Partition statistics counting the Betti numbers of equivariant Hilbert schemes

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I dedicate this thesis to my family. My mother Isabelle, who likes telling people I work on triangles. My father Michael, who tells me if I keep going I'll be able to understand the χ^2 distribution. My husband Pantelis, who says I spend my research time playing tetris. My sister Daisy[1], with whom I have never discussed mathematics outside the three-day window leading up to their A Level exam.

They have all provided me with a huge amount of emotional support and a loving, mathematics-free environment to retreat to during the crazy times. This was undoubtedly been one of the most important factors in the completion of this thesis.

[1]"Why am I in the dedication? All I did was make one sausage casserole."

Declaration

I declare that this thesis is the result of my own work, and includes no work done in collaboration.

It is not substantially the same as any work which has already been submitted before for any degree or other qualification.

Eve Vidalis
May 2024

A relentless sequence of unfortunate events accompanied the completion of this thesis. By early 2024 they amounted to some sort of cosmic joke, best described by Carrie Fisher.

Things were getting worse faster than we could lower our standards.

Abstract

In this thesis, we study a collection of partition statistics $h^{\pm}_{x,c}$ parameterised by a positive real number x, which count the Betti numbers of Hilbert schemes of points on a surface, equivariant under the action of a cyclic group Γ_c, giving a bijective proof that the statistics are equidistributed over partitions of n with a fixed c-core.

We extend techniques introduced by Loehr and Warrington to work with cores and quotients of partitions, and define a map from partitions to multigraphs (where the partition is viewed as an Eulerian tour of the graph) that retains the relevant data on partitions, including the c-core, size, and other statistics closely related to the $h^{\pm}_{x,c}$.

We then define a bijection on partitions with a fixed c-core that preserves the multigraph, which we use to prove bijectively that the distribution of the $h^{\pm}_{x,c}$ is independent of x and the sign. We also compute the $x = 0$ case to obtain a generating function for the distribution.

As a byproduct, we give a combinatorial proof of a partition identity of Buryak-Feigin-Nakajima that can be seen as giving the Betti numbers of $\mathbb{C}^*$-equivariant Hilbert schemes.

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Over the last six months Pantelis has pulled all the stops out to support me first in recovering and then over the last few weeks in writing this thesis. Over the last month he has single-handedly moved us into our house, fed me, and made more cups of tea and coffee than I have been able to drink. He has been a superstar.

Contents

Chapter 1

Introduction

For this chapter, the main character will be the Hilbert Scheme of n points on $\mathbb{A}^2$, $\mathrm{Hilb}_n\left(\mathbb{A}^2\right)$. We will discuss two parallel sets of results. Firstly, we explain a geometric argument due to Haiman that proves that a collection of partition statistics h_x, where x varies over the positive irrationals, are equidistributed with the length of a partition by computing the class of $\mathrm{Hilb}_n\left(\mathbb{A}^2\right)$ in the Grothendieck ring of varieties in multiple different ways. Loehr and Warrington [LW09] take Haiman's result on the equidistribution of h_x over partitions of n and give a bijective proof, also extending x to range over the positive reals. We then run the same geometric argument on the fixed point set of $\mathrm{Hilb}_n\left(\mathbb{A}^2\right)$ under the action of a cyclic group. The main result of this thesis, Theorem A, is to this geometric proof as Loehr and Warrington's result is to Haiman's; we take the resulting equidistribution result, make it bijective, and remove the assumption that x is irrational.

Once we have explained the storyline above, we describe a related result of Buryak, Feigin and Nakajima [BFN15] which gives a geometric proof of a result closely related to our Theorem B (implied by Theorem A taken together with a result of Walsh and Warnaar [WW20]). We conclude the chapter with a description of how our results fit in with those in [BFN15] and [WW20], before comparing and contrasting our methods with Loehr and Warrington's, and finally setting out the structure for the rest of the thesis.

1.1 The Hilbert scheme and the Białynicki-Birula decomposition

To explain the geometric arguments, we briefly describe some basic properties of $\mathrm{Hilb}_n(\mathbb{A}^2)$, describe a torus action on it, and introduce the Białynicki-Birula decomposition which is the main tool in every geometric argument we discuss.

Hilbert schemes were first introduced (much more generally than needed for our purposes) by Grothendieck in [Gro61]. As a set, the Hilbert scheme of n points on $\mathbb{A}^2$ consists of ideals $I \subseteq \mathbb{C}[x,y]$ of colength n:

$$\mathrm{Hilb}_n(\mathbb{A}^2) = \left\{ I \subseteq \mathbb{C}[x,y] \mid \dim_{\mathbb{C}}(\mathbb{C}[x,y])/I) = n \right\}.$$

It comes equipped with the Hilbert-Chow morphism, defined as follows.

Definition 1.1. *The Hilbert-Chow morphism* $\pi : \mathrm{Hilb}_n(\mathbb{A}^2) \to \mathrm{Sym}^n(\mathbb{A}^2)$ *sends an ideal I to the multiset* $\mathrm{Supp}(\mathbb{C}[x,y]/I)$ *consisting of prime ideals* $\mathfrak{p} \in \mathbb{C}[x,y]$ *such that the localisation* $(\mathbb{C}[x,y]/I)_{\mathfrak{p}} \neq 0$*, where the multiplicity of* $\mathfrak{p}$ *is given by the length of the localisation.*

Fogarty [Fog68] showed that the Hilbert-Chow morphism is a resolution of singularities of $\mathrm{Sym}^n(\mathbb{A}^2)$, and that $\mathrm{Hilb}_n(\mathbb{A}^2)$ is smooth, irreducible and connected of dimension $2n$. Other Hilbert schemes are not so well-behaved[1].

The diagonal action of $(\mathbb{C}^*)^2$ on $\mathbb{A}^2$ induces an action on $\mathrm{Hilb}_n(\mathbb{A}^2)$ by

$$(t_1,t_2)\cdot I = \{ f(t_1^{-1}x, t_2^{-1}y) \mid f(x,y) \in I \}.$$

Any ideal fixed under this action must be homogeneous in x and y, and therefore the fixed points are the monomial ideals. Monomial ideals of colength n correspond to partitions of n, as follows.

Definition 1.2. *Let* λ *be a partition of n, so* $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{l(\lambda)}$ *are positive integers with* $\lambda_1 + \cdots + \lambda_{l(\lambda)} = n$*, and* $l(\lambda)$ *is the number of parts, or the length of the partition. This partition corresponds to the monomial ideal* I_λ *generated by* $x^{\lambda_1}, x^{\lambda_2}y, \ldots, x^{\lambda_{l(\lambda)}}y^{l(\lambda)-1}, y^{l(\lambda)}$.

[1] Murphy's law for Hilbert schemes was first formulated by Harris and Morrison [HM98] as follows: "There is no geometric possibility so horrible that it cannot be found generically on some component of some Hilbert scheme."

The generating set may not be minimal; the point is that the property of a monomial *not* being in the ideal is preserved by division by x or y so it suffices to find the monomials that are in some sense the "bottom left corners" of the ideal.

Example 1.3. *The partition $\mu = (4,2,1)$ corresponds to the monomial ideal*

$$I_\mu = (x^4, x^2y, xy^2, y^3).$$

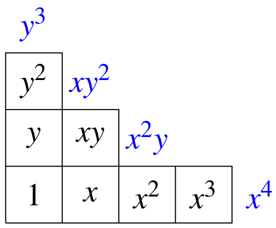


Fig. 1.1 The partition $(4,2,1)$ corresponds to the monomial ideal (x^4, x^2y, xy^2, y^3).

The main geometric technique we will discuss over the next two chapters will use the Białynicki-Birula decomposition (stated below) with different decompositions of $\mathrm{Hilb}_n(\mathbb{A}^2)$ or closely related spaces arising from different choices of $\mathbb{C}^*$-action. As we might expect from the above, the fixed point sets of these actions will often be indexed by partitions.

Theorem 1.4 (Białynicki-Birula)**.** *Let X be a smooth variety with a $\mathbb{C}^*$-action, so that for any $x \in X$, the limit $\lim_{\lambda \to 0} \lambda \cdot x$ exists in X, where the limit is over $\lambda \in \mathbb{C}^*$. Let $F = \cup_{i \in I} F_i$ be the decomposition of the fixed point set into connected components and let*

$$U_i = \{x \in X : \lim_{\lambda \to 0} \lambda x \in F_i\}$$

be the set of points that flow to F_i. Then $X = \sqcup_{i \in I} U_i$.

Moreover, each fibre of the normal bundle of $F_i \subset X$ admits a $\mathbb{C}^*$-action and decomposes into $\mathbb{C}^*$-eigenspaces. Letting d_i be the total dimension of the eigenspaces of positive weight, U_i is a vector bundle over F_i and the fibre is an affine space of dimension d_i.

In the Grothendieck ring of varieties, that is, the ring generated by equivalence classes of varieties $[X]$ where we impose the relation that if $Y \subset X$ is a closed subvariety then $[X] =$

$[Y]+[X\setminus Y]$, the Białynicki-Birula decomposition can be stated as $[X]=\sum[U_i]=\sum[F_i][\mathbb{A}^{d_i}]$. If the fixed points are isolated, $[X]=\sum[\mathbb{A}^{d_i}]$.

Restricting the $(\mathbb{C}^*)^2$ action on $\text{Hilb}_n(\mathbb{A}^2)$ to a generic one-parameter subtorus $\mathbb{C}^*=\{(t^\alpha,t^\beta)\mid t\in\mathbb{C}^*,\frac{\alpha}{\beta}\in\mathbb{R}\backslash\mathbb{Q}\}$, the fixed point set is again the monomial ideals. In particular, $\text{Hilb}_n(\mathbb{A}^2)$ has $|\text{Par}(n)|$-many $\mathbb{C}^*$-fixed points, where $\text{Par}(n)$ denotes the set of partitions of n.

The discussion above tells us that if we choose a generic $\mathbb{C}^*$-subtorus of $(\mathbb{C}^*)^2$, $\text{Hilb}_n(\mathbb{A}^2)$ decomposes into cells indexed by partitions of n. If the subtorus does not have isolated fixed points, then there will be a contribution from the topology of the fixed point sets F_i. Most times, the d_i and the contributions from the topology will be mixed together inextricably. However, Buryak, Feigin and Nakajima [BFN15] use an action with non-isolated fixed points to compute the Betti numbers of a $\mathbb{C}^*$-equivariant Hilbert scheme by controlling the d_i - we will discuss this argument at the end of Chapter 2.

1.2 Ellingsrud-Strømme

Next, we will exhibit different computations of $[\text{Hilb}_n(\mathbb{A}^2)]$, all of which are based on the Białynicki-Birula decomposition. These computations make use of a result of Ellingsrud and Strømme [ES87] on the class of the tangent space $T_\lambda(\text{Hilb}_n(\mathbb{A}^2))$ at a monomial ideal I_λ as a $(\mathbb{C}^*)^2$-representation. This is useful because the Białynicki-Birula decomposition requires that we know the dimension of the positive weight space on the fibre of the normal bundle over the fixed points, and we have seen above that the fixed points are precisely the monomial ideals. So, given Ellingsrud and Strømme's computation of the tangent space as a $(\mathbb{C}^*)^2$-representation, we restrict to a collection of generic $(\mathbb{C}^*)$-representations indexed by an irrational number x and compute the values of d_i that arise in each case. Taken together, the d_i for each action are a statistic on partitions h_x, which a priori are unrelated. However, since each decomposition computes the class $[\text{Hilb}_n(\mathbb{A}^2)]$, this gives an equidistribution result for the h_x as x varies.

Before we begin we recall some definitions from partition combinatorics, and then calibrate our notation with Ellingsrud and Strømme's via Lemma 1.7. In this section we only state and demonstrate some applications of Ellingsrud and Strømme's result. We prove it in Chapter 2.

Definition 1.5 (Arm, leg, hook length)**.** *The arm of a box* $\square$ *in the Young diagram of a partition consists of the boxes that lie strictly to the right of* $\square$ *in the same row, and the leg of* $\square$ *consists of the boxes that lie strictly above*[2] $\square$ *in the same column. We denote the number of boxes in the arm of* $\square$ *by* $a(\square)$ *and the number of boxes in the leg of* $\square$ *by* $l(\square)$*. The hook length of* $\square$ *is* $h(\square) = a(\square) + l(\square) + 1$.

Example 1.6. *The boxes in the arm and leg of the shaded box* $\square$ *in Figure 3.2 are labelled with the corresponding body part. We have* $a(\square) = 5$ *and* $l(\square) = 1$, *so* $h(\square) = 7$.

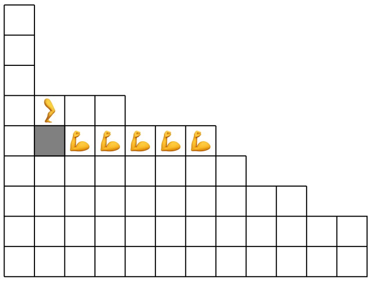


Fig. 1.2 the arm and leg of $\square$.

In [ES87] Ellingsrud and Strømme compute the cohomology of $\mathrm{Hilb}_n\left(\mathbb{A}^2\right)$ and along the way prove a useful formula for the class of the tangent space to $\mathrm{Hilb}_n\left(\mathbb{A}^2\right)$ at a monomial ideal in the representation ring of $(\mathbb{C}^*)^2$. In order to state Ellingsrud and Strømme's result, we first briefly explain a formula to count the arm and leg of the cells in a partition.

Lemma 1.7. *Let* $\lambda_1 \geq \cdots \geq \lambda_{l(\lambda)} > 0$ *and set* $\lambda_{l(\lambda)+1} = 0$*. The multiset* $\{(a(\square), l(\square) \mid \square \in \lambda\}$ *is equal to the multiset*

$$\{(\lambda_i - s, j-i) \mid i,j,s \in \mathbb{Z},\ 1 \leq i \leq j \leq l(\lambda),\ \lambda_j \leq s \leq \lambda_{j-1} - 1\}.$$

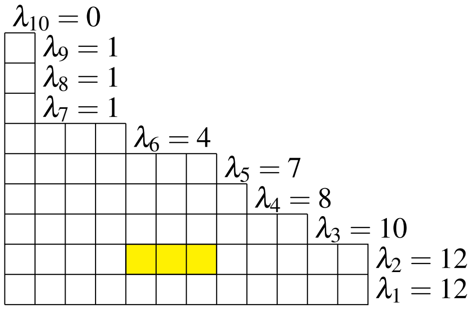
Referring to Example 1.8 may make the proof clearer.

Proof. Let a box $\square$ be in row i of size λ_i, and let row j contain the top box in the same column as $\square$. Then $j \leq i$. All $\lambda_j - \lambda_{j+1}$ of the boxes with this property have $l(\square) = j - i$.

The arm of the rightmost of these boxes is $\lambda_i - \lambda_j$, and as we scan left along the boxes the arm increases by one with each box until we reach the leftmost one which has arm $\lambda_i - \lambda_{j-1} + 1$. $\square$

[2] This convention for drawing diagrams puts the legs above the arms, which is admittedly unusual for humans unless they are in the circus, doing yoga, or in Australia. Offended readers are welcome to adjust their reading position accordingly.

Example 1.8. *The figure shows the $\lambda_5 - \lambda_6 = 3$ boxes in row 2 with $l(\Box) = 6 - 2 - 1 = 3$.*



We are now in a position to state Ellingsrud and Strømme's result.

Theorem 1.9 (Ellingsrud-Strømme)**.** *Let λ be a partition with r parts and exceptionally index λ as $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{r-1} > 0$ and $\lambda_r = 0$.*

In the representation ring of a two-dimensional torus $(\mathbb{C}^*)^2$, the tangent space at a monomial ideal I_λ to $\mathrm{Hilb}_n\left(\mathbb{A}^2\right)$ is given by

$$\left[T_\lambda\left(\mathrm{Hilb}_n\left(\mathbb{A}^2\right)\right)\right] = \sum_{1 \leq i,j \leq r} \sum_{s=\lambda_j}^{\lambda_{j-1}-1} \rho(t)^{i-j-1}\phi(t)^{\lambda_{i-1}-s-1} + \rho(t)^{j-i}\phi(t)^{s-\lambda_{i-1}}$$

where $t \cdot x = \phi(t)x$ and $t \cdot y = \rho(t)y$ for linearly independent characters ϕ and ρ of $(\mathbb{C}^*)^2$.

Since we count λ_0 as the largest part of our partition rather than λ_1, Lemma 1.7 gives us that $\{(a(\Box), l(\Box)) \mid \Box \in \lambda\}$ can be written as

$$\{(\lambda_{i-1} - s - 1, j - i) \mid i, j, s \in \mathbb{Z},\ 1 \leq i \leq j \leq r,\ \lambda_j \leq s \leq \lambda_{j-1} - 1\}.$$

Applying this to Theorem 1.9 gives

$$\left[T_\lambda\left(\mathrm{Hilb}_n\left(\mathbb{A}^2\right)\right)\right] = \sum_{\Box \in \lambda} \rho(t)^{-l(\Box)-1}\phi(t)^{a(\Box)} + \rho(t)^{l(\Box)}\phi(t)^{-a(\Box)-1}.$$

We prove Theorem 1.9 in Chapter 2, and just show here how the main results of interest follow. Loehr and Warrington [LW09] allude to an unpublished argument due to Haiman that restricts to different choices of $\mathbb{C}^*$-action in Ellingsrud and Strømme's result to show the following.

Lemma 1.10 (Haiman, unpublished)**.** *For a positive irrational number x, define*

$$h_x(\lambda) = \left| \left\{ \square \in \lambda : \frac{a(\square)}{l(\square)+1} < x < \frac{a(\square)+1}{l(\square)} \right\} \right|.$$

Then in the Grothendieck ring of varieties,

$$\left[\mathrm{Hilb}_n\left(\mathbb{A}^2\right)\right] = \sum_{\lambda \in \mathrm{Par}(n)} \mathbb{L}^{n+h_x(\lambda)}$$

where $\mathbb{L}$ denotes the class of the affine line. In particular

$$\sum_{\lambda \in \mathrm{Par}} q^{|\lambda|} t^{h_x(\lambda)}$$

is independent of x.

We take the proof sketched in [LW09] (again attributed to Haiman) and fill in the details below.

Proof. We use a Białynicki-Birula decomposition with different choices of $\mathbb{C}^*$-action.

Restrict the $(\mathbb{C}^*)^2$-representation in Theorem 1.9 to a generic one-parameter subtorus $(t^{r+\varepsilon}, t^s)$, where we take r, s to be coprime positive integers and ε to be a small irrational number. At the level of ideals, $(t^{r+\varepsilon}, t^s) \cdot (f_i(x,y)) = (f_i(t^{-(r+\varepsilon)}x, t^{-s}y))$. This leaves us with

$$\left[T_\lambda\left(\mathrm{Hilb}_n\left(\mathbb{A}^2\right)\right)\right] = \sum_{\square \in \lambda} t^{(r+\varepsilon)(l(\square)+1) - sa(\square)} + t^{-(r+\varepsilon)l(\square) + s(a(\square)+1)}.$$

We wish to count the dimension of the positive weight space, that is, the number of boxes $\square \in \lambda$ such that

$$A_\square = (r+\varepsilon)(l(\square)+1) - sa(\square) > 0 \tag{1.1}$$

together with the number of boxes such that

$$B_\square = -(r+\varepsilon)l(\square) + s(a(\square)+1) > 0. \tag{1.2}$$

First, note that $A_\square + B_\square = r + \varepsilon + s > 0$ so for a fixed box $\square$, at least one of $A_\square$ and $B_\square$ must be positive. Both $A_\square$ and $B_\square$ are positive if and only if

$$\frac{a(\square)}{l(\square)+1} < \frac{r+\varepsilon}{s} < \frac{a(\square)+1}{l(\square)}.$$

So, taking $x = \frac{r+\varepsilon}{s}$, the dimension of the positive eigenspace is

$$|\lambda|+\left|\left\{\square\in\lambda : \frac{a(\square)}{l(\square)+1} < \frac{r+\varepsilon}{s} < \frac{a(\square)+1}{l(\square)}\right\}\right| = |\lambda|+h_x(\square).$$

Therefore, by Theorem 1.4, in the Grothendieck ring we have

$$\begin{aligned}\left[\mathrm{Hilb}_n\left(\mathbb{A}^2\right)\right] &= \sum_{\lambda\in\mathrm{Par}(n)}\left[\mathbb{A}^{|\lambda|+h_x(\lambda)}\right]\\ &= \sum_{\lambda\in\mathrm{Par}(n)}\mathbb{L}^{|\lambda|+h_x(\lambda)}.\end{aligned}$$

So,

$$\sum_{n\geq 1}\left[\mathrm{Hilb}_n\left(\mathbb{A}^2\right)\right]q^n = \sum_{\lambda\in\mathrm{Par}}\mathbb{L}^{|\lambda|+h_x(\lambda)}q^{|\lambda|}.$$

This proves the first claim. Identifying t with $\mathbb{L}$, multiplying by $\left(\frac{q}{t}\right)^{|\lambda|}$ and summing over n proves the second claim: the left hand side is independent of x, so the right hand side must be too. □

The next lemma computes the class of $\mathrm{Hilb}_n\left(\mathbb{A}^2\right)$ when x is small, which by the previous lemma in fact computes the generating function for any x.

Lemma 1.11 (Ellingsrud-Strømme, Haiman)**.** *In the Grothendieck ring of varieties,*

$$\left[\mathrm{Hilb}_n\left(\mathbb{A}^2\right)\right] = \sum_{\lambda\in\mathrm{Par}(n)}\mathbb{L}^{|\lambda|+l(\lambda)}$$

where $l(\lambda)$ denotes the number of parts of λ.

Proof. Choosing r, s, ε such that

$$\frac{r+\varepsilon}{s} < \frac{1}{|\lambda|},$$

we have that $s > (r+\varepsilon)|\lambda|$ so for any box

$$\begin{aligned}A_\square &= (r+\varepsilon)(l(\square)+1) - sa(\square)\\ &\leq (r+\varepsilon)|\lambda| - sa(\square)\\ &< (r+\varepsilon)|\lambda|(1-a(\square)).\end{aligned}$$

So, $A_\square$ is positive if and only if $a(\square) = 0$, and the number of boxes with $A_\square > 0$ is $l(\lambda)$. Moreover,

$$\begin{aligned} B_\square &= -(r+\varepsilon)l(\square) + s(a(\square)+1) \\ &> -(r+\varepsilon)|\lambda| + s(a(\square)+1) \\ &> (r+\varepsilon)|\lambda| a(\square), \end{aligned}$$

so $B_\square > 0$ for all $\square \in \lambda$. Hence, the dimension of the positive eigenspace is $|\lambda| + l(\lambda)$. $\square$

Corollary 1.12 (Loehr-Warrington, Haiman)**.** *For a positive irrational number x*

$$\sum_{\lambda \in \mathrm{Par}} q^{|\lambda|} t^{h_x(\lambda)} = \prod_{i=1}^{\infty} \frac{1}{1-tq^i}. \tag{1.3}$$

Proof. Combining the two previous lemmas, we have

$$\sum_{\lambda \in \mathrm{Par}(n)} t^{n+h_x(\lambda)} = \sum_{\lambda \in \mathrm{Par}(n)} t^{n+l(\lambda)}.$$

So,

$$\sum_{\lambda \in \mathrm{Par}(n)} t^{h_x(\lambda)} = \sum_{\lambda \in \mathrm{Par}(n)} t^{l(\lambda)}.$$

Multiplying by $q^{|\lambda|}$ and summing over n,

$$\sum_{\lambda \in \mathrm{Par}} q^{|\lambda|} t^{h_x(\lambda)} = \sum_{\lambda \in \mathrm{Par}} q^{|\lambda|} t^{l(\lambda)} = \prod_{i \geq 1} \frac{1}{1-tq^i}$$

where the final product formula is Euler's formula (proved in Proposition 3.36). $\square$

At this point, the Białynicki-Birula decomposition has told us that the $h_x(\lambda)$ are equidistributed with $l(\lambda)$ over the partitions of n, and one could wonder if a combinatorial proof of this equidistribution could be found. In [LW09], Loehr and Warrington give a bijective proof of Corollary (1.12) and extend the equidistribution to non-negative rational x, with the modification that one of the defining inequalities must be replaced by a soft inequality. This is done by providing a bijection on $\mathrm{Par}(n)$ that interchanges $h_x^+(\lambda)$ with $l(\lambda)$, where

$$h_x^+(\lambda) = \left| \left\{ \square \in \lambda : \frac{a(\square)}{l(\square)+1} \leq x < \frac{a(\square)+1}{l(\square)} \right\} \right|.$$

In particular, $h_x = h_x^+$ when x is irrational.

Theorem 1.13 (Loehr-Warrington)**.** *For any real number x, there is a bijection I_x on partitions of n such that $l(I_x(\lambda)) = h_x^+(\lambda)$ and $h_x^+(I_x(\lambda)) = l(\lambda)$.*

1.3 Throwing in a cyclic action

In this thesis we study a storyline parallel to that laid out in Section 1.2, where instead of $\mathrm{Hilb}_n(\mathbb{A}^2)$ we look at the fixed point set under the action of a cyclic group.

This next section runs the geometric argument of Section 1.2 on the irreducible components of the fixed point sets of $\mathrm{Hilb}_n\left(\mathbb{A}^2\right)$ under the action of a cyclic group

$$\Gamma_c = \left\{ \left(e^{\frac{2k\pi i}{c}}, e^{-\frac{2k\pi i}{c}} \right), 1 \leq k \leq c \right\} \subset \left(\mathbb{C}^*\right)^2.$$

This leads to the equidistribution of a statistic $h_{x,c}$ over a subset $\mathrm{Par}_\mu^c(n)$ of partitions, indexing the irreducible components of the fixed point sets. This allows us to state our main theorem, which takes the geometric proof of the equidistribution result and makes it bijective, just as Theorem 1.13 does with Lemma 1.10. One viewpoint of our main result is that we refine Theorem 1.13 to work with the fixed points of a non-trivial cyclic action.

As with the proof of Theorem 1.13 in [LW09], with a minor modification to the statistic we can also extend x to the rationals, which corresponds to picking a torus action with non-isolated fixed points. A geometric argument (with restrictions on the relationship between the rational x and c) is given in [BFN15]; a special case of our main result makes this purely combinatorial, solving [WW20, Problem 8.9].

1.3.1 Brief interlude on c-cores and quotients

In order to describe the irreducible components of the fixed point set, we need c-core partitions. We treat the background on cores and quotients of partitions thoroughly in Chapter 3, but briefly recall some of the key definitions and properties here to allow us to place our main theorem in context. Proofs are deferred to Chapter 3.

Definition 1.14 (Rimhook)**.** *A rimhook R of length c is a connected set of c boxes in λ such that removing R gives the Young diagram of another partition, and R does not contain a 2×2 box.*

Definition 1.15. *A c-core of a partition λ is a partition obtained by iteratively removing rimhooks of length c from λ until a partition with no rimhooks of length c is obtained. A partition μ is called a c-core if μ has no rimhooks of length c.*

The key properties that we need are that rimhooks of length c are in bijection with boxes of hook length c, and it turns out that a c-core can also be defined to be a partition with no boxes with hook length divisible by c. The c-core of a partition is unique, independent of the order of removing rimhooks.

We defer discussion of quotients to Chapter 3, and just note the size of the quotient $|q_c(\lambda)|$ counts the number of boxes in the partition with hook length divisible by c.

Consider the action of Γ_c on $\mathbb{C}[x,y]/I_\lambda$ for some partition λ. The character of the action of Γ_c on $\mathbb{C}x^a y^b$ is $e^{\frac{2\pi i(a-b)}{c}}$. So, we can colour boxes in the plane according to the character of the Γ_c-representations on the corresponding monomials, with a box with bottom left corner (a,b) given by $(a-b)$ modulo c.

Definition 1.16. *We say that the c-content of a partition is the multiset of colours of boxes in its Young diagram when coloured according to the Γ_c-representation.*

By the discussion above, then, two partitions have the same c-content if and only if the corresponding classes in the representation ring of Γ_c agree.

In particular, a rimhook of length c contains exactly one box of each colour, so adding or removing such a rimhook from a partition λ corresponds to adding or subtracting a copy of the regular representation to $\mathbb{C}[x,y]/I_\lambda$. In fact two partitions λ_1, λ_2 have the same c-core if and only if $\mathbb{C}[x,y]/I_{\lambda_1}$ differs from $\mathbb{C}[x,y]/I_{\lambda_2}$ up to adding and subtracting copies of the regular representation. This is a consequence of [JK81, Thm 2.7.41], which states that two partitions of the same integer n have the same c-core if and only if they have the same c-content.

The real content of this result is that it is enough for two partitions to have the same core if we can obtain one from the other by subtracting and then adding sequences of *any c* boxes of each colour, rather than just sequences of rimhooks (the existence of one of the former sequences implies the existence of the latter). That is, whenever the representations $\mathbb{C}[x,y]/I_{\lambda_1}$ and $\mathbb{C}[x,y]/I_{\lambda_2}$ differ by addition and subtration of copies of the regular representation, the c-cores will agree.

Example 1.17. *See Figure 1.3: iteratively removing rimhooks of length 3 from* $(6,3,3,1)$ *shows that it has 3-core (1).*

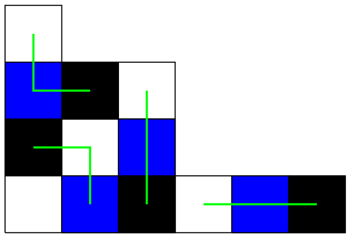


Fig. 1.3 As a Γ_3-representation, an ideal J such that $\mathbb{C}[x,y]/J$ has the same basis as $\mathbb{C}[x,y]/I_{(6,3,3,1)}$ is the trivial representation plus four copies of the regular representation. The green indicates a possible sequence of rimhooks that could be removed (first the horizontal 3 squares, then the vertical 3 squares, then the 3 squares in an L, then the 3 squares in an upside-down L).

Two monomial ideals I_λ, I_μ belong to the same irreducible componenent of $\mathrm{Hilb}_n(\mathbb{C}[x,y])^{\Gamma_c}$ if and only if $\mathbb{C}[x,y]/I_\lambda \cong \mathbb{C}[x,y]/I_\mu$ as Γ_c-representations, that is, if the coloured box count is the same for λ and μ. Therefore, the irreducible components of $\mathrm{Hilb}_n\left(\mathbb{A}^2\right)$ are indexed by the c-cores of partitions of n – in fact these components are smooth and connected [MS10]. We write $\mathrm{Hilb}_n^\mu\left(\mathbb{A}^2\right)^{\Gamma_c}$ for the irreducible component corresponding to the c-core μ: as a set, it consists of Γ_c-invariant ideals I such that $\mathbb{C}[x,y]/I$, as a Γ_c-representation, is the representation corresponding to μ plus $|q_c(\lambda)|$ copies of the regular representation. The monomial ideals are Γ_c-invariant and the tangent space at I_λ is

$$T_\lambda\left(\mathrm{Hilb}_n\left(\mathbb{A}^2\right)^{\Gamma_c}\right) = T_\lambda\left(\mathrm{Hilb}_n\left(\mathbb{A}^2\right)\right)^{\Gamma_c}. \tag{1.4}$$

The next theorem is analogous to Lemma 1.10 (and in fact setting $c=1$ recovers Lemma 1.10). It is unclear where it originates, but it appears in [GZLMH10].

Theorem 1.18. *Let c be a positive integer x a positive irrational number, and μ a c-core partition of n and let $h_{x,c}(\lambda)$ count the boxes of a partition λ that contribute to $h_x(\lambda)$ and have hook length divisible by c. Then,*

$$\left[\mathrm{Hilb}_n^\mu\left(\mathbb{A}^2\right)^{\Gamma_c}\right] = \sum_{\lambda\in \mathrm{Par}_\mu^c(n)} \mathbb{L}^{|q_c(\lambda)|+h_{x,c}(\lambda)}.$$

Proof. Since Γ_c is a subgroup of the torus, the actions commute and Ellingsrud-Strømme's result gives that, as $\mathbb{C}^*\times\Gamma_c$-representations,

$$\left[T_\lambda\left(\mathrm{Hilb}_n\left(\mathbb{A}^2\right)\right)\right] = \sum_{\square\in\lambda}\left(e^{\frac{2k\pi i}{c}}\right)^{a(\square)+l(\square)+1}\rho(t)^{-l(\square)-1}\phi(t)^{a(\square)} + \left(e^{\frac{2k\pi i}{c}}\right)^{-a(\square)-l(\square)-1}\rho(t)^{l(\square)}\phi(t)^{-a(\square)-1},$$

so the Γ_c-invariant directions correspond to boxes where the hook length is divisible by c:

$$\left[T_\lambda\left(\mathrm{Hilb}_n\left(\mathbb{A}^2\right)\right)^{\Gamma_c}\right] = \sum_{\substack{\square\in\lambda\\ c|h(\square)}} t^{(r+\varepsilon)(l(\square)+1)-sa(\square)} + t^{-(r+\varepsilon)l(\square)+s(a(\square)+1)}.$$

Exactly as in the proof of Lemma 1.10, we restrict to a generic one-parameter subtorus with weight $(r+\varepsilon,s)$ as before, the total dimension of the positive eigenspace will count the boxes with hook length divisible by c that contribute to $A_\square$ or $B_\square$ (with multiplicity).

As before, every box contributes to one of $A_\square$ or $B_\square$ and $h_x(\lambda)$ counts the number of boxes that contribute to both, but this time we only want to include boxes with hook length divisible by c, so the dimension of the positive eigenspace is

$$|q_c(\lambda)| + \left|\left\{\square\in\lambda : \frac{a(\square)}{l(\square)+1} < x < \frac{a(\square)+1}{l(\square)} \text{ and } c \mid h(\square)\right\}\right|;$$

we have defined $h_{x,c}$ to be the second term. By Theorem 1.4 the theorem them follows. □

1.4 Fitting the results of this thesis into the literature

So far we have seen that Loehr and Warrington's construction provides a bijective proof of the equidistribution of h_x over the set of partitions, first proved geometrically by Haiman for x irrational. Loehr and Warrington's proof extends to an equidistribution result for the slightly modified statistic h_x^+ as x ranges over the non-negative reals, which is genuinely an extension because the statistic h_x agrees with h_x^+ when x is irrational. In a similar vein, for fixed c we construct a bijection that proves that the $h_{x,c}$ are equidistributed over $\mathrm{Par}^c_\mu(n)$ as x ranges over the positive irrationals, proved geometrically in Theorem 1.18. We also define a slightly modified statistic $h^+_{x,c}$ at the rationals, which agrees with $h_{x,c}$ when x is irrational, and extend the equidistribution result to the $h^+_{x,c}$ as x ranges over the non-negative reals.

Theorem A. *For a real number x and positive integer c there is a bijection $I_{x,c} : \mathrm{Par}^c_\mu(n) \to \mathrm{Par}^c_\mu(n)$ such that $h^+_{x,c}(I_{x,c}(\lambda)) = h^+_{0,c}(\lambda)$.*

Walsh and Warnaar [WW20] gave a combinatorial (but not bijective) proof of a product formula for the $x = 0$ case (analogous to the application of Euler's theorem in Section 1.2), which together with Theorem A allows us to show the following combinatorially.

Theorem B. *For any $x \geq 0$,*

$$\sum_{\lambda \in \mathrm{Par}^c_\mu} q^{|\lambda|} t^{h^+_{x,c}(\lambda)} = q^{|\mu|} \prod_{i \geq 1} \frac{1}{(1-q^{ic})^{c-1}} \prod_{j \geq 1} \frac{1}{1-q^{jc}t}. \tag{1.5}$$

Buryak, Feigin and Nakajima give a geometric proof (outlined in Chapter 2) that when α, β are positive integers $h^+_{\frac{\alpha}{\beta},\alpha+\beta}$ depends only on the sum $\alpha+\beta$, and go on to compute the $\alpha = 0$ case $h^+_{0,\alpha+\beta}$. They then sum over n to obtain the identity

$$\sum_{n \geq 0} \left[\mathrm{Hilb}_n \left(\mathbb{A}^2\right)^{T_{\alpha,\beta} \times \Gamma_{\alpha+\beta}} \right] q^n = \prod_{\substack{i \geq 1 \\ (\alpha+\beta) \nmid i}} \frac{1}{1-q^i} \prod_{i \geq 1} \frac{1}{1-tq^{(\alpha+\beta)i}}$$

in the Grothendieck ring of complex quasi-projective varieties, where t stands in for the class of the affine line. They also compute the singular homology of $\mathrm{Hilb}_n \left(\mathbb{A}^2\right)^{T_{\alpha,\beta}}$ and Borel-Moore homology of $\mathrm{Hilb}_n \left(\mathbb{A}^2\right)^{T_{\alpha,\beta} \times \Gamma_{\alpha+\beta}}$. A combinatorial consequence of this is that we obtain the generating function

$$\sum_{\lambda \in \mathrm{Par}} q^{|\lambda|} t^{h^+_{\frac{\alpha}{\beta},\alpha+\beta}(\lambda)} = \prod_{\substack{i \geq 1 \\ (\alpha+\beta) \nmid i}} \frac{1}{1-q^i} \prod_{i \geq 1} \frac{1}{1-tq^{(\alpha+\beta)i}}, \tag{1.6}$$

although in [BFN15], the statistic $h^+_{\frac{\alpha}{\beta},\alpha+\beta}$ is replaced by a different statistic

$$\mathrm{bf}_{\alpha,\beta}(\lambda) = |\{\square \in \lambda : \alpha l(\square) = \beta a(\square) + \beta \text{ and } (\alpha+\beta) \mid h(\square)\}|.$$

We show that $\mathrm{bf}_{\alpha,\beta} = h^+_{\frac{\alpha}{\beta},\alpha+\beta}$ in the proof of Corollary (4.7).

Walsh and Warnaar [WW20, §8.2] observe that a consequence of Buryak, Feigin and Nakajima's proof is the combinatorial identity

$$\sum_{\lambda \in \mathrm{Par}_\mu} q^{|\lambda|} t^{\mathrm{bf}_{\alpha,\beta}(\lambda)} = q^{|\mu|} \prod_{i \in \mathbb{Z}} \frac{1}{1-tq^{ci}} \prod \frac{1}{\left(1-q^{cj}\right)^{c-1}} \tag{1.7}$$

where $c = \alpha + \beta$.

Walsh and Warnaar observe that the difficulty of providing a combinatorial proof of (1.7) lies in proving that $\{\mathrm{bf}_{\alpha,\beta}(\lambda) : \lambda \in \mathrm{Par}(n)\}$ depends only on $\alpha + \beta$. This is a special case of what Theorem 1.4 achieves, solving problem 8.9 of [WW20]. So, another viewpoint of Theorem B is that it makes Buryak, Feigin and Nakajima's result combinatorial.

1.5 Overview of the construction of $I_{x,c}$

Our construction of the bijection $I_{x,c}$ is similar to Loehr and Warrington's construction of I_x in [LW09], and setting $c = 1$ one recovers Loehr and Warrington's construction. Loehr and Warrington construct a multigraph $M_{r,s}$ by identifying points in the plane (where the Young diagram sits) via $(x_1, y_1) \sim_{r,s} (x_2, y_2)$ when $sx_1 + ry_1 = sx_2 + ry_2$, and view partitions as Eulerian tours of these multigraphs. They go on to prove that some key statistics are independent of the choice of tour by computing explicit formulae for the relevant statistics in terms of the multigraph.

To prove Theorem A, we refine Loehr and Warrington's construction to also account for c-cores by instead imposing the relation $\sim_{r,s,c}$ where $(x_1, y_1) \sim_{r,s,c} (x_2, y_2)$ if $sx_1 + ry_1 = sx_2 + ry_2$ and $y_1 - x_1 \equiv y_2 - x_2 \pmod{c}$. Our proofs that the resulting multigraphs determine the key statistics differ significantly from Loehr and Warrington's. In particular, we do not compute formulae for our statistics in terms of the multigraph (except for one that we need), and instead we make heavy use of an ordering $<_{r,s,c}$ on multigraphs which lifts to an ordering on partitions. We show that the way a statistic changes when taking successors with respect to the ordering is dependent only on the multigraph, and exhibit a family of points that act as base cases for inductive proofs that partition statistics depend only on the multigraph. This makes some of the computations significantly shorter.

1.6 Structure of this thesis

Chapter 2 recalls some of the geometric background. We show that $\mathrm{Hilb}_n(\mathbb{A}^2)$ is a smooth variety (in particular, we can apply the Białynicki-Birula decomposition to it) and build to proving Theorem 1.9 (Ellingsrud-Strømme's description of the tangent space at a monomial ideal) and sketching Buryak, Feigin and Nakajima's proof that the the distribution of $h_{x,c}$ is independent of x where x ranges over a finite set of rational numbers determined by c.

Chapter 3 recalls some definitions from partition combinatorics. In particular, we recall the abacus construction (the standard reference for this is [JK81, §2.7]) and recall some basic generating functions. The chapter builds up to proving Theorem 3.42, which uses a map introduced in [WW20] to compute the distribution of $h^+_{0,c}$ over Par^c_μ, the set of partitions with c-core μ.

$$\sum_{\lambda\in\mathrm{Par}^c_\mu} q^{|\lambda|}t^{\lambda h^+_{0,c}} = q^{|\mu|}\prod_{i\geq 1}\frac{1}{(1-q^{ic})^{c-1}}\prod_{j\geq 1}\frac{1}{1-q^{jc}t}. \tag{1.8}$$

Chapter 4 defines the main partition statistics of interest, $\mathrm{mid}_{x,c}$, $\mathrm{crit}^-_{x,c}$, $\mathrm{crit}^+_{x,c}$, $h^+_{x,c}$ and $h^-_{x,c}$ where $h^\pm_{x,c} = \mathrm{mid}_{x,c} + \mathrm{crit}^\pm_{x,c}$. Then, we introduce our main theorem, Theorem 4.2 which states that for all $x \in [0,\infty)$,

$$\sum_{\lambda\in\mathrm{Par}^c_\mu} q^{|\lambda|}t^{h^+_{x,c}(\lambda)} = q^{|\mu|}\prod_{i\geq 1}\frac{1}{(1-q^{ic})^{c-1}}\prod_{j\geq 1}\frac{1}{1-q^{jc}t}. \tag{1.9}$$

In view of Theorem 3.42, it remains to prove that the left hand side is independent of x. An argument analogous to that in [LW09] is then used to show that the independence of the left hand side from x is implied by the following symmetry property when x is rational,

$$\sum_{\lambda\in\mathrm{Par}^c_\mu} q^{|\lambda|}w^{h^+_{x,c}(\lambda)}y^{h^-_{x,c}(\lambda)} = \sum_{\lambda\in\mathrm{Par}^c_\mu} q^{|\lambda|}w^{h^-_{x,c}(\lambda)}y^{h^+_{x,c}(\lambda)}. \tag{1.10}$$

In Proposition 4.6, we use the symmetry property to give a set of criteria that constitute a sufficient condition for a bijection to prove Theorem 4.2. Finally, the chapter concludes with a proof that the main result of [BFN15] is a consequence of Theorem 4.2.

Chapter 5 defines the multigraph $M_{r,s,c}(\lambda)$ corresponding to a rational $x = \frac{r}{s}$ and positive integer c, defines an ordering $<_{r,s,c}$ on partitions and multigraphs, and a special set of partitions $\lambda_{r,s,k}$. It then goes on to outline the structure of our proofs that $M_{r,s,c}$ remembers partition data. Our proof is structured somewhat differently to Loehr and Warrington's proofs that $M_{r,s}$ remembers partition data in [LW09]. In particular, we do not prove formulae in terms of $M_{r,s,c}$ for any partition statistics except for $\mathrm{crit}^+_{x,c} + \mathrm{crit}^-_{x,c}$. Instead, the chapter works towards providing an inductive framework to prove that $M_{r,s,c}$ remembers partition data by studying how taking successors at the level of partitions and multigraphs are related, culminating in Proposition 5.24. One result of this chapter (Proposition 5.14) is that the map $\lambda \mapsto M_{r,s,c}(\lambda)$ is injective on the preimages of the $M_{r,s,c}(\lambda_{r,s,k})$, so the map does not lose any data at all at these points, allowing the $\lambda_{r,s,k}$ to form a family of base cases. Having outlined the key principles behind the proofs, we then defer the technical checks to Chapter 7.

Chapter 6 defines the involutions $I_{r,s,c} : \mathrm{Par}^c_\mu \to \mathrm{Par}^c_\mu$ that preserve the multigraphs $M_{r,s,c}(\lambda)$, and checks that they are well-defined.

Chapter 7 studies how each statistic of interest in Proposition 4.6 changes when taking successors with respect to the ordering $<_{r,s,c}$, in particular using Proposition 5.24 to prove that the map $\lambda \mapsto M_{r,s,c}(\lambda)$ remembers the statistics $\mathrm{mid}_{x,c}(\lambda)$ and $\mathrm{crit}^+_{x,c} + \mathrm{crit}^-_{x,c}$. It also proves that $I_{r,s,c}$ exchanges the statistics $\mathrm{crit}^+_{x,c}$ and $\mathrm{crit}^-_{x,c}$. Together with the results of Chapter 4, this completes a combinatorial proof of Theorem 4.2.

Chapters 3–7 are a (very slightly modified) version of [Vid23].

Chapter 2

Geometric Background

The main goal of this chapter is to construct $\mathrm{Hilb}_n(\mathbb{A}^2)$ as a variety, prove that it is smooth (in particular, that we can apply the Białynicki-Birula decomposition to it), and compute the tangent space, in so doing proving Ellingsrud-Strømme's result (Theorem 1.9 in the previous chapter).

In order to provide background on applying the Białynicki-Birula decomposition to $\mathrm{Hilb}_n(\mathbb{A}^2)$, we first run through the analogous constructions for the Grassmanian $\mathrm{Gr}_k(V)$. The reasoning for this is two-fold. Firstly, we are able to quickly give "bare hands" example of computations using Białynicki-Birula decomposition with different $\mathbb{C}^*$-actions on the Grassmannian to give a flavour of how the computations work. In particular, we do a computation using a $\mathbb{C}^*$-action with non-isolated fixed points, which we make reference to when discussing Buryak, Feigin, and Nakajima's argument in which the topology of the fixed point set contributes to the decomposition. Secondly, we make use of a map from $\mathrm{Hilb}_n(\mathbb{A}^2)$ into a Grassmannian when describing the structure of $\mathrm{Hilb}_n(\mathbb{A}^2)$ as a quasi-projective variety.

Then, we follow a paper of Haiman [Hai98] to construct $\mathrm{Hilb}_n(\mathbb{A}^2)$ explicitly, describe the natural torus action and the Hilbert-Chow morphism, and compute the tangent space at a monomial ideal, which paves the way for us to prove that $\mathrm{Hilb}_n(\mathbb{A}^2)$ is smooth and prove Ellingsrud-Strømme's description of the tangent space as a $(\mathbb{C}^*)^2$-representation in [ES87].

We conclude with a description of Buryak, Feigin and Nakajima's method in [BFN15].

2.1 Background on Grassmannians

In this section we define Grassmannians and describe their structure as projective varieties, because when we want to view $\mathrm{Hilb}_n(\mathbb{A}^2)$ as a quasi-projective variety, we will make use of an embedding into a Grassmannian.

Then, we run an example of the Białynicki-Birula decomposition on a Grassmannian to showcase an example of how arguments using $\mathbb{C}^*$-actions with non-isolated fixed points work where we can control the normal bundle without extra geometric preparation. We will make reference to this example when we discuss Buryak, Feigin and Nakajima's proof of a special case of our main theorem in [BFN15].

2.1.1 $\mathrm{Gr}_k(V)$ as a projective variety

For V an n-dimensional complex vector space, let $\mathrm{Gr}_k(V)$ be the Grassmannian parameterizing k-dimensional subspaces of V. Let $e_1,\dots,e_n$ be a basis for V.

The Plücker map embeds $\mathrm{Gr}_k(V)$ in the projective space $\mathbb{P}\left(\bigwedge^k V\right)$ by sending a k-dimensional subspace $W \leq V$ with basis $\{w_1,\dots,w_n\}$ to the class of $w_1 \wedge w_2 \wedge \dots \wedge w_k$. The map is well defined: let $B = \{w_1,\dots,w_k\}$ and $B' = \{w'_1,\dots,w'_k\}$ be two choices of basis for W and let M be the change of basis matrix from B' to B. Then, $w_1 \wedge w_2 \wedge \dots \wedge w_k$ and $w'_1 \wedge \dots \wedge w'_k$ differ only by a factor of $\det(M)$ and so coincide in $\mathbb{P}\left(\bigwedge^k V\right)$.

We have defined $\mathrm{Gr}_k(V)$ as a set, but we want to view it as a projective variety. Let $X \leq V$ be a subspace of codimension k, and let $x \in \mathbb{P}\left(\bigwedge^{n-k} V\right)$ be the image of X under the Plücker map, and for a subspace $W \leq V$ of dimension k, let $w \in \bigwedge^k V$ be the image under the Plücker map. Then, the element $x \wedge w \in \bigwedge^n V \cong \mathbb{C}$ which is nonzero if and only if $V = X \oplus W$. Via this mapping, x is a linear form on $\mathbb{P}\left(\bigwedge^k V\right)$, so we may define

$$\begin{aligned} U_X &= \{W \mid W \in G_k(V) \text{ and } w \wedge x \neq 0\} \\ &= \left\{W \mid V = X \oplus W\right\}. \end{aligned}$$

Lemma 2.1. *For any $K_0 \in U_X$, $U_X \cong \mathrm{Hom}(K_0, X) \cong \mathbb{A}^{k(n-k)}$*

Proof. Fix a subspace $K_0 \in \mathrm{Gr}_k(V)$ such that $V = K_0 \bigoplus X$. Then, for any $Y \in U_X$ and $y \in Y$, y can be written as $y = x + k$ for some unique $x \in X$ and $k \in K_0$. Suppose for contradiction that some second $y' = x' + k$ for $y' \in Y$, $x' \in X$ and the same k. Then $y - y' = x - x' \in Y \cap X = \{0\}$. So, for each $k \in K_0$ there is a unique $y \in Y$ and a unique $x \in X$ such that $y = x + k$.

Define $\varphi(k) = x$. Then φ is linear; if $y_1 = x_1 + k_1$ and $y_2 = x_2 + k_2$ then $y_1 + \lambda y_2 = (x_1 + \lambda x_2) + (k_1 + \lambda k_2)$. So, we may associate to each $Y \in U_X$ a linear map $\varphi \in \mathrm{Hom}(K_0, X)$.

Conversely, if $\varphi \in \mathrm{Hom}(K_0, X)$ then define $Y = \{k + \varphi(k) \mid k \in K_0\}$. Then $Y \in \mathrm{Gr}_d(V)$ and if $v \in V$, v can be written uniquely as

$$\begin{aligned} v &= k + x \\ &= (k + \varphi(k)) + (x - \varphi(k)) \end{aligned}$$

and since $x - \varphi(k) \in X$, we can conclude $Y \in U_X$. So,

$$U_X \cong \mathrm{Hom}(K_0, X) \cong \mathbb{A}^{k(n-k)}.$$

□

We will make reference to the following example later when we explain how to view $\mathrm{Hilb}_n(\mathbb{A}^2)$ as a quasi-projective variety.

Example 2.2. *If we regard $G_k(V)$ as the space of k-dimensional quotients of V, and $X = \mathrm{span}_{\mathbb{C}}(\{x_1, \dots, x_{n-k}\})$ has codimension k, we can regard U_X as the set of subspaces Y of V for which $\{q(x_1), \dots, q(x_{n-k})\}$ forms a basis for V/Y.*

2.1.2 Example of the Białynicki-Birula decomposition on the Grassmannian

To prove Lemma 1.10, Haiman computes the class of $\mathrm{Hilb}_n(\mathbb{A}^2)$ in two different ways, using two Białynicki-Birula decompositions resulting from two different choices of generic $\mathbb{C}^*$-action. In [BFN15], Buryak, Feigin, and Nakajima use a similar method. They compute the class of the equivariant Hilbert scheme $\mathrm{Hilb}_n\left(\mathbb{A}^2\right)^{\Gamma_{\alpha+\beta} \times T_{\alpha,\beta}}$ where $T_{\alpha,\beta} = (t^\alpha, t^\beta)$ and α, β are non-negative integers in two different ways. However, the subtorus they use is no longer generic, so the topology of the fixed point sets has to be controlled.

What follows in this section is an illustrative example using different $\mathbb{C}^*$-actions, including one with non-isolated fixed points, to prove combinatorial identities by computing the class of a simpler space (the Grassmannian) in different ways.

The torus $T = (\mathbb{C}^*)^n$ acts on V by $(t_1, \dots, t_n) \cdot \sum v_i e_i = \sum t_i v_i e_i$, and hence acts on $Gr_k(V)$. The fixed points of the $(\mathbb{C}^*)^n$-action on $Gr_k(V)$ consist of spaces spanned by k of the vectors

$e_1, \ldots, e_n$, so there are $\binom{n}{k}$ of them. If we treat a choice of k such vectors as a binary string of length n, with an R in the positions corresponding to the k vectors we choose and a D corresponding to those we do not, this gives a bijection between the $\mathbb{C}^*$-fixed points and partitions that fit inside a $k \times (n-k)$ box (interpreting the string as a boundary path with instructions R for right and D for down).

If we pick any subtorus $\mathbb{C}^*$, the hypotheses we need for Białynicki-Birula will be satisfied. If our subtorus is generic, then the $\mathbb{C}^*$-fixed points will be exactly the T-fixed points. Otherwise, the fixed point loci will be bigger. For instance, if we pick the completely diagonal $\mathbb{C}^*$ with weights (1,1,1,1,1), then we just act by scalar multiplication, and we fix every linear subspace and $Gr_k(n)$ is one big fixed point set. Unsurprisingly, this tells us that $[Gr_k(V)] = [Gr_k(V) \times \mathbb{C}^0] = [Gr_k(V)]$.

In the situation of a generic $\mathbb{C}^*$-action, the fixed points of the action on $Gr_k(V)$ are just points, so each U_i is $\mathbb{A}^{d_i}$, and so in the Grothendieck ring of varieties we will have $[Gr_k(V)] = \sum[\mathbb{C}^{d_i}]$. Write $[\mathbb{C}] = \mathbb{L}$, so we have that

$$[Gr_k(V)] = \sum \mathbb{L}^{d_i}$$

As we understand the fixed point set, to apply the Białynicki-Birula decomposition to the Grassmannian we need to understand how the torus acts on the normal bundle to the fixed point sets.

For $W \in Gr_k(V)$, the tangent space to W in the Grassmannian is

$$T_W Gr_k(V) = \mathrm{Hom}(W, W^{\perp}) \cong W^* \otimes W^{\perp}.$$

For W a T-fixed subspace, reorder the basis so that W is the span of $\{e_1, \ldots, e_k\}$, so that $T_W Gr_k(V)$ has basis $\{e_i^* \otimes e_j \mid 1 \leq i \leq k, k+1 \leq j \leq n\}$. Then, the T-weights on $T_W Gr_k(V)$ on the basis vector $e_i^* \otimes e_j$ are given by 0 in every position except i and j, the original T-weight in the jth position and the negative of the usual T-weight in the ith position.

For our first decomposition (this one with isolated fixed points) let $\mathbb{C}^*$ act on V with weights $(1, 2, \ldots, n)$ so that $t \cdot (e_1, \ldots, e_n) = (t \cdot e_1, t^2 \cdot e_2, \ldots, t^n \cdot e_n)$. Let σ be a permutation of $\{1, \ldots, n\}$ and let W be the span of $\{e_{\sigma(1)}, \ldots, e_{\sigma(k)}\}$. We may take σ to be increasing on the sets $\{1, \ldots, k\}$ and $\{k+1, \ldots, n\}$. Then, W corresponds to the partition λ in a $k \times (n-k)$ rectangle with boundary path a down step in every position not in $\sigma([k])$ and a right step in every position in $\sigma([k])$. Then, the $\mathbb{C}^*$-weights on the tangent space are given by $\sigma(i) - \sigma(j)$

with $k+1 \leq i \leq n$ and $1 \leq j \leq k$. So, the number of positive weights on the tangent space at W is given by the number of pairs (i,j) with $1 \leq j \leq k$ and $k+1 \leq i \leq n$ so that $\sigma(j) < \sigma(i)$. So, the positive weights are counted by pairs of right steps (the $\sigma(j)$) and down steps (the $\sigma(i)$) in the boundary string of λ such that the right step occurs before the down step. That is, the number of positive weights is equal to the total number of boxes in the Young diagram of the corresponding partition. Hence, the Białynicki-Birula decomposition with this action gives us the identity

$$[Gr_k(V)] = \sum_{\lambda \in \mathrm{Par}(k,n-k)} \mathbb{L}^{|\lambda|} \tag{2.1}$$

where $\mathrm{Par}(k,n-k)$ denotes the partitions with Young diagrams that fit inside a $k \times (n-k)$ rectangle.

Let us introduce the notation

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+q^2+\cdots+q^{n-1},$$

$$[n]_q! = \prod_{k=1}^{n} [k]_q,$$

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!}.$$

By induction (using Pascal's identity) we have

$$\binom{n}{k}_q = \sum_{\lambda \in \mathrm{Par}(k,n-k)} q^{|\lambda|}. \tag{2.2}$$

The next proposition computes $[\mathrm{Gr}_k(V)]$ again, this time using a decomposition arising from a $\mathbb{C}^*$-action with non-isolated fixed points.

Proposition 2.3 (q–Vandermonde Identity)**.** *For positive integers n,k,i we have*

$$\binom{n}{k}_q = \sum_{j=0}^{k} \binom{i}{j}_q \binom{n-i}{k-j}_q q^{j(n-i-k+j)}.$$

Proof. Identity (2.1) tells us that

$$[Gr_k(V)] = \sum_{\lambda \in \mathrm{Par}(k,n-k)} \mathbb{L}^{|\lambda|},$$

and using identity (2.2) (by identifying q with $\mathbb{L}$), we can identify $[Gr_k(V)]$ with $\binom{n}{k}_q$. This proof uses the Białynicki-Birula decomposition of a non-generic $\mathbb{C}^*$-action and finds that the decomposition, under the same identification, gives the right hand side of the q-Vandermonde identity.

Let $\mathbb{C}^*$ act on V with weight -1 on the first i coordinates, and weight 0 on the rest, i.e.:

$$\lambda \cdot e_j = \begin{cases} \lambda^{-1} e_j & j \leq i \\ e_j & j > i \end{cases}$$

and consider the induced action on $Gr_k(V)$. For convenience, let V_1 be the span of $\{e_1, \ldots, e_i\}$ and V_2 the span of $\{e_{i+1}, \ldots, e_n\}$, and let π_1 and π_2 denote the projection maps from V to V_1 and V_2 respectively.

We begin by showing that the fixed points $W \in Gr_k(V)$ are the subspaces that live inside $Gr_j(V_1) \times Gr_{k-j}(V_2)$ for some integer j, $0 \leq j \leq k$. Suppose that W is such a space. Then, W has a basis consisting of j vectors in V_1, and $k-j$ vectors in V_2. Acting by λ scales each of these basis vectors (either by λ^{-1} or 1) and hence does not change W. Hence, all spaces of this form are indeed fixed. Now suppose W is fixed and contains a vector of the form $\mathbf{v} + \mathbf{w}$ where $\mathbf{v}$ is in the span of $\{e_1, \ldots, e_i\}$ and $\mathbf{w}$ is in the span of $\{e_{i+1}, \ldots, e_n\}$. Then W also contains the vector $\lambda^{-1}\mathbf{v} + \mathbf{w}$, for all $\lambda \in \mathbb{C}^*$, so in particular W contains $-\mathbf{v} + \mathbf{w}$, and hence contains both $\mathbf{v}$ and $\mathbf{w}$. That is, if W contains a vector $\mathbf{u}$, then it also contains $\pi_1(\mathbf{u})$ and $\pi_2(\mathbf{u})$, so W contains the product $\pi_1(W) \times \pi_2(W)$. The containment in the other direction always holds, so

$$W = \pi_1(W) \times \pi_2(W) \in Gr_{\dim(\pi_1(W))}(V_1) \times Gr_{\dim(\pi_2(W))}(V_2).$$

Finally, note that $\dim(\pi_1(W)) + \dim(\pi_2(W)) = \dim(W) = k$.

In the language of Białynicki-Birula, set $F_j = Gr_j(V_1) \times Gr_{k-j}(V_2)$ for $i = 0, 1, \ldots, k$ and let U_j be the points that flow to F_j. Then we have that U_j is a bundle over F_j with fibres affine spaces of dimension d_j where d_j is the dimension of the positive eigenspace of our

$\mathbb{C}^*$-action. So, we have

$$\begin{aligned}[Gr_k(V)] &= \sum_{j=0}^{k}[Gr_j(V_1) \times Gr_{k-j}(V_2) \times \mathbb{A}^{d_j}] \\ &= \sum_{j=0}^{k}[Gr_j(V_1)][Gr_{k-j}(V_2)][\mathbb{L}]^{d_j}.\end{aligned}$$

Identifying $\mathbb{L}$ with q and using our previous formula for $Gr_k(V)$ where $\dim(V) = n$, the above formula gives us

$$\binom{n}{k}_q = \sum_{j=0}^{k}\binom{i}{j}_q\binom{n-i}{k-j}_q q^{d_j}.$$

Therefore, it remains to show that $d_j = j(n-i-k+j)$. For this, let $W \in Gr_j(V_1) \times Gr_{k-j}(V_2)$ be a fixed point. Let $f_1, \dots, f_j, g_1, \dots g_{k-j}$ be a basis for W with the $f_l \in V_1$ and $g_l \in V_2$, and extend these to bases of V_1 and V_2. Then the tangent space $T_W(Gr_k(V)) \cong W^* \otimes W^{\perp}$ has a basis consisting of:

(i) vectors $f_l^* \otimes f_m$, with $1 \leq l \leq j$ and $j+1 \leq m \leq i$, with $\mathbb{C}^*$-weight 0;

(ii) vectors $f_l^* \otimes g_m$, with $1 \leq l \leq j$ and $k-j+1 \leq m \leq n-i$, with $\mathbb{C}^*$-weight 1;

(iii) vectors $g_l^* \otimes f_m$ with $1 \leq l \leq k-j$ and $j+1 \leq m \leq i$ with $\mathbb{C}^*$-weight -1;

(iv) vectors $g_l^* \otimes g_m$ with $1 \leq l \leq k-j$ and $k-j+1 \leq m \leq n-i$ with $\mathbb{C}^*$-weight 0.

The positive eigenspace is spanned by vectors of type (ii), and counting these basis vectors we have $d_j = j(n-i-k+j)$. □

2.2 Background on the Hilbert scheme

The main aim of this section is to explain how $\mathrm{Hilb}_n(\mathbb{A}^2)$ satisfies the conditions of Theorem 1.4. We begin by defining candidates for affine subvarieties and the corresponding coordinate ring. Then, we briefly recall some early results of Gordan [Gor99] and Macauley [Mac27] in what would later become Gröbner basis theory. We use these results to show how $\mathrm{Hilb}_n(\mathbb{A}^2)$ is a quasi-projective variety by embedding it in a Grassmannian and using the results of the previous section. We then introduce the Hilbert-Chow morphism to contextualise $\mathrm{Hilb}_n(\mathbb{A}^2)$ as a resolution of singularities of $\mathrm{Sym}^n(\mathbb{A}^2)$, and compute the (co)tangent space at a monomial ideal. Once we have computed the tangent space, we prove that $\mathrm{Hilb}_n(\mathbb{A}^2)$

is smooth and prove Ellingsrud and Strømme's characterisation of the tangent space as a $(\mathbb{C}^*)^2$-representation in [ES87].

Finally, we drop some hints about Buryak, Feigin, and Nakajima's proof of a special case of our main theorem in [BFN15].

To develop the background, we largely follow [Hai98].

Let $\mathbb{A}^2 = \operatorname{Spec}\left(\mathbb{C}[x,y]\right)$. As a set,

$$\operatorname{Hilb}_n\left(\mathbb{A}^2\right) = \{I \in \mathbb{C}[x,y] \mid \dim(\mathbb{C}[x,y]/I) = n\}$$

and the diagonal action of $(\mathbb{C}^*)^2$ on $\mathbb{A}^2$ induces an action on $\operatorname{Hilb}_n\left(\mathbb{A}^2\right)$ by

$$(t_1,t_2)\cdot I = \{f(t_1^{-1}x, t_2^{-1}y) \mid f(x,y) \in I\}.$$

Any ideal fixed under this action must be homogeneous in x and y, and therefore the fixed points are the monomial ideals. We will see that there is a monomial ideal in the closure of each torus orbit, so much can be gleaned from the local picture around the monomial ideals. Next, we define our candidates for affine subvarieties.

For a partition μ of n, let $B_\mu = \{(x^h, y^k) \mid (h,k) \in \mu\}$ and define

$$U_\mu = \{I \in \operatorname{Hilb}_n(\mathbb{A}^2) \mid B_\mu \text{ spans } \mathbb{C}[x,y]/I\}.$$

Then, U_μ contains the ideal I_μ generated by monomials *not* in μ.

Since $\dim_{\mathbb{C}}(\mathbb{C}[x,y]/I) = |B_\mu|$, B_μ must be a basis for $\mathbb{C}[x,y]/I$ for any $I \in U_\mu$. So, for each r,s and ideal $I \in U_\mu$ there is a unique expansion

$$x^r y^s = \sum_{(h,k)\in\mu} c_{hk}^{rs}(I) x^h y^k \pmod{I}.$$

The c_{hk}^{rs} define a collection of functions on U_μ, where $(h,k) \in \mu$ and $(r,s) \in \mathbb{Z}^2$.

Example 2.4. *For the ideal $I_\mu \in U_\mu$ we have*

$$c_{hk}^{rs}(I_\mu) = \begin{cases} \delta_{r,s}\delta_{h,k} & (r,s) \in \mu \\ 0 & (r,s) \notin \mu. \end{cases}$$

2.2.1 Gröbner basics

Gröbner basis theory is a thriving area of active research, but we will only need some classical results due to Gordan [Gor99] and Macauley [Mac27]. For the rest of this chapter, let $<$ denote the lexicographic order where $1 < x < x^2 < \ldots < y < xy < \ldots$.

Definition 2.5 (Initial ideal, Gröbner basis)**.** *For a polynomial $p \in \mathbb{C}[x,y]$ let $L_<(p)$ denote the leading monomial of p with respect to $<$. For an ideal $I \subseteq \mathbb{C}[x,y]$ the initial ideal of I is*

$$L_<(I) = \{L_<(p) \mid p \in I\}.$$

A set $\mathscr{G} = \{g_1, g_2, \ldots g_n\} \subset I$ is a Gröbner basis for I if

$$\langle L_<(g_i) \mid g_i \in \mathscr{G} \rangle = L_<(I).$$

We will make reference to the following example over the rest of the section.

Example 2.6. *The ideal $I = (y^2 - y, y - x)$ has Gröbner basis $(x^2 - x, y - x)$.*

Note also that I is the vanishing set of $\{(1,1),(0,0)\}$ so the only polynomials less than $x^2 - x$ that could be in I are $x - 1$ and x. They cannot both be in I, and if just one is in I then the vanishing set reduces to a single point. Hence all leading monomials of polynomials in I are divisible either by y or by x^2.

Note also that $I \in U_{(1,1)} \cap U_{(2)}$, and in particular $I \in U_{(1,1)}$ but $L_<(I) = (x^2, y)$ corresponding to the partition (2), not $(1,1)$.

Gröbner bases exist for any ideal in a multivariate polynomial ring (this is a consequence of Dickson's Lemma [Dic13], although this existence was certainly known by Gordan) and they are useful because they give us a division algorithm for multivariate polynomial rings.

Proposition 2.7 (Gordan 1899)**.** *If $\mathscr{G}$ is a Gröbner basis for I and $p \in \mathbb{C}[x,y]$ then p can be written as*

$$p = \sum_i p_i g_i + h$$

where $p_i \in \mathbb{C}[x,y]$ and no term of h is divisible by any of the $L_<(g_i)$. Moreover, $p \in I$ if and only if $h = 0$.

Proof. Certainly p can be written $p = \sum_i p_i g_i + h$ for some h. Let $L_<(g_1) < L_<(g_2) < \ldots < L_<(g_n)$ and let j be maximal such that g_j divides a term of h. Then for some polynomial q_j,

$p = \sum_{i \neq j} p_i g_i + (p_j + q_j) g_j + (h - q_j g_j)$, and the maximal degree of the terms in h divisible by one of the $L_<(g_k)$ has decreased, so we may take h to not be divisible by any of the $L_<(g_k)$.

It is clear that if $h = 0$ then $p \in I$. If $p \in I$, then $p - \sum_i p_i g_i = h \in I$ so $L_<(h) \in L_<(I)$. But $L_<(h) \notin \langle \mathscr{G} \rangle = L_<(I)$ unless $h = 0$. □

2.2.2 $\mathrm{Hilb}_n(\mathbb{A}^2)$ as a quasi-projective variety

Now we are in a position to show that the U_μ cover $\mathrm{Hilb}_n(\mathbb{A}^2)$. Once we have this, we use a map into the Grassmannian to show that the U_μ are affine subvarieties.

Proposition 2.8 (Macauley 1927)**.** *Let $I \in \mathbb{C}[x,y]$ have colength n. Then, the set of monomials not in $L_<(I)$ are a basis for $\mathbb{C}[x,y]/I$.*

Proof. Let B be the set of monomials not in $L_<(I)$. They are certainly linearly independent: if any $\mathbb{C}$-linear combination of them is in I then one of the monomials has to be the leading term of a polynomial in I. To see that B generates $\mathbb{C}[x,y]/I$, let $\mathscr{G} = \{g_1, \ldots, g_k\}$ be a Gröbner basis for I and let $p \in \mathbb{C}[x,y]$. Then by Proposition 2.7 ,

$$p = \sum_{i=1}^{k} p_i g_i + h \in h + I$$

for some $p_i, h \in \mathbb{C}[x,y]$ where h has no term divisible by any of the $L_<(g_i)$ and so is in B. □

Corollary 2.9. *The U_μ cover* $\mathrm{Hilb}_n(\mathbb{A}^2)$.

Proof. This follows immediately from the previous proposition and the observation that if $x^a y^b$ is *not* in the ideal generated by a set of monomials, then neither are $x^{a-1}y^b$ or $x^a y^{b-1}$, so for an ideal of finite colength the monomials not in $L_<(I)$ must correspond to a partition. □

Proposition 2.10. *The U_μ are open affine subvarieties of* $\mathrm{Hilb}_n(\mathbb{A}^2)$ *and the affine co-ordinate ring is generated by the c_{hk}^{rs}.*

Proof. Let $N = n + 1 = |\mu| + 1$ and let M_N be the set of monomials $\{x^a y^b \mid a + b \leq N\}$ - in particular, M_N contains every monomial in any of the B_μ. We can map each ideal $I \mapsto \mathbb{C}M_N/(\mathbb{C}M_N \cap I)$. Since $B_\mu \subseteq M_N$, $\dim\big(\mathbb{C}M_N/(\mathbb{C}M_N \cap I)\big) = n$ so we have a map

$$\mathrm{Hilb}_n(\mathbb{A}^2) \to \mathrm{Gr}_k(\mathbb{C}M_N)$$

where $k = \frac{N(N-1)}{2} - n$. The above map is injective with locally closed image, and the induced reduced subscheme structure is independent of N (see [Got78]).

The sets U_μ are then the preimages of the affine sets described in Example 2.2, and the c_{hk}^{rs} are the coordinates. So, we may take $\{U_\mu \mid \mu \in \text{Par}\}$ to be an affine cover of $\text{Hilb}_n(\mathbb{A}^2)$ and coordinate ring on U_μ is generated by the c_{hk}^{rs}. □

Now we show that the closure of the torus orbits contain a monomial ideal. We will use this later when we prove that $\text{Hilb}_n(\mathbb{A}^2)$ is smooth.

Lemma 2.11. *Let $I \in \text{Hilb}_n(\mathbb{A}^2)$. Then the limit*

$$\lim_{a\to 0}\lim_{b\to 0}(a,b)\cdot I = \text{span}_{\mathbb{C}}(\{L_<(p) \mid p \in I\}) = I_\mu$$

for some partition μ, where the limits are taken over $\mathbb{C}^*$.

Proof. We have already seen in Proposition 2.8 and Corollary 2.9 that $\text{span}_{\mathbb{C}}\left\{L_<(p) \mid p \in I\right\}$ is a monomial ideal I_μ and that it follows that $I \in U_\mu$, so it suffices to prove the left equality.

By definition of the c_{hk}^{rs}, we have that

$$x^r y^s - \sum_{(h,k)\in\mu} c_{hk}^{rs}(I) x^h y^k \in I.$$

Applying $L_<$ to this polynomial tells us that $x^r y^s$ must be the leading term, because $\text{span}_{\mathbb{C}}\{L_<(p) \mid p \in I\}$ does not contain any monomials in B_μ, ruling out all of the other terms. Hence, $c_{hk}^{rs} = 0$ unless $x^r y^s > x^h y^k$. That is, $c_{hk}^{rs} = 0$ unless $k < s$ or $k = s$ and $h < r$, so either $s - k$ is positive or it is 0 and $r - h$ is positive. Hence

$$\lim_{a\to 0}\lim_{b\to 0}(a,b)c_{hk}^{rs} = \lim_{a\to 0}\lim_{b\to 0} a^{r-h} b^{s-k} c_{hk}^{rs} = 0.$$

So, $\lim_{a\to 0}\lim_{b\to 0}(a,b)\cdot I = I_\mu$. □

2.2.3 The Hilbert-Chow morphism

Next, we define the Hilbert-Chow morphism to connect with the common intuition that $\text{Hilb}_n(\mathbb{A}^2)$ parameterises collections of n points on a surface where, if points coincide, the Hilbert scheme keeps track of how they collided.

Definition 2.12. *The Hilbert-Chow morphism* $\pi : \mathrm{Hilb}_n(\mathbb{A}^2) \to \mathrm{Sym}^n(\mathbb{A}^2)$ *sends an ideal I to the multiset* $\mathrm{Supp}(\mathbb{C}[x,y]/I)$ *consisting of prime ideals* $\mathfrak{p} \in \mathbb{C}[x,y]$ *such that the localisation* $(\mathbb{C}[x,y]/I)_{\mathfrak{p}} \neq 0$*, where the multiplicity of* $\mathfrak{p}$ *is given by the length of the localisation.*

Now, let J be any ideal in $\mathrm{Hilb}_n(\mathbb{A}^2)$. A prime $\mathfrak{p} \in \mathrm{Spec}(\mathbb{C}[x,y])$ is in the support of $\mathbb{C}[x,y]/J$ if and only if it is contains the annihilator, equivalently, it must contain J, so as a set (rather than a multiset) $\pi(J)$ consists of the prime ideals in $\mathbb{C}[x,y]$ containing J. In $\mathbb{C}[x,y]$, the prime ideals are of one of three types:

(i) the zero ideal;

(ii) ideals $(f(x,y))$ generated by a single irreducible polynomial $f(x,y)$;

(iii) maximal ideals of the form $(x-a, y-b)$ for some $(a,b) \in \mathbb{C}^2$.

The only prime ideals that contain ideals of finite colength are those of type (iii).

Example 2.13. *Let I be any monomial ideal in* $\mathrm{Hilb}_n(\mathbb{C}^2)$*. Then I must contain an element of the form* x^a *and another of the form* y^b *(else* $\mathbb{C}[x,y]/I$ *would not be finite dimensional). The only prime ideal containing both* x^a *and* y^b *for* $a,b \geq 1$ *is the maximal ideal* (x,y)*.*

Localising at $\mathbb{C}[x,y]\backslash(x,y)$ *gives all of the rational functions that do not have a pole at* $(0,0)$*. So, as a set,* $\pi(I) = \{(x,y)\}$ *and the multiplicity is given by n, for example when* $I = (x^2, y^3)$ *we have the chain of ideals of length 6*

$$(0) \subsetneq (xy^2) \subsetneq (y^2) \subsetneq (xy, y^2) \subsetneq (y) \subsetneq (x,y) \subsetneq \mathbb{C}[x,y]/I.$$

Example 2.14. *Let* $K = (x^4, y^3 + y^2)$*, contained in the prime ideals* $\mathfrak{p}_1 = (x,y)$ *and* $\mathfrak{p}_2 = (x, y+1)$*. Then,* $\mathbb{C}[x,y]/K$ *as a* $\mathbb{C}$*-vector space has basis*

$$\{1, x, x^2, x^3, xy, x^2y, x^3y, xy^2, x^2y^2, x^3y^2, y, y^2\}$$

so $K \in \mathrm{Hilb}_{12}(\mathbb{A}^2)$*,* $K \in U_{(4,4,4)}$ *and*

$$c^{rs}_{hk}(K) = \begin{cases} (-1)^s \delta_{rh} & r \leq 3, s > 2, k = 2 \\ \delta_{rh}\delta_{sk} & r \leq 3, s \leq 2 \\ 0 & \textit{otherwise.} \end{cases}$$

Localising at $\mathfrak{p}_1$, $\mathbb{C}[x,y]_{\mathfrak{p}_1}$ again consists of rational functions with no pole at $(0,0)$ and so $(x^4, y^3+y^2)_{\mathfrak{p}_1} = (x^4, y^2)$ in $\mathbb{C}[x,y]_{\mathfrak{p}_1}$. The quotient then contains the following chain of ideals,

$$(0) \subsetneq (x^3y) \subsetneq (x^3) \subsetneq (x^3, x^2y) \subsetneq (x^2) \subsetneq (x^2, xy) \subsetneq (x) \subsetneq (x,y) \subsetneq \left(\frac{\mathbb{C}[x,y]}{(x^4, y^3+y^2)}\right)_{\mathfrak{p}_1}.$$

Localising at $\mathfrak{p}_2$, $\mathbb{C}[x,y]_{\mathfrak{p}_2}$ consists of rational functions with no pole at $(0,-1)$ and $(x^4, y^3+y^2)_{\mathfrak{p}_2} = (x^4, y+1)$ in $\mathfrak{p}_2$, $\mathbb{C}[x,y]_{\mathfrak{p}_2}$ The quotient contains the following chain of ideals

$$(0) \subsetneq (x^3) \subsetneq (x^2) \subsetneq (x) \subsetneq \frac{\mathbb{C}[x,y]}{(x^4, y^3+y^2)}_{\mathfrak{p}_2}.$$

So, the Hilbert-Chow morphism sends K to the multiset $\{(x,y)^8, (x,y+1)^4\}$.

2.2.4 Tangent and cotangent space

Next, we compute the cotangent space at the monomial ideal $I_\mu \in U_\mu$ and show that the monomial ideals are smooth points. It will then follow that $\mathrm{Hilb}_n(\mathbb{A}^2)$ is smooth, as the singular locus is stable under the torus action and closed, so by Lemma 2.11 must contain one of the monomial ideals if it is nonempty.

Recall that the cotangent space at a point p is defined to be the vector space $\mathfrak{m}_p/\mathfrak{m}_p^2$ where $\mathfrak{m}_p$ is the ideal in the coordinate ring of X vanishing at p. The tangent space of X at p is the dual vector space, $T_p(X) = (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$ and in general $\dim(T_p(X)) \geq \dim(X)$ with equality when p is a smooth point. See e.g. [Har77] for further details.

As computed in Example 2.4,

$$c_{hk}^{rs}(I_\mu) = \begin{cases} \delta_{r,s}\delta_{h,k} & (r,s) \in \mu \\ 0 & (r,s) \notin \mu. \end{cases}$$

So, $\mathfrak{m}_{I_\mu}$ contains $\{c_{hk}^{rs} \mid (r,s) \notin \mu\} \cup \{c_{hk}^{rs} \mid (r,s) \in \mu, (r,s) \neq (h,k)\}$. The functions in the right bracket are identically zero on U_μ. The next proposition determines two useful relations which we will apply to compute the cotangent space.

Proposition 2.15. *For $(r,s) \notin \mu$ and $(i,j) \in \mu$ we have the relations*

$$c_{ij}^{r,s+1} = c_{i,j-1}^{rs} \pmod{\mathfrak{m}_{I_\mu}^2} \tag{2.3}$$

and

$$c_{ij}^{r+1,s} = c_{i-1,j}^{rs} \pmod{\mathfrak{m}_{I_\mu}^2}. \tag{2.4}$$

Proof. For $I \in U_\mu$ and $(r,s) \notin \mu$ we have

$$x^r y^s = \sum_{(h,k)\in\mu} c_{hk}^{rs}(I) x^h y^k \pmod{I}.$$

Multiplying by y,

$$\begin{aligned} x^r y^{s+1} &= \sum_{(h,k)\in\mu} c_{hk}^{rs}(I) x^h y^{k+1} \pmod{I} \\ &= \sum_{(h,k)\in\mu} \sum_{(i,j)\in\mu} c_{hk}^{rs}(I) c_{ij}^{h,k+1}(I) x^i y^j \pmod{I} \end{aligned}$$

So,

$$c_{ij}^{r,s+1} = \sum_{(h,k)\in\mu} c_{hk}^{rs} c_{ij}^{h,k+1}.$$

Since we have taken $(r,s) \notin \mu$, all of the c_{hk}^{rs} are in $\mathfrak{m}_{I_\mu}$ and the $c_{i,j}^{h,k+1}$ are in $\mathfrak{m}_{I_\mu}$ whenever $(h,k+1) \neq (i,j)$, so reducing modulo $m_{I_\lambda}^2$ gives

$$c_{ij}^{r,s+1} = c_{i,j-1}^{rs} c_{ij}^{ij}.$$

We have $(i,j) \in \mu$ so $c_{ij}^{ij} \equiv 1$ on U_μ and we have

$$c_{ij}^{r,s+1} = c_{i,j-1}^{rs} \pmod{\mathfrak{m}_{I_\mu}^2}.$$

Multiplying through by x and doing a similar computation gives also that

$$c_{ij}^{r+1,s} = c_{i-1,j}^{rs} \pmod{\mathfrak{m}_{I_\mu}^2}.$$

□

We may represent the tangent vector associated to $c_{hk}^{rs} \in \mathfrak{m}_\mu$ as an arrow from the unit square in the plane with bottom left corner (r,s) to the box in μ with bottom left corner (h,k). The relations above then show that we can translate the arrows by unit steps up, down, left or right without changing the tangent vector, as long as the tail of the arrow remains outside the

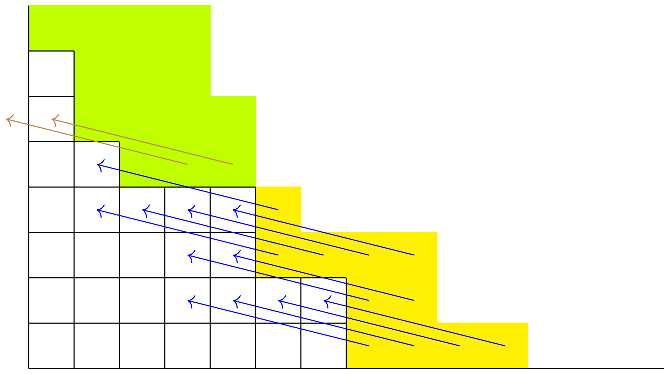


Fig. 2.1 In this diagram for $\mu = (7,7,5,5,2,1,1)$, the blue arrows all represent the same function in $m_{I_\mu}/m_{I_\mu}^2$, and the brown arrows (which could be moved to have tails anywhere in the green area) are all 0.

diagram and the head remains inside the diagram. Moreover, if the head of the arrow can be moved across the coordinate axes then the arrow is zero.

Example 2.16. *Figure 2.1 shows two classes of the c_{hk}^{rs} drawn as arrows: the blue arrows are a nonzero class and the brown arrows are 0, as the heads can be moved across the y axis.*

For arrows to be nonzero, they must point northwest or southeast (there are no northeast arrows, and southwest arrows can trace out a path parallel to the boundary until the head crosses the coordinate axes). Next we define unique representatives of classes of nonzero arrows u_{hk} and d_{hk}. We think of the u_{hk} as the nonzero southeast pointing arrows pushed as far southeast as possible and the d_{hk} as nonzero northwest arrows pushed as far northwest as possible – in fact, this viewpoint will show that the $u_{h,k}$ and $d_{h,k}$ span $\mathfrak{m}/\mathfrak{m}^2$.

Definition 2.17. *Let $\square \in \mu$ have bottom left corner (h,k) and let (f,k), be the bottom left corner of the rightmost box in row k and let (h,g) be the bottom left corner of the topmost box in column h. Define $u_{hk} = c_{f,k}^{h,g+1}$ and $d_{h,k} = c_{h,g}^{f+1,k}$.*

Remark 2.18. *We have borrowed Haiman's notation, but not his diagram convention, so the us point down and the ds point up*[1].

Example 2.19. *Figure 2.2 shows $d_{1,3}$ (red) and $u_{1,3}$ (blue) for the partition $\mu = (7,7,5,5,2,1,1)$ together with the regions the tail could be dragged to without changing the point in the cotangent space (green or yellow). The box with bottom left corner $(1,3)$ is shaded grey.*

[1]This may be some relief to those who prefer to keep their arms above their legs, who are presumably reading upside down.

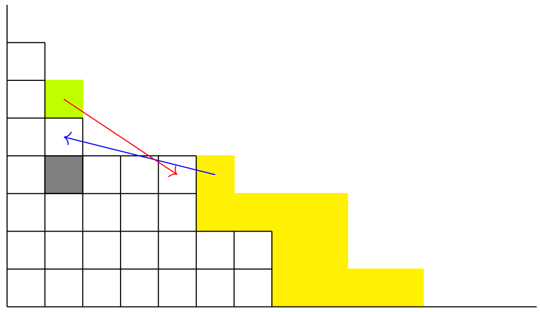


Fig. 2.2 The arrows representing $u_{1,3}$ (red) and $d_{1,3}$ (blue) for $\mu = (7,7,5,5,2,1,1)$.

Proposition 2.20. *The $u_{h,k}$ and $d_{h,k}$ span $\mathfrak{m}/\mathfrak{m}^2$.*

Proof. As we have already observed, any northeast pointing arrow must be zero.

If a southeast-pointing arrow is nonzero, it can be moved so that its tail lies in the southeast-most point of the possible tails of arrows in the same class, and the head of the arrow will be in the partition. Because the arrow cannot be moved further east, the head must be in a box that is the last square in its row. Because the arrow cannot be moved further down, the tail must be directly above a box in the partition. That is, it must be one of the u_{hk}.

The same argument shows that the nonzero northwest pointing arrows all have one of the d_{hk} as a representative. □

Proposition 2.21. *The Hilbert scheme of points on $\mathbb{A}^2$ is smooth of dimension $2n$, and the tangent space at a monomial ideal I_μ is given by*

$$T_\mu\left(\mathrm{Hilb}_n(\mathbb{A}^2)\right) = \mathrm{span}_{\mathbb{C}}\left\{u^*_{h,k}, d^*_{h,k} \mid (h,k) \in \mu\right\}.$$

Proof. It suffices to check smoothness locally at I_μ, since the singular locus is closed and stable under the torus action and therefore contains a fixed point if it is nonempty.

The image of U_μ under the Hilbert-Chow morphism is dense in $\mathrm{Sym}_n(\mathbb{A}^2)$ so U_μ has dimension at least $2n$, and for smoothness it suffices to bound the dimension of the tangent space above by $2n$. Proposition 2.20 shows that the dimension of the cotangent space at I_μ is at most $2n$, so $\mathrm{Hilb}_n(\mathbb{A}^2)$ is indeed smooth at I_μ, and has dimension $2n$. So, the $\{u_{hk}, d_{h,k}\}$ are in fact a basis for the cotangent space, and hence the dual basis is a basis for the tangent space. □

2.2.5 Ellingsrud-Strømme

With this explicit description of the tangent space, we can quickly check Theorem 1.9, which describes the class of the tangent space at a monomial ideal in the representation ring of $(\mathbb{C}^*)^2$. We restate it here for convenience.

Theorem (Ellingsrud-Strømme)**.** *In the representation ring of a two-dimensional torus* $(\mathbb{C}^*)^2$*, the tangent space at a monomial ideal* I_λ *to* $\mathrm{Hilb}_n\left(\mathbb{A}^2\right)$ *is given by*

$$\left[T_\lambda\left(\mathrm{Hilb}_n\left(\mathbb{A}^2\right)\right)\right] = \sum_{\square\in\lambda} \rho(t)^{-l(\square)-1}\phi(t)^{a(\square)} + \rho(t)^{l(\square)}\phi(t)^{-a(\square)-1}.$$

where $t\cdot x = \phi(t)x$ *and* $t\cdot y = \rho(t)y$ *for linearly independent characters* ϕ *and* ρ *of* $(\mathbb{C}^*)^2$.

Proof of Theorem 1.9. Let $\phi(t),\rho(t)$ be as in the statement and fix (h,k) the bottom left corner of some box $\square\in\lambda$. Then, $(\phi(t),\rho(t))$ has the same weight on u^*_{hk} as $(\phi(t)^{-1},\rho(t)^{-1})$ has on u_{hk}, and

$$\begin{aligned}(\phi(t)^{-1},\rho(t)^{-1})\cdot u_{hk} &= (\phi(t)^{-1},\rho(t)^{-1})\cdot c^{h,g+1}_{f,k} &(2.5)\\ &= \phi(t)^{f-h}\rho(t)^{k-g-1}c^{h,g+1}_{f,k} &(2.6)\\ &= \phi(t)^{a(\square)}\rho(t)^{-(l(\square)+1)}u_{h,k}. &(2.7)\end{aligned}$$

Similarly, $(\phi(t),\rho(t))$ has the same weight on d^*_{hk} as $(\phi(t)^{-1},\rho(t)^{-1})$ has on d_{hk} and

$$\begin{aligned}(\phi(t)^{-1},\rho(t)^{-1})\cdot d_{hk} &= (\phi(t)^{-1},\rho(t)^{-1})\cdot c^{f+1,k}_{h,g} &(2.8)\\ &= \phi(t)^{h-f-1}\rho(t)^{g-k}c^{f+1,k}_{h,g} &(2.9)\\ &= \phi(t)^{-(a(\square)+1)}\rho(t)^{l(\square)}d_{h,k}. &(2.10)\end{aligned}$$

Summing over $\square\in\lambda$ then proves the theorem. □

2.2.6 Buryak-Feigin-Nakajima

Finally, we briefly explain Buryak, Feigin and Nakajima's geometric proof in [BFN15] of the special case of our Theorem A when $c=\alpha+\beta$ and x ranges over positive rationals $\frac{r}{s}$ with $r+s=\alpha+\beta$.

We have now seen multiple instances of the Białynicki-Birula decomposition being used as follows: identify a smooth variety with a torus action. Judiciously choose one-parameter subtorus actions so that the corresponding decompositions encode different descriptions of the distribution of combinatorially interesting objects, and conclude an equidistribution result. In most cases we have used a torus action with isolated fixed points.

Buryak, Feigin and Nakajima use the Białynicki-Birula decomposition arising from the action of $T_{\alpha,\beta}$ (which has non-isolated fixed points) on $\mathrm{Hilb}_n(\mathbb{A}^2)^{\Gamma_{\alpha+\beta}}$ not to compute the class of $\mathrm{Hilb}_n(\mathbb{A}^2)^{\Gamma_{\alpha+\beta}}$ itself[2] but to force the class of the $\Gamma_{\alpha+\beta} \times T_{\alpha,\beta}$-fixed points to appear as some of the fixed point sets F_{ij}, just as the class of smaller Grassmannians showed up in our proof of Proposition 2.3. In Proposition 2.3, we were not able to extract any information about the individual smaller Grassmannians from the decomposition (although we could compute them directly) because they were all weighted differently. However, Buryak, Feigin and Nakajima are able to control the contribution from the normal bundle (using a symplectic form) so that the connected components of the $T_{\alpha,\beta}$-fixed sets of $\mathrm{Hilb}_n(\mathbb{A}^2)$ are equally weighted, so the decomposition gives us information about the class of the $\Gamma_{\alpha+\beta} \times T_{\alpha,\beta}$-fixed sets, which appear on the right hand side rather than the left hand side.

Lemma 2.22 (Buryak-Feigin-Nakajima [BFN15, Lemma 3.1])**.** *For α,β non-negative integers, not both zero, then in the Grothendieck ring of complex quasi-projective varieties,*

$$\left[\left(\mathrm{Hilb}_n\left(\mathbb{A}^2\right)\right)^{T_{\alpha,\beta}\times\Gamma_{\alpha+\beta}}\right] = \left[\left(\mathrm{Hilb}_n\left(\mathbb{A}^2\right)\right)^{T_{0,\alpha+\beta}\times\Gamma_{\alpha+\beta}}\right]$$

Proof sketch. The fixed point set of the $\Gamma_{\alpha+\beta}$-action on $\mathrm{Hilb}_n(\mathbb{A}^2)$ is smooth. Let

$$\mathrm{Hilb}_n\left(\mathbb{A}^2\right)^{\Gamma_{\alpha+\beta}} = \bigsqcup_i \left(\mathrm{Hilb}_n\left(\mathbb{A}^2\right)^{\Gamma_{\alpha+\beta}}\right)_i$$

be the decomposition into irreducible components. Let the torus $T_{\alpha,\beta} = \{(t^\alpha, t^\beta) \mid t \in \mathbb{C}^*\}$ act on an irreducible component: Theorem 1.4 then tells us that

$$\left[\left(\mathrm{Hilb}_n(\mathbb{A}^2)^{\Gamma_{\alpha+\beta}}\right)_i\right] = \sum_j [F_{ij}][\mathbb{A}]^{d_{ij}} \tag{2.11}$$

[2]in fact this is relatively straightforward - we already computed it in terms of a family of partition statistics in Theorem 1.18.

where F_{ij} is the jth connected component of the fixed point set of the $T_{\alpha,\beta}$-action on the ith irreducible component,

$$F_{ij} = \left(\left(\mathrm{Hilb}_n(\mathbb{A}^2)^{\Gamma_{\alpha+\beta}} \right)_i^{T_{\alpha,\beta}} \right)_j = \left(\mathrm{Hilb}_n(\mathbb{A}^2)^{\Gamma_{\alpha+\beta} \times T_{\alpha,\beta}} \right)_{i,j}.$$

and d_{ij} is the dimension of the positive eigenspace on the fibre of the normal bundle of $F_{ij} \subset (\mathrm{Hilb}_n(\mathbb{A}^2)^{\Gamma_{\alpha+\beta}})_i$. We *want* to compute the class of the F_{ij} summed over i and j, which we almost have as the right hand side of (2.11), but the contributions are weighted by the d_{ij}. So, is we are to use (2.11) to compute the class we want, we need the d_{ij} to remain constant as j varies.

To control the d_{ij}, Buryak, Feigin and Nakajima use a symplectic form ω on the ith component of $\mathrm{Hilb}_n(\mathbb{A}^2)^{\Gamma_{\alpha+\beta}}$ (lifted from $dx \wedge dy$ on $\mathbb{C}^2$) with respect to which the positive and non-positive eigenspaces of $T_{\alpha,\beta}$ are dual. So, each of the $d_{ij} = \frac{d_i}{2}$ is half of the dimension of the normal bundle. In particular, it only depends on $\alpha+\beta$. Summing over j tells us that

$$\left[\left(\mathrm{Hilb}_n(\mathbb{A}^2)^{\Gamma_{\alpha+\beta}} \right)_i \right] = [\mathbb{A}]^{\frac{d_i}{2}} \sum_j \left[\left(\mathrm{Hilb}_n \left(\mathbb{A}^2 \right)^{\Gamma_{\alpha+\beta} \times T_{\alpha,\beta}} \right)_{i,j} \right] \tag{2.12}$$

$$= [\mathbb{A}]^{\frac{d_i}{2}} \left[\left(\mathrm{Hilb}_n \left(\mathbb{A}^2 \right)^{\Gamma_{\alpha+\beta}} \right)_i^{T_{\alpha,\beta}} \right] \tag{2.13}$$

In particular, the class of the $T_{\alpha,\beta}$-fixed points of the ith component of $\mathrm{Hilb}_n(\mathbb{A}^2)^{\Gamma_{\alpha+\beta}}$ is dependent *only* on $\alpha+\beta$. □

Chapter 3

Combinatorial Background

In this chapter, we recall first definitions in partition combinatorics, including the abacus construction, cores and quotients. The standard reference for the abacus construction is [JK81, §2.7], the abacus was first introduced in [Jam78], cores in [Nak41] and quotients in [Lit51]. We take a nonstandard view of the c-core, and describe it as an equivalence class of complete circuits of a directed multigraph M_c. The language we use to describe the abacus is also nonstandard, but the construction is equivalent. We take this approach so that we have descriptions of Loehr and Warrington's construction in [LW09] and the c-core in terms of directed multigraphs, which allows us to formulate a simultaneous refinement of the two in Chapter 4. Once we have recalled this theory, we will recall a few standard generating functions and define the map G_c previously defined in [WW20] and use these to give a combinatorial proof of Theorem 3.42, which forms our base case.

3.1 Young diagrams and boundary paths

Definition 3.1 (Partition,Young diagram)**.** *A partition of an integer $n \geq 0$ is a sequence of non-increasing positive integers $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_t$ with sum n. The size of λ, denoted $|\lambda|$, is n and the length of λ is the number of summands, written $l(\lambda) = t$. The Young diagram of λ consists of t rows of 1×1 boxes $\square$ in $\mathbb{R}^2$, with λ_i boxes in the ith row for each $1 \leq i \leq t$. The bottom left corner of the diagram sits at $(0,0)$.*

Example 3.2. *The partition $\mu = (12,12,10,8,7,4,1,1,1)$ of 56 has the diagram given in Figure 3.1.*

Informally, the boundary of a partition λ is the bi-infinite path traversing the y-axis from $+\infty$ until it hits a box of the partition, then follows the edge of the Young diagram until it hits the x-axis, before traversing the x-axis to $+\infty$. We split the boundary up into unit steps between lattice points, and view it as a directed multigraph where edges are additionally assigned a label indicating if they are south or east.

Definition 3.3 (SE directed multigraph)**.** *A SE directed multigraph $M = (V, E, s, t, d)$ consists of a vertex set V, an edge set E, and three maps $s : E \to V$, $t : E \to V$ and $d : E \to \{\text{South}, \text{East}\}$, called source, target, and direction respectively. We say the edge e departs from the vertex v if $s(e) = v$ and we say that e arrives at the vertex w if $t(e) = w$. We call e a south edge if $d(e) =$* South *and an east edge if $d(e) =$* East. *We sometimes abbreviate* South *to S and* East *to E in contexts where there is no danger of confusion with the edge set.*

Definition 3.4 (Boundary graph)**.** *The boundary graph $b(\lambda)$ of a partition λ is an SE directed multigraph. The edge set is defined as follows. For natural numbers x, y there is a south edge e with $s(e) = (x, y+1)$, $t(e) = (x, y)$ if either*

- $x = 0$ *and* $y \geq l(\lambda)$, *or*
- $x > 0$ *and* $\lambda_{y+1} = x$.

There is an east edge e with $s(e) = (x, y)$ and $t(e) = (x+1, y)$ if either

- $y = 0$ *and* $x \geq \lambda_1$, *or*
- $y > 0$ *and* $\lambda_{y+1} \leq x < \lambda_y$.

The vertex set $V(b(\lambda))$ is the union of sources and targets of the edges.

Example 3.5. *Let $\mu = (12, 12, 10, 8, 7, 4, 1, 1, 1)$. The boundary graph of μ is given in Figure 3.1, the south edges being the downward arrows and the east edges being the rightward arrows.*

Note that for any edge e in the boundary graph, the value of $x - y$ at the target of e is one greater than at the source, because taking a unit step south or east increases the value of $x - y$ by 1.

So, the value of $x - y$ at the target of an edge indexes an Eulerian tour, or complete circuit, of $b(\lambda)$. For clarity, we recall the definition of a complete circuit.

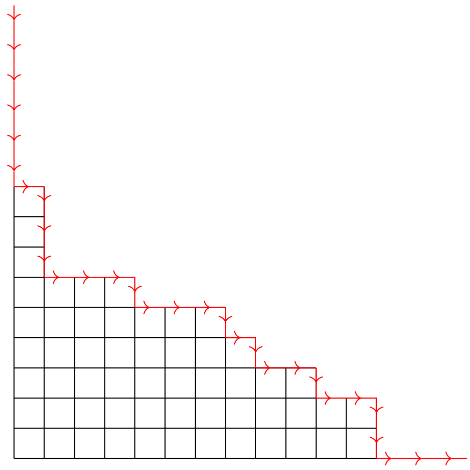


Fig. 3.1 The Young diagram and boundary graph of $(12,12,10,8,7,4,1,1,1)$.

Definition 3.6 (Complete circuit)**.** *Given a directed multigraph M, a complete circuit of M is an ordering of $E(M)$ such that if e_i and e_{i+1} are consecutive with respect to the ordering, then $t(e_i) = s(e_{i+1})$.*

Definition 3.7 (Boundary tour, boundary sequence, index)**.** *If an edge $e \in E(b(\lambda))$ has target (x,y), we say the index of e is $i(e) = x - y$. The boundary tour is the complete circuit of $b(\lambda)$ where the edges are ordered by index. We write the edges in this ordering as $(\ldots, e_{-2}, e_{-1}, e_0, e_1, e_2, \ldots)$. We say an edge e_j occurs before the edge e_k if $j < k$. The boundary sequence is the bi-infinite sequence $(d_i)_{i\in\mathbb{Z}}$ where $d_i = d(e_i)$. We write S and E in place of South and East respectively in the boundary sequence.*

Example 3.8. *The partition $\mu = (12,12,10,8,7,4,1,1,1)$ has boundary sequence*

$$\ldots SSSSSSESSSE_0EESEEESESEESEESSEEE\ldots$$

where d_0 is indicated with a 0 suffix.

3.2 Anatomy of a Young Diagram

Next, we recall some standard partition statistics and how they relate to the boundary sequence, define rimhooks, and connect to cores. We also introduce the notion of an SE directed multigraph homomorphism.

Definition 3.9 (Hand, foot, inversion, hook length, arm, leg)**.** *A box $\square \in \lambda$ can be specified by giving the row and column of the Young diagram that the box sits in. In particular, each box in the Young diagram corresponds to a pair of edges: one south, at the extreme right of the row $\square$ lies in, called the hand of λ, and another east, at the top of the column $\square$ lies in, called the foot of $\square$, where the foot necessarily occurs before the hand. Conversely, given an east edge s_1 departing from (x_1, y_1) and arriving at (x_1+1, y_1) and a south edge s_2 departing from (x_2, y_2) and arriving at (x_2, y_2-1) such that $x_1 - y_1 < x_2 - y_2$,there is a unique box in the Young diagram with bottom left corner (x_1, y_2-1) such that s_1 and s_2 are respectively the foot and hand of $\square$. We call such a pair of south and east edges an inversion. Hence, we may identify a box in the Young diagram with its hand and foot in the boundary sequence. The arm of $\square$ consists of the boxes that lie strictly to the right of $\square$ in the same row, and the leg of $\square$ consists of the boxes that lie strictly above $\square$ in the same column. We denote the number of boxes in the arm of $\square$ by $a(\square)$ and the number of boxes in the leg of $\square$ by $l(\square)$. The hook length of $\square$ is defined to be $h(\square) = a(\square) + l(\square) + 1$.*

Example 3.10. *The boxes in the arm and leg of the shaded box $\square$ in Figure 3.2 are labelled with the corresponding body part. The foot of $\square$ is indicated with a blue arrow, the hand with a red arrow. We have $a(\square) = 5$ and $l(\square) = 1$, so $h(\square) = 7$.*

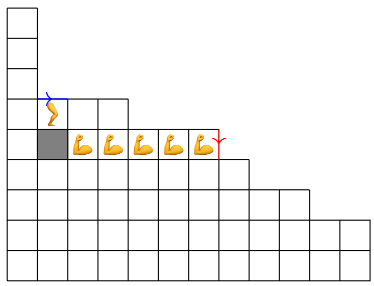


Fig. 3.2 the arm and leg of $\square$.

Proposition 3.11. *Let λ be a partition. A box in the Young diagram of λ with hook length c corresponds to an inversion (d_i, d_j) in the boundary sequence of λ where $j = i + c$.*

Proof. Let h and f be the hand and foot of $\square$ in the boundary respectively. Consider the map from the arm of $\square$ to the boundary sending each box to its foot. The foot of any box in the arm of $\square$ is an east edge that occurs after f and occurs before h. Conversely, each east edge that occurs after f and occurs before h is the foot of a box in the arm of $\square$. So, $a(\square)$ counts east edges that occur after f and before h.

Analogously, $l(\square)$ counts south edges that occur after f and before h. Thus, $a(\square)+l(\square)$ counts the total number of edges that occur after f and before h. There are $h(\square)-1$ such edges. $\square$

We now turn our attention to cores and rimhooks, first introduced by Nakayama [Nak41].

Definition 3.12 (Rimhook)**.** *A rimhook R of length c is a connected set of c boxes in λ such that removing R gives the Young diagram of a partition, and R does not contain a 2×2 box.*

Corollary 3.13. *Rimhooks of length c are in bijection with boxes of hook length c.*

Proof. Let R be a rimhook of length c in the diagram of a partition λ. Then, by the definition of a rimhook, for every box $\square \in R$ there is an edge in the boundary graph of λ arriving at the top right corner of $\square$. Let $e_i, e_{i+1}, \ldots, e_{i+c-1}$ be the the set of all such edges (since R is connected these edges are consecutive in the boundary tour), and let e_{i+c} be the next edge in the boundary tour.

Since R is removable, $d(e_i) = E$.

We now check that $d(e_{i+c}) = S$. Since e_{i+c-1} arrives at the top right corner of the south-eastern-most square $\square$ in R, e_{i+c} departs from the top right corner of $\square$. If e_{i+c} were an east edge, there would be another box to the right of $\square$ in the same row, contradicting that R is removable. Therefore, by Proposition 3.11, e_i and e_{i+c} are the foot and hand respectively of a box of hook length c.

Conversely, if $\square$ is a box of hook length c, with foot e_i and hand e_{i+c} then taking the boxes with top right corners the targets of $e_i, e_{i+1}, \ldots, e_{i+c-1}$ gives a rimhook of length c. $\square$

Definition 3.14. *A c-core of a partition λ is a partition obtained by iteratively removing rimhooks of length c from λ until a partition with no rimhooks of length c is obtained. A partition μ is called a c-core if μ has no rimhooks of length c.*

Applying Corollary 3.13 to c-cores gives the following.

Corollary 3.15. *A partition λ is a c-core if and only if λ has no boxes of hook length c.*

Our aim for now will be to redefine the c-core in the language we wish to use later, and then use it to see that the result of iteratively removing rimhooks of length c is independent of the order in which rimhooks are removed. In order to do so, we need the notion of an SE directed multigraph homomorphism. Informally, these consist of two maps, one between edges, and another between vertices. We require that these maps preserve the direction (S or E) of the edges, and that they be compatible with the source and target maps.

Definition 3.16 (SE directed multigraph homomorphism)**.** *Let $M_1 = (V_1, E_1, s_1, t_1, d_1)$, $M_2 = (V_2, E_2, s_2, t_2, d_2)$ be SE directed multigraphs. A homomorphism of SE directed multigraphs $\varphi : M_1 \to M_2$ is a pair of maps $\varphi_V : V_1 \to V_2$ and $\varphi_E : E_1 \to E_2$ such that for all edges $e \in E_1$,*

$$s_2(\varphi_E(e)) = \varphi_V(s_1(e)) \tag{3.1}$$

$$t_2(\varphi_E(e)) = \varphi_V(t_1(e)) \tag{3.2}$$

$$d_2(\varphi_E(e)) = d_1(e). \tag{3.3}$$

In other words, φ is a quiver homomorphism that preserves direction (S or E).

Example 3.17. *Let M_1 be the boundary graph of $\mu = (12, 12, 10, 8, 7, 4, 1, 1, 1)$ and let φ_V be the map taking each vertex (x, y) to $[x - y]$, the class of $x - y$ modulo 2. This map induces the homomorphism q_2 illustrated in Figure 3.3, with east edges coloured red and south edges coloured blue.*

For ease of reading, we draw edges in the image of q_2 from left to right in order of index as $\ldots, q_2(e_{-2}), q_2(e_{-1}), q_2(e_0), q_2(e_1), q_2(e_2), \ldots$.

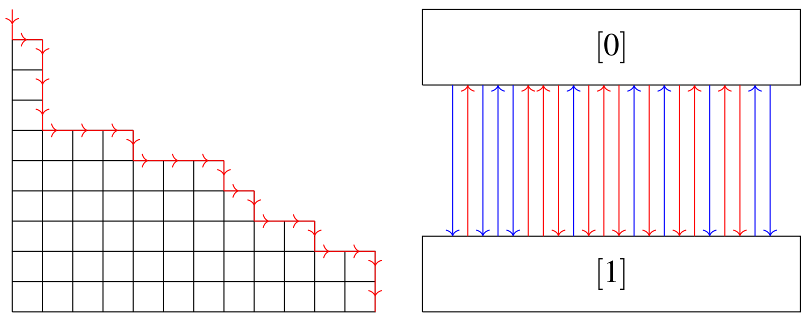


Fig. 3.3 A portion of M_1 and the corresponding edges in $q_2(M_1)$.

We will always work with SE directed multigraph homomorphisms where the edge map φ_E is bijective, so from now on we assume φ_E is bijective for any homomorphism φ. In particular, this assumption allows us to push complete circuits through homomorphisms.

Proposition 3.18. *Let $\varphi : M_1 \to M_2$ be an SE directed multigraph homomorphism. Let $(e_i)_{i \in I}$ be a complete circuit of M_1. Then $(\varphi_E(e_i))_{i \in I}$ is a complete circuit of M_2.*

Proof. Since φ_E is bijective, we need only check that $s_2(\varphi_E(e_{i+1})) = t_2(\varphi_E(e_i))$ for each $i \in I$. By definition,

$$s_2(\varphi_E(e_{i+1})) = \varphi_V(s_1(e_{i+1})) \tag{3.4}$$

$$= \varphi_V(t_1(e_i)) \tag{3.5}$$

$$= t_2(\varphi_E(e_i)). \tag{3.6}$$

□

We have seen already that rimhooks of length c correspond to boxes of hook length c which in turn correspond to inversions in the boundary sequence where, if the first term has index i, the second has index $i+c$. Intuitively enough, then, the useful homomorphism that captures all of this information is the following.

Definition 3.19 (c-abacus tour)**.** *Let $(z,w) \sim_c (x,y)$ if $z-w \equiv x-y \pmod{c}$. Then, $q_c : b(\lambda) \to M_c$ is the SE directed multigraph homomorphism induced by imposing the relation $\sim_c$ on the vertices of $b(\lambda)$. The complete circuit $(q_c(e_i))_{i\in\mathbb{Z}}$ of M_c is called the c-abacus tour associated to λ.*

Proposition 3.11 tells us that the number of boxes with hook length divisible by c can be read off from the c-abacus tour by looking at edges that correspond to a hand and foot arriving at the same vertex $(v,[i])$. So, it is sometimes useful to group the edges in a complete circuit by target. This leads us to arrival words.

Definition 3.20 (Arrival words, departure words)**.** *Let $M = (V,E,s,t,d)$ be a directed SE multigraph and let $(e_i)_{i\in I}$ be a complete circuit of M. For $v \in V$, let $I_v \subset I$ be the subset of indices such that $t(e_i) = v$. The arrival word at v, written v_a, is the sequence of directions $d(e_i)_{i\in I_v}$. The departure word at v is defined analogously, replacing the target map with the source map.*

Notation 3.21. *Given a sequence $(d_i)_{i\in I}$ of Ss and Es, we write* $\mathrm{inv}(d_i)$ *for the number of inversions.*

Proposition 3.22. *Let λ be a partition with boundary tour $(e_i)_{i\in\mathbb{Z}}$ and let M_c have vertex set $\{[i] \mid 1 \le i \le c\}$ where $[i]$ denotes the class of i modulo c. Then, taking arrival words with respect to the complete circuit $(q_c(e_i))_{i\in\mathbb{Z}}$*

$$|\{\Box \in \lambda : c \mid h(\Box)\}| = \sum_{i=0}^{c-1} \mathrm{inv}([i]_a). \tag{3.7}$$

Proof. Apply Proposition 3.11. □

3.3 Alignment and charge

So far, we have associated to every partition a boundary sequence, a bi-infinite sequence of Ss and Es such that if we travel far enough to the left in the sequence every entry is an S, and if we travel far enough to the right, every entry is an E. We will now study these sequences in general, and identify which of them arise as boundary sequences of a partition. Then, we will define an equivalence relation on partitions, which we shall show is equivalent to having the same c-core. We will use this to show that partitions have a unique c-core, to define the c-quotients originally studied by [Lit51], and to give a bijection between partitions of fixed c-core and c-tuples of partitions.

Definition 3.23 (Charge)**.** *Let $D = (d_i)_{i\in\mathbb{Z}}$ be a bi-infinite sequence with $d_i \in \{S, E\}$, for each i such that for some $M \in \mathbb{N}$, $\forall m \geq M$, $d_{-m} = S$ and $d_m = E$. Fix an integer k. Let e_k be the number of Es in $(d_i)_{i\in\mathbb{Z}}$ with index at most k,*

$$e_k = \left|\left\{d_j : d_j = E \text{ and } j \leq k\right\}\right|. \tag{3.8}$$

Similarly, let s_k be the number of Ss with index greater than k,

$$s_k = \left|\left\{d_j : d_j = S \text{ and } j > k\right\}\right|. \tag{3.9}$$

Then, the k-charge of D, written $\mathrm{ch}_k(D)$ is $e_k - s_k - k$.

Proposition 3.24. *If k and l are integers, and D is as in Definition 3.23, then $\mathrm{ch}_k(D) = \mathrm{ch}_l(D)$.*

Proof. We check that $\mathrm{ch}_{k+1}(D) = \mathrm{ch}_k(D)$. The proposition then follows by repeated application of the equality. Suppose $d_{k+1} = E$. Then, $e_{k+1} = e_k + 1$ and $s_{k+1} = s_k$. So,

$$\begin{aligned}
\mathrm{ch}_{k+1}(D) &= e_{k+1} - s_{k+1} - (k+1) && (3.10)\\
&= e_k + 1 - s_k - (k+1) && (3.11)\\
&= e_k - s_k - k && (3.12)\\
&= \mathrm{ch}_k(D). && (3.13)
\end{aligned}$$

Similarly, if $d_{k+1} = S$, then $e_{k+1} = e_k$ and $s_{k+1} = s_k - 1$, so $\mathrm{ch}_{k+1}(D) = \mathrm{ch}_k(D)$. Therefore, $\mathrm{ch}_k(D)$ is independent of k. □

So, in place of $\mathrm{ch}_k(D)$, we may simply write $\mathrm{ch}(D)$.

Proposition 3.25. *A sequence D as in Definition 3.23 is the boundary sequence of a partition if and only if* $\mathrm{ch}(D) = 0$.

Proof. Suppose D is the boundary sequence of a partition. Let (x_1, y_1) be the point on the line $x - y = k$ on the boundary of a partition λ. Since x_1 counts the number of south edges with index greater than k, and y_1 counts the number of east edges with index at most k, $\mathrm{ch}(D) = x_1 - y_1 - k = 0$.

If $\mathrm{ch}(D) = 0$, then we may reconstruct λ from D by placing a point at (e_k, s_k), and drawing the partition boundary in two halves: one as an infinite path departing from (e_k, s_k) taking unit steps with orientations given by $(d_i)_{i>k}$ and the other as an infinite path arriving at (e_k, s_k) taking unit steps with orientations given by $(d_i)_{i \le k}$. □

Definition 3.26 (The relation $\sim_c$)**.** *Let λ and μ be partitions and let the arrival words taken from the c-abacus tours of λ and μ be $[0]_a^\lambda, \ldots, [c-1]_a^\lambda$, and $[0]_a^\mu, \ldots, [c-1]_a^\mu$, respectively. Define the relation $\lambda \sim_c \mu$ if, for all i with $0 \le i \le c-1$,*

$$\mathrm{ch}([i]_a^\lambda) = \mathrm{ch}([i]_a^\mu). \tag{3.14}$$

Example 3.27. *Let $\mu = (12, 12, 10, 8, 7, 4, 1, 1, 1)$ and refer to Figure 3.3. When $c = 2$, $[0]_a^\mu$ is given by*

$$\cdots\ S\ S\ S\ E\ S\ E\ E\ E\ |E\ E\ E\ S\ E\ S\ E\ E \cdots.$$

where the bar separates terms corresponding to edges of negative or zero index from those of positive index.

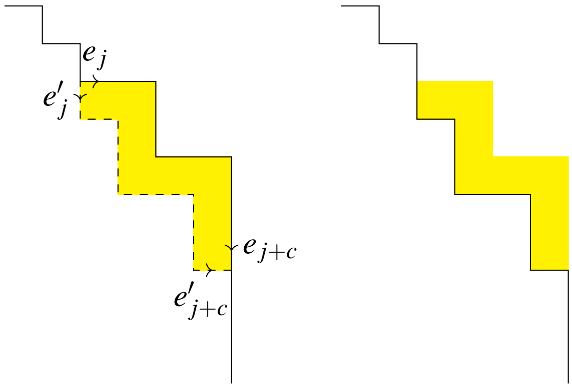
So, $\mathrm{ch}([0]_a^\mu) = 4 - 2 = 2$. *Analogously,* $[1]_a^\mu$ *is*

$$\cdots S\ S\ S\ S\ S\ S\ E\ S\ |E\ S\ S\ E\ E\ S\ E\ E\ \cdots.$$

So, $\mathrm{ch}([1]_a^\mu) = 1 - 3 = -2$.

Proposition 3.28. *If λ is a partition containing a rimhook R of length c and λ' is the partition obtained from λ by removing R, then $\lambda \sim_c \lambda'$.*

Proof. Let the boundary tours of λ and λ' be $(e_i)_{i\in\mathbb{Z}}$ and $(e'_i)_{i\in\mathbb{Z}}$. First, we analyse how the boundary sequences $(d(e_i))$ and $(d(e'_i))$ differ. Let R have south-eastern most box $\square_2$ and north-western most box $\square_1$. Let e_j be the east edge traversing the top edge of $\square_1$, so that e_{j+c} is the south edge traversing the right of $\square_2$.



Since we remove $\square_1$ and $\square_2$, $d(e_j) = E$, $d(e'_j) = S$, and $d(e_{j+c}) = S$ and $d(e'_{j+c}) = E$. Let $j = qc + r$ for $0 \le r \le c-1$. Since the rimhook does not contain a 2×2 box and is connected, the portion of the boundary of λ' between the lines $x - y = j+1$ and $x - y = c + j - 1$ is a translate of the original partition boundary by $(-1,-1)$, so $d(e_i) = d(e'_i)$ for all $i \notin \{j, j+c\}$. So, for all $0 \le s \le c-1$ with $s \ne r$, $[s]_a^{\lambda} = [s]_a^{\lambda'}$, and the arrival word

$$([r]_a^{\lambda'})_i = \begin{cases} ([r]_a^{\lambda})_q & i = q+1 \\ ([r]_a^{\lambda})_{q+1} & i = q \\ ([r]_a^{\lambda})_i & \text{otherwise.} \end{cases} \tag{3.15}$$

So,

$$\operatorname{ch}([r]_a^{\lambda'}) = \operatorname{ch}_{q-1}([r]_a^{\lambda'}) \tag{3.16}$$
$$= \operatorname{ch}_{q-1}([r]_a^{\lambda}) \tag{3.17}$$
$$= \operatorname{ch}([r]_a^{\lambda}). \tag{3.18}$$

$\square$

Corollary 3.29. *The c-core of λ is unique, and $\lambda \sim_c \mu$ if and only if λ and μ have the same c-core.*

Proof. If λ has c-core ν, then ν is obtained from λ by iteratively removing rimhooks of length c from R, so by Proposition 3.28, $\lambda \sim_c \nu$. Every partition has at least one c-

core, so it remains to check that if μ and ν are both c-cores with $\mu \sim_c \nu$ then $\mu = \nu$. By Propositions 3.11 and 3.22, if μ and ν are both c-cores then for each i, the arrival words $[i]_a^\mu$ and $[i]_a^\nu$ do not contain any inversions. So, both consist of a string of Ss up to some index, and a string of Es thereafter. Since $\mu \sim_c \nu$, the charge of both $[i]_a^\mu$ and $[i]_a^\nu$ must be the same, and therefore $[i]_a^\mu = [i]_a^\nu$. □

The important consequence for us will be the following.

Corollary 3.30. *Let λ and μ be partitions. Then λ and μ have the same c-core if there is a value of m with $c \mid m$ such that for each $[i]$, both of the following hold.*

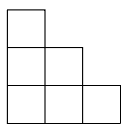
- *the arrival words in the c-abacus tour of λ and μ agree after the entry with index m;*
- *the portion of $[i]_a^\lambda$ with index at most m is a permutation of the portion of $[i]_a^\mu$ with index at most m.*

Example 3.31. *We will calculate the 2-core λ of $\mu = (12,12,10,8,7,4,1,1,1)$. By Corollary 3.30 and the calculation in Example 3.27, λ is the unique 2-core with $\mathrm{ch}([0]_a^\lambda) = 2$ and $\mathrm{ch}([1]_a^\lambda) = -2$.*

So, placing a bar in the bi-infinite string with no inversions to separate edges with positive index from those with negative or 0 index, the arrival words in $M_2(\lambda)$ at 0 and 1 respectively, are

$$\begin{array}{c} \cdots\ S\ S\ S\ S\ S\ S\ S\ S\ \mid S\ S\ E\ E\ E\ E\ E\ E\ \cdots \\ \cdots\ S\ S\ S\ S\ S\ S\ E\ E\ \mid E\ E\ E\ E\ E\ E\ E\ E \cdots. \end{array}$$

So, the 2-core is (3,2,1).



Proposition 3.32. *There is an bijective map f from c-core partitions to a $\mathbb{Z}$-module of length $c-1$.*

Proof. Consider the c-abacus of a c-core partition. The charges $(\mathrm{ch}([0]),\ldots,\mathrm{ch}([c-1]))$ specify the c-core. A c-tuple of integers $(a_0,\ldots,a_{c-1})$ represents the charges of a partition if and only if $\sum_{i=0}^{c-1} a_i = 0$. So, sending a c-core to the c-tuple of charges gives a bijective map with the $\mathbb{Z}$-module $M = \langle e_1,\ldots,e_c : \sum_{i=0}^{c-1} e_i = 0\rangle$. □

Fix a positive integer c, a c-core μ, and a non-negative integer n. Let $\mathrm{Par}^c_\mu(n)$ denote the set of partitions of E with c-core μ. Let Par^c_μ denote the set of all partitions with c-core μ, and let Par denote the set of all partitions.

Definition 3.33 (Quotient)**.** *The c-quotient of λ is the c-tuple $(q_1(\lambda), q_2(\lambda), \dots, q_c(\lambda))$, where $q_i(\lambda)$ is the partition with boundary sequence $[i]_a$, with the index shifted so that the charge is 0.*

Definition 3.34 (Quotient map)**.** *The quotient map $\phi : \mathrm{Par}^c_\mu \to (\mathrm{Par})^c$ sends the partition λ to $(q_1(\lambda), \dots, q_c(\lambda))$.*

Proposition 3.35. *For $\lambda \in \mathrm{Par}^c_\mu$,*

$$|\lambda| = |\mu| + c\sum_{i=1}^{c} |q_i(\lambda)|. \tag{3.19}$$

Proof. By Proposition 3.22, the number of boxes with hook length divisible by c are given by $\sum_{i=1}^{c} \mathrm{inv}[i]_a$. Starting from the c-abacus tour of μ, we can obtain the c-abacus tour of λ by adding these inversions one at a time. Adding each inversion corresponds to adding a rimhook of length c to the diagram, so contributes c to $|\lambda|$. □

3.4 The map G_c

Now we set about proving Theorem 3.42. We first recall three standard generating functions.

Proposition 3.36.

$$\sum_{\lambda \in \mathrm{Par}} q^{|\lambda|} t^{l(\lambda)} = \prod_{m \geq 1} \frac{1}{1 - q^m t} \tag{3.20}$$

$$\sum_{\lambda \in \mathrm{Par}} q^{|\lambda|} = \prod_{m \geq 1} \frac{1}{1 - q^m} \tag{3.21}$$

$$\sum_{\lambda \in \mathrm{Par}^c_\mu} q^{|\lambda|} = q^{|\mu|} \prod_{m \geq 1} \frac{1}{(1 - q^{mc})^c} \tag{3.22}$$

Proof. We may rewrite the right hand side of (3.20) as

$$\prod_{m \geq 1} 1 + q^m t + q^{2m} t^2 + q^{3m} t^3 + \dots,$$

so that picking a term $q^{km}t^k$ for each m corresponds to declaring that λ contains k parts of size m, contributing $|km|$ to λ and k to $l(\lambda)$, giving the left hand side. Setting $t = 1$ in (3.20) gives (3.21).

For (3.22), Proposition 3.35 tells us that the map ϕ gives a bijection between $\lambda \in \mathrm{Par}_\mu^c$ and c-tuples of partitions $(q_1,\dots,q_c)$ where $|\lambda| = |\mu| + c\sum_{i=1}^c |q_i(\lambda)|$. The right hand side of (3.22) corresponds to all choices of c-tuples $q_1,\dots,q_c \in \mathrm{Par}$, and the weighting by c corresponds to each box in q_i corresponding to c boxes in λ. □

Next, we define a partition statistic $\lambda_\square^{c*}$ that arises as a special case of one of the statistics that we study.

For a positive integer d, let $m_d(\lambda)$ denote the number of parts of λ of size d, and for fixed c let $\lambda_\square^{c*}$ denote the weighted sum

$$\lambda_\square^{c*} = \sum_{d=1}^{\infty} \left\lfloor \frac{m_d(\lambda)}{c} \right\rfloor. \tag{3.23}$$

In words, $\lambda_\square^{c*}$ counts the number of rectangles, of any width, of positive height divisible by c in the diagram of λ such that the whole right edge of the rectangle, and at least the rightmost step of the top edge, lies on the boundary of λ, and the left side of the rectangle sits on the y-axis.

Example 3.37. *Let $c = 3$. The partition $\lambda = (7,7,4,4,4,4,4,4,4,3,2,2,2,1)$ has $m_7(\lambda) = 2$, $m_4(\lambda) = 7$, $m_3(\lambda) = 1$, $m_2(\lambda) = 3$ and $m_1(\lambda) = 1$. So, the only nonzero contributions to $\lambda_\square^{3*}$ are when $d = 2$ and $d = 4$, and*

$$\lambda_\square^{3*} = \left\lfloor \frac{m_2(\lambda)}{3} \right\rfloor + \left\lfloor \frac{m_4(\lambda)}{3} \right\rfloor = 1 + 2 = 3.$$

We now define the map G_c, previously defined in [WW20].

Definition 3.38 (The map G_c)**.** *The map $G_c : \mathrm{Par} \to \mathrm{Par} \times K_c$, where $K_c = \{\lambda \in \mathrm{Par} : \lambda_\square^{c*} = 0\}$ is the set of partitions with no parts repeated c or more times, maps a partition λ to (ξ, ν) where for each $d \in \mathbb{N}$,*

$$m_d(\xi) = \left\lfloor \frac{m_d(\lambda)}{c} \right\rfloor, \tag{3.24}$$

and

$$m_d(\nu) = m_d(\lambda) - c\left\lfloor \frac{m_d(\lambda)}{c} \right\rfloor. \tag{3.25}$$

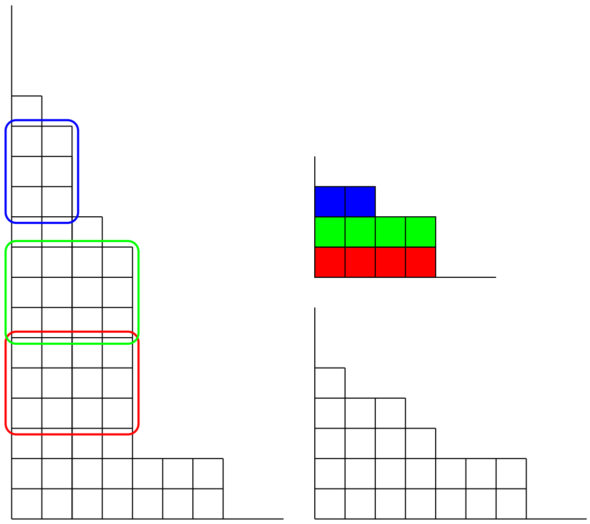


Fig. 3.4 The partition λ has $G_3(\lambda) = ((4,4,2),(7,7,4,3,1))$.

We write $(G_c)^{-1}$ *for the inverse map* $(G_c)^{-1} : K_c \times \mathrm{Par} \to \mathrm{Par}$ *where, for each* $d \in \mathbb{N}$

$$m_d\left((G_c)^{-1}(\xi,\nu)\right) = m_d(\xi)c + m_d(\nu). \tag{3.26}$$

Example 3.39. *As shown in Figure 3.4, the partition* $\lambda = (7,7,4,4,4,4,4,4,4,3,2,2,2,1)$ *has* $G_3(\lambda) = ((4,4,2),(7,7,3,4,1))$.

The next proposition establishes that the c-core of a partition λ is also the c-core of the second argument of $G_c(\lambda)$, so we may restrict G_c to Par^c_μ in a way that interacts sensibly with cores.

Proposition 3.40. *If* $\lambda \in \mathrm{Par}^c_\mu$ *and* $G_c(\lambda) = (\xi,\nu)$, *then* $\nu \in \mathrm{Par}^c_\mu$.

Proof. Suppose the proposition is false for some λ of minimal possible size. Then, we must have $\lambda \neq \nu$, so λ must have some part of some size d repeated at least c times. The rightmost column of the rectangle of width d and height c which has all right edges and the rightmost top edge in the boundary of λ is a rimhook of size c. Let λ' be the partition formed by deleting this rimhook. Then, λ' has c-core μ and $G_c(\lambda') = (\xi',\nu)$ for some ξ'. So, since λ' is smaller than λ, $\nu \in \mathrm{Par}^c_\mu$. □

Therefore, G_c restricts to a bijection $G_c|_{\mathrm{Par}^c_\mu} : \mathrm{Par}^c_\mu \to \mathrm{Par} \times (K_c \cap \mathrm{Par}^c_\mu)$. This allows us to use G_c to prove the following.

Proposition 3.41. *For a positive integer c and a c-core μ, the following product formula holds.*

$$\sum_{\lambda \in K_c \cap \mathrm{Par}_\mu^c} q^{|\lambda|} = q^{|\mu|} \prod_{m \geq 1} \frac{1}{(1-q^{mc})^{c-1}}. \tag{3.27}$$

Proof. Let $\lambda \in \mathrm{Par}_c^\mu$. Then G_μ^c bijectively maps λ to a pair of partitions (ξ, ν) with $|\lambda| = |\xi| + c|\nu|$, because each part of ν corresponds to c parts of λ of the same size. So,

$$\sum_{\lambda \in \mathrm{Par}_\mu^c} q^{|\lambda|} = \sum_{\xi \in K_c \cap \mathrm{Par}_\mu^c} q^{|\xi|} \times \sum_{\nu \in \mathrm{Par}} q^{c|\nu|}. \tag{3.28}$$

Substituting (3.21) and applying Proposition 3.35 to (3.28) gives

$$q^{|\mu|} \prod_{m \geq 1} \frac{1}{(1-q^{mc})^c} = \sum_{\xi \in K_c \cap \mathrm{Par}_\mu^c} q^{|\xi|} \times \prod_{m \geq 1} \frac{1}{(1-q^{mc})}, \tag{3.29}$$

which rearranges to give (3.27). □

We are now in a position to prove the following identity, which forms the base case for Proposition 4.4. This identity follows immediately from Walsh and Waarnar's multiplication formula [WW20, Theorem 6.1] and was derived in the same paper [WW20, 7.1a]. The original proof is uses essentially the same logic (though in much greater generality) than the one we have given here.

Theorem 3.42. *For a fixed positive integer c,*

$$\sum_{\lambda \in \mathrm{Par}_\mu^c} q^{|\lambda|} t^{\lambda_\square^{c*}} = q^{|\mu|} \prod_{i \geq 1} \frac{1}{(1-q^{ic})^{c-1}} \prod_{j \geq 1} \frac{1}{1-q^{jc}t}. \tag{3.30}$$

Proof. Let $\lambda \in \mathrm{Par}_c^\mu$. Then G_μ^c bijectively maps λ to a pair of partitions (ξ, ν) with $|\lambda| = |\xi| + c|\nu|$, where each part of ν of size d corresponds to a $d \times c$ rectangle in λ contributing to $\lambda_\square^{c*}$. So,

$$\sum_{\lambda \in \mathrm{Par}_\mu^c} q^{|\lambda|} t^{\lambda_\square^{c*}} = \sum_{\xi \in K_c \cap \mathrm{Par}_\mu^c} q^{|\xi|} \times \sum_{\nu \in \mathrm{Par}} q^{c|\nu|} t^{l(\nu)}. \tag{3.31}$$

Substituting (3.27) and (3.20) into (3.31) gives (3.30). □

Chapter 4

Further partition statistics

In this section we define the main partition statistics of interest, $h^+_{x,c}$ and $h^-_{x,c}$, where x is a real parameter and c is a positive integer. The main aim of this thesis is to compute the distribution of the statistics $h^+_{x,c}$ and $h^-_{x,c}$ over Par^c_μ, given in Theorem 4.2. The previous chapter computed the distribution of $\lambda^{c*}_\square$ over Par^c_μ, giving the right hand side in Theorem 4.2. In this chapter, we connect to $\lambda^{c*}_\square$ by observing that $\lambda^{c*}_\square = h^+_{0,c}$, and then sketch a framework for piecing together a family of involutions $I_{r,s,c}$ defined on Par^c_μ to prove that the distribution $h^\pm_{x,c}$ over Par^c_μ is independent of both x and the sign. The rest of the thesis will then construct the component bijections $I_{r,s,c}$.

In order to reduce the proof of Theorem 4.2 to the construction of appropriate bijections $I_{r,s,c}$, we first prove that Theorem 4.2 is implied by Theorem 4.3, which states that the $h^+_{x,c}$ and $h^-_{x,c}$ have the same distribution over Par^c_μ. Then, we introduce three other statistics $\mathrm{mid}_{x,c}$, $\mathrm{crit}^-_{x,c}$ and $\mathrm{crit}^+_{x,c}$ and decompose $h^+_{x,c}$ and $h^-_{x,c}$ in terms of these other statistics. Finally, we outline sufficient conditions for the bijections $I_{r,s,c}$ to prove Theorem 4.3 in terms of these three statistics.

We conclude the chapter by explaining how the main result of [BFN15] follows from Theorem 4.2.

Definition 4.1. *For a partition λ, $x \in [0,\infty]$ and a fixed $c \in \mathbb{N}$,*

$$h^+_{x,c}(\lambda) = \left| \left\{ \square \in \lambda \,:\, c \mid h(\square) \text{ and } \frac{a(\square)}{l(\square)+1} \le x < \frac{a(\square)+1}{l(\square)} \right\} \right|, \tag{4.1}$$

and

$$h^-_{x,c}(\lambda) = \left| \left\{ \square \in \lambda \,:\, c \mid h(\square) \text{ and } \frac{a(\square)}{l(\square)+1} < x \le \frac{a(\square)+1}{l(\square)} \right\} \right|. \tag{4.2}$$

We interpret a fraction with denominator 0 *as* $+\infty$*.*

Note that a box $\square$ contributes to $h^+_{0,c}$ if and only if $a(\square) = 0$ and $c \mid (l(\square)+1)$. That is, $\square$ is the rightmost box in its row, and there is some m such that the row containing $\square$ and exactly $mc-1$ rows to the above all have the same height. The number of such boxes is exactly $\lambda^{c*}_{\square}$.

Similarly, $h^-_{\infty,c}(\lambda) = \bar{\lambda}^{c*}_{\square}$, where $\bar{\lambda}$ is the partition conjugate to λ.

We are now in a position to state our main result.

Theorem 4.2. *For all* $x \in [0,\infty)$ *we have*

$$\sum_{\lambda \in \mathrm{Par}^c_{\mu}} q^{|\lambda|} t^{h^+_{x,c}(\lambda)} = q^{|\mu|} \prod_{i \geq 1} \frac{1}{(1-q^{ic})^{c-1}} \prod_{j \geq 1} \frac{1}{1-q^{jc}t}, \tag{4.3}$$

and for all $x \in (0,\infty]$,

$$\sum_{\lambda \in \mathrm{Par}^c_{\mu}} q^{|\lambda|} t^{h^-_{x,c}(\lambda)} = q^{|\mu|} \prod_{i \geq 1} \frac{1}{(1-q^{ic})^{c-1}} \prod_{j \geq 1} \frac{1}{1-q^{jc}t}. \tag{4.4}$$

Proposition 4.4 shows that Theorem 4.2 is a consequence of the following result.

Theorem 4.3. *For all positive rational numbers x and all integers* $n \geq 0$ *we have*

$$\sum_{\lambda \in \mathrm{Par}^c_{\mu}(n)} t^{h^+_{x,c}(\lambda)} = \sum_{\lambda \in \mathrm{Par}^c_{\mu}(n)} t^{h^-_{x,c}(\lambda)}. \tag{4.5}$$

4.1 Reducing to Theorem 4.3

Proposition 4.4. *Theorem 4.3 implies Theorem 4.2.*

Proof. For $x \in (0,\infty)$, $c \in \mathbb{N}$ and $\delta \in \{+,-\}$, or $x=0$ and $\delta = +$, or $x = \infty$ and $\delta = -$, define

$$H^{\delta}_{x,c}(n) = \sum_{\lambda \in \mathrm{Par}^c_{\mu}(n)} t^{h^{\delta}_{x,c}(\lambda)}.$$

Suppose $H^{\delta}_{x,c}(n)$ is independent of both x and δ. Then

$$H^{\delta}_{x,c}(n) = H^+_0(n) = \sum t^{h^+_{0,c}(\lambda)} = \sum t^{\lambda^{c*}_{\square}}.$$

Theorem 4.2 then follows immediately by multiplying by q^n, adding over all $n \geq 0$, and applying Theorem 3.42. So, it suffices to prove that Theorem 4.3 implies that $H^{\delta}_{x,c}(n)$ is independent of x and δ.

For an integer n, we call a positive rational number r a *critical rational for n* if there is a partition $\mu \in \mathrm{Par}(n)$ and a box $\Box \in d(\mu)$ such that $h(\Box)$ is divisible by c, and $\frac{a(\Box)}{l(\Box)+1} = r$ or $\frac{a(\Box)+1}{l(\Box)} = r$. By convention, 0 and $+\infty$ are regarded as critical rationals for all n.

We denote the set of all critical rationals for n by $C(n)$. Since there are finitely many partitions of n each containing finitely many boxes in their diagrams, $C(n)$ is finite for all n. For a fixed n, write $C(n) = \{0 = r_0 < r_1 < \cdots < r_{k-1} < r_k = +\infty\}$. Define open intervals $I_j = (r_{j-1}, r_j)$ for each $1 \leq j \leq k$. Then $[0,\infty]$ decomposes into a disjoint union

$$[0,\infty] = I_1 \cup I_2 \cup \cdots \cup I_k \cup C(n).$$

Let x, x' be two elements of the same interval I_j and let $\delta, \delta' \in \{+,-\}$. Suppose λ is any partition of n. Since there are no critical rationals between x and x', $\Box \in d(\lambda)$ contributes to $h^{\delta}_{x,c}(\lambda)$ if and only if it contributes to $h^{\delta'}_{x',c}(\lambda)$. So, $t^{h^{\delta}_{x,c}(\lambda)} = t^{h^{\delta'}_{x',c}(\lambda)}$. Adding over all λ, we see that if $x, x' \in I_j$,

$$H^{\delta}_{x,c}(n) = H^{\delta'}_{x',c}(n). \tag{4.6}$$

Similarly, for all $x \in I_j$,

$$H^{+}_{r_{j-1},c}(n) = H^{\delta}_{x,c}(n) = H^{-}_{r_j,c}(n). \tag{4.7}$$

On the other hand, Theorem 4.3 implies that

$$H^{+}_{r_j,c}(n) = H^{-}_{r_j,c}(n). \tag{4.8}$$

Therefore, for $\delta, \delta' \in \{+,-\}$ and $y \geq y'$ by applying a chain of these equalities starting with $H^{\delta}_{y,c}(n)$, one can reduce y to a critical rational and change δ to a $+$ using (4.7), or using (4.8) if y is already a critical rational. Then one may iteratively apply (4.8) and (4.7) to change δ to a $-$, and then reduce y to the next lowest critical rational and change δ back to a $+$, until an equality $H^{\delta}_{y,c}(n) = H^{-}_{r_j,c}(n)$ is obtained for $r_{j-1} \leq y' \leq r_j$. Then, applying (4.7) again with $x = y'$ (and (4.8) to flip the sign of δ if $y = r_{j-1}$ and $\delta' = -$), one obtains $H^{\delta}_{y,c}(n) = H^{\delta'}_{y',c}(n)$. $\Box$

4.2 Reducing to a symmetry property

In the case x is rational, where $h^+_{x,c}$ and $h^-_{x,c}$ may differ, it is useful to separate the boxes that contribute to both statistics from those that contribute to just one. In order to do this, we define the following statistics.

Definition 4.5. *For $x = \frac{r}{s}$ a rational number, we have*

$$\mathrm{crit}^+_{x,c}(\lambda) = \left|\left\{\square \in \lambda : c \mid h(\square) \text{ and } \frac{a(\square)}{l(\square+1)} = x\right\}\right|, \tag{4.9}$$

$$\mathrm{crit}^-_{x,c}(\lambda) = \left|\left\{\square \in \lambda : c \mid h(\square) \text{ and } \frac{a(\square)+1}{l(\square)} = x\right\}\right|, \tag{4.10}$$

$$\mathrm{mid}_{x,c}(\lambda) = \left|\left\{\square \in \lambda : c \mid h(\square) \text{ and } -s < sa(\square) - rl(\square) < r\right\}\right|. \tag{4.11}$$

The next proposition shows that a bijection satisfying some constraints on its behaviour with respect to these statistics will give a bijective proof of Theorem 4.3.

Proposition 4.6. *Let r, s, c be positive integers with $(r, s) = 1$ and let $x = \frac{r}{s}$. Suppose there exists a bijection $I_{r,s,c} : \mathrm{Par}^c_\mu \to \mathrm{Par}^c_\mu$ such that*

1. $|\lambda| = |I_{r,s,c}(\lambda)|$,
2. $\mathrm{mid}_{x,c}(\lambda) = \mathrm{mid}_{x,c}(I_{r,s,c}(\lambda))$,
3. $\mathrm{crit}^+_{x,c}(\lambda) + \mathrm{crit}^-_{x,c}(\lambda) = \mathrm{crit}^+_{x,c}(I_{r,s,c}(\lambda)) + \mathrm{crit}^-_{x,c}(I_{r,s,c}(\lambda))$,
4. $\mathrm{crit}^+_{x,c}(\lambda) = \mathrm{crit}^-_{x,c}(I_{r,s,c}(\lambda))$.

Then, Theorem 4.3 is true.

Proof. Assume that $I_{r,s,c}$ exists. Then, property 3 and 4 together imply that

$$\mathrm{crit}^-_{x,c}(\lambda) = \mathrm{crit}^+_{x,c}(I_{r,s,c}(\lambda)) \tag{4.12}$$

so $I_{r,s,c}$ exchanges $\mathrm{crit}^+_{x,c}$ and $\mathrm{crit}^-_{x,c}$ whilst preserving $|\lambda|$ and $\mathrm{mid}_{x,c}$.

Note that a box $\square$ contributes to $\mathrm{mid}_{x,c}$ if and only if $-s < sa(\square) - rl(\square) < r$ and the $c \mid h(\square)$. Adding $s + rl(\square)$, and dividing by $sl(\square)$, the left inequality is equivalent to

$$\frac{a(\square)+1}{l(\square)} > x. \tag{4.13}$$

Similar manipulation of the right inequality together with (4.13) shows that $\square$ contributes to $\mathrm{mid}_{x,c}$ if and only if

$$\frac{a(\square)}{l(\square)+1} < x < \frac{a(\square)+1}{l(\square)}. \tag{4.14}$$

So, comparing the definitions of $\mathrm{crit}^-_{x,c}$, $\mathrm{crit}^+_{x,c}$, $h^+_{x,c}$, $h^-_{x,c}$ and (4.14),

$$h^+_{x,c}(\lambda) = \mathrm{mid}_{x,c}(\lambda) + \mathrm{crit}^+_{x,c}(\lambda) \tag{4.15}$$

and

$$h^-_{x,c}(\lambda) = \mathrm{mid}_{x,c}(\lambda) + \mathrm{crit}^-_{x,c}(\lambda). \tag{4.16}$$

So, $I_{r,s,c}$ exchanges $h^+_{x,c}(\lambda)$ and $h^-_{x,c}(\lambda)$ whilst preserving $|\lambda|$, and hence proves Theorem 4.3. $\square$

4.3 Connecting to Buryak-Feigin-Nakajima

When c is divisible by $r+s$, Theorem 4.2 implies the following product formula. In the case $r+s=c$, this is the main combinatorial result of [BFN15].

Corollary 4.7. *Let r and s be coprime integers, let $x = \frac{r}{s}$ and let $r+s \mid c$. Then*

$$\sum_{\lambda \in \mathrm{Par}} q^{|\lambda|} t^{\mathrm{crit}^+_{x,c}(\lambda)} = \prod_{\substack{i\geq 1 \\ c\nmid i}} \frac{1}{1-q^i} \prod_{i\geq 1} \frac{1}{1-q^{ic}t}. \tag{4.17}$$

Proof. First we show that under the assumption that $r+s \mid c$, then for any partition λ, $\mathrm{mid}_{x,c}(\lambda) = 0$. Suppose $\square$ were to contribute to $\mathrm{mid}_{x,c}(\lambda)$, then $\square$ would have to satisfy

$$-s < sa(\square) - rl(\square) < r. \tag{4.18}$$

Adding $rl + sl + s$,

$$(r+s)l(\square) < s\big(a(\square) + l(\square) + 1\big) < (r+s)\big(l(\square)+1\big) \tag{4.19}$$

However, the upper and lower bound are consecutive multiples of $r+s$, and therefore $s(a(\square)+l(\square)+1)$ cannot be a multiple of $r+s$, so by assumption cannot be a multiple of c. So, $c \nmid h(\square)$ so $\square$ cannot contribute to $\mathrm{mid}_{x,c}(\lambda)$.

So in this case $h^+_{x,c}(\lambda) = \mathrm{crit}^+_{x,c}(\lambda)$ and Theorem 4.2 becomes

$$\sum_{\lambda \in \mathrm{Par}^c_\mu} q^{|\lambda|} t^{\mathrm{crit}^+_{x,c}(\lambda)} = q^{|\mu|} \prod_{i \geq 1} \frac{1}{(1-q^{ic})^{c-1}} \frac{1}{1-q^{ic}t}. \tag{4.20}$$

Summing both sides over all c-cores μ and applying Proposition 3.35,

$$\sum_{\lambda \in \mathrm{Par}} q^{|\lambda|} t^{\mathrm{crit}^+_{x,c}(\lambda)} = \prod_{i \geq 1} \frac{(1-q^{ic})^c}{1-q^i} \frac{1}{(1-q^{ic})^{c-1}} \frac{1}{1-q^{ic}t} \tag{4.21}$$

$$= \prod_{i \geq 1} \frac{(1-q^{ic})}{1-q^i} \frac{1}{1-q^{ic}t} \tag{4.22}$$

$$= \prod_{\substack{i \geq 1 \\ c \nmid i}} \frac{1}{1-q^i} \prod_{i \geq 1} \frac{1}{1-q^{ic}t}. \tag{4.23}$$

□

Chapter 5

The multigraph $M_{r,s,c}$

From now on, $x = \frac{r}{s}$ is a rational number with r coprime to s. In this chapter we take our first key step in the construction of the involution $I_{r,s,c}$. First, Proposition 5.1 relates the statistics $\mathrm{mid}_{x,c}$, $\mathrm{crit}^-_{x,c}$ and $\mathrm{crit}^+_{x,c}$ to the boundary graph. We use this relationship to define a map from the boundary graph to a multigraph $M_{r,s,c}$ that picks out the information relevant to $\mathrm{mid}_{x,c}$ and $\mathrm{crit}^+_{x,c}$, much as the c-abacus tour does for the c-core. The rest of the chapter then outlines the method for proving that $M_{r,s,c}$ retains partition data, including the proofs that $M_{r,s,c}$ retains the c-core and the area. The proof it retains the area is a particularly easy example using the same methodology as used in the more technical proofs in Chapter 7, which check that $M_{r,s,c}$ retains $\mathrm{mid}_{x,c}(\lambda)$ and $\mathrm{crit}^+_{x,c}(\lambda) + \mathrm{crit}^-_{x,c}(\lambda)$.

When we define $I_{r,s,c}$ as a bijection on partitions, we build into the definition that $I_{r,s,c}$ preserves $M_{r,s,c}(\lambda)$ for any partition λ. So, together with these results it is immediate that $I_{r,s,c}$ does map Par^c_μ to Par^c_μ and satisfies hypothesis 1 in Proposition 4.6.

5.1 Partition statistics and the boundary path

Proposition 5.1. *Let λ be a partition and let $\square \in \lambda$. Let e_i be the foot of $\square$, departing from $(x_1 - 1, y_1)$ and arriving at (x_1, y_1) and let e_j be the hand of λ, departing from $(x_2, y_2 + 1)$ and arriving at (x_2, y_2). Let $t = r(y_1 - y_2) + s(x_1 - x_2)$. Then*

1. *$\square$ contributes to $\mathrm{crit}^+_{x,c}$ if and only if $t = 0$ and $x_1 - y_1 \equiv x_2 - y_2 \pmod{c}$;*

2. *$\square$ contributes to $\mathrm{mid}_{x,c}$ if and only if $0 < t < r + s$ and $x_1 - y_1 \equiv x_2 - y_2 \pmod{c}$;*

3. *$\square$ contributes to $\mathrm{crit}^-_{x,c}$ if and only if $t = r + s$ and $x_1 - y_1 \equiv x_2 - y_2 \pmod{c}$.*

Proof. By the definition of index, $x_1 - y_1 = i$ and $x_2 - y_2 = j$. Let $\square \in \lambda$ have bottom left corner $(x_\square, y_\square)$. By Proposition 3.11, $\square$ has hook length divisible by c if and only if $j \equiv i \pmod{c}$, i.e. $x_1 - y_1 \equiv x_2 - y_2 \pmod{c}$. So, assume that $\square$ does have hook length divisible by c.

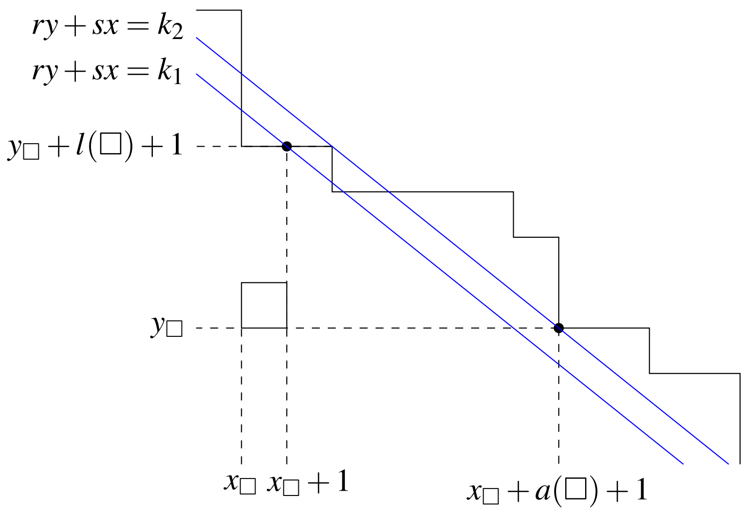


Fig. 5.1 The box $\square$ and the lines $ry + sx = k_1$ and $ry + sx = k_2$.

Let $k_1 = ry_1 + sx_1$ and $k_2 = ry_2 + sx_2$. Then,

$$t = k_1 - k_2, \tag{5.1}$$

$$s(x_\square + 1) + r(y_\square + l(\square) + 1) = k_1, \tag{5.2}$$

and

$$s(x_\square + a(\square) + 1) + ry_\square = k_2. \tag{5.3}$$

Subtracting (5.2) from (5.3), and substituting in (5.1)

$$sa(\square) - rl(\square) = r - t. \tag{5.4}$$

By definition, $\square$ contributes to $\mathrm{crit}^+_{x,c}$ if and only if $sa(\square) - rl(\square) = r$, that is, when $t = 0$, proving the first claim. Similarly, $\square$ contributes to $\mathrm{mid}_{x,c}$ if and only if $-s < sa(\square) - rl(\square) < r$, or equivalently $0 < t < r + s$, proving the second claim.

Finally, note $sa(\square) - rl(\square) = -s$ if and only if $t = r + s$. $\square$

We define the multigraph $M_{r,s,c}(\lambda)$ accordingly.

Definition 5.2 ($M_{r,s,c}$, (r,s,c)-tour)**.** *For a partition λ the SE directed multigraph $M_{r,s,c}(\lambda)$ is obtained from $b(\lambda)$ by imposing the relation $\sim_{r,s,c}$ on the vertices, where $(x_1,y_1) \sim_{r,s,c} (x_2,y_2)$ if $ry_1+sx_1 = ry_2+sx_2$ and $x_2-y_2 \equiv x_1-y_1 \pmod{c}$. Denote the equivalence class with $ry+sx = v$ and $x-y \equiv i \pmod{c}$ by $(v,[i])$. Let $q_{r,s,c} : b(\lambda) \to M_{r,s,c}(\lambda)$ be the induced homomorphism. The (r,s,c)-tour of $M_{r,s,c}(\lambda)$ associated to λ is $(q_{r,s,c}(e_i))_{i\in\mathbb{Z}}$. At each vertex $(v,[i])$, we count the number of east edges arriving at $(v,[i])$ in the (r,s,c)-tour and denote this quantity by $\mathrm{E_{in}}(v,[i])$. Similarly, we count the number of east edges departing from $(v,[i])$ in the (r,s,c)-tour and denote this quantity by $\mathrm{E_{out}}(v,[i])$. We define $\mathrm{S_{in}}(v,[i])$ and $\mathrm{S_{out}}(v,[i])$ analogously.*

Now, we explain a useful way of drawing $M_{r,s,c}(\lambda)$ in the plane. First, we show Proposition 5.3, which says that when the plane is cut into strips of width $\mathrm{lcm}(c,r+s)$ by lines with $sx+ry$ constant, then there is a unique representative of each possible vertex of $M_{r,s,c}(\lambda)$ contained in the strip.

Proposition 5.3. *If there is a lattice point (x,y) satisfying both $sx+ry = v$ and $x-y \equiv i \pmod{c}$, then for any real number m there is exactly one such lattice point satisfying the inequality $m \le x-y < m+\mathrm{lcm}(c,r+s)$.*

Proof. First, note that translating a lattice point (x,y) by $(r,-s)$ does not change the value of $sx+ry$. Moreover, there is no lattice point on the line $sx+ry = v$ between (x,y) and $(x+r,y-s)$, since if $(x+l_1,y-l_2)$ were such a point, we would have $sl_1-rl_2 = 0$, so since r and s are coprime, $s \mid l_2$ and $r \mid l_1$.

Secondly, note that translating by $(r,-s)$ changes the value of $x-y$ by $r+s$. So, the translations that preserve both the value of $sx+ry$ and the residue class of $[y-x]$ modulo c are the translations by $(ar,-as)$ where $a(r+s)$ is divisible by c, that is, $a(r+s)$ is divisible by $\mathrm{lcm}(c,r+s)$. Exactly one of these translates lies in the region $m \le x-y < m+\mathrm{lcm}(c,r+s)$. □

So, for a fixed integer n, we can draw the multigraph by taking the vertices to be lattice points in the portion of $\mathbb{R}^2$ in between the lines $x-y = n$ and $x-y = \mathrm{lcm}(c,r+s)+n$, with an identification along the boundary lines given by

$$(x,y) \sim \left(x+\frac{r\,\mathrm{lcm}(c,r+s)}{r+s}, y-\frac{s\,\mathrm{lcm}(c,r+s)}{r+s}\right).$$

We identify a lattice point (x,y) with the vertex $(sx+ry,[x-y])$. Then, south edges in the multigraph from $(v,[i])$ to $(v-r,[i+1])$ are south edges between lattice points in the region

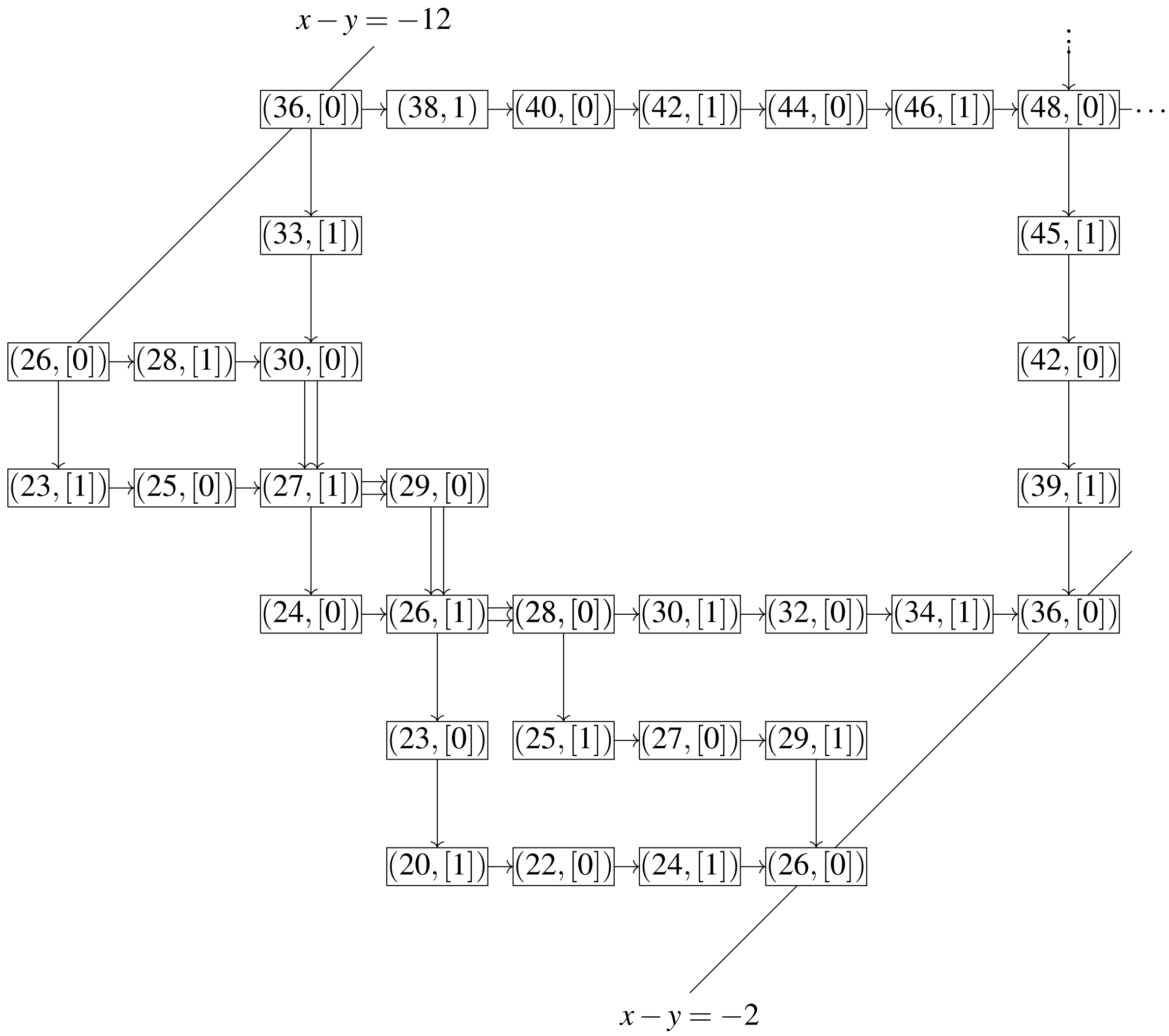


Fig. 5.2 $M_{3,2,2}(12,12,10,8,7,4,1,1,1)$.

described. Similarly, east edges from $(v,[i])$ to $(v+s,[i+1])$ are east edges between lattice points. Moreover, each vertex $(v,[i])$ corresponds to a unique lattice point in the region. We can view the (r,s,c)-tour as the *cylindrical* lattice path tour obtained by collapsing the boundary of the partition onto this cylinder.

Example 5.4. *When* $c=2$*,* $r=3$ *and* $s=2$ *we may draw the* (r,s,c)*-multigraph of* $\mu = (12,12,10,8,7,4,1,1,1)$ *as in Figure 5.2.*

Remark 5.5. *As with the boundary graph, but unlike the c-abacus, the direction of an edge in* $M_{r,s,c}$ *can be read off from its source and target. If an edge e has* $s(e)=(v,[i])$ *and* $t(e)=(w,[i+1])$ *then either* $d(e)=E$ *and* $w=v+s$ *or* $d(e)=S$ *and* $w=v-r$*.*

Remark 5.6. *If we act on $\mathbb{C}[x,y]$ by $T \times \mathbb{Z}/c\mathbb{Z}$ where $T = \{(t^s,t^r) : t \in \mathbb{C}^*\}$, lift to ideals and colour boxes according to the weight of the corresponding monomial with respect to this representation, the colouring carries the same information as the multigraph.*

The first property that we check is that $M_{r,s,c}(\lambda)$ determines the c-core of λ.

Proposition 5.7. *If λ and μ are partitions with $M_{r,s,c}(\lambda) = M_{r,s,c}(\mu)$ then λ and μ have the same c-core.*

Proof. Let v be large enough so that $\left(\left\lceil \frac{v}{s} \right\rceil, 0\right)$ is on the boundary of both λ and μ. Fix $m > \left\lceil \frac{v}{s} \right\rceil$ such that $c \mid m$. Then, in both the boundary tour of μ and the boundary tour of λ, every edge with index at least m is an east edge. These edges account for every E in an arrival word at a vertex $(w,[j])$ with $w \geq sm$.

For each $[i]$, the number of east edges with index less than m, for both λ and μ, is given by

$$\sum_{w<sm} \mathrm{E}_{\mathrm{in}}(w,[i]).$$

Therefore λ, μ and m satisfy the hypotheses of Corollary 3.30, and so λ and μ have the same c-core. □

Next, we show how to read $\mathrm{crit}^+_{x,c}(\lambda)$ and $\mathrm{crit}^-_{x,c}(\lambda)$ off the (r,s,c)-tour of $M_{r,s,c}(\lambda)$. Rephrasing the first part of Proposition 5.1 in terms of the (r,s,c)-tour gives

Corollary 5.8. *Let λ be a partition. Then*

$$\mathrm{crit}^+_{x,c}(\lambda) = \sum_{(v,[i]) \in M_{r,s,c}(\lambda)} \mathrm{inv}(v,[i])_a. \tag{5.5}$$

A similar formula with the departure words holds for $\mathrm{crit}^-_{x,c}(\lambda)$.

Corollary 5.9.

$$\mathrm{crit}^-_{x,c}(\lambda) = \sum_{(v,[i]) \in M_{r,s,c}(\lambda)} \mathrm{inv}(v,[i])_d. \tag{5.6}$$

Proof. By the third part of Proposition 5.1, □ contributes to $\mathrm{crit}^-_{x,c}(\lambda)$ if and only if the foot and hand arrive at points (x_1,y_1) and (x_2,y_2) respectively with

$$r+s = r(y_1-y_2)+s(x_1-x_2) \tag{5.7}$$

and

$$x_1 - y_1 \equiv x_2 - y_2 \pmod{c}. \tag{5.8}$$

The foot and hand arrive at (x_1, y_1) and (x_2, y_2) respectively if and only if they depart from points $(x_1 - 1, y_1)$ and $(x_2, y_2 + 1)$ respectively. The condition (5.8) is equivalent to

$$(x_1 - 1) - y_1 \equiv x_2 - (y_2 + 1) \pmod{c}. \tag{5.9}$$

The condition (5.7) is equivalent to

$$r + s = r(y_1 - (y_2 + 1)) + s((x_1 - 1) - x_2) + r + s, \tag{5.10}$$

so subtracting $r + s$ from both sides,

$$r(y_1 - (y_2 + 1)) + s((x_1 - 1) - x_2) = 0. \tag{5.11}$$

□

We now outline a framework for inductive proofs that the statistics in hypotheses 1–3 of Proposition 4.6 are determined by the multigraph $M_{r,s,c}$, using an ordering $<_{r,s,c}$ on partitions and multigraphs. The key result in this direction is Proposition 5.24.

5.2 The order $<_{r,s,c}$

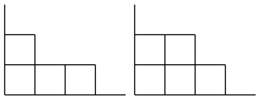
The structure of the proofs that $M_{r,s,c}(\lambda)$ determines each property of λ will be proven by induction on $|\lambda|$, adding a box at each step. Since the structure of $M_{r,s,c}(\lambda)$ is somewhat delicate, we have to be careful when choosing a box to add. The following ordering on partitions gives us a framework for adding boxes.

If (x_1, y_1) and (x_2, y_2) are two points in $\mathbb{N}^2$, say $(x_1, y_1) <_{r,s,c} (x_2, y_2)$ if either of the following hold.

- $sx_1 + ry_1 < sx_2 + ry_2$;
- $sx_1 + ry_1 = sx_2 + ry_2$, and $x_1 - y_1 \equiv x_2 - y_2 \pmod{c}$, and $x_1 - y_1 < x_2 - y_2$.

The partial order $>_{r,s,c}$ on points in the plane induces a partial order $>_{r,s,c}$ on partitions as follows. Say that $\lambda' >'_{r,s,c} \lambda$ if λ' can be obtained from λ by adding a box with bottom left corner (x,y) minimal with respect to $>_{r,s,c}$ over all possible bottom left corners of boxes that can be added to λ to obtain a partition. Then for partitions μ,λ say that $\mu >_{r,s,c} \lambda$ if there is a sequence of partitions $\lambda = \lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_m = \mu$ such that for each i, $\lambda_i <'_{r,s,c} \lambda_{i+1}$. If $\mu >'_{r,s,c} \lambda$, say that μ is a *successor* for λ with respect to $>_{r,s,c}$. Every partition has a successor with respect to $>_{r,s,c}$, but successors are not necessarily unique.

Example 5.10. *Let $r = 3$, $s = 2$, and $c = 2$. There are three boxes that could be added to the Young diagram of $(3,1)$ to give another partition. They have bottom left corners at $(3,0)$, $(1,1)$, and $(0,2)$, with values of $2x+3y$ of $6,5$ and 6 respectively. So $(3,1)$ has a unique successor with respect to $<_{3,2,2}$, which is $(3,2)$.*



For $(3,2)$, the boxes that could be added to the diagram have bottom left corners $(3,0)$, $(2,1)$ and $(0,2)$, with values of $2x+3y$ of $6,7$ and 6 respectively. The values of $x-y$ for $(0,3)$ and $(2,0)$ have different parity so $(2,0) \not<_{3,2,2} (0,3)$, and both $(4,2)$ and $(3,2,1)$ are successors of $(3,2)$. Note that $(4,1) \not>_{3,2,2} (3,1)$.

Note that if $\mu >_{r,s,c} \lambda$, then all boxes of the Young diagram of λ are also boxes of the Young diagram of μ, but as Example 5.10 shows the converse is not true in general.

Definition 5.11 (Accumulation point)**.** *For a partition μ with the property that whenever μ strictly contains λ, we also have $\mu >_{r,s,c} \lambda$, we call μ an accumulation point for $>_{r,s,c}$.*

The next chapter describes a family of accumulation points and proves some key properties.

5.3 The accumulation points $\lambda_{r,s,k}$

Definition 5.12 (The partition $\lambda_{r,s,k}$)**.** *For a given natural number k, the partition $\lambda_{r,s,k}$ is the partition with Young diagram consisting of all boxes with top right corners on or below the line $sx+ry=k$.*

Example 5.13. *The Young diagram for $\lambda_{3,2,54}$ is given in Figure 5.3.*

Proposition 5.14. *Let r,s,k be positive integers. Let μ be a partition with diagram strictly contained in the diagram of $\lambda_{r,s,k}$. Then, any successor μ^+ of μ with respect to $>_{r,s,c}$ has*

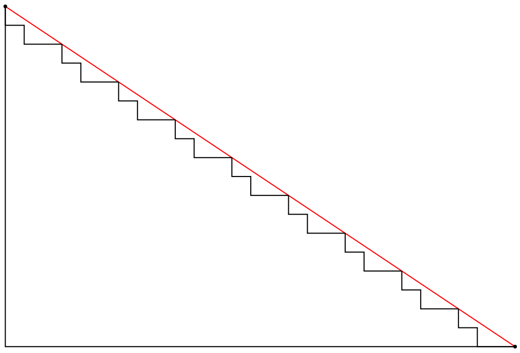


Fig. 5.3 the Young diagram of $\lambda_{3,2,54}$.

diagram contained in the diagram of $\lambda_{r,s,k}$. In particular, $\lambda_{r,s,k}$ is an accumulation point for $\lambda_{r,s,c}$.

Proof. If (x,y) is the top right corner of a box in μ, then since the diagram for μ is contained in the diagram of $\lambda_{r,s,k}$, $sx+ry \leq k$. So, the bottom left corner of the same box is at $(x-1,y-1)$ with $s(x-1)+r(y-1) \leq k-r-s$. Since the containment of μ in λ is strict, there is at least one box in the diagram of λ, not contained in the diagram of μ, with bottom left corner $(x-1,y-1)$ satisfying $s(x-1)+r(y-1) \leq k-r-s$. Moreover, since translating a box with top right corner (x,y) left or down decreases $s(x-1)+r(y-1)$, there is a box $\square_1$ with bottom left corner $(x-1,y-1)$ that can be added to μ to give a valid partition diagram that satisfies $s(x-1)+r(y-1) \leq k-r-s$. Now, if μ^+ is not contained in λ, then μ^+ contains some box $\square_2$ with top right corner (z,w) such that $sz+rw > k$, so the bottom left corner $(z-1,w-1)$ satisfies $s(z-1)+r(w-1) > k-r-s$. This is a contradiction, as $(z-1,w-1) >_{r,s,c} (x-1,y-1)$, and $\square_1$ can be added to μ. $\square$

The accumulation points $\lambda_{r,s,k}$ will be extremely useful for two reasons. Firstly, as we check in Proposition 5.18, $M_{r,s,c}(\lambda_{r,s,k})$ admits a unique (r,s,c)-tour whenever $rsc \mid k$, so that $M_{r,s,c}$ must determine any partition statistic in these cases, as it determines the partition itself. Secondly, as we check in Proposition 5.16, if we take successor with respect to $<_{r,s,c}$ iteratively on a given partition, we will eventually hit an accumulation point. This allows us to use the $\lambda_{r,s,k}$ as a base case for iterative proofs that statistics are independent of the choice of (r,s,c)-tour, and reduces the problem of understanding how a statistic interacts with $M_{r,s,c}$ to understanding how it behaves when we take successor.

The $\lambda_{r,s,k}$ are not necessarily the only accumulation points. However, they suffice for our purposes.

Example 5.15. *The partition* $(1,1)$ *is an accumulation point when* $r = s = 1$ *and* $c = 2$*, but is not a* $\lambda_{1,1,k}$*. Indeed, the only successor of the empty partition is* (1)*, and the only successor of* (1) *is* $(1,1)$ *since the bottom left corners of the boxes addable to* (1) *are* $(1,0)$ *and* $(0,1)$ *with* $1-0 \equiv 0-1 \pmod 2$.

Proposition 5.16. *If the diagram of a partition* μ *is contained in the diagram of* $\lambda_{r,s,k}$ *for some k, then for any sequence*

$$\mu = \mu_0 <'_{r,s,c} \mu_1 <'_{r,s,c} \cdots <'_{r,s,c} \mu_m$$

where $m = |\lambda_{r,s,k}| - |\mu|$*, we must have* $\mu_m = \lambda_{r,s,k}$.

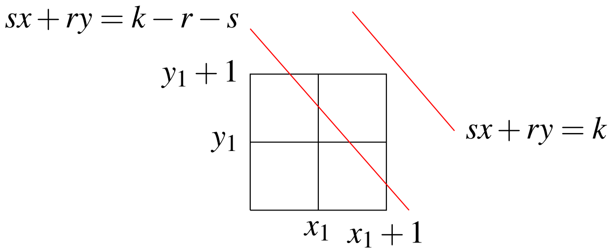
Proof. Applying Proposition 5.14 to $\mu_0, \mu_1, \ldots, \mu_m$, the diagram of μ_m must be contained in the diagram of $\lambda_{r,s,k}$, and $|\mu_m| = |\mu_0| + m = |\lambda_{r,s,k}|$, so $\mu_m = \lambda_{r,s,k}$. □

We now work towards proving that, in the case $rsc \mid k$, if $M_{r,s,c}(\lambda) = M_{r,s,c}(\lambda_{r,s,k})$, then $\lambda = \lambda_{r,s,k}$. First, we collect some restrictions on the arrival words that arise in the (r,s,c)-tour corresponding to $\lambda_{r,s,k}$. The condition that $rsc \mid k$ does not damage the capacity of the $\lambda_{r,s,k}$ to act as base cases, as to contain the diagram of a partition we just need k to be large enough.

Proposition 5.17. *Let* $rsc \mid k$ *and let* $k_1 = \frac{k}{rs}$*. The vertices* $(v,[i])$ *in the multigraph of* $\lambda_{r,s,k}$ *all satisfy* $v > k - r - s$*. Moreover, we have the following constraints on the arrival words at a vertex* $(v,[i])$*.*

- *If* $k - r - s < v \leq k - r$*, then all letters in the arrival word are Ss.*
- *If* $k - r < v < k$*, all letters in the arrival word are Es.*
- *If* $v = k$ *then the arrival word at* $(k,[0])$ *has first letter S and all other letters E. For* $[i] \neq [0]$*, all letters in the arrival word at* $(k,[i])$ *are Es.*

Proof. If a box □ has top right corner (x_1, y_1) with $ry_1 + sx_1 \leq k - r - s$, then the 2×2 box with centre (x_1, y_1) contains □, along with three other boxes with top right corners $(x_1 + 1, y_1)$, $(x_1, y_1 + 1)$ and $(x_1 + 1, y_1 + 1)$.



These points satisfy $ry_1+s(x_1+1)\le k-r<k$, $r(y_1+1)+sx_1\le k-s<k$, and $r(y_1+1)+s(x_1+1)\le k$. Since $\lambda_{r,s,k}$ contains *all* boxes with top right corners on or below the line $sx+ry=k$, the entire 2×2 box with centre (x,y) is contained in $\lambda_{r,s,k}$ so the boundary never visits (x,y).

Suppose $k-r-s<v\le k-r$. Any east letter in the arrival word at a vertex $(v,[i])$ is also an east letter in the departure word of some vertex $(v-s,[i-1])$, but $v-s\le k-r-s$, so there is no such vertex.

Suppose now that $k-r<v\le k$. Any south letter in the arrival word at a vertex $(v,[i])$ arriving at a point (x_1,y_1) with $ry_1+sx_1=v$ is also a south letter in the departure word of some vertex $(v+r,[i-1])$. We have that $v+r>k$, so the south edge cannot be the right edge of a box in the Young diagram of $\lambda_{r,s,k}$ and must be along the y axis. Therefore, $x_1=0$ and $v=ry_1$ is divisible by r. However, by assumption k is divisible by r and therefore $k-r$ and k are consecutive multiples of r. So, this is only possible if $v=k$. Since the value of ry decreases as the boundary progresses south down the y-axis, there is only one such edge, namely, the edge departing from $\left(0,\frac{k}{r}+1\right)$ and arriving at $\left(0,\frac{k}{r}\right)$. □

We are now in a position to check our base case. We will show that, if $rsc\mid k$ and $M_{r,s,c}(\mu)=M_{r,s,c}(\lambda_{r,s,k})$ then $\mu=\lambda_{r,s,k}$. So, the accumulation point $\lambda_{r,s,k}$ act as a base case for a claim that any statistic is independent of the choice of (r,s,c)-tour.

Proposition 5.18. *For fixed integers r,s,c,k with $k=rsk_1$ and $c\mid k_1$, there is a unique (r,s,c)-tour of $M_{r,s,c}(\lambda_{r,s,k})$.*

Proof. Suppose we pick a different (r,s,c)-tour of $M_{r,s,c}(\lambda_{r,s,k})$ corresponding to a partition μ. First, we will show that the partition boundary of μ must leave the y-axis earlier than the boundary of $\lambda_{r,s,k}$. Let $(v,[i])$ be the vertex with v maximal such that the arrival word at $(v,[i])$ changes. Such a vertex certainly exists because any partition boundary differs in finitely many edges from the boundary of the empty partition. Let $(v,[i])_a^{\lambda}$ and $(v,[i])_a^{\mu}$ be the arrival words at $(v,[i])$ in the tour corresponding to $\lambda_{r,s,k}$ and μ respectively. Then, $(v,[i])_a^{\mu}$

must be a permutation of $(v,[i])_a^\lambda$, so since $(v,[i])_a^\lambda \neq (v,[i])_a^\mu$, $(v,[i])_a^\lambda$ must contain both Es and Ss. Proposition 5.17 then tells us that either

- $v > k$, in which case any letter in the arrival word at $(v,[i])$ must correspond to an edge on a co-ordinate axes. Since the value of v decreases as the boundary steps south along the y axis, and increases as it steps east along the x-axis, we must have $(v,[i])_a^{\lambda_{r,s,k}} = SE$.
- $(v,[i]) = (k,[0])$, in which case Proposition 5.17 implies $(v,[i])_a^{\lambda_{r,s,k}}$ is an S followed by a string of Es, where the S corresponds to an edge on the y-axis.

In either case, $(v,[i])_a^\mu$ must begin with an E. So, the boundary of μ must step east off the y-axis before it hits the lattice point on the y-axis corresponding to $(v,[i])$ - otherwise $(v,[i])_a^\mu$ would have first letter S. So, the boundary of μ does step east off the axis earlier than the boundary of $\lambda_{r,s,k}$. In particular, the boundary of μ never visits the point $(0,k_1 s)$.

Now consider the arrival words $(k,[0])_a^{\lambda_{r,s,k}}$ and $(k,[0])_a^\mu$. Let Z be the set of points (x,y) in the plane in the equivalence class $(k,[0])$ with respect to $\sim_{r,s,c}$,

$$Z = \left\{(x,y) : x,y \in \mathbb{Z}_{\geq 0}, sx + ry = k \text{ and } x - y \equiv 0 \pmod{c}\right\}. \tag{5.12}$$

The length of the arrival words $(k,[0])_a^{\lambda_{r,s,k}}$ and $(k,[0])_a^\mu$ count the number of times the boundaries of $\lambda_{r,s,k}$ and μ respectively visit points in Z. Both arrival words have the same length (they are permutations of each other) so the boundaries of $\lambda_{r,s,k}$ and μ must visit the same number of lattice points in Z. By the definition of $\lambda_{r,s,k}$, the boundary of $\lambda_{r,s,k}$ visits *all* of the points in Z, so the boundary of μ must also visit all $|Z|$ of these points. But the boundary of μ does not visit the point $(0,k_1 s)$, a contradiction. □

Next, we check that there is a sensible pull back of the ordering $>_{r,s,c}$ to (r,s,c)-multigraphs, so that taking successor can be understood to mean something at both the level of the partition and at the level of the multigraph. We abuse notation and write $>_{r,s,c}$ for the ordering on multigraphs and partitions.

Proposition 5.19. *Given an (r,s,c)-multigraph M, let V_S be the set of vertices $(w,[i])$ with at least one south edge arriving at $(w,[i])$. Let $(v,[i]) \in V_S$ such that v is minimal. Then there is an edge from $(v,[i])$ to $(v+s,[i+1])$.*

Proof. At least one edge arrives at $(v,[i])$ so at least one edge departs from $(v,[i])$. Any south edge departing from $(v,[i])$ would arrive at $(v-r,[i+1])$, so $(v-r,[i+1])$ would be in V_S, contradicting the minimality of v. Therefore at least one east edge departs from $(v,[i])$, and arrives at $(v+s,[i+1])$. □

Definition 5.20 (Multigraph successors)**.** *Given an (r,s,c)-multigraph M, let $(v,[i]) \in V_S$ as in the previous proposition. Then we say M^+ is a successor of M if M^+ can be obtained from M by deleting one south edge from $(v+r,[i-1])$ to $(v,[i])$ and one east edge from $(v,[i])$ to $(v+s,[i+1])$, and adding one east edge from $(v+r,[i-1])$ to $(v+r+s,[i])$ and one south edge from $(v+r+s,[i])$ to $(v+s,[i+1])$. Sometimes we emphasize the vertex $(v,[i])$ and say M^+ is a successor of M that changes from $(v,[i])$.*

At the level of multigraphs, we will only need the notion of successors, but for completeness we also explicitly define $<_{r,s,c}$ at the level of multigraphs.

Definition 5.21 (Ordering on multigraphs)**.** *Given (r,s,c)-multigraphs $M = M_{r,s,c}(\lambda)$ and $M' = M_{r,s,c}(\lambda')$ we say $M <_{r,s,c} M'$ if there is a sequence of (r,s,c)-multigraphs $M = M_1,\dots,M_n = M'$ such that M_{i+1} is a successor of M_i^+ for each i.*

Corollary 5.22. *If λ is a partition with $M_{r,s,c}(\lambda) = M$ and M^+ is a successor of M changing from $(v,[i])$, then in M, $\mathrm{E_{in}}(v,[i]) = \mathrm{S_{out}}(v,[i]) = 0$.*

Proof. Identical to the proof of Proposition 5.19. □

The next proposition shows that this definition of successors at the level of multigraphs aligns with our definition at the level of partitions.

Proposition 5.23. *Let λ be a partition with $M_{r,s,c}(\lambda) = M$. If M^+ is a successor of M that changes at $(v,[i])$, then there is a unique partition λ^+ such that $\lambda^+ >'_{r,s,c} \lambda$ and $M^+ = M_{r,s,c}(\lambda^+)$.*

Proof. Let λ' be any successor of λ. Then, the Young diagram of λ' consists of all boxes in the Young diagram of λ and one additional box □. Let the bottom left corner of □ have co-ordinate (x_1,y_1), where $x_1 - y_1 \equiv i \pmod{c}$ and $ry_1 + sx_1 = l$. Then by definition of a successor, if we take minima over the points (x,y) in $b(\lambda)$,

$$l = \min(sx+ry) \tag{5.13}$$

and

$$x_1 - y_1 = \min_{\substack{sx+ry=l \\ [x-y]=[i]}} (x-y). \tag{5.14}$$

In particular, l and $[i]$ are sufficent to determine $x_1 - y_1$. Let s_1 and s_2 be the edges in $b(\lambda)$ arriving at and departing from (x_1, y_1) respectively, and let s'_1 and s'_2 be the edges in $b(\lambda')$ arriving at and departing from (x_1+1, y_1+1) respectively, as shown in Figure 5.4. Then, the

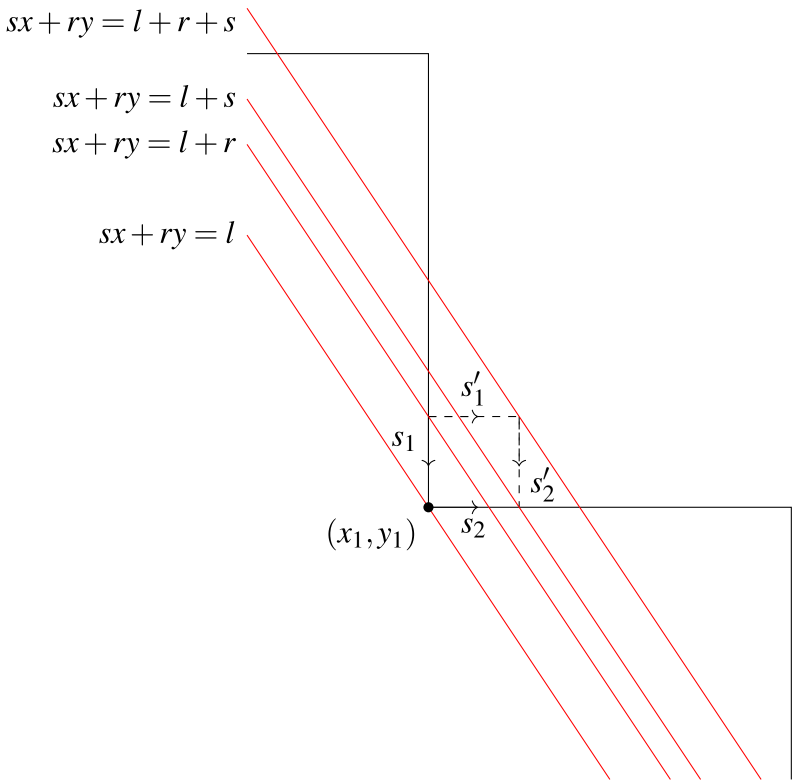


Fig. 5.4 a partition and its successor differ by replacing s_1 and s_2 with s'_1 and s'_2.

multigraph of λ' differs from the multigraph of λ only in that one edge from $(l+r, [i-1])$ to $(l, [i])$, and one edge from $(l, [i])$ to $(l+s, [i+1])$ corresponding to s_1 and s_2 respectively, are deleted, and one edge from $(l+r, [i-1])$ to $(l+r+s, [i])$, and one edge from $(l+r+s, [i])$ to $(l+s, [i+1])$, corresponding to s'_1 and s'_2 respectively are added. That is, $M_{r,s,c}(\lambda')$ is the successor of M changing from $(l, [i])$.

For uniqueness, given that M^+ changes from M at $(l, [i])$, any successor of λ with multigraph M^+ must be λ' by (5.14) because the value of $x - y$ increases by 1 at every consecutive point visited in the boundary.

For existence, if M^+ changes from M at $(v, [j])$ then v is minimal such that there is a south edge into $(v, [j])$ and an east edge out of $(v, [j])$. So, $v = \min_{(x,y)\in b(\lambda)}(sx + ry)$ and there is at least one point (x, y) on the boundary such that $[x - y] = [j]$. So, letting (x_2, y_2)

minimise $x-y$ over all such points, and adding a box with bottom left corner (x_2,y_2) gives a successor λ^+ of λ with multigraph M^+. □

We are now in a position to prove our key structural proposition.

Proposition 5.24. *Let $f:\mathrm{Par}\to\mathbb{R}$. Suppose there is a function $g:\{M_{r,s,c}(\lambda)\mid\lambda\in\mathrm{Par}\}^2\to\mathbb{R}$ such that, if λ is a partition, and λ^+ is a successor of λ, where λ and λ^+ have (r,s,c)-multigraphs M and M^+ respectively,*

$$f(\lambda^+)-f(\lambda)=g\left(M^+,M\right). \tag{5.15}$$

Then, for any partitions μ_1 and μ_2 with $M_{r,s,c}(\mu_1)=M_{r,s,c}(\mu_2)$, $f(\mu_1)=f(\mu_2)$.

Proof. Let $M=M_{r,s,c}(\mu_1)=M_{r,s,c}(\mu_2)$. There is a sequence of multigraphs $M=M_0,M_1,\ldots$ where M_j is a successor of M_{j-1} for each j. Set $\lambda_0=\mu_1$ and $\nu_0=\mu_2$. Then, by Proposition 5.23 there are sequences of partitions $\lambda_0,\lambda_1,\ldots$ and $\nu_0,\nu_1,\ldots$ such that $M_j=M_{r,s,c}(\lambda_j)=M_{r,s,c}(\nu_j)$,

$$\lambda_0<'_{r,s,c}\lambda_1<'_{r,s,c}\cdots, \tag{5.16}$$

and

$$\nu_0<'_{r,s,c}\nu_1<'_{r,s,c}\cdots. \tag{5.17}$$

Let k be divisible by rsc and large enough so that all boxes in the Young diagrams of μ_1 or μ_2 lie below the line $sx+ry=k$. By Proposition 5.16, there is some m such that $M_m=M_{r,s,c}(\lambda_{r,s,k})$. By Proposition 5.18, $\lambda_m=\nu_m=\lambda_{r,s,k}$. Then,

$$\begin{aligned} f(\mu_1) &= f(\lambda_{r,s,k})-\sum_{i=1}^{m}\left(f(\lambda_i)-f(\lambda_{i-1})\right) && (5.18)\\ &= f(\lambda_{r,s,k})-\sum_{i=1}^{m}g(M_i,M_{i-1}) && (5.19)\\ &= f(\lambda_{r,s,k})-\sum_{i=1}^{m}\left(f(\nu_i)-f(\nu_{i-1})\right) && (5.20)\\ &= f(\mu_2). && (5.21)\end{aligned}$$

□

Armed with Proposition 5.24, checking that $M_{r,s,c}$ determines the area of a partition is particularly straightforward.

Corollary 5.25. *If λ and μ are partitions with $M_{r,s,c}(\lambda) = M_{r,s,c}(\mu)$ then $|\lambda| = |\mu|$.*

Proof. Apply Proposition 5.24 with $g(M^+, M) = 1$. □

Having outlined the structure of the proofs that $M_{r,s,c}$ determines partition statistics, we defer the checks that $M_{r,s,c}$ determines $\mathrm{mid}_{x,c}$ and $\mathrm{crit}^+_{x,c} + \mathrm{crit}^-_{x,c}$ to Chapter 7. We now turn our attention to defining $I_{r,s,c}$.

Chapter 6

The involution $I_{r,s,c}$

In this chapter we construct the bijection $I_{r,s,c}$ and check that it is well defined. In order to do so, we first need to understand how to recover a partition from a family of arrival words.

6.1 Recovering a partition from the arrival words

Thus far we have constructed $M_{r,s,c}(\lambda)$ and an (r,s,c)-tour from $b(\lambda)$. We will define $I_{r,s,c}$ as an involution that preserves $M_{r,s,c}$ but changes the (r,s,c)-tour, in fact by changing the order in which some of the letters appear in the arrival words. In order to check the result is well defined, we need to understand how to recover a boundary sequence from a family of arrival words, and indeed have a criterion for when it is possible to do so if the family of arrival words does not a priori arise from a partition.

If v is minimal such that all boxes in the partition have top right corner on or below the line $sx+ry=v$, then we have that for all $w>v$, any arrival at a vertex $(w,[i])$ must be on a co-ordinate axis. So,

$$(w,[i])_a = \begin{cases} SE & \text{if } r \mid w, s \mid w, \frac{w}{s} \equiv \frac{-w}{r} \equiv i \pmod{c} \\ E & \text{if } s \mid w, c \mid \left(\frac{w}{s}-i\right) \text{ and either } r \nmid w \text{ or } c \nmid \left(\frac{-w}{r}-i\right) \\ S & \text{if } r \mid w, c \mid \left(\frac{-w}{r}-i\right) \text{ and either } s \nmid w \text{ or } c \nmid \left(\frac{w}{s}-i\right) \\ \emptyset & \text{otherwise.} \end{cases} \tag{6.1}$$

Moreover, v is uniquely specified as the largest vertex where the arrival word at $(v,[i])$ does *not* satisfy (6.1) for some $i \in \{1,2,\ldots,c\}$.

So, we can identify v and fill in the co-ordinate axes above or to the right of the line $sx + ry = v$ as part of the partition boundary. We may then fill in the remainder working backwards from the arrival words - we outline the method below by example.

Example 6.1. *Suppose we have $r = 3$, $s = 2$, $c = 2$, and the set of arrival words specified below*

(20,[1])	*S*	*(22,[0])*	*E*	*(23,[0])*	*S*	*(23,[1])*	*S*
(24,[0])	*S*	*(24,[1])*	*E*	*(25,[0])*	*E*	*(25,[1])*	*S*
(26,[0])	*ES*	*(26,[1])*	*SSE*	*(27,[0])*	*E*	*(27,[1])*	*SES*
(28,[0])	*EE*	*(28,[1])*	*E*	*(29,[0])*	*EE*	*(29,[1])*	*E*
(30,[0])	*SE*	*(30,[1])*	*E*				

Empty for all other vertices $(w,[j])$ with $w < 30$. Then for $w > 30$,

$$(w,[j])_a = \begin{cases} SE & \textit{if } 6 \mid w, \frac{w}{2} \equiv \frac{-w}{3} \equiv j \pmod 2 \\ E & \textit{if } 2 \mid w, 2 \mid \left(\frac{w}{2} - j\right) \textit{ and either } 3 \nmid w \textit{ or } 2 \nmid \left(\frac{-w}{3} - j\right) \\ S & \textit{if } 3 \mid w, 2 \mid \left(\frac{-w}{3} - j\right) \textit{ and either } 3 \nmid w \textit{ or } 2 \nmid \left(\frac{w}{2} - j\right) \\ \emptyset & \textit{otherwise.} \end{cases}$$

Looking at the vertex $(30,[0])$, with $w = 30$ and $j = 0$ we have $\frac{w}{2} \not\equiv j \pmod 2$, so 30 is maximal such that there is vertex $(30,[i])$ that does not satisfy (6.1) for some i. So, $v = 30$ and we draw a ray along the positive x-axis beginning at $(15,0)$, and a ray along the positive y-axis beginning at $(0,10)$. It then remains to fill in the boundary between the points $\left(\left\lceil \frac{v}{s} \right\rceil, 0\right)$, and $\left(0, \left\lceil \frac{v}{r} \right\rceil\right)$. To do this, we look first at the arrival word at $\left(s\left\lceil \frac{v}{s} \right\rceil, \left[\left\lceil \frac{v}{s} \right\rceil\right]\right)$, $(30,[1])$ in our example, corresponding to the point on the x-axis at which the ray begins. The last letter of this word tells us what kind of edge we should add to the boundary to arrive at $\left(\left\lceil \frac{v}{s} \right\rceil, 0\right)$, in this case an E, so we add an edge from $(14,0)$ to $(15,0)$ and delete the last E from $(30,[1])_a$. The same logic allows the rest of the boundary to be filled out edge by edge, as in Figure 6.1.

6.2 The first arrival tree

Next we lay out a criterion for a family of arrival words to arise from a partition. We already know that any family of arrival words arising from a partition must satisfy (6.1) for $w > v$

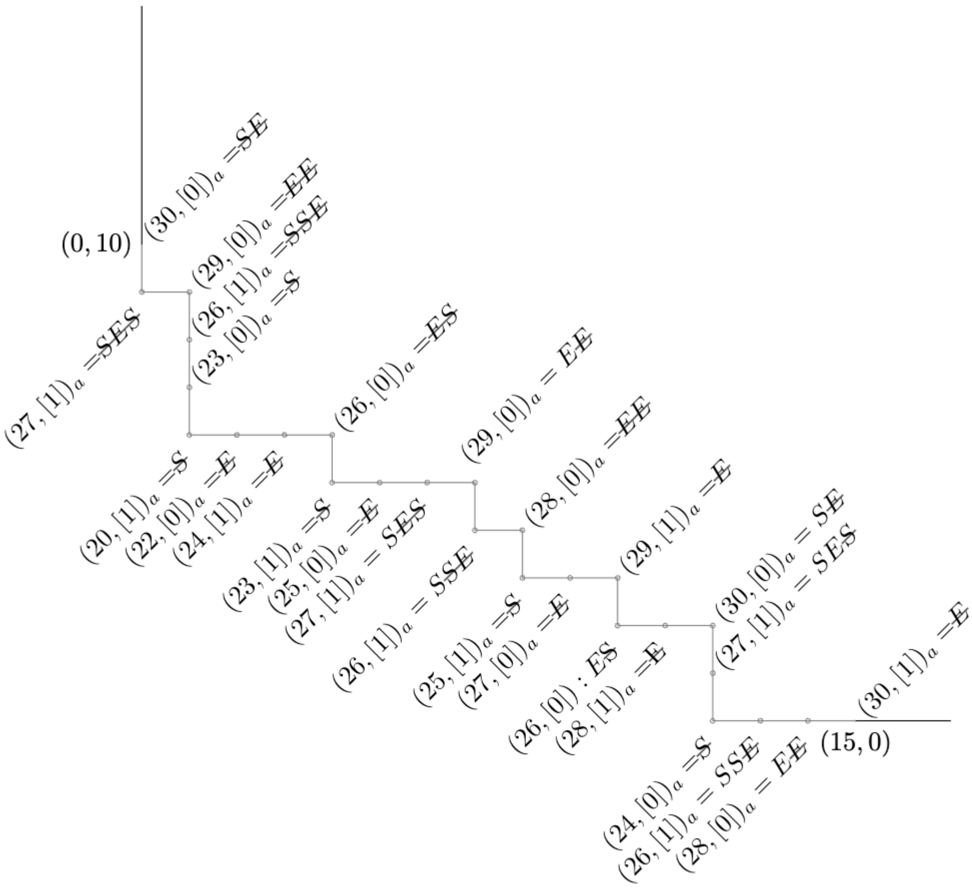


Fig. 6.1 the arrival words in Example 6.1 give the partition $(12, 12, 10, 8, 7, 4, 1, 1, 1)$.

large enough. We now give a criterion on the arrival words at the remaining vertices with $w \leq v$ to arise from a partition.

Definition 6.2 (First arrival graph)**.** *Let λ be a partition and let $M_{r,s,c}(\lambda) = M$. Let V and E be the vertex set and edge set of M respectively. Let the (r,s,c)-tour of M corresponding to λ have arrival word $(v,[i])_a$ at each vertex $(v,[i]) \in V$. Suppose there is another family of arrival words*

$$S = \{(v,[i])'_a : (v,[i]) \in V\}, \tag{6.2}$$

such that for each $(v,[i])$, $(v,[i])'_a$ is a permutation of $(v,[i])_a$. Denote the first letter of the arrival word $(v,[i])'_a$ by $(v,[i])'_1$, and let the first arrival edge $e_1(v,[i])$ with respect to S be any edge e with $t(e) = (v,[i])$ and $d(e) = (v,[i])'_1$. Let T_S be the subgraph of M with vertex set V and directed edge set

$$E(T_S) = \{e_1(v,[i])'_a : (v,[i]) \in M\}. \tag{6.3}$$

In this case we call T_S the first arrival graph with respect to S.

Definition 6.3 (The graphs $M^{\leq k}$ and $T^{\leq k}$)**.** *For an integer k, and an (r,s,c)-multigraph M, let $M^{\leq k}$ be the induced subgraph of M with vertex set*

$$V(M^{\leq k}) = \{(v,[i]) : (v,[i]) \in V(M) \text{ and } v \leq k\}.$$

For a family of arrival words S, let $T_S^{\leq k}$ be the induced subgraph of T_S with vertex set $V(M^{\leq k})$.

We require some preparation before proving Proposition 6.5, as we make use of [vAEdB51, Thm 5]. The proof is not hard, but could possibly be disruptive to the flow, so the interested reader is referred to [vAEdB51] for a full proof. The notion we will need is that of a T-graph.

Definition 6.4 (T-graph)**.** *A T-graph is a finite directed multigraph such that at each vertex, the number of edges arriving is the same as the number of edges departing.*

Theorem 5a of [vAEdB51] says that, given a complete circuit of a T-graph, starting and ending at a vertex v, the set of edges given by the last departures from any given vertex excluding v give a spanning tree of the T-graph rooted at v. Reversing the direction of all edges, equivalently, the first arrival graph arising from a complete circuit of a T-graph is a spanning tree. Conversely, [vAEdB51, Thm 5b] says that any spanning tree rooted at v gives rise to a complete circuit with last departures (or equivalently, first arrivals) agreeing with the edges of the spanning tree.

In order to apply these theorems to our situation, we need to separate M into a T-graph and a well understood complement, which is how we prove Proposition 6.5.

Proposition 6.5. *Let λ be a partition and let $M_{r,s,c}(\lambda) = M$. Suppose there is a family of arrival words $S = \{(v,[i])'_a : (v,[i]) \in V\}$ assigned to M. Let T_S be the first arrival graph with respect to S. Then there is a partition μ with an (r,s,c)-tour having arrival words S if and only if both of the following hold.*

1. *There exists some v such that for all $w > v$, and all j, $(w,[j])'_a$ satisfies* (6.1).
2. *T_S is a spanning tree of M.*

Proof. The first condition has already been shown to be necessary, so we prove that assuming the first condition holds, the second condition is equivalent to the existence of μ. Fix v such that for all $w \geq v$, (6.1) holds for both $(w,[j])_a$ and $(w,[j])'_a$. Let $k \geq v$ be such that $k = rsk_1$ for some integer k_1 with $c \mid k_1$. Let $M^{\leq k}$ and $M^{>k}$ be the induced subgraphs of M with vertex sets given by

$$V(M^{\leq k}) = \{(v,[i]) \in V(M) : v \leq k\}, \tag{6.4}$$

$$V(M^{>k}) = \{(k,[0])\} \cup \{(v,[i]) \in V(M) : v > k\}, \tag{6.5}$$

and let $T_S^{\leq k}$ and $T_S^{>k}$ be the induced subgraphs of T_S with vertex sets $V(M^{\leq k})$ and $V(M^{>k})$. Then $T_S^{>k}$ is a spanning tree of $M^{>k}$.

Let $x = (k_1 r, 0)$ and $y = (0, k_1 s)$ on the boundary. Then, the edges in $M^{>k}$ correspond to the rays along the axes starting at x and y. The (r,s,c)-tour corresponding to λ restricted to $M^{\leq k}$ is a complete circuit starting and finishing at $(k,[0])$, and each edge corresponds to an edge in the boundary of λ that occurs after the south edge arriving at y and occurs before the east edge departing from x. So, $M^{\leq k}$ contains $|x|$ south edges and $|y|$ east edges. Therefore, there is a partition μ with arrival words S if and only if there is a complete circuit of $M^{\leq k}$ such that the arrival words agree with S.

Assume that a complete circuit of $M^{\leq k}$ with arrival words as given in S exists. The (r,s,c)-tour of M corresponding to λ consists of a circuit of $M^{>k}$ and $M^{\leq k}$, so the in-degree of any vertex v of $M^{\leq k}$ is equal to the out-degree of v in $M^{\leq k}$ and $M^{\leq k}$ is connected. In particular, $M^{\leq k}$ is a T-graph. So, [vAEdB51, Thm 5a] implies that $T_S^{\leq k}$ is a tree rooted at $(k,[0])$. Therefore, T_S is a spanning tree of M.

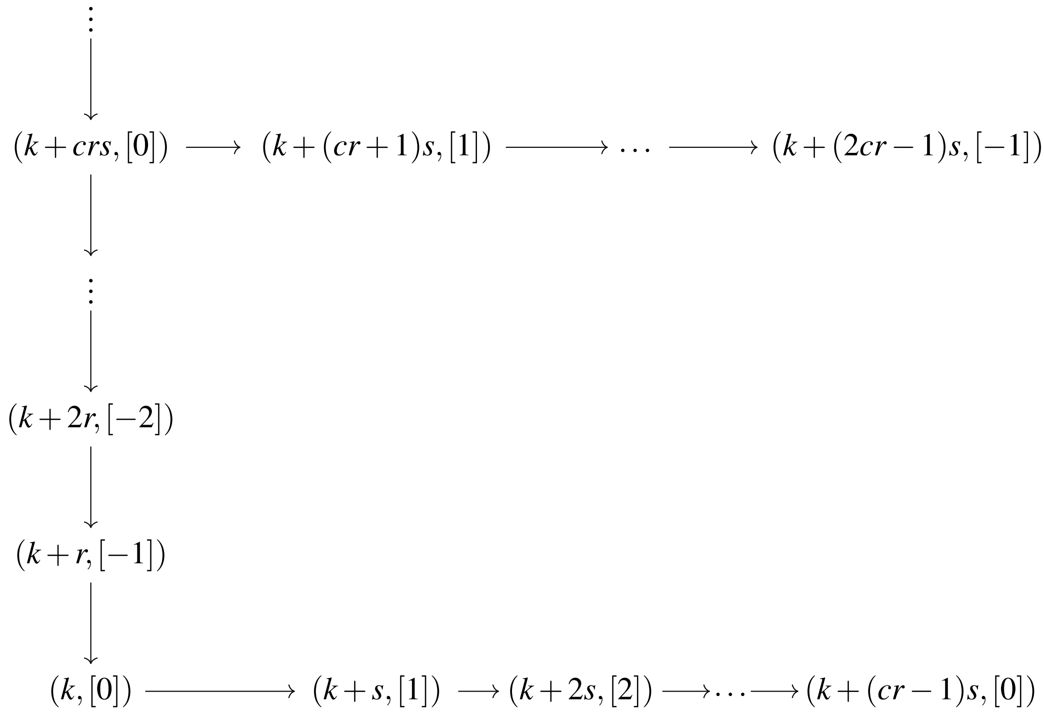


Fig. 6.2 $T_S^{>k}$ in the case $(c,r+s)=1$.

Now assume that T_S is a spanning tree of M. Then, $T_S^{\leq k}$ is a tree rooted at $(k,[0])$ and [vAEdB51, Thm 5b] implies that there is a complete circuit of $M^{\leq k}$ with arrival words agreeing with S. □

From now on, let $\lambda <_{r,s,c} \lambda_{r,s,k}$ where $k = rsk_1$ and $c \mid k_1$. Let $M_{r,s,c}(\lambda) = M$, let S be the family of arrival words corresponding to λ. We will now refer to T_S as the first arrival tree.

When we have a drawing of $M_{r,s,c}(\lambda)$ on the cylinder defined in Proposition 5.3, and a directed path p from $(v,[i])$ to $(w,[j])$, we define the *winding number of p* to be the number of times strictly after leaving $(v,[i])$ and before or on arriving at $(w,[j])$ that p arrives at a vertex on the upper boundary strip. We will be particularly interested in the case where $(v,[i]) = (k,[-k_1 s])$ and p is the unique path in the first arrival tree from $(k,[-k_1 s])$ to $(w,[j])$.

Definition 6.6 (Switch, eastern, southern)**.** *Given a partition λ with $\lambda <_{r,s,c} \lambda_{r,s,k}$, and (r,s,c)-multigraph M, let T denote the first arrival tree of M corresponding to λ. Let $(v,[i]) \in V(M)$ have $v \leq k$ and $(v,[i]) \neq (k,[0])$. Then $(v,[i])$ is a switch if $(v+r,[i-1])$ and $(v-s,[i-1])$*[1] *are both vertices of M, and the distances in T from the vertex $(k,[0])$ to $(v+r,[i-1])$,*

[1] these are the two equivalence classes that, if they are vertices of M, could form a tail of an edge to $(v,[i])$

$(v-s,[i-1])$ *are equal. Now drop the condition that* $v \le k$ *and* $(v,[i]) \neq (k,[0])$. *If* $(v,[i])$ *is not a switch and the first letter in the arrival word is E, say* $(v,[i])$ *is eastern, and let* Ea *be the set of all eastern vertices* $(v,[i])$ *with* $v \le k$. *If* $(v,[i])$ *is not a switch and the first letter in the arrival word is S, say* $(v,[i])$ *is southern, and let* So *be the set of all southern vertices* $(v,[i])$ *with* $v \le k$.

Example 6.7. *For* $\mu = (12,12,10,8,7,4,1,1,1)$, *the first arrival tree of* $M_{3,2,2}(\mu)$ *is as in Figure 6.3.*

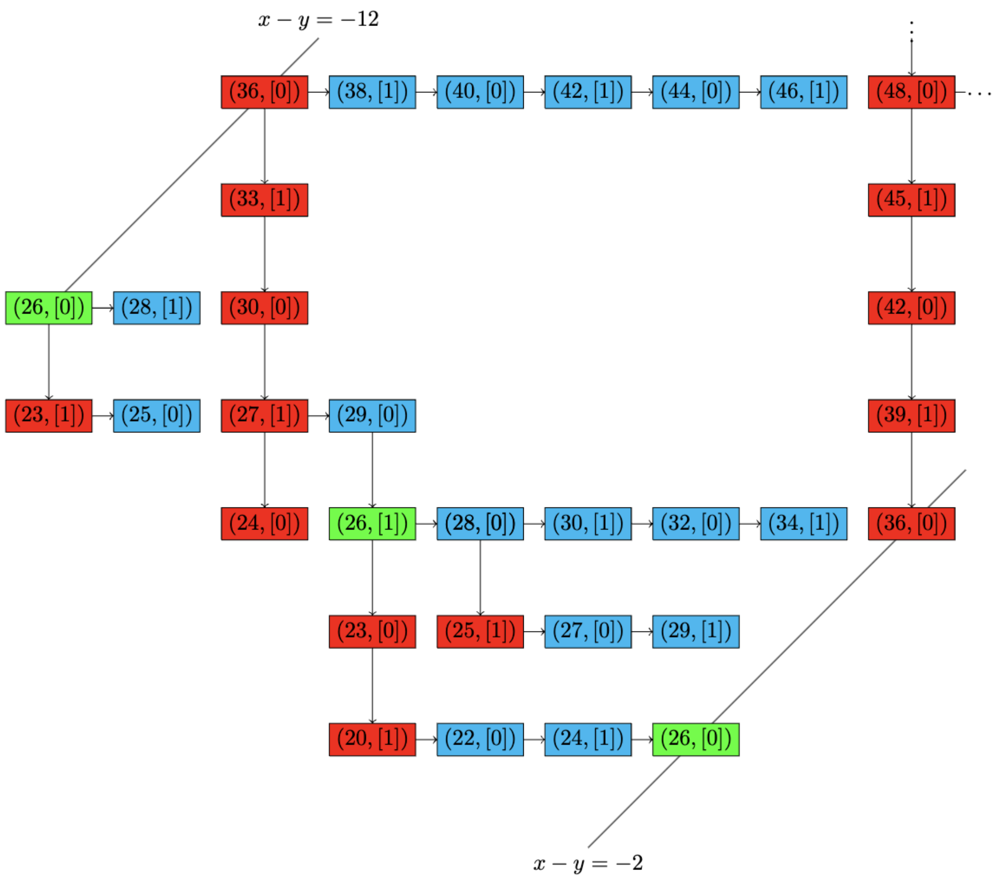


Fig. 6.3 The first arrival tree in $M_{3,2,2}(12,12,10,8,7,4,1,1,1)$ with the eastern vertices coloured blue, the southern vertices coloured red, and the switches coloured green.

The paths in the first arrival tree from $(36,[0])$ *to* $(26,[0])$, $(23,[1])$, $(28,[1])$ *and* $(25,[0])$ *have winding number* 1, *whilst the other vertices* $(v,[i])$ *for which there is a path in the first arrival tree from* $(36,[0])$ *to* $(v,[i])$ *have winding number* 0. *The switches are coloured green,* *the southern vertices red, and the eastern vertices blue (compare with Figure 5.2 to verify the colouring).*

6.3 Definition of $I_{r,s,c}$

Given a partition λ with $\lambda \leq_{r,s,c} \lambda_{r,s,k}$, with multigraph M and first arrival tree T, we define the partition $I_{r,s,c}(\lambda)$ as follows. The multigraph of $I_{r,s,c}(\lambda)$ is also given by M.

Now, we obtain the (r,s,c)-tour of $I_{r,s,c}(\lambda)$ by, at each switch, reversing the arrival word, and at each vertex that is not a switch, fixing the first letter of the arrival word and reversing the rest of the arrival word.

To see that $I_{r,s,c}(\lambda)$ is well defined, we need to check that taking the first arrival at each vertex of M gives a spanning tree. We do this by checking that in T, the move of deleting an east (respectively south) edge arriving at a switch $(v,[i])$ and adding a new south (respectively east) edge arriving at $(v,[i])$ gives another spanning tree T'. There are still edges arriving at every vertex we had edges arriving at before, but now the edge arriving at $(v,[i])$ might be departing from a different vertex. So, it suffices to check that $(v,[i])$ is still connected to each of $(v-s,[i-1])$ and $(v+r,[i-1])$, and that we have not introduced a cycle by adding the new edge. For the former, it suffices to check $(v-s,[i-1])$ and $(v+r,[i-1])$ are still connected to each other. In T, $(v+r,[i-1])$ and $(v-s,[i-1])$ are both connected to $(k,[0])$ by paths. Moreover, the distance in T to $(k,[0])$ strictly decreases with each step along the path we take, so $(v,[i])$ is not a vertex on either of these paths. So, both of these paths exist T', and $(v+r,[i-1])$ and $(v-s,[i-1])$ are connected to one another. To see that the new edge does not introduce a cycle, observe that if we had introduced a cycle, we would now have two distinct paths from $(k,[0])$ to $(v,[i])$. Since the only edge into $(v,[i])$ is from its new neighbour, we must have had two distinct paths from $(k,[0])$ to the new neighbour in T originally. But then we had a cycle in T originally, a contradiction.

Hence, we may permute the letters in any arrival word at any vertex and the result will still correspond to a partition as long as we do not change the first letter in the arrival word at a vertex that is not a switch. Since we defined $I_{r,s,c}(\lambda)$ to fix the first letter in the arrival word at any vertex that is not a switch, $I_{r,s,c}(\lambda)$ is well defined. Moreover, we can recover λ from $I_{r,s,c}(\lambda)$ by doing the same operation again, as each operation is self-inverse and preserves switches.

Since $I_{r,s,c}$ does not change $M_{r,s,c}$, we can apply Proposition 5.7 and Corollary 5.25 respectively to obtain

$$|I_{r,s,c}(\lambda)| = |\lambda|, \tag{6.6}$$

and

$$\text{core}_c(I_{r,s,c}(\lambda)) = \text{core}_c(\lambda). \tag{6.7}$$

Moreover, the map sending λ to $I_{r,s,c}(\lambda)$ is an involution - it is immediate from the definition that a vertex is a switch after this reassignment if and only if it were a switch before the reassignment.

Chapter 7

Further statistics determined by $M_{r,s,c}$

This chapter checks that $I_{r,s,c}$ satisfies hypotheses 2–4 in Proposition 4.6.

First, we use the method introduced in Chapter 5 to prove that $M_{r,s,c}$ determines $\mathrm{mid}_{x,c}$ and $\mathrm{crit}^+_{x,c} + \mathrm{crit}^-_{x,c}$. Then we check that $I_{r,s,c}$ exchanges $\mathrm{crit}^+_{x,c}$ and $\mathrm{crit}^-_{x,c}$, concluding the proof of Theorem 4.2. In the language of Proposition 5.24, Propositions 7.3 and 7.5 calculate $g(M^+, M)$ for $f(\lambda) = \mathrm{mid}_{x,c}(\lambda)$ and $f(\lambda) = \mathrm{crit}^+_{x,c}(\lambda) + \mathrm{crit}^-_{x,c}(\lambda)$ respectively.

7.1 $M_{r,s,c}$ determines $\mathrm{mid}_{x,c}$

Notation 7.1. *Let λ, M, M^+, (x,y), $(l,[i])$, s_1, s_2 s'_1 and s'_2 be as in the proof of Proposition 5.23. For any edge e in the boundary of λ, write $\mathrm{E_{in}}^{\to e}(w,[j])$ for the number of Es in the arrival word at a vertex $(w,[j])$ that correspond to east edges in the boundary of λ that occur before e. Define $\mathrm{S_{in}}^{\to e}(w,[j])$ analogously for the number of Ss. Write $\mathrm{E_{in}}^{e\to}(w,[j])$ for the number of Es in the arrival word at a vertex $(w,[j])$ that correspond to east edges in the boundary of λ that occur after e. Define $\mathrm{S_{in}}^{e\to}(w,[j])$ analogously for the number of Ss. Analogously define $\mathrm{S_{out}}^{\to e}(w,[j])$, $\mathrm{E_{out}}^{\to e}(w,[j])$, $\mathrm{S_{out}}^{\to e}(w,[j])$, and $\mathrm{E_{out}}^{\to e}(w,[j])$ for the departure words. We will use this notation with $e = s_1$ or $e = s_2$. Finally, write $\mathrm{E}^+_{\mathrm{in}}(w,[j])$ for the number of Es in the arrival word at $(w,[j])$ in M^+ and define analogously $\mathrm{S}^+_{\mathrm{in}}, \mathrm{E}^+_{\mathrm{out}}$ and $\mathrm{S}^+_{\mathrm{out}}$.*

We work in the ring R of functions $V(M) \to \mathbb{Z}$. Practically, the only consequence of this is that we write $fg(v,[i])$ for the pointwise product $f(v,[i])g(v,[i])$ and $(f+g)(v,[i])$ for $f(v,[i]) + g(v,[i])$. There should be no confusion between composition and product of functions as functions from $V(M)$ to $\mathbb{Z}$ are not composable.

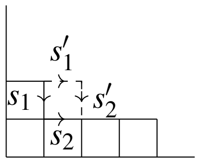


Fig. 7.1 the partition λ with $(l,[i]) = (4,[0])$ and s_1, s_2, s_1' and s_2' labelled

Example 7.2. *Let* $\lambda = (4,1)$, $r = 3$, $s = 1$ *and* $c = 2$. *Then* $\min_{(x,y)\in b(\lambda)}(3y+x) = 4$ *achieved at* $(4,0)$ *and* $(1,1)$. *Since* $4-0 \equiv 1-1 \pmod 2$ *and* $1-1 < 4-0$, *we have* $(1,1) <_{3,1,2} (4,0)$ *so* $(4,2)$ *is the only* $(3,1,2)$*-successor of* λ. *So,* $(l,[i]) = (4,[0])$. *As an example of the use of Notation 7.1,* $(\mathrm{E_{in}}^{\to s_1}\mathrm{S_{out}}^{s_1\to} + \mathrm{E}^{+}_{\mathrm{out}})(7,[1]) = 1\times 1 + 1 = 2$.

Proposition 7.3. *Let* λ *be a partition with* $M_{r,s,c}(\lambda) = M$. *Let* M^+ *be a successor of* M *that changes from* $(l,[i])$. *If* $\lambda^+ >'_{r,s,c} \lambda$ *and* $M^+ = M_{r,s,c}(\lambda^+)$,

$$\mathrm{mid}_{x,c}(\lambda^+) - \mathrm{mid}_{x,c}(\lambda) = \sum_{w=l+1}^{l+s+r-1} (\mathrm{E_{out}} - \mathrm{E_{in}})\,(w,[i]), \tag{7.1}$$

where $x = \frac{r}{s}$.

Proof. By Proposition 5.23, the Young diagram of λ^+ is obtained from that of λ by adding a box with bottom corner (x_1,y_1) where $ry_1 + sx_1 = l$ and $[x_1 - y_1] = [i]$.

Proposition 5.1 gives a formula for $\mathrm{mid}_{x,c}(\lambda)$: it is the number of pairs of edges e_1, e_2 in the boundary sequence such that e_1 is an east edge arriving at a point (x,y) satisfying $sx + ry = v$ and $[x-y] = [j]$ for some j, and e_2 is a south edge occurring after e_1 arriving at a point (x',y') satisfying $ry' + sx' = w$ and $[x'-y'] = [j]$, where w and v satisfy $-s-r < w-v < 0$.

We account for the change in the number of such pairs when changing s_1 to s_1' and s_2 to s_2' below: the only changes to $\mathrm{mid}_{x,c}$ will be when $e_1 \in \{s_2, s_1'\}$ or $e_2 \in \{s_1, s_2'\}$.

By adding s_1' we gain the number of south edges after s_1, arriving at points (x,y) on lines $sx+ry = w$, such that $-s-r < w-(l+r+s) < 0$ and $[x-y] = [i]$. By deleting s_1 we lose the number of east edges occurring before s_1 arriving at points (x,y) on lines $sx+ry = v$ such that $-s-r < l-v < 0$ and $[x-y] = [i]$ So, the contribution to $\mathrm{mid}_{x,c}(\lambda^+) - \mathrm{mid}_{x,c}(\lambda)$ from switching s_1 to s_1' is S_1 where

$$S_1 = \sum_{w=l+1}^{l+r+s-1} (\mathrm{S_{in}}^{s_1\to} - \mathrm{E_{in}}^{\to s_1})\,(w,[i]). \tag{7.2}$$

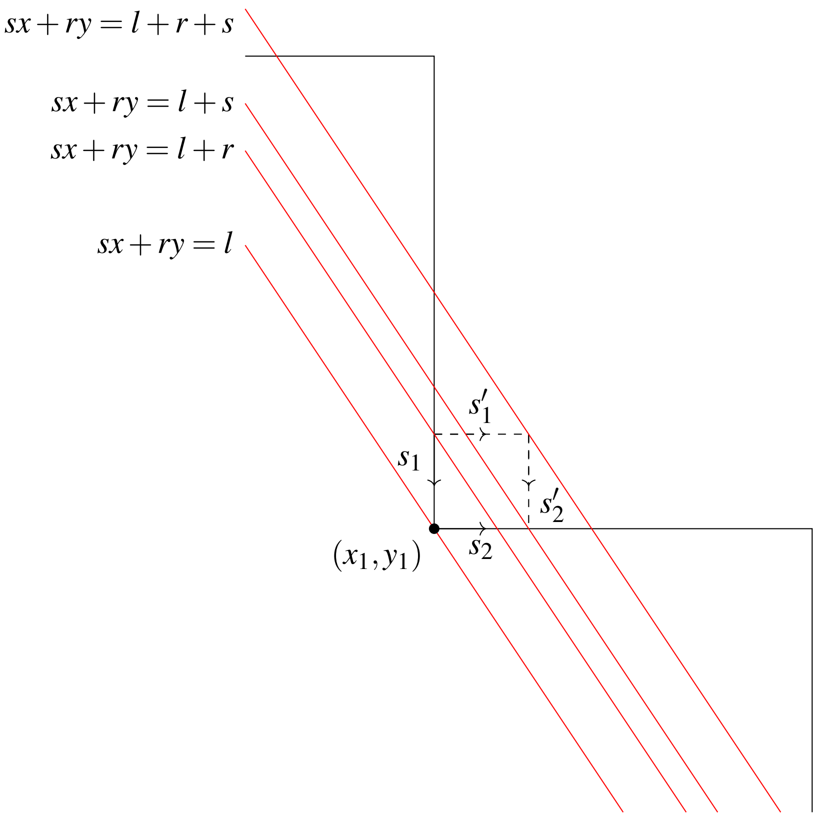


Fig. 7.2 the diagrams of λ and λ^+.

By adding s'_2 we gain the number of east edges before s_2, arriving at points (x,y) on lines $sx+ry=v$, such that $-s-r<l+s-v<0$, and $x-y\equiv i+1 \pmod{c}$. By deleting s_2 we lose the number of south edges occurring after s_2 arriving at points (x,y) on lines $sx+ry=w$ such that $-s-r<w-(l+s)<0$ and $[x-y]=[i+1]$. So, the contribution to $\text{mid}_{x,c}(\lambda^+)-\text{mid}_{x,c}(\lambda)$ from switching s_2 to s'_2 is S_2 where

$$S_2=\sum_{v=l+s+1}^{l+r+2s-1}\text{E}_{\text{in}}{}^{\to s_2}(v,[i+1])-\sum_{v=l-r+1}^{l+s-1}\text{S}_{\text{in}}{}^{s_2\to}(v,[i+1]). \tag{7.3}$$

So,

$$\text{mid}_{x,c}(\lambda^+)-\text{mid}_{x,c}(\lambda)=S_1+S_2. \tag{7.4}$$

Now, note that an east edge into $(v,[i+1])$ is also an east edge out of $(v-s,[i])$, and a south edge into $(w,[i+1])$ is also a south edge out of $(w+r,[i])$. Applying this reasoning to (7.3),

$$S_2 = \sum_{w=l+1}^{l+r+s-1} (\text{E}_{\text{out}}{}^{\to s_2} - \text{S}_{\text{out}}{}^{s_2\to})\,(w,[i]). \tag{7.5}$$

Substituting (7.2) and (7.5) into (7.4),

$$\text{mid}_{x,c}(\lambda^+) - \text{mid}_{x,c}(\lambda) = \sum_{w=l+1}^{l+s+r-1} (\text{S}_{\text{in}}{}^{s_1\to} - \text{E}_{\text{in}}{}^{\to s_1} + \text{E}_{\text{out}}{}^{\to s_2} - \text{S}_{\text{out}}{}^{s_2\to})\,(w,[i]). \tag{7.6}$$

Since s_2 is an east edge occuring immediately after s_1, $\text{S}_{\text{out}}{}^{s_2\to} = \text{S}_{\text{out}}{}^{s_1\to}$, and since s_1 is a south edge immediately preceding s_2, $\text{E}_{\text{out}}{}^{\to s_2} = \text{E}_{\text{out}}{}^{\to s_1}$. So,

$$\text{mid}_{x,c}(\lambda^+) - \text{mid}_{x,c}(\lambda) = \sum_{w=l+1}^{l+s+r-1} (\text{S}_{\text{in}}{}^{s_1\to} - \text{E}_{\text{in}}{}^{\to s_1} + \text{E}_{\text{out}}{}^{\to s_1} - \text{S}_{\text{out}}{}^{s_1\to})\,(w,[i]). \tag{7.7}$$

Now, note that at any vertex $(v,[j])$ except $(l,[i])$, we have that

$$(\text{S}_{\text{in}}{}^{s_1\to} + \text{E}_{\text{in}}{}^{s_1\to})\,(v,[j]) = (\text{S}_{\text{out}}{}^{s_1\to} + \text{E}_{\text{out}}{}^{s_1\to})\,(v,[j]), \tag{7.8}$$

because after s_1 we depart every vertex after we arrive at it, the left hand side counting arrivals at the vertex after s_1 and the right side counting departures. Rearranging gives

$$(\text{S}_{\text{in}}{}^{s_1\to} - \text{S}_{\text{out}}{}^{s_1\to})\,(v,[j]) = (\text{E}_{\text{out}}{}^{s_1\to} - \text{E}_{\text{in}}{}^{s_1\to})\,(v,[j]). \tag{7.9}$$

Substituting (7.9) into (7.7),

$$\text{mid}_{x,c}(\lambda^+) - \text{mid}_{x,c}(\lambda) = \sum_{w=l+1}^{l+s+r-1} (\text{E}_{\text{out}}{}^{s_1\to} - \text{E}_{\text{in}}{}^{s_1\to} - \text{E}_{\text{in}}{}^{\to s_1} + \text{E}_{\text{out}}{}^{\to s_1})\,(w,[i]). \tag{7.10}$$

Since s_1 is a south edge, $\text{E}_{\text{in}}{}^{s_1\to} + \text{E}_{\text{in}}{}^{\to s_1} = \text{E}_{\text{in}}$ and $\text{E}_{\text{out}}{}^{s_1\to} + \text{E}_{\text{out}}{}^{\to s_1} = \text{E}_{\text{out}}$, so

$$\text{mid}_{x,c}(\lambda^+) - \text{mid}_{x,c}(\lambda) = \sum_{w=l+1}^{l+s+r-1} (\text{E}_{\text{out}} - \text{E}_{\text{in}})\,(w,[i]). \tag{7.11}$$

□

Corollary 7.4. *If* λ *and* μ *are partitions with* $M_{r,s,c}(\lambda) = M_{r,s,c}(\mu)$, *then* $\text{mid}_{x,c}(\lambda) = \text{mid}_{x,c}(\mu)$.

Proof. Apply Proposition 5.24 with

$$g(M^+, M) = \sum_{w=l+1}^{l+s+r-1} (\mathrm{E_{out}} - \mathrm{E_{in}})(w, [i]), \tag{7.12}$$

where M^+ is the successor of M that changes from $(l, [i])$. □

7.2 $M_{r,s,c}$ determines $\mathrm{crit}^+_{x,c} + \mathrm{crit}^-_{x,c}$

Proposition 7.5. *Let λ be a partition with $M_{r,s,c}(\lambda) = M$ and let M^+ be a successor of M that changes from $(l, [i])$. Then, if λ^+ is the successor of λ with multigraph M^+,*

$$\mathrm{crit}^+_{x,c}(\lambda^+) + \mathrm{crit}^-_{x,c}(\lambda^+) - \mathrm{crit}^+_{x,c}(\lambda) - \mathrm{crit}^-_{x,c}(\lambda) \tag{7.13}$$

is equal to

$$\mathrm{S_{in}}(l, [i]) - 1 + (\mathrm{S_{in}} - \mathrm{S_{out}})(l + r + s, [i]), \tag{7.14}$$

where $x = \frac{r}{s}$.

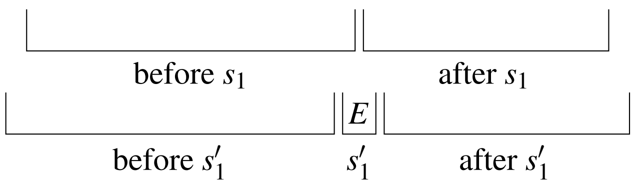
Proof. First, we compute $\mathrm{crit}^+_{x,c}(\lambda^+) - \mathrm{crit}^+_{x,c}(\lambda)$. Corollary 5.8 implies that

$$\mathrm{crit}^+_{x,c}(\lambda^+) - \mathrm{crit}^+_{x,c}(\lambda) = \sum_{v \in M_{r,s,c}(\lambda^+)} \mathrm{inv}(v_a) - \sum_{v \in M_{r,s,c}(\lambda)} \mathrm{inv}(v_a) \tag{7.15}$$

We keep the notation of the previous proposition and reference Figure 7.2 throughout. The only nonzero terms in the difference (7.15) come from $v \in \{(l, [i]), (l + r + s, [i]), (l + s, [i+1])\}$. We work case-by-case through these vertices.

- We delete the first arrival at $(l, [i])$, corresponding to deleting s_1. All arrivals at $(l, [i])$ are Ss by Corollary 5.22, so this does not affect $\mathrm{inv}(l, [i])_a$.

- We add an E to the arrival word at $(l + r + s, [i])$, corresponding to adding s_1'.



This E is the first letter in an inversion with second letter any S occuring after s_1', so $(l+r+s,[i])$ contributes $\mathrm{S_{in}}^{s_1\rightarrow}(l+r+s,[i])$ to (7.15).

- We replace the first E in the arrival word at $(l+s,[i+1])$ (corresponding to s_2) with an S (corresponding to s_2'). Therefore, we lose all inversions with the replaced E edge as their first letter passing from λ to λ^+. There are $\mathrm{S_{in}}^{s_2\rightarrow}(l+s,[i+1])$ such inversions.

$SSSSSSSS\dots SSSS$	E	
before s_2	s_2	after s_2
$SSSSSSSS\dots SSSS$	S	
before s_2'	s_2'	after s_2'

We gain no inversions from the new S edge, because s_2 was the first east departure from $(l,[i])$ in the tour corresponding to λ. So, $(l+s,[i+1])$ contributes $-\mathrm{S_{in}}^{s_2\rightarrow}(l+s,[i+1])$ to (7.15).

So,

$$\mathrm{crit}^+_{x,c}(\lambda^+) - \mathrm{crit}^+_{x,c}(\lambda) = \mathrm{S_{in}}^{s_1\rightarrow}(l+r+s,[i]) - \mathrm{S_{in}}^{s_2\rightarrow}(l+s,[i+1]). \tag{7.16}$$

A south arrival before (respectively after) s_2 at $(l+s,[i+1])$ is a south departure before (respectively after) s_2 from $(l+r+s,[i])$. Combining this logic with (7.16),

$$\mathrm{crit}^+_{x,c}(\lambda^+) - \mathrm{crit}^+_{x,c}(\lambda) = (\mathrm{S_{in}}^{s_1\rightarrow} - \mathrm{S_{out}}^{s_2\rightarrow})\,(l+r+s,[i]). \tag{7.17}$$

We now analyse

$$\mathrm{crit}^-_{x,c}(\lambda^+) - \mathrm{crit}^-_{x,c}(\lambda) = \sum_{v\in M_{r,s,c}(\lambda^+)} \mathrm{inv}(v_d) - \sum_{v\in M_{r,s,c}(\lambda)} \mathrm{inv}(v_d). \tag{7.18}$$

The departure words at every vertex except for $(l,[i])$, $(l+r+s,[i])$, and $(l+r,[i-1])$ are unchanged so the only nonzero terms in (7.18) come from $v \in \{(l,[i]),(l+r+s,[i]),(l+r,[i-1])\}$. An analogous argument to the above shows that the contribution of $(l+r,[i-1])$ to (7.18) is $(\mathrm{S_{out}}^{s_1\rightarrow} - \mathrm{E_{out}}^{\rightarrow s_1})\,(l+r,[i-1])$, the contribution of $(l+r+s,[i])$ is $\mathrm{E_{out}}^{\rightarrow s_2}(l+r+s,[i])$, and $(l,[i])$ does not contribute. So,

$$\mathrm{crit}^-_{x,c}(\lambda^+) - \mathrm{crit}^-_{x,c}(\lambda) = (\mathrm{S_{out}}^{s_1\rightarrow} - \mathrm{E_{out}}^{\rightarrow s_1})\,(l+r,[i-1]) + \mathrm{E_{out}}^{\rightarrow s_2}(l+r+s,[i]). \tag{7.19}$$

An east departure from $(l+r,[i-1])$ is an east arrival at $(l+r+s,[i])$, so

$$\text{crit}^-_{x,c}(\lambda^+) - \text{crit}^-_{x,c}(\lambda) = \text{S}_{\text{in}}{}^{s_1\to}(l,[i]) - (\text{E}_{\text{in}}{}^{\to s_1} - \text{E}_{\text{out}}{}^{\to s_2})(l+r+s,[i]). \tag{7.20}$$

Now, since s_1 is the first edge to arrive at $(l,[i])$,

$$\text{S}_{\text{in}}{}^{s_1\to}(l,[i]) = \text{S}_{\text{in}}(l,[i]) - 1. \tag{7.21}$$

Since s_1 does not arrive at $(l+r+s,[i])$, we leave $(l+r+s,[i])$ before s_1 the same number of times as we arrive before s_1. So,

$$(\text{E}_{\text{in}}{}^{\to s_1} + \text{S}_{\text{in}}{}^{\to s_1})(l+r+s,[i]) = (\text{E}_{\text{out}}{}^{\to s_2} + \text{S}_{\text{out}}{}^{\to s_2})(l+r+s,[i])). \tag{7.22}$$

Rearranging,

$$\text{E}_{\text{in}}{}^{\to s_1}(l+r+s,[i]) = (\text{E}_{\text{out}}{}^{\to s_2} + \text{S}_{\text{out}}{}^{\to s_2} - \text{S}_{\text{in}}{}^{\to s_1})(l+r+s,[i]). \tag{7.23}$$

Substituting (7.23) and (7.21) into (7.20),

$$\text{crit}^-_{x,c}(\lambda^+) - \text{crit}^-_{x,c}(\lambda) = \text{S}_{\text{in}}(l,[i]) - 1 + (\text{S}_{\text{in}}{}^{\to s_1} - \text{S}_{\text{out}}{}^{\to s_2})(l+r+s,[i]). \tag{7.24}$$

Since $(l+r+s,[i])$ is not an endpoint of s_1 or s_2,

$$(\text{S}_{\text{in}}{}^{\to s_1} + \text{S}_{\text{in}}{}^{s_1\to})(l+r+s,[i]) = \text{S}_{\text{in}}(l+r+s,[i]) \tag{7.25}$$

and

$$(\text{S}_{\text{out}}{}^{\to s_2} + \text{S}_{\text{out}}{}^{s_2\to})(l+r+s,[i]) = \text{S}_{\text{out}}(l+r+s,[i]). \tag{7.26}$$

Adding (7.24) and (7.17), and then applying (7.26) and (7.25) completes the proof. □

Corollary 7.6. *If λ and μ are partitions such that $M_{r,s,c}(\mu) = M_{r,s,c}(\lambda)$ then*

$$\text{crit}^+_{x,c}(\lambda) + \text{crit}^-_{x,c}(\lambda) = \text{crit}^+_{x,c}(\mu) + \text{crit}^-_{x,c}(\mu). \tag{7.27}$$

Proof. Apply Proposition 5.24 with

$$g(M^+,M) = \text{S}_{\text{in}}(l,[i]) - 1 + (\text{S}_{\text{in}} - \text{S}_{\text{out}})(l+r+s,[i]). \tag{7.28}$$

where the calculations $\mathrm{S_{in}}$ and $\mathrm{S_{out}}$ are done with respect to the multigraph M, and M^+ is the successor of M changing from $(l,[i])$. □

So, we know that $M_{r,s,c}(\lambda)$ determines the c-core of λ, $|\lambda|$, $\mathrm{mid}_{x,c}(\lambda)$ and $\mathrm{crit}^+_{x,c}(\lambda) + \mathrm{crit}^-_{x,c}(\lambda)$, and that any bijection preserving $M_{r,s,c}$ therefore satisfies hypotheses 1-3 of Proposition 4.6. It will be useful in our final remaining check, that $I_{r,s,c}$ satisfies the fourth criterion in Proposition 4.6, to have a formula for $\mathrm{crit}^+_{x,c}(\lambda) + \mathrm{crit}^-_{x,c}(\lambda)$ in terms of $M_{r,s,c}(\lambda)$. This is what Proposition 7.7 computes.

Proposition 7.7. *Let λ be a partition. If $k = rsk_1$ where $c \mid k_1$ and $\lambda <_{r,s,c} \lambda_{r,s,k}$, then*

$$(\mathrm{crit}^+_{x,c} + \mathrm{crit}^-_{x,c})(\lambda) = \sum_{\substack{(v,[j])\\ v \le k}} \mathrm{E_{in}S_{in}}(v,[j]) - \left\lfloor \frac{k_1(s+r)}{\mathrm{lcm}(c,s+r)} \right\rfloor. \tag{7.29}$$

Proof. First, we prove that (7.29) holds when $\lambda = \lambda_{r,s,k}$.

We will show that for all boxes $\square \in \lambda_{r,s,k}$, $-s < sa(\square) - rl(\square) < r$, and hence that the left hand side of (7.29) is zero at $\lambda_{r,s,k}$. We will then check that the right hand side of (7.29) is zero at $\lambda_{r,s,k}$.

The ith part of $\lambda_{r,s,k}$ corresponds to a row with top right corner (x_i, i) where x_i is maximal such that $sx_i + ri \le k$. So,

$$x_i = \left\lfloor \frac{k - ri}{s} \right\rfloor = \left\lfloor \frac{k_1 rs - ri}{s} \right\rfloor = k_1 r - \left\lceil \frac{ri}{s} \right\rceil. \tag{7.30}$$

Similarly, the number of parts of $\lambda_{r,s,k}$ of size at least j corresponds to a column with top right corner (j, y_j) where y_j is maximal such that $sj + ry_j \le k$, so

$$y_j = \left\lfloor \frac{k - sj}{r} \right\rfloor = \left\lfloor \frac{k_1 rs - sj}{r} \right\rfloor = k_1 s - \left\lceil \frac{sj}{r} \right\rceil. \tag{7.31}$$

Now, let $\square \in \lambda$ be a box with top right corner (i, j). Then, the arm of $\square$ is given by $x_i - j$ and the leg of $\square$ is given by $y_j - i$. So,

$$\begin{aligned} sa(\square) - rl(\square) &= s(x_i - j) - r(y_j - i) && (7.32)\\ &= k_1 rs - s\left\lceil \frac{ri}{s} \right\rceil - sj - k_1 rs + r\left\lceil \frac{sj}{r} \right\rceil + ri && (7.33)\\ &= \left(r\left\lceil \frac{sj}{r} \right\rceil - sj \right) - \left(s\left\lceil \frac{ri}{s} \right\rceil + ri \right). && (7.34) \end{aligned}$$

Now, consider the two bracketed quantities separately, setting $x = \left(r\left\lceil\frac{sj}{r}\right\rceil - sj\right)$ and $y = -\left(s\left\lceil\frac{ri}{s}\right\rceil + ri\right)$. For the first bracket we have that

$$r\left(\frac{sj}{r}\right) \le r\left\lceil\frac{sj}{r}\right\rceil < r\left(\frac{sj}{r}+1\right), \tag{7.35}$$

so

$$0 \le r\left\lceil\frac{sj}{r}\right\rceil - sj < r. \tag{7.36}$$

Similarly for the second bracket,

$$-s < ri - s\left\lceil\frac{ri}{s}\right\rceil \le 0. \tag{7.37}$$

So, $sa(\square) - rl(\square)$ can be written as $x+y$ for $x \in [0,r)$ and $y \in (-s,0]$ and therefore $-s < sa(\square) - rl(\square) < r$.

Therefore,

$$\mathrm{crit}^+_{x,c}(\lambda_{r,s,k}) + \mathrm{crit}^-_{x,c}(\lambda_{r,s,k}) = 0. \tag{7.38}$$

Next we evaluate the right hand side of (7.29) at $\lambda_{r,s,k}$. Proposition 5.17 tells us that for all vertices $(v,[i])$ such that $0 \le v < k$, the arrival word at $(v,[i])$ in $M_{r,s,c}(\lambda_{r,s,k})$ does not contain both an E and a S. So, for all such $(v,[i])$ we have $\mathrm{E_{in}S_{in}}(v,[i]) = 0$. So, the right hand side of (7.29) simplifies to

$$\sum_{i=0}^{c-1} \mathrm{E_{in}S_{in}}(k,[i]) - \left\lfloor \frac{k_1(s+r)}{\mathrm{lcm}(c,s+r)} \right\rfloor \tag{7.39}$$

Proposition 5.17 also tells us that $\mathrm{S_{in}}(k,[i]) = 0$ unless $[i] = [0]$, and that $\mathrm{S_{in}}(k,[0]) = 1$, so we can rewrite (7.39) as

$$\mathrm{E_{in}}(k,[0]) - \left\lfloor \frac{k_1(s+r)}{\mathrm{lcm}(c,s+r)} \right\rfloor. \tag{7.40}$$

So, it suffices to show that $\mathrm{E_{in}}(k,[0]) = \left\lfloor \frac{k_1(s+r)}{\mathrm{lcm}(c,s+r)} \right\rfloor$. The east edges in the boundary of $\lambda_{r,s,k}$ arriving at vertices $(k,[i])$ for some i correspond to points (x,y) with $x > 0$ and $y \ge 0$ such that $sx + ry = k$. These points have coordinates $\{(r, s(k_1-1)), (2r, s(k_1-2)), \ldots, ((k_1-1)r, s), (k_1 r, 0)\}$. Now, $\mathrm{E_{in}}(k,[0])$ counts the number of these points that also lie on a line

$x - y = i$ for $[i] = [0]$. The set of values of $x - y$ for this set of points is $\{r + s - k_1 s, 2(r + s) - k_1 s, \ldots, k_1(r+s) - k_1 s\}$. Letting $l(r+s) = \text{lcm}(c, r+s)$, the values of $x - y$ that give us the same congruence class as 0 when taken modulo c are of the form $ml(r+s) - k_1 s$ for some integer m. The number of values of this form in the given set is indeed $\left\lfloor \frac{k_1(s+r)}{\text{lcm}(c,s+r)} \right\rfloor$.

Now suppose $\lambda <_{r,s,c} \lambda_{r,s,k}$ is maximal with respect to $>_{r,s,c}$ such that the proposition is false. In particular, the proposition holds for any successor $\lambda^+ >'_{r,s,c} \lambda$. Let M^+ be a successor of M that changes from $(l, [i])$, and let λ^+ be the successor of λ with multigraph M^+. Then, $(\text{crit}^+_{x,c} + \text{crit}^-_{x,c})(\lambda^+) - (\text{crit}^+_{x,c} + \text{crit}^-_{x,c})(\lambda)$ can be written as Δ_1, where

$$\Delta_1 = \text{S}_{\text{in}}(l, [i]) - 1 + (\text{S}_{\text{in}} - \text{S}_{\text{out}})(l + r + s, [i]). \tag{7.41}$$

By assumption,

$$(\text{crit}^+_{x,c} + \text{crit}^-_{x,c})(\lambda^+) = \sum_{\substack{(v,[j]) \\ v \le k}} \text{E}^+_{\text{in}} \text{S}^+_{\text{in}}(v, [j]) - \left\lfloor \frac{k_1(s+r)}{\text{lcm}(c, s+r)} \right\rfloor \tag{7.42}$$

So, combining (7.41) and (7.42),

$$(\text{crit}^+_{x,c} + \text{crit}^-_{x,c})(\lambda) = \sum_{\substack{(v,[j]) \\ v \le k}} \text{E}^+_{\text{in}} \text{S}^+_{\text{in}}(v, [j]) - \left\lfloor \frac{k_1(s+r)}{\text{lcm}(c, s+r)} \right\rfloor - \Delta_1. \tag{7.43}$$

First, we note that a vertex $(v, [j])$ contributes the same to the sums

$$\sum_{\substack{(v,[j]) \\ v \le k}} \text{E}_{\text{in}} \text{S}_{\text{in}}(v, [j])$$

taken over the multigraphs M or M^+ unless $(v, [j]) \in \{(l, [i]), (l + r + s, [i], (l + s, [i+1])\}$. In fact, since there are no east edges into $(l, [i])$ in M or M^+, we only need consider terms with $(v, [j]) \in \{(l + r + s, [i], (l + s, [i+1])\}$. So,

$$(\text{crit}^+_{x,c} + \text{crit}^-_{x,c})(\lambda) = \sum_{\substack{(v,[j]) \\ v \le k}} \text{E}_{\text{in}} \text{S}_{\text{in}}(v, [j]) - \left\lfloor \frac{k_1(s+r)}{\text{lcm}(c, s+r)} \right\rfloor - \Delta_1 + \Delta_2 \tag{7.44}$$

where

$$\Delta_2 = (\mathrm{E}^+_{\mathrm{in}}\mathrm{S}^+_{\mathrm{in}} - \mathrm{E}_{\mathrm{in}}\mathrm{S}_{\mathrm{in}})(l+r+s,[i]) + (\mathrm{E}^+_{\mathrm{in}}\mathrm{S}^+_{\mathrm{in}} - \mathrm{E}_{\mathrm{in}}\mathrm{S}_{\mathrm{in}})(l+s,[i+1]). \tag{7.45}$$

Because M^+ changes from M at $(l,[i])$, $\mathrm{E}^+_{\mathrm{in}}(l+s,[i+1]) = \mathrm{E}_{\mathrm{in}}(l+s,[i+1]) - 1$, $\mathrm{S}^+_{\mathrm{in}}(l+s,[i+1]) = \mathrm{S}_{\mathrm{in}}(l+s,[i+1]) + 1$, $\mathrm{E}^+_{\mathrm{in}}(l+r+s,[i]) = \mathrm{E}_{\mathrm{in}}(l+r+s,[i]) + 1$ and $\mathrm{S}^+_{\mathrm{in}}(l+r+s,[i]) = \mathrm{S}_{\mathrm{in}}(l+r+s,[i])$ so (7.45) simplifies to

$$\Delta_2 = \mathrm{E}_{\mathrm{in}}(l+s,[i+1]) - \mathrm{S}_{\mathrm{in}}(l+s,[i+1]) + \mathrm{S}_{\mathrm{in}}(l+r+s,[i]) - 1. \tag{7.46}$$

A south arrival at $(l+s,[i+1])$ is the same as a south departure from $(l+r+s,[i])$, and an east arrival at $(l+s,[i+1])$ is the same as an east departure from $(l,[i])$, so

$$\Delta_2 = \mathrm{E}_{\mathrm{out}}(l,[i]) - (\mathrm{S}_{\mathrm{out}} - \mathrm{S}_{\mathrm{in}})(l+r+s,[i]) - 1. \tag{7.47}$$

By Corollary 5.22, all edges leaving $(l,[i])$ are east edges and all edges arriving are south edges. The same number of edges arrive and leave, so $\mathrm{E}_{\mathrm{out}}(l,[i]) = \mathrm{S}_{\mathrm{in}}(l,[i])$. So,

$$\Delta_2 = (\mathrm{S}_{\mathrm{in}} - \mathrm{S}_{\mathrm{out}})(l+r+s,[i]) + \mathrm{S}_{\mathrm{in}}(l,[i]) - 1 = \Delta_1. \tag{7.48}$$

Substituting (7.48) into (7.44) completes the proof. □

It remains to check that $\mathrm{crit}^+_{x,c}(I_{r,s,c}(\lambda)) = \mathrm{crit}^-_{x,c}(\lambda)$ and $\mathrm{crit}^-_{x,c}(I_{r,s,c}(\lambda)) = \mathrm{crit}^+_{x,c}(\lambda)$.

First, we make some make some straightforward but important observations about $M_{r,s,c}(\lambda)$ and winding numbers in Proposition 7.8. Then, we apply these to the first arrival tree to prove some formulae about distances between consecutive vertices in the (r,s,c)-tour with respect to the first arrival tree, depending on whether the vertex is eastern, southern, or a switch in Proposition 7.9. Finally, we apply these to proving $\mathrm{crit}^+_{x,c}(I_{r,s,c}(\lambda)) = \mathrm{crit}^-_{x,c}(\lambda)$ and $\mathrm{crit}^-_{x,c}(I_{r,s,c}(\lambda)) = \mathrm{crit}^+_{x,c}(\lambda)$ in Proposition 7.10.

Proposition 7.8. *Let $(v,[i])$ and $(w,[j])$ be two vertices of $M_{r,s,c}(\lambda)$, and let p_1 and p_2 be directed paths between $(v,[i])$ and $(w,[j])$. Suppose p_1 is given by the sequence of vertices $(v,[i]) = (v_0,[i_0]),\ldots,(v_{|p_1|},[i_0+|p_1|]) = (w,[j])$. Then,*

1. *$|p_1| - |p_2|$ is divisible by $\mathrm{lcm}(c,r+s)$.*
2. *Let $(v,[i])$ be m lattice steps below the upper boundary of the cylinder, and let $|p_1| = q\,\mathrm{lcm}(c,r+s)+u$ where $-m < u \le \mathrm{lcm}(c,r+s) - m$. The winding number of p_1 is q.*

Proof. The first point follows from Proposition 5.3: p_1 and p_2 are lattice paths from points (x_1, y_1) and $(x_1 + ar, y_1 - as)$ respectively to points (x_2, y_2) and $(x_2 + br, y_2 - bs)$ respectively, where $\mathrm{lcm}(c, r+s)$ divides $a(r+s)$ and $b(r+s)$. We have that $|p_1| = x_2 - x_1 + y_1 - y_2$ and $|p_2| = x_2 + br - x_1 - ar + y_1 - as - y_2 + bs$, so $|p_1| - |p_2| = (r+s)(a-b)$, which is divisible by $\mathrm{lcm}(c, r+s)$.

The second point follows because as we trace out a directed path, the value of $x - y$ moves cyclically through the residue classes modulo $\mathrm{lcm}(c, r+s)$, incrementing by 1 with each step. □

Proposition 7.9. *Let $(k, [0]) = v_0, v_1 \dots, v_{(r+s)k_1} = (k, [0])$ be the vertices visited, in order, by the (r, s, c)-tour, corresponding to the section of the boundary of λ between $(0, k_1 s)$ and $(k_1 r, 0)$. Let d_i denote the distance in the first arrival tree T from $(k, [0])$ to v_i.*

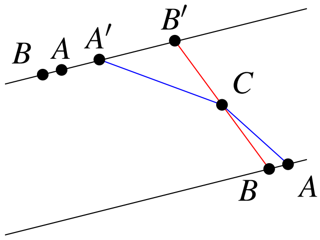
1. *If v_i is a switch, or if there is a copy of the edge (v_{i-1}, v_i) in T, then $d_i - d_{i-1} = 1$.*
2. *If v_i is an eastern vertex and there is no copy of (v_{i-1}, v_i) in $E(T)$, then $d_i - d_{i-1} = 1 + \mathrm{lcm}(c, r+s)$.*
3. *If v_i is a southern vertex and there is no copy of (v_{i-1}, v_i) in $E(T)$, then $d_i - d_{i-1} = 1 - \mathrm{lcm}(c, r+s)$.*

Proof. Write p_i for the path in T from $(k, [0])$ to v_i, so that $|p_i| = d_i$. The first point follows immediately from the definition of a switch and the definition of T.

In general, the winding number of a vertex v is the same as the winding number of the last vertex on the upper boundary strip that T before v. So, drawing T on the cylinder and then forgetting the identification of the two boundary lines, the connected components form sets of vertices of equal winding number.

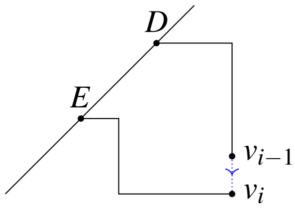
Moreover, if $\mathrm{wind}(p_i) = \mathrm{wind}(p_{i-1})$, then by the second part of Proposition 7.8, $||p_i| - |p_{i-1}|| < \mathrm{lcm}(c, r+s)$. Since there is a path of length 1 (not necessarily in T) connecting v_{i-1} and v_i, then by the first part of Proposition 7.8, $|p_i| - |p_{i-1}| \equiv 1 \pmod{\mathrm{lcm}(c, r+s)}$. Therefore, $d_i - d_{i-1} = 1$, so v_i is a switch or there is a copy of (v_{i-1}, v_i) in $E(T)$.

For 2 and 3, we first prove that as we scan southwest along the upper boundary strip, the winding numbers of the paths from $(k, [0])$ to the vertices on the strip weakly increase. We proceed by induction.



Suppose A and B are vertices on the upper boundary strip and B is southwest of A, and let p_A and p_B be the paths in T from $(k,[0])$ to A and B respectively. We will show $\text{wind}(p_B) \geq \text{wind}(p_A)$. If $A = (k,[0])$ then we are done, so suppose not. There is a copy of both A and B on the lower boundary strip, with B still southwest of A. Moreover, p_A and p_B run from points A' and B' respectively on the upper boundary strip to A and B, where we possibly have $A' = B'$. However, A' cannot be strictly southwest of B', as otherwise p_A and p_B would have to cross at a vertex C, introducing a cycle from $(k,[0])$ following p_A to C and then following p_B back to $(k,[0])$. Let p'_B and p'_A be p_B and p_A shortened to finish at B' and A' respectively. Then, by strong induction, $\text{wind}(p'_B) \geq \text{wind}(p'_A)$. Adding 1 to both sides, $\text{wind}(p_B) \geq \text{wind}(p_A)$.

Now, in the case that v_i is eastern, and there is no copy of (v_{i-1}, v_i) in $E(T)$, (v_{i-1}, v_i) must be a south edge, and v_{i-1} and v_i lie in different connected components. Since $\text{wind}(p_i) \neq \text{wind}(p_{i-1})$, $\text{wind}(p_i) > \text{wind}(p_{i-1})$. Let D and E be the last vertices on the upper boundary strip on p_i and p_{i-1} respectively.



Since all paths in T have vertices at lattice points and do not intersect with each other, there can be no path that starts at a vertex on the upper boundary strip between E and D that crosses all the way to the lower boundary strip. Hence, the copy of E on the lower boundary strip either lies in the same connected component as D or in a component northeast of D. So, $\text{wind}(p_i) = \text{wind}(p_{i-1}) + 1$. Let q_i be the path obtained by extending p_{i-1} by the south edge (v_{i-1}, v_i). Then $|q_i| = d_{i-1} + 1$. The second part of Proposition 7.8 tells us that $|p_i|$ and $|q_i|$ agree modulo $\text{lcm}(c, r+s)$ and therefore $d_i = d_{i-1} + 1 + \text{lcm}(c, r+s)$.

An analogous argument proves the third formula. □

Proposition 7.10. *Let λ be a partition. Then*

$$\mathrm{crit}^+_{x,c}(I_{r,s,c}(\lambda)) = \mathrm{crit}^-_{x,c}(\lambda) \tag{7.49}$$

and

$$\mathrm{crit}^-_{x,c}(I_{r,s,c}(\lambda)) = \mathrm{crit}^+_{x,c}(\lambda). \tag{7.50}$$

Proof. We will check that $\mathrm{crit}^+_{x,c}(\lambda) = (\mathrm{crit}^+_{x,c} + \mathrm{crit}^-_{x,c})(M_{r,s,c}(\lambda)) - \mathrm{crit}^+_{x,c}(I_{r,s,c}(\lambda))$.

Recall that $\mathrm{crit}^+_{x,c}$ counts the total number of inversions in the arrival word at vertices in $M_{r,s,c}(\lambda)$. Suppose the arrival word at vertex $(v,[i])$ has a south edges and b east edges. If v is a switch, then I reverses the arrival word at $(v,[i])$, so the pairs of S,E edges that contribute to $\mathrm{crit}^+_{x,c}(I(\lambda))$ are exactly those that do not contribute to $\mathrm{crit}^+_{x,c}(\lambda)$, so the contributions over $I(\lambda)$ and λ at $(v,[i])$ sum to ab.

Note that if $(v,[i]) \in \mathrm{Ea}$ then $I(\lambda)$ has inversions in the arrival word at $(v,[i])$ using the first E and any S in the arrival word, and then any other pair of south and east edges contribute to $I(\lambda)$ if and only if they do not contribute to λ, so the two contributions sum to $ab+a$. Similarly, if $(v,[i]) \in \mathrm{So}$ then the contributions sum to $ab-b$. Hence, we have that the total $\mathrm{crit}^+_{x,c}(\lambda) + \mathrm{crit}^+_{x,c}(I_{r,s,c}(\lambda))$ can be written as $S_1+S_2+S_3$ where

$$S_1 = \sum_{(v,[i])\text{is a switch}} \mathrm{E_{in}}(v,[i])\mathrm{S_{in}}(v,[i])$$

$$S_2 = \sum_{(v,[i])\in \mathrm{Ea}} \mathrm{E_{in}}(v,[i])\mathrm{S_{in}}(v,[i]) + \mathrm{S_{in}}(v,[i])$$

$$S_3 = \sum_{(v,[i])\in \mathrm{So}} \mathrm{E_{in}}(v,[i])\mathrm{S_{in}}(v,[i]) - \mathrm{E_{in}}(v,[i]).$$

Now, note first that no vertex $(v,[i])$ with $v>k$ contributes to any of these sums. Indeed, no such vertex is a switch, and the arrival word at any such $(v,[i])$ has length $0,1$ or 2, containing at most one S and at most one E. If the arrival word is empty there is nothing to prove. If the arrival word is E then the vertex is eastern, and $\mathrm{E_{in}}(v,[i])\mathrm{S_{in}}(v,[i]) + \mathrm{S_{in}}(v,[i]) = 0$. If the arrival word is S then the vertex is southern and $\mathrm{E_{in}}(v,[i])\mathrm{S_{in}}(v,[i]) - \mathrm{E_{in}}(v,[i]) = 0$. The only other possible arrival word is SE, in which case the vertex is southern and $\mathrm{E_{in}}(v,[i])\mathrm{S_{in}}(v,[i]) - \mathrm{E_{in}}(v,[i]) = 1-1 = 0$. So, we may restrict our sum to vertices $(v,[i])$ with $v \leq k$.

Proposition 7.5 proves (7.29),

$$(\text{crit}^+_{x,c}+\text{crit}^-_{x,c})(M_{r,s,c}(\lambda)) = \sum_{v=0}^{k}\sum_{i=0}^{c-1} \text{E}_{\text{in}}(v,[i])\text{S}_{\text{in}}(v,[i]) - \left\lfloor \frac{k_1(s+r)}{\text{lcm}(c,r+s)} \right\rfloor .$$

We wish to show that (7.29) is equal to $S_1+S_2+S_3$, and therefore it suffices to check that

$$\sum_{(v,[i])\in\text{So}} \text{E}_{\text{in}}(v,[i]) - \sum_{(v,[i])\in\text{Ea}} \text{S}_{\text{in}}(v,[i]) = \left\lfloor \frac{k_1(s+r)}{\text{lcm}(c,s+r)} \right\rfloor . \tag{7.51}$$

Note that the east edges entering southern vertices and the south edges entering eastern vertices are exactly the edges in $M_{r,s,c}(\lambda)$ arriving at non-switch vertices that are *not* a copy of an edge in the first arrival tree T. Hence, if we let n_0 denote the number of edges e entering vertices $(v,[i])$ with $v \leq k$ such that either

- $(v,[i])$ is a switch, or
- $(v,[i])$ is not a switch and there is a copy of e in the first arrival tree T,

then

$$n_0 + \sum_{(v,[i])\in\text{So}} \text{E}_{\text{in}}(v,[i]) + \sum_{(v,[i])\in\text{Ea}} \text{S}_{\text{in}}(v,[i]) = k_1(r+s). \tag{7.52}$$

Now, let $(k,[-k_1s]) = v_0, v_1 \ldots, v_{(r+s)k_1} = (k,[k_1r])$ be the vertices visited, in order, possibly with repetition, by the (r,s,c)-tour. Let d_i denote the distance in the first arrival tree from $(k,[-k_1s])$ to v_i. Now $c \mid k_1$ by assumption, and thus $c \mid (r+s)k_1$, so we have that

$$0 = d_{(r+s)k_1} = \sum_{i=1}^{(r+s)k_1} d_i - d_{i-1}. \tag{7.53}$$

Substituting the formulae for $d_i - d_{i-1}$ proven in Proposition 7.9 into (7.53) and writing l for $\text{lcm}(c,r+s)$,

$$n_0 + (1+l)\sum_{(v,[i])\in\text{Ea}} \text{S}_{\text{in}}(v,[i]) + (1-l)\sum_{(v,[i])\in\text{So}} \text{E}_{\text{in}}(v,[i]) = 0. \tag{7.54}$$

Subtracting (7.54) from (7.52) gives

$$k_1(r+s) = \text{lcm}(c,r+s)\left(\sum_{(v,[i])\in\text{So}} \text{E}_{\text{in}}(v,[i]) - \sum_{(v,[i])\in\text{Ea}} \text{S}_{\text{in}}(v,[i])\right). \tag{7.55}$$

Now, since k is divisible by rsc, $k_1 = \frac{k}{rs}$ is divisible by c, so $\mathrm{lcm}(c, r+s)$ divides $k_1(r+s)$. Therefore,

$$\left\lfloor \frac{k_1(r+s)}{\mathrm{lcm}(c,r+s)} \right\rfloor = \frac{k_1(r+s)}{\mathrm{lcm}(c,r+s)} = \sum_{(v,[i])\in \mathrm{So}} \mathrm{E_{in}}(v,[i]) - \sum_{(v,[i])\in \mathrm{Ea}} \mathrm{S_{in}}(v,[i])),$$

which is (7.51), which completes the proof. $\square$

7.3 Extended Example

Let $c = 2$ and $n = 7$, and $\mu = (2,1)$. Then

$$\mathrm{Par}^2_\mu(7) = \{(6,1),(4,3),(4,1,1,1),(2,2,2,1),(2,1,1,1,1,1)\}. \tag{7.56}$$

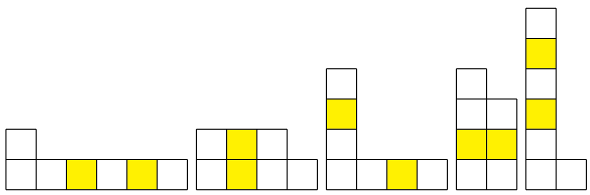


Fig. 7.3 the partitions in $\mathrm{Par}^2_\mu(7)$ with boxes of even hook length coloured yellow.

For the shaded cells, the set of values of $\frac{a(\square)}{l(\square)+1}$ is $\left\{3,1,0,\frac{1}{3}\right\}$, and the set of values of $\frac{a(\square)+1}{l(\square)}$ is $\left\{\infty,3,1,\frac{1}{3}\right\}$. So, the critical rationals are $\left\{0,\frac{1}{3},1,3,\infty\right\}$.

In this example, we will verify that

$$\sum_{\lambda\in \mathrm{Par}^2_\mu(7)} t^{h^+_{4,2}(\lambda)} = \sum_{\mathrm{Par}^2_\mu(7)} t^{\lambda^{2*}_\square}.$$

Recall

$$h^+_{4,2}(\lambda) = \left|\left\{\square\in\lambda : 2 \mid h(\square) \text{ and } \frac{a(\square)}{l(\square)+1} \le 4 < \frac{a(\square)+1}{l(\square)}\right\}\right|. \tag{7.57}$$

From our computation of the critical rationals, given that $2 \mid h(\square)$ for some box in a partition $\lambda \in \mathrm{Par}^2_\mu(7)$, $4 < \frac{a(\square)+1}{l(\square)}$ if and only if $3 < \frac{a(\square)+1}{l(\square)}$, and $\frac{a(\square)}{l(\square)+1} \le 4$ if and only if $\frac{a(\square)}{l(\square)+1} \le 3$. So, $h^+_{4,2}(\lambda) = h^+_{3,2}(\lambda)$. Now we use $I_{3,1,2} : \mathrm{Par}^2_\mu(7) \to \mathrm{Par}^2_\mu(7)$. Because

$\mathrm{mid}_{3,2}(\lambda) = \mathrm{mid}_{3,2}(I_{3,1,2}(\lambda))$ and $\mathrm{crit}^{\pm}_{3,2}(\lambda) = \mathrm{crit}^{\mp}_{3,2}(I_{3,1,2}(\lambda))$, $I_{3,1,2}$ is a bijection exchanging $h^+_{3,2}$ and $h^-_{3,2}$, so

$$\sum_{\lambda \in \mathrm{Par}^2_\mu(7)} t^{h^+_{3,2}(\lambda)} = \sum_{\lambda \in \mathrm{Par}^2_\mu(7)} t^{h^-_{3,2}(\lambda)}.$$

We now explicitly compute $I_{3,1,2}(\lambda)$ for $\lambda = (6,1)$.

The diagram of $(6,1)$ lies below the line $3y+x=9$. So, we choose the smallest value $k \geq 9$ such that $3\times 2\times 1 \mid k$, $k=12$. Then, $k_1 = \frac{12}{3} = 4$.

The (r,s,c)-tour of $M_{3,1,2}((6,1))$ is defined by the following family of arrival words.

(4,[0])	S	(5,[1])	E	(6,[0])	SES
(7,[1])	EEE	(8,[0])	EE	(9,[1])	SEE

and for $w > 9$,

$$(w,[i])_a = \begin{cases} SE & 3 \mid w, w \equiv \frac{-w}{3} \equiv i \pmod 2 \\ E & 2 \mid (w-i) \text{ and either } 3 \nmid w \text{ or } 2 \nmid (\frac{-w}{3} - i) \\ S & 3 \mid w, 2 \mid \left(\frac{-w}{3} - i\right), 2 \nmid (w-i) \\ \text{empty} & \text{otherwise} \end{cases}. \tag{7.58}$$

The multigraph is given in Figure 7.4 with the edges in the first arrival tree in bold.

After applying $I_{3,1,2}$ the arrival words are

(4,[0])	S	(5,[1])	E	(6,[0])	SSE
(7,[1])	EEE	(8,[0])	EE	(9,[1])	SEE

with all arrival words at $(w,[i])$ with $w > 9$ unchanged. These arrival words correspond to the partition $(4,3)$. So, $h^+_{3,2}((6,1)) = h^-_{3,2}((4,3))$.

From our computation of the critical rationals, given that $2 \mid h(\square)$ for some box in a partition $\lambda \in \mathrm{Par}^2_\mu(7)$, $3 \leq \frac{a(\square)+1}{l(\square)}$ if and only if $1 < \frac{a(\square)+1}{l(\square)}$, and $\frac{a(\square)}{l(\square)+1} < 3$ if and only if $\frac{a(\square)}{l(\square)+1} \leq 1$. So, $h^-_{3,2}(\lambda) = h^+_{1,2}(\lambda)$ for all $\lambda \in \mathrm{Par}^2_\mu(7)$. Now, $I_{1,1,2}$ exchanges $h^+_{1,2}$ and $h^-_{1,2}$, and $I_{1,1,2}((4,3)) = (2,2,2,1)$, so $h^+_{4,2}((6,1)) = h^+_{1,2}((2,2,2,1))$. Using the same logic again $h^+_{1,2}(\lambda) = h^-_{\frac{1}{3},2}(\lambda)$ for each $\lambda \in \mathrm{Par}^2_\mu(7)$. Using $I_{1,3,2}$, $I_{1,3,2}((2,2,2,1)) = (2,1,1,1,1,1)$, so $h^+_{4,2}((6,1)) = h^-_{\frac{1}{3},2}((2,1,1,1,1,1))$. Finally, for any partition $\lambda \in \mathrm{Par}^2_\mu(7)$, $\frac{1}{3} \leq \frac{a(\square)+1}{l(\square)}$ if and only if $0 < \frac{a(\square)+1}{l(\square)}$, and $\frac{a(\square)}{l(\square)+1} < \frac{1}{3}$ if and only if $a(\square) = 0$, if and only if $\frac{a(\square)}{l(\square)+1} \leq 0$, so $h^-_{\frac{1}{3},2}(\lambda) = h^+_{0,2}(\lambda)$.

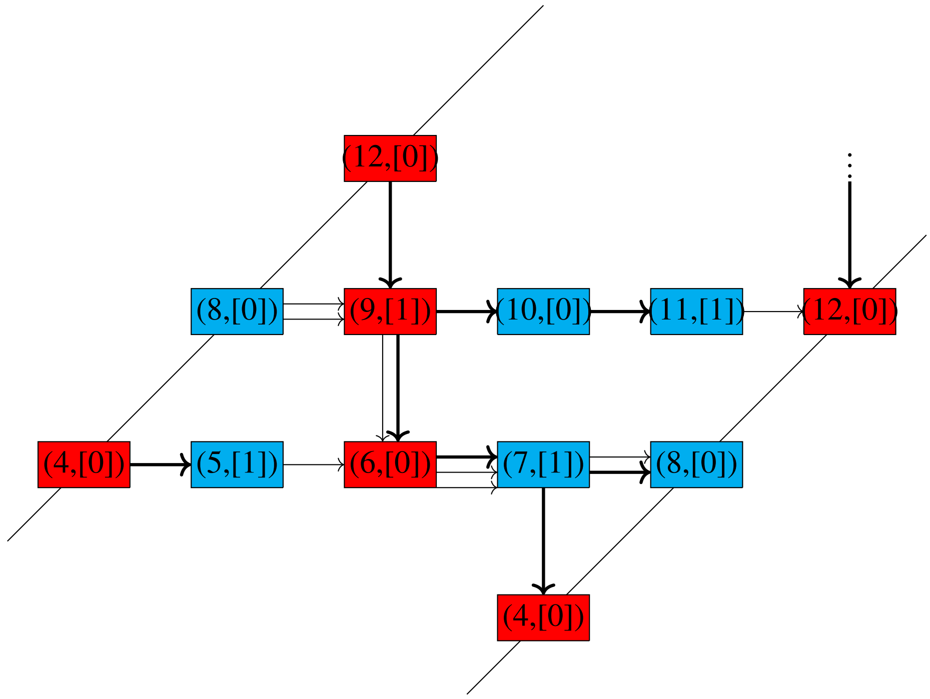


Fig. 7.4 $M_{3,1,2}((6,1))$ with the edges of the first arrival tree in bold.

Therefore, since

$$I_{1,3,2} \circ I_{1,1,2} \circ I_{3,1,2}((6,1)) = (2,1,1,1,1,1),$$

we have that $h^+_{4,2}(6,1) = h^+_{0,2}(2,1,1,1,1,1)$. For the other partitions in $\mathrm{Par}^2_\mu(7)$,

$$I_{1,3,2} \circ I_{1,1,2} \circ I_{3,1,2}(4,3) = I_{1,3,2} \circ I_{1,1,2}((6,1)) = I_{1,3,2}(2,1,1,1,1,1) = (2,2,2,1),$$

$$I_{1,3,2} \circ I_{1,1,2} \circ I_{3,1,2}(4,1,1,1) = I_{1,3,2} \circ I_{1,1,2}((4,1,1,1)) = I_{1,3,2}(4,1,1,1) = (4,1,1,1).$$

$$I_{1,3,2} \circ I_{1,1,2} \circ I_{3,1,2}(2,2,2,1) = I_{1,3,2} \circ I_{1,1,2}((2,2,2,1)) = I_{1,3,2}(4,3) = (4,3).$$

$$I_{1,3,2} \circ I_{1,1,2} \circ I_{3,1,2}(2,1,1,1,1,1) = I_{1,3,2} \circ I_{1,1,2}((2,1,1,1,1,1)) = I_{1,3,2}(6,1) = (6,1).$$

Hence we can verify the equidistribution of $h^+_{x,2}$ with $h^-_{x,2}$ over $\mathrm{Par}^2_\mu(7)$ for each $x \in \mathbb{R}_{>0}$, thus verifying Theorem 3.3 in this case.

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