# T-structures on some local Calabi-Yau varieties

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*In memory of my grandfather*
*Dayi Zhao (1935 - 2020)*

# Abstract

In this thesis we study the bounded t-structures for some local Calabi-Yau varieties, namely the total space of vector bundle over some del Pezzo surface such that its canonical bundle is trivial. The main methods are tilting theory and Happel-Reiten-Smalø's tilting construction. The thesis can be divided into two separated projects.

The first project is about the cotangent bundle of $\mathbb{P}^2$. For any full and strong exceptional collection $\mathbb{E}$ on $\mathbb{P}^2$, we associate a quiver $Q_{\mathbb{E}}$ and construct a functor from $\operatorname{rep} Q_{\mathbb{E}}$ to $\operatorname{Coh}(\Omega_{\mathbb{P}^2})$ such that it sends the tilting object in $\operatorname{rep} Q_{\mathbb{E}}$ to the tilting object in $\operatorname{Coh}(\Omega_{\mathbb{P}^2})$. In this way we obtain a large class of bounded t-structures in the associated Calabi-Yau category. We also calculate the combinatorics of the t-structures given by the tiltings in the sense of Happel-Reiten-Smalø.

The second project is about the canonical bundle of $\mathbb{P}^1 \times \mathbb{P}^1$. This is a mathematical interpretation of the physicists' paper [26], in which they solved the counting problem of the stable objects for certain stability conditions on $\omega_{\mathbb{P}^1 \times \mathbb{P}^1}$. The stability conditions which they consider are invariant under an autoequivalence. We characterize all the stable objects for such stability conditions and describe one connected component of the space of invariant stability conditions.

# Notations

| | |
|---|---|
| $\mathcal{D}$ | Essentially small triangulated category. |
| $D^b(X)$ | Bounded derived category of coherent sheaves on a noetherian and separated scheme $X$ over $\mathbb{C}$. |
| $D^b(A)$ | Bounded derived category of right $A$-modules over a noetherian (possibly graded) $\mathbb{C}$-algebra $A$. |
| $D^b_0(A)$ | Full subcategory of $D^b(A)$ with complexes having nilpotent cohomology modules. |
| $K_0(\mathcal{D})$ (resp. $K_0(\mathcal{A})$). | Grothendieck group of an triangulated category $\mathcal{D}$ (resp. an abelian category $\mathcal{A}$). |
| $\mathrm{supp}(F)$ | Support of a complex of sheaves $F \in D^b(X)$. |
| $\mathrm{thick}(T)$ | Smallest thick subcategory of the object $T$ (or set of objects) in $\mathcal{D}$. |
| $\mathrm{rep}\, Q$ | Category of finite dimensional representations of a quiver $Q$. |
| $\mathrm{mod}_{\mathrm{fd}}$-$A$ | Category of finite dimensional modules over $A$. |

Given a triangulated category $\mathcal{D}$, we write

$$\mathrm{Hom}^i_{\mathcal{D}}(A, B) := \mathrm{Hom}_{\mathcal{D}}(A, B[i]),$$

for $A,\ B \in \mathcal{D}$. We denote by $\mathrm{Hom}^\bullet_{\mathcal{D}}(A, B) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}^i_{\mathcal{D}}(A, B[i])$ the total Hom-space.

Let $Q = (Q_0, Q_1)$ be a quiver specified by a set of vertices $Q_0$, a set of arrows $Q_1$, and source and target maps $s,\ t : Q_1 \to Q_0$. In this thesis we compose the arrows **on the left**, that is for $b,\ a \in Q_1$, $ba = 0$ unless $s(b) = t(a)$.

Denote by $\mathbb{C}Q$ the path algebra of $Q$, and given a two-sided $I \subset \mathbb{C}Q$ generated by linear combinations of paths of length at least 2, let $A = A(Q, I) = \mathbb{C}Q/I$. We write $\operatorname{rep}(Q, I) = \operatorname{mod}_{\mathrm{fd}}\text{-}A(Q, I)$. For each vertex $i \in Q_0$ there is an associated one-dimensional simple module $S_i \in \operatorname{rep}(Q, I)$. Note that we have

$$n_{ij} = \dim_{\mathbb{C}} \operatorname{Ext}^1_A(S_j, S_i)$$

where $n_{ij}$ is the number of arrows from vertex $i$ to $j$ in our notations.

# Acknowledgements

Foremost, I would like to thank my supervisor Tom Bridgeland for a great deal of guidance, patience and encouragement, without whom this thesis will never be finished. Thanks for teaching me lots of beautiful mathematics and always being selfless to share your insights. Your impact are not only academic but extends to many other aspects.

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# Contents

# Chapter 1

# Introduction

There are many known examples of varieties being derived equivalent to non-commutative algebras. The pioneering example is due to Beilinson [3] who proved that $D^b(\mathbb{P}^n)$ is generated by line bundles $\mathcal{O}, \mathcal{O}(1), \cdots \mathcal{O}(n)$. This leads to the result that the functor

$$\mathbb{R}\mathrm{Hom}\left(\bigoplus_{i=0}^{n} \mathcal{O}(i), -\right) : D^b(\mathbb{P}^n) \longrightarrow D^b(A)$$

is a derived equivalence between $\mathbb{P}^n$ and the non-commutative algebra

$$A = \mathrm{End}\left(\bigoplus_{i=0}^{n} \mathcal{O}(i)\right).$$

### Tilting objects

The work of Bondal, Rickard and Keller [7, 50, 38] proposed a general framework for studying such equivalences, called the tilting theory. Suppose $X$ is a smooth quasi-projective variety. Let $\mathcal{D} = D^b(X)$. An object $T$ in $\mathcal{D}$ is called a tilting object if it satisfies the following conditions

(i) $\mathbb{R}\mathrm{Hom}_{\mathcal{D}}(T, T[n]) = 0$, unless $n = 0$;

(ii) $T$ is a classical generator of $\mathcal{D}$, i.e., the smallest thick subcategory containing $T$ which we denote by $\mathrm{thick}(T)$ is $\mathcal{D}$.

Then $\mathbb{R}\mathrm{Hom}(T, -)$ induces a derived equivalence between $X$ and the non-commutative algebra $B = \mathrm{End}(T)$. Tilting objects have become a powerful tool in studying the derived category of varieties and their geometry. For example, if $Y$ and $Y^+$ are 3-dimensional smooth quasi-projective varieties over an affine variety and related by a flop, inspired by the work of Bridgeland [12], Van den Bergh [55] proved there exists a tilting object $T$ in $D^b(Y)$ such that

$$D^b(Y) \cong D^b(\mathrm{End}\, T) \cong D^b(Y^+),$$

which provided a new proof of the Bondal-Orlov conjecture [11] in dimension 3.

In fact the existence of tilting objects for projective varieties are rare, much more progress has been made for non-compact (also called local) varieties, in particular for local models of resolutions of singularities [5].

In this thesis, the example we are interested is when $X = \mathcal{V}$ is the total space of a vector bundle over a del Pezzo surface $Z$ such that the canonical bundle of $X$ is trivial. Such varieties are called local Calabi-Yau: for coherent sheaves (or complexes of sheaves) $F,\ G$ supported on the zero section, we have Serre duality

$$\mathrm{Ext}^i(F, G) = \mathrm{Ext}^{\dim X - i}(G, F)^*.$$

Therefore in this thesis, we usually work with the full subcategory $D^b_0(\mathcal{V})$ consisting of objects which are supported on the zero section.

A first step to study $D^b(\mathcal{V})$ is to study $D^b(Z)$, and since $Z$ is del Pezzo we can generalize Beilinson's approach. We can find a sequence of vector bundles $E_0, \cdots, E_n$ called exceptional and it is strong in the sense of [7] such that they generate $D^b(Z)$. Thereby $D^b(Z)$ is derived equivalent to

$$A := \mathrm{End}_Z(\oplus_i E_i).$$

Then as observed in Bridgeland's paper [14], if we pull back $E_i$ along the projection map $\pi : \mathcal{V} \to Z$, then we get a sequence of vector bundles $T_i := \pi^* E_i$ such that they also generate $D^b(\mathcal{V})$. In good cases, the direct sum $\oplus_i T_i$ is still tilting. Therefore $D^b(\mathcal{V})$ is derived equivalent to

$$\widehat{A} := \mathrm{End}_{\mathcal{V}}(\oplus_i T_i).$$

**T-structures**

The main goal of the thesis is to study the (non-degenerate) bounded t-structures in $D^b_0(\mathcal{V})$ and the combinatorics of them. When we say a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is bounded in $\mathcal{D}$, if for every object $E \in \mathcal{D}$, there exists an integer $n > 0$ such that $E[n] \in \mathcal{D}^{\leq 0}$ and $E[-n] \in \mathcal{D}^{\geq 0}$. And it is non-degenerate, if $\cap_{n\in\mathbb{Z}}\mathcal{D}^{\leq 0}[n] = 0$ and $\cap_{n\in\mathbb{Z}}\mathcal{D}^{\geq 0}[n] = 0$. In this thesis we only consider the non-degenerate t-structures.

The main feature of the bounded t-structure is that it is determined by its heart $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ (Lemma 2.1.1): for any object $E \in \mathcal{D}$ we have a collection of triangles

$$\begin{array}{ccccccccc} 0 = E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 \longrightarrow \cdots \longrightarrow & E_{n-1} & \longrightarrow & E_n = E \\ & \nwarrow \;\; \swarrow & & \nwarrow \;\; \swarrow & & & \nwarrow \;\; \swarrow & \\ & A_1 & & A_2 & & & A_n & \end{array} \tag{1.1}$$

with $A_i[k_i] \in \mathcal{A}$ and $k_1 > k_2 > \cdots > k_n$. For example, given an abelian category $\mathcal{A}$, its bounded derived category $D^b(\mathcal{A})$ admits a standard t-structure where $\mathcal{D}^{\leq 0}$ is the full subcategory of complexes without cohomology in positive degrees, the standard t-structure is a bounded t-structure.

Tilting objects provide an important method for constructing new t-structures, since given a tilting object in $D^b(\mathcal{V})$, the derived equivalence $\mathbb{R}\mathrm{Hom}(T, -) : D^b(\mathcal{V}) \to D^b(B)$ gives us a new bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of $D^b(\mathcal{V})$ by pulling back the standard one on $D^b(B)$:

$$\begin{aligned} \mathcal{D}^{\leq 0} &= \{F \in \mathcal{D} : \mathbb{R}\mathrm{Hom}_{\mathcal{D}}(T, F[i]) = 0,\ i > 0\}, \\ \mathcal{D}^{\geq 0} &= \{F \in \mathcal{D} : \mathbb{R}\mathrm{Hom}_{\mathcal{D}}(T, F[i]) = 0,\ i < 0\}. \end{aligned}$$

The heart $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is equivalent to mod-$B$ by definition. We call $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ the bounded t-structure induced by the tilting object $T$ in this thesis.

Another very useful method for constructing new t-structure from given heart $\mathcal{A}$ of some t-structure is provided by the idea of tilting with respect to a torsion pair, first introduced by Happel-Reiten-Smalø [29]. We will recall the exact definition later (c.f. Section 2.1). The Happel-Reiten-Smalø tilting enables us to find the adjacent t-structures to the given one (see figure 1.1).

A special case of the tilting construction will be important for us: when $\mathcal{B}$ is obtained from $\mathcal{A}$ by tilting at a torsion pair generated by a simple object. In this case, $\mathcal{B}$ is called the left or right simple tilt of $\mathcal{A}$ (see figure 1.1). Then the combinatorics of t-structures

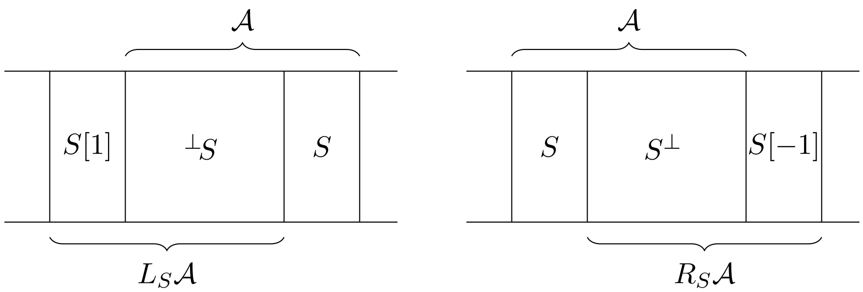



Figure 1.1: Left and right tilts of a heart.

arise by iteration of simple tilts. A useful way to illustrate the combinatorics is by using the tilting exchange graph $\mathrm{EG}(\mathcal{D})$ - that is, the oriented graph whose vertices are hearts in $\mathcal{D}$ and whose edges correspond to simple tilts between them. Tilting exchange graphs will be useful for studying the space of stability conditions $\mathrm{Stab}(\mathcal{D})$, which we will explain later. There is also a notion in cluster theory called the exchange graphs, which has many links to various areas in mathematics (see survey [39]).

Given the heart of a t-structure $\mathcal{A}$ in $\mathcal{D}$ induced by tilting object $T$, a question that one can ask is:

*Can we describe the simple tilts of $\mathcal{A}$ by using tilting objects? More precisely, after simple tilting, can we find another tilting object $T'$ such that the resulting t-structure is induced by $T'$?*

Note that this direction is first suggested and studied in Bridgeland's paper [14, 16]. The first part of this thesis is devoted to studying the above problem for the derived category of the cotangent bundle of $\mathbb{P}^2$.

**Stability conditions**

Inspired by Douglas' work on $\pi$-stability for D-branes, Bridgeland introduced the notion of a stability condition on a triangulated category in [17]. It was shown in [17] that to any triangulated category $\mathcal{D}$, one can associate a complex manifold $\mathrm{Stab}(\mathcal{D})$ which parameterises stability conditions on $\mathcal{D}$. Given the triangulated category $\mathcal{D}$, one can ask the following three questions:

*(i) Can we find a stability condition on $\mathcal{D}$?*

*(ii) What is* $\operatorname{Stab}(\mathcal{D})$ *as a complex manifold?*

*(iii) Given a stability condition $\sigma$, can we count the set of (semi)stable objects in $\mathcal{D}$ for $\sigma$?*

The answer to the first question is closely related with the bounded t-structures. A stability condition on $\mathcal{D}$ is a pair $\sigma = (Z, \mathcal{A})$, where $\mathcal{A}$ is the heart of a bounded t-structure in $\mathcal{D}$, and $Z$ is a group homomorphism called the central charge from the Grothendieck group $K_0(\mathcal{A})$ to $\mathbb{C}$ which satisfies the Harder-Narasimhan property[17]. So far much progress towards this direction has been made for compact and non-compact examples [18, 16, 19, 46, 33, 1]. And it seems that the non-compact example is much easier with the help of tilting object: when $\mathcal{A}$ is induced by a tilting object in the form $\oplus_{i=1}^n T_i$, then $\mathcal{A}$ is of finite-length and consists of a finite set of non-isomorphic simple objects. Then we only need to assign each simple object with a value in $H$ where

$$H := \{z = r\exp(i\pi\phi) \mid r > 0, 0 < \phi \leq 1\} \subset \mathbb{C}.$$

The Harder-Narasimhan property is automatically satisfied since $\mathcal{A}$ is of finite-length. In this way, we get a subset $U(\mathcal{A}) \subset \operatorname{Stab}(\mathcal{D})$ consisting of stability conditions with fixed bounded heart $\mathcal{A}$. By construction $U(\mathcal{A})$ is isomorphic to $H^n$, where $n$ is the number of simple objects.

The answer to the second question in the non-compact case is related to the simple tilts of t-structures. Given finite-length hearts $\mathcal{A}$ and $\mathcal{B}$, $U(\mathcal{A}) \cap \overline{U(\mathcal{B})}$ is non-empty and of codimension one when $\mathcal{B}$ is a simple tilt of $\mathcal{A}$ (Lemma 2.1.7). Therefore one can think of $U(\mathcal{A})$ as a cell of $\operatorname{Stab}(\mathcal{D})$ and $\operatorname{EG}(\mathcal{D})$ as the skeleton of space of stability conditions.

The answer to the final question is usually very hard in both compact and non-compact examples. When $\mathcal{D}$ is a Calabi-Yau category of dimension 3, it is related to the Donaldson-Thomas invariants [36, 43].

The second part of the paper is studying the space of stability conditions for local $\mathbb{P}^1 \times \mathbb{P}^1$, i.e., the total space $X$ of canonical bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. The work is a mathematical interpretation of the work of Closset-Del Zotto [24] and Del Monte-Longhi [26]. In their papers, the physicists found that there were surprisingly complete answers to all questions in the above when we restrict to the invariant subspace of $\mathrm{Stab}(X)$ under an autoequivalence of $D^b(X)$. In physcists' language, we are considering the collimation chamber of stability conditions.

**Results for t-structures on the cotangent bundle of $\mathbb{P}^2$**

We denote by $\pi : X = \mathrm{Tot}\,\Omega_{\mathbb{P}^2} \to \mathbb{P}^2$ the bundle projection map. We consider the full exceptional collection $\mathbb{E} = (E_0, E_1, E_2)$ (see Definition 2.3.1) consisting of sheaves on $\mathbb{P}^2$. Then $\oplus_i E_i$ is a tilting object on $\mathbb{P}^2$, and we try to show that $\pi^*(\oplus_i E_i)$ is a tilting object for $X$.

However, this is not always possible. We need to use the helix theory: for any full exceptional collection $\mathbb{E} = (E_0, E_1, E_2)$ on $\mathbb{P}^2$, one can associate $\mathbb{H} = (\cdots, E_{-1}, E_0, E_1, E_2, E_3, \cdots)$ a collection of exceptional objects satisfying $E_i = E_{i+3} \otimes \omega_{\mathbb{P}^2}$. Any adjacent 3 objects in $\mathbb{H}$ (called a thread) will be a full exceptional collection.

Now our first result on the tilting objects in $D^b(\mathrm{Coh}\,X)$ is stated as the following:

**Theorem 1.0.1** (= Theorem 3.2.7)**.** *For any full and strong exceptional collection (see Definition 2.3.1)* $\mathbb{E} = (E_0, E_1, E_2)$ *on* $\mathbb{P}^2$*, one can always find a thread* $(E_i, E_{i+1}, E_{i+2})$ $i \in \mathbb{Z}$ *in the corresponding helix* $\mathbb{H}$*, such that the pullback* $\bigoplus_i^{i+2} \pi^* E_i$ *is a tilting object in* $D^b(\mathrm{Coh}\,X)$*.*

The tilting objects constructed in the above are not enough for characterizing the simple tilts. Therefore we introduce a quiver associated with any full exceptional collection $\mathbb{E}$ on $\mathbb{P}^2$, whose vertices correspond to $T_i = \pi^* E_i$, and number of arrows between two vertices is given by the dimension of the vector space $\mathrm{Ext}^1_X(T_i, T_j)$. We denote the resulting quiver by $Q_{\mathbb{E}}$ and call it the secondary quiver of $\mathbb{E}$.

After analyzing possible types of the secondary quivers (which are all acyclic), we define a functor $\mathcal{F}_Q$ from mod-$\mathbb{C}Q_{\mathbb{E}}$ to $\mathrm{Coh}\,X$, and prove the following theorem, which gives us many more tilting objects associated with $\mathbb{E}$:

**Theorem 1.0.2** (=Theorem 3.4.5)**.** *For any full and strong exceptional collection* $\mathbb{E}$ *on* $\mathbb{P}^2$*,* $\mathcal{F}_Q$ *sends tilting objects in mod-*$\mathbb{C}Q_{\mathbb{E}}$ *to tilting objects in* $D^b(\mathrm{Coh}\,X)$*.*

Here tilting objects in mod-$\mathbb{C}Q_{\mathbb{E}}$ means they are tilting objects when viewed as objects in $D^b(\mathbb{C}Q_{\mathbb{E}})$.

Let $D^b_0(\operatorname{Coh} X)$ be the full subcategory of $D^b(\operatorname{Coh} X)$ consisting of objects with support on the zero section $\mathbb{P}^2 \subset X$ and $D^b_0(B)$ be the full subcategory of $D^b(B)$ consisting of objects whose cohomology modules are nilpotent over $B$. The derived equivalence $\mathbb{R}\mathrm{Hom}(T, -) : D^b(\operatorname{Coh} X) \to D^b(B = \operatorname{End}_X T)$ descends to equivalence between $D^b_0(\operatorname{Coh} X)$ and $D^b_0(B)$, this induces a t-structure on $D^b_0(\operatorname{Coh} X)$ whose heart $\mathcal{B} \subset D^b_0(\operatorname{Coh} X)$ is an abelian category which is equivalent to the category of nilpotent modules over $B$. In the final section of this part, we calculate simple tilts from a given one. The calculations involve an interesting autoequivalence on $D^b(\operatorname{Coh} X)$ which is induced by the Mukai flop (see Proposition 3.5.5), first studied by Namikawa [48] and Kawamata [37].

We remark that the combinatorics of simple tilts are complicated, and not all of them are induced by the tilting objects we obtained above. However, our methods give many tilting objects on $D^b(\operatorname{Coh} X)$ and should be useful for describing the tilting exchange graph in the future research. We illustrate part of the tilting exchange graph and corresponding Ext-quivers in figure 1.2: the vertices correspond to the simple objects $S_i$ in the heart, the number over the arrows indicates the number of arrows, and black arrows represent the elements in $\operatorname{Ext}^1(S_i, S_j)$, red arrows represent the elements in $\operatorname{Ext}^2(S_i, S_j)$ (for details see Section 3.5).

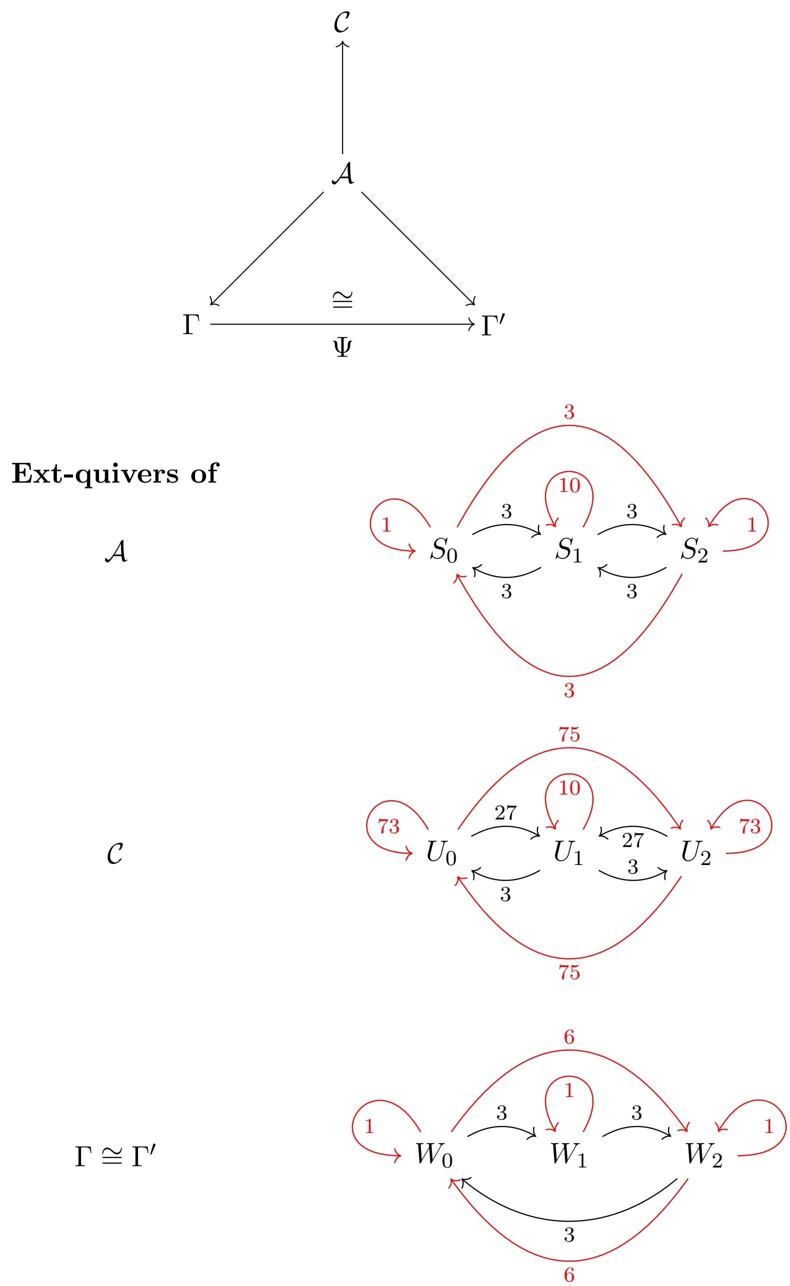



Figure 1.2: Part of the exchange graph and corresponding Ext-quivers of $\Omega_{\mathbb{P}^2}$

**Results for local $\mathbb{P}^1 \times \mathbb{P}^1$**

We again denote by $\pi : \omega \to \mathbb{P}^1 \times \mathbb{P}^1$ the bundle projection map and $p_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ $i = 1,\ 2$ the projection maps to each component. We write $\mathcal{O}(a, b)$ for the line bundle $p_1^*\mathcal{O}(a) \otimes p_2^*\mathcal{O}(b)$. There is a full and strong exceptional sequence of line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$:

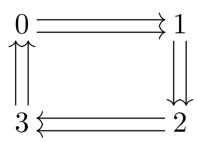
$$\mathbb{E} = (E_0, E_1, E_2, E_3) := (\mathcal{O}(0, 0), \mathcal{O}(1, 0), \mathcal{O}(1, 1), \mathcal{O}(2, 1))\,,$$

which generates $D^b(\mathbb{P}^1 \times \mathbb{P}^1)$, and

$$T = \bigoplus T_i := \bigoplus_i \pi^* E_i$$

is a tilting object in $D^b(\omega)$. Then $\omega$ is derived equivalent to a non-commutative algebra $A = \mathrm{End}(T)$.

Suppose we have a presentation of the algebra $A$ as the path algebra of a quiver $Q$ subject to relations, where the nodes of $Q$ correspond to line bundles $T_i$. The quiver of $A$ will be



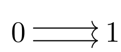

(1.2)

The symmetry of the shape of the quiver suggests that there should be an autoequivalence of $D^b(\omega)$, denoted by $\Phi$, which exchanges the simple modules associated with vertices 0 and 2, 1 and 3. We will realize $\Phi$ explicitly in Section 4.1.2.

Let $\mathcal{C} = D^b_0(\omega)$ be the full subcategory of $D^b(\omega)$ consisting of objects supported on $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathrm{Stab}(X)$ denote the space of stability conditions on $\mathcal{C}$. We consider a subspace of $\Phi$-invariant stability conditions $\mathrm{Stab}(X)^\Phi$.

Let $\varphi$ be the automorphism of Grothendieck group $K_0(\mathcal{C})$ induced by $\Phi$. Then $\mathrm{Stab}(\mathcal{C})^\Phi$ is locally modelled on $\mathrm{Hom}(K_0(\mathcal{C}), \mathbb{C})^\varphi$, the invariant central charges under $\varphi$. Note that $K_0(\mathcal{C}) \cong \mathbb{Z}^4$ has a basis $\gamma_0, \cdots, \gamma_3$ which corresponds to the vertices of $Q$. Therefore $\mathrm{Hom}(K_0(\mathcal{C}), \mathbb{C})^\varphi \cong \mathbb{C}^2$.

Denote by $\mathcal{A}$ the heart in $\mathcal{C}$ induced by the tilting object $T$. We write $\mathcal{U}(\mathcal{A})^\Phi$ for the subset of $\mathrm{Stab}(\mathcal{C})^\Phi$ consisting of $\Phi$-invariant stability conditions with heart $\mathcal{A}$. First we characterize the stable objects for $\sigma \in \mathcal{U}(\mathcal{A})^\Phi$: note that the Kronecker quiver $K_2$



can be embedded into $Q$ (1.2) in 4 different ways. Therefore $\mathrm{rep}(K_2)$ embeds into $\mathcal{A}$ as full subcategories. For stability condition $\sigma \in \mathcal{U}(\mathcal{A})^{\Phi}$, $\sigma$ reduces to be a stability function $\bar{\sigma}$ on $\mathrm{rep}(K_2)$. We are able to show that

**Lemma 1.0.3.** *The stable objects in* $\mathrm{rep}(K_2)$ *with respect to* $\bar{\sigma}$ *are stable in* $\mathcal{A}$ *with respect to* $\sigma$*.*

The stable objects in $\mathrm{rep}(K_2)$ are well known to be the indecomposable representations (with respect to certain stability functions), their images in $\mathcal{A}$ are called objects of special Kronecker types I and II (Definition 4.3.6). We show that these are in fact all the stable objects for $\sigma \in \mathcal{U}(\mathcal{A})^{\Phi}$:

**Theorem 1.0.4** (=Theorem 4.3.23)**.** *Take* $\sigma \in \mathcal{U}(\mathcal{A})^{\Phi}$*. The stable objects for* $\sigma$ *and their classes in Grothendieck group (up to a sign) are as follows:*

*(i) if* $\arg Z(\gamma_0) < \arg Z(\gamma_1)$*, then the classes of stable objects are*

$$\begin{gathered} n\gamma_0 + (n+1)\gamma_1, \quad (n+1)\gamma_0 + n\gamma_1, \\ n\gamma_2 + (n+1)\gamma_3, \quad (n+1)\gamma_2 + n\gamma_3, \\ \gamma_0 + \gamma_1, \qquad \gamma_2 + \gamma_3. \end{gathered}$$

*each of the first 4 classes corresponds to a unique stable object (up to a shift of degree) called the special Kronecker of type I, and each of the last two classes corresponds to a* $\mathbb{P}^1$*-family of stable objects;*

*(ii) if* $\arg Z(\gamma_1) < \arg Z(\gamma_0)$*, then the classes of stable objects are*

$$\begin{gathered} n\gamma_1 + (n+1)\gamma_2, \quad (n+1)\gamma_1 + n\gamma_2, \\ n\gamma_3 + (n+1)\gamma_0, \quad (n+1)\gamma_3 + n\gamma_0, \\ \gamma_1 + \gamma_2, \qquad \gamma_3 + \gamma_0. \end{gathered}$$

*each of the first 4 classes corresponds to a unique stable object(up to a shift of degree) called the special Kronecker of type II, and each of the last two classes corresponds to a* $\mathbb{P}^1$*-family of stable objects;*

*(iii) if* $\arg Z(\gamma_0) = \arg Z(\gamma_1)$*, then the classes of stable objects are*

$$\gamma_0, \ \gamma_1, \ \gamma_2, \ \gamma_3.$$

*each class corresponds to a unique stable object* $S_i$ *(see Corollary 4.1.3).*

Moreover, via the autoequivalences of $\mathcal{C}$ we actually characterize the stable objects for any stability conditions in one connected component (see Corollary 4.4.5). This answers the question of counting the stable objects for the local $\mathbb{P}^1 \times \mathbb{P}^1$. We proceed to characterize a connected component of $\mathrm{Stab}(X)^\Phi$.

Let $K_0(\mathcal{C})^{-\varphi}$ be the subgroup of $K_0(\mathcal{C})$ whose elements are antisymmetric under $\varphi$. We denote by $\overline{K_0(\mathcal{C})} = K_0(\mathcal{C})/K_0(\mathcal{C})^{-\varphi}$ the quotient group. Note that there is an isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(\overline{K_0(\mathcal{C})}, \mathbb{C}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{C}), \mathbb{C})^\varphi.$$

We write $\Delta \subset \overline{K_0(\mathcal{C})}$ for the set of the classes of the stable objects for $\sigma \in \mathcal{U}(\mathcal{A})^\Phi$. The connected component of $\mathrm{Stab}(\mathcal{C})^\Phi$ which contains $\mathcal{U}(\mathcal{A})^\Phi$ is denoted by $\left(\mathrm{Stab}(X)^\Phi\right)_0$, we have

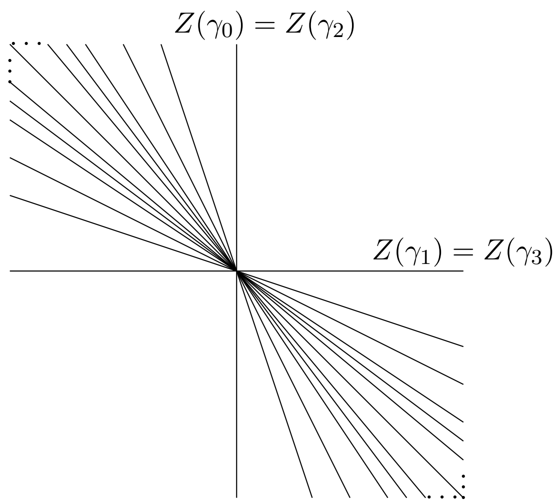



Figure 1.3: Real slice of $\mathcal{H}^{\mathrm{reg}}$

**Theorem 1.0.5** (=Theorem 4.4.6 and 4.4.8 )**.** *The forgetful map*

$$\mathcal{Z} : \left(\mathrm{Stab}(X)^\Phi\right)_0 \to \mathrm{Hom}(\overline{K_0(\mathcal{C})}, \mathbb{C})$$

*factors through*

$$\mathcal{Z} : \left(\mathrm{Stab}(X)^\Phi\right)_0 \to \mathcal{H}^{\mathrm{reg}} \tag{1.3}$$

*where*

$$\mathcal{H}^{\mathrm{reg}} := \mathrm{Hom}(\overline{K_0(\mathcal{C})}, \mathbb{C}) \setminus \bigcup_{\boldsymbol{v} \in \Delta} \boldsymbol{v}^\perp,$$

*and* $\boldsymbol{v}^\perp := \{Z \in \mathrm{Hom}(\overline{K_0(\mathcal{C})}, \mathbb{C}) \mid Z(\boldsymbol{v}) = 0\}$. *Moreover in* (1.3) $\mathcal{Z}$ *is a covering map.*

**Relation with [24, 26]**

Finally we explain the relation between our work with [24, 26]. We keep the notations as above and introduce the notations for normalized stability conditions $\mathcal{U}^n(\mathcal{A})^\Phi \subset \mathcal{U}(\mathcal{A})^\Phi$ by

$$\mathcal{U}^n(\mathcal{A})^\Phi = \{(Z, \mathcal{P}) : Z(\delta) = i\},$$

where $\delta$ is the class of a skyscrapper sheaf $\mathcal{O}_x$, $x \in \mathbb{P}^1 \times \mathbb{P}^1$. Note that $Z(\gamma_0) + Z(\gamma_2) = Z(\gamma_1) + Z(\gamma_3)$ lies on the imaginary axis. It was shown by Closset-Del Zotto [24, Appendix D] that there is a unique stable object in each slicing $\mathcal{P}(\phi)$ where $\phi \neq \frac{1}{2} + n$ ($n \in \mathbb{Z}$), and each such object corresponds to a representation of the Kronecker quiver.

In this thesis we analyze the special slicing $\mathcal{P}(\frac{1}{2})$ in detail. For simplicity, we restrict to the subset $\mathcal{U}^n(\mathcal{A})^\Phi_+ := \big\{(Z, \mathcal{P}) : \arg Z(\gamma_0) < \arg Z(\gamma_1)\big\}$. We show that each stable object in $\mathcal{P}(\frac{1}{2})$ is isomorphic to $s_*\mathcal{O}_{\{y\}\times\mathbb{P}^1}$ or $s_*\mathcal{O}_{\{y\}\times\mathbb{P}^1}(-1)[1]$ for $y \in \mathbb{P}^1$ (Theorem 4.3.22). The case for $\mathcal{U}^n(\mathcal{A})^\Phi_- := \big\{(Z, \mathcal{P}) : \arg Z(\gamma_1) < \arg Z(\gamma_0)\big\}$ is obtained by applying an autoequivalence of $D^b(X)$. The key observation in the proof of the above theorem was taken from [26], which is our Lemma 4.3.15: one can identify the action of the autoequivalence $\mathcal{T}$ on the stability conditions in $\mathcal{U}^n(\mathcal{A})^\Phi_+$ with the action of $\widetilde{g}$ where $\widetilde{g} \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, the universal covering space of $\mathrm{GL}^+(2, \mathbb{R})$. The former autoequivalence $\mathcal{T}$ plays an important role in the tilting process (Theorem 4.2.1).

# Chapter 2

# Preliminaries

In this chapter we assume $\mathcal{D}$ is a triangulated category such that

(i) $\mathcal{D}$ is a $\mathbb{C}$-linear category: the Hom spaces are $\mathbb{C}$-linear spaces, and composition maps are bilinear.

(ii) $\mathcal{D}$ is of finite type, i.e. for any two objects $A$, $B$ of $\mathcal{D}$ the vector space

$$\bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}^i_{\mathcal{D}}(A, B)$$

is finite-dimensional.

(iii) $\mathcal{D}$ is algebraic in the sense of Keller[38].

(iv) $\mathcal{D}$ is saturated [8], i.e., all triangulated functors

$$\mathcal{D} \longrightarrow D(\mathbb{C}), \quad \mathcal{D}^{op} \longrightarrow D(\mathbb{C}),$$

are representable.

### § 2.1 Simple tilts

The reader is assumed to be familiar with the concept of a t-structure [28]. We are only considering the bounded t-structures. Recall that a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is bounded in $\mathcal{D}$, if for every object $E \in \mathcal{D}$, there exists an integer $n > 0$ such that $E[n] \in \mathcal{D}^{\leq 0}$ and $E[-n] \in \mathcal{D}^{\geq 0}$. The bounded t-structure is determined by its heart:

**Lemma 2.1.1** ([17, Lemma 3.2])**.** *Let $\mathcal{A} \subset \mathcal{D}$ be a full additive subcategory of $\mathcal{D}$. Then $\mathcal{A}$ is the heart of a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ if and only if it satisfies the following conditions:*

*(i) if $n_1 > n_2$ then $\mathrm{Hom}_{\mathcal{D}}(A[n_1], B[n_2]) = 0$ for any $A,\ B \in \mathcal{A}$;*

*(ii) for every nonzero object $E \in \mathcal{D}$ there are a finite sequence of integers:*

$$k_1 > k_2 > \cdots > k_n$$

*and a collection of triangles*

$$\xymatrix{ 0 = E_0 \ar[rr] & & E_1 \ar[rr] \ar[dl] & & E_2 \ar[r] \ar[dl] & \dots \ar[r] & E_{n-1} \ar[rr] & & E_n = E \ar[dl] \\ & A_1 \ar@{-->}[ul] & & A_2 \ar@{-->}[ul] & & & & A_n \ar@{-->}[ul] & } \tag{2.1}$$

*with $A_i \in \mathcal{A}[k_i]$ for all $i$.*

In analogy with the standard t-structure on the derived category of an abelian category, given $\mathcal{A}$ the heart of a bounded t-structure and any nonzero object $E$, we denote by $\mathrm{H}^{k_i}_{\mathcal{A}}(E) := A_{k_i}[-k_i] \in \mathcal{A}$ the $i$th-graded cohomology group with respect to $\mathcal{A}$, where $A_i$ appears in (2.1).

A heart of some t-structure will be called finite-length if it is artinian and noetherian as an abelian category.

The following definition comes from Happel-Reiten-Smalø [30].

**Definition 2.1.2** (Torsion pair)**.** *Let $\mathcal{A}$ be a heart of some bounded t-structure in the triangulated category $\mathcal{D}$. A pair of full subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ is called a torsion pair in $\mathcal{A}$ if it satisfies the following conditions*

*(i) $\mathrm{Hom}_{\mathcal{A}}(T, F) = 0$ for $T \in \mathcal{T}$ and $F \in \mathcal{F}$;*

*(ii) for any object $A \in \mathcal{A}$, there exist $M \in \mathcal{T}$ and $N \in \mathcal{F}$ such that they fit into a short exact sequence*

$$0 \longrightarrow M \longrightarrow A \longrightarrow N \longrightarrow 0.$$

The following theorem was proved in [30, Proposition 2.1].

**Theorem 2.1.3** (Happel-Reiten-Smalø)**.** *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in a heart $\mathcal{A}$. Let*

$$\begin{aligned} \mathcal{A}^{\sharp} &:= \left\{E \in \mathcal{D} \mid \mathrm{H}^1_{\mathcal{A}}(E) \in \mathcal{T},\ \mathrm{H}^0_{\mathcal{A}}(E) \in \mathcal{F},\ \mathrm{H}^i_{\mathcal{A}}(E) = 0 \textit{ for } i \neq 0,\ 1\right\}, \\ \mathcal{A}^{\flat} &:= \left\{E \in \mathcal{D} \mid \mathrm{H}^{-1}_{\mathcal{A}}(E) \in \mathcal{F},\ \mathrm{H}^0_{\mathcal{A}}(E) \in \mathcal{T},\ \mathrm{H}^i_{\mathcal{A}}(E) = 0 \textit{ for } i \neq -1,\ 0\right\}, \end{aligned}$$

*then $\mathcal{A}^{\sharp}$ and $\mathcal{A}^{\flat}$ are hearts of bounded t-structures in $\mathcal{D}$.*

A special case of the tilting construction will be particularly important [41, Definition 3.7]. Suppose that $\mathcal{A}$ is a finite-length heart and $S \in \mathcal{A}$ is a simple object. Let $\langle S \rangle$ be the full subcategory consisting of objects $E \in \mathcal{A}$ all of whose simple factors are isomorphic to $S$. Define the full subcategories

$$S^{\perp} := \{E \in \mathcal{A} \mid \ \mathrm{Hom}_{\mathcal{A}}(S, E) = 0\}, \quad {}^{\perp}S := \{E \in \mathcal{A} \mid \ \mathrm{Hom}_{\mathcal{A}}(E, S) = 0\}.$$

Then we can either view $(\langle S \rangle, S^{\perp})$ or $({}^{\perp}S, \langle S \rangle)$ as a torsion pair. Then we can define new tilted hearts

$$L_S\mathcal{A} := \langle S[1], {}^{\perp} S \rangle, \quad R_S\mathcal{A} := \langle S^{\perp}, S[-1] \rangle, \tag{2.2}$$

which we refer to as the left and right simple tilts of the heart $\mathcal{A}$ at the simple object $S$.

**Remark 2.1.4.** *It is easy to see that $S[-1]$ is a simple object of $R_S\mathcal{A}$ and that if the category is of finite-length, then $L_{S[-1]}R_S\mathcal{A} = \mathcal{A}$. Similarly, if $L_S\mathcal{A}$ is of finite-length then $R_{S[1]}L_S\mathcal{A} = \mathcal{A}$, see Figure 1.1.*

The following lemmas will be useful.

**Lemma 2.1.5.** *Let $(D^{\leq 0}, D^{\geq 0})$ and $(\widetilde{D}^{\leq 0}, \widetilde{D}^{\geq 0})$ be two bounded t-structures of $\mathcal{D}$, and we denote by $\mathcal{A}$ and $\mathcal{A}'$ their hearts respectively. If $\mathcal{A} \subset \mathcal{A}'$ then $\mathcal{A} = \mathcal{A}'$.*

*Proof.* Let $E \in \mathcal{A}'$. Since for any object $F \in D^{\leq 0}$, $F$ has a finite filtration by objects in $\mathcal{A}[k_i] \subset \mathcal{A}'[k_i]$ for $k_i \geq 0$ by Lemma 2.1.1, so we have

$$\mathrm{Hom}_{\mathcal{D}}(F, E[-1]) = 0.$$

Therefore $E[-1] \in D^{>0}$.

Similarly for any object $G \in D^{>0}$, $G$ has a finite filtration by objects in $\mathcal{A}[k_i] \subset \mathcal{A}'[k_i]$ for $k_i < 0$, therefore

$$\mathrm{Hom}_{\mathcal{D}}(E, G) = 0.$$

Therefore $E \in D^{\leq 0}$. So we have $E \in D^{\leq 0} \cap D^{>0}[1] = \mathcal{A}$. This proves the lemma. □

**Lemma 2.1.6.** *Take an autoequivalence $\Phi \in \mathrm{Aut}(\mathcal{D})$. Let $\mathcal{A} \subset \mathcal{D}$ be a heart of some bounded t-structure and of finite-length, $S \in \mathcal{A}$ be a simple object. Then we have*

$$\Phi(L_S\mathcal{A}) = L_{\Phi(S)}\Phi(\mathcal{A}), \quad \Phi(R_S\mathcal{A}) = R_{\Phi(S)}\Phi(\mathcal{A}).$$

*Proof.* By the definition of simple tilts (2.2), we have $R_{\Phi(S)}\Phi(\mathcal{A}) = \left\langle (\Phi(S))^{\perp}, \Phi(S)[-1] \right\rangle$. It is easy to check $\Phi(S^{\perp}) = (\Phi(S))^{\perp}$, therefore $R_{\Phi(S)}\Phi(\mathcal{A}) \subset \Phi(R_S\mathcal{A})$ by definition. By Lemma 2.1.5 we have $R_{\Phi(S)}\Phi(\mathcal{A}) = \Phi(R_S\mathcal{A})$. We leave the proof for left tilt to the reader. □

Given a heart of bounded t-structure $\mathcal{A} \subset \mathcal{D}$, we denote by $\mathrm{Sim}\,\mathcal{A}$ the set of all non-isomorphic simple objects in $\mathcal{A}$. The following theorem characterizes the new simple objects in the tilted hearts [42, Proposition 7.6], see also [41, Proposition 5.4].

**Proposition 2.1.7.** *Assume $\mathrm{Sim}\,\mathcal{A}$ is finite and $\mathcal{A}$ is of finite-length. Let $S \in \mathcal{A}$ be such that $\mathrm{Ext}^1_{\mathcal{A}}(S, S) = 0$. Then after taking a left or right simple tilt, the new simple objects are:*

$$\mathrm{Sim}\,R_S\mathcal{A} = \{S[-1]\} \cup \{\phi_S(X) : X \in \mathrm{Sim}\,\mathcal{A},\ X \neq S\} \tag{2.3}$$
$$\mathrm{Sim}\,L_S\mathcal{A} = \{S[1]\} \cup \{\psi_S(X) : X \in \mathrm{Sim}\,\mathcal{A},\ X \neq S\} \tag{2.4}$$

*where*

$$\phi_S(X) = \mathrm{Cone}\ \left(S[-1] \otimes \mathrm{Ext}^1(S, X) \longrightarrow X\right),$$
$$\psi_S(X) = \mathrm{Cone}\ \left(X \longrightarrow S[1] \otimes \mathrm{Ext}^1(X, S)^*\right)[-1].$$

## § 2.2 Stability conditions

We collect some properties and theorems on the space of stability conditions introduced in [17].

**Definition 2.2.1** (Slicing). *A slicing of $\mathcal{D}$ is a collection of full subcategories $\mathcal{P}(\phi)$ indexed by $\phi \in \mathbb{R}$, satisfying the following axioms:*

*(i)* $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$;

*(ii)* $\mathrm{Hom}_D(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0$ *for* $\phi_1 > \phi_2$;

*(iii) for any nonzero object $E \in \mathcal{D}$, we have a collection of triangles*

$$
\begin{array}{ccccccccc}
0 = E_0 & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow \cdots \longrightarrow & E_{n-1} & \longrightarrow & E_n = E \\
 & \nwarrow \;\; \swarrow & & \nwarrow \;\; \swarrow & & & & \nwarrow \;\; \swarrow & \\
 & A_1 & & A_2 & & & & A_n &
\end{array}
$$

*such that $A_i \in \mathcal{P}(\phi_i)$, and*

$$\phi_1 > \phi_2 > \cdots > \phi_n.$$

*For any nonzero object $E$, we denote by $\phi^+(E) = \phi_1$ and $\phi^-(E) = \phi_n$ where $\phi_i$ is defined as above. For any interval $I \subset \mathbb{R}$, $\mathcal{P}(I)$ is defined to be the extension-closed subcategory of $\mathcal{D}$ generated by objects $E \in \mathcal{P}(\phi)$ for $\phi \in I$.*

We denote by $\mathrm{Slice}(\mathcal{D})$ the set of all slicings on $\mathcal{D}$. Bridgeland introduced a generalized metric in $\mathrm{Slice}(\mathcal{D})$:

**Definition 2.2.2** ([17, Section 6] ). *Let $\mathcal{P}_1$, $\mathcal{P}_2 \in \mathrm{Slice}(\mathcal{D})$, then the generalized metric $d : \mathrm{Slice}(\mathcal{D}) \times \mathrm{Slice}(\mathcal{D}) \to [0, +\infty]$ is defined as*

$$d(\mathcal{P}_1, \mathcal{P}_2) := \sup_{E \neq 0 \in \mathcal{D}} \left\{ |\phi_1^+(E) - \phi_2^+(E)|, |\phi_1^-(E) - \phi_2^-(E)| \right\}.$$

Before recalling stability condition on $\mathcal{D}$, we first recall the stability function on an abelian category $\mathcal{A}$ [52].

**Definition 2.2.3.** *A stability function on $\mathcal{A}$ is a group homomorphism $Z : K_0(\mathcal{A}) \to \mathbb{C}$ such that for any nonzero object $A \in \mathcal{A}$, the complex number $Z(A)$ lies in the subset*

$$H = \{z = r\exp(i\pi\phi) | r > 0, 0 < \phi \leq 1\} \subset \mathbb{C}.$$

*The phase of $A$ is defined to be $\phi(A) = \frac{1}{\pi}\arg Z(A) \in (0, 1]$. An object $E \in \mathcal{A}$ is said to be (semi)stable if for any subobject $A \subset E$ we have*

$$\phi(A) < (\leq)\phi(E).$$

**Definition 2.2.4** (Harder-Narasimhan property [17, Definition 2.3])**.** *Let $Z : K_0(\mathcal{A}) \to \mathbb{C}$ be a stability function on the abelian category $\mathcal{A}$. Then $Z$ is said to have Harder-Narasimhan property if for any nonzero object $E \in \mathcal{A}$ there is a filtration*

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots E_{n-1} \subset E_n = E$$

*such that each $F_i = E_i/E_{i-1}$ is a semistable object of phase $\phi_i$ and $\phi_1 > \phi_2 \cdots > \phi_{n-1} > \phi_n$.*

**Definition 2.2.5** (Stability condition)**.** *A stability condition for $\mathcal{D}$ is a pair $\sigma = (Z, \mathcal{A})$ which consists of a heart of a bounded t-structure $\mathcal{A}$ in $\mathcal{D}$, and a stability function (called the central charge of $\sigma$) $Z : K_0(\mathcal{A}) \to \mathbb{C}$ such that $Z$ satisfies the Harder-Narasimhan property.*

The above definition of stability condition is equivalent to the following definition [17, Proposition 5.3]:

**Definition 2.2.6.** *A stability condition is a pair $\sigma = (Z, \mathcal{P})$ which consists of a slicing $\mathcal{P} \in \mathrm{Slice}(\mathcal{D})$ and a group homormorphism called the central charge $Z : K_0(D) \to \mathbb{C}$, such that it satisfies the compatibility condition: if $0 \neq E \in \mathcal{P}(\phi)$ for some $\phi \in \mathbb{R}$, then*

$$Z(E) = r\exp(i\pi\phi), \; r > 0.$$

*The objects in $\mathcal{P}(\phi)$ are called semistable of phase $\phi$, and the simple objects in $\mathcal{P}(\phi)$ are called stable.*

The following lemma will be useful later.

**Lemma 2.2.7** ([17, Lemma 6.4])**.** *If the stability conditions $\sigma = (Z, \mathcal{P})$ and $\tau = (Z, \mathcal{P}')$ have the same central charge and $d(\mathcal{P}, \mathcal{P}') < 1$, then $\sigma = \tau$.*

**Definition 2.2.8** (Support property)**.** *Let $\sigma = (Z, \mathcal{P})$ be a stability condition, by fixing a norm $\|\cdot\|$ on $K_0(\mathcal{D})_{\mathbb{R}} = K_0(\mathcal{D}) \otimes_{\mathbb{Z}} \mathbb{R}$, $\sigma$ is said to have support property if there exists a constant $C > 0$ such that*

$$\|E\| \leq C|Z(E)|$$

*for any stable object $E$.*

We denote by $\mathrm{Stab}(\mathcal{D})$ the set of all stability conditions with the support property. To define the topology on $\mathrm{Stab}(\mathcal{D})$, Bridgeland [17] introduced the following definitions:

**Definition 2.2.9.** *Let $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(\mathcal{D})$. The function $\|\cdot\|_\sigma : \mathrm{Hom}(K_0(\mathcal{D}), \mathbb{C}) \to [0, +\infty]$ is defined as*

$$\|W\|_\sigma := \sup \left\{ \frac{|W(E)|}{|Z(E)|} : \ E \text{ semistable for } \sigma \right\}.$$

**Lemma 2.2.10** ([17, Lemma 6.2])**.** *For $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(\mathcal{D})$ and $0 < \epsilon < \frac{1}{4}$ let*

$$C_\epsilon(\sigma) := \{\tau = (W, \mathcal{Q}) \in \mathrm{Stab}(\mathcal{D}) : \|W - Z\|_\sigma < \sin(\pi\epsilon), d(\mathcal{P}, \mathcal{Q}) < \epsilon\}.$$

*Then by varying $\sigma$, we get a basis for the topology of* $\mathrm{Stab}(\mathcal{D})$*.*

The result on deformation of stability conditions is the main theorem of [17].

**Theorem 2.2.11** ([17, Theorem 7.1])**.** *Let $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(\mathcal{D})$. Suppose there exists $0 < \epsilon_0 < \frac{1}{8}$ such that if $0 < \epsilon < \epsilon_0$ and $W \in \mathrm{Hom}(K_0(\mathcal{D}), \mathbb{C})$ satisfying*

$$|W(E) - Z(E)| < \sin(\pi\epsilon)|Z(E)|$$

*for any semistable object $E \in \mathcal{D}$ with respect to $\sigma$. Then there exists a unique stability condition $\tau = (W, \mathcal{P}')$ such that*

$$d(\mathcal{P}, \mathcal{P}') < \epsilon.$$

**Corollary 2.2.12** ([17, Theorem 1.2])**.** *Let $\mathcal{D}$ be a triangulated category. For each connected component $\Sigma \subset \mathrm{Stab}(\mathcal{D})$ there are a linear subspace $V(\Sigma) \subset \mathrm{Hom}(K_0(\mathcal{D}), \mathbb{C})$, with a well-defined linear topology, and a local homeomorphism $\mathcal{Z} : \Sigma \to V$ which sends a stability condition to its central charge $Z$.*

Since $K_0(\mathcal{D})$ might have infinite rank, in practice we usually assume there is a quotient group $\mathcal{N}$ of finite rank, and the quotient map is denoted by $\mu : K_0(\mathcal{D}) \to \mathcal{N}$. Then let

$\mathrm{Stab}_{\mathcal{N}}(\mathcal{D})$ be the subspace of $\mathrm{Stab}(\mathcal{D})$ consisting of stability conditions whose central charges $Z : K_0(\mathcal{D}) \to \mathbb{C}$ factor through $\mathcal{N}$. Then the following result is an immediate consequence of Corollary 2.2.12.

**Corollary 2.2.13** ([17, Corollary 1.3])**.** *For each connected component $\Sigma \subset \mathrm{Stab}_{\mathcal{N}}(\mathcal{D})$ there are a linear subspace $V(\Sigma) \subset \mathrm{Hom}(\mathcal{N}, \mathbb{C})$, and a local homeomorphism $\mathcal{Z} : \Sigma \to V$ which sends a stability condition to its central charge $Z$. In particular, $\Sigma$ is a finite-dimensional complex manifold.*

We recall some group actions on the space of stability conditions. Let $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ be the universal covering of $\mathrm{GL}^+(2, \mathbb{R})$. Note that an element in $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ can be viewed as a pair $(g, f)$ where $g \in \mathrm{GL}^+(2, \mathbb{R})$ and $f : \mathbb{R} \to \mathbb{R}$ is an increasing map with $f(\phi + 1) = f(\phi) + 1$, such that $g$ and $f$ induce the same action on the circle $S^1 = \{e^{i\pi\phi} : \phi \in \mathbb{R}\} = (\mathbb{R}^2 \setminus \{0\})/\mathbb{R}_{>0}$.

**Definition 2.2.14.** *The space of stability conditions carries a right action by $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. For $\widetilde{g} = (g, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ and $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(\mathcal{D})$, then $\sigma \cdot \widetilde{g} = (Z_g, \mathcal{P}_f)$ where for $[E] \in K_0(\mathcal{D})$*

$$Z_g(E) = g^{-1}Z(E), \quad \mathcal{P}_f(\phi) = \mathcal{P}\left(f(\phi)\right).$$

*The space of stability conditions also carries a left action by $\mathrm{Aut}(\mathcal{D})$. For $T \in \mathrm{Aut}(D)$, denote by $t$ the automorphism of $K_0(\mathcal{D})$ induced by $T$, then $T(\sigma) = (Z_t, \mathcal{P}_T)$ where for $[E] \in K_0(\mathcal{D})$*

$$Z_t(E) = Z(t^{-1}E), \quad \mathcal{P}_T(\phi) = T\left(\mathcal{P}(\phi)\right).$$

**Remark 2.2.15.** *From the definition of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-action, $\sigma$ and $\sigma \cdot \widetilde{g}$ have the same set of semistable objects, but the phases have been relabelled. In particular, note that the additive group $\mathbb{C}$ acts on $\mathrm{Stab}(\mathcal{D})$, via the embedding $\mathbb{C} \hookrightarrow \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$: an element $\lambda \in \mathbb{C}$ acts by*

$$\lambda : (Z, [P]) \mapsto (Z', \mathcal{P}'), \quad Z'(E) = e^{-i\pi\lambda} \cdot Z(E), \quad \mathcal{P}'(\phi) = \mathcal{P}\left(\phi + \mathrm{Re}(\lambda)\right).$$

In the end of this subsection, we recall the following important lemma:

**Lemma 2.2.16** ([22, Proposition 7.6])**.** *Fix $0 \neq E \in D$, then*

*(i) the set of stability conditions $\sigma \in \mathrm{Stab}(\mathcal{D})$ for which $E$ is $\sigma$-stable is open;*

*(ii) the set of stability conditions $\sigma \in \mathrm{Stab}(\mathcal{D})$ for which $E$ is $\sigma$-semistable is closed.*

**Definition 2.2.17.** *Let $\mathcal{A}$ be the heart of a bounded t-structure in $\mathcal{D}$ which is of finite-length, then we introduce a subset of stability conditions*

$$\mathcal{U}(\mathcal{A}) = \{\sigma = (Z, \mathcal{P}) | \mathcal{P}\big((0,1]\big)\}.$$

The relation between simple tilts and stability conditions is the following:

**Lemma 2.2.18** ([15, Lemma 5.5])**.** *Suppose $\mathcal{A}$ is of finite-length. Let $\sigma = (Z, \mathcal{P}) \in \overline{\mathcal{U}}(\mathcal{A})$ the closure of $\mathcal{U}(\mathcal{A})$. Suppose that $Z(S_i) \in \mathbb{R}_{<0}$ for some $i$, also $\mathrm{Im}Z(S_j) > 0$ for $j \neq i$, and $R_{S_i}\mathcal{A}$ is finite length, then there is an open neighborhood $V$ of $\sigma$ such that $V \subset \mathcal{U}(\mathcal{A}) \cup \mathcal{U}(R_{S_i}\mathcal{A})$. Similarly suppose $Z(S_i) \in \mathbb{R}_{>0}$ for some $i$ and $\mathrm{Im}Z(S_j) > 0$ for $j \neq i$, and $L_{S_i}\mathcal{A}$ is finite length, then there is an open neighborhood $V'$ of $\sigma$ such that $V' \subset \mathcal{U}(\mathcal{A}) \cup \mathcal{U}(L_{S_i}\mathcal{A})$.*

## § 2.3 Exceptional collections, helices and mutations

We recall the definitions of exceptional collections, their generalizations called helices and mutations.

**Definition 2.3.1** (Exceptional collection)**.** *An object $E$ in $\mathcal{D}$ is said to be exceptional if*

$$\mathrm{Hom}^k_{\mathcal{D}}(E, E) = \begin{cases} \mathbb{C} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

*An exceptional collection $\mathbb{E} \subset \mathcal{D}$ is a sequence of exceptional objects*

$$\mathbb{E} = (E_0, \cdots, E_n)$$

*such that for all $0 \leq i < j \leq n$, we have $\mathrm{Hom}^\bullet_{\mathcal{D}}(E_j, E_i) = 0$.*

An exceptional collection $\mathbb{E} = (E_0, \cdots, E_n)$ is said to be strong if for all $i,\ j$

$$\mathrm{Hom}^k_{\mathcal{D}}(E_i, E_j) = 0, \quad \text{unless } k = 0.$$

We write $\mathrm{thick}(\mathbb{E}) \subset \mathcal{D}$ for the smallest thick subcategory of $\mathcal{D}$ containing the elements of an exceptional collection $\mathbb{E} \subset \mathcal{D}$. An exceptional collection $\mathbb{E}$ is said to be full if $\mathrm{thick}(\mathbb{E}) = \mathcal{D}$.

From the definitions above, we have that for a full and strong exceptional collection $\mathbb{E}$, the object $\bigoplus_{i=0}^{n} E_i$ is a tilting object in $\mathcal{D}$. Given an exceptional collection $\mathbb{E}$ in $\mathcal{D}$, the right orthogonal subcategory to $\mathbb{E}$ is the full triangulated subcategory

$$\mathbb{E}^{\perp} = \{X \in \mathcal{D} : \ \mathrm{Hom}^{\bullet}_{\mathcal{D}}(E, X) = 0 \text{ for } E \in \mathbb{E}\}.$$

Similarly, the left orthogonal subcategory to $\mathbb{E}$ is

$${}^{\perp}\mathbb{E} = \{X \in \mathcal{D} : \ \mathrm{Hom}^{\bullet}_{\mathcal{D}}(X, E) = 0 \text{ for } E \in \mathbb{E}\}.$$

The subcategory $\langle \mathbb{E} \rangle$ is admissible due to [7, Theorem 3.2], i.e. the inclusion functor $i : \langle \mathbb{E} \rangle \to \mathcal{D}$ has left and right adjoint functors. Thus the fullness of $\mathbb{E}$ is equivalent to $\mathbb{E}^{\perp} = 0$ or ${}^{\perp}\mathbb{E} = 0$.

We suppose $E \in \mathcal{D}$ to be exceptional. Given an object $X \in \mathcal{D}$, the left mutation of $X$ through $E$ is the object $L_E(X)$ defined up to isomorphism by the triangle

$$L_E(X) \longrightarrow \mathrm{Hom}^{\bullet}_{\mathcal{D}}(E, X) \otimes E \xrightarrow{ev} X \longrightarrow L_E(X)[1],$$

where $ev$ denotes the evaluation map. Similarly, given $X \in \mathcal{D}$, the right mutation of $X$ through $E$ is the object $R_E X$ defined by the triangle

$$X \xrightarrow{coev} \mathrm{Hom}^{\bullet}_{\mathcal{D}}(X, E)^* \otimes E \longrightarrow R_E(X) \longrightarrow X[1],$$

where $coev$ denotes the coevaluation map. Moreover, consider the left and right orthogonal subcategories of $E$, these two operations define mutually inverse equivalences of categories (see [23, Appendix B])

$$ {}^{\perp}E \underset{R_E}{\overset{L_E}{\rightleftarrows}} E^{\perp} \tag{2.5}$$

**Definition 2.3.2** (Standard mutation). *Given a full exceptional collection $\mathbb{E} = (E_0, \cdots, E_n)$, the mutation operation $\sigma_i$ for each $0 < i \leq n$ is defined by the rule*

$$\begin{aligned} & \sigma_i(E_0, \cdots, E_{i-2}, E_{i-1}, E_i, E_{i+1}, \cdots, E_n) \\ = \ & (E_0, \cdots, E_{i-2}, L_{E_{i-1}}(E_i), E_{i-1}, E_{i+1}, \cdots, E_n) \end{aligned}$$

This operation takes exceptional collections to exceptional collections [7, Lemma 2.1]. And it takes full collections to full collections [7, Lemma 2.2].

The following definition is due to Bondal[7], and we refer our reader to [23, Appendix B] for the proof.

**Definition-Lemma 2.3.3** (Dual objects)**.** *Let $\mathbb{E} = (E_0, \cdots, E_n)$ be a full exceptional collection and define*

$$F_j = L_{E_0} L_{E_1} \cdots L_{E_{j-1}}(E_j)[j], \quad 0 \leq j \leq n.$$

*Then $F_j$ is called the dual object to $E_j$ and satisfies*

$$\mathrm{Hom}^k_{\mathcal{D}}(E_i, F_j) = \begin{cases} \mathbb{C} & \text{if } i = j \text{ and } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Under our assumptions for the category $\mathcal{D}$, according to [8, Corollary 3.5] $\mathcal{D}$ has a Serre functor $S_D$. By definition of the Serre functor, we have that $S_D$ is an autoequivalence of $\mathcal{D}$ and there are given bi-functorial isomorphisms

$$\phi_{E,G} : \mathrm{Hom}_{\mathcal{D}}(E, G) \longrightarrow \mathrm{Hom}_{\mathcal{D}} \left(G, S_{\mathcal{D}}(E)\right)^*$$

for $E,\ G \in \mathcal{D}$. A motivating example of the Serre functor is the following: when $\mathcal{D} = D^b(Y)$ for a smooth projective variety $Y$, then $S_{\mathcal{D}} := - \otimes \omega_Y[-\dim Y]$ is a Serre functor.

The relation between Serre functor and mutation is the following:

**Lemma 2.3.4** ([23, Corollary 2.10])**.** *Let $\mathbb{E} = (E_0, E_1, \cdots, E_n)$ be a full exceptional collection in $\mathcal{D}$. Then*

$$S_{\mathcal{D}}(E_n) = L_{E_0} \cdots L_{E_{n-1}}(E_n).$$

The following definition first appeared in [51].

**Definition 2.3.5** (Helix)**.** *A sequence of objects $\mathbb{H} = (E_i)_{i \in \mathbb{Z}}$ in $\mathcal{D}$ is a helix if there exist positive integers $(n, d)$ such that*

*(i) for each $i \in \mathbb{Z}$ the collection of objects $(E_i, \cdots, E_{i+n})$ is called a thread and is a full exceptional collection,*

*(ii) for each $i \in \mathbb{Z}$ one has $E_{i-n-1} = S_{\mathcal{D}}(E_i)[-d]$.*

Given a full exceptional collection $\mathbb{E}$, we can generate a helix using the Serre functor $S_{\mathcal{D}}$ by defining $E_{i-m(n+1)} := (S_{\mathcal{D}})^m(E_i)[-dm]$ for every integer $m$. We also have the mutation for helices: given a helix $\mathbb{H} = (E_j)_{j\in\mathbb{Z}}$ and an integer $i$ modulo $n+1$ , then we can get a new helix $\sigma_i(\mathbb{H}) = \mathbb{H}' = (E'_j)_{j\in\mathbb{Z}}$:

$$E'_j = \begin{cases} E_{j-1} & \text{if } j = i \bmod n+1 \\ L_{E_j}(E_{j+1}) & \text{if } j = i-1 \bmod n+1 \\ E_j & \text{otherwise} \end{cases}$$

It is interesting to ask which properties are preserved under mutation, for example, strong exceptional collection may not be preserved under mutation in general.

**Definition 2.3.6** (Geometric helix[9]). *A helix $\mathbb{H} = (E_i)_{i\in\mathbb{Z}}$ is called geometric if for any $i < j$ we have*

$$\mathrm{Hom}^k_{\mathcal{D}}(E_i, E_j) = 0, \quad k \neq 0.$$

*Equivalently, each thread $(E_i, \cdots, E_{i+n})$ is a strong exceptional collection and*

$$\mathrm{Hom}^k_{\mathcal{D}}\left(E_i, S^{-l}_{\mathcal{D}}(E_j)[ld]\right) = 0, \quad \textit{for } k \neq 0,\ l > 0,\ \textit{and all } i,\ j.$$

Given a $t$-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of $\mathcal{D}$, a helix $\mathbb{H} = (E_i)_{i\in\mathbb{Z}}$ is called pure if each element $E_i$ is contained in its heart.

We have the following important result.

**Theorem 2.3.7** ([9, Theorem 2.3 and Lemma 2.5]). *The purity and geometricity of a helix of type $(n, n)$ are preserved under mutations.*

# Chapter 3

# T-structures on the cotangent bundle of $\mathbb{P}^2$

This chapter studies the t-structures for $X = \operatorname{Tot}\Omega_{\mathbb{P}^2}$. We construct tilting objects by using two methods and calculate the simple tilts from the t-structure induced by a tilting object. This chapter is organized in the following way: we first recall some useful properties for exceptional sheaves on $\mathbb{P}^2$, then we show how to obtain the tilting objects for $X$ by using the helix theory on $\mathbb{P}^2$ in section 3.2. As explained in the introduction, we continue constructing more tilting objects by introducing an algebraic structure called the secondary quiver in section 3.3 and construct a functor from the category of representations of the secondary quiver to $\operatorname{Coh} X$, then in section 3.4 we show that the above functor sends tilting objects in the category of representations to tilting objects on $X$. Finally in section 3.5, we mainly calculate some (right) simple tilts from a given t-structure induced by a tilting object.

## § 3.1 Exceptional objects on $\mathbb{P}^2$

First we recall the conception of (semi)stability of sheaves on $\mathbb{P}^n$: fix an ample divisor on $\mathbb{P}^n$, here we can choose the hyperplane $H$, then the slope of a pure sheaf (see [35, Definition 1.1.2]) $\mathcal{F}$ is defined to be

$$\mu(F) := \frac{c_1(\mathcal{F}).H^{n-1}}{r(\mathcal{F})},$$

where $r(\mathcal{F})$ is the rank of $\mathcal{F}$.

**Definition 3.1.1** ((Semi)stability)**.** *A pure sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is called stable (resp. semi-stable), if for any subsheaf $\mathcal{E}$ with strictly smaller rank than $\mathcal{F}$, then we have*

$$\mu(\mathcal{E}) < \mu(\mathcal{F}) \text{ (resp. } \mu(\mathcal{E}) \leq \mu(\mathcal{F}))$$

The following useful lemma is proved in [44, Propositions 2.9 and 2.10].

**Lemma 3.1.2.** *If $E$ is an exceptional object on $\mathbb{P}^2$, then $E$ is a shift of a locally free and stable sheaf.*

A good proof of the following result is given by Bondal and Polishchuk [9, Example 3.2].

**Lemma 3.1.3** (Gorodentsev, Rudakov)**.** *If $(E_0, E_1, E_2)$ is a full and strong exceptional collection in $D^b(\mathbb{P}^2)$, then the positive integers $(a, b, c)$ defined by $a = \dim \mathrm{Hom}_{D^b(\mathbb{P}^2)}(E_0, E_1)$, $b = \dim \mathrm{Hom}_{D^b(\mathbb{P}^2)}(E_1, E_2)$ and $c = \dim \mathrm{Hom}_{D^b(\mathbb{P}^2)}(E_0, E_2)$ satisfy the Markov equation:*

$$a^2 + b^2 + c^2 = abc.$$

Using the above lemmas, we can compare the slopes between the bundles in an exceptional collection in $D^b(\mathbb{P}^2)$:

**Lemma 3.1.4.** *Let $\mathbb{E} = (E_0, E_1, E_2)$ be a full and strong exceptional collection consisting of sheaves in $D^b(\mathbb{P}^2)$, then for $j > i$, we have $\mu(E_j) > \mu(E_i)$.*

*Proof.* If not, we suppose $\mu(E_j) \leq \mu(E_i)$ for some $j > i$. Then we claim that $\mathrm{Hom}_{\mathbb{P}^2}(E_i, E_j) = 0$ since

(i) if $\mu(E_j) = \mu(E_i)$. Any non-trivial homomorphism between stable bundles of equal slopes must be an isomorphism. However, since $\mathrm{Hom}_{\mathbb{P}^2}(E_j, E_i) = 0$ by definition of an exceptional sequence, therefore we have $\mathrm{Hom}_{\mathbb{P}^2}(E_i, E_j) = 0$;

(ii) if $\mu(E_j) < \mu(E_i)$, then $\mathrm{Hom}_{\mathbb{P}^2}(E_i, E_j) = 0$ since there is no map between stable bundles from the one with bigger slope to the smaller slope.

Then by Lemma 3.1.3, we have $a = b = c = 0$. Thus there is no map among $E_0$, $E_1$ and $E_2$.

Since the exceptional collection is full, we see that $D^b(\mathbb{P}^2)$ can be decomposed into two subcategories $\langle E_0 \rangle$ and $\langle E_1,\ E_2 \rangle$. But by [13, Example 3.2], this would imply that $\mathbb{P}^2$ is not connected, which is absurd. □

The following known result will be useful to us:

**Theorem 3.1.5.** *Let $\mathbb{E} = (E_0, E_1, E_2)$ be a full exceptional collection which consists of sheaves in $D^b(\mathbb{P}^2)$. Then $\mathbb{E}$ is strong.*

*Proof.* By [44, Corollary 2.11], for $i < j$ there is at most one $\mathrm{Ext}^k_{\mathbb{P}^2}(E_i, E_j)$ group which doesn't vanish, and such $k \neq 2$. So either $\mathrm{Hom}_{\mathbb{P}^2}(E_i, E_j) \neq 0$ or $\mathrm{Ext}^1_{\mathbb{P}^2}(E_i, E_j) \neq 0$. By using the Riemann-Roch theorem for exceptional sheaves, we have

$$\chi(E_i, E_j) = \frac{1}{2} r(E_i) r(E_j) \left( (\mu(E_i) - \mu(E_j))^2 + 3(\mu(E_j) - \mu(E_i)) + \frac{1}{r^2(E_i)} + \frac{1}{r^2(E_j)} \right).$$

By Lemma 3.1.4, $\mu(E_j) > \mu(E_i)$ when $j > i$. Thus $\chi(E_i, E_j) > 0$, which implies $\mathrm{Ext}^1_{\mathbb{P}^2}(E_i, E_j) = 0$. □

## § 3.2 First method of constructing tilting objects

First, we show for an object which classically generates the derived category of $\mathbb{P}^2$, then the pull back of such an object under the bundle projection map $\pi : X = \mathrm{Tot}\, \Omega_{\mathbb{P}^2} \to \mathbb{P}^2$ will classically generate $D^b(X)$. We say $T \in D^b(X)$ is a generator in the sense of [10, Section 2.1], if $\mathrm{Hom}^{\bullet}_{D^b(X)}(T, A) = 0$ implies $A = 0$ in $D^b(X)$.

**Lemma 3.2.1.** *Let $E$ be an object which classically generates $D^b(\mathbb{P}^2)$, then $\pi^* E$ classically generates $D^b(X)$.*

*Proof.* Let QCoh($X$) be the category of quasi-coherent sheaves on $X$. By [10, Theorems 2.1.2, Theorem 3.1.1], it is equivalent to show that $\pi^*E$ generates the category $D\left(\mathrm{QCoh}(X)\right)$. Since $\pi : X \to \mathbb{P}^2$ is affine and flat, then (non-derived) $\pi_*$ and $\pi^*$ are both exact. Thus we have $\mathbb{L}\pi^* = \pi^*$ and $\mathbb{R}\pi_* = \pi_*$.

For any object $F \in D\left(\mathrm{QCoh}\, X\right)$, we have the adjunction

$$\mathrm{Hom}^\bullet_{D(\mathrm{QCoh}\, X)}\left(\pi^* E, F\right) \cong \mathrm{Hom}^\bullet_{D(\mathrm{QCoh}\, \mathbb{P}^2)}\left(E, \pi_* F\right). \tag{3.1}$$

Then for any $F \in D(\mathrm{QCoh}\, X)$, suppose we have $\mathrm{Hom}^\bullet_{D(\mathrm{QCoh}\, X)}\left(\pi^* E, F\right) = 0$, then due to the adjunction (3.1) and the fact that $E$ generates $D(\mathrm{QCoh}(\mathbb{P}^2))$, we have $\pi_* F = 0$. Since $\pi_*$ has no kernel (this can be verified by taking an open cover of $\mathbb{P}^2$, then applying the definition of $\pi_*$), we have $F \cong 0$.

Applying the same argument in reverse this means that $\pi^*E$ classically generates $D^b(X)$. □

We need the following results on the vanishing of certain cohomology groups.

**Lemma 3.2.2.** *Suppose $\mathbb{E} = (E_0, E_1, E_2)$ is a full exceptional sequence of sheaves on $\mathbb{P}^2$ such that for any $i,\ j \in \{0,\ 1,\ 2\}$, we have $\mu(E_i) - \mu(E_j) \leq 2$. For any object $F \in D^b(\mathbb{P}^2)$, we denote by $F|_H$ the restriction of $F$ to the hyperplane $H \subset \mathbb{P}^2$.*

*Then we have for $n > 0$*

$$\mathrm{H}^1\left(\mathbb{P}^1, E_i^* \otimes E_j(n)|_H\right) = 0, \tag{3.2}$$

*Proof.* Let $i,\ j \in \{0,1,2\}$. By assumptions and Lemma 3.1.4 we know $\mu\left(E_i(-2)\right) = \mu(E_i) - 2 \leq \mu(E_j)$. And since $E_j$ and $E_i(-2)$ are stable, we have $\mathrm{Hom}\left(E_j, E_i(-2)\right) = 0$ or $\mathrm{Hom}\left(E_j, E_i(-2)\right) = \mathbb{C}$.

Using the Serre duality, we have

$$\mathrm{Ext}^2_{\mathbb{P}^2}\left(E_i, E_j(-1)\right) = \mathrm{Hom}_{\mathbb{P}^2}\left(E_j, E_i(-2)\right)^*.$$

Thus we have $\mathrm{Ext}^2_{\mathbb{P}^2}\left(E_i, E_j(-1)\right) = \mathrm{Hom}_{\mathbb{P}^2}\left(E_j, E_i(-2)\right)^* = 0$ or $\mathbb{C}$.

Let $H = \mathbb{P}^1$ be a hyperplane in $\mathbb{P}^2$, then by the Grothendieck splitting theorem, we denote by $E_i|_H = \bigoplus_{m \in I} \mathcal{O}(s_m)^{\oplus k_m}$ the restriction of bundle $E_i$ to $H$, where $I$ is a finite set of indices, $s_m \in \mathbb{Z}$, $k_m > 0$. We write $E_i^* \otimes E_j|_H = \bigoplus_{m \in I'} \mathcal{O}(a_m)^{\oplus u_m}$ for $a_m \in \mathbb{Z}$ and $u_m > 0$. Let $a_0$ be the smallest number of the set $\{a_m\}_{m \in I'}$. Now we consider the tensor product of the restriction sequence with $E_i^* \otimes E_j$:

$$0 \longrightarrow E_i^* \otimes E_j(-1) \longrightarrow E_i^* \otimes E_j \longrightarrow E_i^* \otimes E_j|_H \longrightarrow 0.$$

The long exact sequence in cohomology gives

$$\cdots \longrightarrow \operatorname{Ext}^1_{\mathbb{P}^2}(E_i, E_j) \longrightarrow \operatorname{Ext}^1_H(E_i|_H, E_j|_H) \longrightarrow \operatorname{Ext}^2_{\mathbb{P}^2}(E_i, E_j(-1)) \longrightarrow \operatorname{Ext}^2_{\mathbb{P}^2}(E_i, E_j) \longrightarrow \cdots$$

Now, $\operatorname{Ext}^1_{\mathbb{P}^2}(E_i, E_j) = \operatorname{Ext}^2_{\mathbb{P}^2}(E_i, E_j) = 0$ by the strong property of the exceptional collection due to Theorem 3.1.5, and $\operatorname{Ext}^2_{\mathbb{P}^2}(E_i, E_j(-1)) = 0$ or $\mathbb{C}$ by the previous argument, thus

$$\operatorname{Ext}^1_H(E_i|_H, E_j|_H) \cong \mathrm{H}^1\left(\mathbb{P}^1, E_i^* \otimes E_j|_H\right) = \mathrm{H}^1(\mathbb{P}^1, \bigoplus_{m \in I'} \mathcal{O}(a_m)^{u_m}) = 0 \text{ or } \mathbb{C}.$$

Then we can deduce that $a_0 \geq -2$. Then since $n > 0$, we have $E_i^* \otimes E_j(n)|_H = \bigoplus_{m \in I'} \mathcal{O}(a_m + n)^{u_m}$ where $a_m + n > -2$ for any $m \in I'$, thus

$$\mathrm{H}^1\left(\mathbb{P}^1, E_i^* \otimes E_j(n)|_H\right) = \mathrm{H}^1\left(\mathbb{P}^1, \bigoplus_{m \in I'} \mathcal{O}(a_i + n)\right) = 0.$$

This finishes the proof of the lemma. □

**Proposition 3.2.3.** *Let $\mathbb{E} = (E_0, E_1, E_2)$ be a full exceptional sequence of sheaves on $\mathbb{P}^2$. Suppose for a pair of indices $i,\ j \in \{0, 1, 2\}$ we have that $\mu(E_i) - \mu(E_j) \leq 2$. Then*

$$\mathrm{H}^k\left(\mathbb{P}^2, E_i^* \otimes E_j(n)\right) = 0,$$

*for $k > 0$, $n \geq 0$.*

*Proof.* Let $H = \mathbb{P}^1$ be a hyperplane in $\mathbb{P}^2$. We argue by induction on $n$. For $n = 0$, the claim follows from the strong property of the exceptional collection. Now assume $\mathrm{H}^k\left(\mathbb{P}^2, E_i^* \otimes E_j(n-1)\right) = 0$ for $n \geq 1$ and $k > 0$, then we consider the short exact sequence

$$0 \longrightarrow E_i^* \otimes E_j(n-1) \longrightarrow E_i^* \otimes E_j(n) \longrightarrow E_i^* \otimes E_j(n)|_H \longrightarrow 0.$$

By taking the associated long exact sequence and using the vanishing of $\mathrm{H}^k\left(\mathbb{P}^1, E_i^* \otimes E_j(n)|_H\right)$ for $n > 0$ ($k = 1$ follows from Lemma 3.2 and $k > 1$ always vanish because $\mathbb{P}^1$ is of dimension 1), we have

$$\mathrm{H}^k\left(\mathbb{P}^2, E_i^* \otimes E_j(n-1)\right) = \mathrm{H}^k\left(\mathbb{P}^2, E_i^* \otimes E_j(n)\right)) = 0.$$

The result follows from the induction.

□

Denote by $\mathcal{T}$ the tangent sheaf on $\mathbb{P}^2$, we have the following useful exact sequence:

**Lemma 3.2.4** (Symmetric product of Euler sequence)**.** *We write $\mathbb{P}^2$ as $\mathbb{P}(V)$, where $V$ is a linear space of dimension 3 (or for general $n$). Then we have the short exact sequence of sheaves for $k \geq 1$:*

$$0 \longrightarrow \mathrm{Sym}^{k-1}\left(V \otimes \mathcal{O}(1)\right) \longrightarrow \mathrm{Sym}^k\left(V \otimes \mathcal{O}(1)\right) \longrightarrow \mathrm{Sym}^k \mathcal{T} \longrightarrow 0. \tag{3.3}$$

*Proof.* This follows directly from the following classical result in commutative algebra, for the reader's convenience we include a proof here: suppose there is a short exact sequence of free $A$-modules

$$0 \longrightarrow A \xrightarrow{\ e\ } N \xrightarrow{\ p\ } P \longrightarrow 0,$$

then there is a short exact sequence

$$0 \longrightarrow \mathrm{Sym}^{k-1} N \xrightarrow{\ \widetilde{e}\ } \mathrm{Sym}^k N \xrightarrow{\ \widetilde{p}\ } \mathrm{Sym}^k P \longrightarrow 0$$

where $\widetilde{e} := e \otimes id^{\otimes k-1}$ and $\widetilde{p} := p^{\otimes k}$. For an element $n_1 \otimes \cdots \otimes n_k \in \mathrm{Sym}^k N$ whose image under $\widetilde{p}$ is 0, we have at least one $n_i$ such that $p(n_i) = 0$. Since we are in the symmetric power of modules we can assume that $i = 1$, so $n_1 = e(m)$ for some $m \in M$. Thus $n_1 \otimes \cdots \otimes n_k$ is in the image of $\widetilde{e}$. Finally $\widetilde{e}$ is injective following from that $\mathrm{Sym}^{k-1} N$ is flat and $e$ is injective. □

**Corollary 3.2.5.** *For a pair of vector bundles $E$ and $F$ on $\mathbb{P}^2$, we suppose*

$$\mathrm{Ext}^k_{\mathbb{P}^2}\left(E, F(n)\right) = 0$$

*for $n \geq 0$ and $k \geq 1$, then*

$$\mathrm{Ext}^k_X(\pi^* E, \pi^* F) = 0.$$

*Proof.* We have

$$\mathrm{Ext}^k_X(\pi^*E, \pi^*F) = \mathrm{Ext}^k_{\mathbb{P}^2}(E, F \otimes \mathrm{Sym}^\bullet \mathcal{T}) = \bigoplus_{n=0} \mathrm{H}^k(\mathbb{P}^2, E^* \otimes F \otimes \mathrm{Sym}^n \mathcal{T})$$

by adjunction and $\pi_*\mathcal{O}_X \cong \mathrm{Sym}^\bullet \mathcal{T}$. For each $n \geq 1$, we tensor the exact sequence (3.3) with $E^* \otimes F$, and note that $\mathrm{Sym}^n (V(1)) \cong \mathcal{O}(n)^{\binom{n+2}{n}}$. Thus we get an exact sequence:

$$0 \longrightarrow \mathcal{O}(n-1)^{\binom{n+1}{n-1}} \otimes E^* \otimes F \longrightarrow \mathcal{O}(n)^{\binom{n+2}{n}} \otimes E^* \otimes F \longrightarrow \mathrm{Sym}^n \mathcal{T} \otimes E^* \otimes F \longrightarrow 0.$$

Then we take the long exact sequence of cohomology groups:

$$\cdots \longrightarrow \mathrm{Ext}^1_{\mathbb{P}^2}(E, F(n))^{\oplus\binom{n+2}{n}} \longrightarrow \mathrm{Ext}^1_{\mathbb{P}^2}(E, F \otimes \mathrm{Sym}^n \mathcal{T}_{\mathbb{P}^2}) \longrightarrow$$
$$\longrightarrow \mathrm{Ext}^2_{\mathbb{P}^2}(E, F(n-1))^{\oplus\binom{n+1}{n-1}} \longrightarrow \mathrm{Ext}^2_{\mathbb{P}^2}(E, F(n))^{\oplus\binom{n+2}{n}} \longrightarrow \cdots$$

so $\mathrm{H}^k(\mathbb{P}^2, E^* \otimes F \otimes \mathrm{Sym}^n \mathcal{T})$ vanishes due to the assumptions. □

**Theorem 3.2.6.** *Let $\mathbb{E} = (E_0, E_1, E_2)$ be a full exceptional collection of sheaves on $\mathbb{P}^2$ such that $\mu(E_2) - \mu(E_0) \leq 2$, then $T = \bigoplus_{i=0}^2 \pi^*E_i$ is a tilting object in $D^b(X)$.*

*Proof.* We firstly show that $\mathrm{Ext}^n_X(\pi^*T, \pi^*T) = 0$, for $n > 0$. Since the Ext functor commutes with direct sum, it is enough to show that for any $i,\ j$

$$\mathrm{Ext}^n_X(\pi^*E_i, \pi^*E_j) = 0, \text{ for } n > 0.$$

By Proposition 3.2.3 and Corollary 3.2.5, we have

$$\mathrm{H}^n\left(\mathbb{P}^2, E_i^* \otimes E_j \otimes \mathrm{Sym}^\bullet \mathcal{T}\right) = 0,$$

for $n > 0$. Thus $\mathrm{Ext}^n_X(\pi^*T, \pi^*T) = 0$ for $n > 0$.

Combining with Lemma 3.2.1 that $\pi^*T$ is a classical generator, we conclude that $\pi^*T$ is a tilting object in $D^b(X)$. □

As a corollary, we have

**Theorem 3.2.7.** *Let $\mathbb{E} = (E_0, E_1, E_2)$ be a full exceptional collection of sheaves on $\mathbb{P}^2$. We denote by $\mathbb{H}$ the corresponding helix generated by the Serre functor. There exists a thread $\mathbb{E}' = (E_i, E_{i+1}, E_{i+2})$ in $\mathbb{H}$ such that $\mathbb{E}'$ satisfies the conditions in Proposition 3.2.3. Thus the pull back of the direct sum of the objects in $\mathbb{E}'$ is a tilting object in $D^b(X)$.*

*Proof.* We denote the corresponding helix by $\mathbb{H} = (\cdots, E_0, E_1, E_2, E_3, E_4, \cdots)$. Since the Serre functor on $\mathbb{P}^2$ is $- \otimes \mathcal{O}(-3)[2]$, we have

$$\mu(E_{i-3}) = \mu\left(E_i \otimes \mathcal{O}(-3)\right) = \mu(E_i) - 3.$$

We can consider the numbers $a_0 = \mu(E_2) - \mu(E_0)$, $a_1 = \mu(E_3) - \mu(E_1)$ and $a_2 = \mu(E_4) - \mu(E_2)$. By Lemma 3.1.4, we actually have that the slopes in the helix are strictly increasing, thus $a_i > 0$. The sum $a_0 + a_1 + a_2 = \mu(E_4) - \mu(E_1) + \mu(E_3) - \mu(E_0) = 6$. Then one of the $a_i$ must be $\leq 2$. Then we let $T = \bigoplus_{j=i}^{i+2} \pi^* E_j$, and due to Theorem 3.2.6, we have $T$ is a tilting object in $D^b(X)$. □

Finally we say something about the endomorphism algebra of the tilting object constructed above. By derived Morita theory, given a tilting object $T$. Then there is a derived equivalence:

$$\Phi = \mathbb{R}\mathrm{Hom}_X(T, -) : D^b(X) \longrightarrow D^b\big(B = \mathrm{End}_X(T)\big)$$

sending $T$ to $B$. We take $\mathbb{E}$ to be a full exceptional collection consisting of sheaves and satisfies the condition in Theorem 3.2.6.

**Proposition 3.2.8.** *Let $T = \bigoplus_i \pi^* E_i$ be a tilting object. Then $B = \mathrm{End}_X T$ is noetherian and has finite global dimension.*

*Proof.* Since $\mathcal{T}$ is ample [31], the result follows from [45, Corollary 5.2.9]. □

Now since $B$ is finitely generated, as pointed out by Bridgeland and Stern [23], we can construct a quiver with relations $(Q, I)$ such that $B = \mathbb{C}Q/I$. The quiver is constructed in the following way: the vertex $i$ corresponds to $\pi^* E_i$ (therefore we have three vertices in our case), and the number of arrows from vertex $i$ to $j$ equals the dimension of irreducible maps from $\pi^*E_i$ to $\pi^*E_j$, i.e., the cokernel of the map

$$\bigoplus_{k\neq i,\ j} \mathrm{Hom}_X(\pi^*E_i, \pi^*E_k) \otimes \mathrm{Hom}_X(\pi^*E_k, \pi^*E_j) \longrightarrow \mathrm{Hom}_X(\pi^*E_i, \pi^*E_j).$$

Therefore $B$ can be naturally graded by the length of the path.

Denote by $P_i$ and $S_i$ the projective module and simple module associated with the vertex $i$. We take the collection of dual objects $\mathbb{F}$ of $\mathbb{E}$ (Definition-Lemma 2.3.3) and denote by $s : \mathbb{P}^2 \to X$ the embedding map of the zero section. Then $\Phi$ sends $\pi^*E_i$ to $P_i$ and $s_*F_i$ to $S_i$ by definition [23, Lemma 3.7].

## § 3.3 Secondary quiver and universal extension

Let $\mathbb{E} = (E_0, E_1, E_2)$ be a full exceptional collection of sheaves in $D^b(\mathbb{P}^2)$ (then $\mathbb{E}$ is strong by Theorem 3.1.5), and we denote $T_i := \pi^*E_i$, $T = \oplus_i T_i$. In general, $T$ will not be a tilting object, since we may have $\mathrm{Ext}^n_X(T, T) \neq 0$, for $n > 0$.

**Example 3.3.1.** *Let $\mathbb{E} = \big(\Omega(1), \mathcal{O}, \mathcal{O}(2)\big)$, note that $\mu\left(\mathcal{O}(2)\right) - \mu\left(\Omega(1)\right) = 5/2$. Then*

$$\mathrm{Ext}^1_X\left(\pi^*\mathcal{O}(2), \pi^*\Omega(1)\right) \quad\cong\quad \mathbb{C}^3.$$

*so $T = \pi^*\Omega(1) \oplus \pi^*\mathcal{O} \oplus \pi^*\mathcal{O}(2)$ is not a tilting object.*

However, in general we have

**Lemma 3.3.2.** *For any $i,\ j = 0,\ 1,\ 2$, we have*

*(i)*

$$\mathrm{Ext}^{\geq 2}_X(T_i, T_j) = 0,$$

*(ii) if $i \leq j$ then*

$$\mathrm{Ext}^{\geq 1}_X(T_i, T_j) = 0.$$

*Proof.* (i) By Corollary 3.2.5, it is enough to show that for any $i,\ j$,

$$\mathrm{H}^n\left(\mathbb{P}^2, E_i^* \otimes E_j(k)\right) = 0,\ n \geq 2,$$

where $k \geq 0$. The case $k = 0$ follows from the strong property of $\mathbb{E}$. By restricting the bundle $E_i^* \otimes E_j$ to a degree $k$ curve $C$ in $\mathbb{P}^2$ where $k > 0$, we have the short exact sequence:

$$0 \longrightarrow E_i^* \otimes E_j \longrightarrow E_i^* \otimes E_j(k) \longrightarrow E_i^* \otimes E_j(k)|_C \longrightarrow 0$$

Then we take the cohomology groups, and use the vanishing of $\mathrm{H}^{\geq 2}(C, E_i^* \otimes E_j(k)|_C) = 0$ so we proved the first statement.

(ii) If $i < j$, then $\mu(E_i) < \mu(E_j)$ by Lemma 3.1.4, therefore they satisfies the conditions in Proposition 3.2.3 and we have

$$\mathrm{H}^k\big(\mathbb{P}^2, E_i^* \otimes E_j(n)\big) = 0$$

for $k \geq 1$, $n \geq 0$. By Corollary 3.2.5 we have

$$\mathrm{Ext}_X^k(\pi^* E_i, \pi^* E_j) = \mathrm{Ext}_X^k(T_i, T_j) = 0$$

for $k \geq 1$, this proves the result.

□

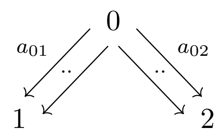
Now we package the non-tilting information of $T$ into the following algebraic structure, and we call it the secondary quiver:

**Definition 3.3.3** (Secondary quiver)**.** *Let $\mathbb{E}$ be a full exceptional collection on $\mathbb{P}^2$ consisting of sheaves and $T_i := \pi^* E_i$. Then let $Q_{\mathbb{E}}$ be the quiver with 3 vertices indexed by $0,\ 1,\ 2$, and the arrows from $i$ to $j$ correspond to a basis of $\mathrm{Ext}_X^1(T_j, T_i)$ as a $\mathbb{C}$-vector space. The quiver $Q_{\mathbb{E}}$ is called the secondary quiver of $\mathbb{E}$. The category of right finitely generated module of $\mathbb{C}Q_{\mathbb{E}}$ is denoted by $\mathrm{rep}\, Q_{\mathbb{E}} =$ mod-$\mathbb{C}Q_{\mathbb{E}}$. We denote the simple module at the vertex $i$ in $\mathrm{rep}\, Q_{\mathbb{E}}$ by $S_i$.*

**Remark 3.3.4.** *Denote by $n_{ij}$ the number of arrows from $i$ to $j$, in our conventions (paths compose on the left and we consider right modules) we have $n_{ij} = \dim \mathrm{Ext}^1(S_j, S_i)$, where $S_i$ is the simple module associated with vertex $i$. With the notations above, we have $\mathrm{Ext}^1_{\mathbb{C}Q_{\mathbb{E}}}(S_i, S_j) \cong \mathrm{Ext}^1_X(T_i, T_j)$.*

**Proposition 3.3.5.** *$Q_{\mathbb{E}}$ takes one of the following forms:*

*(i)*

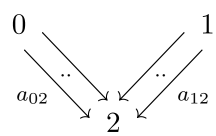



*where* $a_{0i} \geq 0,\ i \in \{1,\ 2\}$ *denote the number of arrows. Then any object* $M$ *in* $\operatorname{rep} Q_{\mathbb{E}}$ *fits into the short exact sequence:*

$$0 \longrightarrow S_0^{m_0} \longrightarrow M \longrightarrow S_1^{m_1} \oplus S_2^{m_2} \longrightarrow 0$$

*where* $m_i \geq 0$.

*(ii)*




*where* $a_{i2} \geq 0,\ i \in \{0,\ 1\}$ *denote the number of arrows. Then any object* $M$ *in* $\operatorname{rep} Q_{\mathbb{E}}$ *fits into the short exact sequence:*

$$0 \longrightarrow S_0^{m_0} \oplus S_1^{m_1} \longrightarrow M \longrightarrow S_2^{m_2} \longrightarrow 0$$

$m_i \in \mathbb{N}$.

From now on, we call the first quiver case 1, and the second quiver case 2.

*Proof.* There is no arrow from $i$ to $j$ for $i \geq j$ due to Lemma 3.3.2. The only case we need to exclude is

$$0 \overset{a_{01}}{\underset{\vdots}{\rightrightarrows}} 1 \overset{a_{12}}{\underset{\vdots}{\rightrightarrows}} 2$$

where $a_{12}, a_{01} > 0$. By definition of the arrows in the secondary quiver, this is equivalently to say

$$\operatorname{Ext}^1_X(T_i, T_{i-1}) \neq 0,\ i = 2,\ 1.$$

By using Proposition 3.2.3, this implies $\mu(E_i) - \mu(E_{i-1}) > 2$, thus

$$\mu(E_2) - \mu(E_0) = (\mu(E_2) - \mu(E_1)) + (\mu(E_1) - \mu(E_0)) > 4. \tag{3.4}$$

However, consider the helix $\mathbb{H} = (\cdots, E_0, E_1, E_2, E_3, \cdots)$ generated by $\mathbb{E}$. We have $\mu(E_3) = \mu(E_0(3)) = \mu(E_0) + 3$, and $\mu(E_2) < \mu(E_3) = \mu(E_0) + 3$ by Lemma 3.1.4. So (3.4) contradicts with this fact. □

Following an idea in [32, Section 3], we construct the tilting objects from any full exceptional collection $\mathbb{E}$. The procedure is called universal extension in loc. cit. and we will use that name.

**Definition 3.3.6** (Universal extension)**.** *Let $\mathbb{E} = (E_0, E_1, E_2)$ be a full exceptional collection consisting of sheaves in $D^b(\mathbb{P}^2)$ (then $\mathbb{E}$ is strong by Theorem 3.1.5) and $T_i := \pi^* E_i$*

*(i) For the secondary quiver of case 1 in Proposition 3.3.5, we consider the objects $\widetilde{T_m} \in \operatorname{Coh} X$ $(m = 1,\ 2)$ which are obtained from the extensions:*

$$0 \longrightarrow T_0 \otimes_{\mathbb{C}} \operatorname{Ext}^1_X(T_m, T_0)^* \xrightarrow{\gamma_m} \widetilde{T_m} \longrightarrow T_m \longrightarrow 0. \tag{3.5}$$

*If we consider the short exact sequence as a triangle in $D^b(X)$, then the connection map is the adjoint of the canonical evaluation map (which is called the coevalutaion map):*

$$coev_m : T_m \longrightarrow T_0 \otimes \operatorname{Ext}^1_X(T_m, T_0)^*[1].$$

*The mapping cone over the coevaluation map defines the exact sequence (3.5) uniquely up to isomorphism.*

*(ii) For the secondary quiver of case 2 in Proposition 3.3.5, we consider the object $\widetilde{T_2} \in \operatorname{Coh} X$, which is obtained from the following extension:*

$$0 \longrightarrow \bigoplus\nolimits_{i=0,1} T_i \otimes_{\mathbb{C}} \operatorname{Ext}^1_X(T_2, T_i)^* \xrightarrow{\delta} \widetilde{T_2} \longrightarrow T_2 \longrightarrow 0. \tag{3.6}$$

*Again if we consider the short exact sequence as a triangle in $D^b(X)$, the connection map is the coevaluation map:*

$$coev : T_2 \xrightarrow{\begin{bmatrix} coev_0 \\ coev_1 \end{bmatrix}} \bigoplus_{i=0,1} T_i \otimes \operatorname{Ext}^1_X(T_2, T_i)^*[1].$$

*Similarly the exact sequence (3.6) is defined unique up to isomorphism.*

**Theorem 3.3.7.** *For the secondary quiver of case 1 in Proposition 3.3.5, the object $\widetilde{T} := T_0 \oplus \widetilde{T_1} \oplus \widetilde{T_2}$ is a tilting object in $D^b(X)$.*

*For the secondary quiver of case 2 in Proposition 3.3.5, the object $\widetilde{T} := T_0 \oplus T_1 \oplus \widetilde{T_2}$ is a tilting object in $D^b(X)$.*

*Proof.* We prove the case 1, the proof of the case 2 is similar.

Firstly, since $\text{thick}\big(T = \oplus_i T_i\big) = D^b(X)$ by Lemma 3.2.1 and $T_i \in \text{thick}\big(\widetilde{T}\big)$ by definition, this shows that $\text{thick}(\widetilde{T}) = D^b(X)$. Now we check the vanishing of $\text{Ext}^{\geq 1}(\widetilde{T}, \widetilde{T})$.

(a) We firstly show that $\text{Ext}_X^{\geq 1}(T_0, \widetilde{T_m}) = 0$ for $m = 1,\ 2$. We take the long exact sequence of (3.5) associated with the functor $\text{Hom}(T_0, -)$:

$$\cdots \text{Ext}_X^1(T_0, T_0) \otimes \text{Ext}_X^1(T_m, T_0)^* \longrightarrow \text{Ext}_X^1(T_0, \widetilde{T_m}) \longrightarrow \text{Ext}_X^1(T_0, T_m) \hookrightarrow \text{Ext}_X^2(T_0, T_0) \otimes \text{Ext}_X^1(T_m, T_0)^* \longrightarrow \text{Ext}_X^2(T_0, \widetilde{T_m}) \longrightarrow \cdots.$$

We have $\text{Ext}_X^k(T_0, T_0) = 0$ and $\text{Ext}_X^k(T_0, T_m) = 0$ $(k \geq 1)$ because of the second statement of Lemma 3.3.2. Thus $\text{Ext}_X^{\geq 1}(T_0, \widetilde{T_m}) = 0$ since the terms on the two sides both vanish.

(b) Now we show that $\text{Ext}_X^{\geq 1}(\widetilde{T_m}, T_0) = 0$ for $m = 1,\ 2$. By taking the long exact sequence associated with the functor $\text{Hom}_X(-, T_0)$:

$$\cdots \longrightarrow \text{Hom}_X(T_0, T_0) \otimes \text{Ext}_X^1(T_m, T_0) \xrightarrow{can_*} \text{Ext}_X^1(T_m, T_0) \longrightarrow \text{Ext}_X^1(\widetilde{T_m}, T_0) \longrightarrow \text{Ext}_X^1(T_0, T_0) \otimes \text{Ext}_X^1(T_m, T_0) \hookrightarrow \text{Ext}_X^2(T_m, T_0) \longrightarrow \text{Ext}_X^2(\widetilde{T_m}, T_0) = 0 \longrightarrow \cdots.$$

Note that the connection map is the Yonneda product[47, Definition 4.16]

$$\begin{aligned} can_* : \text{Hom}_X(T_0, T_0) \otimes \text{Ext}_X^1(T_m, T_0) &\longrightarrow \text{Ext}_X^1(T_m, T_0) \\ id \otimes \eta &\longmapsto \eta. \end{aligned}$$

Thus $can_*$ is surjective, then we can see that $\text{Ext}_X^1(\widetilde{T_m}, T_0) = 0$.

Since $\text{Ext}_X^{\geq 2}(T_m, T_0) = 0$ due to the first statement of Lemma 3.3.2 and $\text{Ext}_X^{\geq 1}(T_0, T_0) = 0$, we have that $\text{Ext}_X^{\geq 2}(\widetilde{T_m}, T_0) = 0$ since terms on the two sides both vanish.

(c) Then we check that $\text{Ext}_X^{\geq 1}(\widetilde{T_m}, \widetilde{T_m}) = 0$. By taking the long exact sequence associated with functor $\text{Hom}_X(\widetilde{T_m}, -)$ $(i \geq 1)$:

$$\cdots \to \text{Ext}_X^i(\widetilde{T_m}, T_0) \otimes \text{Ext}_X^1(T_m, T_0)^* \to \text{Ext}_X^i(\widetilde{T_m}, \widetilde{T_m}) \to \text{Ext}_X^i(\widetilde{T_m}, T_m) \to \cdots. \tag{3.7}$$

Since $\mathrm{Ext}^i_X(\widetilde{T_m}, T_0) = 0$ by part (b), so it is equivalent to check the vanishing of $\mathrm{Ext}^i_X(\widetilde{T_m}, T_m)$.

Again we take the functor $\mathrm{Hom}_X(-, T_m)$ of the universal extension sequence:

$$\cdots \to \mathrm{Ext}^i_X(T_m, T_m) \to \mathrm{Ext}^i_X(\widetilde{T_m}, T_m) \to \mathrm{Ext}^i_X(T_0, T_m) \otimes \mathrm{Ext}^1_X(T_m, T_0) \to \cdots .$$

Note that $\mathrm{Ext}^{\geq 1}_X(T_0, T_m) = 0$ since $0 < m = 1,\ 2$ and $\mathrm{Ext}^i_X(T_m, T_m) = 0$ because of Lemma 3.3.2. The cohomology groups on the both sides vanish, so we have $\mathrm{Ext}^i_X(\widetilde{T_m}, T_m) = 0$ for $i \geq 1$. Return back to (3.7), and we obtain that $\mathrm{Ext}^i_X(\widetilde{T_m}, \widetilde{T_m}) = 0$.

(d) Finally we check that $\mathrm{Ext}^{\geq 1}_X(\widetilde{T_m}, \widetilde{T_n}) = 0$ when $m,\ n \in \{1,\ 2\}$ $(m \neq n)$. Consider the universal extension of $T_n$ and we take the functor $\mathrm{Hom}_X(\widetilde{T_m}, -)$, then we have a long exact sequence:

$$\cdots \to \mathrm{Ext}^i_X(\widetilde{T_m}, T_0) \otimes \mathrm{Ext}^1_X(T_n, T_0)^* \to \mathrm{Ext}^i_X(\widetilde{T_m}, \widetilde{T_n}) \to \mathrm{Ext}^i_X(\widetilde{T_m}, T_n) \to \cdots .$$

Because $\mathrm{Ext}^{\geq 1}_X(\widetilde{T_m}, T_0) = 0$ by part (b), so it is equivalent to check that $\mathrm{Ext}^i_X(\widetilde{T_m}, T_n) = 0$ for $i \geq 1$.

Again we take the functor $\mathrm{Hom}_X(-, T_n)$ to the universal extension of $T_m$:

$$\cdots \to \mathrm{Ext}^i_X(T_m, T_n) \to \mathrm{Ext}^i_X(\widetilde{T_m}, T_n) \to \mathrm{Ext}^i_X(T_0, T_m) \otimes \mathrm{Ext}^1_X(T_m, T_0)^* \to \cdots .$$

Note that $\mathrm{Ext}^{\geq 1}_X(T_0, T_m) = 0$ by Lemma 3.3.2. Since $\mathrm{Ext}^{\geq 1}_X(T_m, T_n) = 0$ by our assumption of case 1 (there is no arrow between vertices $n$ and $m$), thus we obtain that

$$\mathrm{Ext}^i_X(\widetilde{T_m}, T_n) = 0.$$

□

## § 3.4 The functor $\mathcal{F}_Q$ and a second method of constructing tilting objects

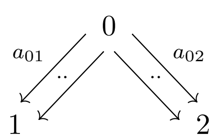
Let $\mathbb{E}$ be a full exceptional collection of sheaves on $\mathbb{P}^2$. The object of this section is to construct a functor from the category of representations of the secondary quiver rep $Q_{\mathbb{E}}$ to $\mathrm{Coh}(X)$, and we prove that the functor preserves the tilting objects and sends the simple module $S_i$ at vertex $i$ to $T_i = \pi^* E_i$.

Let $\widetilde{T}$ be the object obtained by the universal extension as in the Definition 3.3.6 and Theorem 3.3.7. The endomorphism algebra $\operatorname{End}\widetilde{T}$ contains three idempotents $\widetilde{e_i}$ which are projection maps $\widetilde{T} \to \widetilde{T}_i$. We denote by $e_i$ the idempotents in $\mathbb{C}Q_{\mathbb{E}}$.

We first define an map from the path algebra $\mathbb{C}Q_{\mathbb{E}}$ to $\operatorname{End}\widetilde{T}$:

**Definition 3.4.1.** *We discuss the 2 cases in Proposition 3.3.5 separately*

*(i) For case 1, the quiver takes the form*




*We define a map* $\iota : \mathbb{C}Q_{\mathbb{E}} \longrightarrow \operatorname{End}\widetilde{T}$ *which sends* $e_i$ *to* $\widetilde{e_i}$*; and for an arrow* $\alpha \in \operatorname{Ext}^1_X(T_m, T_0)$ *(*$m = 1,\ 2$*)*

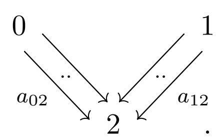
$$\iota(\alpha) : T_0 \longrightarrow \widetilde{T_m}, \quad \iota(\alpha)(t_0) := \gamma_m(t_0 \otimes \alpha^*), \quad t_0 \in T_0$$

*where* $\alpha^*$ *is the dual element of* $\alpha$ *in* $\operatorname{Ext}^1(T_m, T_0)^*$ *and* $\gamma_m$ *is defined in the universal extension sequence (3.5):*

$$0 \longrightarrow T_0 \otimes \operatorname{Ext}^1_X(T_m, T_0)^* \xrightarrow{\gamma_m} \widetilde{T_m} \longrightarrow T_m \longrightarrow 0.$$

*By extending* $\iota$ *linearly to* $\mathbb{C}Q_{\mathbb{E}}$*. Then* $\iota$ *is a map of algebras.*

*(ii) For case 2, the quiver takes the form*




*We define a map* $\iota : \mathbb{C}Q_{\mathbb{E}} \longrightarrow \operatorname{End}\widetilde{T}$ *which sends* $e_i$ *to* $\widetilde{e_i}$ *and for arrow* $\alpha \in \operatorname{Ext}^1_X(T_2, T_m)$ *(*$m = 0,\ 1$*)*

$$\iota(\alpha) : T_m \to \widetilde{T_2}, \quad \iota(\alpha)(t_m) := \delta(t_m \otimes \alpha^*), \quad t_m \in T_m$$

*where again* $\alpha^*$ *is the dual element of* $\alpha$ *in* $\operatorname{Ext}^1_X(T_2, T_m)^*$ *and* $\delta_m$ *appeared in the universal extension sequence (3.6)*

$$0 \longrightarrow \bigoplus_{i=0,1} T_i \otimes_{\mathbb{C}} \operatorname{Ext}^1_X(T_2, T_i)^* \xrightarrow{\delta} \widetilde{T_2} \longrightarrow T_2 \longrightarrow 0.$$

*By extending* $\iota$ *linearly to* $\mathbb{C}Q_{\mathbb{E}}$*. Then* $\iota$ *is a map of algebras.*

**Definition 3.4.2.** *We denote* $A := \mathbb{C}Q_{\mathbb{E}}$ *and* $B := \operatorname{End}\widetilde{T}$*, where* $\widetilde{T}$ *is defined in Theorem 3.3.7. Then we define a functor* $\mathcal{F}_Q$ *from*

$$\mathcal{F}_Q : \textit{mod-}A \quad \longrightarrow \quad \operatorname{Coh}(X)$$

*by the composition:*

$$\textit{mod-}A \xrightarrow{-\otimes_A B} \textit{mod-}B \xrightarrow{-\otimes_B \widetilde{T}} \operatorname{Coh}(X)$$

We have

**Proposition 3.4.3.** $\mathcal{F}_Q$ *is an exact functor.*

*Proof.* We show the result only for case 1 in Proposition 3.3.5, since case 2 is quite similar. In this case $\widetilde{T} = T_0 \oplus \widetilde{T_1} \oplus \widetilde{T_2}$.

Since mod-$A$ is a category of finite length, we only need to show that the derived functor of $\mathcal{F}_Q$ sends $S_i$ to the objects in $\operatorname{Coh} X$, that is, it preserves the t-structure.

We consider the left derived functor of $\mathcal{F}_Q$:

$$\mathbb{L}\mathcal{F}_Q : D^b(\text{mod-}A) \xrightarrow{-\otimes^{\mathbb{L}}_A B} D^b(\text{mod-}B) \xrightarrow{-\otimes^{\mathbb{L}}_B \widetilde{T}} D^b(X).$$

In this case $S_0 = P_0$ where $P_0 := e_0 A$. We also denote by $\widetilde{P_i} = \widetilde{e_i}B \cong \operatorname{Hom}_X(\widetilde{T}, \widetilde{T_i})$ ($i = 0,\ 1,\ 2$) the projective modules in mod-$B$. Then by definition

$$\mathbb{L}\mathcal{F}_Q(P_0) \cong \widetilde{P_0} \otimes^{\mathbb{L}}_B \widetilde{T} \cong T_0$$

in $D^b(X)$. Thus $\mathcal{F}_Q$ is exact on $S_0$.

For $S_m$ where $m = 1,\ 2$. We have a triangle in $D^b(A)$:

$$S_0^{\oplus a_{0m}} = S_0 \otimes \operatorname{Ext}^1_X(S_m, S_0)^* \xrightarrow{\eta = \sum_k x^k_{0m} \cdot} P_m \longrightarrow S_m \longrightarrow S_0^{\oplus a_{0m}}[1]$$

where we denote by $x^k_{0m}$ the arrow from the vertex 0 to $m$.

By taking the functor $- \otimes^{\mathbb{L}}_A B$, we have a triangle in $D^b(B)$:

$$\widetilde{P_0}^{\oplus a_{0m}} \xrightarrow{\iota(\sum_k x^k_{0m} \cdot)_*} \widetilde{P_m} \longrightarrow S_m \otimes^{\mathbb{L}}_A B \longrightarrow \widetilde{P_0}^{\oplus a_{0m}}[1] \qquad (3.8)$$

where $\iota(\sum_k x_{0m}^k)_* : \mathrm{Hom}_X(\widetilde{T}, T_0)^{\oplus a_{0m}} \to \mathrm{Hom}_X(\widetilde{T}, \widetilde{T_m})$ is induced from $\iota(\sum_k x_{0m}^k) : T_0^{\oplus a_{0m}} \to \widetilde{T_m}$. Then by taking the functor $- \otimes_B^{\mathbb{L}} \widetilde{T}$, and under the canonical isomorphisms $\widetilde{P}_i \otimes_B^{\mathbb{L}} \widetilde{T} \cong \widetilde{T}_i$, we have the morphisms between the distinguished triangles in $D^b(X)$:

$$\begin{array}{ccccccc}
\widetilde{P_0}^{\oplus a_{0m}} \otimes \widetilde{T} & \xrightarrow{\iota(\sum_k x_{0m}^k \cdot)} & \widetilde{P_m} \otimes \widetilde{T} & \longrightarrow & S_m \otimes_A B \otimes_B \widetilde{T} & \longrightarrow & \widetilde{P_0}^{\oplus a_{0m}} \otimes \widetilde{T}[1] \\
\downarrow \cong & & \downarrow \cong & & \downarrow f & & \downarrow \\
T_0^{\oplus a_{0m}} & \xrightarrow{\gamma_m} & \widetilde{T_m} & \longrightarrow & T_m & \xrightarrow{coev} & T_0^{\oplus a_{0m}}[1]
\end{array}$$

the diagram is commutative since $\iota(\sum_k x_{0m}^k \cdot) = \gamma_m$ by Definition 3.4.1. Then $f$ is induced from the morphism between the distinguished triangles. Therefore $T_m \cong S_m \otimes_A^{\mathbb{L}} B \otimes_B^{\mathbb{L}} \widetilde{T} = \mathbb{L}\mathcal{F}_Q(S_m)$ in $D^b(X)$. Moreover, since the objects of lower triangle all lie in $\mathrm{Coh}(X)$, we actually have a short exact sequence:

$$0 \longrightarrow \mathbb{L}\mathcal{F}_Q(S_0)^{\oplus a_{0m}} \cong T_0^{\oplus a_{0m}} \longrightarrow \mathbb{L}\mathcal{F}_Q(P_m) \cong \widetilde{T_m} \longrightarrow \mathbb{L}\mathcal{F}_Q(S_m) \cong T_m \longrightarrow 0.$$

So we have tested the exactness on any simple module $S_m$ in mod-$A$, the exactness of $\mathcal{F}_Q$ is obtained. □

The proof in the above theorem also implies

**Corollary 3.4.4.** *For simple modules $S_i \in$ mod-$A$, we have $\mathcal{F}_Q(S_i) = T_i$.*

Here we say $M$ is a tilting object in some abelian category, we mean $M$ is a tilting object in its corresponding derived category.

**Theorem 3.4.5.** *Let $\mathbb{E}$ be a full exceptional collection consisting of sheaves on $\mathbb{P}^2$, $Q_{\mathbb{E}}$ be its secondary quiver and $A$ be the path algebra of $Q_{\mathbb{E}}$. Then $\mathcal{F}_Q$ sends tilting objects in mod-$A$ to tilting objects in* $\mathrm{Coh}(X)$.

*Proof.* We prove case 1, the proof of case 2 is essentially the same. Firstly we show the property of generating. Let $R \in$ mod-$A$ be a generator for $D^b(A)$, then $S_i \in \mathrm{thick}(R)$ for any $i$. Then under the correspondence of $\mathcal{F}_Q$, we have

$$\mathcal{F}_Q(S_i) = T_i \in \mathrm{thick}\big(\mathcal{F}_Q(R)\big)$$

in $D^b(X)$ due to Corollary 3.4.4. Now because $\bigoplus_{i=0}^2 T_i$ generates $D^b(X)$, thus we have that $\mathcal{F}_Q(R)$ generates the whole category.

Given a tilting object $R \in \text{mod-}A$, we assume $R$ fits into

$$0 \longrightarrow S_0^{m_0} \longrightarrow R \longrightarrow S_1^{m_1} \oplus S_2^{m_2} \longrightarrow 0, \tag{3.9}$$

and $\text{Ext}_A^1(R,R) = 0$. For any pair of objects $M,\ N \in \text{mod-}A$, $\text{Ext}_A^{\geq 2}(M,N) = 0$ since $A$ is the path algebra of an acyclic quiver without relations. For the corresponding object $\widetilde{T} = \mathcal{F}_Q(R)$ under the above definition, we have

$$0 \longrightarrow T_0^{m_0} \longrightarrow \widetilde{T} \longrightarrow T_1^{m_1} \oplus T_2^{m_2} \longrightarrow 0. \tag{3.10}$$

By applying the functors $\text{Hom}_X(-, T_1^{m_1} \oplus T_2^{m_2})$, $\text{Hom}_X(-, T_0^{m_0})$ and $\text{Hom}_X(-, \widetilde{T})$ to (3.10), we get the corresponding long exact sequences and put them in the first, second and third row separately. Finally we get the commutative diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{End}_X(T_1^{m_1} \oplus T_2^{m_2}) & \longrightarrow & \text{Hom}_X(\widetilde{T}, T_1^{m_1} \oplus T_2^{m_2}) & \longrightarrow & \text{Hom}_X(T_0^{m_0}, T_1^{m_1} \oplus T_2^{m_2}) \\
\downarrow & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \gamma_1} & & \downarrow \\
\text{End}_X(T_0^{m_0}) & \xrightarrow{\beta_1} & \text{Ext}_X^1(T_1^{m_1} \oplus T_2^{m_2}, T_0^{m_0}) & \xrightarrow{\beta_2} & \text{Ext}_X^1(\widetilde{T}, T_0^{m_0}) & \longrightarrow & \text{Ext}_X^1(T_0^{m_0}, T_0^{m_0}) = 0 \\
\downarrow & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \gamma_2} & & \downarrow \\
\text{Hom}_X(T_0^{m_0}, \widetilde{T}) & \xrightarrow{\theta_1} & \text{Ext}_X^1(T_1^{m_1} \oplus T_2^{m_2}, \widetilde{T}) & \xrightarrow{\theta_2} & \text{Ext}_X^1(\widetilde{T}, \widetilde{T}) & \longrightarrow & \text{Ext}_X^1(T_0^{m_0}, \widetilde{T}) = 0 \\
 & & & & \downarrow & & \\
 & & & & \text{Ext}_X^1(\widetilde{T}, T_1^{m_1} \oplus T_2^{m_2}) = 0 & &
\end{array}$$

where the morphisms on the columns are obtained in the following way: we first apply the functors $\text{Hom}_X(T_0^{m_0}, -)$, $\text{Hom}_X(T_1^{m_1} \oplus T_2^{m_2}, -)$, $\text{Hom}_X(\widetilde{T}, -)$ and $\text{Hom}_X(T_0^{m_0}, -)$ on (3.10) separately, then take the long exact sequences. The diagram is commutative due to the naturality of derived functors.

We have a chain complex (not necessarily exact)

$$\text{End}_X(T_1^{m_1} \oplus T_2^{m_2}) \bigoplus \text{End}_X(T_0^{m_0}) \xrightarrow{\eta = \binom{\alpha_1}{\beta_1}} \text{Ext}_X^1(T_1^{m_1} \oplus T_2^{m_2}, T_0^{m_0}) \xrightarrow[=\gamma_2 \circ \beta_2]{f = \theta_2 \circ \alpha_2} \text{Ext}_X^1(\widetilde{T}, \widetilde{T}) \tag{3.11}$$

where $f$ is surjective. To obtain $\text{Ext}_X^1(\widetilde{T}, \widetilde{T}) = 0$, we only need to show that $\eta$ is surjective, since $f \circ \eta = 0$. So we consider the corresponding diagram in $\text{rep}\, Q_{\mathbb{E}}$ by applying Hom functors similarly to (3.9):

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{End}_A(S_1^{m_1}\oplus S_2^{m_2}) & \xrightarrow{\widetilde{\phi}} & \mathrm{Hom}_A(R, S_1^{m_1}\oplus S_2^{m_2}) & \longrightarrow & \mathrm{Hom}_A(S_0^{m_0}, S_1^{m_1}\oplus S_2^{m_2}) = 0 \\
\downarrow & & \downarrow{\scriptstyle \widetilde{\alpha_1}} & & \downarrow{\scriptstyle \widetilde{\gamma_1}} & & \downarrow \\
\mathrm{End}_A(S_0^{m_0}) & \xrightarrow{\widetilde{\beta_1}} & \mathrm{Ext}^1_A(S_1^{m_1}\oplus S_2^{m_2}, S_0^{m_0}) & \xrightarrow{\widetilde{\beta_2}} & \mathrm{Ext}^1_A(R, S_0^{m_0}) & \longrightarrow & \mathrm{Ext}^1_A(S_0^{m_0}, S_0^{m_0}) = 0 \\
\downarrow & & \downarrow{\scriptstyle \widetilde{\alpha_2}} & & \downarrow{\scriptstyle \widetilde{\gamma_2}} & & \downarrow \\
\mathrm{Hom}_A(S_0^{m_0}, R) & \xrightarrow{\widetilde{\theta_1}} & \mathrm{Ext}^1_A(S_1^{m_1}\oplus S_2^{m_2}, R) & \xrightarrow{\widetilde{\theta_2}} & \mathrm{Ext}^1_A(R, R) & \longrightarrow & \mathrm{Ext}^1_A(S_0^{m_0}, R) = 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Hom}_A(S_0^{m_0}, S_1^{m_1}\oplus S_2^{m_2}) = 0 & & \mathrm{Ext}^1_A(S_1^{m_1}\oplus S_2^{m_2}, S_1^{m_1}\oplus S_2^{m_2}) = 0 & & \mathrm{Ext}^1_A(R, S_1^{m_1}\oplus S_2^{m_2}) = 0 & &
\end{array}
$$

As before, we have a chain complex:

$$\mathrm{End}_A(S_1^{m_1}\oplus S_2^{m_2}) \bigoplus \mathrm{End}_A(S_0^{m_0}) \xrightarrow{\widetilde{\eta}=\binom{\widetilde{\alpha_1}}{\widetilde{\beta_1}}} \mathrm{Ext}^1_A(S_1^{m_1}\oplus S_2^{m_2}, S_0^{m_0}) \xrightarrow{\widetilde{f}=\widetilde{\theta}\circ\widetilde{\alpha_2}} \mathrm{Ext}^1_A(R,R) = 0,$$

but now it is exact at the middle term by diagram chasing: for any $q \in ker(\widetilde{f})$, there exists $s \in \mathrm{Hom}_A(S_0^{m_0}, R)$ such that $\widetilde{\theta_1}(s) = \widetilde{\alpha_2}(q)$. Since $\mathrm{End}_A(S_0^{m_0}) \cong \mathrm{Hom}_A(S_0^{m_0}, R)$, we denote by $p_1$ the corresponding element of $s$ in $\mathrm{End}_A(S_0^{m_0})$. Now consider $q' = q - \widetilde{\beta_1}(p_1)$, then we have $\widetilde{\alpha_2}(q') = 0$. By the exactness we have $p_2 \in \mathrm{End}_A(S_1^{m_1} \oplus S_2^{m_2})$ such that $\widetilde{\alpha_1}(p_2) = q'$. So we proved that there exists an element $(p_1, p_2) \in \mathrm{End}_A(S_1^{m_1} \oplus S_2^{m_2}) \bigoplus \mathrm{End}_A(S_0^{m_0})$ such that $\widetilde{\eta}(p_1, p_2) = q$.

Thus $\widetilde{\eta}$ is surjective since $\mathrm{Ext}^1_A(R,R) = 0$ by assumption. Comparing the above exact sequence with the chain complex (3.11), we have a commutative diagram:

$$
\begin{array}{ccc}
\mathrm{End}_A(S_1^{m_1}\oplus S_2^{m_2}) \bigoplus \mathrm{End}_A(S_0^{m_0}) & \xrightarrow{\widetilde{\eta}} & \mathrm{Ext}^1_A(S_1^{m_1}\oplus S_2^{m_2}, S_0^{m_0}) \\
\downarrow{\scriptstyle \mathcal{F}_Q} & & \downarrow{\scriptstyle \cong} \\
\mathrm{End}_X(T_1^{m_1}\oplus T_2^{m_2}) \bigoplus \mathrm{End}_X(T_0^{m_0}) & \xrightarrow{\eta} & \mathrm{Ext}^1_X(T_1^{m_1}\oplus T_2^{m_2}, T_0^{m_0})
\end{array}
$$

Now $\eta$ is surjective since $\widetilde{\eta}$ is surjective. Thus by the above arguments, we have shown that $\mathrm{Ext}^1_X(\widetilde{T}, \widetilde{T}) = 0$.

The higher cohomology $\mathrm{Ext}^{\geq 1}_X(\widetilde{T}, \widetilde{T})$ vanishes since $\mathrm{Ext}^{\geq 1}_X(T_i, T_j)$ for any $i,\ j$ . We have proved the theorem. $\square$

## § 3.5 Simple tilts and Ext-quivers

### 3.5.1 T-structures in $D_0^b(X)$

In the final section we calculated some simple tilts from a finite length heart in $D_0^b(X)$, where we denote by $D_0^b(X)$ the full subcategory in $D^b(X)$ consisting of objects whose cohomology sheaves are set-theoretically supported on the zero section $\mathbb{P}^2 \subset \operatorname{Tot}\Omega_{\mathbb{P}^2}$. Similarly, for some noetherian graded algebra $W$ we denote by $D_0^b(W)$ the full subcategory of $D^b(W)$ consisting of objects whose cohomology modules are nilpotent. The derived equivalence induced by the tilting object will restrict to the following case:

**Lemma 3.5.1** ([14, Lemma 4.4])**.** *Let $\mathbb{E}$ be a full exceptional collection as in Theorem 3.2.6 and $T = \bigoplus_i \pi^* E_i \in D^b(X)$. Then the equivalence*

$$\Phi = \mathbb{R}\mathrm{Hom}(T, -) : D^b(X) \longrightarrow D^b(B = \mathrm{End}_X T)$$

*will restrict to an equivalence of full subcategories*

$$\mathbb{R}\mathrm{Hom}(T, -) : D_0^b(X) \longrightarrow D_0^b(B).$$

*Proof.* As explained in the end of Section 3.2. $B$ can be written as a path algebra of a quiver subject to relations. We denote by $S_i$ the simple module associated with vertex $i$. The collection of dual objects to $\mathbb{E}$ is denoted by $\mathbb{F}$. Then the result follows from the fact that $D_0^b(X)$ is the smallest triangulated subcategory containing the objects $s_* F_i$ and $D_0^b(B)$ is the smallest triangulated subcategory containing the objects $S_i$, while $\Phi$ sends $s_* F_i$ to $S_i$. □

The standard t-structure on $D^b(B)$ induces one on $D_0^b(B)$ in the obvious way, and pulling this back using the equivalence

$$\mathbb{R}\mathrm{Hom}_X(T, -) : D_0^b(X) \longrightarrow D_0^b(B)$$

of Lemma 3.5.1 gives a bounded t-structure on $D_0^b(X)$. If $T = \bigoplus \pi^* E_i$ for some exceptional collection on $\mathbb{P}^2$ in Theorem 3.2.6, we denote the induced heart by $\mathcal{B}(E_0, E_1, E_2)$. Then $\mathcal{B}(E_0, E_1, E_2)$ is a finite-length abelian category with 3 simple objects: denote the dual exceptional collection of $\mathbb{E}$ by $\mathbb{F} = (F_0, F_1, F_2)$ as in Definition-Lemma 2.3.3. Then the simple objects in $\mathcal{B}(E_0, E_1, E_2)$ will be

$$S_0 := s_* F_0, \quad S_1 := s_* F_1, \quad S_2 := s_* F_2. \tag{3.12}$$

where we denote by $s : \mathbb{P}^2 \to X$ the inclusion map.

Here we introduce a conception which encode the information of simple tilts:

**Definition 3.5.2** (Ext-quivers)**.** *Let $\mathcal{H}$ be a t-structure of finite length in the triangulated category $D$, then the Ext-quiver of $\mathcal{H}$ is the graded quiver $Q(\mathcal{H})$ whose vertexes are labeled by the simple objects $S_i$, and whose graded $k$ arrows $S_i \to S_j$ corresponds to a basis of* $\operatorname{Hom}_D(S_j, S_i[k])$.

We denote by $\mathcal{A}$ the heart of the t-structure which is obtained from the tilting object $T = \pi^*\mathcal{O}(-2) \oplus \pi^*\mathcal{O}(-1) \oplus \pi^*\mathcal{O}$, that is $\mathcal{A} = \mathcal{B}(\mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O})$.

For the heart $\mathcal{A}$, we list the following information:

**Simple objects** By definition of dual collection (Definition-Lemma 2.3.3), we have

$$\mathbb{F} = \big((\mathcal{O}(-2), L_{\mathcal{O}(-2)}\mathcal{O}(-1)[1], \mathcal{O}(-3)[2]\big),$$

the last identity $F_2 = L_{E_1}L_{E_0}E_2 = E_2 \otimes \omega_{\mathbb{P}^2}[2]$ follows from Lemma 2.3.4. For the middle term, by definition

$$0 \longrightarrow L_{\mathcal{O}(-2)}\mathcal{O}(-1) \longrightarrow \operatorname{Hom}_{\mathbb{P}^2}\big(\mathcal{O}(-2), \mathcal{O}(-1)\big)\otimes\mathcal{O}(-2) = \mathcal{O}(-2)^3 \longrightarrow \mathcal{O}(-1) \longrightarrow 0.$$

By comparing the above short exact sequence with the Euler sequence

$$0 \longrightarrow \Omega \longrightarrow \mathcal{O}(-1)^3 \longrightarrow \mathcal{O} \longrightarrow 0$$

we obtain that $L_{\mathcal{O}(-2)}\mathcal{O}(-1) \cong \Omega(-1)$. Therefore

$$S_0 = s_*\mathcal{O}(-2),\ S_1 = s_*\Omega(-1)[1],\ S_2 = s_*\mathcal{O}(-3)[2].$$

**Dual Objects** The dual objects in $D^b(X)$ which are $P_0 = \pi^*\mathcal{O}(-2),\ P_1 = \pi^*\mathcal{O}(-1),\ P_2 = \pi^*\mathcal{O}$ such that $\operatorname{Hom}^\bullet_{D^b(X)}(P_i, S_j) = \delta_{ij}\mathbb{C}$.

**Ext-quiver** The arrows in the Ext-quiver which will be:

$$\begin{aligned}
\operatorname{Ext}^i_X(S_0, S_1) &= \operatorname{Ext}^i_X\left(s_*\mathcal{O}(-2), s_*\Omega(-1)[1]\right)\\
&= \operatorname{Ext}^{i+1}_{\mathbb{P}^2}\left(s^*s_*\mathcal{O}(-2), \Omega(-1)\right)\\
&= \bigoplus_{k=0}^{2}\operatorname{Ext}^{i+1-k}_{\mathbb{P}^2}\Big(\mathcal{O}(-2)\otimes\wedge^k\mathcal{T}, \Omega(-1)\Big)\\
&= \bigoplus_{k=0}^{2}\mathrm{H}^{i+1-k}\Big(\mathbb{P}^2, \Omega(1)\otimes\wedge^k\Omega\Big),
\end{aligned}$$

in the third equation $\mathcal{T}$ is the tangent sheaf and we use the formula obtained from the Koszul resolution $s^*s_*\mathcal{O}_{\mathbb{P}^2} = \bigoplus \wedge^k\mathcal{T}[k]$ in $D^b(\mathbb{P}^2)$ [34, Proposition 11.1]. We firstly tensor the vector bundle $\Omega(1)$ by the exterior powers of the Euler exact sequence:

$$0 \longrightarrow \wedge^k\Omega \longrightarrow \wedge^k V(-1) \longrightarrow \wedge^{k-1}\Omega \longrightarrow 0$$

Then we take the long exact sequence of cohomology groups, and do the inductions on $k$ of $\wedge^k\Omega$. Finally we get

$$\mathrm{H}^p(\mathbb{P}^2, \Omega(1)\otimes\wedge^q\Omega) = \begin{cases} \mathbb{C} & p=1 \text{ and } q=1,\ 2 \\ \mathbb{C}^3 & p=2 \text{ and } q=2 \\ 0 & \text{otherwise.} \end{cases}$$

Thus we have

$$\mathrm{Ext}^i_X(S_0,S_1) = \begin{cases} \mathbb{C}^3 & i=1,\ 3, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly we have

$$\begin{aligned} \mathrm{Ext}^i_X(S_1,S_2) &= \begin{cases} \mathbb{C}^3 & i=1,\ 3, \\ 0 & \text{otherwise.} \end{cases} \\ \mathrm{Ext}^i_X(S_0,S_2) &= \begin{cases} \mathbb{C}^3 & i=2, \\ 0 & \text{otherwise.} \end{cases} \\ (m=0,\ 2)\ \ \mathrm{Ext}^i_X(S_m,S_m) &= \begin{cases} \mathbb{C} & i=0,\ 2,\ 4, \\ 0 & \text{otherwise.} \end{cases} \\ \mathrm{Ext}^i_X(S_1,S_1) &= \begin{cases} \mathbb{C} & i=0,\ 4, \\ \mathbb{C}^{10} & i=2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

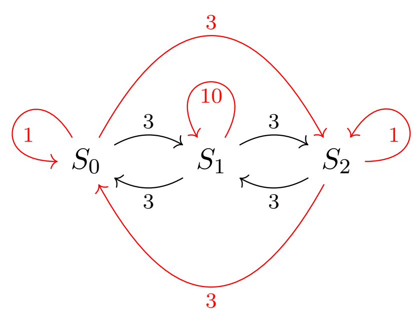
the other arrows can be obtained from the Serre duality

$$\mathrm{Hom}_D(S_i, S_j[k]) \cong \mathrm{Hom}_D(S_j, S_i[4-k])^*.$$

We summarise the dimensions of Ext-groups between the simple objects in the following table:

| $\dim_{\mathbb{C}} \mathrm{Ext}^n(S_i, S_j)$ | $n=0$ | $n=1$ | $n=2$ | $n=3$ | $n=4$ |
|---|---|---|---|---|---|
| $(i,j)=(0,0)$ | 1 | 0 | 1 | 0 | 1 |
| $(i,j)=(0,1)$ | 0 | 3 | 0 | 3 | 0 |
| $(i,j)=(1,0)$ | 0 | 3 | 0 | 3 | 0 |
| $(i,j)=(1,1)$ | 1 | 0 | 10 | 0 | 1 |
| $(i,j)=(1,2)$ | 0 | 3 | 0 | 3 | 0 |
| $(i,j)=(2,1)$ | 0 | 3 | 0 | 3 | 0 |
| $(i,j)=(2,2)$ | 1 | 0 | 1 | 0 | 1 |
| $(i,j)=(0,2)$ | 0 | 0 | 3 | 0 | 0 |
| $(i,j)=(2,0)$ | 0 | 0 | 3 | 0 | 0 |

Finally We draw the Ext-quiver of $\mathcal{A}$ as the following: the black arrows are of degree 1, the red arrows are of degree 2:




(3.13)

we omit the arrows of other degrees since they can be read from the Serre duality.

### 3.5.2 Mukai flops

According to the Ext-quiver (3.13) of $\mathcal{A}$, we see that there is a symmetry between $S_0$ and $S_2$. In fact, we can write down explicitly the auto-equivalence of $D^b(X)$ which interchanges $S_0$ and $S_2$. This is quite useful for simplifying the calculations of the further simple tilts.

**Definition 3.5.3** (Mukai flop)**.** *Let $g: Z \to X$ be a blow-up along the zero section of the projection $\pi: X \to \mathbb{P}^2$. The exceptional locus $E(\subset Z)$ of $g$ is the incidence variety in $\mathbb{P}^2 \times (\mathbb{P}^2)^\vee$ where $(\mathbb{P}^2)^\vee$ is the dual projective space. By blowing down $Z$ along the projection $E \to (\mathbb{P}^2)^\vee$, we obtain a birational map $g^+: Z \to X^+ = \mathrm{Tot}\,\Omega_{(\mathbb{P}^2)^\vee}$. The resulting birational map*

$$\phi = g^+ \circ g^{-1} : X \dashrightarrow X^+$$

*is called a Mukai flop.*

The following result was proved in [48, Theorem 3.1] [37, Corollary 5.7]:

**Theorem 3.5.4** (Derived equivalence for Mukai flop)**.** *Let* $\phi : X \dashrightarrow X^+$ *denote the Mukai flop between* $X$ *and* $X^+$*, then we have a derived equivalence* $\Phi : D^b(X) \to D^b(X^+)$*. Moreover, the derived equivalence can be restricted to* $D^b_0(X) \to D^b_0(X^+)$*.*

*Proof.* Here we adopt the proof from [54, Example 5.3]: put $\mathcal{O}_X(1) := \pi^*\mathcal{O}(1)$. Then we have a tilting object $\varepsilon = \oplus_{i=0}^2 \mathcal{O}_X(-i)$ by Theorem 3.2.6. Since $\phi$ is an isomorphism in codimension one, there is an equivalence between the categories of reflexive sheaves on $X$ and $X^+$. Denote $\varepsilon'$ the sheaf corresponding to $\varepsilon$ on $X^+$. By results in [48, Lemma 1.3], we have

$$\varepsilon' \cong \bigoplus_{i=0}^{2} \mathcal{O}_{X^+}(i).$$

Since we have the isomorphism of rings

$$\phi_* : \mathrm{End}_X(\varepsilon) = A \xrightarrow{\cong} A^+ = \mathrm{End}_{X^+}(\varepsilon')$$

so we have the equivalence of derived categories:

$$\Phi : D^b(X^+) \longrightarrow D^b(A^+) \longrightarrow D^b(A) \longrightarrow D^b(X).$$

By restricting to the full subcategories, we have an equivalence:

$$\overline{\Phi} : D^b_0(X^+) \longrightarrow D^b_0(A^+) \longrightarrow D^b_0(A) \longrightarrow D^b_0(X).$$

□

Then under the canonical isomorphism between $\mathbb{P}^2$ and $(\mathbb{P}^2)^\vee$, we can view $\Phi$ as an autoequivalence of $D^b(X)$ (resp. $D^b_0(X)$).

**Proposition 3.5.5.** *Let* $\mathbb{E} = (\mathcal{O}(-2), \mathcal{O}(-1), \mathcal{O})$*,* $\mathcal{A}$ *be the associated bounded t-structure on* $D^b_0(X)$*, and* $S_i$ *be the simple objects in* $\mathcal{A}$*. Let* $\Psi := \left(- \otimes^{\mathbb{L}} \mathcal{O}_X(-2)\right) \circ \Phi$*, then we have* $\Psi(\mathcal{A}) = \mathcal{A}$*, and* $\Psi(S_j) = S_{2-j}$ *for* $j = 0,\ 1,\ 2$*.*

*Proof.* Since by Theorem 3.5.4, $\mathcal{O}_X(i)$ $(-2 \le i \le 0)$ is sent to $\mathcal{O}_X(-i)$ via $\Phi$, then it follows that $\Psi(\mathcal{O}_X(i)) = \mathcal{O}_X(-i-2)$ for $-2 \le i \le 0$. Thus we have $\Psi\left(\bigoplus_{i=-2}^0 \pi^*\mathcal{O}(i)\right) = \bigoplus_{i=-2}^0 \mathcal{O}_X(i)$, and we get that $\Psi(\mathcal{A}) \cong \mathcal{A}$.

If we denote by $P_0 = \mathcal{O}_X(-2)$, $P_1 = \mathcal{O}_X(-1)$ and $P_2 = \mathcal{O}_X$, then the $\Psi(S_j)$ is uniquely determined by the dual relation with $\Psi(P_i)$, i.e

$$\mathrm{Hom}^{\bullet}_{D^b(X)}(\Psi(P_i), \Psi(S_j)) = \begin{cases} \mathbb{C} & i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\Psi(P_i) = P_{2-i}$, thus we have $\Psi(S_j) = S_{2-j}$ for $j = 0,\ 1,\ 2.$ □

We proceed to calculate the simple tilts at $S_i$ of $\mathcal{A}$, and their Ext-quivers:

### 3.5.3 Simple tilt at $S_1$

Denote by $\mathcal{C} := R_{S_1}\mathcal{A}$ the right simple tilt at $S_1$, we list the following information:

**Simple objects** by Proposition 2.1.7, the simple objects are $U_1 = S_1[-1] = s_*\Omega(-1),\ U_0,\ U_2$ where $U_0$ and $U_2$ fit into the following short exact sequences:

$$0 \to s_*\mathcal{O}(-2) \to U_0 \to s_*\Omega(-1)[1]^3 = S_1 \otimes \mathrm{Ext}^1(S_1, S_0) \to 0, \qquad (3.14)$$

$$0 \to s_*\mathcal{O}(-3)[2] \to U_2 \to s_*\Omega(-1)[1]^3 = S_1 \otimes \mathrm{Ext}^1(S_1, S_2) \to 0. \qquad (3.15)$$

Note that by Proposition 3.5.5 we have $\Psi(U_0) = U_2$ and $\Psi(U_2) = U_0$.

**Dual objects** The dual objects in $D^b(X)$ are $\widetilde{T_0},\ \pi^*\mathcal{O}(-2),\ \pi^*\mathcal{O}$, where $\widetilde{T_0}$ fits into the universal extension sequence

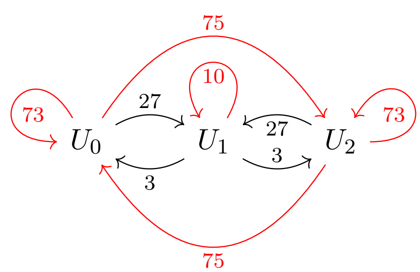
$$0 \longrightarrow \pi^*\Omega(-1) \longrightarrow \widetilde{T_0} \longrightarrow \pi^*\mathcal{O}^3 \longrightarrow 0.$$

Note that $\widetilde{T} := \widetilde{T_0} \oplus \pi^*\mathcal{O}(-2) \oplus \pi^*\mathcal{O}$ is a tilting object, and obtained by the universal extension of the pullback of $\mathbb{E} = (\Omega(-1), \mathcal{O}(-2), \mathcal{O})$.

**Ext-quiver** We calculate the dimensions of Ext-groups between the simple objects in the appendix at the end of this chapter, the readers could refer to them for interest. The dimensions of Ext-groups are summarised in the following table:

| $\dim_{\mathbb{C}} \operatorname{Ext}^n(S_i, S_j)$ | $n=0$ | $n=1$ | $n=2$ | $n=3$ | $n=4$ |
|---|---|---|---|---|---|
| $(i,j)=(0,0)$ | 1 | 0 | 1 | 0 | 1 |
| $(i,j)=(0,1)$ | 0 | 3 | 0 | 3 | 0 |
| $(i,j)=(1,0)$ | 0 | 3 | 0 | 3 | 0 |
| $(i,j)=(1,1)$ | 1 | 0 | 10 | 0 | 1 |
| $(i,j)=(1,2)$ | 0 | 3 | 0 | 3 | 0 |
| $(i,j)=(2,1)$ | 0 | 3 | 0 | 3 | 0 |
| $(i,j)=(2,2)$ | 1 | 0 | 1 | 0 | 1 |
| $(i,j)=(0,2)$ | 0 | 0 | 3 | 0 | 0 |
| $(i,j)=(2,0)$ | 0 | 0 | 3 | 0 | 0 |

Finally the Ext-quiver $Q(\mathcal{C})$ will be:




(3.16)

as before the black arrows are of degree 1, the red arrows are of degree 2, and we omit the arrows of other degrees since they can be read from the Serre duality.

### 3.5.4 Simple tilt at $S_2$

We denote $\Gamma := R_{S_2}\mathcal{A}$ and $\Gamma' := R_{S_0}\mathcal{A}$. Combining the Proposition 3.5.5 with Lemma 2.1.6 we have

$$\Psi\left(R_{S_0}\mathcal{A}\right) \cong R_{\Psi(S_0)}\Psi\left(\mathcal{A}\right) \cong R_{S_2}\mathcal{A}.$$

Thus $\Gamma' = R_{S_0}\mathcal{A} \cong \Gamma = R_{S_2}\mathcal{A}$. Under the equivalence we only need to calculate one of the Ext-quivers (here we choose $\Gamma$):

**Simple objects** the simple objects are $W_0 = s_*\mathcal{O}(-2),\ W_1 = s_*\mathcal{O}(-3)[1],\ W_2 = s_*\mathcal{O}(-4)[2]$.

**Dual objects** The dual objects in $D^b(X)$ are $\pi^*\mathcal{O}(-2),\ \pi^*\Omega,\ \pi^*\mathcal{O}(-1)$, and $T = \pi^*\mathcal{O}(-2) \oplus \pi^*\Omega \oplus \pi^*\mathcal{O}(-1)$ is a tilting object;

**Ext-quiver**

- Calculations of $\mathrm{Ext}^i_X(W_1, W_0)$:

$$\begin{aligned}\mathrm{Ext}^i_X\left(s_*\mathcal{O}(-3)[1], s_*\mathcal{O}(-2)\right) &= \bigoplus_{k=0}^{2}\mathrm{Ext}^{i-k-1}_{\mathbb{P}^2}\left(\mathcal{O}(-3)\otimes\wedge^k\mathcal{T}, \mathcal{O}(-2)\right)\\ &= \bigoplus_{k=0}^{2}\mathrm{H}^{i-k-1}\left(\mathbb{P}^2, \Omega^k\otimes\mathcal{O}(1)\right)\\ &= \begin{cases}\mathbb{C}^3 & i=1,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

- Calculations of $\mathrm{Ext}^i_X(W_2, W_0)$:

$$\begin{aligned}\mathrm{Ext}^i_X\left(s_*\mathcal{O}(-4)[2], s_*\mathcal{O}(-2)\right) &= \bigoplus_{k=0}^{2}\mathrm{Ext}^{i-k-2}_{\mathbb{P}^2}\left(\mathcal{O}(-4)\otimes\wedge^k\mathcal{T}, \mathcal{O}(-2)\right)\\ &= \bigoplus_{k=0}^{2}\mathrm{H}^{i-k-2}\left(\mathbb{P}^2, \Omega^k\otimes\mathcal{O}(2)\right)\\ &= \begin{cases}\mathbb{C}^6 & i=2,\\ \mathbb{C}^3 & i=3,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

- Similarly for $\mathrm{Ext}^i_X(W_2, W_1)$, we have

$$\begin{aligned}\mathrm{Ext}^i_X\left(s_*\mathcal{O}(-4)[2], s_*\mathcal{O}(-3)[1]\right) &= \bigoplus_{k=0}^{2}\mathrm{H}^{i-k-1}\left(\mathbb{P}^2, \Omega^k\otimes\mathcal{O}(1)\right)\\ &= \begin{cases}\mathbb{C}^3 & i=1,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

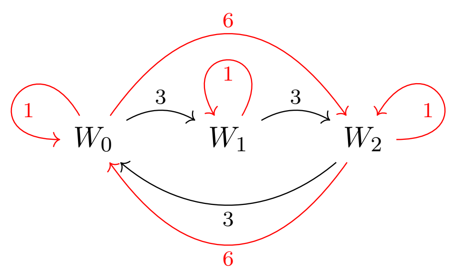
- For $\mathrm{Ext}^i_X(W_j, W_j)$ $(j=0,\ 2)$ we have

$$\begin{aligned}\mathrm{Ext}^i_X\left(W_j, W_j\right) &= \bigoplus_{k=0}^{2}\mathrm{H}^{i-k}\left(\mathbb{P}^2, \Omega^k\right)\\ &= \begin{cases}\mathbb{C} & i=0,\ 2,\ 4,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

We summarise the dimensions of Ext-groups between the simple objects in the following table:

| $\dim_{\mathbb{C}} \operatorname{Ext}^n(S_i, S_j)$ | $n=0$ | $n=1$ | $n=2$ | $n=3$ | $n=4$ |
|---|---|---|---|---|---|
| $(i,j)=(0,0)$ | 1 | 0 | 1 | 0 | 1 |
| $(i,j)=(0,1)$ | 0 | 0 | 0 | 3 | 0 |
| $(i,j)=(1,0)$ | 0 | 3 | 0 | 0 | 0 |
| $(i,j)=(1,1)$ | 1 | 0 | 1 | 0 | 1 |
| $(i,j)=(1,2)$ | 0 | 0 | 0 | 3 | 0 |
| $(i,j)=(2,1)$ | 0 | 3 | 0 | 0 | 0 |
| $(i,j)=(2,2)$ | 1 | 0 | 1 | 0 | 1 |
| $(i,j)=(0,2)$ | 0 | 3 | 6 | 0 | 0 |
| $(i,j)=(2,0)$ | 0 | 0 | 6 | 3 | 0 |

Finally the Ext-quiver $Q(\mathcal{B})$ will be




**Appendix: calculations of Ext-quiver of $R_{S_1}\mathcal{A}$**

We keep using the notations as in Section 3.5.3. We need the following lemma for our calculations:

**Lemma 3.5.6.** *$U_0 \cong s_*P[1]$ where $P \in \operatorname{Coh}\mathbb{P}^2$ fits into the following short exact sequence:*

$$0 \longrightarrow P \longrightarrow \Omega(-1) \otimes \operatorname{Hom}_{\mathbb{P}^2}(\Omega(-1), \mathcal{O}(-2)) \longrightarrow \mathcal{O}(-2) \longrightarrow 0 \qquad (3.17)$$

*that is, $P = L_{\Omega(-1)}\mathcal{O}(-2)$ (the left mutation).*

*Proof.* Since $s$ is a closed immersion, then $s_*$ is exact. Then we push-forward (3.17) by $s_*$, then shift the resulting short exact sequence by degree 1. Note that

$$\begin{aligned} \operatorname{Ext}^1(S_1, S_0) &= \operatorname{Ext}^1_X(s_*\Omega(-1)[1], s_*\mathcal{O}(-2)) \\ &= \operatorname{Hom}_X(s_*\Omega(-1), s_*\mathcal{O}(-2)) \\ &= \operatorname{Hom}_{\mathbb{P}^2}(\Omega(-1), \mathcal{O}(-2)), \end{aligned}$$

by comparing with (3.14), we have $U_0 \cong s_*P[1]$ by the uniqueness of the cone. □

Moreover, $P$ is again an exceptional object and $(\mathcal{O}(-3), \Omega(-1), P)$ forms a full and strong exceptional collection on $\mathbb{P}^2$ by Theorem 2.10.

- To calculate $\mathrm{Ext}^i_X(U_1, U_0)$, we take the long exact sequence of (3.14) associated with $\mathrm{Hom}_X(U_1, -) = \mathrm{Hom}_X(S_1[-1], -)$:

$$\mathrm{Ext}^1_X(U_1, S_0) = 0 \longrightarrow \mathrm{Ext}^1_X(U_1, U_0) \longrightarrow \mathrm{Ext}^1_X(U_1, S_1^{\oplus 3}) = \mathbb{C}^{30}$$
$$\xrightarrow{\delta} \mathrm{Ext}^2_X(U_1, S_0) = \mathbb{C}^3 \longrightarrow \mathrm{Ext}^2_X(U_1, U_0) \longrightarrow \mathrm{Ext}^2_X(U_1, S_1^{\oplus 3}) = 0$$
$$\longrightarrow \mathrm{Ext}^3_X(U_1, S_0) = 0 \longrightarrow \mathrm{Ext}^3_X(U_1, U_0) \longrightarrow \mathrm{Ext}^3_X(U_1, S_1^{\oplus 3}) = \mathbb{C}^3$$
$$\longrightarrow \mathrm{Ext}^4_X(U_1, S_0) = 0 \longrightarrow \mathrm{Ext}^4_X(U_1, U_0) \longrightarrow \mathrm{Ext}^4_X(U_1, S_1^{\oplus 3}) = 0$$

To determine the surjectivity of the connection map $\delta$, we can show that $\mathrm{Ext}^2_X(U_1, U_0) = 0$: by the Serre duality, we have

$$\begin{aligned} \mathrm{Ext}^2_X(U_1, U_0) &= \mathrm{Ext}^2_X(s_*\Omega(-1), s_*P[1]) \\ &= \mathrm{Ext}^1_X(s_*P, s_*\Omega(-1))^*. \end{aligned}$$

Then we use the Koszul resolution again, we have

$$\begin{aligned} \mathrm{Ext}^1_X(s_*P, s_*\Omega(-1)) &= \mathrm{Ext}^1_{\mathbb{P}^2}(s^*s_*P, \Omega(-1)) \\ &= \mathrm{Ext}^1_{\mathbb{P}^2}(P, \Omega(-1)) \oplus \mathrm{Hom}_{\mathbb{P}^2}(P \otimes \mathcal{T}, \Omega(-1)). \end{aligned}$$

The first component vanishes since $P$ and $\Omega(-1)$ form part of a strong exceptional collection. By (3.17) we get that $c_0(P) = 5$ and $c_1(P) = -13$, thus $c_0(P \otimes \mathcal{T}) = 10$ and $c_1(P \otimes \mathcal{T}) = -11$. So we have $\mu(P \otimes \mathcal{T}) = -10/11 > \mu(\Omega(-1)) = -5/2$. Since $P$, $\mathcal{T}$ and $\Omega(-1)$ are slope semistable sheaves, and so is $P \otimes \mathcal{T}$ (see [35, Theorem 3.1.4]), there is no map between the semistable sheaves from the larger slope to the smaller one, so $\mathrm{Hom}_{\mathbb{P}^2}(P \otimes \mathcal{T}, \Omega(-1)) = 0$. We have shown that $\mathrm{Ext}^2_X(U_1, U_0) = 0$, by taking this to the above long exact sequence, we can read that

$$\mathrm{Ext}^i_X(U_1, U_0) = \begin{cases} \mathbb{C}^{27} & i = 1, \\ \mathbb{C}^3 & i = 3, \\ 0 & \text{otherwise.} \end{cases}$$

- For $\mathrm{Ext}^i_X(U_1, U_2)$, we apply the functor $\Psi$, and by using that $\Psi(U_1) = U_1$, $\Psi(U_0) = U_2$, we have

$$\mathrm{Ext}^i_X(U_1, U_2) = \mathrm{Ext}^i_X(U_1, U_0) = \begin{cases} \mathbb{C}^{27} & i = 1, \\ \mathbb{C}^3 & i = 3, \\ 0 & \text{otherwise.} \end{cases}$$

- To get $\mathrm{Ext}^i(U_2, U_0)$, we first take the long exact sequence of (3.15) associated with $\mathrm{Hom}(-, U_0)$ (note that $\mathrm{Ext}^i_X(S_1, U_0) = \mathrm{Ext}^{i-1}_X(U_1, U_0)$):

$$\begin{array}{ccccc} \mathrm{Ext}^1_X(S_1, U_0)^3 = 0 & \longrightarrow & \mathrm{Ext}^1_X(U_2, U_0) & \longrightarrow & \mathrm{Ext}^1_X(S_2, U_0) \\ \hookrightarrow \mathrm{Ext}^2_X(S_1, U_0)^3 = \mathbb{C}^{81} & \longrightarrow & \mathrm{Ext}^2_X(U_2, U_0) & \longrightarrow & \mathrm{Ext}^2_X(S_2, U_0) \\ \hookrightarrow \mathrm{Ext}^3_X(S_1, U_0)^3 = 0 & \longrightarrow & \mathrm{Ext}^3_X(U_2, U_0) & \longrightarrow & \mathrm{Ext}^3_X(S_2, U_0) \\ \hookrightarrow \mathrm{Ext}^4_X(S_1, U_0)^3 = \mathbb{C}^9 & \longrightarrow & \mathrm{Ext}^4_X(U_2, U_0) = 0 & \longrightarrow & \mathrm{Ext}^4_X(S_2, U_0) \end{array} \tag{3.18}$$

We first determine

$$\mathrm{Ext}^i_X(S_2, U_0) = \mathrm{Ext}^i_X(s_*\mathcal{O}(-3)[2], s_*P[1]) = \mathrm{Ext}^{i-1}_X(s_*\mathcal{O}(-3), s_*P),$$

where $P$ was defined in the short exact sequence (3.17).

(i) $i = 1$, then

$$\begin{aligned} \mathrm{Ext}^1_X(S_2, U_0) &= \mathrm{Hom}_X(s_*\mathcal{O}(-3), s_*P) \\ &= \mathrm{Hom}_{\mathbb{P}^2}(\mathcal{O}(-3), P). \end{aligned}$$

Then we take $\mathrm{Hom}(\mathcal{O}(-3), -)$ to (3.17), and note that $\mathrm{Ext}^1(\mathcal{O}(-3), P) = 0$ since $(\mathcal{O}(-3), \Omega(-1), P)$ forms a strong exceptional collection, then we get that

$$\mathrm{Hom}_{\mathbb{P}^2}(\mathcal{O}(-3), P) = \mathbb{C}^6.$$

Thus $\mathrm{Ext}^1_X(S_2, U_0) = \mathbb{C}^6$.

(ii) $i = 2$, then

$$\mathrm{Ext}^2_X(S_2, U_0) = \mathrm{Ext}^1_{\mathbb{P}^2}(\mathcal{O}(-3), P) \oplus \mathrm{Hom}_{\mathbb{P}^2}(\mathcal{O}(-3) \otimes \mathcal{T}, P)$$

then first component vanishes. We have $c_0(\mathcal{T}(-3)) = 2$, $c_1(\mathcal{T}(-3)) = 0$, then $\mu(\mathcal{T}(-3)) = 0 > \mu(P) = -13/5$, so there is no map from $\mathcal{T}(-3)$ to $P$ since they are both semistable sheaves. Thus

$$\mathrm{Ext}^2_X(S_2, U_0) = 0.$$

(iii) $i = 3$, then

$$\begin{aligned}\mathrm{Ext}^3_X(S_2, U_0) &= \mathrm{Ext}^2_{\mathbb{P}^2}(\mathcal{O}(-3), P) \oplus \mathrm{Ext}^1_{\mathbb{P}^2}(\mathcal{O}(-3) \otimes \mathcal{T}, P) \\ &\quad \oplus \mathrm{Hom}_{\mathbb{P}^2}(\mathcal{O}, P)\end{aligned}$$

the first component vanishes, the third component vanishes because $\mu(\mathcal{O}) > \mu(P)$ and they are semistable sheaves. For the second component, we take $\mathrm{Hom}_{\mathbb{P}^2}(\mathcal{T}(-3), -)$ on the short exact sequence (3.17) and obtain the long exact sequence

$$\begin{array}{ccccc}\mathrm{Hom}_{\mathbb{P}^2}(\mathcal{T}(-3), \mathcal{O}(-2)) & \longrightarrow & \mathrm{Ext}^1_{\mathbb{P}^2}(\mathcal{T}(-3), P) & \longrightarrow & \mathrm{Ext}^1_{\mathbb{P}^2}(\mathcal{T}(-3), \Omega(-1))^3 \\ & & & & \downarrow \\ & & & & \mathrm{Ext}^1_{\mathbb{P}^2}(\mathcal{T}(-3), \mathcal{O}(-2))\end{array} \tag{3.19}$$

We have $\mathrm{Hom}_{\mathbb{P}^2}(\mathcal{T}(-3), \mathcal{O}(-2)) = \mathrm{Ext}^1_{\mathbb{P}^2}(\mathcal{T}(-3), \mathcal{O}(-2)) = 0$ by the Bott formula. By taking $\mathrm{Hom}_{\mathbb{P}^2}(\mathcal{T}(-2), -)$ on the Euler sequence, we have

$$\mathrm{Ext}^1_{\mathbb{P}^2}(\mathcal{T}(-2), \Omega) = \mathrm{Ext}^1_{\mathbb{P}^2}(\mathcal{T}(-3), \Omega(-1)) = \mathbb{C}^3.$$

Return to (3.19), we have $\mathrm{Ext}^1_{\mathbb{P}^2}(\mathcal{T}(-3), P) = \mathbb{C}^9$. So finally we have

$$\mathrm{Ext}^3_X(S_2, U_0) = \mathbb{C}^9.$$

Taking the above infomation back to (3.18), we get that $\mathrm{Ext}^3_X(U_2, U_0) = 0$ by the exactness. By Serre duality and applying the autoequivalence $\Psi$, we can also get $\mathrm{Ext}^1_X(U_2, U_0)$:

$$\begin{aligned}\mathrm{Ext}^1_X(U_2, U_0) &= \mathrm{Ext}^3_X(U_0, U_2)^* \\ &= \mathrm{Ext}^3_X(\Psi(U_0), \Psi(U_2))^* \\ &= \mathrm{Ext}^3_X(U_2, U_0)^* \\ &= 0.\end{aligned}$$

$\mathrm{Ext}^2_X(U_2, U_0)$ can be read from the exact sequence:

$$0 \longrightarrow \mathrm{Ext}^1_X(S_2, S_0) = \mathbb{C}^6 \longrightarrow \mathrm{Ext}^2_X(S_1, S_0)^3 = \mathbb{C}^{81} \longrightarrow \mathrm{Ext}^2_X(U_2, U_0) \longrightarrow 0.$$

So we have $\mathrm{Ext}^2_X(U_2, U_0) = \mathbb{C}^{75}$. In summary, we have

$$\mathrm{Ext}^i_X(U_2, U_0) = \begin{cases} \mathbb{C}^{75} & i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

And $\mathrm{Ext}^i_X(U_0, U_2)$ is obtained by the Serre duality:

$$\begin{aligned} \mathrm{Ext}^i_X(U_0, U_2) &= \mathrm{Ext}^{4-i}_X(U_2, U_0)^* \\ &= \begin{cases} \mathbb{C}^{75} & i = 2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

- For $\mathrm{Ext}^i_X(U_0, U_0)$, we have $\mathrm{Hom}_X(U_0, U_0) = \mathrm{Ext}^4_X(U_0, U_0)^* = \mathbb{C}$, and

$$\begin{aligned} \mathrm{Ext}^1_X(U_0, U_0) &= \mathrm{Ext}^1_X(s_*P, s_*P) \\ &= \mathrm{Ext}^1_{\mathbb{P}^2}(P, P) \oplus \mathrm{Hom}_{\mathbb{P}^2}(P \otimes \mathcal{T}, P). \end{aligned}$$

Then the first component vanishes because $P$ is exceptional. In the calculations of $\mathrm{Ext}^i(U_1, U_0)$, we have $\mu(P \otimes \mathcal{T}) = -10/11 > \mu(P) = -13/5$. Then the second component also vanishes.

Also we obtain $\mathrm{Ext}^3_X(U_0, U_0) = \mathrm{Ext}^1_X(U_0, U_0)^* = 0$.

Then we calculate the Euler characteristic $\chi(U_0, U_0)$, since $[U_0] = 3[S_1] + [S_0]$ in the Grothendieck group:

$$\begin{aligned} \chi(U_0, U_0) &= \chi\left(3[S_1] + [S_0], 3[S_1] + [S_0]\right) \\ &= 9\chi(S_1, S_1) + 3\chi(S_1, S_0) + 3\chi(S_0, S_1) + \chi(S_0, S_0) \\ &= 75 \end{aligned}$$

We have $\chi(U_0, U_0) = 2 + \dim \mathrm{Ext}^2_X(U_0, U_0) = 75$, so $\mathrm{Ext}^2_X(U_0, U_0) = \mathbb{C}^{73}$. In summary, we have

$$\mathrm{Ext}^i_X(U_0, U_0) = \begin{cases} \mathbb{C} & i = 0,\ 4, \\ \mathbb{C}^{73} & i = 2, \\ 0 & \text{otherwise.} \end{cases}$$

By applying the functor $\Psi$, we have

$$\begin{aligned} \mathrm{Ext}^i_X(U_2, U_2) &= \mathrm{Ext}^i_X\left(\Psi(U_0), \Psi(U_0)\right) \\ &= \begin{cases} \mathbb{C} & i = 0,\ 4, \\ \mathbb{C}^{73} & i = 2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

- Finally for $\mathrm{Ext}^i_X(U_1, U_1)$ we have

$$\begin{aligned}\mathrm{Ext}^i_X(U_1, U_1) &= \mathrm{Ext}^i_X(S_1, S_1) \\ &= \begin{cases} \mathbb{C} & i = 0,\ 4, \\ \mathbb{C}^{10} & i = 2, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

# Chapter 4

# Invariant stability conditions on local $\mathbb{P}^1 \times \mathbb{P}^1$ (after [24, 26])

This chapter studies a subspace of stability conditions on $X = \operatorname{Tot} \omega_{\mathbb{P}^1 \times \mathbb{P}^1}$ which are invariant under an autoequivalence $\Phi$ of $D^b(X)$. We first construct the autoequivalence $\Phi$ explicitly in section 4.1. Then we calculate the double simple tilts of a given t-structure $\mathcal{A}$ in section 4.2 and relate the tilted heart to $\mathcal{A}$ by certain autoequivalences of $D^b(X)$. Then in section 4.3 we prove one of the main results in this part: we give a complete description of the stable objects (up to a shift of degree) for a set of $\Phi$-invariant stability conditions with the fixed heart. The description depends on the representation theory of Kronecker quiver, which we will recall in the beginning of section 4.3. In the final section, we describe a connected component of the space of $\Phi$-invariant stability conditions on $X$ and show that all such stability conditions are algebraic in the sense of [16, 2]. Throughout this chapter, we write $Z = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{C} = D^b_0(X)$ the full subcategory of objects which are supported on $Z$.

## § 4.1 Quiver symmetry and autoequivalence

### 4.1.1 Quiver

There is a full and strong exceptional collection on $Z$:

$$\mathbb{E} = \big(\mathcal{O}(0,0), \mathcal{O}(1,0), \mathcal{O}(1,1), \mathcal{O}(2,1)\big)^1,$$

which has the dual collection

$$\mathbb{F} = \big(\mathcal{O}(0,0), \mathcal{O}(-1,0)[1], \mathcal{O}(1,-1)[1], \mathcal{O}(0,-1)[2]\big).$$

As in the introduction, we denote by $\pi : X \to Z$ the bundle projection map, $p_i : Z = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, $i = 1,\ 2$ the projection maps to each component. We denote by $s : Z \hookrightarrow X$ the embedding map of the zero section.

**Lemma 4.1.1.** *The pull back $\mathcal{Q} = \bigoplus_i \pi^* E_i$ is a tilting bundle on $X$.*

*Proof.* We have

$$\begin{aligned}
\operatorname{Ext}^i_X\left(\pi^*\mathcal{O}(a,b), \pi^*\mathcal{O}(c,d)\right) &= \operatorname{Ext}^i_Z\left(\mathcal{O}(a,b), \pi_*\pi^*\mathcal{O}(c,d)\right) \\
&= \bigoplus_{n=0} \operatorname{Ext}^i_Z\left(\mathcal{O}(a,b), \mathcal{O}(c,d) \otimes (\omega^*_{\mathbb{P}^1\times\mathbb{P}^1})^n\right) \\
&= \bigoplus_{n=0} \mathrm{H}^i\left(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(c-a+2n, d-b+2n)\right) \\
&= \bigoplus_{n=0} \bigoplus_{s+t=i} \mathrm{H}^s(\mathbb{P}^1, \mathcal{O}(c-a+2n)) \otimes \mathrm{H}^t(\mathbb{P}^1, \mathcal{O}(d-b+2n)).
\end{aligned}$$

For $i > 0$, $\mathrm{H}^i\left(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(c-a+2n, d-b+2n)\right) = 0$ unless

(i) $c-a+2n \le -2,\ d-b+2n \ge 0$ or $d-b+2n \le -2,\ c-a+2n \ge 0$;

(ii) $c-a+2n \le -2,\ d-b+2n \le -2$.

[1] The reason why we choose the above exceptional collection instead of the usual one on $\mathbb{P}^1 \times \mathbb{P}^1$: $\mathbb{E}' = \big(\mathcal{O}(0,0), \mathcal{O}(1,0), \mathcal{O}(0,1), \mathcal{O}(1,1)\big)$ is given below, see the beginning of Chapter 4.2.

Since $-1 \leq d-b \leq 1$ in our case, so only the first part of case 1 is possible. One can easily verify the bundles in our exceptional collection do not belong to this case. Therefore $\mathrm{Ext}^i_X(\mathcal{Q}, \mathcal{Q}) = 0$ for $i \neq 0$.

The proof of the generating property for $\mathcal{Q}$ is the same as in Lemma 3.2.1. We finished the proof. □

The endomorphism algebra $B = \mathrm{End}_X(\mathcal{Q})$ is noetherian [23, Theorem 3.6], therefore we can write it as the path algebra of a quiver $Q$ subject to relations, and grade it by the length of paths. The vertex is indexed by $i$, and the number of arrows from $i$ to $j$ is the dimension of space of irreducible maps from $\pi^*E_i$ to $\pi^*E_j$, i.e., the cokernel of the map

$$\bigoplus_{k\neq i,\ j} \mathrm{Hom}_X(\pi^*E_i, \pi^*E_k) \otimes \mathrm{Hom}_X(\pi^*E_k, \pi^*E_j) \longrightarrow \mathrm{Hom}_X(\pi^*E_i, \pi^*E_j).$$

For a noetherian graded algebra $A$, we denote by $D^b_0(A)$ the full subcategory of $D^b(A)$ consisting of objects whose cohomology modules are nilpotent and $\mathrm{mod}_0$-$A$ the category of nilpotent modules of $A$. The proof of the following result is the same as Lemma 3.5.1.

**Corollary 4.1.2.** *There is a derived equivalence:*

$$\mathcal{R}_{\mathcal{Q}} := \mathbb{R}\mathrm{Hom}(\mathcal{Q}, -) : \mathcal{C} = D^b_0(X) \longrightarrow D^b_0(B = \mathrm{End}_X(\mathcal{Q})).$$

From now on until the end of the paper, we denote by $\mathcal{A}$ the heart of the t-structure in $\mathcal{C}$ induced by $\mathcal{Q}$, i.e. it is equivalent to $\mathrm{mod}_0$-$B$ under $\mathcal{R}_{\mathcal{Q}}$.

Let $I = \{0, 1, 2, 3\}$ be the set of indexes.

**Corollary 4.1.3.** *There are* 4 *simple objects up to isomorphism in* $\mathcal{A}$ *which are* $\{S_i\}_{i\in I} = \{s_*F_i\}_{i\in I}$*, here* $F_i$ *is in the dual collection* $\mathbb{F}$*.*

*Proof.* We write $P_i$ the projective $B$-module and $C_i$ the simple $B$-module associated with vertex $i$. Then $\mathrm{mod}_0$-$B$ is the extension-closed subcategory of mod-$B$ generated by $\{C_i\}_{i\in I}$ and $\{C_i\}_{i\in I}$ is the set of all simple $B$-modules in $\mathrm{mod}_0$-$B$. By definition of $\mathcal{R}_{\mathcal{Q}}$, $\pi^*E_i$ is sent to $P_i$. Then $s_*F_i$ is sent to $C_i$ which follows from the definition of the dual collection:

$$\mathrm{Hom}^\bullet_{D^b(X)}(\pi^*E_i, s_*F_j) = \mathrm{Hom}^\bullet_{D^b(Z)}(E_i, F_j) = \delta_{ij}\mathbb{C}.$$

□

We write down the simple objects explicitly here:

$$S_0 = s_*\mathcal{O}(0,0), \quad S_1 = s_*\mathcal{O}(-1,0)[1], \quad S_2 = s_*\mathcal{O}(1,-1)[1], \quad S_3 = s_*\mathcal{O}(0,-1)[2].$$

**Proposition 4.1.4.** *The quiver $Q = (Q_0, Q_1)$ of the endomorphism algebra* $\mathrm{End}_X(\mathcal{Q})$ *is*

$$\begin{array}{ccc} 0 & \overset{x_1}{\underset{y_1}{\rightrightarrows}} & 1 \\ {\scriptstyle x_3}\upuparrows{\scriptstyle y_3} & & {\scriptstyle y_2}\downdownarrows{\scriptstyle x_2} \\ 3 & \overset{y_4}{\underset{x_4}{\leftleftarrows}} & 2 \end{array} \tag{4.1}$$

*Proof.* The number of arrows from $i$ to $j$ can be calculated by the dimension of the vector space $\mathrm{Ext}^1_X(S_j, S_i)$ (we use the convention that paths compose on the left). By using the Koszul resolution [34, Chapter 11] along the embedding map $s$, for any sheaf $F$ on $Z$ we have

$$s^* s_* F \cong F \oplus (F \otimes \omega_Z^*[1])$$

in $D^b(Z)$. Therefore

$$\mathrm{Ext}^n_X(s_*F_i, s_*F_j) = \mathrm{Ext}^n_Z(F_i, F_j) \oplus \mathrm{Ext}^{3-n}_Z(F_j, F_i)^*.$$

For example

$$\begin{aligned} & \mathrm{Ext}^1_X\left(s_*\mathcal{O}(1,-1)[1], s_*\mathcal{O}(-1,0)[1]\right) \\ = \ & \mathrm{Ext}^1_Z\left(\mathcal{O}(1,-1), \mathcal{O}(-1,0)\right) \oplus \mathrm{Ext}^2_Z\left(\mathcal{O}(-1,0), \mathcal{O}(1,-1)\right)^* \\ = \ & \mathrm{H}^1\left(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(-2,1)\right) \\ = \ & \mathbb{C}^2. \end{aligned}$$

The other calculations are similar. Therefore we get the given quiver $Q$. □

We denote by $I$ the relations of paths in $\mathrm{End}_X(\mathcal{Q})$ and $\mathrm{rep}_{\mathrm{nil}}(Q, I) \cong \mathcal{A}$ the category of nilpotent representations of quiver with relations.

**Remark 4.1.5.** *Since $B = \mathrm{End}_X(\mathcal{Q})$ is graded 3-Calabi-Yau in the sense that the full subcategory consisting of objects with finite dimensional cohomology modules $D^b_{fin}(B)$ has Serre duality:*

$$\mathrm{Ext}^i_B(M, N) = \mathrm{Ext}^{3-i}_B(N, M)^*, \quad M,\ N \in D^b_{fin}(B).$$

*Then work of Bocklandt [6] shows that the relations of* $\mathrm{End}_X(\mathcal{Q})$ *can be encoded in compact form in a potential. Thus we can write*

$$B = B(Q, W) = \mathbb{C}Q/(\partial_a W : a \in Q_1)$$

*for some non-uniquely defined element* $W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$*. In fact we can write down the potential* $W$ *explicitly here: keep the notations as in (4.1). Then*

$$W = x_4x_3x_2x_1 + y_4y_3y_2y_1 - y_4x_3y_2x_1 - x_4y_3x_2y_1.$$

*The corresponding relations are (for* $j \in \mathbb{Z}_4$*):*

$$\partial_{x_j} W = x_{j+3}x_{j+2}x_{j+1} - y_{j+3}x_{j+2}y_{j+1} = 0,$$
$$\partial_{y_j} W = y_{j+3}y_{j+2}y_{j+1} - x_{j+3}y_{j+2}x_{j+1} = 0.$$

### 4.1.2 Autoequivalence and invariant stability conditions

The spherical object and spherical twist were introduced by Seidel and Thomas [53]. We briefly recall the definition and property here: for our use we simply consider $\mathcal{D} := D^b(\mathcal{V})$ where $\mathcal{V}$ is a local Calabi-Yau variety of dimension $n$.

**Definition 4.1.6.** *An object* $S \in \mathcal{D}$ *is called* $n$*-spherical if the following conditions are satisfied:*

*(i) For any* $F \in \mathcal{D}$, $\mathrm{Hom}^{\bullet}_{\mathcal{D}}(F, S)$ *and* $\mathrm{Hom}^{\bullet}_{\mathcal{D}}(S, F)$ *have finite (total) dimension over* $\mathbb{C}$.

*(ii) We have*

$$\mathrm{Ext}^k_{\mathcal{D}}(S, S) = \begin{cases} \mathbb{C} & k = 0,\ n, \\ 0 & \textit{otherwise.} \end{cases}$$

*Let* $S$ *be a spherical object in* $\mathcal{D}$*, then the spherical twist* $\mathrm{Tw}_S(E)$ *of* $E \in \mathcal{D}$ *is defined to be the cone of the canonical evaluation morphism:*

$$\mathrm{Hom}^{\bullet}(S, E) \otimes S \longrightarrow E \longrightarrow \mathrm{Tw}_S(E) \xrightarrow{[1]}$$

The following important lemma is due to Seidel and Thomas.

**Lemma 4.1.7** ([53]). *Let $S$ be a spherical object in $\mathcal{D}$, then $\mathrm{Tw}_S$ is an exact autoequivalence of $\mathcal{D}$.*

**Lemma 4.1.8.** *If $E$ is an exceptional object on $\mathbb{P}^1 \times \mathbb{P}^1$, then $s_*E$ is a 3-spherical object in $D^b(X)$.*

*Proof.* The first condition satisfies since $s_*E$ has compact support, therefore $\mathrm{Hom}^\bullet_{\mathcal{D}}(F, s_*E)$ and $\mathrm{Hom}^\bullet_{\mathcal{D}}(s_*E, F)$ have finite dimensions. By using the Koszul resolution as in Proposition 4.1.4, we have

$$\begin{aligned} \mathrm{Ext}^n_X(s_*E, s_*E) &\cong \mathrm{Ext}^n_Z(E,E) \oplus \mathrm{Ext}^{3-n}_Z(E,E) \\ &= \begin{cases} \mathbb{C} & n = 0,\ 3, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

□

Define $\tau : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$, $\tau(x,y) := (y,x)$, it has a natural extension to an automorphism of $\omega = \mathrm{Tot}\,\mathcal{O}(-2,-2)$ which we also denote by $\tau$. We consider the following functor

$$\Psi := \tau^* \circ \mathrm{Tw}_{S_0} \circ (- \otimes \pi^*\mathcal{O}(0,1))$$

which is an autoequivalence of $D^b(X)$ since it is a composition of autoequivalences.

**Lemma 4.1.9.** *Let $\mathcal{A}$ be the heart of the bounded t-structure induced by $\mathcal{Q}$, and $S_i$ the simple objects in $\mathcal{A}$ defined in Corollary 4.1.3. Then*

$$\Psi(S_i) = S_{i+1}, \quad i \in \mathbb{Z}_4.$$

*Therefore $\Psi$ reduces to be an autoequivalence of $\mathcal{A}$.*

*Proof.* By the projection formula, we have

$$\begin{aligned} s_*\mathcal{O}(a,b) \otimes \pi^*\mathcal{O}(j,k) &= s_*\left(\mathcal{O}(a,b) \otimes s^*\pi^*\mathcal{O}(j,k)\right) \\ &= s_*\left(\mathcal{O}(a,b) \otimes \mathcal{O}(j,k)\right) \\ &= s_*\mathcal{O}(a+j, b+k). \end{aligned}$$

(i) Recall $S_3 = s_*\mathcal{O}(0,-1)[2]$. Thus $S_3 \otimes \pi^*\mathcal{O}(0,1) = s_*\mathcal{O}(0,0)[2] = S_0[2]$. Now

$$\Psi(S_3) = \tau^*\,\mathrm{Tw}_{S_0}(S_0[2]) = \tau^* S_0 = \tau^* s_*\mathcal{O}(0,0) = S_0,$$

where the second equality follows from the standard result, that if $S$ is an $n$-spherical object, then

$$\mathrm{Tw}_S(S) \cong S[1-n],$$

and $n = 3$ in our case.

(ii) For $S_0 = s_*\mathcal{O}(0,0)$, $S_0 \otimes \pi^*\mathcal{O}(0,1) = s_*\mathcal{O}(0,1)$. By using the similar calculations in Proposition 4.1.4, we have

$$\mathrm{Hom}^\bullet(s_*\mathcal{O}(0,0), s_*\mathcal{O}(0,1)) = \mathbb{C}^2,$$

then $\mathrm{Tw}_{S_0}(s_*\mathcal{O}(0,1))$ fits into the triangle:

$$s_*\mathcal{O}(0,0)^{\oplus 2} \longrightarrow s_*\mathcal{O}(0,1) \longrightarrow \mathrm{Tw}_{S_0}(s_*\mathcal{O}(0,1)) \xrightarrow{[1]}$$

By applying the exact functor $s_*$ to the short exact sequence on $Z$:

$$0 \longrightarrow \mathcal{O}(0,-1) \longrightarrow \mathcal{O}^{\oplus 2} \longrightarrow \mathcal{O}(0,1) \longrightarrow 0$$

and comparing with the above triangle, we have

$$\mathrm{Tw}_{S_0}(s_*\mathcal{O}(0,1)) \cong s_*\mathcal{O}(0,-1)[1].$$

Thus $\Psi(S_0) = \tau^* s_*\mathcal{O}(0,-1)[1] = s_*\mathcal{O}(-1,0)[1] = S_1$.

(iii) For $S_1 = s_*\mathcal{O}(-1,0)[1]$, $S_1 \otimes \pi^*\mathcal{O}(0,1) = s_*\mathcal{O}(-1,1)[1]$. Since

$$\mathrm{Hom}^\bullet(s_*\mathcal{O}(0,0), s_*\mathcal{O}(-1,1)[1]) = 0,$$

then $\mathrm{Tw}_{S_0}(s_*\mathcal{O}(-1,1)[1]) \cong s_*\mathcal{O}(-1,1)[1]$. So we have

$$\Psi(S_1) = \tau^* s_*\mathcal{O}(-1,1)[1] = s_*\mathcal{O}(1,-1)[1] = S_2.$$

(iv) For $S_2 = s_*\mathcal{O}(1,-1)[1]$, $S_2 \otimes \pi^*\mathcal{O}(0,1) = s_*\mathcal{O}(1,0)[1]$. Since

$$\mathrm{Hom}^\bullet(s_*\mathcal{O}(0,0), s_*\mathcal{O}(1,0)[1]) = \mathbb{C}^2[1],$$

then $\mathrm{Tw}_{S_0}(s_*\mathcal{O}(1,0))$ fits into the triangle:

$$s_*\mathcal{O}(0,0)^{\oplus 2}[1] \longrightarrow s_*\mathcal{O}(1,0)[1] \longrightarrow \mathrm{Tw}_{S_0}(s_*\mathcal{O}(1,0)) \xrightarrow{[1]}$$

by the same argument as above, we have

$$\mathrm{Tw}_{S_0}(s_*\mathcal{O}(1,0)) \cong s_*\mathcal{O}(-1,0)[2].$$

So we have

$$\Psi(S_2) = \tau^* s_*\mathcal{O}(-1,0)[2] = s_*\mathcal{O}(0,-1)[2] = S_3.$$

□

**Definition 4.1.10.** *We define the autoequivalence of* $\mathcal{C}$

$$\Phi = \Psi^2. \tag{4.2}$$

*We denote by* $\varphi$ *and* $\psi$ *the automorphisms of* $K_0(\mathcal{C})$ *induced by* $\Phi$ *and* $\Psi$ *respectively.*

Let $\mathrm{Stab}(X)$ denote the space of stability conditions satisfying the support condition on $\mathcal{C}$.

**Definition 4.1.11** ($\Phi$-invariant stability conditions)**.** *The space of stability conditions which are invariant under* $\Phi$ *is denoted by* $\mathrm{Stab}(X)^{\Phi}$*. Let* $\mathcal{U}(\mathcal{A})^{\Phi}$ *be the set of* $\Phi$*-invariant stability conditions with the fixed heart* $\mathcal{A}$*. We denote the connected component of* $\mathrm{Stab}(X)^{\Phi}$ *which contains* $\mathcal{U}(\mathcal{A})^{\Phi}$ *by*

$$\left(\mathrm{Stab}(X)^{\Phi}\right)_0.$$

From now on, we denote by $\gamma_i = [S_i]$ the class of $S_i$ in $K_0(\mathcal{A}) = K_0(\mathcal{C})$, $i = 0, \cdots, 3$.

The subgroup of $K_0(\mathcal{A})$ whose elements are antisymmetric under $\varphi$ is generated by $\gamma_0 - \gamma_2$ and $\gamma_1 - \gamma_3$, and is denoted by $K_0(\mathcal{A})^{-\varphi}$. The quotient group is denoted by

$$\overline{K_0(\mathcal{A})} := K_0(\mathcal{A})/K_0(\mathcal{A})^{-\varphi}.$$

The quotient map is denoted by $\nu : K_0(\mathcal{A}) \to \overline{K_0(\mathcal{A})}$. Note that $\overline{K_0(\mathcal{A})}$ is free abelian of rank 2 with basis $\bar{\gamma}_0, \bar{\gamma}_1$ (we will abuse notation and still denote $\gamma_i$ in the quotient group). And there is a natural isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(\overline{K_0(\mathcal{A})}, \mathbb{C}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(K_0(\mathcal{A}), \mathbb{C})^{\varphi}.$$

Therefore we can equally define the $\Phi$-invariant stability conditions to be those whose central charges $Z : K_0(\mathcal{A}) \to \mathbb{C}$ factor through $\overline{K_0(\mathcal{A})}$ and the slicings are invariant under $\Phi$. For technical reason we will work with this definition.

By Corollary 2.2.13, the forgetful map $\left(\mathrm{Stab}(X)^{\Phi}\right)_0 \to \mathrm{Hom}_{\mathbb{Z}}(\overline{K_0(\mathcal{A})}, \mathbb{C}) \cong \mathbb{C}^2$ is a local homeomorphism.

At the end of this section, we recall the following definition from [21].

**Definition 4.1.12.** *Let* $\mathrm{Aut}(\mathcal{C})$ *be the group of exact* $\mathbb{C}$*-linear autoequivalences of the category* $\mathcal{C}$*, then* $\mathrm{Aut}_*(\mathcal{C})$ *is defined to be the subquotient consisting of autoequivalences which preserve the connected component* $\big(\mathrm{Stab}(X)^{\Phi}\big)_0$*, modulo those which acts trivially on it.*

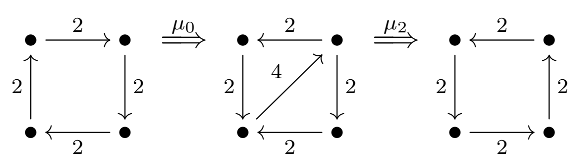
## § 4.2 Simple tilts and autoequivalence

In this section we use Proposition 2.1.7 to figure out the double simple tilts of $\mathcal{A}$, that is we will calculate $\mathcal{A}' = L_{S_{i+2}} L_{S_i}\mathcal{A}$ and $R_{S_{i+2}} R_{S_i}\mathcal{A}$ for $i \in \mathbb{Z}_4$.

We write $\mathcal{A} \cong \mathrm{rep}_{\mathrm{nil}}(Q, W)$ where $W$ is the potential (see Remark 4.1.5). Then by the result of Keller and Yang [40, Theorem 9.2]

$$\mathcal{A}' = \mathrm{rep}_{\mathrm{nil}}\big(\mu_{i+2}\mu_i(Q, W)\big)$$

where $\mu_i$ is the mutation operation given by Derksen, Weyman and Zelvinsky [27]. For simplicity we calculate how the quiver $Q$ changes under the double mutations (take $i = 0$ for example):




The resulted quiver is isomorphic to the original one after relabelling the vertices, which implies that there should be an autoequivalence $\mathcal{T}$ relating $\mathcal{A}$ and $\mathcal{A}'$. We give explicit calculations below.

We simply write $L_i := L_{S_i}$ and $R_i := R_{S_i}$. Recall that there are 4 simple objects in $\mathcal{A}$ up to isomorphism:

$$S_0 = s_*\mathcal{O}(0,0), \quad S_1 = s_*\mathcal{O}(-1,0)[1], \quad S_2 = s_*\mathcal{O}(1,-1)[1], \quad S_3 = s_*\mathcal{O}(0,-1)[2].$$

Proposition 4.1.4 shows that there is no extension between $S_i$ and $S_{i+2}$, for $i \in \mathbb{Z}_4$, therefore $L_i L_{i+2}\mathcal{A} = L_{i+2} L_i \mathcal{A}$.

$L_0\mathcal{A}$: Since the only non-trivial extension to $S_0$ is $\mathrm{Ext}^1(S_1, S_0)$, thus the new simple objects are

$$S_0' = S_0[1], \quad S_1', \quad S_2' = S_2, \quad S_3' = S_3,$$

where $S_1'$ fits into the triangle

$$S_0^{\oplus 2} \to S_1' \to S_1 \to S_0^{\oplus 2}[1].$$

Thus $S_1'$ fits into the short exact sequence

$$0 \to s_*\mathcal{O}(-1,0) \to s_*\mathcal{O}^{\oplus 2} \to S_1' \to 0. \tag{4.3}$$

We already see the above short exact sequence in part 4 of the proof of Lemma 4.1.9, therefore $S_1' = s_*\mathcal{O}(1,0)$.

$L_2L_0\mathcal{A}$: The new simple objects are

$$\widetilde{S_0} = S_0', \quad \widetilde{S_1} = S_1', \quad \widetilde{S_2} = S_2'[1], \quad \widetilde{S_3}$$

where $\widetilde{S_3}$ fits into the triangle

$$S_2^{\oplus 2} \to \widetilde{S_3} \to S_3' \to S_2^{\oplus 2}[1].$$

Thus $\widetilde{S_3}[-1]$ fits into the short exact sequence

$$0 \to s_*\mathcal{O}(0,-1) \to s_*\mathcal{O}(1,-1)^{\oplus 2} \to \widetilde{S_3}[-1] \to 0. \tag{4.4}$$

We obtain $\widetilde{S_3} = s_*\mathcal{O}(2,-1)[1]$.

Therefore in $L_2L_0\mathcal{A}$ we have the following simple objects up to isomorphism:

$$\widetilde{S_0} = s_*\mathcal{O}(0,0)[1], \quad \widetilde{S_1} = s_*\mathcal{O}(1,0), \quad \widetilde{S_2} = s_*\mathcal{O}(1,-1)[2], \quad \widetilde{S_3} = s_*\mathcal{O}(2,-1)[1].$$

**Theorem 4.2.1.** *Let $\mathcal{T} = - \otimes \pi^*\mathcal{O}(1,0)$ and $\mathcal{T}_\Psi = \Psi \circ \mathcal{T} \circ \Psi^{-1}$, then we have*

$$L_2L_0\mathcal{A} = \mathcal{T}\mathcal{A}; \tag{4.5}$$
$$R_3R_1\mathcal{A} = \mathcal{T}^{-1}\mathcal{A}; \tag{4.6}$$
$$L_3L_1\mathcal{A} = \mathcal{T}_\Psi\mathcal{A}; \tag{4.7}$$
$$R_2R_0\mathcal{A} = \mathcal{T}_\Psi^{-1}\mathcal{A}. \tag{4.8}$$

*Proof.* $\mathcal{T}\mathcal{A} \subset L_2L_0\mathcal{A}$ follows directly from the comparison of the simple objects after reordering them. By Lemma 2.1.5 we have $\mathcal{T}\mathcal{A} = L_2L_0\mathcal{A}$. We also have

$$\begin{aligned}\Psi \circ \mathcal{T} \circ \Psi^{-1}(\mathcal{A}) &= \Psi L_2L_0 \circ \Psi^{-1}\mathcal{A} \\ &= \Psi \circ \Psi^{-1}L_3L_1\mathcal{A} \quad \text{by Lemma 2.1.6 and Lemma 4.1.9} \\ &= L_3L_1\mathcal{A}.\end{aligned}$$

This proves the third identity. For the right mutation $R_3R_1\mathcal{A}$, by using Remark 2.1.4 we have

$$R_{S_0[1]}R_{S_2[1]}L_{S_2}L_{S_0}\mathcal{A} = \mathcal{A}.$$

Note that $R_{S_0[1]}R_{S_2[1]}L_{S_2}L_{S_0}\mathcal{A} = R_{S_0[1]}R_{S_2[1]}\mathcal{T}\mathcal{A} = \mathcal{T}R_{S_1}R_{S_3}\mathcal{A}$ by Lemma 2.1.6, and $\mathcal{T}(S_1) = S_0[1]$, $\mathcal{T}(S_3) = S_2[1]$. Therefore combining with the above identities we have

$$R_{S_1}R_{S_3}\mathcal{A} = \mathcal{T}^{-1}\mathcal{A}.$$

Finally for $R_2R_0\mathcal{A}$, the calculation is quite similar to that of $L_3L_1\mathcal{A}$ and we leave it to the reader. □

$\mathcal{T}$ and $\mathcal{T}_\Psi$ induce automorphisms $t$ and $t_\psi$ of the Grothendieck group $K_0(\mathcal{A})$. The following results will be useful later:

**Lemma 4.2.2.** *With respect to the basis $\{\gamma_i\}$ of $K_0(\mathcal{A})$, the automorphisms $t$ and $t_\psi$ have the matrix forms:*

$$t = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad t_\psi = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 2 \end{bmatrix} \tag{4.9}$$

*Proof.* Follows directly from the calculations above. □

It is easy to check that $t$ and $t_\psi$ preserve the subgroup $K_0(\mathcal{A})^{-\varphi}$. Therefore $t$ and $t_\psi$ can be regarded as the actions on $\overline{K_0(\mathcal{A})}$. In fact, when reducing to $\overline{K_0(\mathcal{A})}$, $t$ and $t_\psi$ have the matrix forms with respect to the basis $\{\gamma_0, \gamma_1\}$:

$$t|_{\overline{K_0(\mathcal{A})}} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \qquad t_\psi|_{\overline{K_0(\mathcal{A})}} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \tag{4.10}$$

we have

$$\left(t|_{\overline{K_0(\mathcal{A})}}\right)^{-1} = t_\psi|_{\overline{K_0(\mathcal{A})}}. \tag{4.11}$$

We have the following relation in $K_0(\mathcal{A})$.

**Lemma 4.2.3.** *Let $x = ([a:b],[c:d])$ be a closed point in $\mathbb{P}^1 \times \mathbb{P}^1$, $\delta$ be the class of $\mathcal{O}_x$ in $K_0(\mathcal{A})$. Then we have*

$$\delta = \sum_{i=0}^{3} \gamma_i.$$

*Proof.* We write $F_1 = s_*\mathcal{O}_{[a:b]\times\mathbb{P}^1}$, $F_2 = s_*\mathcal{O}_{[a:b]\times\mathbb{P}^1}(-1)[1]$. First we have the short exact sequences

$$0 \to s_*\mathcal{O}(-1,0) \to s_*\mathcal{O} \to F_1 \to 0,$$
$$0 \to s_*\mathcal{O}(0,-1) \to s_*\mathcal{O}(1,-1) \to F_2[-1] \to 0.$$

This gives $[F_1] = \gamma_0 + \gamma_1$, $[F_2] = \gamma_2 + \gamma_3$. Then we consider the short exact sequence

$$0 \to s_*\mathcal{O}_{[a:b]\times\mathbb{P}^1}(-1) \to s_*\mathcal{O}_{[a:b]\times\mathbb{P}^1} \to s_*\mathcal{O}_x \to 0 \tag{4.12}$$

which gives $\delta = [F_1] + [F_2] = \sum_i \gamma_i$ as required. □

The above lemma shows that $[\mathcal{O}_x]$ does not depend on $x \in Z$.

## § 4.3 Semistable Objects

### 4.3.1 (Semi)stable objects

In this section we describe the set of stable objects for stability conditions $\sigma \in \mathcal{U}(\mathcal{A})^{\Phi}$ . The description relies on the known properties of stability conditions for the Kronecker quiver.

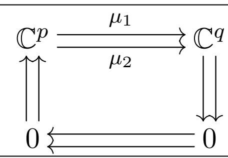
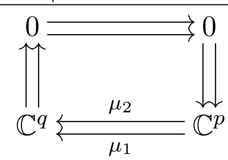
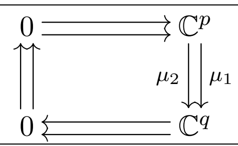
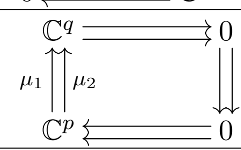
Denote by $K_2$ the Kronecker quiver

$$0 \rightrightarrows 1$$

and $\mathrm{rep}(K_2)$ the category of representations. We denote by $C_0$ and $C_1$ the simple objects at vertices 0 and 1. Recall the underlying quiver $Q$ of $\mathcal{A}$ (Proposition 4.1.4) is

$$\begin{array}{ccc} 0 & \rightrightarrows & 1 \\ \upuparrows & & \downdownarrows \\ 3 & \leftleftarrows & 2 \end{array}$$

**Definition 4.3.1.** *We define full subcategories of* $\mathcal{A} \cong \mathrm{rep}_{\mathrm{nil}}(Q, I)$ *which can naturally be identified with* $\mathrm{rep}\, K_2$*. The objects of the full subcategories are*

| ***Subcategories*** | ***Objects*** | ***Dimension vectors*** | ***Class in*** $K_0(\mathcal{A})$ |
|---|---|---|---|
| $\mathcal{K}_1^I$ |   | $(p, q, 0, 0)$ | $p\gamma_0 + q\gamma_1$ |
| $\mathcal{K}_2^I$ |   | $(0, 0, p, q)$ | $p\gamma_2 + q\gamma_3$ |
| $\mathcal{K}_1^{II}$ |   | $(0, p, q, 0)$ | $p\gamma_1 + q\gamma_2$ |
| $\mathcal{K}_2^{II}$ |   | $(q, 0, 0, p)$ | $p\gamma_3 + q\gamma_0$ |

*The objects in* $\mathcal{K}_i^I$, $i = 1,\ 2$ *are called Kronecker type I, and the objects in* $\mathcal{K}_i^{II}$, $i = 1,\ 2$ *are called Kronecker type II.*

The following lemma is obvious

**Lemma 4.3.2.** *The full subcategories* $\mathcal{K}_i^I$ *and* $\mathcal{K}_i^{II}$, $i = 1,\ 2$ *are equivalent to* $\mathrm{rep}(K_2)$*. They are Serre subcategories of* $\mathcal{A} \cong \mathrm{rep}_{\mathrm{nil}}(Q, I)$*, i.e., they are closed under taking quotients and subobjects.*

We denote by $\Xi_i^I$ and $\Xi_i^{II}$ the corresponding embedding functors from the full subcategories to $\mathcal{A}$.

Recall that for a finite acyclic quiver $Q$, $K_0(\mathrm{rep}\, Q) \cong \mathbb{Z}^{\oplus |Q_0|}$ is generated by the simple modules $S_i$ at each vertex $i$. We denote by $n_{ij}$ the number of arrows from vertex $i$ to $j$. Then the Euler form on $K_0(\mathrm{rep}\, Q)$ is defined by $\chi([S_i], [S_j]) := \delta_{ij} - n_{ji}$. For $\boldsymbol{\alpha} = (\alpha_i)_{i \in Q_0} \in K_0(Q)$, the quadratic form $q(-)$ is defined as $q(\boldsymbol{\alpha}) := \chi(\boldsymbol{\alpha}, \boldsymbol{\alpha})$. The associated matrix of $q$ is a symmetrization of the associated matrix of $\chi$.

When $Q$ is Dynkin or affine Dynkin (for example, the Kronecker quiver), it is well-known that $q(-)$ is positive semi-definite. $\boldsymbol{\alpha}$ is called a real root if $q(\boldsymbol{\alpha}) = 1$ and an imaginary root if $q(\boldsymbol{\alpha}) = 0$. We need the following well-known result (for example, see [4, Theorem 4.3.2]).

**Theorem 4.3.3** (Indecomposable representations of Kronecker quiver)**.** *We identify $K_0(\operatorname{rep} K_2) \cong \mathbb{Z}^2$ using the basis $([C_0], [C_1])$. Then*

*(i) for each real root $(n, n+1)$ or $(n+1, n)$ $(n \geq 0)$, there is a unique indecomposable representation with this class in $K_0(\operatorname{rep} K_2)$, up to isomorphism, which we will denote by $E_{n,n+1}$ or $E_{n+1,n}$;*

*(ii) for each imaginary root $(n, n)$ $(n \geq 1)$, there is a family of indecomposable representations indexed by $\mathbb{P}^1$ with this class in $K_0(\operatorname{rep} K_2)$, which we denote by $E_n^\lambda$ where $\lambda = [a : b] \in \mathbb{P}^1$.*

*The above are all the indecomposable representations in* $\operatorname{rep} K_2$ *up to isomorphism.*

We have the following characterization of stability conditions for Kronecker quivers by Okada [49].

**Lemma 4.3.4.** *Take a stability function $Z : K_0(\operatorname{rep} K_2) \to \mathbb{C}$ and denote by $\phi(E)$ the phase of a nonzero object $E \in \operatorname{rep}(K_2)$*

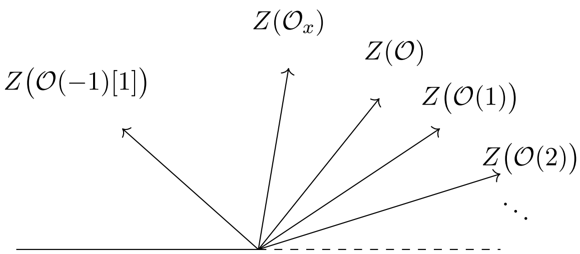
*(i) if $\phi(C_0) < \phi(C_1)$, then every indecomposable representation of $K_2$ is semistable, moreover, all indecomposable representations except for $E_m^\lambda$ when $m > 1$ are stable.*

*(ii) If $\phi(C_1) < \phi(C_0)$, then the only stable objects are $C_0$ and $C_1$. The semistable objects are $C_0^{\oplus k}$, $C_1^{\oplus k}$ for $k > 1$.*

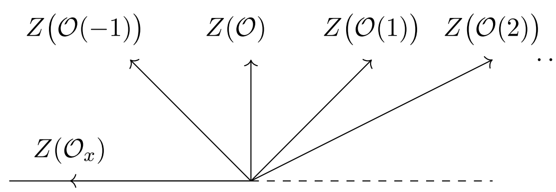
*(iii) If $\phi(C_0) = \phi(C_1)$, then all objects are semistable, and only $C_0$, $C_1$ are stable.*

*Proof.* Since $\operatorname{rep} K_2$ is of finite-length, $Z$ satisfies the Harder-Narasimhan property automatically, therefore $Z$ can be extended to a stability condition for $D^b(K_2) \cong D^b(\mathbb{P}^1)$, and is denoted by $(Z, \mathcal{P})$.

(i) Let $T = \mathcal{O} \oplus \mathcal{O}(1)$ be the tilting object in $D^b(\mathbb{P}^1)$, the functor $\mathbb{R}\mathrm{Hom}(T, -)$ sends $\mathcal{O}$ and $\mathcal{O}(-1)[1]$ to $C_0$ and $C_1$ respectively. If $\phi(C_0) < \phi(C_1)$, after rotating by $\lambda = i\big(\pi - \phi(\mathcal{O}_x)\big)$ where $x$ is a closed point of $\mathbb{P}^1$, the resulting stability condition $\lambda \cdot (Z, \mathcal{P}) = (\overline{Z}, \overline{\mathcal{P}})$ has heart $\overline{\mathcal{P}}(0,1] = \mathrm{Coh}\ \mathbb{P}^1$ (see the following figures). Therefore all line bundles and torsion sheaves are semistable, and in




Central charges of rep $K_2$




Central charges after rotating

fact all line bundles and skyscraper sheaves are stable. They correspond to the indecomposable representations of $K_2$ by the functor $\mathbb{R}\mathrm{Hom}(T, -)$.

(ii) The second statement follows from the fact that $C_0$ is a simple subobject of every indecomposable representation except $C_1$ and $C_1$ is a simple factor object of every indecomposable representation except $C_0$.

(iii) If $\phi(C_0) = \phi(C_1)$, then all nonzero objects in $\mathrm{Rep}(K_2)$ have the same phase and therefore are semistable.

□

**Remark 4.3.5.** *Since $\mathcal{K}_i^I$ and $\mathcal{K}_i^{II}$ are equivalent to* $\mathrm{rep}(K_2)$ *by Lemma 4.3.2, therefore by Theorem 4.3.3 we can describe the indecomposable objects of Kronecker type I and II.*

Since we will only be interested in the stable objects in $\mathrm{rep}(K_2)$ by Lemma 4.3.4, by definition the stable representations are bricks, that is, $\mathrm{End}(M) = \mathbb{C}$ if $M \in \mathrm{rep}(K_2)$ is stable. Then the following definition will be useful:

**Definition 4.3.6** (Special Kronecker type)**.** *We call the indecomposable object of Kronecker type I and II* **special** *if it is a brick, or equivalently it is not isomorphic to the image of $E_m^\lambda$ under $\Xi_i^I$ or $\Xi_i^{II}$ for $m > 1$.*

The following proposition gives us the geometric description of objects of special Kronecker types.

**Proposition 4.3.7.** *Let $l \geq 0$. We have the following correspondences between objects in $\mathcal{A}$ and $\mathrm{rep}_{\mathrm{nil}}(Q, I)$ under the equivalence $\mathcal{R}_Q : \mathcal{A} \to \mathrm{rep}_{\mathrm{nil}}(Q, I)$ (we denote by $x$ a closed point of $\mathbb{P}^1$):*

| ***Objects of special Kronecker type I*** | ***Classes in*** $K_0(\mathcal{A})$ | ***Objects of special Kronecker type II*** | ***Classes in*** $K_0(\mathcal{A})$ |
|---|---|---|---|
| $s_*\mathcal{O}(l,0)$ | $(l+1)\gamma_0 + l\gamma_1$ | $\Psi(s_*\mathcal{O}(l,0))$ | $(l+1)\gamma_1 + l\gamma_2$ |
| $s_*\mathcal{O}(l+1,-1)[1]$ | $(l+1)\gamma_2 + l\gamma_3$ | $\Psi(s_*\mathcal{O}(l+1,-1)[1])$ | $(l+1)\gamma_3 + l\gamma_0$ |
| $s_*\mathcal{O}(-l-1,0)[1]$ | $l\gamma_0 + (l+1)\gamma_1$ | $\Psi(s_*\mathcal{O}(-l-1,0)[1])$ | $l\gamma_1 + (l+1)\gamma_2$ |
| $s_*\mathcal{O}(-l,-1)[2]$ | $l\gamma_2 + (l+1)\gamma_3$ | $\Psi(s_*\mathcal{O}(-l,-1)[2])$ | $l\gamma_3 + (l+1)\gamma_0$ |
| $F_1 = s_*\mathcal{O}_{\{x\}\times\mathbb{P}^1}$ | $\gamma_0 + \gamma_1$ | $\Psi(s_*\mathcal{O}_{\{lx\}\times\mathbb{P}^1})$ | $\gamma_1 + \gamma_2$ |
| $F_2 = s_*\mathcal{O}_{\{x\}\times\mathbb{P}^1}(-1)[1]$ | $\gamma_2 + \gamma_3$ | $\Psi(s_*\mathcal{O}_{\{x\}\times\mathbb{P}^1}(-1)[1])$ | $\gamma_3 + \gamma_0$ |

*Proof.* For example, since $s_*\mathcal{O}(l,0)$ is indecomposable, therefore $\mathcal{R}_Q\,(s_*\mathcal{O}(l,0))$ is an indecomposable representation:

$$\begin{array}{ccc}
\mathbb{R}\mathrm{Hom}^\bullet\,(\pi^*\mathcal{O}, s_*\mathcal{O}(l,0)) & \rightrightarrows & \mathbb{R}\mathrm{Hom}^\bullet\,(\pi^*\mathcal{O}(1,0), s_*\mathcal{O}(l,0)) \\
\upuparrows & & \downdownarrows \\
\mathbb{R}\mathrm{Hom}^\bullet\,(\pi^*\mathcal{O}(2,1), s_*\mathcal{O}(l,0)) & \leftleftarrows & \mathbb{R}\mathrm{Hom}^\bullet\,(\pi^*\mathcal{O}(1,1), s_*\mathcal{O}(l,0))
\end{array}$$

By using the adjuncation isomorphism and $\pi \circ s = id$ we have

$$\begin{aligned}
\mathbb{R}\mathrm{Hom}^\bullet(\pi^*\mathcal{O}(a,b), s_*\mathcal{O}(c,d)) &= \mathbb{R}\mathrm{Hom}^\bullet_{D^b(\mathbb{P}^1\times\mathbb{P}^1)}(\mathcal{O}(a,b), \pi_*s_*\mathcal{O}(c,d)) && (4.13)\\
&= \mathbb{R}\mathrm{Hom}^\bullet_{D^b(\mathbb{P}^1\times\mathbb{P}^1)}(\mathcal{O}(a,b), \mathcal{O}(c,d)) && (4.14)\\
&= \mathrm{H}^\bullet(\mathbb{P}^1, \mathcal{O}(c-a)) \otimes \mathrm{H}^\bullet(\mathbb{P}^1, \mathcal{O}(d-b)) && (4.15)
\end{aligned}$$

The above calculation gives us the class of $\mathcal{R}_Q(s_*\mathcal{O}(l,0))$ in $K_0(\mathcal{A})$: $(l+1)\gamma_0 + l\gamma_1$. Therefore by definition $\mathcal{R}_Q(s_*\mathcal{O}(l,0))$ is of special Kronecker type I. For $s_*\mathcal{O}_{\{x\}\times\mathbb{P}^1}$, we have:

$$\begin{aligned}
\mathbb{R}\mathrm{Hom}^\bullet_D(\pi^*\mathcal{O}(a,b), s_*\mathcal{O}_{\{x\}\times\mathbb{P}^1}) &= \mathbb{R}\mathrm{Hom}^\bullet_{\mathbb{P}\times\mathbb{P}^1}(\mathcal{O}(a,b), \mathcal{O}_{\{x\}\times\mathbb{P}^1})\\
&= \mathrm{H}^\bullet(\mathbb{P}^1, \mathcal{O}_x) \otimes \mathrm{H}^\bullet(\mathbb{P}^1, \mathcal{O}(-b))\\
&= \mathrm{H}^\bullet(\mathbb{P}^1, \mathcal{O}(-b))
\end{aligned}$$

Therefore, the class in $K_0(\mathcal{A})$ of $\mathcal{R}_\mathcal{Q}(s_*\mathcal{O}_{\{x\}\times\mathbb{P}^1})$ is $\gamma_0+\gamma_1$, and $\mathcal{R}_\mathcal{Q}(s_*\mathcal{O}_{\{x\}\times\mathbb{P}^1})$ is of special Kronecker type I. For other objects of special Kronecker type I, the calculations are the same, and we leave them to the reader.

For $F \in \mathcal{A}$, if $\mathcal{R}_\mathcal{Q}(F)$ has dimension vector $(p,q,m,n)$, then by Lemma 4.1.9, the dimension vector of $\mathcal{R}_\mathcal{Q}(\Psi(F))$ will be $(n,p,q,m)$. For example $\mathcal{R}_\mathcal{Q}\left(\Psi(s_*\mathcal{O}(l,0))\right)$ has the class in $K_0(\mathcal{A})$: $(l+1)\gamma_1 + l\gamma_2$. Since it is still indecomposable, it is of special Kronecker type II. For the other objects of special Kronecker type II the verifications are exactly the same. $\square$

From now on, we often identify objects in $\text{rep}_{\text{nil}}(Q,I)$ with the corresponding objects in $\mathcal{A}$ without further comment.

**Definition 4.3.8.** *We introduce the open subsets of* $\mathcal{U}(\mathcal{A})^\Phi$*:*

$$\begin{aligned}\mathcal{U}(\mathcal{A})^\Phi_+ &= \{\sigma \in \mathcal{U}(\mathcal{A})^\Phi : \phi(S_0)=\phi(S_2)<\phi(S_1)=\phi(S_3)\},\\ \mathcal{U}(\mathcal{A})^\Phi_- &= \{\sigma \in \mathcal{U}(\mathcal{A})^\Phi : \phi(S_1)=\phi(S_3)<\phi(S_0)=\phi(S_1)\}\end{aligned}$$

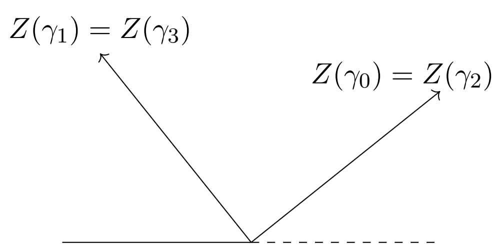



Figure 4.1: Central charges of $\mathcal{U}(\mathcal{A})^\Phi_+$

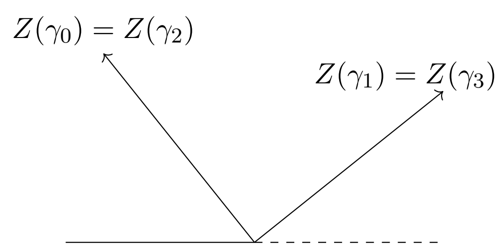



Figure 4.2: Central charges of $\mathcal{U}(\mathcal{A})^\Phi_-$

**Lemma 4.3.9.** *The autoequivalence* $\Psi$ *induces a bijection between* $\mathcal{U}(\mathcal{A})^\Phi_+$ *and* $\mathcal{U}(\mathcal{A})^\Phi_-$*.*

*Proof.* Given $\sigma=(Z,\mathcal{A})\in\mathcal{U}(\mathcal{A})^\Phi_+$, we denote by $\Psi(\sigma)=(Z_\psi,\Psi(\mathcal{A})=\mathcal{A})$. By Lemma 4.1.9, $Z_\psi(\gamma_i)=Z(\gamma_{i-1})$, therefore $\Psi(\sigma)\in\mathcal{U}(\mathcal{A})^\Phi_-$. The statement follows immediately. $\square$

**Theorem 4.3.10.** *Let* $\sigma \in \mathcal{U}(\mathcal{A})^\Phi_+$*, then the objects of special Kronecker type I are stable for* $\sigma$*. For* $\tau \in \mathcal{U}(\mathcal{A})^\Phi_-$*, then the objects of special Kronecker type II are stable for* $\tau$*.*

*Proof.* Let $\sigma = (Z, \mathcal{P}) \in \mathcal{U}(\mathcal{A})^{\Phi}_{+}$. We define a stability function $Z$ on $\operatorname{rep} K_2$ by setting $\overline{Z}(C_0) = Z(\gamma_j), \overline{Z}(C_1) = Z(\gamma_{j+1})$ where $j = 0$ or $2$ and $C_i$ are the simple modules at the vertex $i$. Then $\overline{\phi}(C_0) < \overline{\phi}(C_1)$ by our assumption. Note that by our definition, for $E \in \operatorname{rep} K_2$

$$\overline{\phi}(E) = \phi(\Xi^I_i(E)) \tag{4.16}$$

for $i = 1,\ 2$. Let $M = \Xi(E^{(\lambda)}_{p,q})$ be of special Kronecker type I. Suppose there is a destabilising sequence of $M$ in $\mathcal{A}$:

$$0 \longrightarrow A \longrightarrow M \longrightarrow B \longrightarrow 0$$

with $\phi(A) \geq \phi(M)$.

By Lemma 4.3.2, $A$ and $B$ are both in $\mathcal{K}^I_i$. Therefore $A = \Xi^I_i(A')$ for some $A' \in \operatorname{rep} K_2$, and $\phi(A) = \bar{\phi}(A') > \phi(M) = \bar{\phi}(E^{(\lambda)}_{p,q})$. We get that $E^{(\lambda)}_{p,q}$ is not stable. However, this contradicts with Lemma 4.3.4.

The proof for the second statement is similar. Note that for $\tau = (W, \mathcal{P})$, we define stability function on $\operatorname{rep} K_2$ with $\overline{W}(C_0) = W(\gamma_j)$, and $\overline{W}(C_1) = W(\gamma_{j+1})$ where $j = 1$ or 3, we still have $\overline{\phi}(C_0) < \overline{\phi}(C_1)$ by assumption. □

We recall the following standard fact:

**Lemma 4.3.11.** *Let $Z : K_0(\mathcal{A}) \to \mathbb{C}$ be a stability function. Suppose there is a short exact sequence in $\mathcal{A}$:*

$$0 \to E' \to E \to E'' \to 0,$$

*such that $E'$ and $E''$ are semistable objects of the same phase $\lambda$, then $E$ will also be semistable of phase $\lambda$.*

**Corollary 4.3.12.** *Let $x \in \mathbb{P}^1 \times \mathbb{P}^1$, then $\mathcal{O}_x$ is semistable with respect to the stability condition $\sigma \in \mathcal{U}(\mathcal{A})^{\Phi}_{+}$. For $\sigma \in \mathcal{U}(\mathcal{A})^{\Phi}_{-}$, there is a semistable object whose class in $K_0(\mathcal{A})$ is $\delta = [\mathcal{O}_x]$.*

*Proof.* Let $F_1 = s_*\mathcal{O}_{\{x_1\}\times\mathbb{P}^1},\ F_2 = s_*\mathcal{O}_{\{x_1\}\times\mathbb{P}^1}(-1)[1]$ where $x_1 = p_1(x)$. By (4.12), there is a short exact sequence in $\mathcal{A}$:

$$0 \to F_1 \to \mathcal{O}_x \to F_2 \to 0.$$

By Proposition 4.3.7 and Theorem 4.3.10 $F_1$ and $F_2$ are stable for the stability condition $\sigma \in \mathcal{U}(\mathcal{A})^{\Phi}_{+}$. Moreover, since $Z(F_1) = Z(\gamma_0) + Z(\gamma_1)$ and $Z(F_2) = Z(\gamma_2) + Z(\gamma_3)$ therefore $\phi(F_1) = \phi(F_2)$. So $\mathcal{O}_x$ is (strictly) semistable for $\sigma$ with the same phase as $\phi(F_i)$ by Lemma 4.3.11.

Suppose $\sigma \in \mathcal{U}(\mathcal{A})^{\Phi}_{-}$. We take $\Psi^{-1}(\sigma)$, by Lemma 4.3.9 $\Psi^{-1}(\sigma) \in \mathcal{U}(\mathcal{A})^{\Phi}_{+}$. Therefore $\Psi(\mathcal{O}_x)$ is semistable for $\sigma$. Since

$$\psi(\delta) = \sum_{i=0}^{3} \psi(\gamma_i) = \sum_{i=0}^{3} \gamma_i = \delta,$$

the claim is proved. □

**Remark 4.3.13.** *For $\sigma \in \mathcal{U}(\mathcal{A})^{\Phi}$, we can conclude that there is a semistable object whose class in $K_0(\mathcal{A})$ is $\delta$. By the above corollary, the remaining case we need to verify is that when $\phi(S_i) = \phi(\mathcal{O}_x)$ for all $i$, however, each object in $\mathcal{A}$ is semistable in this case.*

The central charges of $\delta$ and other stable objects for $\sigma \in \mathcal{U}(\mathcal{A})^{\Phi}_{+}$ are depicted in the figure 4.3.

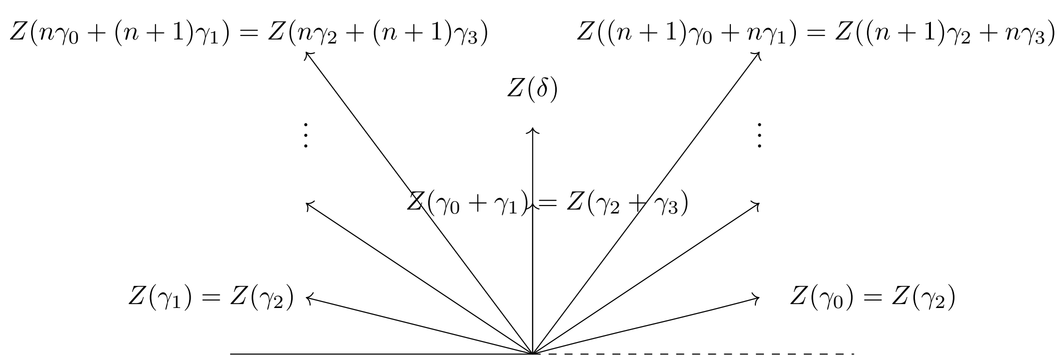



Figure 4.3: Central charges of stable objects and $\mathcal{O}_x$ for $\sigma \in \mathcal{U}(\mathcal{A})^{\Phi}_{+}$

Recall that $\mathcal{T} = - \otimes \pi^*\mathcal{O}(1,0)$ and $\mathcal{T}_{\Psi} = \Psi \circ \mathcal{T} \circ \Psi^{-1}$, the simple objects in $\mathcal{A}$ are $S_0 = s_*\mathcal{O}(0,0)$, $S_1 = s_*\mathcal{O}(-1,0)[1]$, $s_*\mathcal{O}(1,-1)[1]$ and $S_3 = s_*\mathcal{O}(0,-1)[2]$. Finally we mention another description of some objects of special Kronecker types I and II,

**Lemma 4.3.14.** *Let $n \geq 0$*

*(i)* $\mathcal{T}^n(S_0)$, $\mathcal{T}^n(S_2)$, $\mathcal{T}^{-n}(S_1)$, $\mathcal{T}^{-n}(S_3)$ *are objects of special Kronecker type I with classes in* $K_0(\mathcal{A})$*:*

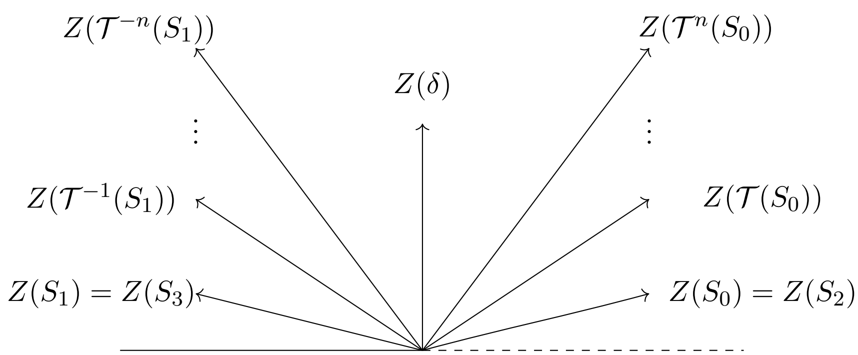
$$(n+1)\gamma_0 + n\gamma_1, \quad (n+1)\gamma_2 + n\gamma_3, \quad n\gamma_0 + (n+1)\gamma_1, \quad n\gamma_2 + (n+1)\gamma_3.$$

*(ii)* $\mathcal{T}_\Psi^n(S_1)$, $\mathcal{T}_\Psi^n(S_3)$, $\mathcal{T}_\Psi^{-n}(S_0)$, $\mathcal{T}_\Psi^{-n}(S_2)$ *are objects of special Kronecker type II with classes in* $K_0(\mathcal{A})$*:*

$$(n+1)\gamma_1 + n\gamma_2, \quad (n+1)\gamma_3 + n\gamma_0, \quad n\gamma_1 + (n+1)\gamma_2, \quad n\gamma_3 + (n+1)\gamma_0.$$

*Proof.* Note that $\mathcal{T}^n\big(s_*\mathcal{O}(a,b)\big) = s_*\mathcal{O}(a+n,b)$ by the projection formula, the result follows directly from the table in Proposition 4.3.7. $\square$

For $\sigma \in \mathcal{U}(\mathcal{A})^\Phi_+$, we can alternatively illustrate the central charges of semistable objects in the complex plane:




### 4.3.2 There are no other stable objects

The goal of this subsection is to prove that for $\sigma \in \mathcal{U}(\mathcal{A})^\Phi$, there are no other stable objects other than the ones in Theorem 4.3.10.

For simplicity, we first restrict ourselves to the normalized stability conditions

$$\big(\mathrm{Stab}(\mathcal{C})^\Phi\big)_n := \{\sigma = (Z,\mathcal{P}) : Z(\delta) = i\} \subset \big(\mathrm{Stab}(X)^\Phi\big)_0,$$

where $\delta$ is the class of skyscrapper sheaf $\mathcal{O}_x$ in $K_0(\mathcal{A})$. Note that $\big(\mathrm{Stab}(\mathcal{C})^\Phi\big)_n$ is a connected submanifold of $\big(\mathrm{Stab}(X)^\Phi\big)_0$.

Let $\mathcal{U}^n(\mathcal{A})^\Phi \subset \left(\mathrm{Stab}(\mathcal{C})^\Phi\right)_n$ which consists of normalized stability conditions $(Z,\mathcal{A})$ with the fixed heart $\mathcal{A}$, and $\mathcal{U}^n(\mathcal{A})^\Phi_\pm = \{(Z,\mathcal{A}) \in \mathcal{U}(\mathcal{A})^\Phi_\pm : Z(\delta) = i\}$. For $\sigma \in \mathcal{U}^n(\mathcal{A})^\Phi_+$, we have

$$\phi(S_0) = \phi(S_2) < \phi(\mathcal{O}_x) = \frac{1}{2} < \phi(S_1) = \phi(S_3). \tag{4.17}$$

Recall the group action on the stability conditions in Definition 2.2.14. Let $\mathcal{T}$, $\mathcal{T}_\Psi$ be the autoequivalences defined in Theorem 4.2.1. We first make the following important observartion

**Lemma 4.3.15.** *(i) For $\sigma \in \mathcal{U}^n(\mathcal{A})^\Phi_+$, then $\mathcal{T}(\sigma) = \sigma \cdot \widetilde{g}$ where $\widetilde{g} = (g,f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ such that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f\left(\frac{1}{2}\right) = \frac{1}{2}$.*

*(ii) For $\sigma \in \mathcal{U}^n(\mathcal{A})^\Phi_-$, then $\mathcal{T}_\Psi(\sigma) = \sigma \cdot \widetilde{g}$ where $\widetilde{g} = (g,f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ such that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f\left(\frac{1}{2}\right) = \frac{1}{2}$.*

*Proof.* Let $\sigma \in \mathcal{U}^n(\mathcal{A})^\Phi_+$. We write $\mathcal{T}(\sigma) = (Z_t, \mathcal{P}_\mathcal{T})$. By viewing $\mathbb{C} \cong \mathbb{R}^2$, we let

$$\boldsymbol{e_0} = Z(\gamma_0) = Z(\gamma_2), \quad \boldsymbol{e_1} = Z(\gamma_1) = Z(\gamma_3).$$

Then by (4.11), we have

$$\begin{aligned} Z_t(\gamma_0) &= Z\left(t^{-1}\gamma_0\right) = Z\left(-\gamma_1\right) = -\boldsymbol{e_1}; \\ Z_t(\gamma_1) &= Z\left(t^{-1}\gamma_1\right) = Z\left(\gamma_0 + 2\gamma_1\right) = \boldsymbol{e_0} + 2\boldsymbol{e_1}. \end{aligned}$$

We define $g \in \mathrm{GL}^+(2,\mathbb{R})$ such that

$$g(\boldsymbol{e_0}) = 2\boldsymbol{e_0} + \boldsymbol{e_1}, \quad g(\boldsymbol{e_1}) = -\boldsymbol{e_0},$$

Note that $2(\boldsymbol{e_0} + \boldsymbol{e_1}) = \sum_i Z(\gamma_i) = Z(\delta) = i$ by Lemma 4.2.3, and $g(\boldsymbol{e_0} + \boldsymbol{e_1}) = \boldsymbol{e_0} + \boldsymbol{e_1}$, therefore we see $g$ preserves the positive imaginary axis. We can take $\widetilde{g} = (g,f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ be the unique lift of $g$ such that $f(\frac{1}{2}) = \frac{1}{2}$. Let $\sigma \cdot \widetilde{g} = (Z_g, \mathcal{P}_f)$. Then by definition $Z_g(\gamma_i) = Z_t(\gamma_i)$ for $i = 0,\ 1$. Therefore $Z_g = Z_t$.

We claim that the bounded hearts $\mathcal{P}_\mathcal{T}(0,1]$ and $\mathcal{P}_f(0,1]$ are the same, thus finishing the proof of the first case. By Theorem 4.2.1 we have

$$\mathcal{P}_\mathcal{T}(0,1] = \mathcal{T}(\mathcal{A}) = L_{S_0}L_{S_2}\mathcal{A}.$$

Suppose $S_1, S_3 \in \mathcal{P}(\phi)$, $\phi \in (0, 1]$. Since $g(\boldsymbol{e_1}) = -\boldsymbol{e_0} = -Z(\gamma_0) = -Z(\gamma_2)$, therefore $f(\phi) = \phi(S_0[i]) = \phi(S_2[i])$ for some odd number $i$. Note that $\phi \in (1/2, 1] \subset (1/2, 3/2)$, so

$$f(\phi) \in \left(f(\frac{1}{2}), f(\frac{3}{2})\right) = (1/2, 3/2).$$

By our assumption,

$$\phi(S_0[1]) = \phi(S_2[1]) \in (1/2, 3/2).$$

Therefore, $i = 1$ and $S_0[1], S_2[1] \in \mathcal{P}_f(\phi)$ which is contained in $\mathcal{A}' = \mathcal{P}_f(0, 1]$.

Suppose $S_0, S_2 \in \mathcal{P}(\omega)$, $\omega \in (0, 1]$. Then $g(\boldsymbol{e_0}) = 2\boldsymbol{e_0} + \boldsymbol{e_1} = Z(\widetilde{S_1}) = Z(\widetilde{S_3})$ where $\widetilde{S_1} = \mathcal{T}(S_0)$ and $\widetilde{S_3} = \mathcal{T}(S_2)$. By Lemma 4.3.14 they are of special Kronecker type I, therefore they are semistable for $\sigma$ by Theorem 4.3.10. We have $f(\omega) = \phi(\widetilde{S_1}[n]) = \phi(\widetilde{S_3}[n])$ for some even number $n$. Since $\omega \in (0, 1/2) \subset (-1/2, 1/2)$, so $f(\omega) \in (-1/2, 1/2)$. By our assumption, $\phi(\widetilde{S_1}) = \phi(\widetilde{S_2}) \in (-1/2, 1/2)$. Therefore $n = 0$ and $\widetilde{S_1}, \widetilde{S_3} \in \mathcal{A}'$.

Recall the simple objects in $L_{S_0}L_{S_1}\mathcal{A}$ are exactly $S_0[1]$, $S_2[1]$, $\widetilde{S_1}$ and $\widetilde{S_3}$ by our computations in Section 4.2. We just proved $L_{S_0}L_{S_2}\mathcal{A} = \langle S_0[1], S_2[1], \widetilde{S_1}, \widetilde{S_3}\rangle \subset \mathcal{A}'$, so by Lemma 2.1.5 the two hearts are equivalent. We finished the proof of the first case.

For $\sigma \in \mathcal{U}^n(\mathcal{A})^{\Phi}_{-}$, note that $\Psi^{-1}(\sigma) \in \mathcal{U}^n(\mathcal{A})^{\Phi}_{+}$ by Lemma 4.3.9, therefore by the first part we have

$$\begin{aligned} \mathcal{T}_{\Psi}(\sigma) &= \Psi \circ \mathcal{T} \circ \Psi^{-1}(\sigma) \\ &= \Psi(\Psi^{-1}(\sigma) \cdot \widetilde{g}) \\ &= \sigma \cdot \widetilde{g}, \end{aligned}$$

where $\widetilde{g} = (g, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ such that $f(1/2) = 1/2$. This finishes the proof. $\square$

**Corollary 4.3.16.** *If $\sigma = (Z, \mathcal{P}) \in \mathcal{U}^n(\mathcal{A})^{\Phi}_{+}$, then $\mathcal{T}^{\pm 1}\left(\mathcal{P}(\frac{1}{2})\right) = \mathcal{P}(\frac{1}{2})$. Similarly, if $\tau = (W, \mathcal{P}') \in \mathcal{U}^n(\mathcal{A})^{\Phi}_{-}$, then $\mathcal{T}_{\Psi}^{\pm 1}\left(\mathcal{P}'(\frac{1}{2})\right) = \mathcal{P}'(\frac{1}{2})$.*

*Proof.* Let $\sigma = (Z, \mathcal{P}) \in \mathcal{U}^n(\mathcal{A})^{\Phi}_{+}$ and $\mathcal{T}(\sigma) = (Z_t, \mathcal{P}_{\mathcal{T}})$. By the above Lemma, $\mathcal{T}(\sigma) = \sigma \cdot \widetilde{g} = (Z_g, \mathcal{P}_f)$ for $\widetilde{g} = (g, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ such that $f(1/2) = 1/2$ , we have

$$\mathcal{P}_f(1/2) = \mathcal{P}(1/2).$$

The proof for the second statement is the same. $\square$

Recall for any interval $I \subset \mathbb{R}$, $\mathcal{P}(I)$ is the extension-closed subcategory of $\mathcal{C}$ generated by the subcategories $\mathcal{P}(\phi)$ for $\phi \in I$. Recall the definition of $\mathrm{Aut}_*(\mathcal{C})$ in Definition 4.1.12. The following proposition will be useful:

**Proposition 4.3.17.** *Let $W$ be an element of $\mathrm{Aut}_*(\mathcal{C})$ such that for a stability condition $\sigma = (Z, \mathcal{P})$ we have $W(\sigma) = \sigma \cdot \widetilde{g}$ for some $\widetilde{g} = (g, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. Suppose that $E,\ F$ are two semistable objects with phases $\phi_E < \phi_F$, then*

$$W\big(\mathcal{P}(\phi_E, \phi_F)\big) = \mathcal{P}\big(\phi_{W(E)}, \phi_{W(F)}\big).$$

*Proof.* We write $\sigma \cdot \widetilde{g} = (Z_g, \mathcal{P}_f)$, then $\mathcal{P}_f(I) = \mathcal{P}\big(f(I)\big)$ for any interval $I \subset \mathbb{R}$ by definition. Therefore by our assumption

$$W\big(\mathcal{P}(\phi_E, \phi_F)\big) = \mathcal{P}\big(f(\phi_E), f(\phi_F)\big).$$

Since $\sigma \cdot \widetilde{g}$ and $\sigma$ contain the same set of semistable objects, therefore $W(E)$ and $W(F)$ are semistable for $\sigma$. So we have $\phi_{W(E)} = f\big(\phi_E\big)$ and $\phi_{W(F)} = f\big(\phi_F\big)$, this proves the result. □

**Lemma 4.3.18.** *Given a stability condition $\sigma = (Z, \mathcal{A})$ such that $\mathcal{A}$ is of finite-length with the finite set of simple objects (up to isomorphism) $\{S_0, S_1, \cdots, S_n\}$. Define the linear cone in $\mathbb{C}$:*

$$C := \{z \in \mathbb{C} : z = \sum_{i=0}^{n} \lambda_i Z(S_i),\ \ \lambda_i \geq 0\} \setminus \{0\}.$$

*Then for any non-zero semistable object $E \in \mathcal{P}(\psi)$, $\psi \in \mathbb{R}$, we have*

$$Z(E) \in C \cup (-C).$$

*Proof.* After shifting $E$, we may assume that $E \in \mathcal{A}$. Since $\mathcal{A}$ is of finite-length, $E$ has a finite filtration by the simple objects $S_i$. Then in $K_0(\mathcal{A})$ we have $[E] = \sum_{i=0}^{n} \lambda_i [S_i]$, $\lambda_i \geq 0$ (at least one $\lambda_j \neq 0$). Therefore the result follows from the linearity of $Z$. □

The following important theorem characterizes the stable objects outside the ray $\phi = \frac{1}{2}$ in the upper half complex plane:

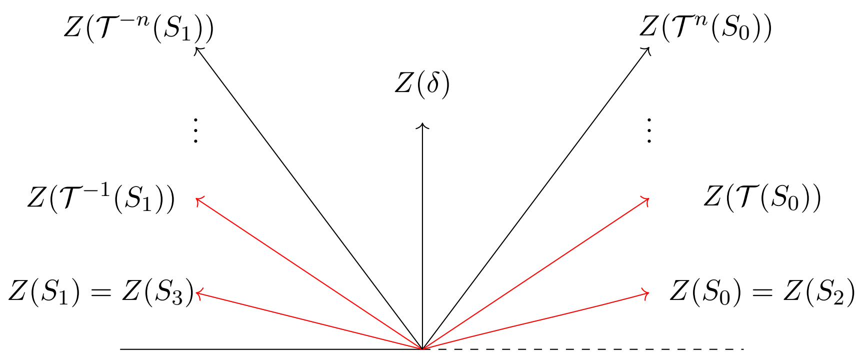
**Theorem 4.3.19.** *Take $\sigma \in \mathcal{U}^n(\mathcal{A})^{\Phi}_{+}$, there is no stable object whose phase lies in the intervals $\big(0, \phi(S_0)\big)$, $\big(\phi(S_1), 1\big)$, nor in the intervals*

$$\Big(\phi\left(\mathcal{T}^m(S_0)\right), \phi\left(\mathcal{T}^{m+1}(S_0)\right)\Big) \text{ or } \Big(\phi\left(\mathcal{T}^{-m-1}(S_1)\right), \phi\left(\mathcal{T}^{-m}(S_1)\right)\Big),$$

*for any integer $m \geq 0$. Moreover, the stable objects of phases $\phi\big(\mathcal{T}^m(S_0)\big)$ and $\phi\big(\mathcal{T}^{-m}(S_1)\big)$ for $m \geq 0$ are of special Kronecker type I.*

*Proof.* Given $\sigma = (Z, \mathcal{P}) \in \mathcal{U}^n(\mathcal{A})^{\Phi}_{+}$, it is clear that there is no stable object of phase in the interval $(0, \phi(S_0)) \cup (\phi(S_1), 1)$ by Lemma 4.3.18.

By Lemma 4.3.15 $\mathcal{T}(\sigma) = \sigma \cdot \widetilde{g}$ for some $\widetilde{g} \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, therefore by Proposition 4.3.17 we only need to check that there is no stable object of phase in the intervals $(\phi(S_0), \phi(\mathcal{T}(S_0)))$ and $\big(\phi(\mathcal{T}^{-1}(S_1)), \phi(S_1)\big)$, then apply $\mathcal{T}^{\pm m}$ we see that there is no stable object of phase in other open intervals.




Suppose $E \in \mathcal{P}\left(\phi(S_0) + \epsilon\right)$ for $0 < \epsilon < \phi(\mathcal{T}(S_0)) - \phi(S_0)$. We will take a $\mathbb{C}$-action on $\sigma$ and reduce to the case in the beginning: we choose $0 < \epsilon' \ll \epsilon$ and let $\lambda = -(\phi(S_0) + \epsilon')$. Then let $\sigma' := \sigma \cdot \lambda = (Z', \mathcal{P}')$. The phase of the semistable object for $\sigma'$ is denoted by $\phi'(-)$. By definition of the $\mathbb{C}$-action (see Remark 2.2.15) we have

$$\begin{aligned}
\phi'(S_0[1]) = \phi'(S_2[1]) &= \phi(S_0[1]) - \phi(S_0) - \epsilon' \\
&= 1 - \epsilon' \in (1/2, 1), \\
\phi'\left(\mathcal{T}(S_0)\right) = \phi'\left(\mathcal{T}(S_2)\right) &= \phi(\mathcal{T}(S_0)) - \phi(S_0) - \epsilon' \in (0, 1/2).
\end{aligned}$$

We see that the heart $\mathcal{A}' = \mathcal{P}'(0, 1]$ contains the simple objects $\{S_0[1], S_2[1], \mathcal{T}(S_0), \mathcal{T}(S_2)\}$ which generate $L_{S_0} L_{S_2} \mathcal{A}$, we have $L_{S_0} L_{S_2}(\mathcal{A}) \subset \mathcal{A}'$ therefore $\mathcal{A}' = L_{S_0} L_{S_2}(\mathcal{A}) = \mathcal{T}\mathcal{A}$ by Theorem 4.2.1 (see figure 4.4).

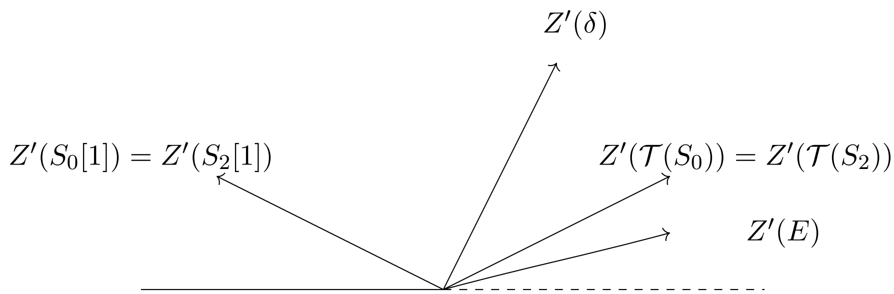



Figure 4.4: Central charges of simple objects and $E$, $\delta$ for $\sigma'$

Since $\mathcal{A}'$ is of finite-length, and the phase of $E$ for $\sigma'$ is $\phi(S_0)+\epsilon-\phi(S_0)-\epsilon'=\epsilon-\epsilon'\in\Big(0,\phi\big(\mathcal{T}(S_0)\big)\Big)$, therefore $E$ is not semistable for $\sigma'$ by Lemma 4.3.18. Since $\sigma'$ and $\sigma$ contain the same set of semistable objects, so $E$ is not semistable for $\sigma$ either.

Similarly, suppose $E\in\mathcal{P}\left(\phi(S_1)-\epsilon\right)$, where $\epsilon\in\big(0,\phi(S_1)-\phi(\mathcal{T}^{-1}(S_1))\big)$, we choose $0<\epsilon'\ll\epsilon$. Let $t=1+\epsilon'-\phi(S_1)$ and $\tau:=\sigma\cdot t=(Z',\mathcal{P}')$. For $\tau$ we have the phases of

$$\begin{aligned}\phi'(S_1[-1])=\phi'(S_3[-1]) &= \epsilon'\in(0,1/2),\\ \phi'\left(\mathcal{T}^{-1}(S_1)\right)=\phi'\left(\mathcal{T}^{-1}(S_3)\right) &= \phi\left(\mathcal{T}^{-1}(S_1)\right)-\phi(S_1)+1+\epsilon'\in(1/2,1).\end{aligned}$$

The heart $\mathcal{A}'=\mathcal{P}'(0,1]$ contains the simple objects $\{S_1[-1],S_3[-1],\mathcal{T}^{-1}(S_1),\mathcal{T}^{-1}(S_3)\}$ which generate $R_{S_1}R_{S_3}\mathcal{A}$, $R_{S_1}R_{S_3}\mathcal{A}\subset\mathcal{A}'$ therefore $\mathcal{A}'=R_{S_1}R_{S_3}\mathcal{A}=\mathcal{T}^{-1}\mathcal{A}$. The phase of $E$ for $\tau$ is $\phi(S_1)-\epsilon+1+\epsilon'-\phi(S_1)=1+\epsilon'-\epsilon\in(\mathcal{T}^{-1}(S_1),1]$, therefore $E$ cannot be semistable for $\tau$ again by Lemma 4.3.18, and is also not semistable for $\sigma$.

For the second statement, let $E\in\mathcal{P}\left(\phi(S_0)\right)$ be a stable object, we take the Jordan-Hölder filtration of $E$:

$$0\subset E_n\subset E_{n-1}\subset\cdots E_1\subset E_0=E,$$

such that $E_i/E_{i-1}=S_j$ for $j\in\{0,\cdots 3\}$. Since $\phi(S_0)=\phi(S_2)\neq\phi(S_1)=\phi(S_3)$, by the linearity of $Z$ the only graded factors appear in the filtrations are $S_0$ and $S_2$. Therefore $S_0$ or $S_2$ is a subobject of $E$, thus must be isomorphic to $E$. Similarly if $E\in\mathcal{P}\left(\phi(S_1)\right)$ is stable, we prove that $E$ is one of $S_1$ and $S_3$ exactly in the same way. Now we apply $\mathcal{T}^{\pm m}$ on $\sigma$ for $m\geq 0$. Using Lemma 4.3.15 again, we see $\mathcal{T}^{\pm m}(\sigma)$ and $\sigma$ contain the same set of stable objects. Therefore the stable objects of phase $\phi\big(\mathcal{T}^m(S_0)\big)$ are $\mathcal{T}^m(S_0)$ and $\mathcal{T}^m(S_2)$, and the stable objects of phase $\phi\big(\mathcal{T}^{-m}(S_1)\big)$ are $\mathcal{T}^{-m}(S_1)$ and $\mathcal{T}^{-m}(S_3)$. By Lemma 4.3.14 they are of special Kronecker type I. □

Let $\sigma \in \mathcal{U}^n(\mathcal{A})^{\Phi}_{-}$, we take $\Psi(\sigma) \in \mathcal{U}^n(\mathcal{A})^{\Phi}_{+}$, then by the above Lemma the stable objects for $\sigma$ outside the ray $\phi = \frac{1}{2}$ are of special Kronecker type II.

The rest of this section is devoted to characterizing the stable objects on the ray $\phi = \frac{1}{2}$.

**Lemma 4.3.20.** *Let $\sigma \in \mathcal{U}^n(\mathcal{A})^{\Phi}_{+}$ and $E \in \mathcal{P}(\frac{1}{2})$, then $\pi_* E$ is (set theoretically) supported on $S \times \mathbb{P}^1$ where $S$ is a finite set of closed points in $\mathbb{P}^1$.*

*Proof.* According to Corollary 4.3.16, $\mathcal{T}^n\left(\mathcal{P}(\frac{1}{2})\right) = \mathcal{P}(\frac{1}{2}) \subset \mathcal{A}$ for any integer $n$. Therefore we have the vanishing of cohomology groups $\mathrm{Ext}^k(\mathcal{Q}, E) = 0$ for $k \neq 0$, in particular:

$$\begin{aligned} 0 &= \mathrm{Ext}^k_X\left(\pi^*\mathcal{O} \oplus \pi^*\mathcal{O}(1,1), E \otimes \pi^*\mathcal{O}(n,0)\right) \\ &= \mathrm{Ext}^k_Z\left(\mathcal{O} \oplus \mathcal{O}(1,1), \pi_* E \otimes \mathcal{O}(n,0)\right) \quad \text{(projection formula)} \\ &= \mathbb{H}^k\big(\mathbb{P}^1, (p_1 \circ \pi)_* E \otimes \mathcal{O}(n)\big) \oplus \mathbb{H}^k\Big(\mathbb{P}^1, (p_1 \circ \pi)_*\big(E(0,-1)\big) \otimes \mathcal{O}(n-1)\Big) \end{aligned}$$

where $\mathbb{H}$ means the hypercohomology of complexes and $k \neq 0$. In general for a complex $F^\star \in D^b(X)$, we have a spectral sequence [34, p.74]

$$E_2^{p,q} = \mathrm{H}^q\left(X, H^p(F^\star)\right) \Rightarrow \mathbb{H}^{p+q}(X, F^\star).$$

We write $E'_0 = (p_1 \circ \pi)_* E$ and $E'_1 = (p_1 \circ \pi)_*\big(E(0,-1)\big)$. By taking $n \gg 0$, then $\mathrm{H}^i\left(\mathbb{P}^1, H^j(E'_m) \otimes \mathcal{O}(n)\right) = 0$ for $i \neq 0$, therefore the spectral sequence degenerates, we have

$$\mathbb{H}^k(\mathbb{P}^1, E'_m \otimes \mathcal{O}(n)) = \bigoplus_j \mathrm{H}^{k-j}\left(\mathbb{P}^1, H^j(E'_m) \otimes \mathcal{O}(n)\right) = 0 \quad \text{for } k \neq 0. \tag{4.18}$$

Fix $j \neq 0$. Then $\mathrm{H}^0\left(\mathbb{P}^1, H^j(E'_m) \otimes \mathcal{O}(n)\right) = 0$ where $n \gg 0$. This implies that $H^j(E'_m) = 0$. Therefore $E'_m$ is concentrated in degree 0 and is indeed a sheaf.

For $m = 0, 1$, now we have

$$\mathrm{H}^k(\mathbb{P}^1, E'_m \otimes \mathcal{O}(n)) = 0, \quad k = 1, \quad n \in \mathbb{Z}. \tag{4.19}$$

By taking $n \ll 0$, and using the fact that every coherent sheaf on $\mathbb{P}^1$ splits into line bundles and torsion sheaves [25], we have $\dim(\mathrm{supp}\ E'_m) = 0$.

We denote $S := \operatorname{supp} E_0' \cup \operatorname{supp} E_1'$. Suppose $s \notin S$, we consider the following fibre product diagram with naturally-defined morphisms:

$$\begin{array}{ccc} p_1^{-1}(s) & \xhookrightarrow{\ i\ } & \mathbb{P}^1 \times \mathbb{P}^1 \\ {\scriptstyle pt}\downarrow & & \downarrow{\scriptstyle p_1} \\ \{s\} & \xhookrightarrow[\ j\ ]{} & \mathbb{P}^1 \end{array} \tag{4.20}$$

We apply the flat base change theorem [34, Chapter 3.3] to $\pi_*E,\ \pi_*\big(E(0,-1)\big) \in D^b(Z)$, for any integer $k$

$$\begin{aligned} m = 0: &\qquad \mathrm{H}^k(\mathbb{P}^1, i^*\pi_*E) = j^*E_0' = 0 \\ m = 1: &\qquad \mathrm{H}^k\left(\mathbb{P}^1, i^*\pi_*(E(0,-1))\right) = \mathrm{H}^k\left(\mathbb{P}^1, (i^*\pi_*E)\otimes\mathcal{O}(-1)\right) = j^*E_1' = 0. \end{aligned}$$

These vanishings imply that $i^*\pi_*E = 0$ (one can again use the structure theorem of coherent sheaf on $\mathbb{P}^1$). Therefore $\pi_*E$ is supported on $S \times \mathbb{P}^1$. □

**Lemma 4.3.21.** *Suppose $E \in \mathcal{A}$ is isomorphic to the shift of a sheaf, and $\operatorname{End}_{\mathcal{A}}(E) \cong \mathbb{C}$ then it is the pushforward $E = s_*F[i]$ for some $F \in \operatorname{Coh} Z$.*

*Proof.* We follow the idea in the proof of [2, Lemma 3.1]: let $Y$ be the scheme-theoretic support of $E$. By definition, $\mathrm{H}^0(\mathcal{O}_Y)$ acts faithfully on $E$, and $\operatorname{End}_{\mathcal{A}}(E) = \mathbb{C}.\mathrm{Id}$, therefore $\mathrm{H}^0(\mathcal{O}_Y) \cong \mathbb{C}$. Take the composition of the embedding of $Y$ with the contraction

$$f : Y \hookrightarrow X \twoheadrightarrow \bar{X} = \operatorname{Spec} \mathrm{H}^0(\mathcal{O}_X),$$

as $\mathrm{H}^0(f_*\mathcal{O}_Y) = \mathrm{H}^0(\mathcal{O}_Y) = \mathbb{C}$, so the scheme-theoretic image of $Y$ under $f$ is a point of $\bar{X}$. By definition of $f$, the point will be the origin (singular point), thus $Y$ is contained scheme-theoretically in the fiber of the contraction map. Since the scheme-theoretic fiber of the origin is exactly $Z = \mathbb{P}^1 \times \mathbb{P}^1$, so $E = s_*F$ for some $F \in \operatorname{Coh} Z$. □

Now we are able to characterize the stable objects on the ray $\phi = \frac{1}{2}$.

**Theorem 4.3.22.** If $\sigma \in \mathcal{U}^n(\mathcal{A})_+^{\Phi}$, *let $E$ be a $\sigma$-stable object in $\mathcal{P}(\frac{1}{2})$, then there exists a point $x \in \mathbb{P}^1$ such that either $E = F_1(x) = s_*\mathcal{O}_{\{x\}\times\mathbb{P}^1}$ or $E = F_2(x) = s_*\mathcal{O}_{\{x\}\times\mathbb{P}^1}(-1)[1]$.*

If $\tau \in \mathcal{U}^n(\mathcal{A})_-^{\Phi}$, *let $E$ be a $\tau$-stable object in $\mathcal{P}(\frac{1}{2})$, then there exists a point $x \in \mathbb{P}^1$ such that either $E = \Psi\left(F_1(x)\right)$ or $E = \Psi\left(F_2(x)\right)$.*

*Proof.* We prove the first statement, the second statement follows since $\Psi$ exchanges the stability conditions in $\mathcal{U}^n(\mathcal{A})_+^\Phi$ and $\mathcal{U}^n(\mathcal{A})_-^\Phi$, as also exchanges the objects of special Kronecker type I and II.

Suppose $E \in \mathcal{P}\left(\frac{1}{2}\right)$ is stable and not isomorphic to $F_1(x_1)$ and $F_2(x_1)$ for any $x_1 \in \mathbb{P}^1$. We will show that there is a vector bundle $F \in \operatorname{Coh} Z$ such that $E \cong s_*F[1]$. We follow the idea of proof in [2, Lemma 3.2]: since $F_i$ are stable, therefore

$$\operatorname{Hom}_X(E, F_n(x_1)) = \operatorname{Hom}_X(F_n(x_1), E) = 0, \ n = 1, \ 2.$$

Note that we have the short exact sequence in $\mathcal{A}$:

$$0 \to F_1(x_1) \to \mathcal{O}_x \to F_2(x_1) \to 0,$$

where $x \in Z$ such that $p_1(x) = x_1$, therefore there cannot be any nonzero map $E \to \mathcal{O}_x$ or $\mathcal{O}_x \to E$. Since $\mathcal{O}_x$ is semistable of phase $1/2$, then $\operatorname{Hom}_X^i(E, \mathcal{O}_x) = 0$ for $i \leq 0$, and Serre duality gives $\operatorname{Hom}_X(\mathcal{O}_x[i], E) = \operatorname{Hom}_X(E, \mathcal{O}_x[i+3]) = 0$ for $i \geq 0$ and $x \in Z$. Since $E$ is supported on $Z$, there will be no homomorphisms with shifts of skyscraper sheaves outside the zero-section. Therefore we can apply [20, Proposition 5.4] and deduce that $E$ is isomorphic to a two-term complex of locally-free sheaves

$$E^{-2} \xrightarrow{d^{-2}} E^{-1}.$$

Hence $H^{-2}(E) \subset E^{-2}$ is torsion free on $X$. However, since $H^{-2}(E)$ is supported on $Z$, therefore it must vanish. The map $d^{-2}$ is injective, so that $E$ is isomorphic to the shift of a sheaf $F'[1]$. Since $E$ is stable, $\operatorname{End}(F') \cong \mathbb{C}.\mathrm{Id}$, therefore we apply Lemma 4.3.21 and show that $F' = s_*F$ where $F \in \operatorname{Coh} Z$. Since $\operatorname{Hom}_X(s_*\mathcal{O}_x, s_*F[1]) \cong \operatorname{Hom}_Z(\mathcal{O}_x, F[1]) \oplus \operatorname{Hom}_Z(\mathcal{O}_x, F) = 0$, therefore $F$ has depth 2 and by Auslander-Buchsbaum formula, $F$ is actually locally free.

However, this contradicts with Lemma 4.3.20 which says that $\pi_*E$ is supported on a $S \times \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$, where $S$ is a finite set of points. Therefore we conclude that $E$ is isomorphic to either $F_1(x)$ or $F_2(x)$ for some $x \in \mathbb{P}^1$. □

In summary, we have completed the description of the stable objects for $\sigma \in \mathcal{U}^n(\mathcal{A})^\Phi$:

**Theorem 4.3.23.** *(i) If $\sigma \in \mathcal{U}^n(\mathcal{A})_+^\Phi$, then the stable objects (up to a shift) are of special Kronecker type I, and the classes of stable objects (up to a sign) in $K_0(\mathcal{A})$*

*are ($n \in \mathbb{N}$)*

$$\begin{aligned} n\gamma_0 + (n+1)\gamma_1, &\quad (n+1)\gamma_0 + n\gamma_1, \\ n\gamma_2 + (n+1)\gamma_3, &\quad (n+1)\gamma_2 + n\gamma_3, \\ \gamma_0 + \gamma_1, &\quad \gamma_2 + \gamma_3. \end{aligned}$$

*(ii) if $\sigma \in \mathcal{U}^n(\mathcal{A})^\Phi_-$, then the stable objects (up to a shift) are of special Kronecker type II, and the classes of stable objects (up to a sign) in $K_0(\mathcal{A})$ are ($n \in \mathbb{N}$)*

$$\begin{aligned} n\gamma_1 + (n+1)\gamma_2, &\quad (n+1)\gamma_1 + n\gamma_2, \\ n\gamma_3 + (n+1)\gamma_0, &\quad (n+1)\gamma_3 + n\gamma_0, \\ \gamma_1 + \gamma_2, &\quad \gamma_3 + \gamma_0. \end{aligned}$$

*(iii) if $\sigma \in \mathcal{U}^n(\mathcal{A})^\Phi$ and $\phi(S_i) = \frac{1}{2}$ for each $i$, then the stable objects (up to a shift) are only $\{S_i\}_i$, and the classes of stable objects (up to a sign) in $K_0(\mathcal{A})$ are*

$$\gamma_0, \ \gamma_1, \ \gamma_2, \ \gamma_3.$$

*Proof.* We have proved the first two cases in the above. For the last case, we do induction on the length $l(E)$ of object $E \in \mathcal{A}$. When $l(E) = 1$, it is obvious. Then for $l(E) = n+1$, by taking the Jordan-Hölder filtration of $E$, we have short exact sequence

$$0 \to E' \to E \to S_i^{\oplus n_i} \to 0.$$

Then $E'$ is semistable by our induction hypothesis, note that $E'$ has the same phase as $S_i$, by using Lemma 4.3.11 we see that $E$ is also semistable of phase of $S_i$. Therefore we proved that there are no other stable objects other that $S_i$, $i = 0, \cdots, 3$. □

By applying $\mathbb{C}$-action we obtain the same description of stable objects for general stability conditions in $\mathcal{U}(\mathcal{A})^\Phi$.

## § 4.4 Space of invariant stability conditions

Recall from the introduction the subset $\Delta \subset \overline{K_0(\mathcal{A})}$ is defined to be the set of classes of stable objects for $\sigma \in \mathcal{U}(\mathcal{A})^\Phi$ in the quotient group $\overline{K_0(\mathcal{A})} = K_0(\mathcal{A})/K_0(\mathcal{A})^{-\varphi}$. By Theorem 4.3.23 $\Delta$ consists of the following elements:

$$\Delta = \big\{n \in \mathbb{Z} : n\gamma_0 + (n+1)\gamma_1, \ (n+1)\gamma_0 + n\gamma_1, \pm(\gamma_0 + \gamma_1)\big\}.$$

Recall also that

$$\mathcal{H}^{\mathrm{reg}} := \mathrm{Hom}(\overline{K_0(\mathcal{A})}, \mathbb{C}) \setminus \bigcup_{\boldsymbol{v} \in \Delta} \boldsymbol{v}^{\perp},$$

where $\boldsymbol{v}^{\perp} := \{Z \in \mathrm{Hom}(\overline{K_0(\mathcal{A})}, \mathbb{C}) | Z(\boldsymbol{v}) = 0\}$ is the hyperplane complement.

Note that $\mathcal{H}^{\mathrm{reg}}$ is the complement of a family of hyperplanes in $\mathrm{Hom}(\overline{K_0(\mathcal{A})}, \mathbb{C}) \cong \mathbb{C}^2$:

$$\begin{aligned} nZ(\gamma_0) + (n+1)Z(\gamma_1) &= 0 \\ (n+1)Z(\gamma_0) + nZ(\gamma_1) &= 0 \\ Z(\gamma_0) + Z(\gamma_1) &= 0 \end{aligned}$$

for $n \geq 0$ (see Figure 1.3).

In the final section we prove Theorem 1.0.5 in the introduction: the forgetful map

$$\mathcal{Z} : \left(\mathrm{Stab}(X)^{\Phi}\right)_0 \to \mathrm{Hom}(\overline{K_0(\mathcal{C})}, \mathbb{C})$$

factors through

$$\mathcal{Z} : \left(\mathrm{Stab}(X)^{\Phi}\right)_0 \to \mathcal{H}^{\mathrm{reg}}.$$

Moreover, the above is a covering map.

Recall that a continuous map $f : A \to B$ between topological spaces is called a covering map, if every point $b \in B$ has an open neighborhood $V \subset B$ such that the restriction of $f$ to each connected component of $f^{-1}(V)$ is a homeomorphism onto $V$.

We first analyze the boundary of $\mathcal{U}^n(\mathcal{A})^{\Phi}$. Recall $\{S_i\}_{i \in \mathbb{Z}_4}$ (Corollary 4.1.3) are the simple objects which generate $\mathcal{A}$ and $\gamma_i = [S_i]$ are their classes. By definition $\partial\, \mathcal{U}^n(\mathcal{A})^{\Phi}$ has four components of codimension-one submanifolds (real lines), which are

$$W_i^+ := \{Z(\gamma_i) = Z(\gamma_{i+2}) \in \mathbb{R}_{>0}\}, \quad W_i^- := \{Z(\gamma_i) = Z(\gamma_{i+2}) \in \mathbb{R}_{<0}\}$$

$i = 0,\ 1$. Though we cannot apply Lemma 2.2.18 directly, however, since we are deforming $\sigma$ while preserving the condition $Z(\gamma_i) = Z(\gamma_{i+2})$, the statement and proof are exactly the same as there.

**Lemma 4.4.1.** *(i) For any stability condition on $W_i^+$ ($i = 0,\ 1$) there exists an open neighborhood $V$ such that $V \subset \mathcal{U}^n(\mathcal{A})^{\Phi} \cup \mathcal{U}^n(L_{S_i} L_{S_{i+2}} \mathcal{A})^{\Phi}$. Similarly, for any stability condition on $W_i^-$ ($i = 0,\ 1$) there exists an open neighborhood $V$ such that $V \subset \mathcal{U}^n(\mathcal{A})^{\Phi} \cup \mathcal{U}^n(R_{S_i} R_{S_{i+2}} \mathcal{A})^{\Phi}$.*

*(ii) We have* $W_i^+ = \overline{\mathcal{U}^n}(\mathcal{A})^\Phi \cap \overline{\mathcal{U}^n}(\tau\mathcal{A})^\Phi$*, where* $\tau = \mathcal{T}$ *when* $i = 0$ *and* $\tau = \mathcal{T}_\Psi$ *when* $i = 1$*. Similarly,* $W_i^- = \overline{\mathcal{U}^n}(\mathcal{A})^\Phi \cap \overline{\mathcal{U}^n}(\tau\mathcal{A})^\Phi$*, where* $\tau = \mathcal{T}_\Psi^{-1}$ *when* $i = 0$ *and* $\tau = \mathcal{T}^{-1}$ *when* $i = 1$*.*

*Proof.* In the following proof we will repeatedly use Lemma 2.1.5 that if $\mathcal{A},\ \mathcal{A}' \subset \mathcal{D}$ are hearts of bounded t-structures and $\mathcal{A} \subset \mathcal{A}'$, then $\mathcal{A} = \mathcal{A}'$.

First we suppose $\sigma \in W_0^+$, that is $Z(\gamma_0) = Z(\gamma_2) \in \mathbb{R}_{>0}$. The objects $\widetilde{S_1} = \mathcal{T}(S_0)$ and $\widetilde{S_3} = \mathcal{T}(S_2)$ lie in $\mathcal{A}$, and are in the short exact sequences by the computations in Section 4.2:

$$0 \to S_{i-1}^{\oplus 2} \to \widetilde{S_i} \to S_i \to 0, \quad i = 1,\ 3 \tag{4.21}$$

where $2 = \dim_{\mathbb{C}} \operatorname{Ext}^1(S_i, S_{i-1})^*$. Since $\operatorname{Hom}(\widetilde{S_1}, S_0) = \operatorname{Hom}(\widetilde{S_3}, S_2) = 0$ the objects $\widetilde{S_i}$ lie in $\mathcal{P}(0,1)$, and by choosing a small enough open neighborhood $V$ of $\sigma$ we can assume this is the case for all stability conditions $(Z, \mathcal{P})$ of $V$. We can split $V$ into two pieces

$$V_+ = \{\mathrm{Im}Z(S_0) = \mathrm{Im}Z(S_2) > 0\}, \quad V_- = \{\mathrm{Im}\ Z(S_0) = \mathrm{Im}Z(S_2) \leq 0\}.$$

For $\sigma \in V_+$, we can shrink $V$ if necessarily such that $S_i \in \mathcal{P}(0,1)$ for all $i$. This shows that $\mathcal{A} \subset \mathcal{P}(0,1]$ for all stability conditions in $V_+$, therefore $\mathcal{P}(0,1] = \mathcal{A}$ and so $V_+ \subset \mathcal{U}^n(\mathcal{A})^\Phi$. On the other hand, for any stability condition $(Z, \mathcal{P}) \in V_-$ the objects $S_0$ and $S_2$ are in $\mathcal{P}(-1/2, 0]$, thus the heart $\mathcal{P}(0,1]$ contains the objects $S_0[1],\ S_2[1],\ \widetilde{S_1}$ and $\widetilde{S_3}$. Since these are the simple objects of the finite length category $L_{S_0}L_{S_2}\mathcal{A}$, therefore $\mathcal{P}(0,1] = L_{S_0}L_{S_2}\mathcal{A}$ and so $V_- \subset \mathcal{U}^n(L_{S_0}L_{S_2}\mathcal{A})^\Phi$. Therefore $V \subset \mathcal{U}^n(\mathcal{A})^\Phi \cup \mathcal{U}^n(L_{S_0}L_{S_2}\mathcal{A})^\Phi$.

By applying $\Psi$ on $\sigma$ then $\Psi(\sigma) \in W_1^+$, that is $Z(\gamma_1) = Z(\gamma_3) \in \mathbb{R}_{>0}$, then there exists an open neighborhood $V$ of $\Psi(\sigma)$ such that $V \subset \mathcal{U}^n(\mathcal{A})^\Phi \cup \mathcal{U}^n(L_{S_1}L_{S_3}\mathcal{A})^\Phi$.

The proof for $\sigma \in W_i^-$ is essentially the same by replacing the left double tilt with the right double tilt.

For the second statement, by the first part we have

$$W_i^+ = \overline{\mathcal{U}^n}(\mathcal{A})^\Phi \cap \overline{\mathcal{U}^n}(L_{S_i}L_{S_{i+2}}\mathcal{A})^\Phi, \quad W_i^- = \overline{\mathcal{U}^n}(\mathcal{A})^\Phi \cap \overline{\mathcal{U}^n}(R_{S_i}R_{S_{i+2}}\mathcal{A})^\Phi.$$

By Theorem 4.2.1 we obtain the results. □

We denote by $H$ the subgroup in $\mathrm{Aut}_*(\mathcal{C})$ (see Definition 4.1.12) generated by $\mathcal{T}$ and $\mathcal{T}_\Psi$.

**Proposition 4.4.2.**

$$\bigcup_{g \in H} \overline{\mathcal{U}^n}(g\mathcal{A})^\Phi = \left(\mathrm{Stab}(\mathcal{C})^\Phi\right)_n. \tag{4.22}$$

*Proof.* Since $t(\delta) = t_\psi(\delta) = \delta$ due to Lemma 4.2.2, therefore $\overline{\mathcal{U}^n}(g\mathcal{A})^\Phi \subset \left(\mathrm{Stab}(\mathcal{C})^\Phi\right)_n$ for any $g \in H$.

Now we show that the left side is open and closed, hence the inclusion is in fact an equality.

First we prove the *openess*. For any $\sigma \in \overline{\mathcal{U}^n}(g\mathcal{A})^\Phi$, considering the preimage of $\sigma$ under the autoequivalence $g$, $\sigma = (Z, \mathcal{P})$ lies in $\overline{\mathcal{U}^n}(\mathcal{A})^\Phi$. Suppose first that $\mathrm{Im}Z(\gamma_i) > 0$ for each $i$, then we can choose an open neighborhood $U$ of $\sigma$ such that each simple object $S_i$ has phase $(0, 1)$ for all stability conditions $(Z, \mathcal{P})$ of $U$. Since $\mathcal{A}$ is the smallest extension-closed subcategory of $\mathcal{D}$ containing $S_i$ it follows that $\mathcal{A} \subset \mathcal{P}(0, 1]$ of all stability conditions in $U$. Therefore $\mathcal{P}(0, 1] = \mathcal{A}$ by Lemma 2.1.5 and so $U$ is contained in $\mathcal{U}^n(\mathcal{A})^\Phi$.

Now suppose $\sigma$ lies on the boundary of $\mathcal{U}^n(\mathcal{A})^\Phi$, according to Lemma 4.4.1, there is an open neighborhood $V$ of $\sigma$ such that $V \subset \mathcal{U}^n(\mathcal{A})^\Phi \cup \mathcal{U}^n(\tau\mathcal{A})^\Phi$ where $\tau$ is one of the autoequivalences $\mathcal{T}^{\pm 1}$ and $\mathcal{T}_\Psi^{\pm 1}$. This finishes the proof of openess.

To check the left side of (4.22) is closed, we only need to show the collection of closed sets is locally finite. Suppose

$$\sigma \in \bigcap_{g \in H' \subset H} \overline{\mathcal{U}^n}(g\mathcal{A})^\Phi.$$

It is obvious that $\mathcal{U}^n(g\mathcal{A})^\Phi \cap \mathcal{U}^n(g'\mathcal{A})^\Phi = \emptyset$ if $g\mathcal{A} \neq g'\mathcal{A}$. Taking the preimage under some autoequivalence $g$, we suppose $\sigma$ lies on the boundary of $\mathcal{U}^n(\mathcal{A})^\Phi$. Then this intersection is finite since by Lemma 4.4.1, each boundary component corresponds to exactly one of the autoequivalences $\mathcal{T}^{\pm 1}$ and $\mathcal{T}_\Psi^{\pm 1}$.

This finishes the proof of the equality (4.22). $\square$

**Lemma 4.4.3.** *For any special stability condition* $\sigma = (Z, \mathcal{P}) \in \left(\mathrm{Stab}(X)^{\Phi}\right)_0$*, we have* $Z(\delta) \neq 0$ *where* $\delta = [\mathcal{O}_x]$ *for* $x \in Z$*. Moreover, there is an exact equivalence* $W \in H$ *and* $\lambda \in \mathbb{C}$ *such that* $\lambda W(\sigma) \in \overline{\mathcal{U}}(\mathcal{A})^{\Phi}$*, the closure of* $\mathcal{U}(\mathcal{A})^{\Phi}$*.*

*Proof.* Suppose there exists $\tau = (Z_1, \mathcal{P}_1) \in \left(\mathrm{Stab}(X)^{\Phi}\right)_0$ such that $Z_1(\delta) = 0$. In particular there is no semistable object in $\tau$ whose class in $K_0(\mathcal{A})$ is $\delta$. By Lemma 2.2.16 this is true in an open neighborhood of $\tau$ in $\left(\mathrm{Stab}(X)^{\Phi}\right)_0$. Consider the stability condition $\sigma = (Z_2, \mathcal{P}_2)$ in such neighborhood such that $Z_2(\delta) \neq 0$. First we act by some $\lambda \in \mathbb{C}$ so that $\lambda \cdot Z_2(\delta) = i$. Then by (4.22) there exists some $W \in H$ such that $\sigma' = \lambda W(\sigma) \in \overline{\mathcal{U}^n}(\mathcal{A})^{\Phi}$. By Remark 4.3.13 and Lemma 2.2.16 there does exist a semistable object whose class is $\delta$ for $\sigma'$. Since $\delta$ is preserved by element of $H$, therefore we conclude that there exists a semistable object $E$ for $\sigma$ whose class in $K_0(\mathcal{A})$ is $\delta$. This gives a contradiction.

Therefore for any stability condition $\sigma = (Z, \mathcal{P})$, we have $Z(\delta) \neq 0$. Then by choosing $\lambda \in \mathbb{C}$ such that $\lambda \cdot Z(\delta) = i$, and by (4.22) we can find $W \in H$ such that $\lambda W(\sigma) \in \overline{\mathcal{U}^n}(\mathcal{A})^{\Phi} \subset \overline{\mathcal{U}}(\mathcal{A})^{\Phi}$. This finishes the proof. □

We let $t$ and $t_{\psi}$ be the automorphisms of $\overline{K_0(\mathcal{A})}$ induced by $\mathcal{T}$ and $\mathcal{T}_{\Psi}$, the following lemma is an easy consequence of Lemma 4.2.2:

**Lemma 4.4.4.** *The automorphisms* $t^{\pm 1}$ *and* $t_{\psi}^{\pm 1}$ *preserve* $\Delta$*.*

*Proof.* There are the following equivalent classes (up to a sign) in $\Delta \subset \overline{K_0(\mathcal{A})}$:

$$n\gamma_0 + (n+1)\gamma_1, \quad (n+1)\gamma_0 + n\gamma_1, \quad \gamma_0 + \gamma_1. \tag{4.23}$$

We apply Lemma 4.2.2 to the following calculations, for $t$, we have

$$\begin{aligned} t\left(n\gamma_0 + (n+1)\gamma_1\right) &= (n-1)\gamma_0 + n\gamma_1; \\ t\left((n+1)\gamma_0 + n\gamma_1\right) &= (n+2)\gamma_0 + (n+1)\gamma_1; \\ t\left(\gamma_0 + \gamma_1\right) &= \gamma_0 + \gamma_1. \end{aligned}$$

Therefore $t\Delta \subset \Delta$. Similarly, for $t_\psi$ we have

$$\begin{aligned} t_\psi\left(n\gamma_0 + (n+1)\gamma_1\right) &= (n+1)\gamma_0 + (n+2)\gamma_1; \\ t_\psi\left((n+1)\gamma_0 + n\gamma_1\right) &= n\gamma_0 + (n-1)\gamma_1; \\ t_\psi\left(\gamma_0 + \gamma_1\right) &= \gamma_0 + \gamma_1. \end{aligned}$$

Therefore $t_\psi\Delta \subset \Delta$. Due to equation (4.11) we actually verified all the cases. □

**Corollary 4.4.5.** *Let $E$ be a stable object for $\sigma \in \left(\mathrm{Stab}(X)^\Phi\right)_0$, then the class $[E] \in \overline{K_0(\mathcal{A})}$ lies in $\Delta$.*

*Proof.* By the above lemma, there is an autoequivalence $W \in H$ and $\lambda \in \mathbb{C}$ such that $\lambda W(\sigma)$ lies in the closure of $\mathcal{U}(\mathcal{A})^\Phi$. Since the stable objects remain stable in an open neighborhood $V$ of $\lambda W(\sigma)$, we choose $\sigma' \in V$ such that $\sigma' \in \mathcal{U}(\mathcal{A})^\Phi$, it follows that $\sigma'$ and $\lambda W(\sigma)$ contain the same set of stable objects. Therefore the classes of stable objects for $\lambda W(\sigma)$ lie in $\Delta$ by Theorem 4.3.23. Since by Lemma 4.4.4 the group element in $H$ preserves $\Delta$, therefore $[E] \in \Delta$. □

**Theorem 4.4.6.** *The image of the local homeomorphism*

$$\mathcal{Z} : \left(\mathrm{Stab}(X)^\Phi\right)_0 \to \mathrm{Hom}(\overline{K_0(\mathcal{A})}, \mathbb{C})$$

*lies in $\mathcal{H}^{\mathrm{reg}}$.*

*Proof.* By Corollary 4.4.5, the set of class of any stable object $E$ for stability condition $\sigma = (Z, \mathcal{P})$ is exactly $\Delta$. Since $Z(E) \neq 0$, therefore $\mathcal{Z}(\sigma) \in \mathcal{H}^{\mathrm{reg}}$. □

We fix a norm $\|\cdot\|$ on $\overline{K_0(\mathcal{A})}_\mathbb{R} = \overline{K_0(\mathcal{A})} \otimes_\mathbb{Z} \mathbb{R}$. The induced norm on $\mathrm{Hom}(\overline{K_0(\mathcal{A})}, \mathbb{C})$ is denoted by $\|\cdot\|^\vee$.

**Lemma 4.4.7.** *Let $Z \in \mathcal{H}^{\mathrm{reg}}$, there exists a constant $C > 0$ (depending on $Z$) such that*

$$\|\boldsymbol{v}\| \leq C|Z(\boldsymbol{v})| \tag{4.24}$$

*for all $\boldsymbol{v} \in \Delta$.*

*Proof.* Since all norms over finite dimensional space are equivalent, we might take

$$\|\boldsymbol{v}\|^2 = v_0^2 + v_1^2,$$

where $v_i$ denotes the $i$-th component of a vector $\boldsymbol{v} \in \overline{K_0(\mathcal{A})}_{\mathbb{R}}$ with respect to the basis $\gamma_0,\ \gamma_1$. For any vector $\boldsymbol{v}$ with the class in $\Delta$, we have

$$\|\boldsymbol{v}\|^2 = n^2 + (n+1)^2 \text{ or } 2,$$

for $n \geq 0$. Suppose $\|\boldsymbol{v}\|^2 = 2$, in this case $\boldsymbol{v} = \gamma_0 + \gamma_1$. Since $Z(\gamma_0 + \gamma_1) \neq 0$ by definition. Therefore we can choose a constant $c_0$ such that (4.24) holds.

Suppose $\arg Z(\gamma_0) \neq \arg Z(\gamma_1)$. Since $Z(\gamma_i) \neq 0$, without loss of generality we take a suitable $\mathrm{GL}(2,\mathbb{R})$-action on the complex plane, such that $Z(\gamma_0) = 1,\ Z(\gamma_1) = i$. Now suppose $\|\boldsymbol{v}\|^2 = n^2 + (n+1)^2$, then in this case $\boldsymbol{v} = n\gamma_0 + (n+1)\gamma_1$ or $(n+1)\gamma_0 + n\gamma_1$ in $\Delta$, therefore $|Z(\boldsymbol{v})|^2 = n^2 + (n+1)^2$. So we have

$$\|\boldsymbol{v}\| = |Z(\boldsymbol{v})|.$$

We can take $c_1 \geq 1$.

Finally we choose the maximum from $c_0$ and $c_1$ such that (4.24) holds for any $\boldsymbol{v} \in \Delta$. $\square$

**Theorem 4.4.8** (Covering property)**.** *$\mathcal{Z} : \left(\mathrm{Stab}(X)^{\Phi}\right)_0 \to \mathcal{H}^{\mathrm{reg}}$ is a covering map.*

*Proof.* We follow the idea in [17, Proposition 8.3]. We first show that $\mathcal{H}^{\mathrm{reg}}$ is open. Let $Z \in \mathcal{H}^{\mathrm{reg}}$. Lemma 4.4.7 shows that there is a constant $C > 0$ (depending on $Z$) such that

$$\|\boldsymbol{v}\| \leq C|Z(\boldsymbol{v})|$$

for all $\boldsymbol{v} \in \Delta$. Given $\epsilon > 0$, we define an open subset

$$B_\epsilon(Z) = \left\{ W \in \mathrm{Hom}(\overline{K_0(\mathcal{A})}, \mathbb{C}) : \|W - Z\|^{\vee} < \epsilon/C \right\} \subset \mathrm{Hom}(\overline{K_0(\mathcal{A})}, \mathbb{C}).$$

Then for $W \in B_\epsilon(Z)$, we have

$$|W(\boldsymbol{v}) - Z(\boldsymbol{v})| \leq \|W - Z\|^{\vee}\|\boldsymbol{v}\| < \epsilon|Z(\boldsymbol{v})|$$

for $\boldsymbol{v} \in \Delta$. Therefore if $\epsilon < 1$ then any $W \in B_\epsilon(Z)$ satisfies $W(\boldsymbol{v}) \neq 0$ for $\boldsymbol{v} \in \Delta$. Hence $W \in \mathcal{H}^{\mathrm{reg}}$, this shows that $\mathcal{H}^{\mathrm{reg}}$ is open.

Now we fix a positive real number $\epsilon_0 < \frac{1}{8}$ and assume that $\epsilon < \sin(\pi\epsilon_0)$. Given any $\sigma = (\mathcal{Z}, \mathcal{P}) \in \left(\mathrm{Stab}(X)^{\Phi}\right)_0$ with $\mathcal{Z}(\sigma) = Z$, we define the open neighborhood of $\sigma$

$$C_\epsilon(\sigma) = \left\{ \tau = (W, \mathcal{Q}) \in \mathcal{Z}^{-1}(B_\epsilon(Z)) : d(\mathcal{P}, \mathcal{Q}) < 1/2 \right\},$$

where $d(-,-)$ is defined in Definition 2.2.2. By Lemma 2.2.7, the map

$$\mathcal{Z} : C_\epsilon(\sigma) \to B_\epsilon(Z) \tag{4.25}$$

is injective. Let $W \in B_\epsilon(Z)$, then for any $E$ stable for $\sigma$, by Corollary 4.4.5, we have

$$|W(E) - Z(E)| < \sin(\pi\epsilon_0)|Z(E)|.$$

Using the deformation result Theorem 2.2.11, we conclude that there is a unique stabilty condition $\tau = (W, \mathcal{P}') \in C_\epsilon(\sigma)$ such that $\mathcal{Z}(\tau) = W$ and $d(\mathcal{P}, \mathcal{P}') < \epsilon$. Thus the map (4.25) is a homeomorphism. For each $\sigma \in \mathcal{Z}^{-1}(Z)$, we prove $C_\epsilon(\sigma)$ is mapped homeomorphically by $\mathcal{Z}$ onto $B_\epsilon(Z)$ exactly in the same way.

Finally we check that

$$\mathcal{Z}^{-1}(B_\epsilon(Z)) = \bigcup_{\sigma \in \mathcal{Z}^{-1}(Z)} C_\epsilon(\sigma) \tag{4.26}$$

is disjoint. Suppose there exists $\tau = (W, \mathcal{Q}) \in C_\epsilon(\sigma) \bigcap C_\epsilon(\sigma')$, where we denote by $\sigma = (Z, \mathcal{P})$ and $\sigma' = (Z, \mathcal{P}')$. Then

$$d(\mathcal{P}, \mathcal{P}') \le d(\mathcal{P}, \mathcal{Q}) + d(\mathcal{Q}, \mathcal{P}') < 1.$$

Therefore by Lemma 2.2.7 again, we have $\sigma = \sigma'$, which means $C_\epsilon(\sigma) = C_\epsilon(\sigma')$. We have finished the proof. □

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