Moduli spaces of curves and integrable hierarchies

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To my beloved family, this work stands as a testament to your unwavering love, support, and sacrifice. Your resilience and wisdom have been the greatest lessons of my life.

To my partner, Lidia, my constant source of strength and inspiration, your endless encouragement and belief in my dreams have been the guiding star of my life. Your patience, understanding, and love have illuminated even the darkest paths of this journey.

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This achievement is not mine alone, but a shared triumph among all of us. It is dedicated to you, with all my love and gratitude.

Intellectual Property Statement

The candidate confirms that the work submitted is their own, except where work which has formed part of jointly authored publications has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

Chapter 4 and 5 contains material from the jointly authored publications:

Oscar Brauer and Alexandr Buryak. Open topological recursion relations in genus 1 and integrable systems. *Journal of High Energy Physics*, 2021(1):1–15, 2021

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Abstract

This work is structured to explore two distinct facets of the moduli space of curves, each inquiring into different aspects: one focuses on topological recursion relations in open Gromov-Witten theory, and the other on the bihamiltonian aspects of the double ramification hierarchy.

The first topic investigates open topological recursion relations in genus 1, represented by a set of partial differential equations (PDEs) that are conjectured to control open Gromov-Witten invariants in genus 1. In this segment, an explicit formula is derived, serving as an analog to the Dijkgraaf-Witten formula, specifically for a descendent Gromov-Witten potential in genus 1. This formula stands as a solution to the recursion relations, and is proven that the exponent of an open descendent potential, when approximated up to genus 1, satisfies a system of linear evolutionary PDEs, explicitly constructed with a single spatial variable.

The second facet of the thesis is based on a conjecture of Buryak, Rossi, and Shadrin [BRS21]. This conjecture proposes a formula for a Poisson bracket associated with any given homogeneous cohomological field theory (CohFT). It is hypothesized that this bracket defines a second Hamiltonian structure for the double ramification hierarchy for the given CohFT. This part of the research validates the conjecture at an approximation up to genus 1 and establishes a relationship between this bracket and the second Poisson bracket of the Dubrovin-Zhang hierarchy by an explicit Miura transformation.

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Organization of the thesis

The first chapter centers on integrable systems, with a specific emphasis on the KdV hierarchy and its dispersionless version. This foundational chapter lays out the mathematical frameworks and terminologies that will be important for the discussions in chapters 4 and 5.

Furthermore, the chapter defines the Hamiltonians associated with the KdV hierarchy and its inherent bihamiltonian structure. This structure is distinguished by the presence of two compatible Poisson structures, a highlight of a very specific set of integrable systems. Such a structure is essential in deriving additional integrals of motion, facilitated by the introduction of a recursion operator (1.1.4). The invertibility of this operator allows for the systematic computation of all related Hamiltonians. This characteristic is postulated to be a featured aspect of the Double Ramification hierarchy (5.3.2), which is elaborated upon in chapter 5.

The second chapter introduces the moduli space of curves, a geometric structure that parametrizes classes of compact Riemann surfaces with marked points. This space possesses an orbifold structure, to facilitate a comprehensive description of singularities that emerge due to the automorphisms in the elements. The chapter further explores the Deligne-Mumford stacks, introduced by Pierre Deligne and David Mumford in 1969 [DM69] . These stacks are central in the study of moduli spaces of algebraic curves and intersection theory. Additionally, the chapter introduces key operations between moduli spaces, specifically the forgetful and attaching maps. These maps induce morphisms in the cohomology ring, laying the foundation for the tautological ring of the moduli space.

The third chapter presents the moduli space of stable maps, evaluation maps, and descendant invariants. It emphasizes their role in defining and computing Gromov-Witten invariants, highlighting their significance in enumerative geometry and the representation of cohomology classes. The connection with integrable systems is illustrated through the topological recursion relations in Gromov-Witten theory, a set of PDEs that govern lower genera invariants. This sets the stage for Dubrovin-Frobenius manifolds, which are geometric solutions to a unique set of PDEs termed the WDVV equations (3.3.2). These manifolds can also be interpreted as a geometric condition for integrability. The chapter also introduces Cohomological Field Theories (CohFTs) in Gromov-Witten theory. Introduced by Maxim Kontsevich and Yuri Manin [KM94], CohFTs consist of a series of linear maps, that encapsulate operations on moduli spaces and to formalize essential tools in Gromov-Witten theory. The chapter enlightens the role of CohFTs in bridging the geometry of moduli spaces of stable curves with intersection numbers and Gromov-Witten invariants. Within the study of CohFTs, the introduction of a quantum product and the notion of a semi-simple CohFT (ssCohFT) are discussed. This specific constraint paves the way for a classification of ssCohFTs by Constantin Teleman [Tel12].

The fourth chapter is devoted to open moduli spaces of curves, emphasizing their significance in extending the theory of moduli spaces of curves. The chapter highlights the effects of boundary terms on intersection numbers and the structure of moduli spaces. A notable segment of the discussion revolves around the Dubrovin-Zhang (DZ) hierarchy, a system of partial differential equations intrinsically linked with total descendant potentials. This hierarchy serves as a link between open moduli spaces and their corresponding integrable systems. The chapter also presents an explicit formula, similar to the Dijkgraaf-Witten formula, for a descendant Gromov-Witten potential in genus 1. The chapter further elaborates on the Gromov-Witten potential, up to genus 1 and the exponential of an open descendant potential satisfying a system of linear partial differential equations with one spatial variable. These results were previously published in the Journal of High Energy Physics [BB21b]. The chapter concludes by exploring the open KdV and Virasoro equations, emphasizing their role in describing the intersection theory within the context of moduli spaces of Riemann surfaces with boundaries.

The fifth chapter focuses on the Double Ramification and Dubrovin-Zhang Hierarchies. The Double Ramification Hierarchy (DR hierarchy) has its foundation in algebraic geometry and acts as a bridge between curve counting theories and integrable systems. This hierarchy expands upon integrable hierarchies associated with a cohomological field theory. The DR hierarchy is defined by the Double Ramification Cycle, a class in the Chow ring of the moduli space of stable curves, representing the condition that a meromorphic function on a Riemann surface has specific orders and residues.

The chapter further examines the moduli space of stable relative maps. This space possesses a virtual fundamental class, and the double ramification cycle can be expressed using basic tautological classes, addressing an issue highlighted by Y. Eliashberg in 2001. A concrete example is provided for the trivial CoFT, which leads to the Hamiltonians of the KdV hierarchy. The main result of this chapter is the validation of a conjecture put forth by Buryak, Rossi, and Shadrin [BRS21]. This conjecture, which suggests a bihamiltonian structure for the DR hierarchy, is proven up to genus one, also establishing a connection between the Poisson structure and the DZ hierarchy through a Miura transformation.

It is worth noting that while the initial three chapters lay the foundational groundwork for understanding the thesis, the fourth and fifth chapters present distinct results and operate independently without overlapping conceptually.

Chapter 1

Integrable Hierarchies

1.1 Integrable systems and bi-Hamiltonian structures

Integrable systems represent mathematical models that describe the dynamical behaviour of physical systems, with the defining characteristic of possessing enough invariants (also called conserved quantities), such as energy, momentum, and angular momentum. The designation *integrable* reflects the unique feature of these systems, which can be exactly solved through a variety of mathematical techniques, encompassing inverse scattering, algebraic-geometric methods, and Lax pairs, among others.

An important attribute of certain integrable systems is their bi-Hamiltonian structure. Franco Magri first introduced the concept of bi-Hamiltonian structures [Mag78], who employed the Korteweg-de Vries (KdV) equation as an example of such a system. These structures allow to find additional integrals of motion and offer a method to analyze the system's dynamics through Hamiltonian flows, thereby offering tools for understanding the system's underlying mechanics.

Characteristically, a bi-Hamiltonian structure exhibits two Poisson structures (see 1.1.1). For these Poisson structures to be deemed compatible, the Poisson brackets of one structure should be preserved under the Hamiltonian flows generated by the other.

This compatibility condition facilitates the construction of a hierarchy of Hamiltonian equations. These equations subsequently generate a series of conserved quantities, enabling the derivation of exact solutions for a wide variety of integrable systems. Some renowned examples of bi-Hamiltonian hierarchies comprise the Korteweg–de Vries (KdV) hierarchy, Toda lattice hierarchy, the Calogero-Moser system, and the Gelfand-Dickey hierarchy.

In this section, we present the formal definitions necessary to describe integrable hierarchies, their space of local functionals, and bi-Hamiltonian structure, we will follow [BRS21] with some modifications to the notation.

It is important to clarify that the focus of this work will not be on the formal solutions of integrable systems or the convergence properties of series representations. Instead, our approach will primarily be algebraic in nature.

Let V be a vector space of dimension N, in order to model the loop space of V, we could describe its ring of functions. We then consider the components of a formal loop $u\colon \mathbb{S}^1 \to V$, where $\mathbb{S}^1$ is the 1-sphere, in a basis $e_1, \ldots, e_N$ of V. So, the variables $u^\alpha := u_0^\alpha$ can be thought of as the components $u^\alpha(x)$ of this formal loop.

We take $u^1, ..., u^N$ to be formal variables, these will be the entries of u^α of the loop u, and attach the additional formal variables u_d^α for $d \geq 0$. We bring forth the ring of differential polynomials, denoted by $\mathcal{A}_u := \mathbb{C}\left[[u^*]\right]\left[u^*_{\geq 0}\right]$. Here, we recognise $u_0^\alpha = u^\alpha$ and denote its derivatives as $u_x^\alpha := u_1^\alpha$, $u_{xx}^\alpha := u_2^\alpha$, and so on. Subsequent sections will develop further our theoretical framework into these concepts and their implications in the context of integrable systems.

We now formally define the following objects:

- The standard degree $\deg(u_i^\alpha) := i$ is defined on $\mathcal{A}_u$.
- The operator $\partial_x := \sum_{i\geq 0} u_{i+1}^\alpha \frac{\partial}{\partial u_i^\alpha}$ increases the standard degree by 1, i.e., $\deg(\partial_x u_i^\alpha) = i + 1$.
- The space $\Lambda_u := \mathcal{A}_u / \left(\mathbb{C} \oplus \operatorname{Im} \partial_x\right)$ is called the space of local functionals.
- $\mathcal{A}_{u;d} \subset \mathcal{A}_u$ and $\Lambda_{u;d} \subset \Lambda_u$ are called the homogeneous components of differential degree d.
- The extended spaces of differential polynomials and local functionals are defined by $\widehat{\mathcal{A}}_u := \mathcal{A}_u[[\varepsilon]]$ and $\widehat{\Lambda}_u := \Lambda_u[[\varepsilon]]$, respectively. Here, $\widehat{\mathcal{A}}_{u;k} \subset \widehat{\mathcal{A}}_u$ and $\widehat{\Lambda}_{u;k} \subset \widehat{\Lambda}_u$ are the subspaces of degree k, where $deg(\varepsilon) = -1$. The parameter ε is called the dispersive parameter.
- The equivalence class of $f(u_*^*; \varepsilon)$ in Λ_u is denoted as $\bar{f} := \int f(u_*^*; \varepsilon) dx$, here the polynomial $f(u_*^*)$ is called the *density* of the *local functional* $\bar{f}$
- Given any variable u^*, we introduce its Fourier transformation as the formal power series

  $$u^\alpha = \sum_{b\in\mathbb{Z}} p_b^\alpha e^{ibx},$$

  we call the variables p_*^* Fourier coordinates.
- The variational derivative $\frac{\delta}{\delta u^\alpha} : \mathcal{A}_u \to \mathcal{A}_u$, $1 \leq \alpha \leq N$, is defined by

  $$\frac{\delta}{\delta u^\alpha} := \sum_{i\geq 0} (-\partial_x)^i \circ \frac{\partial}{\partial u_i^\alpha}.$$

- For $f \in \widehat{\mathcal{A}}_u$, we define the sequence of differential operators

$$L^k_\alpha(f) := \sum_{i\geq k} \binom{i}{k} \frac{\partial f}{\partial u^\alpha_i} \partial_x^{i-k},$$

where $\alpha = 1, \ldots, N$ and $k \geq 0$. We also define $L_\alpha(f) := L^0_\alpha(f)$.

- Let $K = (K^{\mu\nu})$ be an $N \times N$ matrix of differential operators of the form $K^{\mu\nu} = \sum_{j\geq 0} K^{\mu\nu}_j \partial_x^j = \sum_{l,j\geq 0} \varepsilon^l K^{[l],\mu\nu}_j \partial_x^j$, where $K^{[l],\mu\nu}_j \in \mathcal{A}_{u;l-j+1}$. We define a bracket of degree 1 on the space $\widehat{\Lambda}_u$ by

$$\{\bar{f}, \bar{g}\}_K := \int \left(\frac{\delta \bar{f}}{\delta u^\mu} K^{\mu\nu} \frac{\delta \bar{g}}{\delta u^\nu} \right) dx. \tag{1.1.1}$$

- A Poisson operator K is defined to be an operator for which the bracket $\{\cdot, \cdot\}_K$ is skew-symmetric and satisfies the Jacobi identity. We denote the space of Poisson operators by $\mathcal{PO}_u$.

- Two Poisson operators K_1 and K_2 are said to be compatible if the linear combination $K_2 - \lambda K_1$ is a Poisson operator for any $\lambda \in \mathbb{C}$.

- A Miura transformation is a change of variables $u^\alpha \mapsto \widetilde{u}^\alpha(u^*_*, \varepsilon)$ of the form $\widetilde{u}^\alpha(u^*_*, \varepsilon) = u^\alpha + \varepsilon f^\alpha(u^*_*, \varepsilon)$, where $f^\alpha \in \widehat{\mathcal{A}}_{u;1}$. A Poisson operator K rewritten in the new variables $\widetilde{u}^\alpha$ will be denoted by $K_{\widetilde{u}}$.

- For a scalar operator $A = \sum_{m\geq 0} A_m \partial_x^m$, where $A_m \in \mathcal{A}_u$ (finite sum), we define $A^\dagger := \sum_{m\geq 0} (-\partial_x)^m \circ A_m$.

- Let us fix $N \geq 1$, an $N \times N$ symmetric nondegenerate complex matrix $\eta = (\eta_{\alpha\beta})$, and an N-tuple of complex numbers $(A^1, \ldots, A^N)$, not all equal to zero. We will use the notation

$$\frac{\partial}{\partial t^{1\!\!1}_a} := A^\alpha \frac{\partial}{\partial t^\alpha_a}, \quad a \geq 0. \tag{1.1.2}$$

- Let $K = \left(K^{\alpha\beta}\right)$ be a matrix operator defined as $K^{\alpha\beta} = \sum_m K^{\alpha\beta}_m \partial_x^m$ where $K^{\alpha\beta}_m \in \mathcal{A}_u$ is a finite sum. Define $K^\dagger = \left(K^{\dagger;\alpha\beta}\right)$, where $K^{\dagger;\alpha\beta} := \sum_m (-\partial_x)^m \circ K^{\beta\alpha}_m$.

- A Hamiltonian hierarchy of PDEs is a system of the form:

$$\frac{\partial u^\alpha}{\partial t_i} = K^{\alpha\mu} \frac{\delta \bar{h}_i}{\delta u^\mu}, \quad 1 \leq \alpha \leq N, \quad i \geq 1,$$

where $\bar{h}_i$ are local functionals in $\widehat{\Lambda}_{u;0}$, $K = (K^{\mu\nu})$ is a Poisson operator, and $\{\bar{h}_i, \bar{h}_j\}_K = 0$ for $i, j \geq 1$. The local functionals $\bar{h}_i$ are called the Hamiltonians.

- A Hamiltonian hierarchy of the form:

$$\frac{\partial u^\alpha}{\partial t_q^\beta} = K_1^{\alpha\mu} \frac{\delta \bar{h}_{\beta,q}}{\delta u^\mu}, \quad 1 \leq \alpha, \beta \leq N, \quad q \geq 0, \tag{1.1.3}$$

is said to be bi-Hamiltonian if it has N linearly independent Casimirs [1] $\bar{h}_{\alpha,-1}$, for $\alpha = 1, \dots, N$, of the Poisson bracket $\{\cdot,\cdot\}_{K_1}$, and is endowed with a Poisson operator K_2 compatible with K_1 such that:

$$\{\cdot, \bar{h}_{\alpha,i-1}\}_{K_2} = \sum_{j=0}^{i} R_{i,\alpha}^{j,\beta} \{\cdot, \bar{h}_{\beta,i-j}\}_{K_1}, \quad 1 \leq \alpha \leq N, \quad i \geq 0, \tag{1.1.4}$$

where $R_i^j = \left(R_{i,\alpha}^{j,\beta}\right)$, $0 \leq j \leq i$, are constant $N \times N$ matrices. The relation (1.1.4) is called a bi-Hamiltonian recursion and the matrix R_i^j is called the recursion operator, if R_i^0 is invertible, it is possible to compute all the Hamiltonians recursively from the Casimirs $\bar{h}_{\alpha,-1}$.

1.2 The KdV hierarchy

The Korteweg–de Vries (KdV) equation, a nonlinear partial differential equation, serves as a mathematical model for waves propagating in one-dimensional media with weakly nonlinear and dispersive properties. The equation was initially formulated by Korteweg and de Vries in 1895 to describe the propagation of long waves in shallow water [KdV95], and was later rediscovered in the 1960s by Zabusky and Kruskal, who unveiled its soliton solutions [ZK65].

Solitons, which are localized wave packets that retain their shape and speed even after collisions, can propagate without distortion over substantial distances [AC91]. The KdV equation is an example of an integrable system and forms part of an infinite sequence of equations known as the KdV hierarchy.

The KdV hierarchy comprises nonlinear partial differential equations that extend the KdV equation, outlining the evolution of wave amplitude's higher-order terms. These equations are linked by a bi-Hamiltonian recursion operator that allows us to generate an infinite series of conserved quantities for the system.[2]

[1] We refer to a functional as a Casimir if it is annihilated by a Poisson operator. This is mathematically equivalent to having a zero Poisson bracket with any other element in the algebra.

[2] The bi-Hamiltonian operator for the KdV hierarchy is also known as Lenard recursion formula, due to Andrew Lenard, while he did not publish his results, his contributions, particularly the method leading to what is now known as the Lenard recursion scheme, were documented and formalized by others. The main findings and methodologies associated with Lenard were published by others, notably in the works by Peter D. Lax and the joint work of Clifford S. Gardner, John M. Greene, Martin D. Kruskal, and Robert M. Miura.

Our study will be centered on the KdV hierarchy, this system, defined over one spatial variable x, with $u_* \in \mathcal{A}_u$, encompasses the following initial terms of the KdV hierarchy:

$$
\begin{aligned}
u_{t_0} =& u_x, \\
u_{t_1} =& u u_x + \frac{\varepsilon^2}{12} u_{xxx}, \\
u_{t_2} =& \frac{u^2 u_x}{2} + \varepsilon^2 \left(\frac{u u_{xxx}}{12} + \frac{u_x u_{xx}}{6} \right) + \varepsilon^4 \frac{u_5}{240}, \\
u_{t_3} =& \frac{u^3 u_x}{6} + \varepsilon^2 \left(\frac{u^2 u_{xxx}}{24} + \frac{u_x^3}{24} + \frac{u u_x u_{xx}}{6} \right), \\
& + \varepsilon^4 \left(\frac{u u_5}{240} + \frac{u_4 u_x}{80} + \frac{u_{xxx} u_{xx}}{48} \right) + \varepsilon^6 \frac{u_7}{6720}, \\
& \vdots
\end{aligned}
$$

The integrability of the Korteweg–de Vries (KdV) hierarchy was first established by Gardner, Greene, Kruskal, and Miura [GGKM67]. They showed that the KdV hierarchy exhibits an infinite number of conservation laws, a feature characteristic of integrable systems. These laws state that certain quantities associated with the system remain invariant as the system evolves, a property that enables the exact solution of the system. The integrability of the equations in the hierarchy is manifested by the following identities:

$$
(u_{t_i})_{t_j} = (u_{t_j})_{t_i}, \tag{1.2.1}
$$

for all $i, j \geq 0$[1]. This identity illustrates that the time derivatives of the system commute for any pair of indices, a property that characterizes the integrability of the system, moreover it guarantees the existence of a formal solution $u^\alpha(x, t_*^*, \varepsilon) \in \mathbb{C}[[x, t_*^*, \varepsilon]]$ with initial condition $u^\alpha(x, t_*^* = 0, \varepsilon)$ [Lax76].

The KdV hierarchy can be represented in the language of pseudodifferential operators via the Lax operator. The Lax operator simplifies the algebraic structure of the Lax pair, thus exposing some of the properties of the KdV hierarchy. Specifically, the Lax operator for the KdV hierarchy is given by:

$$
L = \frac{1}{2} (\varepsilon \partial_x)^2 + u. \tag{1.2.2}
$$

This operator provides a critical tool for understanding the underlying integrable structure of the KdV hierarchy.

The corresponding Lax pair is then defined in terms of the positive part of the pseu-

[1] Alternatively, integrability can be defined by requiring the commutativity of the Hamiltonian operators within the context of a given Poisson structure K. Specifically, this condition is expressed as $\left\{\bar{h}_{\alpha,d_1}, \bar{h}_{\beta,d_2}\right\}_K = 0$ for any Hamiltonian system characterized by the equation $\frac{\partial u^\alpha}{\partial t_d^\beta} = \left\{u^\alpha, \bar{h}_{\beta,d}\right\}_K$. for $d \geq 0$ and $1 \leq \alpha, \beta \leq N$

dodifferential operator $(2L)^{(2i+1)/2}$, denoted by A_i. This means we only consider terms with $\partial_x^{i\geq 0}$:

$$A_i := \frac{1}{(2i+1)!!}(2L)_+^{\frac{2i+1}{2}} = \frac{1}{(2i+1)!!}\left[(\varepsilon\partial_x)^{2i+1} + (2i+1)u\,(\varepsilon\partial_x)^{2i-1} + \dots\right]_+ . \quad (1.2.3)$$

The pseudodifferential approach to the Lax operator has the advantage of simplifying the algebraic structure of the Lax pair, by virtue of the integrability properties of the KdV hierarchy [Lax68].

The operators A_i in the KdV hierarchy can be explicitly written out for the first few terms. For instance, the first two operators A_1 and A_2 are given by:

$$\begin{aligned} A_1 &= (\varepsilon\partial_x)^3 - \frac{3}{2}\varepsilon\left(u\partial_x + \frac{1}{2}u_x,\right), \\ A_2 &= (\varepsilon\partial_x)^5 - \frac{5\varepsilon^3}{4}u\left(\partial_x^3 + \frac{3}{4}u_{xxx}\right) + \frac{7}{2}u_x\,(\varepsilon\partial_x)^2 + \left(\frac{3}{2}u^2 - 3u_{xx}\right)(\varepsilon\partial_x) + 2uu_x \\ &\quad + \frac{3}{8}u^2 - \frac{1}{8}u_{xx}. \end{aligned}$$

The KdV hierarchy is then obtained by the Lax equation:

$$L_{t_i} = [A_i, L]\,. \quad (1.2.4)$$

This equation represents the evolution of the Lax operator L under the flow t_i generated by the operator A_i. The commutator on the right-hand side ensures that the Lax equation preserves the integrability of the KdV hierarchy.

Finally the Hamiltonians are given by:

$$\bar{h}_i = \frac{1}{(2i+3)!!}\int \mathrm{res} L^{(2i+3)/2}dx,$$

for $i \geq 0$, for a more comprehensive description, we recommend consulting the work by [BBT03].

1.3 Dispersionless KdV hierarchy

It is possible to construct the KdV hierarchy from a simpler system known as the dispersionless KdV hierarchy. This set of nonlinear partial differential equations arises in the limit where the dispersion term vanishes. For the KdV equation, the dispersion term is responsible for the spreading wave packets, meaning that the amplitude and shape of the propagating waves change due to dispersion, by setting this term to zero ($\varepsilon = 0$), we obtain the dispersionless KdV hierarchy, which has the general form:

$$u_{t_i} = \frac{u^i}{i!} u_x.$$

The Hamiltonians $\bar{h}_i$ for the hierarchy are given by:

$$\bar{h}_i = \int \frac{u^{i+2}}{(i+2)!} dx. \tag{1.3.1}$$

This system can be studied using the bi-Hamiltonian formalism, which involves expressing the equations in terms of two Poisson bracket structures:

$$K_1 = \partial_x, \tag{1.3.2}$$

$$K_2 = u\partial_x + \frac{1}{2} u_x, \tag{1.3.3}$$

where K_1 and K_2 are two Poisson operators, known as the first and second Hamiltonian structures respectively.

To obtain the full KdV hierarchy from the dispersionless KdV equation is not a direct process, one can start by studying the Lax representation of both the full KdV hierarchy and the dispersionless KdV hierarchy. The Lax representations of both hierarchies involve different operators, however they are related in the sense that the dispersionless Lax operator can be considered a particular limit of the full KdV Lax operator, where the pseudo-differential part of the operator vanishes, this relation is known as a *quasitriviality transformation* [Dub14].

The main idea behind quasitriviality transformations is to express a given system in terms of a simpler one, with the hope of using the properties of the simpler system to gain insight into the more complicated one. The transformation is called "quasitrivial" because it does not change the underlying integrability of the system, but rather provides a different way of looking at it. This transformation involves introducing an additional dependent variable and a new Hamiltonian that satisfies a certain differential equation, then the quasitriviality transformation can be used to relate the two Hamiltonian structures. The quasitriviality transformation for the dispersionless KdV is given by:

$$u \mapsto v = u + \frac{\varepsilon^2}{24} (\log u_x)_{xx} + \varepsilon^4 \left(\frac{u_{xxxx}}{1152 u_x^2} - \frac{7 u_{xx} u_{xxx}}{1920 u_x^3} + \frac{u_{xx}^3}{360 u_x^4} \right)_{xx} + O\left(\varepsilon^6\right). \tag{1.3.4}$$

This implies that the Hamiltonians for the KdV hierarchy are:

$$\begin{aligned}\bar{H}_i &= \int \frac{1}{(i+2)!}\left[u + \frac{\varepsilon^2}{24}(\log u_x)_{xx} + O\left(\varepsilon^4\right)\right]^{i+2} dx \\ &= \int \left\{\frac{u^{i+2}}{(i+2)!} + \frac{\varepsilon^2}{24}\left[-\frac{u^{i-1}}{(i-1)!}u_x^2 + \partial_x\left(\frac{u^{i+1}}{(i+1)!}\frac{u_{xx}}{u_x}\right)\right] + O\left(\varepsilon^4\right)\right\} dx \\ &= \int \left\{\frac{u^{i+2}}{(i+2)!} - \frac{\varepsilon^2}{24}\frac{u^{i-1}}{(i-1)!}u_x^2 + O\left(\varepsilon^4\right)\right\} dx.\end{aligned}$$

It is desirable to have polynomial densities, and fortunately for the KdV hierarchy, it is possible to add an irrelevant total derivative to the quasitriviality transformation. By adding this expression, it will effectively cancel all non-polynomial terms of the Hamiltonians, this is expressed by $\partial_x(\partial_{t_{i+1}}\Delta\mathcal{F})$, where

$$\Delta\mathcal{F} = \frac{1}{24}\log u_x + \varepsilon^2\left(\frac{u_{xxxx}}{1152{u_x}^2} - \frac{7u_{xx}u_{xxx}}{1920{u_x}^3} + \frac{{u_{xx}}^3}{360{u_x}^4}\right) + O\left(\varepsilon^4\right), \tag{1.3.5}$$

so, the Hamiltonians for the KdV hierarchy can now be expressed as:

$$\bar{H}_i = \int \left[\frac{u^{i+2}}{(i+2)!} + \frac{\varepsilon^2}{24}\left(2\frac{u^i u_{xx}}{i!} + \frac{u^{i-1}u_x^2}{(i-1)!}\right) + \mathcal{O}\left(\varepsilon^4\right)\right] dx. \tag{1.3.6}$$

Example 1.3.1 (Bi-Hamiltonian recursion of the dispersionless KdV hierarchy) *The dispersionless KdV (1.2) in hamiltonian form can be expressed in terms of two different Poisson brackets*

$$K_1 = \partial_x, \tag{1.3.7}$$

$$K_2 = u\partial_x + \frac{1}{2}u_x, \tag{1.3.8}$$

from (1.3.7), we would like to find the recursion operator R (c.f. 1.1.4), the bi-Hamiltonian recursion reads:

$$\begin{aligned}\left\{u(x), \bar{h}_i\right\}_{K_1} &= \partial_x \frac{\delta \bar{h}_i}{\delta u(x)} \\ &= R\left\{u(x), \bar{h}_{i-1}\right\}_{K_2} = \left(u\partial_x + \frac{1}{2}u_x\right)\frac{\delta \bar{h}_{i-1}}{\delta u(x)},\end{aligned}$$

if

$$\bar{h}_i = \int \frac{[u(x)]^{i+2}}{(i+2)!} dx, \tag{1.3.9}$$

the term for the Poisson structure K_1 gives

$$\left\{u(x), \bar{h}_i\right\}_{K_1} = \partial_x\left(\frac{u^{i+1}}{(i+1)!}\right) = \frac{u^i}{i!}u_x,$$

for K_2 we get,

$$R\left(u\partial_x + \frac{1}{2}u_x\right)\frac{u^i}{i!} = R\left(i + \frac{1}{2}\right)\frac{u^i}{i!}u_x,$$

comparing both terms for any field $u(x)$, we find the bi-Hamiltonian recursion matrix is then:

$$R = (i + \frac{1}{2})^{-1}.$$

Theorem 1.3.2 *[Olv93] Consider a bi-Hamiltonian system described by the equation*

$$\frac{\partial u}{\partial t} = K_1\frac{\delta\bar{h}_0}{\delta u} = K_2\frac{\delta\bar{h}_1}{\delta u},$$

where K_1 and K_2 are Poisson operators. Then, the operator $R = K_2K_1^{-1}$, is a recursion operator for the system.

Example 1.3.3 (KdV equation and Bi-Hamiltonian Hierarchy) *Consider a Hamiltonian hierarchy for the KdV equation with simplified variables u, x, and t. Introducing the Hamiltonian functional:*

$$\bar{h}_{1,0} = \int\left(\frac{u^3}{6} + \frac{\varepsilon^2}{24}uu_{xx}\right)dx, \tag{1.3.10}$$

with the Poisson operator $K_1 = \partial_x$. The KdV equation derivation follows the Hamiltonian dynamics:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \{u, \bar{h}_{1,0}\}_{K_1} = \partial_x\frac{\delta\bar{h}_{1,0}}{\delta u}\\ &= \partial_x\left\{\frac{1}{6}\sum_{i\geq 0}(-\partial_x)^i\circ\frac{\partial}{\partial u_i}u^3 + \frac{\varepsilon^2}{24}\sum_{i\geq 0}(-\partial_x)^i\circ\frac{\partial}{\partial u_i}uu_{xx}\right\}\\ &= \partial_x\left\{\frac{u^2}{2} + \frac{\varepsilon^2}{24}(u_{xx} + u_{xx})\right\}\\ &= uu_x + \frac{\varepsilon^2}{12}u_{xxx}.\end{aligned}$$

The Korteweg-de Vries (KdV) hierarchy can be formulated in Hamiltonian form by an additional Poisson bracket [Mag78], (we will use the notation as described in [BRS21]):

$$K_2 = \frac{\varepsilon^2}{8}\partial_x^3 + u\partial_x + \frac{1}{2}u_x. \tag{1.3.11}$$

This system fulfills the conditions of Theorem 1.3.2. Assuming the inverse of K_1 is ∂_x^{-1}, the recursion operator R can be expressed as:

$$R = \left(\frac{\varepsilon^2}{8}\partial_x^3 + u\partial_x + \frac{1}{2}u_x\right)\circ\partial_x^{-1}, \tag{1.3.12}$$

$$= \frac{\varepsilon^2}{8}\partial_x^2 + u + \frac{1}{2}u_x\partial_x^{-1}. \tag{1.3.13}$$

1.4 Integrable Hierarchies in Moduli Spaces: Dubrovin-Zhang and Double Ramification Hierarchies

Recent advancements in integrable systems have been significantly influenced by geometric and topological theories, particularly from the study of moduli spaces of Riemann surfaces. Two notable hierarchies, the Dubrovin-Zhang and Double Ramification hierarchies, have emerged as extensions of the KdV hierarchy and other well known systems. These hierarchies have led to new connections with the geometry of moduli spaces, cohomological field theories, and intersection numbers, and will be elaborated upon in the subsequent chapters.

The Dubrovin-Zhang hierarchy, an integrable system of PDEs, is associated with total descendant potentials. These are formal power series that play a central role in the description of the cohomology of moduli spaces. Motivated in topological field theories, the geometry of moduli spaces and cohomological field theories, the Dubrovin-Zhang hierarchy offers a framework for investigating relations between intersection numbers on moduli spaces, integrable systems, curve counting theories, and algebraic geometry.

On the other hand, the double ramification hierarchy is a Hamiltonian system that arises in the context of moduli spaces of stable curves. This hierarchy has its beginnings in study of the double ramification cycle, a complicated cohomological class in the moduli space of curves. The double ramification hierarchy involves enumerative geometry problems, Gromov-Witten invariants and other geometric invariants.

Despite their distinct origins, the Dubrovin-Zhang and double ramification hierarchies share several similarities and both exhibit connections with the geometry of moduli spaces. The interrelation between these hierarchies, as well as their applications to the theory of integrable systems and algebraic geometry, has been noticed in the DR/DZ equivalence conjecture. This states that the two hierarchies are related by a normal Miura transformation; this means the existence of a change of coordinates preserving the hierarchies' structures. The last chapter is devoted to this conjecture and some new results on the matter.

Chapter 2

Moduli Spaces of Curves

The moduli space of curves is a geometric object that parametrizes classes of compact Riemann surfaces with marked points. Its orbifold structure enables a detailed description of the singularities arising from the presence of automorphisms in the curves [Thu79]. Orbifolds, are spaces that locally resemble the quotient of either $\mathbb{C}^n$ (for complex orbifolds) or $\mathbb{R}^n$ (for real orbifolds) by the actions of finite groups. These actions can introduce singularities into the quotient space. Analogous to manifolds, one can define structures such as an atlas, differential forms, bundles, and cohomology classes on an orbifold, allowing for the application of geometric and topological methods.

However, defining morphisms can be technically challenging, and the category of bundles over an orbifold may be more complex than the category of bundles over a manifold. By defining bundles over an orbifold such that the fibers have a manifold structure, some of these difficulties can be circumvented. The orbifold structure, though, does not provide the necessary tools to describe the boundary of the moduli space. Knudsen introduced the combinatorial boundary, which consists of nodal curves with marked points that represent degenerations of smooth curves [Knu83]. These degenerations can be illustrated using dual graphs that encode the combinatorial data of the nodal curves [Str84].

To study the compactification of the moduli space of curves, one must also examine its Deligne-Mumford stack structure. Introduced by Pierre Deligne and David Mumford in 1969 [DM69], a Deligne-Mumford stack is a generalization of a moduli space that accounts for the presence of automorphisms in the geometric objects being parametrized. Deligne-Mumford stacks are commonly used in algebraic geometry to describe moduli problems where the objects being parametrized possess nontrivial automorphism groups. As a result, Deligne-Mumford stacks are central objects in studying moduli spaces of algebraic curves and intersection theory.

2.1 Orbifolds

Orbifolds are a special class of spaces that have some local structure resembling a quotient of a space by a finite group action. In algebraic geometry, orbifolds arise naturally as quotient spaces of algebraic varieties by finite group actions. Our approach will follow [ALR07], [CJ19], [Dun88], [Kap10] and [CR04]

We start with a connected topological space, denoted as U and a connected $n-$dimensional smooth manifold, V. We also have a finite group G that acts smoothly on V. This means that each element of G corresponds to a transformation of V that is smooth, this action is not necessarily effective. We will assume that for each element of G, the set of points in V that are left fixed by the corresponding transformation, is either the entire space V or a subset of V with codimension at least two.

We now formally introduce the following elements:

- An $n-$dimensional **uniformizing system** of U, is a triple (V, G, π), where π is a continuous map from V to U that induces a homeomorphism between the quotient space V/G and U.

- Two uniformizing systems (V_i, G_i, π_i), $i = 1, 2$, are considered **isomorphic** if there's a diffeomorphism ϕ from V_1 to V_2 and an isomorphism λ from G_1 to G_2 such that ϕ is λ-equivariant (meaning it commutes with the action of the group), and $\pi_2 \circ \phi = \pi_1$.

- Consider an inclusion map $i : U' \to U$, and a uniformizing system (V', G', π') associated with U'. We can establish a relationship between this system and another uniformizing system (V, G, π) of U. Specifically, we say that (V', G', π') is derived from (V, G, π) if the following conditions are met:

  1. There exists a monomorphism $\tau : G' \to G$ which behaves as an isomorphism when it is restricted to the kernels of the actions of G' and G.
  2. There is a τ-equivariant open embedding $\psi : V' \to V$ satisfying the condition $i \circ \pi' = \pi \circ \psi$.

  This relationship is illustrated in the following commuting diagram. In this context, the pair (ψ, τ), which maps (V', G', π') to (V, G, π), is referred to as an **injection**.

$$\begin{array}{ccc} V' & \xrightarrow{\psi} & V \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ U' & \xrightarrow{i} & U \end{array}$$

- Two injections $(\psi_i, \tau_i) : (V_i', G_i', \pi_i') \to (V, G, \pi)$, $i = 1, 2$, are **isomorphic** if there exists an isomorphism (ϕ, λ) between (V_1', G_1', π_1') and (V_2', G_2', π_2'), and an automorphism $(\bar{\phi}, \bar{\lambda})$ of (V, G, π) such that $(\bar{\phi}, \bar{\lambda}) \circ (\psi_1, \tau_1) = (\psi_2, \tau_2) \circ (\phi, \lambda)$.

- Consider a topological space U that is both connected and locally connected. For any given point p within U, we can identify two uniformizing systems, denoted as (V_1, G_1, π_1) and (V_2, G_2, π_2). These systems are associated with the neighborhoods U_1 and U_2 of the point p. We say these two uniformizing systems (V_1, G_1, π_1) and (V_2, G_2, π_2) are **equivalent** at the point p if they induce isomorphic uniformizing systems for some neighborhood U_3 of p.

Definition 2.1.1 (Orbifold) *[CR04] Consider X, a Hausdorff, second countable topological space. An **n-dimensional orbifold structure** on X is characterized by the following data:*

- *For any point p in X, there exists a neighborhood U_p and an n-dimensional uniformizing system (V_p, G_p, π_p) of U_p.*

- *For any point q in U_p, (V_p, G_p, π_p) and (V_q, G_q, π_q) are equivalent at q (i.e., they define the same local structure at q).*

*Given a local structure of orbifold structures, X is called an **orbifold**.*

Definition 2.1.2 *Let X be an orbifold, $p \in X$ and (V, G, π) an uniforming system containing p, the **local group** at p is given by*

$$G_p = \{g \in G \mid gp = p\}.$$

Example 2.1.3 *The complex orbifold $M/\mathbb{Z}_n$ is obtained by taking the quotient of a smooth complex manifold M by an action of the cyclic group $\mathbb{Z}_n$ of order n. We assume that the action is effective, meaning every non-identity element of the group moves at least one point in M.*

We define the equivalence relation on M as follows: $x \sim y$ if and only if there exists $k \in \mathbb{Z}_n$ such that $x = \rho_k(y)$, where in a local chart provided by a map $\phi : U \subset M \to \mathbb{C}^m$, the action ρ_k is given by $\phi(\rho_k(x)) = e^{2\pi ik/n}\phi(x)$ for $x \in U$. This means that the group $\mathbb{Z}_n$ acts on the chart by rotating points in $\mathbb{C}^m$.

The orbifold $M/\mathbb{Z}_n$ has singularities that are locally modeled on the quotient space $\mathbb{C}^m/\mathbb{Z}_n$, where m is the dimension of M. Specifically, near a point $[x]$ in $M/\mathbb{Z}_n$, we can choose an uniforming system $(V, \mathbb{Z}_n, \varphi)$ around x in M such that $\mathbb{Z}_n$ acts on V through ρ_k, and the action in this chart is represented by $\varphi^{-1}(e^{2\pi ik/n} \cdot \varphi(y))$ for $y \in V$, where $e^{2\pi ik/n}$ is a primitive nth root of unity. The quotient space $V/\mathbb{Z}_n$ is then isomorphic to a neighborhood of $[x]$ in $M/\mathbb{Z}_n$, hence locally equivalent to the quotient space $\mathbb{C}^m/\mathbb{Z}_n$.

Example 2.1.4 *The quotient of the 2-sphere by an action of the cyclic group $\mathbb{Z}_2$, which identifies antipodal points, is an orbifold known as the projective plane $\mathbb{P}^1$.*

Example 2.1.5 *Consider the hypersurface X on $\mathbb{CP}^4$ with homogeneous coordinates $[z_0, z_1, z_2, z_3, z_4]$ defined by the homogenous equation:*

$$z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + \alpha z_0 z_1 z_2 z_3 z_4 = 0,$$

for $\alpha \in \mathbb{C}$ constant. X can be realized as an orbifold by considering its action under the group $\mathbb{Z}_5$.

To define the orbifold structure, we begin by constructing an action of $\mathbb{Z}_5$ on $\mathbb{C}^5$ that preserves X. We do this by defining an action of $\mathbb{Z}_5$ on the coordinates $[z_0, z_1, z_2, z_3, z_4]$ by setting:

$$\rho_k[z_0, z_1, z_2, z_3, z_4] = [e^{2\pi ik/5}z_0, e^{2\pi ik/5}z_1, e^{2\pi ik/5}z_2, e^{2\pi ik/5}z_3, e^{2\pi ik/5}z_4],$$

for $k \in \mathbb{Z}_5$. This action clearly preserves X, since each term z_0^5, z_1^5, z_2^5, z_3^5, z_4^5 and $z_0z_1z_2z_3z_4$ is invariant under the action of $\mathbb{Z}_5$.

We can then define an orbifold structure on X as the quotient space $\mathbb{CP}^5/\mathbb{Z}_5$, where $\mathbb{Z}_5$ acts on by the above action. The orbifold has singularities at the points

$$[1:0:0:0:0], \quad [0:1:0:0:0], \quad [0:0:1:0:0], \quad [0:0:0:1:0], \text{ and } \quad [0:0:0:0:1],$$

in $\mathbb{CP}^4$, which are fixed points of the $\mathbb{Z}_5$ action.

Near each singular point, the orbifold is locally modeled on the quotient space $\mathbb{C}^5/\mathbb{Z}_5$, where $\mathbb{Z}_5$ acts on $\mathbb{C}^5$ by the action

$$(w_1, w_2, , w_3, w_4, w_5) \mapsto (e^{2\pi i/5}w_1, e^{2\pi i/5}w_2, e^{2\pi i/5}w_3, e^{2\pi i/5}w_4, e^{2\pi i/5}w_5).$$

These quotient spaces have well-known singularities, which are called cyclic quotient singularities and are the local model for an orbifold point. [HKK+03]

Example 2.1.6 *Consider the hypersurface X in $\mathbb{CP}^4$ with homogeneous coordinates $[z_0, z_1, z_2, z_3, z_4]$ defined by the homogeneous equation:*

$$z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + \alpha z_0 z_1 z_2 z_3 z_4 = 0,$$

for $\alpha \in \mathbb{C}$ constant. X can be realized as an orbifold by considering its action under the group $\mathbb{Z}_5$.

To define the orbifold structure, we begin by constructing an action of $\mathbb{Z}_5$ on $\mathbb{C}^5$ that preserves X. We do this by defining an action of $\mathbb{Z}_5$ on the coordinates $[z_0, z_1, z_2, z_3, z_4]$ by setting:

$$\rho_k[z_0, z_1, z_2, z_3, z_4] = [e^{2\pi ik/5}z_0, e^{2\pi ik/5}z_1, e^{2\pi ik/5}z_2, e^{2\pi ik/5}z_3, e^{2\pi ik/5}z_4],$$

for $k \in \mathbb{Z}_5$*. This action clearly preserves* X*, since each term* z_0^5*,* z_1^5*,* z_2^5*,* z_3^5*,* z_4^5 *and* $z_0z_1z_2z_3z_4$ *is invariant under the action of* $\mathbb{Z}_5$*.*

We can then define an orbifold structure on X *as the quotient space* $\mathbb{CP}^4/\mathbb{Z}_5$*, where* $\mathbb{Z}_5$ *acts as described above. The orbifold has singularities at the points*

$$[1:0:0:0:0], \quad [0:1:0:0:0], \quad [0:0:1:0:0], \quad [0:0:0:1:0], \text{ and } \quad [0:0:0:0:1],$$

in $\mathbb{CP}^4$*, which are fixed points of the* $\mathbb{Z}_5$ *action.*

As X *is a* $3-$*dimensional curve in* $\mathbb{CP}^4$*, near each singular point, the orbifold is locally modeled as the quotient space* $\mathbb{C}^3/\mathbb{Z}_5$*. Here* $\mathbb{Z}_5$ *acts on* $\mathbb{C}^3$ *through the action:*

$$(w_1, w_2, w_3) \mapsto (e^{2\pi i/5}w_1, e^{2\pi i/5}w_2, e^{2\pi i/5}w_3).$$

These quotient spaces exhibit well-known singularities, referred to as cyclic quotient singularities, which serve as the local model for an orbifold point [HKK⁺03]. However, it is important to note that other types of singularities can arise from non-cyclic finite groups.

2.1.1 Bundles on orbifolds

It is possible to introduce fiber bundles on orbifolds in a similar manner to how we do on manifolds. A first approach is by defining vector fields and differential forms, which can be treated as sections of vector bundles. To define a vector bundle over an orbifold, we can use the same definition as for manifolds, which involves assigning a vector space to each point in the orbifold and gluing them together consistently over a cover. Similarly, we can define a section of a vector bundle over an orbifold as a continuous map that assigns a vector to each point in the orbifold, along with a projection map that satisfies certain local triviality conditions adapted for orbifolds.

Definition 2.1.7 (Fiber bundle) *[CR04] Let* X *be an orbifold,* F *and* E *be topological spaces (called the typical fiber and the total space respectively). A fiber bundle on* X *is given by:*

- *A projection map* $\pi_E : E \to X$*, such that for each point* $p \in X$*, there exists a uniformizing system* (V, G, π) *around* $p \in U$*, where* U *is an open connected subset of* X*.*
- *A* G*-equivariant homeomorphism:*

$$\psi\colon \pi_E^{-1}(U) \to (V \times F)/G,$$

where G acts on $V \times F$ by $g \cdot (v, f) = (g \cdot v, g \cdot f)$ and $g \in G$, $v \in V$, $f \in F$. This homeomorphism should satisfy $\pi_E(\pi_E^{-1}(U)) = \varphi^{-1}(V/G)$, where φ is local homeomorphism from the open subset U onto the quotient space V/G, such that the following diagram commutes:

$$\begin{array}{ccc} \pi_E^{-1}(U) & \xrightarrow{\psi} & (V \times F)/G \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \tilde{\varphi}} \\ U & \xrightarrow{\varphi} & V/G \end{array}$$

where $\tilde{\varphi}$ represents the natural projection from $(V \times F)/G$ to V/G induced by the group action.

Definition 2.1.8 *Given an orbifold fiber bundle E we say it is a vector bundle if the typical fiber F has the structure of a vector space.*

Definition 2.1.9 *Let X be an orbifold, and let E be a fiber bundle over X. A section of the fiber bundle over the orbifold is a continuous map $s : X \to E$ such that $\pi(s(x)) = x$ for all $x \in X$ and should satisfy the following:*

- ***Locally compatible with the orbifold structure:*** *For each point $x \in X$, there is an open neighborhood $U_x \subseteq X$ and a uniformizing system (V_x, G_x, ϕ_x) such that the restriction of the section s to U_x can be lifted to a continuous map $s_x : U_x \to V_x$, where V_x is a local model for the fiber bundle in the uniformizing system.*

- ***Consistent with group actions:*** *The lifted maps s_x should be equivariant with respect to the group actions, meaning that $s_x(gx) = g(s_x(x))$ for all $x \in U_x$ and $g \in G_x$, where G_x is the finite group acting on the local model V_x.*

Definition 2.1.10 *Let X be an orbifold, and let $\bigwedge^k T^*X$ denote the vector bundle formed by taking the k-fold exterior product of the cotangent spaces of X. A **differential k-form** on X is defined as a section of the vector bundle $\bigwedge^k T^*X$, we will denote by $\Omega^k(X)$ the set of all differential k-forms over X.*

Definition 2.1.11 *Given an orbifold X and $\omega \in \Omega^k(X)$, the integration of ω over X, denoted by $\int_X \omega$, is defined as follows:*

- *Let (U_p, G_p, φ_p) be a uniformized system $p \in X$, we locally lift the differential form ω to a form $\tilde{\omega}_p$ on $\tilde{U}_p \simeq V_p/G_p$.*

- *Integrate the lifted form $\tilde{\omega}_p$ over the fundamental domain of the group action of G_p, summing the integrals of the transformed forms for each $g \in G_p$.*

- *Patch together the local integrals to obtain a global integral, ensuring compatibility on intersections of open sets and independence of the choice of partition.*

Definition 2.1.12 *Let $X = M/G$ an n-dimensional orbifold, U an open subset of X and $\omega \in \Omega^n(U)$ a G_U-invariant differential form, the integral over U is defined as*

$$\int_U \omega := \frac{1}{|G_U|}\int_{\tilde{U}} \tilde{\omega},$$

where $|G_U|$ is the order of G_U.

Definition 2.1.13 *[Zvo12] The homology and cohomology groups of an orbifold over $\mathbb{Q}$, are defined to be the homology and cohomology groups of the orbifold's underlying topological space, with coefficients in $\mathbb{Q}$.*

Consider an orbifold X and an irreducible sub-orbifold Y[1]. Let $\widehat{X}$ and $\widehat{Y}$ represent the underlying topological spaces of X and Y, respectively. In the context of homology with rational coefficients, the homology class $[Y]$ within $H_*(X,\mathbb{Q})$, which is isomorphic to $H_*(\widehat{X},\mathbb{Q})$, is defined as $\frac{1}{|G_Y|}[\widehat{Y}]$ in $H_*(\widehat{X},\mathbb{Q})$. This convention reflects the quotient structure of the orbifold's homology induced by the action of the stabilizer group.

Example 2.1.14 *We define the orbifold $\mathcal{M}_{1,1}$ as follows:*

$$\mathcal{M}_{1,1} = \frac{\mathbb{H}}{SL(2,\mathbb{Z})} \simeq \left\{ z \in \mathbb{H} \mid \mathrm{Re}(z) \leq \frac{1}{2}, |z| \geq 1 \right\} / \sim .$$

Here, $\sim$ denotes an identification of vertical lines $\mathrm{Re}(z) = \pm\frac{1}{2}$ and unit semi-arcs from $\sigma_\pm = \pm\frac{1}{2} + i\frac{\sqrt{3}}{2}$ to i.

The stabilizer groups are given by:

$$G_i \cong \mathbb{Z}_4, \quad G_{\sigma_+} \cong \mathbb{Z}_6, \quad G_{\hat{D}} \cong \mathbb{Z}_2,$$

where $\hat{D} = \mathcal{M}_{1,1} \setminus \{\sigma_+, i\}$.

If we decompose $\mathcal{M}_{1,1}$ into cells as shown in Figure 2.1, we obtain the following table:

n-cell	c_j^n
0	$c_1^0 = i$ $c_2^0 = \sigma_+$
1	$c_1^1(t) = i + it, t \in \mathbb{R}_{\geq 0}$ $c_2^1(t) = \sigma_+ + it, t \in \mathbb{R}_{\geq 0}$ $c_3^1 = \{z \in \mathbb{H} \mid \lvert z\rvert = 1, 0 < \mathrm{Re}(z) < \frac{1}{2}\}$
2	$c_1^2 = \{z \in \mathbb{H} \mid \lvert z\rvert > 1, 0 < \mathrm{Re}(z) < \frac{1}{2}\}$ $c_2^2 = \{z \in \mathbb{H} \mid \lvert z\rvert > 1, -\frac{1}{2} < \mathrm{Re}(z) < 0\}$

[1]An orbifold is irreducible if it cannot be expressed as a nontrivial connected sum of two orbifolds. This means that there is no way to cut the orbifold into two pieces such that each piece is itself a nontrivial orbifold with boundary, and the original orbifold is the result of gluing these two pieces along their boundary.

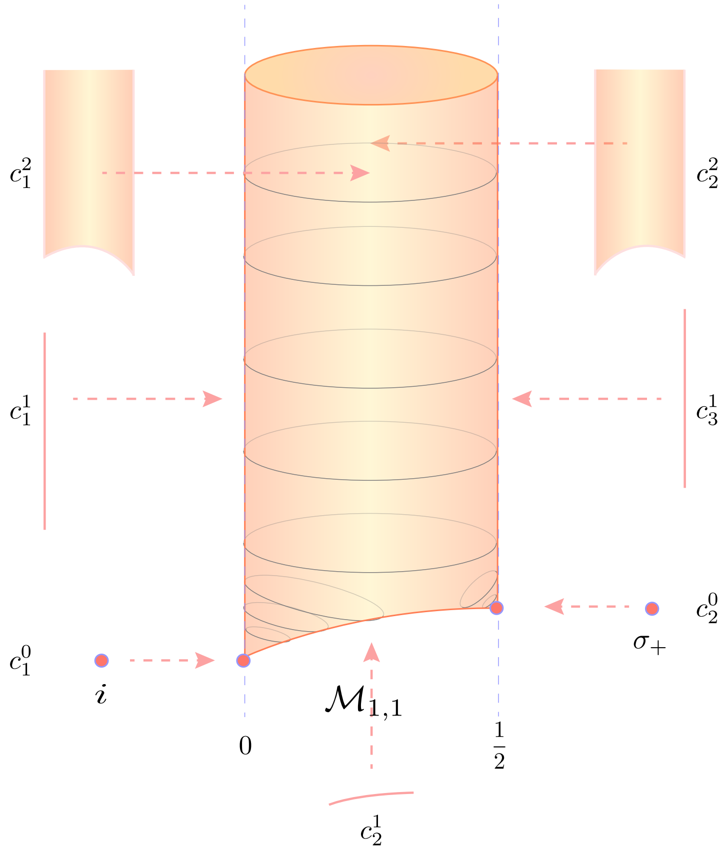

Figure 2.1: Set of n-cells c_j^n of $\mathcal{M}_{1,1}$.

After computing its cell homology, we observe a global 2-to-1 symmetry. This arises because I and $-I$ yield the same quotient topologically. The orbifold Euler characteristic of $\mathcal{M}_{1,1}$, can be computed as follows:

$$\chi(\mathcal{M}_{1,1}) = \frac{1}{2}\left(\frac{c_1^0}{|G_i|} + \frac{c_2^0}{|G_{\sigma_+}|} - \frac{c_1^1 + c_2^1 + c_3^1 - c_1^2 - c_2^2}{|G_{\hat{D}}|}\right) = \frac{1}{24}.$$

2.2 Moduli Spaces of Curves

The moduli space of curves with marked points $\mathcal{M}_{g,n}$ consists of isomorphism classes of smooth projective curves of genus g with n marked points on each curve. Marked points will help to track and study geometric and topological properties of the curves, so the moduli space captures properties related to their geometric structures. The topology of the moduli space has been extensively studied using methods from algebraic geometry and homological algebra [Har86] [Mum83]. This space has found applications in various fields, including:

- Mirror symmetry: relates geometric structures emerging in string theory to elements from algebraic geometry [CdlOGP91] [KM94],

- Enumerative geometry: counting geometric objects with certain properties [Pan06],
- Conformal field theory: conformal 2D systems, including quantum gravity [Dij90].

Definition 2.2.1 *Let g, $n \in \mathbb{Z}_{\geq 0}$. A genus-$g$ curve with n marked points is a connected, smooth projective curve $C_{g,n}$ of genus g together with n distinct points $p_1, \dots, p_n \in C_{g,n}$. Two such curves $(C_{g,n}, p_1, \dots, p_n)$ and $(C'_{g,n}, p'_1, \dots, p'_n)$ are considered isomorphic if there exists an isomorphism of algebraic curves $\phi : C_{g,n} \to C'_{g,n}$ taking p_i to p'_i with $1 \leq i \leq n$. For $2g + n - 2 > 0$, we define the moduli space $\mathcal{M}_{g,n}$ of genus-g curves with n marked points as the space of isomorphism classes of such curves.*

The moduli space $\mathcal{M}_{g,n}$ is a complex orbifold of dimension $3g - 3 + n$. It requires $3g - 3$ parameters, that account for all possible deformations of $C_{g,*}$, and n parameters to represent all possible choices of n distinct points on $C_{g,n}$. [HL97] [ACGH85]

2.2.1 The moduli space $\mathcal{M}_{0,3}$

Let $X = \mathbb{CP}^1$ with three marked points x_1, x_2 and x_3 pairwise different, then there exists a unique element in $PSL(2, \mathbb{C})$ that sends (x_1, x_2, x_3) to $(0, 1, \infty)$ in $\mathbb{CP}^1$ given by:

$$F(x) = \frac{(x - x_1)(x_2 - x_3)}{(x - x_3)(x_2 - x_1)}.$$

This means in particular, we can always find a Möbius transformation that maps $\mathbb{CP}^1$ to itself while sending the three marked points to $(0, 1, \infty)$, so there is only one such curve up to isomorphism, hence $\mathcal{M}_{0,3}$ is a single point.

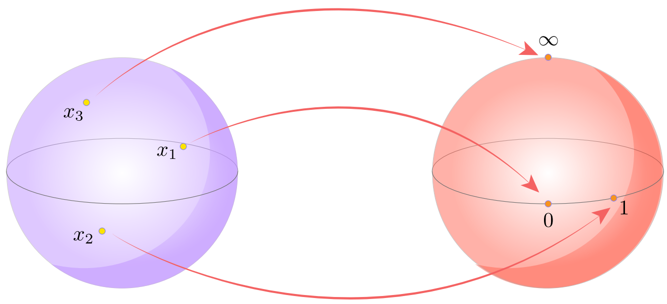

Figure 2.2: Map between $\mathbb{CP}^1$ with three marked points x_1, x_2 and x_3 to $(\mathbb{CP}^1, 0, 1, \infty)$.

2.2.2 The moduli space $\mathcal{M}_{1,1}$

The moduli space $\mathcal{M}_{1,1}$ parameterizes isomorphism classes of elliptic curves (compact Riemann surfaces of genus 1) with a single marked point. In this case, the space has a more complicated description than $\mathcal{M}_{0,3}$. We start by representing an elliptic curve $\mathcal{E}$ as a complex torus, which is the quotient of the complex plane by a lattice Γ, i.e. $\mathcal{E} = \mathbb{C}/\Gamma$.

A lattice Γ can be generated by two linearly independent complex numbers ω_1 and ω_2 as $\Gamma = \{m\omega_1 + n\omega_2 \mid m, n \in \mathbb{Z}\}$. Suppose we have a homothety with a factor of $\lambda \in \mathbb{C}^*$, which scales the lattice as $\Gamma' = \{\lambda m\omega_1 + \lambda n\omega_2 \mid m, n \in \mathbb{Z}\}$. The new lattice Γ' will differ from the original lattice Γ, however, if we consider the complex tori $\mathbb{C}/\Gamma$ and $\mathbb{C}/\Gamma'$, they are isomorphic as complex tori, meaning that the corresponding elliptic curves are isomorphic as well.

Since it is always possible to find a unique basis $\{\omega_1, \omega_2\}$ if we now consider that homotheties do not change the elliptic curve, we might as well consider instead the basis $\{\frac{\omega_1}{\omega_2}, 1\} = \{\tau, 1\}$, with $\tau = \frac{\omega_1}{\omega_2}$. Furthermore, the condition for $\{\omega_1, \omega_2\}$ being an oriented basis translates into $\mathrm{Im}(\omega_1/\omega_2) = \mathrm{Im}(\tau) > 0$. This means that the space of all possible lattices, up to complex multiplication, can be identified with the set of all points in the complex upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. We will denote by Γ_τ the lattice generated by the basis $\{\tau, 1\}$.

Let τ_1 and $\tau_2 \in \mathbb{H}$, two tori: $T_{\tau_1} = \mathbb{C}/\Gamma_{\tau_1}$ and $T_{\tau_2} = \mathbb{C}/\Gamma_{\tau_2}$ are biholomorphic if only if

$$\tau_2 = \frac{a\tau_1 + b}{c\tau_1 + d},$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$.

This means, the modular group $SL(2, \mathbb{Z})$ changes the generators in a specified lattice Γ_τ. Therefore, we examine the action of $SL(2, \mathbb{Z})$ on the upper half-plane, which is characterized by fractional linear transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} := \frac{az + b}{cz + d},$$

where $z \in \mathbb{H}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$.

Two points $z_1, z_2 \in \mathbb{H}$ are considered equivalent if there exists an element of the modular group that maps one to the other, so the moduli space $\mathcal{M}_{1,1}$ can be constructed as the quotient of the upper half-plane $\mathbb{H}$ by the action of the modular group:

$$\mathcal{M}_{1,1} \simeq \mathbb{H}/\,\mathrm{SL}(2, \mathbb{Z}).$$

This means every orbit of $SL(2, \mathbb{Z})$ will be represented as a point in the quotient.

The group $SL(2, \mathbb{Z})$ is generated by the following elements:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Proposition 2.2.2 *The set $\mathbb{H}/SL(2,\mathbb{Z})$ is isomorphic to*

$$D = \left\{ z \in \mathbb{H} \;\middle|\; \frac{-1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2} \text{ and } 1 \leq |z| \right\} / \sim,$$

where for $\mathcal{L}_- = \{z \in \mathbb{H} \mid |z| = 1 \text{ and } -1/2 < \mathrm{Re}(z) < 0\}$, $\mathcal{L}_+ = \{z \in \mathbb{H} \mid |z| = 1 \text{ and } 0 < \mathrm{Re}(z) < 1/2\}$ and $\sigma_\pm = \pm\frac{1}{2} + i\frac{\sqrt{3}}{2}$ we have $\mathcal{L}_+ \sim \mathcal{L}_-$ and $\sigma_+ \sim \sigma_-$

Proof. As $SL(2,\mathbb{Z})$ is generated by the following elements:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We will focus on the effects of T and V on τ (lattice generator)

$$T(\tau) = \tau + 1,$$

$$V(\tau) = -\frac{1}{\tau},$$

this shows T is a translation of the real part of τ, so, the orbit of τ under the action of T can composed of elements whose $|\mathrm{Re}(\tau)| \leq \frac{1}{2}$.

Recall from eq. (B.0.4), that for any $L \in SL(2,\mathbb{Z})$, $\mathrm{Im}(L(z)) \to 0$ so, either $c \to \infty$ or $d \to \infty$, this implies $\mathrm{Im}(T(z))$ is bounded, as neither c or d can be zero simultaneously (because $det(\substack{ab\\dc}) = 1$), so let z be an element in the orbit space whose imaginary part is maximal, as we saw before, the action of T leaves the imaginary part invariant, however the action of V gives:

$$\mathrm{Im}(V(z)) = \frac{\mathrm{Im}(z)}{|z|^2},$$

so, if $|z| < 1$, then $\mathrm{Im}(V(z)) > \mathrm{Im}(z)$. This contradicts the fact that $\mathrm{Im}(z)$ was maximal, hence, we must have $|z| \geq 1$. □

Now, let us consider two elements τ' and τ that lie within the same orbit in the set D, such that

$$\begin{pmatrix} \tau' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix},$$

for $(\substack{ab\\cd}) \in SL(2,\mathbb{Z})$. We consider $c \geq 0$ and $\mathrm{Im}(\tau') \geq \mathrm{Im}(\tau)$, by (B.0.4) we have

$$|c\tau + d| \leq 1, \tag{2.2.1}$$

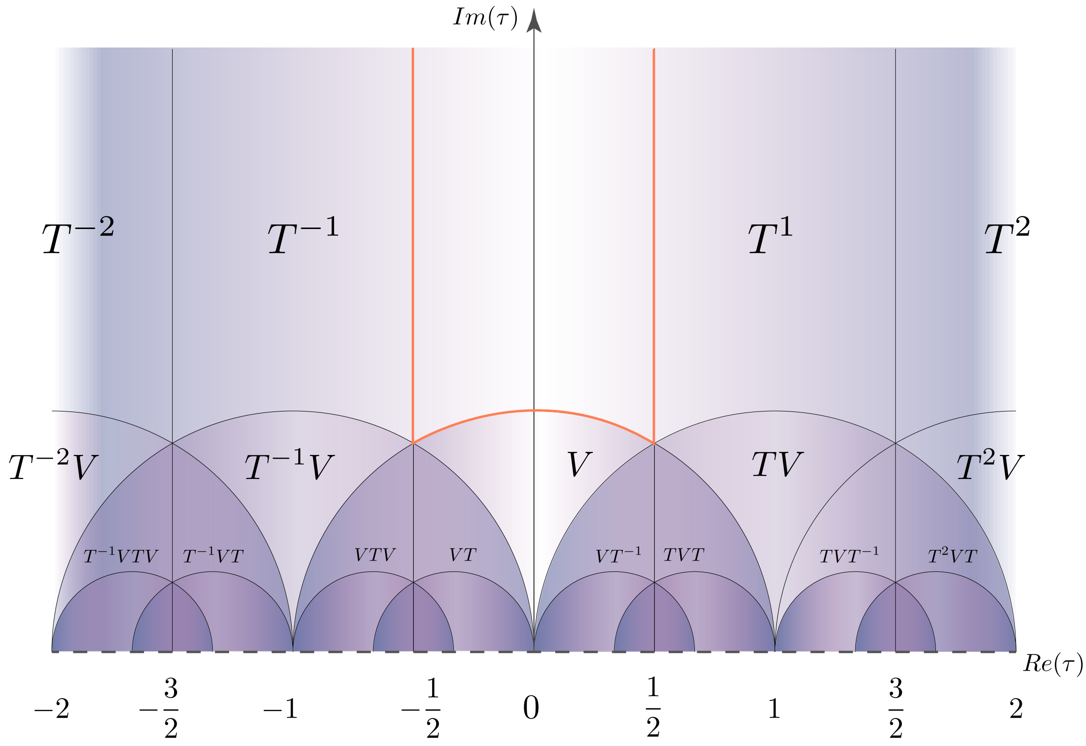

Figure 2.3: The set $\mathbb{H}/SL(2,\mathbb{Z})$ (marked in red) and the image under the composition of T and V.

from $|\text{Re}(\tau)| \leq 1/2$ and $|\tau| \geq 1$ we get $\text{Im}(\tau) \geq \sqrt{3}/2$ so (2.2.1) gives

$$\left|\frac{c\sqrt{3}}{2}\right| \leq 1,$$

as $c \in \mathbb{Z}$, this leaves two options

- $c = 0$, by the determinant condition we get $a = \pm 1 = d$, so

$$A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = T^b.$$

If $|\tau'| \leq 1/2$ and $|\tau| \leq 1/2$ for being in D then $b = \pm 1$, this can only be achieved by points lying on the the vertical lines $|\text{Re}(z)| = 1/2$ this means both lines are identified.

- $c = 1$

Because $|c\tau + d| \leq 1$ and $\tau > 0$, then $|x + d| < 1$ from this we have different possibilities for $d = 0, \pm 1$.

- $c = 1$ and $d = 0$

From $ad - bc = 1$ we have $b = -1$ so

$$A = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix} = T^a V.$$

As T^a is a translation, if we want to stay inside D, either $a = 0$ or $a = \pm 1$, for $a = 0$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so

$$A(\tau) = V(\tau) = -\frac{1}{\tau},$$

if $|\tau| > 1$ then $V(\tau) \notin D$, so $|\tau| = 1$ and the action of V reads

$$V(\tau) = -\text{Re}(\tau) + i\text{Im}(\tau),$$

this identifies the arcs of the unitary circle $\mathcal{L}_-$ and $\mathcal{L}_-$ given by

$$\mathcal{L}_- = \left\{ z \in \mathbb{H} \mid |z| = 1 \text{ and } -\frac{1}{2} < \text{Re}(z) < 0 \right\},$$

$$\mathcal{L}_+ = \left\{ z \in \mathbb{H} \mid |z| = 1 \text{ and } 0 < \text{Re}(z) < \frac{1}{2} \right\}.$$

It is relevant to note the point: i is invariant under such transformation, indeed

$$V(i) = -\frac{1}{i} = i.$$

As for $a = \pm 1$ we will first do a translation T and then the "inversion" V, even though this might seem too restrictive for a given interval of length 1, there is actually a point that satisfies both transformations and is still in D, this point lies at the endpoints of the vertical line that intersects the arc $\mathcal{L}_\pm$, we call it $\sigma_\pm = \pm\frac{1}{2} + i\frac{\sqrt{3}}{2}$, we see indeed:

$$T(\sigma_\pm) = \sigma_\mp \qquad \text{and} \qquad V(\sigma_\pm) = \sigma_\mp,$$

so

$$TV(\sigma_\pm) = \sigma_\mp.$$

Whereas we have $V^2 = (TV)^3 = I$, and by the above arguments we have that the local groups of i and σ_+

$$G_i = \{I, -I, V, -V\} \cong \mathbb{Z}/4\mathbb{Z}, \tag{2.2.2}$$

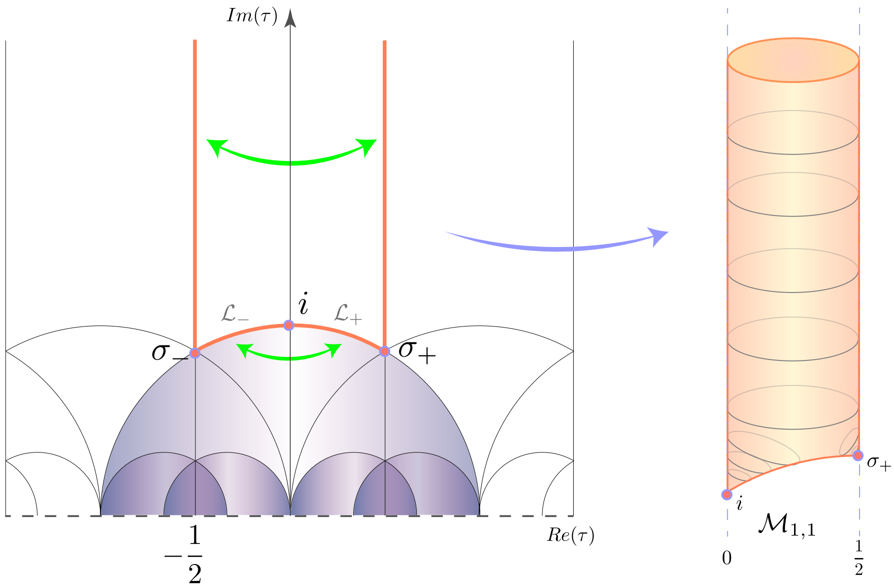

Figure 2.4: Process of identification of the vertical lines $\mathrm{Re}(\tau) = \pm\frac{1}{2}$ and of the arcs $\mathcal{L}_\pm$ resulting in the moduli space $\mathcal{M}_{1,1}$.

$$G_{\sigma_+} = \left\{I, -I, TV, -TV, (TV)^2, -(TV)^2\right\} \cong \mathbb{Z}/6\mathbb{Z}. \tag{2.2.3}$$

This is precisely the reason for which we use orbifolds instead of manifolds, here the local groups are not trivial and the orbifold structure will remember this local information.

This space parameterizes all isomorphism classes of elliptic curves with a marked point. It is worth noting that $\mathcal{M}_{1,1}$ is not compact. Its compactification, known as the Deligne-Mumford compactification (cf. 2.3), is denoted by $\overline{\mathcal{M}}_{1,1}$ and includes stable nodal curves (curves with at most one node) in addition to smooth elliptic curves. The boundary of the compactification corresponds to the cusps of the modular group action on the upper half-plane.

2.2.3 The moduli space $\mathcal{M}_{0,n}$

The moduli space $\mathcal{M}_{0,4}$ parameterizes isomorphism classes of stable rational curves with 4 marked points. As before, a stable 4-pointed rational curve is a Riemann sphere with x_1, x_2, x_3 and x_4 distinct marked points. Since the complex structure of the Riemann sphere is fixed, the only possible parameter in the moduli space comes from the position of the 4th marked point x_4.

By applying a Möbius transformation, we can always fix the positions of x_1, x_2 and x_3 to 0, 1, and ∞ respectively. Then, the position of the fourth marked point can be any complex number t except 0, 1, and ∞, the parameter t can be expressed in terms of the other points as:

$$t = \frac{(x_4 - x_1)(x_2 - x_3)}{(x_4 - x_3)(x_2 - x_1)},$$

because $t \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}$, then:

$$\mathcal{M}_{0,4} = \mathbb{CP}^1 \setminus \{0, 1, \infty\}.$$

Note that this space is also non-compact as the points: $\{0, 1, \infty\}$ are missing. The Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,4}$ adds the points $\{0, 1, \infty\}$ back to $\mathcal{M}_{0,4}$, making it the Riemann sphere $\mathbb{CP}^1$.

As the number of marked points increases, the moduli space $\mathcal{M}_{0,n}$ exhibits combinatorial properties and requires a more detailed description. Consider $\mathbb{CP}^1$ with n pairwise distinct points. By applying a Möbius transformation, we fix the positions of three points, (x_1, x_2, x_3), mapping them to $(0, 1, \infty)$. For the remaining points $(x_4, \ldots, x_n)$, we associate them with $(t_1, \ldots, t_{n-3})$ such that $t_i \neq t_j$ for $i \neq j$. This ensures that each point in the remaining set has a unique corresponding value in the target set, with:

$$t_i = \frac{(x_{i+3} - x_1)(x_2 - x_3)}{(x_{i+3} - x_3)(x_2 - x_1)}$$

This implies that each $t_i \in \mathbb{CP}^1 \setminus \{0, 1, \infty\}$ and $t_i \neq t_j$ for $i \neq j$. Therefore, the moduli space $\mathcal{M}_{0,n}$ can be described as the product of these spaces while avoiding the diagonals, as follows:

$$\mathcal{M}_{0,n} = \{(t_1, \ldots, t_{n-3}) \in (\mathbb{CP}^1)^{n-3} \mid t_i \notin \{0, 1, \infty\},\ t_i \neq t_j \text{ for } i \neq j\},$$

for $n > 3$.

2.3 Deligne-Mumford compactification

We previously observed that the moduli space $\mathcal{M}_{0,4}$ is non-compact, similarly, we can deduce that $\mathcal{M}_{0,n\geq 4}$ is also non-compact. In general, the moduli space $\mathcal{M}_{g,n}$ might not be compact. However, it is possible to compactify the moduli space by including specific sets of curves in a process known as **Deligne-Mumford compactification**. This method extends the moduli space to include stable curves with nodes (singularities). The compactification was introduced by Pierre Deligne and David Mumford in 1969. [DM69]

The idea behind the Deligne-Mumford compactification is to include stable nodal curves in the moduli space. A stable curve is a curve with only nodes as singularities, and finite automorphism group. The nodes can be thought of as "points of attachment" between different components of the curve. By allowing these singularities, the moduli space becomes compact.

Definition 2.3.1 *A **simple nodal point** within a curve $C_{g,n}$ is defined as an isolated singularity characterized by the self-transversal intersection of the curve at a distinct point, near such a nodal point, the curve can be locally modeled by the equation $xy = 0$ in $\mathbb{C}^2$.*[1]

Definition 2.3.2 *A nodal curve is a complex curve $C_{g,n}$ with simple nodal points.*

Definition 2.3.3 *Given a nodal curve $C_{g,n}$, we define its* normalization *to be the smooth, though not necessarily connected, curve $C^S_{g,n}$. This process involves separating the original curve at its nodes, where each singular point, or node, of the curve is replaced with distinct marked points.*

Definition 2.3.4 *A **stable curve** $C_{g,n}$ is a connected nodal curve satisfying the following conditions:*

- *The only singularities of $C_{g,n}$ are simple nodal points.*
- *If x_i and x_j are marked points on $C_{g,n}$, then $x_i \neq x_j$ for all $i \neq j$, and none of the x_i can be nodal points.*
- *Irreducible components of genus 0 must have at least three points, counting both nodal points and marked points.*
- *Irreducible components of genus 1 must have at least one nodal point or marked point.*

The boundary of $\mathcal{M}_{g,n}$ needs to describe the local behaviour when we approach a limit point, in the case of $\mathcal{M}_{0,4}$ this limit corresponds to two marked points colliding within $\mathbb{CP}^1$, so we begin first by studying a stable curve; for a Riemann surface with genus 0 and 4 marked points $C_{0,4}$, we have the isomorphism:

$$(C_{0,4}, x_1, x_2, x_3, x_4) \backsimeq (\mathbb{CP}^1, , 0, 1, \infty, t),$$

so, the boundary elements must represent the limits: $t \to 0$, $t \to 1$ and $t \to \infty$. To visualize this process, consider the following example [Zvo12], take a family of curves in $\mathbb{CP}^2$ denoted by $C(s)$, let $[x, y, x]$ be homogeneous coordinates in $\mathbb{CP}^2$, the family of curves parametrized by $s \in I \subset \mathbb{R}$ is defined as:

$$C(s) = \{[x, y, z] \in \mathbb{CP}^2 \mid xy = sz^2\},$$

this means, each value of s defines a 4−marked curve $\mathbb{CP}^1$, on $C(s)$ we mark the following points:

$$x_1 = [0, 1, 0], \qquad x_2 = [1, s, 1],$$

[1]The equation $xy = 0$ in $\mathbb{C}^2$ represents the standard local model for a simple node, implying that the curve looks like the crossing of two lines near the singularity. In the context of this local model, the curve resembles the transversal intersection of two smooth paths, each referred to as a *branch of the node*.

$$x_3 = [1,0,0], \qquad x_4 = [s,1,1].$$

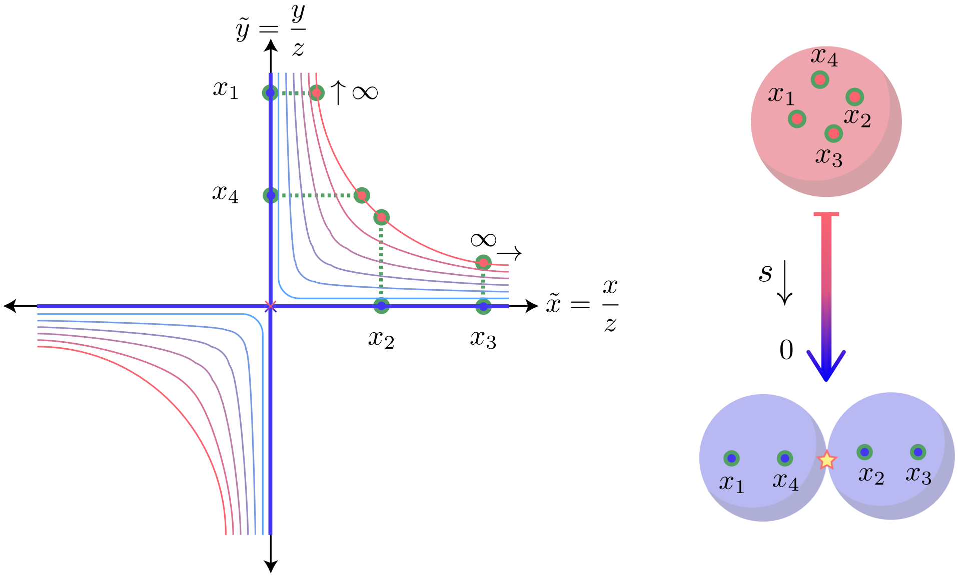

Figure 2.5: A family of curves C_s in $\mathcal{M}_{0,4}$ approaching a limit point as $s \to 0$.

This example shows the behavior of a curve in $\mathcal{M}_{0,4}$ where $x_4 \to x_1$ as $s \to 0$, it is important to keep the marked points separated as we take the limit, the details on this process are very technical but the idea is to apply conformal transformations while keeping the colliding points separated, as a consequence the distance with respect to the other points gets very large and in the limit, the curve collapses a region into one point (the nodal point) creating a separation between the colliding points and the rest of the marked points. This example captures the 'jump' from one sphere into two spheres, and in the process distributing the marked points with the node to obtain a stable surface.

In this example we also notice points on the curve are either smooth marked points or are isomorphic to a neighborhood of $xy = 0$ in $\mathbb{C}^2$, this means, if we take a neighborhood of a node, we can transform it by diffeomorphisms into two open disks with their centers identified, there are two ways to remove the singularity at a nodal points, we will refer to them as:

- **Normalized node**: If the neighborhood near a node can be replaced by two disjoint open discs.
- **Smoothened node**: If the neighborhood near a node can be replaced by a cylinder.

Theorem 2.3.5 *There exists a* $(3g-3+n)$*-dimensional compact complex differential orbifold* $\overline{\mathcal{M}}_{g,n}$*, a* $(3g-2+n)$*-dimensional compact complex differential orbifold* $\overline{\mathcal{C}}_{g,n}$*, and a map* $\pi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ *such that*

- *$\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ is an open dense sub-orbifold and $\pi^{-1}(\mathcal{M}_{g,n}) = \mathcal{C}_{g,n} \subset \overline{\mathcal{C}}_{g,n}$.*
- *The fibers of π are stable surface of genus g with n marked points.*
- *Each stable curve is isomorphic to only one fiber.*
- *The stabilizer of a point $p \in \overline{\mathcal{M}}_{g,n}$ is isomorphic to the automorphism group of the corresponding stable curve $C_{g,n} \simeq [p]$.*

Definition 2.3.6 *We call the space $\overline{\mathcal{M}}_{g,n}$ the **Deligne-Mumford compactification** of $\mathcal{M}_{g,n}$, and the set $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$ its boundary.*

Definition 2.3.7 *The family of curves $\overline{\mathcal{C}}_{g,n}$ such that $\pi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ is called the **universal curve.***

Theorem 2.3.8 *[Zvo12] Let $\pi : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n}$ be the universal curve and $z \in \overline{\mathcal{C}}_{g,n}$ a node in a singular fiber. Then there is a neighborhood of z in $\overline{\mathcal{C}}_{g,n}$ with a system of local coordinates $(w_1, ..., w_{3g-4+n}, x, y)$ and a neighborhood of $\pi(z) \in \overline{\mathcal{M}}_{g,n}$ with a system of local coordinates $(s_1, ..., s_{3g-3+n})$ such that in these coordinates π is given by*

$$s_i = w_i, \qquad \text{for } 1 \leq i \leq 3g-4+n, \qquad s_{3g-3+n} = xy.$$

The Deligne-Mumford compactification provides a method for adding strata with the same dual intersection complex to the limit set and characterizing the topology of $\overline{\mathcal{M}}_{g,n}$.[1] The prevoius theorem also reinforces the idea illustrated in Fig. 2.5, which serves as a model for the behavior near nodal points. Although numerous elements need to be incorporated during the compactification process, the number of elements added will be finite, as each element on the boundary must have the same Euler characteristic. This ensures that the resulting compactified space remains manageable and retains its essential topological and geometric properties.

2.4 Forgetful and attaching maps

There are two primary types of natural maps between moduli spaces: the forgetful maps and the attaching (or gluing) maps. These maps induce relations on the cohomology classes of $\overline{\mathcal{M}}_{g,n}$. The forgetful map removes m marked points from elements in $\overline{\mathcal{M}}_{g,n+m}$, resulting in stable curves in $\overline{\mathcal{M}}_{g,n}$. On the other hand, the attaching map identifies curves in the moduli space by connecting them through a pair of marked points gluing the curves together, or gluing the curve with itself increasing its genus. These operations allow us to construct more complex moduli spaces and also to study some relations between them.

[1]The boundary strata refer to specific sections at the edges of moduli spaces, representing types of degenerate curves, such as those with nodal singularities. These strata help us understanding the compactification and geometric properties of the space, classifying curves by their topological features.

Definition 2.4.1 (Stabilization of a Nodal Curve) *Let* $(C_{g,n+m}, x_1, \ldots, x_{n+m})$ *be a nodal curve in* $\overline{\mathcal{M}}_{g,n+m}$. *The* stabilization *of this curve, denoted by Stab, transforms it into a stable curve* $(C^{stab}_{g,n}, x_1, \ldots, x_n)$. *Should the curve* $(C_{g,n}, x_1, \ldots, x_n)$ *satisfy the condition* $2 - 2g - n < 0$ *yet contain unstable components—particularly, genus 0 components with fewer than three special points—then these components must be contracted into singular points. This process is carried out iteratively, removing each unstable genus 0 component through contraction.*

Definition 2.4.2 (Forgetful Map) *The* forgetful map $f_m : \overline{\mathcal{M}}_{g,n+m} \to \overline{\mathcal{M}}_{g,n}$ *is defined as the morphism that takes an element* $(C_{g,n+m}, p_1, \ldots, p_{n+m})$ *in* $\overline{\mathcal{M}}_{g,n+m}$ *and maps it to* $(C^{stab}_{g,n}, p_1, \ldots, p_n)$ *in* $\overline{\mathcal{M}}_{g,n}$ *by omitting the last* m *marked points and subsequently stabilizing the curve:*

$$f_m(C_{g,m+n}, p_1, \ldots, p_{n+m}) = (C^{stab}_{g,n}, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{g,n}.$$

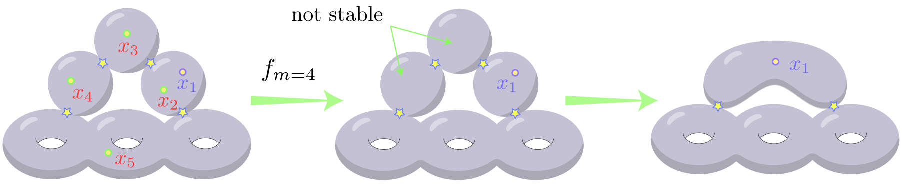

Figure 2.6: Forgetful map of the points $\{x_2, x_3, x_4, x_5\}$ of an element in $\overline{\mathcal{M}}_{4,5}$ to a stable element in $\overline{\mathcal{M}}_{4,1}$.

The motivation for attaching maps originates from the wish to construct new curves by connecting or gluing existing ones, resulting in a different moduli space structure. Given a curve $C'_{g_1,n+1}$ of genus g_1 with $n+1$ marked points and another curve $C''_{g_2,m+1}$ of genus g_2 with $m+1$ marked points, we can identify a marked point in C' with a marked point in C''. This process creates a new curve $C_{g_1+g_2,n+m}$, where the points at which they were attached are replaced by a node.

Another possibility arises when we have a curve $C'_{g_1,n+2}$ of genus g_1 with $n+2$ marked points. By identifying two marked points on the same curve, we can transform it into a curve $C_{g_1+1,n}$. This operation increases the genus of the curve and introduces a new nodal point. With this motivation, we can proceed to formally define the attaching maps and explore their properties:

Definition 2.4.3 *There are two kinds of attaching maps:*

- *Separating attaching map*

Let $\overline{\mathcal{M}}_{g_1,n+1}$ *be the moduli space of stable curves of genus* g_1 *with* $n+1$ *marked points and* $\overline{\mathcal{M}}_{g_2,m+1}$ *be the moduli space of stable curves of genus* g_2 *with* $m+1$ *marked points. The attaching map*

$$\mathrm{gl} : \overline{\mathcal{M}}_{g_1,n+1} \times \overline{\mathcal{M}}_{g_2,m+1} \to \overline{\mathcal{M}}_{g_1+g_2,n+m},$$

maps pairs of equivalence classes of stable curves $(C'_{g_1,n+1}, C''_{g_2,m+1})$ *to an equivalence class of a stable curve* $C_{g_1+g_2,n+m}$ *obtained by identifying a marked point in* $C'_{g_1,n+1}$ *with a marked point in* $C''_{g_2,m+1}$ *and gluing the curves along these points, introducing a nodal point at the gluing location.*

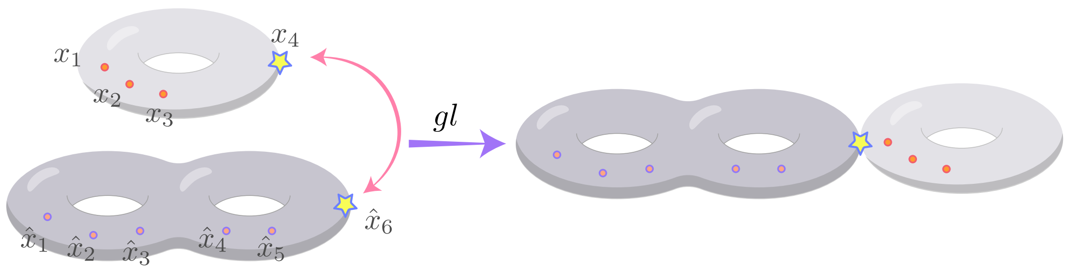

Figure 2.7: Separating attaching map, gluing the points x_4 and $\hat{x}_6$ of elements in $\overline{\mathcal{M}}_{1,4}$ and $\overline{\mathcal{M}}_{2,6}$, the image lies in $\overline{\mathcal{M}}_{3,8}$.

- *Non-separating attaching map.*

Let $\overline{\mathcal{M}}_{g,n+2}$ *be the moduli space of stable curves of genus* g *with* $n+2$ *marked points. The attaching map*

$$\mathrm{gl}^{irr} : \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n},$$

maps an equivalence class of stable curves $C'_{g,n+2}$ *to an equivalence class of a stable curve* $C_{g+1,n}$ *obtained by identifying two marked points on the same curve and gluing them together, increasing the genus of the curve by 1 and introducing a new nodal point.*

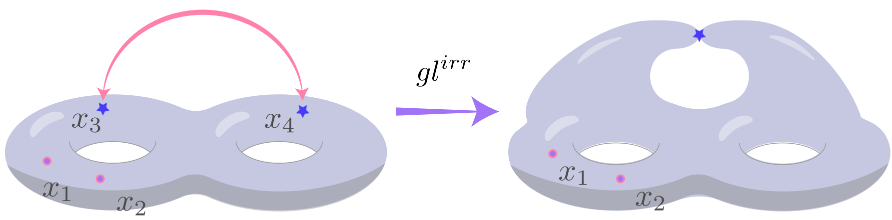

Figure 2.8: Non-separating attaching map , gluing the points x_3 and x_4 of an element in $\overline{\mathcal{M}}_{2,4}$, the image lies in $\overline{\mathcal{M}}_{3,2}$.

2.5 Witten-Kontsevich theorem

The structure of moduli spaces is in general very complicated and performing computations involving these objects is highly non-trivial. Given its geometric structure, it is possible to define cohomology classes on moduli spaces to better understand them. In particular, Chern classes are of great interest, and they can be defined using vector bundles over the moduli space, relating the cohomology classes of the bundle with those on the base space.

One key aspect on the study of moduli spaces are their intersection numbers, which can be obtained through wedge products of Chern classes. In 1990, Edward Witten discovered a connection between algebraic geometry, specifically on intersection numbers of moduli spaces of algebraic curves, and the theory of integrable systems. He observed that intersection numbers played a role in the construction of the free energy function of 2D quantum gravity models, while the function for counting triangulations on surfaces was related to a special tau function of the KdV hierarchy. This led Witten to conjecture a link between algebraic geometry and integrable systems. [Wit90]

Witten's conjecture was later proven by Maxim Kontsevich [Kon92], this remarkable result, known as the Witten-Kontsevich theorem, has been explored and explained in detail in recent years, a modern introduction and proof can be found in the work of B. Eynard [Eyn16]. Witten's conjecture was later reformulated by Robbert Dijkgraaf, Erik Verlinde, and Herman Verlinde [DVV91], who introduced the Virasoro differential operators L_n for $n \geq -1$ and demonstrated the connection between Witten's conjecture and the Virasoro equations. This connection has opened up plenty of new research possibilities and helped develop new tools to describe the structure of moduli spaces and their intersection numbers.

In $\overline{\mathcal{M}}_{g,n}$, each point represents an equivalence class of algebraic curves (up to isomorphism), and the tangent space at a point is related to the infinitesimal deformations of the corresponding curve. The cotangent space then consists of linear functionals that act on these deformations.

As each marked point x_i varies smoothly across curves in $\overline{\mathcal{M}}_{g,n}$, we can define a natural **line bundle** $\mathcal{L}_i$ over $\overline{\mathcal{M}}_{g,n}$, whose fiber at each point is the cotangent space at x_i of the respective curve represented in the moduli space.

Given a vector bundle, we define its characteristic classes as invariant polynomials on the curvature of a connection. Since a line bundle is a specific type of vector bundle, we can consider its first Chern class $c_1(L)$. This class is a 2-form on $\overline{\mathcal{M}}_{g,n}$ and, as a topological invariant, it can be shown its definition does not depend on the choice of connection.

Definition 2.5.1 *The ψ-classes are defined as follows:*

$$\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}). \tag{2.5.1}$$

Remark. By taking the wedge product of a sufficient number of ψ-classes, equal to the dimension of $\overline{\mathcal{M}}_{g,n}$, it is possible to compute the integral of a top form, such as $c_1(\mathcal{L}_1)^{k_1} \wedge \cdots \wedge c_1(\mathcal{L}_n)^{k_n}$, over the space $\overline{\mathcal{M}}_{g,n}$.

Definition 2.5.2 *[Wit90] Let $\mathcal{L}_i \to \overline{\mathcal{M}}_{g,n}$ be the cotangent line bundle associated with the marked point x_i, and let $\psi_i = c_1(\mathcal{L}_i)$ denote its ψ-class. Provided that $2g-2+n>0$, we define the intersection numbers as*

$$\langle \tau_{a_1}\tau_{a_2}\cdots\tau_{a_n}\rangle_g := \begin{cases} \int_{\overline{\mathcal{M}}_{g,n}} c_1(\mathcal{L}_1)^{a_1}\wedge\cdots\wedge c_1(\mathcal{L}_n)^{a_n} & \text{if } \sum_{l=1}^n a_l = \dim(\overline{\mathcal{M}}_{g,n}), \\ 0 & \text{if } \sum_{l=1}^n a_l \neq \dim(\overline{\mathcal{M}}_{g,n}). \end{cases}$$

At times, we might opt for a notation distinct from Witten's bracket notation to denote the intersection numbers:

$$\langle \tau_{a_1}\tau_{a_2}...\tau_{a_n}\rangle_g = \langle \psi_i^{a_1}\cdots\psi_n^{a_n}\rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1}\wedge...\wedge\psi_n^{a_n}.$$

The intersection numbers $\langle \tau_{a_1}\tau_{a_2}\ldots\tau_{a_n}\rangle_g$ are also referred to as descendant integrals. We also notice, once g and n are specified, using the dimension of $\overline{\mathcal{M}}_{g,n}$ and the stability conditions for its elements implies that the set of non-zero intersection numbers must be finite.

Moreover, Witten conjectured that if the stability condition $2g-2+n>0$ is satisfied, then the intersection numbers also fulfills two interconnected systems of equations:

$$\begin{aligned} \left\langle \tau_0 \prod_{i=1}^n \tau_{a_i} \right\rangle_g &= \sum_{j=1}^n \left\langle \tau_{a_{j-1}} \prod_{i\neq j} \tau_{a_i} \right\rangle_g, \\ \left\langle \tau_1 \prod_{i=1}^n \tau_{a_i} \right\rangle_g &= (2g-2+n) \left\langle \prod_{i=1}^n \tau_{a_i} \right\rangle_g. \end{aligned} \tag{2.5.2}$$

These equations are referred to as the string and dilaton equations, respectively. This system offers relations for recursively computing all intersection numbers in genus 0. E. Witten proposed a generating function that combines these intersection numbers into a single object, effectively converting a series of complicated calculations into a study of the properties of a single function. This generating function is now known as the Witten generating function of intersection numbers.

Let $u, t_{i>0}$ be a set of variables. We introduce the generating function of genus g as:

$$F_g^W(t_0, t_1, \ldots; u) := \sum_{\substack{n\geq 1\\ 2g-2+n>0}} \frac{u^{2g-2}}{n!} \sum_{a_1,\ldots,a_n\geq 0} \langle \tau_{a_1}\tau_{a_2}\ldots\tau_{a_n}\rangle_g t_{a_1}t_{a_2}\ldots t_{a_n}. \tag{2.5.3}$$

This expression emerges in the context of two-dimensional quantum gravity, which is a simplified model of quantum gravity that retains many important features of the full theory, this theory is also associated with models of string theories and it can be described in terms of certain topological field theories known as topological gravity.

In this model, Riemann surfaces play the role of the space-time history of a one-dimensional string, and the moduli space of such surfaces is a crucial object of the model. Essentially, the different configurations of the string are parametrized by points in the moduli space. Whenever we try to calculate the quantum amplitudes associated with different configurations, the integral must be computed over the entire moduli space, and the integrand involves the exponential of the so-called action of the theory i.e. the Witten potential.

We now introduce the following differential operators:

$$\begin{aligned} L_{-1} &= -\frac{\partial}{\partial t_0} + \frac{u^{-2}}{2} t_0^2 + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial}{\partial t_i}, \\ L_0 &= -\frac{3}{2} \frac{\partial}{\partial t_1} + \sum_{i=0}^{\infty} \frac{2i+1}{2} t_i \frac{\partial}{\partial t_i} + \frac{1}{16}. \end{aligned} \tag{2.5.4}$$

If $\mathcal{F}^W = \sum_{g \geq 0} F_g^W$, then we can rewrite the string and dilaton equations (2.5.2) by the exponential function $e^{\mathcal{F}^W}$ being annihilated by the operators in (2.5.4).

We can recover the intersection numbers expressions from the generating functions F_g^W as follows:

$$\langle \tau_{a_1} \tau_{a_2} \ldots \tau_{a_k} \rangle_g = \frac{\partial}{\partial t_{a_1}} \frac{\partial}{\partial t_{a_2}} \cdots \frac{\partial}{\partial t_{a_k}} F_g^W \bigg|_{t_i=0, u=1}.$$

In the early 90's, E. Witten [Wit90] conjectured that starting from $\mathcal{F}^W$, $U = \frac{\partial \mathcal{F}^W}{\partial t_0^2}$ satisfies the KdV equations and, along with the string equation (eq. (2.5.2)), enables us to subsequently formulate a relation for the intersection numbers:

$$(2n+1) \langle \tau_n \tau_0^2 \rangle = \langle \tau_{n-1} \tau_0 \rangle \langle \tau_0^3 \rangle + 2 \langle \tau_{n-1} \tau_0^2 \rangle \langle \tau_0^2 \rangle + \frac{1}{4} \langle \tau_{n-1} \tau_0^4 \rangle.$$

This conjecture was later proven by M. Kontsevich [Kon92] and it is known in the literature as the *Witten-Kontsevich theorem.*

Theorem 2.5.3 (Witten-Kontsevich) *The generating function $\mathcal{F}^W(t_0, t_1, \ldots)$ is a solution to the Korteweg-de Vries (KdV) hierarchy.*

Building upon the Witten-Kontsevich theorem, the Virasoro conjecture suggests that not only does the generating function of intersection numbers satisfy the KdV hierarchy, but it is also governed by a larger symmetry group known as the Virasoro algebra, this version will be useful in the following chapters and can be stated as follows:

Theorem 2.5.4 (Virsoro formulation of the Witten-Kontsevich theorem) *Let $\mathcal{F}^W$, be the generating function for the intersection numbers on the moduli space $\overline{\mathcal{M}}_{g,n}$.*

Then, the function $\mathcal{F}^W$ satisfies the Virasoro constraints

$$L_n e^{\mathcal{F}^W} = 0, \quad \text{for all } n \geq -1,$$

where the Virasoro operators L_n are defined by:

$$L_n := \sum_{i\geq 0} \frac{(2i+2n+1)!!}{2^{n+1}(2i-1)!!}(t_i - \delta_{i,1})\frac{\partial}{\partial t_{i+n}} + \frac{u^2}{2}\sum_{i=0}^{n-1}\frac{(2i+1)!!(2n-2i-1)!!}{2^{n+1}}\frac{\partial^2}{\partial t_i \partial t_{n-1-i}} + \delta_{n,-1}\frac{t_0^2}{2u^2} + \delta_{n,0}\frac{1}{16}. \tag{2.5.5}$$

These constraints, containing the string and dilaton equations, fully characterize the generating function $\mathcal{F}^W$. Consequently, this enables a recursive computation of all intersection numbers on the moduli spaces. [Zvo12]

Example 2.5.5 *For $g = 0$, it is possible to obtain an explicit formula for all the intersection numbers for $n \geq 3$, if $\sum a_i = n-3$, then*

$$\langle \tau_{a_1} \cdots \tau_{a_n} \rangle_0 = \frac{(n-3)!}{a_1! \cdots a_n!}. \tag{2.5.6}$$

We will proceed by induction, the last example provides the base case when $n = 3$. For the inductive step, we consider $n+1$ marked points. For some fixed index i, we have $a_i = 0$. Then, the string equation gives:

$$\langle \tau_{a_1} \cdots \tau_{a_{n+1}} \rangle_0 = \sum_{j=1}^{n} \left\langle \tau_{a_{j-1}} \prod_{k\neq j} \tau_{a_k} \right\rangle_0 .$$

Evaluating the right side of the equation, we get:

$$\sum_{j=1}^{n} \left\langle \tau_{a_{j-1}} \prod_{k\neq j} \tau_{a_k} \right\rangle_0 = \sum_{j=1}^{n} \frac{a_j (n-3)!}{a_j! \prod_{k\neq j} a_k!}.$$

Notice that $\sum_{j=1}^{n} a_j + a_i = (n+1)-3$, where $a_i = 0$. Substituting this into the previous equation yields:

$$\sum_{j=1}^{n} \left\langle \tau_{a_{j-1}} \prod_{k\neq j} \tau_{a_k} \right\rangle_0 = \frac{(n-2)(n-3)!}{\prod_{k=1}^{n} a_k!} = \frac{((n+1)-3)!}{\prod_{k=1}^{n+1} a_k!}.$$

Thus, the formula is proven for the case when $n+1$ marked points are considered, given that it holds for n points. This completes the proof.

2.6 Dual graphs

In the study of moduli spaces, some elements can sometimes be difficult to visualize or describe explicitly. To understand these elements, we can associate them a simpler, visually informative, combinatorial structure. Such association is the *dual graph* representation. For example, when examining boundary points, they characterize the behavior of limit points in $\overline{\mathcal{M}}_{g,n}$. Certain elements may decompose into multiple components, resulting in a nodal curve.

For instance, when considering boundary points in $\overline{\mathcal{M}}_{g,n}$, they characterize the behaviour of limit points. Specifically, some configurations may split into multiple components, yielding a nodal curve with some irreducible components intersecting at nodes. While providing a formal description of these curves can be laborious, an alternative, more intuitive method exists. This approach employs dual graphs, which effectively capture the topological features of the curve's degeneration, offering a simplified yet comprehensive representation.

This alternative is known as a dual graph. In this representation, each irreducible component of a nodal curve corresponds to a vertex, and each node translates to an edge. This method transforms potentially complicated geometric structures into a more straightforward combinatorial representation. Consequently, this transformation gives the ability to apply graph theory tools into the combinatorial and geometric aspects of algebraic curves.

Definition 2.6.1 *[Pan18] Consider an element* $[C_{g,n}, p_1, \ldots, p_n] \in \overline{\mathcal{M}}_{g,n}$. *The associated dual graph is constructed as follows:*

- *Each irreducible component of* $C_{g,n}$ *is represented by a vertex, which is labeled by its genus.*
- *Every node of* $C_{g,n}$ *is represented by an edge that connects the vertices corresponding to the two components of the node.*
- *For each marked point* $\{p_i\}_{i=1}^n$, *we attach a half-edge labeled by* i *to the vertex representing the irreducible component containing* p_i.

This construction yields a graph $\Gamma_{g,n}$ *defined by:*

$$\Gamma_{g,n} = (\mathrm{V}, \mathrm{H}, \mathrm{L}, \mathrm{g} : \mathrm{V} \to \mathbb{Z}_{\geq 0}, v : \mathrm{H} \to \mathrm{V}, \iota : \mathrm{H} \to \mathrm{H}),$$

with the following characteristics:

1. V *is the set of vertices, each associated with a genus through the function* $\mathrm{g} : V \to \mathbb{Z}_{\geq 0}$.
2. H *denotes the set of half-edges, which comes with a function* $v : H \to V$ *assigning each half-edge to a vertex and an involution* $\iota : \mathrm{H} \to \mathrm{H}$.

3. *The set of edges, E, is determined by the 2-cycles of ι in H, allowing for self-edges at vertices.*

4. *The set of legs, L, corresponds to the fixed points of ι and is bijectively related to the set of markings $\{1, \ldots, n\}$.*

5. *The graph defined by the pair (V,E) is connected.*

6. *The genus of the graph, denoted by $g(\Gamma_{g,n})$, is given by $g(\Gamma_{g,n}) = \sum_{v \in V} g(v) + h^1(\Gamma)$ and matches g.*

7. *Every vertex v satisfies the stability condition:*

$$2g(v) - 2 + n(v) > 0,$$

where $n(v)$ represents the valence of $\Gamma_{g,n}$ at v, indicating the number of half-edges connected to v.

For each stable graph $\Gamma_{g,n}$, we define its associated moduli space as

$$\overline{\mathcal{M}}_{\Gamma_{g,n}} = \prod_{v \in V} \overline{\mathcal{M}}_{g(v),n(v)}.$$

We use π_v to denote the projection map from $\overline{\mathcal{M}}_{\Gamma_{g,n}}$ to $\overline{\mathcal{M}}_{g(v),n(v)}$, corresponding to the vertex v. A canonical morphism

$$\iota_{\Gamma_{g,n}} : \overline{\mathcal{M}}_{\Gamma_{g,n}} \to \overline{\mathcal{M}}_{g,n} \tag{2.6.1}$$

exists, and its image corresponds to the boundary stratum specifically associated with the graph $\Gamma_{g,n}$.

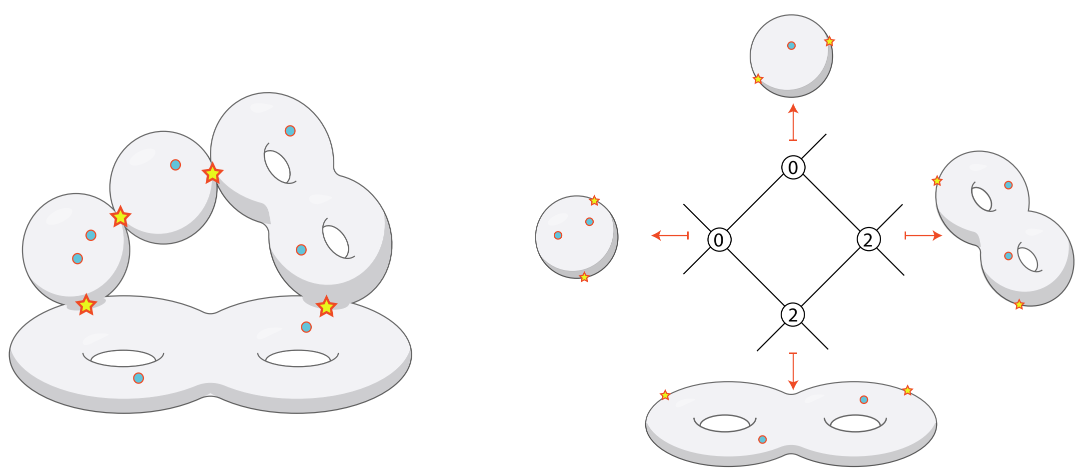

Figure 2.9: A nodal element in $\overline{\mathcal{M}}_{5,7}$ and its dual graph with each component pictured.

Definition 2.6.2 *The moduli space of curves of compact type, denoted as $\mathcal{M}^{ct}_{g,n}$, can be defined as the subset of marked curves $[C_{g,n}, p_1, \ldots, p_n] \in \overline{\mathcal{M}}_{g,n}$ whose associated dual graph is a tree.*

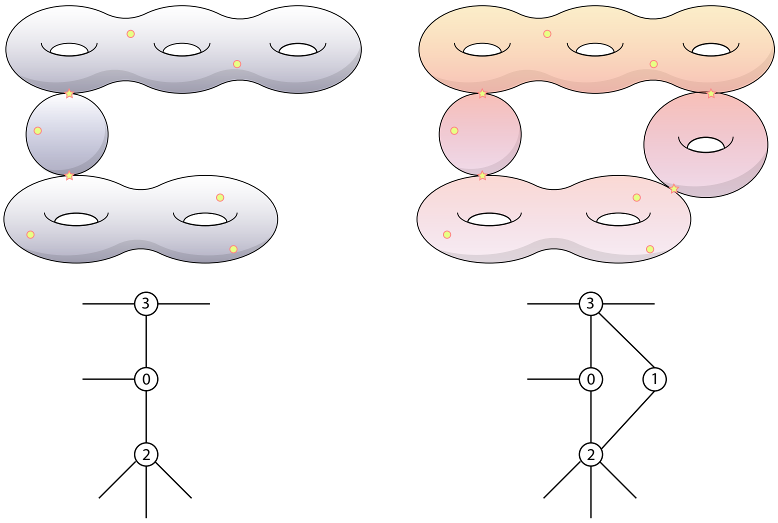

Figure 2.10: Representative elements from $\overline{\mathcal{M}}^{ct}_{5,6}$ and $\overline{\mathcal{M}}_{7,6} \setminus \overline{\mathcal{M}}^{ct}_{7,6}$, illustrated through their respective dual graphs below.

2.7 Tautological Ring

Tautological classes in the moduli spaces of curves are cohomological classes that can be defined naturally from the geometric structure of the moduli space itself. They are intrinsic to the moduli space and are described in terms of the universal properties of the space. The term *tautological* suggests that these classes are self-referential based on the endowed structure of the moduli space.

In this sense, ψ classes (def. 2.5.1) are specific examples of tautological classes, recall ψ classes are associated with the cotangent line bundle over the moduli space of pointed curves. The reason they are considered tautological is that they are constructed directly from the geometric data provided by the moduli space and its universal curve.

Definition 2.7.1 (Tautologial Rings) *The tautological rings are defined as the smallest collection of subrings, denoted by $R^*\left(\overline{\mathcal{M}}_{g,n}\right)$, within $H^*\left(\overline{\mathcal{M}}_{g,n}\right)$, such that it contains $1 \in H^0\left(\overline{\mathcal{M}}_{g,n}\right)$ and it is closed under push-forwards and pullbacks of the forgetful and attaching maps.*

Any cohomology class that resides within a tautological ring is termed a *tautological class.*

Definition 2.7.2 *Let $f_{n+1} : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the forgetful map, that forgets the point $n+1$. The κ-class, denoted κ_j, is defined by:*

$$\kappa_j = f_{n+1*}\left(\psi_{n+1}^{j+1}\right) \in H^{2j}\left(\overline{\mathcal{M}}_{g,n}\right),$$

where ψ_{n+1}^{j+1} represents the power of the ψ-class associated with the marked point $n+1$.

Similarly we can define multi kappa classes by:

$$\kappa_{j_1,\dots,j_m} = (f_*)^m\left(\psi_{n+1}^{j_1+1}\cdots\psi_{n+m}^{j_m+1}\right) \in H^{\sum j_i}\left(\overline{\mathcal{M}}_{g,n}\right).$$

By definition, it is easy to see that κ-classes are also tautological.

In any tautological ring, the generators can be represented using a dual graph, where the corresponding ψ and κ classes are attach to half-edges.

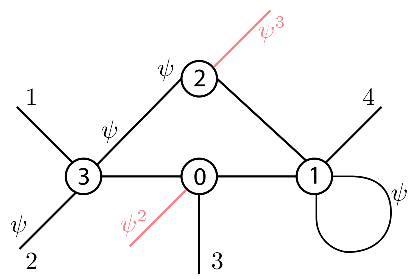

Figure 2.11: Decomposed dual graph in $\overline{\mathcal{M}}_{7,4}$, represented as elements from $\overline{\mathcal{M}}_{3,4} \times \overline{\mathcal{M}}_{2,2+1} \times \overline{\mathcal{M}}_{1,5} \times \overline{\mathcal{M}}_{0,3+1}$. In this representation, the κ classes are visualized as red ψ classes on edges prior to the application of the forgetful map.

Definition 2.7.3 *Consider $\overline{\mathcal{M}}_{g,n}$ and $\overline{\mathcal{C}}_{g,n}$ the associated universal curve. For a curve $C_{g,n}$ in $\overline{\mathcal{C}}_{g,n}$, we define an abelian differential as follows:*

- *If $C_{g,n}$ is smooth, an abelian differential is a holomorphic 1-form on $C_{g,n}$.*
- *If $C_{g,n}$ has a node, an abelian differential is a meromorphic 1-form on $C_{g,n}$ with poles of order at most one at the nodes. Furthermore, the residues at these poles must be equal in magnitude, but with opposite sign on the branches meeting at each node.*

The collection of abelian differentials on $C_{g,n}$ constitutes a vector space of dimension equal to the genus g of the curve.

Remark: The definition of an abelian differential does not depend on the marked points of the curve. Therefore, discussions regarding abelian differentials will focus solely on the genus of the curve.

Definition 2.7.4 (Hodge Bundle on $\overline{\mathcal{M}}_{g,n}$) *Let $(\overline{\mathcal{C}}_{g,n}, \bar{\pi} : \overline{\mathcal{C}}_{g,n} \to \overline{\mathcal{M}}_{g,n})$ be the universal curve over the moduli space $\overline{\mathcal{M}}_{g,n}$. The Hodge bundle, denoted as $\mathbb{E}_g$, is a rank g vector bundle over $\overline{\mathcal{M}}_{g,n}$ defined by:*

$$\mathbb{E}_g := \pi_* \omega_{\bar{\pi}},$$

where $\omega_{\bar{\pi}}$ is the relative cotangent bundle of $\overline{\mathcal{C}}_{g,n}$ over $\overline{\mathcal{M}}_{g,n}$. The fiber of $\mathbb{E}_g$ over a point $[C_{g,n}, p_1, \ldots, p_n]$ in $\overline{\mathcal{M}}_{g,n}$ is the space of holomorphic 1-forms on $C_{g,n}$, i.e.,

$$\mathbb{E}_g|_{[C_{g,n}, p_1, \ldots, p_n]} = H^0(C_{g,n}, \omega_{C_{g,n}}),$$

where $\omega_{C_{g,n}}$ is the canonical bundle of the curve $C_{g,n}$.

Definition 2.7.5 (λ-classes of the Hodge Bundle) *Given the Hodge bundle $\mathbb{E}_g$ over $\overline{\mathcal{M}}_{g,n}$, the λ-classes are defined as the Chern classes of this bundle. For each integer $0 \leq j \leq g$, the j-th λ-class, denoted λ_j, is the j-th Chern class of $\mathbb{E}_g$:*

$$\lambda_j = c_j(\mathbb{E}_g) \in H^{2j}\left(\overline{\mathcal{M}}_{g,n}\right).$$

And the total Chern class of the rank g Hodge bundle $\mathbb{E} \to \overline{\mathcal{M}}_{g,n}$ is expressed as:

$$c(\mathbb{E}_g) = 1 + \lambda_1 + \ldots + \lambda_g \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

In [Mum83], D. Mumford presented a formula that expresses the λ-classes in terms of the more familiar κ- and ψ-classes. This formulation allows the computation and analysis of λ-classes using well-understood classes. Furthermore, it highlights the tautological nature of the λ-classes.

Given the Chern polynomial $\Lambda_g(t) = \sum_{i=0}^{g} t^i \lambda_i$, Mumford's formula can be expressed in terms of this polynomial as a generating function. The formula is then given by:

$$\Lambda_g(t) = \exp\left(\sum_{m\geq 1} t^m \frac{B_{m+1}}{m(m+1)} \left[\kappa_m - \sum_{i=1}^{n} \psi_i^m + \frac{1}{2} j_* \frac{(\psi')^m - (-\psi'')^m}{\psi' + \psi''}\right]\right). \tag{2.7.1}$$

In the formula, B_{m+1} represents the Bernoulli numbers[1]. The morphism j_* and the classes ψ' and ψ'' account for boundary contributions. They also describe the behavior of the λ-classes near the moduli space's boundary, where curves may degenerate and break into multiple components.

Mumford also derived the relation $\Lambda_g(t)\Lambda_g(-t) = 1$. By isolating the coefficients of t^{2g} and t^{2g-1}, obtained the following relations:

[1]The Bernoulli numbers can be defined through the recursive formula: $B_m = 1 - \sum_{k=0}^{m-1} \binom{m}{k} \frac{B_k}{m-k+1}$ for $m \geq 0$, with $B_0 = 1$

$$\lambda_g^2 = 0, \tag{2.7.2}$$

$$\lambda_{g-1}^2 = 2\lambda_g\lambda_{g-2}. \tag{2.7.3}$$

These relations are applicable for all g, n for $2g - 2 + n > 0$. These identities will play a significant role for computations associated with the Double Ramification cycle, detailed in Chapter 5.

Chapter 3

Gromov-Witten Theory and Cohomological Field Theories

Cohomological Field Theories (CohFTs) are systems of linear maps introduced by Maxim Kontsevich and Yuri Manin in the mid-1990s [KM94], they encapsulate essential operations within moduli spaces. Originally conceptualized to formalize tools developed in Gromov-Witten theory, CohFTs have since become instrumental in bridging the geometry of moduli spaces of stable curves with intersection numbers and Gromov-Witten invariants. Moreover, their framework has been extended to encompass broader curve counting theories.

A remarkable classification result for semisimple CohFTs, achieved through the action of the Givental group, was proven by Constantin Teleman in 2012 [Tel12]. The Teleman classification can be used to explicitly calculate the full CohFT in the semi-simple case, extending the range of cases that can be studied by these methods.

In addition to their connection in Gromov-Witten theory, CohFTs significantly contribute to various contexts, including Frobenius manifolds, Dubrovin-Zhang hierarchies, and double ramification hierarchies [BPS12b]. These connections lay the foundations for a comprehensive study of diverse algebraic and geometric structures, particularly those related to the moduli space of curves.

3.1 Moduli Space of Stable Maps $\overline{\mathcal{M}}_{g,n}(X, \beta)$

The moduli space of stable maps was introduced by Maxim Kontsevich as a generalization of the moduli space of curves [Kon95]. This concept aimed to describe Gromov-Witten invariants, which are enumerative invariants in algebraic geometry. The concept of the moduli space of stable maps emerged as a way to formalize the collection of all morphisms from a nonsingular algebraic variety into a specified target space.

3.1.1 Stable Maps

Let us recall definition 2.3.4, in which we introduced the stability condition for nodal curves. We are now interested in understanding the behavior of surfaces with marked points when they are mapped to another projective variety. Such a map might not necessarily be stable, so it is essential to impose a constraint on the maps that guarantees the resulting curve to remain stable. This requirement leads to the concept of stable maps, which offers tools for investigating the geometry and topology of surfaces with marked points, while maintaining the stability condition when mapping them to a target space.

Definition 3.1.1 *[HKK+03] Let X be a non-singular projective variety. A morphism f from a nodal Riemann surface of genus g with n marked nodes, denoted as $\Sigma_{g,n}$, to X is a stable map if every genus 0 contracted component of $\Sigma_{g,n}$ has at least three marked nodes and every genus 1 contracted component has at least one marked node.*

Definition 3.1.2 *[HKK+03] Given the fundamental class $[\Sigma] \in H_2(\Sigma_{g,n}, \mathbb{Z})$, a **stable map** represents a homology class $\beta \in H_2(X, \mathbb{Z})$ if $f_*[\Sigma] = \beta$.*

Definition 3.1.3 *[HKK+03] We denote by $\overline{\mathcal{M}}_{g,n}(X, \beta)$ the **moduli space of stable maps** from nodal curves of genus g with n marked points $\Sigma_{g,n}$ to X representing the class β.*

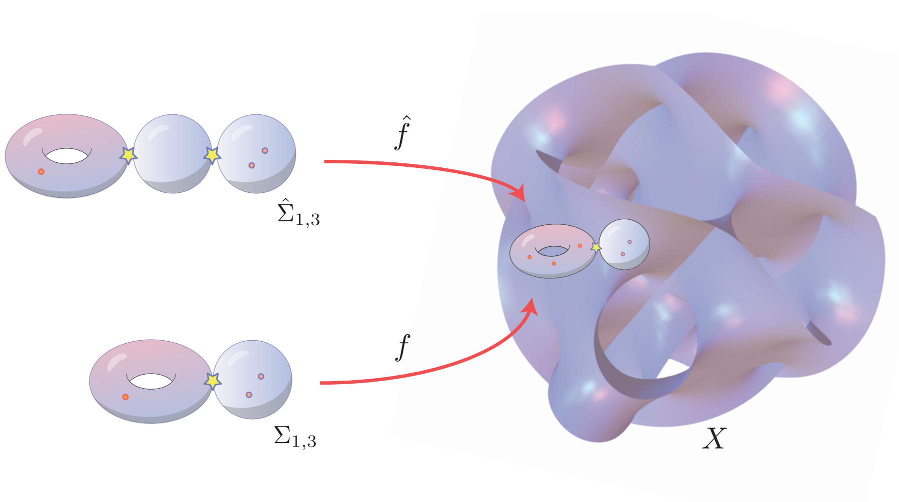

Figure 3.1: Examples of an unstable map $\hat{f} : \hat{\Sigma}_{1,3} \to X$ and stable map $f : \Sigma_{1,3} \to X$.

There are two natural maps [CK99]:

$$\pi_1 : \overline{\mathcal{M}}_{g,n}(X, \beta) \longrightarrow X^n,$$
$$\pi_2 : \overline{\mathcal{M}}_{g,n}(X, \beta) \longrightarrow \overline{\mathcal{M}}_{g,n}.$$

For π_1, consider a stable map $f : (\Sigma, p_1, \ldots, p_n) \to X$. The map π_1 takes this stable map and returns the element obtained by mapping the individual marked points, i.e., $(f(p_1), f(p_2), \ldots, f(p_n)) \in X^n$.

Then π_2 sends the map f to the class associated with the stable curve Σ, represented as $[\Sigma]$ in $\overline{\mathcal{M}}_{g,n}$.

These elements lead to the following induced maps:

$$\pi_1^* : H^*(X^n, \mathbb{Z}) \longrightarrow H^*\left(\overline{\mathcal{M}}_{g,n}(X, \beta), \mathbb{Z}\right),$$
$$\pi_{2*} : H_*(\overline{\mathcal{M}}_{g,n}(X, \beta), \mathbb{Z}) \longrightarrow H_*(\overline{\mathcal{M}}_{g,n}, \mathbb{Z}).$$

For any map $m : X \to Y$ between smooth compact oriented varieties, we introduce the *Gysin* map:

$$m_! : H^*(X, \mathbb{Z}) \to H^*(Y, \mathbb{Z}),$$
$$\alpha \mapsto PD^{-1}(m_*(PD(\alpha))),$$

where PD is the Poincaré dual map, defined by the cap product with the fundamental class of X [Kaj97]. However, the moduli space of stable maps, $\overline{\mathcal{M}}_{g,n}(X, \beta)$, does not possess a well-defined fundamental class in the usual sense. To address this, a virtual fundamental class, denoted as $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{vir}$, is introduced. This virtual class is constructed to be compatible with the Gysin map [LT98] and allows us to formally introduce Gromov-Witten invariants.

The construction of the mentioned virtual cycle is highly technical, and the reader is referred to [LT98] for a detailed explanation. With this in mind, we can now introduce the Gysin map of π_2:

$$\pi_{2!} : H^*(\overline{\mathcal{M}}_{g,n}(X, \beta), \mathbb{Z}) \longrightarrow H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Z}).$$

These Gysin maps are employed to associate cohomology classes from the moduli space of stable maps to the moduli space of stable curves, enabling a more manageable representation of geometric data.

By utilizing the Gysin map, we can effectively translate geometric data from the moduli space of stable maps to the moduli space of stable curves. This process provides a mechanism for associating cohomology classes between the two moduli spaces, which is essential for defining Gromov-Witten invariants.

3.2 Gromov-Witten theory

Initially inspired by Mikhail Gromov's work on pseudo-holomorphic curves in symplectic geometry [Gro85], Gromov-Witten invariants focus on counting the number of solutions to certain geometric problems, specifically the number of rational curves in a given homology class on a complex projective variety. This theory motivated the introduction of Cohomological Field Theories (CohFTs) to formalize and encode the structure of Gromov-Witten invariants, providing a systematic approach to represent the essential properties of Gromov-Witten invariants in terms of linear maps between cohomology groups. [KM94] [BM96]

Given a non-singular projective variety X, consider a fixed homology class $\beta \in H_2(X, \mathbb{Z})$ and n cycles $v_1, \ldots, v_n \subset X$. We are interested in examining the set characterized by the following data:

$$(\Sigma_{g,n} \subset X, \beta) \text{ such that } \Sigma_{g,n} \cap v_i \neq \emptyset \quad \forall i.$$

This set represents the collection of genus g Riemann surfaces with n marked points, $\Sigma_{g,n}$, that are embedded in the projective variety X with the specified homology class β, and intersect each of the given cycles v_i for all i.

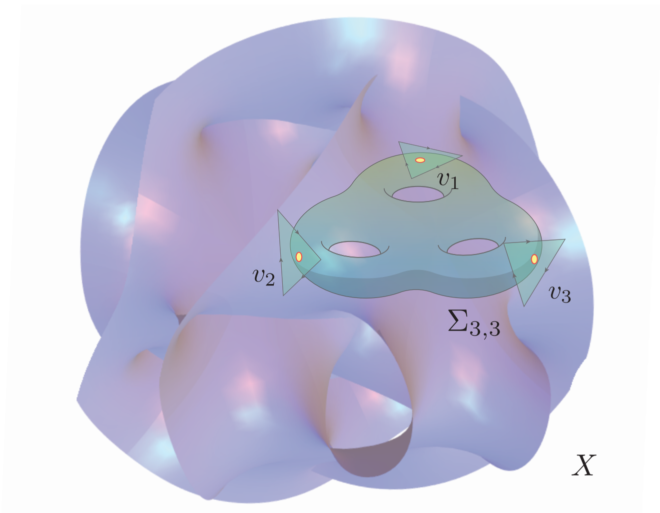

Figure 3.2: Curve $\Sigma_{3,3} \subset X$ with cycles v_1, v_2 and v_3.

If this condition results in a finite set of curves, we refer to them as **Gromov-Witten invariants**; otherwise, they are called **Gromov-Witten classes** [CK99].

Definition 3.2.1 *Let $\alpha_i \in H^*(X)$ be the dual class to v_i. We define the* ***Gromov-Witten class*** *as:*

$$\begin{aligned} GW_{g,n,\beta}(\alpha_1, \ldots, \alpha_n) &= \pi_{2!}\left(\pi_1^*\left(\alpha_1 \otimes \cdots \otimes \alpha_n\right)\right) \\ &= PD^{-1}\left(\pi_{2*}\left(\pi_1^*\left(\alpha_1 \otimes \cdots \otimes \alpha_n\right) \cap \left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{vir}\right)\right) \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}). \end{aligned}$$

This class represents the cohomology class of a set of curves with marked points $(\Sigma_{g,n}, p_1, \ldots, p_n)$ for the stable map f with $f_*[\Sigma_{g,n}] = \beta$.

Definition 3.2.2 *If the degree of $GW_{g,n,\beta}(\alpha_1, \ldots, \alpha_n)$ is $3g-3+n$, then $GW_{g,f,\beta}$ defines a **primary Gromov-Witten invariant** as*

$$\langle \alpha_1, \ldots, \alpha_n \rangle_{g,\beta}^X = \int_{\overline{\mathcal{M}}_{g,n}} GW_{g,n,\beta}(\alpha_1, \ldots, \alpha_n).$$

Definition 3.2.3 *Let $i \in \{1, \ldots, n\}$, and for a stable map $f : (\Sigma, p_1, \ldots, p_n) \to X$, the **evaluation map** ev_i is given by:*

$$\begin{aligned} ev_i : \overline{\mathcal{M}}_{g,n}(X, \beta) &\to X, \\ f &\mapsto f(p_i). \end{aligned}$$

The evaluation maps assist in formulating Gromov-Witten invariants by linking marked points on the curve to specific cohomology classes in the target space. This association imposes constraints on the potential curves in the target space, enumerating curves that intersect specified cycles in the target space at the marked points. This relationship allows us to express the primary Gromov-Witten invariants as follows:

$$\langle \alpha_1, \ldots, \alpha_n \rangle_{g,\beta}^X = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} ev_1(\alpha_1) \cup \cdots \cup ev_n(\alpha_n).$$

The Gromov-Witten invariant measures the number of n-pointed genus-g curves in the class β in the target space X that intersect the cycles representing the cohomology classes $\alpha_1, \ldots, \alpha_n$ at the marked points $p_1, \ldots, p_n$, respectively.

By introducing psi classes on the moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$, we can extend the Gromov-Witten invariants to a more general form known as descendant invariants:

$$\langle \tau_{a_1}(\alpha_1) \cdots \tau_{a_n}(\alpha_n) \rangle_{g,\beta}^X := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\alpha_1) \cup \psi_1^{a_1} \cup \cdots \cup \mathrm{ev}_n^*(\alpha_n) \cup \psi_n^{a_n}, \quad (3.2.1)$$

where $a_i \in \mathbb{Z}_{\geq 0}$ and $\alpha_i \in H^*(X)$.

Similar to the case of $\overline{\mathcal{M}}_{g,n}$, we can introduce the String equation, it arises from considering how Gromov-Witten invariants change when an extra marked point, constrained to lie on a divisor representing the first Chern class of the target space, is added. It reflects a certain recursive structure in the Gromov-Witten invariants. If $e \in H^*(X)$ is the unit, then the string equation for the correlators $\langle \tau_{a_1}(\alpha_1) \cdots \tau_{a_n}(\alpha_n) \rangle_{g,\beta}^X$ reads:

$$\langle \tau_{a_1}(\alpha_1) \cdots \tau_{a_n}(\alpha_n)\, e \rangle_{g,\beta}^X = \sum_{i=1}^{n} \langle \tau_{a_1}(\alpha_1) \cdots \tau_{a_{i-1}}(\alpha_{i-1})\, \tau_{a_i - 1}(\alpha_i)\, \tau_{a_{i+1}}(\alpha_{i+1}) \cdots \tau_{a_n}(\alpha_n) \rangle_{g,\beta}^X .$$

and the dilaton equation relates the Gromov-Witten invariants with an extra marked point carrying the psi class to the invariants without this extra point, but with a shift in the virtual dimension of the moduli space:

$$\langle \tau_{a_1}(\alpha_1)\cdots\tau_{a_n}(\alpha_n)\,\tau_1(e)\rangle_{g,\beta}^X = (2g-2+n)\,\langle \tau_{a_1}(\alpha_1)\cdots\tau_{a_n}(\alpha_n)\rangle_{g,\beta}^X .$$

Definition 3.2.4 *The Gromov-Witten potential in genus g is a generating function that encompasses all descendant invariants. It is defined by the following series:*

$$F_g^{GW}(\mathbf{t},q) = \sum \frac{t_{a_1}^{\alpha_1}\dots t_{a_m}^{\alpha_m}}{m!}\,\langle \tau_{a_1}(\alpha_1)\dots\tau_{a_m}(\alpha_m)\rangle_{g,\beta}^X\, q^\beta. \tag{3.2.2}$$

Here, q^β and $t_{a_i}^{\alpha_i}$ are formal variables, with $d,p \geq 0$ and $1 < i < m$. The total Gromov-Witten potential is obtained by summing over all possible values of g:

$$\mathcal{F}^{GW}(\mathbf{t},q,\varepsilon) = \sum_{g\geq 0} \varepsilon^{2g-2} F_g^{GW}(\mathbf{t},q). \tag{3.2.3}$$

We will later see how the potential $\mathcal{F}^{GW}(\mathbf{t},q,\varepsilon)$ serves as a generating function for a hierarchy of integrable systems. This hierarchy is characterized by time variables t_*^* and $t_0^1 = x$

Gromov-Witten theory establishes a framework to study the enumerative geometry of algebraic varieties by examining curves and their intersections within the target space. The introduction of evaluation maps, psi classes, and descendant invariants allows us to count and categorize these curves in an efficient manner. Consequently, this leads to the formulation of distinct potentials associated with the geometry and topology of the space. These potentials encapsulate the geometric information and their structure can be understood with the theory of integrable systems. This relation between algebraic geometry, represented through these invariants, and integrable systems is an active area of research, with profound implications for both mathematics and theoretical physics, specifically in the context of quantum field theory and string theory.

3.2.1 Topological recursion relations in Gromov-Witten theory

Topological Recursion Relations (TRR) are an important tool for Gromov-Witten potentials, they provide a system of equations that relate invariants of different genera and with different numbers of marked points. Among these potentials, we can include those associated to the intersection theory of moduli spaces of curves, particularly Witten's potential $\mathcal{F}^W$ (eq. (2.5.3)). In genus 0, the Topological Recursion Relations take a particular form and give a relation between total descendent potentials and nonlinear partial differential equations. This relationship is exemplified through a specific system of PDEs that govern the genus 0 descendant potential, denoted as F_0^{GW}. This system, originally traced back to the works of [Get98], [Gat03] and [Lee09], is defined as follows.

Definition 3.2.5 *A formal power series* $F_0 \in \mathbb{C}[[t_*^*]]$ *is called a descendent potential of genus* 0 *if it satisfies the following system of PDEs:*

$$\sum_{a\geq 0} t_{a+1}^{\alpha}\frac{\partial F_0}{\partial t_a^{\alpha}} - \frac{\partial F_0}{\partial t_0^{\mathbb{1}}} = -\frac{1}{2}\eta_{\alpha\beta}t_0^{\alpha}t_0^{\beta}, \tag{3.2.4}$$

$$\sum_{a\geq 0} t_a^{\alpha}\frac{\partial F_0}{\partial t_a^{\alpha}} - \frac{\partial F_0}{\partial t_1^{\mathbb{1}}} = 2F_0, \tag{3.2.5}$$

$$\frac{\partial^3 F_0}{\partial t_{a+1}^{\alpha}\partial t_b^{\beta}\partial t_c^{\gamma}} = \frac{\partial^2 F_0}{\partial t_a^{\alpha}\partial t_0^{\mu}}\eta^{\mu\nu}\frac{\partial^3 F_0}{\partial t_0^{\nu}\partial t_b^{\beta}\partial t_c^{\gamma}}, \qquad 1\leq \alpha,\beta,\gamma\leq N, \qquad a,b,c\geq 0, \tag{3.2.6}$$

$$\frac{\partial^2 F_0}{\partial t_{a+1}^{\alpha}\partial t_b^{\beta}} + \frac{\partial^2 F_0}{\partial t_a^{\alpha}\partial t_{b+1}^{\beta}} = \frac{\partial^2 F_0}{\partial t_a^{\alpha}\partial t_0^{\mu}}\eta^{\mu\nu}\frac{\partial^2 F_0}{\partial t_0^{\nu}\partial t_b^{\beta}}, \qquad 1\leq \alpha,\beta\leq N, \qquad a,b\geq 0.$$

In these equations, the term $(\eta_{\alpha\beta}) = \eta$ denotes a symmetric, nondegenerate $N \times N$ matrix with complex coefficients. The constants $\eta^{\alpha\beta}$ are defined by the inverse matrix $(\eta^{\alpha\beta}) := \eta^{-1}$. Additionally, the Einstein summation convention is applied, where repeated Greek indices in both upper and lower positions imply summation. It is important to note that this framework represents a more general construct than the Gromov-Witten potential as defined in 3.2.4. Specifically, the Gromov-Witten potential in genus 0 is a special case that satisfies this system of equations.

These equations are respectively known as the **string equation**, **dilaton equation** and the **topological recursion relations in genus 0**. These equations can be seen as a specific instance of a more general relation in Gromov-Witten theory, related to the WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) (Equations 3.3.3).

The mathematical meaning of the WDVV equations is best understood in terms of the so-called quantum cohomology ring of a smooth projective variety. This ring helps us understand the structure of a space by studying the functions, forms, and other mathematical objects defined on the space.

The cohomology ring allows us to add and multiply cohomology classes, the structure constants of this quantum product, which encode how to multiply elements in the quantum cohomology ring, are given by certain counts of holomorphic curves that serve as complex analogues of geodesics in the manifold called Gromov-Witten invariants, particularly, the structure is determined by three-point Gromov-Witten invariants of the manifold (genus 0 curves with 3 marked points), and the WDVV equations state that the multiplication in this ring is associative up to quantum corrections.

Recursion relations are important to understand these spaces. Thus, it is desirable to derive such relations for higher genera. Indeed, an analogous expression to Equation (3.2.6) can be found for genus 1, as demonstrated in [EGX00]:

$$\frac{\partial F_1}{\partial t^\alpha_{a+1}} = \frac{\partial^2 F_0}{\partial t^\alpha_a \partial t^\mu_0}\eta^{\mu\nu}\frac{\partial F_1}{\partial t^\nu_0} + \frac{1}{24}\eta^{\mu\nu}\frac{\partial^3 F_0}{\partial t^\mu_0 \partial t^\nu_0 \partial t^\alpha_a}, \quad 1 \le \alpha \le N, \quad a \ge 0. \tag{3.2.7}$$

These are known as the topological recursion relations in genus 1. From these relations, we can have [DW90]:

$$F_1 = \frac{1}{24}\log\det(\eta^{-1}M) + G(v^1, \dots, v^N). \tag{3.2.8}$$

In this context, we introduce the $N \times N$ matrix $M = (M_{\alpha\beta})$, with each element defined as $M_{\alpha\beta} := \frac{\partial^3 F_0}{\partial t^1_0 \partial t^\alpha_0 \partial t^\beta_0}$. Furthermore, we denote v^α as the function of $\eta^{\alpha\mu}$ and the second partial derivative of F_0, such that $v^\alpha := \eta^{\alpha\mu}\frac{\partial^2 F_0}{\partial t^\mu_0 \partial t^1_0}$. We also specify the function $G(t^1, \dots, t^N)$, which is equivalent to the descendent potential F_1 evaluated at $t^{\ge 1} = 0$, denoted as $G(t^1, \dots, t^N) := F_1|_{t^*_{\ge 1=0}}$. These definitions were initially introduced in [DW90] and subsequently extended in [DZ98]. This expression will play a bigger role in the following chapter, as we will use it as motivation to construct similar TRRs for moduli spaces of open Riemann surfaces.

As we move to higher genera, the complexity significantly increases due to the growing number of moduli and the curves' configurations, an example of genus 2 TRRs can be found in [Liu07].

It's also worth noting for any total descendent potential, denoted as $\mathcal{F} = \sum_{g\ge 0}\varepsilon^{2g}F_g$, the function F_0 is inherently a descendent potential in genus 0. However, it remains as an open problem which descendent potentials in genus 0 can be properly extended to total descendent potentials.

3.2.2 Introduction to the Dubrovin-Zhang hierarchy for Gromov-Witten theory

Equations (3.2.4), (3.2.6), and (3.2.7) hold an important property they are universal, which means their form is not restricted to a particular total descendant potential. This universality reveals a structural regularity, thus enabling us to enclose these equations from a more general perspective.

There is a more general theory that describes such systems of equations, suggested by Dubrovin and Zhang in 2001 (see [DZ01]), this theory relates total descendent potentials to a hierarchy of evolutionary partial differential equations with a single spatial variable. This link encapsulates the full information about these potentials, establishing a connection between the geometry of moduli spaces and the dynamics of integrable systems.

Conjecturally, it is believed that for every total descendent potential $\mathcal{F}$, a unique system of Partial Differential Equations (PDEs) can be constructed, related to a specific structural framework. This conjecture remains unproven. If true, it would establish a precise mathematical framework for Gromov-Witten theory, categorizing it as an integrable system. The conjectured system is proposed as follows:

$$\frac{\partial w^\alpha}{\partial t_b^\beta} = P^\alpha_{\beta,b}, \quad 1 \leq \alpha, \beta \leq N, \quad b \geq 0. \tag{3.2.9}$$

In this system, $w^1, \ldots, w^N$ are elements of $\mathbb{C}[[t_*^*, \varepsilon]]$, and $P^\alpha_{\beta,b}$ are differential polynomials in $w^1, \ldots, w^N$. A notable feature of this system is that the flows defined by these equations commute.

The system hypothesizes the existence of a solution to (3.2.9) under the initial condition $w^\alpha|_{t_b^\beta = \delta^{\beta,1}\delta_{b,0}x} = \delta^{\alpha,1}x$, where the solution is defined by $w^\alpha = \eta^{\alpha\mu}\frac{\partial^2 \mathcal{F}}{\partial t_0^\mu \partial t_0^1}$. Should such a system exist, it is referred to as the **Dubrovin–Zhang hierarchy** or the **hierarchy of topological type**.

While the full proof of this conjecture is still unproven, significant progress has been made in specific cases. Notably, the conjecture has been validated for semi-simple CohFTs. Additionally, in the general case, the conjecture has been proven to hold up to an approximation of ε^2 [DZ98].

Though this conjecture remains unproven in its entirety, its validity has been proven for semi-simple CohFTs, and up to an approximation of ε^2 in the general case [DZ98] [DZ01] [BPS12b].

The Dubrovin–Zhang hierarchy for the Witten potential $\mathcal{F}^W = \sum_{g\geq 0} F_g^W$ (cf. 2.5.3), is expressed as the Korteweg–de Vries (KdV) hierarchy, given as:

$$\begin{aligned}
\frac{\partial w}{\partial t_1} &= w w_x + \frac{\varepsilon^2}{12} w_{xxx}, \\
\frac{\partial w}{\partial t_2} &= \frac{w^2 w_x}{2} + \varepsilon^2 \left(\frac{w w_{xxx}}{12} + \frac{w_x w_{xx}}{6} \right) + \varepsilon^4 \frac{w_{xxxxx}}{240}, \\
&\vdots
\end{aligned}$$

This relation with the KdV hierarchy is equivalent to Witten's conjecture (theorem 2.5.4).

We can try to develop a similar construction for the descendent potential of genus 0: F_0. This approach is centered on the introduction and definition of differential polynomials $\Omega^{[0]}_{\alpha,a;\beta,b}$ within the ring of polynimials $\mathcal{A}_{v^1,\ldots,v^N;0}$. Here, $1 \leq \alpha, \beta \leq N$ and $a, b \geq 0$.

The polynomials $\Omega^{[0]}_{\alpha,a;\beta,b}$ are constructed by evaluating second-order derivatives of the potential function, F_0, with respect to variables t_a^α and t_b^β:

$$\Omega^{[0]}_{\alpha,a;\beta,b} := \left. \frac{\partial^2 F_0}{\partial t_a^\alpha \partial t_b^\beta} \right|_{t_c^\gamma = \delta_{c,0} v^\gamma}. \tag{3.2.10}$$

Simultaneously, we define $(v^{\text{top}})^\alpha$ using the metric tensor η and second-order derivatives of the potential F_0:

$$(v^{\text{top}})^\alpha := \eta^{\alpha\mu} \frac{\partial^2 F_0}{\partial t_0^\mu \partial t_0^{\mathbb{1}}} \in \mathbb{C}[[t_*^*]], \quad 1 \leq \alpha \leq N. \tag{3.2.11}$$

Then, using the property found in [Proposition 3] [BPS12b]:

$$\frac{\partial^2 F_0}{\partial t_a^\alpha \partial t_b^\beta} = \Omega^{[0]}_{\alpha,a;\beta,b}\Big|_{v^\gamma = (v^{\text{top}})^\gamma}. \tag{3.2.12}$$

A remarkable outcome of this is the construction of a system of PDEs, whose solutions coincide with an N-tuple of functions derived from $(v^{\text{top}})^\alpha$.

$$\frac{\partial v^\alpha}{\partial t_b^\beta} = \eta^{\alpha\mu} \partial_x \Omega^{[0]}_{\mu,0;\beta,b}, \quad 1 \leq \alpha, \beta \leq N, \quad b \geq 0. \tag{3.2.13}$$

This is known as the principal hierarchy associated with the potential F_0. We will later use a similar construction for open moduli spaces, this will play a crucial role in our discussion for the following chapter.

3.3 Dubrovin–Frobenius manifolds

In the 1990s, B. Dubrovin introduced the concept of Frobenius manifolds, now commonly known as Dubrovin-Frobenius manifolds, which serve as an axiomatic framework for topological field theories (TFTs) and provide a geometric manifestation of the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations. Frobenius manifolds are locally Euclidean spaces with additional algebraic structures. These algebraic structures endow the tangent spaces of Frobenius manifolds with a multiplication operation that satisfies the properties of associativity and potentiality (cf. def. 3.3.5). Such spaces with algebraic operations are referred to as Frobenius algebras.

A remarkable aspect about Dubrovin-Frobenius manifolds is their connection to integrable hierarchies of partial differential equations. Every Dubrovin-Frobenius manifold is associated with a Hamiltonian integrable system called the principal hierarchy. In [DZ01], Dubrovin and Zhang constructed a hierarchy from the partition function with the CohFT associated with a moduli space across all genera, recovering a full dispersive hierarchy in the process.

The connection between Gromov-Witten theory, CohFTs, and Frobenius manifolds arises from the study of quantum cohomology. Quantum cohomology encompasses Gromov–Witten invariants by introducing a new multiplication operation known as the quantum product. In specific scenarios, the quantum cohomology ring of an algebraic variety exhibits the characteristics of a Frobenius manifold. This Frobenius structure gives information about the variety's geometry and enables the computation of enumerative information encoded in Gromov–Witten invariants.

Definition 3.3.1 *Let $(V, \star, e)$ be a finite dimensional algebra over a field $\mathbb{K}$ with unit e, associative and commutative with respect to the product $\star$. A Frobenius algebra $(V, \star, e)$ is an algebra equipped with a non-degenerate bilinear form $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{K}$, such that:*

- *The bilinear form $\eta(\cdot,\cdot)$ (also denoted by $\langle\cdot,\cdot\rangle$) is compatible with the multiplication operation, which means that for all elements $a, b, c \in V$, the following equation holds:*

$$\eta(a \star b, c) = \eta(a, b \star c).$$

Proposition 3.3.2 *Let $(V, \star, e)$ be a finite-dimensional algebra defined as above with V^* its dual space, a Frobenius algebra structure can be equivalently defined by a linear functional $L \in V^*$, meaning this functional can define a non-degenerate bilinear form, if only if, the pairing $\eta : V \times V \to \mathbb{K}$ given by:*

$$\eta(u, v) \equiv L(u \star v),$$

is non-degenerate.

Proof:

(⇒) Take $\eta(u \star e) = L(u)$. Then L is non-trivial, as η defines a Frobenius algebra structure, i.e. is not degenerate.

(⇐) Since $(V, \star, e)$ is associative, we have

$$\eta(u \star v, w) = L((u \star v) \star w) = L(u \star (v \star w)) = \eta(u, v \star w) \quad \square.$$

Example 3.3.3 (Cohomology Ring of a Manifold) *Consider a connected, oriented, compact $2n$-dimensional manifold M. The even degree part of its de Rham cohomology ring forms a vector space V over $\mathbb{C}$, defined as $V = \bigoplus_{k\geq 0} H^{2k}(M, \mathbb{C})$.*

We can construct a symmetric bilinear form on this space using the natural integration pairing given by wedge product and integration over M.

First, let $[M]$ denote the fundamental class of M. For any cohomology class $\alpha \in V$, we can define a linear functional L on V by $L(\alpha) = \int_M \alpha$. This maps each cohomology class to a complex number via integration over the manifold.

Next, we define the symmetric form η as follows:

$$\eta : H^{2k}(M, \mathbb{C}) \times H^{2m}(M, \mathbb{C}) \to \mathbb{C},$$
$$\eta(\alpha, \beta) := \int_M \alpha \wedge \beta.$$

This symmetric form η provides a complex-valued inner product on the cohomology ring, pairing cohomology classes of complementary degrees, we take $e = 1 \in H^0(M, \mathbb{C})$, and the product to be the wedge between forms, then V is a Frobenius algebra.

Example 3.3.4 ($r-$spin Cohomological Field Theory) *Consider $r \geq 2$ and a vector space $V = \mathbb{C}^{r-1}$ with basis $e_1, ..., e_{r-1}$. We can construct a Frobenius algebra structure on this vector space $(V, \star, e)$ as follows:*

1. *The unit of the algebra is chosen to be $e := e_1$.*
2. *The non-degenerate symmetric bilinear form $\eta : V \times V \to \mathbb{C}$ (which gives us the Frobenius structure) is defined by $\eta(e_i, e_j) := \delta_{i+j,r}$. Here, $\delta_{i+j,r}$ is the Kronecker delta, which is 1 if $i + j = r$, and 0 otherwise.*
3. *The product $\star : V \times V \to V$ (giving the algebra structure) is defined as follows:*

$$e_i \star e_j := \begin{cases} e_{i+j-1} & \text{if } i + j \leq r, \\ 0 & \text{otherwise.} \end{cases}$$

Intuitively, a Dubrovin-Frobenius manifold is a geometric structure that blends together the concepts of smooth manifolds with the algebraic structure of associative algebra. Here the algebra multiplication and the metric are not independent; they are deeply related in a specific way defined by the manifold's structure.

Often, a Dubrovin-Frobenius manifold is described using a what is called a prepotential function F, which encodes the manifold's geometric and algebraic properties. In many contexts, especially in mathematical physics, Dubrovin-Frobenius manifolds emerge naturally, specifically in the study of integrable systems and quantum cohomology where the structure of a Frobenius manifold elegantly captures both geometric and algebraic properties.

Definition 3.3.5 (Dubrovin-Frobenius Manifold) *Let V a N-dimensional complex vector space, and U a connected open subset of V with coordinates $\left(t^1, \ldots, t^N\right)$, let $\eta = (\eta_{\alpha\beta})$ be an $N \times N$ complex symmetric nondegenerate matrix and a fixed vector field on U $A^\alpha \frac{\partial}{\partial t^\alpha}$, for $A^\alpha \in \mathbb{C}$, which we will often denote by $\frac{\partial}{\partial t^{\mathbb{1}}}$. Consider a scalar function $F\left(t^1, \ldots, t^N\right) : U \to \mathbb{C}$. Denote*

$$c_{\alpha\beta\gamma} := \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma}. \tag{3.3.1}$$

The function F is known as the prepotential, together with the matrix η and the vector field $\frac{\partial}{\partial t^{\mathbb{1}}}$ define a structure of a Dubrovin-Frobenius manifold on U if the following properties are satisfied:

- *The relation between η and the prepotential F is given by: $\eta_{\alpha\beta} = c_{\mathbb{1}\alpha\beta}$,*
- *$c^{\gamma}_{\alpha\beta}$ give the structure constants of an associative algebra.*

A Dubrovin-Frobenius manifold is said to be conformal if it possesses a specific vector field, known as the Euler field, denoted by E. This field is of the following form:

$$E = (\underbrace{(1 - q^\alpha)\, t^\alpha + r^\alpha}_{E^\alpha}) \frac{\partial}{\partial t^\alpha},$$

satisfying

- *$L_E F = (3 - \delta)F + \frac{1}{2} A_{\alpha\beta} t^\alpha t^\beta + B_\alpha t^\alpha + C$, for some $\delta, A_{\alpha\beta}, B_\alpha, C \in \mathbb{C}$, and L is the Lie derivative.*
- *$L_E \frac{\partial}{\partial t^{\mathbb{1}}} = -\frac{\partial}{\partial t^{\mathbb{1}}}$.*
- *$L_E \eta = (2 - \delta)\eta$.*

The parameter δ is referred to as the conformal dimension or the charge of the manifold[1]*. In a Dubrovin-Frobenius manifold, this parameter gives information on the manifold's conformal properties.*[2]

In a Dubrovin-Frobenius manifold, the algebraic structure is linked to a prepotential function $F(t^1, \ldots, t^n)$, which defines the structure constants for the algebra's multiplication (eq. 3.3.1). This prepotential must fulfill specific conditions, known as the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations:

$$\frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \eta^{\gamma\delta} \frac{\partial^3 F}{\partial t^\delta \partial t^\mu \partial t^\mu} = \frac{\partial^3 F}{\partial t^\nu \partial t^\beta \partial t^\gamma} \eta^{\gamma\delta} \frac{\partial^3 F}{\partial t^\delta \partial t^\mu \partial t^\alpha}. \tag{3.3.2}$$

So, the WDVV equations can be interpreted as the associativity conditions for the Frobenius algebra.

The Euler field plays an important role in defining the geometric and algebraic structure of the manifold, such as:

[1] The term *charge* often refers to its role in describing how the manifold responds to certain transformations, much like how electric charge determines an object's response to an electromagnetic field generated by the abelian symmetry $U(1)$ in gauge theory, where in this context, the symmetry is conformal.

[2] Specifically, the conformal dimension describes the scaling behavior of the manifold under conformal transformations, especially where scale invariance or self-similarity properties need to be considered, examples can be found in topological field theories and conformal gravity in physics.

1. **Conformal Symmetry**: The Euler field represents a vector field on the manifold that encodes the conformal symmetry of the Frobenius structure. It acts as a generator of dilations, scaling the coordinates in a way that reflects the geometric structure of the manifold.

2. **Quantum Cohomology**: In the context of quantum cohomology, which is often modeled by Dubrovin-Frobenius manifolds, the Euler field corresponds to the first Chern class of the target space in Gromov-Witten theory.

3. **Homogeneity of the Prepotential**: The Euler field ensures the homogeneity of the prepotential function, thereby imposing specific constraints on its form and the solutions of the WDVV equations. This implies that the prepotential scales in a particular way under the scaling transformations induced by the action of the Euler field.

4. **Degree Assignment**: The Euler field assigns degrees to the coordinates on the manifold. This degree assignment endows a graded structure of the Frobenius algebra and ensures that the multiplication operation respects this grading.

5. **Control of Deformations**: The Euler field controls the deformations[1] of the Frobenius structure. Specifically, it determines how the structure constants change when the manifold's parameters are varied.

Example 3.3.6 (Trivial DF-manifold) *The most basic example of a DF manifold is a one-dimensional ($N = 1$) manifold $M = \mathbb{C}$ with the following data:*

- *The prepotential function is cubic:* $F = \frac{1}{6}(t^1)^3$.
- *The unit vector field* $e = \frac{\partial}{\partial t^1}$.
- *The metric is simply a scalar,* $\eta = 1$.
- *The Euler vector field is* $E = t^1 \frac{\partial}{\partial t^1}$.
- *The Euler field is conformal with scaling factor 2:* $L_E \eta = 2\eta$. *This tells us the charge or conformal dimension of the manifold is* $2 - \delta = 2$, *which implies* $\delta = 0$.

Example 3.3.7 (Genus 0 Gromov-Witten Theory with Target Variety X) *Let* $\{t^*_*\}$ *be a basis of* $H^*(X, \mathbb{C})$, *consider the Gromov-Witten potential* $\mathcal{F}^{GW}(\mathbf{t}, q)$ (3.2.3). *For genus* $g = 0$, *we set* $t^*_{\geq 1} = 0$, $F_0^{GW}(\mathbf{t}, q)$ *satisfies the WDVV equations* (3.3.2) *[Dub96]:*

$$\frac{\partial^3 F_0^{GW}}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \eta^{\gamma\delta} \frac{\partial^3 F_0^{GW}}{\partial t^\delta \partial t^\mu \partial t^\nu} = \frac{\partial^3 F_0^{GW}}{\partial t^\nu \partial t^\beta \partial t^\gamma} \eta^{\gamma\delta} \frac{\partial^3 F_0^{GW}}{\partial t^\delta \partial t^\mu \partial t^\alpha}. \tag{3.3.3}$$

[1] The deformations might involve changes in the manifold's geometric properties, algebraic relations, or the multiplication rule itself encoded in the structure constants.

Assuming the power series $F_0^{GW}(\mathbf{t},q)|_{t^*_{\geq 1}=0}$ *converges within a non-empty open subset* $M \subseteq H^*(X,\mathbb{C})$. *The subset* M *forms a Dubrovin-Frobenius (DF) manifold, characterized by:*

- *The unit vector field* $e = \frac{\partial}{\partial t^1}$.
- $F_0^{GW}(\mathbf{t},q)|_{t^*_{\geq 1}=0}$ *as the prepotential.*
- *The Euler vector field* $E = c_1(X) + \sum_{\alpha=1}^{N}\left(1 - \frac{1}{2}\deg(\tau_\alpha)\right) t^\alpha \frac{\partial}{\partial t^\alpha}$, *combining the first Chern class and coordinates with classes* τ_α.
- *The metric* η, *defined by the Poincaré pairing on elements of* $H^*(X,\mathbb{C})$.
- *The Euler field's conformality with* $\delta = \dim_{\mathbb{C}} X$, *leading to* $L_E\eta = (2-\delta)\eta$.

Example 3.3.8 (Gromov-Witten theory for $X = \mathbb{CP}^1$) *In this specific case of Gromov-Witten theory for the complex projective line, the DF manifold structure is determined by the following elements:*

- *The prepotential function is* $F_0^{GW}(t^1,t^2) = \frac{1}{2}(t^1)^2 t^2 + e^{t^2}$.
- *The Euler vector field is given by* $E = t^1\frac{\partial}{\partial t^1} + 2t^2\frac{\partial}{\partial t^2}$.
- *The metric is given by an anti-diagonal matrix* $\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- *The Euler field is conformal and preserves the metric, as shown by* $L_E\eta = \eta$.

While Dubrovin-Frobenius Manifolds provide a geometric interpretation to study these structures, there seems to be a disconnection between the geometric foundations of the Dubrovin-Frobenius manifold and the combinatorial complexity of Gromov-Witten invariants. The latter are defined via integrals over the moduli space of stable curves, which is an extremely complex, often singular, space. In contrast, the structure of a Dubrovin-Frobenius manifold is generally defined over the cohomology ring of the target space. Nevertheless, there exists another conceptual structure that aims to bridge this gap. This structure, known as Cohomological Field Theories (CohFTs).

3.4 Cohomological Field theories

Cohomological field theories (CohFTs) were introduced in the 1990 by M. Kontsevich and Y.I. Manin, their motivation was to set in algebraic terms, invariants studied in Gromov-Witten theory, later in 2012, Teleman [Tel12], based on the works of Givental [Giv04], developed a classification for semi-simple CohFTs.

Nowadays, it is used as a tool to study properties of the cohomology ring of $\overline{\mathcal{M}}_{g,n}$, based on a system of linear maps $c_{g,n}$ from $V \otimes \cdots \otimes V \mapsto H^*(\overline{\mathcal{M}}_{g,n})$, where V is a finite dimensional complex vector space, along with these maps we will introduce some axioms to abstract morphisms inspired in the natural maps on $\overline{\mathcal{M}}_{g,n}$. The formal definition requires to fix a basis $e_1, \ldots, e_N$ in V and denote by $\eta = (\eta_{\alpha\beta})$ the matrix of the metric in this basis formally defined by $\eta_{\alpha\beta} := (e_\alpha, e_\beta)$, let A_α be the coordinates of e in this basis: $e = A^\alpha e_\alpha$.

Definition 3.4.1 ([KM94]) *Let V be finite dimensional complex vector space. A* cohomological field theory *(CohFT) is a system of linear maps $c_{g,n} \colon V^{\otimes n} \to H^{\text{even}}(\overline{\mathcal{M}}_{g,n})$, $2g-2+n>0$, such that the following axioms are satisfied:*

1. *The maps $c_{g,n}$ are equivariant with respect to the S_n-action permuting the n copies of V in $V^{\otimes n}$ and the n marked points on classes in $\overline{\mathcal{M}}_{g,n}$.*
2. $f^* c_{g,n}(\otimes_{i=1}^n e_{\alpha_i}) = c_{g,n+1}(\otimes_{i=1}^n e_{\alpha_i} \otimes e)$ *and* $c_{0,3}(e_{\alpha_1} \otimes e_{\alpha_2} \otimes e) = \eta_{\alpha_1\alpha_2}$.[1] *The use of the identity element keeps the map well defined after being pulled back by the forgetful map f, and for the special case of $\overline{\mathcal{M}}_{0,3}$ as it is composed of a point, the class is a scalar that is now chosen to be the inner product.*
3. $\mathrm{gl}^* c_{g_1+g_2,n_1+n_2}(\otimes_{i=1}^{n_1+n_2} e_{\alpha_i}) = c_{g_1,n_1+1}(\otimes_{i\in I_1} e_{\alpha_i} \otimes e_\mu) \otimes c_{g_2,n_2+1}(\otimes_{i\in I_2} e_{\alpha_i} \otimes e_\nu)\eta^{\mu\nu}$. *The metric in this axiom plays the role of the gluing of the extra marked points on $\overline{\mathcal{M}}_{g_1,n_1+1}$ and $\overline{\mathcal{M}}_{g_2,n_2+1}$ respectively.*
4. $\mathrm{gl}^{irr*} c_{g+1,n}(\otimes_{i=1}^n e_{\alpha_i}) = c_{g,n+2}(\otimes_{i=1}^n e_{\alpha_i} \otimes e_\mu \otimes e_\nu)\eta^{\mu\nu}$. *In this case, the metric is used to fill in for the missing points in the pulled back class by the gluing map.*

The vector space V is frequently referred to as the "phase space" of the theory. The term *phase space* refers to the idea that V encompasses all possible states or configurations that the cohomological field theory can assume, and the dimension of V is commonly called the *rank* of the cohomological field theory.

In the framework we've presented for a cohomological field theory, there's a notable characteristic: all the classes $\{c_{g,n}(\otimes_{i=1}^n v_i)\}$, reside in $H^{even}(\overline{\mathcal{M}}_{g,n})$, i.e. they all have an even degree. This isn't a strict necessity, one could indeed conceive a definition of a cohomological field theory without imposing this even-degree constraint, these are often called super CohFTs. However, this approach is considerably more challenging, specifically, when we develop our approach into the associated theory of integrable systems linked to a cohomological field theory (e.g. the double ramification hierarchy). We find that the foundational understanding for supersymmetry in CohFTs is not as robust or well-established. Given these complexities we've chosen to sidestep it in our discussion.

[1] **Remark:** Consider the cohomological class $c_{0,3}(e_{\alpha_1} \otimes e_{\alpha_2} \otimes e)$. This class is essentially a scalar, given that the moduli space $\overline{\mathcal{M}}_{0,3}$ corresponds to a single point. As a result, its cohomology group $H^*(\overline{\mathcal{M}}_{0,3})$ is isomorphic to $\mathbb{C}$. Therefore, the equality is well defined.

Definition 3.4.2 *For a given CohFT* $\{c_{g,n}\}$*, let* $\sum_{i=1}^{n} d_i \leq dim(\overline{\mathcal{M}}_{g,n})$*, we define the* ***correlators*** $\langle \tau_{d_1}(v_1) \cdots \tau_{d_n}(v_n) \rangle_g^{CohFT}$ *on* $\overline{\mathcal{M}}_{g,n}$ *as:*

$$\langle \tau_{d_1}(v_1) \cdots \tau_{d_n}(v_n) \rangle_g^{CohFT} := \int_{\overline{\mathcal{M}}_{g,n}} c_{g,n}(v_1 \otimes \cdots \otimes v_n) \prod_{i=1}^{n} \psi_i^{d_i}, \tag{3.4.1}$$

and $\langle \tau_{d_1}(v_1) \cdots \tau_{d_n}(v_n) \rangle_g^{CohFT} = 0$ *for* $\sum_{i=1}^{n} d_i > dim(\overline{\mathcal{M}}_{g,n})$*.*

Example 3.4.3 (Trivial CohFT) *A basic example of a Cohomological Field Theory (CohFT) is the trivial CohFT, which is defined on* $\mathbb{C}$*. The data* (V, η, e) *for the trivial CohFT is given as follows:*

- *The vector space* V *is the field itself,* $V = \mathbb{C}$*.*
- *The unit of the Frobenius algebra structure is the identity element of the field,* $e = 1$*.*
- *The symmetric bilinear form* η *on* V *is defined by* $\eta(1, 1) = 1$*.*
- *The maps* $c_{g,n}$ *are all set to* 1

Given this data, the correlators (3.4.1) of the trivial CohFT coincide with the intersection numbers on the moduli space of stable curves, $\overline{\mathcal{M}}_{g,n}$*. In other words, the trivial CohFT encapsulates the basic structure of intersection theory on the moduli space of curves.*

Example 3.4.4 (Hodge Class CohFT) *Recall the Hodge bundle (Definition 2.7.4) on* $\overline{\mathcal{M}}_{g,n}$ *and its total Chern class (Definition 2.7.5). Utilizing this information, we can construct a Cohomological Field Theory (CohFT). For this CohFT, we define the vector space* V *as* $\mathbb{Q}$*, the unit element as* 1*, and establish a symmetric bilinear form* η *with* $\eta(1, 1) = 1$*.*

Furthermore, the class $c_{g,n}(1, \ldots, 1)$ *is defined to be the total Chern class of the Hodge bundle, i.e.,* $c_{g,n}(1 \otimes \cdots \otimes 1) = c(\mathbb{E})$*.*

To validate the axioms, we observe that $c_{g,n}(1 \otimes \cdots \otimes 1) = c(\mathbb{E})$ *is independent of the marked points and is equivariant, thereby satisfying Axiom 1. For Axiom 2, the forgetful map leaves the genus unchanged. Additionally, the rank of the space of abelian differentials is zero, which means that the class* $c_{0,3}(1 \otimes 1 \otimes 1) = 1$ *fulfills the requisite condition. Thus, Axiom 2 is also satisfied.*

Axiom 3 can be derived from the properties of the total Chern class, specifically the property that the total Chern class of a direct sum of vector bundles is the product of their individual Chern classes. Consider a separating node class. Since the differential cannot be defined at the node, each connected component of the curve will have its own set of abelian differentials. Let $g = g_1 + g_2$ *be the total genus of the curve, then the pullback of* *the Hodge bundle under the gluing map is given by:*

$$\mathrm{gl}^*\mathbb{E}_g = \mathbb{E}_{g_1} \oplus \mathbb{E}_{g_2}.$$

Consequently, the pullback of the total Chern class is:

$$\mathrm{gl}^* c_{g,*}(1 \otimes \cdots \otimes 1) = c_{g_1,*}(1 \otimes \cdots \otimes 1) \otimes c_{g_2,*}(1 \otimes \cdots \otimes 1),$$

as required.

For Axiom 4, consider a meromorphic differential α with opposite residues around a simple node that is formed. This differential induces a map R_α from $\mathrm{gl}^{irr*}\mathbb{E}_g$ to $\mathbb{C}$, defined by $\alpha \mapsto Res_{n+1}\alpha$. The kernel of this map is $\mathbb{E}_{g-1}$ because, prior to the gluing operation, the marked point lies on the smooth part of the associated universal curve, making its residue zero. This leads to the following sequence of vector bundles:

$$0 \to \mathbb{E}_{g-1} \to \mathbb{E}_g \to \mathbb{C} \to 0,$$

from the splitting property, we have $\mathrm{gl}^{irr*}\mathbb{E}_g \simeq \mathbb{E}_{g-1} \oplus \mathbb{C}$, since the bundles differ in rank and this difference is accounted for by a trivial bundle, Axiom 4 is satisfied by the stability property for Chern classes. [RW23]

3.4.1 Semisimple CohFTs

In recent years, the study of cohomological field theories (CohFTs) has attracted significant interest in mathematics. Among them, semi-simple CohFTs stand out as a special class, exhibiting remarkable properties such as Frobenius algebra structures and compatibility with Dubrovin-Frobenius manifolds, along with a classification given by the Givental-Teleman theorem.

The theorem classifies semisimple Cohomological Field Theories (CohFTs) in terms of certain data, essentially providing a complete classification of these mathematical structures. Specifically, the theorem states that semisimple CohFTs are uniquely determined by their genus-zero data and are thus classified by Frobenius manifolds that arise as their genus-zero correlation functions. This classification is up to a certain equivalence, often called twisting, which involves a change of variables in the underlying Frobenius manifold.

The Teleman classification theorem enables us to recover higher degree classes from the more computationally accessible genus zero data.

A Cohomological Field Theory (CohFT) equipped with a unit $\{c_{g,n}, e\}$ defines a quantum product $\star$ on the complex vector space $V^{\otimes n}$ as follows:

$$\eta(v_1 \star v_2, v_3) = c_{0,3}(v_1 \otimes v_2 \otimes v_3).$$

The quantum product $\star$ is commutative, a property that arises from the equivariance under the action of the symmetric group. The associativity of $\star$ is guaranteed by the gluing map (Axiom 4, Definition 3.4.1). The element $e = 1 \in V$ serves as the identity element for $\star$, as per the definition of the product. Thus, $(V, \star, 1)$ forms a commutative, associative algebra.

Definition 3.4.5 *[Pan18] Given a CohFT $\{c_{g,n}\}_{2g-2+n>0}$, we define its topological part $\{\omega_{g,n}\}$ as the 0-th class part of $\{c_{g,n}\}$,*

$$\omega_{g,n} = [c_{g,n}]^0 \in H^0(\overline{\mathcal{M}}_{g,n}) \otimes V^{*\otimes n}.$$

This simplification does not affect the CohFT structure of $\{\omega_{g,n}\}$, meaning it is also a CohFT.

Example 3.4.6 *Recall the Cohomological Field Theory associated with the Hodge class, which is given by:*

$$c_{g,n}(1 \otimes \cdots \otimes 1) = 1 + \lambda_1 + \ldots + \lambda_g.$$

In this expression, λ_i are the Hodge classes. The topological component of this CohFT denoted as $\omega_{g,n}$ corresponds to:

$$\omega_{g,n}(1 \otimes \cdots \otimes 1) = 1.$$

This implies that the topological part of the CohFT for the Hodge class coincides with the trivial CohFT.

Definition 3.4.7 *[Pan18] A Cohomological Field Theory (CohFT) denoted by $\{c_{g,n}\}$ is said to be semisimple if the associated algebra $(V, \star, 1)$ is a semisimple algebra.* [1]

Lemma 3.4.8 *[Pan18] Let $\{\omega_{g,n}\}$ denote the topological part of $\{c_{g,n}\}$. Then $\{\omega_{g,n}\}$ is uniquely determined by the classes*

$$c_{0,3}(v_1 \otimes v_2 \otimes v_3) \in H^0(\overline{\mathcal{M}}_{g,n}),$$

and their associated quantum product.

Proof. Since the topological part has degree zero, projecting the class associated with a curve $[C_{g,n}, p_1, \ldots, p_n]$ onto its topological part implies that the components of the curve's class are isomorphic to $\mathbb{CP}^1$ with three special points. This allows us to express

[1] In this context, *semisimple* refers to the algebraic structure V equipped with the quantum product $\star$ and the identity element 1. The semisimplicity of the algebra implies that it can be decomposed into a direct sum of simple algebras, which in turn has implications for the structure and properties of the CohFT.

$\omega_{g,n}(v_1 \otimes \cdots \otimes v_n)$ entirely in terms of pull-backs of $c_{0,3}(v_1 \otimes v_2 \otimes v_3)$ by repeatedly applying both the regular and irregular gluing map axioms. This completes the proof of the lemma.

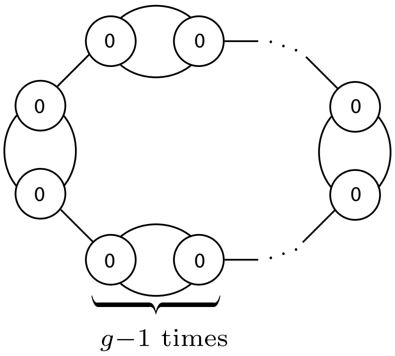

Figure 3.3: The class $\omega_{g,0}$ for $g \geq 2$ represented in terms of dual graphs of the classes $c_{0,3}$.

Definition 3.4.9 *Consider a Cohomological Field Theory (CohFT)* $\{c_{g,n}\}_{2g-2+n>0}$ *equipped with the data* $(V, \eta, 1)$*. We define a translation operator* T *acting on elements of the CohFT as the formal power series:*

$$T(z) = \sum_{k\geq 2} t_k z^k, \quad \textit{where } T \in z^2 V[[z]] \textit{ and } t_k \in V.$$

The action of T *on elements is given by:*

$$T(c_{g,n}(v_1 \otimes \cdots \otimes v_n)) = \sum_{k\geq 0} \frac{1}{k!} f_{n+k*} \left(c_{g,n+k} \left(v_1 \otimes \cdots \otimes v_n \otimes T(\psi_{n+1}) \otimes \cdots \otimes T(\psi_{n+k})\right)\right),$$

where f_{n+k} *is the forgetful map of the point* $n+k$*. Formally, the transformation expands in* ψ*-classes as:*

$$c_{g,n+k}(v_1 \otimes \cdots \otimes v_n \otimes \cdots \otimes T(\psi_-) \otimes \cdots) = \sum_{l\geq 2} \psi_-^l c_{g,n+k}(v_1 \otimes \cdots \otimes v_n \otimes \cdots \otimes t_l \otimes \cdots),$$

where the symbol ψ_- *is used to denote a specific place where evaluation occurs on the* $\psi-$*class within the expression.*

Definition 3.4.10 *[Pan18] Consider a Cohomological Field Theory (CohFT)* $\{c_{g,n}\}_{2g-2+n>0}$ *equipped with the data* $(V, \eta, 1)$*. We define a symplectic operator* R *acting on elements of the CohFT by the matrix series:*

$$R(z) = \sum_{k\geq 0} r_k z^k, \quad \textit{where } R(z) \in End(V)[[z]] \textit{ and } r_k : V \to V,$$

and satisfies the condition

$$R(z) \cdot R^*(-z) = I,$$

here the * *denotes the* $\eta-$*adjoint matrix, i.e.* $R^* = \eta^{-1} R\, \eta$.

The action of R *on elements is given by:*

$$R(c_{g,n}(v_1 \otimes \cdots \otimes v_n)) = \sum_{\{\Gamma_{g,n}\}} \frac{1}{|Aut\ (\Gamma_{g,n})|} \iota_{\Gamma_{g,n}*} \left(\prod_{v \in V} f_{vertex}(v) \prod_{e \in E} f_{edge}(e) \prod_{l \in L} f_{legs}(l) \right),$$

where $\{\Gamma_{g,n}\}$ *denotes the set of all stable graphs of genus* g *with* n *legs*[1]. *We assign the following data to the elements of the graph:*

i) For each vertex v*, we assign the element in* $\{c_{g,n}\}$*:*

$$f_{vertex}(v) = c_{g(v),n(v)},$$

where $g(v)$ *and* $n(v)$ *represent the genus and the number of half-edges and legs associated with the vertex, respectively.*

ii) For each leg l*, we assign the* $End(V)$*-valued cohomology class:*

$$f_{legs}(l) = R(\psi_l),$$

where $\psi_l \in H^2(\overline{\mathcal{M}}_{g(v),n(v)}, \mathbb{Q})$ *is the cotangent class at the marking corresponding to the leg.*

iii) For each edge e*, we assign:*

$$f_{edge}(e) = \frac{\eta^{-1} - R(\psi'_e)\eta^{-1} R(\psi''_e)^{\text{top}}}{\psi'_e + \psi''_e},$$

where $f_{edge}(e) \in V^{\otimes 2}[[\psi', \psi'']]$*,* ψ'_e *and* ψ''_e *are the cotangent classes at the node representing the edge* e*. The symplectic condition ensures that the edge contribution is well-defined.*

Theorem 3.4.11 (Teleman Classification Theorem) *[Tel12] Let* $\{c_{g,n}\}$ *denote a semi-simple CohFT equipped with the quantum product* $\star$*, and let* $\{\omega_{g,n}\}$ *represent its topological component. Then, unique operators* T *and* R *exist such that the following relationship holds:*

$$c_{g,n} = RT\omega_{g,n}$$

This equation establishes a direct link between the CohFT $c_{g,n}$ *and its topological part* $\omega_{g,n}$*, mediated by the operators* T *and* R*.*

[1] For details on the notation and morphisms of stable graphs, please refer to Definition 2.6.1

Example 3.4.12 (Hodge Bundle CohFT) *In the case of the Hodge bundle, the Cohomological Field Theory (CohFT) is characterized by the data* $(V = \mathbb{C}, \eta(1) = 1)$[1]*. The operator* R *in this context is defined as:*

$$R(z) = \exp\left(\sum_{m\geq 1} \frac{1}{m(m+1)} B_{m+1}(-\psi)^m\right),$$

where B_{m+1} *represent the Bernoulli numbers. The operator* R *can be formally derived from Mumford's formula, as expressed in Equation* (2.7.1)*.*

[1]Note that the actions of the R and T operators do not preserve the unit axiom, hence the CohFT of the Hodge class is considered without a unit.

Chapter 4

Moduli Spaces of Riemann surfaces with boundary

The moduli space of Riemann surfaces with boundary represent an important extension of the traditional framework of moduli spaces of curves, with a particular emphasis on how boundary conditions influence the intersection numbers and overall structure of the moduli spaces.

A part of our discussion concerns the Dubrovin-Zhang hierarchy. This system of partial differential equations is inherently linked with total descendant potentials and here also serves as a bridge between the moduli space of Riemann surfaces with boundary and their associated integrable systems. Using these tools, we derive a specific formula for any solution, bearing a resemblance to the Dijkgraaf-Witten formula for a descendant Gromov-Witten potential in genus 1 (3.2.8).

Starting from the Gromov-Witten potential, we demonstrate that when up to genus 1, the exponent of an open descendant potential satisfies a system of linear partial differential equations with a single spatial variable, which we construct explicitly. These results were published in the Journal of High Energy Physics [BB21b].

4.1 Introduction

In 2014, Pandharipande, Solomon, and Tessler [PST14] introduced the theory of intersection numbers on the moduli spaces of Riemann surfaces with boundaries, commonly known as *open moduli spaces*. Unlike traditional moduli spaces, open moduli spaces incorporate additional structures to account for the boundary characteristics of the Riemann surfaces. This provides a more precise framework for examining open Gromov-Witten invariants, essential elements in open string theory. For the necessary definitions relevant to our study, we rely primarily on the works of [PST14] and [Bur15a].

Let $\Delta \subset \mathbb{C}$ denote the open unit disk, and let $\overline{\Delta}$ represent its closure.

Definition 4.1.1 *An* ***extendable embedding*** *of the open disk* Δ *into a closed Riemann surface* C*, specified by*

$$f : \Delta \to C,$$

is a holomorphic map extendable to a holomorphic embedding of an open neighborhood encompassing $\overline{\Delta}$*.*

Two extendable embeddings in C *are deemed* ***disjoint*** *if their images of* $\overline{\Delta}$ *do not intersect, implying that the corresponding closed unit disks are distinctly positioned within the Riemann surface. This differentiation is key to understanding the interplay between various embeddings and their associated regions within the surface.*

Definition 4.1.2 *A* ***Riemann surface with boundary*** $(X, \partial X)$ *is constructed by removing a finite number of disjoint, extendably embedded open disks from a connected, closed Riemann surface* X*.*

This operation results in a surface containing boundaries, which correspond to the extracted disks. In this context, a compact Riemann surface differs from a Riemann surface with boundary.

Given a Riemann surface with boundary $(X, \partial X)$, a canonical process known as the Schwartz reflection, allows us to construct its **double**, $D(X, \partial X)$[AS60]. This 'doubling' transforms $(X, \partial X)$ into a compact Riemann surface. The term 'doubled genus' of $(X, \partial X)$ refers to the standard genus of the resulting Riemann surface $D(X, \partial X)$.

On a Riemann surface with boundary $(X, \partial X)$, we consider two distinct types of marked points. The markings classified as **interior type** are located in $X \setminus \partial X$, while the markings of **boundary type** are located on ∂X.

The moduli space $\mathcal{M}_{g,k,l}$ represents Riemann surfaces with boundary of doubled genus g, containing k unique boundary markings and l unique interior markings. The moduli space $\mathcal{M}_{g,k,l}$ is defined as empty unless the stability condition $2g - 2 + k + 2l > 0$ is satisfied. When this condition is met, the moduli space $\mathcal{M}_{g,k,l}$ is a real orbifold with a real dimension of $3g - 3 + k + 2l$.

The cotangent line classes, denoted as $\psi_i \in H^2(\mathcal{M}_{g,k,l}; \mathbb{Q})$, are defined in the same way as before, which involves using the first Chern classes of the cotangent line bundles that correspond to the markings located within the interior regions (refer to 2.5.1).

However, it is important to note that the cotangent lines situated at the boundaries are not taken into account in [PST14].

In order to formally define the intersection numbers, there are three steps that need to be addressed:

- A compactification of $\mathcal{M}_{g,k,l} \subset \overline{\mathcal{M}}_{g,k,l}$ must be constructed, where $\overline{\mathcal{M}}_{g,k,l}$ is a real orbifold with boundary $\partial\overline{\mathcal{M}}_{g,k,l}$.

- To ensure that integration over $\overline{\mathcal{M}}_{g,k,l}$ is well-defined, the boundary conditions of the integrand along $\partial\overline{\mathcal{M}}_{g,k,l}$ must be specified.

- Orientation issues should be resolved, as the moduli space $\mathcal{M}_{g,k,l}$ is generally non-orientable.

In the works of [PST14], these steps are thoroughly examined and addressed, providing the required foundation. Hence, in this discourse, our focus will be primarily on the computational aspects and attributes of these intersection numbers, along with potential implications for integrable systems.

Definition 4.1.3 ***Open intersection numbers*** *are defined as follows:*

$$\left\langle \tau_{a_1}\tau_{a_2}\dots\tau_{a_l}\sigma^k \right\rangle_g^o := \int_{\overline{\mathcal{M}}_{g,k,l}} \psi_1^{a_1}\psi_2^{a_2}\dots\psi_l^{a_l}. \tag{4.1.1}$$

for $2\sum_{i=1}^{l} a_i = 3g-3+k+2l$*, and* $\left\langle \tau_{a_1}\tau_{a_2}\dots\tau_{a_l}\sigma^k \right\rangle_g^o = 0$ *otherwise. This expression represents the integration of the product of cotangent line classes, raised to their respective powers, over the moduli space* $\overline{\mathcal{M}}_{g,k,l}$*, and* σ *corresponds to boundary markings.*

Following a similar construction to the moduli space $\overline{\mathcal{M}}_{g,n}$ (refer to Equation 2.5.3), we introduce the generalized corresponding total descendant potential:

$$\mathcal{F}^o(t_0,t_1,\dots,s;u) := \sum_{\substack{k>0;g,l\geq 0\\ 2g-2+k+2l>0}} \sum_{a_1,\dots,a_1\geq 0} \frac{u^{g-1}}{k!l!} \left\langle \tau_{a_1}\tau_{a_2}\dots\tau_{a_i}\sigma^k \right\rangle_g^o s^k \prod_{i=1}^{l} t_{a_i}. \tag{4.1.2}$$

In this setup $\mathcal{F}^o(t_0,t_1,\dots,s;u)$ depends on a sequence of variables t_*, a variable s which is associated with the boundary marking, and a variable u typically related to the 'string coupling'.

The double summation goes over all possible values of the genus g, the number of interior markings k, the number of boundary markings l, and the descendant indices a_i. The factor of $(k!l!)^{-1}$ corrects for over-counting of symmetric situations.

4.2 Open KdV and Virasoro Equations

In [Bur15b], A. Buryak demonstrated that the open potential formula (referenced as 4.1.2) conforms to the open Korteweg-de Vries (KdV) equations:

$$\frac{2n+1}{2}\frac{\partial\mathcal{F}^o}{\partial t_n} = u\frac{\partial\mathcal{F}^o}{\partial s}\frac{\partial\mathcal{F}^o}{\partial t_{n-1}} + u\frac{\partial^2\mathcal{F}^o}{\partial s\partial t_{n-1}} + \frac{u^2}{2}\frac{\partial\mathcal{F}^o}{\partial t_0}\frac{\partial^2\mathcal{F}^W}{\partial t_0\partial t_{n-1}} - \frac{u^2}{4}\frac{\partial^3\mathcal{F}^W}{\partial t_0^2\partial t_{n-1}}. \tag{4.2.1}$$

Here, $\mathcal{F}^W = \sum_{g\geq 0} F_g^W$ is the total Witten potential, as defined in equation (2.5.3). The term 'open KdV equations' refers to a system of partial differential equations that are analogous to both the KdV and the Virasoro equations. These equations were developed to describe the intersection theory in the context of the moduli space of Riemann surfaces with boundaries.

The open KdV equations bear a close relationship to the equations governing the wave function of the KdV hierarchy. This connection enables an explicit formula for the open potential $\mathcal{F}^o$ (see (4.1.2)), utilizing Witten's original series generator for intersection numbers on the moduli space of stable curves $\mathcal{F}^W$.

The open potential is found to be in accordance with the open Virasoro equations, offering an equivalent representation of intersection numbers in the moduli space of Riemann surfaces with boundaries.

Furthermore, it's possible to elaborate on the properties of the open potential $\mathcal{F}^o$ by introducing new variables. These variables can be considered as descendants of boundary points. The augmented open potential satisfies a more straightforward set of Virasoro-type equations. In combination, the open KdV equations and these simplified Virasoro equations constitute an analytical setup to study the intersection theory of the moduli space of Riemann surfaces with boundaries.

Moreover, given the following initial condition:

$$\mathcal{F}^o|_{t\geq 1=0} = u^{-1}\left(\frac{s^3}{6} + t_0 s\right), \tag{4.2.2}$$

is possible to fully determinate the open potential $\mathcal{F}^o$. In this case, the open KdV equations, in conjunction with the closed potential F_g^W (2.5.3), provide all necessary information to characterize $\mathcal{F}^o$.

4.2.1 Open Descendent Potentials in Genus 0

In recent years, the field of intersection theory in the context of moduli spaces of Riemann surfaces with boundaries has experienced significant development. These moduli spaces have a corresponding moduli space of closed Riemann surfaces, along with an associated total descendent potential $\mathcal{F}(t_*^*, \varepsilon) = \sum_{g\geq 0} \varepsilon^{2g}\mathcal{F}_g(t_*^*)$ of rank N.

Recent research, as highlighted in [PST14, BCT18, ST23, Che22, CZ21b, CZ21a, Zin20], has prominently featured a genus 0 open descendent potential $F_0^o(t_*^*, s_*) \in \mathbb{C}[[t_*^*, s_*]]$. This potential, which incorporates additional formal variables s_a (for $a \geq 0$) to address boundary contributions, is defined as a solution to a set of partial differential equations that parallel those associated with its closed counterpart. The following definition provides a detailed description of this system.

Definition 4.2.1 *An open descendent potential in genus* 0*, denoted as* $F_0^o \in \mathbb{C}[[t_*^*, s_*]]$*, is a solution of the following system of partial differential equations:*

$$\sum_{b\geq 0} t_{b+1}^{\beta}\frac{\partial F_0^o}{\partial t_b^{\beta}} + \sum_{a\geq 0} s_{a+1}\frac{\partial F_0^o}{\partial s_a} - \frac{\partial F_0^o}{\partial t_0^{\mu}} = -s_0, \tag{4.2.3}$$

$$\sum_{b\geq 0} t_b^{\beta}\frac{\partial F_0^o}{\partial t_b^{\beta}} + \sum_{a\geq 0} s_a\frac{\partial F_0^o}{\partial s_a} - \frac{\partial F_0^o}{\partial t_1^{\mu}} = F_0^o,$$

$$d\left(\frac{\partial F_0^o}{\partial t_{p+1}^{\alpha}}\right) = \frac{\partial^2 F_0}{\partial t_p^{\alpha}\partial t_0^{\mu}}\eta^{\mu\nu} d\left(\frac{\partial F_0^o}{\partial t_0^{\nu}}\right) + \frac{\partial F_0^o}{\partial t_p^{\alpha}} d\left(\frac{\partial F_0^o}{\partial s_0}\right), \tag{4.2.4}$$

$$d\left(\frac{\partial F_0^o}{\partial s_{p+1}}\right) = \frac{\partial F_0^o}{\partial s_p} d\left(\frac{\partial F_0^o}{\partial s_0}\right). \tag{4.2.5}$$

This system is motivated by extending the principles of closed Gromov-Witten theory to the open string sector. The first equation extends the string equation (3.2.4) with the open string parameter s, while the second, like the dilaton equation (3.2.5), describes scaling with respect to gravitational descendants. The third and fourth equations are open analogues of the topological recursion relation (3.2.6), governing the behavior of descendent invariants and reflecting the recursive structure of the moduli space.

To expand the principal hierarchy associated with the potential F_0, let us introduce a new formal variable ϕ. Analogous to the differential polynomials $\Omega^{[0]}_{\alpha,a;\beta,b}$ (refer to Eq. 3.2.10), we define new differential polynomials $\Gamma^{[0]}_{\alpha,a}$ and $\Delta^{[0]}_a$ within the expanded algebraic ring $\mathcal{A}_{v^1,\dots,v^N,\phi;0}$ for $1 \leq \alpha \leq N$ and $a \geq 0$. They are defined as:

$$\Gamma^{[0]}_{\alpha,a} := \left.\frac{\partial F_0^o}{\partial t_a^{\alpha}}\right|_{\substack{t_c^{\gamma}=\delta_{c,0}v^{\gamma} \\ s_c=\delta_{c,0}\phi}},$$

$$\Delta^{[0]}_{a} := \left.\frac{\partial F_0^o}{\partial s_a}\right|_{\substack{t_c^{\gamma}=\delta_{c,0}v^{\gamma} \\ s_c=\delta_{c,0}\phi}},$$

and ϕ^{top} as:

$$\phi^{\text{top}} := \frac{\partial F_0^o}{\partial t_0^{\mu}} \in \mathbb{C}[[t_*^*, s_*]].$$

From these definitions, we arrive at the following relations, analogous to Eq. (3.2.12):

$$\frac{\partial F_0^o}{\partial t_a^{\alpha}} = \Gamma^{[0]}_{\alpha,a}\Big|_{\substack{v^{\gamma}=(v^{\text{top}})^{\gamma} \\ \phi=\phi^{\text{top}}}}, \qquad \frac{\partial F_0^o}{\partial s_a} = \Delta^{[0]}_{a}\Big|_{\substack{v^{\gamma}=(v^{\text{top}})^{\gamma} \\ \phi=\phi^{\text{top}}}}.$$

From these, we can see that the $(N+1)$-tuple of functions $\left((v^{\text{top}})^1,\dots,(v^{\text{top}})^N,\phi^{\text{top}}\right)\big|_{t_0^{\gamma}\mapsto t_0^{\gamma}+A^{\gamma}x}$ satisfy an expanded system of PDEs:

$$\frac{\partial v^\alpha}{\partial t^\beta_b} = \partial_x \eta^{\alpha\mu}\Omega^{[0]}_{\mu,0;\beta,b}, \qquad \frac{\partial v^\alpha}{\partial s_b} = 0,$$
$$\frac{\partial \phi}{\partial t^\beta_b} = \partial_x \Gamma^{[0]}_{\beta,b}, \qquad \frac{\partial \phi}{\partial s_b} = \partial_x \Delta^{[0]}_b,$$

This extended system is referred to as the *extended principal hierarchy*, associated with the pair of potentials (F_0, F_0^o).

4.3 Open descendent potentials in genus 1

In this section, we introduce the concept of an open descendent potential in genus 1 and establish two main theorems, Theorems 4.3.2 and 4.3.7, which are key results of our paper [BB21b].

We define a pair (F_0, F_0^o), which represent the closed and open potentials in genus 0, respectively. Here, F_0 satisfies the string and dilaton equations, whereas F_0^o is established according to Definition 4.2.1.

Definition 4.3.1 *An open descendent potential in genus 1, denoted by $F_1^o \in \mathbb{C}[[t^*_*, s_*]]$, is defined as a solution to the following system of partial differential equations (PDEs):*

$$\frac{\partial F_1^o}{\partial t^\alpha_{a+1}} = \frac{\partial^2 F_0}{\partial t^\alpha_a \partial t^\mu_0}\eta^{\mu\nu}\frac{\partial F_1^o}{\partial t^\nu_0} + \frac{\partial F_0^o}{\partial t^\alpha_a}\frac{\partial F_1^o}{\partial s_0} + \frac{1}{2}\frac{\partial^2 F_0^o}{\partial t^\alpha_a \partial s_0}, \qquad 1 \le \alpha \le N, \qquad a \ge 0, \tag{4.3.1}$$
$$\frac{\partial F_1^o}{\partial s_{a+1}} = \frac{\partial F_0^o}{\partial s_a}\frac{\partial F_1^o}{\partial s_0} + \frac{1}{2}\frac{\partial^2 F_0^o}{\partial s_a \partial s_0}, \qquad a \ge 0. \tag{4.3.2}$$

Consider an open descendent potential in genus 1, represented by F_1^o. We define a formal power series, G^o, belonging to $\mathbb{C}[[v^*, \phi]]$ as follows:

$$G^o := F_1^o\big|_{\substack{t^\alpha_a = \delta_{a,0} v^\alpha \\ s_a = \delta_{a,0}\phi}}.$$

Theorem 4.3.2 *The following formula holds:*

$$F_1^o = \frac{1}{2}\log\frac{\partial^2 F_0^o}{\partial t^{1\!\!1}_0 \partial s_0} + G^o\big|_{\substack{v^\gamma = (v^{\text{top}})^\gamma \\ \phi = \phi^{\text{top}}}}. \tag{4.3.3}$$

Proof 4.3.3 *Firstly, it is important to note that Equation (4.2.3) implies that*

$$\left.\frac{\partial^2 F_0^o}{\partial t^{1\!\!1}_0 \partial s_0}\right|_{t^*_{\ge 1} = s_{\ge 1} = 0} = 1.$$

As such, the logarithm $\log\frac{\partial^2 F_0^o}{\partial t^{1\!\!1}_0 \partial s_0}$ is a well-defined formal power series in the variables t^*_*

and s_*. Furthermore, Equations (3.2.4) and (4.2.3) imply that

$$(v^{\mathrm{top}})^\alpha\big|_{t^*_{\geq 1}=0} = t_0^\alpha, \quad \phi^{\mathrm{top}}\big|_{t^*_{\geq 1}=s_{\geq 1}=0} = s_0.$$

Therefore, Equation (4.3.3) holds when $t^*_{\geq 1} = s_{\geq 1} = 0$.

We define the linear differential operators:

$$\begin{aligned} P^1_{\alpha,a} &:= \frac{\partial}{\partial t^\alpha_{a+1}} - \frac{\partial^2 F_0}{\partial t^\alpha_a \partial t^\mu_0}\eta^{\mu\nu}\frac{\partial}{\partial t^\nu_0} - \frac{\partial F^o_0}{\partial t^\alpha_a}\frac{\partial}{\partial s_0}, && 1 \leq \alpha \leq N, && a \geq 0, \\ P^2_a &:= \frac{\partial}{\partial s_{a+1}} - \frac{\partial F^o_0}{\partial s_a}\frac{\partial}{\partial s_0}, && && a \geq 0. \end{aligned}$$

Under these operators, Equations (4.3.1) and (4.3.2) can be rewritten as

$$\begin{aligned} P^1_{\alpha,a} F^o_1 &= \frac{1}{2}\frac{\partial^2 F^o_0}{\partial t^\alpha_a \partial s_0}, && 1 \leq \alpha \leq N, && a \geq 0, \\ P^2_a F^o_1 &= \frac{1}{2}\frac{\partial^2 F^o_0}{\partial s_a \partial s_0}, && && a \geq 0. \end{aligned}$$

This system of partial differential equations (PDEs) uniquely determines the function F^o_1, given the initial condition $F^o_1|_{t^*_{\geq 1}=s_{\geq 1}=0} = G^o(t^1_0, \ldots, t^N_0, s_0)$.

From Equations (3.2.6), (4.2.4), and (4.2.5), it follows that

$$P^1_{\alpha,a}(v^{\mathrm{top}})^\beta = P^1_{\alpha,a}\phi^{\mathrm{top}} = P^2_a(v^{\mathrm{top}})^\beta = P^2_a\phi^{\mathrm{top}} = 0.$$

Our next step is to verify that the following equations hold:

$$P^1_{\alpha,a} \log\frac{\partial^2 F^o_0}{\partial t^{1\!\!1}_0 \partial s_0} = \frac{\partial^2 F^o_0}{\partial t^\alpha_a \partial s_0}, \tag{4.3.4}$$

$$P^2_a \log\frac{\partial^2 F^o_0}{\partial t^{1\!\!1}_0 \partial s_0} = \frac{\partial^2 F^o_0}{\partial s_a \partial s_0}. \tag{4.3.5}$$

To verify Equation (4.3.4), we compute:

$$\begin{aligned} P^1_{\alpha,a}\log\frac{\partial^2 F^o_0}{\partial t^{1\!\!1}_0\partial s_0} &= \frac{1}{\frac{\partial^2 F^o_0}{\partial t^{1\!\!1}_0 \partial s_0}}\left(\frac{\partial^3 F^o_0}{\partial t^\alpha_{a+1}\partial t^{1\!\!1}_0\partial s_0} - \frac{\partial^2 F_0}{\partial t^\alpha_a \partial t^\mu_0}\eta^{\mu\nu}\frac{\partial^3 F^o_0}{\partial t^\nu_0 \partial t^{1\!\!1}_0 \partial s_0} - \frac{\partial F^o_0}{\partial t^\alpha_a}\frac{\partial^3 F^o_0}{\partial s_0 \partial t^{1\!\!1}_0 \partial s_0}\right) \\ &= \frac{1}{\frac{\partial^2 F^o_0}{\partial t^{1\!\!1}_0 \partial s_0}}\left[\frac{\partial}{\partial s_0}\underline{\left(\frac{\partial^2 F^o_0}{\partial t^\alpha_{a+1}\partial t^{1\!\!1}_0} - \frac{\partial^2 F_0}{\partial t^\alpha_a\partial t^\mu_0}\eta^{\mu\nu}\frac{\partial^2 F^o_0}{\partial t^\nu_0 \partial t^{1\!\!1}_0} - \frac{\partial F^o_0}{\partial t^\alpha_a}\frac{\partial^2 F^o_0}{\partial s_0 \partial t^{1\!\!1}_0}\right)} + \frac{\partial^2 F^o_0}{\partial t^\alpha_a \partial s_0}\frac{\partial^2 F^o_0}{\partial t^{1\!\!1}_0 \partial s_0}\right] \\ &= \frac{\partial^2 F^o_0}{\partial t^\alpha_a \partial s_0}. \end{aligned}$$

The underlined expression vanishes due to Equation (4.2.4). The proof of Equation (4.3.5) follows a similar line of reasoning. Therefore, the theorem is proved.

In this discussion, the computations heavily rely on the structure and properties of the differential operators $P^1_{\alpha,a}$ and P^2_a. Their definitions and applications in the equations demonstrate how the properties of these operators are exploited to simplify the equations and ultimately prove the theorem.

4.3.1 The Role of Differential Operators and PDEs

Consider a differential operator L defined as follows:

$$L = \sum_{i\geq 0} L_i(v^*_*, \varepsilon)(\varepsilon\partial_x)^i, \quad L_i \in \widehat{\mathcal{A}}_{v^1,\ldots,v^N;0}.$$

here v^*_* is a placeholder for the vector variables $v^1, \ldots, v^N$ and its derivatives, $\widehat{\mathcal{A}}_{v^1,\ldots,v^N;0}$ denotes the algebra generated by these variables and their derivatives, evaluated at 0.

Let's introduce f as a formal variable and consider the following partial differential equation (PDE)

$$\frac{\partial}{\partial t}\exp(\varepsilon^{-1}f) = \varepsilon^{-1}L\exp(\varepsilon^{-1}f). \tag{4.3.6}$$

We observe that the differential operator applied to the exponential of a function divided by the same exponential can be represented in terms of a new function $Q_i \in \widehat{\mathcal{A}}_f$:

$$\frac{(\varepsilon\partial_x)^i\exp(\varepsilon^{-1}f)}{\exp(\varepsilon^{-1}f)} = Q_i(f_*, \varepsilon), \quad i \geq 0,$$

which can be recursively computed by the relation

$$Q_i = \begin{cases} 1, & \text{if } i = 0, \\ f_x Q_{i-1} + \varepsilon\partial_x Q_{i-1}, & \text{if } i \geq 1. \end{cases}$$

Remark 4.3.4 *Interestingly, Q_i is independent of f and can be considered a polynomial in the derivatives $f_x, f_{xx}, \ldots$ and ε. In addition, by introducing a new formal variable ψ and substituting $f_{i+1} = \psi_i$ for $i \geq 0$, then Q_i reduces to a differential polynomial of degree* 0.

Equation (4.3.6) can now be rewritten in terms of the new functions Q_i:

$$\frac{\partial f}{\partial t} = \sum_{i\geq 0} L_i(v^*_*, \varepsilon)Q_i(f_*, \varepsilon). \tag{4.3.7}$$

By looking at equation (4.3.7) in greater detail, specifically up to first-order in ε, we have the flowing:

Lemma 4.3.5 *The function Q_i can be approximated as follows up to second order in ε:*

$$Q_i = f_x^i + \varepsilon \frac{i(i-1)}{2} f_x^{i-2} f_{xx} + O(\varepsilon^2).$$

Proof 4.3.6 *The formula is evidently true for $i = 0$. Assuming the inductive hypothesis is true for i, we prove for $i+1$:*

$$\begin{aligned} Q_{i+1} =& f_x Q_i + \varepsilon \partial_x Q_i = f_x^{i+1} + \varepsilon \left(f_x \frac{i(i-1)}{2} f_x^{i-2} f_{xx} + \partial_x \left(f_x^i \right) \right) + O(\varepsilon^2) \\ =& f_x^{i+1} + \varepsilon \frac{(i+1)i}{2} f_x^{i-1} f_{xx} + O(\varepsilon^2), \end{aligned}$$

which concludes the induction step and proves the lemma.

Now consider the expansion of $L_i(v_*^*, \varepsilon)$ in powers of ε:

$$L_i(v_*^*, \varepsilon) = \sum_{j \geq 0} L_i^{[j]}(v_*^*) \varepsilon^j, \quad L_i^{[j]} \in \mathcal{A}_{v^1, \ldots, v^N; j}.$$

With this, equation (4.3.7) can be rewritten in the following form up to the first order in ε:

$$\frac{\partial f}{\partial t} = \sum_{i \geq 0} L_i^{[0]} f_x^i + \varepsilon \sum_{i \geq 0} \left(L_i^{[1]} f_x^i + L_i^{[0]} \frac{i(i-1)}{2} f_x^{i-2} f_{xx} \right) + O(\varepsilon^2). \tag{4.3.8}$$

4.3.2 A Linear PDE for an Open Descendent Potential Up to Genus 1

Let us define the differential operators $L_{\alpha,a}^{\text{int}}$, where $1 \leq \alpha \leq N$, $a \geq 0$, and L_a^{boun}, where $a \geq 0$, as

$$\begin{aligned} L_{\alpha,a}^{\text{int}} &:= \sum_{i \geq 0} \left(L_{\alpha,a,i}^{\text{int};[0]} + \epsilon L_{\alpha,a,i}^{\text{int};[1]} \right) (\epsilon \partial_x)^i, \\ L_a^{\text{boun}} &:= \sum_{i \geq 0} \left(L_{a,i}^{\text{boun};[0]} + \epsilon L_{a,i}^{\text{boun};[1]} \right) (\epsilon \partial_x)^i, \end{aligned}$$

where

$$
\begin{aligned}
L^{\mathrm{int};[0]}_{\alpha,a,i} &:= \mathrm{Coef}_{\phi^i}\Gamma^{[0]}_{\alpha,a} \in \mathcal{A}_{v^1,\dots,v^N;0},\\
L^{\mathrm{int};[1]}_{\alpha,a,i} &:= \mathrm{Coef}_{\phi^i}\left[\left(\frac{\partial G^o}{\partial \phi}\frac{\partial \Gamma^{[0]}_{\alpha,a}}{\partial v^\beta} - \frac{\partial G^o}{\partial v^\beta}\frac{\partial \Gamma^{[0]}_{\alpha,a}}{\partial \phi} + \frac{1}{2}\frac{\partial^2 \Gamma^{[0]}_{\alpha,a}}{\partial v^\beta \partial \phi}\right) v^\beta_x + \frac{\partial G^o}{\partial v^\beta}\eta^{\beta\gamma}\partial_x \Omega^{[0]}_{\gamma,0;\alpha,a}\right]\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \in \mathcal{A}_{v^1,\dots,v^N;1},\\
L^{\mathrm{boun};[0]}_{a,i} &:= \mathrm{Coef}_{\phi^i}\Delta^{[0]}_{a} \in \mathcal{A}_{v^1,\dots,v^N;0},\\
L^{\mathrm{boun};[1]}_{a,i} &:= \mathrm{Coef}_{\phi^i}\left[\left(\frac{\partial G^o}{\partial \phi}\frac{\partial \Delta^{[0]}_{a}}{\partial v^\beta} - \frac{\partial G^o}{\partial v^\beta}\frac{\partial \Delta^{[0]}_{a}}{\partial \phi} + \frac{1}{2}\frac{\partial^2 \Delta^{[0]}_{a}}{\partial v^\beta \partial \phi}\right) v^\beta_x\right] \in \mathcal{A}_{v^1,\dots,v^N;1}.
\end{aligned}
$$

Theorem 4.3.7 *The formal power series* $v^\beta = (v^{\mathrm{top}})^\beta|_{t^\gamma_0 \mapsto t^\gamma_0 + A^\gamma x}$ *and* $f = (F^o_0 + \varepsilon F^o_1)|_{t^\gamma_0 \mapsto t^\gamma_0 + A^\gamma x}$ *satisfy the system of PDEs up to* $O(\varepsilon)$*:*

$$
\frac{\partial}{\partial t^\alpha_a}\exp(\varepsilon^{-1} f) = \varepsilon^{-1} L^{\mathrm{int}}_{\alpha,a}\exp(\varepsilon^{-1} f), \quad \forall\ 1 \le \alpha \le N, \quad a \ge 0, \tag{4.3.9}
$$

$$
\frac{\partial}{\partial s_a}\exp(\varepsilon^{-1} f) = \varepsilon^{-1} L^{\mathrm{boun}}_{a}\exp(\varepsilon^{-1} f), \quad \forall\ a \ge 0. \tag{4.3.10}
$$

Proof: For notational simplicity, let us denote the formal powers series $F^o_0|_{t^\gamma_0 \mapsto t^\gamma_0 + A^\gamma x}$, $F^o_1|_{t^\gamma_0 \mapsto t^\gamma_0 + A^\gamma x}$, $(v^{\mathrm{top}})^\alpha|_{t^\gamma_0 \mapsto t^\gamma_0 + A^\gamma x}$, and $\phi^{\mathrm{top}}|_{t^\gamma_0 \mapsto t^\gamma_0 + A^\gamma x}$ by F^o_0, F^o_1, v^α, and ϕ, respectively. We can then rewrite the statement of Theorem 4.3.2 as

$$
F^o_1 = \frac{1}{2}\log \phi_s + G^o.
$$

Let us prove Equation (4.3.9) up to linear order in ε. We have

$$
\frac{\partial_x (F^o_0 + \varepsilon F^o_1)}{dx} = \phi + \varepsilon\left(\frac{1}{2}\frac{\phi_{xs}}{\phi_s} + \partial_x G^o\right).
$$

By equation (4.3.8), we must verify that

$$
\begin{aligned}
\frac{\partial}{\partial t^\alpha_a}(F^o_0 + \varepsilon F^o_1) &= \sum_{i\ge 0} L^{\mathrm{int};[0]}_{\alpha,a,i}\phi^i + \varepsilon \sum_{i\ge 0} L^{\mathrm{int};[1]}_{\alpha,a,i}\phi^i +\\
&\qquad + \varepsilon \sum_{i \ge 0} L^{\mathrm{int};[0]}_{\alpha,a,i}\left(\frac{i(i-1)}{2}\phi^{i-2}\phi_x + i\phi^{i-1}\left(\frac{1}{2}\frac{\phi_{xs}}{\phi_s} + \partial_x G^o\right)\right)\\
\frac{\partial}{\partial t^\alpha_a}(F^o_0 + \varepsilon F^o_1) &= \Gamma^{[0]}_{\alpha,a} + \varepsilon\left(\sum_{i\ge 0} L^{\mathrm{int};[1]}_{\alpha,a,i}\phi^i + \frac{1}{2}\frac{\partial^2 \Gamma^{[0]}_{\alpha,a}}{\partial \phi^2}\phi_x + \frac{1}{2}\frac{\partial \Gamma^{[0]}_{\alpha,a}}{\partial \phi}\frac{\phi_{xs}}{\phi_s} + \frac{\partial \Gamma^{[0]}_{\alpha,a}}{\partial \phi}\partial_x G^o\right)\\
\frac{\partial F^o_1}{\partial t^\alpha_a} &= \frac{1}{2}\frac{\partial \Gamma^{[0]}_{\alpha,a}}{\partial \phi}\frac{\phi_{xs}}{\phi_s} + \left(\frac{1}{2}\frac{\partial^2 \Gamma^{[0]}_{\alpha,a}}{\partial \phi^2} + \frac{\partial \Gamma^{[0]}_{\alpha,a}}{\partial \phi}\frac{\partial G^o}{\partial \phi}\right)\phi_x + \frac{\partial \Gamma^{[0]}_{\alpha,a}}{\partial \phi}\frac{\partial G^o}{\partial v^\beta} v^\beta_x + \sum_{i \ge 0} L^{\mathrm{int};[1]}_{\alpha,a,i}\phi^i.
\end{aligned}
$$

Using the definition of $L^{\mathrm{int};[1]}_{\alpha,a,i}$, we equate the derivative of F^o_1 with respect to t^α_a to various expressions involving derivatives of $\Gamma^{[0]}_{\alpha,a}$, G^o, and $\Omega^{[0]}_{\gamma,0;\alpha,a}$ as follows:

$$\begin{aligned}\frac{\partial \mathcal{F}^o_1}{\partial t^\alpha_a} =& \frac{1}{2}\frac{\partial \Gamma^{[0]}_{\alpha,a}}{\partial \phi}\frac{\phi_{xs}}{\phi_s} + \frac{1}{2}\frac{\partial^2 \Gamma^{[0]}_{\alpha,a}}{\partial \phi^2}\phi_x + \frac{\partial \Gamma^{[0]}_{\alpha,a}}{\partial \phi}\frac{\partial G^o}{\partial \phi}\phi_x \\ &+ \frac{\partial G^o}{\partial \phi}\frac{\partial \Gamma^{[0]}_{\alpha,a}}{\partial v^\beta}v^\beta_x + \frac{1}{2}\frac{\partial^2 \Gamma^{[0]}_{\alpha,a}}{\partial v^\beta \partial \phi}v^\beta_x + \frac{\partial G^o}{\partial v^\beta}\eta^{\beta\gamma}\partial_x \Omega^{[0]}_{\gamma,0;\alpha,a}.\end{aligned}$$

This can be simplified further as:

$$\frac{\partial F^o_1}{\partial t^\alpha_a} = \frac{1}{2}\frac{\partial \Gamma^{[0]}_{\alpha,a}}{\partial \phi}\frac{\phi_{xs}}{\phi_s} + \frac{1}{2}\partial_x \frac{\partial \Gamma^{[0]}_{\alpha,a}}{\partial \phi} + \frac{\partial G^o}{\partial \phi}\partial_x \Gamma^{[0]}_{\alpha,a} + \frac{\partial G^o}{\partial v^\beta}\eta^{\beta\gamma}\partial_x \Omega^{[0]}_{\gamma,0;\alpha,a}. \tag{4.3.11}$$

To confirm the above, we compute the derivative of F^o_1 with respect to t^α_a:

$$\begin{aligned}\frac{\partial F^o_1}{\partial t^\alpha_a} =& \frac{1}{2}\frac{\left(\phi_{t^\alpha_a}\right)_s}{\phi_s} + \frac{\partial G^o}{\partial v^\beta}\eta^{\beta\gamma}\partial_x \Omega^{[0]}_{\gamma,0;\alpha,a} + \frac{\partial G^o}{\partial \phi}\partial_x \Gamma^{[0]}_{\alpha,a} \\ =& \frac{1}{2}\frac{\partial_x\left(\Gamma^{[0]}_{\alpha,a}\right)_s}{\phi_s} + \frac{\partial G^o}{\partial v^\beta}\eta^{\beta\gamma}\partial_x \Omega^{[0]}_{\gamma,0;\alpha,a} + \frac{\partial G^o}{\partial \phi}\partial_x \Gamma^{[0]}_{\alpha,a} \\ =& \frac{1}{2}\frac{\partial_x\left(\frac{\partial \Gamma^{[0]}_{\alpha,a}}{\partial \phi}\phi_s\right)}{\phi_s} + \frac{\partial G^o}{\partial v^\beta}\eta^{\beta\gamma}\partial_x \Omega^{[0]}_{\gamma,0;\alpha,a} + \frac{\partial G^o}{\partial \phi}\partial_x \Gamma^{[0]}_{\alpha,a} \\ =& \frac{1}{2}\partial_x \frac{\partial \Gamma^{[0]}_{\alpha,a}}{\partial \phi} + \frac{1}{2}\frac{\partial \Gamma^{[0]}_{\alpha,a}}{\partial \phi}\frac{\phi_{xs}}{\phi_s} + \frac{\partial G^o}{\partial v^\beta}\eta^{\beta\gamma}\partial_x \Omega^{[0]}_{\gamma,0;\alpha,a} + \frac{\partial G^o}{\partial \phi}\partial_x \Gamma^{[0]}_{\alpha,a},\end{aligned}$$

which verifies equation (4.3.11) and, by extension, equation (4.3.9) up to the approximation of ϵ.

The proof of equation (4.3.10) follows the same procedure.

The discussion was centered around the series v^β, F^o_0, and F^o_1, along with their interrelation with the system of PDEs. We found that these series and the system are governed by the constraints of certain internal and boundary equations, which we thoroughly examined.

By looking these equations, we notice it is possible to manipulate the formal power series, by employing the notation $L^{\text{int};[n]}_{\alpha,a,i}$ and $L^{\text{boun};[n]}_{a,i}$ we can make the involved expressions more approachable and intuitive.

We also found that certain formal power series are linked with others. The equation $F^o_1 = \frac{1}{2}\log \phi_s + G^o$ is an example of this. By analyzing the derivatives of F^o_0 and F^o_1, we were able to elaborate on how these quantities evolve within the system and its dynamics.

Chapter 5

Double Ramification and Dubrovin-Zhang Hierarchies

5.1 Introduction

The Double Ramification Hierarchy (DR hierarchy) takes as input ideas from algebraic geometry to establish an integrable hierarchy. Its inception can be traced back to the study of moduli spaces of algebraic curves and their intersection theory. The hierarchy is also an elaborated example of the relation between curve counting theories and integrable systems.

The Double Ramification Hierarchy represents an advancement in the field of integrable hierarchies associated with cohomological field theories. The DR hierarchy allows the modification of certain axioms of the usual CohFTs, thereby accommodating a more diverse range of mathematical structures. The hierarchy is characterized by certain initial data coming from the Double Ramification Cycle. This cycle is a class in the homology ring of the moduli space of stable curves, first introduced into an integrable hierarchy by A. Buryak [Bur15a]. The DR cycle, in essence, is a cohomological representation of the condition that a given meromorphic function on a Riemann surface has prescribed orders at the points of the surface.

The DR hierarchy is closely related to total descendant potentials and the theory of nonlinear partial differential equations, providing new connections between geometric invariants and the integrable systems they induce. The DR hierarchy offers a mathematical setting that allows the study of several theories and models, including Gromov-Witten theory and Gauged Linear Sigma Models[1], resulting in a unified theory capable of describing an array of curve counting theories, geometric structures, and physical systems.

The double ramification cycle can be understood from two main perspectives [Pix21]:

[1] Gauged Linear Sigma Model (GLSM) is a type of two-dimensional quantum field theory that describes the dynamics of maps from a Riemann surface to $\mathbb{C}^n$, subject to certain gauge symmetries. Through GLSM, one can compute certain Gromov-Witten invariants, especially in the context of mirror symmetry. [FJR17]

- **Ramification Profiles**: The term *double ramification* is derived from the specific ramification profiles that the cycle aims to parametrize. Consider a genus g curve, denoted as $C_{g,n}$, which admits a ramified map $f : C_{g,n} \to \mathbb{CP}^1$. This map is characterized by predetermined ramification profiles over two distinct points: 0 and ∞. These profiles are encapsulated in a vector $A \in \mathbb{Z}^n$. The positive and negative components of A correspond to two partitions that describe the ramification over 0 and ∞, respectively, and must satisfy $\sum a_i = 0$. The marked points on $C_{g,n}$ that correspond to non-zero entries in A are precisely the pre-images of 0 and ∞ under the map f. Points corresponding to zero entries in A are not subject to this constraint.

  The DR-cycle is then defined using n integers $a_1, \ldots, a_n$ with $\sum a_i = 0$. For a given class $[C_{g,n}, p_1, \ldots, p_n]$ in the moduli space $\mathcal{M}_{g,n}$, we examine the divisor $\sum a_i p_i$ on the curve $C_{g,n}$. If this divisor is principal, it implies the existence of a meromorphic function $f : C_{g,n} \to \mathbb{CP}^1$ such that the sum of the orders of its zeros equals the sum of the orders of its poles. Note that it is permissible for some a_j to be zero, corresponding to unconstrained points p_j.

  Intuitively, the cohomology class $DR_g(a_1, \ldots, a_n)$ is the Poincaré dual to the locus in $\overline{\mathcal{M}}_{g,n}$ that is characterized by curves with the specified principal divisor.

- **Abel-Jacobi Maps**: The double ramification cycle can also be understood through Abel-Jacobi maps. While this perspective is useful within the context of this work, describing the formal details might be excessive for our current objectives. Intuitively it involves a $g-$dimensional universal abelian variety[1] $\mathcal{X}_g$ and the data from A defines a morphism from $j_A : \mathcal{M}_{g,n} \to \mathcal{X}_g$ and the double ramification locus is the inverse image under j_A of a zero section of $\mathcal{X}_g$. This construction can be extended to the moduli space of curves of compact type $\mathcal{M}^{ct}_{g,n}$ (def. 2.6.2).

  In 2016, F. Janda, R. Pandharipande, A. Pixton, and D. Zvonkine provided a formula that expresses the double ramification cycle in terms of basic tautological classes [JPPZ17]. This result addressed a question originally posed by Y. Eliashberg in 2001. [2]

Example 5.1.1 *For $g = 0$, the meromorphic function $f(z) = \prod_{i=1}^{n}(z - p_i)^{a_i}$ satisfies the required ramification profile for any element in $\mathcal{M}_{0,n}$. Therefore, the Double Ramification (DR) cycle encompasses the entire moduli space, and its Poincaré dual is the unit in cohomology:*

[1] An abelian variety is a complex torus that can be embedded into projective space. Given a moduli space that parametrizes abelian varieties of a fixed dimension (with some additional data, like polarization), the universal abelian variety is a family of abelian varieties over this moduli space such that each point of the moduli space corresponds to an abelian variety, which is a fiber of the universal abelian variety over that point.

[2] This is commonly known in the literature as the Eliashberg problem.

$$DR_0(a_1,\ldots,a_n)=1. \tag{5.1.1}$$

Remark: This approach is not universally applicable, especially in the case of nodal curves. While we leveraged the compactness of the space to obtain its Poincaré dual, a more comprehensive definition exists that addresses these special cases. This broader definition serves as the motivation for the topic discussed in the next section.

5.2 Moduli space of stable relative maps

In this section, we will adopt the construction outlined in [Bur17] with minor modifications to the notation.

Definition 5.2.1 *Let* $\left(C_{g,n};p_1,\ldots,p_n,q_1^1,\ldots,q_{n_1}^1,q_1^2,\ldots,q_{n_2}^2\right)$ *be a nodal curve of genus* g *with* n *marked points. Consider* $n_1,n_2,d\in\mathbb{Z}_{\geq 0}$*. Let* $\mu^1=(\mu_1^1,\ldots,\mu_{n_1}^1)\in\mathbb{Z}_{\geq 0}^{n_1}$ *and* $\mu^2=(\mu_1^2,\ldots,\mu_{n_2}^2)\in\mathbb{Z}_{\geq 0}^{n_2}$ *satisfying*

$$|\mu^1|:=\sum_{i=1}^{n_1}\mu_i^1=d,\qquad |\mu^2|:=\sum_{i=1}^{n_2}\mu_i^2=d.$$

A stable relative map (SRM) to the unparametrized $\mathbb{CP}^1$ *is represented by*

$$\left[\left(C_{g,n};p_1,\ldots,p_n,q_1^1,\ldots,q_{n_1}^1,q_1^2,\ldots,q_{n_2}^2\right)\xrightarrow{f_{SRM}}(\mathbb{X};q_1,q_2)\right],$$

where $(\mathbb{X};q_1,q_2)$ *is a nodal curve whose dual graph is of genus 0 (fig. 5.1). In this curve,* q_1 *and* q_2 *are points located on the first and last components of the chain, respectively and consists of the following components:*

- *The pre-image of each* q_j *under* f_{SRM} *is* $\{q_1^j,\ldots,q_{n_j}^j\}_{j=1,2}$*, and the multiplicity of* f_{SRM} *at each* q_i^j *is* μ_i^j*.*
- *The pre-image of any nodal point* q *in* $\mathbb{X}$ *consists of nodal points in* $C_{g,n}$*. Furthermore, if* q *is a nodal point in* $\mathbb{X}$ *and* p *is in* $f_{SRM}^{-1}(q)$*, then local representations of* $C_{g,n}$ *and* $\mathbb{X}$ *at* p *and* q *must exist such that* f_{SRM} *can be expressed as* $(u,v)\mapsto(u^m,v^m)=(x,y)$ *for some integer* $m\geq 1$[1]*.*

Definition 5.2.2 *Two stable relative maps, denoted as* $[C_{g,n}\xrightarrow{f_{SRM}}\mathbb{X}]$ *and* $[C'_{g,n}\xrightarrow{f'_{SRM}}\mathbb{X}']$*, are considered isomorphic if there exist isomorphisms* $\alpha: C_{g,n}\to C'_{g,n}$ *and* $\beta:\mathbb{X}\to\mathbb{X}'$ *that not only preserve the marked points but also commute with* f_{SRM} *and* f'_{SRM}*.*

[1] Recall, any nodal point on a curve with local coordinates (x_1,x_2) can be locally characterized by the equation $x_1x_2=0$.

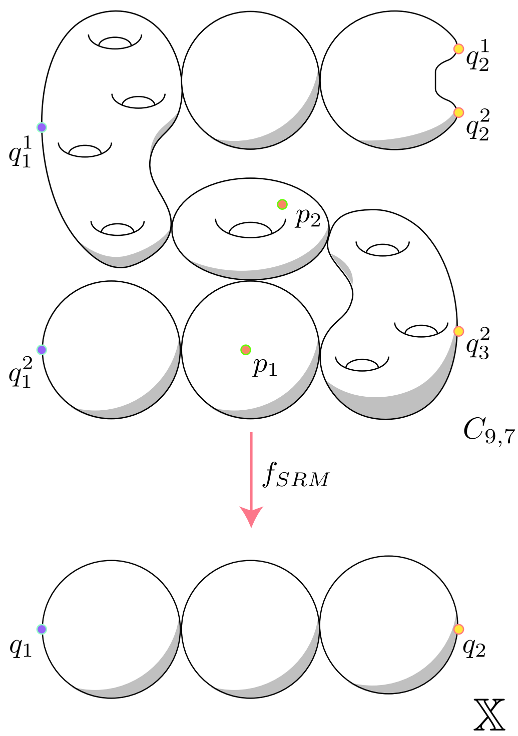

Figure 5.1: Illustration of a stable relative map originating from $C_{9,7}$ targeting $\mathbb{X}$.

Definition 5.2.3 *The moduli space of stable relative maps, denoted by* $\overline{\mathcal{M}}^{\sim}_{g,n}(\mu^1, \mu^2)$*, comprises the isomorphism classes of stable relative maps targeting the unparameterized* $\mathbb{CP}^1$*.*

The space $\overline{\mathcal{M}}^{\sim}_{g,n}(\mu^1, \mu^2)$ is a compact topological space, sharing similarities with the moduli space $\overline{\mathcal{M}}_{g,n}$ in that it is also a Deligne-Mumford stack. This space is endowed with a virtual fundamental class, denoted as

$$\left[\overline{\mathcal{M}}^{\sim}_{g,n}(\mu^1, \mu^2)\right]^{\mathrm{virt}} \in H_{\mathcal{D}}\left(\overline{\mathcal{M}}^{\sim}_{g,n}(\mu^1, \mu^2), \mathbb{Q}\right),$$

where the degree $\mathcal{D}$ is calculated as

$$\mathcal{D} = 2(2g - 3 + n + n_1 + n_2).$$

Assuming that $2g - 2 + n + n_1 + n_2 > 0$, we can associate a stabilized curve to any stable relative map, typically the stabilized curve can be expressed as

$$\left(C_{g,n}; p_1, \ldots, p_n, q^1_1, \ldots, q^1_{n_1}, q^2_1, \ldots, q^2_{n_2}\right).$$

This suggests the existence of a map to $\overline{\mathcal{M}}_{g,n}$, which we denote by $\mathcal{S}$.

For $g, n \geq 0$ and $2g-2+n > 0$, consider a list of integers $a_1, \ldots, a_n$ such that $\sum a_i = 0$. The indices $i_1 < \cdots < i_{n_1}$ and $j_1 < \cdots < j_{n_2}$ label the positive and negative numbers in the list, respectively. The remaining $n_0 = n - n_1 - n_2$ numbers are set to zero. We define

$\mu^1 = (\mu^1_1, \dots, \mu^1_{n_1})$ and $\mu^2 = (\mu^2_1, \dots, \mu^2_{n2})$ where $\mu^1_k = a_{i_k}$ and $\mu^2_k = -a_{j_k}$.

The double ramification cycle is then defined as

$$\mathrm{DR}_g(a_1, \dots, a_n) = \mathrm{PD}\left(\mathcal{S}_* \left[\overline{\mathcal{M}}^{\sim}_{g,n}(\mu^1, \mu^2)\right]^{\mathrm{virt}}\right) \in H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

Here, PD denotes the Poincaré duality map. Consequently, we have

$$\mathrm{DR}_g(a_1, \dots, a_n) \in H^{2g}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}).$$

Recall the forgetful map f, as previously defined in Definition 2.4.2. This map establishes a relationship between the double ramification cycles in different moduli spaces. Specifically, the relationship can be expressed as:

$$f^* \mathrm{DR}_g(a_1, \dots, a_n) = \mathrm{DR}_g(a_1, \dots, a_n, 0). \tag{5.2.1}$$

There exists a generalized version of the relation given in equation (5.1.1), where all the ramification profiles are set to zero:

$$DR_g(0, ..., 0) = (-1)^g \lambda_g \in H^{2g}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q}). \tag{5.2.2}$$

Theorem 5.2.4 *[Hai13][JPPZ17] Consider two non-negative integers h and g such that $0 \leq h \leq g$. Let I be a subset of the set $P = \{1, \dots, n\}$.*

We impose two conditions:

- $2h - 1 + |I| > 0$,
- $2(g - h) - 1 + |I^c| > 0$,

here, $|I|$ and $|I^c|$ denote the cardinalities of the sets I and its complement I^c, respectively.

Finally, we introduce a cohomology class δ^I_h in $H^2(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. This class is defined by the push-forward of the identity classes from both moduli spaces under the gluing map (def. 2.4.3), given by:

$$\delta^I_h := gl_*(1 \times 1).$$

Then, the following formula holds:

$$\mathrm{DR}_g\left(a_1, \dots, a_n\right)|_{\mathcal{M}^{\mathrm{ct}}_{g,n}} = \frac{1}{g!}\left(\sum_{i=1}^{n} \frac{a_i^2 \psi_i}{2} - \frac{1}{2} \sum_{\substack{I \subset \{1, \dots, n\} \\ |I| \geqslant 2}} a_I^2 \delta_0^I - \frac{1}{4} \sum_{I \subset \{1, \dots, n\}} \sum_{h=1}^{g-1} a_I^2 \delta_h^I\right)^g, \tag{5.2.3}$$

where $\mathcal{M}^{\mathrm{ct}}_{g,n}$ is the moduli space of curves of compact type (def. 2.6.2), this theorem implies that the restriction $\mathrm{DR}_g(a_1,\ldots,a_n)|_{\mathcal{M}^{\mathrm{ct}}_{g,n}}$ is a homogeneous polynomial of degree $2g$ in the variables $a_1,\ldots,a_n$ with coefficients in $H^{2g}\left(\mathcal{M}^{\mathrm{ct}}_{g,n},\mathbb{Q}\right)$.

In 2023, A. Pixton proved[1] that the class $\mathrm{DR}_g(a_1,\ldots,a_n)$ is a polynomial in the variables $a_1,\ldots,a_n$ with coefficients in $H^{2g}\left(\overline{\mathcal{M}}_{g,n},\mathbb{Q}\right)$ [Pix23].

5.3 The Double Ramification hierarchy

The most important feature of the Double Ramification (DR) cycle is its behavior under integration. Specifically, for any cohomology class $\theta \in H^*(\overline{\mathcal{M}}_{g,n+1})$, the integral

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \lambda_g \mathrm{DR}_g\left(-\sum a_i, a_1,\ldots,a_n\right)\theta,$$

is a homogeneous polynomial in the variables $a_1,\ldots,a_n$ of degree $2g$.

Given a Cohomological Field Theory (CohFT) denoted by $\{c_{g,n}\}$ (Definition 3.4.1), we introduce differential polynomials $g_{\alpha,d}$ in $\widehat{\mathcal{A}}_{u;0}$ for $1\le\alpha\le N$ and $d\ge 0$ as follows:

$$g_{\alpha,d} := \sum_{\substack{g,n\ge 0\\ 2g-1+n>0}} \frac{\varepsilon^{2g}}{n!} \sum_{\substack{b_1,\ldots,b_n\ge 0\\ b_1+\ldots+b_n=2g}} u^{\alpha_1}_{b_1}\ldots u^{\alpha_n}_{b_n}\times \tag{5.3.1}$$
$$\times \operatorname{Coef}_{a_1^{b_1}\ldots a_n^{b_n}} \int_{\overline{\mathcal{M}}_{g,n+1}} \mathrm{DR}_g\left(-\sum a_i, a_1,\ldots,a_n\right)\lambda_g\psi_1^d c_{g,n+1}\left(e_\alpha\otimes\otimes_{i=1}^n e_{\alpha_i}\right).$$

This construction is proven to yield local functionals $\bar{g}_{\alpha,d}=\int g_{\alpha,d}\,dx$ that commute with respect to the bracket $\{\cdot,\cdot\}_{\eta^{-1}\partial_x}$ [Bur15a].

Definition 5.3.1 *The system of equations*

$$\frac{\partial u^\alpha}{\partial t^\beta_q} = \eta^{\alpha\mu}\partial_x\frac{\delta\bar{g}_{\beta,q}}{\delta u^\mu},\quad 1\le\alpha,\beta\le N,\quad q\ge 0, \tag{5.3.2}$$

is called the Double Ramification Hierarchy.

Example 5.3.2 (Trivial CohFT) *Consider the Double Ramification hierarchy associated with the trivial Cohomological Field Theory, denoted as $\{c_{g,n}=1\}$. The integrals for the coefficients are given by*[2]*:*

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \mathrm{DR}_g\left(-\sum a_i, a_1,\ldots,a_n\right)\lambda_g\psi_1^d. \tag{5.3.3}$$

We observe that the cohomology class $\mathrm{DR}_g(-\sum a_i, a_1,\ldots,a_n)\lambda_g\psi_1^d$ is non zero only when its degree $(g+g+d)$ matches the dimension of $\overline{\mathcal{M}}_{g,n+1}$, leading to the relation:

[1] As of the time of writing, this result has not been officially published and therefore has not undergone peer review. However, it is available online for preliminary examination and discussion.

[2] Note that throughout our computation, we disregarded the α index in equation (5.3.1) for the trivial CohFT, since such term emerges from the free index of the CohFT.

$$d = g + n - 2. \tag{5.3.4}$$

We will focus on genus 0 terms, we simplify (5.3.3) by use of (5.1.1) as follows:

$$\int_{\overline{\mathcal{M}}_{0,n+1}} \mathrm{DR}_0\left(-\sum a_i, a_1, \ldots, a_n\right) \lambda_0 \psi_1^d = \int_{\overline{\mathcal{M}}_{0,n+1}} \psi_1^d,$$

The integral was previously calculated in (2.5.6), so:

$$\int_{\overline{\mathcal{M}}_{0,n+1}} \psi_1^d = \langle \tau_{k_1} \cdots \tau_{k_{n+1}} \rangle_0 \big|_{k_1=d, k_{i>1}=0} = \frac{(n-2)!}{d!} = 1.$$

Substituting this result into (5.3.1) along with the condition (5.3.4), we get the genus 0 Hamiltonian densities:

$$g_d^{[0]} = \sum_{n>1} \frac{1}{n!} u^{\alpha_1} \ldots u^{\alpha_n} \delta^1_{\alpha_1} \cdots \delta^1_{\alpha_n} \Big|_{n=d+2} = \frac{u^{d+2}}{(d+2)!}. \tag{5.3.5}$$

It is interesting to observe the striking similarity of this formulation to the Hamiltonians of the dispersionless KdV hierarchy, as introduced in Equation (1.3.1).

If we were to include higher genus terms, we need to revisit condition (5.3.4), for $d = 0$ the only possible cases are $(g, n) = (0, 1)$ and $(1, 0)$.

The genus 0 is straightforward and follows from (5.3.5). Therefore, our focus shifts to the $(1, 0)$ case:

$$\int_{\overline{\mathcal{M}}_{1,2}} \mathrm{DR}_1(-a_1, a_1)\, \lambda_1.$$

We can replace the DR class utilizing Hain's formula (eq. 5.2.3) and identify the stable nodes. In this instance, stable nodes arise when the genus 0 component has at least 2 markings (fig. 5.2).

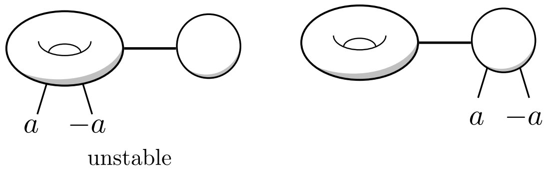

Figure 5.2: Possible boundary Divisors as derived from Hain's Formula, represented from equation (5.2.3).

Applying Hain's formula and considering the symmetry of the ψ classes:

$$DR_1(-a_1, a_1) = \frac{(-a_1)^2}{2}\psi_1 + \frac{a_1^2}{2}\psi_2 = a_1^2 \psi_1. \tag{5.3.6}$$

The $\psi-$class is then eliminated using the dilaton relation (cf. 2.5.2):

$$\int_{\overline{\mathcal{M}}_{1,2}} \mathrm{DR}_1\left(-a_1, a_1\right) \lambda_1 = a_1^2 \int_{\overline{\mathcal{M}}_{1,2}} \psi_1 \lambda_1 = a_1^2 \int_{\overline{\mathcal{M}}_{1,1}} \lambda_1 = \frac{a_1^2}{24}.$$

For the last equality, we used the class computed in Example 2.1.14. According with (5.3.1), the term a_1^2 contributes as u_{xx}, so the Hamiltonian density for $d = 0$ reads:

$$g_0 = \frac{u^2}{2} + \varepsilon^2 \frac{u_{xx}}{24}.$$

For $d = 1$, the non-zero cases are $(g, n) = (0, 3)$, $(1, 2)$, and $(2, 1)$, again the genus 0 case can be derived from (5.3.5), so we move onto $(g, n) = (1, 2)$:

$$\int_{\overline{\mathcal{M}}_{1,3}} \mathrm{DR}_1\left(-a_1 - a_2, a_1, a_2\right) \lambda_1 \psi_1.$$

We can simplify this expression by applying the dilaton equation (see 2.5.2). This yields:

$$(2g - 2 + n)\bigg|_{g=1,n=2} \int_{\overline{\mathcal{M}}_{1,2}} \mathrm{DR}_1\left(a_1, -a_1\right) \lambda_1. \tag{5.3.7}$$

This is the same term as in (5.3.6), so:

$$\int_{\overline{\mathcal{M}}_{1,3}} \mathrm{DR}_1\left(-a_1 - a_2, a_1, a_2\right) \lambda_1 \psi_1 = \frac{a_1^2}{12}. \tag{5.3.8}$$

Following a similar procedure from Hain's formula we find:

$$\int_{\overline{\mathcal{M}}_{2,2}} \mathrm{DR}_2\left(-a_1, a_1\right) \lambda_2 \psi_1 = \frac{1}{2!} \left(\frac{a_1^2}{24}\right)^2.$$

So, the Hamiltonian density for $d = 1$ can be expressed as follows:

$$\begin{aligned} g_1 &= \frac{u^3}{3!} + \frac{\varepsilon^2}{2!} \operatorname{Coef}_{a_1^{b_1}, a_2^{b_2}} \left(\frac{a_1^2}{12} a_2^0\right) u_{b_1} u_{b_2} + \varepsilon^4 \operatorname{Coef}_{a_1^{b_1}} \left(\frac{a_1^4}{1152}\right) u_{b_1}, \\ &= \frac{u^3}{6} + \frac{\varepsilon^2}{24} u u_{xx} + \frac{\varepsilon^4}{1152} u_{xxxx}. \end{aligned}$$

It is noteworthy that this Hamiltonian matches the Korteweg-de Vries (KdV), as previously introduced in (1.3.10). Although deriving the complete terms involves extensive computation, it is a remarkable result that the Double Ramification (DR) hierarchy for the trivial Cohomological Field Theory (CohFT) coincides with the full KdV hierarchy.

We can equip the Double Ramification (DR) hierarchy with N linearly independent Casimirs of its Poisson bracket, denoted as $\{\cdot, \cdot\}_{\eta^{-1}\partial_x}$. Specifically, we define

$$\bar{g}_{\alpha,-1} := \int \eta_{\alpha\beta} u^\beta \, dx,$$

where $1 \leq \alpha \leq N$.

Another crucial object in the context of the DR hierarchy is the local functional $\bar{g} \in \widehat{\Lambda}_{u;0}$. This is determined by the following relation [Bur15a]:

$$\bar{g}_{\mathbb{1},1} = (D-2)\bar{g}, \quad \text{where} \quad D := \sum_{n\geq 0}(n+1)u^\alpha_n \frac{\partial}{\partial u^\alpha_n},$$

and $\bar{g}_{\mathbb{1},1} := A^\alpha \bar{g}_{\alpha,1}$.[1] It's important to note that $\frac{\delta \bar{g}}{\delta u^\alpha} = g_{\alpha,0}$.

The local functional $\bar{g}$ also has an explicit expression up to the approximation ε^2, as given in [BDGR18]:

$$\bar{g} = \int f\, dx - \frac{\varepsilon^2}{48}\int c^\theta_{\theta\xi} c^\varepsilon_{\alpha\beta} u^\alpha_x u^\beta_x\, dx + O\left(\varepsilon^4\right), \tag{5.3.9}$$

where $f := F|_{t^*=u^*}$ and $c^\alpha_{\beta\gamma} := C^\alpha_{\beta\gamma}|_{t^*=u^*}$.

Conjecture 5.3.3 *[BRS21] Consider a homogeneous Cohomological Field Theory along with its associated Double Ramification hierarchy. The conjecture posits the following:*

1. ***Poisson Compatibility of $K^{\rm DR}_2$ and $K^{\rm DR}_1$:** The operator $K^{\rm DR}_2 = \left(K^{{\rm DR};\alpha\beta}_2\right)$ is defined as*

$$K^{{\rm DR};\alpha\beta}_2 := \eta^{\alpha\mu}\eta^{\beta\nu}\left(\left(\frac{1}{2}-\mu_\beta\right)\partial_x \circ L_\nu\left(g_{\mu,0}\right) + \left(\frac{1}{2}-\mu_\alpha\right)L_\nu\left(g_{\mu,0}\right)\circ\partial_x \right.$$
$$\left. + A_{\mu\nu}\partial_x + \partial_x \circ L^1_\nu\left(g_{\mu,0}\right)\circ\partial_x\right). \tag{5.3.10}$$

   This operator is Poisson and is compatible with the operator $K^{\rm DR}_1 := \eta^{-1}\partial_x$. Here, $\mu_\alpha := q_\alpha - \frac{\delta}{2}$.

2. ***Bihamiltonian Structure:** The Poisson brackets $\{\cdot,\cdot\}_{K^{\rm DR}_2}$ and $\{\cdot,\cdot\}_{K^{\rm DR}_1}$ give a bihamiltonian structure for the DR hierarchy. This structure is expressed by the following bihamiltonian recursion relations:*

$$\{\cdot, \bar{g}_{\alpha,d}\}_{K^{\rm DR}_2} = \left(d+\frac{3}{2}+\mu_\alpha\right)\{\cdot,\bar{g}_{\alpha,d+1}\}_{K^{\rm DH}_1} + A^\beta_\alpha\{\cdot,\bar{g}_{\beta,d}\}_{K^{\rm DR}_1}, \quad d \geq -1,$$

   where $A^\alpha_\beta := \eta^{\alpha\nu}A_{\nu\beta}$.

5.4 The Dubrovin-Zhang hierarchy

5.4.1 Homogeneous Cohomological Field Theory and Potential

Consider a homogeneous Cohomological Field Theory (CohFT), denoted by $\{c_{g,n}\}$. We introduce a set of formal variables, which we denote as t^α_a. Here, $1 \leq \alpha \leq N$ and $a \geq 0$. Additionally, we identify $t^\alpha_0 = t^\alpha$.

[1] Recall: $\frac{\partial}{\partial t^{\mathbb{1}}_a} := A^\alpha \frac{\partial}{\partial t^\alpha_a}, \quad a \geq 0$. cf. def. 1.1.2

Definition 5.4.1 *The potential $\mathcal{F}$ associated with a homogeneous CohFT is formally defined as:*

$$\mathcal{F}\left(t_*^*, \varepsilon\right) = \sum_{g\geq 0} \varepsilon^{2g} \mathcal{F}_g\left(t_*^*\right), \tag{5.4.1}$$

$$:= \sum_{\substack{g,n\geq 0\\ 2g-2+n>0}} \frac{\varepsilon^{2g}}{n!} \sum_{\substack{1\leq\alpha_1,\ldots,\alpha_n\leq N\\ d_1,\ldots,d_n\geq 0}} \left(\int_{\overline{\mathcal{M}}_{g,n}} c_{g,n}\left(\otimes_{i=1}^n e_{\alpha_i}\right) \prod_{i=1}^n \psi_i^{d_i}\right) \prod_{i=1}^n t_{d_i}^{\alpha_i} \in \mathbb{C}\left[\left[t_*^*, \varepsilon\right]\right], \tag{5.4.2}$$

We also introduce the formal power series $w^{\text{top};\alpha}$ and $w_n^{\text{top};\alpha}$ as follows:

$$w^{\text{top};\alpha} := \eta^{\alpha\mu} \frac{\partial^2 \mathcal{F}}{\partial t_0^\mu \partial t_0^{\mathbb{1}}},$$
$$w_n^{\text{top};\alpha} := \frac{\partial^n}{\left(\partial t_0^{\mathbb{1}}\right)^n} w^{\text{top};\alpha},$$

where $1 \leq \alpha \leq N$ and $n \geq 0$.

5.4.2 Conjecture on Differential Polynomials and Poisson Operators for the DZ hierarchy

Conjecture 5.4.2 *[DZ01] Consider the ring $\widehat{\mathcal{A}}_w$ consisting of differential polynomials in variables $w^1, \ldots, w^N$.*

1. ***Existence of the Differential Polynomial from the potential:*** *For any $1 \leq \alpha, \beta \leq N$ and $a, b \geq 0$, there exists a differential polynomial $\Omega_{\alpha,a;\beta,b} \in \widehat{\mathcal{A}}_{w;0}$, such that*

$$\frac{\partial^2 \mathcal{F}}{\partial t_a^\alpha \partial t_b^\beta} = \Omega_{\alpha,a;\beta,b}\big|_{w_n^\gamma = w_n^{top;\,\gamma}}. \tag{5.4.3}$$

2. ***Poisson Operator K_1^{DZ}:*** *There exists a Poisson operator $K_1^{\mathrm{DZ}} = \left(K_1^{\mathrm{DZ};\alpha\beta}\right)$ such that the local functionals $\bar{h}_{\alpha,-1} = \int \eta_{ow} w^\nu dx$ are Casimirs. The operator satisfies:*

$$\eta^{\alpha\mu} \partial_x \Omega_{\mu,0;\beta,b} = K_1^{\mathrm{DZ};\alpha\nu} \frac{\delta \bar{h}_{\beta,b}}{\delta w^\nu}, \tag{5.4.4}$$

where $\bar{h}_{\beta,b} := \int \Omega_{\mathbb{1},0;\beta,b+1}\, dx$, and $1 \leq \alpha, \beta \leq N, b \geq 0$.

3. ***Poisson Operator K_2^{DZ} and Bihamiltonian relations:*** *There exists another Poisson operator $K_2^{\mathrm{DZ}} = \left(K_2^{\mathrm{DZ};\alpha\beta}\right)$ that satisfies the following relations:*

$$\{\cdot,\bar{h}_{\alpha,d}\}_{K_2^{\mathrm{DZ}}} = \left(d+\frac{3}{2}+\mu_\alpha\right)\{\cdot,\bar{h}_{\alpha,d+1}\}_{K_1^{\mathrm{DZ}}} + A_\alpha^\beta\{\cdot,\bar{h}_{\beta,d}\}_{K_1^{\mathrm{DZ}}}, \quad 1\le\alpha\le N, \quad d\ge -1. \tag{5.4.5}$$

5.4.3 Uniqueness and Implications of the Conjecture

If the differential polynomials specified in Part 1 of conjecture 5.4.2 exist, then they are unique, as demonstrated in [BDGR18]. Similarly, if the Poisson operators from Parts 2 and 3 of 5.4.2 exist, they are also unique, as shown in [BPS12b].

Commutativity of Local Functionals

Part 2 of conjecture 5.4.2 implies that the local functionals $\bar{h}_{\alpha,d}$ mutually commute with respect to the bracket $\{\cdot,\cdot\}_{K_1^{\mathrm{DZ}}}$.

Bihamiltonian Hierarchy

If conjecture 5.4.2 holds true, the resulting bihamiltonian hierarchy is given by:

$$\frac{\partial w^\alpha}{\partial t_q^\beta} = K_1^{\mathrm{DZ};\alpha\mu}\frac{\delta\bar{h}_{\beta,q}}{\delta w^\mu}, \quad 1\le\alpha,\beta\le N, \quad q\ge 0,$$

and is referred to as the **Dubrovin-Zhang (DZ) hierarchy**.

The N-tuple of formal power series $w^{\mathrm{top};\alpha}$ serves as a solution to this hierarchy. Here, the derivative ∂_x is identified with $\frac{\partial}{\partial t_0^{1}}$. This particular solution is known as the topological solution.

Preliminary Results

Conjecture 5.4.2 has been proven up to genus 1 [DZ98]. Specifically, the differential polynomials and Poisson operators are given by the following approximations:

$$\Omega_{\alpha,a;\beta,b} = \left.\frac{\partial^2\mathcal{F}_0}{\partial t_a^\alpha \partial t_b^\beta}\right|_{t_c^\gamma=\delta_{c,0}w^\gamma} + O\left(\varepsilon^2\right),$$

$$K_1^{\mathrm{DZ};\alpha\beta} = \eta^{\alpha\beta}\partial_x + O\left(\varepsilon^2\right),$$

$$K_2^{\mathrm{DZ};\alpha\beta} = \left.\left(E^\gamma C_\gamma^{\alpha\beta}\right)\right|_{t^*=w^*}\partial_x + \left(\frac{1}{2}-\mu_\beta\right)\left.\left(C_\gamma^{\alpha\beta}\right)\right|_{t^*=w^*} w_x^\gamma + O\left(\varepsilon^2\right).$$

Generalized Proofs

Parts 1 and 2 of conjecture 5.4.2 have been proven for an arbitrary semisimple, not necessarily homogeneous, CohFT [BPS12b]. A simplified proof of Part 2 is presented in [BPS12a].

5.4.4 DR/DZ Equivalence Conjecture

We consider an arbitrary homogeneous CohFT and define the normal coordinates of the DR hierarchy as

$$\widetilde{u}^{\alpha}\left(u_{*}^{*}, \varepsilon\right) := \eta^{\alpha\mu} \frac{\delta \bar{g}_{\mu,0}}{\delta u^{\mathbb{1}}}.$$

Unique Differential Polynomial

In [BDGR18], it is proven that there exists a unique differential polynomial $\mathcal{P} \in \widehat{\mathcal{A}}_{w;-2}$ such that $\mathcal{F}^{\mathrm{red}} \in \mathbb{C}\left[\left[t_{*}^{*}, \varepsilon\right]\right]$ defined by $\mathcal{F}^{\mathrm{red}} := \mathcal{F} + \left.\mathcal{P}\right|_{w_n^{\gamma} = w_n^{\mathrm{top};\gamma}}$ satisfies the following vanishing property:

$$\operatorname{Coef}_{\varepsilon^{2g}} \left.\frac{\partial^n \mathcal{F}^{\mathrm{red}}}{\partial t_{d_1}^{\alpha_1} \ldots \partial t_{d_n}^{\alpha_n}}\right|_{t:=0} = 0, \quad \text{if} \quad \sum_{i=1}^{n} d_i \leq 2g-2.$$

Form of the Differential Polynomial $\mathcal{P}$ and Reduced Potential

The differential polynomial $\mathcal{P}$ takes a specific form, given by:

$$\mathcal{P} = -\varepsilon^2 G(w^1, \ldots, w^N) + O(\varepsilon^4),$$

where the function $G(t^1, \ldots, t^N)$ is defined as the restriction of $\mathcal{F}_1|_{t_{\geq 1}^{*}=0}$.

The power series $\mathcal{F}^{\mathrm{red}}$ is termed the *reduced potential* of our Cohomological Field Theory (CohFT).

Miura Transformations

To establish a relationship between the variables $\widetilde{u}^{\alpha}$ and w^{α}, we introduce a Miura transformation. Specifically, $\widetilde{u}^{\alpha}(w_{*}^{*}, \varepsilon)$ is defined as:

$$\widetilde{u}^{\alpha}(w_{*}^{*}, \varepsilon) = w^{\alpha} + \eta^{\alpha\nu} \partial_x \left\{\mathcal{P}, \bar{h}_{\nu,0}\right\}_{K_1^{DZ}}.$$

Conjecture 5.4.3 *[BDGR18][BRS21] If Conjectures 5.3.3 and 5.4.2 hold true, then the Double Ramification (DR) and Dubrovin/Zhang (DZ) hierarchies, along with their respective bihamiltonian structures, are equivalent when expressed in the normal coordinates $\widetilde{u}^{\alpha}$.*

5.4.5 Main Theorem

Theorem 5.4.4 *The DR and DZ hierarchies, along with their bihamiltonian structures, coincide up to genus 1 in the coordinates $\widetilde{u}^{\alpha}$.*

The proof is given in section 5.6.

Implications: Together with Conjecture 5.4.2, which was proven in [DZ98] up to genus 1, this theorem provides a comprehensive understanding of the bihamiltonian structures of the DR and DZ hierarchies and their relation for an arbitrary homogeneous CohFT.

These results were published in the Journal on Functional Analysis and Its Applications, Springer Verlag [BB21a].

5.5 Extension of the Space of Differential Polynomials by Tame Rational Functions

Before proving Theorem 5.4.4, we introduce several technical lemmas that will be instrumental in our discussion.

Although Conjecture 5.4.2 remains unproven, a weaker form holds true when we extend the space of differential polynomials. Following the work in [BDGR20], let us consider formal variables $v^1, \ldots, v^N$. For any $d \in \mathbb{Z}$, let $\mathcal{A}^{\mathrm{rt}}_{v;d}$ denote the vector space spanned by expressions of the form

$$\sum_{i \geq m} \frac{P_i\left(v^*_*\right)}{\left(v^1_x\right)^i}, \tag{5.5.1}$$

where $m \in \mathbb{Z}$, $P_i \in \mathcal{A}_{v;d+i}$, and $\frac{\partial P}{\partial v^1_x} = 0$.

We define the extended space $\widehat{\mathcal{A}}^{\mathrm{rt}}_v := \mathcal{A}^{\mathrm{rt}}_v[[\varepsilon]]$. A rational function of the form (5.5.1) is termed *tame* if there exists a non-negative integer C such that $\frac{\partial P_i}{\partial v^\alpha_k} = 0$ for $k > C$.

Let $\mathcal{A}^{\mathrm{rt}}_v := \bigoplus_{d \in \mathbb{Z}} \mathcal{A}^{\mathrm{rt}}_{v;d}$. The subspace of tame elements in $\mathcal{A}^{\mathrm{rt}}_v$ is denoted by $\mathcal{A}^{\mathrm{rt,t}}_v$, and its extended space is $\widehat{\mathcal{A}}^{\mathrm{rt,\,t}}_v := \mathcal{A}^{\mathrm{rt,\,t}}_v[[\varepsilon]]$.

We introduce a rational Miura transformation that relates the variables v^α to $\widetilde{v}^\alpha(v^*_*, \varepsilon)$ as follows:

$$\widetilde{v}^\alpha(v^*_*, \varepsilon) = v^\alpha + \varepsilon f^\alpha(v^*_*, \varepsilon),$$

where $f^\alpha \in \widehat{\mathcal{A}}^{\mathrm{rt}}_{v;1}$.

Let us introduce formal power series $v^{\mathrm{top};\alpha} := \eta^{\alpha\mu} \frac{\partial^2 \mathcal{F}_0}{\partial t^\mu_0 \partial t^1_0}$ and $v^{\mathrm{top};\alpha}_n := \frac{\partial^n}{(\partial t^1_0)^n} v^{\mathrm{top};\alpha}$.

It is noteworthy that the mapping $\widehat{\mathcal{A}}^{\mathrm{rt},t}_v \to \mathbb{C}[[t^*_*, \varepsilon]]$, defined by

$$f \in \widehat{\mathcal{A}}^{\mathrm{rt},t}_v \mapsto f|_{v^\gamma_c = v^{\mathrm{top};\gamma}_c} \in \mathbb{C}[[t^*_*, \varepsilon]],$$

is injective. Furthermore, there exists a unique tame rational function $w^\alpha(v^*_*, \varepsilon) \in \widehat{\mathcal{A}}^{\mathrm{rt},t}_{v;0}$ such that $w^\alpha(v^{\mathrm{top};*}_*, \varepsilon) = w^{\mathrm{top};\alpha}$ [BDGR20]. Additionally, $w^\alpha(v^*_*, \varepsilon) - v^\alpha$ belongs to $\operatorname{Im} \partial_x$ [BPS12b].

It is also worth mentioning that the same proof technique used in Proposition 7.6 of [BDGR20] establishes the existence of a unique tame rational function $\Omega_{\alpha,a;\beta,b} \in \widehat{\mathcal{A}}^{\mathrm{rt},t}_{w;0}$ that satisfies equation (5.4.3).

5.5.1 Generalized Poisson operators

In the framework of Poisson operators, we now consider a broader class of operators characterized by structures of the form $K^{\mu\nu} = \sum_{j\geq 0} K_j^{\mu\nu}\partial_x^j$, where each $K_j^{\mu\nu} \in \widehat{\mathcal{A}}^{\mathrm{rt,t}}_{w;j+1}$. We will denote the space of such Poisson operators by $\mathcal{PO}^{\mathrm{rt}}_w$.

Consider K_1^{DZ} and K_2^{DZ} as specific instances in $\mathcal{PO}^{\mathrm{rt}}_w$. These operators are derived from the base operators

$$\eta^{\alpha\beta}\partial_x \quad \text{and} \quad \left(E^\gamma C_\gamma^{\alpha\beta}\right)\Big|_{t^*=v^*}\partial_x + \left(\frac{1}{2}-\mu_\beta\right)\left(C_\gamma^{\alpha\beta}\right)\Big|_{t^*=v^*} v_x^\gamma,$$

respectively, through the rational Miura transformation $v^\alpha \mapsto w^\alpha(v^*_*, \varepsilon)$.

With these definitions, relations (5.4.4) and (5.4.5) hold true, as shown in [BDGR20]. Therefore, Conjecture 5.4.2 can equivalently be stated as $\Omega_{\alpha,a;\beta,b} \in \widehat{\mathcal{A}}_{w;0}$ and $K_1^{\mathrm{DZ}}, K_2^{\mathrm{DZ}} \in \mathcal{PO}_w$.

Lemma 5.5.1 *Let $K \in \mathcal{PO}^{rt}_v$ be a Poisson operator and consider a rational Miura transformation $v^\alpha \mapsto \widetilde{v}^\alpha(v^*_*, \varepsilon)$ such that $\widetilde{v}^\alpha(v^*_*, \varepsilon) - v^\alpha \in \operatorname{Im}\partial_x$. Then, we have*

$$K^{\alpha\beta}_{\widetilde{v};0} = \sum_{m\geq 0} \frac{\partial \widetilde{v}^\alpha}{\partial v^\rho_m}\partial_x^m K_0^{\rho\beta}.$$

Proof. To prove this, we compute the following:

$$K^{\alpha\beta}_{\widetilde{v};0} = \operatorname{Coef}_{\partial_x^0} K^{\alpha\beta}_{\widetilde{v}} \tag{5.5.2}$$

$$= \operatorname{Coef}_{\partial_x^0}\left(\sum_{m,n\geq 0} \frac{\partial \widetilde{v}^\alpha}{\partial v^\rho_m}\partial_x^m \circ K^{\rho\theta} \circ (-\partial_x)^n \circ \frac{\partial \widetilde{v}^\beta}{\partial v^\theta_n}\right) \tag{5.5.3}$$

$$= \operatorname{Coef}_{\partial_x^0}\left(\sum_{m\geq 0} \frac{\partial \widetilde{v}^\alpha}{\partial v^\rho_m}\partial_x^m \circ K^{\rho\theta} \circ \underbrace{\sum_{n\geq 0}(-\partial_x)^n \frac{\partial \widetilde{v}^\beta}{\partial v^\theta_n}}_{=\frac{\delta \tilde{u}^\beta}{\delta v^\theta}=\delta^\beta_\theta}\right) \tag{5.5.4}$$

$$= \operatorname{Coef}_{\partial_x^0}\left(\sum_{m\geq 0} \frac{\partial \widetilde{v}^\alpha}{\partial v^\rho_m}\partial_x^m \circ K^{\rho\beta}\right) \tag{5.5.5}$$

$$= \sum_{m\geq 0} \frac{\partial \widetilde{v}^\alpha}{\partial v^\rho_m}\partial_x^m K_0^{\rho\beta}. \tag{5.5.6}$$

The lemma is now proved.

Lemma 5.5.2 *We have*

$$K_{2;0}^{\mathrm{DZ};\alpha\beta} = \left(\frac{1}{2}-\mu_\beta\right)\eta^{\alpha\theta}\eta^{\beta\nu}\partial_x\Omega_{\theta,0;\nu,0}.$$

Proof. By invoking lemma 5.5.1, we obtain

$$K_{2;0}^{\mathrm{DZ};\alpha\beta} = \left(\frac{1}{2} - \mu_\beta\right) \sum_{m\geq 0} \frac{\partial w^\alpha}{\partial v_m^\rho} \partial_x^m \left(\left(C_\gamma^{\rho\beta} \right)\Big|_{t^*=v^*} v_x^\gamma \right).$$

Note that

$$\left(\left(C_\gamma^{\rho\beta} \right)\Big|_{t^*=v^*} v_x^\gamma \right)\Big|_{v_t^*=v_*^{\mathrm{top}\,\beta;}} = \eta^{\beta\nu} \frac{\partial v^{\mathrm{top};p}}{\partial t_0^\nu}.$$

Therefore, we have

$$\begin{aligned} &\sum_{m\geq 0} \frac{\partial w^\alpha}{\partial v_m^\rho} \partial_x^m \left(\left(C_\gamma^{\rho\beta} \right)\Big|_{t^*=v^*} v_x^\gamma \right)\Bigg|_{v^*=v_*^{\mathrm{top};*}} \\ &= \eta^{\beta\nu} \frac{\partial w^{\mathrm{top};\alpha}}{\partial t_0^\nu} \\ &= \eta^{\alpha\theta}\eta^{\beta\nu}\partial_x \Omega_{\theta,0;\nu,0}\Big|_{v_*^*=v_*^{\mathrm{top};\,*}}, \end{aligned}$$

which confirms that

$$\sum_{m\geq 0} \frac{\partial w^\alpha}{\partial v_m^\rho} \partial_x^m \left(\left(C_\gamma^{\rho\beta} \right)\Big|_{t^*=v^*} v_x^\gamma \right) = \eta^{\alpha\theta}\eta^{\beta\nu}\partial_x \Omega_{\theta,0;\nu,0},$$

as required.

Remark 5.5.3 *The lemma implies that the constant term of the operator K_2^{DZ} is a differential polynomial if $\Omega_{\theta,0;\nu,0}$ is also a differential polynomial. This is particularly true in the semisimple case, as observed in [IS22].*

Lemma 5.5.4 *We have*

$$K_{2;0}^{\mathrm{DR};\alpha\beta} = \left(\frac{1}{2} - \mu_\beta\right) \eta^{\alpha\theta}\eta^{\beta\nu}\partial_x \frac{\delta \bar{g}_{\nu,0}}{\delta u^\theta}.$$

Proof. The lemma directly follows from the definition given by equation (5.3.10).

5.6 Proof of the main Theorem

If we exclude the operators K_2^{DZ} and K_2^{DR} from consideration, the coincidence of the DZ and DR hierarchies in the coordinates $\widetilde{u}^\alpha$, up to genus 1 approximation, has already been established in Theorem 8.4 of [BDGR18]. Therefore, it suffices to prove that

$$K_{2;\tilde{u}}^{\mathrm{DZ}} = K_{2;\tilde{u}}^{\mathrm{DR}} + O\left(\varepsilon^4\right).$$

Given that $K_{2;\bar{u}}^{\mathrm{DZ};[0]} = K_{2;\bar{u}}^{\mathrm{DR};[0]}$ [BRS21], we need to verify that

$$K_{2;\tilde{u};l}^{\mathrm{DZ};[2]} = K_{2;\tilde{u};l}^{\mathrm{DR};[2]} \quad \text{for} \quad l = 0, 1, 2, 3. \tag{5.6.1}$$

We divide the proof into several steps.

Step 1. We first examine equation (5.6.1) for $l=2$ and $l=3$ through direct computation.

For convenience, we introduce the following notations:

$$
\begin{aligned}
\partial_\alpha &:= \frac{\partial}{\partial u^\alpha},\\
c^{\alpha\beta}_{\gamma\delta} &:= \partial_\delta c^{\alpha\beta}_{\gamma},\\
c^{\alpha\beta}_{\gamma\delta\theta} &:= \partial_\theta c^{\alpha\beta}_{\gamma\delta},\\
e^\gamma &:= E^\gamma|_{t^*=u^*} = (1-q_\gamma)\, u^\gamma + r^\gamma,\\
g^{\alpha\beta} &= e^\gamma c^{\alpha\beta}_\gamma.
\end{aligned}
$$

In [DZ98], the authors derived the following formulas:

$$
\begin{aligned}
K^{\mathrm{DZ};[2],\alpha\beta}_{2;\tilde u;3} &= h^{\alpha\beta}\Big|_{u^*=\tilde u^*},\\
K^{\mathrm{DZ};[2],\alpha\beta}_{2;\tilde u;2} &= \left(\frac32 \partial_\gamma h^{\alpha\beta} + \frac{1}{24}\left(\frac32-\mu_\beta\right) c^{\alpha\nu}_\gamma c^{\beta\mu}_{\nu\mu} - \frac{1}{24}\left(\frac32-\mu_\alpha\right) c^{\beta\nu}_\gamma c^{\alpha\mu}_{\nu\mu}\right)\Big|_{u^*=\tilde u^*} \widetilde{u}^\gamma_x,
\end{aligned}
$$

where

$$
h^{\alpha\beta} = \frac{1}{12}\left(\partial_\nu\left(g^{\mu\nu}c^{\alpha\beta}_\mu\right) + \frac12 c^{\mu\nu}_\nu c^{\alpha\beta}_\mu\right).
$$

On the other hand, considering the expansion $\widetilde{u}^\alpha = u^\alpha + \frac{\varepsilon^2}{24}\partial_x^2 c^{\alpha\mu}_\mu + O(\varepsilon^4)$ [BDGR18], we find that

$$
\begin{aligned}
K^{\mathrm{DR};\alpha\beta}_{2;\widetilde u} &= L_\nu\left(u^\alpha + \frac{\varepsilon^2}{24}\partial_x^2 c^{\alpha\lambda}_\lambda\right)\circ K^{\mathrm{DR};\nu\rho}_2 \circ L^\dagger_\rho\left(u^\beta + \frac{\varepsilon^2}{24}\partial_x^2 c^{\beta\theta}_\theta\right) + O\left(\varepsilon^4\right)\\
&= K^{\mathrm{DR};\alpha\beta}_2 + \frac{\varepsilon^2}{24}\left(\partial_x^2\circ L_\nu\left(c^{\alpha\lambda}_\lambda\right)\circ K^{\mathrm{DR};\nu\beta}_2 + K^{\mathrm{DR};\alpha\rho}_2\circ L^\dagger_\rho\left(c^{\beta\theta}_\theta\right)\circ\partial_x^2\right) + O\left(\varepsilon^4\right)\\
&= K^{\mathrm{DR};\alpha\beta}_2 + \varepsilon^2 \underbrace{\frac{1}{24}\left(\partial_x^2\circ c^{\alpha\lambda}_{\nu\lambda}\circ K^{\mathrm{DR};[0],\nu\beta}_2 + K^{\mathrm{DR};[0],\alpha\rho}_2\circ c^{\beta\theta}_{\rho\theta}\circ\partial_x^2\right)}_{=:\sum_{i=0}^3 R^{\alpha\beta}_i\partial_x^i} + O\left(\varepsilon^4\right),
\end{aligned}
$$

where $R^{\alpha\beta}_i \in \mathcal{A}_{u;3-i}$.

Considering the expansion $g_{\mu,0} = \sum_{g\geq 0}\varepsilon^{2g} g^{[2g]}_{\mu,0}, g^{[2g]}_{\mu,0} \in \mathcal{A}_{u;2g}$, from equation (5.3.10) we compute:

$$
\begin{aligned}
K^{\mathrm{DR};[2],\alpha\beta}_{2;3} &= (3-\mu_\alpha-\mu_\beta)\,\eta^{\alpha\mu}\eta^{\beta\nu}\frac{\partial g^{[2]}_{\mu,0}}{\partial u^\nu_{xx}},\\
K^{\mathrm{DR};[2],\alpha\beta}_{2;2} &= \eta^{\alpha\mu}\eta^{\beta\nu}\left[(2-\mu_\alpha-\mu_\beta)\frac{\partial g^{[2]}_{\mu,0}}{\partial u^\nu_x} + \left(\frac52-\mu_\beta\right)\partial_x\frac{\partial g^{[2]}_{\mu,0}}{\partial u^\nu_{xx}}\right].
\end{aligned}
$$

Using equation (5.3.9), we then find:

$$\frac{\partial g^{[2]}_{\mu,0}}{\partial u^{\nu}_{xx}} = \frac{1}{24} c^{\theta}_{\theta\xi} c^{\xi}_{\mu\nu},$$
$$\frac{\partial g^{[2]}_{\mu,0}}{\partial u^{\nu}_{x}} = \frac{1}{24}\left[\partial_\nu\left(c^{\theta}_{\theta\xi} c^{\xi}_{\mu\gamma}\right) + \partial_\gamma\left(c^{\theta}_{\theta\xi} c^{\xi}_{\mu\nu}\right) - \partial_\mu\left(c^{\theta}_{\theta\xi} c^{\xi}_{\nu\gamma}\right)\right] u^{\gamma}_{x},$$

and finally obtain:

$$K^{\mathrm{DR};[2],\alpha\beta}_{2;3} = \frac{3-\mu_\alpha-\mu_\beta}{24} c^{\gamma\sigma}_{\gamma} c^{\alpha\beta}_{\sigma},$$
$$K^{\mathrm{DR};[2],\alpha\beta}_{2;2} = \left[\frac{2-\mu_\alpha-\mu_\beta}{24}\left(c^{\beta\theta}_{\theta\xi} c^{\alpha\xi}_{\gamma} - c^{\alpha\theta}_{\theta\xi} c^{\beta\xi}_{\gamma}\right) + \frac{\frac{9}{2}-\mu_\alpha-2\mu_\beta}{24}\partial_\gamma\left(c^{\theta\xi}_{\theta} c^{\alpha\beta}_{\xi}\right)\right] u^{\gamma}_{x}.$$

Using the expression $K^{\mathrm{DR};[0],\alpha\beta}_{2} = g^{\alpha\beta}\partial_x + \left(\frac{1}{2}-\mu_\beta\right) c^{\alpha\beta}_{\gamma} u^{\gamma}_{x}$, we can also compute:

$$R^{\alpha\beta}_{3} = \frac{1}{24}\left(c^{\alpha\lambda}_{\nu\lambda} g^{\nu\beta} + g^{\alpha\nu} c^{\beta\lambda}_{\nu\lambda}\right),$$
$$R^{\alpha\beta}_{2} = \frac{1}{24}\left[2\partial_\gamma\left(c^{\alpha\lambda}_{\nu\lambda} g^{\nu\beta}\right) + c^{\alpha\lambda}_{\nu\lambda} c^{\nu\beta}_{\gamma}\left(\frac{1}{2}-\mu_\beta\right) + g^{\alpha\nu} c^{\beta\lambda}_{\gamma\nu\lambda} + c^{\alpha\nu}_{\gamma}\left(\frac{1}{2}-\mu_\nu\right) c^{\beta\lambda}_{\nu\lambda}\right] u^{\gamma}_{x}.$$

we aim to prove equation (5.6.1). Specifically, we need to verify the following two equations:

$$\frac{1}{12}\left(\partial_\nu\left(g^{\nu\mu} c^{\alpha\beta}_{\mu}\right) + \underline{\frac{1}{2} c^{\mu\nu}_{\nu} c^{\alpha\beta}_{\mu}}\right) = \frac{3-\mu_\alpha-\mu_\beta}{24}\underline{c^{\gamma\sigma}_{\gamma} c^{\alpha\beta}_{\sigma}} + \frac{1}{24}\left(c^{\alpha\lambda}_{\nu\lambda} g^{\nu\beta} + g^{\alpha\nu} c^{\beta\lambda}_{\nu\lambda}\right), \quad (5.6.2)$$

$$\begin{aligned}
&\frac{1}{8}\partial_\gamma\left(\partial_\nu\left(g^{\mu\nu} c^{\alpha\beta}_{\mu}\right) + \frac{1}{2}\underbrace{c^{\mu\nu}_{\nu} c^{\alpha\beta}_{\mu\beta}}_{*}\right) + \frac{1}{24}\left(\frac{3}{2}-\mu_\beta\right)\underbrace{c^{\alpha\nu}_{\gamma} c^{\beta\mu}_{\nu\mu}}_{**} - \frac{1}{24}\left(\frac{3}{2}-\mu_\alpha\right)\underbrace{c^{\beta\nu}_{\gamma} c^{\alpha\mu}_{\nu\mu}}_{***} \\
&\quad = \left[\frac{2-\mu_\alpha-\mu_\beta}{24}(\underbrace{c^{\beta\theta}_{\theta\xi} c^{\alpha\xi}_{\gamma}}_{**} - \underbrace{c^{\alpha\beta}_{\theta\xi} c^{\beta\xi}_{\gamma}}_{***}) + \frac{\frac{9}{2}-\mu_\alpha-2\mu_\beta}{24}\partial_\gamma(\underbrace{c^{\beta\xi}_{\theta} c^{\alpha\beta}_{\xi}}_{*})\right] \\
&\quad + \frac{1}{24}[2\partial_\gamma\left(c^{\alpha\lambda}_{\nu\lambda} g^{\nu/\beta}\right) + \left(\frac{1}{2}-\mu_\beta\right)\underbrace{c^{\alpha\lambda}_{\nu\lambda} c^{\nu\beta}_{\gamma}}_{***} + g^{\alpha\nu} c^{\beta\lambda}_{\gamma\nu\lambda} + \left(\frac{1}{2}-\mu_\nu\right)\underbrace{c^{\alpha\nu}_{\gamma} c^{\beta\lambda}_{\nu\lambda}}_{**}]. \quad (5.6.3)
\end{aligned}$$

To prove Equation (5.6.2), let us start by focusing on the terms that are underlined. We'll use the identity $g^{\alpha\beta} = e^{\gamma} c^{\alpha\beta}_{\gamma}$ and multiply both sides of the equation by 12. Doing so, we find that Equation (5.6.2) can be rewritten as:

$$(1-q_\nu)\,\underline{c_\nu^{\nu\mu}c_\mu^{\alpha\beta}}+\boxed{e^\gamma\partial_\nu\left(c_\gamma^{\nu\mu}c_\mu^{\alpha\beta}\right)}=\frac{2-\mu_\alpha-\mu_\beta}{2}\underline{c_\gamma^{\gamma\sigma}c_\sigma^{\alpha\beta}}+\frac{1}{2}e^\gamma\left(c_{\nu\lambda}^{\alpha\lambda}c_\gamma^{\nu\beta}+c_\gamma^{\alpha\nu}c_{\nu\lambda}^{\beta\lambda}\right). \quad (5.6.4)$$

Next, let's consider the transformation of the boxed term. Specifically, we have:

$$e^\gamma\partial_\nu\left(c_\gamma^{\nu\mu}c_\mu^{\alpha\beta}\right)=e^\gamma\partial_\nu\left(c_\gamma^{\beta\mu}c_\mu^{\alpha\nu}\right)=e^\gamma\left(c_{\nu\gamma}^{\beta\mu}c_\mu^{\alpha\nu}+c_\gamma^{\beta\mu}c_{\nu\mu}^{\alpha\nu}\right). \quad (5.6.5)$$

Finally, by moving all terms to the left-hand side, we arrive at the following expression:

$$\frac{\mu_\alpha+\mu_\beta-2q_\nu}{2}c_\nu^{\nu\mu}c_\mu^{\alpha\beta}+e^\gamma\left(c_{\nu\gamma}^{\beta\mu}c_\mu^{\alpha\nu}+\underline{c_\gamma^{\beta\mu}c_{\nu\mu}^{\alpha\nu}}-\frac{1}{2}\underline{c_{\nu\lambda}^{\alpha\lambda}c_\gamma^{\nu\beta}}-\frac{1}{2}c_\gamma^{\alpha\nu}c_{\nu\lambda}^{\beta\lambda}\right)$$
$$=\frac{\mu_\alpha+\mu_\beta-2q_\nu}{2}c_\nu^{\nu\mu}c_\mu^{\alpha\beta}+\boxed{e^\gamma c_{\nu\gamma}^{\beta\mu}c_\mu^{\alpha\nu}}+e^\gamma\left(\frac{1}{2}c_\gamma^{\beta\mu}c_{\nu\mu}^{\alpha\nu}-\frac{1}{2}c_\gamma^{\alpha\nu}c_{\nu\lambda}^{\beta\lambda}\right), \quad (5.6.6)$$

To complete the proof, we need to show that a certain expression vanishes. This is where the theory of Dubrovin-Frobenius manifolds comes into play. According to this theory, as outlined in [Dub96], we have the following relationship involving the Lie derivative $\mathcal{L}_E$:

$$\mathcal{L}_E C_{\beta\gamma}^{a}=C_{\beta\gamma}^{\alpha}.$$

This result has an important implication for our problem. Specifically, it leads to the equation:

$$e^\lambda c_{\lambda\gamma}^{\alpha\beta}=(\delta-q_\alpha-q_\beta+q_\gamma)\,c_\gamma^{\alpha\beta}.$$

Upon applying the given relationship to the boxed term, we find that Equation (5.6.6) can be rewritten as a sum of several terms involving the coefficients q_α, q_β, q_ν, and q_μ, as well as the tensor $c_\nu^{\nu\mu}c_\mu^{\alpha\beta}$. Specifically, the equation can be expressed as follows:

$$\frac{q_\alpha+q_\beta-2q_\nu-\delta}{2}\underline{c_\nu^{\nu\mu}c_\mu^{\alpha\beta}}+(\delta-q_\beta-q_\mu+q_\nu)\,\underline{c_\nu^{\beta\mu}c_\mu^{\alpha\nu}}+\frac{e^\gamma}{2}\left(c_\gamma^{\beta\mu}c_{\nu\mu}^{\alpha\nu}-c_\gamma^{\alpha\nu}c_{\nu\lambda}^{\beta\lambda}\right)$$
$$=\frac{q_\alpha-q_\beta+\delta-2q_\mu}{2}c_\nu^{\beta\mu}c_\mu^{\alpha\nu}+\frac{e^\gamma}{2}\left(\underline{c_\gamma^{\beta\mu}c_{\nu\mu}^{\alpha\nu}-c_\gamma^{\alpha\nu}c_{\nu\lambda}^{\beta\lambda}}\right). \quad (5.6.7)$$

To further simplify the equation, we focus on transforming the underlined terms. After some algebraic manipulations, we arrive at the following expression:

$$c^{\beta\mu}_{\gamma} c^{\alpha\nu}_{\nu\mu} - c^{\alpha\nu}_{\gamma} c^{\beta\lambda}_{\nu\lambda} = \left(\underline{\partial_\nu \left(c^{\beta\mu}_{\gamma} c^{\alpha\nu}_{\mu} \right)} - c^{\beta\mu}_{\nu\gamma} c^{\alpha\nu}_{\mu} \right) - \left(\underline{\partial_\lambda \left(c^{\alpha\nu}_{\gamma} c^{\beta\lambda}_{\nu} \right)} - c^{\alpha\nu}_{\lambda\gamma} c^{\beta\lambda}_{\nu} \right)$$
$$= c^{\alpha\nu}_{\mu\gamma} c^{\beta\mu}_{\nu} - c^{\beta\mu}_{\nu\gamma} c^{\alpha\nu}_{\mu}. \quad (5.6.8)$$

Upon simplifying Equation (5.6.7), we find that it can be expressed as follows:

$$\frac{q_\alpha - q_\beta + \delta - 2q_\mu}{2} c^{\beta\mu}_{\nu} c^{\alpha\nu}_{\mu} + \frac{1}{2} e^{\gamma} \left(c^{\alpha\nu}_{\mu\gamma} c^{\beta\mu}_{\nu} - c^{\beta\mu}_{\nu\gamma} c^{\alpha\nu}_{\mu} \right)$$
$$= \frac{q_\alpha - q_\beta + \delta - 2q_\mu}{2} c^{\beta\mu}_{\nu} c^{\alpha\nu}_{\mu} + \frac{\delta - q_\alpha - q_\nu + q_\mu}{2} c^{\alpha\nu}_{\mu} c^{\beta\mu}_{\nu} - \frac{\delta - q_\beta - q_\mu + q_\nu}{2} c^{\beta\mu}_{\nu} c^{\alpha\nu}_{\mu}$$
$$= -\mu_\nu c^{\beta\mu}_{\nu} c^{\alpha\nu}_{\mu} = -\mu_\nu c^{\nu\mu}_{\nu} c^{\alpha\beta}_{\mu}. \quad (5.6.9)$$

To validate the equation, it is sufficient to confirm that $\mu_\nu c^{\nu}_{\alpha\nu} = 0$ for any α. Upon computation, we find:

$$X := \mu_\nu c^{\nu}_{\alpha\nu} = \mu_\nu \eta^{\nu\lambda} \eta_{\nu\theta} c^{\theta}_{\alpha\lambda} = -\mu_\lambda \eta^{\nu\lambda} \eta_{\nu\theta} c^{\theta}_{\alpha\lambda} = -\mu_\lambda c^{\lambda}_{\alpha\lambda} = -X, \quad (5.6.10)$$

which implies $X = 0$, as required.

Next, let us focus on proving Equation (5.6.3). After collecting terms that are alike, the equation can be rewritten as:

$$\frac{1}{8} \partial_\gamma \partial_\nu \left(g^{\mu\nu} c^{\alpha\beta}_{\mu} \right) + \frac{\mu_\alpha + \mu_\nu - 1}{24} c^{\alpha\nu}_{\gamma} c^{\beta\mu}_{\nu\mu}$$
$$= \frac{3 - \mu_\alpha - 2\mu_\beta}{24} \partial_\gamma \left(c^{\nu\mu}_{\nu} c^{\alpha\beta}_{\mu} \right) + \frac{1}{24} \left[2\partial_\gamma \left(c^{\alpha\lambda}_{\nu\lambda} g^{\nu\beta} \right) + g^{\alpha\nu} c^{\beta\lambda}_{\gamma\nu\lambda} \right],$$

which is equivalent to

$$\frac{1 - \frac{\delta}{2} - \mu_\nu}{8} \underbrace{\partial_\gamma \left(c^{\nu\mu}_{\nu} c^{\alpha\beta}_{\mu} \right)}_{*} + \frac{1}{8} \partial_\gamma \left(e^{\theta} \partial_\nu \left(c^{\mu\nu}_{\theta} c^{\alpha\beta}_{\mu} \right) \right) + \frac{\mu_\alpha + \mu_\nu - 1}{24} c^{\alpha\nu}_{\gamma} c^{\beta\mu}_{\nu\mu}$$
$$+ \frac{\mu_\alpha + 2\mu_\beta - 3}{24} \underbrace{\partial_\gamma \left(c^{\nu\mu}_{\nu} c^{\alpha\beta}_{\mu} \right)}_{*} - \frac{1}{24} \left[2\partial_\gamma \left(c^{\alpha\lambda}_{\nu\lambda} g^{\nu\beta} \right) + g^{\alpha\nu} c^{\beta\lambda}_{\gamma\nu\lambda} \right] = 0.$$

Using that $\mu_\nu c^{\mu\mu}_{\nu} = 0$ we see that the left-hand side is equal to

$$\frac{1}{8}\partial_\gamma\left(e^\theta\partial_\nu\left(c_\theta^{\mu\nu}c_\mu^{\alpha\beta}\right)\right)+\frac{\mu_\alpha+\mu_\nu-1}{24}c_\gamma^{\alpha\nu}c_{\nu\mu}^{\beta\mu} + \frac{q_\alpha+2q_\beta-3\delta}{24}\partial_\gamma\left(c_\nu^{\nu\mu}c_\mu^{\alpha\beta}\right)-\frac{1}{24}\left[2\partial_\gamma\left(c_{\nu\lambda}^{\alpha\lambda}g^{\nu\beta}\right)+g^{\alpha\nu}c_{\gamma\nu\lambda}^{\beta\lambda}\right].$$

Transforming the first term in this expression as

$$\begin{aligned}\partial_\gamma\left(e^\theta\partial_\nu\left(c_\theta^{\mu\nu}c_\mu^{\alpha\beta}\right)\right) &= \partial_\gamma\left(e^\theta\partial_\nu\left(c_\theta^{\mu\beta}c_\mu^{\alpha\nu}\right)\right)=\partial_\gamma\left(e^\theta c_{\theta\nu}^{\mu\beta}c_\mu^{\alpha\nu}\right)+\partial_\gamma\left(e^\theta c_\theta^{\mu\beta}c_{\mu\nu}^{\alpha\nu}\right)\\ &=\left(\delta-q_\mu-q_\beta+q_\nu\right)\partial_\gamma\left(c_\nu^{\mu\beta}c_\mu^{\alpha\nu}\right)+\partial_\gamma\left(g^{\mu\beta}c_{\mu\nu}^{\alpha\nu}\right)\\ &=\left(\delta-q_\beta\right)\partial_\gamma\left(c_\nu^{\mu\beta}c_\mu^{\alpha\nu}\right)+\underbrace{\left(-q_\mu+q_\nu\right)\partial_\gamma\left(c_\nu^{\mu\beta}c_\mu^{\alpha\nu}\right)}_{=0}+\partial_\gamma\left(g^{\mu\beta}c_{\mu\nu}^{\alpha\nu}\right)\\ &=\left(\delta-q_\beta\right)\partial_\gamma\left(c_\nu^{\mu\beta}c_\mu^{\alpha\nu}\right)+\partial_\gamma\left(g^{\mu\beta}c_{\mu\nu}^{\alpha\nu}\right),\end{aligned}$$

we come to the expression

$$\begin{aligned}&\frac{\delta-q_\beta}{8}\underline{\partial_\gamma\left(c_\nu^{\mu\beta}c_\mu^{\alpha\nu}\right)}+\frac{1}{8}\boxed{\partial_\gamma\left(g^{\mu\beta}c_{\mu\nu}^{\alpha\nu}\right)}+\frac{\mu_\alpha+\mu_\nu-1}{24}c_\gamma^{\alpha\nu}c_{\nu\mu}^{\beta\mu}\\ &\quad+\frac{q_\alpha+2q_\beta-3\delta}{24}\underline{\partial_\gamma\left(c_\nu^{\nu\mu}c_\mu^{\alpha\beta}\right)}-\frac{1}{24}\left[2\boxed{\partial_\gamma\left(c_{\nu\lambda}^{\alpha\lambda}g^{\nu\beta}\right)}+g^{\alpha\nu}c_{\gamma\nu\lambda}^{\beta\lambda}\right]\\ &=\frac{\mu_\alpha-\mu_\beta}{24}\partial_\gamma\left(c_\nu^{\mu\beta}c_\mu^{\alpha\nu}\right)+\frac{\mu_\alpha+\mu_\nu-1}{24}c_\gamma^{\alpha\nu}c_{\nu\mu}^{\beta\mu}+\frac{1}{24}\left[\underline{\partial_\gamma\left(c_{\nu\lambda}^{\alpha\lambda}g^{\nu\beta}\right)}-g^{\alpha\nu}c_{\gamma\nu\lambda}^{\beta\lambda}\right].\end{aligned}$$

Applying to the underlined term the formula

$$\partial_\gamma g^{\nu\beta}=\left(1-q_\gamma\right)c_\gamma^{\nu\beta}+e^\theta c_{\theta\gamma}^{\nu\beta}=\left(1-\mu_\nu-\mu_\beta\right)c_\gamma^{\nu\beta},$$

we obtain

$$\begin{aligned}&\frac{\mu_\alpha-\mu_\beta}{24}\partial_\gamma\left(c_\nu^{\mu\beta}c_\mu^{\alpha\nu}\right)+\frac{\mu_\alpha+\mu_\nu-1}{24}c_\gamma^{\alpha\nu}c_{\nu\mu}^{\beta\mu}+\frac{1}{24}\left[c_{\nu\lambda\gamma}^{\alpha\lambda}g^{\nu\beta}+\left(1-\mu_\nu-\mu_\beta\right)c_{\nu\lambda}^{\alpha\lambda}c_\gamma^{\nu\beta}-g^{\alpha\nu}c_{\gamma\nu\lambda}^{\beta\lambda}\right]\\ &=\frac{\mu_\alpha-\mu_\beta}{24}\partial_\gamma\left(c_\nu^{\mu\beta}c_\mu^{\alpha\nu}\right)+\frac{\mu_\alpha+\mu_\nu-1}{24}c_\gamma^{\alpha\nu}c_{\nu\mu}^{\beta\mu}+\frac{1-\mu_\nu-\mu_\beta}{24}c_{\nu\lambda}^{\alpha\lambda}c_\gamma^{\nu\beta}+\frac{1}{24}\left[\underline{c_{\nu\lambda}^{\alpha\lambda}g^{\nu\beta}-g^{\alpha\nu}c_{\gamma\nu\lambda}^{\beta\lambda}}\right].\end{aligned}$$

Expressing the underlined terms as follows:

$$\partial_\lambda\partial_\gamma\underbrace{\left(c_\nu^{\alpha\lambda}g^{\nu\beta}-g^{\alpha\nu}c_\nu^{\beta\lambda}\right)}_{=0}-c_{\nu\lambda}^{\alpha\lambda}\partial_\gamma g^{\nu\beta}-c_{\nu\gamma}^{\alpha\lambda}\partial_\lambda g^{\nu\beta}$$

$$-c_\nu^{\alpha\lambda}\partial_\lambda\partial_\gamma g^{\nu\beta}+\partial_\gamma g^{\alpha\nu}c_{\nu\lambda}^{\beta\lambda}+\partial_\lambda g^{\alpha\nu}c_{\nu\gamma}^{\beta\lambda}+\partial_\lambda\partial_\gamma g^{\alpha\nu}c_\nu^{\beta\lambda}$$

$$= (\mu_\beta + \mu_\nu - \mu_\lambda - \mu_\alpha)\, \partial_\gamma \left(c^{\alpha\lambda}_{\nu} c^{\beta\nu}_{\lambda} \right) + (\mu_\beta + \mu_\nu - 1)\, c^{\alpha\lambda}_{\nu\lambda} c^{\beta\nu}_{\gamma} + (1 - \mu_\alpha - \mu_\nu)\, c^{\alpha\nu}_{\gamma} c^{\beta\lambda}_{\nu\lambda},$$

we obtain

$$\frac{\mu_\alpha - \mu_\beta}{24} \underbrace{\partial_\gamma \left(c^{\mu\beta}_{\nu} c^{\alpha\nu}_{\mu\nu} \right)}_{*} + \frac{\mu_\alpha + \mu_\nu - 1}{24} \underbrace{c^{\alpha\nu}_{\nu} c^{\beta\mu}_{\nu\mu}}_{**} + \frac{1 - \mu_\nu - \mu_\beta}{24} \underbrace{c^{\alpha\lambda}_{\nu\lambda} c^{\nu\beta}_{\gamma}}_{***}$$
$$+ \frac{\mu_\beta + \mu_\nu - \mu_\lambda - \mu_\alpha}{24} \underbrace{\partial_\gamma \left(c^{\alpha\lambda}_{\nu} c^{\beta\nu}_{\lambda} \right)}_{*} + \frac{\mu_\beta + \mu_\nu - 1}{24} \underbrace{c^{\alpha\lambda}_{\nu\lambda} c^{\beta\nu}_{\gamma}}_{**} + \frac{1 - \mu_\alpha - \mu_\nu}{24} \underbrace{c^{\alpha\nu}_{\gamma} c^{\beta\lambda}_{\nu\lambda}}_{**} =$$
$$= \frac{\mu_\nu - \mu_\lambda}{24} \partial_\gamma \left(c^{\alpha\lambda}_{\nu} c^{\beta\nu}_{\lambda} \right) = 0,$$

as required.

Step 2. To prove equation (5.6.1) for $l = 0$, we first observe that $\widetilde{u}^\alpha (w^*_*, \varepsilon) - w^\alpha$ belongs to the image of ∂_x, as per its definition. Invoking Lemmas 5.5.1 and 5.5.2, we find that

$$K^{\mathrm{DZ};\alpha\beta}_{2;\tilde{u};0} = \left(\frac{1}{2} - \mu_\beta \right) \eta^{\beta\nu} \sum_{m \ge 0} \frac{\partial \widetilde{u}^\alpha}{\partial w^\rho_m} \eta^{\rho\theta} \partial_x^{m+1} \Omega_{\theta,0;r,0} = \left(\frac{1}{2} - \mu_\beta \right) \eta^{\beta\nu} \sum_{m \ge 0} \left\{ \widetilde{u}^\alpha, \bar{h}_{\nu,0} \right\}_{K_1^{\mathrm{DZ}}}.$$

Similarly, Lemmas 5.5.1 and 5.5.4, along with the fact that $\widetilde{u}^\alpha (u^*_*, \varepsilon) - u^\alpha$ is also in the image of ∂_x [BDGR18], lead us to

$$K^{\mathrm{DR};\alpha\beta}_{2;\tilde{u};0} = \left(\frac{1}{2} - \mu_\beta \right) \eta^{\beta\nu} \sum_{m \ge 0} \frac{\partial \widetilde{u}^\alpha}{\partial u^\rho_m} \eta^{\rho\theta} \partial_x^{m+1} \frac{\delta \bar{g}_{\nu,0}}{\delta u^\theta} = \left(\frac{1}{2} - \mu_\beta \right) \eta^{\beta\nu} \sum_{m \ge 0} \left\{ \widetilde{u}^\alpha, \bar{g}_{\nu,0} \right\}_{K_1^{\mathrm{DR}}}.$$

Finally, considering that in the coordinates $\widetilde{u}^\alpha$ and up to an approximation of ε^2, the local functionals $\bar{h}_{\alpha,a}$ are equivalent to $\bar{g}_{\alpha,a}$, and the Poisson operator K_1^{DZ} is identical to K_1^{DR}, we conclude that $K^{\mathrm{DZ};\alpha\beta}_{2;\tilde{u};0} = K^{\mathrm{DR};\beta}_{2;\tilde{u};0} + O\left(\varepsilon^4\right)$, as required.

Step 3: To complete the proof, we need to show that $K^{\mathrm{DZ}[2]}_{2;\tilde{u}} = K^{\mathrm{DR};[2]}_{2;\tilde{u}}$. We have already established equation (5.6.1) for $l = 0, 2, 3$. Therefore, the difference $K^{\mathrm{DZ};[2]}_{2;\tilde{u}} - K^{\mathrm{DR};[2]}_{2;\tilde{u}}$ can be expressed as $R\partial_x$, where $R = \left(R^{\alpha\beta}\right)$ and $R^{\alpha\beta} \in \mathcal{A}_{\bar{u};2}$.

Given that both $K^{\mathrm{DZ};[2]}_{2;\tilde{u}}$ and $K^{\mathrm{DR};[2]}_{2;\tilde{u}}$ are skew-symmetric operators, we obtain:

$$(R\partial_x)^\dagger = -R\partial_x \quad \Leftrightarrow \quad R^T = R \text{ and } \partial_x R = 0.$$

The condition $\partial_x R = 0$ directly implies that $R = 0$, thereby concluding the proof of the theorem.

Appendix A

Integrability and the WDVV equations

Theorem A.0.1 *[Get02] Consider a Poisson structure $K_u^{\alpha\beta}$ given by:*

$$K_u^{\alpha\beta} = g^{\alpha\beta}(u^*)\partial_x + \Gamma_\gamma^{\alpha\beta}(u^*)u_x^\gamma + O(\varepsilon),$$

where $g^{\alpha\beta}$ is a non-degenerate symmetric matrix.[1]

Then, there exists a Miura transformation, denoted by $u \mapsto \tilde{u}$, such that $K_u^{\alpha\beta}$ can be rewritten as:

$$K_{\tilde{u}}^{\alpha\beta} = \eta^{\alpha\beta}\partial_x,$$

where $\eta^{\alpha\beta}$ is a constant non-degenerate symmetric matrix.

Let K be a Poisson operator as defined in theorem A.0.1, and let $\bar{h}_{\alpha,d_1}$ and $\bar{h}_{\beta,d_2} \in \hat{\Lambda}_u$ such that

$$\{\bar{h}_{\alpha,d_1}, \bar{h}_{\beta,d_1}\}_K = 0$$

by theorem A.0.1, we can express the Poisson operator K as

$$\begin{aligned}\{\bar{h}_{\alpha,d_1}, \bar{h}_{\beta,d_2}\}_K &= \int \frac{\delta \bar{h}_{\alpha,d_1}}{\delta u^\mu} K^{\mu\nu} \frac{\delta \bar{h}_{\beta,d_2}}{\delta u^\nu} dx \\ &= \int \frac{\delta \bar{h}_{\alpha,d_1}}{\delta u^\mu} \eta^{\mu\nu} \partial_x \frac{\delta \bar{h}_{\beta,d_2}}{\delta u^\nu} dx\end{aligned}$$

[1] When $g^{\alpha\beta}$ is interpreted as a metric, its Riemann curvature tensor vanishes due to the Jacobi identities satisfied by K. Additionally, $\Gamma_\gamma^{\alpha\beta}$ are the Christoffel symbols corresponding to the Levi-Civita connection defined by $g^{\alpha\beta}$.

Given that the integral is zero, we have:

$$\frac{\delta}{\delta u^\sigma}\left(\frac{\delta\bar{h}_{\alpha,d_1}}{\delta u^\mu}\eta^{\mu\nu}\partial_x\frac{\delta\bar{h}_{\beta,d_2}}{\delta u^\nu}\right)=0$$

Assuming the local functionals h_{α,d_1} and h_{β,d_2} are defined up to genus 0, they solely depend on u^* (with no u_i^* terms for $i \geq 1$). Thus, we obtain:

$$\begin{aligned}\sum_{i\geq 0}(-\partial_x)^i\circ\frac{\partial}{\partial u_i^\sigma}&\left(\frac{\partial h_{\alpha,d_1}}{\partial u^\mu}\eta^{\mu\nu}\frac{\partial^2 h_{\beta,d_2}}{\partial u^\lambda\partial u^\nu}u_x^\lambda\right)\\ &=\frac{\partial}{\partial u^\sigma}\left(\frac{\partial h_{\alpha,d_1}}{\partial u^\mu}\eta^{\mu\nu}\frac{\partial^2 h_{\beta,d_2}}{\partial u^\lambda\partial u^\nu}u_x^\lambda\right)-\partial_x\left(\frac{\partial h_{\alpha,d_1}}{\partial u^\mu}\eta^{\mu\nu}\frac{\partial^2 h_{\beta,d_2}}{\partial u^\lambda\partial u^\nu}\delta_\sigma^\lambda\right)\\ &=\left(\frac{\partial^2 h_{\alpha,d_1}}{\partial u^\sigma\partial u^\mu}\eta^{\mu\nu}\frac{\partial^2 h_{\beta,d_2}}{\partial u^\lambda\partial u^\nu}u_x^\lambda\right)-\left(\frac{\partial^2 h_{\alpha,d_1}}{\partial u^\lambda\partial u^\mu}\eta^{\mu\nu}\frac{\partial^2 h_{\beta,d_2}}{\partial u^\sigma\partial u^\nu}u_x^\lambda\right)\end{aligned}$$

From the above, we deduce:

$$\left(\frac{\partial^2 h_{\alpha,d_1}}{\partial u^\sigma\partial u^\mu}\eta^{\mu\nu}\frac{\partial^2 h_{\beta,d_2}}{\partial u^\lambda\partial u^\nu}\right)=\left(\frac{\partial^2 h_{\alpha,d_1}}{\partial u^\lambda\partial u^\mu}\eta^{\mu\nu}\frac{\partial^2 h_{\beta,d_2}}{\partial u^\sigma\partial u^\nu}\right)\tag{A.0.1}$$

For $d_{1,2}=0$, these equations are recognized as the WDVV equations. This suggests these equations correspond to a genus 0 relation, and are equivalent to the cohomological relation in $\overline{\mathcal{M}}_{0,4}$:

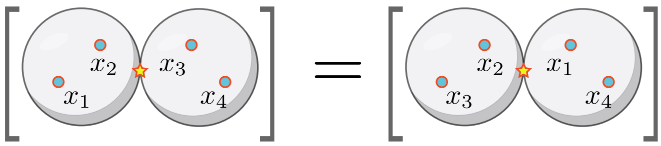

Figure A.1: Cohomological relation represented on $\overline{\mathcal{M}}_{0,4}$ in condition to the WDVV equations.

Appendix B

On the complex structures of tori

Theorem B.0.1 *Consider two tori T_{τ_1} and T_{τ_2} defined over the complex upper half-plane $\mathbb{H}$, such that $T_{\tau_1} = \mathbb{C}/\Gamma_{\tau_1}$ and $T_{\tau_2} = \mathbb{C}/\Gamma_{\tau_2}$. These tori are biholomorphic if and only if their defining parameters τ_1 and τ_2 satisfy the relation*

$$\tau_2 = \frac{a\tau_1 + b}{c\tau_1 + d},$$

where $a, b, c,$ and d are integers such that $ad - bc = 1$.

Proof. ($\Rightarrow$) Consider a biholomorphic map $f : T_{\tau_2} \to T_{\tau_1}$. We can lift this map to a new map, $\tilde{f}$, between the universal covers of T_{τ_2} and T_{τ_1}, such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\tilde{f}} & \mathbb{C} \\ {\scriptstyle \pi_2}\big\downarrow & & \big\downarrow{\scriptstyle \pi_1} \\ T_{\tau_2} & \xrightarrow[f]{} & T_{\tau_1} \end{array}$$

Since f is biholomorphic, $\tilde{f} : \mathbb{C} \to \mathbb{C}$ must also be biholomorphic. We can extend $\tilde{f}$ to a map $\hat{f} : \mathbb{C}_\infty \to \mathbb{C}_\infty$ as follows:

$$\hat{f}(z) = \begin{cases} \tilde{f}(z) & \text{if } z \in \mathbb{C}, \\ \infty & \text{if } z = \infty. \end{cases}$$

Expanding $\hat{f}$ at ∞, we obtain

$$\hat{f} = a_1 z + \sum_{n=0}^{\infty} b_n z^{-n}.$$

Since $\hat{f} - a_1 z$ is holomorphic on $\mathbb{C}_\infty$, it must be a constant, which we denote by κ. Therefore, $\hat{f} = a_1 z + \kappa$ is holomorphic on $\mathbb{C}$. If we require the lattices to match at the origin, then $\kappa = 0$. Moreover, we can express $\tilde{f}(\tau_2)$ and $\tilde{f}(1)$ in terms of elements of Γ_{τ_1}

as follows:

$$\tilde{f}(\tau_2) = a_1\tau_2 = a\tau_1 + b, \tag{B.0.1}$$

and

$$\tilde{f}(1) = a_1 = c\tau_1 + d, \tag{B.0.2}$$

where $a, b, c, d \in \mathbb{Z}$.

In matrix notation, this can be written as:

$$\begin{pmatrix} \tilde{f}(\tau_2) \\ \tilde{f}(1) \end{pmatrix} = a_1 \begin{pmatrix} \tau_2 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau_1 \\ 1 \end{pmatrix}.$$

Using Equation (B.0.2) in (B.0.1), we obtain:

$$\tau_2 = \frac{a\tau_1 + b}{c\tau_1 + d}. \tag{B.0.3}$$

Since $\tau_2 \in \mathbb{H}$ and $\mathrm{Im}(\tau_2) > 0$, we have:

$$\frac{(ad - bc)}{|c\tau_1 + d|^2} \cdot \mathrm{Im}(\tau_1) > 0, \tag{B.0.4}$$

which implies $(ad-bc) > 0$. This shows that $\tilde{f}$ is invertible. Therefore, $\tilde{f}^{-1} \circ \tilde{f}|_{\Gamma_{\tau_2}} = \mathrm{Id}_{\Gamma_{\tau_2}}$, implying that $\frac{1}{\det \tilde{f}} \in \mathbb{Z}$. Combining this with the previous results, we conclude that $ad - bc = 1$.

(⇐) By changing the basis of the lattice using Equation (B.0.3), and noting that $f : T_{\tau_2} \to T_{\tau_1}$ is a biholomorphism, we conclude that $T_{\tau_2} \simeq T_{\tau_1}$.

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