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Decomposition of the unitary representation of SU(1,1) on the unit disk into irreducible components

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Declaration

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Abstract

In this thesis, we decompose the representation of $\mathrm{SU}(1,1)$ on the unit disk into irreducible components. We start with the decomposition over the maximal compact subgroup K, we identify the modules of eigenfunctions which are square integrable with respect to the quasi invariant measure on the unit disk. These modules represent the discrete series representations. Then, we use the induction in stages method to find the principal series representation. The matrix coefficient with the principal series and a K-invariant vector turns to be an important function which is called a spherical function. There is a nice function (Harish Chandra's function) controlling the decay of the spherical function at infinity. Finally, we use a new approach to find the inversion formula which is equivalent to decomposition into irreducible representations using the geometry of cycles with dual numbers and the covariant transform.

Dedicated to my mother, my husband

my son and my daughter

and

to the memory of my father

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Contents

List of Figures

Chapter 1

Introduction

One of the main problems of representation theory is to classify all unitary irreducible representations of a given group G up to isomorphism. A unitary representation is completely reducible, in the sense that for any closed invariant subspace, the orthogonal complement is again a closed invariant subspace. This fundamental property makes this type of representation particularly important in physics, especially in quantum mechanics.

The most significant part of the theory of unitary representations in applications is the theory of unitary representations of locally compact groups. This thesis concerns a classification for the infinite dimensional unitary irreducible representations of the semisimple connected non-compact Lie group $\mathrm{SU}(1,1)$, which is isomorphic to the group $\mathrm{SL}_2(\mathbb{R})$. The main results of irreducible unitary representations of semisimple Lie groups are due to Gelfand-Naimark [13], V. Bargmann [6] and Harish Chandra [15]. Gelfand and Naimark studied $\mathrm{SL}_2(\mathbb{C})$, whereas Bargmann did so only for $\mathrm{SL}_2(\mathbb{R})$. The work of Bargmann is the first to use the Lie algebra method to study representations of semi-simple Lie groups. It contains the crucial ideas of expanding in terms of isotopic components and the use of the Casimir operator. Gelfand and Naimark studied representations π_χ of G induced from a character χ of a Borel subgroup $H \subset G$. Thereafter, they showed how to decompose every unitary representation as a sort of continuous direct sum of irreducible representations. The work in chapter 4 is to find the irreducible modules which are square integrable with respect to the quasi-invariant measure, using the homogeneous space, like in

Gelfand's approach, and the Lie algebra as in Bergmann's works.

Integral transforms establishes a correspondence between functions on a manifold X and functions on some manifold M of submanifolds of X. The main problems are in the description of the images and kernels of these transforms and in the construction of explicit inversion formulas recovering the original objects from their images. The first book devoted to this area was by I. M. Gelfand,M. I. Graev and N. Ya. Vilenkin [12]. From the 1940s, one of the main problems in mathematics was to develop an analog of the Fourier transform for noncommutative Lie group. For the group $SL_2(\mathbb{C})$, I. M. Gelfand and M. A. Naimark constructed a theory in which the role of exponential functions was played by irreducible infinite-dimensional unitary representations of the $SL_2(\mathbb{C})$ group. Obtaining analogs of the inversion formula and the Plancherel formula for the Fourier transform was the most important result of this theory. Our contribution in Chapter 6 is to use a new method starting with the covariant transform to obtain the inversion formula. The covariant transform approach is not restricted to $\mathrm{SL}_2(\mathbb{R})$, it may be successfully used for many other cases. This type of transform uses the representation itself, like in Gelfand's approach [13]. But the eigenvector is selected by the derived representation, like in Bargmann's works [6]. Thus, the new method bridges together the two approaches. Also, covariant transform methods are much used in theoretical physics, and the new approach will be easier to adopt for problems of decomposing a system into smaller blocks corresponding to irreducible representations.

The aim of this thesis is to decompose the induced representation ρ_n of the group $\mathrm{SU}(1,1)$ on the space $\mathrm{SU}(1,1)/K \simeq \mathbb{D}$ into irreducible components using different techniques, where K is the maximal compact subgroup of $\mathrm{SU}(1,1)$ and $\mathbb{D}$ is the unit disk . We use the derived representation of the Lie algebra to obtain the results.

We are interested of the covariant transform [4, 9, 10, 23, 31, 37] or the matrix coefficient from an irreducible square integrable representation and a mother wavelet which is selected to be admissible. This leads to a covariant transform which is an isometry to a space of square integrable functions with respect to the quasi invariant measure on a homogeneous space. Furthermore, let

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R} \setminus \{0\} \right\},$$

$$N = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{R} \right\}.$$

The matrix coefficient with the principal series and a K-invariant vector turns to be an important function associated with the character of the subgroup $P = AN$. This function is called a spherical function [17, § V.3.1; 33, § III.3] and to get it, we find the principal series representatin using the induction in stages method [20, Chap. 2]. There is a nice function (Harish-Chandra's function) controlling the decay of the spherical function at infinity.

Harish-Chandra established the Plancherel formula for semisimple Lie groups. The Plancherel theorem gives the direct integral decomposition into irreducible representations of the regular representation. In this thesis, we find the inversion formula which is equivalent to decomposition of the induced representation of the group $\mathrm{SL}_2(\mathbb{R})$ on the upper half plane into irreducible components, but we use a new approach: the homogeneous spaces $X_1 = \mathrm{SL}_2(\mathbb{R})/K$ and $X_1 = \mathrm{SL}_2(\mathbb{R})/N$ are dual to each other in the sense that points of one manifold are realized as submanifolds of the other. Both X_1 and X_2 are parametrised by the upper half plane. Moreover, when we pass from the space X_1 to the space X_2, we obtain a new homogeneous space with the same action on $\mathrm{SL}_2(\mathbb{R})$. This action becomes a linear-fractional transformation with the hypercomplex number on X_1 and with the dual number $\varepsilon^2 = 0$ on X_2. So, in our method we start with the covariant transform which sends functions on $L_2(X_1)$ to functions on the space $L_2(X_2)$. The main problem is in the construction of explicit inversion formula. Thus, we use the contravariant transform which sends functions to $L_2(X_1)$. To find the measure of the inversion formula, we use the relation between the covariant and contravariant transform.

The structure of this thesis is as follows.

- In chapter 2, we provide some standard notations, definitions and results from group theory and representation theory which are considered in this thesis.

- In chapter 3, we presents some basic facts about the group $\mathrm{SU}(1,1)$ and its Lie algebra.

- In chapter 4, we describe the construction of induced representations of the group $\mathrm{SU}(1,1)$ on Hilbert spaces. Furthermore, we study the classification of the irreducible unitary representations ρ_n of $\mathrm{SU}(1,1)$ on the unit disk. This classification is due to Bargmann [6]. It depends on two components: the constant value of the Casimir operator and the structure of the ladder of Z eigenvectors , where Z is the generator of the subgroup K. According to the Schur's lemma, the Casimir operator C acts as a scalar for the irreducible representation:

$$d\rho_n(C) = \lambda I.$$

Since the Casimir operator is a self-adjoint operator, then it has only real eigenvalues. Depending on these values, there are three classes of the non-trivial unitary irreducible representations [39, Chap. 8,§ 2]:

discrete series represntation	if $\lambda = 1-(n-1)^2 \leq 0,\ n \in \mathbb{Z}$,
principal series represntation	if $\lambda = 1+s^2 \in [1,\infty),\ s \in \mathbb{R}$,
complementary series represntation	if $\lambda = 1-s^2 \in (0,1),\ -1 < s < 1$.

Our new work in this chapter is using the homogeneous space, like in Gelfand approach, and the Lie algebra like in Bergmann works to construct and analyse the irreducible representation on the unit disk. So, first we find the common eigenfunctions of the Casimir operator and the subgroup K. Next, we identify the ladders of these eigenfunctions which are square integrable with respect to the quasi-invariant measure. Then, we find the multiplicity of every type of discrete series unitary irreducible representations. In the last section, we consider the covariant transform which is an isometry to a space of square integrable functions with respect to the invariant measure on a homogeneous space. The image of this transform is a space of an irreducible component, which represents a space of analytic and polyanalytic functions.

- In chapter 5, we construct the induced representation of $\mathrm{SL}_2(\mathbb{R})$ from the trivial character of the subgroup N. This representation is reducible, thus we use the induction in stages method [20, Chap. 2] to obtain the principal representation. Since there is an isomorphism of $\mathrm{SL}_2(\mathbb{R})$ with the group $\mathrm{SU}(1,1)$, the results on $\mathrm{SL}_2(\mathbb{R})$ can be applied to $\mathrm{SU}(1,1)$. Thereafter, we construct the spherical eigenfunction associated with the character of the subgroup $P = AN$. To determine the Plancherel measure, we study the asymptotic behaviour of the spherical function at infinity.

- In chapter 6, we find the inversion formula for the covariant transform $\mathcal{W}_{\varphi_0}^{\rho_k}$. This formula is equivalent to the decomposition into irreducible components of the unitary representation ρ_k, $k \in \mathbb{Z}$. We consider an eigenvalue $1 + s^2$ of the Casimir operator:

$$d\rho_0(C) = -4v^2 \left(\partial_u^2 + \partial_v^2\right).$$

To find this formula, first we study the representations of $\mathrm{SL}_2(\mathbb{R})$, ρ_k and ρ_τ, induced from the complex characters of K and N respectively . Then, we find the induced covariant transform $\mathcal{W}_{\varphi_0}^{\rho_k}$ with N-eigenvector, thus we obtain a transform in the space $L_2(\mathrm{SL}_2(\mathbb{R})/N)$. Thereafter, we compute the contravariant transform with K-eigenvector

$$\mathcal{M}_{\phi_0}^{\rho_\tau} : L_2(\mathrm{SL}_2(\mathbb{R})/N) \to L_2(\mathrm{SL}_2(\mathbb{R})/K).$$

We find these transforms using the representation itself like in Gelfand's approach [13], but the eigenvectors are selected by the derived representation as in Bargmann's works [6]. Finally, we use the relation between the covariant and contravariant transform to find the inversion formula. Thus, the original contribution is using the covariant transform to find the inversion formula with eigenvectors selected by the derived representation. This new method will be easier to adopt for problems of decomposing a system into elementary bits in theoretical physics. Also, it is not restricted to $SL_2(\mathbb{R})$, it can be successfully used for many other cases.

Our new work in this thesis is to use different methods to get the results:

- In the fourth chapter, we found the common eigenfunctions of the Casimir operator and the subgroup K. Our method is to obtain the eigenfunctions of the generator of the subgroup K, Then, we found the restriction of the Casimir operator to these eigenfunctions. Also, we found the irreducible modules which are square integrable with respect to the quasi-invariant measure. Our new work in this chapter is using the homogeneous space, like in Gelfand approach, and the Lie algebra like in Bergmann works to construct and analyse the irreducible representation on the unit disk.

- In chapter 5, we used the induction in stages method and the covariant transform to obtain the spherical eigenfunctions.

- In chapter 6, we identified the horocycles with dual numbers and we used the covariant transform with eigenvectors selected by the derived representation to build the inversion formula, which is equivalent to decomposition of the induced representation of the group $SL_2(\mathbb{R})$ on the upper half plane into irreducible components.

Chapter 2

Preliminaries on representation theory of groups

This chapter is planned to review some known results. We point to the representation theory of matrix groups, which we will use in this thesis.

2.1 Groups and transformations

Group theory is the mathematical theory of symmetry, which can be described in terms of transformations.

Definition 2.1.1. [21, § 2.1; 30, § 2.1, Definition 2.1] A transformation group G is a non-empty set of mappings from some set X to itself with the following properties:

- the identity transformation I belongs to G.
- G is closed under composition: if g_1 and g_2 both belong to G,then so does the composition $g_1 \circ g_2$.
- if $g \in G$, then g^{-1} exists and belongs to G.

Let G_1 is the group of all one-to-one mappings of the set X onto itself, and the group G_2 is the group of the identity mapping alone. Every transformation group G on the space X is contained between these two: $G_2 \subset G \subset G_1$ [21, § 2.1].

Example 2.1.2. [30, § 2.1] The group of linear-fractional transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \mapsto \frac{az+b}{cz+d}, \tag{2.1.1}$$

of the extended complex plane such that $ad - bc \neq 0$.

2.2 Subgroups and homogeneous spaces

Let G be a group, define an operation of G on a set X. We say that a subset $S \subset X$ is G-invariant if $g \cdot s \in S$ for all $g \in G$ and $s \in S$, where $\cdot$ is the action of G on S from the left [30, § 2.2].

Definition 2.2.1. [34, § 1.1.1] A left group action of G on X is a function $G \times X \to X$ that satisfies the following:

- $e \cdot x = x$, where e is the identity element,
- $g \cdot (g_1 \cdot x) = (g * g_1) \cdot x$, where $g, g_1 \in G$, $x \in X$ and $*$ is the multiplication on G.

We say that the action of G on X is transitive if for every $x \in X$, we have

$$Gx := \bigcup_{g \in G} g \cdot x = X, \tag{2.2.1}$$

that is, for every $x, y \in X$, there is $g \in G$ such that $y = g \cdot x$.

Definition 2.2.2. [21, § 2.2; 30, § 2.2] A G-homogeneous space is a space with a transitive group action by G .

Let H be a subgroup of a group G. Define the space of cosets $X = G/H$ by the equivalence relation: $g_1 \sim g_2$ if there exists $h \in H$ such that $g_1 = g_2 * h$.
The space $X = G/H$ is a homogeneous space under the left G-action $g : g_1 H \mapsto (g * g_1)H$. It is convenient to have a parametrisation of X and express the above G-action through those parameters, as shown below [30, § 2.2.2].
We define a section function $\mathbf{s} : X \to G$ such that it is a right inverse to the natural projection $\mathbf{p} : G \to G/H$, i.e. $\mathbf{p}(\mathbf{s}(x)) = x$ for all $x \in X$. Any $g \in G$ has a unique

decomposition of the form $g = \mathbf{s}(x) * h$, where $x = \mathbf{p}(g) \in X$ and $h \in H$. We define a map $\mathbf{r} : G \to H$ associated to $\mathbf{s}$ through the identities:

$$x = \mathbf{p}(g), \quad h = \mathbf{r}(g) := \mathbf{s}(x)^{-1} * g.$$

The set X is a left G-space with the G-action defined in terms of maps $\mathbf{s}$ and $\mathbf{p}$ as follows:

$$g : x \mapsto g \cdot x = \mathbf{p}(g * \mathbf{s}(x)).$$

This is illustrated by the diagram:

$$\begin{array}{ccc} \mathrm{G} & \xrightarrow{g*} & \mathrm{G} \\ \mathbf{s}\big\uparrow\big\downarrow\mathbf{p} & & \mathbf{s}\big\uparrow\big\downarrow\mathbf{p} \\ \mathrm{X} & \xrightarrow{g\cdot} & \mathrm{X} \end{array}$$

Example 2.2.3. [30, § 2.2.1] For any group G, we can define its action on $X = G$ as follows:

- the inner automorphisms or the conjugation

$$g : h \mapsto ghg^{-1}; \tag{2.2.2}$$

- the left shift

$$\Lambda(g) : h \mapsto g^{-1}h; \text{ and} \tag{2.2.3}$$

- the right shift

$$R(g) : h \mapsto hg, \tag{2.2.4}$$

for all $g, h \in G$. The above actions define group homomorphisms from G to the transformation group of G.

2.3 Lie groups and Lie algebras

We introduce the concept of Lie group, which describes symmetries in many physics theories, notably in quantum mechanics and particle physics. It establishes relations between many different areas of mathematics, in particular between algebra, geometry, topology and analysis.

Definition 2.3.1. [21, § 5.1] A topological space M is called a manifold if for every point $x \in M$, there exists a neighbourhood U homeomorphic to an open set in $\mathbb{R}^n$.

A chart (U, ϕ) on a manifold M is an open set $U \subset M$ together with a fixed homemorphism ϕ of this set onto an open set in $\mathbb{R}^n$. A collection of charts which together cover the entire manifold M is called an atlas A [21, § 5.1].

Definition 2.3.2. [21, § 5.1] A manifold M is called a smooth manifold if there is given an atlas A on M consisting of smoothly connected charts.

To show that an atlas is smooth, we need to verify that each transition map $\phi_1 \circ \phi_2^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is smooth (it has continuous derivatives of every order), where (U, ϕ_1) and (V, ϕ_2) are charts in A.

Definition 2.3.3. [21, § 6.1] A Lie group is a topological group G with two structures: G is a group and G is a smooth manifold for which the mapping $f : G \times G \to G$, given by $f(x, y) = xy^{-1}$ is a smooth. Smoothness of the mapping f means that it has continuous derivatives of every order.

Example 2.3.4. [8, § 1.2] The special linear group $\mathrm{SL}_2(\mathbb{R})$ is the group of 2×2 real matrices with determinant is equal to one. The group law coincides with the matrix multiplication:

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}. \tag{2.3.1}$$

The identity is the unit matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the inverse matrix is:

$$g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The group $\mathrm{SL}_2(\mathbb{R})$ is a connected non compact group of three-dimensional as it is defined by four real parameter with one constraint. It is a sub-manifold of $\mathbb{R}^4$, thus, it inherits the differentiable structure and topology from $\mathbb{R}^4$.

To see the topological structure of $\mathrm{SL}_2(\mathbb{R})$, it is convenient to take the substitution

$$a = x + y,\ d = x - y,\ b = u + v,\ c = u - v, \tag{2.3.2}$$

so that the equation $ad - bc = 1$ reads:

$$x^2 - y^2 - u^2 + v^2 = 1 \Rightarrow x^2 + v^2 = 1 + y^2 + u^2. \tag{2.3.3}$$

For each $(y, u) \in \mathbb{R}^2$, the point (x, v) belongs to the circle S^1 with radius equal to $(1 + y^2 + u^2)^{\frac{1}{2}}$. This makes $\mathrm{SL}_2(\mathbb{R})$ diffeomorphic to the direct product $S^1 \times \mathbb{R}^2$.

The study of Lie groups can be considerably facilitated by linearizing the group in the neighbourhood of its identity. This results in a structure called a Lie algebra. For connected, simply connected Lie groups [38, § 4.6], there is a one-to-one correspondence with Lie algebras, thus much of the study of Lie groups can be reduced to the study of their Lie algebras, which is usually easier.

Definition 2.3.5. [21, § 6.2] A Lie algebra is a vector space $\mathfrak{g}$ over a field $\mathbb{F}$ equipped with a bilinear product

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

which we call a Lie bracket, such that the following properties are satisfied:

- it is bilinear: $[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z] \quad$ for $\alpha, \beta \in \mathbb{F}$ and $x, y, z \in \mathfrak{g}$.
- it is skew symmetric: $[x, x] = 0$ which implies $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$.
- it satisfies the Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

For every matrix Lie group G there is an associated matrix Lie algebra $\mathfrak{g}$. An important relation between them is the exponential map [30, § 2.3.1]:

$$\begin{aligned} &\exp : \mathfrak{g} \to G \\ &\exp(tX) = \sum_{k=0}^{\infty} \frac{(tX)^k}{k!}, \quad t \in \mathbb{R}. \end{aligned} \tag{2.3.4}$$

This series is convergent for every point and its behaviour depends on the property of the powers X^k.

There are several ways to construct a Lie algebra from a Lie group [21, § 6.3]. We present two of them as in [30, § 2.3] in the following subsections.

2.3.1 One-parameter subgroups and Lie algebras

Definition 2.3.6. [14, Definition 2.13] A function $x : \mathbb{R} \to G$ is called a one-parameter subgroup of G if

- x is continuous, that is, for every open subset of the topological group G, the inverse image is an open subset of $\mathbb{R}$.
- $x(0) = I$, where I is the identity matrix.
- $x(s + t) = x(s)x(t) \quad \text{for all} \quad s, t \in \mathbb{R}$.

It follows from the well-known property of the exponent that for any element $X \in \mathfrak{g}$, the curve $x(t) = \exp tX$, $t \in \mathbb{R}$ is a one-parameter subgroup of G that satisfies:

$$X = \frac{d}{dt} x(t)|_{t=0}.$$

2.3.2 Invariant vector fields and Lie algebras

In the second realisation of the Lie algebra, $\mathfrak{g}$ is identified with the left(right) invariant vector fields on the group G, that is first order differential operators X defined at every point of G and invariant under the left(right) shifts:

$$X\Lambda = \Lambda X \quad (XR = RX).$$

The left and right derived action can be calculated at any point $g \in G$ as follows:

$$\frac{d}{dt}(x(-t) \cdot g)|_{t=0} \quad \text{and} \quad \frac{d}{dt}(g \cdot x(t))|_{t=0}, \tag{2.3.5}$$

where $x(t)$ is a one-parameter subgroup of G. We see from 2.3.1 that these vector fields define generators of one-parameter subgroups of right (left) shifts.

2.4 Representations of groups

Group representations can be used to realise the ways in which groups act on vector spaces. We begin from linear representations of groups since linear objects are easier to study.

Definition 2.4.1. [21, § 7.1] A representation in the wide sense means a homomorphism of a group G:

$$\rho(g_1 g_2) = \rho(g_1)\rho(g_2), \quad \text{for all}\, g_1, g_2 \in G, \tag{2.4.1}$$

into the group of one-to-one mappings of a certain set $\mathcal{H}$ onto itself. The representation ρ is called linear if $\mathcal{H}$ is a linear space and the mappings $\rho(g)$ are linear operators. The space $\mathcal{H}$ is called the representation space.

Example 2.4.2. [21, § 7.1] Let us consider a linear space $L : X \to \mathbb{C}$, there is a natural linear representation of a group G on $L(x)$ which produced by its action on X:

$$g : f(x) \mapsto \rho_g f(x) = f(g^{-1} \cdot x), \quad \text{where}\, g \in G,\ x \in X. \tag{2.4.2}$$

2.5 Decomposition of representations

Decomposing representations of a group G into the simplest possible components is one of the principal problems and we will illustrate the following notions which are relevant to this topic.

Definition 2.5.1. [21, § 7] Let ρ be a representation of G in $\mathcal{H}$. A linear subspace $L \subset \mathcal{H}$ is invariant subspace for ρ if for any $x \in L$ and any $g \in G$, the vector $\rho(g)x \in L$. We can consider a restriction of ρ to any its invariant subspace. Such a restriction is called subrepresentation.

There are always two trivial invariant subspaces: the null space and entire $\mathcal{H}$.

Definition 2.5.2. [21, § 7.1] If there are only two trivial invariant subspaces, then ρ is irreducible representation. Otherwise it is a reducible representation.

2.5.1 Unitary representations

The theory of unitary representations is one of the most developed parts of the theory of representations of topological groups, which is connected both with the diverse applications and with the presence of a series of properties which facilitate the study of unitary representations. One of these properties is: every unitary

representation is completely reducible (decomposable). That is, for any closed invariant subspace $V \subset \mathcal{H}$, the orthogonal complement $V^{\perp} \subset \mathcal{H}$ is a closed invariant subspace. Thus, $\mathcal{H}$ is a sum of two invariant subspaces: $\mathcal{H} = V \oplus V^{\perp}$.
A bijection $U : \mathcal{H} \to \mathcal{H}$ is a unitary operator if

$$\langle Ux, Uy \rangle = \langle x, y \rangle, \quad x, y \in \mathcal{H}.$$

Definition 2.5.3. [21, § 7.3] A representation ρ in a space $\mathcal{H}$ is called a unitary if $\mathcal{H}$ is a Hilbert space and the operator $\rho(g)$ is a unitary for all $g \in G$.

Definition 2.5.4. [21, § 7.2] Let ρ_1 and ρ_2 be two representations of a Lie group G in spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. An operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ is called an intertwining operator between ρ_1 and ρ_2 if for every $g \in G$,

$$\rho_1(g) = U\rho_2(g)U^{-1}.$$

Furthermore, unitary representations ρ_1 and ρ_2 are unitary equivalent representations if and only if there is a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ intertwining ρ_1 and ρ_2.

Definition 2.5.5. [21, § 7.3] The set of equivalence classes of unitary irreducible representations of a group G is denoted by $\hat{G}$ and called dual object (or dual space) of the group G.

An important fact on representation theory which is used to derive a lot of results on the irreducible representations, is that known as Schur's lemma.

Lemma 2.5.6 (Classical Schur's lemma)**.** [4, § 4.3, Lemma 4.3.1] *Let ρ be a continuous unitary irreducible representation of G on the Hilbert space $\mathcal{H}$. If $\Box : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator on $\mathcal{H}$, and $\Box$ commutes with $\rho(g)$, for all $g \in G$, then*

$$\Box = \lambda I,$$

for some $\lambda \in \mathbb{C}$.

Chapter 3

Preliminaries on the SU(1,1) group

In this chapter, we present some basic facts about the group $\mathrm{SU}(1,1)$ and its Lie algebra, utilizing the relation between $\mathrm{SL}_2(\mathbb{R})$ and $\mathrm{SU}(1,1)$. The main sources for this chapter are [17, 33, 39, 40].

3.1 Structure of $\mathrm{SU}(1,1)$ and Its Parametrisation

The Lie group $\mathrm{SU}(1,1)$ consists of 2×2 complex matrices A of determinant one, such that

$$AGA^* = G, \tag{3.1.1}$$

where

$$G = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.1.2}$$

and A^* is Hermitian conjugated matrix to A.

Let us take a matrix $\begin{pmatrix} \alpha & \iota \\ \beta & \kappa \end{pmatrix}$ from $\mathrm{SU}(1,1)$, since $\det A = 1$, we get

$$\alpha\kappa - \iota\beta = 1. \tag{3.1.3}$$

On the other hand, we have

$$AGA^* = \begin{pmatrix} \alpha & \iota \\ \beta & \kappa \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\iota} & \bar{\kappa} \end{pmatrix} = \begin{pmatrix} \alpha\bar{\alpha} - \iota\bar{\iota} & \alpha\bar{\beta} - \iota\bar{\kappa} \\ \beta\bar{\alpha} - \kappa\bar{\iota} & \beta\bar{\beta} - \kappa\bar{\kappa} \end{pmatrix}, \tag{3.1.4}$$

and so from (3.1.1), we have

$$\begin{aligned} |\alpha|^2 - |\iota|^2 &= 1, \\ |\kappa|^2 - |\beta|^2 &= 1, \\ \alpha\bar{\beta} - \iota\bar{\kappa} &= 0, \\ \beta\bar{\alpha} - \kappa\bar{\iota} &= 0. \end{aligned} \tag{3.1.5}$$

From the first two equations, we deduce that , since $|\alpha|^2 = 1 + |\iota|^2 \geq 1$ and $|\kappa|^2 = 1 + |\beta|^2 \geq 1$, therefore, α and κ are non-zero. From the third equation, dividing by non-zero $\bar{\kappa}$, we obtain

$$\iota = \frac{\alpha\bar{\beta}}{\bar{\kappa}}, \tag{3.1.6}$$

which we substitute into (3.1.3), then multiplying both sides with $\bar{\kappa}$:

$$\alpha|\kappa|^2 - \alpha|\beta|^2 = \bar{\kappa}. \tag{3.1.7}$$

Using $|\kappa|^2 - |\beta|^2 = 1$ from (3.1.5), we get

$$\alpha = \bar{\kappa}.$$

Then, using (3.1.6), we get

$$\iota = \bar{\beta}.$$

So, we obtain that every matrix A of $\mathrm{SU}(1,1)$ is of the form

$$A = \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \tag{3.1.8}$$

where

$$|\alpha|^2 - |\beta|^2 = 1.$$

The group law in $\mathrm{SU}(1,1)$ is provided by matrix multiplication.

Consider a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, the Cayley transform [24, § 8.1; 30, § 10.1; 33, § IX.1] is defined by the matrix conjugation

$$\frac{1}{2}\begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \tag{3.1.9}$$

where

$$\alpha = \frac{1}{2}(a + d - \mathrm{i}c + \mathrm{i}b), \qquad \beta = \frac{1}{2}(c + b - \mathrm{i}a + \mathrm{i}d).$$

The above identities define an isomorphism of $\mathrm{SL}_2(\mathbb{R})$ with the group $\mathrm{SU}(1,1)$. On the other hand, the Poincaré upper half plane $\mathbb{R}^2_+$ [39, Chap. 8] is biholomorphic to the Poincaré disk

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

via the linear fractional transformation

$$T(z) = \frac{z - i}{z + i}. \tag{3.1.10}$$

So, the group $\mathrm{SU}(1,1)$ is more convenient for complex or harmonic analysis in unit disk $\mathbb{D}$, while $\mathrm{SL}_2(\mathbb{R})$ is well suited for the upper half-plane.

The group $\mathrm{SU}(1,1)$ is three-dimensional as it defined by four real parameters bound by one constraint.

Up to the conjugation $g : g' \mapsto g^{-1}g'g$, the group $\mathrm{SU}(1,1)$ has the only non-trivial compact continuous closed subgroup K, namely the group of matrices of the form $k = \begin{pmatrix} e^{\mathrm{i}\omega} & 0 \\ 0 & e^{-\mathrm{i}\omega} \end{pmatrix}$, $-\pi < \omega \leq \pi$, which is parametrised by a point $e^{\mathrm{i}\omega}$ of the unit circle.

Now, any $g \in \mathrm{SU}(1,1)$ has a unique decomposition of the form

$$\begin{aligned} \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} &= |\alpha| \begin{pmatrix} 1 & \bar{\beta}\bar{\alpha}^{-1} \\ \beta\alpha^{-1} & 1 \end{pmatrix} \begin{pmatrix} \frac{\alpha}{|\alpha|} & 0 \\ 0 & \frac{\bar{\alpha}}{|\alpha|} \end{pmatrix} \\ &= \frac{1}{\sqrt{1 - |z|^2}} \begin{pmatrix} 1 & z \\ \bar{z} & 1 \end{pmatrix} \begin{pmatrix} e^{\mathrm{i}\omega} & 0 \\ 0 & e^{-\mathrm{i}\omega} \end{pmatrix}, \end{aligned} \tag{3.1.11}$$

where $\omega = \arg\alpha$, $z = \bar{\beta}\bar{\alpha}^{-1}$, and $|z| < 1$ (because $|\alpha|^2 - |\beta|^2 = 1$). Let $z = re^{\mathrm{i}\phi}$, then the identity (3.1.11) describes an element $g \in \mathrm{SU}(1,1)$

$$\begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} = \frac{1}{\sqrt{1 - |z|^2}} \begin{pmatrix} 1 & z \\ \bar{z} & 1 \end{pmatrix} \begin{pmatrix} e^{\mathrm{i}\omega} & 0 \\ 0 & e^{-\mathrm{i}\omega} \end{pmatrix}, \tag{3.1.12}$$

by a triple of numbers (r, ϕ, ω) where $0 \leq r < 1$ and $-\pi < \phi, \omega \leq \pi$. The connection with the above (α, β) coordinates is as follows:

$$\alpha = \frac{e^{\mathrm{i}\omega}}{\sqrt{1-r^2}}, \qquad \beta = \frac{re^{\mathrm{i}(\omega-\phi)}}{\sqrt{1-r^2}}; \tag{3.1.13}$$

$$r = \left|\frac{\beta}{\alpha}\right|, \qquad \phi = -\arg\frac{\beta}{\alpha}, \qquad \omega = \arg\alpha. \tag{3.1.14}$$

The parametrisation by (r, ϕ, ω) is closely connected with parametrisation by the Euler angles (τ, ϕ, ψ) described in [40, § VI.1.3], yet the former is more suitable for complex analysis in $\mathbb{D}$. Furthermore, we note that (3.1.12) can be also rewritten in the same variables as:

$$\begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} e^{\mathrm{i}\frac{\phi}{2}} & 0 \\ 0 & e^{-\mathrm{i}\frac{\phi}{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1-r^2}} & \frac{r}{\sqrt{1-r^2}} \\ \frac{r}{\sqrt{1-r^2}} & \frac{1}{\sqrt{1-r^2}} \end{pmatrix} \begin{pmatrix} e^{\mathrm{i}(\omega-\frac{\phi}{2})} & 0 \\ 0 & e^{-\mathrm{i}(\omega-\frac{\phi}{2})}. \end{pmatrix}. \tag{3.1.15}$$

The last presentation is decomposition of $\mathrm{SU}(1,1)$ as the product KAK of its subgroups, which is called the *Cartan decomposition*. To present the whole subgroup A in the middle matrix, the parameter r shall belong to $(-1, 1)$.

We also note the Iwasawa decomposition $\mathrm{SL}_2(\mathbb{R}) = ANK'$ [17, § I.3; 30, § 3.2; 33, § III.1] is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & \nu' \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\omega' & -\sin\omega' \\ \sin\omega' & \cos\omega' \end{pmatrix}, \tag{3.1.16}$$

where $\lambda \in \mathbb{R}_+$, $\nu' \in \mathbb{R}$ and $-\pi < \omega' \leq \pi$. The values of parameters in the above decomposition are as follows:

$$\lambda = (c^2 + d^2)^{-1/2}, \qquad \nu' = ac + bd, \qquad \omega' = \arctan\frac{c}{d}. \tag{3.1.17}$$

Consequently, $\cos\omega' = \frac{d}{\sqrt{c^2+d^2}}$ and $\sin\omega' = \frac{-c}{\sqrt{c^2+d^2}}$.

However, in the context of the upper half-plane a slightly different NAK-parametrisation is more convenient:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} \cos\omega' & -\sin\omega' \\ \sin\omega' & \cos\omega' \end{pmatrix}, \tag{3.1.18}$$

with the same defining relations (3.1.17) for λ and ω. Although, the relation for ν is seemingly more complicated:

$$\nu = \frac{ac + bd}{c^2 + d^2},$$

it is preferable in the upper half-plane. Define

$$g \cdot z = \frac{az+b}{cz+d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}), \tag{3.1.19}$$

the mapping $g \mapsto g \cdot i$ from $\mathrm{SL}_2(\mathbb{R})$ into $\mathbb{R}^2_+$ induces a bijection $NA \to \mathbb{R}^2_+$. Indeed, we see that

$$\begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \mapsto x + iy, \tag{3.1.20}$$

with $x = \nu$ and $y = \lambda^2$.

A straightforward calculation shows that the left $d\mu_l$ (right $d\mu_r$) Haar measure on $\mathrm{SL}_2(\mathbb{R})$ can be expressed in terms of the Lebesgue measure for the above parameters as follows [33, § III.1]:

$$d\mu_l(g) = \frac{r\,dr\,d\phi\,d\omega}{(1-r^2)^2} = \frac{d\nu\,d\lambda\,d\omega}{\lambda^3} = \frac{dx\,dy\,d\omega}{y^2}, \tag{3.1.21}$$

$$d\mu_r(g) = \frac{r\,dr\,d\phi\,d\omega}{1-r^2} = \lambda\,d\nu\,d\lambda\,d\omega = \frac{dx\,dy\,d\omega}{y}. \tag{3.1.22}$$

The above identities between invariant measures in different parametrisations are understood up to constant factor.

3.2 The Lie Algebra $\mathfrak{su}(1,1)$

The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$ consists of all 2×2 real matrices of trace zero [39, § 8.1]. This follows from the relation

$$e^{\operatorname{tr} X} = \det e^X, \quad X \in \mathfrak{sl}_2(\mathbb{R}). \tag{3.2.1}$$

One can introduce a basis

$$\mathcal{A} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.2.2}$$

The commutator relations are

$$[Z, \mathcal{A}] = 2B, \qquad [Z, B] = -2\mathcal{A}, \qquad [\mathcal{A}, B] = -\frac{1}{2}Z. \tag{3.2.3}$$

By the mapping

$$g \mapsto \begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix} g \begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix} g \begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix},$$

we get the same base $\mathcal{A}$, B, Z (3.2.2) in the SU(1,1) form:

$$\mathcal{A} = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad B = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}.$$

Of course these $\mathcal{A}$, B, Z satisfy to the same commutation identities (3.2.3). They also generate one-parameter subgroups of SU(1,1):

$$\begin{aligned} \exp(t\mathcal{A}) &= \begin{pmatrix} \cosh\frac{t}{2} & -\sinh\frac{t}{2} \\ -\sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}, \\ \exp(tB) &= \begin{pmatrix} \cosh\frac{t}{2} & -\mathrm{i}\sinh\frac{t}{2} \\ \mathrm{i}\sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}, \\ \exp(tZ) &= \begin{pmatrix} e^{\mathrm{i}t} & 0 \\ 0 & e^{-\mathrm{i}t} \end{pmatrix}. \end{aligned}$$

The important role in the representation theory of SU(1,1) is played by the two operators X_+ and X_- [39, § 8.2]:

$$X_\pm = \mathcal{A} \mp \mathrm{i}B, \tag{3.2.4}$$

which together with Z have commutators derived from (3.2.3):

$$[Z, X_\pm] = \pm 2\mathrm{i}X_\pm, \qquad [X_+, X_-] = -\mathrm{i}Z. \tag{3.2.5}$$

The universal enveloping algebra of $\mathfrak{su}(1,1)$ has centre generated the Casimir [39, Chap. 8, (1.25)] operator C:

$$C = Z^2 - 4\mathcal{A}^2 - 4B^2 = Z^2 - 2(X_+X_- + X_-X_+). \tag{3.2.6}$$

Chapter 4

Induced representations and classification of irreducible unitary representations on the unit disk

In this chapter, we describe the construction of induced representations of the group $\mathrm{SU}(1,1)$ on Hilbert spaces. Furthermore, we study the classification of the irreducible unitary representations of $\mathrm{SU}(1,1)$ on the unit disk. The classification depends on two components: the constant value of the Casimir operator and the structure of the ladder of Z eigenvectors. With the notation of Taylor [39, Chap. 8,§ 2], the non-trivial unitary representations have three classes: discrete series, principal series and complementary series:

holomorphic discrete series	$\rho_n^+, \quad n \in \mathbb{Z}^+,$
conjugate holomorphic discrete series	$\rho_{-n}^-, \quad n \in \mathbb{Z}^+,$
first (even) principal series	$\rho_{\mathrm{i}s}^e, \quad s \in \mathbb{R},$
second (odd) principal series	$\rho_{\mathrm{i}s}^o, \quad s \in \mathbb{R}\backslash\{0\}.$
complementary series	$\rho_s^e, \quad s \in (-1,1)\backslash\{0\}.$

In this chapter, we use the homogeneous space, like in Gelfand approach, and the Lie algebra like in Bergmann works to construct and analyse the irreducible unitary representations on the unit disk. First, we find the common eigenfunctions of the Casimir operator and the subgroup K. Next, we identify the ladders of eigenfunctions which are square integrable with respect to the quasi-invariant measure.

Then, we find the multiplicity of every type of discrete series UIR in the induced representations. Furthermore, we study the Lie derivatives of the ladder operators $\mathcal{L}^{\pm}$ which map an irreducible component induced from a character indexed by an integer n to an irreducible component induced from a character indexed by an integer $n \pm 2$. In the last section, we consider the covariant transform which is an isometry to L_2 space with respect to the invariant measure on a homogeneous space. The image of this transform is a space of an irreducible component, which represents a space of analytic and polyanalytic functions.

4.1 Induced Representations of $\mathrm{SU}(1,1)$ on Homogeneous Spaces

We will describe the construction of induced representations of the $\mathrm{SU}(1,1)$ group on Hilbert spaces of the following form:

1. the left regular representation on the group $\mathrm{SU}(1,1)$ itself which is induced by a character of the trivial subgroup $\{e\}$;

2. the representation on the unit disk which is induced by a character of the subgroup K; and

3. the representation on the unit circle which is induced by a character of the subgroup

$$H = \left\{ \begin{pmatrix} e^{-\mathrm{i}\frac{\theta_1}{2}}\alpha & e^{-\mathrm{i}\frac{\theta_1}{2}}\bar{\beta} \\ e^{\mathrm{i}\frac{\theta_1}{2}}\beta & e^{\mathrm{i}\frac{\theta_1}{2}}\bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, \, |\alpha|^2 - |\beta|^2 = 1 \right\}, \tag{4.1.1}$$

where $e^{\mathrm{i}\theta_1} = \dfrac{\alpha - \mathrm{i}\bar{\beta}}{\bar{\alpha} + \mathrm{i}\beta}$.

[21, § 2.2] Any homogeneous space X with the group G can be identified with the quotient space of G by the subgroup $H = G_x$, the stabilizer of a fixed point $x \in X$, by means of the bijection $y \in X \Leftrightarrow gH \in G/H$, where g is any element of G such that $gx = y$. Thus every left G-homogeneous space is isomorphic to the space of left cosets of G with respect to a certain subgroup.

A construction of induced representations is as follows [21, § 13.2; 22, § 3.1; 39, Chap. 5]. Let H be a subgroup of the $\mathrm{SU}(1,1)$ group and $\mathrm{SU}(1,1)/H$ be the respective homogeneous space. Let χ be a character of H and ${L_2}^{\chi}(\mathrm{SU}(1,1))$ be a space of functions with the H-covariance property:

$$F(gh) = \overline{\chi(h)}F(g), \quad g \in \mathrm{SU}(1,1),\, h \in H. \tag{4.1.2}$$

This space is invariant under left $\mathrm{SU}(1,1)$-shifts

$$\Lambda(g) : F(\acute{g}) \mapsto F(g^{-1}\acute{g}), \quad g, \acute{g} \in \mathrm{SU}(1,1). \tag{4.1.3}$$

Consider a section $\mathbf{s} : \mathrm{SU}(1,1)/H \to \mathrm{SU}(1,1)$, which is a right inverse of the natural projection $\mathbf{p} : \mathrm{SU}(1,1) \to \mathrm{SU}(1,1)/H$. Any element $g \in \mathrm{SU}(1,1)$ can be uniquely decomposed as $g = \mathbf{s}(\mathbf{p}(g))\mathbf{r}(g)$, where the map $\mathbf{r} : G \to H$ is defined by the identity:

$$\mathbf{r}(g) = \mathbf{s}(\mathbf{p}(g))^{-1}g. \tag{4.1.4}$$

Let $L_2(\mathrm{SU}(1,1)/H)$ be the space of square integrable functions on the homogeneous space $X = \mathrm{SU}(1,1)/H$ with the invariant measure $d\nu$ on X (the measure $d\nu$ is invariant if for each $g \in G$ and $x \in X$, $\int f(g{\cdot}x)\, d\nu(x) = \int f(x)\, d\nu(x)$, for $f \in L_2(X)$ [1, § 2.3, Definition 2.3.1]). We define a lifting $\mathcal{L}_\chi : L_2(\mathrm{SU}(1,1)/H) \to L_2^\chi(\mathrm{SU}(1,1))$ as follows:

$$\begin{aligned}[\mathcal{L}_\chi f](g) &= \overline{\chi(\mathbf{r}(g))} f(\mathbf{p}(g)), \qquad f \in L_2(\mathrm{SU}(1,1)/H)\\ &=: F(g).\end{aligned} \tag{4.1.5}$$

Now, we define the pulling $\mathcal{P} : L_2^\chi(\mathrm{SU}(1,1)) \to L_2(\mathrm{SU}(1,1)/H)$ by $[\mathcal{P}F](x) = F(\mathbf{s}(x))$. The induced representation ρ_χ on $L_2(\mathrm{SU}(1,1)/H)$ for the character χ on H is generated by the following formula:

$$\rho_\chi(g) = \mathcal{P} \circ \Lambda(g) \circ \mathcal{L}_\chi. \tag{4.1.6}$$

This can be represented by the following diagram:

$$\begin{array}{ccc} L_2^{\chi}(\mathrm{SU}(1,1)) & \xrightarrow{\Lambda(g)} & L_2^{\chi}(\mathrm{SU}(1,1)) \\ \mathcal{L}_\chi \uparrow \Big\| \downarrow \mathcal{P} & & \mathcal{L}_\chi \uparrow \Big\| \downarrow \mathcal{P} \\ L_2(\mathrm{SU}(1,1)/H) & \xrightarrow{\rho_\chi(g)} & L_2(\mathrm{SU}(1,1)/H) \end{array}$$

Figure 4.1: Induced representation from a character of a subgroup

Thus, the formula of the induced representation on the homogeneous space $\mathrm{SU}(1,1)/H$ (4.1.6) is

$$[\rho_\chi(g)f](x) = \overline{\chi(\mathbf{r}(g^{-1} * \mathbf{s}(x)))} f(g^{-1} \cdot x). \tag{4.1.7}$$

4.1.1 Left regular representation

Let $L_2(\mathrm{SU}(1,1), d\nu)$ be the Hilbert space of all square integrable complex-valued functions F on $\mathrm{SU}(1,1)$ with respect to the left Haar measure $d\nu$ such that:

$$\|F\|^2 = \int_{\mathrm{SU}(1,1)} |F(g)|^2 \, d\nu < \infty.$$

Thus the left regular representation of the $\mathrm{SU}(1,1)$ on $L_2(\mathrm{SU}(1,1), d\nu)$ is given by

$$\begin{aligned} \left[\Lambda \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} F\right] \begin{pmatrix} z & \bar{\omega} \\ \omega & \bar{z} \end{pmatrix} &= F\left[\begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}^{-1} \begin{pmatrix} z & \bar{\omega} \\ \omega & \bar{z} \end{pmatrix}\right] \\ &= F\left[\begin{pmatrix} \bar{\alpha}z - \bar{\beta}\omega & \bar{\alpha}\bar{\omega} - \bar{\beta}\bar{z} \\ \alpha\omega - \beta z & \alpha\bar{z} - \beta\bar{\omega} \end{pmatrix}\right], \quad \begin{pmatrix} z & \bar{\omega} \\ \omega & \bar{z} \end{pmatrix} \in \mathrm{SU}(1,1). \end{aligned} \tag{4.1.8}$$

4.1.2 Induced Representation on the unit disk

For the one-dimensional compact subgroup K of the diagonal matrices, the left homogeneous space $\mathrm{SU}(1,1)/K$ can be identified with the unit disk $\mathbb{D}$.

The natural projection is given by

$$\mathbf{p} : \mathrm{SU}(1,1) \to \mathbb{D}$$

where

$$\mathbf{p}\begin{pmatrix}\alpha & \bar{\beta}\\ \beta & \bar{\alpha}\end{pmatrix} = \frac{\bar{\beta}}{\bar{\alpha}}. \tag{4.1.9}$$

Let us define the section map $\mathbf{s} : \mathbb{D} \to \mathrm{SU}(1,1)$ as follows

$$\mathbf{s} : z \mapsto \frac{1}{\sqrt{1-|z|^2}}\begin{pmatrix}1 & z\\ \bar{z} & 1\end{pmatrix}. \tag{4.1.10}$$

Mapping $\mathbf{r} : \mathrm{SU}(1,1) \to K$ associated to the natural projection $\mathbf{p}$ and the section $\mathbf{s}$ is

$$\mathbf{r} : \begin{pmatrix}\alpha & \bar{\beta}\\ \beta & \bar{\alpha}\end{pmatrix} \mapsto \begin{pmatrix}\frac{\alpha}{|\alpha|} & 0\\ 0 & \frac{\bar{\alpha}}{|\alpha|}\end{pmatrix}. \tag{4.1.11}$$

The set $\mathrm{SU}(1,1)/K$ is a left homogeneous space with the $\mathrm{SU}(1,1)$-action defined in terms of maps $\mathbf{s}$ and $\mathbf{p}$ as follows

$$\begin{aligned} g^{-1}\cdot z &= \mathbf{p}(g^{-1} * \mathbf{s}(z)) \\ &= \frac{\bar{\alpha}z-\bar{\beta}}{\alpha-\beta z}, \qquad g^{-1} = \begin{pmatrix}\bar{\alpha} & -\bar{\beta}\\ -\beta & \alpha\end{pmatrix}, \end{aligned} \tag{4.1.12}$$

where $*$ is the multiplication on $\mathrm{SU}(1,1)$.

The invariant measure $d\mu(z)$ on $\mathbb{D}$ comes from the decomposition $dg = d\mu(z)\,dk$, where dg and dk are Haar measures on G and K respectively, is equal to

$$\mathrm{d}\mu(z) = (1-|z|^2)^{-2}\,\mathrm{d}z\wedge\mathrm{d}\bar{z}, \tag{4.1.13}$$

and $\mathrm{d}z\wedge\mathrm{d}\bar{z}$ is the standard Lebesgue measure on $\mathbb{D}$.

Let $\chi_n : \mathbb{T} \to \mathbb{C}$ be a character of $\mathbb{T}$ defined by :

$$\chi_n(z) = z^{-n}, \qquad n\in\mathbb{Z}, \tag{4.1.14}$$

which induces a linear representation of $\mathrm{SU}(1,1)$ constructed in the Hilbert space $L_2^{\chi_n}(\mathrm{SU}(1,1))$ consisting of the functions $F_n : \mathrm{SU}(1,1) \to \mathbb{C}$ with the property:

$$F_n\left[\begin{pmatrix}\alpha & \bar{\beta}\\ \beta & \bar{\alpha}\end{pmatrix}\right] = \overline{\chi_n\left(\begin{pmatrix}\frac{\alpha}{|\alpha|} & 0\\ 0 & \frac{\bar{\alpha}}{|\alpha|}\end{pmatrix}\right)}\,F\left(\begin{pmatrix}1 & \bar{\beta}\bar{\alpha}^{-1}\\ \beta\alpha^{-1} & 1\end{pmatrix}\right),$$

and the norm

$$\|F_n\|_{\mathbb{D}}^2 = \int_{\mathbb{D}}\left|F\left(\begin{pmatrix}1 & z\\ \bar{z} & 1\end{pmatrix}\right)\right|^2\frac{\mathrm{d}z\wedge\mathrm{d}\bar{z}}{(1-|z|^2)^2}, \qquad z=\bar{\beta}\bar{\alpha}^{-1},$$

where $F \in L_2(\mathbb{D}, \mathrm{d}\mu)$.

For $z \in \mathbb{D}$, we have

$$\begin{aligned} \mathbf{r}\left(g^{-1} * \mathbf{s}(z)\right) &= \mathbf{r}\left(\begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ -\beta & \alpha \end{pmatrix} * \frac{1}{\sqrt{1-|z|^2}}\begin{pmatrix} 1 & z \\ \bar{z} & 1 \end{pmatrix}\right) \\ &= \mathbf{r}\left(\frac{1}{\sqrt{1-|z|^2}}\begin{pmatrix} \bar{\alpha}-\bar{\beta}\bar{z} & \bar{\alpha}z-\bar{\beta} \\ \alpha\bar{z}-\beta & \alpha-\beta z \end{pmatrix}\right) \\ &= \frac{1}{|\alpha-\beta z|}\begin{pmatrix} \bar{\alpha}-\bar{\beta}\bar{z} & 0 \\ 0 & \alpha-\beta z \end{pmatrix}. \end{aligned} \tag{4.1.15}$$

The representation $\rho_n : L_2(\mathbb{D}) \to L_2(\mathbb{D})$, which is induced by the character χ_n, is given by formula (4.1.7), that is

$$[\rho_n(g)f](z) = \overline{\chi(\mathbf{r}(g^{-1} * \mathbf{s}(z)))} f(g^{-1} \cdot z).$$

Substituting the calculation from (4.1.12) and (4.1.15), we get

$$\begin{aligned} [\rho_n(g)f](z) &= \left(\frac{\alpha-\beta z}{|\alpha-\beta z|}\right)^{-n} f\left(\frac{\bar{\alpha}z-\bar{\beta}}{\alpha-\beta z}\right) \\ &= \left(\frac{\bar{\alpha}-\bar{\beta}\bar{z}}{\alpha-\beta z}\right)^{n/2} f\left(\frac{\bar{\alpha}z-\bar{\beta}}{\alpha-\beta z}\right). \end{aligned} \tag{4.1.16}$$

These operators are unitary on the space $L_2(\mathbb{D}, \mathrm{d}\mu)$, where $\mathrm{d}\mu = (1-|z|^2)^{-2}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}$ for the Lebesgue measure $\mathrm{d}z \wedge \mathrm{d}\bar{z}$ on $\mathbb{D}$, that is $\|\rho_n(g)f\|_{\mathbb{D}} = \|f\|_{\mathbb{D}}$. It is common [33, § IX.3; 39, § 8.4] to use the representation which preserves analyticity through the following link.

Definition 4.1.1. [3, § 4; 5, § 4.2; 28, § 2.2] An *n-peeling* is an isometry $\mathcal{P}_n : L_2(\mathbb{D}, (1-|z|^2)^{-2} dz \wedge d\bar{z}) \to L_2(\mathbb{D}, (1-|z|^2)^{n-2}\, dz \wedge d\bar{z})$ defined as:

$$\mathcal{P}_n : f(z) \mapsto [\mathcal{P}_n f](z) = \frac{f(z)}{(1-|z|^2)^{n/2}}, \qquad z = u + \mathrm{i}v. \tag{4.1.17}$$

Representation (4.1.16) is intertwined $\breve{\rho}_n \circ \mathcal{P}_n = \mathcal{P}_n \circ \rho_n$ by the n-peeling with the representation:

$$[\breve{\rho}_n(g)f](z) = \frac{1}{(\alpha-\beta z)^n}\, f\left(\frac{\bar{\alpha}z-\bar{\beta}}{\alpha-\beta z}\right), \tag{4.1.18}$$

which is unitary in $L_2(\mathbb{D}, (1-|z|^2)^{n-2} dz \wedge d\bar{z})$. The demonstration of the intertwining property is based on the following analog of identity for the unit disk (4.1.13):

$$1 - \left|\frac{\bar{\alpha}z-\bar{\beta}}{\alpha-\beta z}\right|^2 = \frac{1-|z|^2}{|\alpha-\beta z|^2}.$$

The derived representations of ρ_n are

$$\tilde{A} = d\rho_n^A = -\tfrac{1}{4}n(z - \bar{z}) \cdot I + \tfrac{1}{2}(1 - z^2) \cdot \partial_z + \tfrac{1}{2}(1 - \bar{z}^2) \cdot \partial_{\bar{z}}\,, \tag{4.1.19}$$

$$\tilde{B} = d\rho_n^B = \tfrac{\mathrm{i}}{4}n(z + \bar{z}) \cdot I + \tfrac{\mathrm{i}}{2}(1 + z^2) \cdot \partial_z - \tfrac{\mathrm{i}}{2}(1 + \bar{z}^2) \cdot \partial_{\bar{z}}\,, \tag{4.1.20}$$

$$\tilde{Z} = d\rho_n^Z = -\mathrm{i}n \cdot I - 2z\mathrm{i} \cdot \partial_z + 2\mathrm{i}\bar{z} \cdot \partial_{\bar{z}}\,. \tag{4.1.21}$$

The ladder operators $L^{\pm} = \tilde{A} \pm \mathrm{i}\tilde{B}$, which can be defined as the operators that increase or decrease the eigenvalue of another operator, are represented by:

$$L^+ = -\tfrac{1}{2}nz \cdot I - z^2 \cdot \partial_z + \partial_{\bar{z}}\,, \tag{4.1.22}$$

$$L^- = \tfrac{1}{2}n\bar{z} \cdot I + \partial_z - \bar{z}^2 \cdot \partial_{\bar{z}}\,. \tag{4.1.23}$$

In quantum mechanics, the operator L^+ (respectively, L^-) is called the creation (respectively, annihilation) operator. There are commutation relations between $\tilde{Z}$ and $L^{\pm}$ given by

$$[\tilde{Z}, L^{\pm}] = \mp 2\mathrm{i}L^{\pm}. \tag{4.1.24}$$

The Casimir operator is:

$$\begin{aligned} d\rho_n(C) &= \tilde{Z}\tilde{Z} - 2\left(L^-L^+ + L^+L^-\right) \\ &= (1 - z\bar{z})\left(-n^2 \cdot I - 2nz \cdot \partial_z + 2n\bar{z} \cdot \partial_{\bar{z}} - 4(1 - z\bar{z}) \cdot \partial_z\partial_{\bar{z}}\right). \end{aligned} \tag{4.1.25}$$

The relations $\partial_z = \dfrac{1}{2}\mathrm{e}^{-\mathrm{i}\theta}(\partial_r - \tfrac{\mathrm{i}}{r}\partial_\theta)$ and $\partial_{\bar{z}} = \dfrac{1}{2}\mathrm{e}^{\mathrm{i}\theta}(\partial_r + \tfrac{\mathrm{i}}{r}\partial_\theta)$ between partial derivatives in the polar coordinates $z = r\mathrm{e}^{\mathrm{i}\theta}$ implies the following form:

$$d\rho_n(C) = (1 - r^2)\left(-n^2 \cdot I + 2\mathrm{i}n \cdot \partial_\theta - (1 - r^2)(r^{-1} \cdot \partial_r + \partial_r^2 + r^{-2} \cdot \partial_\theta^2\right)\,. \tag{4.1.26}$$

Lie derivative (left invariant vector fields) $\mathfrak{L}^X$ for an element X of a Lie algebra is computed through the derived right regular representation of the lifted function:

$$[\mathfrak{L}^X F](g) = \frac{d}{dt}F(g \exp tX)\Big|_{t=0}, \tag{4.1.27}$$

for any differentiable function F on G. For the group $\mathrm{SU}(1,1)$, we find that:

$$\mathfrak{L}^{\mathcal{A}} = -\frac{r}{2}\sin(\phi - 2\omega)\partial_\omega - \frac{1}{2}e^{2\mathrm{i}w}(1 - z\bar{z})\partial_z - \frac{1}{2}e^{-2\mathrm{i}\omega}(1 - z\bar{z})\partial_{\bar{z}}, \tag{4.1.28}$$

$$\mathfrak{L}^B = \frac{r}{2}\cos(\phi - 2\omega)\partial_\omega - \frac{\mathrm{i}}{2}e^{2\mathrm{i}w}(1 - z\bar{z})\partial_z + \frac{\mathrm{i}}{2}e^{-2\mathrm{i}\omega}(1 - z\bar{z})\partial_{\bar{z}}, \tag{4.1.29}$$

$$\mathfrak{L}^Z = \partial_\omega. \tag{4.1.30}$$

The right ladder operators are represented by:

$$\mathfrak{L}^{\mathcal{A}+\mathrm{i}B} = e^{-2\mathrm{i}\omega}\left[\frac{\mathrm{i}z}{2}\partial_\omega - (1-z\bar{z})\partial_{\bar{z}}\right], \tag{4.1.31}$$

$$\mathfrak{L}^{\mathcal{A}-\mathrm{i}B} = -e^{2\mathrm{i}\omega}\left[\frac{\mathrm{i}\bar{z}}{2}\partial_\omega + (1-z\bar{z})\partial_{z}\right]. \tag{4.1.32}$$

4.1.3 Induced Representation on the unit circle

Let $e^{\mathrm{i}\theta_1} = \dfrac{\alpha - \mathrm{i}\bar{\beta}}{\bar{\alpha}+\mathrm{i}\beta}$, the subgroup

$$H = \left\{ \begin{pmatrix} e^{-\mathrm{i}\frac{\theta_1}{2}}\alpha & e^{-\mathrm{i}\frac{\theta_1}{2}}\bar{\beta} \\ e^{\mathrm{i}\frac{\theta_1}{2}}\beta & e^{\mathrm{i}\frac{\theta_1}{2}}\bar{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C},\ |\alpha|^2 - |\beta|^2 = 1 \right\} \tag{4.1.33}$$

is the two-dimensional and the isotropy group of $\mathrm{i} \in \mathbb{T}$. Thus the homogeneous space $\mathrm{SU}(1,1)/H$ can be identified with the unit circle $\mathbb{T}$.

The natural projection map is given by :

$$\mathbf{p} : \mathrm{SU}(1,1) \to \mathbb{T}$$

where

$$\mathbf{p}\left[\begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}\right] = \frac{\alpha - \mathrm{i}\bar{\beta}}{\bar{\alpha}+\mathrm{i}\beta}.$$

Then let

$$\mathbf{s} : \mathbb{T} \to \mathrm{SU}(1,1)$$

where

$$\mathbf{s}(u) = \begin{pmatrix} \sqrt{u} & 0 \\ 0 & \sqrt{\bar{u}} \end{pmatrix},$$

such that

$$[\mathbf{p}\circ\mathbf{s}](u) = u \Rightarrow \mathbf{p}\circ\mathbf{s} = I,$$

where I is the identity map.

The unique decomposition of any $\begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \in \mathrm{SU}(1,1)$ defined by $\mathbf{s}$ is of the form,

$$\begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} e^{\mathrm{i}\frac{\theta_1}{2}} & 0 \\ 0 & e^{-\mathrm{i}\frac{\theta_1}{2}} \end{pmatrix} \begin{pmatrix} e^{-\mathrm{i}\frac{\theta_1}{2}}\alpha & e^{-\mathrm{i}\frac{\theta_1}{2}}\bar{\beta} \\ e^{\mathrm{i}\frac{\theta_1}{2}}\beta & e^{\mathrm{i}\frac{\theta_1}{2}}\bar{\alpha} \end{pmatrix}.$$

The map $\mathbf{r} : \mathrm{SU}(1,1) \to H$ associated to the natural projection $\mathbf{p}$ and the section $\mathbf{s}$ is

$$\begin{aligned}\mathbf{r}(g) &= \mathbf{s}(\mathbf{p}(g))^{-1} g \\ &= \begin{pmatrix} e^{-\mathrm{i}\frac{\theta_1}{2}}\alpha & e^{-\mathrm{i}\frac{\theta_1}{2}}\bar{\beta} \\ e^{\mathrm{i}\frac{\theta_1}{2}}\beta & e^{\mathrm{i}\frac{\theta_1}{2}}\bar{\alpha} \end{pmatrix}.\end{aligned} \tag{4.1.34}$$

The space $\mathrm{SU}(1,1)/H$ is a left homogeneous space under the $\mathrm{SU}(1,1)$-action defined in terms of $\mathbf{p}$ and $\mathbf{s}$ as follows:

$$g^{-1} \cdot u = \frac{\bar{\alpha}u + \mathrm{i}\bar{\beta}}{\alpha - \mathrm{i}\beta u}, \qquad g^{-1} = \begin{pmatrix} \bar{\alpha} & -\bar{\beta} \\ -\beta & \alpha \end{pmatrix}, \quad u \in \mathbb{T}, \tag{4.1.35}$$

and $\cdot$ is the action of $\mathrm{SU}(1,1)$ on $\mathbb{T}$ from the left.

Let $c = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}$, conjugation by c, takes the subgroup H to

$$\frac{1}{2} \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix} H \begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} = \frac{1}{|\bar{\alpha} + \mathrm{i}\beta|} \begin{pmatrix} |\bar{\alpha} + \mathrm{i}\beta|^2 & 2\Re(\beta\alpha) \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}). \tag{4.1.36}$$

Since the character of $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ is a^m [33, § III.3], then the character $\widetilde{\chi} : H \to \mathbb{C}$ can be defined by:

$$\widetilde{\chi}(h) = |\bar{\alpha} + \mathrm{i}\beta|^m, \qquad h \in H,\ m \in \mathbb{R}. \tag{4.1.37}$$

This character induces a linear representation of $\mathrm{SU}(1,1)$ constructed in the Hilbert space $L_2^{\widetilde{\chi}}(\mathrm{SU}(1,1))$ consisting of the functions $\widetilde{F}_h : \mathrm{SU}(1,1) \to \mathbb{C}$ with the property:

$$\widetilde{F}_h\left[\begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}\right] = \overline{\widetilde{\chi}\left(\begin{pmatrix} e^{-\mathrm{i}\frac{\theta_1}{2}}\alpha & e^{-\mathrm{i}\frac{\theta_1}{2}}\bar{\beta} \\ e^{\mathrm{i}\frac{\theta_1}{2}}\beta & e^{\mathrm{i}\frac{\theta_1}{2}}\bar{\alpha} \end{pmatrix}\right)} \cdot \widetilde{F}\left(\begin{pmatrix} e^{\mathrm{i}\frac{\theta_1}{2}} & 0 \\ 0 & e^{-\mathrm{i}\frac{\theta_1}{2}} \end{pmatrix}\right),$$

and the norm

$$\|\widetilde{F}_h\|_{\mathbb{T}}^2 = \frac{1}{2\pi} \int_0^{2\pi} \left| \widetilde{F}\left(\begin{pmatrix} e^{\mathrm{i}\frac{\theta_1}{2}} & 0 \\ 0 & e^{-\mathrm{i}\frac{\theta_1}{2}} \end{pmatrix}\right) \right|^2 d\theta_1,$$

where $\widetilde{F} \in L_2(\mathbb{T})$.

For $u \in \mathbb{T}$, we have

$$\begin{aligned}\mathbf{r}\left(g^{-1} * \mathbf{s}(u)\right) &= \mathbf{r}\left(\begin{pmatrix}\bar{\alpha} & -\bar{\beta}\\ -\beta & \alpha\end{pmatrix} * \begin{pmatrix}\sqrt{u} & 0\\ 0 & \sqrt{\bar{u}}\end{pmatrix}\right)\\ &= \mathbf{r}\left(\begin{pmatrix}\bar{\alpha}\sqrt{u} & -\bar{\beta}\sqrt{\bar{u}}\\ -\beta\sqrt{u} & \alpha\sqrt{\bar{u}}\end{pmatrix}\right)\\ &= \begin{pmatrix}e^{-\mathrm{i}\theta_2}\bar{\alpha}\sqrt{\bar{u}} & -e^{-\mathrm{i}\theta_2}\bar{\beta}\bar{u}\sqrt{\bar{u}}\\ -e^{\mathrm{i}\theta_2}\beta u\sqrt{u} & e^{\mathrm{i}\theta_2}\alpha\sqrt{u}\end{pmatrix},\end{aligned} \tag{4.1.38}$$

where $e^{\mathrm{i}\theta_2} = \dfrac{\bar{\alpha} + \mathrm{i}\bar{\beta}\bar{u}}{\alpha - \mathrm{i}\beta u}$.

Substituting the calculation from (4.1.35) and (4.1.38) into the induced representation formula (4.1.7) implies

$$\left[\rho_{\tilde{\chi}}\begin{pmatrix}\alpha & \bar{\beta}\\ \beta & \bar{\alpha}\end{pmatrix} f\right](u) = |\alpha - \mathrm{i}\beta u|^m f\left(\frac{\bar{\alpha}u + \mathrm{i}\bar{\beta}}{\alpha - \mathrm{i}\beta u}\right), \tag{4.1.39}$$

where $u = e^{\mathrm{i}\theta} \in \mathbb{T}$ and $f \in L_2(\mathbb{T}, d\theta)$.

Proposition 4.1.2. *The induced representation on the unit circle (4.1.39) is isometric on the space $L_2(\mathbb{T}, d\theta)$ if and only if $m = -1$.*

Proof. The representation $\rho_{\tilde{\chi}}$ is isometric if $\|\rho_{\tilde{\chi}}(g)f\|_K = \|f\|_K$:

$$\begin{aligned}\|\rho_{\tilde{\chi}}(g)f\|_K^2 &= \frac{1}{2\pi}\int_0^{2\pi} |\rho_{\tilde{\chi}}(g)f(e^{\mathrm{i}\theta})|^2\, d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi} \left|\alpha - \mathrm{i}\beta e^{\mathrm{i}\theta}\right|^{2m} \left|f\left(\frac{\bar{\alpha}e^{\mathrm{i}\theta} + \mathrm{i}\bar{\beta}}{\alpha - \mathrm{i}\beta e^{\mathrm{i}\theta}}\right)\right|^2 d\theta.\end{aligned} \tag{4.1.40}$$

We use the substitution

$$e^{\mathrm{i}\tilde{\theta}} = \frac{\bar{\alpha}e^{\mathrm{i}\theta} + \mathrm{i}\bar{\beta}}{\alpha - \mathrm{i}\beta e^{\mathrm{i}\theta}} \Rightarrow d\theta = \frac{e^{-\mathrm{i}\theta}e^{\mathrm{i}\tilde{\theta}}}{(\bar{\alpha} + \mathrm{i}\beta e^{\mathrm{i}\tilde{\theta}})^2}d\tilde{\theta}, \quad \alpha - \mathrm{i}\beta e^{\mathrm{i}\theta} = \frac{1}{\bar{\alpha} + \mathrm{i}\beta e^{\mathrm{i}\tilde{\theta}}}.$$

Hence, (4.1.40) becomes

$$\begin{aligned}\|\rho_{\tilde{\chi}}(g)f\|_K^2 &= \frac{1}{2\pi}\int_0^{2\pi} \left|\bar{\alpha} + \mathrm{i}\beta e^{\mathrm{i}\tilde{\theta}}\right|^{-2m} \left|f\left(e^{\mathrm{i}\tilde{\theta}}\right)\right|^2 \frac{d\tilde{\theta}}{\left|\bar{\alpha} + \mathrm{i}\beta e^{\mathrm{i}\tilde{\theta}}\right|^2}\\ &= \frac{1}{2\pi}\int_0^{2\pi} \left|\bar{\alpha} + \mathrm{i}\beta e^{\mathrm{i}\tilde{\theta}}\right|^{-2m-2} \left|f\left(e^{\mathrm{i}\tilde{\theta}}\right)\right|^2 d\tilde{\theta}.\end{aligned} \tag{4.1.41}$$

Since

$$\|f\|_K^2 = \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(e^{\mathrm{i}\tilde{\theta}}\right)\right|^2 d\tilde{\theta}, \tag{4.1.42}$$

then, the representation $\rho_{\tilde{\chi}}$ is isometric if and only if

$$-2m-2=0 \Leftrightarrow m=-1. \tag{4.1.43}$$

□

4.1.4 Casimir eigenfunctions

This subsection is devoted to finding the common eigenfunctions of the Casimir operator (4.1.25) with the eigenvalue λ and the operator $d\rho_n^Z$ (4.1.21).

Lemma 4.1.3. *There are two solutions of the equation $d\rho_n(C)\varphi_{-\frac{m}{2},n} = \lambda\varphi_{-\frac{m}{2},n}$:*

1. *for $-\frac{m}{2} \neq -1,-2,-3,\ldots,$*

$$\varphi_{-\frac{m}{2},n}(z,\bar{z}) = z^{-\frac{m}{2}} \cdot (1-z\bar{z})^{t_\pm} \cdot F\left(\frac{1}{2}(n-m)+t_\pm, -\frac{n}{2}+t_\pm; 1-\frac{m}{2}; z\bar{z}\right), \tag{4.1.44}$$

2. *for $-\frac{m}{2} \neq 1,2,3,\ldots,$*

$$\tilde{\varphi}_{-\frac{m}{2},n}(z,\bar{z}) = \bar{z}^{\frac{m}{2}} \cdot (1-z\bar{z})^{t_\pm} \cdot F\left(\frac{1}{2}(m-n)+t_\pm, \frac{n}{2}+t_\pm; 1+\frac{m}{2}; z\bar{z}\right), \tag{4.1.45}$$

where $t_\pm = \dfrac{1\pm\sqrt{1-\lambda}}{2}$ and F is the hypergeometric function.

Proof. To find the family of eigenfunctions of $d\rho_n^Z$, we solve the equation $\left(d\rho_n^Z\right)\varphi_{-\frac{n}{2}}(z,\bar{z}) = 0$ by the method of characteristic [36, § 2.5] since this is a first-order linear PDE:

$$(-\mathrm{i}n \cdot I - 2z\mathrm{i}\cdot\partial_z + 2\mathrm{i}\bar{z}\cdot\partial_{\bar{z}})\,\varphi_{-\frac{n}{2}}(z,\bar{z}) = 0. \tag{4.1.46}$$

We may write the characteristics for this equation as follows:

$$\frac{dz}{-2z\mathrm{i}} = \frac{d\bar{z}}{2\bar{z}\mathrm{i}} = \frac{d\varphi_{-\frac{n}{2}}}{\mathrm{i}n\varphi_{-\frac{n}{2}}}.$$

$$\frac{dz}{-2z\mathrm{i}} = \frac{d\bar{z}}{2\bar{z}\mathrm{i}} \qquad \Rightarrow \qquad C_1 = z\bar{z}.$$

We need to obtain another integral curve which involves $\varphi_{-\frac{n}{2}}$. This is possible from the following equation

$$\frac{dz}{-2z\mathrm{i}} = \frac{d\varphi_{-\frac{n}{2}}}{\mathrm{i}n\varphi_{-\frac{n}{2}}} \qquad \Rightarrow \qquad \frac{n\cdot dz}{-2z} = \frac{d\varphi_{-\frac{n}{2}}}{\varphi_{-\frac{n}{2}}} \qquad \Rightarrow \qquad C_2 = \varphi_{-\frac{n}{2}} \cdot z^{\frac{n}{2}}.$$

Hence, the general solution of(4.1.46) is of the form $C_2 = \phi(C_1)$, that is

$$\varphi_{-\frac{n}{2}}(z, \bar{z}) = z^{-\frac{n}{2}} \cdot \phi(z\bar{z}).$$

Now, the eigenfunctions

$$\varphi_{-\frac{m}{2}}(z, \bar{z}) = z^{-\frac{m}{2}} \cdot \phi(z\bar{z}) \tag{4.1.47}$$

satisfy

$$(d\rho_n^Z)\varphi_{-\frac{m}{2}}(z, \bar{z}) = \mathrm{i}(m-n)\varphi_{-\frac{m}{2}}(z, \bar{z}).$$

The restriction of the Casimir operator (4.1.26) to eigenfunctions $\varphi_{-\frac{m}{2}}(r, \theta) = \left(re^{\mathrm{i}\theta}\right)^{-\frac{m}{2}} \cdot \phi\left(r^2\right)$ is:

$$\begin{aligned}\left[d\rho_n(C)\right]\varphi_{-\frac{m}{2}}(r, \theta) = \left(re^{\mathrm{i}\theta}\right)^{-\frac{m}{2}} \Bigg[-4r^2\left(1-r^2\right)^2 \cdot \frac{d^2\phi}{dr^2}\left(r^2\right) - 4\left(1-r^2\right)^2\left(1-\frac{m}{2}\right) \\ \cdot \frac{d\phi}{dr}\left(r^2\right) + \left(1-r^2\right)\left(nm-n^2\right) \cdot \phi\left(r^2\right)\Bigg].\end{aligned} \tag{4.1.48}$$

By Schur's lemma, the Casimir operator acts as a scalar for the irreducible unitary representation ρ_n:

$$[d\rho_n(C)]\varphi_{-\frac{m}{2}} = \lambda \cdot \varphi_{-\frac{m}{2}},$$

that is

$$\begin{aligned}r^2\left(1-r^2\right) \cdot \frac{d^2\phi}{dr^2}\left(r^2\right) + \left(1-r^2\right)\left(1-\frac{m}{2}\right) \cdot \frac{d\phi}{dr}\left(r^2\right) - \frac{1}{4}\left[\left(nm-n^2\right) - \frac{\lambda}{1-r^2}\right]\phi\left(r^2\right) \\ = 0.\end{aligned} \tag{4.1.49}$$

Now, put $r^2 = x$, then we get

$$\begin{aligned}x\left(1-x\right) \cdot \frac{d^2\phi}{dx^2}\left(x\right) + \left(1-x\right)\left(1-\frac{m}{2}\right) \cdot \frac{d\phi}{dx}\left(x\right) - \frac{1}{4}\left[\left(nm-n^2\right) - \frac{\lambda}{1-x}\right]\phi\left(x\right) \\ = 0.\end{aligned} \tag{4.1.50}$$

This equation has three regular singular points: 0, 1 and ∞, thus it can be transformed into a hypergeometric differential equation by means of a suitable change of variables.

To transform (4.1.50), we use the substitution

$$\phi(x) = x^l \cdot (1-x)^t \cdot g(x), \tag{4.1.51}$$

thus, the equation (4.1.50) becomes

$$x(1-x)\cdot\frac{d^2g}{dx^2}(x)+\Big[1-\frac{m}{2}+2l-x\big(1-\frac{m}{2}+2(l+t)\big)\Big]\cdot\frac{dg}{dx}\,(x)+\Big[-\frac{1}{4}\big(nm-n^2\big)$$
$$-t\big(1-\frac{m}{2}\big)-2lt+\Big[l\big(1-\frac{m}{2}\big)+l(l-1)\Big]\cdot\frac{1-x}{x}+\frac{1}{4}\frac{\lambda}{1-x}+t(t-1)\frac{x}{1-x}\Big]\cdot g(x)=0. \tag{4.1.52}$$

Now, if

1. $l=0, \qquad t(t-1)=-\frac{\lambda}{4} \qquad \Rightarrow \qquad t_\pm=\frac{1\pm\sqrt{1-\lambda}}{2},$

   the equation (4.1.52) will be represented as

$$\begin{aligned} x(1-x)\cdot\frac{d^2g}{dx^2}(x)+\big[1-\frac{m}{2}-x\big(1-\frac{m}{2}+2t_\pm\big)\big]\cdot\frac{dg}{dx}(x)\\ -\Big[\frac{1}{4}\big(nm-n^2\big)+t_\pm\big(1-\frac{m}{2}\big)+\frac{\lambda}{4}\Big]\cdot g(x)=0. \end{aligned} \tag{4.1.53}$$

   This is a hypergeometric differential equation has the form:

$$x(1-x)\cdot\frac{d^2g}{dx^2}(x)+[\gamma-(\zeta_1+\zeta_2+1)x]\cdot\frac{dg}{dx}(x)-\zeta_1\zeta_2\cdot g(x)=0,$$

   where $\zeta_1=\frac{1}{2}(n-m)+t_\pm, \quad \zeta_2=-\frac{n}{2}+t_\pm, \quad \gamma=1-\frac{m}{2}$.

   The solution for the equation (4.1.53) can be expressed in terms of hypergeometric series:

$$\begin{aligned} g(x)&=F(\zeta_1,\zeta_2;\gamma;x)\\ &=1+\sum_{k=1}^{\infty}\frac{(\zeta_1)_k\cdot(\zeta_2)_k}{(\gamma)_k}\cdot\frac{x^k}{k!}, \end{aligned}$$

   where $(\zeta_1)_k=\zeta_1(\zeta_1+1)...(\zeta_1+k-1)$.

   It is undefined if γ equals a non-positive integer, i.e. if

$$-\frac{m}{2}=-1,-2,-3,\cdots.$$

   Hence, from (4.1.51) the solution for the equation (4.1.49) is

$$\begin{aligned} \phi(r^2)&=\big(1-r^2\big)^{t_\pm}\cdot F(\zeta_1,\zeta_2;\gamma;r^2)\\ &=(1-z\bar{z})^{t_\pm}\cdot F(\zeta_1,\zeta_2;\gamma;z\bar{z}). \end{aligned} \tag{4.1.54}$$

   We substitute (4.1.54) in (4.1.47) to get the eigenfunctions:

$$\varphi_{-\frac{m}{2},n}(z,\bar{z})=z^{-\frac{m}{2}}\cdot(1-z\bar{z})^{t_\pm}\cdot F\Big(\frac{1}{2}(n-m)+t_\pm,-\frac{n}{2}+t_\pm;1-\frac{m}{2};z\bar{z}\Big). \tag{4.1.55}$$

2. $l = \frac{m}{2}, \qquad t(t-1) = -\frac{\lambda}{4} \qquad \Rightarrow \qquad t_{\pm} = \frac{1 \pm \sqrt{1-\lambda}}{2},$

the equation (4.1.52) becomes

$$x(1-x)\cdot\frac{d^2g}{dx^2}(x) + \left[1+\frac{m}{2} - x\left(1+\frac{m}{2}+2t_{\pm}\right]\cdot\frac{dg}{dx}(x)\right. \\ - \left[\frac{1}{4}\left(nm-n^2\right) + t_{\pm}\left(1+\frac{m}{2}\right)+\frac{\lambda}{4}\right]\cdot g(x) = 0, \quad (4.1.56)$$

and the solution for this hypergeometric differential equation can be written as:

$$g(x) = F(\zeta_1', \zeta_2'; \gamma'; x),$$

where $\zeta_1' = \frac{1}{2}(m-n)+t_{\pm}, \quad \zeta_2' = \frac{n}{2}+t_{\pm}, \quad \gamma' = 1+\frac{m}{2}$.
It is undefined if γ' equals a non-positive integer, i.e. if

$$-\frac{m}{2} = 1, 2, 3, \cdots.$$

From (4.1.47) and (4.1.51) we get the eigenfunctions

$$\tilde{\varphi}_{-\frac{m}{2},n}(z,\bar{z}) \quad = \quad z^{-\frac{m}{2}} \;\cdot\; (z\bar{z})^{\frac{m}{2}} \;\cdot\; (1-z\bar{z})^{t_{\pm}} \;\cdot\; F(\zeta_1', \zeta_2'; \gamma'; z\bar{z}). \quad (4.1.57)$$

That is

$$\tilde{\varphi}_{-\frac{m}{2},n}(z,\bar{z}) \;=\; \bar{z}^{\frac{m}{2}}\cdot(1-z\bar{z})^{t_{\pm}}\cdot F\left(\frac{1}{2}(m-n)+t_{\pm}, \frac{n}{2}+t_{\pm}; 1+\frac{m}{2}; z\bar{z}\right). \quad (4.1.58)$$

□

4.2 The classification of irreducible unitary representations of SU(1, 1) group on the unit disk

In this section, we aim to construct and analyse the irreducible unitary representations of the SU(1, 1) group on the unit disk, using Bargmann's [6] classification. We will clarify this classification in the following theorem.

Theorem 4.2.1. [39, Chap. 8] *Any non-trivial irreducible unitary representation of* SU(1, 1) *is unitarily equivalent to one of the following type, for the following values of* $s = \sqrt{1-\lambda}$, *where* λ *is the eigenvalue of the Casimir operator* $d\rho_n(C)$:

1. *The discrete series, with* $s = n - 1$ *denoted by* ρ_n^+ *or* $s = -n + 1$*, denoted by* ρ_{-n}^- *and* n *is an integer* $\geqslant 2$.

2. *The mock discrete series, with* $s = 0$*, denoted by* ρ_1^+, ρ_{-1}^-.

3. *The complementary series with* $-1 < s < 1$, $s \neq 0$*, components which have even parity denoted by* ρ_s^e.

4. *The principal series with* is, $s \in \mathbb{R}\backslash\{0\}$*, both parities, denoted by* $\rho_{\mathrm{i}s}^e$ *and* $\rho_{\mathrm{i}s}^o$ *and for* $s = 0$*, the component of even parity.*

An irreducible unitary representation of $\mathrm{SU}(1,1)$ is determined by the spectrum of $\mathrm{i}d\rho_n^Z$ and $d\rho_n(C)$. We begin with the discrete series where the action of this representation on the Casimir operator is

$$\begin{aligned} d\rho_n^+(C) &= 1 - (n-1)^2, \qquad n \in \mathbb{Z}^+, \\ d\rho_{-n}^-(C) &= 1 - (n-1)^2, \qquad n \in \mathbb{Z}^+. \end{aligned} \tag{4.2.1}$$

4.2.1 The discrete series

Consider the unitary representation $\rho_n(K)$ on the Hilbert space $\mathcal{H}$, then $\mathcal{H}$ splits into a direct sum of irreducible unitary representations H_n of K (Peter-Weyl theorem (Part II) [32, Chap. I]). That is

$$\begin{aligned} \mathcal{H} &= \overline{\bigoplus_{n \in \hat{K}} H_n} \\ &= \overline{\bigoplus_{n \in \mathbb{Z}} H_n}, \end{aligned} \tag{4.2.2}$$

where

$$\rho_n(K)\varphi_{-\frac{m}{2},n} = e^{\mathrm{i}(n-m)}\varphi_{-\frac{m}{2},n}, \quad \text{on } H_n. \tag{4.2.3}$$

Thus,

$$\begin{aligned} d\rho_n^Z \varphi_{-\frac{m}{2},n} &= \mathrm{i}(m-n)\varphi_{-\frac{m}{2},n} \\ &= -2\mathrm{i}\left(-\frac{m}{2} + \frac{n}{2}\right)\varphi_{-\frac{m}{2},n}, \quad \text{on } H_n. \end{aligned} \tag{4.2.4}$$

In the following lemma, we will describe the action of the ladder operators on the spectrum $\mathrm{i}d\rho_n^Z$.

$$\cdots \underset{L^-}{\overset{L^+}{\rightleftarrows}} \varphi_{-\frac{m}{2},n} \underset{L^-}{\overset{L^+}{\rightleftarrows}} \varphi_{-\frac{m}{2}+1,n} \underset{L^-}{\overset{L^+}{\rightleftarrows}} \varphi_{-\frac{m}{2}+2,n} \underset{L^-}{\overset{L^+}{\rightleftarrows}} \cdots$$

Lemma 4.2.2. *The ladder operators $L^{\pm}$ act as*

Proof. From the commutators

$$[d\rho_n^Z, L^{\pm}]\varphi_{-\frac{m}{2},n}(z,\bar{z}) = \mp 2\mathrm{i} L^{\pm}\varphi_{-\frac{m}{2},n}(z,\bar{z}), \tag{4.2.5}$$

we deduce that $L^{\pm}\varphi_{-\frac{m}{2},n}$ are eigenvectors of $d\rho_n^Z$:

$$\begin{aligned} d\rho_n^Z\left(L^{\pm}\varphi_{-\frac{m}{2},n}\right) &= \left(L^{\pm}d\rho_n^Z \mp 2\mathrm{i}L^{\pm}\right)\varphi_{-\frac{m}{2},n} \\ &= L^{\pm}\left(d\rho_n^Z\varphi_{-\frac{m}{2},n}\right) \mp 2\mathrm{i}L^{\pm}\varphi_{-\frac{m}{2},n} \\ &= \mathrm{i}(m-n)L^{\pm}\varphi_{-\frac{m}{2},n} \mp 2\mathrm{i}L^{\pm}\varphi_{-\frac{m}{2},n} \\ &= -2\mathrm{i}\left(\frac{n}{2} - \frac{m}{2} \pm 1\right)L^{\pm}\varphi_{-\frac{m}{2},n}. \end{aligned} \tag{4.2.6}$$

□

Using the relation $[L^+, L^-] = \mathrm{i}d\rho_n^Z$ and the expression for the Casimir operator, $d\rho_n(C) = \tilde{Z}\tilde{Z} - 2\left(L^-L^+ + L^+L^-\right)$, we find that

$$L^-L^+\varphi_{-\frac{m}{2},n}(z,\bar{z}) = -\frac{1}{4}\left[(m-n-1)^2 - 1 + \lambda\right]\varphi_{-\frac{m}{2},n}(z,\bar{z}), \tag{4.2.7}$$

and

$$L^+L^-\varphi_{-\frac{m}{2},n}(z,\bar{z}) = -\frac{1}{4}\left[(m-n+1)^2 - 1 + \lambda\right]\varphi_{-\frac{m}{2},n}(z,\bar{z}). \tag{4.2.8}$$

Moreover, L^+ and L^- are adjoin of each other:

$$(L^+)^* = L^-. \tag{4.2.9}$$

Then, from (4.2.7) and (4.2.8) we see that

$$\|L^+\| = \frac{1}{2}[1 - \lambda - (m-n-1)^2]^{\frac{1}{2}}, \tag{4.2.10}$$

and

$$\|L^-\| = \frac{1}{2}[1 - \lambda - (m-n+1)^2]^{\frac{1}{2}}. \tag{4.2.11}$$

In the case $1-\lambda=(n-1)^2$, for an integer $n>1$, we observe that the eigenfunctions

$$\varphi_{-\frac{m}{2},n}(z,\bar{z}) = z^{-\frac{m}{2}}(1-z\bar{z})^{\frac{n}{2}}, \qquad \text{where} \quad -\frac{m}{2} = 0,1,2,\dots. \tag{4.2.12}$$

These eigenfunctions are square-integrable on $\mathbb{D}$ with respect to the invariant measure $d\mu = \dfrac{dz\wedge d\bar{z}}{(1-|z|^2)^2}$:

$$\begin{aligned}
\|\varphi_{-\frac{m}{2},n}\|^2 &= \int_{\mathbb{D}} |\varphi_{-\frac{m}{2},n}(z,\bar{z})|^2 \frac{dz\wedge d\bar{z}}{(1-|z|^2)^2} \\
&= \int_{\mathbb{D}} (z\bar{z})^{-\frac{m}{2}}(1-z\bar{z})^n \frac{dz\wedge d\bar{z}}{(1-|z|^2)^2} \\
&= \int_0^{2\pi}\int_0^1 r^{-m}(1-r^2)^{n-2}\, r dr d\theta \\
&= 2\pi \int_0^1 r^{-m+1}(1-r^2)^{n-2}\, dr \\
&< 2\pi \int_0^1 r(1-r^2)^{n-2}\, dr \\
&= -\pi \frac{(1-r^2)^{n-1}}{n-1}\Big|_0^1 \\
&= \frac{\pi}{n-1} \\
&< \infty.
\end{aligned} \tag{4.2.13}$$

Looking at the equation (4.2.11), we have $L^- = 0$, if and only if $m=0$, which implies that the vector annihilated by L^- is

$$\varphi_{0,n}(z,\bar{z}) = (1-z\bar{z})^{\frac{n}{2}}. \tag{4.2.14}$$

The vector $\varphi_{0,n}$ is called the lowest weight vector and n is called the lowest weight, because all $\mathrm{i}d\rho_n^Z$ eigenvectors have eigenvalues higher than n.
We can easily obtain the relations:

$$L^+\varphi_{-\frac{m}{2},n} = \left(\frac{m}{2}-n\right)\varphi_{-\frac{m}{2}+1,n}, \quad L^-\varphi_{-\frac{m}{2},n} = -\frac{m}{2}\varphi_{-\frac{m}{2}-1,n}, \tag{4.2.15}$$

which can be visualised by the following diagram:

$$0 \xleftarrow[L^-]{} \varphi_{0,n} \underset{L^-}{\overset{L^+}{\rightleftarrows}} \varphi_{1,n} \underset{L^-}{\overset{L^+}{\rightleftarrows}} \varphi_{2,n} \underset{L^-}{\overset{L^+}{\rightleftarrows}} \cdots$$

In the case $1-\lambda = (-n+1)^2$, for an integer $n > 1$, we have the eigenfunctions

$$\tilde{\varphi}_{-\frac{m}{2},-n}(z,\bar{z}) = \bar{z}^{\frac{m}{2}}(1-z\bar{z})^{\frac{n}{2}}, \qquad \text{where} \quad -\frac{m}{2} = 0, -1, -2, \ldots. \tag{4.2.16}$$

These functions satisfy:

$$\begin{aligned} \|\tilde{\varphi}_{-\frac{m}{2},-n}\|^2 &= \int_{\mathbb{D}} |\tilde{\varphi}_{-\frac{m}{2},-n}(z,\bar{z})|^2 \frac{dz \wedge d\bar{z}}{(1-|z|^2)^2} \\ &= \int_{\mathbb{D}} (z\bar{z})^{\frac{m}{2}}(1-z\bar{z})^n \frac{dz \wedge d\bar{z}}{(1-|z|^2)^2} \\ &= \int_0^{2\pi} \int_0^1 r^m (1-r^2)^{n-2}\, r dr d\theta \\ &= 2\pi \int_0^1 r^m(1-r^2)^{n-2}\, dr \\ &< 2\pi \int_0^1 r(1-r^2)^{n-2}\, dr \\ &= -\pi \frac{(1-r^2)^{n-1}}{n-1}\Big|_0^1 \\ &= \frac{\pi}{n-1} \\ &< \infty. \end{aligned} \tag{4.2.17}$$

Thus, the eigenfunctions $\tilde{\varphi}_{-\frac{m}{2},-n}$, where $-\frac{m}{2} \leqslant 0$ and $n > 1$, are in $L_2(\mathbb{D}, \mathrm{d}\mu)$.

Looking at the equation (4.2.10), we have $L^+ = 0$, if and only if $m = 0$, and the vector which is a null solution of the creation operator L^+, is

$$\tilde{\varphi}_{0,-n}(z,\bar{z}) = (1-z\bar{z})^{\frac{n}{2}}. \tag{4.2.18}$$

The vector $\tilde{\varphi}_{0,-n}$ is called the highest weight vector and $-n$ is called the highest weight, because all $\mathrm{i}d\rho_n^Z$ eigenvectors have eigenvalues lower than $-n$.

We have the relations:

$$L^+\tilde{\varphi}_{-\frac{m}{2},-n} = \frac{m}{2}\tilde{\varphi}_{-\frac{m}{2}+1,-n}, \quad L^-\tilde{\varphi}_{-\frac{m}{2},-n} = -\left(\frac{m}{2}+n\right)\tilde{\varphi}_{-\frac{m}{2}-1,-n}, \tag{4.2.19}$$

which can be visualised by the following diagram:

$$\cdots \underset{L^-}{\overset{L^+}{\rightleftarrows}} \tilde{\varphi}_{-2,-n} \underset{L^-}{\overset{L^+}{\rightleftarrows}} \tilde{\varphi}_{-1,-n} \underset{L^-}{\overset{L^+}{\rightleftarrows}} \tilde{\varphi}_{0,-n} \xrightarrow{L^+} 0$$

Let

$$\begin{aligned} F(\omega, z, \bar{z}) &= \chi_n(e^{-\mathrm{i}\omega})\varphi_{-\frac{m}{2},n} \\ &= e^{in\omega}\varphi_{-\frac{m}{2},n}, \end{aligned} \tag{4.2.20}$$

where $\varphi_{-\frac{m}{2},n}$ is the function (4.2.12). $F(\omega, z, \bar{z})$ is an eigenvector for the Lie derivative of the element Z , that is:

$$\mathfrak{L}^Z F(\omega, z, \bar{z}) = \mathrm{i}nF(\omega, z, \bar{z}). \tag{4.2.21}$$

The commutators

$$[\mathfrak{L}^Z, \mathfrak{L}^{\mathcal{A}\pm\mathrm{i}B}]e^{\mathrm{i}n\omega}\varphi_{-\frac{m}{2},n} = \mp 2\mathrm{i}\mathfrak{L}^{\mathcal{A}\pm\mathrm{i}B}e^{\mathrm{i}n\omega}\varphi_{-\frac{m}{2},n}.$$

We denote $\mathfrak{L}^- = \mathfrak{L}^{\mathcal{A}+iB}$ and $\mathfrak{L}^+ = \mathfrak{L}^{\mathcal{A}-iB}$, then $\mathfrak{L}^\pm$ act as

$$\cdots \underset{\mathcal{L}^-}{\overset{\mathcal{L}^+}{\rightleftarrows}} \varphi_{-\frac{m}{2},n-2} \underset{\mathcal{L}^-}{\overset{\mathcal{L}^+}{\rightleftarrows}} \varphi_{-\frac{m}{2},n} \underset{\mathcal{L}^-}{\overset{\mathcal{L}^+}{\rightleftarrows}} \varphi_{-\frac{m}{2},n+2} \underset{\mathcal{L}^-}{\overset{\mathcal{L}^+}{\rightleftarrows}} \cdots$$

The lowest weight vector (4.2.14), annihilating by the operator $\mathfrak{L}^-$ as well, then all vectors $\varphi_{j,n} = (L^+)^j\varphi_{0,n}$ are lowest weight vectors due to the commutation of the left and right actions:

$$\begin{aligned} \mathfrak{L}^-\varphi_{j,n} &= \mathfrak{L}^-(L^+)^j\varphi_{0,n} \\ &= (L^+)^j\mathfrak{L}^-\varphi_{0,n} \\ &= 0, \qquad j > 0. \end{aligned} \tag{4.2.22}$$

For each $\varphi_{0,n}$, the collection of vectors $\varphi_{j,n} = (L^+)^j\varphi_{0,n}$ form an orthogonal basis of an irreducible component H_n with the respective ladder structure (4.2.15).
Two actions—the left and the right—jointly create the two-dimensional lattice structure with $n = 2$ or $n = 3$:

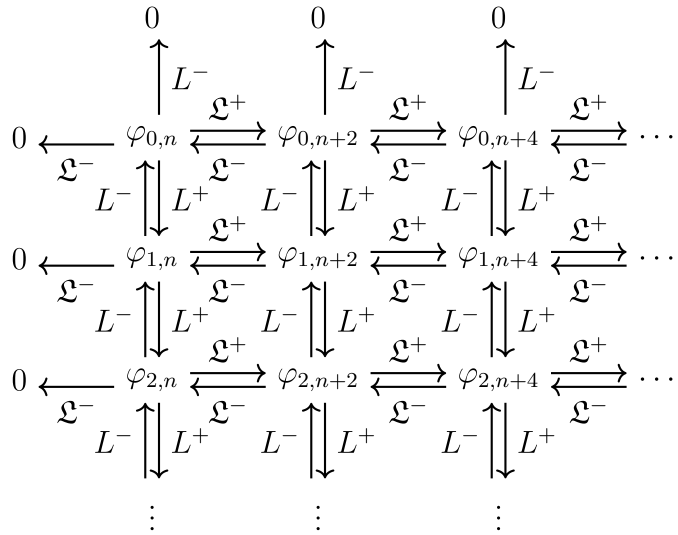

Figure 4.2: The actions of the left and the right ladder operators

Looking to this structure, and the action of the Lie derivative $\mathcal{L}^+$, so we have the orthogonal direct sums

$$\mathcal{H}_{\mathrm{e}}^+ = \overline{\bigoplus_{\substack{n>1 \\ n\,\text{even}}} H_n}, \qquad \mathcal{H}_{\mathrm{o}}^+ = \overline{\bigoplus_{\substack{n>1 \\ n\,\text{odd}}} H_n}, \tag{4.2.23}$$

where $\mathcal{H}^+ = \mathcal{H}_{\mathrm{e}}^+ \oplus \mathcal{H}_{\mathrm{o}}^+$.

In the same manner, we can obtain another two-dimensional structure consists of all vectors $\tilde{\varphi}_{-j,-n}$, where $j \geqslant 0$ and $n > 1$. In this case, for each highest weight vector $\tilde{\varphi}_{0,-n}$, the collection of vectors $\tilde{\varphi}_{-j,-n} = (L^-)^j \tilde{\varphi}_{0,-n}$ form an orthogonal basis of an irreducible component H_{-n} with the respective ladder structure (4.2.19). With the action of the Lie derivative $\mathcal{L}^-$, we get the orthogonal direct sums

$$\mathcal{H}_{\mathrm{e}}^- = \overline{\bigoplus_{\substack{n<-1 \\ n\,\text{even}}} H_n}, \qquad \mathcal{H}_{\mathrm{o}}^- = \overline{\bigoplus_{\substack{n<-1 \\ n\,\text{odd}}} H_n}, \tag{4.2.24}$$

where $\mathcal{H}^- = \mathcal{H}_{\mathrm{e}}^- \oplus \mathcal{H}_{\mathrm{o}}^-$.

Note that the Hilbert space is

$$\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-. \tag{4.2.25}$$

Each H_n is generated by the respective vertical ladder, that is

$$\begin{aligned} H_n &= span\{\varphi_{0,n}, \varphi_{1,n}, \varphi_{2,n}, \dots\}, \quad \text{where}\, n > 1, \\ H_n &= span\{\varphi_{0,n}, \varphi_{-1,n}, \varphi_{-2,n}, \dots\}, \quad \text{where}\, n < -1. \end{aligned} \tag{4.2.26}$$

We provide the following lemma, which shows that each ladder in the discrete series will appear with multiplicity 1.

Lemma 4.2.3. *The space of lowest and highest weight vectors of the same ladder is one-dimensional.*

Proof. The equation

$$d\rho_n(C)\varphi_{-\frac{m}{2},n} = \lambda \varphi_{-\frac{m}{2},n} \tag{4.2.27}$$

is a second order differential equation, thus it has two solutions [2, Theorem 2.1]. We recall that

$$L^- L^+ \varphi_{-\frac{m}{2},n}(z,\bar{z}) = -\frac{1}{4}\left[(m-n-1)^2 - 1 + \lambda\right] \varphi_{-\frac{m}{2},n}(z,\bar{z}),$$

and

$$L^+L^-\varphi_{-\frac{m}{2},n}(z,\bar{z}) = -\frac{1}{4}\left[(m-n+1)^2 - 1 + \lambda\right]\varphi_{-\frac{m}{2},n}(z,\bar{z}).$$

From (4.2.10) and (4.2.11), any lowest or highest weight vector $\varphi_{-\frac{m}{2},n}$ in $L_2(\mathbb{D}, \mathrm{d}\mu)$ satisfies $m = 0$.

If $\tilde{\varphi}_{0,n}$ is a vector such that $L^+\tilde{\varphi}_{0,n}(z,\bar{z}) = 0$, then

$$L^-L^+\tilde{\varphi}_{0,n}(z,\bar{z}) = -\frac{1}{4}\left[(-n-1)^2 - 1 + \lambda\right]\tilde{\varphi}_{0,n}(z,\bar{z}) = 0 \tag{4.2.28}$$

$$\Rightarrow \lambda = 1 - (n+1)^2. \tag{4.2.29}$$

If $\varphi_{0,n}$ is a vector such that $L^-\varphi_{0,n}(z,\bar{z}) = 0$, then

$$L^+L^-\varphi_{0,n}(z,\bar{z}) = -\frac{1}{4}\left[(-n+1)^2 - 1 + \lambda\right]\varphi_{0,n}(z,\bar{z}) = 0$$

$$\Rightarrow \lambda = 1 - (-n+1)^2. \tag{4.2.30}$$

Suppose that these two vectors are solutions of the Casimir operator with the same λ. Then $\lambda = 1 - (-|n|+1)^2$ and n in (4.2.29) must be negative while n in (4.2.30) must be positive, therefore we have two ladders since every lowest or highest weight vector is basis for a different ladder:

$$0 \xleftarrow[L^-]{} \varphi_{0,n} \underset{L^-}{\overset{L^+}{\rightleftarrows}} \varphi_{1,n} \underset{L^-}{\overset{L^+}{\rightleftarrows}} \varphi_{2,n} \underset{L^-}{\overset{L^+}{\rightleftarrows}} \cdots$$

and

$$\cdots \underset{L^-}{\overset{L^+}{\rightleftarrows}} \tilde{\varphi}_{-2,-n} \underset{L^-}{\overset{L^+}{\rightleftarrows}} \tilde{\varphi}_{-1,-n} \underset{L^-}{\overset{L^+}{\rightleftarrows}} \tilde{\varphi}_{0,-n} \xrightarrow{L^+} 0$$

Hence, the space of lowest and highest weight vectors of the same ladder has one dimension.

$\square$

4.2.2 The mock discrete series

In this case, the eigenvalue of the Casimir operator $\lambda = 1$. The representations ρ_1^+ and ρ_{-1}^- are not a discrete series representations, but rather a mock discrete series. The reason for this is that, the eigenfunctions $\varphi_{-\frac{m}{2},\pm 1}$ in the representation space of $\rho_{\pm 1}^{\pm}$ are not in $L_2(\mathbb{D}, \mathrm{d}\mu)$ [39, Chap. 8].

The mock discrete series representations appear in the decomposition, since they are embedded in the principal series [6, § 9].

Suppose $n = 1, 3, \ldots$. From the expression (4.2.11), we see that $L^-\varphi_{-\frac{m}{2},n} = 0$ when $-\frac{m}{2} = \frac{1-n}{2}$. So the corresponding space has an orthogonal basis

$$\left\{\varphi_{-\frac{m}{2},n} : -\frac{m}{2} = \frac{1-n}{2}, \frac{3-n}{2}, \ldots\right\}.$$

Thus (4.2.3) tells us that ρ_1^+ has the lowest weight $\left(n + 2(\frac{1-n}{2})\right) = 1$.

And for $n = -1, -3, \ldots$, from (4.2.10), we see that $L^+\varphi_{-\frac{m}{2},n} = 0$ when $-\frac{m}{2} = \frac{-1-n}{2}$. So, the corresponding space has an orthogonal basis

$$\left\{\varphi_{-\frac{m}{2},n} : -\frac{m}{2} = \frac{-1-n}{2}, \frac{-3-n}{2}, \ldots\right\}.$$

Thus, ρ_{-1}^- has the highest weight $\left(n + 2(\frac{-1-n}{2})\right) = -1$.

4.2.3 The complementary series

In this case, the representation is denotes by

$$\rho_s^e, \quad \lambda = 1 - s^2, \quad s \in (-1, 1)\backslash\{0\}. \tag{4.2.31}$$

The complementary series representations do not appear in the decomposition into irreducible components [33, § VIII.4; 35, § 1].

4.2.4 The principal series

This representation with even parity is denoted by

$$\rho_{\mathrm{i}s}^e, \quad \lambda = 1 + s^2, \quad s \in \mathbb{R}, \tag{4.2.32}$$

and called a member of the first principal series. Note that

$$\rho_{\mathrm{i}s}^e \simeq \rho_{-\mathrm{i}s}^e. \tag{4.2.33}$$

We denote the space corresponding to this type by H_s^e.

And with odd parity, it is denoted by

$$\rho^o_{\mathrm{i}s}, \quad \lambda = 1 + s^2, \quad s \in \mathbb{R}\backslash\{0\}, \tag{4.2.34}$$

and called a member of the second principal series. Note that

$$\rho^o_{\mathrm{i}s} \simeq \rho^o_{-\mathrm{i}s}, \tag{4.2.35}$$

and the space corresponding to this type is denoted by H_s^o.

In the case of principal series, where the spectrum of the Casimir operator is continuous, it is not possible to decompose the Hilbert space as a direct sum of irreducible representations. But each unitary representation of the non-compact group decomposes into a direct integral of irreducible unitary representations. Let $\hat{G}$ is the set of isomorphism classes of unitary representations. We let $d\sigma$ be a measure on $\hat{G}$, the direct integral decomposition [7, 19] is as follows

$$L_2(\mathbb{D}) \cong \int_{\hat{G}} H_\lambda \, d\sigma(s), \tag{4.2.36}$$

where H_λ is the corresponding space to each class. The unique measure $d\sigma$ is called the Plancherel measure. We will find this measure in the following chapters.

In the following lemma, we show that each ladder in the principal series will appear with multiplicity 1.

Lemma 4.2.4. *The space of radial eigenfunctions of the same ladder is one-dimensional.*

Proof. The equation

$$d\rho_n(C)\varphi_{-\frac{m}{2},n} = (1 + s^2)\varphi_{-\frac{m}{2},n} \tag{4.2.37}$$

is a second order differential equation which can be converted to the hypergeometric equation. By lemma 4.1.3, there is one defined solution for every integer $-\frac{m}{2}$. The eigenfunction $\varphi_{-\frac{m}{2},n} = z^{-\frac{m}{2}}\phi(z\bar{z})$ is radial if and only if $m = 0$. Thus, by lemma 4.1.3, the radial eigenfunction takes the form:

$$\varphi_{0,n} = (1 - z\bar{z})^{\frac{1+\mathrm{i}s}{2}} F\left(\frac{1}{2}(n + 1 + \mathrm{i}s), \frac{1}{2}(-n + 1 + \mathrm{i}s); 1; z\bar{z}\right). \tag{4.2.38}$$

Then, every radial eigenfunction is basis for a different ladder:

$$\cdots \underset{L^-}{\overset{L^+}{\rightleftarrows}} \tilde{\varphi}_{-1,n} \underset{L^-}{\overset{L^+}{\rightleftarrows}} \varphi_{0,n} \underset{L^-}{\overset{L^+}{\rightleftarrows}} \varphi_{1,n} \underset{L^-}{\overset{L^+}{\rightleftarrows}} \cdots$$

Hence, the space of radial eigenfunctions of the same ladder has one dimension.

□

4.3 Covariant Transform

Definition 4.3.1. [26, Definition 4.1] Let ρ be a representation of a group G in a space $\mathcal{H}$ and F be an operator acting from $\mathcal{H}$ to a space $\mathbf{U}$. We define a covariant transform $\mathcal{W}_F^\rho$ acting from $\mathcal{H}$ to the space $L(G, \mathbf{U})$ of $\mathbf{U}$- valued functions on G by the formula:

$$\mathcal{W}_F^\rho : v \mapsto \hat{v}(g) = F(\rho(g^{-1})v), \qquad v \in \mathcal{H},\ g \in G. \tag{4.3.1}$$

The operator F is called a fiducial operator.

Example 4.3.2. [26, Example 4.4] Let $\mathcal{H}$ be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and ρ be a unitary representation of a group G in the space $\mathcal{H}$. Let $F : \mathcal{H} \to \mathbb{C}$ be the functional $v \mapsto \langle v, v_0 \rangle$ defined by a vector $v_0 \in \mathcal{H}$. The vector v_0 is called the mother wavelet or the vacuum state. Then, the covariant transform (4.3.1) becomes the wavelet transform:

$$\mathcal{W} : v \mapsto \hat{v}(g) = \langle \rho(g^{-1})v, v_0 \rangle = \langle v, \rho(g)v_0 \rangle, \qquad v \in \mathcal{H},\ g \in G. \tag{4.3.2}$$

Definition 4.3.3. [29, Definition 4.4] Let $F : \mathcal{H} \to U$ intertwines the restriction of ρ to H with a character χ of H : $F(\rho(h)v) = \overline{\chi(h)}F(v)$ for all $h \in H$, $v \in \mathcal{H}$. Then, the induced covariant transform is:

$$[\mathcal{W}_F v](x) = F(\rho(\mathbf{s}(x))v), \qquad v \in \mathcal{H}, \quad x \in G/H,$$

and $\mathbf{s} : G/H \to G$ is a continuous section.

The covariant transform has obtained its name because of the following property.

Theorem 4.3.4. [26, Theorem 6.1] *The covariant transform (4.3.1) intertwines ρ and the left regular representation Λ on $L(G, \mathbf{U})$:*

$$\mathcal{W}\rho(g) = \Lambda(g)\mathcal{W}, \tag{4.3.3}$$

where $\Lambda(g) : f(\acute{g}) \mapsto f(g^{-1}\acute{g})$, $g, \acute{g} \in G$.

Example 4.3.5. For $G = \mathrm{SU}(1,1)$, $H = K$ and the induced representation ρ_2 (4.1.16), a pairing with the vacuum $\upsilon_0(z) = 1 - |z|^2$, has the property

$$\langle \upsilon, \rho_2(h)\upsilon_0 \rangle = e^{2i\theta}\langle \upsilon, \upsilon_0 \rangle, \qquad h = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \in K.$$

Thus, υ_0 can be used for the induced covariant transform:

$$\begin{aligned} [\mathcal{W}\upsilon](\omega) &= \langle \upsilon, \rho_2(\mathbf{s}(\omega))\upsilon_0 \rangle \\ &= (1 - \omega\bar{\omega}) \int_{\mathbb{D}} \frac{\upsilon(z)}{(1 - \omega\bar{z})^2} \frac{dz \wedge d\bar{z}}{(1 - z\bar{z})}, \end{aligned} \tag{4.3.4}$$

where $\mathbf{s}(\omega) = \dfrac{1}{\sqrt{1 - |\omega|^2}} \begin{pmatrix} 1 & \omega \\ \bar{\omega} & 1 \end{pmatrix}$.

Up to the factor $\dfrac{1 - |\omega|^2}{1 - |z|^2}$, this is known as the Bergman integral. The image space $B_2(\mathbb{D})$ of $\mathcal{W}$ is $\mathrm{SU}(1,1)$-invariant subspace of $L_2(\mathbb{D})$, which is called Bergman space. The orthogonal projection:

$$P : L_2(\mathbb{D}) \to B_2(\mathbb{D}) \tag{4.3.5}$$

presented by the Bergman integral (4.3.4) is called the Bergman projection.

Corollary 4.3.6. [26, Corollary 6.8] *Let a fiducial operator F be a null-solution, i.e. $AF = 0$, for the operator $A = \sum_j a_j d\rho_B^{X_j}$, where $X_j \in \mathfrak{g}$ and a_j are constants. Then the covariant transform $[\mathcal{W}_F f](g) = F(\rho(g^{-1})f)$ for any f satisfies*

$$D(\mathcal{W}_F f) = 0, \qquad \textit{where } D = \sum_j \bar{a}_j \mathfrak{L}^{X_j}. \tag{4.3.6}$$

Here, $\mathfrak{L}^{X_j}$ are the left invariant vector fields (Lie derivative) on G corresponding to X_j.

Example 4.3.7. Consider the induced representation ρ_n (4.1.16) of the $\mathrm{SU}(1,1)$ group.

The vacuums $\varphi_{0,n}(z, \bar{z}) = (1 - z\bar{z})^{\frac{n}{2}}$ are null solutions of the operator

$$L^- = d\rho_n^A - i d\rho_n^B. \tag{4.3.7}$$

Therefore, the image of the covariant transform with fiducial operator defined in Example 4.3.2 consists of the null solutions to the operator

$$\mathfrak{L}^{\mathcal{A}} + \mathrm{i}\mathfrak{L}^{B} = e^{-2\mathrm{i}\omega}\left[\frac{\mathrm{i}z}{2}\partial_{\omega} - (1 - z\bar{z})\partial_{\bar{z}}\right]. \tag{4.3.8}$$

We refer to the two-dimensional lattice structure shown in Fig 4.2, the vertical ladders are orthogonal basis of the covariant transform image $\mathcal{W}_{\varphi_{o,n}}$ which generate corresponding invariant irreducible components H_n annihilated by

$$\mathfrak{L}^{\mathcal{A}+\mathrm{i}B}\varphi_{j,n} = 0, \qquad \text{where } j \geq 0,\ n > 1. \tag{4.3.9}$$

Chapter 5

Induction in stages and Plancherel measure

In this chapter, we construct the induced representation of $\mathrm{SL}_2(\mathbb{R})$ from the trivial character of the subgroup N. This representation is reducible, thus we use the induction in stages method [20, Chap. 2] to obtain irreducible representations. Since there is an isomorphism of $\mathrm{SL}_2(\mathbb{R})$ with the group $\mathrm{SU}(1,1)$, the results on $\mathrm{SL}_2(\mathbb{R})$ can be applied to $\mathrm{SU}(1,1)$. Thereafter, we construct the spherical eigenfunction associated with the character of the subgroup $P = AN$. To determine the Plancherel measure, we study the asymptotic behaviour of the spherical function at infinity.

5.1 Induced representations of the group $\mathrm{SL}_2(\mathbb{R})$

In this section, we find the induction to $\mathrm{SL}_2(\mathbb{R})$ from the trivial character of the subgroup N. This representation is reducible, thus we will use the induction in stages method [20, Chap. 2] to get the principal series representation. That is, first we make an induction of the trivial character of N to P. This representation can be decomposed into a one-dimensional representation of P which is a complex character. Then, we induce a representation from this character of P to $\mathrm{SL}_2(\mathbb{R})$.

5.1.1 The induction to $\mathrm{SL}_2(\mathbb{R})$ from the subgroup N

The decomposition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix},$$

can be used to identify the homogeneous space $X = \mathrm{SL}_2(\mathbb{R})/N$ with $KA \simeq \mathbb{T} \times \mathbb{R}_+ \simeq \mathbb{R}^2 \setminus \{0\}$.

There is a natural projection map

$$\mathbf{p} : \mathrm{SL}_2(\mathbb{R}) \to X$$

where

$$\mathbf{p}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = (c, a).$$

Define a section

$$\mathbf{s} : X \to \mathrm{SL}_2(\mathbb{R})$$

as follows

$$\mathbf{s}[(x, y)] = \begin{pmatrix} y & 0 \\ x & y^{-1} \end{pmatrix},$$

which is the right inverse of the natural projection. The map

$$\mathbf{r} : \mathrm{SL}_2(\mathbb{R}) \to N$$

will be

$$\mathbf{r}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}.$$

The left action on the homogeneous space $\mathrm{SL}_2(\mathbb{R})/N$ is defined in terms of $\mathbf{p}$ and $\mathbf{s}$ as follows:

$$\begin{aligned} g^{-1} \cdot (x, y) &= \mathbf{p}\left(g^{-1} \cdot \mathbf{s}(x, y)\right) \\ &= (ax - cy, dy - bx), \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \end{aligned} \tag{5.1.1}$$

and $\cdot$ is the action of $\mathrm{SL}_2(\mathbb{R})$ on X from the left.

Let χ_e be a trivial character of N defined by

$$\chi_e \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = 1, \tag{5.1.2}$$

then it induces a representation of $\mathrm{SL}_2(\mathbb{R})$ in the Hilbert space $L_2^{\chi_e}(\mathrm{SL}_2(\mathbb{R}))$ of functions with the N-covariance property

$$\begin{aligned} F_e \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \overline{\chi_e \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}} F \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix} \\ &= F \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}. \end{aligned} \tag{5.1.3}$$

This space is invariant under left $\mathrm{SL}_2(\mathbb{R})$-shifts

$$\Lambda(g) : F(\acute{g}) \mapsto F(g^{-1}\acute{g}), \qquad g, \acute{g} \in \mathrm{SL}_2(\mathbb{R}).$$

The representation $T : L_2(X) \to L_2(X)$, which is induced by the character χ_e, is given by

$$[T(g)f](x,y) = \overline{\chi_e(\mathbf{r}(g^{-1} \cdot \mathbf{s}(x,y)))} f(g^{-1} \cdot (x,y)).$$

Substituting the calculation from (5.1.1), we get

$$[T(g)f](x,y) = f\,(ax - cy, dy - bx)\,, \tag{5.1.4}$$

where $(x,y) \in X$.

5.1.2 The co-adjoint representation induced from a trivial character of N

The affine group is the set $\mathrm{Aff} = \{(a,b) : a > 0, b \in \mathbb{R}\}$ with the group law $*$ defined by:

$$(a_1, b_1) * (a_2, b_2) = (a_1 a_2, a_1 b_2 + b_1),$$

where $(a_1, b_1), (a_2, b_2) \in \mathrm{Aff}$. The identity element is $(1,0)$ and the inverse is $(a,b)^{-1} = (a^{-1}, -ba^{-1})$.

The group $(\mathrm{Aff}, *)$ is isomorphic to the group $\left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a > 0, b \in \mathbb{R} \right\}$.

For the subgroup $N = \{(1, b) : b \in \mathbb{R}\}$, the homogeneous space $X = \text{Aff}/N \simeq A = \mathbb{R}_+$ generates the co-adjoint representation of Aff on $L_2(\mathbb{R}_+)$.
There is a natural projection

$$\mathbf{p} : \text{Aff} \to \mathbb{R}_+$$

such that

$$\mathbf{p}(a, b) = a.$$

Let us define a section $\mathbf{s} : \mathbb{R}_+ \to \text{Aff}$ as follows

$$\mathbf{s} : a \mapsto (a, 0),$$

such that

$$[\mathbf{p} \circ \mathbf{s}](a) = a \Rightarrow \mathbf{p} \circ \mathbf{s} = I,$$

where I is the identity map.
The decomposition $(a, b) = (a, 0) * (1, \frac{b}{a})$ defines the map $\mathbf{r} : \text{Aff} \to N$ such that

$$\mathbf{r}(a, b) = \mathbf{s}(a)^{-1} * (a, b) = (1, \frac{b}{a}).$$

The space Aff$/N$ is a left homogeneous space under the Aff-action defined in terms of $\mathbf{p}$ and $\mathbf{s}$ as follows:

$$(a, b)^{-1} \cdot u = \mathbf{p}((a, b)^{-1} * \mathbf{s}(u)) = a^{-1}u, \tag{5.1.5}$$

where $(a, b) \in \text{Aff}$, $u \in X$ and $\cdot$ is the action of Aff on X from the left.

The character χ_e induces a representation of Aff constructed in the Hilbert space $L_2^{\chi_e}(\text{Aff})$, which consists of the function $F_e : \text{Aff} \to \mathbb{C}$ with the N-covariance property

$$\begin{aligned} F_e(a, b) &= \overline{\chi_e\left(1, \frac{b}{a}\right)} F(a, 0) \\ &= F(a, 0), \end{aligned} \tag{5.1.6}$$

and the norm

$$\|F_e\|_A^2 = \int_{\mathbb{R}_+} |F(a, 0)|^2 \, \frac{da}{a},$$

where $F \in L_2(\mathbb{R}_+)$.

The representation $\pi^+_{\chi_e} : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$ which is induced by the character χ_e is given by

$$\left[\pi^+_{\chi_e}(a,b)f\right](u) = \overline{\chi_e(\mathbf{r}(g^{-1}\cdot \mathbf{s}(u)))}f(g^{-1}\cdot u).$$

With $u \in \mathbb{R}_+$, we obtain the formula

$$\left[\pi^+_{\chi_e}(a,b)f\right](u) = f\left(\frac{u}{a}\right),$$

where $f \in L_2\left(\mathbb{R}_+, \dfrac{da}{a}\right)$.

By changing the variable $t = u^{-1}, g(t) = t^{-\frac{1}{2}}f(t^{-1})$, we get

$$\left[\pi^+_{\chi_e}(a,b)g\right](t) = \sqrt{a}\cdot g\left(at\right), \tag{5.1.7}$$

where $g \in L_2\left(\mathbb{R}_+, da\right)$.

This representation is unitary reducible, and the decomposition into irreducible components requires finding the eigenfunction of the operator $\pi^+_{\chi_e}(a,b)g$ as follows:

$$\left[\pi^+_{\chi_e}(a,b)g\right](t) = \mu\cdot g(t) \quad \Rightarrow \quad \sqrt{a}\cdot g\left(at\right) = \mu\cdot g(t).$$

The function $g_0(t) = t^\nu$, where $\nu \in \mathbb{C}$, satisfies

$$\left[\pi^+_{\chi_e}(a,b)\right](t^\nu) = \sqrt{a}(at)^\nu = a^{\nu+\frac{1}{2}}\cdot t^\nu.$$

Then, t^ν is the eigenfunction of $\pi^+_{\chi_e}(a,b)$.

The Mellin transform is given by

$$\tilde{g}(s) = \left[Mg\right](s) = \frac{1}{\sqrt{2\pi}}\int_0^\infty t^{-\frac{1}{2}-is}g(t)\,dt,$$

and the inverse Mellin transform is

$$\left[M^{-1}\tilde{g}\right](t) = g(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty t^{-\frac{1}{2}+is}\tilde{g}(s)\,ds,$$

where $\nu = -\frac{1}{2} + is$.

Therefore, we obtain

$$\begin{aligned}
\left[\pi^{+}_{\chi_e}(a,b)g\right](t) &= \sqrt{a}\cdot g\,(at)\\
&= \sqrt{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(at)^{-\frac{1}{2}+is}\tilde{g}(s)\,ds\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}a^{is}t^{-\frac{1}{2}+is}\tilde{g}(s)\,ds\\
&= \left[M^{-1}a^{is}\tilde{g}\right](t)\\
&= \left[M^{-1}a^{is}Mg\right](t)\\
&= \left[M^{-1}Ma^{is}g\right](t)\\
&= a^{is}g(t)\\
&= \chi_s(a,b)\cdot g(t),
\end{aligned}$$

where $\chi_s(a,b)=a^{is}$ is a complex character of the affine group. Hence, an irreducible component of the representation (5.1.7) is a one-dimensional representation which is the character χ_s.

5.1.3 Induction in stages

The subgroup $P=AN$ of $\mathrm{SL}_2(\mathbb{R})$ is defined as follows :

$$P=\left\{\begin{pmatrix} a & b\\ 0 & a^{-1}\end{pmatrix} : a\in\mathbb{R}\backslash\{0\}, b\in\mathbb{R}\right\}.$$

We can find a homomorphism $L: P\to \mathrm{Aff}$ such that $L^{-1}(a,b)$ has two elements one for $a>0$ and the other for $a<0$.

The induced representation of $\mathrm{SL}_2(\mathbb{R})$ from the trivial character of N (5.1.4) is reducible, and to investigate the irreducibility, we use induction in stages technique. That is [20, Chap. 2]

$$\mathrm{ind}_P^{\mathrm{SL}_2(\mathbb{R})}[\mathrm{ind}_N^P\chi_e]=\mathrm{ind}_N^{\mathrm{SL}_2(\mathbb{R})}[\chi_e].$$

First, the trivial character of N induces a representation of the affine group. We get the co-adjoint representation $\pi^{+}_{\chi_e}: L_2(\mathbb{R})\to L_2(\mathbb{R})$ which is given as follows:

$$\left[\pi^{+}_{\chi_e}\begin{pmatrix} a & b\\ 0 & a^{-1}\end{pmatrix} g\right](t)=\sqrt{a}g\,(at)\,. \qquad (5.1.8)$$

This representation is reducible, from subsection 5.1.2 we can decompose it into irreducible component. Therefore, we get the character as follows:

$$\chi_\kappa \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a^\kappa, \quad \kappa \in \mathbb{C}.$$

Hence, for the subgroup $P = AN$, the character is obtained by:

$$\chi_s \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = |a|^s \mathrm{sgn}^\epsilon(a), \quad \epsilon \in \{0,1\},\ s \in \mathbb{C}.$$

Next, the character χ_s induces a representation of $\mathrm{SL}_2(\mathbb{R})$ in the Hilbert space $L_2^{\chi_s}(\mathrm{SL}_2(\mathbb{R}))$ of functions with the P-covariance property

$$\begin{aligned} F_s \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \overline{\chi_s \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}} F \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \\ &= \overline{|a|^s \mathrm{sgn}^\epsilon(a)} F \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix}, \end{aligned} \tag{5.1.9}$$

this space is invariant under left $\mathrm{SL}_2(\mathbb{R})$-shifts.

A useful parametrization of $X = \mathrm{SL}_2(\mathbb{R})/P$ is obtained by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{c}{a} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, \quad a \neq 0$$

and for $b \neq 0$,

$$\begin{pmatrix} 0 & b \\ -b^{-1} & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} b^{-1} & -d \\ 0 & b \end{pmatrix}.$$

This decomposition shows that the coset $\mathrm{SL}_2(\mathbb{R})/P$ either contains matrix of the form $\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$ and , thus, can be parametrised by the real number u; or the coset of matrices $\begin{pmatrix} b^{-1} & -d \\ 0 & b \end{pmatrix}$ which represents ∞.

So, we can identify the space X by the projective real line $\mathbb{P}(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$.

The natural projection map is given by :

$$\mathbf{p} : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{P}(\mathbb{R})$$

where

$$\mathbf{p}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \frac{c}{a}.$$

Then let

$$\mathbf{s} : \mathbb{P}(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$$

where

$$\mathbf{s}(\omega) = \begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix},$$

such that

$$[\mathbf{p} \circ \mathbf{s}](\omega) = \omega \Rightarrow \mathbf{p} \circ \mathbf{s} = I,$$

where I is the identity map.

The map

$$\mathbf{r} : \mathrm{SL}_2(\mathbb{R}) \to P$$

will be

$$\mathbf{r}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}.$$

The left action on the homogeneous space $\mathrm{SL}_2(\mathbb{R})/P \cong \mathbb{P}(\mathbb{R})$ is defined in terms of $\mathbf{p}$ and $\mathbf{s}$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \omega \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \cdot \omega = \mathbf{p}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} * \mathbf{s}(\omega)\right) = \frac{a\omega - c}{d - b\omega}, \qquad (5.1.10)$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, $\omega \in \mathbb{P}(\mathbb{R})$ and $\cdot$ is the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{P}(\mathbb{R})$ from the left.

For $\omega \in \mathbb{P}(\mathbb{R})$, we have

$$\begin{aligned} \mathbf{r}\left(g^{-1} * \mathbf{s}(\omega)\right) &= \mathbf{r}\left(\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \omega & 1 \end{pmatrix}\right) \\ &= \mathbf{r}\left(\begin{pmatrix} d - b\omega & -b \\ -c + a\omega & a \end{pmatrix}\right) \qquad (5.1.11) \\ &= \begin{pmatrix} d - b\omega & -b \\ 0 & (d - b\omega)^{-1} \end{pmatrix} \end{aligned}$$

The representation $\pi_s : L_2(\mathbb{P}(\mathbb{R})) \to L_2(\mathbb{P}(\mathbb{R}))$, which is induced by the character χ_s, is given by :

$$[\pi_s(g)f]\,(\omega) = \overline{\chi_s(\mathbf{r}(g^{-1} * \mathbf{s}(\omega)))} f(g^{-1} \cdot \omega).$$

Substituting the calculation from (5.1.10) and (5.1.11), we obtain the formula

$$[\pi_s(g)f](\omega) = |d - b\omega|^s \mathrm{sgn}^\epsilon(d - b\omega) f\left(\frac{a\omega - c}{d - b\omega}\right), \tag{5.1.12}$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $f \in L_2(\mathbb{P}(\mathbb{R}))$.
This representation is unitary if and only if $s = -1 + it$, $t \in \mathbb{R}$.

The principal series representations of $\mathrm{SL}_2(\mathbb{R})$ are defined by

$$\pi_s^+ = \mathrm{ind}_P^{\mathrm{SL}_2(\mathbb{R})} \chi_s^+, \qquad \pi_s^- = \mathrm{ind}_P^{\mathrm{SL}_2(\mathbb{R})} \chi_s^-,$$

where

$$\chi_s^+ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = |a|^{-1+is}, \qquad s \in \mathbb{R},$$

and

$$\chi_s^- \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = |a|^{-1+is} \mathrm{sgn}(a), \qquad s \in \mathbb{R} \setminus \{0\}.$$

The induction representations to $\mathrm{SL}_2(\mathbb{R})$ from the characters χ_s^+ and χ_s^-, respectively, of the subgroup P are irreducible [20, § 2.5] and given by the formulas

$$[\pi_s^+(g)f](\omega) = |d - b\omega|^{-1+is} f(\frac{a\omega - c}{d - b\omega}), \tag{5.1.13}$$

and

$$[\pi_s^-(g)f](\omega) = \mathrm{sgn}(d - b\omega)|d - b\omega|^{-1+is} f(\frac{a\omega - c}{d - b\omega}). \tag{5.1.14}$$

5.2 Spherical functions

Recall that for a unitary representation ρ of a Lie group G, the matrix coefficient of ρ with respect to two vectors f and h in the Hilbert space $\mathcal{H}$ is the function on G defined by

$$\varphi_{f,h}(g) = \langle f, \rho(g)h \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the unitary inner product on the Hilbert space.

Let

$$A = \left\{ a_r = \begin{pmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{pmatrix} : r \in \mathbb{R} \right\}.$$

We define the spherical functions to be the matrix coefficients of the irreducible representation $\pi_s^+(a_r)$ with a unit vector fixed by the compact group K [17, § V.3.1; 33, § III.3].

The function $f_0(\omega) = 1$ is a unit vector and satisfies $[\pi_s^+(K)f](\omega) = f(\omega)$.

Then, for the unit vector f_0 and the subgroup A, the spherical function is

$$\begin{aligned} \varphi_s(a_r \cdot 0) &= \langle 1 \ , \ \pi_s^+(a_r)(1) \rangle_K \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \cosh r - \sinh r \cdot e^{i\theta} \right|^{-1-is} d\theta \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (\cosh 2r - \sinh 2r \cdot \cos\theta)^{\frac{-1-is}{2}} \, d\theta. \end{aligned} \tag{5.2.1}$$

The main properties of this function are listed as follows:

1. It is continuous and K-bi-invariant, that is

$$\varphi_s(k_1 g k_2) = \varphi_s(g),$$

   for all $k_1, k_2 \in K$ and $g \in \mathrm{SL}_2(\mathbb{R})$.

2. It is a radial eigenfunction of the Casimir operator when $n = 0$:

$$\tilde{\Delta} = \left(1 - |z|^2\right)^2 \frac{\partial^2}{\partial_z \partial_{\bar{z}}}. \tag{5.2.2}$$

3. $\varphi_s(e) = 1$, where e is the unit element of $\mathrm{SL}_2(\mathbb{R})$.

5.2.1 The behaviour of φ_s at ∞

Since $\varphi_s(a_{-r} \cdot 0) = \varphi_s(a_r \cdot 0) = \varphi_{-s}(a_r \cdot 0)$, we can replace s and r in the integral (5.2.1) by $-s$ and $-r$. The replacement is important, so we can study the asymptotic behaviour of φ_s.

Use the substitution

$$u = \tan \frac{1}{2}\theta, \qquad \frac{1}{2} d\theta = (1 + u^2)^{-1} du,$$

we obtain

$$\begin{aligned}\varphi_s(a_r\cdot 0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\cosh 2r+\sinh 2r\cdot\frac{1-u^2}{1+u^2}\right)^{\frac{-1+\mathrm{i}s}{2}}\cdot\frac{2}{1+u^2}\,du\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{(1+u^2)\cdot\cosh 2r+\sinh 2r\cdot(1-u^2)}{1+u^2}\right)^{\frac{-1+\mathrm{i}s}{2}}\cdot\frac{1}{1+u^2}\,du\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(e^{2r}+u^2e^{-2r}\right)^{\frac{-1+\mathrm{i}s}{2}}\left(1+u^2\right)^{\frac{-1-\mathrm{i}s}{2}}\,du\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}e^{(-1+\mathrm{i}s)r}\left(1+u^2e^{-4r}\right)^{\frac{-1+\mathrm{i}s}{2}}\left(1+u^2\right)^{\frac{-1-\mathrm{i}s}{2}}\,du\\ &= \frac{1}{\pi}e^{(-1+\mathrm{i}s)r}\int_{-\infty}^{\infty}\left(1+u^2e^{-4r}\right)^{\frac{-1+\mathrm{i}s}{2}}\left(1+u^2\right)^{\frac{-1-\mathrm{i}s}{2}}\,du.\end{aligned} \tag{5.2.3}$$

Then

$$e^{(1-\mathrm{i}s)r}\varphi_s(a_r\cdot 0)=\frac{1}{\pi}\int_{-\infty}^{\infty}\left(1+u^2e^{-4r}\right)^{\frac{-1+\mathrm{i}s}{2}}\left(1+u^2\right)^{\frac{-1-\mathrm{i}s}{2}}\,du. \tag{5.2.4}$$

The following lemma is similar to the theorem 4.5. in [16]. We start with the spherical function as well, but we find it differently using the covariant transform.

Lemma 5.2.1. *If* $\Re(\mathrm{i}s)>0$*, then the limit*

$$c(s)=\lim_{r\to\infty}e^{(1-\mathrm{i}s)r}\varphi_s(a_r\cdot 0) \tag{5.2.5}$$

exists and

$$c(s)=\frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{\mathrm{i}s}{2}\right)}{\Gamma\left(\frac{\mathrm{i}s+1}{2}\right)}. \tag{5.2.6}$$

Proof. Suppose $s_1=\Re(\mathrm{i}s)>0$, select $0<\varepsilon<\frac{1}{2}$ such that $\varepsilon s_1<\frac{1}{2}$, in absolute value, the integrand in (5.2.4) is bounded from above by

$$\frac{\left(1+u^2e^{-4r}\right)^{\frac{-1+s_1}{2}}}{(1+u^2)^{\frac{1+s_1}{2}}}\leqslant\frac{(1+u^2)^{-\varepsilon s_1+\frac{s_1}{2}}}{(1+u^2)^{\frac{1+s_1}{2}}}=\frac{1}{(1+u^2)^{\frac{1}{2}+\varepsilon s_1}},\qquad e^{4r}\geqslant 1, \tag{5.2.7}$$

and the last expression is integrable. Hence we can apply the dominated convergence theorem [18, § 2.3], to get

$$\lim_{r\to\infty}e^{(1-\mathrm{i}s)r}\varphi_s(a_r\cdot 0)=\frac{1}{\pi}\int_{-\infty}^{\infty}\left(1+u^2\right)^{\frac{-1-\mathrm{i}s}{2}}\,du. \tag{5.2.8}$$

Put $v = (1+u^2)^{-1}$, the integral converts to the Beta function:

$$\begin{aligned} \frac{1}{\pi}\int_0^1 v^{\frac{1+\mathrm{i}s}{2}} v^{-\frac{3}{2}}(1-v)^{-\frac{1}{2}}\,dv &= \frac{1}{\pi}\int_0^1 v^{\frac{\mathrm{i}s}{2}-1}(1-v)^{\frac{1}{2}-1}\,dv \\ &= \frac{1}{\pi}\frac{\Gamma\left(\frac{\mathrm{i}s}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{\mathrm{i}s+1}{2}\right)} \\ &= \frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{\mathrm{i}s}{2}\right)}{\Gamma\left(\frac{\mathrm{i}s+1}{2}\right)} \\ &= c(s). \end{aligned} \tag{5.2.9}$$

□

Thus, we obtain the asymptotic expansion for $\Re(\mathrm{i}s) > 0$ and $r \to \infty$:

$$\varphi_s(a_r \cdot 0) \sim c(s)e^{(\mathrm{i}s-1)r}. \tag{5.2.10}$$

The function $c(s)$ is called Harish-Chandra's c-function [16, Intro. § 4] that appears in the Plancherel measure. We mentioned this measure in the previous chapter. We will use the following formulas to calculate $|c(s)|^{-2}$:

$$\begin{aligned} |\Gamma(\mathrm{i}b)|^2 &= \frac{\pi}{b\sinh(\pi b)}, \\ \left|\Gamma\left(\frac{1}{2}+\mathrm{i}b\right)\right|^2 &= \frac{\pi}{\cosh(\pi b)}, \qquad b \in \mathbb{R}. \end{aligned} \tag{5.2.11}$$

If $s \in \mathbb{R}$, and since the gamma function is defined for all complex numbers except the non-positive integers, and by using the absolute value formulas (5.2.11), we have

$$\begin{aligned} |c(s)|^{-2} &= \pi\left|\Gamma\left(\frac{1+\mathrm{i}s}{2}\right)\right|^2\left|\Gamma\left(\frac{\mathrm{i}s}{2}\right)\right|^{-2} \\ &= \frac{s\pi}{2}\tanh\frac{\pi s}{2}. \end{aligned} \tag{5.2.12}$$

The measure $|c(s)|^{-2}ds$ is called the Plancherel measure [16, Intro. § 4].

If H_s is the corresponding space to the principal series representation π_s^+, the corresponding decomposition is

$$\int_{-\infty}^{\infty} H_s\,|c(s)|^{-2}ds = \int_{-\infty}^{\infty} H_s\,\frac{s\pi}{2}\tanh\frac{\pi s}{2}ds. \tag{5.2.13}$$

Here, we used the covariant transform to find the spherical function and we showed how the decay of this function at infinity is controlled by the Harish-Chandra's function. In the following chapter we will find the explicit inversion formula and the Plancherel measure such that each element is directly linked to the representation theory and the covariant transform.

Chapter 6

Covariant transform and inversion formula

The main purpose of this chapter is to find the inversion formula for the covariant transform $\mathcal{W}_{\varphi_0}^{\rho_k}$. This formula is equivalent to the decomposition of the unitary representation ρ_k into irreducible components. We consider an eigenvalue $1+s^2$ of the Casimir operator:

$$d\rho_k(C) = -4v^2\left(\partial_u^2 + \partial_v^2\right), \quad \text{where}\, k = 0.$$

To find the inversion formula, first we study the representations of $\mathrm{SL}_2(\mathbb{R})$, ρ_k and ρ_τ, induced from the complex characters of K and N respectively. Then, we find the induced covariant transform $\mathcal{W}_{\varphi_0}^{\rho_k}$ with N-eigenvector to obtain a transform in the space $L_2(\mathrm{SL}_2(\mathbb{R})/N)$. Thereafter, we compute the contravariant transform with K-eigenvector

$$\mathcal{M}_{\phi_0}^{\rho_\tau} : L_2(\mathrm{SL}_2(\mathbb{R})/N) \to L_2(\mathrm{SL}_2(\mathbb{R})/K).$$

We find these transforms using the representation itself like in Gelfand's approach [13], but the eigenvectors are selected by the derived representation as in Bargmann's works [6]. Finally, we use the relation between the covariant and contravariant transform to find the inversion formula. Thus, the original contribution is using the covariant transform to find the inversion formula with eigenvectors selected by the derived representation. This new method will be easier to adopt for problems of decomposing a system into elementary bits in theoretical physics. Also, it is not restricted to $SL_2(\mathbb{R})$, it can be successfully used for many other cases.

6.1 Introduction

Action of $SL_2(\mathbb{R})$ by linear-fractional transformation on complex numbers produces isometrical motions of the Lobachevsky geometry. It is less known that there are related actions of $SL_2(\mathbb{R})$ on dual and double numbers which have the form $z = x + \iota y$, $\iota^2 = 0$ or $\iota^2 = 1$, correspondingly. We write ε and j instead of ι within dual and double numbers, respectively.
Three possible values $-1, 0$ and 1 of $\sigma := \iota^2$ will be referred to elliptic, parabolic and hyperbolic cases, respectively.

A generic cycle [30, § 4.2] is the set of points $(u, v) \in \mathbb{R}^2$ defined for all values of σ by the equation

$$k(u^2 - \sigma v^2) - 2lu - 2nv + m = 0. \tag{6.1.1}$$

This equation is represented by a point (k, l, n, m) from a projective space $\mathbb{P}^3$, since for a scaling factor $\lambda \neq 0$, the point $(\lambda k, \lambda l, \lambda n, \lambda m)$ defines an equation equivalent to (6.1.1). We call $\mathbb{P}^3$ the cycle space and refer to the initial $\mathbb{R}^2$ as the point space. In order to obtain a connection with the Möbius action, we arrange numbers (k, l, n, m) into the matrix [30, Definition 4.11]

$$C_{\breve{\sigma}} = \begin{pmatrix} l + \breve{\iota} n & -m \\ k & -l + \breve{\iota} n \end{pmatrix}. \tag{6.1.2}$$

The values of $\breve{\sigma} := \breve{\iota}^2$ are $-1, 0$ or 1 may be chosen to be independent of the values of σ.

Theorem 6.1.1. [30, Theorem 4.13] *Let a matrix* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$*, define a Möbius transformation*

$$g : (u + \iota v) \to \frac{a(u + \iota v) + b}{c(u + \iota v) + d}. \tag{6.1.3}$$

Then the image $\tilde{C}_{\breve{\sigma}}$ of a cycle $C_{\breve{\sigma}}$ under transformation with $g \in \mathrm{SL}_2(\mathbb{R})$ is given by similarity of the matrix (6.1.2):

$$\tilde{C}_{\breve{\sigma}} = gC_{\breve{\sigma}}g^{-1}. \tag{6.1.4}$$

Definition 6.1.2. [30, Definition 5.11] For two cycles C and C_1, define the cycles product by:

$$\langle C, C_1 \rangle = -\operatorname{tr}(C\bar{C}_1), \tag{6.1.5}$$

where tr denotes the trace of a matrix.

We can find the explicit expression of the cycle product (6.1.5) with $\sigma = -1, 0$ and 1:

$$\langle C, C_1 \rangle = km_1 + k_1 m - 2ll_1 + 2\sigma nn_1, \tag{6.1.6}$$

where $C = (k, l, n, m)$ and $C_1 = (k_1, l_1, n_1, m_1)$.

Definition 6.1.3. [11, Chap. 3, § 1.2] On the hyperbolic plane one can define circles of infinitely large radius (horocycles), which are the limits of non-Euclidean circles as the center and the radius of these circles consistently tend to infinity. In the Lobachevsky model, the horocycles are represented either as Euclidean circles tangent to the real axis or as lines parallel to the real axis.

The horizontal line $v-1=0$ as a cycle is represented by the matrix $\begin{pmatrix} -\frac{i}{2} & -1 \\ 0 & -\frac{i}{2} \end{pmatrix}$.
This line is invariant under the subgroup $N = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, that is

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{i}{2} & -1 \\ 0 & -\frac{i}{2} \end{pmatrix} \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{i}{2} & -1 \\ 0 & -\frac{i}{2} \end{pmatrix}.$$

Thus all horocycles obtained by $\mathrm{SL}_2(\mathbb{R})$ action are parametrized by points of the homogeneous space $\mathrm{SL}_2(\mathbb{R})/N$.

The image of $v-1=0$ under the lower triangular matrix $\begin{pmatrix} \xi_1 & 0 \\ \xi_2 & \frac{1}{\xi_1} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ is

$$\begin{pmatrix} \xi_1 & 0 \\ \xi_2 & \frac{1}{\xi_1} \end{pmatrix} \begin{pmatrix} -\frac{i}{2} & -1 \\ 0 & -\frac{i}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\xi_1} & 0 \\ -\xi_2 & \xi_1 \end{pmatrix} = \begin{pmatrix} \xi_1\xi_2 - \frac{i}{2} & -\xi_1^2 \\ \xi_2^2 & -\xi_1\xi_2 - \frac{i}{2}, \end{pmatrix} \tag{6.1.7}$$

that is, cycle $(\xi_2^2, \xi_1\xi_2, \frac{1}{2}, \xi_1^2)$ with the equation

$$\begin{aligned} &\xi_2^2u^2 + \xi_2^2v^2 - 2\xi_1\xi_2u - v + \xi_1^2 = 0 \\ \Leftrightarrow\ &(\xi_2^2u^2 - 2\xi_1\xi_2u + \xi_1^2) + \xi_2^2v^2 = v \\ \Leftrightarrow\ &(\xi_2u - \xi_1)^2 + (\xi_2v)^2 = v \\ \Leftrightarrow\ &|(\xi_2u - \xi_1) + i\xi_2v|^2 = v \\ \Leftrightarrow\ &|\xi_2(u + iv) - \xi_1|^2 = v \\ \Leftrightarrow\ &|\xi_2z - \xi_1|^2 = v, \quad z = u + iv, \quad (\xi_1, \xi_2) \in \mathbb{R}^2 \setminus \{0\}. \end{aligned} \tag{6.1.8}$$

Therefore, the point (ξ_1, ξ_2) of the parabolic upper half plane $\mathrm{SL}_2(\mathbb{R})/N$ parametrizes the space of horocycles. Denote by $h(\xi) = h(\xi_1, \xi_2)$ the horocycle given by (6.1.8).

Every horocycle has a unique common point with the real axis, which is called the center of the horocycle. Horocycles with common center are said to be parallel. Note that a horocycle $h(\xi_1, \xi_2)$ is tangent to the real axis at the point $\frac{\xi_1}{\xi_2}$, hence every parallel horocycle is of the form $\{h(\lambda\xi_1, \lambda\xi_2) : 0 < \lambda < \infty\}$ for some chosen (ξ_1, ξ_2) [11, Chap. 3, § 1.2].

Now, in order to find the invariant distance of a point z in the upper half plane to the horocycle $h(\xi)$, first we calculate the distance from a point $z_1 = (u, v) \in h(\lambda\xi_1, \lambda\xi_2)$ to a horocycle $h(\xi_1, \xi_2)$. The point $z_2 = (u, \lambda^{-2}v)$ is in the horocycle $h(\xi_1, \xi_2)$:

$$|\lambda\xi_2z - \lambda\xi_1|^2 = v \quad \Rightarrow \quad |\xi_2z - \xi_1|^2 = \lambda^{-2}v.$$

Note that the points z_1 and z_2 are on the same vertical line, thus the distance between them is

$$\begin{aligned} \left| \int_{v\lambda^{-2}}^{v} \frac{1}{y}\, dy \right| &= \left| \log v - \log v\lambda^{-2} \right| \\ &= 2\left| \log \lambda \right|. \end{aligned} \tag{6.1.9}$$

Thus, all points of the horocycle $h(\lambda\xi_1, \lambda\xi_2)$ are placed at the same distance $2\left|\log \lambda\right|$ from the parallel horocycle $h(\xi_1, \xi_2)$. Then, the signed distance from a point $z \in h(\lambda\xi)$ to a horocycle $h(\xi)$ is

$$\begin{aligned} \varrho(z; \xi) &= -2\log \lambda \\ &= \log \left(v^{-1}|\xi_2z - \xi_1|^2 \right), \end{aligned} \tag{6.1.10}$$

because $\lambda^{-2} = v^{-1}|\xi_2z - \xi_1|^2$.

6.2 Induced representations of the group $\mathrm{SL}_2(\mathbb{R})$

1. For the subgroup $K = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} : t \in \mathbb{R} \right\}$, the homogeneous space $\mathrm{SL}_2(\mathbb{R})/K$ are parametrised by points of the upper half-plane $\mathbb{H}^+$. The respective maps are:

$$\mathbf{p}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{bd+ac}{c^2+d^2}, \frac{1}{c^2+d^2} \right),$$
$$\mathbf{s}(u,v) = \frac{1}{\sqrt{v}} \begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix}, \qquad (6.2.1)$$
$$\mathbf{r}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{c^2+d^2}} \begin{pmatrix} d & -c \\ c & d \end{pmatrix}.$$

The decomposition defined by the formula $g = \mathbf{s}(\mathbf{p}(g))\mathbf{r}(g)$ takes the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{c^2+d^2} \begin{pmatrix} 1 & bd+ac \\ 0 & c^2+d^2 \end{pmatrix} \begin{pmatrix} d & -c \\ c & d \end{pmatrix}. \qquad (6.2.2)$$

The $\mathrm{SL}_2(\mathbb{R})$-action defined by the formula $g \cdot x = \mathbf{p}(g * \mathbf{s}(x))$ takes the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (u,v) \mapsto \left(\frac{(au+b)(cu+d)+cav^2}{(cu+d)^2+(cv)^2}, \frac{v}{(cu+d)^2+(cv)^2} \right). \qquad (6.2.3)$$

This map preserves the upper half plane $v > 0$. We can simplify this map as a linear-fractional transformation with the complex number unit $i^2 = -1$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : w \mapsto \frac{aw+b}{cw+d}, \quad \text{where } w = u + iv. \qquad (6.2.4)$$

The left invariant measure on the upper half plane $\mathbb{H}^+$ is equal to

$$d\mu(w) = \frac{du dv}{v^2}, \quad w = u + iv. \qquad (6.2.5)$$

The character $\chi_k \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = e^{-ikt}$, $k \in \mathbb{Z}$ of K, induces a linear representation ρ_k on the space of square integrable functions, which is given by the formula:

$$[\rho_k(g)f](w) = \overline{\chi_\tau(\mathbf{r}(g^{-1} * \mathbf{s}(w)))} f(g^{-1} \cdot w), \qquad (6.2.6)$$

where $g \in \mathrm{SL}_2(\mathbb{R})$ and $w \in \mathrm{SL}_2(\mathbb{R})/K$.

By simple calculation we obtain [25, § 8]:

$$[\rho_k(g)f](w) = \frac{|a-cw|^k}{(a-cw)^k} f\left(\frac{dw-b}{a-cw}\right), \quad \text{where}\, w = u+iv. \tag{6.2.7}$$

We consider the basis in the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$:

$$\mathcal{A} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{6.2.8}$$

They generate one-parameter subgroup of $\mathrm{SL}_2(\mathbb{R})$:

$$e^{t\mathcal{A}} = \begin{pmatrix} e^{-\frac{t}{2}} & 0 \\ 0 & e^{\frac{t}{2}} \end{pmatrix}, \; e^{tB} = \begin{pmatrix} \cosh\frac{t}{2} & \sinh\frac{t}{2} \\ \sinh\frac{t}{2} & \cosh\frac{t}{2} \end{pmatrix}, \; e^{tZ} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

The derived representations are:

$$\begin{aligned}
d\rho_k^{\mathcal{A}} &= w\partial_w + \bar{w}\partial_{\bar{w}} && (6.2.9)\\
&= u\partial_u + v\partial_v, && (6.2.10)\\
d\rho_k^{B} &= \frac{1}{4}k(w-\bar{w})\cdot I - \frac{1}{2}(1-w^2)\partial_w - \frac{1}{2}(1-\bar{w}^2)\partial_{\bar{w}} && (6.2.11)\\
&= \frac{1}{2}kvi\cdot I - \frac{1}{2}(1-u^2+v^2)\partial_u + uv\partial_v, && (6.2.12)\\
d\rho_k^{Z} &= -\frac{1}{2}k(w-\bar{w})\cdot I - (1+w^2)\partial_w - (1+\bar{w}^2)\partial_{\bar{w}} && (6.2.13)\\
&= -\mathrm{i}kv\cdot I - (1+u^2-v^2)\partial_u - 2uv\partial_v, && (6.2.14)
\end{aligned}$$

where $w = u + \mathrm{i}v$, $\partial_w = \frac{1}{2}(\partial_u - \mathrm{i}\partial_v)$ and $\partial_{\bar{w}} = \frac{1}{2}(\partial_u + \mathrm{i}\partial_v)$.

The Casimir operator is:

$$\begin{aligned}
d\rho_k(C) &= d\rho_\tau^{Z^2-4\mathcal{A}^2-4B^2}\\
&= 4\mathrm{i}kv\partial_u - 4v^2(\partial_u^2+\partial_v^2).
\end{aligned} \tag{6.2.15}$$

2. For the subgroup $N' = \left\{ \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} : n \in \mathbb{R} \right\}$, the homogeneous space $\mathrm{SL}_2(\mathbb{R})\,/N'$

can be identified with the upper half plane. The respective maps are:

$$\begin{aligned}
\mathbf{p}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \left(\frac{b}{d}, \frac{1}{d^2}\right), \\
\mathbf{s}(u,v) &= \frac{1}{\sqrt{v}}\begin{pmatrix} v & u \\ 0 & 1 \end{pmatrix}, \\
\mathbf{r}\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ \frac{c}{d} & 1 \end{pmatrix}.
\end{aligned} \tag{6.2.16}$$

The maps $\mathbf{p}$ and $\mathbf{s}$ produce the following decomposition $g = \mathbf{s}(\mathbf{p}(g))\mathbf{r}(g)$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{d^2}\begin{pmatrix} 1 & bd \\ 0 & d^2 \end{pmatrix}\begin{pmatrix} d & 0 \\ c & d \end{pmatrix}, \quad \text{where } d \neq 0. \tag{6.2.17}$$

The action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathrm{SL}_2(\mathbb{R})\,/\acute{N}$ defined by the formula $g \cdot x = \mathbf{p}(g * \mathbf{s}(x))$ takes the form:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : (u,v) \mapsto \left(\frac{au+b}{cu+d}, \frac{v}{(cu+d)^2}\right). \tag{6.2.18}$$

It preserves the upper half plane $v > 0$. We can rewrite this map as a linear-fractional transformation with the dual number unit $\varepsilon^2 = 0$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : w \mapsto \frac{aw+b}{cw+d}, \quad \text{where } w = u + \varepsilon v. \tag{6.2.19}$$

The complex character χ_τ of N' is:

$$\chi_\tau \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} = e^{-2\pi \mathrm{i} \tau n}, \quad \text{where } \tau \in \mathbb{R}.$$

This character induces a linear representation ρ_τ on the space of square integrable functions, which is given by the formula:

$$[\rho_\tau(g)f](w) = \overline{\chi_\tau(\mathbf{r}(g^{-1} * \mathbf{s}(w)))} f(g^{-1} \cdot w), \tag{6.2.20}$$

where $g \in \mathrm{SL}_2(\mathbb{R})$ and $w \in \mathrm{SL}_2(\mathbb{R})/\acute{N}$.

A direct calculation shows that [25, § 8]:

$$[\rho_\tau(g)f](w) = \exp\left(-2\pi \mathrm{i} \frac{\tau c v}{a - cu}\right) f\left(\frac{dw - b}{a - cw}\right), \tag{6.2.21}$$

where $w = u + \varepsilon v$ and $f \in L_2(\mathbb{H}^+, d\mu)$.

This representation is unitary on the space of functions on the upper half plane of dual numbers.

The derived representations of the elements $\mathcal{A}$, B and Z (6.2.8) of $\mathfrak{sl}_2(\mathbb{R})$ are:

$$
\begin{aligned}
d\rho_\tau^{\mathcal{A}} &= w\partial_w + \bar{w}\partial_{\bar{w}} && (6.2.22)\\
&= u\partial_u + v\partial_v, && (6.2.23)\\
d\rho_\tau^{B} &= -\pi \mathrm{i} v\tau \cdot I - \frac{1}{2}(1-w^2)\partial_w - \frac{1}{2}(1-\bar{w}^2)\partial_{\bar{w}} && (6.2.24)\\
&= -\pi \mathrm{i} v\tau \cdot I - \frac{1}{2}(1-u^2)\partial_u + uv\partial_v, && (6.2.25)\\
d\rho_\tau^{Z} &= 2\pi \mathrm{i} v\tau \cdot I - (1+w^2)\partial_w - (1+\bar{w}^2)\partial_{\bar{w}} && (6.2.26)\\
&= 2\pi \mathrm{i} v\tau \cdot I - (1+u^2)\partial_u - 2uv\partial_v, && (6.2.27)
\end{aligned}
$$

where $w = u + \varepsilon v$, $\partial_w = \frac{1}{2}(\partial_u + \frac{1}{\varepsilon}\partial_v)$ and $\partial_{\bar{w}} = \frac{1}{2}(\partial_u - \frac{1}{\varepsilon}\partial_v)$.

The Casimir operator is:

$$
\begin{aligned}
d\rho_\tau(C) &= d\rho_\tau^{Z^2 - 4\mathcal{A}^2 - 4B^2}\\
&= -8\pi \mathrm{i} v\tau\partial_u - 4v^2\partial_v^2.
\end{aligned} \qquad (6.2.28)
$$

In the following we will find some eigenfunctions, and the special role of them will become obvious later.

6.2.1 Joint eigenvector of $d\rho_k(C)$ with $d\rho_k^N$

First, we calculate the eigenvector of the derived representation $d\rho_k^N$:

$$
\left[d\rho_k^N f\right](w,\bar{w}) = -\partial_u f(w,\bar{w}) = -(\partial_w + \partial_{\bar{w}}) f(w,\bar{w}) = 0. \qquad (6.2.29)
$$

The solution is $f(w,\bar{w}) = \phi(v)$, where ϕ is an arbitrary function.

Then, we solve the differential equation

$$
d\rho_k(C)\phi(v) = (1+s^2)\phi(v), \quad s \in \mathbb{R},
$$

where $d\rho_k(C)$ is the Casimir operator(6.2.15), and $k = 0$ for simplicity.

This equation becomes

$$
-4v^2\frac{d^2\phi}{dv^2}(v) - (1+s^2)\phi(v) = 0. \qquad (6.2.30)
$$

It is a Cauchy-Euler equation, therefore the solution takes the form $\phi(v) = v^m$. Differentiating gives $\frac{d^2\phi}{dv^2}(v) = m(m-1)v^{m-2}$, and substituting into (6.2.30) leeds to

$$\begin{aligned}
-4m(m-1)v^m - (1+s^2)v^m &= 0 \\
\Rightarrow -4m(m-1) - (1+s^2) &= 0 \\
\Rightarrow m = \frac{1 \pm is}{2}.
\end{aligned}$$

Hence, the set of fundamental solution is

$$\left\{ v^{\frac{1+is}{2}}, v^{\frac{1-is}{2}} \right\}. \tag{6.2.31}$$

6.2.2 Joint eigenvector of $d\rho_\tau(C)$ with $d\rho_\tau^Z$

To begin, we look for an eigenvector of the derived representation $d\rho_\tau^Z$ (6.2.27) with $\tau = 0$. To do that, we solve the equation $d\rho_\tau^Z f(w, \bar{w}) = 0$ using the method of characteristics:

$$\frac{du}{1+u^2} = \frac{dv}{2uv} = \frac{df}{2\pi \mathrm{i}\tau v f}.$$

$$\frac{du}{1+u^2} = \frac{dv}{2uv} \qquad \Rightarrow \qquad \frac{2u du}{1+u^2} = \frac{dv}{v} \qquad \Rightarrow \qquad 2C_1 = \frac{v}{1+u^2}.$$

We need to obtain another integral curve which involves f. Since $\tau = 0$, then

$$\frac{dv}{2uv} = \frac{df}{2\pi \mathrm{i}\tau v f} \qquad \Rightarrow \qquad \frac{df}{f} = 0 \qquad \Rightarrow \qquad C_2 = f.$$

Hence, the general solution is of the form $C_2 = \psi(C_1)$, that is

$$f(w, \bar{w}) = \psi\left(\frac{v}{2(1+u^2)}\right), \qquad w = u + \varepsilon v, \tag{6.2.32}$$

where ψ is an arbitrary function. To specify this function, we solve the equation

$$d\rho_\tau(C)\psi\left(\frac{v}{2(1+u^2)}\right) = (1+s^2)\psi\left(\frac{v}{2(1+u^2)}\right), \tag{6.2.33}$$

where $d\rho_\tau(C)$ is the Casimir operator(6.2.28). This equation turns into

$$-4\frac{v^2}{(1+u^2)^2}\frac{d^2\psi}{dv^2}\left(\frac{v}{2(1+u^2)}\right) - (1+s^2)\psi\left(\frac{v}{2(1+u^2)}\right) = 0. \tag{6.2.34}$$

Using the substitution $t = \frac{v}{2(1+u^2)}$, then we obtain

$$-4t^2\frac{d^2\psi}{dt^2}(t) - (1+s^2)\psi(t) = 0. \tag{6.2.35}$$

It is a Cauchy-Euler equation, so let $\psi(t) = t^m$ and substitute in the differential equation(6.2.35), then $m = \frac{1+is}{2}$ and $m = \frac{1-is}{2}$ are two distinct possible values of m. Therefore, the set of fundamental solution is $\left\{t^{\frac{1+is}{2}}, t^{\frac{1-is}{2}}\right\}$.
Finally, the set of fundamental solution for the equation(6.2.34) is

$$\left\{\left(\frac{v}{2(1+u^2)}\right)^{\frac{1+is}{2}}, \left(\frac{v}{2(1+u^2)}\right)^{\frac{1-is}{2}}\right\}. \tag{6.2.36}$$

6.3 Induced Covariant transform

Definition 6.3.1. [27, § 5.1] Let H be a closed subgroup of G and $f \in \mathcal{H}$ such that

$$\rho(h)f = \chi(h)f \tag{6.3.1}$$

for some character χ of H where $h \in H$ and ρ is a unitary representation of a Lie group G in a Hilbert space $\mathcal{H}$. For a section $\mathbf{s}$ from G/H to G, the induced covariant transform $\mathcal{W}_f^\rho$ is a map from the Hilbert space $\mathcal{H}$ to a space of function on G/H given as follows:

$$\mathcal{W}_f : v \mapsto \tilde{v}(x) = \langle v, \rho(\mathbf{s}(x))f\rangle, \qquad x \in G/H. \tag{6.3.2}$$

The map $v \mapsto \tilde{v}(x) = \tilde{v}(\mathbf{s}(x))$ intertwines ρ with the representation ρ_χ in a certain function space on G/H induced by the character χ of H. That is,

$$\rho_\chi \circ \mathcal{W}_f^\rho = \mathcal{W}_f^\rho \circ \rho. \tag{6.3.3}$$

Example 6.3.2. We will find the induced wavelet transform with N-eigenvector that intertwines respectively the representation ρ_k (6.2.7) where $k = 0$ with the representation ρ_τ (6.2.21). We take the fiducial vector $\varphi_0(w, \bar{w}) = v^{\frac{1+is}{2}}$ (6.2.31) which would be the eigenvector for the representation $\rho_k \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$. That is

$$\rho_k \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \varphi_0 = \chi_\tau \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \varphi_0. \tag{6.3.4}$$

Then, the corresponding induced covariant transform is:

$$\begin{aligned}
\left[\mathcal{W}^{\rho_k}_{\varphi_0} f\right](\xi) &= \langle f, \rho_k(\mathbf{s}(\xi_1,\xi_2))\varphi_0\rangle \\
&= \left\langle f, \rho_k \begin{pmatrix} \xi_1 & 0 \\ \xi_2 & \frac{1}{\xi_1} \end{pmatrix} v^{\frac{1+is}{2}} \right\rangle \\
&= \int_{\mathbb{H}^+} f(w) \left(\frac{v}{|\xi_1 - \xi_2 w|^2}\right)^{\frac{1-is}{2}} \frac{dudv}{v^2} \\
&= \int_{\mathbb{H}^+} f(w) \exp\left\{-\frac{1-is}{2}\varrho(w;\xi)\right\} \frac{dudv}{v^2}, \qquad w = u + iv,
\end{aligned} \tag{6.3.5}$$

where $\varrho(w;\xi)$ (6.1.10) is the signed distance from the point w to the horocycle $h(\xi)$, $\xi = (\xi_1, \xi_2)$.

6.4 Contravariant transform

Definition 6.4.1. [26, § 5] Let ρ be a unitary square integrable representation of the group $\mathrm{SL}_2(\mathbb{R})$ on a Hilbert space $\mathcal{H}$ and H be a closed subgroup of $\mathrm{SL}_2(\mathbb{R})$. Let $X = \mathrm{SL}_2(\mathbb{R})/H$ be a homogeneous space with an invariant measure dx. We define the function $w_{\mathbf{s}(x)} = \rho(\mathbf{s}(x))w_0$, where $w_0 \in \mathcal{H}$ and $\mathbf{s}$ is a section map. The contravariant transform $\mathcal{M}^\rho_{w_0}$ is a map $L_2(X) \to \mathcal{H}$ defined by

$$\mathcal{M}^\rho_{w_0} f = \int_X f(x) w_{\mathbf{s}(x)}\, dx, \qquad x \in X. \tag{6.4.1}$$

For an admissible vector w_0 [4, Definition 8.1.1], the contravariant transform in this setup is known as a reconstruction formula.

Example 6.4.2. For the representation ρ_τ (6.2.21) with $\tau = 0$, we take the K-eigenvector $\phi_0(w,\bar{w}) = \left(\frac{v}{2+2u^2}\right)^{\frac{1+is}{2}}$ (6.2.36). Then, the corresponding contravariant transform is:

$$\begin{aligned}
\left[\mathcal{M}^{\rho_\tau}_{\phi_0} f\right](w) &= \int_{h(\xi)} f(\xi)\rho_\tau(\mathbf{s}(\xi_1,\xi_2))\phi_0\, d\xi \\
&= \int_{\mathbb{R}^2} f(\xi)\rho_\tau\left(\begin{pmatrix} \xi_1 & 0 \\ \xi_2 & \frac{1}{\xi_1} \end{pmatrix}\right) \left(\frac{v}{2(1+u^2)}\right)^{\frac{1+is}{2}} d\xi_1 d\xi_2 \\
&= \int_{\mathbb{R}^2} f(\xi) \left(\frac{v\xi_1^2}{2\xi_1^2(\xi_1 - \xi_2 u)^2 + 2u^2}\right)^{\frac{1+is}{2}} d\xi_1 d\xi_2, \qquad w = u + \varepsilon v.
\end{aligned} \tag{6.4.2}$$

Proposition 6.4.3. [26, Prop. 6.4] *Contravariant transform $\mathcal{M}_{w_0}$ intertwines left regular representation Λ on $L_2(\mathrm{SL}_2(\mathbb{R}))$ and ρ:*

$$\mathcal{M}_{w_0}\Lambda(g) = \rho(g)\mathcal{M}_{w_0}. \tag{6.4.3}$$

Let ρ be an irreducible square integrable representation and φ_0 and w_0 be admissible vectors. The covariant transform intertwines ρ and the left regular representation Λ:

$$\mathcal{W}_{\varphi_0}\rho(g) = \Lambda(g)\mathcal{W}_{\varphi_0}.$$

Combining with (6.4.3), we see that the composition $\mathcal{M}_{w_0} \circ \mathcal{W}_F$ intertwines ρ with itself. That is,

$$(\mathcal{M}_{w_0} \circ \mathcal{W}_{\varphi_0}) \circ \rho(g) = \rho(g) \circ (\mathcal{M}_{w_0} \circ \mathcal{W}_{\varphi_0}). \tag{6.4.4}$$

Thus, from the Schur's lemma we have the relation

$$\mathcal{M}_{w_0} \circ \mathcal{W}_{\varphi_0} = kI, \tag{6.4.5}$$

for some constant $k \in \mathbb{C}$.

On the other hand, and from the orthogonality relations [4, § 8.2]:

$$\langle \mathcal{W}_{\varphi_1} f_1, \mathcal{W}_{\varphi_2} f_2 \rangle = \langle f_1, f_2 \rangle \langle C\varphi_2, C\varphi_1 \rangle, \tag{6.4.6}$$

where C is a unique positive, self adjoint and invertible operator in the Hilbert space. This operator is known as Duflo-Moore operator.

If $f_1, f_2 \in \mathcal{H}$, we have

$$\begin{aligned}\langle \mathcal{M}_{w_0} \circ \mathcal{W}_{\varphi_0} f_1, f_2 \rangle &= \langle \mathcal{W}_{\varphi_0} f_1, \mathcal{W}_{w_0} f_2 \rangle \\ &= \langle f_1, f_2 \rangle \langle Cw_0, C\varphi_0 \rangle \\ &= \langle \langle C\varphi_0, Cw_0 \rangle f_1, f_2 \rangle.\end{aligned} \tag{6.4.7}$$

Thus

$$\mathcal{M}_{w_0} \circ \mathcal{W}_{\varphi_0} = \langle C\varphi_0, Cw_0 \rangle I. \tag{6.4.8}$$

And for non-orthogonal vectors w_0 and φ_0, we get $\langle C\varphi_0, Cw_0 \rangle = k \neq 0$.

6.4.1 Inversion formula

We will find the inversion formula for the covariant transform (6.3.5) from the relation (6.4.8) with the contravariant transform $\mathcal{M}_{\phi_0}^{\rho_\tau}$ (6.4.2):

$$\begin{aligned} f(w) &= \frac{1}{\langle C\varphi_0, C\phi_0\rangle}\left[\mathcal{M}_{\phi_0}^{\rho_\tau}\left(\mathcal{W}_{\varphi_0}^{\rho_k} f\right)\right](w) \\ &= \frac{1}{\langle C\varphi_0, C\phi_0\rangle}\int_{\mathbb{R}^2} \mathcal{W}_{\varphi_0}^{\rho_k} f(\xi)\left(\frac{v\xi_1^2}{2\xi_1^2(\xi_1-\xi_2 u)^2+2u^2}\right)^{\frac{1+is}{2}} d\xi_1 d\xi_2. \end{aligned} \tag{6.4.9}$$

The function $h(s) = \langle C\varphi_0, C\phi_0\rangle$ must be explicitly identified.

The following result is an inversion formula similar to that in Gelfand's book [11, Chap. 3, Theorem 3.2], but with a difference in the eigenvector.

Theorem 6.4.4. *For $f \in L_2(\mathbb{R}_+^2, d\mu)$, we have the inversion formula*

$$f(w) \;=\; \frac{1}{2\pi^2}\int_{\mathbb{R}} s\tanh\frac{\pi s}{2}\left(\int_{\mathbb{R}^2} \mathcal{W}_{\varphi_0}^{\rho_k} f(\xi)\rho_\tau(\mathbf{s}(\xi_1,\xi_2))\phi_0(w)\, d\xi_1 d\xi_2\right) ds, \tag{6.4.10}$$

where

$$\rho_\tau(\mathbf{s}(\xi_1,\xi_2))\phi_0(w) = \left(\frac{v\xi_1^2}{2\xi_1^2(\xi_1-\xi_2 u)^2+2u^2}\right)^{\frac{1+is}{2}},$$

and $\mathcal{W}_{\varphi_0}^{\rho_k} f$ is the covariant transform (6.3.5).

Proof. To find the inversion formula for the covariant transform (6.3.5), we need to identify $\langle C\varphi_0, Cw_0\rangle$ in (6.4.9):

$$\begin{aligned} \langle C\varphi_0, Cw_0\rangle &= \frac{1}{f(w)}\left[\mathcal{M}_{\phi_0}^{\rho_\tau}\left(\mathcal{W}_{\varphi_0}^{\rho_k} f\right)\right](w) \\ &= \frac{1}{f(w)}\int_{\mathbb{R}^2} \mathcal{W}_{\varphi_0}^{\rho_k} f(\xi)\left(\frac{v\xi_1^2}{2\xi_1^2(\xi_1-\xi_2 u)^2+2u^2}\right)^{\frac{1+is}{2}} d\xi_1 d\xi_2. \end{aligned} \tag{6.4.11}$$

To identify this function, it is enough to compute the composition of the covariant transform and the contravariant transform for one particular function. Let

$$f_0(w) = \frac{v^{\mathrm{is}+\frac{1}{2}}(1+v)^{-\frac{\mathrm{is}+1}{2}}}{1+u^2} \in L_2(\mathbb{R}_+^2, d\mu(w)), \quad 0 < \Re(\mathrm{is}) < 1. \tag{6.4.12}$$

We compute the covariant transform for the function f_0:

$$\mathcal{W}_{\varphi_0}^{\rho_k} f_0(\xi) = \int_0^\infty \int_{-\infty}^\infty \frac{v^{\mathrm{is}+\frac{1}{2}}(1+v)^{-\frac{\mathrm{is}+1}{2}}}{1+u^2} v^{\frac{1-\mathrm{is}}{2}}\left(\frac{1}{|\xi_1-\xi_2 w|^2}\right)^{\frac{1-\mathrm{is}}{2}} \frac{du\,dv}{v^2}. \tag{6.4.13}$$

And for $\xi = (\xi_1, 0)$, this value becomes

$$
\begin{aligned}
\mathcal{W}^{\rho_k}_{\varphi_0} f_0(\xi) &= \int_0^\infty \int_{-\infty}^\infty \frac{v^{\mathrm{i}s+\frac{1}{2}}(1+v)^{-\frac{\mathrm{i}s+1}{2}}}{1+u^2} v^{\frac{1-\mathrm{i}s}{2}-2} \left(\xi_1^2\right)^{\frac{\mathrm{i}s-1}{2}} \, dudv \\
&= \left(\xi_1^2\right)^{\frac{\mathrm{i}s-1}{2}} \int_0^\infty v^{\frac{\mathrm{i}s}{2}-1}(1+v)^{-\frac{\mathrm{i}s+1}{2}} \left(\int_{-\infty}^\infty \frac{1}{1+u^2}\, du\right) dv \\
&= \left(\xi_1^2\right)^{\frac{\mathrm{i}s-1}{2}} \int_0^\infty v^{\frac{\mathrm{i}s}{2}-1}(1+v)^{-\frac{\mathrm{i}s+1}{2}} \pi \, dv \\
&= \pi e^{\frac{\mathrm{i}s-1}{2}\varrho(\mathrm{i};\xi)} B(\frac{\mathrm{i}s}{2}, \frac{1}{2}),
\end{aligned} \tag{6.4.14}
$$

where $\varrho(\mathrm{i};\xi)$ is the distance from the point i to the horocycle $h(\xi)$ and B is the Beta function.

Then, we find the function (6.4.11) with $(\xi_1, \xi_2) = (\xi_1, 1)$ and $f(w) = f_0(\mathrm{i})$:

$$
\begin{aligned}
\langle C\varphi_0, Cw_0\rangle &= \frac{1}{f_0(\mathrm{i})} \int_{\mathbb{R}} \mathcal{W}^{\rho_k}_{\varphi_0} f_0((\xi_1, 1)) \left(\frac{\xi_1^2}{2\xi_1^2(\xi_1)^2}\right)^{\frac{1+is}{2}} d\xi_1 \\
&= 2^{\frac{\mathrm{i}s+1}{2}} \pi B(\frac{\mathrm{i}s}{2}, \frac{1}{2}) \int_{\mathbb{R}} e^{\frac{\mathrm{i}s-1}{2}\varrho(\mathrm{i};(\xi_1,1))} \left(2\xi_1^2\right)^{-\frac{1+is}{2}} d\xi_1 \\
&= \pi B(\frac{\mathrm{i}s}{2}, \frac{1}{2}) \int_{\mathbb{R}} (\xi_1^2+1)^{\frac{\mathrm{i}s-1}{2}} \left(\xi_1^2\right)^{-\frac{1+is}{2}} d\xi_1.
\end{aligned} \tag{6.4.15}
$$

Put $u = (\xi_1^2+1)^{-1}$, then (6.4.15) becomes

$$
\begin{aligned}
\langle C\varphi_0, Cw_0\rangle &= \pi B(\frac{\mathrm{i}s}{2}, \frac{1}{2}) \int_0^1 u^{\frac{1}{2}-1}(1-u)^{-\frac{\mathrm{i}s}{2}-1} \, du \\
&= \pi B(\frac{\mathrm{i}s}{2}, \frac{1}{2}) B(-\frac{\mathrm{i}s}{2}, \frac{1}{2}) \\
&= \pi \frac{\Gamma(\frac{\mathrm{i}s}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{\mathrm{i}s+1}{2})} \frac{\Gamma(\frac{-\mathrm{i}s}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{-\mathrm{i}s+1}{2})} \\
&= \pi^2 \left|\Gamma\left(\frac{\mathrm{i}s}{2}\right)\right|^2 \left|\Gamma\left(\frac{\mathrm{i}s+1}{2}\right)\right|^{-2} \\
&= \pi^2 \frac{2}{s} \coth \frac{\pi s}{2}.
\end{aligned} \tag{6.4.16}
$$

Substituting this value in (6.4.9), we obtain

$$
f(w) = \frac{1}{2\pi^2} s \tanh \frac{\pi s}{2} \int_{\mathbb{R}^2} \mathcal{W}^{\rho_k}_{\varphi_0} f(\xi) \left(\frac{v\xi_1^2}{2\xi_1^2(\xi_1 - \xi_2 u)^2 + 2u^2}\right)^{\frac{1+is}{2}} d\xi_1 d\xi_2. \tag{6.4.17}
$$

And for $s \in \mathbb{R}$, we get the inversion formula

$$
f(w) = \frac{1}{2\pi^2} \int_{\mathbb{R}} s \tanh \frac{\pi s}{2} \left(\int_{\mathbb{R}^2} \mathcal{W}^{\rho_k}_{\varphi_0} f(\xi) \rho_\tau(\mathbf{s}(\xi_1, \xi_2))\phi_0(w) \, d\xi_1 d\xi_2\right) ds, \tag{6.4.18}
$$

where

$$
\rho_\tau(\mathbf{s}(\xi_1, \xi_2))\phi_0(w) = \left(\frac{v\xi_1^2}{2\xi_1^2(\xi_1 - \xi_2 u)^2 + 2u^2}\right)^{\frac{1+is}{2}}. \tag{6.4.19}
$$

□

The inversion formula is equivalent to the decomposition of the unitary representation ρ_k, $k = 0$ (6.2.7) into irreducible components. We will describe the irreducible invariant subspaces H_s. Consider the eigenspace

$$\left\{ f \in L_2(\mathbb{H}^+) : \quad d\rho_k(C)f = (1 + s^2)f \right\}. \tag{6.4.20}$$

This space is spanned by the functions (6.4.19). Thus, the elements of this eigenspace can be presented as a continuous linear combination over a set of such functions, that is

$$f_s(w) = \int_{\mathbb{R}^2} \mathcal{W}_{\varphi_0}^{\rho_k} f(\xi) \left(\frac{v\xi_1^2}{2\xi_1^2(\xi_1 - \xi_2 u)^2 + 2u^2} \right)^{\frac{1+is}{2}} d\xi_1 d\xi_2, \tag{6.4.21}$$

where f_s belongs to the space $H_s \subset L_2(\mathbb{H}^+)$.
Introduce the projection operator $P_s : L_2(\mathbb{H}^+) \to H_s$ by

$$P_s f = f_s.$$

Thus, the problem of decomposing the space $L_2(\mathbb{H}^+)$ into irreducible subspaces consists in expanding the functions $f \in L_2(\mathbb{H}^+)$ in their projections f_s.
The solution of this problem is given by (6.4.18), since this formula can be written as follows:

$$f(w) = \frac{1}{2\pi^2} \int_{\mathbb{R}} s \tanh \frac{\pi s}{2} f_s ds, \qquad f_s = P_s f. \tag{6.4.22}$$

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