Pseudo Double Categories of Manifolds, Cobordisms and Diffeomorphisms of Cobordisms.

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Abstract

We construct, for each non-negative integer n, a monoidal pseudo double category of manifolds (closed, smooth and of dimension n), cobordisms, and isotopy classes of certain (collar-compatible) isotopy classes diffeomorphisms between cobordisms. As a first step towards a combinatorialisation of the construction we construct a (strict) double category of nested tuples and permutations, which we believe is equivalent to a full subcategory of our geometric pseudo double category, when $n = 2$.

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My own contributions, fully and explicitly indicated in the thesis, have been as follows.

- The (strict) double category of nested tuples and permutations constructed in chapter 4, as well as the proof that it is a strict double category.
- The details of the proof that the pseudo double category of manifolds, cobordisms and isotopy classes of diffeomorphisms in chapter 5 is in fact a pseudo double category, and that the unquotiented construction does not form a pseudo double category.

Much of the foundational content of chapter 2 is found among the literature, including the fact that gluing topological manifolds along boundary components yields a topological manifold, the details of the proof presented here however are of my own making, as the full details are not always displayed. The non-standard smooth atlas theorem is my own contribution as well, as is its use in constructing smooth atlases on glued manifolds, and verifying that maps defined on such manifolds are smooth with respect to those smooth structures (via the non-standard single representation lemma).

Signed
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Chapter 1

Introduction

In this thesis we construct in detail a monoidal pseudo double category of manifolds, cobordisms, and isotopy classes of diffeomorphisms of cobordisms. Specifically, we construct such a monoidal pseudo double category for each non-negative integer n. We aim to thoroughly verify each component of our construction and provide explicit proofs for all critical results. This pseudo double category of cobordisms is closely related to the bicategory of manifolds, cobordisms and extended cobordisms constructed in [Sch14], however our 2-morphisms are isotopy classes of diffeomorphisms, rather than extended cobordisms. We believe our construction is particularly suited to higher categorical notions of braids, loop braids and generalisations. This served as the core motivation for the construction of this pseudo double category.

One of our main results is the existence of the pseudo double category of manifolds, cobordisms and isotopy classes of diffeomorphisms itself. There are in fact three different versions of the construction, which we call the topological, half-smooth and fully smooth settings. We give full details of only the half-smooth setting in this thesis however as the majority of the construction is the same in all three settings.
We give here a brief overview of the pseudo double category we construct in chapter 5.

- As objects we have n-dimensional smooth manifolds without boundary.
- As tight 1-morphisms we have diffeomorphisms between smooth manifolds.

- As loose 1-morphisms we have smooth collared cobordisms between smooth manifolds.
- As 2-morphisms we have equivalence classes of diffeomorphisms under the relation of collar compatible half-smooth isotopy.

[Sch14] and [Bar+15] give a presentation of the bicategory of manifolds, cobordisms and extended cobordisms by generators and relations, in the (0,1,2) and (1,2,3)-dimensional cases. This motivates the question of whether such a presentation is possible for our pseudo double category, whose 2-morphisms are isotopy classes of diffeomorphisms, rather than diffeomorphism classes of cobordisms with boundaries and corners. This question led to the other main construction of the thesis, that of a (in this case strict) double category defined entirely combinatorially.

This combinatorial double category, of nested tuples and permutations, has close connections to the permutation operad discussed for example in [Fre17]. We believe our construction to be more general, and so we endeavour to include the full details of all results. This double category has:

- As objects non negative integers.
- As tight 1-morphisms from n to n, permutations of n objects. That is, the class of tight 1-morphisms from m to n is S_n if $n = m$, and is empty otherwise.
- As loose 1-morphisms from m to n a nested tuple $(t_1, \ldots, t_m)$ of tuples, containing between them each element of $\{1, \ldots, n\}$ precisely once each.
- 2-morphisms from $(t_1, \ldots, t_m)$ to $(t'_1, \ldots, t'_m)$ will consist of an outer permutation $\sigma \in S_m$, and for each inner tuple t_i an inner permutation of the appropriate length. The conditions for the outer and inner permutations will be discussed in more detail in chapter 4

In the final section, in which we take a somewhat speculative approach, and do not give details, we will discuss the possibility of a (pseudo) double functor between these two constructions, where we consider the pseudo double category of 2-dimensional manifolds, 3-dimensional cobordisms and isotopy classes of diffeomorphisms between cobordisms. We describe the action of this functor on objects and both tight and loose 1-morphisms. We will also discuss the possibility of extending the construction of the double category of nested tuples and permutations to braids, which we believe is connected to the pseudo double category of 1-dimensional manifolds, 2-dimensional cobordisms and isotopy classes of diffeomorphisms between cobordisms.

Chapter 2

Topological and Smooth Manifolds

In this chapter we will specify precisely what we mean by a topological manifold and state some of their basic properties which we will make use of in later sections. We assume the reader is familiar with undergraduate level point set topology, but no other specialist knowledge will be assumed.

2.1 Definitions and Notation.

In this thesis, $\mathbb{R}^n$ will have the Euclidean topology unless specified otherwise, subsets of $\mathbb{R}^n$ will have the subspace topology. Products of spaces will always have the product topology, unless specified otherwise.
By the *disjoint union* two sets X, Y we mean the set:

$$X \sqcup Y := (X \times \{0\}) \cup (Y \times \{1\}).$$

In the case where X, Y are topological spaces, we equip $X \sqcup Y$ with the disjoint union topology.
We begin with the definition of a topological manifold with boundary.

Definition 2.1.1. Let $n \in \mathbb{Z}^+$. An n-dimensional *topological manifold* is a topological space (M, τ) such that for every $p \in M$ there is a triple (U, V, φ), where:

- $U \subseteq M$ is an open neighbourhood of p.
- V is an open subset of $\mathbb{R}^{n-1} \times [0, \infty)$, (with the product topology from $\mathbb{R}^{n-1}$ and $[0, \infty) \subseteq \mathbb{R}$).

- $\varphi\colon U \to V$ is a homeomorphism.

The triple (U, V, φ) is called a *chart* for M at p. The set U is called a *coordinate neighbourhood*, V a *coordinate co-neighbourhood*, and φ a *coordinate map*.

Remarks 2.1.2. We take the convention that $\mathbb{R}^0 = \{0\}$, the one point space, so a one-dimensional manifold is a space locally homeomorphic to $\{0\} \times [0, \infty)$ (which is the same as being locally homeomorphic to $[0, \infty)$). It is worth noting that $\mathbb{R}^{n-1} \times [0, \infty)$ is not actually a subset of $\mathbb{R}^n$. The spaces $\mathbb{R}^{n-1} \times [0, \infty)$ and $\{(x_1, ..., x_n) \in \mathbb{R}^n : x_n \geq 0\}$ are not equal, though they are homeomorphic. It will be useful to be able to take coordinate co-neighbourhoods in either space, and we will address this issue in more detail later.

Zero dimensional topological manifolds are defined separately.

Definition 2.1.3. A zero dimensional topological manifold is any discrete space.

We now define some terminology which we will make frequent use of, particularly when we come to define smooth manifolds.

Definition 2.1.4. Let M be an n-dimensional topological manifold, a set $A = \{(U_\lambda, V_\lambda, \varphi_\lambda) : \lambda \in \Lambda\}$ of charts for M such that $\{U_\lambda : \lambda \in \Lambda\}$ form a cover of M (i.e. A has a chart for every point $p \in M$) is called an *atlas* for M. Given an atlas A for M and two charts $(U, V, \varphi), (U', V', \varphi') \in A$ the map:

$$\varphi' \circ \varphi^{-1}\colon \varphi(U \cap U') \to \varphi'(U \cap U')$$

is called a *transition map* of A.

Remark 2.1.5. Note that in the above definition since $U \cap U'$ is an open subset of U and φ is a homeomorphism, $\varphi(U \cap U')$ is an open subset of V, and hence an open subset of $\mathbb{R}^{n-1} \times [0, \infty)$. Similarly $\varphi'(U \cap U')$ is an open subset of $\mathbb{R}^{n-1} \times [0, \infty)$.

Definition 2.1.6. Let M be a topological manifold, $p \in M$ and (U, V, φ) be a chart for M at p. If $\varphi(p) \in V \cap \mathbb{R}^{n-1} \times \{0\}$ then (U, V, φ) is called a *boundary chart* and p a *boundary point*. Alternately if $V \cap \mathbb{R}^{n-1} \times \{0\} = \emptyset$ then (U, V, φ) is called an *interior chart* and p an *interior point*. The set of boundary points is called the *boundary* of M and denoted ∂M. The set of interior points is called the *interior* of M and denoted $\mathrm{int}(M)$

The definition of boundary and interior make reference to a particular chart, so we need to verify that this choice of chart does not affect the classification of p. For this we call upon a classic theorem of the topology of $\mathbb{R}^n$. A proof of this theorem can be found in [Kul98].

Theorem 2.1.7. *(Invariance of Domain) Let $U \subseteq \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}^n$ be continuous and injective, then $f(U) \subseteq \mathbb{R}^n$ is open and f is a homeomorphism onto its image.*

The invariance of domain theorem allows us to show that the definition of the boundary and interior do not depend on the chosen chart.

Proposition 2.1.8. Let M be a topological manifold, $p \in M$. If one chart for M at p is a boundary chart, then all charts for M at p are boundary charts.

Proof. Suppose that $(U, V, \varphi), (U', V', \varphi')$ are charts for M at p such that $\varphi(p) \in \mathbb{R}^{n-1} \times \{0\}$ but $\varphi'(p) \notin \mathbb{R}^{n-1} \times \{0\}$. Without loss of generality we can assume that $V' \cap \mathbb{R}^{n-1} \times \{0\} = \emptyset$.
Define:

$$\hat{V} := \{(x_1, ..., x_{n-1}, x_n) \in \mathbb{R}^n : ((x_1, ..., x_{n-1}), x_n) \in \varphi(U \cap U')\} \subseteq \mathbb{R}^n,$$
$$\hat{V}' := \{(x_1, ..., x_{n-1}, x_n) \in \mathbb{R}^n : ((x_1, ..., x_{n-1}), x_n) \in \varphi'(U \cap U')\} \subseteq \mathbb{R}^n.$$

Note that $\hat{V}$ is homeomorphic to V and $\hat{V}'$ is homeomorphic to V'. Note that since $\varphi(p) \in \mathbb{R}^{n-1} \times \{0\}$ we have that $\varphi(U \cap U') \cap \mathbb{R}^{n-1} \times \{0\} \neq \emptyset$ and so $\hat{V}$ is not open (contains no open ball around the point 'corresponding' to $\varphi(p)$). However since $\varphi'(U \cap U') \subseteq V' \subseteq \mathbb{R}^{n-1} \times (0, \infty)$, $\hat{V}'$ is open. Now consider the transition map:

$$\varphi' \circ \varphi^{-1}\colon \varphi(U \cap U') \to \varphi'(U \cap U').$$

This is a homeomorphism as it is a composition of restrictions of homeomorphisms, therefore $\varphi(U \cap U')$ is homeomorphic to $\varphi'(U \cap U')$. Thus $\hat{V}$ is homeomorphic to $\hat{V}'$. However since $\hat{V} \subseteq \mathbb{R}^n$ is not open and $\hat{V}' \subseteq \mathbb{R}^n$ is open this contradicts the invariance of domain theorem. Therefore if (U, V, φ) is a boundary chart, then (U', V', φ') must also be a boundary chart. □

In definition 2.1.1 we state that coordinate co-neighbourhoods must be open subsets of $\mathbb{R}^{n-1} \times [0, \infty)$. It will often be useful to take coordinate co-neighbourhoods in other spaces, such as $\mathbb{R}^n$. We make the following definition to formalise this idea.

Definition 2.1.9. Let X be a topological space, $p \in X$. By a *non-standard chart* of dimension n for X at p we mean a quadruple (U, N, V, φ) where:

- $U \subseteq X$ is an open neighbourhood of p.
- N is an n-dimensional topological manifold.
- V is an open subset of N.
- $\varphi \colon U \to V$ is a homeomorphism.

In the same manner as for ordinary charts we call U a *coordinate neighbourhood*, N a *target manifold*, V a *non-standard coordinate neighbourhood*, and φ a *non-standard coordinate map.*

Definition 2.1.10. A set A of non-standard charts (of the same dimension n) for X whose coordinate neighbourhoods cover X is called a *non-standard atlas* of dimension n for X. Given non-standard charts:

$$(U, N, V, \varphi), (U', N', V', \varphi') \in A$$

the map:

$$\varphi' \circ \varphi^{-1} \colon \varphi(U \cap U') \to \varphi'(U \cap U')$$

is called a *non-standard transition map* of A.

The following lemma justifies our use of non-standard charts, and will simplify proofs that spaces are topological manifolds.

Lemma 2.1.11. (The Non-Standard Atlas Lemma) Let X be a topological space and suppose that for every $p \in X$ there exists a non-standard chart (U, N, V, φ) of dimension n for X at p. Then X is an n-dimensional topological manifold.

Proof. To construct (standard) charts for X we simply compose with coordinate maps of N. Let $p \in X$, let (U, N, V, φ) be a non-standard chart for X at p and let (U', V', φ') be a chart for N at $\varphi(p)$. Then the triple:

$$(U \cap \varphi^{-1}(U'), \varphi'(U' \cap \varphi(U)), \varphi' \circ \varphi)$$

is a chart for X at p. □

Now that we have this lemma we are free to take our charts with coordinate co-neighbourhoods in any topological space which we know is itself a topological manifold. Equipped with this result we begin working towards the main result of this section, that we can 'glue' manifolds along boundary components.
We now give some examples of topological manifolds, both with and without boundary.

Examples 2.1.12. Our first family of examples is the n-sphere, defined as follows:

$$S^n := \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \dots, +x_{n+1}^2 = 1\}.$$

To see that this is an n-dimensional topological manifold we take the atlas consisting of charts projecting the open hemispheres into $\mathbb{R}^n$. That is for $k \in \{1, \dots, n+1\}$ we take:

$$U_k^+ = \{(x_1, \dots, x_{n+1}) \in S^n : x_k > 0\},$$
$$\varphi_k^+(x_1, \dots, x_{n+1}) = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}).$$

With U_k^- and φ_k^- defined similarly. It is straightforward to see that each of these coordinate maps are homeomorphisms onto open subsets of $\mathbb{R}^n$, and so S^n is an n-dimensional topological manifold.

Another example of a one-dimensional topological manifold is the line with two origins, obtained by adding an 'extra origin' to the real line. Explicitly the underlying set is $\mathbb{R} \cup \{0'\}$, with the topology being generated by all sets of the form $(U \setminus \{0\}) \cup \{0'\}$ where $U \subseteq \mathbb{R}$ is an open neighbourhood of zero in the Euclidean topology, together with the open subsets of $\mathbb{R}$ itself. For a chart at $0'$ we take the coordinate neighbourhood $((-1,1) \setminus \{0\}) \cup \{0'\}$ and the coordinate map taking $0'$ to 0, leaving other points unchanged.

Remark 2.1.13. Topological manifolds are locally path connected, so the notions of connected and path connected coincide, that is, if a topological manifold is connected, then it is path connected. Further, topological manifolds admit a connected open partition, that is they have connected components, and these components are path connected. Additionally the connected components and unions thereof are precisely the clopen subsets of a topological manifold.

2.2 Atlases and Gluing.

In this section we give a detailed proof of the fact that gluing two topological manifolds along boundary components yields a topological manifold. That this result is true is intuitively clear, however the proof is somewhat more involved than one might expect. To assist us in this proof we first define a special kind of chart with a specific form of coordinate co-neighbourhood.

Definition 2.2.1. (Cylinder Charts) Let M be a topological manifold and $p \in \partial M$. A chart (U, V, φ) for M at p is called a *cylinder chart* if V is of the form $\partial V \times [0, 1)$ (where ∂V is an open subset of $\mathbb{R}^{n-1}$).

Having defined cylinder charts, we show that they always exist.

Lemma 2.2.2. (Cylinder Chart Lemma) Let M be a topological manifold and $p \in \partial M$, then there exists a cylinder chart for M at p.

Proof. Let (U, V, φ) be a chart for M at p. Then since $V \subseteq \mathbb{R}^{n-1} \times [0, \infty)$ is open there exists $\varepsilon > 0$ such that $B_\varepsilon(\varphi(p)) \subseteq V$. Therefore the set:

$$V' := \left\{ \big((x_1, \dots, x_{n-1}), x_n\big) \in \mathbb{R}^{n-1} \times [0, \infty) : \sqrt{x_1^2 + \cdots + x_{n-1}^2} < \frac{\varepsilon}{2}, x_n < \frac{\varepsilon}{2} \right\},$$

is a subset of V. Now letting:

$$V'' := \left\{ \left((x_1, \dots, x_{n-1}), \frac{2x_n}{\varepsilon}\right) : \big((x_1, \dots, x_{n-1}), x_n\big) \in V' \right\},$$

and defining:

$$\psi \colon \mathbb{R}^{n-1} \times [0, \infty) \to \mathbb{R}^{n-1} \times [0, \infty),$$
$$\big((x_1, \dots, x_{n-1}), x_n\big) \mapsto \left((x_1, \dots, x_{n-1}), \frac{2x_n}{\varepsilon}\right),$$

we have that $((\psi \circ \varphi)^{-1}(V''), V'', \psi \circ \varphi)$ is a cylinder chart for M at p. □

Cylinder charts allow us to prove that topological manifolds can be glued.

Theorem 2.2.3. *Let M, M' be n-dimensional topological manifolds, let $N \subseteq \partial M$ and $N' \subseteq \partial M'$ be clopen subsets of the boundaries of M and M' respectively. Let $f \colon N \to N'$ be a homeomorphism and define the relation $\sim$ on $M \sqcup M'$ by $(p, 0) \sim (f(p), 1)$ for $p \in N$. Then $\tilde{M} := M \sqcup M'/\sim$ is an n-dimensional topological manifold.*

Proof. We first find charts for points of the form $[(p, 0)] = [(f(p), 1)]$ for $p \in N$. Let (U, V, φ) be a cylinder chart for M at p and (U', V', φ') be a cylinder chart for M' at $f(p)$. $U' \cap N'$ is an open subset of N' and f is continuous so $U \cap f^{-1}(U' \cap N')$ is an open subset of $U \cap N$. We now define:

$$\hat{V} := \{((x_1, \ldots, x_{n-1}), t) : ((x_1, \ldots, x_{n-1}), 0) \in \varphi(U \cap f^{-1}(U' \cap N')), t \in [0, 1)\},$$
$$\hat{U} := \varphi^{-1}(\hat{V}),$$

and we have that $\hat{V}$ is an open subset of V. Therefore $\hat{U}$ is an open subset of U, and so is an open subset of M. Similarly we have that $U' \cap f(U \cap N)$ is an open subset of $U' \cap N'$, and so we define:

$$\hat{V}' := \{((x_1, \ldots, x_{n-1}), t) : ((x_1, \ldots, x_{n-1}), 0) \in \varphi'(U' \cap f(U \cap N)), t \in [0, 1)\},$$
$$\hat{U}' := \varphi'^{-1}(\hat{V}').$$

As above we have that $\hat{U}'$ is an open subset of M' and $\hat{V}'$ is an open subset of $\mathbb{R}^{n-1} \times [0, \infty)$. By construction we have that $f(\hat{U} \cap N) = \hat{U}' \cap N'$. We claim that the set:

$$\tilde{U} := \{[(p, 0)] : p \in \hat{U}\} \cup \{[(p, 1)] : p \in \hat{U}'\}$$

is an open subset of $\tilde{M}$. Let $q \colon M \sqcup M' \to \tilde{M}$ be the quotient map, we show that $q^{-1}(\tilde{U})$ is open. Since $f(\hat{U} \cap N) = \hat{U}' \cap N'$, for all $p \in N$, if $[(p, 0)] \in \tilde{U}$ then $[(f(p), 1)] \in \hat{U}'$, so we have:

$$\begin{aligned} q^{-1}(\tilde{U}) &= q^{-1}([\hat{U} \times \{0\}] \cup [\hat{U}' \times \{1\}]), \\ &= (\hat{U} \times \{0\}) \cup (\hat{U}' \times \{1\}). \end{aligned}$$

Where we have used the fact that $q^{-1}([\hat{U} \times \{0\}])$ contains, in addition to $\hat{U} \times \{0\}$, the set $f(\hat{U} \cap N) \times \{1\}$, which is precisely the set $(\hat{U}' \cap N') \times \{1\}$. With $q^{-1}([\hat{U}' \times \{1\}])$ defined similarly. Therefore since $\hat{U}$ is an open subset of M and $\hat{U}'$ is an open subset of M', $q^{-1}(\tilde{U})$ is an open subset of $M \sqcup M'$, so $\tilde{U}$ is an open subset of $\tilde{M}$. We now define:

$$g \colon \hat{V} \to \mathbb{R}^{n-1} \times (-\infty, 0],$$
$$((x_1, \ldots, x_{n-1}), t) \mapsto (\pi_1 \circ \varphi' \circ f \circ \varphi^{-1}((x_1, \ldots, x_{n-1}), 0), -t).$$

That is, g extends the action of the map $\varphi' \circ f \circ \varphi^{-1}$ on $\hat{V} \cap \mathbb{R}^{n-1} \times \{0\}$ along each 'fibre' of $\hat{V}$ (recall that $\hat{V}$ is a cylinder, since (U', V', φ') was a cylinder chart), and then reflects in the last coordinate. We now define the map:

$$\tilde{\varphi} \colon \tilde{U} \to \mathbb{R}^{n-1} \times [0, \infty)$$

by:

$$[(p,0)] \mapsto g(\varphi(p)) \text{ for } p \in \hat{U},$$
$$[(p,1)] \mapsto \varphi'(p) \text{ for } p \in \hat{U}'.$$

Note that both cases are homeomorphisms on their respective domains, which are both closed subsets of $\tilde{U}$ (since we have $f(\hat{U} \cap N) = \hat{U}' \cap N'$). To demonstrate that the two definitions agree on the intersection of their domains let $p \in \hat{U} \cap N$, then the first definition maps $[(p,0)]$ to $\varphi'(f(p))$ (note that g is simply $\varphi' \circ f \circ \varphi^{-1}$ on $\hat{V} \cap \mathbb{R}^{n-1} \times \{0\}$, and p is a boundary point). For the second case we have that $[(p,0)] = [(f(p),1)]$, and the second case maps this to $\varphi'(f(p))$, so the two definitions agree on the intersection of their domains. Hence by the glue lemma $\tilde{\varphi}$ is continuous. Because the images of the two cases are disjoint aside from the image of the intersection of their domains, the glued map $\tilde{\varphi}$ is a homeomorphism onto its image. Defining $\tilde{V} := \tilde{\varphi}(\tilde{U})$ we therefore have that the triple $(\tilde{U}, \tilde{V}, \tilde{\varphi})$ is a chart for $\tilde{M}$ at $[(p,0)] = [(f(p),1)]$.

For charts at the other points we note that since the restriction of the quotient map q to the subset $M \setminus N \times \{0\}$ is a homeomorphism the quadruple:

$$([M \setminus N \times \{0\}], M, M \setminus N, q^{-1})$$

is a non-standard chart for $\tilde{M}$ at $[(p,0)]$ for any $p \in M \setminus N$. Similarly we have:

$$([M' \setminus N' \times \{1\}], M', M' \setminus N', q^{-1})$$

for points of the form $[(p,1)]$ for $p \in M' \setminus N'$. Hence $\tilde{M}$ is an n-dimensional topological manifold. □

It is worth noting at this point that theorem 2.2.3 holds even in cases where M and M' are not Hausdorff. For example we have the following.

Example 2.2.4. Let M be the half-line with two origins, and $M' = [0,1] \sqcup [0,1]$ be the disjoint union of two intervals. If we let $N = \{0, 0'\} \subseteq M$ and $N' = \{(0,0),(0,1)\} \subseteq M'$, and let $f \colon N \to N'$ map 0 to $(0,0)$ and $0'$ to $(0,1)$, then we can glue M and M' and obtain a one-dimensional topological manifold.

Though theorem 2.2.3 is true for manifolds which are not Hausdorff, we will require our cobordisms to be Hausdorff, so we show that gluing two Hausdorff manifolds results in a manifold which is also Hausdorff.

Corollary 2.2.5. *Let M, M' be n-dimensional topological manifolds and N, N' be clopen subsets of M and M' respectively. Define $\tilde{M}$ as in theorem 2.2.3. If M and M' are Hausdorff, then $\tilde{M}$ is Hausdorff.*

Proof. Let q_1, q_2 be distinct points in $\tilde{M}$, we seek disjoint open neighbourhoods of q_1 and q_2. There are three subsets of $\tilde{M}$ of interest:

- $A := [(M \setminus N) \times \{0\}]$,
- $B := [(M' \setminus N') \times \{1\}]$,
- $C := [N \times \{0\}] = [N' \times \{1\}]$.

This gives six possible cases to consider. Notice however that A is homeomorphic to $M \setminus N$, and B is homeomorphic to $M' \setminus N'$. Combining the observation that both of these spaces are Hausdorff with the fact that A and B are themselves disjoint open subsets of $\tilde{M}$ eliminates all cases where neither q_1 nor q_2 lies in C.

If $q_1 \in C$ and $q_2 \in A$. Then $q_1 = [(p_1, 0)]$ for $p_1 \in M$ and $q_2 = [(p_2, 0)]$ for $p_2 \in M$. Since M is Hausdorff we may choose disjoint open neighbourhoods U_1, U_2 of p_1, p_2 respectively. Now we take a chart (U, V, φ) for M at p_2 and without loss of generality assume that $U \subseteq U_2$. Using the same method as in the proof of theorem 2.2.3 we find a cylinder chart $(\hat{U}, \hat{V}, \hat{\varphi})$ with $\hat{U} \subseteq U$ and extend this to a chart $(\tilde{U}, \tilde{V}, \tilde{\varphi})$ for $\tilde{M}$ at q_2. Since $q_1 \in A$ we have $p_1 \notin N$ and so we may assume $U \subseteq M \setminus N$, and so $[U \times \{0\}]$ and $\tilde{U}$ are disjoint open neighbourhoods of q_1 and q_2. The proof for the case $q_1 \in B, q_2 \in C$ is similar.

Finally we consider the case $q_1, q_2 \in C$. Since $q_1 \in C$ we have $q_1 = [(p_1, 0)] = [(p_1', 1)]$ for $p_1 \in N, p_1' \in N'$. Similarly $q_2 = [(p_2, 0)] = [(p_2', 1)]$ for $p_2 \in N, p_2' \in N'$. We now take cylinder charts $(U_1, V_1, \varphi_1), (U_2, V_2, \varphi_2)$ for M at p_1, p_2 respectively, and since M is Hausdorff we may assume $U_1 \cap U_2 = \emptyset$. Similarly we take cylinder charts $(U_1', V_1', \varphi_1'), (U_2', V_2', \varphi_2')$ for M' at p_1', p_2' respectively, again we may assume $U_1' \cap U_2' = \emptyset$. We now again use the method from theorem 2.2.3 to combine the charts (U_1, V_1, φ_1) and (U_1', V_1', φ_1') to form a chart $(\tilde{U}_1, \tilde{V}_1, \tilde{\varphi}_1)$ for $\tilde{M}$ at q_1. Similarly we construct a chart $(\tilde{U}_2, \tilde{V}_2, \tilde{\varphi}_2)$ for $\tilde{M}$ at q_2. Since $U_1 \cap U_2 = \emptyset$ and $U_1' \cap U_2' = \emptyset$ we have that $\tilde{U}_1$ and $\tilde{U}_2$ are disjoint open neighbourhoods of q_1, q_2 in $\tilde{M}$. Therefore $\tilde{M}$ is Hausdorff. ∎

2.3 Smooth Atlases and Smooth Structures.

In this section we clarify what we mean by a smooth atlas for a topological manifold, what we mean by a smooth manifold, and what it means for a map between smooth manifolds to be smooth.

Definition 2.3.1. Let M be an n-dimensional topological manifold, an atlas A for M is a *smooth atlas* if all the transition maps of A are smooth.

Remark 2.3.2. Since our manifolds may have boundary we need to take a convention on what it means for a function to be smooth at a boundary point. Specifically if $(U, V, \varphi), (U', V', \varphi') \in A$ with $U \cap U' \cap \partial M \neq \emptyset$ then $\varphi(U \cap U') \subseteq V$ meets $\mathbb{R}^{n-1} \times \{0\}$ and so we need to decide what it means for the transition map:

$$\varphi' \circ \varphi^{-1} \colon \varphi(U \cap U') \to \varphi'(U \cap U')$$

to be smooth on a neighbourhood of $((x_1, ..., x_{n-1}), x_n) \in \mathbb{R}^{n-1} \times \{0\}$. We take the same convention as in [Kos93, Page 2] that such a map is smooth if it is locally a restriction of a smooth map on an open subset.

Definition 2.3.3. Two smooth atlases A, B for M are *smoothly equivalent* if their union is a smooth atlas.

Definition 2.3.4. By a *smooth structure* on M we mean a smooth atlas A for M which is not strictly contained in any larger smooth atlas. A *smooth manifold* is a topological manifold together with a smooth structure.

Remark 2.3.5. The relation of smooth equivalence is easily verified to be an equivalence relation, and so we have the following.

Proposition 2.3.6. Every smooth atlas is contained in a unique smooth structure, and further two smooth atlases are contained in the same smooth structure if and only if they are smoothly equivalent.

Proof. The smooth structure S containing A is the union of all smooth atlases which are smoothly equivalent to A. To see that this is a maximal smooth atlas, suppose that B was a smooth atlas for M containing S, then since $A \subseteq S$ we have that $A \subseteq B$ and so $B \cup A = B$ which is a smooth atlas. Therefore B is smoothly equivalent to A and so by definition $B \subseteq S$, so S is indeed maximal. That two smooth atlases are contained in the same structure if and only if they are smoothly equivalent follows immediately. □

Remark 2.3.7. Note that the empty set (equipped with the topology $\{\emptyset\}$) is an n-dimensional topological manifold for every n. By vacuity the empty set is a smooth atlas (and in fact a smooth structure) for this manifold, hence the pair $((\emptyset, \{\emptyset\}), \emptyset)$ (a set, a topology on that set making it a topological manifold, and a maximal smooth atlas for said manifold) is a smooth manifold of any dimension.

Definition 2.3.8. Let (M, S) and (N, S') be smooth manifolds (not necessarily of the same dimension). A function $f \colon M \to N$ is *smooth* if for every $p \in M$, for every chart $(U, V, \varphi) \in S$ for M at p and every chart $(U', V', \varphi') \in S'$ for N at $f(p)$ the map:

$$\varphi' \circ f \circ \varphi^{-1} \colon \varphi(U \cap f^{-1}(U')) \to \varphi'(f(U) \cap U')$$

is smooth. The map $\varphi' \circ f \circ \varphi$ is called the *local representation* of f in these charts.

Remark 2.3.9. Note that the definition of a smooth map depends on the smooth structures chosen, in general a continuous function $f \colon M \to N$ may be smooth with respect to some smooth structures S, T but not smooth with respect to some different smooth structures S', T'. Given smooth structures however, to verify that a map is smooth it suffices to check a single local representation at every point.

Lemma 2.3.10 (The Single Representation Lemma)**.** Let M, N be smooth manifolds and let:

$$f \colon (M, S_M) \to (N, S_N).$$

Suppose that for every $p \in M$ one local representation of f at p is smooth, then f is smooth. If further f is a homeomorphism and these local representations are diffeomorphisms, then f is a diffeomorphism.

Proof. Let $(U_1, V_1, \varphi_1) \in S_M$ be a chart for M at p and $(U_2, V_2, \varphi_2) \in S_N$ be a chart for N at $f(p)$ such that the local representation $\varphi_2 \circ f \circ \varphi_1$ is smooth. We verify that all other local representations of f are smooth. Let $(U_1', V_1', \varphi_1') \in S_M$ and $(U_2', V_2', \varphi_2') \in S_N$. Then we have:

$$\begin{aligned} \varphi_2' \circ f \circ \varphi_1' &= \varphi_2' \circ (\varphi_2^{-1} \circ \varphi_2) \circ f \circ (\varphi_1 \circ \varphi_1^{-1}) \circ \varphi_1', \\ &= (\varphi_2' \circ \varphi_2^{-1}) \circ (\varphi_2 \circ f \circ \varphi_1) \circ (\varphi_1^{-1} \circ \varphi_1'). \end{aligned}$$

Hence the local representation of f with respect to the charts (U_1', V_1', φ_1') and (U_2', V_2', φ_2') is locally a composition of smooth maps and so is smooth.

Therefore all local representations of f are smooth and so f is smooth with respect to the smooth structures S_M and S_N. If further f is a homeomorphism and these local representations are diffeomorphisms then we apply the above argument to f^{-1} to conclude that f^{-1} is also smooth, hence f is a diffeomorphism. $\square$

Definition 2.3.11. (Smooth Submanifold) A subset N of an n-dimensional smooth manifold (M, S) is a called *smooth submanifold*, also known as a *locally flat submanifold*, of dimension k, if for every $p \in N$ there exists a chart $(U, V, \varphi) \in S$ for M at p such that:

$$\varphi(U \cap N) \subseteq \mathbb{R}^k \times \{0\}^{n-k}.$$

The integer $n - k$ is known as the *co-dimension* of N in M.

Note that the coordinate co-neighbourhood of the chart given in the above definition is not a subset of $\mathbb{R}^{n-1} \times [0, \infty)$, however we have a version of the non-standard atlas lemma which allows us to safely take charts in any smooth manifold, provided we have some compatibility conditions.

Theorem 2.3.12 (Non-Standard Smooth Atlas Theorem)**.** *Let M be an n-dimensional topological manifold and A be a non-standard atlas for M. Let S_N be a smooth structure on N for each of the target manifolds of A. Suppose that for each pair $(U, N, V, \varphi), (U', N', V', \varphi') \in A$ the non-standard transition map:*

$$\varphi' \circ \varphi^{-1} \colon \varphi(U \cap U') \to \varphi'(U \cap U')$$

is a smooth map with respect to $S_N, S_{N'}$. Then there is a unique smooth structure S_A on M such that each of the non-standard coordinate maps is a diffeomorphism.

Proof. Similarly to the proof of lemma 2.1.11 we define:

$$A_M := \big\{(\varphi^{-1}(U_N) \cap U, \varphi_N(U_N \cap \varphi(U)), \varphi_N \circ \varphi) \\ : (U, N, V, \varphi) \in A, (U_N, V_N, \varphi_N) \in S_N\big\}.$$

The transition maps of A_M are precisely the local representations of the non-standard transition maps of A (as shown in the diagram below), so the statement that A_M is a smooth atlas for M is precisely the statement that each of those non-standard transition maps is a diffeomorphism of smooth manifolds. Therefore by proposition 2.3.6 there is a unique smooth structure

S_A on M containing A_M. That each of the coordinate maps of A is a diffeomorphism with respect to this smooth structure is clear, as is the fact that S_A is unique with this property.

$$\begin{array}{ccccc} & & U\cap U' \subseteq M & & \\ & \swarrow^{\varphi} & & \searrow^{\varphi'} & \\ N \supseteq V & & \xrightarrow{\varphi'\circ\varphi^{-1}} & & V' \subseteq N' \\ \downarrow{\scriptstyle \varphi_N} & & & & \downarrow{\scriptstyle\varphi_{N'}} \\ \mathbb{R}^{n-1}\times[0,\infty) \supseteq V_N & & \xrightarrow[\varphi_{N'}\circ(\varphi'\circ\varphi^{-1})\circ\varphi_N^{-1}]{(\varphi_{N'}\circ\varphi')\circ(\varphi_N\circ\varphi)^{-1}} & & V_{N'} \subseteq \mathbb{R}^{n-1}\times[0,\infty) \end{array}$$

The above diagram demonstrates that the transition maps of A_M coincide with the local representations of the non-standard transition maps of A. □

Remark 2.3.13. The non-standard transition maps of A in the above theorem are maps between open subsets of smooth manifolds, which inherit smooth structures by restriction of coordinate maps.

This theorem motivates the following definitions.

Definition 2.3.14. A non-standard atlas A for a manifold M, together with a family of smooth structures on the target manifolds of A satisfying the condition in theorem 2.3.12 is called a *non-standard smooth atlas* and the structure S_A described in theorem 2.3.12 its *induced smooth structure.*

Definition 2.3.15. Let M, N be topological manifolds and A, B be non-standard smooth atlases for M, N respectively. Given a map $f \colon M \to N$ and non-standard charts $(U, N, V, \varphi) \in A, (U', N', V', \varphi') \in B$ the map:

$$\varphi' \circ \varphi^{-1} \colon \varphi(U \cap U') \to \varphi'(U \cap U')$$

is called a *non-standard local representation* of f.

We also have a version of lemma 2.3.10 for non-standard atlases.

Lemma 2.3.16 (The Non-Standard Single Representation Lemma)**.** Let M, M' be topological manifolds, A, B be non-standard smooth atlases for M, N respectively and $f \colon M \to N$. If the non-standard local representations of f in A, B are smooth in the sense of definition 2.3.8, then f is smooth with respect to the induced smooth structures S_A and S_B. If further f is a homeomorphism and the non-standard local representations of f are diffeomorphisms then f is a diffeomorphism with respect to S_A and S_B.

Proof. The local representations of f in the smooth structures S_A, S_B are the local representations of the non-standard local representations of f (as shown below). So the statement that the non-standard local representations are smooth (or diffeomorphisms) in the sense of definition 2.3.8 is precisely the statement that f is smooth (or a diffeomorphism) with respect to S_A, S_B.

$$\begin{array}{ccc}
M & \xrightarrow{\quad f \quad} & M' \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi'} \\
N \supseteq V & \xrightarrow{\quad \varphi' \circ f \circ \varphi^{-1} \quad} & V' \subseteq N' \\
\downarrow{\scriptstyle \varphi_N} & & \downarrow{\scriptstyle \varphi_{N'}} \\
\mathbb{R}^{n-1} \times [0, \infty) \supseteq V_N & \xrightarrow[\varphi'_N \circ (\varphi' \circ f \circ \varphi^{-1}) \circ \varphi_{N'}^{-1}]{(\varphi_{N'} \circ \varphi') \circ f \circ (\varphi_N \circ \varphi)^{-1}} & V_{N'} \subseteq \mathbb{R}^{n-1} \times [0, \infty)
\end{array}$$

The above diagram demonstrates that the local representations of f with respect to S_A, S_B are precisely the local representations with respect to $S_N, S_{N'}$ of the non-standard local representations of f with respect to A, B. □

In the case where at least one of M, N has no boundary, the product $M \times N$, equipped with the product topology, is a smooth manifold with the obvious smooth structure.

Proposition 2.3.17. Let $(M, S), (M', S')$ be smooth manifolds of dimension n and n' respectively and suppose that M has no boundary. Then $M \times M'$ with the product topology is an n-dimensional smooth manifold with the product smooth structure induced by:

$$S \times S' := \{(U \times U', V \times V', \varphi \times \varphi' : (U, V, \varphi) \in S, (U', V', \varphi') \in S'\}.$$

Proof. That $S \times S'$ is a non-standard smooth atlas for $M \times M'$ follows from the fact that $(R^{n-1} \times (0, \infty)) \times (\mathbb{R}^{n-1} \times [0, \infty))$ is an $(n + n')$-dimensional smooth manifold with the obvious smooth structure (coordinate maps simply remove the extra brackets to map into $\mathbb{R}^{n+n'-1} \times [0, \infty)$). The non-standard transition maps of $S \times S'$ are products of transition maps of S with transition maps of S', and so are diffeomorphisms as the product of two diffeomorphisms is a diffeomorphism. Therefore $S \times S'$ induces a smooth structure on $M \times M'$ by the non-standard smooth atlas theorem. □

Remark 2.3.18. The above result relies on the fact that M has no boundary to ensure that $U \times U'$ is always an open subset of $(\mathbb{R}^{n-1} \times (0, 1)) \times (\mathbb{R}^{n-1} \times [0, \infty))$, as a boundary point of M would be mapped onto $\mathbb{R}^{n-1} \times \{0\}$, and we would not get an open coordinate co-neighbourhood.

There is an exactly analogous result to proposition 2.3.17 for the case where M has boundary but M' does not, which we will not state here.

2.4 Gluing of Smooth Manifolds Along Boundary Components

We now perform a construction similar to theorem 2.2.3 for smooth manifolds, however in order to construct a smooth structure on the resulting manifold we will require additional information. This motivates the following definitions.

Definition 2.4.1. In the definition below, by a *smooth embedding* we mean a smooth injective map which is a diffeomorphism onto its image.

Definition 2.4.2. Let M be a smooth manifold with boundary, a *collar* or *collaring* for ∂M is a function $f\colon \partial M \times [0,1) \to M$ which is a smooth embedding.

Remark 2.4.3. A smooth map $f\colon M \to N$ may be a topological embedding, and yet not be a smooth embedding, e.g. $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$.

We now use the collar in conjunction with theorem 2.3.12 to define a smooth structure on a glued manifold.

Proposition 2.4.4. Let M, M' be n-dimensional smooth manifolds, $N \subseteq \partial M$ and $N' \subseteq \partial M'$ be clopen subsets of the boundaries. Let $f\colon N \to N'$ be a diffeomorphism. Let ψ be a collaring of N and ψ' be a collaring of N'. Let $\tilde{M}$ be as in theorem 2.2.3. Then there is a unique smooth structure on $\tilde{M}$ such that the two obvious inclusions of $M \setminus N$ and , $M' \setminus N'$ in $\tilde{M}$ as well as the map:

$$\langle \psi, \psi' \rangle \colon N \times (-1,1) \to \tilde{M}$$

given by:

$$(p,t) \mapsto \begin{cases} [(\psi(p,-t),0)] \text{ if } t \in (-1,0], \\ [(\psi'(f(p),t),1)] \text{ if } t \in [0,1), \end{cases}$$

are smooth embeddings.

Proof. We define the coordinate neighbourhoods:

$$\begin{aligned} U_1 &:= \{[(p,0)] : p \in M \setminus N\}, \\ U_2 &:= \{[(p,1)] : p \in M' \setminus N'\}, \\ U_3 &:= \operatorname{im}\langle \psi, \psi' \rangle. \end{aligned}$$

Then it is easily verified that the three non-standard charts:

$$(U_1, M, M \setminus N, [(p,0)] \mapsto p),$$
$$(U_2, M', M' \setminus N', [(p,1)] \mapsto p),$$
$$(U_3, N \times (-1,1), N \times (-1,1), \langle \psi, \psi', \rangle^{-1}),$$

form a non-standard smooth atlas for $\tilde{M}$. The non-standard transition maps are restrictions of the collaring ψ to $N \times (0,1)$ and the inverse thereof, the same analogous restriction of the collaring ψ', and the empty map. Therefore by theorem 2.3.12 there is a unique smooth structure on $\tilde{M}$ with the stated property. □

2.5 Collared Cobordisms

In this section we introduce cobordisms between smooth manifolds without boundary, as well as what we mean by a collared cobordism and the collar condition which we will require maps between cobordisms to satisfy. From this point forward we will require our object manifolds (M and N in the following definitions) to be compact and our cobordisms (Σ in the following definitions) to be Hausdorff.

Definition 2.5.1. Let M, N be (compact, Hausdorff) n-dimensional smooth manifolds without boundary, a *smooth cobordism* from M to N is a triple (i, Σ, j) where:

- Σ is an (Hausdorff) $(n+1)$-dimensional smooth manifold.
- $i\colon M \to \Sigma$ is a smooth embedding.
- $j\colon N \to \Sigma$ is a smooth embedding.
- $\mathrm{im}(i) \cup \mathrm{im}(j) = \partial M$.
- $\mathrm{im}(i)$ and $\mathrm{im}(j)$ are disjoint clopen subsets of ∂M.

We now introduce cobordisms which come equipped with collars, which we will use to define a smooth structure after gluing as in proposition 2.4.4

Definition 2.5.2 (Collared Cobordism)**.** Let M, N be n-dimensional smooth manifolds without boundary, a *smooth collared cobordism* from M to N is a triple (i, Σ, j) where:

- Σ is an (Hausdorff) $(n+1)$-dimensional smooth manifold.

- $i\colon M \times [0,1) \to \Sigma$ is a smooth embedding.
- $j\colon N \times (-1,0] \to \Sigma$ is a smooth embedding.
- $i(M \times \{0\}) \cup j(N \times \{0\}) = \partial\Sigma$.
- The closures of $\mathrm{im}(i)$ and $\mathrm{im}(j)$ are disjoint.

Remark 2.5.3. The collars are the reason we require M, N to be compact and Σ to be Hausdorff, in order to ensure that sets of the form $i(M \times [0,\varepsilon])$ and $j(N \times [-\varepsilon, 0])$ for $\varepsilon \in [0,1)$ (which we call *closed subcollars*) are in fact closed subsets of Σ. Indeed this property may fail if either condition is dispensed with.

Remark 2.5.4. The source and target manifolds of a cobordism (collared or uncollared) may be the empty manifold (see remark 2.3.7), in which case the embeddings i and j would be the empty map. Indeed we have the empty cobordism $(\emptyset, \emptyset, \emptyset)$ from the empty manifold to itself.

Definition 2.5.5 (Boundary Maps)**.** Given a smooth collared cobordism $(i, \Sigma, j)\colon M \to N$, the *boundary maps* of i and j are the maps:

$$\partial i\colon M \to \Sigma,$$
$$\partial j\colon N \to \Sigma,$$

defined by:

$$\partial i(p) = i(p, 0),$$
$$\partial j(p) = j(p, 0).$$

Remark 2.5.6. Given a smooth collared cobordism (i, Σ, j), we can take the boundary maps and we have that $(\partial i, \Sigma, \partial j)$ is a smooth (uncollared) cobordism from M to N.

From here on, cobordisms will be collared cobordisms unless stated otherwise.

Definition 2.5.7. Let $(i, \Sigma, j)\colon M \to N$ and $(i', \Sigma', j')\colon M' \to N'$ be smooth collared cobordisms, a *map of smooth collared cobordisms* from (i, Σ, j) to (i', Σ', j') is a smooth function $f\colon \Sigma \to \Sigma'$ such that:

- $f(i(M \times \{0\})) = i'(M' \times \{0\})$,
- $f(j(N \times \{0\})) = j'(N' \times \{0\})$.

Definition 2.5.8. Let $f\colon (i,\Sigma,j) \to (i',\Sigma',j')$ be a map of smooth collared cobordisms. Then the *left induced map* of f is the unique function $f_L\colon M \to M'$ such that for all $p \in M$:

$$f(i(p,0)) = i'(f_L(p),0).$$

Similarly the *right induced map* of f is the unique function $f_R\colon N \to N'$ such that for all $p \in N$:

$$f(j(p,0)) = j'(f_R(p),0).$$

Remark 2.5.9. Using the boundary maps we can write the left and right induced maps explicitly:

$$\begin{aligned} f_L &= \partial i'^{-1} \circ f \circ \partial i, \\ f_R &= \partial j'^{-1} \circ f \circ \partial j. \end{aligned}$$

Definition 2.5.10 (Collar Condition and Collar Compatible)**.** Let $\varepsilon > 0$, we say a map $f\colon (i,\Sigma,j) \to (i',\Sigma',j')$ of smooth collared cobordisms satisfies the *collar condition* with parameter ε if the following diagram commutes:

$$\begin{array}{ccccc} M \times [0,\varepsilon) & \xrightarrow{\;i\;} & \Sigma & \xleftarrow{\;j\;} & N \times (-\varepsilon,0] \\ \Big\downarrow{\scriptstyle f_L \times \mathrm{id}} & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle f_R \times \mathrm{id}} \\ M' \times [0,\varepsilon) & \xrightarrow[\;i'\;]{} & \Sigma' & \xleftarrow{\;j'\;} & N' \times (-\varepsilon,0] \end{array}$$

If there is such an ε then we say that f is *collar compatible.*

The primary purpose of the collar condition is to ensure that the horizontal composition is well-defined. Now we use proposition 2.4.4 to define the composition of smooth collared cobordisms.

Definition 2.5.11. Let $(i,\Sigma,j)\colon L \to M$ and $(i',\Sigma',j')\colon M \to N$ be smooth collared cobordisms. The *collar dependent composition*:

$$(i',\Sigma',j') \bigcirc (i,\Sigma,j)\colon L \to N,$$

is the collared cobordism:

$$(\tilde{i},\tilde{\Sigma},\tilde{j})\colon L \to N,$$

where:

$$\tilde{\Sigma} = \Sigma \sqcup \Sigma' / \sim .$$

Here $\sim$ is the equivalence relation given by:

$$(j(p,0),0) \sim (i'(p,0),1),$$

for all $p \in M$. $\tilde{\Sigma}$ is equipped with the unique smooth structure such that the two inclusions of $\Sigma \setminus j(M \times \{0\})$ and $\Sigma' \setminus i'(M \times \{0\})$ as well as the map $\langle j, i' \rangle$ are smooth embeddings (as in proposition 2.4.4).
The collar map:

$$\tilde{i} \colon L \times [0,1) \to \tilde{\Sigma}$$

is given by:

$$(p,t) \mapsto [(i(p,t),0)],$$

and the collar map:

$$\tilde{j} \colon N \times (-1,0] \to \tilde{\Sigma}$$

is given by:

$$(p,t) \mapsto [(j'(p,t),1)].$$

Remark 2.5.12. Proposition 2.4.4 ensures that the collar dependent composition is well-defined, in that it is a smooth collared cobordism from L to N, and that the collars j, i' do induce a unique smooth structure on $\tilde{\Sigma}$.

Since the smooth structure on $\tilde{\Sigma}$ is one induced by a non-standard smooth atlas we will often use lemma 2.3.16 to demonstrate that a map to or from $\tilde{\Sigma}$ is smooth. Here we prove two lemmas which will simplify this process.

Lemma 2.5.13. Let:

$$(i, \Sigma, j) \colon L \to M,$$
$$(i', \Sigma', j') \colon M \to N,$$

be smooth collared cobordisms. Let $(\tilde{i}, \tilde{\Sigma}, \tilde{j})$ be the collar dependent composition:

$$(i', \Sigma', j') \bigcirc (i, \Sigma, j).$$

Let (X, S_X) be a smooth manifold, then a function

$$f \colon \tilde{\Sigma} \to X$$

is smooth if and only if the three functions:

$$p \mapsto f([(p,0)]), \colon \Sigma \setminus j(M \times \{0\}) \to X,$$
$$p \mapsto f([(p,1)]), \colon \Sigma' \setminus i'(M \times \{0\}) \to X,$$
$$f \circ \langle j, i' \rangle \colon M \times (-1,1) \to X,$$

are smooth in the sense of definition 2.3.8.

Proof. First note that the non-standard atlas B consisting of the single chart $(X, (X, S_X), X, \mathrm{id}_X)$ induces the smooth structure S_X on X. In this context the three stated maps are the non-standard representations of f with respect to the non-standard atlas for $\tilde{\Sigma}$ described in definition 2.5.11 and B, so the result follows from lemma 2.3.16. □

We also prove a similar result for maps into a glued cobordism.

Lemma 2.5.14. Let $\tilde{\Sigma}$ and (X, S_X) be as in lemma 2.5.13 and let $f\colon X \to \tilde{\Sigma}$. Let $\iota, : \iota'$ be the two obvious inclusions of $\Sigma' \setminus j(M \times \{0\})$ and $\Sigma' \setminus i'(M \times \{0\})$ into $\tilde{\Sigma}$ and define U, V, W as follows:

$$\begin{aligned} U &:= f^{-1}(\mathrm{im}(\iota)), \\ V &:= f^{-1}(\mathrm{im}(\iota')), \\ W &:= f^{-1}(\mathrm{im}\langle j, i' \rangle). \end{aligned}$$

Then f is smooth if and only if the three maps:

$$\begin{aligned} \iota^{-1} \circ f &\colon U \to \Sigma, \\ \iota'^{-1} \circ f &\colon V \to \Sigma', \\ \langle j, i' \rangle^{-1} \circ f &\colon W \to M \times (-1, 1), \end{aligned}$$

are smooth in the sense of definition 2.3.8.

Proof. As in lemma 2.5.13 we take the non-standard atlas B for X consisting of the single non-standard chart $(X, (X, S_X), X, \mathrm{id}_X)$ and again we see that the three listed maps are the non-standard local representations of f with respect to the non-standard atlases on X and $\tilde{\Sigma}$. So the result follows from lemma 2.3.16. □

Chapter 3

Pseudo Double Categories

In this chapter we define pseudo double categories. Our end goal will be to define a monoidal pseudo double category of manifold and cobordisms. We take our definitions from [HS19].

3.1 Definition of a Pseudo Double Category

Definition 3.1.1. A *pseudo double category* $\mathbb{D}$ consists of two categories $\mathbf{D}_0$ and $\mathbf{D}_1$, together with functors:

- $S, T \colon \mathbf{D}_1 \to \mathbf{D}_0$.
- $U \colon \mathbf{D}_0 \to \mathbf{D}_1$.

The objects and morphisms of $\mathbf{D}_0$ are called the objects and tight 1-morphisms of $\mathbb{D}$, respectively. The objects of $\mathbf{D}_1$ are called the loose 1-morphisms of $\mathbb{D}$ and the morphisms of $\mathbf{D}_1$ are called the 2-morphisms of $\mathbb{D}$. The functors S, T are called the *source* and *target* functors respectively. The functor U is called the *unit functor*. S, T, U are collectively referred to as *structure functors* of $\mathbb{D}$. The structure functors are required to satisfy:

$$S \circ U = T \circ U = \mathrm{id}_{\mathbf{D}_0}.$$

Given objects X, Y of $\mathbf{D}_0$ (objects of $\mathbb{D}$) we will often refer to an object A of $\mathbf{D}_1$, with $S(A) = X, T(A) = Y$, as a loose 1-morphism from X to Y. With the functors S, T we define the pullback category $\mathbf{D}_1 \times_{\mathbf{D}_0} \mathbf{D}_1$ as having objects pairs (X, Y) of objects of $\mathbf{D}_1$ such that $S(Y) = T(X)$, and as morphisms $(X, Y) \to (X', Y')$ pairs $(f \colon X \to X', g \colon Y \to Y')$ of morphisms of $\mathbf{D}_1$ such that $S(g) = T(f)$. From this category we have a functor:

$$\bigcirc \colon \mathbf{D}_1 \times_{\mathbf{D}_0} \mathbf{D}_1 \to \mathbf{D}_1$$

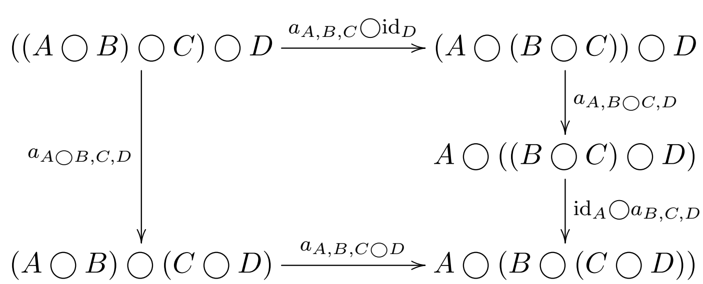
called *loose* or *horizontal composition.* For all loose 1 morphisms $A\colon X \to Y$ and $B\colon Y \to Z$ we must have:

$$S(B \bigcirc A) = S(A),$$
$$T(B \bigcirc A) = T(B).$$

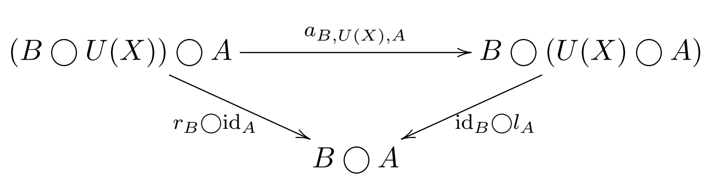
Finally we have natural isomorphisms (given here by their components):

- $a_{A,B,C}\colon (A \bigcirc B) \bigcirc C \to A \bigcirc (B \bigcirc C)$,
- $l_A\colon U(T(A)) \bigcirc A \to A$,
- $r_A\colon A \bigcirc U(S(A)) \to A$.

a is called the *associator morphism* and l, r are called the *left* and *right unitor morphisms* respectively. Collectively a, l and r are called the *structure morphisms* of $\mathbb{D}$. The structure functors and structure morphisms must satisfy the condition that $S(a), T(a), S(l), T(l), S(r)$ and $T(r)$ are all identities. The associator is required to satisfy a pentagon axiom similar to that of a monoidal category. Explicitly for all objects A, B, C, D of $\mathbf{D}_1$ (with the appropriate sources and targets) the diagram:



commutes. Similarly the associator and unitors must satisfy a triangle axiom, specifically for all objects X, Y, Z of $\mathbf{D}_0$ and loose 1-morphisms $A\colon X \to Y$ and $B\colon Y \to Z$ we must have that the diagram:



commutes. A pseudo double category whose structure morphisms are identities is a *(strict) double category.*

For a loose 1-morphism A such that $S(A) = X$ and $T(A) = Y$ we will sometimes write $A\colon X \to Y$ provided it is clear that A is a loose 1-morphism, since we use the same notation for tight 1-morphisms. When confusion may arise we will use $A\colon X \nrightarrow Y$ for loose 1-morphisms. We will similarly employ $f\colon A \Rightarrow B$ for 2-morphisms in cases where confusion may arise.

Remark 3.1.2. The axioms for a pseudo double category are similar to the axioms of a bicategory (see for example [Lei98], or [JY20]), and both have a resemblance to the axioms of a monoidal category. Indeed in the same way that a monoidal category can be though of as a bicategory with only one object, a monoidal category can also be thought of as a pseudo double category with only one object and whose only tight 1-morphism is the identity of that object. Just as with monoidal categories, pseudo double categories can be 'strictified', in that every pseudo double category is equivalent to a strict double category (for a proof, see [GP99], section 7.5). There is a way in which a pseudo double category 'contains' a bicategory, as we will now see.

Definition 3.1.3. Let $\mathbb{D}$ be a pseudo double category, a 2-morphism

$$f\colon A \to B$$

is called *globular* if $S(f)$ and $T(f)$ are both identities (in particular, this means that $S(A) = S(B)$ and $T(A) = T(B)$). The *loose bicategory* $L(\mathbb{D})$ of $\mathbb{D}$ is the bicategory defined as follows:

- The objects of $L(\mathbb{D})$ are the objects of $\mathbb{D}$.
- Given objects X, Y of $L(\mathbb{D})$, the category $L(\mathbb{D})(X, Y)$ has as objects loose 1-morphisms A of $\mathbb{D}$ such that $S(A) = X, T(A) = Y$.
- Morphisms $f\colon A \to B$ of $L(\mathbb{D})(X, Y)$ are globular 2-morphisms of $\mathbb{D}$, with composition the tight composition of 2-morphisms in $\mathbb{D}$.
- Loose or 'horizontal' composition in $L(\mathbb{D})$ is done using the loose composition of $\mathbb{D}$.
- The structure morphisms (associators and unitors for loose composition) are also inherited from $\mathbb{D}$.

3.2 Double Functors and Monoidal Double Categories

In this section we define a double functor, a double natural transformation, and a monoidal pseudo-double category.

Definition 3.2.1. Let $\mathbb{D}, \mathbb{D}'$ be pseudo-double categories. A *double functor* $F\colon \mathbb{D} \to \mathbb{D}'$ consists of functors $F_0\colon \mathbf{D}_0 \to \mathbf{D'}_0$ and $F_1\colon \mathbf{D}_1 \to \mathbf{D'}_1$ such that:

$$S' \circ F_1 = F_0 \circ S,$$
$$T' \circ F_1 = F_0 \circ T.$$

Together with natural transformations:

$$F^{\bigcirc}_{A,B}\colon F_1(A) \bigcirc F_1(B) \to F_1(A \bigcirc B),$$

and:

$$F^U_A\colon U(F_0(A)) \to F_0(U(A)),$$

whose components are globular isomorphisms, and such that the diagram:

$$\begin{array}{ccc}
(F_1(A) \bigcirc F_1(B)) \bigcirc F_1(C) & \xrightarrow{a'_{F_1(A),F_1(B),F_1(C)}} & F_1(A) \bigcirc (F_1(B) \bigcirc F_1(C)) \\
{\scriptstyle F^{\bigcirc}_{A,B} \bigcirc \mathrm{id}_{F_1(C)}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{F_1(A)} \bigcirc F^{\bigcirc}_{B,C}} \\
F_1(A \bigcirc B) \bigcirc F_1(C) & & F_1(A) \bigcirc F_1(B \bigcirc C) \\
{\scriptstyle F^{\bigcirc}_{A \bigcirc B,C}} \downarrow & & \downarrow {\scriptstyle F^{\bigcirc}_{A,B \bigcirc C}} \\
F_1((A \bigcirc B) \bigcirc C) & \xrightarrow{F_1(a_{A,B,C})} & F_1(A \bigcirc (B \bigcirc C))
\end{array}$$

commutes, and for all loose 1-morphisms $A\colon X \to Y$ of $\mathbb{D}$ the diagrams:

$$\begin{array}{ccc}
F_1(A) \bigcirc U'(F_0(X)) & \xrightarrow{r'_{F_1(A)}} & F_1(A) \\
{\scriptstyle \mathrm{id}_{F_1(A)} \bigcirc F^U_X} \downarrow & & \uparrow {\scriptstyle F_1(r_A)} \\
F_1(A) \bigcirc F_1(U(X)) & \xrightarrow{F^{\bigcirc}_{A,U(X)}} & F_1(A \bigcirc U(X))
\end{array}
\qquad
\begin{array}{ccc}
U'(F_0(Y)) \bigcirc F_1(A) & \xrightarrow{l'_{F_1(A)}} & F_1(A) \\
{\scriptstyle F^U_Y \bigcirc \mathrm{id}_{F_1(A)}} \downarrow & & \uparrow {\scriptstyle F_1(l_A)} \\
F_1(U(Y)) \bigcirc F_1(A) & \xrightarrow{F^{\bigcirc}_{U(Y),A}} & F_1(U(Y) \bigcirc A)
\end{array}$$

commute.

We also have a notion of natural transformations between two double functors.

Definition 3.2.2. Given double functors $F, G\colon \mathbb{D} \to \mathbb{D}'$ a *(tight) transformation* η from F to G consists of two natural transformations:

$$\eta^0\colon F_0 \to G_0,$$
$$\eta^1\colon F_1 \to G_1.$$

Such that for all loose 1-morphisms A of $\mathbb{D}$ we have:

$$S'(\eta^1_A) = \eta^0_{S(A)},$$
$$T'(\eta^1_A) = \eta^0_{T(A)},$$

and such that for all loose 1-morphisms $A\colon X \to Y$ and $B\colon Y \to Z$ in $\mathbb{D}$ the following diagram commutes:

$$\begin{array}{ccc}
F_1(B) \bigcirc F_1(A) & \xrightarrow{F^{\bigcirc}_{B,A}} & F_1(B \bigcirc A) \\
\downarrow{\scriptstyle \eta^1_B \bigcirc \eta^1_A} & & \downarrow{\scriptstyle \eta^1_{B \bigcirc A}} \\
G_1(B) \bigcirc G_1(A) & \xrightarrow[G^{\bigcirc}_{B,A}]{} & G_1(B \bigcirc A)
\end{array}$$

The following diagram also commutes for all objects X of $\mathbb{D}$:

$$\begin{array}{ccc}
U'(F_0(X)) & \xrightarrow{F^U_X} & F_1(U(X)) \\
\downarrow{\scriptstyle U'(\eta^0_X)} & & \downarrow{\scriptstyle \eta^1_{U(X)}} \\
U'(G_0(X)) & \xrightarrow{G^U_X} & G_1(U(X))
\end{array}$$

We say that the transformation η is an isomorphism if each of its constituent natural transformations η^0 and η^1 are natural isomorphisms (meaning each of their components are isomorphisms in $\mathbf{D}'_0$ and $\mathbf{D}'_1$ respectively).

We are now able to define a monoidal pseudo double category.

Definition 3.2.3. A *monoidal psuedo double category* is a pseudo double category $\mathbb{D}$ together with the following:

- A double functor $\otimes\colon \mathbb{D} \times \mathbb{D} \to \mathbb{D}$.
- A specified object I_0 and loose 1-morphism $I_1\colon I_0 \nrightarrow I_0$.

As well as natural (double) isomorphisms (given as usual by their components):

- $\alpha_{A,B,C}\colon (A \otimes B) \otimes C \to A \otimes (B \otimes C)$.
- $\lambda_A\colon I \otimes A \to A$.
- $\rho_A\colon A \otimes I \to A$.

Note that since these are double transformations we have two sets of natural isomorphisms here, one in $\mathbf{D}_0$ and one in $\mathbf{D}_1$. The associativity and unit constraints are required to satisfy the usual pentagon and triangle axioms for monoidal categories.

Chapter 4

The (Strict) Double Category of Tuples and Permutations

In this chapter we construct a strict double category from tuples and permutations.

4.1 The Categories of Permutations and Nested Tuples

We begin by defining our category of objects $\mathbf{D}_0$.

Definition 4.1.1. The category of objects is the category of permutations, that is, objects are non-negative integers, and a tight 1-morphism from m to m is a permutation $\sigma \in S_m$ (there are no tight 1-morphisms from m to n if $m \neq n$).

Remark 4.1.2. The group S_0 is the group of permutations of the set $\{n \in \mathbb{Z}^+, n \leq 0\}$, since this set is empty it contains only the empty permutation (the empty map, from the empty set to itself). Hence S_0 is the trivial group.

For the category of morphisms, we define our loose 1-morphisms and 2-morphisms separately.

Definition 4.1.3. Given non-negative integers m, n, a loose 1-morphism from m to n is an m-tuple:

$$t = (t_1, \ldots, t_m) = ((t_{1,1}, t_{1,2}, \ldots, t_{1,k_1}), (t_{2,1}, \ldots, t_{2,k_2}), \ldots, (t_{m,1}, \ldots, t_{m,k_m}))$$

of tuples containing every element of $\{1, \ldots, n\}$ exactly once each. That is, for every $p \in \{1, \ldots, n\}$ there exists a unique $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, k_i\}$ such that $t_{i,j} = p$. The t_i are called the *inner tuples* of t (both inner and outer tuples are permitted to be empty, and the empty tuple $()$ is a loose 1-morphism from 0 to 0).

Definition 4.1.4. Given loose 1-morphisms $t, t' \colon m \nrightarrow n$, a 2-morphism $\eta \colon t \Rightarrow t'$ consists of a permutation $\sigma \in S_m$, called the *outer permutation* of η, such that for all $i \in \{1, \ldots, m\}$:

$$k'_{\sigma(i)} = k_i.$$

That is, the $\sigma(i)^{\text{th}}$ inner tuple of t' is the same length as the i^{th} inner tuple of t. Together with for each $i \in \{1, \ldots, m\}$ a permutation $\sigma_i \in S_{k_i}$, called the i^{th} *inner permutation* of η. We will usually write 2-morphisms as:

$$\eta = (\sigma, (\sigma_1, \ldots, \sigma_m)) \colon t \Rightarrow t'.$$

Examples 4.1.5. Consider the following two loose 1-morphisms from 4 to 8:

$$\begin{aligned} t &:= ((2,6,3), (), (4,1), (7,5,8)), \\ t' &:= ((1,2), (4,7,3), (), (6,8,5)). \end{aligned}$$

An example of a 2-morphism $t \Rightarrow t'$ would be:

$$\eta := \Big((1,4,2,3), \big((1,2)(3), (), (1,2), (3,2,1)\big)\Big).$$

That is, the outer permutation of η is $(1,4,2,3) \in S_4$, and the inner permutations of η are $(1,2)(3) \in S_3$, $() \in S_0$ (the empty permutation of zero elements), $(1,2) \in S_2$ and $(3,2,1) \in S_3$.

Note that since all our tight 1-morphisms are endomorphisms (in fact, automorphisms), there are no 2-morphisms between loose 1-morphisms whose source and target do not match (we have not, strictly speaking, defined our source and target functors yet, though their action on loose 1-morphisms is implicit in definition 4.1.3).

We must also define tight composition of 2-morphisms, which will complete our definition of the category of morphisms.

Definition 4.1.6. Let $t, t', t'' \colon m \nrightarrow n$ be loose 1-morphisms and:

$$\eta = (\sigma, (\sigma_1, \ldots, \sigma_m)) \colon t \Rightarrow t',$$
$$\eta' = (\sigma', (\sigma'_1, \ldots, \sigma'_m)) \colon t' \Rightarrow t'',$$

be 2-morphisms, then their *tight composition* is given by:

$$\eta' \circ \eta := (\sigma' \circ \sigma, (\sigma'_{\sigma(1)} \circ \sigma_1, \ldots, \sigma'_{\sigma(m)} \circ \sigma_m)).$$

That is, the outer permutations are composed, and the i^{th} inner permutation of $\eta' \circ \eta$ is the composition of the $\sigma(i)^{\text{th}}$ inner permutation of η' with the i^{th} inner permutation of η.

We must show that tight composition is associative and has identities (that is, we must show that $\mathbf{D}_1$ is a category).

Proposition 4.1.7. Nested tuples defined in definition 4.1.3 and nested permutations in definition 4.1.4 form a category.

Proof. We will prove associativity of our tight composition first. Let $t, t', t'', t''' \colon m \nrightarrow n$ be loose 1-morphisms and:

$$\eta \colon t \Rightarrow t',$$
$$\eta' \colon t' \Rightarrow t'',$$
$$\eta'' \colon t'' \Rightarrow t''',$$

be 2-morphisms. Then:

$$(\eta'' \circ \eta') \circ \eta = \eta'' \circ (\eta' \circ \eta).$$

That the outer permutations match is trivial. For $i \in \{1, \ldots, m\}$ the i^{th} inner permutation of the left hand side is given by:

$$\begin{aligned}(\eta'' \circ \eta')_{\sigma(i)} \circ \sigma_i &= (\sigma''_{\sigma'(\sigma(i))} \circ \sigma'_{\sigma(i)}) \circ \sigma_i \\ &= \sigma''_{(\sigma' \circ \sigma)(i)} \circ (\sigma'_{\sigma(i)} \circ \sigma_i).\end{aligned}$$

Which is the i^{th} inner permutation of the right hand side (note that $\sigma'_{\sigma(i)} \circ \sigma_i$ is the i^{th} inner permutation of $\eta' \circ \eta$), this completes the proof of associativity. The identity 2-morphism on a loose 1-morphism t has as outer permutation the identity permutation in S_m and as its i^{th} inner permutation the identity permutation in S_{k_i}. That this is an identity is clear from the definition of tight composition. Hence $\mathbf{D}_1$ is a category. □

4.2 The Structure Functors

In this section we define the structure functors:

$$S, T\colon \mathbf{D}_1 \to \mathbf{D}_0$$

and:

$$U\colon \mathbf{D}_0 \to \mathbf{D}_1.$$

To facilitate the definition of the target functor, as well as loose composition, we attach labels to our 2-morphisms indicating their source and target. Explicitly a 2-morphism is strictly speaking a triple (t, η, s) where t and s are loose 1-morphisms and $\eta\colon t \to s$ is a pair $(\sigma, (\sigma_1, \ldots, \sigma_m))$ as described above. Though we will usually omit these labels for brevity.

The source functor is straightforward.

Definition 4.2.1. Given a loose 1-morphism $t = (t_1, \ldots, t_m)$, the *source functor* maps t to the non-negative integer m. The source functor maps a 2-morphism $(\sigma, (\sigma_1, \ldots, \sigma_m))$ to its outer permutation σ.

The target functor is slightly less strightforward.

Definition 4.2.2. Given a loose 1-morphism:

$$t = (t_1, \ldots, t_m)\colon m \not\to n,$$

the *target functor* maps t to n. Given a 2-morphism:

$$(\sigma, (\sigma_1, \ldots, \sigma_m))\colon t \Rightarrow s,$$

the permutation $T(\eta) \in S_n$ is defined by:

$$T(\eta)(t_{i,j}) = s_{\sigma(i),\sigma_i(j)}.$$

That is, $T(\eta)$ maps the j^{th} entry of the i^{th} tuple of t to the $\sigma_i(j)^{\text{th}}$ entry of the $\sigma(i)^{\text{th}}$ tuple of s.

Remarks 4.2.3. The definition of the target functor is one of the reasons that we labelled our 2-morphisms with their sources and targets (in $\mathbf{D}_1$). That the permutation $T(\eta)$ is well-defined follows from the fact that for all i the $\sigma(i)^{\text{th}}$ tuple of s is the same length as the i^{th} tuple of t, as well as the fact that the tuples of t and the tuples of s both contain the integers $\{1, \ldots, n\}$ exactly once each.

Examples 4.2.4. Consider the following loose 1-morphisms from 3 to 5:

$$t = ((2,5,3),(),(1,4)),$$
$$s = ((4,5),(1,3,2),()).$$

We have only one choice for the outer permutation of a 2-morphism $t \to s$, it must be that $1 \mapsto 2$, $2 \mapsto 3$ and $3 \mapsto 1$. That is, the outer permutation must be $\sigma = (1,2,3)$. For our inner permutations, first for $\sigma_1 \in S_3$ we take $(1,3,2)$, σ_2 must be the empty permutation, and we take $\sigma_3 = (1,2)$. Hence our 2-morphism is the pair:

$$\eta = (\sigma, (\sigma_1, \ldots, \sigma_3)) = ((1,2,3), ((1,3,2), (), (1,2))).$$

The source functor maps η to the permutation $\sigma = (1,2,3) \in S_3$.

The permutation $T(\eta) \in S_5$ maps $t_{i,j}$ to $s_{\sigma(i),\sigma_i(j)}$ for each i, j. Since $\sigma(1) = 2$ we read the images of 2,5 and 3 from $s_{\sigma(1)} = s_2$. For which entry of s_2 to map to we use σ_1, That is we have:

$$2 = t_{1,1} \mapsto s_{2,\sigma_1(1)} = s_{2,3} = 2,$$
$$5 = t_{1,2} \mapsto s_{2,\sigma_1(2)} = s_{2,1} = 1,$$
$$3 = t_{1,3} \mapsto s_{2,\sigma_1(3)} = s_{2,2} = 3.$$

Similarly for the images of 1 and 4 we observe that $\sigma(3) = 1$ and hence we read off the images from s_1, in the order specified by σ_3. Thus we obtain:

$$1 = t_{3,1} \mapsto s_{1,\sigma_1(1)} = s_{1,2} = 5,$$
$$4 = t_{3,2} \mapsto s_{1,\sigma_1(2)} = s_{1,1} = 4.$$

So we obtain the permutation $T(\eta) = (1,5)(2)(3)(4) \in S_5$.

Finally we define the unit functor.

Definition 4.2.5. The *unit functor* $U \colon \mathbf{D}_0 \to \mathbf{D}_1$ takes the non-negative integer n to the tuple $((1),(2),\ldots,(n))$ and a permutation $\sigma \in S_n$ to the pair $(\sigma, (\mathrm{id}, \ldots, \mathrm{id}))$. That is, the outer permutation is σ and the inner permutations are all identities in their respective symmetric groups.

We will show that the structure functors are functors.

Proposition 4.2.6. The source functor is a functor.

Proof. This is clear from the definition, since $S(\eta)$ is the outer permutation of η, and (tight) composition of 2-morphisms composes outer tuples (Note that the identity 2-morphism of a tuple t has outer permutations and all inner permutations as identities). $\square$

Proposition 4.2.7. The target functor is a functor.

Proof. To show that T is a functor let $t, t', t'' \colon m \to n$ be loose 1-morphisms and let:

$$\begin{aligned} \eta &\colon t \to t', \\ \eta' &\colon t' \to t'', \end{aligned}$$

be 2-morphisms, then $\eta' \circ \eta = (\hat{\sigma}, (\hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_m))$. Where $\hat{\sigma} = \sigma' \circ \sigma$ and $\hat{\sigma}_i = \sigma'_{\sigma(i)} \circ \sigma_i$. Now evaluating the target functor we obtain, for $i \in \{1, \ldots, m\}, j \in \{1, \ldots, l_i\}$ (where l_i is the length of the i^{th} inner tuple of t):

$$\begin{aligned} T(\eta' \circ \eta)(t_{i,j}) &= t''_{\hat{\sigma}(i), \hat{\sigma}_i(j)} \\ &= t''_{(\sigma' \circ \sigma)(i), (\sigma'_{\sigma(i)} \circ \sigma_i)(j)} \\ &= t''_{\sigma'(\sigma(i)), \sigma'_{\sigma(i)}(\sigma_i(j))} \\ &= T(\eta')(t'_{\sigma(i), \sigma_i(j)}) \\ &= T(\eta')(T(\eta)(t_{i,j})) \\ &= (T(\eta') \circ T(\eta))(t_{i,j}). \end{aligned}$$

So we see that $T(\eta' \circ \eta) = T(\eta') \circ T(\eta)$. That T preserves identities is clear and so T is a functor. $\square$

Remark 4.2.8. It is interesting to note at this point that the target functor is faithful. Given two loose 1-morphisms $t, t' \colon m \to n$ and a permutation $\sigma \in S_n$ there is at most one 2-morphism η such that $T(\eta) = \sigma$. The target functor is not full however, since $T(\eta)$ can only map an element of t_i to an element of $s_{\sigma(i)}$ and hence the images of each element of $\{1, \ldots, n\}$ are not fully independent. We could have characterised our 2-morphisms by their images under the source and target maps, that is, defined a 2-morphism to be a pair $(\sigma_L \in S_m, \sigma_R \in S_n)$ subject to the appropriate conditions. However these conditions would have made it difficult to verify that both tight and loose composition were well-defined, and the format we have used here leads more naturally to the generalisation we will discuss in a later section.

Finally the unit functor.

Proposition 4.2.9. The unit functor is a functor.

Proof. To see that U respects composition let $\sigma, \sigma' \in S_n$, then we have:

$$\begin{aligned} U(\sigma' \circ \sigma) &= (\sigma' \circ \sigma, (\mathrm{id}, \dots, \mathrm{id})) \\ &= (\sigma', (\mathrm{id}, \dots, \mathrm{id})) \circ (\sigma, (\mathrm{id}, \dots, \mathrm{id})) \\ &= U(\sigma') \circ U(\sigma). \end{aligned}$$

That U preserves identities is clear and so U is a functor. □

We must show that the source and target functor are compatible with the unit functor in the required way.

Proposition 4.2.10. $S \circ U$ and $T \circ U$ are both the identity functor on $\mathbf{D}_0$.

Proof. That this is true for objects is clear, for morphisms let $\sigma \in s_n$, then from the definitions we have $U(\sigma) = (\sigma, (\mathrm{id}, \dots, \mathrm{id}))$, and so $S(U(\sigma)) = \sigma$. For the target functor we have that for $k \in \{1, \dots, n\}$, $k = U(n)_{k,1}$, that is, k is the first (and only) entry in the k^{th} inner tuple of $U(n)$. Hence as the inner permutations of $U(\sigma)$ are all identities we have that:

$$\begin{aligned} T(U(\sigma))(k) &= T(U(\sigma))(U(n)_{k,1}) \\ &= U(n)_{\sigma(k),1} \\ &= \sigma(k). \end{aligned}$$

This completes the proof. □

4.3 Loose Composition

Having defined our structure functors S, T we can define our loose composition functor:

$$\bigcirc \colon \mathbf{D}_1 \times_{\mathbf{D}_0} \mathbf{D}_1 \to \mathbf{D}_1.$$

Definition 4.3.1. The *loose composition* is defined on objects as follows. Given loose 1-morphisms $t \colon m \to n$ and $s \colon n \to p$, for $i \in \{1, \dots, m\}$ the i^{th} tuple of $s \bigcirc t$ is the concatenation of the $(t_{i,j})^{\text{th}}$ tuples of s for $j \in \{1, \dots, l_i\}$.

Example 4.3.2. For example if we have the loose 1-morphisms:

$$t = ((2,5,1),(),(4,3))\colon 3 \to 5,$$
$$s = ((6,1,4),(),(2,5),(3),(8,7))\colon 5 \to 8.$$

Then the first tuple of $s \bigcirc t$ would be the concatenation of the 2^{nd}, 5^{th} and 1^{st} tuples of s, so $(s \bigcirc t)_1 = (8,7,6,1,4)$ (note that the 2^{nd} tuple of s is the empty tuple). In full:

$$s \bigcirc t = ((8,7,6,1,4),(),(3,2,5))\colon 3 \to 8.$$

Note that the second tuple of t is empty, and so the second tuple of $s \bigcirc t$ is also empty.

To define loose composition of 2-morphisms we first introduce two operations on symmetric groups, direct sum and block permutation. We take these definitions (adapted slightly) from [Fre17].

Definition 4.3.3. Given permutations $\sigma \in S_m$ and $\sigma' \in S_n$, their *direct sum* is the permutation $\sigma \oplus \sigma' \in S_{m+n}$ given (in matrix notation) by:

$$\begin{pmatrix} 1 & 2 & \dots, & m & m+1 & \dots, & m+n \\ \sigma(1) & \sigma(2) & \dots, & \sigma(m) & m+\sigma'(1) & \dots, & m+\sigma'(n) \end{pmatrix}$$

This is easily verified to be an associative operation, which allows us to define the direct sum of any (finite) number of permutations in the obvious way.

Definition 4.3.4. Let $n_1, \dots, n_m$ be non-negative integers and $\sigma_1 \in S_{n_1}, \dots, \sigma_m \in S_{n_m}$ be permutations, then the *direct sum*:

$$\bigoplus_{i=1}^{m} \sigma_i,$$

is defined by repeated application of the binary direct sum operation.

We also define block permutation.

Definition 4.3.5. Given a permutation $\sigma \in S_m$ and non-negative integers $n_1, \dots, n_m$, the *block permutation* of $n_1 \dots, n_m$ by σ is the permutation $\sigma_*(n_1 \dots, n_m) \in S_{n_1+\dots,+n_m}$ given by permuting the intervals $(1 \dots, n_1), (n_1+1 \dots, n_1+n_2) \dots, (n_1+\dots, n_{m-1}+1 \dots, n_1+\dots,+n_m)$ according to σ. Formulaically, for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n_i\}$ we define:

$$\sigma_*(n_1, \dots, n_m)\Big(\sum_{k=1}^{i-1}(n_k) + j\Big) = \sum_{k=1}^{\sigma(i)-1} n_{\sigma^{-1}(k)} + j.$$

Examples 4.3.6. To give an example of block permutation. In the case where $m = 3$ and σ is the cyclic permutation $(1, 3, 2)$ the block permutation $\sigma_*(n_1, n_2, n_3)$ is given (by listing images) by $n_1+1, n_1+2, \ldots, n_1+n_2, n_1+n_2+1 \ldots, n_1+n_2+n_3, 1, \ldots, n_1$. That is, since $\sigma(1) = 3$ the interval $(1, \ldots, n_1)$ is written third, since $\sigma(2) = 1$ the interval $(n_1+1, \ldots, n_1+n_2)$ is written first, and since $\sigma(3) = 2$ the interval $(n_1+n_2+1, \ldots, n_1+n_2+n_3)$ is written second.

Block permutation is compatible with composition in the following way.

Lemma 4.3.7. Let $\sigma \in S_m$ and $n_1, \ldots, n_m$ be non-negative integers. Similarly let $\sigma' \in S_m$ and $n'_1, \ldots, n'_m$ be non-negative integers such that $n'_{\sigma(i)} = n_i$ for all $i \in \{1, \ldots, m\}$. Then we have:

$$(\sigma' \circ \sigma)_*(n_1, \ldots, n_m) = \sigma'_*(n'_1, \ldots, n'_m) \circ \sigma_*(n_1, \ldots, n_m).$$

Proof. To see that the maps are equal let $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n_i\}$, then evaluating the left hand side we have:

$$\begin{aligned}
(\sigma' \circ \sigma)_*(n_1, \ldots, n_m)(\sum_{k=1}^{i-1}(n_k) + j) &= \sum_{k=1}^{\sigma'\circ\sigma(i)-1} (n_{(\sigma'\circ\sigma)^{-1}(k)}) + j \\
&= \sum_{k=1}^{\sigma'\circ\sigma(i)-1} (n_{\sigma^{-1}\circ\sigma'^{-1}(k)}) + j \\
&= \sum_{k=1}^{\sigma'(\sigma(i))-1} (n'_{\sigma'^{-1}(k)}) + j \\
&= \sigma'_*(n'_1, \ldots, n'_m)\big(\sum_{k=1}^{\sigma(i)-1} (n'_k) + j\big) \\
&= \sigma'_*(n'_1, \ldots, n'_m)\big(\sum_{k=1}^{\sigma(i)-1} (n_{\sigma^{-1}(k)}) + j\big) \\
&= \sigma'_*(n'_1, \ldots, n'_m)\big(\sigma_*(n_1, \ldots, n_m)(\sum_{k=1}^{i-1}(n_k) + j)\big).
\end{aligned}$$

Where in the third line we use the fact that:

$$n_{\sigma^{-1}(\sigma'^{-1}(k))} = n'_{\sigma(\sigma^{-1}(\sigma'^{-1}(k)))} = n'_{\sigma'^{-1}(k)}.$$

Therefore the two permutations are equal. □

Block permutation and direct sum satisfy a commutativity-esque property.

Lemma 4.3.8. Let $\sigma_1 \in S_{n_1}, \ldots, \sigma_m \in S_{n_m}$ and let $\sigma' \in S_m$, then we have:

$$\sigma'_*(n_1, \ldots, n_m) \circ \bigoplus_{i=1}^{m} \sigma_i = (\bigoplus_{i=1}^{m} \sigma_{\sigma'^{-1}(i)}) \circ \sigma'_*(n_1, \ldots, n_m).$$

Proof. To see that the two maps are equal we evaluate the images of $n_1 + \ldots, +n_{k-1} + j$, the j^{th} member of the k^{th} interval (note that the first sum is empty in the case $k = 1$). For the left hand side we have:

$$\begin{aligned}\big(\sigma'_*(n_1, \ldots, n_m) \circ \bigoplus_{i=1}^{m} \sigma_i\big)(n_1 + \ldots, +n_{k-1} + j) &= \sigma'_*(n_1, \ldots, n_m)(n_1 + \ldots, +n_{k-1} + \sigma_k(j)) \\ &= \sum_{i=1}^{\sigma'(k)-1} (n_{\sigma'^{-1}(i)}) + \sigma_k(j).\end{aligned}$$

Similarly for the right hand side we have:

$$\begin{aligned}&\big((\bigoplus_{i=1}^{m} \sigma_{\sigma'^{-1}(i)}) \circ \sigma'_*(n_1, \ldots, n_m)\big)(n_1 + \ldots, +n_{k-1} + j) \\ =& \big(\bigoplus_{i=1}^{m} \sigma_{\sigma'^{-1}(i)}\big)\big(\sum_{l=1}^{\sigma'(k)-1} n_{\sigma'^{-1}(l)} + j\big) \\ =& \sum_{i=1}^{\sigma'(k)-1} (n_{\sigma'^{-1}(i)}) + \sigma_{\sigma'^{-1}(\sigma'(k))}(j) \\ =& \sum_{i=1}^{\sigma'(k)-1} (n_{\sigma'^{-1}(i)}) + \sigma_k(j).\end{aligned}$$

So we see that the maps are equal. $\square$

We now define loose composition for 2-morphisms.

Definition 4.3.9. Let:

$$\begin{aligned}((t_{1,1}, \ldots, t_{1,k_1}), \ldots, (t_{m,n}, \ldots, t_{m,k_m}))&\colon m \nrightarrow n, \\ ((t'_{1,1}, \ldots, t'_{1,k'_1}), \ldots, (t'_{m,n}, \ldots, t'_{m,k'_m}))&\colon m \nrightarrow n, \\ ((s_{1,1}, \ldots, s_{1,l_1}), \ldots, (s_{n,1}, \ldots, s_{n,l_n}))&\colon n \nrightarrow p, \\ ((s'_{1,1}, \ldots, s'_{1,l'_1}), \ldots, (s'_{n,1}, \ldots, s'_{n,l'_n}))&\colon n \nrightarrow p,\end{aligned}$$

be loose 1-morphisms and let:

$$\eta = (\sigma, (\sigma_1, \ldots, \sigma_m)) \colon t \Rightarrow t',$$
$$\eta' = (\sigma', (\sigma'_1, \ldots, \sigma'_n)) \colon s \Rightarrow s',$$

be 2-morphisms such that $S(\eta') = T(\eta)$. Then the *loose composition* $\eta' \bigcirc \eta$ is the 2-morphism $(\hat{\sigma}, (\hat{\sigma}_1, \ldots, \hat{\sigma}_m))$ where $\hat{\sigma} = \sigma$ and for each $i \in \{1, \ldots, m\}$ the permutation $\hat{\sigma}_i$ is defined by:

$$\hat{\sigma}_i = \sigma_{i,*}(l_{t_{i,1}}, \ldots, l_{t_{i,k_i}}) \circ \bigoplus_{j=1}^{k_i} \sigma'_{t_{i,j}}.$$

That is, we take direct sum of the $(t_{i,j})^{\text{th}}$ inner permutations of η', and compose with the block permutation of $(l_{t_{i,1}}, \ldots, l_{t_{i,l_i}})$ by the i^{th} inner permutation of η.

Remark 4.3.10. By lemma 4.3.8 the loose composition of two 2-morphisms is also equal to:

$$\bigoplus_{j=1}^{k_i} \sigma'_{\sigma_i^{-1}(t_{i,j})} \circ \sigma_{i,*}(l_{t_{i,1}}, \ldots, l_{t_{i,k_i}}).$$

We prove that the loose composition is a functor.

Proposition 4.3.11. Let:

$$((t_{1,1}, \ldots, t_{1,k_1}), \ldots, (t_{m,1}, \ldots, t_{m,k_m})) \colon m \nrightarrow n,$$
$$((t'_{1,1}, \ldots, t'_{1,k'_1}), \ldots, (t'_{m,1}, \ldots, t'_{m,k'_m})) \colon m \nrightarrow n,$$
$$((t''_{1,1}, \ldots, t''_{1,k''_1}), \ldots, (t''_{m,1}, \ldots, t''_{m,k''_m})) \colon m \nrightarrow n,$$

and:

$$((s_{1,1}, \ldots, s_{1,l_1}), \ldots, (s_{n,1}, \ldots, s_{n,l_n})) \colon n \nrightarrow p,$$
$$((s'_{1,1}, \ldots, s'_{1,l'_1}), \ldots, (s'_{n,1}, \ldots, s'_{n,l'_n})) \colon n \nrightarrow p,$$
$$((s''_{1,1}, \ldots, s''_{1,l''_1}), \ldots, (s''_{n,1}, \ldots, s''_{n,l''_n})) \colon n \nrightarrow p,$$

be loose 1-morphisms and let:

$$\eta = (\sigma, (\sigma_1, \ldots, \sigma_m)) \colon t \Rightarrow t',$$
$$\eta' = (\sigma', (\sigma'_1, \ldots, \sigma'_m)) \colon t \Rightarrow t'',$$
$$\zeta = (\tau, (\tau_1, \ldots, \tau_n)) \colon s \Rightarrow s',$$
$$\zeta' = (\tau', (\tau'_1, \ldots, \tau'_n)) \colon s' \Rightarrow s'',$$

be 2-morphisms such that:

$$S(\zeta) = T(\eta),$$
$$S(\zeta') = T(\eta').$$

Then we have:

$$(\zeta' \circ \zeta) \bigcirc (\eta' \circ \eta) = (\zeta' \bigcirc \eta') \circ (\zeta \bigcirc \eta).$$

Proof. We expand both expressions. On the left hand side we have:

$$(\zeta' \circ \zeta) \bigcirc (\eta' \circ \eta) = (\tau' \circ \tau, (\tau'_{\tau(1)} \circ \tau_1, \dots, \tau'_{\tau(n)} \circ \tau_n)) \bigcirc (\sigma' \circ \sigma, (\sigma'_{\sigma(1)} \circ \sigma_1, \dots, \sigma'_{\sigma(m)} \circ \sigma_m)).$$

The outer permutation of this expression is simply $\sigma' \circ \sigma$, for $i \in \{1, \dots, m\}$ the i^{th} inner permutation is given by:

$$(\sigma'_{\sigma(i)} \circ \sigma_i)_*(l_{t_{i,1}}, \dots, l_{t_{i,k_i}}) \circ \bigoplus_{j=1}^{k_i} (\tau'_{\tau(t_{i,j})} \circ \tau_{t_{i,j}}).$$

We apply lemma 4.3.7 to the block permutations $\sigma_{i,*}(l_{t_{i,1}}, \dots, l_{t_{i,k_i}})$ and $\sigma'_{\sigma(i),*}(l'_{t'_{\sigma(i),1}}, \dots, l'_{t'_{\sigma(i),k'_{\sigma(i)}}})$, to use lemma 4.3.7 we require that:

$$l'_{t'_{\sigma(i),\sigma_i(j)}} = l_{t_{i,j}}$$

for each $j \in \{1, \dots, k_i\}$. To prove that this is the case, note that:

$$\begin{aligned} t'_{\sigma(i),\sigma_i(j)} &= T(\sigma, (\sigma_1, \dots, \sigma_m))(t_{i,j}) \\ &= S(\tau, (\tau_1, \dots, \tau_n))(t_{i,j}) \\ &= \tau(t_{i,j}). \end{aligned}$$

Now since $(\tau, (\tau_1, \dots, \tau_n))$ is a 2-morphism from s to s' we have that $l'_{\tau(a)} = l_a$ for all $a \in \{1, \dots, n\}$. Therefore we do indeed have that:

$$l'_{t'_{\sigma(i),\sigma_i(j)}} = l_{t_{i,j}}.$$

Therefore by lemma 4.3.7 we have:

$$(\sigma'_{\sigma(i)} \circ \sigma_i)_*(l_{t_{i,1}}, \dots, l_{t_{i,k_i}}) = \sigma'_{\sigma(i),*}(l'_{t'_{\sigma(i),1}}, \dots, l'_{t'_{\sigma(i),k_{\sigma(i)}}}) \circ \sigma_{i,*}(l_{t_{i,1}}, \dots, l_{t_{i,k_i}}).$$

Substituting this into our previous formula we obtain:

$$\sigma'_{\sigma(i),*}(l'_{t'_{\sigma(i),1}}, \dots, l'_{t'_{\sigma(i),k_{\sigma(i)}}}) \circ \sigma_{i,*}(l_{t_{i,1}}, \dots, l_{t_{i,k_i}}) \circ \bigoplus_{j=1}^{k_i} (\tau'_{\tau(t_{i,j})} \circ \tau_{t_{i,j}}).$$

Now by functorality of direct sum this is equal to:

$$\sigma'_{\sigma(i),*}(l'_{t'_{\sigma(i),1}}, \ldots, l'_{t'_{\sigma(i),k_{\sigma(i)}}}) \circ \sigma_{i,*}(l_{t_{i,1}}, \ldots, l_{t_{i,k_i}}) \circ \bigoplus_{j=1}^{k_i}(\tau'_{\tau(t_{i,j})}) \circ \bigoplus_{j=1}^{k_i}(\tau_{t_{i,j}}).$$

Applying lemma 4.3.8 to the block permutation $\sigma_{i,*}$ and the permutations $\tau'_{\tau(t_{i,j})}$ we have that:

$$\sigma_{i,*}(l_{t_{i,1}}, \ldots, l_{t_{i,k_i}}) \circ \bigoplus_{j=1}^{k_i}(\tau'_{\tau(t_{i,j})}) = \bigoplus_{j=1}^{k_i}(\tau'_{\tau(t_{i,\sigma_i^{-1}(j)})}) \circ \sigma_{i,*}(l_{t_{i,1}}, \ldots, l_{t_{i,k_i}}).$$

Note that the permutation σ_i^{-1} is applied directly to the indexing of the summands of the direct sum, which are not in this case the subscripts. Substituting this into our formula we obtain:

$$\sigma'_{\sigma(i),*}(l'_{t'_{\sigma(i),1}}, \ldots, l'_{t'_{\sigma(i),k_{\sigma(i)}}}) \circ \bigoplus_{j=1}^{k_i}(\tau'_{\tau(t_{i,\sigma_i^{-1}(j)})}) \circ \sigma_{i,*}(l_{t_{i,1}}, \ldots, l_{t_{i,k_i}}) \circ \bigoplus_{j=1}^{k_i}(\tau_{t_{i,j}}).$$

Now we use the fact that $S(\zeta) = T(\eta)$, $T(\sigma, (\sigma_1, \ldots, \sigma_m)$ maps $t_{i,\sigma_i^{-1}(j)}$ to $t'_{\sigma(i),j}$, whereas $S(\tau, (\tau_1, \ldots, \tau_n))$ is simply τ. Therefore we have that $\tau(t_{i,\sigma_i^{-1}(j)}) = t'_{\sigma(i),j}$. Substituting this into our formula gives:

$$\big(\sigma'_{\sigma(i),*}(l'_{t'_{\sigma(i),1}}, \ldots, l'_{t'_{\sigma(i),k_{\sigma(i)}}}) \circ \bigoplus_{j=1}^{k_i}(\tau'_{t'_{\sigma(i),j}})\big) \circ \big(\sigma_{i,*}(l_{t_{i,1}}, \ldots, l_{t_{i,k_i}}) \circ \bigoplus_{j=1}^{k_i}(\tau_{t_{i,j}})\big).$$

Comparing this to the definition of loose composition the first two terms form the $\sigma(i)^{\text{th}}$ inner permutation of $(\tau', (\tau'_1, \ldots, \tau'_n)) \bigcirc ((\sigma', \sigma'_1, \ldots, \sigma'_m))$, and the second two terms form the i^{th} inner permutation of $(\tau, (\tau_1, \ldots, \tau_n)) \bigcirc (\sigma, (\sigma_1, \ldots, \sigma_m))$. Therefore the left and right hand side have the same i^{th} inner permutation for all i, and so are equal. The fact that loose composition respects identities is clear from the definition, so loose composition is a functor. □

We now prove that loose composition is compatible with the source and target functors in the required way.

Proposition 4.3.12. Let t, t', s, s' and η, ζ be as in proposition 4.3.11, then we have:

$$S(\zeta \bigcirc \eta) = S(\eta),$$
$$T(\zeta \bigcirc \eta) = T(\zeta).$$

Proof. Since the source functor maps a 2-morphism to its outer permutation the first identity is clear from the definition of loose composition. For the target functor let $i \in \{1, \ldots, m\}$. Since the i^{th} inner tuple of $s \bigcirc t$ is the concatenation of the $t_{i,j}^{\text{th}}$ inner tuples of s, for an arbitrary entry let $j \in \{1, \ldots, k_i\}$ and $j' \in \{1, \ldots, l_{t_{i,j}}\}$, then $s_{t_{i,j},j'}$ is the j'^{th} entry of the $t_{i,j}^{\text{th}}$ inner tuple of s. This element appears at position:

$$\sum_{r=1}^{j-1}(l_{t_{i,r}}) + j',$$

in the i^{th} inner tuple of $s \bigcirc t$. By definition, the i^{th} inner permutation of $\zeta \bigcirc \eta$ is:

$$\sigma_{i,*}(l_{t_{i,1}}, \ldots, l_{t_{i,k_i}}) \circ \bigoplus_{r=1}^{k_i} \tau_{t_{i,r}}.$$

Therefore to evaluate the map $T(\zeta \bigcirc \eta)$ at $s_{t_{i,j},j'}$, we apply this inner permutation to the position we derived above. Thus we have:

$$\begin{aligned}\big(\sigma_{i,*}(l_{t_{i,1}}, \ldots, l_{t_{i,k_i}}) \circ \bigoplus_{r=1}^{k_i} \tau_{t_{i,r}}\big)\big(\sum_{r=1}^{j-1}(l_{t_{i,r}}) + j'\big) &= \sigma_{i,*}(l_{t_{i,1}}, \ldots, l_{t_{i,k_i}})\big(\sum_{r=1}^{j-1}(l_{t_{i,r}}) + \tau_{t_{i,j}}(j')\big) \\ &= \sum_{r=1}^{\sigma_i(j)-1} (l_{t_{i,\sigma_i^{-1}(r)}}) + \tau_{t_{i,j}}(j').\end{aligned}$$

Call this value a (a is the position in the $\sigma(i)^{\text{th}}$ inner tuple of $s' \bigcirc t'$ of the image of $s_{t_{i,j},j'}$ under $T(\zeta \circ \eta)$).

We now evaluate $T(\zeta)$ at $s_{t_{i,j},j'}$. By defnition, $T(\zeta)$ maps $s_{t_{i,j},j'}$ to $s'_{\tau(t_{i,j}),\tau_{t_{i,j}}(j')}$. Since $\tau = S(\zeta) = T(\eta)$, and $T(\eta)$ maps $t_{i,j}$ to $t'_{\sigma(i),\sigma_i(j)}$ we must have that $\tau(t_{i,j}) = t'_{\sigma(i),\sigma_i(j)}$. Therefore we have:

$$\begin{aligned}T(\zeta)(s_{t_{i,j},j'}) &= s'_{\tau(t_{i,j}),\tau_{t_{i,j}}(j')} \\ &= s'_{t'_{\sigma(i),\sigma_i(j)},\tau_{t_{i,j}}(j')}.\end{aligned}$$

This is the $\tau_{t_{i,j}}(j')^{\text{th}}$ entry of the $(t'_{\sigma(i),\sigma_i(j)})^{\text{th}}$ inner tuple of s'. Since the $\sigma(i)^{\text{th}}$ inner tuple of $s' \bigcirc t'$ is the concatenation of the $(t'_{\sigma(i),r})^{\text{th}}$ inner tuples of s', this value appears in the $\sigma_i(j)^{\text{th}}$ component of that concatenation, at position:

$$\sum_{r=1}^{\sigma_i(j)-1} (l'_{t'_{\sigma(i),r}}) + \tau_{t_{i,j}}(j').$$

Call this value b (b is the position in the $\sigma(i)^{\text{th}}$ inner tuple of $\zeta \bigcirc \eta$ at which we find $T(\zeta)(s_{t_{i,j},j'})$).

Finally we use the fact that $(\tau, (\tau_1, \ldots, \tau_n))$ is a 2-morphism from s to s', and so $l'_{\tau(r)} = l_r$, for all $r \in \{1, \ldots, n\}$, and hence $l'_r = l_{\tau^{-1}(r)}$. Noting as above that $\tau = S(\zeta) = T(\eta)$ and so $\tau(t_{u,v}) = t'_{\sigma(u),\sigma_u(v)}$ for all $u \in \{1, \ldots, m\}, v \in \{1, \ldots, k_u\}$, we substitute $v = \sigma_u^{-1}(w)$ into this formula to obtain $t'_{\sigma(u),w} = \tau(t_{u,\sigma_u^{-1}(w)})$ and hence $\tau^{-1}(t'_{\sigma(u),w}) = t_{u,\sigma_u^{-1}(w)}$. Applying these two observations to our formula for b gives us:

$$\begin{aligned}
b &= \sum_{r=1}^{\sigma_i(j)-1} (l'_{t'_{\sigma(i),r}}) + \tau_{t_{i,j}}(j') \\
&= \sum_{r=1}^{\sigma_i(j)-1} (l_{\tau^{-1}(t'_{\sigma(i),r})}) + \tau_{t_{i,j}}(j') \\
&= \sum_{r=1}^{\sigma_i(j)-1} (l_{t_{i,\sigma_i^{-1}(r)}}) + \tau_{t_{i,j}}(j') \\
&= a.
\end{aligned}$$

So $T(\zeta)(s_{t_{i,j},j'})$ and $T(\zeta \bigcirc \eta)(s_{t_{i,j},j'})$ are both found in the same position of the $\sigma(i)^{\text{th}}$ inner tuple of $s' \bigcirc t'$. This completes the proof. $\square$

In order to prove that loose composition of 2-morphisms is strictly associative we first prove a lemma.

Lemma 4.3.13. Let $\sigma \in S_m$, $n_1, \ldots, n_m$ be non-negative integers and for each $i \in \{1, \ldots, m\}$ let k_i be a non-negative integer, define:

$$m' := \sum_{i=1}^{m} k_i$$

and:

$$a_i := \sum_{j=1}^{i-1} k_j.$$

Let $n'_1, \ldots, n'_{m'}$ be non-negative integers satisfying:

$$\sum_{j=a_i+1}^{a_i+k_i} n'_j = n_i.$$

That is, the first k_1 of the n'_j sum to n_1, the next k_2 sum to n_2 and so on (we refer to this condition as the summation condition). Then:

$$[\sigma_*(k_1, \ldots, k_m)]_*(n'_1, \ldots, n'_{m'}) = \sigma_*(n_1, \ldots, n_m).$$

Proof. We evaluate both permutations on an arbitrary element of their domain. To do this let $i \in \{1, \ldots, m\}, j \in \{1, \ldots, n_i\}$. Then evaluating the right hand side we have:

$$\sigma_*(n_1, \ldots, n_m)(\sum_{l=1}^{i-1} n_l + j) = \sum_{l=1}^{\sigma(i)-1} n_{\sigma^{-1}(l)} + j.$$

To evaluate the left hand side we use the summation condition, firstly since $j \in \{1, \ldots, n_i\}$ and:

$$\sum_{p=a_i+1}^{a_i+k_i} n'_p = n_i$$

we have that there exists $i' \in \{1, \ldots, k_i\}$ and $j' \in \{1, \ldots, n'_{a_i+i'}\}$ such that:

$$j = \sum_{p=a_i+1}^{a_i+i'-1} n'_p + j'.$$

Combining this with the fact that for each $l \in \{1, \ldots, \sigma(i) - 1\}$:

$$n_l = \sum_{q=a_l+1}^{a_l+k_l} n'_q$$

we have that:

$$\begin{aligned}
\sum_{l=1}^{i-1} n_l + j &= \sum_{l=1}^{i-1} (\sum_{q=a_l+1}^{a_l+k_l} n'_q) + \sum_{p=a_i+1}^{a_i+i'-1} n'_p + j' \\
&= (\sum_{p=1}^{a_{i-1}+k_{i-1}} n'_p) + (\sum_{p=a_i+1}^{a_i+i'-1} n'_p) + j' \\
&= (\sum_{p=1}^{a_i} n'_p) + (\sum_{p=a_i+1}^{a_i+i'-1} n'_p) + j' \\
&= \sum_{p=1}^{a_i+i'-1} n'_p + j'.
\end{aligned}$$

Substituting this into the left hand side and defining:

$$\varphi := \sigma_*(k_1, \ldots, k_m),$$

we obtain:

$$\begin{aligned}\varphi_*(n'_1, \ldots, n'_{m'})(\sum_{l=1}^{i-1} n_l + j) &= \varphi_*(n'_1, \ldots, n'_{m'})(\sum_{p=1}^{a_i+i'-1} n'_p + j') \\ &= \sum_{p=1}^{\varphi(a_i+i')-1} n'_{\varphi^{-1}(p)} + j'.\end{aligned}$$

To evaluate $\varphi(a_i + i')$ we expand the summation:

$$\begin{aligned}\varphi(a_i + i') &= \sigma_*(k_1, \ldots, k_m)(\sum_{p=1}^{i-1} k_p + i') \\ &= \sum_{p=1}^{\sigma(i)-1} k_{\sigma^{-1}(p)} + i'.\end{aligned}$$

We now we define, for $p \in \{1, \ldots, m\}$:

$$b_p := \sum_{q=1}^{p-1} k_{\sigma^{-1}(q)}.$$

This definition give us:

$$\varphi(a_i + i') = \sum_{p=1}^{\sigma(i)-1} k_{\sigma^{-1}(p)} + i' = b_{\sigma(i)} + i'.$$

We now expand the summation in our expression for the left hand side, giving:

$$\begin{aligned}\sum_{p=1}^{\varphi(a_i+i')-1} n'_{\varphi^{-1}(p)} + j' &= \sum_{p=1}^{b_{\sigma(i)}+i'-1} n'_{\varphi^{-1}(p)} + j' \\ &= \sum_{p=1}^{\sigma(i)-1} \Big(\sum_{q=b_p+1}^{b_p+k_{\sigma^{-1}(p)}} n'_{\varphi^{-1}(q)}\Big) + \Big(\sum_{q=b_{\sigma(i)}+1}^{b_{\sigma(i)}+i'-1} n'_{\varphi^{-1}(q)}\Big) + j' \\ &= \sum_{p=1}^{\sigma(i)-1} \Big(\sum_{q=1}^{k_{\sigma^{-1}(p)}} n_{\varphi^{-1}(b_p+q)}\Big) + \Big(\sum_{q=1}^{i'-1} n'_{\varphi^{-1}(b_{\sigma(i)}+q)}\Big) + j'.\end{aligned}$$

To evaluate terms of the form $\varphi^{-1}(b_p + q)$ we expand the summation:

$$\begin{aligned}
\varphi^{-1}(b_p + q) &= (\sigma_*(k_1, \dots, k_m))^{-1}(\sum_{r=1}^{p-1} k_{\sigma^{-1}(r)} + q) \\
&= \sum_{r=1}^{\sigma^{-1}(p)-1} k_r + q \\
&= a_{\sigma^{-1}(p)} + q.
\end{aligned}$$

Substituting this into our expression for the left hand side we obtain:

$$\begin{aligned}
&\sum_{p=1}^{\sigma(i)-1} \Big(\sum_{q=1}^{k_{\sigma^{-1}(p)}} n_{\varphi^{-1}(b_p+q)}\Big) + \Big(\sum_{q=1}^{i'-1} n'_{\varphi^{-1}(b_{\sigma(i)}+q)}\Big) + j' \\
&\qquad = \sum_{p=1}^{\sigma(i)-1} \Big(\sum_{q=1}^{k_{\sigma^{-1}(p)}} n'_{a_{\sigma^{-1}(p)}+q}\Big) + \Big(\sum_{q=1}^{i'-1} n'_{a_i+q}\Big) + j' \\
&\qquad = \sum_{p=1}^{\sigma(i)-1} \Big(\sum_{q=a_{\sigma^{-1}(p)}+1}^{a_{\sigma^{-1}(p)}+k_{\sigma^{-1}(p)}} n'_q\Big) + \Big(\sum_{q=a_i+1}^{a_i+i'-1} n'_q\Big) + j' \\
&\qquad = \sum_{p=1}^{\sigma(i)-1} \big(n_{\sigma^{-1}(p)}\big) + \Big(\sum_{q=a_i+1}^{a_i+i'-1} n'_q\Big) + j' \\
&\qquad = \sum_{p=1}^{\sigma(i)-1} n_{\sigma^{-1}(p)} + j.
\end{aligned}$$

Which is equal to the right hand side. Therefore the two permutations are equal. □

Remark 4.3.14. Note that in the proof of lemma 4.3.13, we can change the order of the first k_1 of the n' on the left hand side without affecting the overall permutation, since the summation condition would still be satisfied. Indeed we could re-order the next k_2, the k_3 after that and so on. The proof of associativity will make use of this fact.

Finally we prove that loose composition of 2-morphisms is associative (that loose composition of loose 1-morphisms is associative is clear).

Proposition 4.3.15. Let:

$$t, s \colon m \nrightarrow n,$$
$$t', s' \colon n \nrightarrow p,$$
$$t'', s'' \colon p \nrightarrow q,$$

be loose 1-morphisms and:

$$\eta = (\sigma, (\sigma_1, \ldots, \sigma_m)) \colon t \Rightarrow s,$$
$$\eta' = (\sigma', (\sigma'_1, \ldots, \sigma'_n)) \colon t' \Rightarrow s',$$
$$\eta''(\sigma'', (\sigma''_1, \ldots, \sigma''_p)) \colon t'' \Rightarrow s'',$$

be 2-morphisms such that $S(\eta) = T(\eta')$ and $S(\eta'') = T(\eta')$, then we have:

$$(\eta'' \bigcirc \eta') \bigcirc \eta = \eta'' \bigcirc (\eta' \bigcirc \eta).$$

Proof. Let $\hat{\sigma}_i$ denote the i^{th} inner permutation of $\eta \bigcirc \eta'$ and $\hat{\sigma}'_i$ denote the i^{th} inner permutation of $\eta'' \bigcirc \eta'$. Then the i^{th} inner permutation of the left hand side is:

$$\sigma_{i,*}(\hat{k}'_{t_{i,1}}, \ldots, \hat{k}'_{t_{i,k_i}}) \circ \bigoplus_{j=1}^{k_i} \hat{\sigma}'_{t_{i,j}},$$

where $\hat{k}'_{t_{i,j}}$ is the length of the $t^{\text{th}}_{i,j}$ inner tuple of $t'' \bigcirc t$. We now use the fact that the $t^{\text{th}}_{i,j}$ inner permutation of $\eta'' \bigcirc \eta'$ is given by:

$$\sigma'_{t_{i,j},*}(k''_{t'_{t_{i,j},1}}, \ldots, k''_{t'_{t_{i,j},k'_{t_{i,j}}}}) \circ \bigoplus_{r=1}^{k'_{t_{i,j}}} \sigma''_{t'_{t_{i,j},r}}.$$

Therefore our expression for the i^{th} inner permutation of the left hand side becomes:

$$\sigma_{i,*}(\hat{k}'_{t_{i,1}}, \ldots, \hat{k}'_{t_{i,k_i}}) \circ \Big(\bigoplus_{j=1}^{k_i} (\sigma'_{t_{i,j},*}(k''_{t'_{t_{i,j},1}}, \ldots, k''_{t'_{t_{i,j},k'_{t_{i,j}}}}) \circ \bigoplus_{r=1}^{k'_{t_{i,j}}} \sigma''_{t'_{t_{i,j},r}})\Big).$$

The i^{th} inner permutation of the right hand side is:

$$\hat{\sigma}_{i,*}(k''_{\hat{t}_{i,1}}, \ldots, k''_{\hat{t}_{i,\hat{k}_i}}) \circ \bigoplus_{j=1}^{\hat{k}_i} \sigma''_{\hat{t}_{i,j}}.$$

Where $\hat{k}_i$ is the order of the i^{th} inner permutation of $\eta' \bigcirc \eta$, and for $j \in \{1, \dots, \hat{k}_i\}$, $\hat{t}_{i,j}$ is the j^{th} entry of the i^{th} inner tuple of $t' \bigcirc t$. Since the i^{th} inner tuple of $t' \bigcirc t$ is the concatenation of the $t_{i,j}^{\text{th}}$ inner tuples of t', we have that, for $j \in \{1, \dots, k_i\}, r \in \{1, \dots, k'_{t_{i,j}}\}$ the r^{th} entry of the $t_{i,j}^{\text{th}}$ inner tuple of t' appears at position:

$$\sum_{u=1}^{j-1} k'_{t_{i,u}} + r$$

in the i^{th} inner tuple of $t' \bigcirc t$. That is to say, calling the above expression $\alpha(i,j,r)$, we have:

$$\hat{t}_{i,\alpha(i,j,r)} = t'_{t_{i,j},r}.$$

Given also that:

$$\hat{k}_i = \sum_{j=1}^{k_i} k'_{t_{i,j}}.$$

We now expand the direct sum in our above expression:

$$\bigoplus_{j=1}^{\hat{k}_i} \sigma''_{\hat{t}_{i,j}} = \bigoplus_{j=1}^{k_i} (\bigoplus_{r=1}^{k'_{t_{i,j}}} \sigma''_{t'_{t_{i,j},r}}).$$

Substituting this into the expression for the right hand side we obtain:

$$\hat{\sigma}_{i,*}(k''_{\hat{t}_{i,1}}, \dots, k''_{\hat{t}_{i,\hat{k}_i}}) \circ (\bigoplus_{j=1}^{k_i} (\bigoplus_{r=1}^{k'_{t_{i,j}}} \sigma''_{t'_{t_{i,j},r}})).$$

Now expanding $\hat{\sigma_i}$ from definition 4.3.9:

$$\hat{\sigma}_i = \sigma_{i,*}(k'_{t_{i,1}}, \dots, k'_{t_{i,k_i}}) \circ \bigoplus_{j=1}^{k_i} \sigma'_{t_{i,j}}.$$

Substituting this into our expression for the right hand side we obtain:

$$[\sigma_{i,*}(k'_{t_{i,1}}, \dots, k'_{t_{i,k_i}}) \circ \bigoplus_{j=1}^{k_i} \sigma'_{t_{i,j}}]_*(k''_{\hat{t}_{i,1}}, \dots, k''_{\hat{t}_{i,\hat{k}_i}}) \circ (\bigoplus_{j=1}^{k_i} (\bigoplus_{r=1}^{k'_{t_{i,j}}} \sigma''_{t'_{t_{i,j},r}})).$$

By lemma 4.3.7 the nested block permutation expands to:

$$[\sigma_{i,*}(k'_{t_{i,1}}, \dots, k'_{t_{i,k_i}})]_*(k''_{\hat{t}_{i,\beta^{-1}(1)}}, \dots, k''_{\hat{t}_{i,\beta^{-1}(\hat{k}_i)}}) \circ [\bigoplus_{j=1}^{k_i} \sigma'_{t_{i,j}}]_*(k''_{\hat{t}_{i,1}}, \dots, k''_{\hat{t}_{i,\hat{k}_i}}).$$

Where β is the permutation given by:

$$\bigoplus_{j=1}^{k_i} \sigma'_{t_{i,j}}.$$

Consider first the block permutation:

$$[\sigma_{i,*}(k'_{t_{i,1}}, \ldots, k'_{t_{i,k_i}})]_*(k''_{\hat{t}_{i,\beta^{-1}(1)}}, \ldots, k''_{\hat{t}_{i,\beta^{-1}(\hat{k}_i)}}).$$

We claim that:

$$[\sigma_{i,*}(k'_{t_{i,1}}, \ldots, k'_{t_{i,k_i}})]_*(k''_{\hat{t}_{i,\beta^{-1}(1)}}, \ldots, k''_{\hat{t}_{i,\beta^{-1}(\hat{k}_i)}}) = \sigma_{i,*}(\hat{k}'_{t_{i,1}}, \ldots, \hat{k}'_{t_{i,k_i}}).$$

To prove this we use lemma 4.3.13. To verify the summation condition, note that since $\hat{t}_i$ (the i^{th} inner tuple of $t' \bigcirc t$) is the concatenation of the $t_{i,j}^{\text{th}}$ inner tuples of t', the list:

$$(\hat{t}_{i,1}, \ldots, \hat{t}_{i,\hat{k}_i})$$

is in fact the list:

$$(t'_{t_{i,1},1}, \ldots, t'_{t_{i,1},k'_{t_{i,1}}}, t'_{t_{i,2},1}, \ldots, t'_{t_{i,2},k'_{t_{i,2}}}, \ldots, t'_{t_{i,k_i},1}, \ldots, t'_{t_{i,k_i},k'_{t_{i,k_i}}}).$$

Using the fact that β is the direct sum of the $\sigma'_{t_{i,j}}$, each of which is of order $k'_{t_{i,j}}$, we have that the list:

$$(\hat{t}_{i,\beta^{-1}(1)}, \ldots, \hat{t}_{i,\beta^{-1}(\hat{k}_i)})$$

expands to:

$$(t'_{t_{i,1},\sigma'^{-1}_{t_{i,1}}(1)}, \ldots, t'_{t_{i,1},\sigma'^{-1}_{t_{i,1}}(k'_{t_{i,1}})}, \ldots, t'_{t_{i,k_i},\sigma'^{-1}_{t_{i,k_i}}(1)}, \ldots, t'_{t_{i,k_i},\sigma'^{-1}_{t_{i,k_i}}(k'_{t_{i,k_i}})}).$$

Substituting this into the list:

$$(k''_{\hat{t}_{i,1}}, \ldots, k''_{\hat{t}_{i,\hat{k}_i}})$$

of integers we see that the first $k'_{t_{i,1}}$ sum to $\hat{k}'_{t_{i,1}}$ (the $(t_{i,1})^{\text{th}}$ inner tuple of $t'' \bigcirc t'$ is the concatenation of the $(t'_{t_{i,1},j})^{\text{th}}$ inner tuples of t'', so its length is the sum of the lengths of the $(t'_{t_{i,1},j})^{\text{th}}$ inner tuples of t', of which there are $k'_{t_{i,1}}$). Similarly the next $k'_{t_{i,2}}$ sum to $\hat{k}'_{t_{i,2}}$ and so on, so the summation condition is satisfied and by lemma 4.3.13 we have that:

$$[\sigma_{i,*}(k'_{t_{i,1}}, \ldots, k'_{t_{i,k_i}})]_*(k''_{\hat{t}_{i,\beta^{-1}(1)}}, \ldots, k''_{\hat{t}_{i,\beta^{-1}(\hat{k}_i)}}) = \sigma_{i,*}(\hat{k}'_{t_{i,1}}, \ldots, \hat{k}'_{t_{i,k_i}}).$$

Substituting this into our expression for the right hand side we have:

$$\sigma_{i,*}(\hat{k'}_{t_{i,1}},\ldots,\hat{k}'_{t_{i,k_i}})\circ[\bigoplus_{j=1}^{k_i}\sigma'_{t_{i,j}}]_*(k''_{\hat{t}_{i,1}},\ldots,k''_{\hat{t}_{i,\hat{k}_i}})\circ(\bigoplus_{j=1}^{k_i}(\bigoplus_{r=1}^{k'_{t_{i,j}}}\sigma''_{t'_{t_{i,j},r}})).$$

Now consider the block permutation:

$$[\bigoplus_{j=1}^{k_i}\sigma'_{t_{i,j}}]_*(k''_{\hat{t}_{i,1}},\ldots,k''_{\hat{t}_{i,\hat{k}_i}}).$$

Similarly to the above we use the fact that $\hat{t}_i$ is the concatenation of the $(t_{i,r})^{\text{th}}$ inner tuples of t'. That is the list:

$$\hat{t}_{i,1},\ldots,\hat{t}_{i,\hat{k}_i}$$

is in fact that list:

$$(t'_{t_{i,1},1},t'_{t_{i,1},2},\ldots,t'_{t_{i,1},k'_{t_{i,1}}},t'_{t_{i,2},1},\ldots,t'_{t_{i,k_i},1}\ldots,t'_{t_{i,k_i},k'_{t_{i,k_i}}}).$$

Using this to expand the arguments of the block permutation we obtain:

$$[\bigoplus_{j=1}^{k_i}\sigma'_{t_{i,j}}]_*(k''_{\hat{t}_{i,1}},\ldots,k''_{\hat{t}_{i,\hat{k}_i}})=[\bigoplus_{j=1}^{k_i}\sigma'_{t_{i,j}}]_*(k''_{t'_{t_{i,1},1}},\ldots,k''_{t'_{t_{i,1},k'_{t_{i,1}}}},\ldots,k''_{t'_{t_{i,k_i},1}}\ldots,k''_{t'_{t_{i,k_i},k'_{t_{i,k_i}}}}).$$

We can group the $k''_{\hat{t}_{i,j}}$ into k_i groups of integers, the j^{th} of which contains $k'_{t_{i,j}}$ members. Since:

$$\bigoplus_{j=1}^{k_i}\sigma'_{t_{i,j}}$$

is a direct sum of k_i-many permutations, the j^{th} of which is of order $k'_{t_{i,j}}$, we have that:

$$[\bigoplus_{j=1}^{k_i}\sigma'_{t_{i,j}}]_*(k''_{\hat{t}_{i,1}}\ldots,k''_{\hat{t}_{i,\hat{k}_i}})=\bigoplus_{j=1}^{k_i}\sigma'_{t_{i,j},*}(k''_{t'_{t_{i,j},1}},\ldots,k''_{t'_{t_{i,j},k'_{t_{i,j}}}}).$$

Combining this with our previous result we obtain:

$$\hat{\sigma}_{i,*}(k''_{\hat{t}_{i,1}}, \ldots, k''_{\hat{t}_{i,\hat{k}_i}}) = [\sigma_{i,*}(k'_{t_{i,1}}, \ldots, k'_{t_{i,k_i}}) \circ \bigoplus_{j=1}^{k_i} \sigma'_{t_{i,j}}]_*(k''_{\hat{t}_{i,1}}, \ldots, k''_{\hat{t}_{i,\hat{k}_i}})$$

$$= [\sigma_{i,*}(k'_{t_{i,1}}, \ldots, k'_{t_{i,k_i}})]_*(k''_{\hat{t}_{i,\beta^{-1}(1)}}, \ldots, k''_{\hat{t}_{i,\beta^{-1}(\hat{k}_i)}}) \circ [\bigoplus_{j=1}^{k_i} \sigma'_{t_{i,j}}]_*(k''_{\hat{t}_{i,1}}, \ldots, k''_{\hat{t}_{i,\hat{k}_i}})$$

$$= \sigma_{i,*}(\hat{k}'_{t_{i,1}}, \ldots, \hat{k}'_{t_{i,k_i}}) \circ \bigoplus_{j=1}^{k_i} \sigma'_{t_{i,j},*}(k''_{t'_{t_{i,j},1}}, \ldots, k''_{t'_{t_{i,j},k'_{t_{i,j}}}}).$$

Substituting this into our expression for the right hand side we obtain:

$$= \hat{\sigma}_{i,*}(k''_{\hat{t}_{i,1}}, \ldots, k''_{\hat{t}_{i,\hat{k}_i}}) \circ \bigoplus_{j=1}^{\hat{k}_i} \sigma''_{\hat{t}_{i,j}}$$

$$= \Big(\sigma_{i,*}(\hat{k}'_{t_{i,1}}, \ldots, \hat{k}'_{t_{i,k_i}}) \circ \bigoplus_{j=1}^{k_i} \sigma'_{t_{i,j},*}(k''_{t'_{t_{i,j},1}}, \ldots, k''_{t'_{t_{i,j},k'_{t_{i,j}}}})\Big) \circ (\bigoplus_{j=1}^{k_i}(\bigoplus_{r=1}^{k'_{t_{i,j}}} \sigma''_{t'_{t_{i,j},r}}))$$

$$= \sigma_{i,*}(\hat{k}'_{t_{i,1}}, \ldots, \hat{k}'_{t_{i,k_i}}) \circ \Big((\bigoplus_{j=1}^{k_i} \sigma'_{t_{i,j},*}(k''_{t'_{t_{i,j},1}}, \ldots, k''_{t'_{t_{i,j},k'_{t_{i,j}}}})) \circ (\bigoplus_{j=1}^{k_i} \bigoplus_{r=1}^{k'_{t_{i,j}}} \sigma''_{t'_{t_{i,j},r}})\Big)$$

$$= \sigma_{i,*}(\hat{k}'_{t_{i,1}}, \ldots, \hat{k}'_{t_{i,k_i}}) \circ \Big(\bigoplus_{j=1}^{k_i} (\sigma'_{t_{i,j},*}(k''_{t'_{t_{i,j},1}}, \ldots, k''_{t'_{t_{i,j},k'_{t_{i,j}}}}) \circ \bigoplus_{r=1}^{k'_{t_{i,j}}} \sigma''_{t'_{t_{i,j},r}})\Big).$$

Note that when we combine the two direct sums going from $j = 1$ to k_i, we use the fact that for each j, the j^{th} terms in each of the direct sum has the same order. That is to say, the composition

$$\sigma'_{t_{i,j},*}(k''_{t'_{t_{i,j},1}}, \ldots, k''_{t'_{t_{i,j},k'_{t_{i,j}}}}) \circ \bigoplus_{r=1}^{k'_{t_{i,j}}} \sigma''_{t'_{t_{i,j},r}}.$$

Exists for each j (both terms are permutations of $\sum_{r=1}^{k'_{t_{i,j}}} k''_{t'_{t_{i,j},r}}$ elements). This now matches our expression for the i^{th} inner permutation of the left hand side. Therefore the two 2-morphisms share all inner permutations and so are equal (that the outer permutations match is trivial, they are both simply σ). □

We finally prove that loose composition is strictly unital.

Proposition 4.3.16. Let $t, s\colon m \nrightarrow n$ be loose 1-morphisms and $\eta := (\sigma, (\sigma_1, \ldots, \sigma_m))\colon t \Rightarrow s$ be a 2-morphism, then we have:

- $\eta \bigcirc U(S(\eta)) = \eta$,
- $U(T(\eta)) \bigcirc \eta = \eta$.

Proof. As with associativity that the outer permutations match is trivial. For the first inequality the i^{th} inner permutation of $\eta \bigcirc U(S(\eta))$ is given by:

$$\sigma_{i,*}(1, \ldots, 1) \circ \bigoplus_{j=1}^{k_i} (\mathrm{id}_1) = \sigma_i.$$

The proof for the second equality is similar. □

Since loose composition is strictly associative, both on loose 1-morphisms and 2-morphisms, our associator morphisms are identities. Our left and right unitor morphisms are also identities. Therefore we have the following.

Theorem 4.3.17. *The following data define a (strict) double category:*

- *The category* $\mathbf{D}_0$ *is the category of non-negative integers and permutations.*
- *The category* $\mathbf{D}_1$ *has as objects the loose 1-morphisms defined in definition 4.1.3 and as morphisms the 2-morphisms defined in definition 4.1.4. With composition defined in definition 4.1.6.*
- *The structure functors* S, T *and* U *are defined in definitions 4.2.1, 4.2.2 and 4.2.5.*
- *The loose composition functor is as defined in definitions 4.3.1 and 4.3.9.*
- *The associator and unitor morphisms are all identities.*

Proof. The required conditions and their proofs are:

- That $\mathbf{D}_0$ is a category is clear, as it is the category of permutations.
- That $\mathbf{D}_1$ is a category is proposition 4.1.7.
- That the structure functors are functors and satisfy the required conditions are propositions 4.2.6, 4.2.7, 4.2.9 and 4.2.10.

- That loose composition is a functor is proposition 4.3.11.
- That loose composition satisfies the required identities with the source and target functor is proposition 4.3.12.
- That loose composition is strictly associative is proposition 4.3.15.
- That loose composition is strictly unital is proposition 4.3.16

Thus we indeed have a (strict) double category. The associator and unitors are identities, so their naturality as well as the pentagon and triangle axioms are trivial. □

Remark 4.3.18. As mentioned in the introduction, the tight composition in this double category is closely linked with the wreath product of symmetric groups. Specifically in the special case where all the inner tuples of a loose 1-morphism are the same length. That is, given $n = km$ and a tuple:

$$t := ((t_{1,1}, \dots, t_{1,k})), \dots, (t_{m,1}, \dots, t_{m,k}))\colon m \nrightarrow km,$$

the group of 2-morphisms from t to itself is isomorphic to the wreath product of S_m with S_k.

Chapter 5

Manifolds, Cobordisms and Diffeomorphisms

We now use smooth manifolds and smooth collared cobordisms to construct a pseudo double category. This construction extends the usual category of manifolds and diffeomorphism classes of cobordisms to a higher category by taking the cobordisms themselves as loose 1-morphisms and using the diffeomorphisms as 2-morphisms, rather than taking equivalence classes of cobordisms as the morphisms in a category, as is done in for example [CR18] or [Mil65]. Our category D_0 will simply be the category whose objects are n-dimensional smooth manifolds without boundary, and whose morphisms (our tight 1-morphisms) are diffeomorphisms. Our loose 1-morphisms will be smooth collared cobordisms, and our 2-morphisms will be diffeomorphisms of smooth collared cobordisms, up to isotopy. Recall that our object manifolds are required to be compact, and our cobordism manifolds must be Hausdorff, in order to ensure that the closed subcollars are in fact closed subsets (see remark 2.5.3).

Remark 5.0.1. Aspects of the construction of this pseudo double category appear in [HS19], though few details of its construction are given. In particular no isotopy relation on diffeomorphisms between cobordisms is mentioned. Without any such condition, the construction will not yield a pseudo double category, as the unitors will not be natural transformations, that this cannot be avoided will be made precise in proposition 5.1.18.

The majority of the construction can be done without introducing an equivalence relation on the 2-morphisms, so we will begin by constructing the 'almost' pseudo-double category before introducing the necessary relation.

5.1 The Unquotiented Construction

In this section we define the 'almost' pseudo double category, and prove that it has almost all of the required properties of a pseudo double category. At the end of this section we will give a proof that the failure of naturality of the unitor morphisms was unavoidable, in that there is no family of diffeomorphisms we could choose that would form a natural transformation. We begin by defining our two categories. Throughout this chapter we choose and fix a non-negative integer n.

Definition 5.1.1. The category $(n, n{+}1)$-$\mathbf{Cob}_0$ is the category of n-dimensional closed smooth manifolds (that is, compact manifolds without boundary) and diffeomorphisms. The category $(n, n+1)$-$\mathbf{Cob}_1$ has as objects smooth collared cobordisms and as morphisms collar compatible diffeomorphisms of smooth collared cobordisms defined in definition 2.5.2.

We can now define our structure functors S, T, U.

Definition 5.1.2. The *source functor*:

$$S\colon (n, n+1)\text{-}\mathbf{Cob}_1 \to (n, n+1)\text{-}\mathbf{Cob}_0$$

is defined on objects by taking a smooth collared cobordism:

$$(i, \Sigma, j)\colon M \to N$$

to its source manifold M. S is defined on morphisms by taking:

$$f\colon (i, \Sigma, j) \to (i', \Sigma', j')$$

to its left induced map f_L. Similarly the *target functor*:

$$T\colon (n, n+1)\text{-}\mathbf{Cob}_1 \to (n, n+1)\text{-}\mathbf{Cob}_0$$

takes a smooth collared cobordism:

$$(i, \Sigma, j)\colon M \to N$$

to its target manifold N and a diffeomorphism f of diffeomorphisms of smooth collared cobordisms to its right induced map f_R.

Definition 5.1.3. The *unit functor*:

$$U\colon (n, n+1)\text{-}\mathbf{Cob}_0 \to (n, n+1)\text{-}\mathbf{Cob}_1$$

takes a manifold M to the smooth collared cobordism: $(i, M \times [0,3], j)$ where:

$$i\colon M \times [0,1) \to M \times [0,3]$$

is given by:

$$(p,t) \mapsto (p,t)$$

and:

$$j\colon M \times (-1,0] \to M \times [0,3]$$

is given by:

$$(p,t) \mapsto (p, 3+t).$$

U takes a diffeomorphism $f\colon M \to N$ of smooth manifolds to:

$$f \times \mathrm{id}\colon M \times [0,3] \to N \times [0,3].$$

We must now show that the source, target and unit functors are in fact functors.

Proposition 5.1.4. The source and target functors are functors.

Proof. Let:

$$\begin{aligned} f &\colon (i, \Sigma, j) \to (i', \Sigma', j'), \\ g &\colon (i', \Sigma', j') \to (i'', \Sigma'', j''), \end{aligned}$$

be diffeomorphisms of smooth collared cobordisms. Then:

$$\begin{aligned} (g \circ f)_L &= (\partial i'')^{-1} \circ (g \circ f) \circ \partial i \\ &= (\partial i'')^{-1} \circ g \circ (\partial i' \circ (\partial i')^{-1}) \circ f \circ \partial i \\ &= ((\partial i'')^{-1} \circ g \circ \partial i') \circ ((\partial i')^{-1} \circ f \circ \partial i) \\ &= g_L \circ f_L. \end{aligned}$$

Since we also have that the left induced map of the identity map is the identity map we have that the source functor is indeed a functor. The argument for T is similar. □

Proposition 5.1.5. The unit functor is a functor.

Proof. That $(g \circ f) \times \mathrm{id} = (g \times \mathrm{id}) \circ (f \times \mathrm{id})$ is clear, as is the fact that $U(\mathrm{id}_M) = \mathrm{id}_M \times \mathrm{id}_{[0,3]}$ is the identity map on $M \times [0,3]$. □

We are now ready to define our horizontal composition operation.

Definition 5.1.6. Let L, M, N be n-dimensional smooth manifolds without boundary and let:

$$(i, \Sigma, j)\colon L \to M,$$
$$(i', \Sigma', j')\colon M \to N,$$

be smooth collared cobordisms, then their *horizontal composition:*

$$(i', \Sigma', j') \bigcirc (i, \Sigma, j)\colon L \to N$$

is the collar dependent composition defined in definition 2.5.11. Given two (collar compatible) diffeomorphisms:

$$f\colon (i_1, \Sigma_1, j_1) \to (i'_1, \Sigma'_1, j'_1),$$
$$g\colon (i_2, \Sigma_2, j_2) \to (i'_2, \Sigma'_2, j'_2).$$

The horizontal composition $g \bigcirc f$ is the function:

$$\Sigma_1 \sqcup \Sigma_2/ \sim \to \Sigma'_1 \sqcup \Sigma'_2/ \sim'$$

given by:

$$[(p, 0)] \mapsto [(f(p), 0)] \text{ for } p \in \Sigma_1,$$
$$[(p, 1)] \mapsto [(g(p), 1)] \text{ for } p \in \Sigma_2.$$

We must prove that horizontal composition satisfies the required identities with the source and target functors, and is itself a functor.

Proposition 5.1.7. Horizontal composition is a functor.

Proof. Let $L_1, M_1, N_1, L_2, M_2, N_2, L_3, M_3, N_3$ be n-dimensional smooth manifolds without boundary, let:

$$(i_1, \Sigma_1, j_1)\colon L_1 \to M_1,$$
$$(i'_1, \Sigma'_1, j'_1)\colon M_1 \to N_1,$$
$$(i_2, \Sigma_2, j_2)\colon L_2 \to M_2,$$
$$(i'_2, \Sigma'_2, j'_2)\colon M_2 \to N_2,$$
$$(i_3, \Sigma_3, j_3)\colon L_3 \to M_3,$$
$$(i'_3, \Sigma'_3, j'_3)\colon M_3 \to N_3,$$

be smooth collared cobordisms and let:

$$\begin{aligned} f &\colon (i_1, \Sigma_1, j_1) \to (i_2, \Sigma_2, j_2), \\ g &\colon (i_2, \Sigma_2, j_2) \to (i_3, \Sigma_3, j_3), \\ f' &\colon (i'_1, \Sigma'_1, j'_1) \to (i'_2, \Sigma'_2, j'_2), \\ g' &\colon (i'_2, \Sigma'_2, j'_2) \to (i'_3, \Sigma'_3, j'_3), \end{aligned}$$

be collar compatible diffeomorphisms of smooth collared cobordisms. Then that

$$(g' \circ f') \bigcirc (g \circ f) = (g' \bigcirc g) \circ (f' \bigcirc f)$$

is clear from the definition of horizontal composition. That horizontal composition preserves identity maps is similarly clear. □

We now prove that the horizontal composition satisfies the conditions with the source and target functors.

Proposition 5.1.8. The source and target functors are compatible with loose composition. That is given n-dimensional smooth manifolds without boundary K, M, N and smooth collared cobordisms:

$$\begin{aligned} (i, \Sigma, j) &\colon K \to M, \\ (i', \Sigma', j') &\colon M \to N. \end{aligned}$$

We have that:

$$\begin{aligned} S((i', \Sigma', j') \bigcirc (i, \Sigma, j)) &= S((i, \Sigma, j)), \\ T((i', \Sigma', j') \bigcirc (i, \Sigma, j)) &= T((i', \Sigma', j')). \end{aligned}$$

In addition to the analogous results for diffeomorphisms.

Proof. Clear from the definitions. □

Now we are ready to introduce the structure morphisms, the associator and unitors for horizontal composition. The associator is straightforward.

Definition 5.1.9. Let:

$$\begin{aligned} (i_1, \Sigma_1, j_1) &\colon K \to L, \\ (i_2, \Sigma_2, j_2) &\colon L \to M, \\ (i_3, \Sigma_3, j_3) &\colon M \to N, \end{aligned}$$

be smooth collared cobordisms. The *associator morphism*

$$a_{\Sigma_1,\Sigma_2,\Sigma_3}\colon ((i_3,\Sigma_3,j_3)\bigcirc(i_2,\Sigma_2,j_2))\bigcirc(i_1,\Sigma_1,j_1) \to (i_3,\Sigma_3,j_3)\bigcirc((i_2,\Sigma_2,j_2)\bigcirc(i_1,\Sigma_1,j_1))$$

is the map given by:

$$\begin{aligned} [(p,0)] &\mapsto [([(p,0)],0)] \text{ for } p \in \Sigma_1, \\ [([(p,0),1)] &\mapsto [([(p,0)],0)] \text{ for } p \in \Sigma_2, \\ [([(p,1)],1)] &\mapsto [(p,1)] \text{ for } p \in \Sigma_3. \end{aligned}$$

Remark 5.1.10. There is some slight abuse of notation in the above definition, strictly speaking we should write $a_{(i_1,\Sigma_1,j_1),(i_2,\Sigma_2,j_2),(i_3,\Sigma_3,j_3)}$, since the natural transformation a is indexed by triples of objects of $(n, n+1)$-$\mathbf{Cob}_1$ (which are horizontally composable), not just their manifolds $\Sigma_1, \Sigma_2, \Sigma_3$. Since $a_{(i_1,\Sigma_1,j_1),(i_2,\Sigma_2,j_2),(i_3,\Sigma_3,j_3)}$ is unreasonably long however, we stick to $a_{\Sigma_1,\Sigma_2,\Sigma_3}$ for the sake of brevity and readability. We have also reversed the order of the labels of a, since we write loose composition in $(n, n+1)$-$\mathbf{Cob}$ in the same way as in the usual category of cobordisms (that is, we write it left to right), but it is often easier to list the cobordisms from left to right.

Proposition 5.1.11. The associator morphism is a diffeomorphism of smooth collared cobordisms, and satisfies the collar condition with parameter 1 (see definition 2.5.10).

Proof. The non-standard local representations of $a_{\Sigma_1,\Sigma_2,\Sigma_3}$ (or in some cases their non-standard local representations) are identities, which are diffeomorphisms. The fact that $a_{\Sigma_1,\Sigma_2,\Sigma_3}$ satisfies the collar condition with parameter 1 is clear. □

Proposition 5.1.12. The associator morphism a is a natural transformation and satisfies the pentagon axiom.

Proof. Clear from the definition. □

At this point we can define our unitor morphisms.

Definition 5.1.13. First we choose and fix a diffeomorphism:

$$\psi_L\colon (-1,3] \to (-1,0]$$

with the following properties:

- $\psi_L(t) = t$ for $t < -\frac{1}{2}$.

- $\psi_L(t) = t - 3$ for $t > 2 + \frac{3}{4}$.

Now let $(i, \Sigma, j)\colon M \to N$ be a smooth collared cobordism, the *left unitor* of (i, Σ, j) is the map:

$$l_\Sigma \colon U(N) \bigcirc (i, \Sigma, j) \to (i, \Sigma, j)$$

given by:

$$\begin{aligned}
[(q, 0)] &\mapsto q \text{ for } q \in \Sigma \setminus j(N \times [-\frac{3}{4}, 0]), \\
[(j(p, t), 0)] &\mapsto j(p, \psi_L(t)) \text{ for } p \in N, t \in (-1, 0], \\
[((p, t), 1)] &\mapsto j(p, \psi_L(t)) \text{ for } p \in N, t \in [0, 3].
\end{aligned}$$

The right unitor is defined similarly.

Definition 5.1.14. We choose and fix a diffeomorphism:

$$\psi_R \colon [-3, 1) \to [0, 1)$$

such that:

- $\psi_R(t) = t + 3$ for $t < -(2 + \frac{3}{4})$.
- $\psi_R(t) = t$ for $t > \frac{1}{2}$.

Now let $(i, \Sigma, j)\colon M \to N$ be a smooth collared cobordism, the *right unitor* of (i, Σ, j) is the map:

$$r_\Sigma \colon (i, \Sigma, j) \bigcirc U(M) \to (i, \Sigma, j)$$

given by:

$$\begin{aligned}
[((p, t), 0)] &\mapsto i(p, \psi_R(t - 3)) \text{ for } p \in M, t \in [0, 3], \\
[(i(p, t), 1)] &\mapsto i(p, \psi_R(t)) \text{ for } p \in M, t \in [0, 1), \\
[(q, 1)] &\mapsto q \text{ for } q \in \Sigma \setminus i(M \times [0, \frac{3}{4}]).
\end{aligned}$$

Remark 5.1.15. We will not give details here of how diffeomorphisms ψ_L and ψ_R are constructed. The general method is to begin with a smooth function $f\colon \mathbb{R} \to \mathbb{R}$ such that:

- $f(x) = 0$ for $x \leq 0$.
- $f(x) = 1$ for $x \geq 1$.

We then use this f to smoothly interpolate between any two smooth functions of our choice. If we are careful with our choice of such smooth functions, we can be guaranteed a diffeomorphism. Note also that the definitions of the left and right unitors depend on the chosen maps ψ_L and ψ_R, though once we transition to the quotiented construction in order to obtain natural transformations, this dependence will disappear. This will be made explicit in section 5.2.1, specifically proposition 5.2.12.

We must prove that the maps defined above are diffeomorphisms of smooth collared cobordisms and that they are collar compatible.

Proposition 5.1.16. The unitor maps are collar compatible diffeomorphisms of smooth collared cobordisms.

Proof. We demonstrate the proof for the left unitor. Recall that the map l_Σ is given by:

$$\begin{aligned}
[(q,0)] &\mapsto q \text{ for } q \in \Sigma \setminus j(N \times [-\frac{3}{4}, 0]),\\
[(j(p,t),0)] &\mapsto j(p, \psi_L(t)) \text{ for } p \in N, t \in (-1, 0],\\
[((p,t),1)] &\mapsto j(p, \psi_L(t)) \text{ for } p \in N, t \in [0, 3].
\end{aligned}$$

That l_Σ is a homeomorphism follows from two applications of the glue lemma. To see that l_Σ is smooth we use lemma 2.5.13. The first function to consider is:

$$f_1\colon \Sigma \setminus j(N \times \{0\}) \to \Sigma$$

given by:

$$q \mapsto l_\Sigma([(q,0)]).$$

Comparing this to the definition of l_Σ we see that f_1 is given by:

$$\begin{aligned}
q &\mapsto q \text{ for } q \in \Sigma \setminus j(N \times [-\frac{3}{4}, 0]),\\
j(p,t) &\mapsto j(p, \psi_L(t)) \text{ for } p \in N, t \in (-1, 0].
\end{aligned}$$

Since $\psi_L(t) = t$ for $t < -\frac{1}{2}$ we have that this map is well-defined (the two definitions agree on their intersection, namely $j(N \times (-1, \frac{3}{4})))$. Further since the second map is $j \circ (\mathrm{id}_N \times \psi_L) \circ j^{-1}$ both maps are diffeomorphisms and so f_1 is smooth by the glue lemma. The second map is $f_2\colon N \times (0,3] \to \Sigma$

given by $q \mapsto l_\Sigma([(q,1)])$. Again comparing with the definition of l_Σ we see that:

$$\begin{aligned} f_2(p,t) &= l_\Sigma([((p,t),1)]) \\ &= j(p,\psi_L(t)). \end{aligned}$$

So we see that $f_2 = j \circ (\mathrm{id}_N \times \psi_L)$ and so is a diffeomorphism. The third map is $f_3 = l_\Sigma \circ \langle j, i' \rangle \colon M \times (-1,1) \to \Sigma$. Let $p \in M$ and $t \in (-1,1)$, if $t \leq 0$ then we have:

$$\begin{aligned} f_3(p,t) &= l_\Sigma(\langle j, i'\rangle(p,t)) \\ &= l_\Sigma([(j(p,t),0)]) \\ &= j(p,\psi_L(t)). \end{aligned}$$

Similarly if $t \geq 0$ then we have:

$$\begin{aligned} f_3(p,t) &= l_\Sigma(\langle j, i'\rangle(p,t)) \\ &= l_\Sigma([(i'(p,t),1)]) \\ &= l_\Sigma([((p,t),1)]) \\ &= j(p,\psi_L(t)). \end{aligned}$$

So we see that $f_3 = j \circ (\mathrm{id}_N \times \psi_L)$ which is smooth. Therefore by lemma 2.5.13 l_Σ is smooth. To see that l_Σ is a diffeomorphism observe that the map $\Sigma \to \Sigma \sqcup N \times [0,3]/\sim$ given by:

$$\begin{aligned} q &\mapsto [(q,0)] \text{ for } q \in \Sigma \setminus j(N \times [-\frac{3}{4}, 0]), \\ j(p,t) &\mapsto [(j(p,\psi_L^{-1}(t)),0)] \text{ for } p \in N, t \in (-1,\psi_L(0)], \\ j(p,t) &\mapsto [(p,\psi_L^{-1}(t),1)] \text{ for } p \in N, t \in [\psi_L(0),0]. \end{aligned}$$

Is smooth by an argument analogous to the one given for l_Σ and is clearly the inverse of l_Σ. Finally we verify that l_Σ is collar compatible. First let $p \in M, t \in [0,\frac{3}{4})$, then since $i(M \times [0,1))$ does not meet $j(N \times (-1,0])$ we have that:

$$\begin{aligned} l_\Sigma(\tilde{i}(p,t)) &= l_\Sigma([(i(p,t),0)]) \\ &= i(p,t). \end{aligned}$$

Similarly on the other side let $p \in N, t \in (-\frac{1}{4},0]$, then:

$$\begin{aligned} l_\Sigma(\tilde{j}(p,t)) &= l_\Sigma([((p,3+t),1)]) \\ &= j(p,\psi_L(t+3)) \\ &= j(p,t). \end{aligned}$$

Observing that since $t > -\frac{1}{4}, 3+t > 2+\frac{3}{4}$ and therefore $\psi_L(t+3) = t+3-3 = t$ by the definition of ψ_L. Therefore l_Σ satisfies the collar condition with parameter $\frac{1}{4}$. The argument for the right unitor is similar. □

We also require that the structure morphisms are globular, that is the source and target functor map them all to identities

Proposition 5.1.17. The left and right induced maps of both the associator, as well as the left and right unitors are identities. That is to say, $S(a), T(a), S(l), T(l), S(r), T(r)$ are all identities.

Proof. Clear from the definitions. □

This is where the construction fails however, the unitors we have chosen do not form natural transformations. In fact there is no choice of unitor that we could have made which would form a natural transformation, as we shall now prove.

Proposition 5.1.18. Let $(i, \Sigma, j) \colon M \to N$ be a smooth collared cobordism such that N is not the empty manifold. Let:

$$l_\Sigma \colon U(N) \bigcirc (i, \Sigma, j) \to (i, \Sigma, j)$$

be a collar compatible diffeomorphism of smooth collared cobordisms. Then there exists a collar compatible diffeomorphism of smooth collared cobordisms:

$$f \colon (i, \Sigma, j) \to (i, \Sigma, j)$$

such that the diagram:

$$\begin{array}{ccc} U(N) \bigcirc (i, \Sigma, j) & \xrightarrow{U(f_R) \bigcirc f} & U(N) \bigcirc (i, \Sigma, j) \\ \downarrow{\scriptstyle l_\Sigma} & & \downarrow{\scriptstyle l_\Sigma} \\ (i, \Sigma, j) & \xrightarrow{\quad f \quad} & (i, \Sigma, j) \end{array}$$

does not commute. That is to say, l_Σ is not a suitable choice of left unitor for (i, Σ, j), and so such unitor morphism exists.

Proof. Since l_Σ is collar compatible there exists $\varepsilon > 0$ such that l_Σ satisfies the collar condition with parameter ε. We choose and fix a diffeomorphism $\psi \colon (-1, 3] \to (-1, 3]$ such that:

- $\psi(t) = t$ for $t < \frac{-\varepsilon-1}{2}$.

- $\psi(t) = t$ for $t > 3 - \varepsilon$.
- $\psi(-\frac{\varepsilon}{2}) = 1$.

Now let $g\colon \Sigma \sqcup N \times [0,3]/\sim \to \Sigma \sqcup N \times [0,3]/\sim$ be given by:

$$[(q,0)] \mapsto [(q,0)] \text{ for } q \in \Sigma \setminus j(N \times [\frac{-\varepsilon - 1}{2}, 0]),$$
$$[(j(p,t),0)] \mapsto [(j(p,\psi(t)),0)] \text{ for } p \in N, t \in (-1, \psi^{-1}(0)],$$
$$[(j(p,t),0)] \mapsto [((p,\psi(t)),1)] \text{ for } p \in N, t \in [\psi^{-1}(0), 3].$$

To show that g is a diffeomorphism we use the glue lemma, the bottom two definitions define a smooth map by the definition of the smooth structure on the horizontal composition (the non-standard local representation at the gluing boundary is $\mathrm{id} \times \psi$). This leaves two smooth maps on open subsets which agree on the intersection by the properties of ψ. That g satisfies the collar condition with parameter ε is clear, as is the fact that the left and right induced maps of g are identities. Now we let $f\colon \Sigma \to \Sigma$ be given by $l_\Sigma \circ g \circ l_\Sigma^{-1}$, note that the left and right induced maps of f are also identities. Now suppose for a contradiction that the required diagram commutes, that is:

$$f \circ l_\Sigma = l_\Sigma \circ (U(f_R) \bigcirc f)$$
$$(l_\Sigma \circ g \circ l_\Sigma^{-1}) \circ l_\Sigma = l_\Sigma \circ (\mathrm{id} \bigcirc (l_\Sigma \circ g \circ l_\Sigma^{-1}))$$
$$l_\Sigma \circ g = l_\Sigma \circ (\mathrm{id} \bigcirc (l_\Sigma \circ g \circ l_\Sigma^{-1}))$$
$$g = \mathrm{id} \bigcirc (l_\Sigma \circ g \circ l_\Sigma^{-1}).$$

Therefore we see that g is a horizontal composition of two maps. In particular we have that $g([\Sigma \times \{0\}]) = [\Sigma \times \{0\}]$ (g must map the first component of its domain to the first component of its codomain setwise). However this is a contradiction since $\psi(-\frac{\varepsilon}{2}) = 1 > 0$ and so $-\frac{\varepsilon}{2} > \psi^{-1}(0)$. Hence letting $p \in N$ (recall that N is not the empty manifold) we have that $g([(j(p, -\frac{\varepsilon}{2}), 0)]) = [((p, \psi(-\frac{\varepsilon}{2}), 1)] \notin [\Sigma \times \{0\}]$. Therefore the diagram does not commute. $\square$

There is a similar result for the right unitors. There is also a version of this result for diffeomorphisms which are not required to be collar compatible. Therefore closed manifolds, diffeomorphisms, smooth collared cobordisms and diffeomorphisms of smooth collared cobordisms do not form a pseudo double category. It can be shown by a similar argument that the triangle axiom is also impossible to satisfy.

Do not despair however, since we will now prove that if we are careful with our choice of unitor morphism, while the maps represented by two paths of the diagram are not always equal, they are always ambiently isotopic. We first introduce the condition of ambient isotopy which we will use.

5.2 Half-Smooth Isotopy

Definition 5.2.1. Let M, N, M', N' be n-dimensional smooth manifolds without boundary, let:

$$(i, \Sigma, j)\colon M \to N,$$
$$(i', \Sigma', j')\colon M' \to N',$$

be smooth collared cobordisms and let:

$$f, g\colon (i, \Sigma, j) \to (i', \Sigma', j')$$

be collar compatible maps of smooth collared cobordisms. A *collar compatible half-smooth ambient isotopy* from f to g is a continuous map:

$$I\colon \Sigma' \times [0, 1] \to \Sigma'$$

such that:

1. For all $t \in [0, 1]$ the map $I_t\colon \Sigma' \to \Sigma'$ given by $p \mapsto I(p, t)$ is a diffeomorphism of smooth collared cobordisms.

2. The left and right induced maps of I_t are identities for all $t \in [0, 1]$.

3. There exists $\delta > 0$ such that for all $t \in [0, 1]$ the map $I_t\colon (i', \Sigma', j') \to (i', \Sigma', j')$ satisfies the collar condition with parameter δ.

4. I_0 is the identity map on Σ'.

5. $I_1 \circ f = g$.

If such an isotopy exists then we say that f and g are *ambiently isotopic*

Remark 5.2.2. We only require that I is a continuous map on $\Sigma' \times [0, 1]$, rather than a smooth one since both Σ' and $[0, 1]$ have nonempty boundary. Therefore the obvious product charts would not give open coordinate coneighbourhoods and so the meaning of a smooth map in this context is not clear. This will be discussed in more detail in section 6.1.

The first important property of the relation of being ambiently isotopic is that it is an equivalence relation.

Proposition 5.2.3. The relation of being ambiently isotopic is an equivalence relation.

Proof. For reflexivity we claim that:

$$I\colon \Sigma' \times [0,1] \to \Sigma'$$

given by:

$$(p,t) \mapsto p$$

is a collar compatible half-smooth ambient isotopy from f to itself. I is clearly continuous, I_t is the identity map of Σ' for all t, so is a collar compatible diffeomorphism of smooth collared cobordisms satisfying the collar condition with parameter 1.
For symmetry let:

$$f, g\colon (i, \Sigma, j) \to (i, \Sigma, j)$$

be collar compatible diffeomorphisms of smooth collared cobordisms and let:

$$I\colon \Sigma' \to \Sigma'$$

be a collar compatible half-smooth ambient isotopy from f to g. Then:

$$\begin{aligned} J\colon \Sigma' \times [0,1] &\to \Sigma', \\ (p,t) &\mapsto I(p, 1-t), \end{aligned}$$

is clearly a collar compatible half-smooth ambient isotopy from g to f.

Finally for transitivity let:

$$f, g, h\colon (i, \Sigma, j) \to (i', \Sigma', j')$$

be collar compatible diffeomorphisms and let I, I' be collar compatible half-smooth isotopies from f to g and from g to h respectively. Then we claim the map:

$$J\colon \Sigma' \times [0,1] \to \Sigma'$$

given by:

$$(p,t) \mapsto \begin{cases} I(p, 2t) \text{ if } t \leq \frac{1}{2}, \\ I'(I_1(p), 2t-1) \text{ if } t \geq \frac{1}{2}, \end{cases}$$

is a collar compatible half-smooth ambient isotopy from f to h. First note that since the two cases are continuous and correspond to closed subsets of $\Sigma' \times [0,1]$ we have that J is continuous by the glue lemma. Since I is collar compatible there exists $\delta > 0$ such that I_t satisfies the collar condition with parameter δ, similarly there exists $\delta' > 0$ such that I'_t satisfies the collar condition with parameter δ'. Therefore noting that $J_t = I_{2t}$ for $t \leq \frac{1}{2}$ and $J_t = I'_{2t-1} \circ I_1$ for $t \geq \frac{1}{2}$ we have that J_t is a collar compatible diffeomorphism of collared cobordisms and satisfies the collar condition with parameter $\min(\delta, \delta')$ for all $t \in [0,1]$. Thus J is a collar compatible half-smooth ambient isotopy, and we have that:

$$\begin{aligned} J_1 \circ f &= I'_1 \circ (I_1 \circ f) \\ &= I'_1 \circ g \\ &= h. \end{aligned}$$

Therefore f is ambiently isotopic to h as required. Thus the relation of half-smooth ambient isotopy is an equivalence relation. □

Since we want our 2-morphisms to be equivalence classes of diffeomorphisms up to ambient isotopy, we must verify that both vertical and horizontal composition descend to the quotient. We begin with vertical composition.

Proposition 5.2.4. Let M, N, M', N', M'', N'' be smooth n-dimensional manifolds without boundary and let:

$$\begin{aligned} (i, \Sigma, j) &\colon M \to N, \\ (i', \Sigma', j') &\colon M' \to N', \\ (i'', \Sigma'', j'') &\colon M'' \to N'', \end{aligned}$$

be smooth collared cobordisms. Let:

$$\begin{aligned} f, f' &\colon \Sigma \to \Sigma', \\ g, g' &\colon \Sigma' \to \Sigma'', \end{aligned}$$

be diffeomorphisms of smooth collared cobordisms such that f' is ambiently isotopic to f and g' is ambiently isotopic to g, then $g' \circ f'$ is ambiently isotopic to $g \circ f$.

Proof. Let:

$$I \colon \Sigma' \times [0,1] \to \Sigma'$$

be a collar compatible half-smooth ambient isotopy from f to f' and let:

$$I' \colon \Sigma'' \times [0,1] \to \Sigma''$$

be a collar compatible half-smooth ambient isotopy from g to g'. Define:

$$J \colon \Sigma'' \times [0,1] \to \Sigma''$$

by:

$$(p,t) \mapsto I'(g(I(g^{-1}(p),t)),t) = (I'_t \circ g \circ I_t \circ g^{-1})(p).$$

Since $J = I' \circ (g \times \mathrm{id}) \circ ((p,t) \mapsto (I(p,t),t)) \circ (g^{-1} \times \mathrm{id})$ it is clear that J is continuous. We claim that J is a collar compatible half-smooth ambient isotopy from $g \circ f$ to $g' \circ f'$. First note that:

$$\begin{aligned} J(p,0) &= I'(g(I(g^{-1}(p),0)),0) \\ &= (I'_0 \circ g \circ I_0 \circ g^{-1})(p) \\ &= (g' \circ g'^{-1})(p) \\ &= p. \end{aligned}$$

Therefore J_0 is the identity map on Σ''. Further we have:

$$\begin{aligned} J_1 \circ (g \circ f) &= (I'_1 \circ g \circ I_1 \circ g^{-1}) \circ (g \circ f) \\ &= (I'_1 \circ g) \circ (I_1 \circ f) \\ &= g' \circ f'. \end{aligned}$$

Therefore $J_1 \circ (g \circ f) = g' \circ f'$. Since $J_t = I'_t \circ g \circ I_t \circ g^{-1}$ we have that J_t is a diffeomorphism for all $t \in [0,1]$. Finally since I, I' are collar compatible half-smooth ambient isotopies there exists $\delta, \delta' > 0$ such that I_t satisfies the collar condition with parameter δ for all $t \in [0,1]$ and I'_t satisfies the collar condition with parameter δ' for all $t \in [0,1]$. Since g satisfies the collar condition with parameter ε for some $\varepsilon > 0$ we have that for all $t \in [0,1]$ the map $J_t \colon \Sigma'' \to \Sigma''$ satisfies the collar condition with parameter $\min(\delta, \delta', \varepsilon)$. Thus J is a collar compatible half-smooth ambient isotopy from $g \circ f$ to $g' \circ f'$. □

We also prove the analogous result for horizontal composition.

Proposition 5.2.5. Let:

$$\begin{aligned} f, f' &\colon (i_1, \Sigma_1, j_1) \to (i_2, \Sigma_2, j_2), \\ g, g' &\colon (i'_1, \Sigma'_1, j'_1) \to (i'_2, \Sigma'_2, j'_2), \end{aligned}$$

be diffeomorphisms of smooth collared cobordisms such that f is ambiently isotopic to f' and g is ambiently isotopic to g'. Then $g \bigcirc f$ is ambiently isotopic to $g' \bigcirc f'$.

Proof. Let:

$$I \colon \Sigma_2 \times [0,1] \to \Sigma_2$$

be a collar compatible half-smooth ambient isotopy from f to f' and let:

$$I' \colon \Sigma_2' \times [0,1] \to \Sigma_2'$$

be an isotopy from g to g'. Then define the map:

$$J \colon \tilde{\Sigma}_2 \times [0,1] \to \tilde{\Sigma}_2$$

by:

$$\begin{aligned} ([(p,0)],t) &\mapsto [(I(p,t),0)] \text{ for } p \in \Sigma_2, \\ ([(p,1)],t) &\mapsto [(I'(p,t),1)] \text{ for } p \in \Sigma_2'. \end{aligned}$$

That is, $J_t = I_t' \bigcirc I_t$. It is clear that J is a collar compatible half-smooth ambient isotopy from $g \bigcirc f$ to $g' \bigcirc f'$. □

Now we prove that the source and target functors also descend to the quotient.

Proposition 5.2.6. Let $f, g \colon (i, \Sigma, j) \to (i', \Sigma', j')$ be diffeomorphisms of smooth collared cobordisms and suppose that f is ambiently isotopic to g, then $S(f) = S(g)$ and $T(f) = T(g)$.

Proof. Let I be a collar compatible half-smooth ambient isotopy from f to g. Then since the left and right induced maps of I_1 are identities we have that $g_L = f_L$ and $g_R = f_R$. Thus S and T descend to the quotient. □

Now we prove a lemma regarding the collar condition which will assist greatly in proving that the unitors are natural with this isotopy condition.

Lemma 5.2.7 (Collar Exchange Lemma)**.** Let M, N, M', N' be smooth manifolds without boundary, let:

$$\begin{aligned} (i, \Sigma, j) &\colon M \to N, \\ (i', \Sigma', j') &\colon M' \to N', \end{aligned}$$

be smooth collared cobordisms. Let:

$$f\colon (i,\Sigma,j) \to (i',\Sigma',j')$$

be a diffeomorphism of smooth collared cobordism satisfying the collar condition with parameter $\varepsilon > 0$ and let $\varepsilon' \in (\varepsilon, 1)$. Then there exists a diffeomorphism:

$$\hat{f}\colon (i,\Sigma,j) \to (i',\Sigma',j')$$

such that:

1. $\hat{f}$ is ambiently isotopic to f.

2. $\hat{f}$ satisfies the collar condition with parameter ε'.

Proof. We first choose a diffeomorphism $\psi\colon [0,1) \to [0,1)$ such that:

- $\psi(t) = t$ for $t < \frac{\varepsilon}{2}$.
- $\psi(\varepsilon') = \varepsilon$.
- $\psi(t) = t$ for $t > \frac{1+\varepsilon'}{2}$.

Now we define the map $g\colon \Sigma \to \Sigma$ as follows:

$$\begin{aligned} i(p,t) &\mapsto i(p,\psi(t)) \text{ for } p \in M, t \in [0,1), \\ j(p,t) &\mapsto j(p,-\psi(-t)) \text{ for } p \in N, t \in (-1,0], \\ q &\mapsto q \text{ for } q \in \Sigma \setminus (i(M \times [0, \frac{1+\varepsilon'}{2}]) \cup j(N \times [-\frac{1+\varepsilon'}{2}, 0])). \end{aligned}$$

The condition that $\psi(t) = t$ for $t > \frac{1+\varepsilon'}{2}$ ensures that g is a diffeomorphism, and the condition that $\psi(t) = t$ for $t < \frac{\varepsilon}{2}$ ensures that g satisfies the collar condition with parameter $\frac{\varepsilon}{2}$ (note that the left and right induced maps of g are identities). We claim that g is ambiently isotopic to the identity map. We define:

$$I\colon \Sigma \times [0,1] \to \Sigma$$

by:

$$\begin{aligned} (i(p,t),s) &\mapsto i(p,(1-s)t + s\psi(t)) \text{ for } p \in M, t \in [0,1), \\ (j(p,t),s) &\mapsto j(p,(1-s)t - s\psi(-t)) \text{ for } p \in N, t \in (-1,0], \\ (q,s) &\mapsto q \text{ for } q \in \Sigma \setminus (i(M \times [0, \frac{1+\varepsilon'}{2}]) \cup j(N \times [-\frac{1+\varepsilon'}{2}, 0])). \end{aligned}$$

The condition that $\psi(t) = t$ for $t > \frac{1+\varepsilon'}{2}$ ensures that I is continuous (the three cases are all open subsets of $\Sigma \times [0,1]$), I_s is a diffeomorphism for all $s \in [0,1]$ and that I_s satisfies the collar condition with parameter $\frac{\varepsilon}{2}$. It is also clear from the definition that $I_1 = g$, therefore g is ambiently isotopic to the identity map on Σ. We define $g' \colon \Sigma' \to \Sigma'$ similarly, and define:

$$\hat{f} := g'^{-1} \circ f \circ g$$

and we claim that $\hat{f}$ satisfies the collar condition with parameter ε'. To show this, let $p \in M, t \in [0, \varepsilon')$, then we have:

$$\begin{aligned}\hat{f}(i(p,t)) &= (g'^{-1} \circ f \circ g)(i(p,t)) \\ &= (g'^{-1} \circ f)(i(p,\psi(t))) \\ &= g'^{-1}(i'(f_L(p),\psi(t))) \\ &= i'(f_L(p),t) \\ &= i'(\hat{f}_L(p),t).\end{aligned}$$

Similarly for the right side let $p \in N, t \in (-\varepsilon', 0]$, then we have:

$$\begin{aligned}\hat{f}(j(p,t)) &= (g'^{-1} \circ f \circ g)(j(p,t)) \\ &= (g'^{-1} \circ f)(j(p,-\psi(-t))) \\ &= g'^{-1}(j'(f_R(p),-\psi(-t))) \\ &= j'(f_R(p),t) \\ &= j'(\hat{f}_R(p),t).\end{aligned}$$

Hence we see that $\hat{f}$ satisfies the collar condition with parameter ε'. To see that $\hat{f}$ is ambiently isotopic to f note that since g and g' are ambiently isotopic to identities we have that:

$$\begin{aligned}[\hat{f}] &= [g'^{-1} \circ f \circ g] \\ &= [g'^{-1}] \circ [f] \circ [g] \\ &= [\mathrm{id}'_\Sigma] \circ [f] \circ [\mathrm{id}_\Sigma] \\ &= [\mathrm{id}_{\Sigma'} \circ f \circ \mathrm{id}_\Sigma] \\ &= [f].\end{aligned}$$

Hence $\hat{f}$ is ambiently isotopic to f as required (note that we use square brackets around functions to represent equivalence classes under isotopy, similar to the way we represent points in a quotient topology). $\square$

This allows us to prove that our unitors will form a natural transformation, that is, the required diagram of isotopy classes now commutes.

Theorem 5.2.8. *The left and right unitors are natural transformations.*

Proof. We give the proof for the left unitor. Let:

$$(i, \Sigma, j)\colon M \to N,$$
$$(i', \Sigma', j')\colon M' \to N',$$

be smooth collared cobordisms and let:

$$f\colon (i, \Sigma, j) \to (i', \Sigma', j')$$

be a collar compatible diffeomorphism of smooth collared cobordisms. We must show that the maps:

$$l_{\Sigma'} \circ (f \bigcirc U(f_R)),$$
$$f \circ l_{\Sigma},$$

are ambiently isotopic. By lemma 5.2.7 let $\hat{f}$ be ambiently isotopic to f and satisfy the collar condition with parameter $\frac{7}{8}$. We claim that:

$$l_{\Sigma'} \circ (\hat{f} \bigcirc U(\hat{f}_R)) = \hat{f} \circ l_{\Sigma}.$$

There are three cases to consider:

1. $[(q, 0)]$ for $q \in \Sigma \setminus j(N \times [-\frac{3}{4}, 0])$.
2. $[(j(p, t), 0)]$ for $p \in N, t \in (-\frac{7}{8}, 0]$.
3. $[((p, t), 1)]$ for $p \in N, t \in [0, 3]$.

For case 1 let $q \in \Sigma \setminus j(N \times [-\frac{3}{4}, 0])$. Then we have:

$$\begin{aligned}(l_{\Sigma'} \circ (\hat{f} \bigcirc U(\hat{f}_R)))([(q, 0)]) &= l_{\Sigma'}([(\hat{f}(q), 0)]) \\ &= \hat{f}(q) \\ &= \hat{f}(l_{\Sigma}([(q, 0)])).\end{aligned}$$

Where we have used the fact that $\hat{f}$ satisfies the collar condition with parameter $\frac{7}{8}$ and $q \notin j(N \times [-\frac{3}{4}, 0])$, so $\hat{f}(q) \notin j'(N' \times [-\frac{3}{4}, 0])$. For case 2 let $p \in N, t \in (-\frac{7}{8}, 0]$ then we have:

$$\begin{aligned}(l_{\Sigma'} \circ (\hat{f} \bigcirc U(\hat{f}_R)))([(j(p, t), 0)]) &= l_{\Sigma'}([(\hat{f}(j(p, t)), 0)]) \\ &= l_{\Sigma'}([(j'(\hat{f}_R(p), t), 0)]) \\ &= j'(\hat{f}_R(p), \psi_L(t)).\end{aligned}$$

Similarly for the other map:

$$\begin{aligned}(\hat{f} \circ l_\Sigma)([(j(p,t),0)]) &= \hat{f}(j(p,\psi_L(t))) \\ &= j'(\hat{f}_R(p),\psi_L(t)).\end{aligned}$$

Where we use the fact that since $\psi_L(t) = t$ for $t < -\frac{1}{2}$ and ψ_L is increasing, if $t > -\frac{7}{8}$ then $\psi_L(t) > -\frac{7}{8}$, together with the fact that $\hat{f}$ satisfies the collar condition with parameter $\frac{7}{8}$. So the two maps agree in case 2. Finally for case 3 let $p \in N, t \in [0,3]$, then we have:

$$\begin{aligned}(l_{\Sigma'} \circ (\hat{f} \bigcirc U(\hat{f}_R)))([((p,t),1)]) &= l_{\Sigma'}([((\hat{f}_R(p),t),1)]) \\ &= j'(\hat{f}_R(p),\psi_L(t)).\end{aligned}$$

Similarly for the other map:

$$\begin{aligned}(\hat{f} \circ l_\Sigma)([((p,t),1)]) &= \hat{f}(j(p,\psi_L(t))) \\ &= j'(\hat{f}_R(p),\psi_L(t)).\end{aligned}$$

Since we have that $\psi_L(t) > \psi_L(0) > -\frac{7}{8}$ and $\hat{f}$ satisfies the collar condition with parameter $\frac{7}{8}$. Thus we have:

$$l_{\Sigma'} \circ (\hat{f} \bigcirc U(\hat{f}_R)) = \hat{f} \circ l_\Sigma.$$

Therefore since $\hat{f}$ is ambiently isotopic to f and $\hat{f}_R = f_R$ we have:

$$\begin{aligned}[l_{\Sigma'} \circ (f \bigcirc U(f_R))] &= [l_{\Sigma'}] \circ ([f] \bigcirc [U(f_R)]) \\ &= [l_{\Sigma'}] \circ ([\hat{f}] \bigcirc [U(\hat{f}_R)]) \\ &= [l_{\Sigma'} \circ (\hat{f} \bigcirc U(\hat{f}_R))] \\ &= [\hat{f} \circ l_\Sigma] \\ &= [\hat{f}] \circ [l_\Sigma] \\ &= [f] \circ [l_\Sigma] \\ &= [f \circ l_\Sigma].\end{aligned}$$

Therefore $l_{\Sigma'} \circ (f \bigcirc U(f_R))$ is ambiently isotopic to $f \circ l_\Sigma$ as required. Thus the left unitor l given component-wise by the isotopy class of the unitor map l_Σ defined in definition 5.1.13 is a natural transformation in the category $(n, n+1)$-$\mathbf{Cob}_1$. The proof for the right unitor is analogous. □

Finally we verify that the unitors and associator satisfy the triangle axiom. First we prove a lemma.

Lemma 5.2.9 (Middle Shifting Lemma)**.** Let:

$$(i, \Sigma, j) \colon L \to M,$$
$$(i', \Sigma', j') \colon M \to N,$$

be smooth collared cobordisms. Let:

$$\psi \colon (-1, 1) \to (-1, 1)$$

be a diffeomorphism such that $\psi(t) = t$ for $t \notin (-\varepsilon, \varepsilon)$ for some $\varepsilon \in (0, 1)$, then the map:

$$f \colon (i', \Sigma', j') \bigcirc (i, \Sigma, j) \to (i', \Sigma', j') \bigcirc (i, \Sigma, j)$$

given by:

$$q \mapsto \langle j, i' \rangle \circ (\mathrm{id}_M \times \psi) \circ \langle j, i' \rangle^{-1}(q) \text{ for } q \in \mathrm{im}\langle j, i' \rangle,$$
$$q \mapsto q \text{ otherwise},$$

is a collar compatible diffeomorphism and is ambiently isotopic to the identity map.

Proof. The argument that f is a diffeomorphism is simply an application of lemma 2.3.16 (the non-standard local representation at the glued collar is $\mathrm{id}_M \times \psi$, the other two are identities). Define:

$$I \colon (-1, 1) \times (-1, 1) \to (-1, 1)$$
$$(t, s) \mapsto (1 - s)t + s\psi(t).$$

As usual we define:

$$\tilde{\Sigma} := \Sigma \sqcup \Sigma' / \sim$$

then we claim that the map:

$$J \colon \tilde{\Sigma} \times [0, 1] \to \tilde{\Sigma}$$

given by:

$$(q, s) \mapsto \langle j, i' \rangle \circ (\mathrm{id}_M \times I) \circ \langle j, i' \rangle^{-1}(q, s) \text{ if } q \in \mathrm{im}(\langle j, i' \rangle),$$
$$([(p, 0)], s) \mapsto [(p, 0)] \text{ if } p \in \Sigma \setminus j(M \times [-\frac{-1 - \varepsilon}{2}, 0]),$$
$$[(p, 1)] \mapsto [(p, 1)] \text{ if } p \in \Sigma' \setminus i'(M \times [0, \frac{1 + \varepsilon}{2}]),$$

is a collar compatible half-smooth ambient isotopy from the identity map to f. Note that the three cases correspond to open subsets of $\tilde{\Sigma} \times [0,1]$ and since $\psi(t) = t$ for $t \notin (-\varepsilon, \varepsilon)$ the definitions agree on the intersection of the subsets. Thus J is well-defined and continuous by the glue lemma, also by the glue lemma J_s is a diffeomorphism for all $s \in [0,1]$. It is clear from the definition that J_s satisfies the collar condition with parameter 1 for all $s \in [0,1]$. Since $I(t,0) = t$ we have that J_0 is the identity map, and since $I(t,1) = \psi(t)$ we have that $J_1 \circ \mathrm{id}_{\tilde{\Sigma}} = f$. Therefore J is a collar compatible half-smooth ambient isotopy from $\mathrm{id}_{\tilde{\Sigma}}$ to f. □

This allows us to prove that the triangle axiom holds up to isotopy (that is, it holds for the quotiented construction, despite not holding for the unquotiented construction).

Proposition 5.2.10. Let L, M, N be n-dimensional smooth manifolds without boundary and let:

$$(i, \Sigma, j) \colon L \to M,$$
$$(i', \Sigma', j') \colon M \to N,$$

be smooth collared cobordisms, then the diagram:

$$\begin{array}{ccc}
\big((i',\Sigma',j') \bigcirc U(M)\big) \bigcirc (i,\Sigma,j) & & \\
\Big\downarrow{\scriptstyle a_{\Sigma,U(N),\Sigma'}} & \searrow{\scriptstyle r_{\Sigma'} \bigcirc \mathrm{id}_{\Sigma}} & \\
(i',\Sigma',j') \bigcirc \big(U(M) \bigcirc (i,\Sigma,j)\big) & \xrightarrow[\mathrm{id}_{\Sigma'} \bigcirc l_{\Sigma}]{} & (i',\Sigma',j') \bigcirc (i,\Sigma,j)
\end{array}$$

commutes.

Proof. As with naturality of the unitors, the maps are not equal, but they are ambiently isotopic. We use lemma 5.2.9. In particular we will show that there exists $f \colon \tilde{\Sigma} \to \tilde{\Sigma}$ which is of the required form and such that:

$$f \circ (\mathrm{id}_{\Sigma'} \bigcirc l_{\Sigma}) \circ a_{\Sigma,U(M),\Sigma'} = r_{\Sigma'} \bigcirc \mathrm{id}_{\Sigma}.$$

To see that such an f exists we first evaluate both sides of the expression without f. There are five cases of interest, namely:

1. $[(p,0)]$ for $p \in \Sigma \setminus j(M \times [-\frac{3}{4}, 0])$.
2. $[([(p,1)],1)]$ for $p \in \Sigma' \setminus i'(M \times [0, \frac{3}{4}])$.
3. $[(j(p,t),0)]$ for $p \in M, t \in (-1,0]$.

4. $[([(i'(p,t),1)],1)]$ for $p \in M, t \in [0,1)$.

5. $[([((p,t),0)],1)]$ for $p \in M, t \in [0,3]$.

For case 1, if $p \in \Sigma \setminus j(M \times [-\frac{3}{4}, 0])$ then for the left path we have:

$$\begin{aligned}((\mathrm{id}_{\Sigma'} \bigcirc l_\Sigma) \circ a_{\Sigma,U(N),\Sigma'})([(p,0)]) &= (\mathrm{id}_{\Sigma'} \bigcirc l_\Sigma)([([(p,0)],0)]) \\ &= [(l_\Sigma([(p,0)]),0)] \\ &= [(p,0)],\end{aligned}$$

by the definition of l_Σ, and for the right path:

$$\begin{aligned}r_{\Sigma'} \bigcirc \mathrm{id}_\Sigma([(p,0)]) &= [(\mathrm{id}_\Sigma(p),0)] \\ &= [(p,0)].\end{aligned}$$

So we see that the two maps agree for these points. Similarly for case 2 let $p \in \Sigma' \setminus i'(M \times [0, \frac{3}{4}])$, then we have:

$$\begin{aligned}((\mathrm{id}_{\Sigma'} \bigcirc l_\Sigma) \circ a_{\Sigma,U(N),\Sigma'})([([(p,1)],1)]) &= (\mathrm{id}_{\Sigma'} \bigcirc l_\Sigma)([(p,1)]) \\ &= [(\mathrm{id}'_\Sigma(p),1)] \\ &= [(p,1)].\end{aligned}$$

Again for the right path we have:

$$\begin{aligned}(r'_\Sigma \bigcirc \mathrm{id}_\sigma)([([(p,1)],1)]) &= [(r_\Sigma([(p,1)],1)] \\ &= [(p,1)],\end{aligned}$$

by the definition of $r_{\Sigma'}$, and so the maps agree for these points as well. Now for case 3 let $p \in M, t \in (-1,0]$, then:

$$\begin{aligned}((\mathrm{id}_{\Sigma'} \bigcirc l_\Sigma) \circ a_{\Sigma,U(N),\Sigma'})([(j(p,t),0)]) &= (\mathrm{id}_{\Sigma'} \bigcirc l_\Sigma)([([(j(p,t),0)],0)]) \\ &= [(l_\Sigma([(j(p,t),0)]),0)] \\ &= [(j(p,\psi_L(t)),0)].\end{aligned}$$

For the right path:

$$\begin{aligned}(r_{\Sigma'} \bigcirc \mathrm{id}_\Sigma)([(j(p,t),0)]) &= [(\mathrm{id}_\Sigma(j(p,t)),0)] \\ &= [(j(p,t),0)].\end{aligned}$$

As we can see the two maps do not agree on this region. For case 4 let $p \in M, t \in [0,1)$, then we have:

$$\begin{aligned}((\mathrm{id}_{\Sigma'} \bigcirc l_\Sigma) \circ a_{\Sigma,U(N),\Sigma'})([([[(i'(p,t),1)],1)]) &= (\mathrm{id}_{\Sigma'} \bigcirc l_\Sigma)([(i'(p,t),1)]) \\ &= [(\mathrm{id}_{\Sigma'}(i'(p,t)),1)] \\ &= [(i'(p,t),1)].\end{aligned}$$

For the right path:

$$\begin{aligned}(r_{\Sigma'} \bigcirc \mathrm{id}_\Sigma)([([(i'(p,t),1)],1)]) &= [(r_{\Sigma'}(i'(p,t)),1)] \\ &= [(i'(p,\psi_R(t)),1)].\end{aligned}$$

Again the two maps do not agree on this region. Finally for case 5 let $p \in M, t \in [0,3]$, then:

$$\begin{aligned}((\mathrm{id}_{\Sigma'} \bigcirc l_\Sigma) \circ a_{\Sigma,U(N),\Sigma'})([([((p,t),0)],1)]) &= (\mathrm{id}_{\Sigma'} \bigcirc l_\Sigma)([([((p,t),1)],0)]) \\ &= [(l_\Sigma([((p,t),1)]),0)] \\ &= [(j(p,\psi_L(t)),0)].\end{aligned}$$

For the right path:

$$\begin{aligned}(r_{\Sigma'} \bigcirc \mathrm{id}_\Sigma)([([((p,t),0)],1)]) &= [(r_{\Sigma'}([((p,t),0)]),1)] \\ &= [(i'(p,\psi_R(t-3)),1)].\end{aligned}$$

So the two maps disagree on these points as well.

To prove that the two maps are ambiently isotopic we use lemma 5.2.9. We define the map:

$$\psi\colon (-1,1) \to (-1,1)$$

in stages. First define the map:

$$\psi_1\colon (-1,1) \to (-1,4)$$

by:

$$\psi_1(t) = \begin{cases} \psi_L^{-1}(t) \text{ if } t < 0, \\ t+3 \text{ if } t > -\frac{1}{4}, \end{cases}$$

To see that ψ_1 is well-defined note that for $-\frac{1}{4} < t < 0$ we have that $\psi_L^{-1}(t) = t+3$ (since $\psi_L(t) = t-3$ for $t > 2+\frac{3}{4}$). Thus ψ_1 is a diffeomorphism. We now define:

$$\psi_2\colon (-1,4) \to (-4,1)$$

by:

$$t \mapsto t - 3.$$

Now finally we define:

$$\psi_3 \colon (-4, 1) \to (-1, 1)$$

by:

$$\psi_3(t) = \begin{cases} t + 3 \text{ if } t < -(2 + \frac{3}{4}), \\ \psi_R(t) \text{ if } t > -3. \end{cases}$$

To see that ψ_3 is a diffeomorphism note that since $\psi_R(t) = t + 3$ for $t < -2\frac{1}{2}$ we have that for $-3 < t < -(2 + \frac{3}{4})$ the two definitions agree on the overlap. We now define:

$$\psi = \psi_3 \circ \psi_2 \circ \psi_1.$$

To use lemma 5.2.9 we must show that ψ is the identity outside of $(-\frac{3}{4}, \frac{3}{4})$, so let $t < -\frac{3}{4}$, then:

$$\begin{aligned} \psi(t) &= \psi_3 \circ \psi_2 \circ \psi_1(t) \\ &= \psi_3 \circ \psi_2(\psi_L^{-1}(t)) \\ &= \psi_3 \circ \psi_2(t) \\ &= \psi_3(t - 3) \\ &= (t - 3) + 3 \\ &= t. \end{aligned}$$

Similarly let $t > \frac{3}{4}$, then:

$$\begin{aligned} \psi(t) &= (\psi_3 \circ \psi_2 \circ \psi_1)(t) \\ &= (\psi_3 \circ \psi_2)(t) \\ &= \psi_3(t + 3) \\ &= (t + 3) - 3 \\ &= t. \end{aligned}$$

Therefore by lemma 5.2.9 the function f given by:

$$\begin{aligned} &q \mapsto \langle j, i' \rangle \circ (\mathrm{id}_M \times \psi) \circ \langle j, i' \rangle^{-1}(q) \text{ for } q \in \mathrm{im}\langle j, i' \rangle, \\ &q \mapsto q \text{ otherwise}, \end{aligned}$$

is ambiently isotopic to the identity map. We now claim that:

$$f \circ (\mathrm{id}'_\Sigma \bigcirc l_\Sigma) \circ a_{\Sigma, U(M), \Sigma'} = r_{\Sigma'} \bigcirc \mathrm{id}_\Sigma.$$

To show this we will first show that ψ satisfies:

$$\psi(t) = \psi_L^{-1}(t) \text{ if } t \leq \psi_L(0),$$
$$\psi(t) = \psi_R(\psi_L^{-1}(t) - 3) \text{ if } \psi_L(0) \leq t \leq 0,$$
$$\psi(t) = \psi_R(t) \text{ if } t \geq 0.$$

For the first condition, let $t \leq \psi_L(0)$, then since $\psi_L(0) < 0$ we have that $t < 0$ and therefore:

$$\begin{aligned}\psi(t) &= (\psi_3 \circ \psi_2 \circ \psi_1(t)) \\ &= (\psi_3 \circ \psi_2)(\psi_L^{-1}(t)) \\ &= \psi_3(\psi_L^{-1}(t) - 3) \\ &= \psi_L^{-1}(t).\end{aligned}$$

Since $\psi_L^{-1}(t) < 0$ and therefore $\psi_L^{-1}(t) - 3 < -3$. For the second condition let $t \in [\psi_L(0), 0]$, then we have:

$$\begin{aligned}\psi(t) &= (\psi_3 \circ \psi_2 \circ \psi_1)(t) \\ &= \psi_3 \circ \psi_2(\psi_L^{-1}(t)) \\ &= \psi_3(\psi_L^{-1}(t) - 3) \\ &= \psi_R(\psi_L^{-1}(t) - 3).\end{aligned}$$

Since we have that $\psi_L^{-1}(t) - 3 \geq -3$. Finally let $t \in [0, 1)$, then:

$$\begin{aligned}\psi(t) &= (\psi_3 \circ \psi_2 \circ \psi_1)(t) \\ &= (\psi_3 \circ \psi_2)(t + 3) \\ &= \psi_3(t) \\ &= \psi_R(t).\end{aligned}$$

Finally we use these conditions to show that f satisfies:

$$f \circ (\mathrm{id}_{\Sigma'} \bigcirc l_\Sigma) \circ a_{\Sigma, U(M), \Sigma'} = r_{\Sigma'} \bigcirc \mathrm{id}_\Sigma.$$

It is clear that this holds for cases 1 and 2, so we verify case 3. Let $p \in M, t \in (-1, 0]$, then using our previous analysis of this case:

$$\begin{aligned}(f \circ (\mathrm{id}_{\Sigma'} \bigcirc l_\Sigma) \circ a_{\Sigma, U(M), \Sigma'})([(j(p,t), 0)]) &= f([(j(p, \psi_L(t)), 0)]) \\ &= [(j(p, \psi(\psi_L(t))), 0)] \\ &= [(j(p, \psi_L^{-1}(\psi_L(t))), 0)] \\ &= [(j(p,t), 0)] \\ &= (r_{\Sigma'} \bigcirc \mathrm{id}_\Sigma)([(j(p,t), 0)]).\end{aligned}$$

Similarly for case 4 let $p \in M, t \in [0,1)$, then:

$$\begin{aligned}(f \circ (\mathrm{id}_{\Sigma'} \bigcirc l_\Sigma) \circ a_{\Sigma,U(M),\Sigma'})([([[(i'(p,t),1)],1)]) &= f([(i'(p,t),1)]) \\ &= \langle j,i'\rangle(p,\psi(t)) \\ &= \langle j,i'\rangle(p,\psi_R(t)) \\ &= [(i'(p,\psi_R(t)),1)] \\ &= (r_{\Sigma'} \bigcirc \mathrm{id}_\Sigma)([([[(i'(p,t),1)],1)]).\end{aligned}$$

Finally for case 5, let $p \in M, t \in [0,3]$, then:

$$\begin{aligned}(f \circ (\mathrm{id}_{\Sigma'} \bigcirc l_\Sigma) \circ a_{\Sigma,U(M),\Sigma'})([([[((p,t),0)],1)]) &= f([(j(p,\psi_L(t)),0)]) \\ &= \langle j,i'\rangle(p,\psi(\psi_L(t))) \\ &= \langle j,i'\rangle(p,\psi_R(\psi_L^{-1}(\psi_L(t))-3)) \\ &= \langle j,i'\rangle(p,\psi_R(t-3)) \\ &= [(i'(p,\psi_R(t-3)),1)] \\ &= (r_{\Sigma'} \bigcirc \mathrm{id}_\Sigma)([([[((p,t),0)],1)]).\end{aligned}$$

Therefore we do indeed have that:

$$f \circ (\mathrm{id}_{\Sigma'} \bigcirc l_\Sigma) \circ a_{\Sigma,U(M),\Sigma'} = r_{\Sigma'} \bigcirc \mathrm{id}_\Sigma.$$

Finally since by lemma 5.2.9 f is ambiently isotopic to the identity map we have:

$$\begin{aligned}[(\mathrm{id}_{\Sigma'} \bigcirc l_\Sigma) \circ a_{\Sigma,U(M),\Sigma'}] &= [\mathrm{id}_{\tilde{\Sigma}} \circ (\mathrm{id}_{\Sigma'} \bigcirc l_\Sigma) \circ a_{\Sigma,U(M),\Sigma'}] \\ &= [\mathrm{id}_{\tilde{\Sigma}}] \circ [(\mathrm{id}_{\Sigma'} \bigcirc l_\Sigma) \circ a_{\Sigma,U(M),\Sigma'}] \\ &= [f] \circ (\mathrm{id}_{\Sigma'} \bigcirc l_\Sigma) \circ a_{\Sigma,U(M),\Sigma'}] \\ &= [f \circ (\mathrm{id}_{\Sigma'} \bigcirc l_\Sigma) \circ a_{\Sigma,U(M),\Sigma'}] \\ &= [r_{\Sigma'} \bigcirc \mathrm{id}_\Sigma].\end{aligned}$$

So the two maps are ambiently isotopic as required. □

We have now proven all we need for the main theorem of this section and one of the major results of this thesis.

Theorem 5.2.11. *The following data define a pseudo-double category.*

- *The category* $(n, n+1)$-$\mathbf{Cob}_0$ *of compact smooth manifolds without boundary and diffeomorphisms.*

- *The category* $(n, n+1)$-$\mathbf{Cob_1}$ *with objects smooth collared cobordisms and morphisms equivalence classes of collar compatible diffeomorphisms of cobordisms under the relation of collar compatible half-smooth ambient isotopy defined in definition 5.2.1.*
- *The source and target functors* S, T *defined in definition 5.1.2 which take a smooth collared cobordism to its source and target manifold respectively, and take an isotopy class of diffeomorphisms to their shared left and right induced maps respectively.*
- *The horizontal composition* $\bigcirc$ *defined in definition 5.1.6, which takes a pair of smooth collared cobordisms to their collar dependent composition, and a pair of isotopy classes of diffeomorphisms* $[f]$ *and* $[g]$ *to* $[g \bigcirc f]$.
- *The associator transformation defined component-wise as the isotopy class of the associator map defined in definition 5.1.9.*
- *The left and right unitors defined component-wise as the isotopy classes of the left and right unitor maps defined in definitions 5.1.13 and 5.1.14 respectively.*

Proof. The required conditions and their proofs are:

- That $(n, n+1)$-$\mathbf{Cob_0}$ is a category is clear, it is a subcategory of the category $\mathbf{Diff}_n$ of n-dimensional smooth manifolds and smooth maps.
- That the morphisms of $(n, n+1)$-$\mathbf{Cob_1}$ are well-defined is proposition 5.2.3.
- That composition in $(n, n+1)$-$\mathbf{Cob_1}$ (tight composition of 2-morphisms) is well-defined is proposition 5.2.4.
- That $(n, n+1)$-$\mathbf{Cob_1}$ is a category is clear given the above.
- That the source and target functors are well-defined is proposition 5.2.6.
- That the source, target and unit functors are functors is propositions 5.1.4 and 5.1.5.
- That the structure functors satisfy the required identity is clear from the definition of the unit functor.
- That loose (horizontal) composition is well-defined is proposition 5.2.5.

- That loose composition is a functor is clear from the above and proposition 5.1.7.
- That loose composition satisfies the required identities with the source and target functors is proposition 5.1.8.
- That the associator morphism is a natural isomorphism and satisfies the pentagon axioms is clear from propositions 5.1.11 and 5.1.12.
- That the unitor morphisms are isomorphisms is proposition 5.1.16.
- That the source and target functors map the structure morphisms to identities (i.e. the structure morphisms are globular) is clear from proposition 5.1.17.
- That the unitor morphisms are natural transformations is theorem 5.2.8.
- That the structure morphisms satisfy the triangle axiom is proposition 5.2.10.

□

5.2.1 The Dependence on the Choice of ψ_L and ψ_R

In definitions 5.1.13 and 5.1.14 we chose diffeomorphisms ψ_L and ψ_R with a given set of properties. It is of note that once we introduce the condition of isotopy to the 2-morphisms (which we must do due to proposition 5.1.18) these choices cease to affect the construction, in that any two different choices for ψ_L and ψ_R would have resulted in the same isotopy class of diffeomorphism for the unitors. We present here a proof of this.

Theorem 5.2.12. *Let:*

$$\psi_L, \hat{\psi}_L \colon (-1, 3] \to (-1, 0]$$

be diffeomorphisms satisfying the conditions laid out in definition 5.1.13, that is:

- $\psi_L(t) = \hat{\psi}_L(t) = t$ *for* $t < -\frac{1}{2}$.
- $\psi_L(t) = \hat{\psi}_L(t) = t - 3$ *for* $t > 2 + \frac{3}{4}$.

Then for any smooth collared cobordism:

$$(i, \Sigma, j) \colon M \to N$$

the left unitors:

$$l_\Sigma, \hat{l}_\Sigma \colon U(N) \bigcirc (i, \Sigma, j) \to (i, \Sigma, j)$$

defined as in definition 5.1.13 using ψ_L and $\hat{\psi}_L$ respectively are ambiently isotopic.

Proof. We first define the map:

$$J \colon (-1, 3] \times [0, 1] \to (-1, 0]$$

by:

$$(t, s) \mapsto (1 - s)\psi_L(t) + s\hat{\psi}_L(t).$$

As usual for each $s \in [0, 1]$ define:

$$J_s \colon (-1, 3] \to (-1, 0]$$

by:

$$t \mapsto J(t, s).$$

We have that

$$J_s'(t) = (1 - s)\psi_L'(t) + s\hat{\psi}_L'(t) > 0$$

and hence J_s is a diffeomorphism by inverse function theorem.

We now define the map $I \colon \Sigma \times [0, 1] \to \Sigma$ by:

$$(j(p, \psi_L(t)), s) \mapsto j(p, (1 - s)\psi_L(t) + s\hat{\psi}_L(t))$$

for $p \in N, t \in (-1, 3], s \in [0, 1]$ and:

$$q \mapsto q$$

for $q \notin j(N \times (-\frac{3}{4}, 0])$.

The condition that $\psi_L(t) = \hat{\psi}_L(t)$ for $t < -\frac{1}{2}$ ensures that I_s is well-defined and a diffeomorphism for all $s \in [0, 1]$, and that I is continuous. To see that I_0 is the identity map on Σ let $p \in N$ and $t \in (-1, 3]$, then:

$$\begin{aligned} I_0(j(p, \psi_L(t))) &= j(p, (1 - 0)\psi_L(t) + 0\hat{\psi}_L(t)) \\ &= j(p, \psi_L(t)). \end{aligned}$$

Further we have that:

$$\begin{aligned} I_1(j(p,\psi_L(t))) &= j(p,(1-1)\psi_L(t)+1\hat{\psi}_L(t)) \\ &= j(p,\hat{\psi}_L(t)). \end{aligned}$$

So $I_1 \circ l_\Sigma = \hat{l}_\Sigma$. The fact that $\psi_L(t) = \hat{\psi}_L(t) = t-3$ for $t > 2+\frac{3}{4}$ ensures that for all $s \in [0,1]$ the map J_s satisfies the collar condition with parameter $\frac{1}{4}$. It is clear that the left and right induced maps of J_s are identities, so J is a collar compatible half-smooth ambient isotopy from l_Σ to $\hat{l}_\Sigma$. $\square$

The proof of the analogous result that the isotopy class of r_Σ is independent of the choice of ψ_R is similar. Hence the pseudo double category of manifolds, cobordisms and half-smooth isotopy classes of cobordisms does not depend on the choices made in definitions 5.1.13 and 5.1.14, choosing different diffeomorphisms would yield the same pseudo double category.

5.3 The Tensor Product of Disjoint Union.

In this section we define a tensor product which will make $(n,n+1)$-**Cob** a monoidal pseudo double category. We will work almost entirely in the unquotiented construction, since all the required diagrams commute without requiring the quotient by ambient isotopy. We will prove that the disjoint union of diffeomorphisms descends to the quotient, and hence we will obtain a monoidal pseudo double category. In effect, we are constructing a monoidal 'almost' pseudo double category, from which we can transfer the monoidal structure onto the quotiented pseudo double category defined in theorem 5.2.11.
We begin with our category of objects. Recall that by the *disjoint union* of two sets A and B we mean the set $(A \times \{0\}) \cup (B \times \{1\})$.

Definition 5.3.1. The functor:

$$\sqcup^0 \colon (n,n+1)\text{-}\mathbf{Cob_0} \times (n,n+1)\text{-}\mathbf{Cob_0} \to (n,n+1)\text{-}\mathbf{Cob_0}$$

takes a pair M, N of closed smooth manifolds to the manifold $M \sqcup N$ with the disjoint union topology and smooth structure. Given diffeomorphisms:

$$\begin{aligned} f &\colon M \to M', \\ g &\colon N \to N', \end{aligned}$$

we define:

$$f \sqcup g \colon M \sqcup N \to M' \sqcup N'$$

by:

$$(p,0) \mapsto (f(p),0) \text{ for } p \in M,$$
$$(p,1) \mapsto (g(p),1) \text{ for } p \in N.$$

Similarly we define the action of $\sqcup$ on $(n,n+1)$-$\mathbf{Cob}_1$.

Definition 5.3.2. The functor:

$$\sqcup^1\colon (n,n+1)\text{-}\mathbf{Cob}_1 \times (n,n+1)\text{-}\mathbf{Cob}_1 \to (n,n+1)\text{-}\mathbf{Cob}_1$$

acts on objects as follows. Given closed n-dimensional smooth manifolds M, N, M', N' and smooth collared cobordisms:

$$(i,\Sigma,j)\colon M \to N,$$
$$(i',\Sigma',j')\colon M' \to N',$$

the cobordism:

$$(i,\Sigma,j) \sqcup (i',\Sigma',j')\colon M \sqcup M' \to N \sqcup N'$$

is the triple $(i \sqcup i', \Sigma \sqcup \Sigma', j \sqcup j')$ where:

$$i \sqcup i'\colon M \sqcup M' \times [0,1) \to \Sigma \sqcup \Sigma'$$

is given by:

$$((p,0),t) \mapsto (i(p,t),0) \text{ for } p \in M, t \in [0,1),$$
$$((p,1),t) \mapsto (i'(p,t),1) \text{ for } p \in M', t \in [0,1),$$

and similarly:

$$j \sqcup j'\colon N \sqcup N' \times (-1,0] \to \Sigma \sqcup \Sigma'$$

is given by:

$$((p,0),t) \mapsto (j(p,t),0) \text{ for } p \in N, t \in (-1,0],$$
$$((p,1),t) \mapsto (j'(p,t),1) \text{ for } p \in N', t \in (-1,0].$$

As with objects $\Sigma \sqcup \Sigma'$ is equipped with the disjoint union topology and smooth structure. As with tight 1-morphisms we define, for diffeomorphisms of cobordisms:

$$f\colon (i_1,\Sigma_1,j_1) \to (i_2,\Sigma_2,j_2),$$
$$g\colon (i'_1,\Sigma'_1,j'_1) \to (i'_2,\Sigma'_2,j'_2).$$

the map:

$$f \sqcup g \colon \Sigma_1 \sqcup \Sigma_1' \to \Sigma_2 \sqcup \Sigma_2'$$

by:

$$\begin{aligned}(p, 0) &\mapsto (f(p), 0),\\ (p, 1) &\mapsto (g(p), 1).\end{aligned}$$

We begin by verifying that $\sqcup^0$ is a functor.

Proposition 5.3.3. $\sqcup^0$ is a functor.

Proof. Let K, M, N, K', M', N' be closed n-dimensional smooth manifolds and let:

$$\begin{aligned}f &\colon K \to M,\\ g &\colon M \to N,\\ f' &\colon K' \to M',\\ g' &\colon M' \to N',\end{aligned}$$

be diffeomorphisms. Then we must show that:

$$(g \circ f) \sqcup (g' \circ f') = (g \sqcup g') \circ (f \sqcup f').$$

We evaluate both maps, let $p \in K$, then:

$$\begin{aligned}(g \circ f) \sqcup (g' \circ f')(p, 0) &= ((g \circ f)(p), 0)\\ &= (g(f(p)), 0)\\ &= (g \sqcup g')(f(p), 0)\\ &= (g \sqcup g')((f \sqcup f')(p, 0))\\ &= ((g \sqcup g') \circ (f \sqcup f'))(p, 0).\end{aligned}$$

So $\sqcup^0$ respects composition. The proof that $\sqcup^0$ preserves identities is similar. □

To show that $\sqcup^1$ is a functor we must first show that it is well-defined, both on loose 1-morphisms and 2-morphisms.

Proposition 5.3.4. Let M, N, M', N' be closed n-dimensional smooth manifolds and let:

$$\begin{aligned}(i, \Sigma, j) &\colon M \to N,\\ (i', \Sigma', j') &\colon M' \to N',\end{aligned}$$

be smooth collared cobordisms. Then $(i, \Sigma, j) \sqcup (i', \Sigma', j')$ is a smooth collared cobordism from $M \sqcup M'$ to $N \sqcup N'$.

Proof. That $\Sigma \sqcup \Sigma'$ is an $n+1$-dimensional smooth manifold is clear, as is the fact that $i \sqcup i'$ and $j \sqcup j'$ are smooth embeddings (that they are both smooth embeddings follows from the glue lemma). □

We have two results for 2-morphisms. First that the disjoint union of collar compatible diffeomorphisms is a collar compatible diffeomorphism, and that this operation descends to the quotient.

Proposition 5.3.5. Let $M_1, N_1, M_2, N_2, M_1', N_1', M_2', N_2'$ be n-dimensional smooth manifolds without boundary, let:

$$\begin{aligned}(i_1, \Sigma_1, j_1) &\colon M_1 \to N_1,\\ (i_2, \Sigma_2, j_2) &\colon M_2 \to M_2,\\ (i_1', \Sigma_1', j_1') &\colon M_1' \to N_1',\\ (i_2', \Sigma_2', j_2') &\colon M_2' \to N_2',\end{aligned}$$

be smooth collared cobordisms and let:

$$\begin{aligned}f &\colon (i_1, \Sigma_1, j_1) \to (i_2, \Sigma_2, j_2),\\ f' &\colon (i_1', \Sigma_1', j_1') \to (i_2', \Sigma_2', j_2'),\end{aligned}$$

be collar compatible diffeomorphisms of smooth collared cobordisms. Then $f \sqcup f'$ is a collar compatible diffeomorphism from $(i_1, \Sigma_1, j_1) \sqcup (i_1', \Sigma_1', j_1')$ to $(i_2, \Sigma_2, j_2) \sqcup (i_2', \Sigma_2', j_2')$.

Proof. If f satisfies the collar condition with parameter ε and f' satisfies the collar condition with parameter ε' then $f \sqcup f'$ satisfies the collar condition with parameter $\min(\varepsilon, \varepsilon')$. That $f \sqcup f'$ is a diffeomorphism is clear. □

We must also show that the disjoint union of diffeomorphisms descends to the quotient under the relation of isotopy. This is also straightforward.

Proposition 5.3.6. Let $M_1, N_1, M_2, N_2, M_1', N_1', M_2', N_2'$ be n-dimensional smooth manifolds without boundary, let:

$$\begin{aligned}(i_1, \Sigma_1, j_1) &\colon M_1 \to N_1,\\ (i_2, \Sigma_2, j_2) &\colon M_2 \to M_2,\\ (i_1', \Sigma_1', j_1') &\colon M_1' \to N_1',\\ (i_2', \Sigma_2', j_2') &\colon M_2' \to N_2',\end{aligned}$$

be smooth collared cobordisms and let:

$$f, f' \colon (i_1, \Sigma_1, j_1) \to (i_2, \Sigma_2, j_2),$$
$$g, g' \colon (i'_1, \Sigma'_1, j'_1) \to (i'_2, \Sigma'_2, j'_2),$$

be collar compatible diffeomorphisms of smooth collared cobordisms. and suppose f is ambiently isotopic to f' and g is ambiently isotopic to g'. Then $f \sqcup g$ is ambiently isotopic to $f' \sqcup g'$.

Proof. If I is a collar compatible half-smooth ambient isotopy from f to f' and I' is a collar compatible half-smooth ambient isotopy from g to g', we define the map:

$$J \colon (\Sigma_2 \sqcup \Sigma'_2) \times [0,1] \to \Sigma_2 \sqcup \Sigma'_2$$

by:

$$((p,0),t) \mapsto (I(p,t),0) \text{ for } p \in \Sigma_2, t \in [0,1],$$
$$((p,1),t) \mapsto (I'(p,t),1) \text{ for } p \in \Sigma'_2, t \in [0,1].$$

It is clear that J is a collar compatible half-smooth ambient isotopy from $f \sqcup g$ to $f' \sqcup g'$. □

Therefore the disjoint union of smooth collared cobordisms and isotopy classes of diffeomorphisms of smooth collared cobordisms is well-defined. We now prove it is a functor.

Proposition 5.3.7. $\sqcup^1$ is a functor.

Proof. That $\sqcup^1$ is a functor in the unquotiented construction is the same as the proof for $\sqcup^0$. That it is a functor in the quotiented construction follows from proposition 5.3.6. □

Now that we have our functors $\sqcup^0$ and $\sqcup^1$ we must show that they are compatible with the source and target functors.

Proposition 5.3.8. The functors $\sqcup^0$ and $\sqcup^1$ together with the structure functors S and T satisfy the equations in definition 3.2.1.

Proof. That this holds for objects is clear from the definitions, for morphisms let:

$$(i_1, \Sigma_1, j_1) \colon M_1 \to N_1,$$
$$(i_2, \Sigma_2, j_2) \colon M_2 \to N_2,$$
$$(i'_1, \Sigma'_1, j'_1) \colon M'_1 \to N'_1,$$
$$(i'_2, \Sigma'_2, j'_2) \colon M'_2 \to M'_2,$$

be smooth collared cobordisms and let :

$$f\colon (i_1,\Sigma_1,j_1)\to(i_2,\Sigma_2,j_2),$$
$$f'\colon (i'_1,\Sigma'_1,j'_1)\to(i'_2,\Sigma'_2,j'_2).$$

Then we claim that:

$$S(f\sqcup^1 f') = S(f)\sqcup^0 S(f'),$$
$$T(f\sqcup^1 f') = T(f)\sqcup^0 T(f').$$

It is clear from the definitions that the left induced map of $f\sqcup f'$ is the disjoint union of the left induced map of f and the left induced map of f'. That the second equation holds is similarly clear. □

To show that $\sqcup$ is a pseudo double functor we must give natural transformations $\sqcup^{\bigcirc}$ and $\sqcup^{U}$ as in definition 3.2.1.

Definition 5.3.9. Let L, M, N, L', M', N' be n-dimensional smooth manifolds and let:

$$(i_1,\Sigma_1,j_1)\colon L\to M,$$
$$(i_2,\Sigma_2,j_2)\colon M\to N,$$
$$(i'_1,\Sigma'_1,j'_1)\colon L'\to M',$$
$$(i'_2,\Sigma'_2,j'_2)\colon M'\to N'.$$

be smooth collared cobordisms. The morphism $\sqcup^{\bigcirc}_{\Sigma_1,\Sigma'_1,\Sigma_2\Sigma'_2}$ from

$$((i_2,\Sigma_2,j_2)\sqcup(i'_2,\Sigma'_2,j'_2))\bigcirc((i_1,\Sigma_1,j_1)\sqcup(i'_1,\Sigma'_1,j'_1))$$

to

$$((i_2,\Sigma_2,j_2)\bigcirc(i_1,\Sigma_1,j_1))\sqcup((i'_2,\Sigma'_2,j'_2)\bigcirc(i'_1,\Sigma'_1,j'_1))$$

is defined by:

$$[((p,0),0)]\mapsto([(p,0)],0) \text{ for } p\in\Sigma_1.$$
$$[((p,1),0)]\mapsto([(p,0)],1) \text{ for } p\in\Sigma'_1.$$
$$[((p,0),1)]\mapsto([(p,1)],0) \text{ for } p\in\Sigma_2.$$
$$[((p,1),1)]\mapsto([(p,1)],1) \text{ for } p\in\Sigma'_2.$$

Proposition 5.3.10. The components of $\sqcup^{\bigcirc}$ are diffeomorphisms and satisfy the collar condition with parameter 1.

Proof. Clear from the definition. □

Proposition 5.3.11. $\sqcup^{\bigcirc}$ is a natural transformation.

Proof. Clear from the definitions. □

Remarks 5.3.12. Similarly to the associator and unitor morphisms $\sqcup^{\bigcirc}$ is strictly speaking indexed by quadruples of smooth collared cobordisms, though we use only the underlying manifolds in the notation for readability.

We also define the natural transformation $\sqcup^U$.

Definition 5.3.13. Let M, M' be n-dimensional smooth manifolds. The morphism:

$$\sqcup^U_{M,M'} : U(M \sqcup M') \to U(M) \sqcup U(M'),$$

is defined by:

$$((p,0),t) \mapsto ((p,t),0) \text{ for } p \in M, t \in [0,3],$$
$$((p,1),t) \mapsto ((p,t),1) \text{ for } p \in M', t \in [0,3].$$

Proposition 5.3.14. The components of $\sqcup^U$ are diffeomorphisms and satisfy the collar condition with parameter 1.

Proof. Clear from the definition. □

Proposition 5.3.15. $\sqcup^U$ is a natural transformation.

Proof. Clear from the definitions. □

To complete the proof that $\sqcup$ is a double functor we must show that the diagrams in definition 3.2.1 commute.

Proposition 5.3.16. Let $K, L, M, N, K', L', M', N'$ be n dimensional smooth manifolds without boundary, and let:

$$(i_1, \Sigma_1, j_1) \colon K \to L,$$
$$(i_2, \Sigma_2, j_2) \colon L \to M,$$
$$(i_3, \Sigma_3, j_3) \colon M \to N,$$
$$(i'_1, \Sigma'_1, j'_1) \colon K' \to L',$$
$$(i'_2, \Sigma'_2, j'_2) \colon L' \to M',$$
$$(i'_3, \Sigma'_3, j'_3) \colon M' \to N',$$

be smooth collared cobordisms. Then the following diagram:

$$
\begin{array}{ccc}
\big((\Sigma_3 \sqcup \Sigma_3') \bigcirc (\Sigma_2 \sqcup \Sigma_2')\big) \bigcirc (\Sigma_1 \sqcup \Sigma_1') & \xrightarrow{a} & (\Sigma_3 \sqcup \Sigma_3') \bigcirc \big((\Sigma_2 \sqcup \Sigma_2') \bigcirc (\Sigma_1 \sqcup \Sigma_1')\big) \\
{\scriptstyle \sqcup^{\bigcirc}_{\Sigma_2,\Sigma_2',\Sigma_3,\Sigma_3'} \bigcirc \mathrm{id}_{\Sigma_1 \sqcup \Sigma_1'}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{\Sigma_3 \sqcup \Sigma_3'} \bigcirc \sqcup^{\bigcirc}_{\Sigma_1,\Sigma_1',\Sigma_2,\Sigma_2'}} \\
\big((\Sigma_3 \bigcirc \Sigma_2) \sqcup (\Sigma_3' \bigcirc \Sigma_2')\big) \bigcirc (\Sigma_1 \sqcup \Sigma_1') & & (\Sigma_3 \sqcup \Sigma_3') \bigcirc \big((\Sigma_2 \bigcirc \Sigma_1) \sqcup (\Sigma_2' \bigcirc \Sigma_1')\big) \\
{\scriptstyle \sqcup^{\bigcirc}_{\Sigma_1,\Sigma_1',\Sigma_3 \bigcirc \Sigma_2,\Sigma_3' \bigcirc \Sigma_2'}} \downarrow & & \downarrow {\scriptstyle \sqcup^{\bigcirc}_{\Sigma_2 \bigcirc \Sigma_1,\Sigma_2' \bigcirc \Sigma_1',\Sigma_3,\Sigma_3'}} \\
\big((\Sigma_3 \bigcirc \Sigma_2) \bigcirc \Sigma_1\big) \sqcup \big((\Sigma_3' \bigcirc \Sigma_2') \bigcirc \Sigma_1'\big) & \xrightarrow{a \sqcup a} & \big(\Sigma_3 \bigcirc (\Sigma_2 \bigcirc \Sigma_1)\big) \sqcup \big(\Sigma_3' \bigcirc (\Sigma_2' \bigcirc \Sigma_1')\big)
\end{array}
$$

commutes, where the subscripts of the associator morphisms have been ommited for readability.

Proof. As with naturality of $\sqcup^{\bigcirc}$ and $\sqcup^{U}$ this is clear from the definitions. As an example we demonstrate the equality for points of the form:

$$[([((p,0),0)],1)]$$

for $p \in \Sigma_2$. For the top path we have:

$$
\begin{aligned}
\big(\sqcup^{\bigcirc}_{\Sigma_2 \bigcirc \Sigma_1,\Sigma_2' \bigcirc \Sigma_1',\Sigma_3,\Sigma_3'} \circ (\mathrm{id}_{\Sigma_3 \sqcup \Sigma_3'} \bigcirc \sqcup^{\bigcirc}_{\Sigma_1,\Sigma_1',\Sigma_2,\Sigma_2'}) \circ a_{\Sigma_1 \sqcup \Sigma_1',\Sigma_2 \sqcup \Sigma_2',\Sigma_3 \sqcup \Sigma_3'}\big)&([([((p,0),0)],1)]) \\
&= \big(\sqcup^{\bigcirc}_{\Sigma_2 \bigcirc \Sigma_1,\Sigma_2' \bigcirc \Sigma_1',\Sigma_3,\Sigma_3'} \circ (\mathrm{id}_{\Sigma_3 \sqcup \Sigma_3'} \bigcirc \sqcup^{\bigcirc}_{\Sigma_1,\Sigma_1',\Sigma_2,\Sigma_2'})\big)([([((p,0),1)],0)]) \\
&= \sqcup^{\bigcirc}_{\Sigma_2 \bigcirc \Sigma_1,\Sigma_2' \bigcirc \Sigma_1',\Sigma_3,\Sigma_3'}([(\sqcup^{\bigcirc}_{\Sigma_1,\Sigma_1',\Sigma_2,\Sigma_2'}([((p,0),1)]),0)]) \\
&= \sqcup^{\bigcirc}_{\Sigma_2 \bigcirc \Sigma_1,\Sigma_2' \bigcirc \Sigma_1',\Sigma_3,\Sigma_3'}([(([(p,1)],0),0)]) \\
&= ([([(p,1)],0)],0).
\end{aligned}
$$

For the bottom path we have:

$$
\begin{aligned}
\big((a_{\Sigma_1,\Sigma_2,\Sigma_3} \sqcup a_{\Sigma_1,\Sigma_2,\Sigma_3}) \circ (\sqcup^{\bigcirc}_{\Sigma_1,\Sigma_1',\Sigma_3 \bigcirc \Sigma_2,\Sigma_3' \bigcirc \Sigma_2'}) \circ (\sqcup^{\bigcirc}_{\Sigma_2,\Sigma_2',\Sigma_3,\Sigma_3'} \bigcirc \mathrm{id}_{\Sigma_1 \sqcup \Sigma_1'})\big)&([([((p,0),0)],1)]) \\
&= \big((a_{\Sigma_1,\Sigma_2,\Sigma_3} \sqcup a_{\Sigma_1,\Sigma_2,\Sigma_3}) \circ (\sqcup^{\bigcirc}_{\Sigma_1,\Sigma_1',\Sigma_3 \bigcirc \Sigma_2,\Sigma_3' \bigcirc \Sigma_2'})\big)([(\sqcup^{\bigcirc}_{\Sigma_2,\Sigma_2',\Sigma_1,\Sigma_1'}([((p,0),0)]),1)]) \\
&= \big((a_{\Sigma_1,\Sigma_2,\Sigma_3} \sqcup a_{\Sigma_1,\Sigma_2,\Sigma_3}) \circ (\sqcup^{\bigcirc}_{\Sigma_1,\Sigma_1',\Sigma_3 \bigcirc \Sigma_2,\Sigma_3' \bigcirc \Sigma_2'})\big)([(([(p,0)],0),1)]) \\
&= (a_{\Sigma_1,\Sigma_2,\Sigma_3} \sqcup a_{\Sigma_1,\Sigma_2,\Sigma_3})([([(p,0)],1)],0) \\
&= (a_{\Sigma_1,\Sigma_2,\Sigma_3}([([(p,0)],1)]),0) \\
&= ([([(p,1)],0)],0).
\end{aligned}
$$

So the two maps agree for these points. The argument for the other five cases is exactly the same. $\square$

Finally we prove the compatibility with the unitor morphisms.

Proposition 5.3.17. Let M, N, M', N' be n-dimensional smooth manifolds without boundary and let:

$$(i, \Sigma, j)\colon M \to N,$$
$$(i', \Sigma', j')\colon M' \to N',$$

be smooth collared cobordisms. Then the following diagrams commute:

$$\begin{array}{ccc}
(\Sigma \sqcup \Sigma') \bigcirc U(M \sqcup M') & \xrightarrow{r_{\Sigma \sqcup \Sigma'}} & \Sigma \sqcup \Sigma' \\
\Big\downarrow {\scriptstyle \mathrm{id}_{\Sigma \sqcup \Sigma'} \bigcirc \sqcup^U_{M,M'}} & & \Big\uparrow {\scriptstyle r_\Sigma \sqcup r_{\Sigma'}} \\
(\Sigma \sqcup \Sigma') \bigcirc (U(M) \sqcup U(M')) & \xrightarrow{\sqcup^{\bigcirc}_{U(M),U(M'),\Sigma,\Sigma'},} & (\Sigma \bigcirc U(M)) \sqcup (\Sigma' \bigcirc U(M'))
\end{array}$$

$$\begin{array}{ccc}
U(N \sqcup N') \bigcirc (\Sigma \sqcup \Sigma') & \xrightarrow{l_{\Sigma \sqcup \Sigma'}} & \Sigma \sqcup \Sigma' \\
\Big\downarrow {\scriptstyle \sqcup^U_{N,N'} \bigcirc \mathrm{id}_{\Sigma \sqcup \Sigma'}} & & \Big\uparrow {\scriptstyle l_\Sigma \sqcup l_{\Sigma'}} \\
(U(N) \sqcup U(N')) \bigcirc (\Sigma \sqcup \Sigma') & \xrightarrow{\sqcup^{\bigcirc}_{\Sigma,\Sigma',U(N),U(N')}} & (U(N) \bigcirc \Sigma) \sqcup (U(N') \bigcirc \Sigma')
\end{array}$$

Proof. As usual this is clear from the definitions, we present the proof for the top diagram, for points of the form $[(((p,0),t),0)]$ for $p \in M, t \in [0,3]$. For the top path we have:

$$r_{\Sigma \sqcup \Sigma'}([(((p,0),t),0)]) = (i(p, \psi_R(t)), 0).$$

For the bottom path we have:

$$\begin{aligned}
\big((r_\Sigma \sqcup r_{\Sigma'}) \circ \sqcup^{\bigcirc}_{U(M),U(M'),\Sigma,\Sigma'} \circ (\mathrm{id}_{\Sigma \sqcup \Sigma'} \bigcirc \sqcup^U_{M,M'})\big)([(((p,0),t),0)]) \\
= \big((r_\Sigma \sqcup r_{\Sigma'}) \circ \sqcup^{\bigcirc}_{U(M),U(M'),\Sigma,\Sigma'}\big)([(\sqcup^U_{M,M'}((p,0),t),0)]) \\
= \big((r_\Sigma \sqcup r_{\Sigma'}) \circ \sqcup^{\bigcirc}_{U(M),U(M'),\Sigma,\Sigma'}\big)([(((p,t),0),0)]) \\
= (r_\Sigma \sqcup r_{\Sigma'})(([((p,t),0)]),0) \\
= (r_\Sigma([((p,t),0)]),0) \\
= (i(p,\psi_R(t)),0).
\end{aligned}$$

So we see that the two maps agree on these points. The argument for the other three cases is the same, as is the argument for the second diagram. □

We are now ready to define the associativity and unit constraints for both $(n, n+1)$-$\mathbf{Cob}_0$ and $(n, n+1)$-$\mathbf{Cob}_1$.

Definition 5.3.18. Let L, M, N, be n-dimensional smooth manifolds without boundary, then the associativity constraint:

$$\alpha^0_{L,M,N} \colon (L \sqcup M) \sqcup N \to L \sqcup (M \sqcup N)$$

is defined by:

$$\begin{aligned} ((p,0),0) &\mapsto (p,0) \text{ for } p \in L, \\ ((p,1),0) &\mapsto ((p,0),1) \text{ for } p \in M, \\ (p,1) &\mapsto ((p,1),1) \text{ for } p \in N. \end{aligned}$$

Similarly let $M_1, N_1, M_2, N_2, M_3, N_3$ be n-dimensional smooth manifolds without boundary and let:

$$\begin{aligned} (i_1, \Sigma_1, j_1) &\colon M_1 \to N_1, \\ (i_2, \Sigma_2, j_2) &\colon M_2 \to N_2, \\ (i_3, \Sigma_3, j_3) &\colon M_3 \to N_3, \end{aligned}$$

be smooth collared cobordisms. Then the associativity constraint:

$$\alpha^1_{\Sigma_1,\Sigma_2,\Sigma_3} \colon (\Sigma_1 \sqcup \Sigma_2) \sqcup \Sigma_3 \to \Sigma_1 \sqcup (\Sigma_2 \sqcup \Sigma_3)$$

is defined by:

$$\begin{aligned} ((p,0),0) &\mapsto (p,0) \text{ for } p \in \Sigma_1, \\ ((p,1),0) &\mapsto ((p,0),1) \text{ for } p \in \Sigma_2, \\ (p,1) &\mapsto ((p,1),1) \text{ for } p \in \Sigma_3. \end{aligned}$$

Remark 5.3.19. As with the associator, the associativity constraint α^1 is indexed by triples of loose 1-morphisms, and so we should write:

$$\alpha_{(i_1,\Sigma_1,j_1),(i_2,\Sigma_2,j_2),(i_3,\Sigma_3,j_3)},$$

however as for the associator we write $\alpha_{\Sigma_1,\Sigma_2,\Sigma_3}$ for brevity.

The identity object is the empty manifold, the identity loose 1-morphism is the empty cobordism $(\emptyset, \emptyset, \emptyset)$ (see remarks 2.3.7 and 2.5.4). We now define the left and right unit constraints for both $(n, n+1)$-$\mathbf{Cob}_0$ and $(n, n+1)$-$\mathbf{Cob}_1$.

Definition 5.3.20. The left and right unit constraints:

$$\lambda^0_M \colon \emptyset \sqcup M \to M$$

and:

$$\rho^0_M \colon M \sqcup \emptyset \to M$$

are defined by:

$$l_M(p,1) = p,$$
$$r_M(p,0) = p.$$

Similarly the left and right unit constraints:

$$l^1_\Sigma \colon \emptyset \sqcup \Sigma \to \Sigma$$

and:

$$r^1_\Sigma \colon \Sigma \sqcup \emptyset \to \Sigma$$

are given by:

$$l_\Sigma(p,1) = p,$$
$$r_\Sigma(p,0) = p.$$

Remark 5.3.21. It is clear that the associativity and unit constraints are isomorphisms in their respective categories. It is also clear from the definitions that the components of the associativity and unit constraints $\alpha^1, \lambda^1, \rho^1$ satisfy the collar condition with parameter 1.

We now show that the associativity and unit constraints are natural transformations. Since the proofs are similar we present the argument for $(n, n+1)$-$\mathbf{Cob}_1$

Proposition 5.3.22. Let $M_1, N_1, M_2, N_2, M_3, N_3, M'_1, N'_1, M'_2, N'_2, M'_3, N'_3$ be n-dimensional smooth manifolds without boundary, let:

$$(i_1, \Sigma_1, j_1) \colon M_1 \to N_1,$$
$$(i_2, \Sigma_2, j_2) \colon M_2 \to N_2,$$
$$(i_3, \Sigma_3, j_3) \colon M_3 \to N_3,$$
$$(i'_1, \Sigma'_1, j'_1) \colon M'_1 \to N'_1,$$
$$(i'_2, \Sigma'_2, j'_2) \colon M'_2 \to N'_2,$$
$$(i'_3, \Sigma'_3, j'_3) \colon M'_3 \to N'_3,$$

be smooth collared cobordisms and let:

$$\begin{aligned} f &\colon (i_1, \Sigma_1, j_1) \to (i_1', \Sigma_1', j_1'), \\ g &\colon (i_2, \Sigma_2, j_2) \to (i_2', \Sigma_2', j_2'), \\ h &\colon (i_3, \Sigma_3, j_3) \to (i_3', \Sigma_3', j_3'), \end{aligned}$$

be collar compatible diffeomorphisms of smooth collared cobordisms. Then:

$$\alpha^1_{\Sigma_1', \Sigma_2', \Sigma_3'} \circ ((f \sqcup g) \sqcup h) = (f \sqcup (g \sqcup h)) \circ \alpha^1_{\Sigma_1, \Sigma_2, \Sigma_3}.$$

That is, the following diagram commutes:

$$\begin{array}{ccc} \big(\Sigma_1 \sqcup \Sigma_2\big) \sqcup \Sigma_3 & \xrightarrow{\alpha^1_{\Sigma_1,\Sigma_2,\Sigma_3}} & \Sigma_1 \sqcup \big(\Sigma_2 \sqcup \Sigma_3\big) \\ {\scriptstyle (f\sqcup g)\sqcup h}\downarrow & & \downarrow{\scriptstyle f\sqcup(g\sqcup h)} \\ \big(\Sigma_1' \sqcup \Sigma_2'\big) \sqcup \Sigma_3' & \xrightarrow{\alpha^1_{\Sigma_1',\Sigma_2',\Sigma_3'}} & \Sigma_1' \sqcup \big(\Sigma_2' \sqcup \Sigma_3'\big) \end{array}$$

Proof. This is clear from the definitions, we will demonstrate the equality for points of the form $((p, 0), 0)$ for $p \in \Sigma_1$, for the bottom path we have:

$$\begin{aligned} \alpha^1_{\Sigma_1', \Sigma_2', \Sigma_3'} \circ ((f \sqcup g) \sqcup h)((p, 0), 0) &= \alpha^1_{\Sigma_1', \Sigma_2', \Sigma_3'}((f(p), 0), 0) \\ &= (f(p), 0). \end{aligned}$$

Similarly for the top path we have:

$$\begin{aligned} (f \sqcup (g \sqcup h)) \circ \alpha^1_{\Sigma_1, \Sigma_2, \Sigma_3}((p, 0), 0) &= (f \sqcup (g \sqcup h))(p, 0) \\ &= (f(p), 0). \end{aligned}$$

So the maps agree for these points. The proof that the maps agree for points of the form $((p, 0), 1)$ for $p \in \Sigma_2$ and $(p, 1)$ for $p \in \Sigma_3$ is similar. □

That the unit constraints are natural transformations is similarly straightforward.

Proposition 5.3.23. Let M, N, M', N' be n-dimensional smooth manifolds without boundary, let:

$$\begin{aligned} (i, \Sigma, j) &\colon M \to N, \\ (i', \Sigma', j') &\colon M' \to N', \end{aligned}$$

be smooth collared cobordisms and let:

$$f\colon (i,\Sigma,j)\to(i',\Sigma',j')$$

be a diffeomorphism of smooth collared cobordisms. Then:

$$\begin{aligned} f\circ\lambda^1_{\Sigma} &= \lambda^1_{\Sigma'}\circ(\mathrm{id}_{\emptyset}\sqcup f),\\ f\circ\rho^1_{\Sigma} &= \rho^1_{\Sigma'}\circ(f\sqcup\mathrm{id}_{\emptyset}). \end{aligned}$$

Proof. We evaluate both maps for the first equality. Let $p\in\Sigma$, then we have:

$$f\circ\lambda^1_{\Sigma}(p,1)=f(p).$$

Similarly:

$$\begin{aligned} \lambda^1_{\Sigma'}\circ(\mathrm{id}_{\emptyset}\sqcup f)(p,0) &= \lambda^1_{\Sigma'}(f(p),0)\\ &= f(p). \end{aligned}$$

So the first equality holds. The proof for the right unit constraint is similar. □

Finally to show that the associativity and unit constraints are natural transformations between double functors we must show that the diagrams in definition 3.2.2 commute. We begin with the associativity constraint.

Proposition 5.3.24. Let $L_1,L_2,L_3,M_1,M_2,M_3,N_1,N_2,N_3$ be n-dimensional smooth manifolds without boundary and let:

$$\begin{aligned} (i_1,\Sigma_1,j_1)&\colon L_1\to M_1,\\ (i'_1,\Sigma'_1,j'_1)&\colon M_1\to N_1,\\ (i_2,\Sigma_2,j_2)&\colon L_2\to M_2,\\ (i'_2,\Sigma'_2,j'_2)&\colon M_2\to N_2,\\ (i_3,\Sigma_3,j_3)&\colon L_3\to M_3,\\ (i'_3,\Sigma'_3,j'_3)&\colon M_3\to N_3, \end{aligned}$$

be smooth collared cobordisms. Then the following diagram commutes:

$$\begin{array}{ccc} \big((\Sigma'_1\sqcup\Sigma'_2)\sqcup\Sigma'_3)\big)\bigcirc\big((\Sigma_1\sqcup\Sigma_2)\sqcup\Sigma_3\big) & \xrightarrow{F^{\bigcirc}} & \big((\Sigma'_1\bigcirc\Sigma_1)\sqcup(\Sigma'_2\bigcirc\Sigma_2)\big)\sqcup(\Sigma'_3\bigcirc\Sigma_3) \\ \Big\downarrow{\scriptstyle \alpha^1\bigcirc\alpha^1} & & \Big\downarrow{\scriptstyle \alpha^1} \\ \big(\Sigma'_1\sqcup(\Sigma'_2\sqcup\Sigma'_3)\big)\bigcirc\big(\Sigma_1\sqcup(\Sigma_2\sqcup\Sigma_3)\big) & \xrightarrow{G^{\bigcirc}} & (\Sigma'_1\bigcirc\Sigma_1)\sqcup\big((\Sigma'_2\bigcirc\Sigma_2)\sqcup(\Sigma'_3\bigcirc\Sigma_3)\big) \end{array}$$

Where we have omitted the subscripts of $\sqcup^{\bigcirc}$ and α for readability. Note that F here is the double functor mapping the triple $\Sigma_1, \Sigma_2, \Sigma_3$ to $(\Sigma_1 \sqcup \Sigma_2) \sqcup \Sigma_3$, and G the one mapping it to $\Sigma_1 \sqcup (\Sigma_2 \sqcup \Sigma_3)$.

Proof. We demonstrate the equality for points of the form $[(((p,1),0),1)]$ for $p \in \Sigma_2'$. For the top path we have:

$$\begin{aligned}
&\big(\alpha^1_{\Sigma_1'\bigcirc\Sigma_1,\Sigma_2'\bigcirc\Sigma_2,\Sigma_3'\bigcirc\Sigma_3} \circ (F^{\bigcirc}_{\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_1',\Sigma_2',\Sigma_3'})\big)([(((p,1),0),1)]) \\
&\qquad = \alpha^1_{\Sigma_1'\bigcirc\Sigma_1,\Sigma_2'\bigcirc\Sigma_2,\Sigma_3'\bigcirc\Sigma_3}(([(p,1)],1),0) \\
&\qquad = (([(p,1)],0),1).
\end{aligned}$$

For the bottom path we have:

$$\begin{aligned}
&\big(G^{\bigcirc}_{\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_1',\Sigma_2',\Sigma_3'} \circ (\alpha^1_{\Sigma_1',\Sigma_2',\Sigma_3'} \bigcirc \alpha^1_{\Sigma_1,\Sigma_2,\Sigma_3})\big)([(((p,1),0),1)]) \\
&\qquad = G^{\bigcirc}_{\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_1',\Sigma_2',\Sigma_3'}([(\alpha^1_{\Sigma_1',\Sigma_2',\Sigma_3'}((p,1),0),1)]) \\
&\qquad = G^{\bigcirc}_{\Sigma_1,\Sigma_2,\Sigma_3,\Sigma_1',\Sigma_2',\Sigma_3'}([(((p,0),1),1)]) \\
&\qquad = (([(p,1)],0),1).
\end{aligned}$$

So we see that the two maps agree for these points. The other five cases are similar. □

Finally the condition on α^0.

Proposition 5.3.25. Let M_1, M_2, M_3 be n-dimensional smooth manifolds without boundary, then the following diagram commutes:

$$\begin{array}{ccc}
U((M_1 \sqcup M_2) \sqcup M_3) & \xrightarrow{F^U_{M_1,M_2,M_3}} & (U(M_1) \sqcup U(M_2)) \sqcup U(M_3) \\
\downarrow{\scriptstyle U(\alpha^0_{M_1,M_2,M_3})} & & \downarrow{\scriptstyle \alpha^0_{U(M_1),U(M_2),U(M_3)}} \\
U(M_1 \sqcup (M_2 \sqcup M_3)) & \xrightarrow{G^U_{M_1,M_2,M_3}} & U(M_1) \sqcup (U(M_2) \sqcup U(M_3))
\end{array}$$

Where F and G are as in proposition 5.3.24

Proof. We demonstrate the equality for points of the form $((p,1),t)$ for $p \in M_3, t \in [0,3]$. For the top path we have:

$$\begin{aligned}
&\alpha^0_{U(M_1),U(M_2),U(M_3)} \circ F^{\bigcirc}_{M_1,M_2,M_3}((p,1),t) \\
&\qquad = \alpha^0_{U(M_1),U(M_2),U(M_3)}((p,t),1) \\
&\qquad = (((p,t),1),1).
\end{aligned}$$

For the bottom path we have:

$$\begin{aligned}G^U_{M_1,M_2,M_3} &\circ U(\alpha^0_{M_1,M_2,M_3})((p,1),t)\\ &= G^U_{M_1,M_2,M_3}(((p,1),1),t)\\ &= (((p,t),1),1).\end{aligned}$$

So we see that the maps agree for these points. The other two cases are similar. □

This completes the proof that α is a tight double transformation.
Finally we prove that the unit constraints satisfy the analogous conditions. We present the proof for the left unit constraint.

Proposition 5.3.26. Let L, M, N be n-dimensional smooth manifolds without boundary and let:

$$(i,\Sigma,j)\colon L \to M,$$
$$(i',\Sigma',j')\colon M \to N,$$

be smooth collared cobordisms. Then the following diagram commutes:

$$\begin{array}{ccc}(\emptyset \sqcup \Sigma') \bigcirc (\emptyset \sqcup \Sigma) & \xrightarrow{F^{\bigcirc}_{\Sigma,\Sigma'}} & \emptyset \sqcup (\Sigma' \bigcirc \Sigma)\\ \Big\downarrow{\scriptstyle \lambda^1_{\Sigma'}\bigcirc\lambda^1_{\Sigma}} & & \Big\downarrow{\scriptstyle \lambda^1_{\Sigma'\bigcirc\Sigma}}\\ \Sigma' \bigcirc \Sigma & \xrightarrow{\text{id}} & \Sigma' \bigcirc \Sigma\end{array}$$

Note that our 'G' in this case is the identity functor.

Proof. We demonstrate the equality for points of the form $[((p,1),0)]$ for $p \in \Sigma$. For the top path we have:

$$\begin{aligned}\lambda^1_{\Sigma'\bigcirc\Sigma} &\circ F^{\bigcirc}_{\Sigma,\Sigma'}([((p,1),0)])\\ &= \lambda^1_{\Sigma'\bigcirc\Sigma}([(p,0)],1)\\ &= [(p,0)].\end{aligned}$$

For the bottom path we have:

$$\begin{aligned}&\lambda^1_{\Sigma'} \bigcirc \lambda^1_{\Sigma}([((p,1),0)])\\ &= [(\lambda^1_{\Sigma}(p,1),0)]\\ &= [(p,0)].\end{aligned}$$

So the two maps agree for these points. The argument for points of the form $[((p,1),1)]$ for $p \in \Sigma'$ is similar. □

Finally the condition on λ^0.

Proposition 5.3.27. Let M be an n-dimensional smooth manifold without boundary, then the following diagram commutes:

$$\begin{array}{ccc} U(\emptyset \sqcup M) & \xrightarrow{F^U_M} & \emptyset \sqcup U(M) \\ {\scriptstyle U(\lambda^0_M)}\downarrow & & \downarrow{\scriptstyle \lambda^0_{U(M)}} \\ U(M) & \xrightarrow{\mathrm{id}} & U(M) \end{array}$$

Proof. Let $p \in M, t \in [0,3]$, then for the top path we have:

$$\begin{aligned} &\lambda^0_{U(M)} \circ F^U_M((p,1),t) \\ &= \lambda^0_{U(M)}((p,t),1) \\ &= (p,t). \end{aligned}$$

For the bottom path we have:

$$U(\lambda^0_M((p,1),t)) = (p,t).$$

□

The proof for the right unit constraint is similar, and so the left and right unit constraints are tight double transformations.

Theorem 5.3.28. *The following data define a monoidal pseudo double category.*

- *The pseudo double category* $(n, n+1)$-**Cob** *defined earlier in this chapter.*
- *The double functor* $\sqcup$*, with the natural transformations whose components are the isotopy classes of the maps defined in definitions 5.3.9 and 5.3.13.*
- *The associativity constraint has components the isotopy classes of the maps defined in definition 5.3.18.*
- *The left and right unit constraints with components the isotopy classes of the maps defined in definition 5.3.20.*

Proof. The required conditions and their proofs are:

- That $\sqcup^0$ is a functor is proposition 5.3.3.
- That $\sqcup^1$ is well-defined is propositions 5.3.4, 5.3.5 and 5.3.6.
- That $\sqcup^1$ is a functor is proposition 5.3.7.
- That the two functors $\sqcup^0$ and $\sqcup^1$ are compatible with the source and target functors is proposition 5.3.8.
- That $\sqcup^{\bigcirc}$ is well defined is proposition 5.3.10.
- That $\sqcup^{\bigcirc}$ is a natural transformation is proposition 5.3.11.
- That $\sqcup^U$ is well defined is proposition 5.3.14.
- That $\sqcup^U$ is a natural transformation is proposition 5.3.15.
- That the diagrams in definition 3.2.1 commute is propositions 5.3.16 and 5.3.17.
- That the components of the associativity and unit constraints are isomorphisms in their respective categories is remark 5.3.21.
- That the associativity and unit constraints are natural transformations is propositions 5.3.22 and 5.3.23, with exactly analogous results (and proofs) for $(n, n+1)$-$\mathbf{Cob_0}$.
- That the associativity constraint is compatible with $\sqcup^{\bigcirc}$ and $\sqcup^U$ is propositions 5.3.24 and 5.3.25.
- The above condition for the left and right unit constraints is propositions 5.3.26 and 5.3.27, with exactly analogous results for the right unit constraint.

□

Chapter 6

Variations on the Pseudo Double Category of Manifolds, Cobordisms and Diffeomorphisms

The construction of the pseudo double category of manifolds, cobordisms and isotopy classes of diffeomorphisms of cobordisms can be done in (at least) three settings. We have presented, in the previous chapter, what we called the 'half-smooth' setting. In this chapter we discuss the other two settings, which we call the fully smooth setting and the topological setting.

6.1 Fully Smooth Isotopy

In the half-smooth setting we have as objects smooth manifolds, as tight 1-morphisms diffeomorphisms, as loose 1-morphisms smooth collared cobordisms, and as 2-morphisms isotopy classes of diffeomorphisms and in which an isotopy is a *continuously* parameterised family of diffeomorphisms. We could also have taken an isotopy to be a *smoothly* parameterised family of diffeomorphisms, this is what we call the *fully smooth* setting. Specifically, we take an isotopy to be the following.

Definition 6.1.1. Let M, N, M', N' be n-dimensional smooth manifolds without boundary, let:

$$(i, \Sigma, j)\colon M \to N,$$
$$(i', \Sigma', j')\colon M' \to N',$$

be smooth collared cobordisms and let:

$$f, g\colon \Sigma \to \Sigma'$$

be (collar compatible) diffeomorphisms, then a *collar compatible fully smooth ambient isotopy* from f to g is a continuous map

$$I\colon \Sigma' \times [0,1] \to \Sigma'$$

such that:

1. For all $t \in [0,1]$ the map $I_t\colon \Sigma' \to \Sigma'$ given by $p \mapsto I(p,t)$ is a diffeomorphism.

2. The left and right induced maps of I_t are identities for all $t \in [0,1]$

3. There exists $\delta > 0$ such that for all $t \in [0,1]$ the map $I_t\colon (i', \Sigma', j') \to (i', \Sigma', j')$ satisfies the collar condition with parameter δ.

4. I_0 is the identity map on Σ'.

5. $I_1 \circ f = g$.

6. There exists $\delta' > 0$ such that for all $t \in [0, \delta')$ $I_t = I_0$ and for all $t \in (1-\delta', 1], I_t = I_1$.

7. The restriction of I to the open subset $\Sigma' \times (\delta', 1-\delta')$ is smooth with respect to the product smooth structure obtained from that of Σ' and the usual smooth structure on $\mathbb{R}$.

If such an isotopy exists we say that f and g are *fully smoothly ambiently isotopic.*

Observe that the first five conditions are identical to those of a half-smooth ambient isotopy, hence any fully smooth ambient isotopy is also a half-smooth ambient isotopy.
As with half-smooth isotopy it is straightforwardly verified that fully smooth isotopy is an equivalence relation. The sixth condition is essential for this property, specifically transitivity, as it ensures that the 'composition' of two fully smooth isotopies (that is, performing one isotopy and then another) is itself fully smooth, as we see in the following example.

Example 6.1.2. Consider the cobordism:

$$(\emptyset, S^1, \emptyset)\colon \emptyset \to \emptyset$$

from the empty manifold to itself. Define the map:

$$f\colon S^1 \to S^1$$

by:

$$(\cos(\theta), \sin(\theta)) \mapsto (\cos(\theta + \frac{\pi}{2}), \sin(\theta + \frac{\pi}{2})).$$

Define also:

$$g\colon S^1 \to S^1$$

by:

$$(\cos(\theta), \sin(\theta)) \mapsto (\cos(\theta + \frac{3\pi}{2}), \sin(\theta + \frac{3\pi}{2})).$$

It is clear that both f and g are collar compatible diffeomorphisms (the collar condition is vacuous in this case as the collars are empty). We now define:

$$I\colon S^1 \times [0,1] \to S^1$$

by:

$$((\cos(\theta), \sin(\theta)), t) \mapsto (\cos(\theta + \frac{t\pi}{2}), \sin(\theta + \frac{t\pi}{2})).$$

I is a collar compatible half-smooth ambient isotopy from the identity map to f. We also define:

$$I'\colon S^1 \times [0,1] \to S^1$$

by:

$$((\cos(\theta), \sin(\theta)), t) \mapsto (\cos(\theta + t\pi), \sin(\theta + t\pi)).$$

I' is a collar compatible half-smooth ambient isotopy from f to g (observe that $I'_1 \circ f = g$). Both I and I' satisfy all conditions to be fully smooth aside from condition six, if we define their concatenation:

$$J\colon S^1 \times [0,1] \to S^1$$

in the usual manner we obtain the map:

$$J((\cos(\theta), \sin(\theta)), t) = \begin{cases} (\cos(\theta + t\pi), \sin(\theta + t\pi)) \text{ for } t \in [0, \frac{1}{2}], \\ (\cos(\theta + (2t - \frac{1}{2})\pi), \sin(\theta + (2t - \frac{1}{2})\pi)) \text{ for } t \in [\frac{1}{2}, 1]. \end{cases}$$

Note that as usual the second case is $I'_{2t-1} \circ I_1$. We see that J is a collar compatible half-smooth ambient isotopy from the identity map to g, but is not fully smooth since it is not differentiable at points of the form $(p, \frac{1}{2})$ for $p \in S^1$. Intuitively the rotation rate changes abruptly at $t = \frac{1}{2}$, so the isotopy fails to be have partial derivatives with respect to t at these points. This justifies the inclusion of condition six, and serves as an example of a half-smooth isotopy which is not fully smooth.

Remark 6.1.3. The example given in example 6.1.2 could be 'smoothed out' in a fairly straightforward manner, and it is clear that g is fully smoothly ambiently isotopic to the identity map. Whether this is always the case is not clear however, and will be discussed further in section 7.2.

The construction of the pseudo double category of closed smooth manifolds, smooth collared cobordisms and equivalence classes of collar compatible diffeomorphisms of cobordisms under collar compatible half-smooth isotopy performed in the previous chapter can be repeated with the relation of fully smooth isotopy. The unquotiented construction is the same, the constructions differ only in the equivalence relation placed upon the 2-morphisms. The only difficulty is ensuring the lemma 5.2.7 and lemma 5.2.9 continue to hold for the relation of fully smooth ambient isotopy. The isotopy constructed in the proof of lemma 5.2.7 for example is already fully smooth, satisfying condition 6 as a consequence of the chosen map ψ satisfying $\psi(t) = t$ for $t < \frac{\varepsilon}{2}$, hence the isotopy constructed therefrom satisfies condition 6 with $\delta' = \frac{\varepsilon}{2}$.

6.2 The Topological Setting

The construction can also be done in the topological setting, that is, we have as objects n-dimensional topological manifolds, as tight 1-morphisms homeomorphisms, as loose 1-morphisms topological cobordisms, and as 2-morphisms isotopy classes of homeomorphisms. We begin by making clear what we mean by a topological cobordism.

Definition 6.2.1. Let M, N be n-dimensional topological manifolds without boundary, a *topological collared cobordism* from M to N is a triple (i, Σ, j) where:

- Σ is an $(n+1)$-dimensional topological manifold.
- $i\colon M \times [0, 1) \to \Sigma$ is a topological embedding.
- $j\colon N \times (-1, 0] \to \Sigma$ is a topological embedding.

- $i(M \times \{0\}) \cup j(N \times \{0\}) = \partial\Sigma$.
- The closures of the images of i and j are disjoint.

The loose composition of topological cobordisms is simpler than that for smooth cobordisms, since we do not require a smooth structure.

Definition 6.2.2. Let K, M, N be n-dimensional topological manifolds without boundary and let:

$$(i, \Sigma, j) \colon K \to M,$$
$$(i', \Sigma', j') \colon M \to N,$$

be topological collared cobordisms, then the *loose composition*:

$$(i', \Sigma', j') \bigcirc (i, \Sigma, j)$$

is the cobordism:

$$(\tilde{i}, \tilde{\Sigma}, \tilde{j}) \colon K \to N$$

where $\tilde{\Sigma} = \Sigma \sqcup \Sigma' / \sim$ where $\sim$ is the equivalence relation given by $(j(p, 0), 0) \sim (i'(p, 0), 1)$ for $p \in M$, with $\tilde{i}(p, t) = [(i(p, t), 0)]$ and $\tilde{j}(p, t) = [(j'(p, t), 1)]$ as for smooth cobordisms.

The 2-morphisms are also defined similarly to the half-smooth and fully smooth settings.

Definition 6.2.3. Let M, N, M', N' be n-dimensional topological manifolds without boundary and let:

$$(i, \Sigma, j) \colon M \to N,$$
$$(i', \Sigma', j') \colon M' \to N',$$

be topological collared cobordisms. By a *homeomorphism of topological cobordisms* we mean a homeomorphism:

$$f \colon \Sigma \to \Sigma'$$

such that:

$$f(i(M \times \{0\})) = i'(M' \times \{0\}),$$
$$f(j(N \times \{0\})) = j'(N' \times \{0\}).$$

One might expect that we will not require the collar condition, since we do not require loose composition to give a smooth map, only a continuous one. However since the proof of theorem 5.2.8 required that maps satisfied the collar condition with some parameter to ensure that required diagrams commute, and those issues are still present in the topological setting, we still require the collar condition. The same issues with the unitors forming natural transformations (see proposition 5.1.18) and the triangle axiom mean we must still take our 2-morphisms to be isotopy classes. Whether the isotopy condition exempts us from defining the unitors directly, and hence allows us to avoid specifying collars for our cobordisms will be discussed at the end of this section, and further in section 7.1.

We now define the relation of topological isotopy.

Definition 6.2.4. Let M, N, M', N' be n-dimensional topological manifolds without boundary, let:

$$(i, \Sigma, j)\colon M \to N,$$
$$(i', \Sigma', j')\colon M' \to N',$$

be topological collared cobordisms and let:

$$f, g\colon (i, \Sigma, j) \to (i', \Sigma', j')$$

be homeomorphisms of topological collared cobordisms. A *topological ambient isotopy* from f to g is a continuous map

$$I\colon \Sigma' \times [0,1] \to \Sigma'$$

such that:

1. For all $t \in [0,1]$ the map $I_t\colon \Sigma' \to \Sigma'$ given by $p \mapsto I(p,t)$ is a diffeomorphism.

2. The left and right induced maps of I_t are identities for all $t \in [0,1]$.

3. I_0 is the identity map on Σ'.

4. $I_1 \circ f = g$.

If such an isotopy exists then we say that f and g are *ambiently isotopic*

As in the half and fully smooth settings it is easily verified that the relation of isotopy is an equivalence relation and that both tight and loose compositions descend to the quotient. The components of an ambient isotopy do not themselves need to satisfy the collar condition, since their loose composition need not be a smooth map.

Since we take our 2-morphisms up to isotopy, we do not actually need to specify the unitor morphism itself, only its isotopy class. Indeed, this is true in the half and fully smooth settings as well, so the collar's main purpose in those settings is to define the smooth structure after gluing. In the topological setting we do not require a smooth structure (not only do we not need to specify one, it is not necessary that one even exists, we can include so called *non-smoothable* topological manifolds, which do not admit any smooth structure (see for example the manifold defined in [Ker60], beginning on page 257), in our construction). It is possible to question therefore if the topological setting needs the collars at all, if there were a way to construct a canonical isotopy class of unitor morphism from uncollared cobordisms. This is one of several open problems which we will discuss in our final chapter.

Chapter 7

Open Problems and Further Investigation

In this chapter we discuss a number of possible avenues of further investigation related to the two main constructions of this thesis, as well as the variations discussed in chapter 6.

7.1 The Topological Setting for the Pseudo Double Category of Manifolds, Cobordisms and Diffeomorphisms

As mentioned in the previous chapter, we may not require our cobordisms to have collars in the topological setting, in particular if any two collarings of a cobordism were equivalent up to ambient isotopy, in the following sense.

Conjecture 7.1.1. Let:

$$(i, \Sigma, j) \colon M \to N$$

be a (uncollared) topological cobordism, and let:

$$\hat{\imath}, \hat{\imath}' \colon M \times [0, 1) \to \Sigma$$

be two collarings for i. Then there exists an ambient isotopy:

$$I \colon \Sigma \times [0, 1] \to \Sigma$$

such that I_0 is the identity map on Σ and $I_1 \circ \hat{\imath} = \hat{\imath}'$.

With a proof of this conjecture, and a similar result for the other side, we would be able to construct the pseudo double category in the topological setting with uncollared topological cobordisms. When defining the unitors, since we only need an isotopy class of a homeomorphism we can take any collaring, construct the unitor homeomorphism as we normally would, and then take the isotopy class. The above conjecture would, if true, ensure that the isotopy class of the homeomorphism we construct would not depend on the choice of collar. That these isotopy classes would form a natural transformation however is not easy to verify, as the proof of theorem 5.2.8 relies on the fact that the initial diffeomorphism (or homeomorphism in this new context) satisfies the collar condition, and that such a homeomorphism isotopic to any chosen homeomorphism exists is not obvious.

7.2 The Relationship Between the Three Settings, for Pseudo-Double Categories of Manifolds, Cobordisms and Maps between Cobordisms

The fully smooth setting and half-smooth settings share all structure except that the 2-morphisms are considered up to a (possibly) stronger equivalence relation. Formally there is clearly a 'forgetful' pseudo double functor from the fully smooth setting to the smooth setting which is the identity on objects and both kinds of 1-morphism, and which takes an equivalence class of diffeomorphisms under fully smooth isotopy to the equivalence class of diffeomorphisms under half-smooth isotopy which it sits inside. This functor is clearly full, every half-smooth equivalence class is partitioned into some number of fully smooth equivalence classes. The question of whether this functor is faithful, (and the two pseudo double categories are in fact equal) is a question of whether a half-smooth isotopy can be 'smoothed out'. Does the existence of a collar compatible half-smooth ambient isotopy from f to g imply the existence of a fully smooth one?

The relationship between the topological setting and the half-smooth setting is more complicated. Beginning with the half-smooth setting, we can 'forget' the smooth structures. Taking a smooth manifold to its underlying topological manifold, a smooth collared cobordism to the underlying topological collared cobordism (removing the smooth structure from Σ, leaving i, j unchanged) and taking an equivalence class of collar compatible diffeomorphisms of smooth collared cobordisms to the equivalence class of homeomorphisms (again collar compatible) under the relation of topological isotopy which contains it. This functor is not surjective on objects, due to the existence of non-smoothable manifolds, nor is it injective on objects or loose 1-morphisms. Faithfulness of this forgetful functor depends on whether two diffeomorphisms which are connected by a continuously parameterised family of homeomorphisms (a topological isotopy) are also connected by a continuously parameterised family of diffeomorphisms. In other words, can a topological isotopy between two diffeomorphisms be adjusted such that each of its components (I_t in the usual notation) is also a diffeomorphism, or are there some pairs of diffeomorphisms which cannot be connected without passing through a homeomorphism which is not smooth?

7.3 Extending to a Tricategory.

As was discussed briefly at the beginning of chapter 5, diffeomorphisms between cobordisms are typically used to witness an equivalence relation on cobordisms, making the equivalence classes the morphisms in a category, this is necessary to ensure that the composition of cobordisms (which has been our loose composition) is associative and has identities (see [CR18] or [Mil65]). In this thesis we have instead used said diffeomorphisms as 2-morphisms in a higher category, in our case a pseudo double category. In order to ensure that our unitors were natural transformations and that the triangle axiom was satisfied however, we were required to consider those diffeomorphisms up to an equivalence relation, that of ambient isotopy. One might ask then, if we can do the same extension again, and have the isotopies play the role of 3-morphisms in a tricategory, as defined in for example [GPS95]. There are three ways to compose isotopies, corresponding to our tight and loose compositions, as well as 'concatenation', performing one isotopy and then another, and two of these compositions are not strictly associative or unital. In addition our unitors would no longer be natural, they would only be natural up to isomorphism. The number of new natural isomorphisms involved in defining a higher category grows very quickly, and it seems likely that in order to guarantee all necessary diagrams commute one would need to take these isotopies up to some equivalence relation. It is possible that this process could continue indefinitely, leading to an ∞-category of isotopies and isotopies of isotopies and so forth.

7.4 Extending the Nested Tuples and Permumations Double Category From Symmetric Groups to Braid Groups

The construction in chapter 4 uses the direct sum and block permutation operations to define a double category of nested tuples and symmetric groups. Since the direct sum and block permutation operations have unique liftings from symmetric groups to braid groups, one might wonder if the construction in chapter 4 could be repeated with the braid group in place of the symmetric group. This construction would depend on the major results of the chapter (such as lemmas 4.3.7 and 4.3.13) also holding for braids. The loose composition would then operate as follows, given two 2-morphisms:

$$(b, (b_1, \ldots, b_m)) \colon m \nrightarrow n$$
$$(b', (b'_1, \ldots, b'_n)) \colon n \nrightarrow p.$$

The loose composition would have, as its i^{th} inner braid the braid obtained by taking the $t_{i,j}^{\text{th}}$ inner braids of $(b', (b'_1, \ldots, b'_m))$, and placing them inside the strands of the i^{th} inner braid of $(b, (b_1, \ldots, b_m))$.

7.5 Pseudo-Double Functors Connecting the Algebraic and Geometric Pseudo-Double Categories

In this thesis we have constructed two pseudo double categories of very different flavours, those being the (strict) double category of nested tuples and permutations, and that of manifolds, cobordisms, and isotopy classes of diffeomorphisms. In this section we discuss the connection between the two constructions. In particular we conjecture that there exists a (pseudo) double functor $F \colon \mathbf{Perm} \to (n, n+1)\text{-}\mathbf{Cob}$ in the case $n = 2$.

We describe the action of this conjectured functor on the objects, tight and loose 1-morphisms, and 2-morphisms of **Perm**. Given a non negative integer n, $F(n)$ is the disjoint union of n copies of S^2. A permutation $\sigma \in S_n$ is mapped to the diffeomorphism $f \colon \sqcup_{i=1}^n S^2 \to \sqcup_{i=1}^n S^2$ taking (p, i) to $(p, \sigma(i))$ (that is, mapping a point in the i^{th} sphere to the corresponding point in the $\sigma(i)^{\text{th}}$ sphere). For loose 1-morphisms, we take a tuple of length k to the 3-ball with k 'cavities' carved from its interior, evenly spaced along the x-axis. The nested tuple $((t_{1,1}, \ldots, t_{1,k_1}), \ldots, (t_{m,1}, \ldots, t_{1,k_m}))$ is then mapped to the cobordism (i, Σ, j) from m copies of S^2 to n copies of S^2 where Σ is the disjoint union across r of the k_r-cavitied 3-balls. With i taking the r^{th} sphere of $F(m)$ to the outer boundary of the r^{th} 3-ball, and j taking the $(t_{r,s})^{\text{th}}$ 2-sphere of $F(n)$ to the s^{th} inner boundary (the boundary of the s^{th} cavity) of the r^{th} 3-ball. For a 2-morphism $\eta = (\sigma, (\sigma_1, \ldots, \sigma_m))$ we know how $F(\eta)$ must act on the boundary of Σ (that is, we know the left and right induced maps of $F(\eta)$), so we would need an isotopy class of collar compatible diffeomorphisms with those left and right induced maps. That such a diffeomorphism exists is intuitively clear (and indeed is straightforward to verify, though we will not do so here), however a canonical choice of isotopy class of such diffeomorphisms is harder to describe. We intend to leave this for further study, as it is outside the scope of what we have discussed here. Also for further study, we intend to investigate if the braid group version of the double category of nested tuples and permutations, maps to $(n, n+1)$-**Cob** in the case $n = 1$. This is supported by the close relation between braid groups and mapping class groups of punctured disks discussed for example in [BB05].

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