \tag{4.2}$$

Hence $\mathcal{V}$ has at most two H-cuts.

2. *$\mathcal{V}$ has two H-cuts if and only if there exists some $v \in V$ such that (4.2) holds.*

3. *If Ψ has a largest element, then $\mathcal{V}$ has only one H-cut.*

Proof. Suppose $v < v'$ both satisfy (4.2), then set $u := v' - v > 0$. Then since (*) $\psi(u) \in \Psi < v$ and (**) $\psi(u) + u \in (id + \psi)(V^{>0}) > v'$ we have:

$$\psi(u) \underbrace{<}_{*} v = v' - u \underbrace{<}_{**} \psi(u) + u - u = \psi(u)$$

This is a contradiction, hence there is only one v satisfying (4.2). Now suppose there are three H-cuts $C_1 \subset C_2 \subset C_3$. Then pick some $v \in C_2 \setminus C_1$ and $v' \in C_3 \setminus C_2$, we see that both v', v satisfy (4.2), which cannot happen.

Suppose there are two distinct $C_1 \subset C_2$, then any $v \in C_2 \setminus C_1$ will satisfy (4.2). Conversely, if there is some v satisfying (4.2), then we can define two distinct H-cuts $C_1 := (-\infty, v)$ and $C_2 := (-\infty, v]$.

If Ψ has a maximal element a and V two H-cuts, then pick some $b > a$ satisfying (4.2), observe that $\frac{1}{2}(b - a)$ also satisfies (4.2), contradicting (1).

□

Definition 4.2.10. Let $\mathcal{V}$ be an asymptotic triple. We say some $v \in V$ is an **H-point** if it lies between $\psi(V^{\neq 0})$ and $(id + \psi)(V^{>0})$, so:

$$\psi(V^{\neq 0}) < v < (id + \psi)(V^{>0})$$

Subsequently, by Proposition 4.2.9, there can be at most one H-point and it exists if and only if $\mathcal{V}$ has two H-cuts.

Lemma 4.2.11. *Let $\mathcal{G}$ and $\mathcal{H}$ be asymptotic triples, and suppose there exists an $\mathcal{L}'_\psi$-embedding $\psi : \mathcal{G} \hookrightarrow \mathcal{H}$. If $\mathcal{G}$ has only one H-cut, then the map ψ is an $\mathcal{L}_\psi$-embedding.*

Proof. Pick some $g \in G$, we need to show:

$$P_{\mathcal{G}}(g) \iff P_{\mathcal{H}}(\phi(g))$$

Suppose $\mathcal{G} \models P_{\mathcal{G}}(g)$, since $\mathcal{G}$ has only one H-cut, this means there exists some $u \in G^{>0}$ such that $g < \psi(u)$, but since ϕ is an $\mathcal{L}'_\psi$-embedding, we know that $\phi(g) < \psi(\phi(u))$, thus

$\phi(g) \in \mathcal{P}_{\mathcal{H}}$. Conversely, suppose $\mathcal{G} \not\models P_{\mathcal{G}}(g)$. Since $\mathcal{G}$ has only one H-cut, this must mean that $g \in (id+\psi)(G^{>0})$, thus $\phi(g) \in (id+\psi)(H^{>0})$, so $\mathcal{H} \not\models P_{\mathcal{H}}(\phi(g))$. □

Before proceeding further we should mention some notation given in the closed asymptotic couples paper.

Notation 4.2.12 (Aschenbrenner and van den Dries, 2000, p. 337)**.** Let G be an ordered abelian group, $f : G \to G$ be any function, $C \subseteq G$ be a convex subset, $p, q \in G$ and $S \subseteq G$ a downwards closed subset.

1. We say f increases (decreases) on C from p to q if f is increasing (decreasing) on C and $f(C) = [p, q]$ ($f(C) = [q, p]$).
2. f increases (decreases) on C from $-\infty$ to p (from p to $-\infty$) if f is increasing (decreasing) on C and $f(C) = (-\infty, p]$.
3. f increases (decreases) on C from p to S (from S to p) if f is increasing (decreasing) on C, $p \in S$ and

$$f(C) = \{v \in S \mid v \geq p\} = S \cap [p, +\infty)$$

4. f increases (decreases) from $-\infty$ to S (from S to $-\infty$) on C if f is increasing (decreasing) on C and $f(C) = S$.

We will extend this notation as follows:

Notation 4.2.13. Fix the setup as in (4.2.12).

1. We say **the limit of f on the right boundary of C is p** if f increases or decreases **to** p on C. We denote this as:

$$\lim_{x \uparrow C} f(x) = p$$

2. Similarly, we say **the limit of f on the left boundary of C is p** if f increases or decreases **from** p on C, denoted as:

$$\lim_{x \downarrow C} f(x) = p$$

3. We define the following:

$$\lim_{x \uparrow C} f(x) = S \qquad \lim_{x \downarrow C} f(x) = S \qquad \lim_{x \uparrow C} f(x) = -\infty \qquad \lim_{x \downarrow C} f(x) = -\infty$$

   as expected.

Notation 4.2.14. Let $G \subseteq H$ be an extension of divisible ordered abelian groups, and $f : H \to H$ a function such that $f(G) \subseteq G$. Let S_H be a downwards closed subset of H with $S_G \coloneqq S_H \cap G$, and C_H a convex subset of H with $C_G = C_H \cap G$. Let $*$ be an element of $G \cup \{S_H\} \cup \{-\infty\}$.

1. We say **the limit of f on the right boundary of C_H is** $*$ if f increases or decreases to $*$ on C_H, and we will denote it as:

$$\lim_{x \uparrow C_H} f(x) = *$$

   Similarly, define the following as expected:

$$\lim_{x \downarrow C_H} f(x) = * \qquad \lim_{x \uparrow C_G} f(x) = * \qquad \lim_{x \downarrow C_G} f(x) = *$$

2. We define $\lim\limits_{x \uparrow C} = *$ as follows:

$$\lim_{x \uparrow C} f(x) = \begin{cases} p \in G & \text{If } \lim\limits_{x \uparrow C_H} f(x) = \lim\limits_{x \uparrow C_G} f(x) = p \\ -\infty & \text{If } \lim\limits_{x \uparrow C_H} f(x) = \lim\limits_{x \uparrow C_G} f(x) = -\infty \\ = S & \text{If } \lim\limits_{x \uparrow C_H} f(x) = S_H \text{ and } \lim\limits_{x \uparrow C_G} f(x) = S_G \end{cases}$$

3. For some $d \in H$ and $* \in G \cup \{S\} \cup \{-\infty\}$, define:

$$\lim_{x \to^+ d} f(x) = * \iff \text{For sufficiently large } y \in G \text{ with } y < d, \lim_{x \uparrow (y,d)} f(x) = *$$

$$\lim_{x \to^- d} f(x) = * \iff \text{For sufficiently small } y \in G \text{ with } y > d, \lim_{x \downarrow (d,y)} f(x) = *$$

   where $C_H \coloneqq (d, y)_H$, $C_G \coloneqq (y, d)_G$ and $C \coloneqq (y, d)$. We simply write:

$$\lim_{x \to d} f(x) = *$$

   when the direction we approach d is clear or is not relevant.

Lemma 4.2.15. *Let $\mathcal{G} \subseteq \mathcal{H}$ be an extension of asymptotic triples. Suppose further that:*

(i) *$G \subseteq H$ a pure extension of ordered abelian groups*

(ii) *$[G]$ is coinitial in $[H]$*

(iii) *$[G]$ has no minimum*

(iv) *$\mathcal{G}$ has no H-point*

Then $\mathcal{H}$ has no H-point.

Proof. Suppose there exists some $u \in H$ such that:

$$\psi(H^{\neq 0}) < u < (id + \psi)(H^{>0})$$

Then since $[G]$ is coinitial in $[G_b]$ and has no minimum, we must have:

$$\psi(G^{\neq 0}) < u < (id + \psi)(G^{>0})$$

Thus by (iv), we must have $u \in H \setminus G$. But the extension $G \subseteq H$ is pure, so there exists some $g \in G$ such that $[u - g] \notin [G]$. By (ii) and (iii) we can find some $\epsilon \in G$ with $[\epsilon] < [u - g]$. Then $[u - g] = [u - g + \epsilon]$, thus $u - g$ and $u - g + \epsilon$ induce the same cut in G, thus:

$$\psi(G^{\neq 0}) < u < u + \epsilon < (id + \psi)(G^{>0})$$

hence:

$$\psi(H^{\neq 0}) < u < u + \epsilon < (id + \psi)(H^{>0})$$

meaning $\mathcal{H}$ has two H-points, a contradiction. □

4.2.2 Closure operator

We show that every asymptotic triple is embeddable in some closed asymptotic triple.

As a reminder, T_ψ^- is the theory of asymptotic triples, and T_ψ is the theory of closed asymptotic triples. T_ψ^- is the universal theory of T_ψ, and the existential formulas of T_ψ assert that the image of ψ has no max and is everything below the set $x + \psi(x)$ for $x > 0$.

Definition 4.2.16. Aschenbrenner and van den Dries, 2000, p. 324 Let $\mathcal{V}$ be a model of T_ψ^-. We say some $\mathcal{W} \models T_\psi$ is an **H-closure** of $\mathcal{V}$ if it has the factoring property over $\mathcal{V}$ with respect to T_ψ.

An asymptotic couple can fail to be closed in one of three ways, either Ψ has a max, it is not downwards closed or an H-point exists. The following three lemmas show that if any of these occur, we can expand $\mathcal{V}$ in a way which stops it. Fix some asymptotic triple $\mathcal{V} := (V, \psi_\mathcal{V}, P_\mathcal{V})$.

Lemma 4.2.17 (Aschenbrenner and van den Dries, 2000, p. 325)**.** *Suppose there exists some $a \in V^{>0}$ such that*

$$P_{\mathcal{V}} < a < (id + \psi)(V^{>0}) \tag{4.3}$$

(so a is a H-point), then we can extend $\mathcal{V}$ to an asymptotic triple $\mathcal{V}_\epsilon := (V_\epsilon, \psi, P_\epsilon)$, where $V_\epsilon := V + \mathbb{Q}\epsilon$ and $0 < \epsilon < V^{>0}$ is an indeterminate such that:

- *$a = \psi(\epsilon) + \epsilon$, hence there is no H-point in $\mathcal{V}_\epsilon$.*
- *$P_\epsilon := \{x \in V_\epsilon \mid x \leq a - \epsilon\}$*
- *$\mathcal{V}_\epsilon$ has the factoring property over $\mathcal{V}$ with respect to T_ψ.*

Proof. An assortment of simple calculations shows that $\mathcal{V}_\epsilon$ is a model of T_ψ^-. Moreover, since $\psi(V_\epsilon^{\neq 0})$ has a maximum element, by Proposition 4.2.9 we know it has no H-point. It remains to show that the factoring property holds. Suppose we have an $\mathcal{L}_\psi$-embedding $i : \mathcal{V} \hookrightarrow \mathcal{W}$ where $\mathcal{W} \models T_\psi$. Since $a > P_{\mathcal{V}}$ we must also have $i(a) > P_{\mathcal{W}}$. Since $\Psi_{\mathcal{W}}$ has no max, $\mathcal{W}$ can only have one H-cut, hence $i(a) \in (id + \psi)(W^{>0})$. Pick some $\delta > 0$ in W such that $\delta + \psi(\delta) = a$, then if we set $i(q\epsilon) = q\delta$, where $q \in \mathbb{Q}$, then the map $i : V_\epsilon \hookrightarrow W$ is an $\mathcal{L}_\psi$-embedding $\mathcal{V}_\epsilon \hookrightarrow \mathcal{W}$. □

Note that after Lemma 4.2.17, P_ϵ has a maximal element, namely $\psi(\epsilon) = a - \epsilon$.

Lemma 4.2.18 (Aschenbrenner and van den Dries, 2000, p. 325)**.** *Suppose $P_{\mathcal{V}}$ has a maximal element, then we can extend $\mathcal{V}$ to $\mathcal{V}_\epsilon := (V_\epsilon, \psi, P_\epsilon)$, where V_ϵ is as defined in Lemma 4.2.17, such that:*

- *$\psi(\epsilon) = \max(P_{\mathcal{V}}) + \epsilon$, and hence akin to Lemma 4.2.17, if $\mathcal{V}$ has no H-point then $\mathcal{V}_\epsilon$ has no H-point.*
- *$P_\epsilon := \{x \in V_\epsilon \mid x \leq \psi(\epsilon)\}$*
- *$\mathcal{V}_\epsilon$ has the factoring property over $\mathcal{V}$ with respect to T_ψ.*

Proof. Again, an aggregation of similar calculations shows that $\mathcal{V}_\epsilon$ is an asymptotic triple. Conjecture that $i : \mathcal{V} \hookrightarrow W$ is an $\mathcal{L}_\psi$-embedding into a model $\mathcal{W}$ of T_ψ. Set $a := \max P_{\mathcal{V}}$, then $a \in P_{\mathcal{V}}$ hence $i(a) \in P_{\mathcal{W}}$.

Since $P_{\mathcal{W}}$ has no maximum, Lemma 4.2.7 informs us that there exists an element $\delta > 0$ of W such that $-\delta + \psi(\delta) = a$, so $\psi(\delta) = a + \delta$. Extend $i : V \to W$ to V_ϵ by setting $i(\epsilon) := \delta$, then $i : \mathcal{V}_\epsilon \hookrightarrow \mathcal{W}$ is an $\mathcal{L}_\psi$-embedding. □

After Lemma 4.2.18, P_ϵ still has a max. specifically $\max P_\epsilon$ is $\psi(\epsilon) = \max P + \epsilon$. However, since $\psi(V_\epsilon^{\neq 0})$ has a maximal element, Proposition 4.2.9 tells us that $\mathcal{V}_\epsilon$ only has one H-cut hence no H-point.

Lemma 4.2.19 (Aschenbrenner and van den Dries, 2000, p. 326)**.** *Suppose there exists some $a \in V$ with $a \in P_{\mathcal{V}} \setminus \Psi_{\mathcal{V}}$, then we can extend $\mathcal{V}$ to $\mathcal{V}_\epsilon := (V_\epsilon, \psi, P_\epsilon)$ where $V_\epsilon := V + \mathbb{Q}\epsilon$ and $\epsilon > 0$ is an indeterminate (not necessarily infinitesimal!) such that:*

(i) $\{x \in V^{>0} \mid \psi(x) > a\} < \mathbb{Q}^{>0}\epsilon < \{x \in V^0 \mid \psi(x) < a\}$

(ii) $\psi(\epsilon) = a$, *and* $\psi(q\epsilon) = a$ *for all* $q \in \mathbb{Q}^{\neq 0}$.

(iii) If $\mathcal{V}$ *has no H-point and* $P_{\mathcal{V}}$ *has no maximum, then the same holds for* $\mathcal{V}_\epsilon$ *and* P_ϵ.

(iv) $\mathcal{V}_\epsilon$ *has the factoring property over* $\mathcal{V}$ *with respect to* T_ψ.

Proof. First, let us verify that $\mathcal{V}_\epsilon$ is indeed an asymptotic triple. Axiom (A1) is satisfied since we have not changed the values of ψ on V in V_ϵ. Additionally, axioms (A2) and (A4) are fulfilled due to (ii) and (i) respectively. So all that remains is to check (A3), meaning we need to verify that all pairs $v, w \in V_\epsilon$ such that $v \neq 0$ and $w > 0$ satisfy:

$$\psi(v) < (id + \psi)(w)$$

Suppose a is not an H-point, so $a < \psi(b)$ for some $b \in V^{\neq 0}$. Then by condition (i), ϵ cannot be infinitesimal with respect to V, so $V^{>0}$ is coinitial in $V_\epsilon^{>0}$, which combined with the fact that $\mathcal{V}_\epsilon$ satisfies (A4) tells us $\psi(V^{\neq 0})$ is co-final in $V_\epsilon^{\neq 0}$. Then, since ψ on V_ϵ satisfies the conditions of Lemma 4.2.4, $id + \psi$ must be increasing on $V_\epsilon^{>0}$. Since $V^{>0}$ is co-initial in $V_\epsilon^{>0}$, the set $(id + \psi)(V^{>0})$ is also coinitial in $(id + \psi)(V_\epsilon^{>0})$. Since $\mathcal{V}$ satisfies axiom (A3), we then see that $\psi(V_\epsilon^{\neq 0}) < (id + \psi)(V_\epsilon^{>0})$.

Now suppose a is an H-point, so $a > \psi(V^{\neq 0})$. Then ϵ must be infinitesimal in V, so for any $q \in \mathbb{Q}^{>0}$, we have

$$(id + \psi)(q\epsilon) = q\epsilon + \psi(\epsilon) = q\epsilon + a > a$$

Hence axiom (A3) is satisfied on $\mathcal{V}_\epsilon$.

To show that the factoring property holds over T_ψ, note that any $\mathcal{W} \models T_\psi$ extending $\mathcal{V}$ must have some element $\delta \in W$ such that $\psi(\delta) = a$.

Finally, if $\mathcal{V}$ has no H-point and $P_{\mathcal{V}}$ has no maximum, then any $a \in P_{\mathcal{V}} \setminus \psi(V^{\neq 0})$, must be in the downwards closure of $\psi(V^{\neq 0})$, so $[\epsilon] \geq [V]$, hence $[V]$ is coinitial in $[V_\epsilon]$, thus $\psi(V_\epsilon^{\neq 0})$ has no maximum. Moreover, since $V \subseteq V_\epsilon$ is a pure valuational extension, we can apply Lemma 4.2.15 to deduce $\mathcal{V}_\epsilon$ has just one H-cut. □

In the paper, it is claimed that after some (possibly transfinite) induction, we can turn any asymptotic triple into a closed one. It is simple to show that after one application of Lemma 4.2.17 and then ω applications of Lemma 4.2.18 on an arbitrary asymptotic triple, the resulting triple $(V, \psi_{\mathcal{V}}, P_{\mathcal{V}})$ has no H point and $P_{\mathcal{V}}$ has no maximum. Then presumably we keep applying Lemma 4.2.19 until $\Psi_{\mathcal{V}}$ is downwards closed, hence $\mathcal{V}$ must be closed. However, it is not clear that this process will ever terminate. To get around this problem, we can just make sure the contraction χ from Proposition 4.1.1 is well defined, then take 1-contraction hull, and the resulting structure will be a T_ψ-hull.

Lemma 4.2.20. *Let* $\mathcal{V}_0 := (V, \psi_{\mathcal{V}}, P_{\mathcal{V}})$ *be an asymptotic triple where* $P_{\mathcal{V}}$ *has a maximal element. Given* $\mathcal{V}_n$ *let* $\mathcal{V}_{n+1}$ *be the resulting asymptotic triple after applying Lemma 4.2.18, and set:*

$$\mathcal{V}_\omega := \bigcup_{i\in\omega} \mathcal{V}_i$$

Then the asymptotic couple $\mathcal{V}_\omega$ *only has one H-cut and* $\Psi_{\mathcal{V}_\omega}$ *has no maximal element. Moreover, it has the factoring property over* $\mathcal{V}$ *with respect to* T_ψ*.*

Proof. Suppose $\mathcal{V}_\omega$ has two H-cuts, so it must have some H-point a. Pick some $n \in \omega$ such that $a \in V_n$, then a must also be a H-point in $\mathcal{V}_n$, but since $\mathcal{V}_n$ only has one H-cut, this is a contradiction. Similarly, since

$$P_\omega := \bigcup_{i\in\omega} P_{i+1}$$

and each P_{i+1} does not have a maximum element, P_ω also has no max.

Since the factoring property is preserved under infinite chains of extensions, $\mathcal{V}_\omega$ must have the factoring property over $\mathcal{V}$ with respect to T_ψ. ☐

Lemma 4.2.21. *Every asymptotic triple* $\mathcal{V}$ *has an H-closure* $\mathcal{U}$*. Moreover, the domain of* $\mathcal{U}$*,* U *can be written as:*

$$U := V + \sum_{i\in I} \mathbb{Q}\delta_i$$

where $(\delta_i)_{i\in I}$ *is a collection of indeterminates such that* $[\delta_i] \notin [G]$ *and* $[\delta_i] \neq [\delta_j]$ *for all* $i, j \in I$*,* $i \neq J$*.*

Proof. Let $\mathcal{W}$ be an asymptotic triple. We construct some H-closure as follows:

Step 1: Suppose $\mathcal{W}$ has a H-point. Then apply Lemma 4.2.17 to get a new asymptotic $\mathcal{V}$ with no H-point and which factors over $\mathcal{W}$ with respect to T_ψ. Else if $\mathcal{W}$ only had one H-cut, set $\mathcal{V} := \mathcal{W}$.

Step 2: Now suppose P_V has a maximal element, then apply Lemma 4.2.20 to get some asymptotic triple $\mathcal{V}_\omega$ which only has one H-cut but P_{V_ω} has no max, and has the factoring property over $\mathcal{W}$ with respect to T_ψ^-. If P_V does not have a maximal element, then like before set $\mathcal{V}_\omega := \mathcal{V}$.

Step 3: Finally, suppose Ψ_{V_ω} is not downwards closed. Since $\psi(V_\omega^{\neq 0})$ has no maximal element, we can use Proposition 4.1.1 to define the contraction χ well defined on V_ω. Then embed (V_ω, χ) into some 1-contraction hull (U, χ). We know that for all $u \in U$, there exists some $n \in \mathbb{N}$ such that $\chi^n(u) \in V_\omega$, thus extend ψ to U via:

$$\psi(u) = \psi(\chi^n(u)) - \sum_{i=1}^{n} \chi^i(u)$$

for $u < 0$. Then by Equation (4.1), (U, ψ) is an asymptotic couple, and has the factoring property over (V_ω, ψ) with respect to $T_\psi \mid \mathcal{L}'_\psi$. By Remark 3.2.5, $V_\omega \subseteq U$ is a pure valuational extension of ordered abelian groups and $[V_\omega]$ is coinitial in $[U]$. Moreover, since (V_ω, ψ) has no H-point, we can apply Lemma 4.2.15 to deduce that (U, ψ) also has no H-point. Thus $\mathcal{U} := (U, \psi, P_{\mathcal{U}})$, where $P_{\mathcal{U}}$ is the downwards closure of $\psi(U^{\neq 0})$, has the factoring property over $\mathcal{V}_\omega$ with respect to T_ψ. It remains to show that $\mathcal{U}$ is a closed asymptotic triple. Since χ is well defined, $\psi(U^{\neq 0})$ cannot have a maximal element, and since χ is surjective, $\psi(U^{\neq 0})$ is downwards closed, thus $\mathcal{U} \models T_\psi$ and is a T_ψ-hull of $\mathcal{V}_\omega$, hence of $\mathcal{W}$.

The statement about the domain of U follows since in step 1 and step 2, we adjoin linearly independent indeterminates to V, and by Remark 3.2.5, the same is done in step 3. □

4.2.3 Model Completeness

Classifying elementary functions

Fix an extension of closed asymptotic triples $\mathcal{G} \subseteq \mathcal{H}$.

Definition 4.2.22 (Aschenbrenner and van den Dries, 2000, p. 331)**.** Let $\{a_0, a_1, a_2, \ldots\}$ be a collection of constants in some asymptotic couple $\mathcal{G}$.

For some $a \in G$, write:

$$\psi_a(x) := \psi(x - a)$$

Fix some tuple $a_1, \ldots, a_n \in G^n$. The function

$$\psi_{a_1,\ldots,a_n}(x) := \psi_{a_n} \circ \ldots \circ \psi_{a_1}(x)$$

is not defined on the entirety of G, for example when $x = a_1$, $\psi(x - a_1)$ has no value. Let $D_{a_1,\ldots,a_n}(\mathcal{G})$ be the domain of $\psi_{a_1,\ldots,a_n}$, so:

$$D_{a_1}(\mathcal{G}) := \{x \in \mathcal{G} \mid x \neq a_1\}$$
$$D_{a_1,\ldots,a_{n+1}}(\mathcal{G}) := \{x \in D_{a_1,\ldots,a_n}(\mathcal{G}) \mid \psi_{a_1,\ldots,a_n}(x) \neq a_{n+1}\}$$

Note that $D_n(\mathcal{G})$ is made up of at most 2^n convex components. When the context is clear we denote $D_{a_1,\ldots,a_m}(\mathcal{G})$ as $D_n(\mathcal{G})$, ψ_{a_i} as ψ_i and $\psi_{a_1,\ldots,a_n}$ as $\overline{\psi}_n$.

We prove some useful lemmas regarding these functions:

Lemma 4.2.23. *Let $\mathcal{G}$ be an asymptotic triple, $a_1, \ldots, a_n \in G$ and $q_1, \ldots, q_n \in \mathbb{Q}$, where not all the q_i are zero. Then on each convex component C of $D_{1,\ldots,n}(\mathcal{G})$, the function:*

$$x \mapsto x + q_1\psi_1(x) + \ldots + q_m\psi_{1,\ldots,n}(x) : D_{1,\ldots,n} \to G$$

is strictly increasing and has the intermediate value property.

Proof. Fix some convex component C of $D_{1,\ldots,n}(\mathcal{G})$, and define the function $\eta : C \to G$ as:

$$\eta(x) := q_1\overline{\psi}_1(x) + \ldots + q_n\overline{\psi}_n(x)$$

We show that η satisfies the conditions of Lemma 4.2.4, from which the result immediately follows.

Condition (1) : First we show that $[\overline{\psi}_i(x) - \overline{\psi}_i(y)] < [x - y]$ for all distinct $x, y \in C$ and $i \in [1, n]$. For $i = 1$, we have:

$$\begin{aligned}
[\overline{\psi}_1(x) - \overline{\psi}_1(y)] &= [\psi(x - a_1) - \psi(y - a_1)] \\
&< [(x - a_1) - (y - a_1)] && \text{By Proposition 4.2.6} \\
&= [x - y]
\end{aligned}$$

If we assume for some $i \in [1, n-1]$ that $[\overline{\psi}_i(x) - \overline{\psi}_i(y)] < [x - y]$, then:

$$\begin{aligned}
[\overline{\psi}_{i+1}(x) - \overline{\psi}_{i+1}(y)] &= [\psi(\overline{\psi}_i(x) - a_{i+1}) - \psi(\overline{\psi}_i(y) - a_{i+1})] \\
&< [(\overline{\psi}_i(x) - a_{i+1}) - (\overline{\psi}_i(y) - a_{i+1})] && \text{By Proposition 4.2.6} \\
&= [\overline{\psi}_i(x) - \overline{\psi}_i(u)] \\
&< [x - y] && \text{By our inductive assumption}
\end{aligned}$$

Then when we calculate the valuation of $[\eta(x) - \eta(y)]$:

$$\begin{aligned}
[\eta(x) - \eta(y)] &= \left[\sum_{i=0}^{n} q_i(\overline{\psi}_i(x) - \overline{\psi}_i(y))\right] \\
&\leq \max_{0<i\leq n} [\overline{\psi}_i(x) - \overline{\psi}_i(y)] && \text{By the ultrametric inequality} \\
&< [x - y] && \text{By Proposition 4.2.6}
\end{aligned}$$

Condition (2) Without loss of generality, assume $C > a_1$, and fix $x, y, a \in C$. It is sufficient to show that $[x - a_1] = [y - a_1]$, since $\eta(x)$ is determined by $\psi(x - a_1)$. Since $a < a_1$ we have $[x - a_1] > [x - a]$, hence $[x - a_1] > [x - y]$, which means:

$$[x - a_1] = [(x - y) - (y - a_1)] = [y - a_1]$$

Hence η satisfies the conditions of Lemma 4.2.4 so we are done. □

We should verify that the functions $\overline{\psi}_i$ have the intermediate value property when $\mathcal{G}$ is a **closed** asymptotic triple:

Lemma 4.2.24 (Aschenbrenner and van den Dries, 2000, p. 332)**.** *Let $a = (a_1, \ldots, a_n) \in G^n$, then $D_n(\mathcal{G})$ has at most 2^n convex components, and on each the function $\overline{\psi}_n$ is monotone and has the intermediate value property.*

Proof. For $n = 1$, $D_1(\mathcal{G})$ is the union:

$$\{x \in G \mid x < a_1\} \cup \{x \in G \mid x > a_1\}$$

Moreover, the image of each convex component of D_a under $\psi(x - a_1)$ is $\Psi_{\mathcal{G}}$, which itself is convex (since $(G, \psi_{\mathcal{G}}, \Psi_{\mathcal{G}})$ is closed), which combined with the monotonicity of ψ yields us the intermediate value property for $\overline{\psi}_n$.

Suppose the lemma is true for all $a \coloneqq (a_1, \ldots, a_n) \in G^n$ and let $\overline{a} \coloneqq (a_1, \ldots, a_n, a_{n+1}) \in G^{n+1}$, and fix some convex component C of $D_n(\mathcal{G})$. Define:

$$C_1 \coloneqq \{x \in C \mid \overline{\psi}_n(x) < a_{n+1}\}$$
$$C_2 \coloneqq \{x \in C \mid \overline{\psi}_n(x) = a_{n+1}\}$$
$$C_3 \coloneqq \{x \in C \mid \overline{\psi}_n(x) > a_{n+1}\}$$

It is clear that C is the disjoint union of C_1, C_2 and C_3, and the convex components of $D_{n+1}(\mathcal{G})$ arise from the non-empty elements of C_1 and C_3, hence $D_{n+1}(\mathcal{G})$ has at most 2^{n+1} convex components. The intermediate value property follows from the fact that on each C_i, where $i = 1, 3$, $\overline{\psi}_n(C_i)$ is convex and $\psi(x - a_{n+1})$ restricted to $\overline{\psi}_n(C_i)$ is a monotone function with the intermediate value property. □

Properties (A) and (B)

The proof of quantifier elimination for closed asymptotic triples essentially rests on the following two properties holding for any extension $\mathcal{G} \subseteq \mathcal{H}$ of closed asymptotic triples:

Theorem 4.2.25 (Aschenbrenner and van den Dries, 2000, p. 333)**.** *For the extension $\mathcal{G} \subseteq \mathcal{H}$ of closed asymptotic triples, the following hold:*

(A) *For any* $a_1, \ldots, a_n \in G$ *and convex component* $C_H \subseteq H$ *of* $D_n(\mathcal{H})$*, we have* $C_H \cap G \neq \varnothing$

(B) *For any* $x \in H$*,* $a_1, \ldots, a_n, g \in G$ *and* $q_1, \ldots, q_n \in \mathbb{Q}$*, if there exists some* $b \in G$ *such that:*

$$x + q_1\overline{\psi}_1(x) + \ldots + q_n\overline{\psi}_n(x) = b$$

then $x \in G$*.*

Lemma 4.2.26 (Aschenbrenner and van den Dries, 2000, p. 336)**.** *Fix some* $p \in G$*. There exists some* $\gamma \in [G]$ *such that for all sufficiently large* $x \in \Psi_{\mathcal{G}} + p$ *(and thus sufficiently large* $x \in \Psi_{\mathcal{H}} + p$*),* $[x] = \gamma$*.*

Proof. Since $\mathcal{G}$ is closed, either $-p \in \Psi_{\mathcal{G}}$ or $-p > \Psi_{\mathcal{G}}$.

$-p \in \Psi_{\mathcal{G}}$**:** Since p is sufficiently large, we see that $\psi(x_0) + p > 0$ for all sufficiently small $x \in G^{>0}$. Since ψ is decreasing on $G^{>0}$, we further deduce that:

$$[x] < [\psi(x) + p]$$

for all sufficiently small $x \in G^{>0}$. Choose a sufficiently small element $x_0 \in G^{>0}$, then for all $y \in (0, x_0)_H$, we have:

$$\begin{aligned}
[\psi(x_0) + p] &\leq [\psi(y) + p] && \text{As } \psi(y) + p \geq \psi(x_0) + p > 0 \\
&\leq [\psi(x_0) + x_0 + p] && \text{Since } (id + \psi)(G^{>0}) > \Psi_{\mathcal{G}} \\
&\leq [\psi(x_0) + p] && \text{We assumed } [x_0] < [\psi(x_0) + p]
\end{aligned}$$

Thus $[\psi(y) + p] = [\psi(x_0) + p]$, so set γ to be any element of $[G]$ smaller than $[\psi(x_0) + p]$ (Note that $[G]$ has no minimum element since $\Psi_{\mathcal{G}}$ has no maximum).

$-p > \Psi_{\mathcal{G}}$**:** We can find some $x_0 \in G^{>0}$ such that $(id + \psi)(x_0) = -p$. Then for any $y \in (0, x_0)_H$, we have:

$$\begin{aligned}
[x_0] &= [\psi(x_0) + p] && \text{Since } -x_0 = \psi(x_0) + p \\
&\leq [\psi(y) + p] && \text{As } \psi(y) \geq \psi(x_0) > 0 \text{ and } p < 0 \\
&= [(id + \psi)(x_0) - \psi(y)] && \text{We assumed } (id + \psi)(x_0) = -p \\
&= [\psi(x_0) - \psi(y) + x_0] && \text{Rearrange the term} \\
&\leq \max\{[\psi(x_0) - \psi(y)], [x_0]\} && \text{By the ultrametric inequality} \\
&\leq \max\{[x_0 - y], [x_0]\} && \text{Apply Proposition 4.2.6 to } x_0 \text{ and } y \\
&\leq [x_0] && \text{Since } x_0 > y > 0
\end{aligned}$$

Thus any element of $[G]$ smaller than $[x_0]$ works as γ. □

Remark 4.2.27. Lemma 4.2.26 shows us that for all $p \in G$,

$$\lim_{x \downarrow G^{>0}} \psi(\psi(x) + p) = \lim_{x \downarrow H^{>0}} \psi(\psi(x) + p) \in G$$

In the original paper, this is denoted as (see Aschenbrenner and van den Dries, 2000, p. 337):

$$\lim_{y \in \Psi_{\mathcal{G}} + p} \psi(y)$$

Lemma 4.2.28 (Aschenbrenner and van den Dries, 2000, p. 337)**.** *Let $\mathcal{G} \subseteq \mathcal{H}$ be an extension of closed asymptotic triples, with $(a_1, \ldots, a_n) \in G^n$. Let $D_n(\mathcal{G})$ be the domain of ψ_a in $\mathcal{G}$ and $D_n(\mathcal{H})$ be the domain in $\mathcal{H}$.*

1. *Each component C_G of $D_n(\mathcal{G})$ in contained in some unique component C_H of $D_n(\mathcal{H})$. The map $C_G \mapsto C_H$ is a bijection from the convex components of $D_n(\mathcal{G})$ to $D_n(\mathcal{H})$, and $C_G \cap G = C_H$.*

2. *$D_n(\mathcal{G})$ has a unique convex component $C_G^\infty > a_1$ that is unbounded above in G, and the corresponding $C_H^\infty > a_1$ is unbounded above in H. Similarly there is a unique convex component $C_G^{-\infty} < a_1$ that is unbounded below in G with $C_H^\infty < a_1$ unbounded below in H.*

3. *Let C_G be a bounded component of $D_n(\mathcal{G})$. There exists some $p, q \in G$ such that:*

   (a) *$\overline{\psi}_n$ increases on C_G and C_H from p to q*

   (b) *$\overline{\psi}_n$ decreases on C_G and C_H from p to q*

   (c) *$\overline{\psi}_n$ increases on C_G and C_H from p to $\Psi_{\mathcal{G}}$ and $\Psi_{\mathcal{H}}$ respectively*

   (d) *$\overline{\psi}_n$ decreases on C_G and C_H from p to $\Psi_{\mathcal{G}}$ and $\Psi_{\mathcal{H}}$ respectively*

4. *Let C_G^∞ and C_H^∞ be the components of $D_a(\mathcal{G})$ and $D_a(\mathcal{H})$ unbounded above. There exists some $p \in G$ such that either:*

   (a) *$\overline{\psi}_n$ decreases on C_G^∞ and C_H^∞ from $\Psi_{\mathcal{G}}$ and $\Psi_{\mathcal{H}}$ respectively to $-\infty$*

   (b) *$\overline{\psi}_n$ decreases on C_G^∞ and C_H^∞ from p to $-\infty$*

   Similarly on $C_G^{-\infty}$ and C_H^∞, either:

   (a') *$\overline{\psi}_n$ increases on $C_G^{-\infty}$ and C_H^∞ from $-\infty$ to $\Psi_{\mathcal{G}}$ and $\Psi_{\mathcal{H}}$ respectively*

   (b') *$\overline{\psi}_n$ increases on $C_G^{-\infty}$ and C_H^∞ from $-\infty$ to p*

Thus for any component C_G of $D_n(\mathcal{G})$ with corresponding component C_H of $D_n(\mathcal{H})$, we have:

$$\lim_{x \uparrow C_G} \overline{\psi}_n(x) = \lim_{x \uparrow C_H} \overline{\psi}_n(x)$$
$$\lim_{x \downarrow C_G} \overline{\psi}_n(x) = \lim_{x \downarrow C_H} \overline{\psi}_n(x)$$

Following the notation from (4.2.13), we have:

$$\lim_{x \uparrow C} \overline{\psi}_n(x) = -\infty \iff C_G \text{ and } C_H \text{ are unbounded above}$$
$$\lim_{x \uparrow C} \overline{\psi}_n(x) \in G \cup \{\Psi\} \iff C_G \text{ and } C_H \text{ are bounded above}$$

The expected symmetrical statements also hold for $\lim_{x \downarrow C} \overline{\psi}_n(x)$.

Proof. We induct on n. For $n = 1$, then both $D_1(\mathcal{G}) = \{x \in G \mid x \neq a_1\}$ and $D_1(\mathcal{H}) = \{x \in H \mid x \neq a_1\}$ are both made of two unbounded convex components, on the lower components $\overline{\psi}_1$ increases from $-\infty$ to Ψ and on the upper components $\overline{\psi}_1$ decreases from Ψ to $-\infty$.

Suppose the lemma is true for all tuples $(b_1, \dots, b_k) \in G^k$ where $k < n$, and let $(a_1, \dots, a_n) \in G^n$ be a tuple of length n. Pick some component C_G of $D_{n-1}(\mathcal{G})$ and let C_H be the corresponding component of $D_{n-1}(\mathcal{H})$. Without loss of generality, we can assume $C_G > a_1$ (thus $C_H > a_1$ as well). Set:

$$C_G^1 := \{x \in C_G \mid \overline{\psi}_{n-1}(x) < a_n\}$$
$$C_G^2 := \{x \in C_G \mid \overline{\psi}_{n-1}(x) = a_n\}$$
$$C_G^3 := \{x \in C_G \mid \overline{\psi}_{n-1}(x) > a_n\}$$

and define C_H^1, C_H^2 and C_H^3 similarly. Since by inductive assumption $C_H \cap G = C_H$ and:

$$\lim_{x \updownarrow C_G} \overline{\psi}_{n-1}(x) = * \iff \lim_{x \updownarrow C_G} \overline{\psi}_{n-1}(x) = *$$

for any $* \in G \cup \Psi \cup -\infty$, we must have:

$$C_G^i \neq \varnothing \iff C_H^i \neq \varnothing$$

for $i = 1, 3$. Since the components of $D_n(\mathcal{G})$ and $D_n(\mathcal{H})$ arise from the non-empty components of of C_G^1, C_G^3 and C_H^1, C_H^3, we see that (1) holds.

For $i = 1, 3$, C_G^i is unbounded above if and only if C_H^i is unbounded above, and that itself can only happen when C_G is unbounded above. A similar statement follows when C_G is unbounded below, thus we have (2).

Suppose C_G and C_H are bounded, let us analyse the behaviour of ψ_n on each C_G^i, C_H^i for

$i = 1, 3$. We know that either (3a) or (3c) holds for $\overline{\psi}_{n-1}$ on C. Without loss of generality assume $\overline{\psi}_{n-1}$ is increasing on C_G and C_H.

1. Suppose $\overline{\psi}_{n-1}$ increases from p to q on C_G and C_H. Look at the position of a_n compared to p and q:

   (a) $q \leq a_n$: Then C^3_G, C^3_H are both empty, and on both C^1_G, C^1_H, $\overline{\psi}_n$ increases from $\psi(p - a_n)$. If $q = a_n$, then $\overline{\psi}_n$ increases to Ψ, and if $q < a_n$ then $\overline{\psi}_n$ increases to $\psi(q - a_n)$.

   (b) If $p < a_n < q$, then all of C_i, C'_i are non-empty. On C^1_G, C^1_H, $\overline{\psi}_n$ increases from $\psi(p - a_n)$ to Ψ, and on C^3_G, C^3_H, $\overline{\psi}_n$ decreases from Ψ to $\psi(q - a_n)$

   (c) $a_n \leq p < q$: Both C^1_G, C^1_H are empty. If $p = a_n$, then $\overline{\psi}_n$ decreases from Ψ to $\psi(q - a_n)$ on C^3_G, C^3_H and when $p > a_n$ it decreases from $\psi(p - a_n)$ to $\psi(q - a_n)$.

2. Suppose ψ_{n-1} increases from p to Ψ on C and C'.

   (a) If $p = a_n$, then C^1_G, C^1_H are empty and on C^3_G, C^3_H, $\overline{\psi}_n$ decreases from Ψ to $\lim\limits_{x \in \Psi} \psi(x - a_n)$, which exists by Lemma 4.2.26.

   (b) If $p > a_n$ then again C^1_G, C^1_H are empty and $\overline{\psi}_n$ decreases from $\psi(p - a_n)$ to $\lim\limits_{x \in \Psi} \psi(x - a_n)$

   (c) If $p < a_n \in \Psi$ then on C^1_G, C^1_H, $\overline{\psi}_n$ increases from $\psi(p - a_n)$ to Ψ, and on C^3_G, C^3_H, it decreases from Ψ to $\lim\limits_{x \in \Psi} \psi(x - a_n)$

   (d) If $\Psi < a_n$ then C^3_G, C^3_H are empty and on C^1_G, C^1_H, $\overline{\psi}_n$ increases from $\psi(p - a_n)$ to $\lim\limits_{x \in \Psi} \psi(x - a_n)$.

Similarly if $\overline{\psi}_{n-1}$ is decreasing on C_G and C_H, we can apply a symmetrical argument, thus we have proved (3)

Now assume C_G and C_H are unbounded, since they are both bigger than a_1, they must be unbounded above but bounded below. Thus $\overline{\psi}_n$ must be decreasing on both, and $C^1 > C^2 > C^3$.

1. If $\overline{\psi}_{n-1}$ decreases from some $p \in G$ to ∞, then:

   (a) If $a_n \geq p$, then C^3_G and C^3_H are empty, and on C^1_G, C^1_H, $\overline{\psi}_n$ decreases from $\psi(p - a_n)$ to $-\infty$ if $a_n < p$, and when $a_n = p$, $\overline{\psi}_n$ decreases from Ψ to $-\infty$.

   (b) If $a_n < p$, then C^i_G, C^i_H are non-empty for $i = 1, 3$. For $i = 1$, $\overline{\psi}_n$ increases from $\psi(p - a_n)$ to Ψ, and for $i = 3$, $\overline{\psi}_n$ decreases from Ψ to $-\infty$.

2. If $\overline{\psi}_{n-1}$ decreases from some Ψ to ∞, then:

   (a) If $a_n > \Psi$, then C_3, C'_3 are empty, and on C_1, C'_1, ψ_a decreases from $\lim\limits_{x \in \Psi} \psi(x - a_n)$ to $-\infty$.

(b) If $a_n \in \Psi$, then on C_1, C_1', ψ_a decreases from Ψ to $-\infty$ and on C_3, C_3' it increases from $\lim_{x \in \Psi} \psi(x - a_n)$ to Ψ.

A symmetrical argument applies when C_G, C_H are unbounded below, thus we have (4). □

Lemma 4.2.29 (Aschenbrenner and van den Dries, 2000, p. 340)**.** *Let $F(x) := x + \eta(x)$ where:*

$$\eta(x) := \sum_{i=1}^{n} q_i \psi_i \circ \ldots \circ \psi_1(x)$$

Then for any $d \in D_n' \setminus D_n$ we have $F(d) \notin G$

Proof. We induct on n. For $n = 1$, the statement follows since $(id + q_i\psi)(x)$ is strictly increasing and unbounded above on G. Suppose the statement is true for $n - 1$. Let C_H be the component of $D_n(\mathcal{H})$ that d lies in. If d is in the convex hull of C_G in H, then the statement again follows immediately since by Lemma 4.2.4 the function F has the intermediate value property and is strictly increasing on G and H.

So assume d is not in the convex hull of C. We know that $\overline{\psi}_n$ does one of three things on the boundary of C that d lies on:

1. If $\lim_{x \to d} q_n \overline{\psi}_n(x) = p \in G$, then set:

   $$F_{n-1}(x) := x + \sum_{i=1}^{n-1} q_i \psi_i \circ \ldots \circ \psi_1(x)$$

   then by induction we know $F_{n-1}(d) \notin G$, so:

   $$\begin{aligned} F(d) &= F_{n-1}(d) + q_n \overline{\psi}_n(d) \\ &= F_{n-1}(d) + p \notin G & \text{Since } F_{n-1}(d) \notin G \text{ and } p \in G \end{aligned}$$

   So the statement is proved.

2. If $\lim_{x \to d} q_n \overline{\psi}_n(x) = -\infty$, then we know F has the intermediate value property and is unbounded on C, we must have $F(d) \notin G$.

3. If $\lim_{x \to d} q_n \overline{\psi}_n(x) = \Psi$ then the proof is more complicated.

   Claim 4.2.29.1. *There exists some c in G in-between d and a_1.*

   Proof. Suppose $d > a_1$, the converse has a symmetrical proof. Assume there is no c in G. Consider the convex component B_H of $D_{n-1}(\mathcal{H})$ containing d, since we assumed there is no element of G between d and a_1, we deduce that d is not in the convex hull of B_G in H. Moreover since $B_G > a_1$, we must have $d < B_G$. By Lemma 4.2.28, we know what $\overline{\psi}_{n-1}$ does on B_G and B_H, specifically:

$$\lim_{x\to d} \overline{\psi}_{n-1}(x) \in G \cup \Psi$$

(a) If the limit of $\overline{\psi}_{n-1}$ approaching d is Ψ, then by Lemma 4.2.26, we must have:

$$\lim_{x\to d} \overline{\psi}_n(x) = \lim_{x\in\Psi} \psi(x - a_n) \in G$$

which contradicts our assumption that the limit of $\overline{\psi}_n$ approaching d is Ψ.

(b) If the limit of $\overline{\psi}_{n-1}$ approaching d is an element of G, then since $d \notin B_G$, we have:

$$\lim_{x\to d} \overline{\psi}_{n-1}(x) = \overline{\psi}_{n-1}(d) \in G$$

Hence $\overline{\psi}_{n-1}(d) \neq a_n$, else d would not be in $D_n(\mathcal{H})$. Thus:

$$\lim_{x\to d} \overline{\psi}_n(x) = \psi(\overline{\psi}_{n-1}(x) - a_n) \in G$$

which again contradicts our assumption that the limit of $\overline{\psi}_n$ approaching d is Ψ.

■

Claim 4.2.29.2. *Suppose there exists some $c \in G$ between d and a_1. Set $\epsilon := \frac{1}{2}|c - a_1|$, and define:*

$$I = I_c := \{x \in H \mid d - \epsilon \leq x \leq d + \epsilon\}$$

then $I \cap C_G = \varnothing$.

Proof. It can be verified that I is a subset of C_H and $\overline{\psi}_n$ is constant on I. But if there was some $b \in I \cap C_G$, that would means $\lim_{x\uparrow C} \overline{\psi}_n(x) = \overline{\psi}_n(d)$ which is a contradiction since we assumed $\overline{\psi}_n$ tends to Ψ on the boundary of C that d is on. ■

Pick any $c \in G$, then by the claim, $I_c \cap C \neq \varnothing$. Since $\overline{\psi}_n(x)$ approaches Ψ as $x \to d$, we know that:

$$\lim_{x\to d} \overline{\psi}_{n-1}(x) - a_n = 0$$

Thus pick some $b \in C_G$ such that:

$$\left|\overline{\psi}_{n-1}(b) - a_n\right| \leq \epsilon$$

Then:

$$\begin{aligned}[\overline{\psi}_n(d) - \overline{\psi}_n(b)] &< [\overline{\psi}_{n-1}(d) - \overline{\psi}_{n-1}(b)] && \text{By Proposition 4.2.6} \\ &\leq [\overline{\psi}_{n-1}(b) - a_n] && \text{Since } \overline{\psi}_{n-1}(b) < \overline{\psi}_{n-1}(d) < a_n \\ &\leq [\epsilon]\end{aligned}$$

Define $f : I \to H$ as:

$$f(x) := F_{n-1}(x) + q_n \overline{\psi}_n(b)$$

Then:

$$\begin{aligned}F_{n-1}(d) - f(d-\epsilon) &= \epsilon + F(d-\epsilon) - f(d-\epsilon) \\ &= \epsilon + q_n(\overline{\psi}_n(d) - \overline{\psi}_n(b)) > 0\end{aligned}$$

Hence $F(d) > f(d-\epsilon)$, and similarly $F(d) < f(d+\epsilon)$. So by the intermediate value property for $F_{n-1}(x)$, there exists some $x \in I$ such that $F(d) = f(x)$. Since we are assuming $I \cap C_G = \varnothing$, we know that $x \notin G$, hence by induction, we have $f(x) \notin G$, so $F(d) \notin G$.

□

Proof of Theorem 4.2.25. Property (A) follows from point (1) in Lemma 4.2.28, and property (B) is the contrapositive of Lemma 4.2.29. □

4.2.4 Existential Closedness

Lemma 4.2.30 (Aschenbrenner and van den Dries, 2000, p. 332)**.** *Let $G \subseteq H$ be an extension of divisible ordered abelian groups. If some element $b \in H \setminus G$ satisfies:*

1. *For all $\epsilon \in G^{>0}$, there exists some $a, c \in G$ such that $a < b < c$ and $c - a < \epsilon$*
2. *The set $\{a \in G \mid a < b\}$ has no maximum, and $\{c \in G \mid c > b\}$ has no minimum*

Then $[G] = [G + \mathbb{Q}b]$.

The following lemma is proved with respect to the natural valuation in Corollary 4.1 of Aschenbrenner and van den Dries, 2000, p. 333, but that result cannot be applied without the scalar field, thus we prove it with respect to the convex valuation induced by ψ.

Lemma 4.2.31. *Let $\mathcal{G} \subseteq \mathcal{H}$ be an extension of H-triples, where $\Psi_{\mathcal{G}}$ has no maximum. For any $x \in H$ with $\psi(x) > \Psi_{\mathcal{G}}$, we have*

$$[G + \mathbb{Q}\psi(x)] = [G]$$

Proof. Let $b := \psi(x)$ we show it satisfies the conditions in Lemma 4.2.30. For $\epsilon \in G^{>0}$, since ψ is decreasing on positive elements, we have $[x] < [\epsilon]$. Pick ϵ small enough such that $\psi\epsilon > 0$ and:

$$[\psi(x) - \psi(\epsilon)] < [x - \epsilon] = [\epsilon]$$

Set $a := \psi(\epsilon) + \frac{\epsilon}{4}$ and $c := \psi(\epsilon) - \frac{\epsilon}{4}$, then $b \in (a, c)$ and $c - a < \epsilon$, hence the first condition is satisfied, the second holds since $\Psi_{\mathcal{G}} < b < (id + \psi)(G^{>0})$. □

Definition 4.2.32. Let $\mathcal{G} \subseteq \mathcal{H}$ be an extension of closed asymptotic couples, and $b \in H \setminus G$. A sequence $(b_i, g_i)_{i\in\omega}$ is a ψ-characteristic sequence of b in $\mathcal{G}$ if:

- If $[G] \subsetneq [G + \mathbb{Z}b]$ then $b_0 = b - g_0$ and $[b_0] \notin [G]$.
- If $[G] = [G + \mathbb{Z}b]$ then $b_0 = b$ and $g_0 = 0$.
- For any $i \geq 1$, if $b_i \notin G$ then:
  - If $[G + \mathbb{Z}\psi(b_i)] = [G]$ then $g_{i+1} = 0$ and $b_{i+1} = \psi(b_i)$
  - If $[G] \subsetneq [G + \psi(b_i)]$, then $b_{i+1} = \psi(b_i) - g_{i+1}$ and $[b_{i+1}] \notin [G]$.
- If $b_i \in G$ then $b_{j+1}, g_j = 0$ for all $j \geq 1$.

Lemma 4.2.33. *Fix an extension $\mathcal{G} \subseteq \mathcal{H}$ of closed asymptotic couples. For any $b \in H \setminus G$, there exists a ψ-characteristic sequence $(b_i, g_i)_{i<\omega}$ of b in $\mathcal{G}$, and moreover, for any $n < \omega$, if $[b_n] \notin [G]$, then $[b_i] \notin [G]$ and $[b_i] \neq [b_j]$ for all $i, j \leq n$.*

Proof. The construction of a ψ-characteristic sequence is evident from the definition. Suppose $b_i, b_j \notin G$ and $[b_i] = [b_j]$ for $j < i < \omega$. Then $\psi(b_i) = \psi(b_j)$. But from the definition of ψ-characteristic sequences, we know:

$$b_i = \psi_{g_{i-1}} \circ \ldots \circ \psi_{g_{j+1}} \circ \psi(b_j) - g_i$$

Hence:

$$\psi(b_j) - \psi_{g_i}(\psi_{g_{i-1}} \circ \ldots \circ \psi_{g_{j+1}} \circ \psi(b_j)) = 0$$

Which if we set $x := \psi(b_j)$ contradicts property (B), thus $[b_i] \neq [b_j]$. □

Definition 4.2.34. We say a ψ-characteristic sequence $(b_i, g_i)_{i<\omega}$ of b has length $\alpha < \omega$ if α is the least ordinal such that $b_\alpha = 0$. If there is no such α then we say it has length ω.

Write G_b to be the ordered abelian group:

$$G + \sum_{i<\omega} \mathbb{Q}b$$

If b has a finite length characteristic sequence we say it is **ψ-algebraic**, else b is **ψ-transcendental**. If b ψ-algebraic, with characteristic sequence of length $\alpha + 2$, with $[G + \mathbb{Q}b_\alpha] = [G]$, then we say b is **ψ-archimedean-algebraic**, else we say b is **ψ-value-algebraic**.

Lemma 4.2.35. *Let $(b_i, g_i)_{i<\omega}$ be a ψ-characteristic sequence of b. Then $\mathcal{G}\langle b\rangle_{\mathcal{H}}$ is the structure:*

$$(G_b, \psi, G_b \cap P_{\mathcal{H}})$$

Proof. Suppose $(b_i, g_i)_{i<\omega}$ has length α, then by Lemma 4.2.33, we have:

$$[G_b] = [G] \cup \bigcup_{i\in\omega} [b_i]$$

Thus it is sufficient to show for all non-zero b_i that $\psi(b_i) \in G_b$. But this follows since:

$$\psi(b_i) := b_{i+1} + g_{i+1} \in G_b$$

□

Lemma 4.2.36. *Fix an extension $\mathcal{G} \subseteq \mathcal{H}$ of closed asymptotic triples, and $b \in H \setminus G$. If:*

(i) $b \in H \setminus G$ has a characteristic sequence of length two, (so $\psi(G+\mathbb{Q}b) \subseteq G$ and hence $\psi(G + \mathbb{Q}b) \subseteq \psi(G^{\neq 0})$)

(ii) $\mathcal{G}\langle b\rangle$ only has one H-cut.

Then $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b\rangle$.

Proof. Let $p_b(x)$ be the set of formulas:

$$p_b(x) = C_b(x) \cup \{\psi(x - g_0) = \psi(b - g_0)\}$$

Since $\mathcal{G}$ is closed, every finite subset $\Delta(x) \subseteq p_b(x)$ is realised in $\mathcal{G}$ thus p_b is a type of $\mathcal{G}$. Fix some other extension $\overline{\mathcal{H}} \subseteq \mathcal{G}$ of closed asymptotic couples with $\overline{b} \in \overline{H} \setminus G$ and $\overline{b} \models p_b(x)$. By $C_b(x)$ and Lemma 2.3.13, we have as isomorphism ψ of ordered abelian groups:

$$\psi : G + \mathbb{Q}b \to G + \mathbb{Q}\overline{b} : b \mapsto \overline{b}$$

And the formula $\psi(x - g_0) = \psi(b - g_0)$ tells us that it also respects ψ. To show it is an isomorphism of asymptotic couples, it is sufficient to show the asymptotic couple $(G + \mathbb{Q}\bar{b}, \psi)$ only has one H-cut, but this follows from Lemma 4.2.15. □

Lemma 4.2.37. *Fix an extension $\mathcal{G} \subseteq \mathcal{H}$ of closed asymptotic triples, and $b \in H \setminus G$. If:*

(i) $b \in H \setminus G$ has a characteristic sequence of length two.

(ii) $\mathcal{G}\langle b\rangle$ has two H-cuts.

Then $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b\rangle$.

Proof. Suppose some $u := g + qb$ lies in the gap:

$$\psi(G_b) < u < (id + \psi)(G_b^{>0})$$

where $q \in \mathbb{Q}^{\neq 0}$ and $g \in G$. We claim that u uniquely satisfies:

$$\psi(G) < u < (id + \psi)(G^{>0})$$

First, notice that $\psi(G_b) = \Psi_{\mathcal{G}}$, by assumption, which combined with $\Psi_{\mathcal{G}}$ not having a maximal element, tells us that $[G]$ is co-initial in $[G_b]$. Hence $(id + \psi)(G^{>0})$ is co-initial in $(id + \psi)(G_b^{>0})$, meaning for all $w \in \mathcal{G}_b$, we have:

$$\Psi_{\mathcal{G}} < w < (id + \psi)(G^{>0}) \iff \psi(G_b) < w < (id + \psi)(G_b^{>0}) \tag{4.4}$$

Hence by Proposition 4.2.9, u must be unique. Without loss of generality, set $b := u = g + qb$, and $Z(x), C(x), p_b(x)$ as follows:

$$Z(x) := \begin{cases} P(x) & \text{if } P_b(u) \\ \neg P(x) & \text{otherwise} \end{cases}$$
$$C(x) := \{\gamma < x < \overline{\gamma} \mid \gamma, \overline{\gamma} \in G, \gamma < b < \overline{\gamma}\}$$
$$p_b(x) := C(x) \cup \{Z(x)\}$$

Let us verify that the cut $C(x)$ determines the values of ψ on G_b. We claim that $[G_b] = [G]$. Suppose not, so there exists some $a \in G$ such that $[b - a] \notin [G]$. Since $\psi(b - a) \in \Psi_{\mathcal{G}}$ and $\Psi_{\mathcal{G}}$ has no maximum, we can find some $\epsilon \in G^{>0}$ such that $[\epsilon] < [b - a]$. Then there cannot be any $\overline{g} \in G$ between $b - a$ and $b - a + \epsilon$, else we would have $[\overline{g}] = [b - a] \notin [G]$, which means b and $b + \epsilon$ induce the same cut in G, and this contradicts Proposition 4.2.9 via the equivalence in (4.4).

Hence $[G_b] = [G]$, so for all $g' + qb \in G$, we can find some $c, d \in G$ with $c < |g' + qb| < d$ and $[c] = [g' + qb] = [d]$. This tells us that the type $p_b(x)$ determines the quantifier free type of b over $\mathcal{G}$. To show p_b is finitely realised, pick some finite subset Δ, which we can without loss of generality, write as:

$$\Delta(x) \coloneqq a < x < \overline{a} \wedge Z(x)$$

Where $a, \overline{a} \in G$. Since $\mathcal{G}$ is closed, we know $a \in \psi(G^{\neq 0})$ and $\overline{a} \in (id + \psi)(G^{>0})$, thus $(a, \overline{a})_G$ intersects both the sets $\psi(G^{\neq 0})$ and $(id + \psi)(G^{>0})$. Thus we can always find some element in G satisfying Δ, so p_b is a type of $\mathcal{G}$. Then by applying Fact 2.1.12 we deduce that $\mathcal{G}$ is existentially closed in (G_b, ψ, P_b) is existentially closed. □

Lemma 4.2.38. *If b is ψ-algebraic, then $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b\rangle$*

Proof. Fix some ψ-characteristic sequence $(b_i, g_i)_{i<\omega}$, of length $n+2$, so $b_{n+1} \in G^{\neq 0}$ and $b_n \notin G$. We know that by either Lemma 4.2.36 or Lemma 4.2.37 that $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b_n\rangle$. Let $\overline{\mathcal{G}}$ be some H-closure of $\mathcal{G}\langle b_n\rangle$. Recall from Lemma 4.2.21 $\overline{G}$ can be written as the sum:

$$G + \sum_{i\in I} \mathbb{Q}\delta_i$$

where δ_i are indeterminates, we can choose an embedding $\phi : \overline{\mathcal{G}} \hookrightarrow \mathcal{H}$ such that $b_{n-1} \in \phi(\overline{G})$. This is because there must be some $\delta_{n-1} \in \overline{G}$ such that $\psi(\delta_{n-1}) = b_n + g_{n-1} = \psi(b_{n-1})$, and we can set $\phi(\delta_{n-1}) \coloneqq b_{n-1}$. Similarly, there must be some $\delta_{n-1} \in G_n$ such that $\psi(\delta_{n-2}) = \delta_{n-1} = g_{n-2}$, so we can set $\phi(\delta_{n-2}) \coloneqq b_{n-2}$. By repeating this, we deduce that $b \in \phi(\overline{G})$, thus $\mathcal{G}\langle b\rangle$ embeds into $\mathcal{G}\langle b_n\rangle$, thus $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b\rangle$. □

Lemma 4.2.39. *Suppose $[G]$ is coinitial in $[H]$, and b is ψ-transcendental. Then $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b\rangle$.*

Proof. Fix some ψ-characteristic sequence $(b_i, g_i)_{i<\omega}$. Let $p_b(x)$ be the set of formulas:

$$p_b(x) \coloneqq \bigcup_{i<\omega} C_{b_n}(x_n)$$

where x_n is defined as expected:

$$x_n \coloneqq \begin{cases} x - g_n & \text{If } n = 0 \\ \psi(x_{n-1}) - g_n & \text{If } n > 0 \end{cases}$$

Define some finite subset Δ of p_b as:

$$\Delta(x) \coloneqq \bigwedge_{i \le n} a_i < x_i < \overline{a}_i$$

Let C_H be the convex component of $D_n(\mathcal{H})$ that contains b, and let C_G be the corresponding component of $D_n(\mathcal{G})$. Suppose b is in the convex hull of C_G in H.

Claim 4.2.39.1. *There exists some $h, \overline{h} \in C_G$ such that for all $i \le n$:*

1. *If $\overline{\psi}_{i-1}$ is increasing on C_G then*

$$a_i < x_i(h) < b_i < x_i(\overline{h}) < \overline{a}_i$$

2. *If $\overline{\psi}_{i-1}$ is increasing on C_G then*

$$a_i < x_i(\overline{h}) < b_i < x_i(h) < \overline{a}_i$$

Proof. We know each $\overline{\psi}_i$ is monotone and has the intermediate value property on C_G, thus so is $x_i \coloneqq \overline{\psi}_{i+1}(x) - g_i$.

Assuming x_i is increasing, set $h_i \in (a_1, b)_G$ to be some element such that

$$a_i < x_i(h_i) < x_i(b) = b_i$$

If x_i is decreasing, choose h_i so that:

$$x_i(b) < x_i(h_i) < \overline{a}_i$$

Define $\overline{h}_i$ conversely. Then set:

$$h \coloneqq \max\{h_1, \dots, h_n\}$$
$$\overline{h} \coloneqq \min\{\overline{h}_1, \dots, \overline{h}_n\}$$

■

Thus $\frac{h+\overline{h}}{2}$ realises Δ.

Suppose b is not in the convex hull of C_G in H, then one of the following happens:

1. $\lim_{x \to b} \overline{\psi}_n = \Psi$: If this happens, then $\overline{\psi}_{n+1}(b) = \lim_{y \in \Psi - g_{n+1}} \psi(y) \in G$ which is a contradiction since we assumed b was ψ-transcendental.
2. $\lim_{x \to b} \overline{\psi}_n = p \in G$: This is also contraidctory since it implies $\overline{\psi}_n(b) = p \in G$ which means b is ψ-algebraic.

3. $\lim_{x \to b} \overline{\psi}_n = -\infty$. This can only happen if $b > G$ or $b < G$. Either way, a sufficiently large or negatively large element of G will realise Δ.

Thus p_b is a type. We now need to show it determines the quantifier free type of b in $\mathcal{G}$. Pick some other realisation $\overline{b}$ of p_b. By Lemma 2.3.13 we have an isomorphism of ordered abelian groups:

$$\phi : G + \sum_{i \in \omega} \mathbb{Q} b_i \to G + \sum_{i \in \omega} \mathbb{Q} \overline{b}_i : b_i \to \overline{b}_i$$

where $\overline{b}_i := x_i(\overline{b})$. Moreover, ϕ is an isomorphism of asymptotic couples. To show it is an isomorphism of asymptotic triples, it is sufficient to show both asymptotic couples only have one H-cut. Suppose (G_b, ψ) has more than one H-cut, so there is some $g \in G$ and $q_i \in \mathbb{Q}$ not all zero such that:

$$\psi(G_b) < d := g + q_0 b_0 + \ldots q_n b_n < (id + \psi)(G_b^{>0})$$

We assumed $[G]$ is coinitial in $[H]$ and since $\mathcal{G}$ and $\mathcal{H}$ are closed, both have no minimum, thus $\psi(G_b)$ is cofinal in $\Psi_{\mathcal{H}}$ and $(id + \psi)(G_b^{>0})$ is coinitial in $(id + \psi)(H^{>0})$, thus:

$$\Psi_{\mathcal{H}} < d < (id + \psi)(H^{>0})$$

which is a contradiction since $\mathcal{H}$ is closed. By the same argument $(G_{\overline{b}}, \psi)$ also just has one H-cut, thus by Lemma 4.2.11, the map ϕ is an $\mathcal{L}_\psi$-isomorphism.

Hence by Fact 2.1.12, we see that $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b \rangle$ □

Finally, we have:

Theorem 4.2.40 (Aschenbrenner and van den Dries, 2000, p. 355)**.** *The theory of closed asymptotic triples has quantifier elimination.*

Proof. Use the criteria in Theorem 2.1.15. The closure property follows from Lemma 4.2.21, and Lemma 4.2.38, Lemma 4.2.39 give existential closedness for both ψ-algebraic and ψ-transcendental elements respectively. Note that if $b \in H \setminus G$ is infinitesimal with respect to G, then due to Lemma 4.2.30, it must be ψ-algebraic. Put all the infinitesimal elements first in our ordering on $H \setminus G$, thus when proving existential closedness for ψ-transcendental elements we can assume $[G]$ is coinitial in $[H]$, hence we can apply Lemma 4.2.39. The initial structure is the structure generated by the constant 1. □

The main difference between this proof and the two-sorted proof is that in the two sorted structure, the quantifier free type of some $b \in H \setminus G$ is determined purely from the cut of b, and whether or not it is an H-point. Because of (A4'), if b adds a new valuation to

G, then it must immediately be ψ-transcendental and the cuts induced by the elements of the ψ-characteristic sequence are determined just from the cut of b. Thus to determine whether two structures $\mathcal{G}\langle b\rangle$ and $\mathcal{G}\langle \overline{b}\rangle$ are isomorphic over $\mathcal{G}$, we only need to look at the cut induced by b and $\overline{b}$, and can disregard $(b_1, \overline{b}_1, b_2, \overline{b}_2 \ldots)$. Once we weaken the strict inequalities in (A4'), this does not happen, thus we had to look at the cuts induced by the entire characteristic sequence.

4.3 θ-Contraction Groups

Definition 4.3.1. Let $\mathcal{L}_\theta$ be the language $\langle +, -, <, 0, \chi, \theta\rangle$, where χ and θ are unary functions. Let T_θ^- be the universal theory asserting:

1. $+, -, <, 0$ form an ordered abelian group

2. χ, θ form centripetal precontraction groups

(Cθ) $\forall x(\theta\chi x = \theta x - \chi x)$

Let T_θ be the theory T_θ^- along with surjectivity for χ and θ and divisibility for the ordered abelian group. We call a model of T_θ (T_θ^-) a **θ-(pre)contraction group**.

We prove that T_θ has quantifier elimination and is complete.

Notation 4.3.2. Sometimes we may write $\mathcal{L}_1$, T_1 as $\mathcal{L}_\chi$ and T_χ.

4.3.1 Closure Property

Lemma 4.3.3. *Let $\mathcal{G} := (G, \chi, \theta)$ be a θ-precontraction group, and let $\overline{\mathcal{G}} = (\overline{G}, \chi)$ be a 1-contraction hull. There exists a unique extension of θ to $\overline{G}$ such that $(\overline{G}, \chi, \theta)$ is an θ-precontraction group.*

Proof. For any $g \in \overline{G} \setminus G$, then there exists some $n \in \omega$ such that $\chi^n(g) \in G$. Define $\theta(g)$ to be:

$$\theta(g) := \chi(g) + \ldots + \chi^n(g) + \theta\chi^n(g)$$

It is easy to verify that $(\overline{G}, \chi, \theta)$ is a model of T_θ^-, and by (Cθ) this is the only possible extension. $\square$

Lemma 4.3.4. *Let (G, χ, θ) be a θ-precontraction group with 1-contraction-hull $\overline{G}$. Then $(\overline{G}, \chi, \theta)$ has the factoring property over $\mathcal{G}$ with respect to T_θ.*

Proof. Fix some $(H, \chi, \theta) \models T_\theta$ and an embedding $f : (G, \chi, \theta) \to (H, \chi, \theta)$. Since (H, χ) is a contraction group, we already have some $\mathcal{L}_\chi$-homomorphism $h : (\overline{G}, \chi) \to (H, \chi)$ such that $f = h \circ i$. It remains to show that h is a homomorphism of $\mathcal{L}_\theta$, for which it is sufficient to show that h commutes with θ. Pick some $g \in \overline{G}$, and suppose $\chi^n g \in \mathcal{G}$ for some $n \in \omega$, then:

$$\begin{aligned} h(\theta g) &= h(\chi g + \ldots + \chi^n g + \theta(\chi^n g)) && \text{Repeatedly apply (C}\theta\text{)} \\ &= h(\chi g) + \ldots + h(\chi^n g) + h(\theta \chi^n g) && h \text{ is a group homomorphism} \\ &= \chi(h(g)) + \ldots + \chi^n(h(g)) + \theta(\chi^n(h(g))) && h \text{ is an } \mathcal{L}_\chi\text{-homomorphism} \\ &= \theta(h(g)) && \text{Repeatedly un-apply (C}\theta\text{)} \end{aligned}$$

□

Lemma 4.3.5. *Let $\mathcal{G}$ be a θ-precontraction group. If χ is surjective on G, then so is θ.*

Proof. Let $b \in G$ be non-zero. Since χ is surjective there exists some $c \in G$ such that $\chi(c) = b - \theta(b)$. Then we have:

$$\begin{aligned} \theta(c) &= \chi(c) - \theta(\chi(c)) && \text{By (C}\theta\text{)} \\ &= b - \theta(b) + \theta(b - \theta(b)) && \text{Since } \chi(c) = b - \theta(b) \\ &= b - \theta(b) + \theta(b) && \text{Since } [b - \theta(b)] = [b] \text{ and Axiom (CA)} \\ &= b \end{aligned}$$

□

Theorem 4.3.6. *Every model of T_θ^- has a T_θ-hull*

Proof. Let $\mathcal{G}$ be a θ-precontraction group, and let $\overline{\mathcal{G}}$ be a 1-contraction hull. Then by Lemma 4.3.3, $(\overline{G}, \chi, \theta)$ is a $\mathcal{L}_\theta$-structure and a model of T_θ^-, by Lemma 4.3.4, it has the factoring property with respect to $\mathcal{G}$, and by Lemma 4.3.5, it is a model of T_θ. □

We will use the term **θ-contraction hull** when referring to a T_θ-hull.

4.3.2 Initial Structures

Definition 4.3.7. Pick any element a of some θ-contraction group. Then define $H(a)$ as:

$$H(a) := \{q_o a + q_1 \chi a + \ldots + q_n \chi^n a + q_{n+1} \theta \chi^n a \mid q_i \in \mathbb{Z}, n \in \mathbb{N}\}$$

Lemma 4.3.8. *Let $\mathcal{G}$ be a θ-precontraction group, and $a \in G$. Then the domain of the structure $\langle a \rangle$ generated by a is $H(a)$, and hence $H(a)$ is a θ-precontraction group.*

Proof. First we verify that $H(a)$ is an ordered abelian group. Pick any two elements $r, s \in H(a)$, written as:

$$r = r_0 a + r_1 \chi a + \ldots + r_n \chi^m a + r_{m+1} \theta \chi^m a$$
$$s = s_0 a + s_1 \chi a + \ldots + s_n \chi^n a + s_{n+1} \theta \chi^n a$$

Where $r_i, s_i \in \mathbb{Q}$, and $m > n$. Then we can write $\theta \chi^n a$ as

$$\chi^{n+1} a + \ldots + \chi^m a + \theta \chi^m a$$

Hence (by setting $q_i = q_{n+1}$ for $i > n$) we can write $r + s$ as:

$$\sum_{i=0}^{m} (r_i + s_i) \chi^i a + (r_m + s_m) \theta \chi^m a$$

So $H(a)$ is indeed an ordered abelian group.

Note that $[a] > [\chi a] > \ldots > [\chi^n a] > [\theta \chi^n a]$, hence for any element

$$q \coloneqq q_o a + q_1 \chi a + \ldots + q_n \chi^n a + q_{n+1} \theta \chi^n a$$

of $H(a)$, we know $[q] = [\chi^n a]$ where n is the least natural number such that $q_n \neq 0$. Hence to show that $H(a)$ is closed under the contractions, it is sufficient to show that $\delta(\chi^i a) \in H(a)$, for any $\delta \in \{\chi, \delta\}$ and $n \in \mathbb{N}$, but this is evident from the definition of $H(a)$, hence $\langle a \rangle = (H(a), \chi, \theta)$. □

Lemma 4.3.9. *There is a θ-contraction group that embeds into every other θ-contraction group.*

Proof. Let $Q \subseteq \prod_{-\mathbb{N}} \mathbb{Q}$ be the subset of the Hahn product with eventually constant support, so

$$Q \coloneqq \{(r_i)_{i \in \mathbb{N}} \mid \exists j \in \mathbb{N}, r \in \mathbb{Q} \text{ such that } \forall i > j, r_i = r\}$$

We can define contractions on Q in the expected way (where $(r)_\alpha$ is the sequence with r in the α-th spot and 0 everywhere else):

$$\chi(1)_n = (1)_{n+1}$$
$$\theta(1)_n = \sum_{i > n+1} (1)_i$$

From Lemma 4.3.8 we see that (Q, χ, θ) is isomorphic to $\langle a \rangle$ for any element a of some θ contraction group. So by taking a θ-hull, the lemma is proved. □

4.3.3 Existential closedness

Let $\mathcal{G} \subseteq \mathcal{H}$ be an extension of θ-contraction groups, and pick some $b \in H \setminus G$. We need to find a generating set for $\mathcal{G}\langle b\rangle$ as an ordered abelian group. Let $(b_i)_{i\in\omega}, (g_i)_{i\in\omega}$ be a 1-characteristic sequence of b in $\mathcal{G}$.

Lemma 4.3.10. *Suppose b **is not** χ-super-transcendental, then $\mathcal{G}\langle b\rangle$ has domain:*

$$G_b := G + \sum_{i\in\omega} \mathbb{Z} b_i$$

where $(b_i, g_i)_{i<\omega}$ is any χ-characteristic sequence.

Proof. We deal with the 1-transcendental case only, the 1-algebraic case is very similar. By Equation (3.2), we know G_b is closed under χ, so it remains to show it is also closed under θ.

Recall from Definition 3.2.6 that G_b has natural valuation:

$$[G] \cup \bigcup_{i<\omega} [b_i]$$

so we need to show $\theta(b_i) \in G_b$ for all i. Fix some $i \in \omega$, let $j \in \omega$ be the least natural number bigger than i such that $g_j \neq 0$, which means for all $k \in (i, j)$, we have $b_k = \chi^{k-i}(b_i)$, and $b_j = \chi^{j-i}(b_i) - g_j$ thus:

$$\begin{aligned}\theta(b_i) &= \chi(b_i) + \ldots + \chi^{j-i-1}(b_i) + \theta(\chi^{j-i}(b_i)) \\ &= b_{i+1} + \ldots + b_{j-1} + \theta(\chi(b_{j-1}))\end{aligned}$$

But from the definition of χ-characteristic sequences, we know that $[\chi(b_{j-1})] = [g_j]$, hence:

$$\theta(b_i) = b_{i+1} + \ldots + b_{j-1} + \theta(g_j)$$

which is an element of G_b, so we are done. ∎

Lemma 4.3.11. *Let b be 1-transcendental with a zero shift sequence (so $g_i = 0$ for all $i \in \omega$). Then $\mathcal{G}\langle b\rangle$ has domain $G + K(b)$, where:*

$$K(b) := \sum_{\substack{i<2 \\ j<\omega}} \mathbb{Z}\theta^i\chi^j(b)$$

Proof. For certain we know that $G_b := G + K(b) \subseteq |\mathcal{G}\langle b\rangle|$. For the converse we need to show that G_b is closed under χ and θ. First we claim that the natural valuation of $K(b)$ is just $\{[b_i] \mid i < \omega\}$, where $b_i := \chi^i(b)$

Claim 4.3.11.1. *The natural valuation of $K(b)$ is just $\{[b_i] \mid i < \omega\}$.*

Proof. Pick some $x \in K(b)$, written as:

$$\sum_{i=0}^{n} (z_{0,i} b_i + z_{1,i} \theta b_i)$$

By ((Cθ)), we can write:

$$x \coloneqq \theta(b_j) \coloneqq \theta\chi^i(b) = \chi^{i+1}(b_j) + \ldots + \chi^n(b_j) + \theta\chi^n(b_j)$$

Thus x can be written as:

$$x = \sum_{i=0}^{n} \left(z_{0,i} + \sum_{j=0}^{i-1} z_{1,j} \right) \chi^i(b) + \left(\sum_{i=0}^{n} z_{1,i} \right) \theta\chi^n(b)$$

But we know that $[b] > [\chi(b)] > [\chi^2(b)] > \ldots > [\chi^n(b)] > [\theta\chi^n(b)] = [\chi^{n+1}(b)]$, thus $[x] = \chi^i(b)$ for some $i < \omega$. ■

Moreover since b is χ-transcendental, we know that $[\chi^i(b)] \notin [G]$ for all i, thus $[K(b)] \cap [G] = \varnothing$, so:

$$[G + K(b)] = [G] \cup [K(b)]$$

Hence to show $G + K(b)$ is closed under the contractions, we need to show that $\chi(b_i), \theta(b_i) \in G_b$ for all $i < \omega$, but this follows directly from the definition of $K(b)$. □

Remark 4.3.12. The proof of Lemma 4.3.11 shows that $H(a)$ and $K(a)$ coincide for any a, hence we will use them interchangeably. It also shows that:

$$[H(a)] = [K(a)] = \{[\chi^i(a)] \mid i < \omega\}$$

Hence by combining the previous two lemmas, we get the following:

Lemma 4.3.13. *Let b be χ-super-transcendental, with χ-characteristic sequence $(b_i, g_i)_{i<\omega}$. Then $\mathcal{G}\langle b\rangle$ has domain:*

$$G_b \coloneqq G + \sum_{i=0}^{n-1} \mathbb{Z} b_i + H(b_n) \tag{4.5}$$

Where n is the χ-null point of b.

Proof. We know from the definition of χ-characteristic sequences that:

$$[G] \cap H(b_n) = [G] \cap \left[\sum_{i=0}^{n-1} \mathbb{Z} b_i\right] = \varnothing$$

Moreover by (CP), we have:

$$\left[\sum_{i=0}^{n-1} \mathbb{Z} b_i\right] > [H(b_n)]$$

Thus all three summands in Eq. (4.5) have distinct valuations. So we can glue together Lemma 4.3.10 and Lemma 4.3.11, to deduce that G_b is closed under the contractions, thus we have the result. □

So with θ contraction groups, we have two characteristic sequences, which we get from the generators of $K(a)$.

Definition 4.3.14. Let $b \in H \setminus G$, define θ-characteristic sequences as $(b_{i,j}, g_i)_{i<2, j<\omega}$ as:

- $(b_{0,j}, g_j)_{j<\omega}$ is any χ-characteristic sequence
- If b **is not** χ-super-transcendental, then $b_{1,j} = 0$ for all $j < \omega$.
- If b is χ-super-transcendental, with null point $\alpha < \omega$
  - For $j < \alpha$, set $b_{1,j} \coloneqq 0$
  - For $j \geq \alpha$, set $b_{i,j} \coloneqq \theta(b_{0,j})$

Lemma 4.3.15. *Let $b \in H \setminus G$, then it has a θ-characteristic sequence in $\mathcal{G}$. For any θ-characteristic sequence $(b_{i,j}, g_i)_{i<2, j<\omega}$, the structure $\mathcal{G}\langle b\rangle$ has domain:*

$$G_b \coloneqq G + \sum_{\substack{i<2 \\ j<\omega}} \mathbb{Z} b_{i,j}$$

Moreover, G_b has natural valuation:

$$[G] \cup \{[b_{0,j}] \mid j < \omega\}$$

Proof. The construction of a θ-characteristic sequence follows from the definition. Use Lemma 4.3.10 and Lemma 4.3.13 to verify the domain of $\mathcal{G}\langle b\rangle$ is generated by the θ-characteristic sequence, and the natural valuation statement follows since $[b_{0,0}] > [b_{0,1}] > \ldots > [b_{0,n}] > \ldots$ and $[H(b_{0,n})] = \{[b_{0,n}], [b_{0,n+1}], \ldots\}$. □

From this we define the type $p_b(x)$ to be all the cuts induced by the θ-characteristic sequence of b:

Definition 4.3.16. Fix some θ-characteristic sequence $(b_{i,j}, g_j)_{i<2,j<\omega}$ of b in $\mathcal{G}$. For any $\alpha \in \{0,1\} \times \omega$, set $f_\alpha(x)$ to be the function taking $b \mapsto b_\alpha$, so:

- $f_{0,n}(x) = \chi(f_{0,n-1}(x) - g_n)$
- $f_{1,n}(x) = \theta(f_{0,n-1}(x))$, assuming $b_{1,n}$ is non-zero.

Let $p_b(x)$ be the collection of formulas:

$$p_b(x) := \bigcup_{\alpha \in \{0,1\}\times\omega} \{g < f_\alpha(x) < \overline{g} \mid g < b_\alpha < \overline{g}\} \cup \bigcup_{\alpha \in \{0,1\}\times\omega} \{f_\alpha(x) = g \mid b_\alpha = g\}$$

Where $g, \overline{g}$ are elements of G.

Lemma 4.3.17. *Let $\overline{b} \models p_b(x)$. Then $(\overline{b}_{i,j}, g_j)_{i<2,j<\omega}$ is also a θ-characteristic sequence of $\overline{b}$ in $\mathcal{G}$, where $\overline{b}_{i,j} := f_{i,j}(b)$, thus $\mathcal{G}\langle\overline{b}\rangle$ has domain:*

$$G_{\overline{b}} := G + \sum_{\substack{i<2 \\ j<\omega}} \mathbb{Z} b_{i,j}$$

Proof. Use Lemma 2.3.12 and the definition of θ-characteristic sequences. Specifically, the fact that:

$$[G + \mathbb{Z}\chi(b_i)] \neq [G] \iff [G + \mathbb{Z}\chi(\overline{b}_i)] \neq [G]$$

□

Lemma 4.3.18. *Let $\overline{b} \models p_b(x)$, then we have an $\mathcal{L}_\theta$-isomorphism ϕ over $\mathcal{G}$ between the structures generated by b and $\overline{b}$ in $\mathcal{G}$:*

$$\phi : G_b \to G_{\overline{b}} : b_i \mapsto \overline{b}_i$$

Proof. By Lemma 2.3.13, we know that ϕ is an ordered abelian group isomorphism. Recall from Lemma 4.3.17 that G_b and $G_{\overline{b}}$ have natural valuations:

$$[G_b] = [G] \cup \{[b_i] \mid i < \omega\} \qquad [G_{\overline{b}}] = [G] \cup \{[\overline{b}_i] \mid i < \omega\}$$

Claim 4.3.18.1. *For any $x \in G_b, i < \omega$ and $g \in G$, we have:*

$$[x] = [g] \iff [\phi(x)] = [g]$$
$$[x] = [b_{0,i}] \iff [\phi(x)] = [\overline{b}_{0,i}]$$

Proof. When b is not χ-super-transcendental, the claim follows immediately from the extra corresponding result for χ-contraction groups, so assume x is χ-super-transcendental. By Definition 4.3.7, we can write any $x \in G_b$ as:

$$x := g + \sum_{i=0}^{n} z_i b_{0,i} + z_{n+1} b_{1,n} \tag{4.6}$$

where n is past the null point of b. Thus $\phi(x)$ can be written as:

$$\phi(x) = g + \sum_{i=0}^{n} z_i \bar{b}_{0,i} + z_{n+1} \bar{b}_{1,n}$$

Since $[b_{0,0}] > \ldots > [b_{0,n}] > [b_{1,n}]$ and $[\bar{b}_{0,0}] > \ldots > [\bar{b}_{0,n}] > [\bar{b}_{1,n}]$, and each pair $[b_{i,j}], [b_{i,j}]$ induce the same cut in $[G]$, the claim follows. ■

It remains to show that ϕ commutes with χ and θ. Pick some $x \in G_b$ written as in Eq. (4.6). If $[x] = [g]$, then for any $\delta \in \{\chi, \theta\}$:

$$\begin{aligned}
\delta(\phi(x)) &= \delta(g) && \text{By (CA) and since } [\phi(x)] = [g] \\
&= \delta(x) && \text{By (CA) and since } [x] = [g] \\
&= \phi(\delta(x)) && \text{Since } \phi \text{ is constant on } G
\end{aligned}$$

Now suppose $[x] = [b_{0,i}]$, so by Claim 4.3.18.1, we know $[\phi(x)] = [b_{0,i}]$, so:

$$\begin{aligned}
\chi(\phi(x)) &= \chi(\bar{b}_{0,i}) && \text{By (CA) and since } [\phi(x)] = [\bar{b}_{0,i}] \\
&= \bar{b}_{0,i+1} && \text{Since } \bar{b}_{0,i+1} := \chi(\bar{b}_{0,i}) \\
&= \phi(b_{0,i+1}) && \text{By the definition of } \phi \\
&= \phi(\chi(x)) && \text{Reverse the steps above}
\end{aligned}$$

So ϕ commutes with χ. Suppose $i < n$, so i is before the null point of b, then there exists some $j \geq i$ with $g_j \neq 0$ and $b_{0,j} = \chi^{j-i}(b_i) - g_j$, so in particular $[\chi^{j-i}(b_i)] = [\chi^{j-i}(\bar{b}_i)] = [g_j]$ (since $b_{0,j}$ and $\bar{b}_{0,j}$ induce the same cut in G).

$$\begin{aligned}
\theta(\phi(x)) &= \theta(\bar{b}_{0,i}) && \text{By (CA) and since } [\phi(x)] = [\bar{b}_{0,i}] \\
&= \chi(\bar{b}_{0,i}) + \ldots + \chi(\bar{b}_{0,j}) + \theta\chi(\bar{b}_{0,j}) && \text{Repeatedly apply ((C}\theta\text{))} \\
&= \chi(\bar{b}_{0,i}) + \ldots + \chi(\bar{b}_{0,j}) + \theta(g_j) && \text{Since } [\chi^{j-i}(\bar{b}_i)] = [g_j] \\
&= \phi(\chi(b_{0,i})) + \ldots + \phi(\chi(b_{0,j})) + \phi(\theta(g_j)) && \text{Since } \phi \text{ commutes with } \chi \\
&= \phi(\chi(b_{0,i}) + \ldots + \chi(b_{0,j}) + \theta(g_j)) && \text{Since } \phi \text{ is a group isomorphism} \\
&= \phi(\theta(x)) && \text{Reverse the steps above}
\end{aligned}$$

Now suppose $i \geq n$, so i is or is past the null point of b:

$$\begin{aligned} \theta(\phi(x)) &= \theta(\bar{b}_{0,i}) && \text{By (CA) and since } [\phi(x)] = [\bar{b}_{0,i}] \\ &= \bar{b}_{1,i} && \text{Since } \bar{b}_{1,i} := \theta(\bar{b}_{0,i}) \\ &= \phi(b_{1,i}) && \text{By the definition of } \phi \\ &= \phi(\theta(b_{0,i})) && \text{Since } b_{1,i} := \theta(b_{0,i}) \\ &= \phi(\theta(\phi(x))) && \text{By (CA) and since } [x] = [b_{0,i}] \end{aligned}$$

Thus ϕ commutes with χ and θ, hence is an $\mathcal{L}_\theta$-isomorphism. □

Lemma 4.3.19. *The type $p_b(x)$ is a type of $\mathcal{G}$.*

Proof. If b is not χ-super-transcendental, then p_b only contains the cuts from the χ-characteristic sequence, so by the corresponding result for χ-contractions groups, specifically Lemma 3.2.12, we see that p_b must be a type of the reduct of $\mathcal{G}$ to $\mathcal{L}_\chi$, thus a type of $\mathcal{G}$. So suppose b is χ-super-transcendental, with null point n, then we can write p_b as:

$$p_b(x) := \bigcup_{j=0}^{n-1} \{g < f_{0,j}(x) < \bar{g} \mid g < b_j < \bar{g}\} \cup \bigcup_{\substack{i<2 \\ n \leq j < \omega}} \{g < f_{i,j}(x) < \bar{g} \mid g < b_{i,j} < \bar{g}\}$$

So we can write an arbitrary finite subset Δ of p_b as:

$$\Delta(x) := \bigcup_{j=0}^{n-1} \{a_{0,j} < f_{0,j}(x) < \bar{a}_{0,j}\} \cup \bigcup_{j=n}^{m} \{a_{0,j} < f_{0,j}(x) < a_{0,j}, a_{1,j} < f_{1,j}(x) < a_{1,j}\}$$

Since b is χ-transcendental, we know $[b_{i,j}] \notin [G]$ for all i, j with $b_{i,j} \neq 0$, thus the interval $(a_{i,j}, b_{i,j})_G$ is non-empty in G. Since $f_{i,j}$ is increasing and surjective, we can find some $h_{i,j} \in G$, with $a_{0,0} < h_{i,j} < b_{0,0}$ such that:

$$a_{i,j} < f_{i,j}(h_{i,j}) < b_{i,j}$$

Similarly, we can find some $\bar{h}_{i,j} \in (b_{i,j}, a_{i,j})_G$ with

$$b_{i,j} < f_{i,j}(\bar{h}_{i,j}) < a_{i,j}$$

So define h and $\bar{h}$ as:

$$\begin{aligned} h &:= \max\{h_{0,0}, \ldots, h_{0,n-1}, h_{0,n}, h_{1,n}, \ldots, h_{0,m}, h_{1,m}\} \\ \overline{h} &:= \min\{\overline{h}_{0,0}, \ldots, \overline{h}_{0,n-1}, \overline{h}_{0,n}, \overline{h}_{1,n}, \ldots, \overline{h}_{0,m}, \overline{h}_{1,m}\} \end{aligned}$$

Then we must have $h < b_{0,0} < \overline{h}$, thus $\frac{h+\overline{h}}{2} \in G$ realises Δ.

□

4.3.4 Final Result

By the usual criteria mentioned in the introduction, we have:

Theorem 4.3.20. *The theory of θ-contraction groups is complete and has quantifier elimination.*

Proof. Use Lemma 4.3.18 and Lemma 4.3.19 to deduce $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b \rangle$. Apply that and Lemma 4.3.9, Lemma 4.4.4 to Theorem 2.1.14. □

4.4 Hyperlogarithmc-θ contraction groups

Definition 4.4.1. Let $\mathcal{L}_{\theta-L}$ be the language $\langle +, -, <, 0, \chi, \theta, L \rangle$, where χ and θ are unary functions. Let $T^{-}_{\theta-L}$ be the theory asserting:

- $+, -, <, 0$ form a divisible ordered abelian group
- χ, θ, L each form a centripetal precontraction group
- χ, L form a 2-precontraction group
- χ, θ form a θ-precontraction group

Let $T_{\theta-L}$ be the theory $T^{-}_{\theta-L}$ along with surjectivity for χ, L and θ.

We prove that $T_{\theta-L}$ has quantifier elimination and is complete.

4.4.1 Initial Structures

The initial structure will just be ω copies of the structure defined in Lemma 4.3.9.

Lemma 4.4.2. *There is some $\theta - L$-precontraction group that embeds into every other $\theta - L$-precontraction group.*

Proof. Let us remind ourselves about Q, the structure from Lemma 4.3.9:

$$Q := \{(r_i)_{i \in \mathbb{N}} \in \mathbb{Z}^{\omega} \mid \exists j \in \mathbb{N}, r \in \mathbb{Q} \text{ such that } \forall i > j, r_i = r\}$$

Let R be defined as the Hahn **sum** of Q indexed by ω:

$$R := \coprod_{\omega} Q$$

Define the contractions on R as follows:

$$\chi(1_{\omega n+m}) = 1_{\omega n+m+1}$$
$$\theta(1_{\omega n+m}) = \sum_{i>m} 1_{\omega n+i}$$
$$L(1_{\omega n+m}) = 1_{\omega(n+1)+m}$$

It is easy to check that R is a $\theta - L$-precontraction group and moreover, for any $\mathcal{G} \models T^{-}_{\theta-L}$ and $a \in G$, we have $\langle a \rangle \cong (R, \chi, L, \theta)$. □

4.4.2 Closure Operator

Lemma 4.4.3. *Any model of $T^{-}_{\theta-L}$ embeds into some model $\mathcal{K}$ of $T_{\theta-L}$. Moreover, for all $k \in K$, there exists some $n \in \mathbb{N}$ such that $L^n(k) \in G$*

Proof. Let $\mathcal{G}_0 := \mathcal{G}$. Given $\mathcal{G}_{2k}$, set $\mathcal{G}_{2k+1}$ to be the χ-hull of $\mathcal{G}_{2k}$. By Lemma 3.3.3, Lemma 4.3.3 and Lemma 4.3.5, we know θ and L are both defined on $\mathcal{G}_{2k+1}$, θ is surjective on G_{2k+1}, and finally, $L(G_{2k+1}) \subseteq G_{2k}$.

Given $\mathcal{G}_{2k-1}$, we set $\mathcal{G}_{2k} := (\prod_{\Gamma} \mathbb{R}, \chi, L, \theta)$, where:

$$\Gamma := [G_{2k-1}] \cup \big(\omega \times G^{>0}_{2k-1} \setminus L(\mathcal{G})\big)$$

Order Γ in the same way as in Proposition 3.3.4, and define the contractions on G_{2k} as:

- For $\gamma \in [G_{2k-1}]$, and contraction $\delta \in \{\chi, L, \theta\}$, $\delta(1_\gamma) = \delta(a)$ where $a \in G^{>0}_{2k-1}$ and $[a] = \gamma$.
- For $\gamma = (n, a) \in \omega \times G^{>0}_{2k-1} \setminus L(\mathcal{G})$, we define χ and L as we did in Proposition 3.3.4, and set

$$\theta(1_\gamma) := \sum_{\substack{i \in \omega \\ i > n}} 1_{i,a}$$

Again note that $L(G_{2k}) \subseteq G_{2k-1}$. It is easy to see that $\mathcal{K} = \bigcup_{i \in \omega} \mathcal{G}_i$ is a model of $T_{\theta-L}$ containing $\mathcal{G}$ as a substructure. □

Lemma 4.4.4. *(i) For any $T_{\theta-L}$-hull $\overline{\mathcal{G}}$, and $\overline{g} \in \overline{G}$, there exists some $n \in \mathbb{N}$ such that $L^k(\overline{g}) \in G$*

(ii) Any $\mathcal{G} \models T^-_{\theta-L}$ has a $T_{\theta-L}$-hull.

(iii) Moreover, we can choose a hull $\overline{\mathcal{G}}$ such that $\overline{G}$ can be written as:

$$G + \sum_{i \in I} H(\delta_i) + \sum_{j \in J} \mathbb{Q}\epsilon_j \tag{4.7}$$

Where $\overline{H}(x)$ is the divisible hull of $H(x)$ as defined in, Definition 4.3.7 and $\delta_i, \epsilon_j > 0$ are indeterminates such that for all $i, i' \in I$ and $j, j' \in J$:

- $[\overline{H}(\delta_i)] \cap [G] = [\mathbb{Q}\epsilon_j] \cap [G] = \varnothing$
- $[\overline{H}(\delta_i)] \cap [\overline{H}(\delta_{i'})] = [\mathbb{Q}\epsilon_j] \cap [\mathbb{Q}\epsilon_{j'}] = \varnothing$
- $[\overline{H}(\delta_i)] \cap [\mathbb{Q}\epsilon_j] = \varnothing$
- *$\gamma_i < \gamma_j$ if and only if $\chi(\gamma_i) < \chi(\gamma_j)$, where $\gamma_i, \gamma_j \in \{\delta_i \mid i \in I\} \cup \{\epsilon_j \mid j \in J\}$*
- *There exists some $g \in G$ such that $[g] < [\overline{H}(\delta_i)], [\epsilon_j]$.*

Hence $G \subseteq \overline{G}$ is a pure valuational extension of ordered abelian groups, and $[G]$ is coinitial in $[\overline{G}]$.

Proof. Firstly, any $T_{\theta-L}$-hull $\overline{\mathcal{G}}$ of $\mathcal{G}$ embeds into the structure $\mathcal{K}$ defined in Lemma 4.4.3, thus we have (i).

We construct the $T_{\theta-L}$-hull of $\mathcal{G}$ as follows. By Lemma 4.4.3 we can embed it into some $\theta - L$-contraction group (K, χ, θ, L). Then proceed exactly as done in Proposition 3.3.5, which means we induct on $|\overline{\mathcal{G}}|^+$, at the α-th step, we construct some $\mathcal{G}_\alpha$, and stop when $\mathcal{G}_\alpha$ is a $\theta - L$-contraction group.

- If α is an even ordinal, then set $\mathcal{G}_{\alpha+1}$ to be the θ-hull of $\mathcal{G}_\alpha$, and note that by Remark 3.2.5, $G_{\alpha+1}$ keeps the form in Equation (4.7), assuming G_α did.
- If α is a an odd ordinal then pick some $a \in G_\alpha^{>0} \setminus L(\mathcal{G}_\alpha)$, and choose some $b \in K$ such that $L(b) = a$, then set:

  $$G_{\alpha+1} \coloneqq G_\alpha + \overline{H}(b)$$

  It is easy to verify that $[\overline{H}(b)] \cap [G_\alpha] = \varnothing$, and $[H(b)] \geq [G_\alpha]$, so $G_{\alpha+1}$ keeps the form in Equation (4.7), assuming G_α did. To define the contractions on $G_{\alpha+1}$ it is sufficient to define them on $[G_\alpha]$ and $[\overline{H}(b)]$ separately.

  - For $[x] \in [\mathcal{G}_\alpha]$, define all contractions as they are in $\mathcal{G}_\alpha$.
  - For $[x] \in [\overline{H}(b)]$, $L(x)$ goes to b, and $\chi(x), \theta(x)$ are defined as they would be in $\overline{H}(b)$.

Finally, for a limit ordinal β, set:

$$\mathcal{G}_\beta := \bigcup_{\gamma<\beta} \mathcal{G}_\gamma$$

Note that $\mathcal{G}_\alpha$ has the factoring property over $\mathcal{G}_\beta$ for all $\beta < \alpha$. Eventually we reach an ordinal $\gamma < |K|^+$ such that $\mathcal{G}_\gamma$ is a model of $T_{\theta-L}$, this will be a $T_{\theta-L}$-hull satisfying (iii). □

4.4.3 Existential closedness

Fix some extension $\mathcal{G} \subseteq \mathcal{H}$ of models of $T_{\theta-L}$.

Definition 4.4.5. Let $b \in H \setminus G$ have 1-characteristic sequence $(b_i, g_i)_{i\in\omega}$. Define $J(b)$ as the group generated by the θ-characteristic sequence of b in (G, χ, θ). That means:

- If b is not χ-super-transcendental, then:

$$J(b) := \sum_{j<\omega} \mathbb{Z} b_{0,j}$$

- Else, if b has null point n, then

$$J(b) := \sum_{i=0}^{n-1} \mathbb{Z} b_i + H(b_n)$$

Lemma 4.4.6. *Let $b \in H \setminus G$ have 2-characteristic sequence $(b_i, g_i)_{i\in\omega^2}$. Then $\mathcal{G}\langle b\rangle$ has domain:*

$$G_b := G + \sum_{i\in\omega} J_i$$

where $J_i := J(b_{\omega i})$. Moreover, the group G_b has natural valuation:

$$[G_b] := [G] \cup \{b_i \mid i < \omega^2\} \tag{4.8}$$

Proof. First, let us prove Eq. (4.8). Suppose b is 2-algebraic, with characteristic sequence of length $\alpha \in [\omega m, \omega(m+1))$. Then $J_i = 0$ for all $i > m$. By Lemma 4.3.15, we know that the natural valuation of each J_i is:

$$[J_i] = \{[b_{\omega i+j}] \mid j < \omega\}$$

But remember that any 2-characteristic sequence is strictly decreasing in valuation, thus:

$$[J_0] > [J_1] > \ldots > [J_m] \tag{4.9}$$

By definition of 2-characteristic sequences, we know that $[b_\beta] \notin [G]$ for all $\beta < \omega m$, thus

$$[J_i] \cap [G] = \varnothing \tag{4.10}$$

for all $i < m$. So pick any element in $x \in G_b$, and write it as:

$$x = z_0\gamma_0 + \ldots + z_{m-1}\gamma_{m-1} + (z_m\gamma_m + g)$$

where $z_i \in \{0, 1\}$. Then by Eq. (4.9), Eq. (4.10), and the ultrametric inequality, we have:

$$[x] = \max\{[z_0\gamma_0], \ldots, [z_{m-1}\gamma_{m-1}], [z_m\gamma_m + g]\}$$

But again by Lemma 4.3.15, we know that $[\gamma_i] \in [G] \cup [J_i]$, thus:

$$[x] \in [G] \cup \bigcup_{i=0}^{m} [J_i] = [G] \cup \bigcup_{i=0}^{m} \bigcup_{j<\omega} \{[b_{\omega i+j}]\}$$

When b is 2-transcendental, note that $[J_i] \cap [G] = \varnothing$ for all $i < \omega$, so a simpler version of the above arguments proves the statement, thus we have (4.8).

To show $|\mathcal{G}\langle b\rangle|$ is indeed equal to G_b, repeat the same argument as in Lemma 3.3.11. □

From this, we define the $\theta - L$-characteristic sequence:

Definition 4.4.7. Let $b \in \mathcal{H}\backslash\mathcal{G}$. Define the $\theta - L$-characteristic sequence as $(b_{i,j}, g_j)_{i<2, j<\omega^2}$ of b in $\mathcal{G}$ as follows:

- $(b_{0,j}, g_j)_{j<\omega^2}$ is any 2-characteristic sequence of b in $\mathcal{G}$.
- For some fixed $n \in \omega$, $b_{1,\omega n+i}$ is a θ-characteristic sequence of $b_{\omega n}$ created from the χ-characteristic sequence $(b_{\omega n+i}, g_{\omega n+i})_{i<\omega}$.

Lemma 4.4.8. *A $\theta - L$-characteristic sequence of b in $\mathcal{G}$ exists. Given a $\theta - L$-characteristic sequence $(b_{i,j}, g_j)_{i<2, j<\omega^2}$, the structure $\mathcal{G}\langle b\rangle$ has domain:*

$$G_b \coloneqq G + \sum_{i<2, i<\omega^2} \mathbb{Z} b_{i,j}$$

and has natural valuation:

$$[G_b] := \{[b_{0,j}] \mid j < \omega^2\}$$

Proof. A direct consequence of Lemma 4.4.6. □

Let us extend Definition 4.3.16 to $\theta - L$-contraction groups:

Definition 4.4.9. For ordinals $\alpha \in \{0,1\} \times \omega^2$, set $f_\alpha(x)$ to be term $b_\alpha[x/b]$ i.e. the term b_α but with the occurrence of b replaced with the variable x. Stated explicitly:

- $f_{0,n}(x) = \chi(f_{0,n-1}(x) - g_n)$
- $f_{1,n}(x) = \theta(f_{0,n-1}(x))$, assuming $b_{1,n}$ is non-zero.
- $f_{0,n+\omega}(x) = L(f_{0,n_0}(x)) - g_{n+\omega}$, where n_0 is the null point of $b_{0,\omega i}$, where $n \in [\omega i, \omega(i+1))$.

Let $p_b(x)$ be the collection of formulas:

$$p_b(x) := \bigcup_{\alpha \in \{0,1\} \times \omega^2} \{g < f_\alpha(x) < \overline{g} \mid g < b_\alpha < \overline{g}\} \cup \bigcup_{\alpha \in \{0,1\} \times \omega^2} \{f_\alpha(x) = g \mid b_\alpha = g\}$$

Where $g, \overline{g}$ are elements of G.

We want to show that p_b determines the quantifier free type of b over $\mathcal{G}$:

Lemma 4.4.10. *Let $b \in H \setminus G$ have $\theta - L$-characteristic sequence $(b_{i,j}, g_i)_{i<2, j<\omega^2}$. For any $\overline{b} \models p_b(x)$, the sequence $(\overline{b}_{i,j}, g_i)_{i<2,j<\omega^2}$ is a $\theta - L$-characteristic sequence for $\overline{b}$ in $\mathcal{G}$, where $\overline{b}_{i,j} = f_{i,j}(b)$.*

Proof. Simply check that the conditions for $\theta - L$-characteristic sequence are satisfied. □

Lemma 4.4.11. *Let $b \in H \setminus G$, then for any $\overline{b} \in \mathcal{H}$ such that $\overline{b} \models p_b(x)$, there is an $\mathcal{L}_{\theta-L}$ isomorphism $\phi : \mathcal{G}\langle b \rangle \to \mathcal{G}\langle \overline{b} \rangle$ that is the identity on G.*

Proof. By Lemma 4.4.6, we know that $\mathcal{G}\langle \overline{b} \rangle$ has domain:

$$G_{\overline{b}} := G + \sum_{\substack{i<2 \\ j<\omega^2}} \mathbb{Z} b_{i,j}$$

Moreover, G_b and $G_{\overline{b}}$ have natural valuations:

$$[G_b] = [G] \cup \{[b_{0,j}] \mid j < \omega^2\}$$
$$[G_{\overline{b}}] = [G] \cup \{[\overline{b}_{0,j}] \mid j < \omega^2\}$$

Since $\bar{b} \models p_b(x)$, we can use Lemma 2.3.13, to get an isomorphism of ordered abelian groups ϕ over G:

$$\phi : G_b \to G_{\bar{b}} : b_{i,j} \mapsto \bar{b}_{i,j}$$

A routine calculation tells us that for any $\alpha < \omega^2$:

$$[x] = [b_{0,\alpha}] \iff [\psi(x)] = [\bar{b}_{0,j}][x] = [g] \iff [\phi(x)] = [g]$$

Thus to show ϕ commutes with the contractions, it is sufficient to show that $\phi(\eta(b_{0,j})) = \eta(\bar{b}_{0,j})$ for all $\eta \in \{\chi, \theta, L\}$ and $j \in \omega^2$, but this follows from the calculations in Lemma 3.2.13, Lemma 3.3.16 and Lemma 4.3.18. □

Lemma 4.4.12. *The type p_b is a type of $\mathcal{G}$.*

Proof. Suppose b is 2-archimedean algebraic, with characteristic sequence of length $\alpha+2 \in [\omega m+\omega(m+1))$. That means $[b_{0,\alpha}] \notin G$, but $[G+\mathbb{Z}b_{0,\alpha}] = [G]$, hence $b_{0,\alpha+1} \coloneqq \chi(b_{0,\alpha}) \in G$. Then we can write some finite subset Δ of p_b as:

$$\Delta(x) \coloneqq \bigcup_{i=0}^{n} \{a_{0,\alpha_i} < f_{0,\alpha_i}(x) < \bar{a}_{0,i}, a_{1,\alpha_i} < f_{1,\alpha_i}(x) < \bar{a}_{1,i}\}$$
$$\cup \{a_\alpha < f_{0,\alpha}(x) < \bar{a}_\alpha\} \cup \{f_{0,\alpha+1}(x) = a_{\alpha+1}\}$$

Where $0 \leq \alpha_0 < \ldots < \alpha_n < \alpha$ is some sequence of ordinals, and $[a_\alpha] = [\bar{a}_\alpha] = [b_{0,\alpha}]$. To ease notation further assume that $b_{i,\alpha_j} \neq 0$ for all $j \leq n$. Since $[b_{i,\alpha_j}] \notin [G]$, and the function f_{i,α_j} is surjective and increasing on G, we can find some $h_{i,j} \in (a_{0,0}, b_{0,0})_G$ such that:

$$a_{i,\alpha_j} < f_{i,\alpha_j}(h_{i,j}) < b_{i,\alpha_j}$$

Similarly we can find some $\bar{h}_{i,j} \in (b_{0,0}, \bar{a}_{0,0})$ such that:

$$b_{i,\alpha_j} < f_{i,\alpha_j}(h_{i,j}) < \bar{a}_{i,\alpha_j}$$

Let h be the maximum of the $h_{i,j}$'s, and $\bar{h}$ the minimum of the $\bar{h}_{i,j}$'s. Then any element of $(h, \bar{h})_G$ realises the first set of formulas in Δ. Further, note that $h < b < \bar{h}$. We claim that $f_{0,\alpha}((h, \bar{h})_G) \cap (a_\alpha, \bar{a}_\alpha) \neq \varnothing$. Suppose it was, say $f_{0,\alpha}((h, \bar{h})_G) < (a_\alpha, \bar{a}_\alpha)$, then since $f_{0,\alpha}$ is increasing and surjective, we must have:

$$a_\alpha \geq f_{0,\alpha}(\bar{h}) \geq b_{0,\alpha} > a_\alpha$$

hence we have a contradiction. Similarly we cannot have $f_{0,\alpha}((h,\overline{h})_G) > (a_\alpha, \overline{a}_\alpha)$, thus they must intersect. Pick any $y \in (h,\overline{h})_G$ such that $f_{0,\alpha}(y) \in (a_\alpha, \overline{a}_\alpha)$, then $\chi(f_{0,\alpha}(y)) = \chi(b_{0,\alpha})$ thus y realises Δ.

If b is 2-value-algebraic, then we can write Δ as:

$$\Delta(x) := \bigcup_{i=0}^{n} \{a_{0,\alpha_i} < f_{0,\alpha_i}(x) < \overline{a}_{0,i}, a_{1,\alpha_i} < f_{1,\alpha_i}(x) < \overline{a}_{1,i}\} \cup \{f_{0,\alpha+1}(x) = a_{\alpha+1}\}$$

Define h and $\overline{h}$ as before, then since $f_{0,\alpha+1}$ is surjective and increasing, we must have $f_{0,\alpha+1}((h,\overline{h})_G) \cap \{a_{\alpha+1}\} \neq \varnothing$, thus a realisation of Δ exists in $(h,\overline{h})_G$.

If b is 2-transcendental, then we can write Δ as:

$$\Delta(x) := \bigcup_{i=0}^{n} \{a_{0,\alpha_i} < f_{0,\alpha_i}(x) < \overline{a}_{0,i}, a_{1,\alpha_i} < f_{1,\alpha_i}(x) < \overline{a}_{1,i}\}$$

hence any element of $(h,\overline{h})_G$ realises Δ. □

4.4.4 Final Result

As expected, we apply Theorem 2.1.14 to get:

Theorem 4.4.13. *The theory $T_{\theta-L}$ has quantifier elimination, has a prime model and is complete.*

4.5 Asymptotic Triples and Contractions

Consider the structure with the contractions χ, L, θ along with ψ. Let us remind ourselves of the axiomatization of this theory:

Let $\mathcal{L}_{\psi-\chi}$, $T^-_{\psi-\chi}$ and $T_{\psi-\chi}$ be the language and theories from Definition 1.5.9. We call a model of $T^-_{\psi-\chi}$ an **asymptotic contraction** and a model of $T_{\psi-\chi}$ a **closed asymptotic contraction**. Let $\mathcal{L}'_{\psi-\chi}$ be the sublanguage of $\mathcal{L}_{\psi-\chi}$ with the predicate P removed.

In this section, we give a partial proof of quantifier elimination and completeness for $T_{\psi-\chi}$. Recall by Theorem 2.1.14 the three steps needed to get QE. We prove that $T_{\psi-\chi}$-hulls exist, thus the theory has an initial structure (take the $T_{\psi-\chi}$-hull of the structure generated by 1). We then prove an analogue to properties (A) and (B) for closed asymptotic triples (see Section 4.5.2), which we can use to show that any for any $\mathcal{G} \models T_{\psi-\chi}$ and any $\psi-\chi$-algebraic element b (a notion defined in Definition 4.5.8), $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b\rangle$. But we encounter problems when proving $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b\rangle$ for $\psi-\chi$-transcendental b.

Proposition 4.5.1. *The $\mathcal{G}$ be an asymptotic contraction. For any $b \in G^{>0}$ and $n, m \in \mathbb{N}$, we have the following equality:*

$$\psi \circ \chi^n \circ L^m(b) = \psi(b) + \sum_{i=1}^{m} \left(\theta \circ L^{i-1}(b) + L^i(b)\right) + \sum_{i=1}^{n} \chi^i \circ L^m(b) \tag{4.11}$$

Thus for all $b \in G^{<0}$, we have:

$$\psi \circ \chi^n \circ L^m(b) = \psi(b) - \sum_{i=1}^{m} \left(\theta \circ L^{i-1}(b) + L^i(b)\right) - \sum_{i=1}^{n} \chi^i \circ L^m(b) \tag{4.12}$$

Proof. Repeatedly apply (Cψ) and (CI). □

4.5.1 Closure operator

We show that every model $\mathcal{G}$ of $T^-_{\psi\text{-}\chi}$ has a $T_{\psi\text{-}\chi}$-hull.

Lemma 4.5.2. *Let $\mathcal{G} \models T^-_{\psi-\chi}$. There exists some $\overline{\mathcal{G}} \models T^-_{\psi-\chi}$ with no H-point and with the factoring property over $\mathcal{G}$ with respect to $T_{\psi-\chi}$. Moreover, we can write $\overline{G}$ as:*

$$G + \sum_{i \in \omega} \overline{H}(\epsilon_i)$$

where $0 < \epsilon_i < G^{>0}$ are indeterminates, $H(\epsilon_i)$ is from Lemma 4.3.9 and $[\overline{H}(\epsilon_i)] > [\overline{H}(\epsilon_{i+1})]$.

Proof. Situations like these highlight the necessity of the predicate P. Given an H-point u, we do not know whether it 'should' be in the image of ψ or the image of $|id| + \psi$, i.e. there could be two extensions $\mathcal{H}, \overline{\mathcal{H}}$ of $\mathcal{G}$ such that u in in the image of ψ in H and in the image of $|id| + \psi$ in $\overline{H}$, thus we would not have QE. It is the predicate P that fixes the position of u in all extensions of $\mathcal{G}$.

Suppose $\mathcal{G}$ has a H-point $u \in G$. Either $u \in P_{\mathcal{G}}$ or $u > P_{\mathcal{G}}$.

$u \in P_{\mathcal{G}}$**:** This tells us that in any $\mathcal{H} \models T_{\psi-\chi}$ extending $\mathcal{G}$, we must have $u \in \psi(H^{\neq 0}) = P_{\mathcal{H}}$. Let ϵ be an indeterminate satisfying $0 < \epsilon < G^{>0}$, and consider the divisible hull of the $\mathcal{L}_{\theta-L}$ structure generated by it, $R(\epsilon)$ (see Lemma 4.4.2), so:

$$0 < R(\epsilon) < G^{>0}$$

thus $[R(\epsilon)] = \{[\chi^m \circ L^n(\epsilon)] \mid n, m \in \omega\} < [G]$. Set $\overline{G} := G + R(\epsilon)$, then $\overline{G}$ is closed under contractions since χ, L, θ are defined on both $[R(\epsilon)]$ and $[G]$. To close make it an $\mathcal{L}'_{\psi-\chi}$-structure, set:

$$\psi(\epsilon) \coloneqq u$$
$$\psi \circ \chi^n(\epsilon) \coloneqq u + \sum_{i=1}^{n} \chi^i(\epsilon)$$
$$\psi \circ \chi^n \circ L^m(\epsilon) \coloneqq u + \sum_{i=1}^{m} \left(\theta \circ L^{i-1}(\epsilon) + L^i(\epsilon)\right) + \sum_{i=1}^{n} \chi^i \circ L^m(\epsilon)$$
$$P_{\overline{\mathcal{G}}} \coloneqq \{x \in \overline{G} \mid \exists y \in \overline{G}^{\neq 0} \mid x \leq \psi(y)\}$$

We claim that $(\overline{G}, \psi)$ only has one H-cut. Note that $[R(\epsilon)]$ is coinitial in $[\overline{G}]$, and both $\psi(R(\epsilon)), (id+\psi)(R(\epsilon)) \subseteq u+R(\epsilon)$, thus it is sufficient to show $\psi(R(\epsilon))-u, (id+\psi)(R(\epsilon))-u$ induce a disjoint partition of $R(\epsilon)$.

As a reminder, $R(\epsilon)$ is isomorphic to the set of all function $f : \omega^2 \to \mathbb{Q}$ such that for any $i \in \omega$, the sequence $(f_{\omega i+j})_{j\in\omega}$ is eventually constant, with $f > g$ if and only if $f_\alpha > g_\alpha$, where $\alpha = \min(\operatorname{supp}(f) \cup \operatorname{supp}(g))$. Moreover, the contractions are defined as:

$$\chi(1_\alpha) \coloneqq 1_{\alpha+1}$$
$$L(1_\alpha) \coloneqq 1_{\alpha+\omega}$$
$$\theta(1_\alpha) \coloneqq \sum_{i\in\omega} 1_{\alpha+1+i}$$

Thus, if we set:

$$A \coloneqq \{\psi \circ \chi^i \circ L(\epsilon) - u \mid i, j \in \omega\}$$
$$B \coloneqq \{(id + \psi) \circ \chi^i \circ L(\epsilon) - u \mid i, j \in \omega\}$$

Then we can calculate A and B to be:

$$A = \left\{ f \in R(\epsilon) \mid \exists \alpha < \omega^2 \text{ such that } f_\beta = \begin{cases} 1 & \text{If } 0 < \beta \leq \alpha \\ 0 & \text{Otherwise} \end{cases} \right\}$$
$$B = \left\{ f \in R(\epsilon) \mid \exists \alpha < \omega^2 \text{ such that } f_\beta = \begin{cases} 1 & \text{If } 0 < \beta < \alpha \\ 2 & \text{If } \beta = \alpha \\ 0 & \text{Otherwise} \end{cases} \right\}$$

Observe that $A < B$, A is cofinal in $\psi\left(\overline{G}\right)$, B is coinitial in $(id+\psi)\left(\overline{G}^{>0}\right) - u$, and there is no element of $R(\epsilon)$ between A and B, thus $\overline{\mathcal{G}} \coloneqq (\overline{G}, \psi, P_{\overline{\mathcal{G}}})$ only has one H-cut.

Fix some $\mathcal{H} \models T_{\psi-\chi}$ with $\mathcal{G} \subseteq \mathcal{H}$. Since $u \in P_{\mathcal{G}}$, we must have $u \in P_{\mathcal{H}}$, thus there exists some $\delta \in H$ with $\psi(\delta) = u$. By Lemma 2.3.13 and Proposition 4.5.1, the mapping

$$\phi : \overline{G} \to H : g + \theta^i \circ \chi^n \circ L^m(\epsilon) \mapsto g + \theta^i \circ \chi^n \circ L^m(\delta)$$

is an $\mathcal{L}'_{\psi-\chi}$ embedding. Moreover since $\overline{\mathcal{G}}$ only has one H-cut, Lemma 4.2.11 implies ϕ is an $\mathcal{L}_{\psi-\chi}$-embedding. Thus $\overline{\mathcal{G}}$ has the factoring property over $\mathcal{G}$ with respect to $T_{\psi-\chi}$.

$u > P_{\mathcal{G}}$: Repeat the same argument but with $(id + \psi)(\epsilon) = u$, so $\psi(\epsilon) = u - \epsilon$. Note that A and B will be slightly different, for any $f \in A, B$, we would have $f_0 = -1$, since $\psi(u) - u = -\epsilon = -(21)_0 \in R(\epsilon)$.

The final statement about the form of $\overline{G}$ follows since

$$R(\epsilon) = \sum_{i<\omega} \overline{H}(L^i\epsilon)$$

□

Lemma 4.5.3. *Any $\mathcal{G} \models T^-_{\psi-\chi}$ has a $T_{\psi-\chi}$-hull. Moreover, we can choose a $T_{\psi-\chi}$-hull $\overline{\mathcal{G}}$ such that $\overline{G}$ can be written as:*

$$G + \sum_{i\in I} H(\delta_i) + \sum_{j\in J} \mathbb{Q}\epsilon_j \tag{4.13}$$

Where $\delta_i, \epsilon_j > 0$ are indeterminates such that for all $i, i' \in I$ and $j, j' \in J$:

- $[H(\delta_i)] \cap [G] = [\mathbb{Q}\epsilon_j] \cap [G] = \varnothing$
- $[H(\delta_i)] \cap [H(\delta_{i'})] = [\mathbb{Q}\epsilon_j] \cap [\mathbb{Q}\epsilon_{j'}] = \varnothing$
- $[H(\delta_i)] \cap [\mathbb{Q}\epsilon_j] = \varnothing$
- $\gamma_i < \gamma_j$ *if and only if* $\psi(\gamma_i) > \psi(\gamma_j)$*, where* $\gamma_i, \gamma_j \in \{\delta_i \mid i \in I\} \cup \{\epsilon_j \mid j \in J\}$
- *There exists some* $g \in G$ *such that* $[g] < [H(\delta_i)], [\epsilon_j]$.

Proof. By Lemma 4.5.2 we can assume $\mathcal{G}$ only has one H-cut Fix some $\mathcal{H} \models T_{\psi-\chi}$ with $\mathcal{G} \subseteq \mathcal{H}$. Let $(\overline{G}, \chi, L, \theta)$ be the $\theta - L$-contraction hull of (G, χ, L, θ). Then we have an $\mathcal{L}_{\theta-L}$-embedding $\phi : \overline{G} \to H$ over G.

We know for any $\overline{g} \in \overline{G}$, there exists some $n \in \omega$ such that $L^n(\overline{g}) \in G$, thus we can extend ψ to $\overline{G}$ via:

$$\psi(\overline{g}) = \psi(L^n(g)) - \sum_{i=0}^{n} \left(\theta \circ L^{i-1}(g) + L^i(g)\right)$$

It can be verified that $(\overline{G}, \chi, L, \theta, \psi) \models T^-_{\psi-\chi} \mid \mathcal{L}'_{\psi-\chi}$. Moreover, since $G \subseteq \overline{G}$ is a pure extension and $[G]$ is coinitial in $[\overline{G}]$, we can use Lemma 4.2.15 to deduce that $(\overline{G}, \chi, L, \theta, \psi)$

only has one H-cut, thus if we set:

$$P_{\overline{\mathcal{G}}} := \left\{x \in \overline{G} \mid \exists y \in \overline{G}^{\neq 0} \text{ such that } x \leq \psi(y)\right\}$$

then the structure $\overline{\mathcal{G}} := (\overline{G}, \chi, L, \theta, \psi, P_{\overline{\mathcal{G}}})$ is a model of $T^{-}_{\psi-\chi}$, and the map $\phi : \overline{G} \to H$ is an $\mathcal{L}_{\psi-\chi}$-embedding (by Lemma 4.2.11). Since χ is surjective, we deduce that $\psi(\overline{G}^{\neq 0})$ is downwards closed, thus $\overline{\mathcal{G}} \models T_{\psi-\chi}$. Since the structure $\mathcal{H}$ was an arbitrary model of $T_{\psi-\chi}$, we conclude that $\overline{\mathcal{G}}$ is a $T_{\psi-\chi}$-hull of $\mathcal{G}$. The statement about the domain of $\overline{\mathcal{G}}$ follows from the corresponding statements in Lemma 4.4.4 and Lemma 4.5.2. □

4.5.2 Properties (A) and (B) for $T_{\psi-\chi}$

Definition 4.5.4. Fix some closed asymptotic triple $\mathcal{G}$ and some (possibly trivial) extension $\mathcal{H}$. We define ψ-χ-monomials as follows:

1. x is a ψ-χ-monomial in $\mathcal{G}$ with domain $D_x(\mathcal{H}) := H$
2. Let f be a ψ-χ-monomial with domain $D_f(\mathcal{H}) \subseteq H$, then for any $a \in G$
   (a) $\chi(f(x) - a)$ and $L(f(x) - a)$ are ψ-χ-monomials with characteristic domain $D_f(\mathcal{H})$.
   (b) $h(x) := \psi(f(x) - a)$ is a ψ-χ-monomial with characteristic domain:

$$D_h(\mathcal{H}) := \{x \in H \mid f(x) \neq a\}$$

A **successive sequence of ψ-χ-monomials** is a collection $(f_{i,j})_{0\leq i\leq m, 0\leq j\leq n_m}$ of ψ-χ-monomials, where $n_m \in \omega$ and some collection of integers $k_0, \dots, k_{m-1}$ with $0 \leq k_i \leq n_i$ such that:

$$f_{i,j+1} := \gamma_{i,j+1} \circ f_{i,j}$$
$$f_{i+1,0} := \psi_{a_i} \circ f_{i,k_i}$$

where $\gamma_{i,j}$ is of the form $\chi(x - a_{i,j})$ or $L(x - a_{i,j})$, and $f_{0,0} := x$.

Note that for all $i \in [0, m]$, the terms $f_{i,0}, \dots, f_{i,n_i}$ all have the same domain which we denote as $D_i(\mathcal{H})$.

A **ψ-χ-polynomial** is a term of the form:

$$F(x) := \sum_{i\in[0,m]} \sum_{j\in[0,n_m]} z_{i,j} f_{i,j}(x)$$

where $z_{i,j} \in \mathbb{Z}$, $z_{0,0} = 1$ and z_{m,n_m} is non-zero (so any ψ-χ-polynomial leads with the term x). It has domain:

$$D_F(\mathcal{H}) \coloneqq D_F(\mathcal{H})$$

We say F **was generated** by the sequence $(f_{i,j})$ when F was instantiated this way. By Lemma 4.2.4, any ψ-χ-polynomial is strictly increasing and has the intermediate value property on any component of its domain. Moreover, if the component is unbounded then so is the image of F, in the same direction as the component.

Lemma 4.5.5. *Let f be a ψ-χ-monomial in $\mathcal{G}$.*

1. *Both $D_f(\mathcal{G})$ and $D_f(\mathcal{H})$ have the same, and finitely many convex components. For each component C_H of $D_f(\mathcal{H})$, there exists a unique component C_G of $D_f(\mathcal{G})$ such that $C_G \subseteq C_H$, and moreover, it satisfies:*

$$C_G = C_H \cap G$$

2. *f has the intermediate value property on each component of $D_f(\mathcal{G})$ and $D_f(\mathcal{H})$*
3. *Let C_G (and hence C_H) be bounded:*

$$\lim_{x \uparrow C_G} f(x) = \lim_{x \uparrow C_H} f(x) \in G \cup \{\Psi\}$$
$$\lim_{x \downarrow C_G} f(x) = \lim_{x \downarrow C_H} f(x) \in G \cup \{\Psi\}$$

   Moreover, if f is of the form $\gamma \circ f^-(x)$, where $\gamma(x)$ is of the form $\chi(x-a)$ or $L(x-a)$, then the limit on any boundary of C_G is an element of G

   Evenmoreover f is monotone in the same sense on C_G and C_H.

4. *Let C_G (and hence C_H) be unbounded above but bounded below, then f is monotone in the same sense on C_G and C_H, and:*

$$\lim_{x \uparrow C_G} f(x) = \lim_{x \uparrow C_H} f(x) = -\infty$$
$$\lim_{x \downarrow C_G} f(x) = \lim_{x \downarrow C_H} f(x) \in G \cup \{\Psi\}$$

   Similarly, if C_G (and hence C_H) are unbounded below but bounded above, then:

$$\lim_{x \uparrow C_G} f(x) = \lim_{x \uparrow C_H} f(x) \in G \cup \{\Psi\}$$
$$\lim_{x \downarrow C_G} f(x) = \lim_{x \downarrow C_H} f(x) = -\infty$$

   And if C_G (and hence C_H) is unbounded above and below then:

$$\lim_{x\uparrow C_G} f(x) = \lim_{x\uparrow C_H} f(x) = \infty$$
$$\lim_{x\downarrow C_G} f(x) = \lim_{x\downarrow C_H} f(x) = -\infty$$

Proof. The lemma is true for the ψ-χ-monomial x. Assume we have proved the statement for some ψ-χ-monomial $h(x)$, and let $f(x) := \eta(x-a)$ where $a \in G$ and $\eta \in \{\chi, L, \psi\}$.

Suppose $\eta \in \{\chi, L\}$**.** Then $D_f(\mathcal{J}) = D_h(\mathcal{J})$ for $\mathcal{J} = \mathcal{G}, \mathcal{H}$, thus (1).

Let C be some component of $\subseteq D_h(\mathcal{G})$ or $D_h(\mathcal{H})$. We assumed by induction that h has the intermediate value property on C, and $\eta(x-a)$ is surjective and increasing, thus f also has the intermediate value property on C, hence (2).

Let C_G, and hence C_H be bounded. By induction we assume that:

$$\lim_{x\uparrow C} h(x) \in G \cup \{\Psi\}$$

Suppose $\lim_{x\uparrow C} h(x) = p \in G$. By definition (see Notation 4.2.13) this means that the limit p is attained in C_G and C_H, which combined with the monotonicity of h and (C$\leq$) for contraction groups, implies:

$$\lim_{x\uparrow C_G} f(x) = \lim_{x\uparrow C_H} f(x) = \eta(p-a) \in G$$

An identical argument applies to the limits at the left boundaries, thus (3)

Let C_G (and hence C_H) be unbounded above, then by induction we assume:

$$\lim_{x\uparrow C_G} h(x) = \lim_{x\uparrow C_H} h(x) = \pm\infty$$

By (C$\leq$) we then deduce:

$$\lim_{x\uparrow C_G} f(x) = \lim_{x\uparrow C_H} f(x) = \pm\infty$$

If C_G (and hence C_H) is bounded below, then repeat the argument for (3), and a symmetrical argument works if C_G (and hence C_H) is unbounded below, thus (4).

Suppose $\eta \in \{\psi\}$**.** Repeat exactly the same argument as Lemma 4.2.28. □

Lemma 4.5.6. *Let F be a ψ-χ-polynomial of the form:*

$$F(x) \coloneqq \sum_{i=0}^{n} z_i f_{0,i}(x) + z f_{1,0}(x) = x + \sum_{i=1}^{n} z_i f_{0,i}(x) + z\psi(f_{0,n}(x) - a)$$

Then for any $d \in D_F(\mathcal{H}) \setminus D_F(\mathcal{G})$, we have $F(d) \notin G$

Proof. Firstly, observe that for $J \in \{G, H\}$, $D_F(\mathcal{J})$ is the union $C_J^1 < C_J^2$ where:

$$C_J^1 \coloneqq \{x \in J \mid f_{0,n}(x) < a\}$$
$$C_J^2 \coloneqq \{x \in J \mid f_{0,n}(x) > a\}$$

Moreover, by Lemma 4.5.5, we know that $C_H^i \cap G = C_G^i$. On each C^i, F is strictly increasing and has the intermediate value property, as a direct consequence of Lemma 4.2.4, thus if d is in the convex hull of C_G^i in H, then we automatically deduce $F(d) \notin G$.

Note that C_2 is unbounded above in G, moreover, $F(C_2)$ must also be unbounded above, since for sufficiently large $x \in G$,

$$[x] > [f_{0,1}(x)] > \ldots > [f_{0,n}(x)] = [f_{0,n}(x) - a]$$

thus $F(x) > (id + \psi)(f_{0,n}(x))$ which itself is unbounded above. Similarly C_1 and $F(C_1)$ are unbounded below, thus is $d > C_G^2$ or $d < C_G^1$, then again $F(d) \notin G$.

So the only remaining case is when d is on the inner boundary of C_G^i. Suppose $i = 2$, and fix $d \in C_H^2$, $d < C_G^2$. Since f_n is increasing and surjective on G and H, we must have:

$$0 < f_n(d) - a < G^{>0} \tag{4.14}$$

Claim 4.5.6.1. *For all $i \in [0, n]$, $[f_{0,i}(d) - a_{0,i+1}] \notin [G]$, where $a_{0,n+1} \coloneqq a$, and:*

$$[d - a_{0,0}] > [f_{0,1}(d) - a_{0,i+1}] > \ldots > [f_{0,n}(d) - a]$$

Proof. By Equation (4.14), we know that $[f_{0,n}(d) - a] \notin [G]$. Suppose $[f_{0,i}(d) - a_{0,i+1}] \notin [G]$, then $f_{0,i}(d) \notin G$. Since $f_{0,i}(x)$ is defined as:

$$f_{0,i}(x) \coloneqq \gamma(f_{0,i-1}(x) - a_{0,i})$$

for some contraction γ, by applying (CA) and (CP), we must have $[f_{0,i-1}(d) - a_{0,i}] \notin [G]$ and $[f_{0,i-1}(d) - a_{0,i}] > [f_{0,i}(d)]$. If $a_{0,i}$ is non-zero, then we must have $[f_{0,i-1}(d)] = [a_{0,i}] > [[f_{0,i-1}(d) - a_{0,i}]]$, thus the statement is proved. ■

Suppose $F(d) \in G$, then there exists some $g \in G$ such that:

$$F(d) = \sum_{i=0}^{n} z_i f_{0,i}(x) + z\psi(f_{0,n}(d) - a) = g$$

Thus by some rearranging, we get:

$$\sum_{i=0}^{n} z_i \left(f_{0,i}(x) - a_{0,i+1}\right) = g + z\psi(f_{0,n}(d) - a) - \sum_{i=0}^{n} z_{0,i} a_{0,i} \tag{4.15}$$

Note that the left-hand side of Equation (4.15) has valuation outside of $[G]$. But since $0 < f_{0,n}(d) - a < G^{>0}$, we deduce from Lemma 4.2.31 that:

$$[G + \mathbb{Q}\psi(f_{0,n}(d) - a)] = [G]$$

thus the right hand side of Equation (4.15) has valuation within $[G]$, so we have a contradiction, so $F(d) \notin G$. □

Lemma 4.5.7. *Let $F(x)$ be a ψ-χ-polynomial. Let C_G be a component of $D_F(\mathcal{G})$ and C_H be the corresponding component of $D_F(\mathcal{H})$. For any $x \in C_H \setminus C_G$, we have $F(x) \notin G$.*

Proof. The statement is definitely true for the ψ-χ-polynomial x. The actual base case is as follows:

Assume it is true for any ψ-χ-generated by the successive sequence $\overline{f} := (f_{i,j})_{0\leq i\leq m, 0\leq j\leq n_m}$, **where** $m \geq 1$. We show that the statement also holds for ψ-χ-polynomials generated by the successive sequences:

1. $\overline{f}_1 := \overline{f} \cup f_{m,n_m+1}$, where $f_{m,n_m+1} := \eta(f_{m,n_m}(x) - a)$ for some $\eta \in \{\chi, L\}$ and $a \in G$.
2. $\overline{f}_2 := \overline{f} \cup f_{m+1,0}$, where $f_{m+1,0} := \psi(f_{m,k_m}(x) - a)$, $k_m \in [0, n_m]$, and $a \in G$.

By using Lemma 4.5.6 for the base case, the two items above sufficient to prove the statement for all ψ-χ-polynomials.

Case 1: Let F be a ψ-χ-polynomial generated by $\overline{f}_1$, written as:

$$F(x) := \overline{F}(x) + z f_{m,n_m+1}(x)$$

where:

$$\overline{F}(x) := \sum_{i\in[0,m]} \sum_{j\in[0,n_m]} z_{i,j} f_{i,j}(x) \tag{4.16}$$

and $z, z_{i,j} \in \mathbb{Z}$ $z, z_{0,0} \neq 0$. Pick some $d \in D_{\overline{f}_1}(\mathcal{H}) \setminus D_{\overline{f}_1}(\mathcal{G})$, and suppose it is contained in the convex component C_H, with corresponding component C_G. Suppose d is in the convex hull of C_G in H. Then since F has the intermediate value property and is strictly increasing on C_G, C_H, we automatically deduce that $F(x) \notin G$, else F would not be injective. So suppose d is outside C_G. From Lemma 4.5.5, we know that:

$$\lim_{x \to_{C_G} d} f_{m,n_m+1}(x) = \lim_{x \to_{C_H} d} f_{m,n_m+1}(x) \in G \cup \{\pm\infty\}$$

Suppose the limits are $\pm\infty$, then again by Lemma 4.5.5, we deduce that $|d| > G$, which together with the unboundedness of the image of F on C_G, tells us that $|F(d)| > G$, thus $F(d) \notin G$

If the limit is some $p \in G$, then since limits in G are by definition attained in both G, H, we deduce that $f_{m,n_m+1}(d) = p \in G$. By induction we assumed $\overline{F}(d) \notin G$, thus $F(d) := \overline{F}(d) + p \notin G$.

Case 2: Let F be a ψ-χ-generated by $\overline{f}_2$, written as:

$$F(x) := \overline{F}(x) + z\psi(f_{m,k_m}(x) - a)$$

where F is as in Equation (4.16), and $z \in \mathbb{Z}^{\neq 0}$. Pick some $d \in D_F(\mathcal{G}) \setminus D_F(\mathcal{H})$, contained in the convex component $C_H \subseteq H$ with corresponding component $C_G \subseteq G$. Similar to the previous case we can assume d is not in the convex hull of C_G in H, thus by Lemma 4.5.5, we know that:

$$\lim_{x \to d} \psi(f_{m,k_m}(x) - a) \in G \cup \Psi \cup -\infty$$

If the limit is some $p \in G$ or $-\infty$ then repeat the same argument as in Case 1. So suppose the limit is Ψ. Note that $f_{1,0}$ is of the form:

$$\psi(f_{0,k_m}(x) - a_1)$$

where $a_1 \in G$, and remember that f_{0,k_m} is just the composition of contractions and translations in G, ψ does not appear within it. Since $D_F(\mathcal{J}) \subseteq D_{f_{0,k_m}}(\mathcal{J})$ for $\mathcal{J} = \mathcal{G}, \mathcal{H}$, we know that $f_{0,k_m}(d) \neq a_1$. Without loss of generality, assume $f_{0,k_m}(d) \not> a_1$.

Claim 4.5.7.1. *There exists some $c \in G$ such that $a_1 < c < f_{0,k_m}(d)$*

Proof. Let $D_m(\mathcal{H}), D_m(\mathcal{G})$ be the domains of f_{m,n_m} in G and H, Since $m \geq 1$ they both must have at least two convex components. Let C'_H be the component of $D_m(\mathcal{H})$ containing d, and let C'_G be the corresponding component.

Assume there is no such $c \in G$, then d cannot be in the convex hull of C'_G in H, so $d < C'_G$. Consider the limit of f_{m,k_m} as we approach d.

1. Suppose the limit is:

$$\lim_{x \to d} f_{m,k_m}(x) = \Psi$$

then by Lemma 4.2.26, we deduce:

$$\lim_{x \to d} \psi(f_{m,k_m}(x) - a) = \lim_{x \uparrow \Psi} \psi(x - a) \in G$$

which contradicts out assumption that the limit of $f_{m+1,0} := \psi_a \circ f_{m,k_m}$ approaching d is Ψ.

2. Suppose the limit is:

$$\lim_{x \to d} f_{m,k_m}(x) = p \in G$$

Then since the limits are attained, we must have $f_{m,k_m}(d) = p$. Note that $p \neq a$, else d would not be in $D_F(\mathcal{H})$, thus by the monotonicity of $f_{m+1,0}$, we must have:

$$\lim_{x \to d} f_{m+1,0}(x) = \psi(f_{m,k_m}(d) - a) \in G$$

which again contradicts our assumption that the limit is Ψ.

Hence there must be some $c \in G$ in-between a and $f_{0,k_0}(d)$. ■

Claim 4.5.7.2. *For any $c \in G$ in-between a and $f_{0,k_0}(d)$, if we set $\epsilon := \frac{1}{2}|c - a_1|$, and define the convex set I as:*

$$I = I_c := \{x \in C_H \mid |f_{0,k_m}(x) - f_{0,k_m}(d)| \leq \epsilon\}$$

Then $I \cap C_G = \varnothing$.

Proof. Since we made ϵ small enough, we deduce that $I \subseteq D_1(\mathcal{H})$. Furthermore, $f_{1,0}$ is constant on I, since for any $[x] \in I$

$$\begin{aligned} [f_{0,k_m}(x) - a_1] &= [f_{0,k_m}(d) + \gamma - a_1] && \text{Where } |\gamma| \leq \epsilon \\ &= [f_{0,k_m}(d) - a_1] && \text{Since } |\gamma| < \epsilon < \frac{1}{2}|f_{0,k_m}(d) - a_1| \end{aligned}$$

Thus $I \subseteq D_i(\mathcal{H})$ and $f_{i,j}$ is constant on I for all $i \in [0, m+1]$ and $j \in \omega$.

Suppose there was some $b \in I \cap C_G$, then since $f_{m+1,0}$ is constant on I, we deduce that:

$$\lim_{x \to d} f_{m+1,0}(x) = f_{m+1,0}(d) \in G$$

which contradicts our assumption that the limit was Ψ. ■

Since $\lim_{x \to d} f_{m+1,0}(x) = \Psi$, we know that $\lim_{x \to d} f_{m,k_m}(x) = a$, so pick some $b \in C_G$ such that:

$$|f_{m,k_m}(b) - a| \leq \epsilon$$

Then:

$$\begin{aligned}
[f_{m+1,0}(d) - f_{m+1,0}(b)] &< [f_{m,k_m}(d) - f_{m,k_m}(b)] && \text{By fact (4.2.6)} \\
&\leq [f_{m,k_m}(b) - a] && \text{As } f_{m,k_m}(d) \text{ is between } a \text{ and } f_{m,k_m}(b) \\
&\leq [\epsilon]
\end{aligned}$$

Define the function $f : I \to H$ as:

$$f(x) \coloneqq \overline{F}(x) + z f_{m+1,0}(b)$$

Pick $\gamma^+, \gamma^- \in I$ such that $f_{0,k_m}(\gamma^\pm) = f_{0,k_m} \pm \epsilon$. Then $\gamma^- < d < \gamma^+$ and:

$$\begin{aligned}
F(d) - f(\gamma^-) &= \overline{F}(d) - \overline{F}(\gamma^-) + f_{m+1,0}(d) - f_{m+1,0}(b) \\
&= d - \gamma^- + o(d - \gamma^-) + f_{m+1,0}(d) - f_{m+1,0}(b) && (*) \\
&= d - \gamma^- + o(d - \gamma^-) && (**) \\
&> 0
\end{aligned}$$

Where (*) follows since $[x - y] > [f_{i,j}(x) - f_{i,j}(y)]$ for any i, j, and (**) follows since $|f_{m+1,0}(d) - f_{m+1,0}(b)| \leq \epsilon$ and $[\epsilon] < [d - \gamma^-]$. Thus $F(d) > f(\gamma^-)$, and by a symmetrical argument, $F(d) < f(\gamma^+)$. By the intermediate value property for $\overline{F}$ and hence f, we deduce that there exists some $y \in I$ such that $f(y) = F(d)$. Since $I \cap C_G = \varnothing$, we apply the inductive assumption on $\overline{F}$ and thus f to deduce $f(y) \notin G$, thus $F(d) \notin G$. □

4.5.3 Existential Closedness

Definition 4.5.8. Let $b \in H \setminus G$. Define the ψ-χ characteristic sequence $(b_{i,j}, g_j)_{i<2, j<\omega^3}$ of b in $\mathcal{G}$ as follows:

- $(b_{i,\alpha}, g_\alpha)_{\substack{i<2 \\ j<\omega^2}}$ is any $\theta - L$-characteristic sequence of b in $\mathcal{G}$
- Suppose we have defined $(b_{i,j}, g_j)_{i<2, j<\omega^2 m}$

- Suppose $b_{0,\omega^2}(m-1)$ is 2-super-transcendental, with null point $\alpha < \omega^2$. Then set set $(b_{i,\omega^2 m+j}, g_{\omega^2 m+j})_{i<2,j<\omega^2}$ to be a $\theta - L$-characteristic sequence of $\psi(b_{0,\omega^2 m+\alpha})$ in $\mathcal{G}$.
- If $b_{0,\omega^2(m-1)}$ is not 2-super-transcendental, then set $b_{i,j}, g_j := 0$ for all $i < 2, j \geq \omega^2 m$.

We say a ψ-χ-characteristic sequence has length $\alpha < \omega^3$ if α is the least ordinal such that $b_{0,\alpha} = 0$ but $b_{0,\beta} \neq 0$ for all $\beta < \alpha$. If no such α exists we say it has length ω^3. We say some $b \in H$ is $\psi - \chi$-algebraic if it has a characteristic sequence of length less than ω^3, and $\psi - \chi$-transcendental otherwise.

The existence of characteristic sequences are guaranteed by the construction.

Definition 4.5.9. Fix some element $b \in H^{>0} \setminus G$ and some $\psi - \chi$-characteristic sequence $(b_{i,j}, g_j)_{i<2,j<\omega^3}$ of b in $\mathcal{G}$. As usual let $x_{i,j}$ be $b_{i,j}$ but with the occurrence of b replaced with the variable x. Define the set of formulas q_b as:

$$q_b(x) := \bigcup_{\substack{i<2 \\ j<\omega^3}} D_{b_{i,j}}(x_{i,j})$$

4.5.4 Algebraic Elements

Lemma 4.5.10. *Let $\mathcal{G} \subseteq \mathcal{H}$ be an extension of closed asymptotic contractions. Fix some $b \in H \setminus G$ and some $\psi - \chi$-characteristic sequence $B := (b_{i,j}, g_j)_{i<2,j<\omega^3}$ in $\mathcal{G}$. If B has length $\alpha \leq \omega^2 + 1$ (so $b_{0,\omega^2+1} = 0$ and $b_{0,\omega^2} \in G$), then for any $\overline{b} \models q_b(x)$, the structure $\mathcal{G}\langle\overline{b}\rangle$ has domain:*

$$G_{\overline{b}} := G + \sum_{\substack{i<2 \\ j<\omega^2}} \mathbb{Q}\overline{b}_{i,j} \tag{4.17}$$

Moreover, there exists a $\mathcal{L}'_{\psi-\chi}$-isomorphism $\phi : G_b \to G_{\overline{b}}$ over $\mathcal{G}$, with $\phi(b_{i,j}) := \overline{b}_{i,j}$.

Proof. Let p_b be the type from Definition 4.4.9, containing the cuts from the $\theta - L$-characteristic sequence, thus:

$$q_b(x) = p_b(x) \cup \{x_{0,\omega^2} = b_{0,\omega^2} \in G\}$$

By Lemma 4.4.6 it is sufficient to show that $\psi(b_{0,j}) \in G_b$ for all $j < \omega^2$, since by Lemma 4.4.6, we know that:

$$[G_b] = [G] + \{[b_{0,j}] \mid j < \omega^2\}$$

Fix some $j < \omega^2$, we will calculate $\psi(b_{0,j})$. Suppose there exists some least $\alpha \coloneqq \omega m_L + m_\chi \in [1, \omega^2)$ with $g_{j+\alpha} \neq 0$ (where $m_L, m_\chi \in \omega$). Then by construction of 2-characteristic sequences, we must have:

$$b_{0,j+\alpha} = \chi^{m_\chi} \circ L^{m_L}(b_{0,j}) - g_{j+\alpha}$$

Thus $[\chi^{m_\chi} \circ L^{m_L}(b_{0,j})] = [g_{0,j}]$. But by Proposition 4.5.1, we can write $\psi(b_{0,j})$ as:

$$\begin{aligned}\psi(b_{0,j}) &= \psi\left(\chi^{m_\chi} \circ L^{m_L}(b_{0,j})\right) - \sum_{i=1}^{m_L} \left(\theta \circ L^{i-1} + L^i\right)(b_{0,j}) - \sum_{i=0}^{m_\chi} \chi^i(b_{0,j}) \\ &= \psi\left(g_{0,j}\right) - \sum_{i=1}^{m_L} \left(\theta \circ L^{i-1} + L^i\right)(b_{0,j}) - \sum_{i=0}^{m_\chi} \chi^i(b_{0,j}) \\ &\in G_b\end{aligned}$$

This covers the cases when b is not 2-super transcendental, but when b is 2-super transcendental, j could be past the 2-null point γ. In that case, by Definition 4.5.8, we must have $b_{0,\omega^2} = \psi(b_{0,\gamma})$, so $\psi(b_{0,\gamma}) \in G$. Write $j \coloneqq \gamma + \omega m_L + m_\chi$, thus $b_{0,j} = \chi^{m_\chi} \circ L^{m_L}(b_{0,\gamma})$, so by applying Proposition 4.5.1, we have:

$$\begin{aligned}\psi(b_{0,j}) &= \psi(b_{0,\gamma}) + \sum_{i=1}^{m_L} \left(\theta \circ L^{i-1} + L^i\right)(b_{0,\gamma}) + \sum_{i=0}^{m_\chi} \chi^i(b_{0,\gamma}) \\ &\in G_b\end{aligned}$$

Thus G_b is closed under ψ, so $|\mathcal{G}\langle b\rangle| = G_b$. The same also show that $\left|\mathcal{G}\langle \overline{b}\rangle\right| = G_{\overline{b}}$

By Lemma 4.4.11, we have a $\mathcal{L}_{\theta-L}$-isomorphism $\phi : G_b \to G_{\overline{b}}$, where $\phi(b_{i,j}) = \overline{b}_{i,j}$ for $j < \omega^2$. Since $\overline{b} \models q_b$, so $\psi(\overline{b}_{0,\gamma}) = \psi(b_{0,\gamma})$, it is easy to show that ϕ commutes with ψ, thus ϕ is a $\mathcal{L}'_{\psi-\chi}$-isomorphism. □

Note that if the $\psi - \chi$-characteristic sequence of b in $\mathcal{G}$ has length $\alpha \leq \omega^2 + 1$, then one the following happens:

- b is 2-transcendental but not 2-super-transcendental
- b is 2-super-transcendental but $\psi\left(b_{0,\gamma}\right) \in G$, where γ is the 2-null point of b
- b is 2-value-algebraic
- b is 2-archimedean-algebraic

In the first three cases, we can show that $\mathcal{G}\langle b\rangle$ only has one H-cut, thus the map ϕ automatically becomes an $\mathcal{L}_{\psi-\chi}$-isomorphism. When b is 2-archimedean-algebraic, then we may need add a formula to q_b involving the predicate P to make ϕ a $\mathcal{L}_{\psi-\chi}$-isomorphism.

Lemma 4.5.11. *Suppose $\mathcal{G} \subseteq \mathcal{H}$ is an extension of closed asymptotic triples and fix some $b \in H^{>0} \setminus G$. If one of the following occurs:*

- *b is 2-transcendental but not 2-super-transcendental*
- *b is 2-super-transcendental but $\psi(b_{0,\gamma}) \in G$, where γ is the 2-null point of b*
- *b is 2-value-algebraic*

Then $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b\rangle$.

Proof. Let $\bar{b} \models q_b(x)$, then by Lemma 4.5.10 we have an $\mathcal{L}'_{\psi-\chi}$ isomorphism $\phi : \mathcal{G}\langle b\rangle \to \mathcal{G}\langle \bar{b}\rangle$. We want to apply Lemma 4.2.15 to show that $\mathcal{G}\langle b\rangle$ has only one H-cut, thus by Lemma 4.2.11, ϕ will be a $\mathcal{L}_{\psi-\chi}$-isomorphism. Since b is not 2-value-algebraic, we know that the extension $G \subseteq G_b$ is pure valuational, thus it remains to show that $[G]$ is coinitial in $[G_b]$. Suppose not, then it must be the case that b is 2-super-transcendental, with $[b_{0,\gamma}] < [G]$.

Claim 4.5.11.1. *There exists some $g \in G$ with $0 < g < b_{0,\gamma}$*

Proof. Since $\mathcal{G}$ is closed, there exists some $h \in G^{>0}$ with $\psi(h) = \psi(b_{0,\gamma})$. Then by (C$\psi$) we know $\psi(\chi(h)) = \psi(b_{0,\gamma}) + \chi(h) > \psi(h)$, so by (A4), we must have $0 < \chi(h) < b_{0,\gamma}$. ∎

Then because $0 < L^n(g) \leq L^n(b_{0,\gamma})$ for all $n < \omega$, we deduce that $[G]$ is coinitial in $[G_b]$. Hence by the reasoning outlined earlier, we deduce that ϕ is a $\mathcal{L}_{\psi-\chi}$-isomorphism.

It remains to show that q_b is a type of $\mathcal{G}$, after which we can apply Fact 2.1.12 to get existential closedness. If b is not 2-super-transcendental, then q_b is equal to the type p_b from Definition 4.4.9, which by Lemma 4.4.12, we know is a type in the reduct of $\mathcal{G}$ to $\mathcal{L}_{\theta-L}$, so we are done.

So suppose b is 2-super-transcendental, then q_b is the following:

$$q_b(x) = p_b(x) \cup \{x_{0,\omega^2} = b_{0,\omega^2} \in G\}$$

Write some finite subset $\Delta(x) \coloneqq \Delta_0(x) \cup \Delta_1(x)$ of q_b as:

$$\Delta_0(x) \coloneqq \bigcup_{\substack{i<2 \\ j \leq n}} \{a_{i,\alpha_j} < x_{i,\alpha_j} < \bar{a}_{i,\alpha_j}\}$$
$$\Delta_1(x) \coloneqq \{x_{0,\omega^2} = b_{0,\omega^2}\}$$

where $0 \leq \alpha_0 < \alpha_1 < \ldots < a_n < \omega^2$ and $a_{i,\alpha_j}, \bar{a}_{i,\alpha_j} \in G$. Note that $x_{0,\omega^2} \coloneqq \psi(x_{0,\gamma})$ is a $\psi - \chi$-monomial. Let $D_\gamma(\mathcal{H})$ be it's domain in H, let C_H be the component containing b, and let C_G be the corresponding component of $D_\gamma(\mathcal{G})$.

For all $i < 2, j \leq n$, pick some $h_{i,j} \in [a_{0,0}, b)_G$ such that:

$$a_{i,j} \leq x_{i,j}(h_{i,j}) < b_{i,j}$$

We can do this because $x_{i,j}$ is increasing and surjective on G. Similarly define $\overline{h}_{i,j} \in (b, a_{0,0}]_G$ such that:

$$b_{i,j} < x_{i,j}(\overline{h}_{i,j}) \leq a_{i,j}$$

Define $h, \overline{h}$ as:

$$h := \max\{h_{i,j} \mid i < 2, j \leq n\}$$
$$\overline{h} := \min\{\overline{h}_{i,j} \mid i < 2, j \leq n\}$$

Then we must have $h < b < \overline{h}$, and any element of $(h, \overline{h})_G$ realises Δ. By Lemma 4.5.5 we know that $x_{0,\gamma}$ is monotone and has the intermediate value property on C_G, so by the same argument as in Lemma 4.4.12, we can show that $x_{0,\gamma}(h, \overline{h}) \cap \psi^{-1}(b_{0,\omega^2})$ is non-empty. Pick some $y \in x_{0,\gamma}(h, \overline{h}) \cap \psi^{-1}(b_{0,\omega^2})$,then any element of $(h, \overline{h})_G \cap x_{0,\gamma}^{-1}(y)$ will realise Δ. So we are done.

□

Lemma 4.5.12. *Let $\mathcal{G} \subseteq \mathcal{H}$ and $b \in H \setminus G$ be 2-archimedean-algebraic over $\mathcal{G}$. Then $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b \rangle$.*

Proof. Let the $\theta - L$-characteristic sequences of b in $\mathcal{G}$ have length $\alpha + 2 < \omega^2$. That means $b_{0,\alpha+2} = 0$, $b_{0,\alpha+1} \in G^{\neq 0}$, and $b_{o,\alpha} \notin G$ but $[b_{0,\alpha}] \in [G]$.

Let $\overline{b} \models q_b(x)$, then by Lemma 4.5.10, we have an $\mathcal{L}'_{\psi-\chi}$ isomorphism $\phi : \mathcal{G}\langle b \rangle \to \mathcal{G}\langle \overline{b} \rangle$. If $\mathcal{G}\langle b \rangle$ only has one H-cut, then ϕ is immediate a $\mathcal{L}_{\psi-\chi}$-isomorphism, coupled with the fact that q_b is just p_b from Definition 4.4.9 means we can apply Fact 2.1.12 to deduce $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b \rangle$.

So suppose (G_b, ψ) has two H-cuts. Then there exists some unique $v_b \in G_b \setminus G$ such that:

$$\psi(G_b^{\neq 0}) < v_b < (id + \psi)(G_b^{>0})$$

Since b is not 2-super-transcendental, we know that $[G]$ is coinitial in $[G_b]$. Thus $\psi(G^{\neq 0})$ is cofinal in $\psi(G_b^{\neq 0})$ and $(id + \psi)(G^{>0})$ is coinitial in $(id + \psi)(G_b^{>0})$, thus:

$$\psi(G^{\neq 0}) < v_b < (id + \psi)(G^{>0})$$

Claim 4.5.12.1. *The element v_b is of the form $g + qb_{0,\alpha}$, where $g \in G$ and $q \in \mathbb{Q}^{\neq 0}$.*

Proof. Suppose not, then since $[b_{i,j}] \notin [G]$ for all $b_{i,j} \neq 0$ with $i + j < \alpha$, we must have $[v_b - g] \notin [G]$. Thus pick some $\epsilon \in G$ with $[\epsilon] < [v_b - g]$ (which we can do since $[G]$ has no minimum and is coinitial in $[G_b]$), then $v_b, v_b + \epsilon$ induce the same cut in G, thus:

$$\psi(G_b) < v_b, v_b + \epsilon < (id + \psi)(G_b^{>0})$$

which means (G_b, ψ) has at least three H-cuts, which is a contradiction. ■

Thus define $r_b(x)$ as:

$$r_b(x) \coloneqq p_b(x) \cup \{Z(g + qx_{0,\alpha})\}$$

Where $Z(x)$ is as defined in Lemma 4.2.37. Pick some $b \models r_b(x)$, then since $q_b \subseteq r_b$, for any $\overline{b} \models r_b(x)$ we have a $\mathcal{L}'_{\psi-\chi}$-isomorphism ϕ from Lemma 4.5.10. To extend it to $\mathcal{L}_{\psi-\chi}$, note that $v_{\overline{b}} \coloneqq g + \overline{b}_{0,\alpha}$ is the unique element of $G_{\overline{b}}$ such that:

$$\psi(G_{\overline{b}}^{\neq 0}) < v_{\overline{b}} < (id + \psi)(G_{\overline{b}}^{>0})$$

It remains to show r_b is a type of $\mathcal{G}$. Pick some finite subset $\Delta(x) \coloneqq \Delta_0(x) \cup \Delta_1(x)$ of $r_b(x)$ written as:

$$\Delta_0(x) \coloneqq \bigcup_{\substack{i<2 \\ j\leq n}} \{a_{i,\alpha_j} < x_{i,\alpha_j} < \overline{a}_{i,\alpha_j}\}$$
$$\Delta_1(x) \coloneqq \{Z(g + qx_{0,\alpha})\}$$

where $0 \leq \alpha_0 < \alpha_1 < \ldots < a_n < \omega^2$ and $a_{i,\alpha_j}, \overline{a}_{i,\alpha_j} \in G$. Define $h, \overline{h}$ as in Lemma 4.5.11, then any element of $(h, \overline{h})$ realises Δ_0. Consider the interval $(v_h, v_{\overline{h}})_G$, where:

$$v_h \coloneqq g + qx_{0,\alpha}(h)$$
$$v_{\overline{h}} \coloneqq g + qx_{0,\alpha}(\overline{h})$$

Note that $v_h < b_{0,\alpha} < v_{\overline{h}}$, so the interval in non-empty. Moreover, since $\mathcal{G}$ is a closed asymptotic triple, we know that $v_h \in \psi(G^{\neq 0})$ and $v_{\overline{h}} \in (id + \psi)(G^{>0})$, thus $(v_h, v_{\overline{h}})$ intersects both $\psi(G^{\neq 0})$ and $(id + \psi)(G^{>0})$. Thus we can always pick an element y from the realisations of Δ_0 such that $Z(y)$ holds, hence $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b\rangle$. □

By combining Lemma 4.5.11 and Lemma 4.5.12, we have:

Proposition 4.5.13. *Let $\mathcal{G} \subseteq \mathcal{H}$ be an extension of closed asymptotic contractions. If b has a $\psi - \chi$-characteristic sequence in $\mathcal{G}$ of length $\alpha < \omega^2 + 1$, then $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b\rangle$.*

Now we prove that adjoining any $\psi - \chi$-algebraic element to $\mathcal{G}$ preserves existential closedness.

Theorem 4.5.14. *Let $\mathcal{G} \subseteq \mathcal{H}$ be an extension of closed asymptotic contractions. Suppose b is $\psi - \chi$-algebraic with characteristic sequence of length $\alpha < \omega^3$. Then $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b \rangle$.*

Proof. Say $\alpha \in [\omega^2 n, \omega^2(n+1))$. By Proposition 4.5.13, we know $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b_{0,\omega^n} \rangle$, hence $\mathcal{G}$ is existentially closed in any $T_{\psi-\chi}$-hull $\mathcal{G}_n$ of $\mathcal{G}\langle b_{0,\omega^n} \rangle$. We claim that $b_{0,\omega^2(n-1)}$ is $\psi - \chi$-algebraic **over** $\mathcal{G}_n$, with characteristic sequence of length $\alpha_n \leq \omega^2 + 1$. Suppose there exists some $g \in G_n$ such that $[g] = [b_{0,\omega^2(n-1)+\beta}]$ for some $\beta \in [0, \omega^2)$. Then since G_n is of the form:

$$G_{b_{0,\omega^2 n}} + \sum_{i \in I} H(\delta_i) + \sum_{j \in J} \mathbb{Q}\epsilon_j$$

We can assume that $g = b_{0,\omega^2(n-1)+\beta}$. Thus for all $\beta \in [0, \omega^2)$, we must have:

$$[G_n + \mathbb{Q}b_{0,\omega^2(n-1)+\beta}] = [G_n] \cup \{[b_{0,\omega^2(n-1)+\beta}]\}$$

So the $\psi - \chi$-characteristic sequence of $b_{0,\omega^2(n-1)}$ does not increase in length when moving from $\mathcal{G}$ to $\mathcal{G}_n$.

Hence we apply Proposition 4.5.13 again to show $\mathcal{G}_n$ is existentially closed in $\mathcal{G}_n\langle b_{0,\omega^2(n-1)} \rangle$, so in any $T_{\psi-\chi}$-hull $\mathcal{G}_{n-1}$ of $\mathcal{G}_n\langle b_{0,\omega^2(n-1)} \rangle$. Repeat this process to get a chain of structures $\mathcal{G} \subseteq \mathcal{G}_n \subseteq \mathcal{G}_{n-1} \subseteq \ldots \subseteq \mathcal{G}_0$, where $\mathcal{G}_i$ is the $T_{\psi-\chi}$-hull of $\mathcal{G}_{i+1}\langle b_{0,\omega^2 i} \rangle$. Then since $b \in \mathcal{G}_0$, we deduce that $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b \rangle$. □

4.5.5 Transcendental Elements

We proved that for any extension $\mathcal{G} \subseteq \mathcal{H}$ of models of $T_{\psi-\chi}$, if we pick some $\psi - \chi$-algebraic element $b \in H \setminus G$, then $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b \rangle$. By repeatedly applying this result, we can construct some $\overline{\mathcal{G}} \models T_{\psi-\chi}$ with $\mathcal{G} \subseteq \overline{\mathcal{G}} \subseteq \mathcal{H}$ such that any $b \in H \setminus \overline{G}$, is $\psi - \chi$-transcendental over $\overline{\mathcal{G}}$. Note that if b is infinitesimal in G, then it must be $\psi - \chi$-algebraic over $\mathcal{G}$ by Lemma 4.2.31. Thus to prove model completeness for $T_{\psi-\chi}$, it is sufficient to show that any extension $\mathcal{G} \subseteq \mathcal{H}$ of models of $T_{\psi-\chi}$, where $[G]$ is coinitial in $[H]$, is model complete.

The problem arises when trying to find a generating set for $\mathcal{G}\langle b \rangle$ over $\mathcal{G}$ when b is $\psi - \chi$-transcendental. Say $b := b_{0,0}$ is $\psi - \chi$-transcendental, and has 2-null point 0. It is not possible to deduce whether or not $\chi(b)$ and $\psi(b)$ sum together in an unknown way to produce a new archimedean class, and it is difficult to capture that behaviour in a type that is easy to finitely realise.

Suppose, however, that the type q_b does in fact determine this behaviour, i.e for any $\bar{b} \models q_b(x)$ we have an $\mathcal{L}_{\psi-\chi}$ isomorphism $\phi : \mathcal{G}\langle b\rangle \to \mathcal{G}\langle \bar{b}\rangle$. Then to get quantifier elimination, all we would need to do is show that q_b is finitely realised in $\mathcal{G}$. But this is quite simple due to Lemma 4.5.7.

Lemma 4.5.15. *Assume b is $\psi - \chi$-transcendental over $\mathcal{G}$. Then the type $q_b(x)$ is a type of $\mathcal{G}$.*

Proof. Write some finite subset Δ as:

$$\Delta(x) \coloneqq \bigwedge_{\substack{i\in[0,m]\\ j\in[0,n_m]}} (a_{i,j} < f_{i,j}(x) < \bar{a}_{i,j}) \wedge (c_{i,j} < \theta \circ f_{i,j}(x) < \bar{c}_{i,j})$$

where $(f_{i,j})$ is a successive sequence of ψ-χ-monomials. Let f_{m,n_m} have domains $D_m(\mathcal{H})$ and $D_m(\mathcal{G})$ in H and G. Let $C_H \subseteq H$ be the component containing b, and $C_G \subseteq G$ be the corresponding component.

Suppose b is in the convex hull of C_G in H. Then without loss of generality we can assume $a_{0,0}, \bar{a}_0, 0 \in C_G$. By (2) of Lemma 4.5.5, we know that $f_{i,j}$, and hence $\theta \circ f_{i,j}$, are monotone and have the intermediate value property on both C_H and C_G. Thus for all i, j, we can pick some $h_{i,j}, k_{i,j} \in C_G$ with:

$$a_0 < h_{i,j}, k_{i,j} < b$$

satisfying:

$$\begin{array}{lll} a_{i,j} \le f_{i,j}(h_{i,j}) \le f_{i,j}(b) & c_{i,j} \le \theta \circ f_{i,j}(k_{i,j}) \le \theta \circ f_{i,j}(b) & \text{If } f_{i,j} \text{ is increasing on } C \\ f_{i,j}(b) \le f_{i,j}(h_{i,j}) \le \bar{a}_{i,j} & \theta \circ f_{i,j}(b) \le \theta \circ f_{i,j}(k_{i,j}) \le c_{i,j} & \text{If } f_{i,j} \text{ is decreasing on } C \end{array}$$

Define $\bar{h}_{i,j}, \bar{k}_{i,j}$ similarly but from above b, so:

$$b < \bar{h}_{i,j}, \bar{k}_{i,j} < \bar{a}_0$$

Then if we define:

$$\begin{aligned} h &\coloneqq \max\{h_{i,j}, k_{i,j}\} \\ \bar{h} &\coloneqq \max\{\bar{h}_{i,j}, \bar{k}_{i,j}\} \end{aligned}$$

Then $h < b < \bar{h}$, thus $\frac{1}{2}(h + \bar{h})$ realises Δ.

If b is not in the convex hull of C_G in H, say $b > C_G$, then we can, without loss of generality, assume $a_0 \in C_G$. Define h as before, and note that any element of C_G greater than h realised Δ, and such an element must exist since the convex components of $D_M(\mathcal{G})$ are open. $\square$

So assuming that q_b does determine the quantifier free type of b over $\mathcal{G}$, we can show that $\mathcal{G}$ is existentially closed in $\mathcal{G}\langle b \rangle$, hence by Theorem 2.1.15, we would have quantifier elimination for $\mathcal{L}_{\psi-\chi}$.

Chapter 5

Conclusions

To conclude we mention some ways to continue the thesis. Of course, the main continuation would be to finish the proof of quantifier elimination for $T_{\psi-\chi}$, but beyond that, the two main directions would be examining the action of the derivative of L_i on vK, and describing algebraically the models of T_n.

5.1 n-th Hyper-logarithmic derivative

Let K be the Hyperseries mentioned in Theorem 1.2.2, so an ordered differential real closed field with with functions $\log, L_1, L_2, \ldots : K^{>\mathbb{R}} \to K^{>\mathbb{R}}$ satisfying (**L1**) - (**L4**). We know that each L_i induces a well defined map $\chi_{i+1} : cK \to vK$, and we also know that the derivative induced a well defined map $id + \psi : vK^{\neq 0} \to vK$. Thus the action of the derivative of L_i on vK should be given by the map $(id + \psi) \circ \chi_{i+1}$.

Remember that $(id + \psi) \circ \chi_1(x) = \psi(x)$, and $(id + \psi) \circ \chi_2(x) = \psi(x) + \theta(x)$, where $\theta(x)$ was effectively the infinite sum:

$$\theta(x) \approx \sum_{i<\omega} \chi_1^i(x)$$

This suggests that when calculating $(id + \psi) \circ \chi_3$, we would need a symbol for the infinite sum:

$$\sum_{i<\omega} \chi_2^i(x)$$

Ideally, there would be a symmetry between the actions of the derivatives of L_i and L_{i+1} akin to the symmetry between χ_i and χ_{i+1} in T_n.

5.2 Classifying models of T_n

This was a suggestion given to us by Sebastian Krapp. As mentioned in the introduction, the paper Krapp and S. Kuhlmann, 2023 deals with countable models of T_1 and examines when it is possible to recover a trans-exponential field from a countable contraction group. In doing so, they developed some understanding of the countable models of T_1. If we can give some characterisation of the countable models of T_n, this would help us understand when we could recover a trans-exponential field with an n-th Hyper-logarithm.

5.3 Algebraicity

In Section 3.6.2 we compared the notion of n-algebraicity for n-contractin groups to model-theoretic and field-theoretic algebraic closure. We remarked that it was not clear whether such n-algebraicity has the monotonicity property (in the sense of pre-geometries). Thus it would be interesting to see if the n-algebraic closure is indeed a pregeometry. Moreover, the same could be done with $\psi - \chi$-algebraicity for asymptotic contractions.

5.4 A funny Joke

Finally, it was intended for there to be a chapter on elimination of imaginaries in 1-contraction groups, but due to some mistakes, it had to be cut out. All that is left is the following:

> Why did the imaginary tree fall over?
>
> Because we moved to T_r^{eq} and eliminated the fibres of the roots!

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