NONLINEAR WAVE EQUATIONS AND APPLICATIONS FROM CONTROL THEORY

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1. Abstract

In 1961 Joergens [14] first proved that the Cauchy problem to the wave equation with power type nonlinearity

$$u_{tt} - \Delta u + u|u|^{p-1} = 0 \qquad \text{in } \mathbb{R}^n \times \mathbb{R} \tag{1.1}$$

in the case of $\mathbb{R}^3$ and when $p < 5$ admits of a unique regular solution $u \in C^2$ for any Cauchy data $u_0 \in C^3$, $u_1 \in C^2$.

Our research in this paper will be centred on finding the connection between the above results and the closely related equations from the control theory in a more abstract setting. In particular we examine the quasipotential formula in [7] and investigate whether the result is applicable to (1.1) or its generalisations. To this end we first extend the existence theorems in [10] to controlled equations, and then establish the finding that the quasipotential admits of analogous expression for our well-posed problem. Along the way there is also a remarkable discovery concerning the large time behaviour of the solution.

2. Author's declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for a degree or other qualification at this University or elsewhere. All sources are acknowledged as references.

Contents

3. Introduction

In 1961 Joergens [14] first proved that the Cauchy problem to the wave equation with power type nonlinearity

$$u_{tt} - \Delta u + u|u|^{p-1} = 0 \qquad \text{in } \mathbb{R}^n \times \mathbb{R} \tag{3.1}$$

in the case of $\mathbb{R}^3$ and when $p < 5$ admits of a unique regular solution $u \in C^2$ for any Cauchy data $u_0 \in C^3$, $u_1 \in C^2$. The case $p = 5$ was later shown by Rauch [19] to have a regular solution when the initial energy is small. His method reveals that the exponent 5 that arises from Sobolev embedding is the tipping point, on and from either side of which the problem (3.1) requires dramatically different considerations for solution. The critical exponent problem did not receive any further development until in 1988 Struwe [23] was able to remove the small energy assumption in Rauch but additionally assumed radially symmetric data. Two years later Grillakis [12] removed the symmetry assumption and thus solved the problem completely. The method he adopted we will give an analysis below because of its ingenuity and value for inspiration.

Up until this time the arguments were all based on improvements of Rauch's method, and they rely on an explicit representation for the solution of the wave equation. For dimensions higher than three this is not available and a different approach is needed. This approach was found by Kapitanski [17] soon after. It was based on the now well-known Strichartz estimates [25], [25] and their variants obtained by Ginibre and Velo [11]. Euipped with the new method Grillakis in 1992 [13] refined the regularity result to $\mathbb{R}^5$, and to $\mathbb{R}^7$ for radially symmetric data. One year after him, Shatah and Struwe [24] removed the symmetry assumption and solved the problem completely.

Our research in this thesis will be centred upon finding the connections between the above results and the closely related equations from the control theory in the more abstract setting of infinite dimensional space of operators. In particular we examine the quasipotential formula in [7] and investigate whether the result is applicable to (3.1) or its generalisations. To this end we first extend the existence theorems in [10] to controlled equations, and then establish the finding that the quasipotential admits of analogous expression for our problem. Along the way there is also a remarkable finding concerning the large time behaviour of the solution.

The plan for the thesis will be as follows:

- ⋆ In sections 4 and 5, we will discuss Gauss's and Noether's Theorems to pave the way for understanding Grillakis's arguments in his 1990 paper [[12]]. Some examples of multipliers to the wave equation are given in connection to the application of Noether's theorem, these were taken from [[10] Chapter 8.6]. The proof of Gauss's theorem in the plane was from [[6] Chapter V].
- ⋆ In section 6 we utilise the tools from previous sections to discuss elements of Grillakis's paper[[12]], in particular the concept of integrating divergence identities on cones in space-time. The multipliers he used will be slight variations to the ones we have given above, after taking into account the nonlinearity in the wave equation.
- ⋆ Section 7 will contain existence theorems for the general quasi-linear and semi-linear wave equations, taken from [[10] Chapter 12]. Our contribution to these results lies in the addition of a control term to the equation, and the determination

of the conditions it must satisfy in order that the theorems will continue to hold. We will apply those theorems to some generalisations of (3.1) in section 9 as a pretext for considering control-theoretic results.

* In section 8 we introduce control theory in the language of operators and semi-groups. Preliminary material taken from [27], [22] and [18] has been supplied for definitions and theorems used subsequently in the section. For subsection 8.1 on linear problems, we focus our attention on detailing and clarifying the proofs of results that can be found in [7] and [8]. Namely Theorem 8.17 and Theorem 8.20 which correspond to Theorem 2.4 and Theorem 2.3 respectively in [7], and Theorem 8.18 to Theorem 3.1 in [8]. In subsection 8.2 there will be found an original result Theorem 8.25. Example 8.1 is an elaboration of the Example 3.5 in [7]. And finally Theorem 8.24 is complement to Theorem 3.7 in [7].
* Section 9 is the culmination of all that has been gathered in previous sections to produce concrete existence and control theoretic results for the generalisations of (3.1). Specifically we obtained a global weak solution to the controlled Klein-Gordon version of (3.1), with and without a damping term, in Theorem 9.14 and 9.16 respectively. For the damped version we also established the local existence of a unique solution in Theorem 9.15. Furthermore, the question concerning the representation formula of the quasipotential was settled in 9.1, with a time-reversed version in 9.2. Lastly, subsection 9.1 contains miscellaneous findings on the uniqueness of solution to the linear control problem (9.29). These findings are original.

4. Preliminary: Gauss's Divergence Theorem

Theorem 4.1 (Gauss)**.** *Let D be any closed n-dimenstional region with a.e. C^∞ boundary surface ∂D, $F = \{f_1, f_2, \dots, f_n\}$ a C^1 vector field, then*

$$\int_D div\ F\,dx = \int_{\partial D} F \cdot N\,dS \tag{4.1}$$

where $N = \{n_1, n_2, \dots, n_n\}$ is the unit outward normal to the boundary surface, $div\ F = \sum_{i=1}^n (f_i)_{x_i}$, and dS the surface measure.

In particular, for $i = 1, \dots, n$

$$\int_D (f_i)_{x_i} dx = \int_{\partial D} f_i n_i dS.$$

We will give a demonstration of Gauss's theorem in the plane for illustration. Generalizations to higher dimensions follows the same line of idea. The reader is referred to [6] for more details.

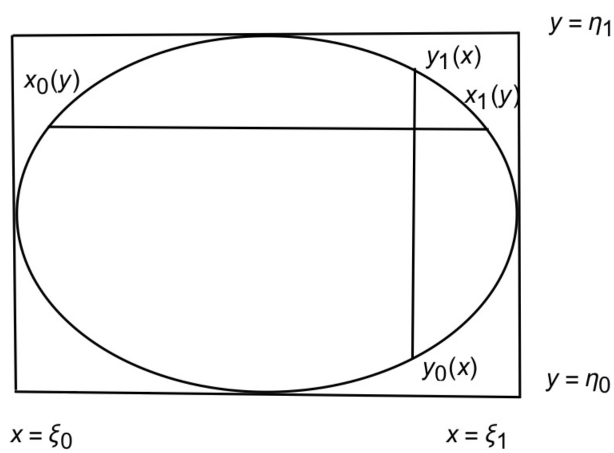
Before that, a generalized notion of the Riemann integral is needed to enable integration along an arbitrary curve in plane instead of just the real axis. This is done by projection.

Suppose we wish the evaluate the integral of function $f(x, y)$ along an arbitray curve C of finite length in plane, which is assumed to have nonoverlapping projections onto the coordinate axes. Projecting the curve onto the x axis to obtain an interval $[a, b]$ which encloses the range of values of x the curve encompasses, subdivide the interval into $\{x_0 = a, x_1, \dots, x_n = b\}$ and with their corresponding projection source on the curve C be called $p_0 = (x_0, y_0), p_1 = (x_1, y_1), \dots, p_n$. Then the sum $\sum_{i=0}^{n-1} f(p_i)(x_{i+1} - x_i)$ on making n infinitely large in such a way that the longest of $x_{i+1} - x_i$ tends to zero gives the Riemann integral which we denote $\int_C f(x, y)dx$, this is usually called the line integral or curvilinear integral of the function f along the curve C with respect to axis x. Similarly the line integral with respect to axis y can be obtained. In particular if the curve C is parametrized by $x(t)$, $y(t)$ with $\alpha \le t \le \beta$ the integral can be written as $\int_\alpha^\beta f(x,y)\dot{x}dt$ as a Riemann integral. If we have a vector field $\mathbf{A}(x,y) = (f(x,y), g(x,y))$ with components integrated on C with respect to x and y respectively and added together, the notation can be abbreviated $\int_\alpha^\beta \mathbf{A}\dot{\mathbf{x}}dt = \int_C \mathbf{A}d\mathbf{x}$.

Suppose $\mathbf{A}$ is a vector field as before with components f and g, each having continuous partial derivatives of first order, the continuity being with respect to both x and y. We attempt to integrate its divergence on a closed and convex plane region D with a boundary curve C. The double integral in question is therefore $\iint_D f_x + g_y dxdy$, because the integrand is continuous with respect to both variables, the double integral is equal to two repeated integrals and their order can be changed at will, so that

$$\iint_D f_x + g_y dxdy = \int \int f_x dxdy + \int \int g_y dydx.$$

In order to assign specific ranges to those integrals, imagine a line perpendicular to the x axis is projected from a point x on the axis upwards to meet the boundary curve C of region D, because D is convex the line can only cut C in two points, let the entry point be $y_0(x)$ and the exit point $y_1(x)$ as is shown in the figure below,



so that the second integral on the right can be firstly integrated with y variable using fundamental theorem of calculus between the limits $y_0(x)$ and $y_1(x)$, and secondly integrate all along the range of x that is within projection from the curve, say from ξ_0 to ξ_1 to get

$$\int_{\xi_0}^{\xi_1} [g(x, y_1(x)) - g(x, y_0(x))]dx$$

After observing carefully we see that the above expression is the same as

$$-\int_{C_+} g(x,y)dx$$

taken counterclockwise around C, where $\int_{\xi_1}^{\xi_0} g(x, y_1(x))dx$ is the upper half, $\int_{\xi_0}^{\xi_1} g(x, y_0(x))dx$ the lower half.

Similarly for f the resulting expression is

$$\int_{C_+} f(x,y)dy$$

the two combined gives us Gauss's theorem in the form

$$\iint_D f_x + g_y dxdy = \int_{C_+} f(x,y)dy - g(x,y)dx.$$

Or the right side can be written $\int_{C_+} [f(x,y)\frac{dy}{ds} - g(x,y)\frac{dx}{ds}]ds$ if we use the curve length s to parametrize the curve, because

$$\frac{dy}{ds} = \frac{\partial x}{\partial n} = \cos\nu, \qquad -\frac{dx}{ds} = \frac{\partial y}{\partial n} = \cos\mu$$

for the outward normal direction n, with $nu(x,y)$, $mu(x,y)$ the angles between the normal and the positive x and y axis respectively. Gauss's theorem then can be written in convienient form

$$\begin{aligned}\iint_D \operatorname{div}\mathbf{A} dxdy &= \int_{C_+} [f(x,y)\cos\nu + g(x,y)\cos\mu]ds \\ &= \int_{C_+} \mathbf{A}\cdot\mathbf{n}\, ds\end{aligned}$$

which are exactly analogous to the formula in higher dimensions. This is the end of proof of the theorem.

5. The Method of Multipliers

Here we intend to give an exposition of some of the ideas M. Grillakis adopted in his decent 1990 paper "Regularity and Asymptotic Behavior of the Wave Equation with a Critical Nonlinearity" which established the existence of a regular solution to the critical power problem

$$u_{tt} - \Delta u + u^5 = 0 \qquad \text{in } \mathbb{R}\times\mathbb{R}^3 \tag{5.1}$$

with prescribed initial data. Our main objective will be to bridge the gap of knowledge in order to understand the multiplier method he used. That is, when multiplying (5.1) by an expression such as

$$I(u) = (t^2 + r^2)u_t + 2tx\cdot Du + 2tu,$$

the result is a divergence identity in the form

$$\partial_t Q_i - \operatorname{div}P_i + R_i = 0. \tag{5.2}$$

This fact may be verified by direct computation, but actually we have a general theorem which can let us apriori determine whether such a transformation is possible and what the resulting identity is.

5.1. **Noether's theorem.** To start with, some notions from the calculus of variations are needed.

Definition 5.1. Assume $U \subset \mathbb{R}^n$ is a bounded, open set with C^∞ boundary ∂U and given a C^∞ function

$$L : \mathbb{R}^n \times \mathbb{R} \times \bar{U} \to \mathbb{R}.$$

L is called a Lagrangian. We use the notations $L = L(p, z, x) = L(p_1, \dots, p_n, z, x_1, \dots, x_n)$, and

$$\begin{cases} D_p L = (L_{p_1}, \dots, L_{p_n}) \\ D_z L = L_z \\ D_x L = (L_{x_1}, \dots, L_{x_n}) \end{cases}$$

The problem of the calculus of variations consists in finding the function $w : U \to \mathbb{R}$ that minimize the value of a functional such as

$$I[w] = \int_U L(Dw, w, x) dx, \tag{5.3}$$

possibly with some constraints on the boundary value. In order to find this minimum, people have the idea that such a function w would be a critical point to the family of values $i(\tau) := I[u + \tau v]$ where v is an arbitrary function and $\tau \in (-\infty, \infty)$. So that we must have $i'(0) = 0$.

For a functional of the above form (5.3)

$$i(\tau) = \int_U L(Du + \tau Dv, u + \tau v, x) dx,$$

and

$$i'(\tau) = \int_U \sum_{i=1}^n L_{p_i}(Du + \tau Dv, u + \tau v, x) v_{x_i} + L_z Du + \tau Dv, u + \tau v, x) v dx.$$

Setting $\tau = 0$ we have

$$0 = i'(0) = \int_U \sum_{i=1}^n L_{p_i}(Du, u, x) v_{x_i} + L_z(Du, u, x) v dx,$$

doing integration by parts supposing v vanishes on the boundary we get

$$0 = \int_U \left[-\sum_{i=1}^n (L_{p_i}(Du, u, x))_{x_i} + L_z(Du, u, x) \right] v dx.$$

Since v is arbitrary we conclude

$$-\sum_{i=1}^n (L_{p_i}(Du, u, x))_{x_i} + L_z(Du, u, x) = 0 \quad \text{in} \quad U.$$

This is the *Euler-Lagrange equation* associated with the functional $I[\cdot]$. It is a second order PDE in divergence form.

Example 5.1. Take

$$L(p, z, x) = \frac{1}{2}|p|^2.$$

Then $L_{p_i} = p_i$, $(i = 1, \dots, n)$, $L_z = 0$, the Euler-Lagrange equation associated with the functional

$$I[w] = \frac{1}{2} \int_U |Dw|^2 dx$$

is

$$\Delta u = 0 \quad \text{in } U.$$

Definition 5.2. Now consider the functional

$$I[w] := \int_U L(Dw, w, x)dx,$$

where $U \subset \mathbb{R}^n$ and $w : U \to \mathbb{R}$.

(1) Let $\mathbf{x} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $\mathbf{x} = \mathbf{x}(x, \tau)$, be a family of C^∞ vector fields with $\mathbf{x}(x, 0) = x$ for all $x \in \mathbb{R}^n$. $x \mapsto \mathbf{x}(x, \tau)$ is called a *domain variation*. Define

$$\mathbf{v}(x) := \mathbf{x}_\tau(x, 0)$$

and

$$U(\tau) := \mathbf{x}(U, \tau).$$

(2) Given a C^∞ function $u : \mathbb{R}^n \to \mathbb{R}$, define the family of *function variations* $w : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $w = w(x, \tau)$, with

$$w(x, 0) = u(x).$$

Writing

$$m(x) := w_\tau(x, 0),$$

m will be called a multiplier. In our previous section the less general definition of $w = u + \tau v$ gives $m = v$.

Definition 5.3. The functional $I[\cdot]$ is said to be invariant with the domain variation $\mathbf{x}$ and the function variations w if

$$\int_U L(Dw(x,\tau), w(x,\tau), x)dx = \int_{U(\tau)} L(Du, u, x)dx \tag{5.4}$$

for every small $|\tau|$ and every open set $U \subset \mathbb{R}^n$, where $D = D_x$. Note that if the Lagrangian does not explicitly depend on the x, then the above equality reduces to a change of variable if we take $w(x, \tau) = u(\mathbf{x}(x, \tau))$.

Theorem 5.4 (Noether's theorem)**.** *Assume that $I[\cdot]$ is invariant with the domain variations $\mathbf{x}$ and the function variations w of u.*

(i) Then

$$\sum_{i=1}^n (mL_{p_i}(Du, u, x) - L(Du, u, x)v^i)_{x_i} = m\left(\sum_{i=1}^n (L_{p_i}(Du, u, x))_{x_i} - L_z(Du, u, x)\right), \tag{5.5}$$

where $\mathbf{v}$ and m are defined as above.

(ii) Especially, if u is a critical point of $I[\cdot]$ and is a solution to the Euler-Lagrange equation, then

$$\sum_{i=1}^n (mL_{p_i}(Du, u, x) - L(Du, u, x)v^i)_{x_i} = 0 \tag{5.6}$$

Proof. See [10] Chapter 8.6.2. □

Example 5.2. Given

$$I[w] = \int_0^T \int_{\mathbb{R}^n} \frac{1}{2} w_t^2 - \left(\frac{1}{2}|Dw|^2 + F(w) \right) dxdt$$

has as its Euler-Lagrange equation the semilinear wave equation

$$u_{tt} - \Delta u + f(u) = 0$$

for $f = F'$. The integrand of the functional does not explicitly depend on t and is consequently invariant under shift in this variable. Noether's theorem then gives us a conservation law of energy, namely

$$\mathbf{x}(x,t,\tau) := (x, t+\tau), \quad w(x,t,\tau) := u(\mathbf{x}(x,t,\tau)) = u(x, t+\tau),$$

so that

$$\mathbf{v} = e_{n+1}, \quad m = u_t.$$

Then (5.6) gives

$$\sum_{i=1}^n (-u_{x_i} u_t)_{x_i} + (u_t^2 - \frac{1}{2}(u_t^2 - |Du|^2) + F(u))_t = 0,$$

which can be written

$$e_t - \operatorname{div}(u_t Du) = 0 \tag{5.7}$$

with the energy density

$$e := \frac{1}{2}(u_t^2 + |Du|^2) + F(u).$$

The divergence is in the x variable. Moreover if u is compactly supported in space, then it follows that the conservation of energy in time

$$\frac{d}{dt} \int_{\mathbb{R}^n} \frac{1}{2}(u_t^2 + |Du|^2) + F(u) dx = 0.$$

Example 5.3. The linear wave equation

$$\square\, u = u_{tt} - \Delta u = 0 \tag{5.8}$$

results from the action functional

$$I[w] = \frac{1}{2} \int_0^T \int_{\mathbb{R}^n} w_t^2 - |Dw|^2 dxdt. \tag{5.9}$$

It is invariant with the scaling

$$(x,t) \mapsto (\lambda x, \lambda t), \quad u \mapsto \lambda^{\frac{n-1}{2}} u(\lambda x, \lambda t).$$

If we write $\lambda = e^\tau$ and define

$$\mathbf{x}(x,t,\tau) := (e^\tau x, e^\tau t), \quad w(x,t,\tau) := e^{\tau \frac{n-1}{2}} u(e^\tau x, e^\tau t).$$

Then

$$\mathbf{v} = (x,t), \quad m = t u_t + x \cdot Du + \frac{n-1}{2} u. \tag{5.10}$$

Noether's theorem (5.6) gives

$$p_t - \operatorname{div}\mathbf{q} = 0$$

for

$$p := \left(t u_t + x \cdot Du + \frac{n-1}{2} u \right) u_t - \frac{1}{2}(u_t^2 - |Du|^2) t,$$

$$\mathbf{q} := \left(tu_t + x \cdot Du + \frac{n-1}{2}u\right)Du + \frac{1}{2}(u_t^2 - |Du|^2)x.$$

Example 5.4. The *hyperbolic inversion* is defined by the mapping

$$(x,t) \mapsto (X,T) = \left(\frac{x}{|x|^2 - t^2}, \frac{t}{|x|^2 - t^2}\right). \tag{5.11}$$

Related is the *hyperbolic Kelvin transform*, defined by

$$R^{\frac{n-1}{2}} U(X,T) = r^{\frac{n-1}{2}} u(x,t),$$

where $r = |x|, R = |X|$. Note that the transformation corresponding to (5.11) really is

$$u(x,t) \mapsto \left(\frac{1}{||x|^2 - t^2|}\right)^{\frac{n-1}{2}} U\left(\frac{x}{|x|^2 - t^2}, \frac{t}{|x|^2 - t^2}\right), \tag{5.12}$$

which is the form useful for computing derivatives. The above symmetric form brings out the invertive character of the transform between the two spaces (x,t) and (X,T).

Kelvin transform preserves the wave operator in the sense that

$$R^{\frac{n+3}{2}}(U_{TT} - \Delta_X U) = r^{\frac{n+3}{2}}(u_{tt} - \Delta_x u). \tag{5.13}$$

A verification of this fact by direct calculation is given in the Appendix.

Now we try to build domain and function variations from the above transformations. First apply the hyperbolic inversion (5.11) to (x,t), add τe_{n+1}, and apply the inversion again. There results the domain variations

$$\mathbf{x}(x,t,\tau) = \gamma(x, t + \tau(|x|^2 - t^2)), \tag{5.14}$$

with

$$\gamma = \frac{|x|^2 - t^2}{|x|^2 - (t + \tau(|x|^2 - t^2))^2}.$$

Similarly for the function variations apply the Kelvin transform (5.12) on u, add τe_{n+1} , and then use the transform again. So that

$$w(x,t,\tau) = \gamma^{\frac{n-1}{2}} u(\mathbf{x}(x,t,\tau)). \tag{5.15}$$

From these variations it follows that the multipliers are of the form

$$\mathbf{v} = (2xt, |x|^2 + t^2), \quad m = (|x|^2 + t^2)u_t + 2tx \cdot Du + (n-1)tu. \tag{5.16}$$

Although (5.9) is not necessarily invariant under (5.16), if we do multiply m to (5.8) there results the Morawetz identity:

$$c_t - \mathrm{div}\mathbf{r} = 0,$$

where

$$c = \frac{1}{2}(|x|^2 + t^2)(u_t^2 + |Du|^2) + 2tx \cdot Duu_t + (n-1)tuu_t - \frac{n-1}{2}u^2 \tag{5.17}$$

and

$$\mathbf{r} = ((|x|^2 + t^2)u_t + 2tx \cdot Du + (n-1)tu)Du + t(u_t^2 - |Du|^2)x.$$

The reason that the above construction works consists in the following: we know that the Kelvin transform preserves the wave operator so that

$$R^{\frac{n+3}{2}}(U_{TT} - \Delta_X U) = r^{\frac{n+3}{2}}(u_{tt} - \Delta_x u),$$

also

$$\frac{R}{r}U_T = (|x|^2 + t^2)u_t + 2tx \cdot Du + (n-1)tu = m,$$

and together with the identity

$$R^4 dX dT = r^4 dx dt$$

there follows

$$\int U_T(U_{TT} - \Delta_X U) dX dT = \int m(u_{tt} - \Delta u) dx dt$$

over any spacelike domain $\mathcal{D}$ with $u \in H^1(\mathcal{D})$. Since the multiplier U_T brings the wave equation to a time-space divergence expression, the integral on the left hand side can be written as a surface integral and therefore so can the right hand side. The function variation computed above (5.15) is actually $\frac{R}{r}U(X, T+\tau)$ inverted back to the (x,t) space.

Finally, the conformal energy c may be written in the following positive definite form involving the radial derivatives $u_r = Du \cdot \frac{x}{|x|}$,

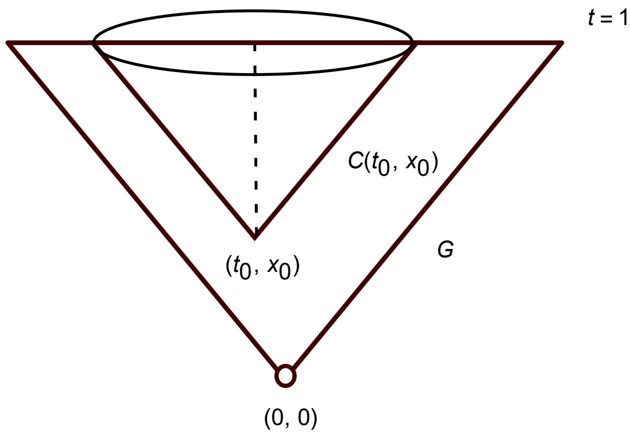
$$\begin{aligned} c = {} & \frac{(t+|x|)^2}{4}\left(u_t + u_r + \frac{n-1}{2|x|}u\right)^2 + \frac{(t-|x|)^2}{4}\left(u_t - u_r - \frac{n-1}{2|x|}u\right)^2 \\ & + \frac{|x|^2+t^2}{2}\left(|Du|^2 - u_r^2 + \frac{(n-3)(n-1)}{4|x|^2}u^2\right) - \frac{n-1}{4}\operatorname{div}\left(\frac{|x|^2+t^2}{|x|^2}u^2 x\right), \end{aligned}$$

where the divergence term can be picked out and combined into $\mathbf{r}$.

6. Integration on Cones

Now we have introduced the multipliers for the wave equations via Noether's theorem, the next step is to apply Gauss's Theorem 4.1 to integrate the resulting divergence identities. The domain of integration that is of interest here is a circular cone in space-time, for example

$$G := \{(t,x) : |x| \le t, \quad 0 < t \le 1\}.$$



In what follows we will explain how to derive by direct computation an integral expression for the solution to

$$u_{tt} - \Delta u + u^5 = 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}^3, \tag{6.1}$$

as was done by Grillakis, in doing so the principle of integrating any divergence identities will be sufficiently illustrated.

6.1. **Integral representation.** Recall the identity (5.7) in the last section, by multiplying equation (6.1) by u_t we have

$$e_t - \operatorname{div}(u_t Du) = 0 \tag{6.2}$$

with the energy density

$$e := \frac{1}{2}(u_t^2 + |Du|^2) + \frac{u^6}{6}.$$

Now let us choose and fix a point (t_0, x_0) in time-space and consider the domain

$$D := \left\{(t,x) : |x - x_0| \le t - t_0, t_0 \le t_2 \le t \le t_1\right\}.$$

In other words, the truncation between t_1 and t_2 of a cone with vertex (t_0, x_0). The boundary of D consists of three parts, namely

$$B_{t_2} := \{(t_2, x) : |x - x_0| \le t_2 - t_0\},$$

$$B_{t_1} := \{(t_1, x) : |x - x_0| \le t_1 - t_0\},$$

and

$$C(t_2, t_1) := \{(t, x) : |x - x_0| = t - t_0, t_2 \le t \le t_1\}.$$

In order to integrate (6.2) on the domain D, recall Gauss's theorem 4.1

$$\int_D div\, F\, dx = \int_{\partial D} F \cdot N\, dS \tag{6.3}$$

Here in our case the vector field $F = (e, -u_t Du)$ and the divergence is with respect to both time and space, therefore on the right hand side there results integrals on each of the boundary surfaces. Note that the unit outward normal vector $N = (-1, 0)$ on B_{t_2} since it is pointing downwards perpendicular to the spatial plane, similarly $N = (1, 0)$ for B_{t_1}. Whereas for $C(t_2, t_1)$ the normal is projecting downwards at 45 degrees with its two components of equal length going towards at the $-t$ and x directions, therefore $N = (-1/\sqrt{2}, 1/\sqrt{2}\hat{y})$ where $\hat{y} = x - x_0/|x - x_0|$ is the unit radial vector in space.

Hence we have as the result of the integration

$$\int_{B_{t_1}} e(u)dx - \int_{B_{t_2}} e(u)dx + \frac{1}{\sqrt{2}} \int_{C(t_2,t_1)} (-e(u) - u_t Du \cdot \hat{y})dz = 0. \tag{6.4}$$

Note that the last integrand can be written as follows

$$\begin{aligned} e(u) + u_t Du \cdot \hat{y} &= \frac{1}{2}(u_t^2 + |Du|^2) + \frac{u^6}{6} + u_t Du \cdot \hat{y} \\ &= \frac{1}{2}|u_t \hat{y} + Du|^2 + \frac{u^6}{6}. \end{aligned} \tag{6.5}$$

In fact this expression can be further abbreviated using the following technique.

Observe that the function $u(t, x)$ when restricted to the side surface of the cone the variations between $t - t_0$ and $|x - x_0|$ are always the same, therefore we can combine the two and regard u as a function of a new variable via the definition

$$u^*(y) := u(t_0 + |y|, x_0 + y)$$

where

$$y := x - x_0.$$

The gradient of this u^* if computed is found to be

$$D^* u = u_t \hat{y} + Du,$$

which is the same expression as in (6.5).

Therefore we have obtained the following energy estimate from (6.4)

$$\int_{B_{t_1}} e(u)dx = \int_{B_{t_2}} e(u)dx + \frac{1}{\sqrt{2}} \int_{C(t_2,t_1)} \left(\frac{1}{2}|D^* u|^2 + \frac{1}{6}u^6\right) dz. \tag{6.6}$$

Now this technique of projecting the argument surface of u onto the base of the cone can be utilised further. If we not only compute the gradient of u^* but also its Laplacian, the result is

$$\Delta^*(u) = \Delta u + 2\hat{y}\cdot Du_t + u_{tt} + \frac{2}{|y|}u_t.$$

Since

$$D^* u_t = u_{tt}\hat{y} + Du_t,$$

we can eliminate the term Du_t to get

$$\Delta^* u - 2\hat{y}\cdot D^* u_t - \frac{2}{|y|}u_t = \Delta u - u_{tt} = u^5,$$

which can be written as

$$\operatorname{div}^*\left[\frac{1}{|y|}Du + \frac{y}{|y|^3}u - \frac{y}{|y|^2}u_t\right] - \frac{u^5}{|y|} = 0. \tag{6.7}$$

This divergence expression can enable us to find the integral formula for the solution if we try to integrate it on the domain $C(t_0, x_0)$ (cf figure on page 10), or in terms of the argument y the equivalent domain $B := \{|y| \le 1 - t_0\}$. But in order to be more careful about the vertex when applying Gauss's theorem, a passage to the limit from a ring-shaped area is prefered, i.e. the area bounded by $S_\epsilon := \{|y| = \epsilon\}$ and $S := \{|y| = 1 - t_0\}$.

When the integration is carried out, there results

$$\begin{aligned}\int_{|y|=1-t_0}\left[\frac{1}{|y|}\left(Du\cdot\hat{y} - u_t + \frac{1}{|y|}u\right)\right]_{t=1} ds - \int_{|y|=\epsilon}\left[\frac{1}{|y|}\left(Du\cdot\hat{y} - u_t + \frac{1}{|y|}u\right)\right]_{t=t_0+\epsilon} ds \\ - \frac{1}{4\pi}\int_{|y|\le 1-t_0}\frac{u^5(t_0+|y|, x)}{|y|}dy = 0.\end{aligned} \tag{6.8}$$

Now for the second integral because of the continuity we have assumed on u and its derivatives, by mean value theorem and sending ϵ to 0 it becomes

$$4\pi\epsilon^2\frac{1}{\epsilon^2}u(t_0+\epsilon, x_0+\epsilon\hat{x_1}) \to 4\pi u(t_0, x_0),$$

where the first two integrands vanish because there is only one $\frac{1}{|y|}$ to cancel with $4\pi|y|^2$ which is the volume of the domain.

Hence we have obtained the integral representation for the solution to equation (6.1)

$$\begin{aligned}u(t_0, x_0) &= u_L(t_0, x_0) + u_N(t_0, x_0) \\ &= \int_{|y|=1-t_0}\left[\frac{1}{|y|}\left(Du\cdot\hat{y} - u_t + \frac{1}{|y|}u\right)\right]_{t=1} ds - \frac{1}{4\pi}\int_{|y|\le 1-t_0}\frac{u^5(t_0+|y|, x)}{|y|}dy.\end{aligned} \tag{6.9}$$

This accomplishes our aim for the section.

For the rest of Grillakis's paper, we have scarcely any words to contribute any insight and would refer the reader to the original paper [12] if he wishes to pursue the subject further.

7. Existence theorems

In the present section a spectrum of existence theorems for nonlinear wave equations will be generalised to the case where there is an external force or control, e.g. instead of

$$u_{tt} - \Delta u + u^5 = 0 \qquad \text{in } \mathbb{R} \times \mathbb{R}^3 \tag{7.1}$$

we consider

$$u_{tt} - \Delta u + u^5 = f \qquad \text{in } \mathbb{R} \times \mathbb{R}^3 \tag{7.2}$$

where f is a function of (t,x) with prescribed regularity. The prototypes of those theorems for equations without a control were considered in [10] Chapter 12. We modified their proofs below by taking into account the extra control term and the conditions it must satisfy for the arguments to remain valid.

Theorem 7.1 ([10] Chapter 12.3.2 Theorem 2)**.** *If $g, h \in C_c^\infty(\mathbb{R}^3)$ and $f \in C^\infty(\mathbb{R}^3 \times [0,T))$ then there is $T > 0$ and a unique C^∞ solution u to*

$$\begin{cases} u_{tt} - \Delta u + u^5 = f & \mathbb{R}^3 \times (0,T) \\ u = g, u_t = h & t = 0 \end{cases} \tag{7.3}$$

A C^∞ function is a function along with all its partial derivatives are continuous with respect to space-time.

Proof. We start with the observation that by the linearity of the wave operater the solution to (7.3) can be represented as the sum of solutions to two other problems, namely

$$\begin{cases} v_{tt} - \Delta v = 0 & \mathbb{R}^3 \times (0,\infty) \\ v = g, v_t = h & t = 0 \end{cases} \tag{7.4}$$

and

$$\begin{cases} w_{tt} - \Delta w = -u^5 + f & \mathbb{R}^3 \times (0,\infty) \\ w = 0, w_t = 0 & t = 0 \end{cases} \tag{7.5}$$

The solutions to which are respectively the well-known Kirchhoff formula

$$v(x,t) = \fint_{\partial B(x,t)} th(y) + g(y) + Dg(y) \cdot (y-x) dS(y) \qquad (x,t) \in \mathbb{R}^3 \times [0,T).$$

and

$$w(x,t) = -\frac{1}{4\pi} \int_{B(x,t)} \frac{u^5(y, t-|y-x|) - f(y, t-|y-x|)}{|y-x|} dy. \tag{7.6}$$

by the Duhamel's principle.

Hence $u = v + w$ must solve the following integral equation

$$u(x,t) = v(x,t) - \frac{1}{4\pi} \int_{B(x,t)} \frac{u^5 - f}{|y-x|} dy. \qquad (x,t) \in \mathbb{R}^3 \times [0,T). \tag{7.7}$$

Introduce the space

$$X = \{u \in C(\mathbb{R}^3; L^\infty[0,T)) : u(\cdot,0) = g, \|u - v\|_{L^\infty} < C\}.$$

so that there is a constant C_1 such that

$$\|u\|_{L^\infty(\mathbb{R}^3\times[0,T))} \le C_1 \tag{7.8}$$

for all $u \in X$.

Define the mapping which corresponds to (7.7) by

$$A[u] = v - \frac{1}{4\pi}\int_{B(x,t)} \frac{u^5 - f}{|y-x|} dy.$$

We will show A is a contraction and that (7.7) has a solution. Firstly A maps X to X since

$$\begin{aligned}\left|\int_{B(x,t)} \frac{u^5 - f}{|y-x|} dy\right| &\le \|u\|^5_{C(\mathbb{R}^3;L^\infty[0,T))} \int_0^t \int_{\partial B(x,t)} \frac{1}{s} dS ds \\ &+ \left\{\int_{B(x,t)} f^2 dy\right\}^{1/2} \left\{\int_{B(x,t)} \frac{1}{|y-x|^2} dy\right\}^{1/2} \\ &\le CT^2 \|u\|^5_{C(\mathbb{R}^3;L^\infty[0,T))} + C\sqrt{T}\|f\|_{L^2(\mathbb{R}^3\times[0,T))} \le C(T)\end{aligned}$$

Then if we have two solutions $u, \hat{u} \in X$,

$$\begin{aligned}\|A[u] - A[\hat{u}]\|_{C(\mathbb{R}^3;L^\infty[0,T))} &\le \sup_{x\in\mathbb{R}^3, 0\le t<T} \left(\frac{1}{4\pi}\int_{B(x,t)} \frac{|u^5 - f - (\hat{u}^5 - f)|}{|y-x|} dy\right) \\ &\le C \sup_{x\in\mathbb{R}^3, 0\le t<T} \int_{B(x,t)} \frac{\|u-\hat{u}\|_{C(\mathbb{R}^3;L^\infty[0,T))}}{|y-x|} dy \\ &\le CT^2 \|u-\hat{u}\|_{C(\mathbb{R}^3;L^\infty[0,T))},\end{aligned}$$

where the third inequality is the result of

$$\int_{B(x,t)} \frac{1}{|y-x|} dy = \int_0^t \int_{\partial B(x,r)} \frac{1}{r} dS dr = \int_0^t 4\pi r dr = 2\pi t^2 \le CT^2.$$

Now take T so small that A a strict contraction, by Banach Fix Point Theorem there is a unique solution $u \in X$ satisfying (7.7) and (7.3).

For the rest of the proof we attempt to show that u is not only continuous but also C^∞. To start with, define the difference quotient for the partial derivative

$$\tilde{u}(x,t) = D_i^h u(x,t) = \frac{u(x+he_k,t) - u(x,t)}{h}, \quad i = 1,2,3,$$

then by subtracting (7.3) from itself and divide by h there results

$$\begin{cases} \tilde{u}_{tt} - \Delta\tilde{u} + c\tilde{u} = \tilde{f} & \mathbb{R}^3\times(0,T) \\ \tilde{u} = \tilde{g}, \tilde{u}_t = \tilde{h} & t = 0 \end{cases} \tag{7.9}$$

where

$$c(x,t) = 5\int_0^1 (su(x+he_k,t) + (1-s)u(x,t))^4 ds$$

and $\tilde{g} = D_k^h g$, $\tilde{h} = D_k^h h$, $\tilde{f} = D_k^h f$.

Now we can derive an integral equation for the solution to (7.9) in the same way as above, that

$$\tilde{u}(x,t) = \tilde{v}(x,t) - \frac{1}{4\pi}\int_{B(x,t)} \frac{c\tilde{u} - \tilde{f}}{|y-x|}dy. \tag{7.10}$$

where

$$\begin{cases} \tilde{v}_{tt} - \Delta\tilde{v} = 0 & \mathbb{R}^3 \times (0,\infty) \\ \tilde{v} = \tilde{g}, \tilde{v}_t = \tilde{h} & t = 0. \end{cases} \tag{7.11}$$

Since

$$\begin{aligned} \left|\int_{B(x,t)} \frac{c\tilde{u}^*}{|y-x|}dy\right| &\le \int_0^t \int_{\partial B(x,s)} \frac{|c||\tilde{u}^*|}{s}dSds \\ &\le 4\pi|c|T\int_0^t \|\tilde{u}(\cdot,s)\|_{L^\infty}ds, \end{aligned}$$

it follows

$$\|\tilde{u}(\cdot,t)\|_{L^\infty} \le C + C\int_0^t \|\tilde{u}(\cdot,s)\|_{L^\infty}ds \quad (0 \le t \le T).$$

By Gronwall's inequality

$$\|\tilde{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^3)} \le C(1 + CTe^{CT}).$$

Therefore $D_i^h u(x,t)$ is bounded and that $D_i^{-h}u(x,t) = \frac{u(x)-u(x-he_i)}{h}$ is also bounded. Then there exists a function $v_i \in L^\infty(\mathbb{R}^3)$ and a subsequence $h_k \to 0$ such that

$$D_i^{-h_k}u(x,t) \rightharpoonup v_i \quad \text{weakly in } L^2_{loc}(\mathbb{R}^3).$$

Consequently

$$\begin{aligned} \int_{\mathbb{R}^3} u\phi_{x_i}dx &= \lim_{h_k\to 0}\int_{\mathbb{R}^3} uD_i^{h_k}\phi dx \\ &= -\lim_{h_k\to 0}\int_{\mathbb{R}^3} D_i^{-h_k}u\phi dx = -\int_{\mathbb{R}^3} v_i\phi dx. \end{aligned}$$

The second equality is the integration by parts for the difference quotients, which follows from

$$\int_{\mathbb{R}^3} u(x)\left[\frac{\phi(x+he_i)-\phi(x)}{h}\right]dx = -\int_{\mathbb{R}^3}\left[\frac{u(x)-u(x-he_i)}{h}\right]\phi(x)dx$$

when h is small. This holds for all $\phi \in C_c^\infty(\mathbb{R}^3)$, and therefore $v_i = u_{x_i}$ in the weak sense $(i = 1,2,3)$. Consequently $u \in W^{1,\infty}(\mathbb{R}^3)$. We can similarly bound u_t, so that $u \in W^{1,\infty}(\mathbb{R}^3 \times [0,T))$

Then by [10] (5.8.2b theorem 4) we have u Lipschitz continuous on every bounded and open $U \in \mathbb{R}^3 \times [0,T)$ with $\partial U \in C^1$. And by Rademacher's theorem u is differentiable almost everywhere in U, and the gradient of u coincides with the weak gradient. Since U is arbitrary we can obtain any compact set by finite cover, and approximate the whole space by compact sets. Therefore $u \in C^1(\mathbb{R}^3 \times [0,T))$.

Writing $\tilde{u} = u_{x_k}$ or u_t, by sending h to 0 and using the continuity of u it follows that $\tilde{u}$ is a weak solution to

$$\tilde{u}_{tt} - \Delta\tilde{u} + 5(u)^4\tilde{u} = f'.$$

Then we can repeat the argument to bound all the higher derivatives and therefore show that $u \in C^\infty(\mathbb{R}^3 \times [0,T))$, the weak solution is identical with the classical solution.

□

In the next theorem and for the rest of this section, we will be switching away from our previous viewpoint of the function u as a function of (t,x), but instead regard it as a function of t only, and for each value of t it is a function of x situated in some Sobolev spaces such as $H^1(\mathbb{R}^n)$. Thus the condition

$$u \in L^2(0,T;H^1(\mathbb{R}^n))$$

means that its norm

$$\begin{aligned}\|u\|_{L^2(0,T;H^1(\mathbb{R}^n))} &= \left\{\int_0^T \|u\|^2_{H^1(\mathbb{R}^n)} dt\right\}^{1/2}\\ &= \left\{\int_0^T \int_{\mathbb{R}^n} |u|^2 + |Du|^2 dx dt\right\}^{1/2}\end{aligned}$$

is finite. In adopting such interpretation of the function u we are expanding the class of permissible solutions to those which only possess the partial derivatives in the distributional or weak sense.

Theorem 7.2 ([10] Chapter 12.2.1 Theorem 1)**.** *(1) Assume $q:\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous with $q(0,0,0)=0$, and if $g\in H^1_{loc}(\mathbb{R}^n)$,$h\in L^2_{loc}(\mathbb{R}^n)$, $f\in L^2(0,T;L^2_{loc}(\mathbb{R}^n))$. Then for each $T>0$ there exists a unique weak solution u to*

$$\begin{cases}u_{tt}-\Delta u+q(Du,u_t,u)=f(x,t) & \mathbb{R}^n\times(0,T)\\ u=g,u_t=h & t=0\end{cases} \tag{7.12}$$

which satisfies

$$\begin{cases}u\in L^\infty(0,T;H^1_{loc}(\mathbb{R}^n))\\ u'\in L^\infty(0,T;L^2_{loc}(\mathbb{R}^n))\\ u''\in L^\infty(0,T;H^{-1}_{loc}(\mathbb{R}^n))\end{cases}$$

(2) If in addition $f\in L^2(0,T;H^1_{loc}(\mathbb{R}^n))\cap L^\infty(0,T;L^2_{loc}(\mathbb{R}^n))$,$g\in H^2_{loc}(\mathbb{R}^n)$,$h\in H^1_{loc}(\mathbb{R}^n)$, then

$$\begin{cases}u\in L^\infty(0,T;H^2_{loc}(\mathbb{R}^n))\\ u'\in L^\infty(0,T;H^1_{loc}(\mathbb{R}^n))\\ u''\in L^\infty(0,T;L^2_{loc}(\mathbb{R}^n))\end{cases}$$

Proof. Define the space

$$X := \{u\in L^\infty(0,T;H^1_0(B(0,R)))|u'\in L^\infty(0,T;L^2(B(0,R)))\},$$

with norm

$$\|u\| := \operatorname*{ess\,sup}_{0\le t\le T}(\|u(t)\|_{H^1_0(B(0,R))}+\|u'(t)\|_{L^2(B(0,R))}).$$

Choose R and T, recast (7.12) in the form

$$\begin{cases}u_{tt}-\Delta u=-q(Dv,v_t,v)+f(x,t) & \text{in } B(0,R)\times(0,T]\\ u=0 & \text{on } \partial B(0,R)\times[0,T]\\ u=g,u_t=h & \text{on } B(0,R)\times\{t=0\},\end{cases} \tag{7.13}$$

from which we define $u=A[v]$ as a map from X to X. This is possible because the linear problem (7.13) admits of a unique solution $u\in X$ whenever $v\in X$.

Then if $w = u - \hat{u}$ for another $\hat{v} \in X$ and $\hat{u} = A[\hat{v}]$,

$$\begin{cases} w_{tt} - \Delta w = q(D\hat{v}, \hat{v}_t, \hat{v}) - q(Dv, v_t, v) & \text{in } B(0,R) \times (0,T] \\ w = 0 & \text{on } \partial B(0,R) \times [0,T] \\ w = g, w_t = h & \text{on } B(0,R) \times \{t = 0\} \end{cases} \tag{7.14}$$

By applying an energy estimate ([10] Chapter 7.2) for linear equations we have

$$\|w\| \leq C \|q(D\hat{v}, \hat{v}_t, \hat{v}) - q(Dv, v_t, v)\|_{L^2(0,T;L^2(B(0,R)))},$$

and because q is Lipschitz continuous,

$$\|w\|^2 \leq C \int_0^T \int_{B(0,R)} |Dv - D\hat{v}|^2 + |v' - \hat{v'}|^2 + |v - \hat{v}|^2 dx dt \leq CT \|v - \hat{v}\|^2.$$

Since $w = u - \hat{u} = A[v] - A[\hat{v}]$, it follows

$$\|A[v] - A[\hat{v}]\| \leq \frac{1}{2} \|v - \hat{v}\|$$

provided $T > 0$ is small enough. Banach fixed point theorem then gives $u \in X$ as the unique solution of (7.12), but only valid within a cylinder of radius R and for a small T.

In order to expand the domain to the whole space, first observe that for all R the solutions are valid for the same T, since the determination of T depends only on the Lipschitz constant. Also by the finite speed of propagation, the solutions overlap on aribitrarily large regions within R^n and their common value determines our solution for the whole space within a fixed T.

Again if we take T as the initial time and repeat the above argument to obtain solutions on the time intervals $[T, 2T]$, $[2T, 3T]$, $\ldots$ the solution is extended to all time.

It remains to prove the second assertion of the theorem. Take $k \in \{1, \ldots .n\}$ and recall that $\tilde{u} := D_k^h u$ is the difference quotient of u. This quotient solves the equation

$$\begin{cases} \tilde{u}_{tt} - \Delta \tilde{u} + b \cdot D\tilde{u} + c\tilde{u} + d\tilde{u}_u = \tilde{f} & \text{in } \mathbb{R}^n \times (0,T) \\ \tilde{u} = \tilde{g}, \tilde{u}_t = \tilde{h} & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases} \tag{7.15}$$

where $\tilde{g} = D_k^h g, \tilde{h} = D_k^h h$, $\tilde{f} = D_k^h(f)$ and

$$\begin{cases} b^j = \int_0^1 q_{p_j}(sDu(x + he_k, t) + (1-s)Du(x,t), su_t(x + he_k, t) \\ +(1-s)u_t(x,t), su(x + he_k, t) + (1-s)u(x,t))ds, \\ c = \int_0^1 q_z(\cdots)ds, d = \int_0^1 q_{p_{n+1}}(\cdots)ds. \end{cases} \tag{7.16}$$

Using the argument above we have

$$\|\tilde{u}\| = \operatorname*{ess\,sup}_{0 \leq t \leq T} (\|\tilde{u}(t)\|_{H^1(B(0,R))} + \|\tilde{u}'(t)\|_{L^2(B(0,R))}) \leq C,$$

where C depends on

$$\begin{cases} \|\tilde{g}\|_{H^1(B(0,2R))} \leq C \|g\|_{H^2(B(0,3R))} \\ \|\tilde{h}\|_{L^2(B(0,2R))} \leq C \|h\|_{H^1(B(0,3R))} \\ \|\tilde{f}\|_{L^2(0,T;L^2(B(0,2R)))} \leq C \|f\|_{L^2(0,T;H^1(B(0,3R)))} \end{cases}$$

Therefore $u \in L^\infty((0,T); H^2_{loc}(\mathbb{R}^n))$ and $u' \in L^\infty((0,T); H^1_{loc}(\mathbb{R}^n))$.

Using the equation $u_{tt} - \Delta u + q(Du, u_t, u) = f$ to estimate $\int_{B(0,R)} u_{tt}^2 dx$ to get $u'' \in L^\infty((0,T); L^2_{loc}(\mathbb{R}^n))$. □

Our next theorem is local in time and without the Lipschitz continuity of q.

Theorem 7.3 ([10] Chapter 12.2.2 Theorem 3)**.** *Assume $q : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ to be C^{k-1}, $q(0,0,0) = 0$. If $g \in H^k(\mathbb{R}^n)$, $h \in H^{k-1}(\mathbb{R}^n)$, $f \in L^\infty((0,T); H^{k-1}(\mathbb{R}^n))$.*

Then there is a $T > 0$ so that

$$\begin{cases} u_{tt} - \Delta u + q(Du, u_t, u) = f(x,t) & \text{in } \mathbb{R}^n \times (0,T] \\ u = g,\ u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\} \end{cases} \tag{7.17}$$

admits a unique weak solution u, with

$$u \in L^\infty(0,T; H^k(\mathbb{R}^n)),\ u' \in L^\infty(0,T; H^{k-1}(\mathbb{R}^n)).$$

Proof. Define

$$X := \{u \in L^\infty(0,T; H^1(\mathbb{R}^n)),\ u' \in L^\infty(0,T; L^2(\mathbb{R}^n))\}$$

with norm

$$\|u\| := \operatorname*{ess\,sup}_{0 \le t \le T} (\|u(t)\|_{H^1(\mathbb{R}^n)} + \|u'(t)\|_{L^2(\mathbb{R}^n)}).$$

and with a stronger norm

$$|||u||| := \operatorname*{ess\,sup}_{0 \le t \le T} (\|u(t)\|_{H^k(\mathbb{R}^n)} + \|u'(t)\|_{H^{k-1}(\mathbb{R}^n)}).$$

For $\lambda > 0$ also define

$$X_\lambda := \{u \in X; |||u||| \le \lambda, u(0) = g, u'(0) = h\}.$$

Recast (7.17) into

$$\begin{cases} u_{tt} - \Delta u = -q(Dv, v_t, v) + f(x,t) & \mathbb{R}^n \times (0,T] \\ u = g,\ u_t = h & t = 0, \end{cases} \tag{7.18}$$

denote as before $u = A[v]$, next we intend to show that A is a strict contraction and is from X_λ to X_λ. For the later assertion it is sufficient to prove $|||u||| \le \lambda$.

To this end we examine the triple norm more closely and try to derive an estimate of $|||u|||$ in terms of $|||v|||$. Firstly define

$$E_k(t) = \|u(t)\|^2_{H^k(\mathbb{R}^n)} + \|u'(t)\|^2_{H^{k-1}(\mathbb{R}^n)}$$

and

$$F_k(t) = \|v(t)\|^2_{H^k(\mathbb{R}^n)} + \|v'(t)\|^2_{H^{k-1}(\mathbb{R}^n)}.$$

Then the following holds

$$E_k(t) \le E_k(0) + C \int_0^t \Psi(F_k(s)) ds + C't \qquad (0 \le t \le T) \tag{7.19}$$

for some continuous and monotone function Ψ which depends on n, k and f. (The derivation of (7.19) can be found in [10] Chapter 12.2.2, we prefer to omit it here.)

Now from (7.19)

$$|||u|||^2 \le \|g\|^2_{H^k(\mathbb{R}^n)} + \|h\|^2_{H^{k-1}(\mathbb{R}^n)} + CT\Psi(|||v|||^2) + C'T \tag{7.20}$$

Fix λ by

$$\lambda^2 := 2(\|g\|^2_{H^k(\mathbb{R}^n)} + \|h\|^2_{H^{k-1}(\mathbb{R}^n)})$$

and by a choice of T so that

$$CT\Psi(\lambda^2) + C'T \le \frac{\lambda^2}{2},$$

(7.20) then becomes $\|\!|A[v]|\!\|^2 \le \lambda^2$, and so $A[v] = u \in X_\lambda$.

To show that A is a strict contraction, or

$$\|A[v] - A[\hat{v}]\| \le \frac{1}{2}\|v - \hat{v}\| \tag{7.21}$$

for every $v, \hat{v} \in X_\lambda$. Denote $w = u - \hat{u} = A[v] - A[\hat{v}]$, integration by parts gives

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n} |Dw|^2 + w_t^2 + w^2 dx \\
&\qquad = 2\int_{\mathbb{R}^n} w_t(q(D\hat{v}, \hat{v}_t, \hat{v}) - q(Dv, v_t, v) + w)dx \\
&\qquad \le \int_{\mathbb{R}^n} w_t^2 + w^2 dx + C\int_{\mathbb{R}^n} |D\hat{v} - Dv|^2 + |\hat{v}_t - v_t|^2 + |\hat{v} - v|^2 dx,
\end{aligned}$$

where C depending on $\|D\hat{v}, \hat{v}_t\hat{v}, Dv, v_t, v\|_{L^\infty}$. By Gronwall lemma

$$\begin{aligned}
&\max_{0\le t\le T}\int |Dw|^2 + |w_t|^2 + w^2 dx \\
&\qquad \le C\int_0^T\int_{\mathbb{R}^n} |Dv - D\hat{v}|^2 + |v_t - \hat{v}_t|^2 + |v - \hat{v}|^2 dxdt \\
&\qquad \le CT\|v - \hat{v}\|^2,
\end{aligned}$$

on taking T small we have (7.21).

Finally choose $u_0 \in X_\lambda$, denote $u_{k+1} := A[u_k]$ for $k = 0, \ldots,$, then $u_k \to u$ in X and

$$A[u] = u.$$

Moreover, $\|\!|u_k|\!\| \le \lambda$ gives $u \in X_\lambda$. (7.21) proves uniqueness.

□

Finally, we have one more existence theorem concerning the simpler form of nonlinearity where q depends on u only.

Theorem 7.4 ([10] Chapter 12.3.1 Theorem 1)**.** *Assume $q : \mathbb{R} \to \mathbb{R}$ continuous and satisfies the sign condition*

$$zq(z) \ge 0 \qquad z \in \mathbb{R} \tag{7.22}$$

with $q(0) = 0$. If $g \in H^1(\mathbb{R}^n)$, $h \in L^2(\mathbb{R}^n)$, $Q(g) \in L^1(\mathbb{R}^n)$, $f \in L^2((0,T); H^1(\mathbb{R}^n)) \cap L^\infty((0,T); L^2(\mathbb{R}^n))$. Then the semilinear problem

$$\begin{cases} u_{tt} - \Delta u + q(u) = f & \mathbb{R}^n \times (0,\infty) \\ u = g, u_t = h & t = 0 \end{cases} \tag{7.23}$$

admits of a weak solution u for all $T > 0$, with

$$\begin{cases} u \in L^2_{loc}((0,T) \times \mathbb{R}^n) \cap L^\infty((0,T); H^1(\mathbb{R}^n)) \\ u' \in L^\infty((0,T); L^2(\mathbb{R}^n)) \\ Q(u) \in L^\infty((0,T); L^1(\mathbb{R}^n)) \end{cases}$$

Moreover, the following energy growth estimate holds

$$E(t) \leq e^T E(0) + C,$$

where

$$E(t) = \int_{\mathbb{R}^n} \frac{1}{2}(u_t^2 + |Du|^2) + Q(u)dx,$$

with $Q' = q$.

Proof. Because of the sign condition on q,

$$Q(z) = \int_0^z q(w)dw \geq 0$$

is increasing for $z \geq 0$ and decreasing for $z \leq 0$.

By Stone-Weierstrass theorem ([21], theorem 7.26) we can select a sequence of continuous functions $Q_k : \mathbb{R} \to \mathbb{R}$ so that

$$Q_k \geq 0, Q_k \to Q \,\text{pointwise}, Q_k \leq Q, Q_k \equiv Q \,\text{on}\, [-k, k]$$

with $q_k = Q_k'$ Lipschitz continuous, and $zq_k(z) \geq 0$ for every $z \in \mathbb{R}$. Also take $\{g_k\}_{k=1}^\infty, \{h_k\}_{k=1}^\infty \in C_0^\infty$ and $g_k, h_k \to g, h$ in H^1, L^2 respectively, and assume $|g_k| \leq |g|$,$|h_k| \leq |h|$, $|Dg_k| \leq |Dg|$ a.e. for all $k = 1, 2, \dots$.

Now the equation

$$\begin{cases} u_{tt}^k - \Delta u^k + q_k(u^k) = f(x,t) & \mathbb{R}^n \times (0,T) \\ u^k = g_k, u_t^k = h_k & t = 0 \end{cases} \tag{7.24}$$

admits of a solution u^k by theorem 7.2 since q_k is Lipschitz continuous. And by the second part of the same theorem gives

$$\begin{cases} u^k, Du^k, u_t^k \in C((0,T); L^2(\mathbb{R}^n)), \\ D_x^2 u^k, D_x u_t^k, u_{tt}^k \in L^\infty((0,T); L^2(\mathbb{R}^n)). \end{cases}$$

If we then calculate

$$\begin{aligned} \dot{E}_k(t) &= \frac{d}{dt} \int_{\mathbb{R}^n} \frac{1}{2}((u_t^k)^2 + |Du^k|^2) + Q_k(u^k)dx = \int_{\mathbb{R}^n} u_t^k f dx \\ &\leq E_k(t) + \frac{1}{2}\|f\|^2_{L^\infty((0,T);L^2(\mathbb{R}^n))} \end{aligned}$$

Gronwall lemma gives

$$E_k(t) \leq e^T E_k(0) + C \tag{7.25}$$

where

$$E_k(0) = \int_{\mathbb{R}^n} \frac{1}{2}(h_k^2 + |Dg_k|^2) + Q_k(g_k)dx.$$

Pass onto a subsequence if necessary such that $Q_k(g_k) \to Q(g)$ a.e., and because $|g_k| \leq |g|$ we have $0 \leq Q_k(g_k) \leq Q(g)$ a.e. for all but finitely many k. Since $Q(g) \in L^1$, the Dominated Convergence Theorem then gives

$$E_k(0) \to \int_{\mathbb{R}^n} \frac{1}{2}(|Dg|^2 + |h|^2) + Q(g)dx = E(0).$$

Inequality (7.25) then implies

$$E_k(t) \leq C(T) \tag{7.26}$$

uniformly in k.

Since $Q_k \geq 0$,

$$\max_{0\leq t\leq T} \|u_t^k, Du^k\|_{L^2(\mathbb{R}^n)} \leq C(T) \tag{7.27}$$

and by Gronwall lemma

$$\max_{0\leq t\leq T} \|u^k\|_{L^2(\mathbb{R}^n)} \leq C(T). \tag{7.28}$$

Therefore

$$u^k \in H^1_{loc}(\mathbb{R}^n \times (0,T)),$$

and by Rellich-Kondrachov compact embedding theorem

$$H^1_{loc}(\mathbb{R}^n \times (0,T)) \subset\subset L^2_{loc}(\mathbb{R}^n \times (0,T)),$$

i.e. each bounded sequence in the former space has a subsequence that converges in the later. Also $L^2_{loc}(\mathbb{R}^n \times (0,T))$ is a reflexive Banach space, so that each bounded sequence has a subsequence that is weakly convergent in the same space. Hence there exists the subsequences

$$\begin{cases} u^k \to u & \text{strongly in } L^2_{loc}(\mathbb{R}^n \times (0,T)) \text{ and } a.e., \\ Du^k, u^k_t \rightharpoonup Du, u_t & \text{weakly in } L^2_{loc}(\mathbb{R}^n \times (0,T)). \end{cases}$$

It remains to show the convergence of the term $q_k(u^k)$ to $q(u)$. As a first step we prove that $\{g^k := q_k(u^k)\}_{k=1}^\infty$ are *uniformly integrable*. i.e. for $\epsilon > 0$, there is $\delta > 0$ so that if $E \subset \mathbb{R}^n \times [0,T)$ and $|E| \leq \delta$, then $\iint_E |g^k| dxdt \leq \epsilon$ for every k.

By multiplying (7.24) by u^k and integrating on $\mathbb{R}^n \times [0,T)$,

$$\int_0^T \int_{\mathbb{R}^n} |Du^k|^2 - (u^k_t)^2 + u^k q_k(u^k) dxdt = \int_{\mathbb{R}^n} u^k_t u^k dx\Big|_{t=0}^{t=T} + \int_0^T \int_{\mathbb{R}^n} u^k f dxdt.$$

Because $u^k q_k(u^k) \geq 0$, (7.27) and (7.28) gives

$$\int_0^T \int_{\mathbb{R}^n} |u^k q_k(u^k)| dxdt \leq C. \tag{7.29}$$

Then

$$\begin{aligned} \iint_E |g^k| dxdt &= \iint_{E\cap\{|u^k|\geq\lambda\}} |q_k(u^k)| dxdt + \iint_{E\cup\{|u^k|\geq\lambda\}} |q_k(u^k)| dxdt \\ &\leq \frac{1}{\lambda} \iint_E |u^k q_k(u^k)| dxdt + |E| \max_{|y|\leq\lambda} |q_k(y)| \\ &\leq \frac{1}{\lambda} C(T) + |E| \max_{|y|\leq\lambda} |q_k(y)|. \end{aligned}$$

By choosing λ so large that $\frac{C(T)}{\lambda} \leq \frac{\epsilon}{2}$, it follows

$$\iint_E |g^k| dxdt \leq \frac{\epsilon}{2} + |E| \max_{|y|\leq\lambda} |q(y)| \leq \epsilon$$

if $|E| \leq \delta$ for small $\delta > 0$. Thus the uniform integrability is proven.

Next we show

$$q_k(u^k) \to q(u) \quad \text{in } L^1_{loc}(\mathbb{R}^n).$$

Denote $g^k := q_k(u^k)$, write $g := q(u)$, and choose $R > 0$, $\epsilon > 0$. Because of uniform integrability there is $\delta > 0$ such that $|E| \le \delta$ gives $\int_E |g^k| dx \le \epsilon$ for $k = 1, \ldots$. By Egoroff's theorem, there exists a E such that $|E| \le \delta$ and $g^k \to g$ uniformly in $B(0,R) - E$. Hence

$$\limsup_{k,l\to\infty} \int_{B(0,R)} |g^k - g^l| dx \le \limsup_{k,l\to\infty} \int_E |g^k - g^l| dx \le 2\epsilon.$$

for every $\epsilon > 0$. Therefore $\{g^k\}_{k=1}^{\infty}$ is a Cauchy sequence in $L^1(B(0,R))$ that has a limit $\hat{g} \in L^1(B(0,R))$. Because $g^k \to g$ a.e., we conclude $g = \hat{g}$.

Since

$$u_{tt}^k - \Delta u^k + q_k(u^k) = f \qquad \text{in } \mathbb{R}^n \times (0,T),$$

by multiplying a test function and passing to the limits, it follows that

$$u_{tt} - \Delta u + q(u) = f \qquad \text{in } \mathbb{R}^n \times (0,T)$$

holds in the weak sense. Moreover by the property of weak convergence

$$\int_{\mathbb{R}^n} u_t^2 + |Du|^2 dx \le \liminf_{k\to\infty} \int_{\mathbb{R}^n} (u_t^k)^2 + |Du^k|^2 dx;$$

and by Fatou's Lemma

$$\int_{\mathbb{R}^n} Q(u) dx \le \liminf_{k\to\infty} \int_{\mathbb{R}^n} Q_k(u^k) dx.$$

Those two inequalities along with (7.26) yield the growth estimate. □

8. Control theory

In this section we present some elements of "Control Theory" which has its origin in the study of centrifugal governors by J.Maxwell in 1868. Problems in this area are usually concerned with the optimization of a mechanical system through an analysis of the underlying equation. The following presentation will be entirely free of physical interpretation and only heed mathematical facts.

To illustrate our aim consider for example the equation

$$\begin{aligned} \dot{y} &= (Ay + BF(y)) + Bu \\ y(0) &= a, \quad y(T) = b \end{aligned} \tag{8.1}$$

where the solution y is interpreted as a function of t with values in function spaces just as we did in the last section. Let it be a Hilbert space H. A and B are some linear operators from H to H. Note that here not only a initial state but also a terminal state is prescribed on the solution y, and the objective is to determine that control u which can accomplish this in an optimal manner. This optimality can be expressed in terms of

$$E_T(a,b) = \frac{1}{2} \inf_u \int_0^T \|u(s)\|_H^2 ds, \tag{8.2}$$

the minimum energy required by the control to transfer y from a to b within a certain time T. Or if we disregard the time frame and require the minimum across all scenarios of the transfer,

$$E_\infty(a,b) = \inf_{T>0} E_T(a,b) \tag{8.3}$$

is the optimum energy which bears the name *quasipotential*.

It is our aim of this section to search for a simple formula which can express $E_\infty(a,b)$ in terms of a and b.

8.1. **Preliminaries.**

Definition 8.1 (Linear operators in Hilbert space)**.** A complex vector space H is called an *inner product space* (or unitary space) if to each ordered pair of vectors x and y in H is associated a complex number (x,y), called the *inner product* or *scalar product* of x and y, such that the following rules hold:

(a) $(y,x) = (\overline{x,y})$.
(b) $(x+y,z) = (x,z) + (y,z)$.
(c) $(\alpha x, y) = \alpha(x,y)$ if $x \in H, y \in H, \alpha \in \mathbb{C}$.
(d) $(x,x) \geq 0$ for all $x \in H$.
(e) $(x,x) = 0$ only if $x = 0$.

Every inner product space can be normed by defining

$$\|x\| = (x,x)^{1/2}.$$

If the resulting normed space is complete, it is called a *Hilbert space.*

A linear map A from a Hilbert space H to itself will be called a *linear operator*. It is bounded if and only if its norm

$$\|A\| = \sup\{\|Ax\| : x \in H, \|x\| \leq 1\}$$

is finite. The family of bounded operators on H will be denoted as $\mathcal{B}(H)$.

The *graph* $\mathcal{G}(A)$ of an operator A in H is the subspace of $H \times H$ that consists of the ordered pairs $\{x, Ax\}$, where x ranges over $\mathcal{D}(A)$, the domain of A. A *closed* operator in H is one whose graph is a closed subspace of $H \times H$. By the closed graph theorem, $A \in \mathcal{B}(H)$ if and only if $\mathcal{D}(A) = H$ and A is closed.

Theorem 8.2 (Positive operator)**.** *Suppose $A \in \mathcal{B}(H)$. Then*

(a) $(Ax,x) \geq 0$ for every $x \in H$ if and only if
(b) $A = A^*$ and $\sigma(A) \subset [0,\infty)$,
where $\sigma(A)$ is the spectrum of A, the set of complex numbers λ for which $\lambda I - A$ is not invertible.

Definition 8.3 (Fractional power operators)**.** The resolvent set $\rho(A)$ of A is the set of all complex numbers λ for which $\lambda I - A$ is invertible, i.e. $(\lambda I - A)^{-1}$ is a bounded linear operator in H. The family $R(\lambda : A) = (\lambda I - A)^{-1}$, $\lambda \in \rho(A)$ of bounded linear operators is called the resolvent of A.

Let A be a densely defined closed linear operator for which

$$\rho(A) \supset \Sigma^+ = \{\lambda : 0 < \omega < |\arg \lambda| \leq \pi\} \cup V \tag{8.4}$$

where V is a neighborhood of zero, and

$$\|R(\lambda : A)\| \leq \frac{M}{1+|\lambda|} \qquad for \quad \lambda \in \Sigma^+.$$

For an operator A satisfying Assumption (8.4) and $\alpha > 0$ we define

$$A^{-\alpha} = \frac{1}{2\pi i}\int_C z^{-\alpha}(A - zI)^{-1} dz \tag{8.5}$$

where the path C runs in the resolvent set of A from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$, $\omega < \theta < \pi$, avoiding the negative real axis and the origin and $z^{-\alpha}$ is taken to be positive for real positive values of z. The integral (8.5) converges in the uniform operator topology for every $\alpha > 0$ and thus defines a bounded linear operator $A^{-\alpha}$.

Let A satisfy Assumption (8.4) with $\omega < \pi/2$. For every $\alpha > 0$ we define

$$A^{\alpha} = (A^{-\alpha})^{-1}. \tag{8.6}$$

For $\alpha = 0$, $A^{\alpha} = I$.

Theorem 8.4. *Let A^{α} be defined as in* (8.6)*, then it has the following properties:*

(a) *A^{α} is a closed operator with domain $D(A^{\alpha}) = R(A^{-\alpha})$ = the range of $A^{-\alpha}$.*
(b) *$\alpha \geq \beta > 0$ implies $D(A^{\alpha}) \subset D(A^{\beta})$.*
(c) *$\overline{D(A^{\alpha})} = H$ for every $\alpha \geq 0$.*
(d) *If α, β are real then*
$$A^{\alpha+\beta}x = A^{\alpha} \cdot A^{\beta}x$$
for every $x \in D(A^{\gamma})$ where $\gamma = \max(\alpha, \beta, \alpha + \beta)$.

Definition 8.5 (Semigroups generated by operators)**.** Let X be a Banach space. A one parameter family $T(t)$, $0 \leq t < \infty$, of bounded linear operators from X into X is a *semigroup of bounded linear operators* on X if

(a) $T(0) = I$, (I is the identity operator on X).
(b) $T(t + s) = T(t)T(s)$ for every $t, s \geq 0$.

A semigroup of bounded linear operators, $T(t)$, is *uniformly continuous* if

$$\lim_{t\downarrow 0} \|T(t) - I\| = 0. \tag{8.7}$$

The linear operator A defined by

$$D(A) = \left\{x \in X : \lim_{t\downarrow 0} \frac{T(t)x - x}{t} \text{exists}\right\} \tag{8.8}$$

and

$$Ax = \lim_{t\downarrow 0} \frac{T(t)x - x}{t} = \left.\frac{d^{+}T(t)x}{dt}\right|_{t=0} \qquad \text{for} \quad x \in D(A) \tag{8.9}$$

is the *infinitesimal generator* of the semigroup $T(t)$, $D(A)$ is the domain of A.

Theorem 8.6. *A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator.*

Theorem 8.7. *Let $T(t)$ be a uniformly continuous semigroup of bounded linear operators. Then*

(a) *There exists a constant $\omega \geq 0$ such that $\|T(t)\| \leq e^{\omega t}$.*
(b) *There exists a unique bounded linear operator A such that $T(t) = e^{tA}$.*
(c) *The operator A in part (b) is the infinitesimal generator of $T(t)$.*

Definition 8.8. A semigroup $T(t)$, $0 \leq t < \infty$, of bounded linear operators on X is a *strongly continuous* semigroup if

$$\lim_{t\downarrow 0} T(t)x = x \qquad \text{for every} \quad x \in X. \tag{8.10}$$

A strongly continuous semigroup of bounded linear operators on X will be called a *semigroup of class* C_0 or simply a C_0 *semigroup.*

Theorem 8.9. *Let* $T(t)$ *be a* C_0 *semigroup. There exist constants* $\omega \geq 0$ *and* $M \geq 1$ *such that*

$$\|T(t)\| \leq Me^{\omega t} \qquad \text{for} \quad 0 \leq t < \infty. \tag{8.11}$$

Definition 8.10. Condition (8.11) is called *uniform boundness* of the C_0 semigroup $T(t)$. If moreover $M = 1$ it is called a C_0 *semigroup of contractions.*

Theorem 8.11 (Hille-Yosida)**.** *A linear (unbounded) operator* A *is the infinitesimal generator of a* C_0 *semigroup of contractions* $T(t)$*,* $t \geq 0$ *if and only if*

(a) A *is closed and* $\overline{D(A)} = X$*.*
(b) *The resolvent set* $\rho(A)$ *of* A *contains* $\mathbb{R}^+$ *and for every* $\lambda > 0$

$$\|R(\lambda : A)\| \leq \frac{1}{\lambda}. \tag{8.12}$$

Definition 8.12. Let $\Delta = \langle z : \varphi_1 < \arg z < \varphi_2, \varphi_1 < 0 < \varphi_2 \rangle$ and for $z \in \Delta$ let $T(z)$ be a bounded linear operator. The family $T(z)$, $z \in \Delta$ is an *analytic semigroup* in Δ if

(a) $z \to T(z)$ is analytic in Δ.
(b) $T(0) = I$ and $\lim_{z \to 0, z \in \Delta} T(z)x = x$ for every $x \in X$.
(c) $T(z_1 + z_2) = T(z_1)T(z_2)$ for z_1, $z_2 \in \Delta$.

Definition 8.13 (Interpolation space $D_A(\theta, p)$)**.** If A generates a C_0 semigroup $T(t)$. Then

$$D_A(\theta, p) = \{x \in X : t \mapsto \psi(t) = t^{-\theta}\|T(t)x - x\| \in L^p_*(0, \infty)\}. \tag{8.13}$$

where L^p_* is the weighted L^p space endowed with the norm

$$\|f\|_{L^p_*(0,\infty)} = \left(\int_0^\infty |f(t)|^p \frac{dt}{t}\right)^{1/p}, \qquad \text{if}\, p < \infty,$$

$$\|f\|_{L^\infty_*(0,\infty)} = \operatorname{ess} \sup_{t \in (0,\infty)} |f(t)|.$$

Definition 8.14 (Compact operators)**.** Let X and Y be real Banach spaces. A bounded linear operator

$$A : X \to Y$$

is called compact provided for each bounded sequence $\{u_k\}_{k=1}^\infty \subset X$, the sequence $\{Au_k\}_{k=1}^\infty$ is precompact in Y; that is, there exists a subsequence $\{u_{k_j}\}_{j=1}^\infty$ such that $\{Au_{k_j}\}_{j=1}^\infty$ converges in Y.

Theorem 8.15. *Let* H *be a Hilbert space. If the operator* $A : H \to H$ *is compact, so is* $A^* : H \to H$*.*

8.2. **Linear systems.** We now assume the operator A to be negative definite and self-adjoint. Let $C : H \to H$ be a bounded operator. Consider the controlled systems:

$$\dot{y} = Ay + u, \qquad y(0) = x \in H \tag{8.14}$$

$$\ddot{y} = Ay + C\dot{y} + u, \qquad y(0) \in D(-A)^{1/2}, \qquad \dot{y}(0) \in H \tag{8.15}$$

Note that the second order equation (8.15) may be viewed as a system of two first order equations:

$$\begin{cases} v = \dot{y} \\ \dot{v} = Ay + C\dot{y} + u \end{cases} \tag{8.16}$$

or equivalently

$$\dot{z} = \mathcal{A}z + \mathcal{B}u, \tag{8.17}$$

where $z = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$, $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & C \end{bmatrix}$, and $\mathcal{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}$. The operator $\mathcal{A}$ with domain $D(A) \times D(-A)^{1/2}$ defines a C_0-semigroup $\mathcal{S}(t)$ on $\mathcal{H} = D(-A)^{1/2} \times H$.

Definition 8.16. The reachable set is the set of possible states $\mathcal{R}_T$ which can be transferred to from the state 0 in T by all controls $u(\cdot) \in L^2[0, T; H]$. If $\mathcal{R}_T$ is the whole space, the system is called exactly controllable. The system is called null controllable for $T > 0$ if any state $a \in H$ can be transferred back to state 0 in T.

Theorem 8.17 ([7] Theorem 2.4)**.** *Suppose A is negative definite and self-adjoint, with A^{-1} compact. If A generates an analytic semigroup on H, then the reachable set for* (8.14) *is independent of time and is $D_A(1/2, 2)$.*

Proof. Let us choose and fix $T > 0$. We will show that the reachable set $\mathcal{R}_T$ at time T for system (8.14) is a subset of $D_A(1/2, 2)$.
For this aim let us take an arbitrary element x which belongs $\mathcal{R}_T$. In other words, there exist $u \in L^2(0, T; H)$ such that

$$x = y(T), \tag{8.18}$$

where $y : [0, T] \to H$ is a mild solution of the following problem

$$\dot{y} = Ay + u, \qquad y(0) = 0 \in H. \tag{8.19}$$

By definition, a mild solution of (8.19) is given by formula

$$y(t) = \int_0^t S(t-s)u(s)\, ds, \ \ t \in [0, T]. \tag{8.20}$$

Moreover, we have that

$$y \in W^{1,2}(0, T; H) \cap L^2(0, T; D(A)). \tag{8.21}$$

Property (8.21) is the so called maximal regularity and it's true under some quite general properties of the operator A. In our case when A is a self-adjoint operator and negative operator in H such that A^{-1} is compact, this property can be proved by direct calculations using the ONB associated with A. Also, when $A = \Delta$, this property can be shown by using

the Fourier transform.
Then the assertion in the theorem follows directly from the definition of $D_A(1/2,2)$:

$$D_A(1/2,2) := \Big\{x \in H : \exists y \in W^{1,2}(0,T;H) \cap L^2(0,T;D(A)) : y(T) = x\Big\}. \tag{8.22}$$

□

Proof that $y \in W^{1,2}(0,T;H) \cap L^2(0,T;D(A))$. We first assume the domain to be $(0,\infty)$, and $u \in L^2(0,\infty;H)$ with compact support in $(0,\infty)$. Under the assumption that $A = A^*$, $-A \geq 0$, A^{-1} is compact, there exists an orthonormal basis $\{e_j\}_{j=1}^\infty$ in H such that

$$-Ae_j = \lambda_j e_j$$

with $\lambda_j > 0$, $\lambda_j \to \infty$ as $j \to \infty$. We can then decompose equation (8.19) in the form

$$\sum_{j=1}^\infty \dot{y}_j(t)e_j = -\sum_{j=1}^\infty \lambda_j y_j(t)e_j + \sum_{j=1}^\infty u_j(t)e_j.$$

Matching the coefficients for e_j and taking into account $y_j(0) = 0$ to get

$$y_j(t) = \int_0^t e^{-\lambda_j(t-s)}u_j(s)ds,$$

and the estimate

$$\begin{aligned}\int_0^\infty \|Ay(t)\|^2dt &= \int_0^\infty \sum_{j=1}^\infty \lambda_j^2|y_j(t)|^2dt\\ &= \sum_{j=1}^\infty \int_0^\infty \left|\int_0^t \lambda_j e^{-\lambda_j(t-s)}u_j(s)ds\right|^2 dt\end{aligned}$$

The last integral is the convolution $\|f * g\|^2_{L^2(0,\infty;H)}$ where

$$f(s) = \begin{cases} \lambda_j e^{-\lambda_j s} & \text{if } s > 0\\ 0 & \text{if } s \leq 0\end{cases}$$

and $g(s) = u_j(s)$. Apply Young's theorem $\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}$ with $1 + 1/r = 1/p + 1/q$ when $r = q = 2$ to get

$$\begin{aligned}\int_0^\infty \|Ay(t)\|^2dt &\leq \sum_{j=1}^\infty \left|\int_0^\infty \lambda_j e^{-\lambda_j s}ds\right|^2 \int_0^\infty |u_j(s)|^2ds\\ &= \|u\|^2_{L^2(0,\infty;H)}.\end{aligned}$$

Thus from equation (8.19) it follows that

$$\|\dot{y}\|_{L^2(0,\infty;H)} \leq \|Ay\|_{L^2(0,\infty;H)} + \|u\|_{L^2(0,\infty;H)} \leq 2\|u\|_{L^2(0,\infty;H)},$$

also

$$\begin{aligned}\int_0^\infty \|y(t)\|^2dt &\leq \int_0^\infty \|A^{-1}Ay(t)\|^2dt\\ &\leq \|A^{-1}\| \int_0^\infty \|Ay(t)\|^2dt\\ &\leq C\|u\|^2_{L^2(0,\infty;H)}.\end{aligned}$$

The orthonormal basis $\{e_j\}_{j=1}^{\infty}$ in H can be normalized with respect to $D(A)$ then the above also holds in $D(A)$. As a result $y \in W^{1,2}(0,\infty;H) \cap L^2(0,\infty;D(A))$.

Finally, it can be shown that the interval $(0,\infty)$ can be replaced by $(0,T)$, so that $u \in L^2(0,T;H)$. Set

$$\tilde{u}(t) = \begin{cases} u(t) & t < T \\ 0 & t \geq T \end{cases}$$

and solve

$$\dot{\tilde{y}} = A\tilde{y} + \tilde{u}, \qquad \tilde{y}(0) = 0 \in H$$

in the interval $(0,\infty)$, and choose

$$y(t) = \begin{cases} \tilde{y}(t) & t < T \\ 0 & t \geq T \end{cases} .$$

Then y is the solution to

$$\dot{y} = Ay + u, \qquad y(0) = 0 \in H$$

in the interval $(0,T)$, and we have

$$\|Ay\|_{L^2(0,T;H)} \leq \|A\tilde{y}\|_{L^2(0,\infty;H)} \leq \|\tilde{u}\|_{L^2(0,\infty;H)} = \|u\|_{L^2(0,T;H)}.$$

□

Theorem 8.18 ([8] Theorem 3.1)**.** *Suppose A is negative definite and self-adjoint, with A^{-1} compact.*

(1) *Equation* (8.14) *is null controllable for any T. The reachable space for* (8.14) *is independent of time and is $D((-A)^{1/2})$.*
(2) *Equation* (8.15) *is exactly controllable.*

Proof of Theorem 8.18. Part (1). By the Lemma on p.208 of [30], null controllability in this case is equivalent to

$$\int_0^T \|S(r)x\|^2 dr \geq k\|S(T)x\|^2, \qquad x \in H \tag{8.23}$$

for some constant $k > 0$. Assume for the moment the operator $-A$ is invertible and bounded, using the orthonormal basis for the decomposition of $-A$, we then have

$$\begin{aligned}\left\langle (e^{-2AT} - I)x, x \right\rangle &= \sum_{j=1}^{\infty} [e^{2\lambda_j T} - 1] x_j^2 \\ &\geq \sum_{j=1}^{\infty} [2\lambda_j T] x_j^2 = \langle -2ATx, x \rangle ,\end{aligned}$$

and $e^{-2AT} - I \geq -2AT$, or

$$(-2A)^{-1}(I - e^{2AT}) \geq Te^{2AT}.$$

The left hand side of the last inequality is equal to $\int_0^T S(2r)dr$ by the calculation

$$\begin{aligned}\left[\int_0^T S(2r)dr\right](x) &= \left[\int_0^T e^{2rA}dr\right](x)\\ &= \int_0^T \sum_{j=1}^{\infty} e^{-2\lambda_j r} x_j e_j dr\\ &= \sum_{j=1}^{\infty} \frac{1}{2\lambda_j}(1-e^{-2\lambda_j T})x_j e_j\\ &= [(-2A)^{-1}(I-e^{2AT})](x),\end{aligned}$$

therefore

$$\int_0^T S(2r)dr \geq TS(2T), \tag{8.24}$$

and this is (8.23) for $k = T$ because

$$\begin{aligned}\left\langle\left[\int_0^T S(2r)dr\right](x), x\right\rangle &= \int_0^T \langle S(r)S(r)x, x\rangle\, dr\\ &= \int_0^T \|S(r)x\|^2 dr\end{aligned}$$

since A is self-adjoint.

Note that

$$R_T := \int_0^T S(2r)dr = (-2A)^{-1}(I - S(2T))$$

and

$$R_T^{1/2} = (I - S(2T))^{1/2}((-2A)^{1/2})^{-1}. \tag{8.25}$$

Since the operator $(I - S(2T))^{1/2}$ transforms $D((-A)^{1/2})$ onto $D((-A)^{1/2})$ for all T the second assertion of Part (1) holds. □

Claim 8.19. *The operator $(I - S(2T))^{1/2}$ transforms $D((-A)^{1/2})$ onto $D((-A)^{1/2})$.*

Proof of Claim 8.19 . Take $x \in D((-A)^{1/2})$ so that $\|x\|^2 = \sum_{j=1}^{\infty} \lambda_j x_j^2 < \infty$, then

$$\|(I - S(2T))^{1/2}(x)\|^2 = \sum_{j=1}^{\infty} \lambda_j x_j^2 (1 - e^{-2\lambda_j T}) \leq \sum_{j=1}^{\infty} \lambda_j x_j^2 < \infty.$$

Therefore the operator maps into $D((-A)^{1/2})$. To show it is also onto, take any $y \in D((-A)^{1/2})$, and set $x_j = (1 - e^{-2\lambda_j T})^{-1/2} y_j$, so that $(I - S(2T))^{1/2}x = y$, and

$$\begin{aligned}\sum_{j=1}^{\infty} \lambda_j x_j^2 &= \sum_{j=1}^{\infty} \lambda_j (1 - e^{-2\lambda_j T})^{-1} y_j^2\\ &\leq \sup(1 - e^{-2\lambda_j T})^{-1} \sum_{j=1}^{\infty} \lambda_j y_j^2 < \infty,\end{aligned}$$

provided $\{\lambda_j\}_{j=1}^{\infty}$ is increasing and strictly positive. Hence $(I - S(2T))^{1/2}$ is also onto. □

Proof of Theorem 8.18. Part (2). Without loss of generality assume $C = 0$. We first find the form of the semigroup $\mathcal{S}(t)$ generated by $\mathcal{A}$. Consider the uncontrolled equation

$$\ddot{y} = Ay, \tag{8.26}$$

decompose using the orthonormal basis to get

$$\frac{d^2 y_j}{dt^2} = -\lambda_j y_j(t) \qquad j = 1, 2, \dots.$$

each of those has a fundamental solution of the form

$$y_j = \alpha \cos(\sqrt{\lambda_j} t) + \beta \sin(\sqrt{\lambda_j} t).$$

Determine α and β by the initial conditions $y_j(0) = y_{0,j}, \dot{y}_j(0) = y_{1,j}$, so that

$$y_j = y_{0,j} \cos(\sqrt{\lambda_j} t) + \frac{y_{1,j}}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j} t)$$

$$\dot{y_j} = -y_{0,j} \sqrt{\lambda_j} \sin(\sqrt{\lambda_j} t) + y_{1,j} \cos(\sqrt{\lambda_j} t)$$

or in the compact form

$$z_j(t) = \begin{bmatrix} \cos(\sqrt{\lambda_j} t) & \frac{1}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j} t) \\ -\sqrt{\lambda_j} \sin(\sqrt{\lambda_j} t) & \cos(\sqrt{\lambda_j} t) \end{bmatrix} \begin{bmatrix} y_{0,j} \\ y_{1,j} \end{bmatrix}.$$

By comparison with $z(t) = \mathcal{S}(t) \left[\begin{smallmatrix} y_0 \\ y_1 \end{smallmatrix} \right]$ we deduce

$$\mathcal{S}(t) = \begin{bmatrix} \cos(\sqrt{-A} t) & \frac{1}{\sqrt{-A}} \sin(\sqrt{-A} t) \\ -\sqrt{-A} \sin(\sqrt{-A} t) & \cos(\sqrt{-A} t) \end{bmatrix}, \tag{8.27}$$

where each of the four operators define the operations

$$\cos(\sqrt{-A} t) x = \sum_{j=1}^{\infty} \cos(\sqrt{\lambda_j} t) x_j e_j \qquad D(-A)^{1/2} \to D(-A)^{1/2} \tag{8.28}$$

$$\frac{1}{\sqrt{-A}} \sin(\sqrt{-A} t) x = \sum_{j=1}^{\infty} \frac{1}{\sqrt{\lambda_j}} \sin(\sqrt{\lambda_j} t) x_j e_j \qquad H \to D(-A)^{1/2} \tag{8.29}$$

$$-\sqrt{-A} \sin(\sqrt{-A} t) x = \sum_{j=1}^{\infty} -\sqrt{\lambda_j} \sin(\sqrt{\lambda_j} t) x_j e_j \qquad D(-A)^{1/2} \to H \tag{8.30}$$

$$\cos(\sqrt{-A} t) x = \sum_{j=1}^{\infty} \cos(\sqrt{\lambda_j} t) x_j e_j \qquad H \to H \tag{8.31}$$

And so $\mathcal{S}(t)$ maps $\mathcal{H} = D(-A)^{1/2} \times H$ into itself. Also because each of its four components are linear, $\mathcal{S}$ is itself linear. To show that it is also bounded, take $z = (x, y) \in \mathcal{H}$, so that

$$\begin{aligned} \|\mathcal{S}(t) z\|_{\mathcal{H}}^2 = & \| \cos(\sqrt{-A} t) x + \frac{1}{\sqrt{-A}} \sin(\sqrt{-A} t) y\|_{D(-A)^{1/2}}^2 \\ & + \| -\sqrt{-A} \sin(\sqrt{-A} t) x + \cos(\sqrt{-A} t) y\|_H^2 \\ & = \|(-A)^{1/2} x\|_H^2 + \|y\|_H^2 \\ & = \|z\|_{\mathcal{H}}^2. \end{aligned}$$

By the addition formulas for the sine and cosine functions,

$$\begin{aligned}
&\begin{bmatrix} \cos(\sqrt{-A}(t_1+t_2)) & \frac{1}{\sqrt{-A}}\sin(\sqrt{-A}(t_1+t_2)) \\ -\sqrt{-A}\sin(\sqrt{-A}(t_1+t_2)) & \cos(\sqrt{-A}(t_1+t_2)) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\sqrt{-A}t_1) & \frac{1}{\sqrt{-A}}\sin(\sqrt{-A}t_1) \\ -\sqrt{-A}\sin(\sqrt{-A}t_1) & \cos(\sqrt{-A}t_1) \end{bmatrix} \cdot \begin{bmatrix} \cos(\sqrt{-A}t_2) & \frac{1}{\sqrt{-A}}\sin(\sqrt{-A}t_2) \\ -\sqrt{-A}\sin(\sqrt{-A}t_2) & \cos(\sqrt{-A}t_2) \end{bmatrix} \\
&= \begin{bmatrix} \cos(\sqrt{-A}t_2) & \frac{1}{\sqrt{-A}}\sin(\sqrt{-A}t_2) \\ -\sqrt{-A}\sin(\sqrt{-A}t_2) & \cos(\sqrt{-A}t_2) \end{bmatrix} \cdot \begin{bmatrix} \cos(\sqrt{-A}t_1) & \frac{1}{\sqrt{-A}}\sin(\sqrt{-A}t_1) \\ -\sqrt{-A}\sin(\sqrt{-A}t_1) & \cos(\sqrt{-A}t_1) \end{bmatrix},
\end{aligned}$$

it follows that $\mathcal{S}(t_1+t_2) = \mathcal{S}(t_1)\mathcal{S}(t_2) = \mathcal{S}(t_2)\mathcal{S}(t_1)$. Moreover observe that $\mathcal{S}(0)$ is the identity operator, and the mapping $t \to \mathcal{S}(t)z$ from $[0,\infty)$ to $\mathcal{H}$ is continuous. Hence we have shown that the above defined family of bounded linear operators $\{\mathcal{S}(t)\}_{t\geq 0}$ is indeed the semigroup generated by $\mathcal{A}$.

To continue with the proof, recall the mild form of solution to the controlled system (9.34) to be

$$z(t) = \mathcal{S}(t)z_0 + \int_0^t \mathcal{S}(t-s)\mathcal{B}u(s)ds,$$

and thus the reachable space $\mathcal{R}_T = \{z(T) : z_0 = 0\}$ is determined by the operator

$$\Lambda_T = \int_0^T \mathcal{S}(T-s)\mathcal{B}ds \qquad L^2[0,T;U] \to \mathcal{H}.$$

To show $\mathcal{R}_T = \mathcal{H}$ it is sufficient to prove $\mathrm{Im}\Lambda_T \supset \mathcal{H}$, this by the lemma in [30] p.208 is equivalent to showing

$$\|\Lambda_T^* z\|_{L^2[0,T;U]} \geq c\|z\|_{\mathcal{H}} \tag{8.32}$$

for every $z \in \mathcal{H}$ and some $c > 0$. To determine the left hand side of (8.32), we first find the forms of $\mathcal{S}^*(t)$ and $\mathcal{B}^*$ since

$$\Lambda_T^*(s) = \mathcal{B}^*\mathcal{S}^*(T-s).$$

Take $z_1 = \left[\begin{smallmatrix} x_1 \\ y_1 \end{smallmatrix}\right], z_2 = \left[\begin{smallmatrix} x_2 \\ y_2 \end{smallmatrix}\right]$ in $\mathcal{H}$, and compute

$$\begin{aligned}
\langle \mathcal{S}(t)z_1, z_2\rangle &= \left\langle \begin{bmatrix} \cos(\sqrt{-A}t) & \frac{1}{\sqrt{-A}}\sin(\sqrt{-A}t) \\ -\sqrt{-A}\sin(\sqrt{-A}t) & \cos(\sqrt{-A}t) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right\rangle_{\mathcal{H}} \\
&= \left\langle \sqrt{-A}\cos(\sqrt{-A}t)x_1 + \sin(\sqrt{-A}t)y_1, \sqrt{-A}x_2 \right\rangle_H \\
&+ \left\langle -\sqrt{-A}\sin(\sqrt{-A}t)x_1 + \cos(\sqrt{-A}t)y_1, y_2 \right\rangle_H \\
&= \left\langle \sqrt{-A}x_1, \sqrt{-A}\cos(\sqrt{-A}t)x_2 - \sin(\sqrt{-A}t)y_2 \right\rangle \\
&+ \left\langle y_1, \sqrt{-A}\sin(\sqrt{-A}t)x_2 + \cos(\sqrt{-A}t)y_2 \right\rangle \\
&= \left\langle \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} \cos(\sqrt{-A}t) & -\frac{1}{\sqrt{-A}}\sin(\sqrt{-A}t) \\ \sqrt{-A}\sin(\sqrt{-A}t) & \cos(\sqrt{-A}t) \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right\rangle_{\mathcal{H}} \\
&= \langle z_1, \mathcal{S}(-t)z_2\rangle,
\end{aligned}$$

similarly

$$\langle \mathcal{B}u, z\rangle = \left\langle \begin{bmatrix} 0 \\ I \end{bmatrix} u, \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle_{\mathcal{H}} = \langle u, y\rangle_H = \left\langle u, \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\rangle_H.$$

Therefore $\mathcal{S}^*(t) = \mathcal{S}(-t)$, $\mathcal{B}^* = \mathcal{B}^\intercal$, and hence

$$\begin{aligned}
&\|\Lambda_T^* z\|_{L^2[0,T;U]} \\
&= \int_0^T \|\mathcal{B}^*\mathcal{S}^*(t)z\|_U^2 dt \\
&= \int_0^T \left\| \begin{bmatrix} \sqrt{-A}\sin(\sqrt{-A}t) & \cos(\sqrt{-A}t) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right\|_U^2 dt \\
&= \int_0^T \{\sum_{j=1}^\infty \lambda_j \sin^2(\sqrt{\lambda_j}t)x_j^2 + 2\sum_{j=1}^\infty \sqrt{\lambda_j}\sin(\sqrt{\lambda_j}t\}\cos(\sqrt{\lambda_j}t)x_j y_j \\
&+ \sum_{j=1}^\infty \cos^2(\sqrt{\lambda_j}t)y_j^2)dt \\
&= \frac{1}{2}\{\sum_{j=1}^\infty \lambda_j \left(T - \frac{\sin(2\sqrt{\lambda_j}T)}{2\sqrt{\lambda_j}}\right)x_j^2 + \sum_{j=1}^\infty \left(T + \frac{\sin(2\sqrt{\lambda_j}T)}{2\sqrt{\lambda_j}}\right)y_j^2 \\
&+ \sum_{j=1}^\infty (1 - \cos(2\sqrt{\lambda_j}T))x_j y_j\}.
\end{aligned}$$

In view of (8.32) it is sufficient to show that there exists $c > 0$ such that for all $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \mathcal{H}$,

$$\begin{aligned}
&\sum_{j=1}^\infty \lambda_j \left(T - c - \frac{\sin(2\sqrt{\lambda_j}T)}{2\sqrt{\lambda_j}}\right)x_j^2 + \sum_{j=1}^\infty \left(T - c + \frac{\sin(2\sqrt{\lambda_j}T)}{2\sqrt{\lambda_j}}\right)y_j^2 \\
&+ \sum_{j=1}^\infty (1 - \cos(2\sqrt{\lambda_j}T))x_j y_j \geq 0.
\end{aligned} \tag{8.33}$$

For this consider the inequality

$$\alpha\|x\|^2 + \beta\|y\|^2 + \gamma\langle x, y\rangle \geq (\sqrt{\alpha}\|x\| - \sqrt{\beta}\|y\|)^2 + (2\sqrt{\alpha\beta} - |\gamma|)\|x\|\|y\| \geq 0$$

which holds whenever $\gamma^2 \leq 4\alpha\beta$,$\alpha, \beta \geq 0$. Applying the inequality to each individual term in the sums (8.33) to obtain

$$\lambda_j\left(T - c - \frac{\sin(2\sqrt{\lambda_j}T)}{2\sqrt{\lambda_j}}\right) \geq 0 \quad \left(T - c + \frac{\sin(2\sqrt{\lambda_j}T)}{2\sqrt{\lambda_j}}\right) \geq 0$$

and

$$(1 - \cos(2\sqrt{\lambda_j}T))^2 \leq 4\lambda_j\left\{(T-c)^2 - \left(\frac{\sin(2\sqrt{\lambda_j}T)}{2\sqrt{\lambda_j}}\right)^2\right\}$$

for all λ_j. All three conditions then reduce to

$$\left|\frac{\sin(2\sqrt{\lambda_j}T)}{2\sqrt{\lambda_j}}\right| \leq T - c$$

Then we can choose T and c small enough so that the inequality holds for all j. Thus we have shown the system (8.15) is exactly controllable for small T. □

Theorem 8.20 ([7] Theorem 2.3). *(1) Assume that the operator A is negative definite and self-adjoint, then for the system* (8.14)

$$E_\infty(0,b) = \begin{cases} 2\|(-A)^{1/2}b\|^2 & \text{if} \quad b \in D((-A)^{1/2}) \\ \infty & \text{otherwise} \end{cases}$$

(2) If in addition the operator C is negative definite, bounded, self-adjoint and that $(-C)^{1/2}$ commutes with $(-A)^{1/2}$, then for the system (8.15)

$$E_\infty\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}a\\b\end{bmatrix}\right) = 2\|(-C)^{1/2}(-A)^{1/2}a\|^2 + 2\|(-C)^{1/2}b\|^2, \quad \begin{bmatrix}a\\b\end{bmatrix} \in \mathcal{H}.$$

Proof.
(1) By formula (8.25)

$$(R_T^{1/2})^{-1} = (-2A)^{1/2}(I - S(2T))^{-1/2},$$

and we have for $b \in D(-A)^{1/2}$

$$\begin{aligned} E_\infty(0,b) &= \lim_{T\to\infty} \|(R_T^{1/2})^{-1}b\|_H^2 \\ &= \lim_{T\to\infty} \|(-2A)^{1/2}(I - S(2T))^{-1/2}b\|_H^2 \\ &= \|(-2A)^{1/2}b\|^2. \end{aligned}$$

On the other hand if $E_\infty(0,b) < \infty$, there exists $T > 0$ such that

$$\|(R_T^{1/2})^{-1}b\| < \infty,$$

or

$$(I - S(2T))^{-1/2}b \in D(-A)^{1/2}$$

which by theorem 8.18 implies $b \in D(-A)^{1/2}$.
(2) Recall that the operator $R = \lim_{T\to\infty} R_T$ satisfies the Liapunov equation:

$$2\langle RA^*x, x\rangle + \langle BB^*x, x\rangle = 0, \qquad x \in D(A^*). \tag{8.34}$$

Let us first consider the function

$$V(y,\dot y) = \|\dot y\|^2 - \langle \dot y, Cy\rangle + \frac{1}{2}\|Cy\|^2 + \|(-A)^{1/2}y\|^2$$

If $y(0) \in D(A)$, $\dot y(0) \in D((-A)^{1/2})$ then along the solution of (8.15) with $u = 0$ we have

$$\begin{aligned} \frac{dV}{dt}(y,\dot y) &= 2\langle \dot y, \ddot y - Ay\rangle - \langle \dot y, C\dot y\rangle + \langle Cy, C\dot y - \ddot y\rangle \\ &= -\|(-C)^{1/2}\dot y\|^2 - \|(-C)^{1/2}(-A)^{1/2}y\|^2. \end{aligned}$$

Thus $V(\cdot,\cdot)$ is a Liapunov function for the uncontrolled system and so $\mathcal{A}$ generates an exponentially stable semigroup.

Then we show that when $C = 0$, $\mathcal{A} = \begin{bmatrix}0 & I\\ A & 0\end{bmatrix}$ is skew-selfadjoint, i.e.

$$\mathcal{A}^* = -\mathcal{A} \quad \text{and } D(\mathcal{A}^*) = D(\mathcal{A}).$$

For $D(\mathcal{A}) \subset D(\mathcal{A}^*)$, since $\mathcal{A}$ is skew-symmetric, or

$$\begin{aligned}\langle \mathcal{A}x, y\rangle_{\mathcal{H}} &= \langle (x_2, Ax_1), (y_1, y_2)\rangle_{\mathcal{H}} \\ &= \left\langle (-A)^{1/2}x_2, (-A)^{1/2}y_1\right\rangle + \langle Ax_1, y_2\rangle \\ &= -\left\langle (-A)^{1/2}x_1, (-A)^{1/2}y_2\right\rangle - \langle x_2, Ay_1\rangle \\ &= -\langle (x_1, x_2), (y_2, Ay_1)\rangle_{\mathcal{H}} \\ &= -\langle x, \mathcal{A}y\rangle_{\mathcal{H}}\end{aligned}$$

for all $x, y \in D(\mathcal{A})$. Fix any y and then

$$|\langle \mathcal{A}x, y\rangle| = |\langle x, \mathcal{A}y\rangle| \le \|\mathcal{A}y\|\|x\|,$$

so the mapping $D(\mathcal{A}) \ni x \to \langle \mathcal{A}x, y\rangle \in \mathbb{R}$ is continuous, and $y \in D(A^*)$ by definition.

The general case follows from the addition of the operator $\tilde{C} = \begin{bmatrix} 0 & 0 \\ 0 & C\end{bmatrix}$, which is bounded and symmetric and therefore self-adjoint. In this case

$$\mathcal{A}^* = \begin{bmatrix} 0 & -I \\ -A & 0\end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & C\end{bmatrix} = \begin{bmatrix} 0 & -I \\ -A & C\end{bmatrix}.$$

Now we can show that the unique solution of (8.34) is

$$R = \frac{1}{2}\begin{bmatrix} C^{-1} & 0 \\ 0 & C^{-1}\end{bmatrix}.$$

Let $\begin{bmatrix} a \\ b\end{bmatrix} \in D(\mathcal{A}^*)$, then $\mathcal{A}^*\begin{bmatrix} a \\ b\end{bmatrix} = \begin{bmatrix} -b \\ -Aa + Cb\end{bmatrix}$. Since $(-A)^{1/2}a \in D((-A)^{1/2})$,

$$\begin{aligned}&\langle b, b\rangle + 2\left\langle R\mathcal{A}^*\begin{bmatrix} a \\ b\end{bmatrix}, \begin{bmatrix} a \\ b\end{bmatrix}\right\rangle \\ &= \left\langle (-A)^{1/2}C^{-1}b, (-A)^{1/2}a\right\rangle - \left\langle C^{-1}(-Aa) + b, b\right\rangle + \langle b, b\rangle \\ &= -\left\langle C^{-1}b, Aa\right\rangle + \left\langle C^{-1}Aa, b\right\rangle = 0\end{aligned}$$

Hence, for arbitrary $\begin{bmatrix} a \\ b\end{bmatrix} \in \mathcal{H}$,

$$\begin{aligned}\|(R^{1/2})^{-1}\begin{bmatrix} a \\ b\end{bmatrix}\|^2 &= \|\sqrt{2}\begin{bmatrix} (-C)^{1/2} & 0 \\ 0 & (-C)^{1/2}\end{bmatrix}\begin{bmatrix} a \\ b\end{bmatrix}\|^2 \\ &= 2\|(-C)^{1/2}(-A)^{1/2}a\|^2 + 2\|(-C)^{1/2}b\|^2.\end{aligned}$$

□

8.3. **Nonlinear systems.** Having introduced the basics from the linear case, it is time to bring forward the corresponding results for nonlinear systems. We choose to omit the part concerning parabolic systems because it has been treated in detail in [7], and also because the hyperbolic systems contain results that are closer to our purpose, namely of applications to nonlinear wave equations.

Consider the equation

$$\ddot{y}(t) = Ay(t) - U'(y(t)) - \beta\dot{y}(t) + u(t), \qquad (8.35)$$
$$y(0) = a_0, \qquad \dot{y}(0) = b_0$$

with the following array of assumptions:

(i) A is a negative definite operator on the Hilbert Space H.
(ii) U is a functional from $V = D((-A)^{1/2})$ into R_+ of class C^1, $U(0) = 0$, $DU(0) = 0$.
(iii) There exists a mapping $U' : V \to H$, Lipschitz on bounded sets such that

$$DU(x;h) = \langle U'(x), h\rangle, \qquad \text{for all}\, x, h \in V,$$

where $DU(x;h)$ denotes the value of the Frechet derivatives at x in the direction h.
(iv) β is a positive constant.
(v) $a_0 \in D(-A)^{1/2}$ and $b_0 \in H$, $u \in L^2(0,T;H)$

We take as an example of a system satisfying those assumptions:

Example 8.1 ([7] Example 3.5)**.** Let $A = \frac{d^2}{dx^2}$, $D(A) = H_0^1(0,L) \cap H^2(0,L)$. Moreover, we set $H_0^1(0,L) = \{u \in H^1(0,L); u(0) = u(L) = 0\}$. Then $V = D(-A)^{1/2} = H_0^1(0,L)$.

Assume that ϕ is a real valued nonnegative function of class C^2, with $\phi(0) = 0, \phi'(0) = 0$. Define

$$U(x) = \int_0^L \phi(x(s))ds, \qquad x \in V,$$

Now we verify assumption (ii) for U.

It is clear that U is well-defined since $x \in C([0,L])$ by the Sobolev embedding. U is also continuous, since if $x_n \to x_0$ in V, by the Sobolev embedding the convergence is uniform on $[0,L]$. Since ϕ is continuous, $\phi(x_n) \to \phi(x_0)$ uniformly on $[0,L]$ so $U(x_n) \to U(x_0)$ in R.

Claim 8.21. *U is Frechet differentiable, and*

$$DU(a,x) = \int_0^L \phi'(a(s))x(s)ds$$

at $a \in V$ in the direction $x \in V$.

Proof. Let us choose and fix $a \in V$. Let the bounded operator $DU(a,\cdot) : V \to \mathbb{R}$ be defined as above, and we wish to show

$$|U(a+x) - U(a) - A(x)| = o(\|x\|_V).$$

Substituting definitions

$$\begin{aligned} LHS &\le \int_0^L |\phi(a(s) - x(s)) - \phi(a(s)) - \phi'(a(s))x(s)|ds \\ &\le \int_0^L \int_0^1 |\phi'(a(s) + rx(s)) - \phi'(a(s))||x(s)|drds \\ &\le C\|x\|_V \int_0^L \int_0^1 |\phi'(a(s) + rx(s)) - \phi'(a(s))|drds \end{aligned}$$

where the second inequality follows from the mean value theorem. By the uniform continuity of ϕ' on bounded sets, the integrand above is small provided $|x(s)|$ or $\|x\|_V$ is small, hence the differentiability follows. □

Similarly the uniform continuity of ϕ' on bounded sets implies that U is of C^1 class, i.e.

$$V \ni a \to DU(a,\cdot) \in \mathcal{L}(V,R)$$

is continuous. Assumption (ii) is thus verified. And it is clear that the mapping U' in assumption (iii) should take the form $V \ni a(s) \to \phi'(a(s)) \in H = L^2(0,L)$.

Claim 8.22. *The mapping $U' : V \to H$ is Lipschitz on bounded sets.*

Proof. Take $a, b \in V$ with $\|a\|_V, \|b\|_V \le R$, by Sobolev embedding, there is $C > 0$ such that $\|a\|_{L^\infty(0,L)}, \|b\|_{L^\infty(0,L)} \le CR$. Then

$$\begin{aligned} \|U'(a) - U'(b)\|_H^2 &= \int_0^L |\phi'(a(x)) - \phi'(b(x))|^2 dx \\ &\le \sup_{|y|\le CR} |\phi''(y)|^2 \int_0^L |a(x) - b(x)|^2 dx \\ &\le C' \|a - b\|_H^2. \end{aligned}$$

□

In the following we denote by $z^{[\substack{a\\b}]}(\cdot)$ the solution of the uncontrolled system:

$$\begin{aligned} \ddot{z} &= Az - U'(z) - \beta \dot{z}, \\ z(0) &= a \in D(-A)^{1/2}, \qquad \dot{z}(0) = b \in H. \end{aligned} \tag{8.36}$$

Lemma 8.23. *With assumptions (i)-(v), the following identity holds:*

$$\begin{aligned} \frac{1}{2}\int_0^T \|u(s)\|^2 ds = \frac{1}{2}\int_0^T &\|u(s) - 2\beta\dot{y}(s)\|^2 ds \\ &+ \beta[\|(-A)^{1/2}y(T)\|^2 + 2U(y(T)) + \|\dot{y}(T)\|^2 \\ &- \|(-A)^{1/2}y(0)\|^2 - 2U(y(0)) - \|\dot{y}(0)\|^2]. \end{aligned} \tag{8.37}$$

Proof. To prove (8.37) observe that U is of C^1 class and is defined for all $D(-A)^{1/2}$. Therefore if $u(\cdot)$ is in $C^1(0,T;D(A))$

$$\begin{aligned} \frac{1}{2}\int_0^T \|u(s)\|^2 ds &= \frac{1}{2}\int_0^T \|u(s) - 2\beta\dot{y}(s) + 2\beta\dot{y}(s)\|^2 ds \\ &= \frac{1}{2}\int_0^T \|u(s) - 2\beta\dot{y}(s)\|^2 ds \\ &+ 2\int_0^T \langle u(s) - 2\beta\dot{y}(s), \beta\dot{y}(s)\rangle + \langle \beta\dot{y}(s), \beta\dot{y}(s)\rangle \, ds \end{aligned}$$

and by equation (8.35)

$$\begin{aligned} &2\int_0^T \langle u(s) - \beta\dot{y}(s), \beta\dot{y}(s)\rangle \, ds \\ &= 2\int_0^T \langle \ddot{y}(s) - Ay(s) + U'(y(s)), \beta\dot{y}(s)\rangle \, ds \\ &= \beta\int_0^T \frac{d}{dt}\left(\|(-A)^{1/2}y(t)\|^2 + 2U(y(t)) + \|\dot{y}(t)\|^2\right) dt. \end{aligned}$$

By approximating functions in $W^{1,2}(0,T;H)\cap L^2(0,T;D(A))$ like in the proof of theorem 8.17 we get (8.37).

□

Theorem 8.24 ([7] Theorem 3.7)**.** *Suppose A is negative definite and self-adjoint. Assume that the assumptions (i)-(v) hold, If $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]\in\mathcal{H}$ and*

$$z^{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}(t)\to 0 \text{ in } \mathcal{H} \text{ as } t\to\infty\ ,$$

then

$$E_\infty\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}a\\b\end{bmatrix}\right)=\beta[\|(-A)^{1/2}a\|^2+2U(a)+\|b\|^2], \tag{8.38}$$

where $\mathcal{H}=D(-A)^{1/2}\times H$.

Proof. To obtain formula (8.38), observe that $y(t)=z(T-t)$ is a solution of (8.35) when $u(t)=2\beta\dot y(t)$, which cancels the first term on the right side of (8.37). Then

$$\begin{aligned}E_T\left(\begin{bmatrix}z^{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}(T)\\ \dot z^{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}(T)\end{bmatrix},\begin{bmatrix}a\\b\end{bmatrix}\right)&=E_T\left(\begin{bmatrix}y(0)\\ \dot y(0)\end{bmatrix},\begin{bmatrix}y(T)\\ \dot y(T)\end{bmatrix}\right)\\ &\le\beta[\|(-A)^{1/2}a\|^2+2U(a)+\|b\|^2\\ &-\|(-A)^{1/2}z^{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}(T)\|^2-2U(z^{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}(T))-\|\dot z^{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}(T)\|^2]\end{aligned}$$

Since by assumption $z^{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}(t)\to 0$ as $t\to\infty$ in $\mathcal{H}$,

$$E_\infty\left(\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}a\\b\end{bmatrix}\right)\le\beta[\|(-A)^{1/2}a\|^2+2U(a)+\|b\|^2]. \tag{8.39}$$

And since the opposite inequality follows from (8.37) we get (8.38).

□

In the last theorem there is a key assumption

$$z^{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}(t)\to 0 \text{ in } \mathcal{H} \text{ as } t\to\infty\ ,$$

where

$$\mathcal{H}=D(-A)^{1/2}\times H,$$

now we show in which conditions this can be true.

Recall the form of the equation to which $z^{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}(t)$ is solution to:

$$\begin{aligned}\ddot z&=Az-U'(z)-\beta\dot z,\\ z(0)&=a\in D(-A)^{1/2},\qquad \dot z(0)=b\in H.\end{aligned} \tag{8.40}$$

or in the more general form

$$\begin{aligned}\ddot z&=Az-U'(z)-\beta\dot z+g(t),\\ z(0)&=a\in D(-A)^{1/2},\qquad \dot z(0)=b\in H.\end{aligned} \tag{8.41}$$

where

$$g\in L^2(0,\infty;H), \tag{8.42}$$

or in the compact matrix form

$$\dot v=\mathcal{M}v(t)+\mathcal{F}(v(t))-\beta\mathcal{K}(v(t))+G(t) \tag{8.43}$$

where

$$z^{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}(t) = v = \begin{bmatrix} z \\ \dot{z} \end{bmatrix}, \tag{8.44}$$

$$\mathcal{M} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix} \tag{8.45}$$

and

$$\mathcal{F}(v) = -\begin{bmatrix} 0 \\ U'(z) \end{bmatrix}, \ \mathcal{K}(v) = \begin{bmatrix} 0 \\ \dot{z} \end{bmatrix}, \ G(t) = \begin{bmatrix} 0 \\ g(t) \end{bmatrix} \tag{8.46}$$

Theorem 8.25. *Suppose A is negative definite and self-adjoint. Assume also that condition* (8.42) *holds. If $U \geq 0$ and $z^{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}(t)$ is a solution to* (8.43)*, then*

$$z^{\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]}(t) \to 0 \ \textit{in } \mathcal{H} \ \textit{as } t \to \infty. \tag{8.47}$$

Proof of Theorem 8.25. Consider the operator $\mathcal{P} : \mathcal{H} \to \mathcal{H}$ defined in the matrix form by

$$\mathcal{P} = \begin{bmatrix} \beta^2 A^{-1} + 2I & \beta A^{-1} \\ \beta I & 2I \end{bmatrix}$$

which according to [4] is self-adjoint isomorphism of $\mathcal{H}$ and which satisfies

$$\|\mathcal{P}\|^{-1}_{\mathcal{L}(\mathcal{H})} \langle \mathcal{P}v, v \rangle_{\mathcal{H}} \leq \|v\|^2_{\mathcal{H}} \leq \langle \mathcal{P}v, v \rangle_{\mathcal{H}} \tag{8.48}$$

$$\langle \mathcal{K}v, \mathcal{P}v \rangle_{\mathcal{H}} = 2\|\dot{z}\|^2 \tag{8.49}$$

$$\langle \mathcal{M}v, \mathcal{P}v \rangle_{\mathcal{H}} = -\beta \|Az\|^2_H + \beta^2 \langle z, \dot{z} \rangle + \beta \|\dot{z}\|^2 \tag{8.50}$$

Next define the Liapunov function $\Phi : \mathcal{H} \to [0, \infty)$ as

$$\Phi(v) = \frac{1}{2} \langle \mathcal{P}v, v \rangle + \frac{1}{2} U(z) \tag{8.51}$$

Then we have [4] for $t \geq s \geq 0$,

$$\begin{aligned} \Phi(v(t))e^{\lambda t} \leq & \Phi(v(s))e^{\lambda s} + \int_s^t e^{\lambda r} \left\{ \lambda \Phi(v(r)) - \beta \|v(r)\|^2 - \beta U(z) \right\} dr \\ & + \int_s^t e^{\lambda r} \langle \beta z(r) + 2\dot{z}(r), g(r) \rangle \, dr \end{aligned} \tag{8.52}$$

We calculate the braced term

$$\lambda \Phi(v) - \beta \|v\|^2_{\mathcal{H}} - \beta U(z) \leq \left(\frac{\lambda}{2} C_p - \beta \right) \|v\|^2 + \left(\frac{\lambda}{2} - \beta \right) U(z)$$

where $C_p := \|\mathcal{P}\|_{\mathcal{L}(\mathcal{H})}$.
and

$$\begin{aligned} |\langle \beta z(r) + 2\dot{z}(r), g(r) \rangle| &\leq \epsilon |z|^2 + \frac{\beta}{4\epsilon^2} |g(r)|^2 + \epsilon |\dot{z}|^2 + \frac{1}{\epsilon} |g(r)|^2 \\ &= \epsilon(|z|^2 + |\dot{z}|^2) + \left(\frac{\beta}{4\epsilon^2} + \frac{1}{\epsilon} \right) |g(r)|^2 \\ &\leq \epsilon \|v\|^2 + \left(\frac{\beta}{4\epsilon^2} + \frac{1}{\epsilon} \right) |g(r)|^2 \end{aligned} \tag{8.53}$$

since

$$\|v\|_{\mathcal{H}}^2 = |z|_{H^1}^2 + |\dot{z}|_{L^2}^2 \geq |z|_{L^2}^2 + |\dot{z}|_{L^2}^2.$$

So we have

(8.54)

$$\Phi(v(t))e^{\lambda t} \leq \Phi(v(s))e^{\lambda s} + \int_s^t e^{\lambda r} \left\{ \left(\frac{\lambda}{2} C_p - \beta + \epsilon \right) \|v\|^2 + \left(\frac{\lambda}{2} - \beta \right) U(z) + \left(\frac{\beta}{4\epsilon^2} + \frac{1}{\epsilon} \right) |g(r)|^2 \right\} dr$$

Now choose λ and ϵ small so that

$$\tfrac{\lambda}{2} C_p - \beta + \epsilon < -\tfrac{\beta}{2} \text{ and } \tfrac{\lambda}{2} - \beta < -\tfrac{\beta}{2},$$

combining property (8.48) of the operator $\mathcal{P}$ we then have

$$\Phi(v(t))e^{\lambda t} \leq \Phi(v(s))e^{\lambda s} - \gamma \int_s^t e^{\lambda r} \Phi(v(r)) dr + \int_s^t e^{\lambda r} C_\epsilon |g(r)|^2 dr \tag{8.55}$$

where $\gamma = \beta\delta$ for $\delta = \min(C_p^{-1}, 1)$.

Provided $U \geq 0$ we can omit the term with γ on the right hand side since it is nonnegative, to obtain

$$\Phi(v(t)) \leq \Phi(v(s))e^{-\lambda(t-s)} + C_\epsilon \int_s^t e^{-\lambda(t-r)} |g(r)|^2 dr \tag{8.56}$$

The first term on the right goes to zero when $t \to \infty$, we show the second term behave in the same way as follows.

Fix $s > 0$ and define

$$k(t) = \begin{cases} e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\phi(t) = \begin{cases} |g(t)|^2 & t \geq s \\ 0 & t < s \end{cases}$$

then we have

$$(k \star \phi)(t) = \int_s^t e^{-\lambda(t-r)} |g(r)|^2 dr.$$

so that $(k \star \phi)(t) \in L^\infty(0, \infty)$ because $g \in L^2(0, \infty)$.

Define the bounded linear operator $\Lambda(\phi) = k \star \phi$ so that

$$\Lambda : X := L^1(0, \infty) \to L^\infty(0, \infty) := Y,$$

also define

$$Y_0 = \{u : \mathbb{R}_+ \to \mathbb{R} \text{ continuous, vanishing at } \infty\}$$

i.e. for all $\epsilon > 0$, there is $K_\epsilon > 0$ such that $|u(t)| \leq \epsilon$, when $t \geq K_\epsilon$.

So we have Y_0 as a closed subset of Y, because when we have a sequence $\{u_k\}_{k=1}^\infty \in Y_0$ so that $u_k \to u$ in $Y = L^\infty(0, \infty)$ the convergence is uniform so u is also continuous. It vanishes at infinity since each of u_k is such.

Also $X_0 := C_c^\infty(0, \infty)$ is dense in $X = L^1(0, \infty)$ by the Sobolev embedding theorem.

Finally if $\phi \in X_0$ then $\Lambda(\phi) \in Y_0$. Since then $\operatorname{supp} \phi \in (a, b)$ for $s < a < b$ and

$$|(k \star \phi)(t)| \leq C e^{-\lambda t} \int_a^b e^{\lambda r} dr \to 0 \text{ for } t \to \infty,$$

and since for any $t_0 \in (0,\infty)$,

$$|(k\star\phi)(t_0+h)-(k\star\phi)(t_0)| = \left| e^{-\lambda h}\int_s^{t_0+h} e^{-\lambda(t_0-r)}|g(r)|^2dr - \int_s^{t_0} e^{-\lambda(t_0-r)}|g(r)|^2dr \right| \to 0,$$

as $h\to 0$.

Combining the above properties, if we have $x\in X$, there is a sequence $\{x_k\}\in X_0$ and $x_k\to x$, $\Lambda(x_k)\to\Lambda(x)$ in Y and in particular in Y_0. Therefore $\Lambda(x)\in Y_0$, which is what we aim to prove.

Therefore

$$z^{\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]}(t)\to 0 \text{ in } \mathcal{H} \text{ as } t\to\infty\ ,$$

always holds provided only U is nonnegative.

□

9. Applications

In this last section we combine the tools from section 7 and 8 to conclude that the quasipotential formula is valid for the equation we started our journey with, i.e.

$$u_{tt}-\Delta u+u|u|^{p-1}=0 \qquad \text{in } \mathbb{R}^n\times\mathbb{R} \tag{9.1}$$

after some modifications and generalisations. The presentation will be swift since most arguments have their core already explained above.

Preliminary material on the definition of weak solutions has been chosen from [10].

Definition 9.1 (Weak derivative)**.** Let U be an open set in R^n. Suppose $u,v\in L^1_{loc}(U)$ and α is a multiindex. We say that v is the α^{th}-weak partial derivative of u, written

$$D^\alpha u = v,$$

provided

$$\int_U uD^\alpha\phi dx = (-1)^{|\alpha|}\int_U v\phi dx \tag{9.2}$$

for all test functions $\phi\in C_c^\infty(U)$.

Lemma 9.2. *A weak α^{th}-partial derivative of u, if it exists, is uniquely defined up to a set of measure zero.*

Definition 9.3 (Sobolev space)**.** The Sobolev space

$$W^{k,p}(U)$$

consists of all locally summable functions $u: U\to\mathbb{R}$ such that for each multiindex α with $|\alpha|\le k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.

Remark 9.4. (a) If $p=2$, we usually write

$$H^k(U)=W^{k,2}(U) \quad (k=0,1,\dots).$$

$H^k(U)$ is a Hilbert space, with $H^0(U)=L^2(U)$.

(b) We henceforth identify functions in $W^{k,p}(U)$ which agree a.e.

Definition 9.5. If $u \in W^{k,p}(U)$, we define its norm to be

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha|\le k} \int_U |D^\alpha u|^p dx\right)^{1/p} & (1 \le p < \infty) \\ \sum_{|\alpha|\le k} \operatorname{ess\,sup}_U |D^\alpha u| & (p = \infty). \end{cases}$$

Let $\{u_m\}_{m=1}^\infty$, $u \in W^{k,p}(U)$. We say u_m converges to u in $W^{k,p}(U)$, written

$$u_m \to u \qquad \text{in } W^{k,p}(U),$$

provided

$$\lim_{m\to\infty} \|u_m - u\|_{W^{k,p}(U)} = 0.$$

We denote by

$$W_0^{k,p}(U)$$

the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$. It is customary to write

$$H_0^k(U) = W_0^{k,2}(U).$$

Definition 9.6 (The space H^{-1})**.** We denote by $H^{-1}(U)$ the dual space to $H_0^1(U)$. We use $\langle\, ,\, \rangle$ to denote the pairing between $H^{-1}(U)$ and $H_0^1(U)$. If $f \in H^{-1}(U)$, define the norm

$$\|f\|_{H^{-1}(U)} := \sup\left\{\langle f, u\rangle \mid u \in H_0^1(U), \|u\|_{H_0^1(U)} \le 1\right\}.$$

Theorem 9.7. *(1) Assume $f \in H^{-1}(U)$. Then there exist functions $f^0, f^1, \dots, f^n$ in $L^2(U)$ such that*

$$\langle f, v\rangle = \int_U f^0 v + \sum_{i=1}^n f^i v_{x_i} dx \quad (v \in H_0^1(U)). \tag{9.3}$$

(b) Furthermore,

$$\|f\|_{H^{-1}(U)} = \inf\left\{\left(\int_U \sum_{i=0}^n |f^i|^2 dx\right)^{1/2} : f \text{ satisfies (9.3) for } f^0, \dots, f^n \in L^2(U)\right\}.$$

(c) In particular, we have

$$(v, u)_{L^2(U)} = \langle v, u\rangle$$

for all $u \in H_0^1(U)$, $v \in L^2(U) \subset H^{-1}(U)$.

We write $f = f^0 - \sum_{i=1}^n f^i_{x_i}$ whenever (9.3) holds.

Definition 9.8. The space

$$L^p(0,T;X)$$

consists of all strongly measurable functions $\mathbf{u} : [0,T] \to X$ with

$$\|\mathbf{u}\|_{L^p(0,T;X)} := \left(\int_0^T \|\mathbf{u}(t)\|^p dt\right)^{1/p} < \infty$$

for $1 \le p < \infty$ and

$$\|\mathbf{u}\|_{L^\infty(0,T;X)} := \operatorname*{ess\,sup}_{0\le t\le T} \|\mathbf{u}(t)\| < \infty.$$

Let $\mathbf{u} \in L^1(0,T;X)$. We say $\mathbf{v} \in L^1(0,T;X)$ is the weak derivative of $\mathbf{u}$, written

$$\mathbf{u}' = \mathbf{v},$$

provided

$$\int_0^T \phi'(t)\mathbf{u}(t)dt = -\int_0^T \phi(t)\mathbf{v}(t)dt$$

for all scalar test functions $\phi \in C_c^\infty(0,T)$.

The Sobolev space

$$W^{1,p}(0,T;X)$$

consists of all functions $\mathbf{u} \in L^p(0,T;X)$ such that $\mathbf{u}'$ exists in the weak sense and belongs to $L^p(0,T;X)$. Furthermore,

$$\|\mathbf{u}\|_{W^{1,p}(0,T;X)} := \begin{cases} \left(\int_0^T \|\mathbf{u}(t)\|^p + \|\mathbf{u}'(t)\|^p dt\right)^{1/p} & (1 \le p < \infty) \\ \operatorname{ess\,sup}_{0\le t\le T}(\|\mathbf{u}(t)\| + \|\mathbf{u}'(t)\|) & (p = \infty). \end{cases}$$

We write $H^1(0,T;X) = W^{1,2}(0,T;X)$.

Definition 9.9 (Hyperbolic equations)**.** Write $U_T = U \times (0,T]$, where $T > 0$ and $U \subset \mathbb{R}^n$ is an open, bounded set. Consider the initial/boundary-value problem

$$\begin{cases} u_{tt} + Lu = f & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g, u_t = h & \text{on } U \times \{t = 0\}, \end{cases} \tag{9.4}$$

where $f : U_T \to \mathbb{R}$, $g, h : U \to \mathbb{R}$ are given, and $u : \bar{U}_T \to \mathbb{R}$ is the unknown, $u = u(x,t)$. The symbol L denotes for each time t a second-order partial differential operator, having either the divergence form

$$Lu = -\sum_{i,j=1}^n (a^{ij}(x,t)u_{x_i})_{x_j} + \sum_{i=1}^n b^i(x,t)u_{x_i} + c(x,t)u \tag{9.5}$$

$$Lu = -\sum_{i,j=1}^n a^{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^n b^i(x,t)u_{x_i} + c(x,t)u \tag{9.6}$$

for given coefficients a^{ij}, b^i, c $(i,j = 1,\dots,n)$.

If $a^{ij} = \delta_{ij}$, $b^i = c = 0$, then $L = -\Delta$.

Definition 9.10. We say the partial differential operator $\frac{\partial^2}{\partial t^2} + L$ is (uniformly) hyperbolic if there exists a constant $\theta > 0$ such that

$$\sum_{i,j=1}^n a^{ij}(x,t)\xi_i\xi_j \ge \theta|\xi|^2$$

for all $(x,t) \in U_T$, $\xi \in \mathbb{R}^n$.

Definition 9.11 (Weak solutions)**.** Assume L to be of divergence form (9.5), we look for an appropriate notion of weak solution for problem (9.4). We will initially suppose that

$$a^{ij}, b^i, c \in C^1(\bar{U}_T) \quad (i,j = 1,\dots,n),$$

$$f \in L^2(U_T),$$

$$g \in H_0^1(U), h \in L^2(U)$$

and always assume $a^{ij} = a^{ji}$ $(i, j = 1, \dots, n)$. Also define the time-dependent bilinear form

$$B[u, v; t] := \int_U \sum_{i,j=1}^{n} a^{ij}(\cdot, t) u_{x_i} v_{x_j} + \sum_{i=1}^{n} b^i(\cdot, t) u_{x_i} v + c(\cdot, t) uv dx \tag{9.7}$$

for $u, v \in H_0^1(U)$ and $0 \leq t \leq T$.

We begin by supposing $u = u(x, t)$ to be a smooth solution of (9.4) and define the associated mapping

$$\mathbf{u} : [0, T] \to H_0^1(U),$$

by

$$[\mathbf{u}(t)](x) := u(x, t) \quad (x \in U, 0 \leq t \leq T).$$

We similarly introduce the function

$$\mathbf{f} : [0, T] \to L^2(U)$$

defined by

$$[\mathbf{f}(t)](x) := f(x, t) \quad (x \in U, 0 \leq t \leq T).$$

Now fix any function $v \in H_0^1(U)$, multiply the PDE $U_{tt} + Lu = f$ by v. and integrate by parts to obtain the identity

$$(\mathbf{u}'', v) + B[\mathbf{u}, v; t] = (\mathbf{f}, v) \tag{9.8}$$

for $0 \leq t \leq T$, where $(\, , \,)$ denotes the inner product in $L^2(U)$. Also we see from the PDE $U_{tt} + Lu = f$ that

$$u_{tt} = g^0 + \sum_{j=1}^{n} g^j_{x_j}$$

for $g^0 := f - \sum_{i=1}^n b^i u_{x_i} - cu$ and $g^j := \sum_{i=1}^n a^{ij} u_{x_i} (j = 1, \dots, n)$. This suggests that we should look for a weak solution $\mathbf{u}$ with $\mathbf{u}'' \in H^{-1}(U)$ for a.e. $0 \leq t \leq T$ and then interpret the first term of (9.8) as $\langle \mathbf{u}'', v \rangle$, $\langle \, , \, \rangle$ denoting as usual the pairing between $H^{-1}(U)$ and $H_0^1(U)$.

Definition 9.12. We say a funtion

$$\mathbf{u} \in L^2(0, T; H_0^1(U)), \mathbf{u}' \in L^2(0, T; L^2(U)), \mathbf{u}'' \in L^2(0, T; H^{-1}(U))$$

is a weak solution of the hyperbolic initial/boundary-value problem (9.4) provided

$$\langle \mathbf{u}'', v \rangle + B[\mathbf{u}, v; t] = (\mathbf{f}, v)$$

for each $v \in H_0^1(U)$ and a.e. time $0 \leq t \leq T$ and

$$\mathbf{u}(0) = g, \mathbf{u}'(0) = h.$$

Definition 9.13. For the mildly quasilinear initial-value problem

$$\begin{cases} u_{tt} - \Delta u + f(Du, u_t, u) = 0 & \text{in } \mathbb{R}^n \times (0, T] \\ u = g, u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases} \tag{9.9}$$

We say that a function $\mathbf{u} \in L^2(0, T; H^1_{loc}(\mathbb{R}^n))$, with $\mathbf{u}' \in L^2(0, T; L^2_{loc}(\mathbb{R}^n))$ and $\mathbf{u}'' \in L^2(0, T; H^{-1}_{loc}(\mathbb{R}^n))$ is a weak solution of the initial-value problem (9.9) provided $\mathbf{u}(0) = g, \mathbf{u}'(0) = h$,

$$\mathbf{f} := -f(D\mathbf{u}, \mathbf{u}', \mathbf{u}) \in L^2(0, T; L^2_{loc}(\mathbb{R}^n)),$$

and

$$\langle \mathbf{u}'', v \rangle + B[\mathbf{u}, v] = (\mathbf{f}, v)$$

for each $v \in H^1(\mathbb{R}^n)$ with compact support and a.e. time $0 \le t \le T$. Here $B[u,v] := \int_{\mathbb{R}^n} Du \cdot Dv dx$.

Theorem 9.14. *The initial value problem*

$$\begin{aligned} &u_{tt} - \Delta u + m^2 u + |u|^{p-1}u = f(x,t) \qquad \mathbb{R}^n \times (0,\infty) \\ &u = g, u_t = h \qquad t = 0 \end{aligned} \tag{9.10}$$

admits of a weak solution u for every $T > 0$, with

$$u \in L^2_{loc}((0,T)\times\mathbb{R}^n) \cap L^\infty((0,T);H^1(\mathbb{R}^n)), \quad u' \in L^\infty(0,T;L^2(\mathbb{R}^n)) \tag{9.11}$$

provided

$$g \in H^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n),\ h \in L^2(\mathbb{R}^n),\ f \in L^2(0,T;H^1(\mathbb{R}^n)) \cap L^\infty((0,T);L^2(\mathbb{R}^n)) \tag{9.12}$$

Proof. By writing

$$q(u) := m^2 u + |u|^{p-1}u.$$

we see that the function $q : \mathbb{R} \to \mathbb{R}$ is continuous. Moreover it satisfies the sign condition $uq(u) \ge 0$ with $q(0) = 0$. Therefore we can apply theorem 7.4 to get the above result. □

Theorem 9.15. *There exists a unique weak solution u for a $T > 0$ to the problem*

$$\begin{aligned} &u_{tt} - \Delta u + m^2 u + \beta u_t + |u|^{p-1}u = f(x,t) \qquad \mathbb{R}^n \times (0,\infty) \\ &u = g, u_t = h \qquad t = 0 \end{aligned} \tag{9.13}$$

with

$$u \in L^\infty(0,T;H^1(\mathbb{R}^n)), \quad u' \in L^\infty(0,T;L^2(\mathbb{R}^n)),$$

provided

$$g \in H^1(\mathbb{R}^n),\ h \in L^2(\mathbb{R}^n),\ f \in L^\infty(0,T;L^2(\mathbb{R}^n)).$$

Proof. By writing

$$m^2 u + \beta u_t + |u|^{p-1}u = q(Du, u_t, u).$$

we see that the function $q : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, or C^{k-1} with $k = 1$. And $q(0,0,0) = 0$. The rest of the proof follows from theorem 7.3. □

Alternatively we can restrict p to take positive integer values to obtain stronger result, so that when p is odd the modulus disappears and q becomes C^∞. For even values the discontinuity only appears at the origin and does not affect the value of the integrals in the proof of theorem 7.3.

Theorem 9.16. *The initial value problem*

$$\begin{aligned} &u_{tt} - \Delta u + m^2 u + \beta u_t + |u|^{p-1}u = f(x,t) \qquad \mathbb{R}^n \times (0,\infty) \\ &u = g, u_t = h \qquad t = 0 \end{aligned} \tag{9.14}$$

admits of a weak solution u for every $T > 0$, with

$$u \in L^2_{loc}((0,T)\times\mathbb{R}^n) \cap L^\infty((0,T);H^1(\mathbb{R}^n)), \quad u' \in L^\infty(0,T;L^2(\mathbb{R}^n)) \tag{9.15}$$

provided

$$g \in H^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n),\ h \in L^2(\mathbb{R}^n),\ f \in L^2(0,T;H^1(\mathbb{R}^n)) \cap L^\infty((0,T);L^2(\mathbb{R}^n)) \tag{9.16}$$

Proof. We transform (9.14) into

$$\begin{aligned} &v_{tt} - \Delta v + (c + m^2)v + e^{-(p-1)\alpha t}|v|^{p-1}v = e^{\alpha t} f(x,t) \qquad \mathbb{R}^n \times (0,\infty) \\ &v = g, v_t = h \qquad t = 0 \end{aligned} \tag{9.17}$$

by defining

$$v = e^{\alpha t}u,\ \alpha = \frac{\beta}{2}, c = -\frac{\beta^2}{4}$$

so that now the equation is without a damping term and is again semilinear.

By writing

$$q(u) := (c + m^2)u + e^{-(p-1)\alpha t}|u|^{p-1}u.$$

we see that the function $q : \mathbb{R} \to \mathbb{R}$ is continuous. Moreover it satisfies the sign condition $uq(u) \geq 0$ if m is chosen large enough. $q(0) = 0$. Therefore we can apply theorem 7.4 to get the above result. □

Question 9.1. Find the quasipotential formula to the system

$$\begin{aligned} &u_{tt} - \Delta u + m^2 u + \beta u_t + |u|^{p-1}u = f(x,t) \qquad \mathbb{R}^n \times (0,\infty) \\ &u = g, u_t = h \qquad t = 0 \end{aligned} \tag{9.18}$$

Answer 9.1. By writing

$$A = \Delta - m^2 I, \tag{9.19}$$

$$U' = |u|^{p-1}u \tag{9.20}$$

and define

$$U : H^1(\mathbb{R}^n) \to \mathbb{R}_+,\ \ U(x) = \int_{R^n} x^p(s)ds,$$

which by [[9] Lecture 9] is of class C^1, with $U(0) = 0$, $DU(0) = 0$, then (9.18) takes the form

$$\ddot{u} = Au - U'(u) - \beta\dot{u} + f, \tag{9.21}$$

which by theorem 8.24 admits of the formula

$$E_\infty\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}a\\b\end{bmatrix}\right) = \beta[\|(-A)^{1/2}a\|^2 + 2U(a) + \|b\|^2], \tag{9.22}$$

Question 9.2. Define the functional

$$Q([\begin{smallmatrix}a\\b\end{smallmatrix}]) := \inf_f \int_{-\infty}^0 |f(t)|^2_{L^2(\mathbb{R}^n)}\,dt, \tag{9.23}$$

find a representation formula in terms of a, b for the system (9.18) defined on $\mathbb{R}^n \times (-\infty, 0)$

Answer 9.2. By a reverse of the time variable

$$u(t) = v(-t),\ \ t \in [0,T], \tag{9.24}$$

we obtain the equivalent equation

$$\begin{aligned} &u_{tt} - \Delta u + m^2 u - \beta u_t + |u|^{p-1}u = f(x,-t) \qquad \mathbb{R}^n \times (0,T), \\ &u(0) = a, u_t(0) = -b. \end{aligned} \tag{9.25}$$

On setting

$$q(Du, u_t, u) = m^2 u - \beta u_t + |u|^{p-1}u, \tag{9.26}$$

theorem 7.3 then establishes the well-posedness of the problem. Moreover, writing $A = \Delta - m^2 I$, $U' = |u|^{p-1}u$ to get

$$\ddot{u} = Au - U'(u) - \beta\dot{u} + f \quad \mathbb{R}^n \times (-\infty, 0) \tag{9.27}$$

Then we find

$$Q([\begin{smallmatrix} a \\ b \end{smallmatrix}]) = \beta[\|(-A)^{1/2}a\|^2 + \|b\|^2 + 2U(a)], \tag{9.28}$$

by the same line of argument as in theorem 8.24.

9.1. **Miscellaneous Discoveries.** Consider the linear version of (8.41) with time on the negative interval

$$\ddot{z} = Az - \beta\dot{z} + g(t), \qquad -\infty < t \le 0 \tag{9.29}$$

we will show that the solution to (9.29) is unique provided it is bounded in the sense that

$$\sup_{t\le 0} \Phi(v) = \frac{1}{2}\langle \mathcal{P}v, v\rangle < \infty.$$

where as before Φ is the Liapunov function and $v = \begin{bmatrix} z \\ \dot{z} \end{bmatrix}$.

Theorem 9.17. *There can be only one bounded solution to* (9.29)*.*

Proof. If there are two such solutions v_1, v_2, let $v = v_1 - v_2$. Then v solves

$$\ddot{z} = Az - \beta\dot{z} \tag{9.30}$$

and by (8.55)

$$\Phi(v(t)) \le \Phi(v(s))e^{-\lambda(t-s)} \tag{9.31}$$

for $s < t \le 0$.

Now fix any $t \le 0$ and let $s \to -\infty$, because of the boundedness of Φ the right hand side vanishes, so we have $\Phi(v(t)) = 0$ for all $t \le 0$, which implies $v = v_1 - v_2 = 0$, so that the two bounded solutions coincide.

Alternatively (9.30) may be written in terms of its spectrally decomposed form to get

$$\ddot{z}_j = -\lambda_j z_j - \beta\dot{z}_j \qquad \text{for } j = 1, 2, \dots \tag{9.32}$$

The solution of which is

$$z_j = C_1 e^{a_1 t} + C_2 e^{a_2 t}$$

where

$$a_1 = \frac{-\beta + \sqrt{\beta^2 - 4\lambda_j}}{2}, a_2 = \frac{-\beta - \sqrt{\beta^2 - 4\lambda_j}}{2}.$$

Because a_1, a_2 are either both real and negative or both complex with a negative real part, when $t \to -\infty$ we get an exploding solution contrary to the assumption that it is bounded. Therefore it must be identically zero. □

Examples 9.1. Consider the equation

$$\dot{z} = -az + g(t), \qquad -\infty < t \le 0 \tag{9.33}$$

with values on $\mathbb{R}$, where $a > 0$ and $g \in L^2(-\infty, 0)$. The solutions to this equation are always bounded provided

$$\lim_{t\to-\infty} z(t) = 0.$$

To see this recall the form of the solution

$$z(t) = e^{-a(t-s)}\left(z(s) + \int_s^t e^{ar} g(r)dr\right) \to \int_{-\infty}^t e^{a(r-t)} g(r)dr$$

as $s \to -\infty$. Applying Cauchy Schwartz inquality we see

$$z(t) \le \frac{1}{2a}\int_{-\infty}^t g^2(r)dr < \infty.$$

Theorem 9.18. *The solutions to* (9.29) *are always bounded provided*

$$\lim_{t\to-\infty} v(t) = 0$$

and if

$$g(t) \in L^1(-\infty, 0).$$

Proof. Recall that (9.29) can be written in the form

$$\dot{v} = \mathcal{A}v + \mathcal{B}g, \tag{9.34}$$

where $v = \begin{bmatrix} z \\ \dot{z} \end{bmatrix}$, $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & -\beta I \end{bmatrix}$, and $\mathcal{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}$. The operator $\mathcal{A}$ with domain $D(A) \times D(-A)^{1/2}$ defines a C_0-semigroup $\mathcal{S}(t)$ on $\mathcal{H} = D(-A)^{1/2} \times H$.

$$\mathcal{S}(t) = \begin{bmatrix} \cos(\sqrt{-A}t) & \frac{1}{\sqrt{-A}}\sin(\sqrt{-A}t) \\ -\sqrt{-A}\sin(\sqrt{-A}t) & \cos(\sqrt{-A}t) \end{bmatrix} \tag{9.35}$$

and

$$\begin{aligned}
\|v(t)\|_{\mathcal{H}} &= \|\mathcal{S}(t-s)v(s) + \int_s^t \mathcal{S}(t-r)\mathcal{B}g(r)dr\|_{\mathcal{H}} \\
&\to \left\| \int_{-\infty}^t \begin{bmatrix} \frac{1}{\sqrt{-A}}\sin(\sqrt{-A}(t-r))g(r) \\ \cos(\sqrt{-A}(t-r))g(r) \end{bmatrix} dr \right\|_{\mathcal{H}} \\
&\le \int_{-\infty}^t \left\| \begin{bmatrix} \sum_{j=1}^\infty \frac{1}{\sqrt{\lambda_j}}\sin(\sqrt{\lambda_j}(t-r))g_j(r)e_j \\ \sum_{j=1}^\infty \cos(\sqrt{\lambda_j}(t-r))g_j(r)e_j \end{bmatrix} \right\|_{\mathcal{H}} dr \\
&= \int_{-\infty}^t \left(\sum_{j=1}^\infty \sin^2(\sqrt{\lambda_j}(t-r))g_j^2(r) + \sum_{j=1}^\infty \cos^2(\sqrt{\lambda_j}(t-r))g_j^2(r) \right)^{\frac{1}{2}} dr \\
&= \int_{-\infty}^t \|g(r)\|_H dr < \infty
\end{aligned} \tag{9.36}$$

□

10. Appendix

We include the calculation that the Kelvin transform preserves the wave operator in the sense that

$$R^{\frac{n+3}{2}}(U_{TT} - \Delta_X U) = r^{\frac{n+3}{2}}(u_{tt} - \Delta_x u). \tag{10.1}$$

For simplicity we assume $|x| > t$, denote $|x|^2 - t^2$ by ρ, so $\frac{\partial}{\partial x_i}\frac{1}{\rho} = -2x_i\rho^{-2}$ and $\frac{\partial}{\partial t}\frac{1}{\rho} = 2t\rho^{-2}$.

Using formula (5.12)

$$u(x,t) \mapsto \left(\frac{1}{|x|^2 - t^2}\right)^{\frac{n-1}{2}} U\left(\frac{x}{|x|^2 - t^2}, \frac{t}{|x|^2 - t^2}\right) = \rho^{-\frac{n-1}{2}} U\left(\frac{x}{\rho}, \frac{t}{\rho}\right)$$

calculate the derivatives:

$$\begin{aligned}\frac{\partial}{\partial x_i} u &= \rho^{-\frac{n-1}{2}}\left(U_{X_i}\cdot\frac{1}{\rho} + \sum_{j=1}^n(-2x_i\rho^{-2}x_j)U_{X_j} + U_T\cdot(-2x_i\rho^{-2}t)\right) + U\cdot\left(-\frac{n-1}{2}\right)\rho^{-\frac{n+1}{2}}2x_i \\ &= \rho^{-\frac{n+3}{2}}(\rho U_{X_i} - \sum_{j=1}^n 2x_i x_j U_{X_j} - 2tx_i U_T - (n-1)\rho x_i U)\end{aligned}$$

(10.2)

$$\begin{aligned}\frac{\partial^2}{\partial x_i^2} u &= -(n+3)\rho^{-\frac{n+5}{2}} x_i(\rho U_{X_i} - \sum_{j=1}^n 2x_i x_j U_{X_j} - 2tx_i U_T - (n-1)\rho x_i U) \\ &+ \rho^{-\frac{n+3}{2}}\Bigg\{2x_i U_{X_i} + \rho U_{X_iX_i}\frac{1}{\rho} + \sum_{j=1}^n \rho U_{X_iX_j}\left(-\frac{2x_i}{\rho^2}\right)x_j + \rho U_{X_iT}\left(-\frac{2x_i t}{\rho^2}\right) \\ &- \left(4x_i U_{X_i} + 2x_i^2 U_{X_iX_i}\frac{1}{\rho} + 2x_i^2\sum_{k=1}^n U_{X_iX_k}\left(-\frac{2x_i}{\rho^2}\right)x_k + 2x_i^2 U_{X_iT}\left(-\frac{2x_i t}{\rho^2}\right)\right) \\ &- \sum_{j\neq i}\left(2x_j U_{X_j} + 2x_j x_i U_{X_jX_i}\frac{1}{\rho} + 2x_j x_i \sum_{k=1}^n U_{X_jX_k}\left(-\frac{2x_i}{\rho^2}\right)x_k + 2x_j x_i U_{X_jT}\left(-\frac{2x_i t}{\rho^2}\right)\right) \\ &- 2tU_T - 2tx_i U_{X_iT}\frac{1}{\rho} - 2tx_i\sum_{j=1}^n U_{X_jT}\left(-\frac{2x_i}{\rho^2}x_j\right) - 2tx_i U_{TT}\left(-\frac{2x_i t}{\rho^2}\right) \\ &- 2(n-1)x_i^2 U - (n-1)\rho U - (n-1)\rho x_i\left(U_{X_i}\frac{1}{\rho} + \sum_{j=1}^n U_{X_j}\left(-\frac{2x_i x_j}{\rho^2}\right) + U_T\left(-\frac{2x_i t}{\rho^2}\right)\right)\Bigg\}\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial t} u &= \rho^{-\frac{n-1}{2}}\sum_{i=1}^n U_{X_i} x_i \frac{2t}{\rho^2} + \rho^{-\frac{n-1}{2}} U_T \frac{1}{\rho} + \rho^{-\frac{n-1}{2}} U_T \frac{2t^2}{\rho^2} + U\frac{n-1}{2}\rho^{-\frac{n-3}{2}}\frac{2t}{\rho^2} \\ &= \rho^{-\frac{n+3}{2}}\left(\sum_{i=1}^n 2tx_i U_{X_i} + \rho U_T + 2t^2 U_T + (n-1)\rho t U\right)\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}u &= (n+3)\rho^{-\frac{n+5}{2}} t\left(\sum_{i=1}^{n} 2tx_iU_{X_i} + \rho U_T + 2t^2U_T + (n-1)\rho t U\right)\\
&+ \rho^{-\frac{n+3}{2}}\left\{\sum_{i=1}^{n} 2x_iU_{X_i} + \sum_{i=1}^{n} 2tx_i \sum_{j=1}^{n} U_{X_iX_j}\frac{2tx_j}{\rho^2} + \sum_{i=1}^{n} 2tx_iU_{X_iT}\left(\frac{1}{\rho}+\frac{2t^2}{\rho^2}\right)\right.\\
&- 2tU_T + (2t^2+\rho)\sum_{i=1}^{n} U_{TX_i}\frac{2tx_i}{\rho^2} + \rho U_{TT}\frac{\rho+2t^2}{\rho^2} + 4tU_T + 2t^2U_{TT}\left(\frac{1}{\rho}+\frac{2t^2}{\rho^2}\right)\\
&\left.- (n-1)2t^2U + (n-1)\rho U + (n-1)\rho t\sum_{i=1}^{n} U_{X_i}\frac{2tx_i}{\rho^2} + (n-1)\rho t U_T\frac{\rho+2t^2}{\rho^2}\right\}\\
&= 2(n+3)t^2\rho^{-\frac{n+5}{2}}x\cdot DU + \rho^{-\frac{n+3}{2}}(n+3)tU_T + \rho^{-\frac{n+5}{2}}2(n+3)t^3U_T + \rho^{-\frac{n+3}{2}}(n-1)(n+3)t^2U\\
&+ \rho^{-\frac{n+3}{2}}2x\cdot DU + \rho^{-\frac{n+7}{2}}4t^2\sum_{i=1}^{n}\sum_{j=1}^{n} x_ix_jU_{X_iX_j} + \rho^{-\frac{n+5}{2}}2tx\cdot DU_T + \rho^{-\frac{n+7}{2}}4t^3x\cdot DU_T\\
&+ 2tU_T\rho^{-\frac{n+3}{2}} + \rho^{-\frac{n+7}{2}}4t^3x\cdot DU_T + \rho^{-\frac{n+5}{2}}2tx\cdot DU_T + \rho^{-\frac{n+3}{2}}U_{TT} + \rho^{-\frac{n+5}{2}}2t^2U_{TT}\\
&- \rho^{-\frac{n+3}{2}}2(n-1)t^2U + \rho^{-\frac{n+1}{2}}(n-1)U + \rho^{-\frac{n+5}{2}}2(n-1)t^2x\cdot DU + \rho^{-\frac{n+3}{2}}(n-1)U_Tt\\
&+ \rho^{-\frac{n+5}{2}}2(n-1)t^3U_T + \rho^{-\frac{n+5}{2}}2t^2U_{TT} + \rho^{-\frac{n+7}{2}}4t^4U_{TT}
\end{aligned}
$$

(10.3)

$$
\begin{aligned}
&= \rho_1(n-1)U + \rho_3(2(n+2)tU_T + (n-1)(n+1)t^2U + 2x\cdot DU + U_{TT})\\
&+ \rho_5(4(n+1)t^2x\cdot DU + 4(n+1)t^3U_T + 4tx\cdot DU_T + 4t^2U_{TT})\\
&+ \rho_7\left(4t^2\sum_{i=1}^{n}\sum_{j=1}^{n} x_ix_jU_{X_iX_j} + 8t^3x\cdot DU_T + 4t^4U_{TT}\right)
\end{aligned}
$$

where ρ_i has replaced $\rho^{-\frac{n+i}{2}}$ in the last equality. Next we continue with (10.2) to obtain the Laplacian,

$$\Delta u = -(n+3)\rho_3 x\cdot DU + 2(n+3)\rho_5|x|^2 x\cdot DU + 2(n+3)\rho_5 tU_T|x|^2 + (n-1)(n+3)\rho_3|x|^2 U$$
$$+ \rho_3 2x\cdot DU + \rho_3\Delta U - \rho_5\sum_{i=1}^n\sum_{j=1}^n 2x_ix_jU_{X_iX_j} - \rho_5 2tx\cdot DU_T + \Big(-2x\cdot DU\rho_3 - 2nx\cdot DU\rho_3$$
$$- \rho_5\sum_{i=1}^n\sum_{j=1}^n 2x_ix_jU_{X_iX_j} + \rho_7\sum_{i=1}^n\sum_{j=1}^n\sum_{k=1}^n 4x_i^2x_jx_kU_{X_jX_k} + \rho_7\sum_{i=1}^n\sum_{j=1}^n 4tx_i^2x_jU_{X_jT}\Big) - 2ntU_T\rho_3$$
$$- \rho_5 2tx\cdot DU_T + \rho_7\sum_{i=1}^n\sum_{j=1}^n 4tx_i^2x_jU_{X_jT} + \rho_7 4t^2U_{TT}|x|^2 - 2(n-1)U|x|^2\rho_3 - n(n-1)U\rho_1$$
$$- \rho_3(n-1)x\cdot DU + \rho_5 2(n-1)|x|^2x\cdot DU + \rho_5 2(n-1)tU_T|x|^2$$

(10.4)

$$= \rho_1(-n(n-1)U) + \rho_3(-(4n+2)x\cdot DU + \Delta U + (n-1)(n+1)U|x|^2 - 2ntU_T)$$
$$+ \rho_5((4n+4)|x|^2x\cdot DU + (4n+4)|x|^2tU_T - 4\sum_{i=1}^n\sum_{j=1}^n x_ix_jU_{X_iX_j} - 4tx\cdot DU_T$$
$$+ \rho_7(\sum_{j=1}^n\sum_{k=1}^n 4|x|^2x_jx_kU_{X_jX_k} + 8t|x|^2x\cdot DU_T + 4t^2U_{TT}|x|^2)$$

Whence upon subtracting (10.4) from (10.3) we get the identity

$$u_{tt} - \Delta u = \frac{1}{(|x|^2 - t^2)^{\frac{n+3}{2}}}(U_{TT} - \Delta U),$$

which is (10.1).

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