On representation-theoretic properties of fermionic fields in de Sitter spacetime and symmetries underlying the conservation of the electromagnetic zilches

Vasileios A. Letsios

PhD

University of York
Mathematics

August 2023

Abstract

This journal-style thesis presents chapters 2-6 in a format suitable for peer-reviewed publication. In chapter 2, we study the quantum spinor field on N-dimensional de Sitter spacetime (dS_N) with $(N-1)$-sphere (S^{N-1}) spatial sections. We construct the mode solutions and study their transformation properties under the de Sitter (dS) algebra, spin$(N,1)$. We reproduce the expression for the massless spinor Wightman two-point function using the mode-sum method. Then, taking advantage of the maximal symmetry of dS_N, we construct the massive Wightman two-point function. In chapter 3, we construct the dictionary between the spaces of mode solutions for totally symmetric spin-$s=3/2, 5/2$ tensor-spinors with any mass parameter on dS_N $(N \geq 3)$ and Unitary Irreducible Representations (UIRs) of spin$(N,1)$. Remarkably, we find that the strictly massless spin-3/2 field, as well as the strictly and partially massless spin-5/2 fields on dS_N, are not unitary unless $N=4$. Chapter 4 provides a technical explanation for the results of chapter 3 by investigating the (non-)existence of positive-definite, dS invariant scalar products for the mode solutions. In chapter 5, we uncover a 'conformal-like' spin$(4,2)$ symmetry for strictly massless spin-$s \geq 3/2$ tensor-spinors on dS_4. We also show that the mode solutions form UIRs of not only the dS algebra but also of spin$(4,2)$. In chapter 6, we shift focus to the 'zilches', a set of little-known conserved quantities for the free electromagnetic (EM) field in four-dimensional Minkowski spacetime. We present, for the first time, the derivation of all zilch conservation laws from 'zilch symmetries' of the standard EM action using Noether's theorem. We also show that the zilch symmetries belong to the enveloping algebra of a "hidden" invariance algebra of free Maxwell's equations in potential form.

Contents

Acknowledgments

I would like to thank my supervisor Atsushi Higuchi for guidance, encouragement, long discussions, and for patiently answering all of my questions and teaching me essentially everything that I know about dS field theory and representation theory. Also, thank you for listening to me and generously sharing your insight. Your contribution to the field, as well as your exceptionally ethical stance towards research, have been - and will be - an endless source of inspiration for me, not only as a researcher but also as a person. It was an honor to be your student.

Special thanks to Harkan and Emma for being great friends and for their heartwarming support during the last months.

I would also like to thank all members of the Department of Mathematics of the University of York. Special thanks to Vila, Rutvij, Andrew, Ambroise, Harkan (again), Ding, Shashaank, Vincenzo, Laura, Diego, David, Ali, Jenny, Max, Cordelia, Sam, Art'om and Kuntal for their friendship and help, as well as for discussions and for being patient while I was talking about de Sitter. Ambroise thank you for preparing this thesis template (and for your great help).

It is also a great pleasure and an honor to thank Mitya Pushkin, Ed Corrigan, Bernard Kay, and Chris Fewster for encouragement, and guidance, as well as for invaluable and intellectually stimulating and engaging discussions. I would also like to thank Benoit Vicedo and Kasia Rejzner for useful discussions.

I am deeply grateful to Charis Anastopoulos for long discussions and continuous encouragement during the last 6 (or even more) years. Thank you for being a great mentor, and a constant source of inspiration, as well as for generously sharing your physical and philosophical insight. Thank you.

I am also grateful to Spyros Konitopoulos for 'infecting' me with the love for symmetries.

Your inspiring lectures at the University of Patras made me fall in love with the representation theory of Lie algebras. Thank you for your friendship, encouragement, help, and hospitality, and for our interesting discussions.

I would also like to thank all of my dear friends from Greece: Kotsos (Konstantinos), Konstantinos (Ntinos) Karv , Aristotelis, Ntinos ('rebetis'), Nikos Kouts, Nikos P., Yannis ('daskalos'), Jason, Ntan, Nikolas, Liana, Savas, Stavros and the VIP's crew. Special thanks to my dear fellow musicians Mitsos, Kir-Yannis, Petropoulos and Malliaris. Special thanks to Christos and Aggelos from Patras for many eye-opening discussions. Also, I thank Casey for being Casey.

Now is the time to be cliche. Spyros, Eva, Litsa and Alexis, thank you from the depths of my heart for your love and patience and for realizing a pleasant superposition of the form: family + friendship. I guess I cannot get luckier than this. I will not thank F.F. John here, because he gets spoiled.

Last, but not least, I would like to thank Alex for reminding me that there exist poetic qualities in life beyond poems, which was at the very least inspiring, for lack of a better wor(l)d. I am deeply grateful for this.

This work is dedicated to my dear friend Vasilis Kordopatis, to whom I did not have the chance to say goodbye. This work is also dedicated to all human and non-human victims of the wildfires burning in Greece as I am typing these words.

Author's declaration

I declare that in this thesis I present my original work and I am the sole author. The present work has not been previously submitted for any degree at the University of York, or any other University. All sources are acknowledged by explicit references.

Chapter 2 corresponds to the published paper: Letsios, V. A. **The eigenmodes for spinor quantum field theory in global de Sitter space-time,** Journal of Mathematical Physics **62**, 032303 (2021), https://doi.org/10.1063/5.0038651.
Chapter 3 corresponds to the published paper: Letsios, V. A. **(Non-)unitarity of strictly and partially massless fermions on de Sitter space,** J. High Energ. Phys. **2023**, 15 (2023) https://doi.org/10.1007/JHEP05(2023)015.
Chapter 4 has not been submitted for publication yet.
Chapters 3 and 4 arose by splitting into two parts a single early preprint: Letsios, V. A. **The (partially) massless spin-3/2 and spin-5/2 fields in de Sitter spacetime as unitary and non-unitary representations of the de Sitter algebra** https://doi.org/10.48550/arXiv.2206.09851.
Chapter 5 has not been submitted for publication yet (it has not been uploaded on the arXiv either).
Chapter 6 corresponds to the published paper: Letsios, V. A. **Conservation of all Lipkin's zilches from symmetries of the standard electromagnetic action and a hidden algebra,** Lett Math Phys **113**, 76 (2023) https://doi.org/10.1007/s11005-023-01697-5.

— 1 —

Introduction

The current thesis adopts a journal-style presentation, integrating the primary chapters (i.e., chapters 2-6) into a format suitable for publication in a peer-reviewed journal. Each of these chapters is self-contained, incorporating its own introduction and discussion sections, as well as background material and explanations for notation and conventions. Additionally, each chapter includes its own reference list.

In chapter 2, the mode solutions of the Dirac equation on N-dimensional de Sitter space-time (dS_N) with $(N-1)$-sphere spatial sections are obtained by analytically continuing the spinor eigenfunctions of the Dirac operator on the N-sphere (S^N). The analogs of flat space-time positive frequency modes are identified and a vacuum is defined. The transformation properties of the mode solutions under the de Sitter group double cover (Spin(N,1)) are studied. We reproduce the expression for the massless spinor Wightman two-point function in closed form using the mode-sum method. By using this closed-form expression and taking advantage of the maximal symmetry of dS_N we find an analytic expression for the spinor parallel propagator. The latter is used to construct the massive Wightman two-point function in closed form.

In chapter 3, we present the dictionary between the one-particle Hilbert spaces of totally symmetric tensor-spinor fields of spin $s = 3/2, 5/2$ with any mass parameter on dS_N and Unitary Irreducible Representations (UIR's) of the de Sitter (dS) algebra spin$(N,1)$. Our approach is based on expressing the eigenmodes on global dS_N in terms of eigenmodes of the Dirac operator on S^{N-1}, which provides a natural way to identify the corresponding representations with known UIR's under the decomposition spin$(N,1) \supset$ spin(N). Remarkably, we find that four-dimensional de Sitter space plays a distinguished role in the case of the gauge-invariant theories. In particular, the strictly massless spin-3/2 field, as well as the strictly and partially massless spin-5/2 fields on

dS_N, are not unitary unless $N = 4$.

In chapter 4, we provide a technical explanation for the results of chapter 3 by studying the (non-)existence of positive-definite, dS invariant scalar products for the spin-3/2 and spin-5/2 eigenmodes of the strictly/partially massless theories on dS_N $(N \geq 3)$. In particular, we show the following. For odd N, any dS invariant scalar product is identically zero. For even $N > 4$, any dS invariant scalar product must be indefinite. This gives rise to positive-norm and negative-norm eigenmodes that mix with each other under spin$(N, 1)$ boosts. In the $N = 4$ case, the positive-norm sector decouples from the negative-norm sector and each sector separately forms a UIR of spin$(4, 1)$. Our analysis makes extensive use of the analytic continuation of tensor-spinor spherical harmonics on S^N to dS_N.

In chapter 5, we present new infinitesimal 'conformal-like' symmetries for the field equations of strictly massless spin-$s \geq 3/2$ totally symmetric tensor-spinors (i.e. gauge potentials) on dS_4. The corresponding symmetry transformations are generated by the five conformal Killing vectors of dS_4, but they are not conventional conformal transformations. We show that the algebra generated by the ten dS symmetries and the five conformal-like symmetries closes on the conformal-like algebra spin$(4, 2)$ up to gauge transformations of the gauge potentials. Furthermore, we demonstrate that the two sets of physical mode solutions, corresponding to the two helicities $\pm s$ of the strictly massless theories, form a direct sum of UIR's of the conformal-like algebra. We also fill a gap in the literature by explaining how these physical modes form a direct sum of Discrete Series UIR's of the dS algebra spin$(4, 1)$.

In chapter 6, our attention shifts to the *zilches*, a set of conserved quantities in free electromagnetism. Among the zilches, *optical chirality* was identified by Tang and Cohen in 2010, serving as a measure of the handedness of light and leading to investigations into light's interactions with chiral matter. While the symmetries underlying the conservation of the zilches have been examined, the derivation of zilch conservation laws from symmetries of the standard free electromagnetic (EM) action using Noether's theorem has only been addressed in the case of optical chirality. We provide the full answer by demonstrating that the *zilch symmetry transformations* of the four-potential, A_μ, preserve the standard free EM action. We also show that the zilch symmetries belong to the enveloping algebra of a "hidden" invariance algebra of free Maxwell's equations. This "hidden" algebra is generated by familiar conformal transformations and certain "hidden" symmetry transformations of A_μ. Generalizations of the "hidden" symmetries are discussed in the presence of a material four-current, as well as in the theory of a complex Abelian gauge field. Additionally, we extend the zilch symmetries of the standard free EM action to the standard interacting action (with a non-dynamical four-current), allowing for a new derivation of the continuity equation for optical chirality in the presence of electric charges and currents. Furthermore, new continuity equations for the remaining zilches are derived.

While each of the main chapters (i.e., chapters 2-6) contains its own discussion section, in chapter 7 we further expand upon these discussions and present questions that could potentially guide future research.

— 2 —

The eigenmodes for spinor quantum field theory in global de Sitter space-time

Abstract

The mode solutions of the Dirac equation on N-dimensional de Sitter space-time (dS_N) with $(N-1)$-sphere spatial sections are obtained by analytically continuing the spinor eigenfunctions of the Dirac operator on the N-sphere (S^N). The analogs of flat space-time positive frequency modes are identified and a vacuum is defined. The transformation properties of the mode solutions under the de Sitter group double cover (Spin(N,1)) are studied. We reproduce the expression for the massless spinor Wightman two-point function in closed form using the mode-sum method. By using this closed-form expression and taking advantage of the maximal symmetry of dS_N we find an analytic expression for the spinor parallel propagator. The latter is used to construct the massive Wightman two-point function in closed form.

2.1 INTRODUCTION

The spinor functions that satisfy the eigenvalue equation of the Dirac operator on S^N

$$\nabla\!\!\!/ \psi = i\lambda\psi \tag{2.1}$$

have been studied by Camporesi and Higuchi [8]. More specifically, the eigenspinors on S^N have been recursively constructed in terms of eigenspinors on S^{N-1} using separation of variables in geodesic polar coordinates and their eigenvalues have been calculated.

The line element for S^N may be written as

$$ds_N^2 = d\theta_N^2 + \sin^2\theta_N ds_{N-1}^2, \tag{2.2}$$

where θ_N is the geodesic distance from the North Pole and ds_{N-1}^2 is the line element of S^{N-1}. Similarly, the line element of S^n $(n = 2, 3, ..., N-1)$ can be expressed as

$$ds_n^2 = d\theta_n^2 + \sin^2\theta_n ds_{n-1}^2, \tag{2.3}$$

while $ds_1^2 = d\theta_1^2$.
The N-dimensional de Sitter space-time is the maximally symmetric solution of the vacuum Einstein field equations with positive cosmological constant Λ [13]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0. \tag{2.4}$$

The cosmological constant is given by

$$\Lambda = \frac{(N-2)(N-1)}{2\,\mathscr{R}^2}, \tag{2.5}$$

where $\mathscr{R}$ is the de Sitter radius. Throughout this paper we use units in which $\mathscr{R} = 1$.
The N-dimensional de Sitter space-time can also be obtained by an "analytic continuation" of S^N. More specifically, by replacing

$$\theta_N \to x \equiv \pi/2 - it \tag{2.6}$$

in the S^N metric (2.2) we find the line element for dS_N with S^{N-1} spatial sections (see Eq. (2.10))

$$ds^2 = -dt^2 + \cosh^2 t\, ds_{N-1}^2. \tag{2.7}$$

Motivated by the above, one can obtain the mode solutions to the Dirac equation on dS_N

$$\nabla\!\!\!/\psi - M\psi = 0 \tag{2.8}$$

just by analytically continuing the eigenmodes of (2.1). The Dirac spinors obtained by analytic continuation can be used to describe spin-1/2 particles in de Sitter space-time and they form a representation of Spin(N,1). The latter has to be unitary to ensure that negative probabilities will not arise. In order to study the unitarity of the representation we are going to introduce a de Sitter invariant inner product among the analytically continued eigenspinors (see Sec. 2.5). Note that this approach has been previously applied for the divergence-free and traceless tensor eigenfunctions of the Laplace-Beltrami operator on S^N [15], where the restriction of unitarity gave rise to the forbidden mass range for the spin-2 field on dS_N.

Main aim. In this paper our main aim is the identification of the mode functions for the free Dirac field on global dS_N with S^{N-1} spatial sections. As a consistency check, we reproduce the expected form for the massless spinor Wightman function [19] using the mode-sum method. We also use this Wightman function to find an analytic expression for the spinor parallel propagator. To our knowledge, such an expression is absent from the literature. Solutions of the free Dirac equation on de Sitter space-time with static charts may be found in Ref. [20], with moving charts in Refs. [6, 24, 10] and with open charts in Ref. [16].

Outline. The rest of this paper is organized as follows. In Sec. 2.2 we discuss the global coordinate system that is relevant to the analytic continuation of S^N and we review the geodesic structure of dS_N. In Sec. 2.3 we present the basics about Dirac spinors and Clifford algebras on dS_N. In Sec. 2.4 we begin by reviewing the eigenspinors of the Dirac operator on S^N following Ref. [8]. Then we obtain the mode solutions of the Dirac equation on dS_N by analytically continuing the eigenmodes on S^N and we give a criterion for generalized positive frequency modes. We also construct spinors satisfying the Dirac equation with the sign of the mass term changed. These spinors are used in Appendix 2.9 for an alternative construction of the negative frequency modes via charge conjugation. In Sec. 2.5 we define a de Sitter invariant inner product among the analytically continued eigenmodes and we show that the associated norm is positive-definite (i.e. the representation is unitary). Using this norm we normalize the analytically continued eigenspinors. Then the transformation properties of the positive frequency solutions under Spin(N,1) are studied using the spinorial Lie derivative [18]. It is shown that the positive frequency solution subspace is Spin(N,1) invariant (hence, so is the corresponding vacuum). In Sec. 2.7, after presenting the negative frequency solutions of the Dirac equation, we perform the canonical quantization procedure for the free Dirac quantum field. Then we review the coordinate independent construction of Dirac spinor Green's functions on dS_N following Ref. [19]. We present a closed-form expression for the massless spinor Wightman two-point function obtained by the mode-sum method. This closed-form expression is in agreement with the construction given in Ref. [19]. Then we find an analytic expression for the spinor parallel propagator and we use it to obtain a closed-form expression for the massive Wightman two-point function in terms of intrinsic geometric objects. Our summary and concluding remarks are given in Sec. 2.8. There are six appendices. In Appendix 2.9 we construct the negative frequency solutions of the Dirac equation on dS_N by charge conjugating our analytically continued eigenspinors. In Appendix 2.14 we compare the mode-sum method for the massive spinor Wightman function with the construction presented in Ref. [19] and we arrive at a closed-form conjecture for a series containing the Gauss hypergeometric function. The rest of the appendices concern technical details. Some minor details omitted in the main text are presented in Appendices 2.10 and 2.11. In Appendix 2.12 we present details about the mode-sum construction of the massless spinor Wightman function. In Appendix 2.13 we demonstrate that our analytic expression for the spinor parallel propagator satisfies the defining properties given in Ref. [19].

Notation and conventions. We use the mostly plus convention for the metric signature. When it comes to tensors, lower case Greek indices refer to components with respect to the "coordinate basis" while Latin ones refer to components with respect to the vielbein (i.e. orthonormal frame) basis. Spinor indices (when not suppressed) are denoted with capital Latin letters. For bitensors (or bispinors) that depend on two space-time points x, x', unprimed indices refer to the tangent space at x while primed ones refer to the tangent space at x'. Summation over repeated indices is understood throughout this paper.

2.2 GEOMETRY OF N-DIMENSIONAL DE SITTER SPACE-TIME

2.2.1 Coordinate system, Christoffel symbols and spin connection

The N-dimensional de Sitter space-time can be represented as a hyperboloid embedded in $(N+1)$-dimensional Minkowski space. The de Sitter hyperboloid is described by

$$\eta_{ab} X^a X^b = 1, \tag{2.9}$$

where $\eta_{ab} = \text{diag}(-1, 1, 1, ..., 1)\,(a, b = 0, 1, ..., N)$ is the flat metric for the embedding space and $X^0, X^1, ..., X^N$ are the standard Minkowski coordinates. The global coordinates used in this paper are given by

$$\begin{aligned} X^0 &= X^0(t, \boldsymbol{\theta}) = \sinh t \\ X^i &= X^i(t, \boldsymbol{\theta}) = \cosh t \; Z^i, \quad i = 1, ..., N, \end{aligned} \tag{2.10}$$

where $t \in \mathbb{R}$, $\boldsymbol{\theta} = (\theta_{N-1}, \theta_{N-2}, ..., \theta_1)$ and the Z^i's are the spherical coordinates for S^{N-1} in N-dimensional Euclidean space

$$
\begin{aligned}
Z^1 &= \sin\theta_{N-1}\, \sin\theta_{N-2} \dots \sin\theta_2\, \sin\theta_1 \\
Z^2 &= \sin\theta_{N-1}\, \sin\theta_{N-2} \dots \sin\theta_2\, \cos\theta_1 \\
&\vdots \\
Z^{N-1} &= \sin\theta_{N-1}\, \cos\theta_{N-2} \\
Z^N &= \cos\theta_{N-1},
\end{aligned} \tag{2.11}
$$

where $0 \le \theta_1 < 2\pi$ and $0 \le \theta_i \le \pi$ $(i \ne 1)$. Using the coordinates (2.10) we obtain the line element (2.7) for dS_N.

The non-zero Christoffel symbols for the coordinates (2.10) are

$$
\begin{aligned}
&\Gamma^{t}_{\theta_i\theta_j} = \cosh t \sinh t\, \tilde{g}_{\theta_i\theta_j}, \quad \Gamma^{\theta_i}_{\theta_j t} = \tanh t\, \tilde{g}^{\theta_i}_{\theta_j}, \\
&\Gamma^{\theta_k}_{\theta_i\theta_j} = \tilde{\Gamma}^{\theta_k}_{\theta_i\theta_j},
\end{aligned} \tag{2.12}
$$

where $\tilde{g}_{\theta_i\theta_j}, \tilde{\Gamma}^{\theta_k}_{\theta_i\theta_j}$ are the metric tensor and the Christoffel symbols, respectively, on S^{N-1}. The vielbein fields are given by

$$
{e^t}_0 = 1, \qquad {e^{\theta_i}}_i = \frac{1}{\cosh t} {\tilde{e}^{\theta_i}}{}_i, \qquad i = 1, ..., N-1, \tag{2.13}
$$

where ${\tilde{e}^{\theta_i}}{}_i$ are the vielbein fields on S^{N-1}. The latter are given by

$$
\begin{aligned}
&{\tilde{e}^{\theta_{N-1}}}{}_{N-1} = 1, \\
&{\tilde{e}^{\theta_j}}{}_j = \frac{1}{\sin\theta_{N-1}\, \sin\theta_{N-2} \dots \sin\theta_{j+1}}, \quad j = 1, ..., N-2.
\end{aligned} \tag{2.14}
$$

The spin connection $\omega_{abc} = \omega_{a[bc]} \equiv (\omega_{abc} - \omega_{acb})/2$ is given by

$$
\omega_{abc} = {e^\mu}_a \Big(\partial_\mu {e^\lambda}_b + \Gamma^\lambda_{\mu\nu} {e^\nu}_b \Big) e_{\lambda c} \tag{2.15}
$$

and its only non-zero components are

$$
\omega_{ijk} = \frac{\tilde{\omega}_{ijk}}{\cosh t}, \quad \omega_{i0k} = \tanh t\, \delta_{ik} \quad i, j, k = 1, ..., N-1, \tag{2.16}
$$

where $\tilde{\omega}_{ijk}$ are the spin connection components on S^{N-1} and δ_{ij} is the Kronecker delta symbol. (Note that the sign convention we use for the spin connection is the opposite of the one used in most supersymmetry texts.)

2.2.2 GEODESICS ON dS_N

Geodesics on dS_N are obtained by intersecting the hyperboloid (2.9) with two-planes passing through the origin [7]. Note that, contrary to the case of maximally symmetric Euclidean spaces $(\mathbb{R}^N, S^N, H^N)$, on pseudo-Riemannian spaces two points cannot always be connected by a geodesic.

Let x, x' be two points on the de Sitter hyperboloid (2.9) and $\mu(x, x')$ the geodesic distance between them. Using the scalar product of the ambient space

$$\mathscr{Z}(x, x') = \eta_{ab} X^a(x) X^b(x') \tag{2.17}$$

one can define the useful quantity

$$z(x, x') = \frac{1}{2}\Big(1 + \eta_{ab} X^a(x) X^b(x')\Big). \tag{2.18}$$

If $-1 \leq \mathscr{Z}(x, x') < 1$ (i.e. $z \in [0, 1)$) the points x, x' are spacelike separated ($\mu \in \mathbb{R}$) and they can be connected by a spacelike geodesic. (The equality sign corresponds to antipodal points.) The geodesic distance is then defined by $\mathscr{Z}(x, x') = \cos(\mu(x, x'))$ or equivalently

$$z = \cos^2 \frac{\mu}{2}. \tag{2.19}$$

If $\mathscr{Z}(x, x') < -1$ (i.e. $z < 0$) the points are spacelike separated but there is no geodesic connecting them. However, the function $\mu(x, x')$ can still be defined by Eq. (2.18) via analytic continuation [3]. (Let $\bar{x}$ be the antipodal point of x and let x' be any point in the interior of the past or future light cone of $\bar{x}$. Then there is no geodesic connecting x and x' [3].) If $\mathscr{Z}(x, x') = 1$ (i.e. $z = 1$) the geodesic distance is zero and the two points can be connected by a null geodesic (or they coincide). If $\mathscr{Z}(x, x') > 1$ (i.e. $z > 1$) the two points are timelike separated ($\mu = i\kappa$, $\kappa \in \mathbb{R}$) and they can be connected by a timelike geodesic. The geodesic distance for timelike separation is given by

$$z = \cos^2 \frac{\mu}{2} = \cosh^2 \frac{\kappa}{2}. \tag{2.20}$$

In the rest of this paper we suppose that the points under consideration can be connected by a spacelike geodesic (unless otherwise stated). The corresponding results for the timelike case can be obtained just by replacing $\mu \to i\kappa$.

The unit tangent vectors at x and x' to the geodesic connecting the two points are defined by

$$n_\kappa(x, x') = \nabla_\kappa \mu(x, x'), \quad n_{\kappa'}(x, x') = \nabla_{\kappa'} \mu(x, x'), \tag{2.21}$$

respectively. Since dS_N is a maximally symmetric space-time, the unit tangents satisfy [3]

$$\nabla_\mu n_\nu = \cot\mu(g_{\mu\nu} - n_\mu n_\nu), \tag{2.22}$$

$$\nabla_{\mu'} n_\nu = -\frac{1}{\sin\mu}(g_{\mu'\nu} + n_{\mu'}\eta_\nu), \tag{2.23}$$

$$\nabla_\kappa g_{\mu\nu'} = \tan\frac{\mu}{2}(g_{\kappa\mu}n_{\nu'} + g_{\kappa\nu'}n_\mu), \tag{2.24}$$

where $g_{\mu\nu}(x)$ is the metric tensor and $g_{\mu\nu'}(x,x')$ is the bivector of parallel transport. The latter is also known as the vector parallel propagator and it performs the parallel transport of a vector field $V^{\nu'}(x')$ from x' to x along the geodesic connecting these points [3]

$$V_{||}^{\mu}(x) = g^{\mu}{}_{\nu'}V^{\nu'}(x'), \tag{2.25}$$

where $V_{||}^{\mu}(x)$ is the parallelly transported vector at x. (In this paper by geodesic we mean the shortest geodesic connecting the two points.) The covariant derivative of a vector field $W^\kappa(x)$ for an infinitesimal interval can be expressed using the vector parallel propagator as [9]

$$\nabla_\mu W^\kappa(x)dx^\mu = g^{\kappa}{}_{\nu'}(x, x+dx)\, W^{\nu'}(x+dx) - W^\kappa(x).$$

It is worth noting the relations [3]

$$n_\mu = -g_\mu{}^{\nu'}\, n_{\nu'}, \quad n_{\mu'} = -g_{\mu'}{}^{\nu}\, n_\nu, \tag{2.26}$$

$$g^{\mu}{}_{\nu'}\, g^{\nu'}{}_{\lambda} = \delta^{\mu}{}_{\lambda}, \quad g^{\mu'}{}_{\kappa}\, g^{\kappa}{}_{\nu'} = \delta^{\mu'}{}_{\nu'}. \tag{2.27}$$

Using the coordinates (2.10) we obtain the following expression for the geodesic distance:

$$\cos\Big(\mu(x,x')\Big) = -\sinh t\sinh t' + \cosh t\cosh t'\cos\Omega_{N-1}, \tag{2.28}$$

where

$$\cos\Omega_n = \cos\theta_n\cos\theta'_n + \sin\theta_n\sin\theta'_n\cos\Omega_{n-1}, \tag{2.29}$$

for $n = 2, ..., N-1$ and

$$\cos\Omega_1 = \cos(\theta_1 - \theta'_1). \tag{2.30}$$

Then the coordinate basis components of the tangent vector $n_\mu(x,x') = (n_t(x,x'), n_{\theta_i}(x,x'))$ $(i = 1, ..., N-1)$ are given by

$$n_t = \frac{1}{\sin\mu}(\cosh t\sinh t' - \sinh t\cosh t'\cos\Omega_{N-1}), \tag{2.31}$$

$$n_{\theta_i} = -\frac{1}{\sin\mu}\cosh t\cosh t'\frac{\partial}{\partial\theta_i}(\cos\Omega_{N-1}), \tag{2.32}$$

where

$$\frac{\partial}{\partial\theta_i}(\cos\Omega_{N-1}) = \left(\prod_{r=1}^{N-(i+1)} \sin\theta_{N-r}\sin\theta'_{N-r}\right) \times (-\sin\theta_i\cos\theta'_i + \cos\theta_i\sin\theta'_i\cos\Omega_{i-1}). \tag{2.33}$$

The components of $n_{\mu'}(x,x')$ are given by analogous expressions with $t \leftrightarrow t', \theta_i \leftrightarrow \theta'_i$. The vielbein basis components of the tangent vector at x, $n_a(x,x') = e^{\mu}{}_a(x)\, n_\mu(x,x')$ $(a = 0, 1, ..., N-1)$, are given by

$$n_0 = n_t, \tag{2.34}$$

$$n_{N-1} = -\frac{\cosh t'}{\sin\mu}(-\sin\theta_{N-1}\cos\theta'_{N-1} + \cos\theta_{N-1}\sin\theta'_{N-1}\cos\Omega_{N-2}), \tag{2.35}$$

$$n_b = -\frac{\cosh t'}{\sin\mu}\left(\prod_{r=1}^{N-(b+1)} \sin\theta'_{N-r}\right) \times (-\sin\theta_b\cos\theta'_b + \cos\theta_b\sin\theta'_b\cos\Omega_{b-1}), \tag{2.36}$$

$(b = 1, ..., N-2)$ while the components of $n_{a'}(x,x') = e^{\mu'}{}_{a'}(x')\, n_{\mu'}(x,x')$ $(a' = 0', 1', ..., (N-1)')$ can be obtained from Eqs. (2.34)-(2.36) with $t \leftrightarrow t'$, $\theta_a \leftrightarrow \theta'_a$. (Note that we define $\cos\Omega_0 \equiv 1$.)

2.3 DIRAC SPINORS AND CLIFFORD ALGEBRA ON N-DIMENSIONAL DE SITTER SPACE-TIME

Dirac spinors are $2^{[N/2]}$-dimensional column vectors that appear naturally in Clifford algebra representations, where $[N/2] = N/2$ if N is even and $[N/2] = (N-1)/2$ if N is odd. A Clifford algebra representation in $(N-1)+1$ dimensions is generated by N gamma matrices satisfying the anti-commutation relations

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbf{1}, \qquad a, b = 0, 1, ..., N-1, \tag{2.37}$$

where $\mathbf{1}$ is the identity matrix and η^{ab} is the inverse of the N-dimensional Minkowski metric $\eta_{ab} = \text{diag}(-1, +1, ..., +1)$. We follow the inductive construction of Ref. [8] where gamma matrices in $(N-1)+1$ dimensions are expressed in terms of spacelike gamma matrices in $(N-1)$ dimensions ($\widetilde{\gamma}^i$) as follows:

- For N even

$$\gamma^0 = i \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & i\widetilde{\gamma}^i \\ -i\widetilde{\gamma}^i & 0 \end{pmatrix}, \quad i = 1, ..., N-1, \tag{2.38}$$

where the lower-dimensional gamma matrices satisfy the Euclidean Clifford algebra anti-commutation relations

$$\{\widetilde{\gamma}^i, \widetilde{\gamma}^j\} = 2\delta^{ij}\mathbf{1}, \quad i, j = 1, ..., N-1. \tag{2.39}$$

- For N odd

$$\gamma^0 = i \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \quad \gamma^{N-1} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},$$

$$\gamma^j = \widetilde{\gamma}^j = \begin{pmatrix} 0 & i\widetilde{\widetilde{\gamma}}^j \\ -i\widetilde{\widetilde{\gamma}}^j & 0 \end{pmatrix}, \quad j = 1, ..., N-2. \tag{2.40}$$

The double-tilde is used to denote gamma matrices in $N-2$ dimensions. For $N=1$ the only (one-dimensional) gamma matrix is equal to 1.

Note that the gamma matrices we use here for dS_N can be obtained by the Euclidean gamma matrices on S^N used in Ref. [8] via the coordinate change (2.6). (Gamma matrices transform as vectors under coordinate transformations and it can be checked that all Euclidean γ^a's remain the same under (2.6) apart from γ^N; the latter transforms into the timelike gamma matrix: $\gamma^N \to i\gamma^N = \gamma^0$.)
Spinors transform under $2^{[N/2]}$-dimensional spinor representations of Spin($N-1$,1) (double cover of SO($N-1$,1)) as

$$\psi(x) \to S(\Lambda(x))\, \psi(x), \tag{2.41}$$

where $S(\Lambda(x)) \in$ Spin($N-1$,1) is a spinorial matrix. The $N(N-1)/2$ generators of Spin$(N-1,1)$ are given by the commutators

$$\Sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b] \tag{2.42}$$

$$= \frac{1}{2}\gamma^a\,\gamma^b - \frac{1}{2}\eta^{ab}, \quad a, b = 0, ..., N-1 \tag{2.43}$$

and they satisfy the Spin$(N-1,1)$ algebra commutation relations

$$[\Sigma^{ab}, \Sigma^{cd}] = \eta^{bc}\Sigma^{ad} - \eta^{ac}\Sigma^{bd} + \eta^{ad}\Sigma^{bc} - \eta^{bd}\Sigma^{ac}. \tag{2.44}$$

The covariant derivative for a spinor along the vielbein is

$$\nabla_a \psi = \boldsymbol{e}_a \psi - \frac{1}{2}\omega_{abc}\Sigma^{bc}\psi, \tag{2.45}$$

where $\boldsymbol{e}_a = e^{\mu}{}_{a}\partial_\mu$. The Dirac adjoint of a spinor is defined as

$$\bar{\psi} \equiv i\psi^\dagger\gamma^0$$

with covariant derivative given by

$$\nabla_a \bar{\psi} = \boldsymbol{e}_a \bar{\psi} + \frac{1}{2}\bar{\psi}\,\omega_{abc}\Sigma^{bc}. \tag{2.46}$$

The covariant derivative of the gamma matrices is

$$\begin{aligned}\nabla_a \gamma^k &= \boldsymbol{e}_a\gamma^k - \omega_a{}^k{}_c\gamma^c - \frac{1}{2}\omega_{abc}[\Sigma^{bc},\gamma^k] \\ &= 0.\end{aligned} \tag{2.47}$$

One can show the following properties of the gamma matrices given by Eqs. (2.38) and (2.40):

$$(\gamma^0)^T = \gamma^0, \;\; (\gamma^r)^T = (-1)^{[\frac{N}{2}]}(-1)^{[\frac{r}{2}]}\gamma^r, \tag{2.48}$$

$$(\gamma^0)^* = -\gamma^0, \;\; (\gamma^r)^* = (-1)^{[\frac{N}{2}]}(-1)^{[\frac{r}{2}]}\gamma^r, \tag{2.49}$$

$(r = 1, ..., N-1)$ and

$$(\gamma^a)^\dagger = \gamma^0\gamma^a\gamma^0, \qquad a = 0, ..., N-1, \tag{2.50}$$

where the star symbol denotes complex conjugation. Note that the timelike gamma matrix is anti-hermitian while the spacelike ones are hermitian.

2.4 SOLUTIONS OF THE DIRAC EQUATION ON N-DIMENSIONAL DE SITTER SPACE-TIME

We first present the basic results from Ref. [8] regarding the eigenmodes of the Dirac operator on S^N and then we perform analytic continuation for the two cases with N even and N odd.

Case 1: N even. The eigenvalue equation for the Dirac operator on S^N is

$$\nabla\!\!\!\!/\,\psi^{(s,\tilde{s})}_{\pm n\ell\sigma} = \pm i(n + \frac{N}{2})\psi^{(s,\tilde{s})}_{\pm n\ell\sigma}, \tag{2.51}$$

where $n = 0, 1, ...$ and $\ell = 0, ..., n$ are the angular momentum quantum numbers on S^N and S^{N-1} respectively. The index s indicates the two different spin projections ($s = \pm$). The symbol σ stands for the angular momentum quantum numbers $\ell_{N-2} \geq \ell_{N-3} \geq ... \geq \ell_2 \geq \ell_1 \geq 0$ on the lower-dimensional spheres while $\tilde{s}$ stands for the $(N/2-1)$ spin projection indices $s_{N-2}, s_{N-4}, ..., s_2$ on the lower-dimensional spheres $S^{N-2}, S^{N-4}, ..., S^2$ respectively. (Note that there exists one spin projection index for each lower-dimensional sphere of even dimension.) For each value of n we have a representation of Spin$(N+1)$ on the space of the eigenspinors $\psi^{(s,\tilde{s})}_{+n\ell\sigma}$ (or $\psi^{(s,\tilde{s})}_{-n\ell\sigma}$) with dimension [8]

$$d_n = \frac{2^{[N/2]}(N+n-1)!}{n!(N-1)!}. \tag{2.52}$$

The solutions of the eigenvalue equation for the Dirac operator on S^N (2.1) are found by writing the spinor ψ in terms of "upper" (φ_+) and "lower" (φ_-) components as follows:

$$\psi \equiv \begin{pmatrix} \varphi_+ \\ \varphi_- \end{pmatrix}. \tag{2.53}$$

By substituting Eq. (2.53) into Eq. (2.1) one obtains two coupled differential equations for φ_+, φ_-. By eliminating φ_+ (or φ_-) one finds [8]

$$\begin{aligned}\Big[\Big(\frac{\partial}{\partial\theta_N} + \frac{N-1}{2}\cot\theta_N\Big)^2 + \frac{1}{\sin^2\theta_N}\tilde{\nabla\!\!\!/}^2 \pm \frac{\cos\theta_N}{\sin^2\theta_N} i\tilde{\nabla\!\!\!/}\Big]\varphi_\pm \\ = -\lambda^2\varphi_\pm,\end{aligned} \tag{2.54}$$

where $\tilde{\nabla\!\!\!/}$ is the Dirac operator on S^{N-1}. (Equations (2.54) are equivalent to $\nabla\!\!\!/^2\psi = -\lambda^2\psi$.) Then, by separating variables, the normalized eigenspinors of $\nabla\!\!\!/|_{S^N}$ are found to be [8]

$$\psi^{(-,\tilde{s})}_{\pm n\ell\sigma}(\theta_N, \Omega_{N-1}) = \frac{c_N(n\ell)}{\sqrt{2}} \begin{pmatrix} \phi_{n\ell}(\theta_N)\chi^{(\tilde{s})}_{-\ell\sigma}(\Omega_{N-1}) \\ \pm i\psi_{n\ell}(\theta_N)\chi^{(\tilde{s})}_{-\ell\sigma}(\Omega_{N-1}) \end{pmatrix} \tag{2.55}$$

and

$$\psi^{(+,\tilde{s})}_{\pm n\ell\sigma}(\theta_N, \Omega_{N-1}) = \frac{c_N(n\ell)}{\sqrt{2}} \begin{pmatrix} i\psi_{M\ell}(\theta_N)\chi^{(\tilde{s})}_{+\ell\sigma}(\Omega_{N-1}) \\ \pm\phi_{M\ell}(\theta_N)\chi^{(\tilde{s})}_{+\ell\sigma}(\Omega_{N-1}) \end{pmatrix}, \tag{2.56}$$

where $\Omega_{N-1} \in S^{N-1}$ and the normalization factor is given by

$$|c_N(n\ell)|^2 = \frac{\Gamma(n-\ell+1)\Gamma(n+N+\ell)}{2^{N-2}|\Gamma(N/2+n)|^2}. \tag{2.57}$$

The eigenspinors on S^{N-1}, $\chi^{(\tilde{s})}_{\pm\ell\sigma}(\Omega_{N-1})$, satisfy the eigenvalue equation

$$\tilde{\nabla\!\!\!/}\chi^{(\tilde{s})}_{\pm\ell\sigma} = \pm i\left(\ell + \frac{N-1}{2}\right)\chi^{(\tilde{s})}_{\pm\ell\sigma}. \tag{2.58}$$

They are normalized by

$$\int_{S^{N-1}} d\Omega_{N-1} \chi^{(\tilde{s})}_{s\ell\sigma}(\Omega_{N-1})^\dagger \chi^{(\tilde{s}')}_{s'\ell'\sigma'}(\Omega_{N-1}) = \delta_{ss'}\delta_{\ell\ell'}\delta_{\sigma\sigma'}\delta_{\tilde{s}\tilde{s}'}, \tag{2.59}$$

while the eigenspinors on S^N are normalized by

$$\int_{S^N} d\Omega_N \psi^{(s,\tilde{s})}_{\pm n\ell\sigma}(\theta_N, \Omega_{N-1})^\dagger \psi^{(s',\tilde{s}')}_{\pm n'\ell'\sigma'}(\theta_N, \Omega_{N-1}) = \delta_{ss'}\delta_{nn'}\delta_{\ell\ell'}\delta_{\sigma\sigma'}\delta_{\tilde{s}\tilde{s}'}, \tag{2.60}$$

where all the ψ_+ eigenspinors are orthogonal to all the ψ_- eigenspinors. The functions $\phi_{n\ell}(\theta_N), \psi_{n\ell}(\theta_N)$ are given in terms of the Gauss hypergeometric function by

$$\begin{aligned}\phi_{n\ell}(\theta_N) = \kappa^{(N)}_\phi(n\ell) &\left(\cos\frac{\theta_N}{2}\right)^{\ell+1} \left(\sin\frac{\theta_N}{2}\right)^{\ell} \\ &\times F\left(n+N+\ell, -n+\ell; N/2+\ell; \sin^2\frac{\theta_N}{2}\right)\end{aligned} \tag{2.61}$$

and

$$\begin{aligned}\psi_{n\ell}(\theta_N) = \frac{\kappa^{(N)}_\phi(n\ell)\,(n+N/2)}{N/2+\ell} &\left(\cos\frac{\theta_N}{2}\right)^{\ell} \left(\sin\frac{\theta_N}{2}\right)^{\ell+1} \\ &\times F\left(n+N+\ell, -n+\ell; N/2+\ell+1; \sin^2\frac{\theta_N}{2}\right),\end{aligned} \tag{2.62}$$

where

$$\kappa^{(N)}_\phi(n\ell) = \frac{\Gamma(n+N/2)}{\Gamma(n-\ell+1)\Gamma(N/2+\ell)}. \tag{2.63}$$

The condition $n \geq \ell$ as well as the quantization of the eigenvalue of the Dirac operator $\lambda^2 = (n+N/2)^2$ $(n = 0, 1, \ldots)$ arise by requiring that the mode functions are not singular [8]. The functions $\phi_{n\ell}, \psi_{n\ell}$ are related to each other by

$$\begin{aligned}&\left[\frac{d}{d\theta_N} + \frac{N-1}{2}\cot\theta_N - \frac{1}{\sin\theta_N}\left(\ell + \frac{N-1}{2}\right)\right]\phi_{n\ell}(\theta_N) \\ &\quad = -\left(n + \frac{N}{2}\right)\psi_{n\ell}(\theta_N),\end{aligned} \tag{2.64}$$

$$\begin{aligned}&\left[\frac{d}{d\theta_N} + \frac{N-1}{2}\cot\theta_N + \frac{1}{\sin\theta_N}\left(\ell + \frac{N-1}{2}\right)\right]\psi_{n\ell}(\theta_N) \\ &\quad = +\left(n + \frac{N}{2}\right)\phi_{n\ell}(\theta_N).\end{aligned} \tag{2.65}$$

As mentioned in the Introduction, we can obtain the Dirac spinors which solve the Dirac equation $(\gamma^a \nabla_a - M)\psi = 0$ on dS_N by analytically continuing the eigenmodes of the

Dirac operator on S^N. The eigenvalues on S^N will be replaced by the spinor's mass M. It is easy to check that under the replacement $\theta_N \to \pi/2 - it$ one finds $\nabla\!\!\!/|_{S^N} \to \nabla\!\!\!/|_{dS_N}$. Without loss of generality, we choose to analytically continue the eigenspinors ψ_+ with the positive sign for the eigenvalue (see Eqs. (2.55)-(2.56)) by making the replacements

$$\theta_N \to x \equiv \pi/2 - it, \qquad n \to -iM - \frac{N}{2}. \tag{2.66}$$

The solutions of the Dirac equation on dS_N are then

$$\psi^{(-,\tilde{s})}_{M\ell\sigma}(t, \Omega_{N-1}) = \frac{c_N(M\ell)}{\sqrt{2}} \begin{pmatrix} \phi_{M\ell}(t)\chi^{(\tilde{s})}_{-\ell\sigma}(\Omega_{N-1}) \\ i\psi_{M\ell}(t)\chi^{(\tilde{s})}_{-\ell\sigma}(\Omega_{N-1}) \end{pmatrix} \tag{2.67}$$

and

$$\psi^{(+,\tilde{s})}_{M\ell\sigma}(t, \Omega_{N-1}) = \frac{c_N(M\ell)}{\sqrt{2}} \begin{pmatrix} i\psi_{M\ell}(t)\chi^{(\tilde{s})}_{+\ell\sigma}(\Omega_{N-1}) \\ \phi_{M\ell}(t)\chi^{(\tilde{s})}_{+\ell\sigma}(\Omega_{N-1}) \end{pmatrix}, \tag{2.68}$$

where $c_N(M\ell)$ is a normalization factor that will be determined later ($\ell = 0, 1, ...$). (Alternatively, we can choose to analytically continue the eigenspinors ψ_- in order to obtain the solutions (2.67)-(2.68) of the Dirac equation. In this case we need to make the replacements $\theta_N \to \pi/2 - it, n \to iM - N/2$ in Eqs. (2.55)-(2.56) instead of the replacements (2.66).) The un-normalized functions that describe the time dependence are

$$\begin{aligned} \phi_{M\ell}(t) =& \Big(\cos\frac{x}{2}\Big)^{\ell+1}\Big(\sin\frac{x}{2}\Big)^{\ell} \\ & \times F\left(\frac{N}{2} + \ell + iM, \frac{N}{2} + \ell - iM; \frac{N}{2} + \ell; \sin^2\frac{x}{2}\right) \end{aligned} \tag{2.69}$$

and

$$\begin{aligned} \psi_{M\ell}(t) = & \frac{-iM}{N/2 + \ell}\Big(\cos\frac{x}{2}\Big)^{\ell}\Big(\sin\frac{x}{2}\Big)^{\ell+1} \\ & \times F\left(\frac{N}{2} + \ell + iM, \frac{N}{2} + \ell - iM; \frac{N}{2} + \ell + 1; \sin^2\frac{x}{2}\right), \end{aligned} \tag{2.70}$$

where

$$\cos\frac{x}{2} = \frac{\sqrt{2}}{2}\left(\cosh\frac{t}{2} + i\sinh\frac{t}{2}\right), \tag{2.71}$$

$$\sin\frac{x}{2} = \frac{\sqrt{2}}{2}\left(\cosh\frac{t}{2} - i\sinh\frac{t}{2}\right), \tag{2.72}$$

$$\sin^2\frac{x}{2} = \frac{1 - i\sinh t}{2}. \tag{2.73}$$

It is clear from Eq. (2.70) that $\psi_{M\ell}(t)$ vanishes in the massless limit. Note the analytically continued version of Eqs. (2.64) and (2.65)

$$\begin{aligned}\left(\frac{d}{dt}+\frac{N-1}{2}\tanh t+\frac{i}{\cosh t}\left(\ell+\frac{N-1}{2}\right)\right)\phi_{M\ell}(t)\\ =+M\psi_{M\ell}(t),\end{aligned} \tag{2.74}$$

$$\begin{aligned}\left(\frac{d}{dt}+\frac{N-1}{2}\tanh t-\frac{i}{\cosh t}\left(\ell+\frac{N-1}{2}\right)\right)\psi_{M\ell}(t)\\ =-M\phi_{M\ell}(t).\end{aligned} \tag{2.75}$$

Using the following relation [11]:

$$F(a,b;c;z)=(1-z)^{c-a-b}F(c-a,c-b;c;z) \tag{2.76}$$

we can rewrite the functions $\phi_{M\ell},\psi_{M\ell}$ as

$$\begin{aligned}\phi_{M\ell}(t)=&\left(\cos\frac{x}{2}\right)^{-N-\ell+1}\left(\sin\frac{x}{2}\right)^{\ell}\\ &\times F\left(iM,-iM;\frac{N}{2}+\ell;\sin^2\frac{x}{2}\right)\end{aligned} \tag{2.77}$$

and

$$\begin{aligned}\psi_{M\ell}(t)=&\frac{-iM}{N/2+\ell}\left(\cos\frac{x}{2}\right)^{-N-\ell+2}\left(\sin\frac{x}{2}\right)^{\ell+1}\\ &\times F\left(iM+1,-iM+1;\frac{N}{2}+\ell+1;\sin^2\frac{x}{2}\right).\end{aligned} \tag{2.78}$$

The short wavelength limit ($\ell\gg 1$) of these functions can be found, by noting that the hypergeometric functions here tend to 1 in this limit, as

$$\frac{d}{dt}\phi_{M\ell}(t)\sim -i\frac{\ell}{\cosh t}\phi_{M\ell}(t), \tag{2.79}$$
$$\frac{d}{dt}\psi_{M\ell}(t)\sim -i\frac{\ell}{\cosh t}\psi_{M\ell}(t). \tag{2.80}$$

We see that the time derivative of our mode solutions (2.67) and (2.68) reproduces locally the positive frequency behaviour of flat space-time. Thus, our modes can serve as the analogs of the positive frequency modes and we can use this criterion as well as de Sitter invariance (see Sec. 2.5) in order to define a vacuum. Our positive frequency conditions (2.79) and (2.80) for the high frequency modes refer to the adiabatic condition [21] [note that $\ell/\cosh t$ is the physical (angular) momentum of the particle with angular momentum quantum number ℓ]. In the adiabatic limit, the high frequency modes experience a slowly varying scale factor $\cosh t$, i.e. $\ell/\cosh t >> |\tanh t|$, where $\tanh t$ is the expansion/contraction rate. Under these assumptions, we can find the approximate WKB solution to the second order equation satisfied by $\phi_{M\ell}(t)$ [8]:

$$\begin{aligned}&\left(\frac{\partial^2}{\partial x^2}+(N-1)\cot x\frac{\partial}{\partial x}+\left(\ell+\frac{N-1}{2}\right)\frac{\cos x}{\sin^2 x}\right.\\&\left.-\frac{(\ell+\frac{N-1}{2})^2-\frac{1}{4}(N-1)(N-3)}{\sin^2 x}-\frac{(N-1)^2}{4}\right)\phi_{M\ell}(t)\\&=M^2\phi_{M\ell}(t),\end{aligned}$$

where $x=\pi/2-it$. The adiabatic positive frequency form of $\phi_{M\ell}$ is the approximate WKB positive frequency solution to this equation and corresponds to Eq. (2.79).

Note that by making the replacements (2.66) in the expressions for the spinors ψ_- with the negative sign for the eigenvalue on S^N (see Eqs. (2.55)-(2.56)), we obtain the spinors

$$\psi^{(-,\tilde{s})}_{-M\ell\sigma}(t,\Omega_{N-1})=\begin{pmatrix}\phi_{M\ell}(t)\chi^{(\tilde{s})}_{-\ell\sigma}(\Omega_{N-1})\\-i\psi_{M\ell}(t)\chi^{(\tilde{s})}_{-\ell\sigma}(\Omega_{N-1})\end{pmatrix}\tag{2.81}$$

and

$$\psi^{(+,\tilde{s})}_{-M\ell\sigma}(t,\Omega_{N-1})=\begin{pmatrix}i\psi_{M\ell}(t)\chi^{(\tilde{s})}_{+\ell\sigma}(\Omega_{N-1})\\-\phi_{M\ell}(t)\chi^{(\tilde{s})}_{+\ell\sigma}(\Omega_{N-1})\end{pmatrix},\tag{2.82}$$

which are not solutions of the Dirac equation (2.8) on dS_N. However, these spinors satisfy the equation $\nabla\!\!\!/\,\psi_{-M}=-M\psi_{-M}$ and they serve as a tool in the construction of the negative frequency solutions of the Dirac equation (2.8) using charge conjugation in Appendix 2.9. (Note that the negative frequency solutions are obtained in two different ways: by separating variables in Sec. 2.6 and via charge conjugation in Appendix 2.9.)

Case 2: N odd. For the construction of the eigenmodes of Eq. (2.1) it is convenient to consider the eigenvalue equation for the iterated Dirac operator $\nabla\!\!\!/^2\psi=-\lambda^2\psi$. The latter may be written as follows [8]:

$$\begin{aligned}&\left[\left(\frac{\partial}{\partial\theta_N}+\frac{N-1}{2}\cot\theta_N\right)^2+\frac{1}{\sin^2\theta_N}\tilde{\nabla\!\!\!/}^2-\frac{\cos\theta_N}{\sin^2\theta_N}\gamma^N\tilde{\nabla\!\!\!/}\right]\psi\\&\qquad=-\lambda^2\psi.\end{aligned}\tag{2.83}$$

By separating variables, the spinor eigenfunctions of the Dirac operator on S^N are found to be [8]

$$\psi^{(s,\tilde{s})}_{\pm n\ell\sigma}(\theta_N,\Omega_{N-1}) = \frac{c_N(n\ell)}{\sqrt{2}} \times (\phi_{n\ell}(\theta_N)\hat{\chi}^{(s,\tilde{s})}_{-\ell\sigma}(\Omega_{N-1}) \pm i\psi_{n\ell}(\theta_N)\hat{\chi}^{(s,\tilde{s})}_{+\ell\sigma}(\Omega_{N-1})), \quad (2.84)$$

where

$$\hat{\chi}^{(s,\tilde{s})}_{-\ell\sigma} = \frac{1}{\sqrt{2}}(\mathbf{1} + i\gamma^N)\chi^{(s,\tilde{s})}_{-\ell\sigma} \quad (2.85)$$

and the eigenvalues are the same as in Eq. (2.51) (i.e. $\lambda = \pm(n + N/2)$ with $n = 0, 1, ...$). The spinors $\chi^{(s,\tilde{s})}_{+\ell\sigma}$ and $\hat{\chi}^{(s,\tilde{s})}_{+\ell\sigma}$ are given by

$$\gamma^N\chi^{(s,\tilde{s})}_{-\ell\sigma} = \chi^{(s,\tilde{s})}_{+\ell\sigma} \quad (2.86)$$

and

$$\hat{\chi}^{(s,\tilde{s})}_{+\ell\sigma} = \gamma^N\hat{\chi}^{(s,\tilde{s})}_{-\ell\sigma}. \quad (2.87)$$

Here s is the spin projection index on S^{N-1} and $\tilde{s}$ stands for the rest of the spin projection indices on the lower-dimensional spheres of even dimensions. The functions $\phi_{n\ell}, \psi_{n\ell}$ are given by Eqs. (2.61) and (2.62), while the spinors $\hat{\chi}^{(s,\tilde{s})}_{\pm\ell\sigma}(\Omega_{N-1})$ are eigenfunctions of the hermitian operator $\gamma^N\tilde{\nabla\!\!\!/}$ (that commutes with the iterated Dirac operator $\nabla\!\!\!/^2$) satisfying [8]

$$\gamma^N\tilde{\nabla\!\!\!/}\hat{\chi}^{(s,\tilde{s})}_{\pm\ell\sigma} = \pm\left(\ell + \frac{N-1}{2}\right)\hat{\chi}^{(s,\tilde{s})}_{\pm\ell\sigma}. \quad (2.88)$$

As in the even-dimensional case, for each value of n the eigenspinors $\psi^{(s,\tilde{s})}_{+n\ell\sigma}$ (or $\psi^{(s,\tilde{s})}_{-n\ell\sigma}$) form a representation of Spin$(N+1)$ with dimension d_n given by Eq. (2.52). (The dimension is half the dimension for the case with N even because there is no contribution from spin projections on S^N.) Notice that on S^1 the Dirac operator is just $\partial/\partial\theta_1$ and the eigenspinors are $\chi_{\pm\ell_1}(\theta_1) = \exp\{(\pm i\,(\ell_1 + 1/2)\theta_1)\}$ (the normalization constant is $(2\pi)^{-1/2}$). The eigenspinors (2.84) are normalized as in the case with N even and the normalization factors are given again by Eq. (2.57).

We choose to analytically continue the ψ_+ eigenmodes. By making the replacements (2.66) in the expression for the eigenspinors $\psi^{(s,\tilde{s})}_{+n\ell\sigma}(\theta_N,\Omega_{N-1})$ (Eq. (2.84)) we obtain the solutions of the Dirac equation on odd-dimensional dS_N

$$\psi^{(s,\tilde{s})}_{M\ell\sigma}(t,\Omega_{N-1}) = \frac{c_N(M\ell)}{\sqrt{2}} \times (\phi_{M\ell}(t)\hat{\chi}^{(s,\tilde{s})}_{-\ell\sigma}(\Omega_{N-1}) + i\psi_{M\ell}(t)\hat{\chi}^{(s,\tilde{s})}_{+\ell\sigma}(\Omega_{N-1})), \quad (2.89)$$

where the normalization factor will be determined later. The functions $\phi_{M\ell}(t), \psi_{M\ell}(t)$ are given again by Eqs. (2.69) and (2.70). Hence, the solutions (2.89) can be used as positive frequency modes.

As in the even-dimensional case, we can analytically continue the eigenspinors ψ_- to obtain

$$\psi^{(s,\tilde{s})}_{-M\ell\sigma}(t,\Omega_{N-1}) = (\phi_{M\ell}(t)\hat{\chi}^{(s,\tilde{s})}_{-\ell\sigma}(\Omega_{N-1}) - i\psi_{M\ell}(t)\hat{\chi}^{(s,\tilde{s})}_{+\ell\sigma}(\Omega_{N-1})), \tag{2.90}$$

which satisfy the Dirac equation (2.8) with $M \to -M$.

2.5 NORMALIZATION FACTORS AND TRANSFORMATION PROPERTIES UNDER SPIN(N,1) OF THE ANALYTICALLY CONTINUED EIGENSPINORS

For each value of M the set of the analytically continued eigenspinors of the Dirac operator $\nabla\!\!\!/|_{S^N}$ forms a representation of the Lie algebra of Spin(N,1) (which is also a representation of the group Spin(N,1)). If we want to use these mode functions to describe spin-1/2 particles on N-dimensional de Sitter space-time, the corresponding representation has to be unitary. Unitarity ensures that no negative probabilities will arise. A representation is unitary if there is a positive definite inner product that is preserved under the action of the group. In this section we show that the representation formed by our analytically continued eigenspinors is unitary by introducing a Spin(N,1) invariant inner product among the solutions of the Dirac equation and by verifying the positive-definiteness of the associated norm for our positive frequency solutions. In addition, we calculate the normalization factors $c_N(M\ell)$ and we show that the positive frequency modes transform among themselves under the action of a boost generator. In view of a mode expansion of the quantum Dirac field using our analytically continued modes, the transformation properties thus obtained imply that the corresponding vacuum is de Sitter invariant.

2.5.1 Unitarity of the Spin(N,1) representation and normalization factors

Let ψ and ψ' be any two Dirac spinors on a globally hyperbolic spacetime (global hyperbolicity is assumed for later convenience). The Dirac inner product of ψ, ψ' is then given by

$$(\psi, \psi') = i \int_\Sigma d\Sigma \, \hat{n}_\mu \; \overline{\psi} \gamma^\mu \psi', \tag{2.91}$$

where the integration is over any Cauchy surface Σ and $\hat{n}_\mu$ is the unit normal to the Cauchy surface with $\hat{n}_0 > 0$. Below we use this inner product in order to show that our positive frequency modes on dS_N have positive norm. (The Dirac inner product is also used in Sec. 2.6 in order to normalize the negative frequency solutions and show that the positive and negative frequency solution subspaces are orthogonal to each other.) Now let ψ, ψ' in Eq. (2.91) be positive frequency solutions of the Dirac equation on global dS_N with same mass M (see Eqs. (2.67)-(2.68)). Then the Dirac inner product (2.91) is written as

$$(\psi^{(s,\tilde{s})}_{M\ell\sigma}, \psi^{(s',\tilde{s}')}_{M\ell'\sigma'}) = i \int d\boldsymbol{\theta} \sqrt{-g} \; \overline{\psi}^{(s,\tilde{s})}_{M\ell\sigma} \gamma^0 \psi^{(s',\tilde{s}')}_{M\ell'\sigma'} \tag{2.92}$$

$$= \int d\boldsymbol{\theta} \sqrt{-g} \; \psi^{(s,\tilde{s})\dagger}_{M\ell\sigma} \psi^{(s',\tilde{s}')}_{M\ell'\sigma'}, \tag{2.93}$$

where the integration is over the Cauchy surface $\Sigma = S^{N-1}$ and $d\boldsymbol{\theta}$ stands for $d\theta_1 d\theta_2 ... d\theta_{N-1}$. The square root of the determinant of the de Sitter metric is

$$\sqrt{-g} = \cosh^{N-1} t \, \sin^{N-2} \theta_{N-1} \ldots \sin \theta_2 = \cosh^{N-1} t \sqrt{\tilde{g}}, \tag{2.94}$$

where $\tilde{g}$ is the determinant of the S^{N-1} metric. First, we show that the inner product (2.92) is both time independent and Spin(N,1) invariant. Let $\psi^{(1)}, \psi^{(2)}$ be two analytically continued eigenspinors which satisfy the Dirac equation (2.8). The Dirac equation and Eq. (2.47) imply that the vector current

$$J^\mu = i \overline{\psi}^{(1)} \gamma^\mu \psi^{(2)} \tag{2.95}$$

is covariantly conserved. Hence, the inner product (2.92) is time independent. As for the invariance under Spin(N,1), we can show that the change in the inner product due to infinitesimal Spin(N,1) transformations vanishes (as in Ref. [15]). Let ξ^μ be a Killing vector of dS_N satisfying

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0. \tag{2.96}$$

The Lie derivative of J^μ with respect to the Killing vector ξ^μ ($\mathscr{L}_\xi J^\mu$) gives the change in J^μ under the corresponding transformation; that is

$$\delta J^\mu = \mathscr{L}_\xi J^\mu = \xi^\nu \nabla_\nu J^\mu - J^\nu \nabla_\nu \xi^\mu$$
$$= \nabla_\nu (\xi^\nu J^\mu - J^\nu \xi^\mu), \tag{2.97}$$

where we used the fact that both J^μ, ξ^μ are divergence free. Then we find

$$\delta J^0 = \nabla_\nu(\xi^\nu J^0 - J^\nu \xi^0) = \frac{1}{\sqrt{-g}}\partial_{\theta_\kappa}\Big[\sqrt{-g}(\xi^{\theta_\kappa} J^0 - J^{\theta_\kappa}\xi^0)\Big], \tag{2.98}$$

where $\kappa = 1, ..., N-1$. By integrating Eq. (2.98) over S^{N-1} we find

$$\delta(\psi^{(1)}, \psi^{(2)}) = \int d\boldsymbol{\theta}\sqrt{-g}\,\delta J^0 = 0. \tag{2.99}$$

Below we study the positive-definiteness of the norm associated with the inner product (2.92) for our positive frequency modes.

Case 1: N even. Substituting the analytically continued eigenspinors (2.67) (or (2.68)) into the inner product (2.92) we find

$$(\psi^{(s,\tilde{s})}_{M\ell\sigma}, \psi^{(s',\tilde{s}')}_{M\ell'\sigma'}) = \frac{|c_N(M\ell)|^2}{2}\cosh^{N-1} t\Big(\phi^*_{M\ell}(t)\phi_{M\ell}(t) + \psi^*_{M\ell}(t)\psi_{M\ell}(t)\Big)\delta_{ss'}\delta_{\tilde{s}\tilde{s}'}\delta_{\ell\ell'}\delta_{\sigma\sigma'}, \tag{2.100}$$

where the positive-definiteness is obvious (i.e. the representation is unitary).
Using Eqs. (2.74) and (2.75) one finds

$$\begin{aligned}\frac{d}{dt}\Big[\cosh^{(N-1)/2} t\,\phi_{M\ell}\Big] = &- i\cosh^{(N-3)/2} t\,\left(\ell + \frac{N-1}{2}\right)\\ &\times \phi_{M\ell} + M\cosh^{(N-1)/2} t\,\psi_{M\ell},\end{aligned} \tag{2.101}$$

and

$$\begin{aligned}\frac{d}{dt}\Big[\cosh^{(N-1)/2} t\psi_{M\ell}\Big] = &+ i\cosh^{(N-3)/2} t\,\left(\ell + \frac{N-1}{2}\right)\\ &\times \psi_{M\ell} - M\cosh^{(N-1)/2} t\,\phi_{M\ell}\end{aligned} \tag{2.102}$$

respectively. Consequently

$$\cosh^{N-1} t\Big(\phi^*_{M\ell}(t)\phi_{M\ell}(t) + \psi^*_{M\ell}(t)\psi_{M\ell}(t)\Big) = K, \tag{2.103}$$

where K is a positive real constant (since the time derivative of the left-hand side vanishes). We can determine the value of K just by letting $t = 0$ in Eq. (2.103). The functions (2.69) and (2.70) for $t = 0$ are

$$\phi_{M\ell}(t=0) = \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right)^\ell F\left(\delta, \delta^*, \frac{\delta+\delta^*}{2}; \frac{1}{2}\right) \tag{2.104}$$

and

$$\psi_{M\ell}(t=0) = \frac{-i\sqrt{2}M}{N+2\ell}\left(\frac{1}{2}\right)^\ell F\left(\delta, \delta^*, \frac{\delta+\delta^*}{2}+1; \frac{1}{2}\right) \tag{2.105}$$

respectively, where

$$\delta = \frac{N}{2} + \ell + iM.$$

Using the following two formulas [1], [2]:

$$F\left(a,b,\frac{a+b}{2};\frac{1}{2}\right) = \sqrt{\pi}\Gamma\left(\frac{a+b}{2}\right)\left[\frac{1}{\Gamma((a+1)/2)\Gamma(b/2)} + \frac{1}{\Gamma((b+1)/2)\Gamma(a/2)}\right], \tag{2.106}$$

$$F\left(a,b,\frac{a+b}{2}+1;\frac{1}{2}\right) = \frac{2\sqrt{\pi}}{a-b}\Gamma\left(\frac{a+b}{2}+1\right)\left[\frac{1}{\Gamma((b+1)/2)} \times \frac{1}{\Gamma(a/2)} - \frac{1}{\Gamma((a+1)/2)\Gamma(b/2)}\right] \tag{2.107}$$

we find

$$\begin{aligned} K &= \phi^*_{M\ell}(0)\phi_{M\ell}(0) + \psi^*_{M\ell}(0)\psi_{M\ell}(0) \\ &= 2^{N-1}\frac{|\Gamma(\frac{N}{2}+\ell)|^2}{|\Gamma(\frac{N}{2}+\ell+iM)|^2}, \end{aligned} \tag{2.108}$$

where we also used the Legendre duplication formula

$$\Gamma(2z) = \frac{1}{\sqrt{\pi}}2^{2z-1}\Gamma(z)\Gamma(z+1/2). \tag{2.109}$$

Since $(\psi^{(s,\tilde{s})}_{M\ell\sigma}, \psi^{(s,\tilde{s})}_{M\ell\sigma}) = |c_N(M\ell)|^2 K/2$, it is straightforward to calculate the normalization factor as

$$|c_N(M\ell)|^2 = 2^{(2-N)}\frac{|\Gamma(\frac{N}{2}+\ell+iM)|^2}{|\Gamma(\frac{N}{2}+\ell)|^2}. \tag{2.110}$$

Our positive frequency solutions are now normalized by

$$(\psi^{(s,\tilde{s})}_{M\ell\sigma}, \psi^{(s',\tilde{s}')}_{M\ell'\sigma'}) = \delta_{ss'}\delta_{\tilde{s}\tilde{s}'}\delta_{\ell\ell'}\delta_{\sigma\sigma'}. \tag{2.111}$$

Case 2: N odd. Substituting the analytically continued eigenspinors (2.89) into the inner product (2.92) we obtain again Eq. (2.100). Thus, the Spin(N,1) representation is unitary (due to the positive-definiteness of the norm) and the normalization is again given by Eqs. (2.110) and (2.111).

2.5.2 Transformation properties of the positive frequency solutions under Spin(N,1)

In this section we use the spinorial Lie derivative [18] with respect to the Killing vector field ξ in order to study the Spin(N,1) transformations of the analytically continued modes of $\nabla\!\!\!/|_{S^N}$ generated by ξ. More specifically, we show that our positive frequency modes transform among themselves under the action of an infinitesimal boost in the θ_{N-1} direction.
The coordinate expression for the spinorial Lie derivative of a spinor field ψ with respect to the Killing vector ξ is [18]

$$\mathscr{L}^s_\xi \psi = \xi^\mu \nabla_\mu \psi + \frac{1}{4}\nabla_\kappa \xi_\lambda \gamma^\kappa \gamma^\lambda \psi. \tag{2.112}$$

(We use the superscript s to distinguish the spinorial Lie derivative from the usual Lie derivative.) We are interested in the transformation generated by the boost Killing vector

$$\xi = \cos\theta_{N-1}\frac{\partial}{\partial t} - \tanh t\, \sin\theta_{N-1}\frac{\partial}{\partial \theta_{N-1}}. \tag{2.113}$$

After a straightforward calculation we find

$$\mathscr{L}^s_\xi \psi = \xi^\mu \partial_\mu \psi + \frac{\sin\theta_{N-1}}{2\cosh t}\gamma^{N-1}\gamma^0\psi \tag{2.114}$$

$$= \cos\theta_{N-1}\partial_t\psi - \tanh t \sin\theta_{N-1}\partial_{\theta_{N-1}}\psi + \frac{\sin\theta_{N-1}}{2\cosh t}\gamma^{N-1}\gamma^0\psi. \tag{2.115}$$

The spinorial Lie derivative with respect to Killing vectors commutes with the Dirac operator [18]. Hence if ψ is an analytically continued eigenspinor of $\nabla\!\!\!/|_{S^N}$ we can express Eq. (2.115) as a linear combination of other such eigenspinors. In order to proceed, it is useful to introduce the ladder operators for the functions $\phi_{M\ell}(t), \psi_{M\ell}(t), \tilde{\phi}_{\ell\,\ell_{N-2}}(\theta_{N-1})$, $\tilde{\psi}_{\ell\,\ell_{N-2}}(\theta_{N-1})$ sending the angular momentum quantum number ℓ to $\ell\pm1$. (The functions $\tilde{\phi}_{\ell\,\ell_{N-2}}, \tilde{\psi}_{\ell\,\ell_{N-2}}$ are given by Eqs. (2.61) and (2.62) respectively, with $N\to N-1$, $n\to\ell$ and $\ell\to\ell_{N-2}$.) The ladder operators are given by the following expressions:

$$T^{(+)}_\phi = \frac{d}{dt} - \left(\ell+\frac{1}{2}\right)\tanh t - \frac{i}{2\cosh t}, \tag{2.116}$$

$$T^{(+)}_\psi = \frac{d}{dt} - \left(\ell+\frac{1}{2}\right)\tanh t + \frac{i}{2\cosh t}, \tag{2.117}$$

$$T^{(-)}_\phi = \frac{d}{dt} + \left(\ell+N-\frac{3}{2}\right)\tanh t + \frac{i}{2\cosh t}, \tag{2.118}$$

$$T^{(-)}_\psi = \frac{d}{dt} + \left(\ell+N-\frac{3}{2}\right)\tanh t - \frac{i}{2\cosh t}, \tag{2.119}$$

$$\tilde{T}^{(+)}_{\tilde{\phi}} = \sin\theta_{N-1}\frac{d}{d\theta_{N-1}} + \left(\ell + N - \frac{3}{2}\right)\cos\theta_{N-1} - \frac{\ell_{N-2} + \frac{N-2}{2}}{2(\ell + \frac{N}{2})}, \tag{2.120}$$

$$\tilde{T}^{(+)}_{\tilde{\psi}} = \sin\theta_{N-1}\frac{d}{d\theta_{N-1}} + \left(\ell + N - \frac{3}{2}\right)\cos\theta_{N-1} + \frac{\ell_{N-2} + \frac{N-2}{2}}{2(\ell + \frac{N}{2})} \tag{2.121}$$

$$\tilde{T}^{(-)}_{\tilde{\phi}} = \sin\theta_{N-1}\frac{d}{d\theta_{N-1}} - \cos\theta_{N-1}\left(\ell + \frac{1}{2}\right) + \frac{\ell_{N-2} + \frac{N-2}{2}}{2(\ell + \frac{N-2}{2})}, \tag{2.122}$$

$$\tilde{T}^{(-)}_{\tilde{\psi}} = \sin\theta_{N-1}\frac{d}{d\theta_{N-1}} - \cos\theta_{N-1}\left(\ell + \frac{1}{2}\right) - \frac{\ell_{N-2} + \frac{N-2}{2}}{2(\ell + \frac{N-2}{2})}. \tag{2.123}$$

The corresponding ladder relations are

$$T^{(+)}_{f} f_{M\ell}(t) = k^{(+)} f_{M\ell+1}(t), \tag{2.124}$$

$$T^{(-)}_{f} f_{M\ell}(t) = k^{(-)} f_{M\ell-1}(t), \tag{2.125}$$

$$\tilde{T}^{(+)}_{\tilde{f}} \tilde{f}_{\ell\ell_{N-2}}(\theta_{N-1}) = \tilde{k}^{(+)} \tilde{f}_{\ell+1\,\ell_{N-2}}(\theta_{N-1}), \tag{2.126}$$

$$\tilde{T}^{(-)}_{\tilde{f}} \tilde{f}_{\ell\ell_{N-2}}(\theta_{N-1}) = \tilde{k}^{(-)} \tilde{f}_{\ell-1\,\ell_{N-2}}(\theta_{N-1}), \tag{2.127}$$

where $f_{M\ell}(t) \in \{\phi_{M\ell}(t), \psi_{M\ell}(t)\}$, $\tilde{f}_{\ell\ell_{N-2}}(\theta_{N-1}) \in \{\tilde{\phi}_{\ell\ell_{N-2}}(\theta_{N-1}), \tilde{\psi}_{\ell\ell_{N-2}}(\theta_{N-1})\}$ and

$$k^{(+)} = -i\frac{(N/2+\ell)^2 + M^2}{N/2+\ell}, \tag{2.128}$$

$$k^{(-)} = -i(N/2+\ell-1), \tag{2.129}$$

$$\tilde{k}^{(+)} = \frac{(\ell + N - 1 + \ell_{N-2})(\ell - \ell_{N-2} + 1)}{(\ell + N/2)}, \tag{2.130}$$

$$\tilde{k}^{(-)} = -\frac{((N-1)/2 + \ell - 1)((N-1)/2 + \ell)}{(N-2)/2 + \ell}. \tag{2.131}$$

The ladder relations (2.124)-(2.127) can be proved using the raising and lowering operators for the parameters of the Gauss hypergeometric function given in Appendix 2.10. Below we describe how to express the spinorial Lie derivative (2.115) of a mode solution $\psi^{(s,\tilde{s})}_{M\ell\sigma}$ as a linear combination of other solutions with the same M.

Case 1: N even (> 2). Using Eq. (2.38) one finds

$$\gamma^{N-1}\gamma^0 = \begin{pmatrix} -\widetilde{\gamma}^{N-1} & 0 \\ 0 & \widetilde{\gamma}^{N-1} \end{pmatrix}. \tag{2.132}$$

Let ψ be the eigenspinor $\psi^{(\pm,\tilde{s})}_{M\ell\ell_{N-2}\tilde{\sigma}}$, where $\tilde{\sigma}$ stands for quantum numbers other than ℓ, ℓ_{N-2}. Since the partial derivatives in Eq. (2.115) refer only to the coordinates

$\{t,\theta_{N-1}\}$ we want to extract the t and θ_{N-1} dependence from our analytically continued eigenspinors. By combining Eqs. (2.67), (2.68) and (2.84) we can express the spinors $\psi^{(\pm,\tilde{s})}_{M\ell\ell_{N-2}\tilde{\sigma}}(t,\Omega_{N-1})$ in terms of eigenspinors on S^{N-2} ($\hat{\tilde{\chi}}^{(\tilde{s})}_{\pm\ell_{N-2}\tilde{\sigma}}(\Omega_{N-2})$) as follows:

$$\psi^{(-,\tilde{s})}_{M\ell\ell_{N-2}\tilde{\sigma}}(t,\Omega_{N-1}) = \frac{c_N(M\ell)}{\sqrt{2}}\frac{c_{N-1}(\ell\,\ell_{N-2})}{\sqrt{2}}\begin{pmatrix} U^{(\tilde{s})}_{-M\ell\ell_{N-2}\tilde{\sigma}}(t,\theta_{N-1},\Omega_{N-2}) \\ D^{(\tilde{s})}_{-M\ell\ell_{N-2}\tilde{\sigma}}(t,\theta_{N-1},\Omega_{N-2}) \end{pmatrix}, \quad (2.133)$$

$$\psi^{(+,\tilde{s})}_{M\ell\ell_{N-2}\tilde{\sigma}}(t,\Omega_{N-1}) = \frac{c_N(M\ell)}{\sqrt{2}}\frac{c_{N-1}(\ell\,\ell_{N-2})}{\sqrt{2}}\begin{pmatrix} D^{(\tilde{s})}_{+M\ell\ell_{N-2}\tilde{\sigma}}(t,\theta_{N-1},\Omega_{N-2}) \\ U^{(\tilde{s})}_{+M\ell\ell_{N-2}\tilde{\sigma}}(t,\theta_{N-1},\Omega_{N-2}) \end{pmatrix}, \quad (2.134)$$

where

$$U^{(\tilde{s})}_{\mp M\ell\ell_{N-2}\tilde{\sigma}}(t,\theta_{N-1},\Omega_{N-2}) = \phi_{M\ell}(t)\Big(\tilde{\phi}_{\ell\ell_{N-2}}(\theta_{N-1})\hat{\tilde{\chi}}^{(\tilde{s})}_{-\ell_{N-2}\tilde{\sigma}}(\Omega_{N-2}) \mp i\tilde{\psi}_{\ell\ell_{N-2}}(\theta_{N-1})\hat{\tilde{\chi}}^{(\tilde{s})}_{+\ell_{N-2}\tilde{\sigma}}(\Omega_{N-2})\Big) \quad (2.135)$$

and $D^{(\tilde{s})}_{\mp M\ell\ell_{N-2}\tilde{\sigma}}$ is given by an analogous expression with $\phi_{M\ell}(t)\to i\psi_{M\ell}(t)$. By substituting Eqs. (2.133)-(2.135) into the expression for the spinorial Lie derivative (2.115) and making use of Eqs. (2.124)-(2.131) we find after a lengthy calculation

$$\mathscr{L}^s_\xi\psi^{(\mp,\tilde{s})}_{M\ell\sigma} = R^{(N)}_{M\ell\ell_{N-2}}\psi^{(\mp,\tilde{s})}_{M\,\ell+1\,\sigma} + L^{(N)}_{M\ell\ell_{N-2}}\psi^{(\mp,\tilde{s})}_{M\,\ell-1\,\sigma} + C^{(N)}_{M\ell\ell_{N-2}}\psi^{(\pm,\tilde{s})}_{M\ell\sigma}, \quad (2.136)$$

where the coefficients on the right-hand side are given by the following expressions:

$$R^{(N)}_{M\ell\ell_{N-2}} = \frac{c_N(M\ell)\,c_{N-1}(\ell\,\ell_{N-2})}{c_N(M\,\ell+1)\,c_{N-1}(\ell+1,\ell_{N-2})}\frac{k^{(+)}\tilde{k}^{(+)}}{2(\ell+\frac{N-1}{2})} \quad (2.137)$$

$$= \frac{-i}{2}\frac{\sqrt{(\frac{N}{2}+\ell)^2+M^2}}{\frac{N}{2}+\ell}\times\sqrt{(\ell-\ell_{N-2}+1)(\ell+\ell_{N-2}+N-1)}, \quad (2.138)$$

$$L^{(N)}_{M\ell\ell_{N-2}} = \frac{(-1)\times c_N(M\ell)\,c_{N-1}(\ell\,\ell_{N-2})}{c_N(M,\ell-1)\,c_{N-1}(\ell-1,\ell_{N-2})}\frac{k^{(-)}\tilde{k}^{(-)}}{2(\ell+\frac{N-1}{2})} \quad (2.139)$$

$$= -\left(R^{(N)}_{M,\ell-1,\ell_{N-2}}\right)^*, \quad (2.140)$$

and

$$C^{(N)}_{M\ell\ell_{N-2}} = -i\frac{M(\ell_{N-2}+\frac{N-2}{2})}{2(\ell+\frac{N-2}{2})(\ell+\frac{N}{2})}. \quad (2.141)$$

Notice that in the last term of the linear combination in Eq. (2.136) the spin projection sign is flipped. We have checked the validity of the above results by using the de Sitter invariance of the inner product (2.92). More specifically, we have verified that $(\mathscr{L}^s_\xi \psi_{M\ell}, \psi_{M\,\ell\pm1}) + (\psi_{M\ell}, \mathscr{L}^s_\xi \psi_{M\,\ell\pm1}) = 0$. (Some details regarding the derivation of Eq. (2.136) can be found in Appendix 2.11 along with the $N = 2$ case.) It is clear from Eq. (2.136) that our positive frequency solutions transform to other positive frequency solutions with the same M under the transformation generated by ξ. Based on this observation we can conclude that the vacuum corresponding to these positive frequency modes is de Sitter invariant (see, e.g. Refs. [26] and [14]).

Case 2: N odd. Using Eq. (2.40) we find

$$\gamma^{N-1}\gamma^0 = i\begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}. \tag{2.142}$$

As in the case with N even, it is convenient to express the analytically continued eigenspinors $\psi^{(s,\tilde{s})}_{M\ell\ell_{N-2}\tilde{\sigma}}(t,\Omega_{N-1})$ (Eq. (2.89)) in terms of eigenspinors on S^{N-2} ($\tilde{\chi}^{(\tilde{s})}_{\pm\ell_{N-2}\tilde{\sigma}}(\Omega_{N-2})$). By combining Eqs. (2.85), (2.55) and (2.56), we can rewrite Eq. (2.89) as

$$\begin{aligned}\psi^{(-,\tilde{s})}_{M\ell\ell_{N-2}\tilde{\sigma}}(t,\theta_{N-1},\Omega_{N-2}) = & \frac{c_N(M\ell)}{\sqrt{2}}\frac{c_{N-1}(\ell\,\ell_{N-2})}{\sqrt{2}}\frac{1}{\sqrt{2}} \\ & \times\begin{pmatrix} (1+i)\tilde{\phi}_{\ell\ell_{N-2}}[\phi_{M\ell}+i\psi_{M\ell}]\tilde{\chi}^{(\tilde{s})}_{-\ell_{N-2}\tilde{\sigma}} \\ (-1+i)i\tilde{\psi}_{\ell\ell_{N-2}}[\phi_{M\ell}-i\psi_{M\ell}]\tilde{\chi}^{(\tilde{s})}_{-\ell_{N-2}\tilde{\sigma}} \end{pmatrix}\end{aligned} \tag{2.143}$$

and

$$\begin{aligned}\psi^{(+,\tilde{s})}_{M\ell\ell_{N-2}\tilde{\sigma}}(t,\theta_{N-1},\Omega_{N-2}) = & \frac{c_N(M\ell)}{\sqrt{2}}\frac{c_{N-1}(\ell\,\ell_{N-2})}{\sqrt{2}}\frac{1}{\sqrt{2}} \\ & \times\begin{pmatrix} (1+i)i\tilde{\psi}_{\ell\ell_{N-2}}[\phi_{M\ell}+i\psi_{M\ell}]\tilde{\chi}^{(\tilde{s})}_{+\ell_{N-2}\tilde{\sigma}} \\ (-1+i)\tilde{\phi}_{\ell\ell_{N-2}}[\phi_{M\ell}-i\psi_{M\ell}]\tilde{\chi}^{(\tilde{s})}_{+\ell_{N-2}\tilde{\sigma}} \end{pmatrix}.\end{aligned} \tag{2.144}$$

Working as in the case with N even, we find after a lengthy calculation

$$\mathscr{L}^s_\xi \psi^{(\mp,\tilde{s})}_{M\ell\sigma} = R^{(N)}_{M\ell\ell_{N-2}}\psi^{(\mp,\tilde{s})}_{M\,\ell+1\,\sigma} + L^{(N)}_{M\ell\ell_{N-2}}\psi^{(\mp,\tilde{s})}_{M\,\ell-1\,\sigma} \pm C^{(N)}_{M\ell\ell_{N-2}}\psi^{(\mp,\tilde{s})}_{M\ell\sigma}. \tag{2.145}$$

Notice that, unlike the even-dimensional case, the two spin projections do not mix with each other. As in the case with N even, we conclude that the vacuum is de Sitter invariant.

2.6 CANONICAL QUANTIZATION

In this section we follow the canonical quantization procedure and give the mode expansion for the free quantum Dirac field on N-dimensional de Sitter space-time with

$(N-1)$-sphere spatial sections using the analytically continued spinor modes of $\nabla\!\!\!\!/|_{S^N}$. As mentioned earlier, our analytically continued eigenspinors can be used as the analogs of the flat space-time positive frequency modes. However, the latter are not the only solutions of the Dirac equation (2.8) on dS_N. New solutions (i.e. the negative frequency modes) can be obtained by separating variables. Below we present the negative frequency solutions before proceeding to the canonical quantization. (Note that the negative frequency solutions can also be obtained using charge conjugation as demonstrated in Appendix 2.9.)

2.6.1 Negative frequency solutions

Case 1: N even. By making the replacements (2.66) in the expression for the iterated Dirac operator on S^N (2.54) one finds

$$\Big[\Big(\frac{\partial}{\partial t}+\frac{N-1}{2}\tanh t\Big)^2-\frac{1}{\cosh^2 t}\tilde{\nabla\!\!\!\!/}^2\pm\frac{\sinh t}{\cosh^2 t}\tilde{\nabla\!\!\!\!/}\Big]\varphi_\pm$$
$$=-M^2\varphi_\pm. \tag{2.146}$$

Then by separating variables (as in Ref. [8]) one finds the negative frequency solutions

$$V^{(-,\tilde{s})}_{M\ell\sigma}(t,\Omega_{N-1})=\frac{c_N(M\ell)}{\sqrt{2}}\begin{pmatrix}\phi^*_{M\ell}(t)\chi^{(\tilde{s})}_{+\ell\sigma}(\Omega_{N-1})\\ i\psi^*_{M\ell}(t)\chi^{(\tilde{s})}_{+\ell\sigma}(\Omega_{N-1})\end{pmatrix}, \tag{2.147}$$

$$V^{(+,\tilde{s})}_{M\ell\sigma}(t,\Omega_{N-1})=\frac{c_N(M\ell)}{\sqrt{2}}\begin{pmatrix}i\psi^*_{M\ell}(t)\chi^{(\tilde{s})}_{-\ell\sigma}(\Omega_{N-1})\\ \phi^*_{M\ell}(t)\chi^{(\tilde{s})}_{-\ell\sigma}(\Omega_{N-1})\end{pmatrix}. \tag{2.148}$$

These are normalized using the inner product (2.92) as

$$(V^{(s,\tilde{s})}_{M\ell\sigma},V^{(s',\tilde{s}')}_{M\ell'\sigma'})=\delta_{ss'}\delta_{\tilde{s}\tilde{s}'}\delta_{\ell\ell'}\delta_{\sigma\sigma'} \tag{2.149}$$

and they are orthogonal to the positive frequency solutions, i.e.

$$(\psi^{(s,\tilde{s})}_{M\ell\sigma},V^{(s',\tilde{s}')}_{M\ell'\sigma'})=0. \tag{2.150}$$

As we can see, the negative frequency modes are given by the positive frequency solutions (2.67) and (2.68) by replacing the functions $\phi_{M\ell}(t),\psi_{M\ell}(t)$ with their complex conjugate functions and by exchanging $\chi_\pm(\Omega_{N-1})$ and $\chi_\mp(\Omega_{N-1})$. The time derivatives of the spinors (2.147)-(2.148) reproduce the flat space-time behaviour in the large ℓ limit, i.e. the complex conjugate of Eqs. (2.79) and (2.80).

Case 2: N odd. Working as in the even-dimensional case, the negative frequency modes are found to be

$$V^{(s,\tilde{s})}_{M\ell\sigma}(t,\Omega_{N-1}) = \frac{c_N(M\ell)}{\sqrt{2}}(\phi^*_{M\ell}(t)\hat{\chi}^{(s,\tilde{s})}_{+\ell\sigma}(\Omega_{N-1}) + i\psi^*_{M\ell}(t)\hat{\chi}^{(s,\tilde{s})}_{-\ell\sigma}(\Omega_{N-1})) \tag{2.151}$$

and they satisfy the conditions (2.149) and (2.150).

2.6.2 Canonical Quantization

The Lagrangian density for a free spinor field Ψ is

$$-L = \sqrt{-g}\,\overline{\Psi}\Big(\gamma^\mu\nabla_\mu - M\Big)\Psi \tag{2.152}$$

$$= \sqrt{-g}\,i\Psi^\dagger_A(\gamma^0)^A{}_B\Big((\gamma^\mu)^B{}_C(\nabla_\mu\Psi)^C - M\Psi^B\Big), \tag{2.153}$$

where we have written out the spinor indices explicitly in the second line ($A,B,C = 1,...,2^{[N/2]}$). The corresponding equation of motion for Ψ is the Dirac equation (2.8). By the standard canonical quantization procedure, we find

$$\{\Psi(t,\boldsymbol{\theta})^A, \Psi^\dagger(t,\boldsymbol{\theta}')_B\} = \frac{1}{\sqrt{-g(t,\boldsymbol{\theta})}}\delta^{(N-1)}(\boldsymbol{\theta}-\boldsymbol{\theta}')\delta^A{}_B, \tag{2.154}$$

$$\{\Psi(t,\boldsymbol{\theta})^A, \Psi(t,\boldsymbol{\theta}')^B\} = \{\Psi^\dagger(t,\boldsymbol{\theta})_A, \Psi^\dagger(t,\boldsymbol{\theta}')_B\} = 0. \tag{2.155}$$

The mode expansion for the free Dirac field is

$$\Psi(t,\boldsymbol{\theta}) = \sum_{\ell,\sigma}\sum_{s,\tilde{s}}\Big(a^{(s,\tilde{s})}_{M\ell\sigma}\psi^{(s,\tilde{s})}_{M\ell\sigma}(t,\boldsymbol{\theta}) + b^{(s,\tilde{s})\dagger}_{M\ell\sigma}V^{(s,\tilde{s})}_{M\ell\sigma}(t,\boldsymbol{\theta})\Big), \tag{2.156}$$

where we are summing over all angular momentum quantum numbers and over all the possible spin projections. (There are $[N/2]$ spin projection indices in total.) Using the normalization conditions (2.111), (2.149) and the orthogonality condition (2.150) we may express the annihilation operators, $a^{(s,\tilde{s})}_{M\ell\sigma}$ and $b^{(s,\tilde{s})}_{M\ell\sigma}$, as

$$a^{(s,\tilde{s})}_{M\ell\sigma} = (\psi^{(s,\tilde{s})}_{M\ell\sigma}(t,\boldsymbol{\theta}), \Psi(t,\boldsymbol{\theta}))$$

$$= \int d\boldsymbol{\theta}\sqrt{-g}\,\psi^{(s,\tilde{s})}_{M\ell\sigma}(t,\boldsymbol{\theta})^\dagger\,\Psi(t,\boldsymbol{\theta}) \tag{2.157}$$

and

$$b^{(s,\tilde{s})}_{M\ell\sigma} = \int d\boldsymbol{\theta}\sqrt{-g}\Psi^\dagger(t,\boldsymbol{\theta})\,V^{(s,\tilde{s})}_{M\ell\sigma}(t,\boldsymbol{\theta}). \tag{2.158}$$

By combining Eqs. (2.157)-(2.158) with the anti-commutation relations (2.154) and (2.155) we obtain

$$\{a^{(s,\tilde{s})}_{M\ell\sigma}, a^{(s',\tilde{s}')\dagger}_{M\ell'\sigma'}\} = \delta_{ss'}\delta_{\tilde{s}\tilde{s}'}\delta_{\ell\ell'}\delta_{\sigma\sigma'}, \tag{2.159}$$

$$\{b^{(s,\tilde{s})}_{M\ell\sigma}, b^{(s',\tilde{s}')\dagger}_{M\ell'\sigma'}\} = \delta_{ss'}\delta_{\tilde{s}\tilde{s}'}\delta_{\ell\ell'}\delta_{\sigma\sigma'}, \tag{2.160}$$

while all the other anti-commutators are zero. The de Sitter invariant vacuum is defined by

$$a^{(s,\tilde{s})}_{M\ell\sigma}\left|0\right\rangle = b^{(s,\tilde{s})}_{M\ell\sigma}\left|0\right\rangle = 0, \tag{2.161}$$

for all $\ell, \sigma, (s,\tilde{s})$. Using the mode expansion of the Dirac field (2.156) we can obtain the mode-sum form for the Wightman two-point function

$$W\Big((t,\boldsymbol{\theta}),(t',\boldsymbol{\theta}')\Big) \equiv \left\langle 0\right| \Psi(t,\boldsymbol{\theta})\overline{\Psi}(t',\boldsymbol{\theta}')\left|0\right\rangle \tag{2.162}$$

$$= \sum_{\ell,\sigma}\sum_{s,\tilde{s}} \psi^{(s,\tilde{s})}_{M\ell\sigma}(t,\boldsymbol{\theta})\overline{\psi}^{(s,\tilde{s})}_{M\ell\sigma}(t',\boldsymbol{\theta}'). \tag{2.163}$$

The high frequency behaviour of our mode solutions (2.79)-(2.80) implies that we should adopt the $-i\epsilon$ prescription (i.e. the time variable t should be understood to have an infinitesimal negative imaginary part: $t \to t - i\epsilon$, $\epsilon > 0$).

2.7 THE WIGHTMAN TWO-POINT FUNCTION

In this section we first review the basics about the construction of Dirac spinor Green's functions on dS_N using intrinsic geometric objects following the work of Mück [19]. (Mück gave the coordinate independent construction of the spinor Green's function in terms of intrinsic geometric objects on maximally symmetric spaces of arbitrary dimensions using Dirac spinors. An analogous construction on 4-dimensional maximally symmetric spaces using two-component spinors was first presented in Ref. [4].) Then using the mode-sum method (2.163) we obtain a closed-form expression for the massless spinor Wightman two-point function on dS_N that agrees with the construction presented in Ref. [19]. Using this massless two-point function we infer the analytic expression for the spinor parallel propagator and then obtain the massive spinor Wightman two-point function in a closed form.

2.7.1 The spinor parallel propagator on dS_N

Let $\left|\psi\right\rangle$ be a state invariant under the action of the de Sitter group. Then two-point functions (such as $\left\langle\psi\right| \Psi(x)\overline{\Psi}(x')\left|\psi\right\rangle$) define maximally symmetric bispinors [3]. These bispinors can be expressed in terms of the following "preferred geometric objects": the geodesic distance (2.19), the unit tangent vectors (2.21) to the geodesic with endpoints x, x' and the bispinor of parallel transport $\Lambda(x,x')$, also known as the spinor parallel propagator [19, 9, 5]. The spinor parallel propagator parallel transports a spinor $\psi(x')$ from x' to x along the (shortest) geodesic joining these points, i.e.

$$\psi_{||}(x)^A = \Lambda(x,x')^A{}_{A'}\, \psi(x')^{A'}, \tag{2.164}$$

where $\psi_{||}(x)$ is the parallelly transported spinor. The covariant derivative of a spinor field for an infinitesimal interval can be expressed using the spinor parallel propagator as [9]

$$\nabla_\mu \psi(x) dx^\mu = \Lambda(x, x+dx)\, \psi(x+dx) - \psi(x).$$

The following relations can be used as the defining properties of the spinor parallel propagator for arbitrary space-time dimension [19]:

$$\Lambda(x',x) = [\Lambda(x,x')]^{-1}, \tag{2.165}$$

$$\gamma^{\nu'}(x') = \Lambda(x',x)\gamma^\mu(x) g^{\nu'}{}_\mu(x',x)\Lambda(x,x'), \tag{2.166}$$

$$n^\mu \nabla_\mu \Lambda(x,x') = 0, \tag{2.167}$$

where the parallel transport equation (2.167) holds along the geodesic connecting x and x'. Equation (2.166) describes the parallel transport of gamma matrices. By contracting Eq. (2.166) with $n_{\nu'}(x,x')$ and using Eqs. (2.26) and (2.165) we find

$$[\Lambda(x,x')]^{-1}\, \not{n}\, \Lambda(x,x') = -\not{n}', \tag{2.168}$$

where $\not{n} \equiv \gamma^\mu(x) n_\mu(x,x')$ and $\not{n}' \equiv \gamma^{\mu'}(x') n_{\mu'}(x,x')$. Equation (2.168) conveniently describes the parallel transport property of $\not{n}$. In Appendix 2.13 we show that our result for the spinor parallel propagator (given by Eq. (2.194)) is consistent with the defining properties (2.165)-(2.168). On dS_N the covariant derivatives of $\Lambda(x.x')$ can be expressed as [19]

$$\nabla_\nu \Lambda(x,x') = -\frac{1}{2}\tan\left(\frac{\mu}{2}\right)(\gamma_\nu \not{n} - n_\nu)\Lambda(x,x'), \tag{2.169}$$

$$\nabla_{\mu'} \Lambda(x,x') = \frac{1}{2}\tan\left(\frac{\mu}{2}\right)\Lambda(x,x')(\gamma_{\mu'} \not{n}' - n_{\mu'}). \tag{2.170}$$

Note that $\not{n}^2 = \mathbf{1}$ and $(\not{n}')^2 = \mathbf{1}'$, where $\mathbf{1}, \mathbf{1}'$ are the identity spinor matrices at x and x', respectively.

2.7.2 Constructing spinor Green's function on dS_N using intrinsic geometric objects

The massive case. The massive spinor Green's function $S_M(x,x')$ on dS_N satisfies the inhomogeneous Dirac equation

$$[\big(\nabla\!\!\!/ - M\big)S_M(x,x')]^A{}_{A'} = \frac{\delta^{(N)}(x-x')}{\sqrt{-g(x)}}\delta^A{}_{A'}. \tag{2.171}$$

The Green's function $S_M(x,x')$ can be expressed in terms of intrinsic geometric objects as follows [19]:

$$S_M(x,x') = (\alpha_M(\mu) + \beta_M(\mu) n\!\!\!/)\Lambda(x,x'), \tag{2.172}$$

where $\alpha_M(\mu), \beta_M(\mu)$ are scalar functions of the geodesic distance. By requiring that $S_M(x,x')$ in Eq. (2.172) satisfies Eq. (2.171) we find the following system of ordinary differential equations for $\alpha_M(\mu), \beta_M(\mu)$:

$$\frac{d\alpha_M}{d\mu} - \frac{N-1}{2}\tan\frac{\mu}{2}\,\alpha_M - M\beta_M = 0, \tag{2.173}$$

$$\frac{d\beta_M}{d\mu} + \frac{N-1}{2}\cot\frac{\mu}{2}\,\beta_M - M\alpha_M = \frac{\delta(x-x')}{\sqrt{-g(x)}}. \tag{2.174}$$

Using the variable $z = \cos^2(\mu/2)$ (see Eq. (2.18)) this system of equations is solved by [19]

$$\begin{aligned}\alpha_M(z) = -M\frac{|\Gamma(\frac{N}{2}+iM)|^2}{\Gamma(\frac{N}{2}+1)(2\pi)^{N/2}2^{N/2}}\sqrt{z}\\ \times F\left(\frac{N}{2}-iM, \frac{N}{2}+iM; \frac{N}{2}+1; z\right)\end{aligned} \tag{2.175}$$

and

$$\beta_M(z) = -\frac{\sqrt{1-z}}{M}\left[\sqrt{z}\frac{d}{dz} + \frac{N-1}{2\sqrt{z}}\right]\alpha_M(z). \tag{2.176}$$

Using Eqs. (2.175) and (2.224) we can rewrite Eq. (2.176) as

$$\begin{aligned}\beta_M(z) =& \frac{|\Gamma(\frac{N}{2}+iM)|^2}{\Gamma(\frac{N}{2}+1)(2\pi)^{N/2}\,2^{N/2}}\\ &\times\sqrt{1-z}\,\frac{N}{2}F\left(\frac{N}{2}-iM, \frac{N}{2}+iM; \frac{N}{2}; z\right).\end{aligned} \tag{2.177}$$

(Note that there is a misprint in the corresponding equation for $\beta_M(z)$ - equation (29) - in Ref. [19]. Equation (2.177) of the present paper and equation (29) of Ref. [19] agree with each other after inserting a missing prefactor.) The proportionality constant for $\alpha_M(\mu)$ (and hence for $\beta_M(\mu)$) has been determined by requiring that the singularity in Eq. (2.175) for $\mu \to 0$ has the same strength as the singularity of the flat space-time Green's function [19]. This ensures that the spinor Green's function (2.172) has the desired short-distance behaviour. (Note that since $\not{n}$, α_M and β_M are known, the only remaining step for obtaining an explicit expression for the two-point function (2.172) is to derive an analytic expression for the spinor parallel propagator.)

The massless case. Letting $M = 0$ in Eqs. (2.175) and (2.177) we find

$$\alpha_0(z) = 0, \tag{2.178}$$

$$\beta_0(z) = \frac{\Gamma(N/2)}{2^{N/2}(2\pi)^{N/2}} \frac{1}{(1-z)^{(N-1)/2}}, \tag{2.179}$$

($z = \cos^2(\mu/2)$) where we used Eq. (2.268). These are just the solutions (with the appropriate singularity strength) of the decoupled system

$$\frac{d\alpha_0}{d\mu} - \frac{N-1}{2}\tan\frac{\mu}{2}\,\alpha_0 = 0, \tag{2.180}$$

$$\frac{d\beta_0}{d\mu} + \frac{N-1}{2}\cot\frac{\mu}{2}\,\beta_0 = \frac{\delta(x-x')}{\sqrt{-g(x)}}. \tag{2.181}$$

The massless Green's function is then given as follows:

$$S_0(x,x') = \beta_0(z)\not{n}\Lambda(x,x') \tag{2.182}$$

$$= \frac{\Gamma(N/2)}{2^{N/2}(2\pi)^{N/2}}(1-z)^{-(N-1)/2}\not{n}\Lambda(x,x'). \tag{2.183}$$

We find that the defining properties of $\Lambda(x,x')$ (Eqs. (2.165)-(2.167)) translate to the following properties for the massless Green's function:

$$[S_0(x,x')]^{-1}\not{n} = \frac{1}{\beta_0^2}\not{n}' S_0(x',x), \tag{2.184}$$

$$[S_0(x,x')]^{-1} = -\frac{1}{\beta_0^2} S_0(x',x), \tag{2.185}$$

$$\left(n^\mu \nabla_\mu + \frac{N-1}{2}\cot\frac{\mu}{2}\right) S_0(x,x') = 0. \tag{2.186}$$

Note that by combining Eqs. (2.184) and (2.185) one obtains

$$\not{n} S_0(x,x') = -S_0(x,x')\not{n}', \tag{2.187}$$

which is equivalent to Eq. (2.168).

2.7.3 Analytic expressions for the massless and massive Wightman two-point function and the spinor parallel propagator

In the massive case the mode-sum approach for the Wightman function (2.163) leads to complicated series involving products of hypergeometric functions and it seems that their corresponding closed-form expressions do not exist in the literature. Fortunately, the situation is simpler in the massless case and we can obtain a closed-form expression for the Wightman two-point function. This directly results in the knowledge of the spinor parallel propagator $\Lambda(x,x')$ due to Eq. (2.182). The spinor parallel propagator $\Lambda(x,x')$ in turn can be used to obtain an analytic expression for the massive spinor Wightman two-point function via Eq. (2.172).

Below we present the closed-form expression we have obtained by the mode-sum method for the massless Wightman two-point function in agreement with Eq. (2.182). We present the details of the lengthy calculation in Appendix 2.12 (as well as the result for the $N=2$ case).

Case 1: N even ($N>2$). By letting $M=0$ in Eqs. (2.67)-(2.68) we obtain the massless positive frequency modes

$$\psi^{(-,\tilde{s})}_{0\ell\sigma}(t,\Omega_{N-1}) = \frac{2^{(2-N)/2}}{\sqrt{2}}\phi_{0\ell}(t)\begin{pmatrix}\chi^{(\tilde{s})}_{-\ell\sigma}(\Omega_{N-1})\\ 0\end{pmatrix}, \tag{2.188}$$

$$\psi^{(+,\tilde{s})}_{0\ell\sigma}(t,\Omega_{N-1}) = \frac{2^{(2-N)/2}}{\sqrt{2}}\phi_{0\ell}(t)\begin{pmatrix}0\\ \chi^{(\tilde{s})}_{+\ell\sigma}(\Omega_{N-1})\end{pmatrix}. \tag{2.189}$$

Now the function describing the time dependence has the following form:

$$\phi_{0\ell}(t) = \frac{(\tan\frac{x}{2})^{\ell}}{(\cos\frac{x}{2})^{N-1}}, \tag{2.190}$$

where $\cos(x/2)$ is given in Eq. (2.71) and

$$\tan\frac{x}{2} = \frac{1-i\sinh t}{\cosh t}. \tag{2.191}$$

Exploiting the rotational symmetry of S^{N-1} we may let $\theta'_{N-1}=\theta'_{N-2}=...=\theta'_2=\theta'_1=0$ in the mode-sum (2.163). After a long calculation we obtain the following $2^{N/2}$-dimensional bispinorial matrix:

$$\begin{aligned} W_0[(t,\boldsymbol{\theta}),(t',\mathbf{0})] =& (\beta_0(\mu)\not{n})|_{\boldsymbol{\theta}'=\mathbf{0}}\ \exp\left\{\frac{\lambda(t,\theta_{N-1},t')}{2}\gamma^0\gamma^{N-1}\right\} \\ &\times\prod_{j=2}^{N-1}\exp\left\{\frac{\theta_{N-j}}{2}\gamma^{N-j+1}\gamma^{N-j}\right\}, \end{aligned} \tag{2.192}$$

where

$$\not{n}|_{\boldsymbol{\theta}'=0} = \gamma^0 n_0[(t,\boldsymbol{\theta}),(t',\mathbf{0})] + \gamma^{N-1} n_{N-1}[(t,\boldsymbol{\theta}),(t',\mathbf{0})] \tag{2.193}$$

(see Eqs. (2.34)-(2.35)). By comparing Eq. (2.192) with Eq. (2.182) we find

$$\Lambda\Big((t,\boldsymbol{\theta}),(t',\mathbf{0})\Big) = \exp\bigg\{\frac{\lambda(t,\theta_{N-1},t')}{2}\gamma^0\gamma^{N-1}\bigg\} \times \prod_{j=2}^{N-1}\exp\bigg\{\frac{\theta_{N-j}}{2}\gamma^{N-j+1}\gamma^{N-j}\bigg\}. \tag{2.194}$$

The biscalar $\lambda(t,\theta_{N-1},t')$ is defined by the following relations:

$$\cosh\frac{\lambda}{2} = \frac{w_+n_+ + w_-n_-}{2i\sin(\mu/2)} = \frac{w_1n_0 + w_2n_{N-1}}{\sin(\mu/2)},$$
$$\sinh\frac{\lambda}{2} = \frac{w_+n_+ - w_-n_-}{2i\sin(\mu/2)} = \frac{w_1n_{N-1} + w_2n_0}{\sin(\mu/2)}, \tag{2.195}$$

where $n_0 \equiv n_0[(t,\boldsymbol{\theta}),(t',\mathbf{0})], n_{N-1} \equiv n_{N-1}[(t,\boldsymbol{\theta}),(t',\mathbf{0})]$ and

$$w_1(t,\theta_{N-1},t') = \sinh\frac{t-t'}{2}\,\cos\frac{\theta_{N-1}}{2}, \tag{2.196}$$
$$w_2(t,\theta_{N-1},t') = \cosh\frac{t+t'}{2}\,\sin\frac{\theta_{N-1}}{2}, \tag{2.197}$$
$$w_\pm(t,\theta_{N-1},t') \equiv i[w_1(t,t',\theta_{N-1}) \pm w_2(t,\theta_{N-1},t')], \tag{2.198}$$
$$n_\pm \equiv n_0 \pm n_{N-1}. \tag{2.199}$$

(This definition of λ is motivated naturally in the mode-sum construction of the massless Wightman function given in Appendix 2.12.) It is worth mentioning that the biscalar functions w_+ and w_- satisfy $w_+w_- = \sin^2(\mu/2)$, i.e. $\beta_0(\mu) \propto (w_+w_-)^{-(N-1)/2}$ (see Eqs. (2.28) and (2.179)). We have verified that Eqs. (2.195) are consistent with the relation $\cosh^2(\lambda/2) - \sinh^2(\lambda/2) = 1$.

It is natural that the spinor parallel propagator (2.194) is given by a product of $N-1$ matrices $\in$ Spin$(N-1,1)$; these correspond to one boost and $N-2$ rotations (see Appendix 2.12).

As mentioned earlier, we do not follow the mode-sum method for the construction of the massive Wightman function. A closed-form expression for the latter can be found using our result for the spinor parallel propagator (2.194). To be specific, by substituting Eq. (2.194) into Eq. (2.172) one can straightforwardly obtain an analytic expression for the massive Wightman function (with $x = (t,\boldsymbol{\theta})$ and $x' = (t',\mathbf{0})$) in terms of intrinsic geometric objects. In Appendix 2.14 we compare the mode-sum form of the massive Wightman function with timelike separated points, $x = (t, \mathbf{0})$ and $x' = (t', \mathbf{0})$, with the expression coming from Eq. (2.172) with $\mu = i(t - t')$. Based on this comparison we make a conjecture for the closed-form expression of a series containing the Gauss hypergeometric function. Note that a closed-form expression for the spinor parallel propagator on anti-de Sitter space-time (along with the construction of the Feynman Green's function for the Dirac field according to Eq. (2.172)) can be found in Ref. [5].

Case 2: N odd. The massless positive frequency solutions (2.89) are given by

$$\psi_{0\ell\sigma}^{(s,\tilde{s})}(t, \Omega_{N-1}) = \frac{2^{(2-N)/2}}{\sqrt{2}}\, \phi_{0\ell}(t)\, \hat{\chi}_{-\ell\sigma}^{(s,\tilde{s})}(\Omega_{N-1}). \tag{2.200}$$

Working as in the even-dimensional case we obtain again Eqs. (2.192) and (2.194) (where γ^0 is given by Eq. (2.40)) and then we can construct the massive two-point function using Eq. (2.172).

2.8 SUMMARY AND CONCLUSIONS

In this paper we analytically continued the eigenspinors of the Dirac operator on S^N to obtain solutions to the Dirac equation on dS_N that serve as analogs of the positive frequency modes of flat space-time. Our generalised positive frequency solutions were used to define a vacuum for the free Dirac field. The negative frequency solutions were also constructed. The de Sitter invariance of the vacuum was demonstrated by showing that the positive frequency solutions transform among themselves under infinitesimal Spin(N,1) transformations.

In order to check the validity of our mode functions, the Wightman function for massless spinors was calculated using the mode-sum method and it was expressed in a form that is in agreement with the construction in terms of intrinsic geometric objects $(\mu, \not{n}, \Lambda)$ given in Ref. [19]. An analytic expression for the spinor parallel propagator was found. This expression was tested using the defining properties of the spinor parallel propagator as presented in Ref. [19] (see Appendix 2.13). Note that it has been checked that the spinor Green's functions expressed in terms of $\mu, \not{n}, \Lambda$ have Minkowskian singularity strength in the limit $\mu \to 0$ [19]. Thus, the conditions for the unique vacuum [17] are satisfied by the vacuum for the free massless Dirac field defined in this paper.

Although we did not obtain a closed-form expression for the massive spinor Wightman function by the mode-sum method using our analytically continued eigenspinors, we constructed it in terms of intrinsic geometric objects. Since the short-distance behaviour has been checked in Ref. [19], the requirements for a preferred vacuum are again satisfied. The mode-sum method and the geometric construction of Ref. [19] should give the same result for the massive Wightman function. This observation leads to the series conjecture of Appendix 2.14.

Acknowledgements. The author is grateful to Atsushi Higuchi for guidance, encouragement and useful discussions. He also thanks Wolfgang Mück for communications and the referee for useful comments. Subsection 2.5.1 was part of the author's MSc thesis at Imperial College London. This work was supported by a studentship from the Department of Mathematics, University of York.

2.9 APPENDIX A - CHARGE CONJUGATION AND NEGATIVE FREQUENCY MODES

In this Appendix we demonstrate how the negative frequency solutions given by Eqs. (2.147)-(2.148) and (2.151) are constructed by charge conjugating our analytically continued eigenspinors. First, let us review charge conjugation for Dirac spinors on dS_N and on spheres following Ref. [25]. For convenience, our discussion will be based on the unitary matrices $B_\pm$ that relate the gamma matrices to their complex conjugate matrices by similarity transformations, i.e.

$$(\gamma^a)^* = B_+ \gamma^a B_+^{-1}, \quad -(\gamma^a)^* = B_- \gamma^a B_-^{-1}, \tag{2.201}$$

and not in terms of the conventional charge conjugation matrices $C_\pm$ that relate γ^a to $(\gamma^a)^T$. These two ways of defining charge conjugation are equivalent [25]. From this point we will refer to the matrices $B_\pm$ as the charge conjugation matrices. (We should note that the representation we use for the gamma matrices (2.38), (2.40) is different from the one used in Ref. [25]. Also, note that charge conjugation matrices are defined up to a phase factor and that $\gamma^N \equiv -i\gamma^0$.)

2.9.1 Charge conjugation on N-dimensional de Sitter space-time and on spheres

For convenience, let us work in $d = \tau + s$ dimensions, with $\tau \in \{0, 1\}$ being the number of timelike dimensions and s being the number of spacelike dimensions.

For d even dimensions there are both B_+ and B_-. For d odd dimensions we can use one of the matrices from the $(d-1)$-dimensional case [25]. (As it will be clear in the next subsections, one needs to modify the charge conjugation matrix on dS_{d-1} before using it on dS_d. This is not the case in Ref. [25], because a different representation for γ^a's is used.) More specifically, on odd-dimensional spaces with Lorentzian (Euclidean) metric signature there is only B_+ (B_-) for $[d/2]$ odd and only B_- (B_+) for $[d/2]$ even. (See Refs. [25] and [12] for more details.)

Let Ψ be a $2^{[d/2]}$-dimensional Dirac spinor transforming under Spin(s,τ). Its charge conjugated spinor is defined with either one of the following two ways:

$$\Psi^{C_+} := B_+^{-1}\Psi^* \quad \text{or} \quad \Psi^{C_-} := B_-^{-1}\Psi^*. \tag{2.202}$$

Suppose now that $\Psi_\pm$ is an eigenspinor of the Dirac operator with eigenvalue $\kappa^{\pm}_{(\tau,s)}$, i.e.

$$\nabla\!\!\!/_{(\tau,s)}\Psi_\pm = \kappa^{\pm}_{(\tau,s)}\Psi_\pm, \tag{2.203}$$

where $\nabla\!\!\!/_{(1,N-1)} \equiv \nabla\!\!\!/|_{dS_N}$ is the Dirac operator on dS_N with $\kappa^{\pm}_{(1,N-1)} \equiv \pm M$ and $\nabla\!\!\!/_{(0,N-1)} \equiv \tilde{\nabla\!\!\!/}$ is the Dirac operator on S^{N-1} with $\kappa^{\pm}_{(0,N-1)} \equiv \pm i(\ell + (N-1)/2)$. The charge conjugated counterparts of the eigenspinors of the Dirac operator are also eigenspinors. This can be understood as follows: taking the complex conjugate of Eq. (2.203) and using Eqs. (2.201) and (2.202) we find

$$\nabla\!\!\!/_{(\tau,s)}\Psi_\pm^{C_+} = +(\kappa^{\pm}_{(\tau,s)})^*\,\Psi_\pm^{C_+}, \tag{2.204}$$

$$\nabla\!\!\!/_{(\tau,s)}\Psi_\pm^{C_-} = -(\kappa^{\pm}_{(\tau,s)})^*\,\Psi_\pm^{C_-}, \tag{2.205}$$

where we also used $(\Sigma^{ab})^* = B_\pm \Sigma^{ab} B_\pm^{-1}$. It is clear from Eqs. (2.204)-(2.205) that performing charge conjugation with B_- changes the sign of the mass term on dS_N. Also, Eqs. (2.204)-(2.205) imply the following relations for the eigenspinors of the Dirac operator on S^n (with $\Psi_\pm = \chi^{(\tilde{s})}_{\pm\ell_n\sigma}$ and $\kappa^{\pm}_{(0,n)} = \pm i(\ell_n + n/2)$):

$$(\chi^{(\tilde{s})}_{\pm\ell_n\sigma})^{C_-} \propto \chi^{(\tilde{s}')}_{\pm\ell_n\sigma}, \quad (\chi^{(\tilde{s})}_{\pm\ell_n\sigma})^{C_+} \propto \chi^{(\tilde{s}')}_{\mp\ell_n\sigma}, \tag{2.206}$$

where n is arbitrary, σ stands for angular momentum quantum numbers other than ℓ_n and $\tilde{s}$ represents the $[n/2]$ spin projection indices that correspond to this eigenspinor. The label $\tilde{s}'$ is no necessarily equal to $\tilde{s}$.

Below we use the tilde notation for quantities defined on S^{N-1}.

2.9.2 NEGATIVE FREQUENCY SOLUTIONS FOR N EVEN

Case 1: $N/2$ even. The charge conjugation matrices $B_{\pm}$, satisfying Eq. (2.201) on dS_N, are given by the following products of gamma matrices:

$$B_+ = \gamma^1 \prod_{r=1}^{(N-4)/4} \gamma^{4r}\gamma^{4r+1}, \tag{2.207}$$

$$B_- = \gamma^0 \prod_{r=1}^{N/4} \gamma^{4r-2}\gamma^{4r-1}. \tag{2.208}$$

On the odd-dimensional spatial part S^{N-1} there is only $\tilde{B}_-$ since $[(N-1)/2]$ is odd. This is given by

$$\tilde{B}_- = \tilde{\gamma}^1 \prod_{r=1}^{(N-4)/4} \tilde{\gamma}^{4r}\tilde{\gamma}^{4r+1}. \tag{2.209}$$

For convenience, we choose to define charge conjugation using B_+, which preserves the sign of the mass term in the Dirac equation. Using the representation (2.38) for the gamma matrices we can express B_+ as follows:

$$B_+ = \begin{pmatrix} 0 & i\tilde{B}_- \\ -i\tilde{B}_- & 0 \end{pmatrix}. \tag{2.210}$$

The charge conjugated counterparts of the positive frequency solutions $\psi^{(-,\tilde{s})}_{M\ell\sigma}$ (Eq. (2.67)) can be constructed using Eqs. (2.206) and (2.210). Then we have (omitting the normalization factors)

$$\begin{aligned}(\psi^{(-,\tilde{s})}_{M\ell\sigma}(t,\Omega_{N-1}))^{C_+} =& (-i)\begin{pmatrix} i\psi^*_{M\ell}(t)(\chi^{(\tilde{s})}_{-\ell\sigma}(\Omega_{N-1}))^{\tilde{C}_-} \\ \phi^*_{M\ell}(t)(\chi^{(\tilde{s})}_{-\ell\sigma}(\Omega_{N-1}))^{\tilde{C}_-}\end{pmatrix} \\ &\propto \begin{pmatrix} i\psi^*_{M\ell}(t)\chi^{(\tilde{s}')}_{-\ell\sigma}(\Omega_{N-1}) \\ \phi^*_{M\ell}(t)\chi^{(\tilde{s}')}_{-\ell\sigma}(\Omega_{N-1})\end{pmatrix}.\end{aligned} \tag{2.211}$$

After normalizing these modes we find the negative frequency solutions (2.148). Similarly, starting from the positive frequency solutions $\psi^{(+,\tilde{s})}_{M\ell\sigma}$ (Eq. (2.68)) we find the negative frequency modes (2.147).

Case 2: $N/2$ odd. The charge conjugation matrices on dS_N are given by

$$B_+ = \gamma^0\gamma^1 \prod_{r=1}^{(N-2)/4} \gamma^{4r}\gamma^{4r+1}, \tag{2.212}$$

$$B_- = \mathbf{1} \times \prod_{r=1}^{(N-2)/4} \gamma^{4r-2}\gamma^{4r-1}. \tag{2.213}$$

Since $[(N-1)/2]$ is even, the only charge conjugation matrix on S^{N-1} is $\tilde{B}_+$. The matrices B_- and $\tilde{B}_+$ are related to each other as follows:

$$B_- = \prod_{r=1}^{(N-2)/4} \begin{pmatrix} \tilde{\gamma}^{4r-2}\tilde{\gamma}^{4r-1} & 0 \\ 0 & \tilde{\gamma}^{4r-2}\tilde{\gamma}^{4r-1} \end{pmatrix} \tag{2.214}$$

$$= \begin{pmatrix} \tilde{B}_+ & 0 \\ 0 & \tilde{B}_+ \end{pmatrix}. \tag{2.215}$$

In order to construct the negative frequency solutions it is convenient to use the charge conjugation matrix B_- that flips the sign of the mass term in the Dirac equation and the "negative mass" spinors $\psi^{(s,\tilde{s})}_{-M\ell\sigma}$ (Eqs. (2.81)-(2.82)). Then, by working as in the case with $N/2$ even, we obtain the negative frequency solutions (2.147)-(2.148) (with $V^{(-,\tilde{s}')}_{M\ell\sigma} \equiv (\psi^{(-,\tilde{s})}_{-M\ell\sigma})^{C_-}$ and $V^{(+,\tilde{s}')}_{M\ell\sigma} \equiv (\psi^{(+,\tilde{s})}_{-M\ell\sigma})^{C_-}$).

2.9.3 Negative frequency solutions for N odd

Case 1: $[N/2]$ even. The only charge conjugation matrix on dS_N is B_-, which changes the sign of the mass term of the Dirac equation. It is given by

$$B_- = \gamma^0 \prod_{r=1}^{(N-1)/4} \gamma^{4r-2}\gamma^{4r-1}. \tag{2.216}$$

Note that this is the matrix (2.208) with $N \to N-1$, where now γ^0 is given by Eq. (2.40). Then Eq. (2.216) may be expressed in terms of the charge conjugation matrix on S^{N-1} as

$$B_- = i\gamma^N \tilde{B}_+ = \tilde{B}_+ i\gamma^N. \tag{2.217}$$

By performing charge conjugation for the spinors $\psi^{(\tilde{s}_{N-1})}_{-M\ell\sigma}$ (Eq. (2.90)) we find

$$\begin{aligned}(\psi^{(\tilde{s}_{N-1})}_{-M\ell\sigma}(t,\Omega_{N-1}))^{C_-} \\ = -\, i\Big[\phi^*_{M\ell}(t)\gamma^N \left(\hat{\chi}^{(\tilde{s}_{N-1})}_{-\ell\sigma}(\Omega_{N-1})\right)^{\tilde{C}_+} + i\psi^*_{M\ell}(t)\gamma^N \left(\hat{\chi}^{(\tilde{s}_{N-1})}_{+\ell\sigma}(\Omega_{N-1})\right)^{\tilde{C}_+}\Big],\end{aligned} \tag{2.218}$$

where $\tilde{s}_{N-1}$ represents the spin projection indices $s_{N-1}, s_{N-3}, ..., s_4, s_2$ on the lower-dimensional spheres and the charge conjugated counterparts of the "hatted" spinors can be found using Eqs. (2.85)-(2.87) and Eq. (2.206). More specifically, by introducing the proportionality constant c, such that $(\chi^{(\tilde{s}_{N-1})}_{-\ell\sigma})^{\tilde{C}_+} = c\chi^{(\tilde{s}'_{N-1})}_{+\ell\sigma}$ we find

$$(\hat{\chi}^{(\tilde{s}_{N-1})}_{\pm\ell\sigma})^{\tilde{C}_+} = -i\, c\hat{\chi}^{(\tilde{s}'_{N-1})}_{\pm\ell\sigma}. \tag{2.219}$$

By substituting this equation into Eq. (2.218) we obtain the negative frequency solution (2.151).

Case 2: $[N/2]$ odd. The only charge conjugation matrix on dS_N is B_+. This is given by

$$B_+ = \gamma^0\gamma^1 \prod_{r=1}^{(N-3)/4} \gamma^{4r}\gamma^{4r+1}, \tag{2.220}$$

$$= \gamma^0 \tilde{B}_- = -\tilde{B}_-\gamma^0. \tag{2.221}$$

As in the case with $[N/2]$ even, we introduce the proportionality constant m, such that $(\chi^{(\tilde{s}_{N-1})}_{-\ell\sigma})^{\tilde{C}_-} = m\chi^{(\tilde{s}'_{N-1})}_{-\ell\sigma}$, and we find

$$(\hat{\chi}^{(\tilde{s}_{N-1})}_{\pm\ell\sigma})^{\tilde{C}_-} = \mp m\hat{\chi}^{(\tilde{s}'_{N-1})}_{\pm\ell\sigma}. \tag{2.222}$$

Then we use the matrix (2.221) in order to find the charge conjugate of the spinors $\psi^{(\tilde{s}_{N-1})}_{M\ell\sigma}$ (Eq. (2.89)) and working as in the previous case we obtain the negative frequency solution (2.151).

2.10 APPENDIX B - SOME RAISING AND LOWERING OPERATORS FOR THE PARAMETERS OF THE GAUSS HYPERGEOMETRIC FUNCTION

The Gauss hypergeometric function $F(a, b; c; z)$ satisfies [11]

$$\frac{d}{dz}F(a, b; c; z) = \frac{ab}{c}F(a + 1, b + 1; c + 1; z), \tag{2.223}$$

$$\left(z\frac{d}{dz} + c - 1\right) F(a, b; c; z) = (c - 1)F(a, b; c - 1; z), \tag{2.224}$$

$$\left(z\frac{d}{dz} + a\right) F(a, b; c; z) = aF(a + 1, b; c; z). \tag{2.225}$$

By combining Eq. (2.225) with the following relation [23]:

$$\begin{aligned}(c - b)F(a + 1, b - 1; c; z) &+ (b - a - 1)(1 - z) \\ &\times F(a + 1, b; c; z) = (c - a - 1)F(a, b; c; z),\end{aligned} \tag{2.226}$$

we find

$$\begin{aligned}&\left(a(b - c) + a(-b + a + 1)z - (-b + a + 1)z(1 - z)\frac{d}{dz}\right) \\ &\quad \times F(a, b; c; z) = a(b - c)F(a + 1, b - 1; c; z).\end{aligned} \tag{2.227}$$

Using Eqs. (2.223) and (2.224) we can show the ladder relations (2.124) and (2.125), while using Eq. (2.227) we can show the ladder relations (2.126) and (2.127).

2.11 APPENDIX C - TRANSFORMATION PROPERTIES OF THE POSITIVE FREQUENCY SOLUTIONS UNDER SPIN(N,1)

2.11.1 TRANSFORMATION PROPERTIES FOR $N > 2$; SOME DETAILS FOR THE DERIVATION OF EQ. (2.136)

Here, we present some details for the derivation of Eq. (2.136) that expresses the spinorial Lie derivative (2.115) of the analytically continued eigenspinors (2.67)-(2.68) as a linear combination of solutions of the Dirac equation. The case with N odd (i.e. Eq. (2.145)) can be proved similarly and its derivation is not presented.

In order to obtain Eq. (2.136) it is useful to introduce the following relations (where $\theta \equiv \theta_{N-1}$):

$$\begin{aligned}\xi^\mu\partial_\mu(\phi_{M\ell}(t)\tilde{\phi}_{\ell\ell_{N-2}}(\theta)) + i\frac{\phi_{M\ell}(t)}{2\cosh t}\sin\theta\,\tilde{\psi}_{\ell\ell_{N-2}}(\theta) \\ =\frac{1}{2(\ell+\frac{N-1}{2})}\Big(T^{(+)}_{\phi}\times\tilde{T}^{(+)}_{\tilde{\phi}} - T^{(-)}_{\phi}\times\tilde{T}^{(-)}_{\tilde{\phi}}\Big)\phi_{M\ell}(t)\tilde{\phi}_{\ell\ell_{N-2}}(\theta) \\ +\frac{M(\ell_{N-2}+\frac{N-2}{2})}{2(\ell+\frac{N}{2})(\ell+\frac{N-2}{2})}\psi_{M\ell}(t)\tilde{\phi}_{\ell\ell_{N-2}}(\theta),\end{aligned} \tag{2.228}$$

$$\begin{aligned}\xi^\mu\partial_\mu(\phi_{M\ell}(t)\tilde{\psi}_{\ell\ell_{N-2}}(\theta)) - i\frac{\phi_{M\ell}(\theta)}{2\cosh t}\sin\theta\,\tilde{\phi}_{\ell\ell_{N-2}}(\theta) \\ =\frac{1}{2(\ell+\frac{N-1}{2})}\Big(T^{(+)}_{\phi}\times\tilde{T}^{(+)}_{\tilde{\psi}} - T^{(-)}_{\phi}\times\tilde{T}^{(-)}_{\tilde{\psi}}\Big)\phi_{M\ell}(t)\tilde{\psi}_{\ell\ell_{N-2}}(\theta) \\ -\frac{M(\ell_{N-2}+\frac{N-2}{2})}{2(\ell+\frac{N}{2})(\ell+\frac{N-2}{2})}\psi_{M\ell}(t)\tilde{\psi}_{\ell\ell_{N-2}}(\theta),\end{aligned} \tag{2.229}$$

$$\begin{aligned}\xi^\mu\partial_\mu(\psi_{M\ell}(t)\tilde{\psi}_{\ell\ell_{N-2}}(\theta)) + i\frac{\psi_{M\ell}(t)}{2\cosh t}\sin\theta\,\tilde{\phi}_{\ell\ell_{N-2}}(\theta) \\ =\frac{1}{2(\ell+\frac{N-1}{2})}\Big(T^{(+)}_{\psi}\times\tilde{T}^{(+)}_{\tilde{\psi}} - T^{(-)}_{\psi}\times\tilde{T}^{(-)}_{\tilde{\psi}}\Big)\psi_{M\ell}(t)\tilde{\psi}_{\ell\ell_{N-2}}(\theta) \\ +\frac{M(\ell_{N-2}+\frac{N-2}{2})}{2(\ell+\frac{N}{2})(\ell+\frac{N-2}{2})}\phi_{M\ell}(t)\tilde{\psi}_{\ell\ell_{N-2}}(\theta),\end{aligned} \tag{2.230}$$

2.11. Appendix C - Transformation properties of the positive frequency solutions under Spin(N,1)

$$\xi^\mu\partial_\mu(\psi_{M\ell}(t)\tilde{\phi}_{\ell\,\ell_{N-2}}(\theta)) - i\frac{\psi_{M\ell}(t)}{2\cosh t}\sin\theta\,\tilde{\psi}_{\ell\,\ell_{N-2}}(\theta)$$
$$= \frac{1}{2(\ell+\frac{N-1}{2})}\Big(T^{(+)}_{\psi}\times\tilde{T}^{(+)}_{\tilde{\phi}} - T^{(-)}_{\psi}\times\tilde{T}^{(-)}_{\tilde{\phi}}\Big)\psi_{M\ell}(t)\tilde{\phi}_{\ell\,\ell_{N-2}}(\theta)$$
$$- \frac{M(\ell_{N-2}+\frac{N-2}{2})}{2(\ell+\frac{N}{2})(\ell+\frac{N-2}{2})}\phi_{M\ell}(t)\tilde{\phi}_{\ell\,\ell_{N-2}}(\theta). \tag{2.231}$$

We can prove relation (2.228) as follows: We express $\tilde{\psi}_{\ell\,\ell_{N-2}}$ on the left-hand side in terms of $\tilde{\phi}_{\ell\,\ell_{N-2}}, d\tilde{\phi}_{\ell\,\ell_{N-2}}/d\theta_{N-1}$ using Eq. (2.64). As for the right-hand side, we expand $T^{(\pm)}_{\phi}, \tilde{T}^{(\pm)}_{\tilde{\phi}}$ using Eqs. (2.124)-(2.127) and then we express $\psi_{M\ell}$ in terms of $\phi_{M\ell}, d\phi_{M\ell}/dt$ using Eq. (2.74). Then it is straightforward to show that the two sides are equal. Relations (2.229), (2.230) and (2.231) can be proved in the same way.

Let us now derive Eq. (2.136) for the negative spin projection solution (the positive spin projection case can be treated in the same way). Substituting Eqs. (2.132) and (2.133) into Eq. (2.115) we find

$$\mathscr{L}^s_\xi\psi^{(-,\tilde{s})}_{M\ell\,\ell_{N-2}\,\tilde{\sigma}} = C_1C_2\begin{pmatrix}\xi^\mu\partial_\mu U^{(\tilde{s})}_{M\ell\,\ell_{N-2}\,\tilde{\sigma}} - \frac{\sin\theta}{2\cosh t}\tilde{\gamma}^{N-1}U^{(\tilde{s})}_{M\ell\,\ell_{N-2}\,\tilde{\sigma}}\\ \xi^\mu\partial_\mu D^{(\tilde{s})}_{M\ell\,\ell_{N-2}\,\tilde{\sigma}} + \frac{\sin\theta}{2\cosh t}\tilde{\gamma}^{N-1}D^{(\tilde{s})}_{M\ell\,\ell_{N-2}\,\tilde{\sigma}}\end{pmatrix}, \tag{2.232}$$

where $C_1 \equiv c_N(M\ell)/\sqrt{2}$ and $C_2 \equiv c_{N-1}(\ell\,\ell_{N-2})/\sqrt{2}$. Then, using

$$\tilde{\gamma}^{N-1}\hat{\tilde{\chi}}^{(\tilde{s})}_{\pm\ell_{N-2}\,\tilde{\sigma}}(\Omega_{N-2}) = \hat{\tilde{\chi}}^{(\tilde{s})}_{\mp\ell_{N-2}\,\tilde{\sigma}}(\Omega_{N-2})$$

(see Eq. (2.87)) and Eq. (2.135), it is straightforward to find

$$\frac{1}{C_1C_2}\mathscr{L}^s_\xi\psi^{(-,\tilde{s})}_{M\ell\,\ell_{N-2}\,\tilde{\sigma}}$$
$$= \begin{pmatrix}\hat{\tilde{\chi}}^{(\tilde{s})}_{-\ell_{N-2}\,\tilde{\sigma}}\Big(\xi^\mu\partial_\mu[\phi_{M\ell}\tilde{\phi}_{\ell\,\ell_{N-2}}] + i\frac{\sin\theta}{2\cosh t}\phi_{M\ell}\tilde{\psi}_{\ell\,\ell_{N-2}}\Big)\\ i\hat{\tilde{\chi}}^{(\tilde{s})}_{-\ell_{N-2}\,\tilde{\sigma}}\Big(\xi^\mu\partial_\mu[\psi_{M\ell}\tilde{\phi}_{\ell\,\ell_{N-2}}] - i\frac{\sin\theta}{2\cosh t}\psi_{M\ell}\tilde{\psi}_{\ell\,\ell_{N-2}}\Big)\end{pmatrix}.$$
$$+ \begin{pmatrix}-i\hat{\tilde{\chi}}^{(\tilde{s})}_{+\ell_{N-2}\,\tilde{\sigma}}\Big(\xi^\mu\partial_\mu[\phi_{M\ell}\tilde{\psi}_{\ell\,\ell_{N-2}}] - i\frac{\sin\theta}{2\cosh t}\phi_{M\ell}\tilde{\phi}_{\ell\,\ell_{N-2}}\Big)\\ \hat{\tilde{\chi}}^{(\tilde{s})}_{+\ell_{N-2}\,\tilde{\sigma}}\Big(\xi^\mu\partial_\mu[\psi_{M\ell}\tilde{\psi}_{\ell\,\ell_{N-2}}] + i\frac{\sin\theta}{2\cosh t}\psi_{M\ell}\tilde{\phi}_{\ell\,\ell_{N-2}}\Big)\end{pmatrix} \tag{2.233}$$

At this point we can use relations (2.228)-(2.231) to find

$$
\begin{aligned}
&\mathscr{L}_\xi^s \psi^{(-,\tilde{s})}_{M\ell\ell_{N-2}\tilde{\sigma}} \\
&\quad = C_1 C_2 \\
&\quad\times \left(\frac{1}{2(\ell+\frac{N-1}{2})} \begin{pmatrix} T^{(+)}_\phi \phi_{M\ell} \\ iT^{(+)}_\psi \psi_{M\ell} \end{pmatrix} \left[\hat{\tilde{\chi}}^{(\tilde{s})}_{-\ell_{N-2}\tilde{\sigma}} \tilde{T}^{(+)}_{\tilde{\phi}} \tilde{\phi}_{\ell\ell_{N-2}} - i\hat{\tilde{\chi}}^{(\tilde{s})}_{+\ell_{N-2}\tilde{\sigma}} \tilde{T}^{(+)}_{\tilde{\psi}} \tilde{\psi}_{\ell\ell_{N-2}} \right] \right. \\
&\quad - \frac{1}{2(\ell+\frac{N-1}{2})} \begin{pmatrix} T^{(-)}_\phi \phi_{M\ell} \\ iT^{(-)}_\psi \psi_{M\ell} \end{pmatrix} \left[\hat{\tilde{\chi}}^{(\tilde{s})}_{-\ell_{N-2}\tilde{\sigma}} \tilde{T}^{(-)}_{\tilde{\phi}} \tilde{\phi}_{\ell\ell_{N-2}} - i\hat{\tilde{\chi}}^{(\tilde{s})}_{+\ell_{N-2}\tilde{\sigma}} \tilde{T}^{(-)}_{\tilde{\psi}} \tilde{\psi}_{\ell\ell_{N-2}} \right] \\
&\quad \left. - i\frac{M(\ell_{N-2}+\frac{N-2}{2})}{2(\ell+\frac{N}{2})(\ell+\frac{N-2}{2})} \begin{pmatrix} i\psi_{M\ell} \\ \phi_{M\ell} \end{pmatrix} \left[\hat{\tilde{\chi}}^{(\tilde{s})}_{-\ell_{N-2}\tilde{\sigma}} \tilde{\phi}_{\ell\ell_{N-2}} + i\hat{\tilde{\chi}}^{(\tilde{s})}_{+\ell_{N-2}\tilde{\sigma}} \tilde{\psi}_{\ell\ell_{N-2}} \right] \right). \qquad (2.234)
\end{aligned}
$$

Then using Eqs. (2.133) and (2.135) as well as the ladder relations (2.124)-(2.127) we obtain Eq. (2.136).

2.11.2 Transformation properties for $N = 2$.

The massive positive frequency solutions (2.67)-(2.68) for $N = 2$ are given by

$$
\psi^{(-)}_{M\ell}(t,\varphi) = \frac{c_2(M\ell)}{2\sqrt{\pi}} \begin{pmatrix} \phi_{M\ell}(t) \\ i\psi_{M\ell}(t) \end{pmatrix} e^{-i(\ell+1/2)\varphi}, \qquad (2.235)
$$

$$
\psi^{(+)}_{M\ell}(t,\varphi) = \frac{c_2(M\ell)}{2\sqrt{\pi}} \begin{pmatrix} i\psi_{M\ell}(t) \\ \phi_{M\ell}(t) \end{pmatrix} e^{+i(\ell+1/2)\varphi}, \qquad (2.236)
$$

where $0 \leq \varphi \equiv \theta_1 < 2\pi$ and $\ell = 0, 1, \ldots$. By calculating the spinorial Lie derivative with respect to the boost Killing vector (2.113) we arrive again at Eq. (2.115), where $\partial\psi^{(\pm)}_{M\ell}/\partial\varphi = \pm i(\ell+\frac{1}{2})\psi^{(\pm)}_{M\ell}$. By expressing $\cos\varphi$ and $\sin\varphi$ in terms of $\exp\{(\pm i\varphi)\}$ and using the ladder operators (2.124), (2.125) with $N = 2$ it is straightforward to find

$$
\begin{aligned}
\mathscr{L}_\xi^s \psi^{(\pm)}_{M\ell} &= \frac{k^{(+)}}{2} \frac{c_2(M\ell)}{c_2(M,\,\ell+1)} \psi^{(\pm)}_{M\,\ell+1} + \frac{k^{(-)}}{2} \frac{c_2(M\ell)}{c_2(M,\,\ell-1)} \psi^{(\pm)}_{M\,\ell-1} &(2.237)\\
&= -\frac{i}{2}(\ell+1-iM)\psi^{(\pm)}_{M\,\ell+1} - \frac{i}{2}(\ell+iM)\psi^{(\pm)}_{M\,\ell-1}. &(2.238)
\end{aligned}
$$

By using Eq. (2.238) we have verified that $(\mathscr{L}_\xi^s \psi_{M\ell}, \psi_{M\,\ell\pm1}) + (\psi_{M\ell}, \mathscr{L}_\xi^s \psi_{M\,\ell\pm1}) = 0$, in agreement with the de Sitter invariance of the inner product (2.92).

2.12 APPENDIX D - DERIVATION OF THE MASSLESS WIGHTMAN TWO-POINT FUNCTION USING THE MODE-SUM METHOD

In this Appendix we present the derivation of the massless Wightman two-point function using the mode-sum method (2.163) for N even. The derivation of the two-point function for N odd has many similarities with the even-dimensional case and therefore is just briefly discussed. The case with $N = 2$ is presented separately at the end.
Let us first introduce the notation and some useful relations used in the calculations. The functions (2.61)-(2.62) used in the recursive construction of the eigenspinors of the Dirac operator on S^{N-r} $(N-r = 1, 2, ..., N-2)$ are denoted as

$$\tilde{\phi}_{\ell_{N-r}\,\ell_{N-r-1}}(\theta_{N-r}) \equiv \tilde{\phi}^{(N-r)}_{\ell_{N-r}\,\ell_{N-r-1}}, \quad \tilde{\psi}_{\ell_{N-r}\,\ell_{N-r-1}}(\theta_{N-r}) \equiv \tilde{\psi}^{(N-r)}_{\ell_{N-i}\,\ell_{N-r-1}}, \tag{2.239}$$

with $\tilde{\phi}^{(N-r)}_{0\,0} = \cos{(\theta_{N-r}/2)}$ and $\tilde{\psi}^{(N-r)}_{0\,0} = \sin{(\theta_{N-r}/2)}$ (see Eqs. (2.251) and (2.252) below). We let $\boldsymbol{\theta}_{N-r} = (\theta_{N-r}, \theta_{N-r-1}, ..., \theta_1)$. The dimension of the Spin($N-1$,1) representation is denoted as $D \equiv 2^{N/2}$. Also, let $\tilde{s}_{N-2}$ represent the spin projection indices $(s_{N-2}, s_{N-4}, ..., s_4, s_2)$, $\tilde{s}_{N-4}$ represent $(s_{N-4}, ..., s_4, s_2)$ and so forth. Similarly, σ_{N-r} represents the angular momentum quantum numbers $(\ell_{N-r}, \ell_{N-r-1}, ..., \ell_2, \ell_1)$ etc. Note that for $\theta'_{N-1} = \theta'_{N-2} = ... = \theta'_1 = 0$ we have

$$\cos\mu|_{\boldsymbol{\theta}'=\mathbf{0}} = -\sinh t \sinh t' + \cosh t \cosh t' \cos\theta_{N-1}, \tag{2.240}$$

(see Eq. (2.28)) while the only non-zero (vielbein basis) components of the tangent vector $n_a|_{\boldsymbol{\theta}'=0}$ (see Eqs. (2.34)–(2.36)) are given by

$$n_0|_{\boldsymbol{\theta}'=\mathbf{0}} = \frac{1}{\sin\mu}(\cosh t \sinh t' - \sinh t \cosh t' \cos\theta_{N-1}), \tag{2.241}$$

$$n_{N-1}|_{\boldsymbol{\theta}'=\mathbf{0}} = \frac{\cosh t'}{\sin\mu}\sin\theta_{N-1} = \frac{1}{\cosh t} n_{\theta_{N-1}}|_{\boldsymbol{\theta}'=\mathbf{0}}. \tag{2.242}$$

(For brevity we will denote $n_0|_{\boldsymbol{\theta}'=\mathbf{0}}$, $n_{N-1}|_{\boldsymbol{\theta}'=\mathbf{0}}$, and $\not{n}|_{\boldsymbol{\theta}'=\mathbf{0}}$ by n_0, n_{N-1} and $\not{n}$ respectively.)
Also, notice that Spin$(N-1,1)$ transformation matrices can be expressed as

$$\exp\Big\{a\,\Sigma^{0j}\Big\} = \exp\Big\{\frac{a}{2}\,\gamma^0\gamma^j\Big\} = \mathbf{1}\,\cosh\frac{a}{2} + \gamma^0\gamma^j\,\sinh\frac{a}{2}, \tag{2.243}$$

$$\exp\Big\{b\,\Sigma^{kj}\Big\} = \exp\Big\{\frac{b}{2}\,\gamma^k\gamma^j\Big\} = \mathbf{1}\,\cos\frac{b}{2} + \gamma^k\gamma^j\,\sin\frac{b}{2}, \tag{2.244}$$

(with $k \neq j$ and $k, j = 1, 2, ..., N-1$) where a, b are the transformation parameters. The corresponding generators are given by Eq. (2.42). Also, many of the following calculations involve the variables $x = \pi/2 - it$, $x' = \pi/2 - it'$ (see Eq. (2.6)).

We can now start deriving the massless Wightman two-point function for N even. By expanding the summation over the spin projections ($s=\pm$) Eq. (2.163) becomes

$$\begin{aligned}&W_0\Big((t,\boldsymbol{\theta}_{N-1}),(t',\mathbf{0})\Big)\\&=\sum_{\ell=0}^{\infty}\sum_{\sigma_{N-2}}\sum_{\tilde{s}_{N-2}}\Big[\psi^{(+,\tilde{s}_{N-2})}_{0\ell\sigma_{N-2}}(t,\boldsymbol{\theta}_{N-1})\overline{\psi}^{(+,\tilde{s}_{N-2})}_{0\ell\sigma_{N-2}}(t',\mathbf{0})+\psi^{(-,\tilde{s}_{N-2})}_{0\ell\sigma_{N-2}}(t,\boldsymbol{\theta}_{N-1})\overline{\psi}^{(-,\tilde{s}_{N-2})}_{0\ell\sigma_{N-2}}(t',\mathbf{0})\Big].\end{aligned} \tag{2.245}$$

Then using Eqs. (2.188)-(2.189) we find

$$\begin{aligned}&W_0\Big((t,\boldsymbol{\theta}_{N-1}),(t',\mathbf{0})\Big)\\&=-\left|\frac{c_N(M=0)}{\sqrt{2}}\right|^2\sum_{\ell=0}^{\infty}\sum_{\sigma_{N-2}}\phi_{0\ell}(t)\phi^*_{0\ell}(t')\\&\times\sum_{\tilde{s}_{N-4}}\sum_{s_{N-2}}\begin{pmatrix}0 & \chi^{(s_{N-2},\tilde{s}_{N-4})}_{-\ell\sigma_{N-2}}(\boldsymbol{\theta}_{N-1})\chi^{(s_{N-2},\tilde{s}_{N-4})}_{-\ell\sigma_{N-2}}(\mathbf{0})^{\dagger}\\ \chi^{(s_{N-2},\tilde{s}_{N-4})}_{+\ell\sigma_{N-2}}(\boldsymbol{\theta}_{N-1})\chi^{(s_{N-2},\tilde{s}_{N-4})}_{+\ell\sigma_{N-2}}(\mathbf{0})^{\dagger} & 0\end{pmatrix},\end{aligned} \tag{2.246}$$

where $\chi^{(s_{N-2},\tilde{s}_{N-4})}_{\pm\ell\sigma_{N-2}}$ are the eigenspinors on S^{N-1}. In order to proceed we need to express the eigenspinors on S^{N-r}, with $N-r$ odd, in terms of eigenspinors on S^{N-r-2}. Therefore, using Eqs. (2.84), (2.55) and (2.56) we derive the following two recursive relations:

$$\begin{aligned}\chi^{(-,\tilde{s}_{N-r-3})}_{\pm\ell_{N-r}\,\sigma_{N-r-1}}(\boldsymbol{\theta}_{N-r})=&\frac{c_{N-r}(\ell_{N-r}\,\ell_{N-r-1})}{\sqrt{2}}\frac{c_{N-r-1}(\ell_{N-r-1}\,\ell_{N-r-2})}{\sqrt{2}}\frac{1}{\sqrt{2}}\\&\times\begin{pmatrix}(1+i)\tilde{\phi}^{(N-r-1)}_{\ell_{N-r-1}\,\ell_{N-r-2}}[\tilde{\phi}^{(N-r)}_{\ell_{N-r}\,\ell_{N-r-1}}\pm i\tilde{\psi}^{(N-r)}_{\ell_{N-r}\,\ell_{N-r-1}}]\\ -(1+i)\tilde{\psi}^{(N-r-1)}_{\ell_{N-r-1}\,\ell_{N-r-2}}[\tilde{\phi}^{(N-r)}_{\ell_{N-r}\,\ell_{N-r-1}}\mp i\tilde{\psi}^{(N-r)}_{\ell_{N-r}\,\ell_{N-r-1}}]\end{pmatrix}\\&\times\chi^{(\tilde{s}_{N-r-3})}_{-\ell_{N-r-2},\,\sigma_{N-r-3}}(\boldsymbol{\theta}_{N-r-2}),\end{aligned} \tag{2.247}$$

$$\begin{aligned}\chi^{(+,\tilde{s}_{N-r-3})}_{\pm\ell_{N-r}\,\sigma_{N-r-1}}(\boldsymbol{\theta}_{N-r})=&\frac{c_{N-r}(\ell_{N-r}\,\ell_{N-r-1})}{\sqrt{2}}\frac{c_{N-r-1}(\ell_{N-r-1}\,\ell_{N-r-2})}{\sqrt{2}}\frac{1}{\sqrt{2}}\\&\times\begin{pmatrix}(-1+i)\tilde{\psi}^{(N-r-1)}_{\ell_{N-r-1}\,\ell_{N-r-2}}[\tilde{\phi}^{(N-r)}_{\ell_{N-r}\,\ell_{N-r-1}}\pm i\tilde{\psi}^{(N-r)}_{\ell_{N-r}\,\ell_{N-r-1}}]\\ (-1+i)\tilde{\phi}^{(N-r-1)}_{\ell_{N-r-1}\,\ell_{N-r-2}}[\tilde{\phi}^{(N-r)}_{\ell_{N-r}\,\ell_{N-r-1}}\mp i\tilde{\psi}^{(N-r)}_{\ell_{N-r}\,\ell_{N-r-1}}]\end{pmatrix}\\&\times\chi^{(\tilde{s}_{N-r-3})}_{+\ell_{N-r-2},\,\sigma_{N-r-3}}(\boldsymbol{\theta}_{N-r-2})\end{aligned} \tag{2.248}$$

(for r odd and $N-3\geq r\geq 1$). Since $\tilde{\psi}^{(N-r)}_{\ell_{N-r}\,\ell_{N-r-1}}(0)=0$ and $\tilde{\phi}^{(N-r)}_{\ell_{N-r}\,\ell_{N-r-1}}(0)$ is non-zero only for $\ell_{N-r-1}=0$, it is clear from the recursive relations (2.247)-(2.248) that the only non-vanishing terms in Eq. (2.246) are the ones with $\ell_{N-2} = \ell_{N-3} = ... = \ell_2 = \ell_1 = 0$. Thus, only the summation over $\ell_{N-1} \equiv \ell$ survives in the mode-sum. Substituting Eqs. (2.247) and (2.248) (with $r = 1$) into Eq. (2.246) one obtains (after some calculations)

$$\begin{aligned} W_0\Big((t,\boldsymbol{\theta}_{N-1}),(t',\mathbf{0})\Big) &= \left|\frac{c_N(M=0)}{\sqrt{2}}\right|^2 \sum_{\ell=0}^{\infty} \phi_{0\ell}(t)\phi^*_{0\ell}(t') \left|\frac{c_{N-1}(\ell 0)}{\sqrt{2}}\right|^2 \left|\frac{c_{N-2}(00)}{\sqrt{2}}\right|^2 \\ &\times \tilde{\phi}^{(N-1)}_{\ell 0}(0)^* \left[i\tilde{\phi}^{(N-1)}_{\ell 0}(\theta_{N-1})\,\gamma^0 + \tilde{\psi}^{(N-1)}_{\ell 0}(\theta_{N-1})\,\gamma^{N-1}\right] \\ &\times \left[\mathbb{I}_2 \otimes \begin{pmatrix} \tilde{\phi}^{(N-2)}_{00} & \tilde{\psi}^{(N-2)}_{00} \\ -\tilde{\psi}^{(N-2)}_{00} & \tilde{\phi}^{(N-2)}_{00} \end{pmatrix} \otimes \mathbb{I}_{D/4}\right] \\ &\times \left[\mathbb{I}_2 \otimes \sum_{\tilde{s}_{N-4}} \begin{pmatrix} \chi^{(\tilde{s}_{N-4})}_{-00}(\boldsymbol{\theta}_{N-3})\chi^{(\tilde{s}_{N-4})}_{-00}(\mathbf{0})^\dagger & 0 \\ 0 & \chi^{(\tilde{s}_{N-4})}_{+00}(\boldsymbol{\theta}_{N-3})\chi^{(\tilde{s}_{N-4})}_{+00}(\mathbf{0})^\dagger \end{pmatrix}\right], \end{aligned} \tag{2.249}$$

where $\mathbb{I}_d$ is the identity matrix of dimension d. Also, we are going to use the following results:

$$|c_{N-1}(\ell 0)|^2 = \frac{\Gamma(\ell+1)\Gamma(\ell+N-1)}{2^{N-3}|\Gamma(\ell+\frac{N-1}{2})|^2}, \tag{2.250}$$

$$\tilde{\phi}^{(N-1)}_{\ell 0}(\theta_{N-1}) = \kappa^{(N-1)}_{\phi}(\ell 0)\,\cos\frac{\theta_{N-1}}{2} F\left(\ell+N-1,-\ell;\frac{N-1}{2};\sin^2\frac{\theta_{N-1}}{2}\right), \tag{2.251}$$

$$\begin{aligned} \tilde{\psi}^{(N-1)}_{\ell 0}(\theta_{N-1}) &= \kappa^{(N-1)}_{\phi}(\ell 0)\frac{(\ell+(N-1)/2)}{(N-1)/2}\,\sin\frac{\theta_{N-1}}{2} \\ &\times F\left(\ell+N-1,-\ell;\frac{N+1}{2};\sin^2\frac{\theta_{N-1}}{2}\right), \end{aligned} \tag{2.252}$$

where

$$\kappa^{(N-1)}_{\phi}(\ell 0) = \frac{\Gamma(\ell+\frac{N-1}{2})}{\ell!\,\Gamma(\frac{N-1}{2})} \tag{2.253}$$

(see Eqs. (2.57) and (2.61)–(2.63)).

We complete the derivation of the massless two-point function in three steps: 1) we calculate the proportionality constant (and we show that it agrees with the proportionality constant in Eq. (2.183); 2) we obtain a closed-form result for the infinite sum over ℓ and we determine the dependence on $\{t,\theta_{N-1},t'\}$; 3) we obtain analytic expressions for the terms of the two-point function that depend only on the angular variables $\theta_{N-2},\theta_{N-3},...,\theta_1$. We call the latter the "angular part" of the two-point function and we denote it as follows:

$$\widetilde{W}_0(\Omega_{N-2}) \equiv \left(\prod_{j=3}^{N-1}\left|\frac{c_{N-j}(00)}{\sqrt{2}}\right|^{-2}\right) \times \left[\mathbb{I}_2 \otimes \begin{pmatrix} \tilde{\phi}_{00}^{(N-2)} & \tilde{\psi}_{00}^{(N-2)} \\ -\tilde{\psi}_{00}^{(N-2)} & \tilde{\phi}_{00}^{(N-2)} \end{pmatrix} \otimes \mathbb{I}_{D/4}\right]$$
$$\times \left[\mathbb{I}_2 \otimes \sum_{\tilde{s}_{N-4}} \begin{pmatrix} \chi_{-00}^{(\tilde{s}_{N-4})}(\boldsymbol{\theta}_{N-3})\chi_{-00}^{(\tilde{s}_{N-4})}(\mathbf{0})^\dagger & 0 \\ 0 & \chi_{+00}^{(\tilde{s}_{N-4})}(\boldsymbol{\theta}_{N-3})\chi_{+00}^{(\tilde{s}_{N-4})}(\mathbf{0})^\dagger \end{pmatrix}\right]. \tag{2.254}$$

After completing these steps it will be clear that the obtained two-point function is of the form (2.182) (i.e. it agrees with the construction presented in Ref. [19]).

2.12.1 THE PROPORTIONALITY CONSTANT

The proportionality constant for the massless two-point function arises from the normalization factors in Eq. (2.249). (Note that apart from $c_N(M=0)$ and $c_{N-1}(\ell 0)$ there are $N-2$ additional normalization factors; one for each lower-dimensional sphere.) The overall contribution from the normalization factors is given by the following product:

$$\left|\frac{c_N(M=0)}{\sqrt{2}}\right|^2 \left|\frac{c_{N-1}(\ell 0)}{\sqrt{2}}\kappa_\phi^{(N-1)}(\ell 0)\right|^2 \prod_{j=2}^{N-1}\left|\frac{c_{N-j}(00)}{\sqrt{2}}\right|^2 \tag{2.255}$$

$$=\frac{1}{2\pi}\left|\frac{c_N(M=0)}{\sqrt{2}}\right|^2 \left|\frac{c_{N-1}(\ell 0)}{\sqrt{2}}\kappa_\phi^{(N-1)}(\ell 0)\right|^2 \prod_{j=2}^{N-2}\left|\frac{c_{N-j}(00)}{\sqrt{2}}\right|^2 \tag{2.256}$$

where $c_1(00) \equiv 1/\sqrt{\pi}$ is the normalization factor for eigenspinors on S^1, while the normalization factors for eigenspinors on higher-dimensional spheres are given by Eq. (2.57). Using Eqs. (2.250) and (2.253) we observe that

$$\left|c_{N-1}(\ell 0)\,\kappa_\phi^{(N-1)}(\ell 0)\right|^2 = \left|c_{N-1}(00)\right|^2 \frac{(N-1)_\ell}{\ell!}, \tag{2.257}$$

where $(N-1)_\ell = \Gamma(N-1+\ell)/\Gamma(N-1)$ is the Pochhamer symbol for the rising factorial. Using Eq. (2.257) we may rewrite Eq. (2.255) as

$$\frac{1}{\pi 2^N 2^{N-2}} \prod_{j=1}^{N-2} \frac{\Gamma(N-j)}{|\Gamma(\frac{N-j}{2})|^2 2^{N-j-2}} \times \frac{(N-1)_\ell}{\ell!}$$
$$= \frac{\Gamma(N/2)}{2^{N/2}\,(2\pi)^{N/2}} \times \frac{(N-1)_\ell}{\ell!}, \tag{2.258}$$

where we also used Eqs. (2.57), (2.110) and the Legendre duplication formula (2.109). Equation (2.258) clearly gives the desired form for the proportionality constant (see Eq. (2.183)). The ℓ-dependence in Eq. (2.258) will be discussed later (it will be used in the summation over ℓ).

2.12.2 Obtaining a closed-form expression for the series

Using Eqs. (2.249), (2.251), (2.252) and (2.258) we collect all the ℓ-dependent terms of the two-point function. Then the mode-sum expression (2.249) can be written as

$$W_0\Big((t,\boldsymbol{\theta}_{N-1}),(t',\mathbf{0})\Big) = \frac{\Gamma(N/2)}{2^{N/2}\,(2\pi)^{N/2}}[iA\gamma^0 + B\gamma^{N-1}]\,\widetilde{W}_0(\Omega_{N-2}), \tag{2.259}$$

where

$$A = \cos\frac{\theta_{N-1}}{2}\sum_{\ell=0}^{\infty}\frac{(N-1)_\ell}{\ell!}\phi_{0\ell}(t)\phi^*_{0\ell}(t')\,F\left(\ell+N-1,-\ell;\frac{N-1}{2};\sin^2\frac{\theta_{N-1}}{2}\right), \tag{2.260}$$

$$B = \frac{2\sin\frac{\theta_{N-1}}{2}}{N-1}\sum_{\ell=0}^{\infty}\frac{(N-1)_\ell}{\ell!}\left(\ell+\frac{N-1}{2}\right)\phi_{0\ell}(t)\phi^*_{0\ell}(t')$$
$$\times F\left(\ell+N-1,-\ell;\frac{N+1}{2};\sin^2\frac{\theta_{N-1}}{2}\right). \tag{2.261}$$

Using Eq. (2.190) for $\phi_{0\ell}(t)$, $\phi^*_{0\ell}(t')$ we find

$$A = \frac{\cos{(\theta_{N-1}/2)}}{\left(\cos(x/2)\,[\cos(x'/2)]^*\right)^{N-1}}\sum_{\ell=0}^{\infty}\frac{(N-1)_\ell}{\ell!}\Big(\rho(t,t')\Big)^\ell$$
$$\times F\left(\ell+N-1,-\ell;\frac{N-1}{2};\sin^2\frac{\theta_{N-1}}{2}\right), \tag{2.262}$$

$$B = \frac{2}{N-1}\frac{\sin{(\theta_{N-1}/2)}}{\left(\cos(x/2)\,[\cos(x'/2)]^*\right)^{N-1}}\sum_{\ell=0}^{\infty}\frac{(N-1)_\ell}{\ell!}(\ell+\frac{N-1}{2})\Big(\rho(t,t')\Big)^\ell$$
$$\times F\left(\ell+N-1,-\ell;\frac{N+1}{2};\sin^2\frac{\theta_{N-1}}{2}\right). \tag{2.263}$$

where

$$\rho(t,t') \equiv \tan\frac{x}{2}\left[\tan\frac{x'}{2}\right]^* = \frac{(1-i\sinh t)(1+i\sinh t')}{\cosh t\,\cosh t'}. \tag{2.264}$$

Let us first find the infinite sum in A. By using the formula [22]

$$\sum_{k=0}^{\infty}\frac{(a)_k}{k!}t^kF(-k,a+k;c;z) = (1-t)^{-a}F(\frac{a}{2},\frac{a+1}{2};c;\frac{-4t}{(1-t)^2}z),\quad |t|<1 \tag{2.265}$$

Eq. (2.262) can be written as

$$A = \frac{\cos{(\theta_{N-1}/2)}}{\left(\cos{(x/2)}\,[\cos{(x'/2)}]^*\right)^{N-1}}(1-\rho(t,t'))^{-N+1}$$
$$\times F\left(\frac{N-1}{2},\frac{N}{2};\frac{N-1}{2};\frac{-4\rho(t,t')}{(1-\rho(t,t'))^2}\sin^2\frac{\theta_{N-1}}{2}\right) \tag{2.266}$$
$$= \frac{\cos{(\theta_{N-1}/2)}}{\left(\cos{(x/2)}\,[\cos{(x'/2)}]^*\right)^{N-1}}\frac{(1-\rho(t,t'))^{-N+1}\left((1-\rho(t,t'))^2\right)^{N/2}}{[(1-\rho(t,t'))^2+4\rho(t,t')\sin^2(\frac{\theta_{N-1}}{2})]^{N/2}}, \tag{2.267}$$

where we also used

$$F(a,b;b;z) = \frac{1}{(1-z)^a}. \tag{2.268}$$

(Note that since $|\rho(t,t')| = 1$ the series in Eq. (2.262) diverges. Therefore, we make the replacement $t \to t - i\epsilon$ with $\epsilon > 0$ before applying (2.265) and then we let $\epsilon \to 0$.) By expressing x, x' and $\rho(t,t')$ in terms of t and t' we can write Eq. (2.267) as

$$\begin{aligned} A =& i \sinh\frac{t-t'}{2}\cos\frac{\theta_{N-1}}{2}(\sin^2\frac{\mu}{2})^{-N/2} && (2.269)\\ =& \, i w_1(t,\theta_{N-1},t')\,(\sin^2\frac{\mu}{2})^{-N/2}. && (2.270)\end{aligned}$$

(The biscalar function w_1 is given in Eq. (2.196), while $\sin^2(\mu/2)$ can be found by Eq. (2.240)).

Let us now find the infinite sum in B. We can rewrite Eq. (2.263) as

$$\begin{aligned} B =& \frac{2}{N-1}\frac{\sin(\theta_{N-1}/2)}{(\cos(x/2)\,[\cos(x'/2)]^*)^{N-1}} \\ & \times\left(\rho\frac{\partial}{\partial\rho}+\frac{N-1}{2}\right) \\ & \times\sum_{\ell=0}^{\infty}\frac{(N-1)_\ell}{\ell!}\,\rho^\ell\,F\left(\ell+N-1,-\ell;\frac{N+1}{2};\sin^2\frac{\theta_{N-1}}{2}\right), \end{aligned} \tag{2.271}$$

where t should be understood as $t - i\epsilon$ ($\epsilon > 0$) in order to achieve convergence in this series. (We take the limit $\epsilon \to 0$ at the end of the calculation.) At this point we use again the formula (2.265) and then we introduce the variable

$$X \equiv \frac{-4\rho}{(1-\rho)^2}\sin^2\frac{\theta_{N-1}}{2}. \tag{2.272}$$

After some calculations we can rewrite B as follows:

$$\begin{aligned} B =& \frac{2}{N-1}\frac{\sin(\theta_{N-1}/2)}{(\cos(x/2)\,[\cos(x'/2)]^*)^{N-1}}\frac{1+\rho}{(1-\rho)^N} \\ & \times[X\frac{\partial}{\partial X}+\frac{N-1}{2}]F(\frac{N-1}{2},\frac{N}{2};\frac{N+1}{2};X), \end{aligned} \tag{2.273}$$

where we notice the appearance of the raising operator for the first parameter of the hypergeometric function (2.225). Then using Eq. (2.268) we obtain

$$\begin{aligned} B =& \frac{\sin(\theta_{N-1}/2)}{(\cos(x/2)\,[\cos(x'/2)]^*)^{N-1}} \\ & \times\frac{1+\rho(t,t')}{[(1-\rho(t,t'))^2+4\rho(t,t')\sin^2(\frac{\theta_{N-1}}{2})]^{N/2}}\frac{((1-\rho(t,t'))^2)^{N/2}}{(1-\rho(t,t'))^N}. \end{aligned} \tag{2.274}$$

After a straightforward calculation Eq. (2.274) can be written as

$$B = \cosh\frac{t+t'}{2}\sin\frac{\theta_{N-1}}{2}\,(\sin^2\frac{\mu}{2})^{-N/2} \tag{2.275}$$
$$= w_2(t,\theta_{N-1},t')(\sin^2\frac{\mu}{2})^{-N/2}. \tag{2.276}$$

(The biscalar function w_2 is given in Eq. (2.197).)

By combining Eqs. (2.270) and (2.276), the two-point function (2.259) can be written in the following form:

$$W_0\Big((t,\boldsymbol{\theta}_{N-1}),(t',\mathbf{0})\Big) = \frac{\beta_0(\mu)}{\sin(\mu/2)}[-w_1\gamma^0 + w_2\gamma^{N-1}]\,\widetilde{W}_0(\Omega_{N-2}), \tag{2.277}$$

where $\beta_0(\mu)$ is given by Eq. (2.179). We can simplify this expression by using $\not{n}^2 = \mathbf{1}$ and observing that

$$\not{n}[-w_1\gamma^0 + w_2\gamma^{N-1}] = (w_1\,n_0 + w_2\,n_{N-1})\mathbf{1} + (w_2\,n_0 + w_1\,n_{N-1})\gamma^0\gamma^{N-1} \tag{2.278}$$
$$= \sin\frac{\mu}{2}\,(\mathbf{1}\,\cosh\frac{\lambda}{2} + \gamma^0\gamma^{N-1}\,\sinh\frac{\lambda}{2}) \tag{2.279}$$
$$= \sin\frac{\mu}{2}\exp\Big\{(\frac{\lambda}{2}\gamma^0\gamma^{N-1})\Big\}, \tag{2.280}$$

where in the last line we used Eq. (2.243). (For the definition of λ see Eq. (2.195).) Substituting Eq. (2.280) into the expression (2.277) of the two-point function we find

$$W_0\Big((t,\boldsymbol{\theta}_{N-1}),(t',\mathbf{0})\Big) = \beta_0(\mu)\not{n}\,\exp\Big\{(\frac{\lambda}{2}\gamma^0\gamma^{N-1})\Big\}\,\widetilde{W}_0(\Omega_{N-2}).$$

The bispinor $\exp\Big\{(\frac{\lambda}{2}\gamma^0\gamma^{N-1})\Big\}\,\widetilde{W}_0(\Omega_{N-2})$ is the spinor parallel propagator (see Eq. (2.194)).

2.12.3 Determining the "angular part" of the two-point function

In this section of the Appendix we show that the "angular part" $\widetilde{W}_0(\Omega_{N-2})$ (which is defined in Eq. (2.254)) can be written as a product of $N-2$ rotation matrices $\in$ Spin$(N-1,1)$ (see Eq. (2.244)). As is well known, these rotation matrices can be constructed by exponentiating the generators (2.42).

It is convenient to express the $2^{N/2}$-dimensional gamma matrices (2.38) using the tensor-product notation as follows:

$$\gamma^0 = i\sigma^2 \otimes \mathbb{I}_{2^{N/2-1}},$$
$$\gamma^{N-r} = [\bigotimes_{i=1}^{[r/2]+1} \sigma^1] \otimes \sigma^3 \otimes \mathbb{I}_{2^{(N-3-r)/2}},$$
$$\gamma^{N-r-1} = [\bigotimes_{i=1}^{[r/2]+1} \sigma^1] \otimes \sigma^2 \otimes \mathbb{I}_{2^{(N-3-r)/2}}, \quad r \text{ odd}, \ 1 \leq r \leq N-3,$$
$$\gamma^1 = \bigotimes_{i=1}^{N/2} \sigma^1, \tag{2.281}$$

where the Pauli matrices are given by

$$\sigma^1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \ \sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.282}$$

Note that they satisfy $\sigma^i \sigma^j = \delta^{ij} + i\sum_k \epsilon^{ijk}\sigma^k$, where ϵ^{ijk} is the totally antisymmetric tensor (the latter equals $+1$ if (i,j,k) is an even permutation of $(1,2,3)$ and -1 if it is an odd permutation). The form of the Pauli matrices we use here is related to their conventional form as follows: $\sigma_1 = \sigma^2, \sigma_2 = -\sigma^1, \sigma_3 = \sigma^3$, where lower indices are used to label the conventional Pauli matrices. For later convenience, consider the rotation generators $\gamma^{N-r+1}\gamma^{N-r}/2$ and $\gamma^{N-r}\gamma^{N-r-1}/2$ (r odd) of Spin$(N-1,1)$ (see Eq. (2.42)). Using Eqs. (2.281) for the gamma matrices the generators can be written as

$$\frac{1}{2}\gamma^{N-r+1}\gamma^{N-r} = \mathbb{I}_{2^{[r/2]}} \otimes (-\frac{i}{2}\sigma^3) \otimes \sigma^3 \otimes \mathbb{I}_{2^{(N-3-r)/2}}, \quad r \text{ odd}, \ N-3 \geq r \geq 3, \tag{2.283}$$

$$\frac{1}{2}\gamma^{N-r}\gamma^{N-r-1} = \mathbb{I}_{2^{[r/2]}} \otimes \mathbb{I}_2 \otimes (-\frac{i}{2}\sigma^1) \otimes \mathbb{I}_{2^{(N-3-r)/2}}, \quad r \text{ odd}, \ N-3 \geq r \geq 1 \tag{2.284}$$

and the corresponding rotation matrices with parameters θ_{N-r}, θ_{N-r-1} are respectively found to be

$$\exp\left\{\frac{\theta_{N-r}}{2}\gamma^{N-r+1}\gamma^{N-r}\right\} = \mathbb{I}_{2^{[r/2]}} \otimes \exp\left[-\frac{i\theta_{N-r}}{2}\sigma^3 \otimes \sigma^3\right] \otimes \mathbb{I}_{2^{(N-3-r)/2}},$$
$$r \text{ odd}, \ N-3 \geq r \geq 3 \tag{2.285}$$

and

$$\exp\left\{\frac{\theta_{N-r-1}}{2}\gamma^{N-r}\gamma^{N-r-1}\right\} = \mathbb{I}_{2^{[r/2]}} \otimes \exp\left\{\mathbb{I}_2 \otimes (-\frac{i\theta_{N-r-1}}{2}\sigma^1)\right\} \otimes \mathbb{I}_{2^{(N-3-r)/2}},$$
$$r \text{ odd}, \ N-3 \geq r \geq 1. \tag{2.286}$$

Similarly, one can show that

$$\exp\left[\frac{\theta_1}{2}\gamma^2\gamma^1\right] = \mathbb{I}_{2^{(N-2)/2}} \otimes \exp\left\{-i\frac{\theta_1}{2}\sigma^3\right\}. \tag{2.287}$$

Our goal is to express the "angular part" of the two-point function (2.254) as a product consisting of rotation matrices such as (2.285), (2.286) and (2.287). By using Eq. (2.286) we can write the "angular part" (2.254) of the two point function as follows:

$$\widetilde{W}_0(\Omega_{N-2}) = \exp\left\{[\frac{\theta_{N-2}}{2}\gamma^{N-1}\gamma^{N-2}]\right\}\left[\mathbb{I}_2 \otimes \begin{pmatrix} X_-^{(N-3)} & 0 \\ 0 & X_+^{(N-3)} \end{pmatrix}\right], \tag{2.288}$$

where we also used $\tilde{\phi}_{0\,0}^{(N-2)} = \cos(\theta_{N-2}/2)$, $\tilde{\psi}_{0\,0}^{(N-2)} = \sin(\theta_{N-2}/2)$ (see Eqs. (2.251)-(2.252)) and we defined

$$\begin{aligned} X_\pm^{(N-r)} \equiv & \prod_{j=r}^{N-1}\left|\frac{c_{N-j}(00)}{\sqrt{2}}\right|^{-2} \\ & \times \left[\sum_{\tilde{s}_{N-r-3}}\sum_{s_{N-r-1}} \chi_{\pm 00}^{(s_{N-r-1},\tilde{s}_{N-r-3})}(\boldsymbol{\theta}_{N-r})\chi_{\pm 00}^{(s_{N-r-1},\tilde{s}_{N-r-3})}(\mathbf{0})^\dagger\right], \\ & r \text{ odd},\ N-3 \geq r \geq 3, \end{aligned} \tag{2.289}$$

with $X_\pm^{(1)} \equiv \exp\{[\pm i\theta_1/2]\}$. In order to proceed we use the recursive relations (2.247)-(2.248) to find the following recursive relation:

$$\begin{aligned} X_\pm^{(N-r)} = & \begin{pmatrix} (\tilde{\phi}_{00}^{(N-r)} \pm i\tilde{\psi}_{00}^{(N-r)})\mathbf{1} & 0 \\ 0 & (\tilde{\phi}_{00}^{(N-r)} \mp i\tilde{\psi}_{00}^{(N-r)})\mathbf{1} \end{pmatrix}\begin{pmatrix} \tilde{\phi}_{00}^{(N-r-1)}\mathbf{1} & \tilde{\psi}_{00}^{(N-r-1)}\mathbf{1} \\ -\tilde{\psi}_{00}^{(N-r-1)}\mathbf{1} & \tilde{\phi}_{00}^{(N-r-1)}\mathbf{1} \end{pmatrix} \\ & \times \begin{pmatrix} X_-^{(N-r-2)} & 0 \\ 0 & X_+^{(N-r-2)} \end{pmatrix} \\ = & \left[\begin{pmatrix} \exp\left\{(\pm i\frac{\theta_{N-r}}{2})\right\} & 0 \\ 0 & \exp\left\{(\mp i\frac{\theta_{N-r}}{2})\right\} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta_{N-r-1}}{2} & \sin\frac{\theta_{N-r-1}}{2} \\ -\sin\frac{\theta_{N-r-1}}{2} & \cos\frac{\theta_{N-r-1}}{2} \end{pmatrix} \otimes \mathbb{I}_{2^{(N-3-r)/2}}\right] \\ & \times \begin{pmatrix} X_-^{(N-r-2)} & 0 \\ 0 & X_+^{(N-r-2)} \end{pmatrix}, \qquad r \text{ odd},\ N-3 \geq r \geq 3, \end{aligned} \tag{2.290}$$

where we expanded the summation over the spin projection index s_{N-r-1} in Eq. (2.289) and we used $\tilde{\phi}_{0\,0}^{(n)} = \cos(\theta_n/2)$, $\tilde{\psi}_{0\,0}^{(n)} = \sin(\theta_n/2)$. Then, by combining Eqs. (2.285), (2.286) and (2.290), we can show that

$$\mathbb{I}_{2^{[r/2]}} \otimes \begin{pmatrix} X_-^{(N-r)} & 0 \\ 0 & X_+^{(N-r)} \end{pmatrix} = \exp\left\{\frac{\theta_{N-r}}{2}\gamma^{N-r+1}\gamma^{N-r}\right\}\exp\left\{\frac{\theta_{N-r-1}}{2}\gamma^{N-r}\gamma^{N-r-1}\right\}$$
$$\times\left[\mathbb{I}_{2^{[(r+2)/2]}} \otimes \begin{pmatrix} X_-^{(N-(r+2))} & 0 \\ 0 & X_+^{(N-(r+2))} \end{pmatrix}\right],$$
$$r \text{ odd}, \; N-3 \geq r \geq 3. \tag{2.291}$$

We can now sequentially apply the recursive relation (2.291) for $r = 3, 5, ..., N-3$ in the expression for the "angular part" of the two-point function (2.288). It is straightforward to find

$$\widetilde{W}_0(\Omega_{N-2})$$
$$= \exp\left\{\frac{\theta_{N-2}}{2}\gamma^{N-1}\gamma^{N-2}\right\}\exp\left\{\frac{\theta_{N-3}}{2}\gamma^{N-2}\gamma^{N-3}\right\}...\exp\left\{\frac{\theta_2}{2}\gamma^3\gamma^2\right\}\exp\left\{\frac{\theta_1}{2}\gamma^2\gamma^1\right\} \tag{2.292}$$
$$= \prod_{j=2}^{N-1}\exp\left\{\frac{\theta_{N-j}}{2}\gamma^{N-j+1}\gamma^{N-j}\right\}. \tag{2.293}$$

2.12.4 Massless Wightman two-point function for N odd

The derivation for N odd shares many similarities with the case with N even. Therefore, we just outline the steps involved in the calculation.

Substituting the massless positive frequency modes (2.200) into the mode-sum expression (2.163) and working as in the case with N even it is straightforward to derive Eq. (2.259) (where A, B and the proportionality constant are calculated in the same way as for N even). The "angular part" of the two-point function is given by

$$\widetilde{W}_0(\Omega_{N-2}) \equiv \left(\prod_{j=2}^{N-1}\left|\frac{c_{N-j}(00)}{\sqrt{2}}\right|^{-2}\right)$$
$$\times\sum_{\tilde{s}_{N-3}}\begin{pmatrix} \chi_{-00}^{(\tilde{s}_{N-3})}(\boldsymbol{\theta}_{N-2})\chi_{-00}^{(\tilde{s}_{N-3})}(\mathbf{0})^\dagger & 0 \\ 0 & \chi_{+00}^{(\tilde{s}_{N-3})}(\boldsymbol{\theta}_{N-2})\chi_{+00}^{(\tilde{s}_{N-3})}(\mathbf{0})^\dagger \end{pmatrix}. \tag{2.294}$$

The gamma matrices (2.40) have dimension $D = 2^{[N/2]}$ and can be expressed in terms of Pauli matrices as follows:

$$\gamma^0 = i\sigma^3 \otimes \mathbb{I}_{2[N/2]-1},$$
$$\gamma^{N-1} = \sigma^2 \otimes \mathbb{I}_{2[N/2]-1},$$
$$\gamma^{N-r-1} = [\bigotimes_{i=1}^{[r/2]+1} \sigma^1] \otimes \sigma^3 \otimes \mathbb{I}_{2(N-4-r)/2},$$
$$\gamma^{N-r-2} = [\bigotimes_{i=1}^{[r/2]+1} \sigma^1] \otimes \sigma^2 \otimes \mathbb{I}_{2(N-4-r)/2}, \quad r = \text{odd}, \ 1 \le r \le N-4, \tag{2.295}$$
$$\gamma^1 = \bigotimes_{i=1}^{[N/2]} \sigma^1. \tag{2.296}$$

Then, as in the case with N even, we can obtain Eqs. (2.285), (2.286) and (2.291) with $N \to N-1$ and r odd, $1 \le r \le N-4$. The recursive relation (2.291) (with $N \to N-1$) can be sequentially applied for $r = 1, 3, ..., N-4$ in Eq. (2.294). Then one obtains the final expression (2.292) for the "angular part" .

2.12.5 Massless Wightman two-point function on dS_2

In this subsection we derive the massless spinor Wightman two-point function on dS_2 using the mode-sum method (2.163) and we show that it agrees with Eq. (2.182). The derivation is slightly different but simpler than the case with $N > 2$. Note that in this subsection we use the same (bi)scalar functions that we introduced for the case with $N > 2$, with $\theta_{N-1} \to \varphi - \varphi'$ and $0 \le \varphi, \varphi' < 2\pi$. (See Eqs. (2.195)-(2.199).) The geodesic distance and the tangent vector components are

$$\cos\mu = -\sinh t \sinh t' + \cosh t \cosh t' \cos(\varphi - \varphi'), \tag{2.297}$$
$$n_0 = \frac{1}{\sin\mu}(\cosh t \sinh t' - \sinh t \cosh t' \cos(\varphi - \varphi')), \tag{2.298}$$
$$n_1 = \frac{1}{\sin\mu} \cosh t' \sin(\varphi - \varphi'), \tag{2.299}$$

where $n_1 = n_\varphi / \cosh t$. (Note that by letting $\varphi - \varphi' \to \theta_{N-1}$ in these expressions we obtain Eqs. (2.241) and (2.242). This makes many steps of the calculation the same as in the case with $N > 2$.) The massless positive frequency solutions (2.188)-(2.189) for

$N = 2$ are given by

$$\psi_{0\ell}^{(-)}(t,\varphi) = \frac{1}{2\sqrt{\pi}}\begin{pmatrix}\phi_{0\ell}(t)\\ 0\end{pmatrix} e^{-i(\ell+1/2)\varphi}, \tag{2.300}$$

$$\psi_{0\ell}^{(+)}(t,\varphi) = \frac{1}{2\sqrt{\pi}}\begin{pmatrix}0\\ \phi_{0\ell}(t)\end{pmatrix} e^{+i(\ell+1/2)\varphi}, \tag{2.301}$$

where $\phi_{0\ell}(t)$ is given as function of $x = \pi/2 - it$ by Eq. (2.190) ($\ell = 0, 1, ...$). After a straightforward calculation the mode-sum method (2.163) gives the following expression for the Wightman two-point function:

$$W_0[(t,\varphi),(t',\varphi')] = -\frac{1}{4\pi}\frac{1}{\cos{(x/2)}\,[\cos{(x'/2)}]^*}\begin{pmatrix}0 & M_-\\ M_+ & 0\end{pmatrix}, \tag{2.302}$$

where

$$\begin{aligned} M_\pm &= e^{\pm i(\varphi-\varphi')/2}\sum_{\ell=0}^{\infty}\left(\tan\frac{x}{2}\left[\tan\frac{x'}{2}\right]^* e^{\pm i(\varphi-\varphi')}\right)^{\ell} \\ &= e^{\pm i(\varphi-\varphi')/2}\left(1-\tan\frac{x}{2}\left[\tan\frac{x'}{2}\right]^* e^{\pm i(\varphi-\varphi')}\right)^{-1}. \end{aligned} \tag{2.303}$$

Since $\left|\tan\frac{x}{2}\left[\tan\frac{x'}{2}\right]^* e^{\pm i(\varphi-\varphi')}\right| = 1$ we let $t \to t - i\epsilon$ (i.e. $x \to x - \epsilon$, where $\epsilon > 0$) in order for the series to converge and then we let $\epsilon \to 0$. By expressing x, x' in terms of t, t' we can show the following relations:

$$M_\pm = \frac{1}{w_\mp} \tag{2.304}$$

$$= \frac{w_\pm}{\sin^2(\mu/2)}, \tag{2.305}$$

where in the second line we used the identity $w_+ w_- = \sin^2(\mu/2)$. By substituting Eq. (2.305) into the two-point function (2.302) it is straightforward to find

$$W_0[(t,\varphi),(t',\varphi')] = \frac{\beta_0(\mu)}{\sin{(\mu/2)}}[-w_1\gamma^0 + w_2\gamma^1]. \tag{2.306}$$

By repeating the same calculation that resulted in Eq. (2.280) we find

$$W_0[(t,\varphi),(t',\varphi')] = \beta_0(\mu)\not{n}\exp\left\{\frac{\lambda}{2}\gamma^0\gamma^1\right\}, \tag{2.307}$$

where the exponential is the spinor parallel propagator (2.194) for dS_2.

2.13 APPENDIX E - TESTING OUR RESULT FOR THE SPINOR PARALLEL PROPAGATOR

In this Appendix we show that our result for the spinor parallel propagator (i.e. Eq. (2.194)) satisfies the defining properties (2.165)− (2.167), as introduced in Ref. [19].

2.13.1 Parallel transport equation

Our result for the spinor parallel propagator (2.194) has to satisfy the parallel transport equation (2.167). Starting from Eq. (2.167) and expressing the spinor parallel propagator in terms of the massless Wightman function (using Eq. (2.182)) one can obtain Eq. (2.186). For convenience, we use Eq. (2.186) rather than Eq. (2.167) in order to test the parallel transport-property of the spinor parallel propagator. Let us express our two-point function in the form (2.277). Since no derivatives act on the "angular part" it is straightforward to write Eq. (2.186) as follows:

$$(D(t,\theta_{N-1},t') + \frac{N-1}{2}\cot\frac{\mu}{2})\Big[\sin^{-N}\frac{\mu}{2}\,(-w_1(t,\theta_{N-1},t')\,\gamma^0 + w_2(t,\theta_{N-1},t')\,\gamma^{N-1})\Big] = 0, \qquad (2.308)$$

where the differential operator $D(t,\theta_{N-1},t')$ is defined as

$$D(t,\theta_{N-1},t') \equiv [n^t\partial_t + n^{\theta_{N-1}}\partial_{\theta_{N-1}} - n^{\theta_{N-1}}\frac{\sinh t}{2}\gamma^0\gamma^{N-1}]_{\boldsymbol{\theta}'=\mathbf{0}}. \qquad (2.309)$$

(The tangent vectors for $\boldsymbol{\theta}' = \mathbf{0}$ are given by Eqs. (2.241)-(2.242).) Now our initial problem has reduced to a partial differential equation involving only the coordinates t, θ_{N-1} and t'. This is expected because geodesics on S^{N-1} lie along the line $(\theta_{N-2},...,\theta_2,\theta_1) = (\theta'_{N-2},...,\theta'_2,\theta'_1) = (0,...,0,0)$. In the rest of this Appendix we implicitly let $\boldsymbol{\theta}' = \mathbf{0}$ in all relevant quantities unless otherwise stated (and hence $n^\mu\partial_\mu$ will stand for $[n^t\partial_t + n^{\theta_{N-1}}\partial_{\theta_{N-1}}]_{\boldsymbol{\theta}'=\mathbf{0}}$). The parallel transport equation (2.308) gives rise to partial differential equations involving just the biscalars $w_1(t,\theta_{N-1},t')$ and $w_2(t,\theta_{N-1},t')$ (see Eqs. (2.196)-(2.197)). Below we derive these differential equations. Their validity has been tested using Mathematica 11.2.

Case 1: N even. Using the expressions for the γ^a's (2.38) we find

$$\frac{-1}{\sin^N(\mu/2)}\left(-w_1(t,\theta_{N-1},t')\,\gamma^0+w_2(t,\theta_{N-1},t')\,\gamma^{N-1}\right)$$
$$=\frac{1}{\sin^N(\mu/2)}\begin{pmatrix}0&0&w_-&0\\0&0&0&w_+\\w_+&0&0&0\\0&w_-&0&0\end{pmatrix} \tag{2.310}$$
$$\equiv\begin{pmatrix}0&W_1\\W_2&0\end{pmatrix}, \tag{2.311}$$

where W_1 and W_2 represent $2^{N/2-1}$-dimensional matrices and their matrix elements can be read from above. Here 0 stands for the matrix having all entries zero. Then Eq. (2.308) can be expanded in matrix-component form as follows:

$$\begin{pmatrix}\mathbf{1}[n^\mu\partial_\mu+\frac{N-1}{2}\cot(\mu/2)\,]-\frac{1}{2}n^{\theta_{N-1}}\sinh t\,\tilde{\gamma}^{N-1} & 0\\ 0 & \mathbf{1}[n^\mu\partial_\mu+\frac{N-1}{2}\cot(\mu/2)\,]+\frac{1}{2}n^{\theta_{N-1}}\sinh t\,\tilde{\gamma}^{N-1}\end{pmatrix}$$
$$\times\begin{pmatrix}0&W_1\\W_2&0\end{pmatrix}$$
$$=\begin{pmatrix}0&0\\0&0\end{pmatrix}. \tag{2.312}$$

After a straightforward calculation we obtain the following two equations for the biscalar functions w_1 and w_2:

$$\left(n^\mu\partial_\mu+\frac{N-1}{2}\cot\frac{\mu}{2}\right)\frac{w_1}{\sin^N(\mu/2)}=-\frac{n^{\theta_{N-1}}}{2}\sinh t\frac{w_2}{\sin^N(\mu/2)},$$
$$\left(n^\mu\partial_\mu+\frac{N-1}{2}\cot\frac{\mu}{2}\right)\frac{w_2}{\sin^N(\mu/2)}=-\frac{n^{\theta_{N-1}}}{2}\sinh t\frac{w_1}{\sin^N(\mu/2)}. \tag{2.313}$$

Then we use $\partial_\alpha\mu=n_\alpha$ in order to simplify Eqs. (2.313). Thus, we obtain the following system of differential equations for w_1 and w_2:

$$(n^t\partial_t+n^{\theta_{N-1}}\partial_{\theta_{N-1}})w_1-\frac{1+\cos\mu}{2\sin\mu}w_1=-w_2\,\frac{n^{\theta_{N-1}}}{2}\sinh t,$$
$$(n^t\partial_t+n^{\theta_{N-1}}\partial_{\theta_{N-1}})w_2-\frac{1+\cos\mu}{2\sin\mu}w_2=-w_1\,\frac{n^{\theta_{N-1}}}{2}\sinh t, \tag{2.314}$$

where we also used $n^\mu n_\mu = 1$, $\cot(\mu/2) = (1+\cos\mu)/\sin\mu$. These equations are expressed in a particularly convenient form since there is a factor of $1/\sin\mu$ (which can be cancelled) in each term (see Eqs. (2.31) and (2.241)). We have verified that the formulas we have derived for w_1, w_2 (given by Eqs. (2.196) and (2.197)) satisfy the differential equations (2.314) using Mathematica 11.2. Thus, our result for the spinor parallel propagator (2.194) satisfies the parallel transport equation (2.167).

Case 2: N odd. Using the gamma matrices (2.40) we find

$$-w_1\,\gamma^0 + w_2\,\gamma^{N-1} = \begin{pmatrix} -iw_1\mathbf{1} & w_2\mathbf{1} \\ w_2\mathbf{1} & iw_1\mathbf{1} \end{pmatrix}, \quad \gamma^0\gamma^{N-1} = \begin{pmatrix} 0 & i\mathbf{1} \\ -i\mathbf{1} & 0 \end{pmatrix}. \tag{2.315}$$

Then, as in the case with N even, we substitute these into Eq. (2.308) and we obtain the system (2.314). The latter can be solved by our results for w_1, w_2 (Eqs. (2.196) and (2.197)). Thus, our result for the spinor parallel propagator (2.194) satisfies the parallel transport equation, as required.

2.13.2 Parallel-transport property of $\not{n}$

In this subsection we show that our result for the spinor parallel propagator (2.194) satisfies Eq. (2.168) describing the parallel-transport property of $\not{n}$. Let L and R denote the left- and right-hand sides of Eq. (2.168) respectively, i.e.

$$L \equiv \Big(\Lambda[(t,\boldsymbol{\theta}),(t',\mathbf{0})]\Big)^{-1}\,\gamma^a n_a|_{\boldsymbol{\theta}'=\mathbf{0}}\;\Lambda[(t,\boldsymbol{\theta}),(t',\mathbf{0})], \tag{2.316}$$

$$R \equiv -\gamma^{a'} n_{a'}|_{\boldsymbol{\theta}'=\mathbf{0}}, \tag{2.317}$$

where the inverse of the spinor parallel propagator can be readily found using Eq. (2.194). The components of $n_a|_{\boldsymbol{\theta}'=\mathbf{0}}$ are given in Eq. (2.241), while the components of $n_{a'}|_{\boldsymbol{\theta}'=\mathbf{0}}$

are found to be (see the paragraph below Eqs. (2.34)- (2.36))

$$\begin{aligned}
n_{0'}|_{\boldsymbol{\theta'}=\mathbf{0}} &= \frac{1}{\sin\mu}(\cosh t'\sinh t - \sinh t'\cosh t\cos\theta_{N-1}),\\
n_{(N-1)'}|_{\boldsymbol{\theta'}=\mathbf{0}} &= -\frac{\cosh t}{\sin\mu}\sin\theta_{N-1}\cos\theta_{N-2},\\
n_{(N-2)'}|_{\boldsymbol{\theta'}=\mathbf{0}} &= -\frac{\cosh t}{\sin\mu}\sin\theta_{N-1}\sin\theta_{N-2}\cos\theta_{N-3},\\
&\vdots\\
n_{2'}|_{\boldsymbol{\theta'}=\mathbf{0}} &= -\frac{\cosh t}{\sin\mu}(\prod_{i=1}^{N-2}\sin\theta_{N-i})\cos\theta_1,\\
n_{1'}|_{\boldsymbol{\theta'}=\mathbf{0}} &= -\frac{\cosh t}{\sin\mu}\prod_{i=1}^{N-1}\sin\theta_{N-i}.
\end{aligned} \tag{2.318}$$

We will show that the two sides of Eq. (2.168) are equal by rearranging the terms in L. Substituting Eqs. (2.193) and (2.194) into Eq. (2.316) we find

$$\begin{aligned}
L =\ & e^{-\frac{\theta_1}{2}\gamma^2\gamma^1}\ldots e^{-\frac{\theta_{N-2}}{2}\gamma^{N-1}\gamma^{N-2}}e^{-\frac{\lambda}{2}\gamma^0\gamma^{N-1}}\left[\gamma^0 n_0+\gamma^{N-1}n_{N-1}\right]_{\boldsymbol{\theta'}=\mathbf{0}}\\
&\times e^{\frac{\lambda}{2}\gamma^0\gamma^{N-1}}e^{\frac{\theta_{N-2}}{2}\gamma^{N-1}\gamma^{N-2}}\ldots e^{\frac{\theta_1}{2}\gamma^2\gamma^1}\\
&= \gamma^0 n_0|_{\boldsymbol{\theta'}=\mathbf{0}}\left(e^{-\frac{\theta_1}{2}\gamma^2\gamma^1}\ldots e^{-\frac{\theta_{N-2}}{2}\gamma^{N-1}\gamma^{N-2}}\right)e^{\lambda\gamma^0\gamma^{N-1}}\left(e^{\frac{\theta_{N-2}}{2}\gamma^{N-1}\gamma^{N-2}}\ldots e^{\frac{\theta_1}{2}\gamma^2\gamma^1}\right)\\
&+\gamma^{N-1}n_{N-1}|_{\boldsymbol{\theta'}=\mathbf{0}}\left(e^{-\frac{\theta_1}{2}\gamma^2\gamma^1}\ldots e^{-\frac{\theta_{N-3}}{2}\gamma^{N-2}\gamma^{N-3}}\right)\\
&\times e^{+\frac{\theta_{N-2}}{2}\gamma^{N-1}\gamma^{N-2}}e^{\lambda\gamma^0\gamma^{N-1}}\left(e^{\frac{\theta_{N-2}}{2}\gamma^{N-1}\gamma^{N-2}}\ldots e^{\frac{\theta_1}{2}\gamma^2\gamma^1}\right),
\end{aligned} \tag{2.319}$$

where we used the fact that if two matrices A, B anti-commute $\exp\{(-A)\}B = B\exp\{(A)\}$. Our goal is to express L as a sum of N terms, where each term will be of the form: $\gamma^a\times$ (scalar) like Eq. (2.317). In order to simplify Eq. (2.319) we use $\exp\left\{\lambda\gamma^0\gamma^{N-1}\right\} = \mathbf{1}\cosh\lambda+\gamma^0\gamma^{N-1}\sinh\lambda$ and find

$$\begin{aligned}
L =&[n_0\cosh\lambda - n_{N-1}\sinh\lambda]_{\boldsymbol{\theta'}=\mathbf{0}}\gamma^0 + [-n_0\sinh\lambda + n_{N-1}\cosh\lambda]_{\boldsymbol{\theta'}=\mathbf{0}}\gamma^{N-1}\\
&\times\left(e^{-\frac{\theta_1}{2}\gamma^2\gamma^1}\ldots e^{-\frac{\theta_{N-3}}{2}\gamma^{N-2}\gamma^{N-3}}\right)e^{\theta_{N-2}\gamma^{N-1}\gamma^{N-2}}\left(e^{\frac{\theta_{N-3}}{2}\gamma^{N-2}\gamma^{N-3}}\ldots e^{\frac{\theta_1}{2}\gamma^2\gamma^1}\right).
\end{aligned} \tag{2.320}$$

Similarly, by expanding $\exp\left\{\theta_{N-2}\gamma^{N-1}\gamma^{N-2}\right\}$ (and then all the other exponentials of the form $\exp\{\theta_j\gamma^{j+1}\gamma^j\}$ that will appear, with $j = N-3, ..., 2, 1$) we find

$$\begin{aligned}
L =\ & [n_0\cosh\lambda - n_{N-1}\sinh\lambda]_{\boldsymbol{\theta'}=\mathbf{0}}\gamma^0 - [-n_0\sinh\lambda + n_{N-1}\cosh\lambda]_{\boldsymbol{\theta'}=\mathbf{0}}\\
&\times\left(\frac{\cosh t}{\sin\mu}\sin\theta_{N-1}\right)^{-1}[n_{(N-1)'}\gamma^{N-1}+n_{(N-2)'}\gamma^{N-2}+\ldots+n_{2'}\gamma^2+n_{1'}\gamma^1]_{\boldsymbol{\theta'}=\mathbf{0}},
\end{aligned} \tag{2.321}$$

where we also used Eq. (2.318). We have verified using Mathematica 11.2 that

$$[n_0 \cosh\lambda - n_{N-1}\sinh\lambda]_{\boldsymbol{\theta}'=\mathbf{0}} = -n_{0'}, \tag{2.322}$$

$$[-n_0 \sinh\lambda + n_{N-1}\cosh\lambda]_{\boldsymbol{\theta}'=\mathbf{0}} = \frac{\cosh t}{\sin\mu}\sin\theta_{N-1}, \tag{2.323}$$

where $\cosh\lambda$ and $\sinh\lambda$ can be found by Eqs. (2.195). By substituting these formulas into Eq. (2.321) we find that $L = R$, i.e. our expression for the spinor parallel propagator (2.194) satisfies Eq. (2.168).

2.13.3 The inverse of the spinor parallel propagator

Finally, we show that our result for the spinor parallel propagator (2.194) satisfies the defining property given by Eq. (2.165). First, let us derive an expression for the two-point function with interchanged points, i.e. $W_0(x',x) = W_0[(t',\mathbf{0}),(t,\boldsymbol{\theta})]$. This can be found by the following relation:

$$W_0(x',x) = -\gamma^0 W_0(x,x')^\dagger \gamma^0, \tag{2.324}$$

(see Eq. (2.163)). Combining this equation with Eq. (2.192) and using $\not{n}^\dagger = \gamma^0 \not{n}\gamma^0$ and $\gamma^0\Lambda(x,x')^\dagger\gamma^0 = -[\Lambda(x,x')]^{-1}$ (this can be verified using Eq. (2.194)) we find

$$W_0[(t',\mathbf{0}),(t,\boldsymbol{\theta})] = -\beta_0(\mu)\Big(\Lambda[(t,\boldsymbol{\theta}),(t',\mathbf{0})]\Big)^{-1}\not{n}|_{\boldsymbol{\theta}'=\mathbf{0}} \tag{2.325}$$

$$= \beta_0(\mu)\,\not{n}'|_{\boldsymbol{\theta}'=\mathbf{0}}\Big(\Lambda[(t,\boldsymbol{\theta}),(t',\mathbf{0})]\Big)^{-1}, \tag{2.326}$$

where in the last line we used Eq. (2.168). Equation (2.182) implies that the massless spinor Green's function with interchanged points, $x \leftrightarrow x'$, has the following form: $S_0(x',x) = \beta_0(\mu)\not{n}'\,\Lambda(x',x)$. Thus, we conclude that our expression for the spinor parallel propagator satisfies: $\Big(\Lambda[(t,\boldsymbol{\theta}),(t',\mathbf{0})]\Big)^{-1} = \Lambda[(t',\mathbf{0}),(t,\boldsymbol{\theta})]$ in agreement with the defining property (2.165).

2.14 APPENDIX F - A CONJECTURE FOR THE CLOSED-FORM EXPRESSION OF A SERIES CONTAINING THE GAUSS HYPERGEOMETRIC FUNCTION

In Sec. 2.7.3 and in Appendix 2.12 we showed that the mode-sum approach (2.163) for the massless Wightman two-point function reproduces the result of Ref. [19] (i.e. Eq. (2.182)).

Motivated by this result, we compare the mode-sum method for the massive Wightman two-point function (2.163) with Eq. (2.172) and we make a conjecture regarding the closed-form expression of a series containing the Gauss hypergeometric function for N even. For simplicity, we specialize to timelike separated points with $\boldsymbol{\theta} = \boldsymbol{\theta}' = \mathbf{0}$. For brevity, we represent the Gauss hypergeometric function as follows:

$$F^{(a,b)}_{(c)}(z) \equiv F(a,b;c;z). \tag{2.327}$$

We first present our conjecture and then we give some details for the reasoning for this conjecture.

The conjecture is

$$F^{(a,b)}_{(c+1)}\left(\cosh^2 t\right) = \sum_{\ell=0}^{\infty} \frac{(a)_\ell (b)_\ell}{(c)_\ell (c+1)_\ell} \frac{(N-1)_\ell}{\ell!} \left(\frac{\cosh^2 t}{4}\right)^\ell F^{(a+\ell,b+\ell)}_{(c+\ell)}\left(\frac{1-i\sinh t}{2}\right) F^{(a+\ell,b+\ell)}_{(c+1+\ell)}\left(\frac{1-i\sinh t}{2}\right), \tag{2.328}$$

where

$$a = \frac{N}{2} + iM, \quad b \equiv \frac{N}{2} - iM, \quad c = \frac{N}{2}. \tag{2.329}$$

By introducing the variable $w \equiv (1 - i\sinh t)/2$ we may rewrite the conjecture as follows:

$$F^{(a,b)}_{(c+1)}\left(4w(1-w)\right) = \sum_{\ell=0}^{\infty} \frac{(a)_\ell (b)_\ell}{(c)_\ell (c+1)_\ell} \frac{(N-1)_\ell}{\ell!} \left(w(1-w)\right)^\ell F^{(a+\ell,b+\ell)}_{(c+\ell)}\left(w\right) F^{(a+\ell,b+\ell)}_{(c+1+\ell)}\left(w\right), \tag{2.330}$$

where $4w(1-w) = 4|w|^2 = \cosh^2 t$. The time variable should be understood as $t - i\epsilon$ with $\epsilon > 0$ (see the paragraph below Eq. (2.163)). This way, the branch cut of the hypergeometric function $F(A,B;C;X)$ along the real axis for $X > 1$ is avoided.

Below we describe the calculations that lead to the conjecture (2.328). For later convenience let C_M be the proportionality constant of the two-point function that appears in Eq. (2.172), i.e.

$$C_M \equiv \frac{|\Gamma(\frac{N}{2} + iM)|^2}{\Gamma(\frac{N}{2}+1)(4\pi)^{N/2}}. \tag{2.331}$$

For $\boldsymbol{\theta} = \boldsymbol{\theta}' = 0$ Eq. (2.172) gives the following expression for the two-point function:

$$S_M[(t,\mathbf{0}),(t',\mathbf{0})] = \alpha_M(\mu)\mathbf{1} + \beta_M(\mu) i\gamma^0, \tag{2.332}$$

where $\mu = i(t-t')$, $\not{n} = i\gamma^0$ and $\Lambda = \mathbf{1}$. The first term in Eq. (2.332) is diagonal while the second is off-diagonal. On the other hand, the mode-sum (2.163) for massive spinors gives the following expression:

$$
\begin{aligned}
&W_M[(t,\mathbf{0}),(t',\mathbf{0})] \\
&= \sum_{\ell m}\left|\frac{c_N(M\ell)}{\sqrt{2}}\right|^2\Bigg[\begin{pmatrix} i\phi_{M\ell}(t)\psi^*_{M\ell}(t') & -\phi_{M\ell}(t)\phi^*_{M\ell}(t') \\ -\psi_{M\ell}(t)\psi^*_{M\ell}(t') & -i\psi_{M\ell}(t)\phi^*_{M\ell}(t')\end{pmatrix}\otimes\left(\chi_{-\ell m}(\mathbf{0})\chi_{-\ell m}(\mathbf{0})^\dagger\right) \\
&+\begin{pmatrix} -i\psi_{M\ell}(t)\phi^*_{M\ell}(t') & -\psi_{M\ell}(t)\psi^*_{M\ell}(t') \\ -\phi_{M\ell}(t)\phi^*_{M\ell}(t') & i\phi_{M\ell}(t)\psi^*_{M\ell}(t')\end{pmatrix}\otimes\left(\chi_{+\ell m}(\mathbf{0})\chi_{+\ell m}(\mathbf{0})^\dagger\right)\Bigg], \quad (2.333)
\end{aligned}
$$

where the functions $\phi_{M\ell}$ and $\psi_{M\ell}$ are given by Eqs. (2.69) and (2.70) and m stands for the angular momentum quantum numbers and spin projection indices on the lower-dimensional spheres. Using relations (2.258) and (2.291) the two-point function (2.333) can be written as

$$
\begin{aligned}
W_M[(t,\mathbf{0}),(t',\mathbf{0})] =& C_M\Gamma(\frac{N}{2})\Gamma(\frac{N}{2}+1) \\
&\times\sum_{\ell=0}^{\infty}\frac{(N-1)_\ell}{\ell!}\left|\frac{(\frac{N}{2}+iM)_\ell}{\Gamma(\frac{N}{2}+\ell)}\right|^2\Big[M_\ell(t,t')\mathbf{1}+N_\ell(t,t')i\gamma^0\Big], \quad (2.334)
\end{aligned}
$$

where $M_\ell(t,t'), N_\ell(t,t')$ are given by

$$M_\ell(t,t') = -i\Big(-\phi_{M\ell}(t)\psi^*_{M\ell}(t')+\psi_{M\ell}(t)\phi^*_{M\ell}(t')\Big), \quad (2.335)$$

$$N_\ell(t,t') = \phi_{M\ell}(t)\phi^*_{M\ell}(t')+\psi_{M\ell}(t)\psi^*_{M\ell}(t'). \quad (2.336)$$

By equating Eqs. (2.332) and (2.334) we find the following conjectured equalities:

$$\sum_{\ell=0}^{\infty}\frac{(N-1)_\ell}{\ell!}\left|\frac{(\frac{N}{2}+iM)_\ell}{\Gamma(\frac{N}{2}+\ell)}\right|^2 M_\ell(t,t') = \frac{\alpha_M(t-t')}{C_M\Gamma(\frac{N}{2})\Gamma(\frac{N}{2}+1)}, \quad (2.337)$$

$$\sum_{\ell=0}^{\infty}\frac{(N-1)_\ell}{\ell!}\left|\frac{(\frac{N}{2}+iM)_\ell}{\Gamma(\frac{N}{2}+\ell)}\right|^2 N_\ell(t,t') = \frac{\beta_M(t-t')}{C_M\Gamma(\frac{N}{2})\Gamma(\frac{N}{2}+1)}, \quad (2.338)$$

where the first relation is obtained by comparing the diagonal parts of Eqs. (2.332) and (2.334), while the second is obtained by comparing the off-diagonal parts. Equations (2.337) and (2.338) are the most general series conjectures we can find for the

time-like case with $\mu = i(t - t')$. (We have checked that these conjectures are true for $t' = i\pi/2$ with $\phi^*_{M\ell}(t' = i\pi/2) = \delta_{\ell 0}$ and $\psi^*_{M\ell}(t' = i\pi/2) = 0$.) By substituting Eqs. (2.69), (2.70) and (2.175) into Eq. (2.337) and letting $t' = -t$ we find our conjecture (2.330). We also made use of the following relation: $\cos(x/2)\ [\sin(x'/2)]^* = \frac{1}{2}(\cosh\frac{t-t'}{2} + i\sinh\frac{t+t'}{2}) = [\sin(x/2)\ [\cos(x'/2)]^*]^*$ (see Eqs. (2.71)-(2.72)).
In this Appendix, we made a series conjecture by letting $t' = -t$ in Eq. (2.337). One can make additional series conjectures from Eqs. (2.337) and (2.338) by giving various values to t' (or t) or by just leaving it arbitrary.

REFERENCES

[1] http://functions.wolfram.com/07.23.03.0024.01. visited on 14th April 2020.

[2] http://functions.wolfram.com/07.23.03.0026.01. visited on 14th April 2020.

[3] B. Allen and T. Jacobson. "Vector two-point functions in maximally symmetric spaces". In: *Comm. Math. Phys.* 103 (1986), p. 669.

[4] B. Allen and C. A. Lütken. "Spinor Two Point Functions in Maximally Symmetric Spaces". In: *Commun. Math. Phys.* 106 (1986), p. 201.

[5] V. E. Ambrus and E. Winstanley. "Thermal expectation values of fermions on anti-de Sitter space-time". In: *Class. Quant. Grav.* 34 (2017), p. 145010. eprint: arXiv:1704.00614.

[6] A. O. Barut and I. H. Duru. "Exact Solutions of the Dirac Equation in Spatially Flat Robertson-Walker Space-Times". In: *Phys. Rev. D* 36 (1987), p. 3705.

[7] J. Bros and U. Moschella. "Two-point Functions and Quantum Fields in de Sitter Universe". In: *Rev. Math. Phys* 08 (1996), p. 327.

[8] R. Camporesi and A. Higuchi. "On the eigenfunctions of the Dirac operator on spheres and real hyperbolic spaces". In: *J. Geom. Phys.* 20 (1996), p. 1.

[9] M. Clutton-Brock. "Quantum gravidynamics. II. Path integrals with spin". In: *J. Phys. A: Math. Gen.* 8.1 (Jan. 1975), pp. 29–35.

[10] I. Cotăescu and C. Crucean. "New Dirac quantum modes in moving frames of the de Sitter space–time". In: *Int. J. Mod. Phys. A* 23 (2012), pp. 3707–3720.

[11] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.0.26 of 2020-03-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds. visited on 14th April 2020.

[12] D. Z. Freedman and A. V. Proeyen. *Supergravity*. Cambridge University Press, 2012.

[13] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge University Press, 1973.

[14] A. Higuchi. "Quantum linearization instabilities of de Sitter spacetime. I". In: *Class. Quant. Grav.* 8 (1991), p. 1961.

[15] A. Higuchi. "Symmetric tensor spherical harmonics on the N-sphere and their application to the de Sitter group $SO(N,1)$". In: *J. Math. Phys.* 28 (1987), p. 1553.

[16] S. Kanno, M. Sasaki, and T. Tanaka. "Vacuum State of the Dirac Field in de Sitter Space and Entanglement Entropy". In: *J. High Energy Phys.* (2017), p. 68.

[17] B. S. Kay and R. M. Wald. "Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate killing horizon". In: *Phys. Rep.* 207 (1991), p. 49.

[18] Y. Kosmann. "Dérivées de Lie des spineurs". In: *Annali di Mat. Pura Appl. (IV)* 91 (1971), p. 317.

[19] W. Mück. "Spinor parallel propagator and Green function in maximally symmetric spaces". In: *J. Phys. A: Math. Gen.* 33 (2000), p. 3021.

[20] V. S. Otchik. "On the Hawking radiation of spin-1/2 particles in the de Sitter spacetime". In: *Class. Quant. Grav.* 2 (1985), p. 539.

[21] L. Parker and D. Toms. *Quantum Field Theory in Curved Spacetime. Quantized Fields and Gravity*. Cambridge University Press, 2009.

[22] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev. *Integrals and Series*. Vol. 3: More Special Functions. Translated from the Russian by G. G. Gould, Gordon and Breach Science Publishers, New York, Philadelphia, London, Paris, Montreux, Tokyo and Melbourne, 1990. Moscow: Nauka, 1986.

[23] M. A. Rakha, A. K. Rathie, and P. Chopra. "On some new contiguous relations for the Gauss hypergeometric function with applications". In: *Comput. Math. with Appl.* 61.3 (2011), pp. 620–629. ISSN: 0898-1221.

[24] G. V. Shishkin. "Some exact solutions of the Dirac equation in gravitational fields". In: *Class. Quant. Grav.* 8 (1991), p. 175.

[25] Y. Tanii. *Introduction to supergravity*. Vol. 1. Springer briefs in mathematical physics. Tokyo, Japan: Springer, 2014.

[26] R. M. Wald. *Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics*. Chicago, IL: University of Chicago Press, 1995.

— 3 —

(Non-)unitarity of strictly and partially massless fermions on de Sitter space

Abstract

We present the dictionary between the one-particle Hilbert spaces of totally symmetric tensor-spinor fields of spin $s = 3/2, 5/2$ with any mass parameter on D-dimensional ($D \geq 3$) de Sitter space (dS_D) and Unitary Irreducible Representations (UIR's) of the de Sitter algebra spin$(D, 1)$. Our approach is based on expressing the eigenmodes on global dS_D in terms of eigenmodes of the Dirac operator on the $(D-1)$-sphere, which provides a natural way to identify the corresponding representations with known UIR's under the decomposition spin$(D, 1) \supset$ spin(D). Remarkably, we find that four-dimensional de Sitter space plays a distinguished role in the case of the gauge-invariant theories. In particular, the strictly massless spin-3/2 field, as well as the strictly and partially massless spin-5/2 fields on dS_D, are not unitary unless $D = 4$.

3.1 INTRODUCTION

3.1.1 Strictly and partially massless field theories in de Sitter space

The de Sitter spacetime, apart from its relevance to inflationary cosmology, is also thought to be a good model for the asymptotic future of our Universe, as suggested by current experimental evidence in favor of a positive cosmological constant [37, 34, 32]. The D-dimensional de Sitter spacetime (dS_D) is the maximally symmetric solution of the vacuum Einstein field equations with positive cosmological constant Λ [17]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0, \tag{3.1}$$

where $g_{\mu\nu}$ is the metric tensor, $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar. Throughout this paper we use units in which the cosmological constant is

$$\Lambda = \frac{(D-2)(D-1)}{2}, \tag{3.2}$$

i.e. the de Sitter radius is one.

Unlike Minkowskian field theories, possible field theories of spin s on dS_D are not restricted to the two usual cases of massive and strictly massless theories, where for $D = 4$ the former has $2s+1$ propagating degrees of freedom (DoF), while the latter has only 2 helicity DoF ($\pm s$) due to the gauge invariance of the theory [40]. On dS_D there also exist intermediate gauge-invariant theories for $s \geq 2$, known as **partially massless** [1] theories [10, 12, 9, 11, 8]. For a given spin $s \geq 1$, there exists one strictly massless theory and $[s]-1$ different partially massless theories, where $[s] = s$ if the spin s is an integer and $[s] = s - 1/2$ if s is a half-odd integer. Partial masslessness was first observed for the spin-2 field by Deser and Nepomechie [5, 6] and for higher integer-spin fields by Higuchi [21]. Partially massless theories with various spins have been discussed further in a series of papers by Deser and Waldron [10, 12, 9, 11, 8, 7]. Note that this paragraph, as well as the rest of the paper, refers only to totally symmetric tensor and tensor-spinor fields. Mixed-symmetry tensor fields on dS_D - for which strict and partial masslessness also occur - have been discussed in Ref. [2].

Each strictly or partially massless theory of spin s is conveniently labeled by a distinct value of the 'depth' $\tau = 1, 2, ..., [s]$ (where the value $\tau = 1$ corresponds to strict masslessness) and in 4 dimensions there are 2τ propagating helicities, namely: $(\pm s, \pm(s-1), ..., \pm(s-\tau+1))$ [9, 10, 11]. For given spin s and depth τ, each of these gauge-invariant theories corresponds to a distinct tuning of the mass parameter to the cosmological constant Λ [21, 9, 10, 7, 11]. Higuchi classified the tunings of the mass parameter for all strictly and partially massless theories with arbitrary integer spin by studying the group-theoretic properties of the eigenmodes of the Laplace-Beltrami operator on dS_D [21, 20]. Deser and Waldron gave an analogous classification for arbitrary integer and half-odd-integer spins by using group representation methods based on the de Sitter/CFT correspondence [7].

[1] Partially massless theories exist also in anti-de Sitter spacetime. Partially and strictly massless theories on both de Sitter and anti-de Sitter spacetimes are discussed in Ref. [9].

3.1.2 Eigenmodes, 'field theory-representation theory' dictionary and purpose of this paper

Unitarity of field theories is very important for physical problems since it ensures the positivity of probabilities. A sufficient condition for field-theoretic unitarity on dS_D is that of the unitarity of the underlying representation of the de Sitter (dS) algebra, spin$(D,1)$. Particles in a D-dimensional dS universe correspond to Unitary Irreducible Representations (UIR's) of spin$(D,1)$.

Representation-theoretic insight from eigenmodes. The interplay between free field theory on dS_D and representation theory of spin$(D,1)$ manifests beautifully itself in the solution space - consisting of eigenmodes - of the corresponding field equation.[2] Let us briefly discuss Higuchi's work [20, 21] in order to demonstrate the great amount of representation-theoretic knowledge that we can obtain for a free field theory on dS_D by studying its eigenmodes. In particular, in Refs. [20, 21] Higuchi studied the group-theoretic properties of totally symmetric tensor eigenmodes of the Laplace-Beltrami operator on dS_D ($D \geq 3$). In these works, he showed that the phenomenon of partial masslessness exists for all totally symmetric tensor fields of spin $s \geq 2$ on dS_D by detecting pure gauge modes (these eigenmodes indicate the gauge invariance of the theory). Also, by calculating the norm of the physical strictly/partially massless eigenmodes using a dS invariant scalar product, he showed that all strictly and partially massless theories with arbitrary integer spin s are unitary for all $D \geq 3$. Moreover, he showed that for all integer spins there exist mass (parameter) ranges where the eigenmodes have negative norm - i.e. the corresponding spin$(D,1)$ representations are non-unitary. The unitary strictly/partially massless theories appear at special tunings of the mass parameter corresponding to the boundaries of the 'forbidden' mass ranges - see Deser and Waldron's works for a detailed analysis and a physical insight into these 'forbidden' ranges [10, 12, 9, 11]. Last, Higuchi's group-theoretic analysis of the eigenmodes showed that there is a lower bound for the mass parameter of integer-spin fields, below which the fields can only be non-unitary[3]. This bound is known as the 'Higuchi bound' in the modern literature - see, e.g Ref. [29, 18].

'Field theory-representation theory' dictionary and a gap in the literature. The

[2] If a dS invariant positive-definite scalar product exists for the eigenmodes, then the vector space of eigenmodes can be identified with the one-particle Hilbert of the corresponding unitary quantum field theory.

[3] The Higuchi bound depends on both the (integer) spin of the field and the spacetime dimension D [21].

basis elements of spin$(D,1)$ correspond to the $(D+1)D/2$ Killing vectors of dS_D and they act on eigenmodes in terms of Lie derivatives (or spinorial generalizations thereof [25, 30]). The (spinorial) Lie derivatives with respect to Killing vectors commute with the field equation of the free theory [25, 30] and the solution space is identified with the representation space of a - often irreducible - representation of spin$(D,1)$ [20, 21]. What we would like to know is whether this representation, which is formed by eigenmodes, is unitary. Fortunately, all UIR's of spin$(D,1)$ have been classified by Ottoson and Schwarz [31, 33] (see also Refs. [41, 22, 23]). Thus, as field theorists, we would like to construct a dictionary between the known UIR's of spin$(D,1)$ and eigenmode spaces (i.e. one-particle Hilbert spaces) of free field theories on dS_D. Such a dictionary was first constructed by Higuchi [20] for totally symmetric integer-spin fields[4] and was later extended to mixed-symmetry integer-spin fields by Basile, Bekaert and Boulanger [2]. However, a detailed study of the dictionary for tensor-spinor fields for arbitrary D is absent from the literature[5].

Main aim. It is the purpose of the present article to construct the dictionary between one-particle Hilbert spaces (consisting of eigenmodes) and UIR's of spin$(D,1)$ for the vector-spinor (i.e. spin-3/2) field and symmetric rank-2 tensor-spinor (i.e. spin-5/2) field on dS_D.

3.1.3 MAIN RESULT FOR STRICTLY AND PARTIALLY MASSLESS THEORIES OF SPIN $s = 3/2, 5/2$

The dictionary between one-particle Hilbert spaces of unitary spin-$s = 3/2, 5/2$ field theories on dS_D and UIR's of spin$(D,1)$ will be given in Section 3.7 (for both massive and strictly/partially massless fields). However, here we would like to draw attention to our remarkable main result concerning the strictly and partially massless theories:

- **Main result:** The strictly massless spin-3/2 field (gravitino field) and the strictly and partially massless spin-5/2 fields on dS_D ($D \geq 3$) are not unitary unless $D = 4$.

(The case with $D = 2$ is not discussed in the present article.) As we will see later, our analysis for the spin-3/2 and spin-5/2 cases suggests that our main result should hold for all strictly and partially massless fields with half-odd-integer spin $s \geq 3/2$.

[4] See also Refs. [36, 35] for more recent discussions concerning the 'field theory-representation theory' dictionary for integer-spin fields on dS_D.

[5] For $D = 4$, a dictionary for half-odd-integer-spin fields has been obtained in Ref. [15].

According to our main result, four-dimensional dS space plays a distinguished role in the unitarity of the strictly massless spin-3/2 field and the strictly and partially massless spin-5/2 fields. This is an example of a remarkable and previously unknown feature of dS field theory that has no known field-theoretic counterparts in anti-de Sitter and Minkowski spacetimes. As will become clear, the significance of four-dimensional dS space is related to the representation theory of spin$(D,1)$, where the latter allows (totally symmetric) fermionic strictly/partially massless UIR's only for $D=4$ (corresponding to a direct sum of spin$(4,1)$ UIR's in the Discrete Series - see Section 3.7). Also, although it might be a mere mathematical coincidence, it is interesting that the dimensionality that plays a special representation-theoretic role happens to correspond to the number of the observed macroscopic dimensions of our Universe.

3.1.4 Strategy

Our strategy in order to construct the dictionary between spin$(D,1)$ UIR's and spin-$s = 3/2, 5/2$ one-particle Hilbert spaces on dS_D is based on constructing the dS eigenmodes using the method of separation of variables [3, 4, 28]. More specifically, we are going to express the spin-3/2 and spin-5/2 eigenmodes on global dS_D in terms of tensor-spinor eigenmodes of the Dirac operator on S^{D-1}. This will help us determine the spin(D) content of the spin$(D,1)$ representations formed by the eigenmodes on dS_D - by spin(D) content we mean the irreducible representations of spin(D) that appear in a spin$(D,1)$ representation under the decomposition spin$(D,1) \supset$ spin(D) [31, 33]. We will also obtain the values of the spin$(D,1)$ quadratic Casimir corresponding to the eigenmodes on dS_D. Once we have determined both the quadratic Casimir and the spin(D) content for the representations formed by the dS eigenmodes, we will be able to construct the dictionary between one-particle Hilbert spaces and UIR's of spin$(D,1)$ by using the known classification of UIR's [31, 33] under the decomposition spin$(D,1) \supset$ spin(D). We also provide the dictionary for the spin-1/2 field (as the group-theoretic properties of the spin-1/2 eigenmodes on global dS_D have been already studied by the author [28]), while our analysis also allows us to propose a dictionary for totally symmetric tensor-spinors of any spin $s \geq 3/2$.
As for our main result concerning the strictly/partially massless theories of spin $s = 3/2, 5/2$, we will show that for $D \neq 4$ there is a mismatch between the values of the quadratic Casimir for the strictly/partially massless eigenmodes and the values corresponding to the UIR's of spin$(D,1)$ and/or another mismatch between the representation

labels of the eigenmodes and the allowed labels in spin$(D,1)$ UIR's. (The spin$(D,1)$ representation labels we use in this paper specify a spin$(D,1)$ representation under the decomposition spin$(D,1) \supset$ spin(D) [31, 33, 20, 21] and their role is similar to the role played by the highest weights in spin$(D+1)$ representations - see Section 3.3.) In other words, we will demonstrate that there are no UIR's of spin$(D,1)$ that correspond to the strictly massless spin-3/2 field and to the strictly and partially massless spin-5/2 fields on dS_D for $D \neq 4$. However, for $D = 4$, both the quadratic Casimir and the representation labels of the strictly/partially massless theories correspond to the Discrete Series UIR's of spin$(4,1)$.

An alternative technical explanation. A technical explanation of all the results reported in this paper can be given by studying the (non-)existence of positive-definite dS invariant scalar products for the spin-3/2 and spin-5/2 eigenmodes on dS_D. Such an analysis has been carried out in detail by the author and will be presented in a separate article [26, 27], in which the author has extended Higuchi's methods [20, 21] to the case of spin-3/2 and spin-5/2 eigenmodes on dS_D $(D \geq 3)$. In particular, in Refs. [26, 27] the author has proved the following results for the strictly/partially eigenmodes of spin $s = 3/2, 5/2$ on dS_D $(D \geq 3)$:

- For odd D all dS invariant scalar products are identically zero.
- For even $D > 4$ all dS invariant scalar products are indefinite giving always rise to positive-norm and negative-norm eigenmodes that mix with each other under spin$(D,1)$ boosts.
- The $D = 4$ case is special as the positive-norm sector decouples from the negative-norm sector. Then, both sectors can be viewed as positive-norm sectors and each sector independently forms a spin$(4,1)$ UIR in the Discrete Series.

Although we have not performed such a technical analysis for the eigenmodes with half-odd-integer spin $s \geq 7/2$, the analysis of our present paper suggests that our main result extends to all strictly and partially massless fields with half-odd-integer spin $s \geq 7/2$ on dS_D.

3.1.5 Outline of the paper, notation and conventions

The rest of the paper is organised as follows. In Section 3.2, we begin by presenting the basics about tensor-spinor fields on dS_D (gamma matrices, vielbein fields, spin connection, and the spinorial generalisation of the Lie derivative) and, then, we specialise to the global slicing of dS_D. In Section 3.3, we review the classification of the spin$(D,1)$ UIR's under the decomposition spin$(D,1) \supset$ spin(D) given originally in Refs. [31, 33]. In Section 3.4, we begin by discussing the totally symmetric tensor-spinor eigenmodes of the Dirac operator on S^{D-1} that are also gamma-traceless and divergence-free, as well as the way they form representations of spin(D) (Subsection 3.4.1). Then, using the aforementioned eigenmodes on S^{D-1}, we present the construction of the TT eigenmodes of the spin-3/2 field on dS_D for both even $D \geq 4$ (Subsecton 3.4.2) and odd $D \geq 3$ (Subsection 3.4.3), in order to illustrate the method of separation of variables for tensor-spinor fields. The spin(D) content of the spin$(D,1)$ representations formed by the spin-3/2 eigenmodes is also identified and the main results are tabulated in Tables 3.1 and 3.2. In Subsection 3.4.4, we present our basic results concerning the TT eigenmodes for the spin-5/2 field on dS_D $(D \geq 3)$. In Section 3.5, we obtain the quadratic Casimir for the spin$(D,1)$ representation formed by eigenmodes with half-odd-integer spin $s \geq 1/2$ on dS_D by using "analytic continuation" techniques that relate dS_D to S^D. In Section 3.6, after identifying the pure gauge and physical modes of our strictly/partially massless theories (Subsection 3.6.1), we prove the main result of this paper, i.e. the strictly massless spin-3/2 field, as well as the strictly and partially massless spin-5/2 fields on dS_D, are not unitary unless $D = 4$ (Subsection 3.6.2). In order to achieve this, we take advantage of both the spin(D) content and the quadratic Casimir corresponding to our physical modes on dS_D and then we show that they do not agree with any UIR of spin$(D,1)$ unless $D = 4$. In Section 3.7, we present our dictionary between spin$(D,1)$ UIR's and (totally symmetric) tensor-spinor fields with arbitrary mass parameters on dS_D $(D \geq 3)$. Although in the main part of the present paper we discuss the spin-3/2 and spin-5/2 fields, our analysis allows us to propose a dictionary for all (totally symmetric tensor-)spinor fields with spin $s \geq 1/2$.

Notation and conventions. We use the term 'tensor-spinor field of rank r' in order to refer to a r^{th}-rank tensor where each one of its tensor components is a spinor. Other authors prefer the name spinor-tensors for these objects - see, e.g., Ref. [4]. We use the mostly plus metric sign convention for dS_D. Lowercase Greek tensor indices refer to components with respect to the 'coordinate basis' on dS_D. Coordinate basis tensor indices on S^{D-1} are denoted as $\tilde{\mu}_1, \tilde{\mu}_2, \ldots$. Lowercase Latin tensor indices are 'flattened', i.e. they refer to components with respect to the vielbein basis (the indices a, b, c, d, f run from 0 to $D-1$, while the indices i, j, k run from 1 to $D-1$). Summation over repeated indices is understood. We denote the symmetrisation of a pair of indices as $A_{(\mu\nu)} \equiv (A_{\mu\nu} + A_{\nu\mu})/2$ and the anti-symmetrisation as $A_{[\mu\nu]} \equiv (A_{\mu\nu} - A_{\nu\mu})/2$. Spinor indices are always suppressed throughout this paper. The rank of tensor-spinors on dS_D is denoted as r, while the rank of tensor-spinors on S^{D-1} as $\tilde{r}$. The complex conjugate of the complex number z is z^*.

3.2 BACKGROUND MATERIAL CONCERNING TENSOR-SPINORS ON dS_D

Fermionic fields with arbitrary half-odd-integer spin $s \equiv r + 1/2$ and mass parameter M on dS_D can be described by totally symmetric tensor-spinors $\Psi_{\mu_1 \dots \mu_r}$ satisfying the onshell conditions [10, 7]:

$$\left(\nabla\!\!\!\!/ + M\right) \Psi_{\mu_1 \dots \mu_r} = 0 \tag{3.3}$$

$$\nabla^{\alpha} \Psi_{\alpha \mu_2 \dots \mu_r} = 0, \quad \gamma^{\alpha} \Psi_{\alpha \mu_2 \dots \mu_r} = 0, \tag{3.4}$$

where $\nabla\!\!\!\!/ = \gamma^{\nu} \nabla_{\nu}$ is the Dirac operator. From now on, we will refer to the divergence-free and gamma-tracelessness conditions in eq. (3.4) as the TT conditions.
The half-odd-integer-spin theories described by eqs. (3.3) and (3.4) become gauge-invariant (i.e. strictly/partially massless) for the following imaginary values of the mass parameter $M = i\tilde{M}$ [7]:

$$\tilde{M}^2 = -M^2 = \left(r - \tau + \frac{D-2}{2}\right)^2 \qquad (\tau = 1, \dots, r) \tag{3.5}$$

for $r \geq 1$ (i.e. $s \geq 3/2$). Real values of M - including $M = 0$ - correspond to non-gauge-invariant theories.

3.2.1 Gamma matrices, vielbein fields, spin connection and Lie-Lorentz derivative on dS_D

The $2^{[D/2]}$-dimensional[6] gamma matrices γ^a (with 'flattened' indices $a = 0, 1, \dots, D-1$) satisfy the anti-commutation relations

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab} \mathbf{1}, \qquad a, b = 0, 1, \dots, D-1, \tag{3.6}$$

where $\mathbf{1}$ is the spinorial identity matrix and $\eta^{ab} = diag(-1, 1, \dots, 1)$. The vielbein fields $\boldsymbol{e}_a = e^{\mu}{}_a \partial_{\mu}$, determining an orthonormal frame, satisfy

$$e_{\mu}{}^a \, e_{\nu}{}^b \eta_{ab} = g_{\mu\nu}, \quad e^{\mu}{}_a \, e_{\mu}{}^b = \delta^b_a, \tag{3.7}$$

[6] For D even we have $[D/2] = D/2$. For D odd we have $[D/2] = (D-1)/2$.

where the co-vielbein fields $\boldsymbol{e}^a = e_\mu{}^a\, dx^\mu$ define the dual coframe. The gamma matrices with coordinate basis indices are defined using the vielbein fields as $\gamma^\mu(x) \equiv e^\mu{}_a(x)\gamma^a$. The covariant derivative for a vector-spinor field is

$$\nabla_\nu \Psi_\mu = \partial_\nu \Psi_\mu + \frac{1}{4}\omega_{\nu bc}\gamma^{bc}\Psi_\mu - \Gamma^\lambda_{\ \nu\mu}\Psi_\lambda, \tag{3.8}$$

where $\omega_{\nu bc} = \omega_{\nu[bc]} = e_\nu{}^a\omega_{abc}$ is the spin connection, $\Gamma^\lambda_{\ \nu\mu}$ are the Christoffel symbols and $\gamma^{bc} = \gamma^{[b}\gamma^{c]}$. The covariant derivatives for higher-spin tensor-spinors are given by straightforward generalisations of eq. (3.8). It is easy to check that the gamma matrices are covariantly constant, as $\nabla_\mu\gamma^a = \frac{1}{4}\omega_{\mu bc}(\gamma^{bc}\gamma^a - \gamma^a\gamma^{bc}) + \omega_\mu{}^a{}_c\gamma^c = 0$. According to our sign convention, we have [7]

$$\partial_\mu e^\rho{}_b + \Gamma^\rho_{\mu\sigma}e^\sigma{}_b - \omega_\mu{}^c{}_b\, e^\rho{}_c = 0. \tag{3.9}$$

For each value of the mass parameter M in eq. (3.3), the set of TT eigenmodes $\Psi_{\mu_1\ldots\mu_r}$ forms a representation of the de Sitter algebra spin$(D,1)$, which - as we will see below - may be unitary or non-unitary depending on both M and the dimension D. The Killing vectors generating spin$(D,1)$ act on tensor-spinors in terms of the spinorial generalisation of the Lie derivative [25, 30] - also known as Lie-Lorentz derivative - as:

$$\begin{aligned}\mathbb{L}_\xi\ \Psi_{\mu_1\ldots\mu_r} =\ & \xi^\nu\nabla_\nu\Psi_{\mu_1\ldots\mu_r} + \Psi_{\nu\mu_2\ldots\mu_r}\nabla_{\mu_1}\xi^\nu + \Psi_{\mu_1\nu\mu_3\ldots\mu_r}\nabla_{\mu_2}\xi^\nu + \ldots + \Psi_{\mu_1\ldots\mu_{r-1}\nu}\nabla_{\mu_r}\xi^\nu \\ & + \frac{1}{4}\nabla_\kappa\xi_\lambda\gamma^\kappa\gamma^\lambda\Psi_{\mu_1\ldots\mu_r},\end{aligned} \tag{3.10}$$

where ξ^μ is any dS Killing vector - i.e. $\nabla_{(\mu}\xi_{\nu)} = 0$. The Lie-Lorentz derivative satisfies [30]

$$\mathbb{L}_\xi\ e_\mu{}^a = 0, \tag{3.11a}$$

$$\mathbb{L}_\xi\ \gamma^a = 0, \tag{3.11b}$$

as well as

$$(\mathbb{L}_\xi\nabla_\mu - \nabla_\mu\mathbb{L}_\xi)\ \Psi_{\mu_1\ldots\mu_r} = 0, \tag{3.12}$$

and hence $\mathbb{L}_\xi$ commutes with the Dirac operator. Moreover, the Lie-Lorentz derivative preserves the Lie bracket between any two vectors $\xi^\mu, X^\mu \in$ spin$(D,1)$ as

$$(\mathbb{L}_\xi\mathbb{L}_X - \mathbb{L}_X\mathbb{L}_\xi)\,\Psi_{\mu_1\ldots\mu_r} = \mathbb{L}_{[\xi,X]}\Psi_{\mu_1\ldots\mu_r}. \tag{3.13}$$

As for the representation of our gamma matrices on dS_D, we choose the following:

[7] The sign convention we use for the spin connection is the opposite of the one used in Refs. [3, 28].

- **For D even**: the $2^{D/2}$-dimensional gamma matrices are

$$\gamma^0 = i \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & i\widetilde{\gamma}^j \\ -i\widetilde{\gamma}^j & 0 \end{pmatrix}, \tag{3.14}$$

$(j = 1, ..., D-1)$ where the $2^{(D-2)/2}$-dimensional gamma matrices $\widetilde{\gamma}^j$ generate a Euclidean Clifford algebra in $D-1$ dimensions, as

$$\{\widetilde{\gamma}^j, \widetilde{\gamma}^k\} = 2\delta^{jk}\mathbf{1}, \qquad j, k = 1, ..., D-1. \tag{3.15}$$

One can construct the extra gamma matrix γ^{D+1} which is given by the product $\gamma^{D+1} \equiv \epsilon\,\gamma^1\gamma^2...\gamma^{D-1}\gamma^0$, where ϵ is a phase factor. The matrix γ^{D+1} anti-commutes with each of the γ^a's in eq. (3.14). We choose the phase factor ϵ such that

$$\gamma^{D+1} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \tag{3.16}$$

For $D = 4$ this is the familiar matrix γ^5.

- **For D odd**: the $2^{(D-1)/2}$-dimensional gamma matrices are

$$\gamma^0 = i \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \qquad \gamma^j = \widetilde{\gamma}^j, \qquad j = 1, ..., D-1, \tag{3.17}$$

where the $\widetilde{\gamma}^j$'s are $2^{(D-1)/2}$-dimensional gamma matrices generating a Euclidean Clifford algebra in $D-1$ dimensions (see eq. (3.15)).

3.2.2 SPECIALISING TO GLOBAL COORDINATES

In order to obtain explicit expressions for the TT eigenmodes of the field equation (3.3), we will choose to work with the global slicing of dS_D. In global coordinates the line element is

$$ds^2 = -dt^2 + \cosh^2 t\, d\Omega_{D-1}^2, \tag{3.18}$$

$(t \in \mathbb{R})$ where $d\Omega_{D-1}^2$ is the line element of S^{D-1}. The line element of S^m can be parameterised as

$$d\Omega_m^2 = d\theta_m^2 + \sin^2\theta_m d\Omega_{m-1}^2, \qquad m = 2, 3, ..., D-1, \tag{3.19}$$

with $0 \leq \theta_m \leq \pi$, while $d\Omega_{m-1}^2$ is the line element of S^{m-1}. For $m = 1$ we have $d\Omega_1^2 = d\theta_1^2$ with $0 \leq \theta_1 \leq 2\pi$. We will use the symbol $\boldsymbol{\theta_{D-1}} \equiv (\theta_{D-1}, \theta_{D-2}, ..., \theta_1)$ to denote a point on S^{D-1}.

The non-zero Christoffel symbols on global dS_D are

$$\Gamma^{t}_{\theta_i\theta_j} = \cosh t \sinh t \, \tilde{g}_{\theta_i\theta_j}, \;\; \Gamma^{\theta_i}_{\theta_j t} = \tanh t \, \tilde{g}^{\theta_i}_{\theta_j},$$
$$\Gamma^{\theta_k}_{\theta_i\theta_j} = \tilde{\Gamma}^{\theta_k}_{\theta_i\theta_j}, \tag{3.20}$$

where $\tilde{g}_{\theta_i\theta_j}$ and $\tilde{\Gamma}^{\theta_k}_{\theta_i\theta_j}$ are the metric tensor and the Christoffel symbols, respectively, on S^{D-1}. We choose the following expressions for the vielbein fields on dS_D:

$$e^{t}{}_{0} = 1, \quad e^{\theta_i}{}_{i} = \frac{1}{\cosh t}\tilde{e}^{\theta_i}{}_{i}, \qquad i = 1, ..., D-1, \tag{3.21}$$

where $\tilde{e}^{\theta_i}{}_{i}$ are the vielbein fields on S^{D-1}. The non-zero components of the spin connection on dS_D are given by

$$\omega_{ijk} = \frac{\tilde{\omega}_{ijk}}{\cosh t}, \qquad \omega_{i0k} = -\omega_{ik0} = -\tanh t \, \delta_{ik}, \qquad i, j, k = 1, ..., D-1, \tag{3.22}$$

where $\tilde{\omega}_{ijk}$ are the spin connection components on S^{D-1}.

3.3 CLASSIFICATION OF THE UIR'S OF SPIN$(D,1)$

Here we review the classification of the spin$(D,1)$ UIR's by Ottoson [31] and Schwarz [33]. These authors have classified the UIR's of spin$(D,1)$ under the decomposition spin$(D,1) \supset$ spin(D) - in the present paper spin(D) denotes the Lie algebra of SO(D). Under this decomposition, an irreducible representation of spin(D) appears at most once in a UIR of spin$(D,1)$ [13]. The case with $D = 2p$ and the case with $D = 2p+1$, where p is a positive integer, are studied separately. Below we will adopt the notation for the labels of UIR's that were used by Higuchi in Ref. [20]. However, we will use the names of the UIR's that are used in the modern literature [2, 36, 35].

Representations of spin(D). Let us review the basics concerning spin(D) representations. As is well-known, a representation of spin$(2p)$ or spin$(2p+1)$ is specified by the highest weight of the representation [1, 14], denoted here as

$$\vec{f} = (f_1, f_2, ..., f_p), \tag{3.23}$$

where

$$f_1 \geq f_2 \geq ... \geq f_{p-1} \geq |f_p|, \qquad \text{for spin}(2p), \tag{3.24}$$
$$f_1 \geq f_2 \geq ... \geq f_{p-1} \geq f_p \geq 0, \qquad \text{for spin}(2p+1). \tag{3.25}$$

The labels f_j $(j = 1, ..., p)$ in eqs. (3.24) and (3.25) are all integers or all half-odd integers. For spin$(2p)$, the label f_p can be negative, while the representation $(f_1, ..., f_{p-1}, -f_p)$ is known as the 'mirror image' of $(f_1, ..., , f_{p-1}, f_p)$ - see, e.g. Ref. [38]. For spin$(2p+1)$, any representation $\vec{f}$ is equivalent to its mirror image [38].
The quadratic Casimir for the representation $\vec{f} = (f_1, ..., f_p)$ is given by [14]

$$c_2\left(\vec{f}\right) = \sum_{j=1}^{p} f_j(f_j + 2p - 2j), \qquad \text{for spin}(2p), \tag{3.26}$$

$$c_2\left(\vec{f}\right) = \sum_{j=1}^{p} f_j(f_j + 2p + 1 - 2j), \qquad \text{for spin}(2p+1). \tag{3.27}$$

UIR's of spin$(2p, 1)$ (even $D = 2p \geq 4$). A UIR of spin$(2p, 1)$ is specified by the set of labels $\vec{F} = (F_0, F_1, ..., F_{p-1})$. The labels $F_1, ..., F_{p-1}$ satisfy

$$F_1 \geq F_2 \geq ... \geq F_{p-1} \geq 0 \tag{3.28}$$

and they are all integers or all half-odd integers. A representation $(f_1, ..., f_p)$ of spin$(2p)$ that is contained in the UIR $(F_0, F_1, ..., F_{p-1})$ satisfies

$$f_1 \geq F_1 \geq f_2 \geq F_2 \geq ... \geq f_{p-1} \geq F_{p-1} \geq |f_p|. \tag{3.29}$$

Ottoson's labels [31] and our labels are related to each other by [20]:

$$f_j = l_{2p-1,j} + j - p, \qquad (j = 1, ..., p), \tag{3.30a}$$

$$F_j = l_{2p,j} + j - p, \qquad (j = 1, ..., p-1), \tag{3.30b}$$

$$F_0 = l_{2p,p} - p. \tag{3.30c}$$

Schwarz's labels [33] and our labels are related to each other by:

$$f_j = m_{2p,p-j+1}, \qquad (j = 1, ..., p), \tag{3.31a}$$

$$F_j = m_{2p+1,p-j}, \qquad (j = 1, ..., p-1), \tag{3.31b}$$

$$F_0 = z_{2p+1,p}. \tag{3.31c}$$

The UIR's of spin$(2p, 1)$ (even $D = 2p \geq 4$) are classified as follows:

- **Principal Series $D_{\text{prin}}(\vec{F})$:**

$$F_0 = -p + \frac{1}{2} + iy = -\frac{D-1}{2} + iy, \qquad (y > 0). \tag{3.32}$$

The labels $F_1, F_2, ..., F_{p-1}$ are all integers or all half-odd integers.

- **Complementary Series $D_{\text{comp}}(\vec{F})$:**

$$-\frac{D-1}{2} = -p + \frac{1}{2} \leq F_0 < -\tilde{n}, \quad (\tilde{n} \text{ is an integer and } 0 \leq \tilde{n} \leq p-1). \tag{3.33}$$

If $0 \leq \tilde{n} < p-1$, then $F_{\tilde{n}+1} = F_{\tilde{n}+2} = ... = F_{p-1} = 0$ and $F_1, F_2, ..., F_{\tilde{n}}$ are all positive integers, while for the spin$(2p)$ content we have $f_{\tilde{n}+2} = f_{\tilde{n}+3} = ... = f_p = 0$. If $\tilde{n} = p-1$, then $F_1, F_2, ..., F_{p-1}$ are all positive integers. [8]

- **Exceptional Series $D_{\text{ex}}(\vec{F})$:**

$$F_0 = -\tilde{n}, \quad (\tilde{n} \text{ is an integer and } 1 \leq \tilde{n} \leq p-1). \tag{3.34}$$

If $1 \leq \tilde{n} < p-1$, then $F_{\tilde{n}+1} = F_{\tilde{n}+2} = ... = F_{p-1} = 0$ and $F_1, F_2, ..., F_{\tilde{n}}$ are all positive integers, while for the spin$(2p)$ content we have $f_{\tilde{n}+1} = f_{\tilde{n}+2} = ... = f_p = 0$. If $\tilde{n} = p-1$, then $F_1, F_2, ..., F_{p-1}$ are all positive integers, while $f_p = 0$. [9]

- **Discrete Series $D^{\pm}(\vec{F})$:** F_0 is real and it is an integer or half-odd integer at the same time as the labels $F_1, F_2, ..., F_{p-1}$.[10] Also, the following conditions have to be satisfied:

$$F_{p-1} \geq f_p \geq F_0 + p \geq \frac{1}{2} \qquad \text{for } D^{+}(\vec{F}), \tag{3.35}$$

$$-F_{p-1} \leq f_p \leq -(F_0 + p) \leq -\frac{1}{2} \qquad \text{for } D^{-}(\vec{F}). \tag{3.36}$$

For a UIR of spin$(2p, 1)$ labelled by $\vec{F} = (F_0, F_1, ..., F_{p-1})$ the quadratic Casimir $C_2(\vec{F})$ is expressed as

$$C_2(\vec{F}) = \sum_{k=0}^{p-1} F_k\,(F_k + 2p - 2k - 1). \tag{3.37}$$

[8]Our Complementary Series is called Exceptional Series $D(e; l_{2p,1}, ..., l_{2p,p})$ in Ottoson's classification [31]. Also, our notation for the Complementary Series is related to Schwarz's notation [33] as follows. The case with $0 \leq \tilde{n} < p-1$ corresponds to $D^k(m_{2p+1,k+1} \, ... \, m_{2p+1,p-1}; x_{2p+1,p})$, where k is related to $\tilde{n}$ by $k = p - \tilde{n} - 1$, while the case with $\tilde{n} = p-1$ corresponds to $D^0(m_{2p+1,1} \, ... \, m_{2p+1,p-1}; x_{2p+1,p})$.

[9]Our Exceptional Series is called Supplementary Series $D(s; l_{2p,1}, ..., l_{2p,p})$ in Ottoson's classification [31, 20]. Also, our notation is related to Schwarz's notation [33] as follows. The case with $1 \leq \tilde{n} < p-1$ corresponds to $D^k(m_{2p+1,k+1} \, ... \, m_{2p+1,p-1}; m_{2p+1,p})$, where Schwarz's label k is related to our label $\tilde{n}$ by $k = p - \tilde{n} - 1$, while the case with $\tilde{n} = p-1$ corresponds to $D^0(m_{2p+1,1} \, ... \, m_{2p+1,p-1}; m_{2p+1,p})$.

[10]Our Discrete Series $D^{\pm}(\vec{F})$ are called Exceptional Series $D(\pm; l_{2p,1}, ..., l_{2p,p})$ in Ottoson's classification [31, 20]. Also, our Discrete Series $D^{\pm}(\vec{F})$ correspond to $D^{\pm}(m_{2p+1,1} \, ... \, m_{2p+1,p-1}; m_{2p+1,p})$ in Schwarz's classification [33].

This expression for the quadratic Casimir can be readily obtained by applying the "analytic continuation" techniques described in Refs. [33, 41] to the quadratic Casimir (3.27) of spin$(2p+1)$. These techniques "analytically continue" $2p$ of the rotation generators of spin$(2p+1)$ to the $2p$ boost generators of spin$(2p,1)$ - for more details see Refs. [33, 41].

Note. In the present paper, following Schwarz [33] and Ottoson [31], for even D the value $F_0 = -(D-1)/2$ is not included in the Principal Series UIR's, but it is included in the Discrete Series UIR's instead. For odd D, the value $F_0 = -(D-1)/2$ is included in the Principal Series UIR's in the present paper. However, in Ref. [2] the value $F_0 = -(D-1)/2$ (corresponding to the weight $\Delta_c = (D-1)/2$) is included in the Principal Series UIR's for arbitrary D. The present note is important for reasons of clarity, as we are going to show that the spin-3/2 and spin-5/2 fields on even-dimensional dS_D with mass parameter $M=0$ have $F_0 = -(D-1)/2$ and they correspond to the Discrete Series UIR's in our paper (i.e. Principal Series in Ref. [2]) - see Section 3.7.

UIR's of spin$(2p+1,1)$ (odd $D = 2p+1 \geq 3$). A UIR of spin$(2p+1,1)$ is labelled by $\vec{F} = (F_0, F_1, ..., F_p)$. The labels $F_1, ..., F_p$ satisfy

$$F_1 \geq F_2 \geq ... \geq F_p \geq 0 \tag{3.38}$$

and they are all integers or half-odd integers. A representation $(f_1, ..., f_p)$ of spin$(2p+1)$ that is contained in the UIR $\vec{F} = (F_0, F_1, ..., F_p)$ satisfies

$$f_1 \geq F_1 \geq f_2 \geq F_2 \geq ... \geq f_p \geq F_p \geq 0. \tag{3.39}$$

Ottoson's labels [31] and our labels are related to each other by [20]:

$$f_j = l_{2p,j} + j - p - 1 \quad (j = 1, ..., p), \tag{3.40a}$$
$$F_j = l_{2p+1,j} + j - p \quad (j = 1, ..., p), \tag{3.40b}$$
$$F_0 = l_{2p+1,p+1} - p, \tag{3.40c}$$

while Schwarz's labels [33] and our labels are related to each other by:

$$f_j = m_{2p+1,p-j+1} \quad (j = 1, ..., p), \tag{3.41a}$$
$$F_j = m_{2p+2,p-j+1} \quad (j = 1, ..., p), \tag{3.41b}$$
$$F_0 = z_{2p+2,p+1}. \tag{3.41c}$$

The UIR's of spin$(2p+1,1)$ (odd $D = 2p+1 \geq 3$) are classified as follows:

- **Principal Series $D_{\text{prin}}(\vec{F})$:**

$$F_0 = -p + iy = -\frac{D-1}{2} + iy, \quad (y \in \mathbb{R}). \tag{3.42}$$

The labels $F_1, F_2, ..., F_p$ are all integers or half-odd integers. If $F_p = 0$, then the UIR with $F_0 = -(D-1)/2 + iy$ and the UIR with $F_0 = -(D-1)/2 - iy$ are equivalent, and thus we can let $y \geq 0$.

- **Complementary Series $D_{\text{comp}}(\vec{F})$:**

$$-\frac{D-1}{2} = -p < F_0 < -\tilde{n}, \quad (\tilde{n} \text{ is an integer and } 0 \leq \tilde{n} \leq p-1), \tag{3.43}$$

while $F_{\tilde{n}+1} = F_{\tilde{n}+2} = ... = F_p = 0$ and $F_1, F_2, ..., F_{\tilde{n}}$ are all positive integers, where for the spin$(2p+1)$ content we have $f_{\tilde{n}+2} = f_{\tilde{n}+3} = ... = f_p = 0$. [11]

- **Exceptional Series $D_{\text{ex}}(\vec{F})$:**

$$F_0 = -\tilde{n}, \quad (\tilde{n} \text{ is an integer and } 1 \leq \tilde{n} \leq p-1), \tag{3.44}$$

where $F_{\tilde{n}+1} = F_{\tilde{n}+2} = ... = F_p = 0$ and $F_1, F_2, ..., F_{\tilde{n}}$ are all positive integers, where for the spin$(2p+1)$ content we have $f_{\tilde{n}+1} = f_{\tilde{n}+2} = ... = f_p = 0$. [12]

For a UIR of spin$(2p+1,1)$ specified by $\vec{F} = (F_0, F_1, ..., F_p)$ the quadratic Casimir $C_2(\vec{F})$ is expressed as[13]

$$C_2(\vec{F}) = \sum_{k=0}^{p} F_k\,(F_k + 2p - 2k). \tag{3.45}$$

3.4 SPIN-3/2 AND SPIN-5/2 EIGENMODES ON dS_D

In this Section, we will obtain the spin-3/2 and spin-5/2 TT eigenmodes on global dS_D satisfying eq. (3.3) using the method of separation of variables - see, e.g., Refs. [21, 3, 4]. Schematically, in this method the spin-$(r+1/2)$ eigenmodes, $\Psi_{\mu_1...\mu_r}(t, \boldsymbol{\theta_{D-1}})$, are expressed as products of two 'parts'; namely a part describing the time-dependence (corresponding to a function of t) and another part describing the $\boldsymbol{\theta_{D-1}}$-dependence

[11]Our Complementary Series corresponds to $D^k(m_{2p+2,k+1} \,...\, m_{2p+2,p}; x_{2p+2,p+1})$ in Schwarz's classification [33], where k is related to $\tilde{n}$ by $k = p - \tilde{n}$.

[12]Our Exceptional Series corresponds to $D^k(m_{2p+2,k+1} \,...\, m_{2p+2,p}; m_{2p+2,p+1})$ in Schwarz's classification [33], where k is related to $\tilde{n}$ by $k = p - \tilde{n}$.

[13]This expression for the quadratic Casimir can be obtained in the same way as in the even-dimensional case - see eq. (3.37).

of the eigenmode (corresponding to tensor-spinor eigenmodes of the Dirac operator on S^{D-1}). In view of the classification of the spin$(D,1)$ UIR's under the decomposition spin$(D,1) \supset$ spin(D), expressing our eigenmodes on dS_D in terms of eigentensor-spinors on S^{D-1} offers an easy way to understand the spin(D) content of our dS eigenmodes. The outline of this Section is:

- In Subsection 3.4.1, we review the necessary material concerning the (totally symmetric) TT tensor-spinor eigenmodes of the Dirac operator on S^{D-1} and the way they form representations of spin(D) [24].
- In Subsections 3.4.2 and 3.4.3, we present the construction of spin-3/2 TT eigenmodes on dS_D in order to illustrate the method of separation of variables for tensor-spinor fields. Some basic results are tabulated in Tables 3.1 and 3.2.
- In Subsection 3.4.4, we summarise our main results concerning the spin-5/2 TT eigenmodes on dS_D.

3.4.1 Tensor-spinor eigenmodes of the Dirac operator on S^{D-1} and representations of spin(D)

The spectrum of the Dirac operator acting on tensor-spinor eigenmodes on spheres, as well as the representations of spin(D) formed by the eigenmodes, have been discussed in Refs. [39, 3, 24, 4] (see also references therein).

Let $\tilde{\nabla\!\!\!/} \equiv \tilde{\gamma}^k \tilde{\nabla}_k$ be the Dirac operator on S^{D-1}, where $\tilde{\nabla}_j$ is the covariant derivative on S^{D-1}. We are interested in rank-$\tilde{r} \geq 0$ totally symmetric TT tensor-spinor eigenmodes $\tilde{\psi}^{(\ell;\underline{m})}_{\pm\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_{\tilde{r}}}(\boldsymbol{\theta_{D-1}})$ on S^{D-1}. The eigenmodes $\tilde{\psi}^{(\ell;\underline{m})}_{\pm\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_{\tilde{r}}}(\boldsymbol{\theta_{D-1}})$ satisfy

$$\tilde{\nabla\!\!\!/}\tilde{\psi}^{(\ell;\underline{m})}_{\pm\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_{\tilde{r}}} = \pm i\left(\ell + \frac{D-1}{2}\right)\tilde{\psi}^{(\ell;\underline{m})}_{\pm\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_{\tilde{r}}} \tag{3.46}$$

$$\tilde{\gamma}^{\tilde{\mu}_1}\tilde{\psi}^{(\ell;\underline{m})}_{\pm\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_{\tilde{r}}} = \tilde{\nabla}^{\tilde{\mu}_1}\tilde{\psi}^{(\ell;\underline{m})}_{\pm\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_{\tilde{r}}} = 0, \tag{3.47}$$

where the angular momentum quantum number on S^{D-1}, ℓ, is allowed to take integer values with $\ell \geq \tilde{r}$. The two sets of eigenmodes, $\{\tilde{\psi}^{(\ell;\underline{m})}_{+\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_{\tilde{r}}}\}$ [with eigenvalue $+i(\ell + \frac{D-1}{2})$] and $\{\tilde{\psi}^{(\ell;\underline{m})}_{-\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_{\tilde{r}}}\}$ [with eigenvalue $-i(\ell + \frac{D-1}{2})$], separately form representations of spin(D). The label $\underline{m}$ represents quantum numbers (other than ℓ) the values of which specify the content of the spin(D) representation concerning the chain of subalgebras spin$(D-1) \supset$ spin$(D-2) \supset ... \supset$ spin(2).

Odd $D \geq 3$ (even-dimensional spheres). For each allowed value of ℓ we have a representation of spin(D) acting on the space of the eigenmodes $\{\tilde{\psi}^{(\ell;\underline{m})}_{+\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_{\tilde{r}}}\}$ (or

$\{\tilde{\psi}^{(\ell;\underline{m})}_{-\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_{\tilde{r}}}\}$) on S^{D-1} with highest weight (3.23) given by [24]

$$\vec{f}_{\tilde{r}} = \left(\ell + \frac{1}{2}, \tilde{r} + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right), \qquad (\ell = \tilde{r}, \tilde{r}+1, ...), \tag{3.48}$$

where we have used the subscript $\tilde{r}$ in order to denote the 'spin' of the representation - e.g. $\vec{f}_0$ corresponds to a spinor representation, $\vec{f}_1$ to a TT vector-spinor representation, $\vec{f}_2$ to a rank-2 (totally symmetric) TT tensor-spinor representation and so forth. The two sets of eigenmodes, $\{\tilde{\psi}^{(\ell;\underline{m})}_{+\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_{\tilde{r}}}\}$ and $\{\tilde{\psi}^{(\ell;\underline{m})}_{-\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_{\tilde{r}}}\}$, form equivalent representations. For $D=5$ the highest weight is $\vec{f}_{\tilde{r}} = (\ell + 1/2, \tilde{r} + 1/2)$. On S^2 - i.e. for $D=3$ - there are no totally symmetric TT eigenmodes satisfying eq. (3.46) with rank $\tilde{r} \geq 1$ - see Refs. [4, 26, 27] and Appendix 3.9. However, eigenmodes with $\tilde{r} = 0$ - i.e. eigenspinors $\tilde{\psi}^{(\ell;\underline{m})}_{\pm}$ of the Dirac operator [3] - exist on S^2 and the corresponding spin(3) representation is labelled by the one-component highest weight $\ell + 1/2$ (with $\ell = 0, 1, ...$).

Even $D \geq 4$ (odd-dimensional spheres). For each allowed value of ℓ the eigenmodes $\{\tilde{\psi}^{(\ell;\underline{m})}_{+\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_{\tilde{r}}}\}$ on S^{D-1} form a spin(D) representation with highest weight (3.23) given by [24]

$$\vec{f}^{+}_{\tilde{r}} = \left(\ell + \frac{1}{2}, \tilde{r} + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right), \qquad (\ell = \tilde{r}, \tilde{r}+1, ...), \tag{3.49}$$

while the eigenmodes $\{\tilde{\psi}^{(\ell;\underline{m})}_{-\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_{\tilde{r}}}\}$ form a representation with highest weight [24]

$$\vec{f}^{-}_{\tilde{r}} = \left(\ell + \frac{1}{2}, \tilde{r} + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}, -\frac{1}{2}\right), \qquad (\ell = \tilde{r}, \tilde{r}+1, ...). \tag{3.50}$$

For $D=4$ the highest weights corresponding to the eigenmodes $\{\tilde{\psi}^{(\ell;\underline{m})}_{\pm\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_{\tilde{r}}}\}$ are $\vec{f}^{\pm}_{\tilde{r}} = (\ell + 1/2, \pm(\tilde{r} + 1/2))$.

For both even D [eqs. (3.49) and (3.50)] and odd D [eq. (3.48)], if the aforementioned irreducible representations of spin(D) are contained in a spin($D,1$) representation, then the allowed values for the angular momentum quantum number ℓ might not just be $\ell = \tilde{r}, \tilde{r}+1, ...$; ℓ might have to satisfy extra conditions because of the branching rules (3.29) and (3.39). This will become clear in the next Subsection as ℓ will have to satisfy $\ell \geq r$, where r is the rank of the tensor-spinor eigenmodes on dS_D.

3.4.2 Separating variables for spin-3/2 eigenmodes on dS_D for even $D \geq 4$

Let us illustrate the method of separation of variables for the TT vector-spinor field $\Psi_\mu = (\Psi_t, \Psi_{\theta_{D-1}}, \Psi_{\theta_{D-2}}, ..., \Psi_{\theta_1})$ with arbitrary mass parameter M on global dS_D for even $D \geq 4$.

The Dirac equation (3.3) is expressed as

$$\left(\frac{\partial}{\partial t}+\frac{D+1}{2}\tanh t\right)\gamma^t\Psi_t+\frac{1}{\cosh t}\begin{pmatrix}0 & i\tilde{\nabla}\\ -i\tilde{\nabla} & 0\end{pmatrix}\Psi_t=-M\Psi_t, \tag{3.51}$$

$$\left(\frac{\partial}{\partial t}+\frac{D-3}{2}\tanh t\right)\gamma^t\Psi_{\theta_j}+\frac{1}{\cosh t}\begin{pmatrix}0 & i\tilde{\nabla}\\ -i\tilde{\nabla} & 0\end{pmatrix}\Psi_{\theta_j}-\tanh t\,\gamma_{\theta_j}\Psi_t=-M\Psi_{\theta_j}, \tag{3.52}$$

$(j=1,2,...,D-1)$, where we have made use of eqs. (3.8), (3.14) and (3.20)-(3.22), while $\gamma_{\theta_j}=e_{\theta_j}{}^{k}\gamma_k$. There are two different ways in which we can separate variables for the TT vector-spinor $\Psi_\mu(t,\boldsymbol{\theta_{D-1}})$ giving rise to two different types of eigenmodes: the type-I modes and the type-II modes. These two different types of eigenmodes correspond to spin(D) representations with different spin. In particular, the spin(D) content that is relevant to type-I modes corresponds to the spinor representation $\vec{f}_0^{\pm}=(\ell+\frac{1}{2},\frac{1}{2},...,\frac{1}{2},\pm\frac{1}{2})$ with $\ell=1,2,...$.[14] The spin(D) content that is relevant to type-II modes corresponds to the vector-spinor representation $\vec{f}_1^{\pm}=(\ell+\frac{1}{2},\frac{3}{2},\frac{1}{2},...,\frac{1}{2},\pm\frac{1}{2})$ with $\ell=1,2,...$.

Type-I modes. Let us denote the type-I modes with spin(D) content given by $\vec{f}_0^{\pm}=(\ell+\frac{1}{2},\frac{1}{2},...,\frac{1}{2},\pm\frac{1}{2})$ as $\Psi_\mu^{(M;\,\tilde{r}=0,\,\pm\ell;\,\underline{m})}(t,\boldsymbol{\theta_{D-1}})$, where the label $\underline{m}$ has the same meaning as in Subsection 3.4.1. We start with the case of $\vec{f}_0^{-}=(\ell+\frac{1}{2},\frac{1}{2},...,\frac{1}{2},-\frac{1}{2})$, i.e. with the type-I modes $\Psi_\mu^{(M;\,\tilde{r}=0,\,-\ell;\,\underline{m})}(t,\boldsymbol{\theta_{D-1}})$. As in Refs. [3, 4, 28], we separate variables for the t-component by expressing it in terms of upper and lower spinor components, as

$$\Psi_t^{(M;\,\tilde{r}=0,\,-\ell;\,\underline{m})}(t,\boldsymbol{\theta}_{D-1})=\begin{pmatrix}-i\Phi_{M\ell}^{(1)}(t)\,\tilde{\psi}_-^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}})\\ -\Psi_{M\ell}^{(1)}(t)\,\tilde{\psi}_-^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}})\end{pmatrix}, \tag{3.53}$$

where $\tilde{\psi}_-^{(\ell;\underline{m})}$ are the $2^{D/2-1}$-dimensional eigenspinors of $\tilde{\nabla}$ on S^{D-1} [see Eq. (3.46)]. Now, we have to determine the functions of time $\Phi_{M\ell}^{(1)}(t)$ and $\Psi_{M\ell}^{(1)}(t)$ - the superscript '(1)' in these functions has been used for later convenience. By substituting eq. (3.53) into the Dirac equation (3.51), we can eliminate the lower component in eq. (3.53). We find in this manner the second order equation for $\Phi_{M\ell}^{(1)}(t)$

$$\mathscr{D}_{(1)}\Phi_{M\ell}^{(1)}=M^2\,\Phi_{M\ell}^{(1)}, \tag{3.54}$$

[14] Under the decomposition spin$(D,1)\supset$ spin(D), the branching rules (3.29) give rise to the restriction $\ell\geq 1$. One can also arrive at this restriction on ℓ by requiring the regularity of type-I eigenmodes, as we will discuss below. See Refs. [26, 27] for more details concerning the explicit form of the eigenmodes.

where the differential operator $\mathscr{D}_{(1)}$ is a special case of the following family of differential operators:

$$\mathscr{D}_{(a)} = \frac{\partial^2}{\partial x^2} + (D+2a-1)\cot x \frac{\partial}{\partial x} + \left(\ell + \frac{D-1}{2}\right)\frac{\cos x}{\sin^2 x} - \frac{(\ell+\frac{D-1}{2})^2 - \frac{1}{4}(D+2a-1)(D+2a-3)}{\sin^2 x} - \frac{(D+2a-1)^2}{4}, \tag{3.55}$$

where we have defined

$$x = x(t) := \frac{\pi}{2} - it \tag{3.56}$$

with $\cos x = i\sinh t$ and $\sin x = \cosh t$. For later convenience, instead of just solving the eigenvalue equation (3.54), we can solve the more general equation

$$\mathscr{D}_{(a)}\Phi^{(a)}_{M\ell} = M^2\,\Phi^{(a)}_{M\ell}, \tag{3.57}$$

for arbitrary integer a. The solution is given by

$$\Phi^{(a)}_{M\ell}(t) = \left(\cos\frac{x(t)}{2}\right)^{\ell+1-a}\left(\sin\frac{x(t)}{2}\right)^{\ell-a} \times F\left(-iM + \frac{D}{2} + \ell, iM + \ell + \frac{D}{2}; \ell + \frac{D}{2}; \sin^2\frac{x(t)}{2}\right), \tag{3.58}$$

where $F(A,B;C;z)$ is the Gauss hypergeometric function [16], while

$$\cos\frac{x(t)}{2} = \left(\sin\frac{x(t)}{2}\right)^* = \frac{\sqrt{2}}{2}\left(\cosh\frac{t}{2} + i\sinh\frac{t}{2}\right). \tag{3.59}$$

Thus, we have now determined the upper component of $\Psi_t^{(M;\,\tilde{r}=0,\,-\ell;\,\underline{m})}$ in eq. (3.53), where $\Phi^{(1)}_{M\ell}$ is given by eq. (3.58) with $a=1$.
In order to determine the lower component in eq. (3.53), we substitute eq. (3.53) into the Dirac equation (3.51) and we straightforwardly find the relations

$$\left(\frac{d}{dt} + \frac{D+1}{2}\tanh t - \frac{i\left(\ell + \frac{D-1}{2}\right)}{\cosh t}\right)\Psi^{(1)}_{M\ell}(t) = -M\,\Phi^{(1)}_{M\ell}(t), \tag{3.60}$$

$$\left(\frac{d}{dt} + \frac{D+1}{2}\tanh t + \frac{i\left(\ell + \frac{D-1}{2}\right)}{\cosh t}\right)\Phi^{(1)}_{M\ell}(t) = M\,\Psi^{(1)}_{M\ell}(t). \tag{3.61}$$

Then, substituting eq. (3.58) (with $a=1$) into eq. (3.61) and using well-known properties of the hypergeometric function [16], we find

$$\begin{aligned}\mathit{\Psi}^{(1)}_{M\ell}(t) = &\frac{-iM}{\ell+\frac{D}{2}}\left(\cos\frac{x(t)}{2}\right)^{\ell-1}\left(\sin\frac{x(t)}{2}\right)^{\ell}\\ &\times F\left(-iM+\frac{D}{2}+\ell, iM+\ell+\frac{D}{2};\ell+\frac{D+2}{2};\sin^2\frac{x(t)}{2}\right).\end{aligned} \tag{3.62}$$

For later convenience, let us note that $\mathit{\Psi}^{(1)}_{M\ell}(t)$ corresponds to a special case (i.e. the case with $a=1$) of the following functions:

$$\begin{aligned}\mathit{\Psi}^{(a)}_{M\ell}(t) = &\frac{-iM}{\ell+\frac{D}{2}}\left(\cos\frac{x(t)}{2}\right)^{\ell-a}\left(\sin\frac{x(t)}{2}\right)^{\ell+1-a}\\ &\times F\left(-iM+\frac{D}{2}+\ell, iM+\ell+\frac{D}{2};\ell+\frac{D+2}{2};\sin^2\frac{x(t)}{2}\right).\end{aligned} \tag{3.63}$$

These functions solve the differential equation $(\hat{\mathscr{D}}_{(a)}-M^2)\mathit{\Psi}^{(a)}_{M\ell}(t)=0$ where the differential operator $\hat{\mathscr{D}}_{(a)}$ is given by eq. (3.55) with x replaced by $\pi-x$. Thus, we have now also determined the lower component of $\Psi_t^{(M;\tilde{r}=0,-\ell;\underline{m})}$ in eq. (3.53).
Now, by following the same procedure as the one described above, we can separate variables for the type-I modes $\Psi_\mu^{(M;\tilde{r}=0,+\ell;\underline{m})}(t,\boldsymbol{\theta_{D-1}})$ corresponding to the spin(D) highest weight $\vec{f}_0^{+}=(\ell+\frac{1}{2},\frac{1}{2},...,\frac{1}{2})$. We find

$$\Psi_t^{(M;\tilde{r}=0,+\ell;\underline{m})}(t,\boldsymbol{\theta_{D-1}}) = \begin{pmatrix}\mathit{\Psi}^{(1)}_{M\ell}(t)\,\tilde{\psi}_{+}^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}})\\ i\mathit{\Phi}^{(1)}_{M\ell}(t)\,\tilde{\psi}_{+}^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}})\end{pmatrix}. \tag{3.64}$$

The rest of the vector components of the type-I modes, $\Psi_{\theta_j}^{(M;\tilde{r}=0,\pm\ell;\underline{m})}$ ($j=1,...,D-1$), can be straightforwardly determined by substituting the known expressions for $\Psi_t^{(M;\tilde{r}=0,\pm\ell;\underline{m})}$ [eqs. (3.53) and (3.64)] into the TT conditions (3.4). By doing so, one finds that there is a proportionality factor of $\frac{1}{\ell}$ in the expressions for each of the $\Psi_{\theta_j}^{(M;\tilde{r}=0,\pm\ell;\underline{m})}$ and, thus, the regularity of type-I eigenmodes gives rise to the restriction $\ell\geq1$. However, here we will not present explicit expressions for $\Psi_{\theta_j}^{(M;\tilde{r}=0,\pm\ell;\underline{m})}$ ($j=1,...,D-1$) as they are lengthy and they are not needed for our analysis. The interested reader can find the explicit expressions in Refs. [26, 27].

Type-II modes. Let us denote the type-II modes with spin(D) content given by $\vec{f}_1^{\pm}=(\ell+\frac{1}{2},\frac{3}{2},\frac{1}{2},...,\frac{1}{2},\pm\frac{1}{2})$ as $\Psi_\mu^{(M;\tilde{r}=1,\pm\ell;\underline{m})}(t,\boldsymbol{\theta_{D-1}})$ ($\ell\geq1$). The type-II modes are TT vector-spinors on S^{D-1} and thus $\Psi_t^{(M;\tilde{r}=1,\pm\ell;\underline{m})}(t,\boldsymbol{\theta}_{D-1})=0$. The components $\Psi_{\theta_j}^{(M;\tilde{r}=1,\pm\ell;\underline{m})}(t,\boldsymbol{\theta}_{D-1})$ can be determined by applying the method of separation of variables as in the case of the type-I modes. However, now we have to express $\Psi_{\theta_j}^{(M;\tilde{r}=1,\pm\ell;\underline{m})}(t,\boldsymbol{\theta}_{D-1})$ in terms of TT eigenvector-spinors on S^{D-1}, instead of eigenspinors on S^{D-1}. By applying the method of separation of variables to the Dirac equation (3.52), we find

$$\Psi_t^{(M;\tilde{r}=1,-\ell;\underline{m})}(t,\boldsymbol{\theta_{D-1}}) = 0, \qquad \Psi_{\theta_j}^{(M;\tilde{r}=1,-\ell;\underline{m})}(t,\boldsymbol{\theta}_{D-1}) = \begin{pmatrix} \Phi_{M\ell}^{(-1)}(t)\,\tilde{\psi}_{-\theta_j}^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}}) \\ -i\Psi_{M\ell}^{(-1)}(t)\,\tilde{\psi}_{-\theta j}^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}}) \end{pmatrix} \tag{3.65}$$

and

$$\Psi_t^{(M;\tilde{r}=1,+\ell;\underline{m})}(t,\boldsymbol{\theta}_{D-1}) = 0, \qquad \Psi_{\theta_j}^{(M;\tilde{r}=1,+\ell;\underline{m})}(t,\boldsymbol{\theta}_{D-1}) = \begin{pmatrix} i\Psi_{M\ell}^{(-1)}(t)\,\tilde{\psi}_{+\theta_j}^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}}) \\ -\Phi_{M\ell}^{(-1)}(t)\,\tilde{\psi}_{+\theta j}^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}}) \end{pmatrix}, \tag{3.66}$$

($j=1,...,D-1$) where $\tilde{\psi}_{\pm\theta j}^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}})$ are the TT eigenvector-spinors (3.46) on S^{D-1}. The functions $\Phi_{M\ell}^{(-1)}(t)$ and $\Psi_{M\ell}^{(-1)}(t)$ are given by eqs. (3.58) and (3.63), respectively, with $a=-1$.

Summary. Some basic results concerning the spin-3/2 eigenmodes for even $D\geq 4$ are tabulated in Table 3.1.

3.4.3 SEPARATING VARIABLES FOR SPIN-3/2 EIGENMODES ON dS_D FOR ODD $D \geq 3$

The Dirac equation (3.3) is expressed as

$$\left(\frac{\partial}{\partial t} + \frac{D+1}{2}\tanh t\right)\gamma^t\Psi_t + \frac{1}{\cosh t}\tilde{\nabla}\Psi_t = -M\Psi_t, \tag{3.67}$$

$$\left(\frac{\partial}{\partial t} + \frac{D-3}{2}\tanh t\right)\gamma^t\Psi_{\theta_j} + \frac{1}{\cosh t}\tilde{\nabla}\Psi_{\theta_j} - \tanh t\,\gamma_{\theta_j}\Psi_t = -M\Psi_{\theta_j}, \tag{3.68}$$

($j=1,2,...,D-1$), where the gamma matrices are now given by eq. (3.17). As in the even-dimensional case, we have two different types of eigenmodes depending on their spin(D) content.

Type-I modes. Let us denote the type-I modes with spin(D) content given by $\vec{f}_0 = (\ell+\frac{1}{2},\frac{1}{2},...,\frac{1}{2})$ as $\Psi_\mu^{(M;\tilde{r}=0,\ell;\underline{m})}(t,\boldsymbol{\theta_{D-1}})$ (with $\ell\geq 1$). As in Refs. [3, 4, 28], we separate variables as

$$\Psi_t^{(M;\tilde{r}=0,\ell;\underline{m})}(t,\boldsymbol{\theta_{D-1}}) = \frac{1}{\sqrt{2}}(\mathbf{1}+\gamma^t)\left\{-i\,\Phi_{M\ell}^{(1)}(t) + i\,\Psi_{M\ell}^{(1)}(t)\gamma^t\right\}\tilde{\psi}_-^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}}), \tag{3.69}$$

where $\tilde{\psi}_{-}^{(\ell;\underline{m})}$ are the eigenspinors (3.46) on S^{D-1}, while $i\tilde{\psi}_{+}^{(\ell;\underline{m})} = \gamma^t \tilde{\psi}_{-}^{(\ell;\underline{m})}$ as γ^t anti-commutes with $\tilde{\nabla}$. Substituting eq. (3.69) into the Dirac equation (3.67), we find that $\Phi^{(1)}_{M\ell}(t), \Psi^{(1)}_{M\ell}(t)$ must satisfy the relations (3.60) and (3.61). Then, we readily find that $\Phi^{(1)}_{M\ell}(t)$ is given by eq. (3.58) with $a = 1$, while $\Psi^{(1)}_{M\ell}(t)$ is given by eq. (3.62). The components $\Psi^{(M;\tilde{r}=0,\ell;\underline{m})}_{\theta_j}(t, \boldsymbol{\theta_{D-1}})$ can be determined with the use of the TT conditions (3.4), as in the even-dimensional case.

Type-II modes. The type-II modes $\Psi^{(M;\tilde{r}=1,\ell;\underline{m})}_{\mu}(t, \boldsymbol{\theta_{D-1}})$ correspond to the following spin(D) representation: $\vec{f}_1 = (\ell + \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, ..., \frac{1}{2})$ with $\ell \geq 1$ and they exist for $D > 3$. We separate variables as

$$\begin{aligned}
&\Psi^{(M;\tilde{r}=1,\ell;\underline{m})}_{t}(t, \boldsymbol{\theta_{D-1}}) = 0, \\
&\Psi^{(M;\tilde{r}=1,\ell;\underline{m})}_{\theta_j}(t, \boldsymbol{\theta_{D-1}}) = \frac{1}{\sqrt{2}}(\mathbf{1} + \gamma^t) \left\{ \Phi^{(-1)}_{M\ell}(t) - \Psi^{(-1)}_{M\ell}(t)\gamma^t \right\} \tilde{\psi}^{(\ell;\underline{m})}_{-\theta_j}(\boldsymbol{\theta_{D-1}}),
\end{aligned} \tag{3.70}$$

$(j = 1, ..., D-1)$ where $\tilde{\psi}^{(\ell;\underline{m})}_{-\theta_j}$ are the eigenvector-spinors (3.46) on S^{D-1}, while $i\tilde{\psi}^{(\ell;\underline{m})}_{+\theta_j} = \gamma^t \tilde{\psi}^{(\ell;\underline{m})}_{-\theta_j}$. Substituting eq. (3.70) into the Dirac equation (3.68), we find that $\Phi^{(-1)}_{M\ell}(t)$ and $\Psi^{(-1)}_{M\ell}(t)$ are given by eqs. (3.58) and (3.63), respectively, with $a = -1$.

Summary. Some basic results concerning the spin-3/2 eigenmodes for odd $D \geq 3$ are tabulated in Table 3.2.

3.4.4 Spin-5/2 eigenmodes on dS_D

In the case of rank-2 totally symmetric tensor-spinors $\Psi_{\mu\nu}$ - which satisfy eqs. (3.3) and (3.4) with $r = 2$ on dS_D - the method of separation of variables can be applied in a way analogous to the case of TT vector-spinors. Depending on the spin(D) content of the spin-5/2 dS eigenmode we can distinguish three types of modes: type-I, type-II and type-III modes (the last two exist for $D > 3$). Here we will just summarise some basic results for the TT spin-5/2 eigenmodes on dS_D. Below we use the same notation for the labels of the eigenmodes as in the spin-3/2 case, while we refer again to the spin(D) content of the eigenmodes using the highest weights $\vec{f}^{\pm}_{\tilde{r}}$ for even D [eqs. (3.49) and (3.50)] and $\vec{f}_{\tilde{r}}$ for odd D [eq. (3.48)].

Even $D \geq 4$. The TT spin-5/2 eigenmodes on dS_D and their spin(D) content are given by:

Type-I: $\vec{f}_0^{\pm} = \left(\ell + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}, \pm\frac{1}{2}\right), \quad \ell = 2, 3, ...$

$$\Psi_{tt}^{(M;\,\tilde{r}=0,\,-\ell;\,\underline{m})} = \begin{pmatrix} -\Phi_{M\ell}^{(2)}(t)\,\tilde{\psi}_{-}^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}}) \\ i\Psi_{M\ell}^{(2)}(t)\,\tilde{\psi}_{-}^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}}) \end{pmatrix}, \Psi_{tt}^{(M;\,\tilde{r}=0,\,+\ell;\,\underline{m})} = \begin{pmatrix} -i\Psi_{M\ell}^{(2)}(t)\,\tilde{\psi}_{+}^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}}) \\ \Phi_{M\ell}^{(2)}(t)\,\tilde{\psi}_{+}^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}}) \end{pmatrix}. \tag{3.71}$$

Type-II: $\vec{f}_1^{\pm} = \left(\ell + \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, ..., \frac{1}{2}, \pm\frac{1}{2}\right), \quad \ell = 2, 3, ...$

$$\Psi_{tt}^{(M;\,\tilde{r}=1,\,-\ell;\,\underline{m})} = \Psi_{tt}^{(M;\,\tilde{r}=1,\,+\ell;\,\underline{m})} = 0,$$

$$\Psi_{t\theta_j}^{(M;\,\tilde{r}=1,\,-\ell;\,\underline{m})} = \begin{pmatrix} -i\Phi_{M\ell}^{(0)}(t)\,\tilde{\psi}_{-\theta_j}^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}}) \\ -\Psi_{M\ell}^{(0)}(t)\,\tilde{\psi}_{-\theta_j}^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}}) \end{pmatrix}, \quad \Psi_{t\theta_j}^{(M;\,\tilde{r}=1,\,+\ell;\,\underline{m})} = \begin{pmatrix} \Psi_{M\ell}^{(0)}(t)\,\tilde{\psi}_{+\theta_j}^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}}) \\ i\Phi_{M\ell}^{(0)}(t)\,\tilde{\psi}_{+\theta_j}^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}}) \end{pmatrix}. \tag{3.72}$$

Type-III: $\vec{f}_2^{\pm} = \left(\ell + \frac{1}{2}, \frac{5}{2}, \frac{1}{2}, ..., \frac{1}{2}, \pm\frac{1}{2}\right), \quad \ell = 2, 3, ...$

$$\begin{aligned} \Psi_{t\mu}^{(M;\,\tilde{r}=2,\,-\ell;\,\underline{m})} &= \Psi_{t\mu}^{(M;\,\tilde{r}=2,\,+\ell;\,\underline{m})} = 0, \\ \Psi_{\theta_j\theta_k}^{(M;\,\tilde{r}=2,\,-\ell;\,\underline{m})} &= \begin{pmatrix} \Phi_{M\ell}^{(-2)}(t)\,\tilde{\psi}_{-\theta_j\theta_k}^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}}) \\ -i\Psi_{M\ell}^{(-2)}(t)\,\tilde{\psi}_{-\theta_j\theta_k}^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}}) \end{pmatrix}, \\ \Psi_{\theta_j\theta_k}^{(M;\,\tilde{r}=2,\,+\ell;\,\underline{m})} &= \begin{pmatrix} i\Psi_{M\ell}^{(-2)}(t)\,\tilde{\psi}_{+\theta_j\theta_k}^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}}) \\ -\Phi_{M\ell}^{(-2)}(t)\,\tilde{\psi}_{+\theta_j\theta_k}^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}}) \end{pmatrix}, \end{aligned} \tag{3.73}$$

where $\tilde{\psi}_{\pm\theta_j\theta_k}^{(\ell;\underline{m})}$ are the rank-2 tensor-spinor eigenmodes (3.46) on S^{D-1}, while $\mu = t, \theta_{D-1}, ..., \theta_2, \theta_1$ and $j, k = 1, ..., D-1$. The components that have not been written down explicitly can be found from the TT conditions (3.4) (for explicit expressions for all the components see Refs. [26, 27]).

Odd $D \geq 3$. The TT spin-5/2 eigenmodes on dS_D and their spin(D) content are given by:

Type-I: $\vec{f}_0 = \left(\ell + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right), \quad \ell = 2, 3, ...$

$$\Psi_{tt}^{(M;\,\tilde{r}=0,\,\ell;\,\underline{m})} = \frac{1}{\sqrt{2}}(\mathbf{1} + \gamma^t)\left\{-\Phi_{M\ell}^{(2)}(t) + \Psi_{M\ell}^{(2)}(t)\gamma^t\right\}\tilde{\psi}_{-}^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}}). \tag{3.74}$$

Type-II (for $D > 3$): $\vec{f}_1 = \left(\ell + \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, ..., \frac{1}{2}\right), \quad \ell = 2, 3, ...$

$$\Psi_{tt}^{(M;\,\tilde{r}=1,\ell;\,\underline{m})} = 0,$$

$$\Psi_{t\theta_j}^{(M;\,\tilde{r}=1,\,\ell;\,\underline{m})} = \frac{1}{\sqrt{2}}(\mathbf{1} + \gamma^t)\left\{-i\Phi_{M\ell}^{(0)}(t) + i\Psi_{M\ell}^{(0)}(t)\gamma^t\right\}\tilde{\psi}_{-\theta_j}^{(\ell;\underline{m})}(\boldsymbol{\theta_{D-1}}). \tag{3.75}$$

Type-III (for $D>3$): $\vec{f}_2 = \left(\ell+\frac{1}{2}, \frac{5}{2}, \frac{1}{2}, ..., \frac{1}{2}\right), \quad \ell = 2, 3, ...$

$$\Psi^{(M;\,\tilde{r}=2,\,\ell;\,\underline{m})}_{t\mu} = 0,$$

$$\Psi^{(M;\,\tilde{r}=2,\,\ell;\,\underline{m})}_{\theta_j\theta_k} = \frac{1}{\sqrt{2}}(\mathbf{1}+\gamma^t)\left\{\Phi^{(-2)}_{M\ell}(t) - \Psi^{(-2)}_{M\ell}(t)\gamma^t\right\}\tilde{\psi}^{(\ell;\underline{m})}_{-\theta_j\theta_k}(\boldsymbol{\theta_{D-1}}), \tag{3.76}$$

where $\mu = t, \theta_{D-1}, ..., \theta_2, \theta_1$ and $j, k = 1, ..., D-1$. As in the even-dimensional case, the components that have not been written down explicitly can be found from the TT conditions (3.4).

3.5 QUADRATIC CASIMIR FOR SPIN-3/2 AND SPIN-5/2 EIGENMODES ON dS_D

In order to find the values of the spin$(D,1)$ quadratic Casimir corresponding to the representation formed by our spin-3/2 and spin-5/2 eigenmodes we will use the "analytic continuation" techniques that have been already used in Refs. [21, 28]. More specifically, we will use the fact that dS_D can be obtained by an "analytic continuation" of S^D. The line element of S^D can be written as

$$d\Omega_D^2 = d\theta_D^2 + \sin^2\theta_D\, d\Omega_{D-1}^2, \tag{3.77}$$

where $0 \leq \theta_D \leq \pi$. By replacing the angle θ_D in $d\Omega_D^2$ as:

$$\theta_D \to x(t) = \frac{\pi}{2} - it, \tag{3.78}$$

($t \in \mathbb{R}$) we find the line element (3.18) for global dS_D ($x(t)$ coincides with the 'useful' variable that we have already introduced in eq. (3.56)).

Quadratic Casimir for tensor-spinor eigenmodes on S^D. Motivated by the aforementioned observation, we can obtain the field equations (3.3) and (3.4) for spin-$(r+1/2)$ fields on dS_D by analytically continuing the equations for totally symmetric TT tensor-spinors of rank r on S^D:

$$\nabla\!\!\!\!/\, \psi_{\pm\mu_1...\mu_r} = \pm i\left(n+\frac{D}{2}\right)\psi_{\pm\mu_1...\mu_r}, \quad (n = r,\, r+1, ...) \tag{3.79}$$

$$\nabla^\alpha\psi_{\pm\alpha\mu_2...\mu_r} = 0, \quad \gamma^\alpha\psi_{\pm\alpha\mu_2...\mu_r} = 0, \tag{3.80}$$

where $\psi_{\pm\mu_1...\mu_r}$ is a tensor-spinor on S^D, while n is the angular momentum quantum number on S^D. Equations (3.79) and (3.80) are essentially the D-dimensional counterparts of eqs. (3.46) and (3.47), while now n on S^D plays the role of ℓ on S^{D-1}. As we discussed in Subsection 3.4.1, the spin$(D+1)$ representations formed by tensor-spinor eigenmodes of the Dirac operator on S^D are known [24]. Using eqs. (3.26) and (3.27), the spin$(D+1)$ quadratic Casimir corresponding to the eigenmodes $\psi_{\pm\mu_1\dots\mu_r}$ on S^D is readily found to be

$$\begin{aligned}\mathscr{C}^{(S^D)}_{eigen} &= \left(n+\frac{D}{2}\right)^2 - r - \frac{D(D-1)}{4} + \frac{(D-2)(D-3)}{8} + s(s+D-2) \qquad (3.81)\\ &= -\nabla^\mu\nabla_\mu + \frac{(D-2)(D-3)}{8} + s(s+D-2) \quad (\text{where } s=r+1/2),\end{aligned}$$

for all $D\geq 3$, while in the second line we used that $\nabla^\mu\nabla_\mu$ acts on $\psi_{\pm\mu_1\dots\mu_r}$ as

$$\nabla^\mu\nabla_\mu = \nabla\!\!\!/^2 + \frac{D(D-1)}{4} + r.$$

Analytic continuation to dS_D. Without loss of generality, we can choose to analytically continue the eigentensor-spinors with either one of the two signs for the eigenvalue in eq. (3.79), since each of the two sets of modes, $\{\psi_{+\mu_1\dots\mu_r}\}$ and $\{\psi_{-\mu_1\dots\mu_r}\}$, forms independently a unitary representation of spin$(D+1)$ labelled by n (see Subsection 3.4.1). Here we choose to analytically continue the eigentensor-spinors $\psi_{-\mu_1\dots\mu_r}$. We perform analytic continuation by making the following replacements in eqs. (3.79) and (3.80)[15]:

$$\theta_D \to x(t) = \frac{\pi}{2} - it, \qquad n \to -iM - \frac{D}{2} \qquad (t\in\mathbb{R}) \tag{3.82}$$

and we obtain eqs. (3.3) and (3.4), respectively, for tensor-spinors $\Psi_{\mu_1\dots\mu_r}$ with mass parameter M on dS_D. Recall that the values of interest for M are: $M\in\mathbb{R}$ (corresponding to massive fermions of spin $s\geq 3/2$), as well as the purely imaginary values of M corresponding to the strictly/partially massless tunings (3.5). The prescription for obtaining the explicit form of dS eigenmodes by analytically continuing eigenmodes on S^D can be found in Refs. [21, 28, 26, 27].

Quadratic Casimir for tensor-spinor eigenmodes on dS_D. With the use of the replacements (3.82), we analytically continue the quadratic Casimir on S^D [Eq. (3.81)], and we find the value of the quadratic Casimir on dS_D:

$$\mathscr{C}^{(dS_D)}_{eigen} = -M^2 - r - \frac{D(D-1)}{4} + \frac{(D-2)(D-3)}{8} + s(s+D-2) \tag{3.83}$$

[15] By making the replacements (3.82), the tensor-spinor $\psi_{-\mu_1\dots\mu_r}$ on S^D is analytically continued to the tensor-spinor $\Psi_{\mu_1\dots\mu_r}$ [eq. (3.3)] on dS_D. Alternatively, we could analytically continue the eigentensor-spinors on S^D by making the replacement $\theta_D\to\pi/2+it$ instead of the replacement (3.78). The analytically continued eigentensor-spinors with $\theta_D\to\pi/2-it$ and the ones with $\theta_D\to\pi/2+it$ are related to each other by charge conjugation. However, these two cases of eigenmodes form equivalent representations of spin$(D,1)$ - see Refs. [26, 27].

Type of eigenmode	Notation	spin(D) content
Type-I	$\Psi_{\mu}^{(M;\tilde{r}=0,\pm\ell;\underline{m})}$	$\vec{f}_0^{\pm}=(\ell+\frac{1}{2},\frac{1}{2},...,\frac{1}{2},\pm\frac{1}{2}),\ \ \ell=1,2,...$
Type-II	$\Psi_{\mu}^{(M;\tilde{r}=1,\pm\ell;\underline{m})}$	$\vec{f}_1^{\pm}=(\ell+\frac{1}{2},\frac{3}{2},\frac{1}{2},...,\frac{1}{2},\pm\frac{1}{2}),\ \ \ell=1,2,...$

Table 3.1: **Spin-3/2 TT eigenmodes with mass parameter M on dS_D (even $D\geq 4$).** For real $M\neq 0$, type-I and type-II modes together form a spin$(D,1)$ Principal Series UIR. For $M=0$ the representation is reducible as the two sets of eigenmodes $\{\Psi_{\mu}^{(M;\tilde{r},-\ell;\underline{m})}\}_{\tilde{r}=0,1}$ and $\{\Psi_{\mu}^{(M;\tilde{r},+\ell;\underline{m})}\}_{\tilde{r}=0,1}$ separately form Discrete Series UIR's of spin$(D,1)$. For $M=\pm i(D-2)/2$ (strictly massless tuning) the type-I modes become pure gauge modes, while the type-II modes are the physical modes forming a non-unitary representation for $D\neq 4$ and a direct sum of two 'chiral' UIR's in the Discrete Series for $D=4$ - see Section 3.7. All these results have been also explained by studying the group-theoretic properties of the eigenmodes in Refs. [26, 27].

Type of eigenmode	Notation	spin(D) content
Type-I	$\Psi_{\mu}^{(M;\tilde{r}=0,\ell;\underline{m})}$	$\vec{f}_0=(\ell+\frac{1}{2},\frac{1}{2},...,\frac{1}{2}),\ \ \ell=1,2,...$
Type-II (for $D>3$)	$\Psi_{\mu}^{(M;\tilde{r}=1,\ell;\underline{m})}$	$\vec{f}_1=(\ell+\frac{1}{2},\frac{3}{2},\frac{1}{2},...,\frac{1}{2}),\ \ \ell=1,2,...$

Table 3.2: **Spin-3/2 TT eigenmodes with mass parameter M on dS_D (odd $D\geq 3$).** Type-II modes exist for $D>3$. For real M, type-I and type-II modes together form a Principal Series UIR of spin$(D,1)$ for $D>3$. For real M and $D=3$, type-I modes form a Principal Series UIR of spin$(3,1)$. For $M=\pm i(D-2)/2$ (strictly massless tuning) and $D>3$, the type-I modes become pure gauge modes, while the type-II modes are the physical modes forming a non-unitary strictly massless representation. At the strictly massless tuning $M=\pm i/2$ for $D=3$, the type-I modes are again pure gauge modes and they form a non-unitary representation of spin$(3,1)$ - see Section 3.7. All these results have been also explained by studying the group-theoretic properties of the eigenmodes in Refs. [26, 27].

(with $s=r+1/2$), which holds for all $D\geq 3$ and for all totally symmetric TT tensor-spinor eigenmodes with spin $s\geq 1/2$ and mass parameter M on dS_D. Specialising to the spin-3/2 TT eigenmodes we find

$$\mathscr{C}_{eigen}^{(dS_D)}=-M^2-\frac{(D-1)(D-8)}{8}, \tag{3.84}$$

while for the spin-5/2 TT eigenmodes we find

$$\mathscr{C}_{eigen}^{(dS_D)}=-M^2-\frac{D(D-17)}{8}. \tag{3.85}$$

3.6 STRICTLY AND PARTIALLY MASSLESS REPRESENTATIONS: NON-UNITARITY FOR $D \neq 4$ AND UNITARITY FOR $D = 4$

Here we will obtain the main result of this paper: the strictly massless spin-3/2 field, as well as the strictly and partially massless spin-5/2 fields, on dS_D $(D \geq 3)$ cannot be unitary unless $D = 4$. Note that we already know the values of the quadratic Casimir [eqs. (3.84) and (3.85)] for the representations formed by our dS eigenmodes for any mass parameter M. By specialising to the strictly/partially massless tunings (3.5), we find

$$\mathscr{C}^{(dS_D)}_{eigen} = \frac{D(D+1)}{8}, \qquad \text{spin-3/2, strictly massless,} \tag{3.86}$$

$$\mathscr{C}^{(dS_D)}_{eigen} = \frac{D(D+17)}{8}, \qquad \text{spin-5/2, strictly massless,} \tag{3.87}$$

$$\mathscr{C}^{(dS_D)}_{eigen} = \frac{(D+1)(D+8)}{8}, \qquad \text{spin-5/2, partially massless.} \tag{3.88}$$

Apart from the values of the quadratic Casimir (3.86)-(3.88), we also know the spin(D) content of the spin$(D,1)$ representations formed by our dS eigenmodes - see Tables 3.1 and 3.2, as well as Subsections 3.4.2-3.4.4. Keeping these results in mind, we can use the classification of the UIR's in Section 3.3 in order to readily deduce the (non-)unitarity of the representations formed by our strictly/partially massless eigenmodes on dS_D.

First, let us identify which types of dS eigenmodes correspond to pure gauge modes and which to physical modes in the strictly/partially massless theories. By 'physical modes' we mean the eigenmodes that form the strictly/partially massless representation of spin$(D,1)$ and that correspond to the (non-gauge) propagating degrees of freedom of the theory. (If the representation formed by the eigenmodes is non-unitary, then the name 'physical modes' could be misleading as the theory is, of course, unphysical due to the appearance of negative probabilities.) The pure gauge modes describe pure gauge degrees of freedom of the theory. If a dS invariant scalar product exists, then the pure gauge modes have zero norm and they are orthogonal to all physical modes [21, 26, 27]. The generators of spin$(D,1)$ act in terms of the Lie-Lorentz derivative (3.10) on equivalence classes of physical modes with equivalence relation given by: "For any two physical modes $\Psi^{(1)}_{\mu_1\ldots\mu_r}$ and $\Psi^{(2)}_{\mu_1\ldots\mu_r}$ we have $\Psi^{(1)}_{\mu_1\ldots\mu_r} \sim \Psi^{(2)}_{\mu_1\ldots\mu_r}$ if and only if their difference $\Psi^{(1)}_{\mu_1\ldots\mu_r} - \Psi^{(2)}_{\mu_1\ldots\mu_r}$ is a linear combination of pure gauge modes".

3.6.1 Pure gauge modes and physical modes

Pure gauge and physical modes for strictly massless spin-3/2 field. The mass parameter for the strictly massless spin-3/2 field is given by $M = \pm i(D-2)/2$ [this is found by letting $r = \tau = 1$ in eq. (3.5)]. The spin-3/2 type-I modes are the pure gauge modes of the theory, while the spin-3/2 type-II modes are the physical modes that form the (strictly massless) representation of spin$(D,1)$. More specifically, we find that for $M = \pm i(D-2)/2$ all type-I modes [see eqs. (3.53) and (3.64) for even $D \geq 4$ and eq. (3.69) for odd $D \geq 3$] are expressed in a pure gauge form as:

$$\Psi^{(PG)}_{\pm\mu}(t, \boldsymbol{\theta_{D-1}}) = \left(\nabla_\mu \pm \frac{i}{2}\gamma_\mu\right)\Lambda_\pm(t, \boldsymbol{\theta_{D-1}}), \tag{3.89}$$

where for convenience we have omitted all quantum number labels from $\Psi^{(PG)}_{\pm\mu}$ and $\Lambda_\pm$. The subscript '$\pm$' in $\Psi^{(PG)}_{\pm\mu}$ denotes the sign of the mass parameter $M = \pm i(D-2)/2$. The spinor gauge functions $\Lambda_\pm(t, \boldsymbol{\theta_{D-1}})$ satisfy

$$\nabla\!\!\!/\,\Lambda_\pm = \mp i\,\frac{D}{2}\,\Lambda_\pm. \tag{3.90}$$

Pure gauge and physical modes for strictly massless spin-5/2 field. The mass parameter for the strictly massless spin-5/2 field is given by $M = \pm iD/2$ [this is found by letting $r = 2$ and $\tau = 1$ in eq. (3.5)]. There are two types of pure gauge modes, namely the type-I and type-II modes. The spin-5/2 type-III modes are the physical modes that form the (strictly massless) representation of spin$(D,1)$. More specifically, we find that for $M = \pm iD/2$ all type-I modes [see eq. (3.71) for even $D \geq 4$ and eq. (3.74) for odd $D \geq 3$] and all type-II modes [see eq. (3.72) for even $D \geq 4$ and eq. (3.75) for odd $D \geq 5$] are expressed in a pure gauge form as:

$$\Psi^{(PG)}_{\pm\mu\nu}(t, \boldsymbol{\theta_{D-1}}) = \left(\nabla_{(\mu} \pm \frac{i}{2}\gamma_{(\mu}\right)\lambda_{\pm\nu)}(t, \boldsymbol{\theta}_{D-1}) \tag{3.91}$$

for some TT vector-spinor gauge functions $\lambda_{\pm\mu}(t, \boldsymbol{\theta_{D-1}})$ with

$$\nabla\!\!\!/\,\lambda_{\pm\mu} = \mp i\,\frac{D+2}{2}\,\lambda_{\pm\mu} \tag{3.92}$$

$$\gamma^\mu\lambda_{\pm\mu} = \nabla^\mu\lambda_{\pm\mu} = 0. \tag{3.93}$$

(The gauge functions for type-I modes are different from the gauge functions for type-II modes - for more details see Refs. [26, 27].)

Pure gauge and physical modes for partially massless spin-5/2 field. The mass parameter for the partially massless spin-5/2 field is given by $M = \pm i(D-2)/2$ [this is found by letting $r = 2$ and $\tau = 2$ in eq. (3.5)]. The type-I modes are the pure gauge modes of the theory. Both type-II and type-III modes are physical modes that form the (partially massless) representation of spin$(D, 1)$. For $M = \pm i(D-2)/2$ all type-I modes are expressed in a pure gauge form as:

$$\Psi^{(PG)}_{\pm\mu\nu}(t, \boldsymbol{\theta_{D-1}}) = \left(\nabla_{(\mu}\nabla_{\nu)} \pm i\gamma_{(\mu}\nabla_{\nu)} + \frac{3}{4}g_{\mu\nu}\right)\varphi_{\pm}(t, \boldsymbol{\theta_{D-1}}), \tag{3.94}$$

where the spinor gauge functions $\varphi_{\pm}(t, \boldsymbol{\theta_{D-1}})$ satisfy

$$\nabla\!\!\!/ \varphi_{\pm} = \mp i\,\frac{D+2}{2}\,\varphi_{\pm}. \tag{3.95}$$

Explicit expressions on global dS_D for the eigenmodes corresponding to the gauge functions in eqs. (3.89), (3.91) and (3.94) can be found in Refs. [26, 27].

Remark 6.1. On dS_3, both spin-$3/2$ and spin-$5/2$ theories with arbitrary mass parameters have only type-I modes. Thus, specialising to the strictly/partially massless theories on dS_3, we conclude that all eigenmodes for these theories are pure gauge modes.

Remark 6.2. In the fermionic strictly/partially massless theories of spin $s = r + 1/2$ and depth $\tau = 1, ..., r$ on global even-dimensional dS_D $(D \geq 4)$, we can deduce which eigenmodes are pure gauge modes and which are physical modes from their spin(D) content. The latter corresponds to the highest weights $\vec{f}^{+}_{\tilde{r}} = (\ell+1/2, \tilde{r}+1/2, 1/2, ..., 1/2)$ and $\vec{f}^{-}_{\tilde{r}} = (\ell + 1/2, \tilde{r} + 1/2, 1/2, ..., 1/2, -1/2)$ with $\tilde{r} \leq r \leq \ell$. The pure gauge modes correspond to the cases with $0 \leq \tilde{r} \leq r - \tau$, while the physical modes correspond to $r - \tau + 1 \leq \tilde{r} \leq r$.

Remark 6.3. In the fermionic strictly/partially massless theories of spin $s = r + 1/2$ and depth $\tau = 1, ..., r$ on global odd-dimensional dS_D $(D \geq 3)$, we can deduce which eigenmodes are pure gauge modes and which are physical modes from their spin(D) content. The latter corresponds to the highest weights $\vec{f}_{\tilde{r}} = (\ell+1/2, \tilde{r}+1/2, 1/2, ..., 1/2)$ with $\tilde{r} \leq r \leq \ell$. As in the even-dimensional case, the pure gauge modes correspond to the cases with $0 \leq \tilde{r} \leq r - \tau$, while the physical modes correspond to $r - \tau + 1 \leq \tilde{r} \leq r$.

The validity of Remarks 6.1-6.3 for the spin-$3/2$ and spin-$5/2$ fields has been demonstrated in this paper, as well as in Refs. [26, 27]. However, we expect that these remarks also hold for all strictly/partially massless fields with half-odd-integer spins $s \geq 3/2$. This expectation is also motivated by the well-studied case of totally symmetric tensors [21].

3.6.2 Studying the (non-)unitarity of the strictly/partially massless theories with spin $s = 3/2, 5/2$

Our 'tools' in order to demonstrate that the unitarity of the strictly/partially massless fields of spin $s = 3/2, 5/2$ occurs only for $D = 4$ are: on the one hand the values of the quadratic Casimir [eqs. (3.86)-(3.88)] and the spin(D) content of the physical modes [see Tables 3.1 and 3.2 and Remarks 6.1-6.3] and, on the other hand, the classification of the UIR's in Section 3.3. Although the readers can readily convince themselves about the non-unitarity for $D \neq 4$ (given our aforementioned tools), we will present here a detailed discussion concerning the strictly massless spin-3/2 field. The cases of the strictly and partially massless spin-5/2 fields can then be treated in the same manner and, therefore, we will not present their details here.

Non-unitarity for odd $D = 2p+1 \geq 5$. Let $\vec{F} = (F_0, F_1, ..., F_p)$ be the spin$(2p+1, 1)$ representation formed by the physical spin-3/2 modes. The corresponding spin(D) content is given by $\vec{f_1} = (\ell + 1/2, 3/2, 1/2, ..., 1/2)$ with $\ell \geq 1$ - see Remark 6.3. The labels $F_1, F_2, ..., F_p$ must all be half-odd-integers. It is clear that these values for $F_1, ..., F_p$ - as well as the spin(D) content - correspond neither to the UIR's of the Exceptional Series (3.44), nor to the UIR's of the Complementary Series (3.43), since these UIR's allow only integer values for $F_1, ..., F_p$. Then, the only remaining candidate that could accommodate the strictly massless spin-3/2 field is the Principal Series (3.42), where $F_0 = -(D-1)/2 + iy$ ($y \in \mathbb{R}$). We will readily show that the Principal Series *cannot* accommodate the strictly massless spin-3/2 field. Suppose, for the sake of contradiction, that the strictly massless spin-3/2 representation $\vec{F} = (F_0, .., F_p)$ belongs to the Principal Series UIR's (3.42). Since we already know the spin(D) content of $\vec{F}$, by using the branching rules (3.39) we find that the following must hold: $F_1 = 3/2$, $F_2 \in \{1/2, 3/2\}$ and $F_3 = ... = F_{p-1} = F_p = 1/2$. Moreover, the quadratic Casimir for the Principal Series $C_2(\vec{F})$ [eq. (3.45)] must coincide with the quadratic Casimir $\mathscr{C}^{(dS_D)}_{eigen} = \frac{D(D+1)}{8}$ [eq. (3.86)] corresponding to the physical modes. By equating these two values for the quadratic Casimir we find that F_0 must satisfy

$$F_0(F_0 + D - 1) + F_2(F_2 + D - 5) + \frac{3}{2} = 0, \quad \text{with } F_2 \in \left\{\frac{1}{2}, \frac{3}{2}\right\}. \tag{3.96}$$

For $F_2 = 1/2$ this equation gives $F_0 = -1/2$ or $F_0 = -D + 3/2$, i.e. we arrive at a contradiction as these values for F_0 do *not* correspond to the Principal Series for odd $D \geq 5$. Similarly, for $F_2 = 3/2$ we arrive again at a contradiction because eq. (3.96) gives $F_0 = -3/2$ or $F_0 = -D + 5/2$ and these values do *not* correspond to the Principal

Series for odd $D \geq 5$. To conclude, we have proved that the strictly massless spin-3/2 field *cannot* be accommodated by any UIR of spin$(D, 1)$ for odd $D \geq 5$.

Non-unitarity for $D = 3$. As we discussed earlier, on dS_3 the strictly massless spin-3/2 field (as well as the strictly and partially massless spin-5/2 fields) has only pure gauge modes - see Remark 6.1. However, it is worth showing here that the spin$(3, 1)$ representation formed by the pure gauge modes of the strictly massless spin-3/2 theory is non-unitary. Let $\vec{F} = (F_0, F_1)$ be the spin$(3, 1)$ representation formed by the pure gauge modes. The spin(3) content for this representation is given by $\ell + 1/2$ with $\ell \geq 1$ - see Remark 6.3. Also, the label F_1 must be a half-odd-integer. Thus, we can rule out both the Complementary Series (3.43) and the Exceptional Series (3.44) [in fact, the Exceptional Series does not exist for $D = 3$ [33]]. Now, as in the case with odd $D \geq 5$, it is easy to show that the quadratic Casimir for the spin$(3, 1)$ Principal Series $C_2(\vec{F})$ [eq. (3.45)] does *not* coincide with the field-theoretic quadratic Casimir $\mathscr{C}^{(dS_3)}_{eigen} = \frac{3}{2}$ [eq. (3.86)] on dS_3.[16]

Non-unitarity for even $D = 2p \geq 6$. Let $\vec{F} = (F_0, F_1, ..., F_{p-1})$ be the spin$(2p, 1)$ representation formed by the physical spin-3/2 modes. The corresponding spin(D) content is given by $\vec{f}_1^{+} = (\ell + 1/2, 3/2, 1/2, ..., 1/2)$ and $\vec{f}_1^{-} = (\ell + 1/2, 3/2, 1/2, ..., 1/2, -1/2)$ with $\ell \geq 1$ (see Remark 6.1), while the labels $F_1, F_2, ..., F_{p-1}$ must all be half-odd-integers. These values are incompatible with both the UIR's of the Exceptional Series (3.34) and the UIR's of the Complementary Series (3.33). Then, the UIR's that are still candidates for accommodating the strictly massless spin-3/2 field are: the Principal Series (3.32) and the Discrete Series (3.35) and (3.36). Now, the following steps are as in the case with odd $D \geq 5$, i.e. we can prove by contradiction that the strictly massless spin-3/2 field corresponds neither to the Principal Series nor to the Discrete Series for even $D \geq 6$. In particular, starting with the contradicting assumption that $\vec{F}$ belongs to the Principal or Discrete Series, and making use of the branching rules (3.29), we equate the field-theoretic Casimir (3.86) with the quadratic Casimir from the UIR's [eq. (3.37)]. By doing so, we find again that F_0 must satisfy eq. (3.96). Then, we readily arrive at a contradiction because the values of F_0 that satisfy eq. (3.96) agree neither with the Principal Series nor with the Discrete Series UIR's for even $D \geq 6$. To conclude, we have proved that the strictly massless spin-3/2 field *cannot* be accommodated by any UIR of spin$(D, 1)$ for even $D \geq 6$.

Unitarity for $D = 4$. The mass parameter for the strictly massless spin-3/2 field on dS_4

[16] For arbitrary D, the physical modes have the same value for the quadratic Casimir as the pure gauge modes.

is $M = \pm i$. However, the physical modes with $M = i$ and the ones with $M = -i$ form equivalent representations[17]. Thus, below we can just let $M = i$. There are two 'chiral' UIR's of spin$(4,1)$ that correspond to the strictly massless spin-3/2 field on dS_4: one UIR for the helicity $+3/2$ and one UIR for the helicity $-3/2$. The physical modes (i.e. the type-II modes) with helicity $\pm 3/2$ have the following spin(4) content: $\vec{f}_1^{\pm} = (\ell + 1/2, \pm 3/2)$ with $\ell \geq 1$. Let $\vec{F} = (F_0, F_1)$ be the spin$(4,1)$ representation formed by the physical modes with helicity $+3/2$. The branching rules (3.29) give $F_1 = 3/2$. Then, by comparing the field-theoretic expression (3.86) for the quadratic Casimir with the UIR expression (3.37), we find that the physical modes with helicity $+3/2$ form the Discrete Series UIR $D^+(\vec{F}) = D^+(-1/2, 3/2)$ [eq. (3.35)]. Similarly, we find that the physical modes with helicity $-3/2$ form the Discrete Series UIR $D^-(\vec{F}) = D^-(-1/2, 3/2)$ [eq. (3.36)]. Thus, the strictly massless spin-3/2 field on dS_4 corresponds to the direct sum of Discrete Series UIR's $D^+(-1/2, 3/2) \bigoplus D^-(-1/2, 3/2)$[18]. More details can be found in the dictionary in Section 3.7.

3.7 DICTIONARY BETWEEN (SYMMETRIC) TENSOR-SPINOR FIELDS ON dS_D AND UIR'S OF SPIN$(D,1)$ FOR $D \geq 3$

Here we present a 'field theory - UIR's dictionary' based on our analysis for the spin-3/2 and spin-5/2 eigenmodes satisfying eq. (3.3) on dS_D. This dictionary relies on the classification of the UIR's under the decomposition spin$(D,1) \supset$ spin(D) given in Section 3.3 and it was constructed by taking advantage of both:

- The values for the spin$(D,1)$ quadratic Casimir corresponding to the eigenmodes [eqs. (3.84), (3.85) and (3.86)-(3.88)].
- The spin(D) content of the eigenmodes (see Section 3.4, Tables 3.1 and 3.2 and Remarks 6.1-6.3).

Although until now we have mainly discussed the spin-3/2 and spin-5/2 fields, our analysis and the classification of the UIR's in Section 3.3 allow us to propose a dictionary for totally

[17]This can be readily understood as follows. If we act with γ^5 on any spin-3/2 physical mode with mass parameter $M = \pm i$ on dS_4, then the resulting eigenmode is a physical mode with the same spin(4) content but with mass parameter $M = \mp i$. Moreover, the matrix γ^5 commutes with the Lie-Lorentz derivative (3.10) with respect to any spin$(4,1)$ Killing vector.

[18]The strictly/partially massless totally symmetric tensors of spin $s = r$ and depth $\tau = 1, ..., r$ on dS_4 also form a direct sum of Discrete Series UIR's corresponding to $D^+(r-\tau-1, r) \bigoplus D^-(r-\tau-1, r)$ [20, 19].

symmetric TT tensor-spinors (3.3) with mass parameter M and any half-odd-integer spin $s = r + 1/2 \geq 1/2$ on dS_D $(D \geq 3)$. However, we note that we have not performed an eigenmode analysis for the fields with half-odd-integer spin $s \geq 7/2$ yet, but this is something that we leave for future work. In our dictionary, we give the explicit values for all representation labels concerning the UIR's under the decomposition spin$(D,1) \supset$ spin(D), and we also translate our results in the representation-theoretic language used in the CFT literature [2]. While reading the following dictionary, one should recall that the spin(D) content is described by the highest weights of the rank-$\tilde{r}$ TT tensor-spinor eigenmodes (3.46) on S^{D-1}: $\vec{f}_{\tilde{r}} = (\ell + 1/2, \tilde{r} + 1/2, 1/2, ..., 1/2)$ for odd D [eq. (3.48)] and $\vec{f}^{\pm}_{\tilde{r}} = (\ell + 1/2, \tilde{r} + 1/2, 1/2, ..., 1/2, \pm 1/2)$ for even D [eqs. (3.49) and (3.50)] - recall also Remarks 6.1-6.3.

Dictionary for fields with half-odd-integer spin $s = r + 1/2 \geq 1/2$ for $D \geq 3$

• **Real $M \neq 0$ for all $D \geq 3$:** Principal Series UIR's.
(This case corresponds to the Principal Series with $so(1,1)$ weight $\Delta_c = \frac{D-1}{2} - iM$ in Ref. [2].) The representation labels are $\vec{F} = (-\frac{D-1}{2} - iM, r + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2})$, while for $D \in \{3, 4\}$ we have $\vec{F} = (-\frac{D-1}{2} - iM, r + \frac{1}{2})$. The spin$(D)$ content corresponds to the highest weights: $\vec{f}_{\tilde{r}}$ (for odd $D \geq 3$) and $\vec{f}^{\pm}_{\tilde{r}}$ (for even $D \geq 4$) with $\ell \geq r \geq \tilde{r} \geq 0$. For even $D \geq 4$, the eigenmodes with opposite values for their mass parameters form equivalent representations.

• **$M = 0$ for odd $D \geq 3$:** Principal Series UIR's.
(This case corresponds to the Principal Series with $\Delta_c = \frac{D-1}{2}$ in Ref. [2].) For $D > 3$ the representation labels are $\vec{F} = (-\frac{D-1}{2}, r + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2})$, while for $D = 3$ we have $\vec{F} = (-1, r + \frac{1}{2})$. The spin$(D)$ content corresponds to the highest weights $\vec{f}_{\tilde{r}}$ with $\ell \geq r \geq \tilde{r} \geq 0$.

• **$M = 0$ for even $D \geq 4$:** Direct sum of two Discrete Series UIR's $D^{+}(\vec{F}) \bigoplus D^{-}(\vec{F})$.
(In Ref. [2], this case corresponds to a direct sum of two Principal Series UIR's with $\Delta_c = \frac{D-1}{2}$ that are related to each other by space reflection.) The eigenmodes with spin(D) content $\vec{f}^{+}_{\tilde{r}}$ (where $\ell \geq r \geq \tilde{r} \geq 0$) form the Discrete Series UIR $D^{+}(-\frac{D-1}{2}, r + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2})$ for $D > 4$ and the UIR $D^{+}(-\frac{3}{2}, r + \frac{1}{2})$ for $D = 4$. The eigenmodes with spin(D) content $\vec{f}^{-}_{\tilde{r}}$ (where $\ell \geq r \geq \tilde{r} \geq 0$) form the Discrete Series UIR $D^{-}(-\frac{D-1}{2}, r + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2})$ for $D > 4$ and $D^{-}(-\frac{3}{2}, r + \frac{1}{2})$ for $D = 4$. The eigenmodes

that form the UIR D^{+} and the ones forming D^{-} belong to different eigenspaces of the matrix γ^{D+1} [eq. (3.16)].

• **Strictly/partially massless fields of depth $\boldsymbol{\tau = 1, ..., r}$ with $\boldsymbol{s \geq 3/2}$ for $\boldsymbol{D \neq 4}$:** Non-unitary.

• **Strictly/partially massless fields of depth $\boldsymbol{\tau = 1, ..., r}$ with $\boldsymbol{s \geq 3/2}$ for $\boldsymbol{D = 4}$:** Direct sum of two Discrete Series UIR's of spin$(4,1)$, $D^{+}(\vec{F}) \bigoplus D^{-}(\vec{F})$.
(In Ref. [2], this case corresponds to a direct sum of two Discrete Series UIR's with $\Delta_c = \frac{5}{2} + r - \tau$ that are related to each other by space reflection.) The physical modes with spin(D) content $\vec{f}^{+}_{\tilde{r}} = (\ell + \frac{1}{2}, \tilde{r} + \frac{1}{2})$ (where $\ell \geq r \geq \tilde{r} \geq r - \tau + 1$) form the Discrete Series UIR $D^{+}(r - \tau - \frac{1}{2}, r + \frac{1}{2})$. The physical modes with spin(D) content $\vec{f}^{-}_{\tilde{r}} = (\ell + \frac{1}{2}, -\tilde{r} - \frac{1}{2})$ (where $\ell \geq r \geq \tilde{r} \geq r - \tau + 1$) form the Discrete Series UIR $D^{-}(r - \tau - \frac{1}{2}, r + \frac{1}{2})$. In particular, the UIR $D^{\pm}(r - \tau - \frac{1}{2}, r + \frac{1}{2})$ corresponds to the depth-τ field with propagating helicities $(\pm s, \pm(s-1), ..., \pm(s - \tau + 1))$. In the strictly massless case ($\tau = 1$), the UIR $D^{+}(r - \frac{3}{2}, r + \frac{1}{2})$ corresponds to the single helicity s, while $D^{-}(r - \frac{3}{2}, r + \frac{1}{2})$ corresponds to the single helicity $-s$. No physical (or pure gauge (3.89)) mode is an eigenfunction of the matrix γ^{5} [eq. (3.16)].

3.8 SUMMARY AND DISCUSSIONS

In the present paper, we demonstrated that four-dimensional dS space plays a distinguished role in the unitarity of the strictly and partially massless (symmetric) tensor-spinor fields of spin $s = 3/2, 5/2$. In particular, the strictly massless spin-3/2 field, as well as the strictly and partially massless spin-5/2 fields on dS_D, are not unitary unless $D = 4$. The explanation relies on the representation theory of spin$(D,1)$, where the latter does not allow strictly/partially massless UIR's for (symmetric) tensor-spinors unless $D = 4$. This is a remarkable feature of dS field theory, while it is also very interesting that the dimensionality that plays a special representation-theoretic role matches the dimensionality of our physical Universe. We also expect that this result should hold for all totally symmetric tensor-spinors with spin $s \geq 7/2$, while this expectation of ours is justified by the classification of the spin$(D,1)$ UIR's. A technical explanation of our results in terms of the (non-)existence of positive-definite dS scalar products for the spin-$3/2$ and spin-$5/2$ eigenmodes has been given in Refs. [26, 27].

In Section 3.7, we presented a dictionary between (totally symmetric) half-odd-integer-spin fields on dS_D and UIR's of spin$(D,1)$ $(D \geq 3)$. The validity of our dictionary for the spin-3/2 and spin-5/2 fields was demonstrated in this paper. Our dictionary for the cases with half-odd-integer spin $s \geq 7/2$ is a 'suggestion' that is motivated by the classification of the UIR's and can be confirmed by performing an eigenmode analysis for half-odd-integer spins $s \geq 7/2$. This is something that we leave for future work.

In the present paper, 'unitarity' of a field theory does not just refer to the positivity of the norm in the Hilbert space. In this paper, unitarity in the one-particle Hilbert space means that: a positive-definite scalar product for the eigenmodes exists that is invariant under spin$(D,1)$. If and only if these conditions are satisfied then the space of eigenmodes can be identified with the representation space of a unitary representation of spin$(D,1)$. For example, consider the strictly massless spin-3/2 field on dS_4 satisfying the onshell conditions

$$\left(\nabla\!\!\!\!/ \pm i\right)\Psi_\mu = 0 \tag{3.97}$$

$$\nabla^\alpha \Psi_\alpha = 0, \quad \gamma^\alpha \Psi_\alpha = 0. \tag{3.98}$$

The physical eigenmodes of this theory are given by eqs. (3.65) and (3.66). It is easy to check that the following (Dirac-like) scalar product is positive-definite

$$\int_{S^3} \sqrt{-g}\, d\boldsymbol{\theta_3}\, g^{\mu\nu} \Psi^{(1)\dagger}_\mu(t,\boldsymbol{\theta_3})\, \Psi^{(2)}_\nu(t,\boldsymbol{\theta_3}) \tag{3.99}$$

for any two physical modes $\Psi^{(1)}_\mu$ and $\Psi^{(2)}_\nu$, where g is the determinant of the dS_4 metric, while $d\boldsymbol{\theta_3}$ stands for $d\theta_3 d\theta_2 d\theta_1$. This is the scalar product for the one-particle Hilbert space that was implicitly used in order to check the positivity property of the equal time anti-commutators in Ref. [9]. However, while the positivity of the norm with respect to the scalar product (3.99) is clearly necessary, it is not sufficient for representation-theoretic unitarity. In particular, it is straightforward to check that the scalar product (3.99) is neither conserved nor dS invariant [26, 27]. The reason is that the conventional (Dirac-like) vector current

$$J^\mu = -\,\Psi^{(1)\dagger}_\nu \gamma^0 \gamma^\mu \Psi^{(2)\nu} \tag{3.100}$$

is not covariantly conserved because of the imaginary mass parameter in eq. (3.97). Thus, we cannot use the scalar product (3.99) in order to check the unitarity of the spin$(4,1)$ representation formed by the physical modes. On the other hand, the (axial) vector current

$$J^\mu_{\text{ax}} = -\,\Psi^{(1)\dagger}_\nu \gamma^0 \gamma^\mu \gamma^5 \Psi^{(2)\nu} \tag{3.101}$$

is covariantly conserved, giving rise to the time-independent and dS invariant scalar product [26, 27]

$$\int_{S^3} \sqrt{-g}\, d\boldsymbol{\theta_3} J^0_{\text{ax}} = \int_{S^3} \sqrt{-g}\, d\boldsymbol{\theta_3}\, g^{\mu\nu} \Psi^{(1)\dagger}_{\mu}(t, \boldsymbol{\theta_3})\, \gamma^5\, \Psi^{(2)}_{\nu}(t, \boldsymbol{\theta_3}). \tag{3.102}$$

This scalar product is a good choice in order to study the unitarity of the corresponding spin$(4,1)$ representation for the reasons mentioned above. In particular, the physical modes (3.65) form the Discrete Series UIR $D^{-}(-\frac{1}{2}, \frac{3}{2})$ with the positive-definite scalar product (3.102), while the physical modes (3.66) form the Discrete Series UIR $D^{+}(-\frac{1}{2}, \frac{3}{2})$ with positive-definite scalar product given by the negative of eq. (3.102). The pure gauge modes (3.89) have zero norm with respect to the scalar product (3.102) and they are orthogonal to all physical modes. For more details concerning the eigenmodes see Refs. [26, 27].

The fermionic strictly and partially massless tunings (3.5) were found in Ref. [7], but the non-unitarity of the corresponding theories for $D \neq 4$ could not be revealed with the methods used in this reference.

Acknowledgments. The author is grateful to Atsushi Higuchi for guidance, suggestions, and encouragement, as well as for invaluable discussions and ideas that inspired the majority of the material presented in this paper and helpful comments on earlier versions of this paper. Also, it is a pleasure to thank Stanley Deser for communications and Andrew Waldron for useful discussions. The author would also like to thank the referee for their useful comments and suggestions. The author also thanks Xavier Bekaert, Nicolas Boulanger, Thomas Basile, Charis Anastopoulos, Lasse Schmieding, Nikolaos Koutsonikos-Kouloumpis, and F. F. John for useful discussions. He also thanks Zoi Taglee for insightful and inspiring discussions. This work was supported by a studentship from the Department of Mathematics, University of York.

3.9 APPENDIX A - THE ONLY TOTALLY SYMMETRIC TT TENSOR-SPINOR EIGENMODES OF THE DIRAC OPERATOR THAT EXIST ON S^2 ARE THE SPINOR EIGENMODES

The spinor eigenmodes of the Dirac operator on S^2 (as well as on spheres of any dimension) have been constructed in Ref. [3]. In Ref. [4], the TT vector-spinor eigenmodes of the Dirac operator on S^d $(d \geq 3)$ were obtained, while it was found that there are no TT

vector-spinor eigenmodes on S^2. In Refs. [26, 27], the author has constructed the rank-2 symmetric TT tensor-spinor eigenmodes of the Dirac operator on S^d $(d \geq 3)$ and he also found that such eigenmodes do not exist on S^2. In this Appendix, we will show that there are no totally symmetric TT tensor-spinor eigenmodes $\tilde{\psi}_{\tilde{\mu}_1 \dots \tilde{\mu}_{\tilde{r}}}$ of rank $\tilde{r} \geq 2$ on S^2. Our proof will closely follow the analogous proof for totally symmetric tensors of rank $\tilde{r} \geq 2$ on S^2 in Ref. [21]. For convenience, we will drop the tildes from the tensor indices and, thus, our tensor-spinor of rank $\tilde{r} \geq 2$ on S^2 will be denoted as $\tilde{\psi}_{\mu_1 \dots \mu_{\tilde{r}}}$.
For later convenience, note that the Riemann tensor on S^2 is

$$\tilde{R}_{\mu\nu\kappa\lambda} = \tilde{g}_{\mu\kappa}\tilde{g}_{\nu\lambda} - \tilde{g}_{\nu\kappa}\tilde{g}_{\mu\lambda}, \tag{3.103}$$

where $\tilde{g}_{\mu\nu}$ is the metric tensor on S^2. The commutator of covariant derivatives acting on a vector-spinor on S^2 is given by

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]\tilde{\psi}_\alpha = \frac{1}{4}\tilde{R}_{\mu\nu\kappa\lambda}\tilde{\gamma}^\kappa\tilde{\gamma}^\lambda\tilde{\psi}_\alpha + \tilde{R}^\lambda{}_{\alpha\nu\mu}\tilde{\psi}_\lambda \tag{3.104}$$

$$= \frac{1}{2}(\tilde{\gamma}_\mu\tilde{\gamma}_\nu - \tilde{g}_{\mu\nu})\tilde{\psi}_\alpha + 2\tilde{g}_{\alpha[\mu}\tilde{\psi}_{\nu]}, \tag{3.105}$$

where $\tilde{\gamma}_\mu$ are the gamma matrices on S^2. The expressions for the commutators of covariant derivatives for tensor-spinors of higher rank are straightforward generalisations of eq. (3.105). Also, let $\tilde{\epsilon}_{\mu\nu}$ be the anti-symmetric tensor on S^2. In the coordinate system (3.19), $\tilde{\epsilon}_{\mu\nu}$ is defined by

$$\begin{aligned} \tilde{\epsilon}_{\theta_1\theta_1} &= \tilde{\epsilon}_{\theta_2\theta_2} = 0 \\ \tilde{\epsilon}_{\theta_2\theta_1} &= -\tilde{\epsilon}_{\theta_1\theta_2} = \sin\theta_2, \end{aligned} \tag{3.106}$$

where $\tilde{\nabla}_\alpha\tilde{\epsilon}_{\mu\nu} = 0$. Now, let us define

$$\tilde{\nabla}_{[\theta_1}\tilde{\psi}_{\theta_2]\mu_2\dots\mu_{\tilde{r}}} = \tilde{\epsilon}_{\theta_1\theta_2}A_{\mu_2\dots\mu_{\tilde{r}}}, \tag{3.107}$$

where $A_{\mu_2\dots\mu_{\tilde{r}}}$ is a totally symmetric tensor-spinor of rank $\tilde{r}-1$ on S^2. Then

$$\tilde{\nabla}_{[\mu}\tilde{\psi}_{\nu]\mu_2\dots\mu_{\tilde{r}}} = \tilde{\epsilon}_{\mu\nu}A_{\mu_2\dots\mu_{\tilde{r}}}. \tag{3.108}$$

By taking the trace of eq. (3.108) with respect to the indices ν and μ_2, and by using the fact that $\tilde{\psi}_{\nu\mu_2\dots\mu_{\tilde{r}}}$ is traceless and divergence-free, we find $A_{\mu_2\dots\mu_{\tilde{r}}} = 0$. In other words,

$$\tilde{\nabla}_{[\mu}\tilde{\psi}_{\nu]\mu_2\dots\mu_{\tilde{r}}} = 0. \tag{3.109}$$

By taking the divergence of this equation with respect to the index μ, and making use of eq. (3.105), we find

$$\tilde{\nabla}_{\mu}\tilde{\nabla}^{\mu}\tilde{\psi}_{\nu\mu_2\dots\mu_{\tilde{r}}} = \left(\tilde{r}+\frac{1}{2}\right)\tilde{\psi}_{\nu\mu_2\dots\mu_{\tilde{r}}}. \tag{3.110}$$

However, as is well-known, $\tilde{\nabla}_{\mu}\tilde{\nabla}^{\mu}$ is negative-definite on compact manifolds. Thus, $\tilde{\psi}_{\nu\mu_2\dots\mu_{\tilde{r}}}$ must be identically zero.

REFERENCES

[1] A. Barut and R. Raczka. *Theory of Group Representations and Applications*. WORLD SCIENTIFIC, 1986.

[2] T. Basile, X. Bekaert, and N. Boulanger. "Mixed-symmetry fields in de Sitter space: a group theoretical glance". In: *Journal of High Energy Physics* 2017 (2016), p. 1.

[3] R. Camporesi and A. Higuchi. "On the eigenfunctions of the Dirac operator on spheres and real hyperbolic spaces". In: *J. Geom. Phys.* 20 (1996), p. 1.

[4] C.-H. Chen, H. T. Cho, A. S. Cornell, and G. Harmsen. "Spin-3/2 fields in D-dimensional Schwarzschild black hole spacetimes". In: *Phys. Rev. D* 94 (2016), p. 044052.

[5] S. Deser and R. I. Nepomechie. "Anomalous propagation of gauge fields in conformally flat spaces". In: *Physics Letters B* 132 (1983), p. 321.

[6] S. Deser and R. I. Nepomechie. "Gauge invariance versus masslessness in de Sitter spaces". In: *Annals of Physics* 154 (1984), p. 396.

[7] S. Deser and A. Waldron. "Arbitrary spin representations in de Sitter from dS / CFT with applications to dS supergravity". In: *Nuclear Physics* 662 (2003), p. 379.

[8] S. Deser and A. Waldron. "Conformal invariance of partially massless higher spins". In: *Physics Letters B* 603 (2004), p. 30.

[9] S. Deser and A. Waldron. "Gauge invariances and phases of massive higher spins in (A)dS". In: *Phys. Rev. Lett.* 87 (2001), p. 031601. eprint: arXiv:hep-th/0102166.

[10] S. Deser and A. Waldron. "Null propagation of partially massless higher spins in (A)dS and cosmological constant speculations". In: *Phys. Lett. B* 513 (2001), p. 137. eprint: hep-th/0105181.

[11] S. Deser and A. Waldron. “Partial masslessness of higher spins in (A)dS”. In: *Nuclear Physics* 607 (2001), p. 577.

[12] S. Deser and A. Waldron. “Stability of massive cosmological gravitons”. In: *Phys. Lett. B* 508 (2001), p. 347. eprint: arXiv:hep-th/0103255.

[13] J. Dixmier. “Sur les représentations de certains groupes orthogonaux”. In: *Compt. Rend.* 250 (1960), p. 3263.

[14] V. K. Dobrev, G. Mack, V. B. Petkova, S. G. Petrova, and I. T. Todorov. “Harmonic Analysis on the n-Dimensional Lorentz Group and Its Application to Conformal Quantum Field Theory”. In: *Lecture Notes in Physics* 63 (1977).

[15] M. Enayati, J. P. Gazeau, H. Pejhan, and A. Wang. “The de Sitter group and its representations: a window on the notion of de Sitterian elementary systems”. In: (2022). eprint: arXiv:2201.11457.

[16] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. seventh. Elsevier/Academic Press, 2007.

[17] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge University Press, 1973.

[18] A. Higuchi. “Forbidden mass range for spin-2 field theory in de Sitter spacetime”. In: *Nuclear Physics B* 282 (1987), pp. 397–436.

[19] A. Higuchi. “Linearized gravity in de Sitter spacetime as a representation of $SO(4,1)$”. In: *Classical and Quantum Gravity* 8 (1991), pp. 2005–2021.

[20] A. Higuchi. “Quantum fields of nonzero spin in De Sitter spacetime”. PhD thesis. Yale University, 1987.

[21] A. Higuchi. “Symmetric tensor spherical harmonics on the N-sphere and their application to the de Sitter group $SO(N,1)$”. In: *J. Math. Phys.* 28 (1987), p. 1553.

[22] T. Hirai. “On infinitesimal operators of irreducible representations of the Lorentz group of n-th order”. In: *Proc. Japan Acad.* 38 (1962), p. 83.

[23] T. Hirai. “The characters of irreducible representations of the Lorentz group of n-th order”. In: *Proc. Japan Acad.* 41 (1965), p. 526.

[24] Y. Homma and T. Tomihisa. “The spinor and tensor fields with higher spin on spaces of constant curvature”. In: *Annals of Global Analysis and Geometry* 60 (2021), pp. 829–861.

[25] Y. Kosmann. “Dérivées de Lie des spineurs”. In: *Annali di Mat. Pura Appl. (IV)* 91 (1971), p. 317.

[26] V. A. Letsios. “On representation-theoretic properties of fermionic fields in de Sitter spacetime and symmetries underlying the conservation of the electromagnetic zilches”. PhD thesis. University of York, 2023. Chap. (Non-)unitarity of strictly and partially massless fermions on de Sitter space II: a technical explanation.

[27] V. A. Letsios. “The (partially) massless spin-3/2 and spin-5/2 fields in de Sitter spacetime as unitary and non-unitary representations of the de Sitter algebra”. In: (2022). eprint: arXiv:2206.09851v3.

[28] V. A. Letsios. “The eigenmodes for spinor quantum field theory in global de Sitter spacetime”. In: *Journal of Mathematical Physics* 62 (2021), p. 032303.

[29] D. Lüst and E. Palti. “A note on string excitations and the Higuchi bound”. In: *Physics Letters B* 799 (2019), p. 135067.

[30] T. Ortín. “A note on Lie-Lorentz derivatives”. In: *Classical and Quantum Gravity* 19 (2002), p. L143.

[31] U. Ottoson. “A Classification of the Unitary Irreducible Representations of $SO_0(N, 1)$”. In: *Commun. Math. Phys.* 8 (1968), p. 228.

[32] PLANCK collaboration. “Planck 2018 results - VI. Cosmological parameters”. In: *Astron. Astrophys.* 641 (2020), A6.

[33] F. Schwarz. “Unitary Irreducible Representations of the Groups $SO_0(n, 1)$”. In: *Journal of Mathematical Physics* 12 (1971), p. 131.

[34] SDSS collaboration. “Baryon acoustic oscillations in the Sloan Digital Sky Survey Data Release 7 Galaxy Sample”. In: *Mon. Not. Roy. Astron. Soc.* 401 (2010), p. 2148.

[35] G. Sengör. “The de Sitter group and its presence at the late-time boundary”. In: *PoS* 2022 (2021), CORFU2021, 356.

[36] Z. Sun. “A note on the representations of $SO(1, d + 1)$”. In: (2021). eprint: arXiv:2111.04591.

[37] SUPERNOVA COSMOLOGY PROJECT collaboration. “Measurements of Ω and Λ from 42 High-Redshift Supernovae”. In: *Astrophys. J.* 517 (1999), p. 565.

[38] I. T. Todorov, M. C. Mintchev, and V. B. Petkova. *Conformal Invariance in Quantum Field Theory*. Pisa: Scuola Normale Superiore, 1978.

[39] A. Trautman. *Spin structures on hypersurfaces and the spectrum of the Dirac operator on spheres*. Kluwer Academic Publishers, 1993.

[40] W.-K. Tung. *Group Theory in Physics: An Introduction To Symmetry Principles, Group Representations, And Special Functions In Classical And Quantum Physics*. World Scientific, 1985.

[41] M. K. F. Wong. "Unitary representations of $SO(n,1)$". In: *Journal of Mathematical Physics* 15 (1974), p. 25.

— 4 —

(Non-)unitarity of strictly and partially massless fermions on de Sitter space II: a technical explanation

Abstract

In our previous article, we showed that the strictly massless spin-3/2 field, as well as the strictly and partially massless spin-5/2 fields, on N-dimensional $(N \geq 3)$ de Sitter spacetime (dS_N) are non-unitary unless $N = 4$. The (non-)unitarity was demonstrated by showing that there is a (mis-)match between the representation-theoretic labels that correspond to the Unitary Irreducible Representations (UIR's) of the de Sitter (dS) algebra spin$(N, 1)$ and the ones corresponding to the space of eigenmodes of the field theories. In this paper, we provide a technical explanation for this fact by studying the (non-)existence of positive-definite, dS invariant scalar products for the spin-3/2 and spin-5/2 eigenmodes on dS_N $(N \geq 3)$. In particular, we show the following. For odd N, any dS invariant scalar product is identically zero. For even $N > 4$, any dS invariant scalar product must be indefinite. This gives rise to positive-norm and negative-norm eigenmodes that mix with each other under spin$(N, 1)$ boosts. In the $N = 4$ case, the positive-norm sector decouples from the negative-norm sector and each sector separately forms a UIR of spin$(4, 1)$. Our analysis makes extensive use of the analytic continuation of tensor-spinor spherical harmonics on the N-sphere (S^N) to dS_N.

4.1 INTRODUCTION

This is a technical sequel to our previous paper [25], in which we constructed a 'field theory-representation theory' dictionary for totally-symmetric spin-$s = 3/2, 5/2$ tensor-spinors on N-dimensional ($N \geq 3$) de Sitter spacetime (dS_N). Totally symmetric tensor-spinors, $\Psi_{\mu_1 \dots \mu_r}$, of spin $s \equiv r + 1/2$ on dS_N satisfy [10, 7]

$$\left(\nabla\!\!\!\!/ + M \right) \Psi_{\mu_1 \dots \mu_r} = 0 \tag{4.1}$$

$$\nabla^{\alpha} \Psi_{\alpha \mu_2 \dots \mu_r} = 0, \quad \gamma^{\alpha} \Psi_{\alpha \mu_2 \dots \mu_r} = 0, \tag{4.2}$$

where $\nabla\!\!\!\!/ = \gamma^{\nu} \nabla_{\nu}$ is the Dirac operator on dS_N. (See Subsection 4.2.2 for our convention for the gamma matrices.) From now on, we will refer to the divergence-free and gamma-tracelessness conditions in eq. (4.2) as the TT conditions.

Main results of our previous paper. In our previous article [25], we constructed the spin-$s = 3/2, 5/2$ eigenmodes of eqs. (4.1) and (4.2) on global dS_N ($N \geq 3$). Then, we investigated the (mis-)match between the representation-theoretic labels that correspond to the Unitary Irreducible Representations (UIR's) of the de Sitter (dS) algebra, spin$(N, 1)$, and the ones corresponding to the eigenmodes. We found that for real values of M the representations are unitary. However, the main interesting result of Ref. [25] concerns the strictly and partially massless theories (i.e. the theories that enjoy a gauge symmetry). In particular, the strictly and partially massless theories, for which the mass parameter is known to be tuned to the special imaginary values [1] $M = i\tilde{M}$ [7]:

$$\tilde{M}^2 = -M^2 = \left(r - \tau + \frac{N-2}{2} \right)^2 \qquad (\tau = 1, \dots, r), \tag{4.3}$$

were found to be non-unitary, unless $N = 4$. (The analysis of the previous [25], as well as of the present, papers focuses only on the cases with $r = 1$ and $r = 2$. However, the 'field theory-representation theory' dictionary of Ref. [25] suggests that our main result extends to all strictly/partially massless (totally symmetric) tensor-spinors of spin $s \geq 7/2$.) The quantity τ is known as the depth of the strictly/partially massless field (i.e. gauge potential). The value $\tau = 1$ corresponds to strict masslessness and the values $\tau = 2, \dots, r$ to partial masslessness - see Refs. [22, 21, 10, 12, 9, 11, 8, 7] for background material concerning strict and partial masslessness.

[1] The imaginary values of M in eq. (4.3) imply that the action functional for strictly/partially massless half-odd-integer-spin theories on dS_N is not hermitian. The fact that the gauge-invariant spin-3/2 field theory in de Sitter spacetime has an imaginary mass parameter had been already observed in cosmological supergravity [28].

4.1.1 Main aim and strategy of the present paper

The main aim of this paper is to provide a technical explanation for the results of our previous paper [25], and, in particular, of the main result:

- **Main result:** The strictly massless spin-3/2 field (gravitino) and the strictly and partially massless spin-5/2 fields on dS_N $(N \geq 3)$ are unitary only for $N = 4$.

Our technical explanation relies on studying the (non-)existence of positive-definite, dS invariant scalar products for the spin-3/2 and spin-5/2 eigenmodes on dS_N $(N \geq 3)$ - see Subsection 4.1.2 for a summary of our technical explanation. Since the strictly/partially massless theories occur for imaginary values (4.3) of the mass parameter, we will focus our group-theoretic analysis on the case where M is an arbitrary imaginary number $M = i\tilde{M}$ $(\tilde{M} \neq 0)$, and we will specialise to the strictly/partially massless values (4.3) when necessary.

Our strategy is as follows:

- We re-obtain the TT vector-spinor eigenmodes Ψ_{μ_1} (spin-3/2 modes) and the TT symmetric tensor-spinor eigenmodes $\Psi_{\mu_1\mu_2}$ (spin-5/2 modes) of eq. (4.1) with arbitrary imaginary mass parameter $M = i\tilde{M}$ $(\tilde{M} \neq 0)$ by taking advantage of the well-known fact that S^N can be analytically continued to dS_N (see Section 4.7). (In Ref. [25], these eigenmodes were constructed directly on dS_N using the method of separation of variables.) In particular, we write down explicitly the mode solutions of the following eigenvalue equation on S^N:

$$\nabla\!\!\!\!/\, \psi_{\mu_1 \dots \mu_r} = i\zeta \psi_{\mu_1 \dots \mu_r} \tag{4.4}$$

$$\nabla^{\alpha} \psi_{\alpha \mu_2 \dots \mu_r} = 0, \quad \gamma^{\alpha} \psi_{\alpha \mu_2 \dots \mu_r} = 0, \tag{4.5}$$

where $\psi_{\mu_1 \dots \mu_r}$ is a totally symmetric tensor-spinor of rank r on S^N which also satisfies the TT conditions (4.5) and $\nabla\!\!\!\!/$ is the Dirac operator on S^N. The eigenvalue in eq. (4.4) is imaginary [24], i.e. $\zeta \in \mathbb{R}$, since, as is well known, $\nabla\!\!\!\!/^2$ is negative semidefinite on compact spin manifolds. We call the eigenmodes satisfying eqs. (4.4) and (4.5) the **symmetric tensor-spinor spherical harmonics (STSSH's)**. In the present work we study only the STSSH's with ranks $r = 1$ and $r = 2$ on S^N $(N \geq 3)$, where we are also going to normalise them, as well as study their transformation properties under a spin$(N + 1)$ transformation, where spin$(N + 1)$ is the Lie algebra of the isometry group of S^N. Note that the unnormalised STSSH's of rank $r = 1$ - i.e. the TT vector-spinor eigenmodes of the Dirac operator $\tilde{\nabla}$ on S^N - have been already constructed in Ref. [6], but no emphasis was given on their group-theoretic properties. To our knowledge, the STSSH's of rank $r = 2$ are constructed in the present paper for the first time (see Section 4.5 and Appendix 4.13). By applying analytic continuation techniques to eqs. (4.4) and (4.5), we will obtain eqs. (4.1) and (4.2), respectively, on dS_N.

- We study the transformation properties of the eigenmodes on dS_N under a spin$(N, 1)$ boost. The corresponding transformation formulae are obtained by analytically continuing the spin$(N + 1)$ transformation formulae for the STSSH's on S^N.
- By exploiting the transformation properties of the eigenmodes on dS_N under the spin$(N, 1)$ boost, we examine when their norm with respect to a dS invariant scalar product is positive-definite.

4.1.2 Technical explanation of the main result

The explanation of our main result is given by the following technical results concerning the spin-3/2 and spin-5/2 TT eigenmodes of eq. (4.1) with arbitrary imaginary mass parameter $M = i\tilde{M}$ ($\tilde{M} \neq 0$):

1. **For even $N > 4$**: all dS invariant scalar products for these eigenmodes must be indefinite for all imaginary $M = i\tilde{M}$ ($\tilde{M} \neq 0$). This is demonstrated by showing that both positive-norm and negative-norm mode solutions exist and they mix with each other under spin$(N, 1)$ for all $\tilde{M} \neq 0$ [including the strictly and partially massless values (4.3)].
2. **For $N = 4$**: all dS invariant scalar products for these eigenmodes must be indefinite unless $\tilde{M}$ is tuned to the strictly/partially massless values (4.3). The solution space of the strictly/partially massless theories is divided into two spin$(4, 1)$ invariant subspaces, denoted as $\mathscr{H}_-$ and $\mathscr{H}_+$, where all mode solutions in $\mathscr{H}_-$ have 'negative helicity', while all mode solutions in $\mathscr{H}_+$ have 'positive helicity'. Then, we introduce a specific dS invariant scalar product [eq. (4.173)] in $\mathscr{H}_-$ and $\mathscr{H}_+$. For this choice of scalar product, it happens that the norm is positive-definite in $\mathscr{H}_-$ and negative-definite in $\mathscr{H}_+$. However, group-theoretically, we are allowed to have a different scalar product for each invariant subspace (since they correspond to different irreducible representations). Thus, by a redefinition of the scalar product in $\mathscr{H}_+$, we can change the sign of the associated norm and make

it positive-definite. This shows that $\mathscr{H}_-$ and $\mathscr{H}_+$ form a direct sum of unitary irreducible representations of spin$(4,1)$.

3. **For N odd**: For all $M = i\tilde{M} \neq 0$ [including the strictly and partially massless values (4.3)], there does not exist any dS invariant scalar product for these eigenmodes. Thus, by definition, the corresponding spin$(N,1)$ representations are not unitary.

These findings provide a technical explanation of the results presented in our previous article [25]. However, our findings seem to contrast with the claims made in Refs. [4, 7]. The non-unitarity of the strictly and partially massless spin-$s = 3/2, 5/2$ fields on dS_N for $N \neq 4$ was missed in Refs. [4, 7], apparently because the norm of the corresponding eigenmodes was not examined. Subsequent discussions with the authors of [4, 7] have clarified that this is indeed the case.

Note on real values of M. In this paper, we do not discuss eigenmodes with real mass parameters on dS_N. However, all of our results (i.e. explicit expressions for the eigenmodes and spin$(N,1)$ transformation formulae) also hold for real M. A crucial difference between the imaginary and the real mass parameter cases on dS_N, is the dS invariant scalar product. As we will demonstrate later, in the case of imaginary mass parameter, a dS invariant scalar product is given by (4.173) for even N, while there is no dS invariant scalar product for odd N. In the case of real mass parameter, the conventional Dirac-like inner product can be always defined, and is dS invariant (this inner product corresponds to the product that results by just removing γ^{N+1} from the scalar product (4.173)). The spin-3/2 and spin-5/2 theories with real mass parameters on dS_N are always unitary [25] - this is easy to check given the tools of the present paper.

4.1.3 Outline of the paper, notation and conventions

The paper is organised as follows. In Section 4.2, we begin by presenting the Christoffel symbols, vielbein fields and spin connection components on S^N in geodesic polar coordinates. Then, we present the basics about gamma-matrices and tensor-spinor fields on S^N. We also review the eigenspinors of the Dirac operator on S^{N-1}. In Section 4.3, we present the functions that describe the dependence of the STSSH's on the geodesic distance (θ_N) from the North Pole of S^N. In Section 4.4, we write down explicitly the unnormalised STSSH's of rank 1 on S^N (which have been constructed in Ref. [6]). In Section 4.5, we write down explicitly the unnormalised STSSH's of rank 2 on S^N (which we construct in Appendix 4.13). In Section 4.6, we use the Lie-Lorentz derivative [27] in order to study the transformation properties of the STSSH's of rank r ($r \in \{1, 2\}$) on S^N under a spin$(N+1)$ transformation and we give their normalisation factors. In Section 4.7, we begin by obtaining the vector-spinor and rank-2 symmetric tensor-spinor TT eigenmodes of the Dirac operator with arbitrary imaginary mass parameter on dS_N by analytically continuing the STSSH's of rank 1 and rank 2, respectively, on S^N. Then, we identify the 'pure gauge' modes of the strictly/partially massless spin-3/2 and spin-5/2 theories on dS_N. In Section 4.8, we derive the main result of this paper (i.e. we prove statements 1, 2 and 3 listed above), by studying the transformation properties of the TT eigenmodes of eq. (4.1) with arbitrary imaginary mass parameter under a spin$(N, 1)$ boost. More specifically, in Subsection 4.8.1, we show that all dS invariant scalar products must be indefinite for even $N > 4$ (i.e. we prove statement 1). Also, for even $N \geq 4$, we show that the 'pure gauge' modes in the strictly/partially massless theories with spin $s \in \{3/2, 5/2\}$ have zero norm with respect to any dS invariant scalar product. Then, for $N = 4$, we show that the requirement for dS invariance of the scalar product does not imply the indefiniteness of the norm if and only if the imaginary mass parameter $M = i\tilde{M}$ (with $\tilde{M} \neq 0$) takes the strictly/partially massless values (4.3). We also find that for the strictly/partially massless theories with spin $s \in \{3/2, 5/2\}$ on dS_4, the eigenmodes with negative helicity and the ones with positive helicity separately form irreducible representations of spin$(4, 1)$ (the unitarity of these irreducible representations is proved in Subsection 4.8.2). In Subsection 4.8.2, we calculate the norms of the eigenmodes on dS_N (for even $N \geq 4$) with respect to a specific dS invariant scalar product and we verify statement 1 (which was proved in the previous Subsection) and we also prove statement 2. Subsection 4.8.3 concerns the case with N odd and we prove statement 3. Finally, in Section 4.9, we give a summary of our results. We also discuss the possible generalisation of our results to higher half-odd-integer spins, as well as to other vacuum spacetimes with positive cosmological constant.

There are six Appendices. In Appendix 4.13, we construct the STSSH's of rank 2 on S^N by making use of the method of separation of variables. In this method, the STSSH's of rank 2 on S^N are expressed in terms of STSSH's of rank $\tilde{r}$ ($0 \leq \tilde{r} \leq 2$) on S^{N-1}. In Appendix 4.14, we present technical details omitted in Section 4.6. To be specific, we first give a detailed derivation of the formulae for the spin$(N+1)$ transformation of the rank-1 STSSH's and we determine their normalisation factors. Then, we discuss briefly the derivation of the transformation formulae and the normalisation factors for the rank-2 STSSH's on S^N. The rest of the Appendices concern other technical details that were omitted in the main text.

Notation and conventions. We use the mostly plus metric sign convention for dS_N. Lowercase Greek tensor indices refer to components with respect to the "coordinate basis". Lowercase Latin tensor indices refer to components with respect to the vielbein basis. Summation over repeated indices is understood. We denote the symmetrisation of a pair of indices as $A_{(\mu\nu)} \equiv (A_{\mu\nu} + A_{\nu\mu})/2$ and the anti-symmetrisation as $A_{[\mu\nu]} \equiv (A_{\mu\nu} - A_{\nu\mu})/2$. Spinor indices are always suppressed throughout this paper. We use the term strictly/partially massless field of spin $s \in \{3/2, 5/2\}$ to refer to either one of the following three cases (unless otherwise stated): the strictly massless spin-3/2 field ($r = \tau = 1$), the strictly massless spin-5/2 field ($r = \tau + 1 = 2$), the partially massless spin-5/2 field ($r = \tau = 2$). The complex conjugate of the complex number z is denoted as z^*. The notation concerning the representation-theoretic labels of the eigenmodes is slightly different than the notation used in our previous article [25]. However, we make sure to explain the representation-theoretic meaning properly such that no confusion will arise.

4.2 GEOMETRY OF THE N-SPHERE AND TENSOR-SPINOR FIELDS

4.2.1 COORDINATE SYSTEM, CHRISTOFFEL SYMBOLS AND SPIN CONNECTION

The N-sphere (S^N) embedded in the Euclidean space $\mathbb{R}^{N+1}$ is described by

$$\delta_{ab} X^a X^b = 1, \tag{4.6}$$

($a, b = 1, 2, ..., N+1$) where δ_{ab} is the Kronecker delta symbol and $X^1, X^2, ..., X^{N+1}$ are the standard coordinates for $\mathbb{R}^{N+1}$. The 'geodesic polar coordinates'[2] are given by

$$\begin{aligned} X^{N+1} &= X^{N+1}(\theta_N) = \cos\theta_N \\ X^i &= X^i(\theta_N, \boldsymbol{\theta}_{N-1}) = \sin\theta_N \ \tilde{X}^i(\boldsymbol{\theta}_{N-1}), \quad i = 1, ..., N, \end{aligned} \tag{4.7}$$

where $0 \leq \theta_N \leq \pi$ is the geodesic distance from the North Pole and $\boldsymbol{\theta}_{N-1} = (\theta_{N-1}, ..., \theta_1)$ (where $0 \leq \theta_1 < 2\pi$ and $0 \leq \theta_i \leq \pi$ for $i = 2, 3, ..., N-1$). The $\tilde{X}^i$'s in eq. (4.7) are the geodesic polar coordinates for S^{N-1} in N-dimensional Euclidean space.

[2] The geodesic polar coordinates are also known as hyperspherical coordinates. They correspond to the straightforward generalisation of the standard spherical coordinates on S^2. The North Pole of S^N is located at $\theta_N = 0$. The geodesic distance, μ_{S^N}, between two points $\boldsymbol{\theta}_{N-1} = (\theta_N, ..., \theta_1)$ and $\boldsymbol{\theta}'_{N-1} = (\theta'_N, ..., \theta'_1)$ on S^N is given by $\cos\mu_{S^N} = \cos\theta_N \cos\theta'_N + \sin\theta_N \sin\theta'_N \cos\mu_{S^{N-1}}$. If we fix $\boldsymbol{\theta}'_N$ to be at the North Pole, then the geodesic distance is given as $\cos\mu_{S^N} = \cos\theta_N$.

The line element for S^N is expressed in coordinates (4.7) as

$$ds_N^2 = d\theta_N^2 + \sin^2\theta_N ds_{N-1}^2, \tag{4.8}$$

where ds_{N-1}^2 is the line element for S^{N-1}. (Note that we define $ds_1^2 \equiv d\theta_1^2$.) The non-zero Christoffel symbols in geodesic polar coordinates are

$$\begin{aligned}
&\Gamma^{\theta_N}_{\theta_i\theta_j} = -\sin\theta_N\cos\theta_N\,\tilde{g}_{\theta_i\theta_j}, \quad \Gamma^{\theta_i}_{\theta_j\theta_N} = \cot\theta_N\,\tilde{g}^{\theta_i}_{\theta_j},\\
&\Gamma^{\theta_k}_{\theta_i\theta_j} = \tilde{\Gamma}^{\theta_k}_{\theta_i\theta_j},
\end{aligned} \tag{4.9}$$

where $\tilde{g}_{\theta_i\theta_j}$ and $\tilde{\Gamma}^{\theta_k}_{\theta_i\theta_j}$ are the metric tensor and the Christoffel symbols, respectively, on S^{N-1}. The vielbein fields $\boldsymbol{e}_a = e^\mu{}_a\partial_\mu$ (where $a = 1, ..., N$ and $\mu = \theta_1, ..., \theta_N$), determining an orthonormal frame, satisfy

$$e_\mu{}^a\, e_\nu{}^b \delta_{ab} = g_{\mu\nu}, \quad e^\mu{}_a\, e_\mu{}^b = \delta^b_a, \tag{4.10}$$

where the co-vielbein fields $\boldsymbol{e}^a = e_\mu{}^a\, dx^\mu$ define the dual coframe. The co-vielbein transforms under local rotations $\Lambda : S^N \to \mathrm{SO}(N)$ as

$$\boldsymbol{e}^a \to \Lambda(x)^a{}_b\, \boldsymbol{e}^b. \tag{4.11}$$

In geodesic polar coordinates the non-zero components of the vielbein fields are given by

$$e^{\theta_N}{}_N = 1, \quad e^{\theta_i}{}_i = \frac{1}{\sin\theta_N}\tilde{e}^{\theta_i}{}_i, \quad i = 1, ..., N-1, \tag{4.12}$$

where $\tilde{e}^{\theta_i}{}_i$ are the vielbein fields on S^{N-1}. The spin connection $\omega_{abc} = \omega_{a[bc]} \equiv (\omega_{abc} - \omega_{acb})/2$ is given by

$$\omega_{abc} = -e^\mu{}_a\Big(\partial_\mu e^\lambda{}_b + \Gamma^\lambda_{\mu\nu} e^\nu{}_b\Big)e_{\lambda c} \tag{4.13}$$

and its only non-zero components are

$$\omega_{ijk} = \frac{\tilde{\omega}_{ijk}}{\sin\theta_N}, \quad \omega_{iNk} = -\omega_{ikN} = -\cot\theta_N\,\delta_{ik}, \quad i,j,k = 1, ..., N-1, \tag{4.14}$$

where $\tilde{\omega}_{ijk}$ are the spin connection components on S^{N-1}. (Note that the sign convention we use for the spin connection is the opposite of the one used in Refs. [5, 26].)

4.2.2 Gamma matrices and tensor-spinor fields on the N-sphere

A Clifford algebra representation in N dimensions is generated by N gamma matrices. These are matrices of dimension $2^{[N/2]}$ - where $[N/2] = N/2$ if N is even and $[N/2] = (N-1)/2$ if N is odd - satisfying the anti-commutation relations

$$\{\gamma^a, \gamma^b\} = 2\delta^{ab}\mathbf{1}, \qquad a, b = 1, 2, ..., N, \tag{4.15}$$

where $\mathbf{1}$ is the identity matrix. We adopt the representation of gamma matrices used in Ref. [5], where gamma matrices in N dimensions are expressed in terms of gamma matrices in $N-1$ dimensions ($\tilde{\gamma}^i$) as follows:

- For N even

$$\gamma^N = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & i\tilde{\gamma}^j \\ -i\tilde{\gamma}^j & 0 \end{pmatrix}, \tag{4.16}$$

  ($j = 1, ..., N-1$) where the lower-dimensional gamma matrices satisfy the Euclidean Clifford algebra anti-commutation relations

$$\{\tilde{\gamma}^j, \tilde{\gamma}^k\} = 2\delta^{jk}\mathbf{1}, \quad j, k = 1, ..., N-1. \tag{4.17}$$

  By using the vielbein fields (4.12) we can express the gamma matrices (4.16) in the "coordinate basis" as $\gamma^\mu(x) = e^\mu{}_a(x)\,\gamma^a$. Note that one can construct the extra gamma matrix γ^{N+1}, which is given by the product $\gamma^{N+1} \equiv \epsilon\,\gamma^1\gamma^2...\gamma^N$, where ϵ is a phase factor. The matrix γ^{N+1} anti-commutes with each of the γ^a's in eq. (4.16). As in Ref. [5], we choose the phase factor ϵ such that

$$\gamma^{N+1} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \tag{4.18}$$

- For N odd

$$\gamma^N = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \qquad \gamma^{N-1} = \tilde{\gamma}^{N-1} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},$$
$$\gamma^j = \tilde{\gamma}^j = \begin{pmatrix} 0 & i\tilde{\tilde{\gamma}}^j \\ -i\tilde{\tilde{\gamma}}^j & 0 \end{pmatrix}, \qquad j = 1, ..., N-2. \tag{4.19}$$

  The double-tilde is used to denote gamma matrices in $N-2$ dimensions. In $N = 1$ dimension the only (one-dimensional) gamma matrix is equal to 1. The gamma matrices (4.19) are expressed in the "coordinate basis" by using the vielbein fields (4.12), as in the case with N even.

Note that all gamma matrices in eqs. (4.16)-(4.19) are hermitian.
The tensor-spinor fields $\psi_{\mu_1...\mu_r}$ of rank r are defined as r^{th}-rank tensors where each one of the tensorial components transforms as a $2^{[N/2]}$-dimensional spinor under Spin(N) (double cover of SO(N)). Tensor-spinors transform under the local rotation of the co-vielbein in eq. (4.11) as

$$\psi_{\mu_1...\mu_r}(x) \rightarrow \Lambda(x)_{\mu_1}^{\ \ \nu_1}...\Lambda(x)_{\mu_r}^{\ \ \nu_r}\, S(\Lambda(x))\, \psi_{\nu_1...\nu_r}(x), \tag{4.20}$$

where the matrix $\Lambda(x) \in$ SO(N) acts on the tensor indices of $\psi_{\mu_1...\mu_r}$, while the matrix $S(\Lambda(x)) \in$ Spin(N) acts on the spinor indices of $\psi_{\mu_1...\mu_r}$ (the spinor indices have been suppressed for convenience). For any $\Lambda(x) \in$ SO(N) we have [13]

$$S(\Lambda(x))^{-1}\, \gamma^a\, S(\Lambda(x)) = \Lambda(x)^a{}_b \gamma^b, \tag{4.21}$$

where $S(\Lambda(x))$ is either one of the two matrices in Spin(N) that correspond to $\Lambda(x)$. (See, e.g., Ref. [5] and Appendix D of Ref. [13] for more detailed discussions on spinor representations of orthogonal groups.)
The covariant derivative for a vector-spinor field is given by

$$\nabla_\nu \psi_\mu = \partial_\nu \psi_\mu + \frac{1}{2}\omega_{\nu bc}\Sigma^{bc}\psi_\mu - \Gamma^{\lambda}_{\ \nu\mu}\psi_\lambda, \tag{4.22}$$

while the covariant derivative for a rank-2 tensor-spinor field is given by

$$\nabla_\nu \psi_{\mu_1\mu_2} = \partial_\nu \psi_{\mu_1\mu_2} + \frac{1}{2}\omega_{\nu bc}\Sigma^{bc}\psi_{\mu_1\mu_2} - \Gamma^{\lambda}_{\ \nu\mu_1}\psi_{\lambda\mu_2} - \Gamma^{\lambda}_{\ \nu\mu_2}\psi_{\mu_1\lambda}, \tag{4.23}$$

where $\omega_{\nu bc} = e_\nu{}^d \omega_{dbc}$ [see eq. (4.14)]. The matrices Σ^{ab} are the generators of the $2^{[N/2]}$-dimensional spinor representation of Spin(N) and they are given by

$$\Sigma^{ab} = \frac{1}{4}[\gamma^a, \gamma^b] \tag{4.24}$$

$$= \frac{1}{2}\gamma^a\,\gamma^b - \frac{1}{2}\delta^{ab}, \qquad a, b = 1, ..., N. \tag{4.25}$$

They satisfy the Spin(N) algebra commutation relations

$$[\Sigma^{ab}, \Sigma^{cd}] = \delta^{bc}\Sigma^{ad} - \delta^{ac}\Sigma^{bd} + \delta^{ad}\Sigma^{bc} - \delta^{bd}\Sigma^{ac}. \tag{4.26}$$

(The gamma matrices are covariantly constant, i.e. $\nabla_a \gamma^b = 0$ - see e.g. Appendix D of Ref. [13].)

Eigenspinors on S^{N-1}. For later convenience, let us introduce the spinor eigenmodes $\chi_{\pm\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})$ of the Dirac operator on S^{N-1} (see also Ref. [5] and Appendix 4.11 of the present paper). These spinor eigenmodes satisfy [5]

$$\tilde{\nabla\!\!\!/}\chi_{\pm\ell\tilde{\rho}} = \pm i\left(\ell + \frac{N-1}{2}\right)\chi_{\pm\ell\tilde{\rho}}, \tag{4.27}$$

where $\tilde{\nabla\!\!\!/} = \gamma^a \tilde{\nabla}_a$ is the Dirac operator on S^{N-1}, $\tilde{\nabla}_a$ is the spinor covariant derivative on S^{N-1} and ℓ is the angular momentum quantum number on S^{N-1}. The symbol $\tilde{\rho}$ represents labels other than ℓ. The requirement for regularity of the spinor eigenmodes (4.27) on S^{N-1} restricts ℓ to take the values $\ell = 0, 1, 2, ...$ [5]. We suppose that the spinor eigenmodes (4.27) are normalised as

$$\int_{S^{N-1}} \sqrt{\tilde{g}}\, d\boldsymbol{\theta}_{N-1}\, \chi_{\pm\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})^{\dagger}\, \chi_{\pm\ell'\tilde{\rho}'}(\boldsymbol{\theta}_{N-1}) = \delta_{\ell\ell'}\delta_{\tilde{\rho}\tilde{\rho}'}, \tag{4.28}$$

where $d\boldsymbol{\theta}_{N-1} = d\theta_{N-1}\, d\theta_{N-2} ... d\theta_1$. The square root of the determinant of the metric on S^{N-1} is

$$\sqrt{\tilde{g}} = \sin^{N-2}\theta_{N-1}\, \sin^{N-3}\theta_{N-2} \dots \sin\theta_2 \tag{4.29}$$

$$= \sin^{N-2}\theta_{N-1}\sqrt{\tilde{\tilde{g}}}, \tag{4.30}$$

where $\tilde{\tilde{g}}$ is the determinant of the metric on S^{N-2}. All the χ_+ eigenspinors are orthogonal to all the χ_- eigenspinors in eq. (4.28) [5]. For each allowed value of ℓ, the eigenspinors $\chi_{+\ell\tilde{\rho}}$ and $\chi_{-\ell\tilde{\rho}}$ separately form irreducible representations of spin(N) [24]. For odd $N = 2p+1$, the spinors $\chi_{+\ell\tilde{\rho}}$ (or $\chi_{-\ell\tilde{\rho}}$) form a spin$(2p+1)$ representation with the (p-component) highest weight

$$\vec{f}_0 = \left(\ell + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right).$$

For even $N = 2p$, the spinors $\chi_{\pm\ell\tilde{\rho}}$ form a spin$(2p)$ representation with the (p-component) highest weight

$$\vec{f}_0^{\pm} = \left(\ell + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}, \pm\frac{1}{2}\right).$$

4.3 TECHNICAL DETAILS FOR THE FUNCTIONS DESCRIBING THE DEPENDENCE OF STSSH'S ON θ_N

Before writing down the explicit form of the STSSH's of rank r $(=1,2)$ on S^N, it is useful to introduce the functions $\phi^{(a)}_{n\ell}(\theta_N)$ [eq. (4.31)] and $\psi^{(a)}_{n\ell}(\theta_N)$ [eq. (4.32)] that describe

the dependence of the STSSH's on θ_N, since they are going to be used extensively in the rest of the paper. The properties of these functions play a crucial role in the normalisation of the STSSH's and in the derivation of the formulae for the spin$(N+1)$ transformation of the STSSH's (see Section 4.6 and Appendix 4.14). Most importantly, in view of the analytic continuation of our STSSH's to dS_N, the properties of the functions $\phi^{(a)}_{n\ell}(\theta_N)$ and $\psi^{(a)}_{n\ell}(\theta_N)$ will play a very important role in studying the unitarity/non-unitarity of the spin$(N,1)$ representations formed by the analytically continued STSSH's.
As we will see in Sections 4.4 and 4.5, the θ_N-dependence of the STSSH's on S^N is described by functions of the following form:

$$\phi^{(a)}_{n\ell}(\theta_N) = \kappa_\phi(n,\ell)\left(\cos\frac{\theta_N}{2}\right)^{\ell+1-a}\left(\sin\frac{\theta_N}{2}\right)^{\ell-a} \times F\left(-n+\ell, n+\ell+N; \ell+\frac{N}{2}; \sin^2\frac{\theta_N}{2}\right), \tag{4.31}$$

$$\psi^{(a)}_{n\ell}(\theta_N) = \kappa_\phi(n,\ell)\frac{n+\frac{N}{2}}{\ell+\frac{N}{2}}\left(\cos\frac{\theta_N}{2}\right)^{\ell-a}\left(\sin\frac{\theta_N}{2}\right)^{\ell+1-a} \times F\left(-n+\ell, n+\ell+N; \ell+\frac{N+2}{2}; \sin^2\frac{\theta_N}{2}\right), \tag{4.32}$$

where the normalisation factor $\kappa_\phi(n,\ell)$ is given by

$$\kappa_\phi(n,\ell) = \frac{\Gamma(n+N/2)}{\Gamma(n-\ell+1)\Gamma(\ell+N/2)}, \tag{4.33}$$

while $F(A,B;C;z)$ is the Gauss hypergeometric function [17]. The number a in eqs. (4.31) and (4.32) is taken to be an integer for the purposes of this paper. The functions in eqs. (4.31) and (4.32) can be expressed in terms of the Jacobi polynomials [17], where $\kappa_\phi(n,\ell)$ plays the role of the conventional normalisation factor for the Jacobi polynomials [17]. (These functions with $a=0$ were used to describe spinors on S^N [5].)
As we will discuss in Section 4.4 and 4.5, the integer n is the angular momentum quantum number of the STSSH's on S^N and it labels the representation of spin$(N+1)$ formed by the STSSH's. The angular momentum quantum number on S^{N-1}, ℓ, is initially assumed to be a positive integer or zero[3]. Furthermore, the requirement for the absence of singularity in the STSSH's on S^N will give rise to the condition

$$n-\ell \in \mathbb{N}_0 \tag{4.34}$$

[3] This requirement on ℓ is motivated naturally in the recursive construction of the STSSH's on S^N in terms of STSSH's on S^{N-1} - see Appendix 4.13.

or equivalently $n \geq \ell$, where $\mathbb{N}_0$ is the set of positive integers including zero. (This condition also arises from the branching rules for spin$(N+1) \supset$ spin(N), as we will see below.) In particular, eq. (4.34) is obtained in Appendix 4.13, by requiring the regularity of $\phi^{(a)}_{n\ell}(\theta_N)$ and $\psi^{(a)}_{n\ell}(\theta_N)$ in the limit $\theta_N \to \pi$.

The functions $\phi^{(a)}_{n\ell}(\theta_N)$ and $\psi^{(a)}_{n\ell}(\theta_N)$ are related to each other by the following formulae:

$$\left(\frac{d}{d\theta_N} + \frac{N+2a-1}{2}\cot\theta_N + \frac{\ell+(N-1)/2}{\sin\theta_N}\right)\psi^{(a)}_{n\ell}(\theta_N) = \left(n+\frac{N}{2}\right)\phi^{(a)}_{n\ell}(\theta_N) \tag{4.35}$$

$$\left(\frac{d}{d\theta_N} + \frac{N+2a-1}{2}\cot\theta_N - \frac{\ell+(N-1)/2}{\sin\theta_N}\right)\phi^{(a)}_{n\ell}(\theta_N) = -\left(n+\frac{N}{2}\right)\psi^{(a)}_{n\ell}(\theta_N). \tag{4.36}$$

Equations (4.35) and (4.36) are proved using the raising and lowering operators for the Gauss hypergeometric function in Appendix 4.10. Note also the relation

$$\psi^{(a)}_{n\ell}(\theta_N) = (-1)^{n-\ell}\phi^{(a)}_{n\ell}(\pi-\theta_N). \tag{4.37}$$

4.4 THE STSSH'S OF RANK 1 ON THE N-SPHERE

In this Section, we write down explicitly the unnormalised STSSH's of rank 1 [i.e. the TT vector-spinor eigenmodes of eq. (4.4)], by following Ref. [6] where these eigenmodes have been originally constructed. However, we will present the results of Ref. [6] in a slightly modified manner that is more suitable for studying the group-theoretic properties of the eigenmodes. We also recommend that the readers refer to our previous article [25], in which the steps in the method of separation of variables are discussed in greater detail for spin-3/2 eigenmodes on dS_N.

4.4.1 STSSH'S OF RANK 1 FOR N EVEN

Representation-theoretic background. The equations (4.4) and (4.5) for the TT vector-spinor eigenmodes on S^N $(N \geq 4)$ are written as

$$\nabla\!\!\!/\, \psi^{(A;\sigma;n\ell;\tilde{\rho})}_{\pm\mu} = \pm i\left(n+\frac{N}{2}\right)\psi^{(A;\sigma;n\ell;\tilde{\rho})}_{\pm\mu}, \tag{4.38}$$

$$\nabla^\alpha\psi^{(A;\sigma;n\ell;\tilde{\rho})}_{\pm\alpha} = \gamma^\alpha\psi^{(A;\sigma;n\ell;\tilde{\rho})}_{\pm\alpha} = 0. \tag{4.39}$$

We have denoted the TT vector-spinor eigenmodes with eigenvalue $\pm i(n+\frac{N}{2})$ as $\psi_{\pm\mu}^{(A;\sigma;n\ell;\tilde{\rho})}$, where $n = 1, 2, ...$ and $\ell = 1, ..., n$ are the angular momentum quantum numbers on S^N and S^{N-1}, respectively.[4]
For each value of n we have a representation of spin$(N+1)$ (i.e. algebra of Spin$(N+1)$) acting on the space of the eigenmodes $\psi_{+\mu}^{(A;\sigma;n\ell;\tilde{\rho})}$ (or $\psi_{-\mu}^{(A;\sigma;n\ell;\tilde{\rho})}$) with highest weight $\vec{\lambda} = (\lambda_1, ..., \lambda_{N/2})$ given by [24]

$$\vec{\lambda} = \left(n + \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, ..., \frac{1}{2}\right), \qquad (n = 1, 2, ...). \tag{4.40}$$

Note that for $N = 4$ we have $\vec{\lambda} = (n + 1/2,\, 3/2)$ $(n = 1, 2, ...)$. The two sets of eigenmodes, $\{\psi_{+\mu}^{(A;\sigma;n\ell;\tilde{\rho})}\}$ and $\{\psi_{-\mu}^{(A;\sigma;n\ell;\tilde{\rho})}\}$, form equivalent representations and they are related to each other by $\psi_{+\mu}^{(A;\sigma;n\ell;\tilde{\rho})} = \gamma^{N+1}\psi_{-\mu}^{(A;\sigma;n\ell;\tilde{\rho})}$.
From a representation-theoretic viewpoint, the construction of eigenmodes on S^N using the method of separation of variables corresponds to specifying the basis vectors of a spin$(N+1)$ representation space in the decomposition spin$(N+1) \supset$ spin(N). For a spin$(N+1)$ representation $\vec{\lambda} = (\lambda_1, ..., \lambda_{N/2})$ (N even), the spin(N) content corresponds to highest weights $\vec{f} = (f_1, ..., f_{N/2})$ with [5, 3, 15]

$$\lambda_1 \geq f_1 \geq \lambda_2 \geq ... \geq \lambda_{N/2} \geq |f_{N/2}|, \tag{4.41}$$

where $f_{N/2}$ can be negative. In the case of TT vector-spinor eigenmodes on S^N, $\vec{\lambda} = \left(n + \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, ..., \frac{1}{2}\right)$, the spin$(N)$ content corresponds to representations with highest weights: $\vec{f}_0^{\sigma} = \left(\ell + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}, \sigma\frac{1}{2}\right)$ and $\vec{f}_1^{\sigma} = \left(\ell + \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, ..., \frac{1}{2}, \sigma\frac{1}{2}\right)$ with $\sigma = \pm$. We call the index σ 'the spin projection index' on S^N. The symbol $\tilde{\rho}$ stands for the representation-theoretic labels concerning the chain of subalgebras spin$(N-1) \supset$ spin$(N-2) \supset ... \supset$ spin(2).
Depending on the 'spin' of the spin(N) representations included in our spin$(N+1)$ representation of interest, the solutions of equations (4.38) and (4.39) are separated into two different types, namely, the **type-*I* modes** and the **type-*II* modes** [6]. In particular, the type-I modes correspond to the spinor representation of spin(N)

$$\vec{f}_0^{\pm} = \left(\ell + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}, \pm\frac{1}{2}\right),$$

while the type-II modes correspond to the TT vector-spinor representation

$$\vec{f}_1^{\pm} = \left(\ell + \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, ..., \frac{1}{2}, \pm\frac{1}{2}\right).$$

[4] The angular momentum quantum numbers for our STSSH's of rank $r \in \{1, 2\}$ on S^N satisfy $n \geq \ell \geq r$. The condition $n \geq \ell$ was discussed in the previous Section - see eq. (4.34). However, as we will see below, the condition $\ell \geq r$ is obtained by using the explicit expressions of the STSSH's.

We assign to the label A the value 'I' in order to indicate the type-I modes ($\psi^{(I;\sigma;n\ell;\tilde{\rho})}_{\pm\mu}$) and the value '$II$-$\tilde{A}$' in order to indicate the type-II modes ($\psi^{(II\text{-}\tilde{A};\sigma;n\ell;\tilde{\rho})}_{\pm\mu}$), where the label $\tilde{A}$ on S^{N-1} corresponds to A on S^N (the label $\tilde{A}$ is discussed further in the passage after eq. (4.53)).

Type-I modes. The type-I modes are expressed in their vector components as

$$\psi^{(I;\sigma;n\ell;\tilde{\rho})}_{\pm\mu} = \left(\psi^{(I;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N}, \psi^{(I;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_j}\right) \tag{4.42}$$

($j = 1, ..., N-1$), where $\psi^{(I;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N}$ is a spinor on S^{N-1}, while $\psi^{(I;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_j}$ is a vector-spinor on S^{N-1} [6]. The type-I modes with negative spin projection ($\sigma = -$) on S^N are given by [6]

$$\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_N}(\theta_N, \boldsymbol{\theta}_{N-1}) = \begin{pmatrix} \phi^{(1)}_{n\ell}(\theta_N)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \pm i\psi^{(1)}_{n\ell}(\theta_N)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix} \tag{4.43}$$

$$\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_j}(\theta_N, \boldsymbol{\theta}_{N-1}) = \begin{pmatrix} C^{(\uparrow)(1)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) + D^{(\uparrow)(1)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \\ \pm iC^{(\downarrow)(1)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \pm iD^{(\downarrow)(1)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix}. \tag{4.44}$$

The type-I modes with positive spin projection ($\sigma = +$) on S^N are given by [6]

$$\psi^{(I;+;n\ell;\rho)}_{\pm\theta_N}(\theta_N, \boldsymbol{\theta}_{N-1}) = \begin{pmatrix} i\psi^{(1)}_{n\ell}(\theta_N)\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \pm\phi^{(1)}_{n\ell}(\theta_N)\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix} \tag{4.45}$$

$$\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\theta_j}(\theta_N, \boldsymbol{\theta}_{N-1}) = \begin{pmatrix} iC^{(\downarrow)(1)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{\theta_j}\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) - iD^{(\downarrow)(1)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{\theta_j}\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \\ \pm C^{(\uparrow)(1)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{\theta_j}\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \mp iD^{(\uparrow)(1)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{\theta_j}\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix}. \tag{4.46}$$

The eigenspinors on S^{N-1}, $\chi_{\pm\ell\tilde{\rho}}$, satisfy eq. (4.27) and they are written down explicitly in Appendix 4.11. The functions $\phi^{(1)}_{n\ell}$ and $\psi^{(1)}_{n\ell}$ are given by eqs. (4.31) and (4.32), respectively. The functions $C^{(\uparrow)(a)}_{n\ell}, C^{(\downarrow)(a)}_{n\ell}$ are expressed in terms of $\phi^{(a)}_{n\ell}$ and $\psi^{(a)}_{n\ell}$ as follows [6]:

$$\begin{aligned} C^{(\uparrow)(a)}_{n\ell}(\theta_N) = \frac{1}{\ell(\ell+N-1)}\Bigg\{ &\sin\theta_N\left[\frac{N-1}{2}\cos\theta_N + \ell + \frac{N-1}{2}\right]\phi^{(a)}_{n\ell}(\theta_N) \\ &- \frac{N-1}{N-2}(n+\frac{N}{2})\sin^2\theta_N\,\psi^{(a)}_{n\ell}(\theta_N)\Bigg\}, \end{aligned} \tag{4.47}$$

$$C^{(\downarrow)(a)}_{n\ell}(\theta_N) = \frac{1}{\ell(\ell+N-1)} \times \Bigg\{ \sin\theta_N \left[\frac{N-1}{2}\cos\theta_N - \ell - \frac{N-1}{2}\right] \psi^{(a)}_{n\ell}(\theta_N) + \frac{N-1}{N-2}\left(n+\frac{N}{2}\right)\sin^2\theta_N \phi^{(a)}_{n\ell}(\theta_N)\Bigg\}, \tag{4.48}$$

while the functions $D^{(\uparrow)(a)}_{n\ell}$ and $D^{(\downarrow)(a)}_{n\ell}$ are given by:

$$D^{(\uparrow)(a)}_{n\ell}(\theta_N) = \frac{-i}{N-1}\left[-\left(\ell+\frac{N-1}{2}\right) C^{(\uparrow)(a)}_{n\ell}(\theta_N) + \sin\theta_N\, \phi^{(a)}_{n\ell}(\theta_N)\right] \tag{4.49}$$

and

$$D^{(\downarrow)(a)}_{n\ell}(\theta_N) = \frac{-i}{N-1}\left[-\left(\ell+\frac{N-1}{2}\right) C^{(\downarrow)(a)}_{n\ell}(\theta_N) - \sin\theta_N\, \psi^{(a)}_{n\ell}(\theta_N)\right], \tag{4.50}$$

respectively. **Allowed values for angular momentum quantum numbers:** The appearance of ℓ in the denominator in eqs. (4.47) and (4.48) reflects the fact that there is no type-I eigenmode if the θ_N-component (4.43) [or (4.45)] has $\ell = 0$ (i.e. ℓ has to satisfy $\ell \geq r = 1$). The condition $n \geq \ell$ and the quantisation of the eigenvalue in eq. (4.38) follow from the requirement of regularity of the functions $\phi^{(a)}_{n\ell}(\theta_N)$ and $\psi^{(a)}_{n\ell}(\theta_N)$ (see Appendix 4.13). Thus, we have verified that the allowed values for the angular momentum quantum numbers are $n = 1, 2, ...$ and $\ell = 1, ..., n$.

Type-II modes. The vector components of the type-II modes are expressed as [6]

$$\psi^{(II\text{-}\tilde{A};\sigma;n\ell;\tilde{\rho})}_{\pm\mu} = \left(0, \psi^{(II\text{-}\tilde{A};\sigma;n\ell;\tilde{\rho})}_{\pm\theta_j}\right), \tag{4.51}$$

($j = 1, ..., N-1$) where $\psi^{(II\text{-}\tilde{A};\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N} = 0$. The type-$II$ modes (4.51) are TT vector-spinors on S^{N-1}. Thus, they can be constructed in terms of TT vector-spinor eigenmodes $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}(\boldsymbol{\theta}_{N-1})$ on S^{N-1} that satisfy

$$\tilde{\nabla\!\!\!/}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j} = \pm i\left(\ell+\frac{N-1}{2}\right)\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j} \tag{4.52}$$

$$\tilde{\gamma}^{\theta_i}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_i} = \tilde{\nabla}^{\theta_i}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_i} = 0, \tag{4.53}$$

where the label $\tilde{A}$ indicates the type of the eigenmode $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}$ [5]. (The TT vector-spinor eigenmodes and the corresponding types of modes on odd-dimensional spheres are

[5]As in the case of the label A for eigenmodes on S^N, the label $\tilde{A}$ in $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}$ refers to the 'spin' of the spin$(N-1)$ representations appearing in the spin$(N-1)$ content of the spin(N) representations formed by $\{\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}\}$.

presented in Subsection 4.4.2.) The requirement for regularity of $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}$ on S^{N-1} gives the allowed values for ℓ, i.e. $\ell = 1, 2, \dots$. This requirement for ℓ follows naturally from the recursive construction of the STSSH's of rank 1 in Ref. [6]. We suppose that the eigenmodes $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}$ are normalised on S^{N-1} as

$$\int_{S^{N-1}} \sqrt{\tilde{g}}\, d\boldsymbol{\theta}_{N-1}\, \tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_i}(\boldsymbol{\theta}_{N-1})^{\dagger}\, \tilde{\psi}^{(\tilde{A}';\ell'\tilde{\rho}')\theta_i}_{\pm}(\boldsymbol{\theta}_{N-1}) = \delta_{\ell\ell'}\delta_{\tilde{\rho}\tilde{\rho}'}\delta_{\tilde{A}\tilde{A}'}, \tag{4.54}$$

where $\sqrt{\tilde{g}}$ is given by eq. (4.29). For each allowed value of ℓ, the set of eigenmodes $\{\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}\}$ forms a spin(N) representation with highest weight
$\vec{f}^{\pm}_1 = \left(\ell + \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2}, \pm\frac{1}{2}\right)$ [24].
The type-II modes $\psi^{(II\text{-}\tilde{A};\sigma;n\ell;\tilde{\rho})}_{\pm\mu}$ on S^N with negative ($\sigma = -$) and positive ($\sigma = +$) spin projections are given by [6]

$$\begin{aligned} \psi^{(II\text{-}\tilde{A};-;n\ell;\tilde{\rho})}_{\pm\theta_N}(\theta_N, \boldsymbol{\theta}_{N-1}) &= 0 \\ \psi^{(II\text{-}\tilde{A};-;n\ell;\tilde{\rho})}_{\pm\theta_j}(\theta_N, \boldsymbol{\theta}_{N-1}) &= \begin{pmatrix} \phi^{(-1)}_{n\ell}(\theta_N)\, \tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}) \\ \pm i\psi^{(-1)}_{n\ell}(\theta_N)\, \tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}) \end{pmatrix} \end{aligned} \tag{4.55}$$

and

$$\begin{aligned} \psi^{(II\text{-}\tilde{A};+;n\ell;\tilde{\rho})}_{\pm\theta_N}(\theta_N, \boldsymbol{\theta}_{N-1}) &= 0 \\ \psi^{(II\text{-}\tilde{A};+;n\ell;\tilde{\rho})}_{\pm\theta_j}(\theta_N, \boldsymbol{\theta}_{N-1}) &= \begin{pmatrix} i\psi^{(-1)}_{n\ell}(\theta_N)\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_i}(\boldsymbol{\theta}_{N-1}) \\ \pm\phi^{(-1)}_{n\ell}(\theta_N)\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_j}(\boldsymbol{\theta}_{N-1}) \end{pmatrix}, \end{aligned} \tag{4.56}$$

$(j = 1, \dots, N-1)$ respectively. The functions $\phi^{(-1)}_{n\ell}$ and $\psi^{(-1)}_{n\ell}$ are given by eqs. (4.31) and (4.32), respectively. As in the case of type-I modes, we find the allowed values $n = 1, 2, \dots$ and $\ell = 1, \dots, n$.

4.4.2 STSSH's of rank 1 for N odd

Representation-theoretic background. The eigenvalue equation and the TT conditions are given again by eqs. (4.38) and (4.39), respectively, while the gamma matrices are now given by eq. (4.19). The TT eigenmodes on S^N are denoted as $\psi^{(A;n\ell;\tilde{\rho})}_{\pm\mu}$. The allowed values for the angular momentum quantum numbers are $n = 1, 2, \dots$ and $\ell = 1, \dots, n$.
For each allowed value of n we have a representation of spin$(N+1)$ acting on the space of the eigenmodes $\psi^{(A;n\ell;\tilde{\rho})}_{\pm\mu}$. The highest weights $\vec{\lambda} = (\lambda_1, \dots, \lambda_{(N+1)/2})$ for these

representations are given by [24]

$$\vec{\lambda}^{\pm} = \left(n + \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, ..., \frac{1}{2}, \pm\frac{1}{2}\right), \qquad (n = 1, 2, ...). \tag{4.57}$$

Unlike the case with N even, for N odd there does not exist any spinorial matrix that relates $\psi_{+\mu}^{(A;n\ell;\tilde{\rho})}$ and $\psi_{-\mu}^{(A;n\ell;\tilde{\rho})}$, since the two sets of modes form inequivalent representations of spin$(N+1)$[6]. Note that for $N = 3$ we have $\vec{\lambda}^{\pm} = (n + 1/2, \pm 3/2)$ $(n = 1, 2, ...)$.

As in the case with N even, the construction of eigenmodes on S^N using the method of separation of variables corresponds to specifying the basis vectors of a spin$(N+1)$ representation space in the decomposition spin$(N+1) \supset$ spin(N). For a spin$(N+1)$ representation $\vec{\lambda} = (\lambda_1, ..., \lambda_{(N+1)/2})$ (N odd), where $\lambda_{(N+1)/2}$ can be negative, the spin(N) content corresponds to highest weights $\vec{f} = (f_1, ..., f_{(N-1)/2})$ with [5, 3, 15]

$$\lambda_1 \geq f_1 \geq \lambda_2 \geq ... \geq \lambda_{(N-1)/2} \geq f_{(N-1)/2} \geq |\lambda_{(N+1)/2}|. \tag{4.58}$$

In the case of TT vector-spinor eigenmodes on S^N, $\vec{\lambda}^{\pm} = \left(n + \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, ..., \frac{1}{2}, \pm\frac{1}{2}\right)$, the spin$(N)$ content corresponds to representations with highest weights: $\vec{f}_0 = \left(\ell + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right)$ and $\vec{f}_1 = \left(\ell + \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, ..., \frac{1}{2}\right)$. As the representation $\vec{f}_{\tilde{r}} = \left(\ell + \frac{1}{2}, \tilde{r} + \frac{1}{2}, ..., \frac{1}{2}, \frac{1}{2}\right)$ is equivalent to $\vec{f}_{\tilde{r}}{}' = \left(\ell + \frac{1}{2}, \tilde{r} + \frac{1}{2}, ..., \frac{1}{2}, -\frac{1}{2}\right)$ [15], there is no need to introduce the notion of the 'spin projection index' for tensor-spinor eigenmodes on odd-dimensional S^N. As in the case with N even, the symbol $\tilde{\rho}$ stands for representation-theoretic labels concerning the chain of subalgebras spin$(N-1) \supset$ spin$(N-2) \supset ... \supset$ spin(2).

As in the even-dimensional case, the label A denotes the type of the mode, i.e. the 'spin' of the corresponding spin(N) representation. In particular, the type-I modes ($\psi_{\pm\mu}^{(I;n\ell;\tilde{\rho})}$) on S^N are constructed in terms of eigenspinors on S^{N-1}, which correspond to the spin(N) highest weight $\vec{f}_0 = \left(\ell + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right)$. The type-$II$ modes ($\psi_{\pm\mu}^{(II\text{-}\tilde{A};n\ell;\tilde{\rho})}$) on S^N are constructed in terms of TT eigenvector-spinors of type-$\tilde{A}$ on S^{N-1}, which correspond to the spin(N) highest weight $\vec{f}_1 = \left(\ell + \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, ..., \frac{1}{2}\right)$. Note that TT eigenvector-spinor modes of any type on S^N (with arbitrary N) exist only for $N \geq 3$, while type-II modes exist only for $N \geq 4$ [6].

Type-I modes. The type-I modes are given by [6]

$$\psi_{\pm\theta_N}^{(I;n\ell;\tilde{\rho})}(\theta_N, \boldsymbol{\theta}_{N-1}) = \frac{1}{\sqrt{2}}(1 + i\gamma^N)\left\{\phi_{n\ell}^{(1)}(\theta_N) \pm i\psi_{n\ell}^{(1)}(\theta_N)\gamma^N\right\}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \tag{4.59}$$

[6] In general, for N odd there does not exist any spinorial matrix that relates two STSSH's of arbitrary rank r with different sign for the eigenvalue.

$$\psi^{(I;n\ell;\tilde{\rho})}_{\pm\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) = \frac{1}{\sqrt{2}}(1+i\gamma^N)\Bigg\{ \left(C^{(\uparrow)(1)}_{n\ell}(\theta_N) \pm iC^{(\downarrow)(1)}_{n\ell}(\theta_N)\gamma^N\right) \tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) + \left(D^{(\uparrow)(1)}_{n\ell}(\theta_N) \pm iD^{(\downarrow)(1)}_{n\ell}(\theta_N)\gamma^N\right) \tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\Bigg\}, \tag{4.60}$$

($j=1,...,N-1$) where $\chi_{-\ell\tilde{\rho}}$ are the eigenspinors on S^{N-1} satisfying eq. (4.27). (Since γ^N anti-commutes with $\tilde{\nabla\!\!\!/}$ we have $\gamma^N\chi_{-\ell\tilde{\rho}} = \chi_{+\ell\tilde{\rho}}$ [5].) As in the case with N even, the functions $\phi^{(1)}_{n\ell}$ and $\psi^{(1)}_{n\ell}$ are given by eqs. (4.31) and (4.32), respectively, while the functions $C^{(\uparrow)(1)}_{n\ell}, C^{(\downarrow)(1)}_{n\ell}, D^{(\uparrow)(1)}_{n\ell}$ and $D^{(\downarrow)(1)}_{n\ell}$ are given by eqs. (4.47), (4.48), (4.49) and (4.50), respectively. As in the even-dimensional case, one finds that the angular momentum quantum numbers are allowed to take the values $n=1,2,...$ and $\ell=1,...,n$.

Type-II modes. The type-II modes are given by [6]

$$\begin{aligned}\psi^{(II\text{-}\tilde{A};n\ell;\tilde{\rho})}_{\pm\theta_N}(\theta_N,\boldsymbol{\theta}_{N-1}) =&0\\ \psi^{(II\text{-}\tilde{A};n\ell;\tilde{\rho})}_{\pm\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) =& \frac{1}{\sqrt{2}}(1+i\gamma^N)\left\{\phi^{(-1)}_{n\ell}(\theta_N) \pm i\psi^{(-1)}_{n\ell}(\theta_N)\gamma^N\right\}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}),\end{aligned} \tag{4.61}$$

where the functions $\phi^{(-1)}_{n\ell}$ and $\psi^{(-1)}_{n\ell}$ are given by eqs. (4.31) and (4.32), respectively, while the rank-1 STSSH's of type-$\tilde{A}$ on S^{N-1}, $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}$, satisfy eqs. (4.52)-(4.54) (where $\gamma^N\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j} = \tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_j}$). As in the case with N even, we find that the angular momentum quantum numbers are allowed to take the values: $n=1,2,...$ and $\ell=1,...,n$.

4.5 THE STSSH'S OF RANK 2 ON THE N-SPHERE

In this Section we write down explicitly the STSSH's of rank 2 on S^N by using the method of separation of variables. In this method the STSSH's of rank 2 on S^N are expressed in terms of STSSH's of rank $\tilde{r}$ (where $\tilde{r}\leq r$) on S^{N-1}. (The 0th rank STSSH's are the eigenspinors of the Dirac operator constructed in Ref. [5].) We present the details of the calculations in Appendix 4.13.

The representation-theoretic background concerning the STSSH's of rank 2 is very similar to the case of STSSH's of rank 1 presented in the previous Section. Therefore, we are not going to discuss the corresponding representation-theoretic details here; we will just focus on the explicit expressions for the STSSH's of rank 2 on S^N. Let us recall the main idea: the construction of eigenmodes on S^N using the method of separation of variables corresponds to specifying the basis vectors of a spin$(N+1)$ representation space in the decomposition spin$(N+1) \supset$ spin(N).

4.5.1 STSSH's of rank 2 for N even

The equations for the STSSH's of rank 2 are given by:

$$\nabla\!\!\!/ \psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\mu\nu} = \pm i|\zeta_{n,N}|\, \psi^{(B;\sigma;\ell_N\ell;\tilde{\rho})}_{\pm\mu\nu}, \tag{4.62}$$

$$\nabla^{\alpha}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\alpha\nu} = \gamma^{\alpha}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\alpha\nu} = 0, \tag{4.63}$$

$$g^{\alpha\beta}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\alpha\beta} = 0, \tag{4.64}$$

[see eqs. (4.4) and (4.5)] where the labels $\sigma, n, \ell, \tilde{\rho}$ have the same meaning as in the case of STSSH's of rank 1 [see the discussion after eqs. (4.38) and (4.39)]. Note that eq. (4.64) is not independent of the gamma-tracelessness condition, as it arises by contracting eq. (4.63) with γ^{ν}. As demonstrated in Appendix 4.13, by requiring our eigenmodes to be non-singular, we find the quantisation condition for the eigenvalue in eq. (4.62),

$$|\zeta_{n,N}| = n + \frac{N}{2}, \qquad n \in \mathbb{N}_0, \tag{4.65}$$

($\mathbb{N}_0$ is the set of positive integers including zero), while the allowed values for the angular momentum quantum numbers are found to be $n = 2, 3, ...$ and $\ell = 2, ..., n$.
For each value of n we have a representation of spin$(N+1)$ acting on the space of the eigenmodes $\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{+\mu\nu}$ (or $\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{-\mu\nu}$). The highest weight $\vec{\lambda} = (\lambda_1, ..., \lambda_{N/2})$ for this representation is given by [24]

$$\vec{\lambda} = \left(n + \frac{1}{2}, \frac{5}{2}, \frac{1}{2}, ..., \frac{1}{2}\right), \qquad (n = 2, 3, ...). \tag{4.66}$$

Note that for $N = 4$ we have $\vec{\lambda} = (n+1/2, 5/2)$. As in the case of STSSH's of rank 1, the two sets of eigenmodes, $\{\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{+\mu\nu}\}$ and $\{\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{-\mu\nu}\}$, form equivalent representations and they are related to each other by $\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{+\mu\nu} = \gamma^{N+1}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{-\mu\nu}$.

spin(N) content and types of eigenmodes. Equations (4.62)-(4.64) have three different types of mode solutions, namely, the **type-*I* modes**, the **type-*II* modes** and the **type-*III* modes**. The label B is used in order to indicate the type of the STSSH $\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\mu\nu}$ on S^N. In analogy with the rank-1 STSSH's discussed in Section 4.4, the rank-2 type-I modes are constructed using the eigenspinors $\chi_{\pm\ell\tilde{\rho}}$ on S^{N-1} [eq. (4.27)], while the type-II modes are constructed using the TT eigenvector-spinors $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_i}$ on S^{N-1} [eqs. (4.52) and (4.53)].

The rank-2 type-III modes are constructed using the STSSH's of rank 2 on S^{N-1} ($\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j}$), satisfying

$$\tilde{\nabla\!\!\!/}\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j} = \pm i\left(\ell+\frac{N-1}{2}\right)\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j} \tag{4.67}$$

$$\tilde{\gamma}^{\theta_i}\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j} = \tilde{\nabla}^{\theta_i}\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j} = 0, \tag{4.68}$$

$$\tilde{g}^{\theta_i\theta_j}\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j} = 0, \tag{4.69}$$

where the label $\tilde{B}$ indicates the type of the STSSH $\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j}$ on S^{N-1}. (The rank-2 STSSH's on odd-dimensional spheres are presented in Subsection 4.5.2.) We require $\ell=2,3,...$ in order for $\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j}$ to be non-singular on S^{N-1}.[7] We suppose that the STSSH's on S^{N-1}, $\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j}$, are normalised as

$$\int_{S^{N-1}}\sqrt{\tilde{g}}\,d\boldsymbol{\theta}_{N-1}\,\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j}(\boldsymbol{\theta}_{N-1})^{\dagger}\,\tilde{\psi}^{(\tilde{B}';\ell'\tilde{\rho}')\theta_i\theta_j}_{\pm}(\boldsymbol{\theta}_{N-1}) = \delta_{\ell\ell'}\delta_{\tilde{\rho}\tilde{\rho}'}\delta_{\tilde{B}\tilde{B}'}, \tag{4.70}$$

where all the $\tilde{\psi}_{+\theta_i\theta_j}$ modes are orthogonal to all the $\tilde{\psi}_{-\theta_i\theta_j}$ modes. For each value of ℓ, the set of eigenmodes $\{\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j}\}$ forms a spin(N) representation with highest weight [24]:

$$\vec{f}_2^{\pm} = \left(\ell+\frac{1}{2},\frac{5}{2},\frac{1}{2},...,\frac{1}{2},\pm\frac{1}{2}\right)$$

for N even, and

$$\vec{f}_2 = \left(\ell+\frac{1}{2},\frac{5}{2},\frac{1}{2},...,\frac{1}{2}\right)$$

for N odd.

Now let us present the explicit form of the STSSH's of rank 2 on S^N (see Appendix 4.13 for the derivation).

Type-I modes. The type-I modes with negative spin projection ($\sigma=-$) on S^N are given by

$$\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N,\boldsymbol{\theta}_{N-1}) = \begin{pmatrix}\phi^{(2)}_{n\ell}(\theta_N)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\ \pm i\psi^{(2)}_{n\ell}(\theta_N)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\end{pmatrix} \tag{4.71}$$

$$\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) = \begin{pmatrix}C^{(\uparrow)(2)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) + D^{(\uparrow)(2)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\ \\ \pm iC^{(\downarrow)(2)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \pm iD^{(\downarrow)(2)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\end{pmatrix} \tag{4.72}$$

[7] This requirement for ℓ is motivated naturally in the recursive construction of the STSSH's of rank 2 in Appendix 4.13.

$$\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) = \begin{pmatrix} K^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{g}_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \\ \pm iK^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{g}_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix} + \begin{pmatrix} W^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{H}_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) + T^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{H}'_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \\ \pm iW^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{H}_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \pm iT^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{H}'_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix}, \quad (4.73)$$

($j,k=1,...,N-1$) where $\chi_{\pm\ell\tilde{\rho}}$ are the eigenspinors on S^{N-1} [see eq. (4.69)] and we have defined

$$\tilde{H}_{\theta_j\theta_k} \equiv \tilde{\nabla}_{(\theta_j}\tilde{\nabla}_{\theta_k)} - \tilde{g}_{\theta_j\theta_k}\frac{\tilde{\Box}}{N-1}, \quad (4.74)$$

$$\tilde{H}'_{\theta_j\theta_k} \equiv \tilde{\gamma}_{(\theta_j}\tilde{\nabla}_{\theta_k)} - \tilde{g}_{\theta_j\theta_k}\frac{\tilde{\nabla\!\!\!/}}{N-1}. \quad (4.75)$$

These differential operators satisfy $\tilde{g}^{\theta_j\theta_k}\tilde{H}_{\theta_j\theta_k} = \tilde{g}^{\theta_j\theta_k}\tilde{H}'_{\theta_j\theta_k} = 0$. Note that $\tilde{\nabla\!\!\!/}\chi_{\pm\ell\tilde{\rho}} = \pm i\left(\ell+\frac{N-1}{2}\right)\chi_{\pm\ell\tilde{\rho}}$ [eq. (4.27)], while $\tilde{\Box}\chi_{\pm\ell\tilde{\rho}} \equiv \tilde{\nabla}^{\theta_k}\tilde{\nabla}_{\theta_k}\chi_{\pm\ell\tilde{\rho}}$ is given by eq. (4.206). The function $\phi^{(2)}_{n\ell}$ is given by eq. (4.31), the function $\psi^{(2)}_{n\ell}$ is given by eq. (4.32), the functions $C^{(\uparrow)(2)}_{n\ell}$ and $C^{(\downarrow)(2)}_{n\ell}$ are given by eqs. (4.47) and (4.48), respectively, while the functions $D^{(\uparrow)(2)}_{n\ell}$ and $D^{(\downarrow)(2)}_{n\ell}$ are given by eqs. (4.49) and (4.50), respectively. The functions describing the dependence on θ_N in eq. (4.73) are given by

$$K^{(\uparrow)}_{n\ell}(\theta_N) = -\frac{\sin^2\theta_N}{N-1}\phi^{(2)}_{n\ell}(\theta_N), \quad (4.76)$$

$$K^{(\downarrow)}_{n\ell}(\theta_N) = -\frac{\sin^2\theta_N}{N-1}\psi^{(2)}_{n\ell}(\theta_N), \quad (4.77)$$

$$T^{(\uparrow)}_{n\ell}(\theta_N) = \frac{-2i}{N+1}\left\{\sin\theta_N\,C^{(\uparrow)(2)}_{n\ell}(\theta_N) - \left(\ell+\frac{N-1}{2}\right)W^{(\uparrow)}_{n\ell}(\theta_N)\right\}, \quad (4.78)$$

$$T^{(\downarrow)}_{n\ell}(\theta_N) = \frac{-2i}{N+1}\left\{-\sin\theta_N\,C^{(\downarrow)(2)}_{n\ell}(\theta_N) - \left(\ell+\frac{N-1}{2}\right)W^{(\downarrow)}_{n\ell}(\theta_N)\right\}, \quad (4.79)$$

$$W^{(\uparrow)}_{n\ell}(\theta_N) = \frac{\sin\theta_N}{(\ell-1)(\ell+N)(N-1)} \times\left\{\left[\frac{N(N-3)\left(\ell+\frac{N-1}{2}\right)}{N-1} + \frac{N(N+1)}{2}\cos\theta_N\right]C^{(\uparrow)(2)}_{n\ell}(\theta_N) - (n+\frac{N}{2})(N+1)\sin\theta_N\,C^{(\downarrow)(2)}_{n\ell}(\theta_N) + \frac{N+1}{N-1}\sin\theta_N\,\phi^{(2)}_{n\ell}(\theta_N)\right\} \quad (4.80)$$

and

$$W_{n\ell}^{(\downarrow)}(\theta_N) = \frac{\sin\theta_N}{(\ell-1)(\ell+N)(N-1)}$$
$$\times \Bigg\{ \Bigg[-\frac{N(N-3)\left(\ell+\frac{N-1}{2}\right)}{N-1} + \frac{N(N+1)}{2}\cos\theta_N\Bigg] C_{n\ell}^{(\downarrow)(2)}(\theta_N)$$
$$+ (n+\frac{N}{2})(N+1)\sin\theta_N\, C_{n\ell}^{(\uparrow)(2)}(\theta_N) + \frac{N+1}{N-1}\sin\theta_N\,\psi_{n\ell}^{(2)}(\theta_N)\Bigg\}. \quad (4.81)$$

The type-I modes with positive spin projection, $\psi_{\pm\mu\nu}^{(I;+;n\ell;\tilde{\rho})}$, are given by expressions similar to the expressions for $\psi_{\pm\mu\nu}^{(I;-;n\ell;\tilde{\rho})}$. To be specific, the expression for $\psi_{\pm\theta_N\theta_N}^{(I;+;n\ell;\tilde{\rho})}$ is found by exchanging $\phi_{n\ell}^{(2)}$ and $i\psi_{n\ell}^{(2)}$ and replacing $\chi_{-\ell\tilde{\rho}}$ by $\chi_{+\ell\tilde{\rho}}$ in eq. (4.71) and the expression for the component $\psi_{\pm\theta_N\theta_j}^{(I;+;n\ell;\tilde{\rho})}$ is found using eq. (4.72) as follows: we exchange $C_{n\ell}^{(\uparrow)(2)}$ and $iC_{n\ell}^{(\downarrow)(2)}$; we also exchange $D_{n\ell}^{(\uparrow)(2)}$ and $iD_{n\ell}^{(\downarrow)(2)}$ and we make the replacements $\tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}} \to \tilde{\nabla}_{\theta_j}\chi_{+\ell\tilde{\rho}}$ and $\tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}} \to -\tilde{\gamma}_{\theta_j}\chi_{+\ell\tilde{\rho}}$. Similarly, $\psi_{\pm\theta_j\theta_k}^{(I;+;n\ell;\tilde{\rho})}$ is found using the expression for $\psi_{\pm\theta_j\theta_k}^{(I;-;n\ell;\tilde{\rho})}$ [eq. (4.73)] as follows: we exchange the functions with superscript '$(\uparrow)$' and the functions with superscript '$(\downarrow)$', i.e., $K_{n\ell}^{(\uparrow)} \leftrightarrow iK_{n\ell}^{(\downarrow)}$, $W_{n\ell}^{(\uparrow)} \leftrightarrow iW_{n\ell}^{(\downarrow)}$ and $T_{n\ell}^{(\uparrow)} \leftrightarrow iT_{n\ell}^{(\downarrow)}$ (the symbol $\leftrightarrow$ denotes the exchange of the functions appearing in the two sides of the 'left-right' arrow) and we also make the replacements $\chi_{-\ell\tilde{\rho}} \to \chi_{+\ell\tilde{\rho}}$ and $\tilde{H}'_{\theta_j\theta_k} \to -\tilde{H}'_{\theta_j\theta_k}$ in eq. (4.73).

Allowed values for angular momentum quantum numbers. Let us now verify that the allowed values for the angular momentum quantum numbers n and ℓ for the type-I modes satisfy $n \geq \ell \geq r = 2$. As in the case of STSSH's of rank 1 (see Subsection 4.4.1), the appearance of ℓ in the denominator in eqs. (4.47) and (4.48) implies that there is no type-I mode if the $\theta_N\theta_N$-component (4.71) has $\ell = 0$. Similarly, as eqs. (4.80) and (4.81) indicate, there is no type-I mode with $\theta_N\theta_N$-component given by eq. (4.71) with $\ell = 1$. Also, as demonstrated in Appendix 4.13, the quantisation condition (4.65) for the eigenvalue, as well as the condition $n - \ell \geq 0$, arise as the requirement for the absence of singularity in the functions $\phi_{n\ell}^{(2)}$ and $\psi_{n\ell}^{(2)}$. Thus, the allowed values for n and ℓ are $n = 2, 3, ...$ and $\ell = 2, ..., n$, respectively.

Type-*II* modes. The type-II modes with negative spin projection ($\sigma = -$) on S^N are given by

$$\psi_{\pm\theta_N\theta_N}^{(II\text{-}\tilde{A};-;n\ell;\tilde{\rho})}(\theta_N, \boldsymbol{\theta}_{N-1}) = 0 \quad (4.82)$$
$$\psi_{\pm\theta_N\theta_j}^{(II\text{-}\tilde{A};-;n\ell;\tilde{\rho})}(\theta_N, \boldsymbol{\theta}_{N-1}) = \begin{pmatrix} \phi_{n\ell}^{(0)}(\theta_N)\,\tilde{\psi}_{-\theta_j}^{(\tilde{A};\ell\tilde{\rho})}(\boldsymbol{\theta}_{N-1}) \\ \pm i\psi_{n\ell}^{(0)}(\theta_N)\,\tilde{\psi}_{-\theta_j}^{(\tilde{A};\ell\tilde{\rho})}(\boldsymbol{\theta}_{N-1}) \end{pmatrix} \quad (4.83)$$

$$\psi^{(II\text{-}\tilde{A};-;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) = \begin{pmatrix} \Gamma^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1}) + \Delta^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1}) \\ \\ \pm i\Gamma^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1}) \pm i\Delta^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1}) \end{pmatrix} \tag{4.84}$$

($j,k=1,...,N-1$), where $\phi^{(0)}_{n\ell}$ is given by eq. (4.31) and $\psi^{(0)}_{n\ell}$ is given by eq. (4.32). The type-$\tilde{A}$ TT vector-spinor eigenmodes $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_k}$ on S^{N-1} satisfy eqs. (4.52)-(4.54) and they are non-singular on S^{N-1} for $\ell=1,2,...$ (see Section 4.4). The functions describing the dependence on θ_N in eq. (4.84) are given by

$$\frac{\Delta^{(\uparrow)}_{n\ell}(\theta_N)}{2} = \frac{-i}{N+1}\left[-\frac{\ell+\frac{N-1}{2}}{2}\Gamma^{(\uparrow)}_{n\ell}(\theta_N) + \sin\theta_N\,\phi^{(0)}_{n\ell}(\theta_N)\right], \tag{4.85}$$

$$\frac{\Delta^{(\downarrow)}_{n\ell}(\theta_N)}{2} = \frac{-i}{N+1}\left[-\frac{\ell+\frac{N-1}{2}}{2}\Gamma^{(\downarrow)}_{n\ell}(\theta_N) - \sin\theta_N\,\psi^{(0)}_{n\ell}(\theta_N)\right] \tag{4.86}$$

and

$$\frac{\Gamma^{(\uparrow)}_{n\ell}(\theta_N)}{2} = \frac{1}{(\ell-1)(\ell+N)}\Bigg\{\sin\theta_N\left[\frac{N+1}{2}\cos\theta_N+\ell+\frac{N-1}{2}\right]\phi^{(0)}_{n\ell}(\theta_N) - \frac{N+1}{N}(n+\frac{N}{2})\sin^2\theta_N\psi^{(0)}_{n\ell}(\theta_N)\Bigg\}, \tag{4.87}$$

$$\frac{\Gamma^{(\downarrow)}_{n\ell}(\theta_N)}{2} = \frac{1}{(\ell-1)(\ell+N)}\Bigg\{\sin\theta_N\left[\frac{N+1}{2}\cos\theta_N-\ell-\frac{N-1}{2}\right]\psi^{(0)}_{n\ell}(\theta_N) + \frac{N+1}{N}(n+\frac{N}{2})\sin^2\theta_N\phi^{(0)}_{n\ell}(\theta_N)\Bigg\}. \tag{4.88}$$

The expressions for the type-II modes with positive spin projection, $\psi^{(II\text{-}\tilde{A};+;n\ell;\tilde{\rho})}_{\pm\mu\nu}$, are similar to the expressions for $\psi^{(II\text{-}\tilde{A};-;n\ell;\tilde{\rho})}_{\pm\mu\nu}$ presented above. More specifically, the expression for $\psi^{(II\text{-}\tilde{A};+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}$ is found by exchanging $\phi^{(0)}_{n\ell}$ and $i\psi^{(0)}_{n\ell}$ and making the replacement $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j} \to \tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_j}$ in eq. (4.83). The steps required in order to find the expression for $\psi^{(II\text{-}\tilde{A};+;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}$ by using eq. (4.84) are: we exchange $\Gamma^{(\uparrow)}_{n\ell}$ and $i\Gamma^{(\downarrow)}_{n\ell}$, as well as $\Delta^{(\uparrow)}_{n\ell}$ and $i\Delta^{(\downarrow)}_{n\ell}$, and we make the replacements $\tilde{\nabla}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)} \to \tilde{\nabla}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_k)}$ and $\tilde{\gamma}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)} \to -\tilde{\gamma}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_k)}$ in eq. (4.84).

Allowed values for angular momentum quantum numbers. Let us now verify that the allowed values for the angular momentum quantum numbers n and ℓ for the type-II modes satisfy $n\geq\ell\geq r=2$. As mentioned in Section 4.4, the eigenvector-spinors on

S^{N-1} ($\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}$) are non-singular for $\ell \geq 1$. Also, since $\ell - 1$ appears in the denominator in eqs. (4.87) and (4.88), there is no type-II mode with $\theta_N\theta_j$-component given by eq. (4.83) with $\ell = 1$. As in the case of the type-I modes, the quantisation condition (4.65) and the condition $n - \ell \geq 0$ arise by demanding $\phi^{(0)}_{n\ell}$ and $\psi^{(0)}_{n\ell}$ to be non-singular. Hence, the allowed values for the angular momentum quantum numbers are $n = 2, 3, ...$ and $\ell = 2, ..., n$.

Type-*III* modes. The type-III modes with negative ($\sigma = -$) and positive ($\sigma = +$) spin projections on S^N are given by

$$\psi^{(III\text{-}\tilde{B};-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N, \boldsymbol{\theta}_{N-1}) = 0 \tag{4.89}$$

$$\psi^{(III\text{-}\tilde{B};-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N, \boldsymbol{\theta}_{N-1}) = 0 \tag{4.90}$$

$$\psi^{(III\text{-}\tilde{B};-;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N, \boldsymbol{\theta}_{N-1}) = \begin{pmatrix} \phi^{(-2)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}) \\ \pm i\psi^{(-2)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}) \end{pmatrix} \tag{4.91}$$

and

$$\psi^{(III\text{-}\tilde{B};+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N, \boldsymbol{\theta}_{N-1}) = 0 \tag{4.92}$$

$$\psi^{(III\text{-}\tilde{B};+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N, \boldsymbol{\theta}_{N-1}) = 0 \tag{4.93}$$

$$\psi^{(III\text{-}\tilde{B};+;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N, \boldsymbol{\theta}_{N-1}) = \begin{pmatrix} i\psi^{(-2)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{+\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}) \\ \pm\phi^{(-2)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{+\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}) \end{pmatrix}, \tag{4.94}$$

($j, k = 1, ..., N-1$) respectively, where $\phi^{(-2)}_{n\ell}$ is given by eq. (4.31) and $\psi^{(-2)}_{n\ell}$ is given by eq. (4.32). The STSSH's of rank 2 on S^{N-1}, $\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{+\theta_j\theta_k}$, satisfy eqs. (4.67)-(4.70) and they are non-singular for $\ell = 2, 3, ...$ (see the next Subsection). By working as in the case of type-I and type-II modes discussed above, we find again that the allowed values for the angular momentum quantum numbers are $n = 2, 3, ...$ and $\ell = 2, ..., n$.

4.5.2 STSSH's of rank 2 for N odd

The equations for the STSSH's of rank 2 are given by eqs. (4.62)-(4.64), where the gamma matrices are given by eq. (4.19). We denote the STSSH's of rank 2 as $\psi^{(B;n\ell;\tilde{\rho})}_{\pm\mu\nu}$ (with $n = 2, ...$ and $\ell = 2, ..., n$), where the label B denotes the type of the mode. Note that for N odd there is no spin projection index on S^N [see also the discussion after eq. (4.57)]. The labels n, ℓ and $\tilde{\rho}$ have the same meaning as in the case of the STSSH's of rank 1 in Subsection 4.4.2.

For each value of n we have a representation of spin$(N+1)$ acting on the space of the eigenmodes $\psi^{(B;n\ell;\tilde{\rho})}_{\pm\mu\nu}$. The highest weights $\vec{\lambda}=(\lambda_1,...,\lambda_{(N+1)/2})$ for these representations are [24]

$$\vec{\lambda}^{\pm}=\left(n+\frac{1}{2},\frac{5}{2},\frac{1}{2},...,\frac{1}{2},\pm\frac{1}{2}\right),\qquad (n=2,3,...). \tag{4.95}$$

Note that for $N=3$ we have $\vec{\lambda}^{\pm}=(n+1/2,\pm5/2)$.

Type-I modes. The type-I modes on S^N are given by

$$\psi^{(I;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N,\boldsymbol{\theta}_{N-1}) =\frac{1}{\sqrt{2}}(\mathbf{1}+i\gamma^N)\left\{\phi^{(2)}_{n\ell}(\theta_N)\pm i\psi^{(2)}_{n\ell}(\theta_N)\gamma^N\right\}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \tag{4.96}$$

$$\begin{aligned}\psi^{(I;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) =&\frac{1}{\sqrt{2}}(\mathbf{1}+i\gamma^N)\Bigg\{\left(C^{(\uparrow)(2)}_{n\ell}(\theta_N)\pm iC^{(\downarrow)(2)}_{n\ell}(\theta_N)\gamma^N\right)\tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\ &+\left(D^{(\uparrow)(2)}_{n\ell}(\theta_N)\pm iD^{(\downarrow)(2)}_{n\ell}(\theta_N)\gamma^N\right)\tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\Bigg\}\end{aligned} \tag{4.97}$$

$$\begin{aligned}\psi^{(I;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) =&\frac{1}{\sqrt{2}}(\mathbf{1}+i\gamma^N)\Bigg\{\left(K^{(\uparrow)}_{n\ell}(\theta_N)\pm iK^{(\downarrow)}_{n\ell}(\theta_N)\gamma^N\right)\tilde{g}_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\ &+\left(W^{(\uparrow)}_{n\ell}(\theta_N)\pm iW^{(\downarrow)}_{n\ell}(\theta_N)\gamma^N\right)\tilde{H}_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\ &+\left(T^{(\uparrow)}_{n\ell}(\theta_N)\pm iT^{(\downarrow)}_{n\ell}(\theta_N)\gamma^N\right)\tilde{H}'_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\Bigg\}\end{aligned} \tag{4.98}$$

$(j,k=1,...,N-1)$ where the eigenspinors $\chi_{-\ell\tilde{\rho}}$ on S^{N-1} satisfy eq. (4.27). The functions $\phi^{(2)}_{n\ell},\psi^{(2)}_{n\ell},C^{(b)(2)}_{n\ell},D^{(b)(2)}_{n\ell},K^{(b)}_{n\ell},W^{(b)}_{n\ell}$ and $T^{(b)}_{n\ell}$ (where $b=\uparrow,\downarrow$), describing the dependence on θ_N, are the same as in the even-dimensional case [see eqs. (4.71)-(4.73)], while $\tilde{H}_{\theta_j\theta_k}$ and $\tilde{H}'_{\theta_j\theta_k}$ are given again by eqs. (4.74) and (4.75), respectively.

Type-II modes. The type-II modes on S^N are given by

$$\psi^{(II\text{-}\tilde{A};n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N,\boldsymbol{\theta}_{N-1})=0 \tag{4.99}$$

$$\psi^{(II\text{-}\tilde{A};n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1})=\frac{1}{\sqrt{2}}(\mathbf{1}+i\gamma^N)\left\{\phi^{(0)}_{n\ell}(\theta_N)\pm i\psi^{(0)}_{n\ell}(\theta_N)\gamma^N\right\}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}) \tag{4.100}$$

$$\begin{aligned}\psi^{(II\text{-}\tilde{A};n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1})=&\frac{1}{\sqrt{2}}(\mathbf{1}+i\gamma^N)\Bigg\{\left(\Gamma^{(\uparrow)}_{n\ell}(\theta_N)\pm i\Gamma^{(\downarrow)}_{n\ell}(\theta_N)\gamma^N\right)\tilde{\nabla}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1})\\ &+\left(\Delta^{(\uparrow)}_{n\ell}(\theta_N)\pm i\Delta^{(\downarrow)}_{n\ell}(\theta_N)\gamma^N\right)\tilde{\gamma}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1})\Bigg\},\end{aligned} \tag{4.101}$$

$(j,k=1,...,N-1)$ where the TT eigenvector-spinors $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k}$ on S^{N-1} satisfy eqs. (4.52)-(4.54). As in the even-dimensional case, the functions $\phi^{(0)}_{n\ell}$ and $\psi^{(0)}_{n\ell}$ are given by eqs. (4.31) and (4.32), respectively. The functions $\Delta^{(\uparrow)}_{n\ell}, \Delta^{(\downarrow)}_{n\ell}, \Gamma^{(\uparrow)}_{n\ell}$ and $\Gamma^{(\downarrow)}_{n\ell}$ are given by eqs. (4.85) , (4.86), (4.87) and (4.88), respectively.

Type-*III* modes. The type-III modes on S^N are given by

$$\psi^{(III\text{-}\tilde{B};n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N,\boldsymbol{\theta}_{N-1})=0 \tag{4.102}$$

$$\psi^{(III\text{-}\tilde{B};n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1})=0 \tag{4.103}$$

$$\psi^{(III\text{-}\tilde{B};n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1})=\frac{1}{\sqrt{2}}(\mathbf{1}+i\gamma^N)\Big\{\phi^{(-2)}_{n\ell}(\theta_N)\pm i\psi^{(-2)}_{n\ell}(\theta_N)\gamma^N\Big\}\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}), \tag{4.104}$$

$(j,k=1,...,N-1)$ where the rank-2 STSSH's on S^{N-1} ($\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}$) satisfy eqs. (4.67)-(4.70), while the functions $\phi^{(-2)}_{n\ell}$ and $\psi^{(-2)}_{n\ell}$ are given by eqs. (4.31) and (4.32), respectively. As in the case with N even, by requiring that the rank-2 STSSH's of all types (i.e. type-I, type-II and type-III) on S^N are non-singular, we obtain the quantisation condition (4.65) for the eigenvalue, while the allowed values for the angular momentum quantum numbers are found to be $n=2,3,...$ and $\ell=2,...,n$.

4.6 NORMALISATION FACTORS AND TRANSFORMATION PROPERTIES UNDER SPIN($N+1$) OF RANK-1 AND RANK-2 STSSH'S

In this Section, we study the transformation properties of a specific class of STSSH's of ranks 1 and 2 on S^N under a spin$(N+1)$ transformation. (This class will be specified by determining the spin(N), as well as the spin$(N-1)$, contents in the decomposition spin$(N+1)\supset$ spin$(N)\supset$ spin$(N-1)$.) We also write down explicitly the normalisation factors for all STSSH's of ranks 1 and 2 and we make a conjecture for the normalisation factors for STSSH's of arbitrary rank r.

In order to derive the transformation formulae and determine the normalisation factors for STSSH's of ranks 1 and 2, we introduce an inner product on the solution space of eqs. (4.4) and (4.5) and we also exploit the spin$(N+1)$ invariance of this inner product. The transformation properties and the normalisation factors that we present in this Section have been obtained after long and tedious calculations. For this reason, in this Section, we simply present the results of our lengthy calculations and provide the necessary mathematical background (for example, we discuss the Lie-Lorentz derivative (4.105) [27]). We refer the reader to Appendix 4.14 for details of the calculations.

4.6.1 Lie-Lorentz derivative and spin$(N+1)$ invariant inner product

Let $\psi_{\mu_1...\mu_r}$ be any tensor-spinor of rank r and ξ be any Killing vector on S^N. The infinitesimal change $\delta_\xi \psi_{\mu_1...\mu_r}$ due to the spin$(N+1)$ transformation generated by ξ is conveniently described by the **Lie-Lorentz derivative** [27]

$$\begin{aligned}\mathbb{L}_\xi\ \psi_{\mu_1...\mu_r} = \ & \xi^\nu \nabla_\nu \psi_{\mu_1...\mu_r} + \psi_{\nu\mu_2...\mu_r}\nabla_{\mu_1}\xi^\nu + \psi_{\mu_1\nu\mu_3...\mu_r}\nabla_{\mu_2}\xi^\nu + ... + \psi_{\mu_1...\mu_{r-1}\nu}\nabla_{\mu_r}\xi^\nu \\ & + \frac{1}{4}\nabla_\kappa \xi_\lambda \gamma^\kappa \gamma^\lambda \psi_{\mu_1...\mu_r}. \end{aligned} \tag{4.105}$$

The Lie-Lorentz derivative satisfies [27]

$$\mathbb{L}_\xi\ e_\mu{}^a = 0, \tag{4.106a}$$

$$\mathbb{L}_\xi\ \gamma^a = 0 \tag{4.106b}$$

and - after a straightforward calculation - one can verify that

$$\left(\mathbb{L}_\xi \nabla_\mu - \nabla_\mu \mathbb{L}_\xi\right)\ \psi_{\mu_1...\mu_r} = 0. \tag{4.107}$$

Thus, if $\psi_{\mu_1...\mu_r}$ satisfies eqs. (4.4) and (4.5) (i.e., if $\psi_{\mu_1...\mu_r}$ is a STSSH of rank r), then $\mathbb{L}_\xi\ \psi_{\mu_1...\mu_r}$ also satisfies eqs. (4.4) and (4.5). Also, the Lie-Lorentz derivative preserves the Lie bracket between two Killing vectors $[\mathbb{L}_\xi, \mathbb{L}_{\xi'}] = \mathbb{L}_{[\xi,\xi']}$, and, thus, generates a spin$(N+1)$ representation in the space of eigenmodes.

Let us introduce the following inner product on the solution space of eqs. (4.4) and (4.5):

$$\left(\psi^{(1)}, \psi^{(2)}\right)_{(r)} = \int_{S^N} \sqrt{g}\, d\boldsymbol{\theta}_N\, \psi^{(1)\dagger}_{\mu_1...\mu_r}\ \psi^{(2)\mu_1...\mu_r}, \tag{4.108}$$

where $d\boldsymbol{\theta}_N$ stands for $d\theta_N ... d\theta_2\, d\theta_1$, while $\psi^{(1)}_{\mu_1...\mu_r}$ and $\psi^{(2)}_{\mu_1...\mu_r}$ are any two STSSH's of rank r with the same angular momentum n on S^N.[8] Since the inner product (4.108) is invariant under spin$(N+1)$, we have

$$\left(\mathbb{L}_\xi \psi^{(1)}, \psi^{(2)}\right)_{(r)} + \left(\psi^{(1)}, \mathbb{L}_\xi \psi^{(2)}\right)_{(r)} = 0 \tag{4.109}$$

[8]Any two STSSH's with different signs for the eigenvalue in eq. (4.4) and/or with different n are orthogonal to each other, since $i\nabla\!\!\!/$ is hermitian with respect to the inner product (4.108).

for any Killing vector ξ on S^N.

We will study the transformation properties of a certain class of STSSH's of ranks 1 and 2 under spin$(N+1)$, by specialising to the case where the Killing vector in eq. (4.105) is given by $\xi = \mathscr{S}$

$$\mathscr{S} = \mathscr{S}^\mu \partial_\mu = \cos\theta_{N-1} \frac{\partial}{\partial \theta_N} - \cot\theta_N \, \sin\theta_{N-1} \frac{\partial}{\partial \theta_{N-1}}. \tag{4.110}$$

Now, let us discuss the certain class of STSSH's of ranks 1 and 2 on S^N $(N \geq 3)$, the transformation properties of which we are interested in.

- **STSSH's of rank $r = 1$.** We will study the transformation properties of the class of STSSH's which comprises: the type-I modes and a certain kind of type-II modes, called type-II-I modes. The type-II-I modes on S^N are defined for $N \geq 4$ and they are constructed in terms of type-I eigenvector-spinors on S^{N-1}. Thus, the type-II-I modes on S^N are given by letting $\tilde{A} = I$ in eqs. (4.55) and (4.56) (for N even) and in eq. (4.61) (for N odd).

  From a representation-theoretic viewpoint, the type-I modes correspond to the following spin(N) and spin$(N-1)$ highest weights:

  Type-I modes for N even:

  $$\vec{f}_0^{\sigma} = \left(\ell + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}, \sigma\frac{1}{2}\right) \text{ for spin}(N)$$

  (with $\sigma = \pm$) and

  $$\vec{l}_0 = \left(m + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right) \text{ for spin}(N-1),$$

  where $\vec{f}_0^{\pm}$ has $N/2$ components, while $\vec{l}_0$ has $N/2 - 1$ components. These highest weights satisfy the branching rules for spin$(N+1) \supset$ spin$(N) \supset$ spin$(N-1)$ [24, 15], which imply $n \geq \ell \geq 1$ and $\ell \geq m \geq 0$.

  Type-I modes for N odd:

  $$\vec{f}_0 = \left(\ell + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}, \frac{1}{2}\right) \text{ for spin}(N)$$

  and

  $$\vec{l}_0^{\sigma_{N-1}} = \left(m + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}, \sigma_{N-1}\frac{1}{2}\right) \text{ for spin}(N-1),$$

  where $\sigma_{N-1} = \pm$, $\vec{f}_0$ has $(N-1)/2$ components, while $\vec{l}_0^{\pm}$ also has $(N-1)/2$ components. We will call the index σ_{N-1} the 'spin projection index' on S^{N-1} $(N$

odd). Again, these highest weights satisfy the branching rules for spin$(N+1) \supset$ spin$(N) \supset$ spin$(N-1)$ [24, 15], which imply $n \geq \ell \geq 1$ and $\ell \geq m \geq 0$.

Similarly, the type-II-I modes correspond to the following spin(N) and spin$(N-1)$ highest weights:

Type-II-I modes for N even:

$$\vec{f}_1^{\sigma} = \left(\ell + \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, ..., \frac{1}{2}, \sigma\frac{1}{2}\right) \ \text{ for spin}(N)$$

(with $\sigma = \pm$) and

$$\vec{l}_0 = \left(m + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}\right) \ \text{ for spin}(N-1).$$

The weight $\vec{f}_1^{\pm}$ has $N/2$ components, while $\vec{l}_0$ has $N/2-1$ components. Again, these highest weights satisfy the branching rules for spin$(N+1) \supset$ spin$(N) \supset$ spin$(N-1)$ [24, 15], which imply $n \geq \ell \geq 1$ and $\ell \geq m \geq 1$.

Type-II-I modes for N odd:

$$\vec{f}_1 = \left(\ell + \frac{1}{2}, \frac{3}{2}, \frac{1}{2}, ..., \frac{1}{2}\right) \ \text{ for spin}(N)$$

and

$$\vec{l}_0^{\sigma_{N-1}} = \left(m + \frac{1}{2}, \frac{1}{2}, ..., \frac{1}{2}, \sigma_{N-1}\frac{1}{2}\right) \ \text{ for spin}(N-1),$$

where $\sigma_{N-1} = \pm$, while $\vec{f}_1$ has $(N-1)/2$ components, and $\vec{l}_0^{\pm}$ also has $(N-1)/2$ components. Again, these highest weights satisfy the branching rules for spin$(N+1) \supset$ spin$(N) \supset$ spin$(N-1)$ [24, 15], which imply $n \geq \ell \geq 1$ and $\ell \geq m \geq 1$.

- **STSSH's of rank $r=2$.** We will study the class of STSSH's which comprises: the type-I modes, the type-II-I modes and the type-III-I modes. As in the case of rank-1 STSSH's, the type-II-I modes on S^N are defined for $N \geq 4$ and they are constructed in terms of type-I eigenvector-spinors on S^{N-1}. Thus, these modes are given by letting $\tilde{A} = I$ in eqs. (4.82)-(4.84) (for N even) and in eqs. (4.99)-(4.101) (for N odd). The type-III-I modes on S^N are defined for $N \geq 4$ and they are constructed in terms of type-I STSSH's of rank 2 on S^{N-1}. Thus, the type-III-I modes on S^N are given by letting $\tilde{B} = I$ in eqs. (4.91) and (4.94) (for N even) and in eq. (4.104) (for N odd).

  The representation-theoretic content of this class of eigenmodes concerning the decomposition spin$(N+1) \supset$ spin$(N) \supset$ spin$(N-1)$ can be found as in the case of STSSH's of rank 1 discussed above.

4.6.2 Normalisation factors and transformation properties under spin$(N+1)$ of STSSH's of ranks 1 and 2

Case 1: N even. Using the inner product (4.108), we define the normalisation factors $c_N^{(B;r)}(n,\ell)$ for the STSSH's of arbitrary rank r and type B on S^N, $\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\mu_1\dots\mu_r}$, as

$$\left(\psi_{\pm}^{(B;\sigma;n\ell;\tilde{\rho})},\psi_{\pm}^{(B';\sigma';n\ell';\tilde{\rho}')}\right)_{(r)} \equiv \left|\frac{c_N^{(B;r)}(n,\ell)}{\sqrt{2}}\right|^{-2} \delta_{BB'}\delta_{\sigma\sigma'}\delta_{\ell\ell'}\delta_{\tilde{\rho}\tilde{\rho}'}. \tag{4.111}$$

The normalised STSSH's are $c_N^{(B;r)}(n,\ell)/\sqrt{2}\ \psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\mu_1\dots\mu_r}$.

As discussed in Sections 4.4 and 4.5 (for $r=1$ and $r=2$, respectively), the STSSH's of rank r on S^N, $\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\mu_1\dots\mu_r}$, are constructed in terms of STSSH's of rank $\tilde{r}\le r$ on S^{N-1}, using the method of separation of variables. The type of the mode $\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\mu_1\dots\mu_r}$ (i.e. the value assigned to the label B) depends on the choice of $\tilde{r}$. For convenience, instead of using the symbol $\tilde{r}$, let us denote the rank of the STSSH's on S^{N-1} as $\tilde{r}_{(B)}$, where the type-I STSSH's ($\psi^{(I;\sigma;n\ell;\tilde{\rho})}_{\pm\mu_1\dots\mu_r}$) have $\tilde{r}_{(I)}=0$, the type-II STSSH's ($\psi^{(II\text{-}\tilde{A};\sigma;n\ell;\tilde{\rho})}_{\pm\mu_1\dots\mu_r}$) have $\tilde{r}_{(II)}=1$, the type-III STSSH's ($\psi^{(III\text{-}\tilde{B};\sigma;n\ell;\tilde{\rho})}_{\pm\mu_1\dots\mu_r}$) have $\tilde{r}_{(III)}=\ 2$ and so forth. As shown in Appendix 4.14, the normalisation factors for STSSH's of rank $r\in\{1,2\}$ are given by

$$\begin{aligned}\left|\frac{c_N^{(B;r)}(n,\ell)}{\sqrt{2}}\right|^2 =\ & \frac{2^{-N-2r+1+4\tilde{r}_{(B)}}}{\binom{r}{\tilde{r}_{(B)}}}\,\frac{\Gamma(n-\ell+1)\Gamma(n+\ell+N)}{|\Gamma(n+\frac{N}{2})|^2}\\ &\times\left(\prod_{j=\tilde{r}_{(B)}}^{r-1}\frac{N+j+\tilde{r}_{(B)}-2}{N+2j-1}\right)\left(\prod_{j=\tilde{r}_{(B)}}^{r-1}(\ell-j)(\ell+N-1+j)\right)\\ &\times\prod_{j=1}^{r-\tilde{r}_{(B)}}\frac{1}{\left(n+\frac{N}{2}\right)^2-\left(r-j+\frac{N-2}{2}\right)^2}\end{aligned} \tag{4.112}$$

($\tilde{r}_{(B)}\le r$) where $\binom{r}{\tilde{r}_{(B)}}$ is the binomial coefficient. Here, if $\nu_1>\nu_2$, then $\prod_{j=\nu_1}^{\nu_2}=1$. We have proved eq. (4.112) only for $r=1$ (where $B=I,II$) and for $r=2$ (where $B=I,II,III$). We make the following conjecture, which is true for $r=1$ and $r=2$:

Conjecture: The normalisation factors for all types of STSSH's (i.e. STSSH's with all possible values of B) of arbitrary rank $r\ge 1$ on S^N are given by eq. (4.112), where $n\ge\ell\ge r\ge\tilde{r}_{(B)}$ and $\tilde{r}_{(B)}\in\{0,1,...,r\}$. (This conjecture will be useful in future attempts to extend our present study to the case with spin $s\ge 7/2$.)

Useful shorthand notation. Before presenting the transformation properties of our STSSH's of rank $r=1,2$ under spin$(N+1)$, let us introduce the shorthand notation

$\psi^{(B;\sigma;n\ell m;\rho)}_{\pm \boldsymbol{N}_r}$ for the STSSH's of ranks 1 and 2, defined as follows:

$$\psi^{(B;\sigma;n\ell m;\rho)}_{\pm \boldsymbol{N}_1} = \psi^{(B;\sigma;n\ell m;\rho)}_{\pm \mu_1} \quad (B = I,\ II\text{-}I), \tag{4.113a}$$

$$\psi^{(III\text{-}I;\sigma;n\ell m;\rho)}_{\pm \boldsymbol{N}_1} = 0, \tag{4.113b}$$

$$\psi^{(B;\sigma;n\ell m;\rho)}_{\pm \boldsymbol{N}_2} = \psi^{(B;\sigma;n\ell m;\rho)}_{\pm \mu_1\mu_2} \quad (B = I, II\text{-}I, III\text{-}I), \tag{4.113c}$$

where we have also written out explicitly the dependence on the angular momentum quantum number on S^{N-2}, m, which corresponds to ℓ on S^{N-1}. The symbol ρ represents labels other than σ, n, ℓ and m. For the type-I modes we have $m = 0, 1, ..., \ell$, for the type-II-I modes we have $m = 1, 2, ..., \ell$ and for the type-III-I modes we have $m = 2, 3, ..., \ell$. (In other words $\ell \geq m \geq \tilde{r}_{(B)}$.)

Transformation formulae for type-I modes. As demonstrated in Appendix 4.14, the spin$(N+1)$ transformation of the type-I modes is expressed as

$$\begin{aligned}\mathbb{L}_{\mathscr{S}}\psi^{(I;\sigma;n\ell m;\rho)}_{\pm \boldsymbol{N}_r} = {} & \mathscr{A}^{(I)}\, \psi^{(I;\sigma;n\,(\ell+1)\,m;\rho)}_{\pm \boldsymbol{N}_r} + \mathscr{B}^{(I)}\, \psi^{(I;\sigma;n\,(\ell-1)\,m;\rho)}_{\pm \boldsymbol{N}_r} - i\varkappa^{(I)}\, \psi^{(I;-\sigma;n\ell m;\rho)}_{\pm \boldsymbol{N}_r} \\ & + \mathscr{K}^{(I\to II)}\, \psi^{(II\text{-}I;\sigma;n\ell m;\rho)}_{\pm \boldsymbol{N}_r},\end{aligned} \tag{4.114}$$

where the coefficients on the right-hand side of eq. (4.114) are

$$\mathscr{A}^{(I)} = -\frac{(n+\ell+N)(\ell+N+r-1)}{2(\ell+\frac{N}{2})(\ell+N-1)} \times \sqrt{(\ell-m+1)(\ell+N-1+m)}, \tag{4.115}$$

$$\mathscr{B}^{(I)} = \frac{(n-\ell+1)(\ell-r)}{2(\ell+\frac{N-2}{2})\ell} \times \sqrt{(\ell-m)(\ell+m+N-2)}, \tag{4.116}$$

$$\varkappa^{(I)} = -\frac{(n+\frac{N}{2})(m+\frac{N-2}{2})(N+2r-2)}{2(\ell+\frac{N-2}{2})(\ell+\frac{N}{2})(N-2)}, \tag{4.117}$$

and

$$\mathscr{K}^{(I\to II)} = -\frac{4\left[\left(n+\frac{N}{2}\right)^2-(N-2)^2/4\right](N+r-2)}{\ell(\ell+N-1)(N-2)} \times \sqrt{\frac{N-3}{N-2}\frac{m(m+N-2)}{(\ell+1)(\ell+N-2)}}. \tag{4.118}$$

Equations (4.114)-(4.118) hold for $r = 1, 2$. Note that the sign of the spin projection index σ is flipped in the third term of the linear combination in eq. (4.114), while $i\varkappa^{(I)}$ is the only imaginary coefficient on the right-hand side of this equation. Also, note that $\mathscr{K}^{(I\to II)}$ vanishes for $m = 0$, i.e. for $m = 0$ there is no mixing between type-I and type-II-I modes in eq. (4.114). This is consistent with the fact that type-II-I modes are defined only for $m = 1, 2, ..., \ell$.

Transformation formulae for type-II-I modes. The spin$(N+1)$ transformation of the type-II-I modes is expressed as

$$\begin{aligned}\mathbb{L}_{\mathscr{S}}\psi^{(II\text{-}I;\sigma;n\ell m;\rho)}_{\pm\boldsymbol{N}_r} =& \mathscr{A}^{(II)}\,\psi^{(II\text{-}I;\sigma;n\,(\ell+1)\,m;\rho)}_{\pm\boldsymbol{N}_r} + \mathscr{B}^{(II)}\,\psi^{(II\text{-}I;\sigma;n\,(\ell-1)\,m;\rho)}_{\pm\boldsymbol{N}_r}\\ &- i\varkappa^{(II)}\psi^{(II\text{-}I;-\sigma;n\ell m;\rho)}_{\pm\boldsymbol{N}_r}\\ &+ \mathscr{K}^{(II\to I)}\,\psi^{(I;\sigma;n\ell m;\rho)}_{\pm\boldsymbol{N}_r} + \mathscr{K}^{(II\to III)}\,\psi^{(III\text{-}I;\sigma;n\ell m;\rho)}_{\pm\boldsymbol{N}_r}\end{aligned} \tag{4.119}$$

where

$$\begin{aligned}\mathscr{A}^{(II)} =& -\frac{(n+\ell+N)(\ell+N+r-1)}{2(\ell+\frac{N}{2})(\ell+N)}\\ &\times\sqrt{\frac{(\ell+2)(\ell+N-2)}{(\ell+1)(\ell+N-1)}(\ell-m+1)(\ell+m+N-1)},\end{aligned} \tag{4.120}$$

$$\mathscr{B}^{(II)} = \frac{(n-\ell+1)(\ell-r)}{2(\ell+\frac{N-2}{2})(\ell-1)}\times\sqrt{\frac{(\ell+1)(\ell+N-3)}{\ell(\ell+N-2)}(\ell-m)(\ell+m+N-2)}, \tag{4.121}$$

$$\varkappa^{(II)} = \frac{-(n+\frac{N}{2})(m+\frac{N-2}{2})(N-4)}{2(\ell+\frac{N-2}{2})(\ell+\frac{N}{2})(N-2)}\times\left(\frac{N+2}{N}\right)^{r-1} \tag{4.122}$$

$$\tag{4.123}$$

$$\mathscr{K}^{(II\to I)} = \frac{r}{4}\times\sqrt{\frac{(N-3)m(m+N-2)}{(N-2)(\ell+1)(\ell+N-2)}}, \tag{4.124}$$

where $r=1,2$ and

$$\mathscr{K}^{(II\to III)} = -\,2^3\frac{\left[\left(n+\frac{N}{2}\right)^2-N^2/4\right](N+1)}{(\ell-1)(\ell+N)N}\times\sqrt{\frac{N-2}{N}\,\frac{(m-1)(m+N-1)}{\ell(\ell+N-1)}} \tag{4.125}$$

[eq. (4.125) is defined only for $r=2$]. The sign of the spin projection index is flipped in the third term of the linear combination in eq. (4.119), while $i\varkappa^{(II)}$ is the only imaginary coefficient on the right-hand side of this equation. **Note** that $\varkappa^{(II)}$ vanishes for $N=4$ and thus type-II-I modes with different spin projections on S^4 do not mix with each other under the transformation (4.119).

Transformation formulae for type-III-I modes. The spin$(N+1)$ transformation of the rank-2 type-III-I modes is expressed as a linear combination of other STSSH's of rank 2, as follows:

$$\begin{aligned}\mathbb{L}_{\mathscr{I}}\psi^{(III\text{-}I;\sigma;n\ell m;\rho)}_{\pm\mu_1\mu_2} = \;& \mathscr{A}^{(III)}\,\psi^{(III\text{-}I;\sigma;n\,(\ell+1)\,m;\rho)}_{\pm\mu_1\mu_2} + \mathscr{B}^{(III)}\,\psi^{(III\text{-}I;\sigma;n\,(\ell-1)\,m;\rho)}_{\pm\mu_1\mu_2} \\ & - i\varkappa^{(III)}\,\psi^{(III\text{-}I;-\sigma;n\ell m;\rho)}_{\pm\mu_1\mu_2} + \mathscr{K}^{(III\to II)}\,\psi^{(II\text{-}I;\sigma;n\ell m;\rho)}_{\pm\mu_1\mu_2},\end{aligned} \tag{4.126}$$

where

$$\mathscr{A}^{(III)} = -\frac{(n+\ell+N)}{2(\ell+\frac{N}{2})} \times \sqrt{\frac{(\ell+2)(\ell+N-2)}{\ell(\ell+N)}(\ell-m+1)(\ell+m+N-1)}, \tag{4.127}$$

$$\mathscr{B}^{(III)} = \frac{(n-\ell+1)}{2(\ell+\frac{N-2}{2})} \times \sqrt{\frac{(\ell+1)(\ell+N-3)}{(\ell-1)(\ell+N-1)}(\ell-m)(\ell+m+N-2)}, \tag{4.128}$$

$$\varkappa^{(III)} = -\frac{(n+\frac{N}{2})(m+\frac{N-2}{2})(N-4)}{2(\ell+\frac{N-2}{2})(\ell+\frac{N}{2})N} \tag{4.129}$$

and

$$\mathscr{K}^{(III\to II)} = \frac{1}{4}\sqrt{\frac{(N-2)(m-1)(m+N-1)}{N\,\ell(\ell+N-1)}}. \tag{4.130}$$

As in eqs. (4.114) and (4.119), the spin projection index σ has flipped sign in the third term of the linear combination in eq. (4.126). **Note** that the STSSH's $\psi^{(III\text{-}I;-;n\ell m;\rho)}_{\pm\mu\nu}$ and $\psi^{(III\text{-}I;+;n\ell m;\rho)}_{\pm\mu\nu}$ do not mix with each other for $N=4$ since the coefficient $\varkappa^{(III)}$ [eq. (4.129)] vanishes for this value of N.

Case 2: N **odd.** As in the case with N even, the normalisation factors for the STSSH's $\psi^{(B;n\ell;\tilde{\rho})}_{\pm\mu_1\dots\mu_r}$ are defined using the inner product (4.108), as

$$\left(\psi^{(B;n\ell;\tilde{\rho})}_{\pm}, \psi^{(B';n\ell';\tilde{\rho}')}_{\pm}\right)_{(r)} \equiv \left|\frac{c^{(B;r)}_{N}(n,\ell)}{\sqrt{2}}\right|^{-2} \delta_{BB'}\delta_{\ell\ell'}\delta_{\tilde{\rho}\tilde{\rho}'}. \tag{4.131}$$

As demonstrated in Appendix 4.14, the normalisation factors for N odd are given again by eq. (4.112). The conjecture for the normalisation factors of the STSSH's in the passage below eq. (4.112) is made for both N odd and N even.

As in the case with N even, we introduce the shorthand notation $\psi^{(B;n\ell;\sigma_{N-1};m;\rho)}_{\pm \boldsymbol{N}_r}$ for the STSSH's of ranks 1 and 2, as

$$\psi^{(B;n\ell;\sigma_{N-1};m;\rho)}_{\pm \boldsymbol{N}_1} = \psi^{(B;n\ell;\sigma_{N-1};m;\rho)}_{\pm \mu_1} \quad (B = I,\ II\text{-}I), \tag{4.132a}$$

$$\psi^{(III\text{-}I;n\ell;\sigma_{N-1};m;\rho)}_{\pm \boldsymbol{N}_1} = 0, \tag{4.132b}$$

$$\psi^{(B;n\ell;\sigma_{N-1};m;\rho)}_{\pm \boldsymbol{N}_2} = \psi^{(B;n\ell;\sigma_{N-1};m;\rho)}_{\pm \mu_1 \mu_2} \quad (B = I, II\text{-}I, III\text{-}I), \tag{4.132c}$$

where we have also written out explicitly the dependence on the angular momentum quantum number on S^{N-2}, m, which corresponds to ℓ on S^{N-1}, as well as the dependence on the spin projection index on S^{N-1} ($\sigma_{N-1} = \pm$). The symbol ρ represents labels other than n, ℓ, σ_{N-1} and m.

Transformation formulae. As shown in Appendix 4.14, the spin$(N+1)$ (N odd) transformation of the type-I, type-II-I and type-III-I modes are expressed as

$$\begin{aligned} \mathbb{L}_{\mathscr{S}} \psi^{(I;n\ell;\sigma_{N-1};m;\rho)}_{\pm \boldsymbol{N}_r} = & \mathscr{A}^{(I)}\, \psi^{(I;n\,(\ell+1);\sigma_{N-1};m;\rho)}_{\pm \boldsymbol{N}_r} + \mathscr{B}^{(I)}\, \psi^{(I;n\,(\ell-1);\sigma_{N-1};m;\rho)}_{\pm \boldsymbol{N}_r} \\ & \pm i\, \sigma_{N-1}\, \varkappa^{(I)}\, \psi^{(I;n\ell;\sigma_{N-1};m;\rho)}_{\pm \boldsymbol{N}_r} + \mathscr{K}^{(I\to II)}\, \psi^{(II\text{-}I;n\ell;\sigma_{N-1};m;\rho)}_{\pm \boldsymbol{N}_r}, \end{aligned} \tag{4.133}$$

$$\begin{aligned} \mathbb{L}_{\mathscr{S}} \psi^{(II\text{-}I;n\ell;\sigma_{N-1};m;\rho)}_{\pm \boldsymbol{N}_r} = & \mathscr{A}^{(II)}\, \psi^{(II\text{-}I;n\,(\ell+1);\sigma_{N-1};m;\rho)}_{\pm \boldsymbol{N}_r} + \mathscr{B}^{(II)}\, \psi^{(II\text{-}I;n\,(\ell-1);\sigma_{N-1};m;\rho)}_{\pm \boldsymbol{N}_r} \\ & \pm i\, \sigma_{N-1}\, \varkappa^{(II)}\, \psi^{(II\text{-}I;n\ell;\sigma_{N-1};m;\rho)}_{\pm \boldsymbol{N}_r} + \mathscr{K}^{(II\to I)}\, \psi^{(I;\sigma;n\ell;\sigma_{N-1};m;\rho)}_{\pm \boldsymbol{N}_r} \\ & + \mathscr{K}^{(II\to III)}\, \psi^{(III\text{-}I;n\ell;\sigma_{N-1};m;\rho)}_{\pm \boldsymbol{N}_r}, \end{aligned} \tag{4.134}$$

and

$$\begin{aligned} \mathbb{L}_{\mathscr{S}} \psi^{(III\text{-}I;n\ell;\sigma_{N-1};m;\rho)}_{\pm \mu_1 \mu_2} = & \mathscr{A}^{(III)}\, \psi^{(III\text{-}I;n\,(\ell+1);\sigma_{N-1};m;\rho)}_{\pm \mu_1 \mu_2} + \mathscr{B}^{(III)}\, \psi^{(III\text{-}I;n\,(\ell-1);\sigma_{N-1};m;\rho)}_{\pm \mu_1 \mu_2} \\ & \pm i\, \sigma_{N-1}\, \varkappa^{(III)}\, \psi^{(III\text{-}I;n\ell;\sigma_{N-1};m;\rho)}_{\pm \mu_1 \mu_2} + \mathscr{K}^{(III\to II)}\, \psi^{(II\text{-}I;n\ell;\sigma_{N-1};m;\rho)}_{\pm \mu_1 \mu_2}, \end{aligned} \tag{4.135}$$

respectively. [In eqs. (4.133) and (4.134) we have $r \in \{1, 2\}$, while eq. (4.135) is relevant only for $r = 2$.] All coefficients in eqs. (4.133)-(4.135) are given by the same expressions as the coefficients in the case with N even [see eqs. (4.114), (4.119) and (4.126)]. Unlike the even-dimensional case, the two spin projections $\sigma_{N-1} = \pm$ do not mix with each other in eqs. (4.133)-(4.135). However, the two spin projections $\sigma_{N-1} = \pm$ mix with each other under spin(N) transformations. Note that the transformation formulae (4.134) and (4.135) are defined only for $N \geq 5$ (N odd), since type-II and type-III modes on S^N do not exist[9] for $N = 3$.

[9] This is consistent with the fact that the coefficient $\mathscr{K}^{(I\to II)}$, given by eq. (4.118), vanishes for $N = 3$.

We are now ready to analytically continue our rank-1 and rank-2 STSSH's to dS_N and study the group representation properties of the analytically continued STSSH's.

4.7 OBTAINING SPIN-3/2 AND SPIN-5/2 MODE SOLUTIONS ON N-DIMENSIONAL DE SITTER SPACETIME BY THE ANALYTIC CONTINUATION OF STSSH'S

4.7.1 Analytic continuation techniques

In this Section, we begin by discussing our analytic continuation techniques for STSSH's of arbitrary rank r and then we specialise to the cases with $r = 1$ and $r = 2$.
It is well known that dS_N can be obtained by an "analytic continuation" of S^N (see, e.g., Ref. [22]). By replacing the angle θ_N in the line element of S^N (4.8) as:

$$\theta_N \to x(t) \equiv \frac{\pi}{2} - it, \tag{4.136}$$

($t \in \mathbb{R}$) we find the line element for global dS_N:

$$ds^2 = -dt^2 + \cosh^2 t \, ds^2_{N-1}. \tag{4.137}$$

Motivated by this observation, we can obtain the field equations (4.1) and (4.2) on dS_N by analytically continuing eqs. (4.4) and (4.5), respectively, for the STSSH's on S^N. For convenience, let us give here again eqs. (4.4) and (4.5) for STSSH's on S^N:

$$\nabla\!\!\!/ \psi_{\pm\mu_1\dots\mu_r} = \pm i\left(n + \frac{N}{2}\right)\psi_{\pm\mu_1\dots\mu_r}, \quad (n = r, \, r+1, \dots) \tag{4.138}$$

$$\nabla^\alpha \psi_{\pm\alpha\mu_2\dots\mu_r} = 0, \quad \gamma^\alpha \psi_{\pm\alpha\mu_2\dots\mu_r} = 0. \tag{4.139}$$

Without loss of generality, we can choose to analytically continue the STSSH's with either one of the two signs for the eigenvalue in eq. (4.138), since each of the two sets of modes, $\{\psi_{+\mu_1\dots\mu_r}\}$ and $\{\psi_{-\mu_1\dots\mu_r}\}$, forms independently a unitary representation of spin$(N+1)$ labelled by n (see the beginning of Sections 4.4 and 4.5). Here we choose to analytically continue the STSSH's $\psi_{-\mu_1\dots\mu_r}$. By making the following replacements in eqs. (4.138) and (4.139):

$$\theta_N \to x(t) \equiv \frac{\pi}{2} - it, \qquad n \to \tilde{M} - \frac{N}{2} \qquad (t \in \mathbb{R}, \; \tilde{M} \in \mathbb{R} \setminus \{0\}) \tag{4.140}$$

we obtain eqs. (4.1) and (4.2), respectively, with imaginary mass parameter $M = i\tilde{M}$ ($\tilde{M} \neq 0$) on dS_N. Recall that we are mainly interested in field equations with imaginary mass parameter because our aim is to study strictly and partially massless representations of spin$(N,1)$, where the mass parameter takes the imaginary values (4.3). Note that the gamma matrices on S^N [eqs. (4.16) and (4.19)] transform under the replacement (4.136) as: $\gamma^N \to i\gamma^N = \gamma^0$, while the γ^j's ($j = 1, ..., N-1$) remain unchanged.[10]

Analytic continuation technicalities.

Let us now give a prescription for obtaining the explicit form of the spin-3/2 and spin-5/2 TT mode functions with mass parameter $M = i\tilde{M}$ on dS_N by analytically continuing the STSSH's of rank 1 and 2, respectively.

Functions describing the time-dependence. The functions describing the time-dependence of the analytically continued STSSH's are found by making the replacements (4.140) in the (unnormalised) functions $\phi^{(a)}_{n\ell}(\theta_N)$ [eq. (4.31)] and $\psi^{(a)}_{n\ell}(\theta_N)$ [eq. (4.32)], as

$$\hat{\phi}^{(a)}_{\tilde{M}\ell}(t) \equiv \left[\kappa_\phi\left(\tilde{M} - \frac{N}{2}, \ell\right)\right]^{-1} \phi^{(a)}_{(\tilde{M}-\frac{N}{2})\,\ell}(x(t)) \tag{4.141}$$

$$= \left(\cos\frac{x(t)}{2}\right)^{\ell+1-a} \left(\sin\frac{x(t)}{2}\right)^{\ell-a} \times F\left(-\tilde{M} + \frac{N}{2} + \ell, \tilde{M} + \ell + \frac{N}{2}; \ell + \frac{N}{2}; \sin^2\frac{x(t)}{2}\right), \tag{4.142}$$

$$\hat{\psi}^{(a)}_{\tilde{M}\ell}(t) \equiv \left[\kappa_\phi\left(\tilde{M} - \frac{N}{2}, \ell\right)\right]^{-1} \psi^{(a)}_{(\tilde{M}-\frac{N}{2})\,\ell}(x(t)) \tag{4.143}$$

$$= \frac{\tilde{M}}{\ell + \frac{N}{2}} \left(\cos\frac{x(t)}{2}\right)^{\ell-a} \left(\sin\frac{x(t)}{2}\right)^{\ell+1-a} \times F\left(-\tilde{M} + \frac{N}{2} + \ell, \tilde{M} + \ell + \frac{N}{2}; \ell + \frac{N+2}{2}; \sin^2\frac{x(t)}{2}\right), \tag{4.144}$$

where $\kappa_\phi(\tilde{M} - \frac{N}{2}, \ell)$ is given by eq. (4.33) with n replaced by $\tilde{M} - \frac{N}{2}$, while

$$\cos\frac{x(t)}{2} = \left(\sin\frac{x(t)}{2}\right)^* = \frac{\sqrt{2}}{2}\left(\cosh\frac{t}{2} + i\sinh\frac{t}{2}\right). \tag{4.145}$$

[10] Alternatively, we could analytically continue the STSSH's on S^N by making the replacement $\theta_N \to \pi/2 + it$ instead of the replacement (4.136). The analytically continued STSSH's with $\theta_N \to \pi/2 - it$ and the ones with $\theta_N \to \pi/2 + it$ are related to each other by charge conjugation. However, these two cases of analytically continued STSSH's form equivalent representations of spin$(N,1)$.

Note that $\hat{\phi}^{(a)}_{(-\tilde{M})\ell} = \hat{\phi}^{(a)}_{\tilde{M}\ell}$ and $\hat{\psi}^{(a)}_{(-\tilde{M})\ell} = -\hat{\psi}^{(a)}_{\tilde{M}\ell}$. The condition $\ell \leq n$ does not hold for dS_N. Now ℓ can be any positive integer with $\ell \geq r$.[11]

Analytic continuation of eigenmodes. For brevity, let us use again the shorthand notation introduced in eqs. (4.113) (for N even) and (4.132) (for N odd). For N even, we denote the analytically continued STSSH's as $\Psi^{(B;\sigma;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r}(t, \boldsymbol{\theta}_{N-1})$ (where $\sigma = \pm$ is the spin projection index on dS_N, while $m \leq \ell$ and $\ell = r,\, r+1, \dots$). We define the modes $\Psi^{(B;\sigma;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r}$ by making the replacements (4.140) in the STSSH's $\psi^{(B;\sigma;n\ell m;\rho)}_{-\boldsymbol{N}_r}$ on S^N, as

$$\Psi^{(B;\sigma;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r}(t, \boldsymbol{\theta}_{N-1}) = \left[\kappa_\phi\left(\tilde{M} - \frac{N}{2}, \ell\right)\right]^{-1} \psi^{\left(B;\sigma;(\tilde{M}-N/2)\,\ell m;\rho\right)}_{-\boldsymbol{N}_r}(\pi/2 - it, \boldsymbol{\theta}_{N-1}) \tag{4.146}$$

where $\left[\kappa_\phi\left(\tilde{M} - \frac{N}{2}, \ell\right)\right]^{-1}$ is essentially the factor used in eqs. (4.141) and (4.143) [it is used in order to cancel the normalisation factor (4.33) of the Jacobi polynomials]. Note that, by viewing the replacement $\theta_N \to \frac{\pi}{2} - it$ as a coordinate change, we find that $\psi^{(B;\sigma;n\ell m;\rho)}_{-\theta_N}$ transforms as

$$\psi^{(B;\sigma;n\ell m;\rho)}_{-\theta_N} \to i\,\psi^{\left(B;\sigma;(\tilde{M}-N/2)\,\ell m;\rho\right)}_{-t}.$$

Similarly, $\psi^{(B;\sigma;n\ell m;\rho)}_{-\theta_N\theta_N}$ and $\psi^{(B;\sigma;n\ell m;\rho)}_{-\theta_N\theta_j}$ transform as

$$\psi^{(B;\sigma;n\ell m;\rho)}_{-\theta_N\theta_N} \to -\psi^{\left(B;\sigma;(\tilde{M}-N/2)\,\ell m;\rho\right)}_{-t\,t}$$

and

$$\psi^{(B;\sigma;n\ell m;\rho)}_{-\theta_N\theta_j} \to i\,\psi^{\left(B;\sigma;(\tilde{M}-N/2)\,\ell m;\rho\right)}_{-t\,\theta_j},$$

respectively.

For N odd, the analytically continued STSSH's are denoted as $\Psi^{(B;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r}$ (where $\sigma_{N-1} = \pm$, $m \leq \ell$ and $\ell = r,\, r+1, \dots$). They are obtained by analytically continuing the STSSH's $\psi^{(B;n\ell;\sigma_{N-1};m;\rho)}_{-\boldsymbol{N}_r}(\theta_N, \boldsymbol{\theta}_{N-1})$ on S^N, as

$$\Psi^{(B;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r}(t, \boldsymbol{\theta}_{N-1}) = \left[\kappa_\phi\left(\tilde{M} - \frac{N}{2}, \ell\right)\right]^{-1} \psi^{\left(B;(\tilde{M}-N/2)\,\ell;\sigma_{N-1};m;\rho\right)}_{-\boldsymbol{N}_r}(\pi/2 - it, \boldsymbol{\theta}_{N-1}). \tag{4.147}$$

[11] In our previous article [25], the function $\hat{\phi}^{(a)}_{\tilde{M}\ell}(t) = \hat{\phi}^{(a)}_{(-iM)\ell}(t)$ is denoted as $\Phi^{(a)}_{M\ell}(t)$ (where $M = i\tilde{M}$). Similarly, the function $\hat{\psi}^{(a)}_{\tilde{M}\ell}(t) = \hat{\psi}^{(a)}_{(-iM)\ell}(t)$ is denoted as $\Psi^{(a)}_{M\ell}(t)$.

Note that, unlike the case with N even [eq. (4.146)], the analytically continued STSSH's (4.147) have a spin projection index (σ_{N-1}) on S^{N-1} instead of a spin projection index on dS_N.

The aforementioned analytically continued eigenmodes have been also constructed directly on dS_N using the method of separation of variables in our previous article [25], where representation-theoretic details concerning the decomposition spin$(N,1) \supset$ spin(N) can also be found.

4.7.2 PURE GAUGE MODES FOR THE STRICTLY/PARTIALLY MASSLESS SPIN-3/2 AND SPIN-5/2 THEORIES

As in Minkowski spacetime, (strictly and partially) massless field theories in dS_N are gauge invariant [9]. In terms of mode solutions of the corresponding field equations, gauge invariance manifests itself through the appearance of 'pure gauge' modes in the set of mode solutions. The 'pure gauge' modes do not describe propagating DoF of the field theory and - assuming that there exists an invariant inner product for the mode solutions - these modes have zero norm (see, e.g. Ref. [22]).

For later convenience, let us present the 'pure gauge' modes that appear among the analytically continued STSSH's of rank r ($r = 1, 2$) when we tune the imaginary mass parameter ($M = i\tilde{M}$) to the strictly/partially massless values $\tilde{M} = \pm [r - \tau + (N-2)/2]$, where $\tau = 1, .., r$ [see eq. (4.3)]. For each strictly/partially massless value of $\tilde{M}$, the analytically continued STSSH's of rank r with $r - \tau \geq \tilde{r} \geq 0$ are 'pure gauge' modes, where $\tilde{r}$ is the rank of the STSSH on S^{N-1} used in the method of separation of variables (see Sections 4.4 and 4.5). In Section 4.8 we will verify that our 'pure gauge' modes have zero norm associated to a spin$(N,1)$ invariant scalar product for N even. We will also demonstrate that for N odd there does not exist any spin$(N,1)$ invariant scalar product for the analytically continued STSSH's with imaginary mass parameter. Thus, for N odd the norm of the 'pure gauge' modes cannot be calculated in a meaningful way, as there is no de Sitter invariant notion of norm.

Strictly massless spin-3/2 field. The mass parameter for the strictly massless spin-3/2 field is given by $M = i\tilde{M} = \pm i(N-2)/2$ [this is found by letting $r = \tau = 1$ in eq. (4.3)]. The analytically continued STSSH's of type-I ($\tilde{r} = 0$) are 'pure gauge' modes. As demonstrated in Appendix 4.15, the analytically continued rank-1 STSSH's (4.146) of type-I with $\tilde{M} = \pm(N-2)/2$ are expressed in a 'pure gauge' form as follows:

$$\Psi_{\mu}^{(I;(\pm\frac{N-2}{2});\tilde{\ell})}(t,\boldsymbol{\theta}_{N-1}) = \left(\nabla_{\mu} \pm \frac{i}{2}\gamma_{\mu}\right) \Lambda_{\pm}^{(\tilde{\ell})}(t,\boldsymbol{\theta}_{N-1}), \tag{4.148}$$

where for brevity we use the symbol $\tilde{\ell}$ to represent all the labels of the analytically continued STSSH's which have not been written down explicitly. The Dirac spinors $\Lambda_{\pm}^{(\tilde{\ell})}(t,\boldsymbol{\theta}_{N-1})$ satisfy

$$\nabla\!\!\!\!/\,\Lambda_{\pm}^{(\tilde{\ell})} = \mp i\,\frac{N}{2}\,\Lambda_{\pm}^{(\tilde{\ell})}. \tag{4.149}$$

The 'pure gauge' expression (4.148) for the type-I modes coincides with the form of the infinitesimal gauge transformation [9] (with a specific gauge condition) that leaves invariant the action for the strictly massless spin-3/2 field in dS_4. In Section 4.8 we show that the 'pure gauge' modes (4.148) have vanishing dS invariant norm for even $N \geq 4$.

Strictly massless spin-5/2 field. The mass parameter for the strictly massless spin-5/2 field is given by $M = i\tilde{M} = \pm iN/2$ [this is found by letting $r = 2$ and $\tau = 1$ in eq. (4.3)]. There are two types of 'pure gauge' modes, namely the analytically continued STSSH's of type-I ($\tilde{r} = 0$) and type-II ($\tilde{r} = 1$). As demonstrated in Appendix 4.15, the analytically continued rank-2 STSSH's (4.146) of type-I and type-II with $\tilde{M} = \pm N/2$ are expressed in the following 'pure gauge' form:

$$\Psi_{\mu\nu}^{(B;(\pm\frac{N}{2});\tilde{\ell})}(t,\boldsymbol{\theta}_{N-1}) = \left(\nabla_{(\mu} \pm \frac{i}{2}\gamma_{(\mu}\right) \lambda_{\pm\nu)}^{(B;\tilde{\ell})}(t,\boldsymbol{\theta}_{N-1}), \qquad B = I, II, \tag{4.150}$$

where the gauge functions $\lambda_{\pm\mu}^{(B;\tilde{\ell})}(t,\boldsymbol{\theta}_{N-1})$ ($B = I, II$) are vector-spinor fields satisfying

$$\nabla\!\!\!\!/\,\lambda_{\pm\mu}^{(B;\tilde{\ell})} = \mp i\,\frac{N+2}{2}\,\lambda_{\pm\mu}^{(B;\tilde{\ell})} \tag{4.151}$$

$$\gamma^{\mu}\lambda_{\pm\mu}^{(B;\tilde{\ell})} = \nabla^{\mu}\lambda_{\pm\mu}^{(B;\tilde{\ell})} = 0. \tag{4.152}$$

The vector-spinors $\lambda_{\pm\mu}^{(B;\tilde{\ell})}(t,\boldsymbol{\theta}_{N-1})$ are given by the analytic continuation of rank-1 STSSH's of type-B ($B = I, II$) - see Appendix 4.15. Note that the 'pure gauge' expressions (4.150) for the type-I and type-II modes coincide with the form of the infinitesimal gauge transformation [9] (with a specific gauge condition) for the gauge-invariant action for the strictly massless spin-5/2 field in dS_4. In Section 4.8 we show that the 'pure gauge' modes (4.150) have zero (dS invariant) norm for even $N \geq 4$.

Partially massless spin-5/2 field. The mass parameter for the partially massless spin-5/2 field is given by $M = i\tilde{M} = \pm i(N-2)/2$ [this is found by letting $r = 2$ and

$\tau = 2$ in eq. (4.3)]. The analytically continued STSSH's of type-I ($\tilde{r} = 0$) are 'pure gauge' modes. As demonstrated in Appendix 4.15, the analytically continued rank-2 STSSH's (4.146) of type-I with $\tilde{M} = \pm(N-2)/2$ are expressed in a 'pure gauge' form as follows:

$$\Psi_{\mu\nu}^{(I;(\pm\frac{N-2}{2});\tilde{\ell})}(t, \boldsymbol{\theta}_{N-1}) = \left(\nabla_{(\mu}\nabla_{\nu)} \pm i\gamma_{(\mu}\nabla_{\nu)} + \frac{3}{4}g_{\mu\nu}\right)\varphi_{\pm}^{(\tilde{\ell})}(t, \boldsymbol{\theta}_{N-1}), \tag{4.153}$$

where the spinor modes $\varphi_{\pm}^{(\tilde{\ell})}(t, \boldsymbol{\theta}_{N-1})$ satisfy

$$\nabla\!\!\!/\, \varphi_{\pm}^{(\tilde{\ell})} = \mp i\,\frac{N+2}{2}\,\varphi_{\pm}^{(\tilde{\ell})}. \tag{4.154}$$

In Section 4.8 we show that the 'pure gauge' modes (4.153) have zero (dS invariant) norm for even $N \geq 4$. We note that we have not constructed a gauge-invariant action for the partially massless spin-5/2 field in dS_N with infinitesimal gauge transformation of the form (4.153). However, we call the modes (4.153) 'pure gauge' modes because we expect that such an action exists and that the expression (4.153) describes infinitesimal gauge transformations (satisfying a specific gauge condition) for this action.
In Appendix 4.15, we discuss the relation between our 'pure gauge' modes (4.153) and the gauge transformation of the partially massless spin-5/2 field in dS_4 given in Ref. [9]. More specifically, we observe the following intriguing fact: for a specific choice for the spinor gauge function in the gauge transformation used in Ref. [9], the gamma-traceless part of this gauge transformation can be expressed in our 'pure gauge' form (4.153).

4.8 (NON)UNITARITY OF THE STRICTLY/PARTIALLY MASSLESS REPRESENTATIONS OF SPIN$(N,1)$ FORMED BY THE ANALYTICALLY CONTINUED STSSH'S

For each value of the imaginary mass parameter $M = i\tilde{M}$ in eq. (4.1), the TT tensor-spinor mode solutions (i.e. the analytically continued STSSH's) form a representation of spin$(N,1)$. If one introduces a dS invariant scalar product among the analytically continued STSSH's, then the unitarity of the representation is equivalent to the positive-definiteness of the associated norm. If there is no dS invariant scalar product, then the corresponding representation of spin$(N,1)$ is, by definition, not unitary.
In this Section, we prove statements 1, 2 and 3 presented in the Introduction, which give the technical explanation of the main result of our paper (which we mention here again

for convenience): the strictly massless spin-3/2 field theory and the strictly and partially massless spin-5/2 field theories on dS_N $(N \geq 3)$ are unitary only for $N = 4$.

4.8.1 The strictly/partially massless spin-3/2 and spin-5/2 representations of spin$(N,1)$ are non-unitary for even $N > 4$

In this Subsection, we show that the representations of spin$(N,1)$ with even $N > 4$ formed by the spin-3/2 and spin-5/2 TT mode solutions of eq. (4.1) with arbitrary imaginary mass parameter $M = i\tilde{M}$ $(\tilde{M} \neq 0)$ are non-unitary (i.e. we prove statement 1). In order to arrive at this result we study the transformation properties of our analytically continued STSSH's under a spin$(N,1)$ boost and then we demonstrate the indefiniteness of the norm associated to a dS invariant scalar product for even $N > 4$. (In this Subsection we work without specifying the form of the dS invariant scalar product. Thus, our results hold for any dS invariant scalar product.) We also find that for $N = 4$ the requirement for dS invariance of the scalar product does not imply the indefiniteness of the norm if and only if the mass parameter $\tilde{M}$ is tuned to the strictly/partially massless values (4.3). Also, for even $N \geq 4$, we show that the 'pure gauge' modes in the strictly/partially massless theories with spin $s \in \{3/2, 5/2\}$ have zero norm with respect to any dS invariant scalar product. Furthermore, for $N = 4$ and $\tilde{M}$ given by eq. (4.3), we show that the TT modes in the strictly/partially massless theories are divided into two spin$(4,1)$ invariant subspaces, denoted as $\mathscr{H}^-$ and $\mathscr{H}^+$ (where each subspace contains modes with definite helicity). The positivity of the norm in each of these subspaces is shown in Subsection 4.8.2 by calculating explicitly the norms of the eigenmodes with respect to a specific dS invariant scalar product. (In Subsection 4.8.2 we also verify the results obtained in the present Subsection for even $N > 4$ by explicit calculation of the norms of the eigenmodes with arbitrary imaginary mass parameter $M = i\tilde{M} \neq 0$.)

The analytic continuation techniques introduced in Section 4.7 can also be applied to the transformation properties of the STSSH's under spin$(N+1)$. By doing so, one obtains the transformation properties of the analytically continued STSSH's on dS_N under spin$(N,1)$. Let us make the replacement (4.136) in the Killing vector $\mathscr{S}^\mu$ [eq. (4.110)] on S^N. One finds that the analytically continued version of $\mathscr{S}^\mu$ is expressed as iX^μ, where X^μ is the following boost generator of spin$(N,1)$:

$$X^\mu \partial_\mu = \cos\theta_{N-1} \frac{\partial}{\partial t} - \tanh t \, \sin\theta_{N-1} \frac{\partial}{\partial \theta_{N-1}}. \tag{4.155}$$

The de Sitter algebra spin$(N,1)$ is generated by the de Sitter boost (4.155) and the generators of spin(N).

spin$(N,1)$ transformation formulae. By making the replacements (4.140) in the spin$(N+1)$ transformation formulae (4.114), (4.119) and (4.126) [and using eq. (4.146)], we find

$$\begin{aligned}\mathbb{L}_X \Psi^{(I;\sigma;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r} = & - i\, c_{(\ell)}\, \mathscr{A}^{(I)}\, \Psi^{(I;\sigma;\tilde{M}\,(\ell+1)\,m;\rho)}_{\boldsymbol{N}_r} - \frac{i}{c_{(\ell-1)}} \mathscr{B}^{(I)}\, \Psi^{(I;\sigma;\tilde{M}\,(\ell-1)\,m;\rho)}_{\boldsymbol{N}_r} \\ & - \varkappa^{(I)}\, \Psi^{(I;-\sigma;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r} - i\mathscr{K}^{(I\to II)}\, \Psi^{(II\text{-}I;\sigma;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r},\end{aligned} \tag{4.156}$$

$$\begin{aligned}\mathbb{L}_X \Psi^{(II\text{-}I;\sigma;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r} = & - i\, c_{(\ell)} \mathscr{A}^{(II)}\, \Psi^{(II\text{-}I;\sigma;\tilde{M}\,(\ell+1)\,m;\rho)}_{\boldsymbol{N}_r} - \frac{i}{c_{(\ell-1)}} \mathscr{B}^{(II)}\, \Psi^{(II\text{-}I;\sigma;\tilde{M}\,(\ell-1)\,m;\rho)}_{\boldsymbol{N}_r} \\ & - \varkappa^{(II)} \Psi^{(II\text{-}I;-\sigma;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r} - i\mathscr{K}^{(II\to I)}\, \Psi^{(I;\sigma;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r} \\ & - i\mathscr{K}^{(II\to III)}\, \Psi^{(III\text{-}I;\sigma;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r}\end{aligned} \tag{4.157}$$

$(r=1,2,)$ and

$$\begin{aligned}\mathbb{L}_X \Psi^{(III\text{-}I;\sigma;\tilde{M}\ell m;\rho)}_{\mu_1\mu_2} = & - i\, c_{(\ell)} \mathscr{A}^{(III)}\, \Psi^{(III\text{-}I;\sigma;\tilde{M}\,(\ell+1)\,m;\rho)}_{\mu_1\mu_2} - \frac{i}{c_{(\ell-1)}} \mathscr{B}^{(III)}\, \Psi^{(III\text{-}I;\sigma;\tilde{M}\,(\ell-1)\,m;\rho)}_{\mu_1\mu_2} \\ & - \varkappa^{(III)} \Psi^{(III\text{-}I;-\sigma;\tilde{M}\ell m;\rho)}_{\mu_1\mu_2} - i\mathscr{K}^{(III\to II)}\, \Psi^{(II\text{-}I;\sigma;\tilde{M}\ell m;\rho)}_{\mu_1\mu_2},\end{aligned} \tag{4.158}$$

respectively, with

$$c_{(\ell)} = \frac{\kappa_\phi(\tilde{M}-\frac{N}{2},\ell+1)}{\kappa_\phi(\tilde{M}-\frac{N}{2},\ell)} = \frac{\tilde{M}-\ell-\frac{N}{2}}{\ell+N/2}, \tag{4.159}$$

where $\kappa_\phi(\tilde{M}-N/2,\ell)$ is found by eq. (4.33) and $\mathbb{L}_X$ is the Lie-Lorentz derivative (4.105) on dS_N. The coefficients $\mathscr{A}^{(B)}$, $\mathscr{B}^{(B)}$, $\varkappa^{(B)}$ (with $B = I,\ II,\ III$), $\mathscr{K}^{(I\to II)}$, $\mathscr{K}^{(II\to I)}$, $\mathscr{K}^{(II\to III)}$ and $\mathscr{K}^{(III\to II)}$ are found by making the replacement $n \to \tilde{M}-N/2$ in the corresponding expressions for the coefficients of STSSH's on S^N [see eqs. (4.114), (4.119) and (4.126)]. Note that we use the same symbols to represent the coefficients in the transformation formulae on S^N and the analytically continued coefficients on dS_N.

Investigating the (non-)existence of positive-definite, dS invariant scalar products.

Let $\left\langle \Psi^{(1)} | \Psi^{(2)} \right\rangle_{(r)}$ be a spin$(N,1)$ invariant scalar product for any two analytically continued rank-r STSSH's $\Psi^{(1)}_{\boldsymbol{N}_r}$, $\Psi^{(2)}_{\boldsymbol{N}_r}$ $(r=1,2)$ with imaginary mass parameter $M = i\tilde{M}$

$(\tilde{M} \neq 0)$. Due to the spin$(N,1)$ invariance of the scalar product we have

$$\left\langle \mathbb{L}_\xi \Psi^{(1)} | \Psi^{(2)}{}_{(r)} \right\rangle + \left\langle \Psi^{(1)} | \mathbb{L}_\xi \Psi^{(2)} \right\rangle_{(r)} = 0 \qquad (4.160)$$

for any Killing vector ξ on dS_N. Then, by letting $\Psi^{(1)}_{\boldsymbol{N}_r} = \Psi^{(B;-;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r}$ and $\Psi^{(2)}_{\boldsymbol{N}_r} = \Psi^{(B;+;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r}$ (with $B = I, II\text{-}I, III\text{-}I$) in eq. (4.160) with $\xi = X$ and using the transformation formulae (4.156)-(4.158), we find that the norms of eigenmodes with opposite spin projections must satisfy:

$$\varkappa^{(I)} \times \Bigg(\left\langle \Psi^{(I;-;\tilde{M}\ell m;\rho)} | \Psi^{(I;-;\tilde{M}\ell m;\rho)} \right\rangle_{(r)} + \left\langle \Psi^{(I;+;\tilde{M}\ell m;\rho)} | \Psi^{(I;+;\tilde{M}\ell m;\rho)} \right\rangle_{(r)} \Bigg) = 0, \qquad (4.161)$$

$$\varkappa^{(II)} \times \Bigg(\left\langle \Psi^{(II\text{-}I;-;\tilde{M}\ell m;\rho)} | \Psi^{(II\text{-}I;-;\tilde{M}\ell m;\rho)} \right\rangle_{(r)} + \left\langle \Psi^{(II\text{-}I;+;\tilde{M}\ell m;\rho)} | \Psi^{(II\text{-}I;+;\tilde{M}\ell m;\rho)} \right\rangle_{(r)} \Bigg) = 0, \qquad (4.162)$$

$$\varkappa^{(III)} \times \Bigg(\left\langle \Psi^{(III\text{-}I;-;\tilde{M}\ell m;\rho)} | \Psi^{(III\text{-}I;-;\tilde{M}\ell m;\rho)} \right\rangle_{(r=2)} + \left\langle \Psi^{(III\text{-}I;+;\tilde{M}\ell m;\rho)} | \Psi^{(III\text{-}I;+;\tilde{M}\ell m;\rho)} \right\rangle_{(r=2)} \Bigg) = 0. \qquad (4.163)$$

Note that, since the scalar product is also spin(N) invariant, analytically continued STSSH's of different type or/and with different values for ℓ are orthogonal to each other because they correspond to inequivalent irreducible representations of spin(N) in the decomposition spin$(N,1) \supset$ spin(N). For convenience, we give here the explicit form of the analytically continued coefficients $\varkappa^{(I)}$ [eq. (4.117)], $\varkappa^{(II)}$ [eq. (4.122)] and $\varkappa^{(III)}$ [eq. (4.129)]:

$$\varkappa^{(I)} = -\frac{\tilde{M}(m + \frac{N-2}{2})(N + 2r - 2)}{2(\ell + \frac{N-2}{2})(\ell + \frac{N}{2})(N-2)} \qquad (r = 1, 2), \qquad (4.164)$$

$$\varkappa^{(II)} = -\frac{\tilde{M}(m + \frac{N-2}{2})(N - 4)}{2(\ell + \frac{N-2}{2})(\ell + \frac{N}{2})(N-2)} \times \left(\frac{N+2}{N}\right)^{r-1} \qquad (r = 1, 2), \qquad (4.165)$$

$$\varkappa^{(III)} = -\frac{\tilde{M}(m + \frac{N-2}{2})(N - 4)}{2(\ell + \frac{N-2}{2})(\ell + \frac{N}{2})N} \qquad (4.166)$$

[eq. (4.166) is relevant only for spin-5/2 modes, i.e. only for $r = 2$]. We also give the explicit form of the analytically continued coefficients $\mathscr{K}^{(I\to II)}$ [eq. (4.118)] and $\mathscr{K}^{(II\to III)}$ [eq. (4.125)]:

$$\mathscr{K}^{(I\to II)} = -\frac{4\left(\tilde{M}^2-(N-2)^2/4\right)(N+r-2)}{\ell(\ell+N-1)(N-2)} \times \sqrt{\frac{N-3}{N-2}\frac{m(m+N-2)}{(\ell+1)(\ell+N-2)}}$$
$$(\text{where } r = 1,2), \tag{4.167}$$

$$\mathscr{K}^{(II\to III)} = -\,2^3\frac{\left(\tilde{M}^2-N^2/4\right)(N+1)}{(\ell-1)(\ell+N)N} \times \sqrt{\frac{N-2}{N}\frac{(m-1)(m+N-1)}{\ell(\ell+N-1)}}, \tag{4.168}$$

where eq. (4.168) is relevant only for $r = 2$. The analytically continued coefficients $\mathscr{K}^{(II\to I)}$ and $\mathscr{K}^{(III\to II)}$ are given by the same expressions as the coefficients on S^N, i.e. eqs. (4.124) and (4.130), respectively.

• **Cases with even $N > 4$.** Let us first discuss the case with even $N > 4$, where $\varkappa^{(I)}$, $\varkappa^{(II)}$ and $\varkappa^{(III)}$ are all non-zero (for all $\tilde{M} \neq 0$). The representation can be unitary only if eqs. (4.161)-(4.163) are consistent with the positive-definiteness of the norm. However, it is clear from eqs. (4.161)-(4.163) that the norm of the modes $\Psi^{(B;-;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r}$ is opposite of the norm of the modes $\Psi^{(B;+;\tilde{M}\ell m;\rho)}_{\boldsymbol{N}_r}$ ($B = I, II\text{-}I, III\text{-}I$) for all $\tilde{M} \neq 0$. Hence, for even $N > 4$, there are negative-norm modes for all values of $\tilde{M} \neq 0$, unless all modes have zero norm. (Not all modes could have zero norm if the field were to describe a physical particle.) Thus, we have proved statement 1.

• **dS invariance requires the norm of 'pure gauge' modes to be zero.** Before discussing the case with $N = 4$, we can show that the 'pure gauge' modes (discussed in Subsection 4.7.2), which appear among the TT mode solutions in the strictly/partially massless theories, have zero norm with respect to any dS invariant scalar product for even $N \geq 4$, as follows [23]. For the strictly massless spin-3/2 theory ($r = \tau = 1$), as well as for the partially massless spin-5/2 theory ($r = \tau = 2$), the mass parameter is $\tilde{M}^2 = (N-2)^2/4$ [see eq. (4.3)], while the type-I modes are 'pure gauge' modes. We observe that the coefficient $\mathscr{K}^{(I\to II)}$ [eq. (4.167)] vanishes for $\tilde{M}^2 = (N-2)^2/4$ (with $r = 1,2$). Then, by letting $\Psi^{(1)}_{\boldsymbol{N}_r} = \Psi^{\left(I;\sigma;(\pm\frac{N-2}{2})\ell m;\rho\right)}_{\boldsymbol{N}_r}$ and $\Psi^{(2)}_{\boldsymbol{N}_r} = \Psi^{\left(II\text{-}I;\sigma;(\pm\frac{N-2}{2})\ell m;\rho\right)}_{\boldsymbol{N}_r}$ in eq. (4.160) with $\xi = X$ and using the transformation formulae (4.156) and (4.157), we straightforwardly find

$$\left\langle \Psi^{\left(I;\sigma;(\pm\frac{N-2}{2})\ell m;\rho\right)} \middle| \Psi^{\left(I;\sigma;(\pm\frac{N-2}{2})\ell m;\rho\right)} \right\rangle_{(r)} = 0$$

(with $r = 1, 2$), i.e. the type-I modes have zero norm for even $N \geq 4$. For the strictly massless spin-5/2 theory ($r = \tau + 1 = 2$) the mass parameter is $\tilde{M}^2 = N^2/4$ [see eq. (4.3)], while both type-I and type-II modes are 'pure gauge' modes. For this value of $\tilde{M}^2$ the coefficient $\mathscr{K}^{(II \rightarrow III)}$ [eq. (4.168)] vanishes. By letting $\Psi^{(1)}_{\boldsymbol{N}_r} = \Psi^{\left(II\text{-}I;\sigma;(\pm\frac{N}{2})\ell m;\rho\right)}_{\mu_1\mu_2}$ and $\Psi^{(2)}_{\boldsymbol{N}_r} = \Psi^{\left(III\text{-}I;\sigma;(\pm\frac{N}{2})\ell m;\rho\right)}_{\mu_1\mu_2}$ in eq. (4.160) with $\xi = X$ and using the transformation formulae (4.157) (with $r = 2$) and (4.158), we find $\left\langle \Psi^{\left(II\text{-}I;\sigma;(\pm\frac{N}{2})\ell m;\rho\right)} | \Psi^{\left(II\text{-}I;\sigma;(\pm\frac{N}{2})\ell m;\rho\right)} \right\rangle_{(r=2)} = 0$. Then, by letting $\Psi^{(1)}_{\boldsymbol{N}_r} = \Psi^{\left(I;\sigma;(\pm\frac{N}{2})\ell m;\rho\right)}_{\mu_1\mu_2}$ and $\Psi^{(2)}_{\boldsymbol{N}_r} = \Psi^{\left(II\text{-}I;\sigma;(\pm\frac{N}{2})\ell m;\rho\right)}_{\mu_1\mu_2}$ in eq. (4.160) with $\xi = X$ and using the transformation formulae (4.156) (with $r = 2$) and (4.157) (with $r = 2$), we find

$$\left\langle \Psi^{\left(I;\sigma;(\pm\frac{N}{2})\ell m;\rho\right)} | \Psi^{\left(I;\sigma;(\pm\frac{N}{2})\ell m;\rho\right)} \right\rangle_{(r=2)} = 0.$$

Thus, in the strictly massless spin-5/2 theory the 'pure gauge' modes have zero norm for even $N \geq 4$.

• **The special case $N = 4$.** Let us now discuss the case with $N = 4$. First, we show that if $N = 4$, then the dS invariance of the scalar product (4.160) (with $\xi = X$) for the analytically continued STSSH's with imaginary mass parameter $M = i\tilde{M} \neq 0$ does not require indefiniteness of the norm if and only if $\tilde{M}$ is tuned to the strictly/partially massless values (4.3). This can be shown as follows. For $N = 4$ eqs. (4.162) and (4.163) are trivial due to the vanishing of $\varkappa^{(II)}$ [eq. (4.165)] and $\varkappa^{(III)}$ [eq. (4.166)], respectively. It is clear that if eq. (4.161) is not trivial, then the indefiniteness of the norm cannot be avoided. Equation (4.161) becomes trivial if we tune $\tilde{M}$ to the strictly/partially massless values (4.3) because for this value of $\tilde{M}$ the type-I modes are pure gauge (i.e. zero-norm modes). Hence, for $N = 4$ the dS invariance of the scalar product does not require the indefiniteness of the norm for the strictly/partially massless theories with spin $s \in \{3/2, 5/2\}$. Note that, since $\varkappa^{(II)}$ and $\varkappa^{(III)}$ are zero, the (non-zero-norm) eigenmodes with negative spin projection do not mix with the eigenmodes with positive spin projection under the spin$(4,1)$ boost in eqs. (4.157) and (4.158). We have also verified that (non-zero-norm) eigenmodes with different spin projections on dS_4 do not mix each other under spin(4).

According to our analysis in the previous paragraph, in the case of strictly/partially massless theories with spin $s = r + 1/2$ ($r \in \{1, 2\}$) on dS_4, we conclude the following:

- The set $\mathscr{H}^{-} = \{\Psi^{(B;-;\tilde{M}\ell;\tilde{\rho})}_{\boldsymbol{N}_r}\}$ of (non-zero-norm) TT eigenmodes with negative spin projection forms an irreducible representation of spin$(4,1)$.

- The set $\mathscr{H}^+ = \{\Psi_{\boldsymbol{N}_r}^{(B;+;\tilde{M}\ell;\tilde{\rho})}\}$ of (non-zero-norm) TT eigenmodes with positive spin projection forms separately an irreducible representation of spin$(4,1)$.[12]

Conclusion concerning the irreducibility of strictly/partially massless representations in $\boldsymbol{N = 4}$ dimensions. The two sets of eigenmodes, $\mathscr{H}^+$ and $\mathscr{H}^-$, form a direct sum of irreducible representations of spin$(4,1)$. In Subsection 4.8.2 we are going to show that these irreducible representations are unitary by demonstrating the positivity of the norm in each subspace. [As we demonstrated in our previous article [25], this is a direct sum of Discrete Series representations of spin$(4,1)$.]

Note. Note that zero-norm modes (i.e. 'pure gauge' modes) transform only into zero-norm modes under spin$(4,1)$ and they can be identified with zero, since, as we discussed above, the coefficient (4.167) (in the transformation formula (4.156) with $r \in \{1,2\}$) vanishes for $\tilde{M}^2 = (N-2)^2/4$, while the coefficient (4.168) (in the transformation formula (4.157) with $r = 2$) vanishes for $\tilde{M}^2 = N^2/4$. For the strictly massless spin-3/2 theory ($r = \tau = 1$, $\tilde{M}^2 = (N-2)^2/4$) and the partially massless spin-5/2 theory ($r = \tau = 2$, $\tilde{M}^2 = (N-2)^2/4$), where the type-I modes have zero norm, the action of spin$(4,1)$ is defined on equivalence classes of the TT modes contained in $\mathscr{H}^\sigma$ ($\sigma = \pm$) with the equivalence relation

$$\Psi_{\boldsymbol{N}_r}^{\left(B;\sigma;(\pm\frac{N-2}{2})\ell;\tilde{\rho}\right)} \sim \Psi_{\boldsymbol{N}_r}^{\left(B;\sigma;(\pm\frac{N-2}{2})\ell;\tilde{\rho}\right)} + \Psi_{\boldsymbol{N}_r}^{\left(I;\sigma';(\pm\frac{N-2}{2})\ell';\tilde{\rho}'\right)}$$

(with $B = \mathit{II}\text{-}I$ for $r = 1$ and $B = \mathit{II}\text{-}I, \mathit{III}\text{-}I$ for $r = 2$), where $\Psi_{\boldsymbol{N}_r}^{\left(I;\sigma';(\pm\frac{N-2}{2})\ell';\tilde{\rho}'\right)}$ is any type-I mode, i.e. the labels σ', ℓ' and $\tilde{\rho}'$ are no necessarily equal to σ, ℓ and $\tilde{\rho}$, respectively. For the strictly massless spin-5/2 theory ($r = \tau + 1 = 2$, $\tilde{M}^2 = N^2/4$), where both type-I and type-II-I modes have zero norm, the action of spin$(4,1)$ is defined on equivalence classes of type-III-I modes in $\mathscr{H}^\sigma$ ($\sigma = \pm$) with the equivalence relation

$$\Psi_{\mu_1\mu_2}^{\left(\mathit{III}\text{-}I;\sigma;(\pm\frac{N}{2})\ell;\tilde{\rho}\right)} \sim \Psi_{\mu_1\mu_2}^{\left(\mathit{III}\text{-}I;\sigma;(\pm\frac{N}{2})\ell;\tilde{\rho}\right)} + \Psi_{\mu_1\mu_2}^{(PG)},$$

where $\Psi_{\mu_1\mu_2}^{(PG)}$ is any (finite or infinite) linear combination of type-I and type-II modes.

Eigenmodes and helicity for strictly/partially massless theories on dS_4

For the strictly massless theories with spin $s \in \{3/2, 5/2\}$ on dS_4, the set $\mathscr{H}^-$ is identified with the set of states with 'negative helicity' ($-s$), while the set $\mathscr{H}^+$ is

[12] This situation is similar to the case of the strictly massless spin-2 field in dS_4 [19], where self-dual and anti-self-dual modes correspond to different irreducible representations of SO$(4,1)$.

identified with the set of states with 'positive helicity' $(+s)$. This can be understood as follows. As in Ref. [19], let us introduce the helicity operator $\tilde{\epsilon}_{\theta_i}{}^{\theta_j\theta_k}\tilde{\nabla}_{\theta_j}$, where $\tilde{\epsilon}_{\theta_i\theta_j\theta_k}$ is the invariant 3-form on S^3 $(i,j,k\in\{1,2,3\})$. For the strictly massless spin-3/2 theory on dS_4, where

$$\mathscr{H}^{\sigma}=\{\Psi^{(B;\sigma;\tilde{M}\ell;\tilde{\rho})}_{\boldsymbol{N}_1}\}=\{\Psi^{(\mathit{II}\text{-}I;\sigma;(\pm1)\ell;\tilde{\rho})}_{\mu}\},$$

it can readily be shown that eigenmodes with different spin projections belong to different eigenspaces of the helicity operator, as

$$\tilde{\epsilon}_{\theta_i}{}^{\theta_j\theta_k}\tilde{\nabla}_{\theta_j}\Psi^{(\mathit{II}\text{-}I;\sigma;(\pm1)\ell;\tilde{\rho})}_{\theta_k}\propto\tilde{\slashed{\nabla}}\Psi^{(\mathit{II}\text{-}I;\sigma;(\pm1)\ell;\tilde{\rho})}_{\theta_i}=i\sigma\left(\ell+\frac{3}{2}\right)\Psi^{(\mathit{II}\text{-}I;\sigma;(\pm1)\ell;\tilde{\rho})}_{\theta_i}. \tag{4.169}$$

(This equation can be readily proved using the fact that $\tilde{\epsilon}_{\theta_i\theta_j\theta_k}\propto\tilde{\gamma}_{\theta_i\theta_j\theta_k}$, where $\tilde{\gamma}_{\theta_i\theta_j\theta_k}$ is the third-rank gamma matrix on S^3 which is given by the anti-symmetrised product of three gamma matrices $\tilde{\gamma}_{\theta_i\theta_j\theta_k}=\tilde{\gamma}_{[\theta_i}\tilde{\gamma}_{\theta_j}\tilde{\gamma}_{\theta_k]}$ - see e.g. Ref. [16].) Similarly, for the strictly massless spin-5/2 theory on dS_4, where

$$\mathscr{H}^{\sigma}=\{\Psi^{(B;\sigma;\tilde{M}\ell;\tilde{\rho})}_{\boldsymbol{N}_2}\}=\{\Psi^{(\mathit{III}\text{-}I;\sigma;(\pm2)\ell;\tilde{\rho})}_{\mu\nu}\},$$

it can readily be shown that

$$\tilde{\epsilon}_{\theta_i}{}^{\theta_j\theta_k}\tilde{\nabla}_{\theta_j}\Psi^{(\mathit{III}\text{-}I;\sigma;(\pm2)\ell;\tilde{\rho})}_{\theta_k\theta_l}\propto\tilde{\slashed{\nabla}}\Psi^{(\mathit{III}\text{-}I;\sigma;(\pm2)\ell;\tilde{\rho})}_{\theta_i\theta_l}=i\sigma\left(\ell+\frac{3}{2}\right)\Psi^{(\mathit{III}\text{-}I;\sigma;(\pm2)\ell;\tilde{\rho})}_{\theta_i\theta_l}. \tag{4.170}$$

In the case of the partially massless spin-5/2 field on dS_4, where

$$\mathscr{H}^{\sigma}=\{\Psi^{(B;\sigma;\tilde{M}\ell;\tilde{\rho})}_{\boldsymbol{N}_2}\}=\{\Psi^{(\mathit{II}\text{-}I;\sigma;(\pm1)\ell;\tilde{\rho})}_{\mu\nu},\Psi^{(\mathit{III}\text{-}I;\sigma;(\pm1)\ell;\tilde{\rho})}_{\mu\nu}\},$$

the helicity operator cannot be defined in the same way. However, it is natural to identify $\mathscr{H}^-$ with the set of states with helicities $(-5/2,-3/2)$ and $\mathscr{H}^+$ with the set of states with helicities $(+5/2,+3/2)$.

Below we choose a specific dS invariant scalar product for the analytically continued STSSH's with imaginary mass parameter. By calculating the associated norms of the modes we will verify the non-unitarity of the spin$(N,1)$ representations for even $N>4$ for arbitrary imaginary mass parameter $M=i\tilde{M}$ $(\tilde{M}\neq0)$. Also, in the case of strictly/partially massless theories on dS_4, we will show that each of the spin$(4,1)$ invariant subspaces, $\mathscr{H}^-$ and $\mathscr{H}^+$, separately forms a unitary representation of spin$(4,1)$ (and, thus, we have a direct sum of UIR's of spin$(4,1)$).

4.8.2 Strictly/partially massless spin-3/2 and spin-5/2 representations of spin$(N,1)$ for N even: norms of the eigenmodes

In this Subsection, by calculating the norms of the analytically continued STSSH's explicitly, we show that the representations of spin$(N,1)$ (even $N \geq 4$) formed by the spin-3/2 and spin-5/2 TT mode solutions of eq. (4.1) with arbitrary imaginary mass parameter $M = i\tilde{M}$ ($\tilde{M} \neq 0$) are non-unitary, unless the following two conditions hold at the same time: i) $N = 4$ and ii) $\tilde{M}$ is tuned to the strictly/partially massless values (4.3). For $N = 4$, we show that the TT modes in the strictly/partially massless theories form a direct sum of UIR's of spin$(4,1)$. In other words, in the present Subsection we verify the results of Subsection 4.8.1 for even $N > 4$ and we prove statement 2.

Let $\Psi^{(1)}_{\mu_1...\mu_r}$ and $\Psi^{(2)}_{\mu_1...\mu_r}$ be any two analytically continued STSSH's [satisfying eqs. (4.1) and (4.2)] with the same imaginary mass parameter $M = i\tilde{M}$ ($\tilde{M} \neq 0$) on dS_N (N even). The (axial) vector current

$$J^{\mu} = i\,\overline{\Psi}^{(1)}_{\mu_1...\mu_r}\gamma^{\mu}\gamma^{N+1}\Psi^{(2)\mu_1...\mu_r} \tag{4.171}$$

is covariantly conserved [23], where $\overline{\Psi}^{(1)}_{\mu_1...\mu_r} = i\Psi^{(1)\dagger}_{\mu_1...\mu_r}\gamma^0$ and we used the fact that gamma matrices are covariantly constant. Then, the scalar product

$$\left\langle \Psi^{(1)}|\Psi^{(2)}\right\rangle_{(r)} = \int_{S^{N-1}} \sqrt{-g}\, d\boldsymbol{\theta}_{N-1}\, J^0 \tag{4.172}$$

is time independent, where $d\boldsymbol{\theta}_{N-1}$ stands for $d\theta_1\, d\theta_2 ... d\theta_{N-1}$, while g is the determinant of the de Sitter metric. This scalar product is equivalently written as

$$\left\langle \Psi^{(1)}|\Psi^{(2)}\right\rangle_{(r)} = \cosh^{N-1} t \int_{S^{N-1}} \sqrt{\tilde{g}}\, d\boldsymbol{\theta}_{N-1}\, \Psi^{(1)\dagger}_{\mu_1...\mu_r}\gamma^{N+1}\Psi^{(2)\mu_1...\mu_r}, \tag{4.173}$$

where we used $(\gamma^0)^2 = -\mathbf{1}$, as well as

$$\sqrt{-g} = \cosh^{N-1} t\, \sqrt{\tilde{g}}, \tag{4.174}$$

while $\sqrt{\tilde{g}}$ is given by eq. (4.29).

Now let us show that the scalar product (4.173) is de Sitter invariant. Let ξ^{μ} be a Killing vector of dS_N satisfying

$$\nabla_{\mu}\xi_{\nu} + \nabla_{\nu}\xi_{\mu} = 0. \tag{4.175}$$

The infinitesimal change $\delta_{\xi}J^{\mu}$ of the current (4.171) under the spin$(N,1)$ transformation generated by ξ^{μ} is described by the Lie derivative

$$\begin{aligned}\delta_{\xi}J^{\mu} = \mathscr{L}_{\xi}J^{\mu} &= \xi^{\nu}\nabla_{\nu}J^{\mu} - J^{\nu}\nabla_{\nu}\xi^{\mu}\\ &= \nabla_{\nu}(\xi^{\nu}J^{\mu} - J^{\nu}\xi^{\mu}),\end{aligned} \tag{4.176}$$

where we used $\nabla_\mu J^\mu = \nabla_\mu \xi^\mu = 0$. Then, it is straightforward to find

$$\delta_\xi J^0 = \frac{1}{\sqrt{-g}} \partial_{\theta_\kappa} \Big[\sqrt{-g} (\xi^{\theta_\kappa} J^0 - J^{\theta_\kappa} \xi^0) \Big], \tag{4.177}$$

where $\kappa = 1, ..., N-1$. By integrating eq. (4.177) over S^{N-1} we find that the scalar product (4.173) is de Sitter invariant, as

$$\delta_\xi \Big\langle \Psi^{(1)} | \Psi^{(2)} \Big\rangle_{(r)} = \int_{S^{N-1}} d\boldsymbol{\theta}_{N-1} \sqrt{-g}\, \delta_\xi J^0 = 0. \tag{4.178}$$

It is possible to calculate the norms of the analytically continued STSSH's of ranks 1 and 2 [the analytically continued STSSH's are defined by eq. (4.146)] using the de Sitter invariant scalar product (4.173). We find in this manner

$$\begin{aligned}
&\Big\langle \Psi^{(B;\sigma;\tilde{M}\ell;\tilde{\rho})} | \Psi^{(B';\sigma';\tilde{M}\ell';\tilde{\rho}')} \Big\rangle_{(r)} \\
&= (-\sigma) \times \binom{r}{\tilde{r}_{(B)}} 2^{N+2r-1-4\tilde{r}_{(B)}} \\
&\quad \times \frac{|\Gamma(\ell + \frac{N}{2})|^2}{\Gamma(\ell + \frac{N}{2} + \tilde{M}) \Gamma(\ell + \frac{N}{2} - \tilde{M})} \\
&\quad \times \left(\prod_{j=\tilde{r}_{(B)}}^{r-1} \frac{N+2j-1}{N+j+\tilde{r}_{(B)}-2} \right) \\
&\quad \times \left(\prod_{j=\tilde{r}_{(B)}}^{r-1} \frac{1}{(\ell-j)(\ell+N-1+j)} \right) \\
&\quad \times \left(\prod_{j=1}^{r-\tilde{r}_{(B)}} \left\{ -\tilde{M}^2 + \left(r - j + \frac{N-2}{2} \right)^2 \right\} \right) \delta_{\sigma\sigma'} \delta_{\ell\ell'} \delta_{\tilde{\rho}\tilde{\rho}'}
\end{aligned} \tag{4.179}$$

for $r \in \{1,2\}$ and $B = I, II, III$ (where $\sigma = \pm$, $\tilde{M} \in \mathbb{R} \setminus \{0\}$, $\tilde{r}_{(B)} \le r$, while $\tilde{r}_{(I)} = 0$, $\tilde{r}_{(II)} = 1$ and $\tilde{r}_{(III)} = 2$). The norms of type-I and type-II spin-3/2 modes, as well as the norms of type-II and type-III spin-5/2 modes, can be determined by direct calculation using the time-independence of the scalar product (4.173). The calculations are simplified by using

$$\Big| \hat{\phi}^{(a)}_{\tilde{M}\ell}(t=0) \Big|^2 - \Big| \hat{\psi}^{(a)}_{\tilde{M}\ell}(t=0) \Big|^2 = \frac{2^{N+2a-1} |\Gamma(\ell + \frac{N}{2})|^2}{\Gamma(\ell + \frac{N}{2} + \tilde{M}) \Gamma(\ell + \frac{N}{2} - \tilde{M})}. \tag{4.180}$$

[This equation can readily be proved using eqs. (4.192) and (4.193).] Once the norms of type-II and type-III spin-5/2 modes have been calculated, the norm of the type-I spin-5/2 modes is readily found using the dS invariance (4.160) of the inner product between type-I and type-II modes (by making use of the transformation formulae (4.156) and (4.157)).

Consistency check. As a consistency check, by using our result for the norms (4.179) of the eigenmodes with spin $s \in \{3/2, 5/2\}$, we can reproduce the strictly/partially massless tunings (4.3) for the imaginary mass parameter as follows. For $r = 1$ (spin-3/2 field), we find that the norm (4.179) of type-I modes ($\tilde{r}_{(I)} = 0$) becomes zero if $\tilde{M}^2 = (N-2)^2/4$, corresponding to the strictly massless spin-3/2 theory. For $r = 2$ (spin-5/2 field), we find that both type-I ($\tilde{r}_{(I)} = 0$) and type-II ($\tilde{r}_{(II)} = 1$) modes have zero norm (4.179) for $\tilde{M}^2 = N^2/4$, corresponding to the strictly massless spin-5/2 theory. Finally, for $r = 2$, we find that type-I ($\tilde{r}_{(I)} = 0$) modes have zero norm (4.179) for $\tilde{M}^2 = (N-2)^2/4$, corresponding to the partially massless spin-5/2 theory.

Note. We observe that the sign of the norm (4.179) depends on the sign of the spin projection index $\sigma = \pm$, as expected from the dS invariance of the scalar product (4.161)-(4.163). Thus, it is easy to understand that representations of spin$(N, 1)$ with spin $s \in \{3/2, 5/2\}$ and arbitrary imaginary mass parameter $M = i\tilde{M} \neq 0$ are non-unitary for even $N > 4$, since positive-norm and negative-norm modes mix with each other under spin$(N, 1)$ [see the transformation formulae (4.157) and (4.158)]. Similarly, we find that for $N = 4$ the representations of spin$(4, 1)$ are not unitary if $\tilde{M}$ is not given by the strictly/partially massless values in eq. (4.3).

StricIty/partially massless theories and direct sum of spin$(4, 1)$ UIRs. Now, let us suppose that the following two conditions are satisfied at the same time: i) $N = 4$ and ii) the imaginary mass parameter is tuned to the strictly/partially massless values (4.3). According to our discussion for the $N = 4$ case in Subsection 4.8.1, each of the solution subspaces, $\mathscr{H}^-$ and $\mathscr{H}^+$, forms separately an irreducible representation of spin$(4, 1)$ with spin $s = r + 1/2$ ($r \in \{1, 2\}$). (The 'pure gauge' modes are identified with zero in each subspace.) We can show that the subspaces $\mathscr{H}^-$ and $\mathscr{H}^+$ form a direct sum of UIR's of spin$(4, 1)$ as follows. By observing that the norms (4.179) of the eigenmodes depend on the spin projection, we have:

- For the set of eigenmodes with negative spin projection (or negative helicity), $\mathscr{H}^- = \{\Psi^{(B;-;\tilde{M}\ell;\tilde{\rho})}_{\boldsymbol{N}_r}\}$, the positive-definite inner product is

$$\left\langle \Psi^{(B;-;\tilde{M}\ell;\tilde{\rho})} | \Psi^{(B';-;\tilde{M}\ell';\tilde{\rho}')} \right\rangle_{(r)}$$
$$= \cosh^3 t \int_{S^3} \sqrt{\tilde{g}}\, d\boldsymbol{\theta}_3\, \Psi^{(B;-;\tilde{M}\ell;\tilde{\rho})\dagger}_{\mu_1 \dots \mu_r}\, \gamma^5\, \Psi^{(B';-;\tilde{M}\ell';\tilde{\rho}')\mu_1 \dots \mu_r}$$

  The explicit expression for the positive-definite norm is given by eq. (4.179).

- For the set of eigenmodes with positive spin projection (or positive helicity), $\mathscr{H}^{+} = \{\Psi^{(B;+;\tilde{M}\ell;\tilde{\rho})}_{\boldsymbol{N}_r}\}$, the positive-definite inner product is

$$-\Big\langle \Psi^{(B;+;\tilde{M}\ell;\tilde{\rho})}\big|\Psi^{(B';+;\tilde{M}\ell';\tilde{\rho}')}\Big\rangle_{(r)}.$$

The explicit expression for the positive-definite norm is given by the negative of eq. (4.179).

4.8.3 The strictly/partially massless spin-3/2 and spin-5/2 representations of spin$(N,1)$ are non-unitary for N odd

In this Subsection, we show that the strictly massless spin-3/2 field theory, as well as the strictly and partially massless spin-5/2 field theories, on dS_N (N odd) are not unitary (i.e. we prove statement 3).

spin$(N,1)$ transformation formulae. As in the case with N even, we study the transformation properties of the analytically continued STSSH's under the de Sitter boost (4.155). By making the replacements (4.140) in the spin$(N+1)$ transformation formulae (4.133), (4.134) and (4.135) [and using eq. (4.147)], we find

$$\begin{aligned}\mathbb{L}_X \Psi^{(I;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r} = & -i\, c_{(\ell)}\, \mathscr{A}^{(I)}\, \Psi^{(I;\tilde{M}\,(\ell+1);\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r} - \frac{i}{c_{(\ell-1)}} \mathscr{B}^{(I)}\, \Psi^{(I;\tilde{M}\,(\ell-1);\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r} \\ & - \sigma_{N-1}\, \varkappa^{(I)}\, \Psi^{(I;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r} - i\mathscr{K}^{(I\to II)}\, \Psi^{(II\text{-}I;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r},\end{aligned} \tag{4.181}$$

$$\begin{aligned}\mathbb{L}_X \Psi^{(II\text{-}I;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r} = & -i\, c_{(\ell)} \mathscr{A}^{(II)}\, \Psi^{(II\text{-}I;\tilde{M}\,(\ell+1);\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r} \\ & - \frac{i}{c_{(\ell-1)}} \mathscr{B}^{(II)}\, \Psi^{(II\text{-}I;\tilde{M}\,(\ell-1);\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r} \\ & - \sigma_{N-1}\, \varkappa^{(II)} \Psi^{(II\text{-}I;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r} - i\mathscr{K}^{(II\to I)}\, \Psi^{(I;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r} \\ & - i\mathscr{K}^{(II\to III)}\, \Psi^{(III\text{-}I;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r}\end{aligned} \tag{4.182}$$

($r = 1, 2,$) and

$$\begin{aligned}& \mathbb{L}_X \Psi^{(III\text{-}I;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\mu_1\mu_2} \\ & = -i\, c_{(\ell)} \mathscr{A}^{(III)}\, \Psi^{(III\text{-}I;\tilde{M}\,(\ell+1);\sigma_{N-1};m;\rho)}_{\mu_1\mu_2} - \frac{i}{c_{(\ell-1)}} \mathscr{B}^{(III)}\, \Psi^{(III\text{-}I;\tilde{M}\,(\ell-1);\sigma_{N-1};m;\rho)}_{\mu_1\mu_2} \\ & \quad - \sigma_{N-1}\, \varkappa^{(III)} \Psi^{(III\text{-}I;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\mu_1\mu_2} - i\mathscr{K}^{(III\to II)}\, \Psi^{(II\text{-}I;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\mu_1\mu_2},\end{aligned} \tag{4.183}$$

respectively, where all the coefficients on the right-hand sides of eqs. (4.181)-(4.183) are the same as the coefficients used in the case with N even [see eqs. (4.156)-(4.158)].

Non-existence of positive-definite, dS invariant scalar products.

Now, we will show that the representations of spin$(N,1)$ (N odd) formed by the spin-3/2 and spin-5/2 TT mode solutions of eq. (4.1) are non-unitary for all values of the imaginary mass parameter $M = i\tilde{M}$ ($\tilde{M} \neq 0$). Let $\left\langle \Psi^{(1)} | \Psi^{(2)} \right\rangle$ be a dS invariant scalar product for any two analytically continued STSSH's $\Psi^{(1)}, \Psi^{(2)}$ [satisfying eqs. (4.1) and (4.2)] with $M = i\tilde{M}$ and $\tilde{M} \neq 0$. We will show that this scalar product must vanish for all eigenmodes. First, let us make the following observation. The infinitesimal transformations $\mathbb{L}_X \Psi^{(B;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r}$ given by eqs. (4.181)-(4.183), always give rise to a term of the form $\varkappa^{(B)}\, \Psi^{(B;\tilde{M}\ell;\sigma_{N-1};m;\rho)}_{\boldsymbol{N}_r}$ in the linear combination on the right-hand sides of each of eqs. (4.181)-(4.183). The coefficients $\varkappa^{(I)}, \varkappa^{(II)}$ and $\varkappa^{(III)}$ are given by eqs. (4.164), (4.165) and (4.166), respectively, and they are all non-zero for N odd. Thus, by combining the dS invariance of the scalar product:

$$\begin{aligned} &\left\langle \mathbb{L}_X \Psi^{(B;\tilde{M}\ell;\sigma_{N-1};m;\rho)} | \Psi^{(B;\tilde{M}\ell;\sigma_{N-1};m;\rho)} \right\rangle \\ &+ \left\langle \Psi^{(B;\tilde{M}\ell;\sigma_{N-1};m;\rho)} | \mathbb{L}_X \Psi^{(B;\tilde{M}\ell;\sigma_{N-1};m;\rho)} \right\rangle = 0 \end{aligned} \tag{4.184}$$

with the transformation formulae (4.181)-(4.183), we find

$$\left\langle \Psi^{(B;\tilde{M}\ell;\sigma_{N-1};m;\rho)} | \Psi^{(B;\tilde{M}\ell;\sigma_{N-1};m;\rho)} \right\rangle = 0 \tag{4.185}$$

for $B = I, \mathit{II}\text{-}I, \mathit{III}\text{-}I$ and for all $\tilde{M} \neq 0$. Then, since the eigenmodes with different labels are orthogonal, we conclude that there is no dS invariant scalar product (which is not identically zero).

4.9 SUMMARY AND DISCUSSIONS

Summary. In this paper, we provided a technical explanation of the results of our previous article [25]. In particular, we showed that the strictly massless spin-3/2 field (i.e. gravitino field) theory, as well as the strictly and partially massless spin-5/2 field theories on dS_N ($N \geq 3$) are unitary only in $N = 4$ dimensions. In order to arrive at this result, we studied the group-theoretic properties of the eigenmodes for the following field theories with imaginary mass parameter on dS_N ($N \geq 3$): the vector-spinor field and the symmetric rank-2 tensor-spinor field. The corresponding eigenmodes satisfy eq. (4.1) with $M = i\tilde{M}$ ($\tilde{M} \neq 0$) and the TT conditions (4.2). These eigenmodes were obtained by analytically continuing STSSH's on S^N. The transformation properties of these eigenmodes under a spin$(N,1)$ boost were studied. By using these transformation properties, we showed that all dS invariant scalar products for even $N > 4$ are indefinite. We also showed that all dS invariant scalar products must vanish identically for odd N. It was found that dS invariant scalar products that are positive-definite are allowed only for strictly and partially massless theories in $N = 4$ dimensions (and, thus, these theories are unitary). Also, for these unitary spin-s ($s \in \{3/2, 5/2\}$) theories in dS_4, we showed that eigenmodes with positive helicity and the ones with negative helicity separately form UIR's of spin$(4,1)$. All the results mentioned in this paragraph are summarised as statements 1, 2 and 3 in the Introduction.

Towards future work. It would also be interesting to investigate whether our result about the non-unitarity of the gauge-invariant spin-3/2 and spin-5/2 theories on dS_N for $N \neq 4$ could be extended to other N-dimensional vacuum spacetimes with positive cosmological constant. As an argument pointing towards the possible generalisation of our result, we would like to mention the forbidden mass range for the symmetric spin-2 field on dS_N [22, 18]. The forbidden mass range for the symmetric spin-2 field on dS_N was explained group-theoretically in Ref. [22] and it was first observed for dS_4 in Ref. [18]. However, it was later shown that the forbidden mass range exists in all 4-dimensional vacuum spacetimes with positive cosmological constant [20].

Acknowledgments. The author is grateful to Atsushi Higuchi for guidance, suggestions and encouragement, as well as for invaluable discussions and ideas that inspired the majority of the material presented in this paper and helpful comments on earlier versions of this paper. Also, it is a pleasure to thank Stanley Deser for communications and Andrew Waldron for useful correspondence. The author would also like to thank Lasse Schmieding, Nikolaos Koutsonikos-Kouloumpis and F. F. John for useful discussions. This work was supported by a studentship from the Department of Mathematics, University of York.

4.10 APPENDIX A - RAISING AND LOWERING OPERATORS FOR THE GAUSS HYPERGEOMETRIC FUNCTION AND OTHER USEFUL FORMULAE

The Gauss hypergeometric function $F(a,b;c;z)$ satisfies [14]

$$\frac{d}{dz}F(a,b;c;z) = \frac{ab}{c}F(a+1,b+1;c+1;z), \tag{4.186}$$

$$\left(z\frac{d}{dz}+c-1\right)F(a,b;c;z) = (c-1)F(a,b;c-1;z), \tag{4.187}$$

$$\left(z\frac{d}{dz}+a\right)F(a,b;c;z) = aF(a+1,b;c;z). \tag{4.188}$$

By combining eq. (4.188) with the following relation [29]:

$$(c-b)F(a+1,b-1;c;z)+(b-a-1)(1-z)F(a+1,b;c;z) = (c-a-1)F(a,b;c;z), \tag{4.189}$$

we find

$$\begin{aligned}\left(a(b-c)+a(-b+a+1)z-(-b+a+1)z(1-z)\frac{d}{dz}\right)F(a,b;c;z)\\ = a(b-c)F(a+1,b-1;c;z).\end{aligned} \tag{4.190}$$

Using eqs. (4.186) and (4.187) we can show the ladder relations (4.286) and (4.287), while using eq. (4.190) we can show the ladder relations (4.288) and (4.289).
The behaviour of the functions (4.31) and (4.32) in the limit $\theta_N \to \pi$ is studied by using the transformation formula [17]

$$\begin{aligned}F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\,F(\alpha,\beta;\alpha+\beta-\gamma+1;1-z)\\ +(1-z)^{\gamma-\alpha-\beta}\frac{\Gamma(\gamma)\Gamma(-\gamma+\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}F(\gamma-\alpha,\gamma-\beta;\gamma-\alpha-\beta+1;1-z).\end{aligned} \tag{4.191}$$

Equation (4.180) is proved using [1]

$$F\left(a,b,\frac{a+b}{2};\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma\left(\frac{a+b}{2}\right)\left[\frac{1}{\Gamma((a+1)/2)\Gamma(b/2)}+\frac{1}{\Gamma((b+1)/2)\Gamma(a/2)}\right] \tag{4.192}$$

and [2]

$$F\left(a,b,\frac{a+b}{2}+1;\frac{1}{2}\right)=\frac{2\sqrt{\pi}}{a-b}\Gamma\left(\frac{a+b}{2}+1\right)\times\left[\frac{1}{\Gamma((b+1)/2)}\frac{1}{\Gamma(a/2)}-\frac{1}{\Gamma((a+1)/2)\Gamma(b/2)}\right]. \quad (4.193)$$

4.11 APPENDIX B - SPINOR EIGENMODES OF THE DIRAC OPERATOR ON S^{N-1}

The spinor eigenmodes of the Dirac operator (i.e. the STSSH's of rank 0) on spheres of arbitrary dimension have been computed in Ref. [5]. Here we write down explicitly the eigenspinors on S^{N-1} that satisfy eq. (4.27). These eigenspinors play an important role in the derivation of the formulae for the spin$(N+1)$ transformation of the STSSH's in Appendix 4.14.

Case 1: $N-1$ odd. We denote the eigenspinors on S^{N-1} as $\chi_{\pm\ell m\rho}(\theta_{N-1},\boldsymbol{\theta}_{N-2})$, where ρ stands for labels other than ℓ and m. These eigenspinors are given by

$$\chi_{\pm\ell m\rho}(\theta_{N-1},\boldsymbol{\theta}_{N-2})=\frac{\tilde{c}_{N-1}(\ell,m)}{\sqrt{2}}\left\{\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2})\pm i\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right\}, \quad (4.194)$$

where $\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})$ and $\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})$ are given by eqs. (4.273) and (4.274), respectively, and

$$\hat{\tilde{\chi}}_{\pm m\rho}(\boldsymbol{\theta}_{N-2})=\frac{1}{\sqrt{2}}(1+i\tilde{\gamma}^{N-1})\tilde{\chi}_{\pm m\rho}(\boldsymbol{\theta}_{N-2}), \quad (4.195)$$

$$\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})=\tilde{\gamma}^{N-1}\tilde{\chi}_{-m\rho}(\boldsymbol{\theta}_{N-2}), \quad (4.196)$$

where the spinors $\tilde{\chi}_{\pm m\rho}(\boldsymbol{\theta}_{N-2})$ are the eigenspinors of the Dirac operator on S^{N-2}. [The gamma matrices on S^{N-1} are denoted as $\tilde{\gamma}^a$ - see eq. (4.16).] In order for the eigenspinors (4.194) to be non-singular we require $\ell\geq m$ and $\ell=0,1,...$ [5]. The eigenspinors (4.194) satisfy the normalisation condition (4.28), while the normalisation factor is given by [5]

$$\left|\frac{\tilde{c}_{N-1}(\ell,m)}{\sqrt{2}}\right|^2=\frac{\Gamma(\ell-m+1)\Gamma(\ell+N-1+m)}{2^{N-2}|\Gamma(\frac{N-1}{2}+\ell)|^2}. \quad (4.197)$$

Case 2: $N-1$ even. We denote the eigenspinors on S^{N-1} as $\chi^{(\sigma_{N-1})}_{\pm\ell m\rho}(\theta_{N-1},\boldsymbol{\theta}_{N-2})$, where $\sigma_{N-1}=\pm$ is the spin projection index on S^{N-1} and ρ stands for labels other than

σ_{N-1}, ℓ and m. The eigenspinors with negative spin projection are given by

$$\chi^{(-)}_{\pm\ell m\rho}(\theta_{N-1}, \boldsymbol{\theta}_{N-2}) = \frac{\tilde{c}_{N-1}(\ell, m)}{\sqrt{2}} \begin{pmatrix} \tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})\, \tilde{\chi}_{-m\rho}(\boldsymbol{\theta}_{N-2}) \\ \pm i\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})\, \tilde{\chi}_{-m\rho}(\boldsymbol{\theta}_{N-2}) \end{pmatrix} \tag{4.198}$$

and those with positive spin projection are given by

$$\chi^{(+)}_{\pm\ell m\rho}(\theta_{N-1}, \boldsymbol{\theta}_{N-2}) = \frac{\tilde{c}_{N-1}(\ell, m)}{\sqrt{2}} \begin{pmatrix} i\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})\, \tilde{\chi}_{+m\rho}(\boldsymbol{\theta}_{N-2}) \\ \pm\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})\, \tilde{\chi}_{+m\rho}(\boldsymbol{\theta}_{N-2}) \end{pmatrix} \tag{4.199}$$

and they both satisfy eq. (4.27). The normalisation factors $\tilde{c}_{N-1}(\ell, m)$, as well as the functions $\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})$ and $\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})$, have the same expressions as in the case with $N-1$ odd.

4.12 APPENDIX C - SOME USEFUL FORMULAE ON S^{N-1}

Let $\tilde{g}_{\mu\nu}$ be the metric tensor on S^{N-1}. The Riemann tensor on S^{N-1} is

$$\tilde{R}_{\mu\nu\kappa\lambda} = \tilde{g}_{\mu\kappa}\tilde{g}_{\nu\lambda} - \tilde{g}_{\nu\kappa}\tilde{g}_{\mu\lambda}. \tag{4.200}$$

Let $\tilde{\psi}, \tilde{\psi}_\mu$ and $\tilde{\psi}_{\mu\nu}$ be any spinor, vector-spinor and rank-2 tensor-spinor field, respectively, on S^{N-1}. The commutator of covariant derivatives acting on these fields is given by

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]\tilde{\psi} = \frac{1}{4}\tilde{R}_{\mu\nu\kappa\lambda}\tilde{\gamma}^\kappa\tilde{\gamma}^\lambda\tilde{\psi} \tag{4.201}$$

$$= \frac{1}{2}(\tilde{\gamma}_\mu\tilde{\gamma}_\nu - \tilde{g}_{\mu\nu})\tilde{\psi}, \tag{4.202}$$

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]\tilde{\psi}_\alpha = \frac{1}{4}\tilde{R}_{\mu\nu\kappa\lambda}\tilde{\gamma}^\kappa\tilde{\gamma}^\lambda\tilde{\psi}_\alpha + \tilde{R}^\lambda{}_{\alpha\nu\mu}\tilde{\psi}_\lambda \tag{4.203}$$

$$= \frac{1}{2}(\tilde{\gamma}_\mu\tilde{\gamma}_\nu - \tilde{g}_{\mu\nu})\tilde{\psi}_\alpha + 2\tilde{g}_{\alpha[\mu}\tilde{\psi}_{\nu]}, \tag{4.204}$$

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu]\tilde{\psi}_{\alpha\beta} = \frac{1}{2}(\tilde{\gamma}_\mu\tilde{\gamma}_\nu - \tilde{g}_{\mu\nu})\tilde{\psi}_{\alpha\beta} + 2\tilde{g}_{\alpha[\mu}\tilde{\psi}_{\nu]\beta} + 2\tilde{\psi}_{\alpha[\nu}\tilde{g}_{\mu]\beta}. \tag{4.205}$$

The Laplace-Beltrami operator on S^{N-1} is defined as $\tilde{\Box} \equiv \tilde{g}^{\kappa\lambda}\tilde{\nabla}_\kappa\tilde{\nabla}_\lambda$. The eigenspinors on S^{N-1} [see eq. (4.27)] satisfy [5]

$$\begin{aligned} \tilde{\Box}\chi_{\pm\ell\tilde{\rho}} &= \left[\tilde{\nabla\!\!\!/}^2 + \frac{(N-1)(N-2)}{4}\right]\chi_{\pm\ell\tilde{\rho}} \\ &= \left[-\left(\ell + \frac{N-1}{2}\right)^2 + \frac{(N-1)(N-2)}{4}\right]\chi_{\pm\ell\tilde{\rho}}. \end{aligned} \tag{4.206}$$

Note also the following relations:

$$\tilde{\gamma}^{\theta_i}\tilde{\nabla}_{(\theta_i}\tilde{\nabla}_{\theta_j)}\chi_{\pm\ell\tilde{\rho}} = \pm i\left(\ell+\frac{N-1}{2}\right)\tilde{\nabla}_{\theta_j}\chi_{\pm\ell\rho}+\frac{N-2}{4}\tilde{\gamma}_{\theta_j}\chi_{\pm\ell\tilde{\rho}}, \tag{4.207}$$

$$\tilde{\gamma}^{\theta_i}\tilde{\gamma}_{(\theta_i}\tilde{\nabla}_{\theta_j)}\chi_{\pm\ell\tilde{\rho}} = \frac{N+1}{2}\tilde{\nabla}_{\theta_j}\chi_{\pm\ell\rho}\mp i\frac{\ell+\frac{N-1}{2}}{2}\tilde{\gamma}_{\theta_j}\chi_{\pm\ell\tilde{\rho}}, \tag{4.208}$$

$$\tilde{\nabla}^{\theta_i}\tilde{\nabla}_{(\theta_i}\tilde{\nabla}_{\theta_j)}\chi_{\pm\ell\tilde{\rho}} = \tilde{\nabla}_{\theta_j}\left(\tilde{\Box}+N-\frac{5}{4}\right)\chi_{\pm\ell\tilde{\rho}}\mp\frac{3}{4}i\left(\ell+\frac{N-1}{2}\right)\tilde{\gamma}_{\theta_j}\chi_{\pm\ell\tilde{\rho}}, \tag{4.209}$$

$$\tilde{\nabla}^{\theta_i}\tilde{\gamma}_{(\theta_i}\tilde{\nabla}_{\theta_j)}\chi_{\pm\ell\tilde{\rho}} = \pm i\frac{\ell+\frac{N-1}{2}}{2}\tilde{\nabla}_{\theta_j}\chi_{\pm\ell\tilde{\rho}}+\frac{1}{2}\tilde{\gamma}_{\theta_j}\left(\tilde{\Box}+\frac{N-2}{2}\right)\chi_{\pm\ell\tilde{\rho}}, \tag{4.210}$$

where in order to prove eqs. (4.207) and (4.210) we have to use eq. (4.202), while in order to prove eq. (4.209) we have to use eqs. (4.202) and (4.204).
The TT vector-spinor eigenmodes [see eqs. (4.52)-(4.53)] satisfy

$$\tilde{\Box}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j} = \left[-\left(\ell+\frac{N-1}{2}\right)^2+\frac{(N-1)(N-2)}{4}+1\right]\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j} \tag{4.211}$$

($j=1,...,N-1$). By combining this equation with eq. (4.204) we can prove the following relation:

$$\begin{aligned}\tilde{\nabla}^{\theta_i}\tilde{\nabla}_{(\theta_i}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_k)} &= \frac{1}{2}\left(\tilde{\Box}+N-\frac{3}{2}\right)\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_k}\\ &= \frac{1}{2}\left(\tilde{\nabla}^2+\frac{N(N+1)}{4}\right)\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_k}.\end{aligned} \tag{4.212}$$

The rank-2 STSSH's on S^{N-1} [see eqs. (4.67)- (4.69)] satisfy

$$\tilde{\Box}\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_j\theta_k} = \left[-\left(\ell+\frac{N-1}{2}\right)^2+\frac{(N-1)(N-2)}{4}+2\right]\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_j\theta_k} \tag{4.213}$$

($j,k=1,...,N-1$).

4.13 APPENDIX D - CONSTRUCTING STSSH'S OF RANK 2 ON THE N-SPHERE

In this Appendix, we construct the STSSH's of rank 2 on S^N. These STSSH's satisfy eqs. (4.62)-(4.64) and we construct them explicitly by using the method of separation of variables in geodesic polar coordinates (4.7), as in Refs. [5, 6]. In the method of separation of variables, the STSSH's of rank 2 on S^N are expressed in terms of STSSH's of rank $\tilde{r}$ (with $\tilde{r}=0,1,2$) on S^{N-1}.

For later convenience, note that the functions $\phi^{(a)}_{n\ell}(\theta_N)$ [eq. (4.31)] satisfy the following differential equation:

$$D_{(a)}\phi^{(a)}_{n\ell}(\theta_N) = -\zeta^2_{n,N}\phi^{(a)}_{n\ell}(\theta_N), \tag{4.214}$$

where $\zeta^2_{n,N} \equiv \zeta^2 = (n+\frac{N}{2})^2$ is the eigenvalue of the STSSH in eq. (4.4), while the differential operator is given by

$$\begin{aligned} D_{(a)} = \frac{\partial^2}{\partial\theta_N^2} + (N+2a-1)\cot\theta_N\frac{\partial}{\partial\theta_N} + \left(\ell+\frac{N-1}{2}\right)\frac{\cos\theta_N}{\sin^2\theta_N} \\ - \frac{(\ell+\frac{N-1}{2})^2 - \frac{1}{4}(N+2a-1)(N+2a-3)}{\sin^2\theta_N} - \frac{(N+2a-1)^2}{4}. \end{aligned} \tag{4.215}$$

One can readily verify that the functions $\phi^{(a)}_{n\ell}(\theta_N)$ [eq. (4.31)] are the unique regular solutions (up to a normalisation constant) of the differential equation (4.214) by using the results of Ref. [5], as follows. By expressing $\phi^{(a)}_{n\ell}$ as

$$\phi^{(a)}_{n\ell}(\theta_N) = \left(\sin\frac{\theta_N}{2}\cos\frac{\theta_N}{2}\right)^{-a}\phi^{(0)}_{n\ell}(\theta_N) \tag{4.216}$$

[see eq. (4.31)] we rewrite eq. (4.214) as $D_{(0)}\phi^{(0)}_{n\ell} = -\zeta^2_{n,N}\phi^{(0)}_{n\ell}$. The latter has been solved in Ref. [5] and it was found that the unique regular solutions $\phi^{(0)}_{n\ell}$ are the ones given by eq. (4.31) (with $a=0$). For the rank-1 STSSH's on S^N the integer a takes the values $a=-1,1$ (see Section 4.4), while for rank-2 STSSH's a takes the values $a=-2,0,2$ (see Section 4.5). The functions $\phi^{(a)}_{n\ell}(\theta_N)$ are regular for $a=1$ and $a=2$ despite the factor $\left(\sin\frac{\theta_N}{2}\cos\frac{\theta_N}{2}\right)^{-a}$ in eq. (4.216) because of the restriction $\ell \geq r$ (this restriction on ℓ is proved in Section 4.4 for $r=1$ and in Section 4.5 for $r=2$).

The differential equation satisfied by the functions $\psi^{(a)}_{n\ell}(\theta_N)$ [eq. (4.32)] is obtained from eq. (4.214) by making the replacement $\theta_N \to \pi-\theta_N$ in the expression (4.215) for the differential operator $D_{(a)}$.

Let us also briefly explain how to obtain the condition $n \geq \ell$ [eq. (4.34)]. By taking the limit $\theta_N \to \pi$ for $\phi^{(a)}_{n\ell}(\theta_N)$ and using the transformation formula (4.191) for the Gauss hypergeometric function, we readily find that the requirement for absence of singularity in $\phi^{(a)}_{n\ell}(\theta_N)$ gives rise to the condition $n \geq \ell$, as well as to the quantisation condition

$$|\zeta_{n,N}| = n+\frac{N}{2}, \quad n\in\mathbb{N}_0. \tag{4.217}$$

4.13.1 Constructing the STSSH's of rank 2 for N even

Our aim is to obtain the STSSH's $\psi_{\mu\nu}^{(B;\sigma;n\ell;\tilde{\rho})}$ that satisfy eqs. (4.62)-(4.64), where the gamma matrices for N even are given by eq. (4.16). As in Ref. [5], we write $\psi_{\mu\nu}^{(B;\sigma;n\ell;\tilde{\rho})}$ in terms of upper and lower $2^{N/2-1}$-dimensional spinor components

$$\psi_{\pm\mu\nu}^{(B;\sigma;n\ell;\tilde{\rho})}(\theta_N,\boldsymbol{\theta}_{N-1}) = \begin{pmatrix} {}^{(\uparrow)}\psi_{\pm\mu\nu}^{(B;\sigma;n\ell;\tilde{\rho})}(\theta_N,\boldsymbol{\theta}_{N-1}) \\ \\ {}^{(\downarrow)}\psi_{\pm\mu\nu}^{(B;\sigma;n\ell;\tilde{\rho})}(\theta_N,\boldsymbol{\theta}_{N-1}) \end{pmatrix}. \tag{4.218}$$

It is clear that eqs. (4.62)-(4.64) - which determine the form of our STSSH's - reduce to a system of equations for the upper and lower components. We will now obtain the system of equations for the upper and lower components. By using eqs. (4.9), (4.14), (4.16), (4.23), (4.24), (4.62) and (4.63) and by expressing $\psi_{\pm\mu\nu}^{(B;\sigma;n\ell;\tilde{\rho})}$ in terms of the upper and lower components as in (4.218), we find that the eigenvalue equation $\nabla\!\!\!\!/\,\psi_{\pm\theta_N\theta_N}^{(B;\sigma;n\ell;\tilde{\rho})} = \pm i|\zeta_{n,N}|\psi_{\pm\theta_N\theta_N}^{(B;\sigma;n\ell;\tilde{\rho})}$ is written as

$$\left(\frac{\partial}{\partial\theta_N} + \frac{N+3}{2}\cot\theta_N + \frac{i}{\sin\theta_N}\tilde{\nabla\!\!\!\!/}\right){}^{(\downarrow)}\psi_{\pm\theta_N\theta_N}^{(B;\sigma;n\ell;\tilde{\rho})} = \pm i|\zeta_{n,N}|\,{}^{(\uparrow)}\psi_{\pm\theta_N\theta_N}^{(B;\sigma;n\ell;\tilde{\rho})}, \tag{4.219a}$$

$$\left(\frac{\partial}{\partial\theta_N} + \frac{N+3}{2}\cot\theta_N - \frac{i}{\sin\theta_N}\tilde{\nabla\!\!\!\!/}\right){}^{(\uparrow)}\psi_{\pm\theta_N\theta_N}^{(B;\sigma;n\ell;\tilde{\rho})} = \pm i|\zeta_{n,N}|\,{}^{(\downarrow)}\psi_{\pm\theta_N\theta_N}^{(B;\sigma;n\ell;\tilde{\rho})}. \tag{4.219b}$$

Similarly, we find that the eigenvalue equation $\nabla\!\!\!\!/\,\psi_{\pm\theta_N\theta_j}^{(B;\sigma;n\ell;\tilde{\rho})} = \pm i|\zeta_{n,N}|\psi_{\pm\theta_N\theta_j}^{(B;\sigma;n\ell;\tilde{\rho})}$ $(j = 1,...,N-1)$ is written as

$$\left(\frac{\partial}{\partial\theta_N} + \frac{N-1}{2}\cot\theta_N + \frac{i}{\sin\theta_N}\tilde{\nabla\!\!\!\!/}\right){}^{(\downarrow)}\psi_{\pm\theta_N\theta_j}^{(B;\sigma;n\ell;\tilde{\rho})} + i\cos\theta_N\tilde{\gamma}_{\theta_j}\,{}^{(\downarrow)}\psi_{\pm\theta_N\theta_N}^{(B;\sigma;n\ell;\tilde{\rho})}$$
$$= \pm i|\zeta_{n,N}|\,{}^{(\uparrow)}\psi_{\pm\theta_N\theta_j}^{(B;\sigma;n\ell;\tilde{\rho})}, \tag{4.220a}$$

$$\left(\frac{\partial}{\partial\theta_N} + \frac{N-1}{2}\cot\theta_N - \frac{i}{\sin\theta_N}\tilde{\nabla\!\!\!\!/}\right){}^{(\uparrow)}\psi_{\pm\theta_N\theta_j}^{(B;\sigma;n\ell;\tilde{\rho})} - i\cos\theta_N\tilde{\gamma}_{\theta_j}\,{}^{(\uparrow)}\psi_{\pm\theta_N\theta_N}^{(B;\sigma;n\ell;\tilde{\rho})}$$
$$= \pm i|\zeta_{n,N}|\,{}^{(\downarrow)}\psi_{\pm\theta_N\theta_j}^{(B;\sigma;n\ell;\tilde{\rho})}, \tag{4.220b}$$

while $\nabla\!\!\!\!/\,\psi_{\pm\theta_j\theta_k}^{(B;\sigma;n\ell;\tilde{\rho})} = \pm i|\zeta_{n,N}|\psi_{\pm\theta_j\theta_k}^{(B;\sigma;n\ell;\tilde{\rho})}$ $(j,k = 1,...,N-1)$ is written as

$$\left(\frac{\partial}{\partial\theta_N} + \frac{N-5}{2}\cot\theta_N + \frac{i}{\sin\theta_N}\tilde{\nabla\!\!\!\!/}\right){}^{(\downarrow)}\psi_{\pm\theta_j\theta_k}^{(B;\sigma;n\ell;\tilde{\rho})} + 2i\cos\theta_N\tilde{\gamma}_{(\theta_j}\,{}^{(\downarrow)}\psi_{\pm\theta_k)\theta_N}^{(B;\sigma;n\ell;\tilde{\rho})}$$
$$= \pm i|\zeta_{n,N}|\,{}^{(\uparrow)}\psi_{\pm\theta_j\theta_k}^{(B;\sigma;n\ell;\tilde{\rho})}, \tag{4.221a}$$

$$\left(\frac{\partial}{\partial\theta_N} + \frac{N-5}{2}\cot\theta_N - \frac{i}{\sin\theta_N}\tilde{\nabla\!\!\!\!/}\right){}^{(\uparrow)}\psi_{\pm\theta_j\theta_k}^{(B;\sigma;n\ell;\tilde{\rho})} - 2i\cos\theta_N\tilde{\gamma}_{(\theta_j}\,{}^{(\uparrow)}\psi_{\pm\theta_k)\theta_N}^{(B;\sigma;n\ell;\tilde{\rho})}$$
$$= \pm i|\zeta_{n,N}|\,{}^{(\downarrow)}\psi_{\pm\theta_j\theta_k}^{(B;\sigma;n\ell;\tilde{\rho})}. \tag{4.221b}$$

By making use of eq. (4.218), we express the gamma-tracelessness condition (4.63) as

$$\begin{cases} {}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\mu} + \dfrac{i}{\sin\theta_N}\tilde{\gamma}^{\theta_i}\;{}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_i\mu} = 0, \\ \\ {}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\mu} - \dfrac{i}{\sin\theta_N}\tilde{\gamma}^{\theta_i}\;{}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_i\mu} = 0, \qquad (\mu = \theta_1, ..., \theta_N \;\text{ and }\; \theta_i = \theta_1, ..., \theta_{N-1}) \end{cases} \tag{4.222}$$

and the tracelessness condition (4.64) as

$$\begin{cases} {}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N} + \dfrac{1}{\sin^2\theta_N}\tilde{g}^{\theta_i\theta_j}\;{}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_i\theta_j} = 0, \\ \\ {}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N} + \dfrac{1}{\sin^2\theta_N}\tilde{g}^{\theta_i\theta_j}\;{}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_i\theta_j} = 0. \end{cases} \tag{4.223}$$

Similarly, by substituting eq. (4.218) into the divergence-free condition (4.63), we may express the condition $\nabla^\alpha \psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\alpha\theta_N} = 0$ as

$$\begin{cases} \left[\frac{\partial}{\partial\theta_N} + (N+\frac{1}{2})\cot\theta_N\right]{}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N} + \dfrac{1}{\sin^2\theta_N}\tilde{\nabla}^{\theta_i}\;{}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_i\theta_N} = 0, \\ \\ \left[\frac{\partial}{\partial\theta_N} + (N+\frac{1}{2})\cot\theta_N\right]{}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N} + \dfrac{1}{\sin^2\theta_N}\tilde{\nabla}^{\theta_i}\;{}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_i\theta_N} = 0, \end{cases} \tag{4.224}$$

while the condition $\nabla^\alpha \psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\alpha\theta_j} = 0$ $(j = 1, ..., N-1)$ is expressed as

$$\begin{cases} \left[\frac{\partial}{\partial\theta_N} + (N-\frac{1}{2})\cot\theta_N\right]{}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j} + \dfrac{1}{\sin^2\theta_N}\tilde{\nabla}^{\theta_i}\;{}^{(\uparrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_i\theta_j} = 0, \\ \\ \left[\frac{\partial}{\partial\theta_N} + (N-\frac{1}{2})\cot\theta_N\right]{}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j} + \dfrac{1}{\sin^2\theta_N}\tilde{\nabla}^{\theta_i}\;{}^{(\downarrow)}\psi^{(B;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_i\theta_j} = 0. \end{cases} \tag{4.225}$$

Type-I STSSH's of rank 2 for N even. Let us start by describing how to obtain the type-I modes, given by eqs. (4.71)-(4.73). The component $\psi^{(I;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}$ is a spinor on S^{N-1}. Thus, in order to solve the system of equations (4.219) we separate variables as in the case of spinor eigenmodes in Ref. [5], i.e.

$$\begin{aligned} {}^{(\uparrow)}\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N, \boldsymbol{\theta}_{N-1}) &= \phi^{(2)}_{n\ell}(\theta_N)\,\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}), \\ {}^{(\downarrow)}\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N, \boldsymbol{\theta}_{N-1}) &= \pm i\psi^{(2)}_{n\ell}(\theta_N)\,\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{aligned} \tag{4.226}$$

$$\begin{aligned} {}^{(\uparrow)}\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N, \boldsymbol{\theta}_{N-1}) &= i\psi^{(2)}_{n\ell}(\theta_N)\,\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}), \\ {}^{(\downarrow)}\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N, \boldsymbol{\theta}_{N-1}) &= \pm\phi^{(2)}_{n\ell}(\theta_N)\,\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}), \end{aligned} \tag{4.227}$$

where $\chi_{\pm\ell\tilde{\rho}}$ are the eigenspinors on S^{N-1} (see eq. (4.27)). By substituting eq. (4.226) [or eq. (4.227)] into the system of equations (4.219) and eliminating ${}^{(\downarrow)}\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}$ (or ${}^{(\downarrow)}\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}$) we find that $\phi^{(2)}_{n\ell}$ has to satisfy the differential equation (4.214) (with $a=2$), while $\psi^{(2)}_{n\ell}$ has to satisfy the differential equation (4.214) ($a=2$) with θ_N replaced by $\pi-\theta_N$ in the differential operator $D_{(2)}$ [eq. (4.215)]. Thus, we find that $\phi^{(2)}_{n\ell}$ and $\psi^{(2)}_{n\ell}$ are given by eqs. (4.31) and (4.32), respectively. As a check, one readily finds that the components defined by eqs. (4.226) and (4.227) satisfy the system of equations (4.219) by making use of the formulae (4.35) and (4.36).
The components $\psi^{(I;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}$ ($j=1,...,N-1$) are vector-spinors on S^{N-1} and thus we may separate variables analogously to eqs. (4.44) and (4.46). Thus, for STSSH's with negative spin projection ($\sigma=-$) we separate variables as

$$
\begin{aligned}
{}^{(\uparrow)}\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) &= C^{(\uparrow)(2)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) + D^{(\uparrow)(2)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}),\\
{}^{(\downarrow)}\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) &= \pm\, iC^{(\downarrow)(2)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \pm iD^{(\downarrow)(2)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}),
\end{aligned}
\tag{4.228}
$$

while for STSSH's with positive spin projection ($\sigma=+$) we separate variables as

$$
\begin{aligned}
{}^{(\uparrow)}\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) &= iC^{(\downarrow)(2)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{\theta_j}\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) - iD^{(\downarrow)(2)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{\theta_j}\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}),\\
{}^{(\downarrow)}\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) &= \pm\, C^{(\uparrow)(2)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{\theta_j}\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \mp D^{(\uparrow)(2)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{\theta_j}\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}).
\end{aligned}
\tag{4.229}
$$

By using the gamma-tracelessness condition (4.222) we readily find that the functions $D^{(b)(2)}_{n\ell}$ and $C^{(b)(2)}_{n\ell}$ ($b=\uparrow,\downarrow$) are related to each other by eqs. (4.49) and (4.50). Then, using the divergence-free condition (4.224), we find that $C^{(\uparrow)(2)}_{n\ell}$ is given by eq. (4.47) and $C^{(\downarrow)(2)}_{n\ell}$ is given by eq. (4.48), where we also have used eqs. (4.35), (4.36) and eq. (4.206). One can straightforwardly verify that the components defined by eqs. (4.228) and (4.229) are solutions of the system of equations (4.220), where the calculations are significantly simplified by using the following formulae:

$$
\begin{aligned}
&\left(\frac{\partial}{\partial\theta_N}+\frac{N-1}{2}\cot\theta_N-\frac{\ell+\frac{N-1}{2}}{\sin\theta_N}\right)C^{(\uparrow)(2)}_{n\ell}(\theta_N)-\frac{2i}{\sin\theta_N}D^{(\uparrow)(2)}_{n\ell}(\theta_N)\\
&\qquad=-\left(n+\frac{N}{2}\right)\,C^{(\downarrow)(2)}_{n\ell}(\theta_N),
\end{aligned}
\tag{4.230}
$$

$$
\begin{aligned}
&\left(\frac{\partial}{\partial\theta_N}+\frac{N-1}{2}\cot\theta_N+\frac{\ell+\frac{N-1}{2}}{\sin\theta_N}\right)C^{(\downarrow)(2)}_{n\ell}(\theta_N)+\frac{2i}{\sin\theta_N}D^{(\downarrow)(2)}_{n\ell}(\theta_N)\\
&\qquad=\left(n+\frac{N}{2}\right)\,C^{(\uparrow)(2)}_{n\ell}(\theta_N),
\end{aligned}
\tag{4.231}
$$

which can be proved by using the formulae (4.35) and (4.36).
The components $\psi^{(I;\sigma;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}$ $(j,k=1,...,N-1)$ are rank-2 symmetric tensor-spinors on S^{N-1}. Let us first discuss the case with negative spin projection ($\sigma=-$). We choose to separate variables for $\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}$ as follows:

$$\begin{aligned}
{}^{(\uparrow)}\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) =& K^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{g}_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\
&+W^{(\uparrow)}_{n\ell}(\theta_N)\left(\tilde{\nabla}_{(\theta_j}\tilde{\nabla}_{\theta_k)}-\tilde{g}_{\theta_j\theta_k}\frac{\tilde{\Box}}{N-1}\right)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\
&+T^{(\uparrow)}_{n\ell}(\theta_N)\left(\tilde{\gamma}_{(\theta_j}\tilde{\nabla}_{\theta_k)}-\tilde{g}_{\theta_j\theta_k}\frac{\tilde{\nabla\!\!\!/}}{N-1}\right)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}),\\
{}^{(\downarrow)}\psi^{(I;-;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) =& \pm\, iK^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{g}_{\theta_j\theta_k}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\
&\pm\, iW^{(\downarrow)}_{n\ell}(\theta_N)\left(\tilde{\nabla}_{(\theta_j}\tilde{\nabla}_{\theta_k)}-\tilde{g}_{\theta_j\theta_k}\frac{\tilde{\Box}}{N-1}\right)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\
&\pm\, iT^{(\downarrow)}_{n\ell}(\theta_N)\left(\tilde{\gamma}_{(\theta_j}\tilde{\nabla}_{\theta_k)}-\tilde{g}_{\theta_j\theta_k}\frac{\tilde{\nabla\!\!\!/}}{N-1}\right)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}), \quad (4.232)
\end{aligned}$$

where $\tilde{\nabla\!\!\!/}\chi_{-\ell\tilde{\rho}}=-i\left(\ell+\frac{N-1}{2}\right)\chi_{-\ell\tilde{\rho}}$ (see eq. (4.27)) and $\tilde{\Box}\chi_{-\ell\tilde{\rho}}\equiv\nabla^{\theta_k}\nabla_{\theta_k}\chi_{-\ell\tilde{\rho}}$ is given by eq. (4.206). By using the tracelessness condition (4.223), we find that $K^{(\uparrow)}_{n\ell}$ and $K^{(\downarrow)}_{n\ell}$ are given by eqs. (4.76) and (4.77), respectively. Then, by using the gamma-tracelessness condition (4.222) (and by making use of eqs. (4.207) and (4.208)) we find that the function $T^{(\uparrow)}_{n\ell}$ ($T^{(\downarrow)}_{n\ell}$) is expressed in terms of $W^{(\uparrow)}_{n\ell}$ ($W^{(\downarrow)}_{n\ell}$) as in eq. (4.78) (eq. (4.79)). Then, by making use of the divergence-free condition (4.225) (and using eqs. (4.209) and (4.210)) we find

$$\begin{aligned}
&\left(\frac{\partial}{\partial\theta_N}+(N-\frac{1}{2})\cot\theta_N\right)C^{(b)(2)}_{n\ell}(\theta_N)+\frac{1}{\sin^2\theta_N}K^{(b)}_{n\ell}(\theta_N)\\
&\quad+\frac{1}{\sin^2\theta_N}W^{(b)}_{n\ell}(\theta_N)\left\{-\frac{\left(\ell+\frac{N-1}{2}\right)^2(N-2)}{N-1}+\frac{N^2-1}{4}\right\}\\
&\quad-i\frac{1}{2\sin^2\theta_N}\frac{\left(\ell+\frac{N-1}{2}\right)(N-3)}{N-1}T^{(b)}_{n\ell}(\theta_N)=0,\qquad b=\uparrow,\downarrow\,. \quad (4.233)
\end{aligned}$$

Finally, by solving the system of equations consisting of eqs. (4.78), (4.79) and (4.233) (and using eqs. (4.230) and (4.231)) we find that $W^{(\uparrow)}_{n\ell}$ is given by eq. (4.80), while $W^{(\downarrow)}_{n\ell}$ is given by eq. (4.81).
By working as in the case with negative spin projection, we find that the components $\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}$ with positive spin projection are expressed in terms of upper and lower spinorial

components as follows:

$$
\begin{aligned}
{}^{(\uparrow)}\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) =& iK^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{g}_{\theta_j\theta_k}\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\
&+iW^{(\downarrow)}_{n\ell}(\theta_N)\left(\tilde{\nabla}_{(\theta_j}\tilde{\nabla}_{\theta_k)}-\tilde{g}_{\theta_j\theta_k}\frac{\tilde{\Box}}{N-1}\right)\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\
&-iT^{(\downarrow)}_{n\ell}(\theta_N)\left(\tilde{\gamma}_{(\theta_j}\tilde{\nabla}_{\theta_k)}-\tilde{g}_{\theta_j\theta_k}\frac{\tilde{\nabla\!\!\!/}}{N-1}\right)\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}),\\
{}^{(\downarrow)}\psi^{(I;+;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) =& \pm K^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{g}_{\theta_j\theta_k}\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\
&\pm W^{(\uparrow)}_{n\ell}(\theta_N)\left(\tilde{\nabla}_{(\theta_j}\tilde{\nabla}_{\theta_k)}-\tilde{g}_{\theta_j\theta_k}\frac{\tilde{\Box}}{N-1}\right)\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\
&\mp T^{(\uparrow)}_{n\ell}(\theta_N)\left(\tilde{\gamma}_{(\theta_j}\tilde{\nabla}_{\theta_k)}-\tilde{g}_{\theta_j\theta_k}\frac{\tilde{\nabla\!\!\!/}}{N-1}\right)\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}).
\end{aligned}
\tag{4.234}
$$

We have verified using Mathematica 11.2 that the components defined by eqs. (4.232) and (4.234) satisfy the system of equations (4.221).

Type-II STSSH's of rank 2 for N even. Now let us describe how to obtain the type-II modes given by eqs. (4.83) and (4.84). The type-II modes satisfy $\psi^{(II\text{-}\tilde{A};\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}=0$ by definition. The components $\psi^{(II\text{-}\tilde{A};\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}$ $(j=1,...,N-1)$ may be expressed as

$$
\begin{aligned}
{}^{(\uparrow)}\psi^{(II\text{-}\tilde{A};-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) &= \phi^{(0)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}),\\
{}^{(\downarrow)}\psi^{(II\text{-}\tilde{A};-;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) &= \pm i\psi^{(0)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}),
\end{aligned}
\tag{4.235}
$$

$$
\begin{aligned}
{}^{(\uparrow)}\psi^{(II\text{-}\tilde{A};+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) &= i\psi^{(0)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_j}(\boldsymbol{\theta}_{N-1})\\
{}^{(\downarrow)}\psi^{(II\text{-}\tilde{A};+;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) &= \pm\phi^{(0)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_j}(\boldsymbol{\theta}_{N-1}).
\end{aligned}
\tag{4.236}
$$

The TT eigenvector-spinors $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}$ $(j=1,...,N-1)$ on S^{N-1} satisfy eqs. (4.52) and (4.53). By working as in the case of type-I modes presented above, we find that $\phi^{(0)}_{n\ell}$ has to satisfy the differential equation (4.214) with $a=0$, while $\psi^{(0)}_{n\ell}$ has to satisfy the differential equation (4.214) $(a=0)$ with θ_N replaced by $\pi-\theta_N$ in the differential operator $D_{(0)}$ [eq. (4.215)]. Thus, we find that $\phi^{(0)}_{n\ell}$ and $\psi^{(0)}_{n\ell}$ are given by eqs. (4.31) and (4.32), respectively. By making use of the formulae (4.35) and (4.36), one can readily verify that the components defined by eqs. (4.235) and (4.236) are solutions of the system of equations (4.220).

The components $\psi^{(II\text{-}\tilde{A};\sigma;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}$ $(j,k=1,...,N-1)$ are symmetric rank-2 tensor-spinors on S^{N-1}. Let us first discuss the case with negative spin projection $(\sigma=-)$. We separate variables as

$$
\begin{aligned}
{}^{(\uparrow)}\psi^{(II\text{-}\tilde{A};-;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) &= \Gamma^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1}) + \Delta^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1}),\\
{}^{(\downarrow)}\psi^{(II\text{-}\tilde{A};-;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) &= \pm\, i\Gamma^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1}) \pm i\Delta^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1}),
\end{aligned}
\tag{4.237}
$$

where we have to determine the functions $\Gamma^{(b)}_{n\ell}$ and $\Delta^{(b)}_{n\ell}$ (with $b=\uparrow,\downarrow$). By using the TT conditions as in the case of type-I modes, we find that $\Delta^{(\uparrow)}_{n\ell}$ and $\Delta^{(\downarrow)}_{n\ell}$ are given by eqs. (4.85) and (4.86), respectively, while $\Gamma^{(\uparrow)}_{n\ell}$ and $\Gamma^{(\downarrow)}_{n\ell}$ are given by eqs. (4.87) and (4.88), respectively, where we also have used eqs. (4.35), (4.36) and (4.212). By using the formulae (4.35) and (4.36), we can also prove the following formulae:

$$
\left(\frac{\partial}{\partial\theta_N}+\frac{N-5}{2}\cot\theta_N-\frac{\ell+\frac{N-1}{2}}{\sin\theta_N}\right)\Gamma^{(\uparrow)}_{n\ell}(\theta_N)-\frac{2i}{\sin\theta_N}\Delta^{(\uparrow)}_{n\ell}(\theta_N)=-(n+\frac{N}{2})\Gamma^{(\downarrow)}_{n\ell}(\theta_N),
\tag{4.238}
$$

$$
\left(\frac{\partial}{\partial\theta_N}+\frac{N-5}{2}\cot\theta_N+\frac{\ell+\frac{N-1}{2}}{\sin\theta_N}\right)\Gamma^{(\downarrow)}_{n\ell}(\theta_N)+\frac{2i}{\sin\theta_N}\Delta^{(\downarrow)}_{n\ell}(\theta_N)=(n+\frac{N}{2})\Gamma^{(\uparrow)}_{n\ell}(\theta_N).
\tag{4.239}
$$

Similarly, we find that the upper and lower components of $\psi^{(II\text{-}\tilde{A};+;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}$ $(j,k=1,...,N-1)$ are given by

$$
\begin{aligned}
{}^{(\uparrow)}\psi^{(II\text{-}\tilde{A};+;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) &= i\Gamma^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_k)}(\boldsymbol{\theta}_{N-1}) - i\Delta^{(\downarrow)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_k)}(\boldsymbol{\theta}_{N-1}),\\
{}^{(\downarrow)}\psi^{(II\text{-}\tilde{A};+;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) &= \pm\,\Gamma^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{\nabla}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_k)}(\boldsymbol{\theta}_{N-1}) \mp \Delta^{(\uparrow)}_{n\ell}(\theta_N)\,\tilde{\gamma}_{(\theta_j}\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_k)}(\boldsymbol{\theta}_{N-1}).
\end{aligned}
\tag{4.240}
$$

By making use of the formulae (4.238) and (4.239), as well as eq. (4.204), one can readily verify that the system of equations (4.221) is satisfied by the type-II modes in eqs. (4.237) and (4.240).

Type-*III* STSSH's of rank 2 for N even. Finally, let us construct the type-III modes, given by eqs. (4.91) and (4.94). The type-III modes satisfy $\psi^{(III\text{-}\tilde{B};\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}=0$ and $\psi^{(III\text{-}\tilde{B};\sigma;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_i}=0$ $(i=1,...,N-1)$ by definition. The components $\psi^{(III\text{-}\tilde{A};\sigma;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}$ $(j,k=1,...,N-1)$ are rank-2 symmetric tensor-spinors on S^{N-1}. Since type-III modes are divergence-free and gamma-traceless, we separate variables in the following way:

$$\begin{aligned}
{}^{(\uparrow)}\psi^{(\textit{III}\text{-}\tilde{B};-;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) &= \phi^{(-2)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}),\\
{}^{(\downarrow)}\psi^{(\textit{III}\text{-}\tilde{B};-;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) &= \pm i\psi^{(-2)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}),
\end{aligned} \tag{4.241}$$

$$\begin{aligned}
{}^{(\uparrow)}\psi^{(\textit{III}\text{-}\tilde{B};+;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) &= i\psi^{(-2)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{+\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}),\\
{}^{(\downarrow)}\psi^{(\textit{III}\text{-}\tilde{B};+;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) &= \pm\phi^{(-2)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{+\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}),
\end{aligned} \tag{4.242}$$

where eq. (4.241) describes the type-III STSSH with negative spin projection, while eq. (4.242) describes the type-III STSSH with positive spin projection. The functions $\phi^{(-2)}_{n\ell}$ and $\psi^{(-2)}_{n\ell}$ are given by eqs. (4.31) and (4.32), respectively. It is straightforward to verify that the type-III modes in eqs. (4.241) and (4.242) are solutions of the system of equations (4.221) (with the use of eqs. (4.35) and (4.36)).

4.13.2 Constructing the STSSH's of rank 2 for N odd

Now the gamma matrices are given by eq. (4.19). By combining eqs. (4.9), (4.14), (4.19), (4.23) and eq. (4.24) we find

$$\nabla\!\!\!/\,\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N} = \left[\left(\frac{\partial}{\partial\theta_N}+\frac{N+3}{2}\cot\theta_N\right)\gamma^N+\frac{1}{\sin\theta_N}\tilde{\nabla\!\!\!/}\right]\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N} = \pm i|\zeta_{n,N}|\,\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}, \tag{4.243}$$

where we have used the gamma-tracelessness condition

$$\gamma^N\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N} = -\gamma^{\theta_j}\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_N}$$

(see eq. (4.63)). Similarly, we find

$$\begin{aligned}
\nabla\!\!\!/\,\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j} &= \left[\left(\frac{\partial}{\partial\theta_N}+\frac{N-1}{2}\cot\theta_N\right)\gamma^N+\frac{1}{\sin\theta_N}\tilde{\nabla\!\!\!/}\right]\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}+\cot\theta_N\gamma_{\theta_j}\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}\\
&= \pm i|\zeta_{n,N}|\,\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}
\end{aligned} \tag{4.244}$$

($j=1,...,N-1$) and

$$\begin{aligned}
\nabla\!\!\!/\,\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k} &= \left[\left(\frac{\partial}{\partial\theta_N}+\frac{N-5}{2}\cot\theta_N\right)\gamma^N+\frac{1}{\sin\theta_N}\tilde{\nabla\!\!\!/}\right]\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}+2\cot\theta_N\gamma_{(\theta_j}\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_k)\theta_N}\\
&= \pm i|\zeta_{n,N}|\,\psi^{(B;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}
\end{aligned} \tag{4.245}$$

$(j,k=1,...,N-1)$. Note that for N odd we have

$$\gamma^N \tilde{\nabla\!\!\!/} + \tilde{\nabla\!\!\!/} \gamma^N = 0, \tag{4.246}$$

since $\{\gamma^N, \tilde{\gamma}^j\} = 0$ $(j=1,...,N-1)$ - see eq. (4.19). Now let us separate variables in eqs. (4.243)-(4.245).

Type-I STSSH's of rank 2 for N odd. As in Ref. [5], since N is odd we choose to express the type-I modes in terms of the following spinors on S^{N-1}:

$$\hat{\chi}_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \equiv \frac{1}{\sqrt{2}}(\mathbf{1}+i\gamma^N)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \tag{4.247}$$

$$\hat{\chi}_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \equiv \gamma^N \hat{\chi}_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) = \frac{1}{\sqrt{2}}(\mathbf{1}+i\gamma^N)\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}), \tag{4.248}$$

where $\chi_{\pm\ell\tilde{\rho}}$ are the eigenspinors on S^{N-1} (satisfying eq. (4.27)). Since N is odd, $\chi_{+\ell\tilde{\rho}}$ and $\chi_{-\ell\tilde{\rho}}$ are related to each other as follows [5]:

$$\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) = \gamma^N \chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}). \tag{4.249}$$

The spinors $\hat{\chi}_{\pm\ell\tilde{\rho}}$ are eigenfunctions of the operator $\gamma^N \tilde{\nabla\!\!\!/}$ (that commutes with $\tilde{\nabla\!\!\!/}^2$) and they satisfy [5]

$$\gamma^N \tilde{\nabla\!\!\!/} \hat{\chi}_{\pm\ell\tilde{\rho}} = \pm\left(\ell + \frac{N-1}{2}\right)\hat{\chi}_{\pm\ell\tilde{\rho}}. \tag{4.250}$$

In order to construct the rank-2 type-I modes on S^N, we separate variables as follows:

$$\psi^{(I;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}(\theta_N,\boldsymbol{\theta}_{N-1}) = \phi^{(2)}_{n\ell}(\theta_N)\hat{\chi}_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \pm i\psi^{(2)}_{n\ell}(\theta_N)\hat{\chi}_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \tag{4.251}$$

$$\begin{aligned}\psi^{(I;n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) =& C^{(\uparrow)(2)}_{n\ell}(\theta_N)\tilde{\nabla}_{\theta_j}\hat{\chi}_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \pm iC^{(\downarrow)(2)}_{n\ell}(\theta_N)\tilde{\nabla}_{\theta_j}\hat{\chi}_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\ &- iD^{(\uparrow)(2)}_{n\ell}(\theta_N)\tilde{\gamma}_{\theta_j}\hat{\chi}_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \mp D^{(\downarrow)(2)}_{n\ell}(\theta_N)\tilde{\gamma}_{\theta_j}\hat{\chi}_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\end{aligned} \tag{4.252}$$

$$\begin{aligned}\psi^{(I;n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) =& \tilde{g}_{\theta_j\theta_k}\left(\hat{\chi}_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})K^{(\uparrow)}_{n\ell}(\theta_N) \pm \hat{\chi}_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\, iK^{(\downarrow)}_{n\ell}(\theta_N)\right)\\ &+\left[\tilde{\nabla}_{(\theta_j}\tilde{\nabla}_{\theta_k)} - \frac{\tilde{g}_{\theta_j\theta_k}}{N-1}\tilde{\Box}\right]\\ &\times\left(\hat{\chi}_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})W^{(\uparrow)}_{n\ell}(\theta_N) \pm \hat{\chi}_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\, iW^{(\downarrow)}_{n\ell}(\theta_N)\right)\\ &+\left[\tilde{\gamma}_{(\theta_j}\tilde{\nabla}_{\theta_k)} - \frac{\tilde{g}_{\theta_j\theta_k}}{N-1}\tilde{\nabla\!\!\!/}\right]\\ &\times\left(-\hat{\chi}_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\, iT^{(\uparrow)}_{n\ell}(\theta_N) \mp \hat{\chi}_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\, T^{(\downarrow)}_{n\ell}(\theta_N)\right),\end{aligned} \tag{4.253}$$

$(j,k=1,...,N-1)$. By working as in the case with N even, we find that the functions $\phi^{(2)}_{n\ell}, \psi^{(2)}_{n\ell}, C^{(b)(2)}_{n\ell}, D^{(b)(2)}_{n\ell}, K^{(b)}_{n\ell}, W^{(b)}_{n\ell}$ and $T^{(b)}_{n\ell}$ (where $b=\uparrow,\downarrow$), describing the dependence on θ_N, are the same functions as the ones used in the even-dimensional case (see eqs. (4.71)-(4.73)). By expressing $\hat{\chi}_{\pm\ell\tilde{\rho}}$ in terms of $\chi_{\pm\ell\tilde{\rho}}$ (by making use of eqs. (4.247) and (4.248)), it is straightforward to show that eqs. (4.251), (4.252) and (4.253) are equal to eqs. (4.96), (4.97) and (4.98), respectively, as presented in Subsection 4.5.2.

Type-II STSSH's of rank 2 for N odd. In order to construct the type-II STSSH's of rank 2 on S^N, we use the following vector-spinors on S^{N-1}:

$$\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}) \equiv \frac{1}{\sqrt{2}}(\mathbf{1}+i\gamma^N)\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}) \tag{4.254}$$

$$\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_j}(\boldsymbol{\theta}_{N-1}) \equiv \gamma^N\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}), \tag{4.255}$$

where $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}$ $(j=1,...,N-1)$ are the TT eigevector-spinors on S^{N-1} (satisfying eqs. (4.52) and (4.53)) and $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_j} = \gamma^N\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}$. The vector-spinors $\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}$ satisfy

$$\gamma^N\tilde{\nabla\!\!\!/}\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j} = \pm\left(\ell+\frac{N-1}{2}\right)\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j} \tag{4.256}$$

$$\tilde{\gamma}^{\theta_i}\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_i} = \tilde{\nabla}^{\theta_i}\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_i} = 0. \tag{4.257}$$

By making use of the vector-spinors $\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}$, we separate variables for the type-II STSSH's $\psi^{(II\text{-}\tilde{A};n\ell;\tilde{\rho})}_{\pm\mu\nu}$ on S^N as follows:

$$\psi^{(II\text{-}\tilde{A};n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}(\theta_N,\boldsymbol{\theta}_{N-1}) = \phi^{(0)}_{n\ell}(\theta_N)\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}) \pm i\psi^{(0)}_{n\ell}(\theta_N)\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_j}(\boldsymbol{\theta}_{N-1}) \tag{4.258}$$

$$\begin{aligned}\psi^{(II\text{-}\tilde{A};n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1}) =& \Gamma^{(\uparrow)}_{n\ell}(\theta_N)\tilde{\nabla}_{(\theta_j}\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1}) \pm i\Gamma^{(\downarrow)}_{n\ell}(\theta_N)\tilde{\nabla}_{(\theta_j}\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_k)}(\boldsymbol{\theta}_{N-1})\\ &- i\Delta^{(\uparrow)}_{n\ell}(\theta_N)\tilde{\gamma}_{(\theta_j}\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{+\theta_k)}(\boldsymbol{\theta}_{N-1}) \mp \Delta^{(\downarrow)}_{n\ell}(\theta_N)\tilde{\gamma}_{(\theta_j}\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_k)}(\boldsymbol{\theta}_{N-1})\end{aligned} \tag{4.259}$$

$(j,k=1,...,N-1)$, while $\psi^{(II\text{-}\tilde{A};n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}=0$ by definition. By working as in the case with N even, we find that the functions $\phi^{(0)}_{n\ell},\psi^{(0)}_{n\ell},\Delta^{(b)}_{n\ell}$ and $\Gamma^{(b)}_{n\ell}$ (where $b=\uparrow,\downarrow$) are given by the same expressions as in the even-dimensional case (see eqs. (4.100) and (4.101)). By expressing $\hat{\tilde{\psi}}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}$ in terms of $\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{\pm\theta_j}$ (with the use of eqs. (4.254) and (4.255)), we straightforwardly find that eqs. (4.258) and (4.259) are equal to eqs. (4.100) and (4.101), respectively.

Type-III STSSH's of rank 2 for N odd. In order to construct the type-III STSSH's of rank 2 on S^N, we use the following rank-2 symmetric tensor-spinors on S^{N-1}:

$$\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}) \equiv \frac{1}{\sqrt{2}}(\mathbf{1}+i\gamma^N)\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}) \tag{4.260}$$

$$\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{+\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}) \equiv \gamma^N\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}), \tag{4.261}$$

where $\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_j\theta_K}$ $(j,k=1,...,N-1)$ are the STSSH's of rank 2 on S^{N-1} (satisfying eqs. (4.67)-(4.69)). Also, note that $\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{+\theta_j\theta_k}=\gamma^N\tilde{\psi}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}$. The tensor-spinors $\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_j\theta_k}$ satisfy

$$\gamma^N\tilde{\nabla}\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_j\theta_k}=\pm\left(\ell+\frac{N-1}{2}\right)\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_j\theta_k} \tag{4.262}$$

$$\tilde{\gamma}^{\theta_i}\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_k}=\tilde{\nabla}^{\theta_i}\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_k}=0 \tag{4.263}$$

$$\tilde{g}^{\theta_i\theta_j}\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_i\theta_j}=0 \tag{4.264}$$

$(i,j,k=1,...,N-1)$.

By making use of the tensor-spinors $\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{\pm\theta_j\theta_k}$, we separate variables for the type-III STSSH's $\psi^{(III\text{-}\tilde{B};n\ell;\tilde{\rho})}_{\pm\mu\nu}$ on S^N as follows:

$$\psi^{(III\text{-}\tilde{B};n\ell;\tilde{\rho})}_{\pm\theta_j\theta_k}(\theta_N,\boldsymbol{\theta}_{N-1})=\phi^{(-2)}_{n\ell}(\theta_N)\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{-\theta_j\theta_k}(\boldsymbol{\theta}_{N-1})\pm i\psi^{(-2)}_{n\ell}(\theta_N)\hat{\tilde{\psi}}^{(\tilde{B};\ell\tilde{\rho})}_{+\theta_j\theta_k}(\boldsymbol{\theta}_{N-1}) \tag{4.265}$$

$(j,k=1,...,N-1)$, while $\psi^{(III\text{-}\tilde{B};n\ell;\tilde{\rho})}_{\pm\theta_N\theta_N}=0$ and $\psi^{(III\text{-}\tilde{B};n\ell;\tilde{\rho})}_{\pm\theta_N\theta_j}=0$ (by definition). By working as in the case with N even, we find that the functions $\phi^{(-2)}_{n\ell}$ and $\psi^{(-2)}_{n\ell}$ are given by eqs. (4.31) and (4.32), respectively [and, thus, eq. (4.265) is equal to eq. (4.104)].

4.14 APPENDIX E - DERIVING THE SPIN$(N+1)$ TRANSFORMATION FORMULAE OF STSSH'S AND DETERMINING THEIR NORMALISATION FACTORS

In Subsections 4.14.1-4.14.3 of this Appendix we derive the transformation formulae (4.114), (4.119), (4.133) and (4.134) for STSSH's of rank 1 on S^N and we calculate the normalisation factors $c_N^{(I;r=1)}(n,\ell)$ and $c_N^{(II;r=1)}(n,\ell)$ [eq. (4.112)]. The derivation of the transformation formulae and the calculation of the normalisation factors for the STSSH's of rank 2 have many similarities with the case of rank-1 STSSH's and, thus, we discuss them in less detail in Subsection 4.14.4.

4.14.1 Calculating $c_N^{(II;r=1)}(n,\ell)$ and making the first step towards the calculation of $c_N^{(I;r=1)}(n,\ell)$

Since it is a quite simple task, let us start by calculating directly the normalisation factor for type-II STSSH's of rank 1 for arbitrary N. For N even, we substitute the unnormalised type-II modes (4.55) (or (4.56)) into the inner product (4.111). Then, by performing the integration over S^{N-1} using eq. (4.54), we find

$$\begin{aligned}\left|\frac{c_N^{(II;r=1)}(n,\ell)}{\sqrt{2}}\right|^{-2} &= \int_0^\pi d\theta_N \sin^{N-3}\theta_N \left[\left(\phi_{n\ell}^{(-1)}(\theta_N)\right)^2 + \left(\psi_{n\ell}^{(-1)}(\theta_N)\right)^2\right] \\ &= \frac{1}{4}\int_0^\pi d\theta_N \sin^{N-1}\theta_N \left[\left(\phi_{n\ell}^{(0)}(\theta_N)\right)^2 + \left(\psi_{n\ell}^{(0)}(\theta_N)\right)^2\right], \end{aligned} \tag{4.266}$$

where the functions $\phi_{n\ell}^{(0)}$ and $\psi_{n\ell}^{(0)}$ are given by eqs. (4.31) and (4.32), respectively. The integral in the last line is the same integral that appears in the normalisation of spinor eigenfunctions on S^N in Ref. [5]. Thus, using the result of Ref. [5] we readily find

$$\left|\frac{c_N^{(II;r=1)}(n,\ell)}{\sqrt{2}}\right|^{2} = \frac{1}{2^{N-3}}\frac{\Gamma(n-\ell+1)\Gamma(n+\ell+N)}{|\Gamma(n+\frac{N}{2})|^2}, \tag{4.267}$$

which is a special case of eq. (4.112). For N odd, the calculation is similar and we find again that the normalisation factor is given by eq. (4.112).

The normalisation factor of the type-I modes can be found by calculating the following integral:

$$\begin{aligned}\left|\frac{c_N^{(I;r=1)}(n,\ell)}{\sqrt{2}}\right|^{-2} &= \int_0^\pi d\theta_N \sin^{N-1}\theta_N \left[\left(\phi_{n\ell}^{(1)}(\theta_N)\right)^2 + \left(\psi_{n\ell}^{(1)}(\theta_N)\right)^2\right] \\ &+ \left[\left(\ell+\frac{N-1}{2}\right)^2 - \frac{(N-1)(N-2)}{4}\right] \\ &\times \int_0^\pi d\theta_N \sin^{N-3}\theta_N \left[\left(C_{n\ell}^{(\uparrow)(1)}(\theta_N)\right)^2 + \left(C_{n\ell}^{(\downarrow)(1)}(\theta_N)\right)^2\right] \\ &+ (N-1)\int_0^\pi d\theta_N \sin^{N-3}\theta_N \left[\left|D_{n\ell}^{(\uparrow)(1)}(\theta_N)\right|^2 + \left|D_{n\ell}^{(\downarrow)(1)}(\theta_N)\right|^2\right] \\ &+ 2i\left(\ell+\frac{N-1}{2}\right) \\ &\times \int_0^\pi d\theta_N \sin^{N-3}\theta_N \\ &\times \left[C_{n\ell}^{(\uparrow)(1)}(\theta_N) D_{n\ell}^{(\uparrow)(1)}(\theta_N) + C_{n\ell}^{(\downarrow)(1)}(\theta_N) D_{n\ell}^{(\downarrow)(1)}(\theta_N)\right], \end{aligned} \tag{4.268}$$

where $C_{n\ell}^{(\uparrow)(1)}, C_{n\ell}^{(\downarrow)(1)}, D_{n\ell}^{(\uparrow)(1)}$ and $D_{n\ell}^{(\downarrow)(1)}$ are given by eqs. (4.47), (4.48), (4.49) and (4.50), respectively. For N even, eq. (4.268) is derived by substituting the expressions (4.43) and (4.44) for type-I modes into the inner product (4.111) and then performing the integration over S^{N-1} (with the use of eqs. (4.28) and (4.206)). For N odd, by working similarly we find again eq. (4.268). Since the integrals in eq. (4.268) are not as simple as in the case of type-II modes, we are going to take an indirect route. To be specific, we first obtain by direct calculation the normalisation factor of the type-I modes with the highest allowed value for ℓ, i.e. $c_N^{(I;r=1)}(n,\ell=n)$. Then, once we have obtained the transformation formulae of the type-I modes under spin$(N+1)$, the normalisation factor $c_N^{(I;r=1)}(n,\ell)$ (for $\ell=1,2,....n-1$) will be constructed in terms of $c_N^{(I;r=1)}(n,n)$ by exploiting the spin$(N+1)$ invariance of the inner product (4.108). To calculate $c_N^{(I;r=1)}(n,n)$ we let $\ell=n$ in eq. (4.268) and by calculating the integrals using Mathematica 11.2 we find

$$\left|\frac{c_N^{(I;r=1)}(n,n)}{\sqrt{2}}\right|^2 = \frac{n(N-2)\Gamma(n+\frac{N}{2}+\frac{1}{2})}{4^{1-n}(1+n)(N-1)\sqrt{\pi}\Gamma(n+\frac{N}{2})}. \tag{4.269}$$

4.14.2 Derivation of the transformation formulae of type-I and type-II-I STSSH's of rank 1 and calculation of the normalisation factor $c_N^{(I;r=1)}(n,\ell)$ for N even

Below we give details for the derivation of the transformation formulae (4.114) and (4.119) for rank-1 ($r=1$) modes with positive spin projection [these modes are given by eqs. (4.45), (4.46) and (4.56)]. The calculations for the rank-1 modes with negative spin projection are not presented here, as they can be performed in the same way.
In order to derive the desired transformation formulae (4.114) and (4.119), it is sufficient to study the following two components of the Lie-Lorentz derivative (4.105): $\mathbb{L}_{\mathscr{S}}\,\psi_{\theta_N}$ and $\mathbb{L}_{\mathscr{S}}\,\psi_{\theta_{N-1}}$. After a straightforward calculation we find

$$\mathbb{L}_{\mathscr{S}}\,\psi_{\theta_N} = \left(\mathscr{S}^\mu\partial_\mu + \frac{\sin\theta_{N-1}}{2\sin\theta_N}\gamma^N\gamma^{N-1}\right)\psi_{\theta_N} + \frac{\sin\theta_{N-1}}{\sin^2\theta_N}\psi_{\theta_{N-1}} \tag{4.270}$$

and

$$\mathbb{L}_{\mathscr{S}}\,\psi_{\theta_{N-1}} = \left(\mathscr{S}^\mu\partial_\mu - \cot\theta_N\,\cos\theta_{N-1} + \frac{\sin\theta_{N-1}}{2\sin\theta_N}\gamma^N\gamma^{N-1}\right)\psi_{\theta_{N-1}} - \sin\theta_{N-1}\,\psi_{\theta_N}, \tag{4.271}$$

where we have substituted eqs. (4.9), (4.14), (4.22) and (4.110) into eq. (4.105). Since N is even, we express $\gamma^N\gamma^{N-1}$ in eqs. (4.270) and (4.271) as

$$\gamma^N\gamma^{N-1} = \begin{pmatrix} -i\tilde{\gamma}^{N-1} & 0 \\ 0 & i\tilde{\gamma}^{N-1}\end{pmatrix}, \tag{4.272}$$

where we have used eq. (4.16).

4.14. Appendix E - Deriving the spin$(N+1)$ transformation formulae of STSSH's and determining their normalisation factors

The partial derivatives in eqs. (4.270) and (4.271) act only on the coordinates $\{\theta_N, \theta_{N-1}\}$. Thus, for later convenience let us introduce the functions $\tilde{\phi}^{(\tilde{a})}_{\ell m}(\theta_{N-1})$ and $\tilde{\psi}^{(\tilde{a})}_{\ell m}(\theta_{N-1})$ describing the θ_{N-1}-dependence of the STSSH's on S^{N-1} . In analogy to eqs. (4.31) and (4.32), these functions are given by

$$\tilde{\phi}^{(\tilde{a})}_{\ell m}(\theta_{N-1}) = \tilde{\kappa}_{\tilde{\phi}}(\ell, m)\left(\cos\frac{\theta_{N-1}}{2}\right)^{m+1-\tilde{a}}\left(\sin\frac{\theta_{N-1}}{2}\right)^{m-\tilde{a}} \times F\left(-\ell+m, \ell+m+N-1; m+\frac{N-1}{2}; \sin^2\frac{\theta_{N-1}}{2}\right), \tag{4.273}$$

and

$$\tilde{\psi}^{(\tilde{a})}_{\ell m}(\theta_{N-1}) = \tilde{\kappa}_{\tilde{\phi}}(\ell, m)\,\frac{\ell+\frac{N-1}{2}}{m+\frac{N-1}{2}}\left(\cos\frac{\theta_{N-1}}{2}\right)^{m-\tilde{a}}\left(\sin\frac{\theta_{N-1}}{2}\right)^{m+1-\tilde{a}} \times F\left(-\ell+m, \ell+m+N-1; m+\frac{N+1}{2}; \sin^2\frac{\theta_{N-1}}{2}\right), \tag{4.274}$$

where the normalisation factor is given by

$$\tilde{\kappa}_{\tilde{\phi}}(\ell, m) = \frac{\Gamma(\ell+\frac{N-1}{2})}{\Gamma(\ell-m+1)\,\Gamma(m+\frac{N-1}{2})}. \tag{4.275}$$

The number $\tilde{a}$ in eqs. (4.273) and (4.274) is an integer and m is the angular momentum quantum number on S^{N-2} [with $\ell \geq m$, in analogy with eq. (4.34)]. The formulae analogous to eqs. (4.35) and (4.36) are given by

$$\left(\frac{d}{d\theta_{N-1}} + \frac{N+2\tilde{a}-2}{2}\cot\theta_{N-1} + \frac{m+\frac{N-2}{2}}{\sin\theta_{N-1}}\right)\tilde{\psi}^{(\tilde{a})}_{\ell m} = \left(\ell+\frac{N-1}{2}\right)\tilde{\phi}^{(\tilde{a})}_{\ell m} \tag{4.276}$$

and

$$\left(\frac{d}{d\theta_{N-1}} + \frac{N+2\tilde{a}-2}{2}\cot\theta_{N-1} - \frac{m+\frac{N-2}{2}}{\sin\theta_{N-1}}\right)\tilde{\phi}^{(\tilde{a})}_{\ell m} = -\left(\ell+\frac{N-1}{2}\right)\tilde{\psi}^{(\tilde{a})}_{\ell m}, \tag{4.277}$$

respectively.

Motivated by the techniques used in Refs. [22] and [26], in order to derive the transformation formulae of our STSSH's we introduce the ladder operators for ℓ, sending ℓ to $\ell \pm 1$ when acting on the functions $\phi^{(a)}_{n\ell}(\theta_N), \psi^{(a)}_{n\ell}(\theta_N), \tilde{\phi}^{(\tilde{a})}_{\ell m}(\theta_{N-1})$ and $\tilde{\psi}^{(\tilde{a})}_{\ell m}(\theta_{N-1})$. The ladder operators are given by the following expressions:

$$T_\phi^{(+;a)} = \frac{d}{d\theta_N} + \left(-\ell + a - \frac{1}{2}\right)\cot\theta_N + \frac{1}{2\sin\theta_N}, \tag{4.278}$$

$$T_\psi^{(+;a)} = \frac{d}{d\theta_N} + \left(-\ell + a - \frac{1}{2}\right)\cot\theta_N - \frac{1}{2\sin\theta_N}, \tag{4.279}$$

$$T_\phi^{(-;a)} = \frac{d}{d\theta_N} + \left(\ell + N + a - \frac{3}{2}\right)\cot\theta_N - \frac{1}{2\sin\theta_N}, \tag{4.280}$$

$$T_\psi^{(-;a)} = \frac{d}{d\theta_N} + \left(\ell + N + a - \frac{3}{2}\right)\cot\theta_N + \frac{1}{2\sin\theta_N}, \tag{4.281}$$

$$\tilde{\Pi}_{\tilde{\phi}}^{(+;\tilde{a})} = \sin\theta_{N-1}\frac{d}{d\theta_{N-1}} + \left(\ell + \tilde{a} + N - \frac{3}{2}\right)\cos\theta_{N-1} - \frac{m + \frac{N-2}{2}}{2(\ell + \frac{N}{2})}, \tag{4.282}$$

$$\tilde{\Pi}_{\tilde{\psi}}^{(+;\tilde{a})} = \sin\theta_{N-1}\frac{d}{d\theta_{N-1}} + \left(\ell + \tilde{a} + N - \frac{3}{2}\right)\cos\theta_{N-1} + \frac{m + \frac{N-2}{2}}{2(\ell + \frac{N}{2})}, \tag{4.283}$$

$$\tilde{\Pi}_{\tilde{\phi}}^{(-;\tilde{a})} = \sin\theta_{N-1}\frac{d}{d\theta_{N-1}} + \left(-\ell + \tilde{a} - \frac{1}{2}\right)\cos\theta_{N-1} + \frac{m + \frac{N-2}{2}}{2(\ell + \frac{N-2}{2})}, \tag{4.284}$$

$$\tilde{\Pi}_{\tilde{\psi}}^{(-;\tilde{a})} = \sin\theta_{N-1}\frac{d}{d\theta_{N-1}} + \left(-\ell + \tilde{a} - \frac{1}{2}\right)\cos\theta_{N-1} - \frac{m + \frac{N-2}{2}}{2(\ell + \frac{N-2}{2})}. \tag{4.285}$$

These operators act as follows:

$$T_f^{(+;a)} f_{n\ell}^{(a)}(\theta_N) = k^{(+)} f_{n\,\ell+1}^{(a)}(\theta_N), \tag{4.286}$$

$$T_f^{(-;a)} f_{n\ell}^{(a)}(\theta_N) = k^{(-)} f_{n\,\ell-1}^{(a)}(\theta_N), \tag{4.287}$$

$$\tilde{\Pi}_{\tilde{f}}^{(+;\tilde{a})} \tilde{f}_{\ell m}^{(\tilde{a})}(\theta_{N-1}) = \tilde{k}^{(+)} \tilde{f}_{\ell+1\,m}^{(\tilde{a})}(\theta_{N-1}), \tag{4.288}$$

$$\tilde{\Pi}_{\tilde{f}}^{(-;\tilde{a})} \tilde{f}_{\ell m}^{(\tilde{a})}(\theta_{N-1}) = \tilde{k}^{(-)} \tilde{f}_{\ell-1\,m}^{(\tilde{a})}(\theta_{N-1}), \tag{4.289}$$

where $f_{n\ell}^{(a)}(\theta_N) \in \{\phi_{n\ell}^{(a)}(\theta_N), \psi_{n\ell}^{(a)}(\theta_N)\}$, $\tilde{f}_{\ell m}^{(\tilde{a})}(\theta_{N-1}) \in \{\tilde{\phi}_{\ell m}^{(\tilde{a})}(\theta_{N-1}), \tilde{\psi}_{\ell m}^{(\tilde{a})}(\theta_{N-1})\}$ and

$$k^{(+)} = -(n + \ell + N), \tag{4.290}$$

$$k^{(-)} = n - \ell + 1, \tag{4.291}$$

$$\tilde{k}^{(+)} = \frac{(\ell + N - 1 + m)(\ell - m + 1)}{\ell + N/2}, \tag{4.292}$$

$$\tilde{k}^{(-)} = -\frac{(\ell + \frac{N-1}{2} - 1)(\ell + \frac{N-1}{2})}{\ell + (N-2)/2}. \tag{4.293}$$

One can straightforwardly prove the ladder relations (4.286)-(4.289) using the raising and lowering operators for the parameters of the Gauss hypergeometric function given in

Appendix 4.10. (Similar ladder relations have been obtained by the author in Ref. [26] while studying the Dirac field on dS_N.)

Let us now proceed to the derivation of the transformation formulae of the type-I and type-II-I modes. It is clear from the expressions (4.270) and (4.271) for the Lie-Lorentz derivative that we need to express the type-I and type-II-I modes in a form where the dependence on both θ_N and θ_{N-1} is written out explicitly. By substituting eq. (4.194) into eqs. (4.45) and (4.46), we express the type-I modes with positive spin projection as

$$\begin{aligned}&\psi^{(I;+;n\ell m;\rho)}_{\pm\theta_N}(\theta_N,\theta_{N-1},\boldsymbol{\theta}_{N-2})\\&=\frac{\tilde{c}_{N-1}(\ell,m)}{\sqrt{2}}\begin{pmatrix}i\psi^{(1)}_{n\ell}(\theta_N)\left[\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2})+i\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right]\\ \\ \pm\phi^{(1)}_{n\ell}(\theta_N)\,\left[\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2})+i\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right]\end{pmatrix}\end{aligned} \tag{4.294}$$

$$\begin{aligned}&\psi^{(I;+;n\ell m;\rho)}_{\pm\theta_{N-1}}(\theta_N,\theta_{N-1},\boldsymbol{\theta}_{N-2})\\&=\frac{\tilde{c}_{N-1}(\ell,m)}{\sqrt{2}}\begin{pmatrix}i\left[E^{(1)}_{n\ell m}(\theta_N,\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2})+i\Sigma^{(1)}_{n\ell m}(\theta_N,\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right]\\ \\ \pm\left[H^{(1)}_{n\ell m}(\theta_N,\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2})+iO^{(1)}_{n\ell m}(\theta_N,\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right]\end{pmatrix},\end{aligned} \tag{4.295}$$

where $\tilde{c}_{N-1}(\ell,m)$ is the normalisation factor (4.197) for the eigenspinors on S^{N-1}, while the spinors $\hat{\tilde{\chi}}_{\pm m\rho}(\boldsymbol{\theta}_{N-2})$ on S^{N-2} are defined by eq. (4.195). Also, we have defined

$$O^{(a)}_{n\ell m}(\theta_N,\theta_{N-1})=C^{(\uparrow)(a)}_{n\ell}(\theta_N)\,\frac{\partial}{\partial\theta_{N-1}}\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})+iD^{(\uparrow)(a)}_{n\ell}(\theta_N)\,\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1}) \tag{4.296}$$

$$H^{(a)}_{n\ell m}(\theta_N,\theta_{N-1})=C^{(\uparrow)(a)}_{n\ell}(\theta_N)\,\frac{\partial}{\partial\theta_{N-1}}\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})-iD^{(\uparrow)(a)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1}) \tag{4.297}$$

$$E^{(a)}_{n\ell m}(\theta_N,\theta_{N-1})=C^{(\downarrow)(a)}_{n\ell}(\theta_N)\,\frac{\partial}{\partial\theta_{N-1}}\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})-iD^{(\downarrow)(a)}_{n\ell}(\theta_N)\,\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1}) \tag{4.298}$$

$$\Sigma^{(a)}_{n\ell m}(\theta_N,\theta_{N-1})=C^{(\downarrow)(a)}_{n\ell}(\theta_N)\,\frac{\partial}{\partial\theta_{N-1}}\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})+iD^{(\downarrow)(a)}_{n\ell}(\theta_N)\,\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1}). \tag{4.299}$$

(Recall that $C^{(\uparrow)(a)}_{n\ell}, C^{(\downarrow)(a)}_{n\ell}, D^{(\uparrow)(a)}_{n\ell}$ and $D^{(\downarrow)(a)}_{n\ell}$ are given by eqs. (4.47), (4.48), (4.49) and (4.50), respectively.) Similarly, the type-I modes with negative spin projection are expressed as

$$\psi^{(I;-;n\ell m;\rho)}_{\pm\theta_N}(\theta_N,\theta_{N-1},\boldsymbol{\theta}_{N-2}) = \frac{\tilde{c}_{N-1}(\ell,m)}{\sqrt{2}} \begin{pmatrix} \phi^{(1)}_{n\ell}(\theta_N)\left[\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2}) - i\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right] \\ \pm i\psi^{(1)}_{n\ell}(\theta_N)\left[\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2}) - i\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right] \end{pmatrix} \tag{4.300}$$

$$\psi^{(I;-;n\ell m;\rho)}_{\pm\theta_{N-1}}(\theta_N,\theta_{N-1},\boldsymbol{\theta}_{N-2}) = \frac{\tilde{c}_{N-1}(\ell,m)}{\sqrt{2}} \begin{pmatrix} H^{(1)}_{n\ell m}(\theta_N,\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2}) - iO^{(1)}_{n\ell m}(\theta_N,\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2}) \\ \pm i\left[E^{(1)}_{n\ell m}(\theta_N,\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2}) - i\Sigma^{(1)}_{n\ell m}(\theta_N,\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right] \end{pmatrix}. \tag{4.301}$$

Similarly, it is straightforward to express the type-II-I modes with positive spin projection (4.56) as follows:

$$\psi^{(II\text{-}I;+;n\ell m;\rho)}_{\pm\theta_N}(\theta_N,\theta_{N-1},\boldsymbol{\theta}_{N-2}) = 0, \tag{4.302}$$

$$\psi^{(II\text{-}I;+;n\ell m;\rho)}_{\pm\theta_{N-1}}(\theta_N,\theta_{N-1},\boldsymbol{\theta}_{N-2}) = \frac{\tilde{c}^{(I;\tilde{r}=1)}_{N-1}(\ell,m)}{\sqrt{2}} \begin{pmatrix} i\psi^{(-1)}_{n\ell}(\theta_N)\left[\tilde{\phi}^{(1)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2}) + i\tilde{\psi}^{(1)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right] \\ \pm\phi^{(-1)}_{n\ell}(\theta_N)\left[\tilde{\phi}^{(1)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2}) + i\tilde{\psi}^{(1)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right] \end{pmatrix}, \tag{4.303}$$

where $\tilde{c}^{(I;r=1)}_{N-1}(\ell,m)$ is the normalisation factor of the STSSH's of rank 1 on S^{N-1} and it will be determined later. The type-II-I modes with negative spin projection (4.55) are expressed as

$$\psi^{(II\text{-}I;-;n\ell m;\rho)}_{\pm\theta_N}(\theta_N,\theta_{N-1},\boldsymbol{\theta}_{N-2}) = 0, \tag{4.304}$$

$$\psi^{(II\text{-}I;-;n\ell m;\rho)}_{\pm\theta_{N-1}}(\theta_N,\theta_{N-1},\boldsymbol{\theta}_{N-2}) = \frac{\tilde{c}^{(I;\tilde{r}=1)}_{N-1}(\ell,m)}{\sqrt{2}} \begin{pmatrix} \phi^{(-1)}_{n\ell}(\theta_N)\left[\tilde{\phi}^{(1)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2}) - i\tilde{\psi}^{(1)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right] \\ \pm i\psi^{(-1)}_{n\ell}(\theta_N)\left[\tilde{\phi}^{(1)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2}) - i\tilde{\psi}^{(1)}_{\ell m}(\theta_{N-1})\,\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\right] \end{pmatrix}, \tag{4.305}$$

4.14.2.1 *Derivation of the transformation formula (4.114) for type-I modes of rank 1 and calculation of the normalisation factor $c_N^{(I;r=1)}(n,\ell)$*

By using the expressions (4.294) and (4.295) for the type-I modes, we express the Lie-Lorentz derivative (4.270) as

$$\mathbb{L}_{\mathscr{S}}\psi_{\pm\theta_N}^{(I;+;n\ell m;\rho)} = \frac{\tilde{c}_{N-1}(\ell,m)}{\sqrt{2}}\begin{pmatrix} i\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2})\,\mathbb{T}_3^{(I)}(\theta_N,\theta_{N-1}) - \hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\,\mathbb{T}_4^{(I)}(\theta_N,\theta_{N-1}) \\ \\ \pm\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2})\,\mathbb{T}_1^{(I)}(\theta_N,\theta_{N-1}) \pm i\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\,\mathbb{T}_2^{(I)}(\theta_N,\theta_{N-1}) \end{pmatrix}, \tag{4.306}$$

where

$$\mathbb{T}_1^{(I)} = \mathscr{S}^\mu\partial_\mu\left[\phi_{n\ell}^{(1)}\tilde{\phi}_{\ell m}^{(0)}\right] - \frac{\sin\theta_{N-1}}{2\sin\theta_N}\phi_{n\ell}^{(1)}\tilde{\psi}_{\ell m}^{(0)} + \frac{\sin\theta_{N-1}}{\sin^2\theta_N}H_{n\ell m}^{(1)}, \tag{4.307}$$

$$\mathbb{T}_2^{(I)} = \mathscr{S}^\mu\partial_\mu\left[\phi_{n\ell}^{(1)}\tilde{\psi}_{\ell m}^{(0)}\right] + \frac{\sin\theta_{N-1}}{2\sin\theta_N}\phi_{n\ell}^{(1)}\tilde{\phi}_{\ell m}^{(0)} + \frac{\sin\theta_{N-1}}{\sin^2\theta_N}O_{n\ell m}^{(1)}, \tag{4.308}$$

$$\mathbb{T}_3^{(I)} = \mathscr{S}^\mu\partial_\mu\left[\psi_{n\ell}^{(1)}\tilde{\phi}_{\ell m}^{(0)}\right] + \frac{\sin\theta_{N-1}}{2\sin\theta_N}\psi_{n\ell}^{(1)}\tilde{\psi}_{\ell m}^{(0)} + \frac{\sin\theta_{N-1}}{\sin^2\theta_N}E_{n\ell m}^{(1)}, \tag{4.309}$$

$$\mathbb{T}_4^{(I)} = \mathscr{S}^\mu\partial_\mu\left[\psi_{n\ell}^{(1)}\tilde{\psi}_{\ell m}^{(0)}\right] - \frac{\sin\theta_{N-1}}{2\sin\theta_N}\psi_{n\ell}^{(1)}\tilde{\phi}_{\ell m}^{(0)} + \frac{\sin\theta_{N-1}}{\sin^2\theta_N}\Sigma_{n\ell m}^{(1)}. \tag{4.310}$$

(Recall that $O_{n\ell m}^{(1)}, H_{n\ell m}^{(1)}, E_{n\ell m}^{(1)}$ and $\Sigma_{n\ell m}^{(1)}$ are given by eqs. (4.296), (4.297), (4.298) and (4.299), respectively.) In order to proceed we need to make use of the following relations:

$$\mathbb{T}_1^{(I)} = \mathscr{R}^{(I)}k^{(+)}\tilde{k}^{(+)}\phi_{n\,\ell+1}^{(1)}\tilde{\phi}_{\ell+1\,m}^{(0)} + \mathscr{L}^{(I)}k^{(-)}\tilde{k}^{(-)}\phi_{n\,\ell-1}^{(1)}\tilde{\phi}_{\ell-1\,m}^{(0)} + \varkappa^{(I)}\psi_{n\ell}^{(1)}\tilde{\phi}_{\ell m}^{(0)}, \tag{4.311}$$

$$\mathbb{T}_2^{(I)} = \mathscr{R}^{(I)}k^{(+)}\tilde{k}^{(+)}\phi_{n\,\ell+1}^{(1)}\tilde{\psi}_{\ell+1\,m}^{(0)} + \mathscr{L}^{(I)}k^{(-)}\tilde{k}^{(-)}\phi_{n\,\ell-1}^{(1)}\tilde{\psi}_{\ell-1\,m}^{(0)} - \varkappa^{(I)}\psi_{n\ell}^{(1)}\tilde{\psi}_{\ell m}^{(0)}, \tag{4.312}$$

$$\mathbb{T}_3^{(I)} = \mathscr{R}^{(I)}k^{(+)}\tilde{k}^{(+)}\psi_{n\,\ell+1}^{(1)}\tilde{\phi}_{\ell+1\,m}^{(0)} + \mathscr{L}^{(I)}k^{(-)}\tilde{k}^{(-)}\psi_{n\,\ell-1}^{(1)}\tilde{\phi}_{\ell-1\,m}^{(0)} - \varkappa^{(I)}\phi_{n\ell}^{(1)}\tilde{\phi}_{\ell m}^{(0)}, \tag{4.313}$$

$$\mathbb{T}_4^{(I)} = \mathscr{R}^{(I)}k^{(+)}\tilde{k}^{(+)}\psi_{n\,\ell+1}^{(1)}\tilde{\psi}_{\ell+1\,m}^{(0)} + \mathscr{L}^{(I)}k^{(-)}\tilde{k}^{(-)}\psi_{n\,\ell-1}^{(1)}\tilde{\psi}_{\ell-1\,m}^{(0)} + \varkappa^{(I)}\phi_{n\ell}^{(1)}\tilde{\psi}_{\ell m}^{(0)}, \tag{4.314}$$

where $k^{(+)}, k^{(-)}, \tilde{k}^{(+)}$ and $\tilde{k}^{(-)}$ are given by eqs. (4.290), (4.291), (4.292) and (4.293), respectively, while $\varkappa^{(I)}$ is the coefficient defined in eq. (4.117) (with $r=1$) and

$$\mathscr{R}^{(I)} = \frac{\ell+N}{2(\ell+\frac{N-1}{2})(\ell+N-1)}, \qquad \mathscr{L}^{(I)} = \frac{1-\ell}{2\ell(\ell+\frac{N-1}{2})}. \tag{4.315}$$

Let us outline the steps required for proving eq. (4.311). (Equations (4.312)-(4.314) are proved similarly.) First, we express $\mathbb{T}_1^{(I)}$ on the left-hand side of eq. (4.311) in terms of $\phi_{n\ell}^{(1)}$, $d\phi_{n\ell}^{(1)}/d\theta_N$, $\tilde{\phi}_{\ell m}^{(0)}$ and $d\tilde{\phi}_{\ell m}^{(0)}/d\theta_{N-1}$ by making use of eqs. (4.307), (4.297), (4.277), (4.47), (4.49) and (4.36). As for the right-hand side, we express $\phi^{(1)}_{n\,\ell\pm1}$ and $\tilde{\phi}^{(0)}_{\ell\pm1\,m}$ in terms of $\phi^{(1)}_{n\ell}$, $d\phi^{(1)}_{n\ell}/d\theta_N$ and $\tilde{\phi}^{(0)}_{\ell m}$, $d\tilde{\phi}^{(0)}_{\ell m}/d\theta_{N-1}$, respectively, by making use of the ladder relations (4.286)-(4.289) and we also express $\psi^{(1)}_{n\ell}$ in terms of $\phi^{(1)}_{n\ell}$ and $d\phi^{(1)}_{n\ell}/d\theta_N$ by making use of eq. (4.36). Then, it is straightforward to show that the two sides of eq. (4.311) are equal. We have verified the calculations using Mathematica 11.2.

Then, by substituting eqs. (4.311)-(4.314) into eq. (4.306), we express the latter as

$$\begin{aligned}\mathbb{L}_{\mathscr{S}}\psi^{(I;+;n\ell m;\rho)}_{\pm\theta_N} = \frac{\tilde{c}_{N-1}(\ell,m)}{\sqrt{2}}\Bigg\{&\mathscr{R}^{(I)}k^{(+)}\tilde{k}^{(+)}\begin{pmatrix} i\psi^{(1)}_{n\,\ell+1}\left[\tilde{\phi}^{(0)}_{\ell+1\,m}\,\hat{\tilde{\chi}}_{-m\rho}+i\tilde{\psi}^{(0)}_{\ell+1\,m}\,\hat{\tilde{\chi}}_{+m\rho}\right]\\ \pm\phi^{(1)}_{n\,\ell+1}\left[\tilde{\phi}^{(0)}_{\ell+1\,m}\,\hat{\tilde{\chi}}_{-m\rho}+i\tilde{\psi}^{(0)}_{\ell+1\,m}\,\hat{\tilde{\chi}}_{+m\rho}\right]\end{pmatrix}\\ &+\mathscr{L}^{(I)}k^{(-)}\tilde{k}^{(-)}\begin{pmatrix} i\psi^{(1)}_{n\,\ell-1}\left[\tilde{\phi}^{(0)}_{\ell-1\,m}\,\hat{\tilde{\chi}}_{-m\rho}+i\tilde{\psi}^{(0)}_{\ell-1\,m}\,\hat{\tilde{\chi}}_{+m\rho}\right]\\ \pm\phi^{(1)}_{n\,\ell-1}\left[\tilde{\phi}^{(0)}_{\ell-1\,m}\,\hat{\tilde{\chi}}_{-m\rho}+i\tilde{\psi}^{(0)}_{\ell-1\,m}\,\hat{\tilde{\chi}}_{+m\rho}\right]\end{pmatrix}\\ &-i\varkappa^{(I)}\begin{pmatrix} \phi^{(1)}_{n\ell}\left[\tilde{\phi}^{(0)}_{\ell m}\,\hat{\tilde{\chi}}_{-m\rho}-i\tilde{\psi}^{(0)}_{\ell m}\,\hat{\tilde{\chi}}_{+m\rho}\right]\\ \pm i\psi^{(1)}_{n\ell}\left[\tilde{\phi}^{(0)}_{\ell m}\,\hat{\tilde{\chi}}_{-m\rho}-i\tilde{\psi}^{(0)}_{\ell m}\,\hat{\tilde{\chi}}_{+m\rho}\right]\end{pmatrix}\Bigg\}\end{aligned} \tag{4.316}$$

and we straightforwardly rewrite this as

$$\mathbb{L}_{\mathscr{S}}\psi^{(I;+;n\ell m;\rho)}_{\pm\theta_N} = \mathscr{A}^{(I)}\psi^{(I;+;n\,(\ell+1)\,m;\rho)}_{\pm\theta_N}+\mathscr{B}^{(I)}\psi^{(I;+;n\,(\ell-1)\,m;\rho)}_{\pm\theta_N}-i\varkappa^{(I)}\psi^{(I;-;n\ell m;\rho)}_{\pm\theta_N}, \tag{4.317}$$

as in eq. (4.114), where we have defined

$$\mathscr{A}^{(I)} \equiv \mathscr{R}^{(I)}k^{(+)}\tilde{k}^{(+)}\frac{\tilde{c}_{N-1}(\ell,m)}{\tilde{c}_{N-1}(\ell+1,m)}, \tag{4.318}$$

$$\mathscr{B}^{(I)} \equiv \mathscr{L}^{(I)}k^{(-)}\tilde{k}^{(-)}\frac{\tilde{c}_{N-1}(\ell,m)}{\tilde{c}_{N-1}(\ell-1,m)}. \tag{4.319}$$

It easy to verify that these expressions for $\mathscr{A}^{(I)}$ and $\mathscr{B}^{(I)}$ agree with the expressions given by eqs. (4.115) (with $r=1$) and (4.116) (with $r=1$), respectively.

Now, we can determine the normalisation factor $c^{(I;r=1)}_N(n,\ell)$ for the type-I modes. By using the spin$(N+1)$ invariance of the inner product (4.109) between $\psi^{(I;\sigma;n\ell m;\rho)}_{\pm\mu}$ and $\psi^{(I;\sigma;n(\ell+1)m;\rho)}_{\pm\mu}$ and using the transformation formula (4.114) we find

$$\left|\frac{c^{(I;r=1)}_N(n,\ell)}{c^{(I;r=1)}_N(n,\ell+1)}\right|^2 = \frac{(n-\ell)\,\ell\,(\ell+N-1)}{(\ell+1)(\ell+N)(n+\ell+N)}. \tag{4.320}$$

By iterating this equation and using eq. (4.269), one can straightforwardly find

$$\left|\frac{c_N^{(I;r=1)}(n,\ell)}{\sqrt{2}}\right|^2 = \frac{1}{2^{N+1}}\frac{\Gamma(n-\ell+1)\Gamma(n+\ell+N)}{|\Gamma(n+\frac{N}{2})|^2} \times \frac{(N-2)\ell(\ell+N-1)}{(N-1)\left([n+N/2]^2-[N-2]^2/4\right)}, \tag{4.321}$$

which is eq. (4.112) with $r=1$ and $\tilde{r}_{(B)}=\tilde{r}_{(I)}=0$. For later convenience, note that we can easily deduce the form of the normalisation factor for the type-I STSSH's of rank 1 on S^{N-1} by making the replacements $N\to N-1$, $n\to\ell$ and $\ell\to m$ in eq. (4.321), as

$$\left|\frac{\tilde{c}_{N-1}^{(I;\tilde{r}=1)}(\ell,m)}{\sqrt{2}}\right|^2 = \frac{1}{2^N}\frac{\Gamma(\ell-m+1)\Gamma(\ell+m+N-1)}{|\Gamma(\ell+\frac{N-1}{2})|^2} \times \frac{(N-3)m(m+N-2)}{(N-2)(\ell+1)(\ell+N-2)}. \tag{4.322}$$

Let us now discuss the mixing between type-I and type-II-I modes under the spin$(N+1)$ transformation. By using the equation $\psi^{(II\text{-}I;\sigma;n\ell m;\rho)}_{\pm\theta_N}=0$ and eqs. (4.294) and (4.303) (or eqs. (4.300) and (4.305)), one readily finds that the component given by (4.270) of the infinitesimal transformation of a type-II-I mode is proportional to a type-I mode, as

$$\begin{aligned}\mathbb{L}_{\mathscr{S}}\psi^{(II\text{-}I;\sigma;n\ell m;\rho)}_{\pm\theta_N} &= \frac{\sin\theta_{N-1}}{\sin^2\theta_N}\psi^{(II\text{-}I;\sigma;n\ell m;\rho)}_{\pm\theta_{N-1}}\\ &= \mathscr{K}^{(II\to I)}\psi^{(I;\sigma;n\ell m;\rho)}_{\pm\theta_N},\end{aligned} \tag{4.323}$$

in agreement with eq. (4.119), where we have defined

$$\mathscr{K}^{(II\to I)} \equiv \frac{1}{2}\frac{\tilde{c}_{N-1}^{(I;\tilde{r}=1)}(\ell,m)}{\tilde{c}_{N-1}(\ell,m)}. \tag{4.324}$$

It is easy to show that this expression for $\mathscr{K}^{(II\to I)}$ is equal to the expression given by eq. (4.124) (with $r=1$). Then, since type-II-I modes transform into type-I modes under the spin$(N+1)$ transformation, the spin$(N+1)$ invariance of the inner product (4.109) (between $\psi^{(I;\sigma;n\ell m;\rho)}_{\pm\mu}$ and $\psi^{(II\text{-}I;\sigma;n\ell m;\rho)}_{\pm\mu}$) implies that

$$\mathbb{L}_{\mathscr{S}}\psi^{(I;\sigma;n\ell m;\rho)}_{\pm\mu} = \ldots + \mathscr{K}^{(I\to II)}\psi^{(II\text{-}I;\sigma;n\ell m;\rho)}_{\pm\mu}, \tag{4.325}$$

where all the STSSH's in '...' are type-I modes, while $\mathscr{K}^{(I\to II)}$ is given by

$$\mathscr{K}^{(I\to II)} = -\mathscr{K}^{(II\to I)*}\left|\frac{c_N^{(II;r=1)}(n,\ell)}{c_N^{(I;r=1)}(n,\ell)}\right|^2, \tag{4.326}$$

where the asterisk denotes complex conjugation. Then, by using the expression for $\mathscr{K}^{(II\to I)}$ [eq. (4.124)] and the expressions for the normalisation factors [eq. (4.112)] we find that $\mathscr{K}^{(I\to II)}$ in eq. (4.326) is equal to the expression given by eq. (4.118) (with $r=1$).

4.14.2.2 *Derivation of the transformation formula (4.119) for type-II-I modes of rank 1*

By substituting the type-II-I mode (4.303) into the Lie-Lorentz derivative (4.271) we find

$$\begin{aligned}
&\mathbb{L}_{\mathscr{S}}\psi^{(II\text{-}I;+;n\ell m;\rho)}_{\pm\theta_{N-1}}\\
&=\frac{\tilde{c}^{(I;\tilde{r}=1)}_{N-1}(\ell,m)}{\sqrt{2}}\begin{pmatrix} i\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2})\,\mathbb{T}^{(II)}_{3}(\theta_N,\theta_{N-1})-\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\,\mathbb{T}^{(II)}_{4}(\theta_N,\theta_{N-1})\\ \\ \pm\hat{\tilde{\chi}}_{-m\rho}(\boldsymbol{\theta}_{N-2})\,\mathbb{T}^{(II)}_{1}(\theta_N,\theta_{N-1})\pm i\hat{\tilde{\chi}}_{+m\rho}(\boldsymbol{\theta}_{N-2})\,\mathbb{T}^{(II)}_{2}(\theta_N,\theta_{N-1})\end{pmatrix},
\end{aligned} \tag{4.327}$$

where

$$\mathbb{T}^{(II)}_1=\left(\mathscr{S}^{\mu}\partial_{\mu}-\cot\theta_N\cos\theta_{N-1}\right)\left[\phi^{(-1)}_{n\ell}\tilde{\phi}^{(1)}_{\ell m}\right]-\frac{\sin\theta_{N-1}}{2\sin\theta_N}\phi^{(-1)}_{n\ell}\tilde{\psi}^{(1)}_{\ell m}, \tag{4.328}$$

$$\mathbb{T}^{(II)}_2=\left(\mathscr{S}^{\mu}\partial_{\mu}-\cot\theta_N\cos\theta_{N-1}\right)\left[\phi^{(-1)}_{n\ell}\tilde{\psi}^{(1)}_{\ell m}\right]+\frac{\sin\theta_{N-1}}{2\sin\theta_N}\phi^{(-1)}_{n\ell}\tilde{\phi}^{(1)}_{\ell m}, \tag{4.329}$$

$$\mathbb{T}^{(II)}_3=\left(\mathscr{S}^{\mu}\partial_{\mu}-\cot\theta_N\cos\theta_{N-1}\right)\left[\psi^{(-1)}_{n\ell}\tilde{\phi}^{(1)}_{\ell m}\right]+\frac{\sin\theta_{N-1}}{2\sin\theta_N}\psi^{(-1)}_{n\ell}\tilde{\psi}^{(1)}_{\ell m}, \tag{4.330}$$

$$\mathbb{T}^{(II)}_4=\left(\mathscr{S}^{\mu}\partial_{\mu}-\cot\theta_N\cos\theta_{N-1}\right)\left[\psi^{(-1)}_{n\ell}\tilde{\psi}^{(1)}_{\ell m}\right]-\frac{\sin\theta_{N-1}}{2\sin\theta_N}\psi^{(-1)}_{n\ell}\tilde{\phi}^{(1)}_{\ell m}. \tag{4.331}$$

Then, as in the case of the type-I modes, we prove the following relations:

$$\begin{aligned}
\mathbb{T}_1^{(II)} &= \mathscr{R}^{(II)} k^{(+)} \tilde{k}^{(+)} \phi^{(-1)}_{n\,\ell+1} \tilde{\phi}^{(1)}_{\ell+1\,m} + \mathscr{L}^{(II)} k^{(-)} \tilde{k}^{(-)} \phi^{(-1)}_{n\,\ell-1} \tilde{\phi}^{(1)}_{\ell-1\,m} + \varkappa^{(II)} \psi^{(-1)}_{n\ell} \tilde{\phi}^{(1)}_{\ell m} \\
&+ \frac{H^{(1)}_{n\ell m}}{2}, && (4.332)\\
\mathbb{T}_2^{(II)} &= \mathscr{R}^{(II)} k^{(+)} \tilde{k}^{(+)} \phi^{(-1)}_{n\,\ell+1} \tilde{\psi}^{(1)}_{\ell+1\,m} + \mathscr{L}^{(II)} k^{(-)} \tilde{k}^{(-)} \phi^{(-1)}_{n\,\ell-1} \tilde{\psi}^{(1)}_{\ell-1\,m} - \varkappa^{(II)} \psi^{(-1)}_{n\ell} \tilde{\psi}^{(1)}_{\ell m} \\
&+ \frac{O^{(1)}_{n\ell m}}{2}, && (4.333)\\
\mathbb{T}_3^{(II)} &= \mathscr{R}^{(II)} k^{(+)} \tilde{k}^{(+)} \psi^{(-1)}_{n\,\ell+1} \tilde{\phi}^{(1)}_{\ell+1\,m} + \mathscr{L}^{(II)} k^{(-)} \tilde{k}^{(-)} \psi^{(-1)}_{n\,\ell-1} \tilde{\phi}^{(1)}_{\ell-1\,m} - \varkappa^{(II)} \phi^{(-1)}_{n\ell} \tilde{\phi}^{(1)}_{\ell m} \\
&+ \frac{E^{(1)}_{n\ell m}}{2}, && (4.334)\\
\mathbb{T}_4^{(II)} &= \mathscr{R}^{(II)} k^{(+)} \tilde{k}^{(+)} \psi^{(-1)}_{n\,\ell+1} \tilde{\psi}^{(1)}_{\ell+1\,m} + \mathscr{L}^{(II)} k^{(-)} \tilde{k}^{(-)} \psi^{(-1)}_{n\,\ell-1} \tilde{\psi}^{(1)}_{\ell-1\,m} + \varkappa^{(II)} \phi^{(-1)}_{n\ell} \tilde{\psi}^{(1)}_{\ell m} \\
&+ \frac{\Sigma^{(1)}_{n\ell m}}{2}, && (4.335)
\end{aligned}$$

where $\varkappa^{(II)}$ is given by eq. (4.122) (with $r=1$) and

$$\mathscr{R}^{(II)} = \frac{\ell+N-2}{2(\ell+\frac{N-1}{2})(\ell+N-1)}, \qquad \mathscr{L}^{(II)} = \frac{-(1+\ell)}{2\ell(\ell+\frac{N-1}{2})}. \tag{4.336}$$

By substituting eqs. (4.332)-(4.335) into eq. (4.327) we find

$$\begin{aligned}
\mathbb{L}_{\mathscr{S}} \psi^{(II\text{-}I;+;n\ell m;\rho)}_{\pm\theta_{N-1}} &= \mathscr{A}^{(II)} \psi^{(II\text{-}I;+;n\,(\ell+1)\,m;\rho)}_{\pm\theta_{N-1}} + \mathscr{B}^{(II)} \psi^{(II\text{-}I;+;n\,(\ell-1)\,m;\rho)}_{\pm\theta_{N-1}} \\
&\quad - i\varkappa^{(II)} \psi^{(II\text{-}I;-;n\ell m;\rho)}_{\pm\theta_{N-1}} + \mathscr{K}^{(II\to I)} \psi^{(I;+;n\ell m;\rho)}_{\pm\theta_{N-1}}, && (4.337)
\end{aligned}$$

in precise agreement with the transformation formula (4.119), where we have defined

$$\mathscr{A}^{(II)} \equiv \mathscr{R}^{(II)} k^{(+)} \tilde{k}^{(+)} \frac{\tilde{c}^{(I;\tilde{r}=1)}_{N-1}(\ell,m)}{\tilde{c}^{(I;\tilde{r}=1)}_{N-1}(\ell+1,m)}, \tag{4.338}$$

$$\mathscr{B}^{(II)} \equiv \mathscr{L}^{(II)} k^{(-)} \tilde{k}^{(-)} \frac{\tilde{c}^{(I;\tilde{r}=1)}_{N-1}(\ell,m)}{\tilde{c}^{(I;\tilde{r}=1)}_{N-1}(\ell-1,m)}. \tag{4.339}$$

It easy to verify that these expressions for $\mathscr{A}^{(II)}$ and $\mathscr{B}^{(II)}$ agree with the expressions given by eqs. (4.120) (with $r=1$) and (4.121) (with $r=1$), respectively.

4.14.3 Derivation of the transformation formulae of type-I and type-II-I STSSH's of rank 1 and calculation of the normalisation factor $c_N^{(I;r=1)}(n,\ell)$ for N odd

The Lie-Lorentz derivative is given by eqs. (4.270) and (4.271), where $\gamma^N\gamma^{N-1}$ is given by

$$\gamma^N\gamma^{N-1} = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}, \tag{4.340}$$

where $\mathbf{1}$ is the identity spinorial matrix of dimension $2^{\frac{N-1}{2}}/2$.

The type-I modes on S^N with positive spin projection index on S^{N-1} ($\sigma_{N-1}=+$) are found by substituting eq. (4.199) into eqs. (4.59) and (4.60), as

$$\begin{aligned}\psi^{(I;n\ell;+;m;\rho)}_{\pm\theta_N}&(\theta_N,\theta_{N-1},\boldsymbol{\theta}_{N-2})\\ &= \frac{\tilde{c}_{N-1}(\ell,m)}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} (1+i)\, i\tilde{\psi}^{(0)}_{\ell m}(\theta_{N-1})\left[\phi^{(1)}_{n\ell}(\theta_N)\pm i\psi^{(1)}_{n\ell}(\theta_N)\right]\tilde{\chi}_{+m\rho}(\boldsymbol{\theta}_{N-2}) \\ (1-i)\,\tilde{\phi}^{(0)}_{\ell m}(\theta_{N-1})\left[-\phi^{(1)}_{n\ell}(\theta_N)\pm i\psi^{(1)}_{n\ell}(\theta_N)\right]\tilde{\chi}_{+m\rho}(\boldsymbol{\theta}_{N-2})\end{pmatrix},\end{aligned} \tag{4.341}$$

$$\begin{aligned}\psi^{(I;n\ell;+;m;\rho)}_{\pm\theta_{N-1}}&(\theta_N,\theta_{N-1},\boldsymbol{\theta}_{N-2})\\ &= \frac{\tilde{c}_{N-1}(\ell,m)}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} (1+i)\left[iO^{(1)}_{n\ell m}(\theta_N,\theta_{N-1})\mp\Sigma^{(1)}_{n\ell m}(\theta_N,\theta_{N-1})\right]\tilde{\chi}_{+m\rho}(\boldsymbol{\theta}_{N-2}) \\ (1-i)\left[-H^{(1)}_{n\ell m}(\theta_N,\theta_{N-1})\pm iE^{(1)}_{n\ell m}(\theta_N,\theta_{N-1})\right]\tilde{\chi}_{+m\rho}(\boldsymbol{\theta}_{N-2})\end{pmatrix}\end{aligned} \tag{4.342}$$

(the functions describing the dependence on θ_N and θ_{N-1} in eq. (4.342) are given by eqs. (4.296)-(4.299)). The component $\psi^{(I;n\ell;-;m;\rho)}_{\pm\theta_N}$ is obtained from eq. (4.341) by making the replacement $\tilde{\chi}_{+m\rho}\to\tilde{\chi}_{-m\rho}$ and exchanging $i\tilde{\psi}^{(0)}_{\ell m}$ and $\tilde{\phi}^{(0)}_{\ell m}$. The component $\psi^{(I;n\ell;-;m;\rho)}_{\pm\theta_{N-1}}$ is obtained from eq. (4.342) by making the replacement $\tilde{\chi}_{+m\rho}\to\tilde{\chi}_{-m\rho}$ and exchanging $iO^{(1)}_{n\ell m}$ and $H^{(1)}_{n\ell m}$, as well as exchanging $\mp\Sigma^{(1)}_{n\ell m}$ and $\pm iE^{(1)}_{n\ell m}$. The ladder relations for the functions $\phi^{(a)}_{n\ell}(\theta_N),\psi^{(a)}_{n\ell}(\theta_N),\tilde{\phi}^{(\tilde{a})}_{\ell m}(\theta_{N-1}),\tilde{\psi}^{(\tilde{a})}_{\ell m}(\theta_{N-1})$ are given again by eqs. (4.286)-(4.289). Equations (4.311)-(4.314) hold as in the even-dimensional case.

The type-II-I modes on S^N with positive spin projection index on S^{N-1} ($\sigma_{N-1}=+$)

are expressed as

$$\psi^{(II\text{-}I;n\ell;+;m;\rho)}_{\pm\theta_{N-1}}(\theta_N,\theta_{N-1},\boldsymbol{\theta}_{N-2}) = \frac{\tilde{c}^{(I;\tilde{r}=1)}_{N-1}(\ell,m)}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix}(1+i)\,i\tilde{\psi}^{(1)}_{\ell m}(\theta_{N-1})\left[\phi^{(-1)}_{n\ell}(\theta_N)\pm i\psi^{(-1)}_{n\ell}(\theta_N)\right]\tilde{\chi}_{+m\rho}(\boldsymbol{\theta}_{N-2})\\ (1-i)\,\tilde{\phi}^{(1)}_{\ell m}(\theta_{N-1})\left[-\phi^{(-1)}_{n\ell}(\theta_N)\pm i\psi^{(-1)}_{n\ell}(\theta_N)\right]\tilde{\chi}_{+m\rho}(\boldsymbol{\theta}_{N-2})\end{pmatrix}, \tag{4.343}$$

while $\psi^{(II\text{-}I;n\ell;-;m;\rho)}_{\pm\theta_{N-1}}$ is obtained from eq. (4.343) by making the replacement $\tilde{\chi}_{+m\rho}\rightarrow\tilde{\chi}_{-m\rho}$ and exchanging $i\tilde{\psi}^{(1)}_{\ell m}$ and $\tilde{\phi}^{(1)}_{\ell m}$. Equations (4.332)-(4.335) hold as in the even-dimensional case.

The rest of the derivation of the transformation formulae is similar to that for the even-dimensional case. We find that the transformation formulae for the type-I and type-II-I modes are given by eqs. (4.133) and (4.134), respectively, while the normalisation factor $c^{(I;r=1)}_N(n,\ell)$ is given by eq. (4.321).

4.14.4 Transformation properties under spin$(N+1)$ and normalisation factors for STSSH's of rank 2 on S^N

As mentioned in the beginning of this Appendix, the calculations needed in order to derive the transformation formulae and determine the normalisation factors for STSSH's of rank 2 on S^N have many similarities with the case of rank-1 STSSH's, which was presented above. Therefore, below we just provide a brief description of the basic steps.

Let us begin by determining the normalisation factor for type-III STSSH's of rank 2, $c^{(III;r=2)}_N(n,\ell)$. In the case with N even, we substitute the rank-2 type-III modes (4.89)-(4.91) into the inner product (4.111), while in the case with N odd we substitute the type-III modes (4.102)-(4.104) into the inner product (4.131). By working as in the case of rank-1 type-II modes, we readily find (with the use of eq. (4.70))

$$\left|\frac{c^{(III;r=2)}_N(n,\ell)}{\sqrt{2}}\right|^2 = \frac{1}{2^{N-5}}\frac{\Gamma(n-\ell+1)\Gamma(n+\ell+N)}{|\Gamma(n+\frac{N}{2})|^2}, \tag{4.344}$$

for both N even and N odd, which is eq. (4.112) with $r=\tilde{r}_{(III)}=2$.

Now we will determine the normalisation factor for type-II STSSH's of rank 2, $c^{(II;r=2)}_N(n,\ell)$. For N even we substitute eqs. (4.82)-(4.84) into the inner product (4.111), while for N odd we substitute eqs. (4.99)-(4.101) into the inner product (4.131). By performing the integration over S^{N-1} (using eqs. (4.212) and (4.54)), we straightforwardly find

$$\begin{aligned}\left|\frac{c_N^{(II;r=2)}(n,\ell)}{\sqrt{2}}\right|^{-2} &= 2\int_0^\pi d\theta_N \sin^{N-3}\theta_N \left[\left(\phi_{n\ell}^{(0)}(\theta_N)\right)^2 + \left(\psi_{n\ell}^{(0)}(\theta_N)\right)^2\right] \\ &+ 2\left[\left(\ell+\frac{N-1}{2}\right)^2 - \frac{N(N+1)}{4}\right] \\ &\times \int_0^\pi d\theta_N \sin^{N-5}\theta_N \left[\left(\frac{\Gamma_{n\ell}^{(\uparrow)}(\theta_N)}{2}\right)^2 + \left(\frac{\Gamma_{n\ell}^{(\downarrow)}(\theta_N)}{2}\right)^2\right] \\ &+ 2(N+1)\int_0^\pi d\theta_N \sin^{N-5}\theta_N \left[\left|\frac{\Delta_{n\ell}^{(\uparrow)}(\theta_N)}{2}\right|^2 + \left|\frac{\Delta_{n\ell}^{(\downarrow)}(\theta_N)}{2}\right|^2\right] \\ &+ i\left(\ell+\frac{N-1}{2}\right) \\ &\times \int_0^\pi d\theta_N \sin^{N-5}\theta_N \left[\Gamma_{n\ell}^{(\uparrow)}(\theta_N)\Delta_{n\ell}^{(\uparrow)}(\theta_N) + \Gamma_{n\ell}^{(\downarrow)}(\theta_N)\Delta_{n\ell}^{(\downarrow)}(\theta_N)\right],\end{aligned} \tag{4.345}$$

where $\Gamma_{n\ell}^{(\uparrow)}, \Gamma_{n\ell}^{(\downarrow)}, \Delta_{n\ell}^{(\uparrow)}$ and $\Delta_{n\ell}^{(\downarrow)}$ are given by eqs. (4.87), (4.88), (4.85) and (4.86), respectively. The calculations can be significantly simplified by making use of the following relations:

$$\begin{aligned}\frac{4}{\sin^2\theta}\phi_{n\ell}^{(0)}(\theta) &= \phi_{n'\ell'}^{(1)}(\theta)\Big|_{N\to N+2}, \\ \frac{4}{\sin^2\theta}\psi_{n\ell}^{(0)}(\theta) &= \psi_{n'\ell'}^{(1)}(\theta)\Big|_{N\to N+2}, \\ \frac{2}{\sin^2\theta}\Gamma_{n\ell}^{(b)}(\theta) &= C_{n'\ell'}^{(b)(1)}(\theta)\Big|_{N\to N+2}, \\ \frac{2}{\sin^2\theta}\Delta_{n\ell}^{(b)}(\theta) &= D_{n'\ell'}^{(b)(1)}(\theta)\Big|_{N\to N+2}, \qquad (b=\uparrow,\downarrow)\end{aligned} \tag{4.346}$$

where $\theta \in [0,\pi]$, $n' = n-1$ and $\ell' = \ell - 1$, while on the right-hand sides of the relations in eq. (4.346) we have denoted the replacement of N by $N+2$ as $N \to N+2$. The relations in eq. (4.346) can be readily proved by using eqs. (4.31), (4.32), (4.47), (4.48), (4.49), (4.50), (4.85), (4.86) (4.87) and (4.88). By comparing eqs. (4.345) and (4.268) and by using eq. (4.346), we straightforwardly find

$$\left|\frac{c_N^{(II;r=2)}(n,\ell)}{\sqrt{2}}\right|^2 = 2^3\left|\frac{c_{N+2}^{(I;r=1)}(n-1,\ell-1)}{\sqrt{2}}\right|^2 \tag{4.347}$$

$$= \frac{1}{2^N}\frac{\Gamma(n-\ell+1)\Gamma(n+\ell+N)}{|\Gamma(n+\frac{N}{2})|^2} \times \frac{N(\ell-1)(\ell+N)}{(N+1)\left([n+N/2]^2-N^2/4\right)}, \tag{4.348}$$

which is eq. (4.112) with $r=2$ and $\tilde{r}_{(B)}=\tilde{r}_{(II)}=1$.

As for the normalisation of rank-2 type-I modes, by working as in the case of rank-1 type-I modes, we calculate the normalisation factor for $\ell=n$ using Mathematica 11.2

$$\left|\frac{c_N^{(I;r=2)}(n,n)}{\sqrt{2}}\right|^2 = \frac{(n-1)(N-2)\Gamma(n+\frac{N}{2}+\frac{1}{2})}{4^{2-n}(n+1)(N+1)\sqrt{\pi}\Gamma(n+\frac{N}{2})}, \tag{4.349}$$

while the normalisation factor $c_N^{(I;r=2)}(n,\ell)$ (for $\ell=2,3,...,n-1$) will be determined using the spin$(N+1)$ invariance of the inner product (4.109).

In order to derive the transformation formulae (4.114), (4.119), (4.126), (4.133), (4.134) and (4.135) for the STSSH's of rank 2 it is sufficient to study the following components of the Lie-Lorentz derivative (4.105):

$$\mathbb{L}_{\mathscr{S}}\psi_{\theta_N\theta_N} = \left(\mathscr{S}^\mu\partial_\mu + \frac{\sin\theta_{N-1}}{2\sin\theta_N}\gamma^N\gamma^{N-1}\right)\psi_{\theta_N\theta_N} + \frac{2\sin\theta_{N-1}}{\sin^2\theta_N}\psi_{\theta_N\theta_{N-1}}, \tag{4.350}$$

$$\mathbb{L}_{\mathscr{S}}\psi_{\theta_N\theta_{N-1}} = \left(\mathscr{S}^\mu\partial_\mu - \cos\theta_{N-1}\,\cot\theta_N + \frac{\sin\theta_{N-1}}{2\sin\theta_N}\gamma^N\gamma^{N-1}\right)\psi_{\theta_N\theta_{N-1}} + \frac{\sin\theta_{N-1}}{\sin^2\theta_N}\psi_{\theta_{N-1}\theta_{N-1}} - \sin\theta_{N-1}\psi_{\theta_N\theta_N} \tag{4.351}$$

and

$$\mathbb{L}_{\mathscr{S}}\psi_{\theta_{N-1}\theta_{N-1}} = \left(\mathscr{S}^\mu\partial_\mu - 2\cos\theta_{N-1}\,\cot\theta_N + \frac{\sin\theta_{N-1}}{2\sin\theta_N}\gamma^N\gamma^{N-1}\right)\psi_{\theta_{N-1}\theta_{N-1}} - 2\sin\theta_{N-1}\,\psi_{\theta_N\theta_{N-1}}. \tag{4.352}$$

By working as in the case of rank-1 STSSH's, we make use of the ladder operators (4.286)-(4.289) and (after a long calculation) we find the transformation formulae (4.114), (4.119) and (4.126) for N even, and the transformation formulae (4.133)-(4.135) for N odd. Then, as in the case of rank-1 type-I modes, the normalisation factor of rank-2 type-I

modes is found by combining the spin$(N+1)$ invariance of the inner product between $\psi^{(I;\sigma;n\ell m;\rho)}_{\pm\mu_1\mu_2}$ and $\psi^{(I;\sigma;n(\ell+1)m;\rho)}_{\pm\mu_1\mu_2}$ with eq. (4.349), as

$$\left|\frac{c_N^{(I;r=2)}(n,\ell)}{\sqrt{2}}\right|^2 = \frac{1}{2^{N+3}}\frac{\Gamma(n-\ell+1)\Gamma(n+\ell+N)}{|\Gamma(n+\frac{N}{2})|^2} \times \frac{N-2}{N+1}\frac{\ell(\ell+N-1)(\ell-1)(\ell+N)}{\left([n+N/2]^2-[N-2]^2/4\right)\left([n+N/2]^2-N^2/4\right)}, \tag{4.353}$$

(for both N even and N odd) which is eq. (4.112) with $r=2$ and $\tilde{r}_{(B)}=\tilde{r}_{(I)}=0$.

4.15 APPENDIX F - PURE GAUGE MODES

In this Appendix, we present details for the derivation of the pure gauge expressions (4.148), (4.150) and (4.153) for N even. The calculations for N odd are similar and, thus, we do not present them here.

For later convenience, note that by making the replacements $\theta_N \to x(t)=\pi/2-it$, $n\to\tilde{M}-N/2$ [eq. (4.140)] in the formulae (4.35) and (4.36) we find

$$\left(\frac{d}{dx}+\frac{N+2a-1}{2}\cot x+\frac{\ell+(N-1)/2}{\sin x}\right)\hat{\psi}^{(a)}_{\tilde{M}\ell}(t)=\tilde{M}\,\hat{\phi}^{(a)}_{\tilde{M}\ell}(t) \tag{4.354}$$

and

$$\left(\frac{d}{dx}+\frac{N+2a-1}{2}\cot x-\frac{\ell+(N-1)/2}{\sin x}\right)\hat{\phi}^{(a)}_{\tilde{M}\ell}(t)=-\tilde{M}\hat{\psi}^{(a)}_{\tilde{M}\ell}(t), \tag{4.355}$$

respectively, where $\cot x = i\tanh t$ and $\sin x=\cosh t$. Also, let us obtain lowering operators for $\tilde{M}$ as follows. By making the replacements $N\to N+1$, $\theta_{N-1}\to x(t)=\pi/2-it$, $\ell\to\tilde{M}-N/2$, $\tilde{a}\to a$ and $m\to\ell$ in the lowering operator (4.284) we find

$$\hat{L}^{(\tilde{M};a)}_{\phi}\hat{\phi}^{(a)}_{\tilde{M}\ell}(t)\equiv\left[\sin x\frac{\partial}{\partial x}+\left(-\tilde{M}+\frac{N-1}{2}+a\right)\cos x+\frac{\ell+\frac{N-1}{2}}{2(\tilde{M}-1/2)}\right]\hat{\phi}^{(a)}_{\tilde{M}\ell}(t) = -\frac{\tilde{M}\left(\tilde{M}-\ell-N/2\right)}{\tilde{M}-1/2}\hat{\phi}^{(a)}_{(\tilde{M}-1)\ell}(t), \tag{4.356}$$

while by making the same replacements in the lowering operator (4.285) we find

$$\hat{L}^{(\tilde{M};a)}_{\psi}\hat{\psi}^{(a)}_{\tilde{M}\ell}(t)\equiv\left[\sin x\frac{\partial}{\partial x}+\left(-\tilde{M}+\frac{N-1}{2}+a\right)\cos x-\frac{\ell+\frac{N-1}{2}}{2(\tilde{M}-1/2)}\right]\hat{\psi}^{(a)}_{\tilde{M}\ell}(t) = -\frac{\tilde{M}\left(\tilde{M}-\ell-N/2\right)}{\tilde{M}-1/2}\hat{\psi}^{(a)}_{(\tilde{M}-1)\ell}(t). \tag{4.357}$$

4.15.1 Pure gauge modes for strictly massless spin-3/2 field, N even

The type-I modes (4.148) for the strictly massless spin-3/2 field (with $\tilde{M} = \pm(N-2)/2$) are 'pure gauge' modes. In this Subsection, we prove explicitly the t-component of eq. (4.148) and we describe the calculations needed in order to prove the rest of the components. Let us denote the spinors $\Lambda_{\pm}^{(\tilde{\ell})}$ in eq. (4.148) as $\Lambda_{\pm}^{(\sigma;\ell;\tilde{\rho})}$, where we have written out explicitly the dependence on the spin projection index $\sigma = \pm$ and the angular momentum quantum number $\ell = 1, 2, \dots$. Since these spinors satisfy the Dirac equation $(\nabla\!\!\!/ \pm iN/2)\Lambda_{\pm}^{(\sigma;\ell;\tilde{\rho})} = 0$, they are given by [26]

$$\Lambda_{\pm}^{(-;\ell;\tilde{\rho})}(t, \boldsymbol{\theta}_{N-1}) = \frac{2}{\ell}\begin{pmatrix} \hat{\phi}^{(0)}_{\frac{N}{2},\ell}(t)\, \chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \mp i\hat{\psi}^{(0)}_{\frac{N}{2},\ell}(t)\, \chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix}, \tag{4.358}$$

$$\Lambda_{\pm}^{(+;\ell;\tilde{\rho})}(t, \boldsymbol{\theta}_{N-1}) = \frac{2}{\ell}\begin{pmatrix} i\hat{\psi}^{(0)}_{\frac{N}{2},\ell}(t)\, \chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \mp\hat{\phi}^{(0)}_{\frac{N}{2},\ell}(t)\, \chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix}, \tag{4.359}$$

where $\hat{\phi}^{(0)}_{\frac{N}{2},\ell}(t)$ and $\hat{\psi}^{(0)}_{\frac{N}{2},\ell}(t)$ are found by letting $\tilde{M} = N/2$ in eqs. (4.142) and (4.144), respectively, while $\chi_{\pm\ell\tilde{\rho}}$ are the eigenspinors (4.27) of the Dirac operator on S^{N-1}. The factor of $2/\ell$ will be motivated naturally below [it arises from the use of the lowering operators (4.356) and (4.357)]. Below we prove the t-component of eq. (4.148) only for negative spin projection $\sigma = -$. The case with $\sigma = +$ can be proved in the same way. The type-I modes $\Psi_{\mu}^{(I;-;(\pm\frac{N-2}{2})\ell;\tilde{\rho})}$ for the strictly massless spin-3/2 field ($\tilde{M} = \pm(N-2)/2$) are found by combining eq. (4.146) with eqs. (4.43) and (4.44), as

$$\Psi_{t}^{(I;-;(\pm\frac{N-2}{2})\ell;\tilde{\rho})}(t, \boldsymbol{\theta}_{N-1}) = -i\begin{pmatrix} \hat{\phi}^{(1)}_{(\frac{N-2}{2})\ell}(t)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \mp i\hat{\psi}^{(1)}_{(\frac{N-2}{2})\ell}(t)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix} \tag{4.360}$$

$$\Psi_{\theta_j}^{(I;-;(\pm\frac{N-2}{2})\ell;\tilde{\rho})}(t, \boldsymbol{\theta}_{N-1}) = \begin{pmatrix} \hat{C}^{(\uparrow)(1)}_{(\frac{N-2}{2})\ell}(t)\, \tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) + \hat{D}^{(\uparrow)(1)}_{(\frac{N-2}{2})\ell}(t)\, \tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \\ \mp i\hat{C}^{(\downarrow)(1)}_{(\frac{N-2}{2})\ell}(t)\, \tilde{\nabla}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \mp i\hat{D}^{(\downarrow)(1)}_{(\frac{N-2}{2})\ell}(t)\, \tilde{\gamma}_{\theta_j}\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix}, \tag{4.361}$$

where the functions $\hat{C}^{(b)(1)}_{\tilde{M}\ell}(t)$ and $\hat{D}^{(b)(1)}_{\tilde{M}\ell}(t)$ $(b =\uparrow, \downarrow)$ are obtained by making the replacements $\theta_N \to \pi/2 - it$, $n \to \tilde{M} - N/2$, $\phi^{(1)}_{n\ell}(\theta_N) \to \hat{\phi}^{(1)}_{\tilde{M}\ell}(t)$, $\psi^{(1)}_{n\ell}(\theta_N) \to \hat{\psi}^{(1)}_{\tilde{M}\ell}(t)$ in the functions $C^{(b)(1)}_{n\ell}(\theta_N)$ and $D^{(b)(1)}_{n\ell}(\theta_N)$ $(b =\uparrow, \downarrow)$, respectively, in eq. (4.44).

Now, let us prove eq. (4.148) for the t-component of $\Psi_\mu^{(I;-;(\pm\frac{N-2}{2})\ell;\tilde{\rho})}$. We will show that the two sides of eq. (4.148) are equal by making use of the lowering operators (4.356) and (4.357). We want to show

$$\Psi_t^{(I;-;(\pm\frac{N-2}{2})\ell;\tilde{\rho})} = \left(\frac{\partial}{\partial t} \pm \frac{i}{2}\gamma_t\right) \Lambda_{\pm}^{(-;\ell;\tilde{\rho})} \tag{4.362}$$

which is expressed in terms of upper and lower components as

$$-i\begin{pmatrix} \hat{\phi}^{(1)}_{(\frac{N-2}{2})\ell}(t)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \\ \mp i\hat{\psi}^{(1)}_{(\frac{N-2}{2})\ell}(t)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix} = \frac{2}{\ell}\begin{pmatrix} \left[\frac{\partial}{\partial t}\hat{\phi}^{(0)}_{\frac{N}{2},\ell}(t) - \frac{i}{2}\hat{\psi}^{(0)}_{\frac{N}{2},\ell}(t)\right]\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \\ \\ \mp\left[i\frac{\partial}{\partial t}\hat{\psi}^{(0)}_{\frac{N}{2},\ell}(t) - \frac{1}{2}\hat{\phi}^{(0)}_{\frac{N}{2},\ell}(t)\right]\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1}) \end{pmatrix}, \tag{4.363}$$

[where we have used eq. (4.16) and $\gamma^t = i\gamma^N$] or equivalently

$$\frac{\ell}{2}\,\hat{\phi}^{(1)}_{(\frac{N-2}{2})\ell}(t) = \frac{\ell}{\sin x}\,\hat{\phi}^{(0)}_{(\frac{N-2}{2})\ell}(t) = \frac{\partial}{\partial x}\hat{\phi}^{(0)}_{\frac{N}{2},\ell}(t) + \frac{1}{2}\hat{\psi}^{(0)}_{\frac{N}{2},\ell}(t) \tag{4.364}$$

$$\frac{\ell}{2}\,\hat{\psi}^{(1)}_{(\frac{N-2}{2})\ell}(t) = \frac{\ell}{\sin x}\,\hat{\psi}^{(0)}_{(\frac{N-2}{2})\ell}(t) = \frac{\partial}{\partial x}\hat{\psi}^{(0)}_{\frac{N}{2},\ell}(t) - \frac{1}{2}\hat{\phi}^{(0)}_{\frac{N}{2},\ell}(t), \tag{4.365}$$

where we have used eqs. (4.142) and (4.144). Then, by using the formulae (4.354) and (4.355) we rewrite eqs. (4.364) and (4.365) as

$$\left(\sin x\frac{d}{dx} - \frac{1}{2}\cot x + \frac{\ell + (N-1)/2}{N-1}\right)\hat{\phi}^{(0)}_{\frac{N}{2},\ell}(t) = \frac{N\,\ell}{N-1}\hat{\phi}^{(0)}_{(\frac{N-2}{2})\ell}(t) \tag{4.366}$$

and

$$\left(\sin x\frac{d}{dx} - \frac{1}{2}\cot x - \frac{\ell + (N-1)/2}{N-1}\right)\hat{\psi}^{(0)}_{\frac{N}{2},\ell}(t) = \frac{N\,\ell}{N-1}\hat{\psi}^{(0)}_{(\frac{N-2}{2})\ell}(t), \tag{4.367}$$

respectively. It is easy to verify that eq. (4.366) is equal to the lowering operator (4.356) acting on $\hat{\phi}^{(0)}_{\frac{N}{2},\ell}(t)$, while eq. (4.367) is equal to the lowering operator (4.357) acting on $\hat{\psi}^{(0)}_{\frac{N}{2},\ell}(t)$. Hence, the two sides of the time component of eq. (4.148) are equal. The rest of the components of eq. (4.148), i.e. $\Psi_{\theta_j}^{(I;-;(\pm\frac{N-2}{2})\ell;\tilde{\rho})} = \left(\nabla_{\theta_j} \pm \frac{i}{2}\gamma_{\theta_j}\right)\Lambda_{\pm}^{(-;\ell;\tilde{\rho})}$ $(j = 1, ..., N-1)$, can be proved straightforwardly just by using eqs. (4.366) and (4.367), as well as formulae (4.354) and (4.355).

4.15.2 Pure gauge modes for strictly massless spin-5/2 field, N even

The type-I and type-II modes for the strictly massless spin-5/2 field (with $\tilde{M} = \pm N/2$ - see eq. (4.150)) are 'pure gauge' modes. In this Subsection, we briefly describe how to obtain the 'pure gauge' expression in eq. (4.150). We denote the vector-spinors $\lambda^{(B;\tilde{\ell})}_{\pm\nu}(t,\boldsymbol{\theta}_{N-1})$ in eq. (4.150) as $\lambda^{(B;\sigma;\ell;\tilde{\rho})}_{\pm\nu}(t,\boldsymbol{\theta}_{N-1})$ ($\sigma=\pm$, $B=I,II$ and $\ell=2,3,...$). Since the calculations for $\sigma=-$ and $\sigma=+$ are similar, below we discuss only the case with $\sigma=-$.

Pure gauge modes of type-$\boldsymbol{I}$. The type-I modes $\Psi^{(I;-;(\pm\frac{N}{2})\ell;\tilde{\rho})}_{\mu\nu}$ for the strictly massless spin-5/2 field ($\tilde{M}=\pm N/2$) are found by combining eq. (4.146) with eqs. (4.71). The 'time-time component' is

$$\Psi^{(I;-;(\pm\frac{N}{2})\ell;\tilde{\rho})}_{tt}(t,\boldsymbol{\theta}_{N-1}) = \ (-1)\times\begin{pmatrix}\hat{\phi}^{(2)}_{\frac{N}{2},\ell}(t)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\ \mp i\hat{\psi}^{(2)}_{\frac{N}{2},\ell}(t)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\end{pmatrix}. \tag{4.368}$$

Similarly, since the TT vector-spinors $\lambda^{(B;-;\ell;\tilde{\rho})}_{\pm\mu}(t,\boldsymbol{\theta}_{N-1})$ in eq. (4.150) satisfy

$$\left(\nabla\!\!\!\!/ \pm i\frac{N+2}{2}\right)\lambda^{(B;-;\ell;\tilde{\rho})}_{\pm\mu}=0,$$

they are given by the analytic continuation of the type-I STSSH's of rank 1 in eqs. (4.43) and (4.44). The 'time component' is given by

$$\lambda^{(I;-;\ell;\tilde{\rho})}_{\pm t}(t,\boldsymbol{\theta}_{N-1}) = \ -\frac{2i}{\ell-1}\begin{pmatrix}\hat{\phi}^{(1)}_{(\frac{N+2}{2})\ell}(t)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\ \mp i\hat{\psi}^{(1)}_{(\frac{N+2}{2})\ell}(t)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\end{pmatrix}. \tag{4.369}$$

(The factor of $2/(\ell-1)$ is inserted for the same reason as the factor of $2/\ell$ in eqs. (4.358) and (4.359).) Then, by using eqs. (4.368) and (4.369), we expand the two sides of $\Psi^{(I;-;(\pm\frac{N}{2})\ell;\tilde{\rho})}_{tt}=\left(\nabla_t\pm\frac{i}{2}\gamma_t\right)\lambda^{(I;-;\ell;\tilde{\rho})}_{\pm t}$ [see eq. (4.150)] and find

$$\frac{\ell-1}{\sin x}\,\hat{\phi}^{(1)}_{\frac{N}{2},\ell}(t)=\frac{\partial}{\partial x}\hat{\phi}^{(1)}_{(\frac{N+2}{2})\ell}(t)+\frac{1}{2}\hat{\psi}^{(1)}_{(\frac{N+2}{2})\ell}(t) \tag{4.370}$$

$$\frac{\ell-1}{\sin x}\,\hat{\psi}^{(1)}_{\frac{N}{2},\ell}(t)=\frac{\partial}{\partial x}\hat{\psi}^{(1)}_{(\frac{N+2}{2})\ell}(t)-\frac{1}{2}\hat{\phi}^{(1)}_{(\frac{N+2}{2})\ell}(t). \tag{4.371}$$

These equations are proved in the same way as eqs. (4.364) and (4.365). Thus, we have verified the 'time-time component' of the 'pure gauge' expression (4.150). The rest of the components of eq. (4.150), i.e. $\Psi^{(I;-;(\pm\frac{N}{2})\ell;\tilde{\rho})}_{t\theta_j}=\left(\nabla_{(t}\pm\frac{i}{2}\gamma_{(t}\right)\lambda^{(I;-;\ell;\tilde{\rho})}_{\pm\theta_j)}$ and $\Psi^{(I;-;(\pm\frac{N}{2})\ell;\tilde{\rho})}_{\theta_k\theta_j}=\left(\nabla_{(\theta_k}\pm\frac{i}{2}\gamma_{(\theta_k}\right)\lambda^{(I;-;\ell;\tilde{\rho})}_{\pm\theta_j)}$, can be proved using eqs. (4.370) and (4.371).

Pure gauge modes of type-$\boldsymbol{II}$. By working as in the case of type-I modes presented above, we find

$$\Psi^{(II\text{-}\tilde{A};-;(\pm\frac{N}{2})\ell;\tilde{\rho})}_{t\theta_j}(t,\boldsymbol{\theta}_{N-1}) = \ (-i)\times\begin{pmatrix}\hat{\phi}^{(0)}_{\frac{N}{2},\ell}(t)\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1})\\ \mp i\hat{\psi}^{(0)}_{\frac{N}{2},\ell}(t)\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1})\end{pmatrix} \tag{4.372}$$

and

$$\lambda^{\left(II\text{-}\tilde{A};-;\ell;\tilde{\rho}\right)}_{\pm\theta_j}(t,\boldsymbol{\theta}_{N-1}) = \frac{4}{\ell-1}\begin{pmatrix} \hat{\phi}^{(-1)}_{(\frac{N+2}{2})\ell}(t)\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}) \\ \mp i\hat{\psi}^{(-1)}_{(\frac{N+2}{2})\ell}(t)\tilde{\psi}^{(\tilde{A};\ell\tilde{\rho})}_{-\theta_j}(\boldsymbol{\theta}_{N-1}) \end{pmatrix}. \tag{4.373}$$

(Recall that for type-II modes we have $\Psi^{\left(II\text{-}\tilde{A};\sigma;(\pm\frac{N}{2})\ell;\tilde{\rho}\right)}_{tt} = 0$ and $\lambda^{\left(II\text{-}\tilde{A};\sigma;\ell;\tilde{\rho}\right)}_{\pm t} = 0$.) Then, we can verify the 'pure gauge' expression (4.150) by working as in the case of type-I modes presented above.

4.15.3 Pure gauge modes for partially massless spin-5/2 field, N even

The type-I modes [eq. (4.153)] for the partially massless spin-5/2 field (with $\tilde{M} = \pm(N-2)/2$) are 'pure gauge' modes. Below we describe briefly how to obtain the 'pure gauge' expression in eq. (4.153) for N even. (We present the proof only for the tt-component of eq. (4.153).) We denote the Dirac spinors $\varphi^{(\tilde{\ell})}_{\pm}(t,\boldsymbol{\theta}_{N-1})$ in eq. (4.153) as $\varphi^{(\sigma;\ell;\tilde{\rho})}_{\pm}(t,\boldsymbol{\theta}_{N-1})$ ($\sigma = \pm$ and $\ell = 2,3,...$). Again, the calculations for $\sigma = -$ and $\sigma = +$ are similar and, thus, we discuss only the case with $\sigma = -$.

For later convenience let us write down explicit expressions for lowering operators that lower the parameter $\tilde{M}$ to $\tilde{M}-2$ of the functions $\hat{f}^{(a)}_{\tilde{M}\ell}(t) \in \{\hat{\phi}^{(a)}_{\tilde{M}\ell}(t), \hat{\psi}^{(a)}_{\tilde{M}\ell}(t)\}$. By applying each of the lowering operators (4.356), (4.357) twice, we find

$$\begin{aligned}\hat{L}^{(\tilde{M}-1;a)}_{f}\hat{L}^{(\tilde{M};a)}_{f}\hat{f}^{(a)}_{\tilde{M}\ell}(t) =& \left[\sin^2 x\,\frac{\partial^2}{\partial x^2} + b_f(x)\,\frac{\partial}{\partial x} + c_f(x)\right]\hat{f}^{(a)}_{\tilde{M}\ell}(t) \\ =& \frac{\tilde{M}(\tilde{M}-1)(\tilde{M}-\ell-\frac{N}{2})(\tilde{M}-1-\ell-\frac{N}{2})}{(\tilde{M}-\frac{1}{2})(\tilde{M}-\frac{3}{2})}\hat{f}^{(a)}_{(\tilde{M}-2)\ell}(t),\end{aligned} \tag{4.374}$$

(recall $x = \pi/2 - it$) where

$$b_f(x) = \sin x\cos x\,\left(-2\tilde{M}+1+2a+N\right) + s_f\,\frac{(\ell+\frac{N-1}{2})(\tilde{M}-1)\sin x}{(\tilde{M}-1/2)(\tilde{M}-3/2)} \tag{4.375}$$

and

$$\begin{aligned}c_f(x) =&\frac{(\ell+\frac{N-1}{2})^2}{4(\tilde{M}-1/2)(\tilde{M}-3/2)}\\&+s_f\frac{(\ell+\frac{N-1}{2})\cos x}{2}\left(\frac{-\tilde{M}+a+\frac{N-1}{2}}{\tilde{M}-3/2}+\frac{1-\tilde{M}+a+\frac{N-1}{2}}{\tilde{M}-1/2}\right)\\&+\left(-\tilde{M}+a+\frac{N-1}{2}\right)\left(1-\tilde{M}+a+\frac{N-1}{2}\right)\\&-\sin^2 x\left(-\tilde{M}+a+\frac{N-1}{2}\right)\left(2-\tilde{M}+a+\frac{N-1}{2}\right),\end{aligned} \tag{4.376}$$

with $s_f=1$ if $\hat{f}^{(a)}_{\tilde{M}\ell}(t)=\hat{\phi}^{(a)}_{\tilde{M}\ell}(t)$ and $s_f=-1$ if $\hat{f}^{(a)}_{\tilde{M}\ell}(t)=\hat{\psi}^{(a)}_{\tilde{M}\ell}(t)$.
Now we will verify the 'time-time component' of eq. (4.153) with negative spin projection ($\sigma=\ -$), i.e.

$$\Psi_{tt}^{\left(I;-;(\pm\frac{N-2}{2})\ell;\tilde{\rho}\right)}(t,\boldsymbol{\theta}_{N-1})=\left(\nabla_t\nabla_t\pm i\gamma_t\nabla_t+\frac{3}{4}g_{tt}\right)\varphi_{\pm}^{(-;\ell;\tilde{\rho})}(t,\boldsymbol{\theta}_{N-1}). \tag{4.377}$$

Since the spinors $\varphi_{\pm}^{(\sigma;\ell;\tilde{\rho})}(t,\boldsymbol{\theta}_{N-1})$ satisfy the Dirac equation $\left[\nabla\!\!\!\!/\pm i(N+2)/2\right]\varphi_{\pm}^{(\sigma;\ell;\tilde{\rho})}=0$, they are given by [26]

$$\varphi_{\pm}^{(-;\ell;\tilde{\rho})}(t,\boldsymbol{\theta}_{N-1})=\frac{4}{\ell(\ell-1)}\begin{pmatrix}\hat{\phi}^{(0)}_{(\frac{N+2}{2})\ell}(t)\,\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\ \mp i\hat{\psi}^{(0)}_{(\frac{N+2}{2})\ell}(t)\,\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\end{pmatrix}, \tag{4.378}$$

$$\varphi_{\pm}^{(+;\ell;\tilde{\rho})}(t,\boldsymbol{\theta}_{N-1})=\frac{4}{\ell(\ell-1)}\begin{pmatrix}i\hat{\psi}^{(0)}_{(\frac{N+2}{2})\ell}(t)\,\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\ \mp\hat{\phi}^{(0)}_{(\frac{N+2}{2})\ell}(t)\,\chi_{+\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\end{pmatrix}, \tag{4.379}$$

where the factor $4/\left(\ell\left[\ell-1\right]\right)$ is motivated naturally below. On the other hand, the tt-component of the type-I mode of the partially massless spin-5/2 field is given by

$$\Psi_{tt}^{\left(I;-;(\pm\frac{N-2}{2})\ell;\tilde{\rho}\right)}(t,\boldsymbol{\theta}_{N-1})=\ (-1)\times\begin{pmatrix}\hat{\phi}^{(2)}_{(\frac{N-2}{2})\ell}(t)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\\ \mp i\hat{\psi}^{(2)}_{(\frac{N-2}{2})\ell}(t)\chi_{-\ell\tilde{\rho}}(\boldsymbol{\theta}_{N-1})\end{pmatrix}. \tag{4.380}$$

By substituting eqs. (4.378) and (4.380) into eq. (4.377) we find

$$\begin{aligned}\frac{\ell(\ell-1)}{4}\hat{\phi}^{(2)}_{(\frac{N-2}{2})\ell}(t)&=\frac{\ell(\ell-1)}{\sin^2 x}\hat{\phi}^{(0)}_{(\frac{N-2}{2})\ell}(t)\\&=\left(\frac{\partial^2}{\partial x^2}+\frac{3}{4}\right)\hat{\phi}^{(0)}_{(\frac{N+2}{2})\ell}(t)+\frac{\partial}{\partial x}\hat{\psi}^{(0)}_{(\frac{N+2}{2})\ell}(t)\end{aligned} \tag{4.381}$$

$$\begin{aligned}\frac{\ell(\ell-1)}{4}\hat{\psi}^{(2)}_{(\frac{N-2}{2})\ell}(t)&=\frac{\ell(\ell-1)}{\sin^2 x}\hat{\psi}^{(0)}_{(\frac{N-2}{2})\ell}(t)\\&=\left(\frac{\partial^2}{\partial x^2}+\frac{3}{4}\right)\hat{\psi}^{(0)}_{(\frac{N+2}{2})\ell}(t)-\frac{\partial}{\partial x}\hat{\phi}^{(0)}_{(\frac{N+2}{2})\ell}(t).\end{aligned} \tag{4.382}$$

Equation (4.381) is proved using the lowering operator (4.374) as follows. First, we express $\partial\hat{\psi}^{(0)}_{(\frac{N+2}{2})\ell}/\partial x$ in eq. (4.381) in terms of $\partial\hat{\phi}^{(0)}_{(\frac{N+2}{2})\ell}/\partial x$ and $\hat{\phi}^{(0)}_{(\frac{N+2}{2})\ell}$ by making use of the formulae (4.354) and (4.355). Then, after a long calculation, we rewrite eq. (4.381) as

$$\frac{\ell(\ell-1)}{\sin^2 x}\hat{\phi}^{(0)}_{(\frac{N-2}{2})\ell} = \frac{(N-1)(N+1)}{N(N+2)\sin^2 x}\left(\hat{L}_f^{(\frac{N}{2};a=0)}\hat{L}_f^{(\frac{N+2}{2};a=0)}\hat{\phi}^{(0)}_{(\frac{N+2}{2})\ell}\right), \tag{4.383}$$

which is readily verified using the lowering relation (4.374). Equation (4.382) is proved in the same way. Thus, we have verified the tt-component of the 'pure gauge' expression (4.153).

Let us now show that our 'pure gauge' expression for the type-I modes $\Psi^{(I;\sigma;(\tilde{M}=+1)\ell;\tilde{\rho})}_{\mu\nu}$ on dS_4 in eq. (4.153) is equal to the gamma-traceless part of the gauge transformation that is proposed in Ref. [9] (for a specific choice of the spinor gauge function in the gauge transformation of Ref. [9]). In order to compare our results with the results of Ref. [9] we let $N=4$ and $\tilde{M}=+(N-2)/2=+1$ in eq. (4.153). [Now, the spinors $\varphi_+^{(\sigma;\ell;\tilde{\rho})}$ in eq. (4.153) satisfy $\nabla\!\!\!/\,\varphi_+^{(\sigma;\ell;\tilde{\rho})} = -3i\varphi_+^{(\sigma;\ell;\tilde{\rho})}$.] By using units in which the cosmological constant is $\Lambda=3$, the gauge transformation for the partially massless spin-5/2 field $\psi_{\mu\nu}$ in Ref. [9] is

$$\delta\psi_{\mu\nu} = \left(\nabla_{(\mu}\nabla_{\nu)} - \frac{1}{4}\gamma_{(\mu}\nabla_{\nu)}\nabla\!\!\!/ + \frac{15}{16}g_{\mu\nu}\right)\epsilon \tag{4.384}$$

$$= \left(\nabla_{(\mu}\nabla_{\nu)} + \frac{3i}{4}\gamma_{(\mu}\nabla_{\nu)} + \frac{15}{16}g_{\mu\nu}\right)\epsilon, \tag{4.385}$$

where we have chosen ϵ to be a solution of the equation $\nabla\!\!\!/\,\epsilon = -3i\,\epsilon$. (For this choice it is clear that our spinors $\varphi_+^{(\sigma;\ell;\tilde{\rho})}$ are the mode functions corresponding to the field ϵ.) Note that for this choice of ϵ the gauge transformation of the auxiliary field is zero - see Ref. [9]. Also, for this choice of ϵ it can be readily verified that $g^{\mu\nu}\delta\psi_{\mu\nu}=0$, but $\gamma^\mu\delta\psi_{\mu\nu}\neq 0$. Let $\delta\psi'_{\mu\nu}$ be the gamma-traceless part of $\delta\psi_{\mu\nu}$, i.e.

$$\delta\psi'_{\mu\nu} = \delta\psi_{\mu\nu} - \frac{\gamma_\mu}{6}\gamma^\alpha\delta\psi_{\alpha\nu} - \frac{\gamma_\nu}{6}\gamma^\alpha\delta\psi_{\alpha\mu}, \tag{4.386}$$

where $\gamma^\alpha\delta\psi'_{\alpha\nu}=0$ and $g^{\mu\nu}\delta\psi'_{\mu\nu}=0$. Then, we can straightforwardly show that

$$\delta\psi'_{\mu\nu} = \left(\nabla_{(\mu}\nabla_{\nu)} + i\gamma_{(\mu}\nabla_{\nu)} + \frac{3}{4}g_{\mu\nu}\right)\epsilon, \tag{4.387}$$

which is in precise agreement with the expression for our type-I modes in eq. (4.153).

REFERENCES

[1] http://functions.wolfram.com/07.23.03.0024.01. visited on 14th April 2020.

[2] http://functions.wolfram.com/07.23.03.0026.01. visited on 14th April 2020.

[3] A. Barut and R. Raczka. *Theory of Group Representations and Applications*. WORLD SCIENTIFIC, 1986.

[4] T. Basile, X. Bekaert, and N. Boulanger. "Mixed-symmetry fields in de Sitter space: a group theoretical glance". In: *Journal of High Energy Physics* 2017 (2016), p. 1.

[5] R. Camporesi and A. Higuchi. "On the eigenfunctions of the Dirac operator on spheres and real hyperbolic spaces". In: *J. Geom. Phys.* 20 (1996), p. 1.

[6] C.-H. Chen, H. T. Cho, A. S. Cornell, and G. Harmsen. "Spin-3/2 fields in D-dimensional Schwarzschild black hole spacetimes". In: *Phys. Rev. D* 94 (2016), p. 044052.

[7] S. Deser and A. Waldron. "Arbitrary spin representations in de Sitter from dS / CFT with applications to dS supergravity". In: *Nuclear Physics* 662 (2003), p. 379.

[8] S. Deser and A. Waldron. "Conformal invariance of partially massless higher spins". In: *Physics Letters B* 603 (2004), p. 30.

[9] S. Deser and A. Waldron. "Gauge invariances and phases of massive higher spins in (A)dS". In: *Phys. Rev. Lett.* 87 (2001), p. 031601. eprint: arXiv:hep-th/0102166.

[10] S. Deser and A. Waldron. "Null propagation of partially massless higher spins in (A)dS and cosmological constant speculations". In: *Phys. Lett. B* 513 (2001), p. 137. eprint: hep-th/0105181.

[11] S. Deser and A. Waldron. "Partial masslessness of higher spins in (A)dS". In: *Nuclear Physics* 607 (2001), p. 577.

[12] S. Deser and A. Waldron. "Stability of massive cosmological gravitons". In: *Phys. Lett. B* 508 (2001), p. 347. eprint: arXiv:hep-th/0103255.

[13] C. Destri, C. A. Orzalesi, and P. Rossi. "Matter fields and metric deformation in multidimensional unified theories". In: *Annals of Physics* 147 (1983).

[14] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.0.26 of 2020-03-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds. visited on 14th April 2020.

[15] V. K. Dobrev, G. Mack, V. B. Petkova, S. G. Petrova, and I. T. Todorov. "Harmonic Analysis on the n-Dimensional Lorentz Group and Its Application to Conformal Quantum Field Theory". In: *Lecture Notes in Physics* 63 (1977).

[16] D. Z. Freedman and A. V. Proeyen. *Supergravity*. Cambridge University Press, 2012.

[17] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. seventh. Elsevier/Academic Press, 2007.

[18] A. Higuchi. "Forbidden mass range for spin-2 field theory in de Sitter spacetime". In: *Nuclear Physics B* 282 (1987), pp. 397–436.

[19] A. Higuchi. "Linearized gravity in de Sitter spacetime as a representation of $SO(4,1)$". In: *Classical and Quantum Gravity* 8 (1991), pp. 2005–2021.

[20] A. Higuchi. "Massive symmetric tensor field in spacetimes with a positive cosmological constant". In: *Nuclear Physics B* 325 (1989), pp. 745–765.

[21] A. Higuchi. "Quantum fields of nonzero spin in De Sitter spacetime". PhD thesis. Yale University, 1987.

[22] A. Higuchi. "Symmetric tensor spherical harmonics on the N-sphere and their application to the de Sitter group $SO(N,1)$". In: *J. Math. Phys.* 28 (1987), p. 1553.

[23] A. Higuchi. Private communication. 2022.

[24] Y. Homma and T. Tomihisa. "The spinor and tensor fields with higher spin on spaces of constant curvature". In: *Annals of Global Analysis and Geometry* 60 (2021), pp. 829–861.

[25] V. A. Letsios. "(Non-)unitarity of strictly and partially massless fermions on de Sitter space". In: *Journal of High Energy Physics* 2023 (2023), p. 15.

[26] V. A. Letsios. "The eigenmodes for spinor quantum field theory in global de Sitter spacetime". In: *Journal of Mathematical Physics* 62 (2021), p. 032303.

[27] T. Ortín. "A note on Lie-Lorentz derivatives". In: *Classical and Quantum Gravity* 19 (2002), p. L143.

[28] K. Pilch, P. van Nieuwenhuizen, and M. F. Sohnius. "De Sitter Superalgebras and Supergravity". In: *Communications in Mathematical Physics* 98 (1985), p. 105.

[29] M. A. Rakha, A. K. Rathie, and P. Chopra. "On some new contiguous relations for the Gauss hypergeometric function with applications". In: *Comput. Math. with Appl.* 61.3 (2011), pp. 620–629. ISSN: 0898-1221.

— 5 —

New conformal-like symmetry of strictly massless fermions in four-dimensional de Sitter space

Abstract

We present new infinitesimal 'conformal-like' symmetries for the field equations of strictly massless spin-$s \geq 3/2$ totally symmetric tensor-spinors (i.e. gauge potentials) on 4-dimensional de Sitter spacetime (dS_4). The corresponding symmetry transformations are generated by the five conformal Killing vectors of dS_4, but they are not conventional conformal transformations. We show that the algebra generated by the ten de Sitter (dS) symmetries and the five conformal-like symmetries closes on the conformal-like algebra $so(4,2)$ up to gauge transformations of the gauge potentials. Furthermore, we demonstrate that the two sets of physical mode solutions, corresponding to the two helicities $\pm s$ of the strictly massless theories, form a direct sum of Unitary Irreducible Representations (UIRs) of the conformal-like algebra. We also fill a gap in the literature by explaining how these physical modes form a direct sum of Discrete Series UIRs of the dS algebra $so(4,1)$.

5.1 INTRODUCTION

Four-dimensional de Sitter spacetime (dS_4) is believed to be a good approximation of the very early epoch of our Universe (Inflation). Also, according to recent data indicating the accelerated expansion of space [50, 48, 43], there is evidence to suggest that our Universe is asymptotically approaching another de Sitter phase.

The D-dimensional de Sitter spacetime (dS_D) is the maximally symmetric solution of the vacuum Einstein equations with positive cosmological constant Λ,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0, \tag{5.1}$$

where $g_{\mu\nu}$ is the de Sitter metric tensor, $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar. In this paper, we use units in which $2\Lambda = (D-1)(D-2)$, while the Riemann tensor is

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}\, g_{\nu\sigma} - g_{\nu\rho}\, g_{\mu\sigma}. \tag{5.2}$$

De Sitter (dS) field theories are known to exhibit characteristics with no Minkowskian analogs. Two such interesting characteristics of integer-spin fields on dS_D - related to the representation theory of the dS algebra $so(D,1)$ - are:

- The existence of unitarily forbidden ranges for the mass parameters of integer-spin fields depending on both D and the spin of the fields [26, 29].
- The existence of exotic unitary "partially massless" fields for spin $s \geq 2$ [12, 13, 29, 16, 17, 18].

For the sake of completeness, let us give here some details about these two field-theoretic characteristics. As discovered by Higuchi [26, 29, 28], massive (i.e. non-gauge-invariant) totally symmetric tensor fields of spin $s \geq 1$, satisfying

$$\begin{aligned} &\Big(\nabla^\alpha \nabla_\alpha - m^2 + (s-2)(s+D-3) - s\Big) h_{\mu_1 \dots \mu_s} = 0, \\ &\nabla^\alpha h_{\alpha\mu_2\dots\mu_s} = 0, \quad g^{\alpha\beta} h_{\alpha\beta\mu_3\dots\mu_s}, \end{aligned} \tag{5.3}$$

are always non-unitary for the following values of the mass parameter:

$$m^2 < (s-1)(s+D-4).$$

Unitary massive theories thus obey the 'Higuchi bound'

$$m^2 > (s-1)(s+D-4). \tag{5.4}$$

Moreover, Higuchi observed that for the following special values of the mass parameter [26, 29, 28]:

$$m^2 = (\tau - 1)(2s + D - 4 - \tau), \qquad (\tau = 1, ..., s), \tag{5.5}$$

the theory is unitary, while at the same time it enjoys a gauge symmetry. A field with mass parameter given by Eq. (5.5) is a gauge potential known as partially massless field of depth τ in the modern literature [16, 17, 18].[1] The case with $\tau = 1$ corresponds to the theory known as strictly massless. In 4 dimensions, a strictly massless field has two propagating helicity degrees of freedom $\pm s$, while a partially massless field of depth τ has 2τ of them: $(\pm s, \pm(s-1), ..., \pm(s-\tau+1))$ [16, 17, 18]. In dS field theory, the strictly massless fields are the closest analogs of Minkowskian massless fields, while partially massless fields of depth $\tau > 1$ have no Minkowsian counterparts. The Unitary Irreducible Representations (UIRs) of the dS algebra $so(D,1)$ corresponding to totally symmetric strictly/partially massless integer-spin fields were first discussed in Higuchi's PhD thesis [29, 28]. More recent discussions concerning both totally symmetric and mixed symmetry integer-spin fields can be found in Ref. [7].

What about the representation theory of fermions on dS_D?

Unlike the integer-spin case, the representation-theoretic properties of fermionic fields on dS_D are not well-studied.

Recent results. Although the phenomena of strict and partial masslessness also occur in the case of spin-$s \geq 3/2$ fermionic fields on dS_D [16, 17, 18], the study of the corresponding unitarity properties was absent from the (mathematical) physics literature for a long time. Interestingly, as the author has recently shown [33, 35], four-dimensional dS space plays a distinguished role in the unitarity of strictly/partially massless (totally symmetric) tensor-spinors on dS_D $(D \geq 3)$. More specifically, the representations of the dS algebra $so(D,1)$, which correspond to strictly/partially massless totally symmetric tensor-spinors of spin $s \geq 3/2$, are non-unitary unless $D = 4$ [33, 34, 35].[2]

[1]The partially massless spin-2 field was first discovered by Deser and Nepomechie [12, 13].

[2]**Note:** For the cases with spin $s = 3/2, 5/2$, these results were obtained following two different approaches: a) on the one hand, by performing a technical analysis of the representation-theoretic properties of the mode solutions on global dS_D (this includes constructing the mode solutions explicitly, studying their transformation properties under $so(D,1)$, and investigating the existence/non-existence of dS invariant, positive definite scalar products in the space of mode solutions) [34, 35], and, b) on the other hand, by carefully examining the list of the dS algebra UIRs in the decomposition $so(D,1) \supset so(D)$ [33] and inferring (non-)unitarity from the (mis-)match between the UIR and the field-theoretic representation labels. For the cases with spin $s \geq 7/2$, the results were obtained in Ref. [33] motivated only by the examination of the $so(D,1)$ UIRs and the (mis-)match of the representation labels. The technical analysis of the representation-theoretic properties of the mode solutions with spin $s \geq 7/2$ is still absent from the literature for $D \neq 4$ (the $D = 4$ case is studied in the present paper). Thus, if we want to be careful - as we should, if we wish to avoid the representation-theoretic confusion that appeared in the past dS literature - the results of Ref. [33] for $s \geq 7/2$ may be viewed as a "suggestion" motivated by the examination of the $so(D,1)$ UIRs. This suggestion can be confirmed by studying the representation-theoretic properties of the spin-$s \geq 7/2$ mode solutions on dS_D, as in the spin-$s = 3/2, 5/2$ cases [34, 35]. This is something that we leave for future work.

5.1. Introduction

In the present paper, we uncover a new group-theoretic feature of all strictly massless totally symmetric spin-$s \geq 3/2$ tensor-spinors on dS_4: these fermionic gauge potentials possess a conformal-like $so(4,2)$ global symmetry algebra. Moreover, we show that the mode solutions with fixed helicity, i.e. the modes forming Unitary Irreducible Representations (UIRs) of the dS algebra $so(4,1)$ [33], also form UIRs of the larger conformal-like $so(4,2)$ algebra.

5.1.1 List of main results and methodology

Here we give some information about our main results and investigations concerning the new conformal-like symmetries of strictly massless fermions on dS_4.

- We present new conformal-like infinitesimal transformations (5.80) for strictly massless totally symmetric tensor-spinors on dS_4. These new transformations are generated by conformal Killing vectors of dS_4 and they are symmetries of the field equations [Eqs. (5.20) and (5.21)], i.e. they preserve the solution space of the field equations. In this paper, by conformal Killing vectors we mean the five genuine conformal Killing vectors of dS_4 with non-vanishing divergence - see Eq. (5.77).
- The conformal-like transformations (5.80), together with the ten known dS transformations (5.14), generate an algebra that is isomorphic to $so(4,2)$. However, this conformal-like algebra closes up to field-dependent gauge transformations.
- We fill a gap in the literature by clarifying the way in which the spin-$s \geq 3/2$ physical (i.e. non-gauge) mode solutions with fixed helicity form a direct sum of Discrete Series UIRs of the dS algebra $so(4,1)$. The modes with opposite helicity correspond to different UIRs - this is also true in the case of strictly massless totally symmetric tensors [28]. (Recall that a strictly massless field has only two propagating helicities $\pm s$ corresponding to two sets of physical mode solutions with opposite helicities.)
- Then, we show that the physical mode solutions also form a direct sum of UIRs of the conformal-like $so(4,2)$ algebra. We arrive at this result by following two basic steps (which stem from the mathematical definitions of representation-theoretic irreducibility and unitarity). First, we show that the mode solutions with fixed helicity transform among themselves under all $so(4,2)$ transformations (this means under the ten dS isometries (5.14), as well as the five conformal-like symmetries (5.80)). Then, we show that there is a $so(4,2)$-invariant, and gauge-

invariant, positive definite scalar product for each set of mode solutions with fixed helicity.

- As the name suggests, our conformal-like symmetry transformations are **not** conventional infinitesimal conformal transformations. This is exemplified as follows. For the cases with spin $s = 3/2, 5/2$, by investigating the conformal-like transformations of the field strength tensor-spinors (i.e. curvatures) of the strictly massless fermions[3], we find that these transformations correspond to the product of two transformations: an infinitesimal axial rotation (i.e. multiplication with γ^5) times an infinitesimal conformal transformation. For the cases with spin $s \geq 7/2$, we present a (justified) conjecture concerning the expressions for the conformal-like transformations of the field-strength tensor-spinors.

We conclude this part of the Introduction with a brief literature review. The UIRs of $so(4,1)$ corresponding to certain fermions on dS_4 have been also discussed in Ref. [21]. The mode solutions and the Quantum Field Theory of spin-1/2 fermions on dS_D have been discussed in various articles, such as Refs. [8, 36, 42, 46, 38, 40, 6, 49, 11, 10, 21, 31, 2]. The invariance of maximal-depth integer-spin partially massless theories on dS_4 under conformal transformations has been investigated in Ref. [15] - however, interestingly, a representation-theoretic study suggests that the associated symmetry algebra does not correspond to the conformal algebra [4].

5.1.2 Outline, notation, and conventions

The rest of this paper is organised as follows. In Section 5.2, we review the basics concerning (strictly massless) tensor-spinors on dS_4. In Section 5.3, we review the classification of the UIRs of the dS algebra $so(4,1)$. In Section 5.4, we discuss the (pure gauge and physical) mode solutions for strictly massless fermions of spin $s \geq 3/2$ on global dS_4. In particular, we use the method of separation of variables to express the physical mode solutions on global dS_4 in terms of tensor-spinor spherical harmonics on S^3 (these spherical harmonics are not constructed explicitly here). We also identify the analogs of the flat-space positive and negative frequency modes. In Section 5.5, we discuss the way in which the (positive frequency) physical modes with fixed helicity form a direct sum of Discrete Series UIRs of $so(4,1)$. In Section 5.6, we present our new

[3] The field strength tensor(-spinor), also known as "generalised Weyl tensor(-spinor)" (see, e.g. [4]), is invariant under gauge transformations. It plays the role of the electromagnetic tensor $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$ in the case of the $U(1)$ gauge potential A_μ - or, likewise, the role of the linearised Weyl tensor in the case of the spin-2 gauge potential (graviton) in linearised gravity.

conformal-like symmetry transformations and we show that the associated symmetry algebra (generated by both dS and conformal-like transformations) closes on $so(4,2)$ up to gauge transformations. In Section 5.7, we show that the physical modes that form a direct sum of $so(4,1)$ UIRs, also form a direct sum of $so(4,2)$ UIRs. In Section 5.8, we discuss the conformal-like transformations of the gauge invariant filed strength tensor-spinors. There are two Appendices, 5.10 and 5.11, in which we include technical details that were omitted in the main text.

Notation and conventions. We use the mostly plus metric sign convention for dS_4. Lowercase Greek tensor indices refer to components with respect to the 'coordinate basis' on dS_4. Coordinate basis tensor indices on S^3 are denoted as $\tilde{\mu}_1, \tilde{\mu}_2, \dots$. Lowercase Latin tensor indices refer to components with respect to the vielbein basis. Repeated indices are summer over. We denote the symmetrisation of indices with the use of round brackets, e.g. $A_{(\mu\nu)} \equiv (A_{\mu\nu} + A_{\nu\mu})/2$, and the anti-symmetrisation with the use of square brackets, e.g. $A_{[\mu\nu]} \equiv (A_{\mu\nu} - A_{\nu\mu})/2$. Spinor indices are always suppressed throughout this paper. The rank of spin-s tensor-spinors on dS_4 is r (i.e. $s = r + 1/2$). The complex conjugate of the number z is denoted as z^*. By conformal Killing vector we mean a genuine conformal Killing vector of dS_4 with non-vanishing divergence - see Eq. (5.77).

5.2 BACKGROUND MATERIAL FOR STRICTLY MASSLESS FERMIONS ON dS_4

5.2.1 Field equations for higher-spin fermions on dS_4

Fermions of spin $s \equiv r + 1/2 \geq 3/2$ on dS_4 can be described by totally symmetric tensor-spinors $\Psi_{\mu_1 \dots \mu_r}$ that satisfy the on-shell conditions [17, 14, 45]:

$$\left(\nabla\!\!\!\!/ + M\right) \Psi_{\mu_1 \dots \mu_r} = 0 \tag{5.6}$$

$$\nabla^\alpha \Psi_{\alpha \mu_2 \dots \mu_r} = 0, \quad \gamma^\alpha \Psi_{\alpha \mu_2 \dots \mu_r} = 0, \tag{5.7}$$

where M is the mass parameter, γ^α are the four gamma matrices and $\nabla\!\!\!\!/ = \gamma^\nu \nabla_\nu$ is the Dirac operator. We call the conditions in Eq. (5.7) the transverse-traceless (TT) conditions.

The 'curved space gamma matrices', $\gamma^\mu(x)$, are defined with the use of the vierbein fields as $\gamma^\mu(x) = e^\mu{}_b(x)\gamma^b$, where γ^b ($b = 0, 1, 2, 3$) are the spacetime-independent gamma matrices. The gamma matrices $\gamma^\mu(x)$ satisfy the anti-commutation relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}\,\mathbf{1}, \tag{5.8}$$

where $\mathbf{1}$ is the spinorial identity matrix. The vierbein and co-vierbein fields satisfy

$$e_\mu{}^a\, e_\nu{}^b \eta_{ab} = g_{\mu\nu}, \quad e^\mu{}_a\, e_\mu{}^b = \delta^b_a, \tag{5.9}$$

where $\eta_{ab} = diag(-1,1,1,1)$. The fifth gamma matrix is determined as [23]

$$\epsilon^{\mu\nu\rho\sigma} = i\gamma^5 \gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]}, \tag{5.10}$$

where $\epsilon_{\mu\nu\rho\sigma}$ are the components of the dS_4 volume element. In the vierbein (i.e. orthonormal frame) basis we have $\epsilon_{0123} = +1$. The matrix γ^5 anti-commutes with the other four gamma matrices, and, hence, with the Dirac operator.
The derivative ∇_ν acts on our totally symmetric tensor-spinors as

$$\nabla_\nu \Psi_{\mu_1 \dots \mu_r} = \left(\partial_\nu + \frac{1}{4}\omega_{\nu bc}\gamma^b\gamma^c\right)\Psi_{\mu_1\dots\mu_r} - r\,\Gamma^\lambda_{\;\nu(\mu_1}\Psi_{\mu_2\dots\mu_r)\lambda}, \tag{5.11}$$

where $\Gamma^\lambda_{\;\nu\mu}$ are the Christoffel symbols. The spin connection, $\omega_{\nu bc} = \omega_{\nu[bc]} = e_\nu{}^a\omega_{abc}$, is determined as

$$\partial_\mu e^\rho{}_b + \Gamma^\rho_{\mu\sigma}e^\sigma{}_b - \omega_\mu{}^c{}_b\, e^\rho{}_c = 0. \tag{5.12}$$

The gamma matrices are covariantly constant, $\nabla_\nu \gamma^\mu = 0$. The commutator of covariant derivatives acting on totally symmetric tensor-spinors is given by

$$\begin{aligned}[\nabla_\mu, \nabla_\nu]\Psi_{\mu_1\dots\mu_r} = & \frac{1}{2}(\gamma_\mu\gamma_\nu - g_{\mu\nu})\Psi_{\mu_1\dots\mu_r} \\ & + r\left(g_{\mu(\mu_1}\Psi_{\mu_2\dots\mu_r)\nu} - g_{\nu(\mu_1}\Psi_{\mu_2\dots\mu_r)\mu}\right).\end{aligned} \tag{5.13}$$

5.2.2 Basics about dS symmetries of the field equations

The dS algebra is generated by the ten Killing vectors of dS_4 satisfying $\nabla_{(\mu}\xi_{\nu)} = 0$. The dS generators act on solutions $\Psi_{\mu_1\dots\mu_r}$ in terms of the spinorial generalisation of the Lie derivative - also known as the Lie-Lorentz derivative [32, 39]. The Lie-Lorentz derivative acts on arbitrary tensor-spinors as follows:

$$\begin{aligned}\mathbb{L}_\xi \Psi_{\mu_1\dots\mu_r} = & \;\xi^\nu \nabla_\nu \Psi_{\mu_1\dots\mu_r} + \nabla_{\mu_1}\xi^\nu\,\Psi_{\nu\mu_2\dots\mu_r} + \nabla_{\mu_2}\xi^\nu\,\Psi_{\mu_1\nu\mu_3\dots\mu_r} + \dots + \nabla_{\mu_r}\xi^\nu\,\Psi_{\mu_1\dots\mu_{r-1}\nu} \\ & + \frac{1}{4}\nabla_\kappa\xi_\lambda\gamma^{\kappa\lambda}\Psi_{\mu_1\dots\mu_r}.,\end{aligned} \tag{5.14}$$

where $\gamma^{\kappa\lambda} = \gamma^{[\kappa}\gamma^{\lambda]}$. The Lie-Lorentz derivative $\mathbb{L}_\xi \Psi_{\mu_1 \dots \mu_r}$ conveniently describes the infinitesimal $so(4,1)$ transformation of $\Psi_{\mu_1 \dots \mu_r}$ generated by the Killing vector ξ^μ. From the properties [39]:

$$\begin{aligned} &\mathbb{L}_\xi \gamma^a = 0, \\ &\mathbb{L}_\xi \nabla_\nu \Psi_{\mu_1 \dots \mu_r} = \nabla_\nu \mathbb{L}_\xi \Psi_{\mu_1 \dots \mu_r}, \end{aligned} \tag{5.15}$$

it follows that if $\Psi_{\mu_1 \dots \mu_r}$ is a solution of Eqs. (5.6) and (5.7), then so is $\mathbb{L}_\xi \Psi_{\mu_1 \dots \mu_r}$. In other words, the Lie-Lorentz derivative is a symmetry of the field equations for any value of M. It is easy to conclude that the associated symmetry algebra is isomorphic to $so(4,1)$ as [39]

$$[\mathbb{L}_\xi, \mathbb{L}_{\xi'}] \Psi_{\mu_1 \dots \mu_r} = \mathbb{L}_{[\xi,\xi']} \Psi_{\mu_1 \dots \mu_r} \tag{5.16}$$

for any two Killing vectors ξ^μ and ξ'^μ.

The dS algebra, $so(4,1)$, has four non-compact generators ('dS boosts') and six compact ones ('dS rotations'). The compact generators generate the $so(4)$ rotational subalgebra of $so(4,1)$. For any fixed value of M, the mode solutions of the field equations (5.6) and (5.7) form an infinite-dimensional representation of $so(4,1)$. The eigenvalue of the quadratic Casimir for this representation is given by [33]

$$\mathscr{C} = \sum_{\text{dS boosts}} \mathbb{L}_\xi \mathbb{L}_\xi - \sum_{\xi \in so(4)} \mathbb{L}_\xi \mathbb{L}_\xi = -M^2 - \frac{9}{4} + s(s+1), \tag{5.17}$$

where $s = r + 1/2 \geq 3/2$ [4]. The unitarity of the representation depends on the value of the mass parameter M [33]. In this paper, we are interested in the strictly massless theories, which appear for special imaginary values of M (see Subsection 5.2.3) - for discussions on arbitrary values of M in any spacetime dimension see Ref. [33].

5.2.3 STRICTLY MASSLESS FERMIONS ON dS_4

For real values of M, Eqs. (5.6) and (5.7) describe a unitary massive theory with $2s+1$ propagating degrees of freedom [17, 14]. The theory enjoys a gauge symmetry for each of the following imaginary tunings of M [17, 14]:

$$M^2 = -\left(r - \tau + 1\right)^2. \tag{5.18}$$

[4] The expression (5.17) for the quadratic Casimir is also true for spin-1/2 fields.

As in the bosonic case discussed in the Introduction, the value $\tau = 1$ corresponds to the strictly massless theory with two propagating helicities. Each of the values $\tau = 2, ..., r$ corresponds to a partially massless field with 2τ helicities: $(\pm s, \pm(s-1), ..., \pm(s-\tau+1))$ [17].

In this paper, we are interested in the equations for strictly massless fermions, i.e. Eqs. (5.6) and (5.7) with mass parameter given by [17, 14]

$$M = \pm i\, r. \tag{5.19}$$

Strict masslessness occurs for either of the two signs for the mass parameter in Eq. (5.19). However, the representations of $so(4,1)$ corresponding to the '$+$' sign are equivalent to the representations corresponding to the '$-$' sign [33]. This is easy to understand as, if $\Psi_{\mu_1...\mu_r}$ satisfies

$$\nabla\!\!\!\!/\, \Psi_{\mu_1...\mu_r} = -M\, \Psi_{\mu_1...\mu_r},$$

then the field $\Psi'_{\mu_1...\mu_r} \equiv \gamma^5 \Psi_{\mu_1...\mu_r}$ satisfies

$$\nabla\!\!\!\!/\, \Psi'_{\mu_1...\mu_r} = +M\, \Psi'_{\mu_1...\mu_r},$$

while, also, γ^5 commutes with all dS transformations (5.14) [33].

Based on the discussion of the previous paragraph, below we will only discuss the field with the '$+$' sign in Eq. (5.19). Thus, from now on, $\Psi_{\mu_1...\mu r}$ denotes the strictly massless field satisfying

$$\left(\nabla\!\!\!\!/ + ir\right) \Psi_{\mu_1...\mu_r} = 0, \tag{5.20}$$

$$\nabla^\alpha \Psi_{\alpha\mu_2...\mu_r} = 0, \quad \gamma^\alpha \Psi_{\alpha\mu_2...\mu_r} = 0. \tag{5.21}$$

Equations (5.6) and (5.7) are invariant under the following restricted gauge transformations:

$$\delta^{res} \Psi_{\mu_1...\mu_r} = \nabla_{(\mu_1} \lambda_{\mu_2...\mu_r)} + \frac{i}{2} \gamma_{(\mu_1} \lambda_{\mu_2...\mu_r)}, \tag{5.22}$$

where the gauge functions $\lambda_{\mu_2...\mu_r}$ are totally symmetric tensor-spinors of rank $r-1$ that satisfy

$$\left(\nabla\!\!\!\!/ + i(r+1)\right) \lambda_{\mu_2...\mu_r} = 0, \tag{5.23}$$

$$\nabla^\alpha \lambda_{\alpha\mu_3...\mu_r} = 0, \quad \gamma^\alpha \lambda_{\alpha\mu_3...\mu_r} = 0. \tag{5.24}$$

For $r = 0$, i.e. in the case of the massless spin-1/2 field satisfying $\nabla\!\!\!\!/\, \Psi = 0$, the theory does not have gauge symmetry.

5.3 CLASSIFICATION OF THE UIRS OF THE DS ALGEBRA

In this Section, we review the classification of the $so(4,1)$ UIRs in the decomposition $so(4,1) \supset so(4)$ [47, 41]. An irreducible representation of $so(4)$ appears at most once in a UIR of $so(4,1)$ [19].
Let us recall that an irreducible representation of $so(4)$ is specified by the highest weight [5, 20]

$$\vec{f} = (f_1, f_2), \tag{5.25}$$

where

$$f_1 \geq |f_2|. \tag{5.26}$$

The numbers f_1 and f_2 are both integers or half-odd integers, while f_2 can be negative. The representation $(f_1, -f_2)$ is the 'mirror image' of (f_1, f_2) [51].
UIRs of $so(4,1)$. A UIR of $so(4,1)$ is specified by two numbers $\vec{F} = (F_0, F_1)$. The number $F_1 \geq 0$ is an integer or half-odd integer. For the $so(4)$ representations $\vec{f} = (f_1, f_2)$ contained in the UIR $\vec{F} = (F_0, F_1)$ we have:

$$f_1 \geq F_1 \geq |f_2|. \tag{5.27}$$

The UIRs of $so(4,1)$ are listed below [47, 41]:

- **Principal Series $D_{\text{prin}}(\vec{F})$:**

  $$F_0 = -\frac{3}{2} + iy, \quad (y > 0). \tag{5.28}$$

  F_1 is an integer or half-odd integer.

- **Complementary Series $D_{\text{comp}}(\vec{F})$:**

  $$-\frac{3}{2} \leq F_0 < -\tilde{n}, \quad \tilde{n} \in \{0, 1\}. \tag{5.29}$$

  If $\tilde{n} = 0$, then $F_1 = 0$, while for the $so(4)$ content we have $f_2 = 0$. If $\tilde{n} = 1$, then F_1 is a positive integer.

- **Exceptional Series $D_{\text{ex}}(\vec{F})$:**

  $$F_0 = -1. \tag{5.30}$$

  F_1 is a positive integer, while $f_2 = 0$.

- **Discrete Series $D^{\pm}(\vec{F})$:** F_0 is real, while F_0 and F_1 are both integers or half-odd integers. The following conditions have to be satisfied:

$$F_1 \geq f_2 \geq F_0 + 2 \geq \frac{1}{2} \qquad \text{for } D^{+}(\vec{F}), \tag{5.31}$$

$$-F_1 \leq f_2 \leq -(F_0 + 2) \leq -\frac{1}{2} \qquad \text{for } D^{-}(\vec{F}). \tag{5.32}$$

For any $so(4,1)$ UIR, $\vec{F} = (F_0, F_1)$, the quadratic Casimir, $C_2(\vec{F})$, is expressed as:

$$C_2(\vec{F}) = F_0\,(F_0 + 3) + F_1(F_1 + 1). \tag{5.33}$$

5.4 MODE SOLUTIONS OF STRICTLY MASSLESS SPIN-$(r + 1/2) \geq 3/2$ FERMIONS

In this Section, we obtain the mode solutions of the spin-$(r+1/2) \geq 3/2$ strictly massless theories [(5.20) and (5.21)]. The spin-3/2 ans spin-5/2 mode solutions (for arbitrary spacetime dimensions) have been already studied in Refs. [33, 34, 35].

5.4.1 GLOBAL COORDINATES AND REPRESENTATION OF GAMMA MATRICES

In order to obtain the mode solutions of Eqs. (5.20) and (5.21) we will work with the global coordinates of dS_4, where the line element is

$$ds^2 = -dt^2 + \cosh^2 t\, d\Omega^2. \tag{5.34}$$

The line element of S^3, $d\Omega^2$, can be parameterised as

$$d\Omega^2 = d\theta_3^2 + \sin^2\theta_3 \left(d\theta_2^2 + \sin^2\theta_2\, d\theta_1^2\right), \tag{5.35}$$

where $0 \leq \theta_j \leq \pi$ (for $j = 2, 3$) and $0 \leq \theta_1 \leq 2\pi$. We will also use the following notation for a point on S^3: $\boldsymbol{\theta_3} \equiv (\theta_3, \theta_2, \theta_1)$.
In global coordinates, the non-zero Christoffel symbols are

$$\begin{aligned} &\Gamma^{t}_{\theta_i\theta_j} = \cosh t \sinh t\; \tilde{g}_{\theta_i\theta_j}, \quad \Gamma^{\theta_i}_{\theta_j t} = \tanh t\; \tilde{g}^{\theta_i}_{\theta_j}, \\ &\Gamma^{\theta_k}_{\theta_i\theta_j} = \tilde{\Gamma}^{\theta_k}_{\theta_i\theta_j}, \end{aligned} \tag{5.36}$$

where $\tilde{g}_{\theta_i\theta_j}$ and $\tilde{\Gamma}^{\theta_k}_{\theta_i\theta_j}$ are the metric tensor and the Christoffel symbols, respectively, on S^3. The vierbein fields on dS_4 can be chosen to be:

$$e^t{}_0 = 1, \qquad e^{\theta_i}{}_i = \frac{1}{\cosh t}\tilde{e}^{\theta_i}{}_i, \qquad i = 1,2,3, \tag{5.37}$$

where $\tilde{e}^{\theta_i}{}_i$ are the dreibein fields on S^3. The non-zero components of the dS spin connection are given by

$$\omega_{ijk} = \frac{\tilde{\omega}_{ijk}}{\cosh t}, \qquad \omega_{i0k} = -\omega_{ik0} = -\tanh t\,\delta_{ik}, \qquad i,j,k \in \{1,2,3\} \tag{5.38}$$

where $\tilde{\omega}_{ijk}$ is the spin connection on S^3.

We will work with the following representation of gamma matrices on dS_4:

$$\gamma^0 = i\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & i\tilde{\gamma}^j \\ -i\tilde{\gamma}^j & 0 \end{pmatrix}, \tag{5.39}$$

$(j = 1,2,3)$ where the lower-dimensional gamma matrices, $\tilde{\gamma}^j$, satisfy

$$\{\tilde{\gamma}^j, \tilde{\gamma}^k\} = 2\delta^{jk}\mathbf{1}, \qquad j,k = 1,2,3. \tag{5.40}$$

The fifth gamma matrix (5.10) is given by

$$\gamma^5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \tag{5.41}$$

5.4.2 Constructing the mode solutions of the strictly massless theories

There are two kinds of spin-$(r+1/2)$ TT mode solutions satisfying the strictly massless field equations [(5.20) and (5.21)] on dS_4:

- The **'physical modes'** describing the propagating degrees of freedom of the theory.
- The **'pure gauge modes'** describing the gauge degrees of freedom of the theory.

In this Subsection, we present some details for the construction of these mode solutions. The mode solutions on global dS_4 can be constructed using the method of separation variables. Schematically, this means that we are looking for solutions that can be expressed as a product "function of t $\times$ function of $\boldsymbol{\theta_3}$". As we will see below, the functions describing the $\boldsymbol{\theta_3}$-dependence are tensor-spinor spherical harmonics on S^3 forming UIRs of $so(4)$. Thus, from a representation-theoretic viewpoint, the solutions on global dS_4 obtained with the method of separation of variables form $so(4,1)$ representations in the decomposition $so(4,1) \supset so(4)$. The method of separation of variables has been applied in Refs. [29, 27, 28] for integer-spin fields, in Refs. [8, 36] for spin-1/2 fields and in Refs. [33, 35] for spin-3/2 and spin-5/2 fields.

5.4.2.1 Physical spin-$(r+1/2) \geq 3/2$ modes on dS_4

Let us start by obtaining the physical mode solutions of Eqs. (5.20) and (5.21). We first discuss the spherical eigenmodes on S^3 that describe the spatial dependence of physical modes. Then, we discuss the time dependence of physical modes and we apply the method of separation of variables.

Spatial dependence and $so(4)$ content of physical modes

The spatial dependence of the spin-$(r+1/2)$ physical mode solutions on dS_4 is expressed in terms of (totally symmetric) tensor-spinor spherical harmonics of rank r on S^3. The latter are the totally symmetric TT tensor-spinor eigenmodes of the Dirac operator on S^3 satisfying [30, 8]

$$\begin{aligned}
&\tilde{\nabla\!\!\!/}\tilde{\psi}^{(\ell;m;k)}_{+\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_r}(\boldsymbol{\theta_3}) = +i\left(\ell+\frac{3}{2}\right)\tilde{\psi}^{(\ell;m;k)}_{+\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_r}(\boldsymbol{\theta_3}), \quad (\ell = r, r+1, ...)\\
&\tilde{\gamma}^{\tilde{\mu}_1}\tilde{\psi}^{(\ell;m;k)}_{+\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_r}(\boldsymbol{\theta_3}) = \tilde{\nabla}^{\tilde{\mu}_1}\tilde{\psi}^{(\ell;m;k)}_{+\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_r}(\boldsymbol{\theta_3}) = 0,
\end{aligned} \tag{5.42}$$

and

$$\begin{aligned}
&\tilde{\nabla\!\!\!/}\tilde{\psi}^{(\ell;m;k)}_{-\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_r}(\boldsymbol{\theta_3}) = -i\left(\ell+\frac{3}{2}\right)\tilde{\psi}^{(\ell;m;k)}_{-\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_r}(\boldsymbol{\theta_3}), \quad (\ell = r, r+1, ...)\\
&\tilde{\gamma}^{\tilde{\mu}_1}\tilde{\psi}^{(\ell;m;k)}_{-\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_r}(\boldsymbol{\theta_3}) = \tilde{\nabla}^{\tilde{\mu}_1}\tilde{\psi}^{(\ell;m;k)}_{-\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_r}(\boldsymbol{\theta_3}) = 0.
\end{aligned} \tag{5.43}$$

The subscripts '$\pm$' in $\tilde{\psi}^{(\ell;m;k)}_{\pm\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_r}$ have been used in order to indicate the sign of the eigenvalue in Eqs. (5.42) and (5.43), while $\tilde{\gamma}_{\tilde{\mu}}, \tilde{\nabla}_{\tilde{\mu}}$ and $\tilde{\nabla\!\!\!/} = \tilde{\gamma}^{\tilde{\mu}}\tilde{\nabla}_{\tilde{\mu}}$ are the gamma matrices, covariant derivative and Dirac operator, respectively, on S^3. The numbers ℓ, m, and k are representation-theoretic labels corresponding to the chain of subalgebras $so(4) \supset so(3) \supset so(2)$. In particular, the number $\ell = r, r+1, ...$ is the angular momentum quantum number on S^3. The numbers m and k are the angular momentum quantum numbers on S^2 and S^1, respectively, and they are allowed to take the values: $m = r, r+1, ..., \ell$ and $k = -(m+1), -m, ..., 0, ..., m$. The explicit form of the tensor-spinors $\tilde{\psi}^{(\ell;m;k)}_{\pm\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_r}(\boldsymbol{\theta_3})$ is not needed for the purposes of this paper[5].
The set of eigenmodes $\{\tilde{\psi}^{(\ell;m;k)}_{+\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_r}\}$ (with fixed ℓ) forms a $so(4)$ representation with highest weight (5.26) given by [30]:

$$\vec{f}^{\,+}_r = \left(\ell+\frac{1}{2}, r+\frac{1}{2}\right), \qquad \ell = r, r+1, \dots . \tag{5.44}$$

[5] See Refs. [52, 8] for explicit expressions for the spinor eigenfunctions of the Dirac operator on spheres and Refs. [9, 35] for the vector-spinor and symmetric rank-2 tensor-spinor cases. The general representation-theoretic properties of tensor-spinor spherical harmonics of arbitrary rank have been discussed in Ref. [30].

Similarly, the set $\{\tilde{\psi}^{(\ell;m;k)}_{-\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_r}\}$ (with fixed ℓ) forms a $so(4)$ representation with highest weight [30]:

$$\vec{f}^{-}_{r} = \left(\ell+\frac{1}{2}, -r-\frac{1}{2}\right), \qquad \ell = r, r+1, ... \tag{5.45}$$

Each of the $so(4)$ UIRs $\vec{f}^{\pm}_{r} = \left(\ell+\frac{1}{2}, \pm(r+\frac{1}{2})\right)$ has the following content concerning its subalgebras: the $so(3)$ content corresponds to the $so(3)$ highest weight $m+\frac{1}{2}$ with $\ell+\frac{1}{2}\geq m+\frac{1}{2}\geq r+\frac{1}{2}$, while the $so(2)$ content corresponds to the $so(2)$ highest weight $k+1/2$ with $m+\frac{1}{2}\geq k+\frac{1}{2}\geq -m-\frac{1}{2}$.
In this paper we assume that the eigenmodes in Eqs. (5.42) and (5.43) are already normalised using the standard inner product on S^3:

$$\int_{S^3}\sqrt{\tilde{g}}\, d\boldsymbol{\theta_3}\; \tilde{g}^{\tilde{\mu}_1\tilde{\nu}_1}\,\tilde{g}^{\tilde{\mu}_2\tilde{\nu}_2}...\tilde{g}^{\tilde{\mu}_r\tilde{\nu}_r}\;\tilde{\psi}^{(\ell';m';k')}_{\sigma'\,\tilde{\mu}_1\tilde{\mu}_2...\tilde{\mu}_r}(\boldsymbol{\theta_3})^{\dagger}\;\tilde{\psi}^{(\ell;m;k)}_{\sigma\,\tilde{\nu}_1\tilde{\nu}_2...\tilde{\nu}_r}(\boldsymbol{\theta_3})$$
$$= \delta_{\sigma\sigma'}\,\delta_{\ell\ell'}\,\delta_{m\,m'}\delta_{kk'}, \tag{5.46}$$

where $\sigma,\sigma'\in\{+,-\}$ and $d\boldsymbol{\theta_3}\equiv d\theta_3 d\theta_2 d\theta_1$, while $\tilde{g}$ is the determinant of the metric on S^3.[6]

Time dependence of physical modes

The physical modes $\Psi_{\mu_1...\mu_r}(t,\boldsymbol{\theta_3})$ on dS_4 are essentially TT tensor-spinors on S^3, and, thus, we have $\Psi_{t\mu_2...\mu_r}=0$, where $\mu_2,\mu_3,...,\mu_r\in\{t,\theta_3,\theta_2,\theta_1\}$ - as will become clear, the TT conditions (5.21) will be automatically satisfied by construction. The only non-zero components of the physical modes are the spatial components $\Psi_{\tilde{\mu}_1...\tilde{\mu}_r}$, where $\tilde{\mu_1},\tilde{\mu}_2,...,\tilde{\mu}_r\in\{\theta_3,\theta_2,\theta_1\}$. These can be determined by solving the Dirac equation (5.20). To be specific, letting $\mu_1=\tilde{\mu}_1$, $\mu_2=\tilde{\mu}_2$, ..., $\mu_r=\tilde{\mu}_r$, the Dirac equation (5.20) for the physical modes is expressed as

$$\left(\frac{\partial}{\partial t}+\frac{3-2r}{2}\tanh t\right)\gamma^t\Psi_{\tilde{\mu}_1...\tilde{\mu}_r}+\frac{1}{\cosh t}\begin{pmatrix}0 & i\tilde{\nabla}\!\!\!/ \\ -i\tilde{\nabla}\!\!\!/ & 0\end{pmatrix}\Psi_{\tilde{\mu}_1...\tilde{\mu}_r}=-ir\Psi_{\tilde{\mu}_1...\tilde{\mu}_r}, \tag{5.47}$$

where we have made use of the expressions for the Christoffel symbols, spin connection, vierbein fields and gamma matrices from Subsection 5.4.1.

[6] In Eq. (5.46), the eigenmodes with different values for $\sigma=\pm$ and/or $\ell=r,r+1,...$ are orthogonal to each other because they belong to different $so(4)$ representations. Similarly, eigenmodes with different values of m and/or k are orthogonal to each other because, in the decomposition $so(4)\supset so(3)\supset so(2)$, they correspond to representations with different content concerning the chain of subalgebras $so(3)\supset so(2)$.

Before proceeding to the construction of the modes, note that the physical modes on dS_4 are naturally split into two classes depending on their $so(4)$ representation-theoretic content - i.e. depending on whether their $\boldsymbol{\theta_3}$-dependence is given by the spherical eigenmodes (5.42) or (5.43). Let us introduce the following notation:

- The physical modes with $so(4)$ content given by $\vec{f}_r^-$ [Eq. (5.45)] are denoted as $\Psi^{(phys,\,-\ell;\,m;k)}_{\mu_1\ldots\mu_r}(t,\boldsymbol{\theta}_3)$. We also refer to these modes as 'physical modes with helicity $-s$' (recall that $s=r+1/2$).
- The physical modes with $so(4)$ content given by $\vec{f}_r^+$ [Eq. (5.44)] are denoted as $\Psi^{(phys,\,+\ell;\,m;k)}_{\mu_1\ldots\mu_r}(t,\boldsymbol{\theta}_3)$. We also refer to these modes as 'physical modes with helicity $+s$'.

Following our previous work [33], we separate variables for $\Psi^{(phys,\,-\ell;\,m;k)}_{\mu_1\ldots\mu_r}(t,\boldsymbol{\theta}_3)$ as:

$$\Psi^{(phys,\,-\ell;\,m;k)}_{t\mu_2\ldots\mu_r}(t,\boldsymbol{\theta}_3)=0,\qquad \Psi^{(phys,\,-\ell;\,m;k)}_{\tilde{\mu}_1\ldots\tilde{\mu}_r}(t,\boldsymbol{\theta}_3)=\begin{pmatrix}\alpha_\ell(t)\,\tilde{\psi}^{(\ell;m;k)}_{-\tilde{\mu}_1\ldots\tilde{\mu}_r}(\boldsymbol{\theta_3})\\ -i\beta_\ell(t)\,\tilde{\psi}^{(\ell;m;k)}_{-\tilde{\mu}_1\ldots\tilde{\mu}_r}(\boldsymbol{\theta_3})\end{pmatrix},\tag{5.48}$$

where $\ell=r,r+1,\ldots$, while $\alpha_\ell(t)$ and $\beta_\ell(t)$ are functions of time that we must determine.[7] Substituting Eq. (5.48) into Eq. (5.47), we find

$$\left(\frac{d}{dt}+\frac{3-2r}{2}\tanh t-\frac{i\left(\ell+\frac{3}{2}\right)}{\cosh t}\right)\beta_\ell(t)=-i\,r\,\alpha_\ell(t),\tag{5.49}$$

$$\left(\frac{d}{dt}+\frac{3-2r}{2}\tanh t+\frac{i\left(\ell+\frac{3}{2}\right)}{\cosh t}\right)\alpha_\ell(t)=\,i\,r\;\beta_\ell(t).\tag{5.50}$$

Using these two relations, and introducing the variable

$$x=\frac{\pi}{2}-it,\tag{5.51}$$

we find two second-order equations:

$$\left[\frac{\partial^2}{\partial x^2}+(3-2r)\cot x\frac{\partial}{\partial x}+\left(\ell+\frac{3}{2}\right)\frac{\cos x}{\sin^2 x}\right.\\ \left.-\frac{(\ell+\frac{3}{2})^2-\frac{1}{4}(3-2r)(1-2r)}{\sin^2 x}-\frac{(3-2r)^2}{4}\right]\alpha_\ell(t)=-r^2\alpha_\ell(t)\tag{5.52}$$

[7]The functions $\alpha_\ell(t)$ and $\beta_\ell(t)$ correspond to $\Phi^{(a)}_{M\ell}$ and $\Psi^{(a)}_{M\ell}$, respectively, with $a=-r$ and $M=ir$ (in four spacetime dimensions) in our previous work [33].

and

$$\Bigg[\frac{\partial^2}{\partial x^2} + (3-2r)\cot x \frac{\partial}{\partial x} - \left(\ell + \frac{3}{2}\right) \frac{\cos x}{\sin^2 x} - \frac{(\ell+\frac{3}{2})^2 - \frac{1}{4}(3-2r)(1-2r)}{\sin^2 x} - \frac{(3-2r)^2}{4} \Bigg] \beta_\ell(t) = -r^2\, \beta_\ell(t), \tag{5.53}$$

where $\cos x = i \sinh t$, $\sin x = \cosh t$ and $\cot x = i \tanh t$.[8] The solutions are given in terms of the Gauss hypergeometric function [25] as:

$$\alpha_\ell(t) = \left(\cos\frac{x(t)}{2}\right)^{\ell+1+r} \left(\sin\frac{x(t)}{2}\right)^{\ell+r} \times F\left(r+2+\ell, -r+\ell+2; \ell+2; \sin^2\frac{x(t)}{2}\right), \tag{5.54}$$

and

$$\beta_\ell(t) = \frac{r}{\ell+2} \left(\cos\frac{x(t)}{2}\right)^{\ell+r} \left(\sin\frac{x(t)}{2}\right)^{\ell+r+1} \times F\left(r+2+\ell, -r+\ell+2; \ell+3; \sin^2\frac{x(t)}{2}\right), \tag{5.55}$$

where

$$\cos\frac{x(t)}{2} = \left(\sin\frac{x(t)}{2}\right)^* = \frac{\sqrt{2}}{2}\left(\cosh\frac{t}{2} + i\sinh\frac{t}{2}\right). \tag{5.56}$$

We have now completely determined the form of the physical modes $\Psi^{(phys,\,-\ell;\,m;k)}_{\mu_1\ldots\mu_r}(t, \boldsymbol{\theta}_3)$ in Eq. (5.48).
Similarly, we find that the physical modes with $so(4)$ content given by $\vec{f}^{\,+}_r$ [Eq. (5.44)] are expressed as

$$\Psi^{(phys,\,+\ell;\,m;k)}_{t\mu_2\ldots\mu_r}(t,\boldsymbol{\theta}_3) = 0, \qquad \Psi^{(phys,\,+\ell;\,m;k)}_{\tilde{\mu}_1\ldots\tilde{\mu}_r}(t,\boldsymbol{\theta}_3) = \begin{pmatrix} i\beta_\ell(t)\,\tilde{\psi}^{(\ell;m;k)}_{+\tilde{\mu}_1\ldots\tilde{\mu}_r}(\boldsymbol{\theta_3}) \\ -\alpha_\ell(t)\,\tilde{\psi}^{(\ell;m;k)}_{+\tilde{\mu}_1\ldots\tilde{\mu}_r}(\boldsymbol{\theta_3}) \end{pmatrix}. \tag{5.57}$$

The functions $\alpha_\ell(t)$ and $\beta_\ell(t)$ are given again by Eqs. (5.54) and (5.55), respectively. The physical modes (5.48) and (5.57) can also be obtained by analytically continuing tensor-spinor spherical harmonics on S^4 (see Refs. [36, 35, 33, 29] for details concerning such analytic continuation techniques).

[8]Note that the third term of the differential operator in Eq. (5.52) has an opposite sign from the third term of the differential operator in Eq. (5.53). This is the only difference between these two differential operators.

Short wavelength limit of physical modes

Using the property [25]:

$$F(A,B;C;z) = (1-z)^{C-A-B}\, F(C-A, C-B; C; z), \tag{5.58}$$

we find that in the limit $\ell >> 1$ (short wavelength limit), the functions $\alpha_\ell(t)$ [Eq. (5.54)] and $\beta_\ell(t)$ [Eq. (5.55)] describe the time dependence of positive frequency Minkowskian modes, as

$$\begin{aligned} \frac{d\,\alpha_\ell(t)}{dt} &\sim -\frac{i\ell}{\cosh t}\,\alpha_\ell(t),\\ \frac{d\,\beta_\ell(t)}{dt} &\sim -\frac{i\ell}{\cosh t}\,\beta_\ell(t). \end{aligned} \tag{5.59}$$

Apart from the physical modes (5.48) and (5.57), there are also physical modes that are the analogs of Minkowskian negative frequency modes. These are given by

$$\Psi^{(phys,\,+\ell;\,m;k)C}_{t\mu_2\dots\mu_r}(t,\boldsymbol{\theta}_3) = 0, \qquad \Psi^{(phys,\,+\ell;\,m;k)C}_{\tilde{\mu}_1\dots\tilde{\mu}_r}(t,\boldsymbol{\theta}_3) = \begin{pmatrix} \alpha^*_\ell(t)\,\tilde{\psi}^{(\ell;m;k)}_{+\tilde{\mu}_1\dots\tilde{\mu}_r}(\boldsymbol{\theta_3}) \\ +i\beta^*_\ell(t)\,\tilde{\psi}^{(\ell;m;k)}_{+\tilde{\mu}_1\dots\tilde{\mu}_r}(\boldsymbol{\theta_3}) \end{pmatrix} \tag{5.60}$$

and

$$\Psi^{(phys,\,-\ell;\,m;k)C}_{t\mu_2\dots\mu_r}(t,\boldsymbol{\theta}_3) = 0, \qquad \Psi^{(phys,\,-\ell;\,m;k)C}_{\tilde{\mu}_1\dots\tilde{\mu}_r}(t,\boldsymbol{\theta}_3) = \begin{pmatrix} i\beta^*_\ell(t)\,\tilde{\psi}^{(\ell;m;k)}_{-\tilde{\mu}_1\dots\tilde{\mu}_r}(\boldsymbol{\theta_3}) \\ +\alpha^*_\ell(t)\,\tilde{\psi}^{(\ell;m;k)}_{-\tilde{\mu}_1\dots\tilde{\mu}_r}(\boldsymbol{\theta_3}) \end{pmatrix}. \tag{5.61}$$

It is straightforward to verify that these modes satisfy Eq. (5.47). In this paper, we do not discuss the representation-theoretic properties of the 'negative frequency' modes $\Psi^{(phys,\,\pm\ell;\,m;k)C}_{\mu_1\dots\mu_r}$, because they form the same $so(4,1)$ UIRs as the ones formed by the 'positive frequency' modes $\Psi^{(phys,\,\pm\ell;\,m;k)}_{\mu_1\dots\mu_r}$.

5.4.2.2 Pure gauge spin-$(r+1/2)\geq 3/2$ modes on dS_4

The pure gauge modes of the stricly massless spin-$(r+1/2)$ equations [(5.20) and (5.21)] satisfy the same conditions as the restricted gauge transformations (5.22). This means that the pure gauge modes are expressed as

$$\Psi^{(pg,\,\tilde{r},\,\pm\ell;\,\underline{m})}_{\mu_1\dots\mu_r}(t,\boldsymbol{\theta}_3) = \left(\nabla_{(\mu_1} + \frac{i}{2}\gamma_{(\mu_1}\right)\lambda^{(\tilde{r},\pm\ell;\,\underline{m})}_{\mu_2\dots\mu_r)}(t,\boldsymbol{\theta}_3). \tag{5.62}$$

The "gauge-function modes", $\lambda^{(\tilde{r},\,\pm\ell;\,\underline{m})}_{\mu_2\dots\mu_r}$, are totally symmetric tensor-spinors of rank $r-1$ and they satisfy Eqs. (5.23) and (5.24). As in the case of physical modes, explicit expressions for $\lambda^{(\tilde{r},\pm\ell;\underline{m})}_{\mu_2\cdots\mu_r}$ can be obtained using the method of separation of variables, but they are not needed for the purposes of this paper[9]. The two labels $\tilde{r},\pm\ell$ in Eq. (5.62) are used to denote the $so(4)$ content of each pure gauge mode; this corresponds to the $so(4)$ highest weights

$$\vec{f}^{\pm}_{\tilde{r}} = (\ell+\tfrac{1}{2},\pm\tilde{r}\pm\tfrac{1}{2}), \quad \tilde{r}\in\{0,1,...,r-1\}, \tag{5.63}$$

with $\ell = r, r+1, ...$ [the value $\tilde{r}=r$ is excluded in Eq. (5.63) since it corresponds to the $so(4)$ content of physical modes - see Eqs. (5.44) and (5.45)]. The label $\underline{m}$ represents angular momentum quantum numbers corresponding to the subalgebras $so(3)\supset so(2)$. The pure gauge modes must have zero norm with respect to any dS invariant scalar product and be orthogonal to all physical modes [28, 29, 35, 33, 27]. Because of these properties, the pure gauge modes can be identified with zero in the solution space of the field equations (5.20) and (5.21). These properties will be demonstrated in Section 5.5 for a specific choice of dS invariant scalar product - see also Refs. [33, 35].

5.5 THE PHYSICAL MODES FORM UIRS OF THE DS ALGEBRA

In this Section, we explain how the 'positive frequency' physical modes (5.48) and (5.57) of the fermionic strictly massless theories form a direct sum of Discrete Series UIRs of the dS algebra $so(4,1)$. In order to identify the $so(4,1)$ UIRs formed by the mode solutions, we follow two basic steps:

- **Irreducibility:** We identify the sets of physical modes that form irreducible representations of $so(4,1)$.

  This means that we need to study the infinitesimal dS transformations of the physical mode solutions. We show that the physical modes with fixed helicity $\pm s$ transform among themselves under all $so(4,1)$ transformations (up to gauge equivalence). Thus, the physical modes form a direct sum of irreducible representations - one corresponding to the helicity $+s$ and one to $-s$. Moreover, it is already easy to see that pure gauge modes transform only into other pure gauge modes under infinitesimal dS transformations, as the Lie-Lorentz derivative (5.14) commutes with the operator $\nabla_\mu + \frac{i}{2}\gamma_\mu$ in Eq. (5.62), while also, it leaves invariant the conditions (5.23) and (5.24), which determine the restricted gauge transformations.

[9] Explicit expressions for the spin-3/2 and spin-5/2 cases can be found in Refs. [34, 35].

- **Unitarity:** We introduce a dS invariant and gauge-invariant scalar product that is positive definite for physical modes of fixed helicity.

  With respect to this scalar product, the pure gauge modes are shown to be orthogonal to themselves, as well as to all physical modes (i.e. it is demonstrated that the pure gauge modes can be identified with zero in the solution space). Interestingly, it turns out that our scalar product is positive definite for the physical modes with helicity $-s$ and negative definite for the physical modes with helicity $+s$. However, as these two sets of fixed-helicity modes form different irreducible $so(4,1)$ representations, we are allowed to use a different scalar product for each set. We thus redefine the scalar product for the $+s$ modes by introducing a factor of -1, in order to achieve positive-definiteness. This peculiarity - i.e. having a different positive definite scalar product for physical modes with different helicity - is already known to appear in the spin-3/2 and spin-5/2 cases on even-dimensional dS_D for $D \geq 4$ [33, 35].

  Note. Although unitarity is often considered to be equivalent to the positive-definiteness of the scalar product in the Hilbert space of mode solutions, this is not a sufficient requirement. For representation-theoretic unitarity, the scalar product must be both positive definite and invariant under the symmetry algebra (or group) of interest. In this Section, the symmetries of interest correspond to the dS algebra, while, in Section 5.6, they correspond to the conformal-like $so(4,2)$ algebra.

Once we ensure both the unitarity and irreducibility of the $so(4,1)$ representations formed by the physical modes with fixed helicity, we will recall the $so(4)$ content [Eqs. (5.44) and (5.45)] of these modes, as well as the value of the field-theoretic quadratic Casimir (5.17). Then, it will be straightforward to identify the UIRs formed by the physical modes with a direct sum of Discrete Series UIRs of $so(4,1)$ [Eqs. (5.31) and (5.32)].

5.5.1 Infinitesimal dS transformations of physical modes and irreducibility of $so(4,1)$ representations

The infinitesimal dS transformations of the mode solutions can be studied with the use of the Lie-Lorentz derivative (5.14) with respect to the dS Killing vectors. Since the $so(4)$ content of the $so(4,1)$ representations formed by mode solutions is already known (see Section 5.4), we just need to study the transformation properties of our mode solutions under dS boosts. In fact, it is sufficient to focus on just one dS boost (the reason is that the Lie bracket between a boost Killing vector and a rotational one is equal to another boost Killing vector). We choose to work with the following boost Killing vector:

$$X = X^{\mu}\partial_{\mu} = \cos\theta_3 \frac{\partial}{\partial t} - \tanh t \sin\theta_3 \frac{\partial}{\partial \theta_3}. \tag{5.64}$$

Our aim is to express $\mathbb{L}_X \Psi^{(phys,\,-\ell;\,m;k)}_{\mu_1\ldots\mu_r}(t,\boldsymbol{\theta}_3)$ and $\mathbb{L}_X \Psi^{(phys,\,+\ell;\,m;k)}_{\mu_1\ldots\mu_r}(t,\boldsymbol{\theta}_3)$ as linear combinations of other mode solutions[10]. There are (at least) two different ways we can follow in order to proceed:

- i) Direct calculation, where in order to express $\mathbb{L}_X \Psi^{(phys,\,\pm\ell;\,m;k)}_{\mu_1\ldots\mu_r}$ as a linear combination of other modes, one has to use the raising and lowering differential operators for the angular momentum quantum number ℓ, as in Refs. [29, 27, 36, 35].

or

- ii) Making use of the matrix elements of $so(5)$ generators obtained by Gelfand and Tsetlin [24]. More specifically, one can use these matrix elements to find explicit expressions for the $so(5)$ transformations of tensor-spinor spherical harmonics on S^4 and then perform analytic continuation to dS_4.

In this paper, we follow approach ii. Here we present the final expressions for $\mathbb{L}_X \Psi^{(phys,\,\pm\ell;\,m;k)}_{\mu_1\ldots\mu_r}$. The reader who is familiar with $so(D+1)$ representations formed by tensor-spinor spherical harmonics on S^D [30] can infer the results from Gelfand and Tsetlin's work [24]. Technical details for the derivation can be found in Appendix 5.10.
Without further ado, following approach ii, the infinitesimal transformation of the physical spin-$(r+1/2) \geq 3/2$ modes under the dS boost (5.64) are found to be:

$$\begin{aligned}\mathbb{L}_X \Psi^{(phys,\,\pm\ell;m;k)}_{\mu_1\ldots\mu_r} = &-\frac{i}{2(\ell+2)}\sqrt{((\ell+2)^2-r^2)\,(\ell-m+1)(\ell+m+3)}\;\Psi^{(phys,\,\pm(\ell+1)\,;m;k)}_{\mu_1\ldots\mu_r} \\ &-\frac{i(\ell+1)}{2}\sqrt{\frac{(\ell-m)(\ell+m+2)}{(\ell+1)^2-r^2}}\;\Psi^{(phys,\,\pm(\ell-1)\,;m;k)}_{\mu_1\ldots\mu_r} + (\text{pure gauge}),\end{aligned} \tag{5.65}$$

where the term '(pure gauge)' is proportional to the pure gauge mode $\Psi^{(pg,\,\tilde{r}=r-1,\,\pm\ell;\,m;k)}_{\mu_1\ldots\mu_r}$ [see Eq. (5.62)].

Conclusion. From the transformation properties (5.65), we conclude that the modes $\{\Psi^{(phys,\,-\ell;m;k)}_{\mu_1\ldots\mu_r}\}$ and $\{\Psi^{(phys,\,+\ell;m;k)}_{\mu_1\ldots\mu_r}\}$ separately form irreducible representations of $so(4,1)$ (up to gauge equivalence).

[10] In the spin-3/2 and spin-5/2 cases, the transformations $\mathbb{L}_X \Psi^{(phys,\,-\ell;\,m;k)}_{\mu_1\ldots\mu_r}(t,\boldsymbol{\theta}_3)$ and $\mathbb{L}_X \Psi^{(phys,\,+\ell;\,m;k)}_{\mu_1\ldots\mu_r}(t,\boldsymbol{\theta}_3)$ have already been expressed as linear combinations of other mode solutions by direct calculation in Refs. [34, 35].

In the next Subsection, by making a choice of a dS invariant scalar product, we will explicitly show that all pure gauge modes have zero associated norm. Thus, the Lie-Lorentz derivatives (5.14) essentially act on equivalence classes of physical modes, i.e. if for any two physical modes, $\Psi^{(1)}_{\mu_1...\mu_r}$ and $\Psi^{(2)}_{\mu_1...\mu_r}$, the difference $\Psi^{(1)}_{\mu_1...\mu_r} - \Psi^{(2)}_{\mu_1...\mu_r}$ is a linear combination of pure gauge modes, then $\Psi^{(1)}_{\mu_1...\mu_r}$ and $\Psi^{(2)}_{\mu_1...\mu_r}$ belong to the same equivalence class.

5.5.2 dS invariant scalar product and unitarity

For any two (physical or pure gauge) solutions $\Psi^{(1)}_{\mu_1...\mu_r}, \Psi^{(2)}_{\mu_1...\mu_r}$ of Eqs. (5.20) and (5.21), define the (axial) vector current $J^\mu(\Psi^{(1)}, \Psi^{(2)})$ as

$$J^\mu(\Psi^{(1)}, \Psi^{(2)}) = -i\, \overline{\Psi}^{(1)}_{\nu_1...\nu_r} \gamma^5 \gamma^\mu \Psi^{(2)\nu_1...\nu_r}, \tag{5.66}$$

where $\overline{\Psi}^{(1)}_{\nu_1...\nu_r} = i \Psi^{(1)\dagger}_{\nu_1...\nu_r} \gamma^0$. This is covariantly conserved, $\nabla^\mu J_\mu(\Psi^{(1)}, \Psi^{(2)}) = 0$ [11]. Thus, it immediately follows that the scalar product

$$\begin{aligned}\left\langle \Psi^{(1)} | \Psi^{(2)} \right\rangle &= \int_{S^3} \sqrt{-g}\, d\boldsymbol{\theta_3}\, J^0(\Psi^{(1)}, \Psi^{(2)}) \\ &= \cosh^3 t \int_{S^3} \sqrt{\tilde{g}}\, d\boldsymbol{\theta_3}\, \Psi^{(1)\dagger}_{\nu_1...\nu_r}(t, \boldsymbol{\theta_3})\, \gamma^5\, \Psi^{(2)\nu_1...\nu_r}(t, \boldsymbol{\theta_3}) \end{aligned} \tag{5.67}$$

is time-independent, where $\cosh^3 t \sqrt{\tilde{g}} = \sqrt{-g}$, while g is the determinant of the dS metric.

dS invariance of the scalar product. The dS invariance of the scalar product (5.67) can be demonstrated as follows. Let $\delta_\xi J^\mu$ be the change of the current (5.66) under the infinitesimal dS transformation generated by a dS Killing vector ξ^μ. Then, we have

$$\begin{aligned}\delta_\xi J^\mu(\Psi^{(1)}, \Psi^{(2)}) &= J^\mu(\mathbb{L}_\xi \Psi^{(1)}, \Psi^{(2)}) + J^\mu(\Psi^{(1)}, \mathbb{L}_\xi \Psi^{(2)}) \\ &= \nabla_\nu \left(\xi^\nu\, J^\mu(\Psi^{(1)}, \Psi^{(2)}) - \xi^\mu\, J^\nu(\Psi^{(1)}, \Psi^{(2)}) \right) \qquad (5.68) \\ &= \frac{1}{\sqrt{-g}} \partial_\nu \left[\sqrt{-g} \left(\xi^\nu\, J^\mu(\Psi^{(1)}, \Psi^{(2)}) - \xi^\mu\, J^\nu(\Psi^{(1)}, \Psi^{(2)}) \right) \right] \qquad (5.69)\end{aligned}$$

where we have used that $\nabla_\nu J^\nu = \nabla_\nu \xi^\nu = 0$. As $\delta_\xi J^\mu$ is equal to the divergence of an anti-symmetric tensor, the following integral vanishes:

$$\delta_\xi \left\langle \Psi^{(1)} | \Psi^{(2)} \right\rangle = \int_{S^3} \sqrt{-g}\, d\boldsymbol{\theta_3}\, \delta_\xi J^0(\Psi^{(1)}, \Psi^{(2)}) = 0. \tag{5.70}$$

[11] I would like to thank Atsushi Higuchi for pointing out that this current is conserved.

In other words, the value of the scalar product (5.67) does not change under infinitesimal dS transformations. This directly implies that

$$\left\langle \mathbb{L}_\xi \Psi^{(1)} | \Psi^{(2)} \right\rangle + \left\langle \Psi^{(1)} | \mathbb{L}_\xi \Psi^{(2)} \right\rangle = 0, \tag{5.71}$$

for any dS Killing vector ξ.

Gauge invariance of the scalar product. Let us show that, with respect to the scalar product (5.67), all pure gauge modes (5.62) are orthogonal to themselves, as well as to all physical modes. In particular, letting $\Psi^{(2)}_{\mu_1...\mu_r}$ be a pure gauge mode (5.62) - i.e. $\Psi^{(2)}_{\mu_1...\mu_r} = \Psi^{(pg)}_{\mu_1...\mu_r} = (\nabla_{(\mu_1} + \frac{i}{2}\gamma_{(\mu_1})\lambda_{\mu_2...\mu_r)}$, where we have omitted the quantum number labels for convenience - the current (5.66) can be expressed as

$$J^\mu(\Psi^{(1)}, \Psi^{(pg)}) = 2i\, \nabla_\lambda \left(\overline{\Psi}^{(1)\nu_2\nu_3...\nu_r[\lambda} \gamma^{\mu]} \gamma^5 \, \lambda_{\nu_2\nu_3...\nu_r} \right), \tag{5.72}$$

where $\Psi^{(1)}$ is any physical or pure gauge mode. As $J^\mu(\Psi^{(1)}, \Psi^{(pg)})$ in Eq. (5.72) is equal to the divergence of an anti-symmetric tensor, the scalar product between any pure gauge mode and any other mode is always zero. Also, this directly implies that the scalar product (5.67) is invariant under restricted gauge transformations (5.22).

Positive-definiteness. Let us now calculate the norm of the physical mode solutions with respect to the scalar product (5.67). Substituting the expressions for the physical modes, (5.48) and (5.57), into the scalar product (5.67), we find

$$\begin{aligned} \left\langle \Psi^{(phys,\,\sigma\ell;m;k)} | \Psi^{(phys,\,\sigma'\ell';m';k')} \right\rangle =& (-\sigma) \times \cosh^{3-2r} t \\ & \times \left(|\alpha_\ell(t)|^2 - |\beta_\ell(t)|^2 \right) \delta_{\sigma\sigma'} \delta_{\ell\ell'} \delta_{mm'} \delta_{kk'}, \end{aligned} \tag{5.73}$$

where $\sigma, \sigma' \in \{-,+\}$, while we have made use of the normalisation condition (5.46) of the tensor-spinor spherical harmonics on S^3. This expression is time-independent and its value has been calculated in equation (8.26) of Ref. [35]. The result is [35]

$$\begin{aligned} &\left\langle \Psi^{(phys,\,\sigma\ell;m;k)} | \Psi^{(phys,\,\sigma'\ell';m';k')} \right\rangle \\ &= (-\sigma) \times 2^{3-2r} \frac{|\Gamma(\ell+2)|^2}{\Gamma(\ell+2+r)\Gamma(\ell+2-r)} \delta_{\sigma\sigma'} \delta_{\ell\ell'} \delta_{mm'} \delta_{kk'}. \end{aligned} \tag{5.74}$$

According to this equation, the physical modes with helicity $-s$, $\{\Psi^{(phys,\,-\ell;m;k)}_{\mu_1...\mu_r}\}$, form a UIR of $so(4,1)$ with positive definite scalar product given by Eq. (5.67), while the physical modes with helicity $+s$, $\{\Psi^{(phys,\,+\ell;m;k)}_{\mu_1...\mu_r}\}$, form a UIR of $so(4,1)$ with positive definite scalar product given by the negative of Eq. (5.67).

5.5.3 Identifying the dS algebra UIRs

The analysis presented in the previous Subsections has demonstrated that the physical modes, $\{\Psi^{(phys,+\ell;m;k)}_{\mu_1...\mu_r}\}$ and $\{\Psi^{(phys,-\ell;m;k)}_{\mu_1...\mu_r}\}$, of the strictly massless theories separately form UIRs of $so(4,1)$. It can be understood that we have a direct sum of Discrete Series UIRs (5.31) and (5.32) as follows. Combining the $so(4)$ content of physical modes [Eqs. (5.44) and (5.45)] with the branching rules (5.27), we find that both $\{\Psi^{(phys,+\ell;m;k)}_{\mu_1...\mu_r}\}$ and $\{\Psi^{(phys,-\ell;m;k)}_{\mu_1...\mu_r}\}$ correspond to UIRs with $F_1 = r + 1/2$ (see Section 5.3). Then, comparing the field-theoretic quadratic Casimir (5.17) (with $M = ir$) with the representation-theoretic one (5.33), we find the following "field theory - representation theory dictionary":

- The set of physical modes with helicity $+s$, $\{\Psi^{(phys,+\ell;m;k)}_{\mu_1...\mu_r}\}$, forms the Discrete Series UIR $D^+(F_0, F_1) = D^+(r - \frac{3}{2}, r + \frac{1}{2})$ [Eq. (5.31)] of $so(4,1)$. The $so(4)$ content is given by Eq. (5.44). The positive definite norm is given by the negative of Eq. (5.74) (with $\sigma = +$).
- The set of physical modes with helicity $-s$, $\{\Psi^{(phys,-\ell;m;k)}_{\mu_1...\mu_r}\}$, forms the Discrete Series UIR $D^-(F_0, F_1) = D^-(r - \frac{3}{2}, r + \frac{1}{2})$ [Eq. (5.32)] of $so(4,1)$. The $so(4)$ content is given by Eq. (5.45). The positive definite norm is given by Eq. (5.74) (with $\sigma = -$).

Thus, the set of all physical mode solutions for the strictly massless spin-$(r+1/2) \geq 3/2$ theory, satisfying Eqs. (5.20) and (5.21), corresponds to the direct sum of Discrete Series UIRs $D^-(r - \frac{3}{2}, r + \frac{1}{2}) \bigoplus D^+(r - \frac{3}{2}, r + \frac{1}{2})$ [12]. This is in agreement with the "field theory - representation theory dictionary" suggested previously by us [33].

5.6 CONFORMAL-LIKE SYMMETRIES FOR STRICTLY MASSLESS FERMIONS

In this Section, we present our main results, i.e. we present and study new conformal-like symmetries for strictly massless spin-$s \geq 3/2$ fermions on dS_4.

Conformal Killing vectors of dS_4. For later convenience, let us review the basics concerning the conformal Killing vectors on dS_4. The five conformal Killing vectors of

[12] This is also true for the massless spin-1/2 field on dS_4, i.e. for $r = 0$ [33].

dS_4 satisfy

$$\nabla_\mu V_\nu + \nabla_\nu V_\mu = g_{\mu\nu} \frac{\nabla^\alpha V_\alpha}{2} \tag{5.75}$$

with $\nabla^\alpha V_\alpha \neq 0$. (The ten dS Killing vectors, ξ^μ, satisfy the same equation, but they are divergence-free.) The 15-dimensional Lie algebra generated by the dS Killing vectors and the conformal Killing vectors is isomorphic to $so(4,2)$. The Lie bracket between a dS Killing vector and a conformal Killing vector is equal to a conformal Killing vector, while the Lie bracket between two conformal Killing vectors closes on $so(4,1)$. These facts can be understood from the $so(4,2)$ commutation relations:

$$[M_{A'B'}, M_{C'D'}] = (\eta'_{B'C'} M_{A'D'} + \eta'_{A'D'} M_{B'C'}) - (A' \leftrightarrow B'), \tag{5.76}$$

with $A', B', C', D' = -1, 0, ..., 4$, where $M_{A'B'} = -M_{B'A'}$ and

$$\eta'_{A'B'} = diag(-1, -1, 1, 1, 1, 1)$$

(with $\eta'_{-1-1} = \eta'_{00} = -1$). The generators $M_{-1A'}$, with $A' = 0, ..., 4$, can be identified with the five conformal Killing vectors of dS_4, while the generators $M_{A'B'}$, with $A', B' = 0, ..., 4$, can be identified with the ten dS Killing vectors.
Each of the five conformal Killing vectors of dS_4, denoted for convenience as $V^{(0)\mu}$, $V^{(1)\mu}$, ..., $V^{(4)\mu}$, can be expressed as a gradient of a scalar function [13]

$$V^{(A)}_\mu = \nabla_\mu \phi_{V^{(A)}}. \tag{5.77}$$

Each of the five scalar functions $\phi_{V^{(A)}}$ $(A = 0, 1, ..., 4)$ satisfies

$$\nabla_\mu \nabla_\nu \phi_{V^{(A)}} = -g_{\mu\nu} \phi_{V^{(A)}}, \tag{5.78}$$

i.e. $\nabla_\mu V^{(A)}_\nu = -g_{\mu\nu}\phi_{V^{(A)}}$. The scalar functions satisfying Eq. (5.78) can be found by analytically continuing the scalar functions that are annihilated by the operator $\nabla_\mu \nabla_\nu + g_{\mu\nu}$ on S^4 [14]. If we embed dS_4 in 5-dimensional Minkowski space as $-(X^0)^2 + \sum_{j=1}^4 (X^j)^2 = 1$, then the five scalar functions $\phi_{V^{(A)}}$ are $\phi_{V^{(A)}} = X^A$ (this equality holds up to a proportionality constant, which we ignore in the present paper). In the case of global

[13] I would like to thank Atsushi Higuchi for pointing this out.
[14] It is known that such functions on S^4 exist [1]. More specifically, they correspond to scalar spherical harmonics on S^4 [29].

coordinates (5.34) we have

$$
\begin{aligned}
X^0 &= \sinh t \\
X^4 &= \cosh t \, \cos\theta_3 \\
X^3 &= \cosh t \, \sin\theta_3 \, \cos\theta_2 \\
X^2 &= \cosh t \, \sin\theta_3 \, \sin\theta_2 \, \cos\theta_1 \\
X^1 &= \cosh t \, \sin\theta_3 \, \sin\theta_2 \, \sin\theta_1 .
\end{aligned} \tag{5.79}
$$

Below we will often drop the label '(A)' from $V^{(A)\mu}$ and $\phi_{V^{(A)}}$. Thus, from now on, we will denote conformal Killing vectors of dS_4 as $V^\mu = \nabla^\mu \phi_V$ or $W^\mu = \nabla^\mu \phi_W$, unless otherwise stated.

5.6.1 Conformal-like symmetry transformations

The main new result of the present paper is:

- If $\Psi_{\mu_1\dots\mu_r}$ is a strictly massless tensor-spinor satisfying Eqs. (5.20) and (5.21), then these equations are also satisfied by $T_V \Psi_{\mu_1\dots\mu_r}$ defined as

$$
\begin{aligned}
T_V \Psi_{\mu_1\dots\mu_r} \equiv \gamma^5 \Big(& V^\rho \nabla_\rho \Psi_{\mu_1\dots\mu_r} + i\, r\, V^\rho \gamma_\rho \Psi_{\mu_1\dots\mu_r} - i\, r\, V^\rho \gamma_{(\mu_1} \Psi_{\mu_2\dots\mu_r)\rho} - \frac{3}{2}\phi_V \, \Psi_{\mu_1\dots\mu_r} \Big) \\
& - \frac{2r}{2r+1} \left(\nabla_{(\mu_1} + \frac{i}{2}\gamma_{(\mu_1} \right) \gamma^5 \Psi_{\mu_2\dots\mu_r)\rho} V^\rho ,
\end{aligned} \tag{5.80}
$$

  for any conformal Killing vector $V^\mu = \nabla^\mu \phi_V$. The latter satisfies $\nabla_\mu V^\mu = -4\phi_V$ [see Eq. (5.78)]. Equation (5.80) describes the new conformal-like infinitesimal symmetry transformations for strictly massless fermions generated by conformal Killing vectors on dS_4.

The differential operator T_V maps solutions of Eqs. (5.20) and (5.21) into other solutions, i.e. T_V corresponds to a symmetry of these equations.

Note. The term in the last line of Eq. (5.80) does **not** correspond to a restricted gauge transformation (5.22). This can be understood by observing that the gauge function, $\lambda_{\mu_2\dots\mu_r}$, in the restricted gauge transformation (5.22) satisfies Eq. (5.23), while $\gamma^5 \Psi_{\mu_2\dots\mu_r\rho} V^\rho$ in the last line of Eq. (5.80) does not; it satisfies the following equation instead [15]:

$$
\nabla\!\!\!/ \gamma^5 \Psi_{\mu_2\dots\mu_r\rho} V^\rho = i r \, \gamma^5 \Psi_{\mu_2\dots\mu_r\rho} V^\rho .
$$

[15] Although the term in the second line of Eq. (5.80) is not a restricted gauge transformation, it still corresponds to an "off-shell" gauge transformation - by "off-shell" gauge transformation we mean any gauge transformation that leaves invariant the Lagrangian for strictly massless fermions (the restricted

In order to prove that the conformal-like transformation (5.80) corresponds to a symmetry we need to show that $T_V\Psi_{\mu_1\ldots\mu_r}$ satisfies the same field equations as $\Psi_{\mu_1\ldots\mu_r}$, i.e. Eqs. (5.20) and (5.21). It is convenient to define the totally symmetric tensor-spinors $\Delta_V\Psi_{\mu_1\ldots\mu_r}$ and $P_V\Psi_{\mu_1\ldots\mu_r}$ as

$$\Delta_V\Psi_{\mu_1\ldots\mu_r} \equiv \gamma^5\Big(V^\rho\nabla_\rho\Psi_{\mu_1\ldots\mu_r} + i\,r\,V^\rho\gamma_\rho\Psi_{\mu_1\ldots\mu_r} - i\,r\,V^\rho\gamma_{(\mu_1}\Psi_{\mu_2\ldots\mu_r)\rho} - \frac{3}{2}\phi_V\,\Psi_{\mu_1\ldots\mu_r}\Big) \tag{5.81}$$

and

$$P_V\Psi_{\mu_1\ldots\mu_r} \equiv -\frac{2r}{2r+1}\left(\nabla_{(\mu_1} + \frac{i}{2}\gamma_{(\mu_1}\right)\gamma^5\Psi_{\mu_2\ldots\mu_r)\rho}V^\rho, \tag{5.82}$$

such that

$$T_V\Psi_{\mu_1\ldots\mu_r} = \Delta_V\Psi_{\mu_1\ldots\mu_r} + P_V\Psi_{\mu_1\ldots\mu_r}. \tag{5.83}$$

We observe that $\Delta_V\Psi_{\mu_1\ldots\mu_r}$ and $P_V\Psi_{\mu_1\ldots\mu_r}$ have opposite gamma traces

$$\gamma^\alpha\Delta_V\Psi_{\alpha\mu_2\ldots\mu_r} = -\gamma^\alpha P_V\Psi_{\alpha\mu_2\ldots\mu_r} = 2i\gamma^5\Psi_{\mu_2\ldots\mu_r\rho}V^\rho. \tag{5.84}$$

Thus, the gamma-tracelessness property of the conformal-like transformation (5.80),

$$\gamma^\alpha\,T_V\Psi_{\alpha\mu_2\ldots\mu_r} = 0,$$

is straightforwardly shown.

Now let us show that, if $\Psi_{\mu_1\ldots\mu_r}$ satisfies Eq. (5.20), then so does $T_V\Psi_{\mu_1\ldots\mu_r}$. In other words, we will show that $T_V\Psi_{\mu_1\ldots\mu_r}$ is an eigenfunction of the Dirac operator with eigenvalue $-ir$. Acting with the Dirac operator on $\Delta_V\Psi_{\mu_1\ldots\mu_r}$ and $P_V\Psi_{\mu_1\ldots\mu_r}$, we find

$$\left(\nabla\!\!\!/ + ir\right)\Delta_V\Psi_{\mu_1\ldots\mu_r} = r\left(\nabla_{(\mu_1} - \frac{i}{2}\gamma_{(\mu_1}\right)\gamma^\alpha\Delta_V\Psi_{\mu_2\ldots\mu_r)\alpha} \tag{5.85}$$

gauge transformations (5.22) correspond to a special case of the "off-shell" transformations). Hermitian and gauge-invariant Lagrangians for strictly massless fermions on AdS_4 have been constructed in Ref. [22] (see also Ref. [44]). By analytically continuing AdS_4 to dS_4, i.e. by replacing the AdS radius as $\mathscr{R}_{AdS} \to i\mathscr{R}_{dS}$, where $\mathscr{R}_{dS}$ is the dS radius ($\mathscr{R}_{dS} = 1$ in our units), one can extend the Lagrangians for strictly massless fermions on AdS_4 [22] to gauge-invariant, but non-hermitian, Lagrangians on dS_4. The field equations derived from these non-hermitian Lagrangians on dS_4 are invariant under "off-shell" gauge transformations that have the form $\delta\Psi_{\mu_1\ldots\mu_r} = (\nabla_{(\mu_1} + \frac{i}{2}\gamma_{(\mu_1})\chi_{\mu_2\ldots\mu_r)}$, where $\chi_{\mu_2\ldots\mu_r}$ is a totally symmetric tensor-spinor with $\gamma^{\mu_2}\chi_{\mu_2\ldots\mu_r} = 0$. If one specialises to the TT gauge, these field equations reduce to Eqs. (5.20) and (5.21), while the initial "off-shell" gauge invariance reduces to the restricted gauge invariance with gauge transformations given by (5.22).

and

$$\left(\nabla\!\!\!\!/ + ir\right) P_V \Psi_{\mu_1 \dots \mu_r} = r \left(\nabla_{(\mu_1} - \frac{i}{2}\gamma_{(\mu_1}\right) \gamma^\alpha P_V \Psi_{\mu_2 \dots \mu_r)\alpha}, \tag{5.86}$$

respectively, where we have used Eq. (5.13). Adding Eqs. (5.85) and (5.86) by parts, and making use of Eqs. (5.83) and (5.84), we find

$$\left(\nabla\!\!\!\!/ + ir\right) T_V \Psi_{\mu_1 \dots \mu_r} = 0,$$

as required. Finally, contracting this equation with γ^{μ_1}, and using the gamma-traceleness property of $T_V \Psi_{\mu_1 \dots \mu_r}$, we find that $T_V \Psi_{\mu_1 \dots \mu_r}$ is also divergence-free.
To conclude, we have proved that the conformal-like transformation $T_V \Psi_{\mu_1 \dots \mu_r}$ [Eq. (5.80)] satisfies

$$\left(\nabla\!\!\!\!/ + ir\right) T_V \Psi_{\mu_1 \dots \mu_r} = 0, \tag{5.87}$$

$$\nabla^\alpha T_V \Psi_{\alpha \mu_2 \dots \mu_r} = 0, \quad \gamma^\alpha T_V \Psi_{\alpha \mu_2 \dots \mu_r} = 0 \tag{5.88}$$

for any conformal Killing vector V^μ and for all spins $s \geq 3/2$. In other words, the operator T_V [Eq. (5.80)] is a symmetry of the field equations (5.20) and (5.21) for strictly massless fermions.

5.6.2 Conformal-like $so(4,2)$ algebra generated by the dS symmetries and the conformal-like symmetries

In order to understand the structure of the algebra generated by the dS transformations (5.14) and the conformal-like transformations (5.80) we need to study the corresponding Lie brackets (i.e. commutators). Below, $V^\mu = \nabla^\mu \phi_V$ and $W^\mu = \nabla^\mu \phi_W$ denote any two conformal Killing vectors of dS_4 [see Eq. (5.77)].
Commutator between dS and conformal-like transformations. After a straightforward calculation, the commutator between a dS transformation (5.14) and a conformal-like transformation (5.80) is found to be

$$\begin{aligned} [\mathbb{L}_\xi, T_V]\Psi_{\mu_1 \dots \mu_r} &= \mathbb{L}_\xi T_V \Psi_{\mu_1 \dots \mu_r} - T_V \mathbb{L}_\xi \Psi_{\mu_1 \dots \mu_r} \\ &= T_{[\xi, V]} \Psi_{\mu_1 \dots \mu_r}, \end{aligned} \tag{5.89}$$

where $[\xi, V]$ is the Lie bracket between the Killing vector ξ and the conformal Killing vector V, i.e. $[\xi, V]^\mu = \mathscr{L}_\xi V^\mu$ ($\mathscr{L}_\xi$ is the usual Lie derivative with respect to ξ).

Commutator between two conformal-like transformations. The calculation of the commutator between two conformal-like transformations, $[T_W, T_V]\Psi_{\mu_1...\mu_r}$, is quite long. Thus, here we present the final result and we refer the reader to Appendix 5.11 for some details of the calculation. The result is

$$[T_W, T_V]\Psi_{\mu_1...\mu_r} = \mathbb{L}_{[W,V]}\Psi_{\mu_1...\mu_r} + \left(\nabla_{(\mu_1} + \frac{i}{2}\gamma_{(\mu_1}\right) L_{\mu_2...\mu_r)}, \tag{5.90}$$

where $[W, V]^\mu = \mathscr{L}_W V^\mu = \phi_W V^\mu - \phi_V W^\mu$ is a Killing vector. The second term on the right-hand side of Eq. (5.90) is a restricted gauge transformation of the form (5.22), where

$$L_{\mu_2...\mu_r} = \frac{4r}{(2r+1)^2}\left((\nabla^\lambda - \frac{i}{2}\gamma^\lambda)\Psi^\rho{}_{\mu_2...\mu_r}\ \nabla_\lambda[W, V]_\rho - (r+1)\ \Psi^\rho{}_{\mu_2...\mu_r}\ [W, V]_\rho\right). \tag{5.91}$$

(We have verified that $L_{\mu_2...\mu_r}$ satisfies Eqs. (5.23) and (5.24).)

Structure of the conformal-like algebra. To conclude, the structure of the conformal-like algebra generated by the ten dS transformations (5.14) and the five conformal-like transformations (5.80) is determined by the following commutation relations:

$$[\mathbb{L}_\xi, \mathbb{L}_{\xi'}]\Psi_{\mu_1...\mu_r} = \mathbb{L}_{[\xi,\xi']}\Psi_{\mu_1...\mu_r}, \tag{5.92a}$$

$$[\mathbb{L}_\xi, T_V]\Psi_{\mu_1...\mu_r} = T_{[\xi,V]}\Psi_{\mu_1...\mu_r}, \tag{5.92b}$$

$$[T_W, T_V]\Psi_{\mu_1...\mu_r} = \mathbb{L}_{[W,V]}\Psi_{\mu_1...\mu_r} + \left(\nabla_{(\mu_1} + \frac{i}{2}\gamma_{(\mu_1}\right) L_{\mu_2...\mu_r)}, \tag{5.92c}$$

where $L_{\mu_2...\mu_r}$ is given by (5.91), ξ^μ and ξ'^μ are any two dS Killing vectors, while $W^\mu = \nabla^\mu\phi_W$ and $V^\mu = \nabla^\mu\phi_V$ are any two conformal Killing vectors. The commutation relations (5.92a)-(5.92c) coincide with the $so(4, 2)$ commutation relations (5.76) up to the restricted gauge transformation in Eq. (5.92c).

Our results demonstrate that there is a representation of $so(4, 2)$ (which closes up to field-dependent gauge transformations) acting on the solution space of Eqs. (5.20) and (5.21). In the following Subsection, we will show that the physical modes, which have been shown to form a direct sum of $so(4, 1)$ UIRs (see Section 5.5), also form a direct sum of $so(4, 2)$ UIRs.

- **Note.** One might think that the closure of the conformal-like algebra up to (field-dependent) gauge transformations is a consequence of the term in the second line of Eq. (5.80). In order to argue that this is **not** the case, let us focus on the strictly massless spin-3/2 field and depart from the TT gauge:

Consider the full Rarita-Schwinger (RS) equation for the strictly massless spin-3/2 field (gravitino) on dS_4 [23]

$$\gamma^{\mu\rho\sigma}(\nabla_\rho + \frac{i}{2}\gamma_\rho)\psi_\sigma = 0, \tag{5.93}$$

where $\gamma^{\mu\rho\sigma} = \gamma^{[\mu}\gamma^\rho\gamma^{\sigma]}$. This equation is invariant under "off-shell" gauge transformations

$$\delta\psi_\mu = (\nabla_\mu + \frac{i}{2}\gamma_\mu)\epsilon, \tag{5.94}$$

where ϵ is an arbitrary spinor. If we choose to work in the TT gauge, then the RS equation reduces to Eqs. (5.20) and (5.21), which have a smaller gauge invariance corresponding to restricted gauge transformations (5.22). After a straightforward calculation, we find that the RS equation (5.93) enjoys the conformal-like symmetry[16]

$$\Delta_V\psi_\mu = \gamma^5\Big(V^\rho\nabla_\rho\psi_\mu + i\,V^\rho\gamma_\rho\psi_\mu - i\,V^\rho\gamma_\mu\psi_\rho - \frac{3}{2}\phi_V\,\psi_\mu\Big). \tag{5.95}$$

In other words, if ψ_μ satisfies the RS equation, then so does $\Delta_V\psi_\mu$. Because of the "off-shell" gauge symmetry (5.94), Eq. (5.95) does not include a part corresponding to the second line of Eq. (5.80). Then, the commutator between two conformal-like transformations (5.95) is found to be

$$[\Delta_W, \Delta_V]\psi_\mu = \mathbb{L}_{[W,V]}\psi_\mu - 2i\left(\nabla_\mu + \frac{i}{2}\gamma_\mu\right)\gamma^\lambda\psi^\rho\,\nabla_\lambda[W,V]_\rho, \tag{5.96}$$

where we notice the appearance of an "off-shell" gauge transformation (which is **not** a restricted gauge transformation (5.22)) on the right-hand side. The rest of the structure of the symmetry algebra is determined by the same commutation relations as in Eqs. (5.92a) and (5.92b) (with T_V replaced by Δ_V).

Conclusion. As in the TT gauge, the full RS equation (5.93) enjoys a conformal-like $so(4,2)$ symmetry and the algebra closes up to "off-shell" gauge transformations (5.94) that do not correspond to restricted gauge transformations (5.22). However, in the TT case (5.92c), the algebra closes up to restricted gauge transformations.

[16]The expression in Eq. (5.95) corresponds just to the the first part (5.81) of the conformal-like transformation in the TT gauge [Eq. (5.80)].

5.7 THE PHYSICAL MODES ALSO FORM UIRS OF THE CONFORMAL-LIKE ALGEBRA

In this Section, we show that the 'positive frequency' physical modes (5.48) and (5.57) of the strictly massless spin-$s \geq 3/2$ fermionic theories form UIRs of the conformal-like $so(4,2)$ algebra. To be specific:

- The **irreducibility** of the $so(4,2)$ representations will be demonstrated by showing that the physical modes with fixed helicity transform among themselves under the infinitesimal conformal-like transformations (5.80). In particular, the physical modes with helicity $+s$ [Eq. (5.57)], and the ones with helicity $-s$ [Eq. (5.48)], will be shown to separately form irreducible representations of $so(4,2)$. (Recall that we have already shown that these modes form a direct sum of UIRs of the dS algebra $so(4,1)$ - see Section 5.5.)
- As for showing the **unitarity** of the two aforementioned irreducible $so(4,2)$ representations, we work as follows. First, we recall from Section 5.5 that the physical modes with helicity $\mp s$ form a $so(4,1)$ UIR with dS invariant and positive definite scalar product given by $(\pm 1)\times$(5.67). Then, since a positive definite and $so(4,1)$-invariant scalar product is known, it is sufficient to show that this scalar product is also invariant under the conformal-like symmetries (5.80).

5.7.1 Conformal-like transformations of physical modes and irreducibility of $so(4,2)$ representations

Let us start with the simple observation that, according to Eq. (5.76), the Lie bracket between a conformal Killing vector and a dS Killing vector is equal to a conformal Killing vector. Similarly, the commutator $[\mathbb{L}_\xi, T_V]\Psi_{\mu_1...\mu_r}$ in Eq. (5.92b) is equal to a conformal-like symmetry transformation. Thus, as the $so(4,1)$ representation-theoretic properties of the physical modes are known (see Section 5.5), it is sufficient to study just one of the five conformal-like transformations (5.80) for our physical modes. Then, the transformation properties of the physical modes under the rest of the conformal-like transformations can be found using the commutation relations (5.92b).

Let us now choose to work with the conformal Killing vector $V^{(0)\mu}$ [Eq. (5.77)] given by

$$V^{(0)}_\mu = \nabla_\mu \sinh t, \tag{5.97}$$

i.e. $(V_t^{(0)}, V_{\theta_3}^{(0)}, V_{\theta_2}^{(0)}, V_{\theta_1}^{(0)}) = (\cosh t, 0, 0, 0)$. The conformal-like transformation (5.80) generated by $V^{(0)\mu}$ is expressed as

$$T_{V^{(0)}} \Psi_{\mu_1 \dots \mu_r} = -\gamma^5 \cosh t \times \left(\frac{\partial}{\partial t} + \left(-r + \frac{3}{2} \right) \tanh t - ir\, \gamma^t \right) \Psi_{\mu_1 \dots \mu_r}. \tag{5.98}$$

Specialising to the physical modes (5.48) and (5.57), and making use of Eqs. (5.39), (5.49) and (5.50), we readily find

$$T_{V^{(0)}} \Psi^{(phys,\, -\ell;\, m;k)}_{\mu_1 \dots \mu_r} = +i \left(\ell + \frac{3}{2} \right) \Psi^{(phys,\, -\ell;\, m;k)}_{\mu_1 \dots \mu_r} \tag{5.99}$$

and

$$T_{V^{(0)}} \Psi^{(phys,\, +\ell;\, m;k)}_{\mu_1 \dots \mu_r} = -i \left(\ell + \frac{3}{2} \right) \Psi^{(phys,\, +\ell;\, m;k)}_{\mu_1 \dots \mu_r}. \tag{5.100}$$

From these equations (and from the discussion at the beginning of this Subsection), it follows that the two sets of modes, $\{\Psi^{(phys,\, +\ell;\, m;k)}_{\mu_1 \dots \mu_r}\}$ and $\{\Psi^{(phys,\, -\ell;\, m;k)}_{\mu_1 \dots \mu_r}\}$, separately form irreducible representations of the conformal-like $so(4,2)$ algebra.

5.7.2 $so(4,2)$-invariant scalar product and unitarity

In the previous Subsection, we showed that the physical modes form a direct sum of irreducible representations of the conformal-like algebra. The only remaining step for showing that this is a direct sum of $so(4,2)$ UIRs is to ensure the existence of a $so(4,2)$-invariant and positive definite scalar product.
Let us show that the dS invariant scalar product (5.67) is also invariant under the conformal-like symmetries (5.80) - and, thus, under the whole conformal-like $so(4,2)$ algebra (recall that this scalar product is also invariant under restricted gauge transformations). Let $\Psi^{(1)}_{\mu_1 \dots \mu_r}$ and $\Psi^{(2)}_{\mu_1 \dots \mu_r}$ be any two solutions of Eqs. (5.20) and (5.21). We consider the change

$$\delta_V J^\mu(\Psi^{(1)}, \Psi^{(2)}) = J^\mu(T_V \Psi^{(1)}, \Psi^{(2)}) + J^\mu(\Psi^{(1)}, T_V \Psi^{(2)})$$

of the vector current (5.66) under the conformal-like transformations (5.80). After a straightforward calculation, we find

$$\delta_V J^\mu(\Psi^{(1)}, \Psi^{(2)}) = -i\, \nabla_\lambda E^{\lambda\mu}, \tag{5.101}$$

where $E^{\lambda\mu}$ is an anti-symmetric tensor given by:

$$\frac{1}{2}E^{\lambda\mu} = -\overline{\Psi}^{(1)}_{\nu_1\dots\nu_r} V^{[\lambda}\gamma^{\mu]}\Psi^{(2)\nu_1\dots\nu_r} + \frac{2r}{2r+1}V^{\rho}\left(\overline{\Psi}^{(1)}_{\nu_2\dots\nu_r\rho}\gamma^{[\mu}\Psi^{(2)\lambda]\nu_2\dots\nu_r} + \overline{\Psi}^{(1)\nu_2\dots\nu_r[\lambda}\gamma^{\mu]}\Psi^{(2)}_{\nu_2\dots\nu_r\rho}\right). \tag{5.102}$$

This ensures that the dS invariant scalar product (5.67) is also invariant under infinitesimal conformal-like transformations, as

$$\delta_V\left\langle\Psi^{(1)}|\Psi^{(2)}\right\rangle = \int_{S^3}\sqrt{-g}\,d\boldsymbol{\theta_3}\,\delta_V J^0(\Psi^{(1)},\Psi^{(2)}) = 0.$$

Based on the discussions in the previous paragraph (and in the previous Subsection), we conclude the following:

- The set of physical modes with helicity $+s$, $\{\Psi^{(phys,+\ell;m;k)}_{\mu_1\dots\mu_r}\}$, forms a UIR of $so(4,2)$ with positive definite norm given by the negative of Eq. (5.74) (with $\sigma = +$).
- The set of physical modes with helicity $-s$, $\{\Psi^{(phys,-\ell;m;k)}_{\mu_1\dots\mu_r}\}$, forms a UIR of $so(4,2)$ with positive definite norm given by Eq. (5.74) (with $\sigma = -$).

5.8 CONFORMAL-LIKE TRANSFORMATIONS OF FIELD STRENGTH TENSOR-SPINORS

In order to gain some insight into the interpretation of the conformal-like transformations $T_V\Psi_{\mu_1\dots\mu_r}$ (5.80), we study the corresponding transformations of the field strength tensor-spinors (i.e. curvatures). In particular, we study the transformations of the spin-$s = 3/2, 5/2$ field strengths explicitly, while in the spin-$s \geq 7/2$ cases we make a conjecture for the expressions of the transformations.

5.8.1 SPIN-3/2 FIELD STRENGTH TENSOR-SPINOR

The field strength tensor-spinor for the strictly massless spin-3/2 field is

$$\mathbb{F}_{\mu_1\nu_1} = -\mathbb{F}_{\nu_1\mu_1} = \left(\nabla_{[\mu_1} + \frac{i}{2}\gamma_{[\mu_1}\right)\Psi_{\nu_1]}. \tag{5.103}$$

For later convenience, we will denote this as $\mathbb{F}_{\mu_1\nu_1}(\Psi)$. The field strength $\mathbb{F}_{\mu_1\nu_1}(\Psi)$ is invariant under not only restricted gauge transformations (5.22) but also "off-shell" gauge transformations (5.94).

Useful properties. Let us discuss some of the properties of $\mathbb{F}_{\mu_1\nu_1}(\Psi)$ that will be useful in studying its conformal-like transformation. Using the field equations (5.20) and (5.21) for Ψ_ν, we find

$$\gamma^{\mu_1}\mathbb{F}_{\mu_1\nu_1}(\Psi) = \nabla^{\mu_1}\mathbb{F}_{\mu_1\nu_1}(\Psi) = 0. \tag{5.104}$$

The dual field strength tensor-spinor is defined as

$${}^*\mathbb{F}_{\mu_1\nu_1}(\Psi) \equiv \frac{1}{2}\epsilon_{\mu_1\nu_1}{}^{\kappa\lambda}\,\mathbb{F}_{\kappa\lambda}(\Psi). \tag{5.105}$$

Expressing $\epsilon_{\mu_1\nu_1}{}^{\kappa\lambda}$ in Eq. (5.105) in terms of gamma matrices [see Eq. (5.10)], and using the gamma-tracelessness of $\mathbb{F}_{\mu_1\nu_1}(\Psi)$, we find

$${}^*\mathbb{F}_{\mu_1\nu_1}(\Psi) = -i\gamma^5\mathbb{F}_{\mu_1\nu_1}(\Psi). \tag{5.106}$$

Also, a straightforward calculation shows that the following identity holds:

$$\nabla_{[\rho}\mathbb{F}_{\mu_1\nu_1]}(\Psi) + \frac{i}{2}\gamma_{[\rho}\mathbb{F}_{\mu_1\nu_1]}(\Psi) = 0. \tag{5.107}$$

It is easy to show that each of the two terms in this equation is zero by observing that[17]

$$\nabla_{[\rho}\,{}^*\mathbb{F}_{\mu_1\nu_1]}(\Psi) = 0. \tag{5.108}$$

It immediately follows from Eqs. (5.106)-(5.108) that

$$\nabla_{[\rho}\mathbb{F}_{\mu_1\nu_1]}(\Psi) = \gamma_{[\rho}\mathbb{F}_{\mu_1\nu_1]}(\Psi) = 0. \tag{5.109}$$

Conformal-like transformation. After a straightforward calculation, the conformal-like transformation of the field strength, $\mathbb{F}_{\mu_1\nu_1}(T_V\Psi)$, is expressed as

$$\begin{aligned}\mathbb{F}_{\mu_1\nu_1}(T_V\Psi) &= \mathbb{F}_{\mu_1\nu_1}(\Delta_V\Psi)\\ &= \gamma^5\left(V^\rho\nabla_\rho - \frac{5}{2}\phi_V\right)\mathbb{F}_{\mu_1\nu_1}(\Psi) + 3\,i\gamma^5\,V^\rho\,\gamma_{[\rho}\mathbb{F}_{\mu_1\nu_1]}(\Psi),\end{aligned} \tag{5.110}$$

where in the first line we have used $T_V\Psi_\mu = \Delta_V\Psi_\mu + P_V\Psi_\mu$ [see Eq. (5.83)] and $\mathbb{F}_{\mu_1\nu_1}(P_V\Psi) = 0$ (the latter follows from the gauge-invariance of the field strength). Then, using Eq. (5.109), we find

$$\mathbb{F}_{\mu_1\nu_1}(T_V\Psi) = \gamma^5\left(V^\rho\nabla_\rho - \frac{5}{2}\phi_V\right)\mathbb{F}_{\mu_1\nu_1}(\Psi), \tag{5.111}$$

[17]**Proof of Eq. (5.108).** In order to prove Eq. (5.108), we contract $\nabla_{[\rho}\,{}^*\mathbb{F}_{\mu_1\nu_1]}(\Psi)$ with $\epsilon_{\alpha\beta}{}^{\mu_1\nu_1}$ and we use the definition (5.105) of the dual field strength. Then, using well-known identities for $\epsilon_{\alpha\beta}{}^{\mu_1\nu_1}$, while also using the divergence-freedom of the field strength, we can show that $\epsilon_{\alpha\beta}{}^{\sigma\kappa}\,\nabla_{[\rho}\,{}^*\mathbb{F}_{\sigma\kappa]}(\Psi) = 0$. Finally, contracting this equation with $\epsilon_{\mu_1\nu_1}{}^{\alpha\beta}$, we arrive at Eq. (5.108). **End of proof.**

or equivalently

$$\mathbb{F}_{\mu_1\nu_1}(T_V\Psi) = \gamma^5\left(\mathbb{L}_V + \frac{\nabla_\kappa V^\kappa}{8}\right)\mathbb{F}_{\mu_1\nu_1}(\Psi), \tag{5.112}$$

where $\mathbb{L}_V$ is the Lie-Lorentz derivative (5.14) with respect to the conformal Killing vector V (5.77)[18].

Conclusion. The expression (5.112) makes clear that the conformal-like transformation of the spin-3/2 field strength tensor-spinor corresponds to the product of two transformations: an infinitesimal axial rotation (i.e. multiplication with γ^5) times an infinitesimal conformal transformation (i.e. Lie-Lorentz derivative plus a conformal weight term).

5.8.2 Spin-5/2 field strength tensor-spinor

The field strength tensor-spinor for the strictly massless spin-5/2 field is a rank-4 tensor-spinor given by

$$\begin{aligned}&\mathbb{F}_{\mu_1\nu_1\mu_2\nu_2}(\Psi)\\ &=\frac{1}{2}\left(\nabla_{\mu_2}\nabla_{[\mu_1} + \frac{3}{4}g_{\mu_2[\mu_1} - \frac{1}{4}\gamma_{\mu_2[\mu_1} + \frac{i}{2}\nabla_{\mu_2}\gamma_{[\mu_1} + \frac{i}{2}\gamma_{\mu_2}\nabla_{[\mu_1}\right)\Psi_{\nu_1]\nu_2} - (\mu_2 \leftrightarrow \nu_2).\end{aligned} \tag{5.113}$$

This is symmetric under the exchange of pairs of indices

$$\mathbb{F}_{\mu_2\nu_2\mu_1\nu_1}(\Psi) = \mathbb{F}_{\mu_1\nu_1\mu_2\nu_2}(\Psi). \tag{5.114}$$

It is also anti-symmetric in its first two and last two indices

$$\mathbb{F}_{\mu_1\nu_1\mu_2\nu_2}(\Psi) = \mathbb{F}_{[\mu_1\nu_1]\mu_2\nu_2}(\Psi) = \mathbb{F}_{\mu_1\nu_1[\mu_2\nu_2]}(\Psi), \tag{5.115}$$

and satisfies the identity

$$\mathbb{F}_{\mu\alpha\beta\gamma}(\Psi) + \mathbb{F}_{\mu\gamma\alpha\beta}(\Psi) + \mathbb{F}_{\mu\beta\gamma\alpha}(\Psi) = 0. \tag{5.116}$$

As in the spin-3/2 case, the field strength is invariant under not only restricted gauge transformations (5.22) but also gauge transformations of the following form:

$$\delta\Psi_{\mu\nu} = \left(\nabla_{(\mu} + \frac{i}{2}\gamma_{(\mu}\right)\epsilon_{\nu)} \tag{5.117}$$

[18] The infinitesimal Lorentz transformation term $\nabla_\alpha V_\beta \gamma^{\alpha\beta}/4$ in the Lie-Lorentz derivative $\mathbb{L}_V$ in Eq. (5.112) vanishes because, according to Eq. (5.77), $\nabla_{[\alpha}V_{\beta]} = 0$.

(i.e. $\mathbb{F}_{\mu_1\nu_1\mu_2\nu_2}(\delta\Psi)=0$), where ϵ_ν is an arbitrary vector-spinor.
Working as in the spin-3/2 case, we can show that the spin-5/2 field strength (5.113) is gamma-traceless and divergence-free with respect to all of its indices, and it also satisfies the identities

$$\nabla_{[\rho}\mathbb{F}_{\mu_1\nu_1]\mu_2\nu_2}(\Psi)=\gamma_{[\rho}\mathbb{F}_{\mu_1\nu_1]\mu_2\nu_2}(\Psi)=0. \tag{5.118}$$

Conformal-like transformation. Let us find the conformal-like transformation of the field strength, $\mathbb{F}_{\mu_1\nu_1\mu_2\nu_2}(T_V\Psi)$. The calculation is similar to the spin-3/2 case, but quite longer. The result is

$$\mathbb{F}_{\mu_1\nu_1\mu_2\nu_2}(T_V\Psi)=\gamma^5\left(V^\rho\nabla_\rho-\frac{7}{2}\phi_V\right)\mathbb{F}_{\mu_1\nu_1\mu_2\nu_2}(\Psi), \tag{5.119}$$

or equivalently

$$\mathbb{F}_{\mu_1\nu_1\mu_2\nu_2}(T_V\Psi)=\gamma^5\left(\mathbb{L}_V-\frac{\nabla_\kappa V^\kappa}{8}\right)\mathbb{F}_{\mu_1\nu_1\mu_2\nu_2}(\Psi). \tag{5.120}$$

Conclusion. As in the spin-3/2 case (5.112), the expression (5.120) makes clear that the conformal-like transformation of the spin-5/2 field strength corresponds to the product: infinitesimal axial rotation times infinitesimal conformal transformation.

5.8.3 A conjecture for the spin-$(r+1/2)\geq 7/2$ field strength tensor-spinors

(Here we do not present explicit expressions for the field strength tensor-spinors $\mathbb{F}_{\mu_1\nu_1....\mu_r\nu_r}(\Psi)$ of the strictly massless spin-$(r+1/2)\geq 7/2$ fields.) We define the field strength $\mathbb{F}_{\mu_1\nu_1....\mu_r\nu_r}(\Psi)$ as the gauge-invariant rank-$2r$ tensor-spinor that satisfies

$$\gamma^{\mu_1}\mathbb{F}_{\mu_1\nu_1...\mu_r\nu_r}(\Psi)=\nabla^{\mu_1}\mathbb{F}_{\mu_1\nu_1...\mu_r\nu_r}(\Psi)=0, \tag{5.121}$$

and it is also anti-symmetric under the exchange of the indices $\mu_l\leftrightarrow\nu_l$ for $l=1,...,r$. It is also symmetric under the exchange of any two pairs of indices as in the following example:

$$\mathbb{F}_{\mu_1\nu_1\mu_2\nu_2....\mu_r\nu_r}(\Psi)=\mathbb{F}_{\mu_2\nu_2\mu_1\nu_1....\mu_r\nu_r}(\Psi)=\mathbb{F}_{\mu_r\nu_r\mu_2\nu_2....\mu_{r-1}\nu_{r-1}\mu_1\nu_1}\quad\text{and so forth,} \tag{5.122}$$

while it also satisfies the identities

$$\mathbb{F}_{[\mu_1\nu_1\mu_2]\nu_2....\mu_r\nu_r}(\Psi)=0 \tag{5.123}$$

and

$$\nabla_{[\rho}\mathbb{F}_{\mu_1\nu_1]\mu_2\nu_2 ... \mu_r\nu_r}(\Psi) = \gamma_{[\rho}\mathbb{F}_{\mu_1\nu_1]\mu_2\nu_2 ... \mu_r\nu_r}(\Psi) = 0. \tag{5.124}$$

Conjecture. The conformal-like transformation of the spin-$(r+1/2) \geq 7/2$ field strength tensor-spinor is given by

$$\mathbb{F}_{\mu_1\nu_1...\mu_r\nu_r}(T_V\Psi) = \gamma^5 \left(V^\rho \nabla_\rho - \left(r + \frac{3}{2}\right)\phi_V \right) \mathbb{F}_{\mu_1\nu_1...\mu_r\nu_r}(\Psi), \tag{5.125}$$

or equivalently

$$\mathbb{F}_{\mu_1\nu_1...\mu_r\nu_r}(T_V\Psi) = \gamma^5 \left(\mathbb{L}_V - (2r-3)\frac{\nabla_\kappa V^\kappa}{8} \right) \mathbb{F}_{\mu_1\nu_1...\mu_r\nu_r}(\Psi). \tag{5.126}$$

This conjecture has been verified for $r = 1$ in Subsection 5.8.1 and for $r = 2$ in Subsection 5.8.2. Our conjecture is further justified by observing that $\mathbb{F}_{\mu_1\nu_1...\mu_r\nu_r}(T_V\Psi)$ [Eq. (5.126)] satisfies Eqs. (5.121)-(5.124).

5.9 SUMMARY AND DISCUSSIONS

In this paper, we uncovered new conformal-like symmetries (5.80) for the field equations [(5.20) and (5.21)] of strictly massless fermions of spin $s \geq 3/2$ on dS_4. The associated symmetry algebra closes on $so(4,2)$ up to gauge transformations [see Eqs. (5.92a)-(5.92c)]. We also showed that the physical (positive frequency) mode solutions (5.48) and (5.57) form a direct sum of UIRs of the conformal-like $so(4,2)$ algebra. As for the interpretation of the conformal-like symmetries, we found that, at the level of the field strength tensor-spinors, each conformal-like transformation is expressed as a product of two transformations: an infinitesimal axial rotation and an infinitesimal conformal transformation (this was shown explicitly for the spin-$s = 3/2, 5/2$ cases and conjectured for the cases with $s \geq 7/2$ - see Section 5.8).
Let us discuss in passing the flat-space limit of the conformal-like symmetries (i.e. the limit of zero cosmological constant). First, we observe that the flat-space limit of the five conformal Killing vectors (5.77) of dS_4 gives rise to the four translation Killing vectors and the generator of dilations of Minkowski spacetime (rather than the five conformal Killing vectors of Minkowski spacetime as one might expect). This can be verified by recovering the dS radius, $\mathscr{R}_{dS}$, such that (5.34) is written as

$$ds^2 = \mathscr{R}_{dS}^2 \Bigg(-dt^2 + \cosh^2 t \left[d\theta_3^2 + \sin^2\theta_3 \left(d\theta_2^2 + \sin^2\theta_2 \, d\theta_1^2 \right) \right] \Bigg),$$

while Eqs. (5.20) and (5.21) are written as

$$\left(\nabla\!\!\!/ + \frac{ir}{\mathscr{R}_{dS}}\right)\Psi_{\mu_1...\mu_r} = 0,$$
$$\nabla^\alpha\Psi_{\alpha\mu_2...\mu_r} = 0, \quad \gamma^\alpha\Psi_{\alpha\mu_2...\mu_r} = 0.$$

Then, defining $t \equiv T/\mathscr{R}_{dS}$ and $\theta_3 \equiv r/\mathscr{R}_{dS}$ and letting $\mathscr{R}_{dS} \to \infty$, we can find the flat-space limit of the dS conformal Killing vectors (5.77). The flat-space version of equations (5.20) and (5.21) corresponds to

$$\begin{aligned} &\partial\!\!\!/\Psi_{\mu_1...\mu_r} = 0, \\ &\partial^\alpha\Psi_{\alpha\mu_2...\mu_r} = 0, \quad \gamma^\alpha\Psi_{\alpha\mu_2...\mu_r} = 0. \end{aligned} \tag{5.127}$$

The five de Sitterian conformal-like symmetries (5.80) reduce to the following flat-space symmetries of Eq. (5.127)

$$T_w^{flat}\Psi_{\mu_1...\mu_r} = \gamma^5 w^\rho \partial_\rho \Psi_{\mu_1...\mu_r}, \tag{5.128}$$

where w^ρ is a translation Killing vector or the generator of dilations (i.e., in the standard Minkowski coordinates x^0, x^1, x^2, x^3 with line element $-(dx^0)^2 + \sum_{j=1}^{3}(dx^j)^2$, we have $w^\rho \in \{\delta^\rho_0, \delta^\rho_1, \delta^\rho_2, \delta^\rho_3, x^\rho\}$). We observe that the transformation (5.128) is a product of two transformations. However, unlike in dS_4, in Minkowski spacetime, each of the two transformations present in the product (5.128) is also a symmetry. In other words, Eqs. (5.127) are invariant under the replacement $\Psi_{\mu_1...\mu_r} \to \gamma^5\Psi_{\mu_1...\mu_r}$ (infinitesimal axial rotations), as well as under $\Psi_{\mu_1...\mu_r} \to w^\rho\partial_\rho\Psi_{\mu_1...\mu_r}$.

In Ref. [54], using the unfolded formalism, Vasiliev presented a $sp(8,\mathbb{R})$ invariant formulation of free massless fields (gauge potentials) of any spin in AdS_4 and showed that the free field equations are invariant under $o(4,2)$ (see also Ref. [53]). Although further study is required, it is likely that the dS version of Vasiliev's conformal invariance [54] is related to the conformal-like symmetries we presented in this paper.

It is worth recalling that unitary superconformal field theories on dS_4 are known to exist [3]. In view of our newly discovered conformal-like symmetries for strictly massless fermions, it is interesting to look for new (and possibly unitary) supersymmetric theories on dS_4 that include strictly massless fermions of any spin $s \geq 3/2$.

Acknowledgments. The author is grateful to Atsushi Higuchi for useful discussions and encouragement. The author also thanks Rakibur Rahman for communications. He would also like to thank Dionysios Anninos, F.F. John, Spyros Konitopoulos, Alan Rios Fukelman,

and Guillermo 'il professore' Silva for useful discussions and encouragement. Also, it is a pleasure and an honor to thank Mikhail Vasiliev for useful correspondence. The author acknowledges financial support from the Department of Mathematics, University of York, and from the WW Smith Fund.
Last, but not least, I would like to thank Alex for reminding me that there exist poetic qualities in life beyond poems, which was at the very least inspiring, for lack of a better wor(l)d.

5.10 APPENDIX A - DERIVING EQ. (5.65) BY ANALYTICALLY CONTINUING $so(5)$ ROTATION GENERATORS AND THEIR MATRIX ELEMENTS TO $so(4,1)$

The aim of this Appendix is to explain how to use group-theoretic tools and analytic continuation techniques in order to derive the transformation properties of physical modes in Eq. (5.65).

5.10.1 Background material for representations of $so(5)$ and Gelfand-Tsetlin patterns

The representations of the algebra $so(D+1)$ - with arbitrary D - and the specification of the matrix elements of the generators have been studied by Gelfand and Tsetlin [24]. The $D(D+1)/2$ generators $I_{AB} = -I_{BA}$ $(A, B = 1, 2, ..., D+1)$ of $so(D+1)$ satisfy the commutation relations

$$[I_{AB}, I_{CD}] = (\delta_{BC} I_{AD} + \delta_{AD} I_{BC}) - (A \leftrightarrow B). \tag{5.129}$$

In Ref. [24], the action of the $so(D+1)$ generators has been determined in the decomposition $so(D+1) \supset so(D)$. In particular, the representation space for a $so(D+1)$ representation is chosen to be the direct sum of the representation spaces of all representations of $so(D)$ that appear in the $so(D+1)$ representation. (If a representation of $so(D)$ appears in a representation of $so(D+1)$, then it appears with multiplicity one.) Similarly, the generators of $so(D)$ are determined in the decomposition $so(D) \supset so(D-1)$ and so forth. In other words, Gelfand and Tsetlin [24] determined a $so(D+1)$ representation in the decomposition $so(D+1) \supset so(D) \supset ... \supset so(2)$.

Focusing on $so(5)$. We now specialise to $so(5)$ - since this is the non-compact partner of the dS algebra $so(4,1)$. Let us review some basic results obtained by Gelfand and Tsetlin [24] (with slightly modified notation).

A (unitary) irreducible representation of $so(5)$ is specified by the highest weight $\vec{s} = (s_1, s_2)$ with $s_1 \geq s_2 \geq 0$, where the numbers s_1 and s_2 are simultaneously integers or half-odd-integers. The 10 anti-hermitian generators $I_{AB} = -I_{BA}$ $(A, B = 1, ..., 5)$ act on a finite-dimensional vector space corresponding to a direct sum of $so(4)$ representation spaces (as described at the beginning of the Subsection). Let v denote the orthonormal basis vectors in the $so(5)$ representation space. Each basis vector is uniquely labelled by a "Gelfand-Tsetlin pattern", α, as follows:

$$v(\alpha) \equiv v \begin{pmatrix} s_1 & s_2 \\ f_1 & f_2 \\ p & \\ q & \end{pmatrix}. \tag{5.130}$$

The labels s_1, s_2 are the same for all basis vectors, since they correspond to the highest weight specifying the $so(5)$ representation. The rest of the labels in Eq. (5.130) specify the content of the $so(5)$ representation concerning the chain of subalgebras $so(4) \supset so(3) \supset so(2)$. In particular, the labels f_1, f_2 correspond to a $so(4)$ highest weight $\vec{f} \equiv (f_1, f_2)$ with $f_1 \geq |f_2|$, where f_1 and f_2 are both integers or half-odd integers, while f_2 can be negative. The $so(3)$ weight $p \geq 0$ is an integer or half-odd integer. The full basis of the representation space is given by all $v(\alpha)$'s in eq. (5.130) - with fixed s_1, s_2 - satisfying:

$$\begin{aligned} &s_1 \geq f_1 \geq s_2 \geq |f_2|, \\ &f_1 \geq p \geq |f_2|, \\ &p \geq q \geq -p. \end{aligned} \tag{5.131}$$

The numbers s_1, s_2, f_1, f_2, p and q are all integers or half-odd integers.

In order to obtain the desired transformation formulae (5.65) using analytic continuation, we need to study the action of the generator I_{54} on the basis vectors (5.130). This is given by [24]:

$$-I_{54}\, v\begin{pmatrix} s_1 & s_2 \\ f_1 & f_2 \\ p & \\ q & \end{pmatrix} = -\frac{1}{2}A(f_1,f_2)\, v\begin{pmatrix} s_1 & s_2 \\ f_1+1 & f_2 \\ p & \\ q & \end{pmatrix} - \frac{1}{2}B(f_1,f_2)\ v\begin{pmatrix} s_1 & s_2 \\ f_1 & f_2+1 \\ p & \\ q & \end{pmatrix}$$
$$+\frac{1}{2}A(f_1-1,f_2)\, v\begin{pmatrix} s_1 & s_2 \\ f_1-1 & f_2 \\ p & \\ q & \end{pmatrix}$$
$$+\frac{1}{2}B(f_1,f_2-1)\ v\begin{pmatrix} s_1 & s_2 \\ f_1 & f_2-1 \\ p & \\ q & \end{pmatrix}, \tag{5.132}$$

where

$$A(f_1,f_2) = \sqrt{\frac{(f_1-p+1)(f_1+p+2)(s_1-f_1)(s_1+f_1+3)(f_1-s_2+1)(f_1+s_2+2)}{(f_1+f_2+1)(f_1+f_2+2)(f_1-f_2+1)(f_1-f_2+2)}} \tag{5.133}$$

and

$$B(f_1,f_2) = \sqrt{\frac{(p-f_2)(f_2+p+1)(s_2-f_2)(s_2+f_2+1)(s_1-f_2+1)(s_1+f_2+2)}{(f_1+f_2+1)(f_1+f_2+2)(f_1-f_2)(f_1-f_2+1)}}. \tag{5.134}$$

(Our matrix elements differ from the matrix elements of Ref. [24] by a factor of $1/2$.) Note that $A(f_1,-f_2)=A(f_1,f_2)$ and $B(f_1,f_2)=B(f_1,-f_2-1)$.

5.10.2 Specialising to $so(5)$ representations formed by tensor-spinor spherical harmonics on S^4

The line element of S^4 can be parametrised as

$$ds^2_{S^4} = d\theta_4^2 + \sin^2\theta_4\, d\Omega^2, \tag{5.135}$$

where $0\le\theta_4\le\pi$ and $d\Omega^2$ is the line element of S^3 (5.35). For later convenience, note that the line element (5.135) can be analytically continued to the dS_4 line element (5.34) by making the replacement

$$\theta_4 \to x = \frac{\pi}{2} - it \tag{5.136}$$

- the variable x has been already introduced in Eq. (5.51).
Let $\nabla\!\!\!\!/ = \gamma^\mu \nabla_\mu$ be the Dirac operator on S^4, where γ^μ and ∇_μ are the gamma matrices and covariant derivative, respectively, on S^4. We are interested in (totally symmetric) rank-r tensor-spinor spherical harmonics $\hat{\psi}^{(n;\tilde{r},\sigma\ell;m;k)}_{\mu_1\dots\mu_r}(\theta_4,\boldsymbol{\theta_3})$ (with $\sigma=\pm$) on S^4 that satisfy [30]

$$\begin{aligned}
&\nabla\!\!\!\!/ \hat{\psi}^{(n;\tilde{r},\sigma\ell;m;k)}_{\mu_1\dots\mu_r} = -i(n+2)\,\hat{\psi}^{(n;\tilde{r},\sigma\ell;m;k)}_{\mu_1\dots\mu_r},\\
&\gamma^{\mu_1}\hat{\psi}^{(n;\tilde{r},\sigma\ell;m;k)}_{\mu_1\dots\mu_r} = \nabla^{\mu_1}\hat{\psi}^{(n;\tilde{r},\sigma\ell;m;k)}_{\mu_1\dots\mu_r} = 0, \quad (n=r,r+1,\dots),
\end{aligned} \tag{5.137}$$

where n is the angular momentum quantum number on S^4 [19]. The representation-theoretic meaning of the labels $n,\sigma,\ell,\tilde{r},m$ and k will be discussed below. The hat has been used in order to indicate that the eigenmodes $\hat{\psi}^{(n;\tilde{r},\sigma\ell;m;k)}_{\mu_1\dots\mu_r}(\theta_4,\boldsymbol{\theta_3})$ are normalised with respect to the standard inner product on S^4

$$\int_{S^4}\sqrt{g_{S^4}}\,d\theta_4\,d\theta_3\,d\theta_2\,d\theta_1\,\hat{\psi}^{(n';\tilde{r}',\sigma'\ell';m';k')}_{\mu_1\dots\mu_r}\,\hat{\psi}^{(n;\tilde{r},\sigma\ell;m;k)\mu_1\dots\mu_r} = \delta_{nn'}\delta_{\ell\ell'}\delta_{\sigma\sigma'}\delta_{\tilde{r}\tilde{r}'}\delta_{mm'}\delta_{kk'}, \tag{5.138}$$

where g_{S^4} is the determinant of the S^4 metric. The indices $\mu_1,\dots,\mu_r$ run from θ_1 to θ_4, while the indices $\tilde{\mu}_1,\dots,\tilde{\mu}_r$ run from θ_1 to θ_3.

Gelfand-Tsetlin patterns and tensor-spinor spherical harmonics. The ten Killing vectors of S^4 act on the solution space of Eqs. (5.137) in terms of the Lie-Lorentz derivatives (5.14), and the latter generate a representation of $so(5)$ on this solution space. In particular, for each allowed value of n, the set of eigenmodes $\{\hat{\psi}^{(n;\tilde{r},\sigma\ell;m;k)}_{\mu_1\dots\mu_r}\}$ forms an irreducible representation of $so(5)$ with highest weight

$$\vec{s} = (s_1,s_2) = \left(n+\frac{1}{2}, r+\frac{1}{2}\right) \tag{5.139}$$

with $n=r,r+1,\dots$. Each eigenmode $\hat{\psi}^{(n;\tilde{r},\sigma\ell;m;k)}_{\mu_1\dots\mu_r}(\theta_4,\boldsymbol{\theta_3})$ corresponds to the following Gelfand-Tsetlin pattern (see Eq. (5.130)):

$$\alpha = \begin{pmatrix} n+\frac{1}{2} & r+\frac{1}{2}\\ \ell+\frac{1}{2} & \sigma(\tilde{r}+\frac{1}{2})\\ m+\frac{1}{2} & \\ k+\frac{1}{2} & \end{pmatrix}. \tag{5.140}$$

[19] There are also tensor-spinor spherical harmonics on S^4 that satisfy Eqs. (5.137) but with an opposite sign for the eigenvalue. We will not discuss these here as they form equivalent $so(5)$ representations with the tensor-spinor spherical harmonics in Eq. (5.137).

The numbers ℓ, m and k are the angular momentum quantum numbers on S^3, S^2 and S^1, respectively, and their allowed values are found from (5.131).

Based on the discussion in the previous paragraph, we can identify each eigenmode $\hat{\psi}^{(n;\tilde{r},\sigma\ell;m;k)}_{\mu_1\ldots\mu_r}(\theta_4,\boldsymbol{\theta_3})$ with a basis vector (5.130) labeled by the pattern (5.140). In particular, we make the identifications:

$$v\begin{pmatrix} n+\frac{1}{2} & r+\frac{1}{2} \\ \ell+\frac{1}{2} & \tilde{r}+\frac{1}{2} \\ m+\frac{1}{2} & \\ k+\frac{1}{2} & \end{pmatrix} \rightarrow \hat{\psi}^{(n;\tilde{r},+\ell;m;k)}_{\mu_1\ldots\mu_r} \tag{5.141}$$

and

$$v\begin{pmatrix} n+\frac{1}{2} & r+\frac{1}{2} \\ \ell+\frac{1}{2} & -(\tilde{r}+\frac{1}{2}) \\ m+\frac{1}{2} & \\ k+\frac{1}{2} & \end{pmatrix} \rightarrow -i\,(-1)^{\tilde{r}}\hat{\psi}^{(n;\tilde{r},-\ell;m;k)}_{\mu_1\ldots\mu_r}, \tag{5.142}$$

where the phase factor $-i(-1)^{\tilde{r}}$ has been introduced for convenience.

5.10.3 Transformation properties of tensor-spinor spherical harmonics on S^4 under $so(5)$

In this Subsection, we find the $so(5)$ transformation formulae for $\mathbb{L}_{\mathscr{S}}\hat{\psi}^{(n;\tilde{r}=r,\pm\ell;m;k)}_{\mu_1\ldots\mu_r}$ that (after analytic continuation) will give rise to the $so(4,1)$ transformation formulae (5.65) for the physical modes of the strictly massless fermions on dS_4. Here $\mathbb{L}_{\mathscr{S}}$ is the Lie-Lorentz derivative on S^4 with respect to the Killing vector

$$\mathscr{S}=\mathscr{S}^{\mu}\partial_{\mu}=\cos\theta_3\,\frac{\partial}{\partial\theta_4}-\cot\theta_4\sin\theta_3\,\frac{\partial}{\partial\theta_3}. \tag{5.143}$$

This Killing vector corresponds to the $so(5)$ generator $I_{45}=-I_{54}$ in Eq. (5.132) and by making the replacement (5.136) it is analytically continued as: $\mathscr{S}\rightarrow iX$, where X is the dS boost Killing vector (5.64).

We focus on the eigenmodes $\hat{\psi}^{(n;\tilde{r}=r,+\ell;m;k)}_{\mu_1\ldots\mu_r}(\theta_4,\boldsymbol{\theta_3})$ and $\hat{\psi}^{(n;\tilde{r}=r,-\ell;m;k)}_{\mu_1\ldots\mu_r}(\theta_4,\boldsymbol{\theta_3})$, as the former will be analytically continued to the physical modes $\Psi^{(phys,+\ell;m;k)}_{\mu_1\ldots\mu_r}(t,\boldsymbol{\theta}_3)$ (5.57) and the latter to the physical modes $\Psi^{(phys,-\ell;m;k)}_{\mu_1\ldots\mu_r}(t,\boldsymbol{\theta}_3)$ (5.48). We will also discuss in passing the eigenmodes $\hat{\psi}^{(n;\tilde{r}=r-1,\pm\ell;m;k)}_{\mu_1\ldots\mu_r}(\theta_4,\boldsymbol{\theta_3})$ as they will be analytically continued to the pure gauge modes $\Psi^{(pg,\tilde{r}=r-1,\pm\ell;m;k)}_{\mu_1\ldots\mu_r}(t,\boldsymbol{\theta}_3)$, which appear in the transformation formulae (5.65).

Explicit expressions for the infinitesimal transformations $\mathbb{L}_{\mathscr{S}}\hat{\psi}^{(n;\,\tilde{r}=r,\,\pm\ell;\,m;k)}_{\mu_1\ldots\mu_r}$ and $\mathbb{L}_{\mathscr{S}}\hat{\psi}^{(n;\,\tilde{r}=r-1,\,\pm\ell;\,m;k)}_{\mu_1\ldots\mu_r}$ are immediately found from Eq. (5.132) with the use of Eqs. (5.141) and (5.142). However, these transformation properties refer to normalised eigenmodes, while the desired dS transformation properties (5.65) refer to un-normalised eigenmodes. Therefore, we will first find the $so(5)$ transformation properties for the un-normalised eigenmodes on S^4 (the un-normalised eigenmodes will be defined below) and then perform analytic continuation to dS_4.

Some useful expressions for eigenmodes on S^3

For later convenience, let us present some expressions for certain tensor-spinor spherical harmonics on S^3 [see Eqs. (5.42) and (5.43)]. These expressions can be easily obtained using the method of separation of variables as has been explained in Refs. [8, 9, 33]. Below, we use the notation $\boldsymbol{\theta_3} = (\theta_3, \theta_2, \theta_1) = (\theta_3, \boldsymbol{\theta_2})$. We only need the following expressions for our computations:

• **Rank-r eigenmodes $\tilde{\psi}^{(\ell;m;k)}_{\pm\tilde{\mu}_1\ldots\tilde{\mu}_r}(\theta_3,\boldsymbol{\theta_2})$ on S^3:** The component $\tilde{\psi}^{(\ell;m;k)}_{\pm\theta_3\theta_3\ldots\theta_3}(\theta_3,\boldsymbol{\theta_2})$ is a spinor on S^2. It is given by

$$\tilde{\psi}^{(\ell;m;k)}_{\pm\underbrace{\theta_3\theta_3\ldots\theta_3}_{r\ \text{times}}}(\theta_3,\boldsymbol{\theta_2}) = \frac{\tilde{c}(r,\ell;m)}{\sqrt{2}}\frac{1}{\sqrt{2}}(\mathbf{1}+i\tilde{\gamma}^3)\Big\{\tilde{\phi}^{(r)}_{\ell m}(\theta_3) \pm i\tilde{\psi}^{(r)}_{\ell m}(\theta_3)\tilde{\gamma}^3\Big\}\tilde{\tilde{\psi}}^{(m;k)}_{-}(\boldsymbol{\theta_2}), \tag{5.144}$$

where $\frac{\tilde{c}(r,\ell;m)}{\sqrt{2}}$ is the normalisation factor, $\tilde{\tilde{\psi}}^{(m;k)}_{-}(\boldsymbol{\theta_2})$ are the spinor eigenfunctions of the Dirac operator $\tilde{\tilde{\nabla\!\!\!/}}$ on S^2 satisfying

$$\tilde{\tilde{\nabla\!\!\!/}}\tilde{\tilde{\psi}}^{(m;k)}_{-} = -i(m+1)\tilde{\tilde{\psi}}^{(m;k)}_{-},$$

while the spinors $\tilde{\tilde{\psi}}^{(m;k)}_{+} \equiv \tilde{\gamma}^3\,\tilde{\tilde{\psi}}^{(m;k)}_{-}$ satisfy

$$\tilde{\tilde{\nabla\!\!\!/}}\tilde{\tilde{\psi}}^{(m;k)}_{+} = +i(m+1)\tilde{\tilde{\psi}}^{(m;k)}_{+}.$$

The functions $\tilde{\phi}^{(r)}_{\ell m}(\theta_3)$ and $\tilde{\psi}^{(r)}_{\ell m}(\theta_3)$ correspond to special cases of the following functions:

$$\begin{aligned}\tilde{\phi}^{(\tilde{a})}_{\ell m}(\theta_3) = \tilde{\kappa}_{\tilde{\phi}}(\ell,m)&\left(\cos\frac{\theta_3}{2}\right)^{m+1-\tilde{a}}\left(\sin\frac{\theta_3}{2}\right)^{m-\tilde{a}}\\ &\times F\left(-\ell+m,\ell+m+3;m+\frac{3}{2};\sin^2\frac{\theta_3}{2}\right),\end{aligned} \tag{5.145}$$

and

$$\tilde{\psi}^{(\tilde{a})}_{\ell m}(\theta_3) = \tilde{\kappa}_{\tilde{\phi}}(\ell, m)\, \frac{\ell + \frac{3}{2}}{m + \frac{3}{2}} \left(\cos\frac{\theta_3}{2}\right)^{m-\tilde{a}} \left(\sin\frac{\theta_3}{2}\right)^{m+1-\tilde{a}} \times F\left(-\ell + m, \ell + m + 3; m + \frac{5}{2}; \sin^2\frac{\theta_3}{2}\right), \tag{5.146}$$

where $\tilde{a}$ is an integer, while the factor $\tilde{\kappa}_{\tilde{\phi}}(\ell, m)$ is given by

$$\tilde{\kappa}_{\tilde{\phi}}(\ell, m) = \frac{\Gamma(\ell + \frac{3}{2})}{\Gamma(\ell - m + 1)\,\Gamma(m + \frac{3}{2})}. \tag{5.147}$$

• **Rank-$(r-1)$ eigenmodes $\tilde{\psi}^{(\ell;m;k)}_{\pm\tilde{\mu}_2\ldots\tilde{\mu}_r}(\theta_3, \boldsymbol{\theta_2})$ on S^3:** The component $\tilde{\psi}^{(\ell;m;k)}_{\pm\theta_3\theta_3\ldots\theta_3}(\theta_3, \boldsymbol{\theta_2})$ is a spinor on S^2. It is given by

$$\tilde{\psi}^{(\ell;m;k)}_{\pm\underbrace{\theta_3\theta_3\ldots\theta_3}_{r-1 \text{ times}}}(\theta_3, \boldsymbol{\theta_2}) = \frac{\tilde{c}(r-1, \ell; m)}{\sqrt{2}} \frac{1}{\sqrt{2}}(\mathbf{1} + i\tilde{\gamma}^3)\left\{\tilde{\phi}^{(r-1)}_{\ell m}(\theta_3) \pm i\tilde{\psi}^{(r-1)}_{\ell m}(\theta_3)\tilde{\gamma}^3\right\}\tilde{\tilde{\psi}}^{(m;k)}_{-}(\boldsymbol{\theta_2}), \tag{5.148}$$

where $\frac{\tilde{c}(r-1,\ell;m)}{\sqrt{2}}$ is the normalisation factor, while $\tilde{\phi}^{(r-1)}_{\ell m}(\theta_3)$ and $\tilde{\psi}^{(r-1)}_{\ell m}(\theta_3)$ are given by Eqs. (5.145) and (5.146), respectively, with $\tilde{a} = r - 1$.

Expressions for the eigenmodes $\hat{\psi}^{(n;\tilde{r}=r,\pm\ell;m;k)}_{\mu_1\mu_2\ldots\mu_r}$ on S^4

Working as in Section 5.4, we separate variables for equations (5.137) on S^4. We find

$$\hat{\psi}^{(n;\tilde{r}=r,+\ell;m;k)}_{\theta_4\mu_2\ldots\mu_r}(\theta_4, \boldsymbol{\theta}_3) = 0,$$
$$\hat{\psi}^{(n;\tilde{r}=r,+\ell;m;k)}_{\tilde{\mu}_1\ldots\tilde{\mu}_r}(\theta_4, \boldsymbol{\theta}_3) = \frac{c(r, n; \tilde{r} = r, \ell)}{\sqrt{2}} \begin{pmatrix} i\psi^{(-r)}_{n\ell}(\theta_4)\, \tilde{\psi}^{(\ell;m;k)}_{+\tilde{\mu}_1\ldots\tilde{\mu}_r}(\boldsymbol{\theta_3}) \\ -\phi^{(-r)}_{n\ell}(\theta_4)\, \tilde{\psi}^{(\ell;m;k)}_{+\tilde{\mu}_1\ldots\tilde{\mu}_r}(\boldsymbol{\theta_3}) \end{pmatrix} \tag{5.149}$$

and

$$\hat{\psi}^{(n;\tilde{r}=r,-\ell;m;k)}_{\theta_4\mu_2\ldots\mu_r}(\theta_4, \boldsymbol{\theta}_3) = 0,$$
$$\hat{\psi}^{(n;\tilde{r}=r,-\ell;m;k)}_{\tilde{\mu}_1\ldots\tilde{\mu}_r}(\theta_4, \boldsymbol{\theta}_3) = \frac{c(r, n; \tilde{r} = r, \ell)}{\sqrt{2}} \begin{pmatrix} \phi^{(-r)}_{n\ell}(\theta_4)\, \tilde{\psi}^{(\ell;m;k)}_{-\tilde{\mu}_1\ldots\tilde{\mu}_r}(\boldsymbol{\theta_3}) \\ -i\psi^{(-r)}_{n\ell}(\theta_4)\, \tilde{\psi}^{(\ell;m;k)}_{-\tilde{\mu}_1\ldots\tilde{\mu}_r}(\boldsymbol{\theta_3}) \end{pmatrix}, \tag{5.150}$$

where $\frac{c(r,n;\tilde{r}=r,\ell)}{\sqrt{2}}$ is a normalisation factor that will be determined below. The functions $\phi^{(-r)}_{n\ell}(\theta_4)$ and $\psi^{(-r)}_{n\ell}(\theta_4)$ belong to the following family of functions:

$$\phi^{(a)}_{n\ell}(\theta_4) = \kappa_\phi(n, \ell) \left(\cos\frac{\theta_4}{2}\right)^{\ell+1-a} \left(\sin\frac{\theta_4}{2}\right)^{\ell-a} \times F\left(-n + \ell, n + \ell + 4; \ell + 2; \sin^2\frac{\theta_4}{2}\right), \tag{5.151}$$

$$\psi^{(a)}_{n\ell}(\theta_4) = \kappa_\phi(n,\ell)\frac{n+2}{\ell+2}\left(\cos\frac{\theta_4}{2}\right)^{\ell-a}\left(\sin\frac{\theta_4}{2}\right)^{\ell+1-a} \times F\left(-n+\ell, n+\ell+4; \ell+3; \sin^2\frac{\theta_4}{2}\right), \tag{5.152}$$

where the factor $\kappa_\phi(n,\ell)$ is given by

$$\kappa_\phi(n,\ell) = \frac{\Gamma(n+2)}{\Gamma(n-\ell+1)\Gamma(\ell+2)}. \tag{5.153}$$

Substituting the eigenmode (5.149) (or (5.150)) into the inner product (5.138), and using the normalisation of the tensor-spinor eigenmodes on S^3 (5.46), we find

$$\left|\frac{c(r,n;\tilde{r}=r,\ell)}{\sqrt{2}}\right|^2 = 2^{2r-3}\frac{\Gamma(n-\ell+1)\Gamma(4+n+\ell)}{|\Gamma(n+2)|^2}. \tag{5.154}$$

Introducing the un-normalised eigenmodes. Now, let us define the un-normalised eigenmodes $\psi^{(n;\tilde{r},\pm\ell;m;k)}_{\mu_1\mu_2\ldots\mu_r}(\theta_4,\boldsymbol{\theta}_3)$ (for any value of $\tilde{r}\in\{0,\ldots,r\}$) as

$$\psi^{(n;\tilde{r},\pm\ell;m;k)}_{\mu_1\mu_2\ldots\mu_r}(\theta_4,\boldsymbol{\theta}_3) \equiv \frac{\sqrt{2}}{c(r,n;\tilde{r},\ell)}\frac{1}{\kappa_\phi(n,\ell)}\hat{\psi}^{(n;\tilde{r},\pm\ell;m;k)}_{\mu_1\mu_2\ldots\mu_r}(\theta_4,\boldsymbol{\theta}_3), \tag{5.155}$$

where the normalisation factors $c(r,n;\tilde{r},\ell)$ that are needed for our computations (and have not been defined yet) will be defined later. (Recall that the un-normalised eigenmodes are the ones that will be analytically continued to dS_4.)

Transformation of the un-normalised eigenmodes $\psi^{(n;\tilde{r}=r,\pm\ell;m;k)}_{\mu_1\mu_2\ldots\mu_r}$. The infinitesimal $so(5)$ transformation of the un-normalised modes $\mathbb{L}_{\mathscr{S}}\psi^{(n;\tilde{r}=r,\pm\ell;m;k)}_{\mu_1\mu_2\ldots\mu_r}$ can be straightforwardly found from the transformation of the normalised modes $\mathbb{L}_{\mathscr{S}}\hat{\psi}^{(n;\tilde{r}=r,\pm\ell;m;k)}_{\mu_1\mu_2\ldots\mu_r}$ (see the discussion at the beginning of this Subsection). We find in this manner

$$\begin{aligned}\mathbb{L}_{\mathscr{S}}\psi^{(n;\tilde{r}=r,\pm\ell;m;k)}_{\mu_1\mu_2\ldots\mu_r} =& -\frac{\kappa_\phi(n,\ell+1)}{2\kappa_\phi(n,\ell)}\sqrt{\frac{(\ell-m+1)(\ell+m+3)}{(\ell+2)^2-r^2}}(n+\ell+4)\\ &\times\psi^{(n;\tilde{r}=r,\pm(\ell+1);m;k)}_{\mu_1\mu_2\ldots\mu_r}\\ &+\frac{\kappa_\phi(n,\ell-1)}{2\kappa_\phi(n,\ell)}\sqrt{\frac{(\ell-m)(\ell+m+2)}{(\ell+1)^2-r^2}}(n-\ell+1)\\ &\times\psi^{(n;\tilde{r}=r,\pm(\ell-1);m;k)}_{\mu_1\mu_2\ldots\mu_r}\\ &+\frac{\sqrt{(n+2)^2-r^2}}{2}K_{\ell m}\frac{c(r,n;\tilde{r}=r-1,\ell)}{c(r,n;\tilde{r}=r,\ell)}\\ &\times\psi^{(n;\tilde{r}=r-1,\pm\ell;m;k)}_{\mu_1\mu_2\ldots\mu_r},\end{aligned} \tag{5.156}$$

where

$$K_{\ell m} = \sqrt{\frac{\left((m+1)^2 - r^2\right)\,(2r+1)}{\left((\ell+1)^2 - r^2\right)\,\left((\ell+2)^2 - r^2\right)}}. \tag{5.157}$$

Note that, under this $so(5)$ transformation, the modes $\psi^{(n;\,\tilde{r}=r,\,+\ell;\,m;k)}_{\mu_1\mu_2\ldots\mu_r}$ do not mix with the modes $\psi^{(n;\,\tilde{r}=r,\,-\ell;\,m;k)}_{\mu_1\mu_2\ldots\mu_r}$. This observation plays a key role when performing analytic continuation to dS_4, as it implies that the strictly massless fermions on dS_4 correspond to a direct sum of irreducible representations of $so(4,1)$ - see Eq. (5.65).

Expressions for the eigenmodes $\hat{\psi}^{(n;\,\tilde{r}=r-1,\,\pm\ell;\,m;k)}_{\mu_1\mu_2\ldots\mu_r}$ on S^4

By separating variables again for equations (5.137) we find

$$\begin{aligned}
&\hat{\psi}^{(n;\,\tilde{r}=r-1,\,+\ell;\,m;k)}_{\theta_4\theta_4\mu_3\ldots\mu_r}(\theta_4,\boldsymbol{\theta}_3) = 0,\\
&\hat{\psi}^{(n;\,\tilde{r}=r-1,\,+\ell;\,m;k)}_{\theta_4\tilde{\mu}_2\ldots\tilde{\mu}_r}(\theta_4,\boldsymbol{\theta}_3) = \frac{c(r,n;\tilde{r}=r-1,\ell)}{\sqrt{2}}\begin{pmatrix} i\psi^{(-r+2)}_{n\ell}(\theta_4)\,\tilde{\psi}^{(\ell;m;k)}_{+\tilde{\mu}_2\ldots\tilde{\mu}_r}(\boldsymbol{\theta_3})\\ -\phi^{(-r+2)}_{n\ell}(\theta_4)\,\tilde{\psi}^{(\ell;m;k)}_{+\tilde{\mu}_2\ldots\tilde{\mu}_r}(\boldsymbol{\theta_3})\end{pmatrix}
\end{aligned} \tag{5.158}$$

and

$$\begin{aligned}
&\hat{\psi}^{(n;\,\tilde{r}=r-1,\,-\ell;\,m;k)}_{\theta_4\theta_4\mu_3\ldots\mu_r}(\theta_4,\boldsymbol{\theta}_3) = 0\\
&\hat{\psi}^{(n;\,\tilde{r}=r-1,\,-\ell;\,m;k)}_{\theta_4\tilde{\mu}_2\ldots\tilde{\mu}_r}(\theta_4,\boldsymbol{\theta}_3) = \frac{c(r,n;\tilde{r}=r-1,\ell)}{\sqrt{2}}\begin{pmatrix} \phi^{(-r+2)}_{n\ell}(\theta_4)\,\tilde{\psi}^{(\ell;m;k)}_{-\tilde{\mu}_2\ldots\tilde{\mu}_r}(\boldsymbol{\theta_3})\\ -i\psi^{(-r+2)}_{n\ell}(\theta_4)\,\tilde{\psi}^{(\ell;m;k)}_{-\tilde{\mu}_2\ldots\tilde{\mu}_r}(\boldsymbol{\theta_3})\end{pmatrix},
\end{aligned} \tag{5.159}$$

where $\frac{c(r,n;\tilde{r}=r-1,\ell)}{\sqrt{2}}$ is the normalisation factor, while the functions $\phi^{(-r+2)}_{n\ell}(\theta_4)$ and $\psi^{(-r+2)}_{n\ell}(\theta_4)$ are given by Eqs. (5.151) and (5.152), respectively, with $a=-r+2$. The components $\hat{\psi}^{(n;\,\tilde{r}=r-1,\,\pm\ell;\,m;k)}_{\tilde{\mu}_1\ldots\tilde{\mu}_r}(\theta_4,\boldsymbol{\theta}_3)$ can be found using the TT conditions in Eq. (5.137).

Now that we know the expressions (5.158) and (5.159), we can perform the following calculation for later convenience. Letting $\mu_1=\theta_4$ and $\mu_2=\ldots=\mu_r=\theta_3$ in $\mathbb{L}_{\mathscr{S}}\psi^{(n;\,\tilde{r}=r,\,\pm\ell;\,m;k)}_{\mu_1\ldots\mu_r}$ [Eq. (5.156)], we find

$$\mathbb{L}_{\mathscr{S}}\psi^{(n;\,\tilde{r}=r,\,\pm\ell;\,m;k)}_{\theta_4\theta_3\ldots\theta_3} = \frac{\sqrt{(n+2)^2-r^2}}{2}K_{\ell m}\frac{c(r,n;\tilde{r}=r-1,\ell)}{c(r,n;\tilde{r}=r,\ell)}\,\psi^{(n;\,\tilde{r}=r-1,\,\pm\ell;\,m;k)}_{\theta_4\theta_3\ldots\theta_3}, \tag{5.160}$$

while using the explicit expressions (5.149), (5.150), (5.158) and (5.159) we rewrite this equation as

$$\mathbb{L}_{\mathscr{S}}\psi^{(n;\,\tilde{r}=r,\,\pm\ell;\,m;k)}_{\theta_4\theta_3\ldots\theta_3} = \frac{1}{2}\frac{\tilde{c}(r,\ell;m)}{\tilde{c}(r-1,\ell;m)}\,\psi^{(n;\,\tilde{r}=r-1,\,\pm\ell;\,m;k)}_{\theta_4\theta_3\ldots\theta_3}, \tag{5.161}$$

Then, comparing Eqs. (5.160) and (5.161) we find

$$c(r,n;\tilde{r}=r-1,\ell) \propto \frac{1}{\sqrt{(n+2)^2-r^2}}. \tag{5.162}$$

Transformation of the un-normalised eigenmodes $\psi^{(n;\,\tilde{r}=r-1,\,\pm\ell;\,m;k)}_{\mu_1\mu_2\dots\mu_r}$**.** Again, the infinitesimal $so(5)$ transformation of the un-normalised modes $\mathbb{L}_{\mathscr{S}}\psi^{(n;\,\tilde{r}=r-1,\,\pm\ell;\,m;k)}_{\mu_1\mu_2\dots\mu_r}$ can be straightforwardly found from the transformation of the normalised modes $\mathbb{L}_{\mathscr{S}}\hat{\psi}^{(n;\,\tilde{r}=r-1,\,\pm\ell;\,m;k)}_{\mu_1\mu_2\dots\mu_r}$ (see the discussion at the beginning of this Subsection). We find

$$\mathbb{L}_{\mathscr{S}}\psi^{(n;\,\tilde{r}=r-1,\,\pm\ell;\,m;k)}_{\mu_1\mu_2\dots\mu_r} = -\frac{\sqrt{(n+2)^2-r^2}}{2}K_{\ell m}\frac{c(r,n;\tilde{r}=r,\ell)}{c(r,n;\tilde{r}=r-1,\ell)}\,\psi^{(n;\,\tilde{r}=r,\,\pm\ell;\,m;k)}_{\mu_1\mu_2\dots\mu_r}+\dots, \tag{5.163}$$

where '...' includes eigenmodes that are orthogonal to both $\psi^{(n;\,\tilde{r}=r,\,\pm\ell;\,m;k)}_{\mu_1\mu_2\dots\mu_r}$ and $\psi^{(n;\,\tilde{r}=r-1,\,\pm\ell;\,m;k)}_{\mu_1\mu_2\dots\mu_r}$.

5.10.4 PERFORMING ANALYTIC CONTINUATION

Let us analytically continue the tensor-spinor spherical harmonics (5.137) on S^4 in order to obtain tensor-spinors satisfying Eqs. (5.6) and (5.7) on dS_4. By making the replacements $\theta_4 \to x(t) = \pi/2 - it$ [see Eq. (5.136)] and

$$n \to -2-iM, \tag{5.164}$$

we analytically continue the un-normalised tensor-spinor spherical harmonics on S^4 to tensor-spinors on dS_4 as

$$\psi^{(n;\,\tilde{r},\,\sigma\ell;\,m;k)}_{\mu_1\dots\mu_r}(\theta_4,\boldsymbol{\theta_3}) \to \psi^{(-2-iM;\,\tilde{r},\,\sigma\ell;\,m;k)}_{\mu_1\dots\mu_r}(x(t),\boldsymbol{\theta_3}).$$

The analytically continued tensor-spinors satisfy Eqs. (5.6) and (5.7) on dS_4, which we rewrite here again for convenience

$$\begin{aligned}\nabla\!\!\!/\,\psi^{(-2-iM;\,\tilde{r},\,\sigma\ell;\,m;k)}_{\mu_1\dots\mu_r} &= -M\,\psi^{(-2-iM;\,\tilde{r},\,\sigma\ell;\,m;k)}_{\mu_1\dots\mu_r},\\ \gamma^{\mu_1}\psi^{(-2-iM;\,\tilde{r},\,\sigma\ell;\,m;k)}_{\mu_1\dots\mu_r} &= \nabla^{\mu_1}\psi^{(-2-iM;\,\tilde{r},\,\sigma\ell;\,m;k)}_{\mu_1\dots\mu_r} = 0 \quad .\end{aligned} \tag{5.165}$$

Let us focus on imaginary values of the mass parameter M. For these values of M, a dS invariant (and time-independent) scalar product is given by (5.67).

By applying the aforementioned analytic continuation techniques to the $so(5)$ transformation formulae (5.156) and (5.163), we find

$$\begin{aligned}
&\mathbb{L}_X \psi^{(-2-iM;\,\tilde{r}=r,\,\pm\ell;\,m;k)}_{\mu_1\mu_2\dots\mu_r} \\
&= i\frac{\kappa_\phi(-2-iM,\ell+1)}{2\kappa_\phi(-2-iM,\ell)}\sqrt{\frac{(\ell-m+1)(\ell+m+3)}{(\ell+2)^2-r^2}}(-iM+\ell+2) \\
&\quad \times \psi^{(-2-iM;\,\tilde{r}=r,\,\pm(\ell+1);\,m;k)}_{\mu_1\mu_2\dots\mu_r} \\
&\quad - i\frac{\kappa_\phi(-2-iM,\ell-1)}{2\kappa_\phi(-2-iM,\ell)}\sqrt{\frac{(\ell-m)(\ell+m+2)}{(\ell+1)^2-r^2}}(-iM-\ell-1) \\
&\quad \times \psi^{(-2-iM;\,\tilde{r}=r,\,\pm(\ell-1);\,m;k)}_{\mu_1\mu_2\dots\mu_r} \\
&\quad - i\frac{\sqrt{-M^2-r^2}}{2}K_{\ell m}\frac{c(r,-2-iM;\tilde{r}=r-1,\ell)}{c(r,-2-iM;\tilde{r}=r,\ell)}\,\psi^{(-2-iM;\,\tilde{r}=r-1,\,\pm\ell;\,m;k)}_{\mu_1\mu_2\dots\mu_r}, \qquad (5.166)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{L}_X \psi^{(-2-iM;\,\tilde{r}=r-1,\,\pm\ell;\,m;k)}_{\mu_1\mu_2\dots\mu_r} \\
&\quad = i\frac{\sqrt{-M^2-r^2}}{2}K_{\ell m}\frac{c(r,-2-iM;\tilde{r}=r,\ell)}{c(r,-2-iM;\tilde{r}=r-1,\ell)}\,\psi^{(-2-iM;\,\tilde{r}=r,\,\pm\ell;\,m;k)}_{\mu_1\mu_2\dots\mu_r} + \dots,
\end{aligned} \qquad (5.167)$$

where

$$c(r,-2-iM;\tilde{r}=r-1,\ell) \propto \frac{1}{\sqrt{-M^2-r^2}}. \qquad (5.168)$$

Recall that we focus on imaginary values of M. For convenience we assume that $-M^2 > r^2$ [the value $-M^2 = r^2$ corresponds to the strictly massless case (5.19)]. Using the dS invariance (5.71) of the scalar product (5.67), we have

$$\begin{aligned}
&\left\langle \mathbb{L}_X \psi^{(-2-iM;\,\tilde{r}=r,\,\pm\ell;\,m;k)} | \psi^{(-2-iM;\,\tilde{r}=r-1,\,\pm\ell;\,m;k)} \right\rangle \\
&\quad + \left\langle \psi^{(-2-iM;\,\tilde{r}=r,\,\pm\ell;\,m;k)} | \mathbb{L}_X \psi^{(-2-iM;\,\tilde{r}=r-1,\,\pm\ell;\,m;k)} \right\rangle = 0. \qquad (5.169)
\end{aligned}$$

Then, using the transformation formulae (5.166) and (5.167), we find

$$\begin{aligned}
&\left\langle \psi^{(-2-iM;\,\tilde{r}=r-1,\,\pm\ell;\,m;k)} | \psi^{(-2-iM;\,\tilde{r}=r-1,\,\pm\ell;\,m;k)} \right\rangle \\
&= -\left|\frac{c(r,-2-iM;\tilde{r}=r,\ell)}{c(r,-2-iM;\tilde{r}=r-1,\ell)}\right|^2 \left\langle \psi^{(-2-iM;\,\tilde{r}=r,\,\pm\ell;\,m;k)} | \psi^{(-2-iM;\,\tilde{r}=r,\,\pm\ell;\,m;k)} \right\rangle \qquad (5.170) \\
&\propto \sqrt{-M^2-r^2}^{\,2}. \qquad (5.171)
\end{aligned}$$

From this equation, we understand that the analytically continued eigenmodes $\psi^{(-2-iM;\,\tilde{r}=r-1,\,\pm\ell;\,m;k)}_{\mu_1\dots\mu_r}$ have zero norm in the strictly massless limit ($M^2=-r^2$). In other words, they become pure gauge modes (5.62) in this limit, i.e. $\psi^{(-2+r;\,\tilde{r}=r-1,\,\pm\ell;\,m;k)}_{\mu_1\dots\mu_r}(x(t),\boldsymbol{\theta_3}) = \Psi^{(pg,\,\tilde{r}=r-1,\,\pm\ell;m;k)}_{\mu_1\dots\mu_r}(t,\boldsymbol{\theta}_3)$.

Specialising to the strictly massless case and, finally, deriving Eq. (5.65). Now we tune the mass parameter to the strictly massless value $M=ir$ (5.19). The physical modes are $\psi^{(-2+r;\,\tilde{r},\,\pm\ell;\,m;k)}_{\mu_1\dots\mu_r}(x(t),\boldsymbol{\theta_3}) \equiv \Psi^{(phys,\,\pm\ell;\,m;k)}_{\mu_1\dots\mu_r}(t,\boldsymbol{\theta_3})$ [see Eqs. (5.48) and (5.57)]. The infinitesimal dS transformation of these modes is found by letting $M=ir$ in Eq. (5.166). By doing so, we straightforwardly arrive at Eq. (5.65), as required.

5.11 APPENDIX B - DETAILS FOR THE COMPUTATION OF THE COMMUTATOR (5.90) BETWEEN TWO CONFORMAL-LIKE TRANSFORMATIONS

We wish to calculate $[T_W,T_V]\Psi_{\mu_1\dots\mu_r}$ in order to arrive at Eq. (5.90). For convenience, we split each of the conformal-like transformations in the commutator in two parts as in Eq. (5.83), i.e. $T_W\Psi_{\mu_1\dots\mu_r}=\Delta_W\Psi_{\mu_1\dots\mu_r}+P_W\Psi_{\mu_1\dots\mu_r}$ and $T_V\Psi_{\mu_1\dots\mu_r}=\Delta_V\Psi_{\mu_1\dots\mu_r}+P_V\Psi_{\mu_1\dots\mu_r}$. Then, we split $[T_W,T_V]\Psi_{\mu_1\dots\mu_r}$ into three parts as

$$[T_W,T_V]\Psi_{\mu_1\dots\mu_r}=[\Delta_W,\Delta_V]\Psi_{\mu_1\dots\mu_r}+\Big([\Delta_W,P_V]-[\Delta_V,P_W]\Big)\Psi_{\mu_1\dots\mu_r}+[P_W,P_V]\Psi_{\mu_1\dots\mu_r}. \tag{5.172}$$

Let us now calculate each of the three parts in this equation. (Recall that we denote the Lie bracket between two vectors as $[W,V]^\mu=\mathscr{L}_W V^\mu$.)

Calculating $[\boldsymbol{\Delta_W},\boldsymbol{\Delta_V}]\boldsymbol{\Psi_{\mu_1\dots\mu_r}}$. Using Eqs. (5.77) and (5.78), we find (after a long calculation):

$$\begin{aligned}[\Delta_W,\Delta_V]\Psi_{\mu_1\dots\mu_r} = \;& \mathbb{L}_{[W,V]}\Psi_{\mu_1\dots\mu_r}-2ir\left(\nabla_{(\mu_1}+\frac{i}{2}\gamma_{(\mu_1}\right)\gamma^\lambda\Psi^\rho{}_{\mu_2\dots\mu_r)}\,\nabla_\lambda[W,V]_\rho\\ &-2ir\,\nabla_\lambda[W,V]_\rho\left(\gamma_{(\mu_1}K^{\lambda\rho}{}_{|\mu_2\dots\mu_r)}+\gamma^\rho K_{(\mu_1}{}^{\lambda}{}_{|\mu_2\dots\mu_r)}+\gamma^\lambda K^\rho{}_{(\mu_1|\mu_2\dots\mu_r)}\right),\end{aligned} \tag{5.173}$$

where we have used that any Killing vector ξ (such as $[W,V]$) satisfies [37]

$$\nabla_{\mu_1}\nabla_\lambda\xi_\rho=R_{\rho\lambda\mu_1\sigma}\xi^\sigma, \tag{5.174}$$

while we have also introduced the rank-$(r+1)$ tensor-spinor

$$K_{\lambda\rho|\mu_2\ldots\mu_r} = -K_{\rho\lambda|\mu_2\ldots\mu_r} = K_{\lambda\rho|(\mu_2\ldots\mu_r)} = \left(\nabla_{[\lambda} + \frac{ir}{2}\gamma_{[\lambda}\right)\Psi_{\rho]\mu_2\ldots\mu_r}, \tag{5.175}$$

which is anti-symmetric in its first two indices and symmetric in its last $r-1$ indices. (For $r=1$, this tensor-spinor coincides with the rank-2 anti-symmetric gauge-invariant field-strength tensor-spinor $\left(\nabla_{[\lambda} + \frac{i}{2}\gamma_{[\lambda}\right)\Psi_{\rho]}$, while for $r \geq 2$, $K_{\lambda\rho|\mu_2\ldots\mu_r}$ is not gauge-invariant.) Note that because of the field equations (5.20) and (5.21), the tensor-spinor (5.175) satisfies

$$\gamma^\lambda K_{\lambda\rho|\mu_2\ldots\mu_r} = 0. \tag{5.176}$$

Now we will show that

$$\gamma_{\mu_1} K^{\lambda\rho}{}_{|\mu_2\ldots\mu_r} + \gamma^\rho K_{\mu_1}{}^{\lambda}{}_{|\mu_2\ldots\mu_r} + \gamma^\lambda K^{\rho}{}_{\mu_1|\mu_2\ldots\mu_r} = 0. \tag{5.177}$$

It is convenient to proceed by defining

$${}^\star K^{\alpha\beta}{}_{|\mu_2\ldots\mu_r} \equiv \frac{1}{2}\epsilon^{\alpha\beta\lambda\rho} K_{\lambda\rho|\mu_2\ldots\mu_r}, \tag{5.178}$$

which satisfies

$$\gamma_{\mu_1}\, {}^*K^{\alpha\beta}{}_{|\mu_2\ldots\mu_r} + \gamma^\beta\, {}^*K_{\mu_1}{}^{\alpha}{}_{|\mu_2\ldots\mu_r} + \gamma^\alpha\, {}^*K^{\beta}{}_{\mu_1|\mu_2\ldots\mu_r} = 0 \tag{5.179}$$

(this is easy to show by contracting with $\epsilon_{\gamma\delta\alpha\beta}$ and using well-known properties of the totally anti-symmetric tensor). Then, using $\epsilon^{\alpha\beta\lambda\rho} = i\gamma^5\gamma^{[\alpha}\gamma^\beta\gamma^\lambda\gamma^{\rho]}$ [see Eq. (5.10)] and the gamma-tracelessness property (5.176), we find that Eq. (5.178) becomes

$${}^\star K^{\alpha\beta}{}_{|\mu_2\ldots\mu_r} = -i\,\gamma^5 K^{\alpha\beta}{}_{|\mu_2\ldots\mu_r}. \tag{5.180}$$

Substituting this into Eq. (5.179), we immediately derive Eq. (5.177), and thus, we have

$$[\Delta_W, \Delta_V]\Psi_{\mu_1\ldots\mu_r} = \mathbb{L}_{[W,V]}\Psi_{\mu_1\ldots\mu_r} - 2ir\left(\nabla_{(\mu_1} + \frac{i}{2}\gamma_{(\mu_1}\right)\gamma^\lambda \Psi^\rho{}_{\mu_2\ldots\mu_r)}\nabla_\lambda [W,V]_\rho. \tag{5.181}$$

Calculating $\Big([\boldsymbol{\Delta_W}, \boldsymbol{P_V}] - [\boldsymbol{\Delta_V}, \boldsymbol{P_W}]\Big)\boldsymbol{\Psi_{\mu_1\ldots\mu_r}}$. We find

$$\Big([\Delta_W, P_V] - [\Delta_V, P_W]\Big)\Psi_{\mu_1\ldots\mu_r} = -\frac{2r}{2r+1} \times \left(\nabla_{(\mu_1} + \frac{i}{2}\gamma_{(\mu_1}\right)\left[2[W,V]^\rho \Psi_{\mu_2\ldots\mu_r)\rho} - i(2r+1)\,\nabla_\lambda[W,V]_\rho\,\gamma^\lambda \Psi^\rho_{\mu_2\ldots\mu_r)}\right]. \tag{5.182}$$

Calculating $[P_W, P_V]\Psi_{\mu_1 \dots \mu_r}$. We find

$$[P_W, P_V]\Psi_{\mu_1 \dots \mu_r} = \frac{4r}{(2r+1)^2} \times \left(\nabla_{(\mu_1} + \frac{i}{2}\gamma_{(\mu_1}\right)\left[r\,[W,V]^{\rho}\Psi_{\mu_2 \dots \mu_r)\rho} + \nabla_{\rho}[W,V]_{\lambda}\,(\nabla^{\rho} - \frac{i}{2}\gamma^{\rho})\Psi^{\lambda}_{\mu_2 \dots \mu_r)}\right]. \tag{5.183}$$

Finally, adding Eqs. (5.181), (5.182) and (5.183) by parts, we arrive at Eq. (5.90), as required.

REFERENCES

[1] B. Allen. "Graviton propagator in de Sitter space". In: *Phys. Rev. D* 34 (1986), p. 3670.

[2] B. Allen and C. A. Lütken. "Spinor Two Point Functions in Maximally Symmetric Spaces". In: *Commun. Math. Phys.* 106 (1986), p. 201.

[3] T. Anous, D. Z. Freedman, and A. Maloney. "de Sitter supersymmetry revisited". In: *J. High Energ. Phys.* 2014 (2014), p. 119.

[4] G. Barnich, X. Bekaert, and M. Grigoriev. "Notes on conformal invariance of gauge fields". In: *Journal of Physics A: Mathematical and Theoretical* 48 (2015), p. 505402.

[5] A. Barut and R. Raczka. *Theory of Group Representations and Applications*. WORLD SCIENTIFIC, 1986.

[6] A. O. Barut and I. H. Duru. "Exact Solutions of the Dirac Equation in Spatially Flat Robertson-Walker Space-Times". In: *Phys. Rev. D* 36 (1987), p. 3705.

[7] T. Basile, X. Bekaert, and N. Boulanger. "Mixed-symmetry fields in de Sitter space: a group theoretical glance". In: *Journal of High Energy Physics* 2017 (2016), p. 1.

[8] R. Camporesi and A. Higuchi. "On the eigenfunctions of the Dirac operator on spheres and real hyperbolic spaces". In: *J. Geom. Phys.* 20 (1996), p. 1.

[9] C.-H. Chen, H. T. Cho, A. S. Cornell, and G. Harmsen. "Spin-3/2 fields in D-dimensional Schwarzschild black hole spacetimes". In: *Phys. Rev. D* 94 (2016), p. 044052.

[10] I. Cotăescu. "Integral representation of the Feynman propagators of the Dirac fermions on the de Sitter expanding universe". In: *Eur. Phys. J. C* 78 (2018), p. 769.

[11] I. Cotăescu and C. Crucean. "New Dirac quantum modes in moving frames of the de Sitter space–time". In: *Int. J. Mod. Phys. A* 23 (2012), pp. 3707–3720.

[12] S. Deser and R. I. Nepomechie. "Anomalous propagation of gauge fields in conformally flat spaces". In: *Physics Letters B* 132 (1983), p. 321.

[13] S. Deser and R. I. Nepomechie. "Gauge invariance versus masslessness in de Sitter spaces". In: *Annals of Physics* 154 (1984), p. 396.

[14] S. Deser and A. Waldron. "Arbitrary spin representations in de Sitter from dS / CFT with applications to dS supergravity". In: *Nuclear Physics* 662 (2003), p. 379.

[15] S. Deser and A. Waldron. "Conformal invariance of partially massless higher spins". In: *Physics Letters B* 603 (2004), p. 30.

[16] S. Deser and A. Waldron. "Gauge invariances and phases of massive higher spins in (A)dS". In: *Phys. Rev. Lett.* 87 (2001), p. 031601. eprint: arXiv:hep-th/0102166.

[17] S. Deser and A. Waldron. "Null propagation of partially massless higher spins in (A)dS and cosmological constant speculations". In: *Phys. Lett. B* 513 (2001), p. 137. eprint: hep-th/0105181.

[18] S. Deser and A. Waldron. "Partial masslessness of higher spins in (A)dS". In: *Nuclear Physics* 607 (2001), p. 577.

[19] J. Dixmier. "Sur les représentations de certains groupes orthogonaux". In: *Compt. Rend.* 250 (1960), p. 3263.

[20] V. K. Dobrev, G. Mack, V. B. Petkova, S. G. Petrova, and I. T. Todorov. "Harmonic Analysis on the n-Dimensional Lorentz Group and Its Application to Conformal Quantum Field Theory". In: *Lecture Notes in Physics* 63 (1977).

[21] M. Enayati, J. P. Gazeau, H. Pejhan, and A. Wang. "The de Sitter group and its representations: a window on the notion of de Sitterian elementary systems". In: (2022). eprint: arXiv:2201.11457.

[22] J. Fang and C. Fronsdal. "Massless, half-integer-spin fields in de Sitter Space". In: *Phys. Rev. D* 22 (1980), p. 1361.

[23] D. Z. Freedman and A. V. Proeyen. *Supergravity*. Cambridge University Press, 2012.

[24] I. M. Gelfand and M. L. Tsetlin. “Finite-dimensional representations of groups of orthogonal matrices”. In: *Dokl. Akad. Nauk SSSR* 71 (1950). English translation in: I. M. Gelfand, “Collected papers”. Vol II, Berlin: Springer-Verlag 1988, pp. 657–661, pp. 1017–1020.

[25] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. seventh. Elsevier/Academic Press, 2007.

[26] A. Higuchi. “Forbidden mass range for spin-2 field theory in de Sitter spacetime”. In: *Nuclear Physics B* 282 (1987), pp. 397–436.

[27] A. Higuchi. “Linearized gravity in de Sitter spacetime as a representation of $SO(4,1)$”. In: *Classical and Quantum Gravity* 8 (1991), pp. 2005–2021.

[28] A. Higuchi. “Quantum fields of nonzero spin in De Sitter spacetime”. PhD thesis. Yale University, 1987.

[29] A. Higuchi. “Symmetric tensor spherical harmonics on the N-sphere and their application to the de Sitter group $SO(N,1)$”. In: *J. Math. Phys.* 28 (1987), p. 1553.

[30] Y. Homma and T. Tomihisa. “The spinor and tensor fields with higher spin on spaces of constant curvature”. In: *Annals of Global Analysis and Geometry* 60 (2021), pp. 829–861.

[31] S. Kanno, M. Sasaki, and T. Tanaka. “Vacuum State of the Dirac Field in de Sitter Space and Entanglement Entropy”. In: *J. High Energy Phys.* 2017 (2016), p. 68.

[32] Y. Kosmann. “Dérivées de Lie des spineurs”. In: *Annali di Mat. Pura Appl. (IV)* 91 (1971), p. 317.

[33] V. A. Letsios. “(Non-)unitarity of strictly and partially massless fermions on de Sitter space”. In: *Journal of High Energy Physics* 2023 (2023), p. 15.

[34] V. A. Letsios. “On representation-theoretic properties of fermionic fields in de Sitter spacetime and symmetries underlying the conservation of the electromagnetic zilches”. PhD thesis. University of York, 2023. Chap. (Non-)unitarity of strictly and partially massless fermions on de Sitter space II: a technical explanation.

[35] V. A. Letsios. “The (partially) massless spin-3/2 and spin-5/2 fields in de Sitter spacetime as unitary and non-unitary representations of the de Sitter algebra”. In: (2022). eprint: arXiv:2206.09851v3.

[36] V. A. Letsios. “The eigenmodes for spinor quantum field theory in global de Sitter spacetime”. In: *Journal of Mathematical Physics* 62 (2021), p. 032303.

[37] C. Misner, K. Thorne, and J. A. Wheeler. *Gravitation*. W.H. Freeman and Company, 1973. ISBN: 9780716703440.

[38] W. Mück. “Spinor parallel propagator and Green function in maximally symmetric spaces”. In: *J. Phys. A: Math. Gen.* 33 (2000), p. 3021.

[39] T. Ortín. “A note on Lie-Lorentz derivatives”. In: *Classical and Quantum Gravity* 19 (2002), p. L143.

[40] V. S. Otchik. “On the Hawking radiation of spin-1/2 particles in the de Sitter spacetime”. In: *Class. Quant. Grav.* 2 (1985), p. 539.

[41] U. Ottoson. “A Classification of the Unitary Irreducible Representations of $SO_0(N,1)$”. In: *Commun. Math. Phys.* 8 (1968), p. 228.

[42] B. Pethybridge and V. Schaub. “Tensors and spinors in de Sitter space”. In: *Journal of High Energy Physics* 2022 (2022), p. 123.

[43] PLANCK collaboration. “Planck 2018 results - VI. Cosmological parameters”. In: *Astron. Astrophys.* 641 (2020), A6.

[44] R. Rahman. “Frame- and Metric-Like Higher-Spin Fermions”. In: *Universe* 4 (2018).

[45] R. Rahman. “The involutive system of higher-spin equations”. In: *Nuclear Physics B* 964 (2021), p. 115325.

[46] V. Schaub. “Spinors in (Anti-)de Sitter Space”. In: (2023). eprint: arXiv:2302.08535v1.

[47] F. Schwarz. “Unitary Irreducible Representations of the Groups $SO_0(n,1)$”. In: *Journal of Mathematical Physics* 12 (1971), p. 131.

[48] SDSS collaboration. “Baryon acoustic oscillations in the Sloan Digital Sky Survey Data Release 7 Galaxy Sample”. In: *Mon. Not. Roy. Astron. Soc.* 401 (2010), p. 2148.

[49] G. V. Shishkin. “Some exact solutions of the Dirac equation in gravitational fields”. In: *Class. Quant. Grav.* 8 (1991), p. 175.

[50] SUPERNOVA COSMOLOGY PROJECT collaboration. "Measurements of Ω and Λ from 42 High-Redshift Supernovae". In: *Astrophys. J.* 517 (1999), p. 565.

[51] I. T. Todorov, M. C. Mintchev, and V. B. Petkova. *Conformal Invariance in Quantum Field Theory*. Pisa: Scuola Normale Superiore, 1978.

[52] A. Trautman. *Spin structures on hypersurfaces and the spectrum of the Dirac operator on spheres*. Kluwer Academic Publishers, 1993.

[53] M. A. Vasiliev. "Conformal higher spin symmetries of 4D massless supermultiplets and $osp(L, 2M)$ invariant equations in generalized (super)space". In: *Phys. Rev. D* 66 (2002), p. 066006.

[54] M. A. Vasiliev. "On conformal, $SL(4, \mathbb{R})$ and $Sp(8, \mathbb{R})$ symmetries of massless fields". In: *Nuclear Physics B* 793 (2008), pp. 469–526.

— 6 —

Conservation of all Lipkin's zilches from symmetries of the standard electromagnetic action and a hidden algebra

Abstract

In 1964, Lipkin discovered the *zilches*, a set of conserved quantities in free electromagnetism. Among the zilches, *optical chirality* was identified by Tang and Cohen in 2010, serving as a measure of the handedness of light and leading to investigations into light's interactions with chiral matter. While the symmetries underlying the conservation of the zilches have been examined, the derivation of zilch conservation laws from symmetries of the standard free electromagnetic (EM) action using Noether's theorem has only been addressed in the case of optical chirality. We provide the full answer by demonstrating that the *zilch symmetry transformations* of the four-potential, A_μ, preserve the standard free EM action. We also show that the zilch symmetries belong to the enveloping algebra of a "hidden" invariance algebra of free Maxwell's equations. This "hidden" algebra is generated by familiar conformal transformations and certain "hidden" symmetry transformations of A_μ. Generalizations of the "hidden" symmetries are discussed in the presence of a material four-current, as well as in the theory of a complex Abelian gauge field. Additionally, we extend the zilch symmetries of the standard free EM action to the standard interacting action (with a non-dynamical four-current), allowing for a new derivation of the continuity equation for optical chirality in the presence of electric charges and currents. Furthermore, new continuity equations for the remaining zilches are derived. The de Sitterian version (for fermionic gauge potentials) of the "hidden" symmetries presented in

this chapter corresponds to the conformal-like symmetries of chapter 5.

6.1 INTRODUCTION

Noether's seminal theorem [20] is the cornerstone in understanding the deep connection between symmetries of physical theories and conservation laws. Starting from continuous symmetries of the action functional of a theory, Noether's theorem can be used to derive conservation laws for the associated Euler-Lagrange equations. In relativistic field theories, such as electromagnetism in Minkowski spacetime, the knowledge of a symmetry leads to a Noether (four-)current, V^μ, which is conserved ($\partial_\mu V^\mu = 0$). This conservation holds for fields satisfying the Euler-Lagrange equations - i.e. for on-shell field configurations - and the corresponding Noether charge, $Q = \int d^3x V^0$, is time-independent.

An example of little-known time-independent quantities in free electromagnetism is given by the ten *zilches* that were discovered by Lipkin in 1964 [16]. One of the zilches, now known as *optical chirality*, started drawing renewed theoretical and experimental interest in 2010, when Tang and Cohen realized that this particular zilch provides a measure of the chirality (or handedness) of light [39]. The optical chirality density for the free electromagnetic (EM) field is [16, 39]

$$C = \frac{1}{2}\left(-\boldsymbol{E}\cdot\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{B}\cdot\frac{\partial \boldsymbol{E}}{\partial t}\right), \tag{6.1}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields, respectively [1]. (Throughout this Letter, we adopt the system of units in which the speed of light and the permittivity of free space are $c = \varepsilon_0 = 1$.) The flux of optical chirality is given by the three-vector

$$\boldsymbol{S} = \frac{1}{2}\boldsymbol{E}\times\frac{\partial \boldsymbol{E}}{\partial t} + \frac{1}{2}\boldsymbol{B}\times\frac{\partial \boldsymbol{B}}{\partial t}, \tag{6.2}$$

while the differential conservation law for optical chirality [16]

$$\frac{\partial}{\partial t}C + \boldsymbol{\nabla}\cdot\boldsymbol{S} = 0 \tag{6.3}$$

[1] An alternative expression for optical chirality density that appears in the literature is $C = \frac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{\nabla}\times\boldsymbol{E} + \boldsymbol{B}\cdot\boldsymbol{\nabla}\times\boldsymbol{B})$ [39]. This expression is equal to Eq. (6.1) only in the absence of electric charges and currents. In this Letter, we use the expression (6.1) when electric charges and currents are present. The justification for our choice is that the expression (6.1) is equal to the 000-component of the zilch tensor (6.10) (up to a factor of $1/2$), while, as we show in this Letter, the zilch tensor is the Noether current corresponding to the zilches in free electromagnetism. Also, arguments in favor of defining the optical chirality density in the presence of electric charges and currents using Eq. (6.1) can be found in Ref. [9].

is satisfied if $\boldsymbol{E}$ and $\boldsymbol{B}$ obey the free Maxwell equations

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{B} &= \frac{\partial \boldsymbol{E}}{\partial t}, \quad & \boldsymbol{\nabla} \times \boldsymbol{E} &= -\frac{\partial \boldsymbol{B}}{\partial t}, \\ \boldsymbol{\nabla} \cdot \boldsymbol{E} &= 0, & \boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0. \end{aligned} \tag{6.4}$$

Optical chirality is given by the integral of C over the space, $\int d^3x\, C$, and is a constant of motion for free electromagnetism [16].

In Ref. [39], Tang and Cohen demonstrated that, in the presence of an EM field, the dissymmetry in the excitation rate of two small chiral molecules that are related to each other by mirror reflection is determined by the optical chirality. These findings have motivated novel investigations into chiral light-matter interactions [39, 42, 10, 38, 11, 32, 31, 33, 36, 27, 28, 18, 25]. Understanding these interactions is very important in various disciplines. For example, it is known that deriving products of a given handedness in chemical reactions can be crucial - because molecules of a given handedness must be used in order to design drugs without negative side-effects [40] - and chiral light has been suggested to serve as a useful tool in order to achieve this [34, 17, 4]. Applications of chiral light to the detection and characterization of chiral biomolecules have been also discussed [10]. As for the other nine zilches, recently, Smith and Strange shed light on the mystery of their physical meaning for certain topologically non-trivial vacuum EM fields [35].

Although the *zilch symmetries* - i.e. the symmetries underlying the zilch conservation laws - and their generalization have been discussed in previous works [12, 6, 1, 14, 43, 29, 37, 5, 2, 15], there are still certain gaps concerning our mathematical understanding of them. Most importantly, there is a gap in the literature concerning the explicit derivation of all zilch conservation laws from symmetries of the standard free EM action using Noether's theorem. In this Letter, we fill this gap and we also provide new insight concerning the zilch symmetries. Before proceeding to the main part of this article, let us discuss what is already known concerning the zilch symmetries in Subsection 6.1.1, as well as review the main findings of the present article in Subsection 6.1.2. For later convenience, we present here our notation and conventions.

Conventions.—Greek tensor indices run from 0 to 3 and Latin tensor indices from 1 to 3. We follow the Einstein summation convention, while indices are raised and lowered with the mostly plus Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. A spacetime point in standard Minkowski coordinates is $x^\mu = (x^0, x^1, x^2, x^3) \equiv (t, x^i)$. The totally antisymmetric tensors in 4 and 3 dimensions are $\epsilon^{\mu\nu\rho\sigma}$ and ϵ^{ijk}, respectively ($\epsilon^{0123} = -\epsilon^{123} = -1$).

Let $A_\mu = (-\phi, \boldsymbol{A})$ denote the EM four-potential. The standard free EM action

$$S = \frac{1}{2}\int d^4x\, (\boldsymbol{E}\cdot\boldsymbol{E} - \boldsymbol{B}\cdot\boldsymbol{B})\,,$$
$$\text{with}\quad \boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \boldsymbol{\nabla}\phi, \quad \boldsymbol{B} = \nabla\times\boldsymbol{A}, \tag{6.5}$$

is expressed as

$$S = -\frac{1}{4}\int d^4x\, F^{\mu\nu}F_{\mu\nu}, \tag{6.6}$$

where the antisymmetric EM tensor is defined as $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ (with $F_{0i} = -E_i$ and $F_{ik} = \epsilon_{ikm}B^m$). We denote the dual EM tensor as ${}^\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. The free Maxwell's equations $\partial^\nu F_{\nu\mu} = 0$ are expressed in potential form as

$$\Box A_\mu - \partial_\mu\partial^\nu A_\nu = 0, \tag{6.7}$$

where $\Box = \partial^\nu\partial_\nu$. Because of the definition of $F_{\mu\nu}$ in terms of the four-potential, the equation

$$\partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} = 0 \tag{6.8}$$

is identically satisfied. Equation (6.7), as well as the action (6.6), are invariant under infinitesimal gauge-transformations

$$\delta^{gauge}A_\mu = \partial_\mu a, \tag{6.9}$$

where a is an arbitrary scalar function.

6.1.1 What is known about the zilch symmetries?

The zilch conservation laws can be conveniently described in terms of the zilch tensor [16, 13]

$$Z^\mu{}_{\nu\rho} = -\,{}^\star F^{\mu\lambda}\partial_\rho F_{\lambda\nu} + F^{\mu\lambda}\,\partial_\rho\,{}^\star F_{\lambda\nu}. \tag{6.10}$$

This is conserved on-shell, $\partial^\rho Z^\mu{}_{\nu\rho} = 0$, and the ten time-independent quantities [16]:

$$\mathscr{Z}^{\mu\nu} = \mathscr{Z}^{\nu\mu} = \int d^3x Z^{\mu\nu 0}$$

are the ten zilches (see Section 6.2 for background material concerning the zilches). The optical chirality density (6.1) is related to the zilch tensor as $Z^{000} = 2C$.

At the level of free Maxwell's equations expressed in terms of the EM tensor, the zilch symmetries are known [14, 2]. More specifically, the *zilch symmetry transformations* of the EM tensor are [14, 2]

$$\Delta F_{\mu\nu} = \tilde{n}^{\alpha} n^{\rho}\, \partial_{\alpha}\partial_{\rho}{}^{\star}F_{\mu\nu}, \tag{6.11}$$

where $\tilde{n}^{\alpha}$ and n^{ρ} are two arbitrary constant four-vectors. These transformations are symmetries of free Maxwell's equations, i.e. if $F_{\mu\nu}$ is a solution, then so is $\Delta F_{\mu\nu}$. In Ref. [2], a complete classification of all independent local conservation laws of Maxwell's equations was given by using the methods described in Refs. [21, 3]. Using these methods, it was shown that the zilch symmetries (6.11) of free Maxwell's equations give rise to the conservation of the zilch tensor (6.10). However, in Ref. [2] the invariance of the standard EM action (6.6) was not discussed.

The zilch symmetries have also been studied in the case of duality-symmetric electromagnetism [1]. The duality-symmetric EM action is [6]

$$\tilde{S} = -\frac{1}{8}\int d^4x\,(F^{\mu\nu}F_{\mu\nu} + G^{\mu\nu}G_{\mu\nu})\,. \tag{6.12}$$

This theory is an extension of the standard EM theory as it has two four-potentials, A_μ and C_μ, and two EM tensors $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$. The duality-symmetric theory coincides with the standard EM theory only after we impose the duality constraint $G_{\mu\nu} = {}^{*}F_{\mu\nu}$. In Ref. [1], following the reverse Noether procedure, it was shown that the 'generalized' version of the zilch tensor:

$$\begin{aligned}\mathscr{Z}^{\mu}{}_{\nu\rho} = &-\frac{1}{2}\,G^{\mu\lambda}\partial_{\rho}F_{\lambda\nu} + \frac{1}{2}F^{\mu\lambda}\,\partial_{\rho}G_{\lambda\nu}\\ &-\frac{1}{2}G_{\nu}{}^{\lambda}\,\partial_{\rho}F_{\lambda}{}^{\mu} + \frac{1}{2}F_{\nu}{}^{\lambda}\,\partial_{\rho}G_{\lambda}^{\ \mu}\end{aligned} \tag{6.13}$$

is the Noether current corresponding to the following zilch symmetry transformations [1]:

$$\begin{aligned}\tilde{\Delta}A_{\nu} &= n^{\rho}\tilde{n}^{\mu}\,\partial_{\rho}G_{\mu\nu}\\ \tilde{\Delta}C_{\nu} &= -n^{\rho}\tilde{n}^{\mu}\,\partial_{\rho}F_{\mu\nu}.\end{aligned} \tag{6.14}$$

It has been shown that these transformations leave invariant the duality-symmetric action (6.12) [1]. Then, the conservation of the zilches follows from the fact that the tensor (6.13) coincides with the zilch tensor (6.20) of the standard EM theory if we apply the duality constraint.

The derivation of the zilch conservation laws from symmetries of alternative actions has been studied in Refs. [37, 14].

6.1.2 Filling a gap in the literature, main results and outline

In order to derive all zilch conservation laws using Noether's theorem in the case of standard electromagnetism, one needs to find the zilch symmetry transformations of the four-potential that leave the standard action (6.6) invariant. It is easy to observe that the zilch symmetry transformations $\Delta F_{\mu\nu}$ [Eq. (6.11)] of free Maxwell's equations are induced by the following zilch transformations of the four-potential:

$$\Delta A_\nu = \ n^\rho \tilde{n}^\mu \, \epsilon_{\mu\nu\sigma\lambda} \, \partial^\sigma \partial_\rho A^\lambda = n^\rho \tilde{n}^\mu \ \partial_\rho {}^*F_{\mu\nu}, \tag{6.15}$$

with $\Delta F_{\mu\nu} \equiv \partial_\mu \Delta A_\nu - \partial_\nu \Delta A_\mu$ for on-shell field configurations. (The transformations (6.15) coincide with $\tilde{\Delta} A_\nu$ in Eq. (6.14) if we apply the duality constraint.) Interestingly, the variation of the standard action (6.6) under the zilch transformations (6.15) has not been studied in the literature. This means that the following question is still open:

How can we derive all zilch conservation laws from symmetries of the standard free EM action using Noether's theorem?

In this Letter, we give the full answer to this question by showing that the zilch transformations (6.15) leave the standard EM action (6.6) invariant, and, then, we derive all zilch conservation laws using the standard Noether procedure (see, e.g. Ref. [41]). Note that the only zilch conservation law that has hitherto been derived from symmetries of the standard action (6.5) is the one concerning the conservation of optical chirality [22]. In particular, Philbin showed that optical chirality is the Noether charge corresponding to the following symmetry transformations [22]:

$$\Delta\phi = 0, \quad \Delta \boldsymbol{A} = \boldsymbol{\nabla} \times \frac{\partial \boldsymbol{A}}{\partial t}. \tag{6.16}$$

This equation corresponds to a special case of the zilch symmetry transformation (6.15) with $\tilde{n}^\mu = n^\mu = \delta^\mu_0$. In this article we provide an alternative (and covariant) derivation of Philbin's [22] result for optical chirality.

Outline and main results. The basics concerning the zilch tensor and the zilches are reviewed in Section 6.2. The derivation of all zilch conservation laws using the invariance of the standard action (6.6) under the zilch symmetries (6.15) is presented in Section 6.3. Then, we proceed by providing new insight concerning the conservation of the zilches and their underlying symmetries. More specifically, the rest of the investigations and findings of this article are summarized as follows:

- **A hidden invariance algebra of free Maxwell's equations and the zilch symmetries (Subsection 6.4.1).**—We show that the zilch symmetry transformations (6.15) of the four-potential belong to the enveloping algebra of a "hidden" invariance algebra of free Maxwell's equations in potential form. This "hidden" algebra closes on the 30-dimensional real Lie algebra $so(6,\mathbb{C})_{\mathbb{R}}$ - i.e. the 'realification' of the complex Lie algebra $so(6,\mathbb{C})$ - up to gauge transformations of the four-potential. (The $so(6,\mathbb{C})_{\mathbb{R}}$ invariance of free Maxwell's equations in terms of the electric and magnetic fields was uncovered in Ref. [23], but the potential form of Maxwell's equations was not discussed.) The 30 generators of the "hidden" algebra correspond to the 15 well-known infinitesimal conformal transformations [Eq. (6.40)] and to 15 little-known ("hidden") infinitesimal transformations [Eq. (6.41)]. The "hidden" transformations (6.41) take a simpler form when acting on the EM tensor; that is a product of a duality transformation with an infinitesimal conformal transformation [14, 43] (see Eq. (6.42)) [2].
- **Hidden symmetries in the presence of matter (Subsection 6.4.2) and in the theory of a complex gauge field (Subsection 6.4.3).**—We show that the "hidden" symmetries [Eq. (6.41)] of free Maxwell's equations persist in the presence of a material four-current [see Eq. (6.50)]. However, unlike the free case, the invariance algebra does not close on $so(6,\mathbb{C})_{\mathbb{R}}$. Then, we observe that the "hidden" symmetries of the real potential A_μ also exist for the free field equations (6.56) of a complex Abelian gauge field $\mathscr{A}_\mu$ - this is related to the complex formulation of duality-symmetric electromagnetism with the complex potential given by $\mathscr{A}_\mu = A_\mu + iC_\mu$ [1]. We show that if we redefine the "hidden" transformations of $\mathscr{A}_\mu$ by multiplying with $i=\sqrt{-1}$, the 30-dimensional algebra becomes $so(4,2)\bigoplus so(4,2)$ (it closes again up to gauge transformations of the complex potential).
- **Zilch continuity equations from symmetries in the presence of matter (Section 6.5) and a new question (Section 6.6).**—We also study the derivation of zilch continuity equations in the presence of electric charges and currents by extending the zilch symmetries of the standard free action (6.6) to zilch symmetries

[2] The fact that the product of a duality transformation with an infinitesimal conformal transformation is a symmetry of Maxwell's equations expressed in terms of the EM tensor was first observed by Krivskii and Simulik [14, 43].

[Eqs. (6.60) and (6.61)] of the standard interacting action (6.59) (in which A_μ couples to a non-dynamical material four-current J^μ). Taking advantage of the invariance of the interacting action under the zilch symmetries, we present a new way to derive the known continuity equation for optical chirality [9]

$$\frac{\partial}{\partial t}C + \boldsymbol{\nabla}\cdot\boldsymbol{S} = \frac{1}{2}\left(\boldsymbol{j}\cdot\frac{\partial\boldsymbol{B}}{\partial t} - \frac{\partial\boldsymbol{j}}{\partial t}\cdot\boldsymbol{B}\right) \tag{6.17}$$

($\boldsymbol{j}$ is the material electric current density). In Ref. [9], the continuity equation (6.17) was obtained from the complementary fields formalism, while a similar continuity equation had been first obtained in Ref. [39]. Apart from Eq. (6.17), in this Letter, we also obtain new continuity equations [Eqs. (6.65) and (6.68)] for the rest of the zilches in the presence of electric charges and currents from symmetries of the interacting EM action (6.59). Then, we pose the interesting open question of whether the aforementioned invariance of the interacting EM action with a non-dynamical material four-current can be extended to the case where the material four-current is dynamical.

6.2 BACKGROUND MATERIAL CONCERNING THE ZILCH TENSOR AND THE ZILCHES

In this Section we review the basics concerning the zilch tensor and the zilches.
The zilch tensor (6.10) can be expressed in various forms [16, 13]. For example, using the following identity [13]:

$$\partial_\rho\left({}^\star F_{\lambda\nu}F^{\mu\lambda}\right) = -\frac{1}{4}\delta^\mu_\nu\,\partial_\rho\left({}^\star F^{\lambda\kappa}F_{\lambda\kappa}\right), \tag{6.18}$$

the zilch tensor (6.10) can be equivalently expressed as

$$Z^\mu{}_{\nu\rho} = \; - \; {}^\star F^{\mu\lambda}\partial_\rho F_{\lambda\nu} - {}^\star F_\nu{}^\lambda\partial_\rho F_\lambda{}^\mu - \frac{1}{2}\delta^\mu_\nu\,{}^\star F^{\lambda\kappa}\partial_\rho F_{\lambda\kappa}. \tag{6.19}$$

This expression makes manifest that the properties $Z^{\mu\nu}{}_\rho = Z^{\nu\mu}{}_\rho$ and $Z^\mu{}_{\mu\rho} = 0$ are identically satisfied. Moreover, by using free Maxwell's equations, it is straightforward to show that the zilch tensor is divergence-free with respect to all of its indices and also satisfies $Z^\rho{}_{\nu\rho} = 0$ [13]. Using the fact that the zilch tensor is symmetric in its first two indices we can rewrite Eq. (6.10) as

$$Z^{\mu}{}_{\nu\rho} = -\frac{1}{2}{}^{\star}F^{\mu\lambda}\partial_{\rho}F_{\lambda\nu} + \frac{1}{2}F^{\mu\lambda}\,\partial_{\rho}{}^{\star}F_{\lambda\nu} - \frac{1}{2}{}^{\star}F_{\nu}{}^{\lambda}\partial_{\rho}F_{\lambda}{}^{\mu} + \frac{1}{2}F_{\nu}{}^{\lambda}\,\partial_{\rho}{}^{\star}F_{\lambda}{}^{\mu}. \tag{6.20}$$

As mentioned in the Introduction, the ten zilches are given by the following ten time-independent quantities [16, 13]:

$$\mathscr{Z}^{\mu\nu} = \mathscr{Z}^{\nu\mu} = \int d^3x Z^{\mu\nu 0}, \tag{6.21}$$

with $\partial\mathscr{Z}^{\mu\nu}/\partial t = 0$. Only nine zilches in Eq. (6.21) are independent since $Z^{\mu}{}_{\mu 0} = 0$. The $\mu\nu 0$-component ($Z^{\mu\nu 0}$) of the zilch tensor is the spatial density of the zilch $\mathscr{Z}^{\mu\nu}$, and the $\mu\nu j$-components ($Z^{\mu\nu j}$) are the components of the three-vector describing the corresponding flux [16]. The time-independence of the ten zilches follows from the ten differential conservation laws described by $\partial_{\rho}Z^{\mu\nu\rho} = 0$. The conservation law (6.3) for optical chirality corresponds to $\frac{1}{2}\left(\partial_0 Z^{000} + \partial_j Z^{00j}\right) = 0$.
For later convenience, note that the integral in Eq. (6.21) has the symmetry property

$$\int d^3x Z^{\mu\nu 0} = \int d^3x Z^{\mu 0\nu} \left(= \int d^3x Z^{0\mu\nu}\right) \tag{6.22}$$

because the difference $Z^{\mu\nu 0} - Z^{\mu 0\nu}$ can always be expressed as a spatial divergence [13]

$$Z^{\mu\nu 0} - Z^{\mu 0\nu} = \partial_j \Lambda^{\mu\nu j},$$

where the explicit expression for the tensor Λ is not needed for the present discussion [3]. It immediately follows that the difference $Z^{\mu 0\nu} - Z^{\nu 0\mu}$ can also be written as a spatial divergence. Hence, the $\mu\nu$-zilch, $\mathscr{Z}^{\mu\nu}$, can be actually interpreted as the time-independent quantity that corresponds to any of the three differential conservation laws: $\partial_{\rho}Z^{\mu\nu\rho} = 0$ (which is the one used by Lipkin [16]), $\partial_{\rho}Z^{\mu\rho\nu} = 0$ and $\partial_{\rho}Z^{\nu\rho\mu} = 0$. These differential conservation laws are not independent of each other. For example, the conservation law $\partial_{\rho}Z^{\mu\nu\rho} = 0$ can be re-written as $\partial_{\rho}Z^{\mu\rho\nu} = 0$ by using the relations

$$\partial_0 Z^{\mu\nu 0} = \partial_0\left(Z^{\mu 0\nu} + \partial_j \Lambda^{\mu\nu j}\right) \tag{6.23}$$

and

$$\partial_j Z^{\mu\nu j} = \partial_j\left(Z^{\mu j\nu} - \partial_0 \Lambda^{\mu\nu j}\right) \tag{6.24}$$

for the corresponding spatial densities and fluxes, respectively.

[3] The interested reader can find the expression for Λ from equation (14) of Ref. [13] or they can set $J^{\mu} = 0$ and let $\rho = 0$ in Eq. (6.66) of the present article.

6.3 CONSERVATION LAWS FOR ALL ZILCHES FROM THE INVARIANCE OF THE STANDARD ACTION UNDER THE ZILCH TRANSFORMATIONS

In this Section we show that the zilch symmetry transformation (6.15), which is given here again for convenience:

$$\Delta A_\nu = \; n^\rho \tilde{n}^\mu \, \epsilon_{\mu\nu\sigma\lambda} \, \partial^\sigma \partial_\rho A^\lambda = n^\rho \tilde{n}^\mu \; \partial_\rho {}^* F_{\mu\nu}, \tag{6.25}$$

is a symmetry of the standard free EM action (6.6). Then, we derive all zilch conservation laws using Noether's theorem.

Let us start by examining the way in which the zilch symmetry transformation (6.25) acts on the EM tensor for off-shell field configurations; that is

$$\begin{aligned} \Delta F_{\mu\nu} \equiv & \; \partial_\mu \Delta A_\nu - \partial_\nu \Delta A_\mu \\ = & \; \tilde{n}^\alpha n^\rho \left(\partial_\alpha \partial_\rho {}^\star F_{\mu\nu} - \epsilon_{\alpha\mu\nu\sigma} \partial_\rho \partial_\lambda F^{\lambda\sigma} \right), \end{aligned} \tag{6.26}$$

where we have made use of the following important off-shell identity [4]:

$$\partial_\alpha {}^\star F_{\mu\nu} + \partial_\nu {}^\star F_{\alpha\mu} + \partial_\mu {}^\star F_{\nu\alpha} = \epsilon_{\alpha\mu\nu\sigma} \partial^\beta F_\beta{}^\sigma. \tag{6.27}$$

We now proceed to demonstrate that the zilch symmetry transformation (6.25) is indeed a symmetry of the action (6.6) and then apply Noether's theorem. We find that the variation

$$\Delta S = -\frac{1}{2} \int d^4 x \, F^{\mu\nu} \, \Delta F_{\mu\nu} \tag{6.28}$$

is given by a total divergence (without making use of the equations of motion), as

$$\Delta S = \int d^4 x \, \partial_\nu D^\nu \tag{6.29}$$

with

$$D^\nu = \frac{1}{2} n^\rho \tilde{n}^\mu \left(2 \, {}^\star F^{\lambda\nu} \partial_\rho F_{\mu\lambda} + Z_\mu{}^\nu{}_\rho + \delta^\nu_\rho \, {}^\star F_{\mu\sigma} \, \partial^\beta F_\beta{}^\sigma \right) \tag{6.30}$$

- see Appendix 6.8 for some details of the calculation. Now, the usual procedure [41] can be followed in order to construct the conserved Noether current, V^ν, associated with the zilch symmetry transformation (6.25), as

$$V^\nu = \frac{\partial \mathscr{L}}{\partial(\partial_\nu A_\mu)} \Delta A_\mu - D^\nu, \tag{6.31}$$

[4] Equation (6.27) can be readily proved by contracting with $\epsilon_{\gamma\delta}{}^{\mu\nu}$ and then using well-known properties of the totally antisymmetric tensor.

where $\mathscr{L} = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}$ is the free EM Lagrangian density. Substituting the expression for D^ν [Eq. (6.30)] into Eq. (6.31) and making use of the identity (6.18), we find

$$V^\nu = \frac{1}{2}n^\rho\tilde{n}^\mu\left(Z_\mu{}^\nu{}_\rho - \delta^\nu_\rho\,{}^\star F_{\mu\sigma}\,\partial^\beta F_\beta{}^\sigma\right). \tag{6.32}$$

The definition of a conserved Noether current is not unique; we are free to add any term that vanishes on-shell and/or any term that is equal to the divergence of any rank-2 antisymmetric tensor to the expression for the Noether current [30]. Thus, we are allowed to express the Noether current in Eq. (6.32) as

$$V^\nu_{zilch} = \frac{1}{2}n^\rho\tilde{n}^\mu Z_\mu{}^\nu{}_\rho \tag{6.33}$$

with $\partial_\nu V^\nu_{zilch} = 0$. Since the constant four-vectors n^ρ and $\tilde{n}^\mu$ in Eq. (6.33) are arbitrary, we conclude that

$$\partial_\nu Z^{\mu\nu\rho} = 0. \tag{6.34}$$

In other words, the zilch tensor is the conserved Noether current corresponding to the zilch symmetries (6.25) of the standard free action (6.6), while the corresponding Noether charges are the zilches (6.21).

6.4 “HIDDEN” SYMMETRIES

6.4.1 “Hidden” invariance algebra of free Maxwell’s equations and the zilch symmetries

Here we investigate the relation of the zilch symmetry transformations (6.25) to a “hidden” $so(6,\mathbb{C})_\mathbb{R}$ invariance algebra of free Maxwell’s equations in potential form (6.7).
Let ξ^μ denote any of the fifteen conformal Killing vectors of Minkowski spacetime satisfying

$$\partial_\mu\xi_\nu + \partial_\nu\xi_\mu = \frac{\partial^\alpha\xi_\alpha}{2}\eta_{\mu\nu}. \tag{6.35}$$

The conformal Killing vectors ξ^μ of Minkowski spacetime consist of [8]: the four generators of spacetime translations,

$$P_{(\alpha)} = P^\mu_{(\alpha)}\partial_\mu = \partial_\alpha, \tag{6.36}$$

the six generators of the Lorentz algebra $so(3,1)$,

$$M_{(\beta,\gamma)} = M^{\mu}_{(\beta,\gamma)}\partial_\mu = x_\beta\partial_\gamma - x_\gamma\partial_\beta, \tag{6.37}$$

the generator of dilations

$$D = D^\mu\partial_\mu = x^\mu\partial_\mu, \tag{6.38}$$

and the four generators of special conformal transformations

$$K_{(\alpha)} = K^{\mu}_{(\alpha)}\partial_\mu = x^\nu x_\nu\partial_\alpha - 2x_\alpha x^\mu\partial_\mu. \tag{6.39}$$

These fifteen vectors form a basis for the algebra of infinitesimal conformal transformations of Minkowski spacetime which is isomorphic to $so(4,2)$.
The "hidden" invariance algebra of free Maxwell's equations (6.7) is generated by two types of infinitesimal symmetry transformations of the four-potential. The first type corresponds to the well-known infinitesimal conformal transformations, conveniently described by the Lie derivative

$$L_\xi A_\mu = \xi^\lambda\partial_\lambda A_\mu + A_\lambda\partial_\mu\xi^\lambda, \quad \xi\in so(4,2). \tag{6.40}$$

These transformations generate a representation of $so(4,2)$ on the solution space of Maxwell's equations (6.7). The second type of transformations corresponds to the little-known ("hidden") transformations [24]

$$T_\xi A_\mu = \xi^\rho\epsilon_{\rho\mu\sigma\lambda}\partial^\sigma A^\lambda, \quad \xi\in so(4,2). \tag{6.41}$$

If A_μ is a solution of Maxwell's equations, then so are $L_\xi A_\mu$ and $T_\xi A_\mu$ for all $\xi\in so(4,2)$ [24]. The effect of the "hidden" transformation (6.41) on $F_{\mu\nu}$ corresponds to the product of a duality transformation with an infinitesimal conformal transformation as

$$T_\xi F_{\mu\nu} \equiv \partial_\mu T_\xi A_\nu - \partial_\nu T_\xi A_\mu = L_\xi{}^\star F_{\mu\nu}, \tag{6.42}$$

where

$$L_\xi{}^\star F_{\mu\nu} = \xi^\rho\partial_\rho{}^\star F_{\mu\nu} + {}^\star F_{\rho\nu}\,\partial_\mu\xi^\rho + {}^\star F_{\mu\rho}\,\partial_\nu\xi^\rho. \tag{6.43}$$

This symmetry transformation of the EM tensor was first found in Refs. [14, 43].
The structure of the "hidden" invariance algebra of Maxwell's equations in potential form is determined by the Lie brackets:

$$[L_{\xi'}, L_\xi]A_\mu = L_{[\xi',\xi]}A_\mu \tag{6.44}$$

$$[L_{\xi'}, T_\xi]A_\mu = \; T_{[\xi',\xi]}A_\mu \tag{6.45}$$

and

$$[T_{\xi'}, T_\xi]A_\mu = -\, L_{[\xi',\xi]}A_\mu + \partial_\mu \left([\xi',\xi]^\sigma A_\sigma - \xi'_\sigma \xi_\rho F^{\sigma\rho}\right), \tag{6.46}$$

where, e.g., $[L_{\xi'}, L_\xi] = L_{\xi'}L_\xi - L_\xi L_{\xi'}$, while ξ and ξ' are any two basis elements of $so(4,2)$ with $[\xi',\xi]^\rho = L_{\xi'}\xi^\rho$. We observe the appearance of a gauge transformation of the form (6.9) in the last term of Eq. (6.46). To the best of our knowledge, the explicit expressions for the commutators (6.45) and (6.46) appear here for the first time. The commutation relations in Eqs. (6.44)-(6.46) coincide with the commutation relations of the 30-dimensional real Lie algebra $so(6,\mathbb{C})_\mathbb{R}$ [23] (up to the gauge transformation in Eq. (6.46)).

Now, let us denote the zilch symmetry transformation (6.25) with associated Noether current corresponding to $Z_\alpha{}^\nu{}_\beta$ (α and β have fixed values) as $\Delta_{(\beta,\alpha)}A_\mu$. The latter is readily expressed as [see Eq. (6.25)]

$$\Delta_{(\beta,\alpha)}A_\mu = \partial_\beta \left(\epsilon_{\alpha\mu\sigma\lambda}\,\partial^\sigma A^\lambda\right) = L_{P_{(\beta)}}T_{P_{(\alpha)}}A_\mu. \tag{6.47}$$

It is obvious from this expression that $\Delta_{(\beta,\alpha)}A_\mu$ is given by the product of a “hidden” transformation (6.41) with respect to the translation Killing vector $P_{(\alpha)} = \partial_\alpha$ and a Lie derivative (6.40) with respect to the translation Killing vector $P_{(\beta)} = \partial_\beta$. This makes clear that the zilch symmetry transformation $\Delta_{(\beta,\alpha)}A_\mu$ belongs to the enveloping algebra of our “hidden” invariance algebra [and so do all transformations of the form (6.25)].

6.4.2 “Hidden” symmetries of Maxwell’s equations in the presence of a material four-current

In the presence of a material four-current Maxwell’s equations are

$$\Box A_\mu - \partial_\mu \partial^\nu A_\nu = -J_\mu, \tag{6.48}$$

where $J^\mu = (\rho, \boldsymbol{j})$ and $\partial_\mu J^\mu = 0$. Maxwell’s equations remain invariant under simultaneous infinitesimal conformal transformations of A_μ and J_μ, i.e. Eq. (6.48) will still be satisfied if we make the following replacements:

$$\begin{aligned} A_\mu &\to L_\xi A_\mu, \\ J_\mu &\to L_\xi J_\mu + \frac{\partial_\alpha \xi^\alpha}{2} J_\mu, \quad \xi \in so(4,2), \end{aligned} \tag{6.49}$$

where L_ξ is the Lie derivative (6.40). It is interesting to investigate whether the "hidden" symmetries (6.41) of free Maxwell's equations also persist in the presence of matter. Indeed, we find that if A_μ and J_μ satisfy Eq. (6.48), then Eq. (6.48) will still be satisfied if we make the following replacements:

$$\begin{aligned} &A_\mu \to T_\xi A_\mu, \\ &J_\mu \to \delta_\xi^{hid} J_\mu = \epsilon_{\rho\mu\sigma\lambda}\partial^\sigma \left(\xi^\rho J^\lambda\right), \quad \xi \in so(4,2), \end{aligned} \tag{6.50}$$

where $T_\xi A_\mu$ is given by Eq. (6.41), while we call $\delta_\xi^{hid} J_\mu$ in the second line the "hidden" transformation of the four-current. Equation (6.50) describes the "hidden" symmetries of Maxwell's equations in the presence of a material four-current.

Unlike the free case, in the presence of matter, the algebra does not close on $so(6,\mathbb{C})_\mathbb{R}$ up to gauge transformations of the four-potential [5]. This can be readily understood from the following example. By calculating the commutator between "hidden" symmetries generated by translation Killing vectors (6.36), we find:

$$\begin{aligned} &[T_{P_{(\alpha)}}, T_{P_{(\beta)}}]A_\mu \\ &= \partial_\mu \left(-P^\sigma_{(\alpha)} P^\rho_{(\beta)} F_{\sigma\rho}\right) + \left(P_{(\alpha)\mu} P^\rho_{(\beta)} - P^\rho_{(\alpha)} P_{(\beta)\mu}\right) \partial^\sigma F_{\sigma\rho} \end{aligned} \tag{6.51}$$

(compare this equation with Eq. (6.46)). In the absence of matter, the second term in Eq. (6.51) vanishes and the algebra closes up to the gauge transformation $\partial_\mu \left(-P^\sigma_{(\alpha)} P^\rho_{(\beta)} F_{\sigma\rho}\right)$ - see Eq. (6.46). However, in the presence of matter, the second term in Eq. (6.51) seems to describe a new symmetry transformation. Similarly, we find the commutator for the four-current:

$$\begin{aligned} [\delta^{hid}_{P_{(\alpha)}}, \delta^{hid}_{P_{(\beta)}}]J_\mu = &\partial_\mu \left(-P^\sigma_{(\alpha)} P^\rho_{(\beta)} (\partial_\sigma J_\rho - \partial_\rho J_\sigma)\right) \\ &+ \left(P_{(\alpha)\mu} P^\rho_{(\beta)} - P^\rho_{(\alpha)} P_{(\beta)\mu}\right) \Box J_\rho. \end{aligned} \tag{6.52}$$

From Eqs. (6.51) and (6.52), it follows that Maxwell's equations (6.48) will still be satisfied (this is easy to verify) if we simultaneously make the replacements:

$$A_\mu \to \left(P_{(\alpha)\mu} P^\rho_{(\beta)} - P^\rho_{(\alpha)} P_{(\beta)\mu}\right) \partial^\sigma F_{\sigma\rho} \tag{6.53}$$

and

$$\begin{aligned} J_\mu \to &\partial_\mu \left(-P^\sigma_{(\alpha)} P^\rho_{(\beta)} (\partial_\sigma J_\rho - \partial_\rho J_\sigma)\right) \\ &+ \left(P_{(\alpha)\mu} P^\rho_{(\beta)} - P^\rho_{(\alpha)} P_{(\beta)\mu}\right) \Box J_\rho. \end{aligned} \tag{6.54}$$

[5] If one considers only the familiar conformal transformations (6.49), then the algebra closes on $so(4,2)$.

The study of the full structure of the algebra in the presence of matter is something that we leave for future work.

6.4.3 “Hidden” symmetries for the complex Abelian gauge field

The free (hermitian) action for the complex Abelian gauge field, $\mathscr{A}_\mu$, is

$$-\frac{1}{8}\int d^4x \mathscr{F}^\dagger_{\mu\nu}\mathscr{F}^{\mu\nu}, \tag{6.55}$$

where † denotes complex conjugation, while $\mathscr{F}_{\mu\nu} = \partial_\mu\mathscr{A}_\nu - \partial_\nu\mathscr{A}_\mu$. Expressing $\mathscr{A}_\mu$ as $\mathscr{A}_\mu = A_\mu + iC_\mu$, the action (6.55) takes the form of the duality-symmetric action (6.12). The field equation for the complex potential is

$$\Box\mathscr{A}_\mu - \partial_\mu\partial^\nu\mathscr{A}_\nu = 0. \tag{6.56}$$

This equation is invariant under the infinitesimal conformal transformations $L_\xi\mathscr{A}_\mu$ in Eq. (6.40) (with A_μ replaced by $\mathscr{A}_\mu$), as well as under the “hidden” transformations $T_\xi\mathscr{A}_\mu$ in Eq. (6.41) (with A_μ replaced by $\mathscr{A}_\mu$). As in the case of the real potential, the structure of the algebra generated by the conformal and the “hidden” transformations is determined by the commutators in Eqs. (6.44)-(6.46) with A_μ replaced by $\mathscr{A}_\mu$ and $F^{\sigma\rho}$ replaced by $\mathscr{F}^{\sigma\rho}$.
Now, we will show that if we redefine the “hidden” transformations $T_\xi\mathscr{A}_\mu$, the “hidden” algebra of Eq. (6.56) will be isomorphic to $so(4,2)\bigoplus so(4,2)$. Let us redefine $T_\xi\mathscr{A}_\mu$ by multiplying with i as

$$T'_\xi\mathscr{A}_\mu \equiv i\,T_\xi\mathscr{A}_\mu = \, i\,\xi^\rho\epsilon_{\rho\mu\sigma\lambda}\partial^\sigma\mathscr{A}^\lambda, \quad \xi\in so(4,2). \tag{6.57}$$

These transformations leave both the action (6.55) and Eq. (6.56) invariant (on the other hand, T_ξ is a symmetry of the field equation only). Now, the “hidden” algebra of the field equation (6.56) is generated by the conformal transformations $L_\xi\mathscr{A}_\mu$ and the redefined “hidden” transformations $T'_\xi\mathscr{A}_\mu$. If we now define the new generators:

$$\mathscr{T}^\pm_\xi\mathscr{A}_\mu \equiv \frac{1}{\sqrt{2}}\left(L_\xi \pm T'_\xi\right)\mathscr{A}_\mu, \quad \xi\in so(4,2), \tag{6.58}$$

it is easy to see that the $\mathscr{T}^+_\xi$'s generate a $so(4,2)$ algebra on their own, and so do the transformations $\mathscr{T}^-_\xi$, while $[\mathscr{T}^+_\xi, \mathscr{T}^-_{\xi'}] = 0$ for any $\xi,\xi'\in so(4,2)$ [these follow directly from Eqs. (6.44)-(6.46)]. Thus, the “hidden” algebra is now isomorphic to $so(4,2)\bigoplus so(4,2)$ and closes up to gauge transformations of the complex potential $\mathscr{A}_\mu$.

6.5 ZILCH CONTINUITY EQUATIONS IN THE PRESENCE OF ELECTRIC CHARGES AND CURRENTS FROM SYMMETRIES OF THE STANDARD INTERACTING ACTION

In the presence of a non-dynamical material four-current, $J^\mu = (\rho, \boldsymbol{j})$, the standard interacting EM action is

$$S' = S + \int d^4x\, J^\nu A_\nu = \int d^4x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + J^\nu A_\nu\right). \tag{6.59}$$

Let us consider the variation of S' under the following simultaneous transformations of A_ν and J^ν:

$$\Delta A_\nu = n^\rho \tilde{n}^\mu\, \epsilon_{\mu\nu\sigma\lambda}\, \partial^\sigma \partial_\rho A^\lambda, \tag{6.60}$$

$$\Delta J^\nu = n^\rho \tilde{n}^\mu\, \epsilon_\mu{}^\nu{}_{\sigma\lambda}\, \partial^\sigma \partial_\rho J^\lambda, \tag{6.61}$$

where n^ρ and $\tilde{n}^\mu$ are two arbitrary constant four-vectors, while Eq. (6.60) coincides with the zilch symmetry transformation (6.25). The variation of the free action, S, is already known to be a total divergence [see Eq. (6.29)]. Also, after a straightforward calculation, we find that the variation of the interaction term is a total divergence, as

$$\begin{aligned}\Delta\Big(\int d^4x\, J^\nu A_\nu\Big) &= \int d^4x\, \left(\Delta J^\nu A_\nu + J^\nu \Delta A_\nu\right) \\ &= \int d^4x\, \partial_\nu D^\nu_{int},\end{aligned} \tag{6.62}$$

where

$$D^\nu_{int} = \tilde{n}^\mu n^\rho (\delta^\nu_\rho\, J^\lambda\, {}^\star F_{\mu\lambda} - \partial_\rho J^\lambda A^\alpha \epsilon_{\mu\lambda}{}^\nu{}_\alpha). \tag{6.63}$$

Thus, the variation of the interacting action is

$$\Delta S' = \int d^4x\, \partial_\nu \left(D^\nu + D^\nu_{int}\right), \tag{6.64}$$

where D^ν is given by Eq. (6.30).

Now, by applying the standard Noether algorithm [41], we find the following continuity equations for the zilch tensor:

$$\partial_\lambda Z^{\mu\lambda\nu} = J_\lambda\, \partial^\nu\, {}^\star F^{\mu\lambda} - {}^\star F^{\mu\lambda}\, \partial^\nu J_\lambda. \tag{6.65}$$

These continuity equations determine the rate of gain or loss of the quantity $\int d^3x Z^{\mu 0\nu}$, with spatial density given by $Z^{\mu 0 \nu}$ and flux components given by $Z^{\mu j \nu}$.

The continuity equations (6.65) can be re-expressed in the form of continuity equations for the zilches (6.21), with spatial density given by $Z^{\mu\nu 0}$ and flux components given by $Z^{\mu\nu j}$, as follows. First, we observe that, although in the presence of electric charges and currents the quantity $\int d^3x Z^{\mu 0 \nu}$ and the $\mu\nu$-zilch, $\int d^3x Z^{\mu\nu 0}$ [Eq. (6.21)] are not equal to each other unless $\nu = 0$ (because the symmetry property (6.22) no longer holds), they are related to each other by [6]

$$Z^{\mu\nu\rho} - Z^{\mu\rho\nu} = \frac{1}{2}\Big(\epsilon^{\kappa\mu\nu\rho}\,\partial_\sigma T^\sigma_{\ \kappa} - \epsilon^{\kappa\lambda\nu\rho}\partial_\lambda T^\mu{}_\kappa - \epsilon^{\kappa\rho\lambda\mu}\partial_\lambda T^\nu{}_\kappa + \epsilon^{\kappa\nu\lambda\mu}\partial_\lambda T^\rho{}_\kappa$$
$$- 2F^\mu{}_\lambda\,\epsilon^{\rho\lambda\nu\sigma} J_\sigma + 2J^\mu\,{}^\star F^{\nu\rho}\Big), \tag{6.66}$$

where

$$T^\alpha{}_\beta = -F^{\alpha\lambda}F_{\lambda\beta} - \frac{1}{4}\delta^\alpha_\beta F^{\kappa\lambda}F_{\kappa\lambda} \tag{6.67}$$

is the Maxwell stress-energy tensor (with $\partial_\alpha T^\alpha{}_\beta = J^\alpha F_{\alpha\beta}$). Then, by taking the divergence of Eq. (6.66) with respect to the index ρ and using the continuity equation (6.65) we find

$$\partial_\rho Z^{\mu\nu\rho} = \eta^{\mu\nu}\,{}^\star F_{\lambda\sigma}\,\partial^\lambda J^\sigma - {}^\star F^{\mu\sigma}\,(\partial^\nu J_\sigma - \partial_\sigma J^\nu)$$
$$- {}^\star F^{\nu\sigma}\,(\partial^\mu J_\sigma - \partial_\sigma J^\mu). \tag{6.68}$$

These are the ten continuity equations determining the rate of gain (or loss) for the ten zilches (6.21) in the presence of electric charges and currents. For $\mu = \nu = 0$, both continuity equations (6.65) and (6.68) coincide with the known equation (6.17) for optical chirality. To the best of our knowledge, the other nine continuity equations in Eq. (6.68) are presented here for the first time.

6.6 AN INTERESTING OPEN QUESTION

Let us suppose that the EM field interacts with a dynamical matter field with corresponding four-current $\tilde{J}^\mu$. Now, the action of the full interacting theory is

$$\int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tilde{J}^\nu A_\nu\right) + S_{\text{matter}}, \tag{6.69}$$

[6] Equation (6.66) is obtained following analogous steps as the ones described by Kibble in order to derive equation (14) (for the free EM field) in Ref. [13]. Our Eq. (6.66) coincides with equation (14) of Ref. [13] if no electric charges and currents are present.

where S_{matter} is the action corresponding to the free matter field. According to our earlier discussion, the simultaneous transformations (6.60) and (6.61) (with J^ν replaced by $\tilde{J}^\nu$) are symmetries of the first two terms in Eq. (6.69). Motivated by this observation, we may pose the question of whether one could identify symmetries of the full interacting theory (i.e. symmetries of all three terms in Eq. (6.69)). In other words, is it possible to identify a transformation of the matter field such that: this transformation is a symmetry of S_{matter}, while the four-current $\tilde{J}^\mu$ transforms as in Eq. (6.61)?

6.7 DISCUSSION

The results of the present Letter establish a clear connection between all zilch continuity equations and symmetries of the standard EM action via Noether's theorem. Having identified all zilches with Noether charges, we can interpret them as the generators of the corresponding symmetry transformations (6.25) of the four-potential in the standard (classical or quantum) EM theory [41, 19, 22]. In the case of optical chirality, the explicit knowledge of the underlying symmetry generator is known to offer physical insight, since it allows the identification of the optical chirality eigenstates with plane waves of circular polarization [22]. Similarly, the symmetry transformations (6.25) can be used to identify the eigenstates of all zilches, which is something that we leave for future work.

A particularly interesting uninvestigated question is the one concerning the role of all zilches in light-matter interactions - the case of optical chirality is the only exception since its role has been studied [39]. The importance of this question becomes manifest by considering the fact that a physical interpretation for all zilches has been recently provided [35]. In particular, in Ref. [35] it was found that the zilches of a certain class of topologically non-trivial EM fields in vacuum can be expressed in terms of energy, momentum, angular momentum and helicity of the fields. Also, it was demonstrated that the zilches of these fields encode information about the topology of the field lines. We hope that the results presented in this Letter will be useful in future attempts to study the role of all zilches in light-matter interactions. More specifically, motivated by the interpretation and applications of the known continuity equation (6.17) for optical chirality [39, 9, 26, 7, 19], it is natural to interpret each of our new zilch continuity equations [Eq. (6.68)] as determining the rate of loss or gain of the corresponding "zilch quantity" of the EM field.

Acknowledgements. The author is grateful to Atsushi Higuchi for useful discussions, encouragement, and comments on earlier versions of this Letter. He also thanks the referee for useful suggestions and comments. Also, it is a pleasure to thank Charis Anastopoulos, Haralampos Geranios, Ed Corrigan, Dmitri Pushkin, Maximilian Ruep, David Serrano Blanco, Nikolaos Koutsonikos-Kouloumpis, Volodimir Simulik and F.F. John for useful discussions. The author acknowledges financial support from the Department of Mathematics, University of York, and from the WW Smith Fund.

6.8 APPENDIX A - INVARIANCE OF THE STANDARD FREE EM ACTION UNDER THE ZILCH SYMMETRIES

Here we present some details for the calculation concerning the invariance of the standard free EM action (6.6) under the zilch symmetry transformation (6.25). For convenience, we focus only on the invariance of the action and not on the derivation of the associated Noether current (6.32). Also, we drop all terms that are total divergences in order to simplify the presentation. However, note that one needs to keep all such terms if they wish to re-derive Eq. (6.29).

Varying the action (6.6) with respect to the zilch symmetry transformation (6.25) we find

$$\begin{aligned}-2\,\Delta S &= \int d^4x\, F^{\mu\nu}\,\Delta F_{\mu\nu} \\ &= \int d^4x\, \left(F^{\mu\nu}\tilde{n}^{\alpha}n^{\rho}\,\partial_\rho\partial_\alpha{}^{\star}F_{\mu\nu} - F^{\mu\nu}\tilde{n}^{\alpha}n^{\rho}\epsilon_{\alpha\mu\nu\sigma}\partial_\rho\partial_\lambda F^{\lambda\sigma}\right), \qquad (6.70)\end{aligned}$$

where in the second line we used Eq. (6.26). The first term is readily shown to be equal to a total divergence as follows:

$$\begin{aligned}\int d^4x\, F^{\mu\nu}\tilde{n}^{\alpha}n^{\rho}\,\partial_\rho\partial_\alpha{}^{\star}F_{\mu\nu} &= -\int d^4x\;\partial_\rho F^{\mu\nu}\tilde{n}^{\alpha}n^{\rho}\,\partial_\alpha{}^{\star}F_{\mu\nu} \\ &= 2\int d^4x\;\partial^{\nu}F_{\rho}{}^{\mu}\;\tilde{n}^{\alpha}n^{\rho}\,\partial_\alpha{}^{\star}F_{\mu\nu} \\ &= 2\int d^4x\;\partial^{\nu}\left(F_{\rho}{}^{\mu}\;\tilde{n}^{\alpha}n^{\rho}\,\partial_\alpha{}^{\star}F_{\mu\nu}\right),\end{aligned}$$

where in the second line we used Eq. (6.8), and in the third line we used that the divergence of ${}^{\star}F_{\mu\nu}$ vanishes identically because of Eq. (6.8). We now drop the first term in Eq. (6.70) (since we showed that it is a total divergence) and we express Eq. (6.70) as

$$-2\,\Delta S = -2\int d^4x\;{}^{\star}F_{\alpha\sigma}\;\tilde{n}^{\alpha}n^{\rho}\;\;\partial_\rho\partial_\lambda F^{\lambda\sigma}. \qquad (6.71)$$

On the other hand, keeping both terms in Eq. (6.70) and using the off-shell identity (6.27), Eq. (6.70) is re-written as

$$-2\,\Delta S = 2\int d^4x\ F^{\nu\sigma}\tilde{n}^{\alpha}n^{\rho}\ \partial_\rho\partial_\nu{}^\star F_{\alpha\sigma}. \tag{6.72}$$

Integrating by parts twice, we find

$$-2\,\Delta S = 2\int d^4x\ {}^\star F_{\alpha\sigma}\ \tilde{n}^{\alpha}n^{\rho}\ \ \partial_\rho\partial_\lambda F^{\lambda\sigma}. \tag{6.73}$$

Comparing this equation with Eq. (6.71), we find $\Delta S = 0$ (i.e. ΔS is equal to the integral of a total divergence), as required.

REFERENCES

[1] S. Aghapour, L. Andersson, and K. Rosquist. "The zilch electromagnetic conservation law revisited". In: *Journal of Mathematical Physics* 61 (2020), p. 122902.

[2] S. C. Anco and G. Bluman. "Direct construction of conservation laws from field equations". In: *Phys. Rev. Lett.* 78 (1997), p. 2869.

[3] S. C. Anco and G. Bluman. "Direct construction of conservation laws from field equations". In: *Phys. Rev. Lett.* 78 (1997), p. 2869.

[4] H. U. Blaser and H. J. Federsel. *Asymmetric Catalysis on Industrial Scale: Challenges, Approaches and Solutions*. Weinheim, Germany: Wiley-VCH Verlag GmbH & Co. KGaA, 2004.

[5] M. G. Calkin. "An invariance property of the free electromagnetic field". In: *American Journal of Physics* 33 (1965), p. 958.

[6] R. P. Cameron and S. M. Barnett. "Electric-magnetic symmetry and Noether's theorem". In: *New Journal of Physics* 14 (2012), p. 123019.

[7] F. Crimin, N. Mackinnon, J. B. Götte, and S. M. Barnett. "Optical Helicity and Chirality: Conservation and Sources". In: *Applied Sciences* 9 (2019).

[8] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory*. Graduate Texts in Contemporary Physics. New York: Springer-Verlag, 1997.

[9] M. Fernández Guasti. "Chirality, helicity and the rotational content of electromagnetic fields". In: *Physics Letters A* 383 (2019), p. 3180.

[10] E. Hendry, T. Carpy, J. Johnston, M. Popland, R. V. Mikhaylovskiy, A. J. Lapthorn, S. M. Kelly, L. D. Barron, N. Gadegaard, and M. Kadodwala. "Ultrasensitive detection and characterization of biomolecules using superchiral fields". In: *Nature Nanotech* 5 (2010), p. 783.

[11] E. Hendry, R. V. Mikhaylovskiy, L. D. Barron, M. Kadodwala, and T. J. Davis. "Chiral Electromagnetic Fields Generated by Arrays of Nanoslits". In: *Nano Letters* 12 (2012), p. 3640.

[12] N. H. Ibragimov. "Symmetries, Lagrangian and Conservation Laws for the Maxwell Equations". In: *Acta Appl. Math.* 105 (2009), p. 157.

[13] T. W. B. Kibble. "Conservation laws for free fields". In: *Journal of Mathematical Physics* 6 (1965), p. 1022.

[14] I. Y. Krivskii and V. M. Simulik. "Noether analysis of zilch conservation laws and their generalization for the electromagnetic field. I. Use of different formulations of the principle of least action". In: *Teor. Mat. Fiz.* 80 (1989), p. 864.

[15] I. Y. Krivsky, V. M. Simulik, and Z. Z. Torich. "A covariant form for the complete set of first-order electromagnetic conservation laws". In: *Physics Letters B* 320 (1994), p. 96.

[16] D. M. Lipkin. "Existence of a new conservation law in electromagnetic theory". In: *Journal of Mathematical Physics* 5 (1964), p. 696.

[17] L. Ma, C. Abney, and W. Lin. "Enantioselective catalysis with homochiral metal-organic frameworks". In: *Chem. Soc. Rev.* 38 (2009), p. 1248.

[18] K. M. McPeak et al. "Ultraviolet Plasmonic Chirality from Colloidal Aluminum Nanoparticles Exhibiting Charge-Selective Protein Detection". In: *Advanced Materials* 27 (2015), p. 6244.

[19] G. Nienhuis. "Conservation laws and symmetry transformations of the electromagnetic field with sources". In: *Phys. Rev. A* 93 (2016), p. 023840.

[20] E. Noether. "Invariante Variationsprobleme". In: *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* (1918), p. 235.

[21] P. Olver. *Applications of Lie Groups to Differential Equations*. Springer-Verlag, 1993.

[22] T. G. Philbin. "Lipkin's conservation law, Noether's theorem, and the relation to optical helicity". In: *Phys. Rev. A* 87 (2013), p. 043843.

[23] J. Pohjanpelto. "First order generalized symmetries of Maxwell's equations". In: *Physics Letters A* 129 (1988), p. 148.

[24] J. Pohjanpelto. "Symmetries, conservation laws, and Maxwell's equations". In: *Advanced Electromagnetism: Foundations, Theory and Applications*. Ed. by T. W. Barrett and D. M. Grimes. Singapore: World Scientific, 1995.

[25] L. V. Poulikakos. "Chiral Light–Matter Interactions in the Near and Far Field". ETH Zürich, Zürich, Switzerland, 2018.

[26] L. V. Poulikakos, J. A. Dionne, and A. García-Etxarri. "Optical Helicity and Optical Chirality in Free Space and in the Presence of matter". In: *Symmetry* 11 (2019).

[27] L. V. Poulikakos, P. Gutsche, K. M. McPeak, S. Burger, J. Niegemann, C. Hafner, and D. J. Norris. "Optical Chirality Flux as a Useful Far-Field Probe of Chiral Near Fields". In: *ACS Photonics* 3 (2016), p. 1619.

[28] L. V. Poulikakos, P. Thureja, A. Stollmann, E. D. Leo, and D. J. Norris. "Chiral Light Design and Detection Inspired by Optical Antenna Theory". In: *Nano Letters* 18 (2018), p. 4633. eprint: https://doi.org/10.1021/acs.nanolett.8b00083.

[29] M. Przanowski, B. Rajca, and J. Tosiek. "On some conservation laws in the Maxwell electrodynamics in vacuum". In: *Acta Phys. Pol. B* 25 (1994), p. 1065.

[30] P. Ramond. *Field Theory: a Modern Primer*. Frontiers in Physics. Second edition, Addison Wesley, 1981.

[31] C. Rosales-Guzmán, K. Volke-Sepulveda, and J. P. Torres. "Light with enhanced optical chirality". In: *Opt. Lett.* 37 (2012), p. 3486.

[32] M. Schäferling, D. Dregely, M. Hentschel, and H. Giessen. "Tailoring Enhanced Optical Chirality: Design Principles for Chiral Plasmonic Nanostructures". In: *Phys. Rev. X* 2 (2012), p. 031010.

[33] G. Schnoering, L. V. Poulikakos, Y. Rosales-Cabara, A. Canaguier-Durand, D. J. Norris, and C. Genet. "Three-Dimensional Enantiomeric Recognition of Optically Trapped Single Chiral Nanoparticles". In: *Phys. Rev. Lett.* 121 (2018), p. 023902.

[34] J. S. Seo, D. Whang, H. Lee, S. I. Jun, J. Oh, Y. J. Jeon, and K. Kim. "A homochiral metal-organic porous material for enantioselective separation and catalysis". In: *Nature* 404 (2000), p. 982.

[35] G. Smith and P. Strange. "Lipkin's conservation law in vacuum electromagnetic fields". In: *Journal of Physics A: Mathematical and Theoretical* 51 (2018), p. 435204.

[36] M. L. Solomon, J. M. Abendroth, L. V. Poulikakos, J. Hu, and J. A. Dionne. "Fluorescence-Detected Circular Dichroism of a Chiral Molecular Monolayer with Dielectric Metasurfaces". In: *Journal of the American Chemical Society* 142 (2020), p. 18304. eprint: https://arxiv.org/abs/https://doi.org/10.1021/jacs.0c07140.

[37] A. Sudbery. "A vector Lagrangian for the electromagnetic field". In: *Journal of Physics A: Mathematical and General* 19 (1986), p. L33.

[38] Y. Tang and A. E. Cohen. "Enhanced enantioselectivity in excitation of chiral molecules by superchiral light". In: *Science (New York, N.Y.)* 332 (2011), p. 333.

[39] Y. Tang and A. E. Cohen. "Optical chirality and its interaction with matter". In: *Phys. Rev. Lett.* 104 (2010), p. 163901.

[40] S. K. Teo, W. A. Colburn, W. G. Tracewell, K. A. Kook, D. I. Stirling, M. S. Jaworsky, M. A. Scheffler, S. D. Thomas, and O. L. Lasklin. "Clinical Pharmacokinetics of Thalidomide". In: *Clin Pharmacokinet* 43 (2004), p. 311.

[41] S. Weinberg. *The Quantum Theory of Fields*. Vol. 1. Cambridge, UK: Cambridge University Press, 1995.

[42] N. Yang and A. E. Cohen. "Local Geometry of Electromagnetic Fields and Its Role in Molecular Multipole Transitions". In: *J. Phys. Chem. B* 115 (2011), pp. 5304–5311.

[43] Yu. Krivskii and M. Simulik. "Noether analysis of zilch conservation laws and their generalization for the electromagnetic field. II. Use of Poincaré-invariant formulation of the principle of least action". In: *Teor. Mat. Fiz.* 80 (1989), p. 912.

— 7 —

Discussion and Future Perspectives

While each of the main chapters (i.e., chapters 2-6) contains its own discussion section, our aim here is to further expand upon these discussions and present intriguing questions that could potentially guide future research.

Discussing the results of chapters 3 and 4

(Here our notation for the physical modes is as in chapter 5.) Let us first recall our main result from chapters 3 and 4: the strictly massless spin-3/2 field (i.e. gravitino), as well as the strictly and partially massless spin-5/2 fields on dS_D $(D \geq 3)$, are not unitary unless $D = 4$. In chapter 3, we also suggested that this result extends to all strictly/partially massless (totally symmetric) tensor-spinors of spin $s \geq 7/2$ - this suggestion was motivated by investigating the (mis-)match between the representation-theoretic labels of the eigenmodes and the representation-theoretic labels corresponding to UIRs of the dS algebra $so(D, 1)$. In order to verify this suggestion, one should extend the technical analysis of chapter 4 to all half-odd-integer spins $s \geq 7/2$. This is expected to be a very cumbersome process that will include[1]:

- Constructing the spin-$s \geq 7/2$ eigenmodes on S^D, and then on dS_D by analytic continuation.
- Determining the transformation formulae for the eigenmodes under an infinitesimal dS boost and investigating whether these are compatible with the existence of dS invariant scalar products that are also positive-definite.

[1] The integer-spin analog of this technical analysis has been carried out by Higuchi [5], and Higuchi's calculations can be characterized, at the very least, as heroic. The haf-odd-integer spin-$s \geq 7/2$ cases are expected to be even more complicated, as it is already evident from the spin-$s = 3/2, 5/2$ cases presented in chapter 3 and 4.

- Identifying the physical and the pure gauge modes of the strictly/partially massless theories. All pure gauge modes are expected to have zero norm with respect to any dS invariant scalar product.

Motivated by our technical findings for the strictly/partially massless spin-$s = 3/2, 5/2$ cases in chapter 4, we expect that for $s \geq 7/2$ there will be no dS invariant scalar products for odd D. We also expect that for even $D > 4$, the physical modes will be separated into a positive-norm and a negative-norm sector that mix with each other under dS boosts. The $D = 4$ case is expected to be special and the negative-norm sector will decouple from the positive-norm one. Then, both sectors will essentially be positive-norm sectors, and the strictly/partially massless theories will correspond to a direct sum of Discrete Series UIRs of the dS algebra $so(4, 1)$ - see our 'field theory - representation theory dictionary' in chapter 3.

However, it recently occurred to us that there are certain group-theoretic tools, which are not widely known among mathematical physicists at the moment, and which we feel can help streamline certain cumbersome calculations. In particular, we are referring to the work of Gelfand and Tsetlin [4], in which the matrix elements of all $so(D)$ generators acting on $so(D-1)$ representation spaces were determined. In Appendix A of chapter 5, we explain in detail how to analytically continue Gelfand's and Tsetlin's results for $so(5)$ (i.e. we translate these results in the language of tensor-spinor spherical harmonics on S^4) in order to find explicit expressions for the $so(4, 1)$ transformation formulae of spin-$s \geq 3/2$ mode solutions for the strictly massless theories on dS_4. We leave the generalisation of our results to arbitrary dimensions, as well as to partially massless fields of any depth, for future work.

Quantisation. In this thesis, we have not discussed the quantization of the spin-$s \geq 3/2$ fermionic fields. However, it is already evident that there are certain complications in the striclty/partially massless cases, which we can exemplify as follows. Consider the strictly massless spin-3/2 field on dS_4. First, we recall that the usual Rarita-Schwinger Lagrangian density [3]

$$-L = \sqrt{-g}\,\overline{\psi}_\mu \gamma^{\mu\rho\sigma}\left(\nabla_\rho + \frac{i}{2}\gamma_\rho\right)\psi_\sigma \tag{7.1}$$

is known to be non-hermitian. This 'reality problem' can be fixed by considering a new hermitian Lagrangian that gives rise to the same equations of motion, as follows:

$$-L' = \sqrt{-g}\,\overline{\psi}_\mu \gamma^5 \gamma^{\mu\rho\sigma}\left(\nabla_\rho + \frac{i}{2}\gamma_\rho\right)\psi_\sigma. \tag{7.2}$$

This Lagrangian density also has the pleasant feature that it gives rise to a dS invariant scalar product[2]. This is a necessary ingredient for unitarity in the one-particle Hilbert space. To be specific, the one-particle Hilbert space we are referring to here coincides with the space of physical mode solutions. This means that we are considering the completely gauge-fixed quantum gravitino field, i.e. we are referring to the field satisfying the equations of motion in the 'Coulomb-like' gauge

$$\psi_t = 0,\;\; \gamma^j \psi_j = 0$$

(j is a spatial index) and

$$(\nabla\!\!\!\!/ + i)\psi_\mu = 0.$$

This gauge is the one that the physical modes come with naturally.
Now, one might think that we can straightforwardly proceed by expanding the gravitino field in terms of the physical modes and introducing annihilation and creation operators, as

$$\psi_\mu(t, \boldsymbol{\theta_3}) = \sum_{\sigma=\pm} \sum_{\ell=1}^{\infty} \sum_{m=1}^{\ell} \sum_{k=-(m+1)}^{m} \sqrt{\frac{\ell+2}{2(\ell+1)}} \times \left(a^{(\sigma)}_{\ell m k}\, \Psi^{(phys,\, \sigma\ell;\, m;k)}_{\mu}(t, \boldsymbol{\theta}_3) + b^{(\sigma)\dagger}_{\ell m k}\, \Psi^{(phys,\, \sigma\ell;\, m;k)C}_{\mu}(t, \boldsymbol{\theta}_3) \right). \tag{7.3}$$

However, recall from chapters 3 and 4 that the physical modes with helicity $-3/2$ have an opposite positive-definite, dS invariant scalar product from the physical modes with helicity $+3/2$. Thus, the quantum field (7.3) that contains both helicity degrees of freedom does not correspond to a unitary theory. On the other hand, one can consider two separate chiral unitary quantum fields, one with helicity $-3/2$ and one with helicity $+3/2$, as

$$\psi^{-}_\mu(t, \boldsymbol{\theta_3}) = \sum_{\ell=1}^{\infty} \sum_{m=1}^{\ell} \sum_{k=-(m+1)}^{m} \sqrt{\frac{\ell+2}{2(\ell+1)}} \left(a^{(-)}_{\ell m k}\, \Psi^{(phys,-\ell;\, m;k)}_{\mu}(t, \boldsymbol{\theta}_3) + b^{(+)\dagger}_{\ell m k}\, \Psi^{(phys,+\ell;\, m;k)C}_{\mu}(t, \boldsymbol{\theta}_3) \right), \tag{7.4}$$

$$\psi^{+}_\mu(t, \boldsymbol{\theta_3}) = \sum_{\ell=1}^{\infty} \sum_{m=1}^{\ell} \sum_{k=-(m+1)}^{m} \sqrt{\frac{\ell+2}{2(\ell+1)}} \left(a^{(+)}_{\ell m k}\, \Psi^{(phys,+\ell;\, m;k)}_{\mu}(t, \boldsymbol{\theta}_3) + b^{(-)\dagger}_{\ell m k}\, \Psi^{(phys,-\ell;\, m;k)C}_{\mu}(t, \boldsymbol{\theta}_3) \right). \tag{7.5}$$

[2]This dS invariant scalar product arises as the Noether charge associated with the invariance under infinitesimal (global) $u(1)$ transformations $\delta\psi_\mu = ia\psi_\mu$, with $a \in \mathbb{R}$.

However, this splitting of helicities cannot be achieved locally at the level of the Lagrangian (or action). On the other hand, at the level of the equations of motion, we are free to focus on solution spaces with definite helicity, as they separately form chiral UIRs of $so(4,1)$. (In order to define the helicity projector one needs to introduce the inverse of the Dirac operator on S^3, which is, of course, a non-local operator.) Does this mean that there is no satisfactory Lagrangian description for the free quantum gravitino field on dS_4?

Some philosophical thoughts. Here I would like to take a short break from the mathematically focused presentation and entertain some philosophical thoughts (I will also shift from the collective "we" to the individual "I" for this paragraph). The starting point of my thought process is the aforementioned observation that four-dimensional de Sitter space, unlike its higher-dimensional counterparts, plays a distinguished role in the unitarity of strictly/partially fermions, and of course, the fact that our four-dimensional Universe was/is/will be approximated by de Sitter space. I feel it is worth wondering whether these findings could have any relation to the following quote by Dirac [2]:

At present we are, of course, very far from this stage [where pure mathematics and physics unify], *even with regard to some of the most elementary questions. For example, only four-dimensional space is of importance in physics, while spaces with other numbers of dimensions are of about equal interest in mathematics. It may well be, however, that this discrepancy is due to the incompleteness of present-day knowledge, and that future developments will show four-dimensional space to be of far greater mathematical interest than all the others.*

In this quote, Dirac highlights the idea that pure mathematics and physics have the potential to unify. But, in order to achieve this unification we have to understand the discrepancy concerning the role of four dimensions in physics and mathematics. The question I would like to pose, assuming Dirac's viewpoint, is: can we view the existence of fermionic strictly/partially massless UIRs of $so(D,1)$ only for $D=4$, as a sign of mathematical interest of four-dimensional space?

Discussing the results of chapter 5

In chapter 5, we uncovered new conformal-like symmetries for the field equations of strictly massless fermions of spin $s = r + 1/2 \geq 3/2$ on dS_4

$$\left(\nabla\!\!\!/ + ir\right) \Psi_{\mu_1 ... \mu_r} = 0,$$
$$\nabla^{\alpha}\Psi_{\alpha\mu_2...\mu_r} = 0, \quad \gamma^{\alpha}\Psi_{\alpha\mu_2...\mu_r} = 0.$$

In particular, we showed that these equations enjoy infinitesimal conformal-like symmetries given by

$$\begin{aligned} & T_W \Psi_{\mu_1 ... \mu_r} \\ &= \gamma^5 \Big(W^{\rho}\nabla_{\rho}\Psi_{\mu_1...\mu_r} + i\,r\,W^{\rho}\gamma_{\rho}\Psi_{\mu_1...\mu_r} - i\,r\,W^{\rho}\gamma_{(\mu_1}\Psi_{\mu_2...\mu_r)\rho} + \frac{3}{8}\nabla_{\alpha}W^{\alpha}\,\Psi_{\mu_1...\mu_r}\Big) \\ &\quad - \frac{2r}{2r+1}\left(\nabla_{(\mu_1} + \frac{i}{2}\gamma_{(\mu_1}\right)\gamma^5\Psi_{\mu_2...\mu_r)\rho}W^{\rho}, \end{aligned}$$

where W is any conformal Killing vector (with non-vanishing divergence) of dS_4. The conformal-like symmetries, together with the dS symmetries, generate the algebra $so(4,2)$, which closes up to gauge transformations. We also showed that the physical (positive frequency) mode solutions form a direct sum of UIRs of the conformal-like $so(4,2)$ algebra.

SUSY in de Sitter. Now, we would like to turn our attention to the topic of supersymmetry (SUSY) in de Sitter space. As is well-known, the role (if any) played by SUSY in a dS Universe is a longstanding question, while the most popular results indicate that unbroken SUSY must be non-unitary in dS_4 (if the bosonic subalgebra closes on the dS algebra [6]). However, interestingly, unitary superconformal field theories (with unbroken SUSY) on dS_4 are known to exist [1]. Motivated by this and by our newly discovered conformal-like symmetries for strictly massless fermions, it is interesting to look for new (and possibly unitary) supersymmetric theories on dS_4 that include strictly massless fermions of spin $s \geq 3/2$. For example, although we are not going to present details here, we have performed calculations suggesting that the following pair of complex strictly massless fields (of spins 1 and 3/2) on dS_4

$$\nabla^{\mu}\left(\nabla_{\mu}A_{\nu} - \nabla_{\nu}A_{\mu}\right) = 0, \tag{7.6}$$

$$\gamma^{\mu\rho\sigma}\left(\nabla_{\rho} + \frac{i}{2}\gamma_{\rho}\right)\psi_{\sigma} = 0 \tag{7.7}$$

realise a representation of unbroken SUSY, where the commutator of two SUSY variations closes on the conformal-like $so(4,2)$ bosonic algebra. The representation is likely to be unitary, but we aim to present more details in a separate article in the future. The bosonic dS symmetries act on ψ_μ in terms of the Lie-Lorentz derivative, while the bosonic conformal-like ones act in terms of $T_W\psi_\mu$. As for the spin-1 field, the bosonic dS symmetries act on it in terms of the familiar Lie derivative, while the conformal-like symmetries correspond to the following expressions:

$$\delta_W A_\mu = iW^\rho \epsilon_{\rho\mu\sigma\lambda}\nabla^\sigma A^\lambda. \tag{7.8}$$

This expression describes new conformal-like symmetries of the spin-1 equation (7.6) on dS_4, and has been discovered collaboratively by Atsushi Higuchi and myself. (This is out of the scope of my PhD work. We aim to present more details in a joint paper in the future.)

Although it plays no role in the aforementioned supersymmetric theory, it is worth mentioning that the strictly massless spin-1 equations (7.6) on dS_4 enjoy more symmetries than the ones in (7.8), namely

$$\delta_K A_\mu = iK^\rho \epsilon_{\rho\mu\sigma\lambda}\nabla^\sigma A^\lambda, \tag{7.9}$$

where K is any Killing or conformal Killing vector of dS_4 (i.e. $K \in so(4,2)$).

Discussing the results of chapter 6

As for the discussions concerning the results of chapter 6, our primary focus has centered around the two main points highlighted in the concluding sections of the same chapter. To briefly recap these points, the first concerns the possibility of extending the zilch symmetries to a classical interacting theory of the electromagnetic and matter (e.g. spinor) fields. The second point concerns the potential application of our new zilch continuity equations to experimental investigations into the role of all zilches in light-matter interactions.

For the sake of completeness, let us also explain how chapter 6 relates to chapter 5 from the viewpoint of symmetries. The answer lies in the "hidden" symmetries of the four-potential in Minkowski spacetime presented in chapter 6, which underlie the conservation of the zilches. These "hidden" symmetries correspond to the symmetries (7.9) of the gauge potential on dS_4 (where, of course, we have to remove the factor of 'i' from Eq. (7.9) if we want to focus on symmetries of the real gauge potential on dS_4). This naturally leads us to pose the question: Do the zilches also exist in dS_4? If so, how can we derive them from the symmetries of the standard Maxwell action in dS_4 using Noether's theorem?

REFERENCES

[1] T. Anous, D. Z. Freedman, and A. Maloney. "de Sitter supersymmetry revisited". In: *J. High Energ. Phys.* 2014 (2014), p. 119.

[2] P. A. M. Dirac. "The Relation between Mathematics and Physics". In: *Proceedings of the Royal Society (Edinburgh)* 59.Part II (1938-39). Lecture delivered on presentation of the JAMES SCOTT prize, February 6, 1939, pp. 122–129.

[3] D. Z. Freedman and A. V. Proeyen. *Supergravity*. Cambridge University Press, 2012.

[4] I. M. Gelfand and M. L. Tsetlin. "Finite-dimensional representations of groups of orthogonal matrices". In: *Dokl. Akad. Nauk SSSR* 71 (1950). English translation in: I. M. Gelfand, "Collected papers". Vol II, Berlin: Springer-Verlag 1988, pp. 657–661, pp. 1017–1020.

[5] A. Higuchi. "Symmetric tensor spherical harmonics on the N-sphere and their application to the de Sitter group $SO(N,1)$". In: *J. Math. Phys.* 28 (1987), p. 1553.

[6] K. Pilch, P. van Nieuwenhuizen, and M. F. Sohnius. "De Sitter Superalgebras and Supergravity". In: *Communications in Mathematical Physics* 98 (1985), p. 105.