Topics in Gromov–Witten theory

Yannik Schüler-Hammer

A thesis submitted in partial fulfilment of the requirements for the degree of
Doctor of Philosophy

The University of Sheffield
Faculty of Science
School of Mathematics and Statistics

April 2024

to Keno

Abstract

This thesis explores different aspects of Gromov–Witten theory and is divided into two parts.

The first investigates conjectures of Bousseau, Brini and van Garrel relating three a priori very different curve counts: Logarithmic Gromov–Witten theory of Looijenga pairs (certain logarithmic Calabi–Yau surfaces), open Gromov–Witten theory of toric Calabi–Yau threefolds and local Gromov–Witten theory of higher dimensional Calabi–Yau varieties. We concentrate on the case where the logarithmic boundary of the initial surface geometry has two components. First we establish the logarithmic-open correspondence in an explicit example where the Looijenga pair is a del Pezzo surface of degree six. The proof relies on a direct calculation using quantum scattering diagrams and involves an intricate identity of q-hypergeometric functions. After this case study we proceed with a more general, geometric approach and ultimately establish the logarithmic-local and all-genus logarithmic-open correspondence for all Looijenga pairs with two boundary components. The proof of the correspondences involves a delicate application the degeneration formula and torus localisation.

In the second part we propose a mathematical interpretation of the so called refined topological string on a Calabi–Yau threefold in terms of equivariant Gromov–Witten theory of an extended Calabi–Yau fivefold geometry. We perform initial checks which indicate that our proposal meets several expectations formulated in the physics literature. We state a refined BPS integrality conjecture and provide evidence in case the threefold is the resolved conifold or a local del Pezzo surface. In the latter case we do so by identifying the Nekrasov–Shatashvili limit with the relative Gromov–Witten theory of the surface relative a smooth anticanonical curve.

Acknowledgements

First and foremost I would like to express my gratitude towards my supervisor Andrea Brini: Thank you for introducing me to the great world of research in enumerative geometry and mathematical physics, our countless discussions, your continuous support and especially that your door was always open whenever I had a question or needed advice.

I would also like to thank the whole Sheffield algebraic geometry and mathematical physics group. I am grateful for numerous stimulating conversations, all that I learned in our reading seminars, our climbing sessions, and in general for a supportive and inspiring environment celebrating the joy in mathematical studies. Thanks to all of you I am proud to call myself Sheffield made!

Special thanks are also due to Michel van Garrel and Navid Nabijou. I learned a great deal from our collaboration and its impact on this thesis cannot be undervalued. I also would like to thank Cristina Manolache for clarifying discussions concerning key steps of this thesis.

Moreover, I would like to thank Gaëtan Borot, Nitin Chidambaram, Martijn Kool, Michel van Garrel, Dhruv Ranganathan, Rahul Pandharipande and Richard Thomas for the opportunity to present parts of this thesis in local research seminars and the resulting discussions.

Thanks are also due to the following people for discussions related to the content of this thesis: Pierrick Bousseau, Tom Bridgeland, Jinwon Choi, Alberto Cobos Rábano, Pierre Descombes, Nadir Fasola, Karoline van Gemst, Alessandro Giacchetto, Luca Giovenzana, Tim Gräfnitz, Samuel Johnston, Jingxiang Ma, Reinder Meinsma, Fabrizio del Monte, Nikita Nikolaev, Georg Oberdieck, Kento Osuga, Maximilian Schimpf, Evgeny Shinder, Pim Spelier, Ajith Urundolil Kumaran, Yirui Xiong, Song Yu and Menelaos Zikidis.

Finally, I wish to thank Laura and my parents Sabine and Dirk for their constant support and Keno for giving me a boost of motivation in the final months of working on this thesis.

Declaration

I, the author, confirm that the Thesis is my own work. I am aware of the University's Guidance on the Use of Unfair Means (www.sheffield.ac.uk/ssid/unfair-means). This work has not been previously been presented for an award at this, or any other, university.

Chapter 2 of this thesis is joint work with Andrea Brini which originally appeared as the article "On quasi-tame Looijenga pairs" in *Communications in Number Theory and Physics*, Volume 17 (2023), Number 2, pp. 313–341 published by International Press. Only the proof of Proposition 2.2.11 got slightly reformulated compared to the published article.

Chapter 3 is joint work together with Michel van Garrel and Navid Nabijou first made available as the arXiv preprint "Gromov–Witten theory of bicyclic pairs" (arXiv: 2310.06058 [math.AG]).

The content of Chapter 4 has also been made available as the arXiv preprint "The log-open correspondence for two-component Looijenga pairs" (arXiv: 2404.15412 [math.AG]) already.

Table of Contents

Chapter

Introduction

1

Gromov–Witten theory is a framework for enumerating parametrised complex curves in a given variety. In the nineties the subject rapidly gained popularity as being the mathematically well defined analogue of the (A-model) topological string studied by physicists. This connection between mathematics and physics catalysed the development of the field since its early beginning leading to discoveries such as mirror symmetry and informed connections to other areas of mathematics.

The first part of this thesis studies correspondences within the area of Gromov–Witten theory. The second part aims at continuing the confluence between mathematics and physics by proposing Gromov–Witten type interpretation for the refined topological string.

1.1 Different flavours of Gromov–Witten Theory

1.1.1 Overview

Given a quasi-projective complex variety X we are interested in enumerating parametrised curves $C \longrightarrow X$ in this geometry. In slightly more precise terms, in Gromov–Witten theory we are studying the moduli space

$$\overline{M}_{g,\beta}(X) \text{ “=” } \left\{ f : C \longrightarrow X \;\middle|\; \begin{array}{c} f \text{ is a stable map,} \\ C \text{ is a genus } g \text{ curve} \\ \text{and } f_*[C] = \beta \end{array} \right\} \Big/ \sim \,.$$

parametrising morphisms from possibly nodal curves with fixed arithmetic genus and image class which are subject to a certain stability condition. This moduli space is equipped with a virtual fundamental class which is of dimension zero if for instance X is a Calabi–Yau threefold. Moreover, the moduli space is proper whenever X is. So if the last two conditions are satisfied one may define the so called Gromov–Witten invariant

$$\mathsf{GW}_{g,\beta}(X) := \int_{[\overline{M}_{g,\beta}(X)]^{\text{virt}}} 1 \qquad \in \mathbb{Q}\,.$$

For further introductory literature to the general topic of Gromov–Witten theory we refer to [51, 62, 87, 103].

This thesis will be concerned with different extensions of the just described moduli problem and investigates relations among these extensions. More precisely, in the first part we will focus on a correspondence relating logarithmic, open and local Gromov–Witten invariants conjectured by Bousseau, Brini and van Garrel [29]. Let us briefly introduce these three different flavours of the initial moduli problem.

Logarithmic/Relative Gromov–Witten theory. The question of how many lines meet a smooth plane cubic in a single point can be treated using standard techniques in algebraic geometry returning the correct answer (nine). Relative Gromov–Witten theory is the natural generalisation of this question to the framework introduced before. It aims at enumerating stable maps $C \to X$ where the image curve has prescribed tangency along a divisor D in X. Here, the main difficulty is how to define tangency in case the image of an irreducible component of the domain curve gets contracted into the divisor.

The issue can be dealt with in different ways. If the divisor D is smooth Li constructs a suitable moduli space parametrising stable maps to expanded degenerations of the target $(X|D)$ [115] (with a similar construction in the symplectic category [91, 118]). Over the years several other approaches emerged: The one of Abramovich–Chen–Gross–Siebert using logarithmic geometry [1, 43, 81], Kim's approach using logarithmic expanded degenerations [100], and one of Abramovich–Fantechi via expanded orbifold targets [4]. As explained in [5] all these approaches essentially yield the same result in the smooth divisor case. Depending on the application, however, one approach might be favourable over another.

One of the central outcomes of relative Gromov–Witten theory is the degeneration formula. It relates the Gromov–Witten theory of a smooth target with the irreducible components of a degeneration thereof. Besides virtual localisation [71] it is certainly one of the most invaluable tools in the field. For double point degenerations the degeneration formula goes back to Li [114] with similar statements in the other approaches to relative Gromov–Witten theory [4, 99]. More recently the degeneration formula was generalised to the simple normal crossings setting in the context of logarithmic Gromov–Witten theory as well [3, 149].

It should also be mentioned that logarithmic Gromov–Witten invariants play a key role in the mirror construction of Gross and Siebert [83] where they are connected to a combinatorial gadget called scattering diagram.

Open Gromov–Witten theory. Suppose we are given a Lagrangian submanifold L in a Calabi–Yau threefold X. To enhance the original moduli problem in a different way, let us view the domain of a stable map as a real two dimensional manifold — a Riemann surface. Open Gromov–Witten theory aims at enumerating maps $(C, \partial C) \to (X, L)$ from Riemann surfaces C with boundary ∂C mapping into L. The fact that makes it hard to define open Gromov–Witten invariants is that the associated moduli space might turn out to be a non-orientable real orbifold with corners unless L is well-behaved. See [155] for an early account on these difficulties which till this day are subject to ongoing research.

There is a way around the these difficulties in case the target variety X is toric and the Lagrangian submanifold $L \cong \mathbb{R}^2 \times S^1$ is preserved by an (sufficiently generic) action of a real torus. A class of such Lagrangian submanifolds was constructed by Aganagic and Vafa [8]. In this case one can argue that the torus fixed locus of the anticipated moduli space is a complex orbifold which allows to define the invariants via localisation as proposed by Katz and Liu [97].

Moreover in this setting, we can make a connection to relative Gromov–Witten theory by observing that a torus fixed open stable map $(C, \partial C_1 \sqcup \ldots \sqcup \partial C_n) \to (X, L)$ can be turned into a relative stable

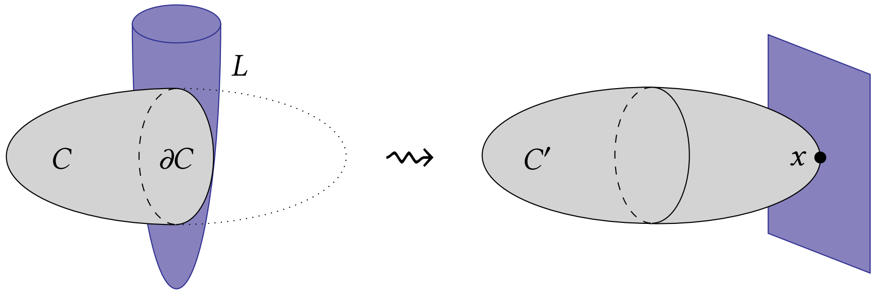


Figure 1.1: Capping the boundary of a torus fixed open stable map yields a relative one.

map $(C', x_1, \ldots, x_n) \to (X, D)$. As illustrated in Figure 1.1, this is achieved by gluing discs with origins $x_1, \ldots, x_n$ along the boundaries $\partial C_1 \sqcup \ldots \sqcup \partial C_n$ whose image in X naturally extends over the toric curve intersected by L. In this process the winding around the Lagrangian L gets replaced with a prescribed tangency along a divisor D nearby L. In [116] Li, Liu, Liu and Zhou pick up on this observation and actually define open Gromov–Witten invariants of a toric Calabi–Yau threefold X via the relative invariants of some partial compactification of the target. Moreover, they prove that these invariants can be computed using the topological vertex method of Aganagic, Klemm, Mariño and Vafa [7]. This method marked one of the most efficient computational tools in the early days of Gromov–Witten theory and is up to today often used in numerical experiments.

Local Gromov–Witten theory. In case the target geometry is a vector bundle N over a projective variety Y one usually calls the count $\mathsf{GW}_{g,\beta}(N)$ a local Gromov–Witten invariant. This is motivated by the fact that for a regularly embedded subvariety $\iota : Y \hookrightarrow X$ with normal bundle $N_Y X$ sufficiently negative the invariant

$$\mathsf{GW}_{g,\beta}(N_Y X)$$

is indeed the local contribution of Y to the overall invariant $\mathsf{GW}_{g,\iota_*\beta}(X)$.

This way local Gromov–Witten invariants can be viewed as building blocks for the general theory. So it does not come as a surprise that they played a crucial role in the development of the field. For instance, the analysis of local curves by Bryan and Pandharipande [35, 36, 37, 146] helped in the early understanding of the MNOP and Gopakumar–Vafa integrality conjecture and later featured as a key ingredient in the proof of the latter [92].

1.1.2 Correspondences

It turns out that these at first sight rather different subtypes of Gromov–Witten theory are related in a surprising way.

1.1.2.1 Logarithmic-local. Experimentally, Takahashi [158] observed that the Gromov–Witten count of rational curves in $\mathbb{P}^2$ with maximum tangency along a smooth cubic D agree with the local Gromov–Witten invariants of $\mathcal{O}_{\mathbb{P}^2}(-D)$ up to an explicit factor. This identity was later proven by Gathmann [67] and extended to cycle valued correspondence by van Garrel, Graber and Ruddat [64] for general pairs $(X|D)$ with D a smooth divisor:

$$\boxed{(X|D)} \xleftrightarrow{g=0} \boxed{\mathcal{O}_X(-D)}.$$

It was also conjectured that a similar relation should hold in the case when $D = D_1 + \ldots + D_l$ has multiple irreducible components D_i with simple normal crossings:

$$\boxed{(X|D_1 + \ldots + D_l)} \xleftrightarrow{g=0} \boxed{\mathcal{O}_X(-D_1) \oplus \ldots \oplus \mathcal{O}_X(-D_l)}. \tag{1.1}$$

However, it was quickly observed by Nabijou and Ranganathan [141] that the expected correspondence cannot hold — at least not *on cycle level*. On the other hand however, there were positive results [28, 29, 30] indicating that (1.1) may hold *on the numerical level* with appropriate incidence conditions and sufficient conditions on the the target geometry $(X\,|\,D)$. Based on their numerical observations Bousseau, Brini and van Garrel in [29] conjecture a numerical (intermediate) logarithmic-local correspondence

$$\begin{array}{llll} \boxed{(S\,|\,D_1+\ldots+D_l)} & & & \\ & \xleftrightarrow[\text{numerical}]{g=0} & \boxed{(\mathcal{O}_S(-D_l)\,|\,D_1+\ldots+D_{l-1})} & \\ & \ddots & & \ddots \\ & & \xleftrightarrow[\text{numerical}]{g=0} & \boxed{\mathcal{O}_S(-D_1)\oplus\ldots\oplus\mathcal{O}_S(-D_l)} \end{array} \tag{1.2}$$

which is supposed to hold for all so called Looijenga pairs $(S\,|\,D_1+\ldots+D_l)$. By this we mean a smooth rational projective surface S together with an anticanonical singular nodal curve $D_1+\ldots+D_l$.

1.1.2.2 Logarithmic-open. We already explained how one can obtain a relative stable map from an open one by capping the boundary with discs. In the process the winding around a Lagrangian submanifold L_i gets replaced by a specific tangency along a nearby divisor D_i. In the context of toric Calabi–Yau threefolds, Fang and Liu turn this intuitive idea into a rigorous statement equating open against logarithmic invariants [59]:

$$\boxed{(X\,|\,L_1,\ldots,L_n)} \xleftrightarrow[\text{numerical}]{\text{all genus}} \boxed{(Y\,|\,D_1,\ldots,D_n)}. \tag{1.3}$$

Now Bousseau, Brini and van Garrel [29] observed a relationship between logarithmic and open Gromov–Witten invariants which is of slightly different flavour than the above correspondence. Starting from a Looijenga pair $(S\,|\,D_1+\ldots+D_l)$ satisfying certain technical conditions they conjecture that one can construct a toric Calabi–Yau threefold X together with Lagrangian submanifolds $L_1,\ldots,L_{l-1}$ so that the maximum contact logarithmic and open Gromov–Witten invariants of the respective targets agree:

$$\boxed{(S\,|\,D_1+\ldots+D_l)} \xleftrightarrow[\text{numerical}]{\text{all genus}} \boxed{(X\,|\,L_1,\ldots,L_{l-1})}$$

This correspondence is clearly of different shape than the one in (1.3) as the dimension of the target geometries and number of divisors and Lagrangian submanifolds differs by one. At least in the case $l=2$ we will see that Bousseau–Brini–van Garrel's logarithmic-open correspondence can be demystified by passing through an intermediate correspondence with the geometry

$$(\mathcal{O}_S(-D_l)\,|\,D_1+\ldots+D_{l-1})$$

which indeed can be linked to the open Gromov–Witten theory of a toric Calabi–Yau threefold with $l-1$ Lagrangian submanifolds as in (1.3) via a correspondence similar to the one of Fang and Liu.

1.1.3 Outline of part I

The objective of the first part of this thesis is a thorough analysis of the web of correspondences between logarithmic, local and open Gromov–Witten invariants in case the initial geometry

$$(S\,|\,D_1+D_2)$$

is a Looijenga pair whose boundary consists of two irreducible curves D_1, D_2. Ultimately, for such geometries we establish all correspondences conjectured by Bousseau, Brini and van Garrel [29]:

$$
\begin{tikzcd}
\boxed{(S\,|\,D_1+D_2)} \arrow[r, leftrightarrow, "\text{all genus}", "\text{numerical}"'] \arrow[dr, leftrightarrow, "\text{``log-open''}"'] & \boxed{(\mathcal{O}_S(-D_2)\,|\,D_1)} \arrow[r, leftrightarrow, "g=0"] \arrow[d, leftrightarrow, "\text{all genus}", "\text{numerical}"'] & \boxed{\mathcal{O}_S(-D_1)\oplus\mathcal{O}_S(-D_2)} \\
& \boxed{(X\,|\,L_1)} &
\end{tikzcd} \tag{1.4}
$$

1.1.3.1 On quasi-tame Looijenga pairs. In Chapter 2 we prove the logarithmic-open correspondence for a specific Looijenga pair $(\mathrm{dP}_3\,|\,D_1+D_2)$. This pair is obtained from $\mathbb{P}^2$ together with a line and a conic by blowing up a point on the line and two on the conic. The example is of special interest since it marks the last remaining case among the class of all so called quasi-tame Looijenga pairs for which the conjecture is still open. For all other quasi-tame Looijenga pairs it has already been proven in [29, 109].

The proof proceeds via a direct calculation of both sides of the correspondence where a closed form solution for the open Gromov–Witten invariants has already been found in [29]. Hence, it remains to prove an analogous formula for the logarithmic invariants which is done via the quantum tropical vertex. This method is due to Bousseau [20] generalising ideas of Gross–Pandharipande–Siebert [80] to higher genus. The result of our calculation is a convoluted sum of products of q-binomial coefficients. Using a trick of Krattenthaler [109] the sum can be simplified to yield a closed form solution which indeed agrees with the one of the open invariants. This is joint work with Andrea Brini which got first published in [33].

1.1.3.2 Gromov–Witten theory of bicyclic pairs. In Chapter 3 we turn from a study of explicit examples to a more conceptual, geometric approach and establish both horizontal arrows in (1.4). We prove the correspondences in a context where the divisor $D_1 + D_2$ is not necessarily assumed to be anticanonical but just a union of two curves intersecting transversely in two points. We call such tuples $(S\,|\,D_1 + D_2)$ bicyclic pairs. The main difficulty is the proof of the left horizontal arrow in (1.4). The proof involves a degeneration to the normal cone argument similar to the one in [64] but features a more intricate analysis of terms in the the degeneration formula [3, 149].

We also provide a detailed case study of $\mathbb{P}^2$ relative a nodal cubic D. We prove a closed form solution for the genus zero maximum tangency logarithmic invariants of $(\mathbb{P}^2\,|\,D)$ via scattering diagrams [80] and as an application of our correspondence theorem we obtain a closed form solution for the genus zero local Gromov–Witten invariants of $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$ as well. Further, we compare the invariants of $\mathbb{P}^2$ relative a nodal cubic with the ones relative a smooth cubic by degenerating both divisors to the toric boundary. Together with our formula for the Gromov–Witten invariants of $(\mathbb{P}^2\,|\,D)$ we use the comparison to prove a conjecture of Barrott and Nabijou [13]. This chapter is joint work with Michel van Garrel and Navid Nabijou which first appeared as [65].

1.1.3.3 The log-open correspondence for two-component Looijenga pairs. In Chapter 4 we establish the vertical arrow in the web of correspondences (1.4). The idea of the proof is to turn the intuitive identification of open and relative invariants illustrated in Figure 1.1 into a solid theorem. We do so by equating the relative invariants of $(\mathcal{O}_S(-D_2)\,|\,D_1)$ via localisation and degeneration against descendant invariants of $\mathcal{O}_S(-D_2)|_{S\setminus D_1}$. The correspondence then follows from a direct comparison with a formula for the open Gromov–Witten invariants in terms of descendant invariants proven by Fang and Liu [59].

As an application of our log-open correspondence we explain how the topological vertex method can applied to determine logarithmic invariants of two component Looijenga pairs. We use the approach to give a new proof of the formula for the logarithmic Gromov–Witten invariants of $(\mathrm{dP}_3|D_1 + D_2)$ proven earlier in [33] (Chapter 2). As a second application, we demonstrate BPS integrality for the logarithmic Gromov–Witten invariants of Looijenga pairs to which our correspondence applies.

1.2 Gromov–Witten and topological string theory

1.2.1 From physics to mathematics

In this section we will briefly recall how Gromov–Witten invariants can be understood as the topological correlators of the A-twisted $\mathcal{N} = (2,2)$ non-linear σ-model coupled to two-dimensional topological gravity [162]. For short we refer to this quantum filed theory as the topological string.

In physics, the partition function of a quantum field theory with action S and fields Φ is often assumed to have a path integral representation

$$\int_{\{\text{fields }\Phi\}} \mathrm{e}^{S(\Phi)} .$$

Mathematically, most of the time this last expression is unfortunately not well defined since the space of fields is often infinite dimensional. However, if the quantum field theory has (enough) supersymmetry there is sometimes a way out. In this case, the above integral heuristically localises on those fields which are invariant under the supersymmetry transformation Q:

$$\int_{\{Q\text{-invariant fields }\Phi\}} \mathrm{e}^{S(\Phi)} . \tag{1.5}$$

Often this last expression does turn out to be well-defined and can hence be taken as the definition for the partition function. Now to get more concrete, this idea can be applied to the A-twisted $\mathcal{N} = (2,2)$ non-linear σ-model on a Riemann surface C with target a complex Calabi–Yau threefold X. In this case the field Φ contain a bosonic component f which is a smooth differentiable map from C to X. The supersymmetry transformation Q on f can be identified with the action of $\overline{\partial}$ and so the kernel of Q is exactly given by holomorphic maps. Moreover, on this fixed locus

$$S(\Phi) = -(\omega, f_*[C])$$

up to neglectable contributions where ω is the Käler form of X. Hence, for this particular example

$$\int_{\{Q\text{-invariant fields }\Phi\}} \mathrm{e}^{S(\Phi)} = \sum_{\beta} \mathrm{e}^{-(\omega,\beta)} \int_{\left\{\substack{f\,:\,C\to X \text{ holom.} \\ \text{and } f_*[C]=\beta}\right\}} 1$$

where we pulled out the integrand by splitting the integration domain into components labelled by the class β of the image.

We now couple this theory to two-dimensional topological gravity. First, this adds the Einstein–Hilbert action to the exponent in (1.5) which for a fixed Riemann surface C of genus g simply evaluates to $2 - 2g$ by Gauss–Bonnet. Hence, we are effectively led to an insertion of u^{2g-2} where u can be identified with the coupling constant. Second, one integrates over all Riemann surfaces C. So up to the caveat of choosing a suitable compactification and the formation of a fundamental class, one can indeed identify the partition function this quantum field theory, which for short we will call the topological string, with the generating series of Gromov–Witten invariants

$$\sum_{\beta} \mathrm{e}^{-(\omega,\beta)} \sum_{g\geq 0} u^{2g-2} \int_{[\overline{M}_{g,\beta}(X)]^{\mathrm{virt}}} 1 =: F^X_{\mathrm{top}}(u)$$

where we view $\mathrm{e}^{-(\omega,\,\cdot\,)}$ and u as formal variables. We refer the interested reader to [51, 87, 102] for more details.

1.2.2 Dualities and open ends

Dualities between the topological string and other quantum field theories have shaped the development of Gromov–Witten theory since the initiation [75, 162] of the research area. For instance at its early beginning the mirror symmetry considerations of Candelas, de la Ossa, Green and Parkes led to a striking prediction for the number of rational curves in a quintic hypersurface in $\mathbb{P}^4$ [40].

Another example is provided by the earlier mentioned topological vertex method. Its proposal due to Aganagic, Klemm, Mariño and Vafa [7] was motivated by a duality between the topological string and Chern–Simons theory. Later this method was put on solid mathematical grounds by Li–Liu–Liu–Zhou [116].

Also a version of the open-local correspondence in (1.4) has already been observed as an open-closed duality of the topological string by Lerche and Mayr [113, 136] and was recently proven by Liu–Yu [119, 122] in the context of algebraic geometry. See also [32] for a discussion of Bousseau–Brini–van Garrel's log-local-open correspondence from the standpoint of mathematical physics.

But the seemingly endless web of dualities and the dictionary between physics and mathematics also features some open ends. For example, it is known that the topological string on a Calabi–Yau threefold can sometimes engineer supersymmetric gauge theories [96]. This is in meant in the sense that if we denote by $Z_{\mathrm{inst}}(\epsilon_1, \epsilon_2)$ the fully equivariant instanton partition function of the gauge theory then

$$Z_{\mathrm{inst}}(\epsilon_1, \epsilon_2)\big|_{\epsilon_1=-\epsilon_2=u} = \exp F^X_{\mathrm{top}}(u) \tag{1.6}$$

for some appropriate non-compact Calabi–Yau threefold X. Motivated by this duality, the existence of a quantum field theory — dubbed the refined topological string on X — was expected whose partition function $F^X_{\mathrm{ref\ top}}(\epsilon_1, \epsilon_2)$ extends the identity (1.6) to general values of ϵ_1, ϵ_2 or in other words which fits into the top right corner of

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{SUSY gauge theory}\\ \text{(fully equivariant)}\\ Z_{\mathrm{inst}}(\epsilon_1,\epsilon_2)\end{array}} \arrow[r, leftrightarrow, dashed] \arrow[d] & \boxed{\begin{array}{c}\text{refined topological string ?}\\ F^X_{\mathrm{ref\ top}}(\epsilon_1,\epsilon_2)\end{array}} \arrow[d, dashed] \\
\boxed{\begin{array}{c}\text{SUSY gauge theory}\\ \text{(equivariant subsector)}\\ Z_{\mathrm{inst}}(u,-u)\end{array}} \arrow[r, leftrightarrow] & \boxed{\begin{array}{c}\text{topological string}\\ F^X_{\mathrm{top}}(u)\end{array}}
\end{tikzcd}$$

One indication for the existence of such a theory is the fact that the refined topological string is enjoying a B-model interpretation as studied extensively in [49, 89, 110]. Nevertheless, a precise formulation of the underlying quantum field theory remained elusive with only partial progress made in topological [10, 11, 142, 147] and physical string theory [86, 150].

1.2.3 Outline of part II

The second part of the thesis aims at resolving some mysteries around the refined topological string by proposing a formulation of

$$F^X_{\text{ref top}}(\epsilon_1, \epsilon_2) \tag{1.7}$$

in terms of equivariant Gromov–Witten theory of $X \times \mathbb{A}^2$ with respect to some appropriate torus action on the extended target. We perform a detailed study of our proposal in case X is the resolved conifold and identify the $\epsilon_2 = 0$ limit for X a local surface with the relative Gromov–Witten theory of the surface together with a smooth anticanonical curve. Moreover, we formulate a refined BPS integrality conjecture motivated by the B-model calculations of Choi–Huang–Katz–Klemm [49, 89] and discuss evidence provided by our earlier case studies. We also mention aspects which are still work in progress [34].

Part I

The Logarithmic-Local-Open correspondence

Chapter 2

On quasi-tame Looijenga pairs

The content of this chapter is joint work with Andrea Brini which already appeared as [33] in *Communications in Number Theory and Physics* published by International Press.

Abstract. We prove a conjecture of Bousseau, van Garrel and the first-named author [Andrea Brini] relating, under suitable positivity conditions, the higher genus maximal contact log Gromov–Witten invariants of Looijenga pairs to other curve counting invariants of Gromov–Witten/Gopakumar–Vafa type. The proof consists of a closed-form q-hypergeometric resummation of the quantum tropical vertex calculation of the log invariants in presence of infinite scattering. The resulting identity of q-series appears to be new and of independent combinatorial interest.

2.1 Introduction

2.1.1 Quasi-tame Looijenga pairs

A Looijenga pair $Y(D) \coloneqq (Y, D)$ is the datum of a smooth rational complex projective surface Y and an anticanonical singular curve $D \in |-K_Y|$. A Looijenga pair $Y(D)$ is called *nef* if the singular curve D is a simple normal crossings divisor $D = \cup_{i=1}^{l} D_i$ with each D_i smooth, irreducible, and nef[1] for all $i = 1, \ldots, l$. A *tame* Looijenga pair is a nef pair with either $l > 2$, or $D_i^2 > 0$ for all i. Writing $E_{Y(D)} \coloneqq \mathrm{Tot}(\oplus_i(\mathcal{O}_Y(-D_i)))$, we will say that a nef pair $Y(D)$ is *quasi-tame* if there exists a tame pair $Y'(D')$ such that $E_{Y(D)}$ is deformation-equivalent to $E_{Y'(D')}$. By definition, there is an obvious sequence of nested inclusions

$$\text{nef Looijenga pairs} \supset \text{quasi-tame Looijenga pairs} \supset \text{tame Looijenga pairs}\,.$$

Looijenga pairs have been the focus of much attention lately due to their intertwined role in mirror

[1]Since we require D to be singular, an l-component nef Looijenga pair must have $l > 1$.

symmetry for surfaces [12, 14, 19, 78, 84, 126, 167, 168] and the study of cluster varieties [77, 125, 170]. In a recent series of papers [28, 29, 30], the log Gromov–Witten theory of quasi-tame pairs was further conjectured to be at the centre of a web of correspondences relating it to several enumerative theories. We recall the relevant context and fix notation below.

2.1.2 Enumerative theories

The authors of [29] consider four different geometries, and associated enumerative invariants, attached to the datum of a quasi-tame Looijenga pair $Y(D)$:

(i) the log Calabi–Yau surface obtained by viewing $Y(D)$ as a log-scheme for the divisorial log structure induced by D. For a given genus g and effective curve class $d \in \mathrm{H}_2(Y, \mathbb{Z})$ with $d \cdot D_i > 0$ for all $i = 1, \ldots, l$, the corresponding set of invariants are the log Gromov–Witten invariants [1, 43, 81] of $Y(D)$ with maximal tangency at each component D_i, $l-1$ point insertions on the surface, and one insertion of the top Chern class of the Hodge bundle:

$$N^{\log}_{g,d}(Y(D)) := \int_{[\overline{M}^{\log}_{g,l-1}(Y(D),d)]^{\mathrm{virt}}} \prod_{i=1}^{l-1} \mathrm{ev}_i^*[\mathrm{pt}_Y](-1)^g \lambda_g,$$

or equivalently, their all-genus generating function

$$\mathbb{N}^{\log}_d(Y(D))(\hbar) := \left(2 \sin\left(\frac{\hbar}{2}\right)\right)^{2-l} \sum_{g \geqslant 0} N^{\log}_{g,d}(Y(D)) \hbar^{2g-2+l};$$

(ii) the quasi-projective Calabi–Yau variety $E_{Y(D)}$, and its genus zero local Gromov–Witten invariants [45, 50]

$$N_d(E_{Y(D)}) := \int_{[\overline{M}_{0,l-1}(E_{Y(D)},d)]^{\mathrm{virt}}} \prod_{i=1}^{l-1} \mathrm{ev}_i^*[\mathrm{pt}_Y]$$

and local Gopakumar–Vafa invariants [92, 101]

$$\mathrm{GV}_d(E_{Y(D)}) := \sum_{k|d} \frac{\mu(k)}{k^{4-l}} N_d(E_{Y(D)})$$

where μ is the Möbius function;

(iii) the quasi-projective Calabi–Yau threefold $\mathrm{Tot}(\mathcal{O}_{Y \setminus \cup_{i<l} D_i}(-D_l))$ equipped with a disjoint union of $l-1$ Lagrangians L_i fibred over real curves in D_i, $i < l$, as defined in [29, Construction 6.4]:

$$Y^{\mathrm{op}}(D) := \left(\mathrm{Tot}\left(\mathcal{O}(-D_l) \to Y \setminus (D_1 \cup \cdots \cup D_{l-1})\right), L_1 \sqcup \cdots \sqcup L_{l-1}\right),$$

The respective invariants, for a given relative homology degree $d \in \mathrm{H}_2(Y^{\mathrm{op}}(D), \mathbb{Z})$, are the open Gromov–Witten counts

$$\begin{aligned} O_{g,d}(Y^{\mathrm{op}}(D)) &:= \int_{[\overline{M}_g(Y^{\mathrm{op}}(D),d)]^{\mathrm{virt}}} 1, \\ \mathbb{O}_d(Y^{\mathrm{op}}(D))(\hbar) &:= \sum_{g \geq 0} \hbar^{2g+l-3} O_{g,d}(Y^{\mathrm{op}}(D)) \end{aligned}$$

virtually enumerating genus-g open Riemann surfaces with $l-1$ connected components of the boundary ending on the Lagrangians L_i, $i=1,\dots l-1$. Under relatively lax conditions[2], $Y^{\rm op}(D)$ can be deformed to a singular Harvey–Lawson (Aganagic–Vafa) Lagrangian pair with $L_i \simeq \mathbb{R}^2 \times S^1$, for which open GW counts can be defined in the algebraic category [97, 117] (see also [59, 116]). Denoting by $w_i(d)$ the winding number of a relative degree-d open stable map to $Y^{\rm op}(D)$ around the non-trivial homology circle in L_i, we will also consider the corresponding genus zero/all-genus Labastida–Mariño–Ooguri–Vafa invariants [111, 112, 145]

$$\begin{aligned}
\mathrm{LMOV}_{0,d}(Y^{\rm op}(D)) &= \sum_{k|d} \frac{\mu(k)}{k^{4-l}} O_{0,d/k}(Y^{\rm op}(D)),\\
\mathbb{LMOV}_d(Y^{\rm op}(D))(\hbar) &= [1]_q^2 \prod_{i<l} \frac{w_i(d)}{[w_i(d)]_q} \sum_{k|d} \frac{\mu(k)}{k} \mathbb{O}_{d/k}(Y^{\rm op}(D))(k\hbar),
\end{aligned}$$

where $[n]_q \coloneqq q^{n/2}-q^{-n/2}$, and $q=\mathrm{e}^{\mathrm{i}\hbar}$;

(iv) for $l=2$, a symmetric quiver $\mathsf{Q}(Y(D))$ with adjacency matrix determined by $Y(D)$ [29, Theorem 7.3]. For a given charge vector d, the corresponding numbers are the numerical Donaldson–Thomas invariants $\mathrm{DT}_d^{\rm num}(\mathsf{Q}(Y(D)))$, defined as the formal Taylor coefficients of (the plethystic logarithm of) the generating series of Euler characteristics on the stack of representations of $\mathsf{Q}(Y(D))$.

The constructions of [29] in particular identify the absolute homology of $Y(D)$, the relative homology of $Y^{\rm op}(D)$, and the free abelian group over the set of vertices of $\mathsf{Q}(Y(D))$,

$$d \in \mathrm{H}_2(Y(D),\mathbb{Z}) \simeq \mathrm{H}_2(E_{Y(D)},\mathbb{Z}) \simeq \mathrm{H}_2^{\rm rel}(Y^{\rm op}(D),\mathbb{Z}) \simeq \mathbb{Z}^{|(\mathsf{Q}(Y(D)))_0|}.$$

Under these identifications, the authors of [29] propose that the invariants above are essentially the same, as follows.

Conjecture 2.A (The genus zero log/local/open correspondence, [29, 64, 119])**.** *The genus zero log, local, and open Gromov–Witten invariants associated to a quasi-tame Looijenga pair $Y(D)$ are related as*

$$N_d(E_{Y(D)}) = O_{0,d}(Y^{\rm op}(D)) = \left(\prod_{j=1}^{l} \frac{(-1)^{d\cdot D_j -1}}{d\cdot D_j}\right) N_{0,d}^{\rm log}(Y(D)),$$

and, for the associated BPS invariants,

$$\mathrm{GV}_d(E_{Y(D)}) = \mathrm{LMOV}_{0,d}(Y^{\rm op}(D)) \in \mathbb{Z}.$$

Moreover, if $l=2$,

$$\mathrm{DT}_d^{\rm num}(\mathsf{Q}(Y(D))) = |\mathrm{GV}_d(E_{Y(D)})|.$$

In [29, Conjecture 1.3], the above is further extended to an identity between all-genus Gromov–Witten generating functions.

[2] This was formalised as "Property O" in [29, Definition 6.3]: this is equivalent to requiring that $E_{Y(D)}$ is deformation-equivalent to $E_{Y'(D')}$ with Y' a toric weak Fano surface, D'_i a prime toric divisor for $i<l$, and D'_l nef. All quasi-tame pairs with $l=2$ satisfy Property O, and all non-tame quasi-tame pairs have $l=2$, so this is safely assumed to hold throughout this paper.

Conjecture 2.B (The higher genus log/open correspondence)**.** *The higher genus log and open Gromov–Witten invariants associated to a quasi-tame Looijenga pair $Y(D)$ are related as*

$$\mathbb{O}_d(Y^{\mathrm{op}}(D))(\hbar) = \left(\prod_{j=1}^{l-1} \frac{(-1)^{d\cdot D_j - 1}}{d \cdot D_j}\right) \frac{(-1)^{d\cdot D_l - 1}}{[d\cdot D_l]_q} \mathbb{N}_d^{\log}(Y(D))(\hbar).$$

Moreover,

$$\begin{aligned} \mathbb{LMOV}_d(Y^{\mathrm{op}}(D))(\hbar) &= [1]_q^2 \left(\prod_{i=1}^{l} \frac{1}{[d\cdot D_i]_q}\right) \\ &\times \sum_{k|d} \frac{(-1)^{d/k\cdot D + l}\mu(k)}{[k]_q^{2-l}\, k^{2-l}} \mathbb{N}_{d/k}^{\log}(Y(D))(k\hbar) \\ &\in \mathbb{Z}\left[q, q^{-1}\right]. \end{aligned} \tag{2.1}$$

Conjecture 2.A was proved in [29, Theorem 1.4–1.6]. Conjecture 2.B was proved in [29, Theorem 1.5 and 1.7] for tame $Y(D)$, and formulated as a conjecture for quasi-tame $Y(D)$ in [29, Conjecture 4.8]. Two non-tame, quasi-tame cases of this conjecture were subsequently proved in [109].

In this paper, we establish a stronger statement from which Conjecture 2.B follows for any quasi-tame Looijenga pair $Y(D)$. We will provide a closed-form calculation and comparison of both sides of (2.B), returning the two equalities in Conjecture 2.B as a corollary. Since the tame setting was already dealt with in [29], we may restrict our attention here to non-tame, quasi-tame pairs alone, for which $l = 2$. We give here a slightly discursive version of the main result of this paper (see Propositions 2.2.3, 2.2.5 and 2.2.7 for precise statements, and e.g. (2.2) and (2.4) for explicit formulas).

Theorem 2.C. *Let $Y(D)$ be a non-tame, quasi-tame Looijenga pair $Y(D)$ and $d \in \mathrm{H}_2(Y, \mathbb{Z})$. The log and open higher genus generating functions $\mathbb{N}_d^{\log}(Y(D))(\hbar)$ and $\mathbb{O}_d(Y^{\mathrm{op}}(D))(\hbar)$ are rational functions of $q = \mathrm{e}^{\mathrm{i}\hbar}$, with zeroes and poles only at $q = 0, \infty$, or at roots of unity. Furthermore, Conjecture 2.B holds.*

2.1.3 Strategy of the proof

Our task is simplified by a number of circumstances, which reduce Theorem 2.C to the computation of *one single example*. As explained in [60] and [29, Proposition 2.2], $Y(D)$ is fully determined by the self-intersections (D_1^2, D_2^2), and when considering specific examples we will shorten notation by writing $Y(D_1^2, D_2^2)$ for $Y(D)$. Let $\pi : \mathrm{dP}_r \to \mathbb{P}^2$ be the blow-up of the plane at $r \geq 0$ points $\{P_1, \dots, P_r\}$, and write $H := \pi^* c_1(\mathcal{O}_{\mathbb{P}^2}(1))$, $E_i := [\pi^{-1}(P_i)]$. Up to deformation and permutation of D_1, D_2 and $E_1, \dots, E_r$, there is a unique non-tame, quasi-tame pair of maximal Picard number given by $Y = \mathrm{dP}_3$ and $D_1 = H - E_1$, $D_2 = 2H - E_2 - E_3$, so that $(D_1^2, D_2^2) = (0, 2)$.

By [29, Proposition 2.2], any other non-tame, quasi-tame pair $Y(D)$ is the result of a toric contraction $\pi' : \mathrm{dP}_3(0,2) \to Y(D)$. Therefore, as we recall in Proposition 2.2.3 below, the blow-up formulas for log and open invariants [29, Proposition 4.3 and 6.9] imply that it suffices to check that Conjecture 2.B holds for the single case $Y(D) = \mathrm{dP}_3(0,2)$. The l.h.s. of (2.B) in that case was computed in

[29, Section 6.3.1] to be

$$\mathbb{O}_d\left(\mathrm{dP}_3^{\mathrm{op}}(0,2)\right) = (-1)^{d_1+d_2+d_3}\frac{[d_1]_q}{d_1[d_0]_q[d_1+d_2+d_3-d_0]_q}\begin{bmatrix} d_3 \\ d_0-d_1\end{bmatrix}_q \times \begin{bmatrix} d_3 \\ d_0-d_2\end{bmatrix}_q\begin{bmatrix} d_0 \\ d_3\end{bmatrix}_q\begin{bmatrix} d_1+d_2+d_3-d_0 \\ d_3\end{bmatrix}_q, \tag{2.2}$$

where we decomposed $d = d_0(H-E_1-E_2-E_3)+d_1E_1+d_2E_2+d_3E_3$ in terms of generators $\{H-E_1-E_2-E_3, E_1, E_2, E_3\}$ of $\mathrm{H}_2(\mathrm{dP}_3,\mathbb{Z})$, and for non-negative integers n, m we denoted[3]

$$[n]_q! := \prod_{i=1}^n [i]_q, \qquad \begin{bmatrix} n \\ m\end{bmatrix}_q := \begin{cases} \frac{[n]_q!}{[m]_q!\,[n-m]_q!} & 0\le m\le n, \\ 0 & \text{otherwise.}\end{cases} \tag{2.3}$$

The first equality (2.B) in Conjecture 2.B is then a consequence of the following

Proposition 2.1.1 (=Propositions 2.2.5 and 2.2.7)**.** *With the conventions above, we have*

$$\mathbb{N}_d^{\log}\left(\mathrm{dP}_3(0,2)\right)(\hbar) = \frac{[d_1]_q[d_2+d_3]_q}{[d_0]_q[d_1+d_2+d_3-d_0]_q}\begin{bmatrix} d_3 \\ d_0-d_1\end{bmatrix}_q\begin{bmatrix} d_3 \\ d_0-d_2\end{bmatrix}_q\begin{bmatrix} d_0 \\ d_3\end{bmatrix}_q\begin{bmatrix} d_1+d_2+d_3-d_0 \\ d_3\end{bmatrix}_q. \tag{2.4}$$

Indeed, comparing (2.2) with (2.4) we see that these generating series are related as in (2.B). The second statement, equation (2.1), then also follows from Proposition 2.1.1 combined with the BPS integrality result of [29, Theorem 8.1] for $l = 2$, whose proof applies identically to this case.

We will show Proposition 2.1.1 in two main steps. We will first construct a toric model for $\mathrm{dP}_3(0,2)$ in the sense of [78], and then compute the λ_g-log Gromov–Witten invariants ((i)) from the corresponding quantum scattering diagram and algebra of quantum broken lines [19, 20, 26, 52, 73, 74, 127, 128]. The lack of tameness is epitomised by the existence of infinite scattering when two quantum walls meet, and the resulting sum over quantum broken lines leads to the intricate-looking multi-variate generalised hypergeometric sum in (2.9). The second step consists of proving that this sum admits a closed-form q-hypergeometric resummation given by (2.4). To our knowledge, this has not previously appeared in the literature, with the exception of the special cases $d_3 = d_2 = d_0$ and $d_3 = d_0 = d_1 + d_2$ considered in [109]. To prove it, we first establish a difference equation satisfied by a 1-parameter deformation of (2.9), and then show inductively that the resulting recursion has a unique closed-form solution compatible with (2.9) by repeated use of Jackson's q-analogue of the Pfaff–Saalschütz summation for the ${}_3\phi_2$-hypergeometric function.

Acknowledgements

We thank P. Bousseau and M. van Garrel for many enlightening discussions surrounding the topic of this paper. We are particularly indebted to C. Krattenthaler for an illuminating e-mail correspondence occurred prior to the appearance of [109]. He exposed us to the idea that our sought-for q-hypergeometric identities were too rigid to be tackled directly, whereas suitable parametric refinements might be amenable to an effective recursive strategy. Special thanks are owed to him for this insight, which is key to the arguments of Section 2.2.2.4. We are also grateful to the anonymous referees for their valuable input.

[3] By [29, Proposition 2.5], the effectiveness of d implies that all arguments of the q-binomial expressions in (2.2) are non-negative integers.

2.2 Proof of Conjecture 2.B

2.2.1 Log GW invariants from the quantum tropical vertex

We start off by giving a summary of the combinatorial setup for the calculation of the higher genus log GW invariants ((i)) from the associated quantum scattering diagram, referring the reader to [29, Section 4.2] for a more extensive treatment.

2.2.1.1 Toric models. Two birational operations on log Calabi–Yau surfaces $Y(D)$ will feature prominently in the rest of the paper.

- If $\widetilde{Y}$ is the blow-up of Y at a node of D and $\widetilde{D}$ is the preimage of D in $\widetilde{Y}$ we say $\widetilde{Y}(\widetilde{D})$ is a **corner blow-up** of $Y(D)$.
- In case $\widetilde{Y}$ is the blow-up of Y at a smooth point in D and $\widetilde{D}$ is the strict transform of D in $\widetilde{Y}$ we say $\widetilde{Y}(\widetilde{D})$ is an **interior blow-up** of $Y(D)$.

Starting from a Looijenga pair $Y(D)$, we will seek to construct pairs $\widetilde{Y}(\widetilde{D})$ and $\overline{Y}(\overline{D})$ fitting into a diagram

$$\begin{tikzcd} & \widetilde{Y}(\widetilde{D}) \arrow[dl, "\phi"'] \arrow[dr, "\pi"] & \\ Y(D) & & \overline{Y}(\overline{D}) \end{tikzcd} \tag{2.5}$$

where ϕ is a sequence of corner blow-ups and π is a *toric model*, meaning that $\overline{Y}$ is toric, $\overline{D}$ is its toric boundary and π is a sequence of interior blow-ups. By [78, Proposition 1.3] such a diagram always exists. We will write ρ_{D_i} for the generator of the ray in the fan $\overline{\sigma}$ of $\overline{Y}$ associated to the toric prime divisor which is the push-forward along π of the strict transform of D_i under ϕ.

Given a toric model $\pi : \widetilde{Y}(\widetilde{D}) \longrightarrow \overline{Y}(\overline{D})$ as in (2.5) we can associate a *consistent quantum scattering diagram* $\mathfrak{D}$ to it, which is what we discuss next.

2.2.1.2 Quantum scattering and higher genus invariants. The scattering diagram $\mathfrak{D}$ is defined on the lattice $N_{\mathbb{Z}} \cong \mathbb{Z}^2$ of integral points of the fan $\overline{\sigma}$ of $\overline{Y}$, as follows: assume that π is a sequence of blow-ups of distinct smooth points $P_1, \ldots, P_s$ of $\overline{D}$ and denote $\mathcal{E}_1, \ldots, \mathcal{E}_s$ the exceptional divisors in $\widetilde{Y}$ associated to these blow-ups. Further, write ρ_j for the primitive generator of the ray associated to the toric prime divisor which P_j is an element of. For each $j \in \{1, \ldots, s\}$ we introduce a wall $\mathfrak{d}_j \coloneqq \mathbb{R}\,\rho_j$ decorated with wall-crossing functions $f_{\mathfrak{d}_j} \coloneqq 1 + t_j z^{-\rho_j} \in \mathbb{C}[\![t_1, \ldots, t_s]\!][N_{\mathbb{Z}}]$. We will often write $z^\rho \eqqcolon x^a y^b$ if $\rho = (a, b)$. Then $\mathfrak{D}$ is the unique (up to equivalence) completion of the initial scattering diagram $\mathfrak{D}_{\rm in} \coloneqq \{(\mathfrak{d}_j, f_{\mathfrak{d}_j})\}_{j\in\{1,\ldots,s\}}$ in the sense of [19, 20, 80, 108]. Such a completion can be found algorithmically by successively adding new rays whenever two walls meet, as we now describe.

First of all, after perturbing the diagram $\mathfrak{D}_{\rm in}$ we may assume that walls intersect in codimension at least one and that no more than two walls meet in a point. Now suppose two walls $\mathfrak{d}^1, \mathfrak{d}^2$ intersect. Denote by $-\rho^i$ a primitive integral direction of $\mathfrak{d}^i$ and assume $f_{\mathfrak{d}^i} = 1 + c_i z^{\rho^i}$. For our purpose the relevant scattering processes are:

- $\det(\rho^1,\rho^2)=\pm 1$ (*simple scattering*): the algorithm tells us to add a ray $\mathfrak{d}$ emanating from the intersection point $\mathfrak{d}^1\cap\mathfrak{d}^2$ in the direction $-\rho^1-\rho^2$ decorated with $1+c_1c_2z^{\rho^1+\rho^2}$.

- $\det(\rho^1,\rho^2)=\pm 2$ (*infinite scattering*): the algorithm creates three types of walls. First, there is a central wall in the direction $-\rho^1-\rho^2$ decorated with a wall-crossing function whose explicit shape is not of interest in the subsequent analysis and hence omitted. Further – and most relevant for us later – one needs to add walls $\mathfrak{d}_1,\dots,\mathfrak{d}_n,\dots$ with slope $-(n+1)\rho^1-n\rho^2$ decorated with
$$1+c_1^{n+1}c_2^nz^{(n+1)\rho^1+n\rho^2}$$
and last a collection ${}_1\mathfrak{d},\dots,{}_n\mathfrak{d},\dots$ of walls respectively having slope $-n\rho^1-(n+1)\rho^2$ and decorated with
$$1+c_1^nc_2^{n+1}z^{n\rho^1+(n+1)\rho^2}\,.$$

Adding new walls to the scattering diagram possibly creates new intersection points. Perturbing the diagram if necessary and repeating the above described process for each newly created intersection point yields a consistent scattering diagram $\mathfrak{D}$.

We now introduce the final combinatorial object we require for our computation of log Gromov–Witten invariants. Let $N_{\mathbb{R}}\coloneqq N_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{R}$. Given $p\in N_{\mathbb{R}}$ and $m\in N_{\mathbb{Z}}$, a **quantum broken line** β **with ends** p **and** m consists of

(i) a continuous piece-wise straight path $\beta:(-\infty,0]\to B\setminus\bigcup_{\mathfrak{d}\in\mathfrak{D}}\partial\mathfrak{d}\cup\bigcup_{\mathfrak{d}^1\neq\mathfrak{d}^2}\mathfrak{d}^1\cap\mathfrak{d}^2$ intersecting walls transversely;

(ii) a labelling $L_1,\dots,L_n$ of the successive line segments with L_n ending at p such that each intersection point $L_i\cap L_{i+1}$ lies on a wall;

(iii) an assignment $a_iz^{m_i}$ to each line segment L_i such that

   - $a_1z^{m_1}=z^m$,
   - if $L_i\cap L_{i+1}\subset\mathfrak{d}$ with $f_{\mathfrak{d}}=\sum_{r\geq 0}c_rz^{r\rho_{\mathfrak{d}}}$ and $\rho_{\mathfrak{d}}$ chosen primitive then $a_{i+1}z^{m_{i+1}}$ is a monomial occurring in the expansion of
$$a_iz^{m_i}\prod_{\ell=-\frac{1}{2}(|\det(\rho_{\mathfrak{d}},m_i)|-1)}^{\frac{1}{2}(|\det(\rho_{\mathfrak{d}},m_i)|-1)}\left(\sum_{r\geq 0}c_rq^{r\ell}z^{r\rho_{\mathfrak{d}}}\right), \tag{2.6}$$
   - L_i is directed in direction $-m_i$.

For such a quantum broken line β call $a_{\beta,\mathrm{end}}\coloneqq a_n$ the end-coefficient and write $m_{\beta,\mathrm{end}}\coloneqq m_n$. Moreover, we will often refer to $a_{\beta,\mathrm{end}}z^{m_{\beta,\mathrm{end}}}$ as the end-monomial and to z^m as the asymptotic monomial of β. We remark that for $f_{\mathfrak{d}}=1+cz^{\rho_{\mathfrak{d}}}$ the product in (2.6) takes the form
$$\prod_{\ell=-\frac{1}{2}(|\det(\rho_{\mathfrak{d}},m_i)|-1)}^{\frac{1}{2}(|\det(\rho_{\mathfrak{d}},m_i)|-1)}\left(1+cq^{\ell}z^{\rho_{\mathfrak{d}}}\right)=\sum_{k=0}^{|\det(\rho_{\mathfrak{d}},m_i)|}\begin{bmatrix}|\det(\rho_{\mathfrak{d}},m_i)|\\ k\end{bmatrix}_q c^kz^{k\rho_{\mathfrak{d}}}\,. \tag{2.7}$$

Now given two lattice vectors $m_1,m_2\in N_{\mathbb{Z}}$, define
$$C^{\mathfrak{D}}_{p;\,m_1,m_2}(q)\coloneqq\sum_{\substack{\beta_1,\beta_2\\ \mathrm{Ends}(\beta_i)=(p,m_i)\\ m_{\beta_1,\mathrm{end}}+m_{\beta_2,\mathrm{end}}=0}}a_{\beta_1,\mathrm{end}}\,a_{\beta_2,\mathrm{end}} \tag{2.8}$$

to be the sum of products of all end-coefficients of quantum broken lines β_1 and β_2 with asymptotic monomials z^{m_1}, resp. z^{m_2}, meeting in a common point p and with opposite exponents of their end-monomials. This sum is a polynomial in the variables t_j with coefficients in $\mathbb{Z}[q^{\pm\frac{1}{2}}]$. It turns out that the definition of $C^{\mathfrak{D}}_{p;\,m_1,m_2}(q)$ is mostly independent of p.

Proposition 2.2.1. *[128, Proposition 2.13 & 2.15] The sum in (2.8) is finite and as long as p is chosen generic, $C^{\mathfrak{D}}_{p;\,m_1,m_2}(q)$ is independent of the choice of p.*

Moreover, we remark that according to the same proposition [128, Proposition 2.15] $C^{\mathfrak{D}}_{p;\,m_1,m_2}(q)$ enjoys the interpretation as being the constant term in the product of two theta functions. However, most crucial for us, this quantity gives us a way to extract the higher genus, maximal contact log Gromov–Witten invariants ((i)) of a Looijenga pair $Y(D)$ from the scattering diagram associated to its toric model, as per the following

Proposition 2.2.2 ([29, 129])**.** *Let $Y(D)$ be a 2-component Looijenga pair, i.e. $D = D_1 + D_2$, $\pi : \widetilde{Y}(\widetilde{D}) \to \overline{Y}(\overline{D})$ a toric model for it as in (2.5), and $\mathfrak{D}$ the corresponding consistent quantum scattering diagram. For $d \in \mathrm{H}_2(Y, \mathbb{Z})$ an effective curve class, set $m_i \coloneqq (d \cdot D_i)\rho_{D_i}$. Then $\mathbb{N}^{\log}_d(Y(D))(\hbar)$ is the coefficient of $\prod_{j=1}^{s} t_j^{d\cdot\phi_*\mathcal{E}_j}$ in*

$$C^{\mathfrak{D}}_{p;\,m_1,m_2}(q)\bigg|_{q=\mathrm{e}^{i\hbar}}.$$

2.2.1.3 Birational invariance. Suppose now that $Y(D)$ is a 2-component quasi-tame Looijenga pair and $\pi : Y'(D') \to Y(D)$ is a sequence of interior blow-ups such that $Y'(D')$ is also quasi-tame. The following compatibility statement explains how the higher genus log-open correspondence interacts with this type of birational transformations.

Proposition 2.2.3. *[29, Proposition 4.3 and 6.9] Let $\pi : Y'(D') \to Y(D)$ be an interior blow-up, with both $Y'(D')$ and $Y(D)$ 2-component quasi-tame Looijenga pairs. Then $\mathbb{N}^{\log}_d(Y(D)) = \mathbb{N}^{\log}_{\pi^* d}(Y'(D'))$, $\mathbb{O}_d(Y^{\mathrm{op}}(D)) = \mathbb{O}_{\pi^* d}(Y'^{\mathrm{op}}(D'))$ for all $d \in \mathrm{H}_2(Y, \mathbb{Z})$.*

The comparison statement of Proposition 2.2.3 for log invariants is easy to visualise in genus 0, where the invariants $N^{\log}_{0,d}(Y(D))$ are enumerative [129]: since blowing up a point away from the curves does not affect the local geometry, the corresponding counts are invariant, a property also reflected in the broken lines calculations of the scattering diagrams of Section 2.2.1.2. The corresponding statement for the open invariants is a non-trivial consequence of the invariance of the topological vertex under flops [93, 104].

By the classification theorem of nef Looijenga pairs in [29, Proposition 2.2], any non-tame, quasi-tame Looijenga pair $Y(D)$ is obtained up to deformation as a sequence of $m \geq 0$ interior blowings-down of $\mathrm{dP}_3(0,2)$. It follows from Proposition 2.2.3 that proving Conjecture 2.B for $Y(D) = \mathrm{dP}_3(0,2)$ implies that the same statement *a fortiori* holds for any other non-tame, quasi-tame pair (and indeed any 2-component quasi-tame pair with Picard number lower than four).

2.2.2 The case of maximal Picard number

Let us then specialise to $Y(D) = \mathrm{dP}_3(0,2)$. Throughout this section we will write $d = d_0(H - E_1 - E_2 - E_3) + \sum_{i=1}^{3} d_i E_i$ for a curve class $d \in \mathrm{H}_2(\mathrm{dP}_3, \mathbb{Z})$. If $d \cdot D_1$ or $d \cdot D_2$ vanishes, then $\mathbb{N}^{\log}_d\big(\mathrm{dP}_3(0,2)\big)(\hbar) = 0$. In case the intersection numbers are strictly positive, we may use the scattering diagram of $\mathrm{dP}_3(0,2)$ to compute the invariants.

2.2.2.1 Constructing the toric model. First, we recall from [29, Section 4.4] the construction of a toric model of $\mathbb{P}^2(D_1 \cup D_2)$, with D_1 a line and D_2 a smooth conic intersecting D_1 in two distinct points P_1 and P_2 . Let $\mathcal{E}$ denote the tangent line through P_1 to D_2. In the following we will always identify D_1, D_2, $\mathcal{E}$, and some yet to be defined divisors F_1, F_2 with their strict transforms, resp. push-forwards, under blow-ups, resp. blow-downs.

Blow up the point P_1, and write F_1 for the exceptional divisor, and then blow up the intersection point of F_1 with D_2 and denote the exceptional curve by F_2. Under these blow-ups, the strict transform of $\mathcal{E}$ is a (-1)-curve intersecting F_2 in one point and can therefore be blown down. This blow-down results in the log Calabi–Yau surface $\left(\overline{\mathbb{P}^2(D)}, \overline{D}\right)$ with anti-canonical divisor $\overline{D} = D_1 \cup F_1 \cup F_2 \cup D_2$ where F_1 is a curve with self-intersection -2, D_2 has self-intersection 2, and D_1, F_2 zero. Therefore, since the irreducible components of $\overline{D}$ form a necklace with the same self-intersections as the toric boundary of $\mathbb{F}_2$, we already must have $\overline{\mathbb{P}^2(D)} = \mathbb{F}_2$ with $\overline{D}$ the toric boundary by [60, Lemma 2.10] and hence we have constructed a toric model for $\mathbb{P}^2(D)$. From the discussion of the previous Section, at a tropical level the fact that we blew down $\mathcal{E}$ amounts to adding a wall $\mathfrak{d}_{F_2}$ emanating from a *focus-focus singularity* on the ray corresponding to F_2 in the fan of $\mathbb{F}_2$.

The above construction results in a toric model for $\mathrm{dP}_3(0,2)$, as displayed in Figure 2.1: we blow up an interior point on D_1 and two interior points on D_2 and take the proper transforms, which at the fan level leads to the addition of focus-focus singularities on the rays corresponding to D_1 and D_2 (indicated with crosses in Figure 2.1). We denote by $\mathcal{E}_1$, resp. by $\mathcal{E}_2$, $\mathcal{E}_3$ the exceptional loci that result from blowing up a point in D_1, resp. two points on D_2.

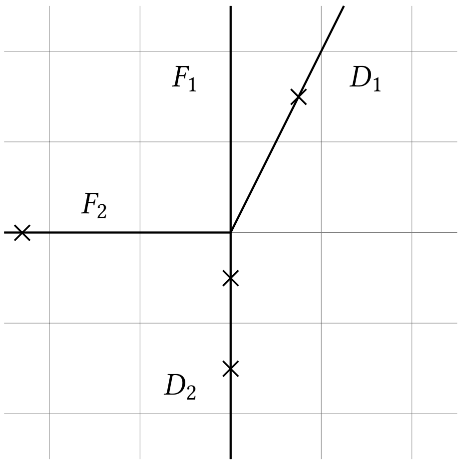


Figure 2.1: The toric model of $\mathrm{dP}_3(0,2)$

2.2.2.2 The quantum scattering diagram. We now follow the construction outlined in Section 2.2.1.2 to derive the quantum scattering diagram $\mathfrak{D}$ of the toric model of $\mathrm{dP}_3(0,2)$ (Figure 2.2). We shoot out walls $\mathfrak{d}_{F_2}$, $\mathfrak{d}_{D_1}$, $\mathfrak{d}_{D_2,1}$, $\mathfrak{d}_{D_2,2}$ emanating from the focus-focus singularities in the direction $-\rho$, where ρ is the respective generator of the ray in the fan. We send the singularities to infinity and perturb the walls as indicated in Figure 2.2. From these initial walls we now want to construct a consistent scattering diagram. However, since in our subsequent analysis we will only be interested in walls with slope lying in the cone generated by $(0,-1)$ and $(1,2)$, we will restrict the discussion to such walls only. As $|\rho_{F_2} \wedge \rho_{D_1}| = 2$ there is infinite scattering in the sense of Section 2.2.1.2 between the walls $\mathfrak{d}_{F_2}$ and $\mathfrak{d}_{D_1}$. This results in walls $\mathfrak{d}_2, \mathfrak{d}_3, \ldots$ with slope $-n\rho_{F_2} - (n-1)\rho_{D_1} = (1, -2(n-1))$ decorated with wallcrossing functions $1 + t^n t_1^{n-1} x^{-1} y^{2(n-1)}$ where $n > 1$. For conformity, let us write $\mathfrak{d}_1 \coloneqq \mathfrak{d}_{F_2}$. Now for all $n \geq 1$ each wall $\mathfrak{d}_n$ intersects both $\mathfrak{d}_{D_2,1}$ and $\mathfrak{d}_{D_2,2}$. Luckily, in this case we only have simple scattering resulting in walls with slope $(1, -2n+3)$ and wallcrossing functions $1+t^n t_1^{n-1} t_i x^{-1} y^{2n-3}$ where $i \in \{2, 3\}$. Lastly, we notice that the wall which is the result of scattering between $\mathfrak{d}_n$ and $\mathfrak{d}_{D_2,1}$ intersects $\mathfrak{d}_{D_2,2}$ thus producing a wall with slope $(1, -2n+4)$ and wallcrossing function $1 + t^n t_1^{n-1} t_2 t_3 x^{-1} y^{2n+4}$ attached to it. Let us call this wall $\mathfrak{d}_n^{D_2,D_2}$. The whole construction is summarised in Figure 2.2.

We collect the walls constructed above into 4-tuples labelled by an integer $n > 0$ as depicted in Figure 2.3. The nth tuple consists of the wall $\mathfrak{d}_n$, the walls which are the result of scattering between $\mathfrak{d}_n$ and $\mathfrak{d}_{D_2,1}$, resp. $\mathfrak{d}_{D_2,2}$, and lastly the wall $\mathfrak{d}_{n+1}^{D_2,D_2}$.

2.2.2.3 Higher genus log GW invariants. In this section we will apply Proposition 2.2.2 to obtain the log Gromov–Witten invariants of $\mathrm{dP}_3(0,2)$. For this we need to determine $C^{\mathfrak{D}}_{p;\,m_1,m_2}(q)$, where $m_i \coloneqq (d \cdot D_i)\rho_{D_i}$ and p is a generically chosen point. First, let us characterise all quantum broken lines which contribute to this sum.

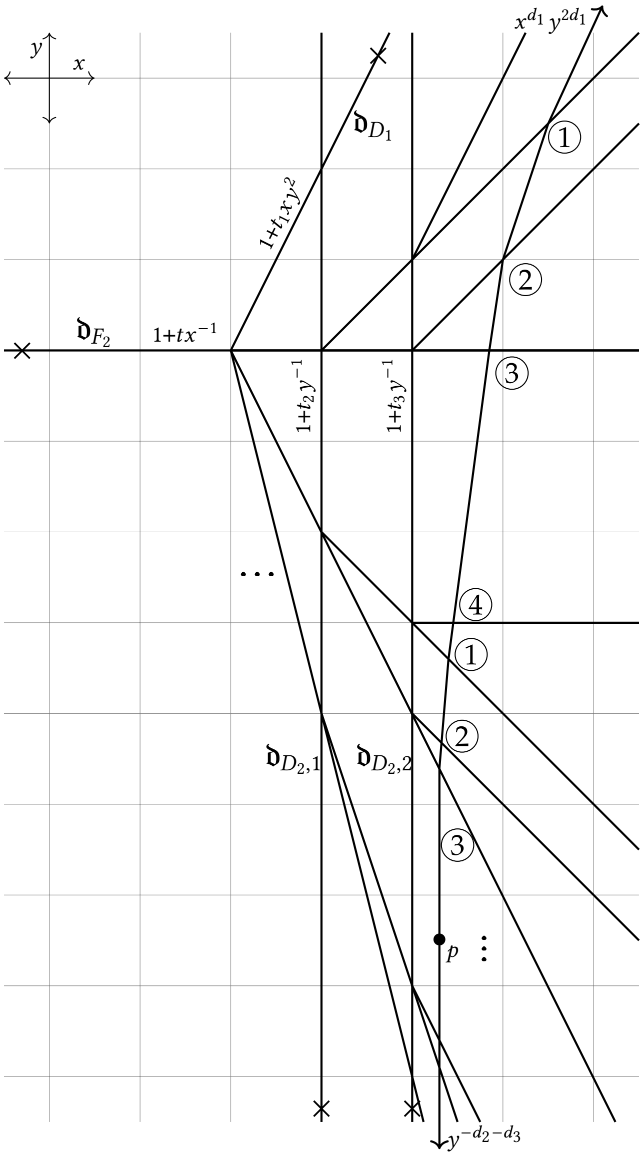


Figure 2.2: The quantum scattering diagram of $\mathrm{dP}_3(0,2)$.

Lemma 2.2.4. *Choose p in the lower right quadrant so that it lies to the right of $\mathfrak{d}_{D_2,2}$ and that below p there are only walls belonging to the nth 4-tuple of walls with $n \geq d \cdot D_2 + 2$. Then the following statements hold:*

(*i*) *If β_2 is a quantum broken line with ends (p, m_2) then either it is a straight line and thus $m_{\beta_2,\mathrm{end}} = m_2$ or $m_{\beta_2,\mathrm{end}}$ lies in the half open cone $\{-a_1\rho_{D_1} - a_2\rho_{D_2} \mid a_1 > 0,\ a_2 \geq 0\}$.*

(*ii*) *If β_1 and β_2 are quantum broken lines with ends (p, m_1) and (p, m_2) respectively such that*

$$m_{\beta_1,\mathrm{end}} + m_{\beta_2,\mathrm{end}} = 0$$

then β_2 is a straight line and β_1 may only bend at walls to the right of $\mathfrak{d}_{D_2,2}$ as in Figure 2.2.

Proof. Statement (i) can be proven by a straightforward, but tedious case-by-case analysis. Since this proof is barely enlightening we omit it here, and only explain how (i) implies (ii). Indeed, suppose β_2 is not a straight line. Then by (i) we must have $m_{\beta_1,\mathrm{end}} \in \{a_1\rho_{D_1} + a_2\rho_{D_2} \mid a_1 > 0,\ a_2 \geq 0\}$. However, this means that β_1 only crosses walls at which it may pick up a contribution that bends it further into the the lower right quadrant. In particular, such a quantum broken line cannot have asymptotic monomial z^{m_1}, leading to a contradiction. Hence, β_2 must be a straight line. □

Having Lemma 2.2.4 at hand, we are now equipped to determine the log Gromov–Witten invariants of $\mathrm{dP}_3(0,2)$ via Proposition 2.2.2.

Proposition 2.2.5. *Let d be an effective curve class with $d \cdot D_1, d \cdot D_2 > 0$. Then*

$$\begin{aligned} &\mathbb{N}_d^{\log}\big(\mathrm{dP}_3(0,2)\big)(\hbar) = \\ &\sum_{\substack{\forall (i,n) \in \{1,2,3,4\} \times \mathbb{Z}_{>0} : k_{i,n} \geq 0 \\ d_0 = \sum_{n\geq 1} \sum_{i=1}^4 (n + \delta_{i,1}) k_{i,n} \\ d_1 = \sum_{n \geq 1} \sum_{i=1}^4 k_{i,n} \\ d_0 - d_2 = \sum_{n\geq 1} (k_{1,n} + k_{4,n}) \\ d_0 - d_3 = \sum_{n \geq 1}(k_{1,n} + k_{3,n})}} \prod_{n \geq 1} \prod_{i=1}^{2} \begin{bmatrix} d_2 + d_3 - \sum_{m\geq 1} \sum_{j=1}^4 (2m - \delta_{j,3} - \delta_{j,4}) k_{j,n+m} \\ k_{i,n} \end{bmatrix}_q \\ &\qquad\qquad \times \begin{bmatrix} d_2 + d_3 - \sum_{m \geq 0} \sum_{j=1}^4 (2m + \delta_{j,1} + \delta_{j,2}) k_{j,n+m} \\ k_{2+i,n} \end{bmatrix}_q . \end{aligned} \tag{2.9}$$

Proof. In order to compute $\mathbb{N}_d^{\log}\big(\mathrm{dP}_3(0,2)\big)$ we choose a point p as specified in Lemma 2.2.4 and consider quantum broken lines β_i with ends $(p, z^{(d \cdot D_i)\rho_{D_i}})$, where $i \in \{1,2\}$, so that the sum of the exponents of their end-monomials at p vanishes. As stated in Proposition 2.2.2, we then obtain the desired log Gromov–Witten invariant by taking the product of the two end-coefficients, summing this over all such pairs of quantum broken lines and extracting the coefficient of the monomial $t^{d \cdot \phi_* \mathcal{E}} \prod_{i=1}^3 t_i^{d \cdot \phi_* \mathcal{E}_i}$.

Let us then analyse a quantum broken line β_1 coming from the direction D_1 with asymptotic monomial $z^{(d \cdot D_1)\rho_{D_1}} = x^{d_1} y^{2d_1}$ ending at p which only bends at walls to the right of $\mathfrak{d}_{D_2,2}$ as illustrated in Figure 2.2. We claim that its end-monomial is of the form

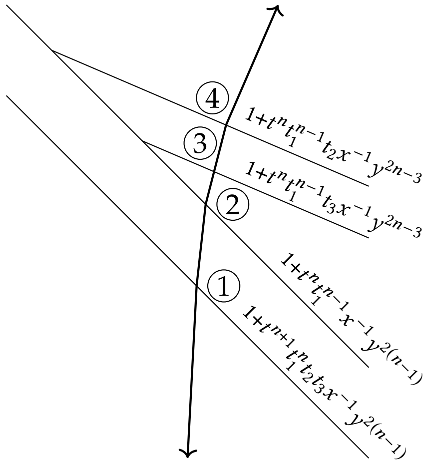


Figure 2.3: The nth 4-tuple of walls.

$$
\begin{aligned}
a_{\beta_1,\mathrm{end}} z^{m_{\beta_1,\mathrm{end}}} &= \prod_{n=1}^{N} \prod_{i=1}^{2} \begin{bmatrix} 2nd_1 - \sum_{l=1}^{n} \sum_{j=1}^{4} (2(n-l) - \delta_{j,3} - \delta_{j,4}) k_{j,l} \\ k_{i,n} \end{bmatrix}_q \\
&\times \begin{bmatrix} (2n-1)d_1 - \sum_{l=1}^{n-1} \sum_{j=1}^{4} (2(n-l) - \delta_{j,1} - \delta_{j,2}) k_{j,l} \\ k_{2+i,n} \end{bmatrix}_q \\
&\times t^{\sum_{n=1}^{N} \sum_{j=1}^{4} (n+\delta_{j,1}) k_{j,n}} t_1^{\sum_{n=1}^{N} \sum_{j=1}^{4} (n-1+\delta_{j,1}) k_{j,n}} \\
&\times t_2^{\sum_{n=1}^{N} (k_{1,n}+k_{4,n})} t_3^{\sum_{n=1}^{N} (k_{1,n}+k_{3,n})} x^{d_1 - \sum_{n=1}^{N} \sum_{j=1}^{4} k_{j,n}} \\
&\times y^{2d_1 + \sum_{n=1}^{N} (2n-2)(k_{1,n}+k_{2,n}) + \sum_{n=1}^{N} (2n-3)(k_{3,n}+k_{4,n})}
\end{aligned}
\tag{2.10}
$$

for some $k_{j,n} \geq 0$ where we set $N = d \cdot D_2 + 1$. Indeed, suppose that after passing the first $(n-1)$ 4-tuples of walls β_2 carries the monomial $a_{n-1} x^{b_{n-1}} y^{c_{n-1}}$. Then crossing wall ④ of the nth 4-tuple displayed in Figure 2.3 we see that by the defining property (iii) of a quantum broken line and formula (2.7) the monomial carried by the quantum broken line must now be

$$
\begin{aligned}
a_{n-1} x^{b_{n-1}} y^{c_{n-1}} \overset{④}{\longmapsto} a_{n-1} &\begin{bmatrix} (2n-3)b_{n-1} + c_{n-1} \\ k_{4,n} \end{bmatrix}_q t^{nk_{4,n}} t_1^{(n-1)k_{4,n}} \\
&\times t_2^{k_{4,n}} x^{b_{n-1} - k_{4,n}} y^{c_{n-1} + (2n-3)k_{4,n}}
\end{aligned}
$$

for some $k_{4,n} \geq 0$. Similarly, after passing the next three walls, which together with ④ form the nth

4-tuple of walls, the quantum broken line carries the monomial

$$
\begin{aligned}
& a_{n-1}\begin{bmatrix}(2n-3)b_{n-1}+c_{n-1}\\ k_{4,n}\end{bmatrix}_q t^{nk_{4,n}} t_1^{(n-1)k_{4,n}} t_2^{k_{4,n}} x^{b_{n-1}-k_{4,n}} y^{c_{n-1}+(2n-3)k_{4,n}} \\
& \qquad \overset{③}{\longmapsto} a_{n-1}\begin{bmatrix}(2n-3)b_{n-1}+c_{n-1}\\ k_{3,n}\end{bmatrix}_q \begin{bmatrix}(2n-3)b_{n-1}+c_{n-1}\\ k_{4,n}\end{bmatrix}_q \\
& \qquad\qquad \times t^{n(k_{3,n}+k_{4,n})} t_1^{(n-1)(k_{3,n}+k_{4,n})} t_2^{k_{4,n}} t_3^{k_{3,n}} \\
& \qquad\qquad \times x^{b_{n-1}-k_{3,n}-k_{4,n}} y^{c_{n-1}+(2n-3)(k_{3,n}+k_{4,n})} \\
& \qquad \overset{②}{\longmapsto} a_{n-1}\begin{bmatrix}(2n-2)b_{n-1}+c_{n-1}+(k_{3,n}+k_{4,n})\\ k_{2,n}\end{bmatrix}_q \\
& \qquad\qquad \times \begin{bmatrix}(2n-3)b_{n-1}+c_{n-1}\\ k_{3,n}\end{bmatrix}_q \begin{bmatrix}(2n-3)b_{n-1}+c_{n-1}\\ k_{4,n}\end{bmatrix}_q \\
& \qquad\qquad \times t^{\sum_{j=2}^4 nk_{j,n}} t_1^{\sum_{j=2}^4 (n-1)k_{j,n}} t_2^{k_{4,n}} t_3^{k_{3,n}} \\
& \qquad\qquad \times x^{b_{n-1}-\sum_{j=2}^4 k_{j,n}} y^{c_{n-1}+(2n-2)k_{2,n}+(2n-3)(k_{3,n}+k_{4,n})} \\
& \qquad \overset{①}{\longmapsto} a_{n-1}\begin{bmatrix}(2n-2)b_{n-1}+c_{n-1}+(k_{3,n}+k_{4,n})\\ k_{1,n}\end{bmatrix}_q \\
& \qquad\qquad \times \begin{bmatrix}(2n-2)b_{n-1}+c_{n-1}+(k_{3,n}+k_{4,n})\\ k_{2,n}\end{bmatrix}_q \\
& \qquad\qquad \times \begin{bmatrix}(2n-3)b_{n-1}+c_{n-1}\\ k_{3,n}\end{bmatrix}_q \begin{bmatrix}(2n-3)b_{n-1}+c_{n-1}\\ k_{4,n}\end{bmatrix}_q \\
& \qquad\qquad \times t^{\sum_{j=1}^4 (n+\delta_{j,1})k_{j,n}} t_1^{\sum_{j=1}^4 (n-1+\delta_{j,1})k_{j,n}} t_2^{k_{1,n}+k_{4,n}} t_3^{k_{1,n}+k_{3,n}} \\
& \qquad\qquad \times x^{b_{n-1}-\sum_{j=1}^4 k_{j,n}} y^{c_{n-1}+(2n-2)(k_{1,n}+k_{2,n})+(2n-3)(k_{3,n}+k_{4,n})} .
\end{aligned}
\tag{2.11}
$$

Since we know that $(b_0, c_0) = m_1 = (d_1, 2d_1)$, by induction we can deduce that

$$
\begin{aligned}
b_n &= d_1 - \sum_{l=1}^n \sum_{j=1}^4 k_{j,l} , \\
c_n &= 2d_1 + \sum_{l=1}^n (2l-2)(k_{1,l}+k_{2,l}) + \sum_{l=1}^n (2l-3)(k_{3,l}+k_{4,l}) .
\end{aligned}
$$

Substituting this into the binomial coefficients in the last line of (2.11) and noting that

$$
\begin{aligned}
(2n-3)b_{n-1}+c_{n-1} &= (2n-1)d_1 - \sum_{l=1}^{n-1} (2(n-l)-1)(k_{1,l}+k_{2,l}) \\
& \quad - \sum_{l=1}^{n-1} 2(n-l)(k_{3,l}+k_{4,l})
\end{aligned}
$$

and

$$\begin{aligned}(2n-2)b_{n-1}+c_{n-1}+(k_{3,n}+k_{4,n}) &= 2nd_1-\sum_{l=1}^{n}2(n-l)(k_{1,l}+k_{2,l}) \\ &- \sum_{l=1}^{n}(2(n-l)-1)(k_{3,l}+k_{4,l})\end{aligned}$$

we indeed see that the end-monomial $a_{\beta_1,\mathrm{end}}z^{m_{\beta_1,\mathrm{end}}}$ is of the form (2.10).
Now by definition, $C^{\mathfrak{D}}_{p;m_1,m_2}(q)$ is the sum of $a_{\beta_1,\mathrm{end}}a_{\beta_2,\mathrm{end}}$ over all quantum broken lines β_i, $i\in\{1,2\}$, with ends (p,m_i) such that

$$m_{\beta_1,\mathrm{end}}+m_{\beta_2,\mathrm{end}}=0\,. \tag{2.12}$$

Since by Lemma 2.2.4 β_2 must be a straight line, we have $m_{\beta_2,\mathrm{end}}=(d\cdot D_2)\rho_{D_2}=(0,-d_2-d_3)$ and thus (2.12) translates into the conditions

$$d_1=\sum_{n=1}^{N}\sum_{i=1}^{4}k_{i,n}, \tag{2.13}$$

$$d_2+d_3=\sum_{n=1}^{N}\big(2n(k_{1,n}+k_{2,n})+(2n-1)(k_{3,n}+k_{4,n})\big)\,. \tag{2.14}$$

Hence, plugging $a_{\beta_2,\mathrm{end}}=1$ and (2.10) into the defining expression (2.8) for $C^{\mathfrak{D}}_{p;m_1,m_2}(q)$ we get

$$\begin{aligned}C^{\mathfrak{D}}_{p;m_1,m_2}(q) &= \\ &\sum_{\substack{\forall(i,n)\in\{1,2,3,4\}\times\{1,\dots,N\}:\,k_{i,n}\geq 0\\ d_1=\sum_{n=1}^{N}\sum_{i=1}^{4}k_{i,n}\\ d_2+d_3=\sum_{n=1}^{N}\sum_{i=1}^{4}(2n-\delta_{i,3}-\delta_{i,4})k_{i,n}}} t^{\sum_{n=1}^{N}\sum_{j=1}^{4}(n+\delta_{j,1})k_{j,n}}t_1^{\sum_{n=1}^{N}\sum_{j=1}^{4}(n-1+\delta_{j,1})k_{j,n}} \\ &\qquad\times t_2^{\sum_{n=1}^{N}(k_{1,n}+k_{4,n})}t_3^{\sum_{n=1}^{N}(k_{1,n}+k_{3,n})} \\ &\times\prod_{n=1}^{N}\prod_{i=1}^{2}\begin{bmatrix}2nd_1-\sum_{l=1}^{n}\sum_{j=1}^{4}(2(n-l)-\delta_{j,3}-\delta_{j,4})k_{j,l}\\ k_{i,n}\end{bmatrix}_q \\ &\times\begin{bmatrix}(2n-1)d_1-\sum_{l=1}^{n-1}\sum_{j=1}^{4}(2(n-l)-\delta_{j,1}-\delta_{j,2})k_{j,l}\\ k_{2+i,n}\end{bmatrix}_q.\end{aligned}$$

Using the sum conditions (2.13) and (2.14) we can simplify the arguments of the q-binomials to bring the sum into the form

$$\begin{aligned}C^{\mathfrak{D}}_{p;m_1,m_2}(q) &= \\ &\sum_{\substack{\forall(i,n)\in\{1,2,3,4\}\times\{1,\dots,N\}:\,k_{i,n}\geq 0\\ d_1=\sum_{n=1}^{N}\sum_{i=1}^{4}k_{i,n}\\ d_2+d_3=\sum_{n=1}^{N}\sum_{i=1}^{4}(2n-\delta_{i,3}-\delta_{i,4})k_{i,n}}} t^{\sum_{n=1}^{N}\sum_{j=1}^{4}(n+\delta_{j,1})k_{j,n}}t_1^{\sum_{n=1}^{N}\sum_{j=1}^{4}(n-1+\delta_{j,1})k_{j,n}} \\ &\qquad\times t_2^{\sum_{n=1}^{N}(k_{1,n}+k_{4,n})}t_3^{\sum_{n=1}^{N}(k_{1,n}+k_{3,n})} \\ &\times\prod_{n=1}^{N}\prod_{i=1}^{2}\begin{bmatrix}d_2+d_3-\sum_{m=1}^{N-n}\sum_{j=1}^{4}(2m-\delta_{j,3}-\delta_{j,4})k_{j,n+m}\\ k_{i,n}\end{bmatrix}_q \\ &\times\begin{bmatrix}d_2+d_3-\sum_{m=0}^{N-n}\sum_{j=1}^{4}(2m-\delta_{j,1}-\delta_{j,2})k_{j,n+m}\\ k_{2+i,n}\end{bmatrix}_q\end{aligned} \tag{2.15}$$

We can now apply Proposition 2.2.2 which states that $\mathbb{N}_d^{\log}\big(\mathrm{dP}_3(0,2)\big)$ is the coefficient of

$$t^{d\cdot\phi_*\mathcal{E}}\prod_{i=1}^{3} t_i^{d\cdot\phi_*\mathcal{E}_i} = t^{d_0} t_1^{d_0-d_1} t_2^{d_0-d_2} t_3^{d_0-d_3}\,. \tag{2.16}$$

in (2.15). Here, ϕ is the sequence of corner blow-ups $\widetilde{\mathrm{dP}_3}(\widetilde{D}) \longrightarrow \mathrm{dP}_3(D)$ and $\mathcal{E},\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3$ are the exceptional curves we introduced in Section 2.2.2.1. The above identity follows from the fact that $[\phi_*\mathcal{E}] = H$ and $[\phi_*\mathcal{E}_i] = E_i$. Now picking the coefficient of (2.16) in (2.15) we obtain that $\mathbb{N}_d^{\log}\big(\mathrm{dP}_3(0,2)\big)$ is the sum of

$$\begin{aligned}\prod_{n\geq 1}\prod_{i=1}^{2} &\begin{bmatrix} d_2+d_3-\sum_{m=1}^{N-n}\big(2m(k_{1,n+m}+k_{2,n+m})+(2m-1)(k_{3,n+m}+k_{4,n+m})\big) \\ k_{i,n}\end{bmatrix}_q \\ &\times\begin{bmatrix} d_2+d_3-\sum_{m=0}^{N-n}\big((2m+1)(k_{1,n+m}+k_{2,n+m})+2m(k_{3,n+m}+k_{4,n+m})\big) \\ k_{2+i,n}\end{bmatrix}_q\end{aligned} \tag{2.17}$$

over $k_{j,n}\geq 0$, $j\in\{1,\ldots,4\}$ and $n\in\{1,\ldots,N\}$ subject to the conditions (2.13), (2.14) and

$$d_0 = \sum_{n=1}^{N}\left(k_{1,n}+n\sum_{i=1}^{4}k_{i,n}\right), \tag{2.18}$$

$$d_0-d_1 = \sum_{n=1}^{N}\left(k_{1,n}+(n-1)\sum_{i=1}^{4}k_{i,n}\right), \tag{2.19}$$

$$d_0-d_2 = \sum_{n=1}^{N}(k_{1,n}+k_{4,n})\,, \tag{2.20}$$

$$d_0-d_3 = \sum_{n=1}^{N}(k_{1,n}+k_{3,n})\,. \tag{2.21}$$

Notice here that (2.18), (2.20), and (2.21) together imply (2.14), and that (2.19) follows from subtracting (2.13) from (2.18). Hence, it is sufficient to impose conditions (2.13), (2.18), (2.20), and (2.21) only. Note that these constraints are exactly the ones appearing in the sum in (2.9). Moreover, (2.17) exactly matches the summand on the right-hand side of (2.9) and hence the claimed formula is proven. □

Remark 2.2.6. It is easy to convince oneself that there are actually only finitely many summands contributing to (2.9), due to the finite number of choices $(k_{i,n})$ satisfying the summation conditions. Moreover, the product in each of these summands is well-defined since the first sum condition forces $k_{i,n}=0$ for all $n>d_0$. Thus, only a finite number of binomials can be different from one and therefore the whole expression becomes well-defined.

To prove that the right-hand side of (2.9) returns (2.4), it will be helpful to consider a 1-parameter deformation of (2.9), as follows. For non-negative integers a,b,c,d,e, we write

$$\begin{aligned}&G(a,b,c,d,e) := \\ &\sum_{\substack{\forall (i,n)\in\{1,2,3,4\}\times\mathbb{Z}_{>0}:\,k_{i,n}\geq 0 \\ a=\sum_{n\geq 1}\sum_{i=1}^{4}(n+\delta_{i,1})k_{i,n} \\ b=\sum_{n\geq 1}\sum_{i=1}^{4}k_{i,n} \\ c=\sum_{n\geq 1}(k_{1,n}+k_{4,n}) \\ d=\sum_{n\geq 1}(k_{1,n}+k_{3,n})}} \prod_{n\geq 1}\prod_{i=1}^{2}\begin{bmatrix} e-\sum_{m\geq 1}\sum_{j=1}^{4}(2m-\delta_{j,3}-\delta_{j,4})k_{j,n+m} \\ k_{i,n}\end{bmatrix}_q \\ &\qquad\qquad\times\begin{bmatrix} e-\sum_{m\geq 0}\sum_{j=1}^{4}(2m+\delta_{j,1}+\delta_{j,2})k_{j,n+m} \\ k_{2+i,n}\end{bmatrix}_q\,.\end{aligned} \tag{2.22}$$

By (2.9), our sought-for log Gromov–Witten generating function is obtained from (2.22) via the restriction

$$\mathbb{N}_d^{\log}\big(\mathrm{dP}_3(0,2)\big) = G(d_0, d_1, d_0 - d_2, d_0 - d_3, d_2 + d_3)\,. \tag{2.23}$$

We claim that (2.22) has a simple closed-form summation, as follows.

Proposition 2.2.7. *For all $a, b, c, d, e \in \mathbb{Z}_{\geq 0}$ we have that*

$$\begin{aligned} G(a,b,c,d,e) = &\begin{bmatrix} b-a+e \\ b-c \end{bmatrix}_q \begin{bmatrix} c-a+d+e \\ c \end{bmatrix}_q \begin{bmatrix} a-c \\ d \end{bmatrix}_q \begin{bmatrix} a-d \\ b-d \end{bmatrix}_q \\ &- \begin{bmatrix} b-a+e-1 \\ b-c \end{bmatrix}_q \begin{bmatrix} c-a+d+e-1 \\ c \end{bmatrix}_q \begin{bmatrix} a-c-1 \\ d \end{bmatrix}_q \begin{bmatrix} a-d-1 \\ b-d \end{bmatrix}_q . \end{aligned} \tag{2.24}$$

In particular, from (2.23), it follows that

$$\begin{aligned} &\mathbb{N}_d^{\log}\big(\mathrm{dP}_3(0,2)\big)(\hbar) = \\ &\quad \frac{[d_1]_q [d_2+d_3]_q}{[d_0]_q [d_1+d_2+d_3-d_0]_q} \begin{bmatrix} d_3 \\ d_0-d_1 \end{bmatrix}_q \begin{bmatrix} d_3 \\ d_0-d_2 \end{bmatrix}_q \begin{bmatrix} d_0 \\ d_3 \end{bmatrix}_q \begin{bmatrix} d_1+d_2+d_3-d_0 \\ d_3 \end{bmatrix}_q . \end{aligned}$$

We will give an inductive proof of Proposition 2.2.7 in the next Section by seeking a suitable recursive relation in the parameter a, broadly following the lead of [109]. The proof is composed of three main steps:

(i) We first establish (2.24) for the base cases $a = 0$ and $b = 0$ by a direct analysis of the quantum broken line sum in (2.22), which in these cases is either vanishing, or reduces to a single summand (Lemma 2.2.8).

(ii) We then establish, for $a, b > 0$, a difference equation satisfied by $G(a,b,c,d,e)$ in the parameters a, b, c and d. This equation recursively and uniquely determines $G(a,b,c,d,e)$ from the knowledge of the base case $G(a',b',c',d',e)$ with $0 \leq a' < a$ (Lemma 2.2.9).

(iii) We finally check that the r.h.s. of (2.24) is also a solution of the difference equation, and conclude by uniqueness (Proposition 2.2.11). The proof is fundamentally based on a classical q-hypergeometric summation result in the form of the q-Pfaff–Saalschütz formula (Lemma 2.2.10).

2.2.2.4 Proof of Proposition 2.2.7. We start by considering the base cases $a = 0$ and $b = 0$ in the induction procedure.

Lemma 2.2.8. *We have that*

$$G(0,b,c,d,e) = \delta_{b,0}\delta_{c,0}\delta_{d,0}\,, \quad G(a,0,c,d,e) = \delta_{a,0}\delta_{c,0}\delta_{d,0}\,. \tag{2.25}$$

In particular, Proposition 2.2.7 holds when $a = 0$ or $b = 0$.

Proof. Both equalities are implied by the conditions on the range of summation of (2.22): setting $a = 0$ or $b = 0$ implies that $k_{i,n} = 0$ for all $(i,n) \in \{1,2,3,4\} \times \mathbb{Z}_{>0}$. In this case the r.h.s. of (2.22) is equal to one for $c = d = 0$, while if either $c \neq 0$ or $d \neq 0$ it vanishes being an empty sum.

To see that (2.25) agrees with (2.24), note that when $a = 0$ the second summand in (2.24) must always vanish, while the first one is non-zero only if $b = c = d = 0$, in which case it is equal to 1. Likewise, when $b = 0$, the right-hand side of (2.24) can only be non-zero if $c = d = 0$: this is equal to 1 for $a = 0$ by the previous analysis, and for $a > 0$ the contributions from the two summands cancel each other, since

$$\begin{aligned}
&\begin{bmatrix} e-a \\ -c \end{bmatrix}_q \begin{bmatrix} c-a+d+e \\ c \end{bmatrix}_q \begin{bmatrix} a-c \\ d \end{bmatrix}_q \begin{bmatrix} a-d \\ -d \end{bmatrix}_q \\
&\quad - \begin{bmatrix} e-a-1 \\ -c \end{bmatrix}_q \begin{bmatrix} c-a+d+e-1 \\ c \end{bmatrix}_q \begin{bmatrix} a-c-1 \\ d \end{bmatrix}_q \begin{bmatrix} a-d-1 \\ -d \end{bmatrix}_q \\
&= \delta_{c,0}\,\delta_{d,0} - \delta_{c,0}\,\delta_{d,0} = 0\,.
\end{aligned}$$

□

Lemma 2.2.9. *For $a, b \in \mathbb{Z}_{>0}$, $c, d, e \in \mathbb{Z}_{\geq 0}$, $G(a, b, c, d, e)$ satisfies the following recursion in (a, b, c, d):*

$$\begin{aligned}
&G(a,b,c,d,e) = \\
&\quad \sum_{k_{j,1}\geq 0} \prod_{i=1}^{2} \begin{bmatrix} e-2a+2b+c+d-k_{3,1}-k_{4,1} \\ k_{i,1} \end{bmatrix}_q \begin{bmatrix} e-2a+b+c+d \\ k_{i+2,1} \end{bmatrix}_q \\
&\quad \times G(a-b-k_{1,1}, b-k_{1,1}-k_{2,1}-k_{3,1}-k_{4,1}, c-k_{1,1}-k_{4,1}, d-k_{1,1}-k_{3,1}, e)\,.
\end{aligned} \tag{2.26}$$

Proof. Looking at the defining equation (2.22) for $G(a, b, c, d, e)$, we see that the sum conditions imply that

$$\begin{aligned}
\sum_{m\geq 1} \big(2m(k_{1,1+m}+k_{2,1+m}) + (2m-1)(k_{3,1+m}+k_{4,1+m})\big) \\
= 2a-2b-c-d+k_{3,1}+k_{4,1}
\end{aligned}$$

and

$$\sum_{m\geq 0} \big((2m+1)(k_{1,1+m}+k_{2,1+m}) + 2m(k_{3,1+m}+k_{4,1+m})\big) = 2a-b-c-d\,.$$

Hence, they allow us to rewrite the four q-binomial coefficients that correspond to the factor $n = 1$ in the product over $n \geq 1$ in (2.22) as

$$\begin{aligned}
&\prod_{i=1}^{2} \begin{bmatrix} e-\sum_{m\geq 1}\big(2m(k_{1,1+m}+k_{2,1+m})+(2m-1)(k_{3,1+m}+k_{4,1+m})\big) \\ k_{i,n} \end{bmatrix}_q \\
&\times \begin{bmatrix} e-\sum_{m\geq 0}\big((2m+1)(k_{1,1+m}+k_{2,1+m})+2m(k_{3,1+m}+k_{4,1+m})\big) \\ k_{2+i,n} \end{bmatrix}_q \\
&\qquad = \prod_{i=1}^{2} \begin{bmatrix} e-2a+2b+c+d-k_{3,1}-k_{4,1} \\ k_{i,1} \end{bmatrix}_q \begin{bmatrix} e-2a+b+c+d \\ k_{i+2,1} \end{bmatrix}_q\,.
\end{aligned}$$

This factor is independent of all $k_{i,n}$ with $n > 1$ and hence in (2.22) we can factor it out of the sum over these integers which gives us

$$
\begin{aligned}
&G(a,b,c,d,e) = \\
&\quad \sum_{k_{j,1}\geq 0} \prod_{i=1}^{2} \begin{bmatrix} e-2a+2b+c+d-k_{3,1}-k_{4,1} \\ k_{i,1} \end{bmatrix}_q \begin{bmatrix} e-2a+b+c+d \\ k_{i+2,1} \end{bmatrix}_q \\
&\qquad \times \sum_{\substack{\forall (i,n)\in\{1,2,3,4\}\times\mathbb{Z}_{>1}:\, k_{i,n}\geq 0 \\ a=\sum_{n\geq 1}\sum_{i=1}^{4}(n+\delta_{i,1})k_{i,n} \\ b=\sum_{n\geq 1}\sum_{i=1}^{4} k_{i,n} \\ c=\sum_{n\geq 1}(k_{1,n}+k_{4,n}) \\ d=\sum_{n\geq 1}(k_{1,n}+k_{3,n})}} \prod_{n\geq 2}\prod_{i=1}^{2} \begin{bmatrix} e-\sum_{m\geq 1}\sum_{j=1}^{4}(2m-\delta_{j,3}-\delta_{j,4})k_{j,n+m} \\ k_{i,n} \end{bmatrix}_q \\
&\qquad\qquad \times \begin{bmatrix} e-\sum_{m\geq 0}\sum_{j=1}^{4}(2m+\delta_{j,1}+\delta_{j,2})k_{j,n+m} \\ k_{2+i,n} \end{bmatrix}_q .
\end{aligned} \tag{2.27}
$$

Now using (2.22) again in order to explicitly write out

$$
\begin{aligned}
&G(a-b-k_{1,1}, b-k_{1,1}-k_{2,1}-k_{3,1}-k_{4,1}, c-k_{1,1}-k_{4,1}, d-k_{1,1}-k_{3,1}, e) = \\
&\sum_{\substack{\forall (i,n)\in\{1,2,3,4\}\times\mathbb{Z}_{>1}:\, k_{i,n}\geq 0 \\ a-b-k_{1,1}=\sum_{n\geq 2}\sum_{i=1}^{4}(n-1+\delta_{i,1})k_{i,n} \\ b-\sum_l k_{l,1}=\sum_{n\geq 2}\sum_{i=1}^{4} k_{i,n} \\ c-k_{1,1}-k_{4,1}=\sum_{n\geq 2}(k_{1,n}+k_{4,n}) \\ d-k_{1,1}-k_{3,1}=\sum_{n\geq 2}(k_{1,n}+k_{3,n})}} \prod_{n\geq 2}\prod_{i=1}^{2} \begin{bmatrix} e-\sum_{m\geq 1}\sum_{j=1}^{4}(2m-\delta_{j,3}-\delta_{j,4})k_{j,n+m} \\ k_{i,n} \end{bmatrix}_q \\
&\qquad\qquad \times \begin{bmatrix} e-\sum_{m\geq 0}\sum_{j=1}^{4}(2m+\delta_{j,1}+\delta_{j,2})k_{j,n+m} \\ k_{2+i,n} \end{bmatrix}_q .
\end{aligned} \tag{2.28}
$$

we notice that the sum conditions in (2.28) are actually equivalent to the ones in the third line of (2.27). Hence, we can identify (2.28) with line three and four of (2.27) and so we arrive at the recursion formula (2.26). □

Define now

$$
\tilde{G}(a,b,c,d,e) := \begin{bmatrix} b-a+e \\ b-c \end{bmatrix}_q \begin{bmatrix} c-a+d+e \\ c \end{bmatrix}_q \begin{bmatrix} a-c \\ d \end{bmatrix}_q \begin{bmatrix} a-d \\ b-d \end{bmatrix}_q , \tag{2.29}
$$

so that the right-hand side of (2.24) equates to $\tilde{G}(a,b,c,d,e)-\tilde{G}(a-1,b,c,d,e-2)$. We shall make extensive use of Jackson's q-analogue of the Pfaff–Saalschütz summation in the form it is stated in [169, Equation (1q)].

Lemma 2.2.10 (The q-Pfaff–Saalschütz Theorem, [169])**.** *For integers $A,B,C,D\geq 0$ we have*

$$
\begin{aligned}
&\sum_{k\geq 0} \frac{[A+B+C+D-k]_q!}{[k]_q!\,[A-k]_q!\,[B-k]_q!\,[C-k]_q!\,[D+k]_q!} = \\
&\qquad\qquad \begin{bmatrix} A+B+D \\ B \end{bmatrix}_q \begin{bmatrix} A+C+D \\ A \end{bmatrix}_q \begin{bmatrix} B+C+D \\ C \end{bmatrix}_q .
\end{aligned}
$$

Proposition 2.2.11. *Let $a,b\in\mathbb{Z}_{>0}$ and $c,d,e\in\mathbb{Z}_{\geq 0}$. Then $\tilde{G}(a,b,c,d,e)$ satisfies the same recursion (2.26) as $G(a,b,c,d,e)$, i.e.*

$$
\begin{aligned}
&\tilde{G}(a,b,c,d,e) = \\
&\quad \sum_{k_1,k_2,k_3,k_4\geq 0} \prod_{i=1}^{2} \begin{bmatrix} e-2a+2b+c+d-k_3-k_4 \\ k_i \end{bmatrix}_q \begin{bmatrix} e-2a+b+c+d \\ k_{i+2} \end{bmatrix}_q \\
&\qquad\qquad \times \tilde{G}(a-b-k_1, b-k_1-k_2-k_3-k_4, c-k_1-k_4, d-k_1-k_3, e) .
\end{aligned} \tag{2.30}
$$

Proof. In order to prove (2.30) we will repeatedly use Lemma 2.2.10. We start from the right-hand side of (2.30), plug in the definition of $\tilde{G}$ given in (2.29), expand the binomials, and collect all factorials involving k_1 and k_2. With the help of a computer algebra system one can check that these steps result in the following expression for the r.h.s. of (2.30):

$$\begin{aligned}
&\sum_{k_3,k_4\geq 0} \begin{bmatrix} b-2a+c+d+e \\ k_3 \end{bmatrix}_q \begin{bmatrix} b-2a+c+d+e \\ k_4 \end{bmatrix}_q \\
&\quad\times \frac{[a-b-d+k_3]_q!\,[2b-2a+c+d+e-k_3-k_4]_q!^2\,[a-b-c+k_4]_q!}{[b-a+d+e-k_3]_q!\,[b-a+c+e-k_4]_q!} \\
&\quad\times \sum_{k_1\geq 0} \frac{[b-a+c+d+e-k_1-k_3-k_4]_q!}{[k_1]_q!\,[c-k_1-k_4]_q!\,[d-k_1-k_3]_q!} \\
&\qquad\quad\times \frac{1}{[2b-2a+c+d+e-k_1-k_3-k_4]_q!\,[a-b-c-d+k_1+k_3+k_4]_q!} \\
&\quad\times \sum_{k_2\geq 0} \frac{[2b-a+e-k_2-k_3-k_4]_q!}{[k_2]_q!\,[b-c-k_2-k_3]_q!\,[b-d-k_2-k_4]_q!} \\
&\qquad\quad\times \frac{1}{[2b-2a+c+d+e-k_2-k_3-k_4]_q!\,[a-2b+k_2+k_3+k_4]_q!}\,.
\end{aligned}$$

We can now use (2.30) to perform the sum over k_1 and k_2. Collecting the factorials depending on k_3 and k_4 in the resulting expression, we get

$$\begin{aligned}
&\frac{[a-b]_q!\,[a-c-d]_q!\,[b-2a+c+d+e]_q!^2}{[b-a+e]_q!\,[c-a+d+e]_q!} \\
&\quad\times \sum_{k_3\geq 0} \frac{[b-a+d+e-k_3]_q!}{[k_3]_q!\,[d-k_3]_q!\,[b-c-k_3]_q!\,[b-2a+c+d+e-k_3]_q!\,[a-b-d+k_3]_q!} \\
&\quad\times \sum_{k_4\geq 0} \frac{[b-a+c+e-k_4]_q!}{[k_4]_q!\,[c-k_4]_q!\,[b-d-k_4]_q!\,[b-2a+c+d+e-k_4]_q!\,[a-b-c+k_4]_q!}\,.
\end{aligned}$$

Thus, we can use (2.30) again to carry out the sums over k_3 and k_4 to deduce that the right-hand side of (2.30) equals

$$\frac{[a-c]_q!\,[a-d]_q!\,[b-a+e]_q!\,[c-a+d+e]_q!}{[a-b]_q!\,[b-c]_q!\,[c]_q!\,[b-d]_q!\,[a-c-d]_q!\,[d]_q!\,[c-a+e]_q!\,[d-a+e]_q!}\,.$$

The above is exactly the expansion of $\tilde{G}(a,b,c,d,e)$ into factorials, proving (2.30). ◻

Using Proposition 2.2.11 we can conclude the proof of Proposition 2.2.7.

Proof of Proposition 2.2.7. By (2.26), for $a>0$ and $b>0$, $G(a,b,c,d,e)$ is determined by the value of $G(a',b',c',d',e)$ for $0\leq a<a'$. This means that $G(a,b,c,d,e)$ is the unique solution of (2.26) compatible with the boundary value in (2.25) for $G(0,b,c,d,e)$. By the second part of Lemma 2.2.8, the r.h.s. of (2.24) indeed returns the correct value of $G(0,b,c,d,e)$ and $G(a,0,c,d,e)$ from (2.22), so all that is left to do is to check that it solves (2.26), and conclude by uniqueness.

To this aim, it is sufficient to argue using Proposition 2.2.11: note first of all that the coefficients of the recursion are invariant under the shift $(a,e)\to(a-1,e-2)$. Therefore, since $\tilde{G}(a,b,c,d,e)$ is a solution of (2.26) by Proposition 2.2.11, then so is $\tilde{G}(a-1,b,c,d,e-2)$. Furthermore, since (2.26) is linear in G, their difference is also a solution. But by (2.29) this is exactly the claimed expression for $G(a,b,c,d,e)$ in (2.24), which concludes the proof. □

Corollary 2.2.12. *Conjecture 2.B holds for any quasi-tame pair $Y(D)$.*

Proof. The tame case having been treated in [29], it suffices to restrict to non-tame, quasi-tame pairs. Taking the specialisation (2.23) of (2.24) leads to (2.4), which together with (2.2) establishes the first equality of Conjecture 2.B for $Y(D) = \mathrm{dP}_3(0,2)$. Since [29, Section 6.3.1]

$$\mathbb{O}_d\left(\mathrm{dP}_3^{\mathrm{op}}(0,2)\right) = \mathbb{O}_d\left(\mathrm{dP}_3^{\mathrm{op}}(1,1)\right) ,$$

the BPS integrality statement in the second equality further follows without any modification from the proof of [29, Theorem 8.1] for $l = 2$. Finally, since every non-tame, quasi-tame pair $Y(D)$ is related to $\mathrm{dP}_3(0,2)$ by a series of $m \geq 0$ iterated interior blow-ups at general points of D [29, Proposition 2.2], Proposition 2.2.3 further implies that Conjecture 2.B holds for any such pair. □

Chapter

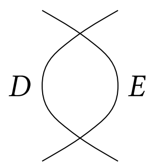
Gromov-Witten theory of bicyclic pairs 3

The entirety of this chapter is joint work with Michel van Garrel and Navid Nabijou which first appeared as [65].

Abstract. A bicyclic pair is a smooth surface equipped with a pair of smooth divisors intersecting in two reduced points. Resolutions of self-nodal curves constitute an important special case. We investigate the logarithmic Gromov–Witten theory of bicyclic pairs. We establish all-genus correspondences with local Gromov–Witten theory and open Gromov–Witten theory, and a genus zero correspondence with orbifold Gromov–Witten theory. For self-nodal curves in $\mathbb{P}(1,1,r)$ we obtain a closed formula for the genus zero invariants and relate these to the invariants of local curves. We also establish a conceptual relationship to the invariants obtained by smoothing the self-nodal curve. The technical heart of the paper is a qualitatively new analysis of the degeneration formula for stable logarithmic maps, complemented by torus localisation and scattering techniques.

3.1 Introduction

A **bicyclic pair** is a pair $(S \,|\, D+E)$ consisting of a smooth surface S and a pair of smooth divisors $D, E \subseteq S$ intersecting in two reduced points:



These arise in two distinct contexts:

(i) **Looijenga pairs** with two boundary components [123]. In this case $D+E \in |-K_S|$ with D and E both rational. When D and E are nef there is a simple classification [29, Table 1]. The general classification proceeds via the minimal model program, see [61, Lemma 3.2] or [60, Section 2].

(ii) **Self-nodal pairs.** Consider a surface containing an irreducible curve with a single nodal singularity. Let S be the blowup of the surface at the node, D the strict transform of the irreducible curve, and E the exceptional divisor. Then $(S \mid D + E)$ is a bicyclic pair. Note that E is not nef.

We study the Gromov–Witten theory of bicyclic pairs; by [6] this includes the Gromov–Witten theory of self-nodal pairs. We investigate connections to local, open and orbifold geometries, obtain closed formulae in important special cases, and probe the behaviour under smoothing of the divisor. The main theories we consider are:

$$\begin{tikzcd}
\boxed{(S \mid D+E)} \arrow[r, leftrightarrow, "\text{Section 3.2}"] & \boxed{(\mathcal{O}_S(-D) \mid E)} \arrow[r, leftrightarrow, "\text{Section 3.2.5}"] \arrow[d, leftrightarrow, "\text{Section 3.3}"] & \boxed{\mathcal{O}_S(-D) \oplus \mathcal{O}_S(-E)} \\
 & \boxed{\mathcal{O}_S(-D)|_{S \setminus E}} \arrow[r, "\text{Section 3.4}"] & \boxed{\mathcal{O}_{\mathbb{P}^1}(r) \oplus \mathcal{O}_{\mathbb{P}^1}(-r-2)}
\end{tikzcd}$$

We work in several different settings. The precise hypotheses are stated clearly at the start of each section. Globally, the paper proceeds from most general to most specific.

3.1.1 Logarithmic-local correspondence (Sections 3.2.1–3.2.4)

Our first main result establishes an all-genus correspondence between the Gromov–Witten theories of the following pairs:

$$(S \mid D + E) \leftrightarrow (\mathcal{O}_S(-D) \mid E).$$

This arises by considering the degeneration to the normal cone of the divisor $D \subseteq S$. This is a well-established technique [21, 64, 160] but additional complications occur due to the non-trivial logarithmic structure on the general fibre (see Remark 3.1.1).

Consider a bicyclic pair $(S \mid D + E)$ and fix a curve class $\beta \in A_1(S)$. We assume:

- $D \cong \mathbb{P}^1$ and $D^2 \geq 0$.
- $D \cdot \beta > 0$ and $E \cdot \beta \geq 0$.[1]

Fix a genus g and tangency data $\mathbf{c}$ with respect to E. From $\mathbf{c}$ we build tangency data $\hat{\mathbf{c}}$ with respect to $D + E$ by appending an additional marked point with tangency $D \cdot \beta$ along D.

Finally let γ be a collection of D-avoidant insertions (Definition 3.2.2) of the correct codimension. This permits markings with no insertions, as well as descendants of evaluation classes disjoint from D.

Theorem 3.A (Theorem 3.2.3)**.** *We have the following equality of generating functions:*

$$\frac{(-1)^{D\cdot\beta-1}}{2\sin\left(\frac{D\cdot\beta}{2}\hbar\right)}\left(\sum_{g\geq 0}\mathsf{GW}_{g,\hat{\mathbf{c}},\beta}(S \mid D+E)\langle(-1)^g\lambda_g\gamma\rangle\cdot\hbar^{2g-1}\right) = \sum_{g\geq 0}\mathsf{GW}_{g,\mathbf{c},\beta}(\mathcal{O}_S(-D) \mid E)\langle\gamma\rangle\cdot\hbar^{2g-2}.$$

Remark 3.1.1. The proof proceeds via the degeneration formula for stable logarithmic maps [3, 149] applied to the degeneration to the normal cone of $D \subseteq S$. The general fibre has non-trivial logarithmic structure corresponding to E, which greatly complicates the analysis.

[1] The case $E\cdot\beta = 0$ can occur, e.g. for resolutions of self-nodal pairs. In this case, the spaces of stable logarithmic maps to $(S \mid D + E)$ and $(S|D)$ have the same virtual dimension, but their logarithmic Gromov–Witten invariants typically differ. See Remark 3.2.4.

We develop novel combinatorial techniques for constraining the shapes of rigid tropical types, and intersection theoretic techniques for establishing the vanishing of certain contributions (see Section 3.2.4). We expect these to serve as a useful toolkit for future calculations.

Having established Theorem 3.A, we explore two proximate results in genus zero.

3.1.2 Nef pairs (Section 3.2.5)

Combining Theorem 3.A with [64, Theorem 1.1] we obtain:

Theorem 3.B (Theorem 3.2.26)**.** *Suppose that $E^2 \geq 0$ and $E \cdot \beta > 0$ and let $\mathbf{c} = (E \cdot \beta)$ be maximal tangency contact data to E. Then we have:*

$$\mathsf{GW}_{0,\hat{\mathbf{c}},\beta}(S \,|\, D+E)\langle\gamma\rangle = (-1)^{(D+E)\cdot\beta}(D\cdot\beta)(E\cdot\beta)\cdot \mathsf{GW}_{0,\mathbf{c},\beta}(\mathcal{O}_S(-D)\oplus\mathcal{O}_S(-E))\langle\gamma\rangle.$$

This gives another instance of the numerical logarithmic-local correspondence for normal crossings pairs [28, 29, 30, 159]. See Remark 3.2.30 for the importance of D-avoidant insertions.

3.1.3 Root stacks and self-nodal pairs (Section 3.2.6)

We next impose that $E \cdot \beta = 0$. This includes the case of self-nodal pairs discussed above. In this setting we provide an alternative proof of Theorem 3.A, by passing through the logarithmic-orbifold [16] and orbifold-local [17] correspondences (Theorem 3.2.28). This allows us to remove the assumptions that D is rational and γ is D-avoidant. The key intermediate result is:

Theorem 3.C (Proposition 3.2.29)**.** *Suppose that $E\cdot\beta = 0$. Then there is an equality between logarithmic and orbifold Gromov–Witten invariants:*

$$\mathsf{GW}^{\mathsf{log}}_{0,\hat{\mathbf{c}},\beta}(S \,|\, D+E)\langle\gamma\rangle = \mathsf{GW}^{\mathsf{orb}}_{0,\hat{\mathbf{c}},\beta}(S \,|\, D+E)\langle\gamma\rangle.$$

This holds without assuming that $D \cong \mathbb{P}^1$ or that γ is D-avoidant.

This result is proved using [16, Theorem X]. The main technical step is to strongly constrain the shapes of tropical types of maps to $(S \,|\, D+E)$.

3.1.4 Toric and open geometries (Section 3.3)

We next specialise to toric Calabi–Yau pairs. We assume:

- S is toric.
- E is a toric hypersurface.
- $D+E \in |{-K_S}|$.
- $E\cdot\beta = 0$.

We do not require that D is toric. In this setting, we obtain a logarithmic-open correspondence:

Theorem 3.D (Theorem 3.3.3)**.** *For all $g \geq 0$ we have:*

$$\mathsf{GW}_{g,0,\beta}(\mathcal{O}_S(-D)|E) = \mathsf{GW}^T_{g,0,\iota^*\beta}(\mathcal{O}_S(-D)|_{S\setminus E})$$

where the Gromov–Witten invariant on the right-hand side is defined by localising with respect to the action of the Calabi–Yau torus T (see Section 3.3.2).

The proof proceeds by localisation on both sides. The difficult step is to establish the vanishing of certain contributions, by isolating a weight zero piece of the obstruction bundle. For this it is crucial that we localise with respect to the Calabi–Yau torus T.

The target $\mathcal{O}_S(-D)|_{S\setminus E}$ is a toric Calabi–Yau threefold and hence its invariants can be computed using the topological vertex formalism [116]. Theorem 3.D combined with Theorem 3.A thus provides a new means to efficiently compute the all-genus logarithmic invariants of $(S \mid D+E)$, otherwise a highly tedious endeavour.

3.1.5 Self-nodal curves: calculations (Section 3.4.1)

We now specialise to our motivating example. Consider $S_r := \mathbb{P}(1,1,r)$ and let $D_r \in |-K_{S_r}|$ be an irreducible curve with a single nodal singularity at the singular point of S_r. We consider degree d curves in the pair $(S_r|D_r)$ which meet D_r in a single point of maximal tangency order $d(r+2)$.

Theorem 3.E (Theorem 3.4.1)**.** *We have:*

$$\mathsf{GW}_{0,(d(r+2)),d}(S_r|D_r) = \frac{r+2}{d^2}\binom{(r+1)^2 d-1}{d-1}.$$

Remark 3.1.2. The numerator $(r+2)$ is the number of flex lines. The remaining factors strongly resemble the multiple cover formula of [80, Proposition 6.1], however for a curve of tangency order $(r+1)^2+1$.

Theorem 3.E is proved via computations on the corresponding local scattering diagram. Passing to the resolution $\mathbb{F}_r \to S_r$ and combining Theorems 3.D and 3.E we immediately obtain the following formula, already known in the physics literature [41, Equation (4.53)]:

Theorem 3.F (Theorem 3.4.2)**.** *We have:*

$$\mathsf{GW}^T_{0,0,d}\big(\mathcal{O}_{\mathbb{P}^1}(r)\oplus\mathcal{O}_{\mathbb{P}^1}(-r-2)\big) = \frac{(-1)^{rd-1}}{d^3}\binom{(r+1)^2 d-1}{d-1}.$$

3.1.6 Self-nodal curves: smoothings (Section 3.4.2)

In the final section we focus on the case $r=1$. We compare the Gromov–Witten invariants of $(\mathbb{P}^2|D)$ and $(\mathbb{P}^2|E)$ where D and E are nodal and smooth plane cubics. Experimentally we observe that the former are always smaller than the latter. We provide a conceptual explanation for this defect, via the enumerative geometry of degenerating hypersurfaces. As in [13] we degenerate both D and E to the toric boundary $\Delta \subseteq \mathbb{P}^2$ and consider the invariants of the logarithmically singular central fibre.

Theorem 3.G (Theorem 3.4.7)**.** *The invariants of* $(\mathbb{P}^2|D)$ *are precisely the central fibre contributions to the invariants of* $(\mathbb{P}^2|E)$ *arising from multiple covers of a single coordinate line.*

Finally (Theorem 3.4.10) we apply Theorem 3.G to settle a conjecture in [13].

Remark 3.1.3. Taken together, Theorems 3.E, 3.F, 3.G relate the Gromov–Witten invariants arising from two non-standard obstruction theories on the space of stable maps to $\mathbb{P}^1$: the local geometry and the degenerated hypersurface.

3.1.7 Context

The present paper fits into the broader body of work on logarithmic-local-open-quiver correspondences [9, 21, 23, 28, 29, 30, 31, 33, 46, 47, 48, 57, 64, 66, 67, 119, 141, 158, 161, 166]. By [17] this is intimately connected to the logarithmic-orbifold correspondence, another area of intense study [2, 16, 38, 39, 160]. This area enjoys close connections to Mirror Symmetry [22, 25, 58, 72, 73, 74, 122, 154, 159, 164, 165].

In [141] it is shown that the naïve logarithmic-local correspondence fails for normal crossings pairs, and a corrected form is established. It is then observed that in many situations, the insertions cap trivially with the correction terms, collapsing the corrected correspondence to the naïve correspondence. The results of this paper provide another instance of this phenomenon.

Our techniques can be used to recover [31, Theorem 1.7], which relates the invariants of $(S|D)$ and $\mathcal{O}_S(-D)$ by applying the more general [21, Theorem 1.1]. Following [23, Proposition 3.1] we pass to a deformation and identify the invariants of $(S|D)$ with the invariants of $(S \mid D+E)$ with tangency orders $(1,\ldots,1)$ along E. Similarly we identify the invariants of $\mathcal{O}_S(-D)$ with the invariants of $(\mathcal{O}_S(-D) \mid E)$. The correspondence [31, Theorem 1.7] then follows from Theorem 3.A.

3.1.8 Prospects

3.1.8.1 Quivers. Two independent results in the literature suggest a relationship between Gromov–Witten invariants of bicyclic pairs and Donaldson–Thomas invariants of quivers:

(i) **Nef pairs.** In [31] the authors equate the Gromov–Witten invariants of $(S|D)$ and $\mathcal{O}_S(-D)$ to the Donaldson–Thomas invariants of certain quivers. The quivers in question differ by explicit framings, exhibiting one quiver moduli space as a projective bundle over the other, see [31, Equation (16)]. This produces a relationship between the Donaldson–Thomas invariants, involving the same correction factor as appears in Theorem 3.A. The direct link to our results (see Section 3.1.7) suggests a similar identification for the invariants of the nef pair $(S \mid D+E)$.

(ii) **Self-nodal pairs.** The Gromov–Witten invariants of a self-nodal pair are governed by the wall-crossing functions attached to the central ray of a local scattering diagram (see Section 3.4.1 and [80]). This is intimately related to quiver invariants by [79, 152].

It would be worthwhile to compare these two cases, and in the case of self-nodal pairs to obtain a relationship between local invariants and quiver invariants.

3.1.8.2 Scattering. Theorem 3.E computes the genus zero Gromov–Witten invariants of $\mathbb{P}(1,1,r)$ relative to a self-nodal anticanonical divisor. We achieve this by equating these invariants with the wall-crossing function of the central ray of a local scattering diagram, with incoming ray directions ρ_1, ρ_2 satisfying $|\rho_1 \wedge \rho_2| = r+2$. We then use [79, Equation (1.4)] proven by Reineke [151] for $\ell_1 = \ell_2$ to arrive at Theorem 3.E, from which we deduce Theorem 3.F.

It may be possible to reverse this logic, proving Theorem 3.F first via a direct analysis of the local invariants (as in [41]) and using this to deduce Theorem 3.E and hence [79, Equation (1.4)]. Speculatively, it may also be possible to study the case $\ell_1 \neq \ell_2$, by extending the arguments of Section 3.4 to weighted projective planes of the form $\mathbb{P}(1,a,b)$.

Parts of Section 3.4 extend readily to higher genus using Bousseau's quantum scattering [20]. An analysis of the central ray of the quantum scattering diagram of the Kronecker quiver, via a correspondence with the all-genus Gromov–Witten generating function of local $\mathbb{P}^1$, is an attractive prospect.

3.1.8.3 Logarithmic-open. Combining Theorems 3.A and 3.D we obtain a logarithmic-open correspondence for two-component Looijenga pairs [29, Conjecture 1.3] in the setting where the curve class pairs trivially with one of the components. If the curve class pairs non-trivially with both components, we still expect Theorem 3.A to lead to progress on the logarithmic-open correspondence. We intend to investigate this in future work.

3.1.8.4 Higher dimensions. We restrict to surface geometries in this paper. This ensures a simplified balancing condition at codimension-2 strata, used crucially in Section 3.2.4.4 to eliminate the contributions of rigid tropical types which are not star-shaped. The resulting Proposition 3.2.18 strongly resembles [21, Theorem 1.1]. Intriguingly, however, the latter result holds in all dimensions.

We speculate that the arguments of Section 3.2.4.4 may be refined to produce a higher-dimensional analogue of Proposition 3.2.18, leading to a higher-dimensional analogue of Theorem 3.A. Continuing in this vein, we speculate that the same arguments may be adapted to treat pairs $(S \mid D + E)$ such that the intersection $D \cap E$ consists of three or more smooth connected components.

Acknowledgements

This project originated in joint discussions with Andrea Brini, Pierrick Bousseau, and Tim Gräfnitz. We are grateful for their generosity and insights. We have benefited from conversations with Luca Battistella, Francesca Carocci, Samuel Johnston, and Dhruv Ranganathan. Special thanks are owed to Qaasim Shafi and Longting Wu, for extremely helpful clarifications at key points.

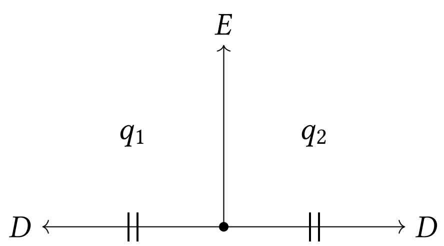
Parts of this work were carried out during research visits at the Isaac Newton Institute, the University of Sheffield, the University of Birmingham, Imperial College London, and Queen Mary University of London. We are grateful to these institutions for hospitality and financial support.

3.2 Bicyclic pairs

We introduce bicyclic pairs $(S \mid D + E)$. The main result of this section (Theorem 3.2.3) establishes a precise correspondence between the logarithmic Gromov–Witten theories of the pairs $(S \mid D + E)$ and $(\mathcal{O}_S(-D) \mid E)$. This occupies Sections 3.2.1–3.2.4. Applications and variations are discussed in Sections 3.2.5 and 3.2.6.

3.2.1 Setup and statement of correspondence

Definition 3.2.1. A **bicyclic pair** $(S \mid D + E)$ consists of a smooth projective surface S and smooth divisors $D, E \subseteq S$ such that D and E intersect in two reduced points, denoted q_1 and q_2. The normal crossings pair $(S \mid D + E)$ has tropicalisation:



(3.1)

Fix a bicyclic pair $(S \mid D + E)$ and a curve class $\beta \in A_1(S)$ such that:

- $D \cong \mathbb{P}^1$ and $D^2 \geq 0$.
- $D \cdot \beta > 0$ and $E \cdot \beta \geq 0$.

We consider stable logarithmic maps to $(S \mid D+E)$ with genus g, class β and markings $x, y_1, \ldots, y_s$ carrying the following tangency conditions:

- x has maximal tangency $D \cdot \beta$ with respect to D.
- y_j has tangency $\alpha_j \geq 0$ with respect to E (with $\Sigma_{j=1}^s \alpha_j = E \cdot \beta$).

We let $\hat{\mathbf{c}}$ denote the matrix of tangency data. The evaluation space Ev_j corresponding to the marking y_j is defined as:

$$\mathrm{Ev}_j \coloneqq \begin{cases} E & \text{if } \alpha_j > 0 \\ S & \text{if } \alpha_j = 0. \end{cases} \tag{3.2}$$

We restrict to the following class of insertions.

Definition 3.2.2. A class $\gamma_j \in A^*(\mathrm{Ev}_j)$ is **D-disjoint** if there exists a regularly embedded subvariety $Z_j \subseteq \mathrm{Ev}_j$ such that $[Z_j] = \gamma_j$ and $Z_j \cap D = \emptyset$. An assembly of insertions

$$\gamma = \prod_{j=1}^{s} \mathrm{ev}_{y_j}^*(\gamma_j)\psi_{y_j}^{k_j}$$

is **D-avoidant** if for each $j \in \{1, \ldots, s\}$ one of the following two conditions holds:

(i) $\gamma_j = \mathbb{1}_{\mathrm{Ev}_j}$ and $k_j = 0$.

(ii) γ_j is D-disjoint.

This provides an enlargement of the stationary sector.

With the above setup, we use the moduli space of stable logarithmic maps [1, 43, 81] to define logarithmic Gromov–Witten invariants with a lambda class insertion

$$\mathsf{GW}_{g,\hat{\mathbf{c}},\beta}(S \mid D+E)\langle(-1)^g \lambda_g \gamma\rangle \coloneqq (-1)^g \lambda_g \gamma \cap [\overline{M}_{g,\hat{\mathbf{c}},\beta}(S \mid D+E)]^{\mathrm{virt}} \in \mathbb{Q}. \tag{3.3}$$

We next consider the pair $(\mathcal{O}_S(-D) \mid E)$.[2] We obtain tangency data $\mathbf{c}$ from $\hat{\mathbf{c}}$ by deleting the marking x, and define logarithmic Gromov–Witten invariants of the local target

$$\mathsf{GW}_{g,\mathbf{c},\beta}(\mathcal{O}_S(-D) \mid E)\langle\gamma\rangle \coloneqq \gamma \cap [\overline{M}_{g,\mathbf{c},\beta}(\mathcal{O}_S(-D) \mid E)]^{\mathrm{virt}} \in \mathbb{Q}. \tag{3.4}$$

Since $D^2 \geq 0$ it follows that D is nef, and then $D \cdot \beta > 0$ implies that $H^0(C, f^*\mathcal{O}_S(-D)) = 0$ for any stable map $f\colon C \to S$ of class β. The local theory of $\mathcal{O}_S(-D)$ is thus well-defined. The main result of this section is a correspondence between the invariants (3.3) and (3.4).

Theorem 3.2.3 (Theorem 3.A)**.** *We have the following equality of generating functions:*

$$\frac{(-1)^{D\cdot\beta-1}}{2\sin\left(\frac{D\cdot\beta}{2}\hbar\right)} \sum_{g\geq 0} \mathsf{GW}_{g,\hat{\mathbf{c}},\beta}(S \mid D+E)\langle(-1)^g\lambda_g\gamma\rangle\hbar^{2g-1} = \sum_{g\geq 0} \mathsf{GW}_{g,\mathbf{c},\beta}(\mathcal{O}_S(-D) \mid E)\langle\gamma\rangle\hbar^{2g-2}.$$

Remark 3.2.4. The case $E \cdot \beta = 0$ is possible, and in fact one of the most interesting (see Section 3.4). In this case there are no markings tangent to E, but we emphasise that the spaces

$$\overline{M}_{g,\mathbf{c},\beta}(S \mid D+E) \qquad \text{and} \qquad \overline{M}_{g,\mathbf{c},\beta}(S|D)$$

[2]For $\pi\colon V \to S$ a vector bundle and $E \subseteq S$ a divisor, we abuse notation and write $(V|E)$ for the pair $(V|\pi^{-1}(E))$.

are *not* the same. The difference is clearly visible for $E \subseteq \mathbb{F}_1$ the (-1)-curve and β a curve class pulled back along the morphism $\mathbb{F}_1 \to \mathbb{P}^2$ contracting E. The moduli space

$$\overline{M}_{0,n,\beta}(\mathbb{F}_1)$$

has a large number of excess components. On the other hand the moduli space

$$\overline{M}_{0,(0,\ldots,0),\beta}(\mathbb{F}_1|E)$$

is irreducible of the expected dimension. Although the markings carry no tangency, the logarithmic structure imposes tangency conditions at the nodes, which cut down the excess components to produce a space of the expected dimension. The Gromov–Witten invariants also differ, with the theory of $(\mathbb{F}_1|E)$ being essentially equivalent to the theory of $\mathbb{P}^2$.

3.2.2 Target degeneration

The proof of Theorem 3.2.3 follows the degeneration argument of [64]. Because our degeneration has logarithmic structure on the general fibre, much greater care is required when enumerating rigid tropical types and performing gluing. This accounts for the significantly more involved proof. Despite this, the shape of the final formula is relatively simple (Theorem 3.2.3), as we strongly constrain the rigid tropical types which contribute.

Consider the degeneration of S to the normal cone of D as illustrated in Figure 3.1. This is a family

$$\mathcal{S} \to \mathbb{A}^1 \tag{3.5}$$

with general fibre S. Let P denote the projective completion of the normal bundle of $D \subseteq S$. The central fibre $\mathcal{S}_0$ of (3.5) is obtained by gluing S and P along the divisors $D \subseteq S$ and the zero section $D_0 \subseteq P$. We write $D_0 \subseteq \mathcal{S}_0$ for the gluing divisor.

Let $\mathcal{E} \subseteq \mathcal{S}$ denote the strict transform of $E \times \mathbb{A}^1$. This intersects the component S of the central fibre in $E \subseteq S$, and the component P of the central fibre in the union of the two fibres $E_1, E_2 \subseteq P$ of the $\mathbb{P}^1$-bundle $P \to D$ over the points $\{q_1, q_2\} = D \cap E$. We equip the total space $\mathcal{S}$ with the divisorial logarithmic structure corresponding to $S + P + \mathcal{E}$.

Now take the strict transform of $D \times \mathbb{A}^1$ under the blowup $\mathcal{S} \to S \times \mathbb{A}^1$ and let $\mathcal{L}$ be the inverse of the corresponding line bundle on $\mathcal{S}$. There is a flat morphism

$$\mathcal{L} \to \mathcal{S}$$

which we use to pull back the logarithmic structure on $\mathcal{S}$. On the general fibre, the resulting logarithmic scheme is

$$(\mathcal{O}_S(-D) \,|\, E).$$

On the central fibre we have $\mathcal{L}|_S = \mathcal{O}_S$ and $\mathcal{L}|_P = \mathcal{O}_P(-D_\infty)$. The central fibre $\mathcal{L}_0$ is therefore obtained by gluing

$$(S \times \mathbb{A}^1 \,|\, D + E) \qquad \text{and} \qquad (\mathcal{O}_P(-D_\infty) \,|\, D_0 + E_1 + E_2)$$

along the divisors $D \times \mathbb{A}^1 \subseteq S \times \mathbb{A}^1$ and $D_0 \times \mathbb{A}^1 \subseteq \mathcal{O}_P(-D_\infty)$.

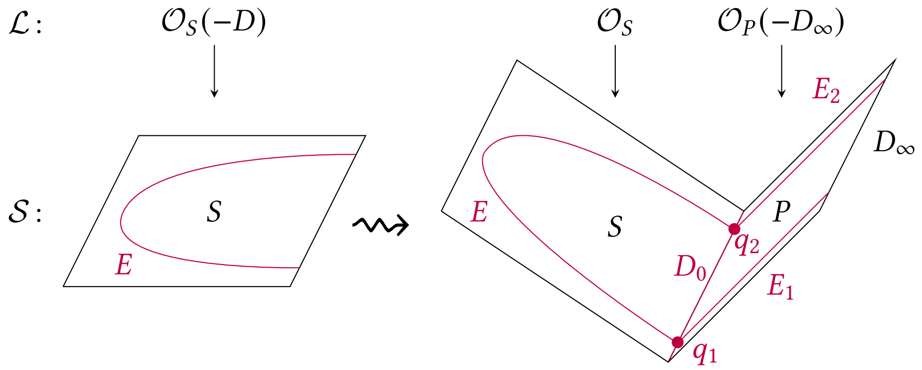


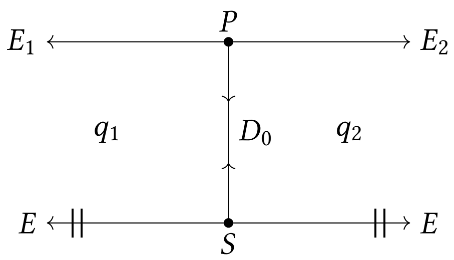
Figure 3.1: The degeneration $\mathcal{S} \to \mathbb{A}^1$ and the line bundle $\mathcal{L}$ on $\mathcal{S}$. The logarithmic structure is indicated in purple.

3.2.3 Tropical types and balancing

The logarithmic morphism $\mathcal{S} \to \mathbb{A}^1$ induces a morphism of tropicalisations

$$\Sigma(\mathcal{S}) \to \Sigma(\mathbb{A}^1) = \mathbb{R}_{\geq 0}.$$

The fibre over $1 \in \mathbb{R}_{\geq 0}$ is a polyhedral complex which we denote Σ:



Notation 3.2.5. We use the notation $S, P, D_0, E, E_1, E_2, q_1, q_2$ to refer both to the strata of $\mathcal{S}_0$ and to the corresponding polyhedra in Σ.

The moduli space of stable logarithmic maps to the central fibre $\mathcal{L}_0$ decomposes into virtual irreducible components indexed by rigid tropical types of maps to Σ.

Definition 3.2.6 ([3, Definition 2.23])**.** A **tropical type of map to** Σ consists of:

(i) **Source graph.** A finite graph Γ with vertices $V(\Gamma)$, finite edges $E(\Gamma)$, unbounded legs $L(\Gamma)$, and a genus assignment $g\colon V(\Gamma) \to \mathbb{N}$.

(ii) **Polyhedra assignments.** An inclusion-preserving function $\sigma\colon V(\Gamma) \sqcup E(\Gamma) \sqcup L(\Gamma) \to \Sigma$. We often write $\mathsf{f}(w) \in \sigma$ instead of $\sigma_w = \sigma$.

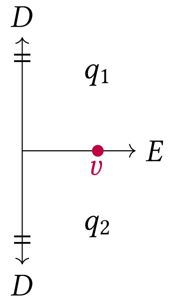
(iii) **Curve classes.** A curve class $\beta_v \in A_1(\mathcal{S}_0(\sigma_v))$ for every $v \in V(\Gamma)$. Here $\mathcal{S}_0(\sigma_v) \subseteq \mathcal{S}_0$ is the closed stratum corresponding to the polyhedron $\sigma_v \in \Sigma$.

(iv) **Slopes.** Vectors $m_{\vec{e}} \in N(\sigma_e)$ for every oriented edge $\vec{e} \in \vec{E}(\Gamma) \sqcup L(\Gamma)$, satisfying $m_{\vec{e}} = -m_{\overleftarrow{e}}$. Here $N(\sigma_e)$ is the lattice associated to the polyhedron σ_e.

A tropical type has an associated tropical moduli space, parametrising choices of edge lengths and vertex positions. A tropical type is **rigid** if its tropical moduli space is a point. See [3, Sections 2.5 and 3.2] for details.

The above data is required to be balanced at each vertex. For vertices in q_1 and q_2 this means that the sum of outgoing slopes is zero. For vertices in S and P it is the usual balancing condition for logarithmic maps to the normal crossings pairs $(S \,|\, D + E)$ and $(P \,|\, D_0 + E_1 + E_2)$ as in [81, Proposition 1.15]. We now explain balancing for vertices in E, D_0, E_1, and E_2.

3.2.3.1 Vertices on E. Local to v the tropical target Σ has the following structure:



For $i \in \{1, 2\}$ let $m_D^i \in \mathbb{N}$ denote the sum of vertical slopes of outgoing edges which enter q_i. Letting $m_D \in \mathbb{Z}$ denote the sum of vertical slopes of all outgoing edges, we have

$$m_D = m_D^1 + m_D^2.$$

On the other hand, let $m_E \in \mathbb{Z}$ denote the total sum of horizontal slopes of all outgoing edges. The curve class $\beta_v \in A_1(E)$ must be of the form

$$\beta_v = kE$$

for some $k \in \mathbb{N}$. We have $D \cdot \beta_v = kDE = 2k$ and $E \cdot \beta_v = kE^2$ where $E^2 \in \mathbb{Z}$ is the self-intersection inside S. The balancing condition therefore gives

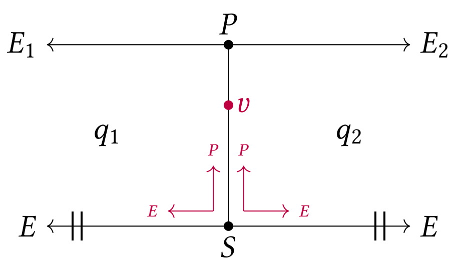
$$m_D = 2k, \qquad m_E = kE^2.$$

However, there is a stronger constraint. Since $f_v \colon C_v \longrightarrow E$ is a degree k cover, the tangency orders m_D^1, m_D^2 give the ramification profiles of f_v over the two intersection points q_1, q_2. Therefore we must have

$$m_D^1 = m_D^2 = k.$$

Geometrically, this means that when we lay the tropicalisation flat along the "spine" E, the sum of outgoing vertical slopes is zero.

3.2.3.2 Vertices on D_0. This is similar to the previous case. Local to v the tropical target Σ has the following structure:



We choose coordinates for the adjacent polyhedra q_1, q_2 as indicated in red. For $i \in \{1, 2\}$ we let $m_E^i \in \mathbb{N}$ denote the sum of horizontal slopes of outgoing edges which enter q_i. Letting $m_E \in \mathbb{Z}$ denote the sum of horizontal slopes of all outgoing edges, we have

$$m_E = m_E^1 + m_E^2.$$

On the other hand, let $m_P \in \mathbb{Z}$ denote the sum of vertical slopes of all outgoing edges (note that if instead we choose coordinates corresponding to S, E then we have $m_S = -m_P$). The curve class $\beta_v \in A_1(D_0)$ necessarily takes the form

$$\beta_v = kD_0$$

for some $k \in \mathbb{N}$. We have $E \cdot \beta_v = 2k$ and $\deg f_v^* N_{D_0|P}^\vee = \deg f_v^* N_{D|S} = kD^2$ where $D^2 \geq 0$ is the self-intersection inside S. The balancing condition is therefore

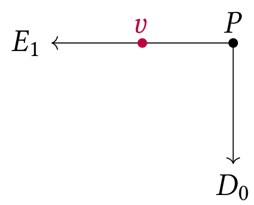
$$m_E = 2k, \qquad m_P = kD^2.$$

As in the previous case, we have the stronger constraint

$$m_E^1 = m_E^2 = k.$$

This means that when we lay the tropicalisation flat along the spine D_0, the sum of outgoing horizontal slopes is zero.

3.2.3.3 *Vertices on E_1, E_2.* This is the simplest case. We restrict to E_1 without loss of generality. Local to v there is a single maximal polyhedron, coordinatised by E_1, D_0:



We let $m_{E_1} \in \mathbb{Z}$ and $m_{D_0} \in \mathbb{N}$ denote, respectively, the sums of horizontal and vertical outgoing slopes. The curve class is $\beta_v = kE_1$ giving $E_1 \cdot \beta_v = 0$ and $D_0 \cdot \beta_v = k$. The balancing condition gives:

$$m_{E_1} = 0, \qquad m_{D_0} = k.$$

Unlike the previous cases there is no stronger constraint, as there is only one adjacent polyhedron.

3.2.4 Degeneration formula analysis

The blowup morphism $p\colon \mathcal{S} \to S \times \mathbb{A}^1$ satisfies $p^{-1}(E \times \mathbb{A}^1) = \mathcal{E}$. Restricting to the central fibre, we obtain a logarithmic morphism $\mathcal{S}_0 \to (S|E)$. Combined with the line bundle projection $\mathcal{L}_0 \to \mathcal{S}_0$ this induces a pushforward morphism

$$\rho\colon \overline{M}_{g,\mathbf{c},\beta}(\mathcal{L}_0) \to \overline{M}_{g,\mathbf{c},\beta}(S|E).$$

We also let ζ denote the morphism forgetting logarithmic structures

$$\zeta\colon \overline{M}_{g,\mathbf{c},\beta}(S|E) \to \overline{M}_{g,s,\beta}(S).$$

The conservation of number principle [3, Theorem 1.1] and the decomposition theorem [3, Theorem 1.2] give the following identity in the Chow homology of $\overline{M}_{g,\mathbf{c},\beta}(S|E)$:

$$[\overline{M}_{g,\mathbf{c},\beta}(\mathcal{O}_S(-D)|E)]^{\mathrm{virt}} = \sum_\tau \frac{m_\tau}{|\mathrm{Aut}(\tau)|} \rho_* \iota_* [\overline{M}_\tau]^{\mathrm{virt}}.$$

The sum runs over rigid tropical types τ of tropical stable maps to Σ of type $(g, \mathbf{c}, \beta)$. Here m_τ is the smallest integer such that scaling Σ by m_τ produces a tropical stable map with integral vertex positions and edge lengths, and $\iota\colon \overline{M}_\tau \hookrightarrow \overline{M}_{g,\mathbf{c},\beta}(\mathcal{L}_0)$ is the inclusion of the virtual irreducible component. Capping with the D-avoidant insertions γ and pushing forward along λ we obtain:

$$\zeta_*\big(\gamma \cap [\overline{M}_{g,\mathbf{c},\beta}(\mathcal{O}_S(-D)|E)]^{\mathrm{virt}}\big) = \sum_\tau \frac{m_\tau}{|\mathrm{Aut}(\tau)|}\zeta_*\big(\gamma \cap \rho_*\iota_*[\overline{M}_\tau]^{\mathrm{virt}}\big). \tag{3.6}$$

We now use the degeneration formula as formulated in [149, Section 6] to describe the terms in the sum. Let Γ denote the source graph of the tropical type τ. There is a subdivision of the product

$$\bigtimes_{v\in V(\Gamma)} \overline{M}_v \to \prod_{v\in V(\Gamma)} \overline{M}_v$$

and for each edge $e \in E(\Gamma)$ a smooth universal divisor $\mathcal{D}_e \to \bigtimes_v \overline{M}_v$ supporting an evaluation section for each flag $(v \in e)$. We consider the universal doubled divisor

$$\mathcal{D}_e^{\{2\}} \coloneqq \mathcal{D}_e \times_{\bigtimes_v \overline{M}_v} \mathcal{D}_e.$$

There is a universal diagonal $\Delta\colon \mathcal{D}_e \hookrightarrow \mathcal{D}_e^{\{2\}}$ which is a regular embedding since $\mathcal{D}_e \to \bigtimes_v \overline{M}_v$ is smooth. We obtain a diagram

$$\begin{tikzcd} \overline{M}_\tau \arrow[r, "\nu"] & \overline{N}_\tau \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \bigtimes_v \overline{M}_v \arrow[d] \\ & \prod_e \mathcal{D}_e \arrow[r, hook, "\Delta"] & \prod_e \mathcal{D}_e^{\{2\}} \end{tikzcd} \tag{3.7}$$

with an equality of classes in the Chow homology of $\overline{N}_\tau$

$$\nu_*[\overline{M}_\tau]^{\mathrm{virt}} = c_\tau\, \Delta^![\bigtimes_v \overline{M}_v]^{\mathrm{virt}}$$

where $c_\tau \in \mathbb{Q}$ is an appropriate combinatorial gluing factor. There is a gluing morphism $\theta : \overline{N}_\tau \to \overline{M}_{g,s,\beta}(S)$ such that the following square commutes

$$\begin{tikzcd} \overline{M}_\tau \arrow[r, "\nu"] \arrow[d, "\iota"] & \overline{N}_\tau \arrow[d, "\theta"] \\ \overline{M}_{g,\mathbf{c},\beta}(\mathcal{S}_0) \arrow[r, "\zeta\circ\rho"] & \overline{M}_{g,s,\beta}(S). \end{tikzcd} \tag{3.8}$$

We thus rewrite the terms appearing on the right-hand side of equation (3.6) as

$$\zeta_*\big(\gamma \cap \rho_*\iota_*[\overline{M}_\tau]^{\mathrm{virt}}\big) = c_\tau\big(\gamma \cap \theta_*\Delta^![\bigtimes_v \overline{M}_v]^{\mathrm{virt}}\big). \tag{3.9}$$

3.2.4.1 Reduction outline. We will prove Theorem 3.2.3 by analysing the contributions of rigid tropical types τ to the degeneration formula (3.6). For the rest of Section 3.2.4 we fix a rigid tropical type τ whose contribution to (3.6) is non-trivial:

$$\zeta_*\big(\gamma \cap \rho_*\iota_*[\overline{M}_\tau]^{\mathrm{virt}}\big) \neq 0.$$

We will show that the shape of τ is tightly constrained. We proceed via four reductions:

- In Section 3.2.4.2 we show that τ is weakly star-shaped (Proposition 3.2.8).

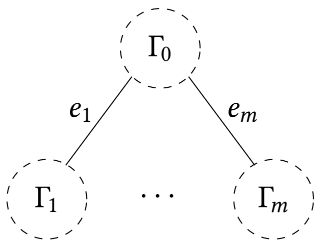
- In Section 3.2.4.3 we show that τ has markings confined to S (Proposition 3.2.9).
- In Section 3.2.4.4 we show that τ is star-shaped (Proposition 3.2.18).
- In Section 3.2.4.5 we show that τ has a single edge (Proposition 3.2.23).

Finally in Section 3.2.4.6 we calculate the contributions of the remaining τ, arriving at Theorem 3.2.3.

Remark 3.2.7. As in [21] we obtain a reduction to star-shaped graphs (Proposition 3.2.18). In fact we reduce further to single edge graphs (Proposition 3.2.23), using crucially the fact that D is rational.

3.2.4.2 First reduction: weakly star-shaped graphs.

Proposition 3.2.8 (First reduction)**.** *The tropical type* τ *is* weakly star-shaped *in the following sense. The source graph* Γ *decomposes into subgraphs*



with the vertices of Γ_0 *mapping to* P, E_1, E_2 *and the vertices of* $\Gamma_1, \ldots, \Gamma_m$ *mapping to* S, E, D_0, q_1, q_2.

Proof. Let $\Gamma' \subseteq \Gamma$ be a maximal connected subgraph contained in the union of the following strata

$$S, E, D_0, q_1, q_2. \tag{3.10}$$

Let $E' \subseteq E(\Gamma)$ denote the set of edges connecting Γ' to the rest of Γ. It is sufficient to show that $|E'| = 1$. The following argument parallels [64, Lemma 3.1].

The line bundle $\mathcal{L}_0$ is trivial when restricted to each of the strata in (3.10). It follows from the definition of Γ' that for each $e \in E'$ the corresponding universal divisor $\mathcal{D}_e$ decomposes as

$$\mathcal{D}_e = \overline{\mathcal{D}}_e \times \mathbb{A}^1$$

where $\overline{\mathcal{D}}_e$ is the universal divisor for the compact degeneration $\mathcal{S}_0$ (since $\mathcal{L}_0 \to \mathcal{S}_0$ is flat and strict, all expansions of $\mathcal{L}_0$ are pulled back from expansions of $\mathcal{S}_0$). We define

$$\overline{\mathcal{D}} \coloneqq \prod_{e \in E(\Gamma) \setminus E'} \mathcal{D}_e \times \prod_{e \in E'} \overline{\mathcal{D}}_e$$

and note that the product of universal divisors appearing in (3.7) decomposes as $\Pi_{e \in E(\Gamma)} \mathcal{D}_e = \overline{\mathcal{D}} \times \mathbb{A}^{E'}$. We now examine evaluations at the nodes corresponding to edges $e \in E'$.

Given $e \in E'$ the corresponding node has an adjacent irreducible component C_e which is mapped to P and has positive intersection with the divisor D_0. Since $D^2 \geq 0$ in S it follows that the pullback of the line bundle $\mathcal{L}_0|_P = \mathcal{O}_P(-D_\infty)$ to this component has negative degree. We conclude that evaluation of $f|_{C_e}$ at the given node factors through the zero section of $\mathcal{L}_0$.

On the other hand, the subcurve C' corresponding to Γ' is connected and maps to S. Since $\mathcal{L}_0|_S = \mathcal{O}_S$ it follows that this subcurve is mapped to a constant section of the bundle $S \times \mathbb{A}^1 \to S$. Therefore the evaluations of $f|_{C'}$ at the nodes corresponding to $e \in E'$ all coincide.

Taken together, we obtain the following cartesian diagram extending (3.7):

$$\begin{tikzcd}
\overline{N}_\tau \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \bigtimes_v \overline{M}_v \arrow[d] \\
\overline{\mathcal{D}} \times \mathbb{A}^0 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{D}}^{\{2\}} \times \mathbb{A}^1 \arrow[d] \\
\overline{\mathcal{D}} \times \mathbb{A}^{E'} \arrow[r, hook, "\Delta"] & \overline{\mathcal{D}}^{\{2\}} \times \mathbb{A}^{2E'}.
\end{tikzcd}$$

If $|E'| \geq 2$ the excess bundle is a trivial vector bundle of positive rank. Consequently the excess class [63, Theorem 6.3] vanishes, and we obtain

$$\Delta^! [\bigtimes_v \overline{M}_v]^{\mathrm{virt}} = 0.$$

From (3.9) we conclude that $|E'| = 1$. $\square$

3.2.4.3 Second reduction: markings confined to S. The next step is to constrain the shape of the subgraphs Γ_i and to confine the unbounded legs.

Proposition 3.2.9 (Second reduction)**.** *Consider a weakly star-shaped tropical type as in Proposition 3.2.8. Then:*

(*i*) *The subgraph Γ_i is a tree for $i \in \{1, \ldots, m\}$.*

(*ii*) *Every leaf vertex $v \in V(\Gamma_i)$ satisfies $\mathsf{f}(v) \in S$.*

(*iii*) *Every marking leg $l \in L(\Gamma)$ is attached to a leaf vertex $v \in V(\Gamma_i)$ for some $i \in \{1, \ldots, m\}$.*

Proof. Combine Propositions 3.2.11, 3.2.14, and 3.2.17 below. $\square$

The proof proceeds by a sequence of intermediate reductions. We fix a weakly star-shaped tropical type τ as in Proposition 3.2.8.

Notation 3.2.10. For each $i \in \{1, \ldots, m\}$ we let $v_i \in V(\Gamma_i)$ and $w_i \in V(\Gamma_0)$ denote the endpoints of e_i.

Proposition 3.2.11. *Each of the graphs $\Gamma_1, \ldots, \Gamma_m$ is a tree.*

Proof. Recall that an edge of a connected graph is **separating** if deleting it produces a graph with two connected components, and that a graph is a tree if and only if every edge is separating.

Fix $i \in \{1, \ldots, m\}$. We will prove that every edge of Γ_i is separating by inducting on the vertices. At each step we prove that the edges adjacent to the current vertex are separating. We pass to the next step by traversing along all adjacent edges, excluding the edge we arrived by. The starting point is the vertex v_i which is connected to w_i along the separating edge e_i. When we arrive at a new vertex, there is by induction a path of separating edges connecting it to w_i.

Suppose that we arrive at a vertex $v \in V(\Gamma)$ via a separating edge $\tilde{e}$. Let $\tilde{e}_1, \ldots, \tilde{e}_r$ be the other adjacent edges and suppose for a contradiction that $\tilde{e}_1$ is not separating. Since $\tilde{e}$ is separating, the subgraph behind $\tilde{e}_1$ must coincide (without loss of generality) with the subgraph behind $\tilde{e}_2$.

Split the graph Γ at the edges $\tilde{e}_1, \tilde{e}_2$. This produces a new combinatorial type τ_{12} with four open half-edges corresponding to the previously closed edges $\tilde{e}_1, \tilde{e}_2$. For $i \in \{1, 2\}$ let $\mathcal{D}_i$ denote the corresponding universal divisor. As in Section 3.2.4.2 this decomposes as $\mathcal{D}_i = \overline{\mathcal{D}}_i \times \mathbb{A}^1$ and we have a diagram

$$\begin{tikzcd}
\overline{M}_\tau \arrow[r, "\nu"] & \overline{N}_\tau \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{\tau_{12}} \arrow[d] \\
& \prod_{i=1}^2 (\overline{\mathcal{D}}_i \times \mathbb{A}^1) \arrow[r, hook, "\Delta"] & \prod_{i=1}^2 (\overline{\mathcal{D}}_i^{\{2\}} \times \mathbb{A}^2).
\end{tikzcd}$$

We now show that the composite $\overline{M}_{\tau_{12}} \to \Pi_{i=1}^2 (\overline{\mathcal{D}}_i^{\{2\}} \times \mathbb{A}^2) \to \mathbb{A}^4$ factors through the linear subspace:

$$\begin{aligned} \epsilon \colon \mathbb{A}^1 &\hookrightarrow \mathbb{A}^4 \\ t &\mapsto (0, t, 0, t). \end{aligned}$$

The inductive argument connects the current vertex v to the vertex w_i via a path of separating edges, which are hence distinct from $\tilde{e}_1$ and $\tilde{e}_2$. The line bundle $f^* \mathcal{L}_0|_{C_{w_i}}$ is negative and so $f|_{C_{w_i}}$ factors through the zero section of $\mathcal{L}_0$. Since C_{w_i} is connected to C_v through nodes which are not split in τ_{12} it follows that $f|_{C_v}$ also factors through the zero section of $\mathcal{L}_0$. This explains the two zero entries. On the other hand, the two t entries occur because the subgraphs behind $\tilde{e}_1$ and $\tilde{e}_2$ coincide. We thus obtain

$$\begin{tikzcd}
\overline{M}_\tau \arrow[r, "\nu"] & \overline{N}_\tau \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{\tau_{12}} \arrow[d] \\
& \left(\prod_{i=1}^2 \overline{\mathcal{D}}_i\right) \times \mathbb{A}^0 \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \left(\prod_{i=1}^2 \overline{\mathcal{D}}_i^{\{2\}}\right) \times \mathbb{A}^1 \arrow[d, hook, "\mathrm{id}\times\epsilon"] \\
& \left(\prod_{i=1}^2 \overline{\mathcal{D}}_i\right) \times \mathbb{A}^2 \arrow[r, hook, "\Delta"] & \left(\prod_{i=1}^2 \overline{\mathcal{D}}_i^{\{2\}}\right) \times \mathbb{A}^4.
\end{tikzcd}$$

Since the codimensions of the lower two horizontal arrows differ, the excess bundle is a trivial bundle of rank one, and hence the excess class vanishes. As in the proof of Proposition 3.2.8, it follows that the contribution of τ vanishes. We conclude that the edges $\tilde{e}_1, \ldots, \tilde{e}_r$ adjacent to v are all separating, and this completes the induction step. □

The arguments now briefly shift from intersection theory to tropical geometry. The background developed in Section 3.2.3 is essential.

By Proposition 3.2.11, each Γ_i is a tree equipped with a root vertex v_i. This defines a canonical flow starting at v_i. From now on we orient each edge of Γ_i according to this flow. The edge e_i is also oriented from w_i to v_i. In this way, every vertex of Γ_i has a unique incoming edge.

A leaf of Γ_i is a vertex adjacent to a single finite edge. Notice that $v \in V(\Gamma_i)$ is a leaf if and only it does not support any outgoing finite edges. Enumerate the leaf vertices in $V(\Gamma_1) \sqcup \ldots \sqcup V(\Gamma_m)$ as

$$\overline{v}_1, \ldots, \overline{v}_\ell$$

and write $\overline{e}_j$ for the edge incoming at $\overline{v}_j$.

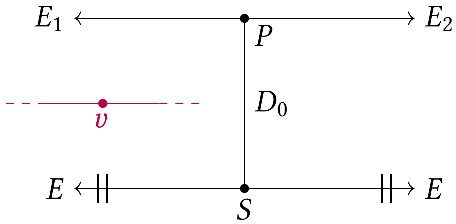
Lemma 3.2.12. *Let $e \in E(\Gamma_i) \cup \{e_i\}$, oriented as above. Suppose that $\mathsf{f}(e)$ is contained in q_1,q_2, or D_0. Then the vertical slope of e is negative.*

Proof. Suppose first that e has positive vertical slope. Since Γ_i has only a single incoming edge and no outgoing edges, the edge e leads to another vertex of Γ_i, which must be contained in q_1, q_2, or

D_0. Since the marking legs have zero vertical slope, the balancing condition ensures that there is a finite outgoing edge with positive vertical slope (see in particular Section 3.2.3.2). Continuing in this way, we produce a path in Γ_i consistent with the flow and with positive vertical slope along each edge. This path continues indefinitely, a contradiction.

It remains to consider the case where e has zero vertical slope. Let Γ_e denote the subgraph of Γ_i behind the oriented edge e. By the previous paragraph, no edge of Γ_e has positive vertical slope. Following the flow, we see by induction and balancing that no edge of Γ_e has negative vertical slope either. Therefore every edge of Γ_e has zero vertical slope, and every vertex and edge is mapped to q_1, q_2, or D_0.

If e has zero horizontal slope then it is contracted by the tropical map f, in which case the tropical type is not rigid. If e has non-zero horizontal slope, we may traverse Γ_e using the balancing condition. It is easy to see that eventually we arrive at a vertex v in q_1 or q_2 around which the image of Γ_e takes the following form:



Varying the position of v horizontally we see that the tropical type is not rigid. □

Lemma 3.2.13. *There is no vertex $v \in V(\Gamma_i)$ with $\mathsf{f}(v) \in E$.*

Proof. Suppose for a contradiction that such a vertex v exists. If it has an adjacent edge with non-zero vertical slope, then by balancing (Section 3.2.3.1) it has at least two adjacent edges with non-zero vertical slope. At most one of these can be the incoming edge, and so at least one is outgoing, i.e. an edge oriented according to the flow with positive vertical slope. This contradicts Lemma 3.2.12. We conclude that all edges adjacent to v have zero vertical slope. It follows immediately that the tropical type is not rigid, as in the proof of Lemma 3.2.12. □

Lemma 3.2.14. *A vertex $v \in V(\Gamma_i)$ is a leaf if and only if $\mathsf{f}(v) \in S$.*

Proof. Suppose $v \in V(\Gamma_i)$ is a leaf. We already argued in Lemma 3.2.13 that $\mathsf{f}(v) \notin E$. If $\mathsf{f}(v) \in q_i$ then by Lemma 3.2.12 the incoming edge must have negative vertical slope. Since there are no outgoing edges, this violates balancing.

It remains to consider the case $\mathsf{f}(v) \in D_0$. The horizontal slope of the incoming edge e must be zero, since otherwise balancing guarantees at least one outgoing finite edge. Hence, both v and e are contained in D_0 which contradicts the assumption that τ is rigid. By process of elimination, we conclude that $\mathsf{f}(v) \in S$.

For the opposite direction, suppose $v \in V(\Gamma_i)$ maps to S but has at least one outgoing finite edge. This edge cannot be contracted (due to rigidity) or flow to a vertex in E (due to Lemma 3.2.13). Therefore it must map to q_1, q_2, or D_0. In particular it has positive vertical slope, contradicting Lemma 3.2.12. □

We now switch from tropical geometry back to intersection theory. We will show that all marking legs are adjacent to vertices mapping to S. From now on we focus on the entire graph Γ rather than the subgraph Γ_i.

Split Γ at the edges $\overline{e}_1, \ldots, \overline{e}_\ell$ incoming to the leaves in $V(\Gamma_1) \sqcup \ldots \sqcup V(\Gamma_m)$. These edges are separating, and we obtain split tropical types:

$$\tau_0, \tau_1, \ldots, \tau_\ell.$$

Each of the types $\tau_1, \ldots, \tau_\ell$ constitutes a single vertex supporting a single finite half-edge and a collection of marking legs.

Passing to a subdivision of Σ we may assume that each edge $\overline{e}_i$ is contained in a one dimensional polyhedron (whose corresponding divisor we denote $D_{\overline{e}_i}$) and that the vertices adjacent to $\overline{e}_i$ map to zero-dimensional polyhedral. The remarks in [149, Section 6.5.3] apply in this setting, and the following is an immediate consequence.

Lemma 3.2.15. *There is a map $\nu\colon \overline{M}_\tau \to \overline{N}_\tau$ with target the fibre product over the unexpanded diagonal*

$$\begin{tikzcd}
\overline{M}_\tau \arrow[r, "\nu"] & \overline{N}_\tau \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{\tau_0} \times \prod_{i=1}^{\ell} \overline{M}_{\tau_i} \arrow[d] \\
& \prod_{i=1}^{\ell} D_{\overline{e}_i} \arrow[r, hook, "\Delta"] & \prod_{i=1}^{\ell} D_{\overline{e}_i}^2
\end{tikzcd}$$

and an equality of virtual classes in the Chow homology of $\overline{N}_\tau$:

$$\nu_*[\overline{M}_\tau]^{\mathrm{virt}} = c_\tau\, \Delta^! \left([\overline{M}_{\tau_0}]^{\mathrm{virt}} \times \prod_{i=1}^{\ell} [\overline{M}_{\tau_i}]^{\mathrm{virt}} \right).$$

As in the proof of Proposition 3.2.8 we have $D_{\overline{e}_i} = \overline{D}_{\overline{e}_i} \times \mathbb{A}^1$ for all edges incoming to a leaf, and the evaluation morphism

$$\overline{M}_{\tau_0} \xrightarrow{\mathrm{ev}_0} \prod_{i=1}^{\ell} (\overline{D}_{\overline{e}_i} \times \mathbb{A}^1)$$

factors through the codimension-ℓ subvariety:

$$\begin{tikzcd}
\overline{M}_{\tau_0} \arrow[r, "\overline{\mathrm{ev}}_0"] \arrow[rr, bend left, "\mathrm{ev}_0"] & \prod_{i=1}^{\ell} \overline{D}_{\overline{e}_i} \arrow[r, hook, "\epsilon"] & \prod_{i=1}^{\ell} (\overline{D}_{\overline{e}_i} \times \mathbb{A}^1).
\end{tikzcd}$$

For $i \in \{1, \ldots, \ell\}$ the moduli space $\overline{M}_{\tau_i}$ parametrises logarithmic maps to some subdivision of $(S \times \mathbb{A}^1 \,|\, D + E)$ of type τ_i. There is a closed embedding

$$\overline{M}_{\tau_i}|_0 \hookrightarrow \overline{M}_{\tau_i}$$

parametrising logarithmic maps which factor through the zero section. Letting δ denote the diagonal $\prod_{i=1}^{\ell} \overline{D}_{\overline{e}_i} \hookrightarrow \prod_{i=1}^{\ell} \overline{D}_{\overline{e}_i}^2$ we obtain the following diagram

$$\begin{tikzcd}
\overline{N}_\tau \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{\tau_0} \times \prod_{i=1}^{\ell} \overline{M}_{\tau_i}|_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{\tau_0} \times \prod_{i=1}^{\ell} \overline{M}_{\tau_i} \arrow[d, "\overline{\mathrm{ev}}_0 \times \Pi_{i=1}^{\ell} \mathrm{ev}_i"] \\
\prod_{i=1}^{\ell} \overline{D}_{\overline{e}_i} \arrow[r, "\delta"] \arrow[d] \arrow[drr, phantom, "\square"] & \prod_{i=1}^{\ell} \overline{D}_{\overline{e}_i} \times \prod_{i=1}^{\ell} \overline{D}_{\overline{e}_i} \arrow[r, "\mathrm{id} \times \epsilon"] & \prod_{i=1}^{\ell} \overline{D}_{\overline{e}_i} \times \prod_{i=1}^{\ell} (\overline{D}_{\overline{e}_i} \times \mathbb{A}^1) \arrow[d, "\epsilon \times \mathrm{id}"] \\
\prod_{i=1}^{\ell} (\overline{D}_{\overline{e}_i} \times \mathbb{A}^1) \arrow[rr, "\Delta"] & & \prod_{i=1}^{\ell} (\overline{D}_{\overline{e}_i} \times \mathbb{A}^1)^2.
\end{tikzcd} \tag{3.11}$$

The following explains the appearance of lambda classes in Theorem 3.2.3.

Lemma 3.2.16. *We have*

$$\nu_*[\overline{M}_\tau]^{\mathrm{virt}} = c_\tau\, \delta^! \left([\overline{M}_{\tau_0}]^{\mathrm{virt}} \times \prod_{i=1}^{\ell} (-1)^{g_i} \lambda_{g_i} \cap [\overline{M}_{\tau_i}|_0]^{\mathrm{virt}} \right).$$

Proof. From (3.11) we obtain

$$\begin{aligned}\Delta^! \left([\overline{M}_{\tau_0}]^{\mathrm{virt}} \times \prod_{i=1}^{\ell} [\overline{M}_{\tau_i}]^{\mathrm{virt}} \right) &= \delta^! (\mathrm{id} \times \epsilon)^! \left([\overline{M}_{\tau_0}]^{\mathrm{virt}} \times \prod_{i=1}^{\ell} [\overline{M}_{\tau_i}]^{\mathrm{virt}} \right) \\ &= \delta^! \left([\overline{M}_{\tau_0}]^{\mathrm{virt}} \times \epsilon^! \left(\prod_{i=1}^{\ell} [\overline{M}_{\tau_i}]^{\mathrm{virt}} \right) \right)\end{aligned}$$

where the first equality follows from [63, Theorems 6.2(c) and 6.5] and the second equality follows from [63, Example 6.5.2]. Since $\mathcal{L}_0$ trivialises over the irreducible component $S \subseteq \mathcal{S}_0$ it continues to do so over the subdivision. Hence, we have

$$\overline{M}_{\tau_i} = \overline{M}_{\tau_i}|_0 \times \mathbb{A}^1$$

for all $i \in \{1, \ldots, \ell\}$. A direct comparison of obstruction theories then produces

$$\epsilon^! \left(\prod_{i=1}^{\ell} [\overline{M}_{\tau_i}]^{\mathrm{virt}} \right) = \prod_{i=1}^{\ell} (-1)^{g_i} \lambda_{g_i} \cap [\overline{M}_{\tau_i}|_0]^{\mathrm{virt}}$$

which we combine with Lemma 3.2.15 to obtain the result. □

We now show that τ_0 supports no marking legs, thus completing the proof of Proposition 3.2.9. Recall that the insertions

$$\gamma = \prod_{j=1}^{s} \mathrm{ev}_{y_j}^*(\gamma_j) \psi_{y_j}^{k_j}$$

are D-avoidant (Definition 3.2.2). The following vanishing result involves a dimension count, for which it is crucial that the codimension of γ coincides with the virtual dimension of $\overline{M}_{g,\mathrm{c},\beta}(\mathcal{O}_S(-D) \mid E)$.

For $i \in \{0, 1, \ldots, \ell\}$ let $J(i) \subseteq \{1, \ldots, s\}$ denote the set indexing those markings y_j supported at τ_i for which $\gamma_j \neq \mathbb{1}_{\mathrm{Ev}_j}$. Define the class

$$\gamma_{J(i)} \coloneqq \prod_{j \in J(i)} \mathrm{ev}_{y_j}^*(\gamma_j) \psi_{y_j}^{k_j}$$

so that $\gamma = \Pi_{i=0}^{\ell} \gamma_{J(i)}$. We continue to write θ for the gluing morphism

$$\overline{N}_\tau \to \overline{M}_{g,s,\beta}(S)$$

making the diagram (3.8) commute.

Proposition 3.2.17. *The cycle*

$$\zeta_*\big(\gamma \cap \rho_* \iota_* [\overline{M}_\tau]^{\mathrm{virt}}\big)$$

vanishes unless τ_0 carries no markings. In this case, we have the following equality in $A_0(\overline{M}_{g,s,\beta}(S))$:

$$\zeta_*\big(\gamma \cap \rho_* \iota_* [\overline{M}_\tau]^{\mathrm{virt}}\big) = c_\tau\, \theta_* \delta^! \left([\overline{M}_{\tau_0}]^{\mathrm{virt}} \times \prod_{i=1}^{\ell} (-1)^{g_i} \lambda_{g_i} \gamma_{J(i)} \cap [\overline{M}_{\tau_i}|_0]^{\mathrm{virt}} \right). \tag{3.12}$$

Proof. It is sufficient to prove the vanishing result. We consider separately the case of markings y_j with trivial and non-trivial insertions. First suppose that τ_0 carries a marking y_j with $\gamma_j \neq \mathbb{1}_{\mathrm{Ev}_j}$. For every marking y_j with $\gamma_j \neq \mathbb{1}_{\mathrm{Ev}_j}$, choose a regularly embedded subvariety $Z_j \subseteq \mathrm{Ev}_j$ with $[Z_j] = \gamma_j$ and $Z_j \cap D = \emptyset$. Let

$$\mathcal{Z}_j \hookrightarrow \mathcal{S} \tag{3.13}$$

denote the strict transform of $Z_j \times \mathbb{A}^1 \hookrightarrow S \times \mathbb{A}^1$. Since $Z_j \cap D = \emptyset$ it follows that on the central fibre the inclusion (3.13) factors through the irreducible component S. In fact:

$$Z_j \hookrightarrow \mathrm{Ev}_j \hookrightarrow S \hookrightarrow \mathcal{S}_0.$$

Consider $J \coloneqq \bigsqcup_{i=0}^{\ell} J(i)$ the set indexing all markings with non-trivial insertion. Take $Z \coloneqq \Pi_{j\in J} Z_j$ and $\mathrm{Ev} \coloneqq \Pi_{j\in J}\mathrm{Ev}_j$. We combine the above morphisms into a regular embedding

$$\eta_Z \colon Z \hookrightarrow \mathrm{Ev}.$$

Similarly, for $i \in \{0, \ldots, \ell\}$ we consider the corresponding inclusion

$$\eta_i \colon Z_{J(i)} \coloneqq \Pi_{j\in J(i)} Z_j \hookrightarrow \Pi_{j\in J(i)} \mathrm{Ev}_j \eqqcolon \mathrm{Ev}_{J(i)}.$$

We perform the pullback along η_Z to produce a closed substack with constrained evaluation:

$$\begin{tikzcd} \overline{N}_\tau|_Z \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{N}_\tau \arrow[d] \\ Z \arrow[r, hook, "\eta_Z"] & \mathrm{Ev}. \end{tikzcd}$$

Applying $\eta_Z^!$ to Lemma 3.2.16 we obtain

$$\begin{aligned} \eta_Z^! \nu_* [\overline{M}_\tau]^{\mathrm{virt}} &= c_\tau\, \delta^! \eta_Z^! \left([\overline{M}_{\tau_0}]^{\mathrm{virt}} \times \prod_{i=1}^{\ell} (-1)^{g_i} \lambda_{g_i} \cap [\overline{M}_{\tau_i}|_0]^{\mathrm{virt}} \right) \\ &= c_\tau\, \delta^! \left(\eta_0^! [\overline{M}_{\tau_0}]^{\mathrm{virt}} \times \prod_{i=1}^{\ell} (-1)^{g_i} \lambda_{g_i} \cap \eta_i^! [\overline{M}_{\tau_i}|_0]^{\mathrm{virt}} \right) \end{aligned} \tag{3.14}$$

where the first equality follows from [63, Theorem 6.2(a) and 6.4] and the second from [63, Example 6.5.2 and Proposition 6.3].

Combining Lemmas 3.2.13 and 3.2.14, we see that for every vertex v of the graph of τ_0, the restriction $f|_{C_v}$ factors through P. It follows that if a marking y_j belongs to τ_0 the associated evaluation map factors through $D \subseteq S$. On the other hand $Z_j \cap D = \emptyset$ for all $j \in J$. We conclude that if $J(0) \neq \emptyset$ then the fibre product

$$\overline{M}_{\tau_0} \times_{\mathrm{Ev}_{J(0)}} Z_{J(0)}$$

is empty. Therefore $\eta_0^! [\overline{M}_{\tau_0}]^{\mathrm{virt}} = 0$ and by (3.14) the contribution vanishes. (This is the only point in the paper where we use the assumption of D-avoidant insertions.)

We conclude that τ_0 only contains markings y_j with $\gamma_j = \mathbb{1}_{\mathrm{Ev}_j}$ and $k_j = 0$. Capping (3.14) with the psi classes appearing in the insertions γ produces:

$$\gamma \cap \nu_* [\overline{M}_\tau]^{\mathrm{virt}} = c_\tau\, \delta^! \left([\overline{M}_{\tau_0}]^{\mathrm{virt}} \times \prod_{i=1}^{\ell} (-1)^{g_i} \lambda_{g_i} \gamma_{J(i)} \cap [\overline{M}_{\tau_i}|_0]^{\mathrm{virt}} \right).$$

We now perform a dimension count. For $i \in \{0, 1, \ldots, \ell\}$ let s_i denote the number of markings y_j contained in τ_i. Then for $i \in \{1, \ldots, \ell\}$ the virtual dimension of $\overline{M}_{\tau_i}|_0$ is $-\beta_i \cdot (K_S + D + E) + g_i + s_i$. It follows that the class

$$\prod_{i=1}^{\ell} (-1)^{g_i} \lambda_{g_i} \gamma_{J(i)} \cap [\overline{M}_{\tau_i}|_0]^{\mathrm{virt}} \tag{3.15}$$

has dimension:

$$\begin{aligned}
&\sum_{i=1}^{\ell} \left(-\beta_i \cdot (K_S + D + E) + s_i - \Sigma_{j \in J(i)}(k_j + \mathrm{codim}(Z_j, \mathrm{Ev}_j))\right) \\
&= \pi_*\beta_0 \cdot (K_S + D + E) - s_0 + \left(-\beta \cdot (K_S + D + E) + s - \Sigma_{i=0}^{\ell} \Sigma_{j \in J(i)}(k_j + \mathrm{codim}(Z_j, \mathrm{Ev}_j))\right) \\
&= \pi_*\beta_0 \cdot (K_S + D + E) - s_0.
\end{aligned} \tag{3.16}$$

Here $\pi_*\beta_0 \in A_1(S)$ is the projection along $\pi : P \to D_0 = D \hookrightarrow S$ of the curve class attached to τ_0. The first equality follows from

$$\beta = \pi_*\beta_0 + \sum_{i=1}^{\ell} \beta_i, \qquad s = \sum_{i=0}^{\ell} s_i$$

and the fact that by the previous arguments $k_j + \mathrm{codim}(Z_j, \mathrm{Ev}_j) = 0$ for all $j \in J(0)$. The second equality holds because of the assumption that the codimension of γ is equal to the virtual dimension of $\overline{M}_{g,\mathbf{c},\beta}(\mathcal{O}_S(-D) \,|\, E)$.

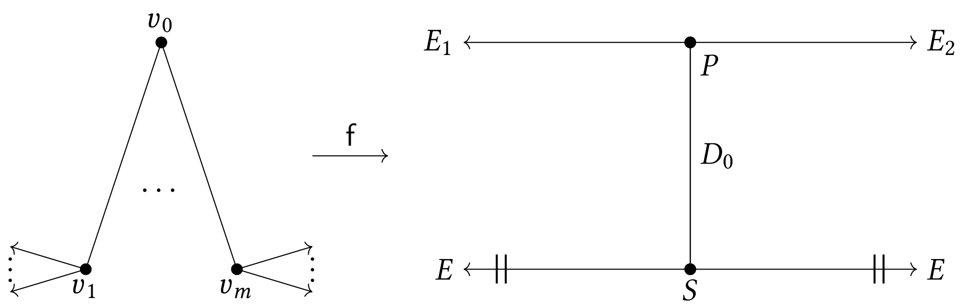
Note that $\pi_*\beta_0$ is necessarily a multiple of D, say $\pi_*\beta_0 = kD$ for some $k \geq 0$. By adjunction and $D \cdot E = 2$ we obtain

$$\pi_*\beta_0 \cdot (K_S + D + E) = kD \cdot (K_S + D) + 2k = 2kg(D) = 0.$$

(This is the first of two points in the argument where we use the assumption that D is rational.) By (3.16) this implies that the cycle (3.15) has dimension $-s_0$. The contribution therefore vanishes unless $s_0 = 0$. □

3.2.4.4 Third reduction: star-shaped graphs. Using tropical arguments we now constrain the shape of the subgraphs Γ_i further.

Proposition 3.2.18 (Third reduction). *For $i \in \{0, 1, \ldots, m\}$ the subgraph Γ_i consists of a single vertex v_i. We have $\mathsf{f}(v_0) \in P$ and $\mathsf{f}(v_1), \ldots, \mathsf{f}(v_m) \in S$. The tropical type τ therefore takes the following form:*



Proof. Combine Lemmas 3.2.21 and 3.2.22 below. □

We prove the statement in several steps.

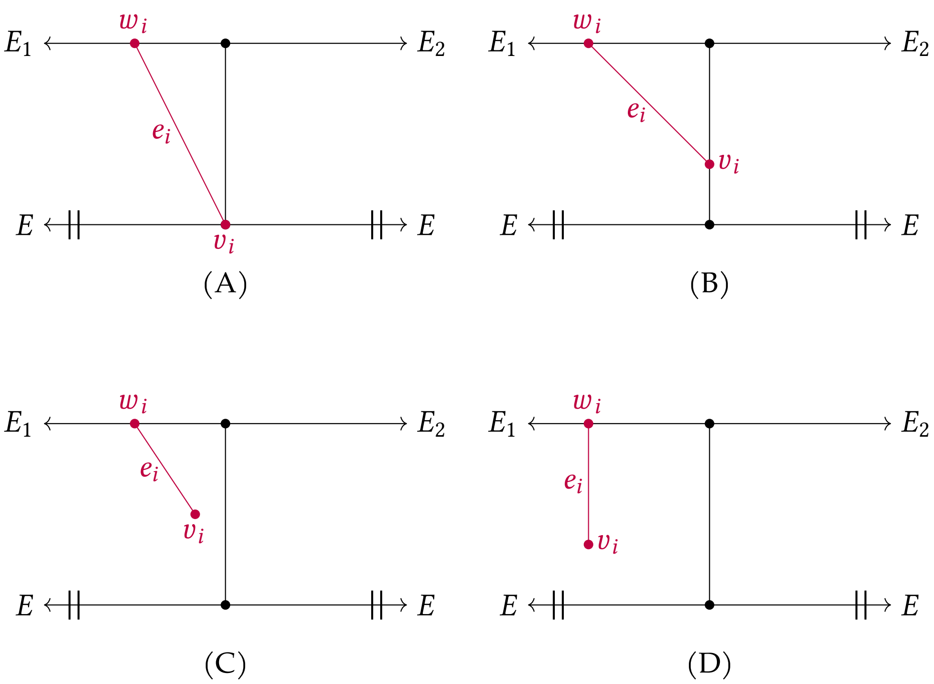
Lemma 3.2.19. *Let $v \in V(\Gamma_i)$ be such that $\mathsf{f}(v)$ belongs to q_1 or q_2. Then the incoming edge at v cannot have positive horizontal slope.*

Proof. Suppose without loss of generality that $\mathsf{f}(v) \in q_1$. By balancing, there must exist an edge outgoing from v with positive horizontal slope. By Proposition 3.2.9 this edge must be finite, and by Lemma 3.2.12 it must have negative vertical slope. Follow this edge to the next vertex. If the vertex also belongs to q_1 we repeat the argument. Eventually, we obtain a vertex w with $\mathsf{f}(w) \in E$. This contradicts Lemma 3.2.13. $\square$

Recall that for $i \in \{1, \dots, m\}$ we denote by e_i the edge connecting Γ_i and Γ_0 and write $v_i \in V(\Gamma_i)$ and $w_i \in V(\Gamma_0)$ for its endpoints (Notation 3.2.10).

Lemma 3.2.20. *We have $\mathsf{f}(w_i) \in P$.*

Proof. It is equivalent to show that $\mathsf{f}(w_i)$ does not belong to E_1 or E_2. Lemmas 3.2.13 and 3.2.19 eliminate many cases. The only ones which remain are



(A) (B) (C) (D)

along with the matching cases for E_2. We deal with cases (A), (B), (C) together. In each case balancing at w_i (Section 3.2.3.3) ensures that there exists a finite outgoing edge with positive horizontal slope (note that w_i cannot support any marking legs by Proposition 3.2.9). If this edge has zero vertical slope, we follow it to the next vertex and repeat the argument. Eventually, we arrive at a vertex on E_1 supporting an outgoing edge with positive horizontal slope and negative vertical slope. This leaves Γ_0 and enters one of the other subgraphs Γ_j. But Lemmas 3.2.13 and 3.2.19 preclude this.

It remains to consider (D). By Lemma 3.2.12, Lemma 3.2.19, and balancing, all edges outgoing from v_i must have zero horizontal slope and negative vertical slope. Inducting along the path, we eventually arrive at a vertex on E, contradicting Lemma 3.2.13. $\square$

Lemma 3.2.21. *For $i \in \{1, \dots, m\}$ the graph Γ_i consists only of the vertex v_i, with $\mathsf{f}(v_i) \in S$.*

Proof. We first show $\mathsf{f}(v_i) \in S$. By Lemma 3.2.13 we have $\mathsf{f}(v_i) \notin E$. On the other hand Lemma 3.2.19 and 3.2.20 together imply that $\mathsf{f}(v_i) \notin q_1$ or q_2. It remains to consider the case $\mathsf{f}(v_i) \in D_0$. By

Lemma 3.2.20 we have $\mathsf{f}(w_i) \in P$. Since Γ_i is a tree, it follows in this case that the corresponding tropical type is not rigid. We conclude that $\mathsf{f}(v_i) \in S$.

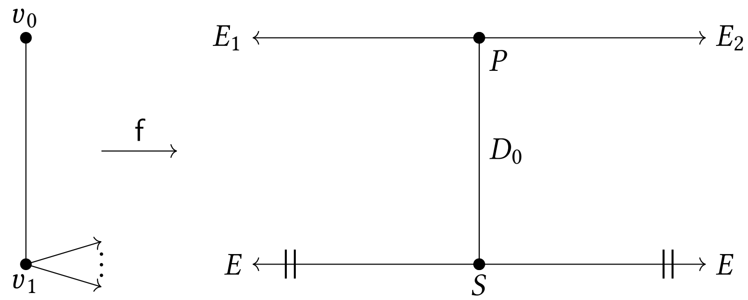
By Lemma 3.2.12 the vertex v_i has no outgoing edges with positive vertical slope. By Lemma 3.2.13 it also has no outgoing edges with positive horizontal slope. It follows that the only outgoing edges have slope zero in both directions. These do not exist because the tropical type is rigid. $\square$

Lemma 3.2.22. *Γ_0 consists only of a single vertex w_0, with $\mathsf{f}(w_0) \in P$.*

Proof. By Lemma 3.2.21 we know that each Γ_i consists of a single vertex v_i with $\mathsf{f}(v_i) \in S$ and that v_i is connected to Γ_0 by an edge e_i with $\mathsf{f}(e_i) \subseteq D_0$. From this we see that if Γ_0 contains any vertices along E_1 or E_2 then the tropical type is not rigid. On the other hand if Γ_0 contains more than one vertex over P then it must contain contracted edges, and again the tropical type is not rigid. $\square$

3.2.4.5 Fourth reduction: single-edge graphs. By Proposition 3.2.18 we know that the tropical type τ is star-shaped, with a vertex v_0 mapping to P and vertices $v_1, \ldots, v_m$ mapping to S and supporting all the marking legs. We now establish the fourth and final reduction:

Proposition 3.2.23 (Fourth reduction)**.** *Let τ be a star-shaped tropical type, as in Proposition 3.2.18. Then $m = 1$ and v_1 contains all of the markings:*



Proof. This follows from Proposition 3.2.24 below. $\square$

The proof necessitates a careful analysis of the gluing appearing in the degeneration formula, similar to Section 3.2.4.3. We first note that if τ is as in Proposition 3.2.18 then there is no need to pass to a subdivision of the target, since all finite edges map to D_0 and all vertices map to S or P. Consequently the identity (3.12) simplifies to[3]

$$\zeta_*\left(\gamma \cap \rho_* \iota_* [\overline{M}_\tau]^{\mathrm{virt}}\right) = \left(\frac{\prod_{i=1}^m m_i}{\mathrm{lcm}(m_i)}\right) \theta_* \delta^! \left([\overline{M}_{\tau_0}]^{\mathrm{virt}} \times \prod_{i=1}^m (-1)^{g_i} \lambda_{g_i} \gamma_{J(i)} \cap [\overline{M}_{\tau_i}(S \,|\, D+E)]^{\mathrm{virt}} \right). \tag{3.17}$$

Note that the graph Γ_0 underlying τ_0 consists of a single vertex v_0 supporting no marking legs. The nodes $q_1, \ldots, q_m$ at this vertex have tangency zero with respect to E_1 or E_2, and the balancing condition determines the associated curve class β_0 as

$$\beta_0 = (D \cdot \beta)\, F$$

[3] There is a typo in the statement of [149, Lemma 6.4.4]: the gluing factor on the right-hand side must be divided by m_τ. We thank Dhruv Ranganathan for confirming this. This typo is corrected in [133]. The above identity is consistent with [76, Theorem 1.1], [44, Corollary 7.10.4], and [99, (1.4)].

where $F \in A_1(P)$ is the class of a fibre. Consequently the evaluation morphism $\overline{M}_{\tau_0} \to \prod_{i=1}^m D_0$ factors through the small diagonal $D_0 \hookrightarrow \Pi_{i=1}^m D_0$. Let

$$\psi : \overline{M}_{\tau_0} \to \overline{M}_{g_0,m} \times D_0$$

denote the morphism remembering only the stabilised source curve and the evaluation. We obtain a diagram

$$\begin{tikzcd}
\overline{M}_{g,s,\beta}(S) & \overline{N}_\tau \arrow[l, "\theta"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{\tau_0} \times \prod_{i=1}^m \overline{M}_{\tau_i}(S \,|\, D+E) \arrow[d, "\psi\times\mathrm{id}"] \\
& \overline{M}_{g_0,m} \times \overline{L}_\tau \arrow[ul, "\phi"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{g_0,m} \times D_0 \times \prod_{i=1}^m \overline{M}_{\tau_i}(S \,|\, D+E) \arrow[d] \\
& \overline{L}_\tau \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & D_0 \times \prod_{i=1}^m \overline{M}_{\tau_i}(S \,|\, D+E) \arrow[d] \\
& \prod_{i=1}^m D_0 \arrow[r, "\delta"] & \prod_{i=1}^m D_0^2
\end{tikzcd}$$

in which $\overline{L}_\tau$ is defined via the bottom cartesian square.[4] It parametrises logarithmic maps $f_i \colon C_i \to (S \,|\, D+E)$ for $i \in \{1, \ldots, m\}$, such that $f_1(q_1) = \ldots = f_m(q_m)$. The gluing morphism ϕ is defined similarly to θ, see (3.8).

For $g, d \geq 0$ consider the moduli space

$$\overline{M}_{g,(d),d}(\mathcal{O}_{\mathbb{P}^1}(-1) \,|\, 0)$$

of stable logarithmic maps with maximal tangency at a single marking, and consider the pushforward of its virtual fundamental class to the moduli space of curves:

$$C_{g,d} \in A_{2g-2}(\overline{M}_{g,1}).$$

Proposition 3.2.24. *Let τ be a star-shaped tropical type as in Proposition 3.2.18. The contribution $\zeta_*(\gamma \cap \rho_* \iota_* [\overline{M}_\tau]^{\mathrm{virt}})$ vanishes unless $m = 1$. In this case $\overline{L}_\tau = \overline{M}_{\tau_1}(S \,|\, D+E)$ and we have*

$$\zeta_*(\gamma \cap \rho_* \iota_* [\overline{M}_\tau]^{\mathrm{virt}}) = (-1)^g \phi_* \left((\lambda_{g_0} \cap C_{g_0, D\cdot\beta}) \times \left(\lambda_{g_1} \gamma \cap [\overline{M}_{\tau_1}(S \,|\, D+E)]^{\mathrm{virt}} \right) \right). \tag{3.18}$$

Proof. Using the compatibility of proper push forward and outer product [63, Proposition 1.10], the identity (3.17) simplifies to

$$\zeta_*(\gamma \cap \rho_* \iota_* [\overline{M}_\tau]^{\mathrm{virt}}) = \left(\frac{\prod_{i=1}^m m_i}{\mathrm{lcm}(m_i)} \right) \phi_* \delta^! \left(\psi_* [\overline{M}_{\tau_0}]^{\mathrm{virt}} \times \prod_{i=1}^m (-1)^{g_i} \lambda_{g_i} \gamma_{J(i)} \cap [\overline{M}_{\tau_i}(S \,|\, D+E)]^{\mathrm{virt}} \right). \tag{3.19}$$

Since D_0 is a rational curve, [63, Example 1.10.2] ensures that the outer product morphism

$$A_*(\overline{M}_{g_0,m}) \otimes A_*(D_0) \to A_*(\overline{M}_{g_0,m} \times D_0)$$

is surjective (this is the second and final point in the argument where we use the assumption that D is rational). We may thus write

$$\psi_* [\overline{M}_{\tau_0}]^{\mathrm{virt}} = (A \times [D_0]) + (B \times [\mathrm{pt}]) \tag{3.20}$$

[4] If $g_0 = 0$ and $m < 3$ we adopt the convention $\overline{M}_{0,1} = \overline{M}_{0,2} = \operatorname{Spec} \mathbb{C}$.

where $A, B \in A_*(\overline{M}_{g_0,m})$ and [pt] is the generator of $A_0(D_0)$. Starting with the second term, we have by [63, Example 6.5.2]:

$$\delta^!\left(B \times [\mathrm{pt}] \times \prod_{i=1}^{m}(-1)^{g_i}\lambda_{g_i}\gamma_{J(i)} \cap [\overline{M}_{\tau_i}(S\,|\,D+E)]^{\mathrm{virt}}\right)$$
$$= B \times \delta^!\left([\mathrm{pt}] \times \prod_{i=1}^{m}(-1)^{g_i}\lambda_{g_i}\gamma_{J(i)} \cap [\overline{M}_{\tau_i}(S\,|\,D+E)]^{\mathrm{virt}}\right).$$

A dimension count similar to (3.16) then shows that

$$\delta^!\left([\mathrm{pt}] \times \prod_{i=1}^{m}(-1)^{g_i}\lambda_{g_i}\gamma_{J(i)} \cap [\overline{M}_{\tau_i}(S\,|\,D+E)]^{\mathrm{virt}}\right) \in A_{-m}(\overline{L}_\tau).$$

We always have $m \geq 1$ and hence this contribution always vanishes. Considering the second term of (3.20), a similar analysis shows that

$$\delta^!\left([D_0] \times \prod_{i=1}^{m}(-1)^{g_i}\lambda_{g_i}\gamma_{J(i)} \cap [\overline{M}_{\tau_i}(S\,|\,D+E)]^{\mathrm{virt}}\right) \in A_{1-m}(\overline{L}_\tau)$$

and so the contribution vanishes unless $m = 1$. This proves the vanishing statement.

Now suppose that $m = 1$. In this case the gluing factor is $(\Pi_{i=1}^m m_i)/\mathrm{lcm}(m_i) = 1$. We obtain:

$$\zeta_*\left(\gamma \cap \rho_*\iota_*[\overline{M}_\tau]^{\mathrm{virt}}\right) = \phi_*\left(A \times \delta^!\left([D_0] \times (-1)^{g_1}\lambda_{g_1}\gamma \cap [\overline{M}_{\tau_1}(S\,|\,D+E)]^{\mathrm{virt}}\right)\right). \tag{3.21}$$

It remains to determine the cycle A. Recall that $\psi_*[\overline{M}_{\tau_0}]^{\mathrm{virt}} = A \times [D_0] + (B \times [\mathrm{pt}])$. Choose a closed point $\xi\colon \operatorname{Spec}\mathbb{C} \hookrightarrow D_0$ and form the fibre product:

$$\begin{tikzcd}
\overline{M}_{g_0,(D\cdot\beta),D\cdot\beta}(\mathcal{O}_{\mathbb{P}^1}(-1)\,|\,0) \arrow[r, hook] \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & \overline{M}_{\tau_0} \arrow[d, "\psi"] \\
\overline{M}_{g_0,1} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{g_0,1} \times D_0 \arrow[d] \\
\operatorname{Spec}\mathbb{C} \arrow[r, hook, "\xi"] & D_0.
\end{tikzcd}$$

We then obtain

$$\begin{aligned}
A &= \xi^!\psi_*[\overline{M}_{\tau_0}]^{\mathrm{virt}} \\
&= \pi_*\xi^![\overline{M}_{\tau_0}]^{\mathrm{virt}} \\
&= \pi_*\left((-1)^{g_0}\lambda_{g_0} \cap [\overline{M}_{g_0,(D\cdot\beta),D\cdot\beta}(\mathcal{O}_{\mathbb{P}^1}(-1)\,|\,0)]^{\mathrm{virt}}\right) \\
&= (-1)^{g_0}\lambda_{g_0} \cap C_{g_0,D\cdot\beta}
\end{aligned}$$

where in the final step we use the fact that lambda classes are preserved by pullback along stabilisation morphisms. The identity (3.21) then immediately implies (3.18). □

We now combine Proposition 3.2.24 with (3.6). For single-edge star-shaped tropical types we have $m_\tau = D \cdot \beta$ and $|\operatorname{Aut}(\tau)| = 1$. We therefore obtain:

Theorem 3.2.25. *The following identity holds in* $A_0(\overline{M}_{g,s,\beta}(S))$

$$
\begin{aligned}
&\zeta_*\left(\gamma \cap [\overline{M}_{g,\mathbf{c},\beta}(\mathcal{O}_S(-D)|E)]^{\mathrm{virt}}\right)\\
&\qquad = (-1)^g (D\cdot\beta) \sum_{g_1+g_2=g} \phi_*\left((\lambda_{g_1} \cap C_{g_1,D\cdot\beta}) \times (\lambda_{g_2}\gamma \cap [\overline{M}_{g_2,\hat{\mathbf{c}},\beta}(S\,|\,D+E)]^{\mathrm{virt}})\right)
\end{aligned}
$$

where the tangency data $\hat{\mathbf{c}}$ *is obtained from* $\mathbf{c}$ *by introducing one additional marked point with maximal tangency along* D *(see Section 3.2.1).*

3.2.4.6 Evaluating the integrals. Finally, we use the cycle-theoretic Theorem 3.2.25 to deduce the numerical Theorem 3.2.3.

Proof of Theorem 3.2.3. Push the formula in Theorem 3.2.25 forward to a point, and form the generating function by summing over g. We obtain:

$$
\begin{aligned}
\sum_{g\geq 0} \mathsf{GW}_{g,\mathbf{c},\beta}(\mathcal{O}_S(-D)\,|\,E)\langle\gamma\rangle\cdot\hbar^{2g-2} = (D\cdot\beta)&\left(\sum_{g_1\geq 0}\mathsf{GW}_{g_1,(D\cdot\beta),D\cdot\beta}(\mathcal{O}_{\mathbb{P}^1}(-1)\,|\,0)\langle(-1)^{g_1}\lambda_{g_1}\rangle\cdot\hbar^{2g_1-1}\right)\\
&\left(\sum_{g_2\geq 0}\mathsf{GW}_{g_2,\hat{\mathbf{c}},\beta}(S\,|\,D+E)\langle(-1)^{g_2}\lambda_{g_2}\gamma\rangle\cdot\hbar^{2g_2-1}\right).
\end{aligned}
$$

The theorem follows immediately from [37, Lemma 6.3] which gives:

$$
\sum_{g_1\geq 0}\mathsf{GW}_{g_1,(D\cdot\beta),D\cdot\beta}(\mathcal{O}_{\mathbb{P}^1}(-1)\,|\,0)\langle(-1)^{g_1}\lambda_{g_1}\rangle\cdot\hbar^{2g_1-1} = \left(\frac{1}{D\cdot\beta}\right)\frac{(-1)^{D\cdot\beta+1}}{2\sin\left(\frac{D\cdot\beta}{2}\hbar\right)}.
$$

□

3.2.5 Nef pairs

In this section, we adopt the setup of Section 3.2.1 and impose the following additional assumptions:

- $E\cdot\beta > 0$.
- $E^2 \geq 0$.

This includes nef Looijenga pairs as studied in [29]. The assumptions ensure that the local theory of $\mathcal{O}_S(-E)$ is well-defined, see Section 3.2.1 for the analogous argument for D.

We consider stable logarithmic maps with two markings of maximal tangency to D and E and possibly additional interior markings. Denote this contact data by $\hat{\mathbf{c}}$ and by $\mathbf{c}$ the result of deleting the two tangency markings. We obtain the following correspondence in genus zero:

Theorem 3.2.26 (Theorem 3.B)**.** *The following identity holds between the genus zero Gromov–Witten invariants with D-avoidant insertions* γ*:*

$$
\mathsf{GW}_{0,\hat{\mathbf{c}},\beta}(S\,|\,D+E)\langle\gamma\rangle = (-1)^{(D+E)\cdot\beta}(D\cdot\beta)(E\cdot\beta)\cdot\mathsf{GW}_{0,\mathbf{c},\beta}(\mathcal{O}_S(-D)\oplus\mathcal{O}_S(-E))\langle\gamma\rangle.
$$

Proof. Take Theorem 3.2.3 in genus zero and apply the logarithmic-local correspondence for smooth pairs [64, Theorem 1.1]. □

This extends [29, Theorem 5.2], which gives the above result when $(S\,|\,D+E)$ is logarithmically Calabi–Yau and has stationary insertions supported at a single interior marking.

Remark 3.2.27. A higher genus analogue of Theorem 3.2.26 may be obtained by combining Theorem 3.2.3 with [21, Theorem 2.7]. The graph sum simplifies, since the logarithmic invariants carry an insertion of λ_g^2 which vanishes unless $g = 0$. The resulting correspondence will thus have limited use, as it compares the genus zero invariants of $(S \mid D+E)$ to the higher genus invariants of the local geometry. We leave a detailed analysis to future work.

3.2.6 Root stacks and self-nodal pairs

In this section we adopt the setup of Section 3.2.1 with the following modifications, which apply only to this section:

- **Stricter:** $g = 0$ and $E \cdot \beta = 0$.
- **Looser:** D is no longer required to be rational, insertions γ are no longer required to avoid D.

Since $E \cdot \beta = 0$ there are no markings with tangency along E. A case of particular interest is resolutions of irreducible self-nodal curves, where β is a curve class pulled back along the blowup.

In this setting, we establish the analogue of Theorem 3.2.3.

Theorem 3.2.28. *The following identity holds between the genus zero Gromov–Witten invariants:*

$$\mathsf{GW}_{0,\hat{c},\beta}(S \mid D+E)\langle\gamma\rangle = (-1)^{D\cdot\beta+1}(D\cdot\beta)\cdot\mathsf{GW}_{0,c,\beta}(\mathcal{O}_S(-D) \mid E)\langle\gamma\rangle.$$

Instead of using the degeneration formula, the result is proved via the enumerative geometry of root stacks. We pass through the following correspondences of genus zero Gromov–Witten theories:

$$\mathsf{Log}(S \mid D+E) \overset{[16]}{\dashleftarrow\dashrightarrow} \mathsf{Orb}(S \mid D+E) \xleftrightarrow{[17]} \mathsf{Orb}(\mathcal{O}_S(-D) \mid E) \xleftrightarrow{[2]} \mathsf{Log}(\mathcal{O}_S(-D) \mid E).$$

The second correspondence follows from [17, Theorem 1.2] (the result is stated for D nef, but in fact β-nef is all that is required in the proof). The fourth correspondence follows from [2, Theorem 1.1]. Only the first correspondence requires further justification. Theorem 3.2.28 thus reduces to the following:

Proposition 3.2.29 (Theorem 3.C)**.** *There is an equality of logarithmic and orbifold Gromov–Witten invariants:*

$$\mathsf{GW}^{\mathsf{log}}_{0,\hat{c},\beta}(S \mid D+E)\langle\gamma\rangle = \mathsf{GW}^{\mathsf{orb}}_{0,\hat{c},\beta}(S \mid D+E)\langle\gamma\rangle.$$

Proof. Let Σ be the tropicalisation of $(S \mid D+E)$. A kaleidoscopic double cover is depicted in Figure 3.2. Given a tropical type of map to Σ, the balancing conditions of Sections 3.2.3.1 and 3.2.3.2 apply verbatim.

By [16, Theorem X] it is sufficient to show that $(S \mid D+E)$ is slope-sensitive with respect to the given numerical data [16, Section 4.1]. Fix a naive type of tropical map to Σ as in [16, Section 3]. We must show that there is no oriented edge $\vec{e}$ of the source graph Γ such that the associated cone $\sigma_e \in \Sigma$ is maximal and the slope $m_{\vec{e}} \in N_{\sigma_e}$ belongs to the positive quadrant.

We begin with a useful construction. Given an oriented edge $\vec{e} \in \vec{E}(\Gamma)$ terminating at a vertex $v \in V(\Gamma)$ with $\sigma_e \in \{q_1, q_2\}$ and $\sigma_v \in \{q_1, q_2, D\}$, we let

$$\Gamma(\vec{e}\,) \subseteq \Gamma$$

denote the maximal connected subgraph which contains $\overleftarrow{e}$ as an outgoing half-edge and is such that all vertices and half-edges have associated cones q_1, q_2, or D. This means that all outgoing

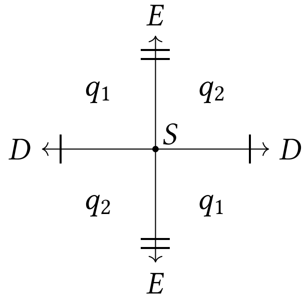


Figure 3.2: Representing Σ as a quotient of a double cover.

half-edges besides $\overleftarrow{e}$ are either unbounded marking legs, or finite edges terminating at E or S (the vertical dividing line in Figure 3.2).

Now suppose for a contradiction that there exists an oriented edge $\vec{e}_1 \in \vec{E}(\Gamma)$ such that $\sigma_{e_1} = q_1$ and $m_{\vec{e}_1} \in N_{q_1}$ belongs to the positive quadrant. The assumption on the slope ensures that the subgraph $\Gamma(\vec{e}_1)$ is well-defined. We claim that $\Gamma(\vec{e}_1)$ contains the (unique) marking leg with positive tangency to D. Since $D^2 \geq 0$ it follows by balancing (Section 3.2.3.2) that at every vertex of $\Gamma(\vec{e}_1)$ the sum of the outgoing slopes in the D-direction is non-negative. Summing over all the vertices, we see that the sum of outgoing slopes from $\Gamma(\vec{e}_1)$ in the D-direction is non-negative. The slope of $\overleftarrow{e}_1$ in the D-direction is negative, hence there exists an outgoing edge whose slope in the D-direction is positive. Such an edge cannot terminate at E or S and so, by the definition of $\Gamma(\vec{e}_1)$, it must be the marking leg with positive tangency to D.

A similar argument shows that $\Gamma(\vec{e}_1)$ also has an outgoing edge with positive slope in the E-direction. Since $E \cdot \beta = 0$ there are no marking legs with tangency to E, and so this must be a finite edge terminating at a vertex v_0 with $\sigma_{v_0} = E$. By balancing (Section 3.2.3.1) we see that v_0 supports an outgoing edge $\vec{e}_2$ with $\sigma_{e_2} = q_2$.

The same argument as above now shows that $\Gamma(\vec{e}_2)$ also contains the marking leg with positive tangency to D. However, $\Gamma(\vec{e}_1)$ and $\Gamma(\vec{e}_2)$ are disjoint: deleting the vertex v_0 separates them, since Γ has genus zero. $\square$

Remark 3.2.30. While restricted to genus zero, Theorem 3.2.28 is strong in that it establishes an equality of virtual fundamental classes. This contrasts with Theorem 3.2.3 which also establishes an equality of Chow classes, but only after capping with suitable insertions. In the latter case, we expect that even in genus zero, the counterexamples of [141, Sections 1 and 3.7] can be adapted to produce pathological insertions (such as naked psi classes) violating the correspondence.

3.3 Toric and open geometries

In this section we restrict to toric targets. The main result (Theorem 3.3.3) equates the Gromov–Witten theories of the local and open geometries:

$$(\mathcal{O}_S(-D) \mid E) \leftrightarrow \mathcal{O}_S(-D)|_{S \setminus E}.$$

The proof proceeds via torus localisation. For the open geometry, the computation is controlled by the topological vertex [116]. The difficult step is to show that the contributions of certain localisation graphs vanish.

3.3.1 Setup

We retain the setup of Section 3.2.1 and introduce the following additional assumptions:

- S is a toric surface.
- E is a toric hypersurface.
- $D + E \in |-K_S|$.
- $E \cdot \beta = 0$.

We do not require that D is toric. An important example is the resolution of an irreducible self-nodal cubic in the plane (Section 3.4).

With these assumptions, the enumerative setup of Section 3.2.1 specialises. There is a single marked point x with tangency $D \cdot \beta$ along D, and no marked points with tangency along E. The space of stable logarithmic maps has virtual dimension g and we consider the Gromov–Witten invariant with a lambda class and no additional insertions:

$$\mathsf{GW}_{g,(D\cdot\beta,0),\beta}(S\,|\,D+E)\langle(-1)^g\lambda_g\rangle \coloneqq (-1)^g\lambda_g \cap [\overline{M}_{g,(D\cdot\beta,0),\beta}(S\,|\,D+E)]^{\mathrm{virt}} \in \mathbb{Q}.$$

The space of stable logarithmic maps to the local target $(\mathcal{O}_S(-D)\,|\,E)$ has virtual dimension zero and we consider the Gromov–Witten invariant with no insertions:

$$\mathsf{GW}_{g,0,\beta}(\mathcal{O}_S(-D)\,|\,E) \coloneqq [\overline{M}_{g,0,\beta}(\mathcal{O}_S(-D)\,|\,E)]^{\mathrm{virt}} \in \mathbb{Q}.$$

Theorem 3.2.3 furnishes a correspondence between these invariants. In this section we relate the latter to the invariants of the open target $\mathcal{O}_S(-D)|_{S\setminus E}$.

3.3.2 Open invariants

We establish conventions for toric geometry. We consider fans Σ not contained in a proper linear subspace of the ambient lattice; these correspond to toric varieties with no torus factors. We write $\Sigma(k)$ for the set of k-dimensional cones. Letting n denote the dimension of the ambient lattice, we introduce notation for closed toric strata appearing in critical dimensions:

- For $\rho \in \Sigma(1)$ we denote the corresponding toric hypersurface D_ρ.
- For $\tau \in \Sigma(n-1)$ we denote the corresponding toric curve L_τ.
- For $\sigma \in \Sigma(n)$ we denote the corresponding torus-fixed point P_σ.

Write Σ_S for the fan of S and $\rho_E \in \Sigma_S(1)$ for the cone corresponding to E. Since $D + E \in |-K_S|$ we have the following identity in the class group of S:

$$D = \sum_{\substack{\rho\in\Sigma_S(1)\\ \rho\neq\rho_E}} D_\rho. \tag{3.22}$$

Set $X \coloneqq \mathcal{O}_S(-D)$ and consider the open subvariety:

$$X^\circ \coloneqq \mathcal{O}_S(-D)|_{S\setminus E}.$$

Equip X° with the trivial logarithmic structure and X with the logarithmic structure induced by E. The open embedding $\iota\colon X^\circ \hookrightarrow X$ is strict.

Following the formalism of the topological vertex [116] we define Gromov–Witten invariants of the open manifold X° by localising with respect to an appropriate torus.

Definition 3.3.1 ([116, Section 3.1])**.** Let $P \in X^\circ$ be a torus-fixed point. Consider the action of the three-dimensional dense torus on $\bigwedge^3 T_P X^\circ$ and let χ_P denote the associated character. The **Calabi–Yau torus** is denoted and defined

$$T \coloneqq \operatorname{Ker} \chi_P.$$

It is a two-dimensional subtorus of the dense torus. The definition is independent of the choice of P because X° is Calabi–Yau.

While the moduli space of stable maps to X° is non-proper, its T-fixed locus is proper. This is used to define Gromov–Witten invariants, via localisation. Let Q_T denote the localisation of $A^*_T(\mathrm{pt})$ at the set of homogeneous elements of non-zero degree, and let $Q_{T,k}$ denote its kth graded piece.

Definition 3.3.2. The **T-localised Gromov–Witten invariant** of X° is denoted and defined:

$$\mathsf{GW}^T_{g,0,\iota^*\beta}(X^\circ) \coloneqq \int_{[\overline{M}_{g,0,\iota^*\beta}(X^\circ)^T]^{\mathrm{virt}}_T} \frac{1}{e^T(N^{\mathrm{virt}})} \in Q_{T,0}.$$

A priori this is a rational function in the equivariant weights, with numerator and denominator homogeneous polynomials of the same degree. However [116, Theorem 4.8] shows that the numerator and denominator are in fact constant, so that:

$$\mathsf{GW}^T_{g,0,\iota^*\beta}(X^\circ) \in \mathbb{Q}.$$

For this it is crucial to restrict to the Calabi–Yau torus.

The main result of this section is the following:

Theorem 3.3.3 (Theorem 3.D)**.** *For all $g \geq 0$ we have:*

$$\mathsf{GW}_{g,0,\beta}(X|E) = \mathsf{GW}^T_{g,0,\iota^*\beta}(X^\circ). \tag{3.23}$$

3.3.3 Localisation calculation

We prove Theorem 3.3.3 via virtual torus localisation [70] which decomposes the left-hand side of (3.23) as a sum over contributions from the T-fixed points of the moduli stack. First, in Section 3.3.3.2 we will identify the contribution of torus fixed points associated to stable maps factoring through $X^\circ \hookrightarrow X$ with the Gromov–Witten invariant $\mathsf{GW}^T_{g,0,\iota^*\beta}(X^\circ)$. To establish equation (3.23) it is consequently sufficient to show that all remaining T-fixed points contribute trivially which we will do in Section 3.3.3.3. We stress that our vanishing argument crucially uses the fact that we localised with respect to the Calabi–Yau torus T (see the proof of Theorem 3.3.3). In general one will not observe such a vanishing when localising with respect to the dense open torus.

3.3.3.1 Fixed loci. The action $T \circlearrowright X$ lifts to actions on $\overline{M}_{g,0,\beta}(X)$ and $\overline{M}_{g,0,\beta}(X|E)$ (in the latter case, this is because T sends E to itself and hence lifts to a logarithmic action $T \circlearrowright (X|E)$).

Restricting from the dense torus of X to the subtorus T does not change the zero- and one-dimensional orbits in X. It follows that it also does not change the fixed locus in the moduli space of stable maps. Since X is a toric variety, this fixed locus is well-understood, see e.g. [156, Section 6], [120, Section 5.2], [70, Section 4], [51, Section 9.2], or [18, Section 4].

Briefly, the fixed locus decomposes into a union of connected components indexed by **localisation graphs**. A localisation graph Γ is a graph equipped with marking legs, degree labelings $d_e > 0$ for every edge $e \in E(\Gamma)$ and genus labelings $g_v \geq 0$ for every vertex $v \in V(\Gamma)$. Furthermore every vertex $v \in V(\Gamma)$ is assigned a cone $\sigma(v) \in \Sigma_X(3)$ and every edge $e \in E(\Gamma)$ is assigned a cone $\sigma(e) \in \Sigma_X(2)$. The corresponding connected component of the fixed locus is denoted

$$F_\Gamma(X)$$

and generically parametrises stable maps with components C_v contracted to torus-fixed points and components C_e forming degree d_e covers of toric curves, totally ramified over the torus-fixed points.

We let $\Omega_{g,0,\beta}(X)$ denote the set of localisation graphs, so that:

$$\overline{M}_{g,0,\beta}(X)^T = \bigsqcup_{\Gamma \in \Omega_{g,0,\beta}(X)} F_\Gamma(X).$$

3.3.3.2 Comparison of fixed loci. Consider the morphism forgetting the logarithmic structures:

$$\overline{M}_{g,0,\beta}(X|E) \to \overline{M}_{g,0,\beta}(X).$$

This is T-equivariant, and hence restricts to a morphism between T-fixed loci. For each $\Gamma \in \Omega_{g,0,\beta}(X)$ we define $F_\Gamma(X|E)$ via the fibre product

$$\begin{tikzcd} F_\Gamma(X|E) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{g,0,\beta}(X|E)^T \arrow[d] \\ F_\Gamma(X) \arrow[r, hook] & \overline{M}_{g,0,\beta}(X)^T \end{tikzcd}$$

and this produces a decomposition of $\overline{M}_{g,0,\beta}(X|E)^T$ into clopen substacks:

$$\overline{M}_{g,0,\beta}(X|E)^T = \bigsqcup_{\Gamma \in \Omega_{g,0,\beta}(X)} F_\Gamma(X|E).$$

Note that we do not claim that each $F_\Gamma(X|E)$ is connected, nor that $F_\Gamma(X|E) \to F_\Gamma(X)$ is virtually birational. Virtual localisation [70] gives:[5]

$$\mathsf{GW}_{g,0,\beta}(X|E) = \sum_{\Gamma \in \Omega_{g,0,\beta}(X)} \int_{[F_\Gamma(X|E)]^{\mathrm{virt}}_T} \frac{1}{e^T(N^{\mathrm{virt}}_{F_\Gamma(X|E)})}. \tag{3.24}$$

Turning to X° we note that there is an inclusion

$$\Omega_{g,0,\iota^*\beta}(X^\circ) \subseteq \Omega_{g,0,\beta}(X)$$

consisting of localisation graphs which do not interact with cones in $\Sigma_X \setminus \Sigma_{X^\circ}$. Since the logarithmic structure on $(X|E)$ is trivial when restricted to X° it follows that for $\Gamma \in \Omega_{g,0,\iota^*\beta}(X^\circ)$ we have

$$F_\Gamma(X|E) = F_\Gamma(X) = F_\Gamma(X^\circ).$$

The perfect obstruction theories coincide when restricted to these loci, producing an identification of the induced virtual fundamental classes and virtual normal bundles. We conclude:

Proposition 3.3.4. *We have:*

$$\mathsf{GW}_{g,0,\beta}(X|E) = \mathsf{GW}^T_{g,0,\iota^*\beta}(X^\circ) + \sum_{\Gamma \in \Omega_{g,0,\beta}(X) \setminus \Omega_{g,0,\iota^*\beta}(X^\circ)} \int_{[F_\Gamma(X|E)]^{\mathrm{virt}}_T} \frac{1}{e^T(N^{\mathrm{virt}}_{F_\Gamma(X|E)})}.$$

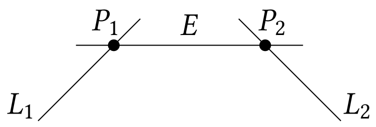
[5] Virtual localisation for spaces of stable logarithmic maps presents conceptual difficulties, as the obstruction theory is defined over the Artin fan which is typically singular. Since the divisor $E \subseteq X$ is smooth, we circumvent these issues by passing to Kim's space of expanded logarithmic maps [18], which has an absolute obstruction theory and arises as a logarithmic modification of the Abramovich–Chen–Gross–Siebert space (see e.g. [15, Section 2.1]). The arguments of this section are insensitive to the choice of birational model of the moduli space. See [138] for a treatment of localisation in this setting.

3.3.3.3 Vanishing of remaining contributions. Fix a localisation graph $\Gamma \in \Omega_{g,0,\beta}(X) \setminus \Omega_{g,0,\iota^*\beta}(X^\circ)$. To prove Theorem 3.3.3 it remains to show

$$\int_{[F_\Gamma(X|E)]^{\mathrm{virt}}_T} \frac{1}{e^T(N^{\mathrm{virt}}_{F_\Gamma(X|E)})} = 0.$$

This requires a detailed analysis of the shape of the localisation graph and its contribution.

Notation 3.3.5. Local to $E \subseteq S$ the toric boundary takes the following form



The zero section gives a closed embedding $S \hookrightarrow X$ as a union of toric boundary strata. Let

$$\tau_E, \tau_1, \tau_2 \in \Sigma_X(2)$$

denote the cones corresponding to the toric curves $E, L_1, L_2 \hookrightarrow S \hookrightarrow X$. Similarly let

$$\sigma_1, \sigma_2 \in \Sigma_X(3)$$

denote the cones corresponding to the torus-fixed points $P_1, P_2 \in S \hookrightarrow X$.

Lemma 3.3.6. *There exists an edge $\tilde{e} \in E(\Gamma)$ with $\sigma(\tilde{e}) = \tau_E$.*

Proof. Suppose for a contradiction that $\sigma(e) \neq \tau_E$ for all $e \in E(\Gamma)$. Since $\Gamma \notin \Omega_{g,0,\iota^*\beta}(X^\circ)$ there exists a vertex $v \in V(\Gamma)$ with $\sigma(v) \in \{\sigma_1, \sigma_2\}$. This vertex is adjacent to an edge $\tilde{e} \in E(\Gamma)$, and since $\sigma(\tilde{e}) \neq \tau_E$ we must have $\sigma(\tilde{e}) \in \{\tau_1, \tau_2\}$. We then find

$$E \cdot \beta = \sum_{e \in E(\Gamma)} d_e \left(E \cdot L_{\sigma(e)}\right) = \sum_{\substack{e \in E(\Gamma) \\ \sigma(e) \in \{\tau_1, \tau_2\}}} d_e \geq d_{\tilde{e}} > 0$$

which contradicts $E \cdot \beta = 0$. □

Lemma 3.3.7. *The following relation holds in $A^1_T(\mathrm{pt})$:*

$$c_1^T(T_{P_1}E) + c_1^T\left(\mathcal{O}_S(-D)|_{P_1}\right) = 0.$$

Proof. Figure 3.3 illustrates the toric skeleton of X in a neighbourhood of $L_1 \cup E$. The horizontal edges index boundary curves contained in the zero section $S \hookrightarrow X$ while the vertical edges index fibres of the projection $X \to S$ over torus-fixed points. We define:

$$u_1 \coloneqq c_1^T(T_{P_1}E), \qquad u_2 \coloneqq c_1^T(T_{P_1}L_1).$$

We now calculate the weights of the T-action on $T_{P_0}L_0$ and $T_{P_0}L_1$. The standard theory of torus actions on projective lines gives:

$$c_1^T(T_{P_0}L_1) = -c_1^T(T_{P_1}L_1) = -u_2. \tag{3.25}$$

Turning to $T_{P_0}L_0$ we have natural identifications:

$$T_{P_0}L_0 = N_{L_1|S}|_{P_0}, \qquad T_{P_1}E = N_{L_1|S}|_{P_1}.$$

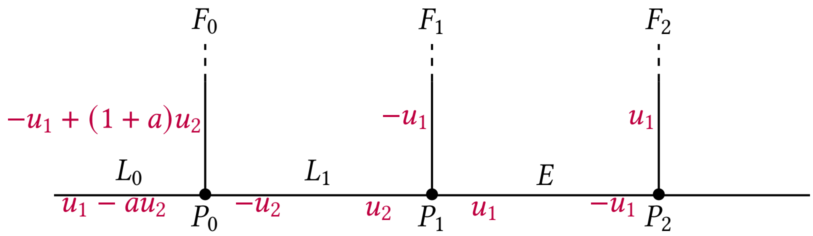


Figure 3.3: The toric skeleton of X locally around $L_1 \cup E$, used in the proof of Lemma 3.3.7. Edges represent boundary curves and vertices represent torus-fixed points. The purple label at a flag (P, L) records the weight $c_1^T(T_P L)$.

Let $a_1 \coloneqq \deg N_{L_1|S}$ denote the self-intersection of the divisor $L_1 \subseteq S$. We then have:

$$c_1^T(T_{P_0}L_0) = c_1^T(N_{L_1|S}|_{P_0}) = c_1^T(N_{L_1|S}|_{P_1}) - a_1 c_1^T(T_{P_1}L_1) = u_1 - a_1 u_2. \tag{3.26}$$

From the definition of the Calabi–Yau torus and (3.25), (3.26) we obtain

$$0 = c_1^T\left(\textstyle\bigwedge^3 T_{P_0}X\right) = c_1^T(T_{P_0}F_0) + (-u_2) + (u_1 - a_1 u_2)$$

from which we deduce:

$$c_1^T(T_{P_0}F_0) = -u_1 + (a_1+1)u_2.$$

By (3.22) we have $\mathcal{O}_S(-D)|_{L_1} \cong \mathcal{O}_{\mathbb{P}^1}(-a_1 - 1)$ from which we conclude

$$c_1^T(T_{P_1}F_1) = c_1^T(T_{P_0}F_0) + (a_1+1)c_1^T(T_{P_0}L_1) = -u_1 = -c_1^T(T_{P_1}E)$$

which completes the proof. □

Proof of Theorem 3.3.3. Since $D^2 \geq 0$ and $D \cdot \beta > 0$ it follows that

$$\overline{M}_{g,0,\beta}(X|E) = \overline{M}_{g,0,\beta}(S|E).$$

Since $(S|E) \hookrightarrow (X|E)$ is strict, there is a short exact sequence

$$0 \longrightarrow T^{\log}_{S|E} \longrightarrow T^{\log}_{X|E}|_S \longrightarrow N_{S|X} \longrightarrow 0$$

and using $N_{S|X} = \mathcal{O}_S(-D)$ we obtain:

$$[\overline{M}_{g,0,\beta}(X|E)]^{\text{virt}}_T = e^T(\mathbf{R}^1\pi_* f^*\mathcal{O}_S(-D)) \cap [\overline{M}_{g,0,\beta}(S|E)]^{\text{virt}}_T.$$

Fix a graph $\Gamma \in \Omega_{g,0,\beta}(X) \setminus \Omega_{g,0,\iota^*\beta}(X^\circ)$. By Proposition 3.3.4 it suffices to show that the contribution of Γ vanishes. We will prove that the T-equivariant vector bundle

$$\mathbf{R}^1\pi_* f^*\mathcal{O}_S(-D)|_{F_\Gamma(X|E)}$$

has a weight zero summand in K-theory. This ensures that the T-equivariant Euler class vanishes, and the claim follows.

Perform the partial normalisation of the source curve at the nodes forced by the localisation graph Γ (such nodes correspond to flags based at either a vertex of valency at least three, or a vertex of valency two which is the intersection of two bounded edges). The normalisation sequence produces a surjection:

$$H^1(C, f^*\mathcal{O}_S(-D)) \twoheadrightarrow \bigoplus_{e \in E(\Gamma)} H^1(C_e, f^*\mathcal{O}_S(-D)).$$

It suffices to show that one summand of the codomain has vanishing equivariant Euler class. By Lemma 3.3.6 there exists an edge $\tilde{e} \in E(\Gamma)$ with $\sigma(\tilde{e}) = \tau_E$. Every point of $F_\Gamma(X|E)$ parametrises a stable logarithmic map whose underlying curve contains an irreducible component $C_{\tilde{e}}$ which maps to E with positive degree $d_{\tilde{e}}$ and is totally ramified over the torus-fixed points. Using Lemma 3.3.7 we write

$$u_1 = -c_1^T\left(\mathcal{O}_S(-D)|_{P_1}\right) = c_1^T(T_{P_1}E).$$

A Riemann–Roch calculation (see e.g. [120, Example 19]) then shows that:

$$\mathrm{ch}_T\left(H^1(C_{\tilde{e}}, f^*\mathcal{O}_S(-D))\right) = \sum_{j=1}^{2d_{\tilde{e}}-1} \exp(-u_1 + j\tfrac{u_1}{d_{\tilde{e}}}).$$

Taking $j = d_{\tilde{e}}$ we see that $H^1(C_{\tilde{e}}, f^*\mathcal{O}_S(-D)$ has a vanishing Chern root, and hence its equivariant Euler class vanishes as claimed. □

3.4 Self-nodal plane curves

In this section we focus on an important special case. Fix $r \geq 1$ and consider the toric variety

$$S_r \coloneqq \mathbb{P}(1, 1, r).^6$$

Let $D_r \in |-K_{S_r}|$ be an irreducible curve with a single nodal toric singularity at the singular point of S_r (or at one of the torus-fixed points if $r = 1$). The pair $(S_r|D_r)$ is logarithmically smooth. By a curve in S_r of degree d we mean a curve whose class is d times the class of the toric hypersurface with self-intersection r. Given a curve in S_r of degree d, its intersection number with D_r is $d(r+2)$. We consider the genus zero maximal tangency Gromov–Witten invariants:

$$\mathsf{GW}_{0,(d(r+2)),d}(S_r|D_r) \in \mathbb{Q}.$$

We begin by deriving an explicit formula for these invariants (Theorem 3.4.1) and applying it to deduce a formula for the invariants of local $\mathbb{P}^1$ (Theorem 3.4.2).

We then specialise to $r = 1$ and establish a relationship between the invariants of $(\mathbb{P}^2|D_1)$ and $(\mathbb{P}^2|E)$ for E a smooth cubic (Theorem 3.4.7). We apply this to prove a conjecture of Barrott and the second-named author (Theorem 3.4.10).

3.4.1 Scattering calculation

The main result of this section is:

Theorem 3.4.1 (Theorem 3.E)**.** *We have:*

$$\mathsf{GW}_{0,(d(r+2)),d}(S_r|D_r) = \frac{r+2}{d^2}\binom{(r+1)^2 d - 1}{d-1}.$$

The following result, already known in the physics literature [41, Equation (4.53)], is a direct consequence of Theorem 3.2.28 and Theorem 3.3.3.

[6] We can also take $r = 0$ in which case we have $S_0 \coloneqq \mathbb{P}^1 \times \mathbb{P}^1$ and D_0 the union of a $(1,0)$ curve and a smooth $(1,2)$ curve. In this case, the results of this section follow from a direct calculation, using [80, Proposition 6.1]. Similarly we can take $r = -1$ which results in the local geometry of a (-1)-curve in a surface.

Theorem 3.4.2 (Theorem 3.F)**.** *We have*

$$\mathsf{GW}^T_{0,0,d}\big(\mathcal{O}_{\mathbb{P}^1}(r) \oplus \mathcal{O}_{\mathbb{P}^1}(-r-2)\big) = \frac{(-1)^{rd-1}}{d^3}\binom{(r+1)^2 d - 1}{d-1}$$

where the left-hand side is defined via localisation with respect to the Calabi–Yau torus, as in Section 3.3.2.

Proof. There is a toric resolution of singularities $\mathbb{F}_r \longrightarrow \mathbb{P}(1,1,r)$. Consider $D_r \subseteq \mathbb{F}_r$ the strict transform and $E \subseteq \mathbb{F}_r$ the exceptional divisor. Realise the Hirzebruch surface as a $\mathbb{P}^1$-bundle

$$\mathbb{F}_r \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(r) \oplus \mathcal{O}_{\mathbb{P}^1})$$

with $E \subseteq \mathbb{F}_r$ the zero section and $E_\infty \subseteq \mathbb{F}_r$ the infinity section. By [6] the Gromov–Witten invariants of $(S_r|D_r)$ are identified with the Gromov–Witten invariants of the bicyclic pair $(\mathbb{F}_r \mid D_r + E)$. We then have

$$\begin{aligned}\mathsf{GW}_{0,(d(r+2),0),d}(S_r|D_r) &= \mathsf{GW}_{0,(d(r+2),0),dE_\infty}(\mathbb{F}_r \mid D_r + E)\\ &= (-1)^{d(r+2)+1} d(r+2)\cdot \mathsf{GW}^T_{0,0,d\,\iota^* E_\infty}(\mathcal{O}_{\mathbb{F}_r}(-D_r)|_{\mathbb{F}_r\setminus E}) \qquad (3.27)\end{aligned}$$

where the second equality follows by combining Theorems 3.2.28 and 3.3.3. Now note that $\mathbb{F}_r \setminus E$ is the total space of $N_{E_\infty|\mathbb{F}_r}$ which is a degree r line bundle on $E_\infty \cong \mathbb{P}^1$. Similarly, we have $\deg \mathcal{O}_{\mathbb{F}_r}(-D_r)|_{E_\infty} = -r-2$. This establishes the identity

$$\mathsf{GW}^T_{0,0,d\,\iota^* E_\infty}(\mathcal{O}_{\mathbb{F}_r}(-D_r)|_{\mathbb{F}_r\setminus E}) = \mathsf{GW}^T_{0,0,d}\big(\mathcal{O}_{\mathbb{P}^1}(r) \oplus \mathcal{O}_{\mathbb{P}^1}(-r-2)\big)$$

and the result follows by combining (3.27) and Theorem 3.4.1. □

It remains to prove Theorem 3.4.1. This proceeds via the connection between the logarithmic Gromov Witten invariants of $(S_r|D_r)$ and local scattering diagrams, following the analyses in [29, 33]. Since the combinatorics of scattering diagrams play only a secondary role, we economise on detail and refer to [29, Section 4] for a full account.

We work on the lattice $M = \mathbb{Z}^2$ equipped with the standard skew-symmetric form $\omega(v,w) = v \wedge w$ and define

$$\rho_1 \coloneqq (-1,0), \qquad \rho_2 \coloneqq (1, r+2).$$

Let $\mathfrak{D}^{\mathrm{in}}_{r+2}$ be the initial scattering diagram with walls $\mathfrak{d}_i \coloneqq \rho_i\mathbb{R}$ decorated with wall-crossing functions

$$f_{\rho_i} \coloneqq 1 + t_i z^{-\rho_i} \in \mathbb{Q}[\![t_1,t_2]\!][M].$$

Write $\mathfrak{D}_{r+2}$ for the consistent scattering diagram obtained by completing $\mathfrak{D}^{\mathrm{in}}_{r+2}$ using the algorithm of [80]. For us, the relevant information is encoded in the wall-crossing function $f_{(0,-1)}$ of the central ray $(0,-1)\mathbb{R}_{\geqslant 0}$.

Proposition 3.4.3. *We have:*

$$f_{(0,-1)} = \exp\left(\sum_{d>0}(r+2)d\ \mathsf{GW}_{0,(d(r+2)),d}(S_r|D_r)\ \left(t_1t_2z^{(0,-(r+2))}\right)^d\right).$$

Proof. We proceed as in [33, Section 2.2.1] and start by constructing a toric model for $(S_r|D_r)$. In the following we will identify a divisor with its strict transform (respectively image) under a blowup (respectively blowdown).

As in the proof of Theorem 3.4.2 we consider the resolution of singularities $\mathbb{F}_r \longrightarrow S_r$, writing $D_r \subseteq \mathbb{F}_r$ for the strict transform of the self-nodal curve and $E \subseteq \mathbb{F}_r$ for the exceptional divisor. We now further blowup at the two intersection points $\{q_1, q_2\} = D_r \cap E$ and let F_1, F_2 denote the resulting exceptional divisors. We write

$$\phi \,:\, (\widetilde{S}_r, \widetilde{D}_r) \coloneqq (\mathrm{Bl}_{q_1,q_2}\mathbb{F}_r \,|\, D_r + F_1 + E + F_2) \longrightarrow (S_r|D_r)$$

for the composition of these blowups. Let $L_1, L_2 \subseteq \widetilde{S}_r$ denote the strict transforms of the tangent lines at the singularity of $D_r \subseteq S_r$. These are (-1)-curves which we blow down to produce:

$$\pi \,:\, (\widetilde{S}_r|\widetilde{D}_r) \to (\overline{S}_r|D_r + F_1 + E + F_2).$$

A quick calculation shows that the self-intersection numbers of $D_r, F_1, E, F_2 \subseteq \overline{S}_r$ are $(r+2), 0, -(r+2), 0$. By [60, Lemma 2.10] this identifies the pair $(\overline{S}_r|D_r + F_1 + E + F_2)$ with $(\mathbb{F}_{r+2} \,|\, \partial\mathbb{F}_{r+2})$.

We observe that $\mathfrak{D}_{r+2}$ as defined above is the scattering diagram associated to the toric model π. By [80, Theorem 5.6] (see also [20, Theorem 3.2]) we have

$$f_{(0,-1)} = \exp\left(\sum_{d>0} d(r+2)\ \mathsf{GW}_{0,(d(r+2),0,0,0),d(E_\infty - L_1 - L_2)}(\widetilde{S}_r|\widetilde{D}_r)\ (t_1t_2)^d\, z^{(0,-d(r+2))}\right)$$

where we write E_∞ for the toric hypersurface in $\mathbb{F}_{r+2}$ with self-intersection $(r+2)$. The result now follows from birational invariance [6], which gives:

$$\mathsf{GW}_{0,(d(r+2),0,0,0),d(E_\infty - L_1 - L_2)}(\widetilde{S}_r|\widetilde{D}_r) = \mathsf{GW}_{0,(d(r+2)),d}(S_r|D_r). \qquad \square$$

It remains to compute $f_{(0,-1)}$. For this, we first identify $\mathfrak{D}_{r+2}$ with the scattering diagram associated to the $(r+2)$-Kronecker quiver. Consider the skew-symmetric form $\widetilde{\omega}(v,w) = (r+2)\,(v \wedge w)$ on $\mathbb{Z}^2$ and denote the resulting symplectic lattice:

$$(\widetilde{M}, \widetilde{\omega}).$$

Set $\widetilde{\rho}_1 = (-1,0), \widetilde{\rho}_2 = (0,1)$, and let $\widetilde{\mathfrak{D}}_{r+2}$ denote the consistent scattering diagram for $(\widetilde{M}, \widetilde{\omega})$ with initial walls $(\widetilde{\rho}_i\mathbb{R}, 1 + t_i z^{-\widetilde{\rho}_i})$.

The resulting scattering pattern is intricate, featuring a subcone in the lower-right quadrant where the support of the walls is dense. However, Reineke has found an explicit description of the wall-crossing function attached to the central wall $(1,-1)\mathbb{R}_{\geq 0}$.

Lemma 3.4.4 ([151, Theorem 6.4])**.** *Let μ be the Möbius function and for $k \geq 1$ set:*

$$N_{k,r} \coloneqq \frac{1}{rk^2} \sum_{i|k} \mu(k/i)\,(-1)^{(r+2)i+1} \binom{(r+1)^2 i - 1}{i}.$$

Then $N_{k,r} \in \mathbb{Z}$ and

$$\widetilde{f}_{(1,-1)} = \prod_{k>0} \left(1 - (-1)^{(r+2)k^2}\left(t_1t_2 z^{(-1,1)}\right)^k\right)^{k(r+2)\,N_{k,r}}.$$

Remark 3.4.5. Strictly speaking, [151, Theorem 6.4] concerns the scattering diagram with initial wall-crossing functions $(1+t_i z^{-\widetilde{\rho}_i})^{r+2}$ on a lattice equipped with the standard skew-symmetric form $(r+2)^{-1}\widetilde{\omega}$. However, the "change of lattice" trick [82, Section C.3, Step IV] translates Reineke's theorem into Lemma 3.4.4. See also [79] for an account using this scattering diagram.

Proof of Theorem 3.4.1. Consider the morphism of lattices $\pi : \widetilde{M} \to M$ given by $\widetilde{\rho}_i \mapsto \rho_i$. The construction of [52, Lemma 2.11] produces a consistent scattering diagram

$$\pi_* \widetilde{\mathfrak{D}}_{r+2}$$

on (M, ω). Under π the initial wall $(\widetilde{\rho}_i\mathbb{R},\ 1 + t_i z^{-\widetilde{\rho}_i})$ is taken to $(\rho_i\mathbb{R},\ 1 + t_i z^{-\rho_i})$. We conclude that up to equivalence $\pi_*\widetilde{\mathfrak{D}}_{r+2} \cong \mathfrak{D}_{r+2}$. Hence, under π the central wall $((1,-1)\mathbb{R}_{\geq 0}, \widetilde{f}_{(1,-1)}) \in \widetilde{\mathfrak{D}}_{r+2}$ is taken to $((0,-1)\mathbb{R}_{\geq 0}, f_{(0,-1)}) \in \mathfrak{D}_{r+2}$. By Lemma 3.4.4 we find

$$\begin{aligned} f_{(0,-1)} &= \prod_{k>0} \left(1 - (-1)^{(r+2)k^2} \left(t_1 t_2 z^{(0,-(r+2))}\right)^k\right)^{k(r+2)\,N_{k,r}} \\ &= \exp\left(-\sum_{l>0} \frac{r+2}{l} \sum_{k>0} (-1)^{(r+2)k^2 l} k\, N_{k,r} \left(t_1 t_2 z^{(0,-(r+2))}\right)^{kl}\right) \\ &= \exp\left(\sum_{d>0} \frac{r+2}{rd} \binom{(r+1)^2 d - 1}{d} \left(t_1 t_2 z^{(0,-(r+2))}\right)^d\right) \end{aligned}$$

where the last equality is proven using the Möbius inversion formula. The statement of Theorem 3.4.1 now follows from Proposition 3.4.3 and the identity:

$$\binom{(r+1)^2 d - 1}{d} = r(r+2)\binom{(r+1)^2 d - 1}{d-1}. \tag{3.28}$$

□

3.4.2 Nodal cubics versus smooth cubics

In this final section we set $r = 1$. We have

$$D_1 = D \subseteq \mathbb{P}^2$$

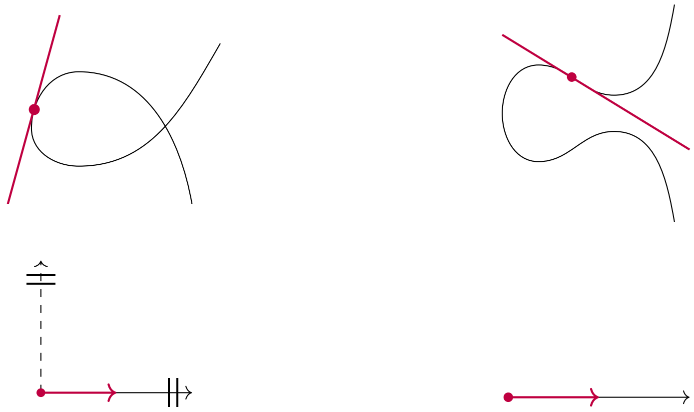
an irreducible cubic with a single nodal singularity. Let $E \subseteq \mathbb{P}^2$ be a smooth cubic. For each of the two pairs we consider the moduli space of genus zero stable logarithmic maps, with maximal tangency at a single marking (see Figure 3.4).

In genus zero, each moduli space has virtual dimension zero and produces a system of enumerative invariants indexed by d. Both theories are completely solved: the case of $(\mathbb{P}^2|D)$ is solved in Theorem 3.4.1, while the case of $(\mathbb{P}^2|E)$ is solved in [67, Example 2.2] with inspiration from [158]. The numbers do not agree, as the following table demonstrates:

d	$\mathsf{GW}_{0,(3d),d}(\mathbb{P}^2\|D)$	$\mathsf{GW}_{0,(3d),d}(\mathbb{P}^2\|E)$
1	3	9
2	$21/4$	$135/4$
3	$55/3$	244
4	$1,365/16$	$36,999/16$
5	$11,628/25$	$635,634/25$
6	$33,649/12$	$307,095$

Experimentally, we always have

$$\mathsf{GW}_{0,(3d),d}(\mathbb{P}^2|D) < \mathsf{GW}_{0,(3d),d}(\mathbb{P}^2|E). \tag{3.29}$$



(a) $(\mathbb{P}^2|D)$ (b) $(\mathbb{P}^2|E)$

Figure 3.4: Tangent curves to plane cubics, nodal and smooth.

In this section we provide a conceptual explanation for this defect, via the geometry of degenerating hypersurfaces (Theorem 3.4.7). We then settle a conjecture posed in [13] (Theorem 3.4.10).

The paper [13] degenerates a smooth cubic to the toric boundary, and studies the resulting logarithmic Gromov–Witten theory on the central fibre. The following construction is similar, except that our starting point is a nodal cubic. We explain how the arguments adapt to this setting, assuming familiarity with [13]. Let $\Delta \subseteq \mathbb{P}^2$ denote the toric boundary and consider a degeneration $D \rightsquigarrow \Delta$, i.e. a divisor

$$\mathcal{D} \subseteq \mathbb{P}^2 \times \mathbb{A}^1$$

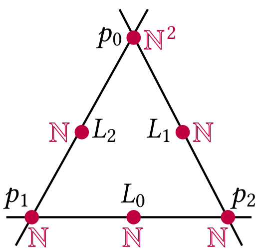
whose general fibre is an irreducible nodal cubic and whose central fibre is Δ. We can choose $\mathcal{D}$ to be irreducible with normal crossings singularities, and such that $\pi^{-1}(t) \cap \mathcal{D}^{\mathrm{sing}}$ is the nodal point of $\mathcal{D}_t$ for $t \neq 0$, and

$$\pi^{-1}(0) \cap \mathcal{D}^{\mathrm{sing}} = p_0$$

where $p_0 = [1, 0, 0]$. Consider the logarithmically regular logarithmic scheme

$$\mathcal{Y} = (\mathbb{P}^2 \times \mathbb{A}^1 \,|\, \mathcal{D}).$$

Equip $\mathbb{A}^1$ with the trivial logarithmic structure and consider the logarithmic morphism $\mathcal{Y} \to \mathbb{A}^1$. This is not logarithmically smooth, but is logarithmically flat; the proof is similar to [13, Lemma 3.7]. The general fibre $\mathcal{Y}_t$ is the logarithmic scheme associated to the smooth pair $(\mathbb{P}^2|\mathcal{D}_t)$. The central fibre, on the other hand, is logarithmically singular. The stalks of the ghost sheaf of $\mathcal{Y}_0$ are



where $L_0, L_1, L_2 \subseteq \mathbb{P}^2$ are the coordinate lines and $p_0, p_1, p_2 \in \mathbb{P}^2$ the coordinate points.

As in [13, Section 2] we construct, in genus zero, a virtual fundamental class on the space of stable logarithmic maps to the central fibre $\mathcal{Y}_0$. Integrals against this class recover the invariants of $(\mathbb{P}^2|D)$

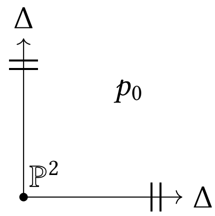
by the conservation of number principle. We now study the moduli space

$$\overline{M}_{0,\mathbf{c},d}(\mathcal{Y}_0).$$

As in [13, Lemma 4.5] we find that every logarithmic map to $\mathcal{Y}_0$ must factor through Δ. In fact, in our new setting we obtain a stronger constraint.

Lemma 3.4.6. *Given a logarithmic map to $\mathcal{Y}_0$ the underlying schematic map to $\mathbb{P}^2$ factors through L_0.*

Proof. Let $C \to \mathcal{Y}_0$ be a logarithmic map and consider its tropicalisation $\mathsf{f}\colon \Sigma C \to \Sigma\mathcal{Y}_0$. The target $\Sigma\mathcal{Y}_0$ is the cone complex:



Following [13, proof of Lemma 4.5] it suffices to show that the image of f does not intersect the interior of the maximal cone p_0.

We first describe the balancing condition. For $v \in V(\Gamma)$ let $C_v \subseteq C$ denote the corresponding irreducible component. Since $f(C_v) \subseteq \Delta$ we have that $\mathsf{f}(v)$ belongs to either Δ or p_0. In the latter case, the balancing condition states that the sum of the outgoing slope vectors is zero. The interesting case is $\mathsf{f}(v) \in \Delta$. We distinguish two possibilities:

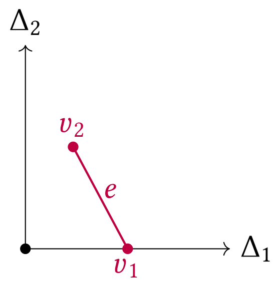
- If $f(C_v) \subseteq L_0$ then all outgoing edges from v are contained in Δ and the sum of their slopes is equal to $3d_v$.

- If $f(C_v) \subseteq L_1$ or L_2 then the slope of each outgoing edge from v can be broken into components tangent to Δ and normal to Δ. The sum of the slopes tangent to Δ is equal to $2d_v$, while the sum of the slopes normal to Δ is equal to d_v. Moreover, edges with positive slope normal to Δ must all enter the same neighbourhood of v in the above chart. Which neighbourhood they enter depends on which of L_1 or L_2 the component C_v maps to.

Due to the balancing condition, every tropical map $\Sigma C \to \Sigma\mathcal{Y}_0$ admits a lift to the standard cover of the target:

$$\begin{tikzcd} & \mathbb{R}^2_{\geqslant 0} \arrow[d] \\ \Sigma C \arrow[r] \arrow[ur] & \Sigma\mathcal{Y}_0. \end{tikzcd}$$

Choose such a lift, and suppose for a contradiction that the image of f intersects the interior of p_0. Then there exists an edge $e \in E(\Gamma)$ such that $\mathsf{f}(e)$ intersects the interior of p_0 and such that one of the end vertices of e is mapped to Δ. On the lift we have:



Define $\Gamma_i \subseteq \Gamma$ by cutting Γ at e and taking the subgraph containing v_i. Start with Γ_1. By balancing we have $d_{v_1} > 0$ and so v_1 supports an outgoing edge with positive slope in the Δ_1-direction. Traversing along this edge to the next vertex, we again conclude by balancing that there exists an outgoing edge with positive slope in the Δ_1-direction. Continuing in this way, we eventually arrive at the marking leg. It follows that the marking leg Γ_1.

Now consider Γ_2. By balancing the vertex v_2 supports an outgoing edge with positive slope in the Δ_2-direction (this occurs both if $\mathsf{f}(v_2) \in p_0$ and if $\mathsf{f}(v_2) \in \Delta_2$). As above, we inductively traverse the graph and produce a path consisting of edges with positive slope in the Δ_2-direction. Eventually we arrive at the marking leg. It follows that the marking leg is contained in Γ_2.

The marking leg is thus contained in both Γ_1 and Γ_2. But these subgraphs are disjoint: since Γ has genus zero, the edge e is separating. □

Using Lemma 3.4.6 we can show as in [13, Proposition 4.11] that

$$\overline{M}_{0,\mathbf{c},d}(\mathcal{Y}_0) = \overline{M}_{0,1,d}(L_0) \cong \overline{M}_{0,1,d}(\mathbb{P}^1).$$

This space carries a virtual fundamental class of dimension zero, arising from logarithmic deformation theory. In [13, Section 4] this is expressed as an obstruction bundle integral, and in [13, Section 5] it is computed via localisation. These calculations apply directly to our new setting, because the logarithmic scheme $\mathcal{Y}_0$ agrees with the logarithmic scheme $\mathcal{X}_0$ of [13, Section 3.1] in a neighbourhood of L_0. We conclude:

Theorem 3.4.7 (Theorem 3.G)**.** *The invariant of $(\mathbb{P}^2|D)$ is precisely the central fibre contribution to the invariant of $(\mathbb{P}^2|E)$ arising from multiple covers of a single line of Δ. In the notation of [13, Section 5.5]:*

$$\mathsf{GW}_{0,(3d),d}(\mathbb{P}^2|D) = \mathrm{C}_{\mathrm{ord}}(d,0,0).$$

Thus the invariants of $(\mathbb{P}^2|D)$ constitute one contribution, amongst many, to the invariants of $(\mathbb{P}^2|E)$. Experimentally, all contributions are positive: this explains the inequality (3.29).

Theorem 3.4.7 also allows us to compute $\mathrm{C}_{\mathrm{ord}}(d,0,0)$. In [13, Section 5] these are computed up to $d = 8$ by computer-assisted torus localisation. Based on these numerics, the following formula is proposed:

Conjecture 3.4.8 ([13, Conjecture 5.9(39)])**.** *We have the following hypergeometric expression for the contribution of degree d covers of L_0:*

$$\mathrm{C}_{\mathrm{ord}}(d,0,0) = \frac{1}{d^2}\binom{4d-1}{d}.$$

It is then shown [13, Proposition 5.13] that Conjecture 3.4.8 is equivalent to the following conjecture in pure combinatorics, which is verified by computer up to $d = 50$:

Conjecture 3.4.9 ([13, Conjecture 5.12])**.** *Fix an integer $d \geq 1$. Then we have*

$$\sum_{(d_1,\ldots,d_r)\vdash d} \frac{2^{r-1}\cdot d^{r-2}}{\#\mathrm{Aut}(d_1,\ldots,d_r)} \prod_{i=1}^{r} \frac{(-1)^{d_i-1}}{d_i}\binom{3d_i}{d_i} = \frac{1}{d^2}\binom{4d-1}{d}$$

where the sum is over strictly positive unordered partitions of d (of any length).

Using Theorem 3.4.7 we can now prove both these conjectures.

Theorem 3.4.10. *Conjecture 3.4.8, and hence also Conjecture 3.4.9, holds.*

Proof. By Theorem 3.4.7 it is equivalent to show that

$$\mathsf{GW}_{0,(3d),d}(\mathbb{P}^2|D) = \frac{1}{d^2}\binom{4d-1}{d}.$$

This follows from Theorem 3.4.1 for $r = 1$ and the identity (3.28). See also [13, Remark 5.10]. $\square$

Chapter

The log-open correspondence for two-component Looijenga pairs

4

The content of this chapter was first made available as the arXiv preprint [153].

Abstract. A two-component Looijenga pair is a rational smooth projective surface with an anticanonical divisor consisting of two transversally intersecting curves. We establish an all-genus correspondence between the logarithmic Gromov–Witten theory of a two-component Looijenga pair and open Gromov–Witten theory of a toric Calabi–Yau threefold geometrically engineered from the surface geometry. This settles a conjecture of Bousseau, Brini and van Garrel in the case of two boundary components. We also explain how the correspondence implies BPS integrality for the logarithmic invariants and provides a new means for computing them via the topological vertex method.

4.1 Introduction

In [29] Bousseau, Brini and van Garrel discovered a surprising relationship between two at first sight quite different curve counting theories:

- **Logarithmic Gromov–Witten theory** of a Looijenga pair

  $$(S|D_1 + \ldots + D_l)$$

  which is a rational smooth projective surfaces S together with an anticanonical singular nodal curve $D_1 + \ldots + D_l$. To this datum one can associate a Gromov–Witten invariant

  $$\mathsf{LGW}_{g,\hat{\mathbf{c}},\beta}(S|D_1 + \ldots + D_l) \in \mathbb{Q}$$

  enumerating genus g, class β stable logarithmic maps to $(S|D_1 + \ldots + D_l)$ with maximum tangency along each irreducible component D_i. Additionally, there are $l-1$ interior markings with a point condition and an insertion of the top Chern class of the Hodge bundle. The tangency order of the markings along $D_1, \ldots, D_l$ is recorded in a matrix $\hat{\mathbf{c}}$.

- **Open Gromov–Witten theory** of a toric triple

$$\big(X, (L_i, \mathsf{f}_i)_{i=1}^k\big)$$

consisting of the toric Calabi–Yau threefold X and a collection of framed Aganagic–Vafa Lagrangian submanifolds (L_i, f_i). To this data one can associate open Gromov–Witten invariants

$$\mathsf{OGW}_{g,((w_1),\dots,(w_k)),\beta'}(X, (L_i, \mathsf{f}_i)_{i=1}^k) \in \mathbb{Q}$$

enumerating genus g, class β' stable maps to X from Riemann surfaces with k boundaries, each wrapping one of the Lagrangian submanifolds L_i exactly w_i times.

It was conjectured in [29] that starting from a Looijenga pair one can geometrically engineer a toric triple so that the above two curve counts turn out to be essentially equal.

Conjecture 4.A. [29, *Conjecture 1.3*] *Let $(S|D_1 + \ldots + D_l)$ be a Looijenga pair and β an effective curve class in S satisfying some technical conditions. Then there exists a toric triple $(X, (L_i, \mathsf{f}_i)_{i=1}^{l-1})$ and an effective curve class β' in X such that*

$$\begin{aligned}\sum_{g\geq 0} \hbar^{2g-2}\, &\mathsf{OGW}_{g,\mathbf{c},\beta'}(X, (L_i, \mathsf{f}_i)_{i=1}^{l-1}) \\ &= \left(\prod_{i=1}^{l-1} \frac{(-1)^{D_i\cdot\beta}}{D_i\cdot\beta}\right) \frac{(-1)^{D_l\cdot\beta+1}}{2\sin\frac{(D_l\cdot\beta)\hbar}{2}} \sum_{g\geq 0} \hbar^{2g-1}\, \mathsf{LGW}_{g,\hat{\mathbf{c}},\beta}(S|D_1 + \ldots + D_l)\end{aligned} \tag{4.1}$$

where $\mathbf{c}$ is obtained from the contact datum $\hat{\mathbf{c}}$ by deleting the $l-1$ interior markings and the marking with tangency along D_l.

For $l = 1$ the right-hand side of (4.1) reduces to the generating series of ordinary Gromov–Witten invariants of X in which case the conjecture has already been proven in [65, Theorem D]. In this paper we investigate the case of two-component Looijenga pairs $(S|D_1 + D_2)$ and as another application of the main result in [65] we will establish Conjecture 4.A for these geometries.

4.1.1 Logarithmic-Open Correspondence

Consider a two-component Looijenga pair $(S|D_1 + D_2)$ together with an effective curve class β in S. We will impose

Assumption $\star$. $(S|D_1 + D_2)$ and β satisfy

- $D_i \cdot \beta > 0$, $i \in \{1, 2\}$,
- $D_2 \cdot D_2 \geq 0$,
- $(S|D_1)$ deforms into a pair $(S'|D_1')$ with S' a smooth projective toric surface and D_1' a toric hypersurface.[1]

Moreover, we denote by $\hat{\mathbf{c}}$ the following contact datum along $D_1 + D_2$

$$\hat{\mathbf{c}} = \begin{pmatrix} 0 & \cdots & 0 & \overbrace{c_1 \quad \cdots \quad c_n}^{=\mathbf{c}} & 0 \\ \underbrace{0 \quad \cdots \quad 0}_{m \text{ times}} & 0 \quad \ldots \quad 0 & D_2\cdot\beta \end{pmatrix}$$

[1] We expand on the last condition in Remark 4.2.7.

where we assume $n \geq m$ and $c_i > 0$ for all $i \in \{1, \ldots, n\}$. This means there are m interior markings, n markings tangent to D_1 and an additional marking with maximum tangency along D_2. We set

$$\mathsf{LGW}_{g,\hat{\mathbf{c}},\beta}(S|D_1+D_2) \coloneqq \int_{[\overline{M}_{g,\hat{\mathbf{c}},\beta}(S|D_1+D_2)]^{\mathrm{virt}}} (-1)^g \lambda_g \prod_{i=1}^{n} \mathrm{ev}_i^*(\mathrm{pt})\,. \tag{4.2}$$

Especially, note that we impose a point condition at all m interior markings. Our main result is the following logarithmic to open correspondence.

Theorem 4.B. (*Theorem 4.2.9*) *Assuming* $\star$ *there is a toric triple* (X, L, f) (*see Construction 4.2.8*) *and a curve class* β' *in* X *satisfying*

$$\sum_{g\geq 0} \hbar^{2g-2}\, \mathsf{OGW}_{g,\mathbf{c},\beta'}(X, L, \mathsf{f}) = \frac{(-1)^{D_1\cdot\beta}}{m!\prod_{i=0}^{m-1} c_{n-i}} \, \frac{(-1)^{D_2\cdot\beta+1}}{2\sin\frac{(D_2\cdot\beta)\hbar}{2}} \sum_{g\geq 0} \hbar^{2g-1}\, \mathsf{LGW}_{g,\hat{\mathbf{c}},\beta}(S|D_1+D_2)\,. \tag{4.3}$$

The specialisation of the above correspondence to the case $m = n = 1$ immediately gives

Corollary 4.C. *Conjecture 4.A holds for all two-component Looijenga pairs* ($l = 2$) *under Assumption* $\star$.

4.1.2 Topological vertex

Computing Gromov–Witten invariants of logarithmic Calabi–Yau surfaces is usually a rather tedious endeavour. In higher genus with the presence of a λ_g insertion the only general tools available are scattering diagrams and tropical correspondence theorems [20, 24, 27, 98].

As an application of Theorem 4.B we obtain a new — highly efficient — means for computing logarithmic Gromov Witten invariants of Looijenga pairs. Since open Gromov–Witten invariants can be computed using the topological vertex [7, 116], the same method can be used to determine the opposite side of our correspondence theorem.

Practical Result 4.D. (*Section 4.5.1*) *The logarithmic Gromov–Witten invariants* (4.2) *of a two-component Looijenga pair satisfying Assumption* $\star$ *are computed by the topological vertex.*

4.1.3 BPS integrality

In general, Gromov–Witten invariants are rational numbers. However, in many cases they are expected to exhibit underlying integral BPS type invariants. For open Gromov–Witten invariants this behaviour was first observed by Labastida–Mariño–Ooguri–Vafa [111, 112, 132, 145], was studied extensively in explicit examples by Luo–Zhu [124, 171, 172] and got recently proven by Yu [166] for general toric targets. Hence, as a corollary of our main result Theorem 4.B we find that logarithmic invariants feature the same property.

Theorem 4.E. (*Theorem 4.5.5*) *The logarithmic Gromov–Witten invariants* (4.2) *of a two-component Looijenga pair satisfying Assumption* $\star$ *exhibit BPS integrality.*

In Section 4.5.2 we spell this statement out in more detail. We conjecture that the above observation holds without imposing Assumption $\star$ as well (Conjecture 4.5.4).

4.1.4 Strategy

The proof of Theorem 4.B splits into several steps which partly have already been carried out elsewhere:

$$\begin{tikzcd} \boxed{(S\,|\,D_1+D_2)} \arrow[r, leftrightarrow, "{[65]}"] & \boxed{(\mathcal{O}_S(-D_2)\,|\,D_1)} \arrow[loop below, "\text{Section 4.4}"] \arrow[r, leftrightarrow, "\text{Section 4.3}"] & \boxed{\mathcal{O}_S(-D_2)|_{S\setminus D_1}} \arrow[r, leftrightarrow, "{[59]}"] & \boxed{(X,L,\mathsf{f})} \end{tikzcd}$$

In [65] van Garrel, Nabijou and the author prove a comparison statement between the Gromov–Witten theory of $(S\,|\,D_1+D_2)$ and $(\mathcal{O}_S(-D_2)\,|\,D_1)$. Together with a result of Fang and Liu [59] expressing the open Gromov–Witten invariants of a toric triple (X,L,f) as descendant invariants of X it is therefore sufficient to prove a formula for relative Gromov–Witten invariants of $(\mathcal{O}_S(-D_2)\,|\,D_1)$ in terms of descendant invariants of $X = \operatorname{Tot}\mathcal{O}_S(-D_2)|_{S\setminus D_1}$ in order to establish Theorem 4.B. We will prove such a relation in Section 4.3 in case there are no interior markings ($m=0$). Later in Section 4.4 we will reduce the general case ($m \geq 0$) to the one proven earlier.

4.1.5 Context & Prospects

Stable maps to Looijenga pairs. The main motivation for this work clearly comes from the conjectures put forward by Bousseau, Brini and van Garrel. Conjecture 4.A was formulated in [29] motivated by direct calculations of both sides of the correspondence. Theorem 4.B covers all two-component Looijenga pairs considered in loc. cit. but also previously unknown cases. Most notably, our correspondence theorem also applies to the Looijenga pairs $\mathrm{dP}_1(0,4)$, $\mathrm{dP}_2(0,3)$, $\mathrm{dP}_3(0,2)$ for which Conjecture 4.A was established by proving an intricate identity of q-hypergeometric functions [33, 109]. We illustrate how our main result can be used to reprove these cases in Section 4.5.1.2.

There is certainly the question whether the techniques that went into the proof of Theorem 4.B can be generalised to cover Looijenga pairs with $l > 2$ boundary components as well. The most difficult part in this endeavour will most likely be to find an appropriate generalisation of [65, Theorem A]. Moreover, in the case $l \geq 3$ the construction of the toric triple proposed in [29, Example 6.5] involves a flop which numerically does the right job but needs to be understood better geometrically first in order to generalise the techniques of this paper.

Open/Closed duality of Liu–Yu. Liu and Yu [119, 122] prove a correspondence between genus zero open Gromov–Witten invariants of a toric threefold and local Gromov–Witten invariants of some Calabi–Yau fourfold constructed from the threefold. If we combine Theorem 4.B with [65, Theorem B] we obtain a similar relation between open/local Gromov–Witten invariants of

$$\boxed{(X,L,\mathsf{f})} \longleftrightarrow \boxed{\mathcal{O}_S(-D_1)\oplus\mathcal{O}_S(-D_2)}. \tag{4.4}$$

It is a combinatorial exercise to check that starting from the toric triple (X,L,f) we define in Construction 4.2.8 the fourfold constructed in [122, Section 2.4] is indeed deformation equivalent (in the sense of Remark 4.2.7) to the right-hand side of (4.4).[2] It should, however, be stressed that the correspondence of Liu and Yu holds in a more general context than just local surfaces.

[2]The key observation is that the toric divisor D we are deleting in Construction 4.2.8 is reinserted in Liu–Yu's construction [122, Section 2.4] of the fourfold as the divisor associated to the ray $\widetilde{b}_{R+1}\mathbb{R}_{\geq 0}$.

BPS invariants. Motivated by Theorem 4.E we conjecture BPS integrality for Gromov–Witten invariants of type (4.2) for two-component logarithmic Calabi–Yau surfaces not necessarily satisfying Assumption ⋆ (Conjecture 4.5.4). This predication overlaps with a conjecture of Bousseau [20, Conjecture 8.3] in special cases. So one may wonder about a simultaneous generalisation of both conjectures. Especially, Bousseau's conjecture also covers logarithmic Calabi–Yau surfaces with more than two components. A geometric proof of Conjecture 4.A in the case of $l > 2$ components in combination with LMOV integrality [166] might allow to make progress towards this direction.

Acknowledgements

First and foremost I would like to thank my supervisor Andrea Brini for suggesting this project, innumerous discussions and his continuous support. Special thanks are also due to Cristina Manolache for clarifying discussions concerning key steps in the proof of our main result. I also benefited from discussions with Alberto Cobos Rábano, Michel van Garrel, Samuel Johnston, Navid Nabijou, Dhruv Ranganathan, Ajith Urundolil Kumaran and Song Yu which I am highly grateful for.

Parts of this work have been carried out during research visits at the University of Cambridge. I thank the university and St John's College for hospitality and financial support.

4.2 Statement of the main result

4.2.1 Logarithmic Gromov–Witten theory

Definition 4.2.1. An **l-component Looijenga pair** $(S\,|\,D_1 + \ldots + D_l)$ is a rational smooth projective surface S together with an anticanonical singular nodal curve $D_1 + \ldots + D_l$ with l irreducible components $D_1, \ldots, D_l$.

Given a two-component Looijenga pair $(S\,|\,D_1 + D_2)$, we denote by

$$\overline{M}_{g,\hat{\mathbf{c}},\beta}(S\,|\,D_1 + D_2)$$

the moduli stack of genus g, class β stable logarithmic maps to $(S\,|\,D_1 + D_2)$ with markings whose tangency along $D_1 + D_2$ is encoded in $\hat{\mathbf{c}}$ [1, 43, 81]. We fix this tangency data to be

$$\hat{\mathbf{c}} = \begin{pmatrix} 0 & \cdots & 0 & c_1 & \ldots & c_n & 0 \\ \underbrace{0 \;\; \cdots \;\; 0}_{m \text{ times}} & & & 0 & \ldots & 0 & D_2 \cdot \beta \end{pmatrix}$$

for some $c_1, \ldots, c_n > 0$ with $\sum_i c_i = D_1 \cdot \beta$. This means there are m interior markings (no tangency), n markings with tangency along D_1 and a marking having maximum tangency with D_2. The virtual dimension of the moduli stack is $g + m + n$ and so we define

$$\mathsf{LGW}_{g,\hat{\mathbf{c}},\beta}(S\,|\,D_1 + D_2) \coloneqq \int_{[\overline{M}_{g,\hat{\mathbf{c}},\beta}(S\,|\,D_1+D_2)]^{\mathrm{virt}}} (-1)^g \lambda_g \prod_{i=1}^{n} \mathrm{ev}_i^*(\mathrm{pt})$$

where λ_g is the top Chern class of the Hodge bundle.

The ultimate goal of this section is to relate the above invariants to the ones of an open subset of (a deformation of) the threefold $Y \coloneqq \mathrm{Tot}\,\mathcal{O}_S(-D_2)$. Let us recall the main result of [65] which serves as a first step towards this direction. Writing π for the projection $Y \to S$, we set $D \coloneqq \pi^{-1}(D_1)$ and define

$$\mathsf{LGW}_{g,\check{\mathbf{c}},\beta}(Y\,|\,D) \coloneqq \int_{[\overline{M}_{g,\check{\mathbf{c}},\beta}(Y\,|\,D)]^{\mathrm{virt}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\pi^*\mathrm{pt})$$

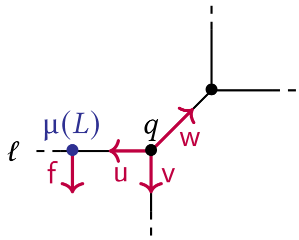


Figure 4.1: The image of the toric skeleton (black) of Tot $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ with an outer brane L (blue) in framing $f = 0$ (red) under the moment map.

where $\tilde{\mathbf{c}}$ is obtained from $\hat{\mathbf{c}}$ by deleting the marking with tangency along D_2.

Theorem 4.2.2. [65, Theorem A] *Suppose $D_2 \cdot D_2 \geq 0$ and β is so that $D_1 \cdot \beta \geq 0$ and $D_2 \cdot \beta > 0$. Then*

$$\sum_{g \geq 0} \hbar^{2g-2} \, \mathsf{LGW}_{g,\tilde{\mathbf{c}},\beta}(Y|D) = \frac{(-1)^{D_2 \cdot \beta + 1}}{2 \sin \frac{(D_2 \cdot \beta)\hbar}{2}} \sum_{g \geq 0} \hbar^{2g-1} \, \mathsf{LGW}_{g,\hat{\mathbf{c}},\beta}(S|D_1 + D_2) \,. \tag{4.5}$$

4.2.2 Open Gromov–Witten theory

Open Gromov–Witten invariants are a delicate topic. We will work in the framework of Fang and Liu [59] which builds on ideas of [97, 117]. We keep details to a minimum since at no point we will be working with open Gromov–Witten invariants directly. We refer the interested reader to [59, 97] for more details.

4.2.2.1 Preliminaries on toric Calabi–Yau threefolds. Let X be a smooth toric Calabi–Yau threefold. We will write $\hat{T} \cong \mathbb{G}_{\mathsf{m}}^3$ for its dense torus and $T \cong \mathbb{G}_{\mathsf{m}}^2$ for its **Calabi–Yau torus**. The latter is defined as the kernel $T = \ker \chi$ of the character χ associated to the induced $\hat{T}$-action on $K_X \cong \mathcal{O}_X$.

If X can be realised as a symplectic quotient, which is the case when X is semiprojective [85], Aganagic and Vafa [8] construct a certain class of Lagrangian submanifolds L in X. Relevant for us here is the fact that these Lagrangian submanifolds are invariant under the action of the maximal compact subgroup $T_{\mathbb{R}} \hookrightarrow T$ and are homeomorphic to $\mathbb{R}^2 \times S^1$. Moreover, these Lagrangian submanifolds intersect a unique one dimensional torus orbit closure of X in a circle S^1. For short we will call a Lagrangian submanifold L of the type considered by Aganagic and Vafa an **outer brane** if it intersects a non-compact one dimensional torus orbit closure.

Let us write μ for the moment map

$$\mu : X \longrightarrow \mathrm{Lie}(T_{\mathbb{R}})^*$$

and fix an isomorphism $\mathrm{Lie}(T_{\mathbb{R}})^* \cong \mathbb{R}^2$. We call the image of the union of all one dimensional torus orbit closures under μ the **toric diagram** of X. It is an embedded trivalent graph since there are always three $\hat{T}$-preserved one dimensional strata meeting in a torus fixed point (see Figure 4.1). Now let L be an outer brane in X. By our earlier discussion, its image $\mu(L)$ under the moment must be a point on a non-compact edge ℓ. Let us write q for the unique torus fixed point contained in ℓ and denote by

$$\mathsf{u}, \mathsf{v}, \mathsf{w} \in \mathrm{H}^2_T(\mathrm{pt}, \mathbb{Z}) \subset \mathrm{Lie}(T_{\mathbb{R}})^*$$

the weights of the induced T action on the tangent spaces of three torus orbit closures at this point. Here, $\mathsf{u} = c_1^T(T_q \ell)$ and v is chosen so that $\mathsf{u} \wedge \mathsf{v} \geq 0$ as indicated in Figure 4.1. A **framing** of L is the choice of an element $\mathsf{f} \in \mathrm{H}^2_T(\mathrm{pt}, \mathbb{Z})$ satisfying[3]

$$\mathsf{u} \wedge \mathsf{f} = \mathsf{u} \wedge \mathsf{v}\,.$$

Equivalently, we have

$$\mathsf{f} = \mathsf{v} - f\mathsf{u} \tag{4.6}$$

for some $f \in \mathbb{Z}$. Moreover, as an element in $\mathrm{H}^2_T(\mathrm{pt}, \mathbb{Z})$ one may view f as a character $T \longrightarrow \mathbb{G}_{\mathrm{m}}$. This defines a one dimensional subtorus $T_{\mathsf{f}} := \ker \mathsf{f} \hookrightarrow T$ which we will refer to as the **framing subtorus**. Finally, let us denote by $|_{\mathsf{f}=0}$ the restriction map

$$\mathrm{H}^T_*(\mathrm{pt}, \mathbb{Z}) \longrightarrow \mathrm{H}^{T_{\mathsf{f}}}_*(\mathrm{pt}, \mathbb{Z})$$

as this morphism essentially sets the weight f equal to zero.

Definition 4.2.3. A **toric triple** (X, L, f) consists of

- a smooth semiprojective toric Calabi–Yau threefold X,
- an outer brane L,
- a framing f.

Remark 4.2.4. To cover certain interesting cases it will actually be necessary to relax the above definition slightly. Concretely, we would like to include the case where X is not semiprojective in our discussion as well. In this case we replace the second point in the above definition with the choice of some non-compact one dimensional torus orbit closure ℓ.

4.2.2.2 Open Gromov–Witten invariants. Given a toric triple (X, L, f) and a collection $\mathbf{c} = (c_1, \ldots, c_n)$ of positive integers we denote by

$$\overline{M}_{g,\mathbf{c},\beta'}(X, L)$$

the moduli space parametrising genus g stable maps $f : (C, \partial C) \longrightarrow (X, L)$ whose domain C is a Riemann surface with n labelled boundary components $\partial C = \partial C_1 \sqcup \ldots \sqcup \partial C_n$ such that for all $i \in \{1, \ldots, n\}$

$$f_*[\partial C_i] = c_i[S^1] \in \mathrm{H}_1(L, \mathbb{Z}) \quad \text{and} \quad f_*[C] = \beta' + \textstyle\sum_i c_i[B] \in \mathrm{H}_2(X, L, \mathbb{Z})\,.$$

Here, β' a curve class in X and B is the unique $T_{\mathbb{R}}$-preserved disk stretching out from q to the circle S^1 in which L and ℓ intersect. The virtual dimension of this moduli problem is zero. We refer to [97] for details.

Moduli spaces parametrising stable maps with boundaries are (if they actually exist) usually notoriously hard to work with. In our case there is however a way out. Since the Calabi–Yau torus $T_{\mathbb{R}}$ leaves L invariant the $T_{\mathbb{R}}$-action on X lifts to an action on $\overline{M}_{g,\mathbf{c},\beta'}(X, L)$. Then as opposed to its ambient space the (anticipated) fixed locus

$$\overline{M}_{g,\mathbf{c},\beta'}(X, L)^{T_{\mathbb{R}}}$$

is a compact complex orbifold. So as in [59] we define open Gromov–Witten invariants via virtual localisation:

$$\mathsf{OGW}_{g,\mathbf{c},\beta'}(X, L, \mathsf{f}) := \int_{[\overline{M}_{g,\mathbf{c},\beta'}(X,L)^{T_{\mathbb{R}}}]^{\mathrm{virt}}_{T_{\mathbb{R}}}} \frac{1}{e^{T_{\mathbb{R}}}(N^{\mathrm{virt}})}\Bigg|_{\mathsf{f}=0} \in \mathbb{Q}\,. \tag{4.7}$$

[3]This definition agrees with the one in [59, Section 2.5] and differs from [29, Definition 6.1] by a minus sign.

The above integral produces a rational function in the two equivariant parameters of homogenous degree zero. Only the restriction to the framing subtorus yields a rational number. This is how definition (4.7) depends on the choice of framing f.

We recall a result of Fang and Liu which allows us to express open Gromov–Witten invariants of (X, L, f) in terms of descendant invariants of X.

Theorem 4.2.5. [59, *Proposition 3.4*] *If (X, L, f) is a toric triple then*

$$\mathsf{OGW}_{g,\mathbf{c},\beta'}(X, L, \mathsf{f}) = \prod_{i=1}^{n} (-1)^{f c_i} \frac{\prod_{k=1}^{c_i - 1}(f c_i + k)}{\mathsf{u} \cdot c_i!} \int_{[\overline{M}_{g,n,\beta'}(X)^T]_T^{\mathrm{virt}}} \frac{1}{e^T(N^{\mathrm{virt}})} \prod_{i=1}^{n} \frac{\mathrm{ev}_i^* \phi}{\left(\frac{\mathsf{u}}{c_i} - \psi_i\right)} \Bigg|_{\mathsf{f}=0} \tag{4.8}$$

where ϕ is the T-equivariant Poincaré dual of the torus fixed point q contained in the one dimensional torus orbit closure ℓ intersected by L.

Remark 4.2.6. In [116] Li, Liu, Liu and Zhou generalise the definition of open Gromov–Witten invariants to the case when X is not semiprojective. (This assumption was necessary for the construction of Aganagic–Vafa Lagrangian submanifolds.) In this case the datum of an outer brane gets replaced by the choice of a non-compact one dimensional torus orbit closure ℓ and open Gromov–Witten invariants are defined via formal relative invariants. It is shown in [59, Corollary 3.3] that these invariants still satisfy Theorem 4.2.5.

4.2.3 Statement of the main correspondence

Let $(S\,|\,D_1 + D_2)$ be a two-component Looijenga pair and β an effective curve class in S. We demand that this data satisfies Assumption $\star$ which we recall means that

- $D_i \cdot \beta > 0$, $i \in \{1, 2\}$,
- $D_2 \cdot D_2 \geq 0$,
- $(S\,|\,D_1)$ deforms into a pair $(S'\,|\,D_1')$ with S' a smooth projective toric surface and D_1' a toric hypersurface.

Remark 4.2.7. By the last condition we mean that there is a logarithmically smooth morphism of fine saturated logarithmic schemes $\mathcal{S} \to T$ with T irreducible and integral together with a line bundle $\mathcal{L}$ on $\mathcal{S}$. Moreover, there are regular points $t, t' : \operatorname{Spec} \mathbb{C} \to T$ with fibres

$$\mathcal{S}_t = (S\,|\,D_1)\,, \qquad \mathcal{S}_{t'} = (S'\,|\,D_1')$$

on which $\mathcal{L}$ restricts to $\mathcal{L}_t = \mathcal{O}_S(-D_2)$ and $\mathcal{L}_{t'} = \omega_{S'}(D_1')$.

Given $(S'\,|\,D_1')$ as above we write $Y \coloneqq \operatorname{Tot} \omega_{S'}(D_1')$ and write D for the preimage of D_1' under the projection $\pi : Y \to S'$. We remark that all compact one dimensional torus orbit closures in Y come from the toric boundary of S' embedded via the zero section $S' \hookrightarrow Y$. This way we may especially view D_1' as a toric curve in Y.

Construction 4.2.8. To $(S\,|\,D_1 + D_2)$ as above we associate the toric triple (X, L, f) where

$$X \coloneqq Y \setminus D$$

and $L \subset X \subset Y$ is an outer brane intersecting a compact one dimensional torus orbit closure $\ell \subset Y$ adjacent to D_1'. The action of the Calabi–Yau torus T of X naturally extends to Y. Writing F for the fibre $\pi^{-1}(p)$ over the point $p = \ell \cap D_1'$ we set $\mathsf{f} \coloneqq c_1^T(F)$ to be the weight of the T-action on F.

We are now able to state our main result. For this we fix $c_1, \dots, c_n > 0$ with $\sum_i c_i = D_1 \cdot \beta$ and denote by $\hat{\mathbf{c}}$ the contact datum

$$\hat{\mathbf{c}} = \begin{pmatrix} 0 & \cdots & 0 & c_1 & \dots & c_n & 0 \\ \underbrace{0 \quad \cdots \quad 0}_{m \text{ times}} & & & 0 & \dots & 0 & D_2 \cdot \beta \end{pmatrix} \tag{4.9}$$

along $D_1 + D_2$. We assume $m \le n$.

Theorem 4.2.9. (*Theorem 4.B*) *Under Assumption* $\star$ *we have*

$$\sum_{g \ge 0} \hbar^{2g-2} \, \mathsf{OGW}_{g,\mathbf{c},\beta'}(X, L, \mathsf{f}) = \frac{(-1)^{D_1 \cdot \beta}}{m! \prod_{i=0}^{m-1} c_{n-i}} \frac{(-1)^{D_2 \cdot \beta + 1}}{2 \sin \frac{(D_2 \cdot \beta)\hbar}{2}} \sum_{g \ge 0} \hbar^{2g-1} \, \mathsf{LGW}_{g,\hat{\mathbf{c}},\beta}(S \,|\, D_1 + D_2) \tag{4.10}$$

where $\beta' = \iota^*(\beta - (D_1 \cdot \beta)[\ell])$ *and* ι *is the open inclusion* $X \hookrightarrow Y$.

See Section 4.5.1.2 for an extended example illustrating Construction 4.2.8 and an application of the above theorem.

Remark 4.2.10. Note that Construction 4.2.8 depends on the choice which of the two torus orbit closures adjacent to D_1 the Lagrangian submanifold L is intersecting. Hence, a priori, one should expect the right-hand side of (4.11) to depend on this choice as well and it should come as a surprise that Theorem 4.2.9 actually demonstrates an independence.

Remark 4.2.11. If X as constructed in 4.2.8 turns out to be non-semiprojective the left-hand side of equation (4.10) can still be defined as described in Remark 4.2.6.

Remark 4.2.12. Let us quickly comment on the importance of Assumption $\star$. The first two points guarantee that the moduli stack of relative stable maps to $(Y \,|\, D)$ is proper which is required in Theorem 4.2.2. The last point in Assumption $\star$ is necessary for X to be toric which is in turn required in the definition of open Gromov–Witten invariants we use. The condition is rather technical but allows us treat interesting cases such as the example we will discuss in Section 4.5.1.2.

We observe that in the light of Theorem 4.2.2 and Theorem 4.2.5 it suffices to prove the following proposition in order to deduce Theorem 4.2.9.

Proposition 4.2.13. *Under Assumption* $\star$ *we have*

$$\begin{aligned} \mathsf{LGW}_{g,\tilde{\mathbf{c}},\beta}(Y \,|\, D) = (-1)^{D_1 \cdot \beta} m! \left(\textstyle\prod_{i=0}^{m-1} c_{n-i} \right) \prod_{i=1}^{n} (-1)^{f c_i} \frac{\prod_{k=1}^{c_i - 1} (f c_i + k)}{\mathsf{u} \cdot c_i!} \\ \times \int_{[\overline{M}_{g,n,\beta'}(X)^T]^{\mathrm{virt}}_T} \frac{1}{e^T(N^{\mathrm{virt}})} \prod_{i=1}^{n} \frac{\mathrm{ev}_i^* \phi}{\frac{\mathsf{u}}{c_i} - \psi_i} \Bigg|_{\mathsf{f}=0} \end{aligned} \tag{4.11}$$

where $\beta' = \iota^*(\beta - (D_1 \cdot \beta)[\ell])$ *and* ϕ *is the* T*-equivariant Poincaré dual of the torus fixed point* q *in* $\ell \cap X$.

We will first prove the proposition in the special case $m = 0$ in Section 4.3 and then deduce the general case from a reduction argument proven in Section 4.4. But before we embark on proving Proposition 4.2.13, let us first argue that it indeed implies Theorem 4.2.9.

Proof of Theorem 4.2.9. Comparing formula (4.8) and (4.11) we deduce

$$\mathsf{OGW}_{g,\mathbf{c},\beta'}(X, L, \mathsf{f}) = \frac{(-1)^{D_1 \cdot \beta}}{m! \prod_{i=0}^{m-1} c_{n-i}} \cdot \mathsf{LGW}_{g,\tilde{\mathbf{c}},\beta}(Y \,|\, D) \,.$$

Now by deformation invariance [81, Theorem 0.3] (see also [127, Appendix A]) the Gromov–Witten invariants of $(S \,|\, D_1)$ and $(S' \,|\, D_1')$ agree. In combination with Theorem 4.2.2 this therefore gives the identity stated in Theorem 4.2.9. $\square$

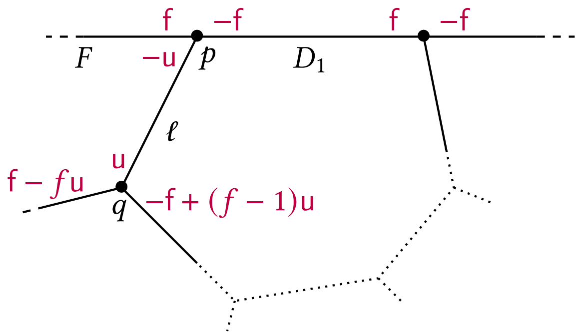


Figure 4.2: The image of the toric skeleton of Y under the moment map.

4.3 Step I: no interior markings

This section is devoted to the proof of a special instance of Proposition 4.2.13.

Proposition 4.3.1. *The statement of Proposition 4.2.13 holds if there are no interior markings* $(m = 0)$*.*

We prove the proposition via virtual localisation [70] which allows us to decompose the Gromov–Witten invariant on the left-hand side of equation (4.11) into contributions labelled by the fixed points of a torus action on the moduli stack of relative stable maps to $(Y|D)$. These fixed points split into two classes: The ones for which the target $(Y|D)$ is generically expanded or unexpanded. A careful analysis of the latter fixed points shows that their contribution to the overall Gromov–Witten count is precisely the expression we find on the right-hand side of equation (4.11) (Proposition 4.3.3). It therefore remains to show that the contribution of all fixed points vanishes for which the target is generically expanded. This last step is carried out in Section 4.3.4.

We start with a careful analysis of the target geometry $(Y|D)$ where we adopt the notation introduced in Section 4.2.3. To unload notation throughout this section we will assume that S is already toric and D_1 is a toric hypersurface which means we can take $(S'|D_1') = (S|D_1)$.

4.3.1 The geometric setup

The Calabi–Yau torus T of X also naturally acts on Y as both varieties share the same dense open torus. The compact one dimensional torus orbit closures of Y are given by the irreducible components of the toric boundary of S embedded into Y by the zero section. We write $\ell \hookrightarrow Y$ for the component intersected by L. By construction ℓ intersects $D_1 \hookrightarrow Y$ transversely in a torus fixed point which we denote p. We write q for the second torus fixed point of ℓ.

Figure 4.2 shows the image of one dimensional torus orbit closures of Y under the moment map locally around D_1. In this sketch dashed lines correspond to non-compact orbit closures while the dotted lines indicate that the toric boundary of $S \hookrightarrow Y$ forms a circle. Moreover, the red labels represent the weights with which the torus T is acting on the tangent spaces of the respective torus orbit closures at a fixed point. The observation (see [65, Lemma 2.7]) that

$$c_1^T(T_pF) = -c_1^T(T_pD_1) = \mathsf{f} \tag{4.12}$$

will be crucial in our localisation calculation later. Moreover, we also note that the framing factor f introduced in (4.6) can be identified with the degrees

$$f = \deg \mathcal{O}_S(-D_2)\big|_\ell = -\deg N_\ell Y - 1\,.$$

The toric diagram of $X = Y \setminus D$ is obtained from the one displayed in Figure 4.2 by erasing the top vertical line. In this process the divisor ℓ gets decompactified and so L is indeed an outer brane.

4.3.2 Initialising the localisation

Construction 4.2.8 involved the choice of a stratum ℓ the outer brane is intersecting. We now do an analogous choice by supporting the cycle whose pushforward gives the Gromov–Witten invariant $\mathsf{LGW}_{g,\mathbf{c},\beta}(Y|D)$ on a substack with evaluations constrained to the fibre $F = \pi^{-1}(p)$ where π is the projection $Y \to S$:

$$\begin{tikzcd} \overline{M}_{g,\mathbf{c},\beta}(Y|D)|_F \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{g,\mathbf{c},\beta}(Y|D) \arrow[d] \\ F^n \arrow[r, hook] & D^n \,. \end{tikzcd} \tag{4.13}$$

Pulling back the obstruction theory along the horizontal arrow we obtain

$$\mathsf{LGW}_{g,\mathbf{c},\beta}(Y|D) = \deg[\overline{M}_{g,\mathbf{c},\beta}(Y|D)|_F]^{\mathrm{virt}} \,.$$

This choice will turn out rather useful when we will calculate the invariant via virtual localisation. Regarding this we remark that the T-action on Y lifts to an action on $\overline{M}_{g,\mathbf{c},\beta}(Y|D)|_F$ since D and F are both preserved by T. Especially, this also provides us with an action of the framing subtorus T_{f} on the moduli stack.

Now equation (4.12) tells us that the divisor D is pointwise fixed under T_{f} since the restriction $T_{\mathsf{f}} \hookrightarrow T$ effectively sets the weight f equal to zero. This means we are in a situation where the analysis of Graber and Vakil [71] of virtual localisation in the context of stable maps relative a smooth divisor applies. Since their analysis was carried out in the setting of relative stable maps to expanded degenerations as introduced by Li [114, 115] we quietly pass to this modification of $\overline{M}_{g,\mathbf{c},\beta}(Y|D)$ until the end of this section without actually changing our notation. Since eventually we push all cycles forward to a point it does not matter which birational model we choose for our moduli stack [5]. See also [138] for an analysis of torus localisation in the context of Kim's logarithmic stable maps to expanded degenerations [100].

Following Graber and Vakil [71] we split the T_{f}-fixed locus $\overline{M}_{g,\mathbf{c},\beta}(Y|D)^{T_{\mathsf{f}}}$ into two distinguished (not necessarily connected or irreducible) components. First, there is the simple fixed locus

$$\overline{M}_{g,\mathbf{c},\beta}(Y|D)^{\mathrm{sim}}$$

whose general points are stable maps with target Y. The complement of this is called composite fixed locus and consequently its general points correspond to maps to non-trivial expanded degenerations of $(Y|D)$. We will denote this component by

$$\overline{M}_{g,\mathbf{c},\beta}(Y|D)^{\mathrm{com}} \,.$$

As we did in (4.13) we form fibre products

$$\overline{M}_{g,\mathbf{c},\beta}(Y|D)|_F^{\mathrm{sim}} \coloneqq \overline{M}_{g,\mathbf{c},\beta}(Y|D)^{\mathrm{sim}} \times_{D^n} F^n \,, \qquad \overline{M}_{g,\mathbf{c},\beta}(Y|D)|_F^{\mathrm{com}} \coloneqq \overline{M}_{g,\mathbf{c},\beta}(Y|D)^{\mathrm{com}} \times_{D^n} F^n \,.$$

Then virtual localisation [70] gives

$$\begin{aligned} \mathsf{LGW}_{g,\mathbf{c},\beta}(Y|D) = {} & \deg_{T_{\mathsf{f}}} \frac{1}{e^{T_{\mathsf{f}}}(N^{\mathrm{virt}}_{\mathrm{sim}})} \cap [\overline{M}_{g,\mathbf{c},\beta}(Y|D)|_F^{\mathrm{sim}}]^{\mathrm{virt}} \\ & + \deg_{T_{\mathsf{f}}} \frac{1}{e^{T_{\mathsf{f}}}(N^{\mathrm{virt}}_{\mathrm{com}})} \cap [\overline{M}_{g,\mathbf{c},\beta}(Y|D)|_F^{\mathrm{com}}]^{\mathrm{virt}} \,. \end{aligned} \tag{4.14}$$

We will inspect the contribution of the simple and composite fixed locus to the Gromov–Witten invariant individually in the next two subsections.

4.3.3 The simple fixed locus

Closely following [71], we start with an analysis of the simple fixed locus. By definition the image of a relative stable map $f : C \to Y$ parametrised by

$$\overline{M}_{g,\mathbf{c},\beta}(Y|D)|_F^{\mathrm{sim}}$$

intersects D transversely, ie. $f^*D = \sum_{i=1}^n c_i x_i$ as Cartier divisors where $x_1, \dots, x_n$ denote the markings. If we write

$$C_1, \dots, C_n$$

for the irreducible components containing these markings then since $f : C \to Y$ lies in the T_{f}-fixed locus and by assumption none of the components C_i get contracted there must be a (non-trivial) lift of the action T_{f} on C_i (possibly after passing to some cover of T_{f}) making $f|_{C_i}$ T_{f}-equivariant. As a consequence, for all $i \in \{1, \dots, n\}$ both C_i and its image under f must be rational curves. Moreover, writing q_i for the second T_{f}-fixed point of the image rational curve, $f|_{C_i}$ is the unique degree c_i cover maximally ramified over $f(x_i)$ and q_i.

Now since we confined the evaluation morphisms to the fibre $F \subset Y$, we deduce that the image of each component C_i is contained in $\pi^{-1}(\ell)$ where π is the projection $Y \to S$ and ℓ is as in Figure 4.2. However, since the pullback of the line bundle $\mathcal{O}_S(-D_2)$ under $\pi \circ f$ is negative we deduce that f has to factor through the zero section $S \hookrightarrow Y$. Hence, the image of each irreducible component C_i has to be the curve ℓ already. This also means that $q_i = q$ for all $i \in \{1, \dots, n\}$.

Now if we write $C' \sqcup \bigsqcup_{i=1}^n C_i$ for the connected components of the partial normalisation of C along the nodes $y_i \in C_i$ mapping to q we observe that $f|_{C'}$ is a genus g, class

$$\beta' = \iota^*\big(\beta - (D_1 \cdot \beta)[\ell]\big)$$

T_{f}-fixed stable map to $X = Y \setminus D$ with markings $y_1, \dots, y_n$ confined to q. Hence, forming the fibre square

$$\begin{tikzcd}
\overline{M}_{g,n,\beta'}(X)|_q \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{g,n,\beta'}(X) \arrow[d, "\Pi_i \mathrm{ev}_{y_i}"] \\
q \arrow[r, hook] & X^n
\end{tikzcd}$$

we can summarise the discussion up to now as follows.

Lemma 4.3.2. *We have*

$$\overline{M}_{g,\mathbf{c},\beta}(Y|D)|_F^{\mathrm{sim}} = \left[\overline{M}_{g,n,\beta'}(X)|_q^{T_{\mathrm{f}}} \Big/ \textstyle\prod_{i=1}^n \mathbb{Z}_{c_i} \right]. \tag{4.15}$$

Using this description of the simple fixed locus we are now able to determine its contribution to the Gromov–Witten invariant.

Proposition 4.3.3. *We have*

$$\begin{aligned}
\deg_{T_{\mathrm{f}}} \frac{1}{e^{T_{\mathrm{f}}}(N_{\mathrm{sim}}^{\mathrm{virt}})} \cap [\overline{M}_{g,\mathbf{c},\beta}(Y|D)|_F^{\mathrm{sim}}]_{T_{\mathrm{f}}}^{\mathrm{virt}} &= (-1)^{D_1 \cdot \beta} \prod_{i=1}^n (-1)^{f c_i} \frac{\prod_{k=1}^{c_i - 1}(f c_i + k)}{\mathsf{u} \cdot c_i!} \\
&\quad \times \int_{[\overline{M}_{g,n,\beta'}(X)^{T_{\mathrm{f}}}]_{T_{\mathrm{f}}}^{\mathrm{virt}}} \frac{1}{e^{T_{\mathrm{f}}}(N^{\mathrm{virt}})} \prod_{i=1}^n \frac{\mathrm{ev}_i^* \phi}{\frac{\mathsf{u}}{c_i} - \psi_i}.
\end{aligned} \tag{4.16}$$

where ϕ is the T_{f}-equivariant Poincaré dual of q.

Proof. With our characterisation (4.15) of the simple fixed locus it remains to analyse the difference between $N^{\mathrm{virt}}_{\mathrm{sim}}$ and the virtual normal bundle N^{virt} of the fixed locus

$$\overline{M}_{g,n,\beta'}(X)^{T_{\mathrm{f}}} \hookrightarrow \overline{M}_{g,n,\beta'}(X)\,.$$

First of all note that we get an overall factor

$$\prod_{i=1}^{n} \frac{1}{c_i}$$

from the quotient. Moreover, partially normalising along the nodes $y_1, \ldots, y_n$ connecting C' with $C_1, \ldots, C_n$ we see that each degree c_i cover

$$f_i \coloneqq f|_{C_i} \ :\ C_i \longrightarrow \ell$$

contributes with factors

$$\frac{1}{e^{T_{\mathrm{f}}}\Big(\mathrm{H}^\bullet\big(C_i, f_i^* T^{\log}_{\ell|p}\big)^{\mathrm{mov}}\Big)} = \frac{1}{c_i!}\left(\frac{\mathsf{u}}{c_i}\right)^{-c_i}$$

$$\frac{1}{e^{T_{\mathrm{f}}}\Big(\mathrm{H}^\bullet\big(C_i, f_i^* N_\ell S\big)^{\mathrm{mov}}\Big)} = \frac{1}{(-(f+1)c_i)!}\left(\frac{\mathsf{u}}{c_i}\right)^{(f+1)c_i}$$

$$\frac{1}{e^{T_{\mathrm{f}}}\Big(\mathrm{H}^\bullet\big(C_i, f_i^* \mathcal{O}_S(-D_2)\big)^{\mathrm{mov}}\Big)} = (-1)^{fc_i+1}\,(-fc_i-1)!\left(\frac{\mathsf{u}}{c_i}\right)^{-fc_i-1}$$

where $f = \deg \mathcal{O}_S(-D_2)|_\ell = -\deg N_\ell Y - 1 < 0$. Note that if we multiply all these factors together we indeed obtain the top line on the right-hand side of (4.16). Regarding the second line let us only remark that the insertions

$$\tfrac{\mathsf{u}}{c_i} - \psi_i$$

for $i \in \{1, \ldots, n\}$ come from smoothings of the nodes $y_1, \ldots, y_n$. □

4.3.4 Vanishing of composite contributions

We observe that the integral on the right-hand side of equation (4.11) is the result of computing the one in (4.16) via T-virtual localisation. Hence, to prove Proposition 4.3.1 it suffices to show

Proposition 4.3.4. *We have*

$$\deg_{T_{\mathrm{f}}} \frac{1}{e^{T_{\mathrm{f}}}(N^{\mathrm{virt}}_{\mathrm{com}})} \cap [\overline{M}_{g,\mathrm{c},\beta}(Y\,|\,D)|^{\mathrm{com}}_F]^{\mathrm{virt}} = 0\,.$$

The proof of this proposition necessitates a careful analysis of the composite fixed locus for which we will closely follow [71, Section 3].

4.3.4.1 The setup. We first set the notation required in this section. Denote by

$$Z \coloneqq \mathbb{P}_D\big(\mathcal{O}_D \oplus N_D Y\big) \tag{4.17}$$

the projective completion of $N_D Y$ and write D_0 and D_∞ for its zero and infinity section. Note that Z is a line bundle over

$$P \coloneqq \mathbb{P}_{D_1}\big(\mathcal{O}_{D_1} \oplus N_{D_1} S\big)\,. \tag{4.18}$$

More precisely, we have $Z \cong \mathcal{O}_P(-2f)$ where f is a fibre of $P \to D_1$.

By definition, the composite fixed locus parametrises relative stable maps whose target is an expanded degeneration of $(Y|D)$ which is the result of gluing $(Y|D)$ and multiple copies of $(Z|D_0 + D_\infty)$ to form an accordion. The composite locus decomposes into a disjoint union of components labelled by certain decorated graphs indicating how a relative stable map distributes over the expanded target.

Definition 4.3.5. A **splitting type** Γ is the data of

(i) a bipartite graph with vertices $V(\Gamma)$, edges $E(\Gamma)$ and legs $L(\Gamma)$;

(ii) an assignment of a curve class β_v to every vertex $v \in V(\Gamma)$;

(iii) a genus label $g_v \in \mathbb{Z}_{\geq 0}$ for every vertex $v \in V(\Gamma)$;

(iv) a contact order $d_e \in \mathbb{Z}_{>0}$ assigned to every edge $e \in E(\Gamma)$;

(v) a labelling $\{1, \ldots, n\} \cong L(\Gamma)$.

This data has to be compatible with $(g, \mathbf{c}, \beta)$ in the usual way. Vertices are separated into Y- and Z-vertices and we will write Γ_Y and Γ_Z for the type of relative stable maps (with possibly disconnected domain) to $(Y|D)$, respectively $(Z|D_0 + D_\infty)$, obtained by cutting Γ along its edges. All legs are adjacent to Z-vertices.

With this notation the composite locus can be written as a union of disconnected components labelled by splitting types:

$$\overline{M}_{g,\mathbf{c},\beta}(Y|D)^{\mathrm{com}} = \bigsqcup_{\Gamma} \overline{F}_\Gamma \,.$$

To characterise the component labelled by a fixed splitting type Γ, let us denote by

$$\overline{M}_{\Gamma_Y}(Y|D)^{\mathrm{sim}}$$

the simple T_{f}-fixed locus of relative stable maps of type Γ_Y and write

$$\overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)^{\sim}$$

for the moduli stack of type Γ_Z relative stable maps to non-rigid $(Z|D_0 + D_\infty)$ and form a substack with constrained evaluations at marking legs

$$\begin{tikzcd} \overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)|_F^{\sim} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)^{\sim} \arrow[d, "\Pi_i \mathrm{ev}_i"] \\ F^n \arrow[r, hook] & D_\infty^n \,. \end{tikzcd}$$

The two moduli stacks support evaluation morphisms to $D \cong D_0$ associated to the edges of Γ. Let Δ denote the diagonal $D^{E(\Gamma)} \hookrightarrow D^{E(\Gamma)} \times D^{E(\Gamma)}$ and form the fibre product

$$\begin{tikzcd} \overline{M}_\Gamma \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{\Gamma_Y}(Y|D)^{\mathrm{sim}} \times \overline{M}_{\Gamma_Z}(Z|D_0 + D_\infty)|_F^{\sim} \arrow[d] \\ D^{E(\Gamma)} \arrow[r, hook, "\Delta"] & D^{E(\Gamma)} \times D^{E(\Gamma)} \,. \end{tikzcd} \tag{4.19}$$

Then the connected component $\overline{F}_\Gamma$ of the composite fixed locus is the quotient of $\overline{M}_\Gamma$ by $\mathrm{Aut}(d_e)$ and moreover, by [71, Lemma 3.2], the Gysin pullback

$$[\overline{M}_\Gamma]^{\mathrm{virt}} \coloneqq \Delta^!\big([\overline{M}_{\Gamma_Y}(Y|D)^{\mathrm{sim}}]^{\mathrm{virt}} \times [\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_{\tilde{F}}]^{\mathrm{virt}}\big)$$

pushes forward to $[\overline{F}_\Gamma]^{\mathrm{virt}}$ under the quotient morphism. This means we get

$$\deg_{T_\mathrm{f}} \frac{1}{e^{T_\mathrm{f}}(N^{\mathrm{virt}}_{\mathrm{com}})} \cap [\overline{M}_{g,\mathrm{c},\beta}(Y|D)|_F^{\mathrm{com}}]^{\mathrm{virt}}_{T_\mathrm{f}} = \sum_\Gamma \frac{1}{|\mathrm{Aut}(d_e)|} \deg_{T_\mathrm{f}} \frac{1}{e^{T_\mathrm{f}}(N^{\mathrm{virt}}_\Gamma)} \cap [\overline{M}_\Gamma]^{\mathrm{virt}} . \tag{4.20}$$

4.3.4.2 First vanishing: restricted curve classes. To prove Proposition 4.3.4 we will show that each term in the sum (4.20) vanishes in several steps. We first recall a lemma from [65].

Lemma 4.3.6. [65, *Section 2.3.3*] *If there is a Z-vertex $v \in V(\Gamma)$ such that β_v is not a multiple of a fibre of $P \to D_1$, then*

$$\deg_{T_\mathrm{f}} \frac{1}{e^{T_\mathrm{f}}(N^{\mathrm{virt}}_\Gamma)} \cap [\overline{M}_\Gamma]^{\mathrm{virt}} = 0 . \tag{4.21}$$

Remark 4.3.7. The above lemma was proven in [65] under the assumption that $D_1 \cdot \beta = 0$ but it is straight forward to generalise the proof to the case at hand where $D_1 \cdot \beta > 0$. The key observation is that after localising the left-hand side of (4.21) with respect to T one can identify a trivial factor in the obstruction bundle guaranteeing the vanishing.

4.3.4.3 Second vanishing: star shaped graphs. From now on we may assume that every Z-vertex v is decorated with curve class β_v which is a multiple of a fibre of $P \to D_1$.

The following results are inspired by vanishing arguments appearing in the localisation calculations [116, 119, 121]. We will present more geometric versions of these arguments here.

Lemma 4.3.8. *Unless every vertex in Γ_Z carries exactly one leg we have*

$$[\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_{\tilde{F}}]^{\mathrm{virt}} = 0 .$$

Proof. Clearly, every Z-vertex v has to have at least one leg for the moduli stack to be non-empty since $D_\infty \cdot \beta_v > 0$. So suppose there is some vertex with at least two legs. Since the curve class carried by this vertex is a fibre class and the restriction of the line bundle $Z \to P$ to a fibre trivialises, the product of the evaluation morphisms of these two markings factors through the diagonal $D_\infty \to D_\infty^2$. Hence, we have the following commuting diagram with Cartesian squares:

$$\begin{tikzcd}
\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_{\tilde{F}} \arrow[r, hook] \arrow[d, "g"] \arrow[dr, phantom, "\square"] & \overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)^\sim \arrow[d] \\
F^{n-1} \arrow[r, hook, "\bar{\eta}"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & D_\infty^{n-1} \arrow[d] \\
F^n \arrow[r, hook, "\eta"] & D_\infty^n .
\end{tikzcd}$$

Note that the excess intersection bundle

$$E = f^* N_{F^n} D_\infty^n \big/ N_{F^{n-1}} D_\infty^{n-1}$$

is trivial as is every vector bundle on $F^{n-1} \cong \mathbb{A}^{n-1}$. Moreover, note that since T_{f} is acting trivially on D_∞ and F, this bundle is also T_{f}-equivariantly trivial. Therefore, by the excess intersection formula [63, Theorem 6.3] we have

$$[\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_F^\sim]^{\mathrm{virt}} = \eta^![\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)^\sim]^{\mathrm{virt}} = c_1^{T_{\mathrm{f}}}(g^*E) \cap \overline{\eta}^![\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)^\sim]^{\mathrm{virt}} = 0\,.$$

□

As another consequence of our assumption that all curve classes of Z-vertices are multiples of fibre classes the evaluation morphism associated to edges

$$\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_F^\sim \longrightarrow D_0^{E(\Gamma)}$$

factors through $F^{E(\Gamma)} \hookrightarrow D_0^{E(\Gamma)}$. If we combine this with the fact that the evaluation morphism

$$\overline{M}_{\Gamma_Y}(Y|D)^{\mathrm{sim}} \longrightarrow D^{E(\Gamma)}$$

factors through $D_1^{E(\Gamma)} \hookrightarrow D^{E(\Gamma)}$ we may restrict the evaluations to the intersection point $p = F \cap D_1$. We get the following commuting diagram with Cartesian squares:

$$\begin{tikzcd}
\overline{M}_{\Gamma_Y}(Y|D)|_F^{\mathrm{sim}} \times \overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)^\sim \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{\Gamma_Y}(Y|D)^{\mathrm{sim}} \times \overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_F^\sim \arrow[d] \\
p \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & (D_1)^{E(\Gamma)} \times F^{E(\Gamma)} \arrow[d] \\
D^{E(\Gamma)} \arrow[r, "\Delta"] & D^{E(\Gamma)} \times D^{E(\Gamma)}\,.
\end{tikzcd} \tag{4.22}$$

Especially, note that we have a fibre square

$$\begin{tikzcd}
\overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)^\sim \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_F^\sim \arrow[d] \\
p \arrow[r, hook, "\xi"] & F^{E(\Gamma)}\,.
\end{tikzcd} \tag{4.23}$$

First, we record the following vanishing.

Lemma 4.3.9. *We have $[\overline{M}_\Gamma]^{\mathrm{virt}} = 0$ unless every Z-vertex has a single adjacent edge.*

Proof. The lemma can be proven with an argument similar to the one in the proof of Lemma 4.3.8 using diagram (4.22). We leave this to the reader. □

Moreover, comparing our definition of $\overline{M}_\Gamma$ in (4.19) with diagram (4.22) we learn that

$$\overline{M}_\Gamma \cong \overline{M}_{\Gamma_Y}(Y|D)|_F^{\mathrm{sim}} \times \overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)^\sim\,.$$

On the level of virtual fundamental classes we find the following identity.

Lemma 4.3.10. *If there is a Z-vertex v with $g_v > 0$ the cycle $[\overline{M}_\Gamma]^{\mathrm{virt}}$ vanishes. Otherwise, the cycle equates to*

$$[\overline{M}_{\Gamma_Y}(Y|D)^{\mathrm{sim}}|_F]^{\mathrm{virt}} \times [\overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)^\sim]^{\mathrm{virt}}\,.$$

Proof. Let ξ be as in (4.23). Then since all squares in (4.22) are Cartesian we have

$$[\overline{M}_\Gamma]^{\mathrm{virt}} = \Delta^!\big([\overline{M}_{\Gamma_Y}(Y|D)^{\mathrm{sim}}]^{\mathrm{virt}} \times [\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_F^{\sim}]^{\mathrm{virt}}\big)$$
$$= [\overline{M}_{\Gamma_Y}(Y|D)|_F^{\mathrm{sim}}]^{\mathrm{virt}} \times \xi^![\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_F^{\sim}]^{\mathrm{virt}}$$

by [63, Theorem 6.2 (c)]. Comparing the usual perfect obstruction theory of $\overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)^\sim$ with the one pulled back from $\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_F^\sim$ we find

$$\xi^![\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_F^{\sim}]^{\mathrm{virt}} = \lambda_{\mathrm{top}}^2 \cap [\overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)^\sim]^{\mathrm{virt}}$$

where λ_{top} is the top Chern class of the Hodge bundle. The statement of the lemma then follows from the fact that $\lambda_{\mathrm{top}}^2 = 0$ if any of the vertices in Γ_Z carries a non-zero genus label and $\lambda_{\mathrm{top}}^2 = 1$ otherwise. □

4.3.4.4 Final vanishing. Having proven Lemmas 4.3.8–4.3.10 we are finally able to deduce Proposition 4.3.4.

Proof of Proposition 4.3.4. According to Lemma 4.3.8 and 4.3.9 the only splitting types Γ whose associated summand on the left-hand side of (4.20) may be non-zero consist of Z-vertices with exactly one adjacent edge and leg. Moreover, by Lemma 4.3.10 every such vertex has to carry a vanishing genus label. Therefore, the virtual dimension of $\overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)^\sim$ is

$$\sum_{v\in V(\Gamma_Z)} (2g_v - 2) + |E(\Gamma)| + |L(\Gamma)| - 1 = -1$$

and so the cycle $[\overline{M}_\Gamma]^{\mathrm{virt}}$ vanishes by Lemma 4.3.10. □

4.4 Step II: reducing interior to contact insertions

4.4.1 The comparison statement

Throughout this section we will assume that $(S|D_1+D_2)$ is a logarithmic Calabi–Yau surface with boundary the union of two transversally intersecting smooth irreducible curves D_1, D_2. Moreover, we fix an effective curve class β in S and assume

- $D_i \cdot \beta > 0$, $i \in \{1,2\}$,
- $D_2 \cdot D_2 \geq 0$.

Especially, in this section we do not necessarily assume $(S|D_1)$ to deform to a toric pair. We set $Y \coloneqq \mathrm{Tot}\,\mathcal{O}_S(-D_2)$ and write D for the preimage of D_1 under the projection morphism.

As before we fix a contact datum $\mathbf{c} = (c_1, \ldots, c_n)$ with D where $c_i > 0$ for all $i \in \{1, \ldots, n\}$. We set

$$\mathsf{LGW}_{g,\mathbf{c},\beta}(Y|D) = \int_{[\overline{M}_{g,\mathbf{c},\beta}(Y|D)]^{\mathrm{virt}}} \prod_{i=1}^n \mathrm{ev}_i^*(\pi^*\mathrm{pt})\,. \tag{4.24}$$

Now denote by

$$\tilde{\mathbf{c}} = (0, \ldots, 0, c_1, \ldots, c_n)$$

the result of adding $m \leq n$ interior markings. Similarly to (4.24) we define $\mathsf{LGW}_{g,\tilde{\mathbf{c}},\beta}(Y|D) \in \mathbb{Q}$ by imposing point conditions at the first n markings. Note that this especially means that there is a point condition at all m interior markings. We have the following comparison statement which reduces computations to the case without any interior markings.

Proposition 4.4.1. *We have*

$$\mathsf{LGW}_{g,\tilde{c},\beta}(Y|D) = m!\left(\prod_{i=0}^{m-1} c_{n-i}\right)\cdot \mathsf{LGW}_{g,c,\beta}(Y|D)\,. \tag{4.25}$$

Together with the special case $m = 0$ of Proposition 4.2.13, which we have already proven in the last section, the above reduction result implies the general case $m \geq 0$ of Proposition 4.2.13 as well. The rest of this section is hence dedicated to the proof of Proposition 4.4.1.

4.4.2 Degeneration

To prove the main result of this section we will use a degeneration argument similar to [64]. Since by now this approach has become a well-established technique [17, 21, 65, 148, 160] we economise on details.

Recall the definition of the line bundle $Z \to P$ from equation (4.17) and (4.18) (where we identify $S = S'$). We defined Z to be the projective completion of $N_D Y$ whose zero and infinity section we denote by D_0 and D_∞ respectively. Let us consider the degeneration to the normal cone of D in Y:

$$\mathcal{Y} \longrightarrow \mathbb{A}^1\,.$$

$\mathcal{Y}$ is a line bundle over the degeneration to the normal cone of D_1 in S. While the general fibre of $\mathcal{Y}$ is Y, the special fibre $\mathcal{Y}_0 = Y \sqcup_D Z$ is the result of gluing Y along D to Z along the zero section D_0. Now let us write $\mathcal{D}$ for the closure of $D \times (\mathbb{A}^1 - \{0\})$ in $\mathcal{Y}$ and equip $\mathcal{Y}$ with the divisorial logarithmic structure with respect to $Y + Z + \mathcal{D}$. The resulting logarithmic scheme has general fibre $(Y|D)$ and its special fibre is obtained by gluing

$$(Y|D) \qquad \text{and} \qquad (Z|D_0 + D_\infty)\,.$$

Finally, for F a fibre of $D \to D_1$ we denote by $\mathcal{F}$ the closure of $F \times (\mathbb{A}^1 - \{0\})$ in $\mathcal{Y}$. The intersection $\mathcal{F}_0$ of $\mathcal{F}$ with the special fibre $\mathcal{Y}_0$ is a fibre of $D_\infty \to D_1$.

4.4.3 Decomposition

Composition with the blowup morphism $\mathcal{Y} \to Y \times \mathbb{A}^1$ produces a pushforward

$$\rho : \overline{M}_{g,\tilde{c},\beta}(\mathcal{Y}_0) \longrightarrow \overline{M}_{g,\tilde{c},\beta}(Y|D)$$

and by the conservation of number principle we have

$$[\overline{M}_{g,\tilde{c},\beta}(Y|D)|_F]^{\mathrm{virt}} = (\rho_{\mathcal{F}_0})_*[\overline{M}_{g,\tilde{c},\beta}(\mathcal{Y}_0)|_{\mathcal{F}_0}]^{\mathrm{virt}} \tag{4.26}$$

where we introduced the fibre products

$$\begin{tikzcd} \overline{M}_{g,\tilde{c},\beta}(Y|D)|_F \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{g,\tilde{c},\beta}(Y|D) \arrow[d, "\Pi_{i=1}^n \mathrm{ev}_i"] \\ F^n \arrow[r, hook] & Y^n \end{tikzcd} \qquad \begin{tikzcd} \overline{M}_{g,\tilde{c},\beta}(\mathcal{Y}_0)|_{\mathcal{F}_0} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{g,\tilde{c},\beta}(\mathcal{Y}_0) \arrow[d, "\Pi_{i=1}^n \mathrm{ev}_i"] \\ \mathcal{F}_0^n \arrow[r, hook] & \mathcal{Y}_0^n\,. \end{tikzcd}$$

and pulled back the respective virtual fundamental classes along horizontal arrows. The pushforward of the left-hand side of (4.26) to $\overline{M}_{g,\tilde{c},\beta}(Y|D)$ gives

$$\Pi_{i=1}^n \mathrm{ev}_i^*(\pi^*\mathrm{pt}) \cap [\overline{M}_{g,\tilde{c},\beta}(Y|D)]^{\mathrm{virt}}$$

and so pushing forward to a point returns the Gromov–Witten invariant we are after:

$$\mathsf{LGW}_{g,\tilde{c},\beta}(Y|D)\,.$$

We combine this with the decomposition formula [3, Theorem 5.4]

$$[\overline{M}_{g,\tilde{c},\beta}(\mathcal{Y}_0)|_{\mathcal{F}_0}]^{\mathrm{virt}} = \sum_{\tau} \frac{m_\tau}{|\operatorname{Aut}\tau|}\iota_*[\overline{M}_\tau(\mathcal{Y}_0)|_{\mathcal{F}_0}]^{\mathrm{virt}} \tag{4.27}$$

where ι denotes the inclusion

$$\overline{M}_\tau(\mathcal{Y}_0)|_{\mathcal{F}_0} \hookrightarrow \overline{M}_{g,\tilde{c},\beta}(\mathcal{Y}_0)|_{\mathcal{F}_0}$$

and the sum runs over all rigid tropical types τ of tropical maps to the polyhedral complex

$$\Sigma\;:\quad \underset{\sigma_Y}{\bullet}\!\!\longrightarrow\!\!\underset{\sigma_Z}{\bullet}\!\!\longrightarrow$$

which is the fibre over $1 \in \mathbb{R}_{\geq 0}$ of the tropicalisation $\Sigma(\mathcal{Y}) \longrightarrow \Sigma(\mathbb{A}^1) = \mathbb{R}_{\geq 0}$.

By the discussion in [3, Section 5.1] τ can only be rigid if all vertices of the underlying graph map to either one of the vertices σ_X or σ_Z. Hence, τ is uniquely reconstructed from the data of its underlying decorated bipartite graph which is exactly the data of a splitting type as introduced in Definition 4.3.5. Hence, to be closer to the notation used in Section 4.3.4 let us write Γ for the splitting type underlying τ and Γ_Y, Γ_Z for the type of stable logarithmic maps to $(Y|D)$, respectively $(Z|D_0 + D_\infty)$, obtained by cutting Γ along edges.

We remark that only such rigid tropical types can contribute non-trivially to the sum (4.27) for which all marking legs are attached to a Z-vertex. Indeed, contact markings necessarily have to be adjacent to a Z-vertex and if an interior marking is carried by a Y-vertex then $\overline{M}_\tau(\mathcal{Y}_0)|_{\mathcal{F}_0}$ is empty (where we crucially use our assumption that $m \leq n$). Moreover, the factor m_τ in the decomposition formula (4.27) equates to $\operatorname{lcm}(d_e)$ where d_e denotes contact order of an edge e.

Since we are in the special setting where Σ is of dimension one the discussion in [76, Section 7] applies. There is a morphism ν with target the fibre product

$$\begin{tikzcd}
\overline{M}_\tau(\mathcal{Y}_0)|_{\mathcal{F}_0} \arrow[r, "\nu"] & \overline{M}_\Gamma \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{\Gamma_Y}(Y|D) \times \overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_{\mathcal{F}_0} \arrow[d] \\
 & D_0^{E(\Gamma)} \arrow[r, "\Delta"] & D_0^{E(\Gamma)} \times D_0^{E(\Gamma)}
\end{tikzcd} \tag{4.28}$$

which is finite and satisfies

$$\nu_*[\overline{M}_\tau(\mathcal{Y}_0)|_{\mathcal{F}_0}]^{\mathrm{virt}} = \frac{\Pi_e\, d_e}{\operatorname{lcm}(d_e)}\,\Delta^!\big([\overline{M}_{\Gamma_Y}(Y|D)]^{\mathrm{virt}} \times [\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_{\mathcal{F}_0}]^{\mathrm{virt}}\big)\,. \tag{4.29}$$

There is a gluing morphism $\theta : \overline{M}_\Gamma \longrightarrow \overline{M}_{g,m+n,\beta}(Y)$ such that

$$\begin{tikzcd}
\overline{M}_\tau(\mathcal{Y}_0)|_{\mathcal{F}_0} \arrow[r, "\nu"] \arrow[d, "\iota"] & \overline{M}_\Gamma \arrow[d, "\theta"] \\
\overline{M}_{g,\tilde{c},\beta}(\mathcal{Y}_0)|_{\mathcal{F}_0} \arrow[r, "\zeta\circ\rho_{\mathcal{F}_0}"] & \overline{M}_{g,m+n,\beta}(Y)
\end{tikzcd} \tag{4.30}$$

commutes where ζ denotes the forgetful morphism

$$\zeta : \overline{M}_{g,\tilde{c},\beta}(Y|D)|_F \longrightarrow \overline{M}_{g,m+n,\beta}(Y)\,.$$

To summarize the discussion so far we combine equation (4.27) and (4.29) and use the fact that diagram (4.30) commutes to deduce

$$\zeta_*[\overline{M}_{g,\tilde{c},\beta}(Y|D)|_F]^{\mathrm{virt}} = \sum_{\Gamma} \frac{\Pi_e\, d_e}{|\operatorname{Aut}\Gamma|} \theta_*[\overline{M}_\Gamma]^{\mathrm{virt}} \tag{4.31}$$

where we introduced the notation

$$[\overline{M}_\Gamma]^{\mathrm{virt}} \coloneqq \Delta^!\big([\overline{M}_{\Gamma_Y}(Y|D)]^{\mathrm{virt}} \times [\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_{\mathcal{F}_0}]^{\mathrm{virt}}\big)\,.$$

4.4.4 Vanishing

In this section we will argue that for most splitting types Γ the cycle $[\overline{M}_\Gamma]^{\mathrm{virt}}$ vanishes. We note that our setup is almost the same as in Section 4.3.4. The only notable differences are as follows.

- $\overline{M}_{\Gamma_Y}(Y|D)$ parametrise logarithmic stable maps to $(Y|D)$. In Section 4.3.4 we considered the simple fixed locus of this moduli stack under some torus action.
- $\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)$ parametrises logarithmic stable maps to $(Z|D_0+D_\infty)$ while in Section 4.3.4 the target was non-rigid $(Z|D_0+D_\infty)$.
- Γ_Z contains interior markings.

We note that all arguments in Section 4.3.4 are essentially insensitive to first two differences which allows us to prove the vanishing of $[\overline{M}_\Gamma]^{\mathrm{virt}}$ in similar steps. Only the third point requires some attention.

4.4.4.1 First vanishing: fibre classes. We start with the analogue of Lemma 4.3.6.

Lemma 4.4.2. *We have $[\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_{\mathcal{F}_0}]^{\mathrm{virt}} = 0$ unless Γ_Z only consists of vertices decorated with a curve class which is a multiple of a fibre class.*

Proof. Let us write β_Z for the overall curve class carried by Γ_Z and suppose its projection $p_*\beta_Z$ along the morphism $p : P \longrightarrow D_1$ is non-zero. In other words we assume $p_*\beta_Z = d[D_1]$ for some $d > 0$. We claim that in this case

$$[\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)]^{\mathrm{virt}} = 0\,.$$

To prove this we use an argument from [64, Section 4]. Set $k = m+n+|E(\Gamma)|$ and consider the morphism

$$\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty) \longrightarrow \overline{M}_{g_Z,k,d[D_1]}(D_0)$$

obtained by forgetting the logarithmic structure and composing with the projection $Z \longrightarrow D_0$. Write $\mathrm{H}_2(P)^+$ and $\mathrm{H}_2(D_1)^+$ for the semigroups of effective curve classes in P, respectively D_1 and denote by $\mathfrak{M}^{(\log)}_{g,n,\mathrm{H}}$ the Artin stack of prestable (logarithmic) curves with irreducible components weighted by an element in a semigroup H. We factor the projection morphism through

$$\begin{tikzcd}
\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty) \arrow[r, "u"] \arrow[dr] & \widetilde{M}_{\Gamma_Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{g_Z,k,d[D_1]}(D_0) \arrow[d] \\
& \mathfrak{M}^{\log}_{g_Z,k,\mathrm{H}_2(P)^+} \arrow[r, "v"] & \mathfrak{M}_{g_Z,k,\mathrm{H}_2(D_1)^+}
\end{tikzcd}$$

Then by [64, Theorem 4.1] we have[4]

$$[\overline{M}_{\Gamma_Z}(Z\,|\,D_0+D_\infty)]^{\mathrm{virt}} = u^!_{\mathfrak{E}_{\overline{M}_{\Gamma_Z}(Z\,|\,D_0+D_\infty)/\widetilde{M}_{\Gamma_Z}}} v^! [\overline{M}_{g_Z,k,d[D_1]}(D_0)]^{\mathrm{virt}}\,.$$

Our claim that $[\overline{M}_{\Gamma_Z}(Z\,|\,D_0+D_\infty)]^{\mathrm{virt}} = 0$ directly follows from the fact that

$$[\overline{M}_{g_Z,k,d[D_1]}(D_0)]^{\mathrm{virt}} = 0$$

since $D_0 \cong \mathrm{Tot}\,\mathcal{O}_{\mathbb{P}^1}(-2)$ carries a nowhere vanishing holomorphic two-form and $d > 0$. □

4.4.4.2 Second vanishing: star shaped graphs. In this section we apply the same arguments we used in Section 4.3.4.3 in order to constrain the shape of Γ to star shaped graphs.

Notation 4.4.3. We introduce the following notation for labelling sets:

- Interior markings with a point condition $I_{\mathrm{pt}} \coloneqq \{1,\ldots,m\}$;
- Contact markings with a point condition $J_{\mathrm{pt}} \coloneqq \{m+1,\ldots,n\}$;
- Contact markings without an insertion $J_{\mathbb{1}} \coloneqq \{n+1,\ldots,m+n\}$.

We may assume that all Z-vertices of Γ are decorated with a curve class which is a multiple of a fibre class. This has the following consequence.

Lemma 4.4.4. *We have $[\overline{M}_{\Gamma_Z}(Z\,|\,D_0+D_\infty)|_{\mathcal{F}_0}]^{\mathrm{virt}} = 0$ unless every Z-vertex*

- *either carries two legs, one labelled in I_{pt} and one in $J_{\mathbb{1}}$;*
- *or carries a single leg with label in J_{pt}.*

Proof. It suffices to show that every Z-vertex is carrying at most one leg with label in $I_{\mathrm{pt}} \cup J_{\mathrm{pt}}$. Indeed, in this case the number of vertices in Γ_Z is bounded from below by $|I_{\mathrm{pt}}| + |J_{\mathrm{pt}}| = n$. On the other hand it is also bounded from above by the number of contact markings n. We deduce that there have to be exactly n vertices each carrying one of the legs labelled by $J_{\mathbb{1}} \cup J_{\mathrm{pt}}$. Since we assumed that every vertex carries at most one leg with label in $I_{\mathrm{pt}} \cup J_{\mathrm{pt}}$, the m interior markings are necessarily distributed among the vertices carrying a leg labelled in $J_{\mathbb{1}}$.

So suppose there is a Z-vertex with more than one leg with label in $I_{\mathrm{pt}} \cup J_{\mathrm{pt}}$. Since the evaluation morphism of an interior marking has target Z let us first define a substack with confined evaluations:

$$\overline{N}_{\Gamma_Z} \coloneqq \overline{M}_{\Gamma_Z}(Z\,|\,D_0+D_\infty) \times_{Z^n} D_\infty^n\,.$$

As every Z-vertex is decorated with a curve class which is a multiple of a fibre class and we assume there is a vertex with two legs labelled in $I_{\mathrm{pt}} \cup J_{\mathrm{pt}}$ the resulting evaluation morphism

$$\overline{N}_{\Gamma_Z} \longrightarrow D_\infty^n$$

[4] In the statement of [64, Theorem 4.1] van Garrel, Graber and Ruddat assume a positivity property for the logarithmic tangent sheaf of the target which guarantees that u is smooth. In our case at hand this positivity condition is in general not satisfied and u is only virtually smooth in the sense of [131, Definition 3.4]. The latter is however sufficient for our purposes thanks to [130, Corollary 4.9].

factors through $D_\infty^{n-1} \longrightarrow D_\infty^n$. We obtain the following commuting diagram with Cartesian squares:

$$\begin{tikzcd}
\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_{\mathcal{F}_0} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{N}_{\Gamma_Z} \arrow[r] \arrow[d] & \overline{M}_{\Gamma_Z}(Z|D_0+D_\infty) \arrow[dd, "\Pi_{i=1}^n \mathrm{ev}_i"] \\
\mathcal{F}_0^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & D_\infty^{n-1} \arrow[d] \arrow[r, phantom, "\square"] & \\
\mathcal{F}_0^n \arrow[r] & D_\infty^n \arrow[r] & Z^n .
\end{tikzcd}$$

The vanishing of $[\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_{\mathcal{F}_0}]^{\mathrm{virt}}$ then immediately follows from an excess bundle calculation as the one we did in Lemma 4.3.8. $\square$

Moreover, by our assumption that all Z-vertices carry curve classes which are multiples of a fibre class the evaluation morphism associated to edges

$$\overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_{\mathcal{F}_0} \longrightarrow D_0^{E(\Gamma)}$$

factors through $F^{E(\Gamma)} \hookrightarrow D_0^{E(\Gamma)}$ where we write F for the image of $\mathcal{F}_0$ under $Z \longrightarrow D_0$. We combine this observation with the fact that stable maps to Y factor through the zero section $S \hookrightarrow Y$ in order to constrain the image of Z-vertices to the fibre of $Z \longrightarrow D_0$ over the point $p = F \cap D_1$. Indeed, as we just argued the evaluation morphism

$$\overline{M}_{\Gamma_Y}(Y|D) \longrightarrow D_0^{E(\Gamma)}$$

factors through $D_1^{E(\Gamma)} \hookrightarrow D_0^{E(\Gamma)}$. Writing $\overline{M}_{\Gamma_Y}(Y|D)|_F$ for the fibre product $\overline{M}_{\Gamma_Y}(Y|D) \times_{D_0^{E(\Gamma)}} F^{E(\Gamma)}$ we find that

$$\begin{tikzcd}
\overline{M}_{\Gamma_Y}(Y|D)|_F \times \overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)|_\infty \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{\Gamma_Y}(Y|D) \times \overline{M}_{\Gamma_Z}(Z|D_0+D_\infty)|_{\mathcal{F}_0} \arrow[d] \\
p \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & D_1^{E(\Gamma)} \times F^{E(\Gamma)} \arrow[d] \\
D_0^{E(\Gamma)} \arrow[r, "\Delta"] & D_0^{E(\Gamma)} \times D_0^{E(\Gamma)} .
\end{tikzcd} \tag{4.32}$$

commutes and both squares are Cartesian. An excess intersection argument as in Lemma 4.3.9 gives

Lemma 4.4.5. *We have $[\overline{M}_\Gamma]^{\mathrm{virt}} = 0$ if there is a Z-vertex with more than one edge.* $\square$

Since all squares in (4.32) are Cartesian, a comparison with (4.28) gives

$$\overline{M}_\Gamma \cong \overline{M}_{\Gamma_Y}(Y|D)|_F \times \overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)|_\infty .$$

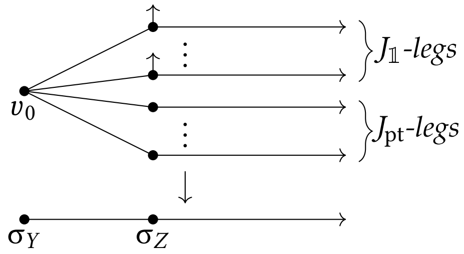
The following lemma comparing the virtual fundamental classes of the above moduli stacks can be proven with the same arguments we used to show Lemma 4.3.10.

Lemma 4.4.6. *If Γ contains a Z-vertex v with $g_v > 0$ the cycle $[\overline{M}_\Gamma]^{\mathrm{virt}}$ vanishes. Otherwise, it equates to*

$$[\overline{M}_{\Gamma_Y}(Y|D)|_F]^{\mathrm{virt}} \times [\overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)|_\infty]^{\mathrm{virt}} .$$ $\square$

Let us summarise the results of this section.

Proposition 4.4.7. *The cycle $[\overline{M}_\Gamma]^{\mathrm{virt}}$ vanishes unless the rigid tropical type τ associated to Γ is of the form*



where v_0 is decorated with $g_{v_0} = g$, $\beta_{v_0} = \beta$ and all other vertices are decorated with multiples of fibre classes. Moreover, in this case

$$[\overline{M}_\Gamma]^{\mathrm{virt}} = [\overline{M}_{\Gamma_Y}(Y|D)|_F]^{\mathrm{virt}} \times [\overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)|_\infty]^{\mathrm{virt}}. \qquad \square$$

4.4.5 Analysis of remaining terms

Now our discussion will deviate from the analysis in Section 4.3.4 because as opposed to Section 4.3.4.4 we will show that all remaining contributions characterised in Proposition 4.4.7 do contribute to the Gromov–Witten invariant and will exactly yield the right-hand side of (4.25).

Let Γ be as in Proposition 4.4.7. We fix a labelling of the markings of $\overline{M}_{\Gamma_Y}(Y|D)|_F$ by fixing a bijection $\{1,\ldots,n\} \cong E(\Gamma)$ where we impose that the ith edge is adjacent to the Z-vertex carrying the $(m+i)$th leg. With this assignment we can naturally identify Γ_Y with the data $(g,\mathbf{c},\beta)$.

Our first goal is to simplify

$$\theta_*[\overline{M}_\Gamma]^{\mathrm{virt}} = \theta_*\big([\overline{M}_{g,\mathbf{c},\beta}(Y|D)|_F]^{\mathrm{virt}} \times [\overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)|_\infty]^{\mathrm{virt}}\big).$$

Under θ all images of stable maps which lie in a fibre of $P \longrightarrow D_1$ get contracted to a point. Hence, the morphism factors into

$$\begin{tikzcd} \overline{M}_{g,\mathbf{c},\beta}(Y|D)|_F \times \overline{M}_{\Gamma_Z}(\mathbb{P}^1|0+\infty)|_\infty \arrow[r, "\mathrm{Id}\times\pi"] \arrow[rr, bend right, "\theta"'] & \overline{M}_{g,\mathbf{c},\beta}(Y|D)|_F \times \big(\overline{M}_{0,3}\big)^m \arrow[r, "\phi"] & \overline{M}_{g,m+n,\beta}(Y) \end{tikzcd}$$

where π is the forgetful morphism which only remembers the stabilised domain curve and ϕ is a morphism which forgets the logarithmic structure and attaches a contracted rational component at the last m markings.

Consequently, a Z-vertex carrying a leg labelled with $i+m \in J_{\mathbb{1}}$ contributes with a factor

$$\deg[\overline{M}_{0,((c_i,0),(0,c_i)),c_i}(\mathbb{P}^1|0+\infty)|_\infty]^{\mathrm{virt}} = \deg[\overline{M}_{0,((c_i,0),(0,c_i)),c_i}(\mathbb{P}^1|0+\infty)]^{\mathrm{virt}} = \frac{1}{c_i}$$

where the denominator is due to the degree c_i automorphism group acting on the unique degree c_i cover $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$ fully ramified over 0 and infinity. Similarly, a Z-vertex with leg labelled by $i+m \in J_{\mathrm{pt}}$ contributes

$$\deg[\overline{M}_{0,(0,(c_i,0),(0,c_i)),c_i}(\mathbb{P}^1|0+\infty)|_\infty]^{\mathrm{virt}} = \deg\big(\mathrm{ev}_1^*(\mathrm{pt}) \cap [\overline{M}_{0,(0,(c_i,0),(0,c_i)),c_i}(\mathbb{P}^1|0+\infty)]^{\mathrm{virt}}\big) = 1$$

because the additional marking kills the automorphism group acting on the domain. We obtain

$$\theta_*[\overline{M}_\Gamma]^{\mathrm{virt}} = \Big(\textstyle\prod_{i=1}^{n-m} \frac{1}{c_i}\Big) \cdot \phi_*\Big([\overline{M}_{g,\mathbf{c},\beta}(Y|D)|_F]^{\mathrm{virt}} \times [\overline{M}_{0,3}]^{\times m}\Big). \qquad (4.33)$$

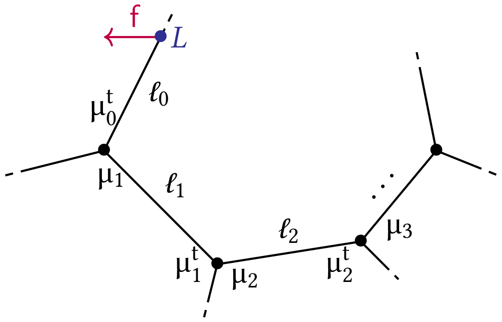


Figure 4.3: Toric diagram of (X, L, f) decorated with partitions.

Note that a splitting type Γ as in Proposition 4.4.7 is uniquely determined by the bijection $I_{\mathrm{pt}} \cong J_{\mathbb{1}}$ it induces. This means there are exactly $m!$ such splitting types each one contributing with (4.33) to the overall Gromov–Witten invariant. Thus, plugging the last equation into (4.31) we find

$$\zeta_*[\overline{M}_{g,\tilde{c},\beta}(Y|D)|_F]^{\mathrm{virt}} = m! \left(\prod_{i=0}^{m-1} c_{n-i}\right) \cdot \phi_*\Big([\overline{M}_{g,c,\beta}(Y|D)|_F]^{\mathrm{virt}} \times [\overline{M}_{0,3}]^{\times m}\Big) .$$

Pushing the identity forward to a point yields formula (4.25) which closes the proof of Proposition 4.4.1.

4.5 Applications

4.5.1 Topological Vertex

Open Gromov–Witten invariants can be computed efficiently using the topological vertex method [7, 116]. Hence, formula (4.10) equating logarithmic Gromov–Witten invariants — which are usually rather cumbersome to compute — against open ones presents a new venue for calculating the former.

4.5.1.1 The general recipe. We refer to [166, Section 2] or [29, Section 6] for a general summary of the topological vertex method which allows to determine the open Gromov–Witten invariants of a toric triple just from the information of its toric diagram. Let (X, L, f) be the outcome of Construction 4.2.8 for some two-component Looijenga pair $(Y|D_1 + D_2)$. We will write

$$\mathcal{W}_\mu(X, L, \mathsf{f})(q, Q) \tag{4.34}$$

for the generating series of disconnected open Gromov–Witten invariants of (X, L, f) in representation basis where the winding around L is encoded in a partition μ. In the special case of a single boundary wrapping the Lagrangian submanifold L we can extract the connected invariant defined in (4.7) via [29, Equation (6.16)][5]

$$\sum_{\beta'}\sum_{g\geq 0} \mathsf{OGW}_{g,(c),\beta'}(X, L, \mathsf{f})\, \hbar^{2g-1}\, Q^{\beta'} = -\mathrm{i} \sum_{k=0}^{c} \frac{(-1)^k}{c} \frac{\mathcal{W}_{(c-k,1^k)}(X, L, \mathsf{f})}{\mathcal{W}_\emptyset(X, L, \mathsf{f})} \tag{4.35}$$

under the change of variables $q = \mathrm{e}^{\mathrm{i}\hbar}$ where we fixed a square root i of -1.

The generating series $\mathcal{W}_\mu(X, L, \mathsf{f})$ can be calculated algorithmically via the topological vertex method by performing a weighted sum over decorations of the toric diagram [7, 116]. As we discussed in

[5] We corrected [29, Equation (6.16)] by an overall factor $-\mathrm{i}$. See [166, Appendix A] for a discussion of this factor.

Section 4.3.1, for our case at hand the toric diagram of $(X, L, \mathfrak{f})$ will have a shape as displayed in Figure 4.3. We label the compact one dimensional torus orbit closures by $\ell_1, \ldots, \ell_k$ starting from the stratum ℓ_0 intersected by L and decorate each edge ℓ_i with a partition μ_i as indicated in the figure. Then every edge ℓ_i, $i \in \{0, \ldots, k\}$, contributes with a factor

$$(-1)^{(\deg N_{\ell_i} S')\cdot|\mu_i|} q^{(\deg N_{\ell_i} S'+1)\cdot\kappa_{\mu_i}/2}$$

to (4.34) where $\kappa_\nu := \sum_j \nu_j(\nu_j - 2j + j)$. Each trivalent vertex corresponding to a torus fixed point $\ell_i \cap \ell_{i+1}$ contributes

$$\mathcal{W}_{\mu_{i+1},\mu_i^t,\emptyset} := s_{\mu_{i+1}}(q^{\rho+\mu_i})\, s_{\mu_i^t}(q^\rho) = q^{-\kappa_{\mu_i}/2} s_{\mu_{i+1}}(q^\rho) s_{\mu_i}(q^{\rho+\mu_{i+1}}) = q^{\kappa_{\mu_{i+1}}/2} \sum_\nu s_{\frac{\mu_i^t}{\nu}}(q^\rho)\, s_{\frac{\mu_{i+1}^t}{\nu}}(q^\rho)$$

where by convention $\mu_0 = \mu$ and $\mu_{k+1} = \emptyset$ and $s_{\alpha/\beta}(q^{\rho+\gamma})$ denotes the skew Schur function

$$s_{\frac{\alpha}{\beta}}(x_1, x_2, \ldots)$$

evaluated at $x_j = -j + \frac{1}{2} - \gamma_j$. As an application of [116, Proposition 7.4] we obtain

Lemma 4.5.1. *For $(X, L, \mathfrak{f})$ the outcome of Construction 4.2.8 we have*

$$\mathcal{W}_\mu(X, L, \mathfrak{f})(q, Q) = \sum_{\mu_1,\ldots,\mu_k} Q^{\sum_{i=1}^k |\mu_i|\cdot[\ell_i]} \prod_{i=0}^{k} (-1)^{(\deg N_{\ell_i} S')\cdot|\mu_i|} q^{(\deg N_{\ell_i} S'+1)\cdot\kappa_{\mu_i}/2} \mathcal{W}_{\mu_{i+1},\mu_i^t,\emptyset}\,.$$

In case the toric triple is a strip geometry the above sum of partitions can be carried out explicitly to yield a closed form solution for $\mathcal{W}_\mu(X, L, \mathfrak{f})$ [93]. Plugged into (4.35) and (4.10) this can be used to give a closed form solution for the logarithmic Gromov–Witten invariants of $(S\,|\,D_1 + D_2)$. We illustrate this method with an explicit example in the following section.

4.5.1.2 Extended example: $\mathrm{dP}_3(0, 2)$. Let us consider $(\mathrm{dP}_3\,|\,D_1 + D_2)$ where dP_3 is a del Pezzo surface which is the result of blowing up three points in $\mathbb{P}^2$. This Looijenga pair was denoted by $\mathrm{dP}_3(0, 2)$ in [29]. If we write E_1, E_2, E_3 for exceptional divisors we take $D_1 \in |H - E_1|$ and $D_2 \in |2H - E_2 - E_3|$ smooth with transverse intersections. We use the following notation for a curve class in dP_3:

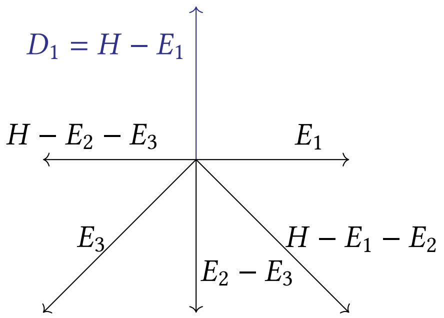
$$\beta = d_0(H - E_1 - E_2 - E_3) + d_1 E_1 + d_2 E_2 + d_3 E_3\,.$$

It was conjectured in [29] and later proven in [33] that the maximum contact logarithmic Gromov–Witten invariants of this geometry have the following closed form solution.

Proposition 4.5.2. [33, *Proposition 1.4*] *For all effective curve classes β in dP_3 with $D_i \cdot \beta > 0$, $i \in \{1, 2\}$, and contact data $\hat{\mathbf{c}} = \begin{pmatrix} 0 & D_1\cdot\beta & 0 \\ 0 & 0 & D_2\cdot\beta \end{pmatrix}$ we have*

$$\begin{aligned}\sum_{g\geq 0} \hbar^{2g-1}\, \mathsf{LGW}_{g,\hat{c},\beta}(\mathrm{dP}_3\,|\,D_1 + D_2) \\ = \frac{[d_1]_q [d_2 + d_3]_q}{[d_0]_q [d_1 + d_2 + d_3 - d_0]_q} \begin{bmatrix} d_1 \\ d_0 - d_3 \end{bmatrix}_q \begin{bmatrix} d_1 \\ d_0 - d_2 \end{bmatrix}_q \begin{bmatrix} d_0 \\ d_1 \end{bmatrix}_q \begin{bmatrix} d_1 + d_2 + d_3 - d_0 \\ d_1 \end{bmatrix}_q\end{aligned} \tag{4.36}$$

as formal Laurent series in $\hbar$ under the identification $q = \mathrm{e}^{\mathrm{i}\hbar}$.



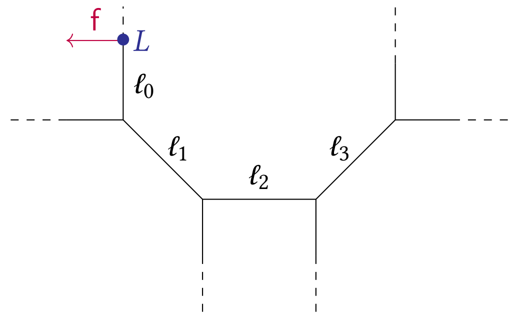


Figure 4.4: The fan of dP_3' on the left and the toric diagram of (X, L, f) on the right [29, Figure 6.5].

In this proposition we use the notation

$$[n]_q \coloneqq q^{n/2} - q^{-n/2}, \quad \begin{bmatrix} n \\ m \end{bmatrix}_q \coloneqq \begin{cases} \prod_{k=1}^m \frac{[n-m+k]_q}{[k]_q} & 0 \le m \le n, \\ 0 & \text{otherwise.} \end{cases}$$

The statement was proven in [33] by extracting the Gromov–Witten invariants from a quantum scattering diagram of a toric model of $(\mathrm{dP}_3 \,|\, D_1 + D_2)$. The outcome of this calculation is a convoluted sum of products of q-binomial coefficients which with a lot of effort can be simplified to yield the above formula. We will give a new proof of this identity using the topological vertex method.

Proof of Proposition 4.5.2. We start by applying Construction 4.2.8 to $(\mathrm{dP}_3 \,|\, D_1 + D_2)$. For this let us write p_1, p_2, p_3 for the three points in $\mathbb{P}^2$ whose blow up gives dP_3. Then D_1 is the strict transform of a line through p_1 and D_2 the strict transform of a conic through p_2 and p_3. We note that we can always choose the $\mathbb{G}_{\mathrm{m}}^2$-action on $\mathbb{P}^2$ in such a way that D_1 is a toric hypersurface and p_1 and p_2 are torus fixed points. Now with this choice of $\mathbb{G}_{\mathrm{m}}^2$-action the blowups at p_1 and p_2 are toric as opposed to the one at p_3 since all points are assumed to be in general position. However, there is a deformation of dP_3 to the blow up of $\mathbb{P}^2$ in p_1, p_2 and a torus fixed point of E_2 leaving D_1 invariant. We denote this deformation by dP_3' which is indeed toric as required. We display its fan in Figure 4.4 where we labelled each ray with its curve class. Since dP_3' is toric with D_1 a toric hypersurface, we can choose

$$X = \operatorname{Tot} \mathcal{O}_{\mathrm{dP}_3'}(-D_2)|_{\mathrm{dP}_3' \setminus D_1}.$$

The toric diagram of (X, L, f) is displayed in Figure 4.4 where we chose L to intersect the torus invariant curve in class $H - E_2 - E_3$. We note that this particular toric triple has already been studied by Bousseau, Brini and van Garrel in [29, Section 6.3.1]. Hence, from here on we basically follow their calculation.

Observe that each compact edge in the toric diagram corresponds to a ray in the fan. Hence, we can read off the degree of the normal bundles of $\ell_0, \dots, \ell_3$ from Figure 4.4 to be $-1, -1, -2$ and -1 respectively. Thus, by Lemma 4.5.1 we have

$$\mathcal{W}_\mu(X, L, \mathsf{f}) = \sum_{\substack{\mu_1,\mu_2,\mu_3 \\ \nu_1,\nu_2}} (-1)^{|\mu|+|\mu_1|+|\mu_3|} \, s_{\mu^t}(q^\rho) \, s_{\mu_1^t}(q^{\rho+\mu}) \, s_{\frac{\mu_1}{\nu_1}}(q^\rho) \, s_{\frac{\mu_2}{\nu_1}}(q^\rho) \, s_{\frac{\mu_2}{\nu_2}}(q^\rho) \, s_{\frac{\mu_3}{\nu_2}}(q^\rho) \, s_{\mu_3^t}(q^\rho) \, Q^{\sum_{i=1}^3 |\mu_i| \cdot [\ell_i]}.$$

This can now be plugged into (4.35) and simplified using the techniques developed in [93]. These steps have already been carried out in [29, Equation (6.18–6.23)] and result in

$$\sum_{\beta'} \sum_{g \ge 0} \mathsf{OGW}_{g,(c),\beta'}(X, L, \mathsf{f}) \, \hbar^{2g-1} \, Q^{\beta'}$$
$$= \sum_{c_1,c_2,c_3} \mathrm{i}(-1)^{c+c_1+c_3+1} \frac{1}{c[c_1]_q} \begin{bmatrix} c \\ c_1 - c_2 \end{bmatrix}_q \begin{bmatrix} c \\ c_2 - c_3 \end{bmatrix}_q \begin{bmatrix} c + c_3 - 1 \\ c_3 \end{bmatrix}_q \begin{bmatrix} c_1 \\ c \end{bmatrix}_q Q^{\sum_{i=1}^3 c_i [\ell_i]}.$$

Extracting the coefficient of $Q^{\iota^*(\beta - d_1[\ell_0])}$ from the above formula we get

$$\sum_{g\geq 0} \hbar^{2g-1} \, \mathsf{OGW}_{g,(d_1),\iota^*(\beta - d_1[\ell_0])}(X, L, \mathsf{f}) \\ = \frac{\mathrm{i}(-1)^{d_1+d_2+d_3+1}[d_1]_q}{d_1[d_0]_q[d_1+d_2+d_3-d_0]_q} \begin{bmatrix} d_1 \\ d_0 - d_3 \end{bmatrix}_q \begin{bmatrix} d_1 \\ d_0 - d_2 \end{bmatrix}_q \begin{bmatrix} d_0 \\ d_1 \end{bmatrix}_q \begin{bmatrix} d_1+d_2+d_3-d_0 \\ d_1 \end{bmatrix}_q$$

which can be plugged into (4.10) to yield the formula stated in Proposition 4.5.2. □

We stress that the above calculation is substantially easier than the one in [33] using scattering diagrams. This hopefully illustrates the usefulness of our main result Theorem 4.2.9.

4.5.2 BPS integrality

It is a general feature rather than a coincidence that the all-genus generating series of logarithmic Gromov–Witten invariants lifts to a rational function in $\mathrm{e}^{\mathrm{i}\hbar}$ as we saw for instance in formula (4.36). This property is usually called BPS integrality.

4.5.2.1 LMOV invariants. In the context of open Gromov–Witten invariants of toric Calabi–Yau threefolds BPS integrality was first systematically studied by Labastida–Marino–Ooguri–Vafa [111, 112, 132, 145]. Just recently Yu gave a proof of this phenomenon by carefully analysing the combinatorics of the topological vertex [166].

To review Yu's result, let (X, L, f) be a toric triple. For a curve class β' in X and a winding profile $\mathbf{c} = (c_1, \ldots, c_n)$ along L we will write

$$k \,|\, (\mathbf{c}, \beta')$$

for $k \in \mathbb{N}$ if k divides c_i for all $i \in \{1, \ldots, n\}$ and there is a curve class β'' satisfying $k\beta'' = \beta'$.

Now for every tuple $(\mathbf{c}, \beta')$ we define

$$\mathsf{LMOV}_{\mathbf{c},\beta'}(\hbar) \in \mathbb{Q}[\![\hbar^2]\!]$$

by demanding that these formal power series satisfy

$$\sum_{g\geq 0} \hbar^{2g-2+n} \, \mathsf{OGW}_{g,\mathbf{c},\beta'}(X, L, \mathsf{f}) =: \prod_{i=1}^{n} \frac{2\sin\frac{\hbar c_i}{2}}{c_i} \cdot \sum_{k|(\mathbf{c},\beta')} k^{n-1} \left(2\sin\tfrac{\hbar k}{2}\right)^{-2} \mathsf{LMOV}_{\mathbf{c}/k,\beta'/k}(k\hbar)\,.$$

Writing μ for the Möbius function one can check that

$$\mathsf{LMOV}_{\mathbf{c},\beta'}(\hbar) = \left(2\sin\tfrac{\hbar}{2}\right)^2 \prod_{i=1}^{n} \frac{c_i}{2\sin\frac{\hbar c_i}{2}} \cdot \sum_{k|(\mathbf{c},\beta')} \frac{\mu(k)}{k} \sum_{g\geq 0} (k\hbar)^{2g-2+n} \, \mathsf{OGW}_{g,\mathbf{c}/k,\beta'/k}(X, L, \mathsf{f})$$

is indeed a solution to the above defining equation.

Theorem 4.5.3 (LMOV integrality)**.** [*166, Theorem 1.1*] *$\mathsf{LMOV}_{\mathbf{c},\beta'}$ lifts to a Laurent polynomial in $q = \mathrm{e}^{\mathrm{i}\hbar}$ with integer coefficients.*

4.5.2.2 BPS integrality of logarithmic invariants. Now let $(S\,|\,D_1+D_2)$ be a logarithmic Calabi–Yau surface with boundary the union of two transversally intersecting smooth curves D_1, D_2. To a tuple $(\hat{\mathbf{c}}, \beta)$, where β is an effective curve class in S satisfying $D_1 \cdot \beta \geq 0$ and $D_2 \cdot \beta > 0$ and $\hat{\mathbf{c}}$ is a contact condition of the form

$$\hat{\mathbf{c}} = \begin{pmatrix} 0 & \cdots & 0 & c_1 & \ldots & c_n & 0 \\ \underbrace{0 \;\; \cdots \;\; 0}_{m \text{ times}} & & & 0 & \ldots & 0 & D_2 \cdot \beta \end{pmatrix}$$

for some $c_1, \ldots, c_n > 0$ and $m \leq n$, we associate a formal power series

$$\mathsf{BPS}_{\hat{\mathbf{c}},\beta}(\hbar) \in \mathbb{Q}[\![\hbar^2]\!]$$

solving the recursive system

$$\begin{aligned} &(-1)^{K_S \cdot \beta + 1} \sum_{g \geq 0} \hbar^{2g-1+n} \, \mathsf{LGW}_{g,\hat{\mathbf{c}},\beta}(S\,|\,D_1+D_2) \\ &=: \left(\prod_{i=1}^{n-m} \frac{2 \sin \frac{\hbar c_i}{2}}{c_i} \right) \left(\frac{1}{m!} \prod_{i=n-m+1}^{n} 2 \sin \tfrac{\hbar c_i}{2} \right) \left(2 \sin \tfrac{\hbar (D_2 \cdot \beta)}{2} \right) \sum_{k \mid (\hat{\mathbf{c}},\beta)} k^{n-1} \left(2 \sin \tfrac{\hbar k}{2} \right)^{-2} \mathsf{BPS}_{\hat{\mathbf{c}}/k,\beta/k}(k\hbar) \,. \end{aligned} \tag{4.37}$$

Again, one can check that

$$\begin{aligned} \mathsf{BPS}_{\hat{\mathbf{c}},\beta} = \left(2 \sin \tfrac{\hbar}{2} \right)^2 \left(\prod_{i=1}^{n-m} \frac{c_i}{2 \sin \frac{\hbar c_i}{2}} \right) \left(\frac{1}{m!} \prod_{i=n-m+1}^{n} \frac{1}{2 \sin \frac{\hbar c_i}{2}} \right) \frac{1}{2 \sin \frac{\hbar (D_2 \cdot \beta)}{2}} \\ \times \sum_{k \mid (\hat{\mathbf{c}},\beta)} \mu(k) \, k^{m-1} \, (-1)^{K_S \cdot \beta/k + 1} \sum_{g \geq 0} (k\hbar)^{2g-1+n} \, \mathsf{LGW}_{g,\hat{\mathbf{c}}/k,\beta/k}(S\,|\,D_1+D_2) \end{aligned}$$

is indeed a solution to this defining equation. Motivated by the last section we expect the following.

Conjecture 4.5.4. $\mathsf{BPS}_{\hat{\mathbf{c}},\beta}$ *lifts to a Laurent polynomial in* $q = e^{i\hbar}$ *with integer coefficients.*

Evidence for the conjecture comes from the following immediate corollary of Yu's LMOV integrality (Theorem 4.5.3) and our logarithmic-open correspondence (Theorem 4.2.9).

Theorem 4.5.5. *Conjecture 4.5.4 holds if* $(S\,|\,D_1+D_2)$ *satisfies Assumption* $\star$. *Especially, if* (X, L, f) *is the toric triple from Construction 4.2.8 we have*

$$\mathsf{BPS}_{\hat{\mathbf{c}},\beta} = \mathsf{LMOV}_{\mathbf{c},\beta'} \,. \qquad \square$$

Note that the pattern in which the sine factors enter the right-hand side of equation (4.37) naturally suggests a generalisation to the case where we permit more than one point of contact with D_2. It remains to be seen whether BPS integrality persists in this case as well.

Part II

Gromov–Witten theory and the refined topological string

Chapter

Refined Gromov–Witten invariants

5

This section contains work in progress and will be made available alongside other results in a joint paper [34] with Andrea Brini once completed.

Abstract. We propose an interpretation of the refined topological string on a Calabi–Yau threefold X in terms of equivariant Gromov–Witten theory of the Calabi–Yau fivefold $X \times \mathbb{A}^2$ in case X admits the action of a torus scaling the canonical bundle non-trivially. We formulate a refined BPS integrality conjecture and provide evidence in case X is the resolved conifold or a local del Pezzo surface. In the latter case we do so by relating the Nekrasov–Shatashvili limit of the refined local surface invariants to the relative Gromov–Witten invariants of the surface relative a smooth anticanonical curve.

5.1 Introduction

The free energy of the A-model topological string on a Calabi–Yau threefold X has a well accepted mathematical formulation as the generating series of Gromov–Witten invariants of this geometry

$$\mathsf{GW}_\beta(X) = \sum_{g \geq 0} (-u^2)^{g-1} \int_{[\overline{M}_{g,\beta}(X)]^{\mathrm{virt}}} 1 \qquad \in \mathbb{Q}((u)) \,. \tag{5.1}$$

This twofold description of essentially one and the same object in a mathematically rigorous and a more intuitive physics setting has been a major catalyser for the fields rapid development.

Motivated by Nekrasov's calculations of instanton partition functions in supersymmetric gauge theory [144] it was expected that $\mathsf{GW}_\beta(X)$ should admit a natural one parameter refinement: a series in two variables ϵ_1 and ϵ_2 which restricts to the original one upon specialising to $\epsilon_1 = -\epsilon_2 = u$. The anticipated quantum field theory giving rise to such a partition function was coined the refined topological string but an explicit formulation of this theory remained elusive to the present day with only partial progress made in topological [10, 11, 142, 147] and physical string theory [86, 150]. At the same time the B-model interpretation found in [110] proves to be highly successful

[49, 88, 89] which clearly hints towards the existence of an A-model, or in mathematical terms, a Gromov–Witten type counterpart.

In this note we aim to close this gap by proposing a formulation of the A-model refined topological string in terms of equivariant Gromov–Witten theory of the extended target geometry $X \times \mathbb{A}^2$. We will present several preliminary results which indicate that our proposal indeed satisfies many properties anticipated in the physics literature. All here mentioned results will appear alongside others in a joint paper [34] with Andrea Brini once completed.

5.1.1 Refined Gromov–Witten invariants

Consider the generating series of equivariant Gromov–Witten invariants of $X \times \mathbb{A}^2$:

$$\mathsf{GW}_\beta(X \times \mathbb{A}^2, T) \text{ ``}\coloneqq\text{''} \sum_{g \geq 0} \int_{[\overline{M}_{g,\beta}(X \times \mathbb{A}^2)]^{\mathrm{virt}}_T} 1 \,. \tag{5.2}$$

Here, the right-hand side integral is defined via localisation with respect to a torus $T \cong \mathbb{G}_{\mathrm{m}}^2$ which we assume is acting on $X \times \mathbb{A}^2$ in a way that

(i) the induced action on $\omega_{X \times \mathbb{A}^2}$ is trivial,

(ii) the induced action on ω_X is non-trivial,

(iii) it contains the anti-diagonal torus action on $\mathbb{A}^2$.

We may expand the series $\mathsf{GW}_\beta(X \times \mathbb{A}^2, T)$ as a formal Laurent series in the T-weights on the two affine directions $\mathbb{A}^2$ which we denote by $-\epsilon_1$ and $-\epsilon_2$. Under mild assumptions on X (see Lemma 5.2.4) it takes the form

$$\mathsf{GW}_\beta(X \times \mathbb{A}^2, T) \in (\epsilon_1\epsilon_2)^{-1}\mathbb{Q}[\![\epsilon_1, \epsilon_2]\!] \,.$$

We claim that this generating series is the Gromov–Witten formulation of the refined topological string free energy. As a first sanity check we verify in Proposition 5.2.6 that it indeed specialises the original generating series upon setting $\epsilon_1 = -\epsilon_1 = u$:

$$\mathsf{GW}_\beta(X \times \mathbb{A}^2, T)\Big|_{\substack{\epsilon_1 = u \\ \epsilon_2 = -u}} = \mathsf{GW}_\beta(X) \,.$$

This result crucially requires condition (iii) on the T-action. Property (ii) guarantees that we obtain a non-trivial refinement (see Remark 5.2.8) and experimentally condition (i) seems to be important for the generating series to satisfy all properties anticipated in the physics literature. However, so far we are lacking a conceptual explanation for why this property has to be imposed. One instance where it seems to be rather crucial is the following.

Conjecture 5.A. (*Conjecture 5.2.9*) *As a formal Laurent series in ϵ_1 and ϵ_2 the generating series*

$$\mathsf{GW}_\beta(X \times \mathbb{A}^2, T)$$

does not depend the choice of T-action on $X \times \mathbb{A}^2$ if (i)–(iii) hold and $\overline{M}_{g,\beta}(X)$ is proper for all $g \geq 0$.

We provide evidence for the above conjecture in the case when X is the resolved conifold and illustrate the importance of the properness assumption in the example $X = \mathrm{Tot}\, \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}$.

5.1.2 BPS integrality

The physics community formulates numerous expected properties for the refined topological string. In most cases these constitute natural refinements of features enjoyed by the usual (unrefined) topological string. One such example is BPS integrality which predicts that the generating series, which is a formal Laurent series with rational coefficients, is governed by a finite number of integer invariants usually called Gopakumar–Vafa or BPS invariants. We formulate this conjecture in three increasing levels of strength and decreasing level of generality:

- **Rationality:** $\mathsf{GW}_\beta(X \times \mathbb{A}^2, T)$ lifts to a rational function in $e^{\epsilon_1/2}, e^{\epsilon_2/2}$ (Conjecture 5.5.1)
- **Integrality:** Up to some resummation this function has integer coefficients (Conjecture 5.5.4)
- **Geometric interpretation:** These integer invariants admit a geometric interpretation in terms of Maulik–Toda type Gopakumar–Vafa invariants (Conjecture 5.5.9)

Specialising to $\epsilon_1 = -\epsilon_2$ this recovers conjectures which are already known in the literature. In this limit the rationality and integrality conjecture go back to Gopakumar and Vafa [68, 69] and were proven in [55, 92].

The third conjecture is formulated in the setting of local del Pezzo surfaces and is very much influenced by the observations of Kononov–Pi–Shen [107] and Bousseau [22]. In the unrefined limit it recovers Maulik and Toda's proposal for Gopakumar–Vafa invariants in the special case of local surfaces [135].

We provide evidence for the integrality conjecture by proving that (for a fairly general class of T-actions) the refined Gromov–Witten generating series of the resolved conifold indeed takes the form predicted in the physics literature [94].

We also make progress when $X = K_S$ is a local del Pezzo surface. In the so called Nekrasov–Shatashvili limit ($\epsilon_2 = 0$) we prove that the rationality conjecture holds for all del Pezzo surfaces and the establish the strongest version of the conjecture for local $\mathbb{P}^2$ in this limit. Our proof utilises a correspondence between

$$\epsilon_2 \, \mathsf{GW}_\beta(X \times \mathbb{A}^2, T) \Big|_{\substack{\epsilon_1 = \hbar \\ \epsilon_2 = 0}}$$

and the Gromov–Witten theory of S relative a smooth anticanonical curve. This is combined with results of Bousseau [20, 22, 25] who proves the respective conjectures in the context of relative Gromov–Witten theory.

5.1.3 The Nekrasov–Shatashvili limit of local surfaces

Let us quickly summarise the just teased correspondence between the Nekrasov–Shatashvili and relative Gromov–Witten invariants in the context of local surfaces. Let S be a smooth del Pezzo surface and D a smooth anticanonical curve. We consider the generating series of relative Gromov–Witten invariants of $(S|D)$ enumerating stable maps with maximum tangency along D together with an insertion of the top Chern class of the Hodge bundle:

$$\mathsf{GW}_\beta(S|D) \coloneqq \frac{(-1)^{D\cdot\beta+1}}{D\cdot\beta} \sum_{g\geq 0} \hbar^{2g-1} \int_{[\overline{M}_{g,0,(D\cdot\beta),\beta}(S|D)]^{\mathrm{virt}}} \lambda_g \, .$$

The following comparison holds.

Theorem 5.B. (*Corollary 5.4.3*) *We have*

$$\epsilon_2 \mathsf{GW}_\beta(K_S, T)\Big|_{\substack{\epsilon_1=\hbar\\ \epsilon_2=0}} = \mathsf{GW}_\beta(S|D)\,.$$

We will actually prove a much stronger version of the above correspondence and establish it as a cycle valued identity in Theorem 5.4.1.

The theorem is remarkable in the regard that for local surfaces the equivariant parameter ϵ_1 exhibits an interpretation as a genus counting variable when we set ϵ_2 equal to zero. A similar observation can be made for the resolved conifold but we do not expect such a feature for general targets.

5.1.4 Structure of this chapter

We carefully define the refined Gromov–Witten generating series $\mathsf{GW}_\beta(X \times \mathbb{A}^2, T)$ and discuss some of its basic properties in Section 5.2. This includes the proof that the unrefined limit correctly reproduces the original generating series of Gromov–Witten invariants of X and a discussion of the dependence on the choice of torus action.

We continue with two case studies. In Section 5.3 we explicitly compute the refined Gromov–Witten generating series for certain local $\mathbb{P}^1$ geometries including the resolved conifold. In 5.4 we then turn towards local surface geometries where we carry out a detailed study of the Nekrasov–Shatashvili limit $\epsilon_2 = 0$. The main result of this section is the correspondence theorem 5.B.

We close with a discussion of refined BPS integrality in Section 5.5. We use our earlier case studies to provide non-trivial evidence for our conjectures.

5.1.5 Context & outlook

5.1.5.1 K-theoretic Donaldson–Thomas theory. Our proposal for a Gromov–Witten type formulation of the refined topological string is very much informed by the five-dimensional point of view that Nekrasov and Okounkov take on Donaldson–Thomas theory [143]. In their K-theoretic refinement of Donaldson–Thomas theory the box counting parameter q should be thought as the character of the anti-diagonal torus acting on the extended target $X \times \mathbb{A}^2$. The parallel in our approach to Gromov–Witten theory is that the the genus counting parameter u is interpreted as the first Chern class of the representation q.

We claim that this parallel is not just an analogy but we conjecture that the Gromov–Witten theory of $X \times \mathbb{A}^2$ is the appropriate counterpart to K-theoretic Pandharipande–Thomas theory refining the PT-GW correspondence. In combination with Conjecture 5.5.1–5.5.9 we expect a triad

$$\begin{tikzcd}[column sep=small] & \boxed{\text{Refined BPS invariants}} \arrow[dl, dash] \arrow[dr, dash] & \\ \boxed{\text{Equivariant GW invariants of } X \times \mathbb{A}^2} \arrow[rr, dash] & & \boxed{\text{K-theoretic PT invariants of } X} \end{tikzcd} \tag{5.3}$$

We will carefully state a refined PT-GW correspondence in [34]. Moreover, we will explain how this correspondence actually implies our rigidity conjecture (Conjecture 5.2.9). Indeed, in K-theoretic

Pandharipande–Thomas theory an analogous phenomenon occurs and is proven by Nekrasov and Okounkov in [143, Theorem 1].

5.1.5.2 Supersymmetric gauge theory. The horizontal correspondence conjectured in (5.3) also allows to make a connection with our initial motivation which was to find an interpretation of Nekrasov's instanton partition function in terms of Gromov–Witten theory. In special instances it is known that via geometric engineering Pandharipande–Thomas theory on a Calabi–Yau threefolds X corresponds to supersymmetric gauge theory on $\mathbb{A}^2$ with gauge group G_X depending on the threefold geometry. In the refined setting this correspondence is for instance reviewed and established for the resolved conifold in [157]. The motivic calculations in loc. cit. and [139] match with the K-theoretic ones (compare e.g. with[105, Section 4]) which in turn are perfectly compatible with our calculation of the refined Gromov–Witten generating series for the resolved conifold (Proposition 5.3.1). It would certainly be interesting to study this correspondence in more complicated examples.

5.1.5.3 B-model. The B-model of the refined topological string has been studied extensively in the physics literature by Choi–Huang–Katz–Klemm [49, 88, 89]. As the refinement only affects the higher genus amplitudes, mirror symmetry is unaffected in genus zero where Gromov–Witten invariants of some target geometry are encoded in periods of the mirror family. The B-model counterpart of the generating series of higher genus refined invariants is expected to enjoy the same modular properties as its unrefined counterpart. Moreover, it is expected to exhibit a slight modification of Bershadsky–Cecotti–Ooguri–Vafa's holomorphic anomaly as suggested by Krefl and Walcher [110]. For several geometries it was observed that these features combined with certain boundary conditions fix the refined B-model invariants uniquely [89]. Conjecturally, these properties should be enjoyed by our refined Gromov–Witten generating series as well and we will partly verify these predictions for local $\mathbb{P}^2$ in [34].

Via the link between refined B-model invariants and motivic stable pair invariants, experimentally observed by Choi–Katz–Klemm in [49], this will provide further evidence for the vertical correspondence in (5.3) since for proper moduli spaces motivic and K-theoretic invariants agree. (Regarding the last statement, compare [143, Equation (64)] with [54, Equation (1.4)]).

In [21] Bousseau–Fan–Guo–Wu prove that $\sum_\beta \mathsf{GW}_\beta(\mathbb{P}^2|D)\, Q^\beta$ enjoys exactly the same modular properties and the holomorphic anomaly equation expected to be satisfied by the Nekrasov–Shatashvili limit of the refined topological string on local $\mathbb{P}^2$. Our correspondence theorem 5.B provides a conceptual explanation for this observation.

It remains to be seen whether our proposal meets all B-model predictions from [49, 88, 89]. A verification of these features would certainly demonstrate a major proof of concept for our proposal.

5.2 The equivariant refinement

In this section we define the refined Gromov–Witten generating series of a Calabi–Yau threefold and prove some of its basic properties. We start with a discussion of the geometric setup.

5.2.1 The geometric setup

Throughout this section let X be a smooth quasi-projective Calabi–Yau threefold. Moreover, we fix the action of a torus T on $X \times \mathbb{A}^2$ satisfying

Assumption $\mathtt{t}$.

(i) the induced T-action on $\omega_{X\times\mathbb{A}^2}$ is trivial,

(ii) the induced T-action on ω_X is non-trivial,

(iii) there is a subtorus $\sigma_- : \mathbb{G}_\mathrm{m} \hookrightarrow T$ for which the restricted action on $\mathbb{A}^2$ is

$$\mathbb{G}_\mathrm{m} \times \mathbb{A}^2 \longrightarrow \mathbb{A}^2,\ \big(t, (y_1, y_2)\big) \longmapsto (t y_1, t^{-1} y_2)\,, \tag{5.4}$$

(iv) T is isomorphic to $\mathbb{G}_\mathrm{m}^2$.

We will refer to $\sigma_- : \mathbb{G}_\mathrm{m} \hookrightarrow T$ as the **anti-diagonal subtorus**.

Remark 5.2.1. Condition (ii) demands that X admits a non-trivial torus action meaning that X cannot be proper.

Construction 5.2.2. If we are given a Calabi–Yau threefold X with an action of a one-dimensional torus

$$\alpha : \mathbb{G}_\mathrm{m} \times X \longrightarrow X$$

satisfying (ii) there is a natural way to extend this action to one satisfying Assumption $\mathtt{t}$. Indeed, by possibly passing to a cover of $\mathbb{G}_\mathrm{m}$ we may assume that the character which $\mathbb{G}_\mathrm{m}$ is acting on ω_X^{-1} with admits a square root q_+. We obtain an action of $T = \mathbb{G}_\mathrm{m}^2$ on $X \times \mathbb{A}^2$ by declaring that the first factor acts via

$$\begin{aligned} \mathbb{G}_\mathrm{m} \times (X \times \mathbb{A}^2) &\longrightarrow X \times \mathbb{A}^2 \\ \big(t, (x, y_1, y_2)\big) &\longmapsto (\alpha(t, x),\ \mathsf{q}_+^{-1}(t) y_1,\ \mathsf{q}_+^{-1}(t) y_2) \end{aligned}$$

and the second factor leaves X invariant and scales the two affine directions as in (5.4).

We now set the notation used throughout this chapter. Given the action of a torus T on $X \times \mathbb{A}^2$ satisfying Assumption $\mathtt{t}$ we write $\mathsf{q}_1, \mathsf{q}_2$ for the inverse of the characters which T acts on the two affine directions with respectively. We identify a character with its induced one dimensional representation and set

$$\epsilon_i \coloneqq c_1^T(\mathsf{q}_i)\ \in \mathrm{CH}_T^*(\mathrm{pt})$$

for $i \in \{1, 2\}$. By assumption (i)–(iii) the weights ϵ_1 and ϵ_2 are linearly independent. Thus, condition (iv) implies that the T-equivariant Chow cohomology of a point is generated by these elements:

$$R_T \coloneqq \mathrm{CH}_T^*(\mathrm{pt}) \cong \mathbb{Q}[\epsilon_1, \epsilon_2]\,. \tag{5.5}$$

Let us write Q_T for the localisation of R_T to the multiplicative subset of homogenous elements of strictly positive degree. Note that this way Q_T inherits the degree grading of R_T. Denote by $\widehat{R}_T$ and $\widehat{Q}_T$ the completion of these rings with respect to the filtration induced by the grading. Of course, under the isomorphism (5.5) we may identify $\widehat{R}_T \cong \mathbb{Q}[\![\epsilon_1, \epsilon_2]\!]$.

5.2.2 Definition of the refinement

As we observed in Remark 5.2.1 any X which admits a torus action on $X \times \mathbb{A}^2$ satisfying Assumption t must be non-proper. In a large number of interesting cases, however, X satisfies the following weaker properness condition with respect to some fixed effective curve class β.

Assumption p. The moduli stack $\overline{M}_{g,\beta}(X)$ is proper for all $g \geq 0$.

Examples of non-proper X satisfying the above condition for all effective curve classes include the resolved conifold and local del Pezzo surfaces.

We fix an effective curve class β in X such that Assumption p is satisfied. We propose that the free energy of the refined topological string is nothing but the generating series of T-equivariant Gromov–Witten invariants of $X \times \mathbb{A}^2$. Since the moduli stack of stable maps to this target is not proper we define the latter invariants via localisation with respect to the action of the anti-diagonal subtorus $\sigma_- : \mathbb{G}_\mathrm{m} \hookrightarrow T$. Indeed, by assumption the fixed locus

$$\overline{M}_{g,\beta}(X \times \mathbb{A}^2)^{\sigma_-} \subseteq \overline{M}_{g,\beta}(X)$$

is proper. Writing N^{virt} for the virtual normal bundle of $\overline{M}_{g,\beta}(X)$ in $\overline{M}_{g,\beta}(X \times \mathbb{A}^2)$ we find that

$$\int_{[\overline{M}_{g,\beta}(X)]^{\mathrm{virt}}_T} \frac{1}{e^T\big(N^{\mathrm{virt}}\big)}$$

is an element of the $(2g-2)$-graded piece of Q_T. Summing over all genera we hence obtain a well defined element in the completion of the ring.

Definition 5.2.3. We define the **refined Gromov–Witten generating series of** X as

$$\mathsf{GW}_\beta(X \times \mathbb{A}^2, T) := \sum_{g \geq 0} \int_{[\overline{M}_{g,\beta}(X)]^{\mathrm{virt}}_T} \frac{1}{e^T(N^{\mathrm{virt}})} \qquad \in \widehat{Q}_T\,.$$

We can be more precise regarding the shape of the above generating series. Note that as an element in equivariant K-theory the virtual normal bundle is given by

$$N^{\mathrm{virt}} = \big(\mathsf{q}_1 - \mathbb{E}_g^\vee \mathsf{q}_1\big) + \big(\mathsf{q}_2 - \mathbb{E}_g^\vee \mathsf{q}_2\big)$$

where $\mathbb{E}_g$ is the Hodge bundle on $\overline{M}_{g,\beta}(X)$. As usual we will denote its Chern classes by $\lambda_k := c_k(\mathbb{E}_g)$. Then

$$\frac{1}{e^T\big(\mathsf{q}_i - \mathbb{E}_g^\vee \mathsf{q}_i\big)} = \sum_{k \geq 0} \lambda_k (-1)^k \epsilon_i^{g-k-1} =: \Lambda(\epsilon_i)$$

and so we may write

$$\mathsf{GW}_\beta(X \times \mathbb{A}^2, T) = \sum_{g \geq 0} \int_{[\overline{M}_{g,\beta}(X)]^{\mathrm{virt}}_T} \Lambda(\epsilon_1)\,\Lambda(\epsilon_2)\,. \tag{5.6}$$

As an immediate consequence we obtain

Lemma 5.2.4. *Under Assumption* t *and* p *we have*

$$\mathsf{GW}_\beta(X \times \mathbb{A}^2, T) \in (\epsilon_1\epsilon_2)^{-1}\mathbb{Q}[\![\epsilon_1, \epsilon_2]\!]_{\mathrm{even}}$$

where the subscript $[\![\cdot]\!]_{\mathrm{even}}$ *indicates that the power series is invariant under* $(\epsilon_1, \epsilon_2) \mapsto (-\epsilon_1, -\epsilon_2)$. *Moreover, if the* T*-action is as in Construction 5.2.2 we have*

$$\mathsf{GW}_\beta(X \times \mathbb{A}^2, T) \in (\epsilon_1\epsilon_2)^{-1}\mathbb{Q}[\![(\epsilon_1 - \epsilon_2)^2, (\epsilon_1 + \epsilon_2)^2]\!]\,.$$

Proof. The product $\epsilon_1\epsilon_2\,\mathsf{GW}_\beta(X\times\mathbb{A}^2,T)$ is a formal power series in ϵ_1 and ϵ_2 by equation (5.6). As each term on the right-hand side sum of equation (5.6) is an evenly graded element of Q_T the first part of the lemma follows.

The second part of the lemma is due to the fact that $\mathsf{GW}_\beta(X\times\mathbb{A}^2,T)$ is symmetric under the exchange $(\epsilon_1,\epsilon_2)\mapsto(\epsilon_2,\epsilon_1)$ since $\Lambda(\epsilon_1)\Lambda(\epsilon_2)$ is and for actions such as the one in Construction 5.2.2 the torus T acts on X (and therefore on $\overline{M}_{g,\beta}(X)$) through the character $\mathsf{q}_+=(\mathsf{q}_1\mathsf{q}_2)^{1/2}$. Consequently, when localising $[\overline{M}_{g,\beta}(X)]_T^{\mathrm{virt}}$ we can only get a dependence on $(\epsilon_1+\epsilon_2)$. □

Remark 5.2.5. In case Assumption p is not satisfied one may still define $\mathsf{GW}_\beta(X\times\mathbb{A}^2,T)$ via localisation with respect to a cocharacter σ whose fixed locus $\overline{M}_{g,\beta}(X\times\mathbb{A}^2)^\sigma$ is proper. However, in this case the generating series may take a more complicated form than the one described in Lemma 5.2.4.

5.2.3 The unrefined limit

As a sanity check we should demonstrate that the refined Gromov–Witten generating series of X indeed specialises to the one encoding the Gromov–Witten invariants of X upon setting $\epsilon_1=-\epsilon_2=u$. We call this specialisation the **unrefined limit**. Note that more formally speaking this specialisation should be interpreted as the restriction to the anti-diagonal subtorus. Indeed, an inclusion of subgroups such as $\sigma_-:\mathbb{G}_\mathsf{m}\hookrightarrow T$ gives rise to a restriction map

$$\mathrm{res}_{T,\mathbb{G}_\mathsf{m}}:\mathrm{CH}_k^T(*)\longrightarrow\mathrm{CH}_k^{\mathbb{G}_\mathsf{m}}(*)$$

in equivariant homology and cohomology. This morphism is compatible with push-forward, formation of virtual fundamental classes and Chern classes. Let us write $\mathsf{q}_i|_{\mathbb{G}_\mathsf{m}}$ for the restriction of q_i to the anti-diagonal subtorus and set

$$u\coloneqq c_1^{\mathbb{G}_\mathsf{m}}(\mathsf{q}_1|_{\mathbb{G}_\mathsf{m}})\,.$$

Then since the anti-diagonal subtorus acts with opposite weight on the second affine direction the restriction of ϵ_2 is

$$c_1^{\mathbb{G}_\mathsf{m}}(\mathsf{q}_2|_{\mathbb{G}_\mathsf{m}})=-u\,.$$

Hence, viewed as a formal power series, the restriction

$$\mathrm{res}_{T,\mathbb{G}_\mathsf{m}}\mathsf{GW}_\beta(X\times\mathbb{A}^2,T)\in\widehat{Q}_{\mathbb{G}_\mathsf{m}}$$

is indeed obtained from $\mathsf{GW}_\beta(X\times\mathbb{A}^2,T)$ by simply substituting $\epsilon_1=-\epsilon_2=u$.

Proposition 5.2.6. *We have*

$$\mathsf{GW}_\beta(X\times\mathbb{A}^2,T)\Big|_{\substack{\epsilon_1=u\\ \epsilon_2=-u}}=\sum_{g\geq0}(-u^2)^{g-1}\int_{[\overline{M}_{g,\beta}(X)]^{\mathrm{virt}}}1\,.$$

The proposition is an immediate consequence of the following observation of Mumford.

Lemma 5.2.7 (Mumford's relation)**.** [140, *Section 5*] *We have*

$$\Lambda(u)\,\Lambda(-u)=(-u^2)^{\mathrm{rk}\,\mathbb{E}.-1}\,.$$

Proof of Proposition 5.2.6. Applying the restriction map to equation (5.6) gives

$$\begin{aligned}\mathrm{res}_{T,\mathbb{G}_\mathrm{m}} \mathsf{GW}_\beta(X\times\mathbb{A}^2, T) &= \sum_{g\geq 0}\int_{[\overline{M}_{g,\beta}(X)]^{\mathrm{virt}}_{\mathbb{G}_\mathrm{m}}} \Lambda(u)\,\Lambda(-u)\\ &= \sum_{g\geq 0}(-u^2)^{g-1}\int_{[\overline{M}_{g,\beta}(X)]^{\mathrm{virt}}_{\mathbb{G}_\mathrm{m}}} 1\end{aligned}$$

where to get the last line we used Mumford's relation. The claim then follows from the fact that the right-hand side integral gives an element in $\mathrm{CH}^0_{\mathbb{G}_\mathrm{m}}(\mathrm{pt}) \cong \mathbb{Q}$ and therefore agrees with its non-equivariant limit. □

Remark 5.2.8. The arguments which go into the above proof also explain the relevance of condition (ii) in Assumption $\mathtt{t}$. If this property is not satisfied, meaning that T fixes ω_X, then (i) implies that $\epsilon_1 = -\epsilon_2$ in R_T. Hence, by Mumford's relation one finds that

$$\mathsf{GW}_\beta(X\times\mathbb{A}^2, T) = \sum_{g\geq 0}(-\epsilon_1^2)^{g-1}\int_{[\overline{M}_{g,\beta}(X)]^{\mathrm{virt}}} 1\,.$$

So in order to obtain a non-trivial refinement we see that it is crucial to impose condition (ii).

5.2.4 Rigidity

Of course, our definition of the refined Gromov–Witten generating series a priori depends on the choice of T-action on $X\times\mathbb{A}^2$. Hence, the following expectation might come as a surprise.

Conjecture 5.2.9 (Rigidity)**.** *Suppose Assumption* $\mathtt{t}$ *and* $\mathtt{p}$ *are satisfied. Then as a formal Laurent series in* ϵ_1 *and* ϵ_2 *the generating series*

$$\mathsf{GW}_\beta(X\times\mathbb{A}^2, T)$$

is independent of the choice of T-action on $X\times\mathbb{A}^2$.

First evidence for Conjecture 5.2.9 readily comes from Proposition 5.2.6. Note here that the observation that the unrefined limit does not depend on the choice of T-action is indeed non-trivial.

Further evidence for the conjecture comes from direct low degree calculations. If X is toric, virtual localisation [70] provides an explicit algorithm to compute $\mathsf{GW}_\beta(X\times\mathbb{A}^2, T)$ from the knowledge of descendant integrals on the moduli space of curves as for instance reviewed in [120]. The latter intersection numbers can be computed with the help of `admcycles` [53] or its predecessors [56, 163]. This method allowed us to compute the refined Gromov–Witten generating series of the resolved conifold, local $\mathbb{P}^2$ and local $\mathbb{P}^1\times\mathbb{P}^1$ to (bi)degree three up to contributions from genus five. We can do so for arbitrary T-actions and find that Conjecture 5.2.9 holds in all considered cases.

Remark 5.2.10. There is an alternative formulation of the above conjecture: Let $\hat{T}$ be the maximal subtorus of the group of automorphisms of $X\times\mathbb{A}^2$ which fixes $\omega_{X\times\mathbb{A}^2}$. Let us still denote by ϵ_1, ϵ_2 the weights of the $\hat{T}$-action on the two affine directions. Similar to Definition 5.2.3 one may define

$$\mathsf{GW}_\beta(X\times\mathbb{A}^2, \hat{T}) \coloneqq \sum_{g\geq 0}\int_{[\overline{M}_{g,\beta}(X)]^{\mathrm{virt}}_{\hat{T}}} \Lambda(\epsilon_1)\,\Lambda(\epsilon_2) \qquad \in \widehat{Q}_{\hat{T}}\,.$$

Then the statement of Conjecture 5.2.9 is equivalent to saying that $\mathsf{GW}_\beta(X\times\mathbb{A}^2, \hat{T})$ lies in the subset

$$(\epsilon_1\epsilon_2)^{-1}\mathbb{Q}[\![\epsilon_1,\epsilon_2]\!] \quad \subset \widehat{Q}_{\hat{T}}\,.$$

Remark 5.2.11. We already discussed the importance of condition (ii) and (iii) of Assumption $\mathtt{t}$ in Section 5.2.3. In explicit localisation calculations we moreover experimentally observe that condition (i) seems to be crucial for the statement of Conjecture 5.2.9 to hold. The relevance of Assumption $\mathtt{p}$ for the conjecture will be illustrated with an example in Section 5.3.2.

5.3 Warm-up calculations: local $\mathbb{P}^1$ geometries

We now shift from a discussion of general features to concrete case studies. In this section we will explicitly compute the refined Gromov–Witten generating series of two local $\mathbb{P}^1$ geometries.

5.3.1 The resolved conifold

We consider

$$\begin{array}{ccc} X \times \mathbb{A}^2 & = & \mathrm{Tot}\left(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}\right) \\ & & \downarrow \\ & & \mathbb{P}^1 \end{array}$$

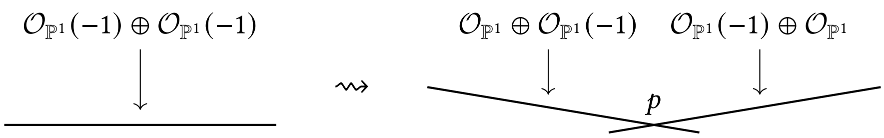
together with an arbitrary action of a two-dimensional torus T satisfying Assumption $\mathtt{t}$. We continue to denote the T-weights on the two trivial directions by ϵ_1 and ϵ_2. Additionally, let us write u_1 and u_2 for the weights of the T-action on the two $\mathcal{O}_{\mathbb{P}^1}(-1)$-fibres over the torus fixed point $0 \in \mathbb{P}^1$. If these weights take a particular form we are able to give a closed form solution for the refined Gromov–Witten generating series.

Proposition 5.3.1. *Suppose $u_i = -\epsilon_j$ for some $i, j \in \{1, 2\}$. Then for all $d > 0$ we have*

$$\mathsf{GW}_d(X \times \mathbb{A}^2, T) = \frac{1}{d}\left(2\sinh\frac{d\epsilon_1}{2}\right)^{-1}\left(2\sinh\frac{d\epsilon_2}{2}\right)^{-1}.$$

Remark 5.3.2. Note that Assumption $\mathtt{p}$ is satisfied in every degree $d > 0$ as stable maps to X factor through the zero section $\mathbb{P}^1 \hookrightarrow X$. Thus, we especially expect Conjecture 5.2.9 to hold for this geometry. In other words, we expect the above stated formula to hold without the assumption that $u_i = -\epsilon_j$ for some $i, j \in \{1, 2\}$.

Proof of Proposition 5.3.1. We will only discuss the case $u_1 = -\epsilon_1$ here since all other cases can be treated similarly. We observe that under this assumption condition (i) fixes the T-weight on the second $\mathcal{O}_{\mathbb{P}^1}(-1)$-fibre over the fixed point $\infty \in \mathbb{P}^1$ to be $-\epsilon_2$. We degenerate the base $\mathbb{P}^1$ into a union of two $\mathbb{P}^1$s in a way respecting the T-action. We write p for the T-fixed point along which the $\mathbb{P}^1$s are glued and assume that the two negative line bundles extend to the special fibre of the degeneration as displayed below



The degeneration formula [114] expresses $\mathsf{GW}_d(X \times \mathbb{A}^2, T)$ as a convolution of the relative Gromov–

Witten invariants of the irreducible components of the special fibre:

$$\begin{aligned}\mathsf{GW}_d(X \times \mathbb{A}^2, T) = \sum_{\Gamma_1,\Gamma_2} &\frac{\prod_i d_i}{|\operatorname{Aut}(\Gamma_1, \Gamma_2)|}(\epsilon_1\epsilon_2)^{2|\mathbf{d}|} \\ &\times \int_{[\overline{M}_{\Gamma_1}(\mathbb{P}^1|p)]^{\mathrm{virt}}_T} \Lambda(-\epsilon_1)\, e^T\big(\mathbf{R}^1\pi_* f^*\mathcal{O}_{\mathbb{P}^1}(-1)\big)\, \Lambda(\epsilon_1)\, \Lambda(\epsilon_2) \\ &\times \int_{[\overline{M}_{\Gamma_2}(\mathbb{P}^1|p)]^{\mathrm{virt}}_T} e^T\big(\mathbf{R}^1\pi_* f^*\mathcal{O}_{\mathbb{P}^1}(-1)\big)\, \Lambda(-\epsilon_2)\, \Lambda(\epsilon_1)\, \Lambda(\epsilon_2)\,.\end{aligned}$$

Here, $\overline{M}_{\Gamma_j}(\mathbb{P}^1|p)$ is Li's moduli stack of type Γ_j stable maps to expanded degenerations of $(\mathbb{P}^1|p)$ [115]. The sum is over all possible types (Γ_1, Γ_2) a stable map can split over the degenerated target and $\mathbf{d}$ is a partition of d encoding the ramification profile over p. The factor $(\epsilon_1\epsilon_2)^{2|\mathbf{d}|}$ is due to the normalisation exact sequence at the contact markings. Now by Mumford's relation (Lemma 5.2.7) we have

$$\Lambda(-\epsilon_i)\, \Lambda(\epsilon_i) = (-\epsilon_i^2)^{-\chi_j/2}$$

where χ_j is the Euler characteristic of the domain curves of relative stable maps of type Γ_j. (This is where we crucially use the assumption that $u_1 = -\epsilon_1$.) The expression hence simplifies to

$$\begin{aligned}\mathsf{GW}_d(X \times \mathbb{A}^2, T) = \sum_{\Gamma_1,\Gamma_2} \frac{\prod_i d_i}{|\operatorname{Aut}(\Gamma_1, \Gamma_2)|}&(-1)^{-\chi_1/2}\, \epsilon_1^{2|\mathbf{d}|-\chi_1} \int_{[\overline{M}_{\Gamma_1}(\mathbb{P}^1|p)]^{\mathrm{virt}}_T} e^T\big(\mathbf{R}^1\pi_* f^*\mathcal{O}_{\mathbb{P}^1}(-1)\big)\, \Lambda(\epsilon_2) \\ \times&(-1)^{-\chi_2/2}\, \epsilon_2^{2|\mathbf{d}|-\chi_2} \int_{[\overline{M}_{\Gamma_2}(\mathbb{P}^1|p)]^{\mathrm{virt}}_T} e^T\big(\mathbf{R}^1\pi_* f^*\mathcal{O}_{\mathbb{P}^1}(-1)\big)\, \Lambda(\epsilon_1)\,.\end{aligned}$$

A quick dimension count as in the proof of [37, Lemma 6.3] shows that almost all terms in the above sum vanish except the ones with a single contact marking, ie. $\mathbf{d} = (d)$. Therefore, the expression further simplifies to

$$\begin{aligned}\mathsf{GW}_d(X \times \mathbb{A}^2, T) = d &\sum_{g_1 \geq 0} (-1)^{g_1-1}\, \epsilon_1^{2g_1} \int_{[\overline{M}_{g_1,(d),d}(\mathbb{P}^1|p)]^{\mathrm{virt}}_T} e^T\big(\mathbf{R}^1\pi_* f^*\mathcal{O}_{\mathbb{P}^1}(-1)\big)\, \Lambda(\epsilon_2) \\ \times &\sum_{g_2 \geq 0} (-1)^{g_2-1}\, \epsilon_2^{2g_2} \int_{[\overline{M}_{g_2,(d),d}(\mathbb{P}^1|p)]^{\mathrm{virt}}_T} e^T\big(\mathbf{R}^1\pi_* f^*\mathcal{O}_{\mathbb{P}^1}(-1)\big)\, \Lambda(\epsilon_1)\,.\end{aligned}$$

The Gromov–Witten invariants appearing on the right-hand side have already been computed by Bryan and Pandharipande in [37, Lemma 6.3] and read

$$\int_{[\overline{M}_{g,(d),d}(\mathbb{P}^1|p)]^{\mathrm{virt}}_T} e^T\big(\mathbf{R}^1\pi_* f^*\mathcal{O}_{\mathbb{P}^1}(-1)\big)\, \Lambda(\epsilon_i) = \frac{(-1)^{d-1}}{d\epsilon_i}\,\big[u^{2g-1}\big]\left(2\sin\frac{du}{2}\right)^{-1}.$$

We insert this into our last expression for $\mathsf{GW}_d(X \times \mathbb{A}^2, T)$ to arrive at the formula stated in the proposition. □

5.3.2 A shifted example

The ideas which went into the proof of Proposition 5.3.1 can readily be used to compute the refined Gromov–Witten generating series of the threefold

$$X = \operatorname{Tot} \mathcal{O}_{\mathbb{P}_1}(-2) \oplus \mathcal{O}_{\mathbb{P}_1}(0)$$

for certain torus actions. Of course the moduli space of stable maps to X is non-proper and so we need to define $\mathsf{GW}_d(X \times \mathbb{A}^2, T)$ via localisation with respect to an appropriate cocharacter. We pick some T-action satisfying Assumption `t` and write ϵ_0 for the weight of the T-action on the affine direction of X.

Proposition 5.3.3. *If the T-action is so that $\epsilon_0 = -\epsilon_i$ for some $i \in \{1,2\}$ then*

$$\mathsf{GW}_d(X \times \mathbb{A}^2, T) = \frac{1}{d}\left(2\sinh\frac{d\epsilon_i}{2}\right)^{-2}.$$

Moreover, $\mathsf{GW}_d(X \times \mathbb{A}^2, T)$ *vanishes if* $\epsilon_0 + \epsilon_1 + \epsilon_2 = 0$.

Remark 5.3.4. We observe that for this particular target X the refined Gromov–Witten generating series does depend on the choice of torus action. This demonstrates the importance of Assumption p in our rigidity conjecture (Conjecture 5.2.9).

Proof of Proposition 5.3.3. The first part of the Proposition 5.3.3 can be proven with the same arguments which went into the proof of Proposition 5.3.1 and is therefore left as an exercise to the reader.

The second part follows from the observation that the assumption $\epsilon_0+\epsilon_1+\epsilon_2 = 0$ and together with (i) implies that the induced T-action on the holomorphic two-form of $\operatorname{Tot}\mathcal{O}_{\mathbb{P}^1}(-2)$ is trivial. Hence, the obstruction bundle has a trivial factor implying the vanishing of $[\overline{M}_{g,d}(\operatorname{Tot}\mathcal{O}_{\mathbb{P}^1}(-2))]^{\mathrm{virt}}_T$. □

Let us formulate a general expectation for the refined Gromov–Witten generating series of X. For this, as in Remark 5.2.10, we denote by $\widehat{T} \cong \mathbb{G}_{\mathsf{m}}^4$ the maximal subtorus of the group of automorphisms of $X\times\mathbb{A}^2$ which fix the canonical bundle. We continue to write ϵ_0, ϵ_1 and ϵ_2 for the respective weights of the $\widehat{T}$-action on the three affine directions of $\operatorname{Tot}\mathcal{O}_{\mathbb{P}^1}(-2) \times \mathbb{A}^3$.

Conjecture 5.3.5. *For all $d > 0$ we have*

$$\mathsf{GW}_d(X \times \mathbb{A}^2, \widehat{T}) = -\frac{1}{d}\frac{2\sinh\frac{d(\epsilon_0+\epsilon_1+\epsilon_2)}{2}}{\prod_{i=0}^{2} 2\sinh\frac{d\epsilon_i}{2}}$$

The conjectured formula is readily seen to be compatible with the special cases computed in Proposition 5.3.3. We also verified the formula in a computer calculation for $d \leq 3$ up to contributions from genus five.

5.4 The Nekrasov–Shatashvili limit for local surfaces

We continue the discussion of explicit examples and consider local surfaces in this section. For these geometries the scaling action on the fibre direction provides us with a natural torus action on the fivefold via Construction 5.2.2. The main result of this section is an identification of the so called Nekrasov–Shatashvili limit ($\epsilon_2 = 0$) with the Gromov–Witten theory of the surface relative a smooth anticanonical divisor. Later in Section 5.5 we will employ this correspondence in order to provide evidence for a refined BPS integrality conjecture.

5.4.1 Statement of the correspondence

5.4.1.1 Fivefold invariants. Let S be a smooth projective surface and D a smooth curve in S. We consider the action of $T = \mathbb{G}_{\mathsf{m}}^2$ on

$$\begin{array}{ccc} Z & = & \operatorname{Tot}\left(\mathcal{O}_S(-D) \oplus \mathcal{O}_S \oplus \mathcal{O}_S\right) \\ & & \downarrow \\ & & S \end{array} \tag{5.7}$$

leaving the base invariant and scaling the fibres with weight $(1,1)$, $(-1,0)$ and $(0,-1)$ respectively. We remark that we do not necessarily assume Z to be Calabi–Yau (or in other words D to be anti-canonical) in what follows.

Under the assumption that $D^2 \geq 0$, all stable maps to $\operatorname{Tot} \mathcal{O}_S(-D)$ of class β intersecting D positively factor through the zero section $S \hookrightarrow \operatorname{Tot} \mathcal{O}_S(-D)$. Hence

$$\overline{M}_{g,n,\beta}(\mathcal{O}_S(-D)) = \overline{M}_{g,n,\beta}(S)$$

which especially means that Assumption p holds for all curve classes β with $D \cdot \beta > 0$. Note that the virtual fundamental classes however differ by an insertion

$$[\overline{M}_{g,n,\beta}(\mathcal{O}_S(-D))]^{\mathrm{virt}}_T = e^T\big(\mathbf{R}^1\pi_* f^* \mathcal{O}_S(-D)\big) \cap [\overline{M}_{g,n,\beta}(S)]^{\mathrm{virt}}\,.$$

We introduce the notation

$$[\overline{M}_{g,n,\beta}(Z)]^{\mathrm{virt}_{\sigma_-}}_T \coloneqq \Lambda(\epsilon_1)\,\Lambda(\epsilon_2)\, e^T\big(\mathbf{R}^1\pi_* f^* \mathcal{O}_S(-D)\big) \cap [\overline{M}_{g,n,\beta}(S)]^{\mathrm{virt}}$$

for the virtual fundamental class of stable maps to the extended target Z localised at the cocharacter $\sigma_- : \mathbb{G}_\mathsf{m} \hookrightarrow T,\ t \mapsto (t, t^{-1})$. In this section will be concerned with a detailed study of the Nekrasov–Shatahvili limit of this cycle. By this we mean the restriction to the inclusion of first factor

$$\mathbb{G}_\mathsf{m} \hookrightarrow \mathbb{G}_\mathsf{m}^2 = T\,.$$

Since the T-action on $\overline{M}_{g,\beta}(S)$ is trivial, the restriction morphism essentially sets ϵ_2 equal to zero. Writing $\hbar$ for the restriction of ϵ_1, the object of study in this section hence is

$$\epsilon_2\,[\overline{M}_{g,n,\beta}(Z)]^{\mathrm{virt}_{\sigma_-}}_T \Big|_{\substack{\epsilon_1=\hbar\\ \epsilon_2=0}} = (-1)^g \lambda_g\, \Lambda(\hbar)\, e^{\mathbb{G}_\mathsf{m}}\big(\mathbf{R}^1\pi_* f^* \mathcal{O}_S(-D)\big) \cap [\overline{M}_{g,n,\beta}(S)]^{\mathrm{virt}}\,.$$

5.4.1.2 Relative invariants. Let $\mathbf{d} = (d_1, \ldots, d_m)$ be an ordered partition of $D \cdot \beta > 0$ where we assume that $d_i > 0$ for all $i \in \{1, \ldots, m\}$. We write

$$\overline{M}_{g,n,\mathbf{d},\beta}(S\,|\,D)$$

for Kim's moduli stack of genus g, class β stable logarithmic maps to expanded degenerations of $(S\,|\,D)$ with n interior markings and m markings with tangency profile $\mathbf{d}$ along D [100]. This moduli stack comes with a forgetful morphism

$$\phi : \overline{M}_{g,n,\mathbf{d},\beta}(S|D) \longrightarrow \overline{M}_{g,n,\beta}(S)\,.$$

If D is anticanonical the virtual dimension of the moduli problem is $g + n + m - 1$. So in the case $n = 0$, $m = 1$ we may cap with the top Chern class of the Hodge bundle and form the following generating series in $\hbar$:

$$\mathsf{GW}_\beta(S\,|\,D) \coloneqq \frac{(-1)^{D\cdot\beta+1}}{D\cdot\beta} \sum_{g\geq 0} \hbar^{2g-1} \int_{[\overline{M}_{g,0,(D\cdot\beta),\beta}(S\,|\,D)]^{\mathrm{virt}}} \lambda_g\,.$$

5.4.1.3 The correspondence. The main result of this section is the following cycle valued identity relating the Nekrasov–Shatashvili limit of equivariant Gromov–Witten theory of $\operatorname{Tot} \mathcal{O}_S(-D) \times \mathbb{A}^2$ with the relative Gromov–Witten theory of $(S\,|\,D)$.

Theorem 5.4.1. *Suppose D is a smooth genus one curve with $D^2 \geq 0$ and β is an effective curve class satisfying $D \cdot \beta > 0$. Then for all $g, n \geq 0$ we have*

$$\epsilon_2 [\overline{M}_{g,n,\beta}(Z)]_T^{\mathrm{virt}_{\sigma_-}} \Big|_{\substack{\epsilon_1=\hbar \\ \epsilon_2=0}} = \sum_{m>0} \hbar^{2g+m-2} \sum_{\substack{d_1,\ldots,d_m>0 \\ \sum_i d_i = D\cdot\beta}} \frac{(-1)^{D\cdot\beta-1}}{m! \prod_{i=1}^m d_i} \phi_*\big(\lambda_g \cap [\overline{M}_{g,n,\mathbf{d},\beta}(S|D)]^{\mathrm{virt}}\big)\,. \tag{5.8}$$

Remark 5.4.2. Specialising the above theorem to $g = 0$ and taking the non-equivariant limit we reproduce a special instance of the log-local correspondence of van Garrel, Graber and Ruddat [64]. Contrary, the log-local correspondence of Bousseau–Fan–Guo–Wu [21, Theorem 1.1] generalising the one of van Garrel–Graber–Ruddat to all genus is in some sense orthogonal to our correspondence as it connects the unrefined limit with the relative Gromov–Witten theory of $(S|D)$. Hence, combining [21, Theorem 1.2] with Theorem 5.4.1 we obtain a correspondence relating the unrefined and the Nekrasov–Shatashvili limit of local surfaces. An instance where a similar constraint relating these two limits appears the physics literature is given by a specialisation of the blow-up equations for the refined topological string studied in [90]. It seems interesting to compare the two relations.

Specifying to the case where D is an anticanonical curve in S and pushing identity (5.8) forward to the point we immediately obtain

Corollary 5.4.3. *Suppose S supports a smooth anticanonical curve D with $D^2 \geq 0$. Then for all curve classes β with $D \cdot \beta > 0$ we have*

$$\epsilon_2\, \mathsf{GW}_\beta(K_S \times \mathbb{A}^2, T) \Big|_{\substack{\epsilon_1=\hbar \\ \epsilon_2=0}} = \mathsf{GW}_\beta(S|D)\,.$$

The rest of this section is dedicated to the proof of Theorem 5.4.1.

5.4.2 Degeneration

Our proof follows the same approach as [64] and uses a degeneration to the normal cone argument. Since this technique has become a well-established approach (see for instance [17, 21, 148, 160]) we economise on details.

Given a surface S together with a smooth curve $D \subset S$, we consider the degeneration to the normal cone of D in S

$$\mathcal{S} = \mathrm{Bl}_{D\times\{0\}}(S \times \mathbb{A}^1) \to \mathbb{A}^1\,.$$

This is a family over $\mathbb{A}^1$ with general fibre S and special fibre $\mathcal{S}_0$ obtained by gluing S along D with $P = \mathbb{P}_D(\mathcal{O}_D \oplus N_D S)$ along the zero section which we denote by D_0.

Further, denote by $\mathcal{D}$ the proper transform of $D \times \mathbb{A}^1$. Its intersection with a general fibre is $D \hookrightarrow S$ while its intersection with the special fibre is the infinity section D_∞ of P. We will write $\mathcal{Z}$ for the total space of

$$\mathcal{O}_{\mathcal{S}}(-\mathcal{D}) \oplus \mathcal{O}_{\mathcal{S}} \oplus \mathcal{O}_{\mathcal{S}}$$

and consider the same fibre wise T-action on $\mathcal{Z}$ as in (5.7).

5.4.3 Decomposition

Composition with the blowup morphism $\mathcal{S}_0 \to S$ contracting the component P gives a pushforward morphism

$$\rho : \overline{M}_{g,n,\beta}(\mathcal{S}_0) \longrightarrow \overline{M}_{g,n,\beta}(S)$$

under which

$$[\overline{M}_{g,n,\beta}(Z)]^{\mathrm{virt}_{\sigma_-}} = \rho_*[\overline{M}_{g,n,\beta}(\mathcal{Z}_0)]^{\mathrm{virt}_{\sigma_-}}\,. \tag{5.9}$$

The degeneration formula for stable logarithmic maps [99] decomposes the right-hand side cycle into contributions labelled by certain decorated graphs Γ which we call splitting types. This is the datum of a bipartite graph with edges $E(\Gamma)$, legs $L(\Gamma)$ and vertices $V(\Gamma) = V_S(\Gamma) \sqcup V_P(\Gamma)$ split into S- and P-vertices. Moreover, each edge e is decorated with a contact order $d_e \in \mathbb{Z}_{>0}$ and to each vertex v we assign a genus $g_v \in \mathbb{Z}_{\geq 0}$ and a curve class β_v in either S or P. Lastly, the datum of a splitting type also includes a bijection $L(\Gamma) \cong \{1,\ldots,n\}$ and for convenience also a labelling $E(\Gamma) \cong \{1,\ldots,m\}$. This datum is subject to the usual compatibility conditions with the discrete datum (g,β).

Following [99], for each Γ as above there is an associated moduli stack $\overline{M}_\Gamma$ parametrising stable logarithmic maps to $\mathcal{S}_0$ of type Γ. It admits a finite morphism $\iota : \overline{M}_\Gamma \longrightarrow \overline{M}_{g,n,\beta}(S)$ and an étale morphism ν to the fibre product

$$\begin{tikzcd}
\overline{M}_\Gamma \arrow[r, "\nu"] & \bigodot_v \overline{M}_v \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \prod_v \overline{M}_v \arrow[d] \\
& \prod_e D_0 \arrow[r, "\Delta"] & \prod_v \prod_{e \ni v} D_0\,.
\end{tikzcd}$$

By the degeneration formula for stable logarithmic maps [99, Theorem 1.5] we have

$$[\overline{M}_{g,n,\beta}(\mathcal{S}_0)]^{\mathrm{virt}} = \sum_\Gamma \frac{\mathrm{lcm}(d_e)}{m!} \iota_* \nu^* \Delta^! \textstyle\prod_v [\overline{M}_v]^{\mathrm{virt}} \tag{5.10}$$

where the sum is over all splitting types. Now observe that there is a morphism $\theta : \bigodot_v \overline{M}_v \to \overline{M}_{g,n,\beta}(S)$ which composes a stable map with the blowup morphism and glues the domain curve along edge markings. This morphism makes the diagram

$$\begin{tikzcd}
\overline{M}_\Gamma \arrow[r, "\nu"] \arrow[d, "\iota"] & \bigodot_v \overline{M}_v \arrow[d, "\theta"] \\
\overline{M}_{g,n,\beta}(\mathcal{S}_0) \arrow[r, "\rho"] & \overline{M}_{g,n,\beta}(S)
\end{tikzcd}$$

commute. In combination with the degeneration formula (5.10) we obtain

$$\rho_*[\overline{M}_{g,n,\beta}(\mathcal{S}_0)]^{\mathrm{virt}} = \sum_\Gamma \frac{\prod_e d_e}{m!} \theta_* \Delta^! \textstyle\prod_v [\overline{M}_v]^{\mathrm{virt}}$$

using that the degree of ν is $(\prod_e d_e)/\mathrm{lcm}(d_e)$ by [99, Equation (1.4)].

Now the cycles $[\overline{M}_{g,n,\beta}(\mathcal{S}_0)]^{\mathrm{virt}}$ and $[\overline{M}_{g,n,\beta}(\mathcal{Z}_0)]^{\mathrm{virt}_{\sigma_-}}$ differ by an insertion

$$\Lambda(\epsilon_1)\,\Lambda(\epsilon_2)\, e^T\big(\mathbf{R}^1\pi_* f^*\mathcal{O}_{\mathcal{S}_0}(-D_\infty)\big)\,.$$

Pulling this insertion back to the product $\prod_v \overline{M}_v$ yields

$$[\overline{M}_{g,n,\beta}(Z)]^{\mathrm{virt}_{\sigma_-}} = \sum_\Gamma \big(\epsilon_1\epsilon_2(-\epsilon_1-\epsilon_2)\big)^{|E(\Gamma)|} \frac{\prod_e d_e}{m!} \theta_* \Delta^! \big(\textstyle\prod_v C_v \cap [\overline{M}_v]^{\mathrm{virt}}\big) \tag{5.11}$$

where we introduce the notation

$$C_v \coloneqq \begin{cases} \Lambda(\epsilon_1)\,\Lambda(\epsilon_2)\,\Lambda(-\epsilon_1-\epsilon_2) & \text{if } v \text{ is an } S\text{-vertex,} \\ \Lambda(\epsilon_1)\,\Lambda(\epsilon_2)\,e^T\big(\mathbf{R}^1\pi_* f^*\mathcal{O}_{S_0}(-D_\infty)\big) & \text{if } v \text{ is a } P\text{-vertex.} \end{cases}$$

We remark that the factor $\big(\epsilon_1\epsilon_2(-\epsilon_1-\epsilon_2)\big)^{|E(\Gamma)|}$ is due to the normalisation exact sequence coming from gluing along the edges of Γ.

From the description of the insertions C_v we observe that the contribution of an S-vertex is polynomial in ϵ_1, ϵ_2 up to a factor $(\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2))^{-1}$. Similarly, the contribution of a P-vertex is polynomial in the equivariant parameters up to a factor $(\epsilon_1\epsilon_2)^{-1}$. This tells us that each term in the sum (5.11) is a cycle valued polynomial in ϵ_1, ϵ_2 times an overall factor

$$\big(\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2)\big)^{|E(\Gamma)|}\,\big(\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2)\big)^{-|V_S(\Gamma)|}\,\big(\epsilon_1\epsilon_2\big)^{-|V_P(\Gamma)|} = (\epsilon_1\epsilon_2)^{h_1(\Gamma)-1}(\epsilon_1+\epsilon_2)^{|E(\Gamma)|-|V_S(\Gamma)|}\,.$$

As a consequence only such splitting types Γ can contribute non-trivially to

$$\epsilon_2[\overline{M}_{g,n,\beta}(Z)]^{\mathrm{virt}_{\sigma_-}}\Big|_{\substack{\epsilon_1=\hbar\\ \epsilon_2=0}}$$

for which $h_1(\Gamma)=0$ or in other words the graph underlying Γ must be a tree. Moreover, in the limit $\epsilon_2=0$ the insertions simplify to

$$C_v^{\mathrm{NS}} \coloneqq \begin{cases} -\hbar^{2g_v-2}\,\lambda_{g_v} & \text{if } v \text{ is an } S\text{-vertex,} \\ (-1)^{g_v}\lambda_{g_v}\,\Lambda(\hbar)\,e^{\mathbb{G}_{\mathrm{m}}}\big(\mathbf{R}^1\pi_* f^*\mathcal{O}_{S_0}(-D_\infty)\big) & \text{if } v \text{ is a } P\text{-vertex} \end{cases}$$

where in the top line we used Lemma 5.2.7. We arrive at

$$\epsilon_2[\overline{M}_{g,n,\beta}(Z)]^{\mathrm{virt}_{\sigma_-}}\Big|_{\substack{\epsilon_1=\hbar\\ \epsilon_2=0}} = \sum_{\text{tree type }\Gamma}\big(-\hbar^2\big)^{|E(\Gamma)|}\frac{\prod_e d_e}{m!}\theta_*\Delta^!\big(\textstyle\prod_v C_v^{\mathrm{NS}}\cap[\overline{M}_v]^{\mathrm{virt}}\big)\,. \tag{5.12}$$

5.4.4 Vanishing

In this section we will prove that most splitting types contribute trivially to the sum (5.12).

Proposition 5.4.4. *The cycle $\theta_*\Delta^!\big(\prod_v C_v^{\mathrm{NS}}\cap[\overline{M}_v]^{\mathrm{virt}}\big)$ associated to a splitting type Γ is non-zero only if*

- *the graph underlying Γ is of star shape with a unique S-vertex $\tilde{v}$;*
- *$\tilde{v}$ carries all marking legs and is decorated with $g_{\tilde{v}}=g$ and $\beta_{\tilde{v}}=\beta$;*
- *all P-vertices v are decorated with $g_v=0$ and β_v is the multiple of fibre class in P.*

Proof. Combine Lemma 5.4.5 and 5.4.6 below. □

5.4.4.1 First reduction: confined curve class and genus label. Let Γ be a splitting type whose underlying graph is a tree. We will constrain its shape in several steps. First we will establish the last condition in Proposition 5.4.4.

Let $v\in V(\Gamma)$ be a P-vertex. The projection $p: P\longrightarrow D_0$ induces a pushforward morphism

$$\overline{M}_v \longrightarrow \overline{M}_{g_v,n_v\cup m_v,p_*\beta_v}(D_0)$$

where $n_v \subseteq L(\Gamma)$ denotes the set of marking legs and $m_v \subseteq E(\Gamma)$ the set of edges adjacent to v. Here of course we need to assume that the right-hand side moduli stack is non-empty. Following [64] we factor the morphism through the fibre product

$$\begin{tikzcd} \overline{M}_v \arrow[r, "u"] \arrow[dr] & \widetilde{M}_v \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_{g_v, n_v \cup m_v, p_*\beta_v}(D_0) \arrow[d] \\ & \mathfrak{M}^{\log}_{g_v, n_v \cup m_v, H_2(P,\mathbb{Z})^+} \arrow[r, "\nu"] & \mathfrak{M}_{g_v, n_v \cup m_v, H_2(D_0,\mathbb{Z})^+} \end{tikzcd} \tag{5.13}$$

where the moduli stacks in the bottom row are the Artin stacks parametrising prestable (log) curves with irreducible components labelled by effective curve classes. Pulling the obstruction theory of $\overline{M}_{g_v, n_v \cup m_v, p_*\beta_v}(D_0)$ back to $\widetilde{M}_v$ we find

$$[\widetilde{M}_v]^{\text{virt}} = \nu^! [\overline{M}_{g_v, n_v \cup m_v, p_*\beta_v}(D_0)]^{\text{virt}}$$

as proven in [64, Section 4]. Moreover, the short exact sequence

$$0 \longrightarrow \mathcal{T}^{\log}_{P(\log D_0)/D_0} \longrightarrow \mathcal{T}^{\log}_{P(\log D_0)} \longrightarrow \mathcal{T}_{D_0} \longrightarrow 0$$

induces a compatible triple for the left-hand side commuting triangle in (5.13). Hence, by [130, Corollary 4.9] we have

$$[\overline{M}_v]^{\text{virt}} = u^! [\widetilde{M}_v]^{\text{virt}} = u^! \nu^! [\overline{M}_{g_v, n_v \cup m_v, p_*\beta_v}(D_0)]^{\text{virt}} . \tag{5.14}$$

We are now equipped to prove the following vanishing.

Lemma 5.4.5. *Suppose D is a genus one curve. Then for every P-vertex v with $p_*\beta_v \neq 0$ or $g_v > 0$ we have*

$$\lambda_{g_v} \cap [\overline{M}_v]^{\text{virt}} = 0 .$$

Proof. Suppose we have $p_*\beta_v \neq 0$. In this case $\overline{M}_{g_v, n_v \cup m_v, p_*\beta_v}(D_0)$ is non-empty and so (5.14) holds. Moreover, since lambda classes are preserved under stabilisation we have

$$\lambda_{g_v} \cap [\overline{M}_v]^{\text{virt}} = u^! \nu^! \big(\lambda_{g_v} \cap [\overline{M}_{g_v, n_v \cup m_v, p_*\beta_v}(D_0)]^{\text{virt}}\big) . \tag{5.15}$$

It therefore suffices to demonstrate the vanishing of the right-hand side cycle. Observe that since we assume that D_0 to be a genus one curve, $D_0 \times \mathbb{A}^1$ is a surface with trivial canonical bundle. Consequently, the virtual fundamental class $[\overline{M}_{g_v, n_v \cup m_v, p_*\beta_v}(D_0 \times \mathbb{A}^1)]^{\text{virt}}$ vanishes. Writing

$$\iota_0 : \overline{M}_{g_v, n_v \cup m_v, p_*\beta_v}(D_0) \hookrightarrow \overline{M}_{g_v, n_v \cup m_v, p_*\beta_v}(D_0 \times \mathbb{A}^1)$$

for the inclusion of the zero section, a comparison of obstruction theories yields

$$\lambda_{g_v} \cap [\overline{M}_{g_v, n_v \cup m_v, p_*\beta_v}(D_0)]^{\text{virt}} = (-1)^{g_v} \iota_0^! [\overline{M}_{g_v, n_v \cup m_v, p_*\beta_v}(D_0 \times \mathbb{A}^1)]^{\text{virt}} = 0$$

which proves the first part of the lemma.

Suppose now that $p_*\beta_v = 0$ and $g_v > 0$. We again use relation (5.15) where now

$$[\overline{M}_{g_v, n_v \cup m_v, 0}(D_0)]^{\text{virt}} = (-1)^{g_v} \lambda_{g_v} \cap [D_0 \times \overline{M}_{g_v, n_v \cup m_v}]$$

since the tangent bundle of D_0 is trivial. The statement of the lemma then follows from the fact that $\lambda_{g_v}^2 = 0$ whenever $g_v > 0$ which can be identified as the leading order vanishing in Lemma 5.2.7. $\square$

5.4.4.2 Second reduction: confined markings. We continue by assuming that all P-vertices of Γ satisfy property three stated in Proposition 5.4.4. In this section we will show that all P-vertices carry at most one marking.

Let v be a P-vertex of Γ. The image of a stable map parametrised by $\overline{M}_v$ gets contracted to a point under composition with the projection $P \longrightarrow D_0$ since we assume that β_v is a multiple of a fibre class. The morphism θ, which we remember composes with the blowup morphism and glues the domain curves, as a consequence factors as follows:

$$\begin{tikzcd}
\overline{M}_{g,n,\beta}(S) & \bigodot_v \overline{M}_v \arrow[l, "\theta"'] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \prod_{v\in V_P(\Gamma)} \overline{M}_v \times \prod_{\tilde{v}\in V_S(\Gamma)} \overline{M}_{\tilde{v}} \arrow[d, "\Pi_v \psi_v \times \mathrm{Id}"] \\
& \overline{N}_\Gamma \arrow[ul, "\phi"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \prod_{v\in V_P(\Gamma)} \left(D_0 \times \overline{M}_{0,m_v\cup n_v}\right) \times \prod_{\tilde{v}\in V_S(\Gamma)} \overline{M}_{\tilde{v}} \arrow[d] \\
& \prod_e D_0 \arrow[r, "\Delta"] & \prod_v \prod_{e\ni v} D_0\,.
\end{tikzcd}$$

Here, we denote by ψ_v the forgetful morphism $\overline{M}_v \longrightarrow D_0 \times \overline{M}_{0,m_v\cup n_v}$ which only remembers the evaluation of the edge marking and the stabilised domain curve. We adopt the convention that $\overline{M}_{0,n} = \operatorname{Spec}\mathbb{C}$ if $|n| \leq 2$. Since the diagram commutes and the top square is Cartesian we find that

$$\theta_*\Delta^!\left(\textstyle\prod_v C_v^{\mathrm{NS}} \cap [\overline{M}_v]^{\mathrm{virt}}\right) = \phi_*\Delta^!\Big(\textstyle\prod_{v\in V_P(\Gamma)} \psi_{v*}\left(C_v^{\mathrm{NS}} \cap [\overline{M}_v]^{\mathrm{virt}}\right) \times \prod_{\tilde{v}\in V_S(\Gamma)} C_{\tilde{v}}^{\mathrm{NS}} \cap [\overline{M}_{\tilde{v}}]^{\mathrm{virt}}\Big)\,. \quad (5.16)$$

A quick dimension count reveals that for all $v \in V_P(\Gamma)$ the cycle $C_v^{\mathrm{NS}} \cap [\overline{M}_v]^{\mathrm{virt}}$ is an element of

$$\mathrm{CH}^{\mathbb{G}_m}_{|m_v|+|n_v|}(\overline{M}_v)$$

times a factor $\hbar^{-1}$. So since $\overline{M}_{0,m_v\cup n_v} \times D_0$ is of dimension $\min(|m_v| + |n_v| - 2, 1)$ the pushforward

$$\psi_{v*}\left(C_v^{\mathrm{NS}} \cap [\overline{M}_v]^{\mathrm{virt}}\right)$$

vanishes whenever $|m_v| + |n_v| \geq 2$. Since necessarily $|m_v| \geq 1$ we deduce the following.

Lemma 5.4.6. *The cycle* $\theta_*\Delta^!\left(\prod_v C_v^{\mathrm{NS}} \cap [\overline{M}_v]^{\mathrm{virt}}\right)$ *vanishes unless every P-vertex has exactly one edge and carries no legs.* □

In combination with Lemma 5.4.5 the above indeed proves the vanishing of all terms described in Proposition 5.4.4.

5.4.5 Remaining terms

Let us analyse the remaining terms of the sum (5.12). We fix a star shaped splitting type Γ as specified in Proposition 5.4.4 and observe that formula (5.16) for its associated term simplifies to

$$\theta_*\Delta^!\left(\textstyle\prod_v C_v^{\mathrm{NS}} \cap [\overline{M}_v]^{\mathrm{virt}}\right) = -\hbar^{2g-2}\,\phi_*\Delta^!\Big(\Big(\textstyle\prod_{v\in V_P(\Gamma)} \psi_{v*}\left(C_v^{\mathrm{NS}} \cap [\overline{M}_v]^{\mathrm{virt}}\right)\Big) \times \left(\lambda_g \cap [\overline{M}_{\tilde{v}}]^{\mathrm{virt}}\right)\Big)$$

where we write $\tilde{v}$ for the unique S-vertex of Γ. Now for every P-vertex v a dimension count as one we did earlier reveals that

$$\psi_{v*}\left(C_v^{\mathrm{NS}} \cap [\overline{M}_v]^{\mathrm{virt}}\right) = \hbar^{-1} a_v [D_0]\,.$$

for some $a_v \in \mathbb{Q}$. We compute a_v as in [64, Proposition 2.4] by considering the fibre square

$$\begin{tikzcd} \overline{M}_{0,0,(d_e),d_e}(\mathbb{P}^1|0) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{M}_v \arrow[d, "\psi_v"] \\ \operatorname{Spec}\mathbb{C} \arrow[r, hook, "\xi"] & D_0 \,. \end{tikzcd}$$

By [37, Lemma 6.3] we find that

$$\hbar^{-1} a_v = \xi^! \psi_{v*}\big(C_v^{\mathrm{NS}} \cap [\overline{M}_v]^{\mathrm{virt}}\big) = \hbar^{-1} \int_{[\overline{M}_{0,0,(d_e),d_e}(\mathcal{O}_{\mathbb{P}^1}(-1)\,|\,0)]^{\mathrm{virt}}_{\mathbb{G}_m}} 1 = \frac{(-1)^{d_e+1}}{\hbar\, d_e^2}$$

where e is the unique edge adjacent to v. Inserting this relation into (5.4.5) gives

$$\theta_* \Delta^! \big(\textstyle\prod_v C_v^{\mathrm{NS}} \cap [\overline{M}_v]^{\mathrm{virt}}\big) = \hbar^{2g-|E(\Gamma)|-2} (-1)^{D\cdot\beta+|E(\Gamma)|+1} \dfrac{1}{\prod_e d_e^2}\, \phi_*\big(\lambda_g \cap [\overline{M}_{\tilde{v}}]^{\mathrm{virt}}\big)\,.$$

Inserted into (5.12) this exactly yields formula (5.8) if we identify $\overline{M}_{\tilde{v}}$ with $\overline{M}_{g,n,\mathbf{d},\beta}(S|D)$. This closes the proof of Theorem 5.4.1.

5.5 Refined BPS integrality

After several explicit case studies we now discuss a general feature expected to be satisfied by the refined Gromov–Witten generating series: BPS integrality. Gopakumar and Vafa predicted the existence of certain integer invariants underlying the (unrefined) topological string [68, 69] and from the the B-model analysis of Choi–Huang–Katz–Klemm [49, 89] one should expect a similar feature for the refined topological string.

In this section we formulate these predictions as precise conjectures in the framework of equivariant Gromov–Witten theory of Calabi–Yau fivefolds. We state the conjectures in three increasing levels of strength and in decreasing level of generality.

5.5.1 Notation

We first set the notation for this section. As before let X be a quasi-projective Calabi–Yau threefold together with the action of a torus T on $X \times \mathbb{A}^2$ satisfying Assumption t. Remember that we write q_1 and q_2 for the respective character which T acts on the two affine directions with. Additionally, we introduce the T-characters

$$\mathsf{q}_+ \coloneqq (\mathsf{q}_1\mathsf{q}_2)^{\frac{1}{2}}\,, \qquad \mathsf{q}_- \coloneqq (\mathsf{q}_1\mathsf{q}_2^{-1})^{\frac{1}{2}}$$

by possibly passing to a double cover of the torus. Under application of the Chern character the associated one dimensional representations get mapped to

$$\mathrm{ch}_T(\mathsf{q}_i) = \exp \epsilon_i\,, \qquad \mathrm{ch}_T(\mathsf{q}_\pm) = \exp \epsilon_\pm\,, \qquad \epsilon_\pm \coloneqq \frac{\epsilon_1 \pm \epsilon_2}{2}\,. \tag{5.17}$$

5.5.2 Rationality

We start with the weakest of our three conjectures which however holds in the most general context.

Conjecture 5.5.1. *For every effective curve class* β *there is a rational function* $\overline{\Omega}_\beta(X,T) \in \mathbb{Q}(\mathsf{q}_+,\mathsf{q}_-)$ *satisfying*

$$\mathrm{ch}_T\,\overline{\Omega}_\beta(X,T) = \mathsf{GW}_\beta(X\times\mathbb{A}^2,T) \tag{5.18}$$

where ch_T *acts via the substitution* (5.17).

Remark 5.5.2. Note that we are not assuming that property $\mathbf{p}$ holds in the above conjecture. We only need to assume that $\mathsf{GW}_\beta(X\times\mathbb{A}^2,T)$ can be defined via localisation with respect to some appropriate cocharacter of T. For instance in case of the local $\mathbb{P}^1$ geometry considered in Section 5.3.2 the above conjecture holds without Assumption $\mathbf{p}$ being satisfied.

We can provide the following evidence for Conjecture 5.5.1.

Proposition 5.5.3. *Let* $X = K_S$ *be a local del Pezzo surface together with the* $T=\mathbb{G}_\mathrm{m}^2$*-action considered in* (5.7). *Then Conjecture 5.5.1 holds in the Nekrasov–Shatashvili limit* ($\epsilon_2 = 0$).

Proof. Let us write y for the restriction of q_1 to the inclusion of the first factor $\mathbb{G}_\mathrm{m} \hookrightarrow T$ and $\hbar$ for the associated weight. In [20, Lemma 8.15] Bousseau shows that the generating series of relative Gromov–Witten invariants of S with maximum tangency along a smooth anticanonical curve D admits a rational lift. To be more precise, Bousseau proves the existence of a rational function $\overline{\Omega}_\beta(S|D) \in \mathbb{Q}(\mathsf{y}^{1/2})$ satisfying

$$\mathrm{ch}_{\mathbb{G}_\mathrm{m}}\,\overline{\Omega}_\beta(S|D) = \mathsf{GW}_\beta(S|D)$$

where $\mathrm{ch}_{\mathbb{G}_\mathrm{m}}\mathsf{y} = \mathrm{e}^{\hbar}$ and we use the notation introduced in Section 5.4.1.2 on the right-hand side. Thus, by Corollary 5.4.3 we have

$$\mathrm{ch}_{\mathbb{G}_\mathrm{m}}\,\overline{\Omega}_\beta(S|D) = \epsilon_2\,\mathsf{GW}_\beta(K_S\times\mathbb{A}^2,T)\Big|_{\substack{\epsilon_1=\hbar\\ \epsilon_2=0}}. \qquad \square$$

Further evidence is discussed below in the context of stronger conjectures.

5.5.3 Integrality

We now present a conjecture which constrains the poles of $\overline{\Omega}_\beta(X,T)$ and expresses the numerators of these rational functions as integer valued Laurent polynomials.

For convenience we fix some saturated additive subset $\mathrm{H} \subseteq \mathrm{H}_2(X,\mathbb{Z})^+$ of effective curve classes such that every $\beta \in \mathrm{H}$ satisfies Assumption $\mathbf{p}$. Note that for instance when X is a local del Pezzo surface we may take $\mathrm{H} = \mathrm{H}_2(X,\mathbb{Z})^+$. Moreover, we assume that Conjecture 5.5.1 holds for all curve classes in H and so we may recursively define

$$\Omega_\beta(X,T)(\mathsf{q}_+,\mathsf{q}_-) \in \mathbb{Q}(\mathsf{q}_+,\mathsf{q}_-)$$

for all $\beta \in \mathrm{H}$ by demanding that these functions satisfy

$$\overline{\Omega}_\beta(X,T)(\mathsf{q}_+,\mathsf{q}_-) = \sum_{k|\beta}\frac{1}{k}\frac{\Omega_{\beta/k}(X,T)(\mathsf{q}_+^k,\mathsf{q}_-^k)}{\left(\mathsf{q}_1^{k/2}-\mathsf{q}_1^{-k/2}\right)\left(\mathsf{q}_2^{k/2}-\mathsf{q}_2^{-k/2}\right)}. \tag{5.19}$$

Indeed, denoting by μ the Möbius function one can check that

$$\Omega_\beta(X,T)(\mathsf{q}_+,\mathsf{q}_-) = \frac{1}{\left(\mathsf{q}_1^{1/2}-\mathsf{q}_1^{-1/2}\right)\left(\mathsf{q}_2^{1/2}-\mathsf{q}_2^{-1/2}\right)}\sum_{k|\beta}\frac{\mu(k)}{k}\,\overline{\Omega}_{\beta/k}(X,T)(\mathsf{q}_+^k,\mathsf{q}_-^k).$$

indeed solves the recursive system.

Conjecture 5.5.4. *For all* $\beta \in \mathrm{H}$ *the rational function* $\Omega_\beta(X,T)$ *is a Laurent polynomial in* $\mathsf{q}_+, \mathsf{q}_-$ *with integer coefficients:*

$$\Omega_\beta(X,T) \in \mathbb{Z}\,[\mathsf{q}_+^{\pm 1}, \mathsf{q}_-^{\pm 1}] \cong K_T(\mathrm{pt})\,.$$

Remark 5.5.5. We remark that it is important to assume property p in the above conjecture in order to guarantee that $\overline{\Omega}_\beta(X,T)$ only features denominators of the form

$$\left(\mathsf{q}_1^{k/2} - \mathsf{q}_1^{-k/2}\right)\left(\mathsf{q}_2^{k/2} - \mathsf{q}_2^{-k/2}\right)$$

for $k \in \mathbb{Z}$. An example where this is not the case can be found in Section 5.3.2. However, even when Assumption p is not satisfied, we still observe that the denominators of $\overline{\Omega}_\beta(X,T)$ appear to have a very constrained form. Experimentally, we always find them to be products of factors $(1 - \mathsf{q}_+^i \mathsf{q}_-^j)$ for some $i, j \in \mathbb{Z}$. In other words conjecturally

$$\overline{\Omega}_\beta(X,T) \in K_T(\mathrm{pt})^{\mathrm{loc}} \otimes \mathbb{Q}$$

where $K_T(\mathrm{pt})^{\mathrm{loc}}$ is the localisation of $K_T(\mathrm{pt})$ at the augmentation ideal. In many cases we also experimentally observe an integrality of numerators similar to the statement of Conjecture 5.5.4. We already saw this for the local $\mathbb{P}^1$ geometry discussed in Section 5.3.2.

Remark 5.5.6. Conjecture 5.5.4 recovers the original Gopakumar–Vafa integrality conjecture by inserting equation (5.19) into (5.18) and restricting to the anti-diagonal torus:

$$\mathsf{GW}_\beta(X) = \mathrm{ch}_{\mathbb{G}_m} \sum_{k|\beta} \frac{1}{k} \frac{\Omega_{\beta/k}(X,T)(1,\mathsf{q}^k)}{\left(\mathsf{q}^{k/2} - \mathsf{q}^{-k/2}\right)\left(\mathsf{q}^{-k/2} - \mathsf{q}^{k/2}\right)} \tag{5.20}$$

Note that $\Omega_\beta(X,T)(1,\mathsf{q})$ is symmetric under $\mathsf{q} \mapsto \mathsf{q}^{-1}$ and so we may expand

$$\Omega_\beta(X,T)(1,\mathsf{q}) =: \sum_{g \geq 0} n_{g,\beta}\,(-1)^g \left(\mathsf{q}^{1/2} - \mathsf{q}^{-1/2}\right)^{2g}$$

for some $n_{g,\beta} \in \mathbb{Z}$ which are non-zero only for finitely many $g \in \mathbb{Z}_{\geq 0}$. This means Conjecture 5.5.4 implies an expansion

$$\mathsf{GW}_\beta(X) = \sum_{k|\beta} \sum_{g \geq 0} \frac{n_{g,\beta/k}}{k}\,(-1)^{g+1} \left(2 \sinh \tfrac{ku}{2}\right)^{2g-2} \tag{5.21}$$

with integer coefficients as predicted by Gopakumar and Vafa [68, 69]. The integrality part of this conjecture was proven by Ionel and Parker [92] while the finiteness of non-zero BPS invariants was established more recently by Doan–Ionel–Walpuski [55].

Remark 5.5.7. Conjecture 5.5.4 also relates to observations made in [49, 88, 89] regarding BPS integrality for B-model invariants. For local surfaces as discussed in loc. cit. or more generally whenever the torus action is as in Construction 5.2.2 the refined generating series satisfies

$$\epsilon_1 \epsilon_2\, \mathsf{GW}_\beta(X \times \mathbb{A}^2, T) \in \mathbb{Q}[\![\epsilon_+^2, \epsilon_-^2]\!]$$

as proven in Lemma 5.2.4. This means that if Conjecture 5.5.4 holds we can expand $\Omega_\beta(X,T)$ as

$$\Omega_\beta(X,T)(\mathsf{q}_+,\mathsf{q}_-) =: \sum_{i,j \geq 0} n_{i,j,\beta}\,(-1)^{i+j} \left(\mathsf{q}_+^{1/2} - \mathsf{q}_+^{-1/2}\right)^{2i} \left(\mathsf{q}_-^{1/2} - \mathsf{q}_-^{-1/2}\right)^{2j}$$

with $n_{i,j,\beta} \in \mathbb{Z}$ non-zero for finitely many $i, j \in \mathbb{Z}_{\geq 0}$. Hence, inserting (5.19) into (5.18) we obtain an expansion

$$\mathsf{GW}_\beta(X \times \mathbb{A}^2, T) = \sum_{k|\beta} \sum_{i,j\geq 0} \frac{n_{i,j,\beta/k}}{k} (-1)^{i+j} \frac{\left(2\sinh\frac{k\epsilon_+}{2}\right)^{2i} \left(2\sinh\frac{k\epsilon_-}{2}\right)^{2j}}{2\sinh\frac{k\epsilon_1}{2}\ 2\sinh\frac{k\epsilon_1}{2}}$$

with integer coefficients which is of the same form predicted for instance in [89, Equation (3.21)].

Example 5.5.8. Our calculation in Proposition 5.3.1 shows that for the resolved conifold

$$X = \operatorname{Tot} \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

(together with a T-action as specified in the proposition) there is a single non-zero BPS invariant

$$n_{0,0,[\mathbb{P}^1]} = 1\,.$$

5.5.4 Geometric interpretation

We now restrict our discussion to the case of local del Pezzo surfaces together with the scaling action on the fibres $K_S \times \mathbb{A}^2 \longrightarrow S$ considered in Section 5.4.1.1. In this section we present a conjecture for a geometric description of the coefficients of $\Omega_\beta(K_S, T)$.

We denote by M_β the moduli space of one dimensional Gieseker semistable sheaves on S (with respect to some fixed polarisation) with support β and Euler characteristic one. The Hilbert–Chow morphism

$$\pi : M_\beta \longrightarrow \mathsf{Chow}_\beta(S)$$

induces a perverse filtration on the cohomology groups of M_β:

$$\mathrm{H}^{i,j} \coloneqq \mathbb{H}^i\left({}^{\mathfrak{p}}\mathcal{H}^j(\mathbf{R}\pi_* \mathbb{Q}_{M_\beta}[\dim M_\beta])\right)\,.$$

We consider a refinement of Maulik and Toda's proposal [135, Definition 1.1] for Gopakumar–Vafa invariants and form a generating series recording the dimension of the individual graded pieces

$$\mathsf{MT}_\beta(S)(\mathsf{q}_+, \mathsf{q}_-) \coloneqq \sum_{i,j\in\mathbb{Z}} (-1)^{i+j} \dim \mathrm{H}^{i,j}\, \mathsf{q}_+^i\, \mathsf{q}_-^j \quad \in \mathbb{Z}\,[\mathsf{q}_+^{\pm 1}, \mathsf{q}_-^{\pm 1}]\,.$$

This refinement has also been investigated by Kononov–Lim–Moreira–Pi–Shen [106, 107] in connection with K-theoretic PT theory. We expect the following relation to Gromov–Witten theory.

Conjecture 5.5.9. *Suppose S is a del Pezzo surface. Then Conjecture 5.5.4 holds for $X = K_S$ and moreover for all $\beta \in \mathrm{H}_2(S, \mathbb{Z})^+$ we have*

$$\Omega_\beta(K_S, T) = \mathsf{MT}_\beta(S)\,.$$

Remark 5.5.10. By Lemma 5.2.4 the refined Gromov–Witten generating series for local surfaces is invariant under $\epsilon_+ \mapsto -\epsilon_+$ and $\epsilon_- \mapsto -\epsilon_-$. This means that if Conjecture 5.5.1 holds, the function $\Omega_\beta(K_S, T)$ has to be invariant under

$$\mathsf{q}_+ \longmapsto \mathsf{q}_+^{-1} \qquad \text{and} \qquad \mathsf{q}_- \longmapsto \mathsf{q}_-^{-1}$$

individually. As a sanity check for Conjecture 5.5.9 we quickly convince ourselves that $\mathsf{MT}_\beta(S)$ indeed enjoys these symmetries: Since M_β is smooth we have

$$\mathrm{H}^{i,j} \cong \mathrm{H}^{-i,j} \qquad \text{and} \qquad \mathrm{H}^{i,j} \cong \mathrm{H}^{i,-j}$$

for all $i, j \in \mathbb{Z}$ by the hard Lefschetz theorem for perverse cohomology groups [42, Theorem 2.1.4] ensuring the anticipated symmetries.

Remark 5.5.11. Restricting to $\mathsf{q}_+ = 1$ we recover a special instance of a conjecture of Maulik and Toda [135, Conjecture 3.18] predicting that the integers $n_{g,\beta}^{\mathsf{MT}} \in \mathbb{Z}$ defined via

$$\sum_{g\geq 0} n_{g,\beta}^{\mathsf{MT}} (-1)^{g+1} \left(\mathsf{q}^{1/2} - \mathsf{q}^{-1/2}\right)^{2g} \coloneqq \mathsf{MT}_\beta(S)(1,\mathsf{q})$$

satisfy equation (5.21). We remark that in the context of local surfaces the BPS sheaf agrees with the IC sheaf as proven by Meinhardt [137, Theorem 1.1]. By [135, Lemma 3.11] the $\epsilon_1 = \epsilon_2 = 0$ limit of our conjecture is therefore also compatible with Katz's proposal for genus zero Gopakumar–Vafa invariants [95].

Remark 5.5.12. In our definition of $\mathsf{MT}_\beta(S)$ the variables q_+, q_- do not enjoy a natural geometric interpretation which makes the definition rather ad hoc. It therefore seems desirable to find an interpretation of $\mathsf{MT}_\beta(S)$ as an honest equivariant index. Such an interpretation also appears necessary in order to generalise Conjecture 5.5.9 to geometries other than local surfaces.

We obtain highly non-trivial evidence for Conjecture 5.5.9 in the case of local $\mathbb{P}^2$ by combining our identification of the Nekrasov–Shatashvili limit and the relative Gromov–Witten theory of $\mathbb{P}^2$ relative a smooth cubic with the following remarkable result of Bousseau [22]. For this observe that by setting $\mathsf{q}_+ = \mathsf{q}_- = \mathsf{y}^{1/2}$ the generating series $\mathsf{MT}_\beta(S)$ specialises to the centred Poincaré polynomial

$$\mathsf{MT}_\beta(S)(\mathsf{y}^{1/2},\mathsf{y}^{1/2}) = \sum_{i\in\mathbb{Z}} (-1)^i \, \dim \mathrm{H}^i\left(M_\beta, \mathbb{Q}_{M_\beta}[\dim M_\beta]\right) \mathsf{y}^{i/2}\,.$$

Theorem 5.5.13. [22, *Theorem 0.4.5*], [134, *Theorem 0.1*] *For D a smooth cubic in $\mathbb{P}^2$ we have*[1]

$$\mathsf{GW}_\beta(\mathbb{P}^2|D) = \mathrm{ch}_{\mathbb{G}_\mathsf{m}} \sum_{k|\beta} \frac{1}{k^2} \frac{\mathsf{MT}_\beta(S)(\mathsf{y}^{k/2},\mathsf{y}^{k/2})}{\mathsf{y}^{k/2}-\mathsf{y}^{-k/2}}$$

where $\mathrm{ch}_{\mathbb{G}_\mathsf{m}}$ *maps* y *to* $\mathrm{e}^{\hbar}$.

Corollary 5.5.14. *Conjecture 5.5.9 holds in the Nekrasov–Shatashvili limit for $S = \mathbb{P}^2$.*

Proof. Combining Corollary 5.4.3 with Theorem 5.5.13 we obtain

$$\epsilon_2\, \mathsf{GW}_\beta(K_{\mathbb{P}^2} \times \mathbb{A}^2, T)\Big|_{\substack{\epsilon_1=\hbar\\ \epsilon_2=0}} = \mathsf{GW}_\beta(\mathbb{P}^2|D) = \mathrm{ch}_{\mathbb{G}_\mathsf{m}} \sum_{k|\beta} \frac{1}{k^2} \frac{\mathsf{MT}_\beta(S)(\mathsf{y}^{k/2},\mathsf{y}^{k/2})}{\mathsf{y}^{k/2}-\mathsf{y}^{k/2}}\,.$$

We observe that the right-hand side expression is nothing but the restriction

$$\epsilon_2 \cdot \mathrm{ch}_T \sum_{k|\beta} \frac{1}{k} \frac{\mathsf{MT}_\beta(S)(\mathsf{q}_+^k,\mathsf{q}_-^k)}{\left(\mathsf{q}_1^{k/2}-\mathsf{q}_1^{k/2}\right)\left(\mathsf{q}_2^{k/2}-\mathsf{q}_2^{k/2}\right)}\Bigg|_{\substack{\epsilon_1=\hbar\\ \epsilon_2=0}}\,.$$ ◻

Remark 5.5.15. The above line of reasoning also demonstrates the compatibility of Conjecture 5.5.9 with [20, Conjecture 8.16] of Bousseau. The latter conjecture asserts that Theorem 5.5.13 generalises to arbitrary del Pezzo surfaces.

Remark 5.5.16. In the case of local $\mathbb{P}^2$ we have further numerical evidence for Conjecture 5.5.9 apart from the unrefined and Nekrasov–Shatashvili limit. In [107] Kononov, Pi and Shen compute $\mathsf{MT}_\beta(\mathbb{P}^2)$ for small degree. Via localisation we computed $\mathsf{GW}_\beta(K_{\mathbb{P}^2} \times \mathbb{A}^2, T)$ for $\beta \leq 3$ up to contributions of genus five and both calculations seem perfectly compatible with Conjecture 5.5.9.

[1] We corrected the factor ℓ^{-1} to ℓ^{-2} in the definition of $\bar{F}^{NS}(y^{1/2}, Q)$ in [22, Theorem 0.4.6].

Bibliography

[1] D. Abramovich and Q. Chen. "Stable logarithmic maps to Deligne–Faltings pairs II". *Asian J. Math.* 18.3 (2014), pp. 465–488. arXiv: 1102.4531 [math.AG].

[2] Dan Abramovich, Charles Cadman, and Jonathan Wise. "Relative and orbifold Gromov-Witten invariants". *Algebr. Geom.* 4.4 (2017), pp. 472–500. arXiv: 1004.0981 [math.AG].

[3] Dan Abramovich, Qile Chen, Mark Gross, and Bernd Siebert. "Decomposition of degenerate Gromov-Witten invariants". *Compos. Math.* 156.10 (2020), pp. 2020–2075. arXiv: 1709.09864 [math.AG].

[4] Dan Abramovich and Barbara Fantechi. "Orbifold thechniques in degeneration formulas". *Annali Scuola Normale Superiore-Classe di Scienze* 16.5 (2016), pp. 519–579. arXiv: 1103.5132 [math.AG].

[5] Dan Abramovich, Steffen Marcus, and Jonathan Wise. "Comparison theorems for Gromov-Witten invariants of smooth pairs and of degenerations". *Ann. Inst. Fourier (Grenoble)* 64.4 (2014), pp. 1611–1667. arXiv: 1207.2085 [math.AG].

[6] Dan Abramovich and Jonathan Wise. "Birational invariance in logarithmic Gromov-Witten theory". *Compos. Math.* 154.3 (2018), pp. 595–620. arXiv: 1306.1222 [math.AG].

[7] Mina Aganagic, Albrecht Klemm, Marcos Mariño, and Cumrun Vafa. "The topological vertex". *Comm. Math. Phys.* 254.2 (2005), pp. 425–478. arXiv: hep-th/0305132 [hep-th].

[8] Mina Aganagic and Cumrun Vafa. "Mirror symmetry, D-branes and counting holomorphic discs". *arXiv preprint* (2000). arXiv: hep-th/0012041 [hep-th].

[9] Konstantin Aleshkin and Chiu-Chu Melissa Liu. "Open/closed Correspondence and Extended LG/CY Correspondence for Quintic Threefolds" (2023). arXiv: 2309.14628 [math.AG].

[10] Carlo Angelantonj, Ignatios Antoniadis, Ioannis Florakis, and Hongliang Jiang. "Refined topological amplitudes from the Ω-background in string theory". *J. High Energy Phys.* 2022.5 (2022). Id/No 143, p. 18. arXiv: 2202.13205 [hep-th].

[11] I. Antoniadis et al. "Worldsheet realization of the refined topological string". *Nucl. Phys., B* 875.1 (2013), pp. 101–133. arXiv: 1302.6993 [hep-th].

[12] S. Bardwell-Evans, M-W. M. Cheung, H. Hong, and Y-S. Lin. "Scattering Diagrams from Holomorphic Discs in Log Calabi-Yau Surfaces". *arXiv preprint* (2021). arXiv: 2110.15234 [math.SG].

[13] L. J. Barrott and N. Nabijou. "Tangent curves to degenerating hypersurfaces". *J. Reine Angew. Math.* 793 (2022), pp. 185–224. arXiv: 2007.05016 [math.AG].

[14] Lawrence Jack Barrott. "Explicit equations for mirror families to log Calabi-Yau surfaces". *Bull. Korean Math. Soc.* 57.1 (2020), pp. 139–165. arXiv: 1810.08356 [math.AG].

[15] L. Battistella, N. Nabijou, and D. Ranganathan. "Curve counting in genus one: elliptic singularities and relative geometry". *Algebr. Geom.* 8.6 (2021), pp. 637–679. arXiv: 1907.00024 [math.AG].

[16] L. Battistella, N. Nabijou, and D. Ranganathan. "Gromov–Witten theory via roots and logarithms". *Geom. Topol.* (in press) (2022). arXiv: 2203.17224 [math.AG].

[17] L. Battistella, N. Nabijou, H.-H. Tseng, and F. You. "The local-orbifold correspondence for simple normal crossings pairs". *J. Inst. Math. Jussieu* 22.5 (2023), pp. 2515–2531. arXiv: 2103.09299 [math.AG].

[18] K. Behrend. "Localization and Gromov–Witten invariants". In: *Quantum cohomology (Cetraro, 1997)*. Vol. 1776. Lecture Notes in Math. Springer, Berlin, 2002, pp. 3–38.

[19] P. Bousseau. "Quantum mirrors of log Calabi–Yau surfaces and higher-genus curve counting". *Compos. Math.* 156.2 (2020), pp. 360–411. arXiv: 1808.07336 [math.AG].

[20] P. Bousseau. "The quantum tropical vertex". *Geom. Topol.* 24.3 (2020), pp. 1297–1379. arXiv: 1806.11495 [math.AG].

[21] P. Bousseau, H. Fan, S. Guo, and L. Wu. "Holomorphic anomaly equation for $(\mathbb{P}^2, E)$ and the Nekrasov–Shatashvili limit of local $\mathbb{P}^2$". *Forum Math. Pi* 9 (2021), Paper No. e3, 57. arXiv: 2001.05347 [math.AG].

[22] Pierrick Bousseau. "A proof of N. Takahashi's conjecture for $(\mathbb{P}^2, E)$ and a refined sheaves/Gromov-Witten correspondence". *Duke Math. J.* 172.15 (2023), pp. 2895–2955. arXiv: 1909.02992 [math.AG].

[23] Pierrick Bousseau. “On an example of quiver Donaldson-Thomas/relative Gromov-Witten correspondence”. *Int. Math. Res. Not. IMRN* 15 (2021), pp. 11845–11888. arXiv: 1811.02448 [math.AG].

[24] Pierrick Bousseau. “Refined floor diagrams from higher genera and lambda classes”. *Selecta Math. (N.S.)* 27.3 (2021), Paper No. 43, 42. arXiv: 1904.10311 [math.AG].

[25] Pierrick Bousseau. “Scattering diagrams, stability conditions, and coherent sheaves on $\mathbb{P}^2$”. *J. Algebraic Geom.* 31.4 (2022), pp. 593–686. arXiv: 1909.02985 [math.AG].

[26] Pierrick Bousseau. “Strong positivity for the skein algebras of the 4-punctured sphere and of the 1-punctured torus”. *Comm. Math. Phys.* 398.1 (2023), pp. 1–58. arXiv: 2009.02266 [math.GT].

[27] Pierrick Bousseau. “Tropical refined curve counting from higher genera and lambda classes”. *Invent. Math.* 215.1 (2019), pp. 1–79. arXiv: 1706.07762 [math.AG].

[28] Pierrick Bousseau, Andrea Brini, and Michel van Garrel. “On the log-local principle for the toric boundary”. *Bull. Lond. Math. Soc.* 54.1 (2022), pp. 161–181. arXiv: 1908.04371 [math.AG].

[29] Pierrick Bousseau, Andrea Brini, and Michel van Garrel. “Stable maps to Looijenga pairs”. *Geom. Topol.* 28.1 (2024), pp. 393–496. arXiv: 2011.08830 [math.AG].

[30] Pierrick Bousseau, Andrea Brini, and Michel van Garrel. “Stable maps to Looijenga pairs: orbifold examples”. *Lett. Math. Phys.* 111.4 (2021), Paper No. 109, 37. arXiv: 2012.10353 [math.AG].

[31] Pierrick Bousseau and Longting Wu. “All-genus WDVV recursion, quivers, and BPS invariants”. *arXiv preprint* (2023). arXiv: 2303.00503 [math.AG].

[32] Andrea Brini. “Enumerative geometry of surfaces and topological strings”. *International Journal of Modern Physics A* 38.09n10 (2023), p. 2330008. arXiv: 2211.11037 [math-ph].

[33] Andrea Brini and Yannik Schuler. “On quasi-tame Looijenga pairs”. *Commun. Number Theory Phys.* 17.2 (2023), pp. 313–341. arXiv: 2201.01645 [math.AG].

[34] Andrea Brini and Yannik Schuler. “Refined Gromov–Witten invariants”. in preperation (2024).

[35] Jim Bryan and Rahul Pandharipande. “Curves in Calabi-Yau threefolds and topological quantum field theory”. *Duke Math. J.* 126.2 (2005), pp. 369–396. arXiv: math/0306316 [math.AG].

[36] Jim Bryan and Rahul Pandharipande. “On the rigidity of stable maps to Calabi-Yau threefolds”. In: *The interaction of finite-type and Gromov-Witten invariants (BIRS 2003)*. Vol. 8. Geom. Topol. Monogr. Geom. Topol. Publ., Coventry, 2006, pp. 97–104. arXiv: math/0405204 [math.AG].

[37] Jim Bryan and Rahul Pandharipande. “The local Gromov-Witten theory of curves”. *J. Amer. Math. Soc.* 21.1 (2008). With an appendix by Bryan, C. Faber, A. Okounkov and Pandharipande, pp. 101–136. arXiv: math/0411037 [math.AG].

[38] Charles Cadman. “Using stacks to impose tangency conditions on curves”. *Amer. J. Math.* 129.2 (2007), pp. 405–427. arXiv: math/0312349 [math.AG].

[39] Charles Cadman and Linda Chen. “Enumeration of rational plane curves tangent to a smooth cubic”. *Adv. Math.* 219.1 (2008), pp. 316–343. arXiv: math/0701406 [math.AG].

[40] Philip Candelas, Xenia C. de la Ossa, Paul S. Green, and Linda Parkes. “A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory”. *Nucl. Phys., B* 359.1 (1991), pp. 21–74.

[41] Nicola Caporaso et al. “Phase transitions, double-scaling limit, and topological strings”. *Phys. Rev. D* 75.4 (2007), pp. 046004, 24. arXiv: hep-th/0606120 [hep-th].

[42] Mark Andrea A. de Cataldo and Luca Migliorini. “The Hodge theory of algebraic maps”. *Ann. Sci. École Norm. Supér. (4)* 38.5 (2005), pp. 693–750. arXiv: math/0306030 [math.AG].

[43] Q. Chen. “Stable logarithmic maps to Deligne–Faltings pairs I”. *Ann. of Math. (2)* 180.2 (2014), pp. 455–521. arXiv: 1008.3090 [math.AG].

[44] Q. Chen. “The degeneration formula for logarithmic expanded degenerations”. *J. Algebraic Geom.* 23.2 (2014), pp. 341–392. arXiv: 1009.4378 [math.AG].

[45] T.M. Chiang, A. Klemm, S-T. Yau, and E. Zaslow. “Local mirror symmetry: Calculations and interpretations”. *Adv. Theor. Math. Phys.* 3 (1999), pp. 495–565. arXiv: hep-th/9903053 [hep-th].

[46] J. Choi, M. van Garrel, S. Katz, and N. Takahashi. “Log BPS numbers of log Calabi–Yau surfaces”. *Trans. Amer. Math. Soc.* 374.1 (2021), pp. 687–732. arXiv: 1810.02377 [math.AG].

[47] J. Choi, M. van Garrel, S. Katz, and N. Takahashi. “Sheaves of maximal intersection and multiplicities of stable log maps”. *Selecta Math. (N.S.)* 27.4 (2021), Paper No. 61, 51. arXiv: 1908.10906 [math.AG].

[48] J. Choi, M. van Garrel, S. Katz, and N. Takahashi. “Local BPS invariants: enumerative aspects and wall-crossing”. *Int. Math. Res. Not.* 2020.17 (2020), pp. 5450–5475. arXiv: 1804.00679 [math.AG].

[49] Jinwon Choi, Sheldon Katz, and Albrecht Klemm. “The refined BPS index from stable pair invariants”. *Comm. Math. Phys.* 328.3 (2014), pp. 903–954. arXiv: 1210.4403 [hep-th].

[50] Tom Coates and Alexander Givental. “Quantum Riemann-Roch, Lefschetz and Serre”. *Ann. of Math. (2)* 165.1 (2007), pp. 15–53. arXiv: math/0110142 [math.AG].

[51] David A. Cox and Sheldon Katz. “Mirror symmetry and algebraic geometry”. Vol. 68. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1999, pp. xxii+469.

[52] Ben Davison and Travis Mandel. "Strong positivity for quantum theta bases of quantum cluster algebras". *Invent. Math.* 226.3 (2021), pp. 725–843. arXiv: 1910.12915 [math.RT].

[53] Vincent Delecroix, Johannes Schmitt, and Jason van Zelm. "admcycles – a Sage package for calculations in the tautological ring of the moduli space of stable curves". *J. Softw. Algebra Geom.* 11 (2021), pp. 89–112. arXiv: 2002.01709 [math.AG].

[54] Pierre Descombes. "Hyperbolic localization of the Donaldson-Thomas sheaf". *arXiv preprint* (2023). arXiv: 2201.12215 [math.AG].

[55] Aleksander Doan, Eleny-Nicoleta Ionel, and Thomas Walpuski. "The Gopakumar-Vafa finiteness conjecture". *arXiv preprint* (2021). arXiv: 2103.08221 [math.SG].

[56] Carel Faber. "Algorithms for computing intersection numbers on moduli spaces of curves, with an application to the class of the locus of Jacobians". In: *New trends in algebraic geometry. Selected papers presented at the Euro conference, Warwick, UK, July 1996*. Cambridge: Cambridge University Press, 1999, pp. 93–109. arXiv: alg-geom/9706006 [math.AG].

[57] H. Fan and L. Wu. "Witten–Dijkgraaf–Verlinde–Verlinde equation and its application to relative Gromov-Witten theory". *Int. Math. Res. Not.* 2021.13 (2021), pp. 9834–9852. arXiv: 1902.05739 [math.AG].

[58] Honglu Fan, Hsian-Hua Tseng, and Fenglong You. "Mirror theorems for root stacks and relative pairs". *Selecta Math. (N.S.)* 25.4 (2019), Paper No. 54, 25. arXiv: 1811.10807 [math.AG].

[59] Bohan Fang and Chiu-Chu Melissa Liu. "Open Gromov-Witten invariants of toric Calabi-Yau 3-folds". *Comm. Math. Phys.* 323.1 (2013), pp. 285–328. arXiv: 1103.0693 [math.SG].

[60] R. Friedman. "On the geometry of anticanonical pairs". *arXiv preprint* (2015). arXiv: 1502.02560 [math.AG].

[61] R. Friedman and R. Miranda. "Smoothing cusp singularities of small length". *Math. Ann.* 263.2 (1983), pp. 185–212.

[62] W. Fulton and R. Pandharipande. "Notes on stable maps and quantum cohomology". In: *Algebraic geometry. Proceedings of the Summer Research Institute, Santa Cruz, CA, USA, July 9–29, 1995*. Providence, RI: American Mathematical Society, 1997, pp. 45–96. arXiv: alg-geom/9608011 [alg-geom].

[63] William Fulton. "Intersection theory". Second. Vol. 2. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 1998.

[64] Michel van Garrel, Tom Graber, and Helge Ruddat. "Local Gromov-Witten invariants are log invariants". *Adv. Math.* 350 (2019), pp. 860–876. arXiv: 1712.05210 [math.AG].

[65] Michel van Garrel, Navid Nabijou, and Yannik Schuler. "Gromov–Witten theory of bicyclic pairs". *arXiv preprint* (2023). arXiv: 2310.06058 [math.AG].

[66] Michel van Garrel, Tony W. H. Wong, and Gjergji Zaimi. "Integrality of relative BPS state counts of toric del Pezzo surfaces". *Commun. Number Theory Phys.* 7.4 (2013), pp. 671–687. arXiv: 1312.4112 [math.AG].

[67] Andreas Gathmann. "Relative Gromov-Witten invariants and the mirror formula". *Math. Ann.* 325.2 (2003), pp. 393–412. arXiv: math/0009190 [math.AG].

[68] Rajesh Gopakumar and Cumrun Vafa. "M-Theory and Topological Strings–I". *arXiv preprint* (1998). arXiv: hep-th/9809187 [hep-th].

[69] Rajesh Gopakumar and Cumrun Vafa. "M-Theory and Topological Strings–II". *arXiv preprint* (1998). arXiv: hep-th/9812127 [hep-th].

[70] T. Graber and R. Pandharipande. "Localization of virtual classes". *Invent. Math.* 135.2 (1999), pp. 487–518. arXiv: alg-geom/9708001 [alg-geom].

[71] Tom Graber and Ravi Vakil. "Relative virtual localization and vanishing of tautological classes on moduli spaces of curves". *Duke Math. J.* 130.1 (2005), pp. 1–37. arXiv: math/0309227 [math.AG].

[72] Tim Graefnitz. "Theta functions, broken lines and 2-marked log Gromov–Witten invariants". *arXiv preprint* (2022). arXiv: 2204.12257 [math.AG].

[73] Tim Graefnitz. "Tropical correspondence for smooth del Pezzo log Calabi-Yau pairs". *J. Algebraic Geom.* 31.4 (2022), pp. 687–749. arXiv: 2005.14018 [math.AG].

[74] Tim Graefnitz, Helge Ruddat, and Eric Zaslow. "The proper Landau-Ginzburg potential is the open mirror map". *arXiv preprint* (2022). arXiv: 2204.12249 [math.AG].

[75] M. Gromov. "Pseudo holomorphic curves in symplectic manifolds". *Invent. Math.* 82 (1985), pp. 307–347.

[76] M. Gross. "Remarks on gluing punctured logarithmic maps". *arXiv preprint* (2023). arXiv: 2306.02661 [math.AG].

[77] M. Gross, P. Hacking, and S. Keel. "Birational geometry of cluster algebras". *Algebr. Geom.* 2.2 (2015), pp. 137–175. arXiv: 1309.2573 [math.AG].

[78] M. Gross, P. Hacking, and S. Keel. "Mirror symmetry for log Calabi–Yau surfaces I". *Publ. Math. Inst. Hautes Études Sci.* 122 (2015), pp. 65–168. arXiv: 1106.4977 [math.AG].

[79] M. Gross and R. Pandharipande. "Quivers, curves, and the tropical vertex". *Port. Math.* 67.2 (2010), pp. 211–259. arXiv: 0909.5153 [math.AG].

[80] M. Gross, R. Pandharipande, and B. Siebert. "The tropical vertex". *Duke Math. J.* 153.2 (2010), pp. 297–362. arXiv: 0902.0779 [math.AG].

[81] M. Gross and B. Siebert. "Logarithmic Gromov–Witten invariants". *J. Amer. Math. Soc.* 26.2 (2013), pp. 451–510. arXiv: 1102.4322 [math.AG].

[82] Mark Gross, Paul Hacking, Sean Keel, and Maxim Kontsevich. "Canonical bases for cluster algebras". *J. Amer. Math. Soc.* 31.2 (2018), pp. 497–608. arXiv: 1411.1394 [math.AG].

[83] Mark Gross and Bernd Siebert. "Intrinsic mirror symmetry and punctured Gromov-Witten invariants". In: *Algebraic geometry: Salt Lake City 2015*. Vol. 97.2. Proc. Sympos. Pure Math. Amer. Math. Soc., Providence, RI, 2018, pp. 199–230. arXiv: 1609.00624 [math.AG].

[84] Paul Hacking and Ailsa Keating. "Homological mirror symmetry for log Calabi-Yau surfaces". *Geom. Topol.* 26.8 (2020). With an appendix by Wendelin Lutz, pp. 3747–3833. arXiv: 2005.05010.

[85] Tamás Hausel and Bernd Sturmfels. "Toric hyperKähler varieties". *Doc. Math.* 7 (2002), pp. 495–534. arXiv: math/0203096 [math.AG].

[86] Simeon Hellerman, Domenico Orlando, and Susanne Reffert. "The omega deformation from string and M-theory". *J. High Energy Phys.* 2012.7 (2012). Id/No 61, p. 36. arXiv: 1204.4192 [hep-th].

[87] Kentaro Hori et al. "Mirror symmetry". Vol. 1. Clay Math. Monogr. Providence, RI: American Mathematical Society (AMS), 2003.

[88] Min-xin Huang, Amir-Kian Kashani-Poor, and Albrecht Klemm. "The Ω deformed B-model for rigid $N = 2$ theories". *Ann. Henri Poincaré* 14.3 (2013), pp. 425–497. arXiv: 1109.5728 [hep-th].

[89] Min-xin Huang and Albrecht Klemm. "Direct integration for general Ω backgrounds". *Adv. Theor. Math. Phys.* 16.3 (2012), pp. 805–849. arXiv: 1009.1126 [hep-th].

[90] Min-xin Huang, Kaiwen Sun, and Xin Wang. "Blowup equations for refined topological strings". *J. High Energy Phys.* 2018.10 (2018). Id/No 196, p. 88. arXiv: 1711.09884 [hep-th].

[91] Eleny-Nicoleta Ionel and Thomas H. Parker. "Relative Gromov-Witten invariants". *Ann. of Math. (2)* 157.1 (2003), pp. 45–96. arXiv: math/0010217 [math.SG].

[92] Eleny-Nicoleta Ionel and Thomas H. Parker. "The Gopakumar-Vafa formula for symplectic manifolds". *Ann. of Math. (2)* 187.1 (2018), pp. 1–64. arXiv: 1306.1516 [math.SG].

[93] Amer Iqbal and Amir-Kian Kashani-Poor. "The vertex on a strip". *Adv. Theor. Math. Phys.* 10.3 (2006), pp. 317–343. arXiv: hep-th/0410174 [hep-th].

[94] Amer Iqbal, Can Kozçaz, and Cumrun Vafa. "The refined topological vertex". *J. High Energy Phys.* 10 (2009), pp. 069, 58. arXiv: hep-th/0701156.

[95] Sheldon Katz. "Genus zero Gopakumar-Vafa invariants of contractible curves". *J. Differ. Geom.* 79.2 (2008), pp. 185–195. arXiv: math/0601193 [math.AG].

[96] Sheldon Katz, Albrecht Klemm, and Cumrun Vafa. "Geometric engineering of quantum field theories". *Nucl. Phys., B* 497.1-2 (1997), pp. 173–195. arXiv: hep-th/9609239 [hep-th].

[97] Sheldon Katz and Chiu-Chu Melissa Liu. "Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc". Geom. Topol. Monogr. 8 (2006), pp. 1–47. arXiv: math/0103074 [math.AG].

[98] Patrick Kennedy-Hunt, Qaasim Shafi, and Ajith Urundolil Kumaran. "Tropical refined curve counting with descendants". *arXiv preprint* (2023). arXiv: 2307.09436 [math.AG].

[99] B. Kim, H. Lho, and H. Ruddat. "The degeneration formula for stable log maps". *Manuscripta Math.* 170.1-2 (2023), pp. 63–107. arXiv: 1803.04210 [math.AG].

[100] Bumsig Kim. "Logarithmic stable maps". In: *New developments in algebraic geometry, integrable systems and mirror symmetry (RIMS, Kyoto, 2008)*. Vol. 59. Adv. Stud. Pure Math. Math. Soc. Japan, Tokyo, 2010, pp. 167–200. arXiv: 0807.3611 [math.AG].

[101] A. Klemm and R. Pandharipande. "Enumerative geometry of Calabi-Yau 4-folds". *Comm. Math. Phys.* 281.3 (2008), pp. 621–653. arXiv: math/0702189 [math.AG].

[102] Albrecht Klemm. "Topological string theory on Calabi-Yau threefolds". English. In: *RTN winter school on strings, supergravity and gauge theories. Papers from the school, Trieste, Italy, January 31–February 2, 2005*. Trieste: Scuola Internazionale Superiore di Studi Avanzati/International School for Advanced Studies (SISSA/ISAS), 2005, p. 100.

[103] Joachim Kock and Israel Vainsencher. "An invitation to quantum cohomology. Kontsevich's formula for rational plane curves". Vol. 249. Prog. Math. Basel: Birkhäuser, 2007.

[104] Y. Konishi and S. Minabe. "Flop invariance of the topological vertex". *Internat. J. Math.* 19.1 (2008), pp. 27–45. arXiv: math/0601352 [math.AG].

[105] Ya. Kononov, A. Okounkov, and A. Osinenko. "The 2-leg vertex in K-theoretic DT theory". *Comm. Math. Phys.* 382.3 (2021), pp. 1579–1599. arXiv: 1905.01523 [math-ph].

[106] Yakov Kononov, Woonam Lim, Miguel Moreira, and Weite Pi. "Cohomology rings of the moduli of one-dimensional sheaves on the projective plane". *arXiv preprint* (2024). arXiv: 2403.06277 [math.AG].

[107] Yakov Kononov, Weite Pi, and Junliang Shen. “Perverse filtrations, Chern filtrations, and refined BPS invariants for local P2”. *Adv. Math.* 433 (2023), p. 109294. arXiv: 2211.06991 [math.AG].
[108] M. Kontsevich and Y. Soibelman. “Affine structures and non-Archimedean analytic spaces”. In: *The unity of mathematics*. Vol. 244. Progr. Math. Birkhäuser Boston, Boston, MA, 2006, pp. 321–385. arXiv: math/0406564 [math.AG].
[109] C. Krattenthaler. “Proof of two multivariate q-binomial sums arising in Gromov–Witten theory”. *arXiv preprint* (2021). arXiv: 2102.02360 [math.CA].
[110] Daniel Krefl and Johannes Walcher. “Extended holomorphic anomaly in gauge theory”. *Lett. Math. Phys.* 95.1 (2011), pp. 67–88. arXiv: 1007.0263 [hep-th].
[111] J. M. F. Labastida and Marcos Mariño. “Polynomial invariants for torus knots and topological strings”. *Comm. Math. Phys.* 217.2 (2001), pp. 423–449. arXiv: hep-th/0004196 [hep-th].
[112] José M. F. Labastida, Marcos Mariño, and Cumrun Vafa. “Knots, links and branes at large N”. *J. High Energy Phys.* 11 (2000), Paper 7, 42. arXiv: hep-th/0010102 [hep-th].
[113] W. Lerche and P. Mayr. “On N=1 Mirror Symmetry for Open Type II Strings”. *arXiv preprint* (2002). arXiv: hep-th/0111113 [hep-th].
[114] Jun Li. “A degeneration formula of GW-invariants”. *J. Differential Geom.* 60.2 (2002), pp. 199–293. arXiv: math/0110113 [math.AG].
[115] Jun Li. “Stable morphisms to singular schemes and relative stable morphisms”. *J. Differential Geom.* 57.3 (2001), pp. 509–578. arXiv: math/0009097 [math.AG].
[116] Jun Li, Chiu-Chu Melissa Liu, Kefeng Liu, and Jian Zhou. “A mathematical theory of the topological vertex”. *Geom. Topol.* 13.1 (2009), pp. 527–621. arXiv: math/0408426 [math.AG].
[117] Jun Li and Yun Song. “Open string instantons and relative stable morphisms”. Geom. Topol. Monogr. 8 (2006), pp. 49–72. arXiv: hep-th/0103100 [hep-th].
[118] An-Min Li and Yongbin Ruan. “Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds”. *Invent. Math.* 145.1 (2001), pp. 151–218. arXiv: math/9803036 [math.AG].
[119] C.-C. M. Liu and S. Yu. “Open/closed correspondence via relative/local correspondence”. *Adv. Math.* 410.part A (2022), Paper No. 108696, 43. arXiv: 2112.04418 [math.AG].
[120] Chiu-Chu Melissa Liu. “Localization in Gromov-Witten theory and orbifold Gromov-Witten theory”. Adv. Lect. Math. (ALM) 25 (2013), pp. 353–425. arXiv: 1107.4712 [math.AG].
[121] Chiu-Chu Melissa Liu, Kefeng Liu, and Jian Zhou. “A proof of a conjecture of Mariño-Vafa on Hodge integrals”. *J. Differential Geom.* 65.2 (2003), pp. 289–340. arXiv: math/0306434 [math.AG].
[122] Chiu-Chu Melissa Liu and Song Yu. “Orbifold Open/Closed Correspondence and Mirror Symmetry”. *arXiv preprint* (2022). arXiv: 2210.11721 [math.AG].
[123] E. Looijenga. “Rational surfaces with an anticanonical cycle”. *Ann. of Math. (2)* 114.2 (1981), pp. 267–322.
[124] Wei Luo and Shengmao Zhu. “Integrality of the LMOV invariants for framed unknot”. *Commun. Number Theory Phys.* 13.1 (2019), pp. 81–100. arXiv: 2106.02882 [math.GT].
[125] T. Mandel. “Classification of rank 2 cluster varieties”. *SIGMA* 15 (2019), Paper 042, 32. arXiv: 1407.6241 [math.AG].
[126] T. Mandel. “Tropical theta functions and log Calabi–Yau surfaces”. *Selecta Math. (N.S.)* 22.3 (2016), pp. 1289–1335. arXiv: 1407.5901 [math.AG].
[127] T. Mandel and H. Ruddat. “Descendant log Gromov–Witten invariants for toric varieties and tropical curves”. *Trans. Amer. Math. Soc.* 373.2 (2020), pp. 1109–1152. arXiv: 1612.02402 [math.AG].
[128] Travis Mandel. “Scattering diagrams, theta functions, and refined tropical curve counts”. *J. Lond. Math. Soc. (2)* 104.5 (2021), pp. 2299–2334. arXiv: 1503.06183 [math.QA].
[129] Travis Mandel. “Theta bases and log Gromov-Witten invariants of cluster varieties”. *Trans. Amer. Math. Soc.* 374.8 (2021), pp. 5433–5471. arXiv: 1903.03042 [math.AG].
[130] Cristina Manolache. “Virtual pull-backs”. *J. Algebraic Geom.* 21.2 (2012), pp. 201–245. arXiv: 0805.2065 [math.AG].
[131] Cristina Manolache. “Virtual push-forwards”. *Geom. Topol.* 16.4 (2012), pp. 2003–2036. arXiv: 1010.2704 [math.AG].
[132] Marcos Mariño and Cumrun Vafa. “Framed knots at large N”. Contemp. Math. 310 (2002), pp. 185–204. arXiv: hep-th/0108064 [hep-th].
[133] Davesh Maulik and Dhruv Ranganathan. “Logarithmic enumerative geometry for curves and sheaves”. *arXiv preprint* (2023). arXiv: 2311.14150 [math.AG].
[134] Davesh Maulik and Junliang Shen. “Cohomological χ-independence for moduli of one-dimensional sheaves and moduli of Higgs bundles”. *Geom. Topol.* 27.4 (2023), pp. 1539–1586. arXiv: 2012.06627 [math.AG].
[135] Davesh Maulik and Yukinobu Toda. “Gopakumar-Vafa invariants via vanishing cycles”. *Invent. Math.* 213.3 (2018), pp. 1017–1097. arXiv: 1610.07303 [math.AG].

[136] P. Mayr. “$N = 1$ mirror symmetry and open/closed string duality”. *Adv. Theor. Math. Phys.* 5.2 (2001), pp. 213–242. arXiv: hep-th/0108229 [hep-th].
[137] Sven Meinhardt. “Donaldson-Thomas invariants vs. intersection cohomology for categories of homological dimension one”. *arXiv preprint* (2015). arXiv: 1512.03343 [math.AG].
[138] S. Molcho and E. Routis. “Localization for logarithmic stable maps”. *Trans. Amer. Math. Soc. Ser. B* 6 (2019), pp. 80–113. arXiv: 1410.7884 [math.AG].
[139] Andrew Morrison, Sergey Mozgovoy, Kentaro Nagao, and Balázs Szendrői. “Motivic Donaldson-Thomas invariants of the conifold and the refined topological vertex”. *Adv. Math.* 230.4-6 (2012), pp. 2065–2093. arXiv: 1107.5017 [math.AG].
[140] David Mumford. “Towards an enumerative geometry of the moduli space of curves”. Progr. Math. 36 (1983), pp. 271–328.
[141] Navid Nabijou and Dhruv Ranganathan. “Gromov-Witten theory with maximal contacts”. *Forum Math. Sigma* 10 (2022), Paper No. e5, 34. arXiv: 1908.04706 [math.AG].
[142] Yu Nakayama and Hirosi Ooguri. “Comments on worldsheet description of the Omega background”. *Nucl. Phys., B* 856.2 (2012), pp. 342–359. arXiv: 1106.5503 [hep-th].
[143] Nikita Nekrasov and Andrei Okounkov. “Membranes and sheaves”. *Algebr. Geom.* 3.3 (2016), pp. 320–369. arXiv: 1404.2323 [math.AG].
[144] Nikita A. Nekrasov. “Seiberg-Witten prepotential from instanton counting”. *Adv. Theor. Math. Phys.* 7.5 (2003), pp. 831–864. arXiv: hep-th/0206161 [hep-th].
[145] Hirosi Ooguri and Cumrun Vafa. “Knot invariants and topological strings”. *Nuclear Phys. B* 577.3 (2000), pp. 419–438. arXiv: hep-th/9912123 [hep-th].
[146] R. Pandharipande. “Hodge integrals and degenerate contributions”. *Comm. Math. Phys.* 208.2 (1999), pp. 489–506. arXiv: math/9811140 [math.AG].
[147] Andrea Prudenziati. “Double genus expansion for general Ω background”. *Int. J. Mod. Phys. A* 31.19 (2016). Id/No 1650114, p. 29. arXiv: 1204.2322 [hep-th].
[148] Alberto Cobos Rabano, Cristina Manolache, and Qaasim Shafi. “The local/logarithmic correspondence and the degeneration formula for quasimaps”. *arXiv preprint* (2024). arXiv: 2404.01381 [math.AG].
[149] Dhruv Ranganathan. “Logarithmic Gromov-Witten theory with expansions”. *Algebr. Geom.* 9.6 (2022), pp. 714–761. arXiv: 1903.09006 [math.AG].
[150] Susanne Reffert. “General omega deformations from closed string backgrounds”. *J. High Energy Phys.* 2012.4 (2012). Id/No 059, p. 14. arXiv: 1108.0644 [hep-th].
[151] Markus Reineke. “Cohomology of quiver moduli, functional equations, and integrality of Donaldson-Thomas type invariants”. *Compos. Math.* 147.3 (2011), pp. 943–964. arXiv: 0903.0261 [math.AG].
[152] Markus Reineke. “Poisson automorphisms and quiver moduli”. *J. Inst. Math. Jussieu* 9.3 (2010), pp. 653–667. arXiv: 0804.3214 [math.RT].
[153] Yannik Schuler. “The log-open correspondence for two-component Looijenga pairs”. *arXiv preprint* (2024). arXiv: 2404.15412 [math.AG].
[154] Qaasim Shafi. “Logarithmic quasimaps”. English. *Adv. Math.* 438 (2024). Id/No 109469, p. 30. arXiv: 2301.01196 [math.AG].
[155] Jake P. Solomon. “Intersection theory on the moduli space of holomorphic curves with Lagrangian boundary conditions”. Thesis (Ph.D.)–Massachusetts Institute of Technology. ProQuest LLC, Ann Arbor, MI, 2006, (no paging). arXiv: math/0606429 [math.SG].
[156] Holger Spielberg. “The Gromov-Witten invariants of symplectic toric manifolds, and their quantum cohomology ring”. English. *C. R. Acad. Sci., Paris, Sér. I, Math.* 329.8 (1999), pp. 699–704. arXiv: math/0006156 [math.AG].
[157] Balázs Szendrői. “Nekrasov’s partition function and refined Donaldson-Thomas theory: the rank one case”. *SIGMA, Symmetry Integrability Geom. Methods Appl.* 8 (2012), paper 088, 16. arXiv: 1210.5181 [math.AG].
[158] Nobuyoshi Takahashi. “Log mirror symmetry and local mirror symmetry”. *Comm. Math. Phys.* 220.2 (2001), pp. 293–299. arXiv: math/0004179 [math.AG].
[159] Hsian-Hua Tseng and Fenglong You. “A mirror theorem for multi-root stacks and applications”. *Selecta Math. (N.S.)* 29.1 (2023), Paper No. 6, 33. arXiv: 2006.08991 [math.AG].
[160] Hsian-Hua Tseng and Fenglong You. “Higher genus relative and orbifold Gromov-Witten invariants”. *Geom. Topol.* 24.6 (2020), pp. 2749–2779. arXiv: 1806.11082 [math.AG].
[161] Yu Wang. “Gross-Siebert intrinsic mirror ring for smooth log Calabi-Yau pairs”. *arXiv preprint* (2022). arXiv: 2209.15365 [math.AG].
[162] Edward Witten. “Topological sigma models”. *Commun. Math. Phys.* 118.3 (1988), pp. 411–449.
[163] Stephanie Yang. “Calculating intersection numbers on moduli spaces of pointed curves”. *arXiv preprint* (2010). arXiv: 0808.1974 [math.AG].

[164] Fenglong You. "Relative Gromov-Witten invariants and the enumerative meaning of mirror maps for toric Calabi-Yau orbifolds". *Trans. Amer. Math. Soc.* 373.11 (2020), pp. 8259–8288. arXiv: 1905.03885 [math.AG].

[165] Fenglong You. "The proper Landau-Ginzburg potential, intrinsic mirror symmetry and the relative mirror map". English. *Commun. Math. Phys.* 405.3 (2024). Id/No 79, p. 44. arXiv: 2209.15371 [math.AG].

[166] Song Yu. "Open/Closed BPS Correspondence and Integrality". *arXiv preprint* (2023). arXiv: 2307.13542 [math.AG].

[167] Tony Yue Yu. "Enumeration of holomorphic cylinders in log Calabi-Yau surfaces. I". *Math. Ann.* 366.3-4 (2016), pp. 1649–1675. arXiv: 1504.01722 [math.AG].

[168] Tony Yue Yu. "Enumeration of holomorphic cylinders in log Calabi-Yau surfaces. II: positivity, integrality and the gluing formula". *Geom. Topol.* 25.1 (2021), pp. 1–46. arXiv: 1608.07651 [math.AG].

[169] D. Zeilberger. "A q-Foata proof of the q-Saalschütz identity". *European J. Combin.* 8.4 (1987), pp. 461–463.

[170] Yan Zhou. "Weyl groups and cluster structures of families of log Calabi-Yau surfaces". *Int. Math. Res. Not. IMRN* 16 (2023), pp. 14290–14333. arXiv: 1910.05762 [math.AG].

[171] Shengmao Zhu. "Integrality structures in topological strings and quantum 2-functions". *J. High Energy Phys.* 2022.5 (2022). Id/No 43, p. 21. arXiv: 2002.09813 [hep-th].

[172] Shengmao Zhu. "Topological strings, quiver varieties, and Rogers-Ramanujan identities". *Ramanujan J.* 48.2 (2019), pp. 399–421. arXiv: 1707.00831 [math.AG].