A Diagrammatic Approach to Constructing Fibrant Double Categories

Submitted for the degree of Doctor of Philosophy of Mathematics
September 2023

Joseph Paul Martin

Supervised by Dr. Simon Willerton

School of Mathematics and Statistics
The University of Sheffield

Abstract

In [Shu08], Shulman describes a way to construct a fibrant double category from a monoidal bifibration. In this thesis, we take an algebraic approach using indexed categories and string diagrams to better understand this construction and the role that the Beck-Chevalley transformation has within it. We give an explicit calculation of the niche-filling morphism arising from cartesian and opcartesian lifting properties, and we use this to give a more intuitive string-diagrammatic proof of a result on conditions equivalent to the Beck-Chevalley conditions. We give a detailed examination of the construction and make explicit calculations—in string diagrammatic language—of the unit loose 1-cell U_f and the loose composition (left) unitor $l^{\odot}$ of the constructed fibrant double category.

Motivating examples are given throughout, including proofs that the forgetful functor **GrpRep** $\rightarrow$ **FinGrp** that maps a G-module V to the group G that acts on it is a weakly Beck-Chevalley and internally closed monoidal bifibration.

Contents

Introduction

The aim of this thesis is to use indexed categories and string diagrams to better understand the Beck-Chevalley condition and the construction of fibrant monoidal double categories by Shulman [Shu08]. The advantage of using indexed categories is that we can make explicit calculations of the data involved in the construction, and the advantage of using string diagrams is that we can see at a glance how the hypotheses of the construction imply the necessary properties of the data.

Fibrant double categories

Double categories were originally introduced by Ehresmann [Ehr63], and the series of papers by Grandis and Paré are a comprehensive study [GP99, GP04, GP08, GP07]. A double category can be defined to be an internal category in the 2-category **Cat** of categories; more explicitly, a double category has 0-cells, two kinds of morphism—called tight 1-cells and loose 1-cells—between the 0-cells and 2-cells between the loose 1-cells. It is the two different kinds of morphism that means certain mathemtaical objects are better understood using double categories. Applications of double categories include the study of dynamical systems [Mye23, BCV22, Cou20], universal 2-algebras [Kel74, Fio07] and derived functors [Shu11], to name just a few examples. A double category that is a focus of this thesis—though this is not made explicit until Example 4.3.4 once we have seen the construction Theorem 4.3.1—is the double category $\mathbb{B}\textbf{imod}$ of bimodules over finite groups. This double category has as 0-cells finite groups, as tight 1-cells group homomorphisms, as loose 1-cells $G \nrightarrow H$ left G- right H-bimodules, and as 2-cells equivariant module maps.

For the history of fibrant double categories, it should be noted that they are essentially the same as proarrow equipments, in the sense that there are inverse constructions that identify proarrow equipments with fibrant double categories whose underlying loose bicategory is a strict 2-category; we don't give the details of these constructions here, but details can be found in [Shu08, Appendix C]. Proarrow equipments were introduced by Wood [Woo82, Woo85] and Street [Str80] and later studied by Carboni and others [CJSV94]. The connection between proarrow equipments and double categories was made current by Shulman [Shu08] and Verity [Ver11], but dates back to the study of Segal spaces by Segal [Seg68].

The construction: from fibrations to double categories

The double category $\mathbb{B}\mathbf{imod}$ can be constructed from the functor

$$\begin{aligned} \mathtt{Rep}\colon \mathbf{GrpRep} &\longrightarrow \mathbf{FinGrp} \\ (G,V) &\longmapsto G \\ (f,\phi) &\longmapsto f \end{aligned}$$

where **GrpRep** is the category of group representations and equivariant maps, and **FinGrp** is the category of finite groups and group homomorphisms; this functor maps a G-module V to the group G and an equivariant map (f,ϕ) to its underlying group homomorphism f. The double category we construct from this functor has as 0-cells and tight 1-cells the objects and morphisms of the category **FinGrp**, and as loose 1-cells $G \nrightarrow H$ the objects in **GrpRep** that the functor **Rep** maps to the group $G \times H$, i.e. $(G \times H)$-modules. There are many common examples of double categories that arise this way, such as the double category $\mathbb{S}\mathbf{pan}$ of spans, the double category $\mathbb{P}\mathbf{rof}$ of profunctors, and the double category $\mathbb{M}\mathbf{at}$ of matrices. We denote by $\mathbb{D}\mathbf{bl}(\Phi)$ the double category constructed from the functor Φ. In order to define the other data of the double category $\mathbb{D}\mathbf{bl}(\Phi)$—such as the composition of loose 1-cells, the associator and unitors—the functor Φ is required to be a bifibration.

Fibrations were introduced by Grothendieck in the context of descent theory [Gro60, Gro71], and some modern textbook references include [Joh02, Chapter B1], [Bor94b, Chapter 8] and [JY21, Chapter 9]. A fibration is a functor $\Phi\colon \mathscr{A} \to \mathscr{B}$ that satisfies a universal property—called a cartesian lifting property—that allows any morphism ψ in $\mathscr{A}$ satisying $\psi\Phi = g \mathbin{⨾} f$ to be factorised as $\psi = \chi \mathbin{⨾} \phi$ for some morphisms ϕ and χ in $\mathscr{A}$ satisfying $\chi\Phi = g$ and $\phi\Phi = f$. A bifibration is a functor that is both a fibration and an opfibration, meaning that it also satisfies the *op*cartesian lifting property. Bifibrations originate from [Gro71]; an early discussion on bifibrations (but which doesn't use the term 'bifibrations') is [Gra66]. It is the two lifting properties of satisfied by bifibrations that Shulman uses to define the other data of the double category $\mathbb{D}\mathbf{bl}(\Phi)$—such as the composition of loose 1-cells, the associator and unitors. However, this thesis aims to give explicit descriptions of this data, something that can't be done using universal properties. We will therefore use indexed categories.

An indexed category consists of a category $\mathscr{B}$ and a pseudofunctor $\mathfrak{I}\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$. They were introduced by Grothendieck in [Gro71] alongside fibrations, and important early treatments include [Bén75] and [PRS+78]. Indexed categories are an algebraicization of fibrations, meaning that, rather than using a universal property, they require a structure to be specified. For example, a vector space V having the property of having dimension n means that there exists a basis for V containing n vectors, but this can be algebraicized by requiring a specific choice of basis $(e_1, \ldots, e_n)$ for V. In the case of fibrations and indexed categories, the factorisation given by the cartesian lifting property is required to be specified. By specifying a factorisation for a fibration $\Phi\colon \mathscr{A} \to \mathscr{B}$, we obtain, for each morphism $f\colon B \to B$ in $\mathscr{B}$, a functor

$$f^*\colon \mathscr{A}_{B'} \longrightarrow \mathscr{A}_B,$$

where, for each object X in $\mathscr{B}$, $\mathscr{A}_X$ denotes the subcategory of $\mathscr{A}$ consisting of all objects M satisfying $M\Phi = X$ and all morphisms ϕ satisfying $\phi\Phi = \mathrm{id}_X$. The indexed category associated to the fibration Φ

is given by

$$\begin{aligned} \mathscr{B}^{\mathrm{op}} &\longrightarrow \mathbf{Cat} \\ B &\longmapsto \mathscr{A}_B \\ f &\longmapsto f^*. \end{aligned}$$

This construction and its inverse—which establish an equivalence between fibrations and indexed categories —are known as the Grothendieck construction; the Grothendieck construction originates in [Gro71] and has textbook accounts in [Bor94b, Section 8.3], [Joh02, Chapters A1 & B1] and [JY21, Chapter 10].

We said above that the loose 1-cells $G \nrightarrow H$ in the double category $\mathbb{D}\mathbf{bl}(\mathsf{Rep})$ are the objects in **GrpRep** that the functor Rep maps to the group $G \times H$, and this is true of the general construction: given a bifibration $\Phi\colon \mathscr{A} \to \mathscr{B}$, the loose 1-cells $A \nrightarrow B$ in the double category $\mathbb{D}\mathbf{bl}(\Phi)$ are objects in $\mathscr{A}$ that the functor Φ maps to $A \times B$. We therefore require the bifibration Φ to be monoidal and for the category $\mathscr{B}$ to be cartesian monoidal. Shulman introduced monoidal fibrations in [Shu08] where he also explicitly constructed, for each cartesian monoidal category $\mathscr{B}$, a 2-equivalence between the 2-category $\mathcal{MonFib}_{\mathscr{B}}$ of monoidal fibrations with base $\mathscr{B}$ and the 2-category $\mathsf{Bicat}^{\mathsf{ps}}(\mathscr{B}^{\mathrm{op}}, \mathbf{MonCat})$ of pseudofunctors $\mathscr{B}^{\mathrm{op}} \to \mathbf{MonCat}$—called indexed (strong) monoidal categories in [HM06]—where **MonCat** denotes the 2-category of monoidal categories, strong monoidal functors and strong monoidal transformations. In [MV20], Moeller and Vasilakopoulou establish the *monoidal* Grothendieck construction—a collection of 2-equivalences involving 2-categories of monoidal fibrations, not restricted to the case of a cartesian base—and perform a thorough investigation into both fibrewise and global monoidal structures of a fibration.

The Beck-Chevalley condition

We saw above that if $\Phi\colon \mathscr{A} \to \mathscr{B}$ is a fibration, then, for each morphism $f\colon B \to B'$ in $\mathscr{B}$, there is a functor $f^*\colon \mathscr{A}_{B'} \to \mathscr{A}_B$. If Φ is a *bi*fibration, then, for each morphism $f\colon B \to B'$ in $\mathscr{B}$, there are functors

$$f^*\colon \mathscr{A}_{B'} \rightleftarrows \mathscr{A}_B : f_!$$

We think of the functor f^* as 'pulling' an object M in the category $\mathscr{A}_{B'}$ back along the morphism $f\colon B \to B'$ in $\mathscr{B}$ to obtain an object Mf^* in the category $\mathscr{A}_B$, and we think of the functor $f_!$ as 'pushing' an object N in the category $\mathscr{A}_B$ forwards along the morphism $f\colon B \to B'$ in $\mathscr{B}$ to obtain an object $Nf_!$ in the category $\mathscr{A}_{B'}$. These functors are sometimes known as the pull-back and push-foward functors associated to f. For the bifibration $\mathsf{Rep}\colon \mathbf{GrpRep} \to \mathbf{FinGrp}$, we pull an H-module W back along a group homomorphism $f\colon G \to H$ to obtain the G-module ${}_fW$ known as the *restricted representation*, and we push a G-module V along $f\colon G \to H$ to obtain the H-module $\mathbb{C}H \otimes_G V$ known as the *induced representation*.

Given a commutative square

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[f]{} & D \end{array}$$

in $\mathscr{B}$, the Beck-Chevalley transformation is a canonical natural transformation

$$\begin{array}{ccc} \mathscr{A}_A & \xleftarrow{h^*} & \mathscr{A}_B \\ {\scriptstyle k_!}\downarrow & \overset{\zeta}{\searrow} & \downarrow{\scriptstyle g_!} \\ \mathscr{A}_C & \xleftarrow[f^*]{} & \mathscr{A}_D \end{array}$$

In some sense, this natural transformation is the difference between pulling then pushing and pushing then pulling. The square in $\mathscr{B}$ is said to satisfying the Beck-Chevalley condition if the Beck-Chevalley transformation ζ assoicated to that square is an isomorphism, i.e. when pulling then pushing is 'the same as' pushing then pulling.

The Beck-Chevalley condition originated from the study of descent when Bénabou and Roubaud used it to prove the Bénabou-Roubaud theorem in [BR70]. Beck and Chevalley studied the Beck-Chevalley condition independently of one another, but neither of them ever appears to have published anything about it. Lawvere mentions the Beck-Chevalley condition in the context of categorical semantics [Law70] and Seely exapnded on this in [See83]. Since then, the Beck-Chevalley condition has been studied extensively in, to name just a few examples, the contexts of subobject lattices [MM12, Chapter IV.9], ∞-categories [HL13, Chapter 4] and quasicategories [Joy08, p. 175].

Our interest in the Beck-Chevalley condition is due to its relevance to the construction of the double category $\mathbb{D}\mathbf{bl}(\Phi)$. A bifibration is called Beck-Chevalley if every pullback square satisfies the Beck-Chevalley condition, and a bifibration is called weakly Beck-Chevalley if every pullback square with a product projection leg satisfies the Beck-Chevalley condition. We can now state an abbreviated version of the construction of the double category $\mathbb{D}\mathbf{bl}(\Phi)$ due to Shulman [Shu08, Theorem 14.2].

Theorem 4.3.1. *If $\Phi\colon \mathscr{A} \to \mathscr{B}$ is a Beck-Chevalley monoidal bifibration, then there is a fibrant double category $\mathbb{D}\mathbf{bl}(\Phi)$ defined as follows.*

(i) The tight category $\mathbb{D}\mathbf{bl}(\Phi)_0$ is equal to $\mathscr{B}$.

(ii) The loose category $\mathbb{D}\mathbf{bl}(\Phi)_1$ and the functors S and T are given by the following pullback in **Cat**:

$$\begin{array}{ccc} \mathbb{D}\mathbf{bl}(\Phi)_1 & \longrightarrow & \mathscr{A} \\ {\scriptstyle (S,T)}\downarrow & \lrcorner & \downarrow{\scriptstyle \Phi} \\ \mathscr{B}\times\mathscr{B} & \xrightarrow[\times]{} & \mathscr{B} \end{array}$$

(iii) *The loose composition of loose 1-cells* $M\colon A \nrightarrow B$ *and* $N\colon B \nrightarrow C$ *is equal to*

$$M \odot N = (M \otimes N)\Delta_B^* \pi_{B!}$$

and the loose composition of 2-cells is similar.

(iv) *The loose unit of the object* A *is equal to*

$$U_A = I\pi_A^* \Delta_{A!}$$

where I *denotes the monoidal unit of* $\mathscr{A}$.

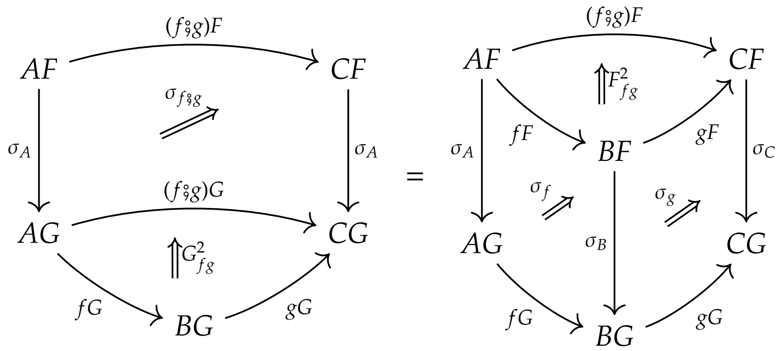
String diagrams

A key focus of this thesis is to make explicit calculations of the certain objects, that are otherwise decribed in the literatue by universal properties. This is of interest in its own right, but taking a diagrammatic approach makes these explicit calculations all the more worthwhile as they allow the reader to see, for example, the presence of the Beck-Chevalley transformation in the definition of the loose composition left unitor

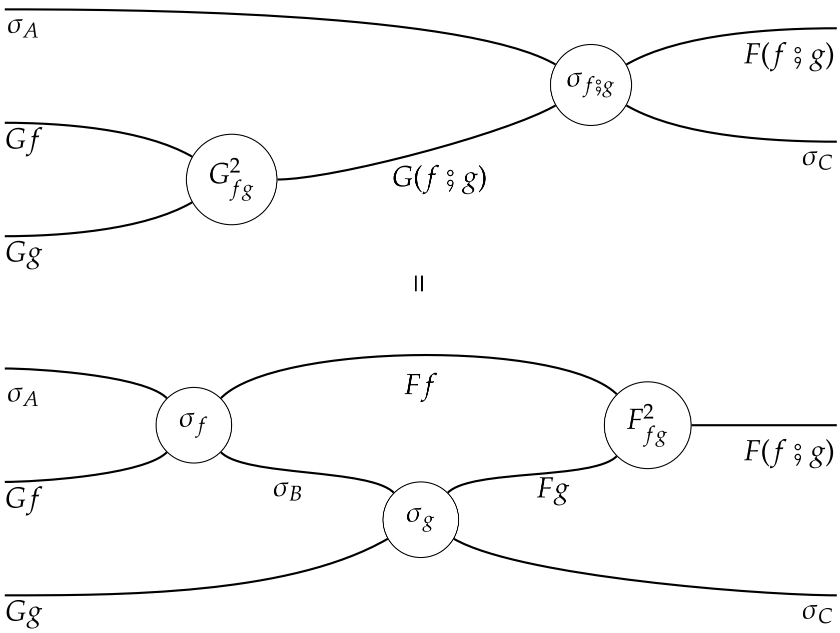
$$l_M^{\odot}\colon U_A \odot M \to M$$

in the double category $\mathbb{D}\mathbf{bl}(\Phi)$. The fact that the Beck-Chevalley transformation being an isomorphism for pullback squares implies that the left unitor $l_M^{\odot}$ is an isomorphism can be seen with ease with the aid of string diagrams.

String diagrams were originally (and still are) used to express operations in a monoidal category; Hotz [Hot65] used string diagrams in the monoidal category of finite sets, and Penrose [Pen71, PR84] used string diagrams in the monoidal category of finite-dimensional vector spaces. String diagrams were fomalised for aribtrary monoidal categories by Joyal and Street [JS91] and they are easily generalised to bicategories. The idea of string diagrams is that pasting diagrams be replaced by their Poincaré duals. In a pasting diagram in a bicategory $\mathscr{B}$, 0-cells are represented as points, 1-cells as lines and 2-cells as regions, whereas, in a string diagram in $\mathscr{B}$, 0-cells are represented as regions, 1-cells as lines and 2-cells as points. For example, the lax multiplicativity axiom



for a lax transformation between bicategories (see Definition 1.1.8) is an equality between two pasting diagrams in the bicategory **Cat** of categories, and this equality is represented using string diagrams as follows.



Outline of the thesis

In Chapter 1, we give some useful definitions and propositions relating to bicategories, adjunctions, and monoidal bicategories. In particular, we discuss mates, which we use in subsequent chapters to describe internal and external closure of monoidal bifibrations and to describe the Beck-Chevalley transformation. We also give the notation and conventions the we use for string diagrams in the subsequent chapters.

In Chapter 2, cover the basic definitions and properties relating to fibrations and indexed categories, as well as a summary of the Grothendieck construction as a 2-equivalence. We also give a summary of theory of monoidal fibrations due to Shulman [Shu08] and Moeller and Vasilakopoulou [MV20]. Throughout this chapter, we give motivating examples, including proofs that together show that the functor $\mathtt{Rep}\colon \mathbf{GrpRep} \to \mathbf{FinGrp}$ is an internally closed monoidal bifibration (see Examples 2.1.16, 2.4.9, 2.6.5 and 2.8.11). While fibrations are well studied, we give thorough proofs which fill gaps in the literature; we provide a definition of the functor f^* (see Proposition 2.1.27), a thorough treatment of which appeared absent from the literature until Johnson and Yau also provided such a definition in [JY21]; we also prove that the pseudofunctoriality morphisms

$$\Phi^{2*}_{fg,P}\colon Pg^*f^* \to P(f \mathbin{;} g)^* \qquad \text{and} \qquad \Phi^{0*}_{B,P}\colon P \to P\mathrm{id}^*_B$$

associated to a fibration Φ are natural in P (see Propositions 2.2.5 and 2.2.7). We also give string diagrammatic proofs of the following results: Lemma 2.3.5, Proposition 2.3.6, Theorem 2.5.13, Proposition 2.5.16.

In Chapter 3, we study the Beck-Chevalley condition. We begin by giving a summary of the theory of integral transfroms, which motivates the study of the Beck-Chevalley condition. We define the Beck-Chevalley morphisms ζ_M, prove that they are natural in M and that the resulting natural transformation is given by a mate (see Proposition 3.2.3) which is the more common definition. In Remarks 3.2.8, 3.2.9 and 3.2.11, we give string diagrammatic arguments which show that the Beck-Chevalley transformation is well-defined. We then prove that the bifibration $\mathtt{Rep}\colon \mathbf{GrpRep} \to \mathbf{FinGrp}$ is weakly Beck-Chevalley (see Proposition 3.3.3), and we show how the Beck-Chevalley condition relates to Mackey's formula of representation theory. We give an explcit calculation of an important morphism arising from a universal property (see Lemma 3.4.3), and, at the end of this chapter, we use this explicit calculation to prove Corollary 3.4.4—a well-known result on conditions equivalent to the Beck-Chevalley conditions—which is made a great deal more intuitive with a string diagrammatic proof.

In Chapter 4, we begin by giving useful definitions and propositions relating to double categories, including fibrant double categories and monoidal double categories. The majority of this chapter contains a detailed examination of Shulman's construction of fibrant double categories from monoidal bifibrations. We conclude with the main results of this thesis: the explicit calculations—in string diagrammatic language—of the unit loose 1-cell U_f and the (left) unitor for loose composition.

Acknowledgements

I'd like to thank those who have supported me during my PhD.

Firstly, thank you to Simon Willerton for being the most patient and understanding person I know. Thank you to Markus Szymik and Joe Grant for their many helpful suggested improvements to this thesis.

Thank you to Callum Reader for being so friendly and approachable. Thank you to David Carey and Lewis Combes for accompanying me through the highs and lows of the past four years; being able to talk to people going through the same thing as me was invaluable. Thank you to Mark Yarrow for his attentive listening and consistent reassurance of care and support. Thank you to Kieran Collins for being my closest and most unconditionally supportive friend; knowing that he always had time for me meant a lot.

Most of all I would like to thank my partner, Faith Reading, without whom this thesis wouldn't have been possible. She devoted an enormous amount of time and energy to supporting me: providing encouragement in the office when I faced writing apprehension, making sure I kept healthy at home, and being there during my lowest moments as the deadline loomed.

Chapter 1

Background

In this chapter, we collect all the background material and conventions we will use in the following chapters. We assume familiarity with basic category theory—e.g. functors, natural transformations, limits and colimts—as well as the basics of monoidal categories.

We define bicategories, pseudofunctors and pseudonatural transformations, and we give brief definitions of monoidal bicategories and pseudomonoids. We recall some terminology, definitions and basic properties of adjunctions. In particular, we provide some of the basic theory of mates relating to adjunctions as they are used frequently in later chapters to describe internal and external closure of monoidal bifibrations and to describe the Beck-Chevalley transformation.

We also give a description of the conventions we use for the string diagrams used throughout this document.

Notation 1.0.1. We will read composition left to right, and we will apply maps and functors on the right. That is, for morphisms $f\colon A \to B$ and $g\colon B \to C$ in a category $\mathscr{A}$, we will denote their composite by $f \fatsemi g\colon A \to C$, and, for a functor $F\colon \mathscr{A} \to \mathscr{B}$, we denote it's application to A and f as AF and fF. ◇

1.1 Bicategories

The basic definitions and properties of bicategories and 2-categories were introduced by Bénabou [Bén65, Bén67].

Definition 1.1.1. A **bicategory** $\mathscr{B}$ consists of the following data:

- a class $\mathrm{ob}\,\mathscr{B}$, whose elements are called **0-cells**;
- for each pair A, B of 0-cells in $\mathscr{B}$, a category $\mathscr{B}(A, B)$ whose objects are called **1-cells** and whose morphisms are called **2-cells**;
- for each 0-cell A in $\mathscr{B}$, a 1-cell id_A in $\mathscr{B}(A, A)$ called the **identity 1-cell** on A;

- for each triple A, B, C of 0-cells in $\mathscr{B}$, a functor

$$\begin{array}{rccll} c_{ABC}: & \mathscr{B}(A,B) \times \mathscr{B}(B,C) & \to & \mathscr{B}(A,C) & \\ & (f,g) & \mapsto & f \fatsemi g & \text{on 1-cells} \\ & (\alpha,\beta) & \mapsto & \alpha \fatsemi \beta & \text{on 2-cells} \end{array} \tag{1.1.2}$$

  called **composition**;

- for each triple $f \in \mathscr{B}(A,B)$, $g \in \mathscr{B}(B,C)$, $h \in \mathscr{B}(C,D)$ of 1-cells in $\mathscr{B}$, an invertible 2-cell

$$a_{fgh}: (f \fatsemi g) \fatsemi h \to f \fatsemi (g \fatsemi h)$$

  called the **associativity constraint**;

- for each 1-cell $f \in \mathscr{B}(A,B)$ in $\mathscr{B}$, invertible 2-cells

$$l_f: \mathrm{id}_A \fatsemi f \to f \quad \text{and} \quad r_f: f \fatsemi \mathrm{id}_B \to f.$$

  called the **left unitality constraint** and the **right unitality constraint**.

These data are required to satisfy the following axioms.

- (**Naturality of the associator**) For all 2-cells $\alpha: f \to f'$ in $\mathscr{B}(A,B)$, $\beta: g \to g'$ in $\mathscr{B}(B,C)$ and $\gamma: h \to h'$ in $\mathscr{B}(C,D)$, the following diagram commutes.

$$\begin{tikzcd} (f \fatsemi g) \fatsemi h \arrow[r, "(\alpha \fatsemi \beta) \fatsemi \gamma"] \arrow[d, "a_{fgh}"'] & (f' \fatsemi g') \fatsemi h' \arrow[d, "a_{f'g'h'}"] \\ f \fatsemi (g \fatsemi h) \arrow[r, "\alpha \fatsemi (\beta \fatsemi \gamma)"'] & f' \fatsemi (g' \fatsemi h') \end{tikzcd} \tag{1.1.3}$$

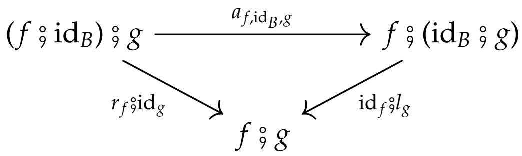
- (**Naturality of the unitors**) For all 2-cells $\alpha: f \to f'$ in $\mathscr{B}(A,B)$, the following diagrams commutes.

$$\begin{tikzcd} \mathrm{id}_A \fatsemi f \arrow[r, "\mathrm{id}_{\mathrm{id}_A} \fatsemi \alpha"] \arrow[d, "l_f"'] & \mathrm{id}_A \fatsemi f' \arrow[d, "l_{f'}"] \\ f \arrow[r, "\alpha"'] & f' \end{tikzcd} \qquad \begin{tikzcd} f \fatsemi \mathrm{id}_B \arrow[r, "\alpha \fatsemi \mathrm{id}_{\mathrm{id}_B}"] \arrow[d, "r_f"'] & f' \fatsemi \mathrm{id}_B \arrow[d, "r_{f'}"] \\ f \arrow[r, "\alpha"'] & f' \end{tikzcd} \tag{1.1.4}$$

- (**Associativity**) For all 1-cells $f \in \mathscr{B}(A,B)$, $g \in \mathscr{B}(B,C)$, $h \in \mathscr{B}(C,D)$, and $k \in \mathscr{B}(D,E)$, the following diagram commutes.

$$\begin{tikzcd} & (f \fatsemi g) \fatsemi (h \fatsemi k) \arrow[dr, "a_{f,g,h \fatsemi k}"] & \\ ((f \fatsemi g) \fatsemi h) \fatsemi k \arrow[ur, "a_{f \fatsemi g,h,k}"] \arrow[d, "a_{f,g,h} \fatsemi \mathrm{id}_k"'] & & f \fatsemi (g \fatsemi (h \fatsemi k)) \\ (f \fatsemi (g \fatsemi h)) \fatsemi k \arrow[r, "a_{f,g \fatsemi h,k}"'] & f \fatsemi ((g \fatsemi h) \fatsemi k) \arrow[ur, "\mathrm{id}_f \fatsemi a_{g,h,k}"'] & \end{tikzcd}$$

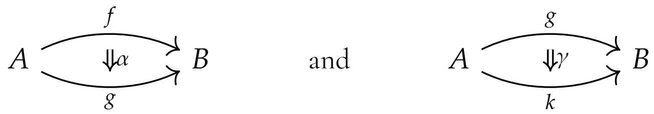
- (**Unitality**) For all 1-cells $f \in \mathscr{B}(A,B)$ and $g \in \mathscr{B}(B,C)$, the following diagram commutes.



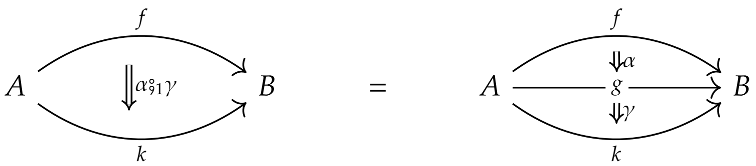
◇

Definition 1.1.5. We call a bicategory $\mathscr{B}$ a **2-category** if all associativity constraints, all left unitality constraints and all right unitality constraints are identities. ◇

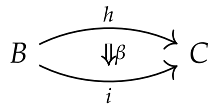
Remark 1.1.6. Let $\mathscr{B}$ be a bicategory. We can compose 2-cells in $\mathscr{B}$ in two different ways. If A and B are 0-cells in $\mathscr{B}$ and if



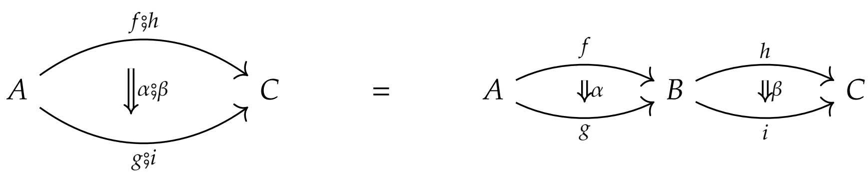
are 2-cells in $\mathscr{B}$, then α and γ are morphisms in the category $\mathscr{B}(A,B)$. We can compose α and γ in $\mathscr{B}(A,B)$ to get another morphism in $\mathscr{B}(A,B)$ which we denote $\alpha \fatsemi_1 \gamma$ because we are composing along a common 1-cell. The following diagrammtic reprsentation of this is called a *pasting diagram*.



The other way of composing 2-cells in $\mathscr{B}$ is by using the functor (1.1.2). Explicitly, if C is another 0-cell in $\mathscr{B}$ and if



is another 2-cell in $\mathscr{B}$, then applying the functor c_{ABC} to the pair (α, β) gives a 2-cell $\alpha \fatsemi \beta$, as shown in the folowing pasting diagram.



Notice that $\alpha \fatsemi \beta$ is the composite of α and β along a common 0-cell. One can therefore use the notation $\fatsemi_0$ to mean $\fatsemi$ so as to clearly distinguish between using $\fatsemi$ to compose along 0-cells and using $\fatsemi_1$ to compose along 1-cells. ◇

Remark 1.1.7. For each tuple A, B, C, D of 0-cells in $\mathscr{B}$, the associativity constraint 2-cells a_{fgh} are the components of a natural isomorphism

$$\begin{array}{ccc} \mathscr{B}(A,B)\times\mathscr{B}(B,C)\times\mathscr{B}(C,D) & \xrightarrow{c_{ABC}\times \mathrm{id}} & \mathscr{B}(A,C)\times\mathscr{B}(C,D) \\ {\scriptstyle \mathrm{id}\times c_{BCD}}\big\downarrow & \overset{a_{ABCD}}{\Longleftarrow} & \big\downarrow{\scriptstyle c_{ACD}} \\ \mathscr{B}(A,B)\times\mathscr{B}(B,D) & \xrightarrow[c_{ABD}]{} & \mathscr{B}(A,D)\end{array}$$

called an **associator**. The naturality axiom for the associator is (1.1.3).

For each pair A, B of 0-cells in $\mathscr{B}$, the left unit constraint 2-cells l_f and the right unit constraint 2-cells r_f are the components of natural isomorphisms

$$\begin{array}{ccccc} & & \mathscr{B}(B,B)\times\mathscr{B}(A,B) & & \\ & \overset{I_B\times\mathrm{id}}{\nearrow} & \Downarrow l_{AB} & \overset{c_{ABB}}{\searrow} & \\ \mathbf{1}\times\mathscr{B}(A,B) & & \xrightarrow[\cong]{\qquad\qquad} & & \mathscr{B}(A,B) \end{array}$$

and

$$\begin{array}{ccccc} & & \mathscr{B}(A,B)\times\mathscr{B}(A,A) & & \\ & \overset{\mathrm{id}\times I_A}{\nearrow} & \Downarrow r_{AB} & \overset{c_{AAB}}{\searrow} & \\ \mathscr{B}(A,B)\times\mathbf{1} & & \xrightarrow[\cong]{\qquad\qquad} & & \mathscr{B}(A,B) \end{array}$$

called the **left unitor** and the **right unitor**. The naturality axioms for the unitors are (1.1.4). ◇

Definition 1.1.8. Let $\mathscr{B}$ and $\mathscr{B}'$ be bicategories. A **lax functor** $F\colon \mathscr{B} \to \mathscr{B}'$ consists of the following data:

- a function $F\colon \operatorname{ob}\mathscr{B} \to \operatorname{ob}\mathscr{B}'$;
- for each pair A, B of 0-cells in $\mathscr{B}$, a functor $F_{AB}\colon \mathscr{B}(A,B) \to \mathscr{B}'(AF, BF)$;
- for each pair $f \in \mathscr{B}(A,B), g \in \mathscr{B}(B,C)$ of 1-cells in $\mathscr{B}$, a 2-cell

  $$fF \fatsemi gF \xrightarrow{F^2_{fg}} (f \fatsemi g)F$$

  called the **lax binary functoriality constraint**;
- for each 0-cell A in $\mathscr{B}$, a 2-cell

  $$\mathrm{id}_{AF} \xrightarrow{F^0_A} \mathrm{id}_A F$$

  called the **lax nullary functoriality constraint**.

These data are required to satisfy the following axioms.

- (**Naturality of F^2**) For all 2-cells $\alpha\colon f \to f'$ in $\mathscr{B}(A,B)$ and $\beta\colon g \to g'$ in $\mathscr{B}(B,C)$, the following diagram commutes.

$$\begin{array}{ccc} fF \fatsemi gF & \xrightarrow{\alpha F \fatsemi \beta F} & f'F \fatsemi g'F \\ {\scriptstyle F^2_{fg}}\big\downarrow & & \big\downarrow{\scriptstyle F^2_{f'g'}} \\ (f \fatsemi g)F & \xrightarrow[(\alpha \fatsemi \beta)F]{} & (f' \fatsemi g')F \end{array}$$

- (**Lax associativity**) For all 1-cells $f \in \mathscr{B}(A,B)$, $g \in \mathscr{B}(B,C)$, and $h \in \mathscr{B}(C,D)$, the following diagram commutes.

$$\begin{array}{ccc} (fF \fatsemi gF) \fatsemi hF & \xrightarrow{a'_{hF,gF,fF}} & fF \fatsemi (gF \fatsemi hF) \\ {\scriptstyle F^2_{fg} \fatsemi \mathrm{id}_{hF}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_{fF} \fatsemi F^2_{gh}} \\ (f \fatsemi g)F \fatsemi hF & & fF \fatsemi (g \fatsemi h)F \\ {\scriptstyle F^2_{f\fatsemi g,h}}\big\downarrow & & \big\downarrow{\scriptstyle F^2_{f,g\fatsemi h}} \\ ((f \fatsemi g) \fatsemi h)F & \xrightarrow[a_{fgh}F]{} & (f \fatsemi (g \fatsemi h))F \end{array} \tag{1.1.9}$$

- (**Lax left unitality**) For all 1-cells $f \in \mathscr{B}(A,B)$, the following diagram commutes.

$$\begin{array}{ccc} \mathrm{id}_{AF} \fatsemi fF & = & \mathrm{id}_{AF} \fatsemi fF \\ {\scriptstyle F^0_A \fatsemi \mathrm{id}_{fF}}\big\downarrow & & \\ \mathrm{id}_A F \fatsemi fF & & \\ {\scriptstyle F^2_{\mathrm{id}_A,f}}\big\downarrow & & \big\downarrow{\scriptstyle l_{fF}} \\ (\mathrm{id}_A \fatsemi f)F & & \\ {\scriptstyle l_f F}\big\downarrow & & \\ fF & = & fF \end{array} \tag{1.1.10}$$

- (**Lax right unitality**) For all 1-cells $f \in \mathscr{B}(A,B)$, the following diagram commutes.

$$\begin{array}{ccc} fF \fatsemi \mathrm{id}_{BF} & = & fF \fatsemi \mathrm{id}_{BF} \\ {\scriptstyle \mathrm{id}_{fF} \fatsemi F^0_B}\big\downarrow & & \\ fF \fatsemi \mathrm{id}_B F & & \\ {\scriptstyle F^2_{f,\mathrm{id}_B}}\big\downarrow & & \big\downarrow{\scriptstyle r'_{fF}} \\ (f \fatsemi \mathrm{id}_B)F & & \\ {\scriptstyle r_f F}\big\downarrow & & \\ fF & = & fF \end{array} \tag{1.1.11}$$

◇

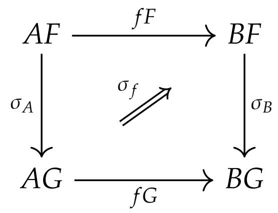
Definition 1.1.12. We call a lax functor $F\colon \mathscr{A} \to \mathscr{B}$ a **pseudofunctor** if all lax binary functoriality constraints and all lax nullary functoriality constraints are invertible. In this case, we call these constraints **pseudofunctoriality constraints**. We call a lax functor between two 2-categories a **strict 2-functor** if all lax binary functoriality constraints and all lax nullary functoriality constraints are idenitites. ⋄

Definition 1.1.13. Let F and G be lax functors $\mathscr{B} \to \mathscr{B}'$. A **lax transformation** $\sigma\colon F \to G$ consists of the following data:

- for each 0-cell A in $\mathscr{B}$, a 1-cell

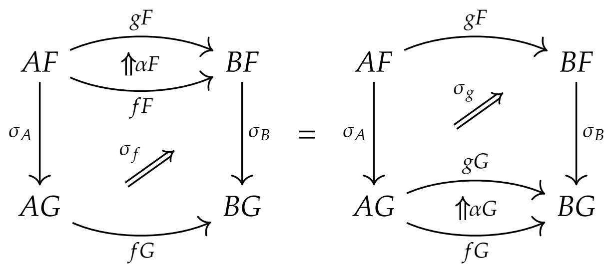
$$\sigma_A\colon AF \to AG;$$

- for each 1-cell $f \in \mathscr{B}(A, B)$, a 2-cell

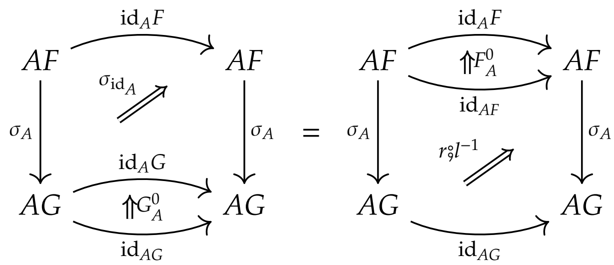


These data are required to satisfy the following axioms.

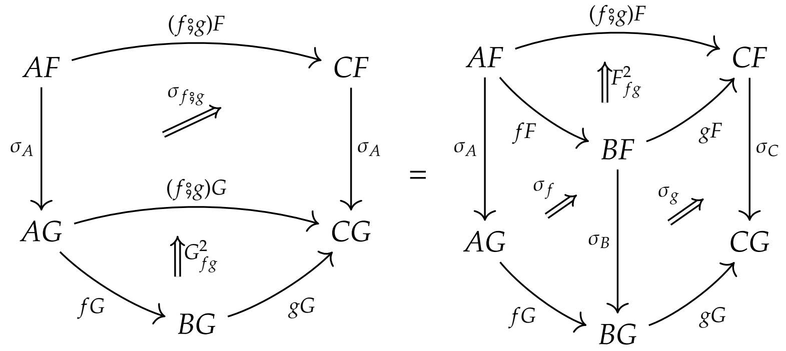
- (**Naturality**) For all 2-cells $\alpha\colon f \to g$ in $\mathscr{B}(A, B)$, the following following equality holds.



- (**Lax unitality**) For all 1-cells $f \in \mathscr{B}(A, B)$, the following equality holds.



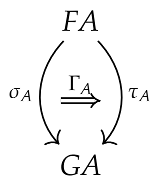
- (**Lax multiplicativity**) For all 1-cells $f \in \mathscr{B}(A, B)$ and $g \in \mathscr{B}(B, C)$, the following equality holds.



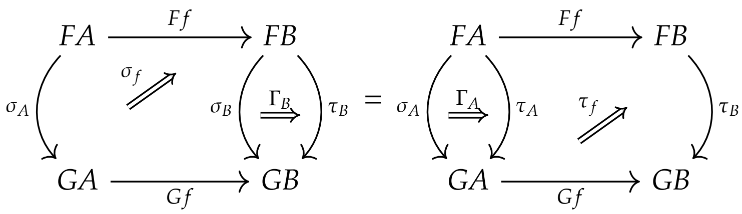
◇

Definition 1.1.14. We call a lax transformation $\sigma\colon F \to G$ a **pseudonatural transformation** if, for every 1-cell f in $\mathscr{B}$, the 2-cell σ_f is invertible. We call a lax transformation $\sigma\colon F \to G$ between strict 2-functors a **strict 2-transformation** if, for every 1-cell f in $\mathscr{B}$, the 2-cell σ_f is an identity. ◇

Definition 1.1.15. Let F and G be lax functors $\mathscr{B} \to \mathscr{B}'$, and suppose that σ and τ are lax transformations $F \to G$. A **modification** $\Gamma\colon \sigma \to \tau$ consists of a 2-cell



for each object A in $\mathscr{B}$. These 2-cells must satisfy the following equality for all 1-cells $f \in \mathscr{B}(A, B)$ in $\mathscr{B}$.



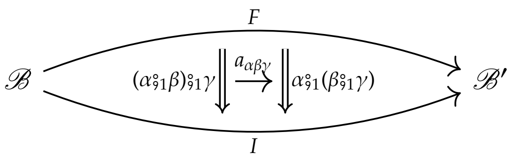
◇

Definition 1.1.16. Let $\mathscr{B}$ and $\mathscr{B}'$ be bicategories. Define the **bicategory of lax functors** $\mathsf{Bicat}(\mathscr{B}, \mathscr{B}')$ by the following data.

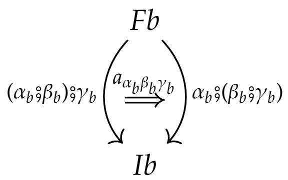
- The 0-cells of $\mathsf{Bicat}(\mathscr{B}, \mathscr{B}')$ are the lax functors $\mathscr{B} \to \mathscr{B}'$.
- For lax functors $F, G\colon \mathscr{B} \to \mathscr{B}'$, the category $\mathsf{Bicat}(\mathscr{B}, \mathscr{B}')(F, G)$ has
  - the lax transformations $F \to G$ as objects,

- the modifications $\alpha \to \beta$ between such lax transformations as morphisms,
- vertical composition of modifications as composition,
- identity modifications as identity morphisms.

- For each lax functor $F\colon \mathscr{B} \to \mathscr{B}'$, the identity 1-cell $\mathbf{id}_F$ is the identity lax transformation $F \to F$.
- Composition is given by $\delta \fatsemi_1 \varepsilon$ on lax transformations and by $\Gamma \fatsemi_1 \Delta$ on modifcations.
- For lax transformations $\alpha\colon F \to G, \beta\colon G \to H$ and $\gamma\colon H \to I$ between lax functors $F, G, H\colon \mathscr{B} \to \mathscr{B}'$, the associator constraint is the modifcation

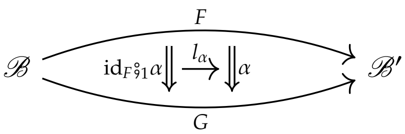


consisting of, for each 0-cell $b \in \mathscr{B}$, the 2-cell

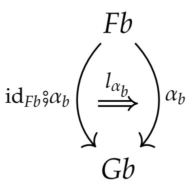


in $\mathscr{B}'$.

- For each lax transformation $\alpha\colon F \to G$ between lax functors $F, G\colon \mathscr{B} \to \mathscr{B}'$, the left unitor constraint is the modifcation



consisting of, for each 0-cell $b \in \mathscr{B}$, the 2-cell



in $\mathscr{B}'$.

◇

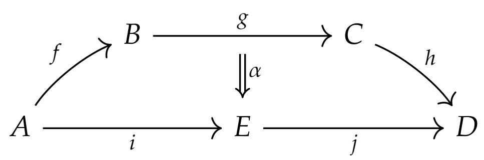
Definition 1.1.17. Let $\mathscr{B}$ and $\mathscr{B}'$ be bicategories. Define the **bicategory of pseudofunctors $\mathsf{Bicat}^{\mathrm{ps}}(\mathscr{B}, \mathscr{B}')$** to be the sub-bicategory of $\mathsf{Bicat}(\mathscr{B}, \mathscr{B}')$ having

- the pseudofunctors $\mathscr{B} \to \mathscr{B}'$ as 0-cells,
- the pseudonatural transformations between such pseudofunctors as 1-cells, and
- the modifications between such pseudonatural transformations as 2-cells.

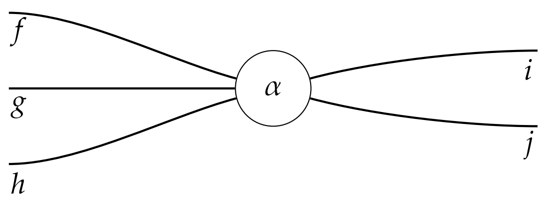
◇

1.2 String diagrams

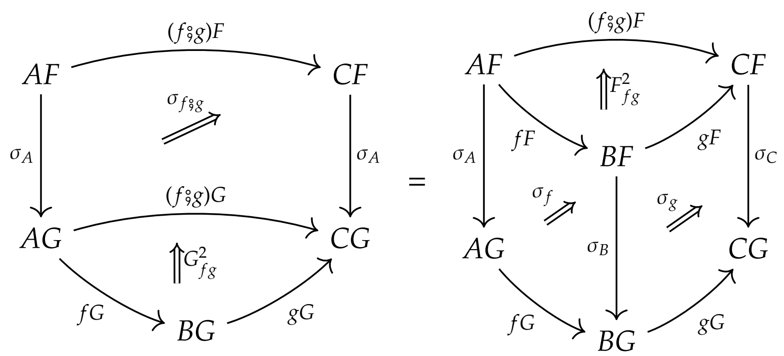
In this section, we introduce the string diagram notation used for 2-cells in a bicategory. As we saw in the introduction, a 2-cell



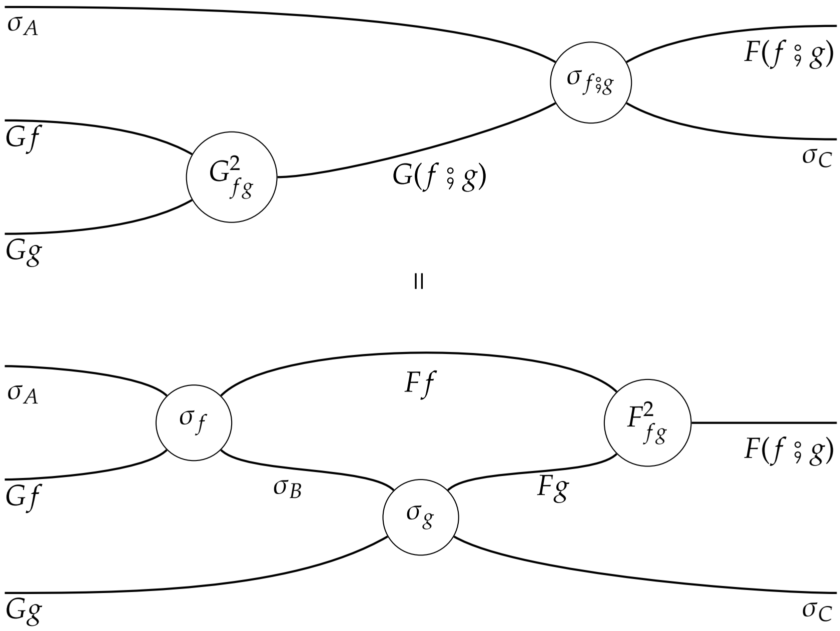
in a bicategory $\mathscr{B}$ is represented using string diagram notation as



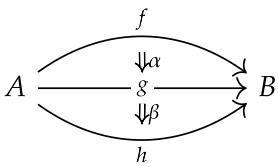
We also saw the lax multiplicativity axiom



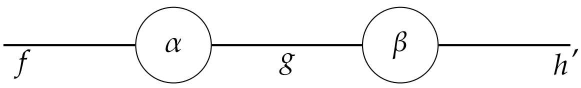
from Definition 1.1.8 represented using a string diagram in the bicategory **Cat** as follows.



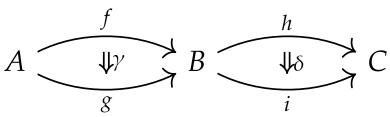
Notice that the composition of 2-cells along a 1-cell



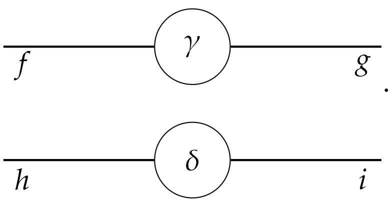
is represented by



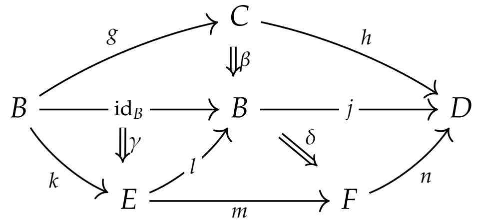
and the composition of 2-cells along a 0-cell



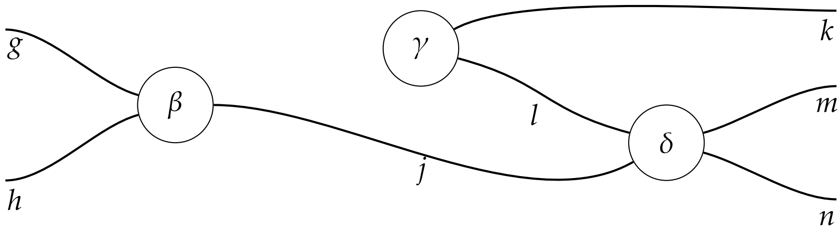
is represented by



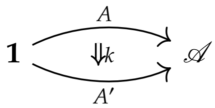
A convention that we will use is to omit drawing identity 1-cells. For example, the pasting diagram



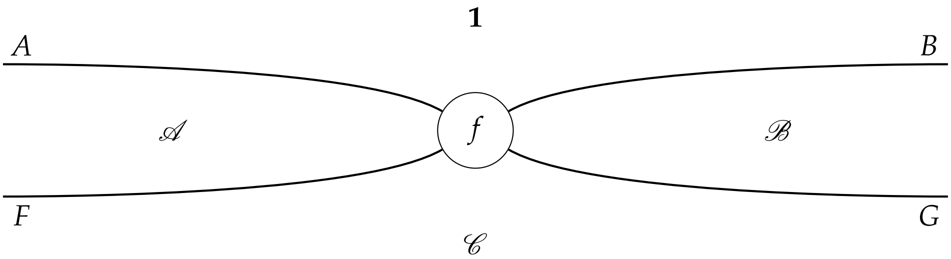
is depicted using string diagrams as



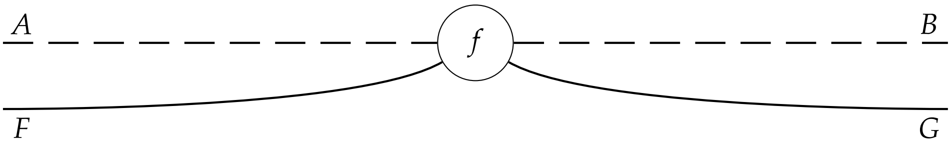
We can also use string diagrams to represent morphisms in a category. An object A in category $\mathscr{A}$ can be thought of as a functor $\mathbf{1} \to \mathscr{A}$ and a morphism $k\colon A \to A'$ in $\mathscr{A}$ can be thought of as a natural transformation



where $\mathbf{1}$ denotes the category with one object and one morphism. With this perspective a string diagram in a category is precisely a string diagram in the bicategory **Cat**. For example, if $F\colon \mathscr{A} \to \mathscr{C}$ and $G\colon \mathscr{B} \to \mathscr{C}$ are functors, then a morphism $f\colon AF \to BG$ in $\mathscr{C}$ is represented using a string diagram in the bicategory **Cat** as



where we have labelled the categories. To clearly distinguish string diagrams in a category from string diagrams in a bicategory (and to avoid labelling the cateogry $\mathbf{1}$ each time) we use the following notation for the morphism $f\colon AF \to BG$ in $\mathscr{C}$.

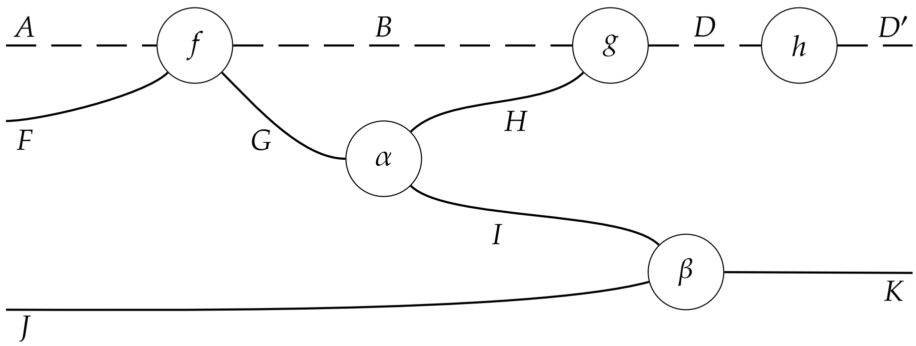


This is the same notation used by Myers in [Mye23].

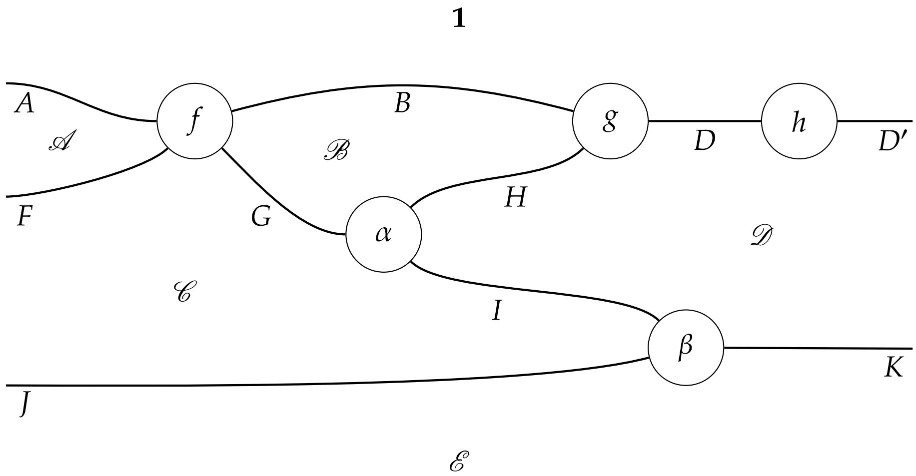
We can introduce natural transformations. Let $H\colon \mathscr{C} \to \mathscr{D}$, $I\colon \mathscr{B} \to \mathscr{D}$, $J\colon \mathscr{E} \to \mathscr{B}$ and $K\colon \mathscr{E} \to \mathscr{D}$ be functors, and let $\alpha\colon G \to H \mathbin{⨾} I$ and $\beta\colon I \mathbin{⨾} J \to K$ be natural transformations. Then the morphism

$$AFJ \xrightarrow{f} BGJ \xrightarrow{\alpha_B J} BHIJ \xrightarrow{gJ} DIJ \xrightarrow{\beta_D} DK \xrightarrow{h} D'K \tag{1.2.1}$$

is depicted using strings diagrams as



As a string diagram in the bicategory **Cat**, the morphism (1.2.1) would be represented as

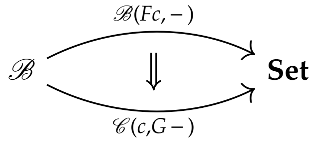


1.3 Adjunctions

Definition 1.3.1. An **adjunction** is a tuple $(\mathscr{C}, \mathscr{B}, F, G, \phi)$ consisting of categories $\mathscr{C}$ and $\mathscr{B}$, functors $F\colon \mathscr{C} \to \mathscr{B}$ and $G\colon \mathscr{B} \to \mathscr{C}$ and a collection of bijections

$$\phi = \left(\phi_{cb}\colon \mathscr{B}(Fc, b) \to \mathscr{C}(c, Gb)\right)_{c\in\mathscr{C}, b\in\mathscr{B}}$$

natural in c and b. By 'natural in c' we mean that, for each $c \in \mathscr{C}$, the morphisms $(\phi_{cb})_{b \in \mathscr{B}}$ are the components of a natural transformation



and similar for 'natural in b', where **Set** denotes the category of sets and maps between them. We will often leave the categories implied and give an adjunction as a triple (F, G, ϕ). ◇

Definition 1.3.2. Let $F\colon \mathscr{C} \to \mathscr{B}$ and $G\colon \mathscr{B} \to \mathscr{C}$ be functor, and let (F, G, ϕ) be an adjunction. The **adjunct** $\overline{h}$ of a morphism $h\colon Fc \to b$ in $\mathscr{B}$ is the morphism $(h)\phi_{cb}\colon c \to Gb$ in $\mathscr{C}$. Also, the **adjunct** $\overline{k}$ of a morphism $k\colon c \to Gb$ in $\mathscr{C}$ is the morphism $(k)\phi_{cb}^{-1}\colon Fc \to b$ in $\mathscr{B}$. ◇

Definition 1.3.3. Let $F\colon \mathscr{C} \to \mathscr{B}$ and $G\colon \mathscr{B} \to \mathscr{C}$ be functors. We say that F is **left adjoint** to G, and that G is **right adjoint** to F, if there exists an adjunction of the form (F, G, ϕ). In this case we write $F \dashv G$. ◇

Definition 1.3.4. Let $F\colon \mathscr{C} \to \mathscr{B}$ and $G\colon \mathscr{B} \to \mathscr{C}$ be functors and let (F, G, ϕ) be an adjunction. The **unit** of the adjunction (F, G, ϕ) is the natural transformation $\eta\colon \mathrm{id}_{\mathscr{C}} \to F \mathbin{⨾} G$ whose component, for each c in $\mathscr{C}$, is the morphism

$$\eta_c = \overline{\mathrm{id}_{Fc}}\colon c \to cFG.$$

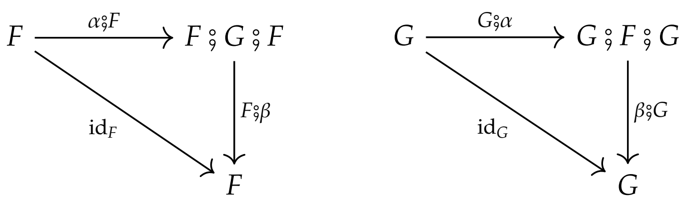
The **counit** of the adjunction $F \dashv G$ is the natural transformation $\varepsilon\colon G \mathbin{⨾} F \to \mathrm{id}_{\mathscr{B}}$ whose component, for each b in $\mathscr{B}$, is the morphism

$$\varepsilon_b = \overline{\mathrm{id}_{Gb}}\colon bGF \to b.$$

◇

The remainder of this section discusses an equivalent definition of adjunction using the unit and counit. This was done first by Huber [Hub61, Theorem 4.1], and Borceux [Bor94a, Chapter 3] is more modern textbook reference.

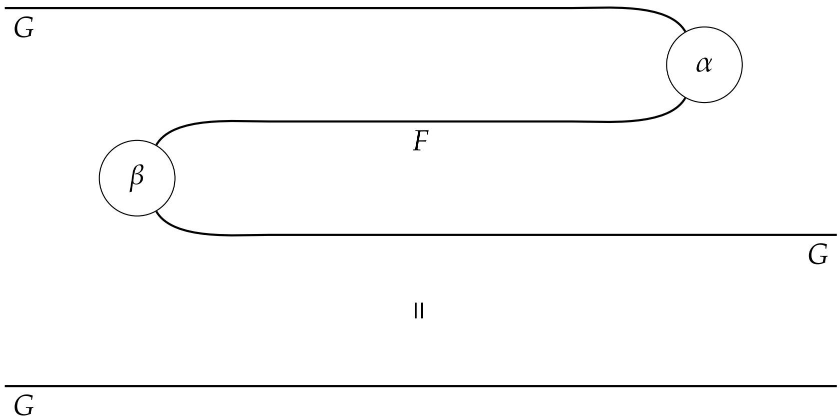
Definition 1.3.5. Let $F\colon \mathscr{C} \to \mathscr{B}$ and $G\colon \mathscr{B} \to \mathscr{C}$ be functors, and let $\alpha\colon \mathrm{id}_{\mathscr{C}} \to F \mathbin{⨾} G$ and $\beta\colon G \mathbin{⨾} F \to \mathrm{id}_{\mathscr{B}}$ be natural transformations. We say that α and β satisfy the **snake identities** if the following diagrams (in the functor categories $\mathbf{Cat}(\mathscr{C}, \mathscr{B})$ and $\mathbf{Cat}(\mathscr{B}, \mathscr{C})$ respectively) commute.



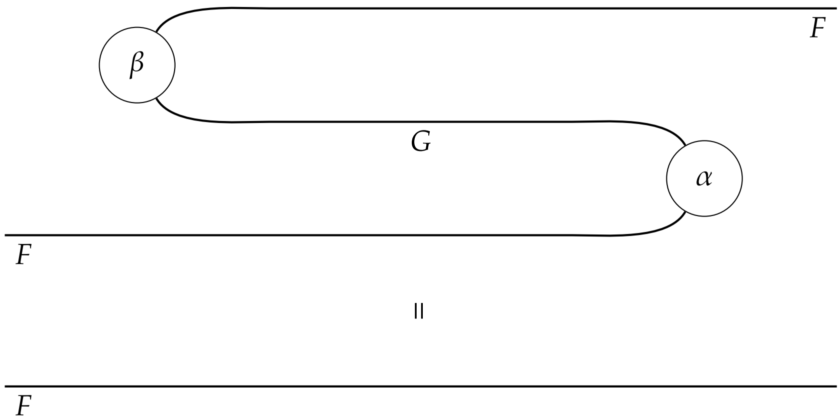
(1.3.6)

◇

If we represent the equalities in (1.3.6) using string diagrams, then we see why they're called the snake identities:



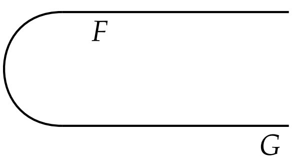
and



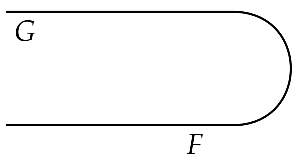
We will sometimes use the work '*yank*' to refer to the application of the snake identities; this terminology comes from thinking about the string diagrams as being literal strings which can be yanked to give a taught, straight piece of string.

Proposition 1.3.7. *Let* $F\colon \mathscr{C} \to \mathscr{B}$ *and* $G\colon \mathscr{B} \to \mathscr{C}$ *be functors, and let* (F, G, ϕ) *be an adjunction. Then the unit and counit of this adjunction satisfy the snake identities.*

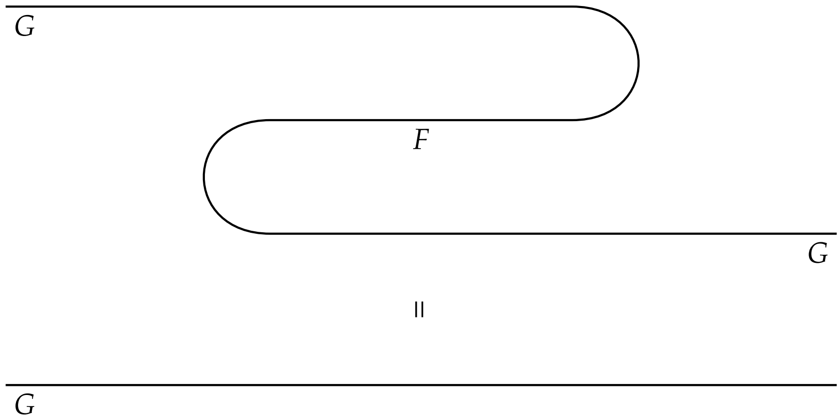
Notation 1.3.8. When drawing the unit and counit of an adjunction (F, G, ϕ) in a string diagram, we will often not write its name—coupled with the convention of not drawing identities (see), the unit and counit will be be drawn as a 'cup' and 'cap':



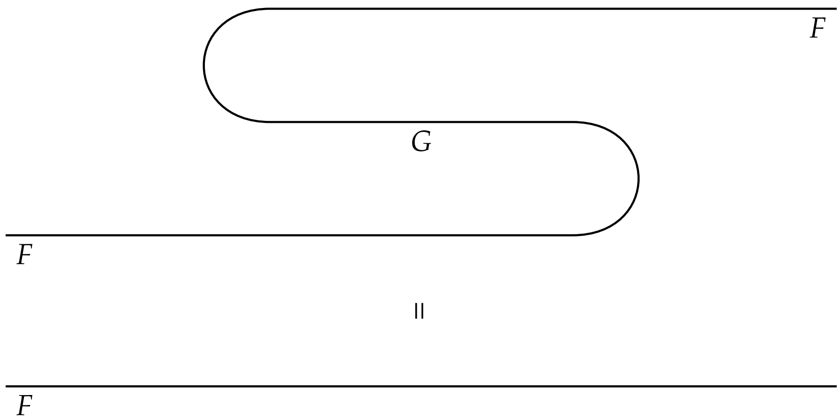
and



For example, with this notation the snake identities are draw as



and



◇

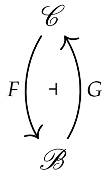
Proposition 1.3.9 ([Bor94a][Theorem 3.1.5]). *Let $F\colon \mathscr{C} \to \mathscr{B}$ and $G\colon \mathscr{B} \to \mathscr{C}$ be functors. There is a bijective correspondence between adjunctions of the form (F, G, ϕ) and tuples $(F, G, \eta, \varepsilon)$ such that $\eta\colon \mathrm{id}_{\mathscr{C}} \to F \fatsemi G$ and $\varepsilon\colon G \fatsemi F \to \mathrm{id}_{\mathscr{B}}$ are natural transformations that satisfy the snake identities.*

Proof. The bijection maps each adjunction (F, G, ϕ) to the tuple $(F, G, \eta, \varepsilon)$ where η and ε are the unit and counit. The inverse of this bijection maps each tuple $(F, G, \eta, \varepsilon)$ to the adjunction (F, G, ϕ) defined as follows.

$$\begin{aligned} \phi\colon \mathscr{B}(Fc, b) &\longrightarrow \mathscr{C}(c, Gb) :\psi \\ x &\longmapsto \eta_c \fatsemi xG \\ yF \fatsemi \varepsilon_b &\longleftarrow y \end{aligned}$$

□

Notation 1.3.10. We can now write an adjunction as either a tuple $(F, G\phi)$ or as a tuple $(F, G, \eta, \varepsilon)$. Sometimes we won't explicitly mention ϕ, η or ε, and we'll just say 'let $F \dashv G$ be an adjunction'. When we want to be particularly clear about the source and target of the functors F and G, it can be helpful to draw an adjunction $F \dashv G$ as follows.



◇

1.3.1 Mates

Maclane discusses conjugate transformations of adjoints in [Mac71], but the first explicit mention of mates is by Kelly and Ross [KS74]. We state the definitions and results in this section for a general bicategory, but this thesis will mostly apply these only to the bicategory **Cat**.

Adjunctions in 2-categories were introduced by Maranda [Mar65], and soon after Kelly [Kel69] studied them further using what is now modern terminology.

Definition 1.3.11. Let $\mathscr{B}$ be a bicategory.

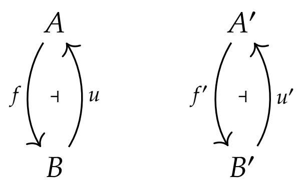
- An **adjunction** in $\mathscr{B}$ is a tuple $(f, u, \eta, \varepsilon)$ consisting of 1-cells $f\colon A \to B$ and $u\colon B \to A$ in $\mathscr{B}$ and 2-cells $\eta\colon \mathrm{id}_A \to f \fatsemi u$ and $\varepsilon\colon u \fatsemi f \to \mathrm{id}_B$ in $\mathscr{B}$ such that the snake identities hold.

- An **equivalence** in $\mathscr{B}$ is a tuple $(f, u, \eta, \varepsilon)$ consisting of 1-cells $f\colon A \to B$ and $u\colon B \to A$ in $\mathscr{B}$ and invertible 2-cells $\eta\colon \mathrm{id}_A \to f \fatsemi u$ and $\varepsilon\colon u \fatsemi f \to \mathrm{id}_B$ in $\mathscr{B}$.

- An **adjoint equivalence** in $\mathscr{B}$ is a tuple $(f, u, \eta, \varepsilon)$ that is an adjunction and an equivalence.

◇

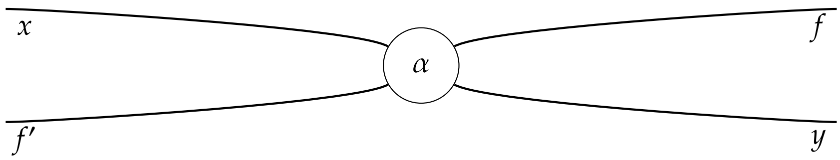
Proposition 1.3.12. *Let $\mathscr{B}$ be a bicatgory, and let $(f, u, \eta, \varepsilon)$ and $(f', u', \eta', \varepsilon')$ be adjunctions in $\mathscr{B}$ as shown in the following diagram.*



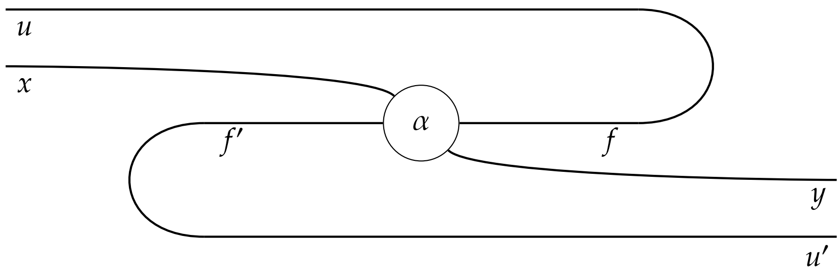
For each pair of 1-cells $x\colon A \to A'$ and $y\colon B \to B'$ in $\mathscr{B}$, the map

$$\mu\colon \mathscr{B}(A, B')(x \fatsemi f', f \fatsemi y) \longrightarrow \mathscr{B}(B, A')(u \fatsemi x, y \fatsemi u')$$

defined by



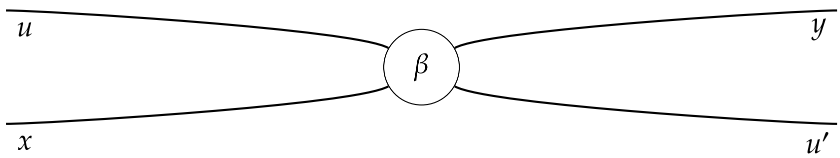
$$\mapsto$$



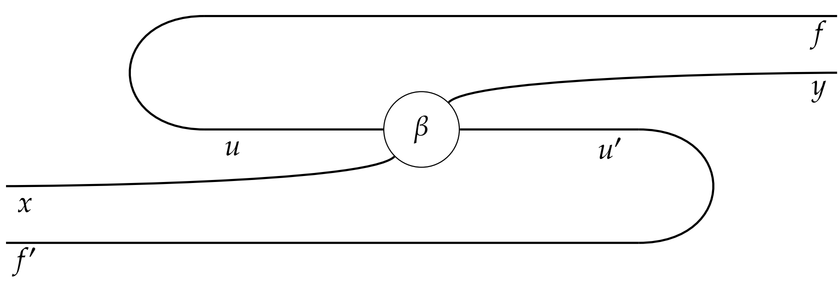
is a bijection with inverse

$$\mu^{-1}\colon \mathscr{B}(B,A')(u \mathbin{;} x, y \mathbin{;} u') \longrightarrow \mathscr{B}(A,B')(x \mathbin{;} f', f \mathbin{;} y)$$

given by

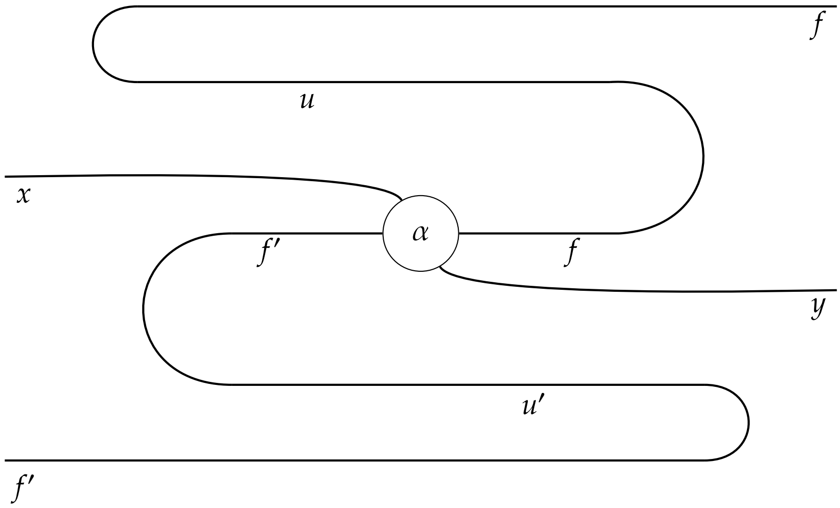


$$\mapsto$$



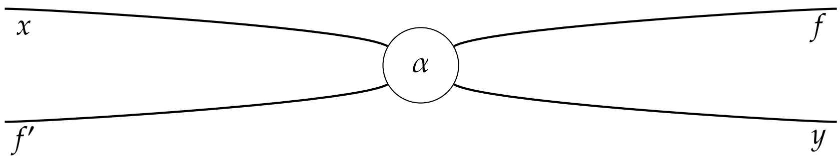
.

Proof. Let $\alpha \in \mathscr{B}(A, B')(x \mathbin{;} f', f \mathbin{;} y)$. Then the two-cell $(\alpha)\mu^{-1}\mu$ is equal to



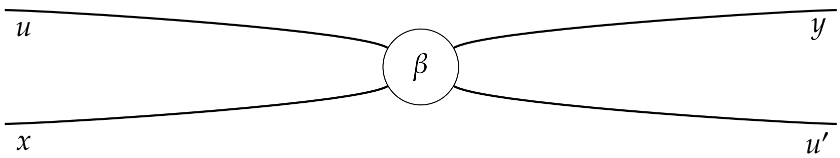
which is equal to α by the snake identities. The proof that $(\alpha)\mu\mu^{-1}$ is similar. □

Definition 1.3.13. If



(1.3.14)

and



(1.3.15)

are 2-cells in $\mathscr{B}$ that correspond to each other under the bijection in Proposition 1.3.12, then we say that β **is the mate of** α, that α is the mate of β and that α and β are **mates**. ◇

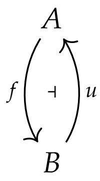
Definition 1.3.16. Let $\mathscr{B}$ be a bicategory. The 2-category $\mathbf{Adj}_{\mathscr{B}}$ of adjucntions in $\mathscr{B}$ is defined as follows.

- The collection of objects in $\mathbf{Adj}_{\mathscr{B}}$ is $\operatorname{ob} \mathbf{Adj}_{\mathscr{B}} = \operatorname{ob} \mathscr{B}$.
- The category $\mathbf{Adj}_{\mathscr{B}}(A, B)$ has
  - as objects adjunctions $(f, u, \eta, \varepsilon)$ in $\mathscr{B}$ such that $f \colon A \to B$ and $u \colon B \to A$ and
  - as morphisms $(f, u, \eta, \varepsilon) \to (f', u', \eta', \varepsilon')$ tuples (x, y, α, β) consisting of 1-cells $x \colon A \to A'$ and $y \colon B \to B'$ in $\mathscr{B}$ and 2-cells α and β in $\mathscr{B}$ of the form (1.3.14) and (1.3.15) such that α and β are mates.

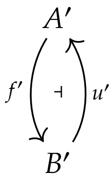
We write **AdjCat** for the 2-category $\mathbf{Adj}_{\mathbf{Cat}}$ of adjunctions in **Cat**. ◇

We write Riehl's proof of the following result using string diagrams.

Proposition 1.3.17 ([RV22, B.3.10]). *Let $\mathscr{B}$ be a bicategory, and let*



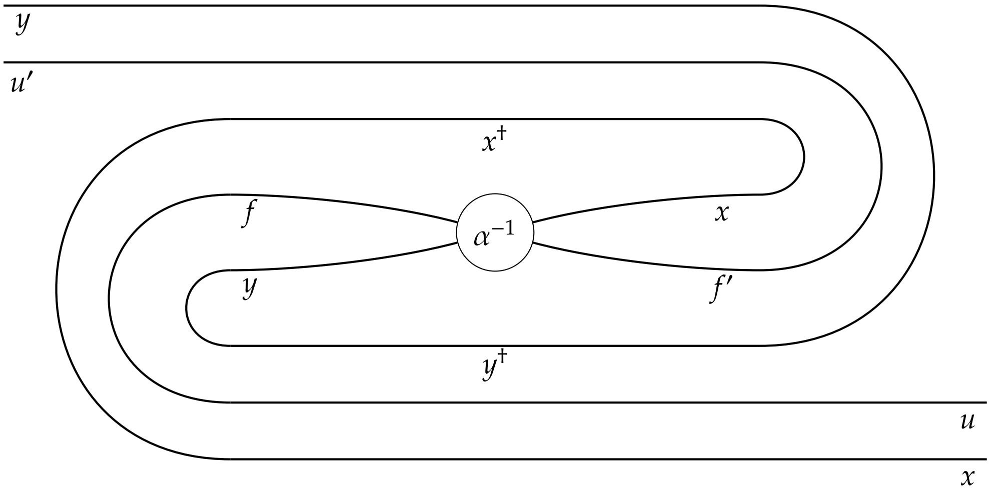
and



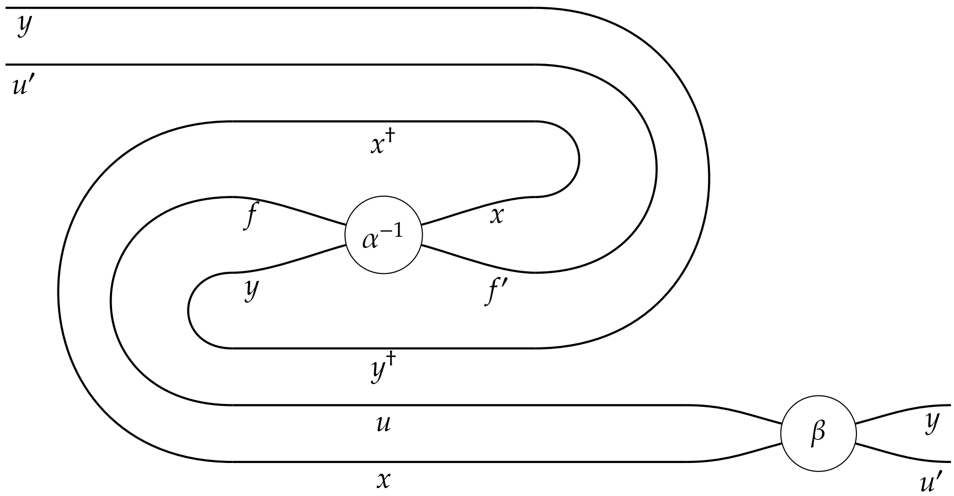
be adjunctions in $\mathscr{B}$, and suppose that α and β are 2-cells in $\mathscr{B}$ of the form (1.3.14) *and* (1.3.15) *that are mates with respect to the above adjunctions.*

If x and y are equivalences, then α is invertible if, and only if, β is invertible.

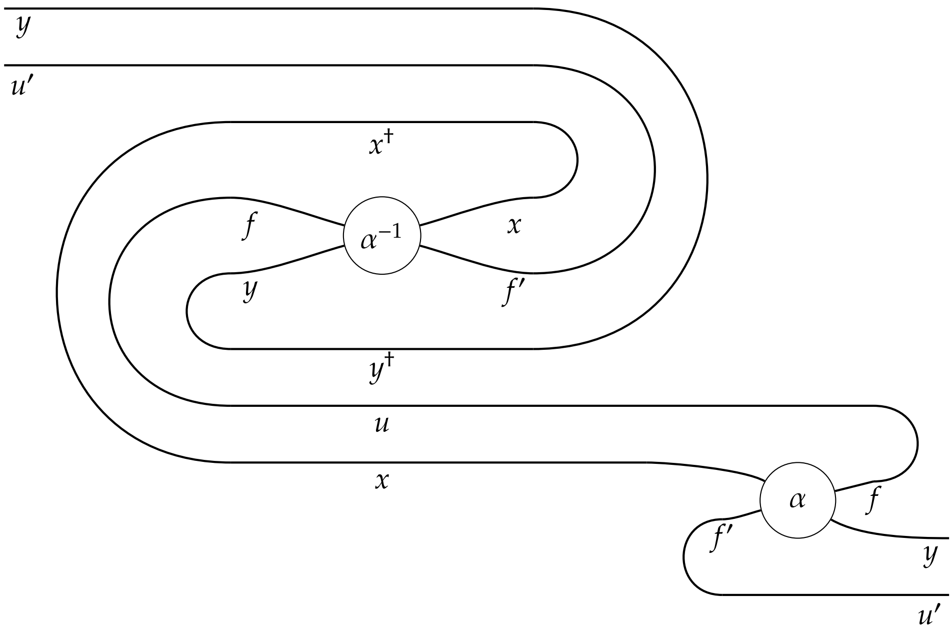
Proof. Suppose that α is invertible. Since x and y are equivalences, there exist adjoint equivalences $(x, x^\dagger, \eta^x, \varepsilon^x)$ and $(y, y^\dagger, \eta^y, \varepsilon^y)$. The following series of equalities shows that the 2-cell



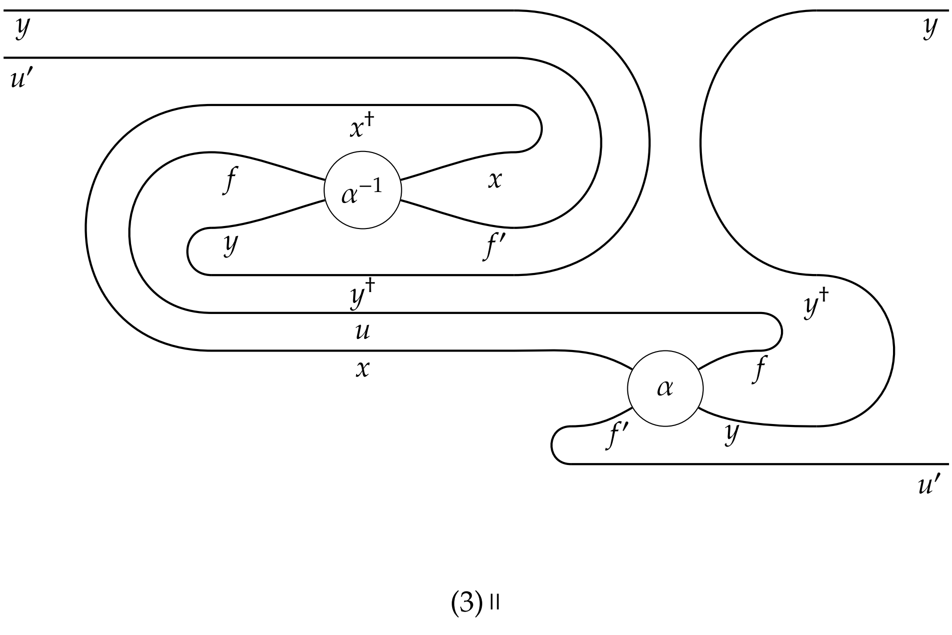
is an inverse to β.



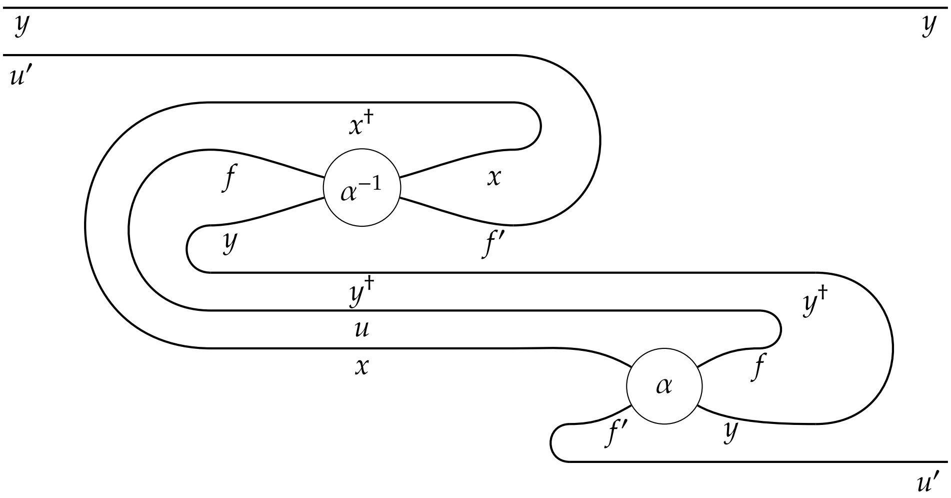
(1) ॥



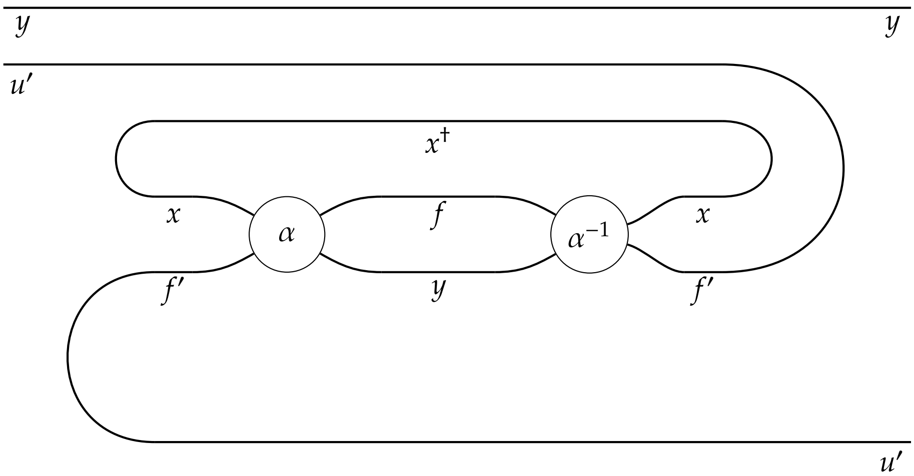
(2) ॥



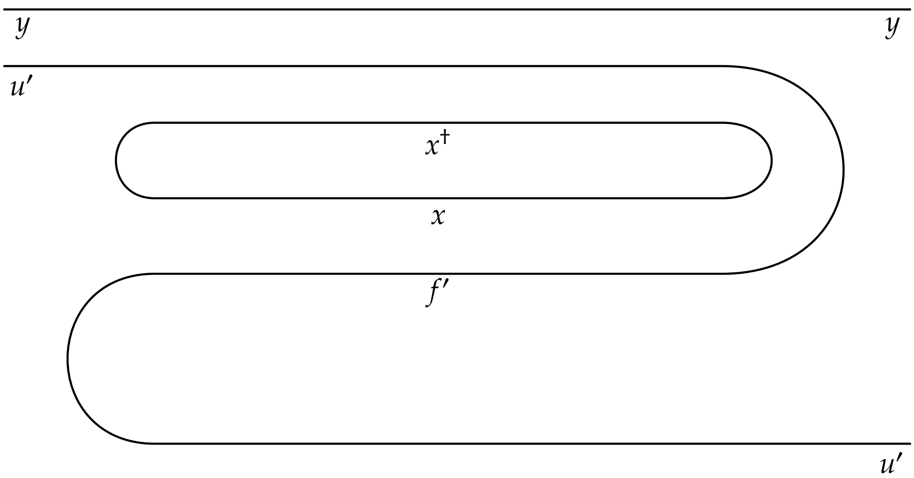
(3) ॥



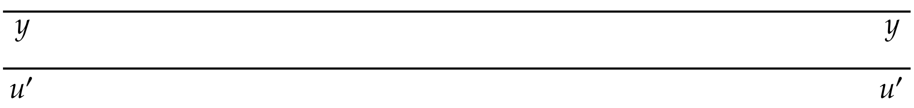
(4) ‖



(5) ‖



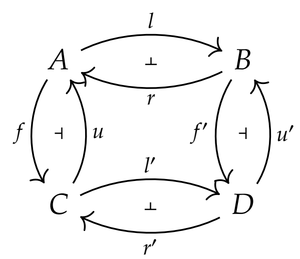
(6) ‖



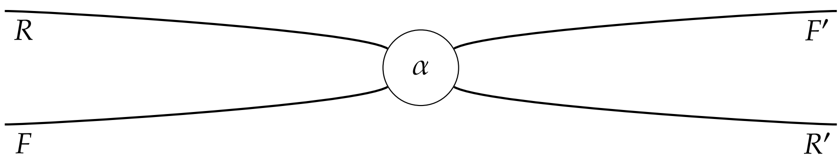
The equality (1) holds because β is the mate of α, the equality (2) holds by the snake identities for the adjunction $(y, y^\dagger, \eta^y, \varepsilon^y)$, the equality (3) holds because η^y and ε^y are part of the adjoint equivalence $(y, y^\dagger, \eta^y, \varepsilon^y)$, the equality (4) holds by the snake identities for the adjunctions $f \dashv u$ and $(y, y^\dagger, \eta^y, \varepsilon^y)$, the equality (5) holds because $\alpha \mathbin{⨾} \alpha^{-1} = \mathrm{id}_{x ⨾ f'}$, and the equality (6) holds by the snake identities for the adjunction $f' \dashv u'$ and because η^x and ε^x are part of the equivalence $(x, x^\dagger, \eta^x, \varepsilon^x)$. □

We prove the following lemma which gives a common situation where one mate is an isomorphism if, and only if, the other is.

Lemma 1.3.18. *Let $\mathscr{B}$ be a bicategory, suppose we have the adjunctions*

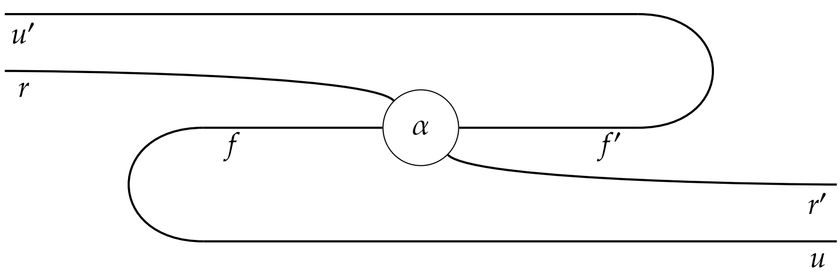


in $\mathscr{B}$, and let the following be a 2-cell in $\mathscr{B}$.

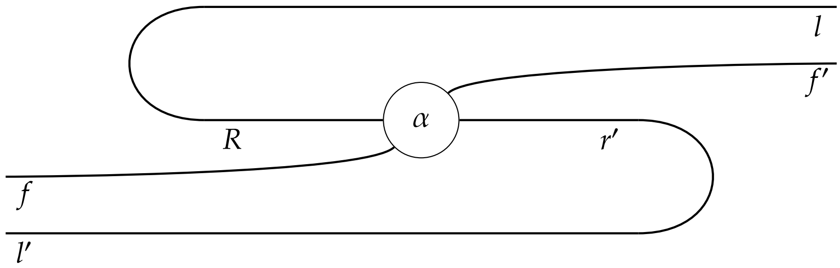


Then, the 2-cell

(1.3.19)



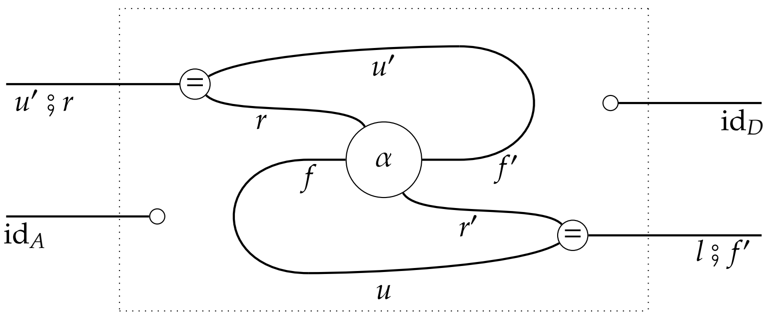
is invertible if, and only if, the 2-cell



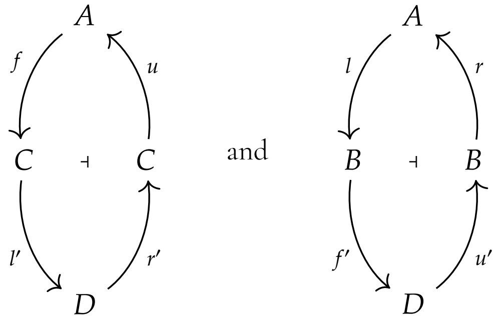
(1.3.20)

is invertible.

Proof. We can rewrite (1.3.19) as follows.

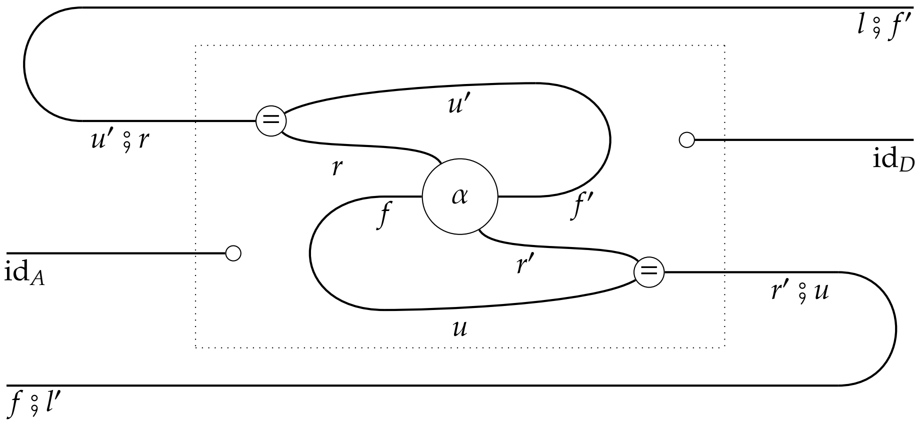


Take the mate of this with respect to the adjunctions



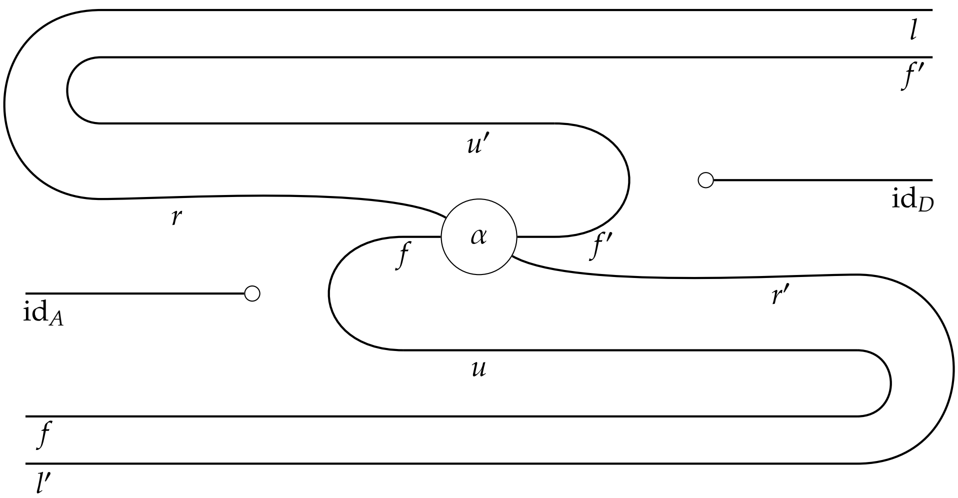
(1.3.21)

to get

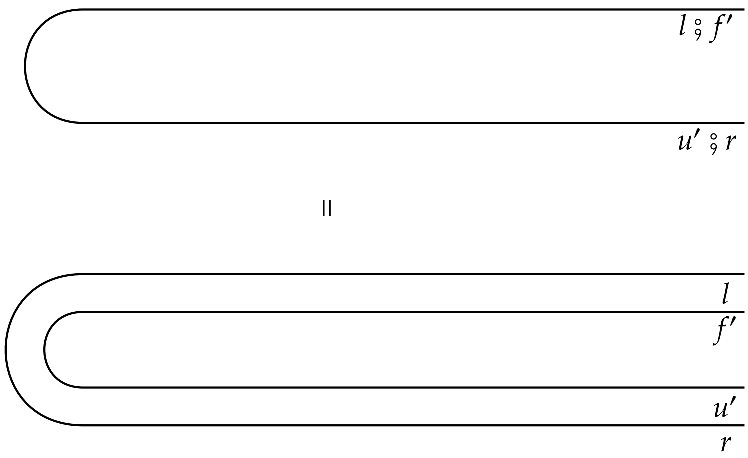


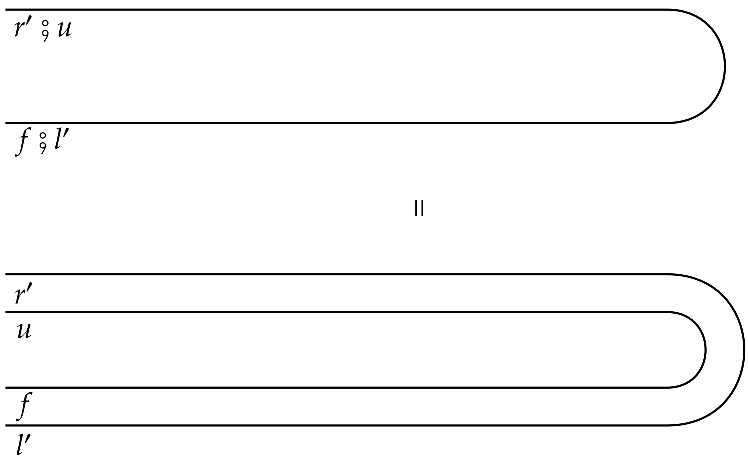
(1.3.22)

By Proposition 1.3.17, the 2-cell (1.3.22) is invertible if, and only if, the 2-cell (1.3.19) is invertible. Therefore, it suffices to show that the 2-cell (1.3.22) is equal to the 2-cell (1.3.20). Consider the 2-cell



This 2-cell is equal to the 2-cell (1.3.19) via the snake identities, and it is equal to the 2-cell (1.3.22) via the following two identities satisfied by the unit and counit of the composite adjunctions (1.3.21).





□

1.4 Monoidal bicategories

In this section, we sketch some basic definitions and constructions relating to monoidal bicategories. References where explicit axioms can be found are [GPS95] and [Gur06].

Definition 1.4.1. A **monoidal bicatgory** consists of the following data:

- a bicategory $\mathscr{B}$;

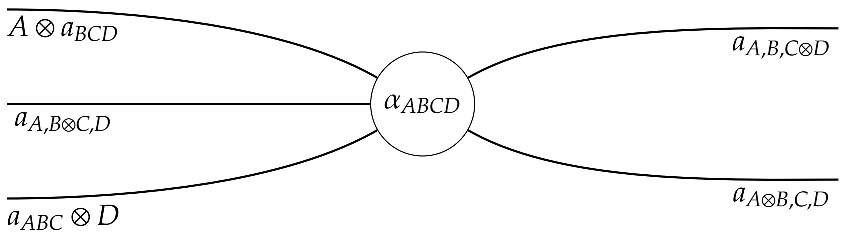
- a pseudofunctor $\otimes\colon \mathscr{B} \times \mathscr{B} \to \mathscr{B}$;
- a 0-cell I in $\mathscr{B}$ called the **unit 0-cell**;
- for each triple A, B, C of 0-cells in $\mathscr{B}$, an adjoint equivalence

$$a_{ABC}\colon (A \otimes B) \rightleftarrows CA \otimes (B \otimes C) : a^{\bullet}_{ABC}$$

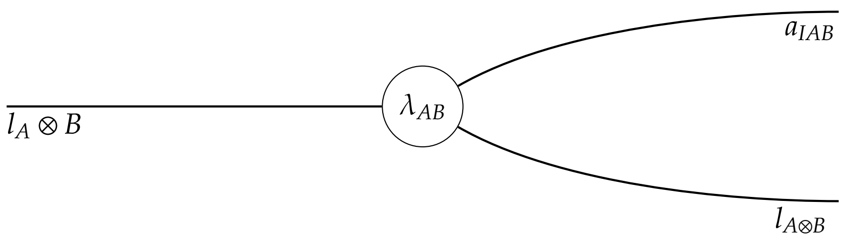
  whose constituent 1-cells are called **monoidal associativity constraints**;
- for each 0-cell A in $\mathscr{B}$, adjoint equivalences

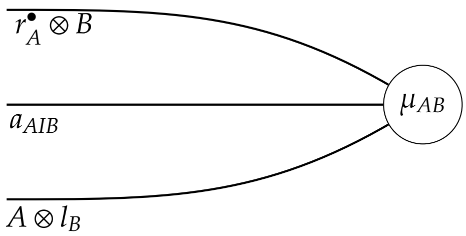
$$l_A\colon I \otimes A \rightleftarrows A : l^{\bullet}_A \quad \text{and} \quad r_A\colon A \otimes I \rightleftarrows A : r^{\bullet}_A$$

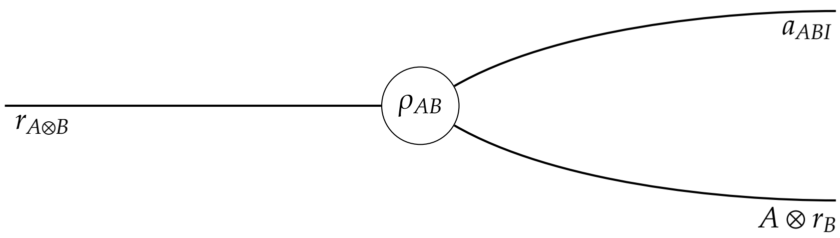
  whose constituent 1-cells are called **monoidal unitality constraints**;
- for each tuple A, B, C, D of 0-cells in $\mathscr{B}$, a 2-cell



  called a **monoidal associativity 2-constraint**;
- for each pair A, B of 0-cells in $\mathscr{B}$, invertible 2-cells







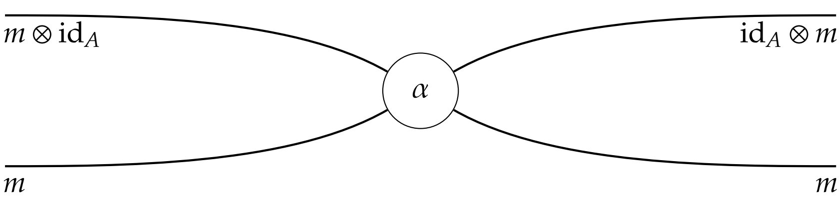
called **monoidal unitality 2-constraints**.

These data are required to satisy axioms, which we choose to omit here but can be found in, for example, Stay's paper on compact closed bicategories [Sta16]. ◇

In Sections 2.6 and 2.7, we will study monoidal fibrations and monoidal indexed categories. These are fibrations and indexed categories with added monoidal structure, and one way in which we add a monoidal structure is using **pseudomonoids**. Pseudomonoids were first introduced by Day and Street [DS97].

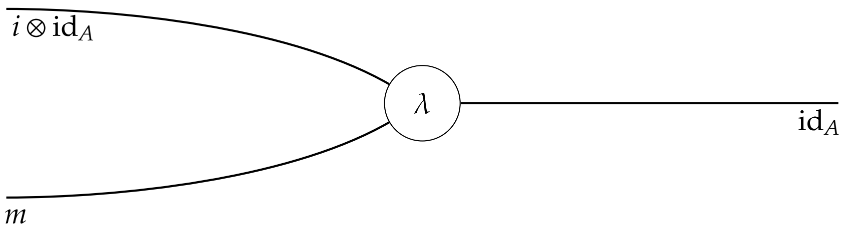
Definition 1.4.2. Let $(\mathscr{B}, \otimes, I)$ be a monoidal bicategory. A **pseudomonoid** in $\mathscr{B}$ consists of the following data:

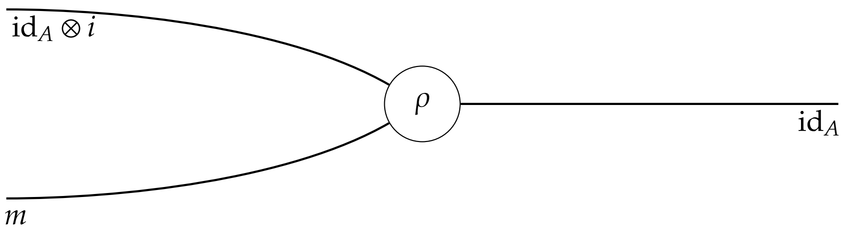
- a 0-cell A in $\mathscr{B}$;
- a 1-cell $m\colon A \otimes A \to A$ called **multiplication**;
- a 1-cell $i\colon I \to A$ called the **unit**;
- an invertible 2-cell



called the **associator**;

- invertible 2-cells





called the left and right **unitors**.

These data are required to satisy axioms, which we choose to omit here but can be found in [DS97, Section 3]. ◇

There are braided and symmetric versions of psedomonoids, definitions of which can be found in [DS97, Sections 4 & 5].

Definition 1.4.3. Let $\mathscr{B}$ be a monoidal bicategory. The 2-category $\mathsf{PsMon}(\mathscr{B})$ of pseudomonoids consists of pseudomonoids, strong pseudomonoid 1-cells and pseudomonoid 2-cells. ◇

There are 2-categories of braided pseudomonoids and of symmetric pseudomonoids which we denote by $\mathsf{BrPsMon}(\mathscr{B})$ and $\mathsf{SymPsMon}(\mathscr{B})$ respectively.

Chapter 2

Fibrations and Indexed Categories

In this chapter, we study the objects from which the construction of fibrant double categories by Shulman start: fibrations.

The first section is dedicated to defining cartesian morphisms and fibrations, as well as establishing notations and conventions for them. We introduce examples that will appear throughout the chapter, and we give a thorough definition of the functor f^* (see Proposition 2.1.27). In the second section, we define indexed categories and give examples. We also define the indexed category associated to a cleaved fibration, and we prove that the pseudofunctoriality morphisms

$$\Phi^{2*}_{fg,P} \colon Pg^*f^* \to P(f \mathbin{⨟} g)^* \qquad \text{and} \qquad \Phi^{0*}_{B,P} \colon P \to P\mathrm{id}^*_B$$

associated to a fibration Φ are natural in P (see Propositions 2.2.5 and 2.2.7). In the third section, we give a summary of the Grothendieck construction as a 2-equivalence. In the fourth section, we give, for opfibrations and opindexed categories, all of the analogous definitions, propositions and examples that we gave for fibrations and indexed categories. In the fifth section, we prove two fundamental results relating to bicleaved bifibrations and their associated indexed and opindexed categories.

In the sixth and seventh sections, we give a summary of theory of monoidal fibrations and monoidal indexed categories due to Shulman [Shu08] and Moeller and Vasilakopoulou [MV20]. In the eighth and final section, we provide string-diagrammatic treatments of internally closed and externally closed bifibrations, and we prove that the bifibration $\mathsf{Rep}\colon \mathbf{GrpRep} \to \mathbf{FinGrp}$ is an internally closed monoidal bifibration (see Examples 2.6.5 and 2.8.11).

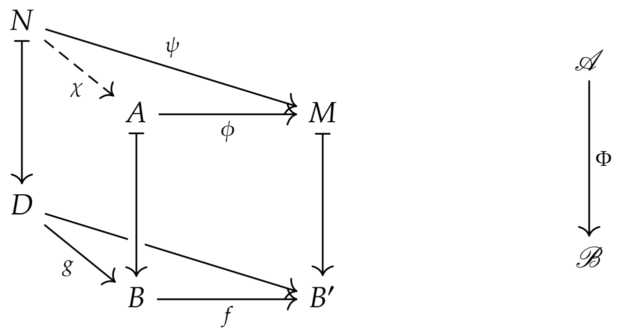
2.1 Fibrations

Suppose that $f \colon G \to H$ is a group homomorphism. Given an H-module W, we can define a G-action on the underlying vector space of W by $g \cdot w := (g)f \cdot w$, and we denote this G-module by ${}_fW$. If G is a subgroup of H and $f \colon G \to H$ is inclusion, then obtatining the G-module ${}_fW$ from the H-module W is done by restricting the action to be just by elementsof the subgroup G. This is why we call the construction of ${}_fW$ **restriction**.

In this section, we'll study fibrations, which we'll see has restriction of G-modules as an example.

Definition 2.1.1. Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a functor and let $f\colon B \to B'$ be a morphism in $\mathscr{B}$. We'll say that an object M in $\mathscr{A}$ **lies over** B if $M\Phi = B$, and we'll say that a morphism ϕ in $\mathscr{A}$ **lies over** f if $\phi\Phi = f$; we'll denote by $\mathscr{A}^h(N, A)$ the set of morphisms $N \to A$ in $\mathscr{A}$ that lie over f. ◇

Definition 2.1.2. Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a functor, let $\phi\colon A \to M$ be a morphism in $\mathscr{A}$, and let $f\colon B \to B'$ denote the morphism $\phi\Phi$ in $\mathscr{B}$. We say that the morphism $\phi\colon A \to M$ is **cartesian** if, for each morphism $g\colon D \to B$ in $\mathscr{B}$ and each morphism $\psi\colon N \to M$ in $\mathscr{A}$ that lies over $g \fatsemi f$, there exists a unique map $\chi\colon N \to A$ that lies over g and that satisfies $\psi = \chi \fatsemi \phi$. This situation is depicted in the following figure.



(2.1.3)

◇

The definition of cartesian morphism is quite verbose, but we can instead phrase it as follows.

Proposition 2.1.4. *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a functor, let $\phi\colon A \to M$ be a morphism in $\mathscr{A}$, and let $f\colon B \to B'$ denote the morphism $\phi\Phi$ in $\mathscr{B}$. Then the morphism $\phi\colon A \to M$ is cartesian if, and only if, for each morphism $g\colon D \to B$ in $\mathscr{B}$ and each object N in $\mathscr{A}$, the map*

$$\begin{aligned} \mathscr{A}^{g}(N, A) &\longrightarrow \mathscr{A}^{g \fatsemi f}(N, M) \\ \chi &\longmapsto \chi \fatsemi \phi \end{aligned} \tag{2.1.5}$$

is a bijection.

The ability to factorise the morphism ψ as $\chi \fatsemi \phi$ using the cartesian morphism ϕ is called **cartesian factorisation**. We have the following notation to describe cartesian factorisation.

Notation 2.1.6. The unique morphism χ in (2.1.3) will be written as $\psi \mathbin{\circ\!\!/} \phi$. This is meant to make the reader think of taking the equation $\chi \fatsemi \phi = \psi$ and 'dividing' both sides on the right by ϕ to get the equation $\chi = \psi \mathbin{\circ\!\!/} \phi$. With this notation we can write the inverse to the bijection (2.1.5) as

$$\begin{aligned} \mathscr{A}^{g \fatsemi f}(N, M) &\longrightarrow \mathscr{A}^{g}(N, A) \\ \psi &\longmapsto \psi \mathbin{\circ\!\!/} \phi \end{aligned}$$

◇

Definition 2.1.7. We say that the functor $\Phi\colon \mathscr{A} \to \mathscr{B}$ is a **fibration** if, for every morphism $f\colon B \to B'$ in $\mathscr{B}$ and every object M in $\mathscr{A}$ lying over B', there exists a cartesian morphism $\phi\colon A \to M$ that lies over f. ◇

Definition 2.1.8. Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a fibration. We call $\mathscr{A}$ the **total category** of Φ and we call $\mathscr{B}$ the **base category** of Φ. ◇

Definition 2.1.9. Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a fibration. A **cleaving** for Φ is a choice, for each morphism $f\colon B \to B'$ in $\mathscr{B}$ and each object M in $\mathscr{A}$ lying over B', of cartesian morphism $\mathsf{cart}_M^f\colon A \to M$ in $\mathscr{A}$ that lies over f. A **cleaved fibration** is a fibration equipped with a cleaving. ◇

Notation 2.1.10. Given a cleaved fibration $\Phi\colon \mathscr{A} \to \mathscr{B}$, we will—unless it's unclear from context—denote by cart_M^f the cartesian morphism in the cleaving that lies over f and has target M. ◇

Remark 2.1.11. A cleaving is an 'algebraicization' of the universal property given in Definition 2.1.2. A property-like structure can be algebraicized by requiring a specific choice of the objects that are required to exist. For example, a vector space V having the property of having dimension n means that there exists a basis for V containing n vectors, but this can be algebraicized by requiring a specific choice of basis $(e_1, \dots, e_n)$ for V. We'll see later (see Theorem 2.3.15) that indexed categories are an algebraicization of fibrations via the notion of cleavings. ◇

Definition 2.1.12. A morphism χ in $\mathscr{A}$ is called **pure** if χ lies over an identity morphism. ◇

Remark 2.1.13. The more standard term for pure morphisms is 'vertical' (see e.g. [Joh02, §B1.3]). David Jaz Myers uses the term 'pure' in [Mye23, Definition 2.6.1.7] so as to avoid a clash in terminology in the context of double categories; we prefer not to use the term 'vertical' here or in the context of double categories (see Definition 4.1.1) so that the name of the morphism doesn't depend on the way you chose to draw it. ◇

Applying the definition of cartesian morphism with $g = \mathrm{id}$, we get the following result called **pure-cartesian** factorisation.

Proposition 2.1.14. *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a fibration. Then, for each morphism $\psi\colon N \to M$ in $\mathscr{A}$, there exists a pure morphism χ and a cartesian morphism ϕ such that $\psi = \chi \mathbin{⨟} \phi$. In particular, if we fix a cleaving for Φ, there exists a unique pure morphism χ and a unique cartesian morphism ϕ in the cleaving such that $\psi = \chi \mathbin{⨟} \phi$.* □

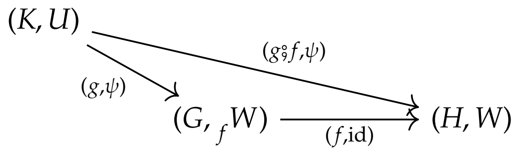
Definition 2.1.15. The category $\mathbf{GrpRep}$ of representations of finite groups is defined as follows. An object in $\mathbf{GrpRep}$ is a pair (G, V) consisting of a finite group G and a G-module V, and a morphism $(G, V) \to (H, W)$ in $\mathbf{GrpRep}$ is a pair (f, ϕ) consisting of a group homomorphism $f\colon G \to H$ and a linear map $\phi\colon V \to W$ satisfying, for every $g \in G$ and every $v \in V$, $(g \cdot v)\phi = (g)f \cdot v$. We call an object in $\mathbf{GrpRep}$ a **representation**, and we call a morphism in $\mathbf{GrpRep}$ a **module map**. ◇

Example 2.1.16. Let $\mathbf{FinGrp}$ denote the category of finite groups. The **acting group functor**

$$\begin{aligned} \mathsf{Rep}\colon \mathbf{GrpRep} &\longrightarrow \mathbf{FinGrp} \\ (G, V) &\longmapsto G \\ (f, \phi) &\longmapsto f \end{aligned}$$

takes a representation (G, V) to its acting group G. We'll show now that Rep is a fibration.

Let $f\colon G \to H$ be a group homomorphism and let (H, W) be a representation lying over H. We'll show that the morphism $(f, \mathrm{id})\colon (G, {}_fW) \to (H, W)$ in **GrpRep**, which lies over f, is cartesian. Given a group homomorphism $g\colon K \to G$ and a module map $(g \mathbin{⨾} f, \psi)\colon (K, U) \to (H, W)$ that lies over $g \mathbin{⨾} f$, the module map $(g, \psi)\colon (K, U) \to (G, W)$ is the only module map that lies over g and makes the following diagram in **GrpRep** commute.



Therefore, $(f, \mathrm{id})\colon (G, {}_fW) \to (H, W)$ is cartesian, which means that $\mathtt{Rep}\colon$ **GrpRep** $\to$ **FinGrp** is a fibration. ◇

Remark 2.1.17. From now on, whenever we refer to a cleaving for the fibration $\mathtt{Rep}$, the reader should assume that this refers to the cleaving $\mathrm{cart}^f_{(H,W)} = (f, \mathrm{id})\colon (G, {}_fW) \to (H, W)$. ◇

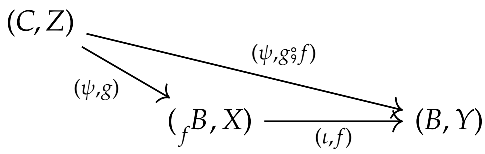
Definition 2.1.18. Let $\mathscr{C}$ be a category. The category $\mathbf{Fam}_{\mathscr{C}}$ of families of objects in $\mathscr{C}$ is defined as follows. An object in $\mathbf{Fam}_{\mathscr{C}}$ is a pair (A, X) consisting of a set X and an X-indexed set $A = \{A_x\}_{x\in X}$ of objects in $\mathscr{C}$, and a morphism $(A, X) \to (B, Y)$ in $\mathbf{Fam}_{\mathscr{C}}$ is a pair (α, f) consisting of a map of sets $f\colon X \to Y$ and an X-indexed set $\alpha = \{\alpha_x\colon A_x \to B_{(x)f}\}_{x\in X}$ of morphisms in $\mathscr{C}$. We call an object in $\mathbf{Fam}_{\mathscr{C}}$ a **family of objects**. ◇

Example 2.1.19. Let $\mathscr{C}$ be a category. The **indexing set functor**

$$
\begin{aligned}
\mathtt{Fam}_{\mathscr{C}}\colon \mathbf{Fam}_{\mathscr{C}} &\longrightarrow \mathbf{Set}\\
(A, X) &\longmapsto X\\
(\alpha, f) &\longmapsto f
\end{aligned}
$$

takes a family of objects (A, X) to its indexing set X. We'll show now that $\mathtt{Fam}_{\mathscr{C}}$ is a fibration.

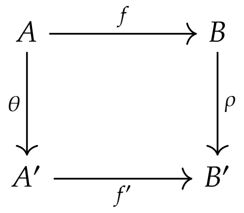
Let $f\colon X \to Y$ be a map of sets, and let (B, Y) be a Y-indexed family of objects. Let ${}_fB$ denote X-indexed family of objects $\{B_{(x)f}\}_{x\in X}$, and let ι denote the X-indexed family of morphisms $\{\mathrm{id}_{B_{(x)f}}\}_{x\in X}$. We'll show that the morphism $(\iota, f)\colon ({}_fB, f) \to (B, Y)$ in $\mathbf{Fam}_{\mathscr{C}}$, which lies over f, is cartesian. Given a map of sets $g\colon Z \to X$ and a morphism $(\psi, g \mathbin{⨾} f)\colon (Z, C) \to (Y, B)$, the morphism $(\psi, g)\colon (C, Z) \to ({}_fB, X)$ is the only morphism that lies over g and makes the folowing diagram in **Set** commute.



Therefore, $(\iota, f)\colon ({}_fB, X) \to (B, Y)$ is cartesian, which means that $\mathtt{Fam}_{\mathscr{C}}\colon \mathbf{Fam}_{\mathscr{C}} \to \mathbf{Set}$ is a fibration. ◇

Remark 2.1.20. From now on, whenever we refer to a cleaving for the fibration $\mathtt{Fam}_{\mathscr{C}}$, the reader should assume that this refers to the cleaving $\mathsf{cart}^{f}_{(B,Y)} = (\iota, f)\colon ({}_{f}B, X) \to (B, Y)$. ◇

Definition 2.1.21. Let $\mathscr{A}$ be a category. The **arrow category** $\mathscr{A}^{\to}$ of $\mathscr{A}$ is defined as follows. An object in $\mathscr{A}^{\to}$ is a morphism $f\colon A \to B$ in $\mathscr{A}$, and a morphism $(f\colon A \to B) \to (f'\colon A' \to B')$ in $\mathscr{A}^{\to}$ is a pair (θ, ρ) of morphisms in $\mathscr{A}$ that makes the following diagram in $\mathscr{A}$ commute.



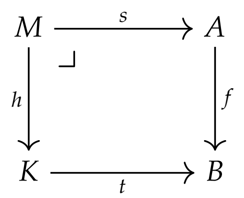
◇

Example 2.1.22. Let $\mathscr{A}$ be a category with pullbacks. The **codomain functor**

$$\begin{aligned} \mathtt{Arr}_{\mathscr{A}}\colon \mathscr{A}^{\to} &\longrightarrow \mathscr{A} \\ (f\colon A \to B) &\longmapsto B \\ (\theta, \rho) &\longmapsto \rho \end{aligned}$$

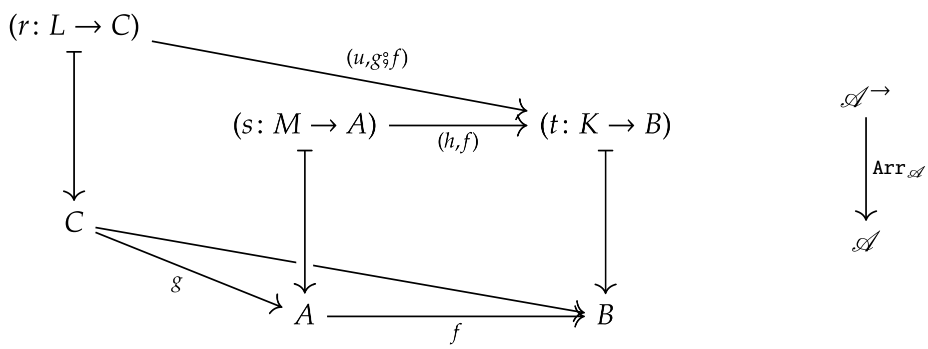
takes a morphism $f\colon A \to B$ to its codomain B. We'll show now that $\mathtt{Arr}_{\mathscr{A}}$ is a fibration.

Let $f\colon A \to B$ and $t\colon K \to B$ be morphisms in $\mathscr{A}$, so t is an object in $\mathscr{A}^{\to}$ and $(t)\mathtt{Arr}_{\mathscr{A}} = B$. Let the following be a pullback square in $\mathscr{A}$.

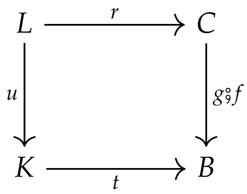


We'll show that the morphism $(h, f)\colon s \to t$ in $\mathscr{A}^{\to}$, which lies over f, is cartesian.

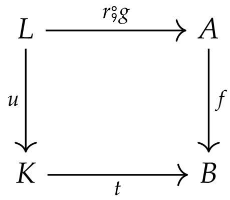
Suppose that $g\colon C \to A$, $r\colon L \to C$ and $u\colon L \to K$ are morphisms in $\mathscr{A}$, as shown in the following figure.



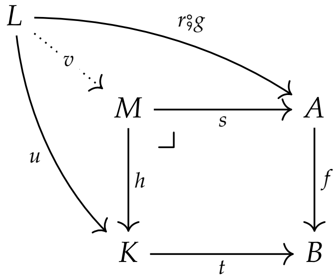
Rewrite the square



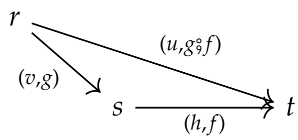
that defines the morphism $(u, g \mathbin{⨾} f)\colon r \to t$ as



The universal property of pullbacks gives a unique morphism $v\colon L \to M$ in $\mathscr{A}$ that makes the following diagram commute.

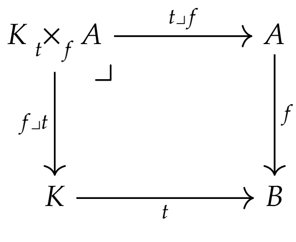


Therefore, $(v, g)\colon r \to s$ is the unique morphism in $\mathscr{A}^{\to}$ that lies over g and that makes the following diagram in $\mathscr{A}^{\to}$ commute.



So $(h, f)\colon s \to t$ is cartesian, which means that $\mathtt{Arr}_{\mathscr{A}}\colon \mathscr{A}^{\to} \to \mathscr{A}$ is a fibration. ◇

Remark 2.1.23. A choice of pullback



for each morphism $f\colon A \to B$ in $\mathscr{B}$ and each object t in $\mathscr{A}/B$, defines a cleaving

$$\mathrm{cart}_t^f = (f \lrcorner t, f)\colon t \lrcorner f \to t$$

for the fibration $\mathtt{Arr}_{\mathscr{A}}$. ◇

We now give some basic properties of cartesian morphisms, for which we give detailed proofs.

Proposition 2.1.24 ([Shu08, Proposition 3.4]). *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a functor.*

(i) Let $\phi\colon A \to M$ and $\phi'\colon M \to P$ be morphisms in $\mathscr{A}$. If ϕ and ϕ' are cartesian, then $\phi \mathbin{⨾} \phi'$ is cartesian.

(ii) Let $\chi\colon N \to A$ and $\phi\colon A \to M$ be morphisms in $\mathscr{A}$. If ϕ and $\chi \mathbin{⨾} \phi$ are cartesian, then χ is cartesian.

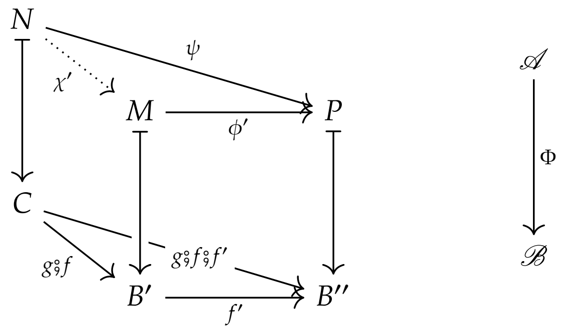
(iii) Let $\phi\colon A \to M$ and $\psi\colon N \to M$ be cartesian morphisms in $\mathscr{A}$. If $\phi\Phi = \psi\Phi$, then there exists a unique pure isomorphism $\chi\colon N \to A$ that satisfies $\psi = \chi \mathbin{⨾} \phi$.

(iv) Every isomorphism in $\mathscr{A}$ is cartesian.

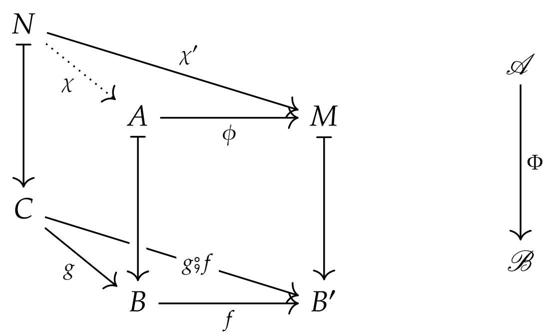
(v) Let ϕ be a morphism in $\mathscr{A}$ and suppose that $\phi\Phi$ is an isomorphism in $\mathscr{B}$. Then, ϕ is cartesian if, and only if, ϕ is an isomorphism.

Proof.

(i) Let $f\colon B \to B'$ and $f'\colon B' \to B''$ denote $\phi\Phi$ and $\phi'\Phi$. Let $g\colon C \to B$ be a morphism in $\mathscr{B}$ and let $\psi\colon N \to P$ be a morphism in $\mathscr{A}$ lying over $g \mathbin{⨾} f \mathbin{⨾} f'$. Since ϕ' is cartesian, there exists a unique morphism χ' that lies over $g \mathbin{⨾} f$ and that satisfies $\psi = \chi' \mathbin{⨾} \phi'$; this situtation is shown in the following figure.

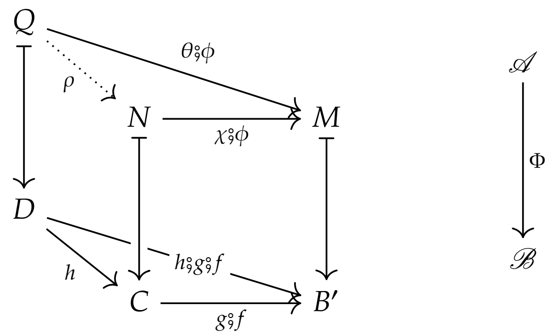


Then, since ϕ is cartesian, there exists a unique morphism χ that lies over g and that satisfies $\chi' = \chi \mathbin{⨾} \phi$; this situtation is shown in the following figure.

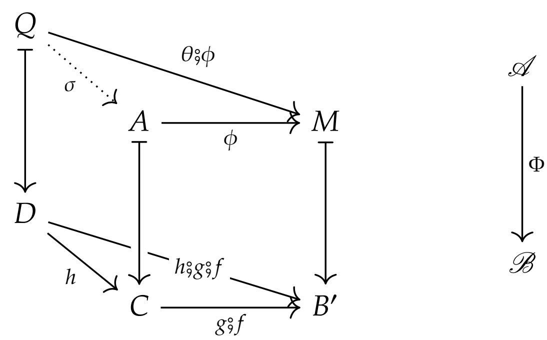


Therefore, χ is the unique morphism that lies over g and satisfies $\psi = \chi \fatsemi (\phi \fatsemi \phi')$. Hence, $\phi \fatsemi \phi'$ is cartesian.

(ii) Let $f\colon B \to B'$ and $g\colon C \to B$ denote $\phi\Phi$ and $\chi\Phi$. Let $h\colon D \to C$ be a morphism in $\mathscr{B}$ and let $\theta\colon Q \to N$ be a morphism in $\mathscr{A}$ lying over h. Since $\chi \fatsemi \phi$ is cartesian, there exists a unique morphism ρ that lies over h and that satisfies $\theta \fatsemi \phi = \rho \fatsemi (\chi \fatsemi \phi)$; this situtation is shown in the following figure.

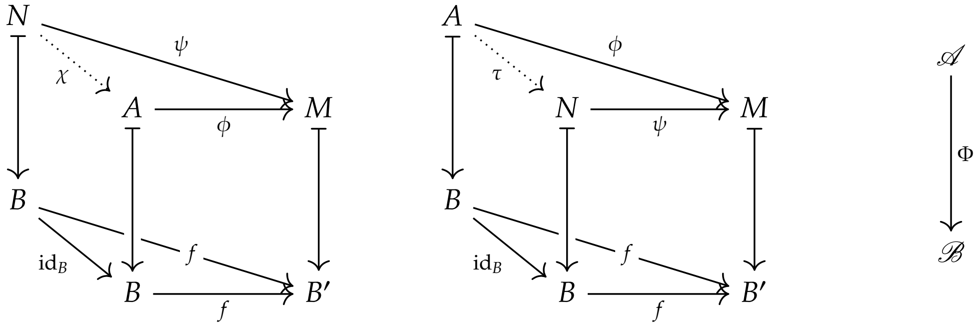


Since ϕ is cartesian, there exists a unique morphism σ that lies over $h \fatsemi g$ and satisfies $\phi \fatsemi \sigma = \theta \fatsemi \phi$; this situtation is shown in the following figure.



Both θ and $\rho \fatsemi \chi$ satisfy the defining properties of σ, so they must be equal. So ρ is the unique morphism that lies over h and that satisfies $\theta = \rho \fatsemi \chi$. Hence, χ is cartesian.

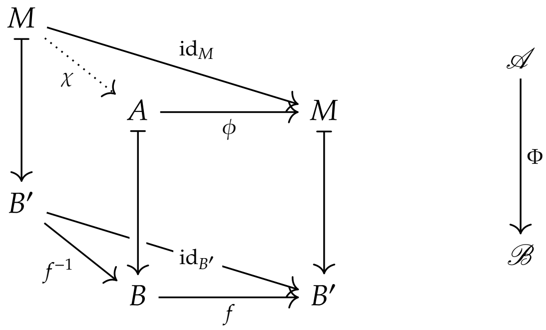
(iii) Let $f\colon B \to B'$ denote the morphism $\phi\Phi = \psi\Phi$. Since ϕ is cartesian, there exists a unique morphism $\chi\colon N \to A$ that lies over $\mathbf{id}_B$ and that satisfies $\psi = \chi \fatsemi \phi$. Since ψ is cartesian, there exists a unique morphism $\tau\colon A \to N$ that lies over $\mathbf{id}_B$ and that satisfies $\phi = \tau \fatsemi \psi$. These two situtations are shown in the following figure.



Since ϕ is cartesian, there is a unique morphism $\xi\colon A \to A$ that lies over $\mathbf{id}_B$ and that satisfies $\phi = \xi \mathbin{⨾} \phi$. Both $\mathbf{id}_A$ and $\tau \mathbin{⨾} \chi$ satisfy the defining properties of ξ, so they must be equal. Similarly, since ψ is cartesian, we can deduce that $\mathbf{id}_N = \chi \mathbin{⨾} \tau$. Therefore, $\chi\colon N \to A$ is an isomorphism, and, since ϕ is cartesian, χ is the unique morphism satisfying $\psi = \chi \mathbin{⨾} \phi$.

(iv) Let ϕ be an isomorphism in $\mathscr{A}$ and let $f\colon B \to B'$ denote $\phi\Phi$. Let $g\colon C \to B$ be a morphism in $\mathscr{B}$ and let $\psi\colon N \to P$ be a morphism in $\mathscr{A}$ lying over $g \mathbin{⨾} f$. Then $\psi \mathbin{⨾} \phi^{-1}$ is the unique morphism $\chi\colon N \to A$ in $\mathscr{A}$ that lies over g and that satisfies $\psi = \chi \mathbin{⨾} \phi$. Hence, ϕ is cartesian.

(v) Let $f\colon B \to B'$ denote $\phi\Phi$. Suppose that ϕ is cartesian. Then there exists a unique morphism $\chi\colon A \to A$ that lies over f^{-1} and that satisfies $\chi \mathbin{⨾} \phi = \mathbf{id}_M$; this situtation is shown in the following figure.



Again, since ϕ is cartesian, there exists a unique morphism $\omega\colon A \to A$ that lies over $\mathbf{id}_B$ and that satisfies $\phi = \omega \mathbin{⨾} \phi$. Both $\mathbf{id}_A$ and $\phi \mathbin{⨾} \chi$ lie over $\mathbf{id}_B$ and satisfy the defining property of ω, so they must be equal. Hence, ϕ is an isomorphism.

□

Given a cleaving of Φ, all other cartesian morphisms can be obtained from those in the cleaving.

Corollary 2.1.25. *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a fibration. A morphism ψ in $\mathscr{A}$ is cartesian if, and only if, there exists a pure isomorphism χ and a cartesian morphism ϕ such that $\psi = \chi \mathbin{⨾} \phi$. In particular, if we fix a cleaving for Φ, there exists a unique pure isomorphism χ and a unique cartesian morphism ϕ in the cleaving such that $\psi = \chi \mathbin{⨾} \phi$.*

Proof. Using pure-cartesian factorisation, there exists a pure morphism χ and a cartesian morphism ϕ such that $\psi = \chi \mathbin{⨾} \phi$. By Proposition 2.1.24 (ii), χ is also cartesian, so, by Proposition 2.1.24 (v), χ is an isomorphism. On the other hand, if ψ is the composite of a pure isomorphism and a cartesian morphism, then, by Proposition 2.1.24 (iv) and (i), ψ is cartesian. □

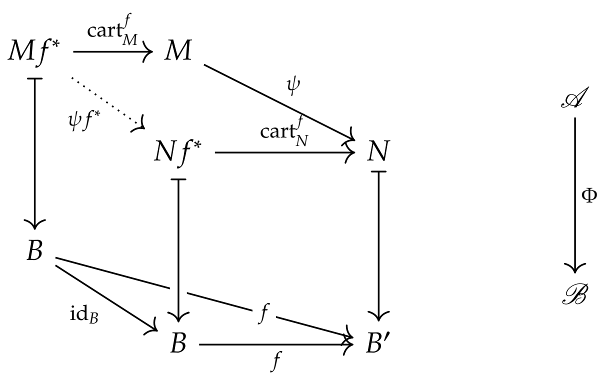
Definition 2.1.26. Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a functor. For each object $B \in \mathscr{B}$, the **fibre category** $\mathscr{A}_B$ is the subcategory of $\mathscr{A}$ given by objects lying over B and morphisms lying over id_B. That is, $\mathsf{ob}\,\mathscr{A}_B = \{M \in \mathsf{ob}\,A \mid M\Phi = B\}$ and $\mathscr{A}_B(M,N) = \{\psi \in \mathscr{A}(M,N) \mid \psi\Phi = \mathsf{id}_B\}$. ◇

If $\Phi\colon \mathscr{A} \to \mathscr{B}$ is a cleaved fibration, then we have, for each morphism $f\colon B \to B'$ in $\mathscr{B}$, a specified cartesian morphism cart^f_M in $\mathscr{A}$ that lies over f. One key piece of information given by the cleaving is the source of the morphism cart^f_M. In general, we denote the source of cart^f_M by Mf^*. We now show that this extends to a functor $f^*\colon \mathscr{A}_{B'} \to \mathscr{A}_B$. We call this functor the **pull-back functor** associated to f because we think of it as pulling an object lying over B' back along $f\colon B \to B'$ to obtain an object lying over B.

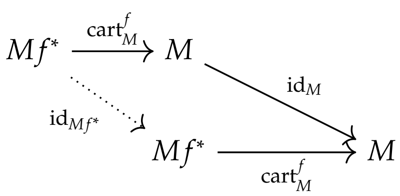
The following is a result is a standard one. We provide detailed proof which seems to have been absent from the literaturee until Johnson and Yau also provided such a proof [JY21, Lemma 10.4.7].

Proposition 2.1.27. *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a cleaved fibration and let $f\colon B \to B'$ be a morphism in $\mathscr{B}$. Then the following defines a functor $f^*\colon \mathscr{A}_{B'} \to \mathscr{A}_B$.*

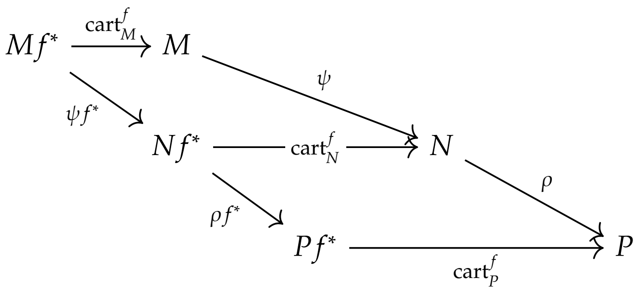
- *For each object M in $\mathscr{A}_{B'}$, define Mf^* to be the domain of cart^f_M.*
- *For each morphism $\psi\colon M \to N$ in $\mathscr{A}_{B'}$, using the fact that cart^f_N is cartesian, define ψf^* to be the unique morphism $Mf^* \to Nf^*$ in $\mathscr{A}$ that lies over id_B and satisfies $\psi f^* \mathbin{⨾} \mathsf{cart}^f_N = \mathsf{cart}^f_M \mathbin{⨾} \psi$. This situtation is shown in the following figure.*



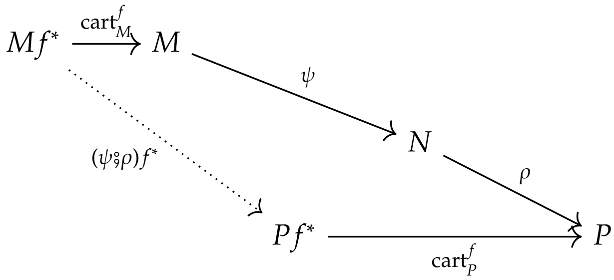
Proof. The fact that $\mathsf{id}_M f^* = \mathsf{id}_{Mf^*}$ follows from the fact that, for each object M in $\mathscr{A}'_B$, the morphism id_{Mf^*} lies over id_B and makes the following diagram in $\mathscr{A}$ commute.



Now let's check, for each pair of composable morphisms $\psi\colon M \to N$ and $\rho\colon N \to P$ in $\mathscr{A}_B$, that $\psi f^* \mathbin{⨾} \rho f^* = (\psi \mathbin{⨾} \rho) f^*$. The following diagram in $\mathscr{A}$ commutes by the defining properties of ψf^* and ρf^*.



But $(\psi \mathbin{⨾} \rho) f^*$ is the unique morphism $Mf^* \to Pf^*$ in $\mathscr{A}$ that lies over $\mathbf{id}_B$ and that makes the diagram



in $\mathscr{A}$ commute, so $\psi f^* \mathbin{⨾} \rho f^* = (\psi \mathbin{⨾} \rho) f^*$. □

Remark 2.1.28. Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a cleaved functor, and let $f\colon B \to B'$ be a morphism in $\mathscr{B}$. Suppose that $\phi\colon A \to M$ is a cartesian morphism in $\mathscr{A}$ that lies of f. Since the morphism

$$\mathsf{cart}^f_M\colon Mf^* \longrightarrow M$$

is also cartesian, there exists, by Proposition 2.1.24(iii), a pure isomorphism $\chi\colon Mf^* \to A$ that satisfies $\mathsf{cart}^f_M = \chi \mathbin{⨾} \phi$. ◇

We'll give examples of pull-back functors soon in Examples 2.1.30, but first we need to define notation that particular fibrations use for their fibre categories.

Definition 2.1.29.

- Let G be a group. We write $\mathbf{Rep}_G$ for the category of G-modules and G-module maps.
- Let $\mathscr{C}$ be a category and let X be a set. The category $\mathscr{C}^X$ of X-indexed family objects in $\mathscr{C}$ is defined as follows. An object in $\mathscr{C}^X$ is an X-indexed set $A = \{A_x\}_{x \in X}$ of objects in $\mathscr{C}$, and a morphism $A \to B$ in $\mathscr{C}^X$ is an X-indexed set $\alpha = \{\alpha_x\colon A_x \to B_x\}_{x\in X}$ of morphisms in $\mathscr{C}$.
- Let $\mathscr{A}$ be a category and let B be an object in $\mathscr{A}$. The **over category** $\mathscr{A}/B$ of $\mathscr{A}$ over B is a subcategory of the arrow category $\mathscr{A}^{\to}$ and is defined as follows: the objects of $\mathscr{A}/B$ are the morphisms in $\mathscr{A}$ with target B, and the morphisms of $\mathscr{A}/B$ are the pairs of the form $(\theta, \mathbf{id}_B)$.

◇

Examples 2.1.30.

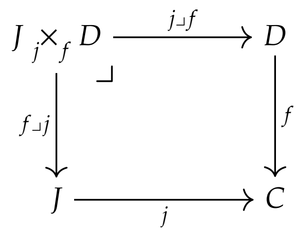
(i) For the fibration $\mathtt{Rep}\colon \mathbf{GrpRep} \to \mathbf{FinGrp}$ and a group homomorphism $f\colon G \to H$, we have

$$\begin{aligned} f^*\colon \mathbf{Rep}_H &\longrightarrow \mathbf{Rep}_G \\ W &\longmapsto {}_fW \\ \alpha &\longmapsto \alpha \end{aligned}$$

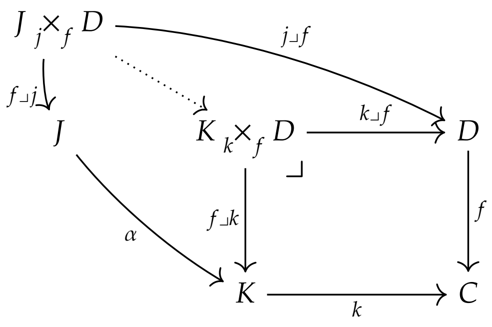
(ii) For the fibration $\mathtt{Fam}_{\mathscr{C}}\colon \mathbf{Fam}_{\mathscr{C}} \to \mathbf{Set}$ and a map of sets $f\colon X \to Y$, we have

$$\begin{aligned} f^*\colon \mathscr{C}^Y &\longrightarrow \mathscr{C}^X \\ (B_y)_{y\in Y} &\longmapsto (B_{f(x)})_{x\in X} \\ (\alpha_y)_{y\in Y} &\longmapsto (\alpha_{f(x)})_{x\in X} \end{aligned}$$

(iii) For the fibration $\mathtt{Arr}_{\mathscr{A}}\colon \mathscr{A}^{\to} \to \mathscr{A}$ and a morphism $f\colon D \to C$ in $\mathscr{A}$, the functor $f^*\colon \mathscr{A}/C \to \mathscr{A}/D$ is given as follows. For each object $j\colon J \to C$ in $\mathscr{A}/C$, take $jf^* = j \lrcorner f$ as in the following pullback square.



For each morphism $\alpha\colon (j\colon J \to C) \to (k\colon K \to C)$ in $\mathscr{A}/C$, the morphism $\alpha f^*\colon j\lrcorner f \to k \lrcorner f$ in $\mathscr{A}/D$ is given by the unique morphism $J\,{}_j\!\times_f D \to K\,{}_k\!\times_f D$ in $\mathscr{C}$ that makes the diagram



in $\mathscr{A}$ commute; this morphism exists by the universal property of pullbacks.

◇

The following standard result follows from the definition of the functor f^* on morphisms (see Proposition 2.1.27).

Proposition 2.1.31. *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a cleaved fibration and let $f\colon B \to B'$ be a morphism in $\mathscr{B}$. Then the bijections*

$$\begin{aligned} \mathscr{A}_B(N, Mf^*) &\longrightarrow \mathscr{A}^f(N, M) \\ \chi &\longmapsto \chi \fatsemi \mathrm{cart}^f_M \end{aligned}$$

are natural in both N and M.

2.2 Indexed categories

Given a fibration $\Phi\colon \mathscr{A} \to \mathscr{B}$ we have fibre categories $\mathscr{A}_B$ and we have functors f^* between them. We'll show that these form part of the structure of an *indexed category*, which we define now.

Definition 2.2.1. An **indexed category** $\mathfrak{I}$ consists of a category $\mathscr{B}$ and a pseudofunctor $\mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$, where we consider $\mathscr{B}$ as a bicategory with identity 2-cells. We call $\mathscr{B}$ the **base category** of the indexed category $\mathfrak{I}$. ◇

Remark 2.2.2. Let's unpack this definition. An indexed category $\mathfrak{I}\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$ consists of the following data:

(i) a base category $\mathscr{B}$;

(ii) for each object B in $\mathscr{B}$, a category $B\mathfrak{I}$;

(iii) for each morphism $f\colon B \to B'$ in $\mathscr{B}$, a functor $f\mathfrak{I}\colon B'\mathfrak{I} \to B\mathfrak{I}$;

(iv) for each pair $f\colon B \to B'$, $g\colon B' \to B''$ of morphisms in $\mathscr{B}$, a natural isomorphism

$$\mathfrak{I}^{2*}_{fg}\colon g\mathfrak{I} \fatsemi f\mathfrak{I} \to (f \fatsemi g)\mathfrak{I} ;$$

(v) for each object B in $\mathscr{B}$, a natural isomorphism

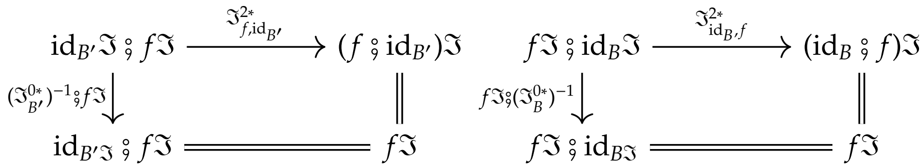
$$\mathfrak{I}^{0*}_B\colon \mathrm{id}_{B\mathfrak{I}} \to \mathrm{id}_B\mathfrak{I} .$$

These data are required to satisfy the following axioms.

- (**Associativity**) For every composable triple $f\colon B \to B'$, $g\colon B' \to B''$, $h\colon B'' \to B'''$ of morphisms in $\mathscr{B}$, the following diagram commutes.

$$\begin{array}{ccc} h\mathfrak{I} \fatsemi g\mathfrak{I} \fatsemi f\mathfrak{I} & \xrightarrow{\mathfrak{I}^{2*}_{gh} \fatsemi f\mathfrak{I}} & (g \fatsemi h)\mathfrak{I} \fatsemi f\mathfrak{I} \\ {\scriptstyle h\mathfrak{I} \fatsemi \mathfrak{I}^{2*}_{fg}} \downarrow & & \downarrow {\scriptstyle \mathfrak{I}^{2*}_{f, g \fatsemi h}} \\ h\mathfrak{I} \fatsemi (f \fatsemi g)\mathfrak{I} & \xrightarrow[\mathfrak{I}^{2*}_{f \fatsemi g, h}]{} & (f \fatsemi g \fatsemi h)\mathfrak{I} \end{array}$$

- (**Unitality**) For every morphism $f: B \to B'$ in $\mathscr{B}$, the following diagrams commute.



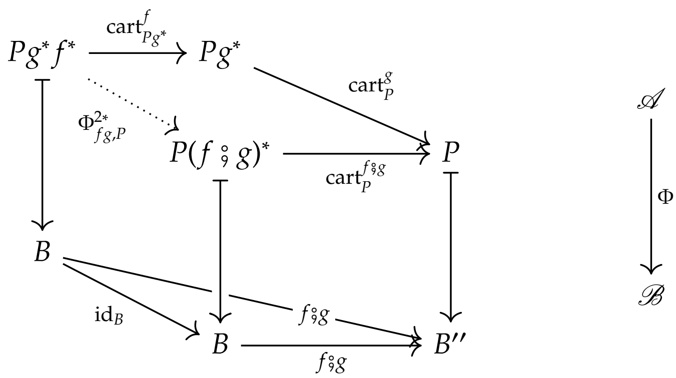
◇

We said we will show that, given a fibration $\Phi: \mathscr{A} \to \mathscr{B}$, the fibre categories $\mathscr{A}_B$ and pull-back functors f^* form part of the structure of an indexed category; namely these are items (ii) and (iii) in Remark 2.2.2. We now define the remaining structure—items (iv) and (v) in Remark 2.2.2—of the indexed category of which the fibre categories $\mathscr{A}_B$ and pull-back functors f^* form part of the structure.

Definition 2.2.3. Let $\Phi: \mathscr{A} \to \mathscr{B}$ be a cleaved fibration. For each pair of composable morphism $f: B \to B'$ and $g: B' \to B''$ in $\mathscr{B}$ and each object P in $\mathscr{A}_{B''}$, define the morphism

$$\Phi^{2*}_{fg,P}: Pg^*f^* \to P(f ⨾ g)^*$$

in $\mathscr{A}_B$ to be the unique pure morphism $Pg^*f^* \to P(f ⨾ g)^*$ in $\mathscr{A}$ that satisfies $\Phi^{2*}_{fg,P} ⨾ \mathsf{cart}^{f⨾g}_P = \mathsf{cart}^f_{Pg^*} ⨾ \mathsf{cart}^g_P$. The existence of this morphism follows from the morphism $\mathsf{cart}^{f⨾g}_P$ being cartesian, and the situtaion is shown in the following figure.



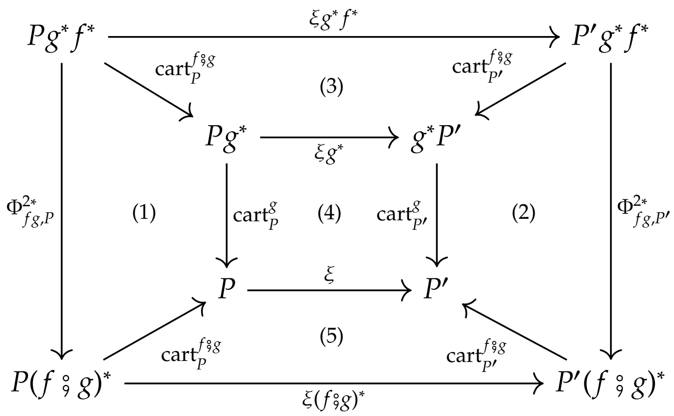
◇

Remark 2.2.4. The morphism $\Phi^{2*}_{fg,P}$ is cartesian since $\mathsf{cart}^{f⨾g}_P$ and $\mathsf{cart}^f_{Pg^*} ⨾ \mathsf{cart}^g_P$ are cartesian; this means that $\Phi^{2*}_{fg,P}$ is both pure and cartesian and is therefore an isomorphism. ◇

We provide a proof of the following result which appears absent from the literature, including the otherwise thorough book by Johnson and Yau [JY21, Lemma 10.4.7].

Proposition 2.2.5. *The morphisms $\Phi^{2*}_{fg,P}$ form the components of a natural isomorphism $\Phi^{2*}_{fg}: g^* ⨾ f^* \to (f ⨾ g)^*$.*

Proof. In the following diagram, the squares (1) and (2) commute by definition of $\Phi^{2*}_{fg,P}$ and $\Phi^{2*}_{fg,P'}$, and the squares (3), (4) and (5) commute by definition of $\xi g^* f^*$, ξg^* and $\xi(f \mathbin{⨾} g)^*$.



Therefore, the outer square—the naturalilty square for $\Phi^{2*}_{fg}\colon g^* \mathbin{⨾} f^* \to (f \mathbin{⨾} g)^*$—commutes. □

Definition 2.2.6. Let Φ be a cleaved fibration. For each object B in $\mathscr{B}$ and each object M in $\mathscr{A}_B$, the morphism $\mathrm{cart}^{\mathrm{id}_B}_M\colon M\mathrm{id}^*_B \to M$ is an isomorphism since it is pure and cartesian. Define the morphism

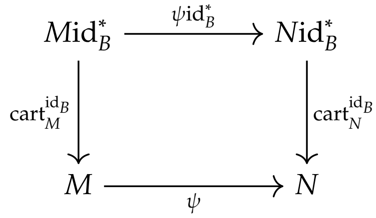
$$\Phi^{0*}_{B,M}\colon M \to M\mathrm{id}^*_B$$

in $\mathscr{A}_B$ to be the inverse of $\mathrm{cart}^{\mathrm{id}_B}_M$. ◇

The following result also appears absent from the literature. Note that the cleaving for the fibration $\Phi\colon \mathscr{A} \to \mathscr{B}$ is often taken to be a *unitary* cleaving—meaning a cleaving for which the cartesian morphism $\mathrm{cart}^{\mathrm{id}_B}_M\colon M\mathrm{id}^*_B \to M$ is the identity for each object B in $\mathscr{B}$—and in this case the following result is trivial [JY21, Convention 10.4.2, Lemma 10.4.7].

Proposition 2.2.7. *The morphisms $\Phi^{0*}_{B,M}$ are the components of a natural isomorphism $\Phi^{0*}_B\colon \mathrm{id}_{\mathscr{A}_B} \to \mathrm{id}^*_B$.*

Proof. Let $\psi\colon M \to N$ be a morphism in $\mathscr{A}_B$. By the definition of the functor id^*_B on morphisms, the following diagram commutes.



Therfore, since $\Phi^{0*}_{B,M}$ and $\mathsf{cart}^{\mathrm{id}_B}_M$ are inverses, the naturality square

$$\begin{array}{ccc} M & \xrightarrow{\psi} & N \\ \downarrow{\scriptstyle \Phi^{0*}_{B,M}} & & \downarrow{\scriptstyle \Phi^{0*}_{B,N}} \\ M\mathrm{id}^*_B & \xrightarrow[\psi\mathrm{id}^*_B]{} & N\mathrm{id}^*_B \end{array}$$

for Φ^{0*}_B commutes. □

We now define the indexed category $\Phi\mathcal{G}^{-1}$ following the unpacked definition of an indexed category (see Remark 2.2.2). This forms one direction of the Grothendieck construction between fibrations and indexed categories; we will see the other direction in the following section.

Definition 2.2.8. Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a cleaved fibration. The indexed category $\Phi\mathcal{G}^{-1}\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$, called the **indexed category associated to** Φ, is given by the following data:

- a base category of $\Phi\mathcal{G}^{-1}$ is $\mathscr{B}$;
- for each object B in $\mathscr{B}$, the category $(B)\Phi\mathcal{G}^{-1}$ is the fibre category $\mathscr{A}_B$;
- for each morphism $f\colon B \to B'$ in $\mathscr{B}$, the functor $f\Phi\mathcal{G}^{-1}$ is the functor $f^*\colon \mathscr{A}_{B'} \to \mathscr{A}_B$ defined in Proposition 2.1.27;
- for each pair of composable morphisms $f\colon B \to B'$ and $g\colon B' \to B''$ in $\mathscr{B}$, the natural isomorphism $\Phi\mathcal{G}^{-1}{}^{2*}_{fg}$ is the natural isomorphism

  $$\Phi^{2*}_{fg}\colon g^* \mathbin{⨾} f^* \to (f \mathbin{⨾} g)^*$$

  defined in Definition 2.2.3 and Proposition 2.2.5.
- for each object B in $\mathscr{B}$, the natural isomorphism $\Phi\mathcal{G}^{-1}{}^{0*}_B$ is the natural isomorphism

  $$\Phi^{0*}_B\colon \mathrm{id}_{\mathscr{A}_B} \to \mathrm{id}^*_B$$

  defined in Definition 2.2.6 and Proposition 2.2.7.

◇

Example 2.2.9. The indexed category $\mathfrak{Rep}\colon \mathbf{FinGrp}^{\mathrm{op}} \to \mathbf{Cat}$ consists of the following data:

- the base category is the category **FinGrp** of finite groups;
- for each group G, $(G)\mathfrak{Rep}$ is the category $\mathbf{Rep}_G$ of representations of G;

- for each group homomorphism $f\colon G \to H$, the functor $f^*\colon \mathbf{Rep}_H \to \mathbf{Rep}_G$ is the restriction functor

$$\begin{aligned} \mathbf{Rep}_H &\longrightarrow \mathbf{Rep}_G \\ W &\longmapsto {}_fW \\ \alpha &\longmapsto \alpha \end{aligned}$$

- for each pair $f\colon G \to H$, $g\colon H \to K$ of group homomorphisms, the natural isomorphism

$$g^* \mathbin{⨾} f^* \to (f \mathbin{⨾} g)^*$$

  is the identity;

- for each group G, the natural isomorphism

$$\mathrm{id}_{\mathbf{Rep}_G} \to \mathrm{id}_G^*$$

  is the identity.

◇

Example 2.2.10. For each category $\mathscr{C}$, the indexed category $\mathfrak{Fam}_{\mathscr{C}}$ consists of the following data:

- the base category is the category **Set** of sets;
- for each set X, $(X)\mathfrak{Fam}$ is the category $\mathscr{C}^X$ of X-indexed families of objects in $\mathscr{C}$;
- for each map of sets $f\colon X \to Y$, the functor $f^*\colon \mathscr{C}^Y \to \mathscr{C}^X$ is given by

$$\begin{aligned} \mathscr{C}^Y &\longrightarrow \mathscr{C}^X \\ (B_y)_{y\in Y} &\longmapsto (B_{f(x)})_{x\in X} \\ (\alpha_y)_{y\in Y} &\longmapsto (\alpha_{f(x)})_{y\in Y} \end{aligned}$$

- for each pair $f\colon X \to Y$, $g\colon Y \to Z$ of maps of sets, the natural isomorphism

$$g^* \mathbin{⨾} f^* \to (f \mathbin{⨾} g)^*$$

  is the identity;

- for each set X, the natural isomorphism

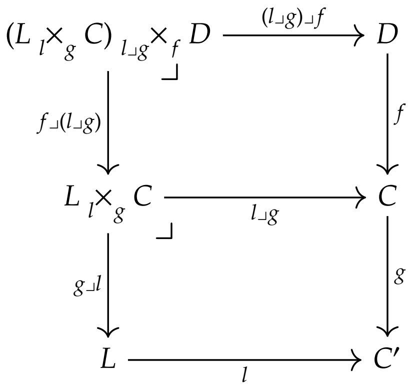
$$\mathrm{id}_{\mathscr{C}^X} \to \mathrm{id}_X^*$$

  is the identity.

◇

Example 2.2.11. For each category $\mathscr{A}$ with finite limits, the indexed category $\mathfrak{Arr}_{\mathscr{A}}$ consists of the following data:

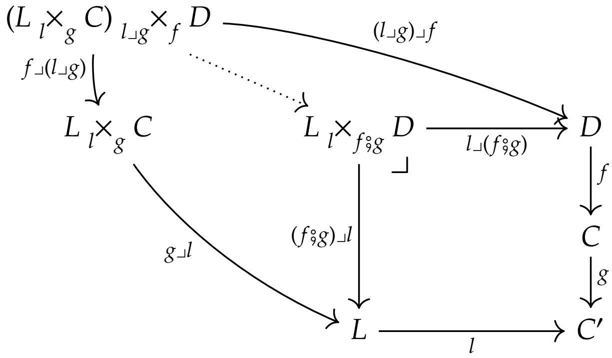
- the base category is the category $\mathscr{A}$;
- for each object A in $\mathscr{A}$, $(A)\mathfrak{Arr}$ is the over category $\mathscr{A}/A$;
- for each morphism $f\colon A \to B$ in $\mathscr{A}$, the functor $f^*\colon \mathscr{A}/B \to \mathscr{A}/A$ is given in Examples 2.1.30 (iii);
- for each composable pair $f\colon D \to C$, $g\colon C \to C'$ of morphisms in $\mathscr{A}$ and each object $l\colon L \to C'$ in $\mathscr{A}/C'$, we compose the pullback squares



  and define the isomorphism

$$(l)(g^* \,⨾\, f^*) \to (l)(f \,⨾\, g)^*$$

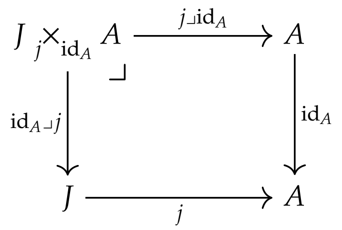
  to be the unique morphism $(L\ {}_l\!\times_g C)\ {}_{l\lrcorner g}\!\times_f D \to L\ {}_l\!\times_{f ⨾ g} D$ in $\mathscr{A}$ that makes the diagram



  commute;
- for each object A in $\mathscr{A}$ and each object $j\colon J \to A$ in $\mathscr{A}/A$, the isomorphism

$$(j)\mathrm{id}_{\mathscr{A}/A} \to (j)\mathrm{id}_A^*$$

is given by the pair $((\mathrm{id}_A \lrcorner j)^{-1}, \mathrm{id}_A)\colon j \to j \lrcorner \mathrm{id}_A$, as shown in the following pullback diagram.



◇

2.3 The Grothendieck construction

We've seen how to construct an indexed category from a fibration, and we'll now see how to construct a fibration from an indexed category. These are the two directions of the Grothendieck construction. The following definition, which define the total category of the fibration obtained by the Grothendieck construction is also sometimes called the Grothendieck construction or the Grothendieck category.

Definition 2.3.1. Let $\mathfrak{I}\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$ be an indexed category. The **total category** of $\mathfrak{I}$, denoted by

$$\int \mathfrak{I},$$

is the category defined as follows.

- An object is a pair $\begin{pmatrix} M \\ B \end{pmatrix}$ consisting of an object B in $\mathscr{B}$ an object M in $B\mathfrak{I}$.
- A morphism $\begin{pmatrix} M \\ B \end{pmatrix} \to \begin{pmatrix} M' \\ B' \end{pmatrix}$ is a pair $\begin{pmatrix} \phi \\ f \end{pmatrix}$ consisting of a morphism $f\colon B \to B'$ in $\mathscr{B}$ and a morphism $\phi\colon M \to M' f^*$ in $B\mathfrak{I}$.
- The composite of the morphisms $\begin{pmatrix} \phi \\ f \end{pmatrix}\colon \begin{pmatrix} M \\ B \end{pmatrix} \to \begin{pmatrix} M' \\ B' \end{pmatrix}$ and $\begin{pmatrix} \psi \\ g \end{pmatrix}\colon \begin{pmatrix} M' \\ B' \end{pmatrix} \to \begin{pmatrix} M'' \\ B'' \end{pmatrix}$ in $\int \mathfrak{I}$ is given by the pair
$$\begin{pmatrix} \phi \mathbin{;} \psi f^* \mathbin{;} \mathfrak{I}^{2*}_{fg,M''} \\ f \mathbin{;} g \end{pmatrix}\colon \begin{pmatrix} M \\ B \end{pmatrix} \to \begin{pmatrix} M'' \\ B'' \end{pmatrix}.$$
- For each object $\begin{pmatrix} M \\ B \end{pmatrix}$ in $\int \mathfrak{I}$, the identity $\mathbf{id}_{\left(\begin{smallmatrix} M \\ B \end{smallmatrix}\right)}$ is given by the pair
$$\begin{pmatrix} \mathfrak{I}^{0*}_{B,M} \\ \mathrm{id}_B \end{pmatrix}\colon \begin{pmatrix} M \\ B \end{pmatrix} \to \begin{pmatrix} M \\ B \end{pmatrix}.$$

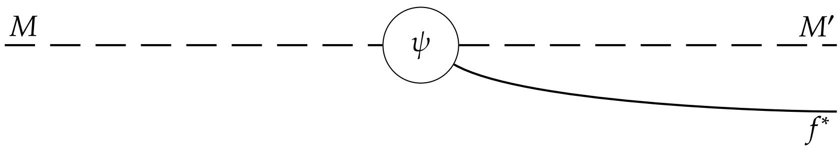
◇

We can use string diagrams to represent categories, functors and natural transformations, and we can use string diagrams to represent morphisms in a category (see Section 1.2). Using string diagrams will be very useful when talking about morphisms in the total category of an indexed category, and we'll explain now how we do this.

Let $\mathfrak{I}\colon \mathscr{B}^{\mathrm{op}}$ be an indexed category and let

$$\begin{pmatrix} \psi\colon M \to M' f^* \\ f\colon B \to B' \end{pmatrix} \colon \begin{pmatrix} M \\ B \end{pmatrix} \to \begin{pmatrix} M' \\ B' \end{pmatrix}$$

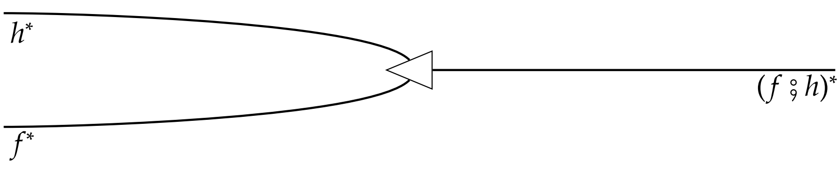
be a morphism in $\int \mathfrak{I}$. The morphism $\psi\colon M \to M' f^*$ in $B\mathfrak{I}$ is called the **total part** of the morphism $\begin{pmatrix} \psi \\ f \end{pmatrix}$ in $\int \mathfrak{I}$ and we can represent it using a string diagram as follows.



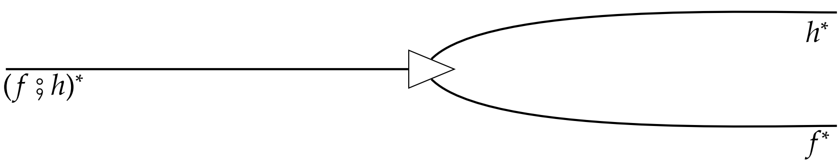
Let

$$\begin{pmatrix} \phi\colon M' \to M'' h^* \\ h\colon B' \to B'' \end{pmatrix} \colon \begin{pmatrix} M' \\ B' \end{pmatrix} \to \begin{pmatrix} M' \\ B'' \end{pmatrix}$$

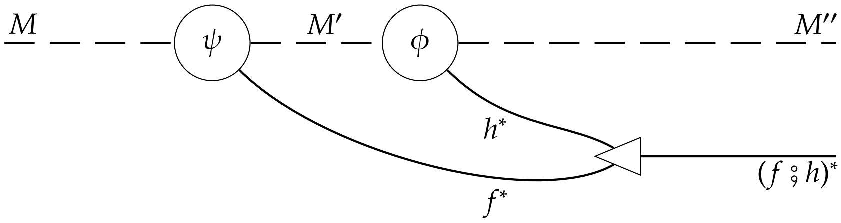
be another morphism in $\int \mathfrak{I}$. The composite of these two morphisms in $\int \mathfrak{I}$ is defined using the functors f^* and h^* as well as the natural transformation $\mathfrak{I}^{2*}_{fh}$. We denote the natural transformation $\mathfrak{I}^{2*}_{fh}$ in string diagrams by



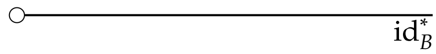
and we denote its inverse $(\mathfrak{I}^{2*}_{fh})^{-1}$ in string diagrams by



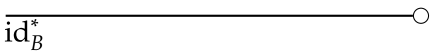
The total part of the composite $\begin{pmatrix}\psi\\ f\end{pmatrix} \fatsemi \begin{pmatrix}\phi\\ h\end{pmatrix}$ can therefore be represented using string diagrams as follows.



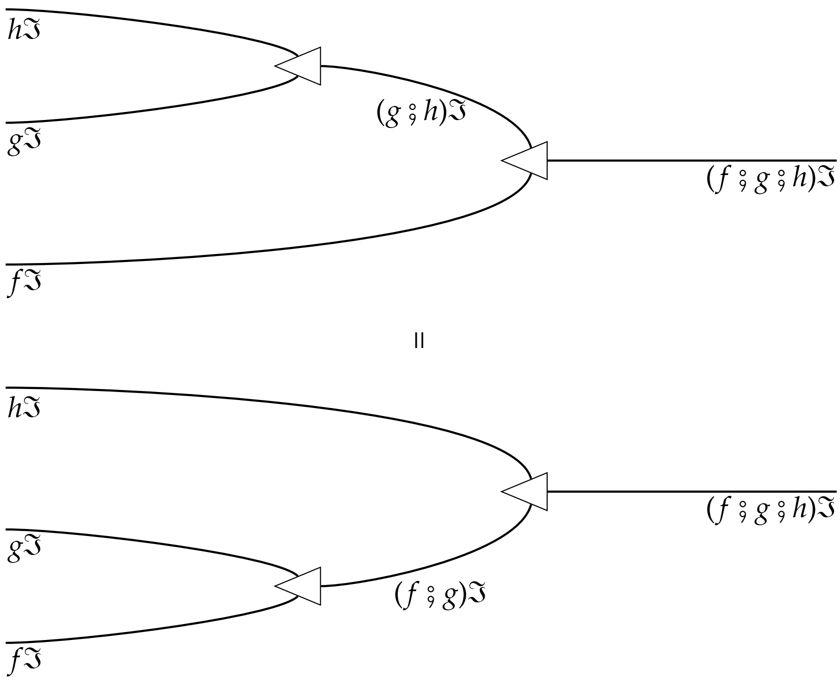
The identity morphisms in $\int \mathfrak{I}$ are defined using the natural transformation $\mathfrak{I}^{0*}_B$. We denote the natural transformation $\mathfrak{I}^{0*}_B$ in string diagrams by



and we denote its inverse $(\mathfrak{I}^{0*}_B)^{-1}$ in string diagrams by

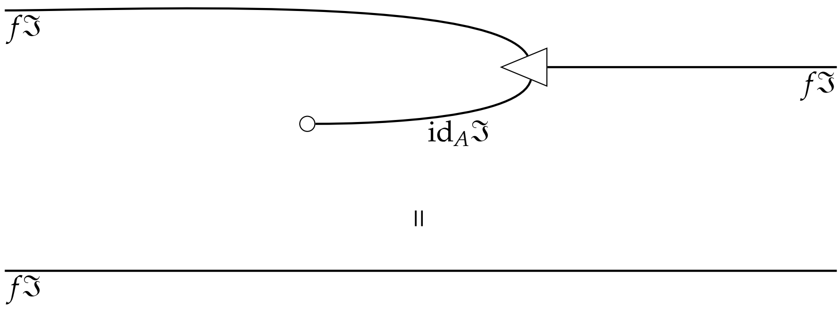


As a demonstration of the notation we've just explained, we will now express the the associativity axiom (see (1.1.9)) and the left and right unitality axioms (see (1.1.10) and (1.1.11)) for the pseudofunctor $\mathfrak{I}\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$ using string diagrams. The associativity axiom is written as the equality



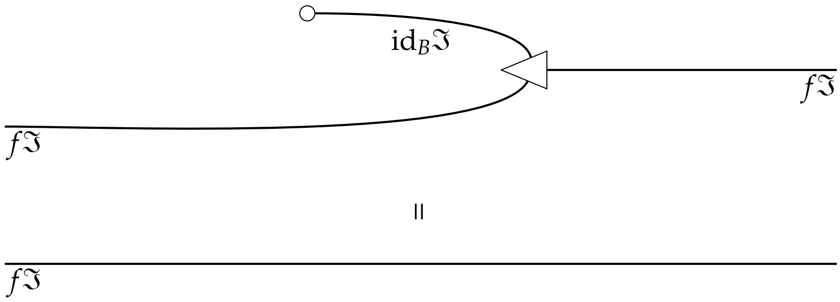
(2.3.2)

the left unitality axiom is written as the equality



(2.3.3)

and the right unitality axiom is written as the equality



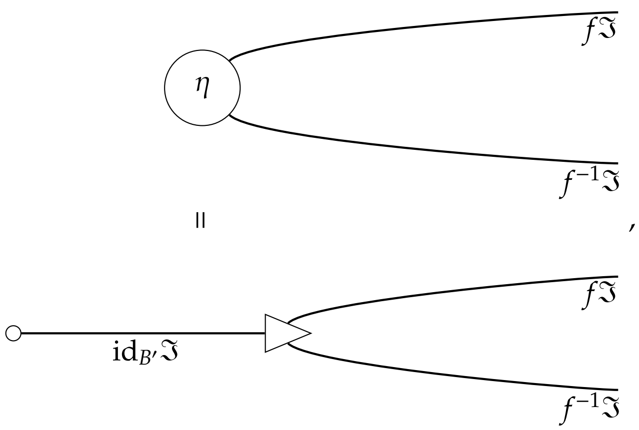
(2.3.4)

We can use these axioms to prove the following lemma. We will then use this lemma to prove Proposition 2.3.6 which will help us to understand isomorphisms in the total category $\int \mathfrak{I}$.

Lemma 2.3.5. *Let* $\mathfrak{I}\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$ *be an indexed category, and let* $f\colon B \to B'$ *be an isomorphism in* $\mathscr{B}$*. Define the natural transformation*

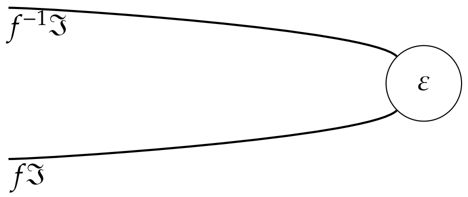
$$\eta\colon \mathrm{id}_{B'\mathfrak{I}} \to f^* \mathbin{⨾} f^{-1*}$$

by

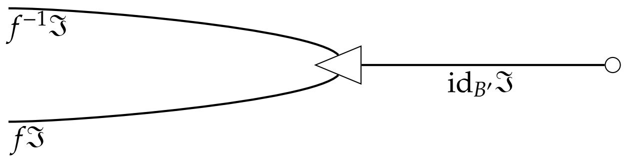


and define the natural transformation

$$\varepsilon\colon f^{-1*} \mathbin{⨾} f^* \to \mathrm{id}_{B\mathfrak{I}}$$



$=$.



In non-string diagrammatic language, η and ε are given by

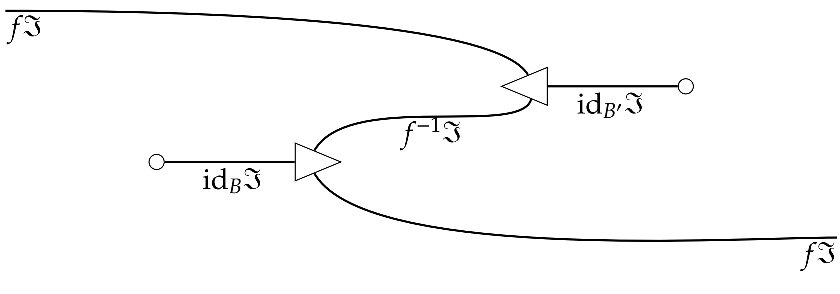
$$\eta = \mathfrak{I}^{0*}_{B'} \fatsemi (\mathfrak{I}^{2*}_{f^{-1}f})^{-1} \quad \textit{and} \quad \varepsilon_N = \mathfrak{I}^{2*}_{ff^{-1}} \fatsemi (\mathfrak{I}^{0*}_{B})^{-1}$$

Then

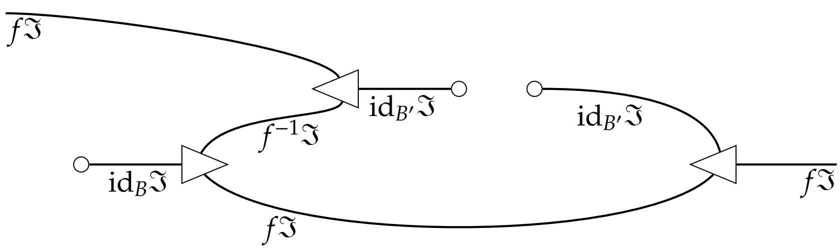
$$(f^*, f^{-1*}, \eta, \varepsilon) \colon B'\mathfrak{I} \to B\mathfrak{I}$$

is an adjoint equivalence.

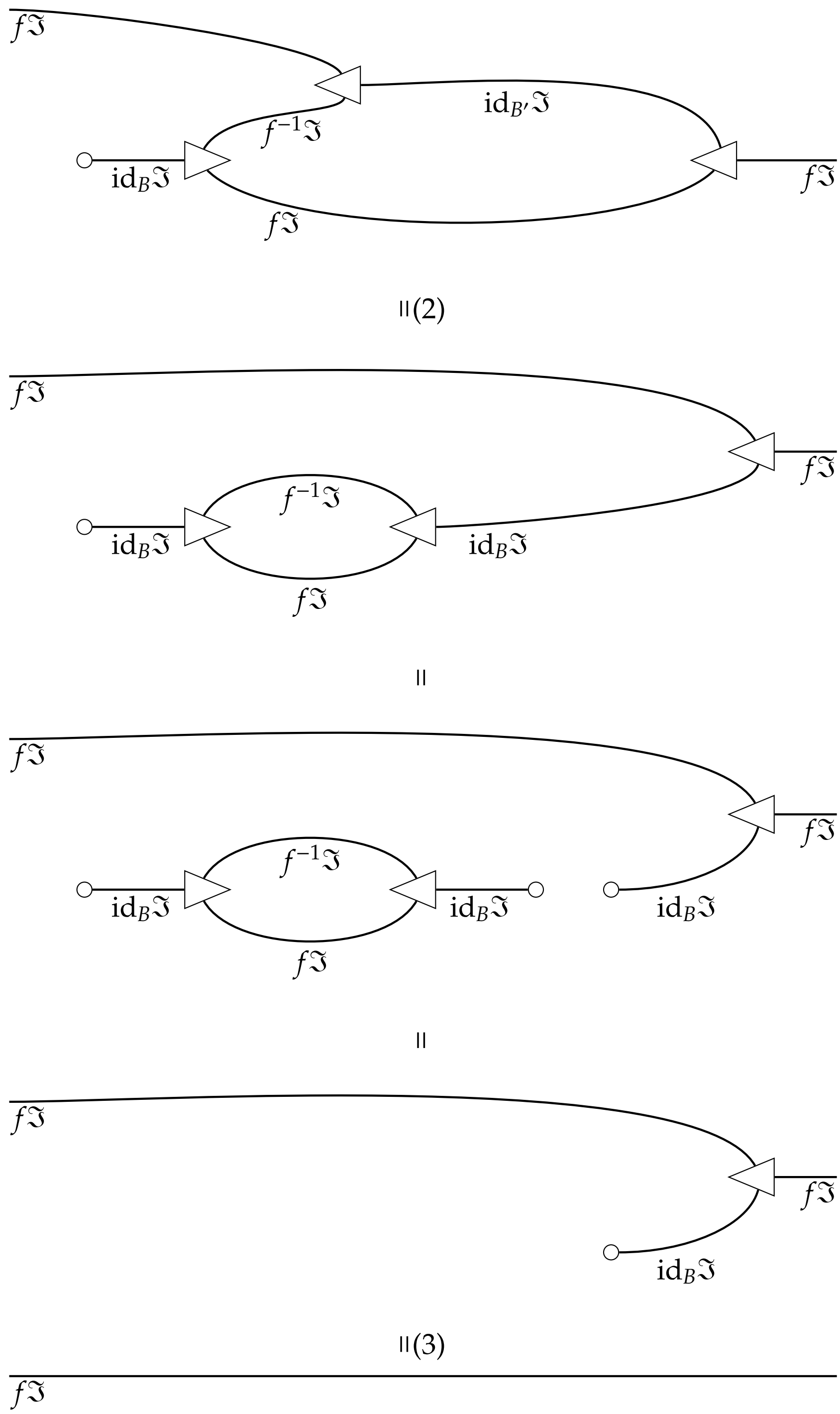
Proof. It's easy to see that η and ε are invertible. The following series of equalities proves one of the snake identities; the proof of the other is similar.



$=(1)$



$=$

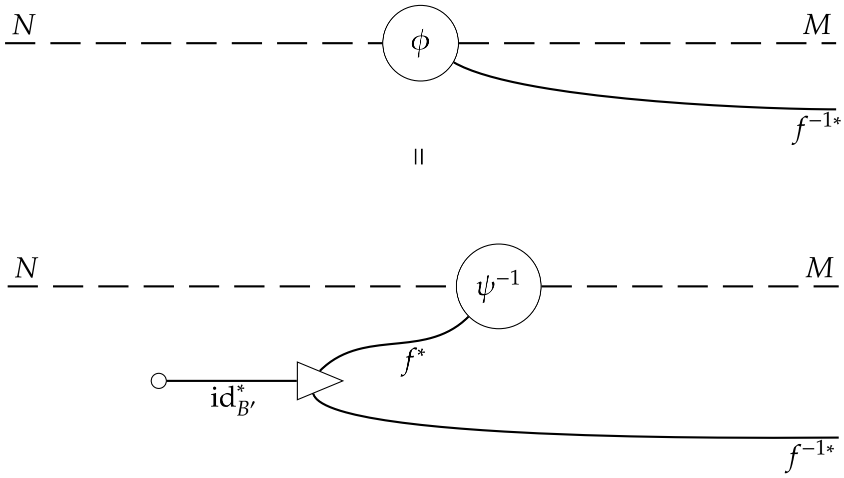


The equality (1) follows by the right unitality axiom (2.3.4), the equality (2) follows by the associativity axiom (2.3.2), and the equality (3) follows by the left unitality axiom (2.3.3). □

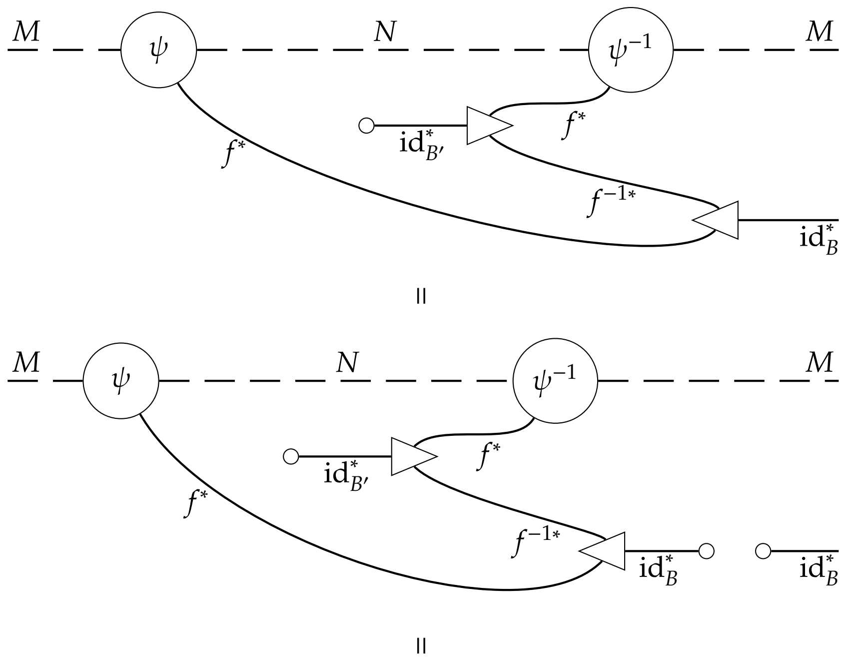
The strict version of the following result is given as Proposition 2.6.1.6 in [Mye23]; we provide a proof using string diagrams.

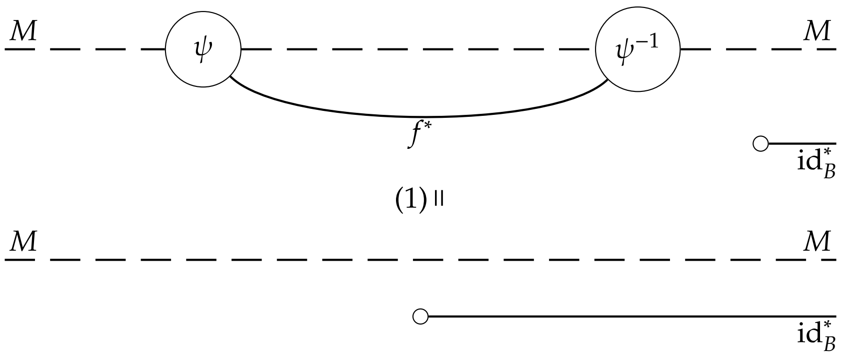
Proposition 2.3.6. *A morphism* $\begin{pmatrix}\psi\\ f\end{pmatrix}\colon \begin{pmatrix}M\\ B\end{pmatrix} \to \begin{pmatrix}N\\ B'\end{pmatrix}$ *in* $\int \mathfrak{I}$ *is an isomorphism if, and only if,* ψ *is an isomorphism in* $B\mathfrak{I}$ *and* f *is an isomorphism in* $\mathscr{B}$.

Proof. Suppose that ψ is an isomorphism in $B\mathfrak{I}$ and that f is an isomorphism in $\mathscr{B}$. Define the morphism $\phi\colon N \to Mf^{-1^*}$ in $B'\mathfrak{I}$ using a string diagram as follows.



We'll show that $\begin{pmatrix}\phi\\ f^{-1}\end{pmatrix}$ is inverse to $\begin{pmatrix}\psi\\ f\end{pmatrix}$ in $\int \mathfrak{I}$, where ϕ is given by the following figure. The total part of $\begin{pmatrix}\psi\\ f\end{pmatrix} ⨾ \begin{pmatrix}\phi\\ f^{-1}\end{pmatrix}$ is equal to $\mathfrak{I}^{0*}_{B,M}\colon M \to M\mathbf{id}^*$, as shown by the following series of equalities.

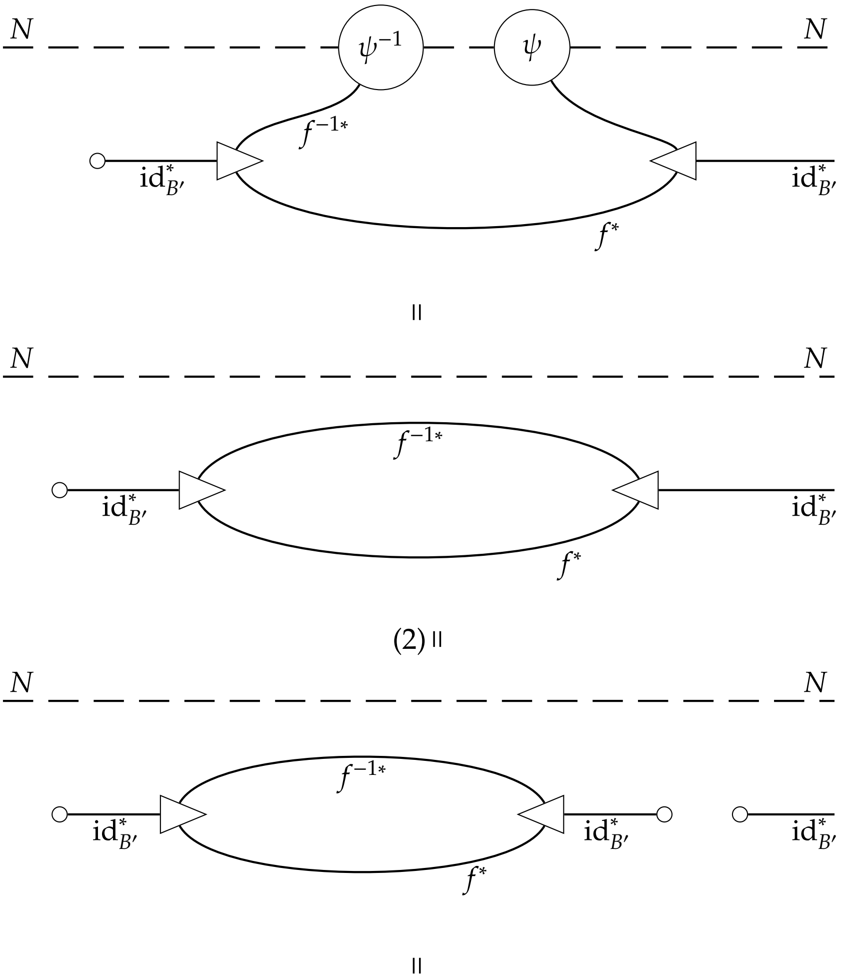


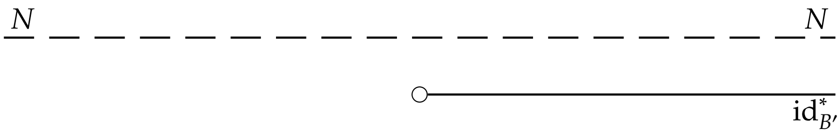


The equality (1) follows from Lemma 2.3.5. Therefore,

$$\begin{pmatrix}\phi\\ f\end{pmatrix} \fatsemi \begin{pmatrix}\psi\\ f^{-1}\end{pmatrix} = \mathrm{id}_{\left(\begin{smallmatrix}M\\ B\end{smallmatrix}\right)}.$$

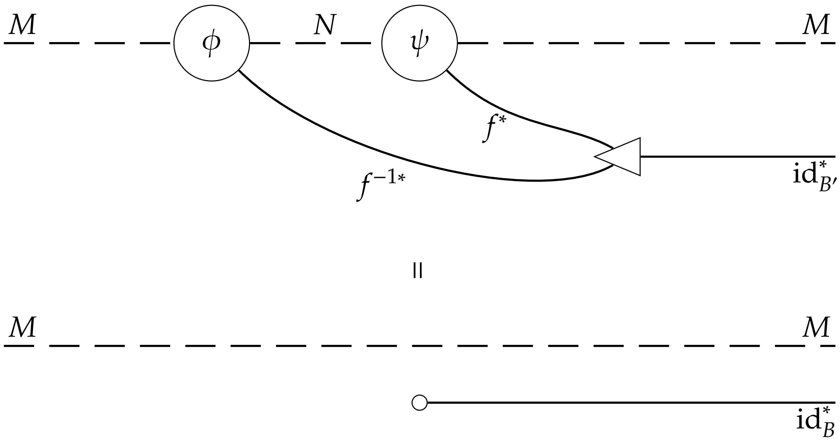
The total part of $\begin{pmatrix}\phi\\ f^{-1}\end{pmatrix} \fatsemi \begin{pmatrix}\psi\\ f\end{pmatrix}$ is equal to $\mathfrak{I}^{0*}_{B',N} \colon N \to N\mathrm{id}^*$, as shown below.



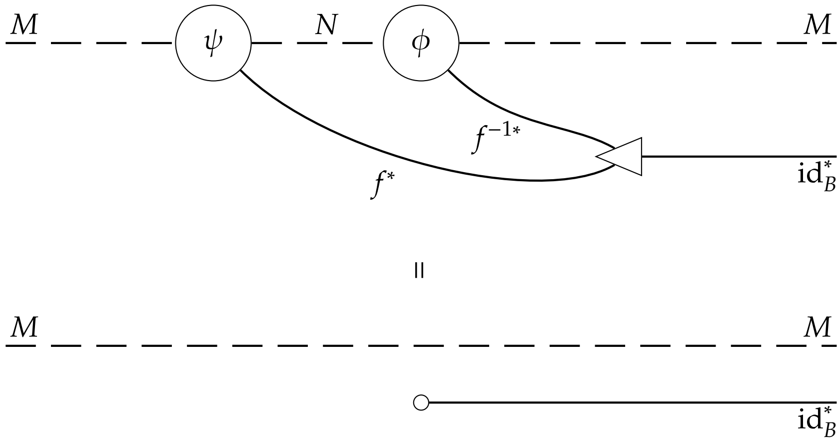


The equality (2) follows from Lemma 2.3.5. Therefore, $\begin{pmatrix}\phi\\ f\end{pmatrix} ⨟ \begin{pmatrix}\psi\\ f^{-1}\end{pmatrix} = \mathbf{id}_{\left(\begin{smallmatrix}N\\ B'\end{smallmatrix}\right)}$.

We've proved that $\begin{pmatrix}\psi\\ f\end{pmatrix}$ is an isomorphism if ψ and f are both isomorphisms. Now we prove the converse. Suppose that $\begin{pmatrix}\psi\\ f\end{pmatrix}$ is an isomorphism in $\int \mathfrak{I}$ with inverse $\begin{pmatrix}\phi\\ g\end{pmatrix}$. So $f ⨟ g = \mathbf{id}_B$, $g ⨟ f = \mathbf{id}_{B'}$ and the following two equalities (2.3.7) and (2.3.8).

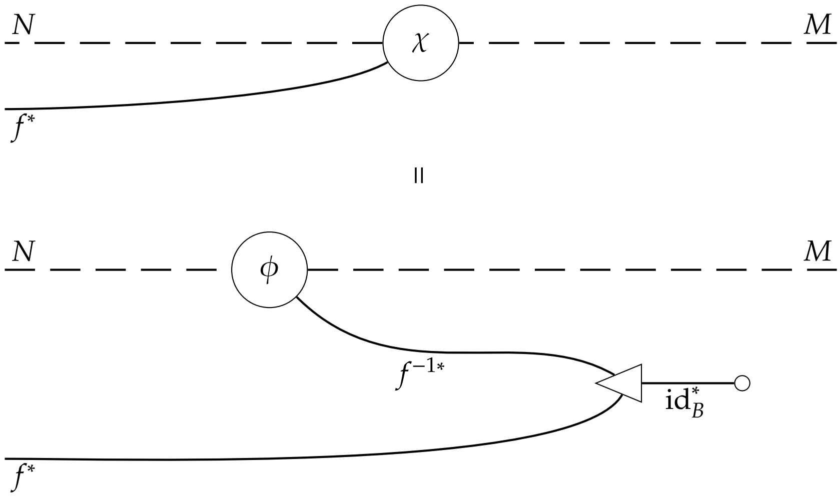


(2.3.7)

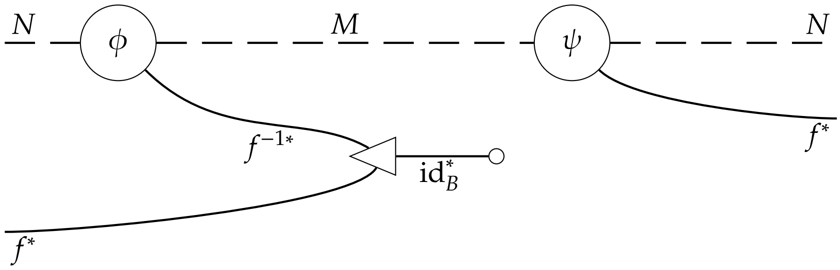


(2.3.8)

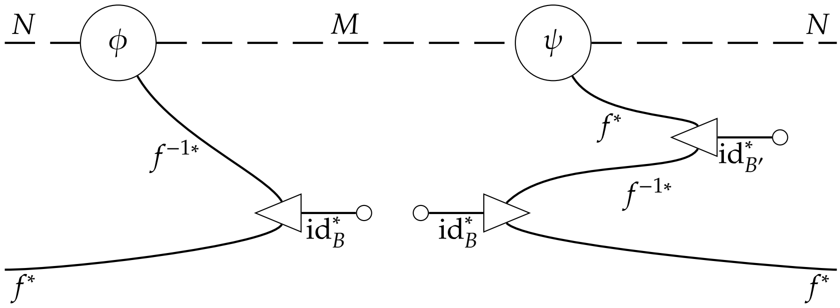
We have immediately that f is an isomorphism in $\mathscr{B}$ with inverse g. We'll now show that ψ is an isomorphism in $B\mathfrak{I}$ with inverse



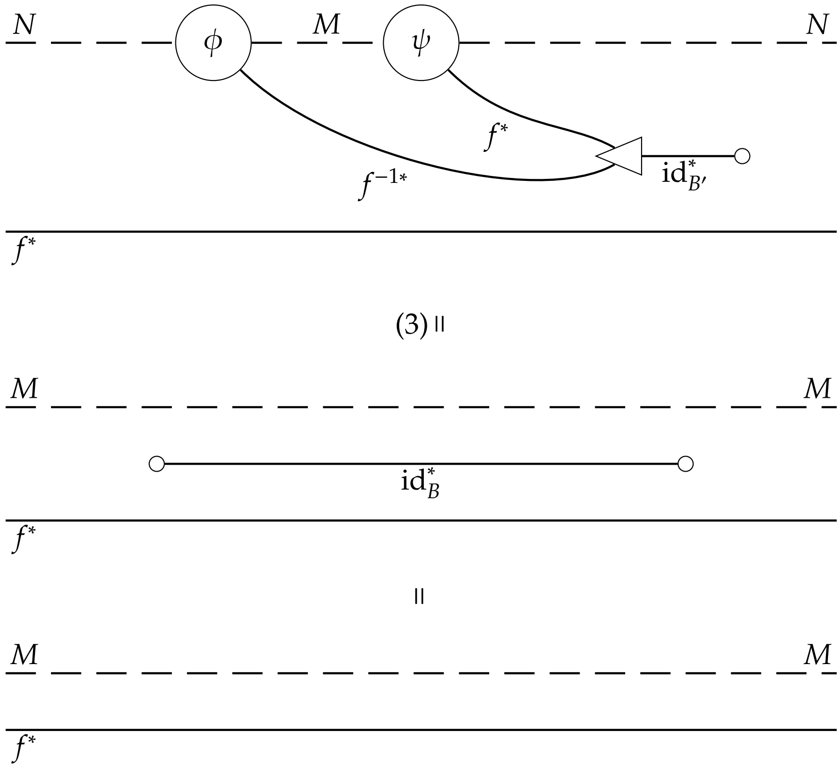
The fact that $\psi \fatsemi \chi = \mathbf{id}$ follows almost immediately from (2.3.7). The fact that $\chi \fatsemi \psi = \mathbf{id}$ is shown by the following series of equalities.



(1) =



(2) =



The equalities (1) and (2) follow by Lemma 2.3.5 and the equality (3) follows by (2.3.7). □

Grothendieck originally used the term 'catégorie fibrée' (fibred category) for indexed categories [Gro71], but, due to Bénabou [Bén75], this later became the standard term for the fibration associated to an indexed category via the Grothendieck construction, the definition of which we give now.

Proposition 2.3.9. *Let* $\mathfrak{I}\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$ *be an indexed category. The following functor is a fibration.*

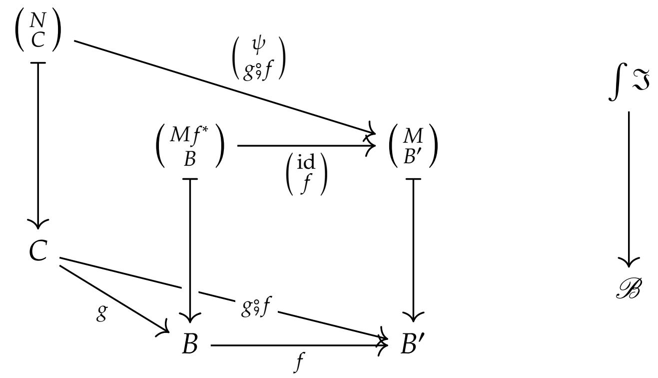
$$\begin{aligned} \mathfrak{I}\mathcal{G}\colon \int \mathfrak{I} &\longrightarrow \mathscr{B} \\ \begin{pmatrix} M \\ B \end{pmatrix} &\longmapsto B \\ \begin{pmatrix} \phi \\ f \end{pmatrix} &\longmapsto f \end{aligned}$$

We call this fibration the ***fibration associated to*** $\mathfrak{I}$.

Proof. For each morphism $f\colon B \to B'$ in $\mathscr{B}$ and each object M in $B'\mathfrak{I}$, we'll see that the morphism

$$\begin{pmatrix} \mathrm{id}\colon Mf^* \to Mf^* \\ f \end{pmatrix} \colon Mf^* \to M$$

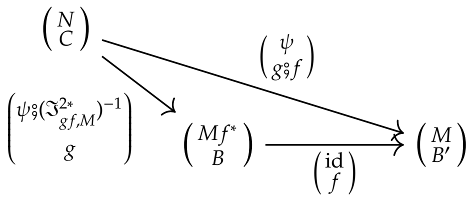
is cartesian. Suppose that $g\colon C \to B$ is a morphism in $\mathscr{B}$ and that $\begin{pmatrix}\psi \\ g \fatsemi f\end{pmatrix}\colon N \to M$ is a morphism in $\int \mathfrak{I}$, as shown in the following figure.



Then the morphism

$$\begin{pmatrix}\psi \fatsemi (\mathfrak{I}^{2*}_{gf,M})^{-1} \\ g\end{pmatrix}\colon N \to Mf^*$$

in $\int \mathfrak{I}$ is the only morphism that lies over g and makes the following diagram commute.



Therefore, the morphism $\begin{pmatrix}\mathrm{id}\colon Mf^* \to Mf^* \\ f\end{pmatrix}\colon Mf^* \to M$ is cartesian, which means that $\mathfrak{I}\mathcal{G}\colon \int \mathfrak{I} \to \mathscr{B}$ is a fibration. □

The following proposition is given as the definition of cartesian morphisms by Johnstone, though he uses the terminology 'prone morphisms'.

Proposition 2.3.10 ([Joh02, Lemma 1.3.2]). *Let $\mathfrak{I}$ be an indexed category. A morphism $\binom{\psi}{f}$ in $\int \mathfrak{I}$ is cartesian with respect to the fibration $\mathfrak{I}\mathcal{G}$ if, and only if, ψ is an isomorphism.*

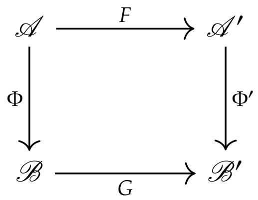
Proof. From Proposition 2.3.9 we have a cleaving $\mathsf{cart}^f_M = \begin{pmatrix}\mathrm{id}_{Mf^*} \\ f\end{pmatrix}$ for the fibration $\mathfrak{I}\mathcal{G}$. By Corollary 2.1.25, $\binom{\psi}{f}$ is cartesian if, and only if, there exists a pure isomorphism $\binom{\chi}{\mathrm{id}}$ and a cartesian morphism $\binom{\mathrm{id}}{f}$ in the cleaving such that $\binom{\psi}{f} = \binom{\chi}{\mathrm{id}} \fatsemi \binom{\mathrm{id}}{f}$. The total part of this equality is $\psi = \chi \fatsemi \mathfrak{I}^{2*}_{\mathrm{id},f,M}$, which can be rearanged to $\chi = \psi \fatsemi (\Im^{2*}_{\mathrm{id},f,M})^{-1}$. Therefore, the morphisms $\begin{pmatrix}\psi\\ f\end{pmatrix}$ for which there exists a pure isomorphism $\begin{pmatrix}\chi\\ \mathrm{id}\end{pmatrix}$ such that $\psi f = \begin{pmatrix}\chi\\ \mathrm{id}\end{pmatrix} \fatsemi \begin{pmatrix}\mathrm{id}\\ f\end{pmatrix}$ are exactly those for which ψ is an isomorphism. □

2.3.1 The Grothendieck construction as a 2-equivalence

In this subsection, we state the Grothendieck construction as a 2-equivalence. This is given in [Bor94b, Theorem 8.3.1] and [Joh02, Theorem 1.3.6]; a more detailed breakdown can be found in [JY21, Section 10.6]. We begin by defining a 2-category of fibrations. This 2-equivalence will be useful in Section 2.7 where we discuss monoidal fibrations and monoidal indexed categories.

Definition 2.3.11 ([JY21, Theorem 10.6.16]).

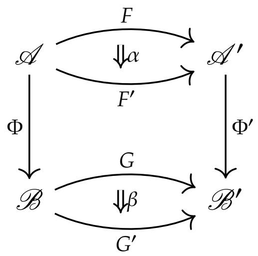
- Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ and $\Phi'\colon \mathscr{A}' \to \mathscr{B}'$ be fibrations. A **fibred 1-cell** $\Phi \to \Phi'$ consists of a pair of functors $F\colon \mathscr{A} \to \mathscr{A}'$ and $G\colon \mathscr{B} \to \mathscr{B}'$ such that the diagram



  in **Cat** commutes and, for every Φ-cartesian morphism ϕ in $\mathscr{A}$, the morphism ϕF in $\mathscr{A}'$ is Φ'-cartesian.

- We call a fibred 1-cell $(F, G)\colon \Phi \to \Phi'$ **pure** if $\mathscr{B} = \mathscr{B}'$ and $G = \mathrm{id}_{\mathscr{B}}$.

- Let (F, G) and (F', G') be fibred 1-cells $\Phi \to \Phi'$. A **fibred 2-cell** consists of a pair of natural transformations $\alpha\colon F \to F'$ and $\beta\colon G \to G'$ such that, for every object M in $\mathscr{A}$, the morphism $\alpha_M\colon MF \to MF'$ in $\mathscr{A}'$ lies over the morphism $\beta_{M\Phi}\colon M\Phi G \to M\Phi G'$ in $\mathscr{B}'$. This situation is shown in the following figure.



- We call a fibred 2-cell $(\alpha, \beta)\colon (F, G) \to (F', G')$ **pure** if (F, G) and (F', G') are pure fibred 1-cells and if, for every object M in $\mathscr{A}$, the morphism $\alpha_M\colon MF \to MF'$ in $\mathscr{A}'$ lies over the identity.

◇

Definition 2.3.12.

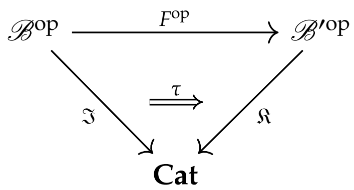
- The 2-category $\mathcal{Fib}$ of fibrations consists of fibrations, fibred 1-cells and fibred 2-cells.
- Let $\mathscr{B}$ be a category. The 2-category $\mathcal{Fib}_{\mathscr{B}}$ of fibrations over $\mathscr{B}$ consists of fibrations over $\mathscr{B}$, pure fibred 1-cells and pure fibred 2-cells.

◇

We now define a 2-category of indexed categories.

Definition 2.3.13.

- Let $\mathfrak{I}\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$ and $\mathfrak{K}\colon \mathscr{B}'^{\mathrm{op}} \to \mathbf{Cat}$ be indexed categories. An **indexed 1-cell** $\mathfrak{I} \to \mathfrak{K}$ consists of a functor $F\colon \mathscr{B} \to \mathscr{B}'$ and a pseudonatural transformation τ:



- We call an indexed 1-cell $(F, \tau)\colon \mathfrak{I} \to \mathfrak{K}$ **pure** if $\mathscr{X} = \mathscr{B}'$ and $F = \mathrm{id}_{\mathscr{B}}$.
- Let (F, τ) and (G, σ) be indexed 1-cells $\mathfrak{I} \to \mathfrak{K}$. An **indexed 2-cell** $\mathfrak{I} \to \mathfrak{K}$ consists of a natural transformation $\alpha\colon F \to G$ and a modification m:

- We call an indexed 2-cell $(\alpha, m)\colon (F, \tau) \to (G, \sigma)$ a **pure** if (F, τ) and (G, σ) are pure indexed 1-cells and $\alpha = \mathrm{id}_{\mathrm{id}_{\mathscr{B}}}$.

◇

Definition 2.3.14.

- The 2-category IndCat of indexed categories consists of indexed categories, indexed 1-cells and indexed 2-cells.
- Let $\mathscr{B}$ be a category. The 2-category $\mathit{IndCat}_{\mathscr{B}}$ of indexed categories over $\mathscr{B}$ consists of indexed categories over $\mathscr{B}$, pure indexed 1-cells and pure indexed 2-cells; this is equal to the 2-category $\mathbf{2Cat}^{\mathrm{ps}}(\mathscr{B}^{\mathrm{op}}, \mathbf{Cat})$.

◇

Now that we've defined 2-categories of fibrations and of indexed categories, we can state the Grothendieck constrction as a 2-equivalence of 2-categories.

Theorem 2.3.15.

- *The function that associates a cleaved fibration $\mathfrak{I}\mathcal{G}\colon \int \mathfrak{I} \to \mathscr{B}$ to an indexed category $\mathfrak{I}\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$ extends to a functor of 2-categories*

$$\mathcal{G}\colon \mathcal{IndCat}_{\mathscr{B}} \to \mathcal{Fib}_{\mathscr{B}}.$$

- *The function that associates an indexed category $\Phi\mathcal{G}^{-1}\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$ to a cleaved fibration $\Phi\colon \mathscr{A} \to \mathscr{B}$ extends to a functor of 2-categories*

$$\mathcal{G}^{-1}\colon \mathcal{Fib}_{\mathscr{B}} \to \mathcal{IndCat}_{\mathscr{B}}.$$

- *The functors $\mathcal{G}$ and $\mathcal{G}^{-1}$ are part of a 2-equivalence $\mathcal{IndCat}_{\mathscr{B}} \simeq \mathcal{Fib}_{\mathscr{B}}$.*

- *The above 2-equivalence extends to a 2-equivalence $\mathcal{IndCat} \simeq \mathcal{Fib}$.*

2.4 Opfibrations and opindexed categories

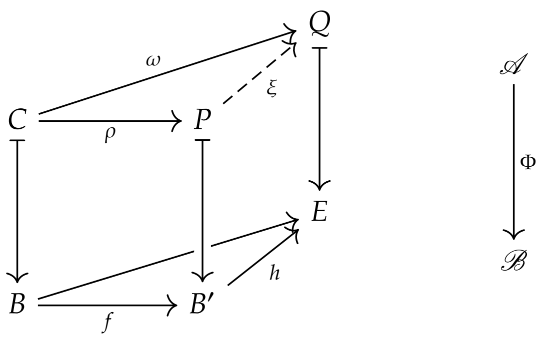
In this section, we look at the dual notions of cartesian morphism and fibration: opcartesian morphisms and opfibrations. The property of a functor $\Phi\colon \mathscr{A} \to \mathscr{B}$ being a fibration enables us to 'pull' an object M in the fibre category $\mathscr{A}_{B'}$ back along a morphism $f\colon B \to B'$ in $\mathscr{B}$ to obtain an object Mf^* in the fibre category $\mathscr{A}_B$. Our first example of this (see Section 2.1) was pulling an H-module W back along a group homomorphism $f\colon G \to H$ to obtain the G-module ${}_fW$ known as the *restriction*. The dual of restriction is **induction**. Given a group homomorphism $f\colon G \to H$ and a G-module V, the induced H-module is $\mathbb{C}H \otimes_G V$, where the group algebra $\mathbb{C}H$ is a G-module via restriction! Notice that induction 'pushes' the G-moudle V along the group homomorphism $f\colon G \to H$ to obtain an H-module. We'll soon use induction in Example 2.4.9 to show that the acting group functor $\mathtt{Rep}\colon \mathbf{GrpRep} \to \mathbf{FinGrp}$ is an opfibration.

Definition 2.4.1. We call a functor $\Phi\colon \mathscr{A} \to \mathscr{B}$ an **opfibration** if the opposite functor $\Phi^{\mathrm{op}}\colon \mathscr{A}^{\mathrm{op}} \to \mathscr{B}^{\mathrm{op}}$ is a fibration. ◇

Definition 2.4.2. We call a morphism $\rho\colon C \to P$ in $\mathscr{A}$ **opcartesian** if it Φ^{op}-cartesian as a morphism in $\mathscr{A}^{\mathrm{op}}$. ◇

The following proposition unpacks what it means for a morphism to be opcartesian.

Proposition 2.4.3. *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a functor, let $\rho\colon C \to P$ be a morphism in $\mathscr{A}$ and let $f\colon B \to B'$ denote the morphism $\rho\Phi$ in $\mathscr{B}$. The morphism $\rho\colon C \to P$ is opcartesian if, and only if, for each morphism $h\colon B' \to E$ in $\mathscr{B}$ and each morphism $\omega\colon C \to Q$ in $\mathscr{A}$ that lies over $f \mathbin{⨾} h$, there exists a unique map $\xi\colon P \to Q$ that lies* *over h and that satisfies $\omega = \rho \fatsemi \xi$. This situation is depicted in the following figure.*



(2.4.4)

□

In analogy to Proposition 2.1.4, the following is a less verbose definition of an opcartesian morphism.

Proposition 2.4.5. *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a functor, let $\rho\colon C \to P$ be a morphism in $\mathscr{A}$, and let $f\colon B \to B'$ denote the morphism $\rho\Phi$ in $\mathscr{B}$. Then the morphism $\rho\colon C \to P$ is opcartesian if, and only if, for each morphism $h\colon B' \to E$ in $\mathscr{B}$ and each object Q in $\mathscr{A}$, the map*

$$\begin{aligned} \mathscr{A}^{h}(P,Q) &\longrightarrow \mathscr{A}^{f\fatsemi h}(C,Q) \\ \xi &\longmapsto \rho \fatsemi \xi \end{aligned} \tag{2.4.6}$$

is a bijection.

Notation 2.4.7. The unique morphism ξ in (2.4.4) will be written as $\rho \searrow \omega$. This is meant to make the reader think of taking the equation $\rho \fatsemi \xi = \omega$ and 'dividing' both sides on the left by ρ to get the equation $\xi = \rho \searrow \omega$. With this notation we can write the inverse to the bijection (2.4.6) as

$$\begin{aligned} \mathscr{A}^{f\fatsemi h}(C,Q) &\longrightarrow \mathscr{A}^{h}(P,Q) \\ \omega &\longmapsto \rho \searrow \omega \end{aligned}.$$

◇

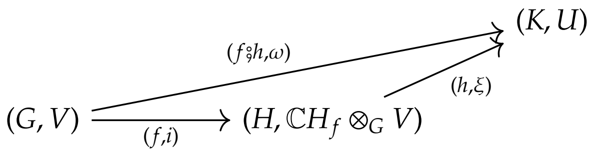
It is easy to check that opcartesian morphisms and opfibratios and are analogous to cartesian morphisms and fibrations:

Proposition 2.4.8. *The functor $\Phi\colon \mathscr{A} \to \mathscr{B}$ is an opfibration if, and only if, for every morphism $f\colon B \to B'$ in $\mathscr{B}$ and every object C in $\mathscr{A}$ lying over B, there exists an opcartesian morphism $\rho\colon C \to P$ that lies over f.* □

In Examples 2.1.16 and 2.1.19, we showed that the functors $\mathtt{Rep}$ and $\mathtt{Fam}_{\mathscr{C}}$ are fibrations. We now show that they are also opfibrations in Examples 2.4.9 and 2.4.11.

Example 2.4.9. We'll show that the forgetful functor $\mathtt{Rep}\colon \mathbf{GrpRep} \to \mathbf{FinGrp}$ is an opfibration. Let $f\colon G \to H$ be a group homomorphism, let (G,V) be a representation lying over G, and define the linear map $i\colon V \to \mathbb{C}H_f \otimes_G V$ by $(v)i = e_H \otimes v$. We'll show that the morphism $(f,i)\colon (G,V) \to (H, \mathbb{C}H_f \otimes_G V)$ in $\mathbf{GrpRep}$, which lies over f, is opcartesian.

Suppose that $x\colon H \to K$ is a group homomorphism and suppose that $(f \mathbin{⨾} x, \omega)\colon (G, V) \to (K, U)$ is a module map that lies over $f \mathbin{⨾} x$. Define the morphism $(h, \xi)\colon (H, \mathbb{C}H_f \otimes_G V) \to (K, U)$ in **GrpRep** by $(h \otimes v)\xi = (h)x \cdot (v)\omega$. The diagram



(2.4.10)

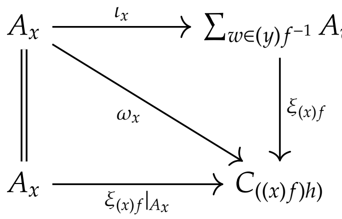
in **GrpRep** commutes because, for each v in V, $(v)\omega = (e_H \otimes v)\xi = (v)(i \mathbin{⨾} \xi)$. If $(h, \zeta)\colon (H, \mathbb{C}H_f \otimes_G V) \to (K, U)$ is a morphism in **GrpRep** making (2.4.10) commute, then

$$(h \otimes v)\zeta = (h \cdot (e_H \otimes v))\zeta = (h)x \cdot (e_H \otimes v)\zeta = (h)x \cdot (v)(i \mathbin{⨾} \zeta) = (h)x \cdot (v)\omega.$$

Therefore, (h, ξ) is the unique morphism $(H, \mathbb{C}H_f \otimes_G V) \to (K, U)$ that lies over h and makes (2.4.10) commute. Hence, (f, i) is opcartesian, and so $\mathtt{Rep}\colon$ **GrpRep** $\to$ **FinGrp** is an opfibration. ⋄

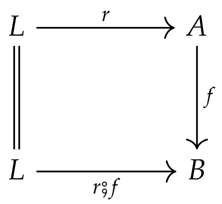
Example 2.4.11. Let $\mathscr{C}$ be a category with arbitrary coproducts; we'll denote coproducts by $\sum$. We'll show that the forgetful functor $\mathtt{Fam}_{\mathscr{C}}\colon \mathbf{Fam}_{\mathscr{C}} \to \mathbf{Set}$ is an opfibration. Let $f\colon X \to Y$ be a map of sets, let (A, X) be an X-indexed family of objects. Let A_f denote the Y-indexed family of objects $(\sum_{w \in (y)f^{-1}} A_w)_{y \in Y}$, and, for each $x \in X$, let ι_x denote the inclusion morphism $A_{f(x)} \to \sum_{w \in f^{-1}(f(x))} A_w$. We'll show that the morphism $(\iota, f)\colon (A, X) \to (A_f, Y)$ in $\mathbf{Fam}_{\mathscr{C}}$, which lies over f, is opcartesian.

Suppose that $h\colon Z \to X$ is a map of sets and suppose that $(\omega, f \mathbin{⨾} h)\colon (A, X) \to (C, Z)$ is morphism in $\mathbf{Fam}_{\mathscr{C}}$ lying over $f \mathbin{⨾} h$. Suppose that $(\xi, h)\colon (A_f, Y) \to (C, Z)$ is a morphism in $\mathbf{Fam}_{\mathscr{C}}$ satisfying $(\iota, f) \mathbin{⨾} (\xi, h) = (\omega, f \mathbin{⨾} h)$. Then the following diagram in $\mathscr{C}$ commutes.



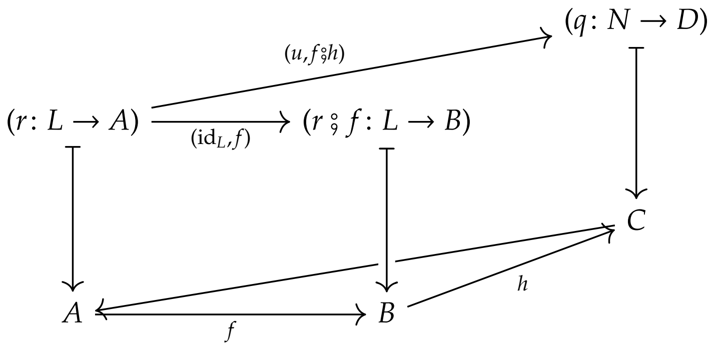
Therefore, if $y \in (f)\,\mathrm{im}$, we have, for each $w \in (y)f^{-1}$, $\xi_y|_{A_w} = \omega_x$, and if $y \notin (f)\,\mathrm{im}$, then we must take ξ_y to be the unique morphism $\sum_{w \in f^{-1}(y)} A_w \to C_{h(y)}$. Therefore, (ξ, h) is the unique morphism that lies over h and that satisfies $(\iota, f) \mathbin{⨾} (\xi, h) = (\omega, f \mathbin{⨾} h)$, so (ι, f) is opcartesian. ⋄

Example 2.4.12. We'll show that the codomain functor $\mathtt{Arr}_{\mathscr{A}}\colon \mathscr{A}^{\to} \to \mathscr{A}$ is an opfibration. Let $f\colon A \to B$ and $r\colon L \to A$ be a morphisms in $\mathscr{A}$, so r is an object in $\mathscr{A}^{\to}$ and $(r)\mathtt{Arr} = A$. Consider the commutative square



in $\mathscr{A}$. We'll show that the morphism $(\mathrm{id}_L, f)\colon r \to r \mathbin{⨟} f$ in $\mathscr{A}^{\to}$, which lies over f, is opcartesian.

Suppose that $h\colon B \to D$, $q\colon N \to D$ and $u\colon L \to N$ are morphisms in $\mathscr{A}$, as shown in the following figure.



Rewrite the square

$$\begin{array}{ccc} L & \xrightarrow{r} & A \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle f ⨟ h} \\ N & \xrightarrow[q]{} & D \end{array}$$

that defines the morphism $(u, f \mathbin{⨟} h)\colon r \to q$ as

$$\begin{array}{ccc} L & \xrightarrow{r} & A \\ \| & & \downarrow{\scriptstyle f} \\ L & \xrightarrow[r ⨟ f]{} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle h} \\ N & \xrightarrow[q]{} & D. \end{array}$$

It then becomes clear that $(u, h)\colon r \mathbin{⨟} f \to q$ is the unique morphism $r \mathbin{⨟} f \to q$ in $\mathscr{A}^{\to}$ that lies over h (that is, of the form $(\_, h)$) and satisfies $(\mathrm{id}_L, f) \mathbin{⨟} (u, h) = (u, f \mathbin{⨟} h)$. Therefore, the morphism $(\mathrm{id}_L, f)\colon r \to r \mathbin{⨟} f$ is opcartesian. ◇

The reader may wish to skip the remainder of this section on first reading as the definitions and propositions stated are directly analogous to those in Sections 2.1 and 2.2.

Definition 2.4.13. Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be an opfibration.

- An **opcleaving** for Φ is a choice, for each morphism $f\colon B \to B'$ in $\mathscr{B}$ and each object C in $\mathscr{A}$ lying over B, of opcartesian morphism $\mathrm{opcart}_C^f\colon C \to P$ in $\mathscr{A}$ that lies over f.
- An **opcleaved fibration** is an opfibration equipped with an opcleaving.

- A morphism ξ in $\mathscr{A}$ is called **pure** if it lies over an identity morphism.
- We call $\mathscr{A}$ the **total category** of Φ and we call $\mathscr{B}$ the **base category** of Φ.

◇

Notation 2.4.14. Given an opcleaved fibration $\Phi\colon \mathscr{A} \to \mathscr{B}$, we will—unless it's unclear from context—denote the opcartesian morphisms the opcleaving by $\mathrm{opcart} = (\mathrm{opcart}^f_C)$. ◇

If $\Phi\colon \mathscr{A} \to \mathscr{B}$ is a fibration, then we have pure-cartesian factorisation of morphism in $\mathscr{A}$ (see Proposition 2.1.14). Similarly, if $\Phi\colon \mathscr{A} \to \mathscr{B}$ is an opfibration, then we have **opcartesian-pure factorisation** of morphisms in $\mathscr{A}$.

Proposition 2.4.15. *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be an opfibration. Then, for each morphism $\omega\colon C \to Q$ in $\mathscr{A}$, there exists an opcartesian morphism ρ and a pure morphism ξ such that $\omega = \rho \fatsemi \xi$. In particular, if we fix an opcleaving for Φ, there exists a unique opcartesian morphism ρ in the opcleaving and a unique pure morphism χ such that $\psi = \chi \fatsemi \phi$.* □

We have the following basic properties of opcartesian morphisms; these are analogous to those for cartesian morphisms we gave in Proposition 2.1.24.

Proposition 2.4.16. *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a functor.*

(i) Let $\rho\colon C \to P$ and $\rho'\colon P \to R$ be morphisms in $\mathscr{A}$. If ϕ and ϕ' are opcartesian, then $\phi \fatsemi \phi'$ is opcartesian.

(ii) Let $\rho\colon C \to P$ and $\xi\colon P \to Q$ be morphisms in $\mathscr{A}$. If ρ and $\rho \fatsemi \xi$ are opcartesian, then ξ is opcartesian.

(iii) Let $\rho\colon C \to P$ and $\omega\colon C \to Q$ be opcartesian morphisms in $\mathscr{A}$. If $\rho\Phi = \omega\Phi$, then there exists a unique isomorphism $\xi\colon P \to Q$ that lies over $\mathrm{id}_{B'}$ and satisfies $\omega = \rho \fatsemi \xi$.

(iv) Every isomorphism in $\mathscr{A}$ is opcartesian.

(v) Let ρ be a morphism in $\mathscr{A}$ and suppose that $\rho\Phi$ is an isomorphism in $\mathscr{B}$. Then, ρ is opcartesian if, and only if, ρ is an isomorphism.

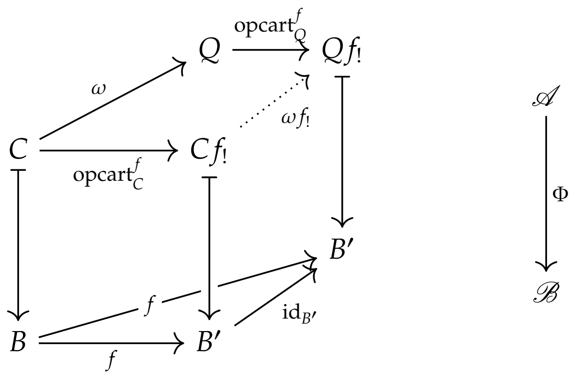
Given an opcleaving of Φ, all other opcartesian morphisms can be obtained from those in the opcleaving.

Corollary 2.4.17. *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a cleaved opfibration. Then, a morphism ω in $\mathscr{A}$ is opcartesian if, and only if, there exists an opcartesian morphism ρ and a pure isomorphism ξ such that $\omega = \rho \fatsemi \xi$. In particular, if we fix an opcleaving for Φ, there exists a unique opcartesian morphism ρ in the opcleaving and a unique pure isomorphism ξ such that $\omega = \rho \fatsemi \xi$.*

Just as a cleaved fibration gives us a pull-back functor functor $f^*\colon \mathscr{A}_{B'} \to \mathscr{A}_B$ for each morphism $f\colon B \to B'$ in $\mathscr{B}$ (see Proposition 2.1.27), an opcleaved opfibration gives us a **push-forward** funtor $f_!\colon \mathscr{A}_B \to \mathscr{A}_{B'}$ for each morphism $f\colon B \to B'$ in $\mathscr{B}$.

Proposition 2.4.18. *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be an opfibration, let opcart be an opcleaving for Φ, and let $f\colon B \to B'$ be a morphism in $\mathscr{B}$. Then the following defines a functor $f_!\colon \mathscr{A}_B \to \mathscr{A}_{B'}$.*

- *For each object* C *in* $\mathscr{A}_B$*, define* $f_! C$ *to be the domain of* opcart^f_C.
- *For each morphism* $\omega\colon C \to Q$ *in* $\mathscr{A}_B$*, using the fact that* opcart^f_C *is cartesian, define* $\omega f_!$ *to be the unique morphism* $Cf_! \to Qf_!$ *in* $\mathscr{A}$ *that lies over* $\mathrm{id}_{B'}$ *and satisfies* $\mathrm{opcart}^f_C \mathbin{⨾} \omega f_! = \omega \mathbin{⨾} \mathrm{opcart}^f_Q$*. This situtation is shown in the following figure.*



□

Examples 2.4.19.

- For the opfibration $\mathtt{Rep}\colon \mathbf{GrpRep} \to \mathbf{FinGrp}$ and a group homomorphism $f\colon G \to H$, we have

$$\begin{aligned} f_!\colon \mathbf{Rep}_G &\longrightarrow \mathbf{Rep}_H \\ V &\longmapsto \mathbb{C}H \otimes_G V \\ \alpha &\longmapsto \mathrm{id}_{\mathbb{C}H} \otimes_G \alpha. \end{aligned}$$

- Let $\mathscr{C}$ be a category with coproducts. For the opfibration $\mathtt{Fam}_{\mathscr{C}}\colon \mathbf{Fam}_{\mathscr{C}} \to \mathbf{Set}$ and a map of sets $f\colon X \to Y$, we have

$$\begin{aligned} f_!\colon \mathscr{C}^X &\longrightarrow \mathscr{C}^Y \\ (A_x)_{x\in X} &\longmapsto \left(\sum_{w\in(y)f^{-1}} A_w\right)_{y\in Y} \\ (\beta_x\colon A_x \to A'_x)_{x\in X} &\longmapsto \left(\sum_{w\in(y)f^{-1}} \beta_x \mathbin{⨾} \iota'_w\right)_{y\in Y}, \end{aligned}$$

where, for each $y \in Y$ and each $w \in (y)f^{-1}$, we denote by ι'_w the inclusion morphism $A'_w \hookrightarrow \sum_{w\in(y)f^{-1}} A'_w$.

◇

For the bifibration $\mathtt{Rep}\colon \mathbf{GrpRep} \to \mathbf{FinGrp}$, we have the following well-known isomorphism between $\pi^{GH}_{H\ !}$ and the *coinvariants* functor.

Example 2.4.20. Let G and H be finite groups and let $\pi_G\colon G\times H \to H$ denote the projection map. The **coinvariants functor** $(-)_{(G)}\colon \mathbf{Rep}_{G\times H} \to \mathbf{Rep}_H$ is defined on objects by

$$V_{(G)} = V/\langle g\cdot v - v \mid g \in G, v \in V\rangle.$$

The space $V_{(G)}$ is called the **space of coinvariants** and it's largest quotient of V on which G acts trivially. We'll show that the following are mutually inverse H-module maps.

$$\begin{aligned} \phi\colon V_{(G)} &\longrightarrow {}_H\mathbb{C}H \otimes_{G\times H} V : \psi \\ [v] &\longmapsto e_H \otimes v \\ [h\cdot v] &\longleftarrow h \otimes v \end{aligned}$$

They are H-module maps since

$$(h\cdot [v])\phi = ([h\cdot v])\phi = e_H \otimes (h\cdot v) = h\otimes v = h\cdot (e_H \otimes v) = h\cdot ([v])\phi,$$

and

$$(h\cdot (x\otimes v))\psi = (hx\otimes v)\psi = [hx\cdot v] = h\cdot [x\cdot v] = h\cdot (x\otimes v)\psi,$$

and they are mutually inverse since

$$(([v])\phi)\psi = (e_H\otimes v)\psi = [e\cdot v] = [v]$$

and

$$((h\otimes v)\psi)\phi = ([h\cdot v])\phi = e_H \otimes (h\cdot v) = h\otimes v.$$

These maps are the components of a natural isomorphism $(\pi_G)_! \cong (-)_{(G)}$. ◇

Proposition 2.4.21. *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be an opcleaved opfibration and let $f\colon B \to B'$ be a morphism in $\mathscr{B}$. Then the bijections*

$$\begin{aligned} \mathscr{A}_{B'}(f_! N, M) &\longrightarrow \mathscr{A}^f(N, M) \\ \xi &\longmapsto \mathrm{opcart}^f_N \,⨾\, \xi \end{aligned}$$

are natural in both N and M.

Now for the analogues of indexed categories and the Grothendieck construction.

Definition 2.4.22. An **opindexed category** $\mathfrak{I}$ consists of a category $\mathscr{B}$ and a pseudofunctor $\mathscr{B} \to \mathbf{Cat}$, where we consider $\mathscr{B}$ as a bicategory with identity 2-cells. We call $\mathscr{B}$ the **base category** of the opindexed category $\mathfrak{I}$. ◇

Remark 2.4.23. Let's unpack this definition. An opindexed category $\mathfrak{I}\colon \mathscr{B} \to \mathbf{Cat}$ consists of the following data:

(i) a base category $\mathscr{B}$;

(ii) for each object B in $\mathscr{B}$, a category $B\mathfrak{I}$;

(iii) for each morphism $f\colon B \to B'$ in $\mathscr{B}$, a functor $f\mathfrak{I}\colon B\mathfrak{I} \to B'\mathfrak{I}$;

(iv) for each pair $f\colon B \to B'$, $g\colon B' \to B''$ of morphisms in $\mathscr{B}$, a natural isomorphism

$$\mathfrak{I}^{2!}_{fg}\colon f\mathfrak{I} \fatsemi g\mathfrak{I} \to (f \fatsemi g)\mathfrak{I}\ ;$$

(v) for each object B in $\mathscr{B}$, a natural isomorphism

$$\mathfrak{I}^{0!}_{B}\colon \mathrm{id}_{B\mathfrak{I}} \to \mathrm{id}_B\mathfrak{I}\ .$$

These data are required to satisfy the following axioms.

- (**Associativity**) For every composable triple $f\colon B \to B'$, $g\colon B' \to B''$, $h\colon B'' \to B'''$ of morphisms in $\mathscr{B}$, the following diagram commutes.

$$\begin{array}{ccc}
f\mathfrak{I} \fatsemi g\mathfrak{I} \fatsemi h\mathfrak{I} & \xrightarrow{f\mathfrak{I} \fatsemi \mathfrak{I}^{2!}_{gh}} & f\mathfrak{I} \fatsemi (g \fatsemi h)\mathfrak{I} \\
{\scriptstyle \mathfrak{I}^{2!}_{fg} \fatsemi h\mathfrak{I}}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{I}^{2!}_{f,g \fatsemi h}} \\
(f \fatsemi g)\mathfrak{I} \fatsemi h\mathfrak{I} & \xrightarrow[\mathfrak{I}^{2!}_{f \fatsemi g,h}]{} & (f \fatsemi g \fatsemi h)\mathfrak{I}
\end{array}$$

- (**Unitality**) For every morphism $f\colon B \to B'$ in $\mathscr{B}$, the following diagrams commute.

$$\begin{array}{ccc}
f\mathfrak{I} \fatsemi \mathrm{id}_{B'}\mathfrak{I} & \xrightarrow{\mathfrak{I}^{2!}_{f,\mathrm{id}_{B'}}} & (f \fatsemi \mathrm{id}_{B'})\mathfrak{I} \\
{\scriptstyle f\mathfrak{I} \fatsemi (\mathfrak{I}^{0!}_{B'})^{-1}}\big\downarrow & & \big\| \\
f\mathfrak{I} \fatsemi \mathrm{id}_{B'\mathfrak{I}} & = & f\mathfrak{I}
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{id}_B\mathfrak{I} \fatsemi f\mathfrak{I} & \xrightarrow{\mathfrak{I}^{2!}_{\mathrm{id}_B,f}} & (\mathrm{id}_B \fatsemi f)\mathfrak{I} \\
{\scriptstyle (\mathfrak{I}^{0!}_{B})^{-1} \fatsemi f\mathfrak{I}}\big\downarrow & & \big\| \\
\mathrm{id}_{B\mathfrak{I}} \fatsemi f\mathfrak{I} & = & f\mathfrak{I}
\end{array}$$

◇

Definition 2.4.24. Let $\mathfrak{I}\colon \mathscr{B} \to \mathbf{Cat}$ be an opindexed category. The **total category** of $\mathfrak{I}$, denoted by

$$\int \mathfrak{I}$$

is the category defined as follows.

- An object is a pair $\begin{pmatrix} M \\ B \end{pmatrix}$ consisting of an object B in $\mathscr{B}$ an object M in $B\mathfrak{I}$.

- A morphism $\begin{pmatrix} M \\ B \end{pmatrix} \to \begin{pmatrix} M' \\ B' \end{pmatrix}$ is a pair $\begin{pmatrix} \rho \\ f \end{pmatrix}$ consisting of a morphism $f\colon B \to B'$ in $\mathscr{B}$ and a morphism $\rho\colon Mf_! \to M'$ in $B\mathfrak{I}$.

- The composite of the morphisms $\begin{pmatrix} \rho \\ f \end{pmatrix}\colon \begin{pmatrix} M \\ B \end{pmatrix} \to \begin{pmatrix} M' \\ B' \end{pmatrix}$ and $\begin{pmatrix} \omega \\ g \end{pmatrix}\colon \begin{pmatrix} M' \\ B' \end{pmatrix} \to \begin{pmatrix} M'' \\ B'' \end{pmatrix}$ in $\int \mathfrak{I}$ is given by the pair

$$\begin{pmatrix} (\mathfrak{I}^{2!}_{fg})^{-1} \mathbin{⨟} \rho g_! \mathbin{⨟} \omega \\ f \mathbin{⨟} g \end{pmatrix}\colon \begin{pmatrix} M \\ B \end{pmatrix} \to \begin{pmatrix} M'' \\ B'' \end{pmatrix}.$$

- For each object $\begin{pmatrix} M \\ B \end{pmatrix}$ in $\int \mathfrak{I}$, the identity $\mathbf{id}_{\left(\substack{M \\ B}\right)}$ is given by the pair

$$\begin{pmatrix} (\mathfrak{I}^{0!}_{B,M})^{-1} \\ \mathrm{id}_B \end{pmatrix}\colon \begin{pmatrix} M \\ B \end{pmatrix} \to \begin{pmatrix} M \\ B \end{pmatrix}.$$

◇

2.5 Bifibrations

We've seen that the functor $\mathtt{Rep}\colon \mathbf{GrpRep} \to \mathbf{FinGrp}$, the functor $\mathtt{Fam}_{\mathscr{A}}\colon \mathbf{Fam}_{\mathscr{A}} \to \mathbf{Set}$ (for a category $\mathscr{A}$ with coproducts), and the functor $\mathtt{Arr}_{\mathscr{A}}\colon \mathscr{A}^{\to} \to \mathscr{A}$ (for a category $\mathscr{A}$ with pullbacks) are both fibrations and opfibrations (see Examples 2.1.16, 2.1.19, 2.1.22, 2.4.9, 2.4.11 and 2.4.12). Such functors are called *bifibrations*.

Definition 2.5.1.

- We call a functor $\Phi\colon \mathscr{A} \to \mathscr{B}$ a **bifibration** if it is both a fibration and an opfibration.

- A **bicleaving** for a bifibration Φ consists of a cleaving and an opcleaving for Φ.

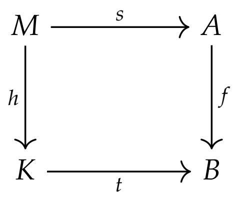
- A **bicleaved bifibration** is a bifibration equipped with a bicleaving.

◇

Before continue this secion where we discuss bifibrations, we'll give a non-example: an opfibration that is not a fibration. In Example 2.1.22, we proved that, if $\mathscr{A}$ is a category with pullbacks, then the codomain functor $\mathtt{Arr}_{\mathscr{A}}\colon \mathscr{A}^{\to} \to \mathscr{A}$ is a fibration. In a moment, we'll prove the converse: if the codomain functor $\mathtt{Arr}_{\mathscr{A}}$ is a fibration then the category $\mathscr{A}$ has pullbacks. Therefore, given any choice of category $\mathscr{A}$ without pullbacks (e.g. the category with two objects and no non-identity morphisms), the codomain functor $\mathtt{Arr}_{\mathscr{A}}$ is an opfibration (this fact doesn't depend on $\mathscr{A}$) but not a fibration.

Proposition 2.5.2. *Let $\mathscr{A}$ be a category, and suppose that the codomain functor $\mathtt{Arr}_{\mathscr{A}}\colon \mathscr{A}^{\to} \to \mathscr{A}$ is a fibration. Then $\mathscr{A}$ has pullbacks.*

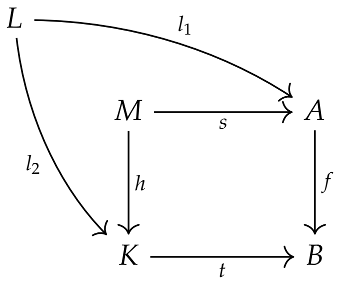
Proof. Let $f\colon A \to B$ and $t\colon K \to B$ be morphisms in $\mathscr{A}$, so t is an object in $\mathscr{A}^{\to}$ and $(t)\mathtt{Arr}_{\mathscr{A}} = B$. Since $\mathtt{Arr}_{\mathscr{A}}$ is a fibration, there exists a cartesian morphism $(h, f)\colon s \to t$ in $\mathscr{A}^{\to}$ that lies over f. We will show that the square



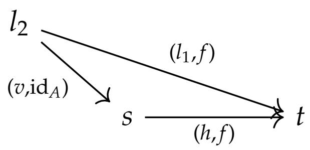
(2.5.3)

in $\mathscr{A}$ is a pullback square.

Suppose that the following is a cone.



Since the morphism $(h, f)\colon s \to t$ is cartesian, there exists a unique morphism $(v, \mathrm{id}_A)\colon l_2 \to s$ in $\mathscr{A}^{\to}$ that lies over id_A and that makes the following diagram in $\mathscr{A}^{\to}$ commute.



In other words, there exists a unique morphism $v\colon L \to M$ in $\mathscr{A}$ such that $v \fatsemi s = l_2 \fatsemi \mathrm{id}_A$ and $v \fatsemi h = l_1$. Therefore, the square (2.5.3) is a pullback square. □

Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a bifibration and let $f\colon B \to B'$ be a morphism in $\mathscr{B}$. By Propositions 2.1.31 and 2.4.21, we have bijections

$$\mathscr{A}_B(N, Mf^*) \to \mathscr{A}^f(N, M) \quad\text{and}\quad \mathscr{A}_{B'}(Nf_!, M) \to \mathscr{A}^f(N, M)$$

natural in N and M. Composing these gives a bijection

$$\mathscr{A}_{B'}(Nf_!, M) \to \mathscr{A}_B(N, Mf^*) \tag{2.5.4}$$

natural in N and M, i.e. an adjunction $f_! \dashv f^*$.

Proposition 2.5.5 ([Jac91, Lemma 9.1.2]). *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a bicleaved bifibration. Then, for each morphism $f\colon B \to B'$ in $\mathscr{B}$, the functor $f_!\colon \mathscr{A}_B \to \mathscr{A}_{B'}$ is left adjoint to the functor $f^*\colon \mathscr{A}_{B'} \to \mathscr{A}_B$.* □

Explicity the bijection (2.5.4) is given by

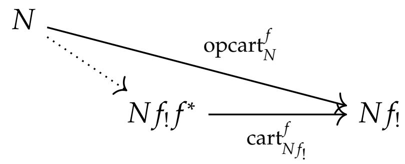
$$\begin{aligned} \mathscr{A}_{B'}(Nf_!, M) &\longrightarrow \mathscr{A}_B(N, Mf^*) \\ \xi &\longmapsto (\mathrm{opcart}^f_N \fatsemi \xi) \mathbin{\circ\!\!/} \mathrm{cart}^f_M. \end{aligned} \tag{2.5.6}$$

Later it'll be useful to have explicit descriptions of the unit and counit of the adjunction $f_! \dashv f^*$ using (2.5.6); this is subject of the following remark.

Remark 2.5.7. Recall from Definition 1.3.4 that, for each object N in $\mathscr{A}_B$, the component $\eta^f_N : N \to Nf_!f^*$ of the unit of the adjunction $f_! \dashv f^*$ is the adjunct of $\mathrm{id}_{f_!N} : Nf_! \to Nf_!$. So, using (2.5.6), we have

$$\mathrm{opcart}^f_N \mathbin{\circ\!\!/} \mathrm{cart}^f_{Nf_!}.$$

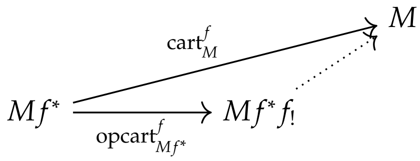
In other words, $\eta^f_N : N \to Nf_!f^*$ is the unique pure morphism $N \to Nf_!f^*$ that makes the following diagram in $\mathscr{A}$ commute.



Similarly, for each object M in $\mathscr{A}_{B'}$, the component $\varepsilon^f_M : Mf^*f_! \to M$ of the counit of the adjunction $f_! \dashv f^*$ is given by

$$\mathrm{cart}^f_M \mathbin{\backslash\!\!\circ} \mathrm{opcart}^f_{Mf^*},$$

or in other words, it is the unique pure morphism $Mf^*f_! \to M$ that makes the following diagram in $\mathscr{A}$ commute.



◇

Now, for the bifibrations $\mathtt{Rep}$ and $\mathtt{Fam}$, we look at examples of the unit and counit of the adjunction $f_! \dashv f^*$.

Example 2.5.8. Consider the bifibration $\mathtt{Rep} : \mathbf{GrpRep} \to \mathbf{FinGrp}$. For each group homomorphism $f : G \to H$, we have an adjunction $f_! \dashv f^*$. For each G-module W, the unit $\eta^f_{(G,W)} : (G, W) \to (G, W)f_!f^*$ is given by the G-module map

$$\begin{aligned} W &\longrightarrow {}_f(\mathbb{C}H \otimes_H W) \\ w &\longmapsto e_H \otimes w \end{aligned}$$

and, for each H-module V, the counit $\varepsilon^{f}_{(H,V)} \colon (H,V)f^{*}f_{!} \to (H,V)$ is given by the H-module map

$$\begin{aligned} \mathbb{C}H \otimes_{G} {}_{f}V &\longrightarrow V \\ h \otimes v &\longmapsto h \cdot v \end{aligned}$$

◇

Example 2.5.9. Consider the bifibration $\mathtt{Fam}_{\mathscr{C}} \colon \mathbf{Fam}_{\mathscr{C}} \to \mathbf{Set}$. For each maps of sets $f \colon X \to Y$, we have an adjunction $f_{!} \dashv f^{*}$. For each X-indexed family of objects $A = (A_x)_{x \in X}$, the unit $\eta^{f}_{(A,X)} \colon (A,X) \to (A,X)f_{!}f^{*}$ is given by the inclusion map

$$A_x \longrightarrow \sum_{\substack{\alpha \in X \\ (\alpha)f = (x)f}} A_\alpha$$

and, for each Y-indexed set $B = (B_y)_{y \in Y}$, the counit $\varepsilon^{f}_{(B,Y)} \colon (B,Y)f^{*}f_{!} \to (B,Y)$ is given by the codiagonal map

$$\sum_{w \in f^{-1}(y)} \mathrm{id}_{B_y} \colon \sum_{w \in f^{-1}(y)} B_y \longrightarrow B_y$$

◇

The following proposition gives a way of upgrading a fibration to a bifibration.

Proposition 2.5.10 ([Jac91, Lemma 9.1.2]). *Let $\Phi \colon \mathscr{A} \to \mathscr{B}$ be a cleaved fibration. If, for every morphism $f \colon B \to B'$, the functor $f^{*} \colon \mathscr{A}_{B'} \to \mathscr{A}_{B}$ has a left adjoint, then $\Phi \colon \mathscr{A} \to \mathscr{B}$ is a bifibration.*

Proof. For each morphism $f \colon B \to B'$ in $\mathscr{B}$, let $f_{!} \colon \mathscr{A}_{B} \to \mathscr{A}_{B'}$ denote a left adjoint to the functor $f^{*} \colon \mathscr{A}_{B'} \to \mathscr{A}_{B}$. The adjunction $f_{!} \dashv f^{*}$ gives the bijection

$$\begin{aligned} \mathscr{A}_{B'}(Nf_{!}, M) &\longrightarrow \mathscr{A}_{B}(N, Mf^{*}) \\ \xi &\longmapsto \eta_N \mathbin{⨾} \xi f^{*} \end{aligned}$$

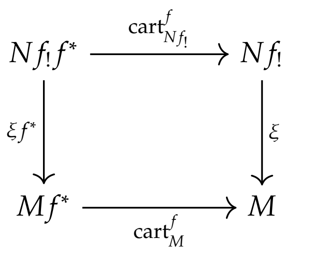
and Proposition 2.1.31 gives the bijection

$$\begin{aligned} \mathscr{A}_{B}(N, Mf^{*}) &\longrightarrow \mathscr{A}^{f}(N, M) \\ \chi &\longmapsto \chi \mathbin{⨾} \mathrm{cart}^{f}_{M} \end{aligned}.$$

Compose these to get the bijection

$$\begin{aligned} \mathscr{A}_{B'}(Nf_{!}, M) &\longrightarrow \mathscr{A}^{f}(N, M) \\ \xi &\longmapsto \eta_N \mathbin{⨾} \xi f^{*} \mathbin{⨾} \mathrm{cart}^{f}_{M} \end{aligned}. \tag{2.5.11}$$

The diagram

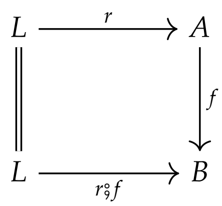


in $\mathscr{A}$ commutes by definition of ξf^*. Therefore, the bijection (2.5.11) is equal to the following bijection.

$$\begin{aligned} \mathscr{A}_{B'}(Nf_!, M) &\longrightarrow \mathscr{A}^f(N, M) \\ \xi &\longmapsto \eta_N \fatsemi \mathrm{cart}^f_{Nf_!} \fatsemi \xi \end{aligned}$$

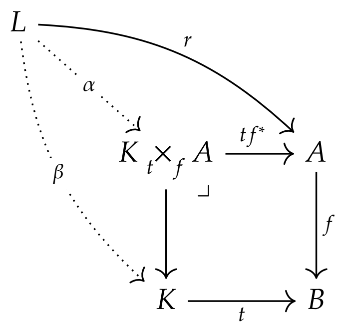
This means that, for each morphism $f\colon B \to B'$ in $\mathscr{B}$, the morphism $\eta_N \fatsemi \mathrm{cart}^f_{Nf_!}\colon N \to Nf_!$ is opcartesian and lies over f. Therefore, $\Phi\colon \mathscr{A} \to \mathscr{B}$ is an opfibration and hence a bifibration. □

In Example 2.4.12, we saw that the codomain functor $\mathtt{Arr}_{\mathscr{A}}\colon \mathscr{A}^{\to} \to \mathscr{A}$ is an opfibration where, for each morphism $f\colon A \to B$ in $\mathscr{A}$ and each morphism $r\colon L \to A$ in $\mathscr{A}$, the opcartesian morphism $r \to rf_!$ we gave was given by the commutative square



in $\mathscr{A}$. The following example shows—assuming only the proof that $\mathtt{Arr}$ is a fibration—how we can use Proposition 2.5.10 to deduce that $\mathtt{Arr}$ is a bifibration.

Example 2.5.12. We showed in Example 2.1.22 that the codomain functor $\mathtt{Arr}$ is a fibration. The following diagram shows, by the universal property of pullbacks, that there is a natural correspondence between morphisms $\alpha\colon L \to K\ {}_t{\times}_f\ A$ that satisfy $\alpha \fatsemi tf^* = r$ and morphisms $\beta\colon L \to K$ that satisfy $\beta \fatsemi t = r \fatsemi f$.



Therefore, there is a natural bijection between commutative squares

$$\begin{array}{ccc} L & \xrightarrow{\quad r \quad} & A \\ {\scriptstyle\alpha}\big\downarrow & & \big\| \\ K \ {}_t{\times}_f \ A & \xrightarrow[\quad tf^* \quad]{} & A \end{array}$$

in $\mathscr{A}$ and commutative squares

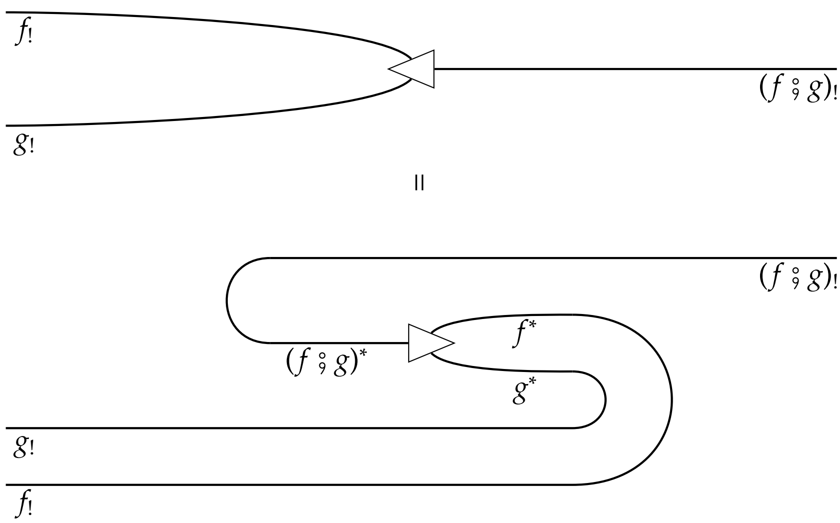
$$\begin{array}{ccc} L & \xrightarrow{\quad r \fatsemi f \quad} & B \\ {\scriptstyle\beta}\big\downarrow & & \big\| \\ K & \xrightarrow[\quad t \quad]{} & B \end{array}$$

in $\mathscr{A}$. In other words, this is a natural bijection between morphisms $r \to tf^*$ in $\mathscr{A}^{\to}_A$ and morphisms $r \fatsemi f \to t$ in $\mathscr{A}^{\to}_B$. So taking $rf_! = r \fatsemi f : L \to B$ makes $\mathtt{Arr}$ a bifibration via Proposition 2.5.10 since we have an adjunction $f_! \dashv f^*$. ◇

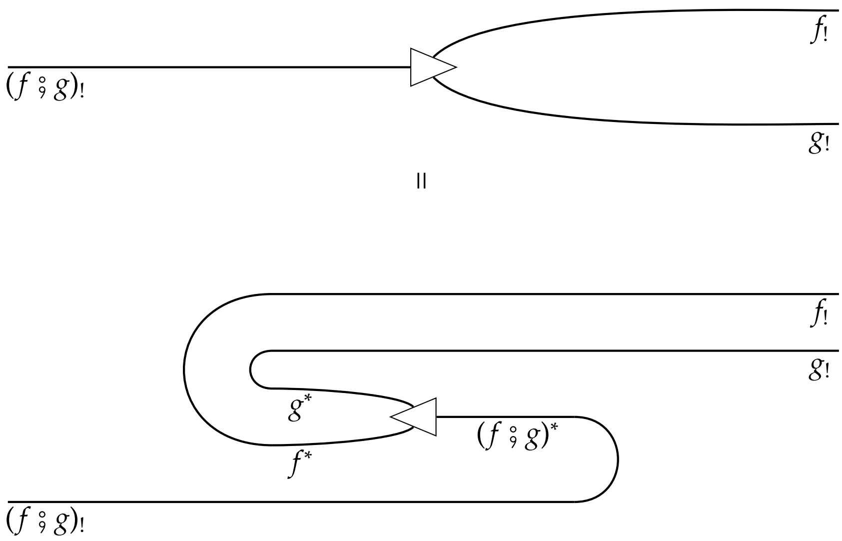
The following theorem states useful identities relating the pseudofunctoriality constraints Φ^{2*}_{fg}, Φ^{0*}_B, $\Phi^{2!}_{fg}$ and $\Phi^{0!}_B$.

Theorem 2.5.13. *Let* $\Phi\colon \mathscr{A} \to \mathscr{B}$ *be a bicleaved bifibration. Then the following equalities holds.*

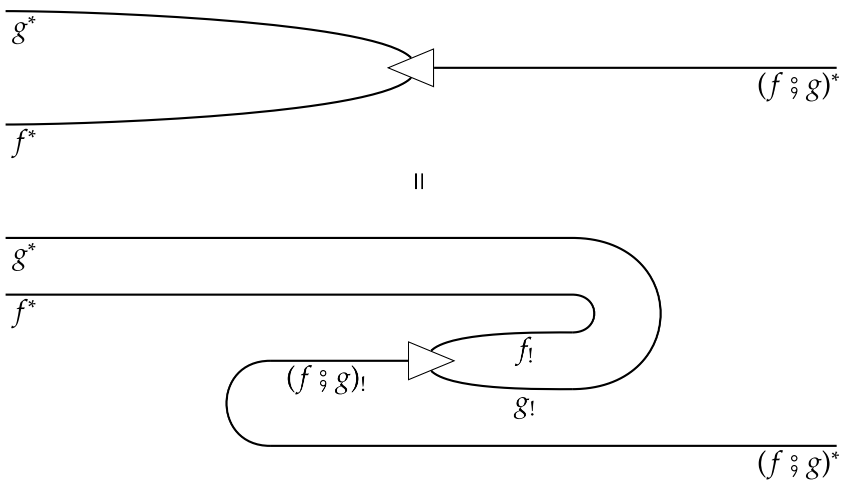
(i)



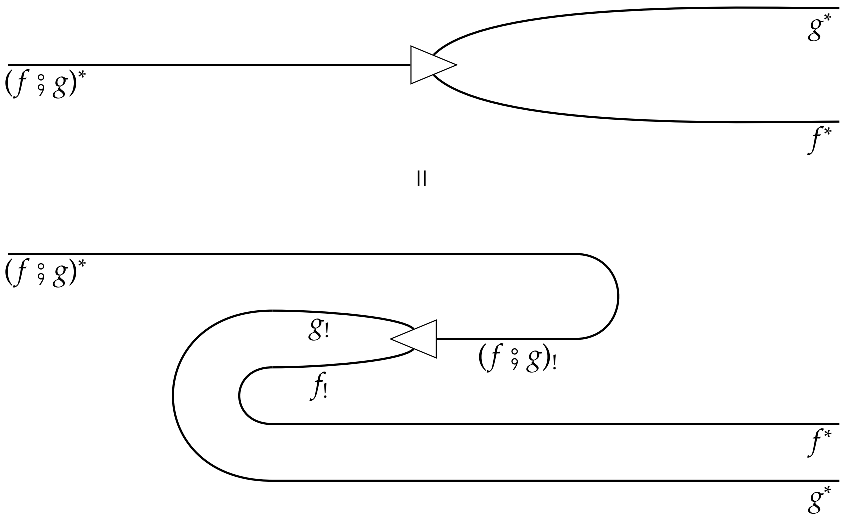
(ii)


$(f \fatsemi g)_!$
$f_!$
$g_!$
$=$
$f_!$
$g_!$
g^*
$(f \fatsemi g)^*$
f^*
$(f \fatsemi g)_!$

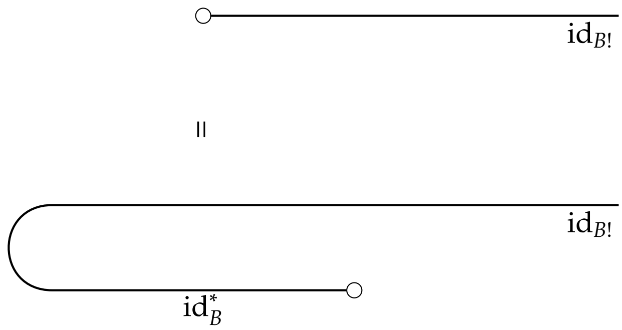
(iii)


g^*
$(f \fatsemi g)^*$
f^*
$=$
g^*
f^*
$f_!$
$(f \fatsemi g)_!$
$g_!$
$(f \fatsemi g)^*$

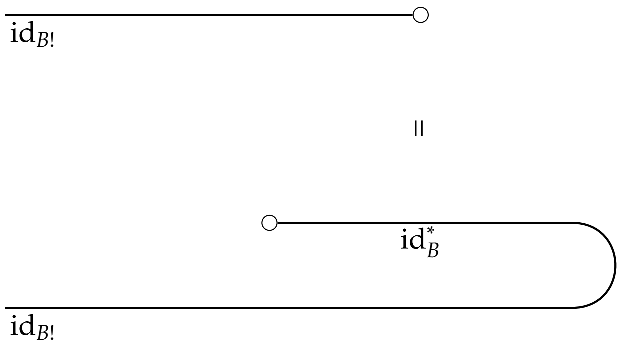
(iv)


g^*
$(f \fatsemi g)^*$
f^*
=
$(f \fatsemi g)^*$
$g_!$
$(f \fatsemi g)_!$
$f_!$
f^*
g^*

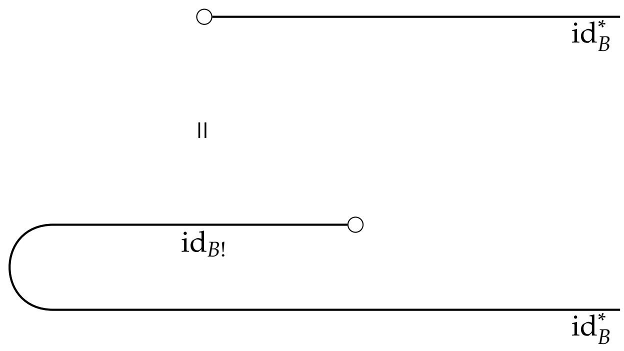
(v)


$\mathrm{id}_{B!}$
=
$\mathrm{id}_{B!}$
id_B^*

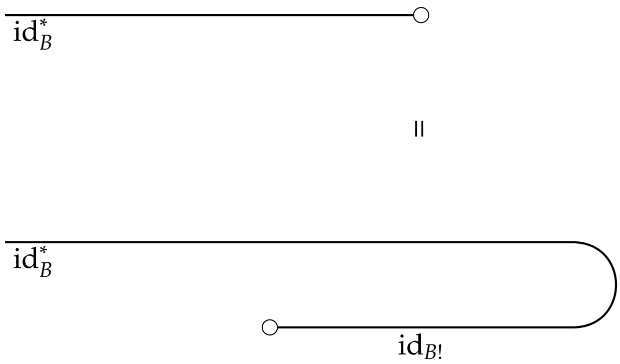
(vi)


$\mathrm{id}_{B!}$
=
id_B^*
$\mathrm{id}_{B!}$

(vii)



(viii)

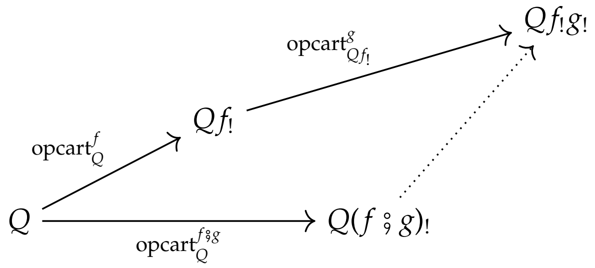


Proof. We'll prove the equalities (i) and (v); the equalities (ii)–(iv) follow by application of the snake identities for the adjunctions $f_! \dashv f^*$ and $g_! \dashv g^*$ to (i), and the equalities (vi)–(vii) follow by application of the snake identities for the adjunction $\mathbf{id}_{B!} \dashv \mathbf{id}_B{}^*$ to (v).

First, the proof of (i). Let Q be an object in $\mathscr{A}_B$. The component morphism $Q(f \mathbin{⨾} g)_! \to Qg_!f_!$ of the right-hand side is equal to the following composite.

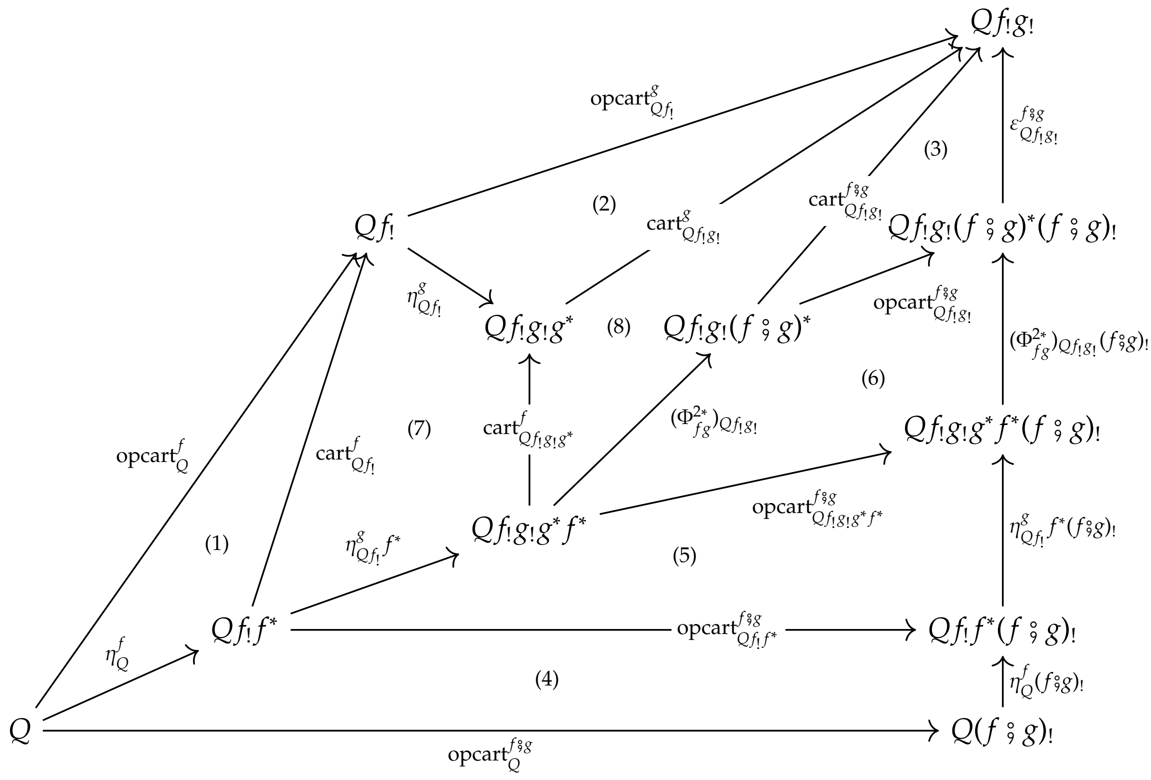
$$\begin{array}{c} Qf_!g_! \\ \big\uparrow {\scriptstyle \varepsilon^{f⨾g}_{Qf_!g_!}} \\ Qf_!g_!(f ⨾ g)^*(f ⨾ g)_! \\ \big\uparrow {\scriptstyle (\Phi^{2*}_{fg})_{Qf_!g_!}(f⨾g)_!} \\ Qf_!g_!g^*f^*(f ⨾ g)_! \\ \big\uparrow {\scriptstyle \eta^g_{Qf_!}f^*(f⨾g)_!} \\ Qf_!f^*(f ⨾ g)_! \\ \big\uparrow {\scriptstyle \eta^f_Q(f⨾g)_!} \\ Q(f ⨾ g)_! \end{array} \tag{2.5.14}$$

The component morphism $Q(f \mathbin{;} g)_! \to Qg_!f_!$ of the left-hand side is defined to be the unique pure morphism $Q(f \mathbin{;} g)_! \to Qg_!f_!$ that makes the following diagram in $\mathscr{A}$ commute.



(2.5.15)

The morphism (2.5.14) is pure since it is the composite of pure morphisms. Therefore, it suffices to show that the morphism (2.5.14) makes the diagram (2.5.15) commute; this follows from the fact that following diagram commutes.

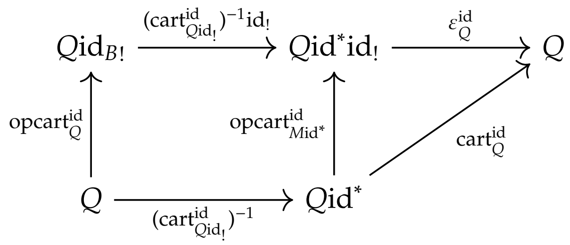


The triangles (1), (2) and (3) commute by the definitions of η^f_Q, $\eta^f_{Qf_!}$ and $\varepsilon^{f;g}_{Qf_!g_!}$. The squares (4), (5) and (6) commute by the definition of $(f \mathbin{;} g)_!$ on morphisms, and (7) commutes by the definition of f^* on morphisms. Finally, the square (8) commutes by the definition of $(\Phi^{2*}_{fg})_{Qf_!g_!}$.

And now, the proof of (v). Let Q be an object in $\mathscr{A}_B$. For ease of reading we'll write $\mathbf{id}_B$ as just $\mathbf{id}$. The component morphism $Q\mathbf{id}_! \to Q$ of the left-hand side is defined to be $(\mathrm{opcart}^{\mathrm{id}}_Q)^{-1}$, and the component morphism $\mathbf{id}_!Q \to Q$ is equal to the following composite.

$$Q\mathrm{id}_! \xrightarrow{(\mathrm{cart}^{\mathrm{id}}_{Q\mathrm{id}_!})^{-1}\mathrm{id}_!} Q\mathrm{id}^*\mathrm{id}_! \xrightarrow{\varepsilon^{\mathrm{id}}_Q} Q$$

The following diagram commutes by the definition of $(\mathbf{cart}^{\mathbf{id}}_{Q\mathrm{id}_!})^{-1}\mathbf{id}_!$ and the definition of $\varepsilon^{\mathrm{id}_B}_Q$, and the result follows.



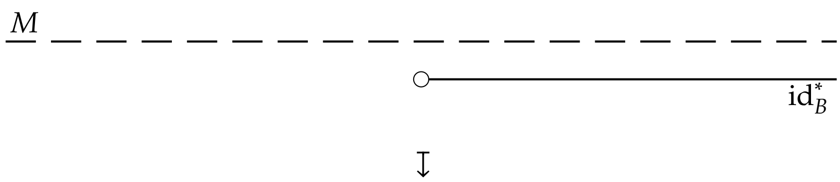
□

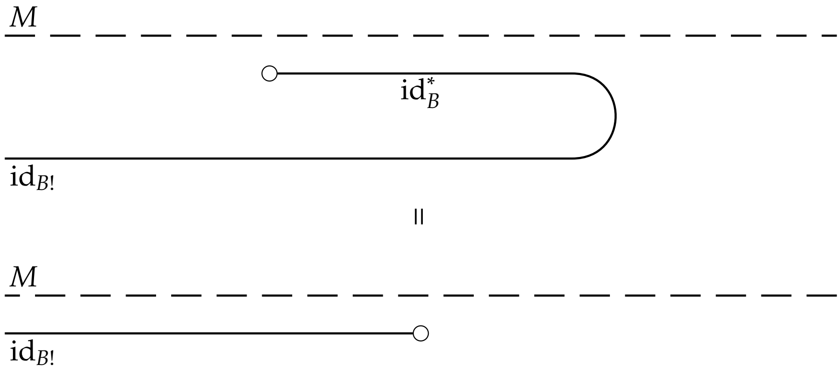
Given a bifibration $\Phi\colon \mathscr{A} \to \mathscr{B}$, we have an indexed category $\mathfrak{I}\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$ and an opindexed category $\mathfrak{K}\colon \mathscr{B} \to \mathbf{Cat}$ that satisfy $B\mathfrak{I} = B\mathfrak{K}$ for each object B in $\mathscr{B}$ and $f\mathfrak{K} \dashv f\mathfrak{I}$ for each morphism $f\colon B \to B'$ in $\mathscr{B}$. It's natural to ask "Are the total categories $\int \mathfrak{I}$ and $\int \mathfrak{K}$ are equivalent?"; we describe an isomorphism between these two categories using string diagrams.

Proposition 2.5.16. *Suppose that $\mathfrak{I}\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$ is an indexed category and that $\mathfrak{K}\colon \mathscr{B} \to \mathbf{Cat}$ is an opindexed category such that $B\mathfrak{I} = B\mathfrak{K}$ for each object B in $\mathscr{B}$ and $f\mathfrak{K} \dashv f\mathfrak{I}$ for each morphism $f\colon B \to B'$ in $\mathscr{B}$. Then the following functor, which is given by taking the adjunct (see Definition 1.3.2), is an isomorphism of categories.*

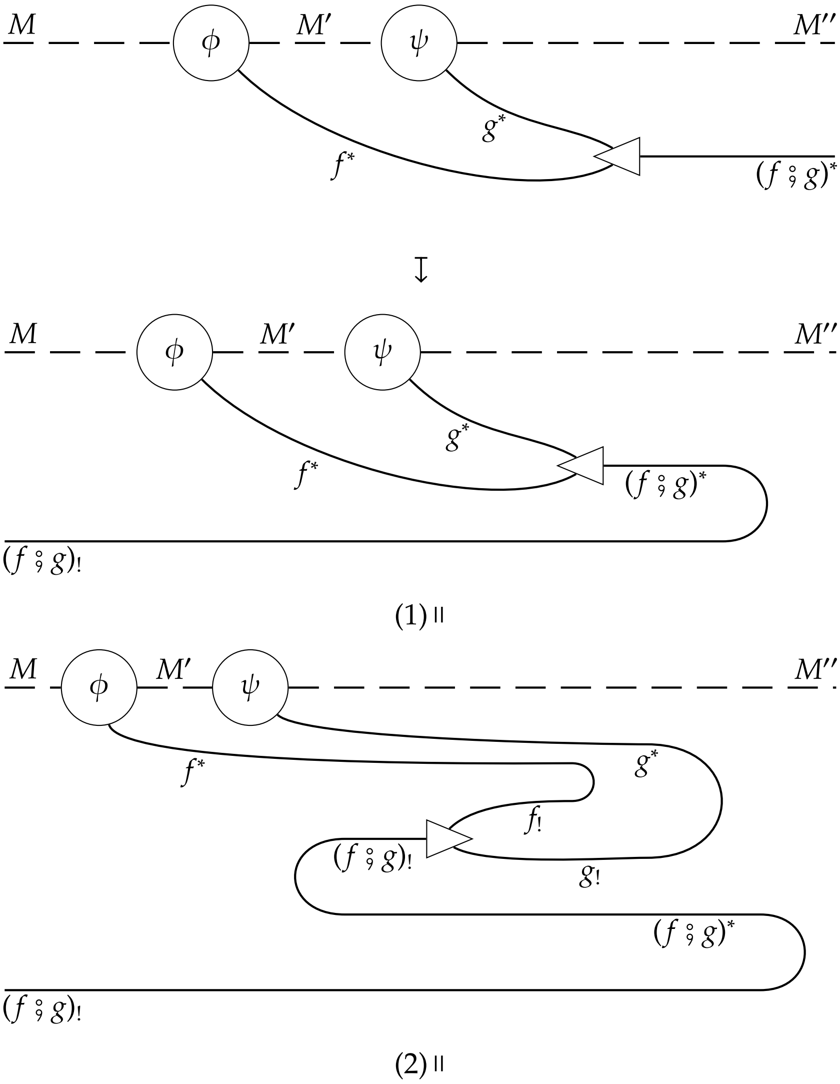
$$\begin{gathered} \int \mathfrak{I} \longrightarrow \int \mathfrak{K} \\ \begin{pmatrix} M \\ B \end{pmatrix} \longmapsto \begin{pmatrix} M \\ B \end{pmatrix} \\ \begin{pmatrix} \phi\colon M \to M'f^* \\ f\colon B \to B' \end{pmatrix} \longmapsto \begin{pmatrix} \overline{\phi}\colon Mf_! \to M' \\ f\colon B \to B' \end{pmatrix} \end{gathered} \tag{2.5.17}$$

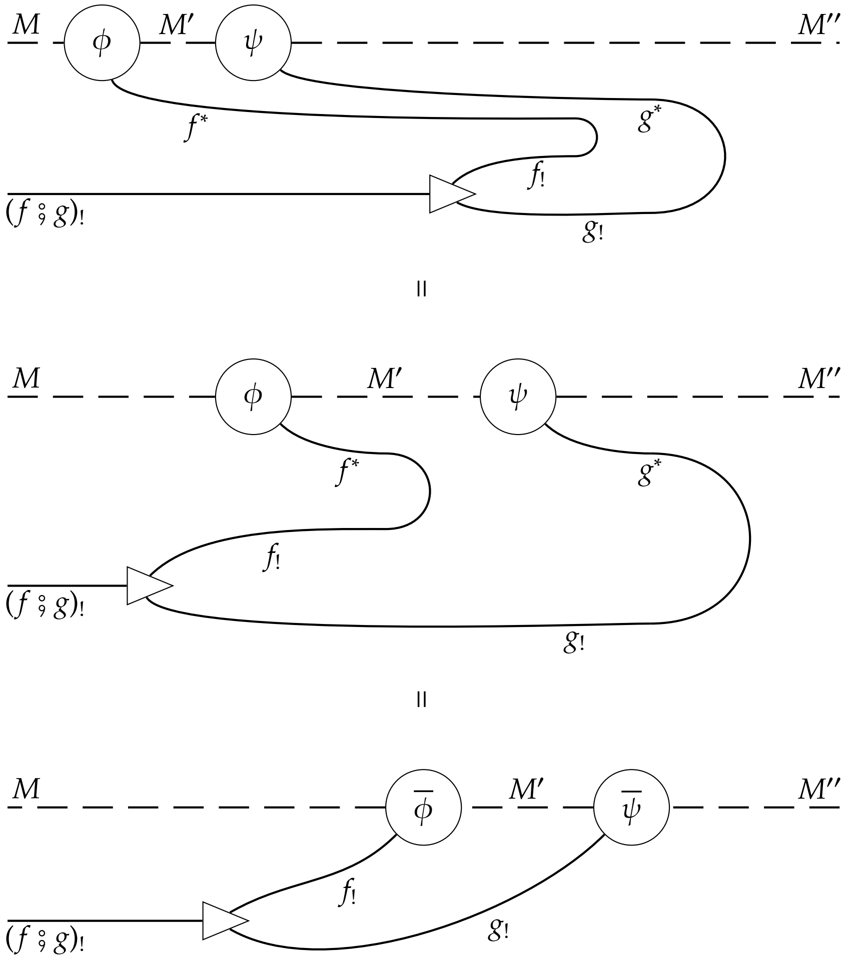
Proof. We'll show that (2.5.17) is a functor; the inverse is also given by taking the adjunct. Let $\begin{pmatrix} M \\ B \end{pmatrix}$ be an object in $\int \mathfrak{I}$. The following string diagrams show that $\mathbf{id}_{\binom{M}{B}} \mapsto \mathbf{id}_{\binom{M}{B}}$ using Theorem 2.5.13.





Let $\begin{pmatrix} \phi\colon M \to M' f^* \\ f\colon B \to B' \end{pmatrix} \colon \begin{pmatrix} M \\ B \end{pmatrix} \to \begin{pmatrix} M' \\ B' \end{pmatrix}$ and $\begin{pmatrix} \psi\colon M' \to M'' g^* \\ g\colon B' \to B'' \end{pmatrix} \colon \begin{pmatrix} M' \\ B' \end{pmatrix} \to \begin{pmatrix} M'' \\ B'' \end{pmatrix}$ be morphisms in $\int \mathfrak{I}$. The following series of equalities proves binary functoriality.



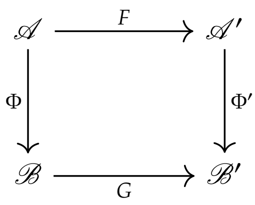


The equality (1) follows from Theorem 2.5.13(iii), and the equality (2) follows from the snake identities for the adjunction $(f \fatsemi g)_! \dashv (f \fatsemi g)^*$. □

So far, we've seen the Grothendieck construction give a 2-equivalence between fibrations and indexed categories and between opfibrations and opindexed categories. Recall from Proposition 2.5.10 that a functor is a bifibration if, and only if, it is a fibration for which each pull-back functor f^* has a left adjoint. This idea is what leads to Theorem 2.5.20, which is analogy of the Grothendieck construction for bifibrations. We first need to give the definitions of the 2-category of bifibrations and the 2-category of bifibrations over a fixed category $\mathscr{B}$, both of which are very similar to the definitions of $\mathcal{Fib}$ and $\mathcal{Fib}_{\mathscr{B}}$ (see Definitions 2.3.11 and 2.3.12).

Definition 2.5.18.

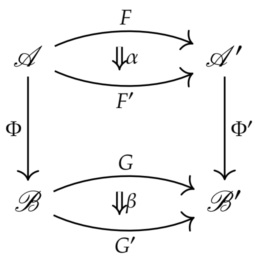
- Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ and $\Phi'\colon \mathscr{A}' \to \mathscr{B}'$ be bifibrations. A **bifibred 1-cell** $\Phi \to \Phi'$ consists of a pair of functors $F\colon \mathscr{A} \to \mathscr{A}'$ and $G\colon \mathscr{B} \to \mathscr{B}'$ such that the diagram



in **Cat** commutes and, for every Φ-cartesian morphism ϕ in $\mathscr{A}$ and every Φ-opcartesian morphism ρ in $\mathscr{A}$, the morphism ϕF in $\mathscr{A}'$ is Φ'-cartesian and the morphism ρF in $\mathscr{A}'$ is Φ'-opcartesian.

- We call a bifibred 1-cell $(F, G)\colon \Phi \to \Phi'$ **pure** if $\mathscr{B} = \mathscr{B}'$ and $G = \mathrm{id}_{\mathscr{B}}$.

- Let (F, G) and (F', G') be bifibred 1-cells $\Phi \to \Phi'$. A **bifibred 2-cell** consists of a pair of natural transformations $\alpha\colon F \to F'$ and $\beta\colon G \to G'$ such that, for every object M in $\mathscr{A}$, the morphism $\alpha_M\colon MF \to MF'$ in $\mathscr{A}'$ lies over the morphism $\beta_{M\Phi}\colon M\Phi G \to M\Phi G'$ in $\mathscr{B}'$. This situation is shown in the following figure.



- We call a bifibred 2-cell $(\alpha, \beta)\colon (F, G) \to (F', G')$ **pure** if (F, G) and (F', G') are pure bifibred 1-cells and if, for every object M in $\mathscr{A}$, the morphism $\alpha_M\colon MF \to MF'$ in $\mathscr{A}'$ lies over the identity.

◇

Definition 2.5.19.

- The 2-category $\mathcal{Bifib}$ of fibrations consists of bifibrations, bifibred 1-cells and bifibred 2-cells.

- Let $\mathscr{B}$ be a category. The 2-category $\mathcal{Bifib}_{\mathscr{B}}$ of bifibrations over $\mathscr{B}$ consists of bifibrations over $\mathscr{B}$, pure bifibred 1-cells and pure bifibred 2-cells.

◇

The following result is an analogy of the Grothendieck construction for bifibrations.

Theorem 2.5.20 ([HP15, Proposition 2.2.1])**.** *There exists a 2-equivalence*

$$\mathcal{G}\colon \mathrm{Bicat}^{\mathrm{ps}}(\mathscr{B}^{\mathrm{op}}, \mathbf{AdjCat}) \to \mathcal{Bifib}_{\mathscr{B}}.$$

Notation 2.5.21. Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a bifibration and let $\mathfrak{I}\colon \mathscr{B}^{\mathrm{op}} \to \mathbf{Cat}$ and $\mathfrak{K}\colon \mathscr{B} \to \mathbf{Cat}$ be the corresponding indexed and opindexed categories. The categories $\int \mathfrak{I}$ and $\int \mathfrak{K}$ are isomorphic and use very similar notation for their morphisms; we therefore use the notation

$$\begin{pmatrix} \phi\colon P \to Qf^* \\ f\colon B \to B' \end{pmatrix}^* \colon \begin{pmatrix} P \\ B \end{pmatrix} \to \begin{pmatrix} Q \\ B' \end{pmatrix}$$

for a morphism in $\int \mathfrak{I}$ and the notation

$$\begin{pmatrix} \overline{\phi}\colon Pf_! \to Q \\ f\colon B \to B' \end{pmatrix}_! \colon \begin{pmatrix} P \\ B \end{pmatrix} \to \begin{pmatrix} Q \\ B' \end{pmatrix}$$

for a morphism in $\int \mathfrak{K}$. ◇

2.6 Monoidal fibrations

The 2-category $\mathcal{M}on\mathcal{F}ib$ of monoidal fibrations is defined to be the 2-category $\mathrm{PsMon}(\mathcal{F}ib)$ of pseudomonoids in the 2-category of fibrations. Unpacking this definition, we get definitions of *monoidal* fibrations, *monoidal* fibred 1-cells and *monoidal* fibred 2-cells that are similar to their non-monoidal counterparts (see Definition 2.3.11). A detalied argument regarding the equivalence of the definition of a monoidal fibration as a pseudomonoid and the following 'unpacked' definition—orignally given by Shulman [Shu08]—can be found in [Vas18].

Definition 2.6.1.

- Let $\mathscr{A}$ and $\mathscr{B}$ be monoidal categories. A **monoidal fibration** $\mathscr{A} \to \mathscr{B}$ is a strict monoidal functor $\Phi\colon \mathscr{A} \to \mathscr{B}$ such that Φ is a fibration and the monoidal product $\otimes_{\mathscr{A}}$ preserves cartesian morphisms—by which we mean that if ϕ and ψ are cartesian morphisms in $\mathscr{A}$, then $\phi \otimes_{\mathscr{A}} \psi$ is cartesian.
- Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ and $\Phi'\colon \mathscr{A}' \to \mathscr{B}'$ be monoidal fibrations. A **monoidal fibred 1-cell $\Phi \to \Phi'$** consists of a pair of monoidal functors $(F, \phi, \phi_0)\colon \mathscr{A} \to \mathscr{A}'$ and $(G, \psi, \psi_0)\colon \mathscr{B} \to \mathscr{B}'$ such that (F, G) is a fibred 1-cell, ϕ lies over ψ and ϕ_0 lies over ψ_0.
- We call a monoidal indexed 1-cell $(F, G)\colon \Phi \to \Phi'$ **pure** if $\mathscr{B} = \mathscr{B}'$ and $(G, \psi, \psi_0) = \mathrm{id}_{\mathscr{B}}$.
- Let (F, G) and (F', G') be monoidal fibred 1-cells $\Phi \to \Phi'$. A **monoidal fibred 2-cell $(F, G) \to (F', G')$** is a fibred 2-cell $(\alpha, \beta)\colon (F, G) \to (F', G')$ such that α and β are both monoidal transformations.
- We call a monoidal fibred 2-cell $(\alpha, \beta)\colon (F, G) \to (F', G')$ **pure** if its underlying fibred 2-cell is pure.

◇

Definition 2.6.2. Let $\mathscr{B}$ be a monoidal category. The 2-category $\mathcal{M}on\mathcal{F}ib_{\mathscr{B}}$ consists of monoidal fibrations over $\mathscr{B}$, pure monoidal fibred 1-cells and pure monoidal fibred 2-cells. ◇

Definition 2.6.3. The 2-categories $\mathcal{BrMonFib}$ of braided monoidal fibrations and $\mathcal{SymMonFib}$ of symmetric monoidal fibrations are respectively defined to be the 2-category $\mathrm{BrPsMon}(\mathcal{Fib})$ of braided pseudomonoids in $\mathcal{Fib}$ and the 2-category $\mathrm{SymPsMon}(\mathcal{Fib})$ of symmetric pseudomonoids in $\mathcal{Fib}$. Of course, for a fixed monoidal category $\mathscr{B}$, these 2-categories have sub-2-categories $\mathcal{BrMonFib}_{\mathscr{B}}$ and $\mathcal{SymMonFib}_{\mathscr{B}}$ for fibrations of the appropriate type with base $\mathscr{B}$. ◇

Definition 2.6.4. If Φ is a monoidal fibration and an opfibration and $\otimes$ preserves opcartesian arrows, then we call Φ a **monoidal bifibration**. ◇

Example 2.6.5. We'll show that the forgetful functor $\mathtt{Rep}\colon \mathbf{GrpRep} \to \mathbf{FinGrp}$ is a monoidal bifibration. Firstly, the category of finite groups **FinGrp** is a cartesian monoidal category, and the category **GrpRep** is a monoidal category with monoidal product

$$(G, V) \otimes (H, W) = (G \times H, V \otimes W)$$

and monoidal unit $(\{1\}, \mathbb{C})$. Secondly, it's easy to check that $\mathtt{Rep}$ is a strict monoidal functor. It's straightforward to check that the monoidal product, $\otimes$, in **GrpRep** of cartesian morphisms is cartesian, It remains to show that the monoidal product also preserves opcartesian morphisms.

Let $f\colon G \to H$ and $g\colon K \to L$ be group homomorphisms, and let (G, U) and (K, X) be objects in **GrpRep**. The monoidal product of the two opcartesian morphisms

$$(f,\ u \mapsto e_H \otimes u)\colon (G, U) \longrightarrow (H, \mathbb{C}H \otimes_G U)$$

and

$$(g,\ x \mapsto e_L \otimes x)\colon (K, X) \longrightarrow (L, \mathbb{C}L \otimes_K X)$$

in **GrpRep** is the morphism

$$(f \times g,\ u \otimes x \mapsto (e_H \otimes u) \otimes (e_L \otimes x))\colon (G \times K, U \otimes X) \longrightarrow (H \times L, (\mathbb{C}H \otimes_G U) \otimes (\mathbb{C}L \otimes_K X)).$$

This morphism is opcartesian because it's the composite of the opcartesian morphism

$$\mathrm{opcart}^{f \times g}_{(G \times K, U \otimes X)}\colon (G \times K, U \otimes X) \longrightarrow (H \times L, \mathbb{C}(H \times K) \otimes_{G \times K} U \otimes X)$$

and the isomorphism

$$\begin{aligned}(\mathrm{id}_{H \times L},\ (h, l) \otimes (u \otimes x) \mapsto (h \otimes u) \otimes (l \otimes x))&\colon\\ (H \times L, \mathbb{C}(H \times K) \otimes_{G \times K} U \otimes X) &\longrightarrow (H \times L, (\mathbb{C}H \otimes_G U) \otimes (\mathbb{C}L \otimes_K X)).\end{aligned}$$

◇

Example 2.6.6. Let $\mathscr{A}$ be a category with pullbacks. Then $\mathscr{A}$ and $\mathscr{A}^{\to}$ are both cartesian monoidal. The codomain functor $\mathtt{Arr}_{\mathscr{A}}\colon \mathscr{A}^{\to} \to \mathscr{A}$ is a monoidal bifibration: the cartesian product in $\mathscr{A}^{\to}$ preserves cartesian morphisms since it preserves pullbacks, and it preserves opcartesian morphism since it's a functor and so preserves composition. ◇

2.7 Monoidal indexed categories

In this section we'll summarise the two approaches to defining monoidal indexed categories over a monoidal category $\mathscr{X}$. We follow the expositions of Shulman [Shu08] and of Moeller and Vasilakopoulou [MV20]; the results of this section also belong to these two papers.

2.7.1 Global monoidal indexed categories

Definition 2.7.1.

- A **monoidal indexed category** consists of a monoidal category $\mathscr{X}$ and a lax monoidal pseudofunctor

$$(\mathfrak{I}, \mu_2, \mu_0) \colon (\mathscr{X}^{\mathrm{op}}, \otimes^{\mathrm{op}}, \mathbb{1}) \to (\mathbf{Cat}, \times, \mathbf{1}).$$

- Let $\mathfrak{I} \colon \mathscr{X}^{\mathrm{op}} \to \mathbf{Cat}$ and $\mathfrak{K} \colon \mathscr{Y}^{\mathrm{op}} \to \mathbf{Cat}$ be monoidal indexed categories. A **monoidal indexed 1-cell** $\mathfrak{I} \to \mathfrak{K}$ is an indexed 1-cell (F, τ) such that the functor F and the pseudonatural transformation τ are both monoidal.

- Let (F, τ) and (G, σ) be monoidal indexed 1-cells. A **monoidal indexed 2-cell** is an indexed 2-cell (α, m) such that the natural transformation α and the modification m are both monoidal.

◇

Definition 2.7.2.

- The 2-category $\mathcal{M}on\mathcal{I}nd\mathcal{C}at$ of monoidal indexed categories consists of monoidal indexed categories, monoidal indexed 1-cells and monoidal indexed 2-cells.

- Let $\mathscr{X}$ be a monoidal category. The 2-category $\mathcal{M}on\mathcal{I}nd\mathcal{C}at_{\mathscr{X}}^{\mathrm{global}}$ of monoidal indexed categories over $\mathscr{X}$ consists of monoidal indexed categories over $\mathscr{X}$, pure monoidal indexed 1-cells and pure monoidal indexed 2-cells; this is equal to the 2-category $\mathbf{Mon2Cat}^{\mathrm{ps}}(\mathscr{X}^{\mathrm{op}}, \mathbf{Cat})$.

◇

Proposition 2.7.3 ([MV20, Section 3.6]). *The 2-category $\mathcal{M}on\mathcal{I}nd\mathcal{C}at$ is equal to the 2-category $\mathrm{PsMon}(\mathcal{I}nd\mathcal{C}at)$ of pseudomonoids in $\mathcal{I}nd\mathcal{C}at$, where the monoidal structure on $\mathcal{I}nd\mathcal{C}at$ defined as follows:*

- *for each pair of indexed categories $\mathfrak{I} \colon \mathscr{X}^{\mathrm{op}} \to \mathbf{Cat}$ and $\mathfrak{K} \colon \mathscr{Y}^{\mathrm{op}} \to \mathbf{Cat}$, their monoidal product $\mathfrak{I} \otimes \mathfrak{K}$ is the following composite*

$$(\mathscr{X} \times \mathscr{Y})^{\mathrm{op}} \xrightarrow{\cong} \mathscr{X}^{\mathrm{op}} \times \mathscr{Y}^{\mathrm{op}} \xrightarrow{\mathfrak{I} \times \mathfrak{K}} \mathbf{Cat} \times \mathbf{Cat} \xrightarrow{\times} \mathbf{Cat}$$

- *the monoidal unit is the indexed category $\Delta\mathbf{1} \colon \mathbf{1}^{\mathrm{op}} \to \mathbf{Cat}$ which has constant value $\mathbf{1}$.*

□

The two 2-categories $\mathcal{F}ib$ and $\mathcal{I}nd\mathcal{C}at$ are both cartesian monoidal 2-categories. Therefore, since 2-equivalences preserve limits, the constituent 2-functors of the 2-equivalence

$$\mathcal{F}ib \simeq \mathcal{I}nd\mathcal{C}at \tag{2.7.4}$$

from Theorem 2.3.15 are monoidal 2-functors. In order for the 2-equivalence (2.7.4) to be a monoidal 2-equivalence, it remains to show that the unit and counit of the equivalence are monoidal, but this follows from the universal property of products.

Lemma 2.7.5 ([MV20, Lemma 3.12]). *The 2-equivalence $\mathcal{F}ib \simeq \mathcal{I}nd\mathcal{C}at$ between the cartesian monoidal 2-categories of fibrations and indexed categories is symmetric monoidal.* □

Every 2-functor between 2-categories preserves equivalences. In particular, this is true from the 2-functors PsMon, $\mathsf{BrPsMon}$ and $\mathsf{SymPsMon}$.

Lemma 2.7.6 ([MV20, Proposition 2.11]). *A monoidal 2-equivalence $\mathcal{K} \simeq \mathcal{L}$ induces 2-equivalences*

$$\begin{aligned}\mathsf{PsMon}(\mathcal{K}) &\simeq \mathsf{PsMon}(\mathcal{L})\\ \mathsf{BrPsMon}(\mathcal{K}) &\simeq \mathsf{BrPsMon}(\mathcal{L})\\ \mathsf{SymPsMon}(\mathcal{K}) &\simeq \mathsf{SymPsMon}(\mathcal{L})\end{aligned}$$

between the 2-categories of pseudomonoids in $\mathcal{L}$ and pseudomonoids in $\mathcal{L}$, as well as their braided and symmetric versions.

The following theorem is just a special case of this lemma with $\mathcal{K} = \mathcal{F}ib$ and $\mathcal{L} = \mathcal{I}nd\mathcal{C}at$.

Theorem 2.7.7 ([MV20, Theorem 3.13]). *There exist 2-equivalences*

$$\begin{aligned}\mathcal{M}on\mathcal{F}ib &\simeq \mathcal{M}on\mathcal{I}nd\mathcal{C}at^{\mathrm{global}}\\ \mathcal{B}r\mathcal{M}on\mathcal{F}ib &\simeq \mathcal{B}r\mathcal{M}on\mathcal{I}nd\mathcal{C}at^{\mathrm{global}}\\ \mathcal{S}ym\mathcal{M}on\mathcal{F}ib &\simeq \mathcal{S}ym\mathcal{M}on\mathcal{I}nd\mathcal{C}at^{\mathrm{global}}\end{aligned}$$

between the 2-categories of monoidal fibrations and monoidal indexed categories, as well as their braided and symmetric versions.

The reason we call these monoidal indexed categories *global* is because they are equivalent to monoidal fibrations where the total category has a monoidal structure as a whole; Section 2.7.2 deals with the alternative which is to instead give a monoidal structure to each of the fibre categories.

2.7.2 Fibrewise monoidal indexed categories

Definition 2.7.8. Let $\mathscr{X}$ be a monoidal category. The 2-category $\mathcal{M}on\mathcal{I}nd\mathcal{C}at^{\mathrm{fibre}}_{\mathscr{X}}$ is defined to be equal to the 2-category $\mathbf{2Cat}^{\mathrm{ps}}(\mathscr{X}^{\mathrm{op}}, \mathbf{MonCat})$. ◇

Street [Str80, 1.34] states, for monoidal 2-categories $\mathcal{A}$, $\mathcal{K}$ and $\mathcal{L}$, equivalences

$$\mathbf{2Cat}^{\mathrm{ps}}(\mathcal{A}, \mathbf{2Cat}^{\mathrm{ps}}(\mathcal{K}, \mathcal{L})) \simeq \mathbf{2Cat}^{\mathrm{ps}}(\mathcal{A} \times \mathcal{K}, \mathcal{L}) \simeq \mathbf{2Cat}^{\mathrm{ps}}(\mathcal{K}, \mathbf{2Cat}^{\mathrm{ps}}(\mathcal{A}, \mathcal{L})).$$

In the following lemma, Moeller and Vasilakopoulou extend these to monoidal structures.

Lemma 2.7.9 ([MV20, Lemma 4.3]). *Let $\mathcal{K}$ and $\mathcal{L}$ be monoidal 2-categories.*

1. *For any monoidal 2-category $\mathcal{A}$,*

$$\mathbf{2Cat}^{\mathrm{ps}}(\mathcal{A}, \mathbf{Mon2Cat}^{\mathrm{ps}}(\mathcal{K}, \mathcal{L})) \simeq \mathbf{Mon2Cat}^{\mathrm{ps}}(\mathcal{K}, \mathbf{2Cat}^{\mathrm{ps}}(\mathcal{A}, \mathcal{L}))$$

2. *For any cocartesian monoidal 2-category $\mathcal{A}$,*

$$\mathbf{2Cat}^{\mathrm{ps}}(\mathcal{A}, \mathbf{Mon2Cat}^{\mathrm{ps}}(\mathcal{K}, \mathcal{L})) \simeq \mathbf{Mon2Cat}^{\mathrm{ps}}(\mathcal{A} \times \mathcal{K}, \mathcal{L}).$$

Corollary 2.7.10 ([MV20, Proposition 4.4]). *Let $\mathscr{X}$ be a monoidal category. There exists a 2-equivalence*

$$\mathcal{MonIndCat}^{\mathrm{fibre}}_{\mathscr{X}} \simeq \mathrm{PsMon}(\mathcal{IndCat}_{\mathscr{X}}).$$

Corollary 2.7.11. *If $\mathscr{X}$ is a cartesian monoidal category, then there exists a 2-equivalence*

$$\mathcal{MonIndCat}^{\mathrm{global}}_{\mathscr{X}} \simeq \mathcal{MonIndCat}^{\mathrm{fibre}}_{\mathscr{X}}.$$

This implies the following theorem which states that, when $\mathscr{X}$ is cartesian monoidal, the two notions of monoidal indexed category coincide and so, if $\Phi\colon \mathscr{A} \to \mathscr{B}$ is a monoidal fibration with cartesian base, we can think of the monoidal structure on $\mathscr{A}$ as on the whole of $\mathscr{A}$ or the fibre categories $\mathscr{A}_B : B \in \mathscr{B}$. This theorem was proved by Shulman in [Shu08], but the way we have presented its deduction mirrors Moeller and Vasilakopoulou's approach in [MV20].

Theorem 2.7.12 ([Shu08, Thm. 12.7]). *If $\mathscr{X}$ is a cartesian monoidal category, then there exist 2-equivalences*

$$\mathcal{MonFib}_{\mathscr{X}} \simeq \mathcal{MonIndCat}^{\mathrm{global}}_{\mathscr{X}} \simeq \mathcal{MonIndCat}^{\mathrm{fibre}}_{\mathscr{X}} \simeq \mathbf{2Cat}^{\mathrm{ps}}(\mathscr{X}^{\mathrm{op}}, \mathbf{MonCat})$$
$$\mathcal{BrMonFib}_{\mathscr{X}} \simeq \mathcal{BrMonIndCat}^{\mathrm{global}}_{\mathscr{X}} \simeq \mathcal{BrMonIndCat}^{\mathrm{fibre}}_{\mathscr{X}} \simeq \mathbf{2Cat}^{\mathrm{ps}}(\mathscr{X}^{\mathrm{op}}, \mathbf{BrMonCat})$$
$$\mathcal{SymMonFib}_{\mathscr{X}} \simeq \mathcal{SymMonIndCat}^{\mathrm{global}}_{\mathscr{X}} \simeq \mathcal{SymMonIndCat}^{\mathrm{fibre}}_{\mathscr{X}} \simeq \mathbf{2Cat}^{\mathrm{ps}}(\mathscr{X}^{\mathrm{op}}, \mathbf{SymMonCat}).$$

There are some important consequences of Theorem 2.7.12 which we note now.

If $\Phi\colon \mathscr{A} \to \mathscr{B}$ is a monoidal fibration, then we have, by definition, a monoidal structure on $\mathscr{A}$, which we denote by $\otimes$, and we call this the **external** monoidal structure. If the base category $\mathscr{B}$ is cartesian monoidal, then, due to the 2-equivalence $\mathcal{MonFib}_{\mathscr{X}} \simeq \mathbf{2Cat}^{\mathrm{ps}}(\mathscr{X}^{\mathrm{op}}, \mathbf{MonCat})$, the fibres $\mathscr{A}_B\colon B \in \mathscr{B}$ each have their own monoidal structure, which we denote by $\boxtimes_B$, and we call this the **internal** monoidal structure.

Recall that the indexed category $\Phi\mathcal{G}^{-1}$ associated to the fibration $\Phi\colon \mathscr{A} \to \mathscr{B}$ maps a morphism $f\colon B \to B'$ in $\mathscr{B}$ to the pull-back functor $f^*\colon \mathscr{A}_{B'} \to \mathscr{A}_B$. Since the 1-cells in $\mathbf{2Cat}^{\mathrm{ps}}(\mathscr{X}^{\mathrm{op}}, \mathbf{MonCat})$ are strong monoidal functors, Theorem 2.7.12 implies that, when the base category $\mathscr{B}$ of a monoidal fibration is *cartesian*, the functor $f^*\colon \mathscr{A}_{B'} \to \mathscr{A}_B$ is strong monoidal.

Lastly, an important fact that arises in the proof of Theorem 2.7.12 is the following relationship between the monoidal product, $\otimes$, on $\mathscr{A}$ and the monoidal product $\boxtimes$ on each fibre category:

- for each pair of objects A, B in $\mathscr{B}$, each object M in $\mathscr{A}_A$ and each object N in $\mathscr{A}_B$,

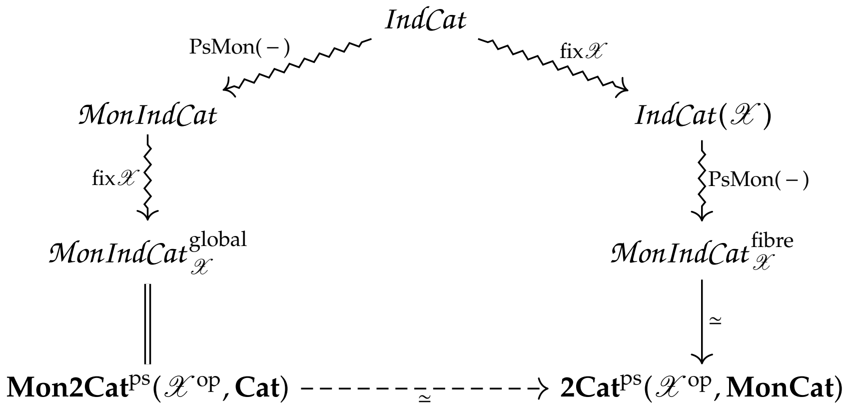
$$M \otimes N \cong M\pi_B^{AB*} \boxtimes_{A\times B} N\pi_A^{AB*}, \text{ and}$$

- for each object A in $\mathscr{B}$ and each pair P, Q of objects in $\mathscr{A}_A$,

$$P \boxtimes_A Q \cong (P \otimes Q)\Delta_A^*. \tag{2.7.13}$$

2.7.3 Summary

The following schematic from [MV20] summarises what we've studied in this section. The squiggly arrows indicate the global and fibrewise approaches: the global approach takes pseudomonoids in $\mathcal{IndCat}$ and then restricts attention to those that are over $\mathscr{X}$, and the fibrewise approach restricts attention to $\mathcal{IndCat}_{\mathscr{X}}$ and then takes pseudomonoids in $\mathcal{IndCat}_{\mathscr{X}}$. The dashed 2-equivalence exists in the case that $\mathscr{X}$ is a cartesian monoidal category.



2.8 Closed monoidal fibrations

This section follows the work of Shulman [Shu08, Section 13]. We take the approach of defining internal and external closure using mates instead of adjuncts, but the two approaches are equivalent. The advantage of using mates is the geometric perspective of string diagrams, and it also enables us to see why one mate might be an isomorphism if, and only if, the other is.

2.8.1 Internal closure

Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a monoidal fibration with $\mathscr{B}$ cartesian monoidal. We saw in Section 2.7 that, for each object B in $\mathscr{B}$, the fibre category $\mathscr{A}_B$ is monoidal, and, for each morphism $f\colon A \to B$ in $\mathscr{B}$, the functor $f^*\colon \mathscr{A}_B \to \mathscr{A}_A$ is strong monoidal. We can ask if, for each object B in $\mathscr{B}$, the fibre category $\mathscr{A}_B$ is left and right closed monoidal; that is, for each object M in $\mathscr{A}_B$, the functors

$$- \boxtimes M\colon \mathscr{A}_B \to \mathscr{A}_B \quad \text{and} \quad M \boxtimes -\colon \mathscr{A}_B \to \mathscr{A}_B$$

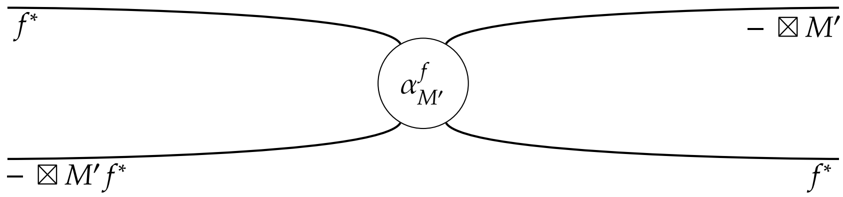
have right adjoints

$$M \blacktriangleright -\colon \mathscr{A}_B \to \mathscr{A}_B \quad \text{and} \quad - \blacktriangleleft M\colon \mathscr{A}_B \to \mathscr{A}_B.$$

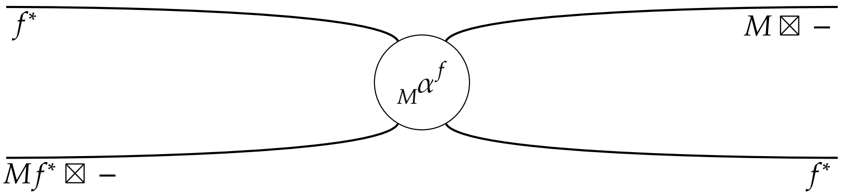
For each object B in $\mathscr{B}$, the isomorphisms

$$Mf^* \boxtimes M'f^* \to (M \boxtimes M')f^*,$$

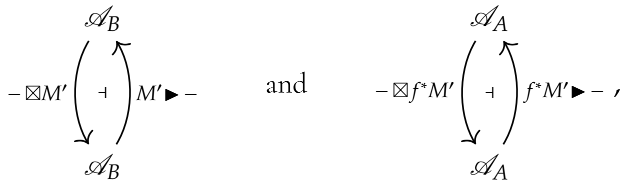
given for each pair of objects M and M' in $\mathscr{A}_B$, define natural transformations



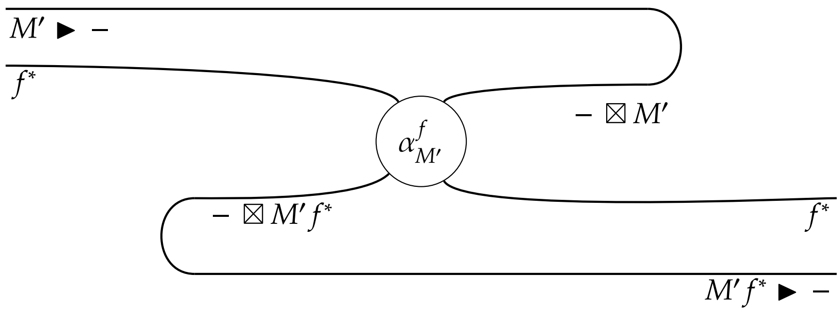
and



With respect to the adjunctions



the transformation $\alpha^f_{M'}$ has mate

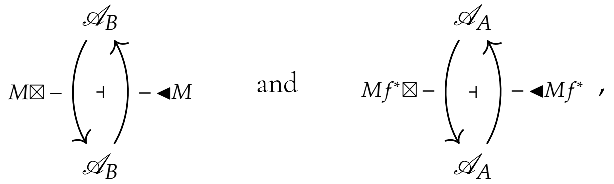


(2.8.1)

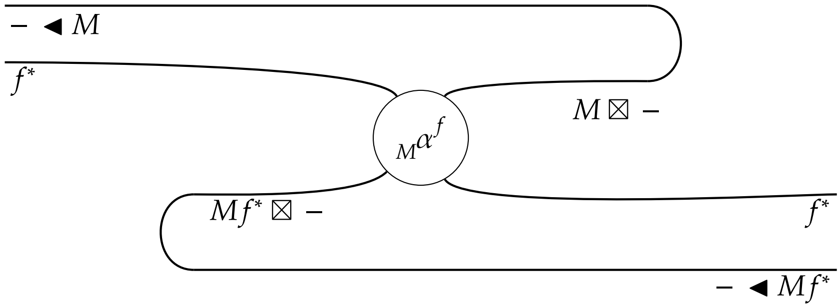
whose component morphism, for each object M in $\mathscr{A}_B$, is of the form

$$(M' \blacktriangleright M)f^* \to M'f^* \blacktriangleright Mf^* \;. \tag{2.8.2}$$

Similarly, with respect to the adjunctions



the transformation ${}_M\alpha^f$ has mate



(2.8.3)

whose component morphism, for each object M' in $\mathscr{A}_B$, is of the form

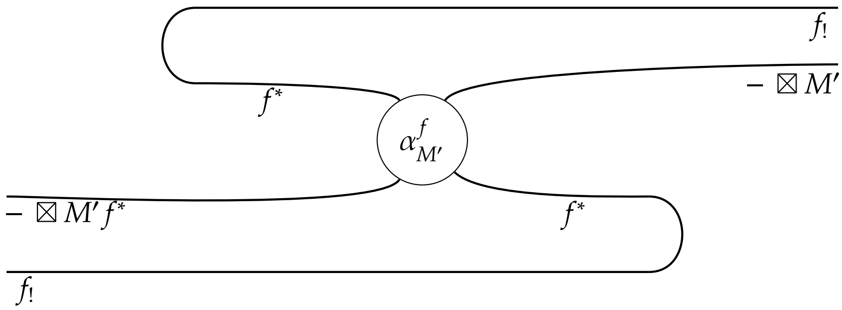
$$(M' \blacktriangleleft M)f^* \to M'f^* \blacktriangleleft Mf^* \,. \tag{2.8.4}$$

Definition 2.8.5. Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a monoidal fibration with $\mathscr{B}$ cartesian monoidal. We call $\Phi\colon \mathscr{A} \to \mathscr{B}$ **internally closed** if, for each object B in $\mathscr{B}$, the fibre category $\mathscr{A}_B$ is closed monoidal and if, for every morphism $f\colon A \to B$ in $\mathscr{B}$ and every pair of objects M, M' in $\mathscr{A}_B$, the morphisms

$$(M' \blacktriangleright M)f^* \to M'f^* \blacktriangleright Mf^* \qquad \text{and} \qquad (M' \blacktriangleleft M)f^* \to M'f^* \blacktriangleleft Mf^*.$$

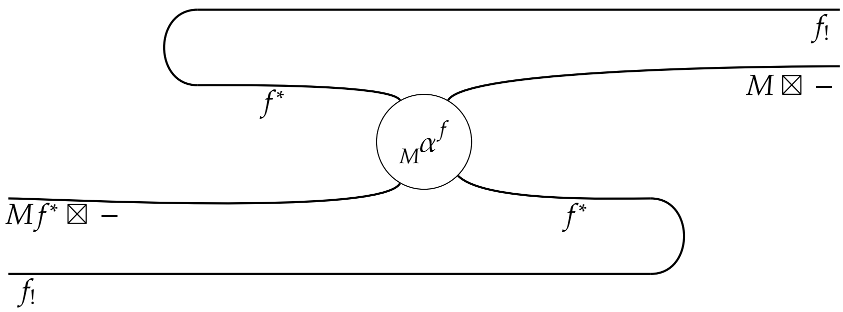
given by (2.8.2) and (2.8.4) are isomorphisms. ◇

Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a monoidal fibration with $\mathscr{B}$ cartesian monoidal and suppose that, for each morphism $f\colon A \to B$ in $\mathscr{B}$, the functor $f^*\colon \mathscr{A}_B \to \mathscr{A}_A$ has a left adjoint $f_!$. The mate of $\alpha^f_{M'}$ with respect to the adjunction $f_! \dashv f^*$ is



(2.8.6)

and the mate of ${}_M\alpha^f$ with respect to the adjunction $f_! \dashv f^*$ is



(2.8.7)

These have component morphisms

$$(M \boxtimes f^*M')f_! \to Mf_! \boxtimes M' \qquad \text{and} \qquad (Mf^* \boxtimes M')f_! \to M \boxtimes M'f_!$$

called the **projection morphisms**, which we call the **projection formulas** if they are isomorphisms.

We have seen that (2.8.1) and (2.8.6) are mates of $\alpha^f_{M'}$. Since the functor

$$f^*\colon \mathscr{A}_B \to \mathscr{A}_A$$

is a right adjoint and the functors

$$- \boxtimes M'\colon \mathscr{A}_B \to \mathscr{A}_B \qquad \text{and} \qquad - \boxtimes M'f^*\colon \mathscr{A}_A \to \mathscr{A}_A$$

are left adjoints, Lemma 1.3.18 gives that the natural transformation (2.8.1) is an isomorphism if, and only if, the natural transformation (2.8.6) is an isomorphism. Similarly, the natural transformation (2.8.3) is an isomorphism if, and only if, the natural transformation (2.8.7) is an isomorphism. So we have shown that

Φ is internally closed if, and only if, the projection formulas hold.

We defined the projection morphisms

$$(M \boxtimes f^*M')f_! \to Mf_! \boxtimes M' \qquad \text{and} \qquad (Mf^* \boxtimes M')f_! \to M \boxtimes M'f_!$$

using mates of the isomorphisms

$$Mf^* \boxtimes M'f^* \to (M \boxtimes M')f^*$$

whihc make f^* strong monoidal. One can also define each projection morphism being the unique morphism satisfying a universal property; this is the subject of the following lemma and its corollary, which will be used later in the proof of Theorem 4.3.1.

Lemma 2.8.8 ([Shu08, Lemma 16.4]). *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a monoidal fibration with $\mathscr{B}$ cartesian monoidal. Let $f\colon B \to B'$ be a morphism in $\mathscr{B}$, let $M \in \mathscr{A}_B$ and let $N \in \mathscr{A}_{B'}$. Then the following diagram*

$$\begin{array}{ccccc}
M \otimes Nf^* & \xleftarrow{\mathrm{cart}^{\Delta_B}_{M\otimes Nf^*}} & (M \otimes Nf^*)\Delta_B^* & \xrightarrow{\cong} & M \boxtimes Nf^* \\
 & & & & \downarrow{\scriptstyle \mathrm{opcart}^f_{M\boxtimes Nf^*}} \\
\downarrow{\scriptstyle \mathrm{opcart}^f_M \otimes \mathrm{cart}^f_N} & & & & (M \boxtimes Nf^*)f_! \\
 & & & & \downarrow{\scriptstyle proj} \\
Mf_! \otimes N & \xleftarrow[\mathrm{cart}^{\Delta_{B'}}_{Mf_!\otimes N}]{} & (Mf_! \otimes N)\Delta_{B'}^* & \xleftarrow[\cong]{} & Mf_! \boxtimes N
\end{array}$$

in $\mathscr{A}$ commutes, where 'proj' denotes the projection morphism and the isomorphisms are given by (2.7.13).

Corollary 2.8.9. *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a moniodal bifibration with $\mathscr{B}$ cartesian monoidal, and let $f\colon B \to B'$ be a morphism in $\mathscr{B}$.*

The niche

$$\begin{array}{ccc} M \otimes N f^* & \xleftarrow{\mathrm{cart}^{\Delta_B}_{M\otimes Nf^*}} & (M \otimes N f^*)\Delta_B^* \\ {\scriptstyle \mathrm{opcart}^f_M \otimes \mathrm{cart}^f_N}\downarrow & & \\ M f_! \otimes N & \xleftarrow[\mathrm{cart}^{\Delta_{B'}}_{Mf_!\otimes N}]{} & (M f_! \otimes N)\Delta_{B'}^*, \end{array} \tag{2.8.10}$$

in $\mathscr{A}$ lies over the niche

$$\begin{array}{ccc} B \times B & \xleftarrow{\Delta_B} & B \\ {\scriptstyle f\times f}\downarrow & & \\ B' \times B' & \xleftarrow[\Delta_{B'}]{} & B' \end{array}$$

in $\mathscr{B}$. If Φ is internally closed, then the unique morphism that fills the niche (2.8.10) *and lies over f is opcartesian.*

Example 2.8.11. In this example we show that the monoidal bifibration $\mathsf{Rep}\colon \mathbf{GrpRep} \to \mathbf{FinGrp}$ is internally closed. Given a G-module V, the dual G-module ${}^{\vee}V = \mathrm{Lin}_{\mathbb{C}}(V, \mathbb{C})$ has G-action $(v)(g \cdot f) = (g^{-1} \cdot v)f$. The fibre category $\mathbf{Rep}_G$ is closed monoidal with internal hom $[{}_GW, {}_GU] = {}_G\mathrm{Lin}_{\mathbb{C}}(W, \mathbb{C}) \otimes {}_GU \cong {}_G\mathrm{Lin}_{\mathbb{C}}(W, U)$ which has G-action given by

$$(w)(g \cdot f) = g \cdot (g^{-1} \cdot w)f.$$

It remains to check that, for each group homomorphism $f\colon G \to H$, the restriction functor $f^*\colon \mathbf{Rep}_H \to \mathbf{Rep}_G$ is closed monoidal; that is, for any H-modules V and W, the canonical morphism

$$\overline{\Gamma}\colon [V, W]f^* \to [Vf^*, Wf^*]$$

is an isomorphism. This morphism is the adjunct of the following composite.

$$\Gamma\colon [V, W]f^* \otimes Vf^* \longrightarrow ([V, W] \otimes V)f^* \xrightarrow{\mathrm{ev} f^*} Wf^*$$

This G-module map is given by $(\phi \otimes v)\Gamma = (v)\phi$. So we'll find the adjunct of this morphism Γ and check if it's an isomorphism. We do this using the following series of isomorphisms.

$$\begin{aligned} [[V, W] \otimes V, W] &\cong W \otimes ([V, W] \otimes V)^{\vee} \\ &\cong W \otimes (V^{\vee} \otimes [V, W]^{\vee}) \\ &\cong (W \otimes V^{\vee}) \otimes [V, W]^{\vee} \\ &\cong [V, W] \otimes [V, W]^{\vee} \\ &\cong [[V, W], [V, W]]. \end{aligned} \tag{2.8.12}$$

Let $(x_i)_{i=1}^n$ be a basis for $[V, W]$ and let $(y_j)_{j=1}^n$ be a basis for V. Then, under the isomorphisms (2.8.12),

$$\begin{aligned}
\Gamma &\mapsto \sum_{i,j} (x_i \otimes y_j)\Gamma \otimes \left(x_i{}^\vee \otimes y_j{}^\vee\right) \\
&\mapsto \sum_{i,j} \left((x_i \otimes y_j)\Gamma \otimes y_j{}^\vee\right) \otimes x_i{}^\vee \\
&\mapsto \sum_{i,j} \left(v \mapsto (v)y_j{}^\vee (x_i \otimes y_j)\Gamma\right) \otimes x_i{}^\vee \\
&\mapsto \sum_{i,j} \left(\phi \mapsto \left(v \mapsto (\phi)x_i{}^\vee (v) y_j{}^\vee (x_i \otimes y_j)\Gamma\right)\right).
\end{aligned}$$

That is, $(v)(\phi)\overline{\Gamma} = \sum_{i,j}(\phi)x_i{}^\vee(v)y_j{}^\vee(x_i \otimes y_j)\Gamma = (\phi \otimes v)\Gamma = (v)\phi$, so $\overline{\Gamma} = \mathrm{id}_{G[V,W]}$. This is an isomorphism, so f^* is closed monoidal. ◇

2.8.2 External closure

In Section 2.8.1, we used the monoidal structure of the fibres categories $\mathscr{A}_B \colon B \in \mathscr{B}$ to define internal closure for monoidal fibrations with cartesian base. In this section, we'll use the monoidal structure of the total category $\mathscr{A}$ to define external closure for monoidal fibrations.

We can ask if, for each pair of objects E and F in $\mathscr{B}$ and each object Q in $\mathscr{A}_E$, the functors

$$- \otimes Q \colon \mathscr{A}_F \to \mathscr{A}_{F\otimes E} \quad \text{and} \quad Q \otimes - \colon \mathscr{A}_F \to \mathscr{A}_{E\otimes F} \tag{2.8.13}$$

have right adjoints

$$Q \rhd - \colon \mathscr{A}_{F\otimes E} \to \mathscr{A}_F \quad \text{and} \quad - \lhd Q \colon \mathscr{A}_{E\otimes F} \to \mathscr{A}_F. \tag{2.8.14}$$

Example 2.8.15. Consider the monoidal fibration $\mathsf{Rep} \colon \mathbf{GrpRep} \to \mathbf{FinGrp}$. The functors

$$- \otimes W \colon \mathbf{Rep}_G \to \mathbf{Rep}_{(G\times H)} \quad \text{and} \quad V \otimes - \colon \mathbf{Rep}_H \to \mathbf{Rep}_{(G\times H)}$$

have right adjoints

$$W \rhd - \colon \mathbf{Rep}_{(G\times H)} \to \mathbf{Rep}_G \quad \text{and} \quad - \lhd V \colon \mathbf{Rep}_{(G\times H)} \to \mathbf{Rep}_H$$

given by $W \rhd U = \mathrm{Hom}_H(W, U)$ and $U \lhd V\, \mathrm{Hom}_G(U, V)$. ◇

Remark 2.8.16. Note that the functors (2.8.13) having right adjoints (2.8.14) is *not* the same as asking that $\mathscr{A}$ be left and right closed monoidal, i.e. that, for each object Q in $\mathscr{A}$, the functors

$$- \otimes Q \colon \mathscr{A} \to \mathscr{A} \quad \text{and} \quad Q \otimes - \colon \mathscr{A} \to \mathscr{A}$$

have right adjoints

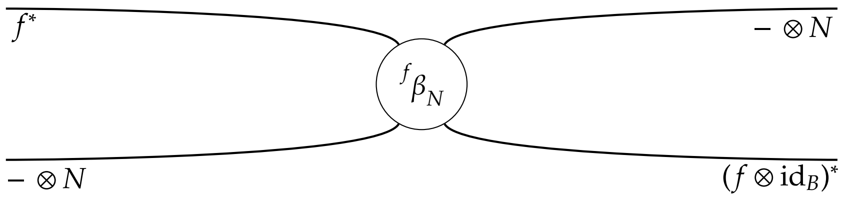
$$Q \rhd - \colon \mathscr{A} \to \mathscr{A} \quad \text{and} \quad - \lhd Q \colon \mathscr{A} \to \mathscr{A}.$$

This is because, given a fixed object R in $\mathscr{A}$, we can't, in general, construct a functor $- \otimes R \colon \mathscr{A} \to \mathscr{A}$ from the functors

$$\left(R \rhd - \colon \mathscr{A}_{F\otimes R\Phi} \to \mathscr{A}_F\right)_{F\in\mathscr{B}}$$

since they are defined only on objects in $\mathscr{A}$ that lie over objects in $\mathscr{B}$ of the form $F \otimes R\Phi$ for some F in $\mathscr{B}$. ◇

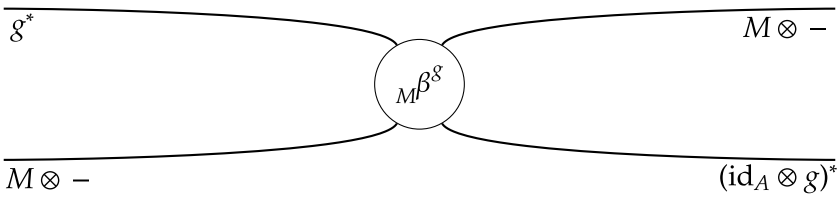
Just as we did for internal closure, we use mates to define the notion of external closure for a monoidal fibration $\Phi\colon \mathscr{A} \to \mathscr{B}$. For each object B in $\mathscr{B}$, each object N in $\mathscr{A}_B$ and each morphism $f\colon C \to A$ in $\mathscr{B}$, define the natural transformation



to have components

$$Mf^* \otimes N \xrightarrow{\mathrm{id}_{Mf^*} \otimes \mathbf{Rep}^{0*}_{B,N}} Mf^* \otimes N\mathrm{id}^*_B \xrightarrow{(\mathrm{cart}^f_M \otimes \mathrm{cart}^{\mathrm{id}_B}_M) \fatsemi \mathrm{cart}^{f\otimes \mathrm{id}_B}_{M\otimes N}} (M \otimes N)(f \otimes \mathrm{id}_B)^* \quad .$$

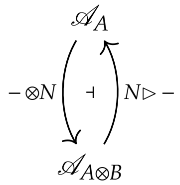
For each object A in $\mathscr{B}$, each object M in $\mathscr{A}_A$ and each morphism $g\colon D \to B$ in $\mathscr{B}$, define the natural transformation



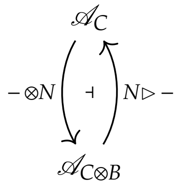
to have components

$$M \otimes Ng^* \xrightarrow{\mathbf{Rep}^{0*}_{A,M} \otimes \mathrm{id}_{Ng^*}} M\mathrm{id}^*_A \otimes Ng^* \xrightarrow{(\mathrm{cart}^{\mathrm{id}_A}_M \otimes \mathrm{cart}^g_M) \fatsemi \mathrm{cart}^{\mathrm{id}_A\otimes g}_{M\otimes N}} (M \otimes N)(\mathrm{id}_A \otimes g)^* \quad .$$

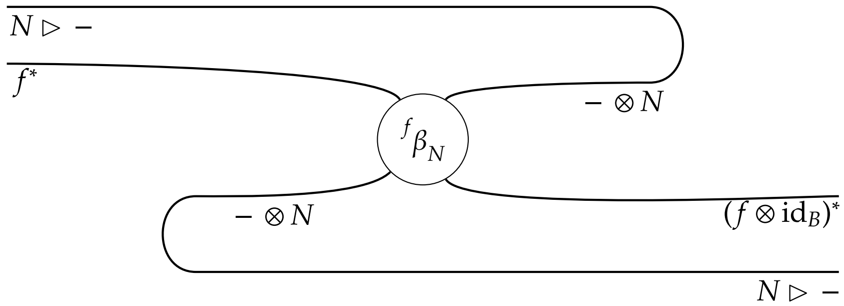
With respect to the adjunctions



and



the mate of ${}^f\beta_N$ is

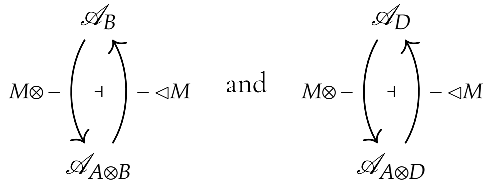


(2.8.17)

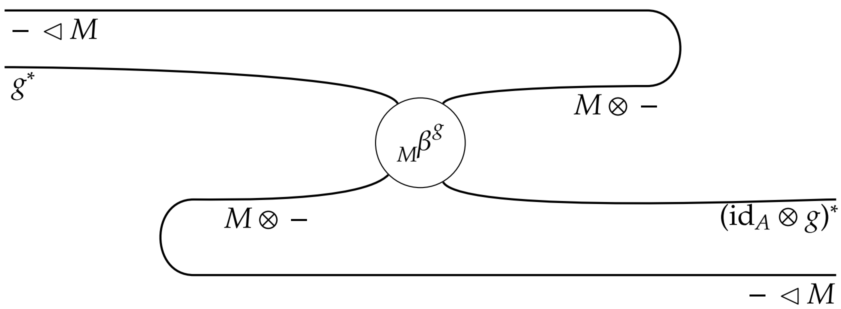
whose component morphism, for each object P in $\mathscr{A}_{A\otimes B}$, is of the form

$$(N \triangleright P)f^* \to N \triangleright P(f \otimes \mathrm{id}_B)^*. \tag{2.8.18}$$

Similarly, with respect to the adjunctions



the mate of ${}_M\beta^g$ is



(2.8.19)

whose component morphism, for each object P in $\mathscr{A}_{A\otimes B}$, is of the form

$$(P \triangleleft M)g^* \to P(\mathrm{id}_A \otimes g)^* \triangleleft M. \tag{2.8.20}$$

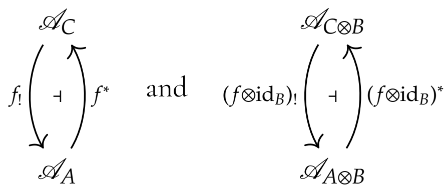
Definition 2.8.21. We call a monoidal fibration $\Phi\colon \mathscr{A} \to \mathscr{B}$ **externally closed** if, for every pair of morphisms $f\colon C \to A$ and $g\colon D \to B$ in $\mathscr{B}$ and every triple of objects $M \in \mathscr{A}_A$, $N \in \mathscr{A}_B$, $P \in \mathscr{A}_{A\otimes B}$, the morphisms

$$(N \triangleright P)f^* \to N \triangleright P(f \otimes \mathrm{id}_B)^* \qquad \text{and} \qquad (P \triangleleft M)g^* \to P(\mathrm{id}_A \otimes g)^* \triangleleft M$$

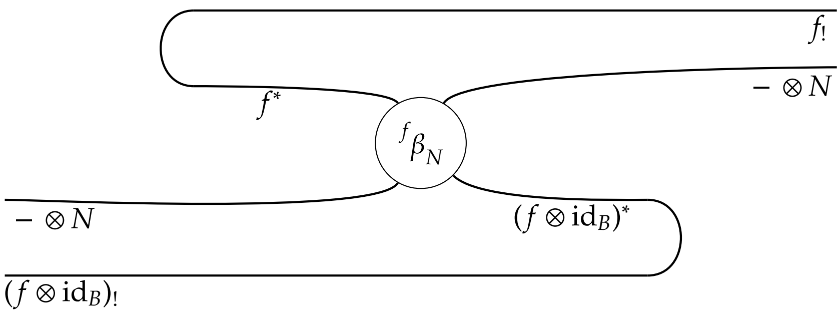
given by (2.8.18) and (2.8.20) are isomorphisms. ◇

We will now work towards Proposition 2.8.26, after which we'll see that the monoidal bifibration $\mathtt{Rep}\colon \mathbf{GrpRep} \to \mathbf{FinGrp}$ is externally closed.

Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a monoidal fibration and suppose that, for each morphism $f\colon C \to A$ in $\mathscr{B}$, the functor $f^*\colon \mathscr{A}_A \to \mathscr{A}_C$ has a left adjoint $f_!$. With respect to the adjunctions



the mate of ${}^{f}\beta_N$ is

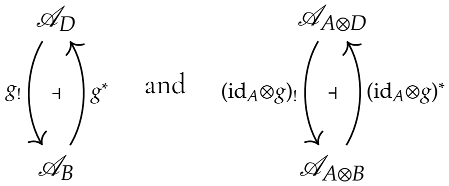


(2.8.22)

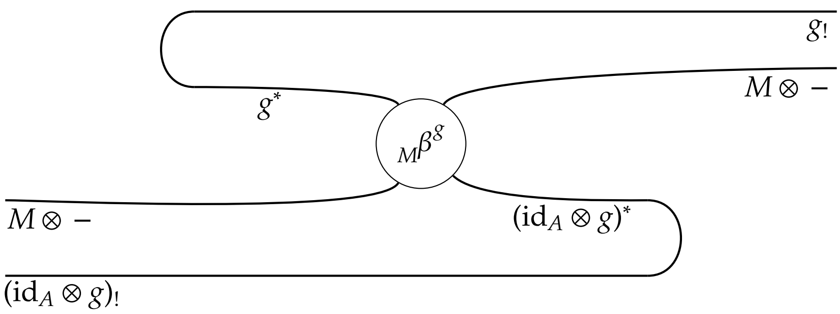
whose component morphism, for each object R in $\mathscr{A}_C$, is of the form

$$(R \otimes N)(f \otimes \mathrm{id}_B)_! \longrightarrow Rf_! \otimes N.$$

With respect to the adjunctions



the mate of ${}_{M}\beta^{g}$ is



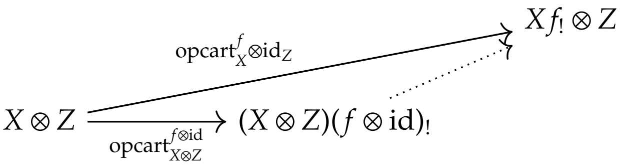
(2.8.23)

whose component morphism, for each object T in $\mathscr{A}_D$, is of the form

$$(M \otimes T)(\mathrm{id}_A \otimes g)_! \longrightarrow M \otimes Tg_!$$

The projection morphism $(M \boxtimes Nf^*)f_! \to Mf_! \boxtimes N$ was defined as a mate (see (2.8.6) and (2.8.7)) and Lemma 2.8.8 stated that it is also given by a universal property. Similarly, we defined the morphism $(R \otimes N)(f \otimes \mathrm{id}_B)_! \to Rf_! \otimes N$ using the mate (2.8.22), and the following proposition follows by the same method as Lemma 2.8.8.

Proposition 2.8.24. *The unique pure morphism* $(X \otimes Z)(f \otimes \mathrm{id})_! \to Xf_! \otimes Z$ *in* $\mathscr{A}$ *that makes the diagram*



commute is (2.8.22). ◻

Corollary 2.8.25. *Let* $\Phi\colon \mathscr{A} \to \mathscr{B}$ *be a monoidal fibration. The functor* $\otimes_{\mathscr{A}}$ *preserves opcartesian morphisms if, and only if, the natural transformations* (2.8.22) *and* (2.8.23) *are natural isomorphisms.*

Proof. Suppose that $\rho\colon X \to Y$ is an opcartesian morphism in $\mathscr{A}$. Using opcartesian-pure factorisation (see Proposition 2.4.15), write $\rho = \mathrm{opcart} \mathbin{⨾} \xi$ for some pure isomorphism ξ in $\mathscr{A}$. So $\rho \otimes Z = (\mathrm{opcart} \otimes Z) \mathbin{⨾} (\xi \otimes Z)$. Note that $\xi \otimes Z$ is opcartesian since it's an isomorphism, so $\rho \otimes Z$ is opcartesian if, and only if, $\mathrm{opcart} \otimes Z$ is opcartesian. By Proposition 2.8.24 and Proposition 2.4.16 (ii), $\mathrm{opcart} \otimes Z$ is opcartesian if, and only if, (2.8.22) is opcartesian, and this occurs if, and only if, (2.8.22) is an isomorphism since it's pure. Therefore, $- \otimes Z$ preserves opcartesian morphisms if, and only if, (2.8.22) is an isomorphism. Similarly, $W \otimes -$ preserves opcartesian morphisms if, and only if, (2.8.23) is an isomorphism.

To finish the proof, we will show that the functor $\otimes$ preserves opcartesian morphisms if, and only if, for all objects Z in $\mathscr{A}$, the functors $- \otimes Z$ and $Z \otimes -$ both preserve cartesian morphisms. Of course if, for all opcartesian morphisms ϕ and ψ, the morphism $\phi \otimes \psi$ is opcartesian, then, for all opcartesian morphisms σ and τ, the morphisms $\sigma \otimes \mathrm{id}$ and $\mathrm{id} \otimes \tau$ are opcartesian. On the other hand, if, for all opcartesian morphisms σ and τ, the morphisms $\sigma \otimes \mathrm{id}$ and $\mathrm{id} \otimes \tau$ are opcartesian, then, for all opcartesian morphisms ϕ and ψ, the morphism

$$\phi \otimes \psi = (\phi \mathbin{⨾} \mathrm{id}) \otimes (\mathrm{id} \mathbin{⨾} \psi) = (\phi \otimes \mathrm{id}) \mathbin{⨾} (\mathrm{id} \otimes \psi)$$

is opcartesian. ◻

Proposition 2.8.26 ([Shu08, Proposition 13.17])**.** *Let* $\Phi\colon \mathscr{A} \to \mathscr{B}$ *be a cleaved monoidal fibration. Suppose that, for each morphism* $f\colon C \to D$ *in* $\mathscr{B}$*, the functor* $f^*\colon \mathscr{A}_D \to \mathscr{A}_C$ *has a left adjoint* $f_!$*. Suppose also that, for each pair of objects* E *and* F *in* $\mathscr{B}$ *and each object* Q *in* $\mathscr{A}_E$*, the functors*

$$- \otimes Q\colon \mathscr{A}_F \to \mathscr{A}_{F \otimes E} \quad \textit{and} \quad Q \otimes -\colon \mathscr{A}_F \to \mathscr{A}_{E \otimes F}$$

have right adjoints

$$Q \rhd -\colon \mathscr{A}_{F \otimes E} \to \mathscr{A}_F \quad \textit{and} \quad - \lhd Q\colon \mathscr{A}_{E \otimes F} \to \mathscr{A}_F.$$

Then Φ *is a monoidal bifibration if, and only if,* Φ *is external closed.*

Proof. For each object B in $\mathscr{B}$, each object N in $\mathscr{A}_B$ and each morphism $f\colon C \to A$ in $\mathscr{B}$, we have defined the natural transformations (2.8.17) and (2.8.22) to be mates of the natural transformation ${}^f\beta_N$. Since the functors

$$f^*\colon \mathscr{A}_A \to \mathscr{A}_C \quad \text{and} \quad (f \otimes \mathrm{id})^*\colon \mathscr{A}_{A \otimes B} \to \mathscr{A}_{C \otimes B}$$

are right adjoints and the functors

$$- \otimes N \colon \mathscr{A}_C \to \mathscr{A}_{C \otimes B} \quad \text{and} \quad - \otimes N \colon \mathscr{A}_A \to \mathscr{A}_{A \otimes B}$$

are left adjoints, we can use Lemma 1.3.18 to get that (2.8.17) is an isomorphism if, and only if, (2.8.22) is an isomorphism. Similarly, (2.8.19) is an isomorphism if, and only if, (2.8.23) is an isomorphism. The result then follows from Corollary 2.8.25. □

Chapter 3

The Beck-Chevalley Condition

In this chapter, we study the Beck-Chevalley condition.

In the first section, we provide an account of the theory of integral transfroms and prove some basic properties of their composition; this will give some motiviation the study of the Beck-Chevalley condition.

In the second section, we define the Beck-Chevalley morphisms ζ_M, prove that they are natural in M and that the resulting natural transformation is given by a mate (see Proposition 3.2.3) which is the more common definition. In Remarks 3.2.8, 3.2.9 and 3.2.11, we give string diagrammatic arguments which show that the Beck-Chevalley transformation is well-defined.

In the third section, we give the definition of Beck-Chevalley and weakly Beck-Chevalley bifibrations. We then prove that the bifibration $\mathsf{Rep}\colon \mathbf{GrpRep} \to \mathbf{FinGrp}$ is weakly Beck-Chevalley (see Proposition 3.3.3), and we show how the Beck-Chevalley condition relates to Mackey's formula of representation theory.

In the fourth section, we give an explcit calculation of the morphism that fills a niche defined by cartesian and opcartesian lifting properties (see Lemma 3.4.3), and, at the end of this chapter, we use this explicit calculation to prove Corollary 3.4.4—a well-known result on conditions equivalent to the Beck-Chevalley condition—which is made a great deal more intuitive with a string diagrammatic proof.

3.1 Integral transforms

The history of integral transforms begins with the Fourier transform. One can see [Tre67] for a discussion of integral transforms in functional analysis, or [Huy06, Chapter 5] for a discussion of Fourier-Mukai kernels: integral transforms in derived algebraic geometry.

Notation 3.1.1. We write $\mathbb{F}$ to denote a field that is $\mathbb{R}$ or $\mathbb{C}$. For topological spaces W and Z, we write $\mathrm{Hom}(W, Z)$ for the set of continuous maps $W \to Z$. For continuous maps f and g in $\mathrm{Hom}(Z, \mathbb{F})$, we write $f \cdot g$ for their pointwise product $z \mapsto (z)f \cdot (z)g$. ◇

Definition 3.1.2. An **integral transform**

$$X \xleftarrow{\;p\;} (S, \kappa) \xrightarrow{\;q\;} Y$$

consists of

- three topological spaces X, Y and S,
- a pair of continuous maps $p\colon S \to X$ and $q\colon S \to Y$,
- a continuous map $\kappa\colon S \to \mathbb{F}$ called a **kernel**,
- for each continuous map $f\colon W \to Z$, maps $f^*\colon \mathrm{Hom}(Z,\mathbb{F}) \to \mathrm{Hom}(W,\mathbb{F})$ and $f_*\colon \mathrm{Hom}(W,\mathbb{F}) \to \mathrm{Hom}(Z,\mathbb{F})$,

such that, for each composable pair of continuous maps $f\colon W \to Z$ and $g\colon Z \to A$, $(f \mathbin{⨾} g)^* = g^* \mathbin{⨾} f^*$ and $(f \mathbin{⨾} g)_* = f_* \mathbin{⨾} g_*$. ◇

We've stated our definition of an integral transform in a more general form than the three examples we'll give below, which are all integral transforms of the following form: the two maps p and q are product projections

$$X \xleftarrow{\;p\;} X \times Y \xrightarrow{\;q\;} Y$$

and, for each map $f\colon W \to Z$, the map f^* is given by precomposing with f and the map f_* is given by an integral— integrals in the examples below are over a discrete space (i.e. a sum), an integral over $(0,\infty) \subset \mathbb{R}$, and an integral over $\mathbb{R}^2$.

The idea of an integral transform

$$X \xleftarrow{\;p\;} (S,\kappa) \xrightarrow{\;q\;} Y$$

is to map scalar maps $X \to \mathbb{F}$ to scalar maps $Y \to \mathbb{F}$, and we do this by defining a map (which is also called an integral transform) $\tau_\kappa\colon (X)F \to (Y)F$ given by

$$\begin{aligned} \tau_\kappa\colon (X)F &\longrightarrow (Y)F \\ h &\longmapsto (hp^* \cdot \kappa)q_*. \end{aligned}$$

Definition 3.1.3. Let p and q denote the following projection maps.

$$\mathbb{R} \xleftarrow{\;p\;} \mathbb{R} \times \mathbb{C} \xrightarrow{\;q\;} \mathbb{C}$$

Define the map $p^*\colon \mathrm{Hom}(\mathbb{R},\mathbb{C}) \to \mathrm{Hom}(\mathbb{R}\times\mathbb{C},\mathbb{C})$ on a map $h\colon \mathbb{R} \to \mathbb{C}$ by

$$(t,s)hp^* = (t)h$$

and define the map $q_*\colon \mathrm{Hom}(\mathbb{R}\times\mathbb{C},\mathbb{R}) \to \mathrm{Hom}(\mathbb{C},\mathbb{C})$ on a map $k\colon \mathbb{R}\times\mathbb{C} \to \mathbb{C}$ by

$$(s)kq_* = \int_0^\infty (t,s)k \,\mathrm{d}t.$$

Define $K\colon \mathbb{R}\times\mathbb{C} \to \mathbb{C}, (t,s) \mapsto e^{-ts}$. The **Laplace transform** is the map $\tau_K\colon \operatorname{Hom}(\mathbb{R},\mathbb{C}) \to \operatorname{Hom}(\mathbb{C},\mathbb{C})$, $h \mapsto (hp^* \cdot K)q_*$. Explicity, this is the map

$$h \mapsto \left(s \mapsto \int_0^\infty (t)h \cdot e^{-ts}\,\mathrm{d}t\right).$$

◇

Definition 3.1.4. Let $\mathbf{n}$, $\mathbf{m}$ and $\mathbf{n}\times\mathbf{m}$ denote the discrete topological spaces with n, m and nm points respectively, and let $\mathbb{R}^n$ deonte the set of maps $\mathbf{n} \to \mathbb{R}$. Of course, an element of $\mathbb{R}^n$ can be thought of as a vector with entries in $\mathbb{R}$ and an element of $\mathbb{R}^{nm}$ can be thought of as an $(n\times m)$-matrices with entries in $\mathbb{R}$.

Fix an $(n \times m)$-matrix K, and let p and q denote the following projection maps.

$$\mathbf{n} \xleftarrow{\;p\;} \mathbf{n}\times\mathbf{m} \xrightarrow{\;q\;} \mathbf{m}$$

Define the map $p^*\colon \mathbb{R}^n \to \mathbb{R}^{nm}$ by

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \longmapsto \begin{pmatrix} v_1 & v_1 & \cdots & v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_1 & v_2 & \cdots & v_n \end{pmatrix}$$

and define the map $q_*\colon \mathbb{R}^{nm} \to \mathbb{R}^m$ by

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \longmapsto \begin{pmatrix} \sum_{i=1}^n a_{1i} \\ \vdots \\ \sum_{i=1}^n a_{mi} \end{pmatrix}.$$

The **matrix mulitplication transform** is the map $\tau_K\colon \mathbb{R}^n \to \mathbb{R}^m$, $v \mapsto (vp^* \cdot K)q_*$. Explicity, this is the map

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \longmapsto Kv = \begin{pmatrix} \sum_{i=1}^n K_{i1}v_i \\ \vdots \\ \sum_{i=1}^n K_{im}v_i \end{pmatrix}.$$

◇

Definition 3.1.5. Fix $\sigma > 0$. The **two-dimensional Gaussian function** with standard deviation σ is the function $G_\sigma\colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined by

$$(x_1, y_1, x_2, y_2) \mapsto \frac{1}{2\pi\sigma^2} e^{-\frac{(x_1-x_2)^2+(y_1-y_2)^2}{2\sigma^2}}.$$

To get an idea of what this function G_σ looks like, we could look at, for a fixed point $(x_1, y_1) \in \mathbb{R}^2$, the graph of the function $(x, y, -, -)G_\sigma\colon \mathbb{R}^2 \to \mathbb{R}$. Figure 3.1 shows a plot of the function

$$(0, 0, -, -)G_{\frac{1}{2}}\colon \mathbb{R}^2 \to \mathbb{R}.$$

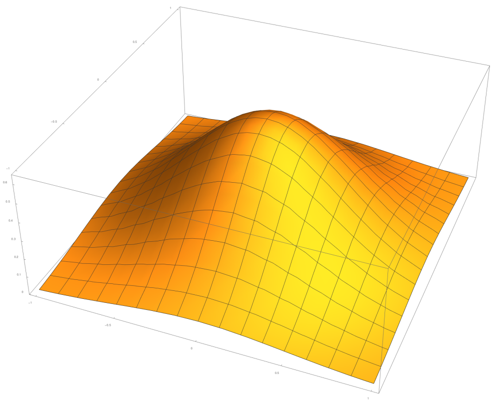


Figure 3.1: A plot of the function $(x, y) \mapsto \frac{1}{2\pi(\frac{1}{2})^2} e^{-\frac{x^2+y^2}{2(\frac{1}{2})^2}}$ created using Wolfram Mathematica.

Let p and q denote the following projection maps.

$$\mathbb{R}^2 \xleftarrow{\;p\;} \mathbb{R}^2 \times \mathbb{R}^2 \xrightarrow{\;q\;} \mathbb{R}^2$$

Define the map $p^* \colon \operatorname{Hom}(\mathbb{R}^2, \mathbb{R}) \to \operatorname{Hom}(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R})$ on a map $h \colon \mathbb{R}^2 \to \mathbb{R}$ by

$$(x_1, y_1, x_2, y_2)hp^* = (x_1, y_1)h$$

and define the map $q_* \colon \operatorname{Hom}(\mathbb{R}^2 \times \mathbb{R}^2, \mathbb{R}) \to \operatorname{Hom}(\mathbb{R}^2, \mathbb{R})$ on a map $k \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$(x_1, y_1)kq_* = \int_{\mathbb{R}^2} (x_1, y_1, x_2, y_2)k \, \mathrm{d}x_2 \mathrm{d}y_2.$$

The **Gaussian blur transform** is the map $\tau_{G_\sigma} \colon \operatorname{Hom}(\mathbb{R}^2, \mathbb{R}) \to \operatorname{Hom}(\mathbb{R}^2, \mathbb{R}), h \mapsto (hp^* \cdot G_\sigma)q_*$. Explicitly, this is the map

$$h \mapsto \left((x_1, y_1) \mapsto \int_{\mathbb{R}^2} (x_1, y_1, x_2, y_2)G_\sigma \cdot (x_2, y_2)h \, \mathrm{d}x_2 \mathrm{d}y_2 \right).$$

◇

The Gaussian blur transform is also known as the two-dimensional Weierstrass transform (after the the transform used to prove the Weierstrass approximation theorem [Wei85]) and as the Gaussian filter in signal analysis. The name 'Gaussian blur' comes from image processing where we can think of the Gaussian blur transform as blurring a two-dimensional image in the following way.

Remark 3.1.6. Given a two-dimensional image, we can define three maps $r, g, b \colon \mathbb{R}^2 \to \mathbb{R}$ as the intensity of red, green and blue of each pixel in the image. Applying the Gaussian blur transform to each of these maps gives three new maps $r\tau_{G_\sigma}, g\tau_{G_\sigma}, b\tau_{G_\sigma} \colon \mathbb{R}^2 \to \mathbb{R}$ which we can think of being the intensities of red, green and blue of each pixel in a new image.

Figure 3.2: An image and the result of apply Gaussian blurs with standard deviations 3 and 10 respectively to that image [Ika15].

But what does the function τ_{G_σ} do? If we fix a point (x_1, y_1) in $\mathbb{R}^2$, then we can think of

$$(x_1, y_1)(r\tau_{G_\sigma}) = \int_{\mathbb{R}^2} (x_1, y_1, x_2, y_2)G_\sigma \cdot (x_2, y_2)r \, \mathrm{d}x_2\mathrm{d}y_2$$

as being the average, across all points (x_2, y_2) in $\mathbb{R}^2$, of the values $(x_2, y_2)r$ weighted by the function $(x_1, y_1, -, -)G_\sigma$. The value of the function $(x_1, y_1, -, -)G_\sigma$ is greatest at (x_1, y_1) and decreases as the distance to (x_1, y_1) increases. So, in a nutshell, the Gaussian blur transform alters the intensity of a pixel based largely on the intensities of nearby pixels and this results in a blurred version of the original image; an example is shown in Figure 3.2. ⋄

What interests us about integral transforms is their connection with the Beck-Chevalley condition; this connection is via the *composition* of integral transforms.

Definition 3.1.7. We say that the integral transform

$$X \xleftarrow{\;p\;} (U, \mu) \xrightarrow{\;q\;} Z$$

is a **composite** of the two integral transforms

$$X \xleftarrow{\;p_1\;} (S, \kappa) \xrightarrow{\;q_1\;} Y \qquad \text{and} \qquad Y \xleftarrow{\;p_2\;} (T, \lambda) \xrightarrow{\;q_2\;} Z$$

if $\tau_\mu = \tau_\kappa \mathbin{⨟} \tau_\lambda \colon \mathrm{Hom}(X, \mathbb{F}) \to \mathrm{Hom}(Z, \mathbb{F})$. ⋄

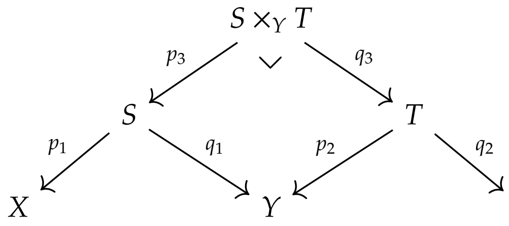
We have the definition of a composite of two integral transforms, and now we'll work towards constructing a composite. Given two integral transforms

$$X \xleftarrow{\;p_1\;} (S, \kappa) \xrightarrow{\;q_1\;} Y \qquad \text{and} \qquad Y \xleftarrow{\;p_2\;} (T, \lambda) \xrightarrow{\;q_2\;} Z$$

we need to construct a space U, a pair of maps

$$X \xleftarrow{\;p\;} U \xrightarrow{\;q\;} Z$$

and a kernel μ in $\mathrm{Hom}(U, \mathbb{F})$. Take $U = S \times_Y T$, $p = p_3 \fatsemi p_1$ and $q = q_3 \fatsemi q_1$, where $S \times_Y T$, p_3 and q_3 denote the pullback and associated maps shown in the following figure.



(3.1.8)

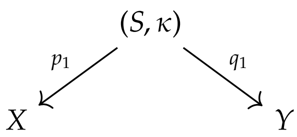
What about the kernel μ? For the chosen U, p and q, the equality $\tau_\mu = \tau_\kappa \fatsemi \tau_\lambda$ holds if, and only if, the equality

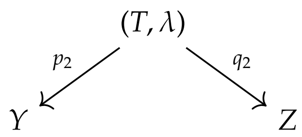
$$(\varphi(p_3 \fatsemi p_1)^* \cdot \mu)(q_3 \fatsemi q_2)_* = ((\varphi p_1^* \cdot \kappa)q_{1*}p_2^* \cdot \lambda)q_{2*}$$

holds for every φ in $\mathrm{Hom}(X, \mathbb{F})$.

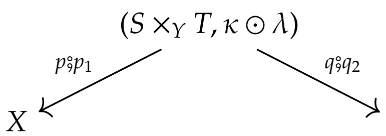
The following result was shown to the author by Simon Willerton.

Proposition 3.1.9. *Consider the following integral transforms.*

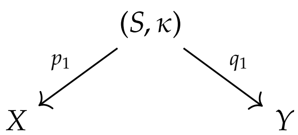




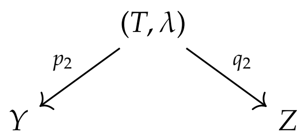
Let $\kappa \odot \lambda$ denote the map in $\mathrm{Hom}(S \times_Y T, \mathbb{F})$ given by $(s, t) \mapsto (s)\kappa \cdot (t)\lambda$. If $p_3^* \fatsemi q_{3*} = q_{1*} \fatsemi p_2^*$ and, for every ψ in $\mathrm{Hom}(S \times_Y T, \mathbb{F})$, $(\psi \cdot \lambda q_{3*})q_{3*} = \psi q_{3*} \cdot \lambda$, then the integral transform



is a composite of



and



Proof. For each φ in $\mathrm{Hom}(X, \mathbb{F})$,

$$\begin{aligned}
(\varphi)(\tau_\kappa \mathbin{⨾} \tau_\lambda) &= ((\varphi p_1^* \cdot \kappa) q_{1*})\tau_\lambda \\
&= \left((\varphi p_1^* \cdot \kappa) q_{1*} p_2^* \cdot \lambda\right) q_{2*} \\
&= \left((\varphi p_1^* \cdot \kappa) p_3^* q_{3*} \cdot \lambda\right) q_{2*} \\
&= \left((\varphi p_1^* p_3^* \cdot \kappa p_3^*) q_{3*} \cdot \lambda\right) q_{2*} \\
&= \left((\varphi (p_3 \mathbin{⨾} p_1)^* \cdot \kappa p_3^*) q_{3*} \cdot \lambda\right) q_{2*} \\
&= \left(\varphi (p_3 \mathbin{⨾} p_1)^* \cdot \kappa p_3^* \cdot \lambda q_3^*\right) q_{3*} q_{2*} \\
&= \left(\varphi (p_3 \mathbin{⨾} p_1)^* \cdot \kappa p_3^* \cdot \lambda q_3^*\right) (q_3 \mathbin{⨾} q_2)_* \\
&= (\varphi (p_3 \mathbin{⨾} p_1)^* \cdot (\kappa \odot \lambda)) (q_3 \mathbin{⨾} q_2)_* \\
&= (\varphi)\tau_{\kappa \odot \lambda}.
\end{aligned}$$

□

If the kernels κ and λ are both trivial—meaning that $(s)\kappa = 1$ for every $s \in S$ and $(t)\lambda = 1$ for every $t \in T$—then the hypotheses of Proposition 3.1.9 simplify, as shown in the following corollary.

Corollary 3.1.10. *Let the following be integral transforms with trivial kernels.*

$$X \xleftarrow{\;p_1\;} (S, 1) \xrightarrow{\;q_1\;} Y \qquad\qquad Y \xleftarrow{\;p_2\;} (T, 1) \xrightarrow{\;q_2\;} Z$$

If $p_3^* \mathbin{⨾} q_{3*} = q_{1*} \mathbin{⨾} p_2^*$, then the integral transform

$$X \xleftarrow{\;p_3 ⨾ p_1\;} (S \times_Y T, 1) \xrightarrow{\;q_3 ⨾ q_2\;} Z$$

is a composite of

$$X \xleftarrow{\;p_1\;} (S, 1) \xrightarrow{\;q_1\;} Y \qquad \textit{and} \qquad Y \xleftarrow{\;p_2\;} (T, 1) \xrightarrow{\;q_2\;} Z$$

Proof. For each φ in $\mathrm{Hom}(X, \mathbb{F})$,

$$\begin{aligned}
(\varphi (p_3 \mathbin{⨾} p_1)^* \cdot 1) (q_3 \mathbin{⨾} q_2)_* &= \varphi p_1^* p_3^* q_{3*} q_{2*} \\
&= \varphi p_1^* q_{1*} p_2^* q_{2*} \\
&= \left((\varphi p_1^* \cdot 1) q_{1*} p_2^* \cdot 1\right) q_{2*}.
\end{aligned}$$

□

The equality $p_3^* \mathbin{⨾} q_{3*} = q_{1*} \mathbin{⨾} p_2^*$ should make the reader think that that pulling a map φ in $\mathrm{Hom}(X, \mathbb{F})$ up along p_3 and then pushing it down along q_3 is equal to pushing φ down along q_1 then pulling it ip along p_2. This is sufficent for the path over the top of (3.1.8) to be equal to the zig-zag path along the bottom of (3.1.8).

3.2 The Beck-Chevalley transformation

Recall that bicleaving for a bifibration $\Phi\colon \mathscr{A} \to \mathscr{B}$ is a choice, for each morphism $j\colon B \to B'$ in $\mathscr{B}$, each object M in $\mathscr{A}$ lying over B' and each object C in $\mathscr{A}$ lying over B, of cartesian morphism $\mathrm{cart}^j_M\colon Mj^* \to M$ in $\mathscr{A}$ that lies over j and of opcartesian morphism $\mathrm{opcart}^j_N\colon N \to Nj_!$ in $\mathscr{A}$ that lies over j. These choices define functors

$$j^*\colon \mathscr{A}_{B'} \underset{\longleftarrow}{\overset{\longrightarrow}{}} \mathscr{A}_B : j_!$$

called the *pull-back* and *push-forward* functors (see Propositions 2.1.27 and 2.4.18). These names come from the idea that the functor j^* pulls an object M in $\mathscr{A}_{B'}$ along j to get an object Mj^* in $\mathscr{A}_B$, and the functor $j_!$ pushes an object N in $\mathscr{A}_B$ along j to get an object $Nj_!$ in $\mathscr{A}_{B'}$. Now we ask "What happens when we pull then push or push then pull?" More formally, if the square

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[f]{} & D \end{array} \tag{3.2.1}$$

in $\mathscr{B}$ commutes, then we can ask if the functors $h^* \mathbin{⨾} k_!$ and $g_! \mathbin{⨾} f^*$ are isomorphic. This a natural question to ask, and we also saw in Section 3.1 the relevance of this property.

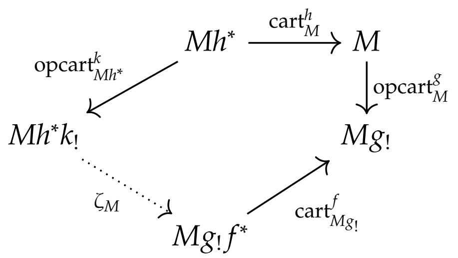
For each object $M \in \mathscr{A}_B$, there are cartesian and opcartesian morphisms as shown in the following diagram.

$$\begin{array}{ccccc} & & Mh^* & \xrightarrow{\mathrm{cart}^h_M} & M \\ & \swarrow{\scriptstyle \mathrm{opcart}^k_{Mh^*}} & & & \downarrow{\scriptstyle \mathrm{opcart}^g_M} \\ Mh^*k_! & & & & Mg_! \\ & & & \nearrow{\scriptstyle \mathrm{cart}^f_{Mg_!}} & \\ & & Mg_!f^* & & \end{array}$$

Intuitively, we're either pulling M along h and then pushing Mh^* along k, or we're pushing M along g and then pulling $Mg_!$ along f. Using the universal property of the opcartesian morphism $\mathrm{opcart}^k_{Mh^*}\colon Mh^* \to Mh^*k_!$, there is a unique morphism $Mh^*k_! \to Mg_!$ that lies over f and that makes the following diagram in $\mathscr{A}$ commute.

$$\begin{array}{ccccc} & & Mh^* & \xrightarrow{\mathrm{cart}^h_M} & M \\ & \swarrow{\scriptstyle \mathrm{opcart}^k_{Mh^*}} & & & \downarrow{\scriptstyle \mathrm{opcart}^g_M} \\ Mh^*k_! & & \cdots\cdots\cdots\cdots\rightarrow & & Mg_! \end{array}$$

Therefore, using the universal property of the cartesian morphism $Mg_!f^* \to Mg_!$, there is unique morphism $\zeta_M : Mh^*k_! \to Mg_!f^*$ that lies over $\mathbf{id}_C$ and that makes the following diagram in $\mathscr{A}$ commute.

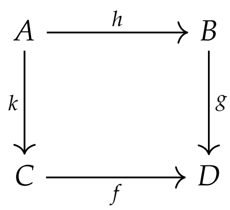


(3.2.2)

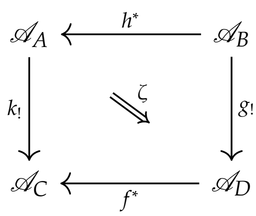
We call ζ_M the **Beck-Chevalley morphism** associated to the square (3.2.1) at M.

We now know that there is a canonical morphism $Mh^*k_! \to Mg_!f^*$. We can use mates to give an explicit description of this morphism. We add a more detailed explanation to Shulman's proof of the following result.

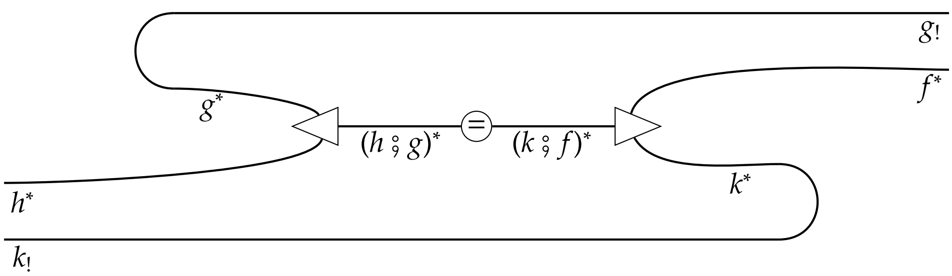
Proposition 3.2.3 ([Shu08, Lemma 16.1]). *Let $\Phi : \mathscr{A} \to \mathscr{B}$ be a bifibration and suppose that the square*



in $\mathscr{B}$ commutes. Then the Beck-Chevalley morphisms $(\zeta_M : k_!h^*M \to f^*g_!M)_{M\in\mathscr{A}_B}$ are the components of a natural transformation

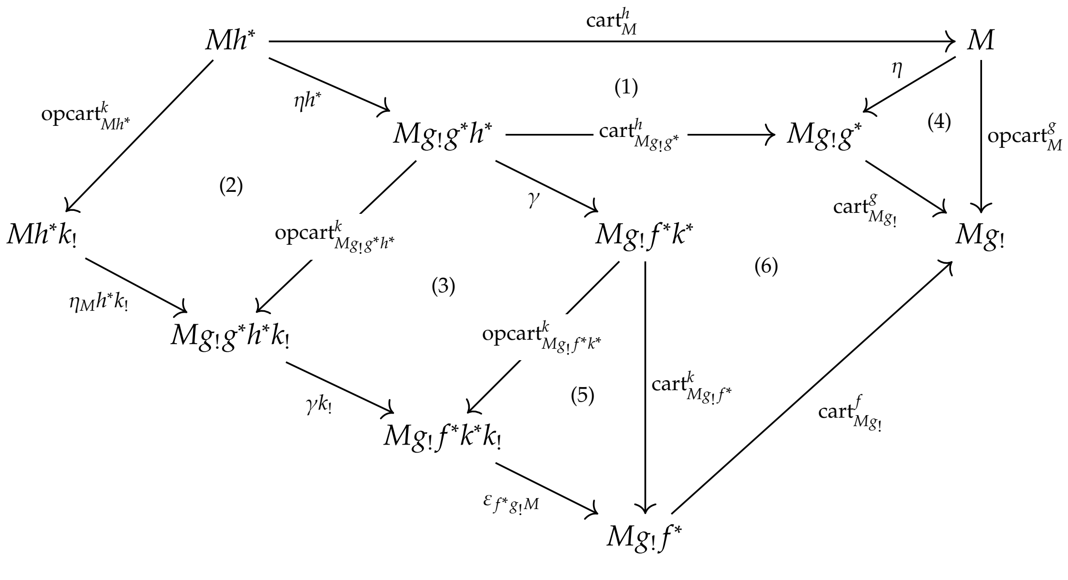


Specifically, ζ is the following natural transformation:

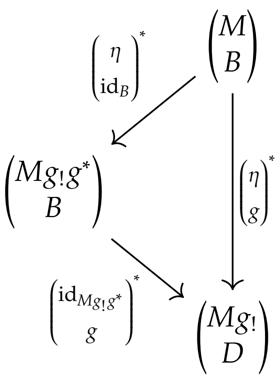


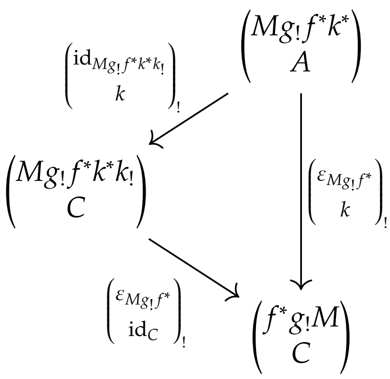
(3.2.4)

Proof. The following diagram adds to (3.2.2).

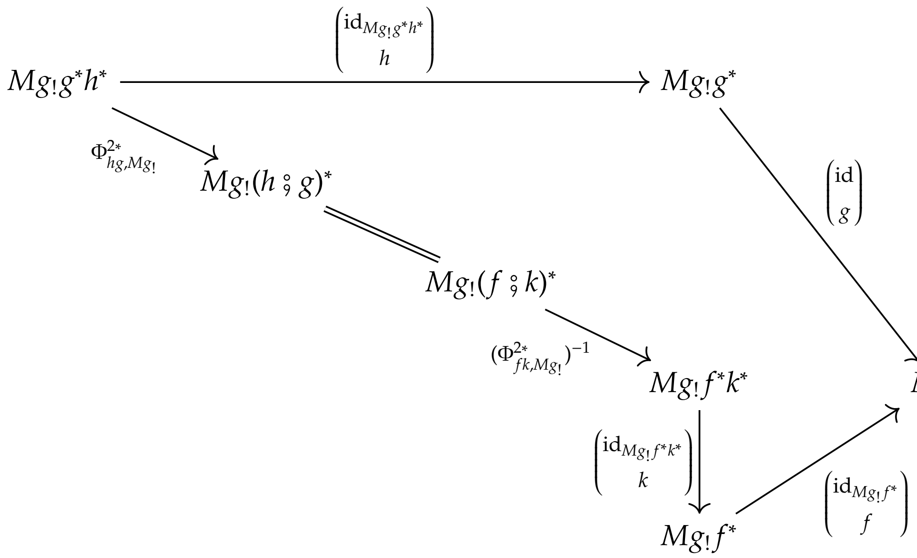


The square (1) commutes by definition of h^* on morphisms, and the squares (2) and (3) commute by definition of $k_!$ on morphisms. It becomes clear that the triangles (4) and (5) commute when we write them in terms of indexed categories:





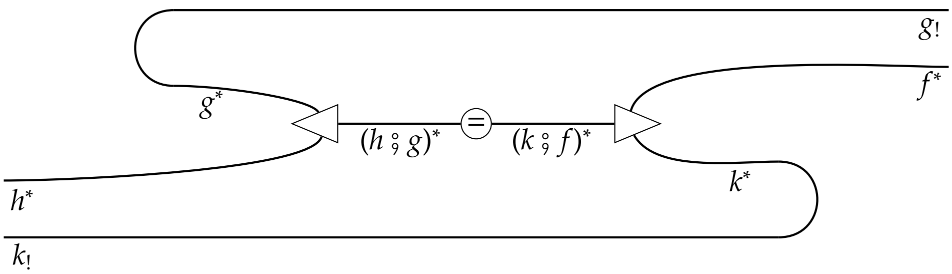
The pentagon (6) can be written in terms of indexed categories as the following.



This diagram commutes by definition of composition in $\mathscr{A}$.

We've shown that the component of the natural trnasformation (3.2.4) at M lies over id_C and makes the diagram (3.2.2) commute. Hence, it is equal to the Beck-Chevalley morphism. □

Definition 3.2.5. The natural transformation



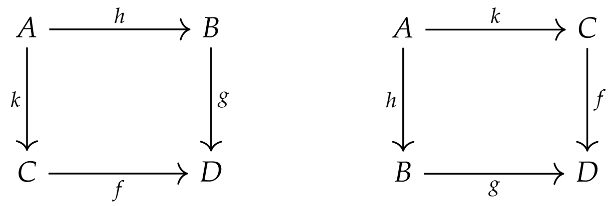
(3.2.6)

is called the **Beck-Chevalley transformation** corresponding to the square

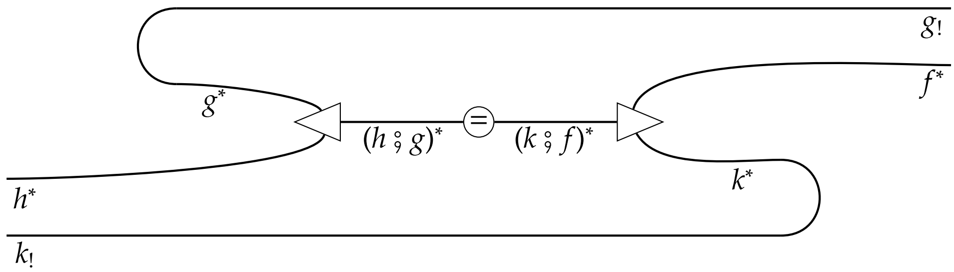
$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[f]{} & D \end{array} \tag{3.2.7}$$

and the bifibration $\Phi\colon \mathscr{A} \to \mathscr{B}$. We say that the square (3.2.7) satisfies the **Beck-Chevalley condition** with respect to Φ if the Beck-Chevalley transformation is an isomorphism. When it's clear from context, we won't say to what bifibration and commuting square a Beck-Chevalley transformation corresponds. ◇

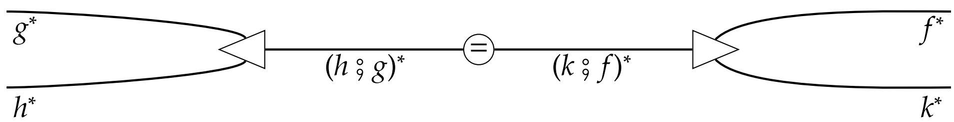
Remark 3.2.8. Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a bifibration. It is important to note that the following two ways of drawing the same square in $\mathscr{B}$ give Beck-Chevalley transformations that are not, in general, equal.



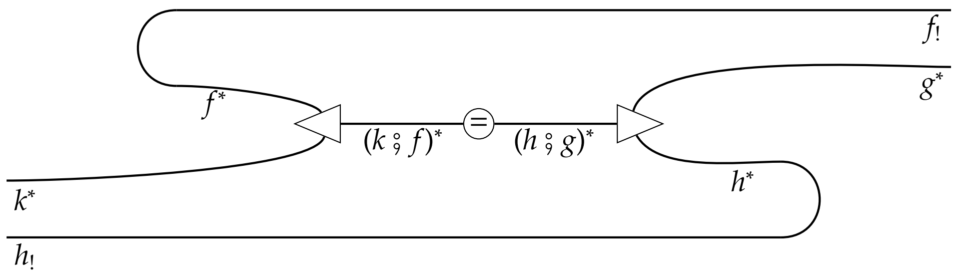
The Beck-Chevalley transformation associated to the square on the left is



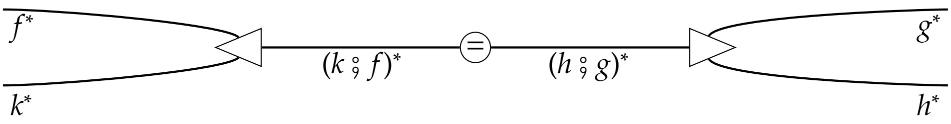
which is the mate of the natural isomorphism



whereas the Beck-Chevalley transformation associated to the square on the right is

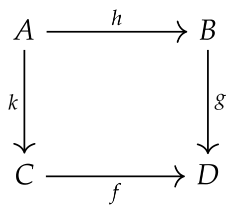


which is the mate of the natural isomorphism

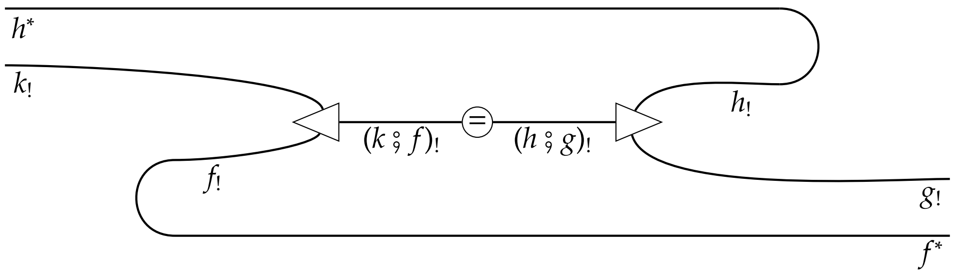


We'll see a clear example of the two not being equal in Example 3.2.12. ◇

Remark 3.2.9. Another candidate for the Beck-Chevalley transformation associated to the square

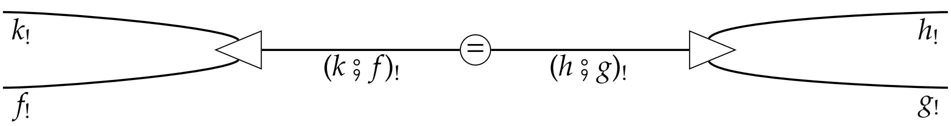


might be

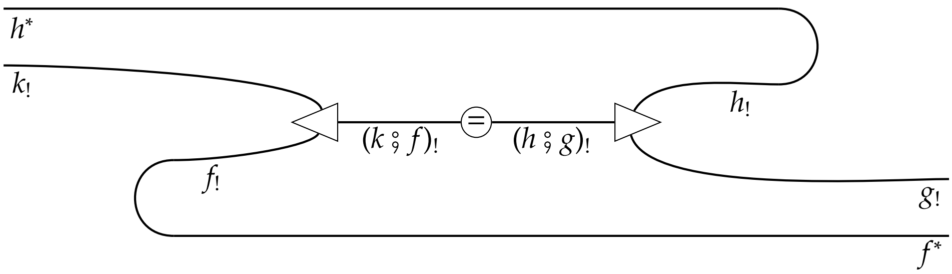


(3.2.10)

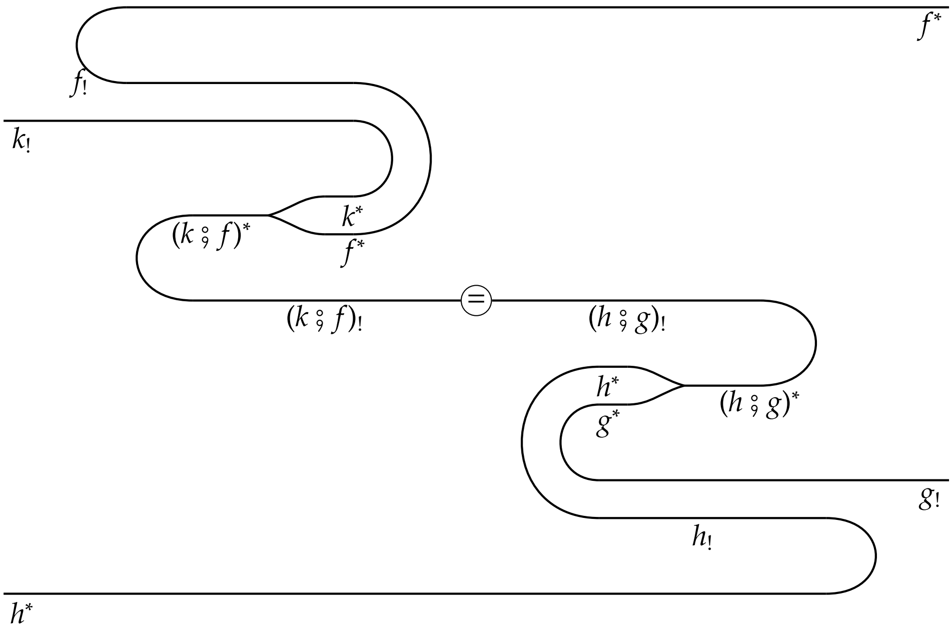
which is the mate of the natural isomorphism



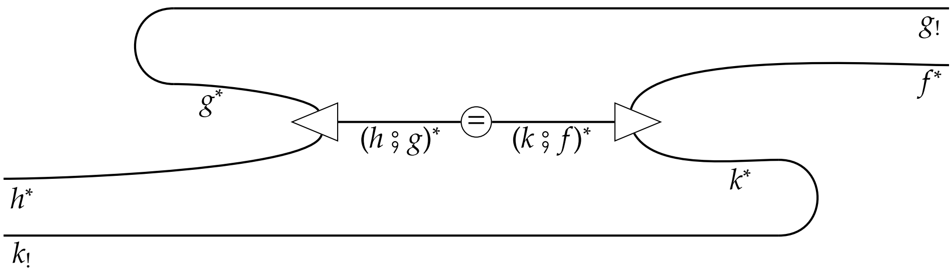
In fact, the natural transformation (3.2.10) is equal to the Beck Chevalley transformation (3.2.6), as shown by the folowing series of equalities.



‖ (1)

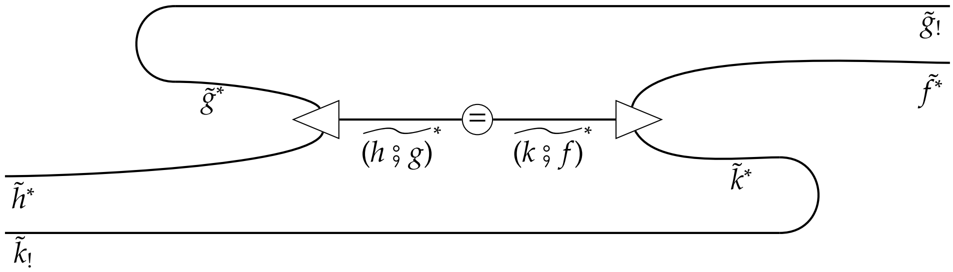


‖ (2)

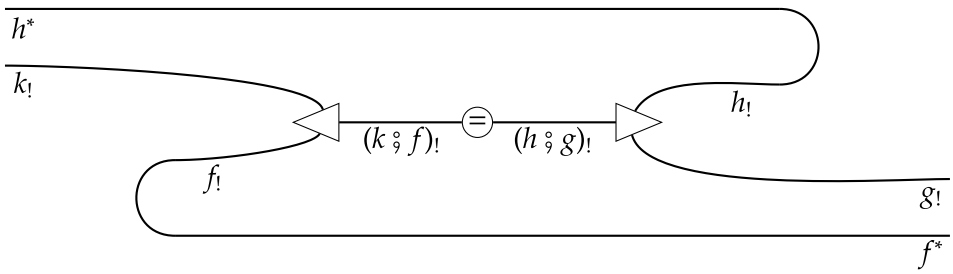


The equality (1) follows from Theorem 2.5.13(i),(ii), and the equality (2) follows from the snake identities for the adjunctions $f_! \dashv f^*$, $(k \fatsemi f)_! \dashv (k \fatsemi f)^*$ and $h_! \dashv h^*$. ⋄

Remark 3.2.11. It is straightforward to check that if $\Phi\colon \mathscr{A} \to \mathscr{B}$ is a bifibration, then the opposite functor $\Phi^{\mathrm{op}}\colon \mathscr{A}^{\mathrm{op}} \to \mathscr{B}^{\mathrm{op}}$ is also a bifibration. Using the notation $\tilde{f}\colon D \to C$ for the morphism in $\mathscr{B}^{\mathrm{op}}$ that corresponds to the morphism $f\colon C \to D$ in $\mathscr{B}$, we have $\tilde{f}^* = f_!$ and $\tilde{f}_! = f^*$, and



$$=$$



We can therefore think of Remark 3.2.9 as proving that the Beck-Chevalley transformation associated to the square

$$\begin{array}{ccc} A & \xrightarrow{\;k\;} & C \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[\;g\;]{} & D \end{array}$$

and the bifibration $\Phi\colon \mathscr{A} \to \mathscr{B}$ is equal to the Beck-Chevalley transformation associated to the square

$$\begin{array}{ccc} D & \xrightarrow{\;\hat{g}\;} & B \\ {\scriptstyle \hat{f}}\downarrow & & \downarrow{\scriptstyle \hat{h}} \\ C & \xrightarrow[\;\hat{k}\;]{} & A \end{array}$$

and the bifibration $\Phi^{\mathrm{op}}\colon \mathscr{A}^{\mathrm{op}} \to \mathscr{B}^{\mathrm{op}}$. ◇

The remainder of this section is devoted to calculating the Beck-Chevalley transformation in some examples.

Example 3.2.12. Consider the bifibration $\mathtt{Rep}\colon \mathbf{GrpRep} \to \mathbf{FinGrp}$, and suppose that the following is a pullback square in **FinGrp**.

$$\begin{array}{ccc} H \times_G K & \xrightarrow{\pi_2} & K \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \beta} \\ H & \xrightarrow[\alpha]{} & G \end{array} \tag{3.2.13}$$

Let V be a K-module. The component morphism ζ_V of the Beck-Chevalley transformation

$$\begin{array}{ccc} \mathbf{Rep}_{(H\times_G K)} & \xleftarrow{\pi_2^*} & \mathbf{Rep}_K \\ {\scriptstyle \pi_{1!}}\downarrow & \overset{\zeta}{\Searrow} & \downarrow{\scriptstyle \beta_!} \\ \mathbf{Rep}_H & \xleftarrow[\alpha^*]{} & \mathbf{Rep}_G \end{array}$$

at V is the morphism

$$V\pi_2^*\pi_{1!} \xrightarrow{\eta_V \pi_2^* \pi_{1!}} V\beta_!\beta^*\pi_2^*\pi_{1!} = V\beta_!\alpha^*\pi_1^*\pi_{1!} \xrightarrow{\varepsilon_{V\beta_!\alpha^*}} V\beta_!\alpha^*$$

The unit $\eta_V\colon V \to V\beta_!\beta^*$ is given by

$$\begin{aligned} V &\longrightarrow \mathbb{C}G \otimes_K V \\ v &\longmapsto e_G \otimes v \end{aligned}$$

and the counit $\varepsilon_{V\beta_!\alpha^*}\colon V\beta_!\alpha^*\pi_1^*\pi_{1!} \to V\beta_!\alpha^*$ is given by

$$\begin{aligned} \mathbb{C}H \otimes_{H\times_G K} (\mathbb{C}G \otimes_K V) &\longrightarrow \mathbb{C}G \otimes_K V \\ h \otimes (g \otimes v) &\longmapsto (h)\alpha g \otimes v \end{aligned}$$

Therefore, the component morphism ζ_V of the Beck-Chevalley transformation associated to the square (3.2.13) and the bifibration $\mathtt{Rep}$ is the K-module map

$$\begin{aligned} \zeta_V\colon \mathbb{C}H \otimes_{H\times_G K} V &\longrightarrow \mathbb{C}G \otimes_K V \\ h \otimes v &\longmapsto (h)\alpha \otimes v. \end{aligned}$$

◇

Example 3.2.14. Consider the bifibration $\mathtt{Fam}_{\mathscr{C}}\colon \mathbf{Fam}_{\mathscr{C}} \to \mathbf{Set}$, and suppose that the following is a pullback square in **Set**.

$$\begin{array}{ccc} Y \times_W Z & \xrightarrow{h} & Y \\ {\scriptstyle k}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ Z & \xrightarrow[f]{} & W \end{array} \tag{3.2.15}$$

Let $B = (B_y)_{y\in Y}$ be a Y-indexed family of objects. The component morphism ζ_B of the Beck Chevalley transformation

$$\begin{array}{ccc} \mathscr{C}^{Y\times_W Z} & \xleftarrow{\;h^*\;} & \mathscr{C}^{Y} \\ {\scriptstyle k_!}\downarrow & \overset{\zeta}{\searrow} & \downarrow{\scriptstyle g_!} \\ \mathscr{C}^{Z} & \xleftarrow[\;f^*\;]{} & \mathscr{C}^{W} \end{array}$$

at B is the morphism

$$Bh^*k_! \xrightarrow{\;\eta_B h^* k_!\;} Bg_!g^*h^*k_! = Bg_!f^*k^*k_! \xrightarrow{\;\varepsilon_{Bg_!f^*}\;} Bg_!f^*.$$

The unit $(\eta_B)_y : B_y \to (Bg_!g^*)_y$ is the inclusion map

$$B_y \longrightarrow \sum_{\substack{\alpha\in Y\\ (\alpha)g=(y)g}} B_\alpha$$

and the counit $(\varepsilon_{Bg_!f^*})_y : (Bg_!g^*h^*k_!)_y \to (Bg_!f^*)_y$ is the map

$$\{\mathrm{id}\}_{y\in((z)f)k^{-1}} : \sum_{\substack{y\in Y\\ (y)g=(z)f}} \sum_{\substack{\alpha\in Y\\ (\alpha)g=(z)f}} B_\alpha \longrightarrow \sum_{\substack{y\in Y\\ (y)g=(z)f}} B_y.$$

Therefore, the component morphism ζ_B of the Beck-Chevalley transformation associated to the square (3.2.15) and the bifibration $\mathtt{Fam}_{\mathscr{C}}$ is the identity

$$\mathrm{id} : \sum_{\substack{y\in Y\\ (y)g=(z)f}} B_y \longrightarrow \sum_{\substack{y\in Y\\ (y)g=(z)f}} B_y.$$

◇

In the following section, we prove that the Beck-Chevalley transformation associated to a square of a particular form and the bifibration $\mathtt{Rep}$: **GrpRep** → **FinGrp** is an isomorphism. Before we do that, we give an example where the the Beck-Chevalley transformation associated a square and the bifibration $\mathtt{Rep}$: **GrpRep** → **FinGrp** is *not* an isomorphism.

Let $\alpha : H \to G$ be a group homomorphism, let $\{e\}$ denote the trivial group, and let V be an $\{e\}$-module (i.e. a vector space). The component morphism ζ_V of the Beck-Chevalley transformation associated to the square

$$\begin{array}{ccc} \ker(\alpha) \cong H \times_G \{e\} & \longrightarrow & \{e\} \\ \downarrow & \lrcorner & \downarrow \\ H & \xrightarrow[\;\alpha\;]{} & G \end{array}$$

and the bifibration **Rep** is the $\{e\}$-module map

$$\begin{aligned} \zeta_V \colon \mathbb{C}H \otimes_{\ker(\alpha)} V &\longrightarrow \mathbb{C}G \otimes_{\{e\}} V \\ h \otimes v &\longmapsto (h)\alpha \otimes v. \end{aligned}$$

This is equal to the linear map

$$\begin{aligned} \zeta_V \colon (\mathbb{C}H/\ker(a)) \otimes V &\longrightarrow \mathbb{C}G \otimes V \\ h \otimes v &\longmapsto (h)\alpha \otimes v \end{aligned}$$

since $\ker(a)$ acts trivially on V. If we choose a group homomorphism $\alpha \colon H \to G$ that is not surjective, then this serves as an example of when the Beck-Chevalley transformation is not an isomorphism.

3.3 Beck-Chevalley bifibrations

Definition 3.3.1. Let $\Phi \colon \mathscr{A} \to \mathscr{B}$ be a bifibration. We say that Φ is **Beck-Chevalley** if, for every pullback square

$$\begin{array}{ccc} B \times_{B'} C & \xrightarrow{\;q\;} & C \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ B & \xrightarrow[\;f\;]{} & B' \end{array}$$

in $\mathscr{B}$, the Beck-Chevalley transformation is an isomorphism. We say that Φ is **weakly Beck-Chevalley** if, for every pullback square of the form

$$\begin{array}{ccc} B \times_{B'} (D \times B') & \xrightarrow{\;q\;} & D \times B' \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_D} \\ B & \xrightarrow[\;f\;]{} & B' \end{array}$$

in $\mathscr{B}$ and every pullback square of the form

$$\begin{array}{ccc} (D \times B') \times_{B'} C & \xrightarrow{\;q\;} & C \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ D \times B' & \xrightarrow[\;\pi_D\;]{} & B' \end{array}$$

in $\mathscr{B}$, the Beck-Chevalley transformation is an isomorphism. ◇

Example 3.3.2. We saw in Example 3.2.14 that the Beck-Chevalley transformation corresponding to a pullback square and the bifibration $\mathtt{Fam} \colon \mathbf{Fam}_{\mathscr{V}} \to \mathbf{Set}$ is always the identity, so $\mathtt{Fam}$ is a Beck-Chevalley bifibration. ◇

Proposition 3.3.3. *The bifibration* $\mathsf{Rep}\colon \mathbf{GrpRep} \to \mathbf{FinGrp}$ *is weakly Beck-Chevalley.*

Proof. Let the following be a pullback square in **FinGrp**.

$$\begin{array}{ccc} H \times_G K & \xrightarrow{\pi_2} & K \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \beta} \\ H & \xrightarrow[\alpha]{} & G \end{array} \tag{3.3.4}$$

Recall from Example 3.2.12 that the Beck-Chevalley transformation

$$\begin{array}{ccc} \mathbf{Rep}_{(H\times_G K)} & \xleftarrow{\pi_2^*} & \mathbf{Rep}_K \\ {\scriptstyle \pi_{1!}}\downarrow & \overset{\zeta}{\searrow} & \downarrow{\scriptstyle \beta_!} \\ \mathbf{Rep}_H & \xleftarrow[\alpha^*]{} & \mathbf{Rep}_G \end{array}$$

is given, for each K-module V, by the H-module map

$$\begin{aligned} \zeta_V\colon \mathbb{C}H \otimes_{H\times_G K} V &\longrightarrow \mathbb{C}G \otimes_K V \\ h \otimes v &\longmapsto (h)\alpha \otimes v. \end{aligned}$$

We will show that ζ_V is an invertible H-module map when (Case 1) α is a product projection, and when (Case 2) β is a product projection. To do this, we'll define, for each K-module V, an H-module map

$$\theta_V\colon \mathbb{C}G \otimes_K V \to \mathbb{C}H \otimes_{H\times_G K} V$$

that is inverse to ζ_V.

(**Case 1**) Suppose that $H = G \times N$ for some group N, and suppose that α is the projection map $G \times N \to G$. This means that the square (3.3.4) is given by

$$\begin{array}{ccc} (G \times N) \times_G K & \xrightarrow{\pi_2} & K \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \beta} \\ G \times N & \xrightarrow[\alpha]{} & G \end{array}$$

Let V be a K-module. Define the $(G \times N)$-module map θ_V by

$$\begin{aligned} \theta_V\colon \mathbb{C}G \otimes_K V &\longrightarrow \mathbb{C}(G \times N) \otimes_{(G\times N)\times_G K} V \\ g \otimes v &\longmapsto (g, e_N) \otimes v. \end{aligned}$$

We just need to check a few things.

- (**θ_V is well-defined**) To check that θ_V is well-defined, we need to prove that, for every $k \in K$, every $g \in G$ and every $v \in V$, $\theta_V(g \cdot k \otimes k^{-1} \cdot v) = \theta_V(g \otimes v)$.

$$\begin{aligned}(g \cdot k \otimes k^{-1} \cdot v)\theta_V &= (g(k)\beta \otimes k^{-1} \cdot v)\theta_V \\ &= (g(k)\beta, e_N) \otimes k^{-1} \cdot v \\ &= (g(k)\beta, e_N) \otimes (k^{-1}, ((k)\beta^{-1}, e_N)) \cdot v \\ &= (g(k)\beta, e_N) \cdot (k^{-1}, ((k)\beta^{-1}, e_N)) \otimes v \\ &= (g, e_N) \otimes v \\ &= \theta_V(g \otimes v).\end{aligned}$$

- (**θ_V is inverse to ζ_V**) Let $g \otimes v \in \mathbb{C}G \otimes_K V$. Then,

$$\begin{aligned}(g \otimes v)\theta_V\zeta_V &= ((g, e_N) \otimes v)\zeta_V \\ &= (g, e_N)\alpha \otimes v \\ &= g \otimes v.\end{aligned}$$

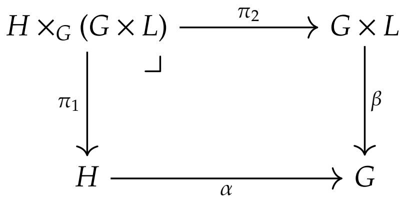
Let $(g, n) \otimes v \in \mathbb{C}(G \times N) \otimes_{(G\times N)\times_G K} V$. Then,

$$\begin{aligned}((g, n) \otimes v)\zeta_V\theta_V &= ((g, n)\alpha \otimes v)\theta_V \\ &= (g \otimes v)\theta_V \\ &= (g, n) \cdot ((e_G, n^{-1}), e_K) \otimes v \\ &= (g, n) \otimes ((e_G, n^{-1}), e_K) \cdot v \\ &= (g, n) \otimes v.\end{aligned}$$

- (**θ_V is a $(G \times N)$-module map**) Let $g \in G$, $v \in V$ and $(g', n) \in G \times N$. Then,

$$\begin{aligned}(g', n) \cdot \theta_V(g \otimes v) &= (g', n) \cdot (g, e_N) \otimes v \\ &= (g', n) \cdot (g, e_N) \otimes v \\ &= (g'g, n) \otimes v \\ &= (g'g, e_N) \cdot ((e_G, n), e_K) \otimes v \\ &= (g'g, e_N) \otimes ((e_G, n), e_K) \cdot v \\ &= (g'g, e_N) \otimes v \\ &= \theta_V(g'g \otimes v) \\ &= \theta_V((g', n) \cdot (g \otimes v)).\end{aligned}$$

(**Case 2**) Suppose that $K = G \times L$ for some group L, and suppose that β is the projection map $G \times L \to G$. This means that the square (3.3.4) is given by



Let V be a $(G \times L)$-module. Define the H-module map θ_V by

$$\begin{aligned} \theta_V \colon \mathbb{C}G \otimes_{G \times L} V &\longrightarrow \mathbb{C}H \otimes_{H \times_G (G \times L)} V \\ g \otimes v &\longmapsto e_H \otimes (g, e_L) \cdot v. \end{aligned}$$

Again, we need to check a few things.

- (θ_V **is well-defined**) To check that θ_V is well-defined, we need to prove that, for every $(g', l) \in G \times L$, every $g \in G$ and every $v \in V$, $\theta_V(g \cdot (g', l) \otimes (g', l)^{-1} \cdot v) = \theta_V(g \otimes v)$.

$$\begin{aligned} (g \cdot (g', l) \otimes (g', l)^{-1} \cdot v)\theta_V &= (gg' \otimes (g', l)^{-1} \cdot v)\theta_V \\ &= e_H \otimes (gg', e_L) \cdot (g', l)^{-1} \cdot v \\ &= e_H \otimes (g, l^{-1}) \cdot v \\ &= e_H \otimes (e_H, (e_G, l^{-1})) \cdot (g, e_L) \cdot v \\ &= e_H \cdot (e_H, (e_G, l^{-1})) \otimes (g, e_L) \cdot v \\ &= e_H \otimes (g, e_L) \cdot v \\ &= \theta_V(g \otimes v). \end{aligned}$$

- (θ_V **is inverse to** ζ_V) Let $g \otimes v \in \mathbb{C}G \otimes_K V$. Then,

$$\begin{aligned} (g \otimes v)\theta_V \zeta_V &= (e_H \otimes (g, e_L) \cdot v)\zeta_V \\ &= e_G \otimes (g, e_L) \cdot v \\ &= e_G \cdot (g, e_L) \otimes v \\ &= g \otimes v. \end{aligned}$$

  Let $h \otimes v \in H \otimes_{H \times_G (G \times L)} V$. Then,

$$\begin{aligned} (h \otimes v)\zeta_V \theta_V &= ((h)\alpha \otimes v)\theta_V \\ &= e_H \otimes ((h)\alpha, e_L) \cdot v \\ &= e_H \otimes (h, ((h)\alpha), e_L) \cdot v \\ &= e_H \cdot (h, ((h)\alpha), e_L) \otimes v \\ &= h \otimes v. \end{aligned}$$

- (θ_V **is a** H**-module map**) Let $g \in G$, $v \in V$ and $h \in H$. Then,

$$\begin{aligned} \theta_V(k \cdot (g \otimes v)) &= \theta_V((k)\alpha g \otimes v) \\ &= e_H \otimes ((k)\alpha g, e_K) \cdot v \\ &= e_H \otimes (((k)\alpha g, e_K), k) \cdot (g, e_K) \cdot v \\ &= e_H \cdot (((k)\alpha, e_K), k) \otimes (g, e_K) \cdot v \\ &= k \cdot e_H \otimes ((g, e_K) \cdot v) \\ &= k \cdot \theta_V(g \otimes v). \end{aligned}$$

□

3.3.1 Mackey's formula

We now have an aside where we'll briefly explain how Mackey's formula relates to the Beck-Chevalley condition of a comma square. Mackey's formula appears as the 'Mackey axiom' in one of the original definitions of a Mackey functor [Bou97].

Let G be a group, let H and K be subgroups of G, and let W be a K-module. For each $g \in G$, let $K_g = H \cap gKg^{-1}$. Since, for each $x \in K_g$, $g^{-1}xg$ is an element of K, we can define the K_g-module W_g to have the same underlying vector space as W and K_g-action

$$x \cdot w := (g^{-1}xg) \cdot w.$$

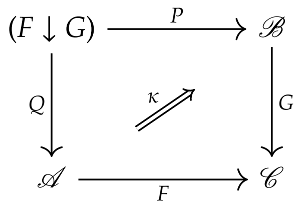
Mackey's formula states

$$W\mathrm{Ind}_K^G \mathrm{Res}_H^G = \bigoplus_{HgK \in H\backslash G/K} W_g \mathrm{Ind}_{K_g}^H.$$

where $H\backslash G/K$ denotes the set of (H,K)-double cosets, **Res** denotes restricted representation, and **Ind** denotes induced representation.

A **comma square** is a generalisation from category theory to 2-category theory of a pullback square. The comma square that relates to Mackey's formula is in the 2-category **Cat**, so we now give the definition of a comma square in **Cat**.

Definition 3.3.5. Let $F\colon \mathscr{A} \to \mathscr{C}$ and $G\colon \mathscr{B} \to \mathscr{C}$ be functors. The **comma square** of F and G consists of a category $(F \downarrow G)$, called the **comma category**, functors $P\colon (F \downarrow G) \to \mathscr{A}$ and $Q\colon (F \downarrow G) \to \mathscr{B}$, and a natural transformation



all of which are defined as follows.

- An object in $(F \downarrow G)$ is a triple (A, B, γ) where A is an object in $\mathscr{A}$, B is an object in $\mathscr{B}$, and $\gamma\colon (A)F \to (B)G$ is a morphism in $\mathscr{C}$. A morphism $(A, B, \gamma) \to (A', B', \gamma')$ in $(F \downarrow G)$ is a pair (α, β) where $\alpha\colon A \to A'$ is a morphism in $\mathscr{A}$ and $\beta\colon B \to B'$ is morphism in $\mathscr{B}$ such that $(\alpha)F \mathbin{⨾} \gamma' = \gamma \mathbin{⨾} (\beta)G$.

- The functor P is given by

$$\begin{aligned} P\colon (F \downarrow G) &\longrightarrow \mathscr{A} \\ (A, B, \gamma) &\longmapsto A \\ (\alpha, \beta) &\longmapsto \alpha, \end{aligned}$$

  and the functor Q is given by

$$\begin{aligned} Q\colon (F \downarrow G) &\longrightarrow \mathscr{B} \\ (A, B, \gamma) &\longmapsto B \\ (\alpha, \beta) &\longmapsto \beta, \end{aligned}$$

- The natural isomorphism $\kappa\colon P \mathbin{⨟} G \to Q \mathbin{⨟} F$ has components

$$\kappa_{(A,B,\gamma)} = \gamma\colon (A)F \to (B)G$$

◇

Let $\mathbf{B}G$, $\mathbf{B}H$, and $\mathbf{B}K$ denote the one-object categories whose morphisms are the elements of the groups G, H and K respectively, and let i_H and i_K denote the functors corresponding to the subgroup inclusions. Let $\mathscr{P}$ denote the comma category $(i_H \downarrow i_K)$ in the comma square below.

$$\begin{array}{ccc} \mathscr{P} & \xrightarrow{P} & \mathbf{B}K \\ {\scriptstyle Q}\downarrow & \overset{\kappa}{\nearrow} & \downarrow{\scriptstyle i_K} \\ \mathbf{B}H & \xrightarrow[i_H]{} & \mathbf{B}G \end{array}$$

After upacking the definition of comma category in this case, one can see that an object in P is an element g in G, and a morphism $g \to g'$ in $\mathscr{P}$ is a pair (h,k) where $h \in H$ and $k \in K$ such that $hg' = gk$. Every morphism (h,k) is an isomorphism with inverse (h^{-1}, k^{-1}), so $\mathscr{P}$ is a groupoid. We say that two objects in $\mathscr{P}$ are in the same **connected component** if there exists an isomorphism between them. Each connected component of $\mathscr{P}$ is a groupoid, and so we can think of $\mathscr{P}$ as being split up into all of these groupoids. In fact, there is an equivalence

$$\mathscr{P} \simeq \coprod_{[g] \in \mathscr{P}\pi_0} \mathbf{B}(\mathrm{Aut}(g)), \tag{3.3.6}$$

where $\mathscr{P}\pi_0$ denotes the set of connected components of $\mathscr{P}$, $\mathrm{Aut}(g)$ denotes the group $\mathscr{P}(g,g)$, and $\coprod$ denotes coproduct in the category **Cat**.

This is beginning to look like Mackey's formula, we just need to make a couple of observations. Firstly, two objects g, g' in $(i_H \downarrow i_K)$ are in the same connected component if, and only only if, there exists elements $h \in H$ and $k \in K$ satisfying $g' = h^{-1}gk$, which is the same as saying that g and g' lie in the same double coset $HgK = Hg'K$. Secondly, $\mathrm{Aut}(g) = \mathscr{P}(g,g) = \{(h,k) \mid h = gkg^{-1}\}$ which is isomorphic to $H \cap gKg^{-1}$. Therefore, the equivalence (3.3.6) becomes

$$\mathscr{P} \simeq \coprod_{HgK \in H\backslash G/K} \mathbf{B}(H \cap gKg^{-1}).$$

We recover the right hand side of Mackey's formula using this form for $\mathscr{P}$ when restricting along P and then inducing along Q, and the left hand side of Mackey's formula comes from inducing along i_K and then restricting along i_H. In this context, restriction is given by precomposition and induction is given by left Kan extension; for more details see [Rie14, Chapter 1].

3.4 Lemmas for the Beck-Chevalley condition

Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a bifibration and let

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle k}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ C & \xrightarrow[f]{} & D \end{array}$$

be a square in $\mathscr{B}$ that commutes. In this section, we will prove some conditions on this square that are equivalent to the Beck-Chevalley condition.

Suppose that the diagram

$$\begin{array}{ccc} M' & \xrightarrow{\phi} & M \\ {\scriptstyle \chi}\big\downarrow & & \big\downarrow{\scriptstyle \xi} \\ M''' & & M'' \end{array} \tag{3.4.1}$$

in $\mathscr{A}$ lies over the diagram

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle k}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ C & & D \end{array}$$

in $\mathscr{B}$ and suppose that ϕ is cartesian and χ and ξ are opcartesian. Since χ is opcartesian, there exists a unique morphism $\psi\colon M''' \to M''$ that lies over f and that makes the following square commute.

$$\begin{array}{ccc} M' & \xrightarrow{\phi} & M \\ {\scriptstyle \chi}\big\downarrow & & \big\downarrow{\scriptstyle \xi} \\ M''' & \xrightarrow[\psi]{} & M'' \end{array} \tag{3.4.2}$$

We call a diagram the same shape as (3.4.1) a **niche**, and we say that a morphism ψ **fills the niche** if it makes the square (3.4.2) commute.

We now give an explicit description of this morphism ψ.

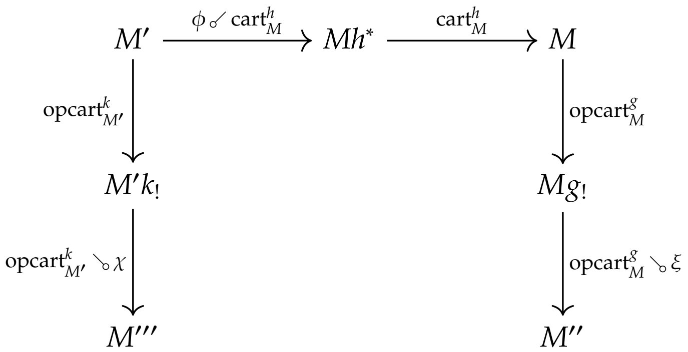
Lemma 3.4.3. *The unique morphism ψ that fills the niche* (3.4.1) *is equal to the following composite.*

$$M''' \xrightarrow{\chi \searrow \mathrm{opcart}^k_{M'}} M'k_! \xrightarrow{(\phi \swarrow \mathrm{cart}^h_M)k_!} Mh^*k_! \xrightarrow{\zeta_M} Mg_!f^* \xrightarrow{\mathrm{cart}^f_{Mg_!}} Mg_! \xrightarrow{\mathrm{opcart}^g_M \searrow \xi} M''$$

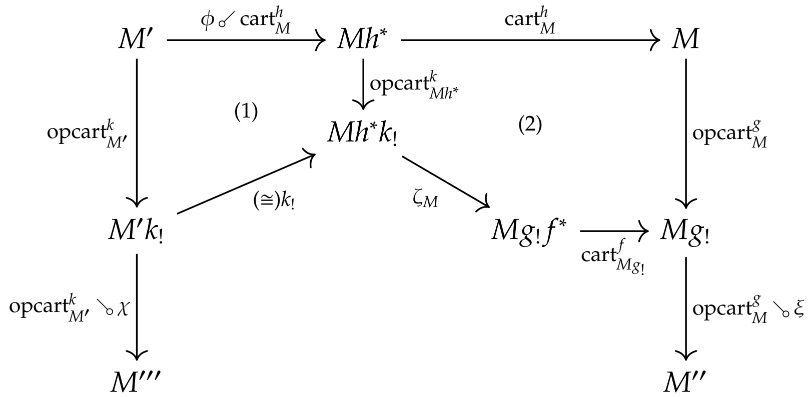
where we have used the notation $\swarrow$ and $\searrow$ as given in Notations 2.1.6 and 2.4.7.

Proof. Recall that we can factorise morphisms in $\mathscr{A}$ using pure-cartesian factorisation (see Proposition 2.1.14) and opcartesian-pure factorisation (see Proposition 2.4.15) and that we can express the results of these factorisations using the the lollipop notation $\multimapdotinv$ and $\searrow\!\circ$ as given in Notations 2.1.6 and 2.4.7. Note in particular that the morphisms that we obtain by using pure-cartesian factorisation and opcartesian-pure factorisation will be isomorphisms (see Corollaries 2.1.25 and 2.4.17).

Factorize the morphisms in (3.4.1) using the cleaving and opcleaving of Φ to get the following diagram



We then add some morphisms to obtain the following diagram.



The square labelled (1) commutes by the definition of $k_!$ on morphisms, and the pentagon labelled (2) commutes by definition of the Beck-Chevalley morphism ζ_M. Therefore the morphism

$$M''' \xrightarrow{\chi \searrow\!\circ\, \mathrm{opcart}^k_{M'}} M'k_! \xrightarrow{(\phi \multimapdotinv \mathrm{cart}^h_M)k_!} Mh^*k_! \xrightarrow{\zeta_M} Mg_!f^* \xrightarrow{\mathrm{cart}^f_{Mg_!}} Mg_! \xrightarrow{\mathrm{opcart}^g_M \searrow\!\circ\, \xi} M''$$

fills the niche (3.4.1) and lies over f. □

The fact that this morphism ψ is constructed using the Beck-Chevalley morphism leads us to the following corollary which is given as Proposition 11 in [Pav91].

Corollary 3.4.4. *Let $\Phi\colon A \to B$ be a bifibration and let*

$$\begin{array}{ccc} A & \xrightarrow{\;h\;} & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[\;f\;]{} & D \end{array} \tag{3.4.5}$$

be a square in $\mathscr{B}$ that commutes. The following are equivalent:

(i) this square (3.4.5) *satisfies the Beck-Chevalley condition;*

(ii) for every niche

$$\begin{array}{ccc} M' & \xrightarrow{\;\phi\;} & M \\ {\scriptstyle \chi}\downarrow & & \downarrow{\scriptstyle \xi} \\ M''' & & M'' \end{array} \tag{3.4.6}$$

in $\mathscr{A}$ that lies above the niche

$$\begin{array}{ccc} A & \xrightarrow{\;h\;} & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ C & & D \end{array}$$

in $\mathscr{B}$ for which ϕ is cartesian and χ and ξ are opcartesian, the unique morphism ψ that fills the niche (3.4.6) *and lies over f is cartesian.*

Proof. The morphism

$$M''' \xrightarrow{\chi \mathbin{\searrow\!\circ} \mathrm{opcart}^k_{M'}} M'k_! \xrightarrow{(\phi \mathbin{\circ\!\swarrow} \mathrm{cart}^h_M)k_!} Mh^*k_! \xrightarrow{\zeta_M} Mg_!f^* \xrightarrow{\mathrm{cart}^f_{Mg_!}} Mg_! \xrightarrow{\mathrm{opcart}^g_M \mathbin{\searrow\!\circ} \xi} M''$$

is cartesian if, and only if, the morphism

$$Mh^*k_! \xrightarrow{\quad\zeta_M\quad} Mg_!f^* \xrightarrow{\quad\mathrm{cart}^f_{Mg_!}\quad} Mg_!$$

is cartesian. Using Proposition 2.1.24(ii), this morphism is cartesian if, and only if, ζ_M is cartesian. Finally, since $\zeta_M\Phi = \mathrm{id}_C$, we can use Proposition 2.1.24(v) to get that ζ_M is cartesian if, and only if, ζ_M is an isomorphism. ◻

In the following remark, we repeat the arguments of Lemma 3.4.3 and Corollary 3.4.4, but this time using string diagrams. The advantage of this visual representation is that we can tell at a glance from the string diagram (3.4.9) below that the unique morphism ψ that fills the niche (3.4.6) and lies over f is cartesian.

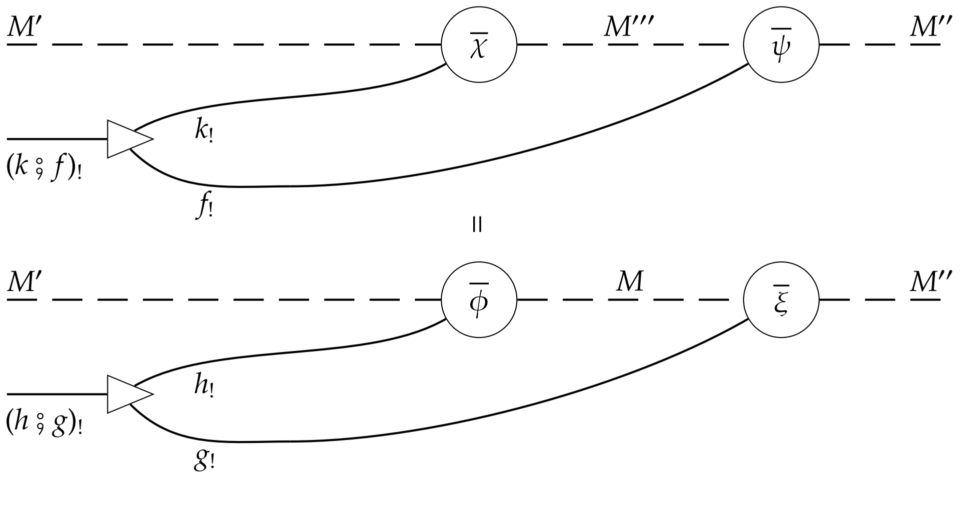
Remark 3.4.7. Suppose that the commuting square

$$\begin{array}{ccc} M' & \xrightarrow{\phi} & M \\ {\scriptstyle \chi}\downarrow & & \downarrow{\scriptstyle \xi} \\ M''' & \xrightarrow[\psi]{} & M'' \end{array} \tag{3.4.8}$$

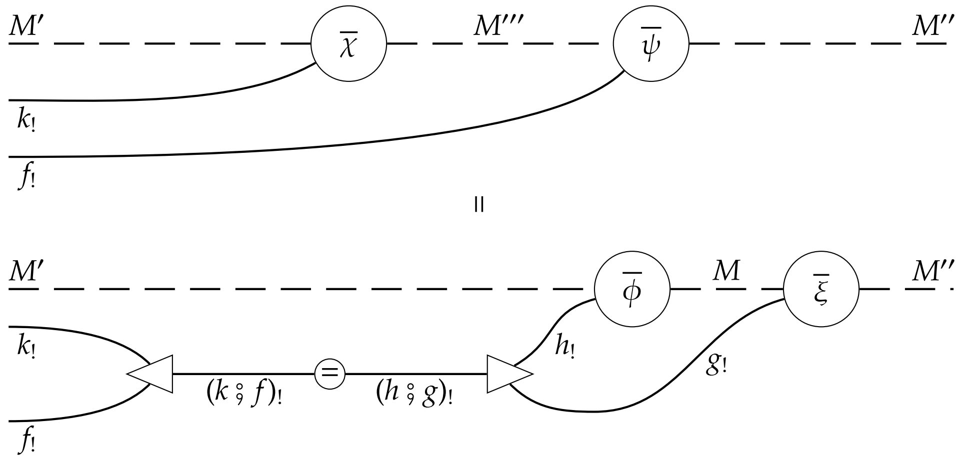
in $\mathscr{A}$ lies over the commuting square

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[f]{} & D \end{array}$$

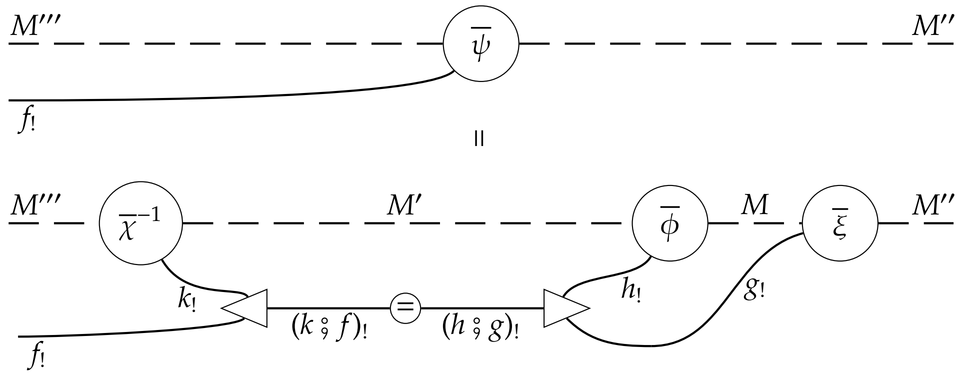
in $\mathscr{B}$, and suppose that ϕ is cartesian and χ and ξ are opcartesian. We can write the square (3.4.8) as the following equality in the fibre category $\mathscr{A}_D$.



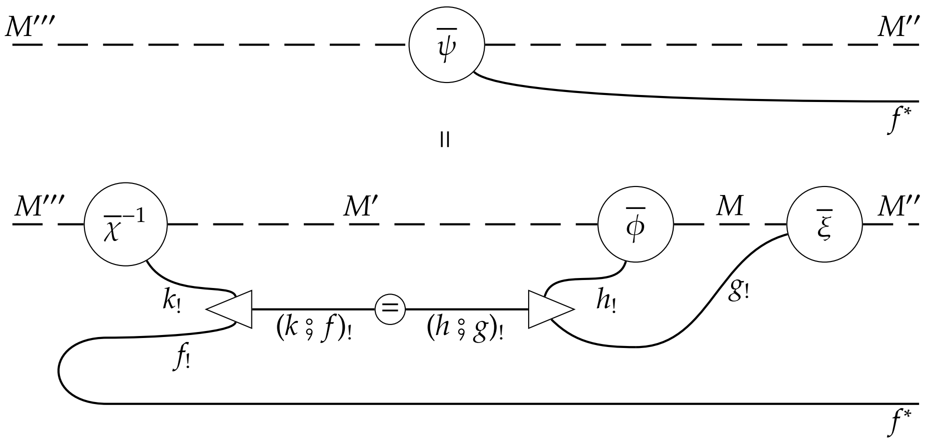
Precompose with the contraint isomorphism $f_! \mathbin{⨾} k_! \to (f \mathbin{⨾} k)_!$ to get



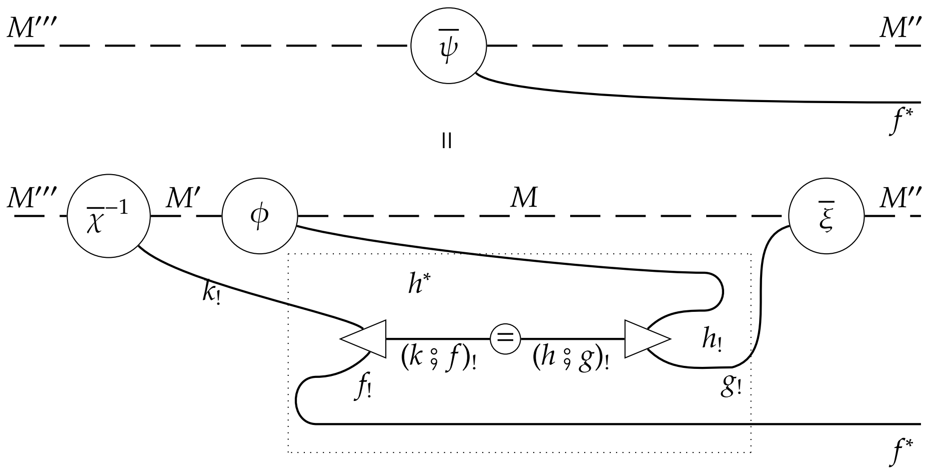
Then precompose with $f_!(\overline{\chi}^{-1})$ to get



Take the adjunct to get

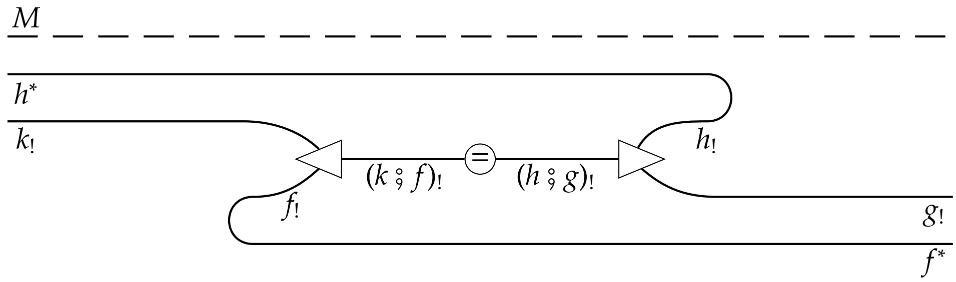


Finally rearrange to get



(3.4.9)

This is the unique morphism $\psi\colon M''' \to M''$ in $\mathscr{A}$ that fills the niche (3.4.6). By Proposition 2.3.10 , ψ is cartesian if, and only if, (3.4.9) is an isomorphism in $\mathscr{A}_C$. Since $\overline{\chi}^{-1}$, ψ and $\overline{\xi}$ are isomorphisms, (3.4.9) is an isomorphism if, and only if, the morphism



is an isomorphism. But, by Proposition 3.2.3, this morphism is equal to the Beck-Chevalley morphism ζ_M. So ψ is cartesian if, and only if, the Beck-Chevalley morphism ζ_M is an isomorphism. ◇

The pull-back and push-forward functors h^*, f^*, $k_!$ and $g_!$ are defined by the cleaving and opcleaving—the cartesian and opcartesian morphisms that we chose. The following corollary of Corollary 3.4.4 says that it doesn't actually matter to the Beck-Chevalley condition which cartesian and opcartesian morphisms we use to pull and push.

Corollary 3.4.10. *Let* $\Phi\colon A \to B$ *be a bifibration and let*

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[f]{} & D \end{array} \tag{3.4.11}$$

be a square in $\mathscr{B}$ *that commutes. The following are equivalent:*

(i) this square (3.4.11) *satisfies the Beck-Chevalley condition;*

(ii) for every $M \in \mathscr{A}_B$*, there exists a commuting square*

$$\begin{array}{ccc} M' & \xrightarrow{\phi} & M \\ {\scriptstyle \chi}\downarrow & & \downarrow{\scriptstyle \xi} \\ M''' & \xrightarrow[\psi]{} & M'' \end{array}$$

in $\mathscr{A}$ *that lies over* (3.4.11) *such that* ϕ *and* ψ *are cartesian and* χ *and* ξ *are opcartesian.*

Proof. $(i) \Rightarrow (ii)$: We have the niche

$$\begin{array}{ccc} Mh^* & \xrightarrow{\mathrm{cart}^h_M} & M \\ {\scriptstyle \mathrm{opcart}^k_{Mh^*}}\downarrow & & \downarrow{\scriptstyle \mathrm{opcart}^g_M} \\ Mh^*k_! & & Mg_! \end{array} \tag{3.4.12}$$

in $\mathscr{A}$ which lies over the niche

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ C & & D \end{array}$$

in $\mathscr{B}$. By Corollary 3.4.4, the unique morphism that fills the niche (3.4.12) is cartesian.

$(ii) \Rightarrow (i)$: Let $M \in \mathscr{A}_B$. Then there exists a commuting square

$$\begin{array}{ccc} M' & \xrightarrow{\phi} & M \\ {\scriptstyle \chi}\downarrow & & \downarrow{\scriptstyle \xi} \\ M''' & \xrightarrow[\psi]{} & M'' \end{array}$$

in $\mathscr{A}$ that lies over (3.4.11) such that ϕ and ψ are cartesian and χ and ξ are opcartesian. Since ψ is cartesian, Corollary 3.4.4 gives that the Beck-Chevalley morphism ζ_M is an isomorphism. □

Chapter 4

Fibrant Double Categories

In this chapter, we study fibrant double categories and give examples, we give a thorough account of Shulman's construction of fibrant double categories from monoidal bifibrations, and we give explicit calculations of data in this construction.

In the first section, we define double categories and give examples. In the second section, we define fibrant double categories to be those double categories for which the functor

$$(S, T) \colon \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$$

given by the source and target functors, S and T, is a bifibration, and we unpack this definition. We then cover some of the basic properties of fibrant double categories, including the fact that that they can be described using companions and conjoints, and that the pull-back and push-forward functors

$$(f, g)^* \colon (\mathbb{D}_1)_{(B,D)} \rightleftarrows (\mathbb{D}_1)_{(A,C)} \colon (f, g)_!.$$

associated to the bifibration (S, T) can be described via loose composotion their action on the unit loose 1-cells (see Theorems 4.2.5 and 4.2.10). At the end of this section, we define a 2-category of double categories and a 2-category of fibrant double categories, we define monoidal double categories and we note how to obtain a (monoidal) bicategory from a (monoidal) fibrant double category; the ability to obtain monoidal bicategories from monoidal fibrant double categories is a key application of Shulman's construction, and it is studied in more detail by Shulman in [Shu10] and the functorially of this process is studied by Hansen and Shulman in [HS19].

The third section contains a detailed examination of Shulman's construction of fibrant double categories from monoidal bifibrations, and we carry out this construction for the weakly Beck-Chevalley and internally closed bifibration $\mathsf{Rep} \colon \mathbf{GrpRep} \to \mathbf{FinGrp}$. In the fouth section, we conclude with the main results of this thesis: the explicit calculations—in string diagrammatic language—of the unit loose 1-cell U_f and the (left) unitor for loose composition.

4.1 Double categories

Strict double categories were introduced by Ehresmann [Ehr63]. Grandis and Paré define *pseudo* double categories in [GP99] and *weak* double categories in [GP19]. The definition we give now is that which Shulman uses in [Shu08].

Definition 4.1.1. A **double category** $\mathbb{D}$ consists of the following data:

- a category $\mathbb{D}_0$ whose objects are called **0-cells** and whose morphisms are called **tight 1-cells**;
- a category $\mathbb{D}_1$ whose objects are called **loose 1-cells** and whose morphisms are called **2-cells**;
- functors $S, T \colon \mathbb{D}_1 \to \mathbb{D}_0$ called the **source and target functors** for which we have the notation $M \colon A \nrightarrow B$ to mean a loose 1-cell M with $SM = A$ and $TM = B$, and the notation

$$\begin{array}{ccc} A & \overset{M}{\nrightarrow} & B \\ {\scriptstyle f}\downarrow & \Downarrow \alpha & \downarrow{\scriptstyle g} \\ A' & \underset{M'}{\nrightarrow} & B' \end{array}$$

  to mean a 2-cell $\alpha \colon M \to M'$ with $S\alpha = f$ and $T\alpha = g$;
- a functor $U \colon \mathbb{D}_0 \to \mathbb{D}_1$ called the **unit functor**;
- a functor

$$\odot \colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$$

  called **loose composition**, where

$$\begin{array}{ccc} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 & \longrightarrow & \mathbb{D}_1 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle T} \\ \mathbb{D}_1 & \underset{S}{\longrightarrow} & \mathbb{D}_0 \end{array}$$

  is a pullback square in **Cat**;
- for each triple $M \colon A \nrightarrow B$, $N \colon B \nrightarrow C$, $P \colon C \nrightarrow D$ of loose 1-cells in $\mathbb{D}$, an invertible 2-cell

$$\begin{array}{ccc} A & \overset{(M \odot N) \odot P}{\nrightarrow} & D \\ {\scriptstyle \mathrm{id}_A}\| & \Downarrow a^{\odot}_{MNP} & \|{\scriptstyle \mathrm{id}_D} \\ A & \underset{M \odot (N \odot P)}{\nrightarrow} & D \end{array}$$

  called a **(loose composition) associator**;

- for each loose 1-cell $M\colon A \nrightarrow B$ in $\mathbb{D}$, invertible 2-cells

$$
\begin{array}{ccc}
A & \xrightarrow{U_A \odot M} & B \\
{\scriptstyle \mathrm{id}_A}\Big\| & \Downarrow {\scriptstyle l^{\odot}_M} & \Big\|{\scriptstyle \mathrm{id}_D} \\
A & \xrightarrow[M]{} & B
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
A & \xrightarrow{M \odot U_B} & B \\
{\scriptstyle \mathrm{id}_A}\Big\| & \Downarrow {\scriptstyle r^{\odot}_M} & \Big\|{\scriptstyle \mathrm{id}_D} \\
A & \xrightarrow[M]{} & B
\end{array}
$$

  called **(loose composition) left and right unitors**.

These data are required to satisfy the following axioms.

- $U \mathbin{⨟} S = \mathrm{id}_{\mathbb{D}_0}$ and $U \mathbin{⨟} T = \mathrm{id}_{\mathbb{D}_0}$.
- For all pairs $M\colon A \nrightarrow B, N\colon B \nrightarrow C$ of 1-cells in $\mathbb{D}$,

$$M \odot N\colon A \nrightarrow C.$$

- (**Naturality of the associator**) For all 2-cells $\alpha\colon M \to M', \beta\colon N \to N', \gamma\colon P \to P'$ in $\mathbb{D}$ satisfying $T(\alpha) = S(\beta)$ and $T(\beta) = S(\gamma)$, the following diagram commutes.

$$
\begin{array}{ccc}
(M \odot N) \odot P & \xrightarrow{(\alpha \odot \beta) \odot \gamma} & (M' \odot N') \odot P' \\
{\scriptstyle a^{\odot}_{MNP}}\Big\downarrow & & \Big\downarrow{\scriptstyle a^{\odot}_{M'N'P'}} \\
M \odot (N \odot P) & \xrightarrow[\alpha \odot (\beta \odot \gamma)]{} & M' \odot (N' \odot P')
\end{array}
\tag{4.1.2}
$$

- (**Naturality of the unitors**) For all pairs $M\colon A \to B$, $M'\colon A' \to B'$ of 1-cells in $\mathbb{D}$ and all 2-cells $\alpha\colon M \to M'$ in $\mathbb{D}$, the following diagrams commute.

$$
\begin{array}{ccc}
U_A \odot M & \xrightarrow{U_{S\alpha} \odot \alpha} & U_{A'} \odot M' \\
{\scriptstyle l^{\odot}_M}\Big\downarrow & & \Big\downarrow{\scriptstyle l^{\odot}_{M'}} \\
M & \xrightarrow[\alpha]{} & M'
\end{array}
\qquad
\begin{array}{ccc}
M \odot U_B & \xrightarrow{\alpha \odot U_{T\alpha}} & M' \odot U_{B'} \\
{\scriptstyle r^{\odot}_M}\Big\downarrow & & \Big\downarrow{\scriptstyle r^{\odot}_{M'}} \\
M & \xrightarrow[\alpha]{} & N
\end{array}
\tag{4.1.3}
$$

- (**Associativity**) For all tuples $M\colon A \nrightarrow B, N\colon B \nrightarrow C, P\colon C \nrightarrow D, Q\colon D \nrightarrow E$ of loose 1-cells in $\mathbb{D}$ the following diagram commutes.

$$
\begin{array}{ccccc}
 & & (M \odot N) \odot (P \odot Q) & & \\
 & {\scriptstyle a^{\odot}_{M \odot N.P,Q}} \nearrow & & \searrow {\scriptstyle a^{\odot}_{M,N,P \odot Q}} & \\
((M \odot N) \odot P) \odot Q & & & & M \odot (N \odot (P \odot Q)) \\
 & {\scriptstyle a^{\odot}_{M,N,P} \odot \mathrm{id}_Q} \searrow & & \nearrow {\scriptstyle \mathrm{id}_M \odot a^{\odot}_{N,P,Q}} & \\
 & (M \odot (N \odot P)) \odot Q & \xrightarrow[a^{\odot}_{M,N \odot P,Q}]{} & M \odot ((N \odot P) \odot Q) &
\end{array}
$$

- (**Unitality**) For all pairs $M\colon A \nrightarrow B$, $N\colon B \nrightarrow C$ of loose 1-cells in $\mathbb{D}$ the following diagram commutes.

$$\begin{array}{ccc} (M \odot U_B) \odot N & \xrightarrow{a^{\odot}_{M,U_B,N}} & M \odot (U_B \odot N) \\ & {\scriptstyle r^{\odot}_M \odot \mathrm{id}_N} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_M \odot l^{\odot}_N} & \\ & M \odot N & \end{array}$$

◇

The terminology 'tight' and 'loose' for 1-cells in a double category was first used by Hansen and Shulman [HS19] after Shulman and Lack used these terms in their study of $\mathscr{F}$-categories [LS12]. Historically, the more standard terminology for 1-cells in a double category is 'vertical' and 'horizontal' [BS76, GP99], and this terminology is still used today. However, just like with our use of the term 'pure' instead of 'vertical' when studying fibrations (see Remark 2.1.13), we favour the terms 'tight' and 'loose' because they don't depend on how things are drawn.

Remark 4.1.4. The left unitor 2-cells $l^{\odot}_M$ and the right unitor 2-cells $r^{\odot}_M$ are the components of natural transformations

$$\begin{array}{ccccc} \mathbb{D}_1 & \xrightarrow{(S,\mathrm{id}_{\mathbb{D}_1})} & \mathbb{D}_0 \times \mathbb{D}_1 & \xrightarrow{U \times \mathrm{id}_{\mathbb{D}_1}} & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \\ & {\scriptstyle \mathrm{id}_{\mathbb{D}_1}} \searrow & & l^{\odot} \Swarrow & \downarrow \odot \\ & & & & \mathbb{D}_1 \end{array}$$

and

$$\begin{array}{ccccc} \mathbb{D}_1 & \xrightarrow{(\mathrm{id}_{\mathbb{D}_1},T)} & \mathbb{D}_1 \times \mathbb{D}_0 & \xrightarrow{\mathrm{id}_{\mathbb{D}_1} \times U} & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \\ & {\scriptstyle \mathrm{id}_{\mathbb{D}_1}} \searrow & & r^{\odot} \Swarrow & \downarrow \odot \\ & & & & \mathbb{D}_1 \end{array}$$

called the **(loose composition) left unitor** and the **(loose composition) right unitor**, where $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$ denotes the category of loose-composable pairs defined by the following pullback in **Cat**.

$$\begin{array}{ccc} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 & \longrightarrow & \mathbb{D}_1 \\ \downarrow & \lrcorner & \downarrow T \\ \mathbb{D}_1 & \xrightarrow[S]{} & \mathbb{D}_0 \end{array}$$

The naturality axioms for the unitors are (4.1.3). ◇

Remark 4.1.5. The associativity 2-cells $a^{\odot}_{MNP}$ are the components of a natural transformation

$$\begin{array}{ccc}
\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 & \xrightarrow{\mathrm{id} \times_{\mathbb{D}_0} \odot} & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \\
{\scriptstyle \odot \times_{\mathbb{D}_0} \mathrm{id}} \big\downarrow & a^{\odot} \Nearrow & \big\downarrow {\scriptstyle \odot} \\
\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 & \xrightarrow[\odot]{} & \mathbb{D}_1
\end{array}$$

called the **(loose composition) associator**, where $\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1$ denotes the category of loose-composable triples defined by the following pullback in **Cat**.

$$\begin{array}{ccccc}
\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 & \longrightarrow & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 & \longrightarrow & \mathbb{D}_1 \\
\big\downarrow & \lrcorner & \big\downarrow & \lrcorner & \big\downarrow {\scriptstyle T} \\
\mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 & \longrightarrow & \mathbb{D}_1 & \xrightarrow[S]{} & \mathbb{D}_0 \\
\big\downarrow & \lrcorner & \big\downarrow {\scriptstyle T} & & \\
\mathbb{D}_1 & \xrightarrow[S]{} & \mathbb{D}_0 & &
\end{array}$$

The naturality axiom for the associator is (4.1.2). ◇

Notation 4.1.6. We have two kinds of composition of 2-cells in a double category $\mathbb{D}$. We have the composition of 2-cells as morphisms in the category $\mathbb{D}_1$, which we call **tight composition**, and we have the composition of 2-cells defined by the loose composition functor, $\odot$, which we unsurprisingly call **loose composition**. We'll depict tight composition of 2-cells as vertically stacking,

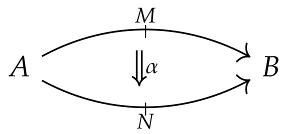
$$\begin{array}{ccc}
A & \xrightarrow{M} & B \\
{\scriptstyle f} \big\downarrow & \Downarrow \alpha & \big\downarrow {\scriptstyle g} \\
A' & \xrightarrow{M'} & B' \\
{\scriptstyle f'} \big\downarrow & \Downarrow \beta & \big\downarrow {\scriptstyle g'} \\
A'' & \xrightarrow[M'']{} & B''
\end{array}$$

and we'll depict loose composition of 2-cells as horizontal stacking,

$$\begin{array}{ccccc}
A & \xrightarrow{M} & B & \xrightarrow{N} & C \\
{\scriptstyle f} \big\downarrow & \Downarrow \alpha & \big\downarrow {\scriptstyle g} & \Downarrow \gamma & \big\downarrow {\scriptstyle h} \\
A' & \xrightarrow[M']{} & B' & \xrightarrow[N']{} & C'
\end{array}$$

◇

Definition 4.1.7. Let $\mathbb{D}$ be a double category. We call a 2-cell α **globular** if $S\alpha$ and $T\alpha$ are both identity tight 1-cells; this name is used because such a 2-cell can be written as



◇

Definition 4.1.8. The category **Bimod** of bimodules has

- as objects triples (A, B, M), called **bimodules**, where A and B are finite groups, and M is a finite dimensional vector space with the structure of a left A-module and a right B-module such that, for all $a \in A$, $b \in B$ and $m \in M$, $(a \cdot m) \cdot b = a \cdot (m \cdot b)$, and
- as morphisms $(A, B, M) \to (A', B', M')$ triples (f, g, α), called **bimodule maps**, where $f \colon A \to A'$ and $g \colon B \to B'$ are group homomorphisms and $\alpha \colon M \to M'$ is a linear map that satisfies, for all $a \in A$, $b \in B$ and $m \in M$, $(a \cdot m \cdot b)\alpha = (a)f \cdot (m)\alpha \cdot (b)g$.

◇

Example 4.1.9. The double category $\mathbb{B}$**imod** consists of the following data.

- The category $\mathbb{B}\mathbf{imod}_0$ is the category **FinGrp** of finite groups.
- The category $\mathbb{B}\mathbf{imod}_1$ is the category **Bimod** of bimodules.
- The source and target functors, S and T, are given by

$$\begin{aligned}(S, T) \colon \mathbf{Bimod} &\longrightarrow \mathbf{FinGrp} \times \mathbf{FinGrp} \\ (A, B, M) &\longmapsto (A, B) \\ (f, g, \alpha) &\longmapsto (f, g).\end{aligned}$$

- The functor unit functor, U, is given by

$$\begin{aligned}U \colon \mathbf{FinGrp} &\longrightarrow \mathbf{Bimod} \\ G &\longmapsto \mathbb{C}G \\ f &\longmapsto (f, f, \mathbb{C}f).\end{aligned}$$

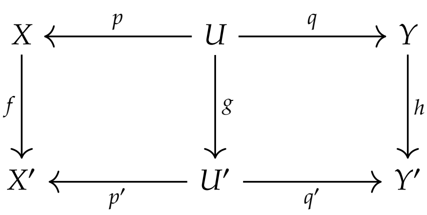
- The loose composition functor, $\odot$, is given by

$$\begin{aligned}\odot \colon \mathbf{Bimod} \times_{\mathbf{FinGrp}} \mathbf{Bimod} &\longrightarrow \mathbf{Bimod} \\ \big((A, B, M), (B, C, N)\big) &\longmapsto (A, C, M \otimes_B N) \\ \big((f, g, \alpha), (g, h, \beta)\big) &\longmapsto (f, h, \alpha \otimes_B \beta).\end{aligned}$$

◇

Definition 4.1.10. Let $\mathscr{A}$ be a category with pullbacks. The category $\mathbf{Span}_{\mathscr{A}}$ of spans in $\mathscr{A}$ has

- as objects tuples (X, Y, U, p, q), called **spans**, where X, Y and S are objects in $\mathscr{A}$ and $p\colon U \to X$ and $q\colon U \to Y$ are morphisms in $\mathscr{A}$, and
- as morphisms $(X, Y, U, p, q) \to (X', Y', U', p', q')$ are triples (f, g, h), called **morphisms of spans**, where $f\colon X \to X'$, $g\colon U \to U'$ and $h\colon Y \to Y'$ are morphisms in $\mathscr{A}$ that make the following diagram in $\mathscr{A}$ commute.



◇

Example 4.1.11. Let $\mathscr{A}$ be a category with pullbacks. The double category $\mathbb{S}\mathbf{pan}_{\mathscr{A}}$ consists of the following data.

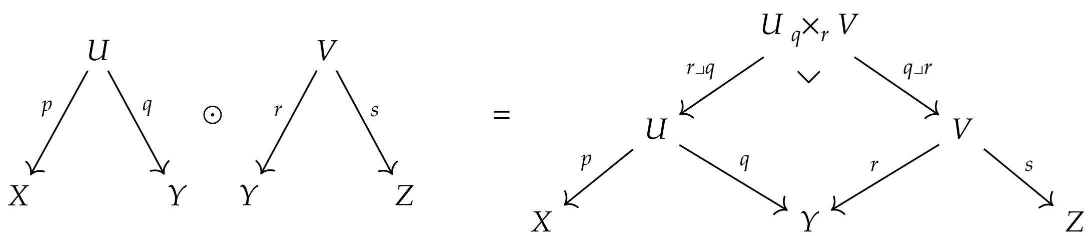
- The category $\mathbb{S}\mathbf{pan}_{\mathscr{A},0}$ is the category $\mathscr{A}$.
- The category $\mathbb{S}\mathbf{pan}_{\mathscr{A},1}$ is the category $\mathbf{Span}_{\mathscr{A}}$ of spans in $\mathscr{A}$.
- The source and target functors, S and T, are given by

$$\begin{aligned}(S, T)\colon \mathbf{Span}_{\mathscr{A}} &\longrightarrow \mathscr{A} \times \mathscr{A}\\ (X, Y, U, p, q) &\longmapsto (X, Y)\\ (f, g, h) &\longmapsto (f, h).\end{aligned}$$

- The functor unit functor, U, is given by

$$\begin{aligned}U\colon \mathscr{A} &\longrightarrow \mathbf{Span}_{\mathscr{A}}\\ A &\longmapsto (A, A, A, \mathrm{id}_A, \mathrm{id}_A)\\ f &\longmapsto (f, f, f).\end{aligned}$$

- The loose composition functor, $\odot$, is defined on loose 1-cells using pullbacks as shown below, and is defined on 2-cells using the universal property of pullbacks.



◇

4.2 Fibrant double categories

Definition 4.2.1. We call a double category $\mathbb{D}$ a **fibrant double category** if the functor

$$(S,T)\colon \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$$

is a bifibration. ◇

Remark 4.2.2. It should be noted that Shulman originally termed these 'framed bicategories' in [Shu08], but later decided on the term fibrant double category (see [Shu10, Remark 3.5]). ◇

What does it mean for $(S,T)\colon \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ to be a fibration? The definition states the following: for each morphism $(f,g)\colon (A,C) \to (B,D)$ in $\mathbb{D}_0$ and each object M in $\mathbb{D}_1$ satisfying $(S,T)(M) = (B,D)$, there exists a cartesian morphism $\phi\colon P \to M$ in $\mathbb{D}_1$ that lies over (f,g). Unpacked this means that, for each pair of tight 1-cells $f\colon A \to B$ and $g\colon C \to D$ in $\mathbb{D}$ and each loose 1-cell $M\colon B \nrightarrow D$ in $\mathbb{D}$, there exists a 2-cell

$$\begin{array}{ccc} A & \overset{P}{\longrightarrow} & C \\ {\scriptstyle f}\downarrow & \Downarrow{\scriptstyle \phi} & \downarrow{\scriptstyle g} \\ B & \underset{M}{\longrightarrow} & D \end{array}$$

such that, for each 2-cell

$$\begin{array}{ccc} E & \overset{Q}{\longrightarrow} & F \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ A & \Downarrow{\scriptstyle \psi} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \underset{M}{\longrightarrow} & D \end{array},$$

there exists a unique 2-cell

$$\begin{array}{ccc} E & \overset{Q}{\longrightarrow} & F \\ {\scriptstyle h}\downarrow & \Downarrow{\scriptstyle \chi} & \downarrow{\scriptstyle k} \\ A & \underset{P}{\longrightarrow} & C \end{array}$$

that satisfies

$$
\begin{array}{ccc}
E & \xrightarrow{Q} & F \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\
A & \Downarrow \psi & B \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
B & \xrightarrow[M]{} & D
\end{array}
\quad = \quad
\begin{array}{ccc}
E & \xrightarrow{Q} & F \\
{\scriptstyle h}\downarrow & \Downarrow \chi & \downarrow{\scriptstyle k} \\
A & \xrightarrow{P} & B \\
{\scriptstyle f}\downarrow & \Downarrow \phi & \downarrow{\scriptstyle g} \\
B & \xrightarrow[M]{} & D
\end{array}
\;.
$$

The description of what it means for $(S, T)\colon \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$ to be an opfibration is similar.

Recall that, if $\Phi\colon \mathscr{A} \to \mathscr{B}$ is a bicleaved bifibration, then we have, for each morphism $f\colon B \to B'$ in $\mathscr{B}$, functors $f^*\colon \mathscr{A}_{B'} \to \mathscr{A}_B$ and $f_!\colon \mathscr{A}_B \to \mathscr{A}_{B'}$ (see Proposition 2.1.27 and Proposition 2.4.18). Given a bicleaving for the bifibration $(S, T)\colon \mathbb{D}_1 \to \mathbb{D}_0 \times \mathbb{D}_0$, we have, for each pair of tight 1-cells $f\colon A \to B$ and $g\colon C \to D$ in $\mathbb{D}$, functors

$$(f, g)^*\colon (\mathbb{D}_1)_{(B,D)} \rightleftarrows (\mathbb{D}_1)_{(A,C)} \colon (f, g)_!.$$

Notation 4.2.3. For loose 1-cells $M\colon B \nrightarrow D$ and $P\colon A \nrightarrow C$, we use the following notation for the functors $(f, g)^*$ and $(f, g)_!$.

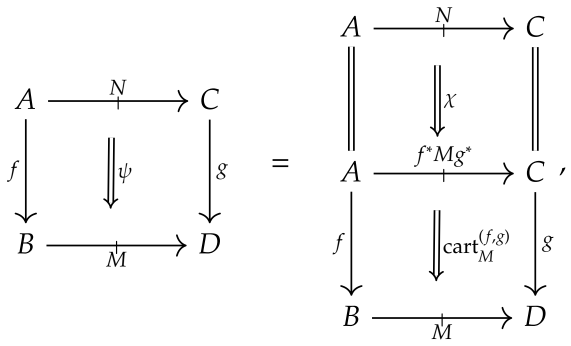
$$
\begin{aligned}
f^*Mg^* &:= (f, g)^*M \\
f_!Pg_! &:= (f, g)_!P \\
f^*M &:= (f, \mathrm{id}_D)^*M \\
Mg^* &:= (\mathrm{id}_B, g)^*M \\
f_!P &:= (f, \mathrm{id}_C)_!P \\
Pg_! &:= (\mathrm{id}_A, g)_!P
\end{aligned}
$$

This notation is due to Shulman [Shu08]. ◇

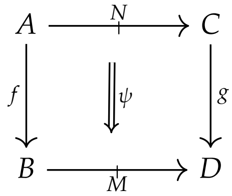
Using this notation, pure-cartesian factorisation says that each 2-cell

$$
\begin{array}{ccc}
A & \xrightarrow{N} & C \\
{\scriptstyle f}\downarrow & \Downarrow \psi & \downarrow{\scriptstyle g} \\
B & \xrightarrow[M]{} & D
\end{array}
$$

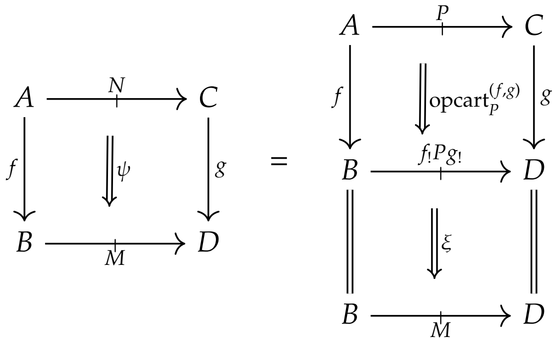
can be uniquely factorised as



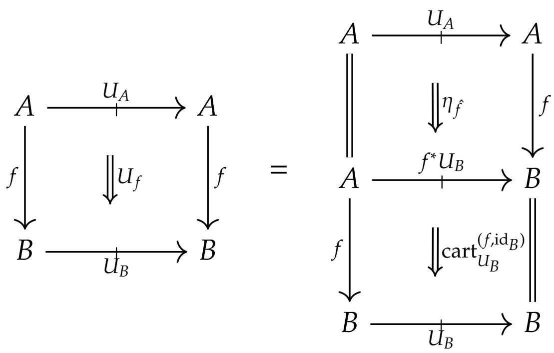
and opcartesian-pure factorisation says that each 2-cell



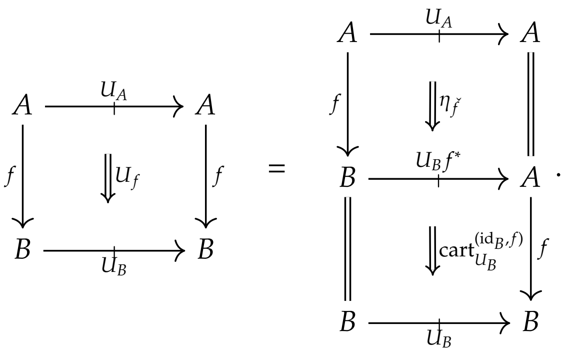
can be uniquely factorised as



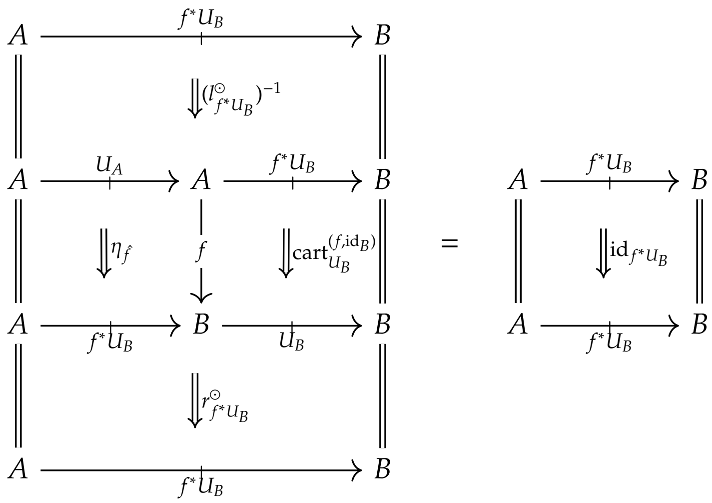
Lemma 4.2.4 ([Shu08, Theorem 4.1]). *Let $\mathbb{D}$ be a fibrant double category and let $f\colon A \to B$ be a tight 1-cell in $\mathbb{D}$. Define the 2-cells $\eta_{\hat{f}}$ and $\eta_{\check{f}}$ using cartesian factorisation:*



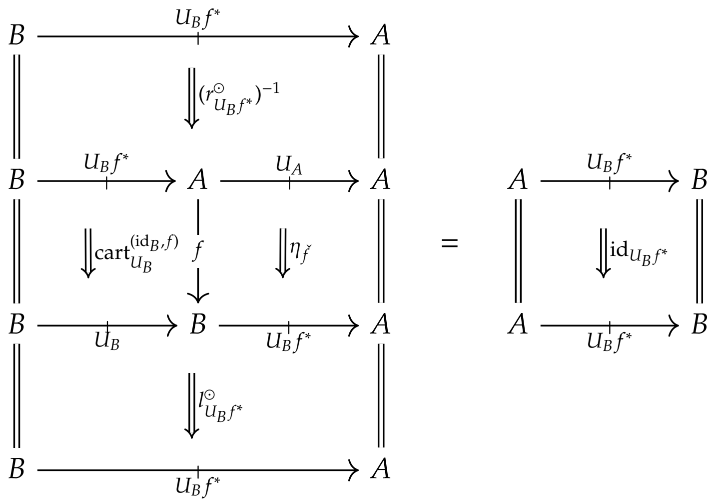
and



Then



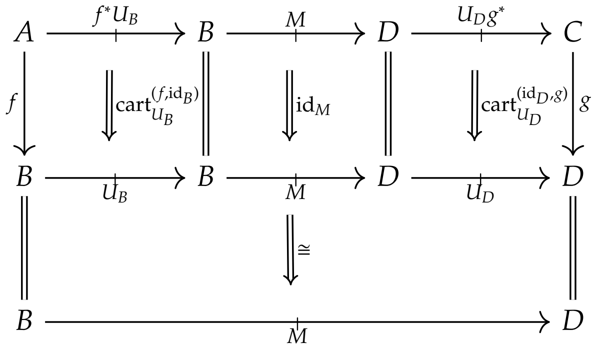
and



Theorem 4.2.5. *Let $\mathbb{D}$ be a fibrant double category. Let $f\colon A \to B$ and $g\colon C \to D$ be tight 1-cells in $\mathbb{D}$, and let $M\colon B \nrightarrow D$ be a loose 1-cell in $\mathbb{D}$. Then there exists an invertible globular 2-cell*

$$f^*Mg^* \cong f^*U_B \odot M \odot U_D g^*.$$

Proof. The 2-cell



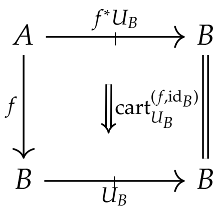
lies over the morphism (f, g) in $\mathscr{B} \times \mathscr{B}$, and it's straightforward to check that it's cartesian. Therefore, by Remark 2.1.28, there exists an isomorphism $f^*Mg^* \cong f^*U_B \odot M \odot U_D g^*$. □

Corollary 4.2.6. *Let $\mathbb{D}$ be a fibrant double category, let $M\colon B \to D$ and $N\colon D \to G$ be loose 1-cells in $\mathbb{D}$, and let $f\colon A \to B$ and $g\colon C \to D$ be tight 1-cells in $\mathbb{D}$. Then there exists an invertible globular 2-cell*

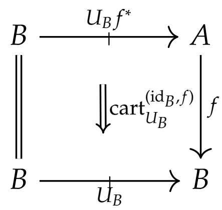
$$f^*(M \odot N)g^* \cong f^*M \odot Ng^*.$$

□

Theorem 4.2.5 tells us that we can construct a cleaving using the cartesian 2-cells



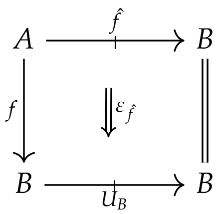
and



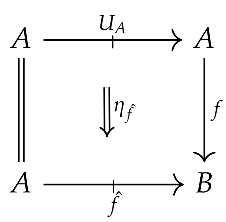
.

We therefore give 2-cells with the necessary properties a name.

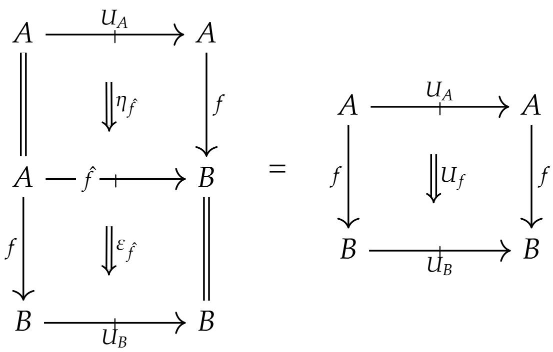
Definition 4.2.7. Let $\mathbb{D}$ be a double category, and let $f\colon A \to B$ be a tight 1-cell in $\mathbb{D}$. A **companion** of f consists of a loose 1-cell $\hat{f}\colon A \nrightarrow B$ and 2-cells



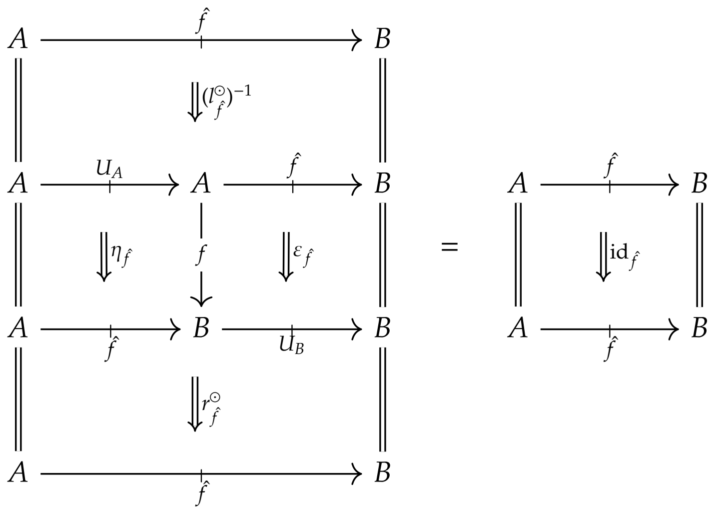
and



such that

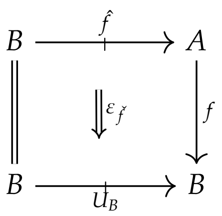


and

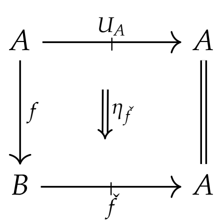


◇

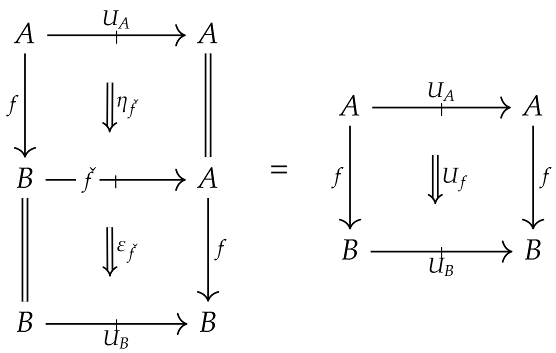
Definition 4.2.8. Let $\mathbb{D}$ be a double category, and let $f\colon A \to B$ be a tight 1-cell in $\mathbb{D}$. A **conjoint** of f consists of a loose 1-cell $\check{f}\colon B \nrightarrow A$ and 2-cells



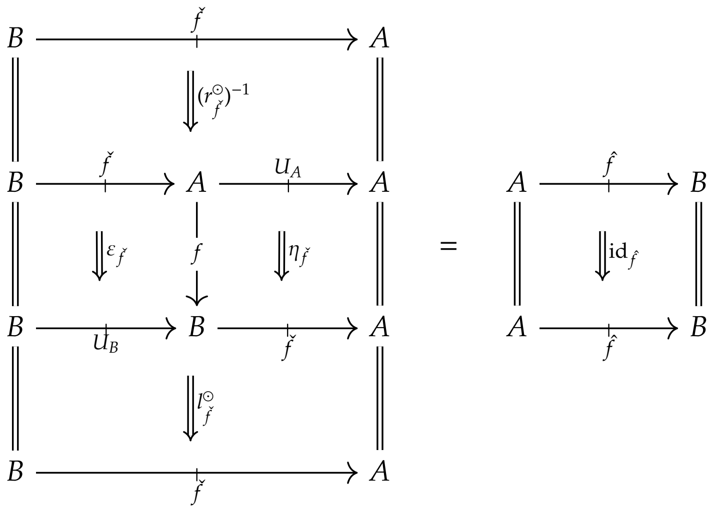
and



such that



and



◇

Compnaions and conjoints are well studied. This definition is due to [GP04], but the idea orginates from Brown and Spencer's work on double groupoids [BS76].

Proposition 4.2.9. *Let $\mathbb{D}$ be a double category. Then the functor $(S, T)\colon \mathbb{D}_0 \to \mathbb{D}_1 \times \mathbb{D}_1$ is a fibration if, and only if, there exists a companion and conjoint for every tight 1-cell in $\mathbb{D}$.* □

One can also use opcartesian factorisation to get a lemma analogous to Lemma 4.2.4 and a theorem analogous to Theorem 4.2.5.

Theorem 4.2.10. *Let $\mathbb{D}$ be a fibrant double category. Let $f\colon A \to B$ and $g\colon C \to D$ be tight 1-cells in $\mathbb{D}$, and let $P\colon A \nrightarrow C$ be a loose 1-cell in $\mathbb{D}$. Then there exists an invertible globular 2-cell*

$$f_!Pg_! \cong f_!U_A \odot P \odot U_C g_!.$$

□

Corollary 4.2.11. *Let $\mathbb{D}$ be a fibrant double category, let $P\colon A \to C$ and $Q\colon C \to H$ be loose 1-cells in $\mathbb{D}$, and let $f\colon A \to B$ and $g\colon C \to D$ be tight 1-cells in $\mathbb{D}$. Then there exists an invertible globular 2-cell*

$$f_!(P \odot Q)g_! \cong f_!P \odot Qg_!.$$

□

The existence of companions and conjoints is self-dual, and so we have the following theorem.

Theorem 4.2.12. *Let $\mathbb{D}$ be a double category. Then the following are equivalent:*

- *$\mathbb{D}$ is a fibrant double category;*
- *the functor (S, T) is a fibration;*
- *the functor (S, T) is an opfibration;*
- *there exists a companion and conjoint for every tight 1-cell in $\mathbb{D}$.*

□

4.2.1 A 2-category of double categories

Definition 4.2.13. Let $\mathbb{D}$ and $\mathbb{E}$ be double categories. A **double functor** consists of the following data:

- two functors $F_0 \colon \mathbb{D}_0 \to \mathbb{E}_0$ and $F_1 \colon \mathbb{D}_1 \to \mathbb{E}_1$ satisfying $F_1 \mathbin{⨟} S = S \mathbin{⨟} F_0$ and $F_1 \mathbin{⨟} T = T \mathbin{⨟} F_0$;
- for each pair M, N of loose composable loose 1-cells in $\mathbb{D}$, an invertible globular 2-cell $F^2_{MN} \colon MF_1 \odot NF_1 \to (M \odot N)F_1$;
- for each 0-cell A in $\mathbb{D}$, an invertible globular 2-cell $F^0_A \colon U_{AF_0} \to U_A F_0$.

These data must satisfy axioms similar to those of a pseudofunctor between bicategories (see Definition 1.1.8). ◇

Definition 4.2.14. Let $F, G \colon \mathbb{D} \to \mathbb{E}$ be double functors. A **double transformation** $\alpha \colon F \to G$ consists of two natural transformations $\alpha_0 \colon F_0 \to G_0$ and $\alpha_1 \colon F_1 \to G_1$ that satisfy, for every loose 1-cell M in $\mathbb{D}$, $\alpha_{1,M} S = \alpha_{1,MS}$ and $\alpha_{1,M} T = \alpha_{1,MT}$, as well as axioms similar to those of a pseudonatural transformation between bicategories (see Definition 1.1.13). ◇

Definition 4.2.15. A **fibrant double functor** is a double functor between fibrant double categories, and a **fibrant double transformation** between two fibrant double functors is a double transformation between their underlying double functors. ◇

It may seem odd that we don't insist that a fibrant double functor preserves (S, T)-cartesian 2-cells and (S, T)-opcartesian 2-cells, but this is automatically true, as proved in [Shu09, Proposition 6.8].

Definition 4.2.16. The 2-category $\mathcal{Dbl}$ of double categories consists of double categories, double functors and double transformations. This has a sub-2-category $\mathcal{FibDbl}$ consisting of fibrant double categories, fibrant double functors and fibrant double transformations. ◇

4.2.2 (Monoidal) fibrant double categories to (monoidal) bicategories

An important application of the property of a double category being fibrant is to monoidal double categories and the monoidal bicategories they induce. A **monoidal double category** is defined to be a pseudomonoid in the 2-category $\mathcal{Dbl}$ of double categories. We give a brief unpacking of this definition below; a full definition of monoidal double category is given in [Shu08, Definition 9.1].

Definition 4.2.17. A **monoidal double category** consists of

- monoidal categories $(\mathbb{D}_0, \otimes_0, I_0)$ and $(\mathbb{D}_1, \otimes_1, I_1)$,
- an invertible 2-cell

$$\mathfrak{x}\colon (M \otimes_1 P) \odot (N \otimes_1 Q) \to (M \odot N) \otimes_1 (P \odot Q),$$

- an invertible 2-cell

$$\mathfrak{u}\colon U_{A \otimes_0 B} \to (U_A \otimes_1 U_B),$$

such that

- U_I is the monoidal unit of $\mathbb{D}_1$,
- the functors $S, T\colon \mathbb{D}_1 \to \mathbb{D}_1$ are strict monoidal,
- $\mathfrak{x}$ and $\mathfrak{u}$ satisfy the appropriate axioms,
- the associativity morphisms for the monoidal products in $\mathbb{D}_0$ and $\mathbb{D}_1$ form a natural transformation of double categories $\mathbb{D} \to \mathbb{D}$,
- the unit morphisms for the monoidal products in $\mathbb{D}_0$ and $\mathbb{D}_1$ form a natural transformation of double categories $\mathbb{D} \to \mathbb{D}$.

◇

Definition 4.2.18. A **monoidal fibrant double category** is monoidal double category for which the constituent double category, double functors and double transformations are all replaced by their fibrant counterparts. ◇

Given a double category $\mathbb{D}$ we can throw away the tight 1-cells and the non-globular 2-cells to get a bicategory.

Definition 4.2.19. The **loose bicategory** $\mathcal{L}\mathbb{D}$ of a double category $\mathbb{D}$ consists of the 0-cells, loose 1-cells, and globular 2-cells of $\mathbb{D}$. ◇

Given a *monoidal* double category $\mathbb{D}$ we can't throw away the tight 1-cells and the non-globular 2-cells to get a *monoidal* bicategory because the monoidal associativity and unitality constraints of $\mathbb{D}$ are tight 1-cells and we don't want to throw away that information. However, if we have a monoidal fibrant double category $\mathbb{D}$, the companions and conjoints mean that the data of the tight 1-cells is also contained in the loose 1-cells. This is the subject of Shulman's [Shu10], and later Hansen and Shulman's [HS19].

Theorem 4.2.20 ([Shu10, Theorem 1.2]). *Let $\mathbb{D}$ be a monoidal fibrant double category. Then the loose bicategory $\mathcal{L}\mathbb{D}$ is a monoidal bicategory.*

The monoidal bicategories that can be expressed as the loose bicategory of a monoidal fibrant double category are called **monoidal equipments**.

4.3 Definition of $\mathbb{D}\mathbf{bl}(\Phi)$

Recall the following two definitions from earlier.

Definition 3.3.1. Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a bifibration. We say that Φ is **Beck-Chevalley** if, for every pullback square

$$\begin{array}{ccc} B \times_{B'} C & \xrightarrow{\;q\;} & C \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ B & \xrightarrow[\;f\;]{} & B' \end{array}$$

in $\mathscr{B}$, the Beck-Chevalley transformation is an isomorphism. We say that Φ is **weakly Beck-Chevalley** if, for every pullback square of the form

$$\begin{array}{ccc} B \times_{B'} (D \times B') & \xrightarrow{\;q\;} & D \times B' \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_D} \\ B & \xrightarrow[\;f\;]{} & B' \end{array}$$

in $\mathscr{B}$ and every pullback square of the form

$$\begin{array}{ccc} (D \times B') \times_{B'} C & \xrightarrow{\;q\;} & C \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ D \times B' & \xrightarrow[\;\pi_D\;]{} & B' \end{array}$$

in $\mathscr{B}$, the Beck-Chevalley transformation is an isomorphism. ◇

Definition 2.8.5. Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a monoidal fibration with $\mathscr{B}$ cartesian monoidal. We call $\Phi\colon \mathscr{A} \to \mathscr{B}$ **internally closed** if, for each object B in $\mathscr{B}$, the fibre category $\mathscr{A}_B$ is closed monoidal and if, for every morphism $f\colon A \to B$ in $\mathscr{B}$ and every pair of objects M, M' in $\mathscr{A}_B$, the morphisms

$$(M' \blacktriangleright M)f^* \to M'f^* \blacktriangleright Mf^* \qquad \text{and} \qquad (M' \blacktriangleleft M)f^* \to M'f^* \blacktriangleleft Mf^*.$$

given by (2.8.2) and (2.8.4) are isomorphisms. ◇

In order to construct a fibrant double category from a moniodal bifibration $\Phi\colon \mathscr{A} \to \mathscr{B}$, we need $\mathscr{B}$ to be cartesian moniodal and we need Φ to be either (i) Beck-Chevalley or (ii) weakly Beck-Chevalley and internally closed. We call such a monoidal bifibration a **doublable** monoidal bifibration.

The following theorem is the main result of Shulman's paper [Shu08]. We give the short version as Shulman does; in the subsections that follow we add some necessary detail.

Theorem 4.3.1 ([Shu08, Theorem 14.2]). *If $\Phi\colon \mathscr{A} \to \mathscr{B}$ is a doublable monoidal bifibration, then there is a fibrant double category $\mathbb{D}\mathbf{bl}(\Phi)$ defined as follows.*

(i) The tight category $\mathbb{D}\mathbf{bl}(\Phi)_0$ is equal to $\mathscr{B}$.

(ii) The loose category $\mathbb{D}\mathbf{bl}(\Phi)_1$ and the functors S and T are given by the following pullback in **Cat***:*

$$\begin{array}{ccc} \mathbb{D}\mathbf{bl}(\Phi)_1 & \longrightarrow & \mathscr{A} \\ {\scriptstyle (S,T)}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \Phi} \\ \mathscr{B}\times\mathscr{B} & \xrightarrow[\times]{} & \mathscr{B} \end{array}$$

(iii) The loose composition of loose 1-cells $M\colon A \nrightarrow B$ and $N\colon B \nrightarrow C$ is equal to

$$M \odot N = (M \otimes N)\Delta_B^{*}\pi_{B!}$$

and the loose composition of 2-cells is similar.

(iv) The loose unit of the object A is equal to

$$U_A = I\pi_A^{*}\Delta_{A!}$$

where I denotes the monoidal unit of $\mathscr{A}$.

Example 4.3.2. Let $\mathscr{A}$ be a category with pullbacks. Then the codomain functor $\mathtt{Arr}_{\mathscr{A}}\colon \mathscr{A}^{\to} \to \mathscr{A}$ is a doublable monoidal bifibration. The tight category of $\mathbb{D}\mathbf{bl}(\mathtt{Arr}_{\mathscr{A}})$ is $\mathscr{A}$, the loose 1-cells $A \nrightarrow B$ in $\mathbb{D}\mathbf{bl}(\mathtt{Arr}_{\mathscr{A}})$ are objects in $\mathscr{A}^{\to}$ that lie over the object $A \times B$ in $\mathscr{B}$, and the 2-cells in $\mathbb{D}\mathbf{bl}(\mathtt{Arr}_{\mathscr{A}})$ are morphisms of $\mathscr{A}^{\to}$ over $f \times g$.

In fact, $\mathbb{D}\mathbf{bl}(\mathtt{Arr}_{\mathscr{A}})$ is isomorphic to the fibrant double category $\mathbb{S}\mathbf{pan}\mathscr{A}$ of spans in the 2-category $\mathcal{FibDbl}$. This follows from the fact that a span $A \leftarrow M \to B$ is equivalently a morphism $\alpha\colon M \to A \times B$, and this is precisely an object α of $\mathscr{A}^{\to}$ satisfying $(\alpha)\mathtt{Arr}_{\mathscr{A}} = A \times B$. ◇

Example 4.3.3. The forgetful functor $\mathtt{Rep}\colon \mathbf{GrpRep} \to \mathbf{FinGrp}$ is doublable since it is weakly Beck-Chevalley (see Proposition 3.3.3) and internally closed (see Example 2.8.11), so we may construct the fibrant double category $\mathbb{D}\mathbf{bl}(\mathtt{Rep})$.

The objects of $\mathbb{D}\mathbf{bl}(\mathsf{Rep})$ are finite groups, and the tight 1-cells of $\mathbb{D}\mathbf{bl}(\mathsf{Rep})$ are group homomorphisms. A loose 1-cell $G \nrightarrow H$ is a $(G \times H)$-module, and a 2-cell

$$\begin{array}{ccc} G & \overset{V}{\nrightarrow} & H \\ {\scriptstyle f}\downarrow & \Downarrow{\scriptstyle \alpha} & \downarrow{\scriptstyle f'} \\ I & \underset{W}{\nrightarrow} & J \end{array}$$

is an (f, f')-equivariant map $\alpha\colon V \to W$, by which we mean a linear map $\alpha\colon V \to W$ such that, for every $(g, h) \in G \times H$ and every $v \in V$,

$$((g,h) \cdot v)\alpha = ((g)f, (h)f') \cdot (v)\alpha.$$

Now for composition of loose 1-cells. Theorem 4.3.1 tells us that this is

$${}_{G\times H}V \odot {}_{H\times K}W = ({}_{G\times H}V \otimes {}_{H\times K}W)\Delta_H^* \pi_{H!}.$$

The obvious part is that $({}_{G\times H}V \otimes {}_{H\times K}W)\Delta_H^*$ is equal to the $G \times H \times K$-module $V \otimes W$ with H acting diagonally. Recall from Example 2.4.20 that induction along a product projection gives the space of coinvariants. So the underlying vector space of ${}_{G\times H}V \odot {}_{H\times K}W$ is

$$V \otimes W_{(H)} = (V \otimes W)/\langle (h \cdot v) \otimes (h \cdot w) - v \otimes w \mid h \in H, v \in V, w \in W \rangle,$$

and G and K act on this space in the obvious way.

Lastly, we define the unit object U_G. Theorem 4.3.1 tells us that this is

$$U_G = \mathbb{C}^{\mathrm{triv}} \pi_G^* \Delta_{G!}.$$

The category $\mathbf{Rep}_{\{e\}}$ is equivalent to the category of vector spaces, and the base-change functor $\pi_G^*\colon \mathbf{Rep}_{\{e\}} \to \mathbf{Rep}_G$ maps a vector space V to the G-module V with trivial G-action; in particular $\mathbb{C}^{\mathrm{triv}}\pi_G^*$ is the trivial G-module ${}_G\mathbb{C}^{\mathrm{triv}}$. Then induction along the diagonal map Δ_G gives

$$U_G = {}_{G\times G}\mathbb{C}G,$$

where $G \times G$ acts on $\mathbb{C}G$ by $(x, y) \cdot g = xgy^{-1}$. ◇

Example 4.3.4. The fibrant double categories $\mathbb{B}\mathbf{imod}$ and $\mathbb{D}\mathbf{bl}(\mathsf{Rep})$ are isomorphic as objects in the 2-category $\mathcal{FibDbl}$ of fibrant double categories. The difference between these two fibrant double categories is only very slight, but we'll define mutually inverse double functors $P\colon \mathbb{D}\mathbf{bl}(\mathsf{Rep}) \rightleftarrows \mathbb{B}\mathbf{imod} : Q$.

To define P, we must define functors

$$\mathbb{D}\mathbf{bl}(\mathsf{Rep})_0 \to \mathbb{B}\mathbf{imod}_0 \quad \text{and} \quad \mathbb{D}\mathbf{bl}(\mathsf{Rep})_1 \to \mathbb{B}\mathbf{imod}_1,$$

as well as, for each pair M, N of loose composable loose 1-cells in $\mathbb{D}\mathbf{bl}(\mathsf{Rep})$, invertible globular 2-cells $P_\odot\colon VP \odot WP \to (V \odot W)P$ and $P_U\colon U_{GP} \to U_G P$. The functor $\mathbb{D}\mathbf{bl}(\mathsf{Rep})_0 \to \mathbb{B}\mathbf{imod}_0$ is the identity on $\mathbf{FinGrp}$. For a loose 1-cell $V\colon G \nrightarrow H$ in $\mathbb{D}\mathbf{bl}(\mathsf{Rep})$ (i.e. a $(G \times H)$-module), the loose

1-cell $VP\colon G \nrightarrow H$ in $\mathbb{B}\mathbf{imod}$ is the (G,H)-bimodule with underlying vector space V, with left G-action $g\cdot v := (g,e_H)\cdot v$, and right H-action $v\cdot h := (e_G,h^{-1})\cdot v$. For a 2-cell

$$\begin{array}{ccc} G & \xrightarrow{\;V\;} & H \\ {\scriptstyle f}\big\downarrow & \Downarrow{\scriptstyle \alpha} & \big\downarrow{\scriptstyle f'} \\ I & \xrightarrow[W]{} & J \end{array}$$

in $\mathbb{D}\mathbf{bl}(\mathsf{Rep})$, the 2-cell

$$\begin{array}{ccc} G & \xrightarrow{\;VP\;} & H \\ {\scriptstyle f}\big\downarrow & \Downarrow{\scriptstyle \alpha P} & \big\downarrow{\scriptstyle f'} \\ I & \xrightarrow[WP]{} & J \end{array}$$

is the map $P\alpha\colon VP \to WP$ which is equal to α as maps of underlying sets $V \to W$.

Now we will show that the natural transformations $P_\odot$ and P_U are both identities. We'll show this for $P_\odot$, but not for P_U as this is immediate from the definitions of U_G in $\mathbb{D}\mathbf{bl}(\mathsf{Rep})$ and in $\mathbb{B}\mathbf{imod}$. Suppose that G, H and K are finite groups, that V is a $(G\times H)$-module and that W is an $(H\times K)$-module. Let

$$X = \Big\langle \Big((e_G,h)\cdot v \otimes (h,e_K)\cdot w\Big) - v\otimes w \mid h\in H, v\in V, w\in W \Big\rangle \leq V\otimes W,$$

and let

$$Y = \Big\langle \Big((e_G,h^{-1})\cdot v \otimes w\Big) - \Big(v\otimes (h,e_K)\cdot w\Big) \mid h\in H, v\in V, w\in W \Big\rangle \leq V\otimes W.$$

The underlying vector space of $VP\odot WP$ is $(V\otimes W)/Y$ and the underlying vector space of $(V\odot W)P$ is $(V\otimes W)/X$, which are equal since $X = Y$. It's easy to see that $VP\odot WP$ and $(V\odot W)P$ have the same left G- and right K-actions.

It's very easy to define an inverse Q to $P\colon \mathbb{D}\mathbf{bl}(\mathsf{Rep}) \to \mathbb{B}\mathbf{imod}$. As with P, the functor of vertical categories $\mathbb{B}\mathbf{imod}_0 \to \mathbb{D}\mathbf{bl}(\mathsf{Rep})_0$ is the identity. For a loose 1-cell V in $\mathbb{B}\mathbf{imod}$ (i.e. a (G,H)-bimodule), the loose 1-cell $VQ\colon G \nrightarrow H$ in $\mathbb{D}\mathbf{bl}(\mathsf{Rep})$ is the $(G\times H)$-module with underlying vector space V, with left G-action $(g,e_H)\cdot := g\cdot v$, and with left H-action $(e_G,h)\cdot v := v\cdot h^{-1}$. Again as with P, for any 2-cell $\alpha\colon V \to W$ in $\mathbb{B}\mathbf{imod}$, the 2-cell $\alpha Q\colon VQ \to WQ$ is equal to α as maps of underlying sets $V \to W$. Lastly, the coherence natural transformations $Q_\odot$ and Q_U are identities. ◇

The following is stated by Shulman as part of the theorem defining the above construction. We don't study this part of the result in this thesis, but it is important to mention; specifying all of the data and checking all of the axioms in order to define a monoidal bicategories is a laborious task, whereas specifying the data and checking the axioms for a monoidal bifibrations is relatively simple.

Theorem 4.3.5 ([Shu08, Theorem 14.2])**.** *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a doublable monoidal bifibration. If Φ is symmetric monoidal, then $\mathbb{D}\mathbf{bl}(\Phi)$ is a symmetric monoidal fibrant double category.*

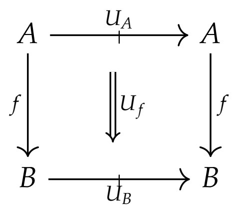
The remainder of this section is a detailed account of how Shulman defines the remaining data needed to define the double category $\mathbb{D}\mathbf{bl}(\Phi)$.

4.3.1 Definition of U

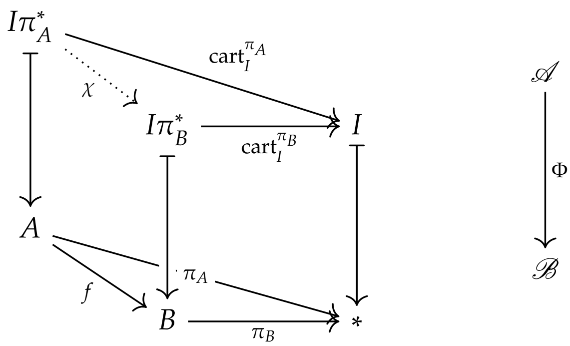
We define the functor $U\colon \mathbb{D}_0 \to \mathbb{D}_1$. For each 0-cell A in $\mathbb{D}$, define

$$U_A = I\pi_A^*\Delta_{A!}.$$

For each tight 1-cell $f\colon A \to B$ in $\mathbb{D}$, define the 2-cell

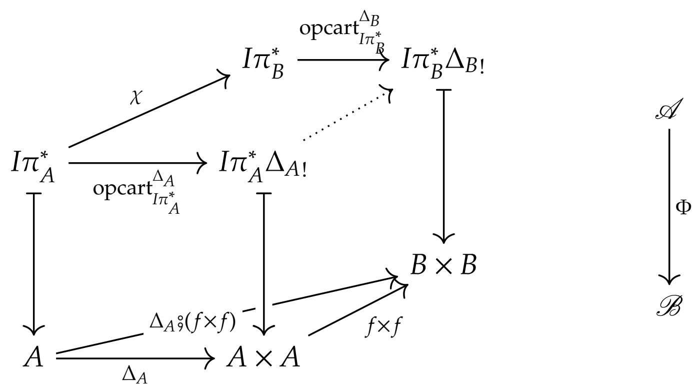


as follows. Define the morphism $\chi\colon I\pi_A^* \to I\pi_B^*$ in $\mathscr{A}$ using cartesian factorisation as shown in the following figure.



(4.3.6)

Define the morphism $U_f\colon I\pi_A^*\Delta_{A!} \to I\pi_B^*\Delta_{B!}$ in $\mathscr{A}$ using opcartesian factorisation as shown in the following figure.



(4.3.7)

Notice that $\chi \mathbin{⨾} \mathrm{opcart}_{I\pi_B^*}^{\Delta_B}$ does indeed lie over $\Delta_A \mathbin{⨾} (f \times f)$ because the following diagram in $\mathscr{B}$ commutes.

$$\begin{array}{ccc}
A & \xrightarrow{\quad f \quad} & B \\
{\scriptstyle \Delta_A}\big\downarrow & & \big\downarrow{\scriptstyle \Delta_B} \\
A \times A & \xrightarrow[f \times f]{} & B \times B
\end{array}$$

4.3.2 Definition of ⊙

We'll define the functor $\odot \colon \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \to \mathbb{D}_1$.

For each pair $M \colon A \nrightarrow B$ and $N \colon B \nrightarrow C$ of loose-composable loose 1-cells in $\mathbb{D}$, define

$$M \odot N = (M \otimes N)\Delta_B^* \pi_{B!}.$$

For each pair

$$\begin{array}{ccc}
A & \xrightarrow{M} & B \\
{\scriptstyle f}\big\downarrow & \Downarrow{\scriptstyle \alpha} & \big\downarrow{\scriptstyle g} \\
X & \xrightarrow[Q]{} & Y
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
B & \xrightarrow{N} & C \\
{\scriptstyle g}\big\downarrow & \Downarrow{\scriptstyle \beta} & \big\downarrow{\scriptstyle h} \\
Y & \xrightarrow[R]{} & Z.
\end{array}$$

of loose-composable 2-cells in $\mathbb{D}$, define the 2-cell

$$\begin{array}{ccc}
A & \xrightarrow{M \odot N} & C \\
{\scriptstyle f}\big\downarrow & \Downarrow{\scriptstyle \alpha \odot \beta} & \big\downarrow{\scriptstyle h} \\
X & \xrightarrow[Q \odot R]{} & Z
\end{array}$$

as follows. The diagram

$$\begin{array}{ccccc}
M \otimes N & \xleftarrow{\mathrm{cart}_{M \otimes N}^{\Delta_B}} & (M \otimes N)\Delta_B^* & \xrightarrow{\mathrm{opcart}_{(M \otimes N)\Delta_B^*}^{\pi_B}} & (M \otimes N)\Delta_B^* \pi_{B!} = M \odot N \\
{\scriptstyle \alpha \otimes \beta}\big\downarrow & & & & \\
Q \otimes R & \xleftarrow[\mathrm{cart}_{Q \otimes R}^{\Delta_Y}]{} & (Q \otimes R)\Delta_Y^* & \xrightarrow[\mathrm{opcart}_{(Q \otimes R)\Delta_Y^*}^{\pi_Y}]{} & \pi_{Y!}\Delta_Y^*(Q \otimes R) = Q \odot R
\end{array}$$

in $\mathscr{A}$ lies over the diagram

$$\begin{array}{ccccc}
ABBC & \xleftarrow{\Delta_B} & ABC & \xrightarrow{\pi_B} & AC \\
{\scriptstyle f \times g \times g \times h}\big\downarrow & & & & \\
XYYZ & \xleftarrow[\Delta_Y]{} & XYZ & \xrightarrow[\pi_Y]{} & XZ
\end{array}$$

in $\mathscr{B}$, and the diagram

$$\begin{array}{ccccc} ABBC & \xleftarrow{\Delta_B} & ABC & \xrightarrow{\pi_B} & AC \\ \downarrow{\scriptstyle f\times g\times g\times h} & & \downarrow{\scriptstyle f\times g\times h} & & \downarrow{\scriptstyle f\times h} \\ XYYZ & \xleftarrow[\Delta_Y]{} & XYZ & \xrightarrow[\pi_Y]{} & XZ \end{array}$$

in $\mathscr{B}$ commutes. As shown in the figure below, we define χ to be the unique morphism that fills the niche (1) and lies over $f \times g \times h$, and we define $\alpha \odot \beta$ to be the unique morphism that fills the niche (2) and lies over $f \times h$.

$$\begin{array}{ccccc} M \otimes N & \xleftarrow{\mathrm{cart}^{\Delta_B}_{M\otimes N}} & (M \otimes N)\Delta_B^* & \xrightarrow{\mathrm{opcart}^{\pi_B}_{(M\otimes N)\Delta_B^*}} & (M \otimes N)\Delta_B^*\pi_{B!} = M \odot N \\ \downarrow{\scriptstyle \alpha\otimes\beta} & (1) & \downarrow{\scriptstyle \chi} & (2) & \downarrow{\scriptstyle \alpha\odot\beta} \\ Q \otimes R & \xleftarrow[\mathrm{cart}^{\Delta_Y}_{Q\otimes R}]{} & (Q \otimes R)\Delta_Y^* & \xrightarrow[\mathrm{opcart}^{\pi_Y}_{(Q\otimes R)\Delta_Y^*}]{} & (Q \otimes R)\Delta_Y^*\pi_{Y!} = Q \odot R \end{array}$$

4.3.3 Definition of $a^{\odot}$

For each triple $M\colon A \nrightarrow B$, $N\colon B \nrightarrow C$ and $Q\colon C \nrightarrow D$ of composbale loose 1-cells, we will define the associator

$$\begin{array}{ccc} A & \xrightarrow{(M\odot N)\odot Q} & D \\ \Big\| {\scriptstyle \mathrm{id}_A} & \Downarrow a^{\odot}_{MNQ} & \Big\| {\scriptstyle \mathrm{id}_D} \\ A & \xrightarrow[M\odot(N\odot Q)]{} & D \end{array}$$

as follows.

We begin by defining an opcartesian morphism χ and a cartesian morphism ψ to form the niche

$$\begin{array}{ccc} (M \odot N) \otimes Q & \xleftarrow{\mathrm{opcart}^{\pi_B}_{(M\otimes N)\Delta_B^*}\otimes Q} & (M \otimes N)\Delta_B^* \otimes Q \\ \uparrow{\scriptstyle \mathrm{cart}^{\Delta_C}_{(M\odot N)\otimes Q}} & & \uparrow{\scriptstyle \mathrm{cart}^{\Delta_C ⨾ \Delta_B}_{(M\otimes N)\otimes Q} \circ\!\!/ (\mathrm{cart}^{\Delta_B}_{(M\otimes N)}\otimes Q)} \\ ((M \odot N) \otimes Q)\Delta_C^* & & ((M \otimes N) \otimes Q)(\Delta_C ⨾ \Delta_B)^* \end{array} \qquad (4.3.8)$$

in $\mathscr{A}$ which lies over the niche

$$\begin{array}{ccc} A\times C\times C\times D & \xleftarrow{\pi_B} & A\times B\times C\times C\times D \\ \uparrow{\scriptstyle \Delta_C} & & \uparrow{\scriptstyle \Delta_C} \\ A\times C\times D & & A\times B\times C\times D \end{array}$$

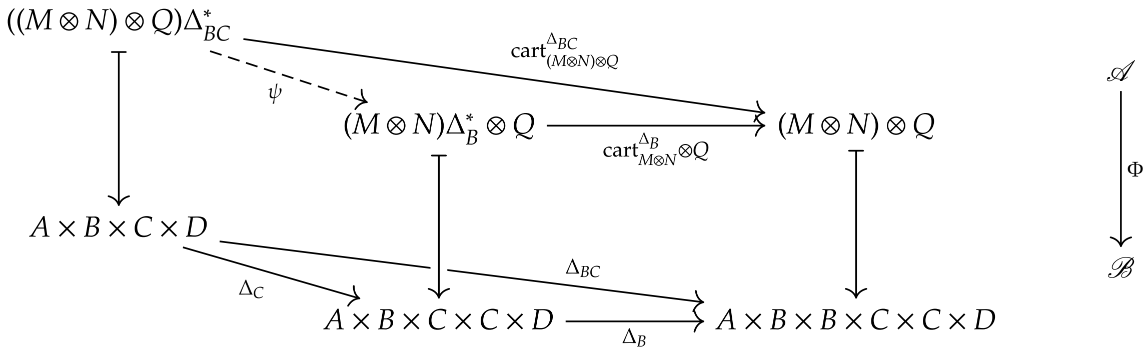
in $\mathscr{B}$. Define χ by

$$\chi = \mathsf{opcart}^{\pi_B}_{(M\otimes N)\Delta_B^*} \otimes Q \colon (M\otimes N)\Delta_B^* \otimes Q \longrightarrow (M\otimes N)\Delta_B^*\pi_{B!} \otimes Q = (M\odot N)\otimes Q$$

and define ψ by

$$\mathsf{cart}^{\Delta_C ⨾ \Delta_B}_{(M\otimes N)\otimes Q} \mathbin{/\!\!\circ} (\mathsf{cart}^{\Delta_B}_{M\otimes N} \otimes Q) \colon ((M\otimes N)\otimes Q)(\Delta_C ⨾ \Delta_B)^* \longrightarrow (M\otimes N)\Delta_B^* \otimes Q$$

as shown in the following figure.



The morphism χ is opcartesian because the functor $- \otimes Q$ preserves opcartesian morphisms. The morphism ψ is cartesian because the morphisms $\mathsf{cart}^{\Delta_C ⨾ \Delta_B}_{(M\otimes N)\otimes Q}$ and $\mathsf{cart}^{\Delta_B}_{M\otimes N} \otimes Q$ are both cartesian, which is because the functor $- \otimes Q$ preserves cartesian morphisms. So we have the niche (4.3.8) that we sought.

The square

$$\begin{array}{ccc} A\times C\times C\times D & \xleftarrow{\pi_B} & A\times B\times C\times C\times D \\ \uparrow{\scriptstyle \Delta_C} & & \uparrow{\scriptstyle \Delta_C} \\ A\times C\times D & \xleftarrow[\pi_B]{} & \ulcorner\, A\times B\times C\times D \end{array}$$

in $\mathscr{B}$ commutes and satisfies the Beck-Chevalley condition since one of its legs is a product projection and the bifibration Φ is at least weakly Beck-Chevalley. Let ξ denote the unique morphism that fills the niche (4.3.8) and that lies over π_B. By Corollary 3.4.4, the morphism ξ is opcartesian.

Let ω_1 denote the opcartesian morphism

$$\omega_1 = \xi ⨾ \mathsf{opcart}^{\pi_C}_{((M\odot N)\otimes Q)\Delta_C^*} \colon ((M\otimes N)\otimes Q)(\Delta_C ⨾ \Delta_B)^* \longrightarrow (M\odot N)\odot Q.$$

We can similarly construct an opcartesian morphism

$$\omega_2 \colon ((M \otimes N) \otimes Q)(\Delta_C \mathbin{⨟} \Delta_B)^* \longrightarrow M \odot (N \odot Q).$$

Define the morphism $a^{\odot}_{M,N,Q} \colon (M \odot N) \odot Q \to M \odot (N \odot Q)$ in $\mathscr{A}$ using opcartesian factorisation as shown in the following figure.

$$\begin{array}{ccccc}
(M \otimes N) \otimes Q & \xleftarrow{\mathrm{cart}^{\Delta_{BC}}_{(M\otimes N)\otimes Q}} & ((M \otimes N) \otimes Q)\Delta^*_{BC} & \xrightarrow{\omega_1} & (M \odot N) \odot Q \\
\downarrow{\scriptstyle a^{\otimes}_{M,N,Q}} & & \downarrow{\scriptstyle (a^{\otimes}_{M,N,Q})\Delta^*_{BC}} & & \vdots{\scriptstyle a^{\odot}_{M,N,Q}} \\
M \otimes (N \otimes Q) & \xleftarrow[\mathrm{cart}^{\Delta_{BC}}_{M\otimes(N\otimes Q)}]{} & (M \otimes (N \otimes Q))\Delta^*_{BC} & \xrightarrow[\omega_2]{} & M \odot (N \odot Q)
\end{array}$$

Notice that $a^{\odot}_{M,N,Q}$ is an isomorphism since it is pure and opcartesian.

4.3.4 Definition of $l^{\odot}_M$

For each loose 1-cell $M \colon A \nrightarrow B$ in $\mathbb{D}$, we will define a 2-cell

$$\begin{array}{ccc}
A & \xrightarrow{M} & B \\
\Vert{\scriptstyle \mathrm{id}_A} & \Downarrow {l^{\odot}_M}^{-1} & \Vert{\scriptstyle \mathrm{id}_D} \\
A & \xrightarrow[U_A \odot M]{} & B
\end{array}$$

We will then show that it is invertible and define the left loose composition unitor $l^{\odot}_M$ to be its inverse. The right loose composition unitor $r^{\odot}_M$ is defined similarly.

We begin by defining an opcartesian morphism χ and a cartesian morphism ψ to form the niche

$$\begin{array}{ccc}
M & & (U_A \otimes M)\Delta^*_A \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \mathrm{cart}^{\Delta_A}_{U_A \otimes M}} \\
I\pi^*_A \otimes M & \xrightarrow[\chi]{} & U_A \otimes M
\end{array} \tag{4.3.9}$$

in $\mathscr{A}$ which lies over the niche

$$\begin{array}{ccc}
A \times B & & A \times A \times B \\
\downarrow{\scriptstyle \Delta_A \times \mathrm{id}_B} & & \downarrow{\scriptstyle \mathrm{id}_A \times \Delta_A \times \mathrm{id}_B} \\
A \times A \times B & \xrightarrow[\Delta_A \times \mathrm{id}_{A \times B}]{} & A \times A \times A \times B
\end{array}$$

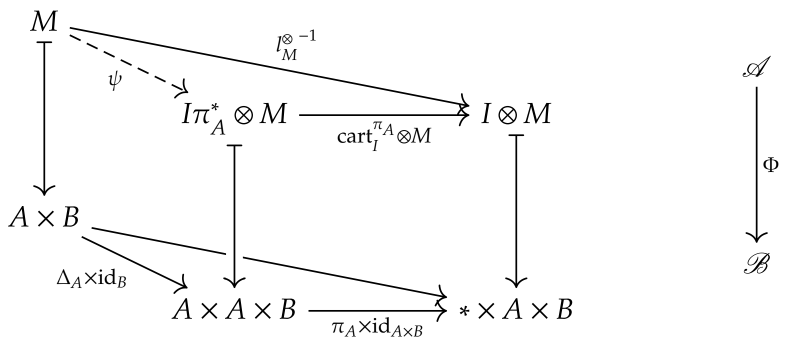
in $\mathscr{B}$. Define χ by

$$\chi = \mathrm{opcart}^{\Delta_A}_{I\pi^*_A} \otimes M \colon I\pi^*_A \otimes M \longrightarrow I\pi^*_A \Delta_{A!} \otimes M = U_A \otimes M$$

and define ψ by

$$\psi = l^{\otimes\,-1}_M \mathbin{⨾} \mathrm{cart}^{\pi_A}_I \otimes M \colon M \longrightarrow I\pi^*_A \otimes M$$

as shown in the following figure.



The morphism χ is opcartesian because the functor $- \otimes M$ preserves opcartesian morphisms. The morphism ψ is cartesian because the morphisms $l^{\otimes\,-1}_M$ and $\mathrm{cart}^{\pi_A}_I \otimes M$ are both cartesian, which is because the functor $- \otimes M$ preserves cartesian morphisms. So we have the niche (4.3.9) that we sought.

The square

$$\begin{array}{ccc} A \times A \times A \times B & \xleftarrow{\Delta_A \times \mathrm{id}_{A\times B}} & A \times A \times B \\ {\scriptstyle \mathrm{id}_A \times \Delta_A \times \mathrm{id}_B}\uparrow & & \uparrow{\scriptstyle \Delta_A \times \mathrm{id}_B} \\ A \times A \times B & \xleftarrow[\Delta_A \times \mathrm{id}_B]{} & A \times B \end{array} \tag{4.3.10}$$

in $\mathscr{B}$ commutes. Let

$$\xi \colon M \longrightarrow (U_A \otimes M)\Delta^*_A \tag{4.3.11}$$

denote the unique morphism that fills the niche (4.3.9) and that lies over $\Delta_A \times \mathbf{id}_B$. Define the morphism $l^{\odot\,-1}_M \colon M \to U_A \odot M$ in $\mathscr{A}$ to be the following composite.

$$M \xrightarrow{\ \xi\ } (U_A \otimes M)\Delta^*_A \xrightarrow{\mathrm{opcart}^{\pi_A}_{(U_A \otimes M)\Delta^*_A}} (U_A \otimes M)\Delta^*_A \pi_{A!} = U_A \odot M$$

4.3.5 Proof that $l^\odot$ is an isomorphism

4.3.5.1 Case (1): Φ is Beck-Chevalley

The square (4.3.10) is a pullback square and therefore satisfies the Beck-Chevalley condition since Φ is Beck-Chevalley. So, by Corollary 3.4.4, the morphism ξ (see (4.3.11)) is opcartesian. Therefore, $l^\odot_M$ is an isomorphism since it is pure and opcartesian.

4.3.5.2 Case (2): Φ is weakly Beck-Chevalley and internally closed

The niche

$$\begin{array}{ccc} M & & (U_A \otimes M)\Delta_A^* \\ \Big\downarrow{\scriptstyle \psi} & & \Big\downarrow{\scriptstyle \mathrm{cart}^{\Delta_A}_{U_A \otimes M}} \\ I\pi_A^* \otimes M & \xrightarrow[\chi]{} & U_A \otimes M \end{array}$$

in $\mathscr{A}$ (see (4.3.9)) doesn't lie over a pullback square in $\mathscr{B}$ that has a product projection as one of its legs, and so, since Φ is only weakly Beck-Chevalley in this case, we can't use Corollary 3.4.4 to show that the morphism ξ (see (4.3.11)) is opcartesian. Instead, we'll use cartesian factorisation to obtain a niche in $\mathscr{A}$ that lies over the niche

$$\begin{array}{ccc} A \times B & & A \times A \times B \\ \Big\downarrow{\scriptstyle \Delta_{A\times B}} & & \Big\downarrow{\scriptstyle \Delta_{A\times A\times B}} \\ A \times B \times A \times B & \xrightarrow[\Delta_A \times \mathrm{id}_B \times \Delta_A \times \mathrm{id}_B]{} & A \times A \times B \times A \times A \times B \end{array}$$

in $\mathscr{B}$, because we can then use Corollary 2.8.9.

Firstly, we will construct a niche

$$\begin{array}{ccc} I\pi_A^*\pi_B^* \otimes M & \xrightarrow{\psi_1 \otimes \psi_2} & U_A\pi_B^* \otimes M\pi_A^* \\ \Big\downarrow{\scriptstyle \mathrm{cart}} & & \Big\downarrow{\scriptstyle \mathrm{cart}} \\ I\pi_A^* \otimes M & \xrightarrow[\chi]{} & U_A \otimes M \end{array}$$

in $\mathscr{A}$, such that the morphisms labelled cart are cartesian and where χ is as in Section 4.3.4.

The niches

$$\begin{array}{ccc} I\pi_A^*\pi_B^* & & U_A\pi_B^* \\ \Big\downarrow{\scriptstyle \mathrm{cart}^{\pi_B}_{I\pi_A^*}} & & \Big\downarrow{\scriptstyle \mathrm{cart}^{\pi_B}_{U_A}} \\ I\pi_A^* & \xrightarrow[\mathrm{opcart}^{\Delta_A}_{I\pi_A^*}]{} & I\pi_A^*\Delta_{A!} = U_A \end{array} \quad \text{and} \quad \begin{array}{ccc} M & & M\pi_A^* \\ \Big\| & & \Big\downarrow{\scriptstyle \mathrm{cart}^{\pi_A}_{M}} \\ M & = & M \end{array} \tag{4.3.12}$$

in $\mathscr{A}$ lie over the niches

$$\begin{array}{ccc} A \times B & & A \times A \times B \\ \Big\downarrow{\scriptstyle \pi_B} & & \Big\downarrow{\scriptstyle \pi_B} \\ A & \xrightarrow[\Delta_A]{} & A \times A \end{array} \quad \text{and} \quad \begin{array}{ccc} A \times B & & A \times A \times B \\ \Big\| & & \Big\downarrow{\scriptstyle \pi_B} \\ A \times B & \xrightarrow[\Delta_A]{} & A \times B \end{array}$$

in $\mathscr{B}$, and the squares

$$\begin{array}{ccc} A\times B & \xrightarrow{\Delta_A\times \mathrm{id}_B} & A\times A\times B \\ {\scriptstyle \pi_B}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_B} \\ A & \xrightarrow[\Delta_A]{} & A\times A \end{array} \quad \text{and} \quad \begin{array}{ccc} A\times B & \xrightarrow{\Delta_A\times \mathrm{id}_B} & A\times A\times B \\ \| & & \downarrow{\scriptstyle \pi_B} \\ A\times B & \xrightarrow[\Delta_A]{} & A\times B \end{array} \tag{4.3.13}$$

in $\mathscr{B}$ commute. Let $\psi_1 \colon I\pi_A^*\pi_B^* \to U_A\pi_B^*$ denote the unique morphism that fills the left niche in (4.3.12) and lies over $\Delta_A \times \mathrm{id}_B$, and let $\psi_2 \colon M \to M\pi_A^*$ denote the unique morphism that fills the right niche in (4.3.12) and lies over $\Delta_A \times \mathrm{id}_B$. We can use Corollary 3.4.4 to get that the morphism ψ_1 is opcartesian since the left square in (4.3.13) is a pullback square with one of its legs a product projection and the bifibration Φ is weakly Beck-Chevalley. The morphism ψ_2 is cartesian by Proposition 2.1.24.

The monoidal product of two cartesian morphisms is cartesian and the monoidal product of two opcartesian morphisms is opcartesian, and here we'll write cart and opcart for the cartesian and opcartesian morphisms obtained this way. The following diagram shows the result of taking the monoidal product of the two niches in (4.3.12); this is the niche we sought.

$$\begin{array}{ccc} I\pi_A^*\pi_B^*\otimes M & \xrightarrow{\psi_1\otimes\psi_2} & U_A\pi_B^*\otimes M\pi_A^* \\ {\scriptstyle \mathsf{cart}}\downarrow & & \downarrow{\scriptstyle \mathsf{cart}} \\ I\pi_A^*\otimes M & \xrightarrow[\chi]{} & U_A\otimes M \end{array} \tag{4.3.14}$$

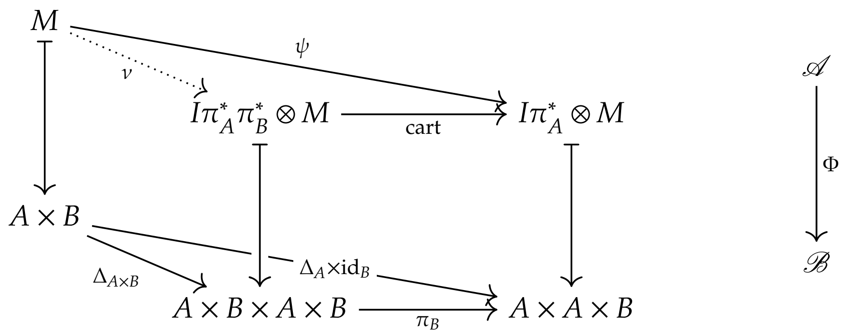
Now, we use these two cartesian morphisms denoted cart to factorise the niche

$$\begin{array}{ccc} M & & (U_A\otimes M)\Delta_A^* \\ {\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \mathsf{cart}^{\Delta_A}_{U_A\otimes M}} \\ I\pi_A^*\otimes M & \xrightarrow[\chi]{} & U_A\otimes M \end{array}$$

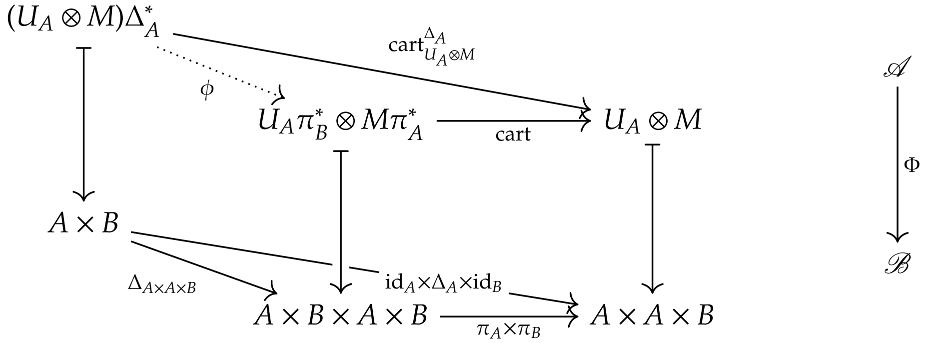
as

$$\begin{array}{ccc} M & & (U_A\otimes M)\Delta_A^* \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \phi} \\ I\pi_A^*\pi_B^*\otimes M & & U_A\pi_B^*\otimes M\pi_A^* \\ {\scriptstyle \mathsf{cart}}\downarrow & & \downarrow{\scriptstyle \mathsf{cart}} \\ I\pi_A^*\otimes M & \xrightarrow[\chi]{} & U_A\otimes M \end{array}$$

That is, the morphism ν is obtained by cartesian factorisation as shown in the figure



and the morphism ϕ is obtained by cartesian factorisation as shown in the figure



By Proposition 2.1.24, both ν and ϕ are cartesian.

The niche

$$\begin{array}{ccc} M & & (U_A\otimes M)\Delta_A^* \\ {\scriptstyle \nu}\downarrow & & \downarrow{\scriptstyle \phi} \\ I\pi_A^*\pi_B^*\otimes M & \xrightarrow[\psi_1\otimes\psi_2]{} & U_A\pi_B^*\otimes M\pi_A^* \end{array} \tag{4.3.15}$$

in $\mathscr{A}$ lies over the niche

$$\begin{array}{ccc} A\times B & & A\times A\times B \\ {\scriptstyle \Delta_{A\times B}}\downarrow & & \downarrow{\scriptstyle \Delta_{A\times A\times B}} \\ A\times B\times A\times B & \xrightarrow[\Delta_A\times\mathrm{id}_B\times\Delta_A\times\mathrm{id}_B]{} & A\times A\times B\times A\times A\times B \end{array}$$

in $\mathscr{B}$, and the square

$$\begin{array}{ccc} A\times B & \xrightarrow{\Delta_A\times\mathrm{id}_B} & A\times A\times B \\ {\scriptstyle\Delta_{AB}}\downarrow & & \downarrow{\scriptstyle\Delta_{AAB}} \\ A\times B\times A\times B & \xrightarrow[\Delta_A\times\mathrm{id}_B\times\Delta_A\times\mathrm{id}_B]{} & A\times A\times B\times A\times A\times B \end{array}$$

in $\mathscr{B}$ commutes. By Corollary 2.8.9, the unique morphism $\rho\colon M \to (U_A\otimes M)\Delta_A^*$ that fills the niche (4.3.15) and lies over $\Delta_A\times\mathbf{id}_B$ is opcartesian. Since ρ fills the niche (4.3.15) and the square (4.3.14) commute, ρ also fills the niche

$$\begin{array}{ccc} M & & (U_A\otimes M)\Delta_A^* \\ {\scriptstyle\nu}\downarrow & & \downarrow{\scriptstyle\phi} \\ I\pi_A^*\pi_B^*\otimes M & & U_A\pi_B^*\otimes M\pi_A^* \\ {\scriptstyle\mathsf{cart}}\downarrow & & \downarrow{\scriptstyle\mathsf{cart}} \\ I\pi_A^*\otimes M & \xrightarrow[\chi]{} & U_A\otimes M \end{array}$$

in $\mathscr{A}$ and lies over $\Delta_A\times\mathbf{id}_B$. But ξ is the unique such morphism, so $\xi=\rho$, and so ξ is opcartesian. Therefore, $l_M^{\odot}$ is an isomorphism since it is pure and opcartesian.

4.3.6 Proof that $\mathbb{D}\mathbf{bl}(\Phi)$ is a fibrant double category

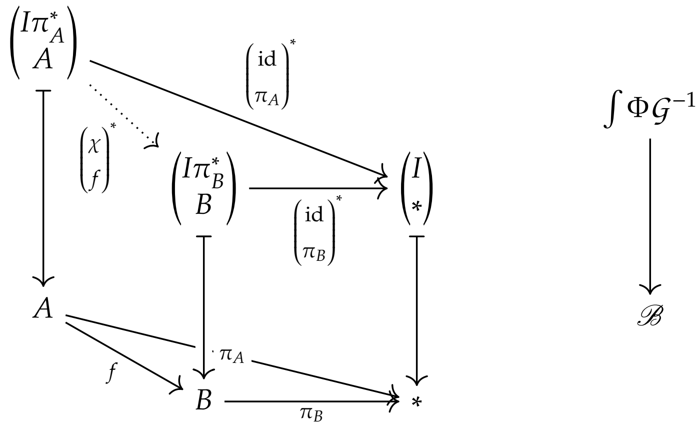
The functor (S,T) is the pullback of Φ along the product of $\mathscr{B}$, and it's easy to check that the pullback of a bifibration is a bifibration, so (S,T) is a bifibration.

4.4 Definition of $\mathbb{D}\mathbf{bl}(\Phi)$ using string diagrams

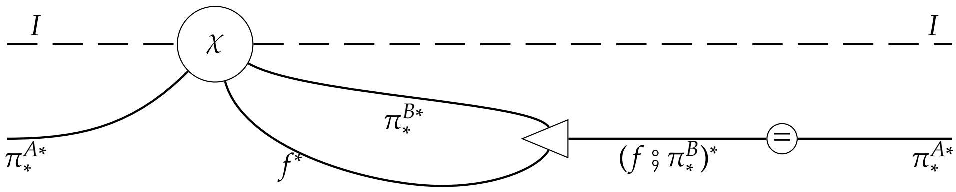
In this section, we have the main results of this thesis: the explicit calculations—in string diagrammatic language—of the unit loose 1-cell U_f and the (left) unitor for loose composition, $l^{\odot}$.

4.4.1 Definition of U using string diagrams

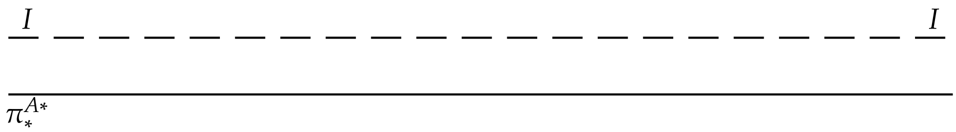
Recall the definition of U_f from Section 4.3.1. Rewrite (4.3.6) as

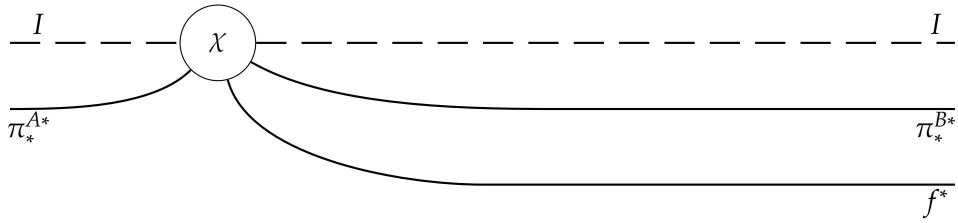


Then χ is the unique morphism satisfying



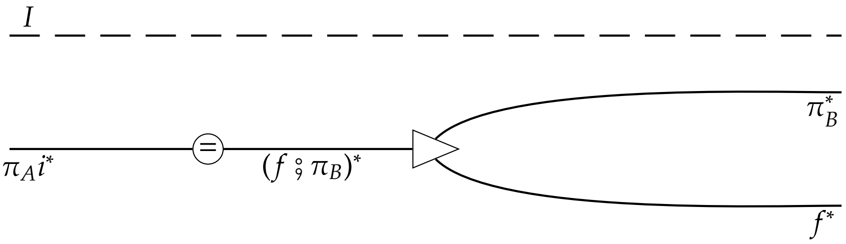
=



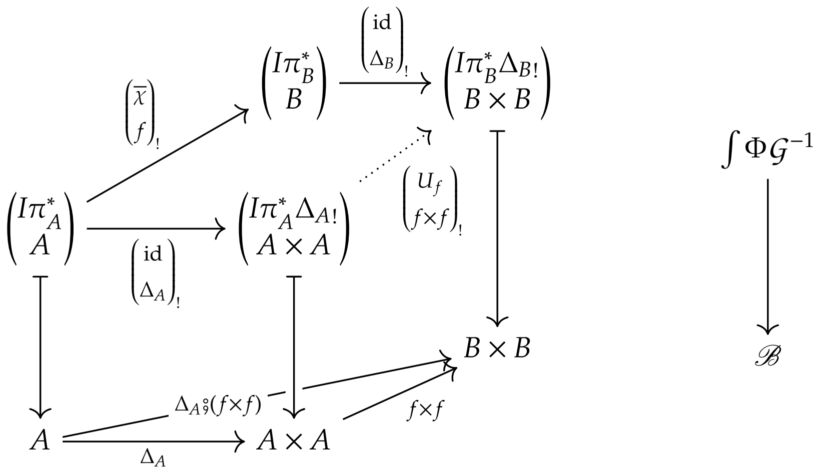


(4.4.1)

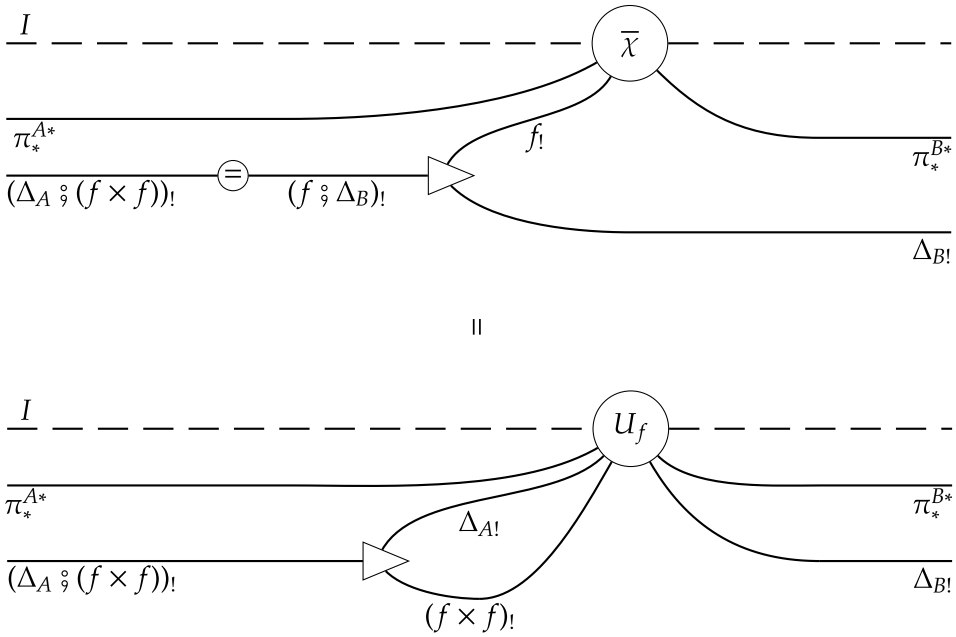
=



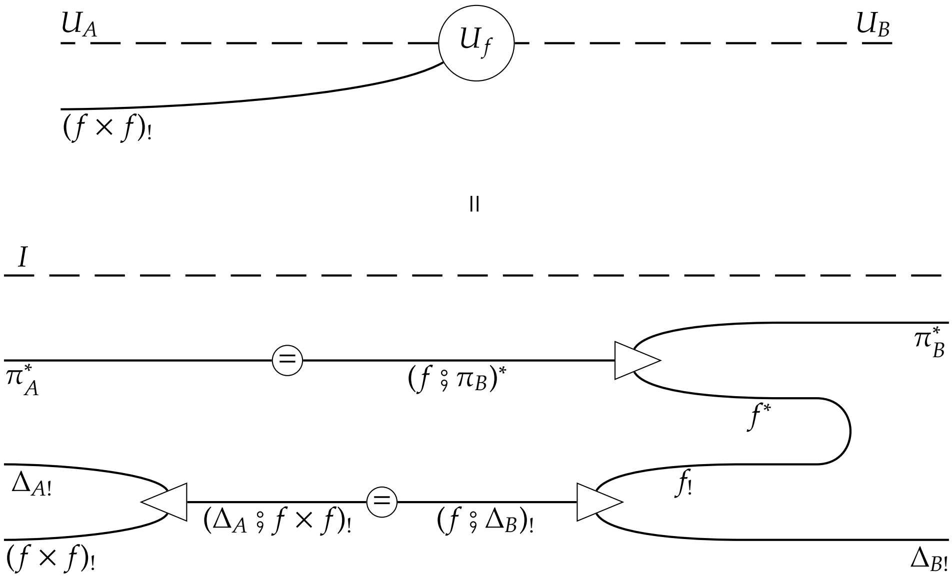
Rewrite (4.3.7) as



Then U_f is the unique morphism satisfying



Therefore, by substituting (4.4.1) into (4.4.1) and rearranging, we get



4.4.2 Definition of $l^{\odot}$ using string diagrams

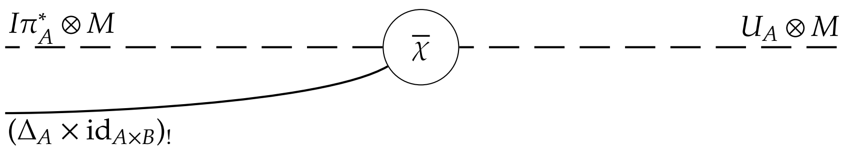
We follow the definition of the unitor $M \to U_A \odot M$ from Section 4.3.4. The niche (4.3.9) in $\mathscr{A}$ is written using indexed category notation as the niche

$$\begin{array}{ccc} U_A \otimes M & \xleftarrow{\begin{pmatrix} \overline{\chi} \\ \Delta_A \times \mathrm{id} A \times B \end{pmatrix}_!} & I\pi^*_A \otimes M \\ \Big\uparrow {\scriptstyle \begin{pmatrix} \mathrm{id} \\ \mathrm{id}_A \times \Delta_A \times \mathrm{id}_B \end{pmatrix}^*} & & \Big\uparrow {\scriptstyle \begin{pmatrix} \psi \\ \Delta_A \times \mathrm{id}_B \end{pmatrix}^*} \\ (U_A \otimes M)\Delta^*_A & & M \end{array} \tag{4.4.2}$$

Since the functor $- \otimes M$ presreves opcartesian morphisms, the morphism

$$\chi = \mathrm{opcart}^{\Delta_A}_{I\pi^*_A} \otimes M \colon I\pi^*_A \otimes M \longrightarrow I\pi^*_A \Delta_{A!} \otimes M = U_A \otimes M$$

is opcartesian, and so there exists a pure isomorphism



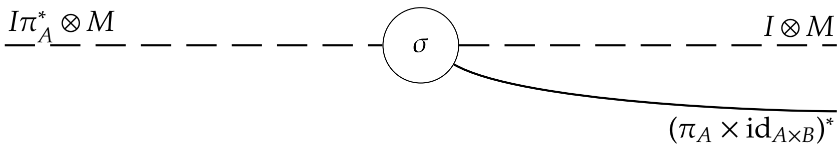
in $\mathscr{A}$ such that

$$\chi = \begin{pmatrix} \overline{\chi} \\ \Delta_A \times \mathrm{id}_{A\times B} \end{pmatrix}_! : I\pi_A^* \otimes M \longrightarrow I\pi_A^*\Delta_{A!} \otimes M = U_A \otimes M.$$

Since the functor $-\otimes M$ presreves cartesian morphisms, the morphism

$$\mathrm{cart}_I^{\pi_A} \otimes M \colon I\pi_A^* \otimes M \longrightarrow I \otimes M$$

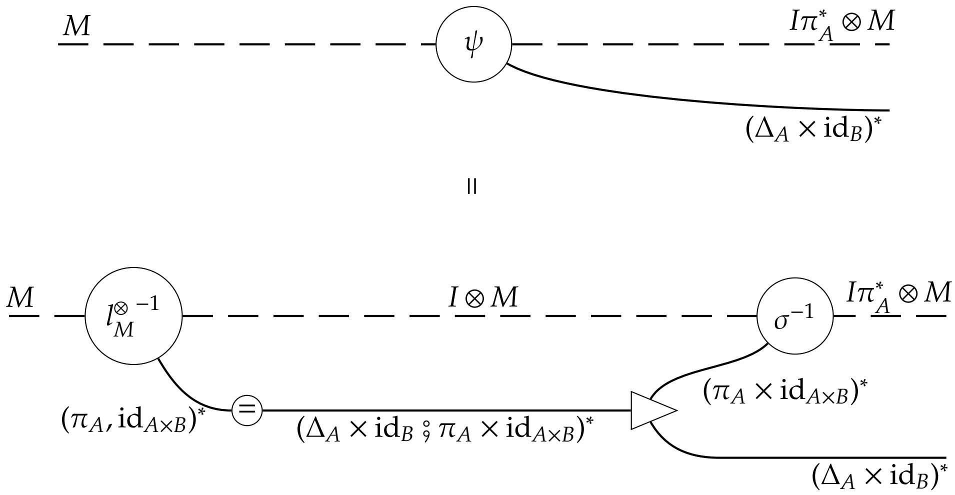
is cartesian, and so there exists a pure isomorphism



in $\mathscr{A}$ such that

$$\mathrm{cart}_I^{\pi_A} \otimes M = \begin{pmatrix} \sigma \\ \pi_A \times \mathrm{id}_{A\times B} \end{pmatrix}^* : I\pi_A^* \otimes M \longrightarrow I \otimes M.$$

With this, we can write the morphism ψ as



Using Remark 3.4.7, we can depict the the total part of the morphism

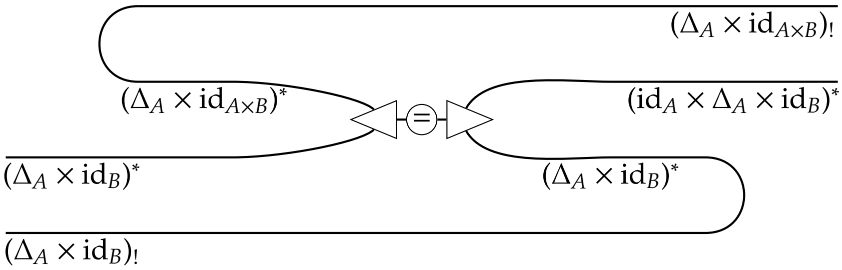
$$\begin{pmatrix} \overline{\xi} \\ \Delta_A \times \mathrm{id}_B \end{pmatrix} : M \longrightarrow (U_A \otimes M)\Delta_A^*$$

that fills the niche (4.4.2) using a string diagram: this diagram is shown in Figure 4.1. The morphism $l_M^{\odot\,-1}$ is given by

$$l_M^{\odot\,-1} = \begin{pmatrix} \overline{\xi} \\ \Delta_A \times \mathrm{id}_B \end{pmatrix} ⨾ \begin{pmatrix} \eta \\ \pi_A^{AAB} B \end{pmatrix} \longrightarrow M \to (U_A \otimes M)(\mathrm{id}_A \times \Delta_A \times \mathrm{id}_B)^* \pi_{AB!}^{AAB} = U_A \odot M.$$

The total part of ${l_M^{\odot}}^{-1}$ is depicted using a string diagram in Figure 4.2.

In Section 4.3.5.1 we proved that if the fibration $\Phi\colon \mathscr{A} \to \mathscr{B}$ is Beck-Chevalley then this morphism, ${l_M^{\odot}}^{-1}$, is an isomorphism. In Figure 4.3 we show the total part of ${l_M^{\odot}}^{-1}$ again, but this time rearranged and with the Beck-Chevalley transformation



associated to the square

$$\begin{array}{ccc} A\times A\times A\times B & \xleftarrow{\Delta_A\times \mathrm{id}_{A\times B}} & A\times A\times B \\ \uparrow{\scriptstyle \mathrm{id}_A\times\Delta_A\times\mathrm{id}_B} & & \uparrow{\scriptstyle \Delta_A\times\mathrm{id}_B} \\ A\times A\times B & \xleftarrow[\Delta_A\times\mathrm{id}_B]{} & A\times B\end{array}$$

shown in a dotted box; this shows at a glance that if the fibration $\Phi\colon \mathscr{A} \to \mathscr{B}$ is Beck-Chevalley then the morphism ${l_M^{\odot}}^{-1}$ is an isomorphism.

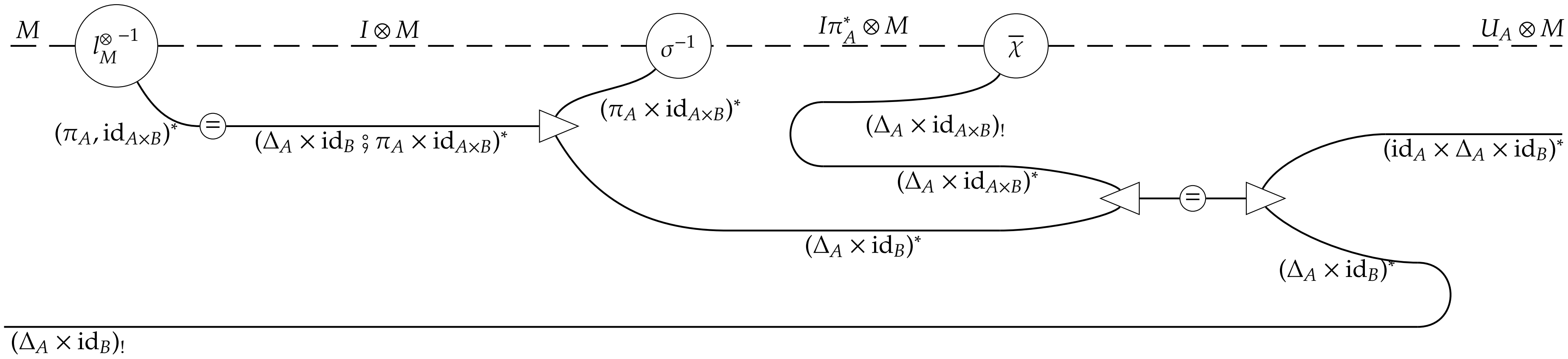


Figure 4.1: The total part of the morphism that fills the niche (4.4.2)

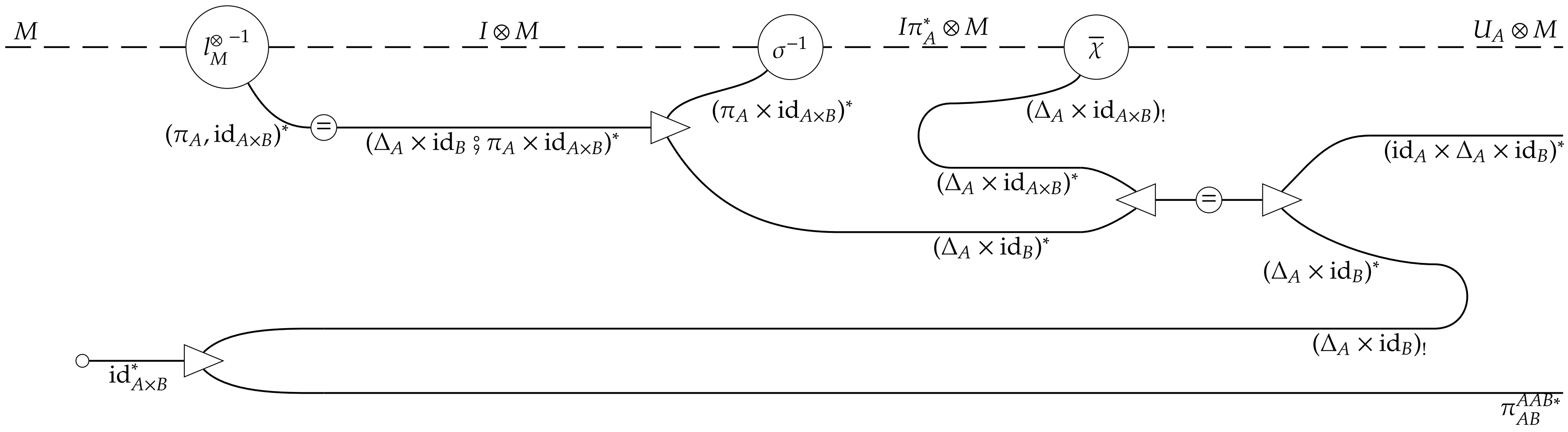


Figure 4.2: The total part of the morphism $l_M^{\odot\,-1}$

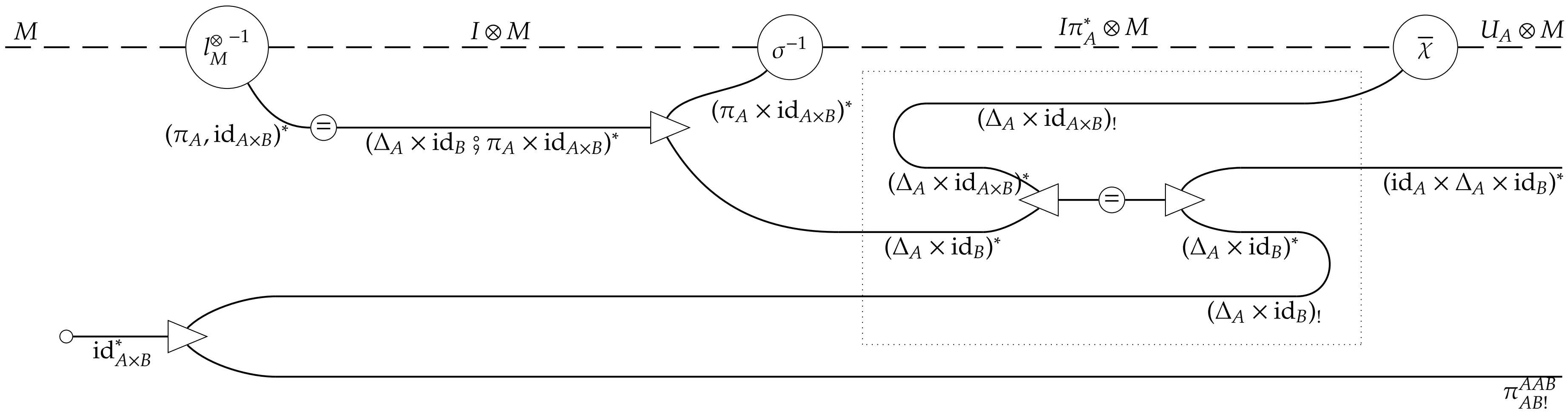


Figure 4.3: The total part of the morphism $l_M^{\odot\,-1}$, rearranged to show the Beck-Chevalley transformation

Bibliography

[BCV22] John Baez, Kenny Courser, and Christina Vasilakopoulou. Structured versus decorated cospans. *Compositionality*, 4, 2022.

[Bén65] Jean Bénabou. Catégories relatives. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences*, 260:3824–3827, 1965.

[Bén67] Jean Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, pages 1–77. Springer Berlin Heidelberg, 1967.

[Bén75] Jean Bénabou. Fibrations petites et localement petites. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences*, 281:A831–A834, 1975.

[Bor94a] Francis Borceux. *Handbook of categorical algebra: volume 1.* Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1994.

[Bor94b] Francis Borceux. *Handbook of categorical algebra: volume 2.* Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1994.

[Bou97] Serge Bouc. *Mackey functors.* Springer Berlin Heidelberg, 1997.

[BR70] Jean Bénabou and Jacques Roubaud. Monades et descente. *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences. Séries A et B*, 270:A96–A98, 1970.

[BS76] Ronald Brown and Christopher B Spencer. Double groupoids and crossed modules. *Cahiers de Topologie et Géométrie Différentielle*, 17(4), 1976.

[CJSV94] Aurelio Carboni, Scott Johnson, Ross Street, and Dominic Verity. Modulated bicategories. *Journal of Pure and Applied Algebra*, 94(3):229–282, 1994.

[Cou20] Kenny Courser. *Open systems: a double categorical perspective.* PhD thesis, University of California, Riverside, 2020. arXiv:2008.02394.

[DS97] Brian Day and Ross Street. Monoidal bicategories and hopf algebroids. *Advances in Mathematics*, 129(1):99–157, 1997.

[Ehr63] Charles Ehresmann. Catégories structurées. *Annales scientifiques de l'École Normale Supérieure*, 80(4):349–426, 1963.

[Fio07] T M Fiore. Pseudo algebras and pseudo double categories. *Journal of Homotopy and Related Structures*, 2(2):119–170, 2007.

[GP99] Marco Grandis and Robert Paré. Limits in double categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 40(3):162–220, 1999.

[GP04] Marco Grandis and Robert Paré. Adjoints for double categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 45(3):193–240, 2004.

[GP07] Marco Grandis and Robert Paré. Lax Kan extensions for double categories (On weak double categories, Part IV). *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 48(3):163–199, 2007.

[GP08] Marco Grandis and Robert Paré. Kan extensions in double categories (On weak double categories, Part III). *Theory and Applications of Categories*, 20(8):152–185, 2008.

[GP19] Marco Grandis and Robert Paré. Persistent double limits. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 60(3):255–297, 2019.

[GPS95] Robert Gordon, Anthony John Power, and Ross Street. *Coherence for tricategories*, volume 558. American Mathematical Society, 1995.

[Gra66] John W Gray. Fibred and cofibred categories. In *Proceedings of the conference on categorical algebra: La Jolla 1965*, pages 21–83. Springer, 1966.

[Gro60] Alexander Grothendieck. Technique de descente et théorèmes d'existence en géométrie algébrique. I. Généralités. Descente par morphismes fidèlement plats. In *Séminaire Bourbaki, Volume 5*, pages Exp. No. 190, 299–327. Société Mathématique de France, Paris, 1960.

[Gro71] Alexandre Grothendieck. Revêtement étales et groupe fondamental. *Lecture Note in Mathematics*, 224, 1971. Originally part of Séminaire de Géométrie Algébrique du Bois Marie in 1960–61.

[Gur06] Michael Nicholas Gurski. *An algebraic theory of tricategories*. PhD thesis, University of Chicago, Department of Mathematics, 2006.

[HL13] Michael Hopkins and Jacob Lurie. Ambidexterity in $K(n)$-local stable homotopy theory. Available at https://people.math.harvard.edu/ lurie/papers/Ambidexterity.pdf, 2013.

[HM06] Pieter Hofstra and Federico De Marchi. Descent for monads. *Theory and Applications of Categories*, 16:668–699, 2006.

[Hot65] Günter Hotz. Eine Algebraisierung des Syntheseproblems von Schaltkreisen I. *Journal of Information Processing and Cybernetics*, 1:185–205, 1965.

[HP15] Yonatan Harpaz and Matan Prasma. The Grothendieck construction for model categories. *Advances in Mathematics*, 281:1306–1363, 2015.

[HS19] Linde Wester Hansen and Michael Shulman. Constructing symmetric monoidal bicategories functorially. arXiv:1910.09240, 2019.

[Hub61] Peter J Huber. Homotopy theory in general categories. *Mathematische Annalen*, 144:361–385, 1961.

[Huy06] Daniel Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2006.

[Ika15] IkamusumeFan. Cappadocia Gaussian blur. https://commons.wikimedia.org/wiki/File:Cappadocia_Gaussian_Blur.svg, 2015.

[Jac91] Bart Jacobs. *Categorical logic and type theory*. Elsevier, 1991.

[Joh02] Peter T Johnstone. *Sketches of an elephant: a topos theory compendium: volume 1*. Oxford University Press, 2002.

[Joy08] André Joyal. The theory of quasi-categories and its applications. In *Advanced Course on Simplicial Methods in Higher Categories*. Centre de Recerca Matemàtica, Barcelona, 2008.

[JS91] André Joyal and Ross Street. The geometry of tensor calculus I. *Advances in mathematics*, 88:55–112, 1991.

[JY21] Niles Johnson and Donald Yau. *2-dimensional categories*. Oxford University Press, 2021.

[Kel69] G M Kelly. Adjunction for enriched categories. In *Reports of the Midwest Category Seminar III*, pages 166–177. Springer Berlin Heidelberg, 1969.

[Kel74] G. M. Kelly. Doctrinal adjunction. In *Proceedings of the Sydney Category Theory Seminar, 1972/1973*, volume 420 of *Lecture Notes in Mathematics*, pages 257–280. Springer, Berlin-New York, 1974.

[KS74] G. M. Kelly and Ross Street. Review of the elements of 2-categories. In *Proceedings of the Sydney Category Theory Seminar, 1972/1973*, volume 420 of *Lecture Notes in Mathematics*, pages 75–103. Springer, Berlin-New York, 1974.

[Law70] F. William Lawvere. Equality in hyperdoctrines and comprehension schema as an adjoint functor. In *Applications of Categorical Algebra*, volume XVII of *Proceedings of Symposia in Pure Mathematics*, pages 1–14. American Mathematical Society, Providence, RI, 1970.

[LS12] Stephen Lack and Michael Shulman. Enhanced 2-categories and limits for lax morphisms. *Advances in Mathematics*, 229(1), 2012.

[Mac71] Saunders MacLane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1971.

[Mar65] J M Maranda. Formal categories. *Canadian Journal of Mathematics*, 17:758–801, 1965.

[MM12] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: a first introduction to topos theory*. Springer Science & Business Media, 2012.

[MV20] Joe Moeller and Christina Vasilakopoulou. Monoidal Grothendieck construction. *Theory and Applications of Categories*, 35(31):1159–1207, 2020.

[Mye23] David Jaz Myers. Categorical systems theory, 2023. Available at http://davidjaz.com/.

[Pav91] Duško Pavlović. Categorical interpolation: Descent and the Beck-Chevalley condition without direct images. In Aurelio Carboni, Maria Cristina Pedicchio, and Guiseppe Rosolini, editors, *Category Theory*, pages 306–325. Springer Berlin Heidelberg, 1991.

[Pen71] Roger Penrose. Applications of negative dimensional tensors. *Combinatorial mathematics and its applications*, 1:221–244, 1971.

[PR84] Roger Penrose and Wolfgang Rindler. *Spinors and space-time: volume 1, two-spinor calculus and relativistic fields*. Cambridge University Press, 1984.

[PRS+78] Robert Pare, RD Rosebrugh, D Schumacher, RJ Wood, and GC Wraith. *Indexed categories and their applications*, volume 661. Springer, 1978.

[Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New mathematical monographs*. Cambridge University Press, 2014.

[RV22] Emily Riehl and Dominic Verity. *Elements of ∞-category theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2022.

[See83] Robert AG Seely. Hyperdoctrines, natural deduction and the beck condition. *Mathematical Logic Quarterly*, 29(10):505–542, 1983.

[Seg68] Graeme Segal. Classifying spaces and spectral sequences. *Institut des Hautes Études Scientifiques. Publications Mathématiques*, 34:105–112, 1968.

[Shu08] Michael Shulman. Framed bicategories and monoidal fibrations. *Theory and Applications of Categories*, 20(18):650–738, 2008.

[Shu09] Michael A. Shulman. Framed bicategories and monoidal fibrations. arXiv:1910.09240, 2009.

[Shu10] Michael Shulman. Constructing symmetric monoidal bicategories. arXiv:1004.0993, 2010.

[Shu11] Michael Shulman. Comparing composites of left and right derived functors. *The New York Journal of Mathematics*, 17:75–125, 2011.

[Sta16] Michael Stay. Compact closed bicategories. *Theory and Applications of Categories*, 31(26):755–798, 2016.

[Str80] Ross Street. Fibrations in bicategories. *Cahiers de Topologie et Géométrie Différentielle*, 21(2):111–160, 1980.

[Tre67] François Treves. *Topological Vector Spaces, Distributions and Kernels*, volume 25 of *Pure and Applied Mathematics*. Elsevier, 1967.

[Vas18] Christina Vasilakopoulou. On enriched fibrations. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 59(4):354–387, 2018.

[Ver11] Dominic Verity. Enriched categories, internal categories and change of base. *Reprints in Theory and Applications of Categories*, 20:1–266, 2011.

[Wei85] Karl Weierstrass. Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin*, (2):709–805, 1885.

[Woo82] R J Wood. Abstract pro arrows I. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 23(3):279–290, 1982.

[Woo85] R J Wood. Proarrows II. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 26(2):135–168, 1985.