On the blob algebra and a Deguchi-Martin quotient of the Hecke algebra

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Abstract

In this thesis, we investigate the representation theory of diagram algebras. We focus on the representation theory of the blob algebra b_n and a certain quotient of Hecke algebra type A which is called Deguchi-Martin quotient, H_n^{PM}. In particular, this research concerns the semisimple case of these algebras.

We use some of the combinatorial results to show that there is a bijection between the canonical basis of the cell modules of the blob algebra b_{n-1} and the Deguchi-Martin quotient of the Hecke Algebra H_n^{PM}. Furthermore, we show that these algebras are isomorphic for $n = 2$ and $n = 3$ at a certain value of parameter m.

Contents

List of Figures

Introduction

In 1991 Deguchi and Martin introduced a quotient of a Hecke algebra of type A, H_n^{PM} [3]. This quotient is a special case of Martin-Rittenberg's hierarchy of 'physical' quotients of the ordinary Hecke algebras corresponding to the $(2,2)$ q-symmetrizer idempotent [13], and they also gave the full description of the representation theory of H_n^{PM}. In this thesis, we called this quotient *the Deguchi-Martin quotient*.

Later in 1993, as a generalization of the Temperley-Lieb algebra, Martin and Saleur introduced blob algebras [14]. It will be denoted by b_n. The blob algebras also generalize the well-known diagram of the Temperley-Lieb algebra [23]. Indeed, it can be defined on the basis of a marked (blobbed) Temperley-Lieb diagram. Because of this, b_n was called the 'blob algebra'. In [15], Martin and Woodcock gave the structure of the standard modules, over any field of characteristic zero. The blob algebra is known as the Temperley-Lieb algebra of type B because it is a quotient of the Hecke algebra type B, similar to how the Temperley-Lieb algebra is a quotient of the Hecke algebra type A. In representation theory, the blob algebra has recently been receiving considerable attention (see for example [16], [19], [11]).

Generically, both algebras b_n and H_n^{PM} are semisimple. For $P = M = 1$, the dimension of the cell module for Deguchi-Martin quotient, R_λ is given via the Pascal triangle [3]. See Figure 2. Shifting n by 1 the dimension of the cell module for the blob algebra, $\Delta_{n-1}(t)$ is also given via the same Pascal triangle [3], [15]. See Figure 1. So, is there a way to construct a bijection between these cell modules? This motivates us to investigate the relation between these algebras.

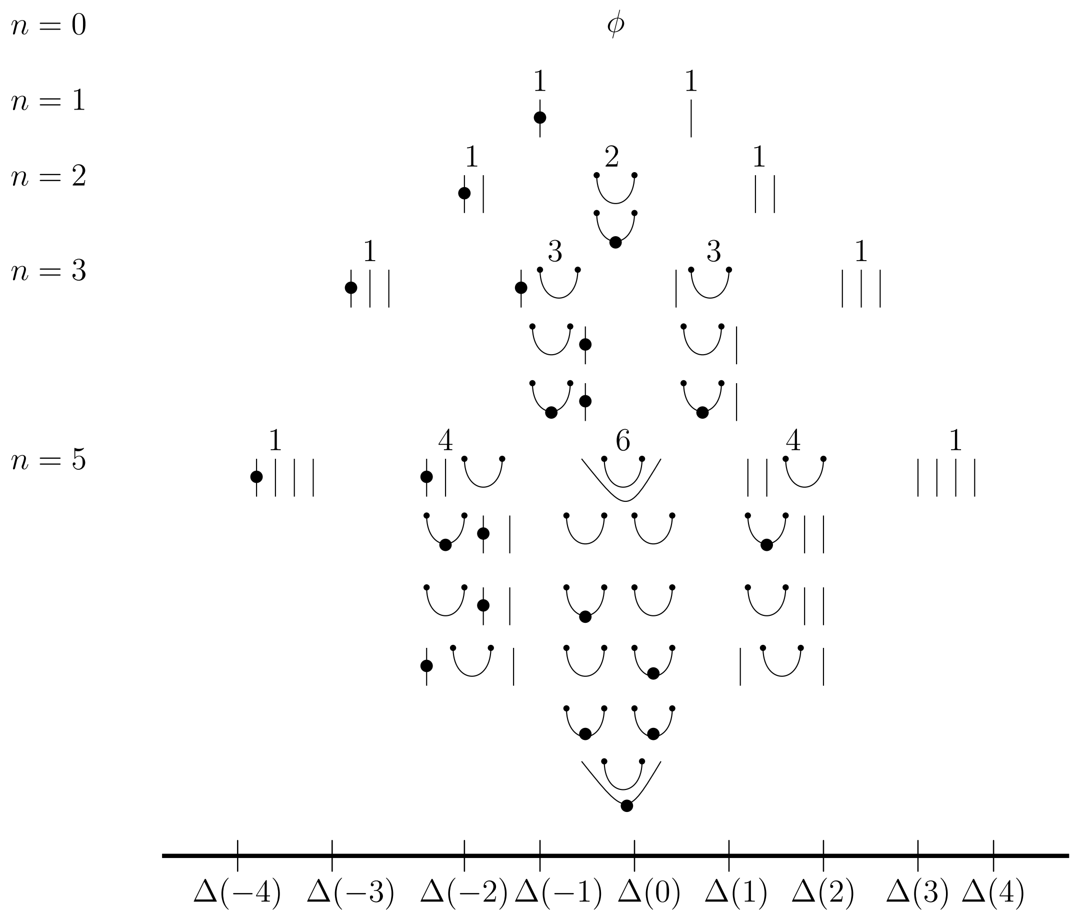


Figure 1: The dimensions and bases for cell modules for the blob algebra

$n = 1$			$\|1>_e$		
$n = 2$		$\|11>_e$		$\|12>_e$	
$n = 3$		$\|111>_e$	$\|112>_e$	$\|111>_e$	
			$\|121>_e$		
$n = 4$	$\|1111>_e$	$\|1121>_e$	$\|1213>_e$	$\|1234>_e$	
		$\|1112>_e$	$\|1123>_e$		
		$\|1211>_e$	$\|1231>_e$		
$n = 5$	$\|11111>_e$	$\|11121>_e$	$\|12113>_e$	$\|12314>_e$	$\|12345>_e$
		$\|12112>_e$	$\|11213>_e$	$\|12341>_e$	
		$\|11112>_e$	$\|12311>_e$	$\|11234>_e$	
		$\|11211>_e$	$\|11123>_e$	$\|12134>_e$	
			$\|12131>_e$		
			$\|11231>_e$		

Figure 2: The Pascal triangle of $H_n^{1|1}$ cell modules

In this work we study the relationship between the simple (cell) basis of the Deguchi-Martin quotient in $(1 \mid 1)$ case and the blob algebra. by the combinatorics of Young tableau. Plaza and Hansen's results inspire this strategy. In [20] they show that the blob algebra is graded cellular algebras, thus making their cell modules, or standard modules, graded modules for the algebra. They introduced a way of determining the blob diagrams, using standard bitableaux of one-line bipartitions. Moreover, they establish a bijection between the set of blob diagrams and pairs of one-line standard bitableaux of the same shape.

We will provide a brief summary of this thesis. The first chapter reviews the fundamental concepts needed to go through the other chapters. Mainly, this will introduce the symmetric group, combinatorial concepts, Hecke algebra, Temperley Lieb algebra, blob algebra, and the general theory of cellular algebras.

In Chapter 2, we review the main algebra that led to the current study. In the first section of this chapter, we introduce a representation $\mathcal{R}_{PM}$ of Hecke algebra. Some notations related to Hecke algebra are reviewed in the second section. In the next section, we recall section 3.3 of [3] which deals with q-permutation representation of $\mathcal{R}_{PM}$. Section four describes the irreducible representations of $\mathcal{R}_{PM}$. After that, we will determine the generic irreducible content of q-permutation representation. Finally, we define the Deguchi-Martin quotient, one of this study's main inputs.

The third chapter presents the main finding of the research. The first section gives a brief overview of the work of Plaza and Hansen including their bijection between the blob diagrams and pairs of one-line standard bitableaux of the same shape. From this point, in section two we define a map between the set of basis cell modules of blob algebra and the Deguchi-Martin quotient and review some definitions required for this. We define a second map in the third section, which will be the reverse of the first. Theorem 3.4.1 in the last section illustrates the main result of this thesis.

In the final chapter, we will begin to investigate isomorphism between the element of blob algebra and the Deguchi-Martin quotient. We look at some small dimensional isomorphism between the algebras themselves.

Chapter 1

Background

The goal of this thesis is to investigate the relationship between blob algebra and the quotient of Hecke algebra using the results of their representation theory. Let us recall the main concepts of Temperley-Lieb algebra and the symmetric group, which is a special case of blob algebra and Hecke algebra. Furthermore, since all of these algebras are cellular algebras, we also recall cellular algebras and related results.

The complex field $\mathbb{C}$ will be used as the ground field throughout the thesis.

1.1 The Symmetric Group

Let $\mathbb{N}$ be the natural numbers. For a fixed $n \in \mathbb{N}$, the symmetric group on n letters, S_n is the group on the set $X = \{1, 2, \ldots, n\}$ whose consisting of all bijections from X to X and whose group multiplication is composition. The elements $\sigma \in S_n$ are called permutations. The permutations of the form $\tau = (i, j)$ are called transpositions which generate S_n. Indeed, the symmetric group is generated by the adjacent transpositions $(1,2),(2,3),\ldots,(n-1,n)$. If $\sigma = \tau_1\tau_2\cdots\tau_k$, where the τ_i are transpositions, then following to section 1.1 in [22] we define the sign of σ to be

$$\operatorname{sgn}(\sigma) = (-1)^k.$$

The representation theory of symmetric groups brings some interesting combinatorial notions such as a partition, Young diagram, Young tableau, and more, which will be used during this thesis. We define these notions in the next section.

1.2 Combinatorics of Tableaux

Definition 1.2.1 ([22, Definition 2.5.3])**.** *A* **composition** *of n is an ordered sequence of non-negative integers*

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$$

such that $\sum_{i=1} \lambda_i = n$. The integers λ_i are called the parts of the composition.

Definition 1.2.2 ([6, p. 1])**. A partition** *λ of a positive integer n is a non-increasing composition λ. Write $\lambda \vdash n$ to indicate that λ is a partition of n.*

For a partition (composition) λ, the Young diagram is another tool that can be used to represent the partition, which is also used to describe many objects in algebra and combinatorics.

Definition 1.2.3 ([6, p. 1])**. A Young diagram** *of size n is an array of n left-justified boxes with a non-increasing number of boxes in each row.*

In particular, there is a clear one-to-one correspondence between partitions and Young diagrams.

Definition 1.2.4 ([22, Definition 3.5.2]). *Let λ be a Young diagram, then* **the conjugate of** λ *is* $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_l)$. *Where λ'_i is the length of the ith column of λ_i. Otherwise put, λ' is just the transpose of the diagram of λ and so it is sometimes denoted by λ^t.*

$$\lambda = (5, 2, 1) \qquad \lambda' = (3, 2, 1, 1, 1)$$

Figure 1.1: Conjugate partition.

Definition 1.2.5 ([6, p. 2]). **Young tableau**, *is obtained by filling in the boxes of a Young diagram of $\lambda \vdash n$ with $1, 2, \dots, n$ such that:*

1. *non-decreasing across each row;*

2. *strictly increasing down each column.*

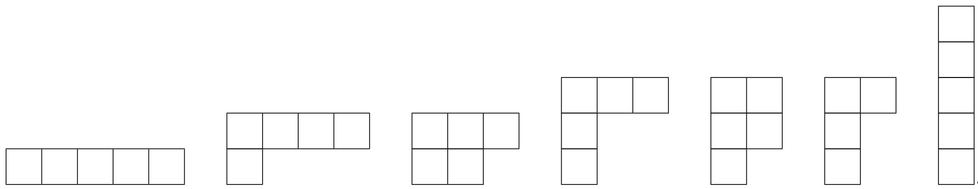
We say that λ is the shape of the tableau for some partition λ.

Definition 1.2.6. A standard tableau *is a Young tableau whose entries are increasing across each row and each column.*

Example 1.2.7. *Let $n = 5$, the set of all partitions of 5 is*

$$\{(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1^5)\}.$$

The Young diagrams corresponding to these partitions are



The standard tableaux of the partition $\lambda = (4, 1)$ are

1	2	3	4
5			

1	2	3	5
4			

1	2	4	5
3			

1	3	4	5
2			

Definition 1.2.8 ([22, Definition 2.2.2])**.** *Suppose* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ *and* $\mu = (\mu_1, \mu_2, \dots, \mu_k)$ *are partitions of* n*. Then* λ **dominates** μ*, written* $\lambda \trianglerighteq \mu$*, if*

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \geq \mu_1 + \mu_2 + \cdots + \mu_i$$

for all $i \geq 1$*. If* $i > l$ *(respectively,* $i > k$*), then we take* λ_i *(respectively,* μ_i*) to be zero. For example if* $\lambda = (3, 2)$ *and* $\mu = (3, 1, 1)$*, then* $\lambda \trianglerighteq \mu$*.*

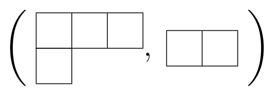
Definition 1.2.9 ([20, Section 4.2])**. A bipartition of** n *is a pair* $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ *of usual (integer) partitions such that* $n = |\lambda^{(1)}| + |\lambda^{(2)}|$*. By the Young diagram of a bipartition* λ *we mean the set*

$$[\lambda] = \{(i, j, k) \in \mathbb{N} \times \mathbb{N} \times \{1, 2\} \,|1 \leq j \leq \lambda_i^{(k)}\}.$$

Its elements are called entries or nodes. We can visualize $[\lambda]$ *as a pair of usual Young diagrams called the components of* $[\lambda]$*.*

Definition 1.2.10 ([20, Section 4.2])**.** *For* λ *a bipartition,* **a** λ**-bitableau** *is a bijection* $t : [\lambda] \to \{1, \dots, n\}$*. We say that* t *has shape* λ *and write shape*$(t) = \lambda$*. A* λ*-bitableau is called standard if in each component its entries increase along each row and down each column. The set of all standard* λ*-bitableaux is denoted* *by* $\mathrm{Std}(\lambda)$ *and the union* $\bigcup_\lambda \mathrm{Std}(\lambda)$ *with* λ *running over all bipartitions of* n *is denoted by* $\mathrm{Std}(n)$*.*

Example 1.2.11. *Let* $\lambda = ((3,1),(2))$ *be a bipartition of* $n = 6$*, then the Young diagram corresponding to the bipartition* $((3,1),(2))$ *is*



an example of $((3,1),(2))$*-bitableau is*

$$\left(\begin{array}{|c|c|c|}\hline 4 & 2 & 5 \\ \hline 3 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 6 & 1 \\ \hline \end{array} \right).$$

A *one-line bipartition* of n is a bipartition λ of n such that $\lambda_i^{(k)} = 0$ for all $i \geq 2$ and $k = 1,2$. The set of all one-line bipartition of n is denoted by $\mathrm{Bip}_1(n)$. For example let $\mu = ((4),(2))$ be a one-line bipartition of 6, one of corresponding standard bitableau of μ is

$$\left(\begin{array}{|c|c|c|c|}\hline 2 & 3 & 4 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 6 \\ \hline \end{array} \right).$$

1.3 Representations of the Symmetric Group

While it is not our goal to cover every aspect of the symmetric group representation theory, we briefly describe some of the key findings that will be relevant to the next section.

Proposition 1.3.1 ([22, Section 1.1])**.** *For a symmetric group* S_n*, there is a natural one-to-one correspondence between partitions of* n *and conjugacy classes of* S_n*.*

Proposition 1.3.2 ([22, Proposition 1.10.1]). *Let G be a finite group, then the number of inequivalent irreducible G-modules equals the number of conjugacy classes of G.*

Definition 1.3.3 ([22, Definition 2.3.1]). *Let λ be a partition of n and let t be a λ-tableau. Suppose that the t has rows R_1,R_2, ..., R_l and columns C_1, C_2,..., C_k. Then*

$$R_t = S_{R_1} \times S_{R_2} \times \cdots \times S_{R_l}$$

and

$$C_t = S_{C_1} \times S_{C_2} \times \cdots \times S_{C_k}$$

are the **row-stabilizer** *and* **column-stabilizer** *of t, respectively.*

Example 1.3.4. *Let $t = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$ be $(2,2)$-tableau, then the row stabilizer and column stabilizer are*

$$R_t = S_{\{1,3\}} \times S_{\{2,4\}} = \{1, (13), (24), (13)(24)\}.$$

and

$$C_t = S_{\{1,2\}} \times S_{\{3,4\}} = \{1, (12), (34), (12)(34)\}.$$

Definition 1.3.5 ([22, Definition 2.1.4]). *Two λ-tableaux t_1 and t_2 are* **row equivalent**, *$t_1 \sim t_2$ if corresponding rows of the two tableaux contain the same elements. A* **tabloid** *of shape λ, or λ-tabloid, is then*

$$\{t\} = \{t_1 \mid t_1 \sim t\}.$$

The λ-tabloid is drawn as the tableaux without vertical lines between the entries in each row.

For example consider t as given in example 1.3.4, then $\{t\}$ is

$$t = \frac{\overline{\begin{array}{cc} 1 & 3 \\ \hline 2 & 4 \end{array}}}{} = \left\{ \begin{array}{cc} 1 & 3 \\ \hline 2 & 4 \end{array}, \begin{array}{cc} 3 & 1 \\ \hline 2 & 4 \end{array}, \begin{array}{cc} 1 & 3 \\ \hline 4 & 2 \end{array}, \begin{array}{cc} 3 & 1 \\ \hline 4 & 2 \end{array} \right\}.$$

For $\sigma \in S_n$, $\sigma\{t\} = \{\sigma t\}$. By this action, we get, in the usual way, an S_n-module.

Definition 1.3.6 ([22, Definition 2.1.5])**.** *Suppose $\lambda \vdash n$, Let*

$$M^\lambda = \mathbb{C}\{\{t_1\}, \{t_2\}, \ldots, \{t_k\}\}$$

where $\{t_1\}$, $\{t_2\}$, $\ldots$, $\{t_k\}$ is a complete list of λ-tabloids. Then M^λ is called the **permutation module** *corresponding to λ.*

Definition 1.3.7 ([22, Definition 2.3.2])**.** *If t is a tableau, then the associated* **polytabloid** *is*

$$e_t = \sum_{\sigma \in C_t} \mathrm{sgn}(\sigma)\sigma\{t\}.$$

Example 1.3.8. *Using the tableau t in Example 1.3.4, then the column stabilizer is $C_t = \{1, (12), (34), (12)(34)\}$ and*

$$e_t = \begin{array}{cc} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} - \begin{array}{cc} \hline 2 & 3 \\ \hline 1 & 4 \\ \hline \end{array} - \begin{array}{cc} \hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array} + \begin{array}{cc} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}.$$

The symmetric group S_n acts on a Young tableau and its associated polytabloid by permuting entries of the tableau.

Lemma 1.3.9 ([22, Lemma 2.3.3])**.** *Let t be a tableau and σ be a permutation, then $e_{\sigma t} = \sigma e_t$.*

Definition 1.3.10 ([22, Definition 2.3.4])**.** *For any partition λ, the corresponding* **Specht module**, *S^λ, is the submodule of M^λ. spanned by the polytabloids e_t where t is of shape λ.*

Theorem 1.3.11 ([22, Theorem 2.4.6]). *The S^λ for $\lambda \vdash n$ form a complete list of irreducible S_n-modules over $\mathbb{C}$ generated by any given polytabloid.*

1.4 The Hecke Algebra of Type A_n

The Hecke algebra was introduced by Erich Hecke [12]. It is a deformation of the group algebra of a Coxeter group. These algebras are defined by generators and relations and depend on a deformation parameter q. Taking $q = 1$ gives us group algebra of a symmetric group corresponding Coxeter group.

For $q \in \mathbb{C}^*$ and an integer k the Gaussian coefficient is defined as

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}} = q^{k-1} + q^{k-3} + \cdots + q^{-(k-3)} + q^{-(k-1)}.$$

Definition 1.4.1 ([17]). *The* Hecke algebra *with parameter $q \in \mathbb{C}^*$, denoted by $H_n(q)$ is defined to be associative algebra over $\mathbb{C}$ generated by $T_1, \ldots, T_{n-1}$ satisfying the relations:*

$$\begin{aligned} T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}, && \text{for } i = 1, 2, \ldots, n-2, \\ T_iT_j &= T_jT_i, && \text{for } |i-j| \geq 2, \\ T_i^2 &= (q-1)T_i + q, && \text{for } i = 1, 2, \ldots, n-1. \end{aligned}$$

The Hecke algebra of the symmetric group S_n can be described as a deformation of the group algebra $\mathbb{C}S_n$. Moreover, in the case $q = 1$, $H_n(q)$ becomes $\mathbb{C}S_n$.

1.5 Representations of the Hecke Algebra

We recall some facts bout the representations of Hecke algebra from [17] and [4].

Theorem 1.5.1 ([17, Theorem 1.13]). *The Hecke algebra is a free as an* $\mathbb{C}$-*module with basis* $\{T_w \mid w \in S_n\}$.

Following to lemma 1.12 in [17] the multiplication rules in this basis is given by

$$T_{s_i}T_w = \begin{cases} T_{s_iw} & \text{if } l(s_iw) > l(w), \\ qT_{s_iw} + (q-1)T_w & \text{if } l(s_iw) < l(w). \end{cases}$$

For a composition λ of n, let t^λ be a row standard tableau. The symmetric group S_n acts on λ-tableaux by permuting the entries of the tableau. We denote the permutation $\sigma \in S_n$ such that $t^\lambda\sigma$ is a row standard by $d(t)$. Let R_t The row stabilizer of a tableau t^λ, it is a subgroup of S_n (Young subgroup), $R_t = R_{\lambda_1} \times R_{\lambda_2} \times \cdots \times R_{\lambda_l}$ (see definition 1.3.3) . Let $H(R_t)$ is a subalgebra of H_n. It is a free as an $\mathbb{C}$-module with basis $\{T_w \mid w \in R_t\}$, then consequently $H(R_t) = H(R_{\lambda_1}) \times H(R_{\lambda_2}) \times \cdots \times H(R_{\lambda_l})$. For given λ we define two elements in H_n play important role in its representation

$$x_\lambda = \sum_{w \in R_t} T_w$$

and

$$y_\lambda = \sum_{w \in R_t} -q^{-l(w)}T_w.$$

Definition 1.5.2 ([17, Chapter 3]). *A* **q-permutation module** M^λ *is the right* H_n*-module* $x_\lambda H_n$*, with basis* $\{x_\lambda T_{d(t)} \mid t \text{ is a row stander } \lambda - \text{tableau}\}$

and the action of H_n on this basis is

$$x_\lambda T_{d(t)}T_{s_i} = \begin{cases} x_\lambda T_{d(s)} & \textit{if } s \textit{ is a row standard and } l(d(s)) > l(d(t)), \\ q x_\lambda T_{d(t)} & \textit{if } s \textit{ is not a row standard,} \\ q x_\lambda T_{d(s)} + (q-1) x_\lambda T_{d(t)} & \textit{if } s \textit{ is a row standard and } l(d(s)) < l(d(t)). \end{cases}$$

Where $s = ts_i$ for some $1 \leq i < n$. Note that M^λ above is the q-analog of the permutation module of the symmetric group.

Now, let w_λ be the permutation in S_n such that $t^\lambda w_\lambda$ is standard columns. Dipper and Jams in [4] defined an element z_λ in H_n by

$$z_\lambda = x_\lambda T_{w_\lambda} y_{\lambda'} = \sum_{u \in R_{\lambda'}} (-q)^{-l(u)} x_\lambda T_{w_\lambda u}.$$

In the next definition, we define certain H_n-submodules S^λ of M^λ.

Definition 1.5.3 ([4, Chapter 4])**.** *The* q-**Specht module** S^λ *is the H_n-module $z_\lambda H_n$.*

Generically, the Hecke algebra $H_n(q)$ is semisimple over the field of characteristic zero. next theorem conclusion the most important facts about the irreducible representation of $H_n(q)$ over $\mathbb{C}$.

Theorem 1.5.4 ([4, Theorem 5.2.])**.** *Let $\lambda \vdash n$, Then $H_n(q)$ is semisimple algebra over $\mathbb{C}$ and $\{S^\lambda \mid \lambda \vdash n\}$ is a complete set of non-isomorphic irreducible H_n-modules, with*

$$\{z_\lambda T_d \mid d \geq w_{\lambda'}\} = \{z_\lambda T_d \mid t^\lambda w_\lambda d \textit{ is standard}\}$$

as a basis. Where $q \neq 1$ and not a root of unity.

Hecke algebra is One of the motivating examples for cellular algebra. Graham and Lehrer introduced cellular algebras as a general framework for investigating modular representation theory [7].

1.6 Cellular Algebras

In this section, we review the definition of cellular algebra and the related results of its representation theory. These algebras have been introduced in 1996 by Graham and Lehrer [7] in terms of a cellular basis.

Definition 1.6.1 ([7, Definition 1.1])**.** *Let R be a commutative ring with 1, the* **cellular algebra** *A is an R-algebra with cell datum (Λ, M, C, i) where*

1. *Λ is a partially ordered set (poset) and for any $\lambda \in \Lambda$, $M(\lambda)$ is a finite set. The algebra A has a R-basis $\{C^{\lambda}_{S,T}\}$ where $(S,T) \in M(\lambda) \times M(\lambda)$ for all $\lambda \in \Lambda$.*

2. *The map i is an R-linear anti-automorphism of A with $i^2 = id$ which sends $C_{S,T}$ to $C_{T,S}$.*

3. *If $\lambda \in \Lambda$ and $S,T \in M(\lambda)$, then for any $a \in A$ the product $aC_{S,T}$ can be written as*

$$aC_{S,T} \equiv \sum_{U \in M(\lambda)} r_a(U,S) C^{\lambda}_{U,T} \quad mod\ A^{<\lambda}, \tag{1.1}$$

where $a \in A$ and $r_a(U,S) \in R$ do not depend on T, and $A^{<\lambda}$ is the R-submodule of A generated by $\{C^{\lambda'}_{VW} : \lambda' < \lambda, V, W \in M(\lambda')\}$. Note that for $S,T \in M(\lambda)$ we write $C(S,T) = C^{\lambda}_{S,T} \in A$.

The basis is called a cellular basis.

Definition 1.6.2 ([7, Definition 3.1]). *For $\lambda \in \Lambda$ the radical of the bilinear form θ_λ is defined to be*

$$\mathrm{rad}(\lambda) := \{x \in W(\lambda) \mid \theta_\lambda(x, y) = 0 \quad \textit{for all} \quad y \in W(\lambda)\} \tag{1.2}$$

Definition 1.6.3 ([7, Definition 2.1]). *Let A be a cellular algebra with cellular datum (Λ, M, C, i). For each $\lambda \in \Lambda$ define the* **cell module** *$W(\lambda)$ as the left A-module which is R-module, with basis $\{C_S^\lambda \mid S \in M(\lambda)\}$ and A-left action given by*

$$aC_S = \sum_{S'} r_a(S', S)C_{S'} \qquad (a \in A, S \in M(\lambda)). \tag{1.3}$$

where r_a is the element in R appeared in Definition1.6.1(3).

For S, T, U and $V \in M(\lambda)$. The cell module $W(\lambda)$ has a bilinear form:

$$\begin{aligned} \theta_\lambda : W(\lambda) \times W(\lambda) &\to R, \\ (C_T, C_U) &\mapsto r(T, U)\, by \\ C_{S,T}C_{U,V} &\equiv \theta(C_T, C_U)C_{S,V}^\lambda \quad mod\ A^\lambda. \end{aligned}$$

Proposition 2.9 in [17] stated that the bilinear form is symmetric and associative.

Proposition 1.6.4 ([7, Proposition 3.2]). *For $\lambda \in \Lambda$ we have*

1. $\mathrm{rad}(\lambda)$ *is an A-submodule of $W(\lambda)$.*

2. *If $\theta_\lambda \neq 0$, the quotient $W(\lambda)/\mathrm{rad}(\lambda)$ is irreducible.*

3. *If $\theta_\lambda \neq 0$, $\mathrm{rad}(\lambda)$ is the radical of $W(\lambda)$.*

Definition 1.6.5 ([7, Definition 3.3]). *For $\lambda \in \Lambda$, $\theta_\lambda \neq 0$, denote the irreducible A-module $W(\lambda)/\mathrm{rad}(\lambda)$ by $L(\lambda)$.*

The following theorem is one of the main conclusions Graham and Lehrer [7] made in order to categorise the irreducible cell modules for a cellular algebra A.

Theorem 1.6.6 ([7, Theorem 3.4]). *Let* $\Lambda_0 = \{\lambda \in \Lambda : \theta_\lambda \neq 0\}$. *Then the set* $\{L(\lambda) := W(\lambda)/\operatorname{rad}(\lambda) : \lambda \in \Lambda_0\}$ *is a complete set of pairwise inequivalent irreducible A-modules.*

1.7 The Temperley-Lieb Algebra

In this section, we will review the two definitions of Temperley-Lieb algebra, one defined by presentations and the other by diagrams, which will be the special case of blob algebra in the following section.

Definition 1.7.1 ([21, p. 964]). **The Temperley-Lieb algebra** $TL_n(\beta)$ *is the associative unital algebra over* $\mathbb{C}$ *with generates elements* $1, e_1, \dots, e_{n-1}$ *satisfying the following relations*

$$\begin{aligned} e_i^2 &= \beta e_i; \\ e_i e_{i\pm1} e_i &= e_i; \\ e_i e_j &= e_j e_i \text{ if } |i-j| > 1. \end{aligned}$$

We can represent this algebra using a basis of diagrams to get a clearer understanding of how the elements of $TL_n(\beta)$ interact. We will describe how this is done briefly here; a more detailed description can be found in [21].

Definition 1.7.2 ([14, Section 2.1]). *A Temperley-Lieb diagram of length* n *is a rectangular frame with* n *points on the top and* n *on the bottom edge. Connecting these points (sometimes referred to as nodes) by non-crossing strings* *that connect one node to exactly one other node. It is convenient to refer to lines that travel from top to bottom as propagating lines and those that double back to the same edge as loop lines or arcs. The multiplication of two diagrams is given by identification of the bottom of one diagram with the top of the other, the exterior rectangles are construction lines only and can be ignored in composition and replacing every closed loop that may arise by a factor* β *(where* $\beta = q + q^{-1}$*).*

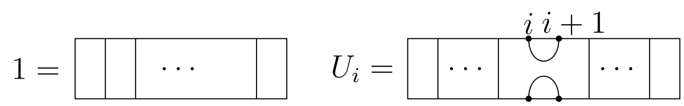


Figure 1.2: The generators 1 and U_i of $TL_n(\beta)$.

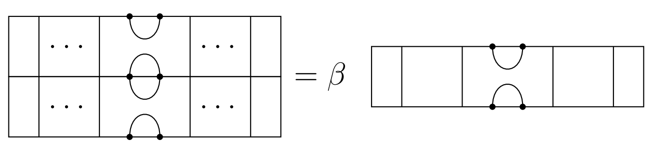
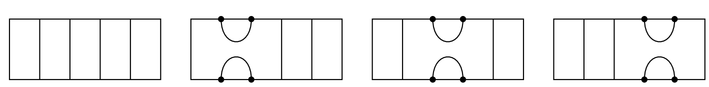


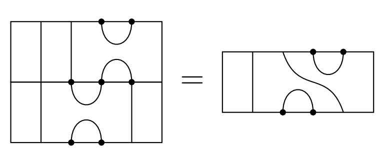
Figure 1.3: Multiplication of two diagrams in TL_n.

Example 1.7.3. *Let* $n = 4$*, The Temperley-Lieb algebra* $TL_4(\beta)$ *is generated by* $\{1, e_1, e_2, e_3\}$ *which corresponding to the following diagrams*



1 U_1 U_2 U_3

The multiplication U_3U_2 *is given by*



Theorem 1.7.4 ([21, Theorem 2.4])**.** *the Temperley-Lieb algebra TL_n has dimension*

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!} \tag{1.4}$$

where C_n is the nth Catalan number.

1.8 The Blob Algebra

Blob algebras are first defined by Martin and Saleur [14] which are generalizations of the Temperley Lieb algebras. These algebras are defined in two ways, as abstract algebras defined by presentations, and as diagram algebras defined by diagrams. In this subsection, we review both definitions.

For $q \in \mathbb{C}^*$ and for an integer k, let $[k]$ be the Gaussian coefficient defined as in Section 1.4.

Definition 1.8.1 ([2, Section 2])**.** *Let $m \in \mathbb{Z}$ and $[m] \neq 0$. The* blob algebra *$b_n(m, q)$ is the $\mathbb{C}$-algebra generated by elements $1, e_0, e_1, \ldots, e_{n-1}$ satisfying the relations*

$$\begin{aligned} e_i^2 &= -[2]e_i; & \\ e_ie_j &= e_je_i, & \text{if } |i-j| > 1; \\ e_ie_je_i &= e_je_ie_j, & \text{if } |i-j| = 1; \\ e_1e_0e_1 &= [m+1]e_1; & \\ e_0^2 &= -[m]e_0. & \end{aligned} \tag{1.5}$$

Throughout this thesis, we will simply refer to $b_n(m, q)$ as b_n.

We shall consider the corresponding diagrammatic definition given in [2].

Definition 1.8.2 ([2, Section 2])**.** *A blob algebra b_n is a $\mathbb{C}$-algebra generated by the Temperley–Lieb diagrams U_i and the additional decoration diagram e of "blobs" on certain lines. See figure 1.4. The multiplication blob diagrams are given as the multiplication on TL_n diagrams with additional rules that we identify any diagram with a double-dots line with $-[m]$ times the same diagram with a single dot, and identify any diagram containing a closed internal loop with no decoration (respectively decorated by a dot) with the same diagram without the loop, multiplied by $-[2]$ (respectively multiplied by $[m+1]$). We call the diagrams that arise generalized Temperley Lieb diagrams.*

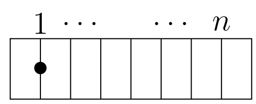
$e =$ 

Figure 1.4: The blob generator e

Lemma 1.8.3 ([9, Lemma 5.7])**.** *The dimension of the blob algebra is given by*

$$dim\ b_n = \binom{2n}{n} = \frac{(2n)!}{(n!)^2}.$$

1.9 Representations of the Blob Algebra

Definition 1.9.1. *Let $\mathbb{B}_n$ be the blob algebra's basis set consisting of the generalized Temperley Lieb diagrams, with the condition that no line to the right of the leftmost propagating line is decorated, and only the outermost arcs to the left of it are decorated and each such line has at most one dot.*

Definition 1.9.2. *For $t \in \{n, n-2, n-4, \ldots, 2-n, 4-n, \ldots, -n\}$, let $\mathbb{B}_n(t)$ be a subset of $\mathbb{B}_n$ consists of blob diagrams with t propagating lines and let $\mathbb{B}_n(-t)$ be a subset of $\mathbb{B}_n$ consists of blob diagrams with t propagating lines with condition that the leftmost propagating line is decorated by the blob.*

Definition 1.9.3. *For any diagram D in $\mathbb{B}_n$, a half diagram is generated by cutting diagram D horizontally from east to west such that only propagating lines are cut. A decorated propagating line with a blob is replaced with a propagating line decorated with two blobs and cut between the blobs.*
By this we get a pair of half diagrams which are the top half diagram and bottom half diagram, as in figure 1.5. We denote the set of all top halves diagrams (bottom halves diagrams) by $\mathbb{B}_n^{top}$ ($\mathbb{B}_n^{bot}$) respectively.

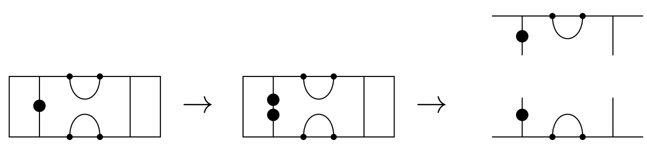


Figure 1.5: Cutting a decorated diagram into two halves

For any $t \in \Lambda$ and $d \in \mathbb{B}_n^{top}$ ($\mathbb{B}_n^{bot}$) the blob algebra acts via multiplication defined in Definition 1.8.2; explicitly, for half diagram d and a diagram $D \in b_n$, the multiplication $D.d$ given by identification of the bottom diagram D with the top of half diagram d. If the final diagram in this multiplication contains lines not connected to the top edge it is sent to zero. The result will therefore either be zero or another top-half n-diagram multiplied by some factor. In other words, the half-top (bottom) diagram gives a basis for left (right) modules over blob algebra b_n which are called standard modules for b_n and are usually dented by $\Delta_n(t)$. see for example [15, Section 4.1] and [14, Section 2.2]. In general the set

$$\{\Delta_n(t) \mid t \in \{-n, -n+2, \ldots, n\}\}$$

is a complete set of standard modules for b_n with $\mathbb{B}_n(t)$ as a set of basis, which are in fact the irreducible representations in generic case [14].

One of the easiest examples of cellular algebra is blob Algebra. We will provide a simple example showing how b_n is cellular.

Proposition 1.9.4 ([8, Section 2]). *The blob algebra b_n is a cellular algebra.*

We now outline the various elements of a cellular structure in sense of definition 1.6.1:

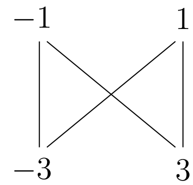
- Index set Λ is given by $\{n, n-2, n-4, \ldots, 2-n, 4-n, \ldots, -n\}$, partially ordered as follows: $t \geq s$ if $|t| < |s|$ or $t = s$. The finite indexing set $M(t)$ is given $M(t) = \{$ half diagrams with t propagating lines $|t \in \Lambda\}$. For $T, S \in M(t)$ define the basis element $C^t_{S,T} = ST^*$, where $*$ denotes reflection in a horizontal axis (which will be explained shortly in the example). If ST^* produces a propagating line with double blobs it is replaced with a propagating line with a single blob.
- Define the map i as follows

$$i : b_n \to b_n$$
$$C^t_{S,T} \mapsto C^t_{T,S}$$

- For any $t \in \Lambda$ and $S, T \in M(t)$, the blob algebra acts via multiplication defined in definition 1.8.2; explicitly, for half diagram S in $M(T)$ and a diagram D in b_n, the multiplication $S.D = SD$ given by identification of half diagram S with the top diagram D if $S.D \in M(T)$, and $S.D = 0$ otherwise.

Now, for any $t \in \{n, n-2, n-4, \ldots, 2-n, 4-n, \ldots, -n\}$ the blob algebra cell module $\Delta_n(t)$ has halves diagrams as a cell basis and the action of b_n as we define above.

Example 1.9.5. *The cellular structure for $b_3(m, q)$ is given as by following: the index set $\Lambda = \{-3, -1, 3, 1\}$, the order of Λ is*



A cell basis elements set $M(t)$ is given by Figure 1.6, Figure 1.7, and Figure 1.8.

Figure 1.6: The cell basis $M(-3)$ (at left) and $M(3)$ (at right).

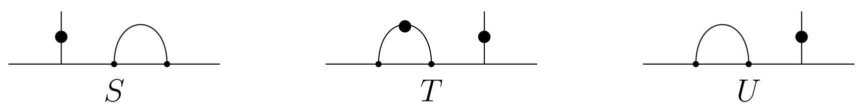


Figure 1.7: The set of cell basis $M(-1)$.

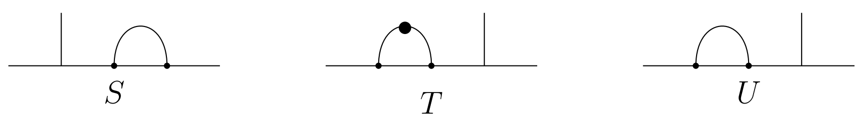
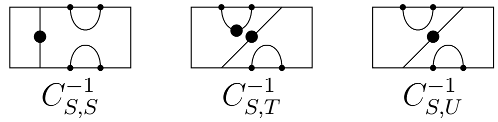


Figure 1.8: The set of cell basis $M(1)$.

The basis elements of $b_3(m,q)$ with one decorated propagating line are given in the figure 1.9 have been constructed from the halves diagrams in the set $M(-1)$ in figure 1.7. For instance, the $C^{-1}_{S,U}$ has been produced by drawing the half diagram S at the south edge and the half diagram U^* at the north edge.



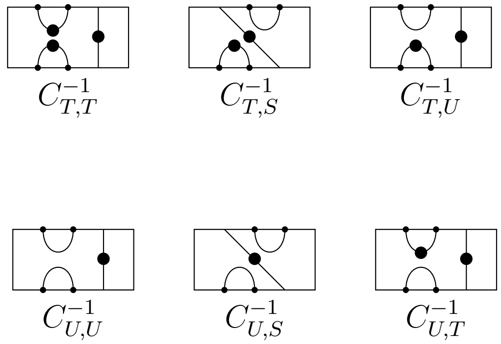


Figure 1.9: The basis elements of $b_3(m,q)$ with one decorated propagating line.

The basis elements of $b_3(m,q)$ *with one propagating line (not decorated) have construed from the halves diagram in set* $M(1)$ *are given in figure 1.10*

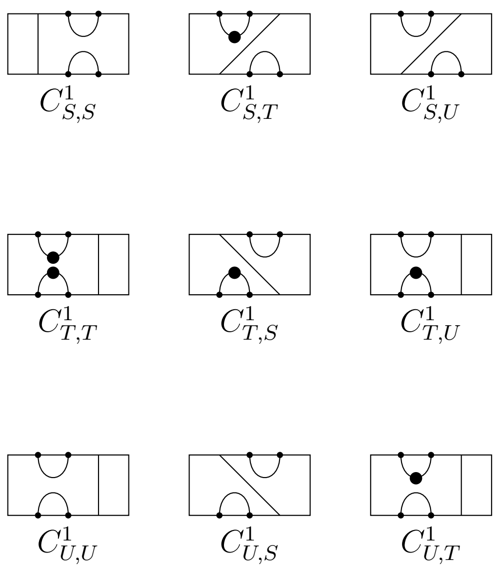
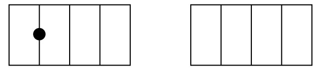


Figure 1.10: The basis elements of $b_3(m,q)$ with one decorated propagating line.

The basis element e *of* $b_3(m,q)$ *has construed from the half diagram in set* $M(-3)$*, and the identity element* I *has construed from the set* $M(3)$ *both are given in figure 1.11.*



$C^{-3}_{S,S} = e$ $\quad$ $C^{3}_{S,S} = I$

Figure 1.11: Basis elements of $b_3(m, q)$ with 3 lines

Therefore, set of all basis elements of $b_3(m, q)$ *can be written as*

$$\mathcal{C} = \{C^{-1}_{S,S}, C^{-1}_{S,T}, C^{-1}_{S,U}, C^{-1}_{T,T}, C^{-1}_{T,S}, C^{-1}_{T,U},$$
$$C^{-1}_{U,U}, C^{-1}_{U,S}, C^{-1}_{U,T}, C^{1}_{S,S}, C^{1}_{S,T}, C^{1}_{S,U}, C^{1}_{T,T},$$
$$C^{1}_{T,S}, C^{1}_{T,U}, C^{1}_{U,U}, C^{1}_{U,S}, C^{1}_{U,T}, C^{-3}_{S,S}, C^{3}_{S,S}\}.$$

Now by considering the order in Λ, *let us find the basis of submodules* $A^{<1}$, $A^{<3}$,

$$\begin{aligned} \textit{basis of } A^{<1} &= \{C^{\mu}_{ST} : S, T \in M(\mu) | \mu < \lambda\} \\ &= \{C^{\mu}_{ST} : S, T \in M(-3)\} \cup \{C^{\mu}_{ST} : S, T \in M(3)\} \\ &= \{C^{-3}_{S,S}, C^{3}_{S,S}\} \\ \textit{basis of } A^{<3} &= \phi. \end{aligned}$$

Chapter 2

Deguchi-Martin Quotient

2.1 Introduction

This chapter is taken from the work of Deguchi and Martin in [3, Chapter 3] and their references. It is the main source of inspiration for the present work. We will provide our own examples to help the reader understand the concepts. The proofs of theorems and propositions can be found in [3, Chapter 3].

We recall the representation $\mathcal{R}_{PM}$ of the Hecke algebra $H_n(q)$ which will be a representation of Deguchi-Martin quotient H_n^{PM}. To define this, we should first define certain elements in $H_n(q)$ which will generate the kernel of this quotient.

In addition to Hecke algebras generators T_i's which are defined in Chapter 1, in this chapter we use alternative generators constructing by $U_i = q^{-1}(1-T_i)$

[3, Section 2.5]. The corresponding relations for U_i are given as follows

$$\begin{aligned} U_iU_{i\pm1}U_i - U_i &= U_{i\pm i}U_iU_{i\pm1} - U_{i\pm1}, \\ U_i^2 &= (q+q^{-1})U_i, \\ U_iU_j &= U_jU_i \text{ for } |i-j| \neq 1. \end{aligned} \tag{2.1}$$

2.2 Representations of the Hecke Algebra $H_n(q)$

In this part, we recall section 3.1 from [3].

For $q \in \mathbb{C}$, $x = q + q^{-1}$. Let P, M be non-negative integers such that $N = P + M$ is positive. Let $V_N = \{e_1, e_2, \ldots, e_N\}$ be shorthand notation for the standard ordered basis for $\mathbb{C}^N$, and I_N be the $N \times N$ identity matrix, and $\mathcal{R}$ be the $N^2 \times N^2$ matrix with action on $u \otimes v \in V_N \otimes V_N$ given by

$$\mathcal{R}(e_u \otimes e_v) = 0 \qquad \text{if } u = v \leq P$$

$$\mathcal{R}(e_u \otimes e_v) = x(e_u \otimes e_v) \qquad \text{if } u = v > P$$

and otherwise, with $p = \text{sign}(v - u)$,

$$\mathcal{R}(e_u \otimes e_v) = q^p(e_u \otimes e_v) + (e_v \otimes e_u).$$

Then for $N < n$ and V the space spanned by V_N^n we can check by direct computation that there is a representation $\mathcal{R}_{PM} : H_n(q) \to \text{End}_{\mathbb{C}}(V)$ given by

$$\mathcal{R}_{PM}(U_i) = I_N \otimes I_N \otimes \cdots \otimes \mathcal{R} \otimes \cdots \otimes I_N \tag{2.2}$$

where $\mathcal{R}$ appears in the i^{th} position in the product.

Example 2.2.1. *Let $N = 2$, $P = M = 1$, $\lambda = (2, 2)$ and let $V_2 = \{e_1, e_2\}$ be a standard basis for $\mathbb{C}^2$. Let $\mathcal{R}$ be 4×4 matrix with action on $V_2 \otimes V_2$ is the* *following*

$$\mathcal{R}(e_1 \otimes e_1) = 0;$$
$$\mathcal{R}(e_1 \otimes e_2) = q(e_1 \otimes e_2) + (e_2 \otimes e_1);$$
$$\mathcal{R}(e_2 \otimes e_1) = q^{-1}(e_2 \otimes e_1) + (e_1 \otimes e_2);$$
$$\mathcal{R}(e_2 \otimes e_2) = x(e_2 \otimes e_2).$$

Now we can write the matrix $\mathcal{R}$ *as follows*

$$\mathcal{R} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 0 & x \end{pmatrix}.$$

2.3 Some Notations of the Hecke Algebra $H_n(q)$

Let us review section 3.2 of [3]

Definition 2.3.1 ([3, Section 3.2, Definition 1])**.** *For* n, m, t *natural numbers and* $n \geq m + t$*, the subalgebra* $H_m^{(t)}(q)$ *is obtained from* $H_n(q)$ *under the translation map*

$$\mathcal{T}^{(t)} : H_n(q) \to H_m^{(t)}(q)$$
$$U_i \mapsto U_{i+t}.$$

We denote by $X^{(t)}$ *the image of* $X \in H_n(q)$ *under this map.*

Definition 2.3.2 ([3, Section 3.2, Definition 2])**.** *For each* $U_i \in H_n(q)$*, we define the 'reflection' involution map*

$$\mathcal{T}^{(-)} : H_n(q) \to H_n(q) \text{ by } U_i \mapsto U_{n-i}$$

Definition 2.3.3 ([3, Section 3.2, Definition 3]). *For $A \in H_n(q)$ define the width $l(A)$ as the minimum number j such that $A^{(t)} \in H_j^{(t)}(q)$ for some t.*

For instance, let $A = U_3U_2 = (U_2U_1)^{(1)}$, that means $U_2U_1 \in H_3^{(1)}$, so the width $l(U_3U_2) = 3$. This is the minimum number of strings, in the braid sense, required to carry the braids involved when A is written out as a linear combination of words in the generators. We may think of it as two plus the minimum distance between the largest indexed generator occurring in A and the smallest indexed generator. For example $l(U_4 - U_1U_3) = 2 + (4 - 1) = 5$.

Definition 2.3.4 ([3, Definition 4]). *For $A \in H_a(q)$ with width $l(A) = a$ and $B \in H_b(q)$ with width $l(B) = b$, we define a product*

$$A \otimes B \in H_{a+b}(q)$$

as the algebra product of A with translation $B^{(a)}$.

Note that $A, B^{(a)}$ commute, but $A \otimes B \neq B \otimes A$ in general.

Example 2.3.5. *Let $A = U_1U_2$, $B = U_5$, $l(U_1U_2) = 2 + 2 - 1 = 3$ and $l(U_2) = 2 + 2 - 0 = 4$. We have that $A \in H_3$ and $B \in H_4$,*

$$(U_1U_2) \otimes U_2 = (U_1U_2)(U_2)^{(3)} = (U_1U_2)(U_5) \in H_7.$$

Definition 2.3.6 ([3, Section 3.2, Definition 5]). *We define an equivalence $\sim$ on $H_n(q)$ by $X \sim Y$ if and only if there exists invertible $G \in H_n$ such that $GXG^{-1} = Y$.*

Proposition 2.3.7. *With A, B as in Definition 2.3.4 we have*

$$A \otimes B \sim B \otimes A.$$

2.4 On q-Permutation Representations

We review some definitions and results from section 3.3 of [3].

Note from the definition that our representation $\mathcal{R}_{PM}$ is block diagonal with blocks labelled by partitions of n, corresponding to fixed numbers of 1's, 2's, 3's,...,N's in the basis vectors. Each one of these partitions (or compositions) corresponding to Young tableau $\lambda = (\lambda_1, \lambda_2, \ldots)$ with i^{th} row length λ_i giving the number of i's. For example, let $\lambda = (\lambda_1, \lambda_2, \lambda_3) = (4, 2, 1)$ that means we have four copy of number 1, two copy of number 2 and one copy of number 3.

We can now break our representation into manageable blocks. For given $(P \mid M)$ we write these direct summands of $\mathcal{R}_{PM}$ as $\mathcal{R}^\lambda$.

Theorem 2.4.1 ([3, Section 3.3, Theorem 1])**.** *For a partition λ, the restriction of the module $\mathcal{R}^\lambda$ to H_{n-1} decomposes into the sum of modules $\mathcal{R}^\mu$, where μ is a partition correspondence to the diagram obtained from the diagram for λ by removing one box.*

Definition 2.4.2 ([3, Section3.3])**.** *For a given λ, the basis state of $\mathcal{R}^\lambda$ are the set $\mathcal{Y}_\lambda$ of all words v in the multiset $\{1^{\lambda_1}, 2^{\lambda_2}, \ldots, N^{\lambda_N}\}$.*

For the partition λ these are the Yamanouchi versions [18] of λ-tableaux.

Example 2.4.3. *Let $\lambda = (2, 2)$, then the basis states of $\mathcal{R}^{(2,2)}$ are all words in the $\{1^2, 2^2\}$*

$$\mathcal{R}^{(2,2)} = \{1122, 1212, 1221, 2112, 2121, 2211\}.$$

Let v_i denote the i^{th} entry in v. If $v_i > v_{i+1}$ then we write v^i for the word obtained by interchanging v_i, v_{i+1}. If $w = v^{i_1 i_2 \ldots i_m}$ for some list of interchanges $i_1 i_2 \ldots i_m$ (not unique in general) we write $w \succeq v$. This defines a partial order on $\mathcal{Y}_\lambda$. There is a unique lowest element, called o, such that $o_i \geq o_{i+1}$ for all i, for instance in Example 2.4.3 the lowest element in $\mathcal{Y}_{(2,2)}$ is $o = 2211$.

Let $v = o^{i_1 i_2 \dots i_m}$ then we write $v \geq w$ if a subset of the list $i_1 i_2 \dots i_m$ in the natural order of subscripts takes o to w. For example $o^{i_1 i_2 i_3} \geq o^{i_1 i_2}$. This defines the 'Bruhat' partial order on $\mathcal{Y}_\lambda$. Note that $v \succeq w$ implies $v \geq w$.

Proposition 2.4.4 ([3, Section 3.3, Proposition 2])**.** *If λ, μ are related by a permutation of rows within the first P and within the last M rows then $\mathcal{R}^\lambda$ and $\mathcal{R}^\mu$ are isomorphic.*

As a result of this proposition equivalence classes of these representations may be characterized by ordered pairs of partitions, the first of at most P parts and the second of at most M parts:

$$\lambda \mapsto (\mu, \nu) \vdash (n_P, n_M) \qquad (n_P + n_M = n)$$

In order to find the irreducible content of these representations $\mathcal{R}^\lambda$ we proceed as follows.

Definition 2.4.5 ([3, Definition 6])**.** *For each positive integer n define k_n, a function of x, by $k_1 = 0$ and*

$$k_{n+1} = \frac{1}{(x - k_n)}.$$

Definition 2.4.6 ([3, Definition 7])**.** *For s an integer and q given*

$$[s] = \frac{q^s - q^{-s}}{q - q^{-1}}$$

and for N a positive integer

$$[N]! = \prod_{s=1}^{N} [s].$$

For example, with $x = q + q^{-1}$:

$$[0] = 0 \qquad [1] = 1 \qquad [2] = x \qquad [3] = x^2 - 1$$

and from Definition 2.4.5

$$k_n = \frac{[n-1]}{[n]}.$$

Note that $[-s] = -s$.

Definition 2.4.7 ([3, Definition 8])**.** *For n a positive integer and with $F_1 = 1$*

$$F_{n+1} = -[n-1]F_n - F_n\Big(\sum_{l=1}^{n}[l-2](U_n U_{n-1} \cdots U_l)\Big)$$

where U_i is the alternative generator defined at the beginning of this chapter.

Definition 2.4.8 ([3, Definition 8])**.** *Define the automorphism via*

$$D : H_n(q) \to H_n(q)$$
$$U_i \mapsto x - U_i$$

under this automorphism we have another element

$$E_l = D(F_l) \qquad (1 \leq l \leq n).$$

Note that F_{n+1} may also constructed by

$$F_{n+1} = -[n-1]F_n + \frac{[n]F_n U_n F_n}{[n]!} \tag{2.3}$$

Example 2.4.9 ([3, Section 3.3]).

$$F_1 = E_1 = 1 \qquad F_2 = U_1 \qquad E_2 = x - U_1 \qquad F_3 = U_1U_2U_1 - U_1.$$

Proposition 2.4.10 ([3, Section 3.3]). *The elements F_n, E_n in H_n satisfy*

(i) $F_n^2 = [n]!F_n$

(ii) $E_n^2 = [n]!E_n$

(iii) $U_iF_n = F_nU_i = xF_n$ *for* $1 \leq i \leq n-1$

(iv) $U_iE_n = E_nU_i = 0$ *for* $1 \leq i \leq n-1$

(v) $F_n/[n]!$ *is a primitive and central idempotent*

Definition 2.4.11 ([3, Definition 9]). *For a non-negative integer P and M. Let λ be a partition of n and (μ, ν) be an ordered pairs of standard partitions of $n_P + n_M = n$ such that μ of at most P parts and ν of at most M parts. Define the elements $\Lambda_\lambda(P, M)$, Λ_λ of H_n by*

$$\Lambda_\lambda(P, M) = \left[(\otimes_{i=1}^{P} E_{\lambda_i}) \otimes (\otimes_{i=P+1}^{P+M} F_{\lambda_i})\right]^{(-)}$$

$$\Lambda_\lambda = \left[(\otimes_{i=1}^{P} E_{\mu_i}) \otimes (\otimes_{i=P+1}^{P+M} F_{\nu_i})\right]^{(-)}.$$

Note from Proposition 2.3.7 that $\Lambda_\lambda \sim \Lambda_\lambda(P, M)$. It is convenient henceforward to ignore all but the ordered paired partitions $\lambda = (\mu, \nu)$ which are representatives of each equivalence class of λ's.

Proposition 2.4.12 ([3, Proposition 3]). *Defining the opposite X^{op} of $X \in H_n$ as the element of the generators written in reverse order, we have*
$\Lambda_\lambda = \Lambda_\lambda^{op}$.

Definition 2.4.13 ([3, Section 3.3]). *Define a subset* $\{|v>: v \in \mathcal{Y}_\lambda\}$ *of the left ideal subalgebra* $H_n\Lambda_\lambda$ *which is in one-to-one correspondence with the basis of* $\mathcal{R}^\lambda$ *as follows:*

- *Define an initial element* $|o>= \Lambda_\lambda$,
- *then generate an element for each* v *by introducing* $T_i = (1 - qU_i)$,
- *repeated application of the rule*

$$|v^i>= \frac{T_i}{q}|v> . \tag{2.4}$$

Example 2.4.14. *Let* $n = 3$ *and* $\lambda = (2, 1)$, *we have*
$\mathcal{Y}_{(2,1)} = \{|112>, |121>, |211>\}$ *and* $\Lambda_{(2,1)}(1|1)$ *given by*

$$\begin{aligned}
\Lambda_{(2,1)}(1|1) &= \left[(\otimes_{i=1}^{1} E_{\lambda_i}) \otimes (\otimes_{i=2}^{2} F_{\lambda_i})\right]^{(-)} \\
&= \left[E_{\lambda_1} \otimes F_{\lambda_2}\right]^{(-)} \\
&= \left[E_2 \otimes F_1\right]^{(-)} \\
&= \left[(x - U_1) \otimes 1\right]^{(-)} \\
&= \left[x - U_1\right]^{(-)} \qquad \textit{(for } (-) \textit{ see definition 2.3.2)} \\
&= x - U_2.
\end{aligned}$$

Now we can construct elements in H_3 *corresponding to* $\mathcal{Y}_{(2,1)}$

$$\begin{aligned}
|211> &= |o>= \Lambda_{(2,1)}(1|1) = x - U_2, \\
|121> &= |o^1>= (q^{-1} - U_1)|o>= (q^{-1} - U_1)(x - U_2), \\
|112> &= |o^{12}>= (q^{-1} - U_2)|o^1>= (q^{-1} - U_2)(q^{-1} - U_1)(x - U_2).
\end{aligned}$$

Proposition 2.4.15 ([3, Proposition 4]). *The elements* $U_i \in H_n$ *act on* $\{|v>\}$ *as*

$$U_i|v>= 0 \qquad \textit{if} \qquad v_i = v_{i+1} \leq P$$

and

$$U_i|v >= (q+q^{-1})|v > \qquad if \qquad v_i = v_{i+1} > P.$$

Proposition 2.4.16 ([3, Proposition 5]). *These elements* $\{|v>\}$ *span* $H_n\Lambda_\lambda$.

Note that this spanning argument holds for all q.

Proposition 2.4.17 ([3, Proposition 6]). *For* q *indeterminate the set* $\{|v>\}$ *is linearly independent in* $H_n(q)$.

Proposition 2.4.18 ([3, Proposition 7]). *The blocks* $\mathcal{R}^\lambda$ *are isomorphic to the left modules* H_n *generated by* Λ_λ.

By analogy with the $q = 1$ case these representations are induced from the q-Young symmetrizes/antisymmetrizers for a smaller algebra. So, we call them q-permutation representations.

2.5 q-Specht Modules

This section follows the notations and results of [3, section 3.4]. It introduces Specht modules. These are H_n-modules which are well defined as submodules of $\mathcal{R}^\lambda$ and they are simple for generic q.

Definition 2.5.1 ([3, Section 3.4]). *For* $(P \mid M)$ *and* $\lambda \vdash n$ *a standard partition, define* $E_\lambda = \Lambda_\lambda$ *and* $F_\lambda = D(E^{(-)}_{\lambda'})$.

Example 2.5.2. *Suppose* $n = 3$*, then for* $\lambda = (1^3)$ *we have*

$$E_{(1^3)} = \Lambda_{(1^3,0)} = [E_1 \otimes E_1 \otimes E_1]^{(-)} = [1 \otimes 1 \otimes 1]^{(-)} = 1.$$

For $\lambda = (3)$

$$\begin{aligned} E_{(3)} = \Lambda_{((3),0)} = [E_3]^{(-)} &= [(x-U_1)(x-U_2)(x-U_1)-(x-U_1)]^{(-)} \\ &= (x-U_2)(x-U_1)(x-U_2)-(x-U_2). \end{aligned}$$

Finlay, for $\lambda = (2,1)$

$$\begin{aligned} E_{(2,1)} = \Lambda_{((2,1),0)} = [E_2 \otimes F_1]^{(-)} &= [(x-U_1)\otimes 1]^{(-)} \\ &= x - U_2. \end{aligned}$$

Now we can find F_λ*, respectively*

$$\begin{aligned} F_{(1^3)} = D(E^{(-)}_{(1^3)'}) = D\Big(E^{(-)}_{(3)}\Big) &= D\Big([(x-U_2)(x-U_1)(x-U_2)-(x-U_2)]^{(-)}\Big) \\ &= D\Big((x-U_1)(x-u_2)(x-U_1)-(x-U_1)\Big) \\ &= U_1U_2U_1 - U_1 \\ &= F_3. \end{aligned}$$

$$F_{(3)} = D\Big(E^{(-)}_{(3)'}\Big) = D\Big(E^{(-)}_{(1^3)}\Big) = D(1) = 1.$$

$$F_{(2,1)} = D\Big(E^{(-)}_{(2,1)}\Big) = D\Big([(x-U_2)]^{(-)}\Big) = U_1 = F_2.$$

Proposition 2.5.3 ([3, Proposition 8])**.** *The subspace* $F_\lambda H_n E_\lambda$ *is one dimensional. A spanning vector for this subspace is*

$$F_\lambda|\lambda'_1\cdots 321\ \lambda'_2\cdots 321\cdots 1 > .$$

This defines through Equation 2.4 (and up to the relations) a minimal product of generators W_λ such that

$$F_\lambda H_n E_\lambda = \mathbb{C} F_\lambda W_\lambda E_\lambda.$$

Definition 2.5.4 ([3, Definition 10]). *The left module $H_n F_\lambda H_n E_\lambda = H_n F_\lambda W_\lambda E_\lambda$ is a left Specht module.*

Corollary 2.5.5 ([3, Corollary 8.1]). *The operator $F_\lambda W_\lambda E_\lambda$ is an elementary operator, i.e.*

$$(FWE)H(FWE) = FHEHFHE = F(HEHFH)E \subseteq FHE = \mathbb{C}FWE.$$

Corollary 2.5.6 ([3, Corollary 8.2]). *The Specht module $H_n F_\lambda H_n E_\lambda$ is a generically simple left module (and well-defined for all q). Running over all λ we generate each generically simple left module of $H_n(q)$.*

We will denote the left module $H_n F_\lambda H_n E_\lambda$ simply by λ. The equivalence class of generically irreducible representations induced from this left ideal will be called R_λ. It is sufficient to specify a representation up to the equivalence class here as the decomposition of a reducible representation does not depend on basis.

Definition 2.5.7 ([3, Section 3.4]). *For $e \in \mathcal{Y}_\lambda$ defined by*

$$e = 123\cdots\lambda'_1\, 123\cdots\lambda'_2 \cdots 123\cdots\lambda'_N$$

the set $\{|v>_e :\ v \geq e\}$, defined as for $|v>$ but starting from $|e>_e = F_\lambda H_n E_\lambda$ is a basis for R_λ.

Example 2.5.8. *Suppose $n = 3$, $\lambda = (2,1)$, then $\mathcal{Y}_{(2,1)} = \{112, 121, 211\}$ (by Definition 2.4.2) and for $\lambda' = (2,1)$, $e = 121$. By the order define on $\mathcal{Y}_\lambda$ in*

Section 2.4 we have one element in $\mathcal{Y}_{(2,1)}$ greater then $e = 121$ which is $v = 112$. So, the basis of $R_{(2,1)}$ is $\{|112>_e, |121>_e\}$.

Corollary 2.5.9 ([3, Corollary 8.3]). *The generically simple module associated to λ occurs exactly once in $H_n E_\lambda$, i.e. the generically irreducible representation R_λ obtained from it occurs exactly once in R_λ.*

Theorem 2.5.10 ([3, Theorem 2]). *(q-Branching theorem) The restriction of R^λ to H_{n-1} is generically a direct sum of the irreducible representations obtained by all deletions of one box from the Young diagram λ consistent with a Young diagram as output. The result of the corresponding induction is determined by Frobenius reciprocity.*

For example let $\lambda = (3,1) \vdash n = 4$, then the restriction of $R_{(3,1)}$ to H_3 is

$$R_{(3,1)} \mid_{H_3} = R_{(3)} \oplus R_{(2,1)}.$$

2.6 The Simple Content of q-Permutation Modules

In this part of this thesis, we recall Section 3.5 of [3]

For given $(P \mid M)$ the generic simple content of left module $H_n \Lambda_\lambda$ may be determined by a special case of the q-Littlewood rule as follows (a trusty old reference of Hamermesh [10]):

Write out λ_1 copies of the letters 1, λ_2 copies of the letters 2, and so on in every possible way consistent with the following rules.

1. write all the 1's in a row (assuming $P \neq 0$);

2. for each number $i = 1, 2, \cdots, P$ take the λ_i copies and add them one at a time to the existing diagram in such a way that each stage is a standard shape Young diagram with non-decreasing rows and columns, and such that no two i's appear in the same column;

3. for each subsequent number $i = P+1, \cdots, N$ take the λ_i copies and add them to the diagram produced in step 2, such that no two i's appear in the same row.

Each diagram constructed in this way gives the tableau shape for an irreducible component of the block $\mathcal{R}^\lambda$.

Example 2.6.1 ([3, Section 3.5])**.**

1. *Consider the case* $P = M = 1$ *and* $\lambda = (2, 2)$*, we have the letters* 1122*. By step* 1*. we have*

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}$$

we have, after step 3*.:*

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array}$$

so that $\mathcal{R}^{(2,2)} = R_{(3,1)} \oplus R_{(2,1^2)}$

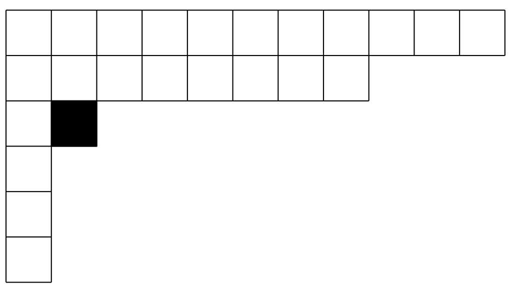
2. *Consider the case the same case in 1 but* $\lambda = (3^2)$*, we have the letters* 111222*, which become*

$$\begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1} 2 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array}$$

so that $\mathcal{R}^{(3^2)} = R_{(4,1^2)} \oplus R_{(3,1^3)}$.

Proposition 2.6.2 ([3, Proposition 9])**.** *The representation* R_{PM} *(equation 2.2), contains every irreducible representation of* $H_n(q)$ *with multiplicity at least one, except those associated to partition shapes containing as a subdiagram the rectangular diagram of height* $P+1$ *and width* $M+1$.

Example 2.6.3 ([3, Section 3.5])**.** *Suppose* $P = 2$ *and* $M = 1$*, the irreducible representation with the following diagram is excluded unless the black box is omitted:*



Proposition 2.6.4 ([3, Proposition 10])**.** *For* $P = M = 1$ *and* q *indeterminate*

$$\mathcal{R}^{(a,b)} = R_{(a,1^b)} \oplus R_{(a+1,1^{b-1})}$$

For example $\mathcal{R}^{(2,1)} = R_{(2,1)} \oplus R_{(3)}$.

Now Theorem 2.4.1 determines the dimension of all R^λ in all $(P \mid M)$, for the $(1 \mid 1)$ case that is given via the Pascal triangle. It then follows from Theorem 2.4.1 and the well-known dimensions and characters of irreducibles that

$$\mathcal{R}^{(n-1,1)} = R_{(n-1,1)} \oplus R_{(n)}.$$

Note that the statement of Proposition 2.6.4 will be a bit different from the original one in the reference [3, Proposition 10] because there is a minor error in the proof of this proposition, then when we fix that error we get the version we stated.

Example 2.6.5. *In case $n = 3$ there are 3 irreducible representations corresponding to the partitions*

$$(3),\ (2,1),\ (1^3).$$

These have dimensions $1, 2$ and 1 respectively. For $\lambda = (3)$, the corresponding basis states for $R_{(3)}$ is $B_{(3)} = \{|111>_e\}$ where

$$|111>_e = |e>_e = F_{(3)} 1 E_{(3)} = (1)\ 1\ ((x-U_1)(x-U_2)(x-U_1) - (x-U_1)) = E_3.$$

$$g_i\ |111>_e = g_i\ E_{(3)} = (1 - qU_i)\ E_3 = E_3 - 0. \qquad \textit{(i=1,2)}$$

And for $\lambda = (1^3)$, the corresponding basis states for $R_{(1^3)}$ is $B_{(1^3)} = \{|123>_e\}$ where

$$|123>_e = |e>_e = F_{(1^3)} 1 E_{(1^2)} = ((U_1)(U_2)(U_1) - (U_1))\ 1\ (1) = F_3.$$

$$g_i\ |123>_e = g_i\ F_{(1^3)} = (1 - qU_i)\ F_3 = F_3 - qxF_3.$$

Finally, for $\lambda = (2,1)$, the corresponding basis states for $R_{(2,1)}$ is $B_{(2,1)} = \{|121>_e,\ g_2|121>_e\}$ where

$$|121>_e = |e>_e = F_{(2,1)} 1 E_{(2,1)} = U_1\ 1\ (x - U_2)\ = F_2 E_{(2,1)}.$$

$$g_1 \begin{pmatrix} |121>_e \\ g_2|121>_e \end{pmatrix} = \begin{pmatrix} g_1|121>_e \\ g_1 g_2|121>_e \end{pmatrix} = \begin{pmatrix} -q^2 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} |121>_e \\ g_2|121>_e \end{pmatrix}$$

$$g_2 \begin{pmatrix} |121>_e \\ g_2|121>_e \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ q & (q-1) \end{pmatrix} \begin{pmatrix} |121>_e \\ g_2|121>_e \end{pmatrix}$$

2.7 The Quotient Algebra $H_n^{P,M}$

This section defines a sequence of quotient algebras of $H_n(q)$.

Definition 2.7.1 ([3, Definition 11])**.** *For* $\chi = ((M+1), (M+1), \dots, (M+1))$ *a partition of* $(M+1)(P+1)$*, define the algebra quotient*

$$H_n^{P,M} = H_n / F_\chi H_{(P+1)(M+1)} E_\chi .$$

In case $P = M = 1$ it will be clear that, among the complete list of generic irreducible representations of H_4 given in Section 2.6 for example, $(2,2)$ never appears as a subdiagram in any irreducible summand (i.e. and so can not be a representation of the $H_4^{1,1}$ quotient), and hence, by Theorem 2.5.10, for any n.

Example 2.7.2. *Let* $n = 4$ *and* $P = M = 1$*, then* $\chi = (2,2) \vdash n$*.*

$$\begin{aligned} E_\chi = E_{(2,2)} = \Lambda_{(2,2)} &= [E_2 \otimes E_2]^{(-)} && \textit{(see Definition 2.5.1)} \\ &= [(x - U_1) \otimes (x - U_1)]^{(-)} && \textit{(see Definitions 2.3.2 and 2.3.4)} \\ &= (x - U_3)(x - U_1). \end{aligned}$$

$$\begin{aligned} F_\chi = D\big(E_{\chi'}^{(-)}\big) &= D\big(((x - U_3)(x - U_1))^{(-)}\big) && (\chi = \chi' = (2,2)) \\ &= D\big((x - U_1)(x - U_3)\big) \\ &= U_3 U_1. \end{aligned}$$

Then in this case the quotient is

$$H_4^{1,1} = H_4/U_3U_1H_4(x - U_3)(x - U_1).$$

Corollary 2.7.3 ([3, Corollary 8.4])**.** *For μ, $\lambda \vdash n$*

$$R_\mu(F_\lambda H_n E_\lambda) = 0$$

unless $\mu = \lambda$.

Chapter 3

A Bijection on Cell Module Bases

In this chapter, we introduce the main theorem of the thesis, a bijection between the cell module bases of the blob algebra b_{n-1} and the Deguchi-Martin quotient $H_n^{1,1}$. A main input to our result comes from the work of Plaza and Hansen that establishes a bijection between blob diagrams and the set of pairs of one-line standard bitableaux of the same shape, see [20].

3.1 A Bijection Between the Set of Blob Diagrams and the Set of Pairs of One-Line Standard Bitableaux

In this section, we will briefly recall the result from [20, section 4.1], where it was proven that there is a bijection between blob diagrams and the set of pairs of one-line standard bitableaux $\mathrm{Std}(\lambda)$.

Let Λ_n be the set $\{-n, -n+2, \ldots, n-2, n\}$. Then the following definition makes Λ_n into a totally ordered set with order relation $\succ$.

Definition 3.1.1 ([20, Definition 4.2])**.** *Suppose $\lambda, \mu \in \Lambda_n$. We then define $\mu \succeq \lambda$ if either $|\mu| < |\lambda|$, or if $|\mu| = |\lambda|$ and $\mu \leq \lambda$. On the other hand, the map f given by*

$$f : \mathrm{Bip}_1(n) \to \Lambda_n, \quad ((a),(b)) \to a - b$$

is a bijection and so we can define a total order $\succeq$ on $\mathrm{Bip}_1(n)$ as follows

Definition 3.1.2 ([20, Definition 4.3])**.** *Suppose $\lambda, \mu \in \mathrm{Bip}_1(n)$. Then we define $\lambda \succeq \mu$ if and only if $f(\lambda) \succeq f(\mu)$.*

For $t \in \mathrm{Std}(\lambda)$ let $t|_k$ be the tableau obtained from t by removing the entries greater than k. We extend the order $\succeq$ to the set of all λ-standard bitableaux as follows.

Definition 3.1.3 ([20, Definition 4.4])**.** *Suppose that $\lambda \in \mathrm{Bip}_1(n)$ and $s, t \in \mathrm{Std}(\lambda)$. We define $s \succeq t$ if Shape$(s|_k) \succeq$ Shape$(t|_k)$ for all $k = 1, \ldots, n$.*

Example 3.1.4. *Let $n = 7$ and $\lambda = ((5),(2)) \in \mathrm{Bip}_1(n)$. Assume s, t as following*

$$s = \left(\boxed{1}\boxed{4}\boxed{5}\boxed{6}\boxed{7}, \boxed{2}\boxed{3} \right), t = \left(\boxed{2}\boxed{3}\boxed{4}\boxed{5}\boxed{6}, \boxed{1}\boxed{7} \right).$$

Then $s \succeq t$.

Note that $\succeq$ is a partial order on $\mathrm{Std}(\lambda)$, but not total. Let t_λ be the unique standard λ-bitableau such that $t^\lambda \succeq t$ for all $t \in \mathrm{Std}(\lambda)$. For $\lambda = (a, b)$, set $m = \min\{a, b\}$. Then in t^λ the numbers $1, 2, \ldots, n$ are located increasingly along the rows according to the following rules:

1. Even numbers less than or equal to $2m$ are placed in the first component.

2. Odd numbers less than $2m$ are placed in the second component.

3. Numbers greater than $2m$ are placed in the remaining boxes.

Definition 3.1.5 ([20, Definition 4.6]). *Suppose that* $\lambda \in \mathrm{Bip}_1(n)$ *and let* $t \in \mathrm{Std}(\lambda)$. *Define a sequence of integers inductively by the rules* $t(0) = 0$ *and for* $1 \le j \le n$

$$t(j) = t(j-1) \pm 1$$

where the $+(-)$ *sign is used if* j *is in the first (second) component of* t.

Using this sequence, we can now describe the order $\succeq$.

Lemma 3.1.6 ([20, Lemma 4.7]). *If* $s, t \in \mathrm{Std}(\lambda)$, *then* $s \succeq t$ *if and only if* $|s(j)| \le |t(j)|$, *for all* $1 \le j \le n$, *and if* $|s(j)| = |t(j)|$ *then* $s(j) \le t(j)$.

Let m be a blob diagram. Given a horizontal line l, in either edge, we put $l = (a, b)$ where a is the left endpoint and b is the right endpoint. Let $l_1 = (a_1, b_1)$ and $l_2 = (a_2, b_2)$ be horizontal lines on the same edge. We say that l_1 covers l_2 if $a_1 < a_2 < b_2 < b_1$. We also say that the leftmost vertical line (if any) covers all lines to the right of it. Now, we say that a node is covered if the line to which it belongs is decorated or the line to which it belongs is covered by a decorated line. If a node is not covered, we call it uncovered [20]. For example in Figure 3.1 the horizontal line l_1 covers the horizontal line l_2 in the diagram d.

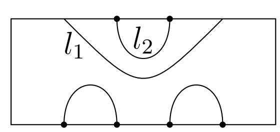
$d =$ 

Figure 3.1: Example of lines covering

Definition 3.1.7 ([20, Definition 4.8]). *Let* m *be a blob diagram. Suppose that* m *has exactly* v *vertical lines and* $h = \frac{n-v}{2}$ *horizontal lines on each edge.*

- *If $v \geq 0$ and the leftmost vertical line is not decorated or there is no vertical lines then we associate to m a pair of λ-bitableaux, $t_{top}(m)$ and $t_{bot}(m)$, with $\lambda = ((h+v),(h))$ by the following rules:*

  1. *k is in the second component of $t_{top}(m)(t_{bot}(m))$ if and only if either k is uncovered and it is the right endpoint of a horizontal line on the top (bottom) edge or it is covered and it is the left endpoint of a horizontal line on the top (bottom) edge;*
  2. *The numbers increase along rows.*

- *If $v > 0$ and the leftmost vertical line is decorated then we associate to m a pair of λ-bitableaux, $t_{top}(m)$ and $t_{bot}(m)$, with $\lambda = ((h),(h+v))$ by the following rules:*

  1. *k is in the first component of $t_{top}(m)(t_{bot}(m))$ if and only if either it is uncovered and it is the left endpoint of a horizontal line on the top (bottom) edge, or it is covered and it is the right endpoint of a horizontal line on the top (bottom) edge;*
  2. *The numbers increase along rows.*

For $\lambda \in \mathrm{Bip}_1(n)$ and $s, t \in \mathrm{Std}(\lambda)$, we let m_{st} denote the unique blob diagram such that $t_{top}(m_{st}) = s$ and $t_{bot}(m_{st}) = t$.

Proposition 3.1.8 ([20, section 4.1])**.** *There is a bijection between blob diagrams and the set of pairs of one-line standard bitableaux.*

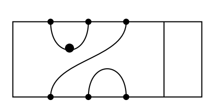
Proof. To prove the bijection it is sufficient to prove a bijection between half-diagrams and one-line standard bitableaux.
($\Rightarrow$) Let m_{top} be the top half diagram of blob diagram m. We show that $t_{top}(m)$ is standard. This is obvious from the way the bijection is defined.

($\Leftarrow$) Let s, t be bitableaux. We show that there is unique diagram m_{st} such that $t_{top}(m_{st}) = s$ and $t_{pot}(m_{st}) = t$. Given the bitableau s, we have two cases:

Case 1 If $\lambda^{(1)} \geq \lambda^{(2)}$, that means the leftmost vertical line is not decorated or there are no vertical lines and all the k nodes in the second component of s is either the left or right endpoint of horizontal line depending on whether the line is covered or not. To avoid crossing the lines, we are forced to connect each of these points, going from left to right, to the node at the left of it (unless the node is the leftmost one it is connected to the next node right it), that is not connected to other nodes yet. Then we see if the k node is the right end of the horizontal line, the line is not decorated, or if k is the left end of the horizontal line, the line will be decorated. So the top horizontal lines are determined by s as are the bottom lines by t. But then there is no choice for vertical lines anymore.

Case 2 If $\lambda^{(1)} < \lambda^{(2)}$, that means the leftmost vertical line is decorated and all the k nodes in the first component of s are either the left or the right endpoint of horizontal line depending on whether the line is covered or not. We connect these nodes as we do in case 1, then we see if the k node is the right end of a horizontal line that is before the decorated vertical line which means the horizontal line is decorated. Note that all the end points of horizontal lines after the decorated vertical line will be a right end node. Then there is no choice for the vertical line anymore. This completes the proof. □

Example 3.1.9. *Let $n = 4$ and let $m \in b_4$ to be*

$m =$ 

Then

$$t_{top}(m) = \left(\boxed{2}\boxed{3}\boxed{4}, \boxed{1} \right), \quad t_{bot}(m) = \left(\boxed{1}\boxed{2}\boxed{4}, \boxed{3} \right).$$

Remark 3.1.10 ([20, Remark 4.9])**.** *For all $t \in \mathrm{Std}(\lambda)$ and $1 \leq j \leq n$, we have*

1. *If $t(j) < 0$ then the node j is covered in the top edge of m_{tt^λ}.*
2. *If the node j is covered in the top edge of m_{tt^λ} then $t(j) \leq 0$.*

3.2 From the Blob Algebra to the Deguchi-Martin Quotient

Our goal in this section is to show that there is a map between the set of the half diagram bases of the cell modules for the blob algebra b_{n-1} and the set of words which give the bases of the cell module for the Deguchi-Martin quotient $H_n^{1,1}$.

Definition 3.2.1. *([6, Section 1.1]) Let x be a positive integer and t be a tableau. The number of boxes in t is denoted by $|t|$. **The row insertion** of x into t is the tableau with $|t|+1$ boxes defined as follows:*

1. *Starting with $i = 1$, if x is greater than or equal to all entries of the i-th row of t (this includes the case where the i-th row has no entries), then add x to the end of the row;*
2. *otherwise find the leftmost entry of the i-th row that is greater than x (call it $\bar{x}$), replace it with x, and insert the $\bar{x}$ into the row $i+1$ by repeating step (1) with $x = \bar{x}$ and i replaced by $i+1$.*

   The tableau obtained by these procedures is denoted by $t \leftarrow x$.

Example 3.2.2. *Let $\lambda = (3, 2, 1)$ and λ-tableau t as follows*

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \; .$$

We insert 2 in the tableau t. The 2 bumps the 4 from the first row and then the 4 bumps the 6 from the second row, which can be added to the end of the third row:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \;\leftarrow 2 \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 6 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \;\leftarrow 4 \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array} \;\leftarrow 6 \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 4 \\ \cline{1-2} 5 & 6 \\ \cline{1-2}\end{array}$$

Definition 3.2.3 ([6, Section 1.1])**.** *Given two tableaux t and u we define* **the product** *$t.u$ to be the result of progressively row-inserting the entries of u into t, starting with the left-most entry in the bottom row of u and going from left to right, bottom to top, so that the bottom row is the first to be emptied until u is empty. Because each row insertion produces a tableau, the final product $t.u$ is also a tableau.*

Example 3.2.4. *Let u and t be two tableaux as following*

$$u = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} \qquad t = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$$

then the product $t.u$ is given as:

$$u.t = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} \cdot \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \cdot \begin{array}{|c|}\hline 2 \\ \hline\end{array} = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}$$

Definition 3.2.5 ([1, Section 3.4.3])**.** *A* **hook shaped tableau** *is a tableau with shape $\lambda = (n-k, 1^k)$ for some $0 \leq k \leq n-1$. See Figure 3.2. The integer k is called the height of the hook. We shall also call the partition λ a hook.*

1	3	4	5	7
2				
6				
8				

Figure 3.2: The hook tableau with $\lambda = (5, 1^3)$.

Note that in the hook standard tableau, 1 must be in the corner cell and n must be in the end cell of either the first row or the first column.

Definition 3.2.6 ([5, Section 2])**.** *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be a partition of n and t be a standard tableau of shape λ.* **Yamanouchi word** *for t is a word $w = w_1, w_2, \dots, w_n$ on the set $\{1^{\lambda_1}, 2^{\lambda_2}, \dots, k^{\lambda_k}\}$, such that the letter w_i be the row index of the box of t containing the number i.*
Notice that once reading Yamanouchi's word from left to right there are never fewer letters i than letters $(i+1)$. Also, according to Sen-Peng in [5] there is a bijection between the set of standard Young tableaux of shape μ and the set of Yamanouchi words of type μ. Given a Yamanouchi word w, it is straightforward to recover the corresponding tableau, i.e., the ith row of which contains the indices of the letters of w that are equal to i.

Let $\mathbb{B}_n^{top}$ (resp., $\mathbb{B}_n^{bot}$) denote the set of upper (lower) halves of blob diagrams which are the bases of cell module for the blob algebra b_n and for $t \in \{-n, 2-n, 4-n, \dots, 0, n-2, n-4, \dots, n\}$. For $\lambda \vdash n$ we let R_λ be the cell module for the Deguchi-Martin quotient $H_n^{P,M}$. Let $Q(n)$ be the set of all words which are the bases of the cell module of $H_n^{1,1}$. Note that $\mathbb{B}_n^{top}$ (resp., $\mathbb{B}_n^{bot}$) is in bijection with Std(n) via $m \to t^{top}(m)$ (resp.,$m \to t^{bot}(m)$). Since $\mathbb{B}_n^{top}$ and $\mathbb{B}_n^{bot}$ are in bijection via the mirror map. We will construct a map from the set $\mathbb{B}_n^{top}$ to the set Q_n.

Definition 3.2.7. *For any half blob diagram $d \in \mathbb{B}_n^{top}$. We define a map*

$$\alpha : \mathbb{B}_{n-1}^{top} \to Q_n$$
$$d \to |w >_e$$

as the following:

1. *Use the Plaza-Hansen bijection in Definition 3.1.7 to get the corresponding one line bitableau $(t_{\lambda_1}, t_{\lambda_2})$ of a the blob diagram d, where t_{λ^1} is a standard tableau t of shape λ^1 and t_{λ^2} is a standard tableau t of shape λ^2;*
2. *insert $n+1$ to the component that contains the number 1;*
3. *takes the product of the transpose of the first component by the second component(i.e. $((t_{\lambda^1})^T.t_{\lambda^2})$ to produce a standard tableau, t;*
4. *use Definition 3.2.6 to construct the Yamanouchi word w from the standard tableau.*
5. *finally, set $\alpha(d) = |w >_e$.*

Each word constructed in this way gives the basis element for an irreducible component of the block representation of Deguchi-Martin quotient $H_n^{1,1}$.

In the following, we will construct a bijection between some sets which will help us to prove our main theorem. Let $\mathrm{Std}_+(\lambda)$ be the set of all one line standard λ-bitableau obtained from the set $\mathrm{Std}(\lambda)$ by inserting $n+1$ into the component of λ-bitableau that contains 1, (i.e., $\mathrm{Std}_+(\lambda) = \{t \leftarrow n+1 \mid t \in \mathrm{Std}(\lambda)\}$). The elements of the set $\mathrm{Std}(\lambda)$ have the form $(t_{\lambda^1}, t_{\lambda^2})$ where t_{λ^1}, (t_{λ^2}) is a standard tableau of shape λ^1, (λ^2).

Example 3.2.8. *Let $P = \left(\boxed{1}\boxed{2}\boxed{4}\boxed{6}\,,\ \boxed{3}\boxed{5}\right) \in \mathrm{Std}(((4),(2)))$.*

Then the tableau T obtain by inserting 7 into P is

$$T = \left(\boxed{1\,|\,2\,|\,4\,|\,6\,|\,7}\,,\ \boxed{3\,|\,5}\right) \in \mathrm{Std}_+(((4),(2))).$$

Lemma 3.2.9. *For each $\lambda \in \mathrm{Bip}_1(n)$ and $n \geq 1$, there is one-to-one correspondence between the set $\mathrm{Std}(\lambda)$ and $\mathrm{Std}_+(\lambda)$.*

Proof. Since both components for each λ-bitableau $P \in \mathrm{Std}(\lambda)$ are single row standard tableau, then the bitableau obtained by inserting $n+1$ into P will be also a standard bitableau. Moreover, the entry $n+1$ must be at the end of either the first component or the second component. On other hand, let $P \in \mathrm{Std}(\lambda)$ and $T = (P \leftarrow (n+1)) \in \mathrm{Std}_+(\lambda)$. Since the entry $(n+1)$ must be at the end of the component which contains 1, then deleting the box containing $n+1$ from bitableau T will give us exactly the bitableau P. □

Now let $\mathrm{Pro}_+(\lambda)$ be the set of all standard tableaux obtained from $\mathrm{Std}_+(\lambda)$ by multiplying the transpose of the first component by the second component, (i.e, $\mathrm{Pro}_+(\lambda) = \{t = (t_{\lambda^1})^T \cdot t_{\lambda^2} \mid L = (t_{\lambda^1}, t_{\lambda^2}) \in \mathrm{Std}_+(\lambda)\}$) (For tableau product see Definition 3.2.3).

Example 3.2.10. *Let $\lambda = ((3),(3)) \in \mathrm{Bip}_1(6)$ and*

$$u = \left(\boxed{2\,|\,5\,|\,6}, \boxed{1\,|\,3\,|\,4}\right) \in \mathrm{Std}(\lambda)$$

then we get

$$u' = \left(\boxed{2\,|\,5\,|\,6}, \boxed{1\,|\,3\,|\,4\,|\,7}\right) \in \mathrm{Std}_+(\lambda)$$

by inserting 7 to the second component of u. Multiplying the transpose of the *first component of u' by the second component, we get*

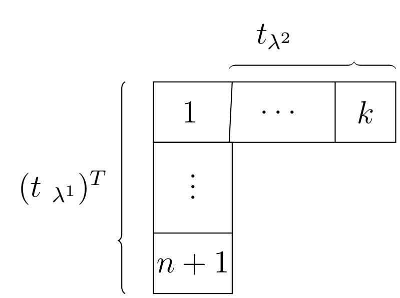
$$u'' = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 7 \\ \hline 2 \\ \cline{1-1} 5 \\ \cline{1-1} 6 \\ \cline{1-1} \end{array} \in \mathrm{Pro}_{+}(\lambda)$$

As shown in the example u'' has hook shape. Moreover, every tableau belonging to $\mathrm{Pro}_{+}(\lambda)$ will be a hook tableau.

Lemma 3.2.11. *For each $\lambda \in \mathrm{Bip}_1(n)$ and $n \geq 1$, if $t \in \mathrm{Pro}_{+}(\lambda)$, then t is a hook tableau.*

Proof. Assume $L \in \mathrm{Std}_{+}(\lambda)$, there are 2 cases to consider:

Case 1 The entry $n+1$ is in the first component t_{λ^1} of L. This means the transpose of this component is a standard single-column tableau with the first entry 1 and the last one $n+1$. Since both components are standard tableau, all entries of the second component are greater than 1, which means all these entries will insert in the first row beside 1 in the standard tableau $t = (t_{\lambda^1})^T \cdot t_{\lambda^2}$. So, t is a standard hook tableau with the following form



where $1 < k < n+1$, and the entries of the first row is subset of $\{1, \ldots, n\}$ that are strictly increasing.

(Note that this case includes when the second component is ϕ, then t will be a single column standard tableau.)

Case 2 The entry $n+1$ is in the second component t_{λ^2} of L. Then the transpose of first component $(t_{\lambda^1})^T$ is a one column tableau containing a subset of $\{2, \ldots, n\}$. Multiplying $(t_{\lambda^1})^T$ by t_{λ^2}, the number 1 in t_{λ^2} will bump the first entry in $(t_{\lambda^1})^T$ which will also bump the second entry and so on until the last entry. So, the second component will be the first row in the final standard hook tableau. Therefore, $t = (t_{\lambda^1})^T \cdot t_{\lambda^2}$ has the form

$$
\begin{array}{c}
\overbrace{\begin{array}{|c|c|c|}\hline 1 & \cdots & n+1 \\ \hline \end{array}}^{t_{\lambda^2}} \\
(t_{\lambda^1})^T \left\{ \begin{array}{|c|}\vdots \\ \hline k \\ \hline \end{array} \right.
\end{array}
$$

where $1 < k < n+1$, and the entries of the first column are a subset of $\{1, \ldots, n\}$ that are strictly increasing down the column.
(Note that this case includes when the first component is ϕ, then t will be a single row standard tableau). □

Lemma 3.2.12. *For each* $\lambda \in \mathrm{Bip}_1(n)$ *and* $n \geq 1$, *there is a one-to-one correspondence between the set* $\mathrm{Std}_+(\lambda)$ *and the set* $\mathrm{Pro}_+(\lambda)$.

Proof. By Lemma 3.2.11, we can define a function

$$
\begin{aligned}
f : \mathrm{Std}_+(\lambda) &\to \mathrm{Pro}_+(\lambda) \\
L = (t_{\lambda^1}, t_{\lambda^2}) &\mapsto t = (t_{\lambda^1})^T \cdot t_{\lambda^2}.
\end{aligned}
$$

Conversely, let

$$g : \mathrm{Pro}_+(\lambda) \to \mathrm{Std}_+(\lambda)$$
$$t \mapsto L = (t_{\lambda^1}, t_{\lambda^2})$$

since each $t \in \mathrm{Pro}_+$ is obtained from the components product of some $L \in \mathrm{Std}_+$. So, to define g we have to consider the previous cases of $(t_{\mu^1})^T \cdot t_{\mu^2}$:

Case i $n+1$ is in the first column of t. Then t_{λ^1} is the transpose of the first column and t_{λ^2} is t with first column deleted.

Case ii $n+1$ is in the first row of t. Then t_{λ^1} is the transpose of t with first row deleted and t_{λ^2} is the first row of t.

Now we prove that $g \circ f = \mathrm{Id}_{\mathrm{Std}_+}$:
let $L \in \mathrm{Std}_+$, applying f on L to get $f(L)$ which is must be one of two cases of f above. Next by applying g on $f(L)$, we will do the following:

- Assume L has the following form

$$\left(\begin{array}{|c|c|c|}\hline 1 & \cdots & n+1 \\ \hline\end{array}\ ,\quad \begin{array}{|c|c|c|c|}\hline b_1 & b_2 & \cdots & b_k \\ \hline\end{array} \right)$$

where the entries of the first component are a subset of the set $\{1, 2, \ldots, n+1\}$ and the entries of the second component are a subset of $\{2, 3, \ldots, n\}$ that are strictly increasing from left to right on both components. Applying f on L we get

$$f(L) = \begin{array}{|c|c|c|c|}\hline 1 & b_1 & \cdots & b_k \\ \hline \vdots \\ \cline{1-1} n+1 \\ \cline{1-1}\end{array}$$

since $n+1$ in the first component of L we applied case 1 of Lemma 3.2.11, Then apply g on $f(L)$. Since $n+1$ contained in the first column of standard tableau $f(L)$, we apply case i of function g to get

$$t_{\lambda_1} = \begin{array}{|c|c|c|}\hline 1 & \cdots & n+1 \\ \hline \end{array}\, , \; t_{\lambda_2} = \begin{array}{|c|c|c|c|}\hline b_2 & b_1 & \cdots & b_k \\ \hline \end{array}$$

that are the same components of L. Therefore $g(f(L)) = L$;

- Assume L has the following form

$$\left(\begin{array}{|c|c|c|c|}\hline a_1 & a_2 & \cdots & a_k \\ \hline \end{array}\, , \quad \begin{array}{|c|c|c|}\hline 1 & \cdots & n+1 \\ \hline \end{array} \right)$$

where the entries of the first component are a subset of $\{2, \ldots, n\}$ and the entries of the second component are a subset of the set $\{1, 2, \ldots, n+1\}$ that are strictly increasing from left to right on both components. Once apply f on L (since $n+1$ in the second component of L we apply case 2 of Lemma 3.2.11) we get

$$f(L) = \begin{array}{|c|c|c|}\hline 1 & \cdots & n+1 \\ \hline a_1 \\ \cline{1-1} \vdots \\ \cline{1-1} a_k \\ \cline{1-1} \end{array} \in \mathrm{Pro}_+(\lambda)$$

Now as we define g above, we apply case 2 of g to get $g(f(L))$

$$g(f(L)) = g\Big(\begin{array}{|c|c|c|}\hline 1 & \cdots & n+1 \\ \hline a_1 \\ \cline{1-1} \vdots \\ \cline{1-1} a_k \\ \cline{1-1} \end{array} \Big) = \Big(\begin{array}{|c|c|c|}\hline a_1 & \cdots & a_k \\ \hline \end{array}\, , \quad \begin{array}{|c|c|c|}\hline 1 & \cdots & n+1 \\ \hline \end{array} \Big) = L$$

Observe that, we have $g(f(L)) = L$ for all $L \in \mathrm{Std}_+$ which means that g is reverse of f. Similarly, we can prove that $f(g(t)) = t$. $\square$

3.3 From the Deguchi-Martin Quotient to the Blob Algebra

In this section, we define the inverse map γ to the previous map α in Definition 3.2.7, which sends every cell module basis of Deguchi-Martin quotient $H_n^{1,1}$. to cell module basis of blob algebra b_{n-1}

Remark 3.3.1. *Note that we study the Deguchi-Martin quotient $H_n^{1,1}$, any diagram of an irreducible representation has to be a hook diagram.*

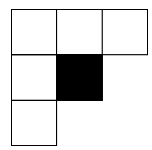


Figure 3.3: The diagram of shape $\lambda = (3, 2, 1)$.

Let $\mathbb{B}_n^{top}$ the sets of upper halves of blob diagrams which are bases of cell module for the blob algebra b_n and Q_n be the set of words which are bases of cell module for the Deguchi and Martin quotient $H_n^{1,1}$. We construct the following map.

Definition 3.3.2. *For each word $|w >_e \in Q_n$, $n \geq 2$, We define a map*

$$\gamma : Q_n \to \mathbb{B}_{n-1}^{top}$$
$$|w >_e \mapsto d$$

as the following:

1. *Use Definition 3.2.6 to construct a standard tableau from a word (note this will always be a hook tableau);*

2. *construct a one line bitableau $L = (t_{\lambda^1}, t_{\lambda^2})$ from the standard tableau t as following*

   (a) *if n in the first row of t, then t_{λ^1} is the transpose of t with first row deleted and t_{λ^2} is the first row of t;*

   (b) *if n in the first column of t, then t_{λ^1} is the transpose of first column and t_{λ^2} is t with the first column deleted;*

   By construction $t_{\lambda^1}, t_{\lambda^2}$ are single line standard tableaux of shape $\lambda^1, (\lambda^2)$, respectively. Then $L = (t_{\lambda^1}, t_{\lambda^2})$ is a one line standard bitableau.

3. *Delete the box from L that contains number n, then we will have a bitableau;*

4. *use the Plaza-Hansen bijection in Definition 3.1.7 to get the corresponding blob diagram d.*

3.4 The Bijection

We get our main theorem by combining the findings of the previous two sections. We aim to prove that the map α is a bijection by showing that the maps α and γ are inverses of each other.

For given n and $t \in \{n, n-2, n-4, \ldots, 0, \ldots 2-n, 4-n\}$ we let $\Delta_n(t)$ be the b_n cell module. Let $\mathbb{B}_n^{top}(t)$ be the set of upper halves of blob diagrams which is the bases for the cell module $\Delta_n(t)$, let $\mathcal{B}(t) \subseteq \mathbb{B}^{top}(n)$ be the basis of the single cell module $\Delta_n(t)$.

For $\lambda \vdash n$ we let R_λ to be the cell modules for the Deguchi-Martin's quotient $H_n^{P,M}$. Let $Q_n(\lambda)$ be the set of words that are the bases of cell modules R_λ, and let $\mathcal{Q}(\lambda) \subseteq Q(n)$ be the basis of the single cell module R_λ.

Theorem 3.4.1. *There is a bijection between the canonical basis of the cell modules of the blob algebra b_{n-1} and the Deguchi-Martin quotient of the Hecke Algebra $H_n^{1,1}$. In particular the map*

$$\alpha : \mathbb{B}_{n-1}^{top}(t) \to Q_n(\lambda)$$
$$d \mapsto |w>_e$$

is a bijection. Furthermore, when we restrict α to $\mathcal{B}$ the image is $\mathcal{Q}$, where λ depends on t.

Note that λ is a hook tableau of shape $(\lambda_1, 1^{n-\lambda_1})$, by saying that λ depends on t we mean that $t = n+1-2\lambda_1$ and $\lambda_1 = \frac{n+1-t}{2}$.

Proof. It is sufficient to show that the map α has an inverse map. Let $d \in \mathbb{B}_n^{top}$, we do the following to get $\alpha(d)$ (see Definition 3.2.7). Firstly, using the Plaza Hansen bijection (Definition 3.1.7) we get the corresponding one line bitableau $L = (t_{\lambda^1}, t_{\lambda^2}) \in \mathrm{Std}(\lambda)$ of d. Secondly, insert $n+1$ to the component of L that contains 1, by this step, we will get an element in $\mathrm{Std}_+(\lambda)$. By Lemma 3.2.9 we ensure that there is bijection between $\mathrm{Std}(\lambda)$ and $\mathrm{Std}_+(\lambda)$. Thirdly, take the product $((t_{\lambda^1})^T \cdot t_{\lambda^2})$ to produce a hook tableau $t \in \mathrm{Pro}_+$ and also, Lemma 3.2.12 guarantees that the bijection between the set $\mathrm{Std}_+(\lambda)$ and the set $\mathrm{Pro}_+(\lambda)$. Fourthly, use Definition 3.2.6 to construct the Yamanouchi word w from t. Finally, set $\alpha(a) = |w>_e$.
Now we prove γ in Definition 3.3.2 is the inverse map of α, i.e. $\gamma(\alpha(d)) = \mathrm{Id}_{B_{n-1}}$. Take $d \in \mathbb{B}_{n-1}^{top}$, apply α (Definition 3.2.7) on d to produce $\alpha(d)$:

1. By the Plaza-Hansen bijection d corresponding to a one line bitableau

$$L = \left(\boxed{a_1}\boxed{a_2}\boxed{\dots}\boxed{a_k} \, , \, \boxed{b_1}\boxed{b_2}\boxed{\dots}\boxed{b_j} \right).$$

There are two possibilities for the location of the entry 1 in L: either $a_1 = 1$ or $b_1 = 1$.

2. By step 2 of the Definition 3.2.7 of α we insert $n+1$ into $L \in \mathrm{Std}(\lambda)$ to produce a bitableau $L' \in \mathrm{Std}_+(\lambda)$. We have two cases depending on what we had in the previous step:

Case 1 If $b_1 = 1$, then

$$L' = \left(\begin{array}{|c|c|c|c|} \hline a_1 & a_2 & \dots & a_k \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|c|c|} \hline 1 & b_2 & \dots & b_j & n+1 \\ \hline \end{array} \right).$$

Case 2 If $a_1 = 1$, then

$$L' = \left(\begin{array}{|c|c|c|c|c|} \hline 1 & a_2 & \dots & a_k & n+1 \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|c|} \hline b_1 & b_2 & \dots & b_j \\ \hline \end{array} \right).$$

3. Apply step 3 of the map α to obtain a standard tableau t of shape $(n-k, 1^k)$, then construct the Yamanouchi word w from t:

Case 1 If $b_1 = 1$ in L', then

$$t = \begin{array}{|c|c|c|c|c|} \hline 1 & b_2 & \dots & b_j & n+1 \\ \hline a_1 \\ \cline{1-1} a_2 \\ \cline{1-1} \dots \\ \cline{1-1} a_k \\ \cline{1-1} \end{array} \iff w = 1 i_2 i_3 \cdots 1.$$

Where w has $n+1$ letters. The first letter is 1, the last one is 1 and i_j is the row label of the box in t containing j.

Case 2 If $a_1 = 1$ in L', then

$$t = \begin{array}{|c|c|c|c|c|} \hline 1 & b_1 & b_2 & \dots & b_j \\ \hline a_1 \\ \cline{1-1} a_2 \\ \cline{1-1} \dots \\ \cline{1-1} a_k \\ \cline{1-1} n+1 \\ \cline{1-1} \end{array} \iff w_2 = 1i_2i_3 \cdots k+1.$$

Where w has $n+1$ letters. The first letter is 1, the last one is $k+1$ and i_j is the row label of the box in t containing j.

4. Finally, set $\alpha(d) = |w>_e$.

Now take $|w>_e \in Q_n$. To apply γ on $|w>_e$ we consider the two cases of $|w>_e$ based on the value of the last letter in $|w>_e$:

Case 1 Last letter is 1.

Case 2 Last letter is $k+1 \geq 2$.

1. Use the Definition 3.2.6 to construct a standard tableau T from the word $|w>_e$. By Remark 3.3.1 the tableau T has hook shape $(n-k, 1^k)$:

Case 1 If the last letter in $|w>_e$ is 1, then

$$|w\rangle = |1i_2i_3\cdots 1\rangle \Longleftrightarrow T = \begin{array}{|c|c|c|c|c|c|}\hline 1 & b_1 & b_2 & \dots & b_j & n \\ \hline a_1 \\ \cline{1-1} a_2 \\ \cline{1-1} \vdots \\ \cline{1-1} a_k \\ \cline{1-1} \end{array}$$

Case 2 If the last letter in $|w >_e$ is $k+1$, then

$$|w\rangle = |1i_2i_3\cdots k+1\rangle \Longleftrightarrow T = \begin{array}{|c|c|c|c|c|}\hline 1 & b_1 & b_2 & \dots & b_j \\ \hline a_1 \\ \cline{1-1} a_2 \\ \cline{1-1} \vdots \\ \cline{1-1} a_k \\ \cline{1-1} n \\ \cline{1-1} \end{array}$$

one has n at the end of the row and the other has n at the end of the column.

2. Construct a one line bitableau $h = (T_{\lambda^1}, T_{\lambda^2})$ from the standard tableau T by applying step 2 of γ's definition

Case 1 If n in the first row of T, then

$$h = \left(\begin{array}{|c|c|c|c|}\hline a_1 & a_2 & \dots & a_k \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|c|c|c|}\hline 1 & b_1 & b_2 & \dots & b_j & n \\ \hline \end{array} \right).$$

Case 2 If n in the first column of T, then

$$h = \left(\begin{array}{|c|c|c|c|c|c|} \hline 1 & a_1 & a_2 & \dots & a_k & n \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|c|} \hline b_1 & b_2 & \dots & b_j \\ \hline \end{array} \right).$$

3. Delete the box that contains number n, we obtain:

Case 1 :

$$h \Leftrightarrow h' = \left(\begin{array}{|c|c|c|c|} \hline a_1 & a_2 & \dots & a_k \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|c|c|} \hline 1 & b_1 & b_2 & \dots & b_j \\ \hline \end{array} \right).$$

Case 2 :

$$h \Leftrightarrow h' = \left(\begin{array}{|c|c|c|c|c|} \hline 1 & a_1 & a_2 & \dots & a_k \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|c|} \hline b_1 & b_2 & \dots & b_j \\ \hline \end{array} \right).$$

4. Using the Plaza-Hansen bijection to get the correspondence half diagram: $h' = L \Leftrightarrow d$.

Observe that, we have $\gamma(\alpha(d)) = \gamma(|w>_e) = d$ which means that γ is reverse of α.

Now we prove that $\alpha(\gamma(|w>_e)) = |w > e$. Take $|w>_e = |1 i_2 i_3 \cdots i_n>_e \in Q_n$. To apply γ on $|w>_e$ we consider the two cases of $|w>_e$ based on the value of the last letter in $|w>_e$:

Case 1 Last letter is 1.

Case 2 Last letter is $k+1 \geq 2$.

Using definition of γ :

1. Use Definition 3.2.6 to construct a standard tableau r from $|w>_e$. By Remark 3.3.1 the tableau r is hook tableau of shape $(n-k, 1^k)$

Case 1 If the last letter in $|w>_e$ is 1, then

$$
r = \begin{array}{|c|c|c|c|c|c|}
\hline
1 & b_1 & b_2 & \dots & b_j & n \\
\hline
a_1 \\
\cline{1-1}
a_2 \\
\cline{1-1}
\vdots \\
\cline{1-1}
a_k \\
\cline{1-1}
\end{array}
$$

Case 2 If the last letter in $|w>_e$ is $k+1$, then

$$
r = \begin{array}{|c|c|c|c|c|}
\hline
1 & b_1 & b_2 & \dots & b_j \\
\hline
a_1 \\
\cline{1-1}
a_2 \\
\cline{1-1}
\vdots \\
\cline{1-1}
a_k \\
\cline{1-1}
n \\
\cline{1-1}
\end{array}
$$

2. Applying step 2 of Definition 3.3.2 to obtain a bitableau u:

Case 1 If n in the first row of r

$$
r \Leftrightarrow u = \left(\begin{array}{|c|c|c|c|} \hline a_1 & a_2 & \dots & a_k \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|c|c|} \hline 1 & b_1 & b_2 & \dots & b_j \\ \hline \end{array} \right).
$$

Case 2 If n in the first row of r

$$
r \Leftrightarrow u = \left(\begin{array}{|c|c|c|c|c|} \hline 1 & a_1 & a_2 & \dots & a_k \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|c|} \hline b_1 & b_2 & \dots & b_j \\ \hline \end{array} \right).
$$

3. Then by the Plaza-Hansen bijection we get the corresponding half diagram: $u \Leftrightarrow d$.

Therefor, We have that $\gamma(|w>_e) = d$.

Now take $d \in \mathbb{B}_{n-1}^{top}$, we apply α (definition 3.2.7) on d:

1. Use the Plaza-Hansen bijection we get the corresponding one-line bitableau u:

$$u = \left(\begin{array}{|c|c|c|c|} \hline a_1 & a_2 & \dots & a_k \\ \hline \end{array} , \begin{array}{|c|c|c|c|} \hline b_1 & b_2 & \dots & b_j \\ \hline \end{array} \right).$$

   There are two possibilities for the location of the entry 1 in u: either $a_1 = 1$ or $b_1 = 1$.

2. Applying step 2 of Definition 3.2.7 by insert $n+1$ to u to obtain u':

Case 1 If $b_1 = 1$ in u:

$$u \Leftrightarrow u' = \left(\begin{array}{|c|c|c|c|} \hline a_1 & a_2 & \dots & a_k \\ \hline \end{array} , \begin{array}{|c|c|c|c|c|} \hline 1 & b_2 & \dots & b_j & n+1 \\ \hline \end{array} \right).$$

Case 2 If $a_1 = 1$ in u:

$$u \Leftrightarrow u' = \left(\begin{array}{|c|c|c|c|c|} \hline 1 & a_2 & \dots & a_k & n+1 \\ \hline \end{array} , \begin{array}{|c|c|c|c|} \hline b_1 & b_2 & \dots & b_j \\ \hline \end{array} \right).$$

3. Applying step 3 of Definition 3.2.7 to produce a standard tableau r of shape $(n-k, 1^k)$ from u':

Case 1 If $b_1 = 1$ in u', then

$$r = \begin{array}{|c|c|c|c|c|c|}\hline 1 & b_1 & b_2 & \ldots & b_j & n+1 \\ \hline a_1 \\ \cline{1-1} a_2 \\ \cline{1-1} \vdots \\ \cline{1-1} a_k \\ \cline{1-1} \end{array}$$

Case 2 If $a_1 = 1$ in u', then

$$r = \begin{array}{|c|c|c|c|c|}\hline 1 & b_1 & b_2 & \ldots & b_j \\ \hline a_1 \\ \cline{1-1} a_2 \\ \cline{1-1} \vdots \\ \cline{1-1} a_k \\ \cline{1-1} n+1 \\ \cline{1-1} \end{array}$$

4. Applying step 4 of Definition 3.2.7 to construct the Yamanouchi word from the standard tableau r:

Case 1 If $n+1$ in the first row of r we get $w = 1i_2i_2 \cdots 1$.

Case 2 If $n+1$ in the first row of r we get $w = 1i_2i_2 \cdots k+1$.

5. Applying step 5 of Definition 3.2.7 we have: $\alpha(d) = |w >_e$.

Observe that, we have $\alpha(\gamma(|w >_e)) = |w >_e$ which means that α is the inverse map of γ.

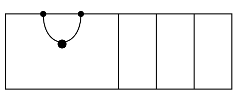
Therefore from the definitions of the maps α and γ we have the following:

- if $d \in \mathbb{B}^{top}_{n-1}$, then $(\gamma \circ \alpha)(d) = d = \mathrm{Id}_{\mathbb{B}^{top}_{n-1}}(d)$, so $\gamma \circ \alpha$ is the identity map on $\mathbb{B}^{top}_{n-1}$ and

- if $|w>_e \in Q_n$, then $(\alpha \circ \gamma)(|w>_e) = |w>_e = \mathrm{Id}_{Q_n}(|w>_e)$, so $\alpha \circ \gamma$ is the identity map on Q_n.

This means that α and γ are inverses of each other. Therefore these two maps produce a bijection between $\mathbb{B}^{top}_{n-1}$ and Q_n. $\square$

Example 3.4.2. *let $d \in \mathbb{B}^{top}(5)$ be as below, applying Theorem 3.4.1 we end with an element in Q_6*

$d =$  $\longleftrightarrow \left(\boxed{2\,|\,3\,|\,4\,|\,5}, \boxed{1} \right) \longleftrightarrow \left(\boxed{2\,|\,3\,|\,4\,|\,5}, \boxed{1\,|\,6} \right)$

$$\longleftrightarrow \begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline 5\\\hline\end{array} \bullet \begin{array}{|c|c|}\hline 1 & 6\\\hline\end{array} \longleftrightarrow \begin{array}{|c|c}\hline 1 & \multicolumn{1}{c|}{6}\\\hline 2 & \\\cline{1-1} 3 & \\\cline{1-1} 4 & \\\cline{1-1} 5 & \\\cline{1-1}\end{array} \longleftrightarrow |w>_e = |123451>_e \in Q_6.$$

Chapter 4

Towards an Algebra Isomorphism

The existence of the bijection in the basis elements of the cell module of blob algebra b_n and the Deguchi-Martin quotient $H_n^{1,1}$ as a result of the previous chapter encouraged us to look for an isomorphism between these algebras' elements. This chapter investigates whether making this bijection an algebra isomorphism is possible.

4.1 In Two Dimensional Algebras

Proposition 4.1.1. *For $n = 2$ we have the following isomorphism*

$$\rho : b_1 \to H_2^{1,1}$$
$$e_0 \mapsto x - U_1;$$

with x in $H_2^{1,1}$ equal to $-[m]$ in b_1.

Proof. This map sends e_0 to $x - U_1$ which differs from the identity. So, to prove it is an isomorphism we need to check that ρ is a homomorphism. Since these two-dimensional algebras b_1 and H_2^{PM} contain one element that differs from the identity element, there is one blob relation to be satisfied:

$$\begin{aligned}(x - U_1)^2 = E_2^2 &= [2]!E_2 \quad \text{(see Proposition 2.4.10)} \\ &= xE_2 \\ &= x(x - U_1).\end{aligned}$$

This shows ρ is an isomorphism. □

4.2 In Six Dimensional Algebras

Now we continue our investigations and look at $n = 3$. Let us begin with the Deguchi-Martin quotient $H_3^{1,1}$ and search for the elements that are isomorphic to the elements of blob algebra b_2. This means we first look for at least two elements in $H_n^{1,1}$ that satisfy the idempotency relation in $H_n^{1,1}$. To find this we set a general element h in $H_n^{1,1}$

$$h = aT_1 + bT_2 + cT_1T_2 + dT_2T_1 + fT_1T_2T_1 + g.$$

Where a, b, c, d, f, g are complex coefficients. Then we solve the equations generated by the relation

$$h^2 = h. \tag{4.1}$$

By setting the relation 4.1 we have a non-linear system of 6 equations in 6 variables:

$$2cdq^2 + g^2 + a^2q + b^2q + f^2q^3 - g = 0$$

$$2ag + bcq + bdq - cdq + cdq^2 + cfq^2 + dfq^2 - a^2 + a^2q - f^2q^3 - a = 0$$

$$2bg + acq + adq - cdq + cdq^2 + cfq^2 + dfq^2 - b^2 + b^2q - f^2q^3 - b = 0$$

$$ab - ac - bc + 2cg + acq + afq + bcq + bfq - 2cfq - 2dfq \\ + 2cfq^2 + 2dfq^2 + d^2q + f^2q - 2f^2q^2 + f^2q^3 - c = 0$$

$$ab - ad - bd + 2dg + adq + afq + bdq + bfq - 2cfq - 2dfq \\ + 2cfq^2 + 2dfq^2 + c^2q + f^2q - 2f^2q^2 + f^2q^3 - d = 0$$

$$ac + ad - 2af + bc + bd - 2bf - 2cd + 2cf + 2df + 2fg \\ + 2afq + 2bfq + 2cdq - 4cfq - 4dfq + 2cfq^2 + 2dfq^2 \\ - c^2 - d^2 - f^2 + c^2q + d^2q + 2f^2q - 2f^2q^2 + f^2q^3 - f = 0$$

To solve this system, using Matlab we set $q = 3$ (we can choose any number in $\mathbb{C}$ as long as it is generic), and we get 16 solutions for this system. These solutions produce 16 idempotent elements in $H_3^{1,1}$:

1. $h_1 = \frac{1}{4}(\mathrm{Id} + T_1)$

2. $h_2 = \frac{1}{4}(\mathrm{Id} + T_2)$

3. $h_3 = \frac{3}{4}\mathrm{Id} - \frac{1}{4}T_2$

4. $h_4 = \frac{3}{4}\mathrm{Id} - \frac{1}{4}T_1$

5. $h_5 = 0$

6. $h_6 = 1$

7. $h_7 = \frac{6}{13}\mathrm{Id} + \frac{2}{13}(T_1 + T_2) - \frac{1}{13}(T_1T_2 + T_2T_1)$

8. $h_8 = \frac{7}{13}\mathrm{Id} - \frac{2}{13}(T_1 + T_2) + \frac{1}{13}(T_1T_2 + T_2T_1)$

9. $h_9 = \frac{1}{52}(\mathrm{Id} + T_1 + T_2 + T_1T_2 + T_2T_1 + T_1T_2T_1)$

10. $h_{10} = \frac{25}{52}\mathrm{Id} + \frac{9}{52}(T_1 + T_2) - \frac{3}{52}(T_1T_2 + T_2T_1) + \frac{1}{52}T_1T_2T_1$

11. $h_{11} = \frac{27}{52}\mathrm{Id} - \frac{9}{52}(T_1 + T_2) + \frac{3}{52}(T_1T_2 + T_2T_1) - \frac{1}{52}T_1T_2T_1$

12. $h_{12} = \frac{51}{52}\mathrm{Id} - \frac{1}{52}(T_1 + T_2 + T_1T_2 + T_2T_1 + T_1T_2T_1)$

13. $h_{13} = \frac{3}{13}(\mathrm{Id} + T_1) - \frac{1}{13}(T_2 + T_1T_2)$

14. $h_{14} = \frac{3}{13}(\mathrm{Id} + T_2) - \frac{1}{13}(T_1 + T_1T_2)$

15. $h_{15} = \frac{10}{13}\mathrm{Id} + \frac{1}{13}(T_1 + T_1T_2) - \frac{3}{13}T_2$

16. $h_{16} = \frac{10}{13}\mathrm{Id} - \frac{3}{13}T_1 + \frac{1}{13}(T_2 + T_1T_2)$

In addition to these elements we noticed that there are more elements at last 4 elements which also satisfy the relation 4.1, so we are not confident we got all elements to satisfy relation 4.1. These additional elements are obtained by swapping 1 and 2 in h_{13}, h_{14}, h_{15} and h_{16} as follows:

1. $h_{17} = \frac{3}{13}(\mathrm{Id} + T_1) - \frac{1}{13}(T_2 + T_2T_1)$

2. $h_{18} = \frac{3}{13}(\mathrm{Id} + T_2) - \frac{1}{13}(T_1 + T_2T_1)$

3. $h_{19} = \frac{10}{13}\mathrm{Id} + \frac{1}{13}(T_1 + T_2T_1) - \frac{3}{13}T_2$

4. $h_{20} = \frac{10}{13}\mathrm{Id} - \frac{3}{13}T_1 + \frac{1}{13}(T_2 + T_2T_1)$

Note that:

$$h_1 + h_4 = 1 \qquad h_9 + h_{12} = 1$$
$$h_2 + h_3 = 1 \qquad h_{10} + h_{11} = 1$$
$$h_1 + h_4 = 1 \qquad h_{13} + h_{16} = 1$$
$$h_7 + h_8 = 1 \qquad h_{14} + h_{15} = 1$$
$$h_{17} + h_{20} = 1 \qquad h_{18} + h_{19} = 1$$

Now, we are going to guess what these elements are for general q:

1. $h_1 = \frac{1}{q+1}(\mathrm{Id} + T_1)$

2. $h_2 = \frac{1}{q+1}(\mathrm{Id} + T_2)$

3. $h_3 = \frac{q}{q+1}\mathrm{Id} - \frac{1}{q+1}T_2$

4. $h_4 = \frac{q}{q+1}\mathrm{Id} - \frac{1}{q+1}T_1$

5. $h_5 = 0$

6. $h_6 = 1$

7. $h_7 = \frac{1}{q^2+q+1}\left((2q)\mathrm{Id} + (q-1)(T_1 + T_2) - (T_1T_2 + T_2T_1)\right)$

8. $h_8 = \frac{1}{1+q+q^2}\left((1 - q + q^2)\mathrm{Id} + (1-q)(T_1 + T_2) + (T_1T_2 + T_2T_1)\right)$

9. $h_9 = \frac{1}{1+2q+2q^2+q^3}(\mathrm{Id} + T_1 + T_2 + T_1T_2 + T_2T_1 + T_1T_2T_1)$

10. $h_{10} = \frac{1}{1+2q+2q^2+q^3}\left((1 + 2q + 2q^2)\mathrm{Id} + (q^2)(T_1 + T_2) - (q)(T_1T_2 + T_2T_1) + T_1T_2T_1\right)$

11. $h_{11} = \frac{1}{1+2q+2q^2+q^3}\,(q^3\,\mathrm{Id} - q^2(T_1+T_2) + q(T_1T_2+T_2T_1) - T_1T_2T_1)$

12. $h_{12} = \frac{2q+2q^2+q^3}{1+2q+2q^2+q^3}\,\mathrm{Id} - \frac{1}{1+2q+2q^2+q^3}(+T_1+T_2+T_1T_2+T_2T_1+T_1T_2T_1)$

13. $h_{13} = \frac{1}{1+q+q^2}\,(q\,\mathrm{Id} + qT_1 - T_2 - T_1T_2)$

14. $h_{14} = \frac{1}{1+q+q^2}\,(q\,\mathrm{Id} + qT_2 - T_1 - T_1T_2)$

15. $h_{15} = \frac{1}{1+q+q^2}\,((1+q^2)\,\mathrm{Id} + T_1 - qT_2 = T_1T_2)$

16. $h_{16} = \frac{1}{1+q+q^2}\,((1+q^2)\,\mathrm{Id} - qT_1 + T_2 = T_1T_2)$

17. $h_{17} = \frac{1}{1+q+q^2}\,(q\,\mathrm{Id} + qT_1 - T_2 - T_2T_1)$

18. $h_{18} = \frac{1}{1+q+q^2}\,(q\,\mathrm{Id} + qT_2 - T_1 - T_2T_1)$

19. $h_{19} = \frac{1}{1+q+q^2}\,((1+q^2)\,\mathrm{Id} + T_1 - qT_2 - T_2T_1)$

20. $h_{20} = \frac{1}{1+q+q^2}\,((1+q^2)\,\mathrm{Id} - qT_1 + T_2 - T_2T_1)$

We prove these generalizations work correctly by using gap3. The Appendix A.1 has details of these calculations.

We substituted these elements into the blob algebra relations when $q = 3$. The result indicates that the only elements that satisfy the blob relations are $e_1 = h_{13}$ with $e_0 = h_{15}$, $e_1 = h_{13}$ with $e_0 = h_{18}$, $e_1 = h_{14}$ with $e_0 = h_{16}$, $e_1 = h_{14}$ with $e_0 = h_{17}$, $e_1 = h_{13}$ with $e_0 = h_{20}$, and $e_1 = h_{14}$ with $e_0 = h_{19}$.

Example 4.2.1. *Multiplying h_{13} by $-[2]$ and h_{20} by $-[m]$ gives two elements in $H_3^{1,1}$ which satisfy the blob relation. Let $e_1 = -[2]h_{13}$ and $e_0 = -[m]h_{20}$:*

- $e_1^2 = -[2]e_1$.
- $e_0^2 = -[m]e_0$.
- $e_1e_0e_1 = \frac{-3[2][m]}{13}e_1$.

The following lemma gives a formal result.

Lemma 4.2.2. *For $n = 3$ and $q = 3$, define two elements in $H_3^{1,1}$*

$$w_1 = -[2]\left(\frac{3}{13}(\mathrm{Id} + T_1) - \frac{1}{13}(T_2 + T_1T_2)\right);$$
$$w_2 = -[m]\left(\frac{10}{13}\,\mathrm{Id} + \frac{1}{13}(T_1 + T_1T_2) - \frac{3}{13}T_2\right).$$

Then for m satisfying the relation $\frac{[m+1]}{[m]} = \frac{[2]([2]+2)}{[2]+1}$ the map

$$\Psi : b_2 \to H_3^{1,1}$$
$$e_1 \mapsto w_1$$
$$e_0 \mapsto w_2$$

is an isomorphism.

Proof. Using gap3 we do the following calculations (see Appendix A.1), we verify that:

- $w_1^2 = -[2]w_1$.
- $w_2^2 = -[m]w_2$.
- $w_1w_2 = [2][m]\left(\frac{48}{169}(\mathrm{Id} + T_1) - \frac{4}{169}(T_2 + T_1T_2 + T_2T_1 + T_1T_2T_1)\right)$.
- $w_2w_1 = [2][m]\left(\frac{48}{169}\,\mathrm{Id} + \frac{39}{169}T_1 - \frac{16}{169}T_2 - \frac{12}{169}(T_1T_2 + T_2T_1) + \frac{4}{169}(T_1T_2T_1)\right)$.
- $w_1w_2w_1 = [m+1]w_1$. Where $[m+1] = \frac{16[2][m]}{13}$.
- $w_2w_1w_2 = -[2][m]^2\left(\frac{48}{169}(\mathrm{Id} + T_1) - \frac{168}{169}(T_2 + T_2T_1)\right)$.

The direct calculation shows that all images satisfy blob relations, and we have six distinct images, proving that the map Ψ is an isomorphism. □

Also for m as in Lemma 4.2.2 we have the next results.

Lemma 4.2.3. *For $n = 3$ and $q = 3$, let w_4 and w_5 be two elements in $H_3^{1,1}$ as following:*

$$w_4 = -[2]\left(\frac{3}{13}(\mathrm{Id} + T_2) - \frac{1}{13}(T_1 + T_1T_2)\right);$$
$$w_5 = -[m]\left(\frac{10}{13}\mathrm{Id} - \frac{3}{13}T_1 + \frac{1}{13}(T_2 + T_1T_2)\right).$$

Then for m satisfying the relation $\frac{[m+1]}{[m]} = \frac{[2]([2]+2)}{[2]+1}$ the map

$$\Phi : b_2 \to H_3^{1,1}$$
$$e_1 \mapsto w_4$$
$$e_0 \mapsto w_5$$

is an isomorphism.

Next, we get other isomorphisms for different values of m.

Lemma 4.2.4. *For $n = 3$ and $q = 3$, define two elements in $H_3^{1,1}$*

$$z_1 = -[2]\left(\frac{3}{13}(\mathrm{Id} + T_1) - \frac{1}{13}(T_2 + T_1T_2)\right);$$
$$z_2 = -[m]\left(\frac{10}{13}\mathrm{Id} + \frac{1}{13}(T_2 + T_2T_1) - \frac{3}{13}T_1\right).$$

Then for m satisfying the relation $\frac{[m+1]}{[m]} = -\frac{[2]}{[2]+1}$ the map

$$\vartheta : b_2 \to H_3^{1,1}$$
$$e_1 \mapsto z_1$$
$$e_0 \mapsto z_2$$

is an isomorphism.

Proof. Also using gap3 we verify that (see Appendix A.1):

1. $z_1^2 = -[2]z_1$.
2. $z_2^2 = -[m]z_2$.
3. $z_1z_2 = \frac{[m][2]}{169}(-9(\mathrm{Id} + T_1 + T_2 + T_1T_2) + 4(T_2T_1 + T_1T_2T_1))$.
4. $z_2z_1 = \frac{[m][2]}{169}(-9\,\mathrm{Id} + 3(T_1 + T_2) - T_1T_2 + 12T_2T_1 - 4T_1T_2T_1)$.
5. $z_1z_2z_1 = [m+1](\frac{3}{13}\mathrm{Id} + \frac{3}{13}T_1 - \frac{1}{13}T_2 - \frac{1}{13}T_1T_2) = [m+1]z_1$. Where $[m+1] = -\frac{3[m][2]}{13}$.
6. $z_2z_1z_2 = \frac{-3[2][m]^2}{13}(\frac{3}{13}\mathrm{Id} - \frac{1}{13}T_1 + \frac{3}{13}T_2 - \frac{1}{13}T_1T_2)$. □

For the same value of m as in Lemma 4.2.4 we have the following result:

Lemma 4.2.5. *For $n = 3$ and $q = 3$, let z_3 and z_4 be two elements in $H_3^{1,1}$ as following:*

$$z_3 = -[2]\left(\frac{3}{13}(\mathrm{Id} + T_2) - \frac{1}{13}(T_1 + T_1T_2)\right);$$
$$z_4 = -[m]\left(\frac{10}{13}\mathrm{Id} + \frac{1}{13}(T_1 + T_2T_1) - \frac{3}{13}T_2\right).$$

For the same value of m as in lemma 4.2.4 the map

$$\Theta : b_2 \to H_3^{1,1}$$
$$e_1 \mapsto z_3;$$
$$e_0 \mapsto z_4.$$

is an isomorphism.

Proof. The proof is similar to the proof for lemma 4.2.4 and also the calculation can be seen in appendix A.1. □

Since we have these nice isomorphisms Ψ and ϑ for $q = 3$, we work to generalize these maps for all q and come to produce the following isomorphisms

Proposition 4.2.6. *Let $n = 3$ for general q we have y_1, y_2 two elements in $H_3^{1,1}$ defined as*

$$y_1 = -\kappa[2]\left(q(\mathrm{Id} + T_1) - (T_2 + T_1T_2)\right)$$
$$y_2 = -\kappa[m]\left((1+q^2)\,\mathrm{Id} + (T_1 + T_1T_2) - qT_2\right),$$

where $\kappa = \frac{1}{1+q+q^2}$. For m satisfying $\dfrac{[m+1]}{[m]} = \dfrac{[2]([2]+2)}{[2]+1}$, we have the following isomorphism :

$$\varphi : b_2 \to H_3^{1,1}$$
$$e_1 \mapsto y_1;$$
$$e_0 \mapsto y_2.$$

Proof. To prove this is an isomorphism it is sufficient to prove the images satisfy the blob relations and produce 6 linearly independent elements. By direct calculations (we use gap3 to do these calculations, see Appendix A.2) we have:

$$y_1^2 = -[2]y_1;$$
$$y_2^2 = -[m]y_2;$$
$$y_1y_2y_1 = \frac{[m][2]([2]+2)}{[2]+1}y_1.$$

That shows the images satisfy the blob relations. Now let us see the remaining three elements:

$$y_1y_2 = \xi[2][m]\left((q+2q^2+q^3)(\mathrm{Id}+T_1) - (1+q)(T_2+T_1T_2+T_2T_1+T_1T_2T_1)\right).$$

$$\begin{aligned} y_2y_1 = {}& \xi[2][m]\left((q+2q^2+q^3)\,\mathrm{Id}+(q^2+q^3)T_1 - (1+2q+q^2)T_2\right) \\ & + \xi[2][m]\left(-(q+q^2)(T_1T_2+T_2T_1) + (1+q)(T_1T_2T_1)\right). \end{aligned}$$

$$y_2y_1y_2 = -\xi[2][m]^2\left((q+2q^2+q^3)(\mathrm{Id}+T_1) - (1+2q+q^2)(T_2+T_2T_1)\right).$$

Where $\xi = \frac{1}{1+2q+3q^2+2q^3+q^4}$. As a result, we get six different images under the map φ that prove φ is an isomorphism. □

Similarity as a generalization of map ϑ for all q we get the following proposition.

Proposition 4.2.7. *For general q and $n = 3$ let s_1, s_2 be elements in $H_3^{1,1}$ defined as*

$$\begin{aligned} s_1 &= -\kappa[2]\left(q(\mathrm{Id}+T_1) - (T_2+T_1T_2)\right) \\ s_2 &= -\kappa[m]\left((1+q^2)\,\mathrm{Id}+(T_2+T_2T_1) - qT_1\right), \end{aligned}$$

where $\kappa = \frac{1}{1+q+q^2}$. For m satisfying $\dfrac{[m+1]}{[m]} = \dfrac{-[2]}{[2]+1}$, we have the following *isomorphism :*

$$\varrho : b_2 \to H_3^{1,1}$$
$$e_1 \mapsto s_1;$$
$$e_0 \mapsto s_2.$$

Proof. First, we show these two elements satisfy the blob relations (see appendix A.3):

- $s_1^2 = -[2]s_1$.
- $s_2^2 = -[m]s_2$.
- $s_1 s_2 s_1 = \frac{-[m][2]}{[2]+1} s_1$.

Then we check that the remaining elements are different:

- $s_1 s_2 = -\kappa^2 [m](q^2+1)\left(q(\mathrm{Id} + T_1 + T_2 + T_1T_2) - (q+1)(T_2T_1 + T_1T_2T_1)\right)$.
- $s_2 s_1 = -\kappa^2 [m](q^2+1)\left(q\,\mathrm{Id} - (T_1 + T_2) + T_1T_2 - (q+1)T_2T_1 + (q+1)T_1T_2T_1\right)$.
- $s_2 s_1 s_2 = q\kappa^2 [m]^2 [2]\left(q(\mathrm{Id} + T_2) - (T_1 + T_1T_2)\right)$ □

4.3 Conclusion

In this thesis, we provide some results on the representation theory of the blob algebras and Deguchi-Martin quotient over $\mathbb{C}$. However, we believe there are areas for future research. We construct a bijection map between the cell bases of blob algebras b_{n-1} and Deguchi-Martin quotient $H_n^{1,1}$. Also, we construct an isomorphism map between the elements of these two algebras for $n = 2$

and $n = 3$ at a certain value of the parameter m. These results may give the possibility to achieve new results in the study of these algebras. The problem of finding a general isomorphism between the blob algebras and the Deguchi-Martin quotient is still open. Future work can be lies in using another technique to generalize the isomorphism that we found between blob algebras and the Deguchi-Martin quotient.

Appendix A

Gap3 Code

In this chapter, we illustrate all of gap3's codes which we use to examine h's in chapter 4 and to verify our results.

A.1 For Lemmas of Chapter 4

First we construct the general element h in H_3^{PM} and then we produce h^2.

```
gap> W := CoxeterGroup( "A", 3);
CoxeterGroup("A",3)
gap> q:=Mvp("q");;

gap> a :=Mvp("a");; b :=Mvp("b");; c :=Mvp("c");;
d :=Mvp("d");; f :=Mvp("f");;
gap> g:=Mvp("g");;
gap> H := Hecke( W,q);;
gap> T := Basis( H, "T" );;
gap> h := g*T( )+a*T(1)+ b*T(2)+ c*T(1,2)+ d*T(2,1)+ f*T(1,2,1);;
```

```
gap> h^2;
(2cdq^2+g^2+a^2q+b^2q+f^2q^3)T()+(2ag+bcq+bdq-cdq+cdq^2\
+cfq^2+dfq^2-a^2+a^2q-f^2q^2+f^2q^3)T(1)+(2bg+acq+adq\
-cdq+cdq^2+cfq^2+dfq^2-b^2+b^2q-f^2q^2+f^2q^3)T(2)+(ab-ac\
-bc+2cg+acq+afq+bcq+bfq-2cfq-2dfq+2cfq^2+2dfq^2+d^2q+f^2q\
-2f^2q^2+f^2q^3)T(1,2)+(ab-ad-bd+2dg+adq+afq+bdq+bfq-2cfq-2dfq\
+2cfq^2+2dfq^2+c^2q+f^2q-2f^2q^2+f^2q^3)T(2,1)+(ac+ad-2af\
+bc+bd-2bf-2cd+2cf+2df+2fg+2afq+2bfq+2cdq-4cfq-4dfq+2cfq^2\
+2dfq^2-c^2-d^2-f^2+c^2q+d^2q+2f^2q-2f^2q^2+f^2q^3)T(1,2,1)
```

Setting the relation $h^2 = h$ (as in chapter 4 p.62) we get 20 solutions. These solutions produce 20 idempotent elements. On the following we test each of h_i on the relation $h_i^2 = h_i$.

```
gap> h1:= 1/(q+1)*(T()+T(1));;
gap> h1*h1-h1;
0
gap> h2:= 1/(q+1)*(T()+T(2));;
gap> h2*h2-h2;
0
gap> h3:=1/(q+1)*(q*T()-T(2));;
gap> h3*h3-h3;
0
gap> h4:= 1/(q+1)*(q*T()-T(1));;
gap> h4*h4-h4;
0
gap> k:= 1/(1+q+q^2);;
gap> h7:=k*((2*q)*T()+(q-1)*T(1)+(q-1)*T(2)-T(1,2)-T(2,1));;
gap> h7*h7-h7;
```

```
0
gap> h8:=T()-h7;;
gap> h8*h8-h8;
0
gap> p:= (1+q)*(1+q+q^2);;
gap> h9:= (1/p)*(T()+T(1)+T(2)+T(1,2)+T(2,1)+T(1,2,1));;
gap> h9*h9-h9;;
0
gap> h11:= (1/p)*(q^3*T()-q^2*T(1)-q^2*T(2)+q*T(1,2)\
+q*T(2,1)-T(1,2,1));;
gap> h11*h11-h11;
0
gap> h10:= T()-h11;;
gap> h10*h10-h10;
0
gap>  h12:= (1/p)*((p-1)*T()-(T(1)+T(2)+T(1,2)+T(2,1)+T(1,2,1)));;
gap> h12*h12-h12;
0
gap> h13:=k*(q*T()+q*T(1)-T(2)-T(1,2));;
gap> h13*h13-h13;
gap> h14:= k*(q*T()+q*T(2)-T(1)-T(1,2));;
gap> h14*h14-h14;
0
gap> h15:= k*((q^2+1)*T()+T(1)-q*T(2)+T(1,2));;
gap> h15*h15-h15;
0
gap> h16:= k*((q^2+1)*T()-q*T(1)+T(2)+T(1,2));;
gap> h16*h16-h16;
```

```
0
gap> h17:=k*(q*T()+q*T(1)-T(2)-T(2,1));;
gap> h17*h17-h17;
0
gap> h18:= k*(q*T()+q*T(2)-T(1)-T(2,1));;
gap> h18*h18-h18;
0
gap> h19:= k*((q^2+1)*T()+T(1)-q*T(2)+T(2,1));;
gap> h19*h19-h19;
0
gap> h20:= k*((q^2+1)*T()-q*T(1)+T(2)+T(2,1));;
gap> h20*h20-h20;
0
```

This shows that $h_i^2 = h_i$ for $1 \leq i \leq 20$. That means we can scale the h_i to get the first two relations of blob algebra (the idempotency relation). Next, we examine each h_i with every others h_j, where $i \neq j$ for $1 \leq i, j \leq 20$. First, if $h_i h_j h_i = c h_i$ for some scalar $c \in \mathbb{C}$. If the candidates satisfy this relation, then we move to check whether $h_i h_j$, $h_j h_i$ and $h_j h_i h_j$ are linearly independent.

In this part, we examine h_{13} with h_{15}, h_{18} and h_{20} for $q = 3$. Note that for $q = 3$ a check was carried out for all h_i but we do not need the result when they dose not work.

```
gap> h13*h15;
(30/169+6/169q)T()+(30/169+6/169q)T(1)+(-22/169+6/169q)T(2)
+(-22/169+6/169q)T(1,2)+(-1/169-1/169q)T(2,1)+(-1/169-1/169q)T(1,2,1)
gap> h15*h13;
(30/169+6/169q)T()+(30/169+2/169q)T(1)+(-22/169+2/169q)T(2)
+(-6/169-2/169q)T(1,2)+(-9/169-1/169q)T(2,1)+(7/169-1/169q)T(1,2,1)
```

```
gap> h13*h15*h13;
(90/2197+10/169q+1/169q^2+1/2197q^3)T()+(90/2197+10/169q\
+1/169q^2+1/2197q^3)T(1)+(-118/2197+6/2197q-15/2197q^2\
+1/2197q^3)T(2)+(-118/2197+6/2197q-15/2197q^2+1/2197q^3)T(1,2)
+(-66/2197+37/2197q-8/2197q^2+1/2197q^3)T(2,1)+(-66/2197\
+37/2197q-8/2197q^2+1/2197q^3)T(1,2,1)
gap> h15*h13*h15;
(300/2197+12/169q-1/169q^2-1/2197q^3)T()+(300/2197+72/2197q\
+15/2197q^2-1/2197q^3)T(1)+(-376/2197+8/169q-1/169q^2\
-1/2197q^3)T(2)+(-168/2197+20/2197q+15/2197q^2-1/2197q^3)T(1,2)
+(-103/2197-50/2197q+8/2197q^2-1/2197q^3)T(2,1)+(105/2197\
-50/2197q+8/2197q^2-1/2197q^3)T(1,2,1)
```

Note that $h13 * h15$,$h15 * h13$ and $h15 * h13 * h15$ are linearly independent and $h13 * h15 * h13 = 16/13 * h13$, thus h_{13} and h_{15} are candidates for an isomorphism.

```
gap> h13*h18;
(9/169+10/169q+1/169q^2)T()+(9/169+10/169q+1/169q^2)T(1)\
+(-7/169+1/169q)T(2)+(-7/169+1/169q)T(1,2)+(-7/169+1/169q)T(2,1)\
+(-7/169+1/169q)T(1,2,1)
gap> h18*h13;
(9/169+10/169q+1/169q^2)T()+(9/169+9/169q)T(1)+(-7/169\
-6/169q+1/169q^2)T(2)+(-3/169-3/169q)T(1,2)+(-3/169-3/169q)T(2,1)\
+(1/169+1/169q)T(1,2,1)
gap> h13*h18*h13;
(27/2197+64/2197q+3/169q^2+2/2197q^3)T()+(27/2197+64/2197q\
+3/169q^2+2/2197q^3)T(1)+(-37/2197-36/2197q-1/2197q^2\
-2/2197q^3)T(2)+(-37/2197-36/2197q-1/2197q^2-2/2197q^3)T(1,2)\
+(-21/2197-11/2197q+9/2197q^2-1/2197q^3)T(2,1)+(-21/2197\
```

```
−11/2197q+9/2197q^2−1/2197q^3)T(1,2,1)
gap> h18*h13*h18;
(27/2197+64/2197q+3/169q^2+2/2197q^3)T()+(27/2197+64/2197q\
+3/169q^2+2/2197q^3)T(1)+(−37/2197−36/2197q−1/2197q^2\
−2/2197q^3)T(2)+(−21/2197−11/2197q+9/2197q^2−1/2197q^3)T(1,2)\
+(−37/2197−36/2197q−1/2197q^2−2/2197q^3)T(2,1)+(−21/2197\
−11/2197q+9/2197q^2−1/2197q^3)T(1,2,1)
```

Note that $h13 * h18$, $h18 * h13$ and $h18 * h13 * h18$ are linearly independent and $h13 * h18 * h13 = 16/13 * h13$, thus h_{13} and h_{18} are candidates for an isomorphism.

```
gap> h13*h20;
(30/169−10/169q−1/169q^2)T()+(30/169−10/169q−1/169q^2)T(1)\
+(−6/169−1/169q)T(2)+(−6/169−1/169q)T(1,2)+(7/169−1/169q)T(2,1)\
+(7/169−1/169q)T(1,2,1)
gap> h20*h13;
(30/169−10/169q−1/169q^2)T()+(30/169−9/169q)T(1)+(−6/169\
+6/169q−1/169q^2)T(2)+(−10/169+3/169q)T(1,2)+(3/169+3/169q)T(2,1)\
+(−1/169−1/169q)T(1,2,1)
gap> h13*h20*h13;
(90/2197+66/2197q−3/169q^2−2/2197q^3)T()+(90/2197\
+66/2197q−3/169q^2−2/2197q^3)T(1)+(−54/2197+10/2197q\
+1/2197q^2+2/2197q^3)T(2)+(−54/2197+10/2197q+1/2197q^2\
+2/2197q^3)T(1,2)+(−18/2197+24/2197q−9/2197q^2+1/2197q^3)T(2,1)\
+(−18/2197+24/2197q−9/2197q^2+1/2197q^3)T(1,2,1)
gap> h20*h13*h20;
(300/2197−196/2197q+1/169q^2+2/2197q^3)T()+(300/2197−183/2197q\
+2/169q^2+2/2197q^3)T(1)+(−24/2197+29/2197q−14/2197q^2\
−2/2197q^3)T(2)+(−60/2197+15/2197q+9/2197q^2−1/2197q^3)T(1,2)\
```

```
+(93/2197-10/2197q-1/2197q^2-2/2197q^3)T(2,1)+(57/2197-37/2197q\
+9/2197q^2-1/2197q^3)T(1,2,1)
```

Note that $h13 * h20$, $h20 * h13$ and $h20 * h13 * h20$ are linearly independent and $h13 * h20 * h13 = -3/13 * h13$, thus h_{13} and h_{20} are candidates for an isomorphism.

Similarly, we examine h_{14} and also we will see that h_{14} with h_{16}, h_{17} and h_{19} will satisfy the blob relations.

```
gap> h14*h16;
(30/169+6/169q)T()+(-22/169+2/169q)T(1)+(30/169+2/169q)T(2)
+(-6/169-2/169q)T(1,2)+(-9/169-1/169q)T(2,1)+(7/169-1/169q)T(1,2,1)
gap> h16*h14;
(30/169+6/169q)T()+(-22/169+6/169q)T(1)+(30/169+6/169q)T(2)
+(-22/169+6/169q)T(1,2)+(-1/169-1/169q)T(2,1)+(-1/169-1/169q)T(1,2,1)
gap> h14*h16*h14;
(90/2197+10/169q+1/169q^2+1/2197q^3)T()+(-118/2197\
+6/2197q-15/2197q^2+1/2197q^3)T(1)+(90/2197+10/169q+1/169q^2\
+1/2197q^3)T(2)+(-118/2197+6/2197q-15/2197q^2+1/2197q^3)T(1,2)
+(-66/2197+37/2197q-8/2197q^2+1/2197q^3)T(2,1)+(-66/2197\
+37/2197q-8/2197q^2+1/2197q^3)T(1,2,1)
gap> h16*h14*h16;
(300/2197+12/169q-1/169q^2-1/2197q^3)T()+(-376/2197+8/169q\
-1/169q^2-1/2197q^3)T(1)+(300/2197+72/2197q+15/2197q^2\
-1/2197q^3)T(2)+(-168/2197+20/2197q+15/2197q^2-1/2197q^3)T(1,2)
+(-103/2197-50/2197q+8/2197q^2-1/2197q^3)T(2,1)+(105/2197\
-50/2197q+8/2197q^2-1/2197q^3)T(1,2,1)
```

Note that $h14 * h16$,$h16 * h14$ and $h16 * h14 * h16$ are linearly independent and $h14 * h16 * h14 = 16/13 * h14$, thus h_{14} and h_{16} are candidates for an isomorphism.

```
gap> h14*h17;
(9/169+10/169q+1/169q^2)T()+(-7/169-6/169q+1/169q^2)T(1)\
+(9/169+9/169q)T(2)+(-3/169-3/169q)T(1,2)+(-3/169-3/169q)T(2,1)\
+(1/169+1/169q)T(1,2,1)
gap> h17*h14;
(9/169+10/169q+1/169q^2)T()+(-7/169+1/169q)T(1)+(9/169\
+10/169q+1/169q^2)T(2)+(-7/169+1/169q)T(1,2)+(-7/169\
+1/169q)T(2,1)+(-7/169+1/169q)T(1,2,1)
gap> h14*h17*h14;
(27/2197+64/2197q+3/169q^2+2/2197q^3)T()+(-37/2197\
-36/2197q-1/2197q^2-2/2197q^3)T(1)+(27/2197+64/2197q\
+3/169q^2+2/2197q^3)T(2)+(-37/2197-36/2197q-1/2197q^2\
-2/2197q^3)T(1,2)+(-21/2197-11/2197q+9/2197q^2-1/2197q^3)T(2,1)\
+(-21/2197-11/2197q+9/2197q^2-1/2197q^3)T(1,2,1)
gap> h17*h14*h17;
(27/2197+64/2197q+3/169q^2+2/2197q^3)T()+(-37/2197\
-36/2197q-1/2197q^2-2/2197q^3)T(1)+(27/2197+64/2197q\
+3/169q^2+2/2197q^3)T(2)+(-21/2197-11/2197q+9/2197q^2\
-1/2197q^3)T(1,2)+(-37/2197-36/2197q-1/2197q^2-2/2197q^3)T(2,1)\
+(-21/2197-11/2197q+9/2197q^2-1/2197q^3)T(1,2,1)
```

Note that $h14 * h17, h17 * h14$ and $h17 * h14 * h17$ are linearly independent and $h14 * h17 * h14 = 16/13 * h14$, thus h_{14} and h_{17} are candidates for an isomorphism.

```
gap> h14*h19;
(30/169-10/169q-1/169q^2)T()+(-6/169+6/169q-1/169q^2)T(1)\
+(30/169-9/169q)T(2)+(-10/169+3/169q)T(1,2)+(3/169+3/169q)T(2,1)\
```

```
+(-1/169-1/169q)T(1,2,1)
gap> h19*h14;
(30/169-10/169q-1/169q^2)T()+(-6/169-1/169q)T(1)\
+(30/169-10/169q-1/169q^2)T(2)+(-6/169-1/169q)T(1,2)\
+(7/169-1/169q)T(2,1)+(7/169-1/169q)T(1,2,1)
gap> h14*h19*h14;
(90/2197+66/2197q-3/169q^2-2/2197q^3)T()+(-54/2197+10/2197q\
+1/2197q^2+2/2197q^3)T(1)+(90/2197+66/2197q-3/169q^2\
-2/2197q^3)T(2)+(-54/2197+10/2197q+1/2197q^2+2/2197q^3)T(1,2)\
+(-18/2197+24/2197q-9/2197q^2+1/2197q^3)T(2,1)+(-18/2197\
+24/2197q-9/2197q^2+1/2197q^3)T(1,2,1)
gap> h19*h14*h19;
(300/2197-196/2197q+1/169q^2+2/2197q^3)T()+(-24/2197\
+29/2197q-14/2197q^2-2/2197q^3)T(1)+(300/2197-183/2197q+2/169q^2\
+2/2197q^3)T(2)+(-60/2197+15/2197q+9/2197q^2-1/2197q^3)T(1,2)\
+(93/2197-10/2197q-1/2197q^2-2/2197q^3)T(2,1)+(57/2197\
-37/2197q+9/2197q^2-1/2197q^3)T(1,2,1)
```

Note that $h14 * h19, h19 * h14$ and $h19 * h14 * h19$ are linearly independent and $h14 * h19 * h14 = -3/13 * h14$, thus h_{14} and h_{19} are candidates for an isomorphism.

A.2 For proposition 4.2.6

In the following, we use gap3 to do the calculations of the proposition 4.2.6. Where $x = q + q^{-1} = [2]$.

```
gap> e1:=q/(q^2+q+1)*T()+q/(q^2+q+1)*T(1) \
```

```
-1/(q^2+q+1)*T(2)-1/(q^2+q+1)*T(1,2);;
gap> e2:= (q^2+1)/(q^2+q+1)*T()+1/(q^2+q+1)*T(1) \
+ 1/(q^2+q+1)*T(1,2) - q/(q^2+q+1) * T(2);;
gap> y1:=-(q+q^-1)*e1;;
gap> y1^2+(q+q^-1)*y1;
0
We use m in place of [m].
gap> y2:= -m*e2;:
gap> y2^2+m*y2;
0
gap> y1*y2*y1;
(-m-2mq-3mq^2-4mq^3-3mq^4-2mq^5-mq^6)/(q+2q^2+3q^3\
+2q^4+q^5)T()+(-m-2mq-3mq^2-4mq^3-3mq^4-2mq^5\
-mq^6)/(q+2q^2+3q^3+2q^4+q^5)T(1)+(m+2mq+3mq^2+4mq^3\
+3mq^4+2mq^5+mq^6)/(q^2+2q^3+3q^4+2q^5+q^6)T(2)\
+(m+2mq+3mq^2+4mq^3+3mq^4+2mq^5+mq^6)/(q^2+2q^3+3q^4+2q^5\
+q^6)T(1,2)
gap> (q^2+2*q^3+3*q^4+2*q^5+q^6)/(m+2*m*q+3*m*q^2+4*m*q^3\
+3*m*q^4+2*m*q^5+m*q^6)*(y1*y2*y1);
-qT()-qT(1)+T(2)+T(1,2)
gap> 1/(1+q^2) * q*(1+q+q^2)*y1;
-qT()-qT(1)+T(2)+T(1,2)
We see
gap> y1*y2*y1-(q+q^2+q^3)/(1+q^2)*(m+2*m*q+3*m*q^2+4*m*q^3\
+3*m*q^4+2*m*q^5+m*q^6)/(q^2 +2*q^3+3*q^4+2*q^5+q^6)*(y1);
0
```

So we need

$$[m{+}1] = \frac{(q+q^2+q^3)([m]+2[m]q+3[m]q^2+4[m]q^3+3[m]q^4+2[m]q^5+[m]q^6)}{(q^2+2q^3+3q^4+2q^5+q^6)(1+q^2)}.$$

I.e.

$$\begin{aligned}\frac{[m+1]}{[m]} &= \frac{q(1+q+q^2)(1+2q+3q^2+4q^3+3q^4+2q^5+q^6)}{q^2(1+2q+3q^2+2q^3+q^4)(1+q^2)}\\ &= \frac{q([2]+1)q^3([4]+2[3]+2[2]+2)}{q.q^2([3]+2[2]+2).q[2]}\\ &= \frac{([2]+1)[2]([3]+2[2]+1)}{([3]+2[2]+2).[2]} = \frac{([2]+1)([2]+2)[2]}{([2]^2+2[2]+1)} = \frac{[2]([2]+2)}{[2]+1}.\end{aligned}$$

The above calculations verify that y_1 and y_2 satisfy the blob relations for m satisfying the above equation. Next, we check that y_1y_2, y_2y_1 and $y_2y_1y_2$ are linear independent elements.

```
gap> y1*y2;
(m+2mq+2mq^2+2mq^3+mq^4)/(1+2q+3q^2+2q^3+q^4)T()\
+(m+2mq+2mq^2+2mq^3+mq^4)/(1+2q+3q^2+2q^3+q^4)T(1)\
+(-m-mq-mq^2-mq^3)/(q+2q^2+3q^3+2q^4+q^5)T(2)+(-m-mq-mq^2-\
mq^3)/(q+2q^2+3q^3+2q^4+q^5)T(1,2)+(-m-mq-mq^2-mq^3)/(q+2q^2\
+3q^3+2q^4+q^5)T(2,1)+(-m-mq-mq^2-mq^3)/(q+2q^2+3q^3+2q^4\
+q^5)T(1,2,1)
gap> y2*y1;
(m+2mq+2mq^2+2mq^3+mq^4)/(1+2q+3q^2+2q^3+q^4)T()\
+(mq+mq^2+mq^3+mq^4)/(1+2q+3q^2+2q^3+q^4)T(1)+(-m-2mq\
-2mq^2-2mq^3-mq^4)/(q+2q^2+3q^3+2q^4+q^5)T(2)+(-m-mq\
-mq^2-mq^3)/(1+2q+3q^2+2q^3+q^4)T(1,2)+(-m-mq-mq^2\
-mq^3)/(1+2q+3q^2+2q^3+q^4)T(2,1)+(m+mq+mq^2+mq^3)/(q+2q^2\
+3q^3+2q^4+q^5)T(1,2,1)
gap> y2*y1*y2;
```

```
(-m^2-2m^2q-2m^2q^2-2m^2q^3-m^2q^4)/(1+2q+3q^2+2q^3+q^4)T()\
+(-m^2-2m^2q-2m^2q^2-2m^2q^3-m^2q^4)/(1+2q+3q^2+2q^3+q^4)T(1)\
+(m^2+2m^2q+2m^2q^2+2m^2q^3+m^2q^4)/(q+2q^2+3q^3+2q^4+q^5)T(2)\
+(m^2+2m^2q+2m^2q^2+2m^2q^3+m^2q^4)/(q+2q^2+3q^3+2q^4+q^5)T(2,1)
```

A.3 For proposition 4.2.7

```
gap>  e1:= q/(q^2+q+1)*T()+q/(q^2+q+1)*T(1)-1/(q^2+q+1)*T(2)\
-1/(q^2+q+1) * T(1,2);;
gap> e2:= (q^2+1)/(q^2+q+1)*T() + 1/(q^2+q+1)*T(2)\
+1/(q^2+q+1)*T(2,1)-q/(q^2+q+1) * T(1);;
gap> s1:=- (q+1/q)*e1;;
gap> s2:=-m*e2;;
gap> s1*s1+(q+q^-1)*s1;
0
gap> s2*s2+m*s2;
0
gap> s1*s2*s1;
(m+2mq^2+mq^4)/(1+2q+3q^2+2q^3+q^4)T()\
+(m+2mq^2+mq^4)/(1+2q+3q^2+2q^3+q^4)T(1)\
+(-m-2mq^2-mq^4)/(q+2q^2+3q^3+2q^4+q^5)T(2)+(-m-2mq^2\
-mq^4)/(q+2q^2+3q^3+2q^4+q^5)T(1,2)
gap>  (q+2*q^2+3*q^3+2*q^4+q^5)/(-m-2*m*q^2-m*q^4)*(s1*s2*s1);
-qT()-qT(1)+T(2)+T(1,2)
gap> 1/(1+q^2) * q*(1+q+q^2)*s1;
-qT()-qT(1)+T(2)+T(1,2)
So:
s1*s2*s1-(q+q^2+q^3)/(1+q^2)*(-m-2*m*q^2-m*q^4)/(q+2*q^2\
```

```
+3*q^3+2*q^4+q^5)*s1;
0
```

So we need

$$[m+1] = \frac{(q+q^2+q^3)(-[m]-2*[m]*q^2-[m]*q^4)}{(1+q^2)(q+2*q^2+3*q^3+2*q^4+q^5)}$$
$$= \frac{-[m]q(1+q+q^2)(1+2q^2+q^4)}{(1+q^2)q(1+2q+3q^2+2q^3+q^4)}$$
$$= \frac{-[m](1+q+q^2)q^2[2][2]}{q[2](1+q+q^2)(1+q+q^2)} = \frac{-[m]q[2]}{q([2]+1)} = \frac{-[m][2]}{[2]+1}.$$

The above shows that s_1 and s_2 satisfy the blob relations, next we check the images are linearly independent.

```
gap> s1*s2;
(-mq-mq^3)/(1+2q+3q^2+2q^3+q^4)T()+(-mq-mq^3)/(1+2q+3q^2\
+2q^3+q^4)T(1)+(-mq-mq^3)/(1+2q+3q^2+2q^3+q^4)T(2)\
+(-mq-mq^3)/(1+2q+3q^2+2q^3+q^4)T(1,2)+(m+mq+mq^2\
+mq^3)/(q+2q^2+3q^3+2q^4+q^5)T(2,1)+(m+mq+mq^2\
+mq^3)/(q+2q^2+3q^3+2q^4+q^5)T(1,2,1)
gap> s2*s1;
(-mq-mq^3)/(1+2q+3q^2+2q^3+q^4)T()+(m+mq^2)/(1+2q+3q^2\
+2q^3+q^4)T(1)+(m+mq^2)/(1+2q+3q^2+2q^3+q^4)T(2)\
+(-m-mq^2)/(q+2q^2+3q^3+2q^4+q^5)T(1,2)+(m+mq+mq^2\
+mq^3)/(1+2q+3q^2+2q^3+q^4)T(2,1)+(-m-mq-mq^2\
-mq^3)/(q+2q^2+3q^3+2q^4+q^5)T(1,2,1)
gap> s2*s1*s2;
(m^2q+m^2q^3)/(1+2q+3q^2+2q^3+q^4)T()\
+(-m^2-m^2q^2)/(1+2q+3q^2+2q^3+q^4)T(1)\
+(m^2q+m^2q^3)/(1+2q+3q^2+2q^3+q^4)T(2)\
+(-m^2-m^2q^2)/(1+2q+3q^2+2q^3+q^4)T(1,2)
```

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