Aspects of 2-dimensional Elementary Topos Theory

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Chapter 1 is based on the solely authored paper *The 2-Set-enriched Grothendieck construction and the lax normal conical 2-limits*, see [36]. Chapter 2 is based on the solely authored paper *Colimits in 2-dimensional slices*, see [35]. Chapter 3 and Chapter 5 are based on the solely authored paper *2-classifiers via dense generators and Hofmann–Streicher universe in stacks*, see [37].

Chapter 4 is based on the jointly authored paper *Indexed Grothendieck construction*, authored by Elena Caviglia and Luca Mesiti, see [12]. The motivating application and the starting ideas that brought to the paper come from the candidate. For the rest of the work, the contribution of the authors is equally distributed.

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Abstract

We contribute to expand 2-dimensional elementary topos theory. We focus on the concept of 2-classifier, which is a 2-categorical generalization of the notion of subobject classifier. The idea is that of a discrete opfibration classifier. Interestingly, a 2-classifier can also be thought of as a Grothendieck construction inside a 2-category. We introduce the notion of good 2-classifier, that captures well-behaved 2-classifiers and is closer to the point of view of logic.

We substantially reduce the work needed to prove that something is a 2-classifier. We prove that both the conditions of 2-classifier and what gets classified by a 2-classifier can be checked just over the objects that form a dense generator. This technique allows us to produce a good 2-classifier in prestacks that classifies all discrete opfibrations with small fibres, and to restrict such good 2-classifier to one in stacks. This is the main part of a proof that Grothendieck 2-topoi are elementary 2-topoi. Our results also solve a problem posed by Hofmann and Streicher when attempting to lift Grothendieck universes to sheaves.

To produce our good 2-classifier in prestacks, we present an indexed version of the Grothendieck construction. This gives a pseudonatural equivalence of categories between opfibrations over a fixed base in the 2-category of 2-copresheaves and 2-copresheaves on the Grothendieck construction of the fixed base. Our result can be interpreted as the result that every (op)fibrational slice of a Grothendieck 2-topos is a Grothendieck 2-topos. We thus generalize what is called the fundamental theorem of elementary topos theory to dimension 2, in the Grothendieck topoi case.

In order to reach our theorems of reduction of the study of a 2-classifier to dense generators, we develop a calculus of colimits in 2-dimensional slices. We generalize to dimension 2 the well-known fact that a colimit in a 1-dimensional slice category

is precisely the map from the colimit of the domains of the diagram which is induced by the universal property. We explain that we need to consider lax slices, and prove results of preservation, reflection and lifting of 2-colimits for the domain 2-functor from a lax slice. We then study the 2-functor of change of base between lax slices.

Our calculus of colimits in 2-dimensional slices is based on an original concept of colim fibration and on the reduction of weighted 2-colimits to essentially conical ones, which is regulated by the 2-category of elements construction. The latter construction is a natural extension of the Grothendieck construction. We study it in detail from an abstract point of view and we conceive it as the 2-$\mathcal{S}et$-enriched Grothendieck construction, via an original notion of pointwise Kan extension. Our work is relevant to higher dimensional elementary topos theory as well as to a generalization of the Grothendieck construction to the enriched setting.

Contents

Introduction

Context and motivation

In this thesis, we contribute to expand 2-dimensional elementary topos theory, which has been introduced by Weber in [51]. Topos theory is a fundamental branch of category theory that has had numerous applications to geometry (e.g. for the calculus of cohomology) and logic (e.g. bringing to the concept of classifying topos and capturing Cohen forcing). It originated in Artin, Grothendieck and Verdier's [3]; we take as main reference Mac Lane and Moerdijk's [34]. A key concept is that of sheaf, which captures the ability of gluing together compatible local data into a global datum, and has had tons of applications in geometry. This is realized thanks to Grothendieck topologies, that generalize classical topologies to the context of categories. Categories of sheaves yield Grothendieck topoi, and they are the basic object of study of topos theory. Lawvere and Tierney [32, 47] have then developed the notion of elementary topos, generalizing that of Grothendieck topos. Lawvere's idea was that an (elementary) topos is a generalized universe of sets. The key observation is that many constructions that can be done with sets can be replicated inside a topos. And elementary topoi have had enormous success in (categorical) logic. Notably, every elementary topos has an internal logic, so that one can do (generalized) logic inside the world of a topos. The most important ingredient of the concept of elementary topos is the subobject classifier, that represents an object of generalized truth values, together with a chosen "true". The archetypal example of elementary topos is the category $\mathcal{S}et$ of sets. Its truth values are the classical true and false, represented by the singleton and the empty set. A fundamental result is that every Grothendieck topos, i.e. every category of sheaves, is an elementary topos. The subobject classifier of presheaves (which are just functors into $\mathcal{S}et$) is given by taking sieves, which are a key ingredient of Grothendieck topologies. Sieves are collections of morphisms into a fixed object that are closed under precomposition, or equivalently subfunctors of representables. They generalize the concept of covering from topology. The subobject classifier of presheaves can then be restricted to one of sheaves, selecting those sieves that behave well with respect to the topology, namely closed sieves. A proof can be found in Mac Lane and Moerdijk's [34, Section III.7]. The definition of elementary topos just adds, to the subobject classifier, cartesian closedness and having finite limits. And it is surprising how these properties imply numerous other ones, making elementary topoi into particularly well-behaved categories and a fundamental concept of mathematics.

We believe it is therefore important to generalize the fruitful 1-dimensional elementary topos theory to dimension 2. The notion of sheaf has been upgraded by Giraud [19] to that of stack in dimension 2. The idea is to only ask the local data to be compatible up to isomorphism. The induced global datum then analogously recovers the starting local data up to isomorphism. Stacks brought to the 2-categorical generalization of Grothendieck topoi, and solved numerous problems in geometry (e.g. moduli problems) that were not solvable using ordinary spaces or 1-dimensional sheaves. The theory of elementary 2-topoi, introduced by Weber in [51], is instead just at its beginning. But it is equally promising, with a high potential of application to categorical logic, geometry and category theory itself. For example, elementary 2-topoi could pave the way towards a 2-categorical logic and offer the tools to study it. The idea is that every elementary 2-topos should have an internal 2-dimensional logic. As an elementary topos is a generalized universe of sets, an elementary 2-topos can be conceived as a generalized universe of categories. Another potential application is a 2-categorical generalization of categories of classes, and the continuation of Weber's [51] work of comparison between structures of "small things". A direct application of the work of this thesis could be a 2-categorical generalization of the fruitful concept of Grothendieck topology, which would bring to an extended notion of stack, with relevance in geometry. Other ideas could be capturing and generalizing classifying topoi as well as geometrical classification processes.

The most important ingredient of the concept of elementary 2-topos is the 2-classifier, which is a 2-categorical generalization of the subobject classifier. Weber's idea, in [51], is that moving to dimension 2 one can and wants to classify morphisms with higher dimensional fibres. While injective functions have as fibres either the singleton or the empty set, discrete opfibrations have as fibres general sets and are thus the perfect notion to use to define 2-classifiers. This idea is closely connected with that of homotopy level in Voevodsky's univalent foundations, see [46, Chapter 7] and Voevodsky's [49]. (Discrete) opfibrations, together with the Grothendieck construction, have been introduced by Grothendieck in [22]; we take Jacobs's book [26] as main reference. The idea of a Grothendieck opfibration is that of a functor which is able to lift morphisms. Discrete opfibrations then require such liftings to be unique. This is similar to the fruitful path lifting properties in geometry, that are relevant to cohomology and homotopy theory. Grothendieck fibrations were born to capture together categories naturally associated to every point of a space. So that one can handle and study all such categories together, taking into account the change of base operations between them. A fundamental result in this framework is the equivalence between Grothendieck fibrations and indexed categories (i.e. prestacks), namely the Grothendieck construction. This process collects the data of an indexed category, i.e. a 2-functor $F\colon \mathcal{B}^{\mathrm{op}} \to \mathcal{C}at$ with $\mathcal{B}$ a small category that is seen as a family of categories, into a total category $\int F$ equipped with a functor $\mathcal{G}(F) : \int F \to \mathcal{B}$ that tells which index each object came from. More precisely, $\mathcal{G}(F)$ is given by the projection on the first component, with $\int F$ built as follows:

an object of $\int F$ is a pair (B, X) with $B \in \mathcal{B}$ and $X \in F(B)$;

a morphism $(B, X) \to (C, X')$ *in* $\int F$ is a pair (f, α) with $f\colon B \to C$ a morphism in $\mathcal{B}$ and $\alpha\colon X \to F(f)(X')$ a morphism in $F(B)$.

And all Grothendieck fibrations can be conceived in this way, as they are precisely the essential image of the fully faithful 2-functor $\mathcal{G}(-)\colon [\mathcal{B}^{\mathrm{op}}, \mathcal{C}at] \to \mathcal{C}at/_{\mathcal{B}}$ that calculates the Grothendieck construction. In algebra, an example of Grothendieck fibration is given by the total category of all modules, that collects together R-modules for every ring R. In geometry, this theory has many applications to bundles. In logic, it managed to expand dependent type theory.

In order to produce the notion of 2-classifier, Weber proposes to upgrade monomorphisms (or subobjects) to discrete opfibrations in a 2-category, which have been introduced by Street in [41]. 2-classifiers are thus discrete opfibration classifiers. In this thesis, we introduce the concept of *good 2-classifier*, that captures well-behaved 2-classifiers and is closer to the point of view of logic. The idea is to still have as classifier an object of generalized truth values equipped with a chosen "true", as in dimension 1. This is realized upgrading the classification process from one regulated by pullbacks (in dimension 1) to one regulated by comma objects (in dimension 2). The archetypal example of (good) 2-classifier is given by the category of elements, which is a restriction of the Grothendieck construction to indexed sets (i.e. presheaves). This construction exhibits the 2-category $\mathcal{C}at$ of small categories as the archetypal elementary 2-topos. More precisely, the inclusion $1\colon \mathit{1} \to \mathcal{S}et$ of the singleton into $\mathcal{S}et$ is a good 2-classifier in $\mathcal{C}at$. Indeed calculating comma objects along $1\colon \mathit{1} \to \mathcal{S}et$ coincides with calculating the category of elements of presheaves, and hence exhibits an equivalence of categories between maps into the universe $\mathcal{S}et$ and the discrete opfibrations with small fibres. This perfectly generalizes to dimension 2 the fact that $\mathcal{S}et$ is the archetypal elementary topos. While truth values in $\mathcal{S}et$ are just true and false, represented by the singleton and the empty set, the generalized truth values of $\mathcal{C}at$ are all sets. In light of this archetypal example of (good) 2-classifier, we can think of a 2-classifier as a Grothendieck construction inside a 2-category.

The main objective of this thesis is to expand 2-dimensional elementary topos theory. We present a novel technique of reduction of the study of 2-classifiers to dense generators. We then apply it to produce a good 2-classifier in prestacks (i.e. 2-dimensional presheaves) and restrict such good 2-classifier to one in stacks. This is the main part of a proof that every Grothendieck 2-topos is an elementary 2-topos. The reason why we focus on 2-classifiers is that the rest of the definition of elementary 2-topos proposed by Weber in [51] is yet to be ascertained. We hope that this thesis will contribute to reach a universally accepted notion of elementary 2-topos. Our good 2-classifiers in prestacks and stacks involve a 2-dimensional generalization of the concepts of sieve and closed sieve, that could be applied to produce a 2-dimensional generalization of Grothendieck topologies. They also involve an indexed version of the Grothendieck construction, which can be interpreted as the result that every (op)fibrational slice of a Grothendieck 2-topos is a Grothendieck 2-topos. Aiming at our technique of reduction of the study of 2-classifiers, we also expand 2-category theory, developing a calculus of colimits in 2-dimensional slices, and the theory of the Grothendieck construction.

Main results

We prove that both the conditions of 2-classifier and what gets classified by a 2-classifier can be checked just over the objects that form a dense generator. So that the whole study of a would-be 2-classifier is substantially reduced (Chapter 3). Dense generators capture the idea of a family of objects that generate all the other ones via nice colimits; the preeminent example is given by representables in categories of presheaves. The following two theorems condense our results of reduction of the study of 2-classifiers to dense generators.

Let $\mathcal{L}$ be a 2-category with pullbacks along discrete opfibrations, comma objects and terminal object (see Section 3.1).

Theorem 1 (Theorem 3.2.5)**.** *Let $I\colon \mathcal{Y} \to \mathcal{L}$ be a fully faithful dense generator of $\mathcal{L}$. Let then $\tau\colon \Omega_\bullet \to \Omega$ be a discrete opfibration in $\mathcal{L}$. If for every $Y \in \mathcal{Y}$*

$$\mathcal{G}_{\tau,I(Y)}\colon \mathcal{L}\left(I(Y),\Omega\right) \to \mathcal{DOpFib}\left(I(Y)\right)$$

is fully faithful, then for every $F \in \mathcal{L}$ also

$$\mathcal{G}_{\tau,F}\colon \mathcal{L}\left(F,\Omega\right) \to \mathcal{DOpFib}\left(F\right)$$

is fully faithful, so that τ is a 2-classifier in $\mathcal{L}$.

Theorem 2 (Corollary 3.2.10)**.** *Let $I\colon \mathcal{Y} \to \mathcal{L}$ be a fully faithful dense generator of $\mathcal{L}$. Assume that $\tau\colon \Omega_\bullet \to \Omega$ is a 2-classifier in $\mathcal{L}$, and let $\varphi\colon G \to F$ be an arbitrary discrete opfibration in $\mathcal{L}$. Consider K and Λ as in Construction 3.2.7. The following properties are equivalent:*

(i) φ is classified by τ, i.e. φ is in the essential image of $\mathcal{G}_{\tau,F}$;

(ii) for every $(C,X) \in \int W$ the change of base $\mathcal{G}_{\varphi,K(C,X)}(\Lambda_{C,X})$ of φ along $\Lambda_{C,X}$ is in the essential image of $\mathcal{G}_{\tau,K(C,X)}$, and the operation of normalization described in Theorem 3.2.8 starting from φ is possible.

To have a hint of the benefits, we can look at the case of $\mathcal{Cat}$. These theorems allow us to deduce all the major properties of the Grothendieck construction from the trivial observation that everything works well over the singleton category (Example 3.2.13). The driving idea behind our results of reduction to dense generators is to express an arbitrary object F as a nice colimit of the dense generators and induce the required data using the universal property of the colimit.

In order to handle such colimits in our 2-categorical setting, we first need to develop a calculus of colimits in 2-dimensional slices (Chapter 2). We generalize to dimension 2 the well-known fact that a colimit in a 1-dimensional slice category is precisely the map from the colimit of the domains of the diagram which is induced by the universal property. We show that the appropriate 2-dimensional slice to consider for this generalization is the lax slice. We obtain the following result, that involves an original concept of colim fibration.

Theorem 3 (Theorem 2.1.21)**.** *Let $\mathcal{E}$ be a 2-category and $M \in \mathcal{E}$. Then the 2-functor $\operatorname{dom}\colon \mathcal{E}/_{\text{lax}}\, M \to \mathcal{E}$ is a 2-colim-fibration. As a consequence,*

$$\begin{array}{c}\operatorname{colim}^W F\\ \downarrow{\scriptstyle q}\\ M\end{array} = \begin{array}{c}\text{oplax}^{\text{cart}}\text{-}\operatorname{colim}^{\Delta 1}\left(F \circ \mathcal{G}\left(W\right)\right)\\ \downarrow{\scriptstyle q}\\ M\end{array} = \text{oplax}^{\text{cart}}\text{-}\operatorname{colim}^{\Delta 1} L^q$$

in the lax slice ${\mathcal{E}}\!\left/_{\!\text{lax}}\, M\right.$. *Here,* L^q *is the* 2*-diagram in* ${\mathcal{E}}\!\left/_{\!\text{lax}}\, M\right.$ *that corresponds to the cartesian-marked oplax cocone* λ^q *on* M *associated to the weighted* 2*-cocylinder on* M *that* q *represents.*

In dimension 1, we also have the useful results that the domain functor from a slice category preserves, reflects and lifts all colimits. We generalize all these results to dimension 2 as well (Chapter 2). We then study the change of base 2-functor between lax slices, laxifying the proof that Conduché functors are exponentiable (Section 2.4). To obtain these results, it is crucial for us to use $\mathcal{F}$-categorical techniques. The idea of $\mathcal{F}$-category theory, introduced in Lack and Shulman's [30], is to consider 2-categories with a selected subclass of morphisms that are called tight.

As shown in the theorem above, a key ingredient for the calculus of colimits in 2-dimensional slices is the reduction of weighted 2-colimits to cartesian-marked oplax conical ones, that are essentially conical. This result was proved in Street's [42]; we present here new, more elementary proofs (Chapter 1). Such reduction is needed, together with $\mathcal{F}$-category theory, to generalize to dimension 2 the bijective correspondence between cocones over an object M and diagrams in the slice over M (Section 2.2). The philosophy behind the reduction of weighted 2-colimits to cartesian-marked oplax conical ones is the following. To capture all data of a category $\mathcal{C}$, we can either consider functors from every possible category into $\mathcal{C}$ or functors from the singleton $\mathit{1}$ into $\mathcal{C}$ together with natural transformations between them. The former idea corresponds with weighted 2-colimits, while the latter corresponds with cartesian-marked oplax conical colimits.

The reduction of weighted 2-colimits to cartesian-marked oplax conical ones is regulated by the 2-category of elements construction. The latter is a natural extension of the Grothendieck construction. We study it in detail from an abstract point of view, using also an original notion of pointwise Kan extension (Chapter 1). This also contributes to expand the theory of cartesian-marked oplax conical colimits, that are crucial in this thesis. Our main results on the 2-category of elements are condensed in the following theorem.

Theorem 4 (Theorem 1.3.8 and Theorem 1.3.12)**.** *Let* $F\colon \mathcal{B} \to \mathcal{C}at$ *be a 2-functor with* $\mathcal{B}$ *a small 2-category. The* 2*-category of elements is equivalently given by the lax comma object*

$$\begin{tikzcd}
\int F \arrow[r] \arrow[d, "\mathcal{G}(F)"'] & \mathit{1} \arrow[d, "\mathit{1}"] \arrow[dl, Rightarrow, shorten=4mm, "\text{lax comma}"] \\
\mathcal{B} \arrow[r, "F"'] & \mathcal{C}at
\end{tikzcd}$$

in $2\text{-}\mathcal{C}at_{\mathrm{lax}}$. *Moreover, this lax comma object square exhibits* F *as the pointwise left Kan extension in* $2\text{-}\mathcal{C}at_{\mathrm{lax}}$ *of* $\Delta 1\colon \int F \to \mathcal{C}at$ *along* $\mathcal{G}(F)$*.*

$$F = \operatorname{Lan}_{\mathcal{G}(F)} \Delta 1.$$

We explain how this exhibits the 2-category of elements as the archetypal 3-dimensional classifier and also as the 2-$\mathcal{S}et$-enriched Grothendieck construction (Chapter 1). So that our results could pave the way towards higher dimensional elementary topos theory and towards a generalization of the Grothendieck construction to the enriched setting.

We then apply our theorems of reduction of the study of 2-classifiers to dense generators to the cases of prestacks and stacks (Chapter 5). Our results offer great benefits there, as they allow us to just consider the classification over representables. The Yoneda lemma determines up to equivalence the construction of a good 2-classifier in prestacks (Section 3.3). We explain how this involves discrete opfibrations over representables, which offer a 2-categorical notion of sieve. We find however a few problems in handling directly discrete opfibrations in prestacks over representables.

We develop an indexed version of the Grothendieck construction to solve all such problems (Chapter 4). This generalizes the classical Grothendieck construction to the case of opfibrations in the 2-category of prestacks. We obtain the following theorem.

Theorem 5 (Theorem 4.3.7 and Theorem 4.3.9)**.** *Let* $\mathcal{A}$ *be a small category and consider the functor* 2*-category* $[\mathcal{A}, \mathcal{C}at]$. *For every* 2*-functor* $F\colon \mathcal{A} \to \mathcal{C}at$, *there* *is an equivalence of categories*

$$\mathcal{OpFib}_{[\mathcal{A},\mathcal{Cat}]}(F) \simeq \left[\int F, \mathcal{Cat}\right]$$

between split opfibrations in the 2*-category* $[\mathcal{A}, \mathcal{Cat}]$ *over* F *and* 2*-(co)presheaves on the Grothendieck construction* $\int F$ *of* F*.*

This restricts to an equivalence of categories

$$\mathcal{DOpFib}^{\mathrm{s}}_{[\mathcal{A},\mathcal{Cat}]}(F) \simeq \left[\int F, \mathcal{Set}\right]$$

between discrete opfibrations in $[\mathcal{A}, \mathcal{Cat}]$ *over* F *with small fibres and* 1*-copresheaves on* $\int F$*.*

Moreover, both the equivalences of categories above are pseudonatural in F*.*

Taking $\mathcal{A} = \mathit{1}$, we recover the classical Grothendieck construction. We can think of the indexed Grothendieck construction as a simultaneous Grothendieck construction on every index $A \in \mathcal{A}$, taking into account the bonds between different indexes (Construction 4.3.8). Our result can also be interpreted as saying that every (op)fibrational slice of a Grothendieck 2-topos is a Grothendieck 2-topos. This generalizes to dimension 2 what is called the fundamental theorem of elementary topos theory (see Mac Lane and Moerdijk's [34, Section IV.7]), in the Grothendieck topoi case.

As a consequence, rather than considering discrete opfibrations with small fibres in prestacks over representables, we can equivalently consider presheaves on slices. This allows us to produce a good 2-classifier in prestacks that classifies all discrete opfibrations with small fibres.

Theorem 6 (Theorem 5.1.14)**.** *The 2-functor*

$$\begin{array}{rccc} \widetilde{\Omega}: & \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathcal{Cat} \\ & C & \mapsto & \left[\left(\mathcal{C}/_{C}\right)^{\mathrm{op}}, \mathcal{Set}\right] \\ & (C \xleftarrow{f} D) & \mapsto & - \circ (f \circ =)^{\mathrm{op}}, \end{array}$$

equipped with the 2-natural transformation $\widetilde{\omega}\colon 1 \to \widetilde{\Omega}$ *that picks the constant at* 1 *presheaf on every component, is a good* 2*-classifier in* $[\mathcal{C}^{\mathrm{op}}, \mathcal{Cat}]$ *that classifies all discrete opfibrations with small fibres.*

Finally, we restrict our good 2-classifier in prestacks to a good 2-classifier in stacks. We achieve this by proving a general result of restriction of good 2-classifiers to nice sub-2-categories, involving factorization arguments and our theorems of reduction to dense generators (Section 5.2). The idea is to select, out of all the presheaves on slices considered in the definition of $\widetilde{\Omega}$, which are the 2-categorical generalization of sieves, the sheaves with respect to the Grothendieck topology induced on the slices, which are the 2-categorical generalization of closed sieves. This restriction of $\widetilde{\Omega}$ is tight enough to give a stack Ω_J, but at the same time loose enough to still host the classification process of prestacks. We obtain the following theorem.

Theorem 7 (Theorem 5.3.22)**.** *The 2-functor*

$$\begin{array}{rccc} \Omega_J: & \mathcal{C}^{\text{op}} & \longrightarrow & \mathcal{C}at \\ & C & \mapsto & \mathcal{S}h\left(\mathcal{C}/_C, J\right) \\ & (C \xleftarrow{f} D) & \mapsto & -\circ(f\circ =)^{\text{op}}, \end{array}$$

equipped with the 2-natural transformation $\omega_J\colon 1 \to \Omega_J$ *that picks the constant at 1 sheaf on every component, is a good* 2*-classifier in* $\mathcal{S}t\left(\mathcal{C}, J\right)$ *that classifies all discrete opfibrations with small fibres.*

This is the main part of a proof that Grothendieck 2-topoi are elementary 2-topoi. Our results also solve a problem posed by Hofmann and Streicher in [25] when attempting to lift Grothendieck universes to sheaves. Indeed, in a different context, they considered the same natural idea to restrict their analogue of $\widetilde{\Omega}$ by taking sheaves on slices. However, this did not work for them, as it does not give a sheaf. Our results show that such a restriction yields nonetheless a stack and a good 2-classifier in stacks.

Outline of the thesis

In Chapter 1, we give a new, more elementary proof of the equivalence between weighted 2-limits and cartesian-marked lax conical ones. We then generalize to dimension 2 the fact that the category of elements can be captured by a comma object that also exhibits a pointwise left Kan extension. For this, we introduce a notion of pointwise Kan extension in 2-$\mathcal{C}at_{\text{lax}}$ and we refine the notion of lax comma object in 2-$\mathcal{C}at_{\text{lax}}$.

In Chapter 2, we present our calculus of colimits in 2-dimensional slices, thanks to an original notion of colim fibration. We also show results of preservation, reflection and lifting of 2-colimits for the domain 2-functor from a lax slice. The result of preservation is shown by proving a general theorem on $\mathcal{F}$-categorical lax left adjoints. Finally, we apply such general theorem to study the change of base 2-functor between lax slices.

In Chapter 3, after recalling the concept of 2-classifier, we introduce good 2-classifiers. We then present our theorems of reduction of the study of 2-classifiers to dense generators. We look at the possibility to apply these theorems to produce a good 2-classifier in prestacks that classifies all discrete opfibrations with small fibres. But using directly the induced 2-categorical sieves presents a few problems that we will solve in Chapter 4.

In Chapter 4, we show an equivalent characterization of opfibrations in $[\mathcal{A}, \mathcal{C}at]$. We then present an indexed version of the Grothendieck construction, proving a pseudonatural equivalence of categories between (split) opfibrations in $[\mathcal{A}, \mathcal{C}at]$ over F and 2-copresheaves on $\int F$. This also restricts to a discrete version. Finally, we obtain a nice candidate for a good 2-classifier in prestacks that classifies all discrete opfibrations with small fibres.

In Chapter 5, we apply our theorems of reduction to dense generators to the cases of prestacks and stacks. We produce a good 2-classifier in prestacks that classifies all discrete opfibrations with small fibres. We then prove a general result of restriction of good 2-classifiers to nice sub-2-categories, involving factorization arguments and our theorems of reduction to dense generators. We use this result to restrict our good 2-classifier in prestacks to one in stacks.

Notations

Throughout this thesis, we fix Grothendieck universes $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$ such that $\mathcal{U} \in \mathcal{V} \in \mathcal{W}$. We denote as $\mathcal{S}et$ the category of $\mathcal{U}$-small sets, as $\mathcal{C}at$ the 2-category of $\mathcal{V}$-small categories (i.e. categories such that both the collections of their objects and of their morphisms are $\mathcal{V}$-small) and as $\mathcal{CAT}$ the 2-category of $\mathcal{W}$-small categories. So that $\mathcal{S}et \in \mathcal{C}at$ and the underlying category $\mathcal{C}at_0$ of $\mathcal{C}at$ is in $\mathcal{CAT}$. *Small category* will mean $\mathcal{V}$-small category. *Small fibres*, for a discrete opfibration in $\mathcal{C}at$, will mean $\mathcal{U}$-small fibres. *2-category* will mean a $\mathcal{W}$-small $\mathcal{C}at$-enriched category. *Small 2-category* will mean $\mathcal{V}$-small 2-category. *Small fibres* for a discrete 2-opfibration will mean $\mathcal{V}$-small fibres.

We will use the following notations.

$\mathcal{C}^{\mathrm{op}}$	the opposite of an (enriched) category
$\mathcal{C}^{\mathrm{co}}$	2-category obtained from $\mathcal{C}$ dualizing the 2-cells; variant $\mathcal{C}^{\mathrm{coop}}$ if both 1-cells and 2-cells are dualized
2-$\mathcal{C}at$	the 3-category of 2-categories, 2-functors, 2-natural transformations and modifications
2-$\mathcal{C}at_{\mathrm{lax}}$	the lax 3-category of 2-categories, 2-functors, lax natural transformations and modifications
$\mathcal{V}$-$\mathcal{C}at$	the 2-category of $\mathcal{V}$-enriched categories
$\mathcal{C}(A, B)$	hom-object of an enriched category $\mathcal{C}$ from C to D (usually hom-category of a 2-category $\mathcal{C}$)
$[\mathcal{C}, \mathcal{D}]$	the strict functor (2-)category from $\mathcal{C}$ to $\mathcal{D}$

$[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]_{\mathrm{lax}}$	the 2-category of 2-functors, lax natural transformations and modifications; variants with $\mathrm{lax}^{\mathrm{cart}}$ (cartesian-marked lax), oplax, $\mathrm{oplax}^{\mathrm{cart}}$ (cartesian-marked oplax), sigma, $\mathrm{oplax}^{\mathrm{mark}}$ (marked oplax) in place of lax
$F \underset{\mathrm{oplax}^{\mathrm{cart}}}{\Longrightarrow} G$	a cartesian-marked oplax natural transformation; variants with pseudo (pseudonatural), sigma, $\mathrm{oplax}^{\mathrm{mark}}$ (marked oplax)
$\mathrm{Ps}\,[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$	the 2-category of pseudofunctors, pseudonatural transformations and modifications
$\mathrm{y}\colon \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$	the 2-categorical Yoneda embedding
$\mathcal{S}t\,(\mathcal{C}, J)$	the full sub-2-category of $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ on stacks with respect to the Grothendieck topology J
$\lim^W F$	the enriched limit of F weighted by W; variant $\mathrm{colim}^W F$ for colimits
$\mathrm{oplax}^{\mathrm{cart}}\text{-}\mathrm{colim}^{\Delta 1} K$	the cartesian-marked oplax conical colimit of the 2-diagram K; variants with limits in place of colimits, lax in place of oplax, other specified markings
$\mathcal{G}(F)\colon \int F \to \mathcal{C}$	the 2-category of elements of a 2-functor $F\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}at$ (with $\mathcal{A}$ a 2-category), or also the classical Grothendieck construction
$(\varphi)_B$	the fibre of an opfibration φ in $\mathcal{C}at$ over an object B of the base
$\mathcal{O}p\mathcal{F}ib_{\mathcal{L}}(F)$	the category of split opfibrations in a 2-category $\mathcal{L}$ over F; variant $\mathcal{DO}p\mathcal{F}ib_{\mathcal{L}}(F)$ for discrete opfibrations
$\mathcal{DO}p\mathcal{F}ib^{\mathrm{s}}(F)$	the full subcategory of $\mathcal{DO}p\mathcal{F}ib_{\mathcal{L}}(F)$ on discrete opfibrations with small fibres; variant $\mathcal{DO}p\mathcal{F}ib^{\mathrm{P}}(F)$ for discrete opfibrations that satisfy a fixed pullback-stable property P
$\mathcal{C}/_{C}$	the slice of a category $\mathcal{C}$ over $C \in \mathcal{C}$
$\mathcal{L}/_{\mathrm{lax}}\, M$	the lax slice of a 2-category $\mathcal{L}$ over $M \in \mathcal{L}$; sometimes, also a general lax comma
$F \mathbin{/\!\!/} G$	the lax comma object in $2\text{-}\mathcal{C}at_{\mathrm{lax}}$ from F to G

dom	the domain (2-)functor from a (lax) slice
$\mathrm{pr}_1\colon A \times B \to A$	the projection from a product to its first component; variant pr_2 for the projection to the second component
1	the terminal object of a (2-)category; variant $\mathit{1}$ for the singleton category
$\Delta 1$	the constant at 1 presheaf; variant ΔU for the functor which is constant at an object U
$f \circ g$	the composite of 1-cells or the vertical composition of 2-cells
$\alpha * \beta$	the horizontal composition of 2-cells; variants $\alpha * f$ and $f * \alpha$ for the whiskerings of a 2-cell α with a 1-cell f
id_A	the identity 1-cell on A; variants $\mathrm{Id}_{\mathcal{C}}$ for the identity (2-)functor on $\mathcal{C}$ and id_f for the identity 2-cell on a 1-cell f
$A \cong B$	an isomorphism between objects A and B; variant $A \xrightarrow{\cong} B$ for an isomorphic arrow from A to B
$\mathcal{C} \simeq \mathcal{D}$	an equivalence of categories between $\mathcal{C}$ and $\mathcal{D}$; variant $\mathcal{C} \xrightarrow{\simeq} \mathcal{D}$ for an equivalence from $\mathcal{C}$ to $\mathcal{D}$
$A \underset{\mathrm{ff}}{\hookrightarrow} B$	a fully faithful morphism in a 2-category or a fully faithful 2-functor
$\mathcal{M} \subseteq \mathcal{L}$	a full sub-2-category, i.e. an injective on objects and fully faithful 2-functor
$\mathcal{A}_\tau$	the tight part of an $\mathcal{F}$-category $\mathcal{A}$; variant $\mathcal{A}_\lambda$ for the loose part
$-$	placeholder; variant $+$ for a second placeholder

1. The 2-$\mathcal{S}et$-enriched Grothendieck construction and cartesian-marked lax limits

This chapter is based on our paper [36].

In this chapter we expand the theory of the 2-category of elements and of the cartesian-marked lax limits. Both these concepts will be very useful to us in the following chapters. Moreover, we believe they are key elements for higher dimensional elementary topos theory, as we explain below. The 2-category of elements is the 2-dimensional generalization of the construction of the category of elements, and it has been introduced by Street in [42]. It is at the same time a natural extension of the usual Grothendieck construction that admits 2-functors from a 2-category $\mathcal{B}$ into $\mathcal{C}at$, and a restriction of the 2-dimensional Grothendieck construction of Baković [5] and Buckley [10] to 2-functors into 2-$\mathcal{C}at$ that factor through $\mathcal{C}at$. Analogously, the corresponding notion of opfibration, introduced by Lambert in [31] with the name discrete 2-opfibration, is at the same time a natural extension of the usual Grothendieck opfibrations and a locally discrete version of Hermida's [23] 2-fibrations. Lambert proved in [31] that split discrete 2-opfibrations with small fibres form the essential image of the 2-functor that calculates the 2-category of elements. We extend this result to 2-equivalences between 2-copresheaves and discrete 2-opfibrations (Theorem 1.3.15).

We study the 2-category of elements from an abstract point of view and we interpret it as the 2-$\mathcal{S}et$-enriched Grothendieck construction. We expect this work to be very useful for generalizations of the Grothendieck construction. In future work, we would like to achieve a generalization to the enriched context. While Vasilakopoulou defined in [48] a notion of fibration enriched over a monoidal fibration, it seems no one has yet proposed a good notion of fibration for a $\mathcal{V}$-enriched functor, with $\mathcal{V}$ a nice enough monoidal category. There would be many applications of such enriched fibrations. Some examples would be additive fibrations, graded fibrations, metric fibrations and general quantale-enriched fibrations. We believe this work is a starting point towards such a theory. As giving an explicit definition of enriched fibration is quite hard, we would like to capture Grothendieck fibrations and the Grothendieck construction from an abstract point of view and try to generalize such abstract theory.

Another motivation that we have in mind is to understand how the various properties of the 2-category of elements are connected with each other. We will show in Section 1.3 that our pointwise Kan extension result for the 2-category of elements (see below) implies many other properties. Among these, the conicalization of weighted 2-limits and the 2-fully faithfulness of the 2-functor that calculates the 2-category of elements.

In dimension 1, it is known that the category of elements can be captured in a more abstract way. Given a copresheaf $F\colon \mathcal{B} \to \mathcal{S}et$, the construction of the category of elements of F is equivalently given by the comma object

$$\begin{tikzcd}
\int^{\mathrm{op}} F \arrow[r] \arrow[d, "\mathcal{G}(F)"'] & 1 \arrow[d, "1"] \arrow[dl, Rightarrow, shorten=12pt, "\text{comma}"] \\
\mathcal{B} \arrow[r, "F"'] & \mathcal{S}et
\end{tikzcd} \tag{1.1}$$

Moreover, this filled square exhibits F as the pointwise left Kan extension of the constant at 1 functor $\Delta 1\colon \int^{\mathrm{op}} F \to \mathcal{S}et$ along the discrete opfibration $\mathcal{G}(F)$.

Our main theorem of this chapter (Theorem 1.3.12, after Theorem 1.3.8) is a 2-dimensional generalization of this result. We prove that an analogous square as the above one exhibits, at the same time, the 2-category of elements $\mathcal{G}(F)$ as a lax comma object in 2-$\mathcal{C}at_{\mathrm{lax}}$ and F as the pointwise left Kan extension in 2-$\mathcal{C}at_{\mathrm{lax}}$

of $\Delta 1$ along the discrete 2-opfibration $\mathcal{G}(F)$. We find the need to consider the lax 3-category 2-$\mathcal{C}at_{\mathrm{lax}}$ of 2-categories, 2-functors, lax natural transformations and modifications, because the analogue of the square above is now only filled by a lax natural transformation. Lax 3-categories, introduced by Lambert in [31], are categories enriched over the cartesian closed category of 2-categories and lax functors (whose tensor product is the ordinary product of 2-categories). As 2-$\mathcal{C}at_{\mathrm{lax}}$ has not yet been studied much, we have to introduce a notion of pointwise Kan extension in 2-$\mathcal{C}at_{\mathrm{lax}}$ (Definition 1.2.16) to achieve our objective. This is one of our main contributions in this chapter. We also prove that pointwise Kan extensions in 2-$\mathcal{C}at_{\mathrm{lax}}$ along a discrete 2-opfibration are always weak ones as well (Proposition 1.2.19). The proof is based on an original generalization of the parametrized Yoneda lemma which is as lax as it can be (Theorem 1.2.18).

Pointwise Kan extensions are actively researched. While it is relatively easy to give notions of weak Kan extension in a categorical framework, it is much harder to give the corresponding pointwise notions. In [41], Street proposes to look at the stability of a Kan extension under pasting with comma objects to obtain a definition of pointwise Kan extension in any 2-category. However, applying Street's definition to the 2-category $\mathcal{V}$-$\mathcal{C}at$ does not give the right notion. For enriched $\mathcal{V}$-functors, the correct notion, that uses $\mathcal{V}$-limits, has been introduced by Dubuc in [16] and later used by Kelly in [28]. However, we needed pointwise Kan extensions in the lax 3-category 2-$\mathcal{C}at_{\mathrm{lax}}$, that we view as 2-$\mathcal{S}et$-$\mathcal{C}at$, with an original idea of *2-$\mathcal{V}$-enrichment*. Pseudo-Kan extensions of Lucatelli Nunes's [33] are a pseudo version of Dubuc's ones, considering weighted bilimits in the place of weighted 2-limits, and quasi-Kan extensions of Gray's [21] are a lax version. Instead, we needed a strict version, using some form of strict 2-limit, but which also takes the 2-$\mathcal{V}$-enrichment into account.

To give our definition of pointwise Kan extension in 2-$\mathcal{C}at_{\mathrm{lax}}$, we take advantage of the connection between the 2-category of elements and cartesian-marked oplax colimits, that we recall in Section 1.1 from a new, more elementary perspective. Cartesian-marked (op)lax conical (co)limits are a particular case of a 2-dimensional notion of limit introduced by Gray in [20, Section I,7] (called there "cartesian quasi-limits"). As proved by Street in [42, Theorem 14 and Theorem 15], and here as well with a more elementary proof, they are an alternative to weighted 2-limits. Indeed they are particular weighted 2-limits and every weighted 2-limit can be reduced to one of them. But cartesian-marked lax conical limits can be much more useful in some situations, as they allow to consider cones rather than cylinders, up to filling the cones with coherent 2-cells. Some of these 2-cells are required to be the identity, whence the adjective "marked". As the choice of such 2-cells comes from the cartesian liftings of a 2-category of elements, we call them "cartesian". Despite their potential, cartesian-marked lax limits have been almost forgotten, until Descotte, Dubuc and Szyld's paper [15], where they use their pseudo version, called by them *sigma-limits*. Later, in [44] and [45], Szyld also considered the strict version that we use here. In Chapter 2, the reduction of weighted 2-limits to *cartesian-marked lax conical* ones will be crucial for us to study colimits in 2-dimensional slices. It will then allow us, in Chapter 3, to reduce the conditions of a 2-classifier to dense generators, and hence, in Chapter 5, to construct a 2-classifier in stacks, towards a proof that Grothendieck 2-toposes are elementary 2-toposes.

We hint at the potential applications of the work of this chapter to higher dimensional elementary topos theory. According to Weber's [51], the filled square of equation (1.1) presents $\mathcal{C}at$ as the archetypal 2-dimensional elementary topos. The classification process is the category of elements construction. This is recalled in Section 3.1. On this line, we believe we should consider 2-$\mathcal{C}at_{\mathrm{lax}}$ as the archetypal 3-dimensional elementary topos. Its classifier would be $\mathit{1}\colon \mathit{1} \to \mathcal{C}at$ and its classification process would be the 2-category of elements construction. We inscribe the sequence of elementary n-topoi

$$\mathcal{S}et \quad \rightsquigarrow \quad \mathcal{C}at \quad \rightsquigarrow \quad 2\text{-}\mathcal{C}at_{\mathrm{lax}}$$

in an original idea of 2-$\mathcal{V}$-*enrichment*. This guided our definition of pointwise Kan extension in 2-$\mathcal{C}at_{\mathrm{lax}}$. And we believe that this observation could be useful also towards an enriched version of the Grothendieck construction. For this, we may also notice that comma objects, that regulate the classification process in $\mathcal{C}at$, are the archetypal example of exact square in $\mathcal{C}at$. And the fact that every copresheaf is the pointwise Kan extension of $\Delta 1$ along its category of elements is actually a consequence of having an exact square, together with $\mathit{1}\colon \mathit{1} \to \mathcal{S}et$ being dense. Moving to 2-$\mathcal{C}at_{\text{lax}}$, we need to upgrade comma objects to lax comma objects. We give a new, refined universal property of lax comma objects to suit the lax 3-dimensional ambient of 2-$\mathcal{C}at_{\text{lax}}$ (Definition 1.3.4). This improves both the universal properties given by Gray in [20, Sections I,2 and I,5] and by Lambert in [31].

Outline of the chapter

In Section 1.1, we recall from an original perspective the explicit 2-category of elements and the cartesian-marked (op)lax conical (co)limits. We give a new, more elementary proof of the equivalence between weighted 2-limits and cartesian-marked lax conical ones.

In Section 1.2, we introduce a notion of pointwise Kan extension in 2-$\mathcal{C}at_{\text{lax}}$ along a discrete 2-opfibration. We prove that such a pointwise Kan extension is always a weak one as well. The proof is based on an original generalization of the parametrized Yoneda lemma.

In Section 1.3, we generalize to dimension 2 the fact that the category of elements can be captured by a comma object that also exhibits a pointwise left Kan extension. For this we use our definition of pointwise Kan extension in 2-$\mathcal{C}at_{\text{lax}}$ and a new refined notion of lax comma object in 2-$\mathcal{C}at_{\text{lax}}$.

1.1. Cartesian-marked (op)lax conical (co)limits

In this section we recall, from an original perspective, the explicit 2-category of elements construction and the cartesian-marked (op)lax conical (co)limits. We originally show how the two concepts arise simultaneously by the wish of giving an essential solution to the problem of conicalization of the weighted 2-limits. This can also be seen as a justification to both the explicit definition of the 2-*category of elements*, introduced by Street in [42], and the *cartesian-marked lax conical limits*, a particular case of a notion introduced by Gray in [20, Section I,7] (called there “cartesian quasi-limits”).

We obtain a new, more elementary proof of the fact that weighted 2-limits and *cartesian-marked lax conical limits* give equivalent theories (Theorem 1.1.15 and Theorem 1.1.13). This has been firstly proved by Street in [42, Theorem 14 and Theorem 15], where weighted 2-limits are called “indexed limits” and (following Gray’s [20]) cartesian-marked lax conical limits are a particular case of ”cartesian quasi-limits”. Cartesian-marked lax conical limits offer huge benefits in many situations, as they have a conical shape, even if with coherent 2-cells inside the triangles that form the cone. Sometimes, it is much easier to handle such 2-cells rather than a non-conical shape.

Indeed such an idea will be crucial to us in Chapters 2, 3 and 5. In Section 1.2, the concept of cartesian-marked (op)lax (co)limit will guide the original definitions of colimit in a 2-$\mathcal{S}et$*-category* and of *pointwise left Kan extension in* 2-$\mathcal{C}at_{\text{lax}}$.

We first recall the notion of weighted limit, from Kelly’s book [28, Chapter 3] (also here weighted limits are called “indexed limits”).

Definition 1.1.1. Let $\mathcal{V}$ be a complete and cocomplete symmetric closed monoidal category. Consider $\mathcal{V}$-functors $F\colon \mathcal{A} \to \mathcal{C}$ (the diagram) and $W\colon \mathcal{A} \to \mathcal{V}$ (the weight, in place of the classical constant at 1 functor $\Delta 1$, that now does no longer suffice), with $\mathcal{A}$ a small $\mathcal{V}$-category. The *$\mathcal{V}$-limit of F weighted by W*, denoted as $\lim^W F$, is (if it exists) an object $L \in \mathcal{C}$ together with an isomorphism in $\mathcal{V}$

$$\mathcal{C}\,(U, L) \cong \left[\mathcal{A}, \mathcal{V}\right] (W, \mathcal{C}\,(U, F(-))) \tag{1.2}$$

$\mathcal{V}$-natural in $U \in \mathcal{C}^{\text{op}}$, where $\left[\mathcal{A}, \mathcal{V}\right]$ is the $\mathcal{V}$-category of $\mathcal{V}$-copresheaves on $\mathcal{A}$ valued in $\mathcal{V}$ enriched over itself. When $\lim^W F$ exists, the identity on L provides a $\mathcal{V}$-natural transformation $\lambda\colon W \Rightarrow \mathcal{C}\,(L, F(-))$ called the *universal cylinder*.

For the notion of *weighted $\mathcal{V}$-colimit*, we start from $F\colon \mathcal{A} \to \mathcal{C}$ and $W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$. The universal property is then

$$\mathcal{C}\left(\operatorname{colim}^W F, U\right) \cong \left[\mathcal{A}^{\mathrm{op}}, \mathcal{V}\right] \left(W, \mathcal{C}\left(F(-), U\right)\right)$$

Weighted $\mathcal{C}at$-(co)limits, i.e. with $\mathcal{V} = \mathcal{C}at$, will be called weighted 2-(co)limits.

Remark 1.1.2. Although the classical constant at 1 weight $\Delta 1$, called the *conical weight*, no longer suffices in the general enriched setting, we pay attention to when a weighted limit can be *reduced to a conical one*. It is well-known (see Kelly's [28, Section 3.4] that in the $\mathcal{S}et$-enriched setting every weighted limit can be conicalized, using the category of elements. A general strategy of conicalization (used by Kelly in [28, Section 3.4]) allows to conicalize just the $\mathcal{V}$-colimits $W \cong \operatorname{colim}^W \mathrm{y}$, for $W\colon \mathcal{A} \to \mathcal{V}$ an enriched presheaf with $\mathcal{A}$ small and $\mathrm{y}\colon \mathcal{A}^{\mathrm{op}} \to \left[\mathcal{A}, \mathcal{V}\right]$ the $\mathcal{V}$-Yoneda embedding. The lemma of continuity of a limit in its weight, whose formula is

$$\lim^{\operatorname{colim}^W H} F \cong \lim^W \left(\lim^{H(-)} F\right), \tag{1.3}$$

then allows to deduce that all weighted limits are conicalized. We will use such a strategy in the proof of Theorem 1.1.15.

Remark 1.1.3. When $\mathcal{V} = \mathcal{C}at$, the conicalization of weighted 2-limits is, strictly speaking, not possible. We would need to encode the universal cocylinder μ (given by the Yoneda lemma) in terms of a universal cocone

$$\widetilde{\mu}\colon \Delta 1 \Rightarrow \left[\mathcal{A}, \mathcal{C}at\right] \left(H(-), W\right)$$

with H some 2-functor $\mathcal{B} \to [\mathcal{A}, \mathcal{C}at]$. And this is not possible because the components of μ are functors rather than mere functions. The idea is to admit 2-cells inside the cocone $\widetilde{\mu}$ in order to encode the extra data. We thus consider lax natural transformations, that have general structure 2-cells inside the naturality squares. But, as we explain in Construction 1.1.4, the right notion of relaxed 2-natural transformation to consider will be a marked version of lax natural transformations.

Construction 1.1.4 (2-category of elements)**.** Following Remark 1.1.3, we search for a relaxed notion of 2-natural transformation, which is however stronger than a lax natural transformation, and for a 2-functor $H\colon \mathcal{B} \to [\mathcal{A}, \mathcal{C}at]$ such that any cocylinder

$$\varphi\colon W \Rightarrow [\mathcal{A}, \mathcal{C}at]\,(\mathrm{y}\,(-), U)$$

with $U\colon \mathcal{A} \to \mathcal{C}at$ can be encoded in terms of a relaxed 2-natural transformation

$$\widetilde{\varphi}\colon \Delta 1 \underset{\text{relaxed}}{\Longrightarrow} [\mathcal{A}, \mathcal{C}at]\,(H(-), U) : \mathcal{B}^{\mathrm{op}} \to \mathcal{C}at.$$

In order to deduce that every weighted 2-limit can be analogously conicalized via the lemma of continuity of a limit in its weight, we need H of the form

$$\left(\int^{\mathrm{op}} W\right)^{\mathrm{op}} \xrightarrow{\mathcal{G}(W)^{\mathrm{op}}} \mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{y}} [\mathcal{A}, \mathcal{C}at]\,.$$

Up to now, $\int^{\mathrm{op}} W$ and $\mathcal{G}\,(A)$ are just symbols, but will be found to be the 2-*category of elements*, as defined explicitly by Street in [42, just above Theorem 15].

For every $A \in \mathcal{A}$ and $X \in W(A)$, we have a morphism $\varphi_A(X)\colon\; \mathrm{y}\,(A) \to U$, and we want to form the cocone $\widetilde{\varphi}$ exactly with these morphisms. So we take the objects of $\int^{\mathrm{op}} W$ to be all pairs (A, X) with $A \in \mathcal{A}$ and $X \in W(A)$, and define $\mathcal{G}\,(W)\,(A, X) \coloneqq A$. We then set $\widetilde{\varphi}_{(A,X)} \coloneqq \varphi_A(X)$.

But $\widetilde{\varphi}$ also needs to encode the assignment of every φ_A on morphisms $\alpha\colon X \to X'$ in $W(A)$. Lax naturality of $\widetilde{\varphi}$ allows to have, for every $\xi\colon (A, X) \to (A', X')$ in $\int^{\mathrm{op}} W$, a 2-cell

$$\begin{tikzcd}
1 \arrow[r, "\widetilde{\varphi}_{(A,X)}"] \arrow[d, equal] & {[\mathcal{A}, \mathcal{C}at]\,(\mathrm{y}\,(A), U)} \arrow[d, "-\circ \mathrm{y}(\mathcal{G}(W)^{\mathrm{op}}(\xi))"] \arrow[dl, Rightarrow, shorten=20pt, "\widetilde{\varphi}_\xi"] \\
1 \arrow[r, "\widetilde{\varphi}_{(A',X')}"'] & {[\mathcal{A}, \mathcal{C}at]\,(\mathrm{y}\,(A'), U)}
\end{tikzcd}$$

For every $A \in \mathcal{A}$ and $\alpha\colon X \to X'$ in $W(A)$, we need a morphism $(A, X) \to (A, X')$ in $\int^{\mathrm{op}} W$ whose image with respect to $\mathcal{G}\,(W)$ is id_A. Wishing to write the action of $\mathcal{G}\,(W)$ as a projection on the first component, we call such morphism $(A, X) \to (A, X')$ as (id_A, α). We set $\widetilde{\varphi}_{(\mathrm{id}_A, \alpha)} \coloneqq \varphi_A(\alpha)$.

We now encode the 2-naturality of φ into the relaxed naturality of $\widetilde{\varphi}$. For every $f\colon A \to A'$ in $\mathcal{A}$ and $X \in W(A)$, the naturality of φ expresses the equality

$$\varphi_{A'}(W(f)(X)) = \varphi_A(X) \circ \mathrm{y}(f)\,.$$

So, for every $f\colon A \to A'$ in $\mathcal{A}$ and $X \in W(A)$, we need a morphism

$$\underline{f}^X\colon (A,X) \to (A', W(f)(X))$$

in $\int^{\mathrm{op}} W$ such that $\mathcal{G}(W)\left(\underline{f}^X\right) = f$ and $\widetilde{\varphi}_{\underline{f}^X} = \mathrm{id}$.

It is natural to take $\underline{\mathrm{id}_A}^X = (\mathrm{id}_A, \mathrm{id}_X)$ for every $A \in \mathcal{A}$ and $X \in W(A)$ and ask any of such equal morphisms to be the identity on (A,X). We then need to close the union of the two kinds of morphisms (id_A, α) and $\underline{f}^X$ under composition. For this, we notice that, given $f\colon A \to A'$ in $\mathcal{A}$ and $\alpha\colon X \to X'$ in $W(A)$, the two morphisms $\underline{f}^{X'} \circ (\mathrm{id}_A, \alpha)$ and $(\mathrm{id}_{A'}, W(f)(\alpha)) \circ \underline{f}^X$ in $\int^{\mathrm{op}} W$ will have the same associated structure 2-cell of $\widetilde{\varphi}$, by lax naturality of $\widetilde{\varphi}$. We then take such two morphisms in $\int^{\mathrm{op}} W$ to be equal, so that we will be able to recover the naturality of φ (on morphisms) starting from $\widetilde{\varphi}$. At this point, every finite composition of morphisms in $\int^{\mathrm{op}} W$ can be reduced to a composite

$$(A,X) \xrightarrow{\underline{f}^X} (A', W(f)(X)) \xrightarrow{(\mathrm{id}_{A'},\alpha)} (A', X')$$

for some $f\colon A \to A'$ in $\mathcal{A}$ and $\alpha\colon W(f)(X) \to X'$ in $W(A')$. We define the morphisms in $\int^{\mathrm{op}} W$ to be all the formal composites $(\mathrm{id}_{A'}, \alpha) \circ \underline{f}^X$, that we call (f, α). And we see that $\underline{f}^X = (f, \mathrm{id}_{W(f)(X)})$. Functoriality forces $\mathcal{G}(W)\,(f,\alpha) = f$, and lax naturality forces

$$\widetilde{\varphi}_{(f,\alpha)} = \widetilde{\varphi}_{(\mathrm{id}_{A'},\alpha)\circ(f,\mathrm{id})} = \varphi_{A'}(\alpha) \circ \mathrm{id} = \varphi_{A'}(\alpha).$$

We now want to encode the 2-dimensional part of the 2-naturality of φ into the 2-dimensional part of the relaxed naturality of $\widetilde{\varphi}$. Lax naturality of $\widetilde{\varphi}$ allows to have, for every 2-cell $\Xi\colon (f,\alpha) \Rightarrow (g,\beta)\colon (A,X) \to (A',X')$ in $\int^{\mathrm{op}} W$,

$$\begin{tikzcd}
1 \arrow[r, "\widetilde{\varphi}_{(A,X)}"] \arrow[d, equal] & {[\mathcal{A}, \mathcal{C}at]\,(\mathrm{y}\,(A), U)} \arrow[d, "-\circ\mathrm{y}(f)"] \arrow[dl, Rightarrow, shorten=1.5em, "\widetilde{\varphi}_{(f,\alpha)}"] \\
1 \arrow[r, "\widetilde{\varphi}_{(A',X')}"'] & {[\mathcal{A}, \mathcal{C}at]\,(\mathrm{y}\,(A'), U)}
\end{tikzcd}
=
\begin{tikzcd}
1 \arrow[r, "\widetilde{\varphi}_{(A,X)}"] \arrow[d, equal] & {[\mathcal{A}, \mathcal{C}at]\,(\mathrm{y}\,(A), U)} \arrow[d, bend right=60, "-\circ\mathrm{y}(g)"', ""{name=G}] \arrow[d, bend left=60, "-\circ\mathrm{y}(f)", ""'{name=F}] \arrow[dl, Rightarrow, shorten=1.5em, "\widetilde{\varphi}_{(g,\beta)}"'] \\
1 \arrow[r, "\widetilde{\varphi}_{(A',X')}"'] & {[\mathcal{A}, \mathcal{C}at]\,(\mathrm{y}\,(A'), U)}
\arrow[from=F, to=G, Rightarrow, "-*\mathrm{y}(\mathcal{G}(W)^{\mathrm{op}}(\Xi))"']
\end{tikzcd}
\tag{1.4}$$

The 2-naturality of φ expresses the following equality, for every 2-cell $\delta\colon f \Rightarrow g\colon A \to A'$ in $\mathcal{A}$ and for every $X \in W(A)$:

$$\varphi_{A'}(W(\delta)_X) = \varphi_A(X)\,\mathrm{y}(\delta) : \varphi_{A'}(W(f)(X)) \Rightarrow \varphi_{A'}(W(g)(X)).$$

So, for every 2-cell $\delta\colon f \Rightarrow g\colon A \to A'$ in $\mathcal{A}$ and every $X \in W(A)$, we need a 2-cell in $\int^{\mathrm{op}} W$, that we call $\underline{\delta}^X$ or just δ, such that

$$\underline{\delta}^X : (f, W(\delta)_X) \Rightarrow (g, \mathrm{id})\colon (A, X) \to (A', W(g)(X))$$

and $\mathcal{G}(W)\left(\underline{\delta}^X\right) = \delta$. These 2-cells are closed under both vertical and horizontal composition, inherited from $\mathcal{A}$, but we have to close them under whiskering with morphisms (id_A, α). Notice that for every $\delta\colon f \Rightarrow g\colon A \to A'$ in $\mathcal{A}$ and every $\alpha\colon X \to X'$ in $W(A)$, we have that the axiom of equation (1.4) of $\widetilde{\varphi}$ on the two whiskerings $\underline{\delta}^{X'}(\mathrm{id}_A, \alpha)$ and $(\mathrm{id}_{A'}, W(g)(\alpha))\underline{\delta}^X$ in $\int^{\mathrm{op}} W$ is exactly the same. So we ask such two whiskerings in $\int^{\mathrm{op}} W$ to be equal. At this point, every horizontal composition of 2-cells in $\int^{\mathrm{op}} W$ can be reduced to a whiskering of the form $(\mathrm{id}, \beta)\underline{\delta}^X$ for some 2-cell $\delta\colon f \Rightarrow g\colon A \to A'$ in $\mathcal{A}$, $X \in W(A)$ and $\beta\colon W(g)(X) \to X'$ in $W(A')$. We define the 2-cells in $\int^{\mathrm{op}} W$ to be precisely such formal whiskerings. Equivalently, a 2-cell $(f, \alpha) \Rightarrow (g, \beta)\colon (A, X) \to (A', X')$ in $\int^{\mathrm{op}} W$ is a 2-cell $\delta\colon f \Rightarrow g$ in $\mathcal{A}$ such that

$$\alpha = \beta \circ W(\delta)_X.$$

For this, we will call such 2-cell just $\delta\colon (f, \alpha) \Rightarrow (g, \beta)$. Compositions are inherited from $\mathcal{A}$. Then 2-functoriality forces $\mathcal{G}(W)(\delta) = \delta$.

It is straightforward to show that $\int^{\mathrm{op}} W$ is a 2-category and that $\mathcal{G}(W) : \int^{\mathrm{op}} W \to \mathcal{A}$ is a 2-functor. Notice that we have also described the right notion of relaxed 2-natural transformation that $\widetilde{\varphi}$ needs to satisfy to encode the 2-naturality of φ. We will define it in Definition 1.1.9. It is a form of marked lax natural transformation, i.e. a lax natural transformation such that certain structure 2-cells are asked to be identities.

We read from Construction 1.1.4 the following explicit definition of the 2-*category of elements*, that coincides with the one of Street's [42, just above Theorem 15] (called there "Grothendieck construction").

Definition 1.1.5. Let $F\colon \mathcal{B} \to \mathcal{C}at$ be a 2-functor with $\mathcal{B}$ a 2-category. The *2-category of elements of F* is the 2-functor $\mathcal{G}(F): \int^{\mathrm{op}} F \to \mathcal{B}$, given by the projection on the first component, with $\int^{\mathrm{op}} F$ such that:

an object of $\int^{\mathrm{op}} F$ is a pair (B,X) with $B \in \mathcal{B}$ and $X \in F(B)$;

a morphism $(B,X) \to (C,X')$ *in* $\int^{\mathrm{op}} F$ is a pair (f,α) with $f\colon B \to C$ a morphism in $\mathcal{B}$ and $\alpha\colon F(f)(X) \to X'$ a morphism in $F(C)$;

a 2-cell $(f,\alpha) \Rightarrow (g,\beta)\colon (B,X) \to (C,X')$ *in* $\int^{\mathrm{op}} W$ is a 2-cell $\delta\colon f \Rightarrow g$ in $\mathcal{B}$ such that $\alpha = \beta \circ F(\delta)_X$;

the compositions and identities are as described in Construction 1.1.4.

Remark 1.1.6. The 2-category of elements is a natural extension of the usual Grothendieck construction, that allows $\mathcal{B}$ to be a 2-category rather than just a 1-category. It is also the restriction of the 2-Grothendieck construction of Baković [5] and Buckley [10] to 2-functors into 2-$\mathcal{C}at$ that factorize through $\mathcal{C}at$. Analogously, the corresponding notion of opfibration is a natural extension of the usual notion, and at the same time a locally discrete version of Hermida's 2-fibrations ([23]). Notice from Construction 1.1.4 that having a 2-cell δ in a 2-category of elements is indeed a property for the underlying 2-cell δ in the base category.

Definition 1.1.7 (Lambert [31])**.** Let $\mathcal{B}$ be a 2-category. A *discrete 2-opfibration over* $\mathcal{B}$ is a 2-functor $P\colon \mathcal{E} \to \mathcal{B}$ such that

(i) the underlying functor P_0 of P is an ordinary Grothendieck opfibration;

(ii) for every pair $X, Y \in \mathcal{E}$ the functor

$$P_{X,Y}\colon \mathcal{E}(X,Y) \to \mathcal{B}(P(X),P(Y))$$

is a discrete fibration.

We say that P is *split* if P_0 is so.

Theorem 1.1.8 (Lambert [31])**.** *Let $\mathcal{B}$ be a 2-category. The essential image of the 2-functor*

$$\mathcal{G}(-): [\mathcal{B}, \mathcal{C}at] \to 2\text{-}\mathcal{C}at/_{\mathcal{B}}$$

is given by the split discrete 2-opfibrations with small fibres.

We will extend Theorem 1.1.8 to a complete 2-equivalence between $[\mathcal{B}, \mathcal{C}at]$ with various laxness flavours on morphisms and corresponding 2-categories of discrete 2-opfibrations in Section 1.2.

From Construction 1.1.4 we also obtain the following definition.

Definition 1.1.9. Let $W\colon \mathcal{A} \to \mathcal{C}at$ be a 2-functor with $\mathcal{A}$ small, and consider 2-functors $M, N\colon \int^{\mathrm{op}} W \to \mathcal{D}$. A *cartesian-marked lax natural transformation α from M to N*, denoted $\alpha\colon M \underset{\mathrm{lax}^{\mathrm{cart}}}{\Longrightarrow} N$, is a lax natural transformation α from M to N such that the structure 2-cell on every morphism of the form

$$\left(f, \mathrm{id}_{W(f)(X)}\right): (A, X) \to (B, W(f)(X))$$

in $\int^{\mathrm{op}} W$ is the identity.

Remark 1.1.10. Cartesian-marked lax natural transformations are a particular case of a more general notion of marked lax natural transformation introduced by Gray in [20, Section I,2]. "Cartesian" refers to the fact that the marking is precisely given by the chosen cartesian liftings of the 2-category of elements $\mathcal{G}(W): \int^{\mathrm{op}} W \to \mathcal{A}$. Street showed in [42] that this less general notion is sufficient to build all the general limits considered by Gray.

Definition 1.1.11. Let $W\colon \mathcal{A} \to \mathcal{C}at$ be a 2-functor with $\mathcal{A}$ small, and let $F\colon \int^{\mathrm{op}} W \to \mathcal{C}$ be a 2-functor. Notice that $\int^{\mathrm{op}} W$ is small, since $\mathcal{A}$ is small. The *cartesian-marked lax conical limit of F*, denoted as $\mathrm{lax}^{\mathrm{cart}}\text{-}\lim^{\Delta 1} F$, is (if it exists) an object $L \in \mathcal{C}$ together with an isomorphism of categories

$$\mathcal{C}(U, L) \cong \left[\int^{\mathrm{op}} W, \mathcal{C}at\right]_{\mathrm{lax}^{\mathrm{cart}}} (\Delta 1, \mathcal{C}(U, F(-)))$$

2-natural in $U \in \mathcal{C}^{\mathrm{op}}$, where $\left[\int^{\mathrm{op}} W, \mathcal{C}at\right]_{\mathrm{lax}^{\mathrm{cart}}}$ is the 2-category of 2-functors, cartesian-marked lax natural transformations and modifications from $\int^{\mathrm{op}} W$ to $\mathcal{C}at$ (it is straightforward to show that this is indeed a 2-category).

When $\mathrm{lax}^{\mathrm{cart}}\text{-}\mathrm{lim}^{\Delta 1} F$ exists, the identity on L provides a cartesian-marked lax natural transformation $\lambda\colon \Delta 1 \underset{\mathrm{lax}^{\mathrm{cart}}}{\Longrightarrow} \mathcal{C}(L, F(-))$ called the *universal cartesian-marked lax cone.*

For the definition of *cartesian-marked lax conical* 2*-colimit* in $\mathcal{C}$, we apply the definition above to a 2-functor $F\colon \int^{\mathrm{op}} W \to \mathcal{C}^{\mathrm{op}}$. As usual, we prefer to consider instead $F^{\mathrm{op}}\colon \left(\int^{\mathrm{op}} W\right)^{\mathrm{op}} \to \mathcal{C}$, but this time we cannot rename $\left(\int^{\mathrm{op}} W\right)^{\mathrm{op}}$ as some $\int^{\mathrm{op}} Z$.

Let $W\colon \mathcal{A} \to \mathcal{C}at$ be a 2-functor with $\mathcal{A}$ small, and let $F\colon \left(\int^{\mathrm{op}} W\right)^{\mathrm{op}} \to \mathcal{C}$ be a 2-functor. The *cartesian-marked lax conical colimit of* F, denoted as $\mathrm{lax}^{\mathrm{cart}}\text{-}\mathrm{colim}^{\Delta 1} F$, is (if it exists) an object $C \in \mathcal{C}$ together with a natural isomorphism of categories

$$\mathcal{C}(C, U) \cong \left[\int^{\mathrm{op}} W, \mathcal{C}at\right]_{\mathrm{lax}^{\mathrm{cart}}} (\Delta 1, \mathcal{C}(F(-), U))$$

Remark 1.1.12. Considering 2-functors F of the form $F\colon \int^{\mathrm{op}} W \to \mathcal{C}$ in Definition 1.1.11 is not restrictive at all. Indeed any 2-category $\mathcal{B}$ can be seen as the 2-category of elements of the 2-functor $\Delta 1\colon \mathcal{B} \to \mathcal{C}at$ constant at *1*. We calculate that $\int^{\mathrm{op}} \Delta 1 \cong \mathcal{B}$ and that $\mathcal{G}(\Delta 1)$ is the identity 2-functor up to this isomorphism.

We now show a new, more elementary proof of the fact that cartesian-marked lax conical limits are particular weighted 2-limits. Street states the analogous result in [42, Theorem 14] for all the general 2-limits introduced by Gray in [20, Section I,7] (cartesian quasi-limits), with a complex proof that gives the weight as the coidentifier of a certain 2-cell with horizontal codomain the weight of lax conical limits. We present an original explicit weight for cartesian-marked lax conical limits that is actually simpler than the one for lax conical limits. Indeed the latter involves quotients of lax 2-dimensional slices (see Street's [42, Theorem 11]), while the former only needs ordinary 1-dimensional slices. The reason is that the laxness of cartesian-marked lax natural transformations is concentrated in the vertical part of $\int^{\mathrm{op}} W$.

Theorem 1.1.13. *Cartesian-marked lax conical limits are particular weighted 2-limits. More precisely, given 2-functors $Z\colon \mathcal{A} \to \mathcal{C}at$ and $F\colon \int^{\mathrm{op}} Z \to \mathcal{C}$ with $\mathcal{A}$ small, the weight that realizes* $\mathrm{lax}^{\mathrm{cart}}\text{-}\lim^{\Delta 1} F$ *is*

$$\begin{tikzcd}
\mathrm{W}^{\mathrm{lax}^{\mathrm{cart}}}\colon & \int^{\mathrm{op}} Z \arrow[r] & \mathcal{C}at \\
& (B, X') \arrow[d, "{(g,\beta)}"] \arrow[r, phantom, "\mapsto"] & Z(B)/_{X'} \arrow[d, "\beta\circ Z(g)(-)"] \\
& (C, X'') & Z(C)/_{X''}
\end{tikzcd}$$

$$\begin{tikzcd}
(B, X') \arrow[d, bend right, "{(g,\beta)}"'] \arrow[d, bend left, "{(h,\gamma)}"] \arrow[d, phantom, "\overset{\delta}{\Rightarrow}"] & \mapsto & Z(B)/_{X'} \arrow[rr, bend left, "\beta\circ Z(g)(-)"] \arrow[rr, bend right, "\gamma\circ Z(h)(-)"'] \arrow[rr, phantom, "{\overset{\|}{\underset{\Downarrow}{Z(\delta)_{\mathrm{dom}(-)}}}}"] & & Z(C)/_{X''} \\
(C, X'') & & & &
\end{tikzcd}$$

where the action of $\beta \circ Z(g)(-)$ on morphisms is given by $Z(g)(\mathrm{dom}(-))$.

Proof. Given $\varphi\colon \Delta 1 \underset{\mathrm{lax}^{\mathrm{cart}}}{\Longrightarrow} N$ a cartesian-marked lax natural transformation, we convert it into a 2-natural transformation $[\varphi]\colon \mathrm{W}^{\mathrm{lax}^{\mathrm{cart}}} \Rightarrow N$ setting, for every $(B, X') \in \int^{\mathrm{op}} Z$,

$$[\varphi]_{(B,X')}(\mathrm{id}_{X'}) := \varphi_{(B,X')}$$

$$[\varphi]_{(B,X')}\left(\begin{tikzcd}[ampersand replacement=\&] X \arrow[rr, "\alpha"] \arrow[dr, "\alpha"'] \& \& X' \arrow[dl, equal] \\ \& X' \& \end{tikzcd}\right) := \varphi_{(\mathrm{id}_B,\alpha)}.$$

$[\varphi]$ extends in a unique way to a 2-natural transformation. □

Corollary 1.1.14. *Cartesian-marked lax conical colimits are particular weighted 2-colimits, and the weight that expresses them is* $\mathrm{W}^{\mathrm{lax}^{\mathrm{cart}}}$.

We now present our new proof of the fact that every weighted 2-limit can be reduced to a cartesian-marked lax conical one. This was first proved by Street in [42, Theorem 15]; another proof can be derived from Proposition 3.18 of Szyld's [44]. Our proof is based on the elementary Construction 1.1.4, allowing it to be understood by a wider audience.

Theorem 1.1.15. *Every weighted 2-limit can be reduced to a cartesian-marked lax conical one. More precisely, given 2-functors $F\colon \mathcal{A} \to \mathcal{C}$ and $W\colon \mathcal{A} \to \mathcal{C}at$*

with $\mathcal{A}$ small,

$$\lim^{W} F \cong \operatorname{lax}^{\mathrm{cart}}\text{-}\lim^{\Delta 1}(F \circ \mathcal{G}(W))$$

either side existing if the other does, where $\mathcal{G}(W)$ is the 2-category of elements of W.

Proof. By Remark 1.1.3, we can just essentially conicalize the weighted 2-colimits

$$W \cong \operatorname{colim}^{W} \mathrm{y}$$

with $W\colon \mathcal{A} \to \mathcal{C}at$ a 2-presheaf with $\mathcal{A}$ small and $\mathrm{y}\colon \mathcal{A}^{\mathrm{op}} \to [\mathcal{A}, \mathcal{C}at]$ the 2-Yoneda embedding. We thus want to extend Construction 1.1.4 to an isomorphism of categories

$$[\mathcal{A}, \mathcal{C}at]\,(W, [\mathcal{A}, \mathcal{C}at]\,(\mathrm{y}(-), U)) \cong \left[\int^{\mathrm{op}} W, \mathcal{C}at\right]_{\operatorname{lax}^{\mathrm{cart}}} (\Delta 1, [\mathcal{A}, \mathcal{C}at]\,((\mathrm{y} \circ \mathcal{G}(W)^{\mathrm{op}})(-), U)) \tag{1.5}$$

2-natural in $U \in [\mathcal{A}, \mathcal{C}at]$, expressing

$$W \cong \operatorname{colim}^{W} \mathrm{y} \cong \operatorname{lax}^{\mathrm{cart}}\text{-}\operatorname{colim}^{\Delta 1}(\mathrm{y} \circ \mathcal{G}(W)^{\mathrm{op}}).$$

This is straightforward, using that in $\int^{\mathrm{op}} W$ we have $(\mathrm{id}, W(f)(\alpha)) \circ (f, \mathrm{id}) = (f, \mathrm{id}) \circ (\mathrm{id}, \alpha)$ for every $f\colon A \to B$ in $\mathcal{A}$ and $\alpha\colon X \to X'$ in $W(A)$, and that any 2-cell δ in $\mathcal{A}$ lifts to a 2-cell $\underline{\delta}^{X}$ in $\int^{\mathrm{op}} W$.

Consider now 2-functors $F\colon \mathcal{A} \to \mathcal{C}$ and $W\colon \mathcal{A} \to \mathcal{C}at$ with $\mathcal{A}$ small. Then by the argument above and Corollary 1.1.14

$$W \cong \operatorname{lax}^{\mathrm{cart}}\text{-}\operatorname{colim}^{\Delta 1}(\mathrm{y} \circ \mathcal{G}(W)^{\mathrm{op}}) \cong \operatorname{colim}^{\mathrm{W}^{\operatorname{lax}^{\mathrm{cart}}}}(\mathrm{y} \circ \mathcal{G}(W)^{\mathrm{op}}).$$

By the lemma of continuity of a limit in its weight (see Remark 1.1.3) and Theorem 1.1.13,

$$\lim^{W} F \cong \lim^{\mathrm{W}^{\operatorname{lax}^{\mathrm{cart}}}} \left(\lim^{(\mathrm{y} \circ \mathcal{G}(W)^{\mathrm{op}})(-)} F\right) \cong \lim^{\mathrm{W}^{\operatorname{lax}^{\mathrm{cart}}}} (F \circ \mathcal{G}(W)) \cong$$
$$\cong \operatorname{lax}^{\mathrm{cart}}\text{-}\lim^{\Delta 1}(F \circ \mathcal{G}(W))$$

where the isomorphism in the middle is easy to prove. □

Remark 1.1.16. The proof of Theorem 1.1.15, together with the proofs of Theorem 1.1.13 and Corollary 1.1.14, also shows how to obtain the correspondence between the universal cylinder of a weighted 2-limit and the associated universal cartesian-marked lax cocone. Calling the two, respectively,

$$\lambda\colon W \Rightarrow \mathcal{C}\,(L, F(-))$$

$$\underset{\smile}{\widehat{\lambda}}\colon \Delta 1 \underset{\mathrm{lax}^{\mathrm{cart}}}{\Longrightarrow} \mathcal{C}\,(L, (F \circ \mathcal{G}\,(W))\,(=))$$

the correspondence is given, for every $(f, \alpha)\colon (A, X) \to (B, X')$ in $\int^{\mathrm{op}} W$, by

$$\underset{\smile}{\widehat{\lambda}}_{(A,X)} = \lambda_A(X) \quad \text{and} \quad \underset{\smile}{\widehat{\lambda}}_{(f,\alpha)} = \lambda_B(\alpha). \tag{1.6}$$

Proposition 1.1.17. *A weighted 2-limit is preserved or reflected precisely when its associated cartesian-marked lax conical limit is so.*

Proof. Clear after Remark 1.1.16. □

Remark 1.1.18. As weighted 2-colimits in $\mathcal{C}$ are just weighted 2-limits in $\mathcal{C}^{\mathrm{op}}$, we automatically obtain from Theorem 1.1.15 the reduction of weighted 2-colimits in $\mathcal{C}$ to cartesian-marked lax conical ones. More precisely, given 2-functors $F\colon \mathcal{A} \to \mathcal{C}$ and $W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}at$ with $\mathcal{A}$ small, we obtain that

$$\operatorname{colim}^W F \cong \mathrm{lax}^{\mathrm{cart}}\text{-}\operatorname{colim}^{\Delta 1}(F \circ \mathcal{G}\,(W)^{\mathrm{op}}),$$

where $\mathcal{G}\,(W)\colon \int^{\mathrm{op}} W \to \mathcal{A}^{\mathrm{op}}$.

But when $W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}at$, there is a more natural 2-category of elements construction that we can do on W, i.e. the one which produces the projection on the first component $\mathcal{G}\,(W)\colon \int W \to \mathcal{A}$, with $\int W$ defined as follows:

an object of $\int W$ is a pair (A, X) with $A \in \mathcal{A}$ and $X \in F(A)$;

a morphism $(A, X) \to (B, X')$ *in* $\int W$ is a pair (f, α) with $f\colon A \to B$ a morphism in $\mathcal{A}$ and $\alpha\colon X \to W(f)(X')$ a morphism in $W(A)$;

a 2-cell $(f, \alpha) \Rightarrow (g, \beta)\colon (A, X) \to (B, X')$ *in* $\int W$ is a 2-cell $\delta\colon f \Rightarrow g$ in $\mathcal{A}$ such that $W(\delta)_{X'} \circ \alpha = \beta$;

the compositions and identities are analogous to the ones described in Construction 1.1.4.

In dimension 1, given $Z\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{S}et$, we have that $\left(\int^{\mathrm{op}} Z\right)^{\mathrm{op}}$ and $\int Z$ coincide, but this is not true in dimension 2.

We believe that it is more natural to reduce weighted 2-limits to cartesian-marked lax conical ones and weighted 2-colimits to *cartesian-marked oplax conical* ones. This idea is original and brings to Theorem 1.1.21 and Theorem 1.1.22.

Definition 1.1.19. Let $W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}at$ be a 2-functor with $\mathcal{A}$ small, and consider 2-functors $M, N\colon \left(\int W\right)^{\mathrm{op}} \to \mathcal{D}$. A *cartesian-marked oplax natural transformation α from M to N*, denoted $\alpha\colon M \underset{\mathrm{oplax}^{\mathrm{cart}}}{\Longrightarrow} N$, is an oplax natural transformation α from M to N such that the structure 2-cell on every morphism $(f, \mathrm{id})\colon (B, X) \leftarrow (A, W(f)(X))$ in $\int W$ is the identity.

Definition 1.1.20. Let $W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}at$ be a 2-functor with $\mathcal{A}$ small, and let $F\colon \int W \to \mathcal{C}$ be a 2-functor. The *cartesian-marked oplax conical 2-colimit of F*, denoted as $\mathrm{oplax}^{\mathrm{cart}}\text{-}\mathrm{colim}^{\Delta 1} F$, is (if it exists) an object $C \in \mathcal{C}$ together with a 2-natural isomorphism of categories

$$\mathcal{C}\left(C, U\right) \cong \left[\left(\int W\right)^{\mathrm{op}}, \mathcal{C}at\right]_{\mathrm{oplax}^{\mathrm{cart}}} \left(\Delta 1, \mathcal{C}\left(F(-), U\right)\right)$$

When $\mathrm{oplax}^{\mathrm{cart}}\text{-}\mathrm{colim}^{\Delta 1} F$ exists, the identity on C provides a cartesian-marked oplax natural transformation $\mu\colon \Delta 1 \underset{\mathrm{oplax}^{\mathrm{cart}}}{\Longrightarrow} \mathcal{C}\left(F(-), C\right)$ called the *universal cartesian-marked oplax cocone.*

Theorem 1.1.21. *Cartesian-marked oplax conical 2-colimits are particular weighted 2-colimits. More precisely, given 2-functors $Z\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}at$ and $F\colon \int Z \to \mathcal{C}$ with $\mathcal{A}$ small, the weight that realizes* $\mathrm{oplax}^{\mathrm{cart}}\text{-}\mathrm{colim}^{\Delta 1} F$ *is*

$$\begin{tikzcd}
\mathrm{W}^{\mathrm{oplax}^{\mathrm{cart}}}\colon & \left(\int Z\right)^{\mathrm{op}} \arrow[r] & \mathcal{C}at \\
& (B, X') & X'/Z(B) \arrow[d, "Z(g)(-)\circ\beta"] \\
& (C, X'') \arrow[u, "{(g,\beta)}"'] \arrow[ur, phantom, "\mapsto"] & X''/Z(C)
\end{tikzcd}$$

where the action of $Z(g)(-) \circ \beta$ on morphisms is given by $Z(g)(\mathrm{cod}(-))$.

Proof. The proof is analogous to the one of Theorem 1.1.13. ☐

Theorem 1.1.22. *Every weighted 2-colimit can be reduced to a cartesian-marked oplax conical one. Given 2-functors $F\colon \mathcal{A} \to \mathcal{C}$ and $W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}at$ with $\mathcal{A}$ small,*

$$\operatorname{colim}^{W} F \cong \operatorname{oplax}^{\mathrm{cart}}\text{-}\operatorname{colim}^{\Delta 1}(F \circ \mathcal{G}(W))$$

where $\mathcal{G}(W) : \int W \to \mathcal{A}$ is the 2-category of elements of W.

Proof. The proof is analogous to the one of Theorem 1.1.15. ☐

Remark 1.1.23. Exactly as in Remark 1.1.16, in the notation of Theorem 1.1.22, we can calculate the correspondence between the universal cocylinder of $\operatorname{colim}^{W} F$ and the universal cartesian-marked oplax cocone of $\operatorname{oplax}^{\mathrm{cart}}\text{-}\operatorname{colim}^{\Delta 1}(F \circ \mathcal{G}(W))$. We find that it is the same as the one in equation (1.6).

Proposition 1.1.24. *A weighted 2-colimit is preserved or reflected precisely when its associated cartesian-marked oplax conical colimit is so.*

Proof. Clear after Remark 1.1.23. ☐

Example 1.1.25. By the proof of Theorem 1.1.22, every 2-presheaf $W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}at$ with $\mathcal{A}$ small can be expressed as

$$W \cong \operatorname{oplax}^{\mathrm{cart}}\text{-}\operatorname{colim}^{\Delta 1}(\mathrm{y} \circ \mathcal{G}(W)).$$

The universal cartesian-marked oplax cocone is given by

$$\forall \begin{tikzcd} (B,X') \\ (A,X) \arrow[u, "{(f,\alpha)}"'] \end{tikzcd} \text{ in } \int W \qquad \begin{tikzcd} \mathrm{y}(A) \arrow[r, "\ulcorner X \urcorner"] \arrow[d, "\mathrm{y}(f)"'] \arrow[dr, phantom, "\Downarrow \ulcorner \alpha \urcorner"] & W \\ \mathrm{y}(B) \arrow[ru, bend right, "\ulcorner X' \urcorner"'] & \end{tikzcd}$$

In particular, taking $\mathcal{A} = \mathit{1}$, W is a small category $\mathcal{D}$ and $\mathcal{G}(W)$ is $\mathcal{D} \to \mathit{1}$. We obtain that $\mathit{1}$ is "cartesian-marked oplax conical dense", building $\mathcal{D}$ with universal cartesian-marked oplax cocone

$$\forall \begin{tikzcd} D \\ C \arrow[u, "f"'] \end{tikzcd} \text{ in } \mathcal{D} \qquad \begin{tikzcd} \mathit{1} \arrow[r, "C"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow f"] & \mathcal{D} \\ \mathit{1} \arrow[ru, bend right, "D"'] & \end{tikzcd}$$

1.2. Pointwise Kan extensions along discrete 2-fibrations

In this section, we propose an original definition of pointwise Kan extension in 2-$\mathcal{C}at_{\text{lax}}$ along a discrete 2-opfibration (Definition 1.2.16). Our motivating application is a 2-dimensional generalization of the fact that the category of elements can be abstractly captured by a comma object that also exhibits a pointwise Kan extension. We will prove it in Theorem 1.3.12 using such definition. We explain why we should consider 2-$\mathcal{C}at_{\text{lax}}$ in order to prove this, and then we inscribe 2-$\mathcal{C}at_{\text{lax}}$ in an original idea of 2-$\mathcal{V}$-enrichment. It is the concept of 2-$\mathcal{V}$-enriched category that guides us to a notion of colimit in a 2-$\mathcal{S}et$-enriched category and then to our notion of pointwise Kan extension in 2-$\mathcal{C}at_{\text{lax}}$. We thus interpret the 2-category of elements as the 2-$\mathcal{S}et$-enriched Grothendieck construction.

Our motivations are to capture the Grothendieck construction in an abstract way, towards an enriched version of the Grothendieck construction, and to understand how the various properties of the 2-category of elements are connected with each other. We will show in Section 1.3 that the pointwise Kan extension result which we prove in Theorem 1.3.12 implies many other properties.

An important ingredient will be that a pointwise Kan extension in 2-$\mathcal{C}at_{\text{lax}}$ is always a weak Kan extension (Definition 1.2.9) as well. The proof (Proposition 1.2.19) will be based on an oplax$^{\text{cart}}$ - lax generalization of the parametrized Yoneda lemma (Theorem 1.2.18), that does not seem to appear in the literature. While a fully lax parametrized Yoneda lemma is not possible, we show that cartesian-marked oplax naturality is strict enough to expand the data on the identities to a lax natural transformation.

The first problem that we encounter is to understand which ambient hosts the 2-category of elements construction. Aiming at a 2-dimensional generalization of the fact that the category of elements can be captured by a comma object that also exhibits a pointwise left Kan extension, we recall the following proposition, due to Bird.

Proposition 1.2.1 (Bird [6])**.** *Let* $F\colon \mathcal{B} \to \mathcal{C}at$ *be a* 2*-functor and consider its* 2*-category of elements. There is a cartesian-marked lax natural transformation* λ *of the form*

$$\begin{tikzcd}
\int^{\mathrm{op}} F \arrow[r] \arrow[d, "\mathcal{G}(F)"'] & \mathit{1} \arrow[d, "\mathit{1}"] \arrow[dl, Rightarrow, shorten=6mm, "\mathrm{lax}^{\mathrm{cart}}"] \\
\mathcal{B} \arrow[r, "F"'] & \mathcal{C}at
\end{tikzcd}$$

Proof. Given a morphism $(f,\alpha)\colon (A,X) \to (B,X')$ in $\int^{\mathrm{op}} F$, $\lambda_{(A,X)}$ is the functor $\mathit{1} \to F(A)$ corresponding to $X \in F(A)$ and $\lambda_{(f,\alpha)}$ corresponds to α. □

Remark 1.2.2. Proposition 1.2.1 forces us to move out of 2-$\mathcal{C}at$ in order to capture the 2-category of elements (but also just the usual Grothendieck construction) from an abstract point of view. Indeed, we need to at least admit lax natural transformations as 2-cells. If we wish to recover the Grothendieck construction of pseudofunctors or of general lax functors into $\mathcal{C}at$, we also need to admit lax functors as 1-cells of our ambient. We will just consider strict 2-functors for simplicity, but we actually expect everything to hold for lax functors as well.

We call 2-$\mathcal{C}at_{\mathrm{lax}}$ the lax 3-category of 2-categories, 2-functors, lax natural transformations and modifications. In [31, Theorem 2.22], Lambert has indeed proved that this forms a lax 3-category, i.e. a category enriched over the cartesian closed 1-category of 2-categories and normalized lax functors (where normalized means that identities are preserved strictly). 2-$\mathcal{C}at_{\mathrm{lax}}$ is actually a restriction of the lax 3-category obtained by enriching the 1-category of 2-categories and normalized lax functors over itself: we consider for simplicity strict 2-functors rather than lax functors. The idea to use 2-$\mathcal{C}at_{\mathrm{lax}}$ is reinforced by the fact that Buckley, in [10] (continuing the work of Baković's [5]), found the need to consider trihomomorphisms $F\colon \mathcal{B}^{\mathrm{coop}} \to$ 2-$\mathcal{C}at_{\mathrm{lax}}$ in order to capture non-split Hermida's 2-fibrations ([23]) via a suitable Grothendieck construction.

We should keep in mind that 2-$\mathcal{C}at_{\mathrm{lax}}$ has no underlying 2-category, since the interchange rule now only holds in a lax version, in the sense that we have a modification between the two possible lax natural transformations. Indeed, consider two lax natural transformations

$$\begin{tikzcd}
\mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{B} \arrow[r, bend left, "H"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathcal{C};
\end{tikzcd}$$

Then for every $A \in \mathcal{A}$, the component α_A is a morphism $F(A) \to G(A)$ in $\mathcal{B}$ and we can consider the structure 2-cell of β on such morphism. We obtain that β_{α_A} is a 2-cell in $\mathcal{C}$ of the form

$$\begin{tikzcd}
H(F(A)) \arrow[r, "\beta_{F(A)}"] \arrow[d, "H(\alpha_A)"'] & K(F(A)) \arrow[d, "K(\alpha_A)"] \arrow[dl, Rightarrow, shorten=10, "\beta_{\alpha_A}"] \\
H(G(A)) \arrow[r, "\beta_{G(A)}"'] & K(G(A))
\end{tikzcd}$$

And the β_{α_A}'s collect into a modification β_α, since the condition we should check on a morphism $f\colon A \to A'$ in $\mathcal{A}$ is guaranteed by the 2-dimensional property of β being a lax natural transformation, applied to the 2-cell α_f in $\mathcal{B}$.

Remark 1.2.3. With the motivating exploration of enriched fibrations and Grothendieck construction in mind, we inscribe 2-$\mathcal{C}at_{\text{lax}}$ in an original idea of 2-$\mathcal{V}$-enrichment. Recall that $\mathcal{S}et$ is the archetypal elementary topos. Its subobject classifier exhibits a bijection between functions into 2 and subsets of the domain. We can consider this as the 0-category of elements construction. Then, according to Weber's [51] (as we will also recall in Section 3.1), $\mathcal{C}at$ is the archetypal 2-dimensional elementary topos, and its 2-dimensional classification process is the category of elements construction. In this chapter, we claim that 2-$\mathcal{C}at_{\text{lax}}$ is the ambient that hosts the 2-category of elements construction. We will say in Remark 1.3.9 how our work could be applied to 3-dimensional elementary topos theory. We believe that the sequence

$$\mathcal{S}et \quad \rightsquigarrow \quad \mathcal{C}at \quad \rightsquigarrow \quad 2\text{-}\mathcal{C}at_{\text{lax}}$$

is best explained by what we call a 2-$\mathcal{V}$-*enrichment*. That is, for $\mathcal{V}$ a general nice enough monoidal category, we will have the sequence

$$\mathcal{V} \quad \rightsquigarrow \quad \mathcal{V}\text{-}\mathcal{C}at \quad \rightsquigarrow \quad 2\text{-}\mathcal{V}\text{-}\mathcal{C}at$$

When we enrich over $\mathcal{V}$-$\mathcal{C}at$, we should take into account the fact that $\mathcal{V}$-$\mathcal{C}at$ is a monoidal 2-category, and then take a *weak enrichment* rather than an ordinary one.

The 2-$\mathcal{V}$-enrichment idea will guide us to propose a notion of colimit and of pointwise Kan extension in 2-$\mathcal{C}at_{\text{lax}}$. We also notice that the further step in the sequence above could bring towards a version of Gray-categories, but we have not investigated this yet.

We recall the following known remark on the ordinary enrichment.

Remark 1.2.4. If $(\mathcal{V}, \otimes, I)$ is a monoidal category with coproducts such that $-\otimes-$ preserves coproducts in each variable, we can define a $\mathcal{V}$-enriched category as pair $(S, \mathcal{A})$ with S a set, that will be the set of objects, and $\mathcal{A}$ a monoid in the monoidal category $\left[S \times S, \mathcal{V}\right]$ of functors (actually given by mere functions) from $S \times S$ to $\mathcal{V}$ and natural transformations (actually given by the mere components), that we think of as the monoidal category of square matrices indexed by S with entries in $\mathcal{V}$, with the tensor product given by matrix multiplication and tensor unit given by the identity matrix (with I all over the main diagonal and the initial object elsewhere). The multiplication of the monoid $\mathcal{A}$ gives indeed the composition of the enriched category, the unit gives the identities and the axioms of monoid precisely ask the composition to be associative and unital.

We can then also define $\mathcal{V}$-enriched functors on this line, using the following construction. Given a $\mathcal{V}$-enriched category $(T, \mathcal{B})$ and a function $F\colon S \to T$, we can define a monoid $F^*\mathcal{B}$ in $\left[S \times S, \mathcal{V}\right]$, taking

$$F^*\mathcal{B} = \left(S \times S \xrightarrow{F \times F} T \times T \xrightarrow{\mathcal{B}} \mathcal{V}\right)$$

and defining the multiplication and the unit by whiskering those of $\mathcal{B}$ with $F \times F$ on the left. Indeed $F^*\mathcal{B}$ is a monoid, by pasting calculations, since all the required axioms just involve cells of strictly higher levels than $F \times F$.

Given $(S, \mathcal{A})$ and $(T, \mathcal{B})$ two $\mathcal{V}$-enriched categories, a $\mathcal{V}$-enriched functor $(S, \mathcal{A}) \to (T, \mathcal{B})$ can be defined as a pair $(F, \overline{F})$ with $F\colon S \to T$ a function and $\overline{F}\colon \mathcal{A} \to F^*\mathcal{B}$ a morphism between monoids in $\left[S \times S, \mathcal{V}\right]$.

We now give a weak 2-categorical generalization of Remark 1.2.4. The concept of *weak enrichment* is explored in Garner and Shulman's [17]. In the terms presented below (on the line of Remark 1.2.4), however, it does not seem to appear in the literature.

We will use the concepts of pseudomonoid in a 2-category and of lax morphism between them. A definition of them can be found in Moeller and Vasilakopoulou's [38].

Construction 1.2.5 (Weak enrichment)**.** Let $(\mathcal{K}, \otimes, I, \alpha, \lambda, \rho)$ be a monoidal 2-category, i.e. a 2-category $\mathcal{K}$ that is monoidal in the 1-dimensional sense but such that the tensor product is a 2-functor $\mathcal{K} \times \mathcal{K} \to \mathcal{K}$. And assume that $\mathcal{K}$ has coproducts and that $- \otimes -$ preserves them in each variable. Then, for every set S, the 2-category $[S \times S, \mathcal{K}]$ is 2-monoidal as well, with tensor product given by matrix multiplication and tensor unit given by the identity matrix. Indeed, the matrix multiplication can be extended to a 2-functor using the 2-dimensional property of the (now enriched) coproducts, with the 2-functoriality given by the fact that everything can be discharged on components and that $- \otimes - \colon \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ is a 2-functor.

We define a $\mathcal{K}$*-weakly enriched category* as a pair $(S, \mathcal{A})$ with S a set, thought as the set of objects, and $\mathcal{A}$ a pseudomonoid in the monoidal 2-category $[S \times S, \mathcal{K}]$ of square S-indexed matrices with entries in $\mathcal{K}$ (whose 1-cells are the 2-natural transformations and whose 2-cells are the modifications). Notice that a strict 2-monoid in $[S \times S, \mathcal{K}]$ is the same thing as a monoid in the monoidal category $[S \times S, \mathcal{K}_0]$, and thus precisely an enriched category over $\mathcal{K}_0$ with object set S.

Now, notice that if $(T, \mathcal{B})$ is a $\mathcal{K}$-weakly enriched category and $F \colon S \to T$ is a function,

$$F^*\mathcal{B} = \left(S \times S \xrightarrow{F \times F} T \times T \xrightarrow{\mathcal{B}} \mathcal{V}\right)$$

is a pseudomonoid in $[S \times S, \mathcal{K}]$, defining all the needed structure cells as those of $\mathcal{B}$ whiskered with $F \times F$ on the left. That $F^*\mathcal{B}$ is a pseudomonoid holds by pasting calculations, since all the required axioms just involve cells of strictly higher levels than $F \times F$.

Given two $\mathcal{K}$-weakly enriched categories $(S, \mathcal{A})$ and $(T, \mathcal{B})$, we define a *$\mathcal{K}$-weakly enriched functor* $(S, \mathcal{A}) \to (T, \mathcal{B})$ as a pair $(F, \overline{F})$ with $F\colon S \to T$ a function and $\overline{F}\colon \mathcal{A} \to F^*\mathcal{B}$ a lax morphism between lax monoids in $[S \times S, \mathcal{K}]$.

Given now $(F, \overline{F}), (G, \overline{G})\colon (S, \mathcal{A}) \to (T, \mathcal{B})$ two $\mathcal{K}$-weakly enriched functors, we define a *$\mathcal{K}$-weakly enriched natural transformation* $\varphi\colon (F, \overline{F}) \underset{\text{lax}}{\Rightarrow} (G, \overline{G})$ as a collection of 1-cells

$$\varphi_A\colon I \to \mathcal{B}\,(F(A), G(A))$$

in $\mathcal{K}$ for every $A \in S$ and 2-cells

$$\begin{tikzcd}
 & \mathcal{A}(A,B)\otimes I \arrow[rr, "G\otimes\varphi_A"] & & \mathcal{B}(GA,GB)\otimes\mathcal{B}(FA,GA) \arrow[dr, "\mathrm{comp}"] & \\
\mathcal{A}(A,B) \arrow[ur, "\rho^{-1}_{\mathcal{A}(A,B)}"] \arrow[dr, "\lambda^{-1}_{\mathcal{A}(A,B)}"'] & & \Downarrow\varphi_{A,B} & & \mathcal{B}(FA,GB) \\
 & I\otimes\mathcal{A}(A,B) \arrow[rr, "\varphi_B\otimes F"'] & & \mathcal{B}(FB,GB)\otimes\mathcal{B}(FA,FB) \arrow[ur, "\mathrm{comp}"'] &
\end{tikzcd}$$

in $\mathcal{K}$ for every pair $(A, B) \in S \times S$ such that, for every $A, B, C \in S$, with notations like $\mathcal{A}_{A,B} \coloneqq \mathcal{A}\,(A, B)$ and $\mathcal{B}^{F,G}_{A,B} \coloneqq \mathcal{B}\,(F(A), G(B))$ and omitting the tensor product of objects, the pasting in $\mathcal{K}$

$$\begin{tikzcd}
 & & \mathcal{B}^{FG}_{AA} \arrow[dr, "\lambda^{-1}"] \arrow[ddrrr, equal, bend left] & & & \\
I \arrow[urr, bend left, "\varphi_A"] \arrow[rr, "\mathrm{id}_{G(A)}"] \arrow[dr, "\mathrm{id}_A"'] & \overline{G}\Downarrow & \mathcal{B}^{GG}_{AA} \arrow[dr, "\rho^{-1}"] \arrow[r, phantom, "||"] & I\mathcal{B}^{FG}_{AA} \arrow[dr, "\mathrm{id}_{G(A)}\otimes 1"] \arrow[r, phantom, "\overset{\mathrm{unit}^{-1}_{\mathcal{B}}}{\cong}"] & & \\
 & \mathcal{A}_{AA} \arrow[ur, "G"] \arrow[r, "\rho^{-1}"] \arrow[dr, "\lambda^{-1}"'] & \mathcal{A}_{AA}I \arrow[u, phantom, "||"] \arrow[r, "G\otimes 1"] & \mathcal{B}^{GG}_{AA}I \arrow[r, "1\otimes\varphi_A"] \arrow[d, phantom, "\Downarrow\varphi_{A,A}"] & \mathcal{B}^{GG}_{AA}\mathcal{B}^{FG}_{AA} \arrow[r, "\mathrm{comp}"] & \mathcal{B}^{FG}_{AA} \\
 & & I\mathcal{A}_{AA} \arrow[r, "1\otimes F"'] & I\mathcal{B}^{FF}_{AA} \arrow[r, "\varphi_A\otimes 1"'] & \mathcal{B}^{FG}_{AA}\mathcal{B}^{FF}_{AA} \arrow[ur, "\mathrm{comp}"'] &
\end{tikzcd}$$

is equal to the pasting

$$\begin{tikzcd}
 & & \mathcal{B}^{FG}_{AA} \arrow[dr, "\rho^{-1}"] \arrow[drrr, equal, bend left] & & & \\
I \arrow[urr, bend left, "\varphi_A"] \arrow[rr, "\rho^{-1}"] \arrow[drr, "\mathrm{id}_{F(A)}"] \arrow[ddr, "\mathrm{id}_A"'] & & II \arrow[u, phantom, "||"] \arrow[d, phantom, "||"] \arrow[r, "\varphi_A\otimes 1"] \arrow[dr, "1\otimes\mathrm{id}_{F(A)}"'] & \mathcal{B}^{FG}_{AA}I \arrow[r, "1\otimes\mathrm{id}_{F(A)}"] \arrow[d, phantom, "||"] \arrow[ur, phantom, "\overset{\mathrm{unit}^{-1}_{\mathcal{B}}}{\cong}"] & \mathcal{B}^{FG}_{AA}\mathcal{B}^{FF}_{AA} \arrow[r, "\mathrm{comp}"'] & \mathcal{B}^{FG}_{AA} \\
 & \overline{F}\Downarrow & \mathcal{B}^{FF}_{AA} \arrow[r, "\lambda^{-1}"'] & I\mathcal{B}^{FF}_{AA} \arrow[ur, "\varphi_A\otimes 1"'] & & \\
 & \mathcal{A}_{AA} \arrow[ur, "F"'] & & & &
\end{tikzcd}$$

and the pasting in $\mathcal{K}$

$$\begin{tikzcd}
& \mathcal{A}_{BC}\left(\mathcal{A}_{AB}I\right) \arrow[rr, "1\otimes G\otimes\varphi_A"] & & \mathcal{A}_{BC}\left(\mathcal{B}^{GG}_{AB}\mathcal{B}^{FG}_{AA}\right) \arrow[rr, "1\otimes\mathrm{comp}"] \arrow[d, phantom, "\Downarrow 1\otimes\varphi_{A,B}"] & & \mathcal{A}_{BC}\mathcal{B}^{FG}_{AB} \arrow[rr, "G\otimes 1"] \arrow[dr, phantom, "\|"] & & \mathcal{B}^{GG}_{BC}\mathcal{B}^{FG}_{AB} \arrow[ddr, "\mathrm{comp}"] & \\
\mathcal{A}_{BC}\mathcal{A}_{AB} \arrow[ur, "1\otimes\rho^{-1}"] \arrow[rr, "1\otimes\lambda^{-1}"] \arrow[drr, "1\otimes F"] \arrow[ddr, "F\otimes F"] \arrow[ddd, "\mathrm{comp}"'] & & \mathcal{A}_{BC}\left(I\mathcal{A}_{AB}\right) \arrow[rr, "1\otimes\varphi_B\otimes F"] & & \mathcal{A}_{BC}\left(\mathcal{B}^{FG}_{BB}\mathcal{B}^{FF}_{AB}\right) \arrow[ur, "1\otimes\mathrm{comp}"] \arrow[rr, "G\otimes 1\otimes 1"] \arrow[d, phantom, "\|"] & & \mathcal{B}^{GG}_{BC}\left(\mathcal{B}^{FG}_{BB}\mathcal{B}^{FF}_{AB}\right) \arrow[ur, "1\otimes\mathrm{comp}"] \arrow[dr, phantom, "\overset{\mathrm{assoc}^{-1}_{\mathcal{B}}}{\cong}"] & & \\
& & \mathcal{A}_{B,C}\mathcal{B}^{FF}_{AB} \arrow[rr, "\rho^{-1}\otimes 1"] \arrow[dr, "\lambda^{-1}\otimes 1"'] & & \left(\mathcal{A}_{BC}I\right)\mathcal{B}^{FF}_{AB} \arrow[rr, "G\otimes\varphi_B\otimes 1"] \arrow[d, phantom, "\Downarrow\varphi_{B,C}\otimes 1"] & & \left(\mathcal{B}^{GG}_{BC}\mathcal{B}^{FG}_{BB}\right)\mathcal{B}^{FF}_{AB} \arrow[u, "\alpha"] \arrow[dr, "\mathrm{comp}\otimes 1"'] & & \mathcal{B}^{FG}_{AC} \\
& \mathcal{B}^{FF}_{BC}\mathcal{B}^{FF}_{AB} \arrow[dr, "\mathrm{comp}"] \arrow[d, phantom, "\overline{F}\Downarrow"] & & \left(I\mathcal{A}_{BC}\right)\mathcal{B}^{FF}_{AB} \arrow[rr, "\varphi_C\otimes F\otimes 1"'] \arrow[dl, phantom, "\overset{\mathrm{assoc}_{\mathcal{B}}}{\cong}"] & & \left(\mathcal{B}^{FG}_{CC}\mathcal{B}^{FF}_{BC}\right)\mathcal{B}^{FF}_{AB} \arrow[rr, "\mathrm{comp}\otimes 1"'] & & \mathcal{B}^{FG}_{BC}\mathcal{B}^{FF}_{AB} \arrow[ur, "\mathrm{comp}"] & \\
\mathcal{A}_{AC} \arrow[rr, "F"'] & & \mathcal{B}^{FF}_{AC} \arrow[rr, "\lambda^{-1}"'] & & I\mathcal{B}^{FF}_{AC} \arrow[rr, "\varphi_C\otimes 1"'] & & \mathcal{B}^{FG}_{CC}\mathcal{B}^{FF}_{AC} \arrow[uurr, bend right, "\mathrm{comp}"'] & &
\end{tikzcd}$$

is equal to the pasting

$$\begin{tikzcd}
& \mathcal{A}_{BC}\left(\mathcal{A}_{AB}I\right) \arrow[rr, "1\otimes G\otimes\varphi_A"] & & \mathcal{A}_{BC}\left(\mathcal{B}^{GG}_{AB}\mathcal{B}^{FG}_{AA}\right) \arrow[rr, "1\otimes\mathrm{comp}"] & & \mathcal{A}_{BC}\mathcal{B}^{FG}_{AB} \arrow[drr, bend left, "G\otimes 1"] & & \\
& \mathcal{B}^{GG}_{BC}\mathcal{B}^{GG}_{AB} \arrow[rr, "\mathrm{comp}"] \arrow[dr, phantom, "\overline{G}\Downarrow"] & & \mathcal{B}^{GG}_{AC} \arrow[rr, "\rho^{-1}"] & & \mathcal{B}^{GG}_{AC}I \arrow[dr, "1\otimes\varphi_A"] \arrow[r, phantom, "\overset{\mathrm{assoc}^{-1}_{\mathcal{B}}}{\cong}"] & & \mathcal{B}^{GG}_{BC}\mathcal{B}^{FG}_{AB} \arrow[dd, "\mathrm{comp}"] \\
\mathcal{A}_{BC}\mathcal{A}_{AB} \arrow[uur, bend left, "1\otimes\rho^{-1}"] \arrow[ur, "G\otimes G"] \arrow[rr, "\mathrm{comp}"'] & & \mathcal{A}_{AC} \arrow[ur, "G"] \arrow[rr, "\rho^{-1}"] \arrow[dr, "\lambda^{-1}"'] & & \mathcal{A}_{AC}I \arrow[u, phantom, "\|"] \arrow[rr, "G\otimes\varphi_A"] \arrow[d, phantom, "\Downarrow\varphi_{A,C}"] & & \mathcal{B}^{GG}_{AC}\mathcal{B}^{FG}_{AA} \arrow[dr, "\mathrm{comp}"] & \\
& & & I\mathcal{A}_{AC} \arrow[rr, "\varphi_C\otimes F"'] & & \mathcal{B}^{FG}_{CC}\mathcal{B}^{FF}_{AC} \arrow[rr, "\mathrm{comp}"'] & & \mathcal{B}^{FG}_{AC}
\end{tikzcd}$$

We notice that Construction 1.2.5 is particularly useful when $\mathcal{K}$ is $\mathcal{V}$-$\mathcal{C}at$, since we obtain a construction that we can apply to a starting ordinary $\mathcal{V}$.

Definition 1.2.6 (The 2-$\mathcal{V}$-enrichment)**.** Let $\mathcal{V}$ be a nice enough monoidal category such that $\mathcal{V}$-$\mathcal{C}at$ with the tensor product of $\mathcal{V}$-categories (so we need at least a braiding in $\mathcal{V}$) becomes a monoidal 2-category with coproducts such that its tensor product preserves them in each variable.

We call *2-$\mathcal{V}$-enriched category (resp. functor, natural transformation)* a $(\mathcal{V}$-$\mathcal{C}at)$-weakly enriched category (resp. functor, natural transformation). We call 2-$\mathcal{V}$-$\mathcal{C}at$ the tridimensional structure they form. A notion of 2-$\mathcal{V}$-enriched modification could be given as well.

Example 1.2.7. Consider $\mathcal{V} = \mathcal{S}et$. Then 2-$\mathcal{S}et$-enriched categories, functors, natural transformations and modifications are, respectively, bicategories, lax functors, lax natural transformations and modifications.

We will think of 2-$\mathcal{C}at_{\mathrm{lax}}$ as the lax 3-category of 2-$\mathcal{S}et$-enriched categories, restricted to strict weakly enriched categories and functors for simplicity.

Remark 1.2.8. We can now think of the 2-category of elements construction as the *2-$\mathcal{S}et$-enriched Grothendieck construction.* It takes enriched presheaves with values in $\mathcal{S}et\text{-}\mathcal{C}at = \mathcal{C}at$, that is the base of the weak enrichment we consider, and converts them into discrete 2-opfibrations, that can be seen as *2-$\mathcal{S}et$-opfibrations.*

In future work, we will explore how we can generalize this to more general monoidal categories $\mathcal{V}$ in the place of $\mathcal{S}et$, thus obtaining an enriched Grothendieck construction. In particular, we believe that for this the 2-$\mathcal{S}et$-enrichment should be distinguished from the ordinary $\mathcal{C}at$-enrichment; we actually expect the latter to be less useful than the former.

We give an original definition of weak Kan extension in a lax 3-category.

Definition 1.2.9. A diagram

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{C} \\
\mathcal{A} \arrow[ru, "L"'] \arrow[from=1-2, to=2-1, phantom, "\Swarrow\lambda" near start] &
\end{tikzcd}$$

in a lax 3-category $\mathcal{Q}$ (that is a category enriched over the 1-category of 2-categories and lax functors), exhibits L as the *weak left Kan extension of F along K*, written $L = \operatorname{lan}_K F$, if pasting with λ gives an isomorphism of categories

$$\mathcal{Q}(\mathcal{A}, \mathcal{C})(L, U) \cong \mathcal{Q}(\mathcal{B}, \mathcal{C})(F, U \circ K) \tag{1.7}$$

for every $U \in \mathcal{Q}(\mathcal{A}, \mathcal{C})$ (the 2-naturality in U is granted automatically).

Remark 1.2.10. Lambert showed in [31, Theorem 2.22] that 2-$\mathcal{C}at_{\mathrm{lax}}$ is a lax 3-category with hom-2-categories

$$2\text{-}\mathcal{C}at_{\mathrm{lax}}(\mathcal{A}, \mathcal{C}) := [\mathcal{A}, \mathcal{C}]_{\mathrm{lax}}$$

where $[\mathcal{A}, \mathcal{C}]_{\mathrm{lax}}$ is the 2-category of 2-functors from $\mathcal{A}$ to $\mathcal{C}$, lax natural transformations and modifications.

So, in the notation of Definition 1.2.9, the isomorphism of categories of equation (1.7) becomes, for every $U \in [\mathcal{A}, \mathcal{C}]_{\text{lax}}$,

$$[\mathcal{A}, \mathcal{C}]_{\text{lax}}\,(L, U) \cong [\mathcal{B}, \mathcal{C}]_{\text{lax}}\,(F, U \circ K)\,.$$

Remark 1.2.11. We aim at a notion of *pointwise Kan extension in* $2\text{-}\mathcal{C}at_{\text{lax}}$ that allows us (Theorem 1.3.12) to prove a 2-dimensional version of the pointwise Kan extension result that we have for the category of elements. For this we need a notion of (co)limit in the setting of $2\text{-}\mathcal{C}at_{\text{lax}}$. There surely is a natural notion of “external limit” in this context, i.e. a limit in the whole $2\text{-}\mathcal{C}at_{\text{lax}}$, which is that of limit enriched over the 1-category of 2-categories and lax functors. Indeed $2\text{-}\mathcal{C}at_{\text{lax}}$ is enriched over such 1-category. For $\mathcal{C}$ a 2-category, this recovers e.g. the natural isomorphism

$$2\text{-}\mathcal{C}at_{\text{lax}}\,(\mathcal{U}, \operatorname{Id}_{\mathcal{C}} /\!\!/ \operatorname{Id}_{\mathcal{C}}) \cong 2\text{-}\mathcal{C}at_{\text{lax}}\,(\mathit{2}, 2\text{-}\mathcal{C}at_{\text{lax}}\,(\mathcal{U}, \mathcal{C}))$$

presented by Lambert in [31] as the universal property of the lax comma object $\operatorname{Id}_{\mathcal{C}} /\!\!/ \operatorname{Id}_{\mathcal{C}}$ (see Definition 1.3.4) being the power of $\mathcal{C}$ by $\mathit{2}$ in $2\text{-}\mathcal{C}at_{\text{lax}}$.

But we are more interested in an “internal” notion of colimit, i.e. a notion of colimit in a $2\text{-}\mathcal{S}et$-enriched category (after Example 1.2.7). This should include the notion of colimit in a 2-category, but be able to express the laxness as well. After Section 1.1, we use (now non-necessarily conical) cartesian-marked oplax colimits as colimits in a $2\text{-}\mathcal{S}et$-category (see Definition 1.2.12). Notice that the marking is a piece of structure. These colimits are a particular case of the sigma-omega-limits (or sigma-s-limits) considered by Szyld in [44] and [45]. By Proposition 3.18 of Szyld’s [44], these colimits can all be reduced to (still marked) conical ones. However, expressing the diagram, the weight and the marking as different pieces of structure makes it easier for us to define *pointwise Kan extensions in* $2\text{-}\mathcal{C}at_{\text{lax}}$.

Definition 1.2.12. Let $M\colon \mathcal{A}^{\text{op}} \to \mathcal{C}at$ (the marking), $F\colon \int M \to \mathcal{C}$ (the diagram) and $W\colon \left(\int M\right)^{\text{op}} \to \mathcal{C}at$ (the weight) be 2-functors with $\mathcal{A}$ small. The *cartesian-marked oplax colimit of* F *marked by* M *and weighted by* W, denoted as $\mathrm{oplax}^{\mathrm{cart}}_{M}\text{-}\mathrm{colim}^{W} F$, is (if it exists) an object $C \in \mathcal{C}$ together with an isomorphism of categories

$$\mathcal{C}\,(C, U) \cong \left[\left(\int M\right)^{\mathrm{op}}, \mathcal{C}at\right]_{\mathrm{oplax}^{\mathrm{cart}}} (W, \mathcal{C}\,(F(-), U))$$

2-natural in $U \in \mathcal{C}$. When $\mathrm{oplax}^{\mathrm{cart}}_{M}\text{-}\mathrm{colim}^{W} F$ exists, the identity on C provides a cartesian-marked oplax natural transformation $\mu\colon W \xRightarrow[\mathrm{oplax}^{\mathrm{cart}}]{} \mathcal{C}\,(F(-), C)$ called the *universal cartesian-marked oplax cocylinder.*

We will also need to consider the case in which the domain of F is expressed as $\int^{\mathrm{op}} M$ for some 2-functor $M\colon \mathcal{A} \to \mathcal{C}at$, and $W\colon \left(\int^{\mathrm{op}} M\right)^{\mathrm{op}} \to \mathcal{C}at$. The *cartesian oplax colimit of F opmarked by M and weighted by W*, denoted as $\mathrm{oplax}^{\mathrm{cart}}_{\mathrm{op}\text{-}M}\text{-}\mathrm{colim}^{W} F$, is (if it exists) an object $C \in \mathcal{C}$ together with a 2-natural isomorphism of categories

$$\mathcal{C}\,(C, U) \cong \left[\left(\int^{\mathrm{op}} M\right)^{\mathrm{op}}, \mathcal{C}at\right]_{\mathrm{oplax}^{\mathrm{cart}}} (W, \mathcal{C}\,(F(-), U))$$

Definition 1.2.13. Recall from Remark 1.1.12 what we can now call *the trivial marking* and denote as triv. That is, given $\mathcal{A}$ a 2-category, we can view $\mathcal{A}$ as the 2-category of elements of $\Delta 1\colon \mathcal{A} \to \mathcal{C}at$. Cartesian-marked oplax with respect to the trivial marking coincides with strict 2-naturality.

Remark 1.2.14. We can now rephrase Theorem 1.1.22 as follows:

In $2\text{-}\mathcal{C}at_{\mathrm{lax}}$ every trivially-marked weighted 2-colimit can be equivalently expressed as a marked trivially-weighted 2-colimit. More precisely

$$\mathrm{oplax}^{\mathrm{cart}}_{\mathrm{triv}}\text{-}\mathrm{colim}^{W} F \cong \mathrm{oplax}^{\mathrm{cart}}_{W}\text{-}\mathrm{colim}^{\Delta 1}(F \circ \mathcal{G}\,(W))$$

Example 1.2.15. Let $F\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}at$ be a 2-functor with $\mathcal{A}$ small. Then by Example 1.1.25

$$F \cong \mathrm{oplax}^{\mathrm{cart}}_{F}\text{-}\mathrm{colim}^{\Delta 1}(\mathrm{y} \circ \mathcal{G}\,(F))$$

In particular, taking $\mathcal{A} = 1$, we obtain that for every small category $\mathcal{D}$

$$\mathcal{D} \cong \mathrm{oplax}^{\mathrm{cart}}_{\mathcal{D}}\text{-}\mathrm{colim}^{\Delta 1}\Delta 1.$$

Notice that the marking given by $\mathcal{D}$ is "chaotic", in the sense that cartesian-marked oplax with respect to it coincides with oplax. In $2\text{-}\mathcal{C}at_{\text{lax}}$ the 2-functor $\mathit{1}\colon \mathit{1} \to \mathcal{C}at$ is thus conically dense, analogously to the fact that the functor $\mathit{1}\colon \mathit{1} \to \mathcal{S}et$ is dense.

We now propose an original notion of pointwise left Kan extension in $2\text{-}\mathcal{C}at_{\text{lax}}$ along a discrete 2-opfibration, using our definition of colimit in such setting (Definition 1.2.12). Our idea is to keep the corresponding diagram and weight considered in the (ordinarily) enriched setting (see Dubuc's [16] or Kelly's [28, Chapter 4] for the classical definition), but adding the marking that we naturally have when we extend along a discrete 2-opfibration. In Theorem 1.3.12, this notion will allow us to prove a 2-dimensional version of the fact that the comma object that captures the category of elements exhibits a pointwise Kan extension.

Definition 1.2.16. Consider a diagram

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "F"] \arrow[d, "K"'] \arrow[rd, phantom, "\Swarrow \lambda"] & \mathcal{C} \\
\mathcal{A} \arrow[ru, "L"'] &
\end{tikzcd}$$

in $2\text{-}\mathcal{C}at_{\text{lax}}$ with $\mathcal{B}$ small and K a discrete 2-opfibration. Then by Theorem 1.1.8, K is isomorphic in the slice $2\text{-}\mathcal{C}at/_{\mathcal{A}}$ to $\mathcal{G}(M)$ for some 2-functor $M\colon \mathcal{A} \to \mathcal{C}at$. We can assume K is in the form $\mathcal{G}(M)$, up to whiskering the diagram with the isomorphism in the slice. Assume further that λ is a cartesian-marked lax natural transformation with respect to M.

We say that λ exhibits L as the *pointwise left Kan extension of F along K*, written $L = \operatorname{Lan}_K F$, if for every $A \in \mathcal{A}$

$$L(A) \cong \text{oplax}^{\text{cart}}_{\text{op-}M}\text{-colim}^{\mathcal{A}(K(-),A)} F$$

with universal cartesian-marked oplax cocylinder

$$\mathcal{A}\left(K(-), A\right) \overset{L}{\Longrightarrow} \mathcal{C}\left((L \circ K)(-), L(A)\right) \xRightarrow[\text{oplax}^{\text{cart}}]{\mathcal{C}(\lambda_-,\text{id})} \mathcal{C}\left(F(-), L(A)\right); \tag{1.8}$$

or equivalently if for every $A \in \mathcal{A}$ and every $C \in \mathcal{C}$ the functor

$$\mathcal{C}\left(L(A), C\right) \longrightarrow [\mathcal{B}^{\text{op}}, \mathcal{C}at]_{\text{oplax}^{\text{cart}}}\left(\mathcal{A}\left(K(-), A\right), \mathcal{C}\left(F(-), C\right)\right)$$

given by the cartesian-marked oplax natural transformation of equation (1.8) is an isomorphism of categories (notice that the 2-naturality in C and A is granted, where the latter is using that L is a 2-functor).

The rest of this section is dedicated to the proof that every pointwise left Kan extension in 2-$\mathcal{C}at_{\text{lax}}$ along a discrete 2-opfibration (as defined in Definition 1.2.16) is a weak left Kan extension in 2-$\mathcal{C}at_{\text{lax}}$ as well.

For this, we need a generalization of the parametrized Yoneda lemma which is cartesian-marked oplax and lax together (Theorem 1.2.18). Such result does not seem to appear in the literature. While a fully lax parametrized Yoneda lemma is not possible, since it is the strict naturality that classically allows to expand the datum on the identity to a natural transformation, our version shows the minimal strictness needed to do so.

Interestingly, such expansion in the fully strict 2-natural case classically depends on the naturality of what will be our parameter A. Instead, we will need to expand via the slight strictness of the cartesian-marked oplax naturality in B. And an expansion through B is harder to achieve than one through A.

Definition 1.2.17. Let $G, H\colon \mathcal{B}^{\text{op}} \times \mathcal{C} \to \mathcal{E}$ be 2-functors. Assume that $\mathcal{B}$ is endowed with the cartesian marking presented by a (split) discrete 2-opfibration $K\colon \mathcal{B} \to \mathcal{A}$ (with small fibres). That is, $\mathcal{B} \cong \int^{\text{op}} M$ for some 2-functor $M\colon \mathcal{A} \to \mathcal{C}at$ and has the cartesian marking. An ***oplax$^{\textit{cart}}$-lax*** *natural transformation* α from G to H is a collection of morphisms

$$\alpha_{B,C}\colon G(B,C) \to H(B,C)$$

in $\mathcal{E}$ for every $(B,C) \in \mathcal{B}^{\text{op}} \times \mathcal{C}$ and, for every $f\colon B' \to B$ in $\mathcal{B}^{\text{op}}$ and $g\colon C \to C'$ in $\mathcal{C}$, structure 2-cells

$$\begin{tikzcd}
G(B',C) \arrow[r, "\alpha_{B',C}"] \arrow[d, "G(f,\mathrm{id})"'] & H(B',C) \arrow[d, "H(f,\mathrm{id})"] \\
G(B,C) \arrow[r, "\alpha_{B,C}"'] \arrow[ur, phantom, "\overset{\alpha_{f,C}}{\Nearrow}"] & H(B,C)
\end{tikzcd}
\qquad
\begin{tikzcd}
G(B,C) \arrow[r, "\alpha_{B,C}"] \arrow[d, "G(\mathrm{id},g)"'] & H(B,C) \arrow[d, "H(\mathrm{id},g)"] \\
G(B,C') \arrow[r, "\alpha_{B,C'}"'] \arrow[ur, phantom, "\underset{\alpha_{B,g}}{\Swarrow}"] & H(B,C')
\end{tikzcd}$$

such that $\alpha_{-,C}$ is cartesian-marked oplax natural in $B \in \mathcal{B}^{\mathrm{op}}$ and $\alpha_{B,-}$ is lax natural in $C \in \mathcal{C}$, and such that the following compatibility axiom holds:

$$\begin{tikzcd}[column sep=small]
 & G(B,C) \arrow[rr, "\alpha_{B,C}"] \arrow[rd, "G(\mathrm{id},g)"] & & H(B,C) \arrow[rd, "H(\mathrm{id},g)"] & & & & G(B,C) \arrow[rr, "\alpha_{B,C}"] & & H(B,C) \arrow[rd, "H(\mathrm{id},g)"] & \\
G(B',C) \arrow[ru, "G(f,\mathrm{id})"] \arrow[rd, "G(\mathrm{id},g)"'] \arrow[rr, phantom, "\sslash"] & & G(B,C') \arrow[rr, "\alpha_{B,C'}"] & & H(B,C') & = & G(B',C) \arrow[ru, "G(f,\mathrm{id})"] \arrow[rd, "G(\mathrm{id},g)"'] \arrow[rr, "\alpha_{B',C}"] & & H(B',C) \arrow[ru, "H(f,\mathrm{id})"] \arrow[rd, "H(\mathrm{id},g)"'] \arrow[rr, phantom, "\sslash"] & & H(B,C') \\
 & G(B',C') \arrow[ru, "G(f,\mathrm{id})"'] \arrow[rr, "\alpha_{B',C'}"'] & & H(B',C') \arrow[ru, "H(f,\mathrm{id})"'] & & & & G(B',C') \arrow[rr, "\alpha_{B',C'}"'] & & H(B',C') \arrow[ru, "H(f,\mathrm{id})"'] &
\arrow[from=1-4, to=2-3, phantom, "\Downarrow\alpha_{B,g}"]
\arrow[from=2-3, to=3-4, phantom, "\Downarrow\alpha_{f,C'}"]
\arrow[from=1-8, to=2-9, phantom, "\alpha_{f,C}\Downarrow"]
\arrow[from=2-9, to=3-8, phantom, "\alpha_{B',g}\Downarrow"]
\end{tikzcd}$$

A *modification* $\Theta\colon \alpha \Rrightarrow \beta\colon G \xRightarrow[\mathrm{opl^c\text{-}lax}]{} H$ between oplax$^{\text{cart}}$-lax natural transformations is a collection of 2-cells

$$\begin{tikzcd}
G(B,C) \arrow[r, bend left=50, "\alpha_{B,C}"] \arrow[r, bend right=50, "\beta_{B,C}"'] \arrow[r, phantom, "\overset{\shortparallel}{\underset{\Downarrow}{\Theta_{B,C}}}"] & H(B,C)
\end{tikzcd}$$

in $\mathcal{E}$ that forms both, fixing C, a modification $\alpha_{-,C} \Rrightarrow \beta_{-,C}$, and, fixing B, a modification $\alpha_{B,-} \Rrightarrow \beta_{B,-}$.

Theorem 1.2.18 (The oplax$^{\text{cart}}$-lax parametrized Yoneda lemma)**.** *Let* $K\colon \mathcal{B} \to \mathcal{A}$ *be a (split) discrete* 2*-opfibration (with small fibres) and* $F\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathcal{C}at$ *be a* 2*-functor. There is a bijection between*

$$\alpha_{B,A}\colon \mathcal{A}(K(B),A) \to F(B,A)$$

oplax$^{\text{cart}}$-lax *natural in* $(B,A) \in \mathcal{B}^{\mathrm{op}} \times \mathcal{A}$ *and*

$$\eta_B\colon 1 \to F(B,K(B))$$

extraordinary lax natural in $B \in \mathcal{B}$ *(see Hirata's [24] for a definition of extraordinary lax natural transformations and modifications between them).*

Moreover this bijection extends to an isomorphism of categories, considering as morphisms of the two categories respectively the modifications between oplax$^{\text{cart}}$-lax *natural transformations and the modifications between extraordinary lax natural transformations.*

Proof. By Theorem 1.1.8, we can assume that K is in the form $\mathcal{G}(M) : \int^{\mathrm{op}} M \to \mathcal{A}$ for a 2-functor $M \colon \mathcal{A} \to \mathcal{C}at$. Given

$$\alpha_{(B,Y),A} \colon \mathcal{A}(K(B,Y), A) \to F((B,Y), A)$$

oplax$^{\mathrm{cart}}$-lax natural in $((B,Y), A) \in \left(\int^{\mathrm{op}} M\right)^{\mathrm{op}} \times \mathcal{A}$ (see Definition 1.2.17), we construct $\eta_{(B,Y)}$ as the composite

$$1 \xrightarrow{\mathrm{id}_B} \mathcal{A}(B,B) \xrightarrow{\alpha_{(B,Y),B}} F((B,Y),B)$$

Then $\eta_{(B,Y)}$ is extraordinary lax natural in $(B,Y) \in \int^{\mathrm{op}} M$, with structure 2-cell on $(g,\gamma) \colon (B,Y) \to (B',Y')$ in $\int^{\mathrm{op}} M$ given by the pasting

$$\begin{tikzcd}
1 \arrow[r, "\mathrm{id}_B"] \arrow[d, "\mathrm{id}_{B'}"'] \arrow[rd, phantom, "="] & \mathcal{A}(B,B) \arrow[r, "\alpha_{(B,Y),B}"] \arrow[d, "g\circ-"] \arrow[rdd, phantom, "\Leftarrow \alpha_{(B,Y),g}"] & F((B,Y),B) \arrow[dd, "F(\mathrm{id},g)"] \\
\mathcal{A}(B',B') \arrow[r, "-\circ g"'] \arrow[d, "\alpha_{(B',Y'),B'}"'] \arrow[rd, phantom, "\Leftarrow \alpha_{(g,\gamma),B'}"] & \mathcal{A}(B,B') \arrow[rd, "\alpha_{(B,Y),B'}"] & \\
F((B',Y'),B') \arrow[rr, "F((g,\gamma),\mathrm{id})"'] & & F((B,Y),B')
\end{tikzcd} \tag{1.9}$$

Indeed it is true in general that, given 2-functors $T, S \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{E}$ (with a cartesian marking on $\mathcal{C}$) the composite

$$J \xrightarrow{\iota_C} T(C,C) \xrightarrow{\beta_{C,C}} S(C,C)$$

is extraordinary lax in $C \in \mathcal{C}$ if ι_C is extraordinary lax in C and $\beta_{C,D}$ is oplax$^{\mathrm{cart}}$-lax in $(C,D) \in \mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, with structure 2-cells given by a pasting like that of equation (1.9) (with now a possibly non-identity 2-cell also in the upper left square). Such structure 2-cells surely respect the identities, and they also respect the composition by the compatibility axiom of oplax$^{\mathrm{cart}}$-lax (and oplax and lax naturality). The two dimensional axiom is satisfied as well by moving the external 2-cell through the diagram using the three 2-dimensional properties that we have. We can then apply this result to $\eta_{(B,Y)}$ since $\mathrm{id}_{K(B,Y)}$ is extraordinary natural in (B,Y) and $\alpha_{(B,Y),K(B',Y')}$ is oplax$^{\mathrm{cart}}$-lax natural in $((B,Y),(B',Y'))$ (as $\alpha(B,Y), A$ is oplax$^{\mathrm{cart}}$-lax in $((B,Y),A)$).

Now, given $\eta_B\colon 1 \to F((B,Y),B)$ extraordinary lax natural in $(B,Y) \in \int^{\mathrm{op}} M$, we expand it to functors

$$\alpha_{(B,Y),A}\colon \mathcal{A}\,(K(B,Y),A) \to F((B,Y),A)$$

oplax$^{\mathrm{cart}}$-lax natural in $((B,Y),A) \in \left(\int^{\mathrm{op}} M\right)^{\mathrm{op}} \times \mathcal{A}$ as follows, using the cartesian-marked oplax naturality in (B,Y) (that is the only strictness we have). Given $u\colon B \to A$ in $\mathcal{A}$, considering $\underline{u}^Y = (u,\mathrm{id})$, the structure 2-cell $\alpha_{(u,\mathrm{id}),A} = \mathrm{id}$ will give us a commutative square

$$\begin{tikzcd}
\mathcal{A}\,(A,A) \arrow[r, "\alpha_{(A,M(u)(Y)),A}"] \arrow[d, "-\circ u"'] & F((A,M(u)(Y)),A) \arrow[d, "F((u,\mathrm{id}),A)"] \\
\mathcal{A}\,(B,A) \arrow[r, "\alpha_{(B,Y),A}"'] & F((B,Y),A)
\end{tikzcd}$$

So, looking at how we constructed η from α, in order to reach the bijection we want, we define

$$\alpha_{(B,Y),A}(u) \coloneqq F((u,\mathrm{id}),A)\left(\eta_{(A,M(u)(Y))}\right).$$

Given $\theta\colon u \Rightarrow v\colon B \to A$ in $\mathcal{A}$, considering

$$\underline{\theta}^Y\colon (u, M(\theta)_Y) \Rightarrow (v,\mathrm{id})\colon (B,Y) \to (A, M(v)(Y))$$

and using that $\alpha_{(v,\mathrm{id}),A} = \mathrm{id}$, we will have by the 2-dimensional axiom of cartesian-marked oplax naturality that

$$\alpha_{(B,Y),A}(\theta) = F(\underline{\theta}^Y, A)_{\alpha_{(A,M(v)(Y)),A}(\mathrm{id}_A)} \circ \left(\alpha_{(u,M(\theta)_Y),A}\right)_{\mathrm{id}_A}$$

So we firstly define the components of the structure 2-cells that express the cartesian-marked oplax naturality of $\alpha_{(B,Y),A}$ in (B,Y) and then we will read how to define the action of $\alpha_{(B,Y),A}$ on morphisms θ.

Looking at the diagram of equation (1.9) applied to $(\mathrm{id}_B,\gamma)\colon (B,Y) \to (B,Y')$ in $\int^{\mathrm{op}} M$, we see that, in order to have a bijection between the α's and the η's, we need to define

$$\left(\alpha_{(\mathrm{id}_B,\gamma),B}\right)_{\mathrm{id}_B} \coloneqq \eta_{(\mathrm{id}_B,\gamma)}.$$

Whence, given arbitrary $(g,\gamma)\colon (B,Y)\to (B',Y')$ in $\int^{\mathrm{op}} M$ and $w\colon B'\to A$ in $\mathcal{A}$, since

$$(w,\mathrm{id})\circ(g,\gamma)=(\mathrm{id}_A,M(w)(\gamma))\circ(w\circ g,\mathrm{id}),$$

we need to define

$$\left(\alpha_{(g,\gamma),A}\right)_w\coloneqq F((w\circ g,\mathrm{id}),A)\left(\eta_{(\mathrm{id}_A,M(w)(\gamma))}\right).$$

And at this point we define, by the argument above,

$$\alpha_{(B,Y),A}(\theta)\coloneqq F(\underline{\theta}^Y,A)_{\eta_{(A,M(v)(Y))}}\circ F((u,\mathrm{id}),A)\left(\eta_{(\mathrm{id}_A,M(\theta)_Y)}\right)$$

for every $\theta\colon u\Rightarrow v\colon B\to A$ in $\mathcal{A}$.

Looking at the diagram of equation (1.9) applied to $(g,\mathrm{id})\colon (B,Y)\to (B',M(g)(Y))$ in $\int^{\mathrm{op}} M$, we see that, in order to have a bijection between the α's and the η's, we need to define

$$\left(\alpha_{(B,Y),g}\right)_{\mathrm{id}_B}\coloneqq \eta_{(g,\mathrm{id})}.$$

Whence, given an arbitrary $f\colon A\to A'$ in $\mathcal{A}$ and $u\colon K(B,Y)\to A$ in $\mathcal{A}$, by the compatibility axiom of oplax$^{\mathrm{cart}}$-lax applied to $(u,\mathrm{id})\colon (B,Y)\to (A,M(u)(Y))$ in $\int^{\mathrm{op}} M$ and $f\colon A\to A'$ in $\mathcal{A}$, we need to define

$$\left(\alpha_{(B,Y),f}\right)_u\coloneqq F((u,\mathrm{id}),A')\left(\eta_{(f,\mathrm{id})}\right).$$

Now, we verify that such assignments work. To show that $\alpha_{(B,Y),A}$ is a functor, consider

$$u\overset{\theta}{\Rightarrow} v\overset{\rho}{\Rightarrow} w\colon B\to A$$

in $\mathcal{A}$. Then

$$\alpha_{(B,Y),A}(\rho\circ\theta)\coloneqq F(\underline{(\rho\circ\theta)}^Y,A)_{\eta_{(A,M(w)(Y))}}\circ F((u,\mathrm{id}),A)\left(\eta_{(\mathrm{id}_A,M(\rho\circ\theta)_Y)}\right)$$

while $\alpha_{(B,Y),A}(\rho)\circ\alpha_{(B,Y),A}(\theta)$ is equal to

$$F(\underline{\rho}^Y,A)_{\eta_{(A,M(w)(Y))}}\circ F((v,\mathrm{id}),A)\left(\eta_{(\mathrm{id}_A,M(\rho)_Y)}\right)\circ F(\underline{\theta}^Y,A)_{\eta_{(A,M(v)(Y))}}\circ F((u,\mathrm{id}),A)\left(\eta_{(\mathrm{id}_A,M(\theta)_Y)}\right)$$

By the extraordinary naturality of η,

$$\eta_{(\mathrm{id}_A,M(\rho\circ\theta)_Y)}=F((\mathrm{id}_A,M(\theta)_Y),A)\left(\eta_{(\mathrm{id}_A,M(\rho)_Y)}\right)\circ\eta_{(\mathrm{id}_A,M(\theta)_Y)}$$

And by the uniqueness of the liftings of 2-cells through $\mathcal{G}(M)$, we have that

$$\underline{(\rho\circ\theta)}^{Y} = \underline{\rho}^{Y}\circ(\mathrm{id}_A, M(\rho)_Y)\underline{\theta}^{Y}.$$

So, by 2-functoriality of F, it suffices to prove that

$$F(\underline{\theta}^{Y}, A)_{F((\mathrm{id}_A,M(\rho)_Y),A)\left(\eta_{(A,M(w)_Y)}\right)}\circ F((u, M(\theta)_Y), A)\left(\eta_{(\mathrm{id}_A,M(\rho)_Y)}\right)$$

is equal to

$$F((v,\mathrm{id}),A)\left(\eta_{(\mathrm{id}_A,M(\rho)_Y)}\right)\circ F(\underline{\theta}^{Y},A)_{\eta_{(A,M(v)(Y))}}.$$

But this is true by naturality of $F(\underline{\theta}^{Y},A)$ applied to the morphism

$$\eta_{(\mathrm{id}_A,M(\rho)_Y)}\colon \eta_{(A,M(v)(Y))}\to F((\mathrm{id}_A,M(\rho)_Y),A)\left(\eta_{(A,M(w)(Y))}\right).$$

The fact that $\left(\alpha_{(B,Y),f}\right)_u$ is a natural transformation is checked with techniques similar to the above ones, noticing that

$$\underline{(f\theta)}^{Y} = (f,\mathrm{id})\underline{\theta}^{Y}.$$

Whereas showing that $\left(\alpha_{(g,\gamma),A}\right)_w$ is a natural transformation uses that for $(g,\gamma)\colon (B,Y)\to(B',Y')$

$$\underline{(\theta g)}^{Y} = \underline{\theta}^{M(g)(Y)}(g,\mathrm{id}) \quad\text{and}\quad \underline{\theta}^{Y'}(\mathrm{id},\gamma) = (\mathrm{id}, M(v)(\gamma))\underline{\theta}^{M(g)(Y)}.$$

At this point, it is straightforward to check that $\alpha_{(B,Y),A}$ is $\mathrm{oplax}^{\mathrm{cart}}$ - lax in $((B,Y),A)$. And it is immediately seen that we obtain a bijection between the α's and the η's by construction.

Finally, we extend such bijection to an isomorphism of categories. Given a modification

$$\Theta_{(B,Y),A}\colon \alpha_{(B,Y),A}\Rightarrow\beta_{(B,Y),A}\colon \mathcal{A}(K(B,Y),A)\to F((B,Y),A)$$

between $\mathrm{oplax}^{\mathrm{cart}}$ - lax natural transformations in $((B,Y),A)$, we send it to the modification

$$\begin{tikzcd}
1 \arrow[r, "\mathrm{id}_B"] & \mathcal{A}(B,B) \arrow[r, bend left, "\alpha_{(B,Y),B}"] \arrow[r, bend right, "\beta_{(B,Y),B}"'] \arrow[r, phantom, "\Downarrow\Theta_{(B,Y),B}"] & F((B,Y),B)
\end{tikzcd}$$

between extraordinary lax natural transformations (that the latter is such is easily checked). And such assignment is surely functorial.

Given a modification

$$\Gamma_{(B,Y)}\colon \eta_{(B,Y)} \Rightarrow {\eta'}_{(B,Y)}\colon 1 \to F((B,Y),B)$$

between extraordinary lax natural transformations, we construct a corresponding modification Θ. We see that, if we want to reach an isomorphism of categories, we need to define

$$\left(\Theta_{(B,Y),B}\right)_{\mathrm{id}_B} \coloneqq \Gamma_{(B,Y)}.$$

Whence, given an arbitrary $u\colon B \to A$ in $\mathcal{A}$, since we want $\Theta_{-,A}$ to be a modification, considering $(u,\mathrm{id})\colon (B,Y) \to (A, M(u)(Y))$ in $\int^{\mathrm{op}} M$, we need to define

$$\left(\Theta_{(B,Y),A}\right)_u \coloneqq F((u,\mathrm{id}),A)\left(\Gamma_{(A,M(u)(Y))}\right).$$

It is straightforward to check that Θ is then a modification between $\text{oplax}^{\text{cart}}$ - lax natural transformations. And such assignment is surely functorial. At this point, it is immediate to see that the two functors are, by construction, inverses of each other, giving the desired isomorphism of categories. □

We are now ready to show that a pointwise left Kan extension in 2-$\mathcal{C}at_{\text{lax}}$ along a discrete 2-opfibration is always a weak left Kan extension.

Proposition 1.2.19. *Consider a diagram*

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "F"] \arrow[d, "K"'] \arrow[dr, phantom, "\Swarrow\lambda" near start] & \mathcal{C} \\
\mathcal{A} \arrow[ur, "L"'] &
\end{tikzcd}$$

in 2-$\mathcal{C}at_{\text{lax}}$ *with* $\mathcal{B}$ *small and* K *a discrete* 2*-opfibration. Assume that* λ *exhibits* $L = \operatorname{Lan}_K F$ *(in the sense of Definition 1.2.16). Then* λ *also exhibits* $L = \operatorname{lan}_K F$.

Proof. Since $L = \operatorname{Lan}_K F$, for every C, the 2-cell

$$\mathcal{A}\left(K(-),A\right) \xRightarrow{L} \mathcal{C}\left((L\circ K)(-),L(A)\right) \xRightarrow[\text{oplax}^{\text{cart}}]{\mathcal{C}(\lambda_-,\mathrm{id})} \mathcal{C}\left(F(-),L(A)\right);$$

is 2-universal, giving an isomorphism of categories

$$\mathcal{C}\left(L(A), C\right) \longrightarrow [\mathcal{B}^{\mathrm{op}}, \mathcal{C}at]_{\mathrm{oplax}^{\mathrm{cart}}}\left(\mathcal{A}\left(K(-), A\right), \mathcal{C}\left(F(-), C\right)\right) \tag{1.10}$$

We need to prove that, for every $U \in [\mathcal{A}, \mathcal{C}]_{\mathrm{lax}}$, pasting with λ gives an isomorphism of categories

$$[\mathcal{A}, \mathcal{C}]_{\mathrm{lax}}\left(L, U\right) \cong [\mathcal{B}, \mathcal{C}]_{\mathrm{lax}}\left(F, U \circ K\right).$$

So consider a lax natural transformation $\varphi \colon L \underset{\mathrm{lax}}{\Longrightarrow} U$. For every $A \in \mathcal{A}$, the component $\varphi_A \colon L(A) \to U(A)$ corresponds to a cartesian-marked oplax natural transformation

$$\alpha_{-,A} \colon \mathcal{A}\left(K(-), A\right) \underset{\mathrm{oplax}^{\mathrm{cart}}}{\Longrightarrow} \mathcal{C}\left(F(-), U(A)\right)$$

via the isomorphism of equation (1.10). And φ_A being lax natural in $A \in \mathcal{A}$ precisely corresponds to the cartesian-marked oplax natural transformations $\alpha_{-,A}$ being lax natural in $\mathcal{A}$, with structure 2-cell on $f \colon A \to A'$ in $\mathcal{A}$ given by the image of φ_f through the isomorphism of equation (1.10). This means that the lax natural transformations φ precisely correspond to functors $\alpha_{B,A}$ $\mathrm{oplax}^{\mathrm{cart}}$ - lax natural in $(B, A) \in \mathcal{B}^{\mathrm{op}} \times \mathcal{A}$.

Consider then a modification $\Sigma \colon \varphi \Rrightarrow \psi \colon L \underset{\mathrm{lax}}{\Longrightarrow} U$. The components Σ_A with $A \in \mathcal{A}$ correspond to modifications $\Theta_{-,A}$ between cartesian-marked oplax natural transformations $\alpha_{-,A}$ and $\beta_{-,A}$. And the modification axiom for Σ corresponds to the modification axiom for $\Theta_{B,-}$ for every fixed $B \in \mathcal{B}$. So the modifications Σ precisely correspond to modifications Θ between $\mathrm{oplax}^{\mathrm{cart}}$ - lax natural transformations α and β. By the functoriality of the isomorphism of equation (1.10), we obtain an isomorphism of categories between $[\mathcal{A}, \mathcal{C}]_{\mathrm{lax}}\left(L, U\right)$ and the category of $\mathrm{oplax}^{\mathrm{cart}}$ - lax natural transformations

$$\alpha_{B,A} \colon \mathcal{A}\left(K(B), A\right) \underset{\mathrm{oplax}^{\mathrm{cart}}}{\Longrightarrow} \mathcal{C}\left(F(B), U(A)\right)$$

in $(B, A) \in \mathcal{B}^{\mathrm{op}} \times \mathcal{A}$ and modifications between them.

By Theorem 1.2.18 (the oplax$^{\text{cart}}$ - lax parametrized Yoneda lemma), the latter category is then isomorphic to the category of extraordinary lax natural transformations

$$1 \to \mathcal{C}\left(F(B), U(K(B))\right)$$

in $B \in \mathcal{B}$ and modifications between them, which is isomorphic (for example, by Hirata's paper [24]) to $[\mathcal{B}, \mathcal{C}]_{\text{lax}}\left(F, U \circ K\right)$. Therefore we have produced an isomorphism of categories

$$[\mathcal{A}, \mathcal{C}]_{\text{lax}}\left(L, U\right) \cong [\mathcal{B}, \mathcal{C}]_{\text{lax}}\left(F, U \circ K\right),$$

and we can read that this is given by pasting with λ. □

1.3. Application to the 2-category of elements

In this section, we explore in detail the 2-category of elements (Definition 1.1.5), conceived as the 2-$\mathcal{S}et$-enriched Grothendieck construction, from an abstract point of view. Our motivation is to introduce, in future work, an enriched version of fibrations and of the Grothendieck construction. It is also useful to understand the connections between the various properties of the 2-category of elements (this speaks in particular of the usual Grothendieck construction as well).

In Theorem 1.3.8, we show that the 2-category of elements can be captured by a *lax comma object in* 2-$\mathcal{C}at_{\text{lax}}$, as defined in Definition 1.3.4. This is original, generalizing a known result due to Bird ([6]). The actual difference between our result and Bird's one is that ours allows to consider also lax natural transformations in the 2-dimensional part of the universal property. This also solves a mismatch in Gray's [20, Sections I,2 and I,5] lax commas. See below for a thorough comparison with the existing literature.

We explain how our work has potential applications to higher dimensional elementary topos theory. Indeed, in our opinion, the 2-category of elements should be seen as the archetypal 3-dimensional classification process, exhibiting 2-$\mathcal{C}at_{\text{lax}}$ as the archetypal elementary 3-topos.

In Theorem 1.3.12, we prove a pointwise Kan extension result for the 2-category of elements, using our original definition of pointwise Kan extension in $2\text{-}\mathcal{C}at_{\mathrm{lax}}$ (Definition 1.2.16). We then show that this result implies many other properties of the 2-category of elements, also thanks to Proposition 1.2.19 (a pointwise Kan extension in $2\text{-}\mathcal{C}at_{\mathrm{lax}}$ is a weak one as well). Among the properties implied, we find the conicalization of weighted 2-limits (Theorem 1.1.15) and the 2-fully faithfulness of the 2-functor that calculates the 2-category of elements (that completes Theorem 1.1.8 to 2-equivalences between 2-copresheaves and discrete 2-opfibrations).

Remark 1.3.1. Proposition 1.2.1 gives a cartesian-marked lax natural transformation λ of the form

$$\begin{tikzcd}
\int^{\mathrm{op}} F \arrow[r] \arrow[d, "\mathcal{G}(F)"'] & \mathit{1} \arrow[d, "\mathit{1}"] \arrow[dl, Rightarrow, phantom, ""] \\
\mathcal{B} \arrow[r, "F"'] & \mathcal{C}at
\arrow[from=1-2, to=2-1, Rightarrow, shorten=6mm, "\mathrm{lax}^{\mathrm{cart}}"]
\end{tikzcd}$$

Bird showed in [6] that this square exhibits a lax comma. The notion of lax comma used by Bird has been introduced by Gray in [20, Section I,2] (with the name "2-comma category") and has then been unravelled by Kelly in [27]. However they did not provide a complete universal property suitable to the lax 3-categorical ambient $2\text{-}\mathcal{C}at_{\mathrm{lax}}$. Lambert attempted in [31] to give a better universal property than the one of Gray and Kelly, but without stating any uniqueness condition in the 2-dimensional part and only giving a partial 3-dimensional part. We present in Definition 1.3.4 (see also Proposition 1.3.5) a complete universal property of the lax comma object, that refines both the ones of Gray (and Kelly) and Lambert.

In order to distinguish the explicit definition (given in Gray's [20]) from the complete universal property of the lax comma object, we will call the former "*lax comma*" and the latter "*lax comma object in* $2\text{-}\mathcal{C}at_{\mathrm{lax}}$". However, we will use the same symbol for both; this is justified by Proposition 1.3.5.

Definition 1.3.2 (Gray [20, Section I,2])**.** Let $F\colon \mathcal{A} \to \mathcal{C}$ and $G\colon \mathcal{B} \to \mathcal{C}$ be 2-functors. The *lax comma from* F *to* G is the 2-category $F \mathbin{/\!/} G$ that is given by the following data:

an object of $F /\!\!/ G$ is a triple (A, B, h) with $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $h\colon F(A) \to G(B)$ a morphism in $\mathcal{C}$;

a 1*-cell* $(A, B, h) \to (A', B', h')$ *in* $F /\!\!/ G$ is a triple (f, g, φ) with $f\colon A \to A'$ in $\mathcal{A}$, $g\colon B \to B'$ in $\mathcal{B}$ and

$$\begin{tikzcd}
F(A) \arrow[r, "h"] \arrow[d, "F(f)"'] & G(B) \arrow[d, "G(g)"] \arrow[dl, Rightarrow, shorten=4mm, "\varphi"'] \\
F(A') \arrow[r, "h'"'] & G(B')
\end{tikzcd}$$

a 2-cell in $\mathcal{C}$;

a 2*-cell* $(f, g, \varphi) \Rightarrow (f', g', \varphi')\colon (A, B, h) \to (A', B', h')$ is a pair (α, β) with $\alpha\colon f \Rightarrow f'$ in $\mathcal{A}$ and $\beta\colon g \Rightarrow g'$ in $\mathcal{B}$ such that

$$\begin{tikzcd}
F(A) \arrow[r, "h"] \arrow[d, bend right=50, "F(f')"'{name=L1}] \arrow[d, bend left=50, "F(f)"{name=L2}] & G(B) \arrow[d, "G(g)"] \arrow[dl, Rightarrow, shorten=4mm, "\varphi"'] & {} \arrow[d, phantom, "="] & F(A) \arrow[r, "h"] \arrow[d, "F(f')"'] & G(B) \arrow[d, bend right=50, "G(g')"'{name=R1}] \arrow[d, bend left=50, "G(g)"{name=R2}] \arrow[dl, Rightarrow, shorten=4mm, "\varphi'"] \\
F(A') \arrow[r, "h'"'] & G(B') & {} & F(A') \arrow[r, "h'"'] & G(B')
\arrow[Rightarrow, from=L2, to=L1, "F(\alpha)"']
\arrow[Rightarrow, from=R2, to=R1, "G(\beta)"']
\end{tikzcd}$$

the composition of 1-cells is given by pasting and that of 2-cells is inherited by the ones in $\mathcal{A}$ and $\mathcal{B}$.

The *oplax comma from* F *to* G is the co of the lax comma from F^{co} to G^{co}.

The following proposition shows the partial universal property of the lax comma object presented by Gray in [20].

Proposition 1.3.3 (Gray [20, Section I,5])**.** *Let* $F\colon \mathcal{A} \to \mathcal{C}$ *and* $G\colon \mathcal{B} \to \mathcal{C}$ *be* 2*-functors. The lax comma from* F *to* G *is equivalently given by the enriched conical limit in* 2*-*$\mathcal{C}at$ *of the diagram*

$$\begin{tikzcd}
\mathcal{A} \arrow[dr, "F"'] & & \mathcal{C}^2_{\mathrm{oplax}} \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "\mathrm{cod}"] & & \mathcal{B} \arrow[dl, "G"] \\
& \mathcal{C} & & \mathcal{C} &
\end{tikzcd}$$

where $\mathcal{C}^2_{\mathrm{oplax}}$ *is the lax comma from* $\mathrm{Id}_{\mathcal{C}}$ *to* $\mathrm{Id}_{\mathcal{C}}$*.*

But there is a better universal property that the lax comma satisfies. Indeed, it is a *lax comma object in the lax* 3*-category* $2\text{-}\mathcal{C}at_{\text{lax}}$, as originally defined here.

Definition 1.3.4. Let $\mathcal{Q}$ be a lax 3-category and consider 1-cells $F\colon \mathcal{A} \to \mathcal{C}$ and $G\colon \mathcal{B} \to \mathcal{C}$ in $\mathcal{Q}$. The *lax comma object in* $\mathcal{Q}$ *from* F *to* G is, if it exists, an object $F /\!\!/ G \in \mathcal{Q}$ together with a 2-cell

$$\begin{tikzcd}
F /\!\!/ G \arrow[r, "\partial_0"] \arrow[d, "\partial_1"'] & \mathcal{A} \arrow[d, "F"] \arrow[ld, Rightarrow, shorten=3mm, "\lambda"] \\
\mathcal{B} \arrow[r, "G"'] & \mathcal{C}
\end{tikzcd}$$

in $\mathcal{Q}$ that is universal in the following lax 3-categorical sense:

(i) for every 2-cell $\gamma\colon F \circ P \Rightarrow G \circ Q\colon \mathcal{M} \to \mathcal{C}$, there exists a unique 1-cell $V\colon \mathcal{M} \to F /\!\!/ G$ such that

$$\begin{tikzcd}
\mathcal{M} \arrow[r, bend left, "P"] \arrow[d, bend right, "Q"'] & \mathcal{A} \arrow[d, "F"] \arrow[ld, Rightarrow, shorten=3mm, "\gamma"] \\
\mathcal{B} \arrow[r, "G"'] & \mathcal{C}
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{M} \arrow[rrd, bend left, "P"] \arrow[rdd, bend right, "Q"'] \arrow[rd, dashed, "V"] & & \\
& F /\!\!/ G \arrow[r, "\partial_0"] \arrow[d, "\partial_1"'] & \mathcal{A} \arrow[d, "F"] \arrow[ld, Rightarrow, shorten=3mm, "\lambda"] \\
& \mathcal{B} \arrow[r, "G"'] & \mathcal{C}
\end{tikzcd}$$

(ii) for every 1-cells $V, W\colon \mathcal{M} \to F /\!\!/ G$ and every 3-cell

$$\begin{tikzcd}
\mathcal{M} \arrow[rd, "V"] \arrow[d, "W"'] & & \\
F /\!\!/ G \arrow[rd, "\partial_1"'] & F /\!\!/ G \arrow[r, "\partial_0"] \arrow[d, "\partial_1"] \arrow[l, Rightarrow, shorten=4mm, "\Delta"'] & \mathcal{A} \arrow[d, "F"] \arrow[ld, Rightarrow, shorten=3mm, "\lambda"] \\
& \mathcal{B} \arrow[r, "G"'] & \mathcal{C}
\end{tikzcd}
\overset{\Xi}{\Rrightarrow}
\begin{tikzcd}
\mathcal{M} \arrow[r, "V"] \arrow[rd, "W"'] & F /\!\!/ G \arrow[rd, "\partial_0"] \arrow[d, Rightarrow, shorten=2mm, "\Gamma"] & \\
& F /\!\!/ G \arrow[r, "\partial_0"] \arrow[d, "\partial_1"'] & \mathcal{A} \arrow[d, "F"] \arrow[ld, Rightarrow, shorten=3mm, "\lambda"] \\
& \mathcal{B} \arrow[r, "G"'] & \mathcal{C}
\end{tikzcd}$$

for 2-cells Γ and Δ, there exists a unique 2-cell $\nu\colon V \Rightarrow W$ such that

$$\partial_0 \nu = \Gamma, \quad \partial_1 \nu = \Delta, \quad \lambda_\nu = \Xi;$$

notice that we are precisely asking that Ξ corresponds to the 3-cell given by the lax interchange rule in $\mathcal{Q}$ of

$$\begin{tikzcd}
\mathcal{M} \arrow[r, bend left, "V"] \arrow[r, bend right, "W"'] \arrow[r, phantom, "\Downarrow\nu"] & F /\!\!/ G \arrow[r, bend left, "F\circ\partial_0"] \arrow[r, bend right, "G\circ\partial_1"'] \arrow[r, phantom, "\Downarrow\lambda"] & \mathcal{C};
\end{tikzcd}$$

(iii) for every 2-cells $\nu, \omega \colon V \Rightarrow W \colon \mathcal{M} \to F \mathbin{/\!\!/} G$ and every pair of 3-cells $\Phi \colon \partial_0\nu \Rrightarrow \partial_0\omega$ and $\Psi \colon \partial_1\nu \Rrightarrow \partial_1\omega$ such that

$$\begin{tikzcd}
 & \mathcal{M} \arrow[dd, "W"'] \arrow[dr, "V"] & & & & \mathcal{M} \arrow[rr, "V"] \arrow[dr, "W"'] & & F \mathbin{/\!\!/} G \arrow[dr, "\partial_0"] \arrow[dl, Rightarrow, bend right, "\partial_0\nu"'] \arrow[dl, Rightarrow, bend left, "\partial_0\omega"] \arrow[dl, phantom, "\overset{\Phi}{\Rrightarrow}"] & & \\
 & & F \mathbin{/\!\!/} G \arrow[r, "\partial_0"] \arrow[dd, "\partial_1"] \arrow[dl, Rightarrow, bend right, "\partial_1\nu"'] \arrow[dl, Rightarrow, bend left, "\partial_1\omega"] \arrow[dl, phantom, "\overset{\Psi}{\Rrightarrow}"] & \mathcal{A} \arrow[dd, "F"] \arrow[ddl, Rightarrow, "\lambda"] & & & F \mathbin{/\!\!/} G \arrow[rr, "\partial_0"'] \arrow[dd, "\partial_1"'] & & \mathcal{A} \arrow[dd, "F"] \arrow[ddll, Rightarrow, "\lambda"] & \\
\lambda_\omega \circ & F \mathbin{/\!\!/} G \arrow[dr, "\partial_1"'] & & & = & & & & & \circ \lambda_\nu \\
 & & \mathcal{B} \arrow[r, "G"'] & \mathcal{C} & & & \mathcal{B} \arrow[rr, "G"'] & & \mathcal{C} &
\end{tikzcd} \tag{1.11}$$

there exists a unique 3-cell $\Theta \colon \nu \Rrightarrow \omega$ such that $\partial_0\Theta = \Phi$ and $\partial_1\Theta = \Psi$.

Proposition 1.3.5. *Let $F \colon \mathcal{A} \to \mathcal{C}$ and $G \colon \mathcal{B} \to \mathcal{C}$ be 2-functors. Then there is a lax natural transformation*

$$\begin{tikzcd}
F \mathbin{/\!\!/} G \arrow[r, "\partial_0"] \arrow[d, "\partial_1"'] & \mathcal{A} \arrow[d, "F"] \arrow[dl, Rightarrow, "\lambda"] \\
\mathcal{B} \arrow[r, "G"'] & \mathcal{C}
\end{tikzcd}$$

that exhibits the lax comma $F \mathbin{/\!\!/} G$ as the lax comma object in 2-$\mathcal{C}at_{\mathrm{lax}}$ from F to G. Moreover, in the condition (ii) of lax comma object in 2-$\mathcal{C}at_{\mathrm{lax}}$, if Γ and Δ (that can be lax natural) are both strict 2-natural (resp. pseudonatural) then also ν is so.

Proof. Firstly, we construct λ. Given $(A, B, h) \in F \mathbin{/\!\!/} G$, we define the component of λ on it to be h. Given a morphism $(f, g, \varphi) \colon (A, B, h) \to (A', B', h')$ in $F \mathbin{/\!\!/} G$, we define the structure 2-cell of λ on it to be φ. It is then straightforward to show that λ is a lax natural transformation.

For condition (i) of lax comma object in 2-$\mathcal{C}at_{\mathrm{lax}}$, since λ picks the third component of objects and morphisms in $F \mathbin{/\!\!/} G$ and the other two components are determined by the projections through ∂_0 and ∂_1, we have to define V as

$$V(M) := (P(M), Q(M), \gamma_M)$$

$$V(m) := (P(m), Q(m), \gamma_m)$$

for every $M \in \mathcal{M}$ and every morphism m in $\mathcal{M}$. This is easily checked to be a 2-functor, and it works by construction.

For (ii), take an arbitrary Ξ as above. We see that the three requests

$$\partial_0 \nu = \Gamma, \quad \partial_1 \nu = \Delta, \quad \lambda_\nu = \Xi$$

force us to construct the component of ν on an arbitrary $M \in \mathcal{M}$ to be the morphism $(\Gamma_M, \Delta_M, \Xi_M)$ in $F \mathbin{/\!\!/} G$. Then the structure 2-cell of ν on a morphism $m\colon M \to M'$ in $\mathcal{M}$, being a 2-cell in $F \mathbin{/\!\!/} G$, is determined by its projections through ∂_0 and ∂_1. So we are forced to define ν_m to be the 2-cell (Γ_m, Δ_m) in $F \mathbin{/\!\!/} G$. This is indeed a 2-cell since Ξ is a modification. It is straightforward to check that ν is a lax natural transformation, since Γ and Δ are so. And we immediately see that if both Γ and Δ are strict 2-natural (resp. pseudonatural) then also ν is so. The observation that ν is then the unique lax natural transformation $V \underset{\text{lax}}{\Longrightarrow} W$ such that the modification corresponding to the lax interchange rule in $2\text{-}\mathcal{C}at_{\text{lax}}$ of ν and λ coincides with Ξ follows from Remark 1.2.2.

For (iii), let $M \in \mathcal{M}$. Since the component Θ_M will be a 2-cell in $F \mathbin{/\!\!/} G$, it is determined by its projections through ∂_0 and ∂_1. So we need to define

$$\Theta_M \coloneqq (\Phi_M, \Psi_M).$$

That this is indeed a 2-cell $\nu_M \Rightarrow \omega_M$ in $F \mathbin{/\!\!/} G$ is guaranteed by equation (1.11), taking components on M. The condition that the Θ_M's need to satisfy in order for them to collect into a modification Θ is then an equality between 2-cells in $F \mathbin{/\!\!/} G$, and thus it suffices to check its projections through ∂_0 and ∂_1. But those two resulting conditions are given by the fact that both Φ and Ψ are modifications. □

Remark 1.3.6. Notice from Definition 1.3.4 that the lax comma object in a lax 3-category really is an upgrade of the comma object to a lax 3-dimensional ambient. Indeed, a lax comma object in a 2-category is precisely a comma object, since any Ξ of Definition 1.3.4 is then forced to be the identity, and the tridimensional part becomes trivial. Interestingly, the uniqueness in the 2-dimensional part of the universal property of the lax comma object in a lax 3-category is obtained by considering the lax interchange rule.

The universal property of Proposition 1.3.3 is obtained precisely by restricting ourselves to consider as Γ and Δ only strict 2-natural transformations.

Remark 1.3.7. The following theorem is original, refining the result of Bird in [6] that the 2-category of elements is given by a lax comma. The actual difference between our result and Bird's one is that ours allows to consider lax natural transformations Γ and Δ in the 2-dimensional part of the universal property. Moreover lax comma objects in 2-$\mathcal{C}at_{\text{lax}}$ solve the mismatch of Gray's lax commas between the use of lax natural transformations and the strict ambient 2-$\mathcal{C}at$ that hosts Gray's universal property.

Theorem 1.3.8. *Let $F\colon \mathcal{B} \to \mathcal{C}at$ be a 2-functor. The 2-category of elements is equivalently given by the lax comma object*

$$\begin{tikzcd}
\int^{\mathrm{op}} F \arrow[r] \arrow[d, "\mathcal{G}(F)"'] & \mathit{1} \arrow[d, "\mathit{1}"] \arrow[dl, Rightarrow, shorten=4mm, "\text{lax comma}"] \\
\mathcal{B} \arrow[r, "F"'] & \mathcal{C}at
\end{tikzcd} \tag{1.12}$$

in 2-$\mathcal{C}at_{\text{lax}}$, exhibited by the cartesian-marked lax natural transformation of Proposition 1.2.1. As a consequence, it is then also given by the strict 3-pullback in 2-$\mathcal{C}at_{\text{lax}}$ (whose universal property can be evidently defined) between F and the replacement τ of $\mathit{1}\colon \mathit{1} \to \mathcal{C}at$ obtained by taking the lax comma object of $\mathit{1}\colon \mathit{1} \to \mathcal{C}at$ along the identity of $\mathcal{C}at$ (that is a lax 3-dimensional version of the lax limit of the arrow $\mathit{1}\colon \mathit{1} \to \mathcal{C}at$):

$$\begin{tikzcd}
\int^{\mathrm{op}} F \arrow[r] \arrow[d, "\mathcal{G}(F)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}at_{\bullet,\text{lax}} \arrow[r] \arrow[d, "\tau"'] & \mathit{1} \arrow[d, "\mathit{1}"] \arrow[dl, Rightarrow, shorten=4mm, "\text{lax comma}"] \\
\mathcal{B} \arrow[r, "F"'] & \mathcal{C}at \arrow[r, equal] & \mathcal{C}at
\end{tikzcd}$$

The domain of τ is a lax pointed version of $\mathcal{C}at$, whence the notation $\mathcal{C}at_{\bullet,\text{lax}}$.

Proof. The proof is a straightforward calculation. The fact that $\mathcal{G}(F)$ is then also the strict 3-pullback of τ is readily checked by showing that such strict 3-pullback satisfies the universal property of the lax comma object $\mathit{1} /\!\!/ F$ in 2-$\mathcal{C}at_{\text{lax}}$ that we have presented in Definition 1.3.4, using the universal properties of $\mathit{1} /\!\!/ \mathrm{Id}_{\mathcal{C}at}$ and of the strict 3-pullback together with some basics of the calculus of pasting. □

Remark 1.3.9. Theorem 1.3.8 shows a potential application of this work to 3-dimensional elementary topos theory. After Theorem 1.3.8, we should indeed view 2-$\mathcal{C}at_{\mathrm{lax}}$ as the archetypal example of a would-be notion of elementary 3-topos. Its classification process is the 2-category of elements, generalizing Weber's idea in [51] that the category of elements is the archetypal 2-dimensional classification process. Towards a definition of elementary 3-topos, one can choose between two ways. We can either regulate the classification process with lax comma objects in a lax 3-category (as originally defined here in Definition 1.3.4) or take pullbacks along discrete 2-opfibrations (that serve as replacement).

Proposition 1.3.10. *By Theorem 1.3.8, the* 2*-category of elements construction canonically extends, for every* 2*-category* $\mathcal{B}$*, to a* 2*-functor*

$$\mathcal{G}\left(-\right)\colon\ [\mathcal{B}, \mathcal{C}at]_{\mathrm{lax}} \to 2\text{-}\mathcal{C}at\,/_{\mathcal{B}}$$

Proof. Given a lax natural transformation $\varphi\colon F \Rightarrow G\colon \mathcal{B} \to \mathcal{C}at$, we define $\mathcal{G}\left(\varphi\right)$ as the unique morphism $\mathcal{G}\left(\varphi\right)\colon \int^{\mathrm{op}} F \to \int^{\mathrm{op}} G$ induced by the universal property of the lax comma object $\int^{\mathrm{op}} G$ in 2-$\mathcal{C}at_{\mathrm{lax}}$ applied to the lax natural transformation

$$\begin{tikzcd}
\int^{\mathrm{op}} F \arrow[r] \arrow[d, "\mathcal{G}(F)"'] & \mathit{1} \arrow[d, "\mathit{1}"] \arrow[dl, Rightarrow, "\lambda^F"'] \\
\mathcal{B} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow_{\varphi}"] & \mathcal{C}at
\end{tikzcd}$$

where λ^F is the lax natural transformation that presents $\int^{\mathrm{op}} F$ as a lax comma object in 2-$\mathcal{C}at_{\mathrm{lax}}$. Explicitly, for every 2-cell $\delta\colon (f,\alpha) \Rightarrow (g,\beta)\colon (B,X) \to (C,X')$ in $\int^{\mathrm{op}} F$

$$\mathcal{G}\left(\varphi\right)(B,X) = (B, \varphi_B(X)) \quad \text{and} \quad \mathcal{G}\left(\varphi\right)(f,\alpha) = (f, \varphi_C(\alpha)) \quad \text{and} \quad \mathcal{G}\left(\varphi\right)(\delta) = \delta.$$

Given a modification $\Theta\colon \varphi \Rrightarrow \psi\colon F \Rightarrow G\colon \mathcal{B} \to \mathcal{C}at$, we define $\mathcal{G}\left(\Theta\right)$ as the unique 2-natural transformation induced by the universal property of the lax comma object $\int^{\mathrm{op}} G$ in 2-$\mathcal{C}at_{\mathrm{lax}}$ applied, in the notation of Definition 1.3.4 to

$V = \mathcal{G}(\varphi)$, $W = \mathcal{G}(\psi)$, $\Gamma = \mathrm{id}$, $\Delta = \mathrm{id}$ and Ξ given by

$$\begin{tikzcd}
\int^{\mathrm{op}} F \arrow[r] \arrow[d, "\mathcal{G}(F)"'] & \mathit{1} \arrow[d, "\mathit{1}"] \arrow[dl, Rightarrow, "\lambda^F"'] \\
\mathcal{B} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\varphi \Downarrow \overset{\Theta}{\Rrightarrow} \Downarrow \psi"] & \mathcal{C}at
\end{tikzcd}$$

Explicitly, the component of $\mathcal{G}(\Theta)$ on an object $(B, X) \in \int^{\mathrm{op}} F$ is

$$\mathcal{G}(\Theta)_{(B,X)} = (\mathrm{id}_B, \Theta_{B,X}).$$

It is straightforward to show that $\mathcal{G}(-)$ is indeed a 2-functor. $\square$

Remark 1.3.11. The following table shows the four co-op versions of the 2-category of elements construction, with the corresponding notions of fibration.

$$\begin{tikzcd}
\int^{\mathrm{op}} F \arrow[r] \arrow[d, "\mathcal{G}(F)"'] & \mathit{1} \arrow[d, "\mathit{1}"] \arrow[dl, Rightarrow, "\text{lax comma}"] \\
\mathcal{B} \arrow[r, "F"'] & \mathcal{C}at
\end{tikzcd}$$

disc 2-opfibrations:

opfibrations, locally

discrete fibrations

$$\begin{tikzcd}
\int F \arrow[r] \arrow[d, "\mathcal{G}(F)"'] & \mathit{1} \arrow[d, "\mathit{1}"] \\
\mathcal{B} \arrow[r, "F"'] \arrow[ur, Rightarrow, "\text{lax comma}"] & \mathcal{C}at^{\mathrm{op}}
\end{tikzcd}$$

disc 2-fibrations:

fibrations, locally

discrete opfibrations

$$\begin{tikzcd}
\int^{\mathrm{coop}} F \arrow[r] \arrow[d, "\mathcal{G}(F)"'] & \mathit{1} \arrow[d, "\mathit{1}"] \arrow[dl, Rightarrow, "\text{oplax comma}"] \\
\mathcal{B} \arrow[r, "F"'] & \mathcal{C}at^{\mathrm{co}}
\end{tikzcd}$$

disc 2-coopfibrations:

opfibrations, locally

discrete opfibrations

$$\begin{tikzcd}
\int^{\mathrm{co}} F \arrow[r] \arrow[d, "\mathcal{G}(F)"'] & \mathit{1} \arrow[d, "\mathit{1}"] \\
\mathcal{B} \arrow[r, "F"'] \arrow[ur, Rightarrow, "\text{oplax comma}"] & \mathcal{C}at^{\mathrm{coop}}
\end{tikzcd}$$

disc 2-cofibrations:

fibrations, locally

discrete fibrations

We now apply Section 1.2 to the 2-category of elements. We show that the same filled square that exhibits a lax comma object in 2-$\mathcal{C}at_{\mathrm{lax}}$ also exhibits a pointwise left Kan extension in 2-$\mathcal{C}at_{\mathrm{lax}}$ (Definition 1.2.16). Such result is original.

Theorem 1.3.12. *Let $F\colon \mathcal{A} \to \mathcal{C}at$ be a 2-functor with $\mathcal{A}$ a small 2-category. Then the lax comma object square in 2-$\mathcal{C}at_{\mathrm{lax}}$*

$$\begin{tikzcd}
\int^{\mathrm{op}} F \arrow[r] \arrow[d, "\mathcal{G}(F)"'] & \mathit{1} \arrow[d, "\mathit{1}"] \arrow[dl, Rightarrow, "\text{lax comma}"] \\
\mathcal{A} \arrow[r, "F"'] & \mathcal{C}at
\end{tikzcd}$$

exhibits

$$F = \mathrm{Lan}_{\mathcal{G}(F)} \Delta 1.$$

Proof. By Proposition 1.2.1, we know that the lax natural transformation λ that presents the lax comma object in $2\text{-}\mathcal{C}at_{\mathrm{lax}}$ is cartesian-marked lax. Given $A \in \mathcal{A}$ and $C \in \mathcal{C}at$, we prove that the cartesian-marked oplax natural transformation

$$\mathcal{A}\left(\mathcal{G}(F)(-),A\right) \overset{F}{\Longrightarrow} \mathcal{C}at\left((F\circ\mathcal{G}(F))(-),F(A)\right) \xRightarrow[\text{oplax}^{\text{cart}}]{\mathcal{C}at(\lambda_-,\mathrm{id})} \mathcal{C}at\left(\Delta 1(-),F(A)\right),$$

that we call μ, is 2-universal. Explicitly, μ has components

$$\begin{tikzcd} \mu_{(B,X)}\colon & \mathcal{B}\left(B,A\right) \arrow[r] & \mathcal{C}at\left(1,F(A)\right) \end{tikzcd}$$

$$\begin{tikzcd} B \arrow[r, bend left, "u"{name=U}] \arrow[r, bend right, "v"'{name=V}] \arrow[from=U, to=V, Rightarrow, phantom, "\Downarrow\theta"] & A \arrow[r, mapsto] & 1 \arrow[r, bend left, "F(u)(X)"{name=FU}] \arrow[r, bend right, "F(v)(X)"'{name=FV}] \arrow[from=FU, to=FV, Rightarrow, "F(\theta)_X"] & F(A) \end{tikzcd}$$

for every $(B,X) \in \int^{\mathrm{op}} F$ and structure 2-cells

$$\left(\mu_{(g,\gamma)}\right)_u = F(u)(\gamma)\colon F(u\circ g)(X') \to F(u)(X)$$

on every $(g,\gamma)\colon (B,X) \leftarrow (B',X')$ in $\int^{\mathrm{op}} F$, for every $u\colon B \to A$ in $\mathcal{A}$. Given

$$\sigma\colon \mathcal{A}\left(\mathcal{G}(F)(-),A\right) \xRightarrow[\text{oplax}^{\text{cart}}]{} \mathcal{C}at\left(\Delta 1(-),C\right),$$

we prove that there exists a unique functor $s\colon F(A) \to C$ such that

$$(s\circ -)\circ\mu = \sigma.$$

We see that there is at most one such s, as we need, for every $\alpha\colon X \to X'$ in $F(A)$,

$$s(X) = s\left(\mu_{(A,X)}(\mathrm{id}_A)\right) = \sigma_{(A,X)}(\mathrm{id}_A)$$
$$s(\alpha) = s\left(\left(\mu_{(\mathrm{id}_A,\alpha)}\right)_{\mathrm{id}_A}\right) = \left(\sigma_{(\mathrm{id}_A,\alpha)}\right)_{\mathrm{id}_A}.$$

And this s works thanks to the fact that σ is cartesian-marked oplax. Indeed it is readily shown to be functorial (using that σ is oplax) and for every $(g,\gamma)\colon (B,X) \to (B',X')$ in $\int^{\mathrm{op}} F$ and every $\theta\colon u \Rightarrow v\colon B \to A$ in $\mathcal{A}$, considering $\underline{u}^X = (u,\mathrm{id})$ and $\underline{\theta}^X\colon (u,F(\theta)_X) \Rightarrow (v,\mathrm{id})$,

$$\sigma_{(B,X)}(u) = \sigma_{(A,F(u)(X))}(\mathrm{id}_A) = s(F(u)(X)) = s\left(\mu_{(B,X)}(u)\right)$$

$$\sigma_{(B,X)}(\theta) = \left(\sigma_{(u,F(\theta)_X)}\right)_{\mathrm{id}} = \left(\sigma_{(\mathrm{id},F(\theta)_X)}\right)_{\mathrm{id}} = s(F(\theta)_X) = s\left(\mu_{(B,X)}(\theta)\right)$$

$$\left(\sigma_{(g,\gamma)}\right)_u = \left(\sigma_{(\mathrm{id},\gamma)}\right)_u = \left(\sigma_{(u,F(u)(\gamma))}\right)_{\mathrm{id}} = \left(\sigma_{(\mathrm{id},F(u)(\gamma))}\right)_{\mathrm{id}} = s(F(u)(\gamma)) = s\left(\left(\mu_{(g,\gamma)}\right)_u\right)$$

We now prove the 2-dimensional universality of μ. Given

$$\Xi\colon \sigma \Rrightarrow \sigma'\colon \mathcal{A}\left(\mathcal{G}\left(F\right)(-), A\right) \underset{\text{oplax}^{\text{cart}}}{\Longrightarrow} \mathcal{C}at\left(\Delta 1(-), C\right),$$

we prove that there exists a unique natural transformation $\xi\colon s \Rightarrow s'\colon F(A) \to C$ such that

$$(\xi * -)\mu = \Xi.$$

We see that there is at most one such ξ, as we need, for every $X \in F(A)$,

$$\xi_X = \xi_{\mu_{(A,X)}(\mathrm{id}_A)} = \Xi_{(A,X),\mathrm{id}_A}$$

And this ξ works since it is readily shown to be natural (using that Ξ is a modification) and for every $(B,X) \in \int^{\mathrm{op}} F$ and $u\colon B \to A$ in $\mathcal{A}$

$$\Xi_{(B,X),u} = \Xi_{(A,F(u)(X)),\mathrm{id}_A} = \xi_{F(u)(X)} = \xi_{\mu_{(B,X)}(u)}.$$

We have thus shown that μ is 2-universal and this concludes the proof. □

Thanks to Proposition 1.2.19, we obtain as a corollary that the 2-category of elements also exhibits a weak left Kan extension in 2-$\mathcal{C}at_{\mathrm{lax}}$. The isomorphism of categories that presents such weak left Kan extension has been proved by Bird in [6]. Moreover, such isomorphism restricts to different flavours of laxness. All these isomorphisms are a particular case of Proposition 3.18 of Szyld's [44].

Corollary 1.3.13 (Bird [6], Szyld [44]). *Let $F\colon \mathcal{A} \to \mathcal{C}at$ be a 2-functor. Then*

$$F = \operatorname{lan}_{\mathcal{G}(F)} \Delta 1.$$

Moreover the isomorphism of categories

$$\left[\mathcal{A}, \mathcal{C}at\right]_{\mathrm{lax}}\left(F, U\right) \cong \left[\int^{\mathrm{op}} F, \mathcal{C}at\right]_{\mathrm{lax}}\left(\Delta 1, U \circ \mathcal{G}\left(F\right)\right),$$

natural in $U\colon \mathcal{A} \to \mathcal{C}at$, that presents the weak left Kan extension in 2-$\mathcal{C}at_{\mathrm{lax}}$ (see Remark 1.2.10) restricts to isomorphisms

$$\left[\mathcal{A}, \mathcal{C}at\right]_{\mathrm{ps}}\left(F, U\right) \cong \left[\int^{\mathrm{op}} F, \mathcal{C}at\right]_{\mathrm{sigma}}\left(\Delta 1, U \circ \mathcal{G}\left(F\right)\right)$$

$$[\mathcal{A}, \mathcal{C}at]\,(F, U) \cong \left[\int^{\mathrm{op}} F, \mathcal{C}at\right]_{\mathrm{lax}^{\mathrm{cart}}} (\Delta 1, U \circ \mathcal{G}(F))\,,$$

where ps *means to restrict to pseudonatural transformations and* sigma *means to restrict to sigma natural transformations, that have been defined in Descotte, Dubuc and Szyld's paper [15] and are a pseudo version of the cartesian-marked lax natural transformations.*

Proof. Clear by Proposition 1.2.19. It is straightforward to see that the restrictions hold. □

Remark 1.3.14. The third isomorphism of Corollary 1.3.13 offers a shorter but less elementary proof to Theorem 1.1.15 (reduction of weighted 2-limits to cartesian-marked lax conical ones). Indeed this is what Street showed in [42, Theorem 15].

We can also deduce the 2-fully faithfulness of the 2-category of elements construction (in three laxness flavours) and extend Lambert's Theorem 1.1.8 to 2-equivalences between 2-copresheaves and discrete 2-opfibrations. The fact that the first 2-functor $\mathcal{G}(-)$ of Theorem 1.3.15 is 2-fully faithful is proved also in Bird's [6], but we show that it is a consequence of the weak Kan extension result. None of the three 2-equivalence results of Theorem 1.3.15 seems to appear in the literature.

Theorem 1.3.15. *Let $\mathcal{A}$ be a 2-category. The 2-category of elements construction (extended to consider lax natural transformations as in the proof of Proposition 1.3.10) produces a 2-equivalence*

$$\mathcal{G}(-):\ [\mathcal{A}, \mathcal{C}at]_{\mathrm{lax}} \xrightarrow{\sim} \mathcal{D}2\mathcal{O}p\mathcal{F}ib(\mathcal{A})$$

where $\mathcal{D}2\mathcal{O}p\mathcal{F}ib(\mathcal{A})$ is the full sub-2-category of $2\text{-}\mathcal{C}at\,/_{\mathcal{A}}$ given by the split discrete 2-opfibrations with small fibres. Moreover this restricts to 2-equivalences

$$\mathcal{G}(-):\ [\mathcal{A}, \mathcal{C}at]_{\mathrm{ps}} \xrightarrow{\sim} \mathcal{D}2\mathcal{O}p\mathcal{F}ib_{\mathrm{cart}}(\mathcal{A})$$

$$\mathcal{G}(-):\ [\mathcal{A}, \mathcal{C}at] \xrightarrow{\sim} \mathcal{D}2\mathcal{O}p\mathcal{F}ib_{\mathrm{clov}}(\mathcal{A})$$

where $\mathcal{D}2\mathcal{O}p\mathcal{F}ib_{\mathrm{cart}}(\mathcal{A})$ and $\mathcal{D}2\mathcal{O}p\mathcal{F}ib_{\mathrm{clov}}(\mathcal{A})$ restrict $\mathcal{D}2\mathcal{O}p\mathcal{F}ib(\mathcal{A})$ respectively *to cartesian functors* (*as underlying functors of the* 1*-cells*) *and to cleavage preserving functors.*

Proof. We already know by Theorem 1.1.8 that the essential image of $\mathcal{G}(-)$ (in each of the three versions) is given by the split discrete 2-opfibrations. So we are missing the 2-fully faithfulness of the three 2-functors.

Let $F, G\colon \mathcal{A} \to \mathcal{C}at$ be 2-functors. Then combining Corollary 1.3.13 and Theorem 1.3.8 (the 2-category of elements exhibits a lax comma object in 2-$\mathcal{C}at_{\text{lax}}$) we obtain the composite isomorphism of categories

$$[\mathcal{A}, \mathcal{C}at]_{\text{lax}}(F, G) \cong \left[\int^{\text{op}} F, \mathcal{C}at\right]_{\text{lax}} (\Delta 1, G \circ \mathcal{G}(F)) \cong 2\text{-}\mathcal{C}at/_{\mathcal{A}}\left(\int^{\text{op}} F, \int^{\text{op}} G\right)$$

where the first functor is given by pasting with the 2-cell λ^F that presents the lax comma object $\int^{\text{op}} F$ and the second functor is the inverse of pasting with λ^G on objects and producing the modification associated to the lax interchange rule (see Remark 1.2.2) on morphisms. Indeed the second functor is surely a bijection on objects, and it is also a bijection on morphisms by part (*ii*) of Definition 1.3.4 (the 2-dimensional part of the universal property of the lax comma object) with $\Gamma = \text{id}$ and $\Delta = \text{id}$ (and so in particular strict 2-natural transformations). Since the composite functor precisely coincides with the functor on morphisms associated to $\mathcal{G}(-)$ between $\mathcal{A}$ and $\mathcal{C}$ (see the proof of Proposition 1.3.10), this completes the proof of the first 2-equivalence.

The composite isomorphism above then restricts to the following two:

$$[\mathcal{A}, \mathcal{C}at]_{\text{ps}}(F, G) \cong \left[\int^{\text{op}} F, \mathcal{C}at\right]_{\text{sigma}} (\Delta 1, G \circ \mathcal{G}(F)) \cong \mathcal{D}2\mathcal{O}p\mathcal{F}ib_{\text{cart}}(\mathcal{A})\left(\int^{\text{op}} F, \int^{\text{op}} G\right)$$

$$[\mathcal{A}, \mathcal{C}at](F, G) \cong \left[\int^{\text{op}} F, \mathcal{C}at\right]_{\text{lax}^{\text{cart}}} (\Delta 1, G \circ \mathcal{G}(F)) \cong \mathcal{D}2\mathcal{O}p\mathcal{F}ib_{\text{clov}}(\mathcal{A})\left(\int^{\text{op}} F, \int^{\text{op}} G\right)$$

by part (*i*) of Definition 1.3.4, since whiskering λ^G on the left with a 2-functor $\int^{\text{op}} F \to \int^{\text{op}} G$ looks at the second component of the morphisms in $\int^{\text{op}} G$. Indeed, if we start for example from $\gamma\colon \Delta 1 \underset{\text{lax}^{\text{cart}}}{\Longrightarrow} G \circ \mathcal{G}(F)$, the associated functor $V^\gamma\colon \int^{\text{op}} F \to \int^{\text{op}} G$ is such that, for every morphism (f, id) in $\int^{\text{op}} F$,

$$\text{pr}_1(V^\gamma(f, \text{id})) = f \quad \text{and} \quad \text{pr}_2(V^\gamma(f, \text{id})) = (\lambda^G V^\gamma)_{(f,\text{id})} = \gamma_{(f,\text{id})} = \text{id}\,.$$

And then the first 2-equivalence restricts to the other two. □

2. Colimits in 2-dimensional slices

This chapter is based on our paper [35].

It is well-known that, in dimension 1, a colimit in a slice category is precisely the map from the colimit of the domains of the diagram which is induced by the universal property of the colimit. This fact, together with the results of preservation, reflection and lifting of all colimits for the domain functor from a slice category, gives a complete calculus of colimits in 1-dimensional slices (see Theorem 2.0.1). And such a calculus has been proven useful in myriads of applications, in particular in the context of locally cartesian closed categories or for general exponentiability of morphisms, in categorical logic, algebraic geometry and topos theory. Indeed, an exponentiable morphism $f : E \to B$ in $\mathcal{C}$, that is an exponentiable object in $\mathcal{C}/_{B}$, can be characterized as a morphism which admits all pullbacks along it and is such that the change of base functor $f^{*}\colon \mathcal{C}/_{B} \to \mathcal{C}/_{E}$ has a right adjoint. The latter condition implies, and by adjoint functor theorems is often implied by, preservation of all colimits for f^{*}. The calculus of colimits in 1-dimensional slices is what allows to apply such preservation of colimits to colimits in the slice that come from colimits in the category $\mathcal{C}$, i.e. the ones that we have in practice.

The main result of this chapter is a generalization to dimension 2 of this fruitful 1-dimensional calculus, including results of preservation, reflection and lifting of 2-colimits for the domain 2-functor from a lax slice. Theorems on suitable change of base 2-functors between lax slices are presented as well. The lax slice is indeed the appropriate 2-dimensional slice to consider in order to achieve such generalization, as we justify with two different approaches.

These results, in combination with Chapter 1, will be crucial for us in the following chapters to expand the theory of 2-classifiers and elementary 2-toposes. In particular, it will be very useful to achieve our theorems of reduction of the study of a 2-classifier to dense generators, in Chapter 3. Indeed the strategy will be to write an arbitrary object as a nice 2-colimit of dense generators (we will briefly recall the theory of dense generators in Section 3.1) and then apply the universal property of such colimits. The work of this chapter will be crucial to handle the relevant 2-dimensional colimits.

The following theorem condenses the calculus of colimits in 1-dimensional slice categories.

Theorem 2.0.1. *Let $\mathcal{C}$ be a category with products and let $M \in \mathcal{C}$. The domain functor* $\operatorname{dom}\colon \mathcal{C}/_{M} \to \mathcal{C}$ *preserves, reflects and lifts uniquely all colimits (and so it creates all colimits).*

Moreover, for every diagram $D\colon \mathcal{A} \to \mathcal{C}$ with $\mathcal{A}$ small that admits a colimit in $\mathcal{C}$, every morphism $q\colon \operatorname{colim}_A D(A) \to M$ in $\mathcal{C}$ is the colimit of a diagram in $\mathcal{C}/_{M}$. More precisely,

$$\begin{array}{c}\operatorname{colim}_A D(A)\\ \big\downarrow{\scriptstyle q}\\ M\end{array} = \operatorname{colim}_A \begin{array}{c}D(A)\\ \big\downarrow{\scriptstyle q\circ i_A}\\ M\end{array} \quad \text{in } \mathcal{C}/_{M}, \tag{2.1}$$

where the $i_A\colon D(A) \to \operatorname{colim}_A D(A)$ are the inclusions that form the universal cocone.

Notice that this theorem recovers the property we mentioned above, that a colimit in $\mathcal{C}/_{M}$ is precisely the map from the colimit of the domains of the diagram which is induced by the universal property. Indeed, half of this fact is captured by the preservation of colimits for $\operatorname{dom}\colon \mathcal{C}/_{M} \to \mathcal{C}$, whereas the other half, that is harder to capture, is represented by equation (2.1). In dimension 1, the latter special property holds because a cocone on M is the same thing as a diagram in the slice over M. But in dimension 2, we need weighted 2-colimits and then weighted 2-cocylinders rather than cocones. This makes it then harder to establish a bijection with diagrams in a 2-dimensional slice, as such diagrams still have a conical shape.

In this chapter, we first focus on generalizing the special property of equation (2.1) to dimension 2 (Theorem 2.1.21), extracting from a weighted 2-cocylinder a diagram in a 2-dimensional slice whose 2-dimensional colimit (of some kind) is the morphism induced by the weighted 2-cocylinder. We show two different approaches to this. The first approach (Construction 2.1.1) is more intuitive, based on the reduction of weighted 2-colimits to essentially conical ones, namely cartesian-marked oplax conical ones. Recall that such reduction is due to Street [42], and that we have described it in Section 1.1 with new, more elementary proofs. Recall also that the reduction of weighted 2-colimits to cartesian-marked oplax conical ones is allowed and regulated by the 2-category of elements, that we have studied in detail in Chapter 1 (both from an elementary and an abstract perspective).

The second approach, culminating with Theorem 2.1.21, is instead more abstract. It is based on an apparently original concept of colim-fibration, that we give both in dimension 1 (Definition 2.1.3) and dimension 2 (Definition 2.1.14), as well as on the 2-category of elements construction. Both the approaches show the need to consider lax slices in order to generalize the special property of equation (2.1) to dimension 2. In the first one, for example, this corresponds to only being able to essentially conicalize weighted 2-colimits, rather than to strictly conicalize them.

A result of reflection of 2-colimits for the domain 2-functor $\operatorname{dom}\colon \mathcal{E}/_{\text{lax}}\, M \to \mathcal{E}$ from a lax slice is the main part of Theorem 2.1.21, as reflection is part of the concept of 2-colim-fibration. Further than reflecting (appropriate) 2-colimits, a 2-colim-fibration is in particular a discrete 2-fibration (Definition 1.1.7), that is, what is classified by the 2-category of elements construction.

We then show a result of lifting of 2-colimits for $\operatorname{dom}\colon \mathcal{E}/_{\text{lax}}\, M \to \mathcal{E}$ in Proposition 2.2.6. This is based on a generalization to dimension 2 of the bijective correspondence between cocones on M and diagrams in the slice over M (Proposition 2.2.1). The 2-dimensional correspondence is captured and justified by $\mathcal{F}$-category theory, also called enhanced 2-category theory and introduced by Lack and Shulman in [30]. We will recall it in Definition 2.2.3 (see also Remark 2.2.4). Roughly, the idea is to consider 2-categories, whose morphisms we think as loose, with a selected subclass of morphisms that we call tight. We then ask 2-functors to preserve the tightness of morphisms.

$\mathcal{F}$-category theory is then crucial in establishing a result of preservation of 2-colimits for $\operatorname{dom}\colon \mathcal{E}/_{\text{lax}}\ M \to \mathcal{E}$ (Theorem 2.3.12). Indeed we guarantee preservation of 2-colimits by an original theorem (Theorem 2.3.10) that states that a lax left adjoint (Definition 2.3.1) preserves appropriate colimits (Definition 2.3.6) if the adjunction is strict on one side and is suitably $\mathcal{F}$-categorical (Definition 2.3.5). Lax adjoints have been firstly introduced by Gray in [20, Section I,7], and the idea is to admit unit and counit to be lax natural transformations (rather than strict or pseudo). At the level of hom-categories, such laxness translates as having an adjunction between them rather than an isomorphism (or an equivalence as for a biadjunction); see Remark 2.3.2. We need such adjunctions because the suitable right adjoint to the domain 2-functor $\operatorname{dom}\colon \mathcal{E}/_{\text{lax}}\ M \to \mathcal{E}$ is still $M \times -$ (as in dimension 1), but the unit is now only lax natural. The idea behind our general $\mathcal{F}$-categorical Theorem 2.3.10 on the preservation of colimits is to move back and forth between the two 2-categories taking advantage of the strictness of the lax adjunction on one side. The condition of having a suitably $\mathcal{F}$-categorical lax adjunction ensures that this idea works. $\mathcal{F}$-categorical adjunctions appear also in Walker's [50]; and another potential source of examples is Bourke's [9]. Remember that, although our preservation result is expressed in an $\mathcal{F}$-categorical language, it is always possible to start from a weighted 2-colimit and view it in this context, after reducing it to a cartesian-marked oplax conical one.

Finally, we apply the general $\mathcal{F}$-categorical theorem of preservation of 2-colimits to the 2-functor of change of base along a split Grothendieck opfibration between lax slices (see Proposition 2.4.2). In dimension 1, the concept of change of base between slice categories is definitely helpful, and it is well-known that the pullback perfectly realizes such a job. For $\mathcal{C}at$, given a functor $\tau\colon \mathcal{E} \to \mathcal{B}$, it is still a good idea to consider the pullback 2-functor $\tau^*\colon \mathcal{C}at/_{\mathcal{B}} \to \mathcal{C}at/_{\mathcal{E}}$ between strict slices. It is well-known that such change of base 2-functor has a right 2-adjoint τ_*, and thus preserves all weighted 2-colimits, precisely when τ is a Conduché functor. This is proved by Conduché in [13], with the ideas already present in Giraud's [18]. However, in order to generalize the calculus of colimits in 1-dimensional slices to dimension 2, we find the need to consider lax slices. And it is then helpful to have a change of base 2-functor between lax slices of a finitely complete 2-category. We believe that the most natural way to achieve this is by calculating comma objects rather than pullbacks. As we have described in Chapter 1, this is connected to the construction of the category of elements, but also, in general, to the concept of 2-dimensional elementary topos (see also Section 3.1). Equivalently to calculating comma objects, we can take pullbacks along split Grothendieck opfibrations, which serve as a kind of fibrant replacement (see Proposition 2.4.1 and Section 3.1). Such a point of view is preferable for us since Grothendieck opfibrations in $\mathcal{C}at$ are always Conduché and we can generalize to lax slices the ideas for finding a right adjoint to the pullback functor $\tau^*\colon \mathcal{C}at/_{\mathcal{B}} \to \mathcal{C}at/_{\mathcal{E}}$ (from Conduché's [13]). Notice that considering lax slices we are fixing a direction and we then need to take opfibrations, while Conduché functors are now too unbiased.

We prove that $\tau^*\colon \mathcal{C}at/_{\text{lax}}\,\mathcal{B} \to \mathcal{C}at/_{\text{lax}}\,\mathcal{E}$ has a loose $\mathcal{F}$-categorical right lax adjoint that is strict on one side, in Theorem 2.4.3. This generalizes also Palmgren's [39], where a similar result for pseudoslices of groupoids is proved, from the comma objects point of view. Our theorem then implies the preservation of appropriate 2-colimits for the 2-functor τ^* between lax slices.

We also show that the 2-functor of change of base along a split Grothendieck opfibration between lax slices makes sense in a general 2-category rather than just in $\mathcal{C}at$ (Proposition 2.4.2). For this we take from Street's [41] and Weber's [51] the needed general notion of opfibration; we will recall it in Section 4.2. We conclude proving that the 2-functor τ^* between lax slices preserves appropriate 2-colimits also in the case of prestacks (Proposition 2.4.5) and more in general of finitely complete 2-categories with a dense generator (Theorem 2.4.6).

Outline of the chapter

In Section 2.1, we present our calculus of colimits in 2-dimensional slices (Theorem 2.1.21), firstly by reducing weighted 2-colimits to cartesian-marked oplax conical ones (Construction 2.1.1) and then via 2-colim-fibrations (Definition 2.1.14). This includes a result of reflection of 2-colimits for $\mathrm{dom}\colon \mathcal{E}/_{\mathrm{lax}}\, M \to \mathcal{E}$.

In Section 2.2, we generalize to dimension 2 the bijective correspondence between cocones on M and diagrams in the slice over M (Proposition 2.2.1). We then show a result of lifting of 2-colimits for $\mathrm{dom}\colon \mathcal{E}/_{\mathrm{lax}}\, M \to \mathcal{E}$ (Proposition 2.2.6).

In Section 2.3, we prove a result of preservation of 2-colimits for the domain 2-functor from a lax slice (Theorem 2.3.12). This is shown by proving a general theorem of $\mathcal{F}$-category theory (Theorem 2.3.10), which states that a lax left adjoint preserves appropriate colimits if the adjunction is strict on one side and is suitably $\mathcal{F}$-categorical.

In Section 2.4, we apply this theorem of preservation of 2-colimits to the 2-functor of change of base along a split Grothendieck opfibration between lax slices (Theorem 2.4.3), laxifying the proof that Conduché functors are exponentiable. We conclude extending such result to prestacks (Proposition 2.4.5) and then to any finitely complete 2-category with a dense generator (Theorem 2.4.6).

2.1. Colimits in 2-dimensional slices

We aim at generalizing to dimension 2 the well-known 1-dimensional result that a colimit in a slice category corresponds to the map from the colimit of the domains of the diagram which is induced by the universal property. Half of such result will be captured by preservation of 2-colimits for the domain 2-functor from a lax slice, that we will address in Section 2.3. In this section, we focus on the other half, that is the generalization to dimension 2 of the special property of equation (2.1) (of Theorem 2.0.1). Namely, we want to prove that a morphism from a 2-colimit to some M can be expressed as a 2-colimit in a 2-dimensional slice over M.

We show two different approaches to this, that lead to the same result (compare Construction 2.1.1 with Theorem 2.1.21). The first one is more intuitive, based on the reduction of the weighted 2-colimits to cartesian-marked oplax conical ones. The second approach is more abstract, based on an apparently original concept of *colim-fibration* (Definition 2.1.3 in dimension 1 and Definition 2.1.14 in dimension 2). This will offer a shorter and more elegant proof, in Theorem 2.1.21. Both the approaches show the need to consider lax slices.

This section also contains a result of reflection of 2-colimits for the domain 2-functor $\operatorname{dom}\colon \mathcal{E}/_{\text{lax}}\, M \to \mathcal{E}$ from a lax slice. Indeed the concept of 2-*colim-fibration* involves reflecting (appropriate) 2-colimits, together with being a discrete 2-fibration.

We now begin exploring the first approach to the generalization to dimension 2 of equation (2.1) (of Theorem 2.0.1).

Construction 2.1.1. Let $\mathcal{E}$ be a 2-category and let $M \in \mathcal{E}$. Consider a 2-diagram $F\colon \mathcal{A} \to \mathcal{E}$ with $\mathcal{A}$ small and a weight $W\colon \mathcal{A}^{\text{op}} \to \mathcal{C}at$ such that the colimit $\operatorname{colim}^W F$ of F weighted by W exists in $\mathcal{E}$. Take then a morphism $q\colon \operatorname{colim}^W F \to M$, or equivalently the corresponding weighted 2-cocylinder

$$\nu^q\colon W \Rightarrow \mathcal{E}\left(F(-), M\right).$$

We would like to express q as a 2-colimit in a 2-dimensional slice of $\mathcal{E}$ over M. So we need to construct from ν^q a 2-diagram in a 2-dimensional slice. In dimension 1, equation (2.1) (of Theorem 2.0.1) is based on the fact that a cocone on M coincides with a diagram in $\mathcal{C}/_{M}$. But here, in dimension 2, we have a weighted 2-cocylinder ν^q instead of a strict cocone, and thus it is not clear how to directly find a corresponding diagram in a slice. We notice that this is essentially a matter of selecting a cocone out of the bunch of cocones that form the weighted 2-cocylinder ν^q. And then we obtain a great help from the reduction of weighted 2-colimits to cartesian-marked oplax conical ones.

As described in Section 1.1, while it is not possible to represent ν^q with a selected strict cocone, it is possible to reduce ν^q to a cartesian-marked oplax cocone. Indeed

$$\operatorname{colim}^W F \cong \text{oplax}^{\text{cart}}\text{-}\operatorname{colim}^{\Delta 1}(F \circ \mathcal{G}(W))$$

where $\mathcal{G}(W)\colon \int W \to \mathcal{A}$ is the 2-category of elements of W. And ν^q corresponds to a cartesian-marked oplax cocone

$$\lambda^q\colon \Delta 1 \underset{\text{oplax}^{\text{cart}}}{\Longrightarrow} \mathcal{E}\left((F \circ \mathcal{G}(W))(-), M\right) : \left(\int W\right)^{\text{op}} \to \mathcal{C}at.$$

It is easy to check that a cartesian-marked oplax cocone on M can be reorganized as a 2-diagram in the lax slice $\mathcal{E}/_{\text{lax}}\, M$ on M (we will see the complete correspondence in Proposition 2.2.1), where a 1-cell in the lax slice from $E \xrightarrow{g} M$ to $E' \xrightarrow{g'} M$ is a filled triangle

$$\begin{tikzcd}
E \arrow[rr, "\widehat{\gamma}"] \arrow[dr, "g"'] & \arrow[d, phantom, "\overset{\gamma}{\Longrightarrow}"] & E' \arrow[dl, "g'"] \\
& M &
\end{tikzcd}$$

More precisely, we can reorganize λ^q as the 2-diagram

$$\begin{array}{rccc}
L^q: & \int W & \longrightarrow & \mathcal{E}/_{\text{lax}}\, M \\[2ex]
& \begin{matrix}(A,X)\\ \big\downarrow{\scriptstyle (f,\alpha)}\\ (B,X')\end{matrix} & \mapsto & \begin{tikzcd}
F(A) \arrow[rr, "F(f)"] \arrow[dr, "\lambda^q_{(A,X)}"'] & \arrow[d, phantom, "\overset{\lambda^q_{f,\alpha}}{\Longrightarrow}"] & F(B) \arrow[dl, "\lambda^q_{(B,X')}"] \\
& M &
\end{tikzcd} \\[2ex]
& \delta & \mapsto & F(\delta)
\end{array}$$

In Theorem 2.1.21, we will prove that

$$\begin{matrix}\operatorname{colim}^W F\\ \big\downarrow{\scriptstyle q}\\ M\end{matrix} \;=\; \begin{matrix}\text{oplax}^{\text{cart}}\text{-}\operatorname{colim}^{\Delta 1}(F \circ \mathcal{G}(W))\\ \big\downarrow{\scriptstyle q}\\ M\end{matrix} \;=\; \text{oplax}^{\text{cart}}\text{-}\operatorname{colim}^{\Delta 1} L^q$$

in the lax slice $\mathcal{E}/_{\text{lax}}\, M$. Of course, one could prove this directly, but our proof will be shorter and more abstract, in Theorem 2.1.21, based on the *colim-fibrations* point of view (that is, the second approach named above).

Remark 2.1.2. Although weighted 2-colimits cannot be conicalized, we can almost conicalize them reducing them to cartesian-marked oplax conical colimits. The price to pay is to have 2-cells inside the cocones. And this then translates as the need to consider lax slices in order to generalize the 1-dimensional Theorem 2.0.1 to dimension 2.

Such need is further justified by the second approach (see Remark 2.1.20), that we now present. The idea is to capture Theorem 2.0.1 from a more abstract point of view, in a way that resembles the property of being a discrete fibration. We will then proceed to generalize such approach to dimension 2, arriving to Theorem 2.1.21.

The following definition does not seem to appear in the literature.

Definition 2.1.3. A functor $p\colon \mathcal{S} \to \mathcal{C}$ is a *colim-fibration* if for every object $S \in \mathcal{S}$ and every universal cocone μ that exhibits $p(S)$ as the colimit of some diagram $D\colon \mathcal{A} \to \mathcal{C}$ with $\mathcal{A}$ small, there exists a unique pair $(\overline{D}, \overline{\mu})$ with $\overline{D}\colon \mathcal{A} \to \mathcal{S}$ a diagram and $\overline{\mu}$ a universal cocone that exhibits $S = \operatorname{colim} \overline{D}$ such that $p \circ \overline{D} = D$ and $p \circ \overline{\mu} = \mu$.

$$\begin{tikzcd}
\overline{D}(A) \arrow[rrd, bend left, "\overline{\mu}_A"] \arrow[rd, "\overline{D}(f)"'] \arrow[dd, "p"] & & \\
& \overline{D}(B) \arrow[r, "\overline{\mu}_B"'] & S \arrow[dd, "p"] \\
D(A) \arrow[rrd, bend left, "\mu_A"] \arrow[rd, "D(f)"'] & & \\
& D(B) \arrow[r, "\mu_B"'] & p(S)
\end{tikzcd} \tag{2.2}$$

Remark 2.1.4. This is actually stronger than the property written in equation (2.1) of Theorem 2.0.1, but it will be clear after Proposition 2.1.10 that $\operatorname{dom}\colon \mathcal{C}/_{M} \to \mathcal{C}$ is also a colim-fibration. The following propositions shed more light on what it means to be a colim-fibration.

Proposition 2.1.5. *Every colim-fibration is a discrete fibration.*

Proof. Let $p\colon \mathcal{S} \to \mathcal{C}$ be a colim-fibration. We firstly show that only identities can be over identities with respect to p. So suppose $v\colon S' \to S$ is a morphism in $\mathcal{S}$ such that $p(v) = \mathrm{id}_{p(S)}$. We have that $p(S)$ is trivially the colimit of the diagram $D\colon \mathit{2} \to \mathcal{C}$ given by the arrow $\mathrm{id}_{p(S)}$, with universal cocone given by just identities. But then both the arrows v and id_S give a diagram $\overline{D}\colon \mathit{2} \to \mathcal{S}$ with a universal cocone that exhibits $S = \operatorname{colim} \overline{D}$ such that it is over the universal cocone given by the identities of $p(S)$. And we conclude that $v = \mathrm{id}_S$.

Take now $S \in \mathcal{S}$ and $u\colon C \to p(S)$ a morphism in $\mathcal{C}$. We want to show that there is a unique lifting of u to S. Consider then the diagram $D\colon \mathit{2} \to \mathcal{C}$ given by the arrow u in $\mathcal{C}$. Then the colimit of D exists trivially and is $p(S)$, with universal cocone

$$\begin{tikzcd}
C \arrow[d, "u"'] \arrow[dr, "u"] & \\
p(S) \arrow[r, equal] & p(S)
\end{tikzcd}$$

As p is a colim-fibration, there exist a unique diagram $\overline{D}\colon \mathit{2} \to \mathcal{S}$ and a unique universal cocone $\overline{\mu}$ that exhibits $S = \operatorname{colim} \overline{D}$ with $p \circ \overline{D} = D$ and $p \circ \overline{\mu} = \mu$. But then we need to have $\overline{D}(1) = S$ and $\overline{\mu}_1 = \mathrm{id}_S$ by the argument above, whence $\overline{D}$ is the unique lifting of u to S. □

Corollary 2.1.6. *Let $p\colon \mathcal{S} \to \mathcal{C}$ be a functor. The following are equivalent:*

- (i) *p is a colim-fibration;*
- (ii) *for every object $S \in \mathcal{S}$ and every universal cocone μ that exhibits $p(S)$ as the colimit of some diagram $D\colon \mathcal{A} \to \mathcal{C}$ with $\mathcal{A}$ small, there exists a unique pair $(\overline{D}, \overline{\mu})$ with $\overline{D}\colon \mathcal{A} \to \mathcal{S}$ a diagram and a $\overline{\mu}$ a cocone for $\overline{D}$ on S such that $p \circ \overline{D} = D$ and $p \circ \overline{\mu} = \mu$; moreover $\overline{\mu}$ is a universal cocone that exhibits $S = \operatorname{colim} \overline{D}$.*

Remark 2.1.7. Corollary 2.1.6 shows that, for a colim-fibration, the liftings $\overline{\mu}$ of universal cocones μ are unique as mere cocone over μ on the starting $S \in \mathcal{S}$.

We notice that the definition of creating colimits (see for example Adámek, Herrlich and Strecker's [1]) and condition (ii) of Corollary 2.1.6 for being a colim-fibration are actually pretty similar, but somehow dual to each other. Indeed, looking at the diagram in equation (2.2), creation of colimits starts from a diagram $\overline{D}$ and produces a colimit S for it, while being a colim-fibration starts from some S and produces a diagram $\overline{D}$ with colimit S.

To further clarify the connection between these two, we recall the following proposition from Adámek, Herrlich and Strecker's [1, Proposition 13.34].

Proposition 2.1.8. *For a functor F, the following are equivalent:*

- (i) *F preserves and lifts [uniquely] all the colimits;*
- (ii) *F preserves and detects all the colimits, and moreover it is a [discrete] iso-fibration.*

Remark 2.1.9. By Proposition 2.1.5, a colim-fibration is always a discrete fibration and so always a discrete iso-fibration. But we still have to clarify the connection between being a colim-fibration and reflecting colimits.

Proposition 2.1.10. *Let $p\colon \mathcal{S} \to \mathcal{C}$ be a functor. The following are equivalent:*

- (i) *p is a colim-fibration;*
- (ii) *p is a discrete fibration that reflects all the colimits.*

Proof. We prove "$(ii) \Rightarrow (i)$". Take $S \in \mathcal{S}$ and a universal cocone μ that exhibits $p(S)$ as the colimit of some diagram $D\colon \mathcal{A} \to \mathcal{C}$ with $\mathcal{A}$ small. Since p is a discrete fibration, there exists a unique pair $(\overline{D}, \overline{\mu})$ with $\overline{D}\colon \mathcal{A} \to \mathcal{S}$ a diagram and $\overline{\mu}$ a cocone for $\overline{D}$ on S such that $p \circ \overline{D} = D$ and $p \circ \overline{\mu} = \mu$. Indeed, for every $A \in \mathcal{A}$, we can define $\overline{D}(A)$ and $\overline{\mu}_A$ by taking the unique lifting of μ_A to S, and, for every $f\colon A \to B$ in $\mathcal{A}$, define $\overline{D}(f)$ to be the unique lifting of $D(f)$ to $\overline{D}(B)$, whose domain needs to be $\overline{D}(A)$ since any discrete fibration is split. Moreover $\overline{\mu}$ needs to be universal since p reflects all the colimits and $p \circ \overline{\mu} = \mu$ is universal.

We now prove "$(i) \Rightarrow (ii)$". So let p be a colim-fibration. After Proposition 2.1.5, we just need to prove that p reflects all the colimits. Take a diagram $H\colon \mathcal{A} \to \mathcal{S}$ with $\mathcal{A}$ small and a cocone ζ for H on some object $S \in \mathcal{S}$; assume then that $p \circ \zeta$ is a universal cocone, exhibiting $p(S) = \operatorname{colim}(p \circ H)$. By Corollary 2.1.6, we know that there exist a unique pair $(\overline{p \circ H}, \overline{p \circ \zeta})$ with $\overline{p \circ H} \colon \mathcal{A} \to \mathcal{S}$ a diagram and $\overline{p \circ \zeta}$ a cocone for $\overline{p \circ H}$ on S such that $p \circ \overline{p \circ H} = p \circ H$ and $p \circ \overline{p \circ \zeta} = p \circ \zeta$, and that moreover $\overline{p \circ \zeta}$ is a universal cocone that exhibits $S = \operatorname{colim} \overline{p \circ H}$. But then we need to have that $\overline{p \circ H} = H$ and $\overline{p \circ \zeta} = \zeta$, whence we conclude. □

Corollary 2.1.11. *Let $p \colon \mathcal{S} \to \mathcal{C}$ be a colim-fibration that preserves and detects all the colimits. Then p (preserves and) creates all the colimits.*

Proof. Clear combining Proposition 2.1.8 and Proposition 2.1.10, since creating all the colimits is equivalent to lifting uniquely and reflecting all the colimits (see for example Adámek, Herrlich and Strecker's [1]). □

Remark 2.1.12. We can now rewrite Theorem 2.0.1 by saying that $\operatorname{dom} \colon \mathcal{C}/_{M} \to \mathcal{C}$ is a colim-fibration that preserves and detects all the colimits. This is actually stronger than Theorem 2.0.1, but we see that dom does satisfy this as it can be expressed as the category of elements of the representable $\operatorname{y}(M) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$ and is thus a discrete fibration. The explicit formula in equation (2.1) then comes from the explicit liftings of dom.

Construction 2.1.13. We want to produce a 2-categorical generalization of the concept of colim-fibration. As described in Chapter 1, what we think most naturally generalizes the notion of discrete fibration to dimension 2 is that of discrete 2-fibration. Recall its definition from Definition 1.1.7 and Remark 1.3.11. Discrete 2-fibrations are a natural extension of the usual Grothendieck fibrations. They are also required to be, locally, discrete opfibrations. Thus they are able to uniquely lift 2-cells to a fixed domain 1-cell. Notice, though, that it would now be much harder to directly generalize Definition 2.1.3 in a way that implies being a discrete 2-fibration. So we think it is more concise to just ask having a discrete 2-fibration.

To reach a 2-categorical notion of colim-fibration, we then need to use a 2-categorical concept of cocone. While weighted 2-cocylinders would be hard to handle, we notice that a discrete 2-fibration has the ability to lift cartesian-marked oplax cocones. Indeed, let $p \colon \mathcal{S} \to \mathcal{E}$ be a cloven discrete 2-fibration.

Consider then $S \in \mathcal{S}$, a marking $W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}at$ with $\mathcal{A}$ small, a 2-diagram $D\colon \int W \to \mathcal{E}$ and a cartesian-marked oplax cocone

$$\theta\colon \Delta 1 \xRightarrow[\text{oplax}^{\text{cart}}]{} \mathcal{E}\left(D(-), p(S)\right)$$

for D on $p(S)$. Then p lifts (D,θ) to a pair $(\overline{D},\overline{\theta})$ with $\overline{D}\colon \int W \to \mathcal{S}$ a 2-diagram and $\overline{\theta}$ a cartesian-marked oplax cocone for $\overline{D}$ on S such that $p \circ \overline{D} = D$ and $p \circ \overline{\theta} = \theta$.

$$\begin{tikzcd}
\overline{D}(A,X) \arrow[rrd, bend left=20, "\overline{\theta}_{(A,X)}"] \arrow[rd, "\overline{D}(f,\alpha)"'] & {} \arrow[d, phantom, "\Swarrow \overline{\theta}_{f,\alpha}"] & \\
& \overline{D}(B,X') \arrow[r, "\overline{\theta}_{(B,X')}"'] & S \arrow[ddd, mapsto, "p"] \\
{} \arrow[d, mapsto, "p"] & & \\
D(A,X) \arrow[rrd, bend left=20, "\theta_{(A,X)}"] \arrow[rd, "D(f,\alpha)"'] & {} \arrow[d, phantom, "\Swarrow \theta_{f,\alpha}"] & \\
& D(B,X') \arrow[r, "\theta_{(B,X')}"'] & p(S)
\end{tikzcd} \tag{2.3}$$

For every $(A,X) \in \int W$, we define $\overline{D}(A,X)$ and $\overline{\theta}_{(A,X)}$ by taking the chosen cartesian lifting of $\theta_{(A,X)}$ to S. For every $(f,\alpha)\colon (A,X) \to (B,X')$ in $\int W$, we then define $\overline{\theta}_{f,\alpha}\colon \overline{\theta}_{(A,X)} \to \xi$ to be the unique lifting of $\theta_{f,\alpha}$ to $\overline{\theta}_{(A,X)}$. Since $\overline{\theta}_{(B,X')}$ is cartesian, ξ factors through $\overline{\theta}_{(B,X')}$; we define $\overline{D}(f,\alpha)$ to be the unique factoring morphism $\overline{D}(A,X) \to \overline{D}(B,X')$, so that $\overline{\theta}_{f,\alpha}\colon \overline{\theta}_{(A,X)} \to \overline{\theta}_{(B,X')} \circ \overline{D}(f,\alpha)$. It remains to define $\overline{D}$ on 2-cells. Given $\delta\colon (f,\alpha) \Rightarrow (g,\beta)\colon (A,X) \to (B,X')$ in $\int W$, we define $\overline{D}(\delta)$ to be the unique lifting of $D(\delta)$ to $\overline{D}(f,\alpha)$. The codomain of $\overline{D}(\delta)$ is $\overline{D}(g,\beta)$ because of the uniqueness of the lifting of

$$\begin{tikzcd}
D(A,X) \arrow[rrd, bend left=20, "\theta_{(A,X)}"] \arrow[rd, "D(f,\alpha)"{name=F}] \arrow[rd, bend right=50, "D(g,\beta)"'{name=G}] & {} \arrow[d, phantom, "\Swarrow \theta_{f,\alpha}"] & \\
& D(B,X') \arrow[r, "\theta_{(B,X')}"'] & p(S),
\arrow[from=F, to=G, Rightarrow, "D(\delta)"']
\end{tikzcd}$$

which coincides with $\theta_{g,\beta}$, to $\overline{\theta}_{(A,X)}$, and the cartesianity of $\overline{\theta}_{(B,X')}$. This argument also proves the 2-dimensional property of the oplax naturality of $\overline{\theta}$. It is straightforward to check that $\overline{D}$ is a 2-functor and that $\overline{\theta}$ is cartesian-marked oplax natural, using the cartesianity of the $\overline{\theta}_{(A,X)}$'s and the uniqueness of the liftings of a 2-cell to a fixed domain 1-cell (with arguments similar to the above one).

Clearly, the $\overline{\theta}_{(A,X)}$'s are not unique above the $\theta_{(A,X)}$, but cartesian. Having fixed them, however, the rest of the cartesian-marked oplax cocone $\overline{\theta}$ is uniquely defined. It is also true that, given another pair $(\widetilde{D},\widetilde{\theta})$ that lifts (D,θ), the unique vertical morphisms $\widetilde{D}(A,X)\to\overline{D}(A,X)$ that produce the factorization of the $\widetilde{\theta}_{(A,X)}$'s through the cartesian $\overline{\theta}_{(A,X)}$'s form a unique vertical 2-natural transformation $j\colon\widetilde{D}\to\overline{D}$ such that $\widetilde{\theta}=(-\circ j)\circ\overline{\theta}$. This can be checked using the uniqueness of the liftings of a 2-cell to a fixed domain 1-cell and the cartesianity of the $\overline{\theta}_{(A,X)}$'s.

The following definition is original.

Definition 2.1.14. A 2-functor $p\colon\mathcal{S}\to\mathcal{E}$ is a 2-*colim-fibration* if it is a cloven discrete 2-fibration such that, for every $S\in\mathcal{S}$, 2-functor $W\colon\mathcal{A}^{\mathrm{op}}\to\mathcal{C}at$ with $\mathcal{A}$ small (the marking), 2-functor $D\colon\int W\to\mathcal{E}$ (the diagram) and universal cartesian-marked oplax cocone

$$\theta\colon\Delta 1\xRightarrow[\text{oplax}^{\text{cart}}]{}\mathcal{E}\left(D(-),p(S)\right)$$

that exhibits $p(S)=\text{oplax}^{\text{cart}}\text{-colim}^{\Delta 1}D$, the pair $(\overline{D},\overline{\theta})$ obtained by lifting (D,θ) through p to S as in Construction 2.1.13 exhibits

$$S=\text{oplax}^{\text{cart}}\text{-colim}^{\Delta 1}\overline{D}.$$

Remark 2.1.15. Remember that every weighted 2-colimit can be reduced to a cartesian-marked oplax conical one, so the property of being a 2-colim-fibration can as well be applied to any universal weighted 2-cocylinder

$$\mu\colon W\Rightarrow\mathcal{E}\left(F(-),p(S)\right)$$

for some 2-diagram $F\colon\mathcal{A}\to\mathcal{E}$, after reducing it to a universal cartesian-marked oplax cocone.

Remark 2.1.16. We would now like to generalize Proposition 2.1.10 to dimension 2. We see, however, that a 2-colim-fibration does not necessarily reflect all the (cartesian-marked oplax conical) 2-colimits, because the lifting $(\overline{D},\overline{\theta})$ of

Construction 2.1.13 is not unique anymore. Indeed, if we start from a cartesian-marked oplax cocone above (at the level of $\mathcal{S}$), project it down and lift, we do not find in general the starting cartesian-marked oplax cocone. We almost find the same, however, if we start from a cartesian-marked oplax cocone that we call *cartesian*, and now define. This will bring us to Proposition 2.1.19.

Definition 2.1.17. Let $p\colon \mathcal{S} \to \mathcal{E}$ be a discrete 2-fibration. Consider then $S \in \mathcal{S}$, a marking $W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}at$ with $\mathcal{A}$ small and a 2-diagram $H\colon \int W \to \mathcal{S}$. A cartesian-marked oplax cocone

$$\zeta\colon \Delta 1 \underset{\mathrm{oplax}^{\mathrm{cart}}}{\Longrightarrow} \mathcal{S}\left(H(-), S\right)$$

is *cartesian* if for every $(A, X) \in \int W$ the component $\zeta_{(A,X)}$ (seen as a morphism in $\mathcal{S}$) is cartesian with respect to p.

We say that p *reflects all the cartesian* (cartesian-marked oplax conical) 2-*colimits* if it reflects the universality of cartesian cartesian-marked oplax cocones.

Example 2.1.18. Let $\mathcal{E}$ be a 2-category and $M \in \mathcal{E}$. The cartesian morphisms in $\mathcal{E}/_{\mathrm{lax}}\ M$ with respect to $\mathrm{dom}\colon \mathcal{E}/_{\mathrm{lax}}\ M \to \mathcal{E}$ are precisely the triangles

$$\begin{tikzcd}
E \arrow[rr, "\widehat{\gamma}"] \arrow[dr, "g"'] & \arrow[d, phantom, "\underset{\gamma}{\overset{\cong}{=}}"] & E' \arrow[dl, "g'"] \\
& M &
\end{tikzcd}$$

with the 2-cell γ an isomorphism. So the cartesian cartesian-marked oplax cocones in $\mathcal{E}/_{\mathrm{lax}}\ M$ are the ones with components triangles filled with isomorphisms.

Proposition 2.1.19. *Let $p\colon \mathcal{S} \to \mathcal{E}$ be a cloven discrete 2-fibration. The following are equivalent:*

(*i*) *p is a 2-colim-fibration;*

(*ii*) *p reflects all the cartesian 2-colimits.*

Proof. We prove "(*ii*) ⇒ (*i*)". In the notation of Definition 2.1.14, the cartesian-marked oplax cocone $\overline{\theta}$ is cartesian by Construction 2.1.13. Since p reflects all the cartesian 2-colimits and $p\circ\overline{\theta}=\theta$ is universal, then $\overline{\theta}$ needs to be universal as well, exhibiting $S=\text{oplax}^{\text{cart}}\text{-}\operatorname{colim}^{\Delta 1}\overline{D}$.

We now prove "$(i)\Rightarrow(ii)$". So consider $S\in\mathcal{S}$, a marking $W\colon\mathcal{A}^{\text{op}}\to\mathcal{C}at$ with $\mathcal{A}$ small, a 2-diagram $H\colon\int W\to\mathcal{S}$ and a cartesian cartesian-marked oplax cocone

$$\zeta\colon\Delta 1\xRightarrow[\text{oplax}^{\text{cart}}]{}\mathcal{S}\left(H(-),S\right).$$

Assume that $p\circ\zeta$ is universal, exhibiting $p(S)=\text{oplax}^{\text{cart}}\text{-}\operatorname{colim}^{\Delta 1}(p\circ H)$. We prove that ζ is universal as well. Consider the lifting $(\overline{p\circ H},\overline{p\circ\zeta})$ of $(p\circ H,p\circ\zeta)$ through p to S, as in Construction 2.1.13. It is straightforward to check that, since cartesian liftings are unique up to a unique vertical isomorphism, there exists a 2-natural isomorphism $j\colon H\cong\overline{p\circ H}$ such that $\zeta=(-\circ j)\circ\overline{p\circ\zeta}$ (see the last part of Construction 2.1.13). Since $\overline{p\circ\zeta}$ is universal, as p is a 2-colim-fibration, and $(-\circ j)$ is a 2-natural isomorphism, we conclude that ζ is universal. ☐

Remark 2.1.20. We have seen in Remark 2.1.12 that we can rephrase the 1-dimensional Theorem 2.0.1 by saying that $\operatorname{dom}\colon\mathcal{C}/_{M}\to\mathcal{C}$ is a colim-fibration that preserves and detects all the colimits. And this latter is actually stronger than Theorem 2.0.1, but true since $\operatorname{dom}\colon\mathcal{C}/_{M}\to\mathcal{C}$ can be obtained as the category of elements of the representable $\mathrm{y}(M)\colon\mathcal{C}^{\text{op}}\to\mathcal{S}et$.

As described in Chapter 1, we believe the most natural categorification of the construction of the category of elements is given by the 2-category of elements. So, to obtain a generalization of Theorem 2.0.1 (or better, the stronger colim-fibration result) to dimension 2, we consider the 2-category of elements of a representable $\mathrm{y}(M)\colon\mathcal{E}^{\text{op}}\to\mathcal{C}at$ (given $\mathcal{E}$ a 2-category and $M\in\mathcal{E}$). This gives the domain functor from the lax slice $\mathcal{E}/_{\text{lax}}\,M$, further justifying Construction 2.1.1:

$$\begin{tikzcd}
\mathcal{E}/_{\text{lax}}\,M \arrow[r] \arrow[d, "\operatorname{dom}"'] & \mathit{1} \arrow[d, "\mathit{1}"] \\
\mathcal{E} \arrow[r, "\mathrm{y}(M)"'] \arrow[ur, Rightarrow, shorten=6mm, "\text{lax comma}"] & \mathcal{C}at^{\text{op}}
\end{tikzcd}$$

We now prove that $\operatorname{dom}\colon\mathcal{E}/_{\text{lax}}\,M\to\mathcal{E}$ is a 2-colim-fibration (Theorem 2.1.21). In particular, considering how liftings along dom are calculated, this will also imply the conclusion of Construction 2.1.1 (first approach) from this abstract point of view. We will then address lifting (that is stronger than detection) of 2-colimits in Proposition 2.2.6 and preservation of 2-colimits in Section 2.3, establishing a full 2-categorical generalization of Theorem 2.0.1.

Theorem 2.1.21. *Let $\mathcal{E}$ be a 2-category and $M \in \mathcal{E}$. Then the 2-functor* $\mathrm{dom}\colon \mathcal{E}/_{\mathrm{lax}}\, M \to \mathcal{E}$ *is a 2-colim-fibration. As a consequence, in the notation of Construction 2.1.1,*

$$\begin{matrix} \operatorname{colim}^{W} F \\ \downarrow^{q} \\ M \end{matrix} = \begin{matrix} \mathrm{oplax}^{\mathrm{cart}}\text{-}\operatorname{colim}^{\Delta 1}(F \circ \mathcal{G}(W)) \\ \downarrow^{q} \\ M \end{matrix} = \mathrm{oplax}^{\mathrm{cart}}\text{-}\operatorname{colim}^{\Delta 1} L^{q}$$

in the lax slice $\mathcal{E}/_{\mathrm{lax}}\, M$. Here, L^{q} is the 2-diagram in $\mathcal{E}/_{\mathrm{lax}}\, M$ that corresponds to the cartesian-marked oplax cocone λ^{q} on M associated to the weighted 2-cocylinder on M that q represents.

Proof. By Remark 2.1.20, we know that $\mathrm{dom}\colon \mathcal{E}/_{\mathrm{lax}}\, M \to \mathcal{E}$ can be obtained as the 2-category of elements of $\mathrm{y}(M) : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}at$. So dom is a discrete 2-fibration with a canonical cleavage (the chosen cartesian lifting of a morphism f is (f, id)).

We prove that the second part of the statement is a consequence of the first one. So assume we have already proved that dom is a 2-colim-fibration. Calling θ the universal cartesian-marked oplax cocone that exhibits $C = \mathrm{oplax}^{\mathrm{cart}}\text{-}\operatorname{colim}^{\Delta 1}(F \circ \mathcal{G}(W))$, we then obtain that the lifting of $(F \circ \mathcal{G}(W), \theta)$ through dom to q (calculated as in Construction 2.1.13)

$$\begin{tikzcd}
\overline{F \circ \mathcal{G}(W)}(A,X) \arrow[dr, "\overline{F\circ\mathcal{G}(W)}(f,\alpha)"'] \arrow[drr, bend left, "\overline{\theta}_{(A,X)}"] \arrow[drr, phantom, "\Downarrow \overline{\theta}_{f,\alpha}"] & & \\
& \overline{F \circ \mathcal{G}(W)}(B,X') \arrow[r, "\overline{\theta}_{(B,X')}"'] \arrow[dd, mapsto, "\mathrm{dom}"] & q \arrow[dd, mapsto, "\mathrm{dom}"] \\
F(A) \arrow[dr, "F(f)"'] \arrow[drr, bend left, "\theta_{(A,X)}"] \arrow[drr, phantom, "\Downarrow \theta_{f,\alpha}"] & & \\
& F(B) \arrow[r, "\theta_{(B,X')}"'] & C
\end{tikzcd}$$

exhibits

$$q = \mathrm{oplax}^{\mathrm{cart}}\text{-}\operatorname{colim}^{\Delta 1}\overline{F \circ \mathcal{G}(W)}$$

in $\mathcal{E}/_{\text{lax}}\, M$. And we can calculate $\overline{F \circ \mathcal{G}(W)}$ and $\overline{\theta}$ explicitly, looking at the action of $\mathrm{y}(M) : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}at$ on 1-cells and 2-cells, since $\mathrm{dom} = \mathcal{G}(\mathrm{y}(M))$. Given $(A, X) \in \int W$,

$$\overline{F \circ \mathcal{G}(W)}(A, X) = \mathrm{y}(M)(\theta_{(A,X)})(q) = q \circ \theta_{(A,X)} = \lambda^q_{(A,X)} = L^q(A, X)$$

$$\begin{tikzcd}
\overline{\theta}_{(A,X)} = & F(A) \arrow[rr, "\theta_{(A,X)}"] \arrow[rd, "\lambda^q_{(A,X)}"'] & \arrow[d, phantom, "\overset{\mathrm{id}}{=}"] & C \arrow[ld, "q"] \\
& & M &
\end{tikzcd}$$

Given $(f, \alpha) \colon (A, X) \to (B, X')$ in $\int W$,

$$\overline{\theta}_{f,\alpha} = \theta_{f,\alpha} \colon (\theta_{(A,X)}, \mathrm{id}) \to (\theta_{B,X'} \circ F(f), \mathrm{y}(M)(\theta_{f,\alpha})_q)$$

whence, since $\mathrm{y}(M)(\theta_{f,\alpha})_q = q * \theta_{f,\alpha} = \lambda^q_{f,\alpha}$,

$$\begin{tikzcd}
\overline{F \circ \mathcal{G}(W)}(f, \alpha) = & F(A) \arrow[rr, "F(f)"] \arrow[rd, "\lambda^q_{(A,X)}"'] & \arrow[d, phantom, "\overset{\lambda^q_{f,\alpha}}{\Longrightarrow}"] & F(B) \arrow[ld, "\lambda^q_{(B,X')}"] & = L^q(f, \alpha). \\
& & M & &
\end{tikzcd}$$

Given $\delta \colon (f, \alpha) \to (g, \beta) \colon (A, X) \to (B, X')$ in $\int W$,

$$\overline{F \circ \mathcal{G}(W)}(\delta) = F(\delta) = L^q(\delta).$$

We now prove that $\mathrm{dom} \colon \mathcal{E}/_{\text{lax}}\, M \to \mathcal{E}$ is a 2-colim-fibration. By Proposition 2.1.19, it suffices to prove that dom reflects all the cartesian 2-colimits. So take $t \colon K \to M$, a marking $W \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}at$ with $\mathcal{A}$ small and a 2-diagram $H \colon \int W \to \mathcal{E}/_{\text{lax}}\, M$. Consider then a cartesian cartesian-marked oplax cocone

$$\zeta \colon \Delta 1 \xRightarrow[\text{oplax}^{\text{cart}}]{} \mathcal{E}/_{\text{lax}}\, M\,(H(-), t)$$

such that $\mathrm{dom} \circ \zeta$ is universal. We prove that ζ is universal as well.

Given $g \colon E \to M$ and

$$\sigma \colon \Delta 1 \xRightarrow[\text{oplax}^{\text{cart}}]{} \mathcal{E}/_{\text{lax}}\, M\,(H(-), g)\,,$$

we need to produce a morphism

$$\begin{tikzcd}
K \arrow[rr, "\widehat{\gamma}"] \arrow[dr, "t"'] \arrow[rr, phantom, shift right=4ex, "\overset{\gamma}{\Longrightarrow}"] & & E \arrow[dl, "g"] \\
& M &
\end{tikzcd}$$

in $\mathcal{E}/_{\text{lax}}\, M$ such that $\sigma = (\gamma \circ -) \circ \zeta$. Then we need

$$\operatorname{dom} \circ \sigma = \operatorname{dom} \circ (\gamma \circ -) \circ \zeta = (\widehat{\gamma} \circ -) \circ \operatorname{dom} \circ \zeta,$$

whence $\widehat{\gamma}$ needs to be the unique morphism $K \to E$ in $\mathcal{E}$ induced by $\operatorname{dom} \circ \sigma$ via universality of $\operatorname{dom} \circ \zeta$. We will now produce the inner 2-cell γ in $\mathcal{E}$ via the 2-dimensional universality of $\operatorname{dom} \circ \zeta$. Indeed γ corresponds to a modification

$$\Xi \colon (t \circ -) \circ \operatorname{dom} \circ \zeta \Rrightarrow (g \circ \widehat{\gamma} \circ -) \circ \operatorname{dom} \circ \zeta.$$

Notice that the target of Ξ coincides with $(g \circ -) \circ \operatorname{dom} \circ \sigma$. Given $(A, X) \in \int W$, we will have that $\Xi_{(A,X)} = \gamma * \widehat{\zeta_{(A,X)}}$, and we want to obtain

$$\begin{tikzcd}
\operatorname{dom}(H(A,X)) \arrow[r, "\widehat{\zeta_{(A,X)}}"] \arrow[dr, "H(A,X)"'] \arrow[dr, phantom, shift left=4ex, "\underset{\zeta_{(A,X)}}{\cong}"] & K \arrow[r, "\widehat{\gamma}"] \arrow[d, "t"] \arrow[r, phantom, shift right=4ex, "\overset{\gamma}{\Longrightarrow}"] & E \arrow[dl, "g"] \\
& M &
\end{tikzcd}
=
\begin{tikzcd}
\operatorname{dom}(H(A,X)) \arrow[rr, "\widehat{\sigma_{(A,X)}}"] \arrow[dr, "H(A,X)"'] \arrow[rr, phantom, shift right=4ex, "\overset{\sigma_{(A,X)}}{\Longrightarrow}"] & & E \arrow[dl, "g"] \\
& M &
\end{tikzcd}$$

The component $\zeta_{(A,X)}$ is indeed a triangle filled with an isomorphism, by Example 2.1.18, since ζ is cartesian. Whence we need to take

$$\Xi_{(A,X)} \coloneqq
\begin{tikzcd}
& \operatorname{dom}(H(A,X)) \arrow[r, "\widehat{\sigma_{(A,X)}}"] \arrow[dl, bend right, "\widehat{\zeta_{(A,X)}}"'] \arrow[d, "H(A,X)"] \arrow[d, phantom, shift right=8ex, "\underset{\zeta^{-1}_{(A,X)}}{\cong}"] \arrow[d, phantom, shift left=8ex, "\overset{\sigma_{(A,X)}}{\Longrightarrow}"] & E \arrow[dl, bend left, "g"] \\
K \arrow[r, "t"'] & M &
\end{tikzcd}$$

It is straightforward to check that Ξ is a modification between cartesian-marked oplax cocones. So Ξ induces a unique 2-cell $\gamma \colon t \Rightarrow g \circ \widehat{\gamma}$ in $\mathcal{E}$ such that $(\gamma * -) * (\operatorname{dom} \circ \zeta) = \Xi$. We check that $\sigma = (\gamma \circ -) \circ \zeta$ (as cartesian-marked oplax natural transformations). It surely holds on object components by construction of Ξ. Given a morphism (f, α) in $\left(\int W\right)^{\mathrm{op}}$, it suffices to check that

$$\sigma_{f,\alpha} = \widehat{\gamma} * \zeta_{f,\alpha}$$

But this holds by construction of $\widehat{\gamma}$.

We now show the uniqueness of γ. So assume there is some $\gamma' \colon t \to g$ in $\mathcal{E}/_{\text{lax}}\, M$ such that $\sigma = (\gamma' \circ -) \circ \zeta$. Then $\widehat{\gamma'} = \widehat{\gamma}$ by the argument above. Since the two 2-cells γ and γ' in $\mathcal{E}$ are then between the same 1-cells, it suffices to prove that

$$(\gamma * -) * (\operatorname{dom} \circ \zeta) = (\gamma' * -) * (\operatorname{dom} \circ \zeta).$$

But this can be checked on components, and $\gamma * \widehat{\zeta_{(A,X)}} = \gamma' * \widehat{\zeta_{(A,X)}}$ holds because

$$(\gamma \circ -) \circ \zeta = \sigma = (\gamma' \circ -) \circ \zeta$$

and ζ is cartesian (essentially, both give the same Ξ).

It remains to prove the 2-dimensional universality of ζ. Given $g \colon E \to M$, two morphisms $\gamma, \gamma' \colon t \to g$ in $\mathcal{E}/_{\text{lax}}\, M$ and a modification

$$\Sigma \colon (\gamma \circ -) \circ \zeta \Rrightarrow (\gamma' \circ -) \circ \zeta \colon \Delta 1 \xRightarrow[\text{oplax}^{\text{cart}}]{} \mathcal{E}/_{\text{lax}}\, M\,(H(-), g)\,,$$

we need to produce a 2-cell $\Gamma \colon \gamma \to \gamma'$ in $\mathcal{E}/_{\text{lax}}\, M$ such that $\Sigma = (\Gamma * -) * \zeta$. But the latter equality is satisfied precisely when it is satisfied after composing with dom. So consider $\operatorname{dom} * \Sigma$; as $\operatorname{dom} \circ \zeta$ is universal, we find a unique $\widehat{\Gamma} \colon \widehat{\gamma} \Rightarrow \widehat{\gamma'}$ such that

$$\operatorname{dom} * \Sigma = (\widehat{\Gamma} * -) * (\operatorname{dom} \circ \zeta).$$

And then $\widehat{\Gamma}$ gives a 2-cell $\Gamma \colon \gamma \to \gamma'$ in $\mathcal{E}/_{\text{lax}}\, M$. Indeed, we need to prove that $g * \widehat{\Gamma} \circ \gamma = \gamma'$ as 2-cells in $\mathcal{E}$, and it suffices to show that

$$((g * \widehat{\Gamma} \circ \gamma) * -) * (\operatorname{dom} \circ \zeta) = (\gamma' * -) * (\operatorname{dom} \circ \zeta),$$

by 2-universality of $\operatorname{dom} \circ \zeta$. This can be checked on components, where it is true because it holds after pasting with the components of ζ (that are isomorphisms, because ζ is cartesian), since $\Sigma \colon (\gamma \circ -) \circ \zeta \Rightarrow (\gamma' \circ -) \circ \zeta$. By construction of Γ, we immediately obtain that $\Sigma = (\Gamma * -) * \zeta$, and our argument has proved the uniqueness of Γ as well. □

Remark 2.1.22. The proof of Theorem 2.1.21 equally works for sigma colimits (of Descotte, Dubuc and Szyld [15], w.r.t. the cartesian marking) in the place of cartesian-marked oplax colimits. Slightly modified, it works as well for sigma bicolimits. Exactly as every 2-colimit can be reduced to a cartesian-marked oplax conical one, every (weighted) pseudo colimit can be reduced to a sigma colimit, and every bicolimit can be reduced to a sigma bicolimit (see [15], where the result is derived from Street's [42]).

We thus have the following corollary.

Corollary 2.1.23.

$$\begin{tikzcd} \operatorname{pseudo-colim}^{W} F \arrow[d, "q"] \\ M \end{tikzcd} = \begin{tikzcd} \operatorname{sigma-colim}^{\Delta 1}(F \circ \mathcal{G}(W)) \arrow[d, "q"] \\ M \end{tikzcd} = \operatorname{sigma-colim}^{\Delta 1} L^q$$

$$\begin{tikzcd} \operatorname{bicolim}^{W} F \arrow[d, "q"] \\ M \end{tikzcd} = \begin{tikzcd} \operatorname{sigma-bicolim}^{\Delta 1}(F \circ \mathcal{G}(W)) \arrow[d, "q"] \\ M \end{tikzcd} = \operatorname{sigma-bicolim}^{\Delta 1} L^q$$

in the lax slice $\mathcal{E}/_{\text{lax}}\, M$*, where* L^q *is the* 2*-diagram in* $\mathcal{E}/_{\text{lax}}\, M$ *that corresponds to the sigma cocone* λ^q *on* M *associated to the weighted pseudo cocylinder on* M *that* q *represents.*

Proof. Construction 2.1.13 equally works to lift any oplax cocone θ to an oplax cocone $\overline{\theta}$. If θ is sigma natural, then $\overline{\theta}$ is sigma natural as well, since a discrete 2-fibration lifts isomorphic 2-cells to isomorphic 2-cells. The same argument of the proof of Theorem 2.1.21 then shows that dom also reflects the universality of cartesian sigma cocones.

Slightly modified, the proof of Theorem 2.1.21 works as well for sigma bi-colimits. Indeed, via bi-universality of $\operatorname{dom}\circ\zeta$, we have that $\operatorname{dom}\circ\sigma$ induces $\widehat{\gamma}$ and an isomorphic modification $\kappa\colon (\widehat{\gamma}\circ -)\circ\operatorname{dom}\circ\zeta \cong \operatorname{dom}\circ\sigma$. We then induce the 2-cell γ from a slightly modified version of Ξ obtained by pasting the assignment of $\Xi_{(A,X)}$ in the proof of Theorem 2.1.21 with $\kappa^{-1}_{(A,X)}$. So that $\kappa_{(A,X)}$ is by construction a 2-cell in the lax slice, and hence becomes a modification between sigma cocones (as the condition of modification holds after applying dom). □

2.2. $\mathcal{F}$-categories and a lifting result for the domain 2-functor

In this section, we address the lifting of colimits of $\operatorname{dom}\colon \mathcal{E}/_{\mathrm{lax}}\, M \to \mathcal{E}$. In dimension 1, the fact that $\operatorname{dom}\colon \mathcal{C}/_{M} \to \mathcal{C}$ lifts all the colimits is based on the correspondence between the diagrams in $\mathcal{C}/_{M}$ and the cocones in $\mathcal{C}$ on M. More precisely, while the fact that dom is a colim-fibration is based on the ability to produce a diagram in the slice over M from a cocone on M, the lifting of colimits is based on the converse.

We have already seen in Construction 2.1.1 (first approach) that, in dimension 2, we can reorganize a cartesian-marked oplax cocone on M as a 2-diagram in the lax slice $\mathcal{E}/_{\mathrm{lax}}\, M$. The second approach allows us to capture this reorganization process from a more abstract point of view (see the proofs of Proposition 2.2.1 and Theorem 2.1.21). However, not every 2-diagram in $\mathcal{E}/_{\mathrm{lax}}\, M$ can produce a cartesian-marked oplax cocone on M, as the cartesian-marked condition may fail.

In this section, we generalize to dimension 2 the bijective correspondence between cocones on M and diagrams in the slice over M (Proposition 2.2.1). We then justify this result via $\mathcal{F}$-category theory (introduced by Lack and Shulman in [30], and recalled in Definition 2.2.3 and Remark 2.2.4). Finally, we show a result of lifting of 2-colimits for $\operatorname{dom}\colon \mathcal{E}/_{\mathrm{lax}}\, M \to \mathcal{E}$ in Proposition 2.2.6.

Proposition 2.2.1. *Let $\mathcal{E}$ be a 2-category and $M \in \mathcal{E}$. Consider then a marking $W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}at$ with $\mathcal{A}$ small and a 2-diagram $D\colon \int W \to \mathcal{E}$. There is a bijection between cartesian-marked oplax cocones*

$$\lambda\colon \Delta 1 \underset{\mathrm{oplax}^{\mathrm{cart}}}{\Longrightarrow} \mathcal{E}(D(-), M)$$

on M and 2-diagrams $\overline{D}\colon \int W \to \mathcal{E}/_{\mathrm{lax}}\, M$ such that for every morphism (f, id) in $\int W$ the triangle $\overline{D}(f, \mathrm{id})$ is filled with an identity.

Proof. Given λ, since dom is a discrete 2-fibration, we can lift (D, λ) to id_M and obtain a pair $(\overline{D}, \overline{\lambda})$ as in Construction 2.1.13. Exactly as in the first part of the proof of Theorem 2.1.21, we can calculate $\overline{D}$ explicitly, looking at the action of $\mathrm{y}(M) : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}at$ on 1-cells and 2-cells, since $\mathrm{dom} = \mathcal{G}(\mathrm{y}(M))$. We obtain the formulas

$$\overline{D}(f,\alpha) = (D(f,\alpha), \lambda_{f,\alpha})\colon \lambda_{(A,X)} \to \lambda_{(B,X')}$$

Starting instead from a 2-diagram $\overline{D}$, we can reorganize its data as a cartesian-marked oplax cocone λ for $\mathrm{dom} \circ \overline{D}$ on M, with formulas

$$\lambda_{(A,X)} \coloneqq \overline{D}(A,X)\colon \ \mathrm{dom}(\overline{D}(A,X)) \to M.$$

$$\lambda_{f,\alpha} \coloneqq \overline{D}(f,\alpha).$$

It is straightforward to check that the 2-functoriality of $\overline{D}$ guarantees that λ is oplax natural. Given (f, id) in $\int W$, we obtain $\lambda_{f,\mathrm{id}} = \overline{D}(f,\mathrm{id}) = \mathrm{id}$, and then λ is also cartesian-marked oplax.

It is clear that the two constructions we have produced are inverses of each other. □

Remark 2.2.2. We believe Proposition 2.2.1 is best captured by $\mathcal{F}$-category theory, also called enhanced 2-category theory, for which we take as main reference Lack and Shulman's [30]. We give a quick recall of $\mathcal{F}$-category theory in Definition 2.2.3 and Remark 2.2.4. We will thus rephrase Proposition 2.2.1 in Remark 2.2.5. $\mathcal{F}$-category theory will then be even more useful for us to prove the preservation of (appropriate) 2-colimits for $\mathrm{dom}\colon \mathcal{E}/_{\mathrm{lax}}\, M \to \mathcal{E}$ in Section 2.3. We will indeed show that dom preserves a large class of 2-colimits (Theorem 2.3.12), despite not every 2-colimit, and that this makes sense from an $\mathcal{F}$-categorical point of view.

Definition 2.2.3. $\mathcal{F}$ is the cartesian closed full subcategory of $\mathcal{C}at^2$ (the category of arrows in $\mathcal{C}at$) determined by the functors which are injective on objects and fully faithful (i.e. full embeddings). It is possible to enrich over $\mathcal{F}$, obtaining $\mathcal{F}$-category theory. An $\mathcal{F}$-category $\mathcal{S}$ is then given by a collection of objects, a hom-category $\mathcal{S}(X,Y)_\tau$ of tight morphisms and a second hom-category $\mathcal{S}(X,Y)_\lambda$ of loose morphisms that give 2-category structures (respectively) $\mathcal{S}_\tau$ and $\mathcal{S}_\lambda$ to $\mathcal{S}$, together with an identity on objects, faithful and locally fully faithful 2-functor $J_{\mathcal{S}}\colon \mathcal{S}_\tau \to \mathcal{S}_\lambda$. An $\mathcal{F}$-functor $F\colon \mathcal{S} \to \mathcal{T}$ is a 2-functor $F_\lambda\colon \mathcal{S}_\lambda \to \mathcal{T}_\lambda$ that restricts to a 2-functor $F_\tau\colon \mathcal{S}_\tau \to \mathcal{T}_\tau$ (forming a commutative square); this is equivalent to F_λ preserving tightness. And an $\mathcal{F}$-natural transformation is a 2-natural transformation α_λ between loose parts that restricts to one between the tight parts; this is equivalent to α_λ having tight components. It is then true that the category $\mathcal{F}$ is enriched over itself, with tight morphisms the morphisms of $\mathcal{F}$, loose morphisms the functors between loose parts and 2-cells the 2-natural transformations between the latter. And for every $\mathcal{F}$-category $\mathcal{S}$ and $S \in \mathcal{S}$ we can build a copresheaf $\mathcal{S}(S,-) : \mathcal{S} \to \mathcal{F}$, that sends S' to the full embedding $\mathcal{S}_\tau(S,S') \to \mathcal{S}_\lambda(S,S')$.

Given $\mathcal{F}$-categories $\mathcal{S}$ and $\mathcal{T}$, there is an $\mathcal{F}$-category $[\mathcal{S},\mathcal{T}]^{\mathcal{F}}$ of $\mathcal{F}$-functors from $\mathcal{S}$ to $\mathcal{T}$, where the tight morphisms are the $\mathcal{F}$-natural transformations, the loose morphisms are the 2-natural transformations between the loose parts and the 2-cells are the modifications between the loose morphisms. But we will need also an oplax version $[\mathcal{S},\mathcal{T}]^{\mathcal{F}}_{\text{oplax}}$ of it, which is the $\mathcal{F}$-category defined as follows:

an object is an $\mathcal{F}$-functor $G\colon \mathcal{S} \to \mathcal{T}$;

a loose morphism $G \underset{\text{loose}}{\Longrightarrow} H$ is an oplax natural transformation α_λ between the loose parts such that the structure 2-cells on tight morphisms are identities, that precisely means that $\alpha_\lambda * J_{\mathcal{S}}$ is (strictly) 2-natural; we call them *loose strict/oplax*;

a tight morphism $G \Rightarrow H$ is a loose one that restricts to a 2-natural transformation between the tight parts, which is equivalent to a loose morphism with tight components; they are usually called *strict/oplax*;

a 2*-cell* is a modification between the loose morphisms.

Remark 2.2.4. We can apply the definitions above to the case $\mathcal{T} = \mathcal{F}$, obtaining two $\mathcal{F}$-categories of copresheaves on $\mathcal{S}$. The strict one, $[\mathcal{S},\mathcal{F}]^{\mathcal{F}}$, can be characterized as follows:

an object is an $\mathcal{F}$-functor $G\colon \mathcal{S} \to \mathcal{F}$, that we can identify with a pair of 2-functors $G_\tau\colon \mathcal{S}_\tau \to \mathcal{C}at$ and $G_\lambda\colon \mathcal{S}_\lambda \to \mathcal{C}at$ together with a 2-natural transformation

$$\begin{tikzcd}
\mathcal{S}_\tau \arrow[rr, "J_{\mathcal{S}}"] \arrow[dr, "G_\tau"'] & \overset{j_G}{\Longrightarrow} & \mathcal{S}_\lambda \arrow[dl, "G_\lambda"] \\
& \mathcal{C}at &
\end{tikzcd}$$

whose components are all full embeddings;

a loose morphism $G \underset{\text{loose}}{\Longrightarrow} H$ is a 2-natural transformation $\alpha_\lambda\colon G_\lambda \Rightarrow H_\lambda\colon \mathcal{S}_\lambda \to \mathcal{C}at$;

a tight morphism $G \Rrightarrow H$ is a loose one with tight components, that precisely means that it induces a 2-natural transformation $\alpha_\tau\colon G_\tau \Rightarrow H_\tau$ such that

$$\begin{tikzcd}
\mathcal{S}_\tau \arrow[r, "J_{\mathcal{S}}"] \arrow[dr, bend right=40, "G_\tau"'] \arrow[dr, phantom, "\overset{j_G}{\Rightarrow}"] & \mathcal{S}_\lambda \arrow[d, bend right=50, "G_\lambda"'] \arrow[d, bend left=50, "H_\lambda"] \arrow[d, phantom, "\overset{\alpha_\lambda}{\Rightarrow}"] & = & \mathcal{S}_\tau \arrow[r, "J_{\mathcal{S}}"] \arrow[d, bend right=50, "G_\tau"'] \arrow[d, bend left=50, "H_\tau"] \arrow[d, phantom, "\overset{\alpha_\tau}{\Rightarrow}"] & \mathcal{S}_\lambda \arrow[dl, bend left=40, "H_\lambda"] \arrow[dl, phantom, "\overset{j_H}{\Rightarrow}"] \\
& \mathcal{C}at & & \mathcal{C}at &
\end{tikzcd}$$

a 2-cell is a modification between the loose morphisms.

Whereas the oplax version $[\mathcal{S},\mathcal{F}]^{\mathcal{F}}_{\text{oplax}}$ can be characterized as follows:

an object is an object of $[\mathcal{S},\mathcal{F}]^{\mathcal{F}}$, that we keep on viewing as a triangle above (in the description of $[\mathcal{S},\mathcal{F}]^{\mathcal{F}}$);

a loose morphism $G \underset{\text{loose}}{\Longrightarrow} H$ is an oplax natural transformation $\alpha_\lambda\colon G_\lambda \to H_\lambda$ that is (strictly) 2-natural on tight morphisms, meaning that $\alpha_\lambda * J_{\mathcal{S}}$ is 2-natural; we call them *marked oplax*;

a tight morphism $G \Rrightarrow H$ is a loose one with tight components, that precisely means that it induces a 2-natural transformation $\alpha_\tau\colon G_\tau \Rightarrow H_\tau$ such that

$$\begin{tikzcd}
\mathcal{S}_\tau \arrow[r, "J_{\mathcal{S}}"] \arrow[dr, bend right=40, "G_\tau"'] \arrow[dr, phantom, "\overset{j_G}{\Rightarrow}"] & \mathcal{S}_\lambda \arrow[d, bend right=50, "G_\lambda"'] \arrow[d, bend left=50, "H_\lambda"] \arrow[d, phantom, "\overset{\alpha_\lambda}{\Rightarrow}"] & = & \mathcal{S}_\tau \arrow[r, "J_{\mathcal{S}}"] \arrow[d, bend right=50, "G_\tau"'] \arrow[d, bend left=50, "H_\tau"] \arrow[d, phantom, "\overset{\alpha_\tau}{\Rightarrow}"] & \mathcal{S}_\lambda \arrow[dl, bend left=40, "H_\lambda"] \arrow[dl, phantom, "\overset{j_H}{\Rightarrow}"] \\
& \mathcal{C}at & & \mathcal{C}at &
\end{tikzcd}$$

a 2-*cell* is modification between the loose morphisms.

Remark 2.2.5. The 2-category of elements of a 2-functor $W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}at$ has a canonical structure of $\mathcal{F}$-functor $\mathcal{G}(W) : \int W \to \mathcal{A}$. Indeed any 2-category $\mathcal{A}$ can be seen as an $\mathcal{F}$-category by taking every morphism to be tight. And we can give $\int W$ a natural $\mathcal{F}$-category structure taking its loose part to be itself (as a 2-category) and as tight morphisms the morphisms of the kind (f, id) (i.e. the morphisms of the cleavage). Then $\left[\left(\int W\right)^{\mathrm{op}}, \mathcal{C}at\right]_{\mathrm{oplax}^{\mathrm{cart}}}$ coincides with the loose part of $\left[\left(\int W\right)^{\mathrm{op}}, \mathcal{F}\right]^{\mathcal{F}}_{\mathrm{oplax}}$, justifying even more the use of the cartesian-marked oplax natural transformations to work with the 2-category of elements (but also with the usual Grothendieck construction).

Since $\mathcal{E}/_{\mathrm{lax}}\, M = \int \mathrm{y}(M)$, we obtain that the lax slice of $\mathcal{E}$ on M has a canonical $\mathcal{F}$-category structure, with loose morphisms the usual ones and tight morphisms the triangles filled with an identity. That is, the tight part of $\mathcal{E}/_{\mathrm{lax}}\, M$ is the strict 2-slice $\mathcal{E}/_{M}$. So we can rephrase Proposition 2.2.1 by saying that there is a bijection between the loose strict/oplax $\mathcal{F}$-cocones on M and the $\mathcal{F}$-diagrams in the lax slice on M.

We can now prove a result of lifting of 2-colimits for $\mathrm{dom}\colon \mathcal{E}/_{\mathrm{lax}}\, M \to \mathcal{E}$. Such result is not about all the cartesian-marked oplax colimits, but this makes sense from an $\mathcal{F}$-categorical point of view.

Proposition 2.2.6. *Let $\mathcal{E}$ be a 2-category and let $M \in \mathcal{E}$. Then the 2-colim-fibration* $\mathrm{dom}\colon \mathcal{E}/_{\mathrm{lax}}\, M \to \mathcal{E}$ *lifts all the cartesian-marked oplax colimits of $\mathcal{F}$-diagrams.*

That is, given a marking $W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}at$ with $\mathcal{A}$ small, an $\mathcal{F}$-diagram $H\colon \int W \to \mathcal{E}/_{\mathrm{lax}}\, M$ and a universal cartesian-marked oplax cocone

$$\theta\colon \Delta 1 \underset{\mathrm{oplax}^{\mathrm{cart}}}{\Longrightarrow} \mathcal{E}\left((\mathrm{dom} \circ H)(-), C\right)$$

that exhibits $C = \mathrm{oplax}^{\mathrm{cart}}\text{-}\mathrm{colim}^{\Delta 1}(\mathrm{dom} \circ H)$ in $\mathcal{E}$, there exist $q \in \mathcal{E}/_{\mathrm{lax}}\, M$ over C and a universal cartesian-marked oplax cocone

$$\overline{\theta}\colon \Delta 1 \underset{\mathrm{oplax}^{\mathrm{cart}}}{\Longrightarrow} \mathcal{E}/_{\mathrm{lax}}\, M\,(H(-), C)$$

for H *on* q *exhibiting* $q = \operatorname{oplax}^{\mathrm{cart}}\text{-}\operatorname{colim}^{\Delta 1} H$ *in* $\mathcal{E}/_{\mathrm{lax}}\, M$ *such that* $\operatorname{dom} \circ \overline{\theta} = \theta$ *(i.e.* $\overline{\theta}$ *is over* θ *).*

Proof. By Proposition 2.2.1 (together with its proof) and Remark 2.2.5, the $\mathcal{F}$-diagram $H \colon \int W \to \mathcal{E}/_{\mathrm{lax}}\, M$ corresponds to a cartesian-marked oplax cocone λ for $\operatorname{dom} \circ H$ on M. As θ is universal, then λ induces a unique morphism $q \colon C \to M$ such that $\lambda = (q \circ -) \circ \theta$.

Consider now the lifting of the pair $(\operatorname{dom} \circ H, \theta)$ through the discrete 2-fibration dom to q, as in Construction 2.1.13. Since dom is a 2-colim-fibration, by Theorem 2.1.21, the pair $(\overline{\operatorname{dom} \circ H}, \overline{\theta})$ that we obtain over $(\operatorname{dom} \circ H, \theta)$ exhibits

$$q = \operatorname{oplax}^{\mathrm{cart}}\text{-}\operatorname{colim}^{\Delta 1} \overline{\operatorname{dom} \circ H}.$$

But we can calculate $\overline{\operatorname{dom} \circ H}$ explicitly, as in the first part of the proof of Theorem 2.1.21, looking at the action of $\mathrm{y}(M) : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}at$ on 1-cells and 2-cells, since $\operatorname{dom} = \mathcal{G}(\mathrm{y}(M))$. Given $\delta \colon (f, \alpha) \Rightarrow (g, \beta) \colon (A, X) \to (B, X')$ in $\int W$, using also the proof of Proposition 2.2.1 we obtain

$$\overline{\operatorname{dom} \circ H}(A, X) = q \circ \theta_{(A,X)} = \lambda_{(A,X)} = H(A, X)$$

$$\overline{\operatorname{dom} \circ H}(f, \alpha) = (\operatorname{dom}(H(f, \alpha)), q * \theta_{f,\alpha}) = (\operatorname{dom}(H(f, \alpha)), \lambda_{f,\alpha}) = H(f, \alpha)$$

$$\overline{\operatorname{dom} \circ H}(\delta) = \operatorname{dom}(H(\delta)) = H(\delta)$$

So $\overline{\operatorname{dom} \circ H} = H$, whence we conclude. □

We have thus already proved that $\operatorname{dom} \colon \mathcal{E}/_{\mathrm{lax}}\, M \to \mathcal{E}$ is a 2-colim fibration (Theorem 2.1.21, which includes a result of reflection of 2-colimits) that also lifts a large class of colimits (Proposition 2.2.6, justified via $\mathcal{F}$-category theory). In order to reach a complete generalization of the 1-dimensional Theorem 2.0.1, it only remains to address preservation of 2-colimits for dom.

2.3. Lax $\mathcal{F}$-adjoints and preservation of colimits

Aiming at a full generalization of the 1-dimensional Theorem 2.0.1 to dimension 2, we would like to prove that $\operatorname{dom} \colon \mathcal{E}/_{\mathrm{lax}}\, M \to \mathcal{E}$ preserves 2-colimits, assuming that $\mathcal{E}$ has products. $\mathcal{F}$-category theory will be crucial for this. Remember also that it helped justify the lifting result and that it could also justify more the reflection result.

In dimension 1, given a category $\mathcal{C}$ with products, the functor $\operatorname{dom}\colon \mathcal{C}/_{M} \to \mathcal{C}$ preserves colimits because it has a right adjoint, namely $\operatorname{dom} \dashv M \times -$. In dimension 2, assuming to have products, the 2-functor $M \times -$ is only a lax right adjoint to $\operatorname{dom}\colon \mathcal{E}/_{\text{lax}}\, M \to \mathcal{E}$, and more precisely a right-semi-lax right adjoint, as we prove in Theorem 2.3.12. However, this is not enough for the preservation of 2-colimits.

In this section, we prove (Theorem 2.3.10) that having a *right-semi-lax right* $\mathcal{F}$*-adjoint* is enough to guarantee the preservation of all *tight strict/oplax* $\mathcal{F}$*-colimits* (see Definition 2.3.6). Furthermore, having only a *right-semi-lax loose right* $\mathcal{F}$*-adjoint* is enough to guarantee the preservation of a large class of 2-colimits. We then prove that dom has such a (tight) $\mathcal{F}$-categorical right adjoint (Theorem 2.3.12), and thus preserves all *tight strict/oplax* $\mathcal{F}$*-colimits*, as well as other (more loose) 2-colimits.

In Section 2.4, we will prove that also the 2-functor of change of base along a split Grothendieck opfibration between lax slices has a suitable $\mathcal{F}$-categorical right adjoint (so that also such 2-functor preserves a large class of 2-colimits).

We begin recalling the concept of lax adjunction and the universal mapping property that characterizes it (Definition 2.3.1, Remark 2.3.2 and Proposition 2.3.3), for which we take as references Gray's [20, Section I,7] and Bunge's [11].

Definition 2.3.1. A lax adjunction is, for us, what Gray calls a strict weak quasi-adjunction in [20, Section I,7]. That is, a *lax adjunction* from a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ to a 2-functor $U\colon \mathcal{B} \to \mathcal{A}$ is given by a lax natural unit $\eta\colon \operatorname{Id} \Rightarrow U \circ F$, a lax natural counit $\varepsilon\colon F \circ U \Rightarrow \operatorname{Id}$ and modifications

$$\begin{tikzcd}
F \arrow[rr, equal] \arrow[dr, "F\eta"'] & {} \arrow[d, phantom, "\Downarrow s"] & F \\
& F \circ U \circ F \arrow[ur, "\varepsilon F"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& U \circ F \circ U \arrow[dr, "U\varepsilon"] \arrow[d, phantom, "\Downarrow t"] & \\
U \arrow[ur, "\eta U"] \arrow[rr, equal] & {} & U
\end{tikzcd}$$

that express lax triangular laws, such that both the swallowtail modifications

$$\begin{tikzcd}
\mathrm{id} \arrow[r, "\eta"] \arrow[d, "\eta"'] & UF \arrow[d, "UF\eta"] \arrow[ddr, equal, bend left=40] \arrow[dl, phantom, "\Swarrow \eta_\eta"] \arrow[ddr, phantom, "\Swarrow Us"] & \\
UF \arrow[r, "\eta UF"'] \arrow[drr, equal, bend right=30] & UFUF \arrow[dr, "U\varepsilon F"] \arrow[d, phantom, "\Downarrow tF"] & \\
& & UF
\end{tikzcd}
\qquad
\begin{tikzcd}
FU \arrow[dr, "F\eta U"] \arrow[drr, equal, bend left=30] \arrow[ddr, equal, bend right=40] \arrow[r, phantom, "\Downarrow sU"] \arrow[dr, phantom, "\Swarrow Ft"'] & & \\
& FUFU \arrow[r, "\varepsilon FU"] \arrow[d, "FU\varepsilon"'] \arrow[dr, phantom, "\Swarrow \varepsilon_\varepsilon"] & FU \arrow[d, "\varepsilon"] \\
& FU \arrow[r, "\varepsilon"'] & \mathrm{id}
\end{tikzcd}$$

are identities.

A *right-semi-lax adjunction* is a lax adjunction in which the counit ε is strictly 2-natural and the modification s is the identity.

We call a lax adjunction *strict* when s and t are both identities, making the triangular laws to hold strictly.

Remark 2.3.2. Using lax comma objects, firstly introduced by Gray in [20, Section I,2] and refined in Definition 1.3.4, we can reduce the study of lax adjunctions to ordinary adjunctions between hom-sets. Indeed, according to Gray, a lax adjunction is equivalently given by homomorphic (see below) 2-adjoint functors

$$\begin{tikzcd}
F \mathbin{/_{\mathrm{lax}}} \mathcal{B} \arrow[r, bend left, "S"] \arrow[r, phantom, "\perp"] & \mathcal{A} \mathbin{/_{\mathrm{lax}}} U \arrow[l, bend left, "T"]
\end{tikzcd}$$

over $\mathcal{A}\times\mathcal{B}$ with unit $\chi\colon \mathrm{id} \Rightarrow T\circ S$ and counit $\xi\colon S\circ T \Rightarrow \mathrm{id}$ (that are automatically 2-natural if assumed natural) over $\mathcal{A}\times\mathcal{B}$. Here S and T homomorphic means that they are given uniquely by lax natural η and ε (it can be defined precisely as in 1.5.10 of Gray's [20], asking for example T to transform precomposition with cells in $\mathcal{A}$ into precomposition with the image through F of those cells; see also below how we produce T from ε).

Strictness corresponds to

$$\chi * i_F = \mathrm{id} \qquad \text{and} \qquad \xi * i_U = \mathrm{id},$$

where $i_F\colon \mathcal{A} \to F \mathbin{/_{\mathrm{lax}}} \mathcal{B}$ is the 2-functor induced by the identity on F and analogously for i_U.

Such a 2-adjunction $S \dashv T$ means, in particular, that we have ordinary adjunctions between homsets

$$\begin{tikzcd}
\mathcal{B}\left(F(A), B\right) \arrow[r, phantom, "\perp"] \arrow[r, bend left, "S"] & \mathcal{A}\left(A, U(B)\right) \arrow[l, bend left, "T"]
\end{tikzcd}$$

for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. And we can rephrase such ordinary adjunctions in terms of having universal units. The global adjunction $S \dashv T$ corresponds, then, to such units satisfying a broader universal property, that captures the possibility of $h\colon F(A) \to B$ to vary in the whole lax comma object $F \,/_{\text{lax}}\, \mathcal{B}$ rather than in just $\mathcal{B}\left(F(A), B\right)$. This is the idea behind Proposition 2.3.3, that shows the universal mapping property that characterizes lax adjunctions, except that in general the lax right-adjoint produced is only oplax functorial. In our examples, however, such characterization will produce a strict (right-semi-) lax adjunction between 2-functors.

Before that, it is helpful to see explicitly how a lax adjunction $(F, U, \eta, \varepsilon, s, t)$ produces the adjunctions (S, T, χ, ξ) between the homsets on $A \in \mathcal{A}$ and $B \in \mathcal{B}$, as we will use this later. S and T are defined as usual as

$$S = (- \circ \eta_A) \circ U$$

$$T = (\varepsilon_B \circ -) \circ F$$

And the lax naturality of η and ε gives χ and ξ; precisely, given $h\colon F(A) \to B$ in $\mathcal{B}$ and $k\colon A \to U(B)$ in $\mathcal{A}$

$$\chi_h = \begin{tikzcd}
 & \Downarrow s_A & F(A) \arrow[rd, "h"] & \\
F(A) \arrow[r, "F(\eta_A)"'] \arrow[rru, equal, bend left] & F(U(F(A))) \arrow[ru, "\varepsilon_{F(A)}"] \arrow[rd, "F(U(h))"'] & \Downarrow \varepsilon_h & B \\
 & & F(U(B)) \arrow[ru, "\varepsilon_B"'] &
\end{tikzcd}$$

$$\xi_k = \begin{tikzcd}
 & U(F(A)) \arrow[rd, "U(F(k))"] & & \\
A \arrow[ru, "\eta_A"] \arrow[rd, "k"'] & \Downarrow \eta_k & U(F(U(B))) \arrow[r, "U(\varepsilon_B)"] & U(B) \\
 & U(B) \arrow[ru, "\eta_{U(B)}"'] \arrow[rru, equal, bend right] & \Downarrow t_B &
\end{tikzcd}$$

In particular, we see that for a right-semi-lax adjunction we obtain $\chi = \mathrm{id}$ and then $T \circ S = \mathrm{Id}$ (this is where the name comes from).

Proposition 2.3.3 (dual of Proposition I,7.8.2 in Gray's [20] and of Theorem 4.1 in Bunge's [11]). *Let $F \colon \mathcal{A} \to \mathcal{B}$ be a 2-functor. Suppose that for every $B \in \mathcal{B}$ there is an object $U(B) \in \mathcal{A}$ and a morphism $\varepsilon_B \colon F(U(B)) \to B$ in $\mathcal{B}$ that is universal in the following sense: for every $h \colon F(A) \to B$ in $\mathcal{B}$ there is an $\overline{h} \colon A \to U(B)$ in $\mathcal{A}$ and a 2-cell*

$$\begin{tikzcd}
F(A) \arrow[d, "F(\overline{h})"'] \arrow[rd, bend left, "h"] \arrow[rd, phantom, shift right=2ex, "\overset{\lambda_h}{\Swarrow}"] & \\
F(U(B)) \arrow[r, "\varepsilon_B"'] & B
\end{tikzcd}$$

in $\mathcal{B}$ such that, given any other $g \colon A \to U(B)$ and $\sigma \colon h \Rightarrow \varepsilon_B \circ F(g)$, there is a unique $\delta \colon \overline{h} \Rightarrow g$ such that

$$\begin{tikzcd}
F(A) \arrow[d, bend right=50, "F(g)"'{name=G}] \arrow[d, bend left=50, "F(\overline{h})"{name=H}] \arrow[from=H, to=G, Rightarrow, "F(\delta)"'] \arrow[rd, bend left, "h"] \arrow[rd, phantom, shift right=2ex, "\overset{\lambda_h}{\Swarrow}"] & \\
F(U(B)) \arrow[r, "\varepsilon_B"'] & B
\end{tikzcd}
=
\begin{tikzcd}
F(A) \arrow[d, "F(g)"'] \arrow[rd, bend left, "h"] \arrow[rd, phantom, shift right=2ex, "\overset{\sigma}{\Swarrow}"] & \\
F(U(B)) \arrow[r, "\varepsilon_B"'] & B
\end{tikzcd}$$

Assume then that, for every $h \colon F(A) \to B$ in $\mathcal{B}$, we have $\overline{h} = \overline{h \circ \varepsilon_{F(A)}} \circ \overline{\mathrm{id}_{F(A)}}$ and

$$\begin{tikzcd}
F(A) \arrow[d, "F(\overline{\mathrm{id}})"'] \arrow[rd, bend left, equal] \arrow[rd, phantom, shift right=2ex, "\overset{\lambda_{\mathrm{id}}}{\Swarrow}"] & \\
F(U(F(A))) \arrow[r, "\varepsilon_{F(A)}"'] \arrow[d, "F(\overline{h \circ \varepsilon_{F(A)}})"'] \arrow[rd, phantom, "\overset{\lambda_{h \circ \varepsilon_{F(A)}}}{\Swarrow}"] & F(A) \arrow[d, "h"] \\
F(U(B)) \arrow[r, "\varepsilon_B"'] & B
\end{tikzcd}
=
\begin{tikzcd}
F(A) \arrow[d, "F(\overline{h})"'] \arrow[rd, bend left, "h"] \arrow[rd, phantom, shift right=2ex, "\overset{\lambda_h}{\Swarrow}"] & \\
F(U(B)) \arrow[r, "\varepsilon_B"'] & B
\end{tikzcd}
\tag{2.4}$$

and also that, for every $B \in \mathcal{B}$, we have $\overline{\varepsilon_B} = \mathrm{id}$ and $\lambda_{\varepsilon_B} = \mathrm{id}$.

Then U extends to an oplax functor, ε extends to a lax natural transformation and there exist a lax natural transformation η and modifications s, t such that U is a lax right-adjoint to F, except that in general U is only an oplax functor (and the swallowtail identities need to be slightly modified accordingly).

In particular, if $\lambda_h = \mathrm{id}$ for every $h \colon F(A) \to B$, we obtain a right-semi-lax adjunction.

Proof (constructions). Given $g\colon B \to B'$ in $\mathcal{B}$, define $U(g) := \overline{g \circ \varepsilon_B}$ and $\varepsilon_g := \lambda_{g \circ \varepsilon_B}$. Given a 2-cell $\mu\colon g \Rightarrow g'$ in $\mathcal{B}$, define $U(\mu)$ as the unique 2-cell induced by $\varepsilon_{g'} \circ \mu\varepsilon_B$. Given composable morphisms g and g' in $\mathcal{B}$, pasting ε_g and $\varepsilon_{g'}$ induces a unique coassociator for U, while the identity 2-cell induces a unique counitor.

We then define $\eta_A := \overline{\mathrm{id}_{F(A)}}$ and $s_A := \lambda_{\mathrm{id}_{F(A)}}$.

$$\begin{tikzcd}
F(A) \arrow[d, "F(\eta_A)"'] \arrow[dr, bend left, equal, ""{name=U}] & \\
F(U(F(A))) \arrow[r, "\varepsilon_{F(A)}"'] & F(A)
\arrow[Rightarrow, from=U, to=2-1, shorten <=6pt, shorten >=20pt, "s_A"]
\end{tikzcd}$$

And for every $f\colon A \to A'$ in $\mathcal{A}$ we take η_f to be the unique 2-cell that is induced from

$$\begin{tikzcd}
F(A) \arrow[d, "F(\eta_A)"'] \arrow[dr, bend left, equal, ""{name=U}] & \\
F(U(F(A))) \arrow[r, "\varepsilon_{F(A)}"'] \arrow[d, "F(U(F(f)))"'] & F(A) \arrow[d, "F(f)"] \\
F(U(F(A'))) \arrow[r, "\varepsilon_{F(A')}"'] & F(A')
\arrow[Rightarrow, from=U, to=2-1, shorten <=6pt, shorten >=20pt, "s_A"]
\arrow[Rightarrow, from=2-2, to=3-1, shorten <=12pt, shorten >=20pt, "\varepsilon_{F(f)}"]
\end{tikzcd}$$

considering $s_{A'} * F(f)$, thanks to the assumption in equation (2.4). Finally, for every $B \in \mathcal{B}$, we induce t_B from the identity 2-cell on ε_B, using again the assumption in equation (2.4), with $h = \varepsilon_B$. ☐

Remark 2.3.4. In our two examples, i.e. with F equal to $\mathrm{dom}\colon \mathcal{E}/_{\mathrm{lax}}\, M \to \mathcal{E}$ (Theorem 2.3.12) and F equal to the 2-functor of pullback along a split Grothendieck opfibration between lax slices of $\mathcal{C}at$ (Theorem 2.4.3), Proposition 2.3.3 will produce a strict right-semi-lax adjunction between 2-functors. But this would not be enough to guarantee preservation of colimits. It is enough if the right-semi-lax adjunction is $\mathcal{F}$-categorical, as we prove in Theorem 2.3.10. But we have to restrict the attention to the (*tight*) *strict/oplax* $\mathcal{F}$*-colimits*, defined in Definition 2.3.6 (that we believe are the suitable colimits to consider in this context). It might be helpful to look at Definition 2.2.3 and Remark 2.2.4 (that recall $\mathcal{F}$-category theory).

The concept of lax $\mathcal{F}$-adjunction appears in Walker's [50], but in a pseudo/lax version and with the stronger request that s and t are isomorphisms. Moreover, only what for us is the tight version is considered there, asking the unit η and the counit ε to be (tight) pseudo/oplax $\mathcal{F}$-natural rather than loose ones. The latter request means that η and ε are tight morphisms in some $[\mathcal{S},\mathcal{S}]^{\mathcal{F}}_{\text{oplax}}$ of Definition 2.2.3 rather than loose ones. Such request is not necessary to guarantee the preservation of the "loose part" of tight strict/oplax $\mathcal{F}$-colimits. Moreover, it is not satisfied by our example of change of base along a split Grothendieck opfibration between lax slices.

Definition 2.3.5. A *loose lax $\mathcal{F}$-adjunction* is a lax adjunction $(F,U,\eta,\varepsilon,s,t)$ between the loose parts in which F and U are $\mathcal{F}$-functors and η and ε are loose strict/lax $\mathcal{F}$-natural transformations (i.e. loose morphisms in $[\mathcal{S},\mathcal{S}]^{\mathcal{F}}_{\text{lax}}$ of Definition 2.2.3 for suitable $\mathcal{S}$).

A *(tight) lax $\mathcal{F}$-adjunction* is a loose one such that η and ε are (tight) strict/lax $\mathcal{F}$-natural transformations (that is, have tight components).

A *right-semi-lax loose $\mathcal{F}$-adjunction* is a loose lax $\mathcal{F}$-adjunction such that ε is strictly 2-natural (i.e. a loose morphism in $[\mathcal{S},\mathcal{S}]^{\mathcal{F}}$ of Definition 2.2.3) and s is the identity.

We call a loose lax $\mathcal{F}$-adjunction *strict* if both s and t are identities.

Definition 2.3.6. Let $\mathcal{A}$ be a small $\mathcal{F}$-category and consider $\mathcal{F}$-functors $W\colon \mathcal{A}^{\text{op}}\to\mathcal{F}$ (the weight) and $H\colon\mathcal{A}\to\mathcal{S}$ (the $\mathcal{F}$-diagram). The *strict/oplax $\mathcal{F}$-colimit of H weighted by W*, denoted as $\text{oplax}^{\mathcal{F}}\text{-colim}^{W} H$, is (if it exists) an object $C\in\mathcal{S}$ together with an isomorphism in $\mathcal{F}$

$$\mathcal{S}\left(C,Q\right)\cong[\mathcal{A}^{\text{op}},\mathcal{F}]^{\mathcal{F}}_{\text{oplax}}\left(W,\mathcal{S}\left(H(-),Q\right)\right)$$

$\mathcal{F}$-natural in $Q\in\mathcal{S}$, where $[\mathcal{A}^{\text{op}},\mathcal{F}]^{\mathcal{F}}_{\text{oplax}}$ is the $\mathcal{F}$-category described in Remark 2.2.4.

Remark 2.3.7. The natural isomorphism of Definition 2.3.6 is equivalently a 2-natural isomorphism between the loose parts, that is,

$$\mathcal{S}_{\lambda}\left(C,Q\right)\cong[\mathcal{A}^{\text{op}}_{\lambda},\mathcal{C}at]_{\text{oplax}^{\text{mark}}}\left(W_{\lambda},\mathcal{S}_{\lambda}\left(H_{\lambda}(-),Q\right)\right), \tag{2.5}$$

where the right hand side denotes the marked oplax natural transformations, which restricts to a 2-natural isomorphism between the tight parts. Such tight parts are respectively $\mathcal{S}_\tau\left(C,Q\right)$ and those marked oplax natural transformations α_λ that restrict to 2-natural ones $\alpha_\tau\colon W_\tau \Rightarrow \mathcal{S}_\tau\left(H_\tau(-),Q\right)$, i.e. those forming a commutative square

$$\begin{tikzcd}
W_\tau(A) \arrow[r, "(\alpha_\tau)_A"] \arrow[d, "(j_W)_A"'] & \mathcal{S}_\tau\left(H_\tau(A),Q\right) \arrow[d, "(J_{\mathcal{S}} * H_\tau)_A"] \\
W_\lambda(A) \arrow[r, "(\alpha_\lambda)_A"'] & \mathcal{S}_\lambda\left(H_\lambda(A),Q\right)
\end{tikzcd}$$

for every $A \in \mathcal{A}$ (where $J_{\mathcal{S}}\colon \mathcal{S}_\tau\left(-,Q\right) \Rightarrow \mathcal{S}_\lambda\left(-,Q\right) \circ J_{\mathcal{S}}$)

Remember that identities are always tight and tight morphisms are closed under composition. So the request that the 2-natural isomorphism of equation (2.5) restricts to one between the tight parts equivalently means that the universal marked oplax cocylinder μ^λ (corresponding to id_C) satisfies the following two conditions. For every $A \in \mathcal{A}$ and $X \in W_\tau(A)$, the morphism

$$\mu^\lambda_A(X)\colon H(A) \to C$$

is tight, and, for every $q\colon C \to Q$ in $\mathcal{S}$, if $q \circ \mu^\lambda_A(X)$ is tight for every $A \in \mathcal{A}$ and $X \in W_\tau(A)$ then q needs to be tight. We say that the (*cocylinder*) *τ-components* $\mu^\lambda_A(X)$'s are tight and *jointly detect tightness.*

Proposition 2.3.8. *Let $\mathcal{A}$ be a small $\mathcal{F}$-category and consider $\mathcal{F}$-functors $W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{F}$ (the weight) and $H\colon \mathcal{A} \to \mathcal{S}$ (the $\mathcal{F}$-diagram). The strict/oplax $\mathcal{F}$-colimit of H weighted by W is, equivalently, an object $C \in \mathcal{S}$ together with an marked oplax cocylinder*

$$\mu^\lambda\colon W_\lambda \xRightarrow[\mathrm{oplax}^{\mathrm{mark}}]{} \mathcal{S}_\lambda\left(H_\lambda(-),C\right)$$

that is universal in the 2-categorical sense, giving a 2-natural isomorphism

$$\mathcal{S}_\lambda\left(C,Q\right) \cong \left[\mathcal{A}^{\mathrm{op}}_\lambda, \mathcal{C}at\right]_{\mathrm{oplax}^{\mathrm{mark}}}\left(W_\lambda, \mathcal{S}_\lambda\left(H_\lambda(-),Q\right)\right),$$

and has τ-components that are tight and jointly detect tightness.

Proof. The proof is clear after Remark 2.3.7. Since the loose limit is 2-categorical, it can indeed be characterized as having a universal marked oplax cocylinder. □

Definition 2.3.9. We call a strict/oplax $\mathcal{F}$-colimit *tight* if it is exhibited by a marked oplax cocylinder μ^{λ} as in Proposition 2.3.8 such that all *cocylinder* λ-*components* $\mu^{\lambda}_{A}(X)$, for $A \in \mathcal{A}$ and $X \in W_{\lambda}(A)$, are tight. Notice that this condition is automatic in the case of cartesian-marked oplax cocones, that is the one we are mostly interested in (as every weighted 2-cocylinder can be reduced to one of this form).

We are now ready to prove that having a right-semi-lax right $\mathcal{F}$-adjoint guarantees the preservation of all tight strict/oplax $\mathcal{F}$-colimits. We will actually see that the property of the universal marked oplax cocylinder to have τ-components that jointly detect tightness is preserved when we have a right-semi-lax (tight) left $\mathcal{F}$-adjoint, but not necessary to prove the preservation of the rest of the structure, for which a loose adjunction is enough.

The following theorem does not seem to appear in the literature.

Theorem 2.3.10. *Right semi-lax loose left $\mathcal{F}$-adjoints preserve all the universal marked oplax cocylinders for an $\mathcal{F}$-diagram which have tight λ-components* (*i.e. in some sense the "loose part" of all the tight strict/oplax $\mathcal{F}$-colimits, even if the τ-components do not jointly detect tightness*).

Right semi-lax (*tight*) *left $\mathcal{F}$-adjoints preserve all tight strict/oplax $\mathcal{F}$-colimits.*

Proof. Let $(F, U, \eta, \varepsilon, s, t)$ be a right-semi-lax loose $\mathcal{F}$-adjunction between $\mathcal{F}$-categories $\mathcal{S}$ and $\mathcal{E}$. That is, a lax adjunction between the loose parts where F and U are $\mathcal{F}$-functors, η is a loose strict/lax $\mathcal{F}$-natural transformation, ε is strictly 2-natural and s is the identity. Let then $\mathcal{A}$ be a small $\mathcal{F}$-category and consider $\mathcal{F}$-functors $W \colon \mathcal{A}^{\mathrm{op}} \to \mathcal{F}$ and $H \colon \mathcal{A} \to \mathcal{S}$ such that the strict/oplax $\mathcal{F}$-colimit of H weighted by W exists in $\mathcal{S}$ and is tight. Call C such colimit; we want to show that F preserves it. By Proposition 2.3.8, it suffices to consider the universal marked oplax cocylinder

$$\mu^{\lambda}\colon W_{\lambda} \xRightarrow[\text{oplax}^{\text{mark}}]{} \mathcal{S}_{\lambda}\left(H_{\lambda}(-), C\right)$$

with tight λ-components that exhibits $C = \text{oplax}^{\mathcal{F}}\text{-colim}^{W} H$ and the marked oplax cocylinder

$$W_{\lambda} \xRightarrow[\text{oplax}^{\text{mark}}]{\mu^{\lambda}} \mathcal{S}_{\lambda}\left(H_{\lambda}(-), C\right) \xRightarrow{F} \mathcal{E}_{\lambda}\left(\left(F_{\lambda} \circ H_{\lambda}\right)(-), F(C)\right)$$

obtained applying F to the former.

We prove that $F \circ \mu^{\lambda}$ is universal in the 2-categorical sense and such that the $F(\mu^{\lambda}_{A}(X))$'s, for $A \in \mathcal{A}$ and $X \in W_{\lambda}(A)$, are all tight, without using that the τ-components jointly detect tightness. Moreover, we show that if η and ε have tight components, giving a right-semi-lax (tight) $\mathcal{F}$-adjunction, then having τ-components that jointly detect tightness is preserved as well.

Since a right-semi-lax loose $\mathcal{F}$-adjunction is in particular a right-semi-lax adjunction between the loose parts, we know by Remark 2.3.2 that $(F, U, \eta, \varepsilon, \mathrm{id}, t)$ induces an adjunction between homsets

$$\begin{tikzcd}
\mathcal{E}_{\lambda}\left(F_{\lambda}(Y), Z\right) \arrow[r, bend left, "S_{Y,Z}"] \arrow[r, phantom, "\perp"] & \mathcal{S}_{\lambda}\left(Y, U_{\lambda}(Z)\right) \arrow[l, bend left, "T_{Y,Z}"]
\end{tikzcd}$$

for every $Y \in \mathcal{S}$ and $Z \in \mathcal{E}$ with unit the identity, showing $T \circ S = \mathrm{Id}$, and counit $\xi\colon S \circ T \Rightarrow \mathrm{id}$. The strategy will be to make use of the equality $T \circ S = \mathrm{Id}$ to move back and forth between $\mathcal{E}$ and $\mathcal{S}$, recovering, after $T \circ S$, the original starting data of $\mathcal{E}$ but with new information gathered in $\mathcal{S}$.

The λ-components $F(\mu^{\lambda}_{A}(X))$'s are surely tight, since F is an $\mathcal{F}$-functor. Take then $q\colon F(C) \to Z$ in $\mathcal{E}$ such that $q \circ F(\mu^{\lambda}_{A}(X))$ is tight for every $A \in \mathcal{A}$ and every $X \in W_{\tau}(A)$. We show that if η and ε have tight components, then q needs to be tight as well. Notice that $S_{-,Z}$ is marked oplax natural in $Y \in \mathcal{S}_{\lambda}^{\mathrm{op}}$, with structure 2-cell on $y\colon Y \leftarrow Y'$ in $\mathcal{S}_{\lambda}$ given by $(- * \eta_{y}) * U$, since η is loose strict/lax $\mathcal{F}$-natural. Moreover, if η has tight components then $S_{-,Z}$ is tight as well, since

U is an $\mathcal{F}$-functor. Since $\mu^{\lambda}_{A}(X)\colon H(A) \to C$ is tight, we obtain $S_{\mu^{\lambda}_{A}(X),Z} = \mathrm{id}$ and hence

$$\begin{tikzcd}
\mathcal{E}_{\lambda}\left(F_{\lambda}(C), Z\right) \arrow[r, "S_{C,Z}"] \arrow[d, "-\circ F_{\lambda}(\mu^{\lambda}_{A}(X))"'] & \mathcal{S}_{\lambda}\left(C, U_{\lambda}(Z)\right) \arrow[d, "-\circ\mu^{\lambda}_{A}(X)"] \\
\mathcal{E}_{\lambda}\left(F_{\lambda}(H(A)), Z\right) \arrow[r, "S_{H(A),Z}"'] & \mathcal{S}_{\lambda}\left(H(A), U_{\lambda}(Z)\right)
\end{tikzcd}$$

$$S_{H(A),Z}(q \circ F(\mu^{\lambda}_{A}(X))) = S_{C,Z}(q) \circ \mu^{\lambda}_{A}(X)$$

So if η is tight then the left hand side of the equality here above is tight, and since the $\mu^{\lambda}_{A}(X)$'s jointly detect tightness we obtain that $S_{C,Z}(q)$ is tight. If we also assume that ε is tight, then $T_{C,Z}$ is tight, whence $q = T(S(q))$ is tight.

We now prove that $F \circ \mu^{\lambda}$ is universal, assuming only a right-semi-lax loose $\mathcal{F}$-adjunction and never using that the τ-components $\mu^{\lambda}_{A}(X)$ jointly detect tightness. Everything below will be loose, so we abuse the notation dropping the loose subscripts. The following figure condenses the strategy.

$$\begin{tikzcd}
W \arrow[r, Rightarrow, "\mu"] \arrow[d, Rightarrow, "\sigma"'] & \mathcal{S}\left(H(-), C\right) \arrow[r, Rightarrow, "F"] \arrow[d, Rightarrow, dashed, "\gamma\circ-"] & \mathcal{E}\left((F\circ H)(-), F(C)\right) \arrow[d, Rightarrow, dashed, "T(\gamma)\circ-"] \\
\mathcal{E}\left((F\circ H)(-), Z\right) \arrow[r, Rightarrow, "S"'] & \mathcal{S}\left(H(-), U(Z)\right) \arrow[r, Rightarrow, "T"'] & \mathcal{E}\left((F\circ H)(-), Z\right)
\end{tikzcd} \tag{2.6}$$

Given a marked oplax cocylinder

$$\sigma\colon W \xRightarrow[\text{oplax}^{\text{mark}}]{} \mathcal{E}\left((F\circ H)(-), Z\right),$$

we want to prove that there is a unique $\delta\colon F(C) \to Z$ in $\mathcal{E}$ such that

$$(\delta \circ -) \circ F \circ \mu = \sigma.$$

Postcomposing σ with $S_{H(-),Z}$, we obtain a marked oplax cocylinder for H. Indeed $S_{H(-),Z}$ is marked oplax natural in $A \in \mathcal{A}^{\mathrm{op}}$ with structure 2-cell on $a\colon A \leftarrow A'$ in $\mathcal{A}$ given by $(- * \eta_{H(a)}) * U$, since η is loose strict/lax $\mathcal{F}$-natural and H is an $\mathcal{F}$-functor. So, by universality of μ, $S \circ \sigma$ induces a unique $\gamma\colon C \to U(Z)$ in $\mathcal{S}$ such that

$$(\gamma \circ -) \circ \mu = S \circ \sigma.$$

Notice then that the right square of the figure in equation (2.6) is commutative, since it is equivalent to

$$(\varepsilon_Z \circ F(\gamma)) \circ F(-) = \varepsilon_Z \circ F(\gamma \circ -),$$

that holds since F is a 2-functor. So $\delta \coloneqq T(\gamma)$ in $\mathcal{E}$ is such that

$$(\delta \circ -) \circ F \circ \mu = T \circ (\gamma \circ -) \circ \mu = T \circ S \circ \sigma = \sigma.$$

We now show the uniqueness of δ. So consider another $\delta' \colon F(C) \to Z$ in $\mathcal{E}$ such that $(\delta' \circ -) \circ F \circ \mu = \sigma$. Postcomposing with S we obtain

$$S \circ (\delta' \circ -) \circ F \circ \mu = S \circ \sigma.$$

But we notice that

$$S \circ (\delta' \circ -) \circ F \circ \mu = (S(\delta') \circ -) \circ \mu$$

as marked oplax natural transformations. Indeed, given $A \in \mathcal{A}$ and $\alpha \colon X \to X'$ in $W(A)$,

$$(S \circ (\delta' \circ -) \circ F \circ \mu)_A(X) = U\left(\delta' \circ F\left(\mu_A(X)\right)\right) \circ \eta_{H(A)} = U(\delta') \circ U\left(F\left(\mu_A(X)\right)\right) \circ \eta_{H(A)}.$$

Since $\mu_A(X)$ is tight and η is loose strict/lax $\mathcal{F}$-natural, $\eta_{\mu_A(X)} = \mathrm{id}$ and hence

$$U(\delta') \circ U\left(F\left(\mu_A(X)\right)\right) \circ \eta_{H(A)} = U(\delta') \circ \eta_C \circ \mu_A(X) = S(\delta') \circ \mu_A(X).$$

And it works similarly for the images on α, using the 2-dimensional property of η being oplax natural. Given $a \colon A \leftarrow A'$ in $\mathcal{A}$ and $X \in W(A)$,

$$(S \circ (\delta' \circ -) \circ F \circ \mu)_{a,X} = U\left(\delta' \circ F\left(\mu_A(X)\right)\right) * \eta_{H(a)} \circ U\left(\delta' * F\left(\mu_{a,X}\right)\right) * \eta_{H(A')}.$$

Considering $\mu_{a,X} \colon \mu_{A'}(W(a)(X)) \Rightarrow \mu_A(X) \circ H(a)$ in $\mathcal{S}$, since η is loose strict/lax $\mathcal{F}$-natural and both $\mu_A(X)$ and $\mu_{A'}(W(a)(X))$ are tight, we obtain

$$U\left(F\left(\mu_A(X)\right)\right) * \eta_{H(a)} \circ U\left(F\left(\mu_{a,X}\right)\right) * \eta_{H(A')} = \eta_C * \mu_{a,X},$$

whence we conclude that

$$S \circ (\delta' \circ -) \circ F \circ \mu = (S(\delta') \circ -) \circ \mu$$

Therefore, we have

$$(S(\delta') \circ -) \circ \mu = S \circ \sigma = (\gamma \circ -) \circ \mu,$$

and by universality of μ we conclude that $S(\delta') = \gamma$, whence

$$\delta' = T(S(\delta')) = T(\gamma) = \delta.$$

It only remains to prove the 2-dimensional universal property of $F \circ \mu$. Given a modification

$$\Sigma \colon \sigma \Rrightarrow \sigma' \colon W \xRightarrow[\text{oplax}^{\text{mark}}]{} \mathcal{E}\left((F \circ H)(-), Z\right),$$

we want to prove that there is a unique $\Delta \colon \delta \Rightarrow \delta' \colon F(C) \to Z$ in $\mathcal{E}$ such that

$$(\Delta * -) * (F \circ \mu) = \Sigma.$$

By universality of μ, whiskering Σ with S on the right induces a unique $\Gamma \colon \gamma \Rightarrow \gamma'$ in $\mathcal{S}$ such that

$$(\Gamma * -) * \mu = S * \Sigma.$$

Notice then that

$$T * (\Gamma * -) = (T(\Gamma) * -) * F$$

because F is a 2-functor, and thus $\Delta \coloneqq T(\Gamma)$ in $\mathcal{E}$ is such that

$$(\Delta * -) * (F \circ \mu) = T * (\Gamma * -) * \mu = T * S * \Sigma = \Sigma.$$

To show the uniqueness of Δ, take another Δ' such that $(\Delta' * -) * (F \circ \mu) = \Sigma$. Whiskering with S on the right, we obtain

$$S * (\Delta' * -) * (F \circ \mu) = S * \Sigma.$$

But notice that

$$S * (\Delta' * -) * (F \circ \mu) = (S(\Delta') * -) * \mu$$

Indeed it suffices to check it on components, where it holds since $\mu_A(X)$ is tight and hence $\eta_{\mu_A(X)} = \mathrm{id}$ (analogously to what we have shown for the 1-dimensional universal property). So

$$(S(\Delta') * -) * \mu = S * \Sigma = (\Gamma * -) * \mu,$$

whence $S(\Delta') = \Gamma$ by universality of μ and thus

$$\Delta' = T(S(\Delta')) = T(\Gamma) = \Delta.$$

Therefore $F \circ \mu$ is universal in the 2-categorical sense. $\square$

We can now conclude the generalization of the 1-dimensional Theorem 2.0.1 to dimension 2, by showing that, when $\mathcal{E}$ has products, $\operatorname{dom}\colon \mathcal{E}/_{\text{lax}}\, M \to \mathcal{E}$ has a strict right-semi-lax (tight) right $\mathcal{F}$-adjoint (Theorem 2.3.12) and hence preserves all tight strict/oplax $\mathcal{F}$-colimits. More importantly in the context of this thesis, Theorem 2.3.10 then also guarantees that dom preserves all the universal cartesian-marked oplax cocones (which are marked oplax with respect to the Grothendieck construction) for an $\mathcal{F}$-diagram which have tight components. Remember that any weighted 2-colimit can be reduced to a cartesian-marked oplax conical one.

Remark 2.3.11. Let $\mathcal{E}$ be a 2-category with products. After Theorem 2.3.12 we will have proved that, considering a marking $W\colon \mathcal{A}^{\text{op}} \to \mathcal{C}at$ with $\mathcal{A}$ small and an $\mathcal{F}$-diagram $H\colon \int W \to \mathcal{E}/_{\text{lax}}\, M$ (that is, a 2-functor that sends every morphism of the kind (f, id) to a triangle filled with an identity), if

$$\zeta\colon \Delta 1 \xRightarrow[\text{oplax}^{\text{cart}}]{} \mathcal{E}/_{\text{lax}}\, M\,(H(-), C)$$

is a universal cartesian-marked oplax 2-cocone for H on $q \in \mathcal{E}/_{\text{lax}}\, M$ exhibiting $q = \text{oplax}^{\text{cart}}\text{-}\operatorname{colim}^{\Delta 1} H$ such that $\zeta_{(A,X)}$ is a tight morphism for every $(A, X) \in \int W$ (which means that it is a triangle filled with an identity), then $\operatorname{dom} \circ \zeta$ is universal as well, exhibiting

$$\operatorname{dom}(q) = \text{oplax}^{\text{cart}}\text{-}\operatorname{colim}^{\Delta 1}(\operatorname{dom} \circ H).$$

Theorem 2.3.12. *Let $\mathcal{E}$ be a 2-category with products and let $M \in \mathcal{E}$. Then the 2-colim-fibration* $\operatorname{dom}\colon \mathcal{E}/_{\text{lax}}\, M \to \mathcal{E}$ *has a strict right-semi-lax (tight) right $\mathcal{F}$-adjoint.*

As a consequence, by Theorem 2.3.10, dom *preserves all tight strict/oplax $\mathcal{F}$-colimits, but also all the universal cartesian-marked oplax cocylinders for an $\mathcal{F}$-*

diagram which have tight λ-components (see Remark 2.3.11 for what this means explicitly in practice).

Proof. First of all, notice that dom is surely an $\mathcal{F}$-functor, as every morphism of $\mathcal{E}$ is tight (see Remark 2.2.5). We use the universal mapping property Proposition 2.3.3 that characterizes a lax adjunction to build a right-semi-lax right adjoint U to $\mathrm{dom}\colon \mathcal{E}/_{\mathrm{lax}}\, M \to \mathcal{E}$. For every $E \in \mathcal{E}$, we define $U(E) \coloneqq (M \times E \xrightarrow{\mathrm{pr}_1} E)$ and $\varepsilon_E\colon M \times E \xrightarrow{\mathrm{pr}_2} E$, that is tight in $\mathcal{E}$, remembering that in dimension 1 the domain functor from $\mathcal{C}/_M$ is left adjoint to $M \times -$.

We show that such counit is universal in the lax sense. Given $h\colon \mathrm{dom}(K \xrightarrow{t} M) \to E$ in $\mathcal{E}$, take $\overline{h} \coloneqq ((t,h),\mathrm{id})\colon (K \xrightarrow{t} M) \to (M \times E \xrightarrow{\mathrm{pr}_1} M)$, which is tight in $\mathcal{E}/_{\mathrm{lax}}\, M$ (see Remark 2.2.5), and $\lambda_h \coloneqq \mathrm{id}$.

$$\begin{tikzcd}
\mathrm{dom}(K \xrightarrow{t} M) \arrow[d, "{\mathrm{dom}((t,h),\mathrm{id})}"'] \arrow[rrd, bend left, "h"] \arrow[rrd, phantom, "="{description}, "\lambda_h"'] & & \\
\mathrm{dom}(M \times E \xrightarrow{\mathrm{pr}_1} M) \arrow[rr, "\mathrm{pr}_2"'] & & E
\end{tikzcd}$$

This guarantees that we will find a right-semi-lax adjunction in the end (see Proposition 2.3.3). Given then another morphism

$$\begin{tikzcd}
K \arrow[rr, "\widehat{\gamma}"] \arrow[rdd, "t"'] & & M \times E \arrow[ldd, "\mathrm{pr}_1"] \\
& \overset{\gamma}{\Longrightarrow} & \\
& M &
\end{tikzcd}$$

in $\mathcal{E}/_{\mathrm{lax}}\, M$ and another $\sigma\colon h \Rightarrow \mathrm{pr}_2 \circ \widehat{\gamma}$ in $\mathcal{E}$, there is a unique $\delta\colon ((t,h),\mathrm{id}) \Rightarrow (\widehat{\gamma},\gamma)$ in $\mathcal{E}/_{\mathrm{lax}}\, M$ such that

$$\begin{tikzcd}
\mathrm{dom}(K \xrightarrow{t} M) \arrow[d, bend right=60, "{\mathrm{dom}(\widehat{\gamma},\gamma)}"'{name=L}] \arrow[d, "{\mathrm{dom}((t,h),\mathrm{id})}"{name=R}] \arrow[rrd, bend left, "h"] \arrow[rrd, phantom, "="{description}, "\lambda_h"'] \arrow[from=R, to=L, Rightarrow, "\delta"] & & & = & \mathrm{dom}(K \xrightarrow{t} M) \arrow[d, "{\mathrm{dom}(\widehat{\gamma},\gamma)}"'] \arrow[rrd, bend left, "h"] \arrow[rrd, phantom, "\overset{\sigma}{\Longleftarrow}"{description}] & & \\
\mathrm{dom}(M \times E \xrightarrow{\mathrm{pr}_1} M) \arrow[rr, "\mathrm{pr}_2"'] & & E & & \mathrm{dom}(M \times E \xrightarrow{\mathrm{pr}_1} M) \arrow[rr, "\mathrm{pr}_2"'] & & E
\end{tikzcd}$$

Indeed δ is determined by

$$\mathrm{dom}(\delta) = \begin{tikzcd}
K \arrow[r, bend left, "{(t,h)}"{name=U}] \arrow[r, bend right, "\widehat{\gamma}"'{name=D}] \arrow[from=U, to=D, Rightarrow, "\delta"] & M \times E
\end{tikzcd},$$

that needs to satisfy $\mathrm{pr}_1 * \delta = \gamma$ in order to be a 2-cell $((t,h),\mathrm{id}) \Rightarrow (\widehat{\gamma},\gamma)$ and $\mathrm{pr}_2 * \delta = \sigma$ by the condition above. So δ needs to be (γ,σ), and this works.

We then see that, for every $E \in \mathcal{E}$, $\overline{\varepsilon_E} = \mathrm{id}$ since in this case $(t,h) = (\mathrm{pr}_1,\mathrm{pr}_2) = \mathrm{id}$ (and λ is always the identity). Moreover, for every $h\colon \mathrm{dom}(K \xrightarrow{t} M) \to E$ in $\mathcal{E}$,

$$\overline{h \circ \varepsilon_{F(A)}} \circ \overline{\mathrm{id}_{F(A)}} = ((\mathrm{pr}_1, h \circ \mathrm{pr}_2), \mathrm{id}) \circ ((t,\mathrm{id}),\mathrm{id}) = ((t,h),\mathrm{id}) = \overline{h},$$

making the assumption of equation (2.4) (of Proposition 2.3.3) hold.

By Proposition 2.3.3, as λ_h is always the identity, U extends to an oplax functor, ε extends to a 2-natural transformation and there exist a lax natural transformation η and a modification t such that U is a right-semi-lax right adjoint to $\mathrm{dom}\colon \mathcal{E}/_{\mathrm{lax}}\, M \to \mathcal{E}$. But it is easy to see, following the explicit construction of Proposition 2.3.3, that U is the (strict) 2-functor

$$\begin{array}{rccc}
M \times - : & \mathcal{E} & \longrightarrow & \mathcal{E}/_{\mathrm{lax}}\, M \\
& E & \mapsto & \left(M \times E \xrightarrow{\mathrm{pr}_1} E\right) \\
& E \xrightarrow{e} E' & \mapsto & (\mathrm{id} \times e, \mathrm{id}) \\
& e \overset{\beta}{\Rightarrow} e' & \mapsto & \mathrm{id} \times \beta
\end{array}$$

Then, for every $(K \xrightarrow{t} M) \in \mathcal{E}/_{\mathrm{lax}}\, M$,

$$\eta_t = \begin{tikzcd} K \arrow[rr, "{(t,\mathrm{id})}"] \arrow[dr, "t"'] & {\overset{\mathrm{id}}{=}} & M \times K \arrow[dl, "\mathrm{pr}_1"] \\ & M & \end{tikzcd}$$

The fact that $\overline{h}$ is always tight implies that $U = M \times -$ is an $\mathcal{F}$-functor and that η has tight components (both also immediate to check directly). Given a morphism

$$\begin{tikzcd} K \arrow[rr, "\widehat{\gamma}"] \arrow[dr, "t"'] & {\overset{\gamma}{\Longrightarrow}} & K' \arrow[dl, "t'"] \\ & M & \end{tikzcd}$$

in $\mathcal{E}/_{\mathrm{lax}}\, M$, we find that

$$\eta_{(\widehat{\gamma},\gamma)} = \begin{tikzcd} & M\times K \arrow[rd, "\mathrm{id}\times\widehat{\gamma}"] \arrow[dd, phantom, "\Downarrow(\gamma{,}\mathrm{id})"] & \\ K \arrow[ru, "(t{,}\mathrm{id})"] \arrow[rd, "\widehat{\gamma}"'] & & M\times K' \\ & K' \arrow[ru, "(t'{,}\mathrm{id})"'] & \end{tikzcd}$$

whence it is clear that η is (tight) strict/lax $\mathcal{F}$-natural (since $\eta_{(\widehat{\gamma},\mathrm{id})} = \mathrm{id}$). Finally, $t = \mathrm{id}$, giving a strict right-semi-lax adjunction.

We have also already checked that $\mathrm{dom}\colon \mathcal{E}/_{\mathrm{lax}}\, M \to \mathcal{E}$ and $M\times -$ are $\mathcal{F}$-functors, η is (tight) strict/lax $\mathcal{F}$-natural and ε has tight components, giving a strict right-semi-lax (tight) $\mathcal{F}$-adjunction. □

Remark 2.3.13. We can actually obtain a sharper result of preservation of 2-colimits for $\mathrm{dom}\colon \mathcal{E}/_{\mathrm{lax}}\, M \to \mathcal{E}$ (when $\mathcal{E}$ has products), as we show in Proposition 2.3.14. Namely, we can omit the assumption that the universal cartesian-marked oplax cocones for an $\mathcal{F}$-diagram have tight λ-components. Indeed, in the proof of Theorem 2.3.10, the preservation of the universal marked oplax cocylinder uses the assumption that the $\mu^{\lambda}_{A}(X)$'s are tight only to guarantee the uniqueness part of the 1- and 2-universal property. But we can prove both uniqueness results in another way, taking advantage of the simple description of the strict right-semi-lax right $\mathcal{F}$-adjoint $U = M\times -$ of dom.

Proposition 2.3.14. *Let $\mathcal{E}$ be a 2-category with products and let $M\in\mathcal{E}$. Then $\mathrm{dom}\colon \mathcal{E}/_{\mathrm{lax}}\, M \to \mathcal{E}$ preserves all the universal cartesian-marked oplax cocones for an $\mathcal{F}$-diagram (without assuming them to have tight λ-components).*

Proof. We only need to prove the uniqueness part of the 1- and 2-dimensional universal property, by Remark 2.3.13. Following the proof of Theorem 2.3.10 with $F = \mathrm{dom}$ and considering $\delta'\colon \mathrm{dom}(K\xrightarrow{t} M)\to Z$ in $\mathcal{E}$ such that $(\delta'\circ -)\circ \mathrm{dom}\circ\mu = \sigma$, rather than considering $S(\delta') = ((t,\delta'),\mathrm{id})$, we define

$$\gamma' := \begin{tikzcd} K \arrow[rr, "(\mathrm{pr}_1\circ\mathrm{dom}(\gamma){,}\delta')"] \arrow[rd, "t"'] & & M\times Z \arrow[ld, "\mathrm{pr}_1"] \\ & \overset{\gamma}{\Longrightarrow} & \\ & M & \end{tikzcd}$$

Then γ' satisfies $(\gamma' \circ -) \circ \mu = S \circ \sigma$, by the universal property of the product, since γ satisfies the analogous equation and $(\delta' \circ -) \circ \operatorname{dom} \circ \mu = \sigma$. By the uniqueness of γ, we obtain that $\gamma' = \gamma$ and hence $\delta' = \operatorname{pr}_2 \circ \operatorname{dom}(\gamma) = T(\gamma) = \delta$.

Analogously, we can prove also the uniqueness of the 2-dimensional universal property, producing from Δ' the 2-cell $(\operatorname{pr}_1 * \operatorname{dom}(\Gamma), \Delta')$ between the two suitable triangles γ' here above. We indeed obtain $\Delta' = \operatorname{pr}_2 * \operatorname{dom}(\Gamma) = T(\Gamma) = \Delta$. ☐

2.4. Change of base between lax slices

In this section, we present and study a change of base 2-functor between lax slices of a finitely complete 2-category. It is actually enough to only assume pullbacks along split opfibrations and comma objects. We believe that it is natural to consider a change of base 2-functor that calculates comma objects. Equivalently, we can take pullbacks along split Grothendieck opfibrations (Proposition 2.4.1). This is preferable in the context of this section, since Grothendieck opfibrations in $\mathcal{C}at$ are always Conduché and we can then generalize to lax slices the ideas of Conduché's [13] for finding a right adjoint to the pullback functor $\tau^* \colon \mathcal{C}at/_{\mathcal{B}} \to \mathcal{C}at/_{\mathcal{E}}$.

We prove that if $\tau \colon \mathcal{E} \to \mathcal{B}$ is a split opfibration in a 2-category $\mathcal{K}$ (see Section 4.2), then pulling back along τ extends to a 2-functor $\tau^* \colon \mathcal{K}/_{\text{lax}}\, \mathcal{B} \to \mathcal{K}/_{\text{lax}}\, \mathcal{E}$. Furthermore, we prove that when $\mathcal{K} = \mathcal{C}at$ such 2-functor has a strict right-semi-lax loose right $\mathcal{F}$-adjoint. As a consequence, by Theorem 2.3.10, $\tau^* \colon \mathcal{C}at/_{\text{lax}}\, \mathcal{B} \to \mathcal{C}at/_{\text{lax}}\, \mathcal{E}$ preserves all the universal cartesian-marked oplax cocylinders for an $\mathcal{F}$-diagram which have tight λ-components (see also Remark 2.3.11). Remember that the context of universal cartesian-marked oplax cocylinders includes the one of weighted 2-colimits, by Section 1.1.

We then extend this result of preservation of 2-colimits for τ^* to more general 2-categories other than $\mathcal{C}at$: firstly to prestacks (Proposition 2.4.5) and then to any finitely complete 2-category with a dense generator (Theorem 2.4.6). This will be crucial in Section 3.2 to reduce the study of a 2-classifier to dense generators.

Throughout this section, we fix $\mathcal{K}$ a 2-category with pullbacks along split opfibrations and comma objects. We also fix a choice of all pullbacks. Reading the proofs, it will be clear that if one is just interested in τ a discrete opfibration, then pullbacks along discrete opfibrations are enough.

Proposition 2.4.1. *Let $\rho\colon \mathcal{J} \to \mathcal{B}$ be a morphism in $\mathcal{K}$. Then taking comma objects along ρ is equivalent to taking (strict 2-)pullbacks along the free Grothendieck opfibration $\partial_1\colon \rho/\mathcal{B} \to \mathcal{B}$ on ρ, which is split.*

$$\begin{tikzcd}
\mathcal{P} \arrow[r] \arrow[d] & \mathcal{J} \arrow[d, "\rho"] \arrow[dl, Rightarrow, shorten=4mm, "\text{comma}"'] \\
\mathcal{A} \arrow[r, "F"'] & \mathcal{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P} \arrow[r, "\partial_1^* F"] \arrow[d, "F^*\partial_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \rho/\mathcal{B} \arrow[r, "\partial_0"] \arrow[d, "\partial_1"'] & \mathcal{J} \arrow[d, "\rho"] \arrow[dl, Rightarrow, shorten=4mm, "\text{comma}"'] \\
\mathcal{A} \arrow[r, "F"'] & \mathcal{B} \arrow[r, equal] & \mathcal{B}
\end{tikzcd}$$

Proof. It suffices to check that the diagram on the right above has the universal property of the comma object on the left, for every $F\colon \mathcal{A} \to \mathcal{B}$ in $\mathcal{K}$. It is known that ∂_1 is the free Grothendieck opfibration on ρ and that it is split, see Street's [41]. □

Proposition 2.4.2. *Let $\tau\colon \mathcal{E} \to \mathcal{B}$ be a split Grothendieck opfibration in $\mathcal{K}$. Then pulling back along τ extends to a 2-functor*

$$\tau^*\colon \mathcal{K}/_{\text{lax}}\,\mathcal{B} \to \mathcal{K}/_{\text{lax}}\,\mathcal{E}.$$

Moreover, considering the canonical $\mathcal{F}$-category structure on the lax slice described in Remark 2.2.5 (that is, the loose part of the lax slice is itself and its tight part is given by the strict slice), τ^* is an $\mathcal{F}$-functor.

Proof. Given a morphism $F\colon \mathcal{A} \to \mathcal{B}$ in $\mathcal{K}$, we define $\tau^* F$ as the upper morphism of the chosen pullback square in $\mathcal{K}$ on the left below. Given then a morphism in $\mathcal{K}/_{\text{lax}}\,\mathcal{B}$ as in the middle below, we can lift the 2-cell in $\mathcal{K}$ on the right below

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "\tau^* F"] \arrow[d, "F^*\tau"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "\tau"] \\
\mathcal{A} \arrow[r, "F"'] & \mathcal{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A} \arrow[rr, "\widehat{\alpha}"] \arrow[dr, "F"'] & \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}" very near start] & \mathcal{A}' \arrow[dl, "F'"] \\
& \mathcal{B} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P} \arrow[rr, "\tau^* F"] \arrow[d, "F^*\tau"'] & & \mathcal{E} \arrow[d, "\tau"] \\
\mathcal{A} \arrow[rr, "F"] \arrow[dr, "\widehat{\alpha}"'] & \arrow[d, phantom, "\Downarrow\alpha" very near start] & \mathcal{B} \\
& \mathcal{A}' \arrow[ur, "F'"'] &
\end{tikzcd}$$

along the Grothendieck opfibration τ, producing the chosen cartesian 2-cell $\tau^*\alpha\colon \tau^*F \Rightarrow V\colon \mathcal{P} \to \mathcal{E}$ (in the cleavage) with $\tau \circ V = F' \circ \widehat{\alpha} \circ F^*\tau$ and $\tau * \tau^*\alpha = \alpha * F^*\tau$. Using then the universal property of the pullback $\mathcal{P}'$ of τ and F' we can factorize V through τ^*F', obtaining a morphism $\widehat{\tau^*\alpha}\colon \mathcal{P} \to \mathcal{P}'$. We define $\tau^*\alpha$ to be the upper triangle in the following commutative solid:

$$\begin{tikzcd}
\mathcal{P} \arrow[rrd, bend left, "\tau^*F"] \arrow[rd, "\widehat{\tau^*\alpha}"'] \arrow[dd, "F^*\tau"'] & \arrow[d, phantom, "\Downarrow \tau^*\alpha"] & \\
& \mathcal{P}' \arrow[r, "\tau^*F'"] \arrow[dd, "(F')^*\tau"'] & \mathcal{E} \arrow[dd, "\tau"] \\
\mathcal{A} \arrow[rrd, bend left, "F"] \arrow[rd, "\widehat{\alpha}"'] & \arrow[d, phantom, "\Downarrow \alpha"] & \\
& \mathcal{A}' \arrow[r, "F'"'] & \mathcal{B}
\end{tikzcd}$$

It is straightforward to check that τ^* is functorial, since τ is a split Grothendieck opfibration. For this, remember that a cleavage is the choice of a left adjoint to $\eta_\tau\colon \mathcal{E} \to \tau/_{\mathcal{B}}$, where the latter is the morphism induced by the identity 2-cell on τ. Such a choice then determines the liftings of the Grothendieck opfibrations $(\tau \circ -)\colon \mathcal{K}(\mathcal{X}, \mathcal{E}) \to \mathcal{K}(\mathcal{X}, \mathcal{B})$ in $\mathcal{C}at$ that we have for every $\mathcal{X} \in \mathcal{K}$, by using the universal property of $\tau/_{\mathcal{B}}$ (to factorize the 2-cells we want to lift). So notice that taking $(\widehat{\alpha}, \alpha)\colon F \to F'$ as above and $(\widehat{\beta}, \beta)\colon F' \to F''$ in $\mathcal{K}/_{\text{lax}}\,\mathcal{B}$ we have that the chosen cartesian lifting of $\beta * (\widehat{\alpha} \circ F^*\tau) = (\beta * (F')^*\tau) * \widehat{\tau^*\alpha}$ needs to coincide with $\tau^*\beta * \widehat{\tau^*\alpha}$.

Given a 2-cell $\delta\colon (\widehat{\alpha}, \alpha) \to (\widehat{\beta}, \beta)\colon F \to F'$ in $\mathcal{K}/_{\text{lax}}\,\mathcal{B}$, we define $\tau^*\delta$ to be the chosen cartesian lifting of the 2-cell $\delta * F^*\tau$ along the Grothendieck opfibration $(F')^*\tau$, where the latter has the cleavage induced by the cleavage of τ. Using then that τ is split, together with the definition of 2-cell in $\mathcal{K}/_{\text{lax}}\,\mathcal{B}$ and the universal property of the pullback $\mathcal{P}'$, we obtain that the codomain of $\tau^*\delta$ is indeed $\widehat{\tau^*\beta}$ and that $\tau^*\delta$ is a 2-cell in $\mathcal{K}/_{\text{lax}}\,\mathcal{E}$ from $\tau^*\alpha$ to $\tau^*\beta$. It is then straightforward to check that τ^* is a 2-functor, using that $(F')^*\tau$ is split.

Finally, consider on both $\mathcal{K}/_{\text{lax}}\,\mathcal{B}$ and $\mathcal{K}/_{\text{lax}}\,\mathcal{E}$ the canonical $\mathcal{F}$-category structure described in Remark 2.2.5. Since the lifting of an identity 2-cell through a split Grothendieck opfibration is always an identity, then τ^* is an $\mathcal{F}$-functor. $\square$

Theorem 2.4.3. *Let $\tau\colon \mathcal{E} \to \mathcal{B}$ be a split Grothendieck opfibration in $\mathcal{C}at$. Then the $\mathcal{F}$-functor*

$$\tau^*\colon \mathcal{C}at\,/_{\text{lax}}\,\mathcal{B} \to \mathcal{C}at\,/_{\text{lax}}\,\mathcal{E}$$

has a strict right-semi-lax loose right $\mathcal{F}$-adjoint.

As a consequence, by Theorem 2.3.10, τ^* preserves all the universal cartesian-marked oplax cocylinders for an $\mathcal{F}$-diagram which have tight λ-components.

Proof. We use Proposition 2.3.3 (universal mapping property that characterizes a lax adjunction) to build a right-semi-lax right adjoint $\tau_*\colon \mathcal{C}at\,/_{\text{lax}}\,\mathcal{E} \to \mathcal{C}at\,/_{\text{lax}}\,\mathcal{B}$ to τ^*. We will generalize the ideas of the construction of a right adjoint to the pullback between strict slices (see Conduché's [13] and Palmgren's [39]), using that τ is Conduché (being a Grothendieck opfibration). To suit the lax context, we will fill the relevant triangles with general 2-cells.

So, given a morphism $f\colon X \to X'$ in $\mathcal{B}$, we will need to consider the following pullbacks in $\mathcal{C}at$

$$\begin{tikzcd}
\tau^{-1}(X) \arrow[r, "U"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "\tau"] \\
\mathit{1} \arrow[r, "X"'] & \mathcal{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
\tau^{-1}(f) \arrow[r, "V"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "\tau"] \\
\mathit{2} \arrow[r, "f"'] & \mathcal{B}
\end{tikzcd}
\qquad
\begin{tikzcd}
\tau^{-1}(X) \arrow[r, "\widetilde{0}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \tau^{-1}(f) \arrow[r, "V"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "\tau"] \\
\mathit{1} \arrow[r, "0"'] & \mathit{2} \arrow[r, "f"'] & \mathcal{B}
\end{tikzcd}$$

Notice that $\tau^{-1}(X)$ is the fibre of τ over X. Whereas $\tau^{-1}(f)$ has three kinds of morphisms, namely the morphisms in $\mathcal{E}$ over id_X, those over $\mathrm{id}_{X'}$ and those over $f\colon X \to X'$.

Given a functor $H\colon \mathcal{D} \to \mathcal{E}$, we define $\tau_* H$ as the projection on the first component $\mathrm{pr}_1\colon \mathcal{H} \to \mathcal{B}$, where the category $\mathcal{H}$ is defined as follows:

an object is a pair $(X, (\widehat{\alpha}, \alpha))$ with $X \in \mathcal{B}$ and $(\widehat{\alpha}, \alpha)$ a morphism in $\mathcal{C}at\,/_{\text{lax}}\,\mathcal{E}$

$$\begin{tikzcd}
\tau^{-1}(X) \arrow[rr, "\widehat{\alpha}"] \arrow[dr, "U"'] \arrow[rr, phantom, shift right=4ex, "\overset{\alpha}{\Longrightarrow}"] & & \mathcal{D} \arrow[dl, "H"] \\
& \mathcal{E} &
\end{tikzcd}$$

a morphism $(X,(\widehat{\alpha},\alpha)) \to (X',(\widehat{\beta},\beta))$ is a pair $(f,(\widehat{\Phi},\Phi))$ with $f\colon X \to X'$ in $\mathcal{B}$ and $(\widehat{\Phi},\Phi)$ a morphism in $\mathcal{C}at\,/_{\text{lax}}\,\mathcal{E}$ as on the left below such that $\Phi * \widetilde{0} = \alpha$ and $\Phi * \widetilde{1} = \beta$

$$\begin{tikzcd}
\tau^{-1}(f) \arrow[rr, "\widehat{\Phi}"] \arrow[rd, "V"'] & \arrow[d, phantom, "\overset{\Phi}{\Longrightarrow}"] & \mathcal{D} \arrow[ld, "H"] \\
& \mathcal{E} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\tau^{-1}(X) \arrow[rrr, bend left, "\widehat{\alpha}"] \arrow[r, "\widetilde{0}"] \arrow[rrd, bend right, "U"'] & \tau^{-1}(f) \arrow[rr, "\widehat{\Phi}"] \arrow[rd, "V"'] & \arrow[d, phantom, "\overset{\Phi}{\Rightarrow}"] & \mathcal{D} \arrow[ld, "H"] \\
& & \mathcal{E} &
\end{tikzcd}$$

So the only data of $(\widehat{\Phi},\Phi)$ that are not already determined by its domain and its codomain are the assignments of $\widehat{\Phi}$ on the morphisms of $\tau^{-1}(f)$ that correspond to morphisms $g\colon E \to E'$ in $\mathcal{E}$ over $f\colon X \to X'$, and we have that such assignments produce a morphism in $\mathcal{H}$ precisely when they organize into a functor $\widehat{\Phi}$ such that for every $g\colon E \to E'$ in $\mathcal{E}$ over $f\colon X \to X'$ the following square is commutative:

$$\begin{tikzcd}
E \arrow[r, "\alpha_E"] \arrow[d, "g"'] & H(\widehat{\alpha}(E)) \arrow[d, "H(\widehat{\Phi}(0\to 1,g))"] \\
E' \arrow[r, "\beta_{E'}"'] & H(\widehat{\beta}(E'))
\end{tikzcd}$$

the identity on $(X,(\widehat{\alpha},\alpha))$ is the pair $(\mathrm{id}_X,(\widehat{\mathrm{id}_\alpha},\mathrm{id}_\alpha))$ determined by

$$\widehat{\mathrm{id}_\alpha}(0 \to 1, g) = \widehat{\alpha}(g);$$

the composition of $(f,(\widehat{\Phi},\Phi))$ *and* $(f',(\widehat{\Phi'},\Phi'))$ has first component $f' \circ f$ and second component determined by sending $g\colon E \to E'$ over $X \xrightarrow{f} X' \xrightarrow{f'} X''$ to

$$\widehat{\Phi'}(1 \to 2, g_2) \circ \widehat{\Phi}(0 \to 1, g_1)$$

where $E \xrightarrow{g_1} Z \xrightarrow{g_2} E'$ is a factorization of g over $X \xrightarrow{f} X' \xrightarrow{f'} X''$ obtained by the fact that τ is a Conduché functor. Notice that such assignment is independent from the choice of the factorization because $\widehat{\Phi}$ and $\widehat{\Phi'}$ need to agree on any morphism $(1 = 1, h)$, since the codomain of the former, that equals the domain of the latter, determines their images. Moreover it is immediate to check that this gives a morphism in $\mathcal{H}$, pasting the two commutative squares for g_1 and g_2.

It is straightforward to check that $\mathcal{H}$ is a category, and $\tau_* H$ is then surely a functor.

We define the counit ε on H as the morphism in $\mathcal{C}at /_{\text{lax}}\, \mathcal{E}$

$$\begin{tikzcd}
\mathcal{N} \arrow[rr, "\widehat{\varepsilon_H}"] \arrow[rd, "\tau^*\tau_* H"'] & \overset{\varepsilon_H}{\Longrightarrow} & \mathcal{D} \arrow[ld, "H"] \\
& \mathcal{E} &
\end{tikzcd}$$

given by the evaluation, as follows. An object of $\mathcal{N}$ is a pair $((X,(\widehat{\alpha},\alpha)),E)$ with $(X,(\widehat{\alpha},\alpha)) \in \mathcal{H}$ and $E \in \tau^{-1}(X)$, whereas a morphism in $\mathcal{N}$ is a pair $((f,(\widehat{\Phi},\Phi)),g)$ with $(f,(\widehat{\Phi},\Phi))$ a morphism in $\mathcal{H}$ and $g\colon E \to E'$ in $\mathcal{E}$ over f. We define

$$\widehat{\varepsilon_H}((X,(\widehat{\alpha},\alpha)),E) \coloneqq \widehat{\alpha}(E)$$

$$\widehat{\varepsilon_H}((f,(\widehat{\Phi},\Phi)),g) \coloneqq \widehat{\Phi}(0 \to 1, g)$$

$$(\varepsilon_H)_{((X,(\widehat{\alpha},\alpha)),E)} \coloneqq \alpha_E$$

Then $\widehat{\varepsilon_H}$ is readily seen to be a functor, and ε_H is a natural transformation thanks to the commutative square that a morphism in $\mathcal{H}$ needs to satisfy. Notice, however, that ε_H is not tight, so that we can only hope to obtain a loose adjunction.

We prove that ε_H is universal in the lax sense. So take a functor $F\colon \mathcal{A} \to \mathcal{B}$ and a morphism in $\mathcal{C}at /_{\text{lax}}\, \mathcal{E}$

$$\begin{tikzcd}
\mathcal{P} \arrow[rr, "\widehat{\gamma}"] \arrow[rd, "\tau^* F"'] & \overset{\gamma}{\Longrightarrow} & \mathcal{D} \arrow[ld, "H"] \\
& \mathcal{E} &
\end{tikzcd}$$

Wishing to obtain a right-semi-lax loose $\mathcal{F}$-adjunction, we search for a tight morphism in $\mathcal{C}at /_{\text{lax}}\, \mathcal{B}$ as on the left below that satisfies the equality of diagrams on the right

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\widehat{\overline{\gamma}}"] \arrow[rd, "F"'] & \overset{\overline{\gamma}}{=} & \mathcal{H} \arrow[ld, "\tau_* H"] \\
& \mathcal{B} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P} \arrow[r, "\widehat{\tau^*\overline{\gamma}}"] \arrow[rd, bend right=40, "\tau^* F"'] \arrow[rd, phantom, near start, "="] & \mathcal{N} \arrow[r, "\widehat{\varepsilon_H}"] \arrow[d, "\tau^*\tau_* H" description] \arrow[rd, phantom, "\overset{}{\underset{\varepsilon_H}{\Nearrow}}"] & \mathcal{D} \arrow[ld, bend left=40, "H"] \\
& \mathcal{E} &
\end{tikzcd}
=
\begin{tikzcd}
\mathcal{P} \arrow[rr, "\widehat{\gamma}"] \arrow[rd, "\tau^* F"'] & \overset{\gamma}{\Longrightarrow} & \mathcal{D} \arrow[ld, "H"] \\
& \mathcal{E} &
\end{tikzcd}
\tag{2.7}$$

so that we can take $\lambda_{(\widehat{\gamma},\gamma)} = \mathrm{id}$. Given $a\colon A \to A'$ in $\mathcal{A}$, we have $\widehat{\overline{\gamma}}(A) = (F(A),(\widehat{\alpha},\alpha))$ and $\widehat{\overline{\gamma}}(a) = (F(a),(\widehat{\Phi},\Phi))$ with

$$\begin{tikzcd}
\tau^{-1}(F(A)) \arrow[rr, "\widehat{\alpha}"] \arrow[dr, "U"'] \arrow[rr, phantom, shift right=5ex, "\overset{\alpha}{\Longrightarrow}"] & & \mathcal{D} \arrow[dl, "H"] \\
& \mathcal{E} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\tau^{-1}(F(a)) \arrow[rr, "\widehat{\Phi}"] \arrow[dr, "V"'] \arrow[rr, phantom, shift right=5ex, "\overset{\Phi}{\Longrightarrow}"] & & \mathcal{D} \arrow[dl, "H"] \\
& \mathcal{E} &
\end{tikzcd}$$

And given $g\colon E \to E'$ in $\mathcal{E}$ over $F(a)\colon F(A) \to F(A')$, then $(a,g)\colon (A,E) \to (A',E')$ is a morphism in $\mathcal{P}$. So we want to define

$$\widehat{\alpha}(E) = \widehat{\varepsilon_H}(\widehat{\overline{\gamma}}(A),E) = \widehat{\varepsilon_H}(\widehat{\tau^*\overline{\gamma}}(A,E)) = \widehat{\gamma}(A,E)$$

$$\widehat{\Phi}(0 \to 1, g) = \widehat{\varepsilon_H}(\widehat{\overline{\gamma}}(a),g) = \widehat{\varepsilon_H}(\widehat{\tau^*\overline{\gamma}}(a,g)) = \widehat{\gamma}(a,g)$$

$$\alpha_E = (\varepsilon_H)_{(\widehat{\overline{\gamma}}(A),E)} = (\varepsilon_H)_{(\widehat{\tau^*\overline{\gamma}}(A,E))} = \gamma_{(A,E)}$$

Taking a morphism $g'\colon E \to E'$ in $\tau^{-1}(F(A))$,

$$\widehat{\alpha}(g') = \widehat{\mathrm{id}}_{\alpha}(0 \to 1, g') = \widehat{\varepsilon_H}(\widehat{\overline{\gamma}}(\mathrm{id}_A),g') = \widehat{\varepsilon_H}(\widehat{\tau^*\overline{\gamma}}(\mathrm{id}_A,g')) = \widehat{\gamma}(\mathrm{id}_A,g')$$

It is straightforward to check that this defines a functor $\widehat{\overline{\gamma}}$ as in the left part of equation (2.7); $\widehat{\overline{\gamma}}$ satisfies the equality in the right part of the same equation by construction. Take then $\lambda_{(\widehat{\gamma},\gamma)} = \mathrm{id}$.

Given another morphism in $\mathcal{C}at /_{\mathrm{lax}} \mathcal{B}$

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\widehat{\xi}"] \arrow[dr, "F"'] \arrow[rr, phantom, shift right=5ex, "\overset{\xi}{\Longrightarrow}"] & & \mathcal{H} \arrow[dl, "\tau_* H"] \\
& \mathcal{B} &
\end{tikzcd}$$

and a 2-cell $\Xi\colon (\widehat{\gamma},\gamma) \Rightarrow (\widehat{\varepsilon_H},\varepsilon_H) \circ (\widehat{\tau^*\xi},\tau^*\xi)$ in $\mathcal{C}at /_{\mathrm{lax}} \mathcal{E}$, we prove that there is a unique 2-cell $\delta\colon (\widehat{\overline{\gamma}},\mathrm{id}) \Rightarrow (\widehat{\xi},\xi)$ in $\mathcal{C}at /_{\mathrm{lax}} \mathcal{B}$ such that

$$(\widehat{\varepsilon_H},\varepsilon_H) * \tau^*\delta \circ \mathrm{id} = \Xi. \tag{2.8}$$

In order for δ to be a 2-cell $(\widehat{\overline{\gamma}},\mathrm{id}) \Rightarrow (\widehat{\xi},\xi)$,

$$\tau_* H * \delta = \xi.$$

Whereas the request of equation (2.8) translates as

$$\widehat{\varepsilon_H} * \tau^*\delta = \Xi.$$

So, for every $A \in \mathcal{A}$, the component $\delta_A \colon \widehat{\widehat{\gamma}}(A) \to \widehat{\xi}(A)$ needs to be the morphism in $\mathcal{H}$ with first component ξ_A and second component

$$\begin{tikzcd}
\tau^{-1}(\xi_A) \arrow[rr, "\widehat{\delta_A}"] \arrow[dr, "V"'] & \arrow[d, phantom, "\overset{\delta_A}{\Longrightarrow}"] & \mathcal{D} \arrow[dl, "H"] \\
& \mathcal{E} &
\end{tikzcd}$$

given as follows. For every $g \colon E \to E'$ over ξ_A, factorizing g as the cartesian morphism $\mathrm{Cart}\,(\xi_A, E)$ in the cleavage of the Grothendieck opfibration τ over ξ_A to E followed by the unique induced vertical morphism g_{vert},

$$\begin{aligned}
&\widehat{\delta_A}(0 \to 1, g) = \widehat{\delta_A}(1 = 1, g_{\mathrm{vert}}) \circ \widehat{\delta_A}(0 \to 1, \mathrm{Cart}\,(\xi_A, E)) = \\
&= \widehat{\widehat{\xi}(A)}(g_{\mathrm{vert}}) \circ \widehat{\varepsilon_H}(\delta_A, \mathrm{Cart}\,(\xi_A, E)) = \widehat{\widehat{\xi}(A)}(g_{\mathrm{vert}}) \circ \widehat{\varepsilon_H}\,((\tau^*\delta)_{A,E}) = \\
&= \widehat{\widehat{\xi}(A)}(g_{\mathrm{vert}}) \circ \Xi_{A,E}
\end{aligned}$$

It is straightforward to prove that δ_A is a morphism in $\mathcal{H}$ and that δ is a natural transformation, using the uniqueness of the morphisms induced by cartesian liftings. δ is then a 2-cell in $\mathcal{C}at\,/_{\mathrm{lax}}\,\mathcal{B}$ such that

$$(\widehat{\varepsilon_H}, \varepsilon_H) * \tau^*\delta \circ \mathrm{id} = \Xi$$

by construction.

Considering $(\widehat{\gamma}, \gamma) = (\widehat{\varepsilon_H}, \varepsilon_H)$, we immediately see that we obtain $\widehat{\widehat{\varepsilon_H}} = \mathrm{id}$, because $(\widehat{\varepsilon_H}, \varepsilon_H)$ is the evaluation.

Moreover, for every functor $F \colon \mathcal{A} \to \mathcal{B}$ and morphism in $\mathcal{C}at\,/_{\mathrm{lax}}\,\mathcal{E}$

$$\begin{tikzcd}
\mathcal{P} \arrow[rr, "\widehat{\gamma}"] \arrow[dr, "\tau^*F"'] & \arrow[d, phantom, "\overset{\gamma}{\Longrightarrow}"] & \mathcal{D} \arrow[dl, "H"] \\
& \mathcal{E} &
\end{tikzcd}$$

we prove that

$$\overline{((\widehat{\gamma}, \gamma) \circ (\widehat{\varepsilon_H}, \varepsilon_H))} \circ \overline{\mathrm{id}_{\tau^*F}} = \overline{(\widehat{\gamma}, \gamma)}. \tag{2.9}$$

$\overline{\mathrm{id}_{\tau^*F}} = (\widehat{\eta_F}, \mathrm{id})$, that will be the unit η_F, is such that, for every $a\colon A \to A'$ in $\mathcal{A}$, morphism g' in $\tau^{-1}(F(A))$ and $g\colon E \to E'$ in $\mathcal{E}$ over $F(a)\colon F(A) \to F(A')$,

$$\widehat{\widehat{\eta_F}(A)}(E) = (A, E) \qquad \widehat{\widehat{\eta_F}(A)}(g') = (\mathrm{id}_A, g')$$

$$\widehat{\widehat{\eta_F}(a)}(0 \to 1, g) = (a, g) \qquad \widehat{\eta_F}(A)_E = \mathrm{id}$$

Whereas for a general $(\widehat{\psi}, \psi)\colon G \to H$ in $\boldsymbol{\mathit{Cat}}\,/_{\mathrm{lax}}\ \mathcal{E}$, the morphism

$$\overline{\left((\widehat{\psi}, \psi) \circ (\widehat{\varepsilon_H}, \varepsilon_H)\right)} = (\widehat{\tau_* \psi}, \mathrm{id})$$

will be the action of τ_* on the morphism $(\widehat{\psi}, \psi)$, and is such that $\widehat{\tau_* \psi}$ acts by postcomposing the triangles with $(\widehat{\psi}, \psi)$. Thus equation (2.9) holds.

By Proposition 2.3.3, as λ is always the identity, then τ_* extends to an oplax functor, that can be easily checked to be a 2-functor (it acts by postcomposition), ε extends to a 2-natural transformation, η extends to a lax natural transformation and there exists a modification t such that τ_* is a right-semi-lax right adjoint to τ^*. It is easy to check that t is the identity.

Since $\overline{(\widehat{\gamma}, \gamma)}$ is always tight, then τ_* is an $\mathcal{F}$-functor and η has tight components. It remains to show that η is loose strict/lax $\mathcal{F}$-natural. Given a morphism in $\boldsymbol{\mathit{Cat}}\,/_{\mathrm{lax}}\ \mathcal{B}$

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\widehat{\sigma}"] \arrow[dr, "F"'] & {} \arrow[d, phantom, "\overset{\sigma}{\Longrightarrow}"] & \mathcal{A}' \arrow[dl, "F'"] \\
& \mathcal{B} &
\end{tikzcd}$$

the component on $A \in \mathcal{A}$ of the structure 2-cell $\eta_{(\widehat{\sigma}, \sigma)}$ is the morphism in the domain of $\tau_* \tau^* F'$ with first component $\sigma_A\colon F(A) \to F'(\widehat{\sigma}(A))$ in $\mathcal{B}$ and second component given by

$$\widehat{\eta_{(\widehat{\sigma}, \sigma), A}}(0 \to 1, g) = \widehat{\eta_{F'}}\,\widehat{(\widehat{\sigma}(A)})(g_{\mathrm{vert}}) = \left(\mathrm{id}_{\widehat{\sigma}(A)}, g_{\mathrm{vert}}\right).$$

When $(\widehat{\sigma}, \sigma)$ is tight, so when the 2-cell σ is the identity, then $g = g_{\mathrm{vert}}$ because τ has a normal cleavage, and we find $\eta_{(\widehat{\sigma}, \sigma)} = \mathrm{id}$. Thus η is strict/lax $\mathcal{F}$-natural. We conclude that τ_* is a strict right-semi-lax loose right $\mathcal{F}$-adjoint to τ^*. □

Remark 2.4.4. We have actually proved in Theorem 2.4.3 that τ_* sends every morphism in $\mathcal{C}at/_{\text{lax}}\ \mathcal{E}$ to a tight one in $\mathcal{C}at/_{\text{lax}}\ \mathcal{B}$. So $\tau_*\colon \mathcal{C}at/_{\text{lax}}\ \mathcal{E} \to \mathcal{C}at/_{\text{lax}}\ \mathcal{B}$ is still an $\mathcal{F}$-functor if we take the trivial $\mathcal{F}$-category structure on the domain, i.e. taking everything to be tight, and the canonical one in the codomain. Of course, with such a choice of $\mathcal{F}$-category structures, τ^* remains an $\mathcal{F}$-functor and η remains with tight components. But ε becomes tight, having now tight components trivially.

So we find a strict right-semi-lax (tight) $\mathcal{F}$-adjunction between τ^* and τ_*. But Theorem 2.3.10 does not add anything to the preservation of 2-colimits we have already proved in Theorem 2.4.3, since it would consider strict/oplax $\mathcal{F}$-colimits in an $\mathcal{F}$-category with trivial $\mathcal{F}$-category structure.

We now extend the result of preservation of 2-colimits that we have proved for the 2-functor $\tau^*\colon \mathcal{K}/_{\text{lax}}\ \mathcal{B} \to \mathcal{K}/_{\text{lax}}\ \mathcal{E}$ when $\mathcal{K} = \mathcal{C}at$ (Theorem 2.4.3) to $\mathcal{K} = [\mathcal{L}^{\text{op}}, \mathcal{C}at]$ a 2-category of 2-dimensional presheaves.

Proposition 2.4.5. *Let $\mathcal{L}$ be a small 2-category and let $\tau\colon \mathcal{E} \to \mathcal{B}$ be a split Grothendieck opfibration in $[\mathcal{L}^{\text{op}}, \mathcal{C}at]$. Then the $\mathcal{F}$-functor*

$$\tau^*\colon [\mathcal{L}^{\text{op}}, \mathcal{C}at]/_{\text{lax}}\ \mathcal{B} \to [\mathcal{L}^{\text{op}}, \mathcal{C}at]/_{\text{lax}}\ \mathcal{E}$$

preserves all the universal cartesian-marked oplax cocones for an $\mathcal{F}$-diagram which have tight components.

Proof. Consider a marking $W\colon \mathcal{A}^{\text{op}} \to \mathcal{C}at$ with $\mathcal{A}$ a small 2-category and an $\mathcal{F}$-diagram $H\colon \int W \to [\mathcal{L}^{\text{op}}, \mathcal{C}at]/_{\text{lax}}\ \mathcal{B}$ (that is, a 2-functor that sends every morphism of the kind (f, id) to a triangle filled with an identity). Let then

$$\zeta\colon \Delta 1 \underset{\text{oplax}^{\text{cart}}}{\Longrightarrow} [\mathcal{L}^{\text{op}}, \mathcal{C}at]/_{\text{lax}}\ \mathcal{B}(H(-), C)$$

be a universal cartesian-marked oplax cocone that exhibits $C = \text{oplax}^{\text{cart}}\text{-colim}^{\Delta 1} H$ such that $\zeta_{(A,X)}$ is tight for every $(A, X) \in \int W$ (which means that it is a triangle filled with an identity). We want to prove that $\tau^* \circ \zeta$ is universal as well, exhibiting $\tau^*(C) = \text{oplax}^{\text{cart}}\text{-colim}^{\Delta 1}(\tau^* \circ H)$.

Since the $\zeta_{(A,X)}$'s are all cartesian, as they are tight, and τ^* is an $\mathcal{F}$-functor, by Theorem 2.1.21 (the domain 2-functor from a lax slice is a 2-colim-fibration), we know that $\operatorname{dom}\colon [\mathcal{L}^{\mathrm{op}}, \mathcal{C}at] /_{\text{lax}}\, \mathcal{E} \to [\mathcal{L}^{\mathrm{op}}, \mathcal{C}at]$ reflects the universality of $\tau^* \circ \zeta$. But the 2-functors $(-)(L)\colon [\mathcal{L}^{\mathrm{op}}, \mathcal{C}at] \to \mathcal{C}at$ of evaluation on $L \in \mathcal{L}$ jointly reflect 2-colimits (as 2-colimits in 2-presheaves are calculated pointwise). Therefore, in order to prove that $\tau^* \circ \zeta$ is universal, it suffices to show that, for every $L \in \mathcal{L}$, the cartesian-marked oplax cocone $(-)(L) \circ \operatorname{dom} \circ \tau^* \circ \zeta$ is universal. Notice now that the diagram of 2-functors

$$\begin{tikzcd}
{[\mathcal{L}^{\mathrm{op}}, \mathcal{C}at] /_{\text{lax}}\, \mathcal{B}} \arrow[r, "\tau^*"] \arrow[d, "(-)_L"] & {[\mathcal{L}^{\mathrm{op}}, \mathcal{C}at] /_{\text{lax}}\, \mathcal{E}} \arrow[r, "\operatorname{dom}"] \arrow[d, "(-)_L"] & {[\mathcal{L}^{\mathrm{op}}, \mathcal{C}at]} \arrow[d, "(-)(L)"] \\
\mathcal{C}at /_{\text{lax}}\, \mathcal{B}(L) \arrow[r, "(\tau_L)^*"'] & \mathcal{C}at /_{\text{lax}}\, \mathcal{E}(L) \arrow[r, "\operatorname{dom}"'] & \mathcal{C}at
\end{tikzcd}$$

is commutative, where $(-)_L$ is the $\mathcal{F}$-functor that takes components on L, because pullbacks in $[\mathcal{L}^{\mathrm{op}}, \mathcal{C}at]$ are calculated pointwise and the components of the liftings along τ are the liftings of the components of τ. Indeed every component τ_L of τ is a split Grothendieck opfibration in $\mathcal{C}at$ because $\tau \circ -\colon [\mathcal{L}^{\mathrm{op}}, \mathcal{C}at]\,(\mathrm{y}(L), \mathcal{E}) \to [\mathcal{L}^{\mathrm{op}}, \mathcal{C}at]\,(\mathrm{y}(L), \mathcal{B})$ is so, taking on the former the cleavage induced by the latter. And since a cleavage for τ is the choice of a left adjoint to $\eta_\tau\colon \mathcal{E} \to \tau/_{\mathcal{B}}$ (where the latter is the morphism induced by the identity 2-cell on τ), the cleavages determined on the Grothendieck opfibrations $(\tau \circ -)\colon [\mathcal{L}^{\mathrm{op}}, \mathcal{C}at]\,(\mathcal{X}, \mathcal{E}) \to [\mathcal{L}^{\mathrm{op}}, \mathcal{C}at]\,(\mathcal{X}, \mathcal{B})$ in $\mathcal{C}at$ (by applying the universal property of $\tau/_{\mathcal{B}}$) are compatible.

We prove that $\operatorname{dom} \circ (\tau_L)^* \circ (-)_L \circ \zeta$ is universal. We have that $(-)_L \circ \zeta$ is universal because it suffices to check that $\operatorname{dom} \circ (-)_L \circ \zeta = (-)(L) \circ \operatorname{dom} \circ \zeta$ is so, by Theorem 2.1.21, as ζ has tight components and $(-)_L$ is an $\mathcal{F}$-functor. And dom preserves the universality of ζ by Theorem 2.3.12 (thanks to the hypotheses), while $(-)(L)$ preserves every 2-colimit. Then $\operatorname{dom} \circ (\tau_L)^* \circ (-)_L \circ \zeta$ is universal applying Theorem 2.4.3 and Theorem 2.3.12, thanks to the hypothesis and to the fact that both $(-)_L$ and $(\tau_L)^*$ are $\mathcal{F}$-functors. □

We conclude extending again the result of preservation of 2-colimits for $\tau^*\colon \mathcal{K}/_{\text{lax}}\, \mathcal{B} \to \mathcal{K}/_{\text{lax}}\, \mathcal{E}$, to $\mathcal{K}$ endowed with a dense generator. For this, we will need to restrict to relatively absolute 2-colimits. But remember that any object of $\mathcal{K}$ can be expressed as a relatively absolute 2-colimit of dense generators, so that our assumption is not much restrictive in practice. We take Kelly's [28, Chapter 5] as the main reference for dense generators; we will recall them in Section 3.1.

Theorem 2.4.6. *Let $J\colon \mathcal{L} \to \mathcal{K}$ be a fully faithful dense 2-functor. Consider $\tau\colon \mathcal{E} \to \mathcal{B}$ a split Grothendieck opfibration in $\mathcal{K}$. Then the $\mathcal{F}$-functor*

$$\tau^*\colon \mathcal{K}/_{\text{lax}}\ \mathcal{B} \to \mathcal{K}/_{\text{lax}}\ \mathcal{E}$$

preserves all the universal cartesian-marked oplax cocones for an $\mathcal{F}$-diagram which have tight components and whose domain is J-absolute.

Proof. Let $\mathcal{A}$ be a small 2-category and consider a marking $W\colon \mathcal{A}^{\text{op}} \to \mathcal{C}at$ and an $\mathcal{F}$-diagram $H\colon \int W \to \mathcal{K}/_{\text{lax}}\ \mathcal{B}$. Let then

$$\zeta\colon \Delta 1 \underset{\text{oplax}^{\text{cart}}}{\Longrightarrow} \mathcal{K}/_{\text{lax}}\ \mathcal{B}\,(H(-), C)$$

be a universal cartesian-marked oplax cocone that exhibits $C = \text{oplax}^{\text{cart}}\text{-colim}^{\Delta 1} H$ such that $\zeta_{(A,X)}$ is tight for every $(A, X) \in \int W$. Assume also that $\text{dom} \circ \zeta$ is J-absolute, i.e. preserved by $\widetilde{J}\colon \mathcal{K} \to [\mathcal{L}^{\text{op}}, \mathcal{C}at]$. Notice that $\text{dom} \circ \zeta$ is indeed universal by Theorem 2.3.12. We want to prove that $\tau^* \circ \zeta$ is universal as well, exhibiting $\tau^*(C) = \text{oplax}^{\text{cart}}\text{-colim}^{\Delta 1}(\tau^* \circ H)$.

Since the $\zeta_{(A,X)}$'s are all cartesian (as they are tight) and τ^* is an $\mathcal{F}$-functor, by Theorem 2.1.21, we know that $\text{dom}\colon \mathcal{K}/_{\text{lax}}\ \mathcal{E} \to \mathcal{K}$ reflects the universality of $\tau^* \circ \zeta$. Moreover, by definition of dense functor, $\widetilde{J}$ is fully faithful and hence reflects any 2-colimit; and the 2-functors $(-)(L)\colon [\mathcal{L}^{\text{op}}, \mathcal{C}at] \to \mathcal{C}at$ of evaluation on $L \in \mathcal{L}$ jointly reflect 2-colimits. Therefore, in order to prove that $\tau^* \circ \zeta$ is universal, it suffices to show that, for every $L \in \mathcal{L}$, the cartesian-marked oplax cocone $(-)(L) \circ \widetilde{J} \circ \text{dom} \circ \tau^* \circ \zeta$ is universal.

Notice now that the diagram of 2-functors

$$\begin{tikzcd}
\mathcal{K}/_{\text{lax}}\,\mathcal{B} \arrow[r, "\tau^*"] \arrow[d, "\widetilde{J}/_{\text{lax}}"] & \mathcal{K}/_{\text{lax}}\,\mathcal{E} \arrow[r, "\text{dom}"] \arrow[d, "\widetilde{J}/_{\text{lax}}"] & \mathcal{K} \arrow[d, "\widetilde{J}"] \\
{[\mathcal{L}^{\text{op}}, \mathcal{C}at]}/_{\text{lax}}\,\widetilde{J}(\mathcal{B}) \arrow[d, "(-)_L"] & {[\mathcal{L}^{\text{op}}, \mathcal{C}at]}/_{\text{lax}}\,\widetilde{J}(\mathcal{E}) \arrow[r, "\text{dom}"] \arrow[d, "(-)_L"] & {[\mathcal{L}^{\text{op}}, \mathcal{C}at]} \arrow[d, "(-)(L)"] \\
\mathcal{C}at\,/_{\text{lax}}\,\widetilde{J}(\mathcal{B})(L) \arrow[r, "(\tau\circ -)^*"'] & \mathcal{C}at\,/_{\text{lax}}\,\widetilde{J}(\mathcal{E})(L) \arrow[r, "\text{dom}"'] & \mathcal{C}at
\end{tikzcd}$$

is commutative, where $\widetilde{J}/_{\text{lax}}$ is the $\mathcal{F}$-functor that applies $\widetilde{J}$ on morphisms and triangles. Indeed $\mathcal{K}(J(L), -)$ preserves pullbacks, and since a cleavage for τ is the choice of a left adjoint to $\eta_\tau \colon \mathcal{E} \to \tau/_{\mathcal{B}}$, the cleavages determined on the Grothendieck opfibrations $(\tau \circ -) \colon [\mathcal{L}^{\text{op}}, \mathcal{C}at](\mathcal{X}, \mathcal{E}) \to [\mathcal{L}^{\text{op}}, \mathcal{C}at](\mathcal{X}, \mathcal{B})$ in $\mathcal{C}at$ (by applying the universal property of $\tau/_{\mathcal{B}}$) are compatible.

We prove that $\text{dom} \circ (\tau \circ -)^* \circ (-)_L \circ \widetilde{J}/_{\text{lax}} \circ \zeta$ is universal. We have that $\widetilde{J}/_{\text{lax}} \circ \zeta$ is universal, since it suffices to check that $\text{dom} \circ \widetilde{J}/_{\text{lax}} \circ \zeta$ is so, by Theorem 2.1.21, as ζ has tight components and $\widetilde{J}/_{\text{lax}}$ is an $\mathcal{F}$-functor. And $\text{dom} \circ \widetilde{J}/_{\text{lax}} \circ \zeta = \widetilde{J} \circ \text{dom} \circ \zeta$ is universal because $\text{dom} \circ \zeta$ is J-absolute by hypothesis. Then $(-)_L$ preserves the universality of $\widetilde{J}/_{\text{lax}} \circ \zeta$ because $\text{dom} \circ (-)_L = (-)(L) \circ \text{dom}$ does so. Finally, we obtain that $\text{dom} \circ (\tau \circ -)^* \circ (-)_L \circ \widetilde{J}/_{\text{lax}} \circ \zeta$ is universal applying Theorem 2.4.3 and Theorem 2.3.12, thanks to the hypothesis and to the fact that $\widetilde{J}/_{\text{lax}}$, $(-)_L$ and $(\tau \circ -)^*$ are all $\mathcal{F}$-functors. □

3. 2-classifiers via dense generators

This chapter is based on the first half of our [37].

The notion of 2-classifier has been proposed by Weber in [51]. It is a 2-categorical generalization of the concept of subobject classifier and thus the main ingredient of a 2-dimensional notion of elementary topos. In [51], Weber proposes as well a definition of elementary 2-topos. Although 2-dimensional elementary topos theory is still at its beginning, we believe it has a great potential. Indeed, for example, elementary 2-topoi could pave the way towards a 2-categorical logic and offer the right tools to study it. We believe they could also be fruitful for theories of bundles in geometry.

In this chapter, we contribute to expand 2-dimensional elementary topos theory. We substantially reduce the work needed to prove that something is a 2-classifier. This will allow us, in Chapter 5, to present the main part of a proof that Grothendieck 2-topoi are elementary 2-topoi. Weber's idea for 2-categorical classifiers is that, moving to dimension 2, one can and wants to classify morphisms with higher dimensional fibres. So monomorphisms (or subobjects) are upgraded to discrete opfibrations in a 2-category, which have been introduced by Street in [41]. Interestingly, a 2-classifier can also be thought of as a Grothendieck construction inside a 2-category. Indeed the archetypal example of 2-classifier is given by the category of elements (or Grothendieck construction), that exhibits the 2-category $\mathcal{C}at$ of small categories as the archetypal elementary 2-topos. We introduce the notion of *good 2-classifier* in Definition 3.1.12, which captures well-behaved 2-

classifiers. The idea is to keep as classifier a morphism with domain the terminal object, and upgrade the classification process from one regulated by pullbacks to one regulated by comma objects. Good 2-classifiers are closer to the point of view of logic, as they can still be interpreted as an object of generalized truth values together with a chosen "true". Moreover, a classification process regulated by comma objects is sometimes more natural and easier to handle. We also ask good 2-classifiers to classify all discrete opfibrations that satisfy a fixed pullback-stable property P. In our examples, such a P will be the property of having small fibres. Of course, the construction of the category of elements, hosted by $\mathcal{C}at$, is a good 2-classifier, classifying all discrete opfibrations (in $\mathcal{C}at$) with small fibres. A problem with 2-classifiers is that it is quite hard and lengthy to prove that something is a 2-classifier.

We prove that both the conditions of 2-classifier and what gets classified by a 2-classifier can be checked just over the objects that form a dense generator. So that the whole study of a would-be 2-classifier is substantially reduced. We also give a concrete recipe to build the characteristic morphisms (i.e. the morphisms into the universe that encode what gets classified). This is organized in the three Theorems 3.2.2, 3.2.5 and 3.2.8; see also Corollaries 3.2.10 and 3.2.11. Dense generators capture the idea of a family of objects that generate all the other ones via colimits in a nice way. The preeminent example is given by representables in categories of presheaves. To have a hint of the benefits offered by our theorems of reduction to dense generators, we can look at the case of $\mathcal{C}at$. We have that the singleton category alone forms a dense generator of $\mathcal{C}at$. All the major properties of the Grothendieck construction are hence deduced from the trivial observation that everything works well over the singleton category (Example 3.2.13). The proof of our theorems of reduction to dense generators uses our calculus of colimits in 2-dimensional slices, developed in Chapter 2. Recall that such calculus generalizes to dimension 2 the well-known fact that a colimit in a 1-dimensional slice category is precisely the map from the colimit of the domains of the diagram which is induced by the universal property. The calculus is based on the reduction of weighted 2-colimits to cartesian-marked oplax conical ones (Section 1.1), and on $\mathcal{F}$-category theory (recalled in Section 2.2).

In Chapter 5, we will apply our theorems of reduction of the study of 2-classifiers to dense generators to the cases of 2-presheaves, i.e. prestacks, and stacks. Our theorems will offer great benefits there, as they allow us to consider the classification just over representables. We start looking at this in Section 3.3. Yoneda lemma determines up to equivalence the construction of a good 2-classifier in prestacks that classifies all discrete opfibrations with small fibres. We explain how this involves discrete opfibrations over representables, which offer a 2-categorical notion of sieve. Exactly as sieves are a key element for the subobject classifier in presheaves, the 2-dimensional generalization of sieves described above is a key element for the 2-classifier in prestacks. The only problem, described in Section 3.3 is that taking discrete opfibrations over representables only gives a pseudofunctor Ω which is not a 2-functor and that a priori only lands in large categories. Such a problem will be solved by the work of Chapter 4; see also Section 3.3.

Outline of the chapter

In Section 3.1, we recall 2-classifiers (3.1.1) and dense generators (3.1.2). We also introduce good 2-classifiers (Definition 3.1.12).

In Section 3.2, we present a reduction of the study of a 2-classifier to dense generators (Theorems 3.2.2, 3.2.5 and 3.2.8; see also Corollaries 3.2.10 and 3.2.11).

In Section 3.3, we look at the possibility to apply our theorems of reduction to dense generators to produce a good 2-classifier in prestacks that classifies all discrete opfibrations with small fibres. We find a problem that will be solved by the work of Chapter 4.

Notations

Recall the size notations fixed in the Notations chapter (right after the Introduction).

Throughout this chapter, we fix an arbitrary 2-category $\mathcal{L}$ with pullbacks along discrete opfibrations (see Definition 3.1.2), comma objects and terminal object. We also fix a choice of such pullbacks in $\mathcal{L}$ such that the change of base of an identity is always an identity. Following the proofs, it will be clear that some results of this chapter involving 2-classifiers à la Weber work also without assuming comma objects, and that some results involving *good 2-classifiers* (see Definition 3.1.12) work also without assuming pullbacks along discrete opfibrations.

3.1. Preliminaries

In this section, we recall some important concepts and results that we will use throughout the chapter. These include 2-classifiers and dense generators. Moreover, we introduce the notion of *good 2-classifier* (Definition 3.1.12).

In 3.1.1, we recall the notion of 2-classifier. Weber's idea (in [51]) is that a 2-classifier should be a discrete opfibration classifier. The definition of opfibration in a 2-category is due to Street [41], in terms of pseudo-algebras for a 2-monad. It is known that opfibrations in a 2-category can be equivalently defined by representability, as in Weber's [51]. In Definition 3.1.12, we introduce the notion of *good 2-classifier.*

In 3.1.2, we briefly recall from Kelly's [28, Chapter 5] (2-categorical) dense generators and the preeminent example of representables in 2-presheaves. The idea is that every object of a 2-category can be written as a nice colimit of a small family of objects.

3.1.1. 2-classifiers

Weber's idea is to define 2-classifiers by looking at the following well-known equivalent definition of subobject classifier.

Definition 3.1.1. Let $\mathcal{E}$ be a category. A *subobject classifier* is a monomorphism

$i\colon I \hookrightarrow \Omega$ in $\mathcal{E}$ such that for every $F \in \mathcal{E}$ the function

$$G_{i,F}\colon\ \mathrm{Hom}_{\mathcal{E}}(F,\Omega) \to \mathrm{Sub}(F)$$

given by pulling back i is a bijection, where $\mathrm{Sub}(F)$ is the set of subobjects of F. When this holds, I is forced to be the terminal object of $\mathcal{E}$.

Towards a notion of 2-classifier, Weber proposed in [51] to upgrade the concept of monomorphism to the one of discrete opfibration. The idea is that, moving to dimension 2, i.e. increasing by 1 the dimension of the ambient, we want to increase by 1 also the dimension of the fibres of the morphisms to classify. While injective functions have as fibres either the singleton or the empty set, discrete opfibrations have as fibres general sets. Exactly as the notion of injective function extends to the one of monomorphism in any category, the notion of discrete opfibration extends to the one inside any 2-category. This idea is closely connected with that of homotopy level in Voevodsky's univalent foundations, see [46, Chapter 7] and Voevodsky's [49].

Exactly as $\mathcal{Set}$ is the archetypal elementary topos, $\mathcal{Cat}$ needs to be the archetypal elementary 2-topos. And $\mathcal{Cat}$ hosts indeed a nice classification of discrete opfibrations, given by the category of elements (or Grothendieck construction).

Definition 3.1.2 (Street [41], Weber [51])**.** A morphism $s : G \to F$ in $\mathcal{L}$ is a *discrete opfibration in* $\mathcal{L}$ (*over* F) if for every $X \in \mathcal{L}$ the functor

$$\mathcal{L}(X,G) \xrightarrow{s\circ -} \mathcal{L}(X,F)$$

induced between the hom-categories is a discrete opfibration in $\mathcal{Cat}$.

We denote as $\mathcal{DOpFib}_{\mathcal{L}}(F)$ or just as $\mathcal{DOpFib}(F)$ the full subcategory of the strict slice $\mathcal{L}/_{F}$ on the discrete opfibrations over F. That is, a morphism between discrete opfibrations $s\colon G \to F$ and $s'\colon G' \to F$ is a morphism $G \to G'$ that makes the triangle with s and s' commute. Given P a pullback-stable property for discrete opfibrations, we denote as $\mathcal{DOpFib}^{\mathrm{P}}(F)$ the full subcategory of $\mathcal{DOpFib}(F)$ on the discrete opfibrations that satisfy P.

Remark 3.1.3. By definition, a discrete opfibration $s\colon G \to F$ in $\mathcal{L}$ is required to lift every 2-cell $\theta\colon s \circ a \Rightarrow b$ to a unique 2-cell $\overline{\theta}^{a}\colon a \Rightarrow \theta_{*}a$. We can draw the following diagram to say that $s \circ \theta_{*}a = b$ and $s \star \overline{\theta}^{a} = \theta$.

$$\begin{tikzcd}
X \arrow[rr, bend left, "a"{name=U}] \arrow[rr, bend right, "\theta_{*}a"'{name=D}] \arrow[dd, equal] & & G \arrow[dd, "s"] \\
& G \arrow[dr, "s"] & \\
X \arrow[ur, "a"] \arrow[rr, bend right, "b"'{name=B}] & & F
\arrow[from=U, to=D, Rightarrow, "\overline{\theta}^{a}"]
\arrow[from=2-2, to=B, Rightarrow, "\theta"]
\end{tikzcd}$$

Remark 3.1.4. Discrete opfibrations in $\mathcal{L}$ are stable under pullbacks. Indeed $\mathcal{L}(X, -)$ preserves pullbacks (as it preserves all limits, because it is a representable) and discrete opfibrations in $\mathcal{C}at$ are stable under pullbacks.

Remark 3.1.5. Applying Definition 3.1.2 to $\mathcal{L} = \mathcal{C}at$, we obtain a notion that is equivalent to the usual one of discrete opfibration. This is essentially because for $\mathcal{L} = \mathcal{C}at$ it suffices to require the above liftings for $X = 1$. We are then able to lift entire natural transformations θ componentwise. In Proposition 4.2.5, we will extend this idea to the case of 2-presheaves (i.e. prestacks).

Proposition 3.1.6. *Let $p\colon E \to B$ be a discrete opfibration in $\mathcal{L}$. For every $F \in \mathcal{L}$, pulling back p extends to a functor*

$$\mathcal{G}_{p,F}\colon \mathcal{L}(F, B) \to \mathcal{DO}p\mathcal{F}ib(F)\,.$$

Proof. Given a morphism $z\colon F \to B$ in $\mathcal{L}$, consider the chosen pullback in $\mathcal{L}$

$$\begin{tikzcd}
G^{z} \arrow[r, "\widetilde{z}"] \arrow[d, "s^{z}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\
F \arrow[r, "z"'] & B
\end{tikzcd}$$

We define $\mathcal{G}_{p,F}(z)$ to be s^{z}, which is a discrete opfibration in $\mathcal{L}$ by Remark 3.1.4.

Given a 2-cell $\alpha\colon z \Rightarrow z'\colon F \to B$, we induce $\mathcal{G}_{p,F}(\alpha)$ by lifting the 2-cell

$$\begin{tikzcd}
G^{z} \arrow[r, "s^{z}"] & F \arrow[r, bend left, "z"{name=U}] \arrow[r, bend right, "z'"'{name=D}] & B
\arrow[from=U, to=D, Rightarrow, "\alpha"]
\end{tikzcd}$$

to $\widetilde{z}$ along the discrete opfibration p. We obtain a unique 2-cell

$$\begin{tikzcd}
G^z \arrow[r, bend left=40, "\widetilde{z}"] \arrow[r, bend right=40, "v"'] \arrow[r, phantom, "\Downarrow\widetilde{\alpha}"] & E
\end{tikzcd}$$

such that $p \circ v = z' \circ s^z$ and $p\widetilde{\alpha} = \alpha s^z$. We define $\mathcal{G}_{p,F}(\alpha)$ to be the morphism from G^z to $G^{z'}$ induced by s^z and v via the universal property of the pullback $G^{z'}$. We may represent this with the following diagram:

$$\begin{tikzcd}
G^z \arrow[rrd, bend left, "\widetilde{z}"] \arrow[rd, "\mathcal{G}_{p,F}(\alpha)"] \arrow[dd, "s^z"'] \arrow[rr, phantom, "\Downarrow\widetilde{\alpha}"] & & \\
& G^{z'} \arrow[r, "\widetilde{z}'"] \arrow[dd, "s^{z'}"] & E \arrow[dd, "p"] \\
F \arrow[rrd, bend left, "z"] \arrow[rd, equal] \arrow[rr, phantom, "\Downarrow\alpha"] & & \\
& F \arrow[r, "z'"'] & B
\end{tikzcd} \tag{3.1}$$

It is straightforward to show that $\mathcal{G}_{p,F}$ is a functor, using the universal property of the pullback and the uniqueness of the liftings through the discrete opfibration p. □

Notation 3.1.7. Given a morphism $z\colon F \to B$ in $\mathcal{L}$, we will denote as $\mathrm{G}^z_{p,F}$ or just as G^z the domain of the discrete opfibration $\mathcal{G}_{p,F}(z)$. We will also often draw the action of $\mathcal{G}_{p,F}$ on morphisms as in the diagram of equation (3.1). Sometimes, we will denote the functor $\mathcal{G}_{p,F}$ as $\mathcal{G}_p$.

We will also need the following result from our joint work with Caviglia [12]. It is a particular case of a proposition that we will report in Proposition 4.2.9.

Proposition 3.1.8 ([12], Proposition 4.2.9)**.** *The assignment* $F \in \mathcal{L} \mapsto \mathcal{DO}p\mathcal{F}ib\,(F) \in \mathcal{CAT}$ *extends to a pseudofunctor*

$$\mathcal{DO}p\mathcal{F}ib\,(-) : \mathcal{L}^{\mathrm{op}} \to \mathcal{CAT}.$$

Moreover this restricts to a pseudofunctor $\mathcal{DO}p\mathcal{F}ib^{\mathrm{P}}(-)$ *that sends* $F \mapsto \mathcal{DO}p\mathcal{F}ib^{\mathrm{P}}(F)$.

Proof. (Definition of the assignment). On the underlying category of $\mathcal{L}^{\mathrm{op}}$, we define $\mathcal{DO}p\mathcal{F}ib\,(-)$ as the restriction of the pseudofunctor given by the pullback (thanks to Remark 3.1.4). So, given a morphism $H \xrightarrow{y} F$ in $\mathcal{L}$, we define

$$\mathcal{DOpFib}(y) := y^* \colon \mathcal{DOpFib}(F) \to \mathcal{DOpFib}(H)$$

Given a 2-cell $\alpha \colon y \Rightarrow y' \colon H \to F$ in $\mathcal{L}$, we define $\mathcal{DOpFib}(\alpha) = \alpha^* \colon y^* \to (y')^*$ as the natural transformation whose component on a discrete opfibration $s \colon G \to F$ in $\mathcal{L}$ is $\mathcal{G}_{s,H}(\alpha)$. □

Proposition 3.1.9. *Let $p \colon E \to B$ be a discrete opfibration in $\mathcal{L}$. The functors $\mathcal{G}_{p,F}$ are pseudonatural in $F \in \mathcal{L}$.*

Proof. The proof is straightforward, using the universal property of the pullback and the uniqueness of the liftings through a discrete opfibration. □

Definition 3.1.10 (Weber [51]). A 2-*classifier in* $\mathcal{L}$ is a discrete opfibration $p \colon E \to B$ in $\mathcal{L}$ such that for every $F \in \mathcal{L}$ the functor

$$\mathcal{G}_{p,F} \colon \mathcal{L}(F, B) \to \mathcal{DOpFib}(F)$$

is fully faithful.

In that case, we say that a discrete opfibration $s \colon G \to F$ in $\mathcal{L}$ is *classified by* p (or that p *classifies* s) if s is in the essential image of $\mathcal{G}_{p,F}$, and we call *characteristic morphism of* s a morphism $z \colon F \to B$ such that $\mathcal{G}_{p,F}(z) \cong s$.

Remark 3.1.11. While Definition 3.1.1 asks for a universal monomorphism, Definition 3.1.10 asks for a universal discrete opfibration. The classification process is kept to be regulated by pullbacks. The condition to have a bijection is upgraded in dimension 2 to ask $\mathcal{G}_{p,F}$ to be an equivalence of categories with its essential image. Notice that Definition 3.1.10 allows for a classification of a smaller class of discrete opfibrations. In dimension 1, this idea brought for example to the concept of quasitopos.

However, Definition 3.1.1 loses the interpretation of the subobject classifier as picking "true" inside an object of generalized truth values. Indeed the domain of a 2-classifier is not forced at all to be the terminal object. In order to keep such point of view, which is useful for categorical logic, we propose to upgrade the 1-dimensional classification process, which is regulated by pullbacks, to one regulated by comma objects. This is slightly less general than Definition 3.1.10, but with better properties (see the following two remarks).

Definition 3.1.12. Let P be a fixed pullback-stable property P for discrete opfibrations in $\mathcal{L}$. A *good 2-classifier in* $\mathcal{L}$ (with respect to P) is a morphism $\omega\colon 1 \to \Omega$ in $\mathcal{L}$ such that for every $F \in \mathcal{L}$ the functor

$$\widehat{\mathcal{G}}_{\omega,F}\colon \mathcal{L}\,(F,\Omega) \to \mathcal{DOpFib}\,(F)$$

given by taking comma objects from ω is fully faithful and forms an equivalence of categories when restricting the codomain to $\mathcal{DOpFib}^{\mathrm{P}}(F)$. (In particular, we are asking that $\widehat{\mathcal{G}}_{\omega,F}$ lands in $\mathcal{DOpFib}^{\mathrm{P}}(F)$).

In the following two remarks, we show that $\widehat{\mathcal{G}}_{\omega,F}$ is indeed a functor which lands in $\mathcal{DOpFib}\,(F)$ and that good 2-classifiers are 2-classifiers enjoying better properties.

Remark 3.1.13. Taking comma objects from ω is equivalent to pulling back the lax limit τ of the arrow ω, which serves as a replacement.

$$\begin{tikzcd}
G \arrow[r] \arrow[d, "\widehat{\mathcal{G}}_{\omega,F}(z)"'] & 1 \arrow[d, "\omega"] \arrow[dl, phantom, "\swarrow\text{comma}"] \\
F \arrow[r, "z"'] & \Omega
\end{tikzcd}
=
\begin{tikzcd}
G \arrow[r] \arrow[d, "\mathcal{G}_{\tau,F}(z)"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \Omega_{\bullet} \arrow[r] \arrow[d, "\tau"'] & 1 \arrow[d, "\omega"] \arrow[dl, phantom, "\swarrow\text{comma}"] \\
F \arrow[r, "z"'] & \Omega \arrow[r, equal] & \Omega
\end{tikzcd}$$

Moreover, by Weber's [51], the span with vertex $\Omega_{\bullet}$ formed by τ and the map to 1 is a bisided discrete fibration. And we get that τ is a discrete opfibration. (In $\mathcal{L} = \mathcal{Cat}$, it is also known that such a τ is the free opfibration on the functor ω.) Explicitly, the lifting of $\theta\colon \tau \circ a \to b$ to a is calculated by applying the universal property of the comma object. Indeed $\theta_{*}a$ is induced by

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[dr, bend right, "b"'] & \Omega_{\bullet} \arrow[r] \arrow[d, "\tau"] \arrow[d, phantom, shift right=4, "\swarrow\theta"] & 1 \arrow[d, "\omega"] \arrow[dl, phantom, "\swarrow\text{comma}"] \\
& \Omega \arrow[r, equal] & \Omega
\end{tikzcd}$$

and $\overline{\theta}^{a}\colon a \Rightarrow \theta_{*}a$ is then induced by the pair of 2-cells formed by the identity (between X and 1) and θ (between X and Ω).

By Remark 3.1.4, it follows that $\widehat{\mathcal{G}}_{\omega,F}$ lands in $\mathcal{DOpFib}(F)$. Moreover, $\widehat{\mathcal{G}}_{\omega,F}$ lands in $\mathcal{DOpFib}^{\mathrm{P}}(F)$ if and only if τ satisfies P. It is easy to see that, up to choosing appropriate representatives of the comma objects, $\widehat{\mathcal{G}}_{\omega,F} = \mathcal{G}_{\tau,F}$. So that if ω is a good 2-classifier, τ is a 2-classifier.

Remark 3.1.14. Good 2-classifiers enjoy better properties than 2-classifiers. They are closer to the point of view of categorical logic, as they can still be thought of as the inclusion of "true" inside an object of generalized truth values. Moreover, a classification process regulated by taking comma objects from a morphism that has the terminal object as domain is sometimes easier to handle. As an example, the assignment of $\widehat{\mathcal{G}}_{\omega,F}$ on morphisms is just induced by the universal property of the comma object, while for $\mathcal{G}_{p,F}$, in Proposition 3.1.6, we had to consider liftings along a discrete opfibration. Using good 2-classifiers rather than 2-classifiers will be very useful for us in Section 5.2. Indeed, we will restrict 2-dimensional classifiers to nice sub-2-categories via factorization arguments, and factorizing good 2-classifiers is much easier than factorizing general 2-classifiers.

We also notice that a classification process regulated by comma objects offers another justification for the idea of upgrading subobjects to discrete opfibrations. Indeed discrete opfibrations are what gets produced by taking comma objects from a morphism that has the terminal object as domain.

In our examples, P will be the property of having small fibres. In some sense, our good 2-classifiers will classify "all possible morphisms", as the $\widehat{\mathcal{G}}_{\omega,F}$'s are required to be equivalences of categories.

Example 3.1.15. $\mathcal{C}at$ is the archetypal example of 2-category endowed with a (good) 2-classifier. Consider indeed $\omega = 1 \colon \mathit{1} \to \mathcal{S}et$. For every $\mathcal{B} \in \mathcal{C}at$, the functor

$$\widehat{\mathcal{G}}_{\omega,\mathcal{B}} \colon \mathcal{C}at\,(\mathcal{B}, \mathcal{S}et) \to \mathcal{DOpFib}\,(\mathcal{B})$$

is precisely the Grothendieck construction (or category of elements). It is well-known that this forms an equivalence of categories when restricting the codomain to be the full subcategory $\mathcal{DOpFib}^{\mathrm{s}}(\mathcal{B})$ of $\mathcal{DOpFib}\,(\mathcal{B})$ on the discrete opfibrations with small fibres. So that $1\colon \mathit{1} \to \mathcal{S}et$ is a good 2-classifier in $\mathcal{C}at$.

$$\begin{tikzcd}
\mathcal{E} \arrow[r] \arrow[d, "\substack{\forall p\\ \text{disc opfib}\\ \text{small fibres}}"'] & \mathit{1} \arrow[d, "\mathit{1}"] \arrow[ld, Rightarrow, shorten=10pt, "\text{comma}"] \\
\mathcal{B} \arrow[r, dashed, "\substack{\exists \chi_p\\ \text{taking fibres}}"'] & \mathcal{S}et
\end{tikzcd}$$

Notice that the lax limit of the arrow ω is given by the forgetful functor $\tau\colon \mathcal{S}et_{\bullet} \to \mathcal{S}et$ from pointed sets to sets.

Remark 3.1.16. In light of this archetypal example, we can think of a 2-classifier as a Grothendieck construction inside a 2-category.

Notation 3.1.17. We will often write as $\tau\colon \Omega_{\bullet} \to \Omega$ any 2-classifier or would-be 2-classifier, having in mind the archetypal example of $\mathcal{C}at$.

Remark 3.1.18. Upgrading monomorphisms to discrete opfibrations, one could try to upgrade $\operatorname{Sub}(F)$ to a category of isomorphism classes of discrete opfibrations over F. It is possible to form such a category and almost the entire reduction to dense generators of the study of 2-classifiers would equally work (if accordingly adjusted). However, there is one point, in Theorem 3.2.5, that does not seem to work well with this choice. We will give more details in Remark 3.2.6. We believe it is more natural and fruitful to work without isomorphism classes.

3.1.2. Dense generators

In Section 3.2, we will reduce to dense generators the study of 2-classifiers. Here we briefly recall what a (2-dimensional) dense generator is. The main reference we take for this is Kelly's [28, Chapter 5].

The basic idea behind the concept of generator of a 2-category $\mathcal{L}$ is that of a family of objects that builds all the objects of $\mathcal{L}$ via weighted 2-colimits.

Definition 3.1.19. A fully faithful 2-functor $I\colon \mathcal{K} \to \mathcal{L}$ is a *naive generator* if every $F \in \mathcal{L}$ is a weighted 2-colimit in $\mathcal{L}$ of a diagram which factors through $\mathcal{K}$.

Definition 3.1.20 ([28, Chapter 5])**.** A 2-functor $I\colon \mathcal{Y} \to \mathcal{L}$ with $\mathcal{Y}$ small is *a dense generator* (or also just *dense*) if the restricted Yoneda embedding

$$\begin{aligned}\widetilde{I}\colon\ \mathcal{L} &\longrightarrow [\mathcal{Y}^{\mathrm{op}}, \mathcal{C}at]\\ F &\mapsto \mathcal{L}(I(-), F)\end{aligned}$$

is (2-)fully faithful.

Remark 3.1.21. Of course the Yoneda embedding is fully faithful. And we may interpret this by saying that considering all morphisms from any object into F we get the whole information of F. The idea of a dense generator is that morphisms with source in a smaller family of objects are enough.

Definition 3.1.22. Let $I\colon \mathcal{Y} \to \mathcal{L}$ be a 2-functor. A weighted 2-colimit in $\mathcal{L}$ is called *I-absolute* if it is preserved by $\widetilde{I}\colon \mathcal{L} \to [\mathcal{Y}^{\mathrm{op}}, \mathcal{C}at]$.

When $I\colon \mathcal{Y} \to \mathcal{L}$ is fully faithful, we can characterize density of I in terms of weighted 2-colimits in $\mathcal{L}$.

Theorem 3.1.23 ([28, Theorem 5.19])**.** *Let $I\colon \mathcal{Y} \to \mathcal{L}$ be a fully faithful 2-functor with $\mathcal{Y}$ small. The following are equivalent:*

(i) *I is a dense generator;*

(ii) *every $F \in \mathcal{L}$ is an I-absolute weighted 2-colimit in $\mathcal{L}$ of a diagram which factors through $\mathcal{Y}$.*

Remark 3.1.24. We can thus interpret density of a fully faithful $I\colon \mathcal{Y} \to \mathcal{L}$ as the request that all objects of $\mathcal{L}$ are nice weighted 2-colimits of objects of $\mathcal{Y}$. So this is stronger than being a naive generator.

Example 3.1.25. Let $\mathcal{C}$ be a small 2-category. The Yoneda embedding $\mathrm{y}\colon \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ is a dense generator. That is, representables form a dense generator of the 2-category of 2-presheaves. Indeed it is well-known that every 2-presheaf is a weighted 2-colimit of representables, weighted by itself. And y-absoluteness is automatic as $\widetilde{\mathrm{y}}$ is essentially the identity.

In particular the singleton category $\mathit{1}$ is a dense generator of $\mathcal{C}at$.

3.2. Reduction of 2-classifiers to dense generators

In this section, we present a novel reduction of the study of a 2-classifier to dense generators. This is organized in the three Theorems 3.2.2, 3.2.5 and 3.2.8. More precisely, we prove that all the conditions of 2-classifier (see Definition 3.1.10) can just be checked on those objects F that form a dense generator (see Definition 3.1.20). The study of what is classified by a 2-classifier is similarly reduced to a study over the objects that form a dense generator. We also give a concrete recipe to build the characteristic morphisms.

This result offers great benefits. For example, applied to $\mathcal{C}at$, it reduces all the major properties of the Grothendieck construction to the trivial observation that everything works well over the singleton category (see Example 3.2.13).

In Chapter 5, we will apply our theorems of reduction to dense generators of this section to the cases of prestacks and stacks; see also Section 3.3. This will allow us to find a good 2-classifier in 2-presheaves that classifies all discrete opfibrations with small fibres, and to restrict it to a good 2-classifier in stacks.

Throughout this section, we fix a discrete opfibration $\tau\colon \Omega_{\bullet} \to \Omega$ in $\mathcal{L}$. Recall from Definition 3.1.10 that τ is a 2-classifier if for every $F \in \mathcal{L}$ the functor

$$\mathcal{G}_{\tau,F}\colon \mathcal{L}(F,\Omega) \to \mathcal{DOpFib}(F)$$

is fully faithful. And that, provided that this is the case, the essential image of such functors precisely represents which discrete opfibrations are classified.

The following proposition will often be useful.

Proposition 3.2.1. *Let $p\colon E \to B$ be a discrete opfibration in $\mathcal{L}$. For every pair of composable morphisms $H \xrightarrow{y} F \xrightarrow{z} B$ in $\mathcal{L}$, the pseudonaturality of $\mathcal{G}_{p,-}$ (see Proposition 3.1.9) gives isomorphisms*

$$\mathcal{G}_{p,H}(z \circ y) \cong y^{*}\mathcal{G}_{p,F}(z) = \mathcal{G}_{\mathcal{G}_{p,F}(z),H}(y), \tag{3.2}$$

where y^* *is the functor* $\mathcal{D}\mathcal{O}p\mathcal{F}ib\,(y)$ *defined in the proof of Proposition 3.1.8.*

Moreover, given a diagram

$$\begin{tikzcd} H \arrow[r, bend left, "y"] \arrow[r, bend right, "y'"'] \arrow[r, phantom, "\Downarrow\alpha"] & F \arrow[r, bend left, "z"] \arrow[r, bend right, "z'"'] \arrow[r, phantom, "\Downarrow\beta"] & B \end{tikzcd}$$

in $\mathcal{L}$*, the isomorphisms above form the following two commutative squares:*

$$\begin{tikzcd} \mathcal{G}_{p,H}(z \circ y) \arrow[r, "\simeq"] \arrow[d, "\mathcal{G}_{p,H}(z\alpha)"'] & \mathcal{G}_{\mathcal{G}_{p,F}(z),H}(y) \arrow[d, "\mathcal{G}_{\mathcal{G}_{p,F}(z),H}(\alpha)"] \\ \mathcal{G}_{p,H}(z \circ y') \arrow[r, "\simeq"] & \mathcal{G}_{\mathcal{G}_{p,F}(z),H}(y') \end{tikzcd} \qquad \begin{tikzcd} \mathcal{G}_{p,H}(z \circ y) \arrow[r, "\simeq"] \arrow[d, "\mathcal{G}_{p,H}(\beta y)"'] & \mathcal{G}_{\mathcal{G}_{p,F}(z),H}(y) \arrow[d, "y^*\,\mathcal{G}_{p,F}(\beta)"] \\ \mathcal{G}_{p,H}(z' \circ y) \arrow[r, "\simeq"] & \mathcal{G}_{\mathcal{G}_{p,F}(z'),H}(y) \end{tikzcd}$$

Proof. The proof is straightforward. The first square is given by the 2-dimensional part of the pseudonaturality of $\mathcal{G}_{p,-}$ applied to the 2-cell α, whereas the second square is given by the naturality in z of the isomorphisms of equation (3.2). ☐

We now present the first of our three theorems of reduction to dense generators. This reduces the study of the faithfulness of the functors $\mathcal{G}_{\tau,F}$. Such first theorem is much easier than the other two and actually works with any naive generator (Definition 3.1.19).

We also notice that injectivity on objects of the functors $\mathcal{G}_{\tau,F}$ can be reduced in a similar way, although this is less interesting for us.

Theorem 3.2.2. *Let* $\mathcal{Y}$ *be a full sub-2-category of* $\mathcal{L}$*. If for every* $Y \in \mathcal{Y}$

$$\mathcal{G}_{\tau,Y} \colon \mathcal{L}\,(Y, \Omega) \to \mathcal{D}\mathcal{O}p\mathcal{F}ib\,(Y)$$

is faithful, then for every F *in the closure of* $\mathcal{Y}$ *in* $\mathcal{L}$ *under weighted* 2*-colimits,*

$$\mathcal{G}_{\tau,F} \colon \mathcal{L}\,(F, \Omega) \to \mathcal{D}\mathcal{O}p\mathcal{F}ib\,(F)$$

is faithful.

In particular, if $\mathcal{Y}$ *is a naive generator of* $\mathcal{L}$ *and* $\mathcal{G}_{\tau,Y}$ *is faithful for every* $Y \in \mathcal{Y}$*, then* $\mathcal{G}_{\tau,F}$ *is faithful for every* $F \in \mathcal{L}$*.*

Proof. Let $K\colon \mathcal{I} \to \mathcal{L}$ be a 2-diagram which factors through $\mathcal{Y}$ and has a weighted 2-colimit F in $\mathcal{L}$, with weight $W : \mathcal{I}^{\mathrm{op}} \to \mathcal{C}at$. Call $\Lambda\colon W \Rightarrow \mathcal{L}\left(K(-), F\right)$ the universal cocylinder of such colimit.

In order to prove that $\mathcal{G}_{\tau,F}$ is faithful, take two arbitrary 2-cells in $\mathcal{L}$

$$\begin{tikzcd} F \arrow[r, bend left, "z"{name=U}] \arrow[r, bend right, "z'"'{name=D}] & \Omega \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F \arrow[r, bend left, "z"{name=U}] \arrow[r, bend right, "z'"'{name=D}] & \Omega \arrow[Rightarrow, from=U, to=D, "\alpha'"] \end{tikzcd}$$

such that $\mathcal{G}_{\tau,F}(\alpha) = \mathcal{G}_{\tau,F}(\alpha')\colon \mathrm{G}^{z} \to \mathrm{G}^{z'}$. We prove that $\alpha = \alpha'$.

As $F = \operatorname{colim}^{W} K$ with universal cocylinder Λ, it suffices to show that the two modifications

$$\begin{tikzcd} W \arrow[r, Rightarrow, "\Lambda"] & \mathcal{L}\left(K(-), F\right) \arrow[r, Rightarrow, bend left, "z\circ -"{name=U}] \arrow[r, Rightarrow, bend right, "z'\circ -"'{name=D}] & \mathcal{L}\left(K(-), \Omega\right) \arrow[Rrightarrow, from=U, to=D, "\alpha * -"] \end{tikzcd}$$

and $(\alpha' * -)\Lambda$ are equal, as we then conclude by the (2-dimensional) universal property of the weighted 2-colimit F.

It then suffices to prove that, given arbitrary $i \in \mathcal{I}$ and $X \in W(i)$,

$$\left(\begin{tikzcd} K(i) \arrow[r, "\Lambda_i(X)"] & F \arrow[r, bend left, "z"{name=U}] \arrow[r, bend right, "z'"'{name=D}] & \Omega \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd} \right) = \left(\begin{tikzcd} K(i) \arrow[r, "\Lambda_i(X)"] & F \arrow[r, bend left, "z"{name=U}] \arrow[r, bend right, "z'"'{name=D}] & \Omega \arrow[Rightarrow, from=U, to=D, "\alpha'"] \end{tikzcd} \right)$$

But as $K(i) \in \mathcal{Y}$ and $\mathcal{G}_{\tau,K(i)}$ is faithful by assumption, it then suffices to show that

$$\mathcal{G}_{\tau,K(i)}\left(\alpha\, \Lambda_i(X)\right) = \mathcal{G}_{\tau,K(i)}\left(\alpha'\, \Lambda_i(X)\right). \tag{3.3}$$

By Proposition 3.2.1, we can write $\mathcal{G}_{\tau,K(i)}\left(\alpha\, \Lambda_i(X)\right)$ as the composite

$$\mathcal{G}_{\tau,K(i)}(z\circ\Lambda_i(X)) \cong \Lambda_i(X)^{*}\left(\mathcal{G}_{\tau,F}(z)\right) \xrightarrow{\Lambda_i(X)^{*}\left(\mathcal{G}_{\tau,F}(\alpha)\right)} \Lambda_i(X)^{*}\left(\mathcal{G}_{\tau,F}(z')\right) \cong \mathcal{G}_{\tau,K(i)}(z'\circ\Lambda_i(X))$$

and analogously for α'. Since such composites for α and for α' are equal, we conclude that equation (3.3) holds. □

Remark 3.2.3. In order to prove our second and third theorems of reduction to dense generators, we apply our calculus of colimits in 2-dimensional slices, that we explored in Chapter 2.

We first need to compare, given $p\colon E\to B$ a discrete opfibration in $\mathcal{L}$, the 2-functor $\operatorname{dom}\circ\mathcal{G}_{p,F}$ with $\operatorname{dom}\circ p^{*}$, where

$$p^{*}\colon \mathcal{L}/_{\text{lax}}\, B \to \mathcal{L}/_{\text{lax}}\, E$$

is the change of base between lax slices introduced in Proposition 2.4.2.

For every $z\colon F\to B$ in $\mathcal{L}$,

$$\operatorname{dom}(p^{*}z)=\operatorname{dom}(\mathcal{G}_{p,F}(z)).$$

Given $(\mathrm{id},\alpha)\colon z\to z'$ in $\mathcal{L}/_{\text{lax}}\, B$, which is just $\alpha\colon z\Rightarrow z'\colon F\to B$, we have that

$$\operatorname{dom}(p^{*}(\mathrm{id},\alpha))=\operatorname{dom}(\mathcal{G}_{p,F}(\alpha))$$

by comparing the diagram of equation (3.1) with the similar one in the proof of Proposition 2.4.2.

Given a general $(\widehat{\alpha},\alpha)\colon z\to z'$ in $\mathcal{L}/_{\text{lax}}\, B$, we can still express $\operatorname{dom}(p^{*}(\widehat{\alpha},\alpha))=\widehat{p^{*}\alpha}$ in terms of $\operatorname{dom}(\mathcal{G}_{p,F}(\alpha))$. Indeed consider the total pullback R of p with the composite $z'\circ\widehat{\alpha}$ and the composite pullback S as below

$$\begin{tikzcd}
S \arrow[r, "\mathcal{G}_{p}(z')^{*}\widehat{\alpha}"] \arrow[d, "\mathcal{G}_{\mathcal{G}_{p}(z')}(\widehat{\alpha})"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathrm{G}^{z'} \arrow[r, "p^{*}z'"] \arrow[d, "\mathcal{G}_{p}(z')"'] \arrow[dr, phantom, very near start, "\lrcorner"] & E \arrow[d, "p"] \\
F \arrow[r, "\widehat{\alpha}"'] & F' \arrow[r, "z'"'] & B
\end{tikzcd}$$

Call i the induced isomorphism between R and S. Comparing again the diagrams of equations (3.1) and of the proof of Proposition 2.4.2, we obtain that

$$\operatorname{dom}(p^{*}(\widehat{\alpha},\alpha))=\mathcal{G}_{p,F}(z')^{*}\widehat{\alpha}\circ i\circ\operatorname{dom}(\mathcal{G}_{p,F}(\alpha))$$

The following construction will be useful to prove our second and third theorems of reduction to dense generators. It is based on our calculus of colimits in 2-dimensional slices, explored in Chapter 2.

Construction 3.2.4. Let $I\colon\mathcal{Y}\to\mathcal{L}$ be a fully faithful dense generator of $\mathcal{L}$, and let $F\in\mathcal{L}$. By Theorem 3.1.23, there exist a 2-diagram $J\colon\mathcal{I}\to\mathcal{L}$ which factors through $\mathcal{Y}$ and a weight $W\colon\mathcal{I}^{\mathrm{op}}\to\mathcal{C}at$ such that

$$F=\operatorname{colim}^{W}J$$

in $\mathcal{L}$ and this colimit is I-absolute. By Theorem 1.1.22, the 2-diagram $K \coloneqq J \circ \mathcal{G}(W) : \int W \to \mathcal{L}$ factors through $\mathcal{Y}$ and is such that

$$F = \text{oplax}^{\text{cart}}\text{-colim}^{\Delta 1} K.$$

Moreover this colimit is still I-absolute by Proposition 1.1.24. Call

$$\Lambda\colon \Delta 1 \underset{\text{oplax}^{\text{cart}}}{\Longrightarrow} \mathcal{L}\left(K(-), F\right)$$

the universal cartesian-marked oplax cocone that presents such colimit.

Consider now a discrete opfibration $p\colon E \to B$ in $\mathcal{L}$, a morphism $z\colon F \to B$ and the chosen pullback in $\mathcal{L}$

$$\begin{tikzcd} \mathrm{G}^z \arrow[r, "p^* z"] \arrow[d, "\mathcal{G}_{p,F}(z)"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "p"] \\ F \arrow[r, "z"'] & B \end{tikzcd}$$

We want to exhibit G^z as a cartesian-marked oplax conical colimit of a diagram constructed from K and Λ.

By Theorem 2.1.21, we can construct from K and Λ a 2-diagram $K'_z = K'\colon \int W \to \mathcal{L}/_{\text{lax}}\, B$ and a universal cartesian-marked oplax cocone $\Lambda'_z = \Lambda'$ which exhibits

$$\begin{matrix} F \\ \downarrow z \\ B \end{matrix} = \begin{matrix} \text{oplax}^{\text{cart}}\text{-colim}^{\Delta 1} K \\ \downarrow z \\ B \end{matrix} = \text{oplax}^{\text{cart}}\text{-colim}^{\Delta 1} K'$$

in the lax slice $\mathcal{L}/_{\text{lax}}\, B$. Explicitly, K' is the 2-diagram that corresponds to the cartesian-marked oplax cocone

$$\lambda^z\colon \Delta 1 \underset{\text{oplax}^{\text{cart}}}{\Longrightarrow} \mathcal{L}\left(K(-), B\right) : \left(\int W\right)^{\text{op}} \to \mathcal{C}at$$

associated to z. That is,

$$K' : \quad \int W \longrightarrow \mathcal{E}/_{\text{lax}}\, B$$

$$\begin{tikzcd} (C,X) \arrow[d, "(f,\nu)"] & \mapsto & K(C,X) \arrow[rr, "K(f,\nu)"] \arrow[dr, "\lambda^z_{(C,X)}"'] \arrow[rr, phantom, shift right=3, "\overset{\lambda^z_{f,\nu}}{\Rightarrow}"'] & & K(D,X') \arrow[dl, "\lambda^z_{(D,X')}"] \\ (D,X') & & & B & \end{tikzcd}$$

$$\delta \quad \mapsto \quad F(\delta)$$

Considering the canonical $\mathcal{F}$-category structure on $\int W$, with the tight part given by the morphisms of type (f, id), we have that K' is an $\mathcal{F}$-diagram.

The universal cartesian-marked oplax cocone Λ' has component on (C, X) given by the identity filled triangle (which is thus a tight morphism in $\mathcal{L}/_{\text{lax}}\, B$)

$$\begin{tikzcd}
K(C,X) \arrow[rr, "\Lambda_{(C,X)}"] \arrow[dr, "\lambda^z_{(C,X)}"'] & & F \arrow[dl, "z"] \\
& B &
\end{tikzcd}$$

Then $\Lambda'_{f,\nu} = \Lambda_{f,\nu}$ for every morphism (f, ν) in $\int W$.

By Theorem 2.4.6, the 2-functor

$$p^* \colon \mathcal{L}/_{\text{lax}}\, B \to \mathcal{L}/_{\text{lax}}\, E$$

preserves the colimit $z = \text{oplax}^{\text{cart}}\text{-colim}^{\Delta 1} K'$, since K' is an $\mathcal{F}$-diagram, Λ' has tight components and the domain of such colimit is $F = \text{oplax}^{\text{cart}}\text{-colim}^{\Delta 1} K$, which is I-absolute. Then by Proposition 2.3.14 (and Theorem 2.3.12) the domain 2-functor $\mathrm{dom} \colon \mathcal{L}/_{\text{lax}}\, E \to \mathcal{L}$ preserves the latter colimit $p^*(z)$, since p^* is an $\mathcal{F}$-functor. We obtain that $\mathrm{dom} \circ p^* \circ \Lambda'$ is a universal cartesian-marked oplax cocone that presents

$$\mathrm{G}^z = \text{oplax}^{\text{cart}}\text{-colim}^{\Delta 1}(\mathrm{dom} \circ p^* \circ K').$$

Explicitly, given $(D, X') \xleftarrow{(f,\nu)} (C, X)$ in $\int W$, we have that $\mathrm{dom}\left(p^*\left(\Lambda'_{(C,X)}\right)\right)$ is the unique morphism into the pullback G^z induced by

$$\begin{tikzcd}
Q^{(C,X)} \arrow[drr, bend left, "p^*K'(C,X)"] \arrow[dr, dashed] \arrow[dd, "\mathcal{G}_p(K'(C,X))"'] & & \\
& \mathrm{G}^z \arrow[r, "p^*z"] \arrow[dd, "\mathcal{G}_p(z)"'] & E \arrow[dd, "p"] \\
K(C,X) \arrow[dr, "\Lambda_{(C,X)}"'] \arrow[drr, bend left, "K'(C,X)"] & & \\
& F \arrow[r, "z"'] & B
\end{tikzcd}$$

Then $\mathrm{dom}\left(p^*\left(\Lambda'_{f,\nu}\right)\right)$ is the unique lifting along the discrete opfibration $\mathcal{G}_{p,F}(z)$ of the 2-cell $\Lambda_{f,\nu} * \mathcal{G}_{p,K(C,X)}\,(K'(C,X))$ to $\mathrm{dom}\left(p^*\left(\Lambda'_{(C,X)}\right)\right)$.

Notice that in particular this construction can be applied to $z = \mathrm{id}_F$, exhibiting the domain of any discrete opfibration over F as a cartesian-marked oplax conical colimit (starting from K and Λ). Remember that we chose pullbacks in $\mathcal{L}$ such that the change of base of an identity is always an identity. Notice also that, given $z\colon F \to B$,

$$\lambda^z = (z \circ -) \circ \lambda^{\mathrm{id}}$$

Whence we can express the $\mathcal{F}$-functor K'_z constructed from z in terms of the one constructed from id:

$$K'_z = (z \circ -) \circ K'_{\mathrm{id}}$$

Then the components and the structure 2-cells of Λ'_z and Λ'_{id} have the same domains, which determine them.

We now present the second of our three theorems of reduction to dense generators. This reduces the study of the fullness of the functors $\mathcal{G}_{\tau,F}$, provided that we already know their faithfulness. Indeed such a theorem builds over Theorem 3.2.2.

Theorem 3.2.5. *Let $I\colon \mathcal{Y} \to \mathcal{L}$ be a fully faithful dense generator of $\mathcal{L}$. If for every $Y \in \mathcal{Y}$*

$$\mathcal{G}_{\tau,I(Y)}\colon \mathcal{L}\,(I(Y),\Omega) \to \mathcal{DOpFib}\,(I(Y))$$

is fully faithful, then for every $F \in \mathcal{L}$ also

$$\mathcal{G}_{\tau,F}\colon \mathcal{L}\,(F,\Omega) \to \mathcal{DOpFib}\,(F)$$

is full, and hence fully faithful by Theorem 3.2.2, so that τ is a 2-classifier in $\mathcal{L}$.

Proof. Let $F \in \mathcal{L}$; we prove that $\mathcal{G}_{\tau,F}$ is full. Consider then two morphisms $z, z'\colon F \to \Omega$ in $\mathcal{L}$ and a morphism $h\colon \mathcal{G}_{\tau,F}(z) \to \mathcal{G}_{\tau,F}(z')$ in $\mathcal{DOpFib}\,(F)$. We search for a 2-cell $\alpha\colon z \Rightarrow z'\colon F \to \Omega$ such that $\mathcal{G}_{\tau,F}(\alpha) = h$. The idea is to write F as a colimit and define α by giving its "components", which we can produce by the fullness of the functors $\mathcal{G}_{\tau,I(Y)}$ with the Y's in $\mathcal{Y}$ that generate F.

By Theorem 3.1.23, Theorem 1.1.22 and Proposition 1.1.24, there exists a 2-diagram $K\colon \int W \to \mathcal{L}$ which factors through $\mathcal{Y}$ and a universal cartesian-marked oplax cocone

$$\Lambda\colon \Delta 1 \xRightarrow[\mathrm{oplax}^{\mathrm{cart}}]{} \mathcal{L}\,(K(-),F)$$

that exhibits F as the I-absolute cartesian-marked oplax conical colimit of K.

$$F = \text{oplax}^{\text{cart}}\text{-colim}^{\Delta 1} K$$

Then, in order to produce α, it suffices to give a modification

$$\begin{tikzcd}[column sep=huge]
\Delta 1 \arrow[r, Rightarrow, bend left=40, "(z\circ -)\circ\Lambda", ""{name=U, below}] \arrow[r, Rightarrow, bend right=40, "(z'\circ -)\circ\Lambda"', ""{name=D, above}] & \mathcal{L}(K(-),\Omega) \arrow[from=U, to=D, Rightarrow, "\theta"]
\end{tikzcd}$$

Given $(C,X) \in \int W$, we will have that

$$\left(\begin{tikzcd} K(C,X) \arrow[r, "\Lambda_{(C,X)}"] & F \arrow[r, bend left=40, "z", ""{name=U, below}] \arrow[r, bend right=40, "z'"', ""{name=D, above}] & \Omega \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}\right) = \theta_{(C,X)}$$

So, we will need to have that

$$\mathcal{G}_{\tau,K(C,X)}(\theta_{(C,X)}) = \mathcal{G}_{\tau,K(C,X)}(\alpha\Lambda_{(C,X)})$$

and, by Proposition 3.2.1 and the request that $\mathcal{G}_{\tau,F}(\alpha) = h$, the right hand side of such equation is equal to the composite

$$\mathcal{G}_{\tau,K(C,X)}(z\circ\Lambda_{(C,X)}) \cong {\Lambda_{(C,X)}}^{*}\left(\mathcal{G}_{\tau,F}(z)\right) \xrightarrow{{\Lambda_{(C,X)}}^{*}(h)} {\Lambda_{(C,X)}}^{*}\left(\mathcal{G}_{\tau,F}(z')\right) \cong \mathcal{G}_{\tau,K(C,X)}(z'\circ\Lambda_{(C,X)})$$

Call $h^{C,X}$ such composite morphism in $\mathcal{DOpFib}(K(C,X))$. Since $K(C,X) \in \mathcal{Y}$, the functor $\mathcal{G}_{\tau,K(C,X)}$ is fully faithful by assumption, and hence there exists a unique 2-cell

$$\begin{tikzcd}[column sep=huge]
K(C,X) \arrow[r, bend left=40, "z\circ\Lambda_{(C,X)}", ""{name=U, below}] \arrow[r, bend right=40, "z'\circ\Lambda_{(C,X)}"', ""{name=D, above}] & \Omega \arrow[from=U, to=D, Rightarrow, "\gamma^{C,X}"]
\end{tikzcd}$$

in $\mathcal{L}$ such that $\mathcal{G}_{\tau,K(C,X)}(\gamma^{C,X}) = h^{C,X}$. We define the component $\theta_{(C,X)}$ of θ on (C,X) to be such 2-cell $\gamma^{C,X}$. The faithfulness of the functors $\mathcal{G}_{\tau,K(C,X)}$ guarantees that θ becomes a modification. Indeed, given a morphism $(f,\nu)\colon (D,X') \to (C,X)$ in $\int W$, we need to prove that

$$\begin{tikzcd}[column sep=huge, row sep=large]
1 \arrow[r, bend left=30, "z'\circ\Lambda_{(C,X)}"] \arrow[d] & \mathcal{L}(K(C,X),\Omega) \arrow[d, "-\circ K(f,\nu)"] \\
1 \arrow[r, bend left=30, "z'\circ\Lambda_{(D,X')}", ""{name=U, below}] \arrow[r, bend right=30, "z\circ\Lambda_{(D,X')}"', ""{name=D, above}] \arrow[ur, phantom, "\overset{z'\Lambda_{(f,\nu)}}{\Rightarrow}" description] & \mathcal{L}(K(D,X'),\Omega) \arrow[from=D, to=U, Rightarrow, "\theta_{(D,X')}"]
\end{tikzcd}
=
\begin{tikzcd}[column sep=huge, row sep=large]
1 \arrow[r, bend left=30, "z'\circ\Lambda_{(C,X)}", ""{name=U, below}] \arrow[r, bend right=30, "z\circ\Lambda_{(C,X)}"', ""{name=D, above}] \arrow[d] & \mathcal{L}(K(C,X),\Omega) \arrow[d, "-\circ K(f,\nu)"] \arrow[from=D, to=U, Rightarrow, "\theta_{(C,X)}"] \\
1 \arrow[r, bend right=30, "z\circ\Lambda_{(D,X')}"'] \arrow[ur, phantom, "\overset{z\Lambda_{(f,\nu)}}{\Rightarrow}" description] & \mathcal{L}(K(D,X'),\Omega)
\end{tikzcd}$$

But since $\mathcal{G}_{\tau,K(D,X')}$ is faithful, it suffices to prove that

$$\mathcal{G}_{\tau,K(D,X')}\big(z'\Lambda_{(f,\nu)}\circ\theta^{D,X'}\big)=\mathcal{G}_{\tau,K(D,X')}\big(\theta_{(C,X)}K(f,\nu)\circ z\Lambda_{(f,\nu)}\big)$$

and hence that

$$\mathcal{G}_{\tau,K(D,X')}\big(z'\Lambda_{(f,\nu)}\big)\circ\mathcal{G}_{\tau,K(D,X')}\big(\theta^{D,X'}\big)=\mathcal{G}_{\tau,K(D,X')}\big(\theta_{(C,X)}K(f,\nu)\big)\circ\mathcal{G}_{\tau,K(D,X')}\big(z\Lambda_{(f,\nu)}\big)$$

At this point, it is straightforward to see that such equality is given by the naturality square of $\mathcal{DOpFib}\big(\Lambda_{(f,\nu)}\big)$ (see Proposition 3.1.8) obtained considering the morphism $h\colon \mathcal{G}_{\tau,F}(z)\to\mathcal{G}_{\tau,F}(z')$ in $\mathcal{DOpFib}(F)$. For this, one needs the pseudonaturality of $\mathcal{G}_{\tau,-}$ (Proposition 3.1.9), the equation $\mathcal{G}_{\tau,K(C,X)}(\theta_{(C,X)})=k^{C,X}$ and the analogous one for (D,X'), together with both the squares of Proposition 3.2.1. The square on the right of Proposition 3.2.1 allows us to calculate $\mathcal{G}_{\tau,K(D,X')}\big(\theta_{(C,X)}K(f,\nu)\big)$, whereas the square on the left allows us to compare the components of $\mathcal{DOpFib}\big(\Lambda_{(f,\nu)}\big)$ on $\mathcal{G}_{\tau,F(z)}$ and $\mathcal{G}_{\tau,F(z')}$ with $\mathcal{G}_{\tau,K(D,X')}\big(z\Lambda_{(f,\nu)}\big)$ and the analogous one with z' instead of z.

Thus we conclude that θ is a modification, and by the universal property of the colimit F we obtain an induced 2-cell $\begin{tikzcd}F \arrow[r, bend left, "z"] \arrow[r, bend right, "z'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \Omega\end{tikzcd}$ in $\mathcal{L}$. It remains to prove that $\mathcal{G}_{\tau,F}(\alpha)=h$.

The idea is to conclude such equality by using the uniqueness part of the universal property of a colimit. As $\mathcal{G}_{\tau,F}(\alpha)$ and h are morphisms $\mathcal{G}_{\tau,F}(z)\to\mathcal{G}_{\tau,F}(z')$ in $\mathcal{DOpFib}(F)$, both are totally determined by a morphism in $\mathcal{L}$ from G^z to $\mathrm{G}^{z'}$; we call respectively $\mathcal{G}_{\tau,F}(\alpha)$ and h such morphisms in $\mathcal{L}$. By Construction 3.2.4, applied to $z=\mathrm{id}_F$,

$$\mathrm{G}^z=\operatorname{oplax}^{\mathrm{cart}}\text{-}\operatorname{colim}^{\Delta 1}(\operatorname{dom}\circ\mathcal{G}_{\tau,F}(z)^*\circ K'),$$

presented by the universal cartesian-marked oplax cocone $\operatorname{dom}\circ\mathcal{G}_{\tau,F}(z)^*\circ\Lambda'$, with Λ' and K' constructed from Λ and K as in Construction 3.2.4.

Then, in order to conclude that $\mathcal{G}_{\tau,F}(\alpha)$ and h are equal, it suffices to show that

$$(\mathcal{G}_{\tau,F}(\alpha)\circ-)\circ\operatorname{dom}\circ\mathcal{G}_{\tau,F}(z)^*\circ\Lambda'=(h\circ-)\circ\operatorname{dom}\circ\mathcal{G}_{\tau,F}(z)^*\circ\Lambda' \qquad (3.4)$$

as cartesian-marked oplax natural transformations. And the last part of Construction 3.2.4 gives us an explicit calculation of $\operatorname{dom}\circ\mathcal{G}_{\tau,F}(z)^* \circ \Lambda'$ in terms of Λ. Precisely, given an arbitrary morphism $(D, X') \xleftarrow{(f,\nu)} (C, X)$ in $\int W$,

$$\operatorname{dom}\left(\mathcal{G}_{\tau,F}(z)^*\left(\Lambda'_{(C,X)}\right)\right) = \mathcal{G}_{\tau,F}(z)^*\left(\Lambda_{(C,X)}\right)$$

and $\operatorname{dom}\left(\mathcal{G}_{\tau,F}(z)^*\left(\Lambda'_{f,\nu}\right)\right)$ is the unique lifting of the 2-cell $\Lambda_{f,\nu} * \mathcal{G}_{\mathcal{G}_{\tau,F}(z),K(C,X)}\left(\Lambda_{(C,X)}\right)$ to $\mathcal{G}_{\tau,F}(z)^*\left(\Lambda_{(C,X)}\right)$ along $\mathcal{G}_{\tau,F}(z)$. We now notice that the following two squares are commutative:

$$\begin{tikzcd}
\mathrm{G}^{\Lambda_{(C,X)}}_{\mathcal{G}_{\tau,F}(z),K(C,X)} \arrow[r, "\mathcal{G}_{\tau,F}(z)^*\left(\Lambda_{(C,X)}\right)"] \arrow[d, "\Lambda_{(C,X)}{}^*\left(\mathcal{G}_{\tau,F}(\alpha)\right)"'] & \mathrm{G}^{z} \arrow[d, "\mathcal{G}_{\tau,F}(\alpha)"] \\
\mathrm{G}^{\Lambda_{(C,X)}}_{\mathcal{G}_{\tau,F}(z'),K(C,X)} \arrow[r, "\mathcal{G}_{\tau,F}(z')^*\left(\Lambda_{(C,X)}\right)"'] & \mathrm{G}^{z'}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{G}^{\Lambda_{(C,X)}}_{\mathcal{G}_{\tau,F}(z),K(C,X)} \arrow[r, "\mathcal{G}_{\tau,F}(z)^*\left(\Lambda_{(C,X)}\right)"] \arrow[d, "\Lambda_{(C,X)}{}^*(h)"'] & \mathrm{G}^{z} \arrow[d, "h"] \\
\mathrm{G}^{\Lambda_{(C,X)}}_{\mathcal{G}_{\tau,F}(z'),K(C,X)} \arrow[r, "\mathcal{G}_{\tau,F}(z')^*\left(\Lambda_{(C,X)}\right)"'] & \mathrm{G}^{z'}
\end{tikzcd}$$

Then, to prove that equation (3.4) holds on component (C, X), it suffices to show that

$$\Lambda_{(C,X)}{}^*\left(\mathcal{G}_{\tau,F}(\alpha)\right) = \Lambda_{(C,X)}{}^*(h)$$

as morphisms in $\mathcal{DOpFib}(K(C, X))$. Using Proposition 3.2.1, we see that $\Lambda_{(C,X)}{}^*\left(\mathcal{G}_{\tau,F}(\alpha)\right)$ is equal to

$$\Lambda_{(C,X)}{}^*\left(\mathcal{G}_{\tau,F}(z)\right) \cong \mathcal{G}_{\tau,F}(z\circ\Lambda_{(C,X)}) \xrightarrow{\mathcal{G}_{\tau,F}\left(\alpha\Lambda_{(C,X)}\right)} \mathcal{G}_{\tau,F}(z'\circ\Lambda_{(C,X)}) \cong \Lambda_{(C,X)}{}^*\left(\mathcal{G}_{\tau,F}(z')\right)$$

and thus is equal to $\Lambda_{(C,X)}{}^*(h)$ since $\alpha\Lambda_{(C,X)} = \theta_{(C,X)}$, by construction of $\theta_{(C,X)}$.

It only remains to prove that equation (3.4) holds on morphism component (f, ν). But this is straightforward using the uniqueness of the liftings through a discrete opfibration, that equation (3.4) holds on object components and the fact that both $\mathcal{G}_{\tau,F}(\alpha)$ and h are morphisms of discrete opfibrations. □

Remark 3.2.6. As anticipated in Remark 3.1.18, the possibility of defining the functors $\mathcal{G}_{\tau,F}$ to have as codomain a category of isomorphism classes of discrete opfibrations does not work well with the reduction of the study of 2-classifiers to generators. Theorem 3.2.5 is however the only delicate point one encounters. We can define a category $\mathcal{DOpFib}(F)/_{\cong}$ which has as objects isomorphism classes of discrete opfibrations and as morphisms collections of morphisms compatible with every possible change of representative for the domain or the codomain. Notice that we then have a full functor

$$\mathcal{DOpFib}(F) \xrightarrow[\text{full}]{q} \mathcal{DOpFib}(F)/\cong$$

The problem with Theorem 3.2.5 is that, from the assumption that for every $Y \in \mathcal{Y}$ the composite

$$\mathcal{L}(Y,\Omega) \xrightarrow{\mathcal{G}_{\tau,Y}} \mathcal{DOpFib}(Y) \xrightarrow[\text{full}]{q} \mathcal{DOpFib}(Y)/\cong$$

is fully faithful, we cannot deduce the analogue of this for every $F \in \mathcal{L}$. Indeed, we can no longer have

$$\mathcal{G}_{\tau,K(C,X)}(\theta_{(C,X)}) = h^{C,X}$$

in $\mathcal{DOpFib}(F)$, but only in $\mathcal{DOpFib}(F)/\cong$. Of course we then only need $\mathcal{G}_{\tau,F}(\alpha) = h$ in $\mathcal{DOpFib}(F)/\cong$, but it seems that there is no way to find the isomorphisms that regulate $\mathcal{G}_{\tau,F}(\alpha) = h$ starting from the ones that regulate $\mathcal{G}_{\tau,K(C,X)}(\theta_{(C,X)}) = h^{C,X}$ for every (C,X). One strategy could be to induce them using the universal property of the colimit, but we cannot guarantee that the isomorphisms that regulate all the $\mathcal{G}_{\tau,K(C,X)}(\theta_{(C,X)}) = h^{C,X}$ form a cocone.

We aim at the third of our three theorems of reduction to dense generators. Theorem 3.2.2 and Theorem 3.2.5 allow to check the conditions for $\tau\colon \Omega_\bullet \to \Omega$ to be a 2-classifier just on a dense generator. We now want to similarly reduce to dense generators the study of what τ classifies. The following construction will be very important for this.

Recall that we denote as $\mathcal{DOpFib}^{\mathrm{P}}(F)$ the full subcategory of $\mathcal{DOpFib}(F)$ on the discrete opfibrations that satisfy a fixed pullback-stable property P (in our examples, P will be the property of having small fibres).

Construction 3.2.7. Let $I\colon \mathcal{Y} \to \mathcal{L}$ be a fully faithful dense generator of $\mathcal{L}$. Assume that τ satisfies a pullback-stable property P and that for every $Y \in \mathcal{Y}$

$$\mathcal{G}_{\tau,I(Y)}\colon \mathcal{L}(I(Y),\Omega) \to \mathcal{DOpFib}^{\mathrm{P}}(I(Y))$$

is an equivalence of categories. Let then $\varphi\colon G \to F$ be a discrete opfibration in $\mathcal{L}$ that satisfies the property P. We would like to construct a characteristic morphism $z\colon F \to \Omega$ for φ. That is, a z such that $\mathcal{G}_{\tau,F}(z)$ is isomorphic to φ, so that φ gets classified by τ.

There exist a 2-diagram $K\colon \int W \to \mathcal{L}$ which factors through $\mathcal{Y}$ and a universal cartesian-marked oplax cocone Λ that exhibits F as the I-absolute

$$F = \text{oplax}^{\text{cart}}\text{-colim}^{\Delta 1} K$$

We would like to induce z via the universal property of the colimit F. The idea is condensed in the following diagram:

$$\begin{tikzcd}
H^{(C,X)} \arrow[r] \arrow[d, "\mathcal{G}_{\varphi}\left(\Lambda_{(C,X)}\right)"'] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d, "\varphi"] & \Omega_{\bullet} \arrow[d, "\tau"] \\
K(C,X) \arrow[r, Rightarrow, "\Lambda_{(C,X)}"] \arrow[rr, bend right, "\mathcal{G}_{\tau}^{-1}\left(\mathcal{G}_{\varphi}\left(\Lambda_{(C,X)}\right)\right)"'] & F \arrow[r, dashed, "z"] & \Omega
\end{tikzcd}$$

For every (C,X) in $\int W$, the change of base $\mathcal{G}_{\varphi,K(C,X)}(\Lambda_{C,X})$ of φ along $\Lambda_{C,X}$ satisfies the property P and is thus in the essential image of $\mathcal{G}_{\tau,K(C,X)}$. We can then consider the oplax natural transformation χ given by the composite

$$\Delta 1 \xRightarrow[\text{oplax}^{\text{cart}}]{\Lambda} \mathcal{L}\left(K(-),F\right) \xRightarrow[\text{pseudo}]{\mathcal{G}_{\varphi,K(-)}} \mathcal{DO}p\mathcal{F}ib^{\,\text{P}}(K(-)) \xRightarrow[\text{pseudo}]{{\mathcal{G}_{\tau,K(-)}}^{-1}} \mathcal{L}\left(K(-),\Omega\right),$$

where every ${\mathcal{G}_{\tau,K(C,X)}}^{-1}$ is a right adjoint quasi-inverse of $\mathcal{G}_{\tau,K(C,X)}$ giving an adjoint equivalence. The action of such ${\mathcal{G}_{\tau,K(C,X)}}^{-1}$ on morphisms $h\colon \psi \to \psi'$ is exhibited by the naturality squares of the counit ε

$$\begin{tikzcd}
\mathcal{G}_{\tau,K(C,X)}\left({\mathcal{G}_{\tau,K(C,X)}}^{-1}(\psi)\right) \arrow[r, phantom, "\underset{\varepsilon_{\psi}}{\cong}"] \arrow[d, "\mathcal{G}_{\tau,K(C,X)}\left({\mathcal{G}_{\tau,K(C,X)}}^{-1}(h)\right)"'] & \psi \arrow[d, "h"] \\
\mathcal{G}_{\tau,K(C,X)}\left({\mathcal{G}_{\tau,K(C,X)}}^{-1}(\psi')\right) \arrow[r, phantom, "\underset{\varepsilon_{\psi'}}{\cong}"] & \psi'
\end{tikzcd}$$

We are also using that for every morphism $(f,\nu)\colon (D,X') \leftarrow (C,X)$ in $\int W$ the functor $\mathcal{DO}p\mathcal{F}ib\,(K(f,\nu)) = K(f,\nu)^{*}$ restricts to a functor between the essential image of $\mathcal{G}_{\tau,K(D,X')}$ and the essential image of $\mathcal{G}_{\tau,K(C,X)}$. Moreover, using Proposition 3.1.9, we have that $\mathcal{G}_{\tau,K(-)}^{-1}$ extends to a pseudonatural transformation. Its structure 2-cell on any $(f,\nu)\colon (D,X') \leftarrow (C,X)$ in $\int W$ is given by the pasting

$$\begin{tikzcd}
\mathcal{DOpFib}^{\mathrm{P}}\left(K^{D,X'}\right) \arrow[r, "\mathcal{G}_\tau{}^{-1}"] \arrow[dr, equal] & \mathcal{L}\left(K^{D,X'},\Omega\right) \arrow[r, "-\circ K(f,\nu)"] \arrow[d, "\mathcal{G}_\tau"] \arrow[dl, phantom, "\overset{\cong}{\varepsilon^{-1}}" near start] & \mathcal{L}\left(K^{C,X},\Omega\right) \arrow[d, "\mathcal{G}_\tau"] \arrow[dr, equal] & \\
& \mathcal{DOpFib}^{\mathrm{P}}\left(K^{D,X'}\right) \arrow[r, "K(f,\nu)^*"'] \arrow[ur, Rightarrow, shorten=20pt, "\left(\mathcal{G}_{\tau,K(f,\nu)}\right)^{-1}"'] & \mathcal{DOpFib}^{\mathrm{P}}\left(K^{C,X}\right) \arrow[r, "\mathcal{G}_\tau{}^{-1}"'] \arrow[ur, phantom, "\underset{\cong}{\eta^{-1}}" near start] & \mathcal{L}\left(K^{C,X},\Omega\right)
\end{tikzcd}$$

where η is the unit of the adjoint equivalence, and we denoted $K(C,X)$ as $K^{C,X}$.

The composite χ above is readily seen to be a sigma natural transformation (of Descotte, Dubuc and Szyld's [15], w.r.t. the cartesian marking). That is, χ is an oplax natural transformation with the structure 2-cells $\chi_{f,\mathrm{id}}$ being isomorphisms for every morphism of type (f,id) in $\left(\int W\right)^{\mathrm{op}}$. If we were in a bicategorical context, this would then be enough to induce a morphism $z\colon F \to \Omega$, as we will explore in detail in future work. In our strict 2-categorical context, we further need to be able to "normalize" such sigma natural transformation, ensuring that the structure 2-cells on the morphisms (f,id) are identities. So that the morphisms $\mathcal{G}_\tau^{-1}\left(\mathcal{G}_\varphi\left(\Lambda_{(C,X)}\right)\right)$ yield a cartesian-marked oplax cocone. Essentially, this means that we can choose good quasi-inverses of the $\mathcal{G}_{\tau,K(C,X)}$'s. Such an extra hypothesis is satisfied by 2-dimensional presheaves (i.e. prestacks), see Theorem 5.1.14. By Theorem 5.2.9, it is then satisfied by nice sub-2-categories of prestacks, such as the 2-category of stacks (Theorem 5.3.22).

We now present the third of our three theorems of reduction to dense generators. Building over Theorem 3.2.2 and Theorem 3.2.5, we reduce to dense generators the study of what a 2-classifier $\tau\colon \Omega_\bullet \to \Omega$ classifies. The proof is constructive, based on Construction 3.2.7, so that we also give a concrete recipe for the characteristic morphisms.

Theorem 3.2.8. *Let $I\colon \mathcal{Y} \to \mathcal{L}$ be a fully faithful dense generator of $\mathcal{L}$. Assume that τ satisfies a pullback-stable property* P *and that for every $Y \in \mathcal{Y}$*

$$\mathcal{G}_{\tau,I(Y)}\colon \mathcal{L}\left(I(Y),\Omega\right) \to \mathcal{DOpFib}^{\mathrm{P}}(I(Y))$$

is an equivalence of categories. Assume further that, for every discrete opfibration $\varphi\colon G \to F$ in $\mathcal{L}$ that satisfies the property P*, the sigma natural transformation χ*

produced in Construction 3.2.7 (starting from φ and some K and Λ) is isomorphic to a cartesian-marked oplax natural transformation $\aleph$ (i.e. can be normalized).

Then for every $F \in L$

$$\mathcal{G}_{\tau,F} \colon \mathcal{L}(F, \Omega) \to \mathcal{DOpFib}^{\mathrm{P}}(F)$$

is essentially surjective, and hence an equivalence of categories by Theorem 3.2.5.

Proof. Let $\varphi \colon G \to F$ be a discrete opfibration in $\mathcal{L}$ that satisfies the property P. Consider the associated χ and $\aleph$ as in the statement. Let $z \colon F \to \Omega$ be the unique morphism induced by $\aleph$ via the universal property of the colimit $F = \text{oplax}^{\text{cart}}\text{-colim}^{\Delta 1} K$. We prove that z is a characteristic morphism for φ with respect to τ.

Consider the pullback

$$\begin{tikzcd}
V \arrow[r, "\widetilde{z}"] \arrow[d, "\mathcal{G}_{\tau,F}(z)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega_{\bullet} \arrow[d, "\tau"] \\
F \arrow[r, "z"'] & \Omega
\end{tikzcd}$$

We want to prove that there is an isomorphism $j \colon G \cong V$ such that $\mathcal{G}_{\tau,F}(z) \circ j = \varphi$. Applying Construction 3.2.4 to φ and id_F, we construct $K' = K'_{\mathrm{id}}$ and $\Lambda' = \Lambda'_{\mathrm{id}}$ that exhibits

$$G = \text{oplax}^{\text{cart}}\text{-colim}^{\Delta 1}(\operatorname{dom} \circ \varphi^* \circ K').$$

Applying again Construction 3.2.4 to τ and $z \colon F \to \Omega$ we obtain

$$V = \text{oplax}^{\text{cart}}\text{-colim}^{\Delta 1}(\operatorname{dom} \circ \tau^* \circ (z \circ -) \circ K').$$

We show that

$$\operatorname{dom} \circ \tau^* \circ (z \circ -) \circ K' \cong \operatorname{dom} \circ \varphi^* \circ K'.$$

Notice that $(z \circ -) \circ K'$ is the 2-functor $\int W \to \mathcal{L}/_{\text{lax}}\, \Omega$ associated to the oplax natural transformation $\aleph$, as described in Construction 3.2.4. Since $\aleph \cong \chi$, we obtain that $(z \circ -) \circ K' \cong U$ where U is the 2-functor $\int W \to \mathcal{L}/_{\text{lax}}\, \Omega$ associated to the sigma natural transformation χ. Moreover the general component $t_{C,X}$

on $(C, X) \in \int W$ of such isomorphism has first component equal to the identity. This produces an isomorphism

$$\operatorname{dom} \circ \tau^* \circ (z \circ -) \circ K' \cong \operatorname{dom} \circ \tau^* \circ U$$

whose general component on (C, X) is over $K(C, X)$. Indeed by Remark 3.2.3

$$\operatorname{dom} \left(\tau^* \left(t_{(C,X)}\right)\right) = \operatorname{dom} \left(\mathcal{G}_{\tau, K(C,X)} \left(\operatorname{pr}_2 \left(t_{C,X}\right)\right)\right)$$

and is thus an isomorphism that makes the following triangle commute:

$$\begin{tikzcd}
\operatorname{dom} \left(\mathcal{G}_{\tau, K(C,X)} \left(\aleph_{(C,X)}\right)\right) \arrow[rr, "\simeq"] \arrow[dr, swap, "\mathcal{G}_{\tau, K(C,X)}\left(\aleph_{(C,X)}\right)"] & & \operatorname{dom} \left(\mathcal{G}_{\tau, K(C,X)} \left(\chi_{(C,X)}\right)\right) \arrow[dl, "\mathcal{G}_{\tau, K(C,X)}\left(\chi_{(C,X)}\right)"] \\
& K(C, X) &
\end{tikzcd}$$

In this way, we have handled the isomorphisms given by the normalization process. We now show that

$$\operatorname{dom} \circ \tau^* \circ U \cong \operatorname{dom} \circ \varphi^* \circ K'.$$

Given $(C, X) \in \int W$, by Remark 3.2.3 and by construction of χ we have

$$\operatorname{dom} \left(\tau^* \left(\chi_{(C,X)}\right)\right) = \operatorname{dom} \left(\mathcal{G}_{\tau, K(C,X)} \left(\chi_{(C,X)}\right)\right) \cong \operatorname{dom} \left(\mathcal{G}_{\varphi, K(C,X)} \left(\Lambda_{(C,X)}\right)\right) =$$
$$= \operatorname{dom} \left(\varphi^* \left(\Lambda_{(C,X)}\right)\right).$$

So the counits of the adjoint equivalences $\mathcal{G}_{\tau, K(C,X)} \dashv {\mathcal{G}_{\tau, K(C,X)}}^{-1}$ give isomorphisms

$$\xi_{(C,X)} \colon Q^{(C,X)} = \left(\operatorname{dom} \circ \tau^* \circ U\right)(C, X) \xrightarrow{\simeq} \left(\operatorname{dom} \circ \varphi^* \circ K'\right)(C, X) = H^{(C,X)}$$

over $K(C, X)$, where the map to $K(C, X)$ from the right hand side is $\mathcal{G}_{\varphi, K(C,X)} \left(\Lambda_{(C,X)}\right)$. We prove that these isomorphisms form a 2-natural transformation ξ. Take then $(f, \nu) \colon (C, X) \to (D, X')$ in $\int W$. By Remark 3.2.3, we can express the action of $\operatorname{dom} \circ \tau^*$ on the morphism

$$U(f, \nu) = \begin{tikzcd}
K(C, X) \arrow[rr, "K(f,\nu)"] \arrow[dr, swap, "\chi_{(C,X)}"] & \arrow[d, phantom, "\overset{\chi_{f,\nu}}{\Longrightarrow}"] & K(D, X') \arrow[dl, "\chi_{(D,X')}"] \\
& \Omega &
\end{tikzcd}$$

in terms of $\operatorname{dom}\left(\mathcal{G}_{\tau,K(C,X)}(\chi_{f,\nu})\right)$. Analogously, we express $\operatorname{dom}\left(\varphi^*\left(K(f,\nu),\Lambda_{f,\nu}\right)\right)$ in terms of $\operatorname{dom}\left(\mathcal{G}_{\varphi,K(C,X)}\left(\Lambda_{f,\nu}\right)\right)$. Consider the composite pullbacks

$$\begin{tikzcd}
S^H \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & H^{(D,X')} \arrow[r] \arrow[d, "\mathcal{G}_\varphi\left(\Lambda_{(D,X')}\right)"'] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d, "\varphi"] & S^Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Q^{(D,X')} \arrow[r] \arrow[d, "\mathcal{G}_\tau\left(\chi_{(D,X')}\right)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Omega_\bullet \arrow[d, "\tau"] \\
K(C,X) \arrow[r, "K(f,\nu)"'] & K(D,X') \arrow[r, "\Lambda_{(D,X')}"'] & F & K(C,X) \arrow[r, "K(f,\nu)"'] & K(D,X') \arrow[r, "\chi_{(D,X')}"'] & \Omega
\end{tikzcd}$$

and call R^H and R^Q respectively the pullbacks of φ and of τ along the composites. By construction of χ and by the action of ${\mathcal{G}_{\tau,K(C,X)}}^{-1}$ on morphisms, we calculate that $\mathcal{G}_{\tau,K(C,X)}(\chi_{f,\nu})$ is precisely the composite

$$Q^{(C,X)} \xrightarrow[\xi_{(C,X)}]{\simeq} H^{(C,X)} \xrightarrow{\mathcal{G}_{\varphi,K(C,X)}\left(\Lambda_{f,\nu}\right)} R^H \cong S^H \xrightarrow[K(f,\nu)^*\xi^{-1}_{(D,X')}]{\simeq} S^Q \cong R^Q$$

Notice that we also need one triangular equality of the adjoint equivalence $\mathcal{G}_{\tau,K(C,X)}$ to handle the η^{-1} part of $\mathcal{G}^{-1}_{\tau,K(f,\nu)}$. In order to obtain $\operatorname{dom}\left(\tau^*\left(K(f,\nu),\chi_{f,\nu}\right)\right)$, by Remark 3.2.3, we compose $\mathcal{G}_{\tau,K(C,X)}(\chi_{f,\nu})$ with

$$R^Q \cong S^Q \to \; Q^{(D,X')}$$

We conclude the naturality of ξ by definition of $K(f,\nu)^*\xi^{-1}_{(D,X')}$. To prove that ξ is 2-natural, consider a 2-cell $\delta\colon (f,\nu) \Rightarrow (f',\nu')\colon (C,X) \to (D,X')$ in $\int W$. Both

$$\xi_{(D,X')} * \left(\operatorname{dom}\circ\tau^*\circ U\right)(\delta)$$

$$\left(\operatorname{dom}\circ\varphi^*\circ K'\right)(\delta) * \xi_{C,X}$$

give the unique lifting of $K(\delta) * \mathcal{G}_{\tau,K(C,X)}\left(\chi_{(C,X)}\right)$ to $\xi_{(D,X')}\circ\left(\operatorname{dom}\circ\tau^*\circ U\right)(f,\nu)$ along $\mathcal{G}_{\varphi,K(D,X')}\left(\Lambda_{(D,X')}\right)$. Thus ξ is 2-natural. We then obtain a 2-natural isomorphism

$$\zeta\colon \operatorname{dom}\circ\tau^*\circ(z\circ -)\circ K' \cong \operatorname{dom}\circ\varphi^*\circ K'$$

whose general component on $(C,X) \in \int W$ is over $K(C,X)$.

As a consequence, there is an isomorphism $j\colon G \cong V$, respecting the universal cartesian-marked oplax cocones Θ^G and Θ^V that exhibit the two as colimits:

$$\begin{tikzcd}
\Delta 1 \arrow[r, Rightarrow, "\Theta^G", "\text{oplax}^{\text{cart}}"'] \arrow[d, Rightarrow, "\Theta^V"', "\text{oplax}^{\text{cart}}"] & \mathcal{L}\left(\left(\operatorname{dom}\circ\varphi^*\circ K'\right)(-), G\right) \arrow[d, Rightarrow, "j\circ -"] \\
\mathcal{L}\left(\left(\operatorname{dom}\circ\tau^*\circ(z\circ -)\circ K'\right)(-), V\right) \arrow[r, Rightarrow, "-\circ\zeta^{-1}_{(-)}"'] & \mathcal{L}\left(\left(\operatorname{dom}\circ\varphi^*\circ K'\right)(-), V\right)
\end{tikzcd}$$

We want to show that the following triangle is commutative:

$$\begin{tikzcd}
G \arrow[rr, "\simeq", "j"'] \arrow[dr, "\varphi"'] & & V \arrow[dl, "\mathcal{G}_{\tau,F}(z)"] \\
& F &
\end{tikzcd} \tag{3.5}$$

Since G is a cartesian-marked oplax conical colimit, it suffices to show that

$$(\varphi\circ -)\circ\Theta^G = \left(\mathcal{G}_{\tau,F}(z)\circ -\right)\circ(j\circ -)\circ\Theta^G$$

Whence it suffices to show that

$$(\varphi\circ -)\circ\Theta^G = \left(\mathcal{G}_{\tau,F}(z)\circ - \circ\zeta^{-1}_{(-)}\right)\circ\Theta^V$$

Given $(C,X)\in\int W$, the two have equal components on (C,X) since by Construction 3.2.4

$$\mathcal{G}_{\tau,F}(z)\circ\Theta^V_{(C,X)}\circ\zeta^{-1}_{(C,X)} = \Lambda_{(C,X)}\circ\mathcal{G}_{\tau,K(C,X)}\left(\aleph_{(C,X)}\right)\circ\zeta^{-1}_{(C,X)} = \Lambda_{(C,X)}\circ\mathcal{G}_{\varphi,K(C,X)}\left(\Lambda_{(C,X)}\right),$$

using that $\zeta_{(C,X)}$ is over $K(C,X)$. Given $(f,\nu)\colon (D,X')\leftarrow (C,X)$ in $\int W$, also the structure 2-cells of the two cartesian-marked oplax natural transformations on (f,ν) are equal, since by Construction 3.2.4

$$\mathcal{G}_{\tau,F}(z)*\Theta^V_{(f,\nu)}*\zeta^{-1}_{(C,X)} = \Lambda_{f,\nu}*\left(\mathcal{G}_{\tau,K(C,X)}\left(\aleph_{(C,X)}\right)\circ\zeta^{-1}_{(C,X)}\right) = \Lambda_{f,\nu}*\mathcal{G}_{\varphi,K(C,X)}\left(\Lambda_{(C,X)}\right).$$

Therefore the triangle of equation (3.5) is commutative and φ is in the essential image of $\mathcal{G}_{\tau,F}$. □

Remark 3.2.9. In the proof of Theorem 3.2.8, we have actually proved the following sharper result, that involves the classification of single discrete opfibrations φ. We also show that the operation of normalization described in Theorem 3.2.8 is necessary.

Corollary 3.2.10 (of the proof of Theorem 3.2.8). *Let $I\colon \mathcal{Y} \to \mathcal{L}$ be a fully faithful dense generator of $\mathcal{L}$. Assume that $\tau\colon \Omega_\bullet \to \Omega$ is a 2-classifier in $\mathcal{L}$, and let $\varphi\colon G \to F$ be an arbitrary discrete opfibration in $\mathcal{L}$. Consider K and Λ as in Construction 3.2.7. The following properties are equivalent:*

(i) φ is classified by τ, i.e. φ is in the essential image of $\mathcal{G}_{\tau,F}$;

(ii) for every $(C,X) \in \int W$ the change of base $\mathcal{G}_{\varphi,K(C,X)}(\Lambda_{C,X})$ of φ along $\Lambda_{C,X}$ is in the essential image of $\mathcal{G}_{\tau,K(C,X)}$, and the operation of normalization described in Theorem 3.2.8 starting from φ is possible.

Proof. The proof of $(ii) \Rightarrow (i)$ is exactly as the proof of Theorem 3.2.8, using the essential image of $\mathcal{G}_{\tau,K(C,X)}$ in place of $\mathcal{DOpFib}^{\mathrm{P}}(K(C,X))$.

We prove $(i) \Rightarrow (ii)$. By assumption, there exists a characteristic morphism z for φ. For every (C,X), we then have that $z \circ \Lambda_{(C,X)}$ is a characteristic morphism for $\mathcal{G}_{\varphi,K(C,X)}(\Lambda_{C,X})$. It remains to prove that the operation of normalization described in Theorem 3.2.8 starting from φ is possible. We can choose the quasi-inverse of $\mathcal{G}_{\tau,K(C,X)}$ (restricted to its essential image) so that for every $b\colon K(C,X) \to F$ in $\mathcal{L}$

$$\mathcal{G}^{-1}_{\tau,K(C,X)}(\mathcal{G}_{\varphi,K(C,X)}(b)) = z \circ b$$

and the component of the counit on $\mathcal{G}_{\varphi,K(C,X)}(b)$ is given by the pseudofunctoriality of the pullback. Then for every morphism $(f,\mathrm{id})\colon (D,F(f)(X)) \leftarrow (C,X)$ in $\int W$

$$\chi_{(C,X)} = z \circ \Lambda_{(C,X)} = z \circ \Lambda_{(D,F(f)(X))} \circ K(f,\mathrm{id}) = \chi_{(D,F(f)(X))} \circ K(f,\mathrm{id}).$$

In order to prove that $\chi_{f,\mathrm{id}} = \mathrm{id}$, it suffices to prove that $\mathcal{G}_{\tau,K(C,X)}(\chi_{f,\mathrm{id}}) = \mathrm{id}$. It is straightforward to see that this holds, using the recipe described in the proof of Theorem 3.2.8. Indeed it is just given by the compatibilities of a pullback along a composite of three morphisms with the composite pullbacks. □

Corollary 3.2.11. *Let $I\colon \mathcal{Y} \to \mathcal{L}$ be a fully faithful dense generator of $\mathcal{L}$. Let $\omega\colon 1 \to \Omega$ be a morphism in $\mathcal{L}$ such that the lax limit of the arrow ω satisfies a* *fixed pullback-stable property* P. *If for every* $Y \in \mathcal{Y}$

$$\widehat{\mathcal{G}}_{\omega,I(Y)}\colon \mathcal{L}\,(I(Y),\Omega) \to \mathcal{DOpFib}^{\mathrm{P}}(I(Y))$$

is an equivalence of categories and the operation of normalization described in Theorem 3.2.8 (*starting from every* φ) *is possible, then* ω *is a good 2-classifier in* $\mathcal{L}$ *with respect to* P.

Proof. By Remark 3.1.13, taking τ to be the lax limit of the arrow ω, we have that $\widehat{\mathcal{G}}_{\omega,F} = \mathcal{G}_{\tau,F}$ for every $F \in \mathcal{L}$. We conclude by Theorem 3.2.8. □

Remark 3.2.12. The theorems of reduction of the study of 2-classifiers to dense generators offer great benefits. To have an idea of this, we can look at the following example.

In Chapter 5, we will then apply the theorems of reduction to dense generators to the cases of 2-presheaves (i.e. prestacks) and stacks; see also Section 3.3.

Example 3.2.13. The theorems of reduction to dense generators allow us to deduce all the major properties of the Grothendieck construction (or category of elements) from the trivial observation that everything works well over the singleton category.

Indeed the singleton category $\mathit{1}$ is a dense generator in $\mathcal{C}at$. So we can just look at the discrete opfibrations over $\mathit{1}$. Let $\omega = 1\colon \mathit{1} \to \mathcal{S}et$. The lax limit $\tau\colon \mathcal{S}et_{\bullet} \to \mathcal{S}et$ certainly has small fibres. In fact one immediately sees that any comma object from ω, say to $F\colon \mathcal{B} \to \mathcal{S}et$, gives a discrete opfibration with small fibres, since the fibre over $B \in \mathcal{B}$ is isomorphic to $\mathcal{S}et\,(1, F(B))$. Let $\mathrm{P} = \mathrm{s}$ be the property of having small fibres.

The functor

$$\widehat{\mathcal{G}}_{\omega,\mathit{1}}\colon \mathcal{C}at\,(\mathit{1},\mathcal{S}et) \to \mathcal{DOpFib}^{\mathrm{s}}(1) \cong \mathcal{S}et$$

sends a functor $\mathit{1} \to \mathcal{S}et$ to the set it picks, so it is clearly an equivalence of categories. By the theorems of reduction to dense generators, we deduce that the construction of the category of elements is fully faithful and classifies all discrete opfibrations with small fibres (deducing thus the whole Example 3.1.15). Indeed, let $p\colon \mathcal{E} \to \mathcal{B}$ be a discrete opfibration with small fibres. The density of $\mathit{1}$ allows us to express $\mathcal{B}$ as a cartesian-marked oplax conical colimit of the constant at 1 functor $\Delta 1$. By Example 1.1.25 we know that the universal cartesian-marked oplax cocone Λ is given by

$$\forall \begin{array}{c} B' \\ \uparrow f \\ B \end{array} \text{ in } \mathcal{B} \qquad \begin{tikzcd} \mathit{1} \arrow[r, "B"] \arrow[d, equal] \arrow[d, phantom, shift left=5ex, "\Downarrow f"] & \mathcal{B} \\ \mathit{1} \arrow[ru, bend right, "B'"'] & \end{tikzcd}$$

Following Construction 3.2.7 and the proof of Corollary 3.2.11, we consider the sigma natural transformation χ given by the composite

$$\Delta 1 \xRightarrow[\text{oplax}^{\text{cart}}]{\Lambda} \mathcal{C}at\,(\Delta 1(-), \mathcal{B}) \xRightarrow[\text{pseudo}]{\mathcal{G}_{p,\Delta 1(-)}} \mathcal{DO}p\mathcal{F}ib^{\,\mathrm{s}}(\Delta 1(-)) \xRightarrow[\text{pseudo}]{\widehat{\mathcal{G}}^{-1}_{\omega,\Delta 1(-)}} \mathcal{C}at\,(\Delta 1(-), \mathcal{S}et)\,.$$

But $\mathcal{G}_{p,\Delta 1(-)}$ is strict 2-natural (thanks to our choice of pullbacks). Similarly, also $\widehat{\mathcal{G}}_{\omega,\Delta 1(-)}$ and hence its quasi-inverse are strict 2-natural. So that χ is already cartesian-marked oplax natural. Explicitly, χ is given by

$$\forall \begin{array}{c} B' \\ \uparrow f \\ B \end{array} \text{ in } \mathcal{B} \qquad \begin{tikzcd} \mathit{1} \arrow[r, "(p)_B"] \arrow[d, equal] \arrow[d, phantom, shift left=5ex, "\Downarrow f_*"] & \mathcal{S}et \\ \mathit{1} \arrow[ru, bend right, "(p)_{B'}"'] & \end{tikzcd}$$

where $(p)_B$ is the fibre of p on B, since the pullback of p along each $B\colon \mathit{1} \to \mathcal{B}$ gives precisely the fibre over B. This induces the known characteristic morphism $\mathcal{B} \to \mathcal{S}et$ for p, collecting together the fibres of p. So the concrete recipe for characteristic morphisms described in the proof of Theorem 3.2.8 recovers the usual recipe for the quasi-inverse of the category of elements construction.

3.3. Towards a 2-classifier in prestacks

In this section, we start looking at the possibility to apply our theorems of reduction to dense generators to the case of prestacks. Our theorems offer great benefits here, allowing us to consider the classification just over representables. We find a problem that will be solved by the work of Chapter 4. In Chapter 5, we will then produce a good 2-classifier in prestacks and restrict it to a good 2-classifier in stacks.

Remark 3.3.1. We consider the 2-category $\mathcal{L} = [\mathcal{C}^{\text{op}}, \mathcal{C}at]$ of 2-presheaves on a small category $\mathcal{C}$ (that is, prestacks on $\mathcal{C}$). Notice that this 2-category is complete and cocomplete, since $\mathcal{C}at$ is so.

Construction 3.3.2. We search for a good 2-classifier $\omega\colon 1 \to \Omega$ in $[\mathcal{C}^{\text{op}}, \mathcal{C}at]$. Looking at the archetypal example of $\mathcal{C}at$, we expect such a good 2-classifier to classify all discrete opfibrations in $[\mathcal{C}^{\text{op}}, \mathcal{C}at]$ that have, in some sense, *small fibres*. The work of Section 4.2, in Chapter 4, will allow us to define *having small fibres* for a discrete opfibration in prestacks. We will denote such a pullback-stable property as s. Anyway, another problem appears, as we show below. This problem will as well be solved in Chapter 4.

By Example 3.1.25, representables form a dense generator

$$I = \text{y}\colon \mathcal{C} \to [\mathcal{C}^{\text{op}}, \mathcal{C}at]\,.$$

Then, by our theorems of reduction to dense generators (see Corollary 3.2.11), we will be able to look just at the functors

$$\widehat{\mathcal{G}}_{\omega,\text{y}(C)}\colon [\mathcal{C}^{\text{op}}, \mathcal{C}at]\,(\text{y}\,(C), \Omega) \to \mathcal{DOpFib}^{\text{s}}(\text{y}\,(C))$$

with $C \in \mathcal{C}$. Notice that the left hand side is isomorphic to $\Omega(C)$, by the Yoneda lemma. As we want all the $\widehat{\mathcal{G}}_{\omega,\text{y}(C)}$'s with $C \in \mathcal{C}$ to be equivalences of categories (forming then a pseudonatural equivalence by Proposition 3.1.9), the assignment of Ω is forced up to equivalence to be

$$C \overset{\Omega}{\mapsto} \mathcal{DOpFib}^{\text{s}}(\text{y}\,(C))\,.$$

Remark 3.3.3. This is a nice generalization of what happens in dimension 1. Indeed recall that the subobject classifier in 1-dimensional presheaves sends C to the set of sieves on C. And sieves on C are equivalently the subfunctors of $\text{y}\,(C)$. In line with the philosophy to upgrade subobjects to discrete opfibrations (discussed in 3.1.1), discrete opfibrations over $\mathrm{y}(C)$ generalize the concept of sieve to dimension 2. Notice that Ω coincides with the composite

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{y}^{\mathrm{op}}} [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]^{\mathrm{op}} \xrightarrow{\mathcal{DO}p\mathcal{F}ib^{\mathrm{s}}(-)} \mathcal{CAT}$$

and is thus a pseudofunctor by Proposition 3.1.8. We then take as $\omega\colon 1 \to \Omega$ the pseudonatural transformation with component on $C \in \mathcal{C}$ that picks the identity $\mathrm{id}_{\mathrm{y}(C)}$ on $\mathrm{y}(C)$. However, Ω is not a strict 2-functor and, a priori, does not land in $\mathcal{C}at$; so Ω cannot be a 2-classifier in $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$. Thanks to Chapter 4, we will be able to produce a nice concrete strictification of Ω. Although it was already known that any pseudofunctor can be strictified, by the theory developed by Power in [40] and later by Lack in [29], the work of Chapter 4 can be applied to produce an explicit and easy to handle strictification of Ω, which in addition lands in $\mathcal{C}at$. Moreover, as we will present in Chapter 5, such strictification can also be restricted in a natural way to a good 2-classifier in stacks.

4. Indexed Grothendieck construction

This chapter is based on our joint work with Caviglia [12].

In this chapter, we produce an indexed version of the Grothendieck construction, that does not seem to appear in the literature. The Grothendieck construction is a fundamental tool in category theory that has had numerous applications in geometry, logic and algebra. It gives the possibility to exploit the advantages of both indexed categories and Grothendieck fibrations. Moreover, also the construction itself has important and useful consequences, such as the conicalization of all weighted $\mathcal{S}et$-enriched (i.e. ordinary) limits and the famous explicit formula for ordinary Kan extensions.

So we believe that an indexed version of the Grothendieck construction can be very fruitful as well. Our motivating application is to produce a nice candidate for a good 2-classifier in the 2-category $[\mathcal{A}^{\mathrm{op}}, \mathcal{C}at]$ of 2-presheaves (i.e. prestacks) that classifies all discrete opfibrations with small fibres, towards a 2-dimensional elementary topos structure on prestacks and stacks. Recall from Section 3.1 that a 2-classifier, which is a generalization of the concept of subobject classifier to dimension 2, can also be thought of as a Grothendieck construction inside a 2-category. So it is natural to expect an indexed version of the Grothendieck construction to give a 2-classifier in the 2-category of 2-presheaves. Our indexed Grothendieck construction will allow us to solve the problem we found in Section 3.3, producing a concrete and easy to handle strictification $\widetilde{\Omega}$ of the pseudofunctor Ω of Construction 3.3.2 that clearly lands in small categories. We describe this in Example 4.4.9. We will show in Chapter 5 that such $\widetilde{\Omega}$ is a good 2-classifier in prestacks and that it can be restricted to a good 2-classifier in stacks.

Our main results (Theorem 4.3.7 and Theorem 4.3.9) are condensed in the following theorem. Op and non-split variations are also considered in Remark 4.3.12 and Remark 4.3.13.

Theorem 4.0.1. *Let $\mathcal{A}$ be a small category and consider the functor 2-category $[\mathcal{A}, \mathcal{C}at]$. For every 2-functor $F\colon \mathcal{A} \to \mathcal{C}at$, there is an equivalence of categories*

$$\mathcal{O}p\mathcal{F}ib_{[\mathcal{A},\mathcal{C}at]}(F) \simeq \left[\int F, \mathcal{C}at\right]$$

between split opfibrations in the 2-category $[\mathcal{A}, \mathcal{C}at]$ over F and 2-(co)presheaves on the Grothendieck construction $\int F$ of F.

This restricts to an equivalence of categories

$$\mathcal{DO}p\mathcal{F}ib^{\mathrm{s}}{}_{[\mathcal{A},\mathcal{C}at]}(F) \simeq \left[\int F, \mathcal{S}et\right]$$

between discrete opfibrations in $[\mathcal{A}, \mathcal{C}at]$ over F with small fibres and 1-copresheaves on $\int F$.

Moreover, both the equivalences of categories above are pseudonatural in F.

When $\mathcal{A} = \mathit{1}$, we recover the usual Grothendieck construction. Indeed $[\mathcal{A}, \mathcal{C}at]$ reduces to $\mathcal{C}at$, a 2-functor $F\colon \mathit{1} \to \mathcal{C}at$ is just a small category $\mathcal{C}$ and $\int F = \mathcal{C}$. So we find

$$\mathcal{O}p\mathcal{F}ib(\mathcal{C}) \simeq [\mathcal{C}, \mathcal{C}at]\,.$$

But we introduce an indexed version of the Grothendieck construction that allows $\mathcal{A}$ to be an arbitrary small category and $F\colon \mathcal{A} \to \mathcal{C}at$ to be an arbitrary 2-functor. Interestingly, the data of the opfibrations in $[\mathcal{A}, \mathcal{C}at]$ over F are still packed in a $\mathcal{C}at$-valued copresheaf, now on the Grothendieck construction of F.

We can think of the indexed Grothendieck construction as a simultaneous Grothendieck construction on every index $A \in \mathcal{A}$, taking into account the bonds between different indexes. Indeed, an opfibration φ in $[\mathcal{A}, \mathcal{C}at]$ is, in particular, a natural transformation such that every component φ_A is a Grothendieck opfibration (in $\mathcal{C}at$); see Section 4.2. Our construction essentially applies the quasi-inverse of the usual Grothendieck construction to every component of φ at the same time. All the obtained copresheaves in $\mathcal{C}at$ are then collected into a single total copresheaf, exploiting the usual Grothendieck construction on F.

The restricted equivalence of categories

$$\mathcal{DOpFib}^{\mathrm{s}}{}_{[\mathcal{A},\mathcal{C}at]}(F) \simeq \left[\textstyle\int F, \mathcal{S}et\right]$$

further reduces, when $F\colon \mathcal{A} \to \mathcal{S}et$, to the well-known equivalence

$$[\mathcal{A}, \mathcal{S}et]/_{F} \simeq \left[\textstyle\int F, \mathcal{S}et\right].$$

When F is a representable $\mathrm{y}(A) : \mathcal{A} \to \mathcal{S}et$, this is the famous equivalence

$$[\mathcal{A}, \mathcal{S}et]\Big/_{\mathrm{y}(A)} \simeq [\mathcal{A}/_{A}, \mathcal{S}et]$$

between slices of (co)presheaves and (co)presheaves on slices. Our theorem also guarantees its pseudonaturality in A, which does not seem to be stated in the literature.

The last equivalence between slices of (co)presheaves and (co)presheaves on slices had many applications in geometry and logic. In particular, it is the archetypal case of the fundamental theorem of elementary topos theory, showing that every slice of a Grothendieck topos is a Grothendieck topos. Our equivalence

$$\mathcal{OpFib}_{[\mathcal{A},\mathcal{C}at]}(F) \simeq \left[\textstyle\int F, \mathcal{C}at\right]$$

gives a 2-dimensional generalization of this, and we thus expect it to be very fruitful. Indeed, the concept of (op)fibrational slice has recently been proposed as the correct upgrade of slices to dimension 2. Rather than taking all maps into a fixed element, we restrict to the (op)fibrations over that element. This idea appears in Ahrens, North and van der Weide's [2], where it is attributed to Shulman, but was already implicit in previous literature (e.g. in the work of

Quillen). Our equivalence can be thought of as saying that every (op)fibrational slice of a Grothendieck 2-topos is again a Grothendieck 2-topos.

The strategy to prove our main theorem will be to use that the Grothendieck construction of a 2-functor $F\colon \mathcal{A} \to \mathcal{C}at$ is equivalently the oplax colimit of F. So that we will be able to apply the usual Grothendieck construction on every index $A \in \mathcal{A}$. We will then show that all the opfibrations produced for each A can be collected into an opfibration in $[\mathcal{A}, \mathcal{C}at]$ over F. For this, we will also need to prove that the usual Grothendieck construction is pseudonatural in the base category. The chain of abstract processes above will then be very useful to conclude the pseudonaturality in F of the indexed Grothendieck construction.

We will also give an explicit description of the indexed Grothendieck construction, in Construction 4.3.8, so that it can be applied more easily.

We will conclude showing some interesting examples, choosing particular $\mathcal{A}$'s and F's in Theorem 4.0.1. Among them, we will consider the cases $\mathcal{A} = \mathit{2}$ (arrows between opfibrations) and $\mathcal{A} = \Delta$ (cosimplicial categories).

Outline of the chapter

In Section 4.1, we recall that the Grothendieck construction can be equivalently expressed as an oplax colimit. We prove that the equivalence of categories given by the Grothendieck construction is pseudonatural in the base category.

In Section 4.2, after recalling the notion of opfibration in a 2-category, we show an equivalent characterization of opfibrations in $[\mathcal{A}, \mathcal{C}at]$. This also allows us to define *having small fibres* for a discrete opfibration in $[\mathcal{A}, \mathcal{C}at]$. We produce a pseudofunctor $F \mapsto \mathcal{O}p\mathcal{F}ib_{[\mathcal{A}, \mathcal{C}at]}(F)$.

In Section 4.3, we present our main theorems, proving an equivalence of categories between (split) opfibrations in $[\mathcal{A}, \mathcal{C}at]$ over F and 2-copresheaves on $\int F$. We also show that such equivalence is pseudonatural in F. We present the explicit indexed Grothendieck construction.

In Section 4.4, we show some interesting examples. In particular, we obtain a nice candidate for a good 2-classifier in prestacks that classifies all discrete opfibrations with small fibres.

4.1. Some properties of the Grothendieck construction

In this section, we recall that the Grothendieck construction can be equivalently expressed as an oplax colimit. As we could not find a proof of this in the literature, we show a proof below (Theorem 4.1.7). It will also be important to recall from Theorem 1.3.8 that the Grothendieck construction is as well equivalently given by a lax comma object (or by a strict 3-pullback) in 2-$\mathcal{C}at_{\text{lax}}$.

We then prove that the equivalence of categories given by the Grothendieck construction is pseudonatural in the base category.

Remark 4.1.1. In this chapter, we focus on the Grothendieck construction of (strict) 2-functors $F\colon \mathcal{C} \to \mathcal{C}at$ with $\mathcal{C}$ a small category, which correspond with split opfibrations over $\mathcal{C}$. We will consider variations of this setting in Remark 4.3.12 (fibrations) and Remark 4.3.13 ($\mathcal{C}$ a 2-category and non-split opfibrations).

We denote as $\mathcal{O}p\mathcal{F}ib(\mathcal{C})$ the subcategory of $\mathcal{C}at/_{\mathcal{C}}$ on split opfibrations over $\mathcal{C}$ and cleavage preserving morphisms. We denote as $\mathcal{DO}p\mathcal{F}ib^{\mathrm{s}}(\mathcal{C})$ its full subcategory on discrete opfibrations over $\mathcal{C}$ with small fibres; recall that any functor over $\mathcal{C}$ between discrete opfibrations over $\mathcal{C}$ is automatically cleavage preserving.

Remark 4.1.2. The pullback $H^*p\colon H^*\mathcal{E} \to \mathcal{D}$ of a split opfibration $p\colon \mathcal{E} \to \mathcal{C}$ along $H\colon \mathcal{D} \to \mathcal{C}$ is a split opfibration. We can choose the cleavage of H^*p to make the universal square that exhibits the pullback into a cleavage preserving morphism.

Remark 4.1.3. Let $F\colon \mathcal{C} \to \mathcal{C}at$ be a functor with $\mathcal{C}$ small. Recall that every morphism $(f,\alpha)\colon (C,X) \to (D,X')$ in the Grothendieck construction $\int F$ of F

can be factorized as

$$(C,X)\xrightarrow{(f,\mathrm{id})}(D,F(f)(X))\xrightarrow{(\mathrm{id},\alpha)}(D,X')$$

That is, as a cartesian morphism of the cleavage followed by a morphism which is over the identity (also called vertical morphism).

This means that the Grothendieck construction of F is somehow given by collecting all $F(C)$ together, where we have the morphisms (id,α), and adding the morphisms (f,id) to handle change of index. This idea will be made precise in Theorem 4.1.7.

The following fundamental theorem is due to Grothendieck [22] (see also Borceux's [8]).

Theorem 4.1.4 ([22])**.** *The Grothendieck construction extends to an equivalence of categories*

$$\mathcal{G}(-):\ [\mathcal{C},\mathcal{C}at]\xrightarrow{\sim}\mathcal{O}p\mathcal{F}ib(\mathcal{C})$$

Given a natural transformation $\gamma\colon F\Rightarrow G\colon\mathcal{C}\to\mathcal{C}at$*, the functor* $\mathcal{G}(\gamma):\int F\to\int G$ *is defined to send* (C,X) *to* $(C,\gamma(X))$ *and* $(f,\alpha)\colon(C,X)\to(D,X')$ *to* $(f,\gamma_D(\alpha))$*.*

The quasi-inverse is given by taking fibres on every $C\in\mathcal{C}$*.*

Moreover, the equivalence above restricts to an equivalence of categories

$$\mathcal{G}(-):\ [\mathcal{C},\mathcal{S}et]\xrightarrow{\sim}\mathcal{DO}p\mathcal{F}ib^{\mathrm{s}}(\mathcal{C})$$

Aiming at proving that the Grothendieck construction is equivalently given by an oplax colimit, we recall the definition of oplax colimit.

Definition 4.1.5. Let $F\colon\mathcal{C}\to\mathcal{D}$ be a 2-functor with $\mathcal{C}$ small. The *oplax (conical) colimit of* F, denoted as $\operatorname{oplax-colim}F$, is (if it exists) an object $K\in\mathcal{D}$ together with an isomorphism of categories

$$\mathcal{D}(K,U)\cong[\mathcal{C}^{\mathrm{op}},\mathcal{C}at]_{\mathrm{oplax}}(\Delta 1,\mathcal{D}(F(-),U))$$

2-natural in $U \in \mathcal{D}$, where $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]_{\mathrm{oplax}}$ is the 2-category of 2-functors, oplax natural transformations and modifications from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{C}at$. $\Delta 1$ is the functor which is constant at singleton category 1 and the right hand side of the isomorphism above should be thought of as the category of oplax cocones on F with vertex U. Indeed we also have an isomorphism

$$[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]_{\mathrm{oplax}}\,(\Delta 1, \mathcal{D}\,(F(-), U)) \cong [\mathcal{C}, \mathcal{D}]_{\mathrm{lax}}\,(F, \Delta U)$$

2-natural in U, where $[\mathcal{C}, \mathcal{D}]_{\mathrm{lax}}$ is the 2-category of 2-functors, lax natural transformations and modifications, and ΔU is the functor which is constant at U.

Remark 4.1.6. When $\text{oplax-colim} F$ exists, taking $U = K$ and considering the identity on K gives us in particular a lax natural transformation

$$\lambda \colon F \underset{\mathrm{lax}}{\Longrightarrow} \Delta K$$

which is called the *universal oplax cocone* on F.

An equivalent way to show that $K = \text{oplax-colim} F$ is to exhibit such a lax natural transformation λ that is universal in the following 2-categorical sense:

(i) for every lax natural transformation $\sigma \colon F \underset{\mathrm{lax}}{\Longrightarrow} \Delta U$, there exists a unique morphism $s \colon K \to U$ in $\mathcal{D}$ such that $\Delta s \circ \lambda = \sigma$;

(ii) for every $s, t \colon K \to U$ in $\mathcal{D}$ and every modification $\Xi \colon \Delta s \circ \lambda \Rrightarrow \Delta t \circ \lambda$, there exists a unique 2-cell $\chi \colon s \Rightarrow t$ in $\mathcal{D}$ such that $\Delta\chi \star \lambda = \Xi$.

We will need the following known characterization of the Grothendieck construction. As we could not find a proof of this in the literature, we show a proof here.

Theorem 4.1.7. *Let $\mathcal{C}$ be a small category and let $F \colon \mathcal{C} \to \mathcal{C}at$ be a 2-functor. The Grothendieck construction $\int F$ of F is equivalently the oplax (conical) colimit of the 2-diagram F:*

$$\int F = \text{oplax-colim} F$$

Proof. Following Remark 4.1.6, we produce a lax natural transformation $\mathrm{inc}\colon F \underset{\mathrm{lax}}{\Longrightarrow} \Delta \int F$ and prove that it is universal in the 2-categorical sense. For every $C \in \mathcal{C}$ we define the component of inc on C to be the functor

$$\begin{tikzcd} \mathrm{inc}_C\colon & F(C) \arrow[r] & \int F \\ & X \arrow[d, "\alpha"] \arrow[r, mapsto] & (C,X) \arrow[d, "(\mathrm{id},\alpha)"] \\ & X' & (C,X') \end{tikzcd}$$

For every morphism $f\colon C \to D$ in $\mathcal{C}$, we define the structure 2-cell of inc on f to be the natural transformation

$$\begin{tikzcd} F(C) \arrow[r, "\mathrm{inc}_C"] \arrow[d, "F(f)"'] \arrow[rd, phantom, "\Downarrow \mathrm{inc}_f"] & \int F \\ F(D) \arrow[ru, bend right, "\mathrm{inc}_D"'] & \end{tikzcd}$$

that has components $(\mathrm{inc}_f)_X = (f, \mathrm{id})$ for every $X \in F(C)$. The naturality of inc_f expresses

$$(f, \mathrm{id}) \circ (\mathrm{id}, \alpha) = (\mathrm{id}, F(f)(\alpha)) \circ (f, \mathrm{id}).$$

As explained with more detail in Construction 1.1.4, to get the whole $\int F$ we just need the two kinds of morphisms (id, α) and (f, id) as building blocks. This is what will ensure the universality of inc. Composition of morphisms of type (id, α) corresponds with the functoriality of inc_C. While composition of morphisms of type (f, id) corresponds with the lax naturality of inc. We could then define general morphisms to be formal composites $(\mathrm{id}, \alpha) \circ (f, \mathrm{id})$, following the factorization of morphisms in $\int F$ described in Remark 4.1.3. And the equation above, that swaps the two kinds of morphisms, tells how to reduce every composition to this form.

We prove that inc is universal in the 2-categorical sense. Given a lax natural transformation $\sigma\colon F \underset{\mathrm{lax}}{\Longrightarrow} \Delta U$, we show that there exists a unique $s\colon \int F \to U$ such that $\Delta s \circ \mathrm{inc} = \sigma$. These conditions impose to define for every $(f, \alpha)\colon (C, X) \to (D, X')$ in $\int F$

$$s(C, X) = (s \circ \mathrm{inc}_C)(X) = \sigma_C(X)$$

$$s(\mathrm{id}_D, \alpha) = (s \circ \mathrm{inc}_D)(\alpha) = \sigma_D(\alpha)$$

$$s(f,\mathrm{id}) = s\left((\mathrm{inc}_f)_X\right) = (\sigma_f)_X$$

So by the factorization described in Remark 4.1.3, we need to define

$$s(f,\alpha) = s(\mathrm{id},\alpha)\circ s(f,\mathrm{id}) = \sigma_D(\alpha)\circ(\sigma_f)_X.$$

s is a functor by naturality of σ_g, functoriality of σ_E and lax naturality of σ. And $\Delta s\circ\mathrm{inc} = \sigma$ by construction.

Take now $s,t\colon \int F\to U$ and a modification $\Xi\colon \Delta s\circ\mathrm{inc}\Rrightarrow\Delta t\circ\mathrm{inc}$. Ξ has as components on C natural transformations $\Xi_C\colon s\circ\mathrm{inc}_C\Rightarrow t\circ\mathrm{inc}_C$. We show that there exists a unique natural transformation $\chi\colon s\Rightarrow t$ such that $\Delta\chi\star\mathrm{inc} = \Xi$. We need to define

$$\chi_{(C,X)} = (\chi\star\mathrm{inc}_C)_X = \Xi_{C,X}$$

and this works. So inc is universal. $\square$

Remark 4.1.8. Let $\mathcal{C}$ be a small category and let $F\colon\mathcal{C}\to\mathcal{C}at$ be a 2-functor. $\int F$ is also the oplax (conical) colimit, with respect to the enrichment over $\mathcal{CAT}$, of the 2-diagram $\mathcal{C}\xrightarrow{F}\mathcal{C}at\hookrightarrow\mathcal{CAT}$. Indeed the Grothendieck construction of the latter composite is clearly just $\int F$.

We can now prove that the Grothendieck construction is pseudonatural in the base category. Such result does not seem to appear in the literature. We will use Theorem 1.3.8 from Chapter 1.

Proposition 4.1.9. *The equivalence of categories*

$$\mathcal{G}_{\mathcal{C}}\colon\ [\mathcal{C},\mathcal{C}at]\xrightarrow{\sim}\mathcal{O}p\mathcal{F}ib(\mathcal{C})$$

of Theorem 4.1.4 given by the Grothendieck construction is pseudonatural in $\mathcal{C}\in\mathcal{C}at^{\mathrm{op}}$.

Proof. The assignment $\mathcal{C}\mapsto\mathcal{O}p\mathcal{F}ib(\mathcal{C})$ extends to a pseudofunctor

$$\mathcal{O}p\mathcal{F}ib(-)\colon\mathcal{C}at^{\mathrm{op}}\to\mathcal{CAT}$$

that on the underlying category of the domain $\mathcal{C}at^{\mathrm{op}}$ is a restriction of the pseudofunctor that does the pullback. So given $H\colon\mathcal{D}\to\mathcal{C}$ and a split opfibration

$p\colon \mathcal{E} \to \mathcal{C}$, we define the action of $\mathit{OpFib}(-)$ on H to be the pullback functor H^*.

$$\begin{tikzcd}
H^*\mathcal{E} \arrow[r, "\widetilde{H}"] \arrow[d, "H^*p"'] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{D} \arrow[r, "H"'] & \mathcal{C}
\end{tikzcd}$$

Given a natural transformation $\alpha\colon H \Rightarrow K\colon \mathcal{D} \to \mathcal{C}$, we use the cleavage of p to define $\mathit{OpFib}(\alpha) = \alpha^*$ as the natural transformation that has as component on p the functor

$$\alpha^*p\colon H^*\mathcal{E} \to K^*\mathcal{E}$$

that sends $(D, E) \in H^*\mathcal{E}$ to $(D, (\alpha_D)_*E) \in K^*\mathcal{E}$. We will prove that $\mathit{OpFib}(-)$ is indeed a pseudofunctor in Proposition 4.2.9, for general split opfibrations in a 2-category. In that general setting, we can define α^* by lifting a 2-cell along an opfibration. This point of view is helpful to apply below the universal property of the lax comma object, using Theorem 1.3.8.

We define a pseudonatural transformation

$$\mathcal{G}_-\colon\ [-, \mathit{Cat}] \xRightarrow[\text{pseudo}]{} \mathit{OpFib}(-)$$

that has component on $\mathcal{C}$ given by $\mathcal{G}_{\mathcal{C}}$. Given a functor $H\colon \mathcal{D} \to \mathcal{C}$, we define the structure 2-cell $\mathcal{G}_H$ to be the natural isomorphism

$$\begin{tikzcd}
{[\mathcal{C}, \mathit{Cat}]} \arrow[r, phantom, "\simeq"] \arrow[d, "-\circ H"'] & \mathit{OpFib}(\mathcal{C}) \arrow[d, "H^*"] \arrow[dl, phantom, "\mathcal{G}_H \, \Swarrow \cong"] \\
{[\mathcal{D}, \mathit{Cat}]} \arrow[r, phantom, "\simeq"] & \mathit{OpFib}(\mathcal{D})
\end{tikzcd}$$

that is given by the pseudofunctoriality of the pullback (or actually by the fact that the pullback of a lax comma is isomorphic to the lax comma with the composite), thanks to Theorem 1.3.8:

$$\begin{tikzcd}
\int(F\circ H) \arrow[rrrd, bend left] \arrow[rdd, bend right, "\mathcal{G}(F\circ H)"'] \arrow[rd, phantom, "\cong\,(\mathcal{G}_H)_F^{-1}"] & & & \\
& H^*\int F \arrow[r] \arrow[d, "H^*\mathcal{G}(F)"'] & \int F \arrow[r] \arrow[d, "\mathcal{G}(F)"'] & \mathit{Cat}_{\bullet,\mathrm{lax}} \arrow[d, "\tau"] \\
& \mathcal{D} \arrow[r, "H"'] & \mathcal{C} \arrow[r, "F"'] & \mathit{Cat}
\end{tikzcd}$$

$\mathcal{G}_H$ is indeed a natural transformation thanks to the universal property of the lax comma object. And $\mathcal{G}_-$ satisfies the 1-dimensional condition of pseudonatural transformation by the pseudofunctoriality of the pullback (choosing the pullbacks along identities to be the identity).

Take now a natural transformation $\alpha\colon H \Rightarrow K\colon \mathcal{D} \to \mathcal{C}$. In order to prove the 2-dimensional condition of pseudonatural transformation for $\mathcal{G}_-$, we need to show that the following square is commutative for every $F\colon \mathcal{C} \to \mathcal{C}at$:

$$\begin{tikzcd}
H^* \int F \arrow[r, "\alpha^*_{\mathcal{G}(F)}"] \arrow[d, "(\mathcal{G}_H)_F"'] & K^* \int F \arrow[d, "(\mathcal{G}_K)_F"] \\
\int (F \circ H) \arrow[r, "\mathcal{G}(F \star \alpha)"'] & \int (F \circ K)
\end{tikzcd}$$

This is shown by the universal property of the lax comma object (or of the pullback) $\int F$. For this we use the fact that the chosen cleavage on $\mathcal{G}(F): \int F \to \mathcal{C}$ (with $\overline{f}^{(C,X)} = (f, \mathrm{id})$) makes the square

$$\begin{tikzcd}
\int F \arrow[r] \arrow[d, "\mathcal{G}(F)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C}at_{\bullet,\mathrm{lax}} \arrow[d, "\tau"] \\
\mathcal{C} \arrow[r, "F"'] & \mathcal{C}at
\end{tikzcd}$$

into a cleavage preserving morphism. ☐

4.2. Opfibrations in the 2-category of 2-presheaves

In this section, after recalling the notion of opfibration in a 2-category, we characterize the opfibrations in the functor 2-category $[\mathcal{A}, \mathcal{C}at]$ with $\mathcal{A}$ a small category. We will show that such characterization restricts to one of discrete opfibrations as well. This will allow us to define *having small fibres* for a discrete opfibration in $[\mathcal{A}, \mathcal{C}at]$ (Definition 4.2.7).

The definition of (op)fibration in a 2-category is due to Street [41], in terms of algebras for a 2-monad. It is known that we can equivalently define (op)fibrations in a 2-category by representability, as done in Weber's [51].

Definition 4.2.1. Let $\mathcal{L}$ be a 2-category. A *split opfibration in* $\mathcal{L}$ is a morphism $\varphi\colon G \to F$ in $\mathcal{L}$ such that for every $X \in \mathcal{L}$ the functor

$$\varphi \circ -\colon \mathcal{L}\,(X, G) \to \mathcal{L}\,(X, F)$$

induced by φ between the hom-categories is a split Grothendieck opfibration (in $\mathcal{C}at$) and for every morphism $\lambda\colon K \to X$ in $\mathcal{L}$ the commutative square

$$\begin{tikzcd}
\mathcal{L}\,(X,G) \arrow[r, "-\circ\lambda"] \arrow[d, "\varphi\circ-"'] & \mathcal{L}\,(K,G) \arrow[d, "\varphi\circ-"] \\
\mathcal{L}\,(X,F) \arrow[r, "-\circ\lambda"'] & \mathcal{L}\,(K,F)
\end{tikzcd}$$

is cleavage preserving.

We call φ a *discrete opfibration in* $\mathcal{L}$ if for every X the functor $\varphi \circ -$ above is a discrete opfibration (in $\mathcal{C}at$). In this case, the second condition is automatic. This is in line with Definition 3.1.2.

Given split opfibrations $\varphi\colon G \to F$ and $\psi\colon H \to F$ in $\mathcal{L}$ over F, a cleavage preserving morphism from φ to ψ is a morphism $\xi\colon \varphi \to \psi$ in $\mathcal{L}/_{F}$ such that for every $X \in \mathcal{L}$ the triangle

$$\begin{tikzcd}
\mathcal{L}\,(X,G) \arrow[rr, "\xi\circ-"] \arrow[dr, "\varphi\circ-"'] & & \mathcal{L}\,(X,H) \arrow[dl, "\psi\circ-"] \\
& \mathcal{L}\,(X,F) &
\end{tikzcd}$$

is cleavage preserving.

If φ and ψ are discrete opfibrations, any morphism in $\mathcal{L}/_{F}$ is cleavage preserving.

Split opfibrations in $\mathcal{L}$ over F and cleavage preserving morphisms form a category $\mathcal{O}p\mathcal{F}ib_{\mathcal{L}}(F)$. We denote the full subcategory on discrete opfibrations in $\mathcal{L}$ as $\mathcal{DO}p\mathcal{F}ib_{\mathcal{L}}(F)$ (again, this is in line with Definition 3.1.2).

Remark 4.2.2. By definition, a (split) opfibration $\varphi\colon G \to F$ in $\mathcal{L}$ is required to lift every 2-cell $\theta\colon \varphi \circ \alpha \Rightarrow \beta$ to a cartesian 2-cell $\overline{\theta}^{\alpha}\colon \alpha \Rightarrow \theta_{*}\alpha$. We can draw the following diagram to say that $\varphi \circ \theta_* \alpha = \beta$ and $\varphi \star \overline{\theta}^{\alpha} = \theta$.

$$\begin{tikzcd}
X \arrow[rr, bend left, "\alpha"] \arrow[rr, bend right, "\theta_*\alpha"'] \arrow[rr, phantom, "\Downarrow \overline{\theta}^{\alpha}"] \arrow[dd, equal] & & G \arrow[dd, "\varphi"] \\
& G \arrow[dr, "\varphi"] & \\
X \arrow[ur, "\alpha"] \arrow[rr, bend right, "\beta"'] \arrow[rr, phantom, "\Downarrow \theta"] & & F
\end{tikzcd}$$

$\overline{\theta}^{\alpha}$ cartesian means that for every 2-cell $\rho \colon \alpha \Rightarrow \alpha'$ and 2-cell $\sigma \colon \beta \Rightarrow \varphi \circ \alpha'$, there exists a unique 2-cell $\nu \colon \theta_*\alpha \Rightarrow \alpha'$ such that $\varphi \star \nu = \sigma$ and $\nu \circ \overline{\theta}^{\alpha} = \rho$. Analogously, we can express being split in these terms.

The second condition of Definition 4.2.1 then requires the chosen lifting of $\theta \star \lambda$ to be $\overline{\theta}^{\alpha} \star \lambda$ (i.e. the chosen lifting of θ whiskered with λ).

φ is a discrete opfibration in $\mathcal{L}$ when the liftings $\overline{\theta}^{\alpha}$ are unique.

Remark 4.2.3. Pullbacks of split opfibrations are split opfibrations, because $\mathcal{L}(X, -)$ preserves pullbacks (as it preserves all limits) and pullbacks of split opfibrations in $\mathcal{Cat}$ are split opfibrations in $\mathcal{Cat}$. We are also using (for the second condition) that we can choose the cleavage of the pullback of a split opfibration in $\mathcal{L}$ so that the universal square that exhibits the pullback is cleavage preserving.

Remark 4.2.4. We can of course apply Definition 4.2.1 to $\mathcal{L} = \mathcal{Cat}$. The produced notion is equivalent to the usual notion of Grothendieck opfibration. This is essentially because for $\mathcal{L} = \mathcal{Cat}$ it suffices to ask the above liftings for $X = 1$. We are then able to lift entire natural transformations θ as a consequence, componentwise. Analogously with discrete opfibrations in $\mathcal{L} = \mathcal{Cat}$.

We extend this idea below and characterize opfibrations in $[\mathcal{A}, \mathcal{Cat}]$.

The following proposition does not seem to appear in the literature.

Proposition 4.2.5. *Let $\mathcal{A}$ be a small category and consider a morphism $\varphi \colon G \to F$ in $[\mathcal{A}, \mathcal{Cat}]$ (i.e. a natural transformation). The following facts are equivalent:*

(*i*) *$\varphi \colon G \to F$ is a split opfibration in $[\mathcal{A}, \mathcal{Cat}]$;*

(*ii*) *for every $A \in \mathcal{A}$ the component $\varphi_A \colon G(A) \to F(A)$ of φ on A is a split opfibration* (*in* $\mathcal{C}at$) *and for every morphism $h \colon A \to B$ in $\mathcal{A}$ the naturality square*

$$\begin{tikzcd} G(A) \arrow[r, "G(h)"] \arrow[d, "\varphi_A"'] & G(B) \arrow[d, "\varphi_B"] \\ F(A) \arrow[r, "F(h)"'] & F(B) \end{tikzcd}$$

is cleavage preserving.

Analogously with discrete opfibrations in $[\mathcal{A}, \mathcal{C}at]$, *where the condition on the naturality square of* (*ii*) *is automatic.*

Proof. We prove (*i*) ⇒ (*ii*). Let $A \in \mathcal{A}$. Taking $X = \mathrm{y}(A)$ in Definition 4.2.1 with $\mathcal{L} = [\mathcal{A}, \mathcal{C}at]$, we obtain that

$$\varphi \circ - \colon [\mathcal{A}, \mathcal{C}at]\,(\mathrm{y}(A), G) \to [\mathcal{A}, \mathcal{C}at]\,(\mathrm{y}(A), F)$$

is a split opfibration in $\mathcal{C}at$. By Yoneda lemma, we have isomorphisms that form a commutative square

$$\begin{tikzcd} {[\mathcal{A}, \mathcal{C}at]\,(\mathrm{y}(A), G)} \arrow[r, phantom, "\cong"] \arrow[d, "\varphi\circ -"'] & G(A) \arrow[d, "\varphi_A"] \\ {[\mathcal{A}, \mathcal{C}at]\,(\mathrm{y}(A), F)} \arrow[r, phantom, "\cong"] & F(A) \end{tikzcd}$$

We can then choose a cleavage on φ_A that makes it into a split opfibration in $\mathcal{C}at$ such that the above square is cleavage preserving. Given $h \colon A \to B$ in $\mathcal{A}$ we have that the naturality square of φ on h is equal to the pasting

$$\begin{tikzcd} G(A) \arrow[r, phantom, "\cong"] \arrow[d, "\varphi_A"'] & {[\mathcal{A}, \mathcal{C}at]\,(\mathrm{y}(A), G)} \arrow[r, "-\circ \mathrm{y}(h)"] \arrow[d, "\varphi\circ -"'] & {[\mathcal{A}, \mathcal{C}at]\,(\mathrm{y}(B), G)} \arrow[r, phantom, "\cong"] \arrow[d, "\varphi\circ -"] & G(B) \arrow[d, "\varphi_B"] \\ F(A) \arrow[r, phantom, "\cong"] & {[\mathcal{A}, \mathcal{C}at]\,(\mathrm{y}(A), F)} \arrow[r, "-\circ \mathrm{y}(h)"'] & {[\mathcal{A}, \mathcal{C}at]\,(\mathrm{y}(B), F)} \arrow[r, phantom, "\cong"] & F(B) \end{tikzcd}$$

and is thus cleavage preserving.

When φ is a discrete opfibration, φ_A is discrete as well for every $A \in \mathcal{A}$.

We now prove $(ii) \Rightarrow (i)$. Let $X \in [\mathcal{A}, \mathcal{C}at]$, $\alpha\colon X \to G$, $\beta\colon X \to F$ and consider $\theta\colon \varphi \circ \alpha \Rightarrow \beta$. We need to produce a cartesian lifting $\overline{\theta}^{\alpha}\colon \alpha \Rightarrow \theta_*\alpha$ of θ to α.

$$\begin{tikzcd}
X \arrow[rr, bend left, "\alpha"] \arrow[rr, bend right, "\theta_*\alpha"'] \arrow[dd, equal] & \Downarrow \overline{\theta}^{\alpha} & G \arrow[dd, "\varphi"] \\
& G \arrow[dr, "\varphi"] & \\
X \arrow[ur, "\alpha"] \arrow[rr, bend right, "\beta"'] & \Downarrow \theta & F
\end{tikzcd}$$

As $\theta_*\alpha$ is a natural transformation and $\overline{\theta}^{\alpha}$ is a modification, we can define them on components. Given $A \in \mathcal{A}$ and $Z \in X(A)$, we define the image of the functor $(\theta_*\alpha)_A$ on Z and the morphism $(\overline{\theta}^{\alpha})_{A,Z}$ in $G(A)$ to be given by the chosen cartesian lifting along φ_A of $\theta_{A,Z}$ to $\alpha_A(Z)$:

$$\begin{tikzcd}
\alpha_A(Z) \arrow[r, "(\overline{\theta}^{\alpha})_{A,Z}"] \arrow[d, |->, "\varphi_A"'] & (\theta_*\alpha)_A(Z) \arrow[d, |->, "\varphi_A"] \\
\varphi_A(\alpha_A(Z)) \arrow[r, "\theta_{A,Z}"'] & \beta_A(Z)
\end{tikzcd}$$

Given a morphism $f\colon Z \to Z'$ in $X(A)$, we define $(\theta_*\alpha)_A(f)$ by cartesianity of $(\overline{\theta}^{\alpha})_{A,Z}$, making by construction $(\overline{\theta}^{\alpha})_A$ into a natural transformation. $(\theta_*\alpha)_A$ is then automatically a functor. In order to prove that $\theta_*\alpha$ is a natural transformation, we need to show that for every $h\colon A \to B$ in $\mathcal{A}$ the following square is commutative:

$$\begin{tikzcd}
X(A) \arrow[r, "(\theta_*)_A"] \arrow[d, "X(h)"'] & G(A) \arrow[d, "G(h)"] \\
X(B) \arrow[r, "(\theta_*)_B"'] & G(B)
\end{tikzcd}$$

This is straightforward using the hypothesis that $(G(h), F(h))$ is cleavage preserving. The argument shows at the same time that $\overline{\theta}^{\alpha}$ is a modification. $\overline{\theta}^{\alpha}$ is then a lifting of θ to α by construction, as this can be checked on components. It is straightforward to show that it is cartesian as well, inducing the required morphism on components by the cartesianity of all the $(\overline{\theta}^{\alpha})_{A,Z}$. Coherences are shown using again that $(G(h), F(h))$ is cleavage preserving. φ is split because all φ_A are split.

Given $\lambda\colon K \to X$ in $[\mathcal{A}, \mathcal{C}at]$, we prove that

$$\begin{tikzcd}
\mathcal{L}(X, G) \arrow[r, "-\circ\lambda"] \arrow[d, "\varphi\circ-"'] & \mathcal{L}(K, G) \arrow[d, "\varphi\circ-"] \\
\mathcal{L}(X, F) \arrow[r, "-\circ\lambda"'] & \mathcal{L}(K, F)
\end{tikzcd}$$

is cleavage preserving. This means that

$$\begin{tikzcd}
K \arrow[r, "\lambda"] \arrow[d, equal] & X \arrow[rr, bend left, "\alpha"{name=U}] \arrow[rr, bend right, "\theta_*\alpha"'{name=D}] \arrow[d, equal] & & G \arrow[d, "\varphi"] \\
K \arrow[r, "\lambda"] & X \arrow[r, "\alpha"] \arrow[rr, bend right, "\beta"'{name=B}] & G \arrow[r, "\varphi"] & F
\arrow[from=U, to=D, Rightarrow, "\overline{\theta}^{\alpha}"]
\arrow[from=2-3, to=B, Rightarrow, "\theta"]
\end{tikzcd}$$

exhibits the chosen cartesian lifting of $\theta \star \lambda$ to $\alpha \circ \lambda$. This works by construction, as the lifting of every 2-cell along φ is reduced to lift morphisms of $F(A)$ along φ_A for every $A \in \mathcal{A}$.

When φ_A is a discrete opfibration for every $A \in \mathcal{A}$, the argument above produces the needed cartesian liftings. We only need to show that such liftings are unique. But any lifting $\overline{\theta}^{\alpha}$ needs to have as component $(\overline{\theta}^{\alpha})_{A,Z}$ on $\mathcal{A} \in \mathcal{A}$ and $Z \in X(A)$ the unique lifting of $\theta_{A,Z}$ to $\alpha_A(Z)$ along φ_A. □

Proposition 4.2.6. *Let $\mathcal{A}$ be a small category and consider $\varphi, \psi \in \mathcal{O}p\mathcal{F}ib_{[\mathcal{A},\mathcal{C}at]}(F)$. Let then $\xi\colon \varphi \to \psi$ be a morphism in $[\mathcal{A}, \mathcal{C}at]/_F$. The following facts are equivalent:*

(i) $\xi\colon \varphi \to \psi$ is cleavage preserving;

(ii) for every $A \in \mathcal{A}$, the component $\xi_A\colon \varphi_A \to \psi_A$ is cleavage preserving (between split opfibrations in $\mathcal{C}at$).

Proof. We prove $(i) \Rightarrow (ii)$. Given $A \in \mathcal{A}$ we have that

$$\begin{tikzcd}
G(A) \arrow[r, phantom, "\cong"] \arrow[d, "\varphi_A"'] & {[\mathcal{A}, \mathcal{C}at]\,(\mathrm{y}(A), G)} \arrow[r, "\xi\circ-"] \arrow[d, "\varphi\circ-"'] & {[\mathcal{A}, \mathcal{C}at]\,(\mathrm{y}(A), H)} \arrow[r, phantom, "\cong"] \arrow[d, "\psi\circ-"] & H(A) \arrow[d, "\psi_A"] \\
F(A) \arrow[r, phantom, "\cong"] & {[\mathcal{A}, \mathcal{C}at]\,(\mathrm{y}(A), F)} \arrow[r, equal] & {[\mathcal{A}, \mathcal{C}at]\,(\mathrm{y}(A), F)} \arrow[r, phantom, "\cong"] & F(A)
\end{tikzcd}$$

is cleavage preserving.

We prove $(ii) \Rightarrow (i)$. The equality of modifications that we need to prove can be checked on components, where it holds by hypothesis. $\square$

Thanks to Proposition 4.2.5, we can define *having small fibres* for a discrete opfibration in $[\mathcal{A}, \mathcal{C}at]$.

Definition 4.2.7. Let $\mathcal{A}$ be a small category. A discrete opfibration $\varphi\colon G \to F$ in $[\mathcal{A}, \mathcal{C}at]$ *has small fibres* if for every $A \in \mathcal{A}$ the component φ_A of φ on A has small fibres.

We denote as $\mathcal{DO}p\mathcal{F}ib^{\mathrm{s}}{}_{[\mathcal{A},\mathcal{C}at]}(F)$ the full subcategory of $\mathcal{DO}p\mathcal{F}ib_{[\mathcal{A},\mathcal{C}at]}(F)$ on the discrete opfibrations with small fibres.

Remark 4.2.8. The property of having small fibres for a discrete opfibration in $[\mathcal{A}, \mathcal{C}at]$ is stable under pullbacks. Indeed taking components on $A \in \mathcal{A}$ preserves 2-limits in 2-presheaves and discrete opfibrations in $\mathcal{C}at$ with small fibres are stable under pullbacks.

We will also need the following result.

Proposition 4.2.9. *Let $\mathcal{L}$ be a 2-category. The assignment $F \in \mathcal{L} \mapsto \mathcal{O}p\mathcal{F}ib_{\mathcal{L}}(F) \in \mathcal{CAT}$ extends to a pseudofunctor*

$$\mathcal{O}p\mathcal{F}ib_{\mathcal{L}}(-) : \mathcal{L}^{\mathrm{op}} \to \mathcal{CAT}.$$

Moreover, this pseudofunctor restricts to a pseudofunctor

$$\mathcal{O}p\mathcal{F}ib^{\mathrm{P}}{}_{\mathcal{L}}(-) : \mathcal{L}^{\mathrm{op}} \to \mathcal{CAT}$$

that sends $F \in \mathcal{L}$ to the full subcategory of $\mathcal{O}p\mathcal{F}ib_{\mathcal{L}}(F)$ on the split opfibrations that satisfy a fixed pullback-stable property P *(e.g. being discrete opfibrations).*

Proof. On the underlying category of $\mathcal{L}^{\mathrm{op}}$, we define $\mathcal{O}p\mathcal{F}ib_{\mathcal{L}}(-)$ as the restriction of the pseudofunctor given by the pullback (to consider opfibrations rather than general morphisms). So given $\alpha\colon F' \to F$ in $\mathcal{L}$, we have

$$\mathcal{O}p\mathcal{F}ib_{\mathcal{L}}(\alpha) = \alpha^*\colon \mathcal{O}p\mathcal{F}ib_{\mathcal{L}}(F) \to \mathcal{O}p\mathcal{F}ib_{\mathcal{L}}(F')$$

We are also using Remark 4.2.3. We immediately get also the isomorphisms that regulate the image of identities and compositions.

Given a 2-cell $\delta\colon \alpha \Rightarrow \beta\colon F' \to F$ in $\mathcal{L}$, we define $\mathcal{OpFib}_{\mathcal{L}}(\delta) = \delta^*$ as the natural transformation with component on a split opfibration $\varphi\colon G \to F$ in $\mathcal{L}$ given by the morphism $\delta^*_{\varphi}\colon \alpha^*\varphi \to \beta^*\varphi$ induced by lifting $\delta \star \alpha^*\varphi$ along φ:

$$\begin{tikzcd}
\alpha^*G \arrow[rrd, bend left, "\overline{\delta\star\alpha^*\varphi}"] \arrow[rd, dashed, "\delta^*_{\varphi}"] \arrow[rdd, bend right, "\alpha^*\varphi"'] \arrow[rrd, phantom, "\Swarrow"] & & \\
& \beta^*G \arrow[r] \arrow[d, "\beta^*\varphi"'] & G \arrow[d, "\varphi"] \\
& F' \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] \arrow[r, phantom, "\Downarrow\delta"] & F
\end{tikzcd}$$

Indeed the codomain of the lifting of $\delta \star \alpha^*\varphi$ along φ induces the morphism δ^*_{φ} by the universal property of the pullback β^*G. δ^* is a natural transformation because the morphisms in $\mathcal{OpFib}_{\mathcal{L}}(F)$ are cleavage preserving.

$\mathcal{OpFib}_{\mathcal{L}}(-)$ preserves identity 2-cells and vertical compositions of 2-cells because the objects of $\mathcal{OpFib}_{\mathcal{L}}(F)$ are split. We already know that the isomorphisms that regulate the image of identities and compositions satisfy the 1-dimensional coherences. It only remains to prove their naturality (actually, only the one for compositions). This essentially means that it preserves whiskerings, up to pasting with the isomorphisms that regulate the image of compositions. For whiskering on the left, this is true by the second condition of Definition 4.2.1. For whiskerings on the right, we use that the universal square that exhibits a pullback is cleavage preserving.

Thus we conclude that $\mathcal{OpFib}_{\mathcal{L}}(-)$ is a pseudofunctor. It then readily restricts to a pseudofunctor $\mathcal{OpFib}^{\mathrm{P}}_{\mathcal{L}}(-)$. □

4.3. Indexed Grothendieck construction

In this section, we present our main results. We prove an equivalence of categories between split opfibrations in $[\mathcal{A}, \mathcal{Cat}]$ over F and 2-copresheaves on $\int F$. This equivalence restricts to one between discrete opfibrations in $[\mathcal{A}, \mathcal{C}at]$ over F with small fibres and $\mathcal{S}et$-valued copresheaves on $\int F$. We also show that both such equivalences are pseudonatural in F.

We introduce the explicit indexed Grothendieck construction and show how our results recover known useful results. In particular, we recover the equivalence between slices of presheaves over $F\colon \mathcal{A} \to \mathcal{S}et$ and presheaves on $\int F$, that shows how the slice of a Grothendieck topos is a Grothendieck topos. We interpret our main theorem as a 2-dimensional generalization of this.

Let $\mathcal{A}$ be a small category and consider the functor 2-category $[\mathcal{A}, \mathcal{C}at]$.

Remark 4.3.1. We aim at proving that for every 2-functor $F\colon \mathcal{A} \to \mathcal{C}at$, there is an equivalence of categories

$$\mathcal{O}p\mathcal{F}ib_{[\mathcal{A},\mathcal{C}at]}(F) \simeq \left[\int F, \mathcal{C}at\right]$$

between split opfibrations in $[\mathcal{A}, \mathcal{C}at]$ over F (see Proposition 4.2.5) and 2-copresheaves on the Grothendieck construction $\int F$ of F.

Our strategy will be to use Theorem 4.1.7, that states that the Grothendieck construction $\int F$ of F is equivalently the oplax colimit of the 2-diagram $F\colon \mathcal{A} \to \mathcal{C}at$. Notice that a (strict) 2-functor from a category to $\mathcal{C}at$ is the same thing as a functor into the underlying category $\mathcal{C}at_0$ of $\mathcal{C}at$. In Remark 4.3.13, we will say what we could do to extend our results to $\mathcal{A}$ a 2-category or F a pseudofunctor.

Proposition 4.3.2. *There is an isomorphism of categories*

$$\left[\int F, \mathcal{C}at_0\right] \cong [\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}]_{\mathrm{oplax}}(\Delta 1, [F(-), \mathcal{C}at_0])$$

which is (*strictly*) *2-natural in F.*

Proof. We obtain the isomorphism of categories in the statement by Theorem 4.1.7, that proves that $\int F = \text{oplax-colim}\, F$ (see also Remark 4.1.8 and Definition 4.1.5). The isomorphism is 2-natural in F by a general result on weighted colimits, see Kelly's [28, Section 3.1]. We can apply this result on an oplax colimit as well because by Street's [42, Theorem 11] any oplax colimit is also a weighted one. □

Remark 4.3.3. Thanks to Proposition 4.3.2, we can reduce ourselves to apply the usual Grothendieck construction on every index. For this we also need the pseudonaturality of the Grothendieck construction (Proposition 4.1.9).

Proposition 4.3.4. *There is an equivalence of categories*

$$[\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}]_{\mathrm{oplax}}(\Delta 1, [F(-), \mathcal{C}at_0]) \simeq \mathrm{Ps}\,[\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}]_{\mathrm{oplax}}\left(\Delta 1, \mathcal{O}p\mathcal{F}ib_{\mathcal{C}at}(F(-))\right)$$

which is pseudonatural in F, where $\mathrm{Ps}\,[\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}]_{\mathrm{oplax}}$ is the 2-category of pseudofunctors, oplax natural transformations and modifications.

Proof. Notice that

$$[\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}]_{\mathrm{oplax}}(\Delta 1, [F(-), \mathcal{C}at_0]) \cong \mathrm{Ps}\,[\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}]_{\mathrm{oplax}}(\Delta 1, [F(-), \mathcal{C}at_0])$$

So it suffices to exhibit an equivalence

$$[F(-), \mathcal{C}at_0] \simeq \mathcal{O}p\mathcal{F}ib(F(-))$$

in the (large) 2-category $\mathrm{Ps}\,[\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}]_{\mathrm{oplax}}$. Indeed, for a general (large) 2-category, postcomposing with a morphism that is an equivalence in the 2-category gives a functor between hom-categories that is an equivalence of categories. The left hand side is certainly a 2-functor, while the right hand side is a pseudofunctor by Proposition 4.2.9. We have that the Grothendieck construction gives a pseudonatural adjoint equivalence

$$\mathcal{G}_{-}\colon\ [-, \mathcal{C}at_0] \simeq \mathcal{O}p\mathcal{F}ib(-),$$

by Proposition 4.1.9. Whiskering it with F^{op} on the left gives another pseudonatural adjoint equivalence, that is then also an equivalence in the 2-category $\mathrm{Ps}\,[\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}]_{\mathrm{oplax}}$ as needed. The quasi-inverse is given by extending to a pseudonatural transformation the quasi-inverses of the Grothendieck construction on every component. This can always be done by choosing as structure 2-cells the pasting of the inverse of the structure 2-cells of the Grothendieck construction with unit and counit of the adjoint equivalences on components. The triangular equalities then guarantee that we have an equivalence in $\mathrm{Ps}\,[\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}]_{\mathrm{oplax}}$ (we have that the two composites are isomorphic to the identity).

We now prove that the equivalence of categories

$$\mathcal{G}_{F(-)}\circ - \colon [\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}]_{\mathrm{oplax}}(\Delta 1, [F(-), \mathcal{C}at_0]) \simeq \mathrm{Ps}\,[\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}]_{\mathrm{oplax}}\left(\Delta 1, \mathcal{O}p\mathcal{F}ib_{\mathcal{C}at}(F(-))\right)$$

that we have produced is pseudonatural in F.

We show that $\mathrm{Ps}\,[\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}]_{\mathrm{oplax}}\left(\Delta 1, \mathcal{O}p\mathcal{F}ib_{\mathcal{C}at}(+(-))\right) : [\mathcal{A}, \mathcal{C}at]^{\mathrm{op}} \to \mathcal{CAT}$ is a pseudofunctor. Given $\alpha \colon F' \to F$ in $[\mathcal{A}, \mathcal{C}at]$, we define the image on α to be $\alpha^*_{\_} \circ -$, where

$$\alpha^*_{\_} : \mathcal{O}p\mathcal{F}ib(F(-)) \underset{\text{pseudo}}{\Longrightarrow} \mathcal{O}p\mathcal{F}ib(F'(-)) : \mathcal{A}^{\mathrm{op}} \to \mathcal{CAT}$$

is the pseudonatural transformation described as follows. For every $A \in \mathcal{A}$, we define $(\alpha^*_{\_})_A \coloneqq \alpha^*_A$ (see Proposition 4.2.9). For every $h \colon A \to B$ in $\mathcal{A}$, we define the structure 2-cell $(\alpha^*_{\_})_h$ to be the pasting

$$\begin{tikzcd}
\mathcal{O}p\mathcal{F}ib(F(B)) \arrow[r, "\alpha^*_B"] \arrow[d, "F(h)^*"'] \arrow[rd, "(F(h)\circ\alpha_A)^*" description] & \mathcal{O}p\mathcal{F}ib(F'(B)) \arrow[d, "F'(h)^*"] \arrow[ld, phantom, "\cong"', near start] \\
\mathcal{O}p\mathcal{F}ib(F(A)) \arrow[r, "\alpha^*_A"'] \arrow[ru, phantom, "\cong", near start] & \mathcal{O}p\mathcal{F}ib(F'(A))
\end{tikzcd}$$

where the two isomorphisms are the ones given by the pseudofunctoriality of $\mathcal{O}p\mathcal{F}ib(-)$ (see Proposition 4.2.9). We are using that $F(h) \circ \alpha_A = \alpha_B \circ F'(h)$ by naturality of α. Then $\alpha^*_{\_}$ is a pseudonatural transformation because $\mathcal{O}p\mathcal{F}ib(-)$ is a pseudofunctor. As $\alpha^*_{\_}$ is a morphism in the (large) 2-category $\mathrm{Ps}\,[\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}]_{\mathrm{oplax}}$, we have that $\alpha^*_{\_} \circ -$ is a functor. Considering $F'' \xrightarrow{\alpha'} F' \xrightarrow{\alpha} F$ in $[\mathcal{A}, \mathcal{C}at]$, there is a an invertible modification

$$(\alpha'_{\_})^* \circ \alpha^*_{\_} \cong (\alpha \circ \alpha')^*_{\_}$$

with components given by the pseudofunctoriality of $\mathcal{O}p\mathcal{F}ib(-)$. And then whiskering with this gives the natural isomorphism that regulates the image on the composite $\alpha \circ \alpha'$.

Given $\delta \colon \alpha \Rightarrow \beta \colon F' \to F$ in $[\mathcal{A}, \mathcal{C}at]$, we define the image on δ to be $\delta^*_{\_} \star -$, where $\delta^*_{\_}$ is the modification that has components δ^*_A on every $A \in \mathcal{A}$ (see Proposition 4.2.9). This forms indeed a modification by pseudofunctoriality of $\mathcal{O}p\mathcal{F}ib(-)$.

It is straightforward to check that $\mathrm{Ps}\left[\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}\right]_{\mathrm{oplax}}\left(\Delta 1, \mathcal{O}p\mathcal{F}ib_{\mathcal{C}at}\left(+(-)\right)\right)$ is a pseudofunctor.

We prove that $\mathcal{G}_{F(-)} \circ -$ is pseudonatural in $F \in [\mathcal{A}, \mathcal{C}at]^{\mathrm{op}}$. Given $\alpha\colon F' \to F$ in $[\mathcal{A}, \mathcal{C}at]$, we define the structure 2-cell on α to be $\mathcal{G}_{\alpha_-} \star -$, where $\mathcal{G}_{\alpha_-}$ is the invertible modification

$$\begin{tikzcd}
{[F(-), \mathcal{C}at_0]} \arrow[r, phantom, "\overset{\mathcal{G}_{F(-)}}{\simeq}"] \arrow[d, "-\circ\alpha"'] & \mathcal{O}p\mathcal{F}ib\,(F(-)) \arrow[d, "\alpha^*_-"] \arrow[dl, phantom, "\overset{\mathcal{G}_{\alpha_-}}{\cong}"] \\
{[F'(-), \mathcal{C}at_0]} \arrow[r, phantom, "\underset{\mathcal{G}_{F'(-)}}{\simeq}"] & \mathcal{O}p\mathcal{F}ib\,(F'(-))
\end{tikzcd}$$

with components defined by the pseudonaturality of the Grothendieck construction in the base (see Proposition 4.1.9). The latter pseudonaturality also guarantees that $\mathcal{G}_{\alpha_-}$ is a modification. Whence $\mathcal{G}_{\alpha_-} \star -$ is a natural isomorphism. We then conclude that $\mathcal{G}_{F(-)} \circ -$ is pseudonatural in $F \in [\mathcal{A}, \mathcal{C}at]^{\mathrm{op}}$ because the needed equalities of modifications can be checked on components, where everything holds because the Grothendieck construction is pseudonatural in the base (we also need the 2-dimensional condition of this pseudonaturality). □

Remark 4.3.5. An object of $\mathrm{Ps}\left[\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}\right]_{\mathrm{oplax}}\left(\Delta 1, \mathcal{O}p\mathcal{F}ib_{\mathcal{C}at}\left(F(-)\right)\right)$ is essentially a collection of opfibrations on every index $A \in \mathcal{A}$ together with a compact information on how to move between different indexes. The last ingredient that we need in order to prove our main result is that we can pack these data in terms of an opfibration in $[\mathcal{A}, \mathcal{C}at]$ over F.

Proposition 4.3.6. *There is an isomorphism of categories*

$$\mathrm{Ps}\left[\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}\right]_{\mathrm{oplax}}\left(\Delta 1, \mathcal{O}p\mathcal{F}ib_{\mathcal{C}at}\left(F(-)\right)\right) \cong \mathcal{O}p\mathcal{F}ib_{[\mathcal{A}, \mathcal{C}at]}\left(F\right)$$

which is pseudonatural in F (with the structure 2-cells being identities).

Proof. Given $\varphi\colon G \to F$ an opfibration in $[\mathcal{A}, \mathcal{C}at]$, we produce an oplax natural

$$[\varphi]\colon \Delta 1 \underset{\mathrm{oplax}}{\Longrightarrow} \mathcal{O}p\mathcal{F}ib_{\mathcal{C}at}\left(F(-)\right)\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{CAT}.$$

For every $A \in \mathcal{A}$, we define $[\varphi]_A \coloneqq \varphi_A$, thanks to Proposition 4.2.5. For every $h\colon A \to B$ in $\mathcal{A}$, the structure 2-cell $[\varphi]_h$ is the functor $\varphi_A \to F(h)^*(\varphi_B)$ defined by the universal property of the pullback in $\mathcal{C}at$:

$$\begin{tikzcd}
G(A) \arrow[rrd, bend left, "G(h)"] \arrow[rd, dashed, "{[\varphi]_h}"] \arrow[rdd, bend right, "\varphi_A"'] & & \\
& F(h)^*(G(B)) \arrow[r] \arrow[d, "F(h)^*(\varphi_B)"'] \arrow[rd, phantom, "\lrcorner", very near start] & G(B) \arrow[d, "\varphi_B"] \\
& F(A) \arrow[r, "F(h)"'] & F(B)
\end{tikzcd}$$

$[\varphi]_h$ is cleavage preserving because $(G(h), F(f))$ and the universal square that exhibits the pullback are cleavage preserving, thanks to Proposition 4.2.5. $[\varphi]$ is an oplax natural transformation by the universal property of the pullback, using also the pseudofunctoriality of the pullback.

Given

$$\gamma\colon \Delta 1 \underset{\text{oplax}}{\Longrightarrow} \mathcal{O}p\mathcal{F}ib_{\mathcal{C}at}(F(-)),$$

we produce an opfibration $\widehat{\gamma}\colon G \to F$ in $[\mathcal{A}, \mathcal{C}at]$. We define the (2-)functor G sending $A \in \mathcal{A}$ to $\operatorname{dom}(\gamma_A)$ and $h\colon A \to B$ to the composite above of the diagram

$$\begin{tikzcd}
\operatorname{dom}(\gamma_A) \arrow[r, "\gamma_h"] \arrow[d, "\gamma_A"'] & F(h)^*\operatorname{dom}(\gamma_B) \arrow[r] \arrow[d, "F(h)^*(\gamma_B)"'] \arrow[rd, phantom, "\lrcorner", very near start] & \operatorname{dom}(\gamma_B) \arrow[d, "\gamma_B"] \\
F(A) \arrow[r, equal] & F(A) \arrow[r, "F(h)"'] & F(B)
\end{tikzcd}$$

G is a functor because γ is oplax natural. For every $A \in \mathcal{A}$, we define $\widehat{\gamma}_A \coloneqq \gamma_A$. Then $\widehat{\gamma}$ is a natural transformation by construction of G. And the naturality squares of $\widehat{\gamma}$ are cleavage preserving because every γ_h and every universal square that exhibits a pullback are cleavage preserving. By Proposition 4.2.5, we conclude that $\widehat{\gamma}$ is a split opfibration in $[\mathcal{A}, \mathcal{C}at]$.

We can extend both constructions to functors, that will be inverses of each other. Given a cleavage preserving morphism $\xi\colon \varphi \to \psi$ between split opfibrations in $[\mathcal{A}, \mathcal{C}at]$ over F, we produce a modification $[\xi]\colon [\varphi] \Rrightarrow [\psi]$. For every $A \in \mathcal{A}$, we define $[\xi]_A \coloneqq \xi_A$, thanks to Proposition 4.2.6. It is straightforward to prove that this is a modification using the universal property of the pullback. Then $[-]$ is readily seen to be a functor, because the conditions can be checked on components. Given a modification

$$\zeta\colon \gamma \Rrightarrow \delta\colon \Delta 1 \underset{\text{oplax}}{\Longrightarrow} \mathcal{O}p\mathcal{F}ib_{\mathcal{C}at}\,(F(-)),$$

we produce a cleavage preserving morphism $\widehat{\zeta}\colon \widehat{\gamma} \to \widehat{\delta}$. For every $A \in \mathcal{A}$, we define $\widehat{\zeta}_A \coloneqq \zeta_A$, and this is then clearly cleavage preserving. $\widehat{\zeta}$ is a natural transformation because ζ is a modification. By Proposition 4.2.6, we conclude that $\widehat{\zeta}$ is a cleavage preserving morphism. Then $\widehat{-}$ is readily seen to be a functor because the conditions can be checked on components. It is straightforward to check that $[-]$ and $\widehat{-}$ are inverses of each other.

We now prove the pseudonaturality in F, with the structure 2-cells being identities, of the isomorphism of categories we have just produced. The left hand side extends to a pseudofunctor by the proof of Proposition 4.3.4, while the right hand side extends to a pseudofunctor by Proposition 4.2.9. Given a morphism $\alpha\colon F' \to F$ in $[\mathcal{A}, \mathcal{C}at]$, we show that the following square is commutative:

$$\begin{tikzcd}
\mathcal{O}p\mathcal{F}ib_{[\mathcal{A},\mathcal{C}at]}\,(F) \arrow[r, "\simeq"] \arrow[d, "\alpha^*"'] & \mathrm{Ps}\left[\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}\right]_{\mathrm{oplax}} \left(\Delta 1, \mathcal{O}p\mathcal{F}ib_{\mathcal{C}at}\,(F(-))\right) \arrow[d, "\alpha^*_{-}\circ -"] \\
\mathcal{O}p\mathcal{F}ib_{[\mathcal{A},\mathcal{C}at]}\,(F') \arrow[r, "\simeq"] & \mathrm{Ps}\left[\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}\right]_{\mathrm{oplax}} \left(\Delta 1, \mathcal{O}p\mathcal{F}ib_{\mathcal{C}at}\,(F'(-))\right)
\end{tikzcd}$$

Let $\varphi\colon G \to F$ be a split opfibration in $[\mathcal{A}, \mathcal{C}at]$. For every $A \in \mathcal{A}$, since pullbacks in $[\mathcal{A}, \mathcal{C}at]$ are calculated pointwise,

$$(\alpha^*_{-} \circ [\varphi])_A = \alpha^*_A(\varphi_A) = (\alpha^*\varphi)_A = [\alpha^*\varphi]_A.$$

For every $h\colon A \to B$ in $\mathcal{A}$, we have that $[\alpha^*\varphi]_h$ is equal to the pasting

$$\begin{tikzcd}
1 \arrow[r, "\varphi_B"] \arrow[d, equal] & \mathcal{O}p\mathcal{F}ib\,(F(B))] \arrow[r, "\alpha^*_B"] \arrow[d, "F(h)^*"'] \arrow[dr, "(F(h)\circ\alpha_A)^*" description] \arrow[dr, phantom, "\cong"', shift right=4] & \mathcal{O}p\mathcal{F}ib\,(F'(B)) \arrow[d, "F'(h)^*"] \arrow[from=1-2, to=2-3, phantom, "\cong", shift left=4] \\
1 \arrow[r, "\varphi_A"'] \arrow[ur, Rightarrow, "{[\varphi]_h}"] & \mathcal{O}p\mathcal{F}ib\,(F(A)) \arrow[r, "\alpha^*_A"'] & \mathcal{O}p\mathcal{F}ib\,(F'(A))
\end{tikzcd}$$

by the universal property of the pullback, using again that pullbacks in $[\mathcal{A}, \mathcal{C}at]$ are calculated pointwise. It is then easy to see that the square above is commutative on morphisms $\xi\colon \varphi \to \psi$ as well, since it can be checked on components $A \in \mathcal{A}$.

We prove that identities are the structure 2-cells of an isomorphic pseudonatural transformation

$$\mathcal{O}p\mathcal{F}ib_{[\mathcal{A},\mathcal{C}at]}(+) \cong \mathrm{Ps}\left[\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}\right]_{\mathrm{oplax}}\left(\Delta 1, \mathcal{O}p\mathcal{F}ib_{\mathcal{C}at}(+(-))\right) : [\mathcal{A}, \mathcal{C}at]^{\mathrm{op}} \to \mathcal{CAT}$$

This means that the isomorphisms that regulate the image of the two pseudofunctors on identities and compositions are compatible, and that the two pseudofunctors agree on 2-cells. The first condition holds because it can be checked on components and pullbacks in $[\mathcal{A}, \mathcal{C}at]$ are calculated pointwise (choosing pullbacks along identities to be the identity). The second condition is, for every $\delta \colon \alpha \Rightarrow \beta \colon F' \to F$ in $[\mathcal{A}, \mathcal{C}at]$,

$$[-] \star \delta^* = (\delta^*_- \star -) \star [-]$$

This can be checked on components $\varphi \colon G \to F$ (split opfibration in $[\mathcal{A}, \mathcal{C}at]$). On such components we need to prove an equality of modifications, that can be then checked on components $A \in \mathcal{A}$. So we need to show

$$(\delta^* \varphi)_A = \delta^*_A(\varphi_A).$$

This holds because the components of the liftings along φ are the liftings along the components of φ. Indeed, for a general θ as below,

$$\begin{tikzcd}
\mathrm{y}(A) \arrow[r, "x"] \arrow[dd, equal] & X \arrow[rr, bend left, "\alpha"] \arrow[rr, bend right, "\theta_*\alpha"'] \arrow[rr, phantom, "\Downarrow \overline{\theta}^{\alpha}"] \arrow[dd, equal] & & G \arrow[dd, "\varphi"] \\
& & G \arrow[rd, "\varphi"] & \\
\mathrm{y}(A) \arrow[r, "x"] & X \arrow[ru, "\alpha"] \arrow[rr, bend right, "\beta"'] \arrow[rr, phantom, "\Downarrow \theta"] & & F
\end{tikzcd}$$

the second condition of Definition 4.2.1 ensures that the lifting of $\theta_{A,x}$ along φ is equal to $\overline{\theta}^{\alpha}_{A,x}$. But the former is also the lifting of $\theta_{A,x}$ (seen as a morphism in $F(A)$) along φ_A, thanks to Proposition 4.2.5. And everything works on morphisms $f \colon x \to x'$ in $X(A)$ as well by cartesianity arguments, using the naturality of $\overline{\theta}^{\alpha}_{A}$. □

We are now ready to prove our main result.

Theorem 4.3.7. *Let* $\mathcal{A}$ *be a small category and consider the functor* 2*-category* $[\mathcal{A}, \mathcal{C}at]$*. For every* 2*-functor* $F\colon \mathcal{A} \to \mathcal{C}at$*, there is an equivalence of categories*

$$\mathcal{O}p\mathcal{F}ib_{[\mathcal{A},\mathcal{C}at]}(F) \simeq \left[\textstyle\int F, \mathcal{C}at\right]$$

between split opfibrations in $[\mathcal{A}, \mathcal{C}at]$ *over* F *and* 2*-copresheaves on the Grothendieck construction* $\int F$ *of* F*. Moreover this equivalence is pseudonatural in* F*.*

Proof. It suffices to compose the equivalences of categories of Proposition 4.3.2, Proposition 4.3.4 and Proposition 4.3.6.

$$\begin{aligned}&\left[\textstyle\int F, \mathcal{C}at_0\right] \cong [\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}]_{\mathrm{oplax}}(\Delta 1, [F(-), \mathcal{C}at_0]) \simeq \\ &\simeq \mathrm{Ps}\,[\mathcal{A}^{\mathrm{op}}, \mathcal{CAT}]_{\mathrm{oplax}}\left(\Delta 1, \mathcal{O}p\mathcal{F}ib_{\mathcal{C}at}(F(-))\right) \cong \mathcal{O}p\mathcal{F}ib_{[\mathcal{A},\mathcal{C}at]}(F)\end{aligned}$$

Notice that a 2-functor from a category into $\mathcal{C}at$ is the same thing as its underlying functor. As all three equivalences are pseudonatural in $F \in [\mathcal{A}, \mathcal{C}at]^{\mathrm{op}}$, so is the composite. □

We can extract the explicit indexed Grothendieck construction from the proof of Theorem 4.3.7.

Construction 4.3.8 (Indexed Grothendieck construction)**.** We can follow the chain of equivalences of the proof of Theorem 4.3.7 to get its explicit action. Let $\varphi\colon G \to F$ be a split opfibration in $[\mathcal{A}, \mathcal{C}at]$ over F. We first produce the oplax natural transformation

$$[\varphi]\colon \Delta 1 \xRightarrow[\mathrm{oplax}]{} \mathcal{O}p\mathcal{F}ib_{\mathcal{C}at}(F(-))$$

with $[\varphi]_A \coloneqq \varphi_A$ for every $A \in \mathcal{A}$ and

$$\begin{tikzcd}
G(A) \arrow[rrd, bend left, "G(h)"] \arrow[rd, dashed, "{[\varphi]_h}"] \arrow[rdd, bend right, "\varphi_A"'] & & \\
& F(h)^*(G(B)) \arrow[r] \arrow[d, "F(h)^*(\varphi_B)"'] \arrow[rd, phantom, "\lrcorner", very near start] & G(B) \arrow[d, "\varphi_B"] \\
& F(A) \arrow[r, "F(h)"'] & F(B)
\end{tikzcd}$$

for every $h\colon A \to B$ in $\mathcal{A}$. Then we produce the oplax natural transformation with components on $A \in \mathcal{A}$ and structure 2-cells on $h\colon A \to B$ defined by the pasting

$$\begin{tikzcd}
1 \arrow[r, "\varphi_B"] \arrow[d, equal] & \mathcal{O}p\mathcal{F}ib\,(F(B)) \arrow[r, "\mathcal{G}'_{F(B)}"] \arrow[d, "F(h)^*"] & {[F(B), \mathcal{C}at_0]} \arrow[d, "-\circ F(h)"] \\
1 \arrow[r, "\varphi_A"'] \arrow[ur, Rightarrow, "{[\varphi]_h}"] & \mathcal{O}p\mathcal{F}ib\,(F(A)) \arrow[r, "\mathcal{G}'_{F(A)}"'] \arrow[ur, phantom, "\cong"', "\mathcal{G}'_{F(h)}"] & {[F(A), \mathcal{C}at_0]}
\end{tikzcd}$$

where $\mathcal{G}'_{-}$ is the quasi-inverse of the Grothendieck construction. We have that $\mathcal{G}'_{F(A)}(\varphi_A)$ sends every $X \in F(A)$ to the fibre $(\varphi_A)_X$ of φ_A over X and every morphism $\alpha\colon X \to X'$ in $F(A)$ to the functor

$$\alpha_*\colon (\varphi_A)_X \to (\varphi_A)_{X'}$$

that lifts α (on morphisms, it is defined by cartesianity). The structure 2-cell on h is the natural transformation with component on $X \in F(A)$ given by

$$(\varphi_A)_X \xrightarrow{[\varphi]_h} (F(h)^*(\varphi_B))_X \cong (\varphi_B)_{F(h)(X)}$$

which coincides with $G(h)$.

Finally, we induce the 2-functor

$$\begin{tikzcd}
\mathcal{G}'(\varphi) : & \int F & \longrightarrow & \mathcal{C}at \\
& (A,X) \arrow[d, "{(h,\mathrm{id})}"] \arrow[dd, bend right=60, "{(h,\alpha)}"'] & & (\varphi_A)_X \arrow[d, "G(h)"] \\
& (B, F(h)(X)) \arrow[d, "{(\mathrm{id},\alpha)}"] & \mapsto & (\varphi_B)_{F(h)(X)} \arrow[d, "\alpha_*"] \\
& (B, X') & & (\varphi_B)_{X'}
\end{tikzcd}$$

using the universal property of the oplax colimit $\int F$, as in the proof of Theorem 4.1.7.

Let $Z\colon \int F \to \mathcal{C}at$ be a 2-functor. We produce the oplax natural transformation γ with components on $A \in \mathcal{A}$ and structure 2-cells on $h\colon A \to B$ in $\mathcal{A}$ defined by the pasting

$$\begin{tikzcd}
\mathit{1} \arrow[r, "Z\circ\mathrm{inc}_B"] \arrow[d, equal] & {[F(B),\mathcal{C}\mathit{at}_0]} \arrow[r, "\mathcal{G}_{F(B)}"] \arrow[d, "-\circ F(h)"] \arrow[dr, phantom, "\cong\ \mathcal{G}_{F(h)}"] & \mathit{Op}\mathcal{F}\mathit{ib}(F(B)) \arrow[d, "F(h)^*"] \\
\mathit{1} \arrow[r, "Z\circ\mathrm{inc}_A"'] \arrow[ur, Rightarrow, "Z\star\mathrm{inc}_h"] & {[F(A),\mathcal{C}\mathit{at}_0]} \arrow[r, "\mathcal{G}_{F(A)}"'] & \mathit{Op}\mathcal{F}\mathit{ib}(F(A))
\end{tikzcd}$$

We have that $\mathcal{G}_{F(A)} \circ Z \circ \mathrm{inc}_A \colon G(A) \to F(A)$ is the Grothendieck construction of the 2-functor $F(A) \to \mathcal{C}\mathit{at}$ that sends every object $X \in F(A)$ to $Z(A,X)$ and every morphism $\alpha\colon X \to X'$ in $F(A)$ to $Z(\mathrm{id},\alpha)$. Its domain $G(A)$ has the following description:

an object is a pair (X,ξ) with $X \in F(A)$ and $\xi \in Z(A,X)$;

a morphism $(X,\xi) \to (X',\xi')$ is a pair (α,Ξ) with $\alpha\colon X \to X'$ in $F(A)$ and $\Xi\colon Z(\mathrm{id},\alpha)(\xi) \to \xi'$ in $Z(A,X')$.

These are then collected as a split opfibration $\mathcal{G}(Z) : G \to F$ in $[\mathcal{A},\mathcal{C}\mathit{at}]$ over F whose components on every A are the projections $G(A) \to F(A)$ on the first component. For every $h\colon A \to B$ in $\mathcal{A}$, the functor $G(h)$ is defined by the composite above in the diagram

$$\begin{tikzcd}
G(A) \arrow[r, "\gamma_h"] \arrow[d, "\mathcal{G}(Z)_A"'] & F(h)^*G(B) \arrow[r] \arrow[d, "F(h)^*(\mathcal{G}(Z)_B)"'] \arrow[dr, phantom, very near start, "\lrcorner"] & G(B) \arrow[d, "\mathcal{G}(Z)_B"] \\
F(A) \arrow[r, equal] & F(A) \arrow[r, "F(h)"'] & F(B)
\end{tikzcd}$$

Explicitly,

$$\begin{array}{cccc}
G(h): & G(A) & \longrightarrow & G(B) \\
& (X,\xi) & & (F(h)(X), Z(h,\mathrm{id})(\xi)) \\
& \downarrow{\scriptstyle(\alpha,\Xi)} & \mapsto & \downarrow{\scriptstyle(F(h)(\alpha),Z(h,\mathrm{id})(\Xi))} \\
& (X',\xi') & & (F(h)(X'), Z(h,\mathrm{id})(\xi'))
\end{array}$$

We can see how this construction is indeed an indexed Grothendieck construction. We essentially collect together triples (A,X,ξ) with $A \in \mathcal{A}$, $X \in F(A)$ and $\xi \in Z(A,X)$.

Theorem 4.3.9. *Let $\mathcal{A}$ be a small category. For every 2-functor $F\colon \mathcal{A} \to \mathcal{C}at$, the equivalence of categories*

$$\mathcal{O}p\mathcal{F}ib_{[\mathcal{A},\mathcal{C}at]}(F) \simeq \left[\int F, \mathcal{C}at\right]$$

of Theorem 4.3.7 restricts to an equivalence of categories

$$\mathcal{DO}p\mathcal{F}ib^{\mathrm{s}}_{[\mathcal{A},\mathcal{C}at]}(F) \simeq \left[\int F, \mathcal{S}et\right]$$

between discrete opfibrations in $[\mathcal{A}, \mathcal{C}at]$ over F with small fibres and $\mathcal{S}et$-valued copresheaves on $\int F$. Moreover this equivalence is pseudonatural in F.

Proof. The isomorphism of Proposition 4.3.2 restricts to one with $\mathcal{S}et$ on both sides in the place of $\mathcal{C}at_0$ by 2-naturality in U of the isomorphism given by an oplax colimit (see Definition 4.1.5). Pseudonaturality in F still holds by the same general argument that guaranteed it with $\mathcal{C}at_0$ on both sides.

The equivalence of Proposition 4.3.4 restricts to one with $\mathcal{S}et$ in the place of $\mathcal{C}at_0$ on the left hand side and discrete opfibrations with small fibres in the place of opfibrations in the right hand side. Indeed the following is a commutative square of pseudonatural transformations:

$$\begin{tikzcd}
{[F(-), \mathcal{S}et]} \arrow[r, "\mathcal{G}_{F(-)}"] \arrow[d] & \mathcal{DO}p\mathcal{F}ib^{\mathrm{s}}(F(-)) \arrow[d] \\
{[F(-), \mathcal{C}at_0]} \arrow[r, "\mathcal{G}_{F(-)}"'] & \mathcal{O}p\mathcal{F}ib(F(-))
\end{tikzcd}$$

On components $A \in \mathcal{A}$, this is true by the classical Theorem 4.1.4. And it is straightforward to check that it is true on structure 2-cells as well, since structure 2-cells are given by the pseudofunctoriality of the pullback. Then pseudonaturality in F holds for the restricted equivalence as well, as one can readily check.

The isomorphism of Proposition 4.3.6 restricts to one with discrete opfibrations with small fibres on both sides in the place of opfibrations, because it suffices to look at the components. Then pseudonaturality in F holds as well, precomposing the pseudonatural transformation produced in the proof of Proposition 4.3.6 with the inclusion of discrete opfibrations with small fibres into opfibrations. □

Remark 4.3.10. When $F\colon \mathcal{A} \to \mathcal{S}et$, the equivalence of categories

$$\mathcal{DO}p\mathcal{F}ib^{\mathrm{s}}_{[\mathcal{A},\mathcal{C}at]}(F) \simeq \left[\int F, \mathcal{S}et\right]$$

becomes the well-known

$$[\mathcal{A}, \mathcal{S}et]/_F \simeq \left[\int F, \mathcal{S}et\right].$$

Indeed any discrete opfibration $\varphi\colon G \to F$ in $[\mathcal{A}, \mathcal{C}at]$ over $F\colon \mathcal{A} \to \mathcal{S}et$ with small fibres needs to have $G\colon \mathcal{A} \to \mathcal{S}et$, and all functors $G \to F$ in $[\mathcal{A}, \mathcal{S}et]$ are discrete opfibrations with small fibres. Our theorem guarantees that this equivalence is pseudonatural in F, which does not seem to appear in the literature.

When F is a representable $\mathrm{y}(A) : \mathcal{A} \to \mathcal{S}et$, we obtain the famous equivalence

$$[\mathcal{A}, \mathcal{S}et]\Big/_{\mathrm{y}(A)} \simeq [\mathcal{A}/_A, \mathcal{S}et]$$

between slices of (co)presheaves and (co)presheaves on slices. We will apply its pseudonaturality in F in Example 4.4.9 to get a nice candidate for a Hofmann–Streicher universe (see [25]) in 2-presheaves.

Remark 4.3.11. The equivalence

$$[\mathcal{A}, \mathcal{S}et]/_F \simeq \left[\int F, \mathcal{S}et\right]$$

is very useful in topos theory. It has also been applied to geometry, for example by Artin, Grothendieck and Verdier in [3, Section IV.5] (where they call "topos induit" the slice of a topos). It is the archetypal case of the fundamental theorem of elementary topos theory, as it shows that every slice of a Grothendieck topos is a Grothendieck topos.

We can interpret our main theorem as a 2-dimensional generalization of this. Indeed it has recently been proposed that, in dimension 2, the correct analogue of the slice is an (op)fibrational slice (e.g. Ahrens, North and van der Weide's [2]). Our equivalence

$$\mathcal{O}p\mathcal{F}ib_{[\mathcal{A},\mathcal{C}at]}(F) \simeq \left[\int F, \mathcal{C}at\right]$$

says that every opfibrational slice of a Grothendieck 2-topos is again a Grothendieck 2-topos.

We can now explore some variations on the indexed Grothendieck construction.

Remark 4.3.12. We can change $\mathcal{A}$ to $\mathcal{A}^{\mathrm{op}}$ and get the 2-category $[\mathcal{A}^{\mathrm{op}}, \mathcal{C}at]$ of 2-presheaves. Then $F\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}at$. Be careful that, for opfibrations in $[\mathcal{A}^{\mathrm{op}}, \mathcal{C}at]$, we still need to apply the Grothendieck construction to F as if we did not know that the domain of F is an opposite category. We write $\int^{\mathrm{op}} F$ for this Grothendieck construction on F, to emphasize that it is not the most natural one for a 2-functor $\mathcal{A}^{\mathrm{op}} \to \mathcal{C}at$. We obtain

$$\mathcal{O}p\mathcal{F}ib_{[\mathcal{A}^{\mathrm{op}},\mathcal{C}at]}(F) \simeq \left[\int^{\mathrm{op}} F, \mathcal{C}at\right]$$

The most natural Grothendieck construction $\int F$ of a contravariant 2-functor $F\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{C}at$ appears instead to handle fibrations in the place of opfibrations. Such Grothendieck construction $\int F$ is the lax colimit of F. Then $-^{\mathrm{op}}\colon \mathcal{C}at \to \mathcal{C}at^{\mathrm{co}}$, where $\mathcal{C}at^{\mathrm{co}}$ is the dualization on 2-cells of $\mathcal{C}at$, preserves this colimit. We obtain that $\left(\int F\right)^{\mathrm{op}}$ is the lax colimit in $\mathcal{C}at^{\mathrm{co}}$ of $F(-)^{\mathrm{op}}$, which means that

$$\left(\int F\right)^{\mathrm{op}} = \text{oplax-colim}\left(F(-)^{\mathrm{op}}\right)$$

in $\mathcal{C}at$. Then we have the following chain of equivalences of categories:

$$\left[\left(\int F\right)^{\mathrm{op}}, \mathcal{C}at_0\right] \cong [\mathcal{A}, \mathcal{CAT}]_{\mathrm{oplax}}\left(\Delta 1, [F(-)^{\mathrm{op}}, \mathcal{C}at_0]\right) \simeq$$
$$\simeq \mathrm{Ps}\,[\mathcal{A}, \mathcal{CAT}]_{\mathrm{oplax}}\left(\Delta 1, \mathcal{F}ib_{\mathcal{C}at}\left(F(-)\right)\right) \cong \mathcal{F}ib_{[\mathcal{A}^{\mathrm{op}},\mathcal{C}at]}(F)$$

Remark 4.3.13. We believe that, when $\mathcal{A}$ is a 2-category, one can still obtain an equivalence of categories

$$\mathcal{O}p\mathcal{F}ib_{[\mathcal{A},\mathcal{C}at]}(F) \simeq \left[\int F, \mathcal{C}at\right]$$

where $\int F$ is now the 2-category of elements (or 2-$\mathcal{S}et$-enriched Grothendieck construction), introduced by Street in [42] and explored more in detail in Chapter 1. In order to adapt our proof of Theorem 4.3.7 to this setting, one would need $\int F$ to be a kind of oplax colimit in 2-$\mathcal{C}at$. We believe that F is the 2-oplax colimit of F followed by the inclusion i of $\mathcal{C}at$ into 2-$\mathcal{C}at$, where a 2-oplax natural transformation is a Crans's [14] oplax 1-transfor. Such transformations have the same

1-dimensional conditions of an oplax natural transformation but now also have structure 3-cells on every 2-cell in $\mathcal{A}$. Having as codomain 2-$\mathcal{C}at$, they compose well. The added structure 3-cells are precisely what one needs in order to encode the 2-cells

$$\underline{\delta}^X : (f, F(\delta)_X) \Rightarrow (g, \mathrm{id}) : (A, X) \to (B, F(g)(X))$$

in $\int F$ for every $\delta : f \Rightarrow g : A \to B$ in $\mathcal{A}$. As explained in Construction 1.1.4, every 2-cell in $\int F$ is a whiskering of such particular 2-cells (in some sense, these 2-cells are the only ones we need). The middle equivalence of the chain that proves our Theorem 4.3.7 would then be given by the 2-category of elements construction. Finally, the last part of the chain would probably work as well, with the structure 3-cells managing to encode the action of G on 2-cells. However, we have not checked these details.

Such generalization would be helpful also to handle non-split opfibrations and pseudofunctors from $\int F$ into $\mathcal{C}at$, for which we cannot reduce to functors into $\mathcal{C}at_0$. Of course, for this, one could also extend the explicit indexed Grothendieck construction.

For the restriction to copresheaves and discrete opfibrations, we need to be careful that

$$2\text{-}\mathcal{C}at\left(\int F, i(\mathcal{S}et)\right) \cong \mathcal{C}at\left(\pi_* \int F, \mathcal{S}et\right)$$

where π^* is the left adjoint of $i : \mathcal{C}at \to 2\text{-}\mathcal{C}at$. So a quotient of $\int F$ by its 2-cells appears: morphisms in $\int F$ that were connected via a 2-cell becomes equal.

4.4. Examples

In this section, we show some interesting examples. We can vary both $\mathcal{A}$ and F in our main results. We start with $\mathcal{A} = \mathit{1}$, that recovers the usual Grothendieck construction. $\mathcal{A} = \mathit{2}$ represents the simultaneous Grothendieck construction of two opfibrations connected by an arrow. While $\mathcal{A} = \Delta$ considers (co)simplicial categories. We also explore other examples. In particular, we obtain a nice candidate for a Hofmann–Streicher universe in 2-presheaves.

Remark 4.4.1 ($\mathcal{A} = \mathit{1}$)**.** When $\mathcal{A} = \mathit{1}$, the 2-category $[\mathcal{A}, \mathcal{C}at]$ reduces to $\mathcal{C}at$. A 2-functor $F\colon \mathit{1} \to \mathcal{C}at$ is just a small category $\mathcal{C}$ and $\int F = \mathcal{C}$. So Theorem 4.3.7 gives the classical

$$\mathcal{O}p\mathcal{F}ib\,(\mathcal{C}) \simeq [\mathcal{C}, \mathcal{C}at]\,.$$

The explicit indexed Grothendieck construction becomes the usual Grothendieck construction. Indeed the first part and the last part of the chain become trivial, while the middle part is the Grothendieck construction on the unique index $* \in \mathit{1}$.

Example 4.4.2 ($\mathcal{A}$ discrete)**.** When $\mathcal{A}$ is discrete, the 2-category $[\mathcal{A}, \mathcal{C}at]$ is a product of copies of $\mathcal{C}at$. A 2-functor $F\colon \mathcal{A} \to \mathcal{C}at$ just picks as many categories as the cardinality of $\mathcal{A}$, without bonds. Since the diagram F is parametrized by a discrete category, we have that $\int F = \text{oplax-colim}\, F$ becomes the coproduct of the categories picked by F. And $\left[\int F, \mathcal{C}at\right]$ is then a collection of functors from every such category into $\mathcal{C}at$. On the other hand, by Proposition 4.2.5, a split opfibration in $[\mathcal{A}, \mathcal{C}at]$ is just a collection of as many opfibrations as the cardinality of $\mathcal{A}$, without bonds. The indexed Grothendieck construction

$$\left[\int F, \mathcal{C}at\right] \simeq \mathcal{O}p\mathcal{F}ib_{[\mathcal{A},\mathcal{C}at]}\,(F)$$

is the simultaneous Grothendieck construction of all the functors into $\mathcal{C}at$ that are collected as a single functor from the coproduct. This shows the indexed nature of the indexed Grothendieck construction.

Example 4.4.3 ($\mathcal{A} = \mathit{2}$)**.** When $\mathcal{A} = \mathit{2}$, the 2-category $[\mathcal{A}, \mathcal{C}at]$ is the arrow category of $\mathcal{C}at$ and $F\colon \mathit{2} \to \mathcal{C}at$ is a functor $\widetilde{F}\colon \mathcal{C} \to \mathcal{D}$. The Grothendieck construction $\int F$ has as objects the disjoint union of the objects of $\mathcal{C}$ and of $\mathcal{D}$, denoted respectively $(0, C)$ and $(1, D)$ with $C \in \mathcal{C}$ and $D \in \mathcal{D}$. The morphisms of $\int F$ are of three kinds: morphisms in $\mathcal{C}$ (over 0), morphisms in $\mathcal{D}$ (over 1) and morphisms over $0 \to 1$ that represents the objects $(C, D, \widetilde{F}(C) \to D)$ of the comma category $\widetilde{F}/\mathcal{D}$. On the other hand, given $G\colon \mathit{2} \to \mathcal{C}at$ corresponding to $\widetilde{G}\colon \mathcal{E} \to \mathcal{L}$, a split opfibration $\varphi\colon G \to F$ in $[\mathit{2}, \mathcal{C}at]$ is a cleavage preserving morphism

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "\widetilde{G}"] \arrow[d, "\varphi_0"'] & \mathcal{L} \arrow[d, "\varphi_1"] \\
\mathcal{C} \arrow[r, "\widetilde{F}"'] & \mathcal{D}
\end{tikzcd}$$

between split opfibrations φ_0 and φ_1. So Theorem 4.3.7 gives an equivalence of categories between morphisms of (classical) split opfibrations (in $\mathcal{C}at$) which have $\widetilde{F}$ as second component and 2-copresheaves on a category that collects together $\mathcal{C}$, $\mathcal{D}$ and the comma category ${\widetilde{F}}/{\mathcal{D}}$.

Following Construction 4.3.8, we get the explicit (quasi-inverse of the) indexed Grothendieck construction in this case. The arrow above between split opfibrations φ_0 and φ_1 can be reorganized as the functor $\int F \to \mathcal{C}at$ that sends

(i) $(0, C)$ to the fibre of φ_0 on C and every morphism f in $\mathcal{C}$ to the functor f_* that lifts it along φ_0;

(ii) $(1, D)$ to the fibre of φ_1 on D and every morphism g in $\mathcal{D}$ to the functor g_* that lifts it along φ_1;

(iii) every morphism corresponding to an object $(C, D, \alpha\colon \widetilde{F}(C) \to D)$ of the comma category ${\widetilde{F}}/{\mathcal{D}}$ to the composite functor

$$(\varphi_0)_C \xrightarrow{\widetilde{G}} (\varphi_1)_{\widetilde{F}(C)} \xrightarrow{\alpha_*} (\varphi_1)_D.$$

In the particular case in which $F\colon \mathit{2} \to \mathcal{S}et$, we have that $\widetilde{F}\colon S \to T$ is a function between sets. Then $\int F$ is a poset with objects the disjoint union of the objects of S and of T and such that $(0, s) \leq (1, t)$ with $s \in S$ and $t \in T$ if and only if $\widetilde{F}(s) = t$. On the other hand, a split opfibration in $[\mathit{2}, \mathcal{C}at]$ over F is precisely a commutative square in $\mathcal{S}et$ with bottom leg equal to $\widetilde{F}$.

Example 4.4.4 ($\mathcal{A} = \mathcal{I}$)**.** When $\mathcal{A}$ is the walking isomorphism $\mathcal{I}$, we have that $F\colon \mathcal{I} \to \mathcal{C}at$ is an invertible functor $\widetilde{F}\colon \mathcal{C} \to \mathcal{D}$. Then $\int F$ is similar to the one of Example 4.4.3, but there is now a fourth kind of morphisms, that represents the objects $(D, C, \widetilde{F}^{-1}(D) \to C)$ of the comma category ${\widetilde{F}^{-1}}/{\mathcal{C}}$.

If $F\colon \mathcal{I} \to \mathcal{S}et$, the partial order of the poset constructed as in Example 4.4.3 now becomes an equivalence relation. Every object is in relation precisely with itself and with its copy in the other set.

Example 4.4.5 ($\mathcal{A} = \Delta$)**.** When $\mathcal{A}$ is the simplex category Δ, we have that $F\colon \Delta \to \mathcal{C}at$ is a cosimplicial category. This is equivalently a cosimplicial object in $\mathcal{C}at$ or an internal category in cosimplicial sets. The Grothendieck construction $\int F$ collects together all the cosimplexes in a total category, taking into account faces and degeneracies. Theorem 4.3.7 gives an equivalence of categories between split opfibrations between cosimplicial categories over F and functors into $\mathcal{C}at$ from the total category that collects all the cosimplexes given by F.

Example 4.4.6 ($F = \Delta 1$)**.** Given any small category $\mathcal{A}$, we can consider $F = \Delta 1\colon \mathcal{A} \to \mathcal{C}at$ the functor constant at the terminal $\mathit{1}$. We have that $\int \Delta 1 = \mathcal{A}$. So Theorem 4.3.7 gives an equivalence of categories

$$\mathcal{O}p\mathcal{F}ib_{[\mathcal{A},\mathcal{C}at]}(\Delta 1) \simeq [\mathcal{A}, \mathcal{C}at]\,.$$

Indeed, as being opfibred over $\mathit{1}$ means nothing, a split opfibration $\varphi\colon G \to \Delta 1$ is a collection of categories $G(A)$ and of functors $G(h)$ for every $h\colon A \to B$ in $\mathcal{A}$. This forms a functor $\mathcal{A} \to \mathcal{C}at$ because φ is split.

Putting together this equivalence with that of Example 4.4.1, we obtain

$$\mathcal{O}p\mathcal{F}ib_{[\mathcal{A},\mathcal{C}at]}(\Delta 1) \simeq \mathcal{O}p\mathcal{F}ib_{\mathcal{C}at}(\mathcal{A})$$

Example 4.4.7 ($F = \Delta\mathcal{B}$)**.** Given any small category $\mathcal{A}$, we can consider $F = \Delta\mathcal{B}\colon \mathcal{A} \to \mathcal{C}at$ the functor constant at a fixed category $\mathcal{B}$. We have that $\int \Delta\mathcal{B} = \mathcal{A} \times \mathcal{B}$ and $\mathcal{G}(\Delta\mathcal{B})$ is the projection $\mathcal{A} \times \mathcal{B} \to \mathcal{A}$. Theorem 4.3.7 characterizes functors $\mathcal{A} \times \mathcal{B} \to \mathcal{C}at$, and hence the $\mathcal{C}at$-enriched profunctors, in terms of split opfibrations in $[\mathcal{A}, \mathcal{C}at]$ over $\Delta\mathcal{B}$.

Example 4.4.8 (semidirect product of groups)**.** Let $\mathcal{A}$ be the one-object category $\mathcal{B}G$ corresponding to a group G. Consider then $F\colon \mathcal{B}G \to \mathcal{C}at$ that sends the unique object of $\mathcal{B}G$ to the one-object category that corresponds to a group H. Functoriality of F corresponds precisely to giving a group homomorphism

$\rho\colon G \to \mathrm{Aut}(H)$ where $\mathrm{Aut}(H)$ is the group of automorphisms of H. Then the Grothendieck construction $\int F$ is a one-object category corresponding to the semidirect product $H \rtimes_\rho G$. Thus Theorem 4.3.7 characterizes functors $H \rtimes_\rho G \to \mathcal{C}at$ in terms of opfibrations in $[\mathcal{B}G, \mathcal{C}at]$ over the functor F that corresponds with $\rho\colon G \to \mathrm{Aut}(H)$.

Example 4.4.9 (Hofmann–Streicher universe in 2-presheaves)**.** We apply Theorem 4.3.7 (and Theorem 4.3.9) to solve the problem we found in Section 3.3 and get a nice candidate for a Hofmann–Streicher universe (see [25]) in the 2-category $[\mathcal{A}^{\mathrm{op}}, \mathcal{C}at]$ of 2-presheaves (i.e. prestacks). In Chapter 5, we will then show that such candidate is indeed a good 2-classifier in $[\mathcal{A}^{\mathrm{op}}, \mathcal{C}at]$ that classifies all discrete opfibrations with small fibres, towards a 2-dimensional elementary topos structure on $[\mathcal{A}^{\mathrm{op}}, \mathcal{C}at]$. This was the starting motivation for the work of this chapter.

Recall from Construction 3.3.2 that, by our theorems of reduction of the study of a 2-classifier to dense generators, a good 2-classifier in $[\mathcal{A}^{\mathrm{op}}, \mathcal{C}at]$ that classifies all discrete opfibrations with small fibres is forced up to equivalence to be the pseudofunctor

$$\mathcal{A}^{\mathrm{op}} \xrightarrow{\mathrm{y}^{\mathrm{op}}} [\mathcal{A}^{\mathrm{op}}, \mathcal{C}at]^{\mathrm{op}} \xrightarrow{\mathcal{DO}p\mathcal{F}ib^{\mathrm{s}}(-)} \mathcal{CAT},$$

that assigns $A \overset{\Omega}{\mapsto} \mathcal{DO}p\mathcal{F}ib^{\mathrm{s}}_{[\mathcal{A}^{\mathrm{op}}, \mathcal{C}at]}(\mathrm{y}(A))$. By Remark 3.3.3, such Ω nicely generalizes to dimension 2 the subobject classifier of 1-dimensional presheaves . However, it has the problem of not being a strict 2-functor and not clearly landing in $\mathcal{C}at$, so that it cannot be a 2-classifier in $[\mathcal{A}^{\mathrm{op}}, \mathcal{C}at]$.

Theorem 4.3.7 (together with Theorem 4.3.9 and Remark 4.3.12) offers a nice way to replace such pseudofunctor Ω with a concrete strict 2-functor $\widetilde{\Omega}$ that lands in $\mathcal{C}at$. Although it was already known that any pseudofunctor can be strictified, by the theory developed by Power in [40] and later by Lack in [29], the work of this chapter provides an explicit and easy to handle strictification of Ω, which in addition lands in $\mathcal{C}at$. Moreover, as we will present in Chapter 5, such strictification can also be restricted in a natural way to a good 2-classifier in stacks.

Theorem 4.3.9 gives an equivalence of categories

$$\mathcal{DOpFib}^{\mathrm{s}}_{[\mathcal{A}^{\mathrm{op}},\mathcal{Cat}]}(\mathrm{y}(A)) \simeq \left[\int^{\mathrm{op}} \mathrm{y}(A), \mathcal{Set}\right]$$

that is pseudonatural in $A \in \mathcal{A}^{\mathrm{op}}$, by precomposing the equivalence that is pseudonatural in $F \in [\mathcal{A}^{\mathrm{op}}, \mathcal{Cat}]^{\mathrm{op}}$ with $\mathrm{y}^{\mathrm{op}} \colon \mathcal{A}^{\mathrm{op}} \to [\mathcal{A}^{\mathrm{op}}, \mathcal{Cat}]^{\mathrm{op}}$. So the right hand side of the equivalence above gives a strict 2-functor $\widetilde{\Omega}$ that is pseudonaturally equivalent to Ω and lands in $\mathcal{Cat}$.

When $\mathcal{A}$ is a 1-category,

$$\int^{\mathrm{op}} \mathrm{y}(A) = \left(\mathcal{A}/_{A}\right)^{\mathrm{op}} \quad \text{and} \quad \widetilde{\Omega}(A) = \left[\left(\mathcal{A}/_{A}\right)^{\mathrm{op}}, \mathcal{Set}\right].$$

$\widetilde{\Omega}$ acts on morphisms by postcomposition. Notice that in this case $\mathrm{y}(A) : \mathcal{A}^{\mathrm{op}} \to \mathcal{Set}$ and so the left hand side of the equivalence above simplifies to $[\mathcal{A}^{\mathrm{op}}, \mathcal{Set}] \big/_{\mathrm{y}(A)}$, but we still need the pseudonaturality in A, that does not seem to appear in the literature. Our Theorem 4.3.9 (with Theorem 4.3.7) guarantees such pseudonaturality and therefore that we get a strict 2-functor $\widetilde{\Omega}$ pseudonaturally equivalent to Ω. This is a Hofmann–Streicher universe, in line with the ideas of [25] and with Awodey's recent work [4]. In Chapter 5, we will show that $\widetilde{\Omega}$ is indeed a good 2-classifier in $[\mathcal{A}^{\mathrm{op}}, \mathcal{Cat}]$ that classifies all discrete opfibrations with small fibres, using our theorems of reduction of the study of a 2-classifier to dense generators. We will then restrict this good 2-classifier to one in stacks.

When $\mathcal{A}$ is a 2-category, the 2-category of elements gives

$$\int^{\mathrm{op}} \mathrm{y}(A) = \left(\mathcal{A}/_{\mathrm{oplax}}\ A\right)^{\mathrm{op}}$$

Checking the details of the strategy proposed in Remark 4.3.13, we would get a refined strict 2-functor $\widetilde{\Omega}$ defined by

$$\widetilde{\Omega}(A) = \left[\pi_*\left(\mathcal{A}/_{\mathrm{oplax}}\ A\right)^{\mathrm{op}}, \mathcal{Set}\right]$$

(recall the definition of π_* from Remark 4.3.13). Interestingly, such quotients of (op)lax slices give the right weights to represent (op)lax (co)limits as weighted ones, by Street's [42, Theorem 11].

5. Hofmann–Streicher universe in stacks

This chapter is based on the second half of our [37].

In this chapter, we contribute to expand further 2-dimensional elementary topos theory, introduced by Weber in [51]; see also Chapter 3. Recall from Definition 3.1.12 the notion of good 2-classifier, that captures well-behaved Weber's 2-classifiers (Definition 3.1.10) and generalizes the concept of subobject classifier to dimension 2. The archetypal good 2-classifier is given by the construction of the category of elements (or Grothendieck construction), that exhibits $\mathcal{C}at$ as the archetypal elementary 2-topos.

We present a good 2-classifier in prestacks that classifies all discrete opfibrations with small fibres. We then restrict this good 2-classifier to one in stacks. This is the main part of a proof that Grothendieck 2-topoi are elementary 2-topoi. The reason why we focus on 2-classifiers is that the rest of the definition of elementary 2-topos proposed by Weber is yet to be ascertained. We hope that this thesis will contribute to reach a universally accepted notion of elementary 2-topos.

In order to produce a good 2-classifier in prestacks, we apply our theorems of reduction of the study of a 2-classifier to dense generators, presented in Chapter 3. This allows us to consider the classification just over representables. As explained in Section 3.3, Yoneda lemma determines up to equivalence the construction of a good 2-classifier in prestacks that classifies all discrete opfibrations with small fibres. Recall from Remark 3.3.3 that the assignment involves a natural notion of 2-dimensional sieve, generalizing the subobject classifier of 1-dimensional presheaves. The work of Chapter 4 has then solved all the problems we described in Remark 3.3.3. Indeed, the indexed Grothendieck construction allowed us to produce a nice candidate $\widetilde{\Omega}$ for a good 2-classifier in prestacks that is strictly 2-functorial and moreover lands in $\mathcal{C}at$ rather than just in large categories. Explicitly, the 2-functor $\widetilde{\Omega}$ takes presheaves on slices (see Example 4.4.9). Recall also that Section 4.2 allowed us to define the property of having small fibres for a discrete opfibration in prestacks, in Definition 4.2.7.

In Theorem 5.1.14, we prove that the candidate $\widetilde{\Omega}$ produced in Example 4.4.9 is indeed a good 2-classifier in prestacks that classifies all discrete opfibrations with small fibres. A partial result on this direction is already in Weber's [51]. Our result is in line with Hofmann and Streicher's [25], that uses a similar idea to lift Grothendieck universes to presheaves, in order to interpret Martin-Löf type theory in a presheaf topos. It is also in line with the recent Awodey's [4], that captures the construction of the Hofmann and Streicher's universe in presheaves in a conceptual way. Our proof goes through a bicategorical classification process that, over representables, is exactly the Yoneda lemma. Although some points would be smoother in a bicategorical context, we believe that it is important to show how strict the theory can be. In the case of prestacks, the strict classification process, which involves an indexed Grothendieck construction, actually seems more interesting than the bicategorical one, which reduces to the Yoneda lemma. In future work, we will adapt the results of this chapter to the bicategorical context, using a suitable bicategorical notion of classifier.

Finally, in Theorem 5.3.22, we restrict our good 2-classifier in prestacks to a good 2-classifier in stacks, classifying again all discrete opfibrations with small fibres. We achieve this by proving a general result of restriction of good 2-classifiers to nice sub-2-categories, in Theorem 5.2.9, involving factorization arguments and our theorems of reduction to dense generators. Stacks are a bicategorical generalization of sheaves and they were introduced by Giraud in [18]. Like sheaves, they are able to glue together families of objects that are compatible under descent. But such descent compatibilities are only asked up to isomorphism. And the produced global data then equally recovers the starting local data up to isomorphism. Stacks are the right notion to use to generalize Grothendieck topoi to dimension 2. Our result is thus the main part of a proof that Grothendieck 2-topoi are elementary 2-topoi. As explained in Remark 5.3.10, we consider strictly functorial stacks with respect to a subcanonical Grothendieck topology. So that they form a full sub-2-category of the 2-category of 2-presheaves. While our good 2-classifier in prestacks involves a 2-dimensional notion of sieves, our good 2-classifier in stacks involves a 2-dimensional notion of closed sieves. The idea is to select, out of all the presheaves on slices considered in the definition of the good 2-classifier in prestacks, the sheaves with respect to the Grothendieck topology induced on the slices. This restriction is tight enough to give a stack Ω_J but at the same time loose enough to still host the classification process of prestacks. We prove that Ω_J is a good 2-classifier in stacks. Our result solves a problem posed by Hofmann and Streicher in [25]. Indeed, in a different context, they considered the same natural idea to restrict their analogue of $\widetilde{\Omega}$ by taking sheaves on slices. However, this did not work for them, as it does not give a sheaf. Our results show that such a restriction yields nonetheless a stack and a good 2-classifier in stacks. The idea is that, in order to increase the dimension of the fibres of the morphisms to classify, one should also increase the dimension of the ambient. And thus stacks behave better than sheaves for the classification of small families.

Outline of the chapter

In Section 5.1, we apply our theorems of reduction of the study of a 2-classifier to dense generators to the case of prestacks. We thus produce a good 2-classifier in prestacks that classifies all discrete opfibrations with small fibres (Theorem 5.1.14). We also show a concrete recipe for the characteristic morphisms (Remark 5.1.16).

In Section 5.2, we prove a general result of restriction of good 2-classifiers to nice sub-2-categories (Theorem 5.2.9), involving factorization arguments and our theorems of reduction to dense generators.

In Section 5.3, we apply the results of Section 5.2 to restrict our good 2-classifier in prestacks to a good 2-classifier in stacks, classifying again all discrete opfibrations with small fibres (Theorem 5.3.22).

5.1. A 2-classifier in prestacks

In this section, we apply our theorems of reduction of the study of 2-classifiers to dense generators to the case of prestacks. Our theorems allow us to consider just the classification over representables, which is essentially given by the Yoneda lemma (see Proposition 5.1.12). We show that the normalization process required by Theorem 3.2.8 is possible in prestacks (Theorem 5.1.14 and Remark 5.1.15). Whence, by Theorem 5.2.9, it is also possible in any nice sub-2-category of prestacks (such as the 2-category of stacks, see Theorem 5.3.22).

We thus produce a good 2-classifier in prestacks that classifies all discrete opfibrations with small fibres, in Theorem 5.1.14 (see also Definition 3.1.12). Our result is in line with Hofmann and Streicher's [25], and with the recent Awodey's [4], see Remark 5.1.4. We conclude the section extracting from the constructive proof of Theorem 3.2.8 a concrete recipe for the characteristic morphisms in prestacks (Remark 5.1.16).

In Section 5.3, we will restrict the good 2-classifier in prestacks to a good 2-classifier in stacks (Theorem 5.3.22, using Theorem 5.2.9).

Throughout the rest of this chapter, we consider the 2-category $\mathcal{L} = [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ of 2-presheaves on a small category $\mathcal{C}$ (that is, prestacks on $\mathcal{C}$). Notice that this 2-category is complete and cocomplete, since $\mathcal{C}at$ is so. Recall from Section 4.2 the characterization of discrete opfibrations in $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ and the definition of discrete opfibration in $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ with small fibres.

Notation 5.1.1. Given $p\colon \mathcal{E} \to \mathcal{B}$ a discrete opfibration in $\mathcal{C}at$ with small fibres, we denote as $(p)_B$ the fibre of p on $B \in \mathcal{B}$.

Remark 5.1.2. Recall from Construction 3.3.2 that a good 2-classifier in prestacks that classifies all discrete opfibrations with small fibres needs to be pseudonaturally equivalent to the the pseudofunctor Ω defined by the composite

$$\mathcal{C}^{\mathrm{op}} \xrightarrow{\mathrm{y}^{\mathrm{op}}} [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]^{\mathrm{op}} \xrightarrow{\mathcal{DO}p\mathcal{F}ib^{\,\mathrm{s}}(-)} \mathcal{CAT}$$

We also consider the pseudonatural transformation $\omega\colon 1 \to \Omega$ with component on $C \in \mathcal{C}$ that picks the identity $\mathrm{id}_{\mathrm{y}(C)}$ on $\mathrm{y}(C)$.

Recall then that the work of Chapter 4 solved all the problems shown in Remark 3.3.3. We summarize this in the following proposition.

Proposition 5.1.3 (Chapter 4)**.** *The pseudofunctor* $\Omega\colon C \mapsto \mathcal{DO}p\mathcal{F}ib^{\,\mathrm{s}}(\mathrm{y}(C))$ *is pseudonaturally equivalent to the 2-functor*

$$\begin{array}{rccl} \widetilde{\Omega}: & \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathcal{C}at \\ & C & \mapsto & \left[\left(\mathcal{C}/_{C}\right)^{\mathrm{op}}, \mathcal{S}et\right] \\ & (C \xleftarrow{f} D) & \mapsto & -\circ (f\circ =)^{\mathrm{op}} \end{array}$$

The pseudonatural equivalence $j\colon \widetilde{\Omega} \simeq \Omega$ *is given by the indexed Grothendieck construction. Explicitly, a discrete opfibration* $\psi\colon H \to \mathrm{y}(C)$ *with small fibres corresponds with the presheaf*

$$\begin{array}{rccl} j_C^{-1}(\psi): & \left(\mathcal{C}/_{C}\right)^{\mathrm{op}} & \longrightarrow & \mathcal{S}et \\ & (D \xrightarrow{f} C) & \mapsto & (\psi_D)_f \\ & (f \xleftarrow{g} f\circ g) & \mapsto & H(g) \end{array}$$

Remark 5.1.4. Call j^{-1} the quasi-inverse of j described in Construction 4.3.8 and hinted above (that transforms ψ into $j_C^{-1}(\psi)$). The composite $1 \xrightarrow{\omega} \Omega \xrightarrow{j^{-1}} \widetilde{\Omega}$ is (isomorphic to) a 2-natural transformation $\widetilde{\omega}$. Explicitly, the component $\widetilde{\omega}_C\colon \mathit{1} \to \widetilde{\Omega}(C)$ of $\widetilde{\omega}$ on $C \in \mathcal{C}$ picks the constant at 1 presheaf $\Delta 1\colon \left(\mathcal{C}/_{C}\right)^{\mathrm{op}} \to \mathcal{S}et$.

We will prove in Theorem 5.1.14 that $\widetilde{\omega}\colon 1 \to \widetilde{\Omega}$ is a good 2-classifier in $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ that classifies all discrete opfibrations with small fibres. This is in line with Hofmann and Streicher's [25], where a similar idea is used to construct a universe in 1-dimensional presheaves for small families, in order to interpret Martin-Löf type theory in a presheaf topos. See also the recent Awodey's [4], that constructs Hofmann and Streicher's universe in 1-dimensional presheaves in a more conceptual way.

Remark 5.1.5. In Example 4.4.9 (using Remark 4.3.13), we also suggest a strategy to prove that, for $\mathcal{C}$ a 2-category, what works is $\widetilde{\Omega}\colon C \mapsto \left[\pi_*\left(\mathcal{C}/_{\text{oplax}}\, C\right)^{\text{op}}, \mathcal{S}et\right]$, where π_* is the left adjoint of the inclusion $\mathcal{C}at \hookrightarrow 2\text{-}\mathcal{C}at$.

Proposition 5.1.6. *The lax limit $\widetilde{\tau}\colon \widetilde{\Omega}_\bullet \to \widetilde{\Omega}$ of the arrow $\widetilde{\omega}\colon 1 \to \widetilde{\Omega}$ is a discrete opfibration in $[\mathcal{C}^{\text{op}}, \mathcal{C}at]$ with small fibres.*

As a consequence, for every $F \in [\mathcal{C}^{\text{op}}, \mathcal{C}at]$, the functor

$$\widehat{\mathcal{G}}_{\widetilde{\omega},F}\colon [\mathcal{C}^{\text{op}}, \mathcal{C}at]\left(F, \widetilde{\Omega}\right) \to \mathcal{DO}p\mathcal{F}ib(F)$$

lands in $\mathcal{DO}p\mathcal{F}ib^{\,\text{s}}(F)$.

Proof. By Remark 3.1.13 any comma object from $\widetilde{\omega}$ can be expressed as a pullback of $\widetilde{\tau}$, and by Remark 4.2.8 the property of having small fibres is stable under pullback. So we can just look at $\widetilde{\tau}$. Since comma objects in $[\mathcal{C}^{\text{op}}, \mathcal{C}at]$ are calculated pointwise, for every $C \in \mathcal{C}$ the component $\widetilde{\tau}_C$ of $\widetilde{\tau}$ on C is given by the comma object in $\mathcal{C}at$ from $\widetilde{\omega}_C = \Delta 1$ to $\text{id}_{\widetilde{\Omega}(C)}$. Given $Z \in \widetilde{\Omega}(C) = \left[\left(\mathcal{C}/_{C}\right)^{\text{op}}, \mathcal{S}et\right]$,

$$(\widetilde{\tau}_C)_Z \cong \widetilde{\Omega}(C)\,(\Delta 1, Z) = \left[\left(\mathcal{C}/_{C}\right)^{\text{op}}, \mathcal{S}et\right](\Delta 1, Z) \cong Z(\text{id}_C)$$

and thus $\widetilde{\tau}_C$ has small fibres. Indeed a natural transformation from $\Delta 1$ to Z is the same thing as an element in $Z(\text{id}_C)$, by the naturality condition. This is similar to the proof of the Yoneda lemma; see also Remark 5.1.9. □

Remark 5.1.7. $\widetilde{\Omega}_\bullet$ is a pointed version of $\widetilde{\Omega}$. The "points" of $Z \in \widetilde{\Omega}(C)$ are the elements of $Z(\text{id}_C)$.

It will be useful to consider the bicategorical classification process produced by the pseudofunctor Ω.

Remark 5.1.8. We need to consider the 2-category (actually $\mathcal{CAT}$-enriched category) $\text{Ps}\,[\mathcal{C}^{\text{op}}, \mathcal{CAT}]$ of pseudofunctors from $\mathcal{C}^{\text{op}}$ to $\mathcal{CAT}$, pseudonatural transformations and modifications. We can of course extend the definition of discrete opfibration in a 2-category to one in any $\mathcal{CAT}$-enriched category, considering $\mathcal{CAT}$ in the place of $\mathcal{C}at$.

By Bird, Kelly, Power and Street's [7, Remark 7.4], $\operatorname{Ps}[\mathcal{C}^{\mathrm{op}}, \mathcal{CAT}]$ has all bilimits and all flexible limits, calculated pointwise. In particular, it has all comma objects, the terminal object and all pullbacks along discrete opfibrations, calculated pointwise. Indeed, for the latter, recall that in $\mathcal{CAT}$ pullbacks along discrete opfibrations exhibit bi-iso-comma objects (the idea is similar to that of the diagram of equation (3.1)). So, given $p\colon E \to B$ and $z\colon F \to B$ in $\operatorname{Ps}[\mathcal{C}^{\mathrm{op}}, \mathcal{CAT}]$ with p a discrete opfibration, we can construct (pointwise) a bi-iso-comma object G of p and z whose universal square is filled with an identity. This is obtained by choosing the pullbacks as representatives for the bi-iso-comma objects in $\mathcal{CAT}$ on every component. It follows that G is also a pullback in $\operatorname{Ps}[\mathcal{C}^{\mathrm{op}}, \mathcal{CAT}]$, using that discrete opfibrations lift identities to identities.

Remark 5.1.9. As hinted in the proof of Proposition 5.1.6, the Yoneda lemma is the reason why that proposition holds. Consider the pseudofunctor Ω and the lax limit τ of the arrow $\omega\colon 1 \to \Omega$ in $\operatorname{Ps}[\mathcal{C}^{\mathrm{op}}, \mathcal{CAT}]$. For every $C \in \mathcal{C}$ and every discrete opfibration $\psi\colon H \to \mathrm{y}(C)$ in $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ with small fibres, by the Yoneda lemma

$$(\tau_C)_{\psi} \cong \Omega(C)\left(\mathrm{id}_{\mathrm{y}(C)}, \psi\right) \cong (\psi_C)_{\mathrm{id}_C}.$$

Thus τ_C has small fibres.

Proposition 5.1.10. *For every $F \in [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$, taking comma objects from $\omega\colon 1 \to \Omega$ extends to a functor*

$$\widehat{\mathcal{G}}_{\omega,F}\colon \operatorname{Ps}[\mathcal{C}^{\mathrm{op}}, \mathcal{CAT}](F, \Omega) \to \mathcal{DO}p\mathcal{F}ib^{\mathrm{s}}{}_{[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]}(F)$$

Proof. Taking comma objects from the morphism $\omega\colon 1 \to \Omega$ in $\operatorname{Ps}[\mathcal{C}^{\mathrm{op}}, \mathcal{CAT}]$ certainly extends to a functor

$$\widehat{\mathcal{G}}_{\omega,F}\colon \operatorname{Ps}[\mathcal{C}^{\mathrm{op}}, \mathcal{CAT}](F, \Omega) \to \mathcal{DO}p\mathcal{F}ib_{\operatorname{Ps}[\mathcal{C}^{\mathrm{op}}, \mathcal{CAT}]}(F)$$

by Remark 3.1.13. But, given $z\colon F \to \Omega$, the comma object

$$\begin{tikzcd}
L \arrow[r] \arrow[d, "s"'] & 1 \arrow[d, "\omega"] \arrow[dl, Rightarrow, shorten=4mm, "\text{comma}"] \\
F \arrow[r, "z"'] & \Omega
\end{tikzcd}$$

in $\mathrm{Ps}\left[\mathcal{C}^{\mathrm{op}}, \mathcal{CAT}\right]$ is calculated pointwise. Since 1 and F are both strict 2-functors, the universal property of the comma object induces a strict 2-functor L and a strict 2-natural transformation s. The functor $\widehat{\mathcal{G}}_{\omega,F}$ also sends modifications between F and Ω to strict 2-natural transformations over F. Moreover, every component s_C of s on $C \in \mathcal{C}$ needs to be a discrete opfibration in $\mathcal{CAT}$ with small fibres, by Remark 5.1.9 and the fact that the property of having small fibres is pullback-stable. Since $F(C) \in \mathcal{C}at$, it follows that s_C is a discrete opfibration in $\mathcal{C}at$ with small fibres. And then s is a discrete opfibration in $\left[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at\right]$ with small fibres, by Proposition 4.2.5 (and Definition 4.2.7). $\square$

Remark 5.1.11. The following proposition shows how the bicategorical classification process in prestacks is essentially given by the Yoneda lemma.

Proposition 5.1.12. *For every $C \in \mathcal{C}$, the functor*

$$\widehat{\mathcal{G}}_{\omega,\mathrm{y}(C)}\colon \mathrm{Ps}\left[\mathcal{C}^{\mathrm{op}}, \mathcal{CAT}\right](\mathrm{y}(C), \Omega) \to \mathcal{DOpFib}^{\mathrm{s}}{}_{\left[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at\right]}(\mathrm{y}(C)) = \Omega(C)$$

is isomorphic to the Yoneda lemma's equivalence of categories.

Thus $\widehat{\mathcal{G}}_{\omega,\mathrm{y}(C)}$ is an equivalence of categories.

Proof. Given $z\colon \mathrm{y}(C) \to \Omega$, call $\overline{z}\colon G \to \mathrm{y}(C)$ the corresponding element in $\Omega(C)$ via the Yoneda lemma and $s\colon L \to \mathrm{y}(C)$ the morphism on the left of the comma object

$$\begin{tikzcd}
L \arrow[r] \arrow[d, "s"'] & 1 \arrow[d, "\omega"] \arrow[dl, Rightarrow, shorten=4mm, "\text{comma}"] \\
\mathrm{y}(C) \arrow[r, "z"'] & \Omega
\end{tikzcd}$$

in $\mathrm{Ps}\left[\mathcal{C}^{\mathrm{op}}, \mathcal{CAT}\right]$. We show that there is a 2-natural isomorphism $L \cong G$ over $\mathrm{y}(C)$. Given $D \in \mathcal{C}$, we have that $L(D) \cong G(D)$ over $\mathcal{C}(D, C)$ because of the following bijection between the fibres, which is natural in $f\colon D \to C$

$$(s_D)_f \cong \Omega(D)\left(\mathrm{id}_{\mathrm{y}(D)}, z_D(f)\right) \cong \Omega(D)\left(\mathrm{id}_{\mathrm{y}(D)}, \mathrm{y}(f)^* \,\overline{z}\right) \cong ((\mathrm{y}(f)^* \,\overline{z})_D)_{\mathrm{id}_D} \cong (\overline{z}_D)_f$$

The first isomorphism is given by the explicit construction of comma objects in $\mathcal{CAT}$. It is natural by construction of the structure of discrete opfibration on s induced by the comma object (see Remark 3.1.13). The second natural isomorphism is given by pseudonaturality of z. The third one is given by the Yoneda lemma and trivially natural. The fourth one is given by the explicit construction of pullbacks in $\mathcal{C}at$ and is natural by construction of $\mathcal{G}_{\overline{z},\mathrm{y}(D)}$ on morphisms. It is straightforward to show that the isomorphism $L(D) \cong G(D)$ is 2-natural over $\mathrm{y}(C)$ and to conclude the proof. □

We are ready to prove that, at least over representables, $\widetilde{\omega}\colon 1 \to \widetilde{\Omega}$ satisfies the conditions of a good 2-classifier in prestacks that classifies all discrete opfibrations with small fibres.

Proposition 5.1.13. *For every $C \in \mathcal{C}$, the functor*

$$\widehat{\mathcal{G}}_{\widetilde{\omega},\mathrm{y}(C)}\colon [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]\left(\mathrm{y}(C), \widetilde{\Omega}\right) \to \mathcal{DOpFib}^{\mathrm{s}}(\mathrm{y}(C))$$

is an equivalence of categories.

Proof. We prove that there is an isomorphism

$$\begin{tikzcd}
{[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]\left(\mathrm{y}(C), \widetilde{\Omega}\right)} \arrow[rr, "\widehat{\mathcal{G}}_{\widetilde{\omega},\mathrm{y}(C)}"] \arrow[dr, "j\circ-"'] & \cong & \mathcal{DOpFib}^{\mathrm{s}}_{[\mathcal{C}^{\mathrm{op}},\mathcal{C}at]}(\mathrm{y}(C)) \\
& \mathrm{Ps}\,[\mathcal{C}^{\mathrm{op}}, \mathcal{CAT}]\,(\mathrm{y}(C), \Omega) \arrow[ur, "\widehat{\mathcal{G}}_{\omega,\mathrm{y}(C)}"'] &
\end{tikzcd} \tag{5.1}$$

Given $z\colon \mathrm{y}(C) \to \widetilde{\Omega}$, consider the comma objects

$$\begin{tikzcd}
\widetilde{L} \arrow[r] \arrow[d, "\widetilde{s}"'] & 1 \arrow[d, "\widetilde{\omega}"] \arrow[dl, Leftarrow, "\widetilde{\lambda}"] \\
\mathrm{y}(C) \arrow[r, "z"'] & \widetilde{\Omega}
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[rr] \arrow[dd, "s"'] & & 1 \arrow[d, "\widetilde{\omega}"] \\
& & \widetilde{\Omega} \arrow[d, "j"] \arrow[dll, phantom, "\Leftarrow \lambda"] \\
\mathrm{y}(C) \arrow[r, "z"'] & \widetilde{\Omega} \arrow[r, "j"'] & \Omega
\end{tikzcd}$$

respectively in $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ and in $\mathrm{Ps}\,[\mathcal{C}^{\mathrm{op}}, \mathcal{CAT}]$. Notice that the left hand side comma is also a comma object in $\mathrm{Ps}\,[\mathcal{C}^{\mathrm{op}}, \mathcal{CAT}]$ because commas are calculated pointwise in both 2-categories. We have that $\widetilde{L}(D) \cong L(D)$ over $\mathcal{C}(D, C)$ for every $D \in \mathcal{C}$, since j_D is an equivalence of categories. Indeed $(j * \widetilde{\lambda})_D$ exhibits the comma object on the right hand side in component D. It is straightforward to show that the isomorphism $\widetilde{L}(D) \cong L(D)$ is 2-natural in $D \in \mathcal{C}$ over $\mathrm{y}(C)$ and to conclude the isomorphism of equation (5.1). Notice that $j \circ \widetilde{\omega}$ is isomorphic to ω, whence $\widehat{\mathcal{G}}_{\omega,\mathrm{y}(C)}$ is isomorphic to $\widehat{\mathcal{G}}_{j\circ\widetilde{\omega},\mathrm{y}(C)}$.

By Proposition 5.1.12, the functor $\widehat{\mathcal{G}}_{\omega,\mathrm{y}(C)}$ is an equivalence of categories. By the Yoneda lemma, also $j \circ -$ is an equivalence of categories. Indeed the following square is commutative:

$$\begin{tikzcd}
{[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]\left(\mathrm{y}(C), \widetilde{\Omega}\right)} \arrow[r, "j\circ-"] \arrow[d, phantom, "\wr\|"] & {\mathrm{Ps}\,[\mathcal{C}^{\mathrm{op}}, \mathcal{CAT}]\,(\mathrm{y}(C), \Omega)} \arrow[d, phantom, "\wr"] \\
\widetilde{\Omega}(C) \arrow[r, "j_C"'] & \Omega(C)
\end{tikzcd}$$

And thus $j \circ -$ is an equivalence of categories by the two out of three property. Therefore also $\widehat{\mathcal{G}}_{\widetilde{\omega},\mathrm{y}(C)}$ is an equivalence of categories. ☐

We now apply the theorems of reduction of 2-classifiers to dense generators to prove that $\widetilde{\omega}\colon 1 \to \widetilde{\Omega}$ is a good 2-classifier in prestacks that classifies all discrete opfibrations with small fibres. The partial result that (the lax limit of the arrow) $\widetilde{\omega}$ gives a 2-classifier can be obtained from Weber's [51, Example 4.7]. However, Weber's paper does not address the problem of which discrete opfibrations get classified.

Theorem 5.1.14. *The 2-natural transformation $\widetilde{\omega}$ from 1 to*

$$\begin{array}{rrcl}
\widetilde{\Omega}: & \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathcal{C}at \\
& C & \mapsto & \left[\left(\mathcal{C}/_{C}\right)^{\mathrm{op}}, \mathcal{S}et\right] \\
& (C \xleftarrow{f} D) & \mapsto & - \circ (f \circ =)^{\mathrm{op}}
\end{array}$$

that picks the constant at 1 presheaf on every component is a good 2-classifier in $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ that classifies all discrete opfibrations with small fibres.

Proof. Consider the fully faithful dense generator $\mathrm{y}\colon \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ formed by representables. By Proposition 5.1.6, the lax limit of the arrow $\widetilde{\omega}$ has small fibres.

By Proposition 5.1.13, we have that for every $C \in \mathcal{C}$ the functor

$$\widehat{\mathcal{G}}_{\widetilde{\omega},\mathrm{y}(C)} \colon [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]\left(\mathrm{y}(C), \widetilde{\Omega}\right) \to \mathcal{DO}p\mathcal{F}ib^{\mathrm{s}}(\mathrm{y}(C))$$

is an equivalence of categories. In order to prove that $\widetilde{\omega} \colon 1 \to \widetilde{\Omega}$ is a good 2-classifier in $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ with respect to the property of having small fibres, by Corollary 3.2.11, it only remains to prove that the operation of normalization described in Theorem 3.2.8 is possible.

So let $\varphi \colon G \to F$ be a discrete opfibration in $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ with small fibres. Using the dense generator $\mathrm{y} \colon \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$, we express F as a cartesian-marked oplax colimit of representables. By Example 1.1.25,

$$F \cong \operatorname{colim}^{F} \mathrm{y} \cong \operatorname{oplax}^{\mathrm{cart}}\text{-}\operatorname{colim}^{\Delta 1}(\mathrm{y} \circ \mathcal{G}(F)),$$

whence $K = \mathrm{y} \circ \mathcal{G}(F)$, with the universal cartesian-marked oplax cocone Λ given by

$$\begin{tikzcd}[row sep=small] & (D, X') & & & \mathrm{y}(C) \arrow[r, "\ulcorner X \urcorner"] \arrow[d, "\mathrm{y}(f)"'] \arrow[dr, phantom, "\Downarrow \ulcorner \nu \urcorner"] & F \\ \forall & & \text{in } \int F & & & \\ & (C, X) \arrow[uu, "{(f,\nu)}"'] & & & \mathrm{y}(D) \arrow[ur, bend right, "\ulcorner X' \urcorner"'] & \end{tikzcd}$$

Looking at the proof of Corollary 3.2.11 (and Construction 3.2.7), we consider the sigma natural transformation χ given by the composite

$$\Delta 1 \xRightarrow[\operatorname{oplax}^{\mathrm{cart}}]{\Lambda} [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at](K(-), F) \xRightarrow[\text{pseudo}]{\mathcal{G}_{\varphi, K(-)}} \mathcal{DO}p\mathcal{F}ib^{\mathrm{s}}(K(-)) \xRightarrow[\text{pseudo}]{\widehat{\mathcal{G}}^{-1}_{\widetilde{\omega}, K(-)}} [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]\left(K(-), \widetilde{\Omega}\right).$$

We can visualize it as follows:

$$\begin{tikzcd} H^{(C,X)} \arrow[r] \arrow[d, "\mathcal{G}_{\varphi}\left(\Lambda_{(C,X)}\right)"'] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d, "\varphi"] & 1 \arrow[d, "\widetilde{\omega}"] \\ \mathrm{y}(C) \arrow[r, "\Lambda_{(C,X)}"] \arrow[rr, bend right, "\widehat{\mathcal{G}}^{-1}_{\widetilde{\omega}}\left(\mathcal{G}_{\varphi}\left(\Lambda_{(C,X)}\right)\right)"'] & F \arrow[r, dashed, "z"] & \widetilde{\Omega} \end{tikzcd}$$

For this, we need to choose an adjoint quasi-inverse of $\widehat{\mathcal{G}}_{\widetilde{\omega},K(C,X)} = \widehat{\mathcal{G}}_{\widetilde{\omega},\mathrm{y}(C)}$ for every $(C, X) \in \int F$. By Proposition 5.1.13, we can construct such a quasi-inverse by taking quasi-inverses of $j \circ -$ and $\widehat{\mathcal{G}}_{\omega,\mathrm{y}(C)}$. Both the latter are given by the Yoneda lemma, respectively by the proof of Proposition 5.1.13 and by Proposition 5.1.12. So given $\psi \colon H \to \mathrm{y}(C)$, we can take $\widehat{\mathcal{G}}^{-1}_{\widetilde{\omega},\mathrm{y}(C)}(\psi)$ to be the morphism

$\mathrm{y}(C) \to \widetilde{\Omega}$ which corresponds to $j_C^{-1}(\psi)$ (see Proposition 5.1.3). With this choice, $\chi_{(C,X)}$ would be the morphism $\mathrm{y}(C) \to \widetilde{\Omega}$ which corresponds to

$$\begin{array}{rrcl} j_C^{-1}\left(\mathcal{G}_\varphi\left(\Lambda_{(C,X)}\right)\right): & \left(\mathcal{C}/_C\right)^{\mathrm{op}} & \longrightarrow & \mathcal{S}et \\ & (D \xrightarrow{f} C) & \mapsto & \left(\mathcal{G}_\varphi\left(\Lambda_{(C,X)}\right)_D\right)_f \\ & (f \xleftarrow{g} f \circ g) & \mapsto & H^{(C,X)}(g) \end{array}$$

Using the explicit construction of pullbacks in $\mathcal{C}at$ to calculate $\left(\mathcal{G}_\varphi\left(\Lambda_{(C,X)}\right)_D\right)_f$, we do not obtain a cartesian-marked oplax natural transformation χ. This is due to the unnecessary keeping track of the morphisms $f\colon D \to C$ other than the objects of $G(D)$.

Instead, we choose $\widehat{\mathcal{G}}^{-1}_{\widetilde{\omega},\mathrm{y}(C)}$ on the objects $\mathcal{G}_\varphi\left(\Lambda_{(C,X)}\right)$ so that $\chi_{(C,X)}$ is the morphism $\mathrm{y}(C) \to \widetilde{\Omega}$ which corresponds to

$$\begin{array}{rrcl} \ulcorner\chi_{(C,X)}\urcorner: & \left(\mathcal{C}/_C\right)^{\mathrm{op}} & \longrightarrow & \mathcal{S}et \\ & (D \xrightarrow{f} C) & \mapsto & (\varphi_D)_{F(f)(X)} \\ & (f \xleftarrow{g} f \circ g) & \mapsto & G(g) \end{array}$$

This is the operation of normalization that we need. Notice that

$$(\varphi_D)_{F(f)(X)} = (\varphi_D)_{\Lambda_{(C,X)}(f)} \cong \left(\mathcal{G}_\varphi(\Lambda_{(C,X)})_D\right)_f$$

and that such isomorphism is natural in $f \in \left(\mathcal{C}/_C\right)^{\mathrm{op}}$. So that, thanks to the argument above, $\widehat{\mathcal{G}}_{\widetilde{\omega},\mathrm{y}(C)}(\chi_{(C,X)})\colon Q^{(C,X)} \to \mathrm{y}(C)$ is indeed isomorphic to $\mathcal{G}_\varphi\left(\Lambda_{(C,X)}\right) : H^{(C,X)} \to \mathrm{y}(C)$. We then extend $\widehat{\mathcal{G}}^{-1}_{\widetilde{\omega},\mathrm{y}(C)}$ to a right adjoint quasi-inverse of $\widehat{\mathcal{G}}_{\widetilde{\omega},\mathrm{y}(C)}$, choosing the components of the counit on the objects $\mathcal{G}_\varphi\left(\Lambda_{(C,X)}\right)$ to be the just obtained isomorphisms.

We prove that χ is cartesian-marked oplax natural. Given a morphism $(f,\mathrm{id})\colon (D,X') \leftarrow (C,F(f)(X'))$ in $\int F$, it is straightforward to show that

$$\chi_{(D,X')} \circ \mathrm{y}(f) = \chi_{(C,F(f)(X'))}$$

using that F is a strict 2-functor. We still need to show that $\chi_{f,\mathrm{id}} = \mathrm{id}$. For this, it is straightforward to prove that

$$\widehat{\mathcal{G}}_{\widetilde{\omega},\mathrm{y}(C)}(\chi_{f,\mathrm{id}}) = \mathrm{id},$$

following the recipe given in the proof of Theorem 3.2.8. So we conclude using the fully faithfulness of $\widehat{\mathcal{G}}_{\widetilde{\omega},\mathrm{y}(C)}$. □

Remark 5.1.15. Looking at the proof of Theorem 5.1.14, we see that the idea of the operation of normalization in prestacks is the following. Rather than considering the "local fibres" of the $\mathcal{G}_{\varphi}(\Lambda_{(C,X)})$'s, that are not compatible with each other, we express all of them in terms of the "global fibres" of φ.

Remark 5.1.16. The proof of Theorem 5.1.14 also gives us a recipe for the characteristic morphism $z\colon F \to \widetilde{\Omega}$ of a discrete opfibration $\varphi\colon G \to F$ in $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ with small fibres.

$$\begin{tikzcd}
G \arrow[r] \arrow[d, "\varphi"'] & 1 \arrow[d, "\widetilde{\omega}"] \arrow[dl, phantom, "\Swarrow \text{comma}"] \\
F \arrow[r, "z"'] & \widetilde{\Omega}
\end{tikzcd}$$

We obtain that z is the 2-natural transformation whose component on $C \in \mathcal{C}$ is the functor z_C that sends $X \in F(C)$ to

$$\begin{array}{rccc}
z_C(X): & \left(\mathcal{C}/_{C}\right)^{\mathrm{op}} & \longrightarrow & \mathcal{S}et \\
& (D \xrightarrow{f} C) & \mapsto & (\varphi_D)_{F(f)(X)} \\
& (f \xleftarrow{g} f \circ g) & \mapsto & G(g)
\end{array}$$

Given $\nu\colon X \to X'$ in $F(C)$, we have that $z_C(\nu)$ is the natural transformation whose component on $f\colon D \to C$ is the function

$$F(f)(\nu)_*\colon (\varphi_D)_{F(f)(X)} \to (\varphi_D)_{F(f)(X')}$$

that calculates the codomain of the liftings along φ_D of $F(f)(\nu)$.

It is interesting to compare our result with what happens in dimension 1. The characteristic morphism for a subobject $G \hookrightarrow F$ in 1-dimensional presheaves has component on C that sends $X \in F(C)$ to

$$\bigcup_{D \in \mathcal{C}} \left\{ D \xrightarrow{f} C \mid F(f)(X) \in G(D) \right\}.$$

While in dimension 1 the fibre on $F(f)(X)$ can only be either empty or a singleton, in dimension 2 we need to handle the general sets formed by such fibres.

5.2. Restricting good 2-classifiers to nice sub-2-categories

In this section, we present two general results (Proposition 5.2.6 and Theorem 5.2.9) that describe a strategy to restrict a good 2-classifier in $\mathcal{L}$ to a good 2-classifier in a nice sub-2-category $\mathcal{M}$ of $\mathcal{L}$. Such a strategy will involve factorization arguments and our theorems of reduction of the study of 2-classifiers to dense generators. The key idea is to restrict $\Omega \in \mathcal{L}$ to some $\Omega_{\mathcal{M}} \in \mathcal{M}$ so that the characteristic morphisms in $\mathcal{L}$ of discrete opfibrations in $\mathcal{M}$ factor through $\Omega_{\mathcal{M}}$. This is the strategy that we will follow in Section 5.3 to restrict our good 2-classifier in prestacks to one in stacks.

An advantage of producing a good 2-classifier following the results of this section is that the normalization process described in Theorem 3.2.8 is not required. Theorem 5.2.9 shows indeed that the normalization process is automatic for nice sub-2-categories of 2-categories with a good 2-classifier. Since the normalization process is possible in prestacks (Theorem 5.1.14 and Remark 5.1.15), then by Theorem 5.2.9 it is also possible in any nice sub-2-category of prestacks, such as in particular the 2-category of stacks, see Theorem 5.3.22.

Notation 5.2.1. Throughout this section, we fix an arbitrary 2-category $\mathcal{L}$ with pullbacks along discrete opfibrations, comma objects and terminal object. We then fix a choice of such pullbacks in $\mathcal{L}$ such that the change of base of an identity is always an identity.

We also fix P an arbitrary pullback-stable property P for discrete opfibrations in $\mathcal{L}$. We assume of course that P only depends on the isomorphism classes of discrete opfibrations.

Definition 5.2.2. A fully faithful 2-functor $i\colon \mathcal{M} \underset{\text{ff}}{\hookrightarrow} \mathcal{L}$ will be called *nice* if it lifts pullbacks along discrete opfibrations, comma objects and the terminal object (that is, such limits exist in $\mathcal{M}$ and are calculated in $\mathcal{L}$) and preserves discrete opfibrations.

A sub-2-category $i\colon \mathcal{M} \subseteq \mathcal{L}$ (that is, an injective on objects and fully faithful 2-functor $i\colon \mathcal{M} \underset{\text{ff}}{\hookrightarrow} \mathcal{L}$) will be called *nice* if i is nice.

Example 5.2.3. Any reflective sub-2-category is nice. Indeed notice that any right 2-adjoint preserves discrete opfibrations thanks to the natural isomorphism between hom-categories given by the adjunction.

Remark 5.2.4. Given a nice fully faithful 2-functor $i\colon \mathcal{M} \underset{\text{ff}}{\hookrightarrow} \mathcal{L}$, we will say that a discrete opfibration φ in $\mathcal{M}$ satisfies P if $i(\varphi)$ does so.

We will need the notions of fully faithful morphism and of chronic morphism in $\mathcal{L}$.

Definition 5.2.5. A morphism $l\colon F \to B$ in $\mathcal{L}$ is *fully faithful* if for every $X \in \mathcal{L}$ the functor $l \circ -\colon \mathcal{L}(X, F) \to \mathcal{L}(X, B)$ is fully faithful. l is *chronic* if every $l \circ -$ is injective on objects and fully faithful.

We are ready to present the first result of restriction. Rather than restricting a good 2-classifier, we start from a morphism $\omega\colon 1 \to \Omega$ in $\mathcal{L}$ such that its lax limit τ is a 2-classifier in $\mathcal{L}$. By Remark 3.1.13, this condition is weaker than being a good 2-classifier. Indeed it means that for every $F \in \mathcal{L}$

$$\widehat{\mathcal{G}}_{\omega,F}\colon \mathcal{L}(F, \Omega) \to \mathcal{DOpFib}_{\mathcal{L}}(F)$$

is fully faithful. Of course, the result can then be applied to a starting good 2-classifier in $\mathcal{L}$ as well.

Proposition 5.2.6. *Let $i\colon \mathcal{M} \underset{\text{ff}}{\hookrightarrow} \mathcal{L}$ be a nice fully faithful 2-functor (Definition 5.2.2). Let then $\omega\colon 1 \to \Omega$ in $\mathcal{L}$ such that its lax limit τ is a 2-classifier in $\mathcal{L}$. Finally, let $\Omega_{\mathcal{M}} \in \mathcal{M}$ such that there exists a fully faithful morphism $\ell\colon i(\Omega_{\mathcal{M}}) \underset{\text{ff}}{\hookrightarrow} \Omega$ in $\mathcal{M}$ and ω factors through ℓ; call $\omega_{\mathcal{M}}\colon 1 \to \Omega_{\mathcal{M}}$ the resulting morphism. Then the lax limit $\tau_{\mathcal{M}}$ of the arrow $\omega_{\mathcal{M}}$ is a 2-classifier in $\mathcal{M}$.*

In addition to this, if τ satisfies P *then also $\tau_{\mathcal{M}}$ satisfies* P.

Moreover, given φ a discrete opfibration in $\mathcal{M}$, if $i(\varphi)$ is classified by τ via a characteristic morphism z that factors through ℓ then φ is classified by $\tau_{\mathcal{M}}$.

$$\begin{tikzcd}
1 \arrow[d, "i(\omega_{\mathcal{M}})"'] \arrow[dr, "\omega"] & \\
i(\Omega_{\mathcal{M}}) \arrow[r, hook, "\ell"'] & \Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega_{\mathcal{M},\bullet} \arrow[r] \arrow[d, "\tau_{\mathcal{M}}"'] & 1 \arrow[d, "\omega_{\mathcal{M}}"] \arrow[dl, Rightarrow, shorten=4mm, "\text{comma}"] \\
\Omega_{\mathcal{M}} \arrow[r, equal] & \Omega_{\mathcal{M}}
\end{tikzcd}
\qquad
\begin{tikzcd}
i(G) \arrow[rr] \arrow[d, "i(\varphi)"'] \arrow[drr, phantom, "\lrcorner", very near start] & & \Omega_{\bullet} \arrow[d, "\tau"] \\
i(F) \arrow[rr, "z"] \arrow[dr, dashed, "\exists i(z_{\mathcal{M}})"'] & & \Omega \\
& i(\Omega_{\mathcal{M}}) \arrow[ur, hook, "\ell"'] &
\end{tikzcd}$$

Proof. We prove that, for every $F' \in \mathcal{L}$, there is an isomorphism

$$\begin{tikzcd}
\mathcal{L}\,(F', i(\Omega_{\mathcal{M}})) \arrow[rr, "\widehat{\mathcal{G}}_{i(\omega_{\mathcal{M}}),F'}"] \arrow[dr, "\ell\circ-"'] & \cong & \mathcal{DOpFib}_{\mathcal{L}}\,(F') \\
& \mathcal{L}\,(F', \Omega) \arrow[ur, "\widehat{\mathcal{G}}_{\omega,F'}"'] &
\end{tikzcd} \tag{5.2}$$

Given $z\colon F' \to i(\Omega_{\mathcal{M}})$, consider the comma objects

$$\begin{tikzcd}
L_{\mathcal{M}} \arrow[r] \arrow[d, "s_{\mathcal{M}}"'] & 1 \arrow[d, "i(\omega_{\mathcal{M}})"] \arrow[dl, Rightarrow, shorten=4mm, "\lambda_{\mathcal{M}}"] \\
F' \arrow[r, "z"'] & i(\Omega_{\mathcal{M}})
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[rr] \arrow[ddd, "s"'] & & 1 \arrow[d, "i(\omega_{\mathcal{M}})"] \\
& & i(\Omega_{\mathcal{M}}) \arrow[d, "\ell"] \\
& & \\
F' \arrow[r, "z"'] \arrow[from=1-3, to=4-1, Rightarrow, shorten=8mm, "\lambda"] & i(\Omega_{\mathcal{M}}) \arrow[r, "\ell"'] & \Omega
\end{tikzcd}$$

in $\mathcal{L}$. It is straightforward to see that, since ℓ is a fully faithful morphism, $\ell * \lambda_{\mathcal{M}}$ exhibits the comma object on the right hand side. Then $L_{\mathcal{M}} \cong L$ over F', and such isomorphism is natural in z by the universal property of the comma object. $\ell \circ -$ is fully faithful by definition of fully faithful morphism in $\mathcal{L}$. Hence $\widehat{\mathcal{G}}_{i(\omega_{\mathcal{M}}),F'}$ is fully faithful.

Moreover, we obtain that if τ satisfies P then the lax limit ξ of the arrow $i(\omega_{\mathcal{M}})$ satisfies P. Since i lifts comma objects and the terminal objects, the lax limit $\tau_{\mathcal{M}}$ of the arrow $\omega_{\mathcal{M}}$ can be calculated in $\mathcal{L}$. So $i(\tau_{\mathcal{M}}) = \xi$ satisfies P as well.

Let now $F \in \mathcal{M}$. Since i preserves discrete opfibrations, i induces a fully faithful functor

$$i\colon \mathcal{DOpFib}_{\mathcal{M}}\,(F) \to \mathcal{DOpFib}_{\mathcal{L}}\,(i(F))\,.$$

Notice then that the following square is commutative:

$$\begin{tikzcd}
\mathcal{M}\left(F,\Omega_{\mathcal{M}}\right) \arrow[r, "\widehat{\mathcal{G}}_{\omega_{\mathcal{M}},F}"] \arrow[d, equal, "i"', "\wr"] & \mathcal{DOpFib}_{\mathcal{M}}\left(F\right) \arrow[d, "i"] \\
\mathcal{L}\left(i(F),i(\Omega_{\mathcal{M}})\right) \arrow[r, "\widehat{\mathcal{G}}_{i(\omega_{\mathcal{M}}),i(F)}"'] & \mathcal{DOpFib}_{\mathcal{L}}\left(i(F)\right)
\end{tikzcd} \tag{5.3}$$

Indeed by assumption i lifts comma objects and the terminal object. Whence $\widehat{\mathcal{G}}_{\omega_{\mathcal{M}},F}$ is fully faithful and $\tau_{\mathcal{M}}$ is a 2-classifier.

Consider now a discrete opfibration φ in $\mathcal{M}$. Following the diagrams of equations (5.2) and (5.3), we obtain that if $i(\varphi)$ is classified by τ via a characteristic morphism z that factors through ℓ then φ is classified by $\tau_{\mathcal{M}}$. This can also be seen from the diagram on the right in the statement, using the pullbacks lemma, after Remark 5.2.7. □

Remark 5.2.7. $\tau_{\mathcal{M}}$ can be equivalently produced, in $\mathcal{L}$, as the pullback of τ along the fully faithful $\ell\colon i(\Omega_{\mathcal{M}}) \underset{\text{ff}}{\hookrightarrow} \widetilde{\Omega}$. Indeed the lax limit $\tau_{\mathcal{M}}$ of the arrow $\omega_{\mathcal{M}}$ corresponds with the lax limit of the arrow $i(\omega_{\mathcal{M}})$ in $\mathcal{L}$. By whiskering with ℓ, we then see that the latter is equivalently given by the comma object from ω to ℓ, which is also the pullback along ℓ of the lax limit of the arrow ω. However, by producing $\tau_{\mathcal{M}}$ as the lax limit of the arrow $\omega_{\mathcal{M}}$ in $\mathcal{M}$, it is guaranteed that $\tau_{\mathcal{M}}$ is a morphism in $\mathcal{M}$.

We would like to show that we can check the factorizations of the characteristic morphisms in $\mathcal{L}$ of discrete opfibrations in $\mathcal{M}$ just on a dense generator. The following construction helps with this.

Construction 5.2.8. Let $i\colon \mathcal{M} \underset{\text{ff}}{\hookrightarrow} \mathcal{L}$ be a fully faithful 2-functor. Consider then $I\colon \mathcal{Y} \to \mathcal{M}$ a fully faithful 2-functor such that $i \circ I\colon \mathcal{Y} \to \mathcal{L}$ is a dense generator of $\mathcal{L}$. By Kelly's [28, Theorem 5.13], then I is a fully faithful dense generator of $\mathcal{M}$.

Moreover, let $F \in \mathcal{M}$. We want to exhibit F as a nice colimit of the objects that form the dense generator I. By Construction 3.2.4, there exist a 2-diagram

$J\colon \mathcal{A} \to \mathcal{L}$ which factors through $i \circ I$ and a weight $W\colon \mathcal{A}^{\mathrm{op}} \to \mathcal{Cat}$ such that, calling $K \coloneqq J \circ \mathcal{G}(W)$,

$$i(F) = \operatorname{oplax}^{\mathrm{cart}}\text{-}\operatorname{colim}^{\Delta 1} K$$

and this colimit is $(i \circ I)$-absolute. Call Λ the universal cartesian-marked oplax cocone that presents such colimit. Notice that, as K factors through $i \circ I$, it also factors through i. Call $K_{\mathcal{M}}\colon \int W \to \mathcal{M}$ the resulting diagram; so that $i \circ K_{\mathcal{M}} = K$. It is clear that $K_{\mathcal{M}}$ factors through I. Take $\Lambda_{\mathcal{M}}$ to be the unique cartesian-marked oplax cocone such that $i \circ \Lambda_{\mathcal{M}} = \Lambda$. Then, since a fully faithful 2-functor reflects colimits (see also Proposition 1.1.24),

$$F = \operatorname{oplax}^{\mathrm{cart}}\text{-}\operatorname{colim}^{\Delta 1} K_{\mathcal{M}},$$

exhibited by $\Lambda_{\mathcal{M}}$. Moreover, this colimit is I-absolute, as $\widetilde{I} \cong \widetilde{(i \circ I)} \circ i$.

Building over Proposition 5.2.6, we now present a general result of restriction of good 2-classifiers in $\mathcal{L}$ to nice sub-2-categories $\mathcal{M}$ of $\mathcal{L}$. We show that the factorization of the characteristic morphisms in $\mathcal{L}$ of discrete opfibrations in $\mathcal{M}$ can be checked just on a dense generator of the kind described in Construction 5.2.8. Then our theorems of reduction of the study of a 2-classifier to dense generators (Corollary 3.2.11) guarantee that we find a good 2-classifier in $\mathcal{M}$. For this, we need to ensure that the operation of normalization described in Theorem 3.2.8 (starting from every φ) is possible. We show that we can just do the normalization process in $\mathcal{L}$, where it is certainly possible since we have a good 2-classifier; see Corollary 3.2.10.

Theorem 5.2.9. *Let $i\colon \mathcal{M} \subseteq \mathcal{L}$ be a nice sub-2-category (Definition 5.2.2). Let $\omega\colon 1 \to \Omega$ in $\mathcal{L}$ be a good 2-classifier in $\mathcal{L}$ with respect to* P*. Let then $\Omega_{\mathcal{M}} \in \mathcal{M}$ such that there exists a chronic arrow (Definition 5.2.5) $\ell\colon i(\Omega_{\mathcal{M}}) \underset{\mathrm{ff}}{\hookrightarrow} \Omega$ in $\mathcal{M}$ and ω factors through ℓ; call $\omega_{\mathcal{M}}\colon 1 \to \Omega_{\mathcal{M}}$ the resulting morphism. Finally, let $I\colon \mathcal{Y} \to \mathcal{M}$ be a fully faithful 2-functor such that $i \circ I$ is a dense generator of $\mathcal{L}$. Assume that for every $\psi\colon H \to I(Y)$ a discrete opfibration in $\mathcal{M}$ that satisfies* P *over $I(Y)$ with $Y \in \mathcal{Y}$, every characteristic morphism of $i(\psi)$ with respect to ω factors through ℓ. Then $\omega_{\mathcal{M}}$ is a good 2-classifier in $\mathcal{M}$ with respect to* P.

Proof. By Construction 5.2.8, $I\colon \mathcal{Y} \to \mathcal{M}$ is a fully faithful dense generator of $\mathcal{M}$. By Proposition 5.2.6, the lax limit of the arrow $\omega_{\mathcal{M}}$ satisfies P and for every $Y \in \mathcal{Y}$

$$\widehat{\mathcal{G}}_{\omega_{\mathcal{M}},I(Y)}\colon \mathcal{M}\left(I(Y),\Omega_{\mathcal{M}}\right) \to \mathcal{DOpFib}^{\mathrm{P}}(I(Y))$$

is an equivalence of categories. Indeed $\widehat{\mathcal{G}}_{\omega_{\mathcal{M}},I(Y)}$ is fully faithful and, given $\psi\colon H \to I(Y)$ a discrete opfibration in $\mathcal{M}$ that satisfies P, any characteristic morphism of $i(\psi)$ in $\mathcal{L}$ factors through $\ell\colon i(\Omega_{\mathcal{M}}) \underset{\mathrm{ff}}{\hookrightarrow} \Omega$. In order to prove that $\omega_{\mathcal{M}}\colon 1 \to \Omega_{\mathcal{M}}$ is a good 2-classifier in $\mathcal{M}$, by Corollary 3.2.11, it only remains to prove that the operation of normalization described in the proof of Theorem 3.2.8 is possible.

So let $\varphi\colon G \to F$ be a discrete opfibration in $\mathcal{M}$ that satisfies P. By Construction 5.2.8, we express

$$F = \mathrm{oplax}^{\mathrm{cart}}\text{-}\mathrm{colim}^{\Delta 1} K_{\mathcal{M}},$$

exhibited by $\Lambda_{\mathcal{M}}$. Looking at the proof of Corollary 3.2.11 (and Construction 3.2.7), we consider the sigma natural transformation $\chi_{\mathcal{M}}$ given by the composite

$$\Delta 1 \underset{\mathrm{oplax}^{\mathrm{cart}}}{\overset{\Lambda_{\mathcal{M}}}{\Longrightarrow}} \mathcal{M}\left(K_{\mathcal{M}}(-),F\right) \underset{\mathrm{pseudo}}{\overset{\mathcal{G}_{\varphi,K_{\mathcal{M}}(-)}}{\Longrightarrow}} \mathcal{DOpFib}^{\mathrm{P}}{}_{\mathcal{M}}\left(K_{\mathcal{M}}(-)\right) \underset{\mathrm{pseudo}}{\overset{\widehat{\mathcal{G}}^{-1}_{\omega_{\mathcal{M}},K_{\mathcal{M}}(-)}}{\Longrightarrow}} \mathcal{M}\left(K_{\mathcal{M}}(-),\Omega_{\mathcal{M}}\right).$$

We can visualize it as follows:

$$\begin{tikzcd}
H^{(C,X)}_{\mathcal{M}} \arrow[r] \arrow[d, swap, "\mathcal{G}_{\varphi}\left(\Lambda_{\mathcal{M},(C,X)}\right)"] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d, "\varphi"] & 1 \arrow[d, "\omega_{\mathcal{M}}"] \\
K_{\mathcal{M}}(C,X) \arrow[r, "\Lambda_{\mathcal{M},(C,X)}"] \arrow[rr, bend right, swap, "\widehat{\mathcal{G}}^{-1}_{\omega_{\mathcal{M}}}\left(\mathcal{G}_{\varphi}\left(\Lambda_{\mathcal{M},(C,X)}\right)\right)"] & F \arrow[r, dashed, "z_{\mathcal{M}}"] & \Omega_{\mathcal{M}}
\end{tikzcd}$$

For this, we need to choose an adjoint quasi-inverse of $\widehat{\mathcal{G}}_{\omega_{\mathcal{M}},K_{\mathcal{M}}(C,X)}$ for every $(C,X) \in \int W$. Let then $\psi\colon H \to K_{\mathcal{M}}(C,X)$ be a discrete opfibration in $\mathcal{M}$ that satisfies P. By assumption, the "normalized" characteristic morphism $\widehat{\mathcal{G}}^{-1}_{\omega,i(K_{\mathcal{M}}(C,X))}(i(\psi)) = t$ defined as in the proof of Corollary 3.2.10 starting from $i(\varphi)$ and K and Λ, factors through $\ell\colon i(\Omega_{\mathcal{M}}) \underset{\mathrm{ff}}{\hookrightarrow} \Omega$. We define $\widehat{\mathcal{G}}^{-1}_{\omega_{\mathcal{M}},K_{\mathcal{M}}(C,X)}(\psi) = t_{\mathcal{M}}$ to be the morphism in $\mathcal{M}$ corresponding to the resulting morphism $i(K_{\mathcal{M}}(C,X)) \to i(\Omega_{\mathcal{M}})$ in $\mathcal{L}$ given by the factorization. So that

$\ell \circ i(t_{\mathcal{M}}) = t$. We then extend $\widehat{\mathcal{G}}^{-1}_{\omega_{\mathcal{M}}, K_{\mathcal{M}}(C,X)}$ to a right adjoint quasi-inverse of $\widehat{\mathcal{G}}_{\omega_{\mathcal{M}}, K_{\mathcal{M}}(C,X)}$, choosing the components of the counit on the objects ψ to be the isomorphism corresponding to the one in $\mathcal{L}$ from the comma of $i(\omega_{\mathcal{M}})$ and $i(t_{\mathcal{M}})$ to the comma of $\omega = \ell \circ i(\omega_{\mathcal{M}})$ and $t = \ell \circ i(t_{\mathcal{M}})$ composed with the counit of $\widehat{\mathcal{G}}_{\omega, K(C,X)} \dashv \widehat{\mathcal{G}}^{-1}_{\omega, K(C,X)}$.

We prove that $\chi_{\mathcal{M}}$ is cartesian-marked oplax natural. It suffices to prove that

$$(\ell \circ -) \circ i \circ \chi_{\mathcal{M}} = \chi$$

where χ is the cartesian-marked (i.e. "normal") oplax natural transformation produced as in the proof of Corollary 3.2.10, starting from $i(\varphi)$ and K and Λ. Indeed ℓ is a chronic arrow and i is injective on objects and fully faithful. So we show that the following diagram of oplax natural transformations is commutative:

$$\begin{tikzcd}
\Delta 1 \arrow[r, Rightarrow, "\Lambda_{\mathcal{M}}"] \arrow[d, equal] & \mathcal{M}\left(K_{\mathcal{M}}(-), F\right) \arrow[r, Rightarrow, "\mathcal{G}_{\varphi, K_{\mathcal{M}}(-)}"] \arrow[d, Rightarrow, "i"] & \mathcal{DOpFib}^{\mathrm{P}}_{\mathcal{M}}\left(K_{\mathcal{M}}(-)\right) \arrow[r, Rightarrow, "\widehat{\mathcal{G}}^{-1}_{\omega_{\mathcal{M}}, K_{\mathcal{M}}(-)}"] \arrow[d, Rightarrow, "i"] & \mathcal{M}\left(K_{\mathcal{M}}(-), \Omega_{\mathcal{M}}\right) \arrow[d, Rightarrow, "(\ell \circ -)\circ i"] \\
\Delta 1 \arrow[r, Rightarrow, "\Lambda"'] & \mathcal{L}\left(K(-), i(F)\right) \arrow[r, Rightarrow, "\mathcal{G}_{i(\varphi), K(-)}"'] & \mathcal{DOpFib}^{\mathrm{P}}_{\mathcal{L}}\left(K(-)\right) \arrow[r, Rightarrow, "\widehat{\mathcal{G}}^{-1}_{\omega, K(-)}"'] & \mathcal{L}\left(K(-), \Omega\right).
\end{tikzcd}$$

The square on the left is commutative by construction of $\Lambda_{\mathcal{M}}$. The square in the middle is commutative because pullbacks along opfibrations in $\mathcal{M}$ are calculated in $\mathcal{L}$, by assumption. We also use that the structure of a discrete opfibration in $\mathcal{M}$ is just given by the structure of the underlying discrete opfibration in $\mathcal{L}$. We prove that the square on the right is commutative as well. Let $(C, X) \in \int W$. Given $\psi \colon H \to K_{\mathcal{M}}(C, X)$ a discrete opfibration in $\mathcal{M}$ that satisfies P,

$$\ell \circ i\left(\widehat{\mathcal{G}}^{-1}_{\omega_{\mathcal{M}}, \mathrm{y}(C)}(\psi)\right) = \widehat{\mathcal{G}}^{-1}_{\omega, i(K_{\mathcal{M}}(C,X))}(i(\psi))$$

by construction of $\widehat{\mathcal{G}}^{-1}_{\omega_{\mathcal{M}}, K_{\mathcal{M}}(C,X)}$. Given $\theta \colon \psi \to \psi'$ in $\mathcal{DOpFib}^{\mathrm{P}}_{\mathcal{M}}\left(K_{\mathcal{M}}(C, X)\right)$,

$$\ell * i\left(\widehat{\mathcal{G}}^{-1}_{\omega_{\mathcal{M}}, K_{\mathcal{M}}(C,X)}(\theta)\right) = \widehat{\mathcal{G}}^{-1}_{\omega, i(K_{\mathcal{M}}(C,X))}(i(\theta))$$

because they are equal after applying the fully faithful $\widehat{\mathcal{G}}_{\omega, i(K_{\mathcal{M}}(C,X))}$, by construction of the counit of $\widehat{\mathcal{G}}_{\omega_{\mathcal{M}}, K_{\mathcal{M}}(C,X)} \dashv \widehat{\mathcal{G}}^{-1}_{\omega_{\mathcal{M}}, K_{\mathcal{M}}(C,X)}$ (together with the proof of Proposition 5.2.6). Finally, let $(f, \nu) \colon (D, X') \leftarrow (C, X)$ in $\int W$. The two composite oplax natural transformations of the square on the right also have the same structure 2-cells on (f, ν). Indeed it suffices to show it after applying the fully faithful $\widehat{\mathcal{G}}_{\omega,i(K_{\mathcal{M}}(C,X))}$. And this is straightforward to prove, following (part of) the recipe for $\mathcal{G}_{\tau}(\chi_{f,\nu})$ described in the proof of Theorem 3.2.8 (together with the construction of the counit of $\widehat{\mathcal{G}}_{\omega_{\mathcal{M}},K_{\mathcal{M}}(C,X)} \dashv \widehat{\mathcal{G}}^{-1}_{\omega_{\mathcal{M}},K_{\mathcal{M}}(C,X)}$ and the proof of Proposition 5.2.6). We conclude that $\chi_{\mathcal{M}}$ is cartesian-marked oplax natural. □

Remark 5.2.10. Theorem 5.2.9 offers another strategy to produce a good 2-classifier in a 2-category via a dense generator. Indeed, this is what we will do in Section 5.3 to produce our good 2-classifier in stacks. An advantage of this strategy is that we do not have to do the normalization process described in Theorem 3.2.8. By Kelly's [28, Proposition 5.16], any 2-category $\mathcal{M}$ equipped with a fully faithful dense generator $I\colon \mathcal{Y} \to \mathcal{M}$ is equivalent to a full sub-2-category of $[\mathcal{Y}^{\text{op}}, \mathcal{C}at]$ containing the representables. So, after Section 5.1, the strategy described in Theorem 5.3.22 can be very helpful.

Notice that the proof of Theorem 5.2.9 shows that we just need to be able to factorize the "normalized" characteristic morphisms produced as in the proof of Corollary 3.2.10 (starting from every φ).

5.3. A 2-classifier in stacks

In this section, after recalling the notion of stack, we restrict our good 2-classifier in prestacks (Theorem 5.1.14) to a good 2-classifier in stacks that classifies all discrete opfibrations with small fibres (Theorem 5.3.22). We follow the strategy described in the general Theorem 5.2.9 to restrict a good 2-classifier to a nice sub-2-category. The idea is to select, out of all the presheaves on slice categories involved in the definition of $\widetilde{\Omega}$, the sheaves with respect to the Grothendieck topology induced on the slices. This restriction of $\widetilde{\Omega}$ is tight enough to give a stack Ω_J, but at the same time loose enough to still host the classification process of prestacks.

Our result solves a problem posed by Hofmann and Streicher in [25]. Indeed, in a different context, they considered the same natural idea to restrict their analogue of $\widetilde{\Omega}$ by taking sheaves on slices. However, this did not work for them, as it does not give a sheaf. Our results show that such a restriction yields nonetheless a stack and a good 2-classifier in stacks.

Remark 5.3.1. As explained in Remark 5.3.10, we take strictly functorial stacks with respect to a subcanonical topology J. We consider $\mathcal{C}at$-valued stacks that, for simplicity, have a 1-category as domain. The stacks we consider have the usual gluing condition that gives an equivalence of categories between the image on an object C and each category of descent data on C. We recall below the explicit gluing conditions.

We will use the language of sieves, rather than the one of covering families. This simplifies the form of the conditions of stack and will make it easier for us to prove that our 2-classifier in stacks is indeed a stack. Moreover, sieves are also what forms the subobject classifier of sheaves, in dimension 1. The less standard equivalent definition of stack that we write below can be obtained unravelling Street's [43] abstract definition of stack, in the (more usual) case of a 1-dimensional Grothendieck topology.

We fix $\mathcal{C}$ a small category.

Definition 5.3.2. A *sieve* S on $C \in \mathcal{C}$ is a collection of morphisms with codomain C that is closed under precomposition with any morphism of $\mathcal{C}$.

Equivalently, a sieve S on C is a subfunctor of the representable $\mathrm{y}(C)$.

The *maximal sieve* is the collection of all morphisms with codomain C, or equivalently the identity on $\mathrm{y}(C)$.

Notation 5.3.3. Given a pseudofunctor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at$ and a morphism $g\colon D' \to D$ in $\mathcal{C}$, we denote as g^* the functor $F(g)$.

The following definition upgrades the concept of matching family to dimension 2. The compatibility under descent of the local data is up to isomorphism.

Definition 5.3.4. Let $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at$ be a pseudofunctor and let S be a sieve on $C \in \mathcal{C}$. A *descent datum for F with respect to S* is an assignment

$$(D \xrightarrow{f} C) \in S \quad \overset{m}{\longmapsto} \quad M_f \in F(D)$$

together with, for all composable morphisms $D' \xrightarrow{g} D \xrightarrow{f} C$ with $f \in S$, an isomorphism

$$\varphi^{f,g}\colon g^* M_f \xrightarrow{\simeq} M_{g \circ f}$$

such that, for all composable morphisms $D'' \xrightarrow{h} D' \xrightarrow{g} D \xrightarrow{f} C$ with $f \in S$, the following cocycle condition holds:

$$\begin{tikzcd}
h^* g^* M_f \arrow[r, "h^* \varphi^{f,g}"] \arrow[d, phantom, "\wr\|"] & h^* M_{f \circ g} \arrow[d, "\varphi^{f \circ g, h}"] \\
(g \circ h)^* M_f \arrow[r, "\varphi^{f, g \circ h}"'] & M_{f \circ g \circ h}.
\end{tikzcd}$$

The following definition upgrades the concept of amalgamation for a matching family to dimension 2. The global data produced only recovers the starting local data up to isomorphism.

Definition 5.3.5. In the notation of Definition 5.3.4, a descent datum m for F with respect to S is *effective* if there exists an object $M \in F(C)$ together with, for every morphism $f\colon D \to C$ in S, an isomorphism

$$\psi^f\colon f^* M \xrightarrow{\simeq} M_f$$

such that, for all composable morphisms $D' \xrightarrow{g} D \xrightarrow{f} C$ with $f \in S$

$$\begin{tikzcd}
g^* f^* M \arrow[r, "g^* \psi^f"] \arrow[d, phantom, "\wr\|"] & g^* M_f \arrow[d, "\varphi^{f,g}"] \\
(f \circ g)^* M \arrow[r, "\psi^{f \circ g}"'] & M_{f \circ g}.
\end{tikzcd}$$

Remark 5.3.6. Notice that the square in Definition 5.3.5 is very similar to the one of Definition 5.3.4. An object M that makes a descent datum m effective plays the role of an M_{id}, although the identity belongs to the sieve if and only if the sieve is maximal.

Definition 5.3.7. Let $\mathcal{C}$ be a category equipped with a Grothendieck topology J. A pseudofunctor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}at$ is a *stack* (*with respect to* J) if it satisfies the following three conditions for every $C \in \mathcal{C}$ and covering sieve in J on C:

(i) (gluing of objects) every descent datum for F with respect to S is effective;

(ii) (gluing of morphisms) for all $X, Y \in F(C)$ and for every assignment to each $f\colon D \to C$ in S of a morphism $h_f\colon f^*X \to f^*Y$ in $F(D)$ such that $g^*(h_f) = h_{f\circ g}$ for all composable morphisms $D' \xrightarrow{g} D \xrightarrow{f} C$, there exists a morphism $h\colon X \to Y$ such that $f^*h = h_f$;

(iii) (uniqueness of gluings of morphisms) for all $X, Y \in F(C)$ and morphisms $h, k : X \to Y$ such that $f^*h = f^*k$ for every $f\colon D \to C$ in S, it holds that $h = k$.

Remark 5.3.8. Conditions (*ii*) and (*iii*) of Definition 5.3.7 may be interpreted as saying that F is a sheaf on morphisms.

Theorem 5.3.9 (Street [43, Section 2])**.** *Stacks form a bireflective sub-2-category of the* 2*-category* $\mathrm{Ps}\,[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ *of pseudofunctors, pseudonatural transformations and modifications.*

Remark 5.3.10. Throughout the rest of this section, as the notion of 2-classifier is rather strict, we will consider strictly functorial stacks, so that they form a full sub-2-category of the functor 2-category $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$. We will also take a subcanonical Grothendieck topology J, so that all representables are sheaves (and hence stacks). We keep however the usual gluing conditions written above, that give an equivalence of categories between $F(C)$ and each category of descent data on C with respect to a covering sieve S. In future work, we will produce a suitable classifier for the usual pseudofunctorial stacks.

Notation 5.3.11. We denote as $i\colon \mathcal{S}t\,(\mathcal{C}, J) \subseteq [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ the full sub-2-category of $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ on stacks. Notice that i is indeed an injective on objects and fully faithful 2-functor.

We want to show that i satisfies all the assumptions of Theorem 5.2.9, so that we can restrict our good 2-classifier in prestacks to one in stacks.

The following proposition does not seem to appear in the literature.

Proposition 5.3.12. *The 2-category* $\mathcal{S}t_{\mathrm{Ps}}(\mathcal{C}, J)$ *of pseudofunctorial stacks has all bilimits and all flexible limits, calculated in* $\mathrm{Ps}\,[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ *and hence pointwise.*

$\mathcal{S}t\,(\mathcal{C}, J)$ *has all flexible limits (thus all comma objects and the terminal object) and all pullbacks along discrete opfibrations, calculated in* $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ *and hence pointwise.*

Proof. By Theorem 5.3.9, $\mathcal{S}t_{\mathrm{Ps}}(\mathcal{C}, J)$ is a bireflective sub-2-category of $\mathrm{Ps}\,[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$. So by Remark 5.1.8 it has all bilimits, calculated in $\mathrm{Ps}\,[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$. Consider then a flexible weight W and a 2-diagram F in $\mathcal{S}t_{\mathrm{Ps}}(\mathcal{C}, J)$. Then the flexible limit of F weighted by W exists in $\mathrm{Ps}\,[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$, by Remark 5.1.8. In particular, by flexibility of W, it satisfies the universal property of a bilimit of a 2-diagram that factors through $\mathcal{S}t_{\mathrm{Ps}}(\mathcal{C}, J)$. And it is then a pseudofunctorial stack. It follows that it is the flexible limit of F weighted by W in $\mathcal{S}t_{\mathrm{Ps}}(\mathcal{C}, J)$, since fully faithful 2-functors reflect 2-limits.

Consider now a 2-diagram F in $\mathcal{S}t\,(\mathcal{C}, J)$ and a flexible weight W. The flexible limit of F weighted by W exists in $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$, calculated pointwise. Then it is also the flexible limit in $\mathrm{Ps}\,[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$, as the latter is as well calculated pointwise. So it is a stack, as the 2-diagram in $\mathrm{Ps}\,[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ factors through $\mathcal{S}t_{\mathrm{Ps}}(\mathcal{C}, J)$. Whence we have produced the flexible limit of F weighted by W in $\mathcal{S}t\,(\mathcal{C}, J)$.

Finally, consider $p\colon E \to B$ and $z\colon F \to B$ in $\mathcal{S}t\,(\mathcal{C}, J)$, with p a discrete opfibration. The pullback of p and z exists in $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$, calculated pointwise. Then it is also the bi-iso-comma object of p and z in $\mathrm{Ps}\,[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$, by Remark 5.1.8. We conclude that it is a stack and hence the pullback of p and z in $\mathcal{S}t\,(\mathcal{C}, J)$, by the argument above. □

Proposition 5.3.13. *A morphism in* $\mathcal{S}t\,(\mathcal{C}, J)$ *is a discrete opfibration if and only if its underlying morphism in* $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ *is so. In particular,* $i\colon \mathcal{S}t\,(\mathcal{C}, J) \underset{\mathrm{ff}}{\hookrightarrow} [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ *preserves discrete opfibrations.*

Proof. The "if" part is clear by the fact that $\mathcal{S}t(\mathcal{C},J) \underset{\text{ff}}{\hookrightarrow} [\mathcal{C}^{\mathrm{op}},\mathcal{C}at]$ is fully faithful. The "only if" part follows from Proposition 4.2.5, since J is subcanonical. □

Remark 5.3.14. Putting together Proposition 5.3.12 and Proposition 5.3.13, we have thus proved that $i\colon \mathcal{S}t(\mathcal{C},J) \subseteq [\mathcal{C}^{\mathrm{op}},\mathcal{C}at]$ is a nice sub-2-category (Definition 5.2.2).

Definition 5.3.15. We say that a discrete opfibration φ in $\mathcal{S}t(\mathcal{C},J)$ *has small fibres* if $i(\varphi)$ has small fibres. Notice that this is in line with Remark 5.2.4.

Proposition 5.3.16. *Let $l\colon F \to B$ be a morphism in $[\mathcal{C}^{\mathrm{op}},\mathcal{C}at]$ (that is, a 2-natural transformation l). l is a fully faithful morphism if and only if for every $C \in \mathcal{C}$ the component l_C of l on C is a fully faithful functor.*

l is chronic (Definition 5.2.5) if and only if for every $C \in \mathcal{C}$ the component l_C of l on C is an injective on objects and fully faithful functor.

Proof. The proof is straightforward. □

Construction 5.3.17. We want to produce the object $\Omega_{\mathcal{M}}$ of Theorem 5.2.9 in our case with $i\colon \mathcal{S}t(\mathcal{C},J) \subseteq [\mathcal{C}^{\mathrm{op}},\mathcal{C}at]$. That is, a stack, which we will call Ω_J, that is a nice restriction of the good 2-classifier $\widetilde{\Omega}$ in prestacks. Recall that, in dimension 1, the subobject classifier in sheaves is given by taking closed sieves. We produce a 2-categorical notion of closed sieve.

We have already said in Remark 3.3.3 that discrete opfibrations over representables generalize the concept of sieve to dimension 2; we call them 2-sieves. Thanks to Proposition 5.1.3 (indexed Grothendieck construction, explored in Chapter 4), we can equivalently consider presheaves on slice categories. We now need to generalize closedness of a sieve to dimension 2. The indexed Grothendieck construction can be restricted to a bijection between 1-dimensional sieves on $C \in \mathcal{C}$ and presheaves $\left(\mathcal{C}/_{C}\right)^{\mathrm{op}} \to \mathit{2}$. It can be shown that closed 1-sieves correspond with sheaves $\left(\mathcal{C}/_{C}\right)^{\mathrm{op}} \to \mathit{2}$, and that the closure of a sieve corresponds to the sheafification of the corresponding presheaves. So we define *closed 2-sieves* to be the sheaves $\left(\mathcal{C}/_{C}\right)^{\mathrm{op}} \to \mathcal{S}et$ (with respect to the Grothendieck topology induced by J on the slices, that we call again J). And we use them to restrict our good 2-classifier $\omega\colon 1 \to \widetilde{\Omega}$ in prestacks to a good 2-classifier $\omega_J\colon 1 \to \Omega_J$ in stacks (Theorem 5.3.22).

Remark 5.3.18. The "maximal" 2-sieve $\mathrm{id}_{\mathrm{y}(C)}$, associated with $\widetilde{\omega}_C = \Delta 1\colon \left(\mathcal{C}/_{C}\right)^{\mathrm{op}} \to \mathcal{S}et$ (see Remark 5.1.4), is a closed 2-sieve.

Closed 2-sieves are stable under pullbacks. Indeed if $F\colon \left(\mathcal{C}/_{C}\right)^{\mathrm{op}} \to \mathcal{S}et$ is a sheaf and $f\colon D \to C$ is a morphism in $\mathcal{C}$ then also $F \circ (f\circ =)\colon \left(\mathcal{C}/_{D}\right)^{\mathrm{op}} \to \mathcal{S}et$ is a sheaf.

Proposition 5.3.19. *The 2-functor*

$$\begin{array}{rccc} \Omega_J : & \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathcal{C}at \\ & C & \mapsto & \mathcal{S}h\left(\mathcal{C}/_{C}, J\right) \\ & (C \xleftarrow{f} D) & \mapsto & - \circ (f\circ =)^{\mathrm{op}} \end{array}$$

is a stack with respect to the Grothendieck topology J.

Moreover, the inclusions $\mathcal{S}h\left(\mathcal{C}/_{C}, J\right) \underset{\mathrm{ff}}{\hookrightarrow} \left[\left(\mathcal{C}/_{C}\right)^{\mathrm{op}}, \mathcal{S}et\right]$ form a chronic arrow $\ell\colon i(\Omega_J) \underset{\mathrm{ff}}{\hookrightarrow} \widetilde{\Omega}$ in $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$. And $\widetilde{\omega}\colon 1 \to \widetilde{\Omega}$ factors through ℓ; call $\omega_J\colon 1 \to \Omega_J$ the resulting morphism.

Proof. The second part of the statement is clear after Proposition 5.3.16 and Remark 5.3.18.

We prove that Ω_J is a stack (recall from Definition 5.3.7 the definition). So let $C \in \mathcal{C}$ and $S \in J(C)$ a covering sieve on C.

We first prove the uniqueness of gluings of morphisms. Let $M, N \in \mathcal{S}h\left(\mathcal{C}/_{C}, J\right)$ and let $\alpha, \beta\colon M \Rightarrow N$ two natural transformations such that $f^*\alpha = f^*\beta$ for every $(D \xrightarrow{f} C) \in S$. We show that $\alpha = \beta$. Given $(D \xrightarrow{f} C) \in S$,

$$\alpha_f = (f^*\alpha)_{\mathrm{id}_D} = (f^*\beta)_{\mathrm{id}_D} = \beta_f.$$

Let now $g\colon E \to \mathcal{C}$ in $\mathcal{C}$ and consider $g^*S \in J(g) = J(E)$. Since M is a sheaf,

$$M(g) \cong \mathrm{Match}^M(g^*S)$$

where the right hand side denotes the set of matching families for M with respect to the covering sieve g^*S. And this holds analogously for N. We produce a commutative square

$$\begin{tikzcd}
M(g) \arrow[r, "\alpha_g"] \arrow[d, phantom, "\wr\parallel"] & N(g) \arrow[d, phantom, "\wr\parallel"] \\
\operatorname{Match}^M(g^*S) \arrow[r, "{[\alpha_g]}"'] & \operatorname{Match}^N(g^*S)
\end{tikzcd}$$

and analogously for β. We define $[\alpha_g]$ to send a matching family $(h \in g^*S) \overset{m}{\mapsto} (x_h \in M(g \circ h))$ to the matching family $(h \in g^*S) \mapsto (\alpha_{g\circ h}(x_h) \in N(g \circ h))$. The latter is indeed a matching family by naturality of α. The square above commutes since, for every $m \in \operatorname{Match}^M(g^*S)$, calling X the amalgamation of m, we have that $\alpha_g(X)$ is an amalgamation of $[\alpha_g](m)$. Notice now that $[\alpha_g] = [\beta_g]$, as for every $h \in g^*S$ we have that $g \circ h \in S$ and hence $\alpha_{g\circ h} = \beta_{g\circ h}$. Thus $\alpha_g = \beta_g$.

We prove that we have the gluings of morphisms. Let $M, N \in \mathcal{Sh}\left(\mathcal{C}/_C, J\right)$ and consider a matching family

$$(D \xrightarrow{f} C) \in S \mapsto (\alpha_f \colon f^*M \Rightarrow f^*N) \text{ in } \mathcal{Sh}\left(\mathcal{C}/_D, J\right)$$

So that for every $D' \xrightarrow{l} D \xrightarrow{f} C$ with $f \in S$ it holds that $l^*\alpha_f = \alpha_{f\circ l}$. We produce a natural transformation $\lambda \colon M \Rightarrow N$ such that $f^*\lambda = \alpha_f$ for every $(D \xrightarrow{f} C) \in S$. Given $(D \xrightarrow{f} C) \in S$, we would like to define $\lambda_f \coloneqq (\alpha_f)_{\mathrm{id}_D}$. Let $g \colon E \to C$ in $\mathcal{C}$. We define λ_g to be the composite

$$M(g) \cong \operatorname{Match}^M(g^*S) \xrightarrow{[\lambda_g]} \operatorname{Match}^N(g^*S) \cong N(g)$$

where $[\lambda_g]$ sends a matching family $(h \in g^*S) \overset{m}{\mapsto} (x_h \in M(g\circ h))$ to the matching family $(h \in g^*S) \mapsto ((\alpha_{g\circ h})_{\mathrm{id}}(x_h) \in N(g \circ h))$. The latter is indeed a matching family by naturality of $\alpha_{g\circ h}$. It is then straightforward to prove that λ is natural. Given $(D \xrightarrow{f} C) \in S$, we have that $\lambda_f = (\alpha_f)_{\mathrm{id}_D}$, since $\mathrm{id}_D \in f^*S$ and hence the amalgamation of any matching family on f^*S is just the datum on id_D. So $f^*\lambda = \alpha_f$. Indeed for every $l \colon D' \to D$ in $\mathcal{C}$

$$(f^*\lambda)_l = \lambda_{f\circ l} = (\alpha_{f\circ l})_{\mathrm{id}_E} = (l^*\alpha_f)_{\mathrm{id}_E} = (\alpha_f)_l.$$

It remains to prove that we have the gluing of objects. So consider a descent datum

$$(D \xrightarrow{f} C) \in S \mapsto M_f \in \mathcal{S}h\left(\mathcal{C}/_D, J\right)$$

with $\varphi^{f,h}\colon h^*M_f \cong M_{f\circ h}$ such that the cocycle condition

$$\begin{tikzcd}
k^*h^*M_f \arrow[r, "k^*\varphi^{f,h}"] \arrow[d, phantom, "\wr\|"] & k^*M_{f\circ h} \arrow[d, "\varphi^{f\circ h,k}"] \\
(h\circ k)^*M_f \arrow[r, "\varphi^{f,h\circ k}"'] & M_{f\circ h\circ k}.
\end{tikzcd}$$

holds for every $D'' \xrightarrow{k} D' \xrightarrow{h} D \xrightarrow{f} C$ with $f \in S$. We produce $M \in \mathcal{S}h\left(\mathcal{C}/_C, J\right)$ and for every $(D \xrightarrow{f} C) \in S$ isomorphisms $\psi^f\colon f^*M \cong M_f$ such that

$$\begin{tikzcd}
h^*f^*M \arrow[r, "h^*\psi^f"] \arrow[d, phantom, "\wr\|"] & h^*M_f \arrow[d, "\varphi^{f,h}"] \\
(f\circ h)^*M \arrow[r, "\psi^{f\circ h}"'] & M_{f\circ h}.
\end{tikzcd}$$

for every $D' \xrightarrow{h} D \xrightarrow{f} C$ with $f \in S$. We construct the presheaf

$$Z\colon \qquad \left(\mathcal{C}/_C\right)^{\mathrm{op}} \longrightarrow \mathcal{S}et$$

$$\begin{tikzcd}
D \arrow[r, "f\in S"] & C & & M_f(\mathrm{id}_D) \arrow[d, "Z(\underline{h})"] \\
D' \arrow[u, "h"] \arrow[ur, "f'"'] & & \longmapsto & M_{f'}(\mathrm{id}_{D'})
\end{tikzcd}$$

$$\left(D \xrightarrow{g\notin S} C\right) \longmapsto \emptyset$$

where $Z(\underline{h})$ is the composite

$$M_f(\mathrm{id}_D) \xrightarrow{M_f(\underline{h})} M_f(h) = (h^*M_f)(\mathrm{id}_{D'}) \xrightarrow[\varphi^{f,h}_{\mathrm{id}_{D'}}]{\simeq} M_{f\circ h}(\mathrm{id}_{D'}).$$

Z is indeed a functor, by the cocycle condition. Moreover, it is straightforward to show that $f^*Z \cong M_f$ for every $(D \xrightarrow{f} C) \in S$. However, Z is not a sheaf. So we define $M := Z^{++}$, where Z^+ is the plus construction of Z and hence Z^{++} is the sheafification of Z. It is straightforward to check that $(f^*Z)^+ \cong f^*(Z^+)$, by the explicit plus construction. Thus, using that $f^*Z \cong M_f$, we define ψ^f to be the composite

$$f^*(Z^{++}) \cong (f^*Z^+)^+ \cong (f^*Z)^{++} \cong (M_f)^{++} \cong M_f,$$

where the last isomorphism is given by the fact that M_f is a sheaf. It is then straightforward to show that the isomorphisms ψ^f satisfy the required condition. □

Remark 5.3.20. Representables form a fully faithful dense generator $\mathrm{y}\colon \mathcal{C} \to \mathcal{S}t\,(\mathcal{C}, J)$ of the kind described in Construction 5.2.8. We want to apply Theorem 5.2.9 on such a dense generator. So we need to factorize the characteristic morphisms in $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ of discrete opfibrations with small fibres in $\mathcal{S}t\,(\mathcal{C}, J)$ over representables.

Proposition 5.3.21. *For every $\psi\colon H \to \mathrm{y}\,(C)$ a discrete opfibration in $\mathcal{S}t\,(\mathcal{C}, J)$ with small fibres, with $C \in \mathcal{C}$, every characteristic morphism of $i(\psi)$ with respect to $\widetilde{\omega}$ factors through $\ell\colon i(\Omega_J) \underset{\mathrm{ff}}{\hookrightarrow} \widetilde{\Omega}$.*

Proof. It suffices to prove that the characteristic morphism z for $i(\varphi)$ with respect to $\widetilde{\omega}$ produced in Remark 5.1.16 (and Theorem 5.1.14) factors through ℓ. Indeed such factorization only depends on the isomorphism class of z, by the Yoneda lemma, as any presheaf isomorphic to a sheaf is a sheaf. By Remark 5.1.16, z is the 2-natural transformation $\mathrm{y}\,(C) \to \widetilde{\Omega}$ that corresponds with the functor

$$\begin{array}{rccc} z_C(\mathrm{id}_C): & \left(\mathcal{C}/_C\right)^{\mathrm{op}} & \longrightarrow & \mathcal{S}et \\ & (D \xrightarrow{f} C) & \mapsto & (\varphi_D)_f \\ & (f \xleftarrow{g} f\circ g) & \mapsto & H(g) \end{array}$$

So it suffices to prove that such functor is a sheaf. Let $f\colon D \to C$ in $\mathcal{C}$ and R a covering sieve on $f\colon D \to C$, i.e. $R \in J(D)$. Consider then a matching family

$$\begin{tikzcd} D' \arrow[d, "g"'] \arrow[rd, dashed] & \\ D \arrow[r, dashed, "f"'] & C \end{tikzcd} \in R \quad \overset{m}{\longmapsto} \quad X_g \in (\varphi_{D'})_{f\circ g}$$

on R for $z_C(\mathrm{id}_C)$. We need to show that there is a unique $X \in (\varphi_D)_f$ such that $H(g)(X) = X_g$ for every $\left(D' \xrightarrow{g} D\right) \in R$. Notice that m is also a matching family on $R \in J(D)$ for H, as $(\varphi_{D'})_{f\circ g} \subseteq H(D')$ and the action of $z_C(\mathrm{id}_C)$ on morphisms is given by the action of H. Since $\mathrm{y}\,(C) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$, also $H\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}et$. As H

is a stack, it then needs to be a sheaf. So there is a unique $X \in H(D)$ such that $H(g)(X) = X_g$ for every $\left(D' \xrightarrow{g} D\right) \in R$. It remains to prove that $\varphi_D(X) = f$. Since $\mathrm{y}(C)$ is separated, it suffices to prove that $\varphi_D(X) \circ g = f \circ g$ for every $\left(D' \xrightarrow{g} D\right) \in R$. But by naturality of φ

$$\varphi_D(X) \circ g = \varphi_{D'}(H(g)(X)) = \varphi_{D'}(X_g) = f \circ g.$$

We thus conclude that $z_C(\mathrm{id}_C)$ is a sheaf. ☐

We can now apply Theorem 5.2.9 (based on our theorems of reduction of the study of a 2-classifier to dense generators) to guarantee that we have produced a good 2-classifier in stacks that classifies all discrete opfibrations with small fibres. The following theorem is original.

Theorem 5.3.22. *The 2-natural transformation ω_J from 1 to*

$$\begin{array}{rccc} \Omega_J: & \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathcal{C}at \\ & C & \mapsto & \mathcal{S}h\left(\mathcal{C}/_C, J\right) \\ & (C \xleftarrow{f} D) & \mapsto & - \circ (f \circ =)^{\mathrm{op}} \end{array}$$

that picks the constant at 1 sheaf on every component is a good 2-classifier in $\mathcal{S}t\left(\mathcal{C}, J\right)$ that classifies all discrete opfibrations with small fibres.

Proof. By Theorem 5.2.9, the restriction ω_J of the good 2-classifier $\widetilde{\omega}$ in $[\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ along $i\colon \mathcal{S}t\left(\mathcal{C}, J\right) \underset{\mathrm{ff}}{\hookrightarrow} [\mathcal{C}^{\mathrm{op}}, \mathcal{C}at]$ is a good 2-classifier in $\mathcal{S}t\left(\mathcal{C}, J\right)$ with respect to the property of having small fibres. We can apply Theorem 5.2.9 thanks to Remark 5.3.14, Theorem 5.1.14, Proposition 5.3.19, Remark 5.3.20 and Proposition 5.3.21. ☐

Remark 5.3.23. We can extract from Theorem 5.2.9 and Remark 5.1.16 a recipe for the characteristic morphism $z_J\colon F \to \Omega_J$ of a discrete opfibration $\varphi\colon G \to F$ in $\mathcal{S}t\left(\mathcal{C}, J\right)$ with small fibres.

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\varphi"'] & 1 \arrow[d, "\omega_J"] \arrow[dl, phantom, "\Swarrow \text{comma}"] \\ F \arrow[r, "z_J"'] & \Omega_J \end{tikzcd}$$

Recall that we denote as i the inclusion $\mathcal{St}(\mathcal{C}, J) \subseteq [\mathcal{C}^{\mathrm{op}}, \mathcal{Cat}]$. We obtain that z_J corresponds to the 2-natural transformation $i(z_J)\colon i(F) \to i(\Omega_J)$ whose component on $C \in \mathcal{C}$ is the functor $i(z_J)_C$ that sends $X \in i(F)(C)$ to the sheaf

$$\begin{array}{rrcl} i(z_J)_C(X): & \left(\mathcal{C}/_C\right)^{\mathrm{op}} & \longrightarrow & \mathcal{Set} \\ & (D \xrightarrow{f} C) & \mapsto & \left(i(\varphi)_D\right)_{i(F)(f)(X)} \\ & (f \xleftarrow{g} f \circ g) & \mapsto & i(G)(g) \end{array}$$

Conclusions

We conclude with a summary of the results of this thesis and some possible directions for future work.

In Chapter 1, we have studied the 2-category of elements from an abstract point of view, thanks to an original notion of pointwise Kan extension in $2\text{-}\mathcal{C}at_{\text{lax}}$ and a refined notion of lax comma object. We have conceived the 2-category of elements as the archetypal 3-dimensional classification process and at the same time as the $2\text{-}\mathcal{S}et$-enriched Grothendieck construction. It would be very interesting to study 3-dimensional classifiers further, extracting a definition from the ideas of Chapter 1. We should thus upgrade discrete opfibrations to discrete 2-opfibrations and comma objects to lax comma objects. Interestingly, the archetypal 3-dimensional classifier has been crucial in this thesis to study 2-dimensional elementary topos theory. It would be very interesting as well to generalize fibrations and the Grothendieck construction to the general enriched setting. This is work in progress, and we believe it would have many fruitful applications. Some examples of enriched fibrations would be additive fibrations, graded fibrations, metric fibrations and general quantale-enriched fibrations.

In Chapter 2, we have developed a calculus of colimits in 2-dimensional slices, thanks to an original notion of colim fibration and the reduction of weighted 2-colimits to cartesian-marked oplax conical ones. This calculus has been a crucial tool in this thesis to reach our theorems of reduction of the study of 2-classifiers to dense generators. We believe that it could be fruitfully applied to other contexts as well.

In Chapter 3, we have introduced the notion of good 2-classifier, that captures well-behaved 2-classifiers and is closer to the point of view of logic. We believe this will be helpful to explore the 2-dimensional logic internal to an elementary 2-topos. We intend to investigate this in the near future. The main result of Chapter 3 is a novel technique of reduction of the study of 2-classifiers to dense generators. Our results substantially reduce the effort needed to prove that something is a 2-classifier or to study which morphisms get classified by a 2-classifier. In the cases of prestacks and stacks, our theorems have allowed us to look just at the classification over representables. In future work, we will explore more examples of elementary 2-topoi, and we believe that our technique of reduction will be an asset for this.

In Chapter 4, we have presented an indexed version of the Grothendieck construction. This helped us reach a nice candidate for a good 2-classifier in prestacks. It would be surely interesting to explore further applications of this construction. We believe it could be useful in geometry and algebra. We can interpret our indexed Grothendieck construction as the the result that every (op)fibrational slice of a Grothendieck 2-topos is a Grothendieck 2-topos. This generalizes to dimension 2 the Grothendieck topoi case of the fundamental theorem of elementary topos theory. In future work, we will extend our results to consider arbitrary elementary 2-topoi, reaching a complete 2-dimensional generalization of the fundamental theorem of elementary topos theory.

In Chapter 5, we have applied our theorems of reduction of the study of 2-classifiers to dense generators to produce a good 2-classifier in prestacks that classifies all discrete opfibrations with small fibres. This involves a 2-categorical generalization of the notion of sieve. In future work, we would like to apply our good 2-classifier in prestacks to reach a 2-categorical generalization of the fruitful concept of Grothendieck topology. This would bring to an extended notion of stack, with relevance in geometry. In Chapter 5, we have also restricted our good 2-classifier in prestacks to one in stacks, classifying again all discrete opfibrations with small fibres. The strategy has been to use factorization arguments and our theorems of reduction to dense generators. Our result is the main part of a proof that Grothendieck 2-topoi are elementary 2-topoi. We believe that this thesis could contribute to reach a universally accepted notion of elementary 2-topos. We can indeed look at the preeminent example of stacks to understand which conditions an elementary 2-topos should satisfy. In future work, we would like to adapt the results of this thesis to the bicategorical context, and thus to pseudofunctorial stacks, using a suitable bicategorical notion of classifier. Other ideas for future research are capturing and generalizing classifying topoi as well as geometrical classification processes.

Bibliography

[1] J. Adámek, H. Herrlich, and G. E. Strecker. Abstract and concrete categories: the joy of cats. *Repr. Theory Appl. Categ.*, 2006(17):1–507, 2006.

[2] B. Ahrens, P. R. North, and N. van der Weide. Bicategorical type theory: semantics and syntax. *Mathematical Structures in Computer Science*, 33(10):868–912, 2023.

[3] M. Artin, A. Grothendieck, J. L. Verdier, N. Bourbaki, P. Deligne, and B. Saint-Donat, editors. *Séminaire de géométrie algébrique du Bois-Marie 1963–1964. Théorie des topos et cohomologie étale des schémas. (SGA 4). Tome 1: Théorie des topos. Exposés I à IV. 2e éd.*, volume 269 of *Lect. Notes Math.* Springer, Cham, 1972.

[4] S. Awodey. On Hofmann-Streicher universes. *arXiv: https://arxiv.org/abs/2205.10917*, 2023.

[5] I. Baković. Fibrations of bicategories. https://www2.irb.hr/korisnici/ibakovic/groth2fib.pdf, last retrieved December 2022.

[6] G. J. Bird. Limits in 2-categories of locally-presented categories. *Ph.D. Thesis, University of Sydney*, 1984.

[7] G. J. Bird, G. M. Kelly, A. J. Power, and R. H. Street. Flexible limits for 2-categories. *J. Pure Appl. Algebra*, 61(1):1–27, 1989.

[8] F. Borceux. *Handbook of Categorical Algebra*, volume 2 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 1994.

[9] J. Bourke. Two-dimensional monadicity. *Adv. Math.*, 252:708–747, 2014.

[10] M. Buckley. Fibred 2-categories and bicategories. *J. Pure Appl. Algebra*, 218(6):1034–1074, 2014.

[11] M. C. Bunge. Coherent extensions and relational algebras. *Trans. Am. Math. Soc.*, 197:355–390, 1974.

[12] E. Caviglia and L. Mesiti. Indexed Grothendieck construction. *arXiv: https://arxiv.org/abs/2307.16076*, 2023.

[13] F. Conduché. Au sujet de l'existence d'adjoints à droite aux foncteurs "image reciproque" dans la categorie des categories. *C. R. Acad. Sci., Paris, Sér. A*, 275:891–894, 1972.

[14] S. E. Crans. A tensor product for Gray-categories. *Theory Appl. Categ.*, 5:12–69, 1999.

[15] M. E. Descotte, E. J. Dubuc, and M. Szyld. Sigma limits in 2-categories and flat pseudofunctors. *Adv. Math.*, 333:266–313, 2018.

[16] E. J. Dubuc. *Kan extensions in enriched category theory*, volume 145 of *Lect. Notes Math.* Springer, Cham, 1970.

[17] R. Garner and M. Shulman. Enriched categories as a free cocompletion. *Adv. Math.*, 289:1–94, 2016.

[18] J. Giraud. Méthode de la descente. *Bull. Soc. Math. Fr., Suppl., Mém.*, 2:115, 1964.

[19] J. Giraud. Cohomologie non abelienne. *C. R. Acad. Sci., Paris*, 260:2666–2668, 1965.

[20] J. W. Gray. *Formal category theory: Adjointness for 2-categories*, volume 391 of *Lect. Notes Math.* Springer, 1974.

[21] J. W. Gray. Quasi-Kan extensions for 2-categories. *Bull. Am. Math. Soc.*, 80:142–147, 1974.

[22] A. Grothendieck and M. Raynaud. *Revêtements Etales et Groupe Fondamental - Séminaire de Géométrie Algébrique du Bois Marie 1960/61 (SGA 1)*. Lect. Notes Math. Springer Berlin Heidelberg, 1971.

[23] C. Hermida. Some properties of Fib as a fibred 2-category. *J. Pure Appl. Algebra*, 134(1):83–109, 1999.

[24] K. Hirata. Notes on lax ends. *arXiv: https://arxiv.org/abs/2210.01522*, 2022.

[25] M. Hofmann and T. Streicher. Lifting Grothendieck universes. *preprint: https://www2.mathematik.tu-darmstadt.de/~streicher/NOTES/lift.pdf*, 1997.

[26] B. Jacobs. *Categorical Logic and Type Theory*, volume 141 of *Stud. Logic Found. Math.* Amsterdam: Elsevier, 1999.

[27] G. M. Kelly. On clubs and doctrines. Category Sem., Proc., Sydney 1972/1973, Lect. Notes Math. 420, 181-256 (1974)., 1974.

[28] G. M. Kelly. Basic concepts of enriched category theory. *Repr. Theory Appl. Categ.*, 2005(10):1–136, 2005.

[29] S. Lack. Codescent objects and coherence. 175(1):223–241, 2002.

[30] S. Lack and M. Shulman. Enhanced 2-categories and limits for lax morphisms. *Adv. Math.*, 229(1):294–356, 2012.

[31] M. Lambert. Discrete 2-fibrations. *arXiv: https://arxiv.org/abs/2001.11477*, 2020.

[32] F. W. Lawvere. An elementary theory of the category of sets. *Proc. Natl. Acad. Sci. USA*, 52:1506–1511, 1964.

[33] F. Lucatelli Nunes. On biadjoint triangles. *Theory Appl. Categ.*, 31:217–256, 2016.

[34] S. Mac Lane and I. Moerdijk. *Sheaves in geometry and logic: a first introduction to topos theory.* Universitext. New York etc.: Springer-Verlag, 1992.

[35] L. Mesiti. Colimits in 2-dimensional slices. *arXiv:* https://arxiv.org/abs/2305.01494, 2023.

[36] L. Mesiti. The 2-Set-enriched Grothendieck construction and the lax normal conical 2-limits. *arXiv:* https://arxiv.org/abs/2302.04566, 2023.

[37] L. Mesiti. 2-classifiers via dense generators and Hofmann-Streicher universe in stacks. *arXiv:* https://arxiv.org/abs/2401.16900, 2024.

[38] J. Moeller and C. Vasilakopoulou. Monoidal Grothendieck construction. *Theory Appl. Categ.*, 35:1159–1207, 2020.

[39] E. Palmgren. Groupoids and local cartesian closure. *Preprint:* http://www2.math.uu.se/~palmgren/gpdlcc.pdf, 2003.

[40] A. Power. A general coherence result. 57(2):165–173, 1989.

[41] R. Street. Fibrations and Yoneda's Lemma in a 2-category. Category Sem., Proc., Sydney 1972/1973, Lect. Notes Math. 420, 104-133, Springer, 1974.

[42] R. Street. Limits indexed by category-valued 2-functors. *J. Pure Appl. Algebra*, 8:149–181, 1976.

[43] R. Street. Characterization of bicategories of stacks. Category theory. Applications to algebra, logic and topology, Proc. int. Conf., Gummersbach 1981, Lect. Notes Math. 962, 282-291 (1982)., 1982.

[44] M. Szyld. A general limit lifting theorem for 2-dimensional monad theory. *J. Pure Appl. Algebra*, 222(8):2359–2373, 2018.

[45] M. Szyld. Lifting PIE limits with strict projections. *Theory Appl. Categ.*, 34:1–12, 2019.

[46] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics.* https://homotopytypetheory.org/book, Institute for Advanced Study, 2013.

[47] M. Tierney. Axiomatic sheaf theory: some constructions and applications. C.I.M.E., III Ciclo Varenna 1971, Categories commut. Algebra, 249-326 (1973)., 1973.

[48] C. Vasilakopoulou. On enriched fibrations. *Cah. Topol. Géom. Différ. Catég.*, 59(4):354–387, 2018.

[49] V. Voevodsky. Univalent foundations of mathematics. In *Logic, language, information and computation. 18th international workshop, WoLLIC 2011, Philadelphia, PA, USA, May 18–20, 2011. Proceedings*, page 4. Berlin: Springer, 2011.

[50] C. Walker. Lax familial representability and lax generic factorizations. *Theory Appl. Categ.*, 35:1424–1475, 2020.

[51] M. Weber. Yoneda structures from 2-toposes. *Appl. Categ. Struct.*, 15(3):259–323, 2007.