State Extension and Detector Response for Quantum Fields in Curved Spacetime

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Abstract

The contents of this thesis involve two different areas of research pertaining to the dynamics of quantum fields on a classical and curved spacetime.

In the first, we analyse under which circumstances it is possible to extend a Hadamard State defined on a region of a Globally Hyperbolic Spacetime onto a larger region within the same spacetime, while keeping it Hadamard. We find that it is possible to do so, as long as we sacrifice knowledge of the state near the boundary of the original region. Our method can be employed in any locally covariant theory with a suitable notion of state space. This method can be made constructive for conformally ultrastatic spacetimes, employing a modified version of the *Alcubierre warp drive*. Furthermore, we were able to verify our ideas via numerical simulation in the context of 2-dimensional Minkowski spacetime.

The second area is linked to an experimental endeavour. We study how long it takes for an *Unruh-DeWitt detector* to thermalise while being accelerated in a uniform circular trajectory in a 3-dimensional spacetime. This is as opposed to the usual setting in the Unruh effect, where the detector is accelerated in uniform linear motion and the waiting time for thermalisation is infinite. However, the linear accelerations needed for this to happen are not attainable in a laboratory. So, the alternative of exploring finite times in a circular trajectory seems more approachable for experimental settings – in particular, within the *analogue gravity* research programme, which inspired this research. Under the assumption that the waiting time λ and the detector's energy gap E are inversely related, we found that the temperature is non-linear in E. Yet, further study is needed as if E and λ are inversely related, thermalisation is not obtained in finite time.

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Acknowledgments

Esta tesis está dedicada Elsa y Paco, mis queridos padres.
A Ewa, mi amada esposa.
Y a Olga, Elvira, Mario y José. Mis antepasados.

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which is crucial to this thesis. Furthermore, his proofreading was fundamental to bringing all of the forthcoming into an acceptable and understandable shape.

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Author's declaration

I declare that the work presented in this thesis, except where otherwise stated, is based on joint research with Prof. Chris Fewster carried out at the University of York and has not been submitted previously for any degree at this or any other university. Sources are acknowledged by explicit references.

— 1 —

Introduction

> Therefore Ilúvatar gave to their vision Being, and set it amid the Void, and the Secret Fire was sent to burn at the heart of the World; and it was called Eä.
>
> Valaquenta, J. R. R. Tolkien

During the 20th century, two major theoretical frameworks of physics were formulated: Quantum Field Theory (QFT) and General Relativity (GR). The first gave us a detailed insight into our description of matter and its interactions while the second helped us to understand the nature of spacetime and its relationship to gravitation.

However, QFT presents certain issues of its own and also when we try to make it compatible with GR. It is precisely this last problem that started the quest towards formulating a Quantum Theory of Gravity (QG), but all of the attempts into finding such theory have not been successful as of yet. Nonetheless, these attempts were not in vain since there is an intermediate point in common: semiclassical gravity, which is described by considering quantum fields propagating on curved spacetimes, giving origin to the so-called Quantum Field Theory in Curved Spacetime (QFTCS).

Given that the spacetime has a classical treatment, QFTCS can not provide a fundamental theory of the observable nature, as opposed to what is expected from QG. Nevertheless, QFTCS remains relevant when studying quantum phenomena taking place when the curvature of the spacetime is significant but not to the extent that QG is needed. The basic idea of QTFCS is to reconcile the postulates of GR with those of Quantum Field Theory (QFT).

The foundations of GR are easy to identify: the spacetime structure is encoded in a manifold equipped with a Lorentzian metric. Usually, this metric and any other field are dynamical objects whose evolution is governed by a well-posed Cauchy problem [43, 15] provided by the Einstein Field Equations (EFE). So, their dynamical evolution can be determined uniquely once initial data is provided on a Cauchy surface. However, it must be noted that often in QFTCS this metric is taken as given and might not necessarily be a solution to the EFE.

On the other hand, the foundations of QFT are not so easy to identify. A good way to illustrate this is when one considers the vacuum state in Minkowski Spacetime. Choosing this state relies on the notion of Poincaré covariance as the vacuum is defined as the state with maximal symmetry and lowest energy. Therefore, selecting a vacuum state in a generic spacetime seems like a rather difficult task, and as a matter of fact it can be shown that it is not possible [25, 30]. Hence, it seems that the concept of vacuum state is not a fundamental element of QFT. This goes against the usual dictum of QFT in Minkowski spacetime since in this particular setting the vacuum seems to be the bedrock of the theory.

Nonetheless, it is possible to find a class of states that have the same singular structure (i.e. high-energy behaviour) as the Minkowski vacuum state. This is the so-called Hadamard class and it allows us to extend the action of the state to an algebra containing polynomials of the field and its derivatives evaluated at a particular point–such as the energy density. Hadamard states are guaranteed to exist on Globally Hyperbolic Spacetimes, which are often deemed as physically reasonable spacetimes as they do not allow for naked singularities nor closed-timelike curves. In addition to this, Hadamard states guarantee the existence of Wick polynomials and hence of a well-defined perturbation theory. Because of the aforementioned, the Hadamard class is often deemed as *physically reasonable* and its members generalise the notion of a vacuum state to a generic curved spacetime.

Another considerable problem with QFT in Minkowski spacetime is the presence of infinities. In the original formulation of the theory, quantities such as the energy density of the field are infinite, which clearly indicates that something is wrong with the theory. Although the renormalisation programme has been successful at providing a method to extract physically meaningful and experimentally verifiable quantities from QFT predictions–with the agreement between theory and experiment being quite remarkable–it remains to check whether the theory is mathematically consistent or not. Checking for mathematical consistency is not merely a scientific ethics exercise, it is important because it might help us to identify the foundations of QFT.

The algebraic [42] and axiomatic [59] methods are certainly taking steps in the right direction and have made exceptional progress in recent years. The Algebraic QFT approach transfers the attention from the Hilbert space representations of a field to the algebraic relations satisfied by it. This is very convenient for QFTCS as in the usual treatment of QFT symmetries are used to select states, and every state gives rise to a representation. Hence, by avoiding the use of representations, the formulation is valid in a generic spacetime.

This thesis contains two different lines of research that will add to the QFTCS literature. The first studies under which circumstances a Hadamard state can be extended from one region onto a larger one in a globally hyperbolic spacetime. Our findings indicate that this is possible to do if one gives up knowledge of the state on a region near the boundary that can be made arbitrarily small. Our method is constructive as it involves building a commutative diagram in the category of globally hyperbolic spacetimes. We will give an explicit construction for the class of conformally ultrastatic spacetimes. Moreover, we carry out analytical and numerical computations to show that this can be done for the Minkowski vacuum state in two-dimensional spacetime, and we also study the properties of the extended state.

The second line of research studies the response of a detector moving in a uniform circular trajectory. As it is well-known, Unruh [62, 63] found that an observer undergoing uniform linear acceleration starts thermalising. As the acceleration needed to observe a temperature of $1K$ is of the order of $10^{20} m/s^2$, verifying this experimentally has proven rather difficult. That is why a new proposal arose from the University of Nottingham analogue gravity experimental programme: instead of linear acceleration let us put a detector in uniform circular motion and wait for it to thermalise. Then, the question shifts to: how long does one have to wait to get a thermal response in the detector? Our findings show that, assuming that the detector's energy gap E and the interaction time λ are related as $E\lambda = S_0$ (where S_0 is real constant), then one can not detect this response after some finite time. However, if one modifies this power law, then the response function becomes very similar to that of the finite time case. Their difference is a term $O(\lambda^{-4})$. In this sense, our work combines some of the results found in [26] and [9]. In the first, the waiting time is addressed while in the second a small energy gap is considered to approach the near-sonic limit when the speed of the detector is large. Moreover, we also obtained some evidence indicating that a deeper study of the parameter space might improve the size of the detected temperature, as the ranges obtained for it thus far are still beyond experimental verification.

Both lines of research are very different and have little in common, but they add to the knowledge of two fundamental notions for the theoretical and experimental understanding of QFTCS, that of a state and a detector. Moreover, it is worth mentioning that having such different research projects was beneficial to the author as it required learning a larger breadth of techniques and theory. Without a doubt, this will be useful for any future endeavours in research and allows to have more fluency in the large area of knowledge that is QFTCS.

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Preliminaries

The beginning is the most
important part of the work.

Plato, The Republic

In this section we will lay out all the framework needed to deal with globally hyperbolic spacetimes, quantum fields defined on them, and states associated to said fields. Unless indicated otherwise, these are our conventions for the rest of this text. $C_0^\infty(S)$ is the real vector space of compactly-supported and real-valued smooth functions over S. We denote Minkowski spacetime by $\mathbb{M}$. The Fourier transform of a function f is denoted by $\mathscr{F}[f]$ and its definition is $\mathscr{F}[f](u) = \int_{-\infty}^{\infty} dx\, e^{-iux} f(x)$. The symmetric tensor product for tensors of the same type is taken to be $A \otimes_s B = (A \otimes B + B \otimes A)/2$. Most of the geometrical definitions are taken from [53] and [65].

2.1 GLOBALLY HYPERBOLIC SPACETIMES

Definition 2.1.1. Consider a spacetime (M, g). Then, a curve is a smooth mapping $\gamma : I \longrightarrow M$ where $I \subseteq \mathbb{R}$ is an open interval. A curve is said to be causal if its tangent vector v is timelike ($g(v, v) > 0$) or null ($g(v, v) = 0$) at all the points of the curve. Also, p is said to be a future endpoint of γ if $p = \lim_{t\to\sup I} \gamma(t)$. A past endpoint is defined in a similar fashion but in this case using the infimum. A curve is said to be future (past) inextendible if it has no future (past) endpoints.

DEFINITION 2.1.2. The set of events that may be reached by a future directed causal curve emerging from $p \in M$ is called the causal future $J^+(p)$, more precisely,

$$J^+(p) = \{q \in M \mid \gamma(s) \text{ is a future directed causal curve with } \gamma(0) = p \text{ and } \gamma(1) = q\}.$$

By convention $p \in J^+(p)$. If $S \subset M$, its causal future is the union of the causal futures of its points, i.e. $J^+(S) := \cup_{p \in S} J^+(p)$. The causal past of a point $J^-(p)$ or a set $J^-(S)$ may be defined in a similar fashion.

DEFINITION 2.1.3. A subset S is said to be achronal if there are no points within it that can be joined by a timelike curve.

DEFINITION 2.1.4. Consider a spacetime (M, g) and let $S \subset M$ be an achronal subset. Then, its future Cauchy development $D^+_g(S)$ with respect to the metric g is given by

$$D^+_g(S) = \{p \in M | \text{ Any past inextendible causal curve through } p \text{ intersects } S.\}$$

To define its past Cauchy development $D^-_g(S)$ just exchange past for future in the definition. Moreover, the Cauchy development $D_g(S)$ is defined as their union, i.e. $D_g(S) = D^+_g(S) \cup D^-_g(S)$

DEFINITION 2.1.5. A subset S in a spacetime M is said to be causally convex in M if every causal curve with endpoints in S has image entirely contained in O as well. Henceforth, we will denote by $O(M)$ the set of all open causally convex subsets of M.

It is worth mentioning that in [49][Prop 3.43] it is proven that if a set S is causally convex, then $D^{\pm}_g(S)$ (and hence $D_g(S)$) will be causally convex as well.

DEFINITION 2.1.6. If Σ is a closed, achronal set in a spacetime (M, g) such that $D_g(\Sigma) = M$, then it is called a Cauchy surface. If a spacetime has a Cauchy surface, then we say that it is globally hyperbolic.

Definition 2.1.7. We say that diffeomorphism $\alpha : M \to M$ is an isometry if it maps the metric into itself. Hence, $\alpha_* g = g$ for every $p \in M$. An upshot of this is that $\alpha_* : T_p M \to T_{\alpha(p)} M$ will preserve inner products, so for $u,\ v \in T_p M$ we will have

$$g(u, v)|_p = \alpha_* g(\alpha_* u, \alpha_* v)|_{\alpha(p)} = g(\alpha_* u, \alpha_* v)|_{\alpha(p)}.$$

If the one-parameter group of diffeomorphisms α_τ generated by a vector ξ corresponds to a group of isometries, then we will refer to ξ as a Killing vector field.

Intuitively speaking, a Killing vector field ξ keeps distances intact in a rod as long as said rod is moving along the direction of ξ. The Lie derivative £ of the metric g vanishes along ξ, as it can be seen from

$$\pounds_\xi g = \lim_{\tau \to 0} \frac{1}{\tau}(g - \alpha_* g) = 0. \tag{2.1}$$

From this and the identity $\pounds_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$ we can deduce the so-called Killing equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0.$$

Definition 2.1.8. A Killing horizon is a null hypersurface in M where a Killing vector becomes null. See [44] for further reference.

For the sake of brevity, from now on we will refer to a Killing horizon simply as a horizon.

2.2 Quantum Fields and States

The algebraic approach to QFT in curved spacetime is powerful since it allows us to study the algebraic relations of quantum fields without referencing any specific Hilbert space representation. For the purposes of this thesis, we shall consider the simplest model available: the linear Klein-Gordon scalar field ϕ on a globally hyperbolic spacetime M. The classical field ϕ satisfies the equation

$$P_g \phi := \left(\Box_g + m^2 + \xi R_g \right) \phi = 0 \tag{2.2}$$

with mass $m \geq 0$, scalar curvature R_g, coupling ξ and d'Alembert operator $\Box_g$.

DEFINITION 2.2.1. Let us denote by $\mathscr{A}(M,g)$ the unital $*$-algebra over $\mathbb{C}$ that is generated by the following choice of generators and relations. The generators are the field operators $\hat{\phi}(f)$, where the functions $f,\ g \in C_0^\infty(M)$ act as labels. For these generators the following relations hold:

$\mathbb{R}$-linearity: $\hat{\phi}(af+bg) = a\hat{\phi}(f) + b\hat{\phi}(g)$.

Hermiticity: $\hat{\phi}(f)^* = \hat{\phi}(f)$.

Solution: $\hat{\phi}(P_g f) = 0$.

Commutation relations: $[\hat{\phi}(f), \hat{\phi}(g)] = iE_M(f,g)\mathbf{1}_{\mathscr{A}(M)}$.

The bilinear functional $E_M(f,g)$ is formed from the fundamental solution $E_M := E_M^- - E_M^+$ where $E_M^\mp$ is the advanced/retarded Green operator – c.f. [13][Chapter 5, Remark 5.2.2]. They satisfy the relations $P_g E_M^\pm f = 0 = E_M^\pm P_g f$ and thus $P_g E_M f = 0 = E_M P_g f$. Its form is given by $E_M(f,g) = \int_M \mathrm{dvol}_M f(E_M g)$.

The algebra $\mathscr{A}(M,g)$ is also known as the Canonical Commutation Relations (CCR) algebra and it contains a lot of elements; those which are Hermitian are said to be the *elementary observables* of our theory.

DEFINITION 2.2.2. (Timeslice axiom) If O is a causally convex set in (M,g), we will denote by $\mathscr{A}(M,g;O)$ the subalgebra of $\mathscr{A}(M,g)$ generated by $\hat{\phi}(f)$ with $f \in C_0^\infty(O)$. Then, if O contains a Cauchy surface of (M,g) there exists a unital $*$-algebra isomorphism between $\mathscr{A}(M,g)$ and $\mathscr{A}(M,g;O)$. Furthermore, in particular, we have

$$\mathscr{A}(M,g;O) = \mathscr{A}(M,g;D_g(O)). \tag{2.3}$$

So far, we have defined the field stressing the importance of its algebraic relations. Next, we will define states on the algebra of the fields.

Using the GNS construction [42] we can build a Hilbert space for each given state and therefore obtain representations for the field algebra, yielding the usual understanding of QFT.

DEFINITION 2.2.3. A state ω is a positive normalised linear functional ω : $\mathscr{A}(M,g) \to \mathbb{C}$, i.e. ω is an element of the dual space of the algebra $\mathscr{A}(M,g)^*$ such that $\omega(\mathbf{1}) = 1$ and the positivity condition $\omega(A^*A) \geq 0$ is fulfilled for $A \in \mathscr{A}(M,g)$. Also, if O is a causally convex subset of (M,g), we introduce the following notation

$$\omega_O := \omega|_{\mathcal{A}(M,g;O)} \tag{2.4}$$

The action of a state on an element of the algebra will be known as soon as we determine how it behaves for each monomial of the field, which introduces the notion of a n-point function.

DEFINITION 2.2.4. Let $f_i \in C_0^\infty(M)$ for natural number i such that $0 < i \leq n$. The n-point function W^n of a state ω is defined by

$$W^n(f_1,\ldots,f_n) := \omega(\hat{\phi}(f_1)\cdots\hat{\phi}(f_n)).$$

It should be stressed that if for all $n \in \mathbb{N}$, two states have the same n-point functions, then they are identical. Also, from Definition 2.2.1 it is easy to see that the n-point functions are n-fold solutions to the Klein-Gordon equation, that is

$$W^n(P_g f_1,\ldots,f_n) = W^n(f_1,P_g f_2,\ldots,f_n) = W^n(f_1,\ldots,P_g f_n) = 0.$$

If we assume that W^n is continuous with respect to the test-function topology on $C_0^\infty(M)$, then, in virtue of the Schwartz kernel theorem we can write W^n using its distributional kernel $W^n \in \mathcal{D}'(M^{\times n})$, where $\mathcal{D}'(U)$ denotes the space of all distributions on U:

$$W^n(f_1,\ldots,f_n) = \int_{M^{\times n}} W^n(p_1,\ldots,p_n) f_1(p_1)\cdots f_n(p_n) \mathrm{dvol}_{M^{\times n}}.$$

2.3 HADAMARD STATES

Clearly, Definition 2.2.3 is rather general, since there are many states satisfying it and there is not a clear way to select a physically meaningful one. Even in Minkowski spacetime, there are many states of physical interest such as Poincaré invariant vacuum, thermal and coherent states. However, if one reduces the scope of the search to that of states that make physical sense, then a class of states emerges naturally: the Hadamard class [21]. Let us be more precise about what we mean when we say *states that make physical sense*. We are referring to states in an arbitrary spacetime whose ultraviolet behaviour is similar to the one present in states with finite energy density in Minkowski spacetime which share a common singularity structure with the vacuum. In this sense, we say that these states generalise the notion of the Minkowski vacuum state to a generic spacetime.

Recall that to obtain a finite energy density for a state in Minkowski spacetime one has to build Wick polynomials. The usual approach is to reorder the annihilation and creation operators to get a well-defined expression. However, in order to do so, one must choose a vacuum state and this is not possible in a covariant setting [30, 25]. So, if we ought to develop a fully covariant theory we need to use a different approach where instead of reordering, we subtract divergences. To see that this is effectively what one does in Minkowski spacetime when building Wick polynomials consider the following example: the distributional kernel two-point function $W_0^2(x, y)$ for the Minkowski vacuum state ω_0 has a well-known singular structure [37, 35] (see [12] for a precise statement). To define a regular object one usually introduces the Wick polynomial as

$$:\hat{\phi}(x)\hat{\phi}(y):=\hat{\phi}(x)\hat{\phi}(y)-W_0^2(x,y)\mathbf{1}_{\mathscr{A}(M)}. \tag{2.5}$$

So, this entails that $\omega_0(:\hat{\phi}(x)\hat{\phi}(y):)$ will have no singularities. To illustrate this, we will consider interactions which at the same time will allow us to have a look at the perturbative treatment of the theory. One of the simplest examples is the scalar field in Minkowski spacetime with the ϕ^4 potential. Within this setting, we need to calculate certain products of polynomials, such as $:\hat{\phi}(x)^2::\hat{\phi}(y)^2:$. To do this we make use of the Wick's theorem(see [13][Section 5.3.1]), which yields

$$:\hat{\phi}(x)^2::\hat{\phi}(y)^2:=:\hat{\phi}(x)^2\hat{\phi}(y)^2:+4:\hat{\phi}(x)\hat{\phi}(y):W_0^2(x,y)+2W_0^2(x,y)^2.$$

In order to have a well-defined perturbation theory, the expression above needs to form an algebra where the product is given by Wick's theorem. Nevertheless, $W_0^2(x, y)$ exhibits a singular behaviour for null-related x and y since it can be written as

$$W_0^2(x,y) = \frac{1}{4\pi^2}\frac{1}{\sigma(x,y) - 2i\varepsilon 0^+}$$
$$= \lim_{\varepsilon\to 0^+}\int \frac{d^4k}{(2\pi)^3}\Theta(k_0)\delta(k^2)e^{-ik\cdot(x-y)}e^{-\varepsilon k_0}, \tag{2.6}$$

where $\sigma(x,y) = t^2 - (\vec{x} - \vec{y})^2$, the quantity t is the time difference at said points and Θ is the Heaviside step function. Next, we will check if pointwise products of this two-point function happen to be a distribution. This is not the case in general but is needed for Wick's theorem to make sense. If we observe the last integral in (2.6) we realise that it only has support when $\Theta(k_0)\delta(k^2) \neq 0$, which is precisely the *future light cone*. Furthermore, calculating $W_0^2(x,y)^2$–after using the convolution theorem– yields

$$W_0^2(x,y)^2$$
$$= \lim_{\varepsilon\to 0^+}\int \frac{d^4k}{(2\pi)^6}\int d^4q\ \Theta(q_0)\delta(q^2)\Theta(k_0 - q_0)\delta((k-q)^2)e^{-ik\cdot(x-y)}e^{-\varepsilon k_0}. \tag{2.7}$$

Due to the oscillatory behaviour in k, we only need to check whether the integral over q converges or not. Thus, we need to verify that the integrand is rapidly decreasing in q. To this end, we will follow closely the argument given in [7]. Fix any k and consider a large q in the sense of the Euclidean norm for $\mathbb{R}^4$. Then, at some point, we will have $\delta(q^2) = 0$ and the Heaviside function $\Theta(k_0 - q_0)$ will also vanish as we will have $k_0 - q_0 < 0$ for large q_0. Hence, we conclude that the integral must be rapidly decreasing in q. In other words, due to the high-energy behaviour (large q) of Minkowski's vacuum state, the integral above is well defined which implies that the Wick theorem yields a finite answer and therefore, perturbation theory is consistent.

This nice behaviour at high energies exhibited by the vacuum state in Minkowski is enough to guarantee that all of the products that may arise due to normal-ordering are well-defined. Thus, it seems reasonable to ask for this same behaviour for any other physical state in Minkowski, not just the vacuum. This is the so-called *Hadamard condition*, which dictates that any state that obeys it must have the same singular structure as the Minkowski vacuum.

As a closing remark before defining the Hadamard condition in a more formal way, we would like to point out the importance of having a well-defined notion of objects such as $:\hat{\phi}(x)^2:$ in curved spacetimes. Consider the semi-classical Einstein Field Equations (SEFE), which read

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi\omega(T_{\mu\nu}),$$

where $R_{\mu\nu}$ and R are the *Ricci tensor* and *scalar*, respectively. The left-hand side (the geometric terms) is considered to be classical. On the other hand, the right-hand side contains the stress-energy tensor $T_{\mu\nu}$ which will contain our quantum fields and in many cases (such as the scalar field) involves terms that are quadratic in the field and its derivatives. If one chooses a state that satisfies the Hadamard condition, then $\omega(T_{\mu\nu})$ can be regularised which in turn implies that the SEFE will be well-defined. Note that in particular, the energy density contains terms that are quadratic in the field. To promote the procedure sketched above to a covariant setting, instead of subtracting the two-point function of the vacuum state we subtract a Hadamard parametrix (properly explained in Definition 2.3.3).

Finally, it is worth mentioning that Radzikowski [54, 55] reformulated the Hadamard condition in terms of microlocal analysis, which allows one to prove results for Hadamard states on general backgrounds. We shall also have a brief overlook of this reformulation. In their famous paper [44] Kay and Wald introduced a rigorous definition of Hadamard states specifying their short-scale behaviour via the notion of a global Hadamard parametrix. Their definition relies on the possibility of assigning a unique geodesic between two points, which is known to be possible if said points lie in a convex normal neighbourhood.

However, it was noted recently that this notion might run into problems if the points are spacelike separated as it is possible to find many convex normal neighbourhoods containing them. Moretti addressed and amended this technical issue in [52]. Small but significant changes need to be made to the original definition and so, fortunately, it seems that all of the results proven before still hold.

Definition 2.3.1. Let (M, g) be a spacetime. A strong convex covering of M is a covering C of M made of normal convex open sets such that $C \cap C'$ is normal convex if $C,\ C' \in \mathcal{C}$ and $C \cap C' \neq \emptyset$.

DEFINITION 2.3.2. Let $\mathcal{C}$ be a strong convex covering of (M, g) and $\mathcal{A}$ be an open neighbourhood of the diagonal $\Delta_M := \{(p, p)|p \in M\}$ defined as

$$\mathcal{A} := \bigcup_{C \in \mathcal{C}} C \times C. \tag{2.8}$$

Then, the signed squared geodesic distance $\sigma \in C^\infty(\mathcal{A})$ is

$$\sigma(p, p') := g(\dot{\gamma_{pq}}(0), \dot{\gamma_{pq}}(0))|_p = \pm \left(\int_0^1 \sqrt{|g(\dot{\gamma_{pq}}(t), \dot{\gamma_{pq}}(t))|dt} \right)^2 \tag{2.9}$$

for $p,\ q \in \mathcal{A}$ where $\gamma_{pq} : [0, 1] \to M$ is the unique geodesic segment between p and q.

DEFINITION 2.3.3. Consider a four-dimensional spacetime (M, g), let $\mathcal{C}$ be a strong convex covering of it, and construct $\mathcal{A}$ as in 2.3.2. For any natural number n, let $v_n \in C^\infty(\mathcal{A})$ and define $t(p, p') := T(p) - T(p')$, where T is a smooth global time function increasing towards the future. Furthermore, let $u \in C^\infty(\mathcal{A})$ be a function such that $u(p, p) = 1$, this function is known in the literature as the Van Vleck-Morette determinant. Then, the global Hadamard parametrix of order N is defined via the following boundary-valued distribution

$$\begin{aligned} H_g^{(N)}(p, p') = & \frac{1}{4\pi^2} \frac{u(p, p')}{\sigma(p, p') + 2it0^+} \\ & + \frac{1}{4\pi^2} \sum_{n=0}^{N} v_n(p, p') \sigma(p, p')^n \log \left[\frac{\sigma(p, p') + 2it0^+}{\lambda^2} \right] \end{aligned}$$

where $\lambda > 0$ is a length scale and the branch cut of the logarithm is taken along the negative real axis.

The Hadamard parametrix is a bisolution to the Klein-Gordon equation, and so the u and v_n functions are determined by the local geometry. In particular, the v_n functions in the definition above can be calculated from recursive integration of $\sigma(p, p')$ (c.f. Appendix A of [51]). It must be noted that if we fix p', then $H_g^{(N)}(p, p')$ is an approximate solution to the Klein-Gordon equation (3.1). The error in this approximation is of order σ^N. So, in order to arrive at a true parametrix one has to take the limit $N \to \infty$. This can be done. However, as we will see in a moment some caution is needed.

Let $\chi : \mathbb{R} \to [0,1]$ be a smooth function so that $\chi(x) \equiv 1$ for $|x| \leq 1/2$ and $\chi \equiv 0$ for $|x| \geq 1$. Then, as noted in [33][Sect. 4.3], there exists a strictly increasing sequence of positive numbers c_n such that

$$\sum_{n=0}^{\infty} v_n(p,p')\sigma(p,p')^n \chi(c_n\sigma(p,p'))$$

converges uniformly to a function $v(p,p') \in C^\infty(\mathcal{A})$.

DEFINITION 2.3.4. With this, we can define the global Hadamard parametrix as the following boundary-valued distribution

$$H_g(p,p') = \frac{1}{4\pi^2}\left(\frac{u(p,p')}{\sigma(p,p') + 2it0^+} + v(p,p')\log\left[\frac{\sigma(p,p') + 2it0^+}{\lambda^2}\right]\right),$$

where λ and t are as in Definition 2.3.3.

As noted above, the Hadamard parametrix is a distributional bisolution to the Klein-Gordon equation – up to smooth functions of both arguments. That is, there exists $s \in C^\infty(\mathcal{A})$ such that for all $f,\ h \in C_0^\infty(M)$ we have

$$\begin{aligned}
&\int_{\mathcal{A}} H_g(p,p')(P_g f)(p)h(p')\mathrm{dvol}_{M\times M} \\
&= \int_{\mathcal{A}} H_g(p,p')f(p)(P_g h)(p')\mathrm{dvol}_{M\times M} \\
&= \int_{\mathcal{A}} s(p,p')f(p)h(p')\mathrm{dvol}_{M\times M}.
\end{aligned}$$

where P_g is the differential operator associated to the general Klein-Gordon equation as in (2.2). The volume form $\mathrm{dvol}_{M\times M}$ is defined in $M \times M$ and each spacetime is equipped with the metric g.

DEFINITION 2.3.5. A state ω is said to have the Hadamard form in (M,g) if for $f \in C^\infty(M \times M)$, its two point W^2 function can be written as

$$W^2(p,p') = H_g(p,p') + f(p,p').$$

It must be noted that the parametrix is completely determined by the local geometry whereas f will depend on the specific choice of state and that is why this defines a class of states instead of a particular state. Note that $W^2(p, p') - H_g(p, p') \in C^\infty(M \times M)$ and therefore this procedure mimics the divergence removal that takes place in the Wick polynomials in (2.5). Retaking the discussion we had for the two-point function for Minkowski spacetime in (2.7) leads us to realise that in order to have a well-defined perturbation theory, we need rapidly decreasing Fourier integrals. This implies that we need to consider the high-energy part of the spectrum and see whether it satisfies the Hadamard condition or not.

There is a rather remarkable reformulation that allows us to characterise this high-energy behaviour in a coordinate-independent way which can be done by making use of tools from microlocal analysis, namely the *wavefront set*. We will proceed to explain these notions following closely the treatment in [50]. The wavefront set tells us how a distribution fails to be smooth via the decay of the Fourier transform of the product of this distribution with a test function. To see this, recall the Fourier transform for any $f \in C_0^\infty(\mathbb{R}^n)$ has rapid decay, i,e. for every N there exists a constant C_N such that $|\mathscr{F}[f](k)| \leq \frac{C_N}{(1+|k|^N)}$ for sufficiently large k.

Then if we take a distribution u and multiply it by a test function f with $f(x_0) \neq 0$ for $x_0 \in \mathrm{supp}(f)$ we will obtain a compactly supported distribution fu. If the resulting distribution is smooth, then it will have a Fourier transform $\mathscr{F}[fu]$ with rapid decay; failure to do so in a neighbourhood of x_0 will indicate in which directions k of Fourier space we do not have rapid decay. Summarising, the wavefront set tells us where a distribution is singular and in which directions we should expect this singular behaviour to propagate.

Definition 2.3.6. Let $k_0 \in \mathbb{R}^n$ for $n \in \mathbb{N}$. We say that a neighbourhood O of k_0 is conic if $k \in O$ implies $\lambda k \in O$ for all $\lambda > 0$. A regular directed point for the distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ is the pair $(x_0, k_0) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$ that satisfies: there exists a smooth compactly supported function f with $f(x_0) \neq 0$ and a fixed C_N such that for k in a conic neighbourhood of k_0, the following holds

$$|\mathscr{F}[fu](k)| \leq \frac{C_N}{(1 + |k|)^N}, \qquad N \in \mathbb{N}, \qquad k \in \mathrm{Con}(k_0).$$

The wavefront set $WF(u)$ is the complement in $\mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$ of the set all of the regular directed points of u.

Although we defined the wavefront set for $\mathbb{R}^n$, this definition can be generalised to smooth manifolds [61], and the wavefront set becomes a subset of the cotangent bundle of the manifold, i.e. $WF(u) \subset T^*M\backslash\{0\}$ where $\{0\}$ denotes the zero section. The wavefront set allows us to formulate the Hadamard condition without making any allusion to coordinates. A diffeomorphism $\psi : M \to N$ defines a pull-back $\psi^*u \in \mathcal{D}'(N)$ via $\psi^*u(f) := u(\psi_* f)$ for all $f \in C_0^\infty(M)$. The wavefront set behaves nicely under pull-backs as it satisfies

$$WF(\psi^*u) = \psi^*WF(u) := \{(\psi^{-1}(x), \psi^*k) | (x,k) \in WF(u)\},$$

where ψ^*k is to be understood in the usual sense of pull-back of covectors.

DEFINITION 2.3.7. Let ω be a state on the algebra of quantum linear scalar fields. We say that the state satisfies the *Hadamard condition* if its two-point function has the following wavefront set

$$WF(W^2) = \{(x, y; k_x, -k_y) \in T^*M^2\backslash\{0\} \mid (x, k_x) \sim (y, k_y),\ k_x \triangleright 0\},$$

where $(x, k_x) \sim (y, k_y)$ indicates that x and y are connected by a null geodesic γ such that k_x is coparallel (in the sense of parallel transport) and cotangent to γ at x, k_y is its parallel transport from x to y along γ and $k_x \triangleright 0$ denotes that the covector is future directed.

The next result due to Radzikowski [54] establishes a connection between Hadamard condition and form (see Definition 2.3.5).

THEOREM 2.3.1 (Radzikowski). *Let W^2 be the two-point function for a state ω, then W^2 satisfies the Hadamard condition if and only if it is of Hadamard form.*

We would like to point out that Hadamard states may be defined for any globally hyperbolic spacetime, which brings us to a full circle since they are considered to be physically reasonable spacetimes. Some examples of Hadamard states include all vacuum and thermal states on ultrastatic spacetimes, and asymptotic vacuum and thermal states in Friedmann-Lemaître-Robertson-Walker spacetimes [29].

2.4 KMS STATES

Thermodynamic equilibrium states are of paramount importance in quantum theory. The Gibbs states are an example of these equilibrium states and the textbook literature usually tells us that they are characterised by a density matrix $\rho = e^{-\beta H}/Z$, where H is the Hamiltonian associated with the system under consideration and Z is a normalisation constant. If the Cauchy surface is non-compact then the spectrum of H is usually continuous, so H can not be a trace class operator.

The KMS states were created to generalise the notion of a thermodynamic equilibrium state –henceforth, we will refer to it as a thermal state– and overcome some technical challenges as the one we just mentioned. If the spacetime under consideration happens to have a complete timelike Killing vector field, then it is possible to define a thermal state whose time evolution will be associated to this vector field.

Following the philosophy adopted throughout this text, we will seek to define this notion without making any reference to any Hilbert space representation. The KMS states obey a condition that precisely does this, as the only elements needed to formulate it are the algebra, isometries of the spacetime and the expectation value provided by a state ω. It is worth mentioning that the *KMS condition* is usually stated for C^*-algebras [41], so we will adapt it to the context of the algebra $\mathcal{A}(M, g)$ in Definition 2.2.1 .

Definition 2.4.1 (KMS condition)**.** Let M be a globally hyperbolic spacetime and $\alpha_\tau : M \to M$ be a 1-parameter group of isometries generated by a timelike Killing vector field (c.f Definition 2.1.7). Then, the action on the algebra $\mathscr{A}(M, g)$ is defined as $\alpha_\tau(\hat{\phi}(f_1) \cdots \hat{\phi}(f_n)) = \hat{\phi}(f_{1,\tau}) \cdots \hat{\phi}(f_{n,\tau})$ where $f_\tau(p) = f(\alpha_{-\tau}(p))$. Since the isometries are a 1-parameter group, we have $\alpha_\tau \circ \alpha_\sigma = \alpha_{\tau+\sigma}$ and it can be shown that they are automorphisms of $\mathscr{A}(M, g)$. For our purposes, it is just necessary to introduce the KMS condition for the two-point function. We say that a state ω is a KMS state with inverse temperature β if two conditions are met:

1. For the observables $a, b \in \mathscr{A}(M)$, the function

$$F_{a,b}(\tau_1, \tau_2) = \omega(\alpha_{\tau_1}(a)\alpha_{\tau_2}(b))$$

has an analytic continuation to the strip $S = \{(\zeta_1, \zeta_2) \in \mathbb{C}^2 | 0 < \mathrm{Im}(\zeta_2) - \mathrm{Im}(\zeta_1) < \beta\}$.

2. This function is bounded and continuous at the boundary of the strip. Also, on the boundary $F_{a,b}(\tau_1, \tau_2 + i\beta) = F_{b,a}(\tau_2, \tau_1)$.

As we mentioned before, the usual thermal equilibrium states such as the Gibbs state obey the KMS condition (in its usual formulation for C^*-algebras). As a matter of fact, if the chosen algebra happens to be a matrix algebra, then a KMS state will turn out to be a Gibbs state. Also, as the time evolution of said state is defined by a Killing vector field, the state will be static with respect to it – that is, it will be invariant with respect to it.

In general, showing that a state is KMS involves calculations that can get somewhat cumbersome. However, it is a criterion that is deeply connected to symmetries, as a consequence of the definition is that a KMS state will have a *time translation invariant* two-point function. In consequence, there is an alternative way to show that it satisfies the KMS condition:

DEFINITION 2.4.2. We say that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the detailed balance condition at temperature $T > 0$ (in units where the Boltzmann constant equals one) if

$$f(-u) = e^{u/T} f(u).$$

Then, a state will satisfy the KMS condition at $\beta = 1/T$ if and only if the Fourier transform of its two-point function restricted to a time-like worldline satisfies the detailed balance condition at temperature T (c.f. [26][Prop. 4.3]).

2.5 LOCALLY COVARIANT QUANTUM FIELD THEORY

In this section, we shall outline the basic features of the Locally Covariant formulation of a Quantum Field Theory (LCQFT) by following the canonical references [29, 32]. Our exposition will be in close agreement to that found in [29], where the scalar field is used as an example that highlights the main notions of this framework. The main idea of LCQFT is to think of a QFT as a functor $\mathscr{A} : \mathbf{Loc} \to \mathbf{Alg}$ between the category of globally hyperbolic spacetimes **Loc** and the category of unital $*$-algebras **Alg**, which will be defined below; *a grosso modo*, this relates spacetimes and their isometries to algebras and their homomorphisms. More precisely, the categories are:

Loc The **objects** are globally hyperbolic spacetimes $\mathbf{M} = (M, g, \mathfrak{o}, \mathfrak{t})$ where M is a manifold of dimension n, equipped with metric g, orientation $\mathfrak{o}$ and a time orientation $\mathfrak{t}$. By orientation, we refer to choosing one of the components of the set of non-zero smooth n-forms on M, whereas a time orientation is choosing a component of the set of non-zero timelike 1-forms on M. The **morphisms** are smooth isometric embeddings[1] with causally convex image (c.f. Def. 2.1.5), that preserve both the orientation and the time-orientation.

Alg The **objects** are unital $*$-algebras. The **morphisms** are the unit-preserving $*$-monomorphisms.

At this stage, this description might seem too abstract, so let us elaborate further on why these categories were chosen in such a way. To this end, we will modify slightly the notation in Definition 2.2.1 for the remainder of this section and Chapter 4. Basically, one does a relabelling for the algebra $\mathscr{A}(M) \to \mathscr{A}(\mathbf{M})$. This induces corresponding relabellings for the field $\hat{\phi} \to \hat{\phi}_{\mathbf{M}}$, the Klein-Gordon operator $P_g \to P_{\mathbf{M}}$ and fundamental solution $E_M \to E_{\mathbf{M}}$.

The aim of these categories is to encode the principle of locality, that is, that causal relations between points in $\mathbf{M}$ should be preserved under the image $\psi(\mathbf{M})$ (with respect to $\mathbf{N}$) of a map $\psi : \mathbf{M} \to \mathbf{N}$. Of course this is not the case in general and because of this we incorporated very specific choices in the definition of our categories. To gain some insight into this, let us consider the CCR algebras $\mathscr{A}(\mathbf{M})$ and $\mathscr{A}(\mathbf{N})$. As test functions label the generators of these algebras, we would like to push forward them from $C_0^\infty(\mathbf{M})$ to $C_0^\infty(\mathbf{N})$. To do so, we need ψ to be

[1] An isometric embedding between manifolds M and M' is a smooth embedding $\psi : M \to M'$ which preserves the metric in the sense that g is equal to the pullback of g' by ψ, i.e. $g = \psi^* g'$.

smoothly invertible, so that we can define $(\psi_* f)(p) := f(\psi^{-1}(p))$ for $p \in \psi(\mathbf{M})$ and $(\psi_* f)(p) = 0$ otherwise. So, in algebraic terms for $f \in C_0^\infty(\mathbf{M})$, we define

$$\mathscr{A}(\psi)\hat{\phi}_{\mathbf{M}}(f) := \hat{\phi}_{\mathbf{N}}(\psi_* f), \tag{2.10}$$

which only makes sense if the algebraic relations in Definition 2.2.1 hold for both $\mathscr{A}(\mathbf{M})$ and $\mathscr{A}(\mathbf{N})$. Clearly, linearity and hermiticity follow immediately. Nevertheless, the solution relation along with (2.10) yields $0 = \mathscr{A}(\psi)\hat{\phi}_{\mathbf{M}}(P_{\mathbf{M}} f) = \hat{\phi}_{\mathbf{N}}(\psi_* P_{\mathbf{M}} f)$, and so, for every $f \in C_0^\infty(\mathbf{M})$ we must have $\hat{\phi}_{\mathbf{N}}(\psi_* P_{\mathbf{M}} f) = 0$. In other words, we need $\psi_* P_{\mathbf{M}} f = P_{\mathbf{N}} \psi_* f$, which is indeed the case as by assumption ψ is an isometry. The commutation relations hold at a formal level as the only morphisms allowed in **Alg** are unit preserving $*$-homomorphisms, from this we obtain

$$\begin{aligned} iE_{\mathbf{N}}(\psi_* f, \psi_* g)\mathbf{1}_{\mathscr{A}(\mathbf{N})} &= [\hat{\phi}_{\mathbf{N}}(\psi_* f), \hat{\phi}_{\mathbf{N}}(\psi_* g)] = [\mathscr{A}(\psi)\hat{\phi}_{\mathbf{M}}(f), \mathscr{A}(\psi)\hat{\phi}_{\mathbf{M}}(g)] \\ &= \mathscr{A}(\psi)[\hat{\phi}_{\mathbf{M}}(f), \hat{\phi}_{\mathbf{M}}(g)] = iE_{\mathbf{M}}(f, g)\mathbf{1}_{\mathscr{A}(\mathbf{N})}. \end{aligned}$$

From this we may conclude that $E_{\mathbf{N}}(\psi_* f, \psi_* g) = E_{\mathbf{M}}(f, g)$ for all smooth f and g compactly supported in $\mathbf{M}$ whereupon we find that

$$E_{\mathbf{M}} = (\psi \times \psi)^* E_{\mathbf{N}}. \tag{2.11}$$

Given that we know how the wavefront set (see Definition 2.3.6) behaves under pull-back, we can conclude that $WF(E_{\mathbf{M}}) \subset (\psi \times \psi)^* WF(E_{\mathbf{N}})$. Since the wavefront set of the fundamental solution contains null-covectors sitting at future or past related points, ψ must necessarily map null-covectors from $\mathbf{N}$ to $\mathbf{M}$ and preserve time-orientation, which is granted since ψ is an isometry.

So far, we have motivated the need for specific choices of objects and morphisms in our categories. We see that if we want to map test functions from $\mathbf{M}$ to $\mathbf{N}$, while preserving locality and fulfilling the CCR relations, then our choices for ψ are rather restricted. Not only this but also we can not just simply use any algebra morphism, as we need them to be unit-preserving. It must be noted that in the discussion above we used the scalar field to motivate and illustrate the main ideas. However, these categories are valid in much more general settings such as the Maxwell or Dirac fields. This explains why we have specified the categories above in such a way, although it must be mentioned that this is not specific to the scalar field, as it just served as an example to illustrate the main points of our definitions.

Finally, to account for the need for the image of ψ being causally convex, recall that the singularities of both $E_{\mathbf{M}}$ and $E_{\mathbf{N}}$ travel along geodesics, so if we want (2.11) to hold, we will need every causal curve in $\mathbf{N}$ with endpoints in $\psi(\mathbf{M})$ to also lie within $\psi(\mathbf{M})$. A nice feature emerges from our categorical approach to QFT, that is, *functoriality*. Consider we have a spacetime $\mathbf{L}$ such that $\tau : \mathbf{L} \to \mathbf{M}$, then if we use ψ as defined above, we find $\psi \circ \tau : \mathbf{L} \to \mathbf{N}$. Inserting this into the functor relation yields

$$\mathscr{A}(\psi \circ \tau)\phi_{\mathbf{L}}(f) = \phi_{\mathbf{N}}((\psi \circ \tau)_* f) = \mathscr{A}(\psi)\phi_{\mathbf{M}}(\tau_* f)) = \mathscr{A}(\psi)(\mathscr{A}(\tau)\phi_{\mathbf{L}}(f)),$$

since this is for an arbitrary generator, we conclude

$$\mathscr{A}(\psi \circ \tau) = \mathscr{A}(\psi) \circ \mathscr{A}(\tau). \tag{2.12}$$

Now that we have made acquainted ourselves with the categories, we can be more specific about what it means for a QFT to be a functor. However, before doing so, we need to introduce some definitions that will allow us to choose useful classes of objects and morphisms in the categories defined above.

Definition 2.5.1. Let O be a causally convex subset of $\mathbf{M} \in \mathbf{Loc}$. Then, a region is the restriction $\mathbf{M}|_O = (O, g|_O, \mathfrak{o}|_O, \mathfrak{t}|_O)$. We note that since it is also an object in **Loc**, it is a spacetime in its own right. Furthermore, the inclusion $O \hookrightarrow M$ induces the morphism $\mathbf{M}|_O \xrightarrow{\iota_{(M;O)}} \mathbf{M}$ which embeds the region into the original spacetime.

Definition 2.5.2 (Cauchy morphism)**.** We say that a morphism $\psi : \mathbf{M} \to \mathbf{N}$ in **Loc** is Cauchy if its image $\psi(M)$ contains a Cauchy surface for N.

Definition 2.5.3 (Kinematic algebra)**.** Let $\iota_{M;O} : \mathbf{M}\big|_O \to \mathbf{M} \in \mathbf{Loc}$ be the inclusion morphism. Note that $\mathbf{M}|_O = (O, g|_O, \mathfrak{o}|_O, \mathfrak{t}|_O)$ is an object in **Loc**, and thus, a spacetime. Then, the functor $\mathscr{A}$ defines the morphism $\mathscr{A}(\iota_{M;O})$: $\mathscr{A}(\mathbf{M}\big|_O) \to \mathscr{A}(\mathbf{M})$, where $\mathscr{A}(\mathbf{M})$ and $\mathscr{A}(\mathbf{M}\big|_O)$ being objects in **Alg**. Since this morphism is mapping towards $\mathscr{A}(\mathbf{M})$ we define the kinematic algebra as the image of $\mathscr{A}(\iota_{M;O})$, i.e. $\mathscr{A}^{\mathrm{kin}}(\mathbf{M};O) = \mathscr{A}(\iota_{M;O})(\mathscr{A}(\mathbf{M}\big|_O))$ which describes the physics lying within O, for a certain theory.

With this, we are in a position to introduce the three main ingredients of a Locally Covariant QFT.

DEFINITION 2.5.4. A Locally Covariant QFT is a functor

$$\mathscr{A} : \mathbf{Loc} \to \mathbf{Alg}.$$

that satisfies the following:

- **[Einstein Causality]** For O_1 and O_2 causally disjoint, i.e. $O_1 \subset O_2'$, we have

$$[\mathscr{A}^{\mathrm{kin}}(\mathbf{M};O_1), \mathscr{A}^{\mathrm{kin}}(\mathbf{M};O_2)] = \{0\},$$

  where we introduced $O' := M \backslash \overline{J_M(O)}$, the **open causal complement** of a region $O \subset M$.

- **[Timeslice axiom]** If $\psi : M \to N$ is a Cauchy morphism, then $\mathscr{A}(\psi)$ is an isomorphism, which is a morphism that has a two-sided inverse. In other words, if ψ is a Cauchy morphism, then $\mathscr{A}(\psi)^{-1}$ exists.

It can be checked (c.f. [29][Section 4.3]) that the free Klein-Gordon field satisfies the requirements above.

— 3 —

Hadamard State Extension in Two-dimensional Minkowski Spacetime

> Se vogliamo che tutto rimanga
> come è, bisogna che tutto cambi.
> *If we want everything to remain as it is, everything must change.*
>
> Giuseppe Tomasi di Lampedusa, Il Gattopardo

The vacuum of Quantum Field Theory (QFT) in flat spacetime is defined as the state of minimal energy and maximal symmetry specified via Poincaré invariance. This state is of paramount importance to QFT in flat spacetime because every inertial observer recognises the state as distinguished. Also, several general properties of QFT such as the spin-statistics theorem and PCT symmetry can be derived from the high degree of symmetry of this state in combination with other axioms [60]. Because of this, when one deals with curved spacetimes it may be quite tempting to find a preferred state mimicking the aforementioned procedure.

However, a generic spacetime possesses no symmetries at all and so, no single state can be distinguished as in flat spacetime. As a matter of fact, a no-go theorem excluding the possibility of a local and covariant distinguished state can be found in [25, 30]. To overcome this problem one has to give up the search for a single distinguished state and instead, study a class of physically acceptable states, the most notable being the class of Hadamard states [44, 52]. These states have been studied exhaustively because they possess very nice mathematical properties that permit the evaluation of Wick polynomials, including the stress-energy tensor and time-ordered products, which are essential to the study of the semi-classical Einstein Field Equations and perturbative QFT models [50].

Since Hadamard states are very relevant to QFT in curved spacetime, it is natural to address the following question:

> Suppose one has a globally hyperbolic spacetime M equipped with a metric g_0 containing regions T and $\tilde{S}$ enclosed within as shown in Figure 3.1. Then, can one extend an arbitrary Hadamard state from the smaller region (T) to the larger one ($\tilde{S}$) while keeping it Hadamard?

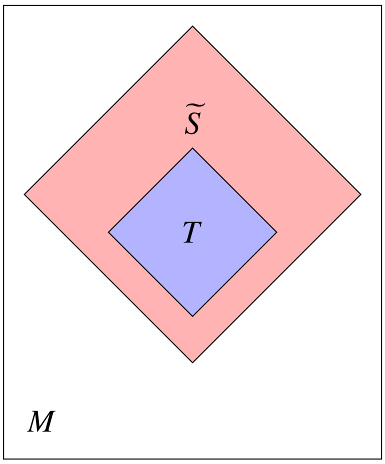


Figure 3.1: We want to extend the state from T to $\tilde{S}$ while keeping it Hadamard.

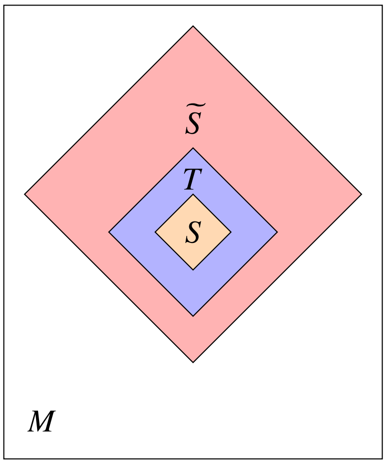


Figure 3.2: We are giving up the region $T \setminus S$.

Unfortunately, there are examples that show that the answer is no. In [22] it is shown that the so-called S-J States [3, 4] fail to be Hadamard at the boundary of T. Moreover, it is known from the lore of QFT in curved spacetime that the renormalised stress-energy tensor with respect to the Rindler vacuum, diverges at the boundary of the Rindler wedge. Instead one can shift the focus to a relaxed version of it to obtain a positive answer.

> Suppose that you have the same spacetime with (M, g_0), T, $\tilde{S}$ as before, suppose $S \subset T$ is specified as shown in Figure 3.2. Then, can one extend an arbitrary Hadamard state from the smaller region (S) to the larger one ($\tilde{S}$) while keeping it Hadamard?

Our findings indicate that this can be done for conformally ultrastatic spacetime using methods from Locally Covariant QFT [29] and can be found in Section 4.3. Our method is constructive in nature and it relies on finding a modified metric g for the spacetime M such that at time $t = 0$ we can specify a region $\mathbf{P}$ of the original one (T) and tilt its lightcones so that they follow a trajectory ρ until at time $t = t_F$ they reach a region $\mathbf{F}$, where their tilt will agree with that of the cones of $\mathbf{P}$. Then, the state ω will be extended to the state $\tilde{\omega}$ which will be defined in the region $\tilde{S}$ that is taken to be the Cauchy development of $\mathbf{F}$. All of this is summarised in Figure 3.3.

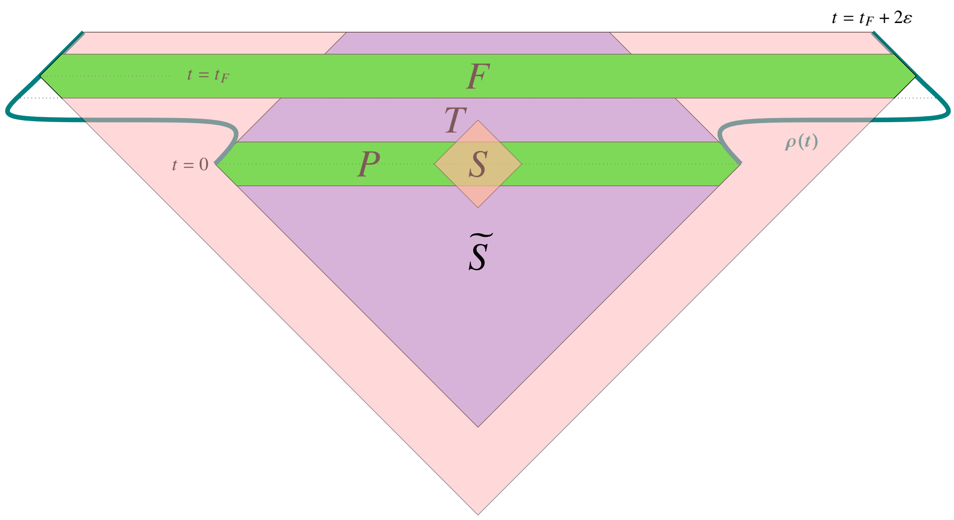


Figure 3.3: The metric is g_0 for T, S, $\widetilde{S}$ and the green regions. The spacetime enclosed by the teal lines has metric g. Note that its lightcones agree with those of g_0 at $t = 0$ and $t = t_F$.

This takes care of the extension, but we also intend to preserve the state i.e. we want the restriction of the extended state $\tilde{\omega}$ to the region S to match the restriction of ω to the same region. To this end, we need to introduce another region K such that the metric g restricted to it will coincide with g_0. If this is the case, then one can take initial data surface $\{0\} \times \sigma \subset S$ at time $t = 0$ and propagate it to time $t = t_F$ using the advanced Green operator of g_0. This will yield another data surface $\{t_F\} \times \lambda \subset F$. Then, we do the same for the data surface $\{t_F\} \times \lambda \subset F$ at $t = t_F$ and propagate the data into the past to time $t = 0$ using the retarded Green operator of Minkowski, which yields yet another surface $\{0\} \times \kappa$. This procedure is illustrated in Figure 3.4. It is worth mentioning that the metric g that allows for

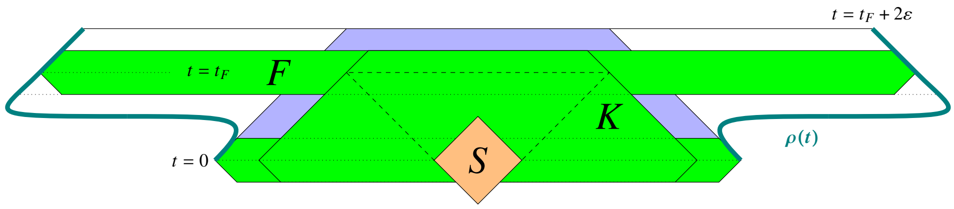


Figure 3.4: As $g = g_0$ within K, the advanced/retarded Green operators of (M, g) will be those of (M, g_0). Hence, the propagation of boundaries of initial data surfaces will be along the null-trajectories of g_0 which are denoted by diagonal lines. First, we propagate an initial data surface σ from $t = 0$ to $t = t_F$ using the advanced Green operator of (M, g). Then, we propagate the resulting surface at $t = t_F$ into the past to time $t = 0$. The resulting surface κ is such that its Cauchy development under either g or g_0 renders K.

this to happen resembles a lot the famous Alcubierre metric [5]. However, in this instance, we are warping null trajectories instead of timelike ones.

The main objective of this Chapter is to provide a proof of concept of our ideas in a simple scenario: we expand the vacuum state ω_0 defined over a region T of the two-dimensional Minkowski spacetime onto a state $\tilde{\omega}$ defined over a larger region $\tilde{S}$ and show that: (a) it is still Hadamard and (b) it agrees with ω_0 in S where $S \subset T$.

Although it is known that massless QFT in two dimensions runs into complications (see [16]), this can be resolved if one modifies the algebra. One option is to choose derivatives of the field instead of the field itself. Another option is to smear the field against a test function with support in the region under consideration. We will choose the latter as we are interested in extending the state from a compact region T onto a larger compact region $\tilde{S}$. To do this, we study the energy density of the extended state $\tilde{\omega}$ in the expanded region of spacetime $\tilde{S}$ and see that: (a) it is Hadamard and (b) it matches the energy density of the original state ω when restricted to the region S.

3.1 preliminaries

In this section, we will adapt the general framework provided in Chapter 2 to the two-dimensional and conformally invariant case. Since in this case we have $m = 0$, the field equation simplifies to

$$P_g\phi = \Box_g\phi = 0. \tag{3.1}$$

Remark 3.1.1. (Timeslice axiom) If O is a causally convex set in (M, g), we will denote by $\mathscr{A}(M, g; O)$ the subalgebra of $\mathscr{A}(M, g)$ generated by $\hat{\phi}(f)$ with $f \in C_0^\infty(O)$. Then, if O contains a Cauchy surface of (M, g) there exists a unital $*$-algebra isomorphism between $\mathscr{A}(M, g)$ and $\mathscr{A}(M, g; O)$. Furthermore, in particular, we have

$$\mathscr{A}(M, g; O) = \mathscr{A}(M, g; D_g(O)).$$

The Hadamard parametrix introduced in Definition 2.3.3 has a different form in two-dimensional spacetimes, which is the content of the following definition.

Definition 3.1.1. Consider a two-dimensional spacetime (M, g). Then using the conventions in Definition 2.3.3, we define the Hadamard parametrix of order N as

$$H_g^{(N)}(p, p') = \frac{1}{4\pi}\sum_{n=0}^{N} v_n(p, p')\sigma(p, p')^n \log\left[\frac{\sigma(p, p') + 2it0^+}{\lambda^2}\right].$$

Following the same procedure as in Definition 2.3.4, we can define the global Hadamard parametrix as the following boundary-valued distribution

$$H_g(p, p') = \frac{v(p, p')}{4\pi}\log\left[\frac{\sigma(p, p') + 2it0^+}{\lambda^2}\right],$$

where λ and t are as in Definition 2.3.3. As noted in Definition 2.3.4, the Hadamard parametrix is a distributional bisolution to the Klein-Gordon equation up to smooth functions of both arguments. So, in particular, this covers the conformally invariant case given by (3.1).

3.2 STATE EXTENSION

In this section we will analyse all the geometrical features that are necessary to expand the region T into $\tilde{S}$ while preserving a region K. We are going to set our manifold to be the two-dimensional Minkowski spacetime $(\mathbb{M}, g_0)$ where $g_0 = \text{diag}(1, -1)$ in standard coordinates $(t, x) \in \mathbb{R}^2$. Roughly speaking, the expansion procedure consists of defining various regions within $\mathbb{M}$ and finding a globally hyperbolic metric g over $\mathbb{R}^2$ which coincides with g_0 in some of these regions while also expanding a part of the region T on which the state was originally defined. By expansion, we mean that if P is a subregion of T, then $D_{g_0}(P) \subset D_g(P)$.

The metric g is a slight modification of that of Alcubierre [5], which is a solution for Einstein Field Equations known for allowing a massive particle to travel faster than speed of light. This solution has been well-studied and it is known that the massive particle is not travelling faster than light in reference to a local frame, but rather that the geometry of spacetime is warping around the particle within a finite distance known as the *warp bubble*, thus allowing for it to reach a point before a light-ray. We modified this metric to consider not a massive particle but a photon, which will allow us to specify a trajectory for an ingoing or outgoing null worldline. To perform the expansion, one needs to follow the steps described below.

THE STATE EXTENSION PROCEDURE

(SEP.I) We start by studying the original region. Given $0 < r_T$, define the following diamond

$$T = D_{g_0}(\{0\} \times (-r_T, r_T)).$$

(SEP.II) Define the past region P to be the slice

$$P = ((-\varepsilon, \varepsilon) \times \mathbb{R}) \cap T$$

of T for some $0 < \varepsilon$. Choose ε so that $(11/2)\varepsilon < r_T$.

(SEP.III) As $(11/2)\varepsilon < r_T$ we may choose t_F so that $t_F > 2\varepsilon$ and $r_T > 2t_F + (3/2)\varepsilon$. Then there is an $r_S > 0$ so that $r_S + 2t_F + (3/2)\varepsilon < r_T$. With this, we define a diamond

$$S = D_{g_0}(\{0\} \times (-r_S, r_S)),$$

which will be known as the matching region. See Figure 3.5.

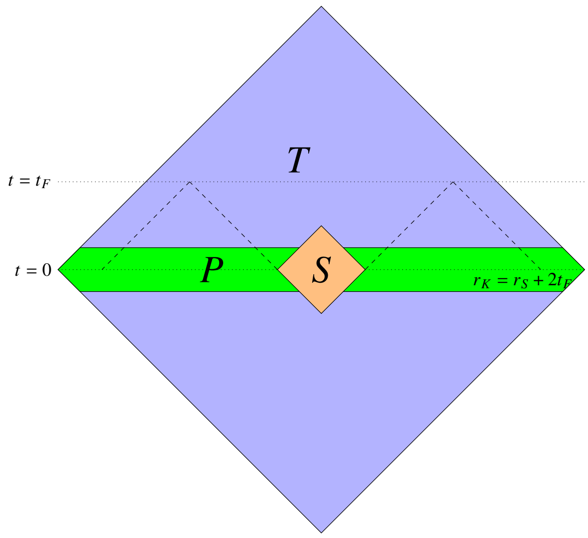


Figure 3.5: The original (T), past (P) and matching (S) regions. Dashed lines are light rays under g_0.

(SEP.IV) Define also $r_K = r_S + 2t_F$ and

$$K = D_{g_0}(\{0\} \times (-r_K, r_K)) \cap (-\varepsilon, t_F + 2\varepsilon) \times \mathbb{R},$$

which will be referred to as the protected region. Note that P contains a Cauchy surface of K.

(SEP.V) Choose $r_F > r_T + t_F$, and define the future region as the slice

$$F = D_{g_0}(\{t_F\} \times (-r_F - r_T, r_F + r_T)) \cap (t_F - \varepsilon, t_F + \varepsilon) \times \mathbb{R}. \tag{3.2}$$

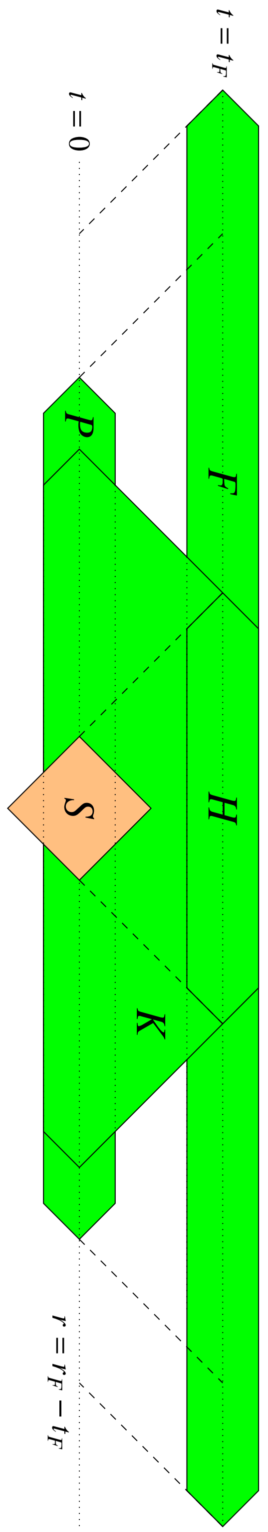


Figure 3.6: The regions P, K, S, F and H. Dashed lines are light rays under g_0.

(SEP.VI) Introduce the final region via the diamond

$$\widetilde{S} = D_{g_0}(F). \tag{3.3}$$

(SEP.VII) Define $r_H = r_K - t_F$ and introduce

$$H = D_{g_0}(\{t_F\} \times (-r_H, r_H)) \cap (t_F - \varepsilon, t_F + \varepsilon) \times \mathbb{R}.$$

By using the definition of r_K above we see that $r_H = r_S + t_F$. See Figure 3.6.

At this point is worth noting that with the definitions of the regions introduced so far we can deduce the following nestings:

$$S \subset T \subset \tilde{S}, \qquad S \subset D_{g_0}(H), \qquad \text{and} \qquad H \subset K. \tag{3.4}$$

The nesting $S \subset T$ is immediate. To see why $T \subset \tilde{S}$, see Appendix A.4. The other relations are proven in a similar fashion.

(SEP.VIII) Now seek a globally hyperbolic metric g on $\mathbb{R}^2$ so that (a) $g = g_0$ on $P \cup K \cup F$, (b) if we define

$$E = D_g(P) \cap (-\infty, t_F + 2\varepsilon) \times \mathbb{R},$$

then $F \subset E$ and (c) $F \cap (\{t_F\} \times \mathbb{R}) = E \cap (\{t_F\} \times \mathbb{R})$. Note that because of (b), (c) and the definition of F in (3.2) we can see that E contains a Cauchy surface for F with respect to both g and g_0. See Figure 3.7.

To specify the metric g we introduce $\mathcal{T} = (-\varepsilon, t_F + \varepsilon)$ and specify a smooth function $\rho : \mathcal{T} \to \mathbb{R}$ obeying

$$\rho(0) = r_T, \quad \rho(t_F) = r_F + r_T, \quad \frac{d\rho}{dt} \geq -1, \tag{3.5}$$

where $d\rho/dt|_{\mathcal{I}} = -1$ for $\mathcal{I} = (-\varepsilon, \varepsilon) \cup (t_F - \varepsilon, t_F + \varepsilon)$. Next define $v \in C^\infty(\mathbb{R})$ by

$$v = \begin{cases} \frac{d\rho}{dt} + 1 & t \in \mathcal{T} \\ 0 & t \in \mathbb{R} \setminus \mathcal{T} \end{cases} \tag{3.6}$$

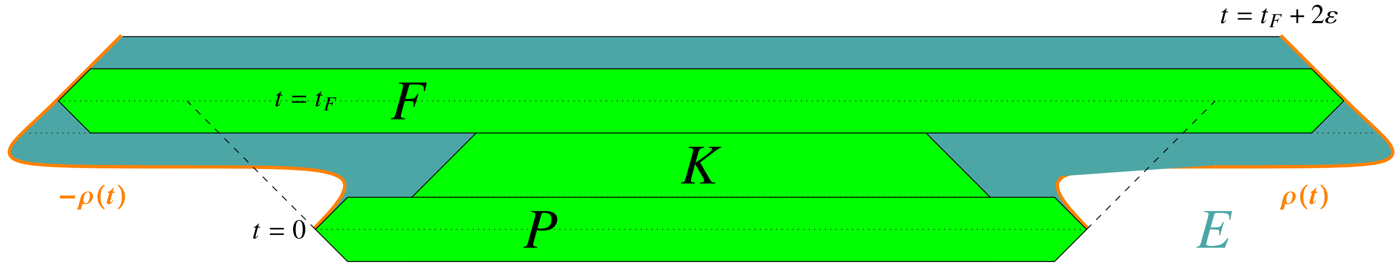


Figure 3.7: Note that E is defined as the Cauchy development of P under the metric g. This metric was designed so the lightcones would get tilted as prescribed by $\rho(t)$ and $-\rho(t)$. The metric g matches g_0 in the green regions.

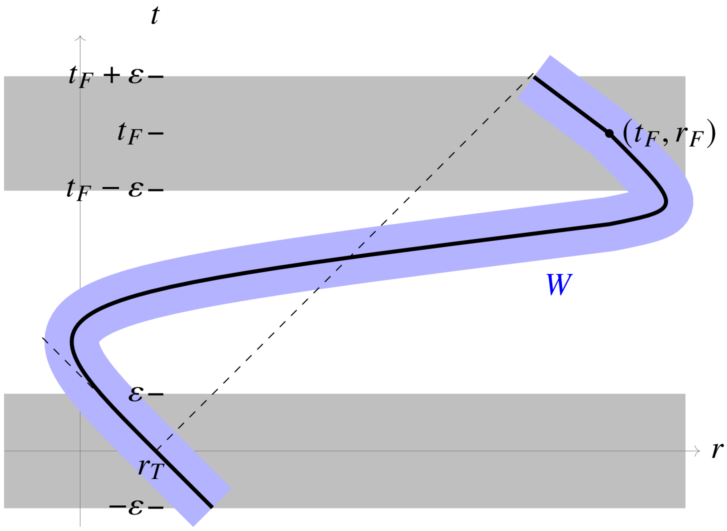


Figure 3.8: Plot of $\rho(t)$ and W_R. We have $g = g_0$ in the shaded regions and outside of W. Dashed lines are light rays under g_0.

for which we clearly have $0 \leq v$. Then, choose r_B such that $0 < r_B - \varepsilon < \rho_{min} - r_K$ where ρ_{min} is the minimum of ρ. Then, define the right and left warp bubbles at time t as the intervals

$$B_{R,t} = (\rho(t) - r_B, \rho(t) + r_B), \qquad B_{L,t} = (-\rho(t) - r_B, -\rho(t) + r_B).$$

Their corresponding warp zones are defined as

$$W_{L/R} = \bigcup_{t \in \mathcal{T}} \{t\} \times B_{L/R,t}. \tag{3.7}$$

One more thing is needed to build the metric, choose $f_{L/R} \in C_0^\infty(\mathbb{R}^2)$ to take values in $[0,1]$ such that $f_R \equiv 1$ on a neighbourhood of $\bigcup_{t \in [0,t_F]} (t, \rho(t))$ and $f_R \equiv 0$ outside W_R. The function f_L obeys the same, but in this case we have $f_L \equiv 1$ on a neighbourhood of $\bigcup_{t \in [0,t_F]} (t, -\rho(t))$ and $f_L \equiv 0$ outside W_L. (These functions certainly exist, c.f. [1][Prop. 6.5.8].) Next, define $f = f_R - f_L$ the metric g on $\mathbb{R}^2$ by

$$g = dt \otimes dt - (dx - f v dt) \otimes (dx - f v dt). \tag{3.8}$$

As noted above, (t, x) are the inertial coordinates in Minkowski spacetime $(\mathbb{M}, g_0)$ where $g_0 = dt \otimes dt - dx \otimes dx$. Note that we have $g = g_0$ outside of W and in $\mathbb{R} \setminus \mathcal{T}$, as can be seen in Figure 3.8. Next, we show that $(\mathbb{R}^2, g)$ is globally hyperbolic.

LEMMA 3.2.1. *Let g be as in* (3.8) *then,* $(\mathbb{R}^2,\ g)$ *is globally hyperbolic.*

Proof. See Appendix A.1. □

Next, we prove one of our main results. Which is, that the metric g meets the requirements from (**SEP.VIII**).

PROPOSITION 3.2.1. *Let F be as in* (3.2)*, then*

$$F \cap (\{t_F\} \times \mathbb{R}) = E \cap (\{t_F\} \times \mathbb{R}) \qquad \text{and} \qquad F \subset E.$$

*this proves (b) and (c) in (**SEP.VIII**).*

Proof. See Appendix A.3. □

PROPOSITION 3.2.2. *The metric g coincides with* g_0 *on* P, K *and* F. *This proves (a) in (**SEP.VIII**).*

Proof. To prove that $g = g_0$ when restricted to P and F, observe that P and F are contained within $\mathcal{I} \times \mathbb{R}$. Given that $v = 0$ in $\mathcal{I}$, it follows that (3.8) becomes $g = dt \otimes dt - dx \otimes dx = g_0$. Next, we prove that regions K and $W_L \cup W_R$ are disjoint, which is equivalent to proving that

$$\rho(t) - r_B > r_K - |t| \qquad \text{for all } t \in [-\varepsilon, t_F + 2\varepsilon]. \tag{3.9}$$

Note that from (**SEP.VII**) we see that $\rho_{\min} > r_B + r_K$, so it follows immediately that (3.9) is satisfied when $t = 0$. Moreover, as $d\rho/dt + 1 = v \geq 0$, we deduce that it also holds for all positive t in the range. Likewise, $d\rho/dt - 1 < 0$ for $t < 0$ in the range, so the inequality also holds for negative t. Therefore, (3.9) holds for all t in the range. □

In order to prove (b) in (**SEP.VIII**) we need an intermediate result, which is to prove that E really is the set contained within the curves $\rho(t)$ and $-\rho(t)$ for the time interval $(-\infty, t_F + 2\varepsilon)$.

PROPOSITION 3.2.3. *Let E^+ and E^- be regions in $\mathbb{R}^2$ defined by*

$$E^- = \{(t,x) : t \le 0,\ |x| < r_T + t\}, \quad E^+ = \{(t,x) : 0 \le t < t_F + 2\varepsilon,\ |x| < \rho(t)\},$$

observe that they intersect on $\{0\} \times (-r_T, r_T)$. We claim that $E = E^+ \cup E^-$.

Proof. See Appendix A.2. □

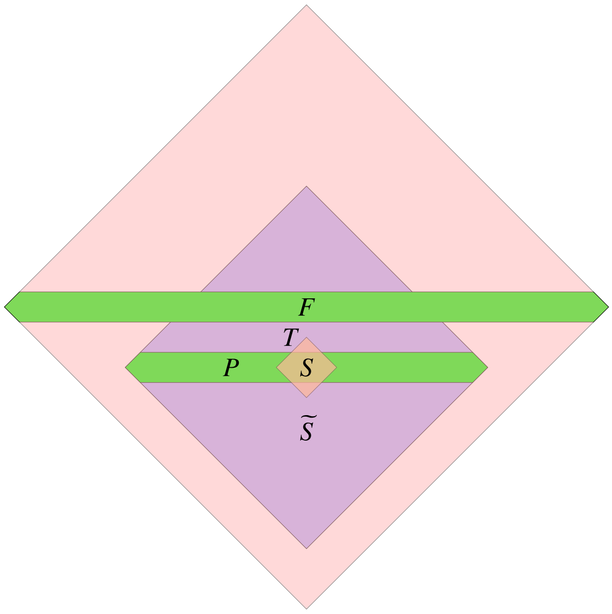


Figure 3.9: The $T,\ P,\ S,\ F$ and $\widetilde{S}$ regions.

The final thing we need to check is that $T \subset \tilde{S}$.

PROPOSITION 3.2.4. *The region T lies inside $\tilde{S}$.*

Proof. See Appendix A.4. □

So, we can conclude that this metric does expand the region T into the region E whilst leaving the regions P, K, and F unaltered. Now our procedure will stop dealing with regions and will focus on aspects regarding quantum states. It must be noted that if U in a spacetime M is causally convex for two different metrics g and g_0, and if g and g_0 are equal on U, then $\mathscr{A}(M, g; U) \cong \mathscr{A}(M, g_0; U)$. From now on we will identify these algebras if this is the case, and so, a state can be defined seamlessly on each.

(SEP.IX) Consider the **original state** ω defined in T. With the notation introduced in (2.4), this is ω_T. Note that the restriction of this state to P, K and S will also determine the corresponding state within those regions, which is not difficult to see given that $\mathscr{A}(M;P)$, $\mathscr{A}(M;K)$ and $\mathscr{A}(M;S)$ are subalgebras of $\mathscr{A}(M;T)$.

(SEP.X) By making use of the timeslice axiom on (M,g), the state ω_T determines a state ω_P on $\mathscr{A}(M,g;P)$ and since P is causally convex this also determines a state on $\mathscr{A}(M,g_0;P)$. As $E = D_g(P)$, by timeslice we see that ω_P determines ω_E as

$$\mathscr{A}(M,g;P) = \mathscr{A}(M,g;D_gP) = \mathscr{A}(M,g;E).$$

Since $F \subset E$, we see that ω_E determines ω_F on $\mathscr{A}(M,g;F)$, since F is causally convex and $g|_F = g_0|_F$, this state will also be defined on $\mathscr{A}(M,g_0;F)$ as well.

Note that $\mathscr{A}(M;P) \subset \mathscr{A}(M;T)$, so, ω_T determines ω_E, which, as indicated above, determines ω_F.

(SEP.XI) Observe that $\widetilde{S} = D_{g_0}(F)$ and hence by timeslice, ω_F on

$$\mathscr{A}(M,g;F) \cong \mathscr{A}(M,g_0;F) \cong \mathscr{A}(M,g_0;D_{g_0}(F))$$

determines a **extended state** $\widetilde{\omega}$ on $\mathscr{A}(M,g_0,\widetilde{S})$ which we will denote as $\widetilde{\omega}_{\widetilde{S}}$. This is the end of the procedure.

By using the definition of $\widetilde{S}$ in (3.3) found within **(SEP.VII)** along with the timeslice axiom once more, we arrive to $\mathscr{A}(M;F) \cong \mathscr{A}(M;D_{g_0}F) \cong \mathscr{A}(M;\widetilde{S})$. Consequently ω_F determines $\widetilde{\omega}$. So far, it is not difficult to see that the procedure that we have just described does take care of extending the state. However, it remains unclear if it preserves it, so, proving this will be our next task.

Theorem 3.2.1. *Consider the original (ω) and extended ($\widetilde{\omega}$) states as given in **(SEP.IX)**-**(SEP.XI)**. Then, following the notation introduced in* (2.4)*, we have that*

$$\omega_S = \widetilde{\omega}_S.$$

Proof. Suppose U (resp. U_0) is an open causally convex subset of (M, g) resp. (M, g_0). Then an n-point function of a state on $\mathscr{A}(M, g; U)$ is an n-fold solution to $\Box_g$ on $U^{\times n}$, while an n-point function of a state on $\mathscr{A}(M, g_0; U_0)$ is an n-fold solution to $\Box_{g_0}$ on $U_0^{\times n}$. The n-point functions of a state restricted to a smaller region are the restrictions of the n-point functions. So, let us denote the n-point function of a state ω over a region U by W_U^n, e.g.: W_T^n is the n-point function of ω_T and so, it is an n-fold $\Box_{g_0}$-solution on $T^{\times n}$.

We would like to draw the reader's attention onto two things that are entailed in (**SEP.VIII**). The first is that P, K and F are contained within E; the second is that $g = g_0$ on $P \cup K \cup F$. This means that W_E^n is an n-fold $\Box_g$-solution on $E^{\times n}$ that also agrees with W_T^n on $P^{\times n}$.

Noticing that (**SEP.VI**) indicates that $F \subset \widetilde{S}$ and as noted above $F \subset E$, a similar argument leads us to deduce that $W_{\tilde{S}}^n$ is an n-fold $\Box_{g_0}$-solution on $\tilde{S}^{\times n}$ that agrees with W_E^n on $F^{\times n}$. Also from above we know that $g|_K = g_0$, then $\Box_g$ and $\Box_{g_0}$ agree on K, so $W_E^n|_{K^{\times n}}$ and $W_T^n|_{K^{\times n}}$ obey the same equations and agree on $(K \cap P)^{\times n}$; therefore they agree on all of $K^{\times n}$ as $K \cap P$ contains a Cauchy surface of K. In (**SEP.VII**) it is noted that $H \subset K$, so in particular, W_E^n and W_T^n agree on the subset $H^{\times n}$.

Since according to (**SEP.VI**) we also have $H \subset F$, then W_T^n and $W_{\tilde{S}}^n$ agree on $H^{\times n}$; as they are both n-fold $\Box_{g_0}$-solutions, they agree on $D_{g_0}(H)^{\times n}$ which contains $S^{\times n}$. Therefore the states ω_T and $\widetilde{\omega}_{\tilde{S}}$ have identical n-point functions on $S^{\times n}$ and therefore restrict to the same state on $\mathscr{A}(M, g_0; S)$. □

Remark 3.2.1. For purposes that will be clearer later on, we want to point out that the only assumptions needed to prove the theorem above are the relevant set inclusions and that the metric g becomes g_0 in selected regions.

For the purpose of numerical investigation, we need to specify a trajectory ρ in (3.5) and the test function f in (3.8). To this end, for $t \in (\varepsilon, t_F - \varepsilon)$ we introduce the function

$$I_{\varepsilon,t_F,a}(t) := \frac{r_F + r_T}{\sqrt{\pi}} \frac{\Gamma(a + 3/2)}{\Gamma(a + 1)} \int_{-1}^{-(t_F-2t)/(t_F-2\varepsilon)} (1 - x^2)^a \, dx \tag{3.10}$$

and choose our trajectory to be

$$\rho(t) = \begin{cases} -t & -\varepsilon < t \leq \varepsilon \\ -t + I_{\varepsilon,t_F,a}(t) & \varepsilon < t < t_F - \varepsilon \\ r_F + r_T - t & t_F - \varepsilon \leq t < t_F + \varepsilon. \end{cases}$$

Next, for $s \in (0,1)$, $z \in \mathbb{R}$ and making use of the beta function $B(x,y)$ we define

$$\beta_b(s) := \frac{1}{B(b,b)} \int_0^s x^{b-1}(1-x)^{b-1}\, dx \tag{3.11}$$

$$J_{r_B,b}(z) := \begin{cases} 0 & z \leq -1 - r_B \\ \beta_b(r_B + z + 1) & -1 - r_B < z < -r_B \\ 1 & -r_B \leq z \leq r_B \\ \beta_b(r_B - z + 1) & r_B < z < r_B + 1 \\ 0 & r_B + 1 \leq z \end{cases}, \tag{3.12}$$

with this, we choose our test functions to be

$$f_R(t,x) = J_{r_B,b}(x - \rho(t) - r_T), \qquad f_L(t,x) = J_{r_B,b}(x + \rho(t) + r_T). \tag{3.13}$$

It must be noted that although the **SEP** procedure requires f and ρ to be smooth, our choices are not. This should not cause any trouble given that after some point, the computer is not able to distinguish between a smooth function and a function of finite differentiability class. It is not hard to find out how many continuous derivatives we have for each function, which will depend on the parameters a and b.

Proposition 3.2.5. *Let $I_{\varepsilon,t_F,a}$ and β_b be as in* (3.10) *and* (3.11)*. Also, assume that a, $b \in \mathbb{N}$ with $a < b$. If we introduce $N = 2a$ and $M = \min\{2a, 2(b-1)\}$, then $\rho \in C^N(\mathcal{T})$ (where $\mathcal{T}$ is as in* (3.5)*) and $f \in C^M(\mathbb{R}^2)$.*

Proof. See Appendix A.5. □

If we set the following values for the parameters

$$r_T = 1, \qquad r_S = \frac{1}{2}, \qquad r_F = 2, \qquad r_K = \frac{9}{10},$$
$$t_F = \frac{1}{5}, \qquad r_B = \varepsilon = \frac{1}{20}, \qquad a = 2, \qquad b = 3, \tag{3.14}$$

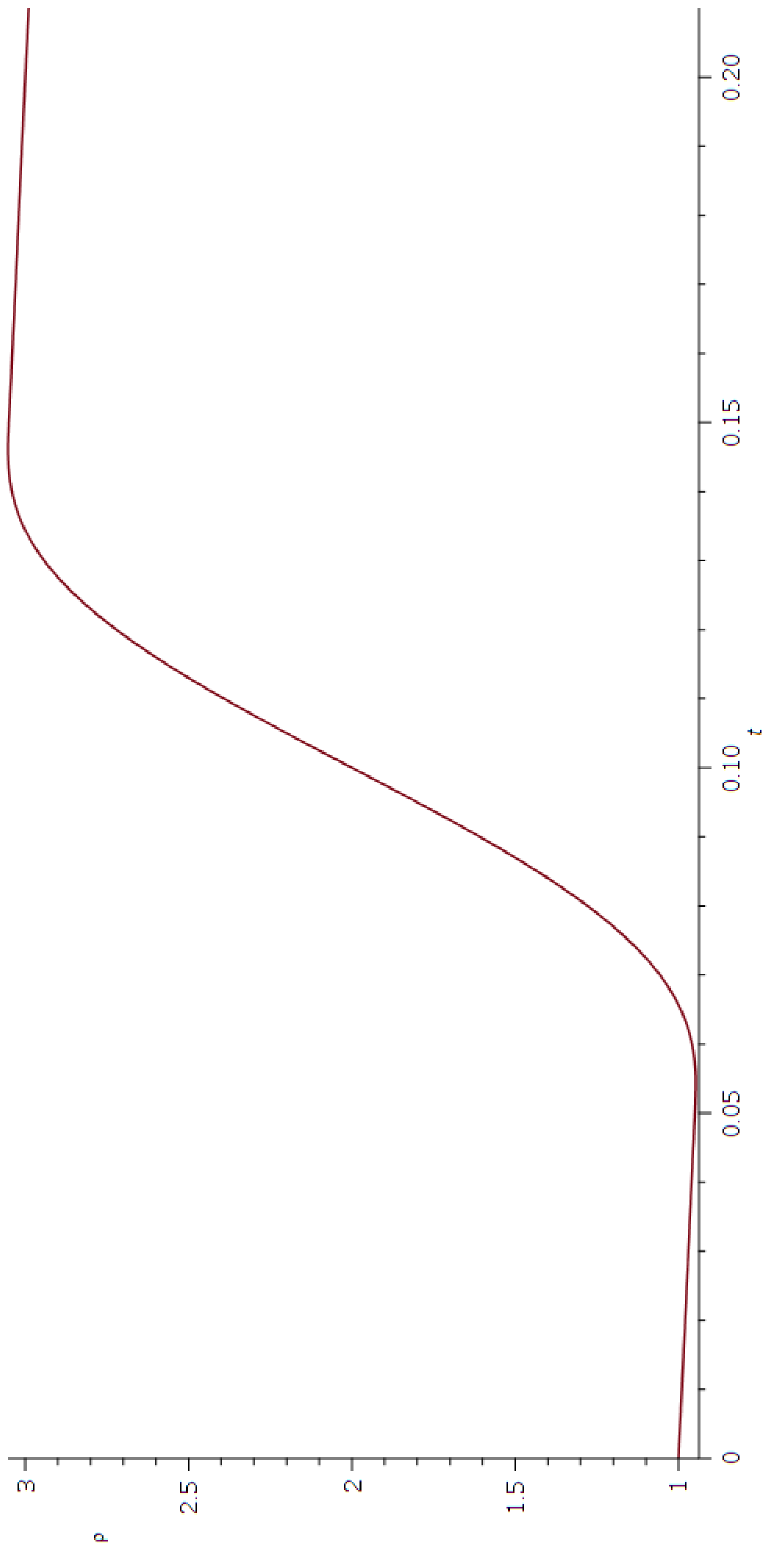


Figure 3.10: Plot of $\rho(t)$ once we have chosen parameters.

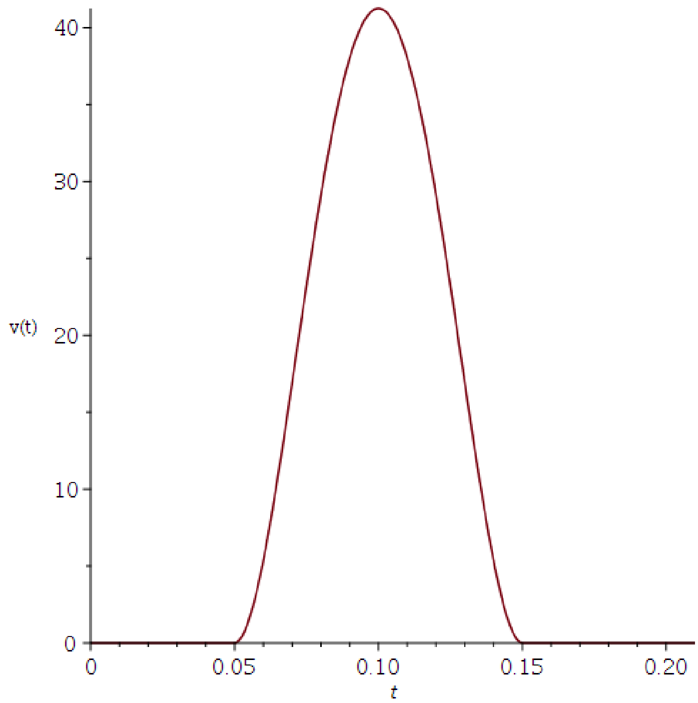


Figure 3.11: Plot of $v(t)$ for the chosen parameters.

we obtain a concrete form of the functions that we will use for our numeric investigations. By setting $a = 2$ and $b = 3$ in Proposition 3.2.5 we note that $\rho \in C^4(\mathcal{T})$, $v \in C^3(\mathcal{T})$ and $f \in C^4(\mathbb{R}^2)$; their plots can be found in Figures 3.10, 3.11 and 3.12.

3.3 NULL COORDINATES

Although the metric (3.8) in (t, x) coordinates carries out the expansion, it is useful to write it in term of other coordinates. These new coordinates acquire constant values along null-geodesics, and in consequence are called null coordinates. To obtain the null-geodesics we have to solve the following equations

$$\frac{dx}{d\lambda} = [k + f(r(\lambda), t(\lambda)) \cdot v(t(\lambda))]\frac{dt}{d\lambda}, \qquad (\text{ for } k \in \{-1, 1\}), \tag{3.15}$$

setting $t = \lambda$ and integrating numerically using Maple with a continuous 7th-8th order Runge-Kutta method we can plot the ingoing and the outgoing null geodesics.

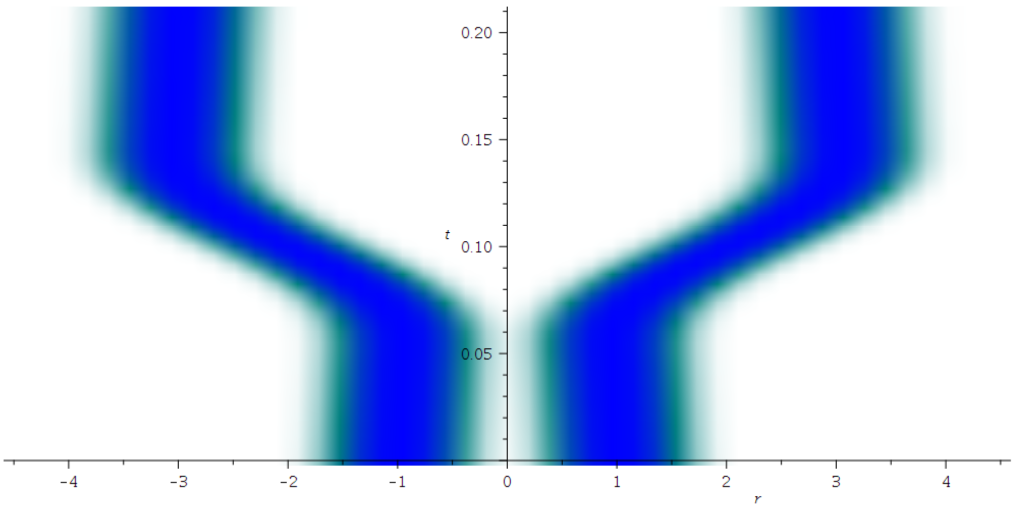


Figure 3.12: Plot of f_L and f_R .

Our results are in Figures 3.13 and 3.14 where the black trajectories correspond to the images of $\rho(t)$ and $-\rho(t)$, respectively. Before continuing, we will outline the *ray-tracing* procedure. By ray-tracing, we mean that we need to specify a point (t', x') in the future, find the solution to (3.15) that passes through it, and then see what is the value of U or V in the far past for that solution – for our purposes, it suffices to check the value of that solution at $t = 0$. This yields surfaces for U and V as functions of (t, x), its plots can be found in Figures 3.15 and 3.17.

PROPOSITION 3.3.1. *Via ray-tracing it is possible to construct global coordinates* (U, V) *and a smooth function* Ω *such that*

$$g = e^{2\Omega(U,V)} dU \otimes_s dV. \tag{3.16}$$

The conformal factor is given by

$$\frac{e^{2\Omega(U,V)}}{2} = \left| \frac{\partial(t, x)}{\partial(U, V)} \right|. \tag{3.17}$$

Proof. It is well-known that any two-dimensional metric is locally conformal to a metric of the form $dU \otimes_s dV$. As a matter of fact, this result holds globally for a globally hyperbolic spacetime (c.f. [31]). Also, we know that under a

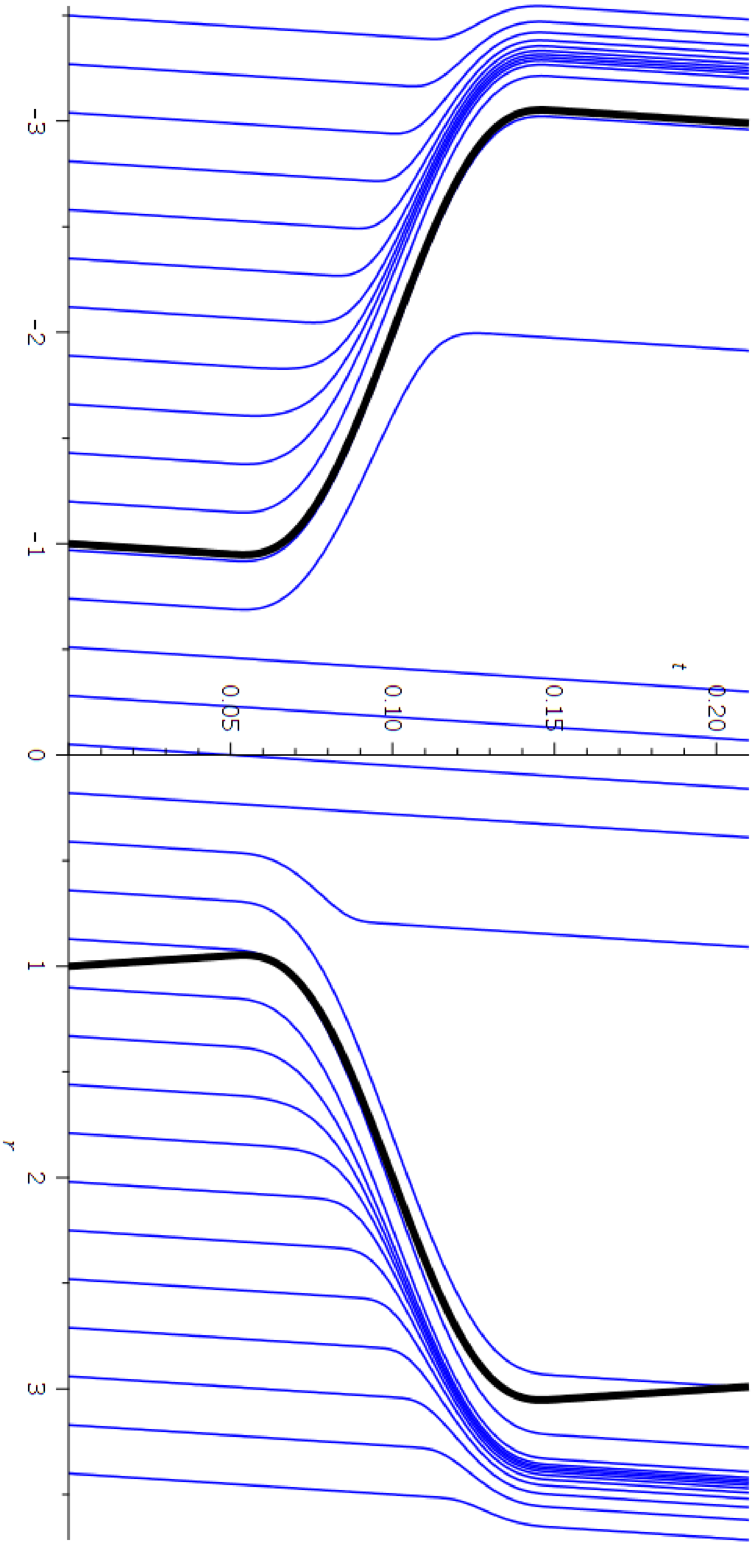


Figure 3.13: Right-moving null geodesics.

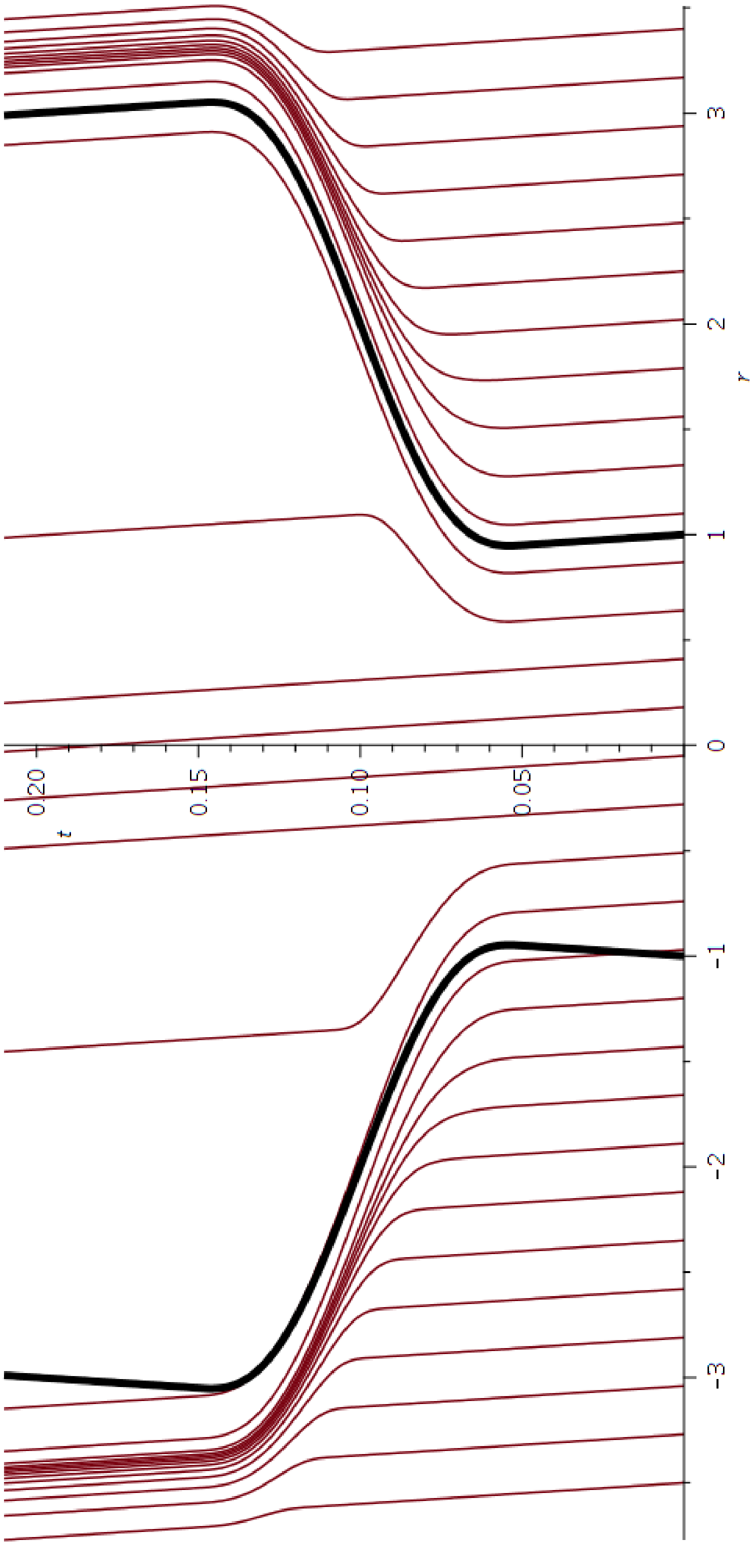


Figure 3.14: Left-moving null geodesics.

transformation $g \mapsto \overline{g}$, we will have $\det(\overline{g}) = \left|\partial(t,x)/\partial(U,V)\right|^2 \det(g)$, from (3.8) we deduce that $\det g = -1$ and so, we conclude that

$$g = e^{2\Omega(U,V)} dU \otimes_s dV. \tag{3.18}$$

Proposition 4.2 in [31] gives a conformal map Φ from M onto a relatively compact region in Minkowski (i.e. its closure is compact in Minkowski), which we can write as $\Phi(p) = (T(p), X(p))$. Then $U(p) = T(p) - X(p)$ and $V(p) = T(p) + X(p)$ give global coordinates on M so that (3.18) holds for some smooth Ω. The coordinates U and V are harmonic, so they must satisfy the wave equation $\Box_{(g,M)} U = \Box_{(g,M)} V = 0$ and since they are *light-like* coordinates, both dU and dV are null. Also, to avoid degeneracy in the metric (3.18) dU and dV must be non-zero.

As U satisfies the wave equation and in the region $t < \varepsilon$ we have $g = g_0$, we find that for smooth f and h with non-vanishing derivatives, a solution is $U(t,x) = f(t-x) + h(t+x)$. Then $dU = (f' + h')dt - (f' - h')dx$ and since its null, we find that either f' or h' must vanish. Without loss of generality assume that h' does, then h is a constant that can be absorbed into f and so, $U(t,x) = f(t-x)$. The same reasoning can be followed to see that $V(t,x) = h(t+x)$, if we introduce $u = t - x$ and $v = t + x$, we find that $U = f(u)$ and $V = h(v)$.

On the grounds that u and v each map the $t < 0$ region onto the whole real line, we see that f and h are defined on $\mathbb{R}$. Moreover, u and v have non-vanishing derivatives, so, they are invertible onto their images. Also, by construction, U and V are each constant along appropriate families of null geodesics which by global hyperbolicity, must reach the $t < 0$ region; therefore U and V will never take any value that they can not take on $t < 0$. Lastly, for an open set O define $\hat{U} = f^{-1}(U)$ and $\hat{V} = h^{-1}(V)$ which are coordinates mapping M to $O \times O \subset \mathbb{R}^2$, in which

$$g = e^{2\hat{\Omega}} d\hat{U} \otimes_s d\hat{V}$$

and $\hat{U} = u,\ \hat{V} = v$ on $t < 0$. Because $\hat{U}$ and $\hat{V}$ are constant along appropriate nulls, this means that we can determine $\hat{U}(t,x)$ and $\hat{V}(t,x)$ by ray tracing. To avoid confusion, from now on we will drop the hats, and the coordinates U and V are to be understood as global coordinates on $\mathbb{R}^2$. $\Box$

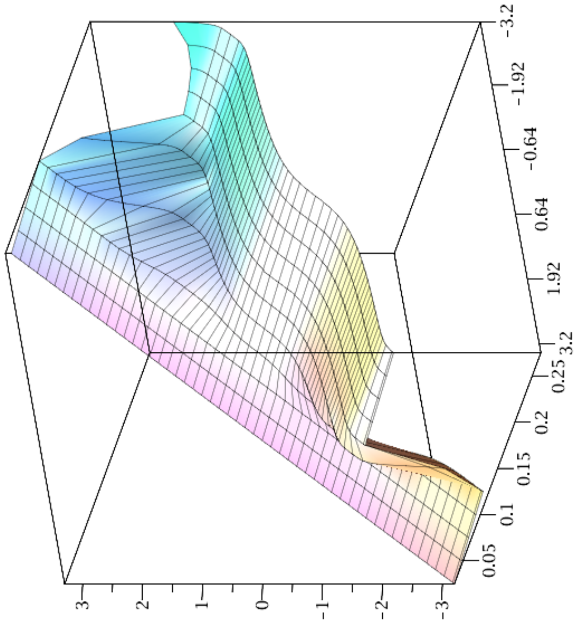


Figure 3.15: U as a function of (t, x).

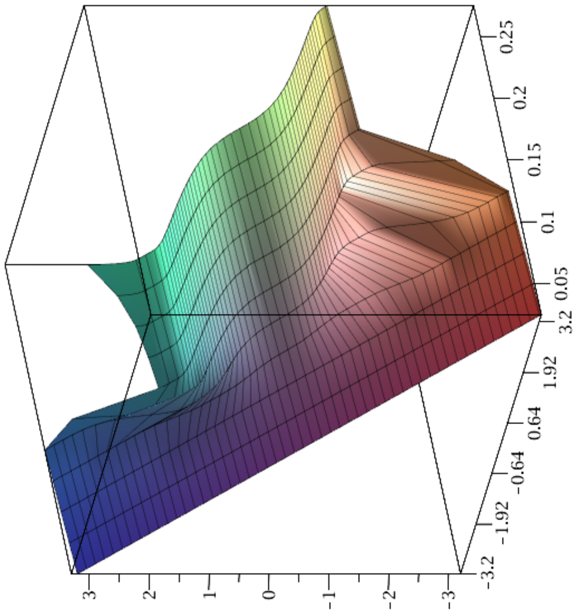


Figure 3.16: U as a function of (t, x) rotated by an angle of $\pi/2$.

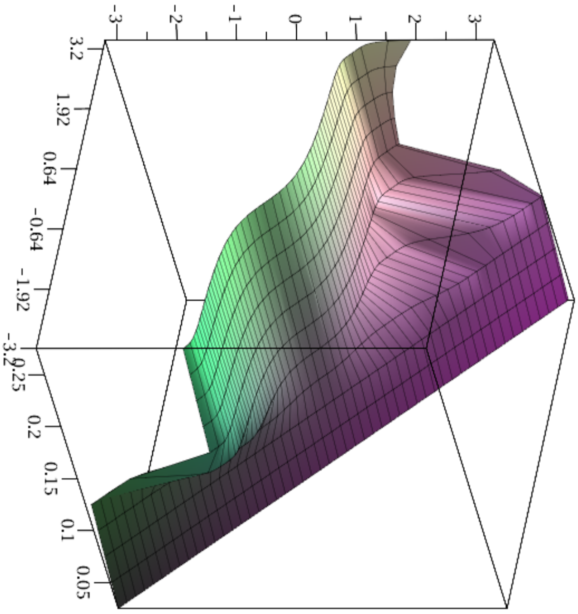


Figure 3.17: V as a function of (t, x).

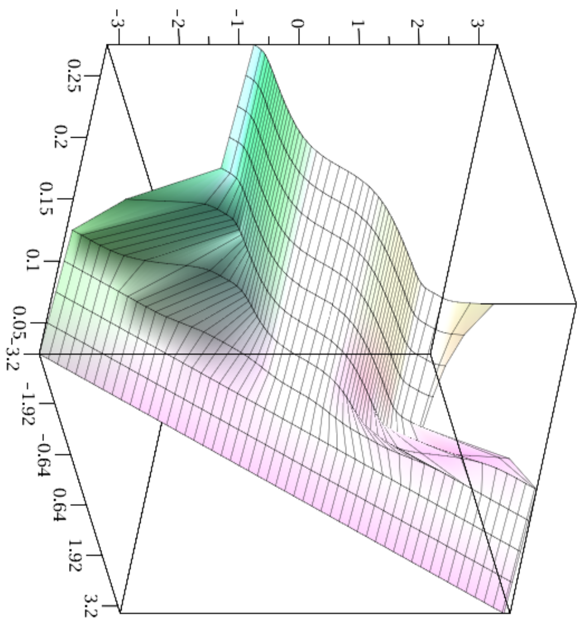


Figure 3.18: V as a function of (t, x) rotated by an angle of $\pi/2$.

The fact that we can write the metric globally as in (3.16) will be very useful when we compute the stress-energy tensor as it acquires a rather simplified form in these coordinates - c.f. Proposition 3.4.1. Furthermore, one of the main advantages of working in two dimensions is that if one can find U and V as functions of (t,x) then one can do the same for the conformal factor $\exp(2\Omega)$. The following Proposition and Corollary elaborate on this.

PROPOSITION 3.3.2. *The following expressions relate the original coordinates* (t,x) *with the new coordinates* (U,V)

$$\left|\frac{\partial t}{\partial U}\frac{\partial t}{\partial V}\right| = \frac{1}{4}e^{2\Omega(U,V)}, \tag{3.19}$$

$$\frac{\partial x}{\partial U} = (fv-1)\frac{\partial t}{\partial U}, \tag{3.20}$$

$$\frac{\partial x}{\partial V} = (fv+1)\frac{\partial t}{\partial V}. \tag{3.21}$$

Proof. Because of Proposition 3.3.1 we know that it is possible to write the metric (3.8) making use of $dt = (\partial t/\partial U)dU + (\partial t/\partial V)dV$ and $dx = (\partial x/\partial U)dU + (\partial x/\partial V)dV$, this results in

$$\begin{aligned} g = &\left(\left(\frac{\partial t}{\partial U}\right)^2 - \left(\frac{\partial x}{\partial U} - fv\frac{\partial t}{\partial U}\right)^2\right) dU \otimes dU \\ &+ \left(\left(\frac{\partial t}{\partial V}\right)^2 - \left(\frac{\partial x}{\partial V} - fv\frac{\partial t}{\partial V}\right)^2\right) dV \otimes dV \\ &+ 2\left(\frac{\partial t}{\partial U}\frac{\partial t}{\partial V} - \left(\frac{\partial x}{\partial U} - fv\frac{\partial t}{\partial U}\right)\left(\frac{\partial x}{\partial V} - fv\frac{\partial t}{\partial V}\right)\right) dU \otimes_s dV. \end{aligned} \tag{3.22}$$

Comparing (3.22) with (3.18) we conclude that for $k,\ k' \in \{-1,1\}$ we will have

$$\frac{\partial x}{\partial U} = (k+fv)\frac{\partial t}{\partial U} \tag{3.23}$$

$$\frac{\partial x}{\partial V} = (k'+fv)\frac{\partial t}{\partial V} \tag{3.24}$$

$$e^{2\Omega} = 2\frac{\partial t}{\partial U}\frac{\partial t}{\partial V} - 2\left(\frac{\partial x}{\partial U} - fv\frac{\partial t}{\partial U}\right)\left(\frac{\partial x}{\partial V} - fv\frac{\partial t}{\partial V}\right). \tag{3.25}$$

Substituting (3.23) and (3.24) into (3.17) yields

$$e^{2\Omega} = 2|k-k'||(\partial t/\partial U)(\partial t/\partial V)|,$$

so, we necessarily must have (a) $k = -k'$ and (b) $dt \neq 0$ for any value of (U, V) [from this and (3.23-3.24) it follows that $dx \neq 0$ as well]. Clearly, (a) implies that

$$e^{2\Omega} = 4\frac{\partial t}{\partial U}\frac{\partial t}{\partial V}. \tag{3.26}$$

We choose $k = -1$ to adhere to the usual convention of null coordinates. Making this substitution in (3.23), (3.24) and (3.25) leads to our result. □

Corollary 3.3.1. *The following identities for the derivatives*

$$\frac{\partial t}{\partial U} = \frac{e^{2\Omega}}{2}\frac{\partial V}{\partial x}, \quad \frac{\partial t}{\partial V} = -\frac{e^{2\Omega}}{2}\frac{\partial U}{\partial x}, \quad \frac{\partial x}{\partial U} = -\frac{e^{2\Omega}}{2}\frac{\partial V}{\partial t}, \quad \frac{\partial x}{\partial V} = \frac{e^{2\Omega}}{2}\frac{\partial U}{\partial t}, \tag{3.27}$$

lead us to deduce that the conformal factor acquires the following form

$$\Omega(U(t,x), V(t,x)) = -\frac{1}{2}\ln\left|\frac{\partial U}{\partial x}\frac{\partial V}{\partial x}\right|. \tag{3.28}$$

Proof. Since $|\partial(t,x)/\partial(U,V)| = e^{2\Omega}/2$ we know that $(\partial(t,x)/\partial(U,V)) = (\partial(U,V)/\partial(t,x))^{-1}$ must hold everywhere. Also, as $|(\partial(U,V)/\partial(t,x))| = 2e^{-2\Omega}$, we will have

$$\begin{pmatrix} \frac{\partial t}{\partial U} & \frac{\partial t}{\partial V} \\ \frac{\partial x}{\partial U} & \frac{\partial x}{\partial V} \end{pmatrix} = \begin{pmatrix} \frac{\partial U}{\partial t} & \frac{\partial U}{\partial x} \\ \frac{\partial V}{\partial t} & \frac{\partial V}{\partial x} \end{pmatrix}^{-1} = \frac{e^{2\Omega}}{2}\begin{pmatrix} \frac{\partial V}{\partial x} & -\frac{\partial U}{\partial x} \\ -\frac{\partial V}{\partial t} & \frac{\partial U}{\partial t} \end{pmatrix},$$

from which it follows that (3.27) hold. To deduce (3.28), just substitute (3.27) into (3.19). □

As it will be shown later in Section 3.1, this will allow us to find the stress-energy tensor in terms of the original coordinates (t, x).

3.3.1 States in null coordinates

Our choice of coordinates in Proposition 3.3.1 is particularly useful when computing the renormalised stress-energy tensor because it acquires a rather simple form. However, before going into this, we will discuss some necessary aspects of the states of the quantum field under different coordinates, namely Cartesian (t, x)

and null (U, V). Then, we consider Minkowski spacetime with the following systems of global coordinates

$$(t, x) \in \mathbb{R}^2 \qquad \text{and} \qquad (u = t - x, v = t + x) \in \mathbb{R}^2. \tag{3.29}$$

Consider the global system of coordinates in Proposition 3.3.1 given by

$$(U(t, x), V(t, x)) \in \mathbb{R}^2 \text{ such that } U(t, x) = u \text{ and } V(t, x) = v \text{ for } t \leq \varepsilon. \tag{3.30}$$

Next, let us write the Minkowski metric as

$$g_0 = dt \otimes dt - dx \otimes dx = du \otimes_s dv \tag{3.31}$$

and introduce two conformally related metrics, given by

$$\tilde{g}_0 = dU \otimes_s dV \qquad g = e^{2\Omega}\tilde{g}_0, \tag{3.32}$$

the conformal factor is as defined in Eq. (3.18) and the metric g is given in (t, x) coordinates in Eq. (3.8). Observe that g_0 and $\tilde{g}_0$ coincide in the region where $t \leq \varepsilon$ as both are the Minkowski metric in this region. Furthermore, we remind the reader that we can obtain U, V and Ω as functions of (t, x) using the methods found in Section 3.3.

In the remainder of this section, we will show that the two-point function for g in the (t, x) coordinates, can be written in terms of the two-point function for g_0 in terms of the (U, V) coordinates, which as we mentioned before, are functions of (t, x). In order to get us to this result, let us recall some basic facts from the methods of characteristics in partial differential equations.

The main idea is as follows: suppose we that ϕ_0 is a solution to the Klein-Gordon equation for Minkowski spacetime and we know its form on the region where $t \leq \varepsilon$. Furthermore, suppose that ϕ is a solution to the Klein-Gordon equation for our warped spacetime. Then, we can determine ϕ from ϕ_0.

Also, we would like to stress that as noted in (3.30), we have $(U, V) = (u, v)$ for $u + v \leq \varepsilon$. And so, a solution to $\square_{\tilde{g}_0}\phi_0 = 0$ can be written in terms of the (u, v) coordinates.

PROPOSITION 3.3.3. *Let ϕ_0 and ϕ be solutions to the massless wave equation in $(\mathbb{R}^2, \tilde{g}_0)$ and $(\mathbb{R}^2, g)$, respectively. Impose the boundary condition $\phi = \phi_0$ in*

$(-\infty, \mathbb{R}) \times \mathbb{R}$. *Then, let us consider the region where* $u + v \leq \varepsilon$ *and denote the solution to* $\Box_{\tilde{g}_0} \phi_0 = 0$ *in the* (u, v) *coordinate system (written in terms of* u *and* v*) as* ${}_{(u,v)}\phi_0(u, v)$. *Then, if we do the same for* $\Box_g \phi = 0$ *in the* (U, V) *coordinates by writing* ${}_{(U,V)}\phi(U, V)$, *we find that*

$$ {}_{(U,V)}\phi(U, V) = {}_{(u,v)}\phi_0(U, V). \tag{3.33} $$

Proof. Making use of (3.31) and (3.32) we deduce that $\tilde{g}_0 = \exp(-2\Omega)g$. Additionally, as ϕ is a solution to the massless wave equation in $(\mathbb{R}^2, g)$, then $\Box_g \phi = 0$ which in turn implies that $\exp(-2\Omega)\Box_{\tilde{g}_0}\phi = 0$ and consequently that $\partial^2({}_{(U,V)}\phi)/\partial U \partial V = 0$. Hence, $\phi(t, x) = f(U(t, x)) + h(V(t, x))$ for appropriate functions f and h.

Moreover, as we are considering the region $t = u + v \leq \varepsilon$, then (3.29) and (3.30) imply that $\phi(t, x) = f(t-x)+h(t+x)$. Therefore f and h can be determined everywhere by knowing ϕ in $t \leq \varepsilon$. In fact, we find that

$$ \begin{aligned} &\text{if } {}_{(u,v)}\phi_0(u, v) = f(u) + h(v) && \text{on } u + v \leq \varepsilon \\ &\text{then } {}_{(U,V)}\phi(U, V) = f(U) + h(V) && \text{for all } U,\ V, \\ &\text{so } {}_{(U,V)}\phi(U, V) = {}_{(u,v)}\phi_0(U, V). \end{aligned} $$

□

Equation (3.33) reads as: the solution for $\Box_g \phi = 0$ in (U, V) coordinates is the same as the solution obtained for $\Box_{\tilde{g}_0}\phi = 0$ in (u, v) coordinates upon substitution of u with U and v with V. Note that (3.33) is not a consequence of a coordinate transformation since this would be ${}_{(U,V)}\phi(U, V) = {}_{(u,v)}\phi_0(u(U, V), v(U, V))$. Let us follow the same train of thought with bisolutions; this will allow us to make contact with our main goal: finding the two-point function for $(\mathbb{R}^2, g)$ with (t, x) coordinates in terms of the two-point function for $(\mathbb{R}^2, g_0)$ with (U, V) coordinates.

PROPOSITION 3.3.4. *Let* ω_0 *be a bisolution to the Klein-Gordon equation in* $(\mathbb{R}^2, \tilde{g}_0)$*, that is,* $(\Box_{\tilde{g}_0} \otimes 1)\omega_0 = 0$ *and* $(1 \otimes \Box_{\tilde{g}_0})\omega_0 = 0$. *Also, let us denote by* ω *the bisolution to* $\Box_g$ *with boundary condition* $\omega = \omega_0$ *in* $((-\infty, \varepsilon) \times \mathbb{R})^{\times 2}$. *Then, following the notation from Proposition 3.3.3, we claim that*

$$ {}_{(U,V)\times(U,V)}\omega(U, V; U', V') = {}_{(u,v)\times(u,v)}\omega_0(U, V; U', V'). $$

Proof. We begin by noting that because of the boundary condition, the claim holds in $((-\infty, \varepsilon) \times \mathbb{R})^{\times 2}$. Also, this fixes ω completely in $\mathbb{R}^2 \times \mathbb{R}^2$ in a similar fashion as f and h were fixed in Proposition 3.3.3. Next, we need to check that under these assumptions, ω will be a bisolution to $\Box_g$. As it was noted in Proposition 3.3.3 we have $\Box_g = \exp(-2\Omega)\Box_{\tilde{g}_0}$, from this same Proposition it also follows that

$$\begin{aligned}
&{}_{(U,V)\times(U,V)}(\Box_g \otimes 1)\omega(U, V; U', V') \\
&= \exp(-2\Omega)\frac{\partial^2}{\partial U \partial V}\left({}_{(U,V)\times(U,V)}\omega\right)(U, V; U', V') \\
&= \exp(-2\Omega)\frac{\partial^2}{\partial U \partial V}\left({}_{(u,v)\times(u,v)}\omega_0\right)(U, V; U', V') \\
&= \exp(-2\Omega)\ {}_{(u,v)\times(u,v)}(\Box_{g_0} \times 1)\omega_0(u, v : u', v') \\
&= 0,
\end{aligned}$$

following a similar approach it is possible to show that ${}_{(U,V)\times(U,V)}(1 \otimes \Box_g)\omega = 0$. □

Therefore, the two-point functions are related as follows

$$\begin{aligned}
&{}_{(U,V)\times(U,V)}\omega(U(t,x), V(t,x); U(t',x'), V(t',x')) \\
&= {}_{(u,v)\times(u,v)}\omega_0(U(t,x), V(t,x); U(t',x'), V(t',x')),
\end{aligned}$$

which can be rephrased as

$$\omega(t,x;t',x') = {}_{(u,v)\times(u,v)}\omega_0(U(t,x), V(t,x); U(t',x'), V(t',x'))$$

or yet, in a conceptually clearer although visually intricate formulation, we find

$$\begin{aligned}
&\omega(t,x;t',x') = \\
&\omega_0\left(\frac{V(t,x)+U(t,x)}{2}, \frac{V(t,x)-U(t,x)}{2}; \frac{V(t',x')+U(t',x')}{2}, \frac{V(t',x')-U(t',x')}{2}\right).
\end{aligned}$$

The following Corollary is just an immediate step towards a bigger result.

COROLLARY 3.3.2. *Let ω and ω_0 be bisolutions as in Proposition 3.3.4. Also, let $H_{\tilde{g}_0}$ and H_{g_0} be the Hadamard parametrix associated to $(\mathbb{R}^2, \tilde{g}_0)$ and $(\mathbb{R}^2, g_0)$,* *respectively. Then,*

$$\begin{aligned}
&{}_{(U,V)\times(U,V)}\left(\left(\nabla_{\partial/\partial U}\otimes 1\right)(\omega - H_{\tilde{g}_0})\right)(U,V;U',V')\\
&= {}_{(u,v)\times(u,v)}\left(\left(\nabla_{\partial/\partial u}\otimes 1\right)(\omega_0 - H_{g_0})\right)(U,V;U',V').
\end{aligned}$$

A similar result holds for $\partial/\partial V$ and $\partial/\partial v$. Furthermore, the same result for the second slot is proven in a similar way.

Proof. We begin by observing that $(\mathbb{R}^2, g_0)$ is isometric to $(\mathbb{R}^2, \tilde{g}_0)$ under the map $(u,v)\mapsto(U,V)$ and in consequence, the Hadamard parametrices obey the following property

$${}_{(U,V)\times(U,V)}H_{\tilde{g}_0}(U,V;U',V') = {}_{(u,v)\times(u,v)}H_{g_0}(U,V;U',V').$$

Using this, we calculate

$$\begin{aligned}
&{}_{(U,V)\times(U,V)}\left(\left(\nabla_{\partial/\partial U}\otimes 1\right)(\omega - H_{\tilde{g}_0})\right)(U,V;U',V')\\
&= \frac{\partial}{\partial U}\,{}_{(U,V)\times(U,V)}(\omega - H_{\tilde{g}_0})(U,V;U',V')\\
&= {}_{(u,v)\times(u,v)}\left(\left(\nabla_{\partial/\partial U}\otimes 1\right)(\omega_0 - H_{g_0})\right)(U,V;U',V').
\end{aligned}$$

□

In order to make contact with the usual physics vernacular, we will refer to bisolutions as two-point functions from now on. Also, for a state ω we will denote its corresponding two-point function by $\omega(p,p')$, unless indicated otherwise.

3.4 The stress-energy tensor of an extended state

A good way to see if we actually extended the state according to the requirements stated in Section 3.1, is to calculate the stress-energy tensors of the original and extended states. We have argued in Section 3.3.1 that the regions where these states are defined on are conformally related spacetimes. So, if one takes the original state ω to be the Minkowski vacuum in T, the stress-energy tensor of the extended state $\widetilde{\omega}$ should be conformally related to it in E (hence on $\widetilde{S}$) and agree with that of ω on S.

It must be noted that one needs to be careful when computing the regularised stress-energy tensor when the dimension of the spacetime is even (our case) because proceeding in a naïve way will lead to a stress-energy tensor that is not conserved and that does not account for the conformal anomaly. However, all of these problems can be fixed (c.f. [64] or [51] for a more rigorous approach) by subtracting a term of the form $g_{\mu\nu}(z)Q(z)$–where Q is a local curvature term–from $\omega(T_{\mu\nu}))$.

The previous section lays the foundation needed to compute the stress-energy tensor. Usually one has to consider the point-split version of a differential operator, examine its action on the two-point function and take the coincidence limit making use of the so-called Synge's rule (c.f. Chapter 8 in [36]). In the author's viewpoint, this procedure is rather obscure and can easily lead to mistakes when not done with extreme precaution. Because of this, we will introduce a slightly different formulation of this.

As it is widely known, the bisolutions considered in the previous section often happen to be distributions that have a very particular singular structure. This structure is completely determined by the state and the local geometry. Furthermore, it can be encoded into the Hadamard parametrix 3.1.1. As it contains the singularities from the two-point function, their difference introduces the notion of regularised two-point function.

DEFINITION 3.4.1. Let $f \in C^\infty(M \times M)$ and denote its value on the diagonal by $[f]$, that is, $[f](p) = f(p, p)$.

A famous example where this notation shows its usefulness is when one defines regularised expectation values of two-point functions at a point p. Suppose we are given a Hadamard parametrix $H_g(p, p')$ associated to (M, g). Then, for a two-point function $\omega(p, p')$ of the Hadamard form, we have the regularised object

$$w(p) = \left[\omega - H_g\right](p).$$

Note that $w \in C^\infty(M)$. This follows from Definition 2.3.5 of Hadamard form, which implies that $(\omega - H_g) \in C^\infty(M \times M)$. The notation introduced above can be extended to tensors. For instance, if we consider the smooth covector field given by $(\nabla \times \mathbf{1})f$, then, $[(\nabla \times \mathbf{1})f](p)$ will be a covector at p, which is a rank

$(0, 1)$ tensor. Recalling that a bi-covector field is an element of the tensor product of smooth sections of cotangent bundles, we can apply the same logic to the smooth bi-covector field $(\nabla \times \nabla)f$ to obtain the rank $(0, 2)$ tensor $[(\nabla \times \nabla)f](p)$. Since the stress-energy tensor is a rank $(0, 2)$, we are on good grounds to define it.

DEFINITION 3.4.2. Let $\omega(p, p')$ be a two-point function and $H_g(p, p')$ the Hadamard parametrix associated to (M, g). Consider the rank $(0, 2)$ tensor $[(\nabla \times \nabla)(\omega - H_g)](p)$, then, using its components we define

$$D_{\mu\nu}(p) = [(\nabla \times \nabla)(\omega - H_g)]_{\mu\nu}(p)$$

and introduce the regularised stress-energy tensor for a conformally-invariant massless scalar field in two dimensions, which is given by

$$\langle : T_{\mu\nu}[g] : \rangle_\omega(p) = D_{\mu\nu}(p) - \frac{1}{2} g_{\mu\nu}(p) g^{\rho\sigma}(p) D_{\rho\sigma}(p). \tag{3.34}$$

Clearly, if we consider Minkowski spacetime and the two-point function of the vacuum state, we will have $\langle : T_{\mu\nu}[\tilde{g}_0] : \rangle_{\omega_0}(p) \equiv 0$.

COROLLARY 3.4.1. *Let ω and ω_0 be two-point functions as specified in Section 3.3.1. If we consider components of the stress-energy tensor following the notation introduced in that Section, we will have*

$$\begin{aligned}
\langle : {}_{(U,V)\times(U,V)}T_{UU}[\tilde{g}_0] : \rangle_\omega(U, V) &= \langle : {}_{(u,v)\times(u,v)}T_{uu}[g_0] : \rangle_{\omega_0}(U, V) \\
\langle : {}_{(U,V)\times(U,V)}T_{UV}[\tilde{g}_0] : \rangle_\omega(U, V) &= \langle : {}_{(u,v)\times(u,v)}T_{uv}[g_0] : \rangle_{\omega_0}(U, V) \\
\langle : {}_{(U,V)\times(U,V)}T_{VV}[\tilde{g}_0] : \rangle_\omega(U, V) &= \langle : {}_{(u,v)\times(u,v)}T_{vv}[g_0] : \rangle_{\omega_0}(U, V).
\end{aligned}$$

Note that these formulae are not that of a coordinate transformation. So, in particular, we observe that $\langle : {}_{(U,V)\times(U,V)}T_{\mu\nu}[\tilde{g}_0] : \rangle_\omega(U, V) \equiv 0$.

Proof. As (3.34) indicates, the components of the stress-energy tensor can be found by taking derivatives of the regularised two-point function. So, by using Corollary 3.3.2, the result follows immediately. □

Let us adapt the previous results to the case where the original data is the two-point function ω_0 for the Minkowski vacuum state with the g_0 metric in (u, v) coordinates. Thus, from the start we can calculate quantities such as $\langle :$

${}_{(u,v)\times(u,v)}T_{uu}[g_0] :\rangle_{\omega_0}$, etc. So, provided we can compute $\langle :\ {}_{(U,V)\times(U,V)}T_{\mu\nu}[g_0] :\rangle_{\omega}$ from them for a certain state ω, we can make use of the conformal transformation rule (see Section 6 in [10]) and obtain $\langle :\ {}_{(t,x)\times(t,x)}T_{\mu\nu}[g] :\rangle_{C\omega}$, the regularised two-point function for the conformally related state $C\omega$.

It must be noted that this conformal transformation rule is remarkably simple in the two-dimensional case, hence our choice to study it. It might not be evident why we can go from $\langle :\ {}_{(U,V)\times(U,V)}T_{\mu\nu}[g] :\rangle_{C\omega}$ to $\langle :\ {}_{(t,x)\times(t,x)}T_{\mu\nu}[g] :\rangle_{C\omega}$ but this is certainly the case as it has been proven in Section 3.3.

To summarise: in principle, the expectation value $\langle :\ {}_{(t,x)\times(t,x)}T_{\mu\nu}[g] :\rangle_{C\omega}$ can be computed easily using a conformal transformation rule once $\langle :\ {}_{(U,V)\times(U,V)}T_{\mu\nu}[\tilde{g}_0] :\rangle_{\omega}$ is known in terms of $\langle :\ {}_{(u,v)\times(u,v)}T_{uu}[g_0] :\rangle_{\omega_0}$. Henceforth, to prevent hand fatigue, we will denote $\langle :\ {}_{(t,x)\times(t,x)}T_{\mu\nu}[g] :\rangle_{C\omega}$ by $\langle : T_{\mu\nu}[g] :\rangle_{C\omega}$, the same subscripts will be omitted for $\langle :\ {}_{(U,V)\times(U,V)}T_{\mu\nu}[\tilde{g}_0] :\rangle_{\omega}$ unless it is indicated otherwise.

PROPOSITION 3.4.1. *Let $\langle T_{\mu\nu}[\tilde{g}_0]\rangle_{\omega}$ be the regularised stress-energy tensor with respect to a reference state ω (not necessarily the vacuum) in Minkowski spacetime $(\mathbb{M}, g)$. Then, this defines a conformally related regularised stress-energy tensor given by*

$$\langle T_{\mu\nu}[g]\rangle_{C\omega} = e^{-2\Omega}\langle T_{\mu\nu}[\tilde{g}_0]\rangle_{\omega} + \theta_{\mu\nu} + \frac{e^{-2\Omega}}{24\pi}\frac{\partial^2\Omega}{\partial U\partial V}g_{\mu\nu} \tag{3.35}$$

where the $\theta_{\mu\nu}$ symbols are as follows

$$\begin{aligned} \theta_{UU} &:= \frac{1}{12\pi}\left[\frac{\partial^2\Omega}{\partial U^2} - \left(\frac{\partial\Omega}{\partial U}\right)^2\right] \\ \theta_{VV} &:= \frac{1}{12\pi}\left[\frac{\partial^2\Omega}{\partial V^2} - \left(\frac{\partial\Omega}{\partial V}\right)^2\right] \\ \theta_{UV} &= \theta_{VU} = 0. \end{aligned}$$

Proof. According to (6.136) in [10] the coordinates in Proposition 3.3.1 lead to

$$\langle T_{\mu\nu}[g]\rangle_{C\omega} = \left(\frac{\det(g)}{\det(\tilde{g}_0)}\right)^{-1/2}\langle T_{\mu\nu}[\tilde{g}_0]\rangle_{\omega} + \theta_{\mu\nu} - \frac{R}{48\pi}g_{\mu\nu}. \tag{3.36}$$

In our particular setting $R = -2e^{-2\Omega}(\partial^2\Omega/\partial U\partial V)$ and $\det(g) = -e^{4\Omega}$ whilst $\det(\tilde{g}_0) = -1$, substituting this into (3.36) yields our final result. □

If the reference state used in evaluating the expectation value in the RHS is the vacuum state, then the resulting state in the LHS will be a conformal vacuum. However, the first term will vanish only if the spacetime is conformal to all of Minkowski. If not, it will contribute. As our reference state is Minkowski vacuum, this result simplifies further into:

Corollary 3.4.2. *The components of the stress-energy tensor* (3.35) *are given by*

$$\begin{aligned} T_{UU} &= \frac{1}{12\pi}\left(\frac{\partial^2\Omega}{\partial U^2} - \left(\frac{\partial\Omega}{\partial U}\right)^2\right), \\ T_{VV} &= \frac{1}{12\pi}\left(\frac{\partial^2\Omega}{\partial V^2} - \left(\frac{\partial\Omega}{\partial V}\right)^2\right), \\ T_{UV} &= \frac{1}{24\pi}\frac{\partial^2\Omega}{\partial U\partial V}. \end{aligned}$$

Proof. Making use of (3.35) we evaluate

$$\langle : T_{UU}[g] : \rangle_{C\omega} = e^{-2\Omega}\langle : T_{UU}[\tilde{g}_0] : \rangle_\omega + \theta_{UU},$$

from which it is clear that a similar result holds for $\langle : T_{VV}[g] : \rangle_{C\omega}$. for the remaining component, we obtain

$$\langle : T_{UV}[g] : \rangle_{C\omega} = e^{-2\Omega}\langle T_{UV}[\tilde{g}_0]\rangle_\omega + \frac{1}{24\pi}\frac{\partial^2\Omega}{\partial U\partial V}g_{\mu\nu}$$

Because of Corollary 3.4.1 we know that $\langle : T_{\mu\nu}[\tilde{g}_0] : \rangle_{C\omega} \equiv 0$, this along with the definition of the θ symbols introduced in Proposition 3.4.1 proves the claim. □

3.5 Discussion

To obtain any of the components of the stress-energy tensor, we need to compute the second derivatives of Ω. This has been done numerically and we have set up our numerical scheme so that the following quantities are easy to obtain

$$\frac{\partial U}{\partial t}, \frac{\partial U}{\partial x}, \frac{\partial V}{\partial t}, \frac{\partial V}{\partial x}, \frac{\partial \Omega}{\partial t}, \frac{\partial \Omega}{\partial x}.$$

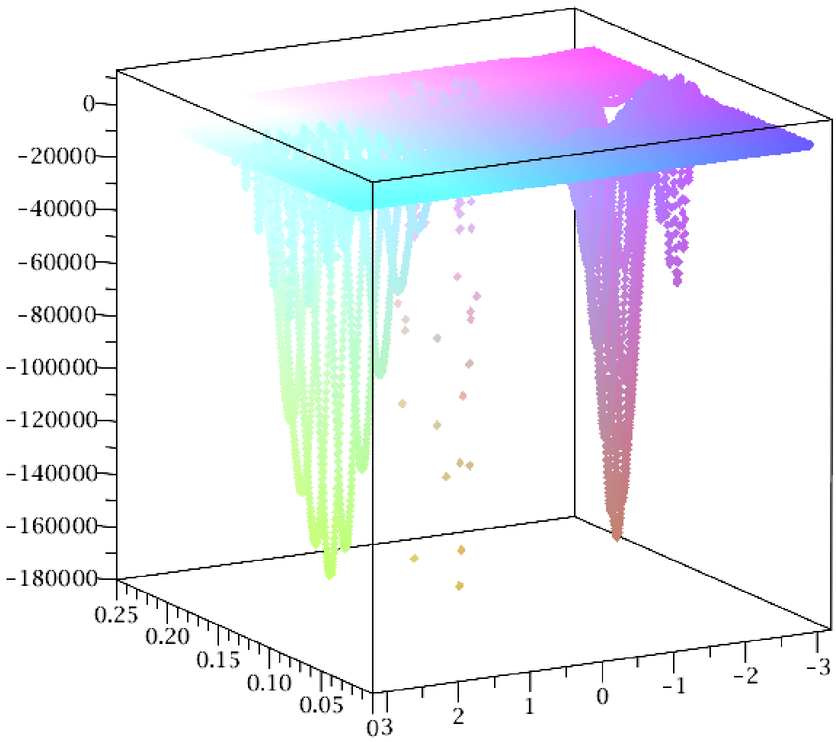


Figure 3.19: Energy density $\mathcal{E}$ (given in (3.37)) for our timelike observer.

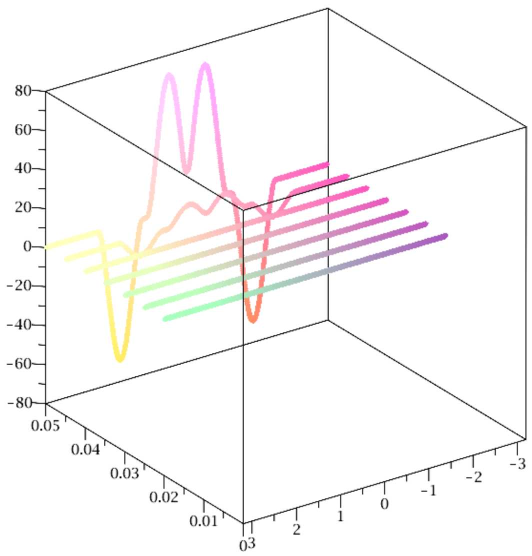


Figure 3.20: Detail of $\mathcal{E}$ (given in (3.37)) for our timelike observer with $0 \le t \le 0.055$. Note that as $\varepsilon = 1/20$, then $v > 0$ so we expect to have non-vanishing $\mathcal{E}$.

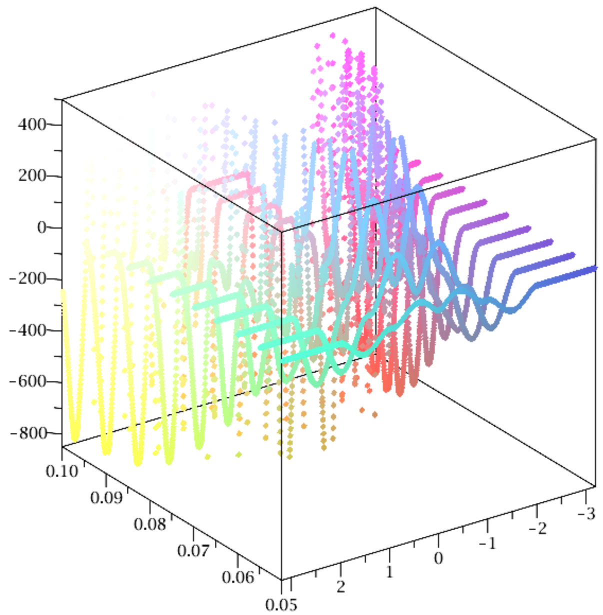


Figure 3.21: Detail of $\mathcal{E}$ (given in (3.37)) for our timelike observer with $0.05 \le t \le 0.105$. The outlying points are due to our numerical scheme.

Because of this, it is convenient to write the second derivatives of Ω in terms of these first derivatives. According to Corollary 3.3.1 we have

$$\frac{\partial}{\partial U} = \frac{e^{2\Omega}}{2}\left(\frac{\partial V}{\partial x}\frac{\partial}{\partial t} - \frac{\partial V}{\partial t}\frac{\partial}{\partial x}\right), \qquad \frac{\partial}{\partial V} = -\frac{e^{2\Omega}}{2}\left(\frac{\partial U}{\partial x}\frac{\partial}{\partial t} - \frac{\partial U}{\partial t}\frac{\partial}{\partial x}\right),$$

which in turn implies that

$$\frac{\partial^2\Omega}{\partial U^2} = \frac{e^{2\Omega}}{2}\left(\frac{\partial V}{\partial x}\frac{\partial}{\partial t} - \frac{\partial V}{\partial t}\frac{\partial}{\partial x}\right)\left[\frac{\partial \Omega}{\partial U}\right].$$

In order to gain further insight into the state expansion, we will study the energy density for a timelike observer in the (t, x) coordinates. The work done in Proposition 3.3.2 and Corollary 3.3.1 guarantees that all of the relevant quantities can be written as a function of these coordinates, so its usefulness will become clearer in a moment.

Since our goal is to compute the energy density for a timelike observer, we choose one with a constant timelike vector tangent to its trajectory given by $W = \partial/\partial U + \partial/\partial V$. For this observer, the energy density is given by

$$\mathcal{E} = T_{UU} + 2T_{UV} + T_{VV} \tag{3.37}$$

and its plots can be found in Figures 3.19, 3.24 and 3.25. More detailed plots for selected times are found in Figures 3.20, 3.21, 3.22 and 3.23 Moreover, the energy density at time $t = t_F$ for this observer is in Figure 3.26. Our current choice of parameters

$$r_T = 1,\ r_S = 1/2,\ t_F = 1/5,\ r_K = 9/10,$$

led to all of these plots. Note that from Figure 3.19 we can observe that the energy density remains zero in the protected region defined by $D_{g_0}(\{0\} \times (-r_K, r_K))$, as $r_K = 9/10$. In particular, our choice of parameters leads to $r_H = r_K - t_F = 7/10$ as it can be seen from (**SEP.VII**). Hence, from Figure 3.20 we can deduce that the energy density at time $t = t_F$ is zero within $(-r_H, r_H)$ for $r_H = 7/10$.

The numerical evidence provided by these plots therefore indicate that we can conclude that the Minkowski vacuum state was indeed extended to another

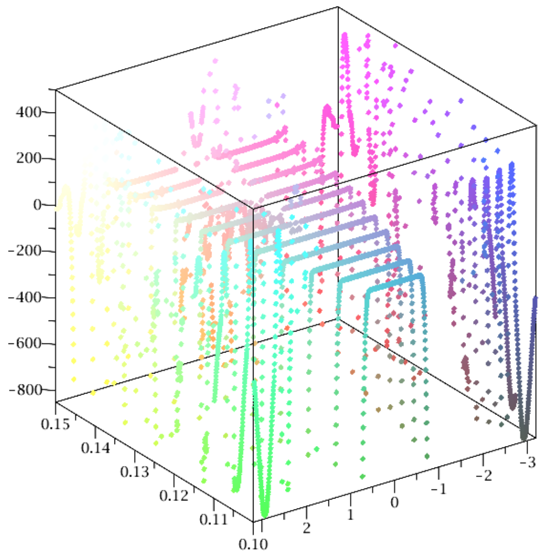


Figure 3.22: Detail of $\mathcal{E}$ for our timelike observer with $0.10 \leq t \leq 0.15$. The outlying points are due to our numerical scheme.

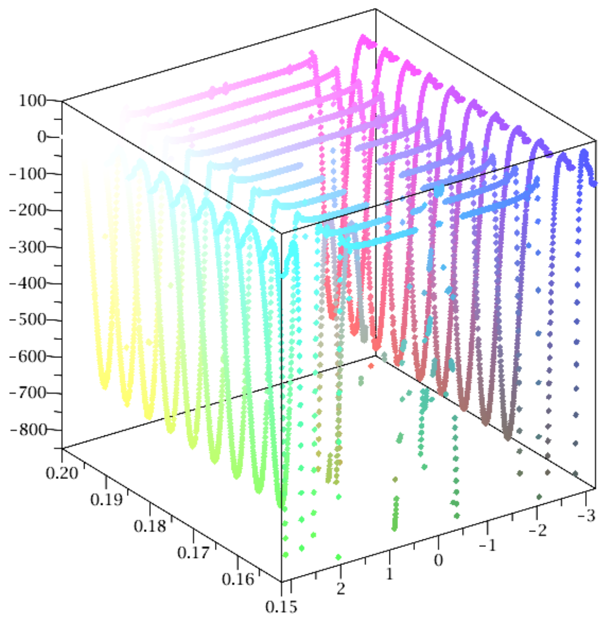


Figure 3.23: Detail of $\mathcal{E}$ for our timelike observer with $0.15 \leq t \leq< 0.20$. Note that the energy density is indeed zero within the protected zone $(-r_H, r_H)$ where $r_H = 7/10$.

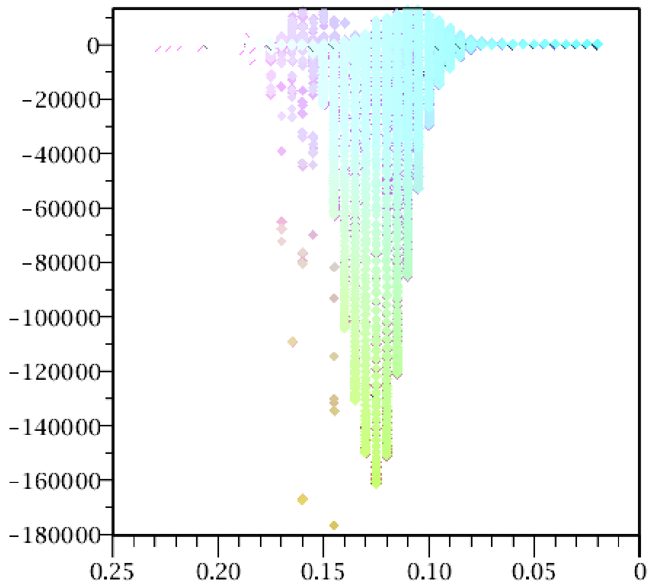


Figure 3.24: $\mathcal{E}$ for our timelike observer projected into the $t - \mathcal{E}$ plane.

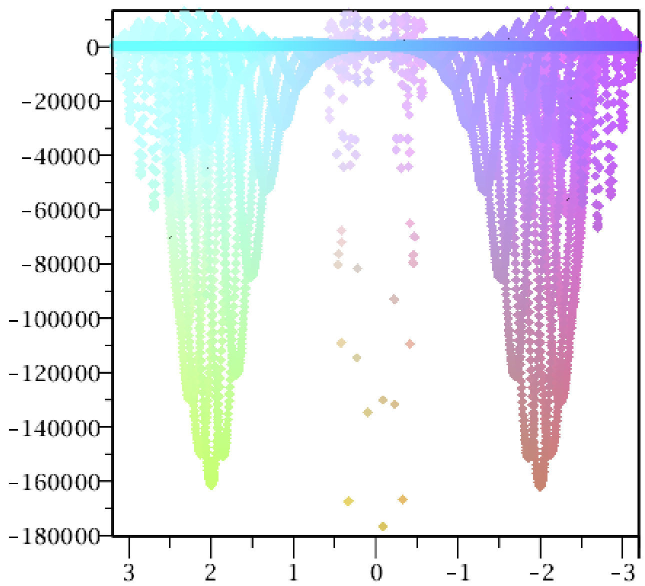


Figure 3.25: $\mathcal{E}$ for our timelike observer projected into the $R - \mathcal{E}$ plane.

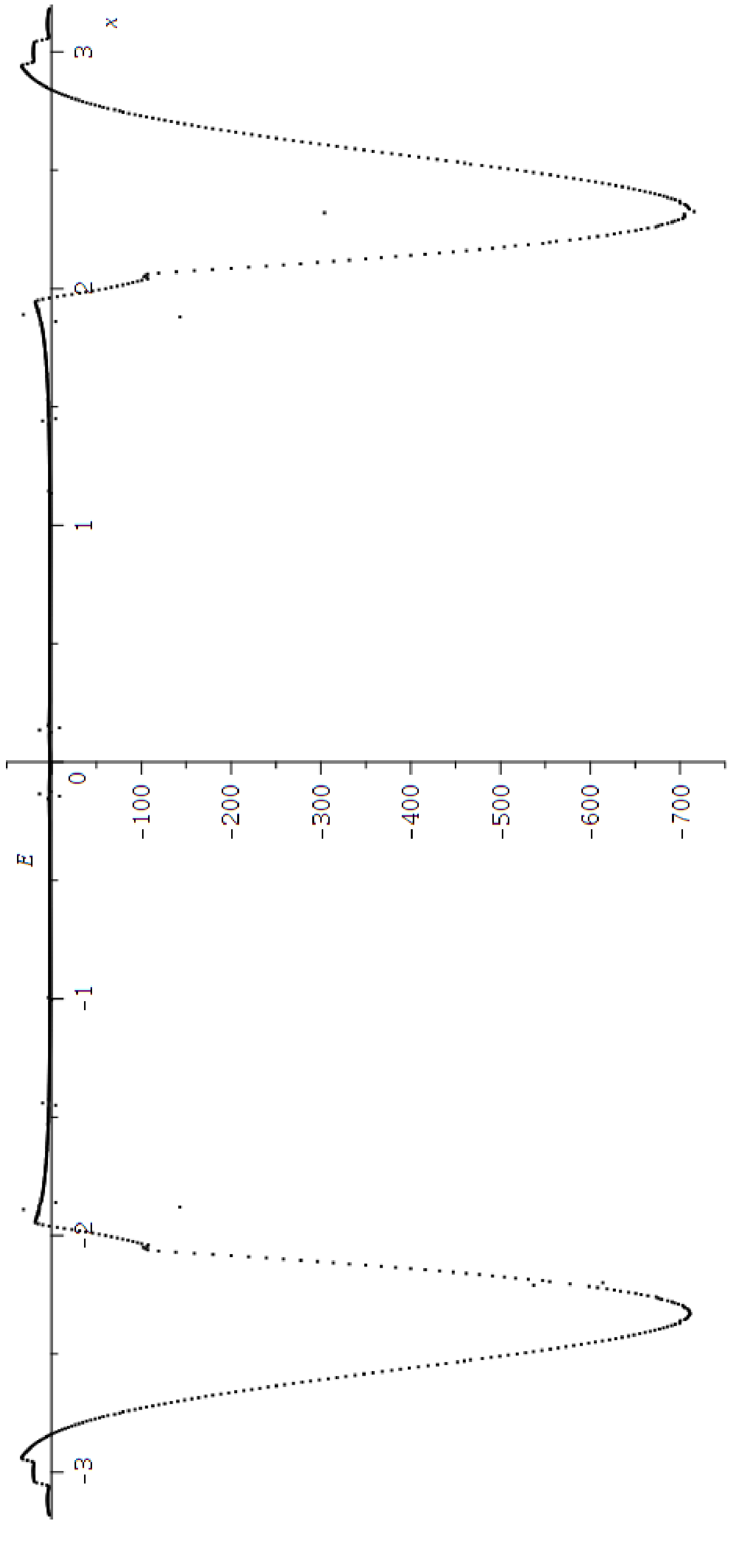


Figure 3.26: $\mathcal{E}$ for our timelike observer at $t = t_F$.

Hadamard state defined in a larger region $D_{g_0}(\{t_F\} \times (-r_F - r_T, r_F + r_T))$, while being preserved in $D_{g_0}(\{0\} \times (-r_K, r_K))$. Moreover, there is agreement with the extended state and the Minkowski vacuum at $D_{g_0}(\{0\} \times (-r_S, r_S))$.

It might seem that we have found a contraption to extend the two-dimensional Minkowski vacuum onto a larger region while keeping it Hadamard and in agreement with the original state in a subset of the original region. However, this is not a special feature of $1 + 1$ dimensional QFT. The purpose of this chapter was to show that it can be done and to illustrate the main ideas behind this extension procedure. In the next chapter we will show why this can be done and not only with diamonds in two-dimensional Minkowski spacetime, but rather more general settings.

— 4 —

Hadamard State Extension

There are things known and there are thing unknown, and in between are the doors of perception.

Aldous Huxley, The Doors of Perception

In Chapter 3 we showed that a state can be extended from a region T onto a larger region $\tilde{S}$ while keeping it Hadamard, if one introduces another region $S \subset T$ and gives up knowledge of the states on $T \setminus S$, which can be made arbitrarily small. This involved the construction of some additional auxiliary spacetimes with very specific characteristics and the use of a modified warp-drive metric. In our exposition of this, it seems that we just made some clever decisions that are exclusive to vacuum states in two-dimensional Minkowski spacetime. However, this is not the case as there are deeper reasons as to why the regions and the metric had to be specified in such a way. From the two-dimensional example one can observe that treating each region as a spacetime on its own right and the embeddings relating these regions are crucial elements of our construction.

The LCQFT framework is precisely the ideal one to deal with these notions and in this chapter we will explain why each region was specified with such characteristics and how this fits in a more general setting, not only for two-dimensional spacetimes but rather globally hyperbolic spacetimes in an arbitrary number of dimensions. Also, we will discuss some possible applications to quasifree states, symmetries and multiple extension–in some cases it is possible to extend to the whole of the spacetime. Finally, it must be noted that as it will be seen soon, spacetimes will be denoted in bold font and this will mean that they are objects in a category. However, if one wants to make any comparison between what is written in this Chapter and our two-dimensional model (c.f. Chapter 3), this makes little difference.

4.1 PRELIMINARIES

From now on we will assume that the **timeslice property** (see 2.5.4) holds, that is: if ψ happens to be a Cauchy morphism, then $\mathscr{A}(\psi)$ will be an isomorphism, i.e. a morphism for which a two-sided inverse $\mathscr{A}(\psi)^{-1}$ exists. Also, in Chapter 2 we introduced some concepts that now will have to be made compatible with the LCQFT framework, we will reformulate this into the category theory language. We begin by introducing the notion of a *state space*.

DEFINITION 4.1.1. Let us denote the set of all states on the algebra by $\mathscr{A}^*_{+,1}$. Then, a *state space* **S** for an algebra $\mathscr{A}$, is a set S of states on $\mathscr{A}$ that is algebraically closed under operations in $\mathscr{A}$ and under finite convex sums. In more concrete terms, let $\omega \in \mathbf{S}$ be a state, then for any $A,\ B_i \in \mathscr{A}$ and $\lambda_i > 0$ (with $\omega(B_i^* B_i) > 0$ and $\sum_i \lambda_i = 1$), we have that the state ω' defined by

$$\omega'(A) = \sum_{i=1}^{N} \lambda_i \frac{\omega(B_i^* A B_i)}{\omega(B_i^* B_i)}$$

is also in **S**. With the additional condition that $\sum_i \lambda_i \omega_i \in \mathbf{S}$ for a family of states $\omega_i \in \mathbf{S}$.

Next, we will define the category of *topological convex spaces* **TCvx**. First, we recall that a convex space is a set admitting abstract form of convex combinations subject to axioms (see [34, Section 3] for discussion and references, and [40] for the original definitions).

DEFINITION 4.1.2 (Topological Convex Space)**.** Let X and Y be convex spaces. A map f is said to respect convex combinations $pX + (1-p)Y$ (with $p \in [0,1]$) if, for all p we have $f(pX + (1-p)Y) = pf(X) + (1-p)f(Y)$. Then, the category of topological convex spaces **TCvx** is defined as:

TCvx Each **object** is a convex space C equipped with a topology in which forming convex combinations is continuous as a function $[0,1] \times C \times C \to C$ with respect to that topology. The **morphisms** are the continuous functions respecting convex combinations.

There is a contravariant functor $\mathbf{Alg} \xrightarrow{\mathscr{T}} \mathbf{TCvx}$ defined by

$$\mathscr{A} \mapsto \mathscr{A}^*_{+,\mathbf{1}} \text{ (equipped the with weak-} * \text{ topology)}$$
$$\alpha \mapsto \alpha^*|,$$

where $\mathscr{A}$ and $\mathscr{A}^*_{+,\mathbf{1}}$ denote a $*$- algebra and its full set of states, respectively. We remind the reader that the weak-$*$ topology on $\mathscr{A}^*$ is the weakest topology that makes all elements of $\mathscr{A}$ continuous as linear functionals on $\mathcal{A}^*$. Furthermore, the bar in $\alpha^*|$ denotes the restriction of α^* to a map between $\mathscr{B}^*_{+,\mathbf{1}} \to \mathscr{A}^*_{+,\mathbf{1}}$ that is both automatically convex and continuous under the weak-$*$ topology.

DEFINITION 4.1.3. A **state space** for a theory $\mathscr{A} : \mathbf{Loc} \to \mathbf{Alg}$ is any subfunctor $\mathcal{S}$ of $\mathscr{T} \circ \mathscr{A} : \mathbf{Loc} \to \mathbf{TCvx}$ such that $\mathcal{S}(\mathbf{M}) \subset \mathscr{A}^*_{+,\mathbf{1}}(\mathbf{M})$ is closed with respect to operations induced by $\mathscr{A}(\mathbf{M})$. In other words, for a morphism $\psi : \mathbf{M} \to \mathbf{N}$ in **Loc**, we have

$$\mathcal{S}(\psi) : \mathcal{S}(\mathbf{N}) \to \mathcal{S}(\mathbf{M})$$
$$\mathcal{S}(\psi) = \mathscr{T}(\mathscr{A}(\psi)) = \mathscr{A}(\psi)^*|_{\mathcal{S}(\mathbf{N})}.$$

Remark 4.1.1. There are two possibilities when mapping states, depending on whether the morphism $\psi : \mathbf{M} \to \mathbf{N}$ is Cauchy or not.

- Let ψ be **any** morphism and $\omega_{\mathbf{N}} \in \mathcal{S}(\mathbf{N})$, then $\mathcal{S}(\psi)\omega_{\mathbf{N}} \in \mathcal{S}(\mathbf{M})$. More concretely, $\mathcal{S}(\psi) : \mathcal{S}(\mathbf{N}) \to \mathcal{S}(\mathbf{M})$.
- Let ψ be a **Cauchy** morphism and $\omega_{\mathbf{M}} \in \mathcal{S}(\mathbf{M})$, then $\left(\mathscr{A}(\psi)^{-1}\right)^* |_{\mathcal{S}(\mathbf{M})}\omega_{\mathbf{M}} \in \mathcal{S}(\mathbf{N})$, i.e. $\left(\mathscr{A}(\psi)^{-1}\right)^* |_{\mathcal{S}(\mathbf{M})} : \mathcal{S}(\mathbf{M}) \to \mathcal{S}(\mathbf{N})$. Hence, $\left(\mathscr{A}(\psi)^{-1}\right)^* |_{\mathcal{S}(\mathbf{M})} = \mathcal{S}(\psi)^{-1}$.

4.1.1 The General State Extension Problem

All of the above lays the groundwork required for stating in detail the problem introduced in Chapter 3. For readability let us write it once more:

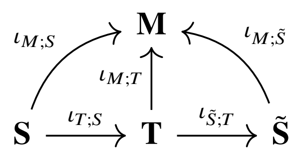


Figure 4.1: Original spacetimes provided in the GSEP.

> Suppose that you have the same spacetime with (M, g_0), T, $\tilde{S}$ as before, suppose $S \subset T$ is specified as shown in Figure 3.2. Then, can one extend an arbitrary Hadamard state from the smaller region (S) to the larger one ($\tilde{S}$) while keeping it Hadamard?

The question above can be reformulated in more formal and detailed terms as follows:

General state extension problem

Given a locally covariant theory $\mathscr{A}$ and a state space $\mathcal{S}$ for it, consider the following question: Fixing $\mathbf{M} \in \mathbf{Loc}$ and a causally convex open set (c.f. Def 2.1.5) $T \subset M$, can one find causally convex subsets S, $\tilde{S} \subset M$ with $S \subset T \subset \tilde{S}$ and a continuous convex map $\mathcal{E} : \mathcal{S}(\mathbf{T}) \to \mathcal{S}(\tilde{\mathbf{S}})$ such that $\mathcal{S}(\iota_{\tilde{S};S}) \circ \mathcal{E} = \mathcal{S}(\iota_{T;S})$? If so, we say that the triple $(S, \tilde{S}, \mathcal{E})$ solves the state extension problem for $(\mathscr{A}, \mathcal{S}, \mathbf{M}, T)$.

Next, we prove that the *General State Extension Problem* (GSEP) can be rewritten as a geometrical problem if we construct additional auxiliary spacetimes and suitable morphisms between them.

Theorem 4.1.1 (General State Extension). *Let $(\mathscr{A}, \mathcal{S})$ be a locally covariant theory with state space satisfying the timeslice condition. For any fixed $\mathbf{M} \in \mathbf{Loc}$ and causally convex open subset T in $\mathbf{M}$, suppose that there exist commuting diagrams in $\mathbf{Loc}$ such as the one in Figure 4.1 for the original regions provided in the GSEP. Moreover, for some additional auxiliary regions we have a Diagram such as in Figure 4.2, where the arrows marked with "c" are Cauchy and $\mathbf{T} = \mathbf{M}|_T$, $\mathbf{S} = \mathbf{M}|_S$ and $\tilde{\mathbf{S}} = \mathbf{M}|_{\tilde{S}}$ for some open causally convex sets $S, \tilde{S}$ of $\mathbf{M}$. Then, setting*

$$\mathcal{E} = \mathcal{S}(\iota_{\tilde{S};F})^{-1} \circ \mathcal{S}(\iota_{E;F}) \circ \mathcal{S}\left(\iota_{E;P}\right)^{-1} \circ \mathcal{S}\left(\iota_{T;P}\right),$$

the triple $(S, \tilde{S}, \mathcal{E})$ solves the General State Extension Problem for $(\mathscr{A}, \mathcal{S}, T, M)$.

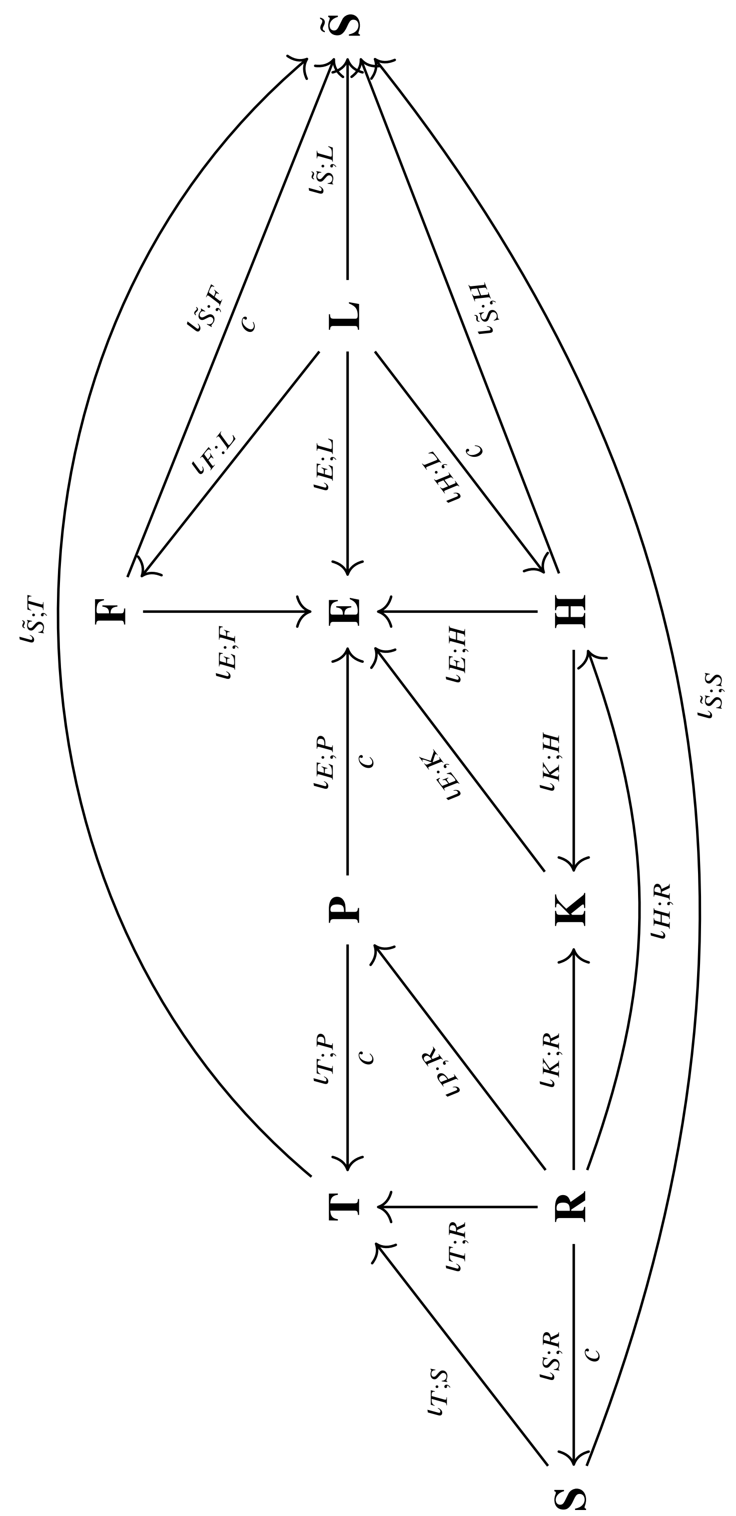


Figure 4.2: Additional auxiliary spacetimes needed to solve the GSEP.

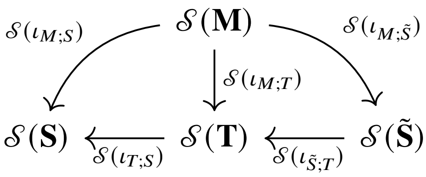


Figure 4.3: Original spacetimes provided in the GSEP.

Proof. Consider the Diagram in Figure 4.2, which is comprised by objects and morphisms in **Loc**. Next, in virtue of Definition 4.1.3 we apply the contravariant functor $\mathcal{S}$ to each object and morphism in Diagrams 4.1 and 4.2 which yields the following commuting diagrams where $\simeq$ indicates the existence of $\mathcal{S}^{-1}$–opening the possibility to reverse the arrow–since there is an algebra morphism due to the fact that $\iota_{E;P}$, $\iota_{H;L}$ and $\iota_{\tilde{S};F}$ are Cauchy morphisms. Hence, due to Remark 4.1.1 the morphisms $\mathcal{S}(\iota_{E;P})^{-1}$, $\mathcal{S}(\iota_{H;L})^{-1}$ and $\mathcal{S}(\iota_{\tilde{S};F})^{-1}$ must exist. Next, we will find the formula for the convex map by making use of the tools we have introduced so far: the LCQFT framework, Diagram 4.4 and, the ability to move states discussed in Remark 4.1.1. Then, making use of the aforementioned tools, a state $\omega_{\mathbf{T}} \in \mathscr{A}(\mathbf{T})^*$ may be mapped to a state $\omega_{\tilde{\mathbf{S}}} \in \mathscr{A}(\tilde{\mathbf{S}})$ by making use of $\omega_{\tilde{\mathbf{S}}} = \mathcal{E} \circ \omega_{\mathbf{T}}$, where $\mathcal{E}$ is given by following formula

$$\mathcal{E} = \mathcal{S}(\iota_{\tilde{S};F})^{-1} \circ \mathcal{S}(\iota_{E;F}) \circ \mathcal{S}\left(\iota_{E;P}\right)^{-1} \circ \mathcal{S}\left(\iota_{T;P}\right). \tag{4.1}$$

Now, we want to check that the restriction of $\omega_{\mathbf{T}}$ to $\mathbf{S}$ is the same as the restriction of $\omega_{\tilde{\mathbf{S}}}$ to $\mathbf{S}$, i.e. $\mathcal{S}(\iota_{\tilde{S};S})(\mathcal{E}(\omega_{\mathbf{T}})) = \mathcal{S}(\iota_{T;S})(\omega_{\mathbf{T}})$. For this purpose, we turn our attention to Diagram 4.4 and notice that all of the following relations hold due to commutativity

$$\mathcal{S}(\iota_{T;S}) = \mathcal{S}(\iota_{S;R})^{-1} \circ \mathcal{S}(\iota_{P;R}) \circ \mathcal{S}(\iota_{T;P}) \tag{4.2}$$

$$\mathcal{S}(\iota_{P;R}) = \mathcal{S}(\iota_{H;R}) \circ \mathcal{S}(\iota_{E;H}) \circ \mathcal{S}(\iota_{E;P})^{-1} \tag{4.3}$$

$$\mathcal{S}(\iota_{E;H}) = \mathcal{S}(\iota_{H;L})^{-1} \circ \mathcal{S}(\iota_{E;L}) \tag{4.4}$$

$$\mathcal{S}(\iota_{E;L}) = \mathcal{S}(\iota_{F;L}) \circ \mathcal{S}(\iota_{E;F}) \tag{4.5}$$

$$\mathcal{S}(\iota_{F;L}) = \mathcal{S}(\iota_{\tilde{S};L}) \circ \mathcal{S}(\iota_{\tilde{S};F})^{-1}. \tag{4.6}$$

Next, we insert (4.6) into (4.5) and then substitute this in (4.4) to obtain

$$\mathcal{S}(\iota_{E;H}) = \mathcal{S}(\iota_{H;L})^{-1} \circ \mathcal{S}(\iota_{\tilde{S};L}) \circ \mathcal{S}(\iota_{\tilde{S};F})^{-1} \circ \mathcal{S}(\iota_{E;F}), \tag{4.7}$$

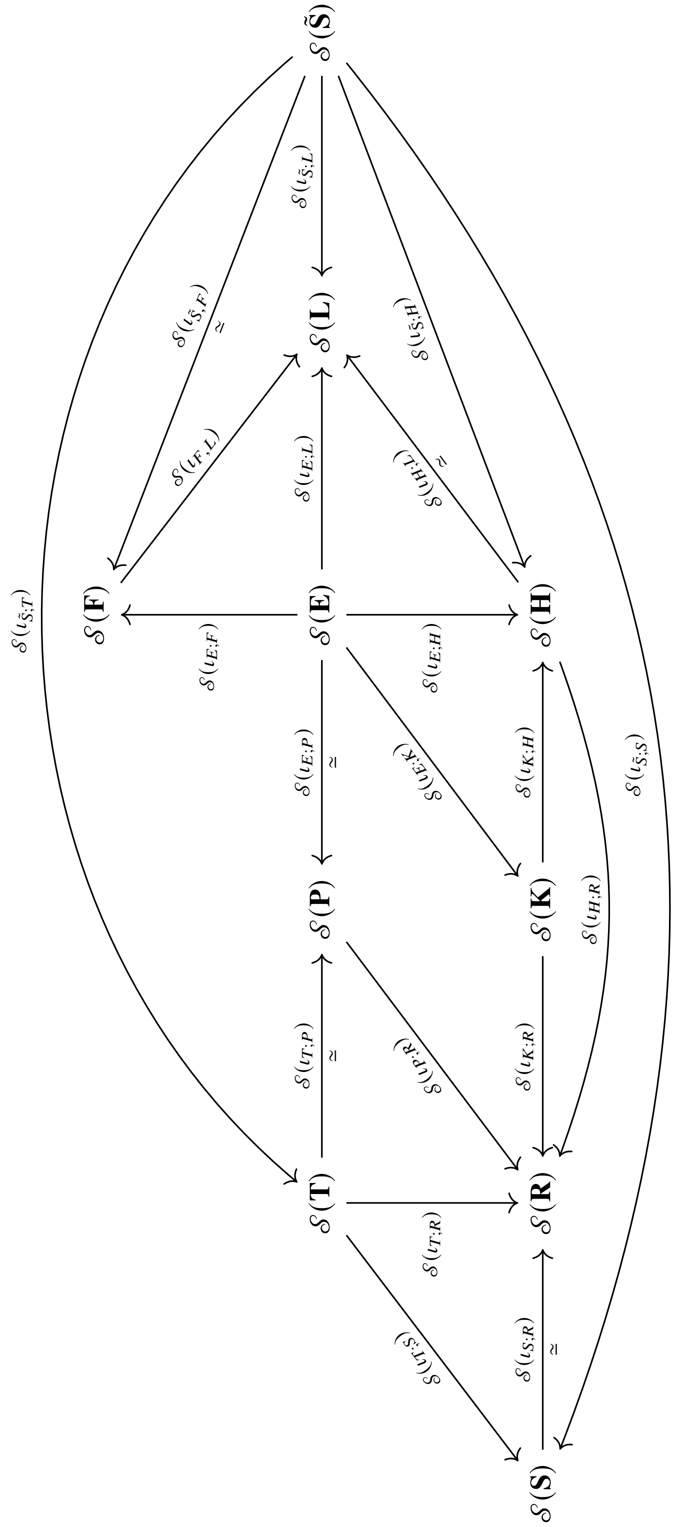


Figure 4.4: Diagram (4.2) after applying the State Space functor $\mathcal{S}$.

substituting (4.7) into (4.3) yields

$$\mathscr{S}(\iota_{P;R}) = \mathscr{S}(\iota_{H;R}) \circ \mathscr{S}(\iota_{H;L})^{-1} \circ \mathscr{S}(\iota_{\tilde{S};L}) \circ \mathscr{S}(\iota_{\tilde{S};F})^{-1} \circ \mathscr{S}(\iota_{E;F}) \circ \mathscr{S}(\iota_{E;P})^{-1}. \tag{4.8}$$

Finally, putting (4.8) into (4.2) gives

$$\begin{aligned}\mathscr{S}(\iota_{T;S}) =& \mathscr{S}(\iota_{S;R})^{-1} \circ \mathscr{S}(\iota_{H;R}) \circ \mathscr{S}(\iota_{H;L})^{-1} \circ \mathscr{S}(\iota_{\tilde{S};L}) \\ & \circ \mathscr{S}(\iota_{\tilde{S};F})^{-1} \circ \mathscr{S}(\iota_{E;F}) \circ \mathscr{S}(\iota_{E;P})^{-1} \circ \mathscr{S}(\iota_{T;P}),\end{aligned}$$

since the first four terms obey $\mathscr{S}(\iota_{S;R})^{-1} \circ \mathscr{S}(\iota_{H;R}) \circ \mathscr{S}(\iota_{H;L})^{-1} \circ \mathscr{S}(\iota_{\tilde{S};L}) = \mathscr{S}(\iota_{\tilde{S};S})$ and the remaining terms are $\mathcal{E}$, our result is

$$\mathscr{S}(\iota_{T;S}) = \mathscr{S}(\iota_{\tilde{S};S}) \circ \mathcal{E},$$

which completes our proof. □

Note that, while the diagram (4.4) implies that

$$\mathscr{S}(\iota_{T;S}) = \mathscr{S}(\iota_{\tilde{S};S}) \circ \mathcal{E} = \mathscr{S}(\iota_{T;S}) \circ \mathscr{S}(\iota_{\tilde{S};T}) \circ \mathcal{E},$$

the morphism $\mathscr{S}(\iota_{T;S})$ is not expected to be monic, so we may not conclude that $\mathscr{S}(\iota_{\tilde{S};T}) \circ \mathcal{E}$ is the identity on $\mathscr{S}(\mathbf{T})$; that is, the extended state will generally differ from the original outside S. Let us comment on the roles of some of these regions. As we mentioned before our original region is **T**. To expand it into a larger region we may take a lozenge **P** at time $t = 0$ and then consider its Cauchy development under the metric g, which we denote by **E**. This metric is built so that its light-cones tilt smoothly in such a way that (a) at $t = 0$ and $t = t_F$ they agree with those of (M, g_0), (b) the light-cones also tilt for **K** and all of the spacetimes within it. Since the light cones in **E** tilt back to those of **M** after time t_F, we may take another lozenge F and define the extended region $\tilde{S}$ as its Cauchy development under g_0. The region **S** is that where the extended state agrees with the one originally specified on **T**.

Regions such as **K**, **L** and **H** are needed to make the diagram commutative so that the restriction of the original and extended to **S** is well-defined. This is because the restriction to **S** of extended state $\mathcal{E}\omega$ is determined by the restriction of this same state to **H** and in particular to **L**. This, by uniqueness of solutions of the Klein-Gordon equation (c.f. Theorem 3.2.1 and its companion Remark 3.2.1) is determined by the restriction of the original state ω to restricted to K, which in turn is determined by the original state in **T**. Therefore, **K**, **L** and **H** act as interpolators that guarantee that $\mathcal{E}\omega|_{\mathcal{A}(S)} = \omega|_{\mathcal{A}(S)}$.

After inspecting the proof it becomes clear that due to functoriality, we can solve the (GSEP) as long as we can build Diagram 4.2 in **Loc**. Finding this construction is not a minor undertaking and it is a problem on its own right: the *Region Expansion Problem* (REP), which is explained in more detail and solved in section 4.2.

4.1.2 The G-Equivariant State Extension Problem

Now suppose that **M** has a nontrivial group of symmetries Aut(**M**). We will formulate a symmetric version of the general state extension problem and show how it may also be reduced to a geometric problem. In particular, this allows symmetries of the original state to be reflected in symmetries of the extended one. Let T be any open causally convex subset of **M** and suppose G is a subgroup of Aut(**M**). We say that T is G-symmetric relative to **M** if $\gamma(T) = T$ for each $\gamma \in G$, which implies that there is a homomorphism $\gamma \mapsto \gamma_T$ from G to Aut(**T**) such that

$$\iota_{M;T} \circ \gamma_T = \gamma \circ \iota_{M;T} \tag{4.9}$$

for all $\gamma \in G$. Of course, the underlying map of γ_T is simply the restriction of γ to T. This allows us to formulate a symmetric version of our previous problem:

THE G-EQUIVARIANT STATE EXTENSION PROBLEM

Suppose T is a causally convex open subset of M that is G-symmetric relative to M. A solution $(S, \tilde{S}, \mathcal{E})$ to the General State Extension Problem for $(\mathscr{A}, \mathscr{S}, T, M)$ is G-equivariant if S and $\tilde{S}$ are G-symmetric relative to **M** *and the extension map obeys*

$$\mathcal{E} \circ \mathscr{S}(\gamma_T) = \mathscr{S}(\gamma_{\tilde{S}}) \circ \mathcal{E} \qquad (\forall \gamma \in G). \tag{4.10}$$

In this case, we say that $(S, \tilde{S}, \mathcal{E})$ solves the G-Equivariant State Extension Problem for $(\mathscr{A}, \mathscr{S}, T, M, G)$.

In particular, a G-equivariant extension map extends G-invariant states on T to G-invariant states on $\widetilde{S}$. To proceed, it is useful to extend the notion of symmetry G from objects to diagrams. Namely, a diagram in **Loc** is said to be G-symmetric if there is a homomorphism $G \to \mathrm{Aut}(O)$, $\gamma \mapsto \gamma_O$ for every object O in the diagram, and that for every arrow $f : O \to O'$ in the diagram one has the compatibility condition

$$\gamma_{O'} \circ f = f \circ \gamma_O. \tag{4.11}$$

In particular, T is G-symmetric relative to $\mathbf{M}$ if and only if the diagram $\mathbf{T} \xrightarrow{\iota_{\mathbf{M};T}} \mathbf{M}$ is G-symmetric. More abstractly, any diagram in **Loc** can be identified with a functor $\Delta : \mathbf{I} \to \mathbf{Loc}$ for some small (in our case, finite) indexing category $\mathbf{I}$; G-symmetry of the diagram then means that there is a homomorphism $G \to \mathrm{Aut}(\Delta)$ (i.e., into the natural isomorphisms of Δ to itself). In our case we want to require additionally that $\gamma_{\mathbf{M}} = \gamma$ for all $\gamma \in G$, which is equivalent to saying that the symmetry of the diagram is compatible with/extends the action of G on $\mathbf{M}$.

THEOREM 4.1.2 (G-Equivariant State Extension)**.** *If there exists a commuting Diagram of the form 4.2 that satisfies the conditions stated in Theorem 4.1.1 and is additionally G-symmetric then $(S, \tilde{S}, \mathcal{E})$ solves the G-Equivariant State Extension Problem for $(\mathscr{A}, \mathcal{S}, T, M)$ with $\mathcal{E}$ defined as in* (4.1)*.*

Proof. We begin by noticing that $\gamma_O \in \mathrm{Aut}(O)$ are isomorphisms (in particular, Cauchy morphisms) for any $\gamma \in G$. This implies the existence of $\mathscr{A}(\gamma_O)^{-1}$ and hence of $\mathcal{S}(\gamma_O)^{-1}$. Next, consider γ_O, $\gamma_{O'}$ and an arrow $f : O \to O'$ in **Loc**. By virtue of the compatibility condition (4.11) and the existence of $\mathcal{S}(\gamma_O)^{-1}$, $\mathcal{S}(\gamma'_O)^{-1}$, we have

$$\mathcal{S}(\gamma_O) \circ \mathcal{S}(\iota_{O';O}) = \mathcal{S}(\iota_{O';O}) \circ \mathcal{S}(\gamma_{O'}) \tag{4.12}$$

$$\mathcal{S}(\iota_{O';O}) \circ \mathcal{S}(\gamma_{O'})^{-1} = \mathcal{S}(\gamma_O)^{-1} \circ \mathcal{S}(\iota_{O';O}). \tag{4.13}$$

From this one can see that,

$$\begin{aligned}\mathcal{E} \circ \mathcal{S}(\gamma_T) &= \mathcal{S}(\iota_{\tilde{S};F})^{-1} \circ \mathcal{S}(\iota_{E;F}) \circ \mathcal{S}\left(\iota_{E;P}\right)^{-1} \circ \mathcal{S}\left(\iota_{T;P}\right) \circ \mathcal{S}(\gamma_T) \\ &= \mathcal{S}(\iota_{\tilde{S};F})^{-1} \circ \mathcal{S}(\iota_{E;F}) \circ \mathcal{S}\left(\iota_{E;P}\right)^{-1} \circ \mathcal{S}(\gamma_P) \circ \mathcal{S}\left(\iota_{T;P}\right) \\ &= \mathcal{S}(\iota_{\tilde{S};F})^{-1} \circ \mathcal{S}(\iota_{E;F}) \circ \left(\mathcal{S}(\gamma_P)^{-1} \circ \mathcal{S}\left(\iota_{E;P}\right)\right)^{-1} \circ \mathcal{S}\left(\iota_{T;P}\right) \\ &= \mathcal{S}(\iota_{\tilde{S};F})^{-1} \circ \mathcal{S}(\iota_{E;F}) \circ \left(\mathcal{S}\left(\iota_{E;P}\right) \circ \mathcal{S}(\gamma_E)^{-1}\right)^{-1} \circ \mathcal{S}\left(\iota_{T;P}\right) \\ &= \mathcal{S}(\iota_{\tilde{S};F})^{-1} \circ \mathcal{S}(\iota_{E;F}) \circ \mathcal{S}(\gamma_E) \circ \mathcal{S}\left(\iota_{E;P}\right)^{-1} \circ \mathcal{S}\left(\iota_{T;P}\right),\end{aligned}$$

where in the first equality we just substituted $\mathcal{E}$, in the second we used (4.12) setting $O = P$ and $O' = T$, the third and fifth are just identities and, in the fourth we used (4.13) setting $O = P$ and $O' = E$. Continuing in a similar fashion, always trying to shift the symmetry term to the left, by using either (4.12) or (4.13) with the appropriate source and target regions, we get

$$\begin{aligned}\mathcal{E} \circ \mathcal{S}(\gamma_T) &= \mathcal{S}(\iota_{\tilde{S};F})^{-1} \circ \mathcal{S}(\iota_{E;F}) \circ \mathcal{S}(\gamma_E) \circ \mathcal{S}\left(\iota_{E;P}\right)^{-1} \circ \mathcal{S}\left(\iota_{T;P}\right) \\ &= \mathcal{S}(\iota_{\tilde{S};F})^{-1} \circ \mathcal{S}(\gamma_F) \circ \mathcal{S}(\iota_{E;F}) \circ \mathcal{S}\left(\iota_{E;P}\right)^{-1} \circ \mathcal{S}\left(\iota_{T;P}\right) \\ &= \left(\mathcal{S}(\gamma_F)^{-1} \circ \mathcal{S}(\iota_{\tilde{S};F})\right)^{-1} \circ \mathcal{S}(\iota_{E;F}) \circ \mathcal{S}\left(\iota_{E;P}\right)^{-1} \circ \mathcal{S}\left(\iota_{T;P}\right) \\ &= \left(\mathcal{S}(\iota_{\tilde{S};F}) \circ \mathcal{S}(\gamma_{\tilde{S}})^{-1}\right)^{-1} \circ \mathcal{S}(\iota_{E;F}) \circ \mathcal{S}\left(\iota_{E;P}\right)^{-1} \circ \mathcal{S}\left(\iota_{T;P}\right) \\ &= \mathcal{S}(\gamma_{\tilde{S}}) \circ \mathcal{S}(\iota_{\tilde{S};F})^{-1} \circ \mathcal{S}(\iota_{E;F}) \circ \mathcal{S}\left(\iota_{E;P}\right)^{-1} \circ \mathcal{S}\left(\iota_{T;P}\right) \\ &= \mathcal{S}(\gamma_{\tilde{S}}) \circ \mathcal{E},\end{aligned}$$

which is the required equivariance property (4.10). □

4.2 The Region Expansion Problem

In the previous section we demonstrated that the GSEP reduces to constructing auxiliary spacetimes and morphisms between them in the form of Diagrams 4.1 and 4.2 in the category **Loc**. In this section we first reduce that construction to a geometric *Region Expansion Problem* (REP), which we will then solve for conformally ultrastatic globally hyperbolic spacetimes, thereby solving the GSEP for these spacetimes.

Before starting, we remind the reader that if O is a causally convex set (c.f. Def 2.1.5), then its *Cauchy development* $D_g(O) \in \mathcal{O}(M)$ is the set of all the points in the manifold for which any inextendible piecewise-smooth causal curve that passes through them intersects O. Furthermore, we can define the future (past) Cauchy development in an analogous fashion but for past (future) directed causal curves. We denote this by $D_g^+(O)$ ($D_g^-(O)$).

Also, we introduce the *regular domains* and *standard form* concepts. A *regular domain* (c.f. [47][Proposition 5.46]) is an embedded codimension-0 manifold with boundary. A globally hyperbolic spacetime $\mathbf{M}$ with metric g_0 is said to be in standard form [8] if it is diffeomorphic to $\mathbb{R} \times \Sigma$ with metric $g_0 = \beta dt \otimes dt - h_t$, where for each $t \in \mathbb{R}$, $\{t\} \times \Sigma$ is a smooth spacelike Cauchy surface, h_t is the induced metric on it and $\beta : \mathbb{R} \times \Sigma \to (0, \infty)$ is smooth. Henceforth, we will require that dt is future-pointing.
The main idea in the REP is to make a geometrical construction that generalises the diagrams in Figures 3.6 and 3.7 in the setting of a globally hyperbolic spacetime.

THE REGION EXPANSION PROBLEM

Let Σ be a smooth spacelike Cauchy surface in globally hyperbolic spacetime $\mathbf{M}$ and suppose τ is the interior of a regular domain in Σ. Assume without loss of generality that Σ is identified with the $t = 0$ hypersurface in a standard form representation of $\mathbf{M}$ with metric g_0. The Region Expansion Problem for $(\tau, \Sigma, \mathbf{M})$ is to find a globally hyperbolic metric g on $\mathbb{R} \times \Sigma$, positive constants t_F, ε with $t_F > 2\varepsilon$, and subsets $\sigma, \lambda, \kappa, \phi$ of Σ such that the following conditions are met.

REP 1 Each set $\sigma, \lambda, \kappa, \phi$ is the interior of a regular domain in Σ, and the following nesting is satisfied:

$$\sigma \sqsubset \lambda \sqsubset \kappa \sqsubset \tau \sqsubset \phi \sqsubset \Sigma, \tag{4.14}$$

where $A \sqsubset B$ means that the closure of A is contained in B.

REP 2 The following relations between Cauchy developments hold:

$$D_g(\{0\} \times \tau) \cap (\{t_F\} \times \Sigma) = \{t_F\} \times \phi, \tag{4.15}$$

$$D_{g_0}(\{0\} \times \kappa) \cap (\{t_F\} \times \Sigma) = \{t_F\} \times \lambda, \tag{4.16}$$

$$D_{g_0}(\{t_F\} \times \lambda) \cap (\{0\} \times \Sigma) = \{0\} \times \sigma \tag{4.17}$$

$$D_{g_0}(\{0\} \times \tau) \subset D_{g_0}(\{t_F\} \times \phi). \tag{4.18}$$

REP 3 Let $\mathcal{I} = (-\varepsilon, \varepsilon) \cup (t_F - \varepsilon, t_F + \varepsilon)$, then $g = g_0$ when restricted to $\mathcal{I} \times \Sigma$ and $D_{g_0}(\{0\} \times \kappa) \cap ((-\varepsilon, t_F + \varepsilon) \times \Sigma)$.

REP 4 The hypersurfaces $\{0\} \times \tau$ and $\{t_F\} \times \phi$ are achronal with respect to g.

Moreover, we can extend this problem to include symmetries of the Cauchy surface in the following way: let G be any group of isometries of (Σ, h_t) where each $\gamma \in G$ induces an automorphism $\gamma_M = \mathrm{id} \times \gamma$ of $\mathbf{M}$ and $\gamma(\tau) = \tau$. Then, we demand in addition that

$$\gamma(\bullet) = \bullet, \quad (\text{for } \sigma, \lambda, \kappa, \phi) \tag{4.19}$$

$$g = \gamma_M^* g. \tag{4.20}$$

for all $\gamma \in G$. In this case we say that the symmetric region expansion problem has been solved.

THEOREM 4.2.1 (Region Expansion)**.** *Let $(\mathscr{A}, \mathscr{S})$ be a locally covariant theory with state space satisfying the timeslice condition. Let Σ be a smooth spacelike Cauchy surface in a globally hyperbolic spacetime $\mathbf{M}$, and suppose τ is the interior of a regular domain in Σ. Then any solution to the REP for $(\tau, \Sigma, \mathbf{M})$ may be used to construct a solution to the GSEP for $(\mathscr{A}, \mathscr{S}, T, \mathbf{M})$, where T is the Cauchy development of τ in $\mathbf{M}$. Moreover, if (Σ, h_t) admits an isometry group, then the symmetric REP can be solved.*

Proof. Consider any solution to the REP (using the notation introduced in the statement). We begin by noticing that, as $\mathbf{M}$ is in standard form, then

$$T = D_{g_0}(\{0\} \times \tau).$$

In a similar fashion, we introduce $\mathcal{T} := (-\varepsilon, t_F + \varepsilon)$ and the *past-originated sets*

$$P = ((-\varepsilon,\varepsilon)\times\Sigma)\cap T, \qquad S = D_{g_0}(\{0\}\times\sigma), \qquad R = S\cap P \tag{4.21}$$

$$K = D_{g_0}(\{0\}\times\kappa)\cap(\mathcal{T}\times\Sigma)\,. \tag{4.22}$$

Defining $\mathcal{T}' := (t_F - \varepsilon, t_F + \varepsilon)$, we can designate the *future-originated sets*

$$L = D_{g_0}\left(\{t_F\}\times\lambda\right)\cap(\mathcal{T}'\times\Sigma)\,, \qquad F = D_{g_0}(\{t_F\}\times\phi)\cap(\mathcal{T}'\times\Sigma) \tag{4.23}$$

$$H = D_{g_0}\left(\{t_F\}\times\lambda\right)\cap(\mathcal{T}\times\Sigma)\,, \qquad \tilde{S} = D_{g_0}\left(\{t_F\}\times\phi\right)\cap(\mathcal{T}\times\Sigma)\,. \tag{4.24}$$

Recalling that the Cauchy development of an acausal topological hypersurface is open and globally hyperbolic [53, Lem. 14.43], and that the intersection of two globally hyperbolic subsets is globally hyperbolic, the sets

$$T,\ S,\ P,\ R,\ K,\ F,\ L,\ H \quad \text{and} \quad \tilde{S}$$

are globally hyperbolic when equipped with the metric g_0 and serve as the underlying manifolds for the following objects in **Loc**

$$\mathbf{T},\ \mathbf{S},\ \mathbf{P},\ \mathbf{R},\ \mathbf{K},\ \mathbf{F},\ \mathbf{L},\ \mathbf{H} \quad \text{and} \quad \tilde{\mathbf{S}}.$$

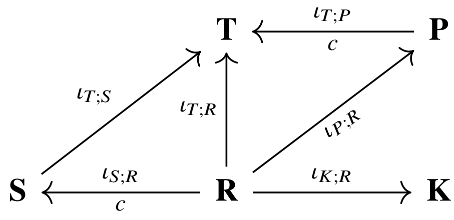
These spacetimes are endowed with the metric, orientation and time orientation induced from **M**. From (4.14) in **REP 1** it follows that

$$R\subset S\subset T, \quad R\subset P\subset T, \quad R\subset K\subset T, \quad L\subset F\subset\tilde{S}, \quad L\subset H\subset\tilde{S}. \tag{4.25}$$

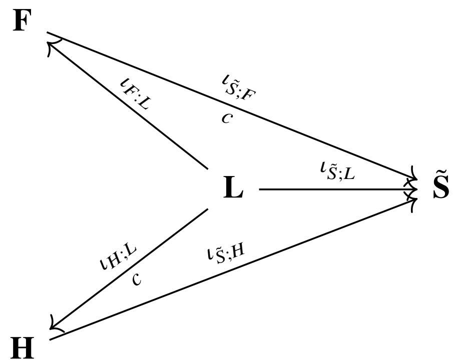
Next, we recall that if a set A is closed and achronal, then $\mathrm{Int}D(A)$, $\mathrm{Int}D^+(A)$ and $\mathrm{Int}D^-(A)$ are causally convex [49, Prop. 3.43]. Also, as any achronal topological hypersurface B is a Cauchy surface for its Cauchy development, we will have $D(B) = \mathrm{Int}D(B)$.

So, from this we note that all of the sets above are causally convex and since all of them share metric, orientation and time orientation and thus the obvious inclusions are morphisms in **Loc**. The latter fact also implies that $\{0\}\times\tau$ is a Cauchy surface for **T** and **P**, while $\{0\}\times\sigma$ is a Cauchy surface for **R** and **S**, and $\{t_F\}\times\phi$ is a Cauchy surface for **L**, **H**, **F** and $\tilde{\mathbf{S}}$. This means that the inclusions

$\iota_{T;P}$, $\iota_{S;R}$, $\iota_{\tilde{S};F}$ and $\iota_{H;L}$ are Cauchy morphisms. All this gives rise to the following parts of Diagram 4.2:



(4.26)



(4.27)

which commute trivially because all the morphisms are inclusions. Next, we define an open set E in terms of the Cauchy development of $\{0\} \times \tau$ under the modified metric g, by

$$E = D_g(\{0\} \times \tau) \cap (\mathcal{T} \times \Sigma), \tag{4.28}$$

As $\{0\} \times \tau$ is achronal with respect to g, by **REP 4**, we deduce that E is globally hyperbolic when endowed with g and has $\{0\} \times \tau$ as a Cauchy surface. Equipping E with metric g, orientation and time orientation agreeing with those of $\mathbf{M}$ on P, we obtain a spacetime $\mathbf{E} \in \mathbf{Loc}$.

Recalling the definition of $\mathcal{I} \subset \mathcal{T}$ in **REP 3**, we note that we have the nesting

$$\{0\} \times \tau \subset P \subset \mathcal{I} \times \Sigma. \tag{4.29}$$

Looking at **REP 3**, we observe that

$$g = g_0, \quad \text{(when restricted to } P,\ K,\ F). \tag{4.30}$$

Before proceeding any further, we would like to establish an intermediate result concerning Cauchy developments.

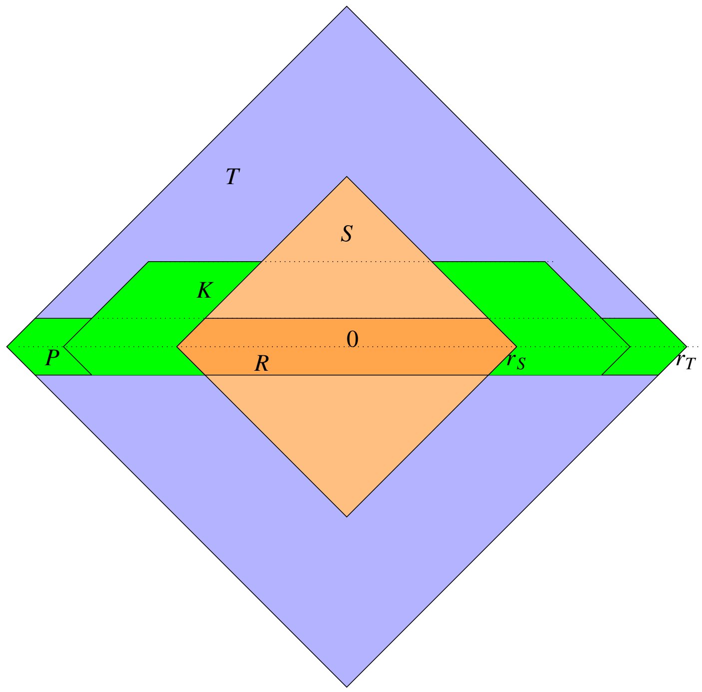


Figure 4.5: The regions defined so far.

LEMMA 4.2.1. *Let A and B be subsets of M. (a) If A and B are acausal obeying $A \subset D^+(B)$, then*

$$D(A) \subset D(B). \tag{4.31}$$

(b) If B is achronal and $B \subset A$ is such that $D_{g_0}(B) \cap A$ is timelike compact [58] and $g = g_0$ when restricted to $D_{g_0}(B) \cap A$, the following holds

$$D_{g_0}(B) \cap A \subset D_g(B) \cap A. \tag{4.32}$$

Proof. See Appendix B.1.1. □

Applying (4.32) in turn to $A = (-\varepsilon, \varepsilon) \times \Sigma$, $B = \{0\} \times \tau$ and $A = \mathcal{T} \times \Sigma$, $B = \{0\} \times \kappa$ yields

$$P = D_{g_0}(\{0\} \times \tau) \cap ((-\varepsilon, \varepsilon) \times \Sigma) \subset D_g(\{0\} \times \tau) \cap ((-\varepsilon, \varepsilon) \times \Sigma) \subset E, \tag{4.33}$$

$$K = D_{g_0}(\{0\} \times \kappa) \cap (\mathcal{T} \times \Sigma) \subset D_g(\{0\} \times \kappa) \cap (\mathcal{T} \times \Sigma) \subset E, \tag{4.34}$$

where the last inclusion in the first relation follows from the definition of $\mathcal{I} \subset \mathcal{T}$ in **REP 3** and in the second it was obtained by making use of the fact that $\kappa \sqsubset \tau$ according to (4.14) in **REP 1**.

From (4.30) we see that **P** and **K** share metric, orientation and time-orientation with **E**, this means that these inclusions induce the morphisms $\iota_{E;K},\ \iota_{E;P}$. Next, noting that (4.16) in **REP 3** implies that $\{t_F\} \times \phi \subset D_g(\{0\} \times \tau)$, we make use of (4.31) to obtain $D_g(\{t_F\} \times \phi) \subset D_g(\{0\} \times \tau)$. After taking the intersection with $\mathcal{T} \times \Sigma$ and using (4.32) with $A = \mathcal{T} \times \Sigma,\ B = \{t_F\} \times \phi$ this becomes

$$\begin{aligned} F = D_{g_0}(\{t_F\} \times \phi) \cap (\mathcal{T} \times \Sigma) &\subset D_g(\{t_F\} \times \phi) \cap (\mathcal{T} \times \Sigma) \\ &\subset D_g(\{0\} \times \tau) \cap (\mathcal{T} \times \Sigma) = E. \end{aligned} \tag{4.35}$$

Using the same argument we used before, we note that this inclusion induces the morphism $\iota_{E;F}$. Considering the definition of H given in (4.24) we can write $H = D_{g_0}(\{t_F\} \times \lambda) \cap (\mathcal{T} \times \Sigma)$. Moreover, using (4.16) in **REP 2**, (4.31) and (4.32) we deduce $D_{g_0}(\{t_F\} \times \lambda) \subset D_{g_0}(\{0\} \times \kappa)$, after taking the intersection with $\mathcal{T} \times \Sigma$ we obtain

$$H \subset K \quad \Longrightarrow \quad H \subset E, \quad L \subset K \quad \text{and} \quad L \subset E, \tag{4.36}$$

which in the same fashion as before, gives the morphisms $\iota_{E;L},\ \iota_{K;H},\ \iota_{E;H}$. The same argument can be used with (4.18) from **REP 2** to show that $S \subset H$, this along with (4.25) lets us conclude that

$$R \subset H, \tag{4.37}$$

using the same argument one last time this results in the morphism $\iota_{H;R}$.

Exploiting the achronality of $\{0\} \times \tau$ once more, we realise that it is also a Cauchy surface for **E**. Recalling that it was also a Cauchy surface for **P**, we conclude that the inclusion $\iota_{E;P}$ is also a Cauchy morphism. The inclusions between these regions and **E** can be summarised in the following diagram, which

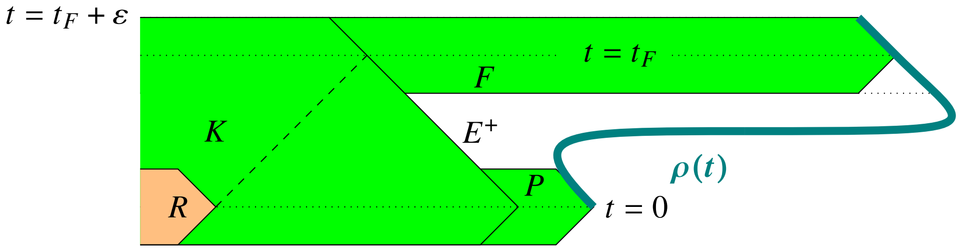
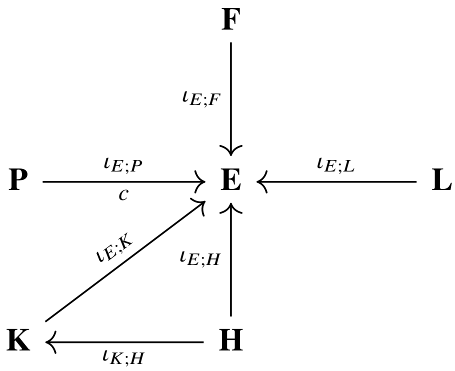


Figure 4.6: Detail of the construction for both cases. We have $g = g_0$ in green regions and to the future of the dotted line extending the past boundary of F.

is also part of Diagram 4.2



(4.38)

which also commutes because as in the previous diagram, all of the morphisms involved are inclusions. Next, we join the diagrams (4.26, 4.27) and (4.38) by introducing the inclusions $\iota_{\tilde{S};S}$, $\iota_{H;R}$ and $\iota_{\tilde{S};T}$.

As all of the regions defined by these sets share metric, orientation and time-orientation, we see that it suffices to prove that $T \subset \widetilde{S}$ and $R \subset H$–as $S \subset T$ implies $S \subset \widetilde{S}$. Considering condition (4.18) from **REP 2** and intersecting both sides with $\mathcal{T} \times \Sigma$ we obtain $T \subset \widetilde{S}$, which means that

$$S \subset T \subset \widetilde{S}$$

and that the morphisms $\iota_{\tilde{S};S}$, $\iota_{\tilde{S};T}$ are well-defined and hence we have completed Diagram 4.2. The only morphisms that we need to complete the Diagram 4.1 are the trivial inclusions $\iota_{M;S}$, $\iota_{M;T}$ and $\iota_{M;\tilde{S}}$. Given that all the morphisms appearing in Diagrams 4.1 and 4.2 correspond to subset inclusions, the diagram commutes in full.

Furthermore, each isometry γ of (Σ, h_t) induces an automorphism γ_M, its action on T is

$$\gamma_M\,[T] = \gamma_M\left[D_{g_0}(\{0\}\times\tau)\right] = D_{\gamma_M^* g_0}\,(\gamma_M[\{0\}\times\tau)]) = D_{g_0}\,(\{0\}\times\tau) = T,$$

where to obtain the last equality we used (4.19) and that

$$\gamma_M^* g_0 = \gamma_M^*(1\oplus -h) = (\mathrm{id}\times\gamma)^*(1\oplus -h) = 1\oplus -h = g_0$$

because γ is an isometry of (Σ, h). The same formula holds if τ and T are replaced by (σ, S), (κ, K), (λ, L) and (ϕ, F). This implies that

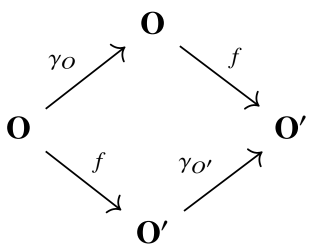
$$\gamma_M(\bullet) = \bullet, \quad (\text{for } S,\ T,\ K,\ L,\ F),$$

which is precisely the notion of symmetry of a set introduced in Section 4.1.2. Moreover, the symmetry is carried on to E, as it can be seen from

$$\gamma_M\,[E] = \gamma_M\left[D_g(\{0\}\times\tau)\right] = D_{\gamma_M^* g}\,(\gamma_M[\{0\}\times\tau)]) = D_g\,(\{0\}\times\tau) = E,$$

where we have used (4.19) and (4.20) to get the last equality. Given that the definition of the remaining spacetimes in Diagram 4.2 depends on the ones we just mentioned, all of its objects are symmetric.

Next, we need to prove that the diagram itself is symmetric, for this we note for any $\mathbf{O},\ \mathbf{O}'$ in said diagram such that $f:\mathbf{O}\to\mathbf{O}'$, we have



which implies that we have $f\circ\gamma_O = \gamma_{O'}\circ f$, which corresponds to the compatibility condition (4.11) that is needed for a diagram to be G-symmetric. □

4.3 SOLUTION TO THE REP IN CONFORMALLY ULTRASTATIC SPACETIMES

In this section we will solve the region expansion problem in spacetimes with conformally ultrastatic metrics, that is g_0 as in Section 4.2 with $\beta \equiv 1$ and

$\pounds_{\partial/\partial t} h_t = 0$ and for any regular domain interior τ whose boundary has nonzero normal injectivity radius–the largest radius such that the normal exponential map at τ is still a diffeomorphism. Moreover, as the Cauchy developments in the REP are conformally invariant, we will present a solution in ultrastatic spacetimes that is valid for their conformally related counterparts. As it will be seen shortly, if among other things we assume that $\partial\tau$ has non-zero injectivity radius, then we can solve the REP for ultrastatic spacetimes.

Our solution relies on the metric h adopting a block diagonal form near $\partial\tau$. This is analogous to adopting Gaussian normal coordinates [45][Proposition 3.2], which diagonalises the metric locally. However, the method we will introduce does not require to adopt any coordinate system on $\partial\tau$. This means that this diagonalisation will be valid not only locally, but for neighbourhoods contained within the normal injectivity radius.

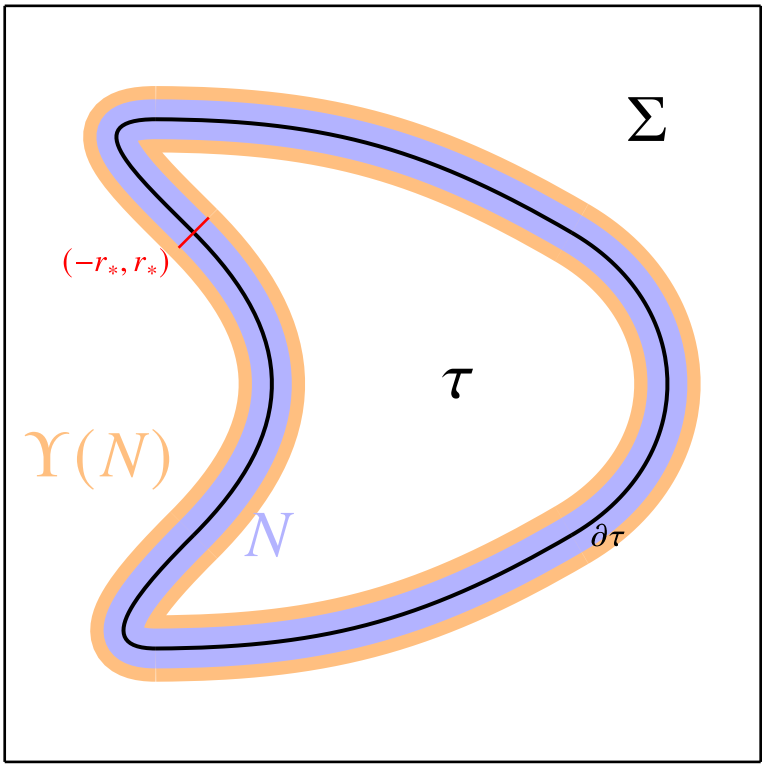


Figure 4.7: The original region τ with the neighbourhoods N and $\Upsilon(N)$. Note that in this particular instance, the injectivity radius of $\partial\tau$ is finite.

LEMMA 4.3.1. *Suppose τ is the interior of a regular domain in Σ and suppose its boundary $\partial\tau$ has nonzero normal injectivity radius r_0, whereupon*

$$\Upsilon(s, y) = \exp^{\perp}(s n_y), \tag{4.39}$$

defines a diffeomorphism from $(-r_0, r_0) \times \partial\tau$ onto its image (see Figure 4.7), where n_y is the outward normal unit vector to $\partial\tau$ at y and $\exp^{\perp}$ is the normal exponential map. For any $r_* \in (0, r_0)$, choose $\zeta \in C^{\infty}(\mathbb{R})$ with $0 \leq \zeta' < 1$ obeying

$$\zeta(s) = \begin{cases} s & |s| < r_* \\ (r_0 + 2r_*)/3 & s \geq (r_0 + r_*)/2 \\ -(r_0 + 2r_*)/3 & s \leq -(r_0 + r_*)/2. \end{cases} \tag{4.40}$$

Then, setting $N = (-r_*, r_*) \times \partial\tau$ there is a smooth function $r : \Sigma \to \mathbb{R}$ given by

$$r(\exp^{\perp} s n_y) = \zeta(s) \tag{4.41}$$

for $(s, y) \in (-r_0, r_0) \times \partial\Sigma$, and satisfying $dr \equiv 0$ outside $\Upsilon((-r_0, r_0) \times \partial\tau)$. In particular, $\partial\tau = r^{-1}(0)$ and $r \leq -r_*$ on $\tau \setminus \Upsilon(N)$, while $r \geq r_*$ on $(\Sigma \setminus \tau) \setminus \Upsilon(N)$. And so, the metric h_t pulls back to N as

$$\Upsilon^* h_t|_{\{s\}\times\partial\tau} = 1 \oplus \tilde{h}_s \tag{4.42}$$

where $s \mapsto \tilde{h}_s$ is a smooth map from $(-r_*, r_*)$ to the smooth Riemannian metrics on $\partial\tau$.

Proof. Note that as r and ζ are smooth, so will be the metric. Also, as $r_* < r_0$ this metric is well-defined on N. Then, the only thing we need to prove is that the pull-back of the metric to N has a block diagonal form. To this end, consider an arbitrary curve $t \mapsto y(t)$ in $\partial\tau$ and the smooth 2-parameter map $(s, t) \mapsto \mu(s, t) = \Upsilon(s, y(t))$, with tangent fields $U = \partial\mu/\partial s$, $V = \partial\mu/\partial t$. Then U is a unit geodesic vector field and $[U, V] = 0$. From this we see that the absolute derivative is $\frac{D}{ds} h(U, V) = h(U, \nabla_U V) = h(U, \nabla_V U) = \frac{1}{2}\nabla_V h(U, U) = 0$, which means that for a constant c we have $h(U, V) = c$. However, given that U is normal to V at $s = 0$, we deduce that this constant is zero and thus $h(U, V) = 0$.

Moreover, since $U = \Upsilon_*(1,0)$ and $V = \Upsilon_*(0, \dot{y}(0))$, we obtain

$$(\Upsilon^* h)|_{(s,y(0))}((1,0),(0,\dot{y}(0))) = 0$$

and as $y(t)$ was an arbitrary smooth curve we have the desired block diagonal form; finally, as U is a unit field we already know $(\Upsilon^* h)((1,0),(1,0)) = 1$. □

This is the required block diagonal form of h_t on $\Upsilon(N)$. Using the diagonal form, it is easily seen that the length of any curve between two level sets $r^{-1}(r_1)$ and $r^{-1}(r_2)$ of r contained in $\Upsilon(N)$ is bounded below by $|r_1 - r_2|$, which is attained by radial geodesics.

Remark 4.3.1. Note that r has the role of radial coordinate in $\Upsilon(N)$ since $r(p) \in (-r_*, r_*)$ if and only if $p \in \Upsilon(N)$ and $\Upsilon^{-1}(p) \in \{r(p)\} \times \partial\tau$. Outside N it need not be a valid coordinate.

So, as long as we choose an open neighbourhood of $\partial\tau$ the metric g can be written in a block diagonal form. This gives us some freedom in choosing the size of this neighbourhood, which will be encoded on a variable r_* of radial nature.

The main idea is to carry an expansion within N. At some point, we will like to make use of a generalised version of the uniqueness argument found in Theorem 3.2.1. So, the expansion has to be done in such a way we can specify a region σ inside a protected region κ where the metric g becomes g_0. In our current setting, these regions will play the roles, respectively, of S and K found in (**SEP.III**) (see Figures 3.6 and 3.7 for further reference). The size of these regions will depend on parameters r_S and r_K. They have to be chosen so the following happens: a radial null geodesic to travel from $\partial\sigma$ at time $t = 0$ to a time t_F and back to t_F while always staying in the protected region. Because of this, we will have specific choices for them found in the following Theorem.

THEOREM 4.3.1. *Let τ be the interior of a regular domain in Σ with nonzero normal injectivity radius r_0, and for any $r_* \in (0, r_0)$ let N, Υ and r be as in Lemma 4.3.1. Solutions to the REP may be found as follows: choose $r_S \in (0, r_*)$, $t_F \in (0, r_S/2)$, and an arbitrary $0 < \delta \ll 1$. Then, fix r_F so that*

$$t_F < r_F < \frac{r_*}{1+\delta} \tag{4.43}$$

and define (necessarily) positive

$$r_L = r_S - t_F, \qquad r_K = r_S - 2t_F. \tag{4.44}$$

Recalling that $0 < r_K < r_L < r_S$ *we define the nested sets* $\sigma \subset \lambda \subset \kappa \subset \phi$ *as*

$$\sigma = r^{-1}((-\infty, -r_S)), \quad \lambda = r^{-1}((-\infty, -r_L)), \quad \kappa = r^{-1}((-\infty, -r_K)) \tag{4.45}$$

and

$$\phi = r^{-1}((-\infty, r_F)), \tag{4.46}$$

which correspond to the interior of regular domains in Σ*. Note the absence of a minus sign in front of* r_F *in the definition of* ϕ*.*

To specify a metric, we remind the reader that $\mathcal{T} = (-\varepsilon, t_F + \varepsilon)$ *and* $\mathcal{I} = (-\varepsilon, \varepsilon) \cup (t_F - \varepsilon, t_F + \varepsilon)$*, then we choose a smooth function* $\rho : \mathcal{T} \to (-r_*, r_*)$ *obeying*

$$\rho(0) = 0, \quad \rho(t_F) = r_F, \quad \frac{d\rho}{dt} \geq -1, \tag{4.47}$$

where equality is achieved in $\mathcal{I}$*, i.e.* $d\rho/dt|_{\mathcal{I}} = -1$*. Next set,*

$$v = \begin{cases} \frac{d\rho}{dt} + 1 & t \in \mathcal{T} \\ 0 & t \in \mathbb{R} \setminus \mathcal{T} \end{cases} \tag{4.48}$$

for which we clearly have $0 \leq v$*. Also, choose*

$$0 < r_B < \min(r_K, r_* - \rho_{max}, r_* + \rho_{min}) \tag{4.49}$$

where ρ_{max} *and* ρ_{min} *are the maximum and minimum of* ρ*, this minimum is necessarily positive because* $r_K > 0$ *and due to the codomain of* ρ*. Let* $\rho_\pm(t) = \rho(t) \pm r_B$*, then, we define the warp bubble* $B(t)$ *and the warp zone* W *as*

$$B(t) = r^{-1}((\rho_-(t), \rho_+(t))), \qquad W = \bigcup_{t \in \mathcal{T}} \{t\} \times B(t).$$

From this, now it is easy to explain why r_B *was chosen as in* (4.49)*. This prevents the warp bubble from meeting the protected zone specified by* r_K *or going out from* $\mathcal{T} \times N$*. Furthermore, choose* $f \in C_0^\infty(\mathbb{R} \times \Sigma)$ *to take values in* $[0, 1]$ *such*

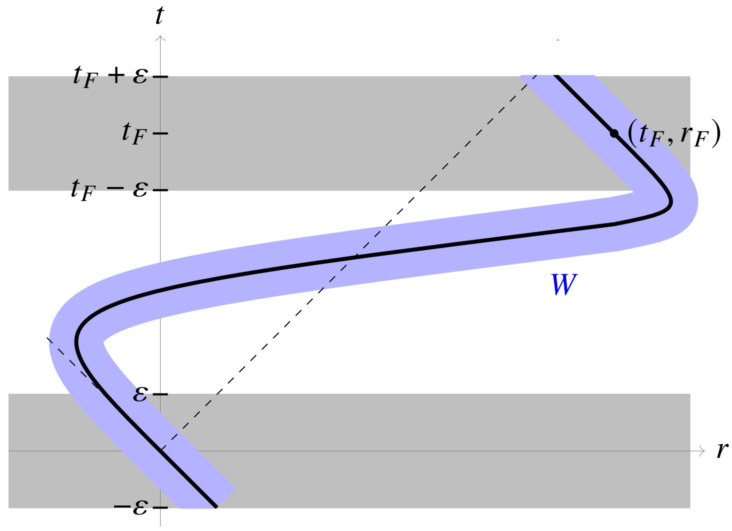


Figure 4.8: Plot of $\rho(t)$, $g = g_0$ in the shaded regions.

that $f \equiv 1$ *on a neighbourhood of* $\bigcup_{t\in[0,t_F]}\{t\}\times r^{-1}(\rho(t))$ *and* $f \equiv 0$ *outside* W. *(This function certainly exists, c.f. [1][Prop. 6.5.8].) Next, define the metric* g *on* $\mathbb{R}\times\Sigma$ *by*

$$g = g_0 - (fv)^2 dt \otimes dt + 2fv dt \otimes_s dr \tag{4.50}$$

where r *is the function defined in Lemma 4.3.1.*

Proof. We are going to show that all of the parameters, sets and metric provided above are enough to satisfy the requirements given in **REP 1**-**REP 4**. Almost all the work goes into checking **REP 2**. However, before doing so we need to check that the metric (4.50) is globally hyperbolic indeed.

LEMMA 4.3.2. *Let* g *be the metric* (4.50) *then,* $(\mathbb{R}\times\Sigma,\ g)$ *is globally hyperbolic.*

Proof. See Appendix B.1.2. □

Remark 4.3.2. The metric (A.2) can be written as

$$g = dt \otimes dt - (dr - fvdt) \otimes (dr - fvdt) - \tilde{h}_r, \tag{4.51}$$

which resembles the famous Alcubierre warp drive [5], the difference being that, in this case, we are specifying photon trajectories rather than that of a spaceship.

The trajectory is given by ρ (implicitly defined by $v = 1 + d\rho/dt$) while f defines the warp bubble around it.

REP 1. Note that $-r_S, -r_L, -r_K, -r_F$ are regular values of r because $dr \neq 0$ on $\Upsilon(N)$, given that $\inf r = -r_0$ we know that $r^{-1}([-r_0, b]) = r^{-1}((-\infty, b])$ and by [47][Prop. 5.47] we deduce that $r^{-1}([-r_0, b])$ is a regular domain for $b \in \{-r_S, -r_L, -r_K, -r_F\}$. To see that its interior is $r^{-1}((-r_0, b))$ note that points in $r^{-1}(c)$ for a regular value c have nearby points intersecting both $r^{-1}((-\infty, c))$ and $r^{-1}((c, \infty))$, so any interior point p of $r^{-1}([a, b)])$, where a, b are regular values, has $r(p) \in (a, b)$; conversely, any point with $r(p) \in (a, b)$ has a neighbourhood in $r^{-1}([a, b])$ by continuity of r.

Next, from (4.45-4.46) and the definition of r in Lemma 4.3.1 we observe that

$$\partial\sigma = r^{-1}(-r_S), \quad \partial\lambda = r^{-1}(-r_L), \quad \partial\kappa = r^{-1}(-r_K) \tag{4.52}$$

and

$$\partial\phi = r^{-1}(r_F). \tag{4.53}$$

so because of [47][Corollary 5.14] the boundaries are indeed codimension-1 embedded submanifolds of Σ. In addition, we note that the radii of these regions given in (4.44) obey $0 < r_K < r_L < r_S$, and together with the observation that $\tau \sqsubset \phi$ we see that (4.14) is satisfied.

Remark 4.3.3. Recall that in an ultrastatic spacetime, a curve $t \mapsto (t, s(t))$ is an affine null geodesic if and only if $t \mapsto s(t)$ is a unit speed geodesic. Using this and the definition of σ given in (4.45), we deduce that r_S is the distance in Σ between $\partial\sigma$ and $\partial\tau$, this is the same as the infimum of the light-travel time between $\mathbb{R} \times \partial\sigma$ and $\mathbb{R} \times \partial\tau$. The same holds for λ, κ and ϕ.

As noted above, most of the work comes in verifying that **REP 2** is satisfied. Because of this, we will prove **REP 3** and **REP 4** first, as this is rather straightforward. To this end we will first obtain a preliminary result.

LEMMA 4.3.3. *Let* $I_K = (0, r_* - r_K)$*, then* $D^+_{g_0}(\{0\} \times \kappa) \cap (I_K \times \Sigma) = \cup_{t \in I_K} \{t\} \times r^{-1}(-\infty, -r_K - t)$.

Proof. First, we prove that $D^+_{g_0}(\{0\}\times\kappa)\cap(I_K\times\Sigma)\supset\cup_{t\in I_K}\{t\}\times r^{-1}(-\infty,-r_K-t)$, that is, any past-directed inextendible causal curve $c(t)=(t,x(t))$ passing through a point in $\cup_{t\in I_K}\{t\}\times r^{-1}(-\infty,-r_K-t)$ meets $\{0\}\times\kappa$, for which it is sufficient that if

$$r(x(t_*)) < -r_K - t_* \tag{4.54}$$

for some $t_*\in I_K$, then $r(x(t)) < -r_K - t$ must hold for any $0\le t<t_*$. To show this, note that causality demands that $0\le g_0(\dot c,\dot c)$, which in turn implies $|dr/dt|\le 1$. Integrating it and making use of (4.54) yields, after some rearrangement, the result we were seeking.

Next, we prove the other inclusion, that is $D^+_{g_0}(\{0\}\times\kappa)\cap(I_K\times\Sigma)\subset\cup_{t\in I_K}\{t\}\times r^{-1}(-\infty,-r_K-t)$, or conversely: any point outside $\cup_{t\in I_K}\{t\}\times r^{-1}(-\infty,-r_K-t)$ lies on some past-directed causal curve that avoids $\{0\}\times\kappa$. Note that we need not to worry about points (t,x) for which $r(x)\ge -r_K$ because g_0 is ultrastatic and thus $D^+_{g_0}(\{0\}\times\kappa)\subset\mathbb{R}^+\times\kappa$. So consider (t_*,x_*) with

$$-r_K > r(x_*) \ge r_K - t_* \tag{4.55}$$

for $t_*\in I_K$. A radially inward future-directed null geodesic through (t_*,x_*) obeys $r(x(t))=r(x(0))-t$, making use of (4.55) leads to $r(x(0))\ge -r_K$. □

Note that in the proof presented above we need r to measure radial geodesic distance. That is, we need $r(x(t))$ to be within $(-r_*,r_*)$, which is actually the case.

Corollary 4.3.1. *Let $I_L=(0,t_F)$, then $D^-_{g_0}(\{t_F\}\times\lambda)\cap(I_L\times\Sigma)=\cup_{t\in I_L}\{t\}\times r^{-1}(-\infty,-r_L-t_F+t)$.*

Proof. The proof follows from the same argument we used in the previous Lemma, but on this occasion, the RHS of inequalities (4.54) and (4.55) must read $-r_L+t-t_F$, and the first holds for $t_*<t\le t_F$. □

REP 3. From the definition of v in (4.48), it is clear that $g=g_0$ when restricted to $\mathcal{I}\times\Sigma$. To see that $g=g_0$ in K, we just need to show that the sets K and W are disjoint. To see this, note that because of the definition of r_B, $\rho(t)$ and $\rho_-(t)$

in Theorem 4.3.1, we will have $-r_K - t < -r_B - t \leq -r_B + \rho(t) = \rho_-(t)$, so if we consider a point $p \in K$ with coordinates (t, x), because of Lemma 4.3.3 we conclude that $r(x) \leq -r_K - t$ and thus $r(x) < \rho_-(t)$, which means that $p \notin W$. Because of this, we see that $g = g_0$ in K, which ends our proof for **REP 3**.

REP 4. Consider a causal curve $c(s) = (t(s), x(s))$. The metric (4.50) yields

$$g(\dot{c}, \dot{c}) = \dot{t}\left[(1 - f^2 v^2)\dot{t} + 2 f v \dot{r}\right] - h_t(\dot{x}, \dot{x}) \geq 0, \tag{4.56}$$

given that h_t is Riemannian, we see that the causal vector defined by $\dot{c} = (\dot{t}, \dot{x})$ has nonzero time component and therefore has constant sign along the curve. Hence any hypersurface of the form $\{t\} \times \Sigma$ is acausal, so in particular is achronal. As $\tau,\ \phi \subset \Sigma$, we conclude that $\{0\} \times \tau$ and $\{t_F\} \times \phi$ are achronal as well.

REP 2. Before proving our main result we need to prove an intermediate Lemma.

LEMMA 4.3.4. *For the modified metric g and $I_E = [0, t_F + \varepsilon)$, we have $D_g^+(\{0\} \times \tau) \cap (I_E \times \Sigma) = \cup_{t \in I_E} \{t\} \times r^{-1}((-\infty, \rho(t)))$.*

Proof. First, we will show $D_g^+(\{0\} \times \tau) \cap (I_E \times \Sigma) \supset \cup_{t \in I_E} \{t\} \times r^{-1}((-\infty, \rho(t)))$. Making use of Lemma 4.2.1 (a) and the fact that $\kappa \sqsubset \tau$, we have $D_g(\{0\} \times \kappa) \subset D_g(\{0\} \times \tau)$. In addition to this, using Lemma 4.2.1 (b) with $A = I_E \times \Sigma$ and $B = \{0\} \times \kappa$ we obtain

$$D_{g_0}^+(\{0\} \times \kappa) \cap (I_E \times \Sigma) \subset D_g^+(\{0\} \times \tau) \cap (I_E \times \Sigma), \tag{4.57}$$

where we have used that $W \cap K = \emptyset$ (as shown in the proof of **REP3**), and so $g = g_0$ in K. By Lemma 4.3.3 we know that $\bigcup_{t \in I_K} \{t\} \times r^{-1}(-\infty, -r_K - t)$ is contained in the RHS of (4.57), so we just need to consider the remaining points.

Hence, it is just necessary to show that every past-directed inextendible causal curve $c(t) = (t, x(t))$ passing through a point in

$$X = \cup_{t \in I_E} \{t\} \times r^{-1}([-r_K - t, \rho(t))) \tag{4.58}$$

necessarily intersects $\{0\} \times \tau$. Clearly, if the curve remains in X then it certainly meets $\{0\} \times \tau$. By the same logic, if at some time we have $r(x(t)) < -r_K - t$, then it must intersect $\{0\} \times \kappa \subset \{0\} \times \tau$. Thus, we only need to show that $c(t)$ can not ever have $r(x(t)) > \rho(t)$.

We will prove by contradiction, to this end let us assume that the curve can leave X at a certain time. Thus, for some time $0 < t_*$ the curve is in X, which means that we have $r(x(t_*)) < \rho(t_*)$. Then, at some point before $c(t_*)$ the curve leaves X, which means that it must necessarily intersect $(t, \rho(t))$ at some time $t_c = \sup\{t < t_* : r(x(t)) > \rho(t)\}$. Hence, at this contact time $r(x(t_c)) = \rho(t_c)$.

As it was noted in Remark 4.3.2, because of (4.42) the metric (4.50) pulls back to (4.51) in $\Upsilon(N)$, then, for $c(t)$ we will have $g(dc/dt, dc/dt) \geq 0$ which can be written as

$$\left| \frac{dr}{dt} - fv \right| \leq 1. \tag{4.59}$$

By continuity, we may assume without loss of generality that $c(t)$ is in the vicinity of $(t, \rho(t))$ for all $t \in [t_c, t_*]$ such that $f \equiv 1$ on this portion of $c(t)$. Hence, causality (4.59) demands that $d\rho/dt \leq dr/dt$. By hypothesis we have that $r(x(t_*)) < \rho(t_*)$ and $r(x(t_c)) = \rho(t_c)$. So, we deduce that $dr/dt < d\rho/dt$ for some $t \in (t_c, t_*)$ (actually for a set of nonzero measure). But given that $f(c(t)) = 1$, this contradicts causality of $c(t)$. This contradiction arose from the assumption that our causal curve can cross ρ at some time t, therefore, we must have that $r(t) < \rho(t)$ for every t, which proves our claim. Consequently, we deduce that

$$D_g^+(\{0\} \times \tau) \cap (I_E \times \Sigma) \supset \bigcup_{t \in I_E} \{t\} \times r^{-1}(-\infty, \rho(t)). \tag{4.60}$$

Next, we need to check that this set inclusion also happens in the opposite direction, or conversely, that every point outside the RHS of (4.60) lies on some past-directed inextendible causal curve that avoids $\{0\} \times \tau$. This is tantamount to proving that

$$\text{if } \; r(x(t)) \geq \rho(t) \text{ for some } t > 0 \implies r(x(0)) \geq \rho(0) = 0. \tag{4.61}$$

As (4.59) just depends on the radial coordinate, it is enough to consider that a causal radially inward trajectory passing through a point outside X never meets $\{0\} \times \tau$. Also, as it is saturated for null trajectories, we just need to consider this case as the rest will follow from continuity. So, consider the radially inward null curve that passes through a point outside X at time t_*, that is, $\rho(t_*) \leq r(x(t_*))$ for some $t_* \in I_E$. For this trajectory, (4.59) demands

$$\frac{dr}{dt} - 1 - \frac{d\rho}{dt} = \frac{dr}{dt} - v \leq \frac{dr}{dt} - fv = -1 \tag{4.62}$$

where the inequality follows from v being positive and we have used its definition (3.6) in the equality to the left. Integrating this yields $r(x(t_*)) - \rho(t_*) \leq r(x(0)) - \rho(0)$, by hypothesis the LHS is positive or zero, hence $\rho(0) \leq r(x(0))$. This shows that the null geodesic lies outside $\{0\} \times \tau$ for all $t \in I_E$, which means that (4.61) is true, thus

$$D_g^+(\{0\} \times \tau) \cap (I_E \times \Sigma) \subset \bigcup_{t \in I_E} \{t\} \times r^{-1}(-\infty, \rho(t)),$$

this together with (4.60) proves our claim. □

Next, we make use of Corollary 4.3.1 along with Lemmas 4.3.3 and 4.3.4 to prove that the relations (4.15), (4.16) and (4.17) in **REP 2** hold. For the first, we observe that

$$\begin{aligned} D_g^+(\{0\} \times \tau) \cap (I_E \times \Sigma) \cap (\{t_F\} \times \Sigma) &= \left(\bigcup_{t \in I_E} \{t\} \times r^{-1}(-\infty, \rho(t)) \right) \cap (\{t_F\} \times \Sigma) \\ &= \{t_F\} \times r^{-1}(-\infty, r_F) = \{t_F\} \times \phi, \end{aligned} \tag{4.63}$$

where we used Lemma 4.3.4 in the first equality, (4.47) for the second and (4.46) for the third. Because $D_g^+(\{0\} \times \tau) \cap (\{t_F\} \times \Sigma) = D_g(\{0\} \times \tau) \cap (\{t_F\} \times \Sigma)$, we find that (4.63) proves (4.15). For the second, that is (4.16), we have

$$\begin{aligned} &D_{g_0}^+(\{0\} \times \kappa) \cap (I_K \times \Sigma) \cap (\{t_F\} \times \Sigma) \\ &= \left(\bigcup_{t \in I_K} \{t\} \times r^{-1}(-\infty, -r_K - t) \right) \cap (\{t_F\} \times \Sigma) \end{aligned}$$

which follows from Lemma (4.3.3). Resuming our computation yields

$$\begin{aligned} D_{g_0}^+(\{0\} \times \kappa) \cap (I_K \times \Sigma) \cap (\{t_F\} \times \Sigma) &= \{t_F\} \times r^{-1}(-\infty, -r_K - t_F) \\ &= \{t_F\} \times r^{-1}(-\infty, -r_L) = \{t_F\} \times \lambda, \end{aligned} \tag{4.64}$$

these equalities follow from (4.44) and (4.45), respectively. Noting that $D^+_{g_0}(\{0\}\times\kappa)\cap(\{t_F\}\times\Sigma) = D_{g_0}(\{0\}\times\kappa)\cap(\{t_F\}\times\Sigma)$, we deduce that (4.64) proves (4.16). Finally, for the third relation (4.16), we obtain

$$\begin{aligned} &D^+_{g_0}(\{t_F\}\times\lambda)\cap(I_L\times\Sigma)\cap(\{0\}\times\Sigma)\\ &=\left(\bigcup_{t\in I_L}\{t\}\times r^{-1}(-\infty,-r_L-t_F+t)\right)\cap(\{0\}\times\Sigma)\\ &=\{0\}\times r^{-1}(-\infty,-r_L-t_F)=\{0\}\times r^{-1}(-\infty,-r_S)=\{0\}\times\sigma, \end{aligned} \tag{4.65}$$

the first, third and fourth equality follow from Corollary 4.3.1, (4.44) and (4.45), respectively.

In a similar fashion as before, $D^-_{g_0}(\{t_F\}\times\lambda)\cap(\{0\}\times\Sigma) = D_{g_0}(\{t_F\}\times\lambda)\cap(\{0\}\times\Sigma)$, and thus we see that implies that (4.17) holds. For the last relation (4.18), it suffices to prove that $\{0\}\times\tau\subset D^-_{g_0}(\{t_F\}\times\phi)$ and then use Lemma 4.2.1 (a). We will do this by contradiction. That is, we will assume there is a future-directed inextendible g_0 causal curve $c(t)=(t,x(t))$ that passes through $p\in\{0\}\times\tau$ that fails to meet $\{t_F\}\times\phi$, so $r(x(0))<0$ and $r(x(t_F))\geq r_F$. Moreover, there must be a $t_*\in(0,t_F)$

$$(r\circ x)'(t_*)=((r\circ x)(t_F)-(r\circ x)(0))/t_F > r_F/t_F. \tag{4.66}$$

From causality we get $1\geq(r\circ x)'(t)^2+h(dc/dt,dc/dt)$, which along with (4.66) leads to $1\geq(r_F/t_F)^2+h(dc/dt,dc/dt)|_{t=t_*}$, which is a contradiction since from (4.43) we know that that $r_F>t_F$. Therefore, we have $\{0\}\times\tau\subset D^-_{g_0}(\{t_F\}\times\phi)$, making use of this together with (4.31) implies that

$$D_{g_0}(\{0\}\times\tau)\subset D_{g_0}(\{t_F\}\times\phi).$$

□

4.4 DISCUSSION

As noted in [24], Hadamard States form a State Space. Hence, as we have found a solution to the REP (see 4.2), this implies we found a solution to the GSEP (4.1) for the State Space of Hadamard States. Consequently, we have achieved extending a state while keeping its Hadamard property. So, the next logical question is: what other properties of the state can kept after extending?

4.4.1 Quasifree states

Recalling that a state ω_0 on $\mathbf{M}$ is quasifree if its n-point functions are determined by the 2-point function via

$$\omega_0(e^{i\lambda\phi_{\mathbf{M}}(f)}) = e^{-\lambda^2 W(f,f)/2} \tag{4.67}$$

and vanish for odd n. We will investigate if a quasifree state remains quasifree after being extended. For $\omega \in \mathscr{S}(M)$, $f \in C_0^\infty(\mathbf{M})$ and $\lambda \in \mathbb{R}$, we write

$$\mathcal{Z}_{\mathbf{M}}[\omega; f; \lambda] = \omega(e^{i\lambda\phi_{\mathbf{M}}(f)}) \tag{4.68}$$

in the sense of a formal series in λ. Let $\mathbf{M}$, $\mathbf{N} \in \mathbf{Loc}$, then for a morphism $\psi : \mathbf{M} \to \mathbf{N}$ in $\mathbf{Loc}$ we can define the push forward $\psi_* : C_0^\infty(\mathbf{M}) \to C_0^\infty(\mathbf{N})$ as follows

$$\psi_* f = (f \circ \psi^{-1}) \cdot \mathcal{I}_{\psi(\mathbf{M})}, \tag{4.69}$$

where $\cdot$ is the usual function pointwise product and $\mathcal{I}_B$ is the characteristic function of the set B. With this, we can exchange mappings between states for mappings between test functions in the functional (4.68) in a formal way via

$$\begin{aligned}\mathcal{Z}_{\mathbf{M}}[\mathscr{S}(\psi)\omega; f; \lambda] &= (\mathscr{S}(\psi) \circ \omega)(e^{i\lambda\phi_{\mathbf{M}}(f)}) = \omega(\mathscr{A}(\psi)e^{i\lambda\phi_{\mathbf{M}}(f)}) = \omega(e^{i\lambda\phi_{\mathbf{N}}(\psi_* f)}) \\ &= \mathcal{Z}_{\mathbf{N}}[\omega; \psi_* f; \lambda]. \end{aligned}\tag{4.70}$$

If $\psi : \mathbf{M} \to \mathbf{N}$ is Cauchy, there exist maps $\zeta : C_0^\infty(\mathbf{N}) \to C_0^\infty(\mathbf{M})$ so that

$$E_{\mathbf{M}}\zeta f = \psi^* E_{\mathbf{N}} f \qquad (f \in C_0^\infty(\mathbf{N})). \tag{4.71}$$

Any such morphism ζ will be called a "timeslice map", this notion was first introduced in [27]. An example is obtained by setting $\zeta = (P\chi E)_{\mathbf{N}}$, so

$$\psi^*(P\chi E)_{\mathbf{N}} f = (P\chi E)_{\mathbf{N}} f \circ \psi, \quad \text{with} \quad (P\chi E)_{\mathbf{N}} f \in C_0^\infty(\psi(\mathbf{M})), \tag{4.72}$$

where we have introduced the shorthand notation $(P\chi E)_{\mathbf{N}}$ for $P_{\mathbf{N}}\chi_{\mathbf{N}}E_{\mathbf{N}}$ and $\chi_{\mathbf{N}} \in C^\infty(\mathbf{N})$ is identically zero to the future of a Cauchy surface $\Sigma^+_{\mathbf{N}}$ of $\psi(\mathbf{M})$ and equals unity to the past of another. In the following, we write $¥_\psi$ to denote a specific (though arbitrary) choice of timeslice map corresponding to a Cauchy morphism ψ. Although there is no canonical timeslice map, note that $\phi_{\mathbf{M}}(¥_\psi f) = \mathscr{A}(\psi)^{-1}\phi_{\mathbf{N}}(f)$ holds for every such maps and all test functions f on $\mathbf{N}$. With this, we can do something similar to (4.70) but in the opposite direction,

$$\mathcal{Z}_{\mathbf{N}}[\mathcal{S}(\psi)^{-1}\omega; f; \lambda] = \mathcal{Z}_{\mathbf{M}}[\omega; ¥_\psi f; \lambda]. \tag{4.73}$$

Next, to see the effect of the extension map we compute $\mathcal{Z}_{\tilde{\mathbf{S}}}[\mathcal{E}\omega; f; \lambda]$ using Eq. (4.1) for $\mathcal{E}$. This yields

$$\begin{aligned}
&\mathcal{Z}_{\tilde{\mathbf{S}}}[\mathcal{S}(\iota_{\tilde{S};F})^{-1}\mathcal{S}(\iota_{E;F})\mathcal{S}\left(\iota_{E;P}\right)^{-1}\mathcal{S}\left(\iota_{T;P}\right)\omega; f; \lambda] \\
&= \mathcal{Z}_{\mathbf{F}}[\mathcal{S}(\iota_{E;F})\mathcal{S}\left(\iota_{E;P}\right)^{-1}\mathcal{S}\left(\iota_{T;P}\right)\omega; ¥_{\iota_{\tilde{S};F}} f; \lambda] \\
&= \mathcal{Z}_{\mathbf{E}}[\mathcal{S}\left(\iota_{E;P}\right)^{-1}\mathcal{S}\left(\iota_{T;P}\right)\omega; (\iota_{E;F})_* ¥_{\iota_{\tilde{S};F}} f; \lambda] \\
&= \mathcal{Z}_{\mathbf{P}}[\mathcal{S}\left(\iota_{T;P}\right)\omega; ¥_{\iota_{E;P}}(\iota_{E;F})_* ¥_{\iota_{\tilde{S};F}} f; \lambda] \\
&= \mathcal{Z}_{\mathbf{T}}[\omega; (\iota_{T;P})_* ¥_{\iota_{E;P}}(\iota_{E;F})_* ¥_{\iota_{\tilde{S};F}} f; \lambda],
\end{aligned}$$

where in the first line we just substituted the definition of $\mathbf{E}$ given in (4.1) and in the rest of the lines we exchange maps to test function maps with (4.70) and (4.73), accordingly. The last expression suggests introducing the map

$$\mathrm{ext} = (\iota_{T;P})_* ¥_{\iota_{E;P}}(\iota_{E;F})_* ¥_{\iota_{\tilde{S};F}}, \quad \mathrm{ext} : C_0^\infty(\tilde{\mathbf{S}}) \to C_0^\infty(\mathbf{T}),$$

which allows us to write our result as

$$\mathcal{Z}_{\tilde{\mathbf{S}}}[\mathcal{E}\omega; f; \lambda] = \mathcal{Z}_{\mathbf{T}}[\omega; \mathrm{ext}\, f; \lambda]. \tag{4.74}$$

Let us denote a quasifree state by ω_0, in this case the functional (4.68) becomes $\mathcal{Z}_{\mathbf{M}}[\omega_0; f; \lambda] = e^{-\lambda^2 W(f,f)/2}$, where W is the two-point function. Using this and (4.74) it is not difficult to see that

$$\mathcal{Z}_{\tilde{\mathbf{S}}}[\mathcal{E}\omega_0; f; \lambda] = \mathcal{Z}_{\mathbf{T}}[\omega_0; \mathrm{ext}\, f; \lambda] = e^{-\lambda^2 W(\mathrm{ext}\, f, \mathrm{ext}\, f)/2}, \tag{4.75}$$

hence, $\omega_{\tilde{S}}$ will also be quasifree.

It is worth mentioning that although ext depends on the choice of the two timeslice maps used in its construction, the 2-point function $(f, g) \mapsto W(\mathrm{ext}\, f, \mathrm{ext}\, g)$ is independent of these choices – this is immediate because the LHS of (4.75) does not depend on them. The fact that the quasifree property is kept after the extension is remarkable and should not be taken for granted. As a matter of fact, it relies crucially in the existence of Cauchy morphisms between some of our regions; which were originally needed for quite different reasons. This might be of interest for applications in the context of perturbative QFT as this alongside the fact that the extended state is also Hadamard, implies that the structure of the Wick algebra structure of the linear scalar field is preserved.

4.4.2 Symmetries

On first glance, it might seem that our current formulation of symmetries is not well-suited to deal with the whole of a Globally Hyperbolic Spacetime. However, it is not possible to have full symmetries once we have introduced regular domains such as $\tau \subset \Sigma$. This is due to the fact that in general, an isometry of $(\mathbb{R} \times \Sigma, g)$ will not preserve the Cauchy surface Σ.

Recall that if we want the **REP** to be compatible with symmetries, we need to preserve the regular domains τ, σ, κ, λ and ϕ under the action of all the members of an isometry group G. But, as all of the relevant regions are diamonds defined via the Cauchy developments of regular domains, it would be odd to expect that the symmetries of the whole spacetime are preserved. This can be illustrated via the following example: consider our spacetime to be Minkowski and let τ be the ball of radius r_T sitting at the origin. Clearly, the Poincaré group can not be an appropriate isometry group, nonetheless, $SO(3)$ (which is a subgroup) is.

So, as introducing the regular domains in M may already break pre-existing symmetries, the only thing one can hope for is to prove that isometries of Σ are preserved under the **REP**, which is precisely what we have done. This is good for applications since it tells us that the extension not only preserves the Hadamard property, but also some important characteristics of the original state that will help us to quickly identify familiar features in the extended state.

4.4.3 Extension to the whole spacetime

So far, we have considered the problem of extending a state from one region to a slightly expanded region. A natural question is: can we repeat this procedure?

If so, one could ask: how many more times? This leads to inquiring whether or not we can extend to the whole spacetime. Some work has been done in this direction, however, it requires to be formulated in more precise terms. The main idea is to have a sequence of extensions ω_n with nested protected regions S_n so that $\bigcup S_n = M$. Then, assuming additivity for this sequence, it is possible to obtain a state on $\mathscr{A}(M)$ that agrees with the original state on S_1.

It is also worth mentioning that we are working on this subject with the same methodology used in **REP**. That is, we solve the abstract problem to see what is needed from the sequence of ω_n and S_n. After finding this, then we would go on to building a specific solution using a construction similar to the one found in Section 4.3. This certainly restricts the class of spacetimes and possible choices for τ, σ and ϕ.

This is due to the fact that we need each of the boundaries $\partial\tau_n$ to always have non-zero (outward) normal injectivity radius for each n. So, this restricts our choices of τ and Σ. In particular, note that Σ can not be compact, as in this case there will be $m \geq 0$ such that $\partial\tau_m$ has zero injectivity radius. Please note that this does not mean that one can not find such a sequence if Σ is compact, but rather that our construction can not be applied as it makes use of the notion of injectivity radius. A good way to guarantee that $\partial\tau_{n+1}$ will never have a vanishing injectivity radius, is to require that the extrinsic curvature of $\partial\tau_n$ is non-negative. With this, we rule out the existence of conjugate points thus guaranteeing that Υ in Lemma 4.3.1 is always a diffeomorphism.

— 5 —

Detector Response in Uniform Circular Motion

-Let's go.
-We can't.
-Why not?
-We're waiting for Godot.

Samuel Beckett, Waiting for Godot

Quantum Field Theory has many predictions and the Unruh Effect [62, 63] is one of the most astonishing as it predicts that an observer moving with linear acceleration a in empty space will start thermalising with temperature $T_U = a/(2\pi)$. Unfortunately, verifying this experimentally is not an easy task as an observer would need to attain a linear acceleration of the order of $10^8 m/s^2$ to reach a temperature of $10^{-12} K$, which is the lowest temperature measured in a laboratory as of today.

Also, in the original presentation of the Unruh effect, it was not easy to discern whether or not it was just a result of the mathematical formulation of QFT or a physical effect. However, Unruh and DeWitt could demonstrate that if one introduces a detector, then its response: (a) depends on its proper time and (b) its response matches the one given by T_U.

Since then, many experimental proposals have been put forth [14, 11, 2, 39] and our research intends to propose yet another one with a novel approach. Instead of a linearly accelerated detector, we study one following a uniform circular trajectory. Additionally, we drop the usual assumption that the detection time is very large and introduce the following question: how long does one have to wait for thermalisation? Our findings indicate that a thorough study of the space of experimental parameters must be made in order to increase the size of the detected temperature and decrease waiting times.

5.1 PRELIMINARIES

Throughout this section, we will make use of standard Minkowski coordinates $x = (t, \vec{x})$ with $\vec{x} \in \mathbb{R}^2$ for which the metric reads $g = -\mathbf{1} \oplus e$, where e is the 2-dimensional Euclidean metric.

DEFINITION 5.1.1. If for $U \subseteq \mathbb{R}$ and measure $\mu : U \to \mathbb{R}$, the measurable function $f : U \to \mathbb{C}$ satisfies $\int_U |f| d\mu < \infty$, then we say that f belongs to the vector space $L^1(U, d\mu)$ and we define its L^1-norm as $||f||_{1,U} := \int_U |f| d\mu$. If $U = \mathbb{R}$ we will drop the U subscript from the notation.

Also, for $f,\ g : U \to \mathbb{C}$ let us introduce the inner-product over U defined as $\langle f, g\rangle_U := \int_U \overline{f} g\ d\mu$ which induces the L^2-norm $||f||_{2,U} = \sqrt{\langle f, f\rangle}$; we say that $f \in L^2(U, d\mu)$ if $||f||_{2,U} < \infty$, note that $L^2(U, d\mu)$ is also a vector space and we will also drop the subscript if $U = \mathbb{R}$.

DEFINITION 5.1.2. The Fourier transform of a function f is denoted by $\mathscr{F}[f]$ and its definition is $\mathscr{F}[f](u) = \int_{-\infty}^{\infty} dx\ e^{-iux} f(x)$. Following this definition, Plancherel's formula becomes $\langle \mathscr{F}[f], \mathscr{F}[g]\rangle = 2\pi\langle f, g\rangle$ where $f,\ g \in L^1(\mathbb{R}, dx) \cap L^2(\mathbb{R}, dx)$.

The sine and cosine transforms are defined as $\mathscr{S}[f](u) = \int_0^\infty dx\ \sin(ux) f(x)$ and $\mathscr{C}[f](u) = \int_0^\infty dx\ \cos(ux) f(x)$, respectively. The convolution of two functions f and g is given by $[f * g](x) := \int_{\mathbb{R}} dy\ f(y) g(x - y)$.

The symmetric tensor product is taken to be $A \otimes_s B = (A \otimes B + B \otimes A)/2$. It also is important to note that in this Chapter we have changed our sign convention to $(-, +, +, +)$, as opposed to the rest of this thesis. Also, as we will only be working with the two-point function of the Minkowski vacuum, we will denote it with $\mathcal{W}$.

5.2 THE UNRUH EFFECT

The Unruh effect is the phenomenon where a uniformly linearly-accelerated detector interacting with a quantum field becomes thermalised with a temperature proportional to the acceleration a. The wordline of an observer in Minkowski spacetime with uniform linear acceleration a in the z direction is given by $c(\tau) = (\sinh(a\tau)/a, x_0, y_0, \cosh(a\tau)/a)$ where x_0 and y_0 are constants. This observer is at rest if with respect to coordinates (τ, ξ) (following the conventions established in [17]) such that

$$t = \frac{e^{a\xi}}{a}\sinh(a\tau), \qquad z = \frac{e^{a\xi}}{a}\cosh(a\tau). \tag{5.1}$$

This is not the most obvious choice of coordinates, but it has the advantage of rendering the metric into a conformally flat form

$$g = e^{2a\xi}\left(-d\tau^2 + d\xi^2\right) + dx_\perp^2,$$

which has a Killing vector given by $v = (\partial/\partial\tau) = a[z(\partial/\partial t) + t(\partial/\partial z)]$, whereupon we deduce that the Killing horizon is at

$$\partial W = \{(t, z) \in \mathbb{R}^2 | t + z = 0 \text{ or } t - z = 0\}.$$

Also, it must be noted that these coordinates do not cover all of Minkowski spacetime, but they do cover the right wedge sitting at the origin defined by $W = \{(t, z, \vec{x}_\perp) \in \mathbb{R}^d,\ \vec{x}_\perp \in \mathbb{R}^{d-2} \,|\, z > |t|\}$ also known as the Rindler wedge. We would like to stress the fact that τ plays a privileged role: first, surfaces of

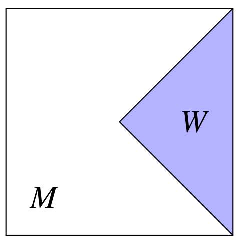


Figure 5.1: The Rindler wedge W.

constant τ are Cauchy surfaces, from which we deduce that W is indeed globally hyperbolic and therefore can be considered a spacetime in its own right. Further, the translation $\alpha_{\tau'} : \tau \to \tau + \tau'$ corresponds to a Lorentz boost and is an isometry of W. Thus, we see that this spacetime is static with Lorentz boosts now playing the role of time translations. The orbits of these boosts are timelike everywhere and complete in W, which means that $\tau \in (-\infty, \infty)$. Said orbits correspond to the wordlines of uniformly accelerated observers in Minkowski spacetime and they become null in the boundary of the Rindler wedge.

Next, we turn our attention to the quantum theory of a scalar field in W. Consider a unital $*$-algebra $\mathscr{A}(W)$ over the Rindler wedge whose generators $\hat{\phi}(f)$ are labelled by $f \in C_0^\infty(W)$. The field satisfies the usual CCR relations (2.2.1) which are identical to those of Minkowski spacetime because E_M the fundamental solution, is locally the same. However, the main difference with the Minkowski case comes from the fact that the algebra $\mathscr{A}(W)$ only contains elements of the form $\hat{\phi}(f)$ with f compactly supported in W, in particular, those away from the boundary of the Rindler wedge. And so, $\mathscr{A}(W)$ can be seen as a proper subalgebra of the algebra of Minkowski spacetime.

Because of this, we can obtain a state ω_W on the Rindler wedge by restricting the Minkowski vacuum ω to W. However, this restriction does not yield a pure state. This can be seen from the following argument, which we will formulate in d-dimensions ($2 < d$) as little effort to do so is needed. The d-dimensional two-point function for the Minkowski vacuum ω in the massless case is given by

$$\mathcal{W}(x, x') = \lim_{\varepsilon \to 0^+} \frac{\Gamma(d/2 - 1)}{4\pi^{d/2} \sigma_\varepsilon (x - x')^{(d-2)/2}}, \qquad \sigma_\varepsilon(x - x') := (x - x' - i\varepsilon\Delta t)^2, \tag{5.2}$$

where Δt is a (fixed) future-directed timelike vector. If we consider $x \in W$ and $z' \in W_L = \{(t, z, \vec{x}_\perp) \in \mathbb{R}^d,\ \vec{x}_\perp \in \mathbb{R}^{d-2} \mid -z > |t|\}$, where the region W_L is known as the left wedge, we find that $\mathcal{W}(x, x') \neq 0$.

Thus, there must be correlations between these disjoint wedges and in consequence, the restriction of ω to $\mathscr{A}(W)$ is not a pure state. A state that is not pure is known as mixed, and the usual description of said states is given by density matrices. However, as intuitive as it may be, the idea of density matrices is not totally accurate in the given setting, as they do not have a Hilbert space representation. This is why we will adapt these ideas to the level of algebraic states.

Usually, when one encounters a mixed state, certain questions regarding the thermal equilibrium properties associated with said state arise quite naturally. As we mentioned in 2.4.1, KMS states allow us to formulate the notions pertaining to thermal equilibrium in terms of the state and isometries of the spacetime. It turns out that the restriction of ω to $\mathscr{A}(W)$ is a KMS state with respect to the isometries $\alpha_{\tau'} : \tau \to \tau + \tau'$, given by Lorentz boosts in Minkowski spacetime. Furthermore, its inverse temperature is $\beta = 2\pi/a$, where a is the uniform acceleration of the observer along the orbits generated by the Lorentz boosts, as we will show now explicitly.

THEOREM 5.2.1. *When restricted to the algebra of the Rindler wedge, the Minkowski vacuum state satisfies the KMS condition with respect to isometries defined by the Lorentz boost $\alpha_{\tau'} : \tau \to \tau + \tau'$. The inverse temperature of this state is $\beta = 2\pi/a$.*

Proof. To prove this we need to see that the Minkowski vacuum state satisfies the KMS condition (c.f. Definition 2.4.1) for $a = \hat{\phi}(f),\ b = \hat{\phi}(g)$ with $f,\ g$ supported in the interior of the Rindler wedge. This holds if for $x,\ x' \in \mathcal{W}$ the distribution $F_{x,x'}(\theta + i\rho) = \omega(\hat{\phi}(x), \alpha_{\theta+i\rho}\hat{\phi}(x'))$ has the distributional boundary values

$$\lim_{\rho\to 0^+} F_{x,x'}(\theta + i\rho) = F_{x,x'}(\theta), \qquad \lim_{\rho\to\beta^-} F_{x,x'}(\theta + i\rho) = F_{x',x}(\theta). \tag{5.3}$$

To prove this, we write the $\sigma_\varepsilon(x, x')$ in (5.2) in terms of the Rindler coordinates (5.1) which results in

$$\begin{aligned}\sigma_\varepsilon(x, x') = a^{-2}\{&e^{2a\xi} + e^{2a\xi'} - 2e^{a(\xi+\xi')}\cosh(a(\tau - \tau')) \\ &+ 2i\varepsilon[e^{a\xi}\sinh(a\tau) - e^{a\xi'}\sinh(a\tau')] + \varepsilon^2\} + (\vec{x}_\perp - \vec{x}'_\perp)^2.\end{aligned} \tag{5.4}$$

For our purposes, it is convenient to slightly modify the $i\varepsilon$ prescription, to do this we are going to use the same method as in [23][Section 2.5]. After a minor rearrangement of terms, this allows us to write (5.4) as

$$\sigma_\varepsilon(x, x') = (\vec{x}_\perp - \vec{x}'_\perp)^2 + a^{-2}\{(e^{a\xi} - e^{a\xi'})^2 - 4e^{a(\xi+\xi')}\sinh\left(\tfrac{a}{2}(\tau - \tau' - 2i\varepsilon)\right)^2\}. \tag{5.5}$$

This is possible because when x and x' are null-separated, both (5.4) and (5.5) have positive imaginary parts. If we introduce the functions $I(x,x') = a^{-2}(e^{a\xi} - e^{a\xi'})^2 + (\vec{x}_\perp - \vec{x}'_\perp)^2$ we can write

$$\sigma_\varepsilon(x,x') = I(x,x') - \frac{4e^{a(\xi+\xi')}}{a^2} \sinh\left(\tfrac{a}{2}(\tau - \tau' - 2i\varepsilon)\right)^2,$$

from which it follows that the distribution $F_{x,x'}$ can be written as

$$F_{x,x'}(\theta + i\rho) = \lim_{\varepsilon\to 0^+} \frac{\Gamma(d/2-1)}{4\pi^{d/2}} [\sigma_{\varepsilon+(\rho+i\theta)/2}(x,x')]^{-(d-2)/2}.$$

To take the limit we note that the function $f(\varepsilon,\rho) := \sigma_{\varepsilon+\rho/2}(x,x')$ is continuous, hence $\lim_{\rho\to 0^+} f(\varepsilon,\rho)$ exists pointwise for each $\varepsilon \neq 0$. Moreover, observe that $\lim_{\varepsilon\to 0^+} f(\varepsilon,\rho)$ converges uniformly for $\rho \neq 0$, so by making use of the Moore-Osgood theorem [57] we deduce that we can exchange the limits. In particular, if we take $\rho \longrightarrow 0^+$ we deduce that

$$\lim_{\rho\to 0^+} F_{x,x'}(\theta + i\rho) = \lim_{\varepsilon\to 0^+} \frac{\Gamma(d/2-1)}{4\pi^{d/2}} [\sigma_{\varepsilon+i\theta/2}(x,x')]^{-(d-2)/2} = F_{x,x'}(\theta)$$

which verifies the first equation in (5.3). For the limit $\rho \longrightarrow \beta^-$ we recall that $\beta = 2\pi/a$ so, we have $\lim_{\rho\to\beta^-} f(\varepsilon,\rho) = \lim_{\rho\to\beta^-} \sigma_{\varepsilon+\rho}(x,x') = \sigma_{-\varepsilon}(x',x)$ where we used the fact that $I(x,x')$ is symmetric in its arguments. Note that this limit exists for each $\varepsilon \neq 0$, also $\lim_{\varepsilon\to 0^+} f(\varepsilon,\rho) = \sigma_\varepsilon(x,x')$ converges uniformly for $\rho \neq \beta$ so we can exchange limits again. And so, we obtain

$$\lim_{\rho\to\beta^-} F_{x,x'}(\theta + i\rho) = \lim_{\varepsilon\to 0^+} \frac{\Gamma(d/2-1)}{4\pi^{d/2}} \sigma_{-\varepsilon}(x',x)^{-(d-2)/2} = F_{x',x}(\theta),$$

which verifies the second condition in (5.3). □

It must be noted that we have proved that the KMS condition holds in the case where the observables a and b are single fields. The general result follows from using the quasifree property of the Minkowski vacuum state, which allows us to deduce all n-point functions from 2-point functions.

Furthermore, the upshot of Theorem 5.2.1 is that if an observer is in the Minkowksi vacuum state ω, after undergoing uniform acceleration a, it will thermalise with temperature

$$k_B T_U = \frac{\hbar a}{2\pi c} \tag{5.6}$$

where we have restored constants: $k_B = 1.380649 \times 10^{-23} J \cdot K^{-1}$ is the Boltzmann constant, $\hbar = 1.054571817... \times 10^{-34} J \cdot s$ the (reduced) Planck constant and $c = 2.99792458 \times 10^8 m \cdot s^{-1}$ is the speed of light.

Note that this observer corresponds to a static observer in Rindler spacetime. This might seem a little puzzling since if the state is in Minkowski vacuum, an observer at rest will say that there are no particles present. However, it is important to mention that the KMS temperature is not necessarily the physical temperature of the vacuum. Hence, this does not contradict the fact that accelerated observers thermalise, it just shows that the notion of vacuum state or particles is not fundamental. Both observers will agree on the fact that the two-point function of the field is given by $\mathcal{W}$ and in consequence, they will calculate the same probabilities for measuring any field observables.

5.3 UNRUH-DEWITT DETECTOR

To measure the temperature experimentally, one needs to introduce a theoretical model for a detector. This is the so-called Unruh-DeWitt detector. From now on we will follow closely the exposition of the theory given in Section 3 of [26]. We will consider a detector to be a pointlike two-level quantum system with Hilbert space $\mathcal{H}_D = \mathbb{C}^2$ that moves along the wordline $x(\tau)$. This Hilbert space is spanned by the orthonormal basis $\{|0\rangle, \; |1\rangle\}$ with $H_D|0\rangle = 0$ and $H_D|1\rangle = E_{gap}|1\rangle$, where H_D is the detector's Hamiltonian and E_{gap} is its energy gap – from now on we will write E when no confusion might arise.

The quantum field $\hat{\phi}$ is the minimally-coupled Klein-Gordon scalar field with mass $m \geq 0$ whose Hilbert space $\mathcal{H}_{\hat{\phi}}$ is known to contain Hadamard state vectors[1] and admit a unitary time evolution generated by a Hamiltonian $H_{\hat{\phi}}$. Therefore, the Hilbert space of the total system is $\mathcal{H} = \mathcal{H}_{\hat{\phi}} \otimes \mathcal{H}_D$ and the total Hamiltonian is

$$H = H_{\hat{\phi}} \otimes \mathbf{1}_D + \mathbf{1}_{\hat{\phi}} \otimes H_D + H_{int},$$

where $H_{int}(\tau) = c\chi(\tau)\Phi(x(\tau)) \otimes \mu$ is the interaction Hamiltonian which is constituted by c, a coupling constant; $\chi \in C_0^\infty(\mathbb{R})$, the detector's switching function and the detector's monopole moment operator is $\mu = |0\rangle\langle 1| + |1\rangle\langle 0|$.

[1]Vectors $|\phi_i\rangle$ for which the two-point function $\langle \phi_i|\hat{\phi}(x')\hat{\phi}(x'')|\phi_i\rangle$ satisfies the Hadamard condition.

Suppose that before the interaction starts, the detector is in $|0\rangle$ and the field in some Hadamard state $|\phi_i\rangle$. Then, after the interaction has ceased, the probability of finding the detector in $|1\rangle$, regardless of the final state of the field, is (according to first order perturbation theory) [62, 10, 64]

$$P = c^2|\langle 1|\mu(0)|0\rangle|^2\mathcal{F}(E),$$

where

$$\mathcal{F}(E) = \int_{\mathbb{R}^2} d\tau'\, d\tau'' \chi(\tau')\chi(\tau'')e^{-iE(\tau'-\tau'')}\langle\phi_i|\hat{\phi}(x(\tau'))\hat{\phi}(x(\tau''))|\phi_i\rangle \quad (5.7)$$

is called the **response function**. The internal structure of the detector is encoded by the constant factor $c^2|\langle 1|\mu(0)|0\rangle|^2$, hence, the transition probability will depend only on the response function (c.f. [20]). Note that as $|\phi_i\rangle$ is by assumption Hadamard, $\mathcal{W}(\tau', \tau'') := \langle\phi_i|\hat{\phi}(x(\tau'))\hat{\phi}(x(\tau''))|\phi_i\rangle$ is a well-defined distribution.

From now on we are going to consider two-point functions whose pull-back is *time translation independent*, i.e. they satisfy the property $\mathcal{W}(\tau', \tau'') = \mathcal{W}(\tau' - \tau'')$, the upshot of this is that $\mathcal{F}(E)$ will be invariant under translations in χ. If we make the additional assumption that the detector is switched on for a long time, we will arrive at the usual Unruh effect. So, if we divide (5.7) by the interaction time and let this time tend to infinity, we find that the response function $\mathcal{F}(E)$ is proportional to the *stationary response function* $\mathcal{F}_s(E)$ independent of χ defined by

$$\mathcal{F}_s(E) = \mathscr{F}(\mathcal{W})(E).$$

This Fourier transform can be calculated via contour methods (c.f. [28][Sect III.B]) whereupon we find that for the Minkowski vacuum state, we will have

$$\mathcal{F}_s(E) = \frac{1}{2\pi}\left(\frac{E}{e^{2\pi E/a} - 1}\right). \quad (5.8)$$

We can find the usual result for Unruh temperature $T_U = a/(2\pi)$ by substituting the stationary response function $\mathcal{F}_s(E)$ in (2.4.2), which agrees with Theorem 5.2.1 and is clearly independent of the energy gap E.

5.4 DETECTOR IN UNIFORM CIRCULAR MOTION

Next, we study a detector that is in uniform circular motion, rather than the uniform linear acceleration in the usual Unruh effect formulation. It is natural to think that one would study the properties of the *rotating vacuum* in order to understand the detector response. However, this introduces some remarkable differences with respect to the usual Unruh effect.

An initial approach would be to think of cylindrical coordinates (t, r, ϕ, z) for $\mathbb{R}^4$ and implement the rotation for an angular velocity Ω via the coordinate transformation $\phi \to \phi + \Omega t$. Note that this is rather naïve as this transformation does not define a Lorentz frame. However, it illustrates the point we want to make. In these coordinates the metric becomes

$$g = (-1 + \Omega^2 r^2) dt \otimes dt + dr \otimes dr + r^2 d\phi \otimes d\phi + 2\Omega r^2 dt \otimes_s d\phi + dz \otimes dz$$

and so the wordline of a detector will be the integral curve of the Killing vector $\xi = \partial/\partial t$ which will be timelike if $\Omega r < 1$, null if $\Omega r = 1$ and spacelike if $\Omega r > 1$. Also, observe that in this case there is no event horizon as opposed to the Rindler coordinates (5.1).

The form of the solutions to the Klein-Gordon equation in the rotating frame are the same (after relabelling modes) to those of the nonrotating frame (c.f. [48]). Consequently, all of the n-point functions of the ground state with respect to translations in t, will also be identical to those of the Minkowski vacuum and therefore the states will be the same (c.f. 2.2.4). This is already very different to the usual Unruh effect where clearly the Minkowski vacuum is different to the Rindler vacuum as accelerated detectors give no response in the Rindler vacuum state but thermalise in the Minkowski vacuum. Yet, in our present case, since the rotating and nonrotating states are the same, a rotating detector will give no response in the Minkowksi vacuum state.

This is certainly puzzling, more so if we consider that the detector response per unit time can be calculated (c.f. [18]) and is different from zero when $\Omega r < 1$

since it is given by

$$\mathcal{F}_{rot}(E) = \frac{\sqrt{-1+\Omega^2 r^2}}{4\pi^2}\mathscr{F}[\mathcal{W}_{rot}]\left(\sqrt{-1+\Omega^2 r^2}E\right), \tag{5.9}$$

$$\mathcal{W}_{rot}(s) = \lim_{\varepsilon\to 0^+}(-(s-i\varepsilon)^2 + 4r^2\sin(\Omega s/2)^2)^{-1}. \tag{5.10}$$

It may be argued that the problem arises from doing a Galilean coordinate transformation and that one instead must use a local Lorentz frame [19] given by $t \to \gamma(t+\Omega r^2\phi)$, $\phi \to \gamma(\phi+\Omega t)$. Unfortunately, this renders the metric multi-valued. This happens because the constant time surfaces are helicoidal with jumps in time proportional to Ωr^2, and so, there is no Cauchy surface for this geometry.

This seems even more strange than the previous case, nonetheless, a rather elegant solution was found by Korsbakken and Leinaas [46] by the means of a clever reformulation: instead of trying to build a rotating vacuum state, let us consider a detector rotating in the unit circle ($r = 1$) in the Minkowski vacuum state. The difference might be subtle but is remarkably important, as now the emphasis is on understanding the detector response.

What Korsbakken and Leinaas do, is analyse a detector in a stationary trajectory that admits a frame that is both accelerated uniformly (with acceleration a) and rotating (with angular velocity Ω) with respect to an inertial rest frame. Within this reformulation, (5.9) could be understood as a limiting case when a is very small and Ω is kept fixed. When $a > \Omega$ there will be an event horizon and a static limit, which is defined by certain values of a and Ω for which the detector's trajectory becomes timelike.

Beyond the static limit the field will have negative-energy modes which implies that in the detectors frame the Minkowski vacuum is no longer a ground state. As a matter of fact, the detector's response comes from radiating into negative-energy modes, or absorbing positive-energy ones. The other case they study is when $a < \Omega$, for which the event horizon disappears and it can be interpreted as uniform circular motion. Since there is no longer a horizon, Minkowski vacuum can no longer be seen as a thermal state, and so the interpretation of the detector response might seems elusive and ill-defined.

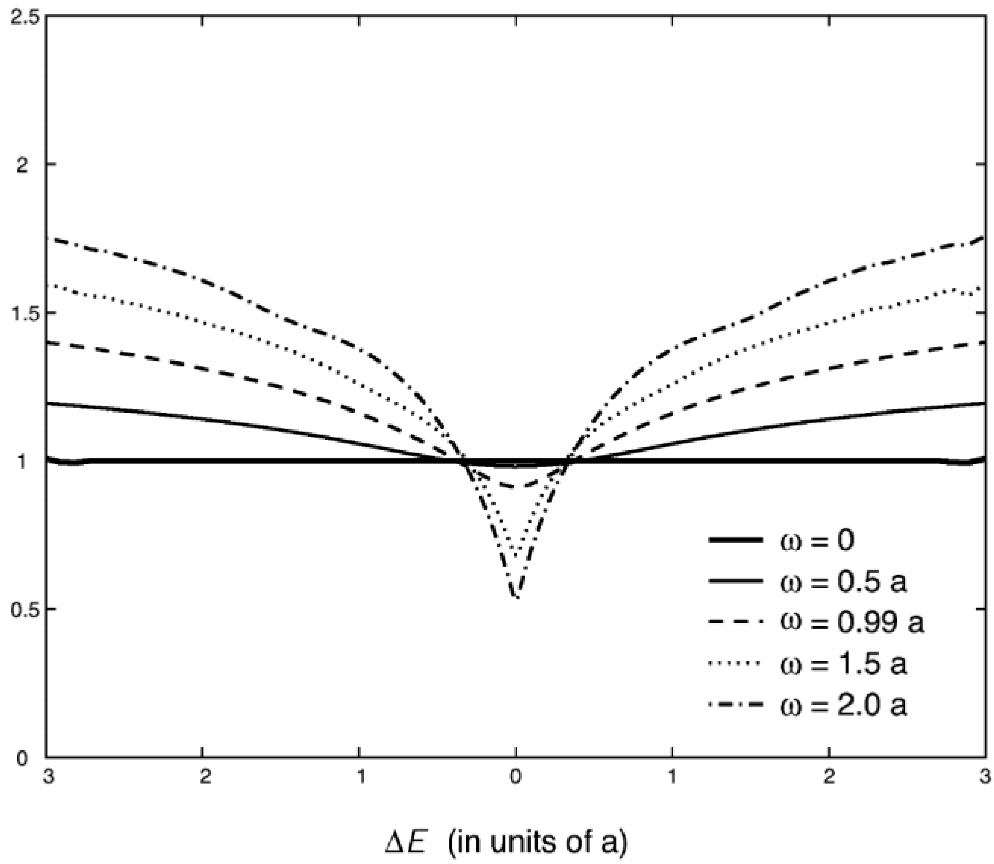


Figure 5.2: T as a function of E published by Korsbakken and Leinaas in [46]. In their notation ω is the angular velocity and ΔE the energy gap.

However, Korsbakken and Leinaas also found that if we assume that the response function always satisfies the detailed balance condition at temperature $T = 1/\beta$, then one can show that this temperature varies continuously with respect to changes the parameter Ω/a. See Figure 5.2 to see the plot of T as a function of E, originally published in [46]. Therefore, by assuming that the detector's response function satisfies the detailed balance condition (2.4.2) we can still expect thermalisation. This thermalisation is due to different causes from the one observed in the Unruh effect, but both of them can be related by a continuous transition when Ω/a varies continuously. Also, now the temperature will depend on the energy gap E and will be independent of it only when $\Omega = 0$, which is the usual Unruh effect.

5.5 RESPONSE FUNCTION FOR A DETECTOR IN UNIFORM CIRCULAR MOTION

In the previous section, we mentioned that assuming that the detailed balance condition still holds for a rotating detector in a spacetime with no horizon is of paramount importance to deem the detector response as thermal. This assumption is somewhat puzzling as there are many questions on the interpretation of temperature in this case, or even of the validity of the perturbation theory used to arrive to this result. As the conceptual side is rather opaque, experimental proposals to test this assumption have been envisaged.

Initially this seems like a challenging task, nevertheless the analogue spacetime programme [6] simplifies the situation as it allows us to simulate relativistic phenomena in non-relativistic laboratory environments. In particular, in [39] it has been proposed to use a Bose-Einstein Condensate (BEC) as an analogue spacetime because its phonons are the fastest perturbations within the condensate. This allows us of thinking of them as analogues of photons, as they are the fastest signals that can be propagated in the BEC, but with the added advantage that the analogue speed of light defined by them is 12 orders of magnitude lower than the usual one. A laser coupled to this condensate serves as the detector and is in uniform circular motion in $2 + 1$ spacetime dimensions.

With this experimental setting in mind, we will analyse a detector in planar uniform circular motion. More specifically, we ask:

> How long does one has to wait for a 2-level detector in circular motion to become thermalised, as a function of its speed v and the energy gap E in the detector?

In the possible experimental settings, the energy gap is usually small while the waiting time is large. To this end, we will assume that the energy gap E and the waiting time λ are related as $E\lambda = S_0$ for a positive constant S_0. This assumption was made on the grounds that we are expecting physical scenarios where the energy gap of the detector is very small compared to the waiting time in the laboratory. We will adapt the results of [26] and [9] under this assumption to find an asymptotic expression (as $E \to 0$) for the transition function. It is important to note that in [26] they were interested in large E. So, will follow their approach in considering a finite observation time (encoded in the support of χ) and modify it for the small E case.

One point worth noting is that this expression can be taken up to an arbitrary order in E and the upshot of this is that the thermalisation temperature T of the detector will depend non-linearly on E (as we expected) as $E \to 0$. In other work (see [9][Sect. 4.3]) it is more normal to have a linear relation between E and T for small E. Thus, a nonlinear relation could have implications for the observability of the thermalisation and this will be a focus of the work as it develops further.

DEFINITION 5.5.1. For a continuous function g, we define its *dilation* by μ as $\mathscr{D}_\mu[g(x)] = g(\mu \cdot x)$. Note that when $\mu = -1$ we have a reflection and it shall be denoted by $\mathscr{R}$.

Hence, we are interested in studying the case where the response function is time translation independent but the detector may switch on and off. So (5.7) remains as it is, which corresponds to the general case. In this form, it is not obvious that $\mathcal{F}(E)$ will be invariant under translations in time. To see that this indeed happens, make the change of variable $(\tau', \tau'') \to (s = \tau'' - \tau', t = \tau'')$ and define $\chi_E(\tau) := e^{-iE\tau}\chi(\tau)$, with this (5.7) becomes

$$\begin{aligned}\mathcal{F}(E) &= \int_{\mathbb{R}^2} dt\; ds\; \chi(t)\chi(t-s)e^{-iEs}\mathcal{W}(s) = \int_{\mathbb{R}^2} dt\; ds\; \overline{\chi_E(t-s)}\chi_E(t)\mathcal{W}(s) \\ &= \int_{\mathbb{R}} ds\; [\overline{\mathscr{R}\chi_E} * \chi_E](s)\mathcal{W}(s) = \frac{1}{2\pi}\int_{\mathbb{R}} d\omega\; \mathscr{F}[\overline{\mathscr{R}\chi_E} * \chi_E](\omega)\mathscr{F}[\mathcal{W}](\omega)\end{aligned}$$

where in the last line we made use of Parseval's identity. Next we make use of another identity, $\mathscr{F}[\overline{\mathscr{R}\chi_E}](\omega) = \overline{\mathscr{F}[\chi_E](\omega)}$ to obtain

$$\begin{aligned}\mathcal{F}(E) &= \frac{1}{2\pi}\int_{\mathbb{R}} d\omega\; \mathscr{F}[\overline{\mathscr{R}\chi_E}](\omega)\mathscr{F}[\chi_E](\omega)\mathscr{F}[\mathcal{W}](\omega) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} d\omega\; |\mathscr{F}[\chi_E](\omega)|^2\mathscr{F}[\mathcal{W}](\omega).\end{aligned}$$

Finally, we observe that as $\mathscr{F}[\chi_E](\omega) = \mathscr{F}[\chi](\omega - E)$, we can make the change of variable $\omega \to \omega + E$ and arrive at

$$\mathcal{F}(E) = \frac{1}{2\pi}\int_{\mathbb{R}} d\omega\; |\mathscr{F}[\chi](\omega)|^2\mathscr{F}[\mathcal{W}](E+\omega). \tag{5.11}$$

Note that (5.8) and (5.11) are quite different, so allowing the detector to switch on and off has an striking consequence in the final form of the response function. Also, observe that time is implicitly included in the support of the switching function χ, nonetheless, this is not the only relevant time scale that can be studied. This is because in this particular instance, the support of χ just tells us for how long the detector is switched on.

Still, we also need to account for the interaction time λ of the detector with the field, regardless of whether the former is making measurements or not. In order to study this interaction time, we introduce an adiabatically scaled switching function χ_λ (see [26]) which in turn will lead us to define an interaction time-dependent response function $\mathcal{F}_\lambda(E)$.

DEFINITION 5.5.2. Define the adiabatically scaled switching function as $\chi_\lambda(\tau) := \chi(\tau/\lambda)$. Here $\lambda > 0$ is the interaction time of the field and the detector. Then, we define

$$\mathcal{F}_\lambda(E) := \frac{\mathcal{F}(E)}{\lambda}.$$

Note that one expects that under the scaling $\chi_\lambda(\tau)$ introduced above, the response function $\mathcal{F}(E)$ will be proportional to λ. As we will be interested in studying its behaviour for large λ, it makes sense to take the ratio. Moreover, as $\mathscr{F}[\chi_\lambda](\omega) = \lambda\mathscr{F}[\chi](\lambda\omega)$, from (5.11) we obtain

$$\begin{aligned}\mathcal{F}_\lambda(E) &= \frac{\lambda}{2\pi}\int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi](\lambda\omega)|^2\mathscr{F}[\mathcal{W}](E+\omega)\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2\mathscr{F}[\mathcal{W}](E+\omega/\lambda). \qquad (5.12)\end{aligned}$$

Recall that the Bochner-Schwartz theorem states that a continuous function f on $\mathbb{R}$ is positive definite if and only if it is the Fourier transform of a finite non-negative measure μ of at most polynomial growth–c.f. [56]. Thus, f is a tempered distribution.

PROPOSITION 5.5.1. *Let* $\mathcal{F}_\infty(E) := \lim_{\lambda\to\infty}\mathcal{F}_\lambda(E)$*, then*

$$\mathcal{F}_\infty(E) = \mathscr{F}[\mathcal{W}](E)||\chi||_2^2. \qquad (5.13)$$

Proof. As argued in [26][Section 4.1], we note that as the distribution $\mathcal{W}$ is of positive type, the Bochner-Schwartz theorem indicates that it must be a tempered distribution, whereupon we deduce that $\mathscr{F}[\mathcal{W}]$ must be a polynomially bounded measure. Hence, there must be positive constants A, B and $n \in \mathbb{N}_0$ so that for $1 \leq \lambda$ we will have

$$\begin{aligned} |\mathscr{F}[\mathcal{W}](E+\omega/\lambda)| &\leq A + B(E+\omega/\lambda)^{2n} \leq A + B(|E| + |\omega|/\lambda)^{2n} \\ &\leq A + B(|E| + |\omega|)^{2n}. \end{aligned}$$

Also, as $\chi \in C_0^\infty(\mathbb{R})$, its Fourier transform $\mathscr{F}[\chi](\omega)$ decays faster that any inverse power of ω, we deduce that the integrand in (5.12) is dominated by an integrable function that does not depend on λ. Therefore, we can take the limit under the integral to obtain our result. □

From now on we will assume that $\mathcal{F}_\infty(E)$ satisfies detailed balance, i.e. $\mathcal{F}_\infty(-E) = e^{E/T}\mathcal{F}_\infty(E)$. This follows from the fact that if $\mathscr{F}[\mathcal{W}]$ satisfies the detailed balance condition at temperature T (see Definition 2.4.2), then the response function $\mathcal{F}(E)$ will do so as well. Expecting $\mathcal{F}_\infty(E)$ to satisfy detailed balance is reasonable since after an infinite interaction time we should have thermalisation–as in the usual Unruh effect– with temperature given as a function of the energy gap E in the following fashion

$$\frac{1}{T} = \frac{1}{E}\ln\left(\frac{\mathcal{F}(-E)}{\mathcal{F}(E)}\right).$$

However, as it was mentioned above, we are not interested in studying the usual Unruh effect since we are studying situations where the temperature depends on E. Given that we want to find an asymptotic expansion as $E \longrightarrow 0$ for the response function (5.7), we want to work more on its current form to end up with a more manageable expression.

5.6 ASYMPTOTIC EXPANSION OF THE RESPONSE FUNCTION

We begin by studying the scenario where the interaction time between the field and the detector is taken to be very large and thus not relevant to the other time scale relevant to the experiment, the support of the switching function.

Recalling that we are using standard Minkowski coordinates, we introduce the notation $(\Delta\vec{x})^2 := (x(\tau') - x(\tau''))^2 + (y(\tau') - y(\tau''))^2$, $\Delta t = t(\tau') - t(\tau'')$ and $(\Delta x)^2 := -(\Delta t)^2 + (\Delta\vec{x})^2$. The two-point function in $2+1$ dimensions is given by

$$\mathcal{W}(x', x'') = \lim_{\varepsilon\to 0^+} \frac{1}{4\pi}\frac{1}{\sqrt{(\Delta\vec{x})^2 - (\Delta t - i\varepsilon)^2}} = \lim_{\varepsilon\to 0^+} \frac{1}{4\pi}\frac{1}{\sqrt{(\Delta x)^2 + 2i\varepsilon\Delta t + \varepsilon^2}}. \tag{5.14}$$

Note that this two-point function depends only on the difference of its arguments $s := \tau' - \tau''$, and so, $\mathcal{W}(\tau', \tau'') = \mathcal{W}(\tau' - \tau'') = \mathcal{W}(s)$. Because of this, from now on, $\mathcal{W}$ will denote the single variable distribution. The worldline for uniform circular motion in $2+1$ dimensions is $x(\tau) = (\gamma\tau, R\cos(\frac{\gamma v\tau}{R}), R\sin(\frac{\gamma v\tau}{R}))$, where

$$k = \frac{2R}{\gamma v}, \tag{5.15}$$

and R is the radius of the trajectory, $v < 1$ its speed and γ the Lorentz factor. In this case we have

$$(\Delta x)^2 = (x(s) - x(0))^2 = -4R^2\left(\frac{z^2}{v^2} - \sin(z)^2\right), \qquad \left(\text{with } z = \frac{s}{k}\right)$$

and so, the pullback two-point function to a circular trajectory in Minkowski spacetime is given by

$$\mathcal{W}_{\mathbb{S}^1}(\tau' - \tau'') = \lim_{\varepsilon\to 0^+} \frac{1}{4\pi}\frac{1}{\sqrt{4R^2\sin((\tau - \tau')/k)^2 - (\gamma(\tau - \tau') - i\varepsilon)^2}}.$$

For brevity's sake, from now on we will denote the pullback of this function by $\mathcal{W}$.

We will find the asymptotically small E expansion of $\mathcal{F}(E)$ keeping χ and v arbitrary but fixed. Also, initially, we will not consider interaction time as we want to compare this with the result found in "2+1" [9].

DEFINITION 5.6.1. Define the functions

$$R_v(z) \doteq \frac{1}{\sqrt{z^2 - v^2\sin(z)^2}}, \quad Q_v(z) \doteq 1 - zR_v(z), \quad S_v(z) \doteq \frac{Q_v(z)}{z} - \frac{1-\gamma}{z(1+z^2)}, \tag{5.16}$$

with $\mathbb{R}^+$ serving as their domain. Henceforth we will consider the odd extension of these functions. That is, extending the domain to $\mathbb{R}$ with $f(-x) = -f(x)$.

The Fourier transform $\mathscr{F}[\mathcal{W}]$ can be read off from (2.5) and (4.3) in [9]. Making use of the definitions above and, reminding the reader that $\mathcal{S}$ denotes the sine transform (c.f. Definition 5.1.2), we obtain

$$\mathscr{F}[\mathcal{W}](E) = \frac{1}{4} - \frac{1}{2\pi\gamma}\mathcal{S}[R_v](kE). \tag{5.17}$$

and so, we can substitute (5.17) into (5.11) and use Parseval's identity to obtain

$$\mathcal{F}(E) = \frac{||\chi||_2^2}{4} - \frac{1}{4\pi^2\gamma}\int_{-\infty}^{\infty} d\omega\ |\mathscr{F}[\chi](\omega)|^2\mathcal{S}[R_v](k(E+\omega)). \tag{5.18}$$

Our strategy to simplify this expression is to write the integral associated with the sine transform in the expression above. Then, make use Fubini's theorem in the second term to exchange the integration order and then expand the sine term using Taylor's theorem. However, for large z, $R_v(z) = 1/z + (v^2/2)\sin(z)^2/z^3 + O(1/z^5)$, which would not allow us to use Fubini. Subtracting the divergence leads to the following decomposition

$$\mathcal{S}[R_v](\zeta) = \mathcal{S}[R_v - 1/z](\zeta) + \mathcal{S}[1/z](\zeta) = -\mathcal{S}[Q_v(z)/z](\zeta) + \frac{\pi}{2}\mathrm{sgn}(\zeta) \tag{5.19}$$

where we have used the definition of Q_v in (5.6.1) and the identity $\mathcal{S}[1/z](u) = (\pi/2)\mathrm{sgn}(u)$.

Next, we take the sine in the remaining integral, expand it as $\sin(k(E+\omega)z) = \sin(kEz)\cos(k\omega z) + \cos(kEz)\sin(k\omega z)$ and drop the last term due to the fact that $|\mathscr{F}[\chi](\omega)|^2$ is even. This leads to

$$\int_{-\infty}^{\infty} d\omega\ |\mathscr{F}[\chi](\omega)|^2\mathcal{S}[R_v](k(E+\omega)) = \frac{\pi}{2}\int_{-\infty}^{\infty} d\omega\ |\mathscr{F}[\chi](\omega)|^2\mathrm{sgn}(E+\omega)$$
$$-\int_{-\infty}^{\infty} d\omega\ |\mathscr{F}[\chi](\omega)|^2\int_0^{\infty} dz\ \sin(kEz)\cos(k\omega z)\frac{Q_v(z)}{z} \tag{5.20}$$

where we have made us of the fact that $k > 0$ to simplify the sgn term.

Remark 5.6.1. From (5.16) we see that, as $Q_v(z) = 1 - zR_v(z)$, then

$$Q_v(z) = 1 - \frac{1}{\sqrt{1 - v^2\mathrm{sinc}(z)^2}} < 0 \tag{5.21}$$

Its L^1 norm is an analytic function in $v < 1$, experimental considerations (sonic limit) amount to consider $v \leq 0.99$, which in turn leads to the bound $||Q_v||_1 \leq M \approx 3.809$ (c.f. Proposition C.2.1). Moreover, for large z, we have $Q_v(z)/z = -(v^2/2)\sin(z)^2/z^3 + O(1/z^5)$, while for z near the origin, $Q_v(z)/z = (1-\gamma)/z + O(z)$.

Because of this, we see that the inner integrand is integrable over z. Let us introduce two definitions,

$$J_1[\phi](E) = \int_{-\infty}^{\infty} d\omega\ \phi(\omega)\mathrm{sgn}(\omega + E) \tag{5.22}$$

$$J_2[f](E) = \int_{-\infty}^{\infty} d\omega\ |\mathscr{F}[\chi](\omega)|^2 \int_0^{\infty} dz\ \sin(kEz)\cos(k\omega z) f(z) \tag{5.23}$$

this allows us to write

$$\int_{-\infty}^{\infty} d\omega\ |\mathscr{F}[\chi](\omega)|^2 \mathcal{S}[R_v](k(E+\omega)) = \frac{\pi}{2} J_1[|\mathscr{F}[\chi]|^2](E) - J_2\left[\frac{Q_v(z)}{z}\right](E). \tag{5.24}$$

DEFINITION 5.6.2. Define the linear functional for bounded f and fixed (but arbitrary) $\chi \in C_0^\infty(\mathbb{R})$

$$M_p[f] := \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dz'\ \chi(z) f(z-z')(z-z')^p \chi(z')$$

with $p \in \mathbb{N}_0$.

The asymptotic expansions of J_1 and J_2 (c.f. Corollary C.2.1 and Proposition C.2.4) in our case are given by

$$J_1[|\mathscr{F}[\chi]|^2](E) = 2\sum_{n=0}^{l-1}(-1)^n \frac{M_{2n}[1]}{(2n+1)!} E^{2n+1} + O(E^{2l+1}), \tag{5.25}$$

$$J_2[Q_v(z)/z](E) = \frac{\pi}{k}\sum_{n=0}^{l-1}(-1)^n \frac{M_{2n+1}[\mathscr{D}_{1/k}\{Q_v(z)/z\}]}{(2n+1)!} E^{2n+1} + O(E^{2l+1}). \tag{5.26}$$

And so, if we make of use (5.25) and (5.26), we can write (5.24) as

$$\begin{aligned}&\int_{-\infty}^{\infty} d\omega\ |\mathscr{F}[\chi](\omega)|^2 \mathcal{S}[R_v](k(E+\omega)) \\ =&\frac{\pi}{k}\sum_{n=0}^{l-1}(-1)^n \frac{M_{2n}[k] - M_{2n+1}\left[\mathscr{D}_{1/k}\{Q_v(z)/z\}\right]}{(2n+1)!} E^{2n+1} + O(E^{2l+1}).\end{aligned} \tag{5.27}$$

Noting that $M_p[f(z)] = M_{p+1}[f(z)/z]$, we deduce

$$M_{2n}[k] - M_{2n+1}\left[\mathscr{D}_{1/k}\left\{\frac{Q_v(z)}{z}\right\}\right] = M_{2n+1}\left[\frac{k}{z} - \mathscr{D}_{1/k}\left\{\frac{Q_v(z)}{z}\right\}\right]$$
$$= M_{2n+1}[\mathscr{D}_{1/k}R_v]. \tag{5.28}$$

And so, if we make use of Plancherel's formula, (5.27) and (5.28), we can write the response function (5.18) as

$$\mathcal{F}(E) = \frac{||\chi||_2^2}{4} - \frac{1}{4\pi\gamma k}\sum_{n=0}^{l-1}(-1)^n \frac{M_{2n+1}[\mathscr{D}_{1/k}R_v]}{(2n+1)!}E^{2n+1} + O(E^{2l+1}), \tag{5.29}$$

where $||\cdot||_2$ denotes the L^2 norm in $L^2(\mathbb{R}, dz)$. At this stage a question arises: how far one must go in the expansion? The detailed balance condition along with some experimental considerations will shed some light on this. Using (2.4.2) we can obtain the temperature T as a function of the energy gap E as follows

$$\frac{1}{T(E)} = \frac{1}{E}\ln\left(\frac{\mathcal{F}(-E)}{\mathcal{F}(E)}\right),$$

from (5.29) we note that $\mathcal{F}(E)$ only contains odd powers of E above zeroth order. Hence, the response function can be written as

$$\mathcal{F}(E) = a_0 + a_1 E + a_3 E^3 + O(E^5). \tag{5.30}$$

where the coefficients are given by

$$a_0 = \frac{||\chi||_2^2}{4}, \qquad a_1 = -\frac{M_1[\mathscr{D}_{1/k}R_v]}{4\pi\gamma k}, \qquad a_3 = \frac{M_3[\mathscr{D}_{1/k}R_v]}{24\pi\gamma k} \tag{5.31}$$

And so, if we assume that $a_0 \neq 0$ and $a_1 \neq 0$, the temperature for small energies is

$$T(E) = -\frac{1}{2}\frac{a_0}{a_1} + \frac{1}{6}\left(\frac{a_1}{a_0} + 3\frac{a_0 a_3}{a_1^2}\right)E^2 + O(E^4). \tag{5.32}$$

With this, we can explain why we decided to truncate at $O(E^5)$. The formula above is nonlinear in E. This could be very convenient for experimental purposes as it gives room to modify the parameters so that a_i yields a significant contribution in the E^2 term. Note that the coefficient for a fifth power and above will not affect the second order term in the temperature, and so, the truncation at this point will give us all the information we need for the simplest case of nonlinear behaviour.

Next, we define $T_0 := -a_0/(2a_1)$ and study the nature of this quantity, in particular whether if it is positive-definite or negative-definite for fixed χ and v.

PROPOSITION 5.6.1. *Consider that $\chi \in C_0^\infty(\mathbb{R})$ is a positive-definite and even function. Then, for odd p, we have*

$$M_{2n+1}[\mathscr{D}_{1/k}R_v] \geq 0.$$

Proof. From Definition 5.6.2 and (5.6.1) we obtain

$$\begin{aligned} M_{2n+1}[\mathscr{D}_{1/k}R_v] &= \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dz' \, \chi(z) R_v((z-z')/k)(z-z')^p \chi(z') \\ &= k \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dz' \, \chi(z) \frac{(z-z')^{p-1}}{\sqrt{1-v^2 \mathrm{sinc}(z-z')^2}} \chi(z'). \end{aligned}$$

If we choose odd p, then $p-1$ will be even and since χ is positive-definite in $\mathbb{R}$, the integrand will also be. This concludes the proof. □

Because of this, we can conclude that for fixed χ and v, where the former is arbitrary but even and positive-definite, we will have $a_1 < 0$. From this and the fact that regardless of our choice of χ we will have $a_0 = ||\chi||_2^2 > 0$, it follows that $T_0 > 0$.

5.7 ASYMPTOTIC EXPANSION OF THE RESPONSE FUNCTION FOR LARGE TIMES

Next, to study the case that depends on the interaction time we consider the adiabatically scaled switching function χ_λ and its corresponding response function $\mathcal{F}_\lambda$ from Definition 5.5.2, with the additional choice that the energy gap and the interaction time are related via

$$E\lambda = S_0, \tag{5.33}$$

where $S_0 > 0$ is a constant. In this scenario, we analyse the behaviour of $\mathcal{F}_\lambda(E)$ for large λ. Using (5.18), (5.19), the power law (5.33) and the behaviour of the

Fourier transform under scaling we can define $\mathcal{F}_\lambda(E) := \mathcal{F}(E)/\lambda$ to obtain:

$$\begin{aligned}\mathcal{F}_\lambda(E) := \frac{\|\chi\|_2^2}{4} - \frac{1}{8\pi\gamma}\int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2 \operatorname{sgn}((S_0+\omega)/\lambda) \\ + \frac{1}{4\pi^2\gamma}\int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2 \mathcal{S}[Q_v(z)/z]((S_0+\omega)/\lambda).\end{aligned}$$

Using (5.22) to write the second term using J_1 and introducing $\chi_e(z) = e^{izS_0}\chi(z)$ leads to

$$\mathcal{F}_\lambda(E) := \frac{\|\chi\|_2^2}{4} - \frac{J_1[|\mathscr{F}[\chi]|^2](E)}{8\pi\gamma} + \int_{-\infty}^{\infty} \frac{d\omega}{4\pi^2\gamma} |\mathscr{F}[\chi_e](\omega)|^2 \mathcal{S}\left[\frac{Q_v(z)}{z}\right]\left(\frac{k\omega}{\lambda}\right), \tag{5.34}$$

Our current goal now is to compute this expression. To deal with the third term, we need to go further in the asymptotics of $Q_v(z)/z$ for large z, these are

$$\begin{aligned}\frac{Q_v(z)}{z} &= -\frac{v^2}{2}\frac{\sin(z)^2}{z^3} - \frac{3v^4}{8}\frac{\sin(z)^4}{z^5} + \ldots \\ &= \frac{1}{z}\sum_{n=1}^{l-1}\binom{n-1/2}{n}\left(v\frac{\sin(z)}{z}\right)^{2n} + O\left(\frac{1}{z^{2l+1}}\right).\end{aligned} \tag{5.35}$$

Recalling the asymptotics of Q_v near the origin given in Remark 5.6.1, we remove the singularity at the origin and improve decay at infinity, this leads to introduce the odd function

$$S_v(z) = \frac{Q_v(z)}{z} - \frac{1-\gamma}{z(1+z^2)^3} + \frac{v^2}{2}\frac{\sin(z)^2}{z(1+z^2)^2} + \frac{v^2}{2}\frac{\sin(z)^2}{z(1+z^2)} + \frac{3}{8}v^4\frac{\sin(z)^4}{z(1+z^2)^2}. \tag{5.36}$$

Let it be noted that we have chosen terms of the form $\sin(z)^{2n}/z(1+z^2)^n$ instead of $\sin(z)^{2n}/z^{2n+1}$ to avoid introducing new singularities for small z. The asymptotics of (5.36) can be deduced from those of Q_v, for z near the origin we will have

$$S_v(z) = O(z) \quad (\text{as } z \to 0), \qquad S_v(z) = O(1/z^7) \quad (\text{as } z \to \infty), \tag{5.37}$$

from which we deduce that $S_v \in L^1(\mathbb{R}, dz)$.

We would like to make a small note about what we mean by improving the decay at infinity. As it will be seen later in Proposition C.3.1, we want an asymptotic expansion that requires of us to differentiate the sine transform at least five times. Because of (C.2.3), we see that this amounts to $\mathcal{S}^{(5)}[S_v(z)](\zeta) = \mathcal{C}[z^5 S_v(z)](\zeta)$.

So, if we want this expression to be well-defined, we need a decay at infinity of at least $O(1/z^7)$. Consequently, the original sine transform in (5.34) can be written as

$$\mathcal{S}\left[\frac{Q_v(z)}{z}\right](\zeta) = \mathcal{S}[S_v(z)](\zeta) + \mathcal{S}\left[\frac{1-\gamma}{z(1+z^2)^3}\right](\zeta) - \frac{v^2}{2}\mathcal{S}\left[\frac{\sin(z)^2}{z(1+z^2)^2}\right](\zeta) - \frac{1}{2}v^2\mathcal{S}\left[\frac{\sin(z)^2}{z(1+z^2)}\right](\zeta) - \frac{3}{8}v^4\mathcal{S}\left[\frac{\sin(z)^4}{z(1+z^2)^2}\right](\zeta). \tag{5.38}$$

As $S_v \in L^1(\mathbb{R}^+, dz)$ we can conclude that the $\mathcal{S}[S_v(z)](\zeta)$ term in (5.38) is continuous in ζ. The other terms are continuous as well. For the second one this is seen by direct calculation and the others are all transforms of elements in $L^1(\mathbb{R}^+, dz)$. Thus, $\mathcal{S}\left[\frac{Q_v(z)}{z}\right](\zeta)$ will be continuous as well.

Our goal is to compute (5.34), which means that this quantity must be integrated when its argument is small. Because of this, the following claim is one of the most relevant parts of our analysis

LEMMA 5.7.1. *Consider* (5.38). *Then, (a) the following asymptotic expansion is valid for ζ near the origin*

$$\mathcal{S}\left[\frac{Q_v(z)}{z}\right](\zeta) = q_1\zeta + q_2\operatorname{sgn}(\zeta)\zeta^2 + q_3\zeta^3 + O(\zeta^4). \tag{5.39}$$

The explicit form of these constants, for fixed v, is

$$q_1 = \int_0^\infty dz\, Q_v(z)$$
$$q_2 = \frac{\pi}{4}\left(\frac{v}{2}\right)^2$$
$$q_3 = -\frac{1}{6}\left[\int_0^\infty dz\, z^2\left(Q_v(z) + \frac{v^2}{2}\frac{\sin(z)^2}{1+z^2}\right) + \frac{\pi}{2}\left(1-\frac{1}{e^2}\right)\left(\frac{v}{2}\right)^2\right].$$

(b) for large ζ the asymptotic expansion is

$$\mathcal{S}\left[\frac{Q_v(z)}{z}\right](\zeta) = \operatorname{sgn}(\zeta)\frac{\pi}{2}(1-\gamma)\left(1 + e^{-|\zeta|}P_2(|\zeta|)\right) + o(1/\zeta^5).$$

Proof. See Appendix C.3. ◻

Recall that the power law (5.33) indicates that $E = S_0/\lambda$ and that we have introduced $\chi_e(z) = e^{izS_0}\chi(z)$. This sine transform is an integrand in our original expression (5.34), computing the integral yields the following asymptotic expansion:

PROPOSITION 5.7.1. *Let $q : \mathbb{R} \to \mathbb{C}$ be bounded with $q(z) = q_1 z + q_2 \mathrm{sgn}(z) z^2 + q_3 z^3 + O(z^4)$ as $z \to 0$. Then, for E near the origin, the following asymptotic expansion is valid*

$$\int_{-\infty}^{\infty} d\omega \, |\mathscr{F}[\chi_e](\omega)|^2 q(k\omega/\lambda) = \tilde{q}_1 E + \tilde{q}_2 \mathrm{sgn}(E) E^2 + \tilde{q}_3 E^3 + O(E^4).$$

where the coefficients are defined as

$$\tilde{q}_1 := 2\pi k ||\chi||_2^2 q_1, \quad \tilde{q}_2 := \frac{k^2}{S_0^2} \left(\int_{-|S_0|}^{|S_0|} d\omega \, |\mathscr{F}[iS_0\chi + \chi'](\omega)|^2 \right) q_2$$
$$\tilde{q}_3 := 2\pi \frac{k^3}{S_0^2} \left(S_0^2 ||\chi||_2^2 + 3||\chi'||_2^2 \right) q_3$$

Proof. See Appendix C.4. □

And so, we can conclude that the response function for large λ (5.34) has the following form

$$\mathcal{F}_\lambda(E) = a_0 + a_1 E + a_2 \mathrm{sgn}(E) E^2 + a_3 E^3 + O(E^4). \tag{5.40}$$

Recalling that according to Definition 5.6.2 we have

$$M_n[1] = \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dz' \chi(z)(z - z')^n \chi(z')$$

which along with (5.25) and Proposition 5.7.1 implies that the a_i coefficients in (5.40) are given by

$$a_0 = \frac{||\chi||_2^2}{4}, \quad a_1 = -\frac{M_0[1] - \tilde{q}_1}{4\pi\gamma}, \quad a_2 = \frac{\tilde{q}_2}{4\pi^2\gamma}, \quad a_3 = \frac{1}{4\pi\gamma} \left(\frac{M_2[1]}{6} + \frac{\tilde{q}_3}{\pi} \right).$$

and can be written in a more explicit form as follows

$$
\begin{aligned}
a_0 &= \frac{||\chi||_2^2}{4}, \quad a_1 = -\frac{M_0[1] - 2k||\chi||_2^2 q_1}{4\pi\gamma}, \\
a_2 &= \frac{k^2}{4\pi^2\gamma S_0^2}\left(\int_{-|S_0|}^{|S_0|} d\omega \, |\mathscr{F}[iS_0\chi + \chi'](\omega)|^2\right) q_2, \\
a_3 &= \frac{1}{2\pi\gamma}\left(\frac{M_2[1]}{12} + \frac{k^3}{S_0^2}\left(S_0^2||\chi||_2^2 + 3||\chi'||_2^2\right) q_3\right).
\end{aligned} \tag{5.41}
$$

Defining $T_0 := -a_0/(2a_1)$, we deduce that the detailed balance condition in (2.4.2) yields the following temperature

$$
T_\lambda(E) = T_0 + \frac{a_2 a_0}{2\,a_1{}^2}|E| + \frac{3\left(a_1 a_3 - a_2{}^2\right) a_0{}^2 + a_1{}^4}{6\,a_0 a_1{}^3} E^2 + O\left(E^3\right). \tag{5.42}
$$

Although T_0 might seem to be a non-positive quantity, it is not. Using the definition of a_0 and a_1 we can show that $T_0 > 0$, and that if the support of the test function χ is small compared to $|q_1|$, then T_0 is a constant determined by v.

PROPOSITION 5.7.2. *Denote the support of χ by $l(S)$. Then, T_0 is positive and obeys the bound*

$$
\left|\frac{1}{T_0} - \frac{4k|q_1|}{\pi\gamma}\right| \leq \frac{2l(S)^2}{\pi\gamma}. \tag{5.43}
$$

Moreover if $l(S) << \sqrt{2k|q_1|}$, then

$$
T_0 \approx \frac{\pi\gamma}{4k|q_1|}. \tag{5.44}
$$

Proof. First, from the definition of q_1 in Lemma 5.7.1 and the fact that Q_v is never positive, we conclude that q_1 is negative. Because of this we write $q_1 = -|q_1|$ and see that a_1 must be negative. Moreover, as a_0 is positive, we conclude that T_0 is positive. Next, as χ is compactly supported, let us denote its support by S and its length by $l(S)$. Then, $M_0[1] = \left(\int_S \chi\right)^2 = \langle 1_S, \chi\rangle^2$ where 1_S is the indicator function for S. Hence,

$$
\frac{M_0[1]}{||\chi||_2^2} = \frac{\langle 1_S, \chi\rangle^2}{||\chi||_2^2} \leq ||1_S||_2^2 = l(S)^2.
$$

Making use of this we get

$$\left|\frac{1}{T_0} - \frac{4k|q_1|}{\pi\gamma}\right| = \frac{2}{\pi\gamma}\frac{M_0[1]}{||\chi||_2^2} \leq \frac{2l(S)^2}{\pi\gamma}.$$

Which proves the first part. For the second, we observe that if the RHS in the previous inequality is significantly smaller than the LHS, then the inequality saturates, and our result follows immediately. □

5.8 THE MODEL

Our main objective is to know how long does it take for a detector to thermalise in terms of its speed v and energy gap E. Since the energy gap obeys the relation $E\lambda = S_0$, we can deduce the waiting time from it. The switching function χ is yet to be determined, so now we will specify it and all of the other relevant parameters, that is, we are going to choose a model.

This model will be studied via numerical analysis and has to be chosen keeping in mind that the range $90/100 < v < 99/100$ is the one relevant to some of the experimental proposals. We think that this could indicate what are the energy gap ranges (hence waiting times) for which we could observe thermalisation in a laboratory setting. More concretely, choosing a model will allow us to evaluate the a_i coefficients of the time-dependent (5.41) case.

As an initial approximation, we choose χ to be the even, positive-definite and non-smooth function supported on $[-\beta, \beta]$ defined as

$$\chi(z) = \begin{cases} 1 + z/\beta, & -\beta \leq z \leq -\beta/2 \\ 1/2 & -\beta/2 < z < \beta/2 \\ 1 - z/\beta & \beta/2 \leq z \leq \beta \end{cases}$$

for which we have

$$||\chi||_2^2 = \frac{\beta}{3}, \quad ||\chi'||_2^2 = \frac{1}{\beta}, \quad M_0[1] = \frac{9}{16}\beta^2, \quad M_2[1] = \frac{15}{64}\beta^4. \tag{5.45}$$

It must be noted that we have relaxed the condition that χ has to be smooth because we are performing a numerical analysis. To determine the a_i coefficients, we start by analysing the q_i ones in Lemma 5.7.1.

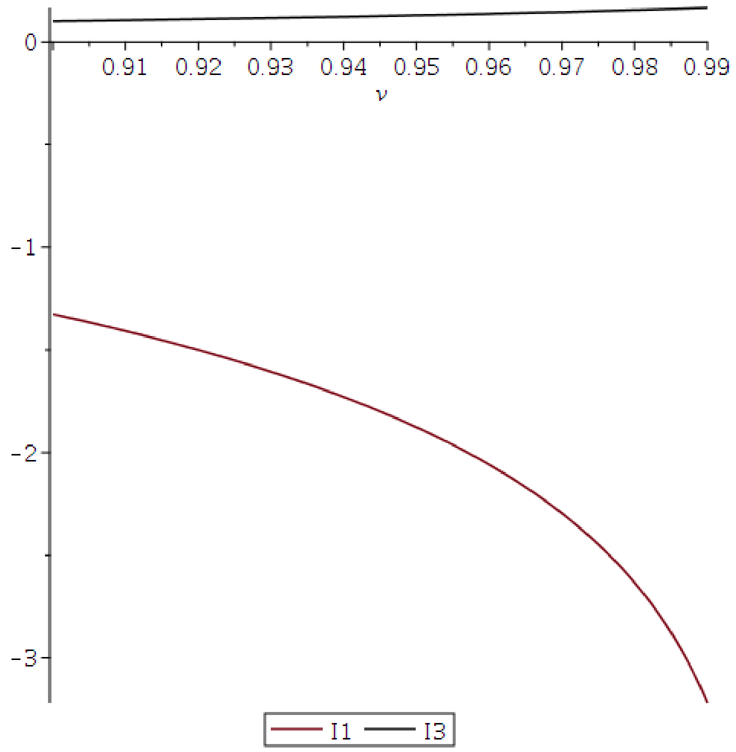


Figure 5.3: Plot of $I_1(v)$ and $I_3(v)$.

In contrast the simplicity of q_2, both q_1 and q_3 contain integrals, which shall be evaluated by numerical means. For this purpose, we define the following functions

$$I_1(v) = \int_0^\infty dz\, Q_v(z),$$
$$I_3(v) = \int_0^\infty dz\, z^2 \left(Q_v(z) + \frac{v^2}{2}\frac{\sin(z)^2}{1+z^2}\right) + \frac{\pi}{2}\left(1 - \frac{1}{e^2}\right)\left(\frac{v}{2}\right)^2$$

and integrate numerically using Maple. Their plots for different values of v within the range $90/100 < v < 99/100$ can be found in Figure 5.3. This is consistent with the fact that these integrals do exist.

To check the validity of the asymptotic expansion of $\mathcal{S}\left[Q_v(z)/z\right]$ presented in Lemma 5.7.1, we chose the particular value $v = 99/100$ and compare this to the sine transform obtained via numerical integration. We plotted both outcomes results in Figure 5.4. Moreover, as the coefficients q_1, q_2 and q_3 in Lemma

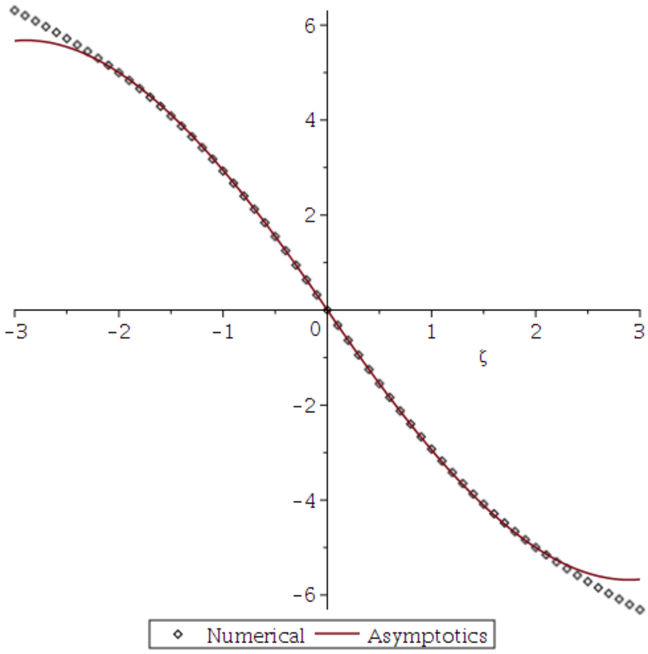


Figure 5.4: Plot of $\mathcal{S}[Q_v(z)/z](\zeta)$.

5.7.1 just depend in the values that v can take, they naturally define the following functions:

$$q_1(v) = I_1(v), \quad q_2(v) = \frac{\pi}{4}\left(\frac{v}{2}\right)^2, \quad q_3(v) = -\frac{I_3(v)}{6} - \frac{\pi}{12}\left(1 - \frac{2}{e^2}\right)\left(\frac{v}{2}\right)^2;$$

which are plotted for several values of $v < 1$ in Figure 5.5.

Before we continue, we note that as χ is even then χ' will be odd. This vastly simplifies the calculation of the Fourier transform:

$$\begin{aligned}\mathcal{F}[iS_0\chi + \chi'](\omega) &= 2i\left(S_0\mathcal{C}[\chi](\omega) - \mathcal{S}[\chi'](\omega)\right)\\ &= 2i(S_0+\omega)\frac{\cos(\beta\omega/2) - \cos(\beta\omega)}{\beta\omega^2}.\end{aligned} \tag{5.46}$$

The integral can be evaluated in closed form, however the final result is too long and it is not immediate to extract anything useful to our purposes from it. Next, we need to choose the remaining parameters; first, we set

$$S_0 = 1, \qquad R = 1$$

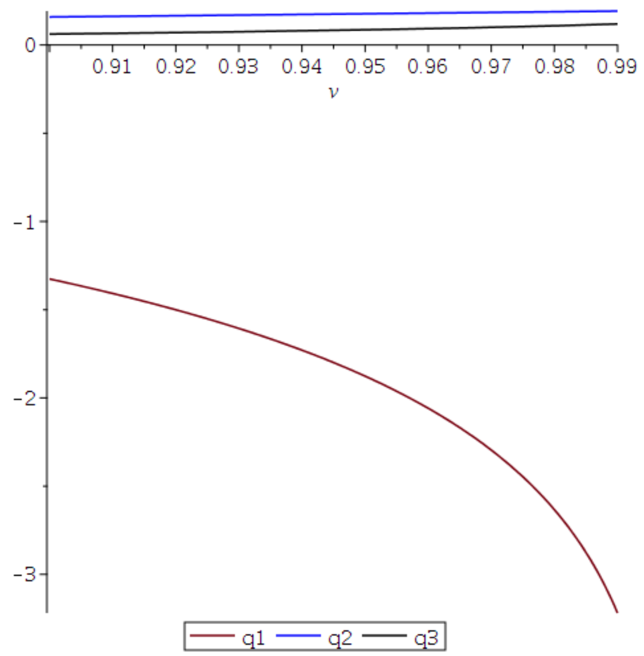


Figure 5.5: Plot of $q_1(v)$, $q_2(v)$ and $q_3(v)$.

and hence, in virtue of (3.23), $k = 2/(\gamma v)$; thus it just remains to choose β.

To bring some clarity on some convenient values for β, we write (5.42) as $T_\lambda(E) = T_0 + T_1|E| + T_2E^2 + O(E^3)$. If we truncate the error term, we see that a stationary point for this function is[2] $E_* = -T_1/(2T_2)$. If we denote by T_* the value of the temperature at this point and note that according to (5.42) we have $T_1 := -(a_2/a_1)T_0$, we find

$$T_* = T_0\left(1 + \frac{3}{4}\frac{a_2^2}{a_1^2}\frac{T_0}{T_2}\right).$$

So, regardless of T_* being a maximum or a minimum, it depends on T_0. Therefore, if T_0 is large, the possibility of measuring thermalisation in a laboratory improves.

To this end, we focus our attention on fixing the remaining parameters in such a way T_0 becomes large. From Proposition 5.7.2 it follows that as $l(S)$ (or, equivalently β) decreases, then the lower bound on T_0 increases. This is precisely what we want to do. Hence, we set $\beta = 1/100$ and compare it with the bound on

[2]Modulo a sgn(E) term that is irrelevant due to the fact that $T_\lambda(E)$ is even

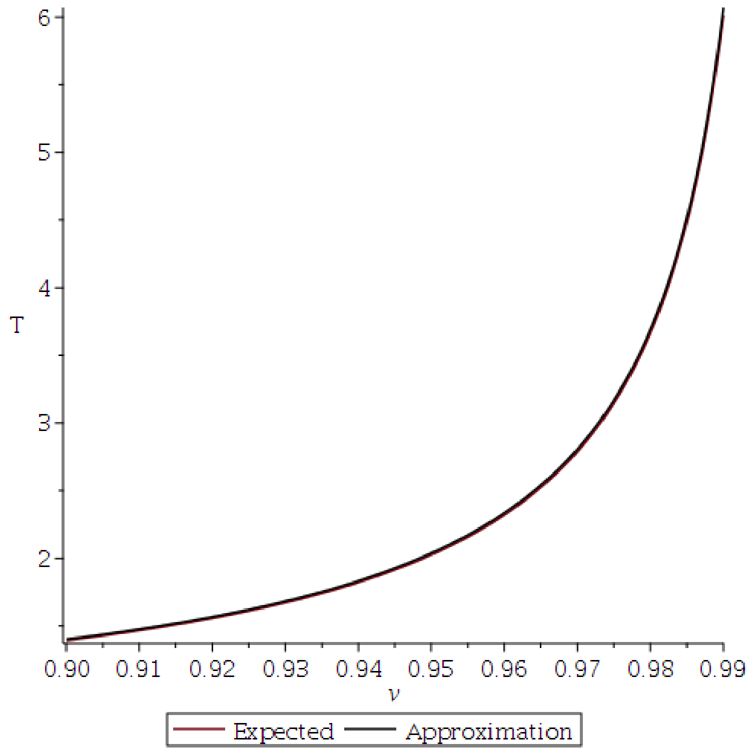


Figure 5.6: Expected and approximated behaviours of T_0.

the aforementioned Proposition. This yields

$$2\beta = l(S) = 1/50 << 1.602740577 = \inf_{0.9<\nu<0.99} \sqrt{2k|q_1|}, \tag{5.47}$$

where the infimum was computed using Maple. Conveniently so, (5.47) indicates that according to Proposition 5.7.2, T_0 acquires a remarkably simple (approximate) form, which is

$$T_0 = \frac{\pi\gamma}{4k|q_1|}.$$

To test whether our choice for β is reasonable or not, we present a plot in Figure 5.6 of the expected $T_0 = -a_0/(2a_1)$ and the approximation $T_0 = \pi\gamma/(4k|q_1|)$ as a function of ν; this shows that both expressions are in close agreement, thus verifying the chosen value for β.

The aforementioned parameter choices allows us to write the a_i coefficients in terms of q_i. Particularly, the integral in the a_2 coefficient of (5.41) can be

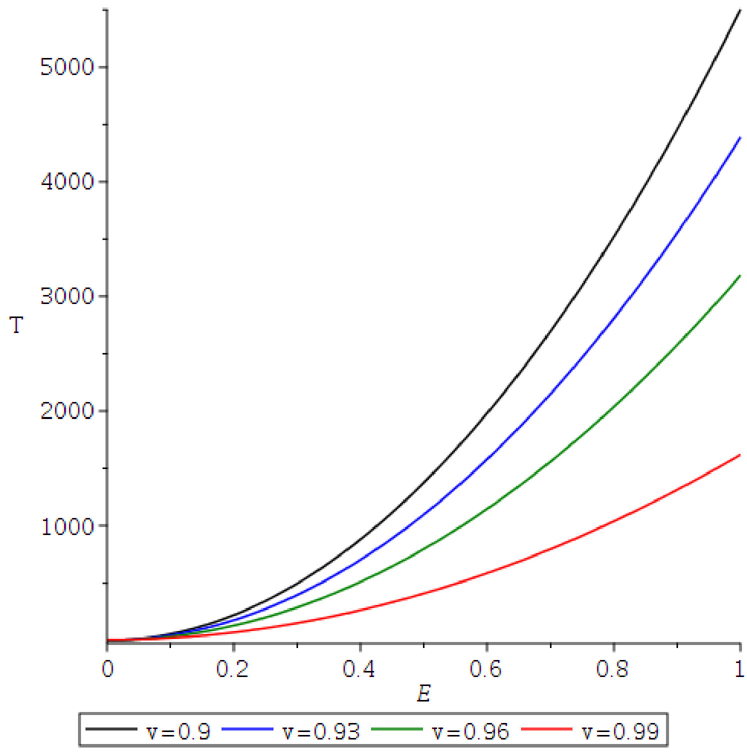


Figure 5.7: Plot of T for different values of v.

calculated numerically by substituting the values of the parameters into the integrand in (5.46). Hence, we find that the coefficients are

$$a_0 = \frac{1}{1200}, \qquad a_1(v) = -\frac{1}{400\pi\gamma}\left(\frac{(3/4)^2}{100} + \frac{4}{3}\frac{|q_1(v)|}{v\gamma}\right),$$
$$a_2(v) = \frac{4(1.124992 \times 10^{-4})}{\pi^2\gamma^3 v^2} q_2(v),$$
$$a_3(v) = \frac{1}{600\pi\gamma}\left(\frac{5}{12}\frac{(3/4)^2}{100} + 8\frac{q_3(v)}{\gamma^3 v^3}(1 + 300^2)\right).$$

5.9 DISCUSSION

In order to obtain several plots of $T(E)$ for different values of v, we make use of the coefficients found above and (5.42). Our results can be found in Figure 5.7. Making use of these coefficients and (5.42) to produce several plots of $T(E)$ for different values of v; our findings are in Figure 5.7. First, we observe that increasing the speed does not imply more temperature; on the contrary, the greater the speed the lower the observable temperature becomes. If we restore constants, then the temperature must be multiplied by a factor of $\hbar/(k_B \cdot c) = 2.547841002.. \times 10^{-20} K$, where c is the speed of the phonons in the analogue spacetime. This indicates that the observable temperatures T for these values of v have the range $4.130639743 \times 10^{-5} pK \leq T \leq 1.402276341 \times 10^{-4} pK$ where p stands for *pico*, which is a factor of 10^{-12}. Unfortunately, according to [39] the smallest temperatures we can detect are in the order of picoKelvins, so this choice of parameters is not convenient for experimental purposes.

Also, since the assumed power law is $E = S_0/\lambda$, from Figure 5.7 we infer that waiting will not result in a detectable thermalisation. To understand why the observable temperature is so low we will study the same range $0.9 \leq v \leq 0.99$. However, on this occasion, we will keep the value of S_0 unassigned and will assume the constant R to be small, which implies that k (3.23) will also be. This assumption is reasonable for experimental purposes as one can not expect to have the detector following a circular trajectory with a very large radius in an experimental setting.

Under this assumption, we will not necessarily have the conditions to use the approximation of T_0 in (5.44). This is due to the fact that for $v \approx 0.99$, we have $\gamma \approx 7$ and consequently, $k = 2R/(\gamma v) \approx (2/7)R$. So, for small R it is difficult to get $l(S) << \sqrt{2k|q_1|}$. However, for small k, the a_i coefficients in (5.41) simplify to

$$a_0 = \frac{\beta}{12}, \qquad a_1 = -\frac{(3/8)^2}{\pi\gamma}\beta^2, \qquad a_2 = 0, \qquad a_3 = \frac{5/8^3}{\pi\gamma}\beta^4. \tag{5.48}$$

where we have used (5.45) to calculate the β dependency of these coefficients. We would like to draw the reader's attention onto the fact that β determines the plateau length of χ. With this simplification, the temperature $T_\lambda(E)$ in (5.42) is now approximated by

$$T_\lambda(E) = T_0 - \left(T_0 \frac{a_3}{a_1} + \frac{1}{12\, T_0}\right) E^2 + O\left(E^3\right).$$

For fixed E, this can be understood as a function of β by substituting the values of the coefficients according to (5.48), and so we obtain

$$T_\lambda(\beta; E) = \frac{8\pi\gamma}{3^3\beta} + \beta\left(\frac{5\pi\gamma}{3^5} - \frac{9}{2^5\pi\gamma}\right)E^2 + O\left(E^3\right).$$

The term in parenthesis multiplying E^2 is a strictly increasing function of v, it is also positive and of order unity for our chosen range. We want to find the value of β that maximises the detected temperature for fixed E and v. Since $\beta > 0$ we find that $T_\lambda(\beta; E)$ only has a global minimum, however it will increase as long as

$$\beta > \frac{2}{3|E|}\sqrt{\frac{2\pi\gamma}{3}\left(\frac{5\pi\gamma}{3^5} - \frac{9}{2^5\pi\gamma}\right)^{-1}}.$$

To have a better understanding of this, now we shall address the other time scale in action: the total interaction time λ. Clearly, scenarios that have the support of the switching function larger than the interaction time are of no experimental interest. As in such scenarios, the detector would be switched on even when it is not detecting with the quantum field.

Bearing this in mind, we address the original question: *how long does it take for a detector in uniform circular motion to thermalise?* We know that $\mathcal{F}_\infty(E) := \lim_{\lambda\to\infty}\mathcal{F}_\lambda(E)$ corresponds to a situation where a detector with fixed energy gap E, has attained thermalisation after $\lambda \to \infty$. So, we will assume that a necessary condition for thermalisation after a large–but finite–interaction time λ, is that if for some $\varepsilon > 0$ and fixed E we have

$$|\mathcal{F}_\lambda(E) - \mathcal{F}_\infty(E)| = O(\lambda^{-\varepsilon}) \qquad (\text{as } |\lambda| \to \infty). \tag{5.49}$$

In addition, let us assume that $\mathcal{F}_\infty(E) \le \mathcal{F}_\lambda(E) \le \mathcal{F}_\infty(E)[1 + O(\lambda^{-\varepsilon})]$, which in particular implies that $\mathcal{F}_\infty(-E) \le \mathcal{F}_\lambda(-E)$. This does not need to hold in general, but (5.57) motivates this assumption. Using this along with the fact that $\mathcal{F}_\infty(E)$ satisfies detailed balance–that is $\mathcal{F}_\infty(-E) = e^{E/T}\mathcal{F}_\infty(E)$– leads us to find

$$\ln\left(\frac{\mathcal{F}_\lambda(-E)}{\mathcal{F}_\lambda(E)}\right) \ge \ln\left(\frac{\mathcal{F}_\infty(-E)}{\mathcal{F}_\lambda(E)}\right) = \frac{E}{T} - \ln\left(\frac{\mathcal{F}_\lambda(E)}{\mathcal{F}_\infty(E)}\right) \ge \frac{E}{T} - \ln\left(1 + O(\lambda^{-\varepsilon})\right),$$

a similar argument leads to the upper bound from which it follows that

$$\frac{E}{T} - \ln\left(1 + O(\lambda^{-\varepsilon})\right) \le \ln\left(\frac{\mathcal{F}_\lambda(-E)}{\mathcal{F}_\lambda(E)}\right) \le \frac{E}{T} + \ln\left(1 + e^{E/T}O(\lambda^{-\varepsilon})\right).$$

Observe that thermalisation is attained when the logarithm terms vanish, and so, as long as E is near the origin and λ is large, this will happen. Note that this even holds if we consider $0 < \alpha < 1$ in a generalised power law of the form $E = S_0/\lambda^{\alpha-1}$.

However, this choice of power law is not as free as one would expect. As a matter of fact, it turns out that our choice of power law was not suitable for thermalisation. To see why this is the case, let us make use of (5.13), (5.40) and (5.17) to study the difference

$$\begin{aligned}\mathcal{F}_\lambda(E) - \mathcal{F}_\infty(E) &= a_0 + a_1\left(\frac{S_0}{\lambda}\right) + a_2\mathrm{sgn}\left(\frac{S_0}{\lambda}\right)\left(\frac{S_0}{\lambda}\right)^2 \\ &\quad + a_3\left(\frac{S_0}{\lambda}\right)^3 - \mathscr{F}[\mathcal{W}](E)||\chi||_2^2 + O(\lambda^{-4}).\end{aligned}$$

Making use of the power law (5.33), decomposition (5.19) and the value of a_0 as specified in (5.41), the expression above yields

$$\begin{aligned}\mathcal{F}_\lambda(E) - \mathcal{F}_\infty(E) &= a_1\left(\frac{S_0}{\lambda}\right) + a_2\mathrm{sgn}\left(\frac{S_0}{\lambda}\right)\left(\frac{S_0}{\lambda}\right)^2 + a_3\left(\frac{S_0}{\lambda}\right)^3 \\ &\quad + \frac{||\chi||_2^2}{2\pi\gamma}\left(\frac{\pi}{2}\mathrm{sgn}\left(\frac{S_0}{\lambda}\right) - \mathcal{S}\left[\frac{Q_\nu(z)}{z}\right]\left(\frac{kS_0}{\lambda}\right)\right) + O(\lambda^{-4}).\end{aligned}$$

Furthermore, making use of Lemma 5.7.1 and simplifying we find

$$\begin{aligned}\mathcal{F}_\lambda(E) - \mathcal{F}_\infty(E) &= \left(a_1 - k\frac{q_1||\chi||_2^2}{2\pi\gamma}\right)\left(\frac{S_0}{\lambda}\right) + \left(a_2 - k^2\frac{q_2||\chi||_2^2}{2\pi\gamma}\right)\mathrm{sgn}\left(\frac{S_0}{\lambda}\right)\left(\frac{S_0}{\lambda}\right)^2 \\ &\quad + \left(a_3 - k^3\frac{q_3||\chi||_2^2}{2\pi\gamma}\right)\left(\frac{S_0}{\lambda}\right)^3 + \frac{||\chi||_2^2}{4\gamma}\mathrm{sgn}(E) + O(\lambda^{-4}).\end{aligned}$$

In virtue of the remaining values of a_i in (5.41) along with the model relations in (5.45), after simplifying and reordering we reach

$$\begin{aligned}|\mathcal{F}_\lambda(E) - \mathcal{F}_\infty(E)| &\leq \frac{\beta}{12\gamma} + \frac{9}{16}\frac{S_0\beta^2}{\lambda} + \frac{1}{2\pi\gamma}\left(\frac{5\beta^4}{2^8} + \frac{3q_3}{\beta S_0^2}\right)\left(\frac{S_0}{\lambda}\right)^3 \\ &\quad + \frac{k^2q_2}{4\pi^2\gamma}\left(\frac{1}{S_0^2}\int_{-|S_0|}^{|S_0|} d\omega\, |\mathscr{F}[iS_0\chi + \chi'](\omega)|^2 + \frac{\beta}{3}\right)\left(\frac{S_0}{\lambda}\right)^2 + O(\lambda^{-4}).\end{aligned}$$

Finally, the integral might be estimated by making use of (5.46) from which we can see that $|\mathscr{F}[iS_0\chi + \chi'](\omega)|^2 \leq 4(S_0 + \omega)^2/(\beta\omega^2)^2$. After another rearrangement, this we find that

$$|\mathcal{F}_\lambda(E) - \mathcal{F}_\infty(E)| \leq \frac{\beta}{12\gamma} + \frac{9}{16}\frac{S_0\beta^2}{\lambda} + \frac{k^2 q_2}{12\pi^2\gamma}\left(\frac{7\cdot 8}{\beta^2 S_0^3} + \beta\right)\left(\frac{S_0}{\lambda}\right)^2 + \frac{1}{2\pi\gamma}\left(\frac{5\beta^4}{2^8} + \frac{3q_3}{\beta S_0^2}\right)\left(\frac{S_0}{\lambda}\right)^3 + O(\lambda^{-4}). \tag{5.50}$$

From the expression above it is easy to deduce that thermalisation can be hardly reached as $|\mathcal{F}_\lambda(E) - \mathcal{F}_\infty(E)| \neq O(\lambda^{-\varepsilon})$ due to the presence of the first term. If we wanted to attain thermalisation the first term in the bound has to be very small with respect to $|\mathcal{F}_\lambda(E) - \mathcal{F}_\infty(E)|$ and the other terms. However, if this were the case, then the terms containing $1/\beta$ would be very large.

This argues why the right-hand side can not vanish for any large λ and any choice of β. This is due to the current choice of power law $E\lambda = S_0$. Let us drop the power law assumption (5.33). If we use the response functions (5.12) and (5.13) along with the decompositions (5.17) and (5.19). Hence, the difference becomes

$$\mathcal{F}_\lambda(E) - \mathcal{F}_\infty(E) = -\frac{1}{8\pi\gamma}\int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2\, (\mathrm{sgn}(E + \omega/\lambda) - \mathrm{sgn}(E)) + \frac{1}{4\pi^2\gamma}\int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2\, [\mathcal{S}[Q_v(z)/z](k(E + \omega/\lambda)) - \mathcal{S}[Q_v(z)/z]](kE)]\,, \tag{5.51}$$

define $f(\omega) := \mathrm{sgn}(E + \omega/\lambda) - \mathrm{sgn}(E)$. Note that $\mathrm{supp}(f) = (-\infty, -\lambda|E|)$ and that $f \equiv 2$ on its support. With this, we see that the first integral in (5.51) becomes

$$\int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2\, (\mathrm{sgn}(E + \omega/\lambda) - \mathrm{sgn}(E)) = -2\mathrm{sgn}(\lambda E)\int_{\lambda|E|}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2, \tag{5.52}$$

which can only vanish if as $\lambda \to \infty$, we have $\lambda|E| \to \infty$. This certainly is not the case for our choice of power law in (5.33) which is $E\lambda = S_0$ and explains why there are non-vanishing terms in (5.50) as λ grows large. Note that this power law implies that it is not possible to take the limit $\lambda \to \infty$ with E fixed.

The source of this problem was assuming that E and λ are inversely related via the chosen power law. So, to avoid this, we will now study what happens if

$$E(\lambda) = \frac{g(\lambda)}{\lambda}, \tag{5.53}$$

where g is yet to be determined but is even and satisfies $g(\lambda)/\lambda \to 0$ as $|\lambda| \to \infty$. With this we see that (5.52) acquires the form

$$\int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2 \left(\text{sgn}(E + \omega/\lambda) - \text{sgn}(E)\right)$$
$$= -2\text{sgn}(g(\lambda)) \int_{|g(\lambda)|}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2, \tag{5.54}$$

So, if we want small energy gaps, large waiting times and thermalisation, it seems that we need to demand that g satisfies

$$\frac{g(\lambda)}{\lambda} \to 0 \qquad \text{as} \qquad |\lambda| \to \infty \tag{5.55}$$
$$g(\lambda) \to \infty \qquad \text{as} \qquad |\lambda| \to \infty. \tag{5.56}$$

Note that under these assumptions, (5.52) vanishes. Next, we make use of the small arguments for $\mathcal{S}[Q_\nu(z)/z]](\zeta)$ in Lemma 5.7.1, with this we find that

$$\mathcal{S}[Q_\nu(z)/z](k(E + \omega/\lambda)) - \mathcal{S}[Q_\nu(z)/z]](kE)$$
$$= q_1 \frac{k\omega}{\lambda} + q_2 \left(\text{sgn}(E + \omega/\lambda) - \text{sgn}(E)\right) E^2 + q_2 \text{sgn}(E + \omega/\lambda) \frac{k\omega}{\lambda} \left(2E + \frac{k\omega}{\lambda}\right)$$
$$+ q_3 \frac{k\omega}{\lambda} \left(3E^2 + 3E\frac{k\omega}{\lambda} + \frac{k^2\omega^2}{\lambda^2}\right) + O(\lambda^{-4}).$$

We will drop the terms that are odd in ω due to the fact that in (5.51) they are under the ω integral and $|\mathscr{F}[\chi](\omega)|^2$ is even, this yields

$$\int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2 \left(\mathcal{S}[Q_\nu(z)/z](k(E + \omega/\lambda)) - \mathcal{S}[Q_\nu(z)/z]](kE)\right)$$
$$= q_2 E^2 \int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2 \left(\text{sgn}(E + \omega/\lambda) - \text{sgn}(E)\right)$$
$$+ \int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2 \left[q_2 \text{sgn}(E + \omega/\lambda) \frac{k\omega}{\lambda} \left(2E + \frac{k\omega}{\lambda}\right) + 3Eq_3 \frac{k^2\omega^2}{\lambda^2}\right] + O(\lambda^{-4}).$$

Next, in the expression above we will substitute (5.52) and (5.53). Moreover, making use of $i^s\omega^s\mathscr{F}[h](\omega) = \mathscr{F}[h^{(n)}](\omega)$ for h differentiable enough–with $s \in \mathbb{N}_0$–alongside Plancherel's formula allows us to further simplify this to

$$\begin{aligned}
&\int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2 \left(\mathscr{S}[Q_\nu(z)/z](k(E+\omega/\lambda)) - \mathscr{S}[Q_\nu(z)/z]](kE)\right) \\
&= -2\text{sgn}(g(\lambda))\frac{q_2 g(\lambda)^2}{\lambda^2}\int_{|g(\lambda)|}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2 + 6\pi q_3 \frac{k^2 g(\lambda)}{\lambda^3}||\chi'||_2^2 \\
&+ \frac{k^2 q_2}{\lambda^2}\text{sgn}(g(\lambda)/\lambda)\int_{-|g(\lambda)|}^{|g(\lambda)|} d\omega\, |\mathscr{F}[\chi'](\omega)|^2 \\
&+ \frac{2g(\lambda)kq_2}{\lambda^2}\int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2 \text{sgn}((g(\lambda)+\omega)/\lambda)\omega + O(\lambda^{-4})
\end{aligned}$$

where we have made use of an argument similar to those within Proposition C.2.3. We want to study these terms for large λ. The first two terms vanish. Regarding the third term, note that the integral obeys the bound

$$\frac{1}{\lambda^2}\left|\int_{-|g(\lambda)|}^{|g(\lambda)|} d\omega\, |\mathscr{F}[\chi'](\omega)|^2\right| \leq \frac{2|g(\lambda)|}{\lambda^2}\sup|\mathscr{F}[\chi'](\omega)|^2,$$

the supremum on the right hand side certainly exists because$\chi \in C_0^\infty(\mathbb{R})$, then $\mathscr{F}[\chi']$ must be smooth and has fast-decay. Because of the fast-decay, we can deduce that the fourth term is bounded by a constant independent of λ that we shall denote as V. Therefore, we can conclude that

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2 \left(\mathscr{S}[Q_\nu(z)/z](k(E+\omega/\lambda)) - \mathscr{S}[Q_\nu(z)/z]](kE)\right)\right| \\
&\leq 2\frac{q_2 g(\lambda)^2}{\lambda^2}\int_{|g(\lambda)|}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2 + 6\pi q_3\frac{k^2 g(\lambda)}{\lambda^3}||\chi'||_2^2 \\
&+ \frac{2k^2 q_2|g(\lambda)|}{\lambda^2}\sup|\mathscr{F}[\chi'](\omega)|^2 + \frac{2g(\lambda)kq_2}{\lambda^2}V + O(\lambda^{-4}),
\end{aligned}$$

and making use of this result in combination (5.51) and (5.54) we can deduce that

$$\begin{aligned}
|\mathcal{F}_\lambda(E) - \mathcal{F}_\infty(E)| &\leq \frac{1}{4\pi\gamma}\int_{|g(\lambda)|}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2 \\
&+ \frac{|g(\lambda)|}{2\pi^2\gamma\lambda}\left(\frac{q_2|g(\lambda)|}{\lambda}\int_{|g(\lambda)|}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2 + 3\pi q_3\frac{k^2}{\lambda^2}||\chi'||_2^2\right. \\
&\left. + \frac{k^2 q_2}{\lambda}\sup|\mathscr{F}[\chi'](\omega)|^2 + \frac{kq_2}{\lambda}V\right) + O(\lambda^{-4}).
\end{aligned} \tag{5.57}$$

This seems to indicate that a good candidate is

$$g(\lambda) = \lambda^{\alpha} \qquad 0 < \alpha < 1.$$

Note that this makes the integrals vanish as $|\lambda| \to \infty$, and so we end up with the sought form specified in (5.49), from which we can conclude that this will achieve thermalisation.

From this we can see that the proposed power law (5.33) will not lead to thermalisation under the assumption that the switching function is of adiabatic nature. Moreover, as we have motivated throughout this section, we need to have a more dedicated study of the parameter space as it is yet not clear under which regimes a continuous variation of the parameters will lead to a continuous family of temperature curves. Also, there might be an ideal range which makes more feasible the detection of thermalisation in a laboratory.

— 6 —

Conclusions

> Por la mañana escribir, por la tarde corregir, por las noches leer y en las horas muertas ejercer la diplomacia, el disimulo, el encanto dúctil. (*Write in the morning, revise in the afternoon, read at night, and spend the rest of your time exercising your diplomacy, stealth, and ductile charm.*)
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>
> Roberto Bolaño, Los Detectives Salvajes (The Savage Detectives)

All of the aforementioned has made very clear that this thesis deals with two lines of research relevant to the study of quantum fields in curved spacetime. Due to their content, both are autonomous for all pragmatic instances. Because of this, it is not easy to establish a connection between these lines of research unless we want to dwell into forced arguments. Therefore, we will conclude on the overall work first and then we will present specific conclusions concerning each topic.

At this stage the reader might be wondering why the present document is deemed as a thesis instead of two theses. However, it is the author's firm belief that the strength of this text resides on containing two different lines of work, as they contribute towards its breadth. Although both lines of research make ample use of the conceptual foundations envisaged by General Relativity and Quantum Field

Theory, each project required very different frameworks and specific approaches.

The Hadamard State Extension project is abstract in nature, but we were able to do some numerical computations to prove that our claims hold in a simple model. This was very satisfying, as it is always pleasant seeing how abstract and general underpinnings coalesce into explicit and concrete results. During the investigation of this problem, we implemented techniques pertaining to Locally Covariant QFT, Algebraic QFT and Lorentzian Geometry. On the mathematical side, this makes use of notions related to Category Theory, Operator Algebras and Semi-Riemannian Geometry.

After we were convinced that a Hadamard state could be extended, we did some numerical investigations in which we had to process non-linear ODEs in data grids of approximately $50,000$ points, which is a considerable size. Even though this is a simple numerical problem at the conceptual level, the technical implementation needed some work and had several modifications. Therefore, some notions of scientific computing were also needed for this project.

Contrastingly so, the Detector Response project is very concrete as it was motivated by an ongoing experimental venture. For its research, we implemented methods commonly used both in Fourier and Asymptotic Analysis. There was also some numerical analysis, but it was rather straightforward as it was only used to verify the validity of certain asymptotic approximations. Still, doing asymptotic analysis is not a minor undertaking, as a matter of fact, Prof. Chris Fewster once sagely said that "*...asymptotic analysis is an art*". This is certainly true, as every situation requires a bespoken approach depending on what is needed of it.

Our findings regarding the extension of Hadamard states may be summarised as follows. If one wants to extend a Hadamard state from a region onto a larger one, then one can do so as long as knowledge of the state is sacrificed near the boundary of the original region. This is not to be taken for granted and has profound implications. Imagine you have a Hadamard state in your laboratory, then one could ask: why should it be Hadamard outside the laboratory? The results obtained in our research show that the state can indeed be extended to a Hadamard state outside the laboratory and since one cannot see it outside the laboratory it may as well be taken without loss of generality to be Hadamard.

Moreover, not being able to probe it near the boundary does not introduce any problem whatsoever, as it is known that one can not probe arbitrarily close to the boundary of a region anyway.

The numerical exploration carried out afterwards confirms these ideas in a simple setting where it is easy to see clearly the impact of our choice of parameters. As we mentioned earlier, the numerical problem needed to be solved is a highly non-linear one over an extensive data grid. As it is known, numerical differentiation is somewhat unstable, which led to some outlying points in the surface plots. Although one can still distinguish the main features of said surfaces, we hope to find a better method to get rid of the outlying points. At some stage, we tried averaging over certain regions, but this eroded important aspects of the plots and on some occasions generated incorrect results. For instance, regions with negative energy density would be misrepresented, which can give rise to incorrect interpretations.

The more abstract results regarding Hadamard states show that the extension can be performed between regions of a globally hyperbolic spacetime. However, our explicit construction is just for conformally ultrastatic spacetimes. So, finding an explicit construction for general globally hyperbolic spacetimes seems like the next thing to do. Another thing to be done is formulating the extension to the whole of the spacetime in more precise terms. As for now, the general direction is clear but some technical details need to be revised carefully. Finally, an interesting future line of research would be considering the case when the original region T is not simply connected or even if its components are disconnected.

Although the Detector Response is a more concrete problem, some of its aspects remain unclear. As we discussed at the beginning of the corresponding Chapter, the notion of a rotating vacuum is problematic in this context. Hence, instead of using it, we shifted our focus to a detector following a circular trajectory in the usual Minkowski vacuum state. However, in this setting, something similar to a *ergosphere* emerges as beyond it the time axis becomes spacelike. The upshot of this is there will be both positive and negative energy modes. This is problematic as we usually associated the detector response was due to emission into the positive energy modes. Nonetheless in the present case, there might be a mixing of modes, which complicates interpreting this as temperature. So, the detector will certainly exhibit a response, nonetheless, it is yet unclear if said response will be consistent with a thermal spectrum.

In other words, it still is uncertain if it is correct to think of the detector as an accelerated thermometer or if it is detecting something else. So, if there is an experiment where the detector exhibits response, can we really say we have detected the Unruh effect? Yet, the work by Korsbakken and Leinaas shows something interesting regarding this matter. If we consider a detector that is rotating uniformly with speed Ω and moving with linear acceleration a, then we can see that when the ratio Ω/a vanishes we obtain the Unruh temperature, as one would expect. Moreover, continuous variation of this ratio will lead to a continuous family of temperature curves. So, this supports the idea that if a rotating detector has a response, then it corresponds to what we call the Unruh effect. Nevertheless, more clarification and further study on this topic is necessary as many of the details are still rather opaque.

Aside from these considerations, the results that we obtained indicate although it is reasonable to assume that the interaction time λ and the energy gap E are related, one has to be careful when choosing their relation. This is because not all choices will lead to thermalisation in a finite interaction time. As a matter of fact, our initial choice $E\lambda = S_0$ (where S_0 is a constant) can not attain thermalisation for any value of λ. We actually discussed under which circumstances the relation between E and λ are expected to lead to thermalisation, but it was at a late stage during research. So, a future line of work that looks promising is studying relations of the type $E(\lambda) = S_0\lambda^{\alpha-1}$ where $0 < \alpha < 1$. Other subject worth studying in the future, is how the choices of the switching function χ affect the observed temperature. We mention this because we chose a χ that led to a temperature that is too low to observe in an experimental setting. Therefore, more study on the parameter space needs to be done in order to increase the chances of detection in a laboratory. Nevertheless, the asymptotic approximations that we obtained are valid and accurate to the sought level. So one can use them to study the two aforementioned questions.

In conclusion, this thesis involved the understanding of quantum fields in curved spacetimes from many perspectives, from theory to experiment and from abstraction to application. This compelled me to learn a wide range of techniques that certainly augmented and enhanced my research skillset. It is particularly pleasant to me that part of this work might have potential to influence forthcoming experiments. The breadth of technique and applications presented in this work certainly indicates that the research area of Quantum Field Theory in Curved Spacetime is a deeply rich one.

— A —

Hadamard State Extension for Two-Dimensional Minkowski Spacetime

A.1 PROOF OF LEMMA 3.2.1

LEMMA A.1.1. *Let g be as in* (3.8)*, that is*

$$g = dt \otimes dt - (dx - fvdt) \otimes (dx - fvdt). \tag{A.1}$$

Then, $(\mathbb{R}^2,\ g)$ *is globally hyperbolic.*

Proof. We start by showing that g is Lorentzian on $\mathbb{R}^2$, which we divide into W and $\mathbb{R}^2 \setminus W$ with $W = W_L \cup W_R$. On the second region $f \equiv 0$ and so $g = g_0$, which is clearly Lorentzian. The metric g can be written as

$$g = (1 - f^2v^2)dt \otimes dt + 2fvdt \otimes_s dx - dx \otimes dx, \tag{A.2}$$

since the tangent bundle can be decomposed as $T\mathbb{R}^2 = T\mathbb{R} \oplus T\mathbb{R}$ any vector at p can be written as $v = v^t\partial/\partial t + v^x\partial/\partial x$. A basis is given by

$$\left\{\frac{\partial}{\partial t} + (1 + fv)\frac{\partial}{\partial x}, \frac{\partial}{\partial t} - (1 - fv)\frac{\partial}{\partial x}\right\},$$

letting $k \in \{-1, 1\}$, we can write $v_k = \partial/\partial t + k(1 + kfv)\partial/\partial r$ and calculate the norm of the these elements easily

$$\begin{aligned} g\,(v_k, v_k) &= (1 - f^2v^2) + 2kfv(1 + kfv) - (1 + kfv)^2 \\ &= -2f^2v^2 + 2fv(k + fv) - 2kfv = 0 \end{aligned}$$

which indicates they are null and given that they constitute a basis, we conclude that the metric is also Lorentzian in W (c.f. p.141 [53]) and thus in $\mathbb{R}^2$.

Finally, to show that g is globally hyperbolic we will use a domination argument. We say that a metric g_1 dominates the metric g_2 if $g_1 \geq g_2$ as quadratic forms, a consequence of this is that if g_1 is globally hyperbolic then g_2 must be globally hyperbolic as well, this can be seen from the fact that every g_2-causal curve is necessarily g_1-causal. First, note that since $0 \leq f \leq 1$ is smooth and $v \geq 0$ is compactly supported, the supremum of fv must exist. Denote it by F_+ and introduce

$$g' = g_0 + F_+ dt \otimes dt + \frac{F_+}{1+F_+} dx \otimes dx = (1+F_+) dt \otimes dt - \frac{1}{1+F_+} dx \otimes dx.$$

We claim that $g' \geq g$; to prove this, we see that $g' - g = [F_+ + f^2 v^2] dt \otimes dt - 2 f v dt \otimes_s dx + F_+ [1+F_+]^{-1} dx \otimes dx$, thinking of $g' - g$ as a 2×2 matrix it follows that

$$\det(g' - g) = \frac{F_+^2 - f^2 v^2}{1+F_+} \geq 0, \qquad \operatorname{tr}(g' - g) = \frac{F_+}{1+F_+}(2+F_+) + F^2 > 0.$$

And so, we can see that its eigenvalues are non-negative and conclude that $g' \geq g$ everywhere in $\mathbb{R}^2$.

Given that g' is conformal to $(1+F_+)^2 dt \otimes dt - dx \otimes dx$ and this is globally hyperbolic, it implies that it and the metrics that it dominates must be globally hyperbolic as well, in particular g, which concludes our proof. □

A.2 PROOF OF PROPOSITION 3.2.3

PROPOSITION A.2.1. *Let E^+ and E^- be regions in $\mathbb{R}^2$ defined by*

$$E^- = \{(t,x) : t \leq 0,\ |x| < r_T + t\}, \quad E^+ = \{(t,x) : 0 \leq t < t_F + 2\varepsilon,\ |x| < \rho(t)\},$$

observe that they intersect on $\{0\} \times (-r_T, r_T)$. We claim that $E = E^+ \cup E^-$.

Proof. We will begin by showing that $E^\pm \subset E$. As $g = g_0$ in E^- and K, and both these sets lie in $D_{g_0}(P)$, it is enough to show that every past-inextendible curve passing through a point in $E^+ \setminus K$ necessarily intersects P. To do this, we consider a causal curve, that is, a curve $\gamma(s) = (t(s), x(s))$ for which $g(\dot{\gamma}, \dot{\gamma}) \geq 0$. So, from (3.8) it follows that

$$\dot{t}\left((1 - f^2 v^2)\dot{t} + 2fv\dot{x}\right) \geq \dot{x}^2 \tag{A.3}$$

which in turn, can be rewritten as

$$\left|\frac{dx}{dt} - fv\right| \leq 1. \tag{A.4}$$

Next, we prove that if $|x(t_*)| < \rho(t_*)$ for some $0 < t < t_*$, then this inequality holds all earlier times. First, we analyse the case when $0 < t$, and prove by contradiction. Let us do this separately for points either in the right warp bubble W_R (hence $f_L \equiv 0$) or the left warp bubble W_L (hence $f_R \equiv 0$).

To this end, let us assume that the causal curve can leave E at positive time. This means that at some time $\tau \in (0, t)$ we will have $\rho(\tau) = |x(\tau)|$. So in W_R this is $\rho(\tau) = x(\tau)$ while in W_L it is $-\rho(\tau) = x(\tau)$. Without loss of generality, we choose the largest of the "contact" times, hence, $\tau := \sup_{\mathbb{R}_+}\{t : x(t) = \rho(t)\}$ and, immediately after this the curve lie inside E. This means that if x is near W_R, we will have $x(\tau + \eta) < \rho(\tau + \eta)$ for some $0 < \eta$ small enough so that $x(\tau + \eta)$ is in the region where $f_R = 1$. For x near W_L we have $-\rho(\tau + \eta) < x(\tau + \eta)$ and η is such that x is at a region where $f_L = 1$. From $f = f_R - f_L$ and by virtue of the bounds in (A.4), it follows that $-1 + v \leq dx/dt$ for x near W_R and $dx/dt \leq 1 - v$ for x near W_L for all times between τ and $\tau + \eta$.

Moreover, by definition $d\rho/dt = -1 + v$, substituting this into the previous inequalities yields $d\rho/dt \leq dx/dt$ and $dx/dt \leq -d\rho/dt$, which integrate to

$$\rho(\tau + \eta) - \rho(\tau) \leq x(\tau + \eta) - x(\tau) \qquad \text{for } x \text{ near } W_R, \tag{A.5}$$

$$x(\tau + \eta) - x(\tau) \leq -\rho(\tau + \eta) + \rho(\tau) \qquad \text{for } x \text{ near } W_L. \tag{A.6}$$

On the other hand, by hypothesis if x is near W_R then $\rho(\tau) = x(\tau)$, while if it is near W_L then $\rho(\tau) = -x(\tau)$. So (A.5) and (A.6) become $\rho(\tau + \eta) \leq x(\tau + \eta)$ and $x(\tau + \eta) \leq -\rho(\tau + \eta)$, respectively. This contradicts the remaining parts of our hypothesis: $x(\tau + \eta) < \rho(\tau + \eta)$ for W_R and $-\rho(\tau + \eta) < x(\tau + \eta)$ for W_L.

Said contradiction arose on the assumption that our causal curve can cross ρ at some time t. Therefore, we must have that $|x(t)| < \rho(t)$ for every t, from this, it is easy to see that $E^+ \subset D_g^+(P)$. Also, when $t \leq 0$ we have that $g = g_0$, which in turn implies that $D_g^-(P) = E^-$. Taking these two last equations, we see that $E^+ \cup E^- \subset E$.

Next, we need to check that $D_g^+(P) \cap [0, t_F + 2\epsilon) \times \mathbb{R} \subset E^+$, or conversely that every point outside E^+ lies on some past-directed causal curve that avoids P. As before, we will separate this into two separate cases depending on whether the point under consideration is near W_R or W_L. For the W_R (W_L) case, consider a right(left)-moving future-directed null geodesic through a point outside E^+, this is respectively, $\rho(t_1) < x(t_1)$ and $x(t_1) < -\rho(t_1)$ for some $t_1 \in [0, t_F + 2\epsilon)$. Making use of (A.3) for this trajectory, we get $dx/dt - f_R d\rho/dt = -1 + f_R \leq 0$ for W_R and $dx/dt + f_L d\rho/dt = -1 + f_L \leq 0$ for W_L. In the case of W_R, integration yields the following for $0 \leq t_0 < t_1$

$$\begin{aligned} x(t_1) - x(t_0) + \rho(t_0) - \rho(t_1) &\leq r(t_1) - r(t_0) - \int_{\rho(t_0)}^{\rho(t_1)} f_R \, d\rho' \\ &= -(t_1 - t_0) + \int_{t_0}^{t_1} f_R \, dt' \leq 0, \end{aligned} \tag{A.7}$$

while for W_L we have

$$\begin{aligned} x(t_1) - x(t_0) + \rho(t_1) - \rho(t_0) &\geq x(t_1) - x(t_0) + \int_{\rho(t_0)}^{\rho(t_1)} f_L \, d\rho' \\ &= t_1 - t_0 - \int_{t_0}^{t_1} f_L \, dt' \geq 0 \end{aligned} \tag{A.8}$$

from which we get $x(t_1) - \rho(t_1) \leq x(t_0) - \rho(t_0)$ and respectively $-x(t_1) - \rho(t_1) \leq -x(t_0) - \rho(t_0)$. Since in both cases the point lies outside E^+, both LHS are greater than zero, which leads us to conclude that the null geodesic lies outside P for all $t_0 \in [0, t_F + 2\varepsilon)$. Therefore, we have the inclusion $D_g^+(P) \cap [0, t_F + 2\epsilon) \times \mathbb{R} \subset E^+$, which demonstrates our claim: $D_g(P) \cap (-\infty, t_F + 2\epsilon) \times \mathbb{R} = E$. □

A.3 PROOF OF PROPOSITION 3.2.1

PROPOSITION A.3.1. *Let F be as in* (3.2)*, then*

$$F \cap (\{t_F\} \times \mathbb{R}) = E \cap (\{t_F\} \times \mathbb{R}) \qquad \text{and} \qquad F \subset E.$$

*this proves (b) and (c) in (**SEP.VIII**).*

Proof. Because of Proposition 3.2.3, we know that E lies within the curves ρ and $-\rho$, so, at time $t = t_F$ we obtain

$$E \cap (\{t_F\} \times \mathbb{R}) = \{t_F\} \times (-\rho(t_F), \rho(t_F)) = \{t_F\} \times (-r_F - r_T, r_F + r_T)$$

where in the second equality we have used that $\rho(t_F) = r_F + r_T$ as indicated in (3.5). From the definition of F in (3.2), we see that

$$\begin{aligned} F \cap (\{t_F\} \times \mathbb{R}) &= D_{g_0}(\{t_F\} \times (-r_F - r_T, r_F + r_T)) \cap (\{t_F\} \times \mathbb{R}) \\ &= \{t_F\} \times (-r_F - r_T, r_F + r_T), \end{aligned} \tag{A.9}$$

proving the first identity. To prove that $F \subset E$, we observe that from the definition of F in (3.2), we can infer that

$$F = \{(t,x) : t_F - \varepsilon < t < t_F + \varepsilon,\ |x| < r_F + r_T - |t - t_F|\}.$$

If we recall that the definition of ρ in (3.5) indicates that $d\rho/dt = -1$ for the interval $(t_F - \varepsilon, t_F + \varepsilon)$. So, if we choose t in said interval, integrate over (t_F, t) and rearrange, we will get $\rho(t) = r_F + 2t_F - t$. Also, as it is straightforward to show that $r_F + r_T - |t - t_F| \le r_F + 2t_F - t = \rho(t)$, one may deduce from (A.9) that any element inside F must necessarily be inside E and so, we conclude that $E \subset F$. □

A.4 PROOF OF PROPOSITION 3.2.4

PROPOSITION A.4.1. *The region T lies inside $\tilde{S}$.*

Proof. Note that the surface $\{t_F\} \times (-r_F - r_T, r_F + r_T)$ is a Cauchy surface for F and so, $F \subset D_{g_0}(\{t_F\} \times (-r_F - r_T, r_F + r_T))$. This along with the definition of $\tilde{S}$ in (3.3) leads us to conclude that $\tilde{S} = D_{g_0}(\{t_F\} \times (-r_F - r_T, r_F + r_T))$. Since a Cauchy development under the Minkowski metric is well-known, it is not difficult to deduce that

$$\begin{aligned} T &= \{(t,x) : |t| < r_T,\ |x| < r_T - |t - r_T|\}, \\ \tilde{S} &= \{(t,x) : |t| < r_F + r_T,\ |x| < r_F + r_T - |t - t_F|\}. \end{aligned}$$

Since $r_T - |t - r_T| < r_F + r_T - |t - t_F|$ for any $t \in (-r_T, r_T)$, we see that $T \subset \tilde{S}$. □

A.5 PROOF OF PROPOSITION 3.2.5

PROPOSITION A.5.1. *Let $I_{\varepsilon,t_F,a}$ and β_b be as in* (3.10) *and* (3.11). *Also, assume that $a,\ b \in \mathbb{N}$ with $a < b$. If we introduce $N = 2a$ and $M = \min\{2a, 2(b-1)\}$, then $\rho \in C^N(\mathcal{T})$ (where $\mathcal{T}$ is as in* (3.5)*) and $f \in C^M(\mathbb{R}^2)$.*

Proof. Consider in (3.10), as the integrand is a polynomial of degree $2a$ which we will denote by $P_{2a}(t)$ then $I_{\varepsilon,t_F,a}(t) = P_{2a+1}(t)$. Differentiating this polynomial $2a+1$ times will yield a constant $c_{2a+1} \neq 0$. the same argument can be used to infer that taking $2b-1$ derivatives of β_b in (3.11) will result in a constant $d_{2b-1} \neq 0$. Hence, we deduce that

$$\rho^{(2a+1)}(t) = \begin{cases} c_{2a+1} & \text{for } t \in (\varepsilon, t_F - \varepsilon) \\ 0 & \text{otherwise} \end{cases},$$

$$J^{(2b-1)}_{r_B,b}(z) = \begin{cases} d_{2b-1} & -1 < z + r_B < 0 \\ (-1)^{2b-1} d_{2b-1} & 0 < z - r_B < 1 \\ 0 & \text{otherwise}. \end{cases}$$

So, if we set $N = 2a$ it follows that $\rho \in C^N(\mathcal{T})$. For f a little more works needs to be done. Taking M derivatives of f_R with respect to r leads to

$$\frac{\partial^M}{\partial r^M} f_R(t,r) = \frac{d^M}{dr^M} J_{r_B,b}(r - \rho(t) - r_T),$$

however, if we do the same but in this turn with respect to t we obtain

$$\frac{\partial^M}{\partial t^M} f_R(t,r) = \sum_{m=0}^{M-1} \binom{M-1}{m} (-1)^{m+1} \frac{d^{(M-m)}}{dt^{(M-m)}} [\rho(t)] \frac{d^{(m+1)}}{dt^{(m+1)}} \left[J_{r_B,b}(r - \rho(t) - r_T)\right].$$

Similar results hold for f_L. Because of this and given that $f = f_R - f_L$, we see that derivatives of f will be continuous as long as both derivatives of ρ and $J_{r_B,b}$ are. Thus, we conclude that $f \in C^M(\mathbb{R}^2)$ where $M = \min\{2a, 2(b-1)\}$. □

— B —

Hadamard State Extension

B.1 PROOFS OF SECTION 4.2

B.1.1 Proof of Lemma 4.2.1

Lemma B.1.1. *Let A and B be subsets of M. (a) If A and B are acausal obeying $A \subset D^+(B)$, then*

$$D(A) \subset D(B). \tag{B.1}$$

(b) If B is achronal and $B \subset A$ is such that $D_{g_0}(B) \cap A$ is timelike compact [58] and $g = g_0$ when restricted to $D_{g_0}(B) \cap A$, the following holds

$$D_{g_0}(B) \cap A \subset D_g(B) \cap A. \tag{B.2}$$

Proof. To prove (a), take a point $p \in D(A)$ and exclude the trivial case when $p \in B$. Then this point must lie either in $D^+(A)$ or $D^-(A)$. In the first case, we know that any inextendible past-directed causal curve that passes through p must meet A and by hypothesis also B. Since we excluded the case $p \in B$, if $p \in D^-(A)$ then there are two options: $p \in I^+(B)$ or $p \in I^-(B)$.

If $p \in I^+(B)$ take any inextendible past directed causal inextendible curve through p and make it future directed inextendible as well, then it will certainly meet A, as $p \in D^-(A)$. Since $A \subset D^+(B)$, this curve also hits B and this must happen to the past of p, from which we deduce that $p \in D^+(B)$. If $p \in I^-(B)$ take any future directed causal inextendible curve through p and make it past inextendible as well; thus it must meet A and therefore also B (as $A \subset D^+(B)$). It must meet B in the future of p, hence $p \in D^-(B)$.

A similar argument is valid for $A \subset D^-(B)$, from this we conclude a more general statement: for acausal sets $A,\ B$, we can write $D(A) = D(A^+) \cup D(A^-)$, where $A^\pm = A \cap D^\pm(B)$, hence $D(A) \subset D(B)$.

To prove (b), consider any inextendible past [future] directed g-causal curve through a point p of $D_{g_0}(B)$ in $I^+(B) \cap A$ $[I^-(B) \cap A]$. This curve eventually leaves $D_{g_0}(B) \cap A$ by timelike compactness and inextendibility, but the portion inside it, is a g_0-causal curve that is intextendible within $D_{g_0}(B) \cap A$–as $g = g_0$ in $D_{g_0}(B) \cap A$, and even on its boundary by continuity. Thus the curve must hit B, and since the curve was arbitrary, we deduce that $p \in D^+_g(B) \cap A$ $[D^-_g(B) \cap A]$. As p is arbitrary as well, we obtain the inclusion. □

B.1.2 Proof of Lemma 4.3.2

Lemma B.1.2. *Let g be the metric* (4.50)*, that is*

$$g = dt \otimes dt - (dr - fvdt) \otimes (dr - fvdt) - \tilde{h}_r. \tag{B.3}$$

Then, $(\mathbb{R} \times \Sigma,\ g)$ *is globally hyperbolic.*

Proof. We start by showing that g is Lorentzian on $\mathbb{R} \times \Sigma$, which we divide into W and $(\mathbb{R} \times \Sigma) \setminus W$. On the second region $f \equiv 0$ which indicates $g = g_0$, which is clearly Lorentzian. From the definition of r_B we note that $r_B < r_* - \rho_{max}$ hence $r_B + \rho_{max} < r_*$, which leads us to $|\rho_+| < r_*$. A similar argument shows that $|\rho_-| < r_*$, therefore W is contained in $\Upsilon(N)$ whereupon we conclude that W never touches the region where $dr = 0$.

Then, r is a valid coordinate in W, so if we consider $p \in W$, we can choose (t, r, y) with $y \in \partial\tau$ as coordinates for p. Furthermore, as the metric h_t takes a block diagonal form given by (4.42), the metric g becomes, after pull back by Υ,

$$g = (1 - f^2v^2)dt \otimes dt + 2fvdt \otimes_s dr - dr \otimes dr - \tilde{h}_r. \tag{B.4}$$

With these coordinates, there is a mapping $\mathrm{id} \times \Upsilon : \mathbb{R} \times (-r_*, r_*) \times \partial\tau \to M$ so that the tangent bundle may be decomposed as $TM = T\mathbb{R} \oplus T(-r_*, r_*) \oplus T(\partial\tau)$

and thus any vector at p can be written as $v = v^t\partial/\partial t + v^r\partial/\partial r + v_\tau$ where $\partial/\partial t = (\mathrm{id}\times\Upsilon)_*(1,0,0)$, $\partial/\partial r = (\mathrm{id}\times\Upsilon)_*(0,1,0)$ and $v_\tau \in T_p(\partial\tau)$. A basis is given by

$$\left\{\frac{\partial}{\partial t} + (1+fv)\frac{\partial}{\partial r}, \frac{\partial}{\partial t} - (1-fv)\frac{\partial}{\partial r}, v_\tau\right\},$$

letting $k \in \{-1,1\}$, we can write $v_k = \partial/\partial t + k(1+kfv)\partial/\partial r$ and calculate the norm of the two first elements easily

$$\begin{aligned} g\left(v_k, v_k\right) &= (1-f^2v^2) + 2kfv(1+kfv) - (1+kfv)^2 \\ &= -2f^2v^2 + 2fv(k+fv) - 2kfv = 0 \end{aligned}$$

which indicates they are null. All of the other elements are clearly spacelike and given that they constitute a basis, we conclude that the metric is also Lorentzian in W (c.f. p.141 [53]) and thus in $\mathbb{R}\times\Sigma$.

Finally, to show that g is globally hyperbolic we will use a domination argument. We say that a metric g_1 dominates the metric g_2 if $g_1 \geq g_2$ as quadratic forms, a consequence of this is that if g_1 is globally hyperbolic then g_2 must be globally hyperbolic as well, this can be seen from the fact that every g_2-causal curve is necessarily g_1-causal.

First, note that since $0 \leq f \leq 1$ is smooth and $v \geq 0$ is compactly supported, the supremum of fv must exist, denote it by F_+ and introduce

$$g' = g_0 + F_+ dt \otimes dt + \frac{F_+}{1+F_+} dr \otimes dr,$$

then, we claim that

$$\left[(1+F_+)dt\otimes dt - \frac{h_t}{1+F_+}\right] \geq g' \geq g. \tag{B.5}$$

To prove the second inequality, we see that $g' - g = [F_+ + f^2v^2]dt\otimes dt - 2fv dt \otimes_s dr + F_+[1+F_+]^{-1} dr\otimes dr$, thinking of $g'-g$ as a 2×2 matrix it follows that

$$\det(g'-g) = \frac{F_+^2 - f^2v^2}{1+F_+} \geq 0, \qquad \mathrm{tr}(g'-g) = \frac{F_+}{1+F_+}(2+F_+) + f^2v^2 > 0.$$

From this, we can see that its eigenvalues are non-negative and conclude that $g' \geq g$ everywhere in $\mathbb{R}\times\Sigma$. To prove the first inequality, we need an intermediate result first. Consider the curve $\gamma(u) = \Upsilon(s(u), y(u))$ and its tangent vector $v = \dot{\gamma}(0)$, making use of (4.40) and (4.41), inside $\Upsilon(N)$ we have $\nabla_v r = \zeta' \frac{ds}{du}|_{u=0} \leq \frac{ds}{du}|_{u=0}$ and h_t pulls back to $1 \oplus \tilde{h}_r$ so that $(\frac{ds}{du}|_{u=0})^2 \leq h_t(v, v)$ from which we deduce that $(\nabla_v r)^2 \leq h_t(v, v)$ inside $\Upsilon(N)$. Outside $\Upsilon(N)$, we know that $dr = 0$ and hence $\nabla_v r = 0$, which along with the fact that h_t is Riemannian implies that $(\nabla_v r)^2 \leq h_t(v, v)$ in all of $\mathbb{R} \times \Sigma$ and thereby it follows that $dr \otimes dr \leq h_t$. From this we deduce that

$$g' = (1 + F_+)dt \otimes dt - h_t + \tfrac{F_+}{1+F_+} dr \otimes dr \leq (1 + F_+)dt \otimes dt - (1 + F_+)^{-1} h_t,$$

which is the remaining inequality. Given that the l.h.s. of (B.5) is conformal to $(1 + F_+)^2 dt \otimes dt - h_t$ and this is globally hyperbolic, it implies that it and the metrics that it dominates must be globally hyperbolic as well, in particular g, which concludes our proof.

□

— C —

Detector Response

Throughout this appendix we will always assume that $\chi \in C_0^\infty(\mathbb{R})$ and is real-valued. Also, to make what follows more understandable, we remind the reader of our conventions from 5.1. Throughout this section, we will make use of standard Minkowski coordinates $x = (t, \vec{x})$ with $\vec{x} \in \mathbb{R}^2$ for which the metric reads $g = -\mathbf{1} \oplus e$, where e is the 2-dimensional Euclidean metric.

C.1 CONVENTIONS

Definition C.1.1. If for $U \subseteq \mathbb{R}$ and measure $\mu : U \to \mathbb{R}$, the measurable function $f : U \to \mathbb{C}$ satisfies $\int_U |f| d\mu < \infty$, then we say that f belongs to the vector space $L^1(U, d\mu)$ and we define its L^1-norm as $||f||_{1,U} := \int_U |f| d\mu$. If $U = \mathbb{R}$ we will drop the U subscript from the notation.

Also, for $f,\ g : U \to \mathbb{C}$ let us introduce the inner-product over U defined as $\langle f, g \rangle_U := \int_U \overline{f} g\ d\mu$ which induces the L^2-norm $||f||_{2,U} = \sqrt{\langle f, f \rangle}$; we say that $f \in L^2(U, d\mu)$ if $||f||_{2,U} < \infty$, note that $L^2(U, d\mu)$ is also a vector space and we will also drop the subscript if $U = \mathbb{R}$.

Definition C.1.2. The Fourier transform of a function f is denoted by $\mathscr{F}[f]$ and its definition is $\mathscr{F}[f](u) = \int_{-\infty}^{\infty} dx\ e^{-iux} f(x)$. Following this definition, Plancherel's formula becomes $\langle \mathscr{F}[f], \mathscr{F}[g] \rangle = 2\pi \langle f, g \rangle$ where $f,\ g \in L^1(\mathbb{R}, dx) \cap L^2(\mathbb{R}, dx)$.

The sine and cosine transforms are defined as $\mathcal{S}[f](u) = \int_0^\infty dx\ \sin(ux) f(x)$ and $\mathscr{C}[f](u) = \int_0^\infty dx\ \cos(ux) f(x)$, respectively. The convolution of two functions f and g is given by $[f * g](x) := \int_{\mathbb{R}} dy\ f(y) g(x - y)$.

Note that in this Appendix we have changed our sign convention to $(-,+,+,+)$, as opposed to the rest of the Chapter and Appendices in this thesis. Also, as we will only be working with the two-point function of the Minkowski vacuum, we will denote it with $\mathcal{W}$.

C.2 MISCELLANEOUS RESULTS

PROPOSITION C.2.1. *The quantity* $||Q_v||_1$ *is an analytic function for* $v < 1$ *and can be expressed as* $||Q_v||_1 = \sum_{n=1}^{\infty} a_n v^{2n}$ *for* $a_n = \frac{\pi}{n!(n-1)!}\frac{A(2n-1,n-1)}{2^{2n}}$ *where* $A(2n-1, n-1)$ *are the Eulerian numbers [38]. Furthermore, as experimental considerations dictate that* $v^2 \leq 99/100$*, the bound* $||Q_v||_1 \leq M$ *is satisfied where* $M \approx 7.617235504$.

Proof. Note that

$$\begin{aligned} ||Q_v||_1 &= \int_{-\infty}^{\infty} dz\, |1 - [1 - v^2 \mathrm{sinc}(z)^2]^{-1/2}| \\ &= \int_{-\infty}^{\infty} dz\, \sum_{n=1}^{\infty} \binom{n-1/2}{n} v^{2n} \mathrm{sinc}(z)^{2n}. \end{aligned}$$

We can exchange the sum and the integral using $U(k) = e^{k-1/2} - 1$ as dominating function for the partial sums. To see why this is the case, let $f_n(z) = \binom{n-1/2}{n} v^{2n}\mathrm{sinc}(z)^{2n}$ and $s_k(z)$ be its k-th partial sum, then

$$\begin{aligned} |s_k(z)| &\leq \sum_{n=1}^{k} \left|\binom{n-1/2}{n}\right| \leq \sum_{n=1}^{k} \frac{(n-1/2)^n}{n!} \leq \sum_{n=1}^{k} \frac{(k-1/2)^n}{n!} \\ &< \sum_{n=1}^{\infty} \frac{(k-1/2)^n}{n!} = U(k), \end{aligned}$$

where we have made use of the inequality $\binom{n}{k} \leq n^k/k!$. Making use of this and the identity $\int_{-\infty}^{\infty} dz\, \mathrm{sinc}(z)^{2n} = \frac{\pi}{(2n-1)!} A(2n-1, n-1)$ (where $A(n,m)$ are the Eulerian numbers, c.f. [38]), we arrive at

$$||Q_v||_1 = \pi \sum_{n=1}^{\infty} \binom{n-1/2}{n} \frac{v^{2n}}{(2n-1)!} A(2n-1, n-1) =: \sum_{n=1}^{\infty} a_n v^{2n}.$$

Where we have made use of the identity $\binom{n-1/2}{n} = 2^{-2n}(2n)!/(n!)^2$, to define the a_n the coefficients as

$$a_n := \frac{\pi}{n!(n-1)!}\frac{A(2n-1, n-1)}{2^{2n}}.$$

To verify that the series converges we will make use of two asymptotic forms. The first is Stirling's approximation: $n!(n-1)! \to (n!)^2 \to 2\pi n(n/e)^{2n}$ as $n \to \infty$. The second can be found making use of (6.3) in [38], which leads to $A(2n-1, n-1) \to c_1(2n)^{2n-1}\exp(-c_2 n)$ as $n \to \infty$ where $c_1 \sim O(\ln(1))$ and $2 < c_2$.

Thus, for large n, we have $a_n \to c_1 e^{-(c_2-2)n}/n^2$ which leads to $\lim_{n\to\infty} |a_{n+1}/a_n| = e^{-(c_2-2)} < 1$, whereupon we conclude that the series converges as long as $v^2 < 1$. Note that due to experimental considerations $\gamma \leq 10$, hence $v^2 \leq 99/100$ which leads to the following bound

$$||Q_v||_1 \leq \sum_{n=1}^{\infty} a_n\,(99/100)^n \approx 7.617235504$$

□

PROPOSITION C.2.2. *Let $\chi \in C_0^\infty(\mathbb{R})$ and f be a bounded odd function, then, the following identities hold for $n \in \mathbb{N}_0$:*

$$\int_{-\infty}^{\infty} dz\, z^{2n}[\chi * \mathscr{R}[\chi]](kz) = \frac{1}{k^{2n+1}} M_{2n}[1]$$
$$\int_{-\infty}^{\infty} dz\, z^{2n+1}[\chi * \mathscr{R}[\chi]](kz) f(z) = \frac{1}{k^{2n+2}} M_{2n+1}[\mathscr{D}_{1/k} f]$$

Proof. For bounded g and $s \in \mathbb{N}_0$, we write

$$\int_{-\infty}^{\infty} dz\, z^s[\chi * \mathscr{R}[\chi]](kz) g(z) = \frac{1}{k^{s+1}} \int_{-\infty}^{\infty} dz\, z^s[\chi * \mathscr{R}[\chi]](z) g(z/k),$$

where in the last equality we made the change of variable $z \to z/k$, next, we write the convolution explicitly, which leads to

$$\frac{1}{k^{s+1}} \int_{-\infty}^{\infty} dz\, z^s g\left(\frac{z}{k}\right) \int_{-\infty}^{\infty} dz'\, \chi(z')\chi(-z+z')$$
$$= \frac{(-1)^s}{k^{s+1}} \int_{-\infty}^{\infty} dz\, z^s g\left(-\frac{z}{k}\right) \int_{-\infty}^{\infty} dz'\, \chi(z')\chi(z+z'),$$

where we made the change of variable $z \to -z$. Making a last change of variable $z \to z - z'$ leads to the formula

$$\int_{-\infty}^{\infty} dz\, z^s [\chi * \mathscr{R}[\chi]](kz) g(z)) \\ = \frac{(-1)^s}{k^{s+1}} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dz'\, (z - z')^s g\left(-\frac{z - z'}{k}\right) \chi(z')\chi(z) \tag{C.1}$$

So, if we set $g = 1$, $s = 2n$ in (C.1), we will obtain

$$\int_{-\infty}^{\infty} dz\, z^{2n} [\chi * \mathscr{R}[\chi]](kz) = \frac{1}{k^{2n+1}} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dz'\, (z - z')^{2n} \chi(z')\chi(z) \\ = \frac{1}{k^{2n+1}} M_{2n}[1],$$

where we made use of Definition 5.6.2, this is the first identity. For the second, we set $s = 2n + 1$ and $g = f$ where f is odd in (C.1) and use Definition 5.6.2 once more, this yields

$$\int_{-\infty}^{\infty} dz\, z^{2n+1} [\chi * \mathscr{R}[\chi]](kz) f(z) \\ = \frac{1}{k^{2n+2}} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dz'\, (z - z')^{2n+1} f\left(\frac{z - z'}{k}\right) \chi(z')\chi(z) \\ = \frac{1}{k^{2n+2}} M_{2n+1}[\mathscr{D}_{1/k} f].$$

□

PROPOSITION C.2.3. *Let ϕ be smooth and even function with $\phi \in L^1(\mathbb{R}, dz)$. If we define*

$$J_1[\phi](E) = \int_{-\infty}^{\infty} d\omega\, \phi(\omega) \mathrm{sgn}(\omega + E),$$

then, the following asymptotic expansion is valid for E near the origin and $0 \leq l$

$$J_1[\phi](E) = 2 \sum_{n=0}^{l-1} \frac{\phi^{(2n)}(0)}{(2n+1)!} E^{2n+1} + O(E^{2l+1}).$$

Proof. We begin by splitting the integral defined by $J_1[\phi](E)$ into

$$\int_{|\omega|>|E|} d\omega\, \phi(\omega) \mathrm{sgn}(E + \omega) + \int_{-|E|}^{|E|} d\omega\, \phi(\omega) \mathrm{sgn}(E + \omega).$$

Observe that the first integral vanishes because the integrand agrees with an odd function on the integration region. Furthermore, we note that

$$\int_{-|E|}^{|E|} d\omega\ \phi(\omega)\mathrm{sgn}(E+\omega) = \mathrm{sgn}\ E \int_{-|E|}^{|E|} d\omega\ \phi(\omega) = 2\mathrm{sgn}(E)f(|E|), \quad \text{(C.2)}$$

where the last equality is due to the fact that ϕ is even and we have introduced $f(x) = \int_0^x d\omega\ \phi(\omega)$. Note that for $0 \leq n$, we have $f^{(n+1)}(x) = \phi^{(n)}(x)$, and so, even derivatives of f will vanish at the origin.

Because of this, for $\lambda_E \in [0,1]$, we make use of Taylor's theorem with remainder to write

$$2\mathrm{sgn}(E)f(|E|) = 2\mathrm{sgn}(E)\left(\sum_{n=0}^{2l} \frac{f^{(n)}(0)}{n!}|E|^n + \frac{f^{(2l+1)}(\lambda_E E)}{(2l+1)!}|E|^{2l+1}\right)$$

$$= 2\mathrm{sgn}(E)\left(\sum_{n\ \mathrm{odd}}^{2l} \frac{\phi^{(n-1)}(0)}{n!}|E|^n + \frac{\phi^{(2l)}(\lambda_E E)}{(2l+1)!}|E|^{2l+1}\right)$$

where we have discarded the even terms of the sum due to the fact that f has even derivatives at the origin. Since we are assuming that ϕ is smooth, we can see that the second term is bounded, whereupon we conclude

$$2\mathrm{sgn}(E)f(|E|) = 2\mathrm{sgn}(E)\sum_{n\ \mathrm{odd}}^{2l} \frac{\phi^{(n-1)}(0)}{n!}|E|^n + O(E^{2l+1}).$$

Making the change $n \longrightarrow 2n+1$ leads to

$$2\mathrm{sgn}(E)\sum_{n=0}^{l-1} \frac{\phi^{(2n)}(0)}{(2n+1)!}|E|^{2n+1} + O(E^{2l+1}) = 2\sum_{n=0}^{l-1} \frac{\phi^{(2n)}(0)}{(2n+1)!}E^{2n+1} + O(E^{2l+1}),$$

which is our result. □

COROLLARY C.2.1. *Consider Proposition C.2.3 and set* $\phi = |\mathscr{F}[\chi]|^2$*, then*

$$J_1[|\mathscr{F}[\chi]|^2](E) = 2\sum_{n=0}^{l-1}(-1)^n \frac{M_{2n}[1]}{(2n+1)!}E^{2n+1} + O(E^{2l+1}).$$

Proof. The convolution theorem reads

$$|\mathscr{F}[\chi]|^2 = \mathscr{F}[\chi * \mathscr{R}[\chi]], \quad \text{(C.3)}$$

also, recall that differentiating the Fourier transform yields $d^s/d\omega^s(\mathscr{F}[f](\omega)) = \mathscr{F}[(-i)^s z^s f(z)](\omega)$ for $s \in \mathbb{N}$. So, setting $s = 2n$ leads us to obtain

$$\frac{d^{2n}}{d\omega^{2n}}\left(|\mathscr{F}[\chi](\omega)|^2\right)\Big|_{\omega=0} = (-1)^n \mathscr{F}[z^{2n}[\chi * \mathscr{R}[\chi]](z)](0)$$
$$= (-1)^n \int_{-\infty}^{\infty} dz\, z^{2n}[\chi * \mathscr{R}[\chi]](z) = (-1)^n k^{2n+1} \int_{-\infty}^{\infty} dz\, z^{2n}[\chi * \mathscr{R}[\chi]](kz).$$

Writing the convolution explicitly and making use of the notation introduced in Proposition C.2.2, we conclude that

$$\frac{d^{2n}}{d\omega^{2n}}\left(|\mathscr{F}[\chi](\omega)|^2\right)\Big|_{\omega=0} = (-1)^n M_{2n}[1], \tag{C.4}$$

which proves our result. □

Proposition C.2.4. *Let $f \in L^1(\mathbb{R}, dz)$ be an odd function and define*

$$J_2[f](E) = \int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2 \int_0^{\infty} dz\, \cos(k\omega z)\sin(kEz) f(z).$$

We claim that the following asymptotic expansion holds for E near the origin

$$J_2[f](E) = \frac{\pi}{k}\sum_{n=0}^{l-1}(-1)^n \frac{M_{2n+1}[\mathscr{D}_{1/k} f]}{(2n+1)!} E^{2n+1} + O(E^{2l+1}).$$

Proof. Define $F_E(\omega, z)[f] \doteq |\mathscr{F}[\chi](\omega)|^2 \cos(k\omega z)\sin(kEz) f(z)$, note that since $f \in L^1(\mathbb{R}, dz)$, then $F_E \in L^1(\mathbb{R}\times\mathbb{R}, d\omega\, dz)$ and thus, we can use Fubini's theorem to swap the integration order. The integral over ω is

$$\int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2 \cos(k\omega z) = 2\pi[\chi * \mathscr{R}[\chi]](kz),$$

where the equality is due to Fourier's inversion theorem and the fact that since $\chi \in C_0^\infty(\mathbb{R})$, then its Fourier transform will have rapid decay and thus will be integrable. For the remaining integral we use the fact that f is odd to arrive to

$$2\pi \int_0^{\infty} dz\, [\chi * \mathscr{R}[\chi]](kz)\sin(kEz) f(z)$$
$$= \pi \int_{-\infty}^{\infty} dz\, [\chi * \mathscr{R}[\chi]](kz)\sin(kEz) f(z).$$

Using Taylor's theorem with remainder, we write

$$\sin(kEz) = \sum_{n=0}^{2l} \sin^{(n)}(0)\frac{(kEz)^n}{n!} + \sin^{(2l+1)}(\lambda_{Ez}kEz)\frac{(kEz)^{2l+1}}{(2l+1)!}$$

for some $\lambda_{Ez} \in [0,1]$, note that the last term in the sum vanishes and that all of the sine derivatives are bounded by the unit. Furthermore, as χ is compactly supported, the integration range becomes finite, without loss of generality assume it is contained within the $[-L, L]$ interval, with this we see that the remainder obeys

$$\Big|\int_{-\infty}^{\infty} dz\ [\chi * \mathscr{R}[\chi]](kz)\sin^{(2l+1)}(\lambda_{Ez}kEz)f(z)\frac{(kEz)^{2l+1}}{(2l+1)!}\Big| \\ \leq \frac{|kEL|^{2l+1}}{(2l+1)!}\int_{-L}^{L} dz\ \Big|[\chi * \mathscr{R}[\chi]](kz)f(z)\Big|.$$

The RHS is bounded because f is absolutely integrable on the real line, with this we can write J_2 as the following expansion

$$J_2(E)[f] = \pi\sum_{n=0}^{l-1}(-1)^n\frac{(kE)^{2n+1}}{(2n+1)!}\int_{-\infty}^{\infty} dz\ z^{2n+1}[\chi * \mathscr{R}[\chi]](kz)f(z) + O(E^{2l+1})$$

which in virtue of Proposition C.2.2 can be written in its final form

$$J_2(E)[f] = \frac{\pi}{k}\sum_{n=0}^{l-1}(-1)^n\frac{M_{2n+1}[\mathscr{D}_{1/k}f]}{(2n+1)!}E^{2n+1} + O(E^{2l+1}).$$

□

PROPOSITION C.2.5. *For $n \in \mathbb{N}$ define the following family of integrals*

$$\alpha_n(\zeta) = \int_0^{\infty} dz\ \frac{\sin(\zeta z)}{z(1+z^2)^n}.$$

Then, if we introduce

$$P_n(u) = \sum_{m=0}^{n} c_{n,m}u^m, \quad with \quad c_{n,m} = -\frac{(2n-m)!}{m!(n-m)!}\frac{{}_2F_1(1, m-n; m-2n; 2)}{2^{2n-m+1}}$$

(a) it can be shown that a closed form expression for this family of integrals is

$$\alpha_n(\zeta) = \operatorname{sgn}(\zeta)\frac{\pi}{2} + \frac{\pi\ \operatorname{sgn}(\zeta)e^{-|\zeta|}}{(n-1)!}P_{n-1}(|\zeta|).$$

And (b) this function is continuous at the origin for every n and acquires the value $\alpha_n(0) = 0$.

Proof. To prove (a), we note that as the $\alpha_n(\zeta)$ is odd in ζ we can write $\alpha_n(\zeta) = \mathrm{sgn}(\zeta)\alpha_n(|\zeta|)$ and use the evenness of the integrand in z to extend the integration domain to the real line then extend further into the complex plane. Let C_1 be the upper semi-circle closing $[-R, R]$ with a bump that avoids the pole at the origin and C_2 the lower semi-circle closing $[-R, R]$ with a bump that encloses this pole. Then, we have

$$\alpha_n(|\zeta|) = \lim_{R\to\infty} \frac{1}{4i} \left(\oint_{C_1} dz\, \frac{e^{i|\zeta|z}}{z(1+z^2)^n} - \oint_{C_2} dz\, \frac{e^{-i|\zeta|z}}{z(1+z^2)^n} \right).$$

Note that each of these contours surrounds a pole of order n at $z = i$ and $z = -i$, respectively. A straightforward calculation yields

$$\begin{aligned} \alpha_n(|\zeta|) &= \frac{\pi}{2} + \frac{\pi}{2(n-1)!} \left(\lim_{z\to i} \frac{d^{n-1}}{dz^{n-1}} \left[\frac{e^{i|\zeta|z}}{z(z+i)^n} \right] + \lim_{z\to -i} \frac{d^{n-1}}{dz^{n-1}} \left[\frac{e^{-i|\zeta|z}}{z(z-i)^n} \right] \right) \\ &= \frac{\pi}{2} + \frac{\pi}{(n-1)!} \lim_{z\to i} \frac{d^{n-1}}{dz^{n-1}} \left[\frac{e^{i|\zeta|z}}{z(z+i)^n} \right], \end{aligned} \tag{C.5}$$

where we made the change of variable $z \to -z$ in the second limit. These derivatives can be further simplified making use of the Leibniz rule and the fact that complex exponentials are eigenfunctions of iterated derivatives, which yields

$$\begin{aligned} \frac{d^{n-1}}{dz^{n-1}} \left[\frac{e^{i|\zeta|z}}{z(z+i)^n} \right] &= \sum_{m=0}^{n-1} \binom{n-1}{m} \frac{d^m e^{i|\zeta|z}}{dz^m} \frac{d^{n-1-m}}{dz^{n-1-m}} \left[\frac{1}{z(z+i)^n} \right] \\ &= e^{i|\zeta|z} \sum_{m=0}^{n-1} \binom{n-1}{m} (i|\zeta|)^m \frac{d^{n-1-m}}{dz^{n-1-m}} \left[\frac{1}{z(z+i)^n} \right]. \end{aligned} \tag{C.6}$$

As we are mainly interested with the behaviour in $|\zeta|$, we introduce the following coefficients

$$c_{n,m} = i^m \binom{n}{m} \lim_{z\to i} \frac{d^{n-m}}{dz^{n-m}} \left[\frac{1}{z(z+i)^{n+1}} \right], \tag{C.7}$$

and the following identity

$$\frac{d^N}{dz^N} \left[\frac{1}{(z+z_0)^M} \right] = (-1)^N \frac{(M+N-1)!}{(M-1)!} \frac{1}{(z+z_0)^{M+N}}. \tag{C.8}$$

Making use of Leibniz in (C.7) we find

$$c_{n,m} = i^m \binom{n}{m} \lim_{z\to i} \sum_{j=0}^{n-m} \binom{n-m}{j} \frac{d^j}{dz^j} \left(\frac{1}{z} \right) \frac{d^{n-m-j}}{dz^{n-m-j}} \left(\frac{1}{(z+i)^{n+1}} \right)$$

differentiating $1/z$ j-times and using the identity (C.8) leads to

$$\begin{aligned} c_{n,m} &= i^m(-1)^{n-m}\binom{n}{m}\frac{(n-m)!}{n!}\lim_{z\to i}\sum_{j=0}^{n-m}\frac{(2n-m-j)!}{(n-m-j)!}\frac{1}{z^{j+1}}\frac{1}{(z+i)^{2n-m-j+1}} \\ &= -\frac{1}{2^{2n-m+1}}\frac{1}{m!}\sum_{j=0}^{n-m}\frac{(2n-m-j)!}{(n-m-j)!}2^j, \end{aligned} \tag{C.9}$$

the last equality is obtained after a straightforward simplification of terms. Next, observe that the hypergeometric function reduces to a polynomial when one of its two first arguments is a negative integer $-N$, in which case we have

$$_2F_1(\alpha,-N;\gamma;z) = \sum_{j=0}^{N}(-1)^j\binom{N}{j}\frac{(\alpha)_j}{(\gamma)_j}z^j$$

where $(\lambda)_j$ denotes the rising factorial defined by

$$(\lambda)_j = \begin{cases} 1 & j=0 \\ \lambda(\lambda+1)\cdots(\lambda+j-1) & j>0. \end{cases}$$

For our purposes we want the reader to note that

$$\begin{aligned} _2F_1(1,-L;-L-n;2) &= \sum_{j=0}^{L}\frac{(-1)^jL!}{(L-j)!}\frac{2^j}{(-L-n)_j} \\ &= \sum_{j=0}^{L}\frac{L!}{(L-j)!}\frac{2^j}{(L+n+1-j)_j}, \end{aligned} \tag{C.10}$$

where the last equality follows from the identity $(-M)_N = (-1)^N(M+1-N)_N$. Setting $L = n-m$ in (C.10) leads to

$$\begin{aligned} _2F_1(1,-L;-L-n;2) &= \sum_{j=0}^{n-m}\frac{(n-m)!}{(n-m-j)!}\frac{2^j}{(2n-m+1-j)_j} \\ &= \frac{(n-m)!}{(2n-m)!}\sum_{j=0}^{n-m}\frac{(2n-m-j)!}{(n-m-j)!}2^j. \end{aligned} \tag{C.11}$$

And so, in order to go back to the the coefficients $c_{n,m}$ we substitute (C.11) into (C.9), which leads to

$$c_{n,m} = -\frac{(2n-m)!}{m!(n-m)!}\frac{_2F_1(1,m-n;m-2n;2)}{2^{2n-m+1}}. \tag{C.12}$$

Thus, taking the limit in (C.5) by making use of (C.6) and (C.12) results in

$$\alpha_n(|\zeta|) = \frac{\pi}{2} + \frac{\pi e^{-|\zeta|}}{(n-1)!} \sum_{m=0}^{n-1} c_{n-1,m} |\zeta|^m.$$

We define the following polynomial with real coefficients

$$P_n(u) = \sum_{m=0}^{n} c_{n,m} u^m, \tag{C.13}$$

with this, our expression becomes

$$\alpha_n(\zeta) = \text{sgn}(\zeta)\frac{\pi}{2} + \frac{\pi \ \text{sgn}(\zeta) e^{-|\zeta|}}{(n-1)!} P_{n-1}(|\zeta|).$$

To prove (b) consider (C.9), set $m = 0$ and make use of the identity $\sum_{j=0}^{n} 2^j (2n - j)!/(n-j)! = 2^{2n} n!$. This results in

$$c_{n,0} = -\frac{1}{2^{2n+1}} \sum_{j=0}^{n} \frac{(2n-j)!}{(n-j)!} 2^j = -\frac{2^{2n}}{2^{2n+1}} n! = -\frac{1}{2} n!$$

and so, $P_n(|\zeta|) = -n!/2 + \sum_{m=1}^{n} c_{n,m} u^m$. Consequently, we find

$$\alpha_n(\zeta) = \frac{\pi}{2} \text{sgn}(\zeta) \left(1 - e^{-|\zeta|}\right) + \frac{\pi \text{sgn}(\zeta) e^{-|\zeta|}}{(n-1)!} \sum_{m=1}^{n-1} c_{n-1,m} |\zeta|^m.$$

From which we can easily see that for every n this function is continuous at the origin. □

PROPOSITION C.2.6. *Consider α_n as given in Proposition C.2.5, then, the following asymptotic expansions hold for ζ near the origin*

$$\alpha_1(\zeta) = \frac{\pi}{2}\zeta \left(1 + \frac{\zeta^2}{6}\right) - \frac{\pi}{4}\text{sgn}(\zeta)\zeta^2 + O(\zeta^4)$$

$$\alpha_2(\zeta) = \frac{\pi}{4}\zeta \left(1 - \frac{\zeta^2}{6}\right) + O(\zeta^4)$$

$$\alpha_3(\zeta) = \frac{\pi}{16}\zeta \left(3 - \frac{\zeta^2}{6}\right) + O(\zeta^5).$$

Proof. First, we begin by calculating all of the relevant polynomials using (C.13), this yields

$$P_0(u) = -\frac{1}{2}, \qquad P_1(u) = -\frac{1}{2} - \frac{u}{4}, \qquad P_2(u) = -1 - \frac{5u}{8} - \frac{u^2}{8}$$

Next we calculate the alpha terms by making use of (C.16), for the first we have

$$\alpha_1(\zeta) = \frac{\pi}{2}\mathrm{sgn}(\zeta)(1 - e^{-|\zeta|}) = \frac{\pi}{2}\zeta\left(1 + \frac{\zeta^2}{6}\right) - \frac{\pi}{4}\mathrm{sgn}(\zeta)\zeta^2 + O(\zeta^4),$$

the second equality corresponds to its asymptotic expansion for ζ near the origin. For the next term we obtain

$$\alpha_2(\zeta) = \frac{\pi}{2}\mathrm{sgn}(\zeta)\left(1 - e^{-|\zeta|}\left(\frac{|\zeta|}{2} + 1\right)\right) = \frac{\pi}{4}\zeta\left(1 - \frac{\zeta^2}{6}\right) + O(\zeta^4),$$

Proceeding in a similar fashion we find the last term

$$\alpha_3(\zeta) = \frac{\pi}{2}\mathrm{sgn}(\zeta)\left(1 - e^{-|\zeta|}\left(1 + \frac{5|\zeta|}{8} + \frac{\zeta^2}{8}\right)\right) = \frac{\pi}{16}\zeta\left(3 - \frac{\zeta^2}{48}\right) + O(\zeta^5).$$

□

COROLLARY C.2.2. *Let $\zeta_0 \neq 0$, then, the following asymptotic expansions are valid for ζ near the origin and $\zeta_0 > |\zeta|$*

$$\alpha_1(\zeta + \zeta_0) + \alpha_1(\zeta - \zeta_0) = \pi\zeta(1 + \zeta^2/3!)e^{-|\zeta_0|} + O(\zeta^5)$$
$$\alpha_2(\zeta + \zeta_0) + \alpha_2(\zeta - \zeta_0) = \frac{\pi e^{-|\zeta_0|}}{2}\zeta\left(1 + |\zeta_0| + \frac{|\zeta_0| - 1}{6}\zeta^2\right) + O(\zeta^5)$$

Proof. Since α_n is odd and smooth away from the origin, from Taylor's theorem it follows that

$$\alpha_n(\zeta + \zeta_0) + \alpha_n(\zeta - \zeta_0) = 2\alpha_n'(\zeta_0)\zeta + \frac{2}{3!}\alpha_n^{(3)}(\zeta_0)\zeta^3 + O(\zeta^5).$$

The derivatives of α_1 can be calculated with relative ease, we find that $\alpha_1'(\zeta_0) = \alpha_1^{(3)}(\zeta_0) = \pi e^{-|\zeta_0|}/2$, which proves the first expansion. Likewise, for α_2 we obtain $\alpha_2'(\zeta_0) = \pi(1 + |\zeta_0|)e^{-|\zeta_0|}/4$ and $\alpha_2^{(3)}(\zeta_0) = -\pi(1 - |\zeta_0|)e^{-|\zeta_0|}/4$, which proves the second expansion. □

PROPOSITION C.2.7. *Let $f(z)$ and $z \cdot f(z)$ be absolutely integrable functions on $\mathbb{R}$. Then $\mathcal{S}[f(z)]'(\zeta) = \mathcal{C}[z \cdot f(z)](\zeta)$ and $\mathcal{C}[f(z)]'(\zeta) = -\mathcal{S}[z \cdot f(z)](\zeta)$.*

Proof. We will use a dominated convergence argument. Define the function $F(z,\zeta) = f(z)\sin(z\zeta)$ in $\mathbb{R} \times \mathbb{R}$, then $|\partial F(z,\zeta)/\partial\zeta| \leq |z \cdot f(z)|$. For every z, $F(z,\zeta)$ has continuous derivatives with respect to ζ. This, in combination with the mean value theorem gives

$$\left|\frac{F(z,\zeta+h) - F(z,\zeta)}{h}\right| = \left|\frac{\partial F(z,\zeta+\lambda_z h)}{\partial\zeta}\right| \leq |z \cdot f(z)|, \qquad \lambda_z \in [0,1].$$

Since the RHS is absolutely integrable, we can use it as dominating function to take the sought derivatives under the integral sign, which proves the claim. □

COROLLARY C.2.3. *Let $f \in L^1(\mathbb{R}, dz)$ and suppose there is $m \in \mathbb{N}_0$ so that $z^k \cdot f(z) \in L^1(\mathbb{R}, dz)$ for all $0 \leq k \leq m$. Then, the following identities are valid for $\zeta \in \mathbb{R}$*

$$\mathcal{S}[f(z)]^{(2n)}(\zeta) = (-1)^n \mathcal{S}[z^{2n} \cdot f(z)](\zeta),$$
$$\mathcal{S}[f(z)]^{(2n+1)}(\zeta) = (-1)^n \mathcal{C}[z^{2n+1} \cdot f(z)](\zeta)$$

and

$$\mathcal{C}[f(z)]^{(2n)}(\zeta) = (-1)^n \mathcal{C}[z^{2n} \cdot f(z)](\zeta),$$
$$\mathcal{C}[f(z)]^{(2n+1)}(\zeta) = (-1)^{n+1} \mathcal{S}[z^{2n+1} \cdot f(z)](\zeta).$$

Proof. This follows immediately from the fact that $z^m \cdot f(z) \in L^1(\mathbb{R}, dz)$ and Proposition C.2.7. □

C.3 PROOF OF LEMMA 5.7.1

We will compute asymptotic expansions of $\mathcal{S}\left[Q_v(z)/z\right](\zeta)$ for both small and large arguments. Since $z \cdot Q_v(z)/z = Q_v(z)$ and $Q_v \in L^1(\mathbb{R}, dz)$, according to Proposition (C.2.1) we have $\mathcal{S}\left[Q_v(z)/z\right]'(\zeta) = \mathcal{C}\left[Q_v(z)\right](\zeta)$. In particular, this derivative exists at the origin and so, we can read off q_1 which is given by

$$q_1 = \mathcal{S}\left[\frac{Q_v(z)}{z}\right]^{(1)}(0) = \int_0^\infty dz\, Q_v(z).$$

For the next terms more work is needed as the LHS will no longer be absolutely integrable. However, the decomposition that (5.38), recall that according to it we have

$$\mathcal{S}\left[\frac{Q_v(z)}{z}\right](\zeta) = \mathcal{S}[S_v(z)](\zeta) + \mathcal{S}\left[\frac{1-\gamma}{z(1+z^2)^3}\right](\zeta) - \frac{v^2}{2}\mathcal{S}\left[\frac{\sin(z)^2}{z(1+z^2)^2}\right](\zeta)$$
$$- \frac{1}{2}v^2\mathcal{S}\left[\frac{\sin(z)^2}{z(1+z^2)}\right](\zeta) - \frac{3}{8}v^4\mathcal{S}\left[\frac{\sin(z)^4}{z(1+z^2)^2}\right](\zeta). \tag{C.14}$$

Fro now we will focus in the last four terms of (C.14). This makes us consider the following family of integrals for $n \in \mathbb{N}$

$$\alpha_n(\zeta) = \int_0^\infty dz\, \frac{\sin(\zeta z)}{z(1+z^2)^n}. \tag{C.15}$$

In Proposition C.2.5 we show that this can be calculated in closed form,

$$\alpha_n(\zeta) = \operatorname{sgn}(\zeta)\frac{\pi}{2} + \frac{\pi\, \operatorname{sgn}(\zeta)e^{-|\zeta|}}{(n-1)!}P_{n-1}(|\zeta|), \tag{C.16}$$

where $P_n(\zeta)$ is a polynomial with real coefficients. Making use of the indentities

$$\begin{aligned}
\sin(z)^2\sin(\theta) &= (1/2)\sin(\theta) - (1/4)(\sin(\theta+2z) + \sin(\theta-2z)) \\
\sin(\theta)\sin(z)^4 &= \frac{3}{8}\sin(\theta) - \frac{1}{4}(\sin(\theta+2z) + \sin(\theta-2z)) \\
&\quad + \frac{1}{16}(\sin(\theta+4z) + \sin(\theta-4z)),
\end{aligned}$$

we find simplifications of some sine transforms in terms of α_n:

$$\mathcal{S}\left[\frac{1}{z(1+z^2)^n}\right](\zeta) = \alpha_n(\zeta) \tag{C.17}$$

$$\mathcal{S}\left[\frac{\sin(z)^2}{z(1+z^2)^n}\right](\zeta) = \frac{\alpha_n(\zeta)}{2} - \frac{\alpha_n(\zeta+2)+\alpha_n(\zeta-2)}{4} \tag{C.18}$$

$$\mathcal{S}\left[\frac{\sin(z)^4}{z(1+z^2)^n}\right](\zeta) = \frac{3}{8}\alpha_n(\zeta) - \frac{\alpha_n(\zeta+2)+\alpha_n(\zeta-2)}{4} + \frac{\alpha_n(\zeta+4)+\alpha_n(\zeta-4)}{16}. \tag{C.19}$$

Thus, we can make use of (C.17), (C.18) and (C.19) to write the last three terms of (C.14) as

$$\begin{aligned}\mathcal{S}\left[\frac{Q_v(z)}{z}\right](\zeta) &= \mathcal{S}[S_v(z)](\zeta) + (1-\gamma)\alpha_3(\zeta)\\ &-\left(\alpha_1(\zeta) - \frac{\alpha_1(\zeta+2)+\alpha_1(\zeta-2)}{2} + \alpha_2(\zeta) - \frac{\alpha_2(\zeta+2)+\alpha_2(\zeta-2)}{2}\right)\left(\frac{v}{2}\right)^2\\ &-3\left(\frac{3}{4}\alpha_2(\zeta) - \frac{\alpha_2(\zeta+2)+\alpha_2(\zeta-2)}{2} + \frac{\alpha_2(\zeta+4)+\alpha_2(\zeta-4)}{8}\right)\left(\frac{v}{2}\right)^4.\end{aligned} \tag{C.20}$$

Since our goal is to obtain the next terms in the asymptotic expansion for ζ near the origin, we present those of the α terms, which can be found in Proposition C.2.6 and Corollary C.2.2. They are

$$\alpha_1(\zeta) = \frac{\pi}{2}\zeta\left(1+\frac{\zeta^2}{6}\right) - \frac{\pi}{4}\operatorname{sgn}(\zeta)\zeta^2 + O(\zeta^4) \tag{C.21}$$

$$\alpha_2(\zeta) = \frac{\pi}{4}\zeta\left(1-\frac{\zeta^2}{6}\right) + O(\zeta^4) \tag{C.22}$$

$$\alpha_3(\zeta) = \frac{\pi}{16}\zeta\left(3-\frac{\zeta^2}{6}\right) + O(\zeta^5). \tag{C.23}$$

$$\alpha_1(\zeta+\zeta_0) + \alpha_1(\zeta-\zeta_0) = \pi\zeta\left(1+\frac{\zeta^2}{6}\right)e^{-|\zeta_0|} + O(\zeta^5) \tag{C.24}$$

$$\alpha_2(\zeta+\zeta_0) + \alpha_2(\zeta-\zeta_0) = \frac{\pi e^{-|\zeta_0|}}{2}\zeta\left(1+|\zeta_0|+\frac{|\zeta_0|-1}{6}\zeta^2\right) + O(\zeta^5). \tag{C.25}$$

So, to calculate the second order term, we note two things that vastly simplify the calculation. The first is: as $\mathcal{S}[S_v(z)](\zeta)$ is odd in ζ, all of its even derivatives will vanish at the origin, in particular the second derivative. The second, is that the only alpha term with a quadratic contribution is α_1. Then, remembering we already have taken into account a signum factor in (5.39), we conclude that

$$q_2 = \frac{\pi}{4}\left(\frac{v}{2}\right)^2.$$

For the last term we need to calculate the third derivative, this is done in Proposition C.3.1 where it is found that

$$\begin{aligned}\mathcal{S}[S_\nu(z)]^{(3)}(0) = &-\int_0^\infty dz\, z^2 \left(Q_\nu(z) + \frac{\nu^2}{2}\frac{\sin(z)^2}{1+z^2}\right) \\ &+ (1-\gamma)\frac{\pi}{16} - \frac{\pi}{4}\left(1+\frac{1}{e^2}\right)\left(\frac{\nu}{2}\right)^2 - \frac{3\pi}{4}\left(\frac{3}{4}+\frac{1}{e^2}-\frac{3}{4e^4}\right)\left(\frac{\nu}{2}\right)^4.\end{aligned} \tag{C.26}$$

With this, we conclude that the third constant is

$$\begin{aligned}q_3 = &\frac{1}{6}\mathcal{S}[S_\nu(z)]^{(3)}(0) - (1-\gamma)\frac{\pi}{6\cdot 16} - \frac{\pi}{6\cdot 4}\left(1-\frac{3}{e^2}\right)\left(\frac{\nu}{2}\right)^2 \\ &+ \frac{\pi}{2\cdot 4}\left(\frac{3}{4}+\frac{1}{e^2}-\frac{3}{4e^4}\right)\left(\frac{\nu}{2}\right)^4\end{aligned}$$

substituting the value of $\mathcal{S}[S_\nu(z)]^{(3)}(0)$ given in C.26 we find the last coefficient

$$q_3 = -\frac{1}{6}\int_0^\infty dz\, z^2 \left(Q_\nu(z) + \frac{\nu^2}{2}\frac{\sin(z)^2}{1+z^2}\right) - \frac{\pi}{12}\left(1-\frac{1}{e^2}\right)\left(\frac{\nu}{2}\right)^2.$$

Finally, to see that the error is $O(\zeta^4)$, observe that although some of the asymptotic expansions in (C.25) and (C.29) are up to $O(\zeta^5)$, two of them are $O(\zeta^4)$. To prove (b), we recall that $S_\nu \in L^1(\mathbb{R}, dz)$ and make use of the Riemann-Lebesgue Lemma to observe that, as $S_{\nu,l}$ can be differentiated at least five times, for large ζ we will have the estimate

$$\mathcal{S}[S_\nu](\zeta) = o(1/\zeta^5). \tag{C.27}$$

Additionally, making use of (C.16) we find that for large $|\zeta|$

$$\alpha_n(\zeta+\zeta_0) = \operatorname{sgn}(\zeta)\frac{\pi}{2} + \frac{\pi\operatorname{sgn}(\zeta)e^{-|\zeta|}}{(n-1)!}P_{n-1}(|\zeta|). \tag{C.28}$$

And so, making use of (C.27) and (C.28) it follows that for (C.20) we will have

$$\mathcal{S}\left[\frac{Q_\nu(z)}{z}\right](\zeta) = \operatorname{sgn}(\zeta)\frac{\pi}{2}(1-\gamma)\left(1+e^{-|\zeta|}P_2(|\zeta|)\right) + o(1/\zeta^4),$$

which is the sought expression.

PROPOSITION C.3.1. *For ζ near the origin, we find*

$$\mathcal{S}[S_\nu(z)](\zeta) = \mathcal{S}[S_\nu(z)]^{(1)}(0)\zeta + \mathcal{S}[S_\nu(z)]^{(3)}(0)\frac{\zeta^3}{6} + O(\zeta^5), \qquad \text{(C.29)}$$

where the first and third derivatives are given by

$$\begin{aligned}
\mathcal{S}[S_\nu(z)]^{(1)}(0) &= \int_0^\infty dz\, Q_\nu(z) - (1-\gamma)\frac{3\pi}{16} + \frac{\pi}{4}\left(3 - \frac{5}{e^2}\right)\left(\frac{\nu}{2}\right)^2 \\
&\quad + \frac{3\pi}{4}\left(\frac{3}{4} - \frac{3}{e^2} + \frac{5}{4e^4}\right)\left(\frac{\nu}{2}\right)^4, \\
\mathcal{S}[S_\nu(z)]^{(3)}(0) &= -\int_0^\infty dz\, z^2\left(Q_\nu(z) + \frac{\nu^2}{2}\frac{\sin(z)^2}{1+z^2}\right) \\
&\quad + (1-\gamma)\frac{\pi}{16} - \frac{\pi}{4}\left(1 + \frac{1}{e^2}\right)\left(\frac{\nu}{2}\right)^2 - \frac{3\pi}{4}\left(\frac{3}{4} + \frac{1}{e^2} - \frac{3}{4e^4}\right)\left(\frac{\nu}{2}\right)^4.
\end{aligned}$$

Proof. To find an expansion for ζ near the origin, we need to prove rigorously that we can make use of Taylor's theorem. This can be proven by noting that $z^m S_\nu(z) \in L^1(\mathbb{R}^+, dz)$ for $m \leq 5$ and by making use of Corollary C.2.3. Then, we can write the derivatives of $\mathcal{S}[S_z(z)](\zeta)$ up to the fifth order. This alongside Taylor's theorem with remainder results in

$$\mathcal{S}[S_\nu(z)](\zeta) = \mathcal{S}[S_\nu(z)]^{(1)}(0)\zeta^1 + \frac{\mathcal{S}[S_\nu(z)]^{(3)}(0)}{3!}\zeta^3 + \frac{\mathcal{S}[S_\nu(z)]^{(5)}(\lambda_\zeta\zeta)}{5!}\zeta^5,$$

for some $\lambda_\zeta \in [0,1]$. Note that for the remainder we have $|\mathcal{S}[S_\nu(z)]^{(5)}(\lambda_\zeta\zeta)| \leq ||S_\nu(z)z^5||_1$. And so, because of the aforementioned asymptotics of S_ν we conclude that we can write

$$\mathcal{S}[S_\nu(z)](\zeta) = \mathcal{S}[S_\nu(z)]^{(1)}(0)\zeta + \mathcal{S}[S_\nu(z)]^{(3)}(0)\frac{\zeta^3}{6} + O(\zeta^5)$$

Since our goal is to calculate q_1, q_2 and q_3 in C.3, there is no need to calculate the first derivative.

Nonetheless, we will do it as it sheds some light on the method and will quench the curiosity of the inquisitive reader. Consider S_ν as given in (5.36), substitute that into (C.20) and take the derivative of the sine transform using

Corollary C.2.3, this at $\zeta = 0$ results in

$$\begin{aligned}
&\mathscr{S}[S_\nu(z)]^{(1)}(0)\\
&= \mathscr{C}\left[Q_\nu(z) - \frac{1-\gamma}{(1+z^2)^3} + \frac{\nu^2}{2}\frac{\sin(z)^2}{(1+z^2)^2} + \frac{\nu^2}{2}\frac{\sin(z)^2}{(1+z^2)} + \frac{3}{8}\nu^4\frac{\sin(z)^4}{(1+z^2)^2}\right](0)\\
&= \int_0^\infty dz\, Q_\nu(z) - (1-\gamma)\alpha_3^{(1)}(0) + \left(\alpha_1^{(1)}(0) + \alpha_2^{(1)}(0) - \alpha_1^{(1)}(2) - \alpha_2^{(1)}(2)\right)\left(\frac{\nu}{2}\right)^2\\
&+ 3\left(\frac{3}{4}\alpha_2^{(1)}(0) - \alpha_2^{(1)}(2) + \frac{\alpha_2^{(1)}(4)}{4}\right)\left(\frac{\nu}{2}\right)^4.
\end{aligned}$$

Substituting the values of the derivatives of α_n according to each case leads to

$$\begin{aligned}
\mathscr{S}[S_\nu(z)]^{(1)}(0) = &\int_0^\infty dz\, Q_\nu(z) - (1-\gamma)\frac{3\pi}{16} + \frac{\pi}{4}\left(3 - \frac{5}{e^2}\right)\left(\frac{\nu}{2}\right)^2\\
&+ \frac{3\pi}{4}\left(\frac{3}{4} - \frac{3}{e^2} + \frac{5}{4e^4}\right)\left(\frac{\nu}{2}\right)^4.
\end{aligned}$$

Because of Proposition C.2.1 we know that the integral exists. The first derivative for the α_n terms also exist as it can be seen in Proposition C.2.5 and can be computed with relative ease.

To calculate the third derivative, we proceed as above. That is: we substitute (5.36) into (C.20), take the third derivative of the sine transform using Corollary C.2.3 and evaluate at $\zeta = 0$. Note that according to Proposition C.2.5 the third derivative of α_2 and α_3 exists at the origin. We wont differentiate α_1 as it will help us to get rid of the divergence in the integral. Hence,

$$\begin{aligned}
\mathscr{S}[S_\nu(z)]^{(3)}(0) = &-\int_0^\infty dz\, z^2\left(Q_\nu(z) + \frac{\nu^2}{2}\frac{\sin(z)^2}{1+z^2}\right) - (1-\gamma)\alpha_3^{(3)}(0)\\
&+ \left(\alpha_2^{(3)}(0) - \alpha_2^{(3)}(2)\right)\left(\frac{\nu}{2}\right)^2 + 3\left(\frac{3}{4}\alpha_2^{(3)}(0) - \alpha_2^{(3)}(2) + \frac{\alpha_2^{(3)}(4)}{4}\right)\left(\frac{\nu}{2}\right)^4.
\end{aligned}$$

To see that the integral exists, just consider the asymptotics for large z of Q_ν given in (5.35). With this, one can deduce that the integrand will be $(\nu/2)^2\sin(z)^2/(1+z^2)$ plus a term $O(1/z^2)$. Next, we substitute the α_n derivatives, which leads to

$$\begin{aligned}
\mathscr{S}[S_\nu(z)]^{(3)}(0) = &-\int_0^\infty dz\, z^2\left(Q_\nu(z) + \frac{\nu^2}{2}\frac{\sin(z)^2}{1+z^2}\right) + (1-\gamma)\frac{\pi}{16}\\
&- \frac{\pi}{4}\left(1 + \frac{1}{e^2}\right)\left(\frac{\nu}{2}\right)^2 - \frac{3\pi}{4}\left(\frac{3}{4} + \frac{1}{e^2} - \frac{3}{4e^4}\right)\left(\frac{\nu}{2}\right)^4.
\end{aligned}$$

This concludes our proof. □

C.4 PROOF OF PROPOSITION 5.7.1

For the first term we have to calculate the integral

$$\frac{k}{\lambda}\int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi_e](\omega)|^2\omega = \frac{k}{\lambda}\int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi](\omega)|^2(\omega + S_0) = 2\pi k\|\chi\|_2^2 E, \tag{C.30}$$

where we have used the definition of χ_e, Plancherel's formula, the evenness of $|\mathscr{F}[\chi](\omega)|^2$ and the relation $E = S_0/\lambda$. Before we analyse the remaining terms, we would like to note that $|f_e|^2 = |f|^2$ for real-valued f and the following identities are a consequence of this

$$\chi_e' = (iS_0\chi + \chi')_e, \qquad |\chi_e'|^2 = S_0^2|\chi|^2 + |\chi'|^2. \tag{C.31}$$

Thus, for the next order term we will have

$$\frac{k^2}{\lambda^2}\int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi_e](\omega)|^2\omega^2\operatorname{sgn}\left(\frac{\omega}{\lambda}\right) = \frac{k^2}{\lambda^2}\operatorname{sgn}(\lambda)\int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi_e'](\omega)|^2\operatorname{sgn}(\omega)$$

where we have made use of the well-known identity $(i\omega)^n\mathscr{F}[f](\omega) = \mathscr{F}[f^{(n)}](\omega)$, which is valid for an arbitrary function f and $n \in \mathbb{N}_0$. Next, in virtue of (C.31) we deduce

$$\begin{aligned}&\frac{k^2}{\lambda^2}\int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi_e](\omega)|^2\omega^2\operatorname{sgn}\left(\frac{\omega}{\lambda}\right)\\ =&\frac{k^2}{\lambda^2}\operatorname{sgn}(\lambda)\int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[iS_0\chi + \chi'](\omega)|^2\operatorname{sgn}(\omega + S_0).\end{aligned}$$

Since $|\mathscr{F}[f](\omega)|^2$ is even for real-valued f, we use the same argument from Proposition C.2.3 to find

$$\begin{aligned}&\frac{k^2}{\lambda^2}\int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi_e](\omega)|^2\omega^2\operatorname{sgn}\left(\frac{\omega}{\lambda}\right)\\ &= \frac{k^2}{\lambda^2}\operatorname{sgn}(\lambda)\operatorname{sgn}(S_0)\int_{-|S_0|}^{|S_0|} d\omega\, |\mathscr{F}[iS_0\chi + \chi'](\omega)|^2.\end{aligned}$$

The power law $E\lambda = S_0$ leads us to the final expression for the second term

$$\begin{aligned}&\frac{k^2}{\lambda^2}\int_{-\infty}^{\infty} d\omega\, |\mathscr{F}[\chi_e](\omega)|^2\omega^2\operatorname{sgn}\left(\frac{\omega}{\lambda}\right)\\ &= \frac{k^2E^2}{S_0^2}\operatorname{sgn}(E)\int_{-|S_0|}^{|S_0|} d\omega\, |\mathscr{F}[iS_0\chi + \chi'](\omega)|^2.\end{aligned} \tag{C.32}$$

For the third order term, we use the definition of χ_e once more, this yields

$$\frac{k^3}{\lambda^3}\int_{-\infty}^{\infty} d\omega \, |\mathscr{F}[\chi_e](\omega)|^2\omega^3 = \frac{k^3}{\lambda^3}\int_{-\infty}^{\infty} d\omega \, |\mathscr{F}[\chi](\omega)|^2(\omega + S_0)^3$$

dropping the terms that are odd in ω results in

$$\begin{aligned}&\frac{k^3}{\lambda^3}\int_{-\infty}^{\infty} d\omega \, |\mathscr{F}[\chi_e](\omega)|^2\omega^3 \\ &= 3S_0\frac{k^3}{\lambda^3}\int_{-\infty}^{\infty} d\omega \, |\mathscr{F}[\chi](\omega)|^2\omega^2 + S_0^3\frac{k^3}{\lambda^3}\int_{-\infty}^{\infty} d\omega \, |\mathscr{F}[\chi](\omega)|^2.\end{aligned}$$

Finally, we use Plancherel's formula to obtain the final form of this term

$$\frac{k^3}{\lambda^3}\int_{-\infty}^{\infty} d\omega \, |\mathscr{F}[\chi_e](\omega)|^2\omega^3 = 2\pi S_0\frac{k^3}{\lambda^3}\left(3||\chi'||_2^2 + S_0^2||\chi||_2^2\right). \qquad \text{(C.33)}$$

And so, we make use of (C.30), (C.32) and (C.33) to deduce that our final expression is

$$\begin{aligned}&\int_{-\infty}^{\infty} d\omega \, |\mathscr{F}[\chi_e](\omega)|^2 q\left(k\frac{\omega}{\lambda}\right) \\ &= 2\pi k q_1||\chi||_2^2 E + \frac{k^2}{S_0^2}q_2\left(\int_{-|S_0|}^{|S_0|} d\omega \, |\mathscr{F}[iS_0\chi + \chi'](\omega)|^2\right)\operatorname{sgn}(E)E^2 \\ &+ 2\pi\frac{k^3}{S_0^2}q_3\left(S_0^2||\chi||_2^2 + 3||\chi'||_2^2\right)E^3 + I(\lambda).\end{aligned}$$

Where we have introduced the error $I(\lambda)$, our next goal is to prove that this term is $O(1/\lambda^4)$ as $\lambda \to \infty$. Said error is defined as

$$I(\lambda) = \int_{-\infty}^{\infty} d\omega \, |\mathscr{F}[\chi_e](\omega)|^2\left(q\left(k\frac{\omega}{\lambda}\right) - q_1\cdot\left(k\frac{\omega}{\lambda}\right) - q_2\cdot\left(k\frac{\omega}{\lambda}\right)^2 - q_3\cdot\left(k\frac{\omega}{\lambda}\right)^3\right).$$

Introduce a fixed $\varepsilon > 0$ that is arbitrary but small and define $V_\varepsilon = (-\infty, \varepsilon) \cup (\varepsilon, \infty)$. Making the change of variable $\omega \mapsto \lambda\omega$, we find that

$$\begin{aligned}|I(\lambda)| &\le |\lambda|\int_{-\varepsilon}^{\varepsilon} d\omega \, |\mathscr{F}[\chi_e](\lambda\omega)|^2\left|q(k\omega) - q_1\cdot(k\omega) - q_2\cdot(k\omega)^2 - q_3\cdot(k\omega)^3\right| \\ &+ |\lambda|\int_{V_\varepsilon} d\omega \, |\mathscr{F}[\chi_e](\lambda\omega)|^2\left(|q(k\omega)| + |q_1\cdot(k\omega)| + |q_2\cdot(k\omega)^2| + |q_3\cdot(k\omega)^3|\right).\end{aligned} \qquad \text{(C.34)}$$

By hypothesis on q, we observe for small z we have $|q(z) - q_1 z - q_2 z^2 - q_3 z^3| \leq C|z|^4$ for some $C > 0$. Moreover, $q(z)$ is bounded, i.e. $|q(z)| \leq q_0$ for some $q_0 > 0$. With this, we deduce that

$$
\begin{aligned}
|I(\lambda)| \leq Ck^4 |\lambda| \int_{-\varepsilon}^{\varepsilon} d\omega\, |\mathscr{F}[\chi_e](\lambda\omega)|^2 |\omega|^4 \\
+ |\lambda| \int_{V_\varepsilon} d\omega\ |\mathscr{F}[\chi_e](\lambda\omega)|^2 \left(q_0 + |q_1 \cdot (k\omega)| + |q_2 \cdot (k\omega)^2| + |q_3 \cdot (k\omega)^3| \right),
\end{aligned}
$$

reversing the change of variable in the first integral and recalling that $\chi_e(z) = e^{izS_0}\chi(z)$, we obtain

$$
\begin{aligned}
&\lambda^5 \int_{-\varepsilon}^{\varepsilon} d\omega\, |\mathscr{F}[\chi_e](\lambda\omega)|^2 |\omega|^4 = \int_{-\lambda\varepsilon}^{\lambda\varepsilon} d\omega\, |\mathscr{F}[\chi](\omega - S_0)|^2 |\omega|^4 = \\
&= \int_{-\lambda\varepsilon}^{\lambda\varepsilon} d\omega\, |\mathscr{F}[\chi](\omega)|^2 (\omega^4 + 6\omega^2 S_0^2 + S_0^4)
\end{aligned}
$$

where as usual, we have dropped terms that are odd in ω. Observing that for a positive-valued function f and $U \subset \mathbb{R}$ we have $\int_U f(\omega) d\omega \leq \int_{\mathbb{R}} f(\omega) d\omega$, we deduce that the following bound is satisfied

$$
\begin{aligned}
&\lambda^5 \int_{-\varepsilon}^{\varepsilon} d\omega\, |\mathscr{F}[\chi_e](\lambda\omega)|^2 |\omega|^4 \\
&= \int_{-\lambda\varepsilon}^{\lambda\varepsilon} d\omega\, |\mathscr{F}[\chi''](\omega)|^2 + 6S_0^2 \int_{-\lambda\varepsilon}^{\lambda\varepsilon} d\omega\, |\mathscr{F}[\chi'](\omega)|^2 + S_0^4 \int_{-\lambda\varepsilon}^{\lambda\varepsilon} d\omega\, |\mathscr{F}[\chi](\omega)|^2 \\
&\leq 2\pi \left(||\chi''||_2^2 + 6S_0^2 ||\chi'||_2^2 + S_0^4 ||\chi||_2^2 \right).
\end{aligned}
$$

Introduce the λ-independent term $C' := 2\pi C k^4 \left(||\chi''||_2^2 + 6S_0^2 ||\chi'||_2^2 + S_0^4 ||\chi||_2^2 \right)$. As ε is fixed, in the $\lambda \to \infty$ regime, we conclude that I is bounded as follows

$$
\begin{aligned}
&|I(\lambda)| \\
&\leq \frac{C'}{\lambda^4} + |\lambda| \int_{V_\varepsilon} d\omega\, |\mathscr{F}[\chi](\lambda(\omega - S_0))|^2 \left(q_0 + |q_1(k\omega)| + |q_2(k\omega)^2| + |q_3(k\omega)^3| \right).
\end{aligned}
\tag{C.35}
$$

We want to show that the last integral goes faster to zero than $O(1/\lambda^4)$ as $\lambda \to \infty$.

To do so, we make the change of variable $\omega \to \omega + S_0$, this yields another fourth-order polynomial in ω (with coefficients p_i) multiplying the squared modulus of the Fourier transform. This means that we need to show that terms of the form

$$K_{n,\varepsilon}(\lambda) = \int_{\varepsilon}^{\infty} d\omega \, |\mathscr{F}[\chi](\lambda\omega)|^2 \omega^n \qquad (n \in \mathbb{N})$$

go faster to zero than $O(1/\lambda^4)$ as $\lambda \to \infty$. As χ is a compactly-supported smooth function, its Fourier transform has fast-decay.

In consequence, for an arbitrary fixed $N \in \mathbb{N}$ (which will be taken so that $N > n$) we have $|\mathscr{F}[\chi_e](\lambda\omega)|^2 \leq M(1 + (\lambda\omega)^2)^{-N}$ where $M > 0$ is a constant. With this, we deduce the following bound

$$|K_{n,\varepsilon}(\lambda)| \leq \frac{M}{|\lambda|^{n+1}} \int_{\varepsilon}^{\infty} \lambda d\omega \, \frac{(\lambda\omega)^n}{(1 + (\lambda\omega)^2)^N} \leq \frac{M}{|\lambda|^{n+1}} \int_{\lambda\varepsilon}^{\infty} dx \, x^{n-2N} = O\left(\frac{1}{\lambda^{2N}}\right)$$

where we have introduced the new variable $x = \lambda\omega$. And so, we see that the second integral in our original expression (C.35) vanishes faster that $O(1/\lambda^4)$. With this, we conclude that when $|\lambda| \to \infty$, (or equivalently $|E| \to 0$) we will have

$$|I(\lambda)| \leq O\left(\frac{1}{\lambda^4}\right) = O(E^4).$$

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