A Generalised *abc* Conjecture and Quantitative Diophantine Approximation

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PhD

University of York
Mathematics

May 2023

Abstract

The *abc* Conjecture and its number field variant have huge implications across a wide range of mathematics. While the conjecture is still unproven, there are a number of partial results, both for the integer and the number field setting. Notably, Stewart and Yu have exponential *abc* bounds for integers, using tools from linear forms in logarithms [51][52], while Győry has exponential *abc* bounds in the number field case, using methods from S-unit equations [20]. In this thesis, we aim to combine these methods to give improved results in the number field case. These results are then applied to the effective Skolem-Mahler-Lech problem, and to the smooth *abc* conjecture [27].

The smooth *abc* conjecture concerns counting the number of solutions to $a+b=c$ with restrictions on the values of a, b and c. this leads us to more general methods of counting solutions to Diophantine problems. Many of these results are asymptotic in nature due to use of tools such as Lemmas 1.4 and 1.5 of [23]. We make these lemmas effective rather than asymptotic other than on a set of size $\delta > 0$, where δ is arbitrary. From there, we apply these tools to give an effective Schmidt's Theorem, a quantitative Koukoulopoulos-Maynard Theorem (also referred to as the Duffin-Schaeffer Theorem), and to give effective results on inhomogeneous Diophantine Approximation on M_0-sets, normal numbers and give an effective Strong Law of Large Numbers. We conclude this thesis by giving general versions of Lemmas 1.4 and 1.5 of [23].

Contents

Acknowledgments

This thesis was possible due to the support of many.

I would like to thank my supervisor, Dr. Evgeniy Zorin for his unwavering support, encouragement, patience, guidance and kindness; I could not have asked for a better supervisor. I struggle to articulate my admiration and gratitude for him.

I would also like to thank the Number Theory group at York; I don't think there is anyone I haven't asked a question to at some point. I would particularly like to thank Sanju Velani for all his support and guidance.

I'd like to thank all my fellow PhDs, but a few deserve special mention. Ali, thank you for all your patience answering my physics questions, the PhD and especially the Covid lockdown has been made far better by your friendship; Brad, for some early starts to watch rugby, and I do have a better bench now due to you; Diego, I know no one who can match your passion, thank you for sharing some of it with me; my (extended) housemates, for getting me through lockdown, putting up with more MMA than you ever thought you would and putting up with me in general.

The Golden Ball and its staff and patrons, for being the best pub to read maths in, play bar billiards, enjoy Latin nights and everything in between. To everyone not yet mentioned from the Friday night regulars; thank you for your commitment to the cause!

Professor Mike Bennett, for letting me visit him at UBC, and all the people I met in Vancouver; thank you for your warm welcome.

My parents deserve all the thanks I can gather, for their endless support and encouragement; thank you for everything you have done, and thank you in advance for everything you will continue to do.

Author's declaration

I declare that the work presented in this thesis, except where otherwise stated, is based on my own research carried out at the University of York and has not been submitted previously for any degree at this or any other university. Sources are acknowledged by explicit references.

Chapters 2, 3 and 4 are taken from or refer heavily to [48], which has been submitted for publication at Mathematica.

Chapters 5, 6, 7 and 8 are taken from or heavily refer to joint work with Ying Wai Lee, which will be uploaded to ArXiV and submitted for publication imminently.

— 1 —

Introduction

Perhaps the most simple equation one could consider is the equation

$$a + b = c,$$

where a, b, $c \in \mathbb{N}$. This clearly has infinitely many solutions, but one might ask about the size of the c in terms of the primes dividing a, b and c. Under some mild conditions that we will discuss later, the *abc* Conjecture then gives a sharp upper bound for c in terms of the primes dividing the product abc [16]. We discuss this in more depth in Chapter 2.

Despite there being no commonly accepted proof of the *abc* Conjecture, it is natural to generalise the concepts from working over the integers to working over arbitrary number fields; that is, considering the above equation over a finite extension of $\mathbb{Q}$. This leads to the Uniform *abc* Conjecture [16], which we also discuss in some depth in Chapter 2.

Despite the simplicity of the equation, the *abc*-conjecture is of deep interest to mathematicians, both due to its own deep statement about the natures of addition and multiplication, but also due to its wide ranging implications should it be proven. I mention only a few here (and expand a little further in Chapter 2), and refer the reader to the *abc* Conjecture Homepage [39], created and maintained by Nitaj, for a wide range of applications and further literature on the subject.

- Fermat's Last Theorem: Despite having been proven by Wiles, an effective version of the *abc* Conjecture gives a very efficient proof; we discuss this further in Chapter 2.
- Erdös' conjecture on consecutive powerful numbers: a natural number n is called powerful if for every prime $p \mid n$, we have that $p^2 \mid n$. Erdös conjectures

that there are not three consecutive powerful numbers; the *abc* conjecture provides a weaker answer, that the set of powerful triples $\{n,\, n+1,\, n+2\}$ is finite.

- Roth's Theorem: It is known that the *abc* Conjecture implies Roth's Theorem on Diophantine Approximation, and further that an effective *abc* result leads to an effective version of Roth's Theorem [54].
- Szpiro's conjecture for elliptic curves: While being a motivation for the *abc* Conjecture, the *abc* Conjecture implies Szpiro's conjecture for elliptic curves [40].

Much work has thus been done on the *abc* Conjecture, with Stewart and Yu obtaining exponential bounds in the integer case using methods in linear forms in logarithms [51][52], and Győry obtaining exponential bounds in the number field case using methods in S-unit equations [20]' these results will be discussed in detail in Chapter 2.

The first half of this thesis brings these ideas together to provide an improved exponent for an *abc* style bound in the number field case; we note in some cases, we are able to achieve a sub-exponential bound. We then go on to give applications of these results to the effective Skolem-Mahler-Lech problem (which concerns finding zeroes in a linear recurrence sequence) and to Lagarias' and Soundararajan's smooth *abc* Conjecture [27], details of which are given in Chapter 2.

We remark here that both the *abc* Conjecture and the smooth *abc* Conjecture are about counting the number of solutions to a given Diophantine Equation under certain restrictions. This heavily inspires the second topic of this thesis. There are many results in Diophantine Approximation about counting the number of solutions to a given Diophantine inequality, many of which use results akin to Lemmas 1.4 and 1.5 of [23]. A prime example of this is Schmidt's Theorem, which we discuss in Chapter 5, though we give some details here. A more historically accurate exposition of the following is given in Chapter 5; here we give sufficient details to motivate our discussion.

Khintchine's Theorem tells us that given a non-increasing approximation function $\psi : \mathbb{N} \to [0,\, \infty)$, then the finiteness of the number of solutions $(p,\, q)$ in positive integers to

$$\left|\alpha - \frac{p}{q}\right| < \frac{\psi(q)}{q}$$

for almost all $\alpha \in \mathbb{R}$ depends entirely on the convergence or divergence of the sum

$$\sum_{q=1}^{\infty} \psi(q).$$

Notably, when the sum above diverges, then there are infinitely many solutions to the above for almost all α. However, one may be interested in having a formula for the number of solutions to the above, given restrictions on the size of q. Schmidt's Theorem gives a precise asymptotic formula for the number of solutions with $q < Q$ for almost all α [23]. Indeed, in the literature one can find many asymptotic formulas for the number of solutions to various Diophantine inequalities, the proofs of which use results like Lemmas 1.4 and 1.5 of [23]; a notable recent such result is given in [1] in reference to the Duffin Schaeffer conjecture, proved recently by Harper and Koukouopolous [26]. we return to this in more detail in Chapters 5 and 7 of this thesis.

The asymptotic nature of these results come from the fact that Lemmas 1.4 and 1.5 of [23] are asymptotic. However, in some situations one may wish for explicit results, for example in wireless communications [4], and this is the focus of the second half of this thesis. We make Lemmas 1.4 and 1.5 of [23] effective other than on a set of size δ (where $\delta > 0$ and can be chosen to be arbitrarily small). After this, we prove an effective Schmidt's Theorem, give an effective quantitative Koukouopolous-Maynard Theorem [26][1], give some results on inhomogeneous Diophantine approximation on M_0 sets with restricted denominators [43], give a result related to normal numbers [23] and give an effective Law of Large Numbers. We conclude this by giving general versions of Lemmas 1.4 and 1.5 of [23].

The structure of this thesis is as follows. In Chapter 2 we discuss the *abc* Conjecture, giving background information on absolute values and heights, linear forms in logarithms and S-unit equations, before discussing the effective Skolem-Mahler-Lech problem and the smooth *abc*-Conjecture. In Chapter 3 we prove some new *abc* style bounds, before giving applications of them in Chapter 4. In Chapter 5 we give some background to Diophantine Approximation, and discuss Lemmas 1.4 and 1.5 of [23] in detail, before discussing the topics we will apply our effective theorems to. In Chapter 6 we give the proofs of the effective versions of Lemmas 1.4 and 1.5, before giving the applications mentioned above in Chapter 7. In Chapter 8 we give a very general version of Lemmas 1.4 and 1.5, before briefly discussing potential future work in Chapter 9.

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The *abc* Conjecture: Background and Preliminaries

In this chapter we will give background for the first half of this thesis, discussing the *abc* Conjecture and some background theory.

2.1 THE *abc* CONJECTURE

Before giving the *abc* conjecture, we shall discuss a related theorem, the Mason-Stothers Theorem, published independently by Stothers in 1981 [53] and Mason in 1984 [33]. Given a polynomial $f \in \mathbb{C}[x]$, we define the radical of f to be

$$G(f) := \prod_{\substack{g \mid f \\ g \text{ irreducible}}} g.$$

Theorem (Mason-Stothers Theorem)**.** *Let f, g, $h := f + g \in \mathbb{C}[x]$ be relatively prime non-constant polynomials. Then*

$$\max\{\deg f,\, \deg g,\, \deg h\} < \deg G(fgh) - 1$$

This theorem leads to some nice applications, for example a neat proof of Fermat's Last Theorem for polynomials as follows:

Corollary. *Let $n \geq 3$. Then there is no solution in non-constant, relatively prime polynomials x, y, $z \in \mathbb{C}[t]$ satisfying*

$$x(t)^n + y(t)^n = z(t)^n.$$

We prove this to demonstrate how simple the application of the Mason-Stothers Theorem is. We follow the proof given in [30].

Proof. Assume we have such non-constant relatively prime polynomials such that $x(t)^n + y(t)^n = z(t)^n$. Then the Mason-Stother's Theorem states that

$$\deg\left(x(t)^n\right) = n\deg(x^t) \leq \deg G\left(x(t)^n y(t)^n z(t)^n\right) - 1.$$

We note that by the definition of G, we have that

$$\begin{aligned} G\left(x(t)^n y(t)^n z(t)^n\right) &= G\left(x(t)y(t)z(t)\right) \\ &= G\left(x(t)\right) + G\left(y(t)\right) + G\left(z(t)\right). \end{aligned}$$

We note further that the definition of G implies that for all $f \in \mathbb{C}[t]$, we have that $\deg G\left(f\right) \leq \deg\left(f\right)$. It thus follows that

$$n\deg(x^t) \leq \deg\left(x(t)\right) + \deg\left(y(t)\right) + \deg\left(z(t)\right) - 1.$$

We repeat this argument for $y(t)$ and $z(t)$ to obtain that

$$n\deg(y^t) \leq \deg\left(x(t)\right) + \deg\left(y(t)\right) + \deg\left(z(t)\right) - 1,$$

and that

$$n\deg(z^t) \leq \deg\left(x(t)\right) + \deg\left(y(t)\right) + \deg\left(z(t)\right) - 1.$$

Adding these inequalities together, we obtain that

$$n\left(\deg\left(x(t)\right) + \deg\left(y(t)\right) + \deg\left(z(t)\right)\right) \leq 3\left(\deg\left(x(t)\right) + \deg\left(y(t)\right) + \deg\left(z(t)\right)\right) - 3,$$

so

$$(n-3)\left(\deg\left(x(t)\right) + \deg\left(y(t)\right) + \deg\left(z(t)\right)\right) \leq -3.$$

This is impossible for $n \geq 3$, and we have proved the corollary. □

This was a known result, but the proof using Mason-Stothers Theorem is much more elegant. Having seen the strength of this theorem, it is natural to want to extend it, and a natural way to do so is to try and find an analogous inequality over the integers. We note that this is not historically how the conjecture was made, but hope the above discussion gives some motivation for the following conjecture.

Let a, b, $c := a + b$ be positive, pairwise coprime integers and define the radical

$$G(a,\, b,\, c) = G = \prod_{\substack{p \mid abc \\ p \text{ a prime}}} p.$$

In 1988, based on the Mason-Stothers Theorem in function fields and on a conjecture of Szpiro about elliptic curves, Osterlé conjectured the following:

Conjecture (Osterlé's Formulation of the *abc* Conjecture [40]). *Given the set up above, there exists a positive constant $\mathcal{C}_1$ such that $c < G^{\mathcal{C}_1}$.*

Further, Masser conjectured a stronger statement, as follows:

Conjecture (Masser's Formulation of the *abc* Conjecture [35]). *Given the set up above, for all positive ϵ there exists a constant $\mathcal{C}_2(\epsilon)$ such that $c < \mathcal{C}_2(\epsilon)\, G^{1+\epsilon}$.*

While both of these conjectures are referred to as the *abc* conjecture, generally the second form by Masser is focused on in the literature.

It is important to note that the coprimeness condition is necessary and we cannot remove the dependence on ϵ in the above. To see the coprimeness condition is necessary, consider the triple $(a,\, b,\, c) = (2^n,\, 2^n,\, 2^{n+1})$. This triple trivially satisfies $a + b = c$ for all $n \in \mathbb{N}$, but $G(abc) = 2$, providing infinitely many (non-coprime) triples that would give contradictions to the statement above. Comparing the theorem with the Mason-Stothers Theorem, one may wonder about the significance of the ϵ in the exponent of the radical. To see we cannot remove the reliance on ϵ, consider the following statement:

Lemma. *There is no constant K such that for all coprime triples $(a,\, b,\, c) \in \mathbb{Z}$ satisfying $a + b = c$ we have that*

$$|c| \le KG(abc).$$

To prove this, we use an example from [30].

Proof. Consider the triple $(a_n,\, b_n,\, c_n) = \left(1,\, 3^{2^n} - 1,\, 3^{2^n}\right)$. This satisfies $a + b = c$, and we can show by induction that 2^n divides $3^{2^n} - 1$. It thus follows that

$$G(a_n b_n c_n) \le 3 \cdot 2 \cdot \frac{c_n}{2^n}.$$

We see that for any K, we cannot have that for all n,

$$c_n = 3^{2^n} \le KG(a_n b_n c_n) \le K3 \cdot 2 \cdot \frac{3^{2^n}}{2^n}.$$

□

This in fact gives an infinite family of examples showing that the *abc* Conjecture as stated without the ϵ in the exponent does not hold.

These conjectures have far-reaching implications across a range of topics. We shall give only a brief discussion here; see [8] and the references within for further discussion on this matter. We also refer the reader to Chapter 14 of [6] and Chapter 5 of [55] for a discussion of Vojta's conjectures, a generalisation of the second formulation of the *abc* conjecture given above.

We discussed how the Mason-Stothers Theorem allows us to prove a version of Fermat's Last Theorem for polynomials; here we will show how the *abc* conjecture directly proves a weaker version Fermat's Last Theorem. That is, we shall show that the equation

$$x^n + y^n = z^n$$

has no solutions in integers for sufficiently large $n \in \mathbb{N}$. We assume that there are solutions; applying the *abc* conjecture we see that

$$\begin{aligned} z^n &\leq \mathcal{C}(\epsilon) G\left((xyz)^n\right)^{1+\epsilon} \\ &= \mathcal{C}(\epsilon) G\left(xyz\right)^{1+\epsilon} \\ &\leq \mathcal{C}(\epsilon)\left(xyz\right)^{1+\epsilon} \end{aligned}$$

Similarly we obtain $x^n \leq \mathcal{C}(\epsilon) G\left(xyz\right)^{1+\epsilon}$ and $y^n \leq \mathcal{C}(\epsilon) G\left(xyz\right)^{1+\epsilon}$. Multiplying these three inequalities together we see that

$$(xyz)^n \leq \mathcal{C}(\epsilon)\left(xyz\right)^{3+3\epsilon}.$$

Taking logarithms and rearranging it follows that

$$(n - 3 - 3\epsilon)\log\left(xyz\right) \leq \log\left(\mathcal{C}(\epsilon)\right).$$

As by assumption $xyz \geq 2$ we get an upper bound for n depending on the value of the constant $\mathcal{C}(\epsilon)$. Given an explicit choice of ϵ, say 1 for example, we would be left with only finitely many n to do computations for; indeed, Fermat's Last Theorem was shown to hold true for many small n before Wiles' proved the result in full. Indeed, to prove the theorem it suffices to show the theorem is true when n is an odd prime; using ideal factorisation, Kummer was able to show Fermat's Last Theorem was true for all prime exponents less than 100 with the exceptions of 37, 59 and 67 [14]. For this application, we need an effective version of the *abc* Conjecture;

that is, we need to be able to find the value of $\mathcal{C}(\epsilon)$. Regarding Wiles' method of proof of Fermat's Last Theorem, we note that there are many equations where Wiles' methods are not sufficient to give full results about finiteness of solutions. The *abc* conjecture, once proved, has the potential to be used to give results on many equations where Wiles' method is not applicable.

2.1.0.1 Absolute Values and Heights

Throughout this thesis we shall use properties of absolute values and heights regularly. Before continuing our discussion of the *abc* Conjecture, we shall give the basic definitions and properties. Much of the material in this section is standard; we follow [6], and in places Chapter 3 of [56], and refer the reader to references mentioned within these books.

Definition. An absolute value on a field K is a function $|\cdot| : K \to \mathbb{R}$ satisfying

1. $|x| \geq 0$ and $|x| = 0$ if and only if $x = 0$,
2. $|xy| = |x|\,|y|$,
3. $|x + y| \leq |x| + |y|$.

Condition 3 is famously referred to as the triangle inequality. It may be an absolute value satisfies a stronger condition than this, the ultrametric triangle inequality. That is, for all $x,\, y \in K$,

$$|x + y| \leq \max\{|x|\,,\, |y|\}\,.$$

In this case we call the absolute value non-archimedian. Should the ultrametric triangle inequality fail to hold for a pair $(x,\, y) \in K^2$ then the absolute value is archimedian.

The trivial absolute value is equal to 1 at all $x \in K \backslash \{0\}$, and is 0 for $x = 0$. Generally we will not consider this absolute value, or omit it in considerations; for example see the definition of a place below.

We call two absolute values on a field K equivalent if they define the same topology on a field. It turns out two absolute values $|\cdot|_1\,, \,|\cdot|_2$ are equivalent if and only if there is some positive real number s such that for $x \in K$, $|x|_1^s = |x|_2$ [6].

Definition. A place v of a field K is an equivalence class of non-trivial absolute values.

We denote an absolute value in the equivalence class of the place υ by $|\cdot|_\upsilon$. Given a field extension L/K and a place υ of K, we say that ω, a place of L, extends υ and write $\omega \mid \upsilon$ if and only if the restriction of any representative of ω to K is a representative of υ. In this case, we also say that ω lies over υ.

Remark. The notation and language used here corresponds to the fact that the non-archimedian places in number fields correspond to prime ideals of the ring of integers of the number field.

Given a place υ of a filed K, we can consider the completion of this field with respect to this place. This is an extension field denoted K_υ, with a place ω satisfying

- $\omega \mid \upsilon$,
- The topology induced by ω on K_υ is complete,
- K is a dense subset of K_υ with respect to the previously mentioned topology.

Given a field K and place υ, we have that K_υ exists and is unique up to isometric isomorphism [38].

We consider the places on the rationals $\mathbb{Q}$. There is only one arichmedian absolute value, that is the ordinary absolute value $|\cdot|$, which we sometimes denote by $|\cdot|_\infty$ [56].

Given a prime number p, we define the p-adic absolute value $|\cdot|_p$ to be

$$|x|_p = p^{-\mathrm{ord}_p(x)},$$

where $\mathrm{ord}_p(x)$ denotes the largest exponent e such that p^e divides x.. That is, given $x = \frac{m}{n} = p^\alpha \frac{m'}{n'}$ where m, $n \in \mathbb{Z}$, $n \neq 0$, $\alpha \in \mathbb{Z}$ and m', n' are coprime to p; we define $\mathrm{ord}_p(x) = \alpha$. We denote the completion of $\mathbb{Q}$ with respect to a p-adic absolute value by $\mathbb{Q}_p$. The algebraic closure of $\mathbb{Q}_p$ will be denoted by $\overline{\mathbb{Q}}_p$, and by $\mathbb{C}_p$ we shall denote the completion of $\overline{\mathbb{Q}}_p$ with respect to $|\cdot|_p$. We note further that $\mathbb{C}_p$ is also algebraically closed [25]

The p-adic absolute values give us representatives for all the inequivalent non-archimedian absolute values on $\mathbb{Q}$, and Ostrowski's Theorem tells us these are all the places on the rationals.

Theorem (Ostrowski, 1916)**.** *Every non-trivial absolute value on the rationals is equivalent either to $|\cdot|_\infty$ or to $|\cdot|_p$ for some prime p.*

For a proof of this, we refer the reader to [25]. Another theorem sometimes called Ostrowski's Theorem tells us that the only complete archimedian fields are $\mathbb{R}$ and $\mathbb{C}$ [10].

We now explicitly consider absolute values over a number field. We begin with the archimedian case. Consider a number field K; the archimedian absolute values are determined entirely by the embeddings $\sigma : K \to \mathbb{C}$ in the following way. There are $d = [K : \mathbb{Q}]$ such embeddings. An embedding is said to be real if $\sigma(K) \subset \mathbb{R}$, and complex otherwise. Denote the number of real embeddings by r and the number of complex embeddings by $2s$, so that $d = r + 2s$. Then there are $d + s$ distinct places, as the complex embeddings come in conjugate pairs and these pairs generate the same topology on K. The absolute values are defined to be $|\sigma(x)|_\infty$ for a given embedding σ.

We now consider the non-archimedian places of a number field. Given a rational prime p, the absolute value $|\cdot|_p$ on $\mathbb{Q}_p$ has a unique extension to K, where K is any finite extension of $\mathbb{Q}_p$. This is because $\mathbb{Q}_p$ is complete [38]. The extension of the absolute value is as follows: given an element $\alpha \in K$, let $\mathrm{N}_{K/\mathbb{Q}_p}(\alpha)$ denote the norm of α; more explicitly, the determinant of the matrix associated to the linear $\mathbb{Q}_p$-endomorphism of K mapping x to αx. Writing $d = [K : \mathbb{Q}_p]$, the extension of the p-adic absolute value of $\mathbb{Q}_p$ to K is

$$|\alpha|_p = \left|\mathrm{N}_{K/\mathbb{Q}_p}(\alpha)\right|_p^{1/d},$$

where we have somewhat abused notation and let $|\cdot|_p$ refer to both the p-adic absolute value of $\mathbb{Q}_p$ and its extension to K.

Now let $K = \mathbb{Q}(\alpha)$ be a number field of degree d, where f is the minimal polynomial of α over $\mathbb{Q}$. We write $\alpha_1^{(p)}, \ldots, \alpha_d^{(p)}$ for the roots of f in $\mathbb{C}_p$. This gives us d embeddings of K into $\mathbb{C}_p$, by $\sigma_i : K : \gamma \longmapsto \alpha_i^{(p)}$, for $1 \leq i \leq d$. To each embedding, we can associate an ultrametric absolute value $\upsilon_\sigma \mid p$, where $|x|_{\upsilon_\sigma} = |\sigma(x)|_p$.

We note that this is similar to the archimedian case above, but to be more precise about the number of equivalent absolute values we need to consider the decomposition of $f \in \mathbb{Q}[X]$ into irreducible factors in $\mathbb{Q}_p[X]$. We will write $f = f_1 \cdots f_r$, with $d_i = \deg(f_i)$. Assume that σ_1 and σ_2 are distinct embeddings of K into $\mathbb{C}_p$. They give rise to the same ultrametric absolute value if and only if $\sigma_1(\alpha)$ and $\sigma_2(\alpha)$ are conjugate over $\mathbb{Q}_p$ and thus are roots of the same irreducible factor f_i [56][38]. Thus the number of ultrametric absolute values on the field K is r.

Remark. We can start to see here how ultrametric absolute values correspond to prime ideals of the ring of integers $\mathcal{O}_K$ of the field. Maintaining the notation used above, we can often use the Dedekind-Kummer Theorem [11], to determine the prime ideals lying over a rational prime p by considering the factorisation of the minimal polynomial f into its irreducible factors in $\mathbb{Q}_p[X]$ [29][38]. Each factor corresponds to a prime ideal lying above p, and the polynomial f_i can be used to explicitly find said prime ideals. If $K/\mathbb{Q}$ is galois, then the Galois group permutes prime ideals [11]. This is what is happening in the discussion above; that is, ultrametric places correspond to prime ideals of $\mathcal{O}_K$. Explicitly, given a place represented by an absolute value $|\cdot|_p$, one can define the corresponding prime ideal to be

$$\mathfrak{I}_p := \{a \in \mathcal{O}_k \, : \, |a|_p = 0\} .$$

In light of this remark, we could define these non-archimedian absolute values as follows. Given a field K, let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$. Define $\mathrm{ord}_{\mathfrak{p}}(x)$ analogously to $\mathrm{ord}_p(x)$, but this time considering the decomposition into prime ideals of the principal ideal $x\mathcal{O}_K$; that is, define $\mathrm{ord}_{\mathfrak{p}}(x)$ to be the exponent of the prime ideal $\mathfrak{p}$ in the prime decomposition of the ideal $x\mathcal{O}_K$. We then define $|\cdot|_{\mathfrak{p}} = \mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{p})^{-\mathrm{ord}_{\mathfrak{p}}(x)}$ [12]. Recall that the norm of a prime ideal $\mathfrak{p}$ of the ring of integers $\mathcal{O}_K$ of a number field K is defined to be

$$\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{p}) = p^{f_{\mathfrak{p}}},$$

where $f_{\mathfrak{p}}$ is the inertia degree of $\mathfrak{p}$ over p [38]; that is

$$f_{\mathfrak{p}} := [\mathcal{O}_K/\mathfrak{p} : \mathbb{Z}/p] .$$

Ostrowski's Theorem, above, tells us that the only non-trivial absolute value on number fields are the archimedian and non-archimedian absolute values we have discussed, up to equivalence.

We now explicitly choose representatives of places in a number field that will make later theorems easier to state. Given a field K with a fixed non-trivial absolute value $|\cdot|_v$, we consider a finite dimensional separable extension L/K and a place ω of L where $\omega \mid v$. For any $x \in L$, write

$$||x||_\omega := \left|\mathrm{N}_{L_\omega/K_v}(x)\right|_v$$

and

$$|x|_\omega = ||x||_\omega^{1/[L:K]} ,$$

where we consider the norm of the element x over the extensions of the completions of the field with respect to the given places, L_ω/K_v.

Remark. We note throughout this thesis we will be discussing number fields; these have characteristic 0 and thus the extensions are separable. Over $\mathbb{Q}$, we take the representative of each place to be the ones as defined; that is, the normal absolute value for $|\cdot|_\infty$ and $|\cdot|_p = p^{-\mathrm{ord}_p(\cdot)}$.

Remark 1. By M_K we shall refer to the set of places of K normalised as above, and by M_K^∞ the set of archimedean places. We shall always construct places as given above; that is, M_K is the set constructed from $M_\mathbb{Q}$, where over $\mathbb{Q}$, if $p = \infty$ then $|\cdot|_p$ is the ordinary absolute value on $\mathbb{Q}$, and if p is prime then $|p|_p = \frac{1}{p}$. This condition gives us $|\cdot|_p$ as previously defined. In either case, for $v \in M_K$, $v \mid p$, we have that

$$|x|_v = \left|\mathrm{N}_{L_v/\mathbb{Q}_p}(x)\right|_p^{\frac{1}{[K:\mathbb{Q}]}};$$

this fits with the notation and representatives of places chosen above.

This is all done so that the places on a number field K satisfy the product formula; that is

$$\prod_{v \in M_K} |x|_v = 1$$

for any $x \in K \setminus \{0\}$ [6].

We also note we shall refer to archimedean places as infinite places, and refer to the set of infinite places of a number field K as M_K^∞; this explains the notation $|\cdot|_\infty$. The non-archimedean places are then referred to as finite places.

We are now in a position to defined heights on projective space, before defining heights on elements of number fields. Roughly speaking, a height function is a measure of the algebraic complexity needed to describe a point $P \in \mathbb{P}^n_{\overline{\mathbb{Q}}}$.

We define the multiplicative height of the point $P \in \mathbb{P}^n_{\overline{\mathbb{Q}}}$, represented by a homogeneous non-zero vector $\boldsymbol{x} = (x_0 : \cdots : x_n)$ with coordinates in number field K by

$$H(\boldsymbol{x}) := \prod_v \max_j \left\{|x_j|_v\right\},$$

where we take the maximum over the coordinates of $\boldsymbol{x}$. We then define the absolute logarithmic height (or simply *height*) of $\boldsymbol{x}$ to be

$$h(\boldsymbol{x}) := \log H(\boldsymbol{x}) = \sum_{v \in M_K} \max_j \log |x_j|_v.$$

We note that $h(\boldsymbol{x})$ does not depend on the choice of K or the choice of coordinates. We are also quickly able to define both multiplicative and absolute logarithmic height for affine space; let $\mathbb{A}^n_{\overline{\mathbb{Q}}}$ be the affine space of dimension n over $\overline{\mathbb{Q}}$; this could be embedded into $\mathbb{P}^n_{\overline{\mathbb{Q}}}$ by mapping

$$P = (x_1, \dots, x_n) \to [1 : x_1 : \dots : x_n]$$

We then define $H(P)$ and $h(P)$ to be the height of the image of P in $\mathbb{P}^n_{\overline{\mathbb{Q}}}$.

It is convenient to introduce some notation that makes calculations with heights easier to follow. Let $\log^+ x := \max\{0, \log x\}$ on the positive real numbers, and extend this by setting $\log^+ 0 = 0$. Then one can immediately see that for a point $P = (x_1, \dots, x_n) \in \mathbb{A}^n_{\overline{Q}}$, we have that

$$h(P) = \sum_{v \in M_K} \max_j \log^+ |x_j|_v \, .$$

In the case we're thinking about, that is, an algebraic number field $K \cong \mathbb{A}^1_K$, the height of an algebraic number $\alpha \in K$ is

$$h(\alpha) = \sum_{v \in M_K} \log^+ |\alpha|_v \, .$$

We give the main properties of the height function that we will use throughout the thesis.

Lemma. *For algebraic numbers α_1, α_2, and for any algebraic number $\alpha \neq 0$ and $n \in \mathbb{Z}$, we have that*

- $h(\alpha_1\alpha_2) \leq h(\alpha_1) + h(\alpha_2)$,
- $h(\alpha_1 + \alpha_2) \leq \log 2 + h(\alpha_1) + h(\alpha_2)$,
- $h(\alpha^n) = |n|\, h(\alpha)$.

We note further that by definition, it follows that $h(\alpha) = h(-\alpha)$. The proof of the above Lemma is given in [56].

We also have Kronecker's Theorem, which gives an exact characterisation of the algebraic numbers which have a height of 0.

Theorem (Kronecker's Theorem)**.** *The absolute logarithmic height of $x \in \overline{\mathbb{Q}}$ is 0 if and only if x is a root of unity.*

The proof of this is contained in [6].

We now briefly consider polynomials in order to give a formula for absolute logarithmic height of an algebraic number in terms of its Galois conjugates. Given a polynomial $f(t_1, \ldots, t_n) \in \mathbb{C}[t_1, \ldots, t_n]$, we define the Mahler Measure

$$M(f) := \exp\left(\int_{\mathbb{T}^n} \log\left|f\left(e^{i\theta_1}, \ldots, e^{i\theta_n}\right)\right| \mathrm{d}\mu_1 \cdots \mathrm{d}\mu_n\right),$$

there $\mathbb{T}^n$ is the n dimensional unit torus (that is, the cross product of n unit circles), equipped with the standard measure $\mathrm{d}\mu = \frac{1}{2\pi} d\theta_i$.

This is a multiplicative function, that is $M(fg) = M(f)M(g)$ [6]. One can show that if $f(t) = \sum_{i=0}^{d} a_i t^i = a_d \prod_{j=1}^{d} (t - \alpha_j)$, where α_j, $1 \leq j \leq d$ are the roots of $f(t)$, then we have that

$$\log M(f) = \log|a_d| + \sum_{j=1}^{d} \log^+ |\alpha_j|.$$

We note that this is a special case of Jensen's Formula for analytic functions.

We now relate the height of an algebraic number and the Mahler measure of its minimal polynomial.

Lemma. *Let $\alpha \in \overline{\mathbb{Q}}$, with f, its minimal polynomial over $\mathbb{Z}$. Then*

$$\log M(f) = \deg(\alpha) \cdot h(\alpha).$$

This is also shown in [6]. This lemma allows us to find the height of an algebraic number by considering only archimedian absolute values and the conjugates of the algebraic number.

We finish our discussion of heights by giving an important property of heights that we shall use ubiquitously in this thesis.

Theorem 2.1.0.2 (Northcott's Theorem)**.** *There are only finitely many algebraic numbers of bounded degree and height.*

The proof can also be found in [6].

Remark. We note it is necessary to bound both degree and height; if we just bound height, then we have already seen there are infinitely many algebraic numbers with height less than any given positive upper bound. A clear example here, given Kronecker's Theorem (above) are the roots of unity, which all have height 0.

We are now in a position to continue our discussion of the *abc* conjecture.

2.1.1 Existing Results over The Integers

While the *abc* conjecture is yet to be proven in full generality, there has been progress made in finding *abc*-style bounds. In [51], Stewart and Yu prove that there exists an effectively computable positive constant $\mathcal{C}_3$ such that for all positive integers a, b, $c = a + b$ with $(a, b, c) = 1$ and $c > 2$,

$$\log c < G^{\frac{2}{3}+\frac{\mathcal{C}_3}{\log\log G}}.$$

In [52], they were able to improve this result to

$$\log c < \mathcal{C}_4 G^{\frac{1}{3}} (\log G)^3 .$$

To do this required the use of Yu's work extending lower bounds for linear forms in logarithms to the p-adic setting [59][57]. We note Yu further improved these bounds in a series of papers [58][60][61][62]; indeed, we will use results from [62], which make use of group varieties to strengthen the relevant bounds.

2.1.1.1 Linear Forms in Logarithms

Explicit bounds for Linear Forms in Logarithms were first given by Baker in [2], and was followed up by 3 related papers, work for which he was awarded the Fields Medal.

To motivate this topic a little, we begin by recalling the celebrated Gelfond-Schneider Theorem.

Theorem (Gelfond-Schneider, 1934)**.** *If a and b are complex algebraic numbers with $a \neq 0, 1$ and b not rational, then a^b is transcendental.*

This theorem solved Hilbert's seventh problem. We could equivalently formulate the theorem as follows: for any algebraic number $\alpha \neq 0, 1$, the logarithm of α to any algebraic base other than 0 or 1 is either rational or transcendental. It is natural to wish to generalise this result; consider the following. Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers and let $\beta_1, \ldots, \beta_n$ be algebraic numbers. Can we determine whether $\beta_1 \log \alpha_1 + \cdots \beta_n \log \alpha_n$ is transcendental? Baker, using his theorem (which we give below) was able to prove this sum is either 0 or transcendental, which along with his other results solved the multi-dimensional analogue of Hilbert's seventh problem.

We now give some definitions before stating Baker's Theorems. As normal, define the complex logarithm by $\log z = \log|z| + i\arg z$, where for ease we shall choose the argument such that $-\pi \leq \arg z \leq \pi$. Let $\alpha_1, \ldots, \alpha_n \in \mathbb{A}\setminus\{0, 1\}$, $\beta_0 \in \mathbb{A}$ and $\beta_1, \ldots, \beta_n \in \mathbb{A}\setminus\{0\}$, where $\mathbb{A}$ denotes the set of algebraic numbers.

Define

$$\Lambda := \beta_0 + \beta_1 \log\alpha_1 + \cdots + \beta_n \log\alpha_n.$$

We are now in a position to state Baker's results.

Theorem (Baker, 1967). *If* $\log\alpha_1, \ldots, \log\alpha_n$ *are linearly independent over* $\mathbb{Q}$ *then* $\Lambda \neq 0$.

That is, $1, \log\alpha_1, \ldots, \log\alpha_n$ *are linearly independent over* $\mathbb{A}$.

The next thing Baker was able to do was to give a lower bound for the value of $|\Lambda|$. This lower bound has been much improved in time; the initial result by Baker involved huge constants and had a factor of n^n. In many cases this constant was computationally too large to effectively use in proofs; for example Baker's method was applied to Catalan's Conjecture but the constants were too large to be able to computationally verify the conjecture, so the conjecture remained open [44]. The version I give here is due to Matveev and is the best current result at the time of writing; I give the version as presented at Theorem 7.1.5 of [37]. I note this result is in terms of the absolute logarithmic height, discussed above.

Theorem (Matveev, 2000). *Let* $\alpha_1, \ldots, \alpha_n \in \mathbb{A}\setminus\{0, 1\}$ *and let* $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ *be a number field of degree at most* d *over* $\mathbb{Q}$. *Let*

$$\kappa = \begin{cases} 1 \text{ if } K \subset \mathbb{R}, \\ 2 \text{ if } K \subset \mathbb{C}. \end{cases}$$

Finally, let

$$h(\alpha_j) := \sum_{v \in M_K} \log^+ |\alpha_j|_v$$

denote the absolute logarithmic height, where M_K *is the set of places on* K *normalised to satisfy the product formula and* $\log^+ x = \max\{0, \log x\}$ *for* $x \in \mathbb{R}$, $x > 0$. *Consider*

$$\Lambda = \beta_1 \log\alpha_1 + \cdots + \beta_n \log\alpha_n,$$

with $\beta_j \in \mathbb{Z}$ *for all* $1 \leq j \leq n$.

Set

$$B = \max\{|b_1, \ldots, |b_n||\}$$

and for $1 \leq j \leq n$ *let* A_j *be real numbers satisfying*

$$A_j \geq \max\left\{dh\left(\alpha_j\right), \left|\log \alpha_j\right|, 0.16\right\}.$$

Then either $\Lambda = 0$ *of*

$$\log|\Lambda| \geq -\mathcal{C}d^2 A_1 \ldots A_n \log(ed)\log(eB),$$

where

$$\mathcal{C} = \min\left\{\kappa^{-1}\left(\frac{en}{2}\right)^{\kappa} 30^{n+3} n^{3.5},\ 2^{6n+30}\right\}.$$

We note here that the constant does not have the n^n factor present in Baker's earlier work; this aids us computationally in many cases. A number of problems in Diophantine Equations can be solved using linear forms in logarithms where this smaller constant helps us in these proofs; for a survey see Chapter 12 of [12].

Work has also been done on the p-adic analogue of Baker's Theorem; the problem in this setting is as follows. Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers and let p be a rational prime such that the norm of α_j is not divisible by p for all $1 \leq j \leq n$. We then wish to find an upper bound for $\mathrm{ord}_p(\alpha_1 \cdots \alpha_n - 1)$; this is then a lower bound for $|\alpha_1 \cdots \alpha_n - 1|_p$. The bounds we have for this are now comparable to those of the Archimedean case. We note that we can generalise the set-up above to consider the $\mathfrak{p}$-adic anologue for a prime ideal $\mathfrak{p}$ of the ring of integer $\mathcal{O}_K$ of a number field K.

We give a result due to Yu as stated in [37]; we will give a more precise bound at Lemma 3.1.0.3. Before stating the theorem, we recall some facts and definitions. For any algebraic number α, there exits a $d \in \mathbb{Z}$ such that $d\alpha$ is an algebraic integer. We denote the least such d by $d(\alpha)$ and refer to this as the denominator of α. Further, we define $\overline{\alpha}$ to be the maximum of the absolute values of α and its conjugates, and call this the house of α. Finally, the size of α is defined to be $s(\alpha) = d(\alpha) + \overline{\alpha}$.

Theorem (Yu)**.** *Let* p *be a rational prime and let* K *be an algebraic number field of degree* d*. Let* $\alpha_1, \ldots, \alpha_n \in K$ *such that* $s(\alpha_i) \leq H$ *for* $1 \leq i \leq n$*. Further, let* $b_1, \ldots, b_n$ *be rational integers such that*

$$\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1 \neq 0.$$

Set $B = \max\{|b_1|, \ldots, |b_n|\}$ *and let* $\mathfrak{p}$ *be a prime ideal of* K *lying over the rational prime* p. *Then*

$$|\alpha_1 \cdots \alpha_n - 1|_{\mathfrak{p}} \geq (eB)^{-\mathcal{C}}$$

where $\mathcal{C} = \mathcal{C}(H, n, d, \mathfrak{p})$.

We give an easy application of this result, as also given in [37].

Theorem. *Let* S *be a finite set of rational primes. Consider the solutions*

$$(a, b, c) \in \mathbb{Z}^3$$

to $a + b = c$ *such that the only primes* p *dividing* a, b, c *are contained in* S *and* a, b *and* c *are coprime. Then the number of such solutions is finite.*

Remark. This is our first look at the idea of smooth solutions to a Diophantine equation; solutions such that the primes dividing them belong to a fixed set S. Later in this chapter we will discuss the smooth *abc* conjecture and later in the thesis we give some improvements on current results. This theorem also introduces the idea of counting the number of solutions to a given Diophantine problem; the second part of my thesis expands on these concepts.

Proof. Let $|S| = s$ and begin by writing

$$\begin{aligned} a &= \pm p_1^{q_{1,1}} \cdots p_s^{q_{1,s}}, \\ b &= \pm p_1^{q_{2,1}} \cdots p_s^{q_{2,s}}, \\ c &= \pm p_1^{q_{3,1}} \cdots p_s^{q_{3,s}}, \end{aligned}$$

where $0 \leq q_{i,j} \in \mathbb{Z}$ for all i, j. We let

$$Z = \max\{|a|, |b|, |c|\}.$$

For $1 \leq i \leq s$, we automatically have that

$$2^{q_{i,j}} \leq p_i^{q_{i,j}} \leq Z,$$

and it immediately follows that specifically $q_{1,i} \leq 2 \log Z$ and $q_{2,i} \leq 2 \log Z$.

We note that

$$|c|_{p_i} = \frac{1}{p_i^{q_{3,i}}} = |a + b|_{p_i} = \left| p_1^{q_{1,1}} \cdots p_s^{q_{1,s}} \pm p_1^{q_{2,1}} \cdots p_s^{q_{2,s}} \right|_{p_i}.$$

By assumption, a, b and c are coprime, so p_i cannot divide both a and b; without loss of generality we shall assume that $p_i \not| b$ so that $|b|_{p_i} = 1$. It thus follows from these comments and the equation above that

$$|c|_{p_i} = \frac{|c|_{p_i}}{|b|_{p_i}} = \left|\pm p_1^{q_{1,1}-q_{2,1}} \cdots p_s^{q_{1,s}-q_{2,s}} - 1\right|_{p_i}.$$

From our comments above, we further note that $|q_{1,i} - q_{2,i}| \leq 4 \log Z$.

We now apply Yu's Theorem on linear forms in p-adic logarithms on the above to determine that

$$|c|_{p_i} = \left|\pm p_1^{q_{1,1}-q_{2,1}} \cdots p_s^{q_{1,s}-q_{2,s}} - 1\right|_{p_i} > \exp\left(-\mathcal{C}(s,\, p_i) \log\log Z\right).$$

We know however that $|c|_{p_i} = \frac{1}{p_i^{q_{3,i}}}$, so combining this with the lower bound obtained gives us that

$$p_i^{a_{3,i}} < \exp\left(\mathcal{C} \log\log Z\right),$$

for all $1 \leq i \leq s$ and for some constant $\mathcal{C} > 0$. We now apply this to the factorisation for c, finding that

$$c = \prod_{i=1}^{s} p_i^{q_{3,i}} < \exp\left(\mathcal{C}' \log\log Z\right),$$

where $\mathcal{C}' > 0$ is some other constant.

If $Z = |c|$, then we are done; else suppose $Z = \max\{|a|,\, |b|\}$. Then, modifying slightly what we have done already we find that

$$\begin{aligned} |c| = |a+b| &= |b| \times \left|\frac{a+b}{b}\right| \\ &= |b| \times \left|\pm p_1^{q_{1,1}-q_{2,1}} \cdots p_s^{q_{1,s}-q_{2,s}} - 1\right| \\ &\geq Z \exp(-\mathcal{C}'' \log\log Z), \end{aligned}$$

where $\mathcal{C}'' > 0$ is another constant.

From our upper bound on $|c|$ we find that

$$Z \exp(-\mathcal{C}'' \log\log Z) = \exp\left(\log Z - \mathcal{C}'' \log\log Z\right) < \exp\left(\mathcal{C}' \log\log Z\right).$$

This gives us that $Z \leq \mathcal{C}'''$ for some constant $\mathcal{C}''' > 0$ so $\max\{|a|,\, |b|\}$ is bounded and as $|c| \leq |a| + |b|$, this gives a bound on $|c|$. It thus follows that the number of coprime solutions to $a + b = c$ in S is bounded. $\square$

Remark. We will use many aspects of this proof throughout the thesis; namely, we will freely use that if $a + b = c$ then by dividing through by (in this case) $-b$ we obtain that $\frac{a}{-b} - 1 = \frac{c}{-b}$, and we can use linear forms in logarithms in the forms given above. Further, coprimeness conditions will allow us to apply p-adic linear forms in logarithms as above.

Remark. If we divide the above through by c, we are considering the equation

$$x + y = 1$$

where the primes dividing x and y belong to the set S. This is referred to as an S-unit equations, more on which we shall discuss in Section 2.1.2.1.

Having discussed linear forms in logarithms, we now return to our discussion of the *abc* Conjecture.

2.1.2 THE *abc* CONJECTURE AND ALGEBRAIC NUMBER FIELDS

Much work has also been done generalising the *abc* conjecture to algebraic number fields. Browkin discusses this direction of research in [7], while in [34] Masser discusses some issues regarding adapting the radical G to the case of number fields. Let

$$N_K\left(a,\, b,\, c\right) = \prod_{v} \mathrm{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{p}\right)^{\mathrm{ord}_{\mathfrak{p}}(p)},$$

where $v \in M_K$ and v is taken over all finite places such that $\left|a\right|_v, \left|b\right|_v, \left|c\right|_v$ are not all equal, $\mathfrak{p}$ is the prime ideal of $\mathcal{O}_K$ corresponding to v and p is a rational prime such that $\mathfrak{p}$ lies over p (we note that this is the same as the modified support (1.11) of [34]). The Uniform *abc* Conjecture for number fields is then given as follows:

Conjecture (The Uniform *abc* Conjecture for Number Fields [16])**.** *For every $\epsilon > 0$ there exists a $\mathcal{C}(\epsilon) > 0$ such that if $a+b+c = 0$, $a,\, b,\, c \in K^{\times}$ where K is an algebraic number field of degree d, then*

$$H_K(a,\, b,\, c) < \mathcal{C}(\epsilon)^d \left(\left|D_K\right| \cdot N_K\left(a,\, b,\, c\right)\right)^{1+\epsilon},$$

where D_K is the discriminant of K and H_K is defined in Section 2.1.0.1.

We note that for $K = \mathbb{Q}$, this reduces to the *abc* Conjecture given previously.

In [20], Győry shows that given a number field K and $a,\, b,\, c \in K^*$ with $a+b+c = 0$ and any $\epsilon > 0$, there is an effectively computable $\mathcal{C}_5\left(\epsilon\right)$ such that

$$\log\left(H_K\left(a,\, b,\, c\right)\right) < \mathcal{C}_5 N_K\left(a,\, b,\, c\right)^{1+\epsilon}.$$

The key methods Győry uses to prove his results are bounds to the number of solutions to an S-unit equation.

2.1.2.1 S-integers and S-units

We have already seen S-units in passing, here we shall give more details on this subject. Most of this material is covered in [16].

Given a number field K, let S be a subset of M_K that contains all the infinite places and at most finitely many finite places. At this point, we recall the canonical choice of representatives we work with as given at Remark 1. We then say that $\alpha \in K$ is an S-integer if $|\alpha|_v \leq 1$ for all $v \in M_K \backslash S$; equivalently, if the finite places in S correspond to the prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$, then the S-integers are the elements $\alpha \in K$ such that $\operatorname{ord}_{\mathfrak{p}}(\alpha) \geq 0$ for all prime ideals not equal to one of $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$. We note throughout this thesis we will somewhat abuse notation here and let $\mathfrak{p}$ refer to both the prime ideal and its corresponding place.

The set of S-integers forms a ring $\mathcal{O}_S$, and the units of $\mathcal{O}_S$ form a group $\mathcal{O}_S^*$. We see that the ring of S-integers where $S = M_K^\infty$ is just the set of algebraic integers in K, and the units correspond to the units of $\mathcal{O}_K$. If $S = M_K^\infty \bigcup \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$, then $\mathcal{O}_S = \mathcal{O}_K\left[(\mathfrak{p}_1 \cdots \mathfrak{p}_t)^{-1}\right]$ and $\mathcal{O}_S^*$ consists of elements α such that the principal ideal $\langle \alpha \rangle = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_t^{n_t}$, where $n_i \in \mathbb{Z}$ for $1 \leq i \leq t$.

Many results about units of a ring of integers extent to S-integers; for example, there is an S-unit theorem extending Dirichlet's Unit Theorem.

Theorem (S-unit Theorem)**.** *Let S be a finite subset of M_K containing all the infinite places, where $|S| = s$. Then*

$$\mathcal{O}_S^* \cong \mu_K \times \mathbb{Z}^{s-1},$$

where μ_K is the set of roots of unity contained in K.

This means there are $\epsilon_1, \ldots, \epsilon_{s-1} \in \mathcal{O}_S^*$ and $\zeta \in \mu_K$ such that all elements $x \in \mathcal{O}_S^*$ can be written in the form $x = \zeta \times \epsilon_1^{a_1} \cdots \epsilon_{s-1}^{a_{s-1}}$, where $a_i \in \mathbb{Z}$ for $1 \leq i \leq s-1$. We prove this similarly to how we prove Dirichlet's Unit Theorem; that is, given the set S containing the places $v_1, \ldots, v_s$, we can show that the map

$$LOG_S \,:\, \epsilon \to \left(\log |\epsilon|_{v_1}, \ldots, \log |\epsilon|_{v_s}\right)$$

defines a surjective homomorphism from $\mathcal{O}_S^*$ to a full lattice in the real vector space

$$\left\{(x_1, \ldots, x_s) \,:\, \sum_{i=1}^{s} x_i = 0\right\},$$

where the kernel of the homomorphism is μ_K. A proof of this is given in [29].

We call such a system $\{\epsilon_1, \ldots, \epsilon_s\}$ satisfying the S-unit theorem a system of fundamental S-units, and we note such a system is not unique.

Continuing extending ideas about the ordinary ring of integers to S-integers, there is an analogous S-regulator. Given a system of fundamental S-units $\{\epsilon_1, \ldots, \epsilon_{s-1}\}$, we define this to be

$$R_S = \left| \det \left(\log |\epsilon_i|_{\upsilon_j} \right)_{i,\, j=1,\ldots,\, s-1} \right| .$$

This definition does not depend on the choice of fundamental S-units, nor the choice of $\upsilon_1, \ldots, \upsilon_{s-1} \in S$.

We give a formula that will make it easier to calculate and bound the S-regulator. It is known that

$$R_S = R_K \left[I(S) : P(S) \right] \cdot \prod_{i=1}^{t} \log \mathrm{N}_{K/\mathbb{Q}} \left(\mathfrak{p}_{\mathrm{i}} \right),$$

where R_K is the regulator of the field K, $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ are the prime ideals of S, $I(S)$ is the group of fractional ideals of $\mathcal{O}_K$ composed of prime ideals from $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$, and $P(S)$ is the group of principal fractional ideals of $\mathcal{O}_K$ composed of prime ideals from $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$. From here, we can see that $[I(S) : P(S)]$ must divide the class number h_K of K, and thus we have that

$$R_S \leq h_K R_K \prod_{i=1}^{t} \log \mathrm{N}_{K/\mathbb{Q}} \left(\mathfrak{p}_{\mathrm{i}} \right)$$

We now consider S-unit equations. These take the form

$$a_1 x_1 + \cdots + a_n x_n = 1,$$

where $0 \neq a_i \in K$ for some algebraic number field K, and $x_1, \ldots, x_n$ are unknowns from the ring of units of K, or are S-units. More generally, we can consider these unknowns to be elements of a finitely generated multiplicative subgroup $\Gamma \subset K^*$. We will focus mostly on the case in two unknowns, that is

$$a_1 x_1 + a_2 x_2 = 1.$$

This is the only case that is relevant to this thesis; we shall briefly consider the general case using n terms for completeness at the end of this section.

It was proved by Siegel in 1921 that the above equation has only finitely many solutions for units of a number field, and by Mahler in 1933 for S-units in $\mathbb{Q}$. The general case for a number field K follows from the work of Parry in 1950. In the rest of this section, we discuss effective upper bounds for the heights of the solutions to S-unit equations over number fields K; the results generally follow from applications of Baker's method. There are now many effective bounds for the numbers of solutions to S-unit equations; the result we shall use most in this thesis is as follows.

Lemma 2.1.2.2 ([20]). *Let F be an algebraic number field of degree d with set of normalised places M_F, and let S be a finite subset of M_F which contains S_∞, the set of infinite places. Let s be the cardinality of S, $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ the prime ideals corresponding to finite places of S and let $P = \max_i Nm_{\mathbb{Q}}^F(\mathfrak{p}_i)$. Further let R_S be the S-regulator of F. Given α, β, non-zero elements of F, we consider the S-unit equation $\alpha x + \beta y = 1$ in x, y, where x, y are S-units. Let r denote the unit rank of F and let $\mathcal{R} = \max\{h_F, \mathcal{C}_6(r, d)R\}$, where $\mathcal{C}_6(r, d)$ is given explicitly in [20] and R denotes the regulator of the field F. Let $H = \max\{h(\alpha), h(\beta), 1\}$ Then, if $t = 0$, all solutions x, y of the above equation satisfy*

$$\max\{h(x), h(y)\} \leq \mathcal{C}_7(r, d)\, R \log^*(R) H,$$

where $\mathcal{C}_7$ is also given explicitly in [20] and $\log^* x = \max\{\log x, 1\}$.

Maintaining the notation above, if $t > 0$ then we obtain

$$\max\{h(x), h(y)\} \leq \mathcal{C}_8(r, d, t)\, h_F R\, (\log^* R)\, \mathcal{R}^{t+1} (\log^* \mathcal{R}) \left(\frac{P}{\log^* P}\right) R_S H,$$

where $\mathcal{C}_8(r, d, t)$ is again explicitly given in [20].

This result has recently been improved; in [31], Le Fourn is able give bounds for the heights of solutions of S-unit equations in terms of the norm of the third largest ideal in the set S, as follows.

Let K be an algebraic number field of degree d, and S a subset of M_K consisting of all infinite places and finitely many finite places. We will let $s = |S|$. Let P_S denote the norm of the largest prime in S, and let P'_S denote the norm of the third largest prime in S. Let R denote the regulator of K, h_K the class number of K and let r be the unit rank. Further, let R_S be the S-regulator of K, as defined above.

Theorem (Theorem 1.4 of [31]). *Given the notation above, consider the S-unit equation*

$$\alpha x + \beta y = 1,$$

with $\alpha,\, \beta \in K^*$ *and* $x,\, y \in \mathcal{O}_S$.

If S *contains at most two finite places, all solutions of the above satisfy*

$$\max\left\{h\left(x\right),\, h\left(y\right)\right\} \leq \mathcal{C}(d,\, s) R_S \log^*(R_S) \max\left\{h\left(\alpha\right),\, h\left(\beta\right),\, 1,\, \frac{\pi}{d}\right\},$$

where $\mathcal{C}(d,\, s)$ *is the constant* $c_{26}(s,\, d)$ *in formula (30) of [21].*

For any set of places S, *all solutions of the above satisfy*

$$\max\left\{h\left(x\right),\, h\left(y\right)\right\} \leq \mathcal{C}'(d,\, s) P'_S R_S \left(1 + \frac{\log^* R_S}{\log^* P'_S}\right) \max\left\{h\left(\alpha\right),\, h\left(\beta\right),\, 1,\, \frac{\pi}{d}\right\},$$

where $\mathcal{C}'(d,\, s)$ *is* $c_1(s,\, d)$ *from Theorem 1 of [21].*

We note that this bound cannot be directly used to prove Theorem 2.1.3.3 of this thesis (which we will state later) as the constants $\mathcal{C}(d,\, s)$ and $\mathcal{C}'(d,\, s)$ contain terms of the form s^s, where $s = |S|$. In [19], Győry is able to give an improved constant where the dependence of s in the constant is s^5, which gives us a bound that leads to the result given. Before stating Győry's bound, we define some terms.

We maintain the notation above, and let $\mathcal{R} = \max\left\{h_K,\, \mathcal{C}(d,\, r) R_K\right\}$, where $\mathcal{C}$ is given in [19], the value of which depends on whether $r = 0$, 1 or ≥ 2. Further, in the case that S contains 2 or fewer finite places, we set P'_S to be equal to 1.

We are now in a position to give the lemma.

Lemma 2.1.2.3 ([19]). *Let* $t > 0$, *and consider the* S*-unit equation is* x, y

$$\alpha x + \beta y = 1,$$

where $\alpha,\, \beta \in K^*$ *and* $x,\, y \in \mathcal{O}_S^*$. *Again, let* $H = \max\left\{h\left(\alpha\right),\, h\left(\beta\right),\, 1\right\}$ *Every solution to this* S*-unit equation satisfies*

$$\max\left\{h(x),\, h(y)\right\} < \mathcal{C}(d,\, r,\, s,\, t) \mathcal{R}^{t+4} \frac{P'_S}{\log^* P'_S} \left(1 + \frac{\log^* \log P_S}{\log^* P'_S}\right) R_S H,$$

where $\mathcal{C}(d,\, r,\, s,\, t) = s^5 \left(16e\right)^{3r+4t+7} d^{4r+2t+7}$.

2.1.3 abc STYLE RESULTS

Initially we introduce some notation we will use throughout this thesis and give some further comments on the notation. Let K be a number field of degree d and let $a,\, b,\, c \in \mathcal{O}_K \setminus \{0\}$ be such that $a + b + c = 0$. Further, assume that $a\mathcal{O}_K$, $b\mathcal{O}_K$

and $c\mathcal{O}_K$ are pairwise coprime; that is $a\mathcal{O}_K + b\mathcal{O}_K = \mathcal{O}_K$, and similarly for all other pairs. Let $L = HCF(K)$ be the Hilbert Class Field of K (that is, the maximal abelian unramified extension of K [11]) and let

$$G = \prod_{\substack{\mathfrak{P} \text{ prime ideal} \\ \mathfrak{P} \subset \mathcal{O}_L \\ \mathfrak{P} | (abc)\mathcal{O}_L}} \mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{P}).$$

Let $\mathfrak{p}_a$ be the prime ideal of $\mathcal{O}_L$ of greatest norm dividing $a\mathcal{O}_L$, and similarly for $\mathfrak{p}_b$ and $\mathfrak{p}_c$. If a is a unit, then we write that $\mathfrak{p}_a = 1$ with norm 1, and similarly for b and c. Write $\mathfrak{p}_{\max}$ for the prime ideal of $\mathcal{O}_L$ of greatest norm dividing G. A priori, this is equal to one of $\mathfrak{p}_a$, $\mathfrak{p}_b$, $\mathfrak{p}_c$.

In Section 3.2 we will show that we can write $a = u_a a'$ where u_a is a unit and a' satisfies

$$\mathcal{C}_9 \log \left| \mathrm{N}_{L/\mathbb{Q}}(a') \right| \leq h(a') \leq \mathcal{C}_{10} \log \left| \mathrm{N}_{L/\mathbb{Q}}(a') \right|,$$

where $\mathcal{C}_9$, $\mathcal{C}_{10}$ are computable constants, and similarly for b and c. We assume without loss of generality that

$$h(a') \leq h(b') \leq h(c'). \tag{2.1}$$

Given these definitions, we are in a position to state our main theorems.

Theorem 2.1.3.1. *Given the set up above, relabeling a, b and c if necessary to satisfy* (2.1)*, there exists an effectively computable constant $\mathcal{C}_{11}$ depending only on the field K such that*

$$\log H_L(a,\, b,\, c) < \left(Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_a)\, Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_b)\, Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_c)^2 \max\left\{ Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_b),\, Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_c) \right\} \right)^{\frac{1}{3}} \cdot \tag{2.2}$$
$$\cdot\, G^{\mathcal{C}_{11} \frac{\log\log\log G}{\log\log G}}.$$

We will then give various corollaries to put the product of norms of prime ideals in terms of the radical G, namely Corollaries 3.2.1.1-3.2.1.6. Importantly, in Corollary 3.2.1.6 we will give conditions that allow us to attain a sub-exponential bound.

In later parts we will give related results that can be easier to manipulate due to fewer prime ideals on the right hand side of the inequality, attaining the following theorem.

Theorem 2.1.3.2. *Given the set up above, there exists an effectively computable number $\mathcal{C}_{12}$ depending only on the field K such that*

$$\begin{aligned}\log H_L(a,\, b,\, c) &< \left(Nm_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right) Nm_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)^2\right)^{\frac{1}{2}} G^{\mathcal{C}_{12}\frac{\log\log\log G}{\log\log G}} \\ &= Nm_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right)^{\frac{1}{2}} Nm_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right) G^{\mathcal{C}_{12}\frac{\log\log\log G}{\log\log G}}. \end{aligned} \tag{2.3}$$

We will then deduce Corollaries 3.3.1.1-3.3.1.7, again giving conditions in Corollary 3.3.1.4 that give a sub-exponential bound in terms of the radical G.

We will then discuss how exploiting the bound of Győry [19] on the solutions of S-unit equations enables us to reduce the dependency on prime ideals to give the following result with no further conditions:

Theorem 2.1.3.3. *Given the set up above, there exists an effectively computable constant $\mathcal{C}_{13}$ depending on K such that*

$$\begin{aligned}\log H_L(a,\, b,\, c) < \Big(&Nm_{\mathbb{Q}}^{L}\left(\mathfrak{p}_a\right) Nm_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right) Nm_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right) Nm_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c'\right) Nm_{\mathbb{Q}}^{L}\left(\mathfrak{q}\right)\Big)^{\frac{1}{3}} \cdot \\ &G^{\mathcal{C}_{13}\frac{\log\log\log G}{\log\log G}}, \end{aligned} \tag{2.4}$$

where $\mathfrak{p}_c'$ is the prime ideal of third largest norm dividing $c\mathcal{O}_L$ and $\mathfrak{q}$ is the prime ideal of $\mathcal{O}_L$ of third largest norm dividing $bc\mathcal{O}_L$.

From Theorem 2.1.3.3 we will deduce that

$$\log H_L(a,\, b,\, c) < G^{\frac{1}{3}+\mathcal{C}_{14}\frac{\log\log\log G}{\log\log G}}. \tag{2.5}$$

The results given in this thesis, in particular Theorem 2.1.3.3, allow us to give a new method of solving the effective Skolem-Mahler-Lech problem [42] of order 3. Additionally, we use Corollary 3.3.1.4 to expand on results by Lagarias and Soundararajan regarding smooth solutions to the *abc* equation [27]. We briefly discuss both these problems here.

2.1.4 Background for the Applications of Results

The results given in this paper allow us to give a new method of solving the effective Skolem-Mahler-Lech problem [42] of order 3. We note that this problem has been resolved, but we give a new method to resolve this problem [42]. Additionally, we are able to expand on results by Lagarias and Soundararajan regarding smooth solutions to the *abc* equation [27]. We briefly discuss both these problems here.

2.1.4.1 The Effective Skolem-Mahler-Lech Problem

First we discuss the effective Skolem-Mahler-Lech problem. The problem is, given a linear recurrence sequence, to decide whether said sequence contains zeroes. Before stating the problem, we shall state some definitions. The material that follows is adapted mostly from [15].

We recall that a linear recurrence sequence is a sequence (a_x) of elements of a commutative ring with 1, R, satisfying a homogeneous linear recurrence relation (also called a difference equation)

$$a(x+n) = c_1 a(x+n-1) + \cdots + c_n a(x), \tag{2.6}$$

where $c_1, \ldots, c_n \in R$ [15]. We note, we will take R to be an algebraic number field, which is the case in the majority of applications of linear recurrence sequences.

The polynomial

$$f(x) = x^n - c_1 x^{n-1} - \cdots \ c_n$$

associated to the recurrence relation above is referred to as its characteristic polynomial, and the recurrence relation is said to be of order n. If the ring R has no zero divisors (which will always be the case in this thesis), then all linear recurrence sequences satisfy a recurrence relation of minimal length; in this case the characteristic polynomial of the minimal length relation is called the minimal polynomial; the degree of the minimal polynomial is called the degree of the sequence.

For a sequence of the type above, the values $a_1, \ldots, a_n$ are the initial values and they determine the rest of the sequence; we note different sequences of numbers can satisfy the same recurrence relation if the initial values are different. Indeed, given a polynomial f defined over a field, define $\mathcal{L}(f)$ to be the set of all possible linear recurrence sequences satisfying (2.6), and $\mathcal{L}^*(f)$ the set of all sequences for which f is the characteristic polynomial of the sequence. We see that if g divides f then $\mathcal{L}(g) \subset \mathcal{L}(f)$. Further, if f is irreducible then $\mathcal{L}^*(f)$ consists of all elements of $\mathcal{L}(f)$ other than the identically zero sequences.

We note that $\mathcal{L}(f)$ is in fact a finite dimensional vector space of dimension n; to see this consider the following. Take n sequences a_i satisfying (2.6) we shall refer to as impulse sequences, with initial values $a_i(j) = \delta_{ij}$, where $i, j \in \{1, \ldots, n\}$. Then any linear recurrence sequence satisfying (2.6) can be uniquely represented as a linear combination

$$a(x) = \sum_{i=1}^{n} a(i) a_i(x)$$

for $x \in \mathbb{N}$. To see this, we note that the right hand side of the above satisfies (2.6), as do linear combinations of solutions. Further, the right hand side has the same initial values as the recurrence relation a.

From this point, we will explicitly consider difference equations over a number field; the following comments have analogues in various different fields and rings. For full details, we refer the reader to [15].

Recall, given a linear recurrence relation

$$a_{x+n} = c_1 a_{x+n-1} + \cdots + c_n a_x,$$

and initial terms $a_1, \ldots, a_n \in R$, we can find a formula for the m'th term. Given the recurrence relation, we find the characteristic polynomial

$$f(X) = X^n - c_1 X^{n-1} - \cdots - c_{n-1} X - c_n,$$

with roots $r_1, \ldots, r_l$ with multiplicities $m_1, \ldots, m_l$ respectively. The x'th term of the sequence then is given by

$$a_x = g_1(x) r_1^x + \cdots + g_l(x) r_l^x,$$

where $g_i(x)$ are polynomials with $\deg(g_i) \leq m_i - 1$ which depend on the initial values $a_1, \ldots, a_n$.

Given a linear recurrence equation, it is natural to ask whether the sequence contains zeroes, and if so what structure they take; this is the content of the Skolem-Mahler-Lech Theorem, which is as follows.

Theorem (Skolem-Mahler-Lech)**.** *If a sequence of numbers satisfies a linear recurrence relation over a field of characteristic zero, then the zeroes of this sequence can be decomposed into the union of a finite set, and finitely many arithmetic progressions.*

Remark. We note that there exists an algorithm to tell us if there are infinitely many zeroes, and if so to find the decomposition of these zeros into periodic sets guaranteed to exist by the Skolem–Mahler–Lech Theorem [5]. The effective Skolem-Mahler-Lech problem then is to find an algorithm to determine whether there exists at least one zero in a given linear recurrence sequence (importantly in the case where the only zeroes are non-periodic, as an algorithm to find the periodic zeroes exists) [42]. This would allow us to effectively answer whether a given linear recurrence relation contains any zeroes.

In Chapter 4 we will give a new method of determining if a linear recurrence sequence of order 3 contains zeroes. For further references of preexisting methods and results on this problem and related problems we refer the reader to [22] [49] [41] and the references contained within them.

2.1.4.2 The smooth abc conjecture

We now discuss the smooth *abc* Conjecture, also referred to as the xyz conjecture given by Lagarias and Soundararajan in [27]. Given a Diophantine equation, we may consider the set of solutions such that all solutions have a decompositions consisting of primes from a given set S; in this section we consider smooth solutions of the equation $a + b = c$ over the rational integers.

Given a triple a, b, $c := a + b \in \mathbb{N}$, define the smoothness of the triple

$$S(a, b, c) := \max \{p \,:\, p \mid abc\} .$$

In [27], Lagarias and Soundararajan give the following conjecture, which they refer to as the xyz conjecture.

Conjecture 2.1.4.3 (xyz conjecture)**.** *There exists a positive constant κ such that the following hold.*

a) For each $\epsilon > 0$ there are only finitely many integer solutions (X, Y, Z) to the equation $X + Y = Z$ with $(X, Y, Z) = 1$ and

$$S\left(X, Y, Z\right) < \left(\log H\left(X, Y, Z\right)\right)^{\kappa-\epsilon} .$$

b) For each $\epsilon > 0$ there are infinitely many integer solutions (X, Y, Z) to the equation $X + Y = Z$ with $(X, Y, Z) = 1$ and

$$S\left(X, Y, Z\right) < \left(\log H\left(X, Y, Z\right)\right)^{\kappa+\epsilon} .$$

When a triple (X, Y, Z) satisfies $X + Y = Z$ and $(X, Y, Z) = 1$, we will call the triple a primitive solution. Lagarias and Soundararajan go on to conjecture that $\kappa = \frac{3}{2}$. They note however that to prove the above conjecture, one need only prove that there exists a $\kappa_0 > 0$ satisfying part a) and a $\kappa_1 < \infty$ satisfying part b). As a) and b) are independent, monotonicity would then imply the existence of a unique constant κ.

Lagarias and Soundararajan prove part b) assuming the Generalised Riemann Hypothesis, and show that the *abc* conjecture implies part a). Further, in Corollary

1 of [24], Harper is able to show unconditionally that the xyz-smoothness exponent κ is finite, and showed that part b) of the conjecture holds.

Unconditionally, Lagarias and Soundararajan are able to give the following result.

Theorem (Theorem 2.2 of [27]). *For each $\epsilon > 0$ there are only finitely many solutions to $X + Y = Z$ satisfying $(X, Y, Z) = 1$ and*

$$S(X, Y, Z) \leq (3 - \epsilon) \log\log H(X, Y, Z).$$

The proof of this, and the improvement we will give below depend heavily on Northcott's Theorem, as given at Theorem 2.1.0.2. Using results from this paper, we will improve this bound with the following theorem.

Theorem 2.1.4.4. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a function such that $\phi(x) < \log\log x$ with*

$$\lim_{x \to +\infty} \phi(x) = +\infty.$$

Then there are finitely many integer solutions to $X + Y = Z$ satisfying $(X, Y, Z) = 1$ and

$$S(X, Y, Z) \leq \log\log H(X, Y, Z) \frac{\log\log\log H(X, Y, Z)}{\log\log\log\log H(X, Y, Z)\, \phi(\log\log H(X, Y, Z))}. \tag{2.7}$$

We note that this result implies that there are only finitely many primitive integer triples (X, Y, Z) satisfying $X + Y = Z$ with

$$S(X, Y, Z) < c \log\log H(X, Y, Z)$$

for any constant $c \in \mathbb{R}$, $c > 0$. This is because for any such c, there is a value H such that for any $H(X, Y, Z) > H$,

$$\frac{\log\log\log H(X, Y, Z)}{\log\log\log\log H(X, Y, Z)\, \phi(\log\log H(X, Y, Z))} > c,$$

and by Northcott's Theorem, there are only finitely many triples (X, Y, Z) satisfying $H(X, Y, Z) < H$. The statement above then follows from Theorem 2.1.4.4, and along with the result given in Theorem 2.1.4.4 is also an improvement on the $3 - \epsilon$ in Theorem 2.2 of [27].

Remark. We make a small remark that Mochizuki claims to have proven the *abc* conjecture; however, this result relies on Mochizuki's results in Inter-universal Teichmüller Theory [36], the veracity of which is currently being debated [47]. Due to the status this result has within the mathematical community at the time of writing, we will make limited reference to this work in the rest of this thesis.

When this manuscript was completed, Professor Győry informed me about a sharper *abc* inequality over $\mathbb{Q}$ and imaginary quadratic number fields by Mochizuki, Fesenko, Hoshi, Minamide and Porowski (submitted for publication). However, this result also relies on Mochizuki's results in Inter-universal Teichmüller Theory.

We further note that independently, using different methods, Győry has been able to show a similar result to that of these results, but over the base field K [18]. We discuss this further in Chapter 3.

— 3 —

Proofs of the Main *abc* Theorems

3.1 PRELIMINARY DEFINITIONS AND LEMMAS

We begin by recalling some of the ideas already mentioned in the last chapter, before giving some key lemmas we will use throughout the following two chapters.

Given a number field K, recall the definitions of $h(x)$ and $H_K(x_1, \ldots, x_n)$; we note that

$$h(x) = d \log H_K(x, 1),$$

where $H_K(x, 1)$ is defined in the previous chapter and $[F : \mathbb{Q}] = d$.

It is worth pointing out that $H_K(x_1, \ldots, x_n)$ is the projective height, so it gives the same value for any representative of $[x_1 : \cdots : x_n] \in \mathbb{P}^{n-1}(K)$. Explicitly, this means that for any $(a, b, c) \in \mathbb{P}^2(K)$ and any $k \in K^\times$ we have that

$$H_K(a, b, c) = H_K(ka, kb, kc). \tag{3.1}$$

In particular, since in the set up of this article $c \neq 0$, we have that

$$H_K(a, b, c) = H_K\left(\frac{a}{c}, \frac{b}{c}, 1\right).$$

We will generally be considering the height over the Hilbert Class Field L. In this case, as $[K : \mathbb{Q}] = d$,

$$h(x) = dh_K \log H_L(1, x),$$

where h_K is the class number of K. This follows as $[L : K] = h_K$[6] [56]

We note that for any x, y, $z \in K$ where K is an algebraic number field of degree d,

$$\begin{aligned}\log H_F(x,\, y,\, z) &= \log H_F\left(\frac{x}{z},\, \frac{y}{z},\, 1\right) \\ &\leq 2d \max\left(h\left(\frac{x}{z}\right),\, h\left(\frac{y}{z}\right)\right).\end{aligned} \tag{3.2}$$

This follows directly from (4.3) of [20].

We now state some pre-existing lemmas which we will repeatedly use throughout the proof of Theorem 2.1.3.1. In the rest of the paper, $\mathcal{C}_1$, $\mathcal{C}_2, \ldots$ denote effectively computable constants, and we will, where relevant, state what these constants depend on. In many cases, the constants depend on properties determined by a certain field; in these cases we will sometimes explicitly give which properties of the field the constants depend on.

We first give a result about the existence of a set of fundamental S-units which will make computation throughout the paper easier.

Lemma 3.1.0.1. *Given a set of places S of size s consisting of all infinite places and finitely many finite places of the field F, we can find a system of fundamental units $\eta_1, \ldots, \eta_{s-1}$ such that*

(i) $\displaystyle\prod_{i=1}^{s-1} h(\eta_i) \leq \mathcal{C}_3 R_S$,

(ii) $\displaystyle\max_{1\leq i\leq s-1} h(\eta_i) \leq \mathcal{C}_4 R_S$ *if* $s \geq 3$,

(iii) if $\upsilon_1, \ldots, \upsilon_{s-1}$ are any distinct places from S, then the absolute values of the entries of the inverse matrix of $(\log |\eta_i|_{\upsilon_j})_{i,\, j=1,\ldots,\, s-1}$ *do not exceed* $\mathcal{C}_5$,

where R_S is the S-unit regulator of F.

Proof. This is Proposition 4.1.8 of [17] □

Throughout this paper we will use such a system of fundamental units, and often refer to them as "the fundamental units" of the field in question.

Lemma 3.1.0.2. *Let F be a number field of degree d, and let $\alpha \in \mathcal{O}_F \backslash \mathcal{O}_F^*$. Then there is an effectively computable number $\mathcal{C}_6\,(F)$, depending on the fundamental units of $\mathcal{O}_F$, and an $\epsilon \in \mathcal{O}_F^*$ such that*

$$\overline{|\epsilon\alpha|} \leq \mathcal{C}_6 \left|N_{F/\mathbb{Q}}\,(\alpha)\right|^{1/d}$$

where $\overline{|\alpha|}$ denotes the house of α.

We recall that $\lceil\overline{\alpha}\rceil$, the house of α, is defined to be the maximal absolute value of the conjugates of α over $\mathbb{C}$.

Proof. This is Lemma 1.3.8 from [37]. □

Lemma 3.1.0.3. *Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers and F a number field of degree d containing $\alpha_1, \ldots, \alpha_n$. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_F$ lying above the rational prime p with ramification index $e_{\mathfrak{p}}$ and residue class degree $f_{\mathfrak{p}}$. For $\alpha \in F$, $\alpha \neq 0$, we write $ord_{\mathfrak{p}}(\alpha)$ for the exponent to which $\mathfrak{p}$ divides the principal fractional ideal generated by α in F, and we set $ord_{\mathfrak{p}}(0) = +\infty$. Let $b_1, \ldots b_n$ be integers, and set $\Theta = \alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1$. Assume that $\Theta \neq 0$. Finally, set $h'(\alpha_j) = \max\left\{h(\alpha_j), \frac{1}{16e^2d^2}\right\}$. Then*

$$ord_{\mathfrak{p}}(\Theta) < (16ed)^{2(n+1)} n^{5/2} \log(2nd) \log(2d) \cdot e_{\mathfrak{p}}^{n} \frac{\mathrm{Nm}_{\mathbb{Q}}^{F}(\mathfrak{p})}{\left(\log \mathrm{Nm}_{\mathbb{Q}}^{F}(\mathfrak{p})\right)^2} \prod_{i=1}^{n} h'(\alpha_i) \log B,$$

where $B = \max\{|b_1|, \ldots, |b_n|, 3\}$.

Proof. This is a consequence of the main theorem of [62], given on page 190. □

Lemma 3.1.0.4. *Let F be a number field with ring of integers $\mathcal{O}_F$. Apply a total ordering to the prime ideals, so that if $\mathrm{Nm}_{\mathbb{Q}}^{F}(\mathfrak{x}) > \mathrm{Nm}_{\mathbb{Q}}^{F}(\mathfrak{y})$, then $\mathfrak{x} \succ \mathfrak{y}$. Arbitrarily order ideals of the same norm. Then there is an effectively computable positive constant $\mathcal{C}_7$ such that for every positive integer r we have*

$$\prod_{i=1}^{r} \frac{\mathrm{Nm}_{\mathbb{Q}}^{F}(\mathfrak{p}_i)}{\log \mathrm{Nm}_{\mathbb{Q}}^{F}(\mathfrak{p}_i)} > \left(\frac{r}{\mathcal{C}_7}\right)^r.$$

Proof. Let $\pi_F(x)$ denote the number of prime ideals in number field F of norm less than or equal to x. By the Landau Prime Ideal Theorem [28], we know that $\pi_F(x) \sim \frac{x}{\log x}$. Partially order the prime ideals as in the statement of the Lemma. Then by Landau,

$$\pi_F\left(\mathrm{Nm}_{\mathbb{Q}}^{F}(\mathfrak{p}_j)\right) \sim \frac{\mathrm{Nm}_{\mathbb{Q}}^{F}(\mathfrak{p}_j)}{\log\left(\mathrm{Nm}_{\mathbb{Q}}^{F}(\mathfrak{p}_j)\right)}.$$

Thus by Landau, there exists an effectively computable number $\mathcal{C}_8$ such that

$$\frac{\mathrm{Nm}_{\mathbb{Q}}^{F}(\mathfrak{p}_j)}{\log\left(\mathrm{Nm}_{\mathbb{Q}}^{F}(\mathfrak{p}_j)\right)} > j/\mathcal{C}_8.$$

Thus, using the inequality $r! \geq (r/e)^r$, we see that

$$\prod_{j=1}^{r} \frac{\mathrm{Nm}_{\mathbb{Q}}^{F}(\mathfrak{p}_j)}{\log\left(\mathrm{Nm}_{\mathbb{Q}}^{F}(\mathfrak{p}_j)\right)} > \frac{r!}{\mathcal{C}_{20}^{r}} \geq \left(\frac{r}{\mathcal{C}_9 e}\right)^r \geq \left(\frac{r}{\mathcal{C}_7}\right)^r,$$

which proves the claim in the lemma. □

We finally state a lemma that we will use to tidy our end arguments.

Lemma 3.1.0.5. *If $x,\ a \in \mathbb{R},\ a \neq e$, with $\frac{a}{\log a} < x$, then $a < \max\{e,\ 2x\log x\}$.*

Proof. For $a < e$ the statement is automatically true, so we consider only $a > e$. If $\frac{a}{\log a} < x$, then

$$a < x \log a. \tag{3.3}$$

As $\frac{a}{\log a} < x$, we see that $\log a - \log\log a < \log x$. Further, as $\log a < \sqrt{a}$, we can show that $\frac{\log a}{2} < \log a - \log\log a$.

Combining these we get that

$$\log a < 2\log x. \tag{3.4}$$

Multiplying together (3.3) and (3.4) and cancelling $\log a$, we obtain that $a < 2x\log x$. □

3.2 PROOF OF THE MAIN THEOREM

Let K be a number field with ring of integers $\mathcal{O}_K$, class number h_K, and Hilbert Class Field L. Let $[K : \mathbb{Q}] = d$, so by the tower law, $[L : \mathbb{Q}] = h_K d$.

Take $a,\ b,\ c := -a - b \in \mathcal{O}_K$ so that

$$a + b + c = 0, \tag{3.5}$$

with the assumption that $a\mathcal{O}_K$, $b\mathcal{O}_K$ and $c\mathcal{O}_K$ are coprime. We write

$$\begin{aligned} a\mathcal{O}_K &= \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_t^{e_t} \\ b\mathcal{O}_K &= \mathfrak{q}_1^{f_1} \cdots \mathfrak{q}_u^{f_u} \\ c\mathcal{O}_K &= \mathfrak{r}_1^{g_1} \cdots \mathfrak{r}_v^{g_v}, \end{aligned} \tag{3.6}$$

where $\mathfrak{p}_i,\ \mathfrak{q}_j,$ and $\mathfrak{r}_k$ are prime ideals of $\mathcal{O}_K$ and $e_i,\ f_j,\ g_k$ are integers.

A key property of $L = HCF(K)$ is that every ideal $\mathfrak{I}$ of $\mathcal{O}_K$ is principal in $\mathcal{O}_L$; that is $\mathfrak{I}\mathcal{O}_L = \alpha\mathcal{O}_L$ for some $\alpha \in \mathcal{O}_L$. By Lemma 3.1.0.2, we can pick the generator of each ideal so that if α is the generator, then

$$h(\alpha) \leq \log \overline{|\alpha|} \leq \log\left(\mathcal{C}_{10}(K)\left|\mathrm{N}_{L/\mathbb{Q}}(\alpha)\right|^{1/d}\right) = \mathcal{C}_{11}(K)\log\left|\mathrm{N}_{L/\mathbb{Q}}(\alpha)\right|. \tag{3.7}$$

We note that the dependence of the constants is on K rather than L, as L is uniquely determined by K. Further, for such algebraic α we have

$$\log\left|\mathrm{N}_{L/\mathbb{Q}}(\alpha)\right| \leq dh(\alpha),$$

giving us that

$$\mathcal{C}_{12}(K)\log\left|\mathrm{N}_{L/\mathbb{Q}}(\alpha)\right| \leq h(\alpha) \leq \mathcal{C}_{13}(K)\log\left|\mathrm{N}_{L/\mathbb{Q}}(\alpha)\right|. \tag{3.8}$$

Recalling this, we write

$$\begin{aligned}
\mathfrak{p}_i\mathcal{O}_L &= a_i\mathcal{O}_L \\
\mathfrak{q}_j\mathcal{O}_L &= b_j\mathcal{O}_L \\
\mathfrak{r}_k\mathcal{O}_L &= c_k\mathcal{O}_L,
\end{aligned} \tag{3.9}$$

where a_i, b_j, c_k satisfy (3.7).

We can also write $a\mathcal{O}_L$, $b\mathcal{O}_L$ and $c\mathcal{O}_L$ as a product of prime ideals of $\mathcal{O}_L$. Knowing this, we will write

$$G = \prod_{\substack{\mathfrak{P} \text{ prime ideal} \\ \mathfrak{P} \subset \mathcal{O}_L \\ \mathfrak{P} \mid (abc)\mathcal{O}_L}} \mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{P}). \tag{3.10}$$

Note that by our assumptions this is equivalent to taking the field to be L in equation (1.7) of [34]. Further we will denote the prime ideal of $\mathcal{O}_L$ of largest norm dividing $a\mathcal{O}_L$ by $\mathfrak{p}_a$, and similarly for b and c.

From (3.9) we can write a, b, c as follows:

$$\begin{aligned}
a &= u_a a_1^{e_1} \cdots a_t^{e_t} \\
b &= u_b b_1^{f_1} \cdots b_u^{f_u} \\
c &= u_c c_1^{g_1} \cdots c_v^{g_v}
\end{aligned} \tag{3.11}$$

where u_a, u_b and u_c are units of $\mathcal{O}_L$. We will also often write $u_a a' + u_b b' + u_c c' = 0$ where $a' = \prod_{i=1}^{s} a_i$ and similarly for b' and c'. After relabeling if needed, we can assume that (2.1) holds. We note that if $h\left(c'\right) \leq 1$ then straightforwardly the claim holds. This is because necessarily $h(a') \leq h(b') \leq h(c') \leq 1$ and Theorem 2.1.3.1 readily follows from (3.2). Similarly, after some work we do below finding a bound on $h(c')$, we see that if $h(b') \leq 1$ then again we'll find the claim straightforwardly follows, and the same logic will hold if $h(a') \leq 1$. Thus we assume in the following that

$$1 < h(a') \leq h(b') \leq h(c'). \tag{3.12}$$

Dividing through by $u_c c' = c$ in (3.5) we obtain that

$$-\frac{u_a a'}{u_c c'} - \frac{u_b b'}{u_c c'} = 1, \tag{3.13}$$

so we are in a position to apply Lemma 2.1.2.2.

Before doing this, we note that by the remarks around (3.1), $\log H_L\left(a,\, b,\, c\right) = \log H_L\left(\frac{a}{c},\, \frac{b}{c},\, 1\right)$. This allows us to move between representatives of the projective point $[a : b : c] \in \mathbb{P}^2\left(L\right)$.

First, following the notation from Lemma 2.1.2.2, initially take S to be the set of infinite places of L. Applying this to (3.13), we obtain that

$$\begin{aligned} \max\left\{h\left(-\frac{u_a}{u_c}\right),\, h\left(-\frac{u_b}{u_c}\right)\right\} &= \max\left\{h\left(\frac{u_a}{u_c}\right),\, h\left(\frac{u_b}{u_c}\right)\right\} \\ &\leq \mathcal{C}_{14}\left(K\right) \max\left\{h\left(\frac{a'}{c'}\right),\, h\left(\frac{b'}{c'}\right),\, 1\right\}. \end{aligned} \tag{3.14}$$

We note similar bounds also hold if we divide (3.5) through by $u_a a'$ or $u_b b'$. Recall that $h\left(xy\right) \leq h\left(x\right) + h\left(y\right)$ [6][56]. By this fact and our assumptions on $h(a')$, $h(b')$ and $h(c')$, we deduce that

$$\max\left\{h\left(\frac{a'}{c'}\right),\, h\left(\frac{b'}{c'}\right),\, 1\right\} \leq \max\left\{h\left(a'\right) + h\left(c'\right),\, h\left(b'\right) + h\left(c'\right)\right\} \leq 2h\left(c'\right).$$

It thus follows that

$$\begin{aligned} h\left(\frac{u_a a'}{u_c c'}\right) &\leq h\left(\frac{u_a}{u_c}\right) + h\left(a'\right) + h\left(c'\right) \\ &\leq \mathcal{C}_{15} h\left(c'\right) + 2h\left(c'\right) \\ &= \mathcal{C}_{16} h\left(c'\right). \end{aligned} \tag{3.15}$$

Similarly, we also obtain that

$$h\left(\frac{u_b b'}{u_c c'}\right) \leq \mathcal{C}_{17} h\left(c'\right). \tag{3.16}$$

We again apply Lemma 2.1.2.2, but this time with a different set of S-units. Now let $S = S_\infty \cup \{\mathfrak{p} : \mathfrak{p} \mid c\mathcal{O}_L\}$ where $\mathfrak{p}$ refers both to the prime ideal $\mathfrak{p}$ and its corresponding place. Applying Lemma 2.1.2.2 in this case to (3.13), we obtain that

$$\begin{aligned}
\max\left\{h\left(\frac{u_a}{u_c c'}\right), h\left(\frac{u_b}{u_c c'}\right)\right\} &\leq \mathcal{C}_{18}\left(K\right)\left(2^{13.32}\mathcal{R}d\right)^t \left(\frac{\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)}{\log^* \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)}\right) \\
&\quad \left(\prod_{\substack{\mathfrak{p}\in\mathcal{O}_L \\ \mathfrak{p}\mid c\mathcal{O}_L}} \log \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right)\right) \max\left\{h\left(a'\right), h\left(b'\right), 1\right\} \\
&\leq \mathcal{C}_{18}\left(K\right)\left(2^{13.32}\mathcal{R}d\right)^t \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right) \\
&\quad \left(\prod_{\substack{\mathfrak{p}\in\mathcal{O}_L \\ \mathfrak{p}\mid c\mathcal{O}_L \\ \mathfrak{p}\neq\mathfrak{p}_c}} \log \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right)\right) h\left(b'\right).
\end{aligned}$$

We note that, to make the argument easier to follow, we have brought out the parts of the constant from Lemma 2.1.2.2 that depend on t. Again, L is defined by K uniquely, hence the dependence of the constants here on K rather than L. Further, by (3.12), we have that $\max\left\{h\left(a'\right), h\left(b'\right), 1\right\} = h\left(b'\right)$. We further note that by (4.21) of [20] and the arguments that follow in that paper, we have that

$$\left(2^{13.32}\mathcal{R}d\right)^t \leq \left(2^{13.32}d\right)^{t^2} \mathcal{R}^t \leq G^{\frac{\mathcal{C}_{19}(K)}{\log\log G}}.$$

Applying this to the above we obtain that

$$\max\left\{h\left(\frac{u_a}{u_c c'}\right), h\left(\frac{u_b}{u_c c'}\right)\right\} \leq \mathcal{C}_{18}\left(K\right) G^{\frac{\mathcal{C}_{19}(K)}{\log\log G}} \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right) \left(\prod_{\substack{\mathfrak{p}\in\mathcal{O}_L \\ \mathfrak{p}\mid c\mathcal{O}_L \\ \mathfrak{p}\neq\mathfrak{p}_c}} \log \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right)\right) h\left(b'\right). \tag{3.17}$$

From (3.17) we obtain that

$$\begin{aligned} h\left(\frac{u_a a'}{u_c c'}\right) &\leq h\left(\frac{u_a}{u_c c'}\right) + h\left(a'\right) \\ &\leq \mathcal{C}_{18}\left(K\right) G^{\frac{\mathcal{C}_{19}(K)}{\log\log G}} \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right) \left(\prod_{\substack{\mathfrak{p}\in\mathcal{O}_L \\ \mathfrak{p}\mid c\mathcal{O}_L \\ \mathfrak{p}\neq\mathfrak{p}_c}} \log \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right) \right) h\left(b'\right) + h\left(a'\right) \\ &\leq \mathcal{C}_{20}\left(K\right) G^{\frac{\mathcal{C}_{19}(K)}{\log\log G}} \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right) \left(\prod_{\substack{\mathfrak{p}\in\mathcal{O}_L \\ \mathfrak{p}\mid c\mathcal{O}_L \\ \mathfrak{p}\neq\mathfrak{p}_c}} \log \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right) \right) h(b'). \end{aligned} \tag{3.18}$$

Similarly, we find that

$$h\left(\frac{u_b b'}{u_c c'}\right) \leq \mathcal{C}_{21}\left(K\right) G^{\frac{\mathcal{C}_{22}(K)}{\log\log G}} \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right) \left(\prod_{\substack{\mathfrak{p}\in\mathcal{O}_L \\ \mathfrak{p}\mid c\mathcal{O}_L \\ \mathfrak{p}\neq\mathfrak{p}_c}} \log \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right) \right) h(b'). \tag{3.19}$$

We now choose another set S, this time containing the infinite places and the finite places corresponding to the prime ideals dividing $bc\mathcal{O}_L$; that is, $S = S_\infty \cup \{\mathfrak{p} : \mathfrak{p} \mid bc\mathcal{O}_L\}$. Applying Lemma 2.1.2.2 to (3.13) with this S we obtain that

$$\begin{aligned} \max\left\{ h\left(\frac{u_a}{u_c c'}\right), h\left(\frac{u_b b'}{u_c c'}\right) \right\} &\leq \mathcal{C}_{23}\left(K\right) G^{\frac{\mathcal{C}_{24}(K)}{\log\log G}} \max\left\{ \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right), \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right) \right\} \\ &\qquad \left(\prod_{\substack{\mathfrak{p}\in\mathcal{O}_L \\ \mathfrak{p}\mid bc\mathcal{O}_L}} \log \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right) \right) \max\left\{ h\left(a'\right), 1 \right\}. \end{aligned} \tag{3.20}$$

From (3.12) and (3.20) we obtain that

$$\begin{aligned} h\left(\frac{u_a a'}{u_c c'}\right) &\leq h\left(\frac{u_a}{u_c c'}\right) + h\left(a'\right) \\ &\leq \mathcal{C}_{25}\left(K\right) G^{\frac{\mathcal{C}_{26}(K)}{\log\log G}} \max\left\{\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right), \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)\right\} \cdot \qquad (3.21) \\ &\cdot \left(\prod_{\substack{\mathfrak{p}\in\mathcal{O}_L \\ \mathfrak{p}\mid bc\mathcal{O}_L}} \log \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right)\right) h\left(a'\right) + h\left(a'\right) \\ &\leq \left(\mathcal{C}_{27}\left(K\right) G^{\frac{\mathcal{C}_{26}(K)}{\log\log G}} \max\left\{\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right), \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)\right\}\right) \cdot \qquad (3.22) \\ &\cdot \left(\left(\prod_{\substack{\mathfrak{p}\in\mathcal{O}_L \\ \mathfrak{p}\mid bc\mathcal{O}_L}} \log \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right)\right)\right) h\left(a'\right). \qquad (3.23) \end{aligned}$$

Similarly, we find that

$$h\left(\frac{u_b b'}{u_c c'}\right) \leq \mathcal{C}_{28} G^{\frac{\mathcal{C}_{29}(K)}{\log\log G}} \max\left\{\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right), \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)\right\} \left(\prod_{\substack{\mathfrak{p}\in\mathcal{O}_L \\ \mathfrak{p}\mid bc\mathcal{O}_L}} \log \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right)\right) h(a'). \qquad (3.24)$$

By consideration of (3.15), (3.16), we see that

$$\begin{aligned} \max\left\{h\left(\frac{u_a a'}{u_c c'}\right), h\left(\frac{u_b b'}{u_c c'}\right)\right\} &< \max\left\{\mathcal{C}_{16}, \mathcal{C}_{17}\right\} h\left(c'\right) \\ &= \mathcal{C}_{30} h\left(c'\right), \qquad (3.25) \end{aligned}$$

while (3.18) and (3.19) show that

$$\max\left\{h\left(\frac{u_a a'}{u_c c'}\right),\, h\left(\frac{u_b b'}{u_c c'}\right)\right\} < \max\left\{\mathcal{C}_{20}, \mathcal{C}_{21}\right\} G^{\frac{\max\{\mathcal{C}_{22}(K),\, \mathcal{C}_{24}(K)\}}{\log\log G}} \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right) \cdot \tag{3.26}$$

$$\left(\prod_{\substack{\mathfrak{p}\in\mathcal{O}_L\\ \mathfrak{p}|c\mathcal{O}_L\\ \mathfrak{p}\neq\mathfrak{p}_c}} \log \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right)\right) h(b')$$

$$= \mathcal{C}_{31} G^{\frac{\mathcal{C}_{32}(K)}{\log\log G}} \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right) \left(\prod_{\substack{\mathfrak{p}\in\mathcal{O}_L\\ \mathfrak{p}|c\mathcal{O}_L\\ \mathfrak{p}\neq\mathfrak{p}_c}} \log \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right)\right) h(b'). \tag{3.27}$$

In the same way, it follows from (3.21) and (3.24) that

$$\max\left\{h\left(\frac{u_a a'}{u_c c'}\right),\, h\left(\frac{u_b b'}{u_c c'}\right)\right\} < \max\left\{\mathcal{C}_{27}, \mathcal{C}_{28}\right\} G^{\frac{\max\{\mathcal{C}_{26}(K),\, \mathcal{C}_{29}(K)\}}{\log\log G}} \cdot$$

$$\cdot \max\left\{\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right),\, \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)\right\} \left(\prod_{\substack{\mathfrak{p}\in\mathcal{O}_L\\ \mathfrak{p}|bc\mathcal{O}_L}} \log \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right)\right) h(a')$$

$$= \mathcal{C}_{33} G^{\frac{\mathcal{C}_{34}(K)}{\log\log G}} \max\left\{\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right),\, \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)\right\} \cdot$$

$$\cdot \left(\prod_{\substack{\mathfrak{p}\in\mathcal{O}_L\\ \mathfrak{p}|bc\mathcal{O}_L}} \log \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right)\right) h(a'). \tag{3.28}$$

We next prove the following lemma.

Lemma 3.2.0.1. *Let $\alpha \in \{a,\, b,\, c\}$. Then*

$$h\left(\alpha'\right) \leq \mathcal{C}_{35}\left(\max_{\mathfrak{p}|\langle\alpha\rangle_L} \mathit{ord}_{\mathfrak{p}}\left(\alpha\right)\right) \log G.$$

Proof. If α is a unit, then we define $\max \mathrm{ord}_{\mathfrak{p}}\left(\alpha\right) := 1$.

By construction of α', we have that $h\left(\alpha'\right) \leq \mathcal{C}_{36} \log \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\alpha\mathcal{O}_L\right)$. This follows from the fact that

$$\left|\mathrm{N}_{L/\mathbb{Q}}\left(\alpha'\right)\right| = \left|\mathrm{N}_{L/\mathbb{Q}}\left(\alpha\right)\right| = \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\alpha\mathcal{O}_L\right).$$

We write a factorisation of $\alpha\mathcal{O}_L = \mathfrak{P}_{1,L}^{g_{1,L}} \cdots \mathfrak{P}_{u_L,L}^{g_{u_L,L}}$ into prime ideals of $\mathcal{O}_L$. Note this may be different to the ideals given in (3.9), as the ideals in (3.9) may not be prime. Working with this prime factorisation, we obtain that

$$\begin{aligned}\log \mathrm{Nm}_{\mathbb{Q}}^{L}(\alpha\mathcal{O}_L) &= \log\left(\prod_{i=1}^{u_L} \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_{i,L}^{u_{i,L}}\right)\right)\\ &= \sum_{i=1}^{u_L} u_{i,L}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_{i,L})\right)\\ &\le \left(\max_{\mathfrak{p}\mid\langle\alpha\rangle_L} \mathrm{ord}_{\mathfrak{p}}(\alpha)\right)\log G.\end{aligned}$$

The claim then follows. □

It follows immediately from (3.2), (3.25) and Lemma 3.2.0.1 that

$$\frac{\log H_L(a,\, b,\, c)}{\mathcal{C}_{30}\log G} \le \max_{\mathfrak{p}\mid\langle c\rangle_L} \mathrm{ord}_{\mathfrak{p}}(c)\,. \tag{3.29}$$

Similarly, it follows from (3.2), (3.26) and Lemma 3.2.0.1 that

$$\frac{\log H_L(a,\, b,\, c)}{\mathcal{C}_{31}G^{\frac{\mathcal{C}_{32}(K)}{\log\log G}}\log G\,\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c)\left(\prod_{\substack{\mathfrak{p}\in\mathcal{O}_L\\ \mathfrak{p}\mid c\mathcal{O}_L\\ \mathfrak{p}\neq\mathfrak{p}_c}}\log \mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p})\right)} \le \max_{\mathfrak{p}\mid\langle b\rangle_L} \mathrm{ord}_{\mathfrak{p}}(b)\,. \tag{3.30}$$

Further, it follows from (3.2), (3.28) and Lemma 3.2.0.1 that

$$\begin{aligned}&\frac{\log H_L(a,\, b,\, c)}{\mathcal{C}_{33}G^{\frac{\mathcal{C}_{34}(K)}{\log\log G}}\log G\,\max\left\{\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_b)\,,\, \mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c)\right\}\left(\prod_{\substack{\mathfrak{p}\in\mathcal{O}_L\\ \mathfrak{p}\mid bc\mathcal{O}_L}}\log \mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p})\right)}\\ &\le \max_{\mathfrak{p}\mid\langle b\rangle_L} \mathrm{ord}_{\mathfrak{p}}(a)\,.\end{aligned} \tag{3.31}$$

We will use Lemma 3.1.0.3 to establish upper bounds for the right-hand sides of (3.29), (3.30) and (3.31). In order to do this we need to write $\mathrm{ord}_{\mathfrak{p}}(c)$, $\mathrm{ord}_{\mathfrak{p}}(b)$ and $\mathrm{ord}_{\mathfrak{p}}(a)$ in a form where we're able to use Lemma 3.1.0.3.

By the coprimeness of $a\mathcal{O}_L$, $b\mathcal{O}_L$ and $c\mathcal{O}_L$ we see that

$$\begin{aligned}\mathrm{ord}_{\mathfrak{p}}(c) &= \mathrm{ord}_{\mathfrak{p}}\left(\frac{c}{b}\right)\\ &= \mathrm{ord}_{\mathfrak{p}}\left(\frac{-a-b}{b}\right)\\ &= \mathrm{ord}_{\mathfrak{p}}\left(-\frac{a}{b}-1\right)\\ &= \mathrm{ord}_{\mathfrak{p}}\left(-\frac{u_a}{u_b}a_1^{e_1}\cdots a_t^{e_t}b_1^{f_1}\ldots b_u^{f_u}-1\right).\end{aligned} \tag{3.32}$$

Similarly we find that

$$\operatorname{ord}_{\mathfrak{p}}(b) = \operatorname{ord}_{\mathfrak{p}}\left(-\frac{u_c}{u_a}c_1^{g_1}\cdots c_v^{g_v}a_1^{-e_1}\dots a_t^{-e_t}-1\right), \tag{3.33}$$

and

$$\operatorname{ord}_{\mathfrak{p}}(a) = \operatorname{ord}_{\mathfrak{p}}\left(-\frac{u_b}{u_c}b_1^{f_1}\cdots b_u^{f_u}c_1^{-g_1}\dots c_v^{-g_v}-1\right). \tag{3.34}$$

To apply Lemma 3.1.0.3 we need to bound exponents e_i, f_j, g_k and bound the heights of the units $h(u_a)$, $h(u_b)$ and $h(u_c)$.

First note that

$$\max\left\{\operatorname{ord}_{\mathfrak{p}}(a), \operatorname{ord}_{\mathfrak{p}}(b), \operatorname{ord}_{\mathfrak{p}}(c)\right\} \le \log H_L(a, b, c), \tag{3.35}$$

directly from the definition of H_L.

This follows from the definitions of projective and absolute logarithmic heights.

In order to use Yu's bound, we need to manage the heights of u_a, u_b and u_c. To do this, we will use fundamental units of $\mathcal{O}_L^*$. By Dirichlet's Unit Theorem, there exist fundamental units $\xi_1, \dots, \xi_r$ of $\mathcal{O}_L$, where r is the unit rank of $\mathcal{O}_L$ such that all units u of $\mathcal{O}_L$ can be written $u = \mu\xi_1^{\delta_1}\cdots\xi_r^{\delta_r}$, with μ a root of unity. We note again that finding a set of fundamental units is computable, for example see [9], and we can find a nice system satisfying (3.1.0.1). Indeed, we could use any system of fundamental units, but this choice is helpful should one wish to explicitly compute the constants. Thus once found, the product $\prod_{i=1}^r h'(\xi_i)$ we will obtain applying Lemma 3.1.0.3 can be upper bounded constants depending on the field. It remains to find an upper bound for $\max_i \delta_i$.

Note that (3.14) gives us that

$$h\left(\frac{u_a}{u_c}\right) \le \mathcal{C}_{37}h(c').$$

Further,

$$h(c') \le \mathcal{C}_{38}(K)\log\left|\mathrm{N}_{L/\mathbb{Q}}(c)\right| \le \mathcal{C}_{39}(K)\,h(c) \le \mathcal{C}_{40}(K)\log H_L(a, b, c).$$

It follows from the above comments that

$$h\left(\frac{u_a}{u_c}\right) \le \mathcal{C}_{41}(K)\log H_L(a, b, c). \tag{3.36}$$

Let L have r_1 real embeddings $\epsilon_1, \ldots, \epsilon_{r_1}$ and $2r_2$ complex embeddings $\epsilon_{r_1+1}, \overline{\epsilon_{r_1+1}}, \ldots, \epsilon_{r_2}, \overline{\epsilon_{r_2}}$. By Dirichlet's Unit Theorem, there are $r := r_1 + r_2 - 1$ fundamental units $\xi_1, \ldots, \xi_r$, such that for any unit $u \in \mathcal{O}_L^*$, $u = \mu\xi_1^{\delta_1} \cdots \xi_r^{\delta_r}$ where μ is a root of unity in L and $\delta_i \in \mathbb{Z}$ for all i.

We now prove a lemma that gives an upper bound for $\max_i |\delta_i|$.

Lemma 3.2.0.2. *Given the set up above,*

$$\max_i \{|\delta_i|, 3\} \leq \mathcal{C}_{42}(K) \log H_L(a, b, c). \tag{3.37}$$

Proof. We consider the unit $u = \frac{u_a}{u_c}$, but this choice is arbitrary and the logic that follows holds for all relevant quotients of the units u_a, u_b and u_c. Write, as we do above, $u = \mu\xi_1^{\delta_1} \cdots \xi_r^{\delta_r}$ where μ is a root of unity in L and $\delta_i \in \mathbb{Z}$ for all i. As shown in the proof of inequality 4.3.2 in [17], it is shown that

$$\max\{3, |\delta_1|, \ldots, |\delta_r|\} \leq \mathcal{C}_{43}(L)h(u).$$

As L depends only on K, the dependency in the constant is really only on K. We remark that in the notation of [17], S consists only of the infinite places of the field L.

We have shown above at (3.36) that

$$h(u) = h\left(\frac{u_a}{u_c}\right) \leq \mathcal{C}_{41}(K) \log H_L(a, b, c).$$

Combining these inequalities gives the result.

□

Out of interest, we give an alternative proof, which gives us one way to see where inequality 4.3.2 of [17] comes from.

Second proof of Lemma 3.2.0.2. Recall that there are $r+1$ distinct embeddings of L into $\mathbb{C}$. Let us denote these embeddings by e_i, $i = 1 \ldots, r+1$. Furthermore for all $i = 1 \ldots, r+1$, let us define

$$\begin{aligned} \varepsilon_i : L &\to \mathbb{R} \\ \alpha &\to \log|e_i(\alpha)|. \end{aligned}$$

Let u be an element of L. We see that

$$h(u) = \frac{1}{2} \sum_{i=1}^{r+1} N_i |\varepsilon_i(u)|,$$

where $N_i = 1$ if the image of e_i is a subset of $\mathbb{R}$, and $N_i = 2$ in the complimentary case, when the image of e_i is not a subset of $\mathbb{R}$. This is because

$$h(u) = \sum_{i=1}^{r+1} N_i \log\max\left(|e_i(u), 1|\right) = -\sum_{i=1}^{r+1} N_i \log\min\left(|e_i(u), 1|\right),$$

where N_i is defined as above. Thus

$$\begin{aligned} 2h(u) &= \sum_{i=1}^{r+1} N_i \log\max\left(|e_i(u), 1|\right) - \sum_{i=1}^{r+1} N_i \log\min\left(|e_i(u), 1|\right) \\ &= \sum_{i=1}^{r+1} N_i \left|\log|e_i(u)|\right| \\ &= \sum_{i=1}^{r+1} N_i \left|\varepsilon_i(u)\right|, \end{aligned}$$

and thus the identity follows.

We now take advantage of some properties of u_a, u_b and u_c so we will write them explicitly. In what follows we use the case $u = \frac{u_a}{u_c}$, but it is true for the other relevant quotients of u_a, u_b and u_c. Write $\frac{u_a}{u_c} = \mu\xi_1^{\delta_1}\cdots\xi_r^{\delta_r}$ where μ is a root of unity. From the comments above it follows that

$$\begin{aligned} h\left(\frac{u_a}{u_c}\right) &= \frac{1}{2}\sum_{i=1}^{r+1} N_i \left|\varepsilon_i\left(\mu\prod_{j=1}^{r}\xi_j^{\delta_j}\right)\right| \\ &= \frac{1}{2}\sum_{i=1}^{r+1} N_i \left|\sum_{j=1}^{r}\delta_j\varepsilon_i(\xi_j)\right|, \end{aligned} \tag{3.38}$$

where we lose the μ as it is a root of unity, so for all i, $\varepsilon_i(\mu) = 0$.

From (3.36), we know that $h\left(\frac{u_a}{u_c}\right) \leq \mathcal{C}_{41}\log H_L(a,\, b,\, c)$, giving us an upper bound for the absolute logarithmic height of the unit. Along with (3.38), this implies that, for all $i = 1, \ldots, r+1$, we have

$$\left|\sum_{j=1}^{r}\delta_j\varepsilon_i(\xi_j)\right| < \mathcal{C}_{44}(K)\log H_L(a,\, b,\, c). \tag{3.39}$$

That is, (3.36) implies that all the exponents δ_j, $j = 1, \ldots, r$ satisfy (3.39). This holds for all ε_i, $i = 1, \ldots,\, r+1$, so (3.39) gives us a system of $r+1$ inequalities.

We pick any r inequalities of $r+1$ in the system (3.39). For the sake of concreteness, let us take the first r inequalities. We are going to deduce the upper bound for the system of inequalities (3.39) where $i = 1, \ldots, r$. Note that the left-hand side of these inequalities are coordinates of the vector

$$M \begin{pmatrix} \delta_1 \\ \vdots \\ \delta_r \end{pmatrix},$$

where the matrix M is defined by

$$M := (\varepsilon_i(\xi_j))_{1 \le i,j \le r}.$$

By definition, the absolute value of determinant of M is equal to the regulator of the number field L, and is thus non-zero. Hence M is non-degenerate. Importantly, the matrix M depends only on the number field L. Further, as the value of the regulator is independent of the choice of the r inequalities we picked, it shows that our choice of inequalities is irrelevant and we obtain the same result given a different choice of r inequalities from the $r+1$ in (3.39).

It follows that the solutions to the system of inequalities (3.39) for $i = 1, \ldots, r$ are given by $M^{-1}\mathcal{B}$, where $\mathcal{B}$ is an r-dimensional cube
$[-\mathcal{C}_{44}(K) \log H_L(a,\, b,\, c), \mathcal{C}_{44}(K) \log H_L(a,\, b,\, c)]^r$. Thus these solutions form a parallepiped, the form of which depends on M (hence eventually on L only, which is uniquely determined by K) and the linear size is given by $\mathcal{C}_{45}(K) \log H_L(a,\, b,\, c)$. This means that the solutions δ_i have an upper bound of the form

$$\mathcal{C}_{46}\mathcal{C}_{45} \log H(a,\, b,\, c)$$

, where the constant $\mathcal{C}_{46}$ depends on M (hence actually depends on K) only. We thus conclude that

$$\max_i |\delta_i| \le \mathcal{C}_{47}(K) \log H_L(a,\, b,\, c),$$

as claimed. □

Importantly, as commented during the proof, the method is not changed if we choose a different unit such as $\frac{u_a}{u_b}$ and so on. Thus this lemma holds for all relevant units in this paper.

We return to considering (3.32). We can now use the above after writing the unit in terms of fundamental units as follows:

$$\begin{aligned}\operatorname{ord}_{\mathfrak{p}}(c) &= \operatorname{ord}_{\mathfrak{p}}\left(-\frac{u_a}{u_b}a_1^{e_1}\cdots a_t^{e_t}b_1^{f_1}\ldots b_u^{f_u}-1\right)\\ &= \operatorname{ord}_{\mathfrak{p}}\left(\mu\xi_1^{\delta_1}\cdots\xi_r^{\delta_r}a_1^{e_1}\cdots a_t^{e_t}b_1^{f_1}\cdots b_u^{f_u}-1\right)\end{aligned}\tag{3.40}$$

where μ is a root of unity. We are in a position to apply Lemma 3.1.0.3. Using the notation of Lemma 3.1.0.3, from (3.35) and Lemma 3.2.0.2 we obtain that $\log B \leq \mathcal{C}_{48}(K)\log\log H_L(a,\, b,\, c)$.

Applying Lemma 3.1.0.3 on (3.40), we obtain that

$$\begin{aligned}\operatorname{ord}_{\mathfrak{p}}(c) <&\mathcal{C}_{49}(K)^{r+t+u+2}(r+t+u+1)^{5/2}\log\left(2d\left(r+t+u+1\right)\right)\frac{\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c)}{\left(\log \mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c)\right)^2}\\ &h'(\mu)h'(\xi_1)\cdots h'(\xi_r)h'(a_1)\cdots h'(a_t)h'(b_1)\cdots h'(b_u)\log\log H_L(a,\, b,\, c).\end{aligned}\tag{3.41}$$

Note that $h(\mu)=0$ as μ is a root of unity so $h'(\mu)=\frac{1}{16e^2d^2}$, which we take into the constant. Further, we recall our system of fundamental units satisfies (3.1.0.1) so we take $\prod_{i=1}^{r}h'(\xi_i)$ into the constant.

We further note that $\left|\mathrm{N}_{L/\mathbb{Q}}(a_i)\right| = \mathrm{Nm}_{\mathbb{Q}}^{L}(\langle a_i\rangle) = \left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{p}_i)\right)^{f_K}$, and similarly for b_j and c_k [29]. We recall that by definition, $f_K \leq d$ [38]. Further, the norms of all these prime ideals are greater than 1, so for all $x \in \{a_1,\ldots,a_s,\, b_1,\ldots,b_t\}$, if $\mathfrak{a}_x$ is the prime ideal associated with x, $h'(x)\leq \mathcal{C}_{50}(K)\log \mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{a}_x)$. Putting this together with the inequality above, with other bounds used as necessary, we obtain that

$$\begin{aligned}\operatorname{ord}_{\mathfrak{p}}(c) \leq&\mathcal{C}_{51}(K)^{t+u}(r+t+u+1)^{7/2}\log\log H_L(a,\, b,\, c)\\ &\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c)\prod_{i=1}^{t}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{p}_i)\right)\cdot\prod_{j=1}^{u}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{q}_j)\right).\end{aligned}\tag{3.42}$$

Similarly, we see that by considering (3.33) in the same way as above, we obtain that

$$\begin{aligned}\operatorname{ord}_{\mathfrak{p}}(b) &= \operatorname{ord}_{\mathfrak{p}}\left(-\frac{u_c}{u_a}c_1^{g_1}\cdots c_v^{g_v}a_1^{-e_1}\ldots a_t^{-e_t}-1\right)\\ &= \operatorname{ord}_{\mathfrak{p}}\left(\mu'\xi_1^{\delta_1'}\cdots\xi_r^{\delta_r'}c_1^{g_1}\cdots c_v^{g_v}a_1^{-e_1}\ldots a_t^{-e_t}-1\right).\end{aligned}\tag{3.43}$$

Following the same line of reasoning as above we obtain that

$$\begin{aligned}\operatorname{ord}_{\mathfrak{p}}(b) \leq&\mathcal{C}_{52}(K)^{t+v}(r+t+v+1)^{7/2}\log\log H_L(a,\, b,\, c)\\ &\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_b)\prod_{i=1}^{t}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{p}_i)\right)\cdot\prod_{j=1}^{v}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{r}_j)\right).\end{aligned}\tag{3.44}$$

In the same way, by considering (3.34) we further obtain that

$$\begin{aligned}\operatorname{ord}_{\mathfrak{p}}(a) &= \operatorname{ord}_{\mathfrak{p}}\left(-\frac{u_b}{u_c} b_1^{f_1}\cdots b_u^{f_u} c_1^{-g_1}\dots c_v^{-g_v} - 1\right) \\ &= \operatorname{ord}_{\mathfrak{p}}\left(\mu'' \xi_1^{\delta_1'}\cdots \xi_r^{\delta_r'} b_1^{f_1}\cdots b_u^{f_u} c_1^{-g_1}\dots c_v^{-g_v} - 1\right), \end{aligned} \tag{3.45}$$

and as before, applying Lemma 3.1.0.3 gives us that

$$\begin{aligned}\operatorname{ord}_{\mathfrak{p}}(a) \leq &\mathcal{C}_{53}(K)^{u+v}(r+u+v+1)^{7/2}\log\log H_L(a,\, b,\, c) \\ &\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_a)\prod_{i=1}^{u}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{q}_i)\right)\cdot\prod_{j=1}^{v}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{r}_j)\right). \end{aligned} \tag{3.46}$$

From this point, all constants depend on the field K, in particular on the degree of the field d, so we omit these dependencies.

By combining (3.29) and (3.42) we obtain that

$$\begin{aligned}\frac{\log H_L(a,\, b,\, c)}{\log\log H_L(a,\, b,\, c)} < &\mathcal{C}_{54}^{t+u}(r+t+u+1)^{7/2}\log G\cdot\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c) \\ &\prod_{i=1}^{t}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{p}_i)\right)\cdot\prod_{j=1}^{u}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{q}_j)\right). \end{aligned} \tag{3.47}$$

Similarly, combining (3.30) and (3.44) gives us that

$$\begin{aligned}\frac{\log H_L(a,\, b,\, c)}{\log\log H_L(a,\, b,\, c)} < &\mathcal{C}_{55}^{t+v} G^{\frac{\mathcal{C}_{32}(K)}{\log\log G}}(r+t+v+1)^{7/2}\log G\cdot\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_b)\cdot\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c) \\ &\prod_{i=1}^{t}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{p}_i)\right)\cdot\prod_{j=1}^{v}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{r}_j)\right)\cdot\prod_{\substack{\mathfrak{p}\in\mathcal{O}_L \\ \mathfrak{p}\mid c\mathcal{O}_L \\ \mathfrak{p}\neq\mathfrak{p}_c}}\log\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}). \end{aligned} \tag{3.48}$$

Applying the same idea, combining (3.31) and (3.46) gives us that

$$\begin{aligned}\frac{\log H_L(a,\, b,\, c)}{\log\log H_L(a,\, b,\, c)} < &\mathcal{C}_{56}^{u+v} G^{\frac{\mathcal{C}_{34}(K)}{\log\log G}}(r+u+v+1)^{7/2}\log G\cdot\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_a)\cdot \\ &\cdot\max\left\{\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_b),\,\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c)\right\}\prod_{i=1}^{u}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{p}_i)\right)\cdot \\ &\cdot\prod_{j=1}^{v}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{r}_j)\right)\cdot\prod_{\substack{\mathfrak{p}\in\mathcal{O}_L \\ \mathfrak{p}\mid bc\mathcal{O}_L}}\log\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}). \end{aligned} \tag{3.49}$$

Multiplying together (3.47), (3.48) and (3.49) and bounding some terms for ease, we obtain that

$$\begin{aligned}\left(\frac{\log H_L(a,\,b,\,c)}{\log\log H_L(a,\,b,\,c)}\right)^3 <&\mathcal{C}_{57}^{t+u+v} G^{\frac{\mathcal{C}_{58}}{\log\log G}} \left(r+t+u+v\right)^{21/2} (\log G)^3 \cdot \\ &\cdot \left(\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_a\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)^2\right) \cdot \\ &\cdot \left(\max\left\{\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right),\,\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)\right\}\right)\left(\prod_{i=1}^{t}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}\right)\left(\mathfrak{p}_i\right)\right)^3 \cdot \\ &\cdot \left(\prod_{j=1}^{u}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{q}_j\right)\right)\right)^3 \cdot \left(\prod_{k=1}^{v}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{r}_k\right)\right)\right)^3 \cdot \\ &\cdot \prod_{\substack{\mathfrak{p}\in\mathcal{O}_L\\ \mathfrak{p}\mid c\mathcal{O}_L\\ \mathfrak{p}\neq\mathfrak{p}_c}} \log\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right) \cdot \prod_{\substack{\mathfrak{p}\in\mathcal{O}_L\\ \mathfrak{p}\mid bc\mathcal{O}_L}} \log\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right). \end{aligned} \tag{3.50}$$

We note that r depends on the field, so we can write $(r+(t+u+v))^{21/2} \leq \mathcal{C}_{59}\,(t+u+v)^{21/2}$. Further, for sufficiently large $\mathcal{C}_{60}$ this will absorb $(t+u+v)^{21/2}$, so we can move this into the constant. We thus obtain that

$$\begin{aligned}\left(\frac{\log H_L(a,\,b,\,c)}{\log\log H_L(a,\,b,\,c)}\right)^3 <&\mathcal{C}_{61}^{t+u+v} G^{\frac{\mathcal{C}_{58}}{\log\log G}} (\log G)^3 \cdot \\ &\cdot \left(\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_a\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right)\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)^2 \cdot \\ &\cdot \max\left\{\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right),\,\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)\right\} \cdot \left(\prod_{i=1}^{t}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{p}_i\right)\right)\right)^3 \cdot \\ &\cdot \left(\prod_{j=1}^{u}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{q}_j\right)\right)\right)^3 \cdot \left(\prod_{k=1}^{v}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{r}_k\right)\right)\right)^3 \cdot \\ &\cdot \prod_{\substack{\mathfrak{p}\in\mathcal{O}_L\\ \mathfrak{p}\mid c\mathcal{O}_L\\ \mathfrak{p}\neq\mathfrak{p}_c}} \log\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right) \cdot \prod_{\substack{\mathfrak{p}\in\mathcal{O}_L\\ \mathfrak{p}\mid bc\mathcal{O}_L}} \log\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right). \end{aligned} \tag{3.51}$$

Next we aim to deal with

$$\prod_{i=1}^{t}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{p}_i\right)\right) \cdot \prod_{j=1}^{u}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{q}_j\right)\right) \cdot \prod_{k=1}^{v}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{r}_k\right)\right).$$

First note that $\mathrm{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{P}\right)^{h_K} = \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{P}\mathcal{O}_L\right)$, where $\mathfrak{P}$ is a prime ideal of $\mathcal{O}_K$. We follow an idea from the first part of Section 3 of [52]. Let N be the number of prime ideals of $\mathcal{O}_L$ such that the prime ideal $\mathfrak{P} \mid (abc)\,\mathcal{O}_L$. By definition, these all lie above primes $\mathfrak{p}$ of $\mathcal{O}_K$, so $N \geq t+u+v$. Thus from these comments and Lemma 3.1.0.4 we obtain that

$$\left(\frac{t+u+v}{\mathcal{C}_{62}}\right)^{t+u+v} \leq \left(\frac{N}{\mathcal{C}_{63}}\right)^{N} < G, \tag{3.52}$$

where $\mathcal{C}_{63}$ is the constant given by Lemma 3.1.0.4. It follows that

$$t+u+v < \mathcal{C}_{64}\frac{\log G}{\log\log G}. \tag{3.53}$$

By the arithmetic-geometric mean inequality we obtain that

$$\begin{aligned}
&\prod_{i=1}^{t} \log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{p}_i\right)\right) \cdot \prod_{j=1}^{u} \log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{q}_j\right)\right) \cdot \prod_{k=1}^{v} \log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{r}_k\right)\right) \\
&\leq \left(\frac{1}{t+u+v}\left(\sum_{i=1}^{t} \log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{p}_i\right)\right) + \sum_{j=1}^{u} \log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{q}_j\right)\right) + \sum_{k=1}^{v} \log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{r}_k\right)\right)\right)\right)^{t+u+v} \\
&\leq \left(\frac{1}{t+u+v} \sum_{\substack{\mathfrak{P}\subset\mathcal{O}_L \\ \mathfrak{P}\text{prime} \\ \mathfrak{P}|(abc)\mathcal{O}_L}} \log\left(\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{P}\right)\right)\right)^{t+u+v} \\
&\leq \left(\frac{\log G}{t+u+v}\right)^{t+u+v}.
\end{aligned} \tag{3.54}$$

It follows from (3.53) and (3.54) that

$$\prod_{i=1}^{t} \log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{p}_i\right)\right) \cdot \prod_{j=1}^{u} \log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{q}_j\right)\right) \cdot \prod_{k=1}^{v} \log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{r}_k\right)\right) < G^{\mathcal{C}_{65}\frac{\log\log\log G}{\log\log G}}. \tag{3.55}$$

The same logic can be used to show that

$$\prod_{\substack{\mathfrak{p}\in\mathcal{O}_L \\ \mathfrak{p}|c\mathcal{O}_L \\ \mathfrak{p}\neq\mathfrak{p}_c}} \log \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right) < G^{\mathcal{C}_{66}\frac{\log\log\log G}{\log\log G}}, \tag{3.56}$$

and that

$$\prod_{\substack{\mathfrak{p}\in\mathcal{O}_L \\ \mathfrak{p}|bc\mathcal{O}_L}} \log \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right) < G^{\mathcal{C}_{67}\frac{\log\log\log G}{\log\log G}}.$$

We note that for large enough constant $\mathcal{C}_{68}$ or large enough G,

$$G^{\frac{\mathcal{C}_{69}}{\log\log G}} < G^{\mathcal{C}_{68}\frac{\log\log\log G}{\log\log G}}.$$

Applying these to (3.51), we obtain that

$$\left(\frac{\log H_L(a,\,b,\,c)}{\log\log H_L(a,\,b,\,c)}\right)^3 < \mathcal{C}_{70}^{t+u+v}\left(\log G\right)^3 \left(\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_a\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)^2\right)\cdot \cdot \max\left\{\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right),\,\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)\right\}\cdot G^{\mathcal{C}_{71}\frac{\log\log\log G}{\log\log G}}. \tag{3.57}$$

Further, from Lemma 3.1.0.4 we obtain that

$$\mathcal{C}_{70}^{t+u+v} < G^{\frac{\mathcal{C}_{72}}{\log\log G}}. \tag{3.58}$$

Further, note that $\log G = G^{\frac{\log\log G}{\log G}}$. Thus we obtain that

$$\left(\frac{\log H_L(a,\,b,\,c)}{\log\log H_L(a,\,b,\,c)}\right)^3 < \left(\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_a\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)^2\right)\cdot \cdot \max\left\{\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right),\,\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)\right\} G^{\mathcal{C}_{73}\left(\frac{\log\log\log G}{\log\log G}+\frac{1}{\log\log G}+\frac{\log\log G}{\log G}\right)}. \tag{3.59}$$

We take the cube root of both sides, before applying Lemma 3.1.0.5, obtaining

$$\log H_L(a,\,b,\,c) < \left(\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_a\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)^2\max\left\{\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right),\,\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)\right\}\right)^{\frac{1}{3}} G^{\mathcal{C}_{74}\left(\frac{\log\log\log G}{\log\log G}+\frac{1}{\log\log G}+\frac{\log\log G}{\log G}\right)}. \tag{3.60}$$

Note, the dominant term in the power of G is $\frac{\log\log\log G}{\log\log G}$. Combining this with the above proves Theorem 2.1.3.1.

3.2.1 Corollaries of Theorem 2.1.3.1

In this section we show various corollaries of Theorem 2.1.3.1. The first two corollaries depend on the Class Group of K and the ideals that $\mathfrak{p}_b$ and $\mathfrak{p}_c$ lie above.

Corollary 3.2.1.1. *Assume that $Nm_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right) > Nm_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)$ and that $\mathfrak{p}_b$ and $\mathfrak{p}_\mathfrak{c}$ both lie over prime ideals of $\mathcal{O}_K$ that do not generate the class group of K. Then*

$$\log H_L(a,\,b,\,c) < G^{\frac{1}{3}+\mathcal{C}_{75}\frac{\log\log\log G}{\log\log G}}. \tag{3.61}$$

Proof. By assumption,

$$\begin{aligned}&\left(\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_a\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)^2\max\left\{\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right),\,\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)\right\}\right)\\&=\left(\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_a\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right)^2\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)^2\right).\end{aligned}$$

Recall that in the Hilbert Class Field L, a prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$ splits into $\frac{h_K}{P}$ prime ideals of $\mathcal{O}_L$, where P is the order of $[\mathfrak{p}]$ in the Class Group of K. By assumption, there must be at least two prime ideals dividing $b\mathcal{O}_L$ with the same norm $\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right)$, and similarly for $\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)$.

It follows that

$$\left(\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_a\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right)^2\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)^2\right)\leq G,$$

and the claim follows. □

Corollary 3.2.1.2. *Assume that $Nm_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right) < Nm_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)$ and that $\mathfrak{p}_c$ lies above a prime ideal of $\mathcal{O}_K$ that has order greater than 2 in the class group of K. Then*

$$\log H_L(a,\,b,\,c) < G^{\frac{1}{3}+\mathcal{C}_{76}\frac{\log\log\log G}{\log\log G}}. \tag{3.62}$$

Proof. By assumption,

$$\begin{aligned}&\left(\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_a\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)^2\max\left\{\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right),\,\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)\right\}\right)\\&=\left(\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_a\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)^3\right).\end{aligned}$$

By the comments in the proof of the previous corollary, our assumption here gives us that there are at least 3 prime ideals of $\mathcal{O}_L$ dividing $c\mathcal{O}_L$ with the same norm, $\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)$. It follows that

$$\left(\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_a\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)^3\right)\leq G,$$

and the claim follows. □

The following corollary holds regardless of the class field of K.

Corollary 3.2.1.3. *Assume that $Nm_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right) > Nm_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)$. Then*

$$\log H_L(a,\,b,\,c) < G^{\frac{2}{3}+\mathcal{C}_{77}\frac{\log\log\log G}{\log\log G}}.$$

Proof. Note that $\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_a\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)\leq G$. Further, by assumption

$$\begin{aligned}\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)\max\left\{\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right),\,\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)\right\}&=\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)\\&\leq G.\end{aligned}$$

Thus the corollary follows. □

Corollary 3.2.1.4. *Assume that* $Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_a) > Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_b) > Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_c)$. *Then*

$$\log H_L(a,\,b,\,c) < G^{\frac{5}{9}+\mathcal{C}_{78}\frac{\log\log\log G}{\log\log G}}.$$

If $\max\left\{Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_b),\, Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_c)\right\} = Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_c)$ *then we obtain that*

$$\log H_L(a,\,b,\,c) < G^{\frac{2}{3}+\mathcal{C}_{79}\frac{\log\log\log G}{\log\log G}}.$$

Proof. By assumption, $\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_b)\,\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c) \leq G^{\frac{2}{3}}$ and $\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_b) \leq G^{\frac{1}{2}}$, $\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c) \leq G^{\frac{1}{3}}$. Applying this to Theorem 2.1.3.1 gives both parts of the corollary. $\square$

Remark. If we assume that none of $a,\,b,\,c$ are units of $\mathcal{O}_K$ then the only assumption we need to obtain the first inequality in the corollary above is that $\max\left\{\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_a),\, \mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_b),\, \mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c)\right\} = \mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_a)$. This follows as then by assumption, $\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_b) \leq G^{\frac{1}{3}}$, $\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c) \leq G^{\frac{1}{3}}$. The argument follows.

We now present some corollaries that depend on the value of

$$\max\left\{\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_b),\, \mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c)\right\}.$$

Corollary 3.2.1.5. *Assume that* $\max\left\{Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_b),\, Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_c)\right\} < (\log H_L(a,\,b,\,c))^{\alpha}$ *for* $0 < \alpha < \frac{2}{3}$. *Then*

$$\log H_L(a,\,b,\,c) < G^{\frac{1}{3-2\alpha}+\mathcal{C}_{80}\frac{\log\log\log G}{\log\log G}}.$$

Proof. Consider (3.59). Applying the assumption, we can rewrite this as

$$\left(\frac{\log H_L(a,\,b,\,c)}{\log\log H_L(a,\,b,\,c)}\right)^3 < G\,(\log H_L(a,\,b,\,c))^{2\alpha}\,G^{\mathcal{C}_{81}\frac{\log\log\log G}{\log\log G}}.$$

Dividing through by $(\log H_L(a,\,b,\,c))^{2\alpha}$ we obtain that

$$\frac{(\log H_L(a,\,b,\,c))^{3-2\alpha}}{(\log\log H_L(a,\,b,\,c))^3} < G^{1+\mathcal{C}_{82}\frac{\log\log\log G}{\log\log G}}.$$

Taking the $3-2\alpha$'th root and applying a variant of Lemma 3.1.0.5 gives the result. $\square$

Corollary 3.2.1.6. *Assume that* $Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_{max}) < (\log H_L(a,\,b,\,c))^{\alpha}$ *for* $0 < \alpha < \frac{3}{5}$. *Then*

$$\begin{aligned}\log H_L(a,\,b,\,c) &< G^{\frac{\mathcal{C}_{83}}{3-5\alpha}\frac{\log\log\log G}{\log\log G}}\\ &= G^{\mathcal{C}_{84}\frac{\log\log\log G}{\log\log G}}.\end{aligned} \tag{3.63}$$

Remark. We note that we can write this in the following terms. For any given $\varepsilon > 0$, given the assumptions of the theorem and corollary there is a computable number $\mathcal{C}_{85}$ such that

$$\log H_L(a,\, b,\, c) < G^{\mathcal{C}_{85}\cdot\varepsilon}.$$

Proof. Consider Theorem 2.1.3.1. By assumption,

$$\log H_L(a,\, b,\, c) < (\log H_L(a,\, b,\, c))^{\frac{5\alpha}{3}}\, G^{\frac{\log\log\log G}{\log\log G}}.$$

□

Dividing through, we obtain that

$$(\log H_L(a,\, b,\, c))^{1-\frac{5\alpha}{3}} < G^{\frac{\log\log\log G}{\log\log G}}.$$

Take the $\frac{1}{1-\frac{5\alpha}{3}}$'th root and the result follows.

3.3 METHOD ONLY USING TWO S-UNIT BOUNDS

Part of the difficulty in analysing cases in the previous section comes from the number of prime ideals on the right hand side of (3.60). If we only use two S-unit bounds then, while in general the bound is worse, it is easier to analyse for corollaries. We now prove Theorem 2.1.3.2, as stated in the Chapter 2.

We follow the main text until (3.26). We then do not use the S-unit bound obtained by letting S be equal to the infinite places and finite places corresponding to the prime ideals of $\mathcal{O}_L$ dividing $bc\mathcal{O}_L$. Following the argument of the main text, we obtain (3.47) and (3.48). Multiplying these together we obtain that

$$\begin{aligned}\left(\frac{\log H_L(a,\, b,\, c)}{\log\log H_L(a,\, b,\, c)}\right)^2 <& \mathcal{C}_{86}^{t+u+v} G^{\frac{\mathcal{C}_{32}}{\log\log G}} (r+t+u+v)^7 (\log G)^2 \\ & \left(\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_b)\, \mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c)^2\right) \left(\prod_{i=1}^{t} \log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{p}_i)\right)\right)^2 \\ & \left(\prod_{j=1}^{u} \log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{q}_j)\right)\right)^2 \cdot \left(\prod_{k=1}^{v} \log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{r}_k)\right)\right)^2 \cdot \\ & \prod_{\substack{\mathfrak{p}\in\mathcal{O}_L \\ \mathfrak{p}\mid c\mathcal{O}_L \\ \mathfrak{p}\neq\mathfrak{p}_c}} \log \mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}).\end{aligned} \tag{3.64}$$

From here we follow the arguments of the proof of the main theorem, obtaining

$$\left(\frac{\log H_L(a,\, b,\, c)}{\log\log H_L(a,\, b,\, c)}\right)^2 < \left(\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)^2\right) G^{\mathcal{C}_{87}\left(\frac{\log\log\log G}{\log\log G}+\frac{1}{\log\log G}+\frac{\log\log G}{\log G}\right)}. \tag{3.65}$$

We take the square root of both sides, before applying Lemma 3.1.0.5, obtaining

$$\begin{aligned}\log H_L(a,\, b,\, c) &< \left(\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)^2\right)^{\frac{1}{2}} G^{\mathcal{C}_{88}\left(\frac{\log\log\log G}{\log\log G}+\frac{1}{\log\log G}+\frac{\log\log G}{\log G}\right)} \\ &= \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right)^{\frac{1}{2}}\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right) G^{\mathcal{C}_{89}\left(\frac{\log\log\log G}{\log\log G}+\frac{1}{\log\log G}+\frac{\log\log G}{\log G}\right)}. \end{aligned} \tag{3.66}$$

Again, $\frac{\log\log\log G}{\log\log G}$ is the dominant term in the exponent of G. This completes the proof of Theorem 2.1.3.2.

From this point, there are many corollaries we can find, similarly to in the previous section. However, given that there are fewer prime ideal on the right hand side of (3.66), they are generally easier to prove. Further, Theorem 2.1.3.1 and Theorem 2.1.3.2 are independent, so if $\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_a\right)$ is sufficiently large in comparison to $\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right)$ and $\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)$, Theorem 2.1.3.2 could give a better bound.

3.3.1 COROLLARIES OF THEOREM 2.1.3.2

This corollary relies on the class group of $\mathcal{O}_K$.

Corollary 3.3.1.1. *Assume that the prime ideal $\mathfrak{r} \subset \mathcal{O}_K$ that $\mathfrak{p}_c \subset \mathcal{O}_L$ lies above does not generate the class group of K. Then there exits an effectively computable constant $\mathcal{C}_{90}$ such that*

$$\log H_L(a,\, b,\, c) < G^{\frac{1}{2}+\mathcal{C}_{90}\frac{\log\log\log G}{\log\log G}}.$$

Proof. Let $\mathfrak{r}$ be a prime ideal of $\mathcal{O}_K$ dividing $c\mathcal{O}_K$ such that $\mathfrak{p}_\mathfrak{c} \subset \mathcal{O}_L$ lies above $\mathfrak{r}$. Assume that $\mathfrak{r}$ does not generate the class group of K. Then in $L = HCF(K)$, $\mathfrak{r}$ splits into h_K/P prime ideals, where h_K is the class number of K and P is the order of $[\mathfrak{r}]$ in C_K [11] [38]. As $\mathfrak{r}$ does not generate the class group of K, then the order of $[\mathfrak{r}]$ is at least 2. As $\mathfrak{r}$ splits in $\mathcal{O}_L$, we know that all prime ideals lying above $\mathfrak{r}$ in $\mathcal{O}_L$ have the same norm. By assumption, we have at least two such ideals in $\mathcal{O}_L$, so $\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right)\left(\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)\right)^2 < G$. More explicitly, there is another prime ideal $\mathfrak{p}_c'$ of $\mathcal{O}_L$ lying above $\mathfrak{r}$ such that $\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right) = \mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c'\right)$. It then follows from Theorem 2.1.3.2 that

$$\log H_L(a,\, b,\, c) < G^{\frac{1}{2}+\mathcal{C}_{90}\frac{\log\log\log G}{\log\log G}}.$$

□

The following corollaries give different bounds depending on $\mathfrak{p}_{max}$ or $\max \mathrm{ord}_{\mathfrak{p}}(c)$.

Corollary 3.3.1.2. *Assume that*

$$Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_c) < G^{\alpha}$$

with $0 < \alpha < 1$, *or that*

$$\max ord_{\mathfrak{p}}(c) < G^{\alpha}$$

with $0 < \alpha < 1$. *Then there exits an effectively computable constant* $\mathcal{C}_{91}$ *such that*

$$\log H_L(a,\, b,\, c) < G^{\frac{1+\alpha}{2}+\mathcal{C}_{91}\frac{\log\log\log G}{\log\log G}}. \tag{3.67}$$

Further, if $\max\left\{Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_b)\,,\, Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_c)\right\} < G^{\alpha}$ *then*

$$\log H_L(a,\, b,\, c) < G^{\frac{3\alpha}{2}+\mathcal{C}_{92}\frac{\log\log\log G}{\log\log G}}. \tag{3.68}$$

Remark. We note that $\frac{3\alpha}{2} < 1$ for $\alpha < \frac{2}{3}$, and further that $\frac{3\alpha}{2} < \frac{1+\alpha}{2}$ for $\alpha < \frac{1}{2}$. Thus, our second bound is better than our first given in this corollary for $\alpha < \frac{1}{2}$.

Corollary 3.3.1.3. *Assume that* $Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_{\max}) > G^{\alpha}$ *for* $\alpha > \frac{1}{3}$, *and that* $\mathfrak{p}_a = \mathfrak{p}_{\max}$. *Then*

$$\log H_L(a,\, b,\, c) < G^{\frac{3-3\alpha}{2}+\mathcal{C}_{93}\frac{\log\log\log G}{\log\log G}}.$$

If $Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_{\max}) \leq G^{\frac{1}{3}}$ *it follows directly from Theorem 2.1.3.2 that*

$$\log H_L(a,\, b,\, c) < G^{\frac{1}{2}+\mathcal{C}_{94}\frac{\log\log\log G}{\log\log G}}.$$

Remark. Note we have the assumption that $\alpha > \frac{1}{3}$ in order to make sure that $\frac{3-3\alpha}{2} < 1$.

Further, the second inequality given is the same case as $\alpha = \frac{1}{3}$ in the last part of Corollary 3.3.1.2.

Proof. We first assume that $\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c) < G^{\alpha}$ where $\alpha \in (0,\, 1)$. Thus

$$\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_{\mathrm{b}})\,\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_{\mathrm{c}})^2 < G^{1+\alpha} < G^2.$$

Thus from Theorem 2.1.3.2, we obtain that

$$\log H_L(a,\, b,\, c) < G^{\frac{1+\alpha}{2} + \mathcal{C}_{95} \frac{\log\log\log G}{\log\log G}}. \tag{3.69}$$

We now assume that $\max \operatorname{ord}_{\mathfrak{p}}(c) < G^{\alpha}$ for some $\alpha \in (0,\, 1)$. Then, in place of (3.42), we have that for all $\mathfrak{p} \mid c\mathcal{O}_L$, $\operatorname{ord}_{\mathfrak{p}}(c) < G^{\alpha}$. It follows from (3.29) that

$$\log H_L(a,\, b,\, c) < \mathcal{C}_{96} \left(\log G\right) G^{\alpha}.$$

We note that for $\alpha < \frac{1}{2}$, this bound is actually better than the bound that follows.

As in the proof for the main theorem, (3.48) still holds. Multiplying the above and (3.48) we obtain that

$$\begin{aligned} \frac{\left(\log H_L(a,\, b,\, c)\right)^2}{\log\log H_L(a,\, b,\, c)} <& \mathcal{C}_{97}^{t+v} \left(r+t+v+1\right)^{\frac{7}{2}} \left(\log G\right)^2 G^{\alpha} \operatorname{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right) \cdot \operatorname{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right) \\ & \prod_{i=1}^{t} \log\left(\operatorname{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{p}_i\right)\right) \cdot \prod_{j=1}^{v} \log\left(\operatorname{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{r}_j\right)\right) \cdot \prod_{\substack{\mathfrak{p}\in\mathcal{O}_L \\ \mathfrak{p}\mid c\mathcal{O}_L \\ \mathfrak{p}\neq\mathfrak{p}_c}} \log \operatorname{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right). \end{aligned} \tag{3.70}$$

We note that $\operatorname{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right) \cdot \operatorname{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right) \leq G$. Further, we can use the techniques from above to tidy terms in the same way as we did for the main theorem to obtain

$$\frac{\left(\log H_L(a,\, b,\, c)\right)^2}{\log\log H_L(a,\, b,\, c)} < G^{1+\alpha} G^{\mathcal{C}_{98} \frac{\log\log\log G}{\log\log G}}. \tag{3.71}$$

Taking the square root and applying a variant of Lemma 3.1.0.5, we obtain that

$$\log H_L(a,\, b,\, c) < G^{\frac{1+\alpha}{2} + \mathcal{C}_{99} \frac{\log\log\log G}{\log\log G}}. \tag{3.72}$$

This proves the first part of Corollary 3.3.1.2. The further comments follow directly from Theorem 2.1.3.2 when we bound $\operatorname{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_b\right)$ and $\operatorname{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)$ above by G^{α}. This gives us that

$$\log H_L(a,\, b,\, c) < G^{\frac{3\alpha}{2} + \mathcal{C}_{100} \frac{\log\log\log G}{\log\log G}},$$

as claimed, where $\frac{3\alpha}{2} < 1$ for $\alpha < \frac{2}{3}$, and is a better bound than given above for $\alpha < \frac{1}{2}$.

It also follows directly from Theorem 2.1.3.2 that if

$$\max\left\{\operatorname{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_{\mathrm{b}}\right),\, \operatorname{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_{\mathrm{c}}\right)\right\} < G^{\alpha},$$

then

$$\log H_L(a,\, b,\, c) < G^{\frac{3\alpha}{2} + \mathcal{C}_{101} \frac{\log\log\log G}{\log\log G}}.$$

□

Corollary 3.3.1.4. *Assume now that* $Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_c) < (\log H_L(a,\, b,\, c))^{\alpha}$ *with* $0 < \alpha < 1$, *or that*
$\max ord_{\mathfrak{p}}(c) < (\log H_L(a,\, b,\, c))^{\alpha}$ *with* $0 < \alpha < 1$. *Then there exits an effectively computable constant* $\mathcal{C}_{102}$ *such that*

$$\log H_L(a,\, b,\, c) < G^{\frac{1}{2-\alpha} + \mathcal{C}_{102} \frac{\log\log\log G}{\log\log G}}.$$

Furthermore, if $\max\left\{Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_b),\, Nm_{\mathbb{Q}}^{L}(\mathfrak{p}_c)\right\} < (\log H_L(a,\, b,\, c))^{\alpha}$ *for* $\alpha < \frac{2}{3}$, *then directly from Theorem 2.1.3.2 we obtain that*

$$\begin{aligned} \log H_L(a,\, b,\, c) &< G^{\frac{\mathcal{C}_{103}}{2-3\alpha} \frac{\log\log\log G}{\log\log G}} \\ &= G^{\mathcal{C}_{104} \frac{\log\log\log G}{\log\log G}}. \end{aligned} \tag{3.73}$$

This is the best bound we achieve in this text.

Remark. We note that the second inequality in this corollary gives a sub-exponential bound, an improvement on the bounds given in [51][52].

To more easily compare with existing results, we note we can slightly weaken this upper bound. Inequality (3.73) implies that given any $\varepsilon > 0$ there exists some computable $\mathcal{C}_{105}$ such that

$$\log H_L(a,\, b,\, c) < G^{\mathcal{C}_{105} \cdot \epsilon},$$

where importantly $\mathcal{C}_{105}$ does not depend on ϵ.

Proof. We first assume that $\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c) < (\log H_L(a,\, b,\, c))^{\alpha}$ with $\alpha \in (0, 1)$. Then from this assumption and (3.65), we obtain that

$$\frac{(\log H_L(a,\, b,\, c))^{2-\alpha}}{(\log\log H_L(a,\, b,\, c))^{2}} < G \cdot G^{\mathcal{C}_{106} \frac{\log\log\log G}{\log\log G}}. \tag{3.74}$$

By assumption, $2 - \alpha > 1$, and we take this root to obtain that

$$\frac{\log H_L(a,\, b,\, c)}{(\log\log H_L(a,\, b,\, c))^{\frac{2}{2-\alpha}}} < G^{\frac{1}{2-\alpha} + \mathcal{C}_{107} \frac{\log\log\log G}{\log\log G}}. \tag{3.75}$$

Note that $1 > \frac{1}{2-\alpha} > \frac{1}{2}$. Applying a variant of Lemma 3.1.0.5, we obtain that

$$\log H_L(a,\, b,\, c) < G^{\frac{1}{2-\alpha} + \mathcal{C}_{108} \frac{\log\log\log G}{\log\log G}}. \tag{3.76}$$

Now instead assume that $\max \operatorname{ord}_{\mathfrak{p}}(c) < (\log H_L(a, b, c)^{\alpha}$ for some $\alpha \in (0, 1)$. Then as above, from (3.29) we obtain that

$$\log H_L(a, b, c) < \mathcal{C}_{109} \log G \left(\log H_L(a, b, c)\right)^{\alpha}.$$

It immediately follows that

$$\left(\log H_L(a, b, c)\right)^{1-\alpha} < \mathcal{C}_{109} \log G.$$

Again, we still have (3.48), as obtained by following the main argument. We multiply (3.48) by the above to obtain

$$\begin{aligned} \frac{\left(\log H_L(a, b, c)\right)^{2-\alpha}}{\log\log H_L(a, b, c)} <& \mathcal{C}_{110}^{t+v} (r+t+v+1)^{\frac{7}{2}} (\log G)^2 \operatorname{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_b) \cdot \operatorname{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c) \\ & \prod_{i=1}^{t} \log\left(\operatorname{Nm}_{\mathbb{Q}}^{K}(\mathfrak{p}_i)\right) \cdot \prod_{j=1}^{v} \log\left(\operatorname{Nm}_{\mathbb{Q}}^{K}(\mathfrak{r}_j)\right) \cdot \prod_{\substack{\mathfrak{p}\in\mathcal{O}_L \\ \mathfrak{p}\mid c\mathcal{O}_L \\ \mathfrak{p}\neq\mathfrak{p}_c}} \log \operatorname{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}). \end{aligned} \tag{3.77}$$

As before, we can use the same method of tidying as in the proof of the main theorem to show that

$$\frac{\left(\log H_L(a, b, c)\right)^{2-\alpha}}{\log\log H_L(a, b, c)} < G^{1+\mathcal{C}_{111}\frac{\log\log\log G}{\log\log G}}.$$

Taking the $2-\alpha$'th root and applying a variant of Lemma 3.1.0.5, we obtain that

$$\log H_L(a, b, c) < G^{\frac{1}{2-\alpha}+\mathcal{C}_{112}\frac{\log\log\log G}{\log\log G}}.$$

This concludes the proof of the first part of Corollary 3.3.1.4.

To see the strongest case, we appeal directly to Theorem 2.1.3.2. Assume that

$$\operatorname{Nm}(\mathfrak{p}_{\max}) < \log\left(H_L(a, b, c)\right)^{\alpha}$$

with $\alpha < \frac{2}{3}$. Then we can bound $\operatorname{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_b)$ and $\operatorname{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c)$ above by $\log\left(H_L(a, b, c)\right)^{\alpha}$, obtaining

$$\log H_L(a, b, c) < \left(\log H_L(a, b, c)\right)^{\frac{3\alpha}{2}} G^{\mathcal{C}_{113}\left(\frac{\log\log\log G}{\log\log G}+\frac{1}{\log\log G}+\frac{\log\log G}{\log G}\right)}.$$

It then follows that

$$\left(\log H_L(a, b, c)\right)^{1-\frac{3\alpha}{2}} < G^{\mathcal{C}_{113}\left(\frac{\log\log\log G}{\log\log G}+\frac{1}{\log\log G}+\frac{\log\log G}{\log G}\right)}.$$

Taking the $\frac{1}{1-\frac{3\alpha}{2}}$'th root gives us the result, namely that

$$\log H_L(a,\, b,\, c) < G^{\frac{\mathcal{C}_{113}}{2-3\alpha}\left(\frac{\log\log\log G}{\log\log G}+\frac{1}{\log\log G}+\frac{\log\log G}{\log G}\right)}.$$

Finally, we recall that the dominant term in the exponent is $\frac{\log\log\log G}{\log\log G}$, so we obtain that

$$\log H_L(a,\, b,\, c) < G^{\frac{\mathcal{C}_{114}}{2-3\alpha}\left(\frac{\log\log\log G}{\log\log G}\right)}.$$

As commented in the statement of the theorem, this is of the form

$$\log H_L(a,\, b,\, c) < G^{\mathcal{C}_{115}\cdot\epsilon}.$$

□

Remark. While the assumptions are hard to compare due to their different natures, we can see that for all $\alpha \in (0,\, 1)$, $\frac{1+\alpha}{2} \geq \frac{1}{2-\alpha}$. Thus, generally speaking, the bound of Corollary 3.3.1.4 is better than that of Corollary 3.3.1.2. More concretely, given $(a,\, b,\, c)$ that satisfy the assumptions of both Corollary 3.3.1.2 and 3.3.1.4, Corollary 3.3.1.4 gives a better bound in terms of the radical G than that of Corollary 3.3.1.2.

Corollary 3.3.1.5. *Assume that* $\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_{\max}) > G^{\alpha}$ *for* $\alpha > \frac{1}{3}$*, and that* $\mathfrak{p}_a = \mathfrak{p}_{\max}$*. Then*

$$\log H_L(a,\, b,\, c) < G^{\frac{3-3\alpha}{2}+\mathcal{C}_{116}\frac{\log\log\log G}{\log\log G}}.$$

If $\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_{\max}) \leq G^{\frac{1}{3}}$ *it follows directly from Theorem 2.1.3.2 that*

$$\log H_L(a,\, b,\, c) < G^{\frac{1}{2}+\mathcal{C}_{117}\frac{\log\log\log G}{\log\log G}}.$$

Remark. Note we have the assumption that $\alpha > \frac{1}{3}$ in order to make sure that $\frac{3-3\alpha}{2} < 1$.

Further, the second inequality given is the same case as $\alpha = \frac{1}{3}$ in the last part of Corollary 3.3.1.2.

Proof. Assume that $\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_{\max}) > G^{\alpha}$, and assume that $\mathfrak{p}_a = \mathfrak{p}_{\max}$. Then considering (3.66), we note that

$$\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_b)\,\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c)^2 < \left(G^{1-\alpha}\right)^3,$$

so

$$\left(\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_b)\,\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c)^2\right)^{\frac{1}{2}} < G^{\frac{3-3\alpha}{2}}.$$

It follows that

$$\log H_L(a,\, b,\, c) < G^{\frac{3-3\alpha}{2}+\mathcal{C}_{118}\frac{\log\log\log G}{\log\log G}}. \tag{3.78}$$

We note that $\frac{3-3\alpha}{2} < 1$ for $\alpha > \frac{1}{3}$. □

Corollary 3.3.1.6. *Assume* $ord_{\mathfrak{p}_c}c < G^\alpha$ *for* $0 < \alpha \leq 1$. *Then*

$$\log H_L\left(a,\, b,\, c\right) < G^{\max\left\{\alpha,\, \frac{3}{4}\right\}+\mathcal{C}_{119}\frac{\log\log\log G}{\log\log G}}.$$

Proof. Assume that $\mathrm{ord}_{\mathfrak{p}_c}c < G^\alpha$.

Note that we can write

$$\max_{\mathfrak{p}|\langle c\rangle_L}\mathrm{ord}_{\mathfrak{p}}\left(c\right) = \max\left\{\max_{\substack{\mathfrak{p}|c\mathcal{O}_L\\ \mathfrak{p}\neq\mathfrak{p}_c}}\mathrm{ord}_{\mathfrak{p}}\left(c\right),\, \mathrm{ord}_{\mathfrak{p}_c}\left(c\right)\right\}.$$

By assumption, we attain the bound

$$\max_{\mathfrak{p}|\langle c\rangle_L}\mathrm{ord}_{\mathfrak{p}}\left(c\right) = \max\left\{\max_{\substack{\mathfrak{p}|c\mathcal{O}_L\\ \mathfrak{p}\neq\mathfrak{p}_c}}\mathrm{ord}_{\mathfrak{p}}\left(c\right),\, G^\alpha\right\}. \tag{3.79}$$

From the above and (3.29), it follows directly that

$$\log H_L\left(a,\, b,\, c\right) < \mathcal{C}_{120}\log G\max\left\{\max_{\substack{\mathfrak{p}|c\mathcal{O}_L\\ \mathfrak{p}\neq\mathfrak{p}_c}}\mathrm{ord}_{\mathfrak{p}}\left(c\right),\, G^\alpha\right\}. \tag{3.80}$$

We now consider cases depending on $\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_c\right)$.

First assume that $\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_{\max}\right) < G^{\frac{1}{2}}$. Then we can directly use Corollary 3.3.1.2 to obtain the bound given there, namely

$$\log H_L(a,\, b,\, c) < G^{\frac{3}{4}+\mathcal{C}_{121}\frac{\log\log\log G}{\log\log G}}.$$

Thus from the above and (3.80), we obtain that

$$\log H_L\left(a,\, b,\, c\right) < G^{\max\left\{\frac{3}{4},\alpha\right\}+\mathcal{C}_{122}\frac{\log\log\log G}{\log\log G}}$$

Assume now instead that $\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}_{\max}\right) \geq G^{\frac{1}{2}}$. It immediately follows that for all other prime ideals $\mathfrak{p}$ contributing to G, we have that $\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right) < G^{\frac{1}{2}}$.

Consider now $\max_{\substack{\mathfrak{p}|c\mathcal{O}_L\\ \mathfrak{p}\neq\mathfrak{p}_c}}\mathrm{ord}_{\mathfrak{p}}\left(c\right)$. We apply Yu's bound as before on this, taking the above comments into consideration. It follows that

$$\begin{aligned}\max_{\substack{\mathfrak{p}|c\mathcal{O}_L\\ \mathfrak{p}\neq\mathfrak{p}_c}}\mathrm{ord}_{\mathfrak{p}}\left(c\right) <&\mathcal{C}_{123}^{t+u}\left(r+t+u+1\right)^{7/2}\log\log H_L(a,\, b,\, c)\mathrm{Nm}_{\mathbb{Q}}^{L}\left(\mathfrak{p}\right)\\ &\prod_{i=1}^{t}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{p}_i\right)\right)\cdot\prod_{j=1}^{u}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}\left(\mathfrak{q}_j\right)\right),\end{aligned} \tag{3.81}$$

where $\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}) < G^{\frac{1}{2}}$. Following the same logic as the main text, we obtain that

$$\frac{\log H_L(a,\, b,\, c)}{\log\log H_L(a,\, b,\, c)} < \mathcal{C}_{124}^{t+u}\,(r+t+u+1)^{7/2}\log G\cdot G^{\frac{1}{2}} \prod_{i=1}^{t}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{p}_i)\right)\cdot\prod_{j=1}^{u}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{q}_j)\right). \tag{3.82}$$

Note that (3.48) still holds, and $\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_b)\,\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c) \leq G$. Multiplying (3.48) and (3.82), tidying terms as we do in the text, and considering (3.80), we obtain that

$$\frac{\log H_L(a,\, b,\, c)}{\log\log H_L(a,\, b,\, c)} < \max\left\{G^{\frac{3}{4}+\mathcal{C}_{125}\frac{\log\log\log G}{\log\log G}},\, G^{\alpha+\mathcal{C}_{126}\frac{\log\log\log G}{\log\log G}}\right\}. \tag{3.83}$$

More concisely, after applying Lemma 3.1.0.5, we obtain that

$$\log H_L(a,\, b,\, c) < G^{\max\left\{\alpha,\,\frac{3}{4}\right\}+\mathcal{C}_{127}\frac{\log\log\log G}{\log\log G}}. \tag{3.84}$$

Thus, in either case depending on $\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_{\max})$, we obtain that

$$\log H_L(a,\, b,\, c) < G^{\max\left\{\alpha,\,\frac{3}{4}\right\}+\mathcal{C}_{128}\frac{\log\log\log G}{\log\log G}}. \tag{3.85}$$

$\square$

Corollary 3.3.1.7. *Assume that* $\mathrm{ord}_{\mathfrak{p}_c}c < (\log H_L(a,\, b,\, c))^{\alpha}$ *for* $0<\alpha<1$. *Then*

$$\log H_L(a,\, b,\, c) < \max\left\{G^{\frac{3}{4}+\mathcal{C}_{129}\left(\frac{\log\log\log G}{\log\log G}\right)},\, \mathcal{C}_{130}(\log G)^{\frac{1}{1-\alpha}}\right\}.$$

Proof. Assume that $\mathrm{ord}_{\mathfrak{p}_c}(c) < (\log H_L(a,\, b,\, c))^{\alpha}$ for some $0<\alpha<1$. As in Corollary 3.3.1.6 it immediately follows that

$$\begin{aligned}\max_{\mathfrak{p}|\langle c\rangle_L}\mathrm{ord}_{\mathfrak{p}}(c) &= \max\left\{\max_{\substack{\mathfrak{p}|c\mathcal{O}_L\\ \mathfrak{p}\neq\mathfrak{p}_c}}\mathrm{ord}_{\mathfrak{p}}(c),\, \mathrm{ord}_{\mathfrak{p}_c}(c)\right\}\\ &\leq \max\left\{\max_{\substack{\mathfrak{p}|c\mathcal{O}_L\\ \mathfrak{p}\neq\mathfrak{p}_c}}\mathrm{ord}_{\mathfrak{p}}(c),\, (\log H_L(a,\, b,\, c))^{\alpha}\right\}.\end{aligned} \tag{3.86}$$

This along with (3.29) implies that

$$\log H_L(a,\, b,\, c) < \max\left\{\max_{\substack{\mathfrak{p}|c\mathcal{O}_L\\ \mathfrak{p}\neq\mathfrak{p}_c}}\mathrm{ord}_{\mathfrak{p}}(c)\log G,\, \mathcal{C}_{131}(\log H_L(a,\, b,\, c))^{\alpha}\log G\right\}.$$

If

$$\max\left\{\max_{\substack{\mathfrak{p}|c\mathcal{O}_L\\ \mathfrak{p}\neq\mathfrak{p}_c}} \operatorname{ord}_{\mathfrak{p}}(c)\log G,\ (\log H_L(a,\, b,\, c))^{\alpha}\log G\right\} = \mathcal{C}_{132}\,(\log H_L(a,\, b,\, c))^{\alpha}\log G,$$

then we can see that

$$\log H_L(a,\, b,\, c) < \mathcal{C}_{133}\,(\log G)^{\frac{1}{1-\alpha}}\,.$$

We now consider two cases.

In the first case we assume that $\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_{\max}) < G^{\frac{1}{2}}$. In this case we can appeal directly to Corollary 3.3.1.2, obtaining that

$$\log H_L(a,\, b,\, c) < G^{\frac{3}{4}+\mathcal{C}_{134}\left(\frac{\log\log\log G}{\log\log G}\right)}.$$

For the second case we assume that $\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_{\max}) \geq G^{\frac{1}{2}}$ and follow the same argument as in Case 2 in Corollary 3.3.1.6.

As before, we see that for all prime ideals $\mathfrak{p}\neq\mathfrak{p}_{\max}$ contributing to G, we have that $\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}) < G^{\frac{1}{2}}$. Applying Yu's bound on $\max_{\substack{\mathfrak{p}|c\mathcal{O}_L\\ \mathfrak{p}\neq\mathfrak{p}_c}} \operatorname{ord}_{\mathfrak{p}}(c)$ again, we find that

$$\begin{aligned}\max_{\substack{\mathfrak{p}|c\mathcal{O}_L\\ \mathfrak{p}\neq\mathfrak{p}_c}} \operatorname{ord}_{\mathfrak{p}}(c) <& \mathcal{C}_{135}^{s+t}\,(r+t+u+1)^{7/2}\log\log H_L(a,\, b,\, c)\cdot\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p})\\ &\prod_{i=1}^{t}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{p}_i)\right)\cdot\prod_{j=1}^{u}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{q}_j)\right),\end{aligned}$$

Again, we know that $\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}) < G^{\frac{1}{2}}$. Following the logic of the main argument and the proof of Corollary 3.3.1.6, it again follows that

$$\begin{aligned}\frac{\log H_L(a,\, b,\, c)}{\log\log H_L(a,\, b,\, c)} <& \mathcal{C}_{136}^{s+t}\,(r+t+u+1)^{7/2}\log G\cdot G^{1-\beta}\prod_{i=1}^{t}\\ &\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{p}_i)\right)\cdot\prod_{j=1}^{u}\log\left(\mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{q}_j)\right).\end{aligned}$$

After tidying as we have previously and applying Lemma 3.1.0.5, it follows that

$$\log H_L(a,\, b,\, c) < G^{\frac{3}{4}+\mathcal{C}_{137}\frac{\log\log\log G}{\log\log G}}.$$

Combining these results, in both cases we obtain that

$$\log H_L(a,\, b,\, c) < \max\left\{G^{\frac{3}{4}+\mathcal{C}_{138}\frac{\log\log\log G}{\log\log G}},\ \mathcal{C}_{113}\,(\log G)^{\frac{1}{1-\alpha}}\right\}.$$

□

3.4 APPLICATION OF GYŐRY'S BOUND

In this section, we prove Theorem 2.1.3.3. We begin by considering Lemma 2.1.2.3. In the statement of the Lemma, we see that the constant depends on t; we note that by (4.14) of [20] gives us that

$$t < \mathcal{C}_{139} \frac{\log G}{\log \log G}. \tag{3.87}$$

Similarly as in Section 3.2, we can deduce that

$$\left(2^{13.32} d\mathcal{R}\right)^t \leq G^{\frac{\mathcal{C}_{140}}{\log \log G}},$$

so

$$\mathcal{R}^t \leq G^{\frac{\mathcal{C}_{141}}{\log \log G}}$$

Further, using 3.87, we can bound

$$\mathcal{C}_{142}(d,\, r,\, s,\, t) = s^5 \left(16e\right)^{3r+4t+7} d^{4r+2t+7} \leq \mathcal{C}_{143} G^{\frac{\mathcal{C}_{144}}{\log \log G}}$$

This, as done in the proof of the earlier theorems, allows us to remove the dependency on t when we apply Lemma 2.1.2.3. We note this is the same line of reasoning as used in [18].

When we apply Lemma 2.1.2.3 in the places we previously applied Lemma 2.1.2.2 we attain (after moving things into the constant) essentially the same bounds with $\mathfrak{p}_a$, $\mathfrak{p}_b$, $\mathfrak{p}_c$ replaced by $\mathfrak{p}_a'$, $\mathfrak{p}_b'$ and $\mathfrak{p}_c'$ where $\mathfrak{p}_a'$ is the prime ideal of third largest norm dividing $a\mathcal{O}_L$ and similarly for $\mathfrak{p}_b'$ and $\mathfrak{p}_c'$. If fewer than three prime ideals divide a, b or c then we define the corresponding norm to be 1.

We follow the proof of the main theorem, but we replace any use of Lemma 2.1.2.2 with Lemma 2.1.2.3. For the most part, all that changes is any occurrence of $\mathfrak{p}_a$, $\mathfrak{p}_b$ and $\mathfrak{p}_c$ arising from the use of Lemma 2.1.2.2 is replaced by $\mathfrak{p}_a'$, $\mathfrak{p}_b'$ and $\mathfrak{p}_c'$. We follow the line of reasoning from the main text up until (3.14). For this first application of S-units, where we have no finite places, we continue to use Lemma (2.1.2.2) as it is simpler in this case than Lemma 2.1.2.3. As before, we obtain (3.25).

As before, now let $S = S_\infty \cup \{\mathfrak{p} \,:\, \mathfrak{p} \mid c\mathcal{O}_L\}$. Applying Lemma 2.1.2.3 to

$$-\frac{u_a a'}{u_c c'} - \frac{u_b b'}{u_c c'} = 1,$$

we obtain that

$$\begin{aligned}
\max\left\{h\left(-\frac{u_a}{u_c c'}\right), h\left(-\frac{u_b}{u_c c'}\right)\right\} &< \mathcal{C}_{145} G^{\frac{\mathcal{C}_{146}}{\log\log G}} P'_S R_S \left(1+\frac{\log^+ P_S}{\log^+ \log P'_S}\right) \cdot \\
&\quad \cdot \max\left\{h\left(a'\right), h\left(b'\right), 1\right\} \\
&< \mathcal{C}_{147} G^{\frac{\mathcal{C}_{148}}{\log\log G}} \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}'_c\right) R_S \log^+\left(\mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_c\right)\right) \cdot \\
&\quad \cdot \max\left\{h\left(a'\right), h\left(b'\right), 1\right\} \\
&< \mathcal{C}_{149} G^{\frac{\mathcal{C}_{148}}{\log\log G}} \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}'_c\right) \left(\prod_{\substack{\mathfrak{p}\subset\mathcal{O}_L \\ \mathfrak{p}\mid c\mathcal{O}_L}} \log \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}\right)\right) \cdot \\
&\quad \cdot \log G \cdot h\left(b'\right) \\
&\leq \mathcal{C}_{150} G^{\mathcal{C}_{151}\frac{\log\log\log G}{\log\log G}} \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}'_c\right) \cdot \\
&\quad \left(\prod_{\substack{\mathfrak{p}\subset\mathcal{O}_L \\ \mathfrak{p}\mid c\mathcal{O}_L}} \log \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}\right)\right) h\left(b'\right), \qquad (3.88)
\end{aligned}$$

where the line of reasoning about $\max\left\{h\left(a'\right), h\left(b'\right), 1, \frac{\pi}{d}\right\}$ follows from assumption (3.12), and the last line follows as

$$\log G = G^{\frac{\log\log G}{\log G}} \leq G^{\mathcal{C}_{152}\frac{\log\log\log G}{\log\log G}}$$

for a sufficiently large constant and large enough G. We are thus able to replace (3.26) with

$$\begin{aligned}
\max\left\{h\left(\frac{u_a a'}{u_c c'}\right), h\left(\frac{u_b b'}{u_c c'}\right)\right\} <& \mathcal{C}_{153} G^{\mathcal{C}_{154}\frac{\log\log\log G}{\log\log G}} \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}'_c\right) \cdot \\
&\cdot \left(\prod_{\substack{\mathfrak{p}\subset\mathcal{O}_L \\ \mathfrak{p}\mid c\mathcal{O}_L}} \log \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}\right)\right) h\left(b'\right).
\end{aligned}$$

As before, we now let $S = S_\infty \cup \{\mathfrak{p} : \mathfrak{p} \mid bc\mathcal{O}_L\}$. Applying Lemma 2.1.2.3 again, following the same method as above, in place of (3.28) we obtain that

$$\begin{aligned}
\max\left\{h\left(-\frac{u_a}{u_c c'}\right), h\left(-\frac{u_b b'}{u_c c'}\right)\right\} <& \mathcal{C}_{155} G^{\mathcal{C}_{156}\frac{\log\log\log G}{\log\log G}} \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{q}\right) \cdot \\
&\cdot \left(\prod_{\substack{\mathfrak{p}\subset\mathcal{O}_L \\ \mathfrak{p}\mid bc\mathcal{O}_L}} \log \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}\right)\right) h\left(a'\right),
\end{aligned}$$

where $\mathfrak{q}$ is the prime ideal of $\mathcal{O}_L$ of third largest norm dividing $bc\mathcal{O}_L$. We note that this is not necessarily $\mathfrak{p}_b$ or $\mathfrak{p}_c$, though it may have the same norm as one of them, and could indeed be either of them.

We now follow the argument of the main text again, using the above inequalities in place of (3.26) and (3.28) as necessary, and we end up obtaining

$$\begin{aligned}\log H_L(a,\, b,\, c) < \Big(\mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_a\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_b\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_c\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}'_c\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{q}\right)\Big)^{\frac{1}{3}} \cdot \\ \cdot\, G^{\mathcal{C}_{157}\left(\frac{\log\log\log G}{\log\log G}+\frac{1}{\log\log G}+\frac{\log\log G}{\log G}\right)}\end{aligned} \tag{3.89}$$

in place of (3.60). As before, $\frac{\log\log\log G}{\log\log G}$ is the dominant term in the exponent of G, so we can write

$$\log H_L(a,\, b,\, c) < \Big(\mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_a\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_b\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_c\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}'_c\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{q}\right)\Big)^{\frac{1}{3}} G^{\mathcal{C}_{158}\frac{\log\log\log G}{\log\log G}}. \tag{3.90}$$

We explore some cases. If $\mathfrak{q} = \mathfrak{p}'_b$ then

$$\mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_a\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_b\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_c\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}'_c\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}'_b\right) \leq G.$$

If $\mathfrak{q} = \mathfrak{p}'_c$ then there exists a prime ideal $\mathfrak{p}''_c$, the prime of second largest norm dividing $c\mathcal{O}_L$. Note that $\mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}'_c\right) \leq \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_c\right)'' \leq \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_c\right)$, and all these primes divide $abc\mathcal{O}_L$ so their norms contribute to G. Thus, in this case we obtain that

$$\begin{aligned}\mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_a\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_b\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_c\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}'_c\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}'_c\right) &< \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_a\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_b\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_c\right) \cdot \\ &\quad \cdot \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}'_c\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}''_c\right) \\ &\leq G.\end{aligned}$$

We have dealt with the cases where $\mathfrak{q} = \mathfrak{p}'_b$ and $\mathfrak{q} = \mathfrak{p}'_c$. Using the notation above, there are four further possibilities for $\mathfrak{q}$, namely $\mathfrak{p}_b$, $\mathfrak{p}_c$, $\mathfrak{p}''_b$, $\mathfrak{p}''_c$. If $\mathfrak{q} = \mathfrak{p}''_b$ or $\mathfrak{p}''_c$ then substituting into the above expression, it follows from the definition of G that $\mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_a\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_b\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_c\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}'_c\right) \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{q}\right) < G$.

On the other hand, if $\mathfrak{q} = \mathfrak{p}_b$ or $\mathfrak{p}_c$, we can still upper bound this by G. Assume $\mathfrak{q} = \mathfrak{p}_b$. Then we deduce that $\mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_b\right) \leq \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}'_c\right) \leq \mathrm{Nm}^L_{\mathbb{Q}}\left(\mathfrak{p}_c\right)$. It follows then that

$$\begin{aligned}\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_a)\,\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_b)\,\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c)\,\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c')\,\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_b) &< \mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_a)\,\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_b)\,\cdot \\ &\quad \cdot \mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c)\,\mathrm{Nm}_{\mathbb{Q}}^{L}(\mathfrak{p}_c')^2 \\ &\leq G.\end{aligned}$$

The argument is symmetric so applies if $\mathfrak{q} = \mathfrak{p}_b$. Thus in all cases we obtain that

$$\log H_L(a,\, b,\, c) < G^{\frac{1}{3}+\mathcal{C}_{159}\frac{\log\log\log G}{\log\log G}}. \tag{3.91}$$

We note that for given a, b, c, once we know the prime ideals dividing $a\mathcal{O}_K$, $b\mathcal{O}_K$ and $c\mathcal{O}_K$, the inequality (3.90) may be stronger than that given in (3.91).

3.5 SOME REMARKS

A combination of methods and results by Győry and Yu [21] and Győry [20], [19] with the method of Le Fourn [31] can be used directly to find results over the base field, as done by Győry in [18]. Further, in terms of S, Győry improved the S-unit bound given by Le Fourn. Győry's result regarding the *abc* conjecture is as follows.

Let K be a number field and let a, b, $c := a + b$ belong to K^*. Define

$$N_K = \prod_{v} \mathrm{Nm}_{\mathbb{Q}}^{K}(\mathfrak{p})^{\mathrm{ord}_p(\mathfrak{p})},$$

where v is taken from the set of finite places such that $|a|_v$, $|b|_v$ and $|c|_v$ are not all equal, and p is the rational prime such that $\mathfrak{p} \cap \mathbb{Z} = p$. Then there is a computable constant $\mathcal{C}_{160}$ depending only on $d = [K : \mathbb{Q}]$ and Δ_K such that

$$\log H_K(a,\, b,\, c) < \mathcal{C}_{160} N_K^{\frac{1}{3}+\frac{\log\log\log N_K}{\log\log N_K}}.$$

This is an immediate consequence of the sub-exponential inequality in [18] before estimating P' from above by $N_K^{\frac{1}{3}}$, where P' denotes the third largest norm of prime ideals involved.

Győry's combination of his method with that of Le Fourn's enables him to state his results entirely over the base field K, rather than over the Hilbert Class Field. We note that for the case where the base field K has class number 1, the results are in essence the same. On the other hand, the dependence on the norms of prime ideals in this thesis allows us to state corollaries depending on these norms, leading to the sub-exponential bound for example. Further, we believe they may allow some attack at open problems such as the smooth *abc* conjecture [27].

For all these results, we have considered them in terms of $\log H_L(a, b, c)$. We note that $H_L(a, b, c) = H_K(a, b, c)^{h_K}$, as $h_K = [L : K]$. Thus, as h_K depends on the field, after taking the logarithm we can incorporate the h_K into our computable constant and have the height in terms of the base field K. However, so far we have been unable to do the same for the radical G.

— 4 —

Applications of the *abc* Results

4.1 APPLICATION TO EFFECTIVE SKOLEM-MAHLER-LECH PROBLEM

In this section we will use our main result to allow us to determine whether a linear recurrence sequence of degree three with no repeated roots of the characteristic polynomial has zeroes. As noted in the introduction, there exists an algorithm to determine whether there are periodic zeroes, so we are concerned with the case when there are only potentially finitely many zeroes.

4.1.1 Case Where all Terms are Coprime

Consider a linear recurrence sequence of the following form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3},\ n = 3,\ 4,\ 5, \ldots$$

where the values of a_0, a_1 and a_2 are known. We form the characteristic polynomial of the sequence

$$x^3 - c_1 x^2 - c_2 x - c_3$$

and assume that this has distinct roots r_1, r_2, r_3. Let $K = \mathbb{Q}(r_1,\ r_2,\ r_3)$. We further assume the roots are pairwise coprime when considered as principal ideals of the ring of integers $\mathcal{O}_K$.

By our assumptions, we know that we can write

$$a_n = k_1 r_1^n + k_2 r_2^n + k_3 r_3^n$$

where k_1, k_2, k_3 are constants depending on a_0, a_1 and a_2. We further assume k_1, k_2 and k_3 are coprime, but we will look at ways to try and deal with this when not coprime later.

For ease, we use the result obtained by using Győry's bound, that is the inequality given at (3.91). Assume there exists an n such that $a_n = 0$. Explicitly,

$$0 = k_1 r_1^n + k_2 r_2^n + k_3 r_3^n.$$

We are in a position to use the result. Let $L = HCF(K)$ and define G as above. Then,

$$\log H\left(k_1 r_1^n,\, k_2 r_2^n,\, k_3 r_3^n\right) < G^{\frac{1}{3} + \mathcal{C}_1 \frac{\log\log\log G}{\log\log G}}.$$

Without loss of generality, assume that

$$h\left(r_1\right) \leq h\left(r_2\right) \leq h\left(r_3\right).$$

Note that

$$H\left(k_1 r_1^n,\, k_2 r_2^n,\, k_3 r_3^n\right) = H\left(\frac{k_1}{k_3} r_1^n,\, \frac{k_2}{k_3} r_2^n,\, r_3^n\right).$$

Further, by comparing definitions,

$$h\left(r_3^n\right) \leq \log H\left(\frac{k_1}{k_3} r_1^n,\, \frac{k_2}{k_3} r_2^n,\, r_3^n\right).$$

Moreover, $h\left(r_3^n\right) = nh\left(r_3\right)$ [56]. Combining all this we obtain that

$$nh\left(r_3\right) < G^{\frac{1}{3} + \mathcal{C}_2 \frac{\log\log\log G}{\log\log G}}.$$

It follows that

$$n < \frac{G^{\frac{1}{3} + \mathcal{C}_2 \frac{\log\log\log G}{\log\log G}}}{h\left(r_3\right)},$$

giving an upper bound for n.

Explicitly, given a recurrence relation satisfying the given conditions, we first check whether there are any periodic zeroes in arithmetic progressions; as noted in Chapter 2, there exists and algorithm to do this [42]. If so, we are done. If not, we apply the above method, which gives an upper bound for the maximal value of n such that $a_n = 0$. We numerically check the values of a_x for x less than the obtained upper bound. This answers the question as to whether the recurrence sequence has a zero.

Example 1. Consider the linear recurrence sequence with $a_0 = 9$, $a_1 = 46$, $a_2 = 254$ and

$$a_n = 10a_{n-1} - 31a_{n-2} + 30a_{n-3}.$$

This sequence has characteristic polynomial

$$x^3 - 10x^2 + 31x - 30,$$

with roots 2, 3 and 5. Thus,

$$a_n = k_1 2^n + k_2 3^n + k_3 5^n,$$

where k_1, k_2 and k_3 are to be found. They are found to be $k_1 = 7$, $k_2 = -11$ and $k_3 = 13$, so

$$a_n = 7 \cdot 2^n - 11 \cdot 3^n + 13 \cdot 5^n.$$

This means $G = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 = 30030$. The largest logarithmic height of the roots is $h(5) = \log 5$. It follows that if $a_n = 0$, then

$$\begin{aligned} n &< \frac{30030^{\frac{1}{3} + \mathcal{C}_3 \frac{\log\log\log 30030}{\log\log 30030}}}{\log 5} \\ &< 20 \cdot 43^{\mathcal{C}_4}. \end{aligned}$$

In principle, $\mathcal{C}_3$ and $\mathcal{C}_4$ can be computed following the proof given in this paper. This gives an upper bound for n.

In this example, once we derive $a_n = 7 \cdot 2^n + 11 \cdot 3^n + 13 \cdot 5^n$, it is clear there are no zeroes. With more complicated examples, it may not be so obvious.

4.1.2 Case Where Terms are Not Coprime

Assume we have a linear recurrence relation as above with characteristic polynomial $f(x)$ with roots r_1, r_2, r_3. There are constants k_1, k_2, k_3 such that

$$a_n = k_1 r_1^n + k_2 r_2^n + k_3 r_3^n.$$

We assume nothing about coprimeness. If they are all coprime, we're done as above. We thus assume $k_1 r_1^n$, $k_2 r_2^n$, $k_3 r_3^n$ are not coprime.

If there exists an n such that $a_n = 0$, then the same prime ideal must divide all 3 terms. We can see this as if

$$0 = k_1 r_1^n + k_2 r_2^n + k_3 r_3^n,$$

then

$$-k_1 r_1^n = k_2 r_2^n + k_3 r_3^n,$$

and it follows if a prime ideal divides two of these as ideals, it has to divide the third.

More rigorously,

$$\operatorname{ord}_{\mathfrak{p}}(a+b) \geq \min\{\operatorname{ord}_{\mathfrak{p}}(a), \operatorname{ord}_{\mathfrak{p}}(b)\}, \tag{4.1}$$

with equality when $\operatorname{ord}_{\mathfrak{p}}(a) \neq \operatorname{ord}_{\mathfrak{p}}(b)$. The claim directly follows from this. It also follows from this that at least two of the terms are divisible by the prime ideal to the same order.

We consider these as ideals of $\mathcal{O}_K$. Assume that $\mathfrak{q}$ is a prime ideal of $\mathcal{O}_K$ dividing all three terms, and that $\operatorname{ord}_{\mathfrak{q}}(k_1 r_1^n) = \operatorname{ord}_{\mathfrak{q}}(k_2 r_2^n) = l$. We write

$$\begin{aligned}
k_1 r_1^n \mathcal{O}_K &= \mathfrak{p}_{k_1,1}^{e_{k_1,1}} \cdots \mathfrak{p}_{k_1,a}^{e_{k_1,a}} \mathfrak{p}_{r_1,1}^{n\cdot e_{r_1,1}} \cdots \mathfrak{p}_{r_1,b}^{n\cdot e_{r_1,b}} \mathfrak{q}^l \\
k_2 r_2^n \mathcal{O}_K &= \mathfrak{p}_{k_2,1}^{e_{k_2,1}} \cdots \mathfrak{p}_{k_2,a}^{e_{k_2,c}} \mathfrak{p}_{r_2,1}^{n\cdot e_{r_2,1}} \cdots \mathfrak{p}_{r_2,b}^{n\cdot e_{r_2,d}} \mathfrak{q}^l \\
k_3 r_3^n \mathcal{O}_K &= \mathfrak{p}_{k_3,1}^{e_{k_3,1}} \cdots \mathfrak{p}_{k_3,f}^{e_{k_3,f}} \mathfrak{p}_{r_3,1}^{n\cdot e_{r_3,1}} \cdots \mathfrak{p}_{r_3,b}^{n\cdot e_{r_3,g}} \mathfrak{q}^m,
\end{aligned}$$

where these ideals are prime ideals of $\mathcal{O}_K$.

Note, it may be the case that $l = m$. Also, if $\mathfrak{q} \mid r_1^n$, this implies that $l = an$ for some a, but we will see this doesn't matter for the argument. Finally, it may be that there is a further prime ideal that divides all three terms; if so, we apply the following process iteratively on all prime ideals dividing all three terms.

We now move to the Hilbert Class Field L. All the ideals above are principal as ideals of $\mathcal{O}_L$, so we can write

$$\begin{aligned}
k_1 r_1^n &= u_1 p_{k_1,1}^{e_{k_1,1}} \cdots p_{k_1,a}^{e_{k_1,a}} p_{r_1,1}^{n\cdot e_{r_1,1}} \cdots p_{r_1,b}^{n\cdot e_{r_1,b}} q^l \\
k_2 r_2^n &= u_2 p_{k_2,1}^{e_{k_2,1}} \cdots p_{k_2,a}^{e_{k_2,c}} p_{r_2,1}^{n\cdot e_{r_2,1}} \cdots p_{r_2,b}^{n\cdot e_{r_2,d}} q^l \\
k_3 r_3^n &= u_3 p_{k_3,1}^{e_{k_3,1}} \cdots p_{k_3,f}^{e_{k_3,f}} p_{r_3,1}^{n\cdot e_{r_3,1}} \cdots p_{r_3,b}^{n\cdot e_{r_3,g}} q^m,
\end{aligned}$$

where the terms on the right hand side are all elements of L that generate the relevant principal ideals. Thus, we can now write

$$\begin{aligned}
a_n =& u_1 p_{k_1,1}^{e_{k_1,1}} \cdots p_{k_1,a}^{e_{k_1,a}} p_{r_1,1}^{n\cdot e_{r_1,1}} \cdots p_{r_1,b}^{n\cdot e_{r_1,b}} q^l \\
&+ u_2 p_{k_2,1}^{e_{k_2,1}} \cdots p_{k_2,a}^{e_{k_2,c}} p_{r_2,1}^{n\cdot e_{r_2,1}} \cdots p_{r_2,b}^{n\cdot e_{r_2,d}} q^l \\
&+ u_3 p_{k_3,1}^{e_{k_3,1}} \cdots p_{k_3,f}^{e_{k_3,f}} p_{r_3,1}^{n\cdot e_{r_3,1}} \cdots p_{r_3,b}^{n\cdot e_{r_3,g}} q^m.
\end{aligned}$$

By (4.1), we see that $m \geq l$. We thus divide through the above equation by q^l. After repeating this for all prime ideals dividing all three terms, the remaining terms on the right hand side will all be coprime. We now assume that there exists an n such that $a_n = 0$, and we are in the same position as Section 4.1.1, and the argument follows identically.

Remark. We note that this application also follows from Győry's result [18].

4.2 SMOOTH SOLUTIONS TO THE *abc* CONJECTURE

In this section we will prove Theorem 2.1.4.4. We will first prove the following lemma.

Lemma 4.2.0.1. *Let $(X, Y, Z) \in \mathbb{Z}^3$ be a triple with smoothness $S(X, Y, Z)$ and radical $G(X, Y, Z)$ defined as above. Then*

$$G(X, Y, Z) \leq e^{3S(X,Y,Z)}. \tag{4.2}$$

Proof. From [45] and [46], we know that for $n > 2$,

$$n \log n < p_n < 2n \log n.$$

where p_n denotes the n'th rational prime.

It follows that for $n \geq 6$,

$$p_n \leq 2n \log n.$$

Indeed, computationally we can check primes 2, 3, 5, 7, 11 and we find that for $n > 2$ the above inequality holds.

Further, by Rosser's Theorem [46],

$$p_n > n \log n.$$

Thus, it follows that the product of the first k primes satisfies the following inequality:

$$\begin{aligned}
\prod_{i=1}^{k} p_i &\leq 2 \cdot 3 \cdot \prod_{i=3}^{k} 2i \log i \\
&= 2 \cdot 3 \cdot 2^{k-2} \left(4 \cdot 5 \cdots k\right) \cdot \prod_{i=3}^{k} \log i \\
&= 2^{k-2} \cdot k! \cdot \prod_{i=3}^{k} \log i \\
&\leq 2^{k} \cdot k^{k} \cdot \prod_{i=3}^{k} \log i. \qquad (4.3)
\end{aligned}$$

We now take logs of each side of the inequality attaining

$$\begin{aligned}
\log\left(\prod_{i=1}^{k} p_i\right) &\leq \log\left(2^{k} \cdot k^{k} \cdot \prod_{i=3}^{k} \log i\right) \\
&= k \log 2 + k \log k + \log\left(\prod_{i=3}^{k} \log i\right) \\
&= k \log 2 + k \log k + \sum_{i=3}^{k} \log \log i \\
&\leq k \log 2 + k \log k + k \log \log k \\
&\leq k \log 2 + k \log k + k \log k \\
&\leq 3k \log k \\
&\leq 3p_k \qquad (4.4)
\end{aligned}$$

where the last line follows from Rosser's Theorem.

It thus follows that for a triple of pairwise coprime integers satisfying $X + Y = Z$ with smoothness $S\left(X,\, Y,\, Z\right)$,

$$\begin{aligned}
G\left(X,\, Y,\, Z\right) &\leq \prod_{\substack{p \text{ prime} \\ p \leq S(X,\, Y,\, Z)}} p \\
&\leq e^{3S(X,\, Y,\, Z)}, \qquad (4.5)
\end{aligned}$$

by the above inequality. □

We are now in a position to prove Theorem 2.1.4.4.

Proof. By Northcott's Theorem, we can assume in the following that $H\left(X,\, Y,\, Z\right) > B$ for any given bound B as we will only be excluding finitely many possible solutions $\left(X,\, Y,\, Z\right)$ satisfying (2.7).

For ease of notation, write

$$T := \log\log H\left(X,\, Y,\, Z\right). \tag{4.6}$$

By assumption, we have that

$$S\left(X,\, Y,\, Z\right) < T\frac{\log T}{\log\log T\phi\left(T\right)}. \tag{4.7}$$

We study triples $(X,\, Y,\, Z)$ satisfying (4.7), and note that for a sufficiently large $H\left(X,\, Y,\, Z\right)$,

$$T\frac{\log T}{\log\log T\phi\left(T\right)} < \left(\log H\left(X,\, Y,\, Z\right)\right)^{\frac{1}{2}},$$

so we can apply Corollary 3.3.1.4. We note the choice of the exponent to be $\frac{1}{2}$ is incidental; indeed any exponent less than $\frac{2}{3}$ could have been chosen. Further, the Hilbert Class Field of $\mathbb{Q}$ is itself $\mathbb{Q}$ [11], so the radical G in this case is defined over $\mathbb{Q}$ and coincides with the radical given in [27].

By Corollary 3.3.1.4, we know that

$$\log H\left(X,\, Y, Z\right) < G^{\mathcal{C}_5 \frac{\log\log\log G}{\log\log G}}.$$

From the upper bound for G given at (4.2) in Lemma 4.2.0.1, we obtain from the above that

$$\begin{aligned}\log H\left(X,\, Y,\, Z\right) &< e^{3\mathcal{C}_5 S(X,\, Y,\, Z)\frac{\log\log\log e^{3S(X,\, Y,\, Z)}}{\log\log e^{3S(X,\, Y,\, Z)}}}\\ &= e^{\mathcal{C}_6 S(X,\, Y,\, Z)\frac{\log\log 3S(X,\, Y,\, Z)}{\log 3S(X,\, Y,\, Z)}}.\end{aligned} \tag{4.8}$$

It follows from the above that

$$\log\log H\left(X,\, Y,\, Z\right) = T < \mathcal{C}_6 S\left(X,\, Y,\, Z\right)\frac{\log\log 3S\left(X,\, Y,\, Z\right)}{\log 3S\left(X,\, Y,\, Z\right)}. \tag{4.9}$$

We recall that by Northcott's Theorem again, we can assume that $S\left(X,\, Y,\, Z\right)$ can be larger than any given constant while only dropping finitely many solutions to $X + Y = Z$. Thus, only loosing finitely many solutions, for sufficiently large $S\left(X,\, Y,\, Z\right)$ it follows from (4.9) that

$$T < \mathcal{C}_7 S\left(X,\, Y,\, Z\right)\frac{\log\log S\left(X,\, Y,\, Z\right)}{\log S\left(X,\, Y,\, Z\right)}. \tag{4.10}$$

Taking logarithms on both side of inequality (4.10), we can assume $S\,(X,\,Y,\,Z)$ is large enough to give that

$$\begin{aligned}
\log T &< \log\left(\mathcal{C}_7 S\,(X,\,Y,\,Z)\,\frac{\log\log S\,(X,\,Y,\,Z)}{\log S\,(X,\,Y,\,Z)}\right)\\
&= \log \mathcal{C}_7 + \log S\,(X,\,Y,\,Z) + \log\log\log S\,(X,\,Y,\,Z) - \log\log S\,(X,\,Y,\,Z)\\
&< 2\log S\,(X,\,Y,\,Z)\,. \qquad (4.11)
\end{aligned}$$

For ease later, we divide the above by 2 to give that

$$\frac{1}{2}\log T < \log S\,(X,\,Y,\,Z)\,. \qquad (4.12)$$

For triples $(X,\,Y,\,Z)$ satisfying (4.7), taking logarithms in inequality (4.7) we deduce that

$$\begin{aligned}
\log S\,(X,\,Y,\,Z) &< \log\left(T\frac{\log T}{\log\log T\phi\,(T)}\right)\\
&= \log T + \log\log T - \log\log\log T - \log\phi\,(T)\\
&< 2\log T. \qquad (4.13)
\end{aligned}$$

We note that this also implies that for sufficiently large $S\,(X,\,Y,\,Z)$,

$$\log\log S\,(X,\,Y,\,Z) < 2\log\log T.$$

Substituting this and (4.12) into (4.7) we obtain that

$$\begin{aligned}
S\,(X,\,Y,\,Z) &< T\frac{\log T}{\log\log T\phi\,(T)}\\
&< T\frac{2\log S\,(X,\,Y,\,Z)}{\frac{1}{2}\log\log\left(S\,(X,\,Y,\,Z)\right)\phi\,(T)}\\
&= 4T\frac{\log S\,(X,\,Y,\,Z)}{\log\log\left(S\,(X,\,Y,\,Z)\right)\phi\,(T)}. \qquad (4.14)
\end{aligned}$$

Rearranging the above we obtain that

$$\frac{1}{4}\frac{S\,(X,\,Y,\,Z)\log\log S\,(X,\,Y,\,Z)}{\log S\,(X,\,Y,\,Z)}\phi\,(T) < T \qquad (4.15)$$

We now have two inequalities relating S and T, namely (4.10) and (4.15) given above. We compare these directly to find that

$$\frac{1}{4}\frac{S\,(X,\,Y,\,Z)\log\log S\,(X,\,Y,\,Z)}{\log S\,(X,\,Y,\,Z)}\phi\,(T) < T < \mathcal{C}_7\frac{S\,(X,\,Y,\,Z)\log\log S\,(X,\,Y,\,Z)}{\log S\,(X,\,Y,\,Z)}, \qquad (4.16)$$

which we rewrite as

$$\frac{S(X,Y,Z)\log\log S(X,Y,Z)}{\log S(X,Y,Z)}\phi(T) < T < \mathcal{C}_8\frac{S(X,Y,Z)\log\log S(X,Y,Z)}{\log S(X,Y,Z)}. \tag{4.17}$$

Cancelling terms on both sides gives us that

$$\phi(T) < \mathcal{C}_8. \tag{4.18}$$

However, $\phi(T)$ tends to $+\infty$ as T tends to $+\infty$, and as $T = \log\log H(X,Y,Z)$, this happens as $H(X,Y,Z)$ gets arbitrarily large. Thus, there is a value B such that if $H(X,Y,Z) > B$, then (4.18) cannot hold. This gives an upper bound for values of $H(X,Y,Z)$ such that the triple satisfies the assumptions of the theorem. It thus follows by Northcott's Theorem that there are only finitely many primitive triples (X,Y,Z) satisfying $(X+Y=Z)$ with

$$S(X,Y,Z) \leq \log\log H(X,Y,Z) \frac{\log\log\log H(X,Y,Z)}{\log\log\log\log H(X,Y,Z)\,\phi(\log\log H(X,Y,Z))}$$

□

We note that we could also directly prove that there are only finitely many primitive integer triples (X,Y,Z) satisfying $X+Y=Z$ with

$$S(X,Y,Z) < c\log\log H(X,Y,Z)$$

for any constant $c \in \mathbb{R}$, $c > 0$ using the same method of proof as above, though this result follows from Theorem 2.1.4.4 as stated previously.

We further note that this method of proof could be employed in certain number fields (hence why we appeal to Northcott's Theorem as opposed to simpler techniques); however, the Conjecture currently only holds for integers. It is an open problem to generalise the Conjecture to number fields [27]

— 5 —

Quantitative Diophantine Approximation: Background and Preliminaries

5.1 DIOPHANTINE APPROXIMATION AND METRIC NUMBER THEORY

Roughly speaking, we can think of Diophantine approximation as a branch of number theory that studies quantitatively the density of the rationals within the reals. This area is very associated with Metric Number Theory, where we are often interested in studying the size (in terms of, for example, the Lebesgue measure or Hausdorff dimension) of Diophantine sets satisfying certain properties. One problem here is, given an approximating function $\psi : \mathbb{N} \to [0, \infty)$, to determine the size of the set

$$W(\psi) := \limsup_{q\to\infty} \left\{ x \in [0,1) \, : \, \left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q}, \text{ for some } p \in \mathbb{N} \right\}.$$

We say that the elements of $W(\psi)$ are ψ-approximable; and note that $x \in W(\psi)$ if there exist infinitely many positive integers (p, q) satisfying

$$\left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q}. \tag{5.1}$$

It is clear that every $x \in \mathbb{R}$ is within $\frac{1}{2q}$ of a rational number with denominator q; that is, we can take $\psi(q) = \frac{1}{2}$ for all q and it is clear that $\mu\left(W\left(\psi\right)\right) = 1$, where μ denotes the Lebesgue measure on $\mathbb{R}$. It is natural to ask how far this can be improved.

We now consider $\psi(q) = 1/q$; in this case, the theory of continued fractions tells us that every x is $\frac{1}{q}$-approximable. In fact, this is the content of a theorem of Dirichlet.

Theorem (Dirichlet's Theorem, 1842)**.** *For any $x \in \mathbb{R}$ and $N \in \mathbb{N}$, there exist $p, q \in \mathbb{Z}$ such that*

$$\left| x - \frac{p}{q} \right| < \frac{1}{qN},$$

with $1 \leq q \leq N$.

This has the following immediate corollary; we note the proof of the following can also be given via continued fractions, the theory of which is older than Dirichlet's Theorem.

Theorem. *Let $x \in \mathbb{R} \backslash \mathbb{Q}$. Then there exist infinitely many p, q such that* $\gcd(p, q) = 1$ *and*

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}. \tag{5.2}$$

We note that this theorem is true for all $x \in \mathbb{R}$ if we remove the condition that $\gcd(p, q) = 1$. Further, to connect this with our discussion above, we note we have shown that $\mu\left(W\left(\frac{1}{q}\right)\right) = 1$; we recall in our definition of $W(\psi)$ we do not insist that $\gcd(p, q) = 1$.

We make some brief comments on the connection between these approximations and continued fractions; in fact we can find best approximations to real numbers α by considering convergents of continued fractions. More explicitly, given $\alpha \in \mathbb{R}$, we call the rational number $\frac{p}{q}$, $(p, q) \in \mathbb{Z} \times \mathbb{N}$ with $\gcd(p, q) = 1$ a best approximation to α if $|q\alpha - p| \leq \frac{1}{2}$, and for all $(p', q') \in \mathbb{Z} \times \mathbb{N}$ with $q' < q$, we have that

$$|q'\alpha - p'| > |q\alpha - p|.$$

We now consider the following Theorem:

Theorem (Legendre)**.** *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let p, q be coprime integers with $q > 0$ and let*

$$\left| \alpha - \frac{p}{q} \right| \leq \frac{1}{2q^2}.$$

Then $\frac{p}{q}$ is a best approximation to α.

The connection to continued fractions becomes clear in with the next theorem.

Theorem (Vahlen, 1895)**.** *Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Then at least one of any two consecutive convergents of the continued fraction of α satisfies the inequality*

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{2q^2}.$$

We have shown through the theory of continued fractions that we can improve inequality (5.2) by a factor of a half; that is, we can replace $\frac{1}{q^2}$ with $\frac{1}{2q^2}$ in Dirichlet's Theorem; removing the coprime condition also allows this improvement for rationals. We note that this had been shown before Vahlen, and will show this below.

Now it is natural to ask how much we can improve inequality (5.2), and a theorem by Hurwitz shows that we can only improve this inequality for all $x \in \mathbb{R}$ so far.

Theorem (Hurwitz's Theorem, 1891)**.** *Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Then there exist infinitely many p, q such that* $\gcd(p, q) = 1$ *and*

$$\left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}. \tag{5.3}$$

We note that the constant $1/\sqrt{5}$ is the best constant possible; for all $\varepsilon > 0$, we can find an irrational x such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{(\sqrt{5}+\varepsilon)q^2} \tag{5.4}$$

has only finitely many coprime solution pairs (p, q). Namely, if we take x to be the Golden Ratio $\phi = \frac{1+\sqrt{5}}{2}$ then for any ε, we can only find finitely many integer solution pairs (p, q) with $\gcd(p, q) = 1$ to (5.4)..

This comment means that if we take $\psi(q) < \frac{1}{\sqrt{5}q}$ then (5.1) will not hold for all $x \in \mathbb{R}$. This naturally leads us to ask under what conditions on ψ can we say that $W(\psi)$ has full measure? Khintchine gave the following elegant theorem as an answer to this question:

Theorem (Khintchine, 1924)**.** *Let $\psi : \mathbb{N} \to [0, +\infty)$ be a function. Suppose ψ is (eventually) non-increasing. Then*

$$\mu(W(\psi)) = \begin{cases} 0, & \text{if } \sum_{q=1}^{\infty} \psi(q) < +\infty, \\ 1, & \text{if } \sum_{q=1}^{\infty} \psi(q) = +\infty, \end{cases}$$

where μ is the Lebesgue measure on $[0, 1)$.

One may wonder why $W(\psi)$ must have either full or zero measure; this is due to a zero-one law, further discussion of which can be found in [23][3].

Khintchine's Theorem tells us that, if ψ is non-increasing, the convergence or divergence of the sum

$$\sum_{q=1}^{\infty} \psi(q) \tag{5.5}$$

entirely determines whether or not almost all $x \in [0,1)$ are ψ-approximable; for further details we refer the reader to [23]. We note that Khintchine's Theorem lets us give immediate improvements on the example $\psi(q) = \frac{1}{q}$ given above; for example, as

$$\sum_{q=1}^{\infty} \frac{1}{q \log(q+1)} = +\infty,$$

we know that $W\left(\frac{1}{q\log(q+1)}\right)$ has full Lebesgue measure; that is we can improve the function given previously by a logarithmic factor.

One may ask whether we can remove the condition that ψ is monotonic in Khintchine's Theorem. In the convergence case, the condition can be removed as the proof is an application of the Borel-Cantelli Lemma. The divergence case is somewhat more tricky, and monotonicity is required. In [13], Duffin and Schaeffer consider the function ϑ which is non-monotonic, such that $\sum_q \vartheta(q)$ diverges, but $\mu\left(W\left(\vartheta\right)\right) = 0$; we briefly give some details below and refer the reader to [13] for further information. We follow the ideas from [13], and follow the presentation of [3].

We recall two well known facts; for any $N \in \mathbb{N}$, prime p and $s > 0$ we have

$$\sum_{q|N} q = \prod_{p|N} (1+p),$$

and that

$$\prod_{p} (1 + p^{-s} = \frac{\zeta(s)}{\zeta(2s)}.$$

The second equality here gives us that

$$\prod_{p} \left(1 + p^{-1}\right) = \infty,$$

so it follows that we can find a sequence of square-free positive integers $(N_i)_i$, $i = 1,\, 2,\, \ldots$ such that $\gcd\left(N_i,\, N_j\right) = 1$ for $i \neq j$, and

$$\prod_{p|N_i} \left(1 + p^{-1}\right) > 2^i + 1. \tag{5.6}$$

We now define

$$\vartheta(q) = \begin{cases} \frac{2^{-i-1}q}{N_i} \text{ if } q > 1 \text{ and } q \mid N_i \text{ for some i,} \\ 0 \text{ else.} \end{cases}$$

One can show using the convergence Borel-Cantelli Lemma that $\mu\left(W\left(\vartheta\right)\right) = 0$, but it is shown by the first fact and (5.6) that

$$\sum_{q=1}^{\infty} \vartheta(q) = \sum_{i=1}^{\infty} 2^{-i-1} \frac{1}{n_i} \sum_{\substack{q>1 \\ q|N_i}} q = \infty.$$

This shows that the assumption that ψ is monotonic is necessary in Khintichine's Theorem.

Duffin and Schaeffer went on to consider a generalisation of this problem, considering arbitrary approximation functions $\psi : \mathbb{N} \to \mathbb{R}^{+}$. We note that, contrary to Dirichlet's Theorem above, we do not have the condition that $\gcd(p, q) = 1$ in the definition of $W(\psi)$. Duffin and Schaeffer also added this condition to relate the rational $\frac{p}{q}$ with the error of approximation $\frac{\psi(q)}{q}$ uniquely. They thus considered the set

$$W'(\psi) := \limsup_{q \to \infty} \left\{ x \in [0, 1) : \left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q}, \text{ for some } p \in \mathbb{N}, \gcd\left(p, q\right) = 1. \right\}.$$

They went on to give the following conjecture (previously known as the Duffin-Schaeffer Conjecture) which was proven by Koukoulopoulos and Maynard in 2019.

Theorem (Koukoulopoulos-Maynard, 2019)**.** *Let $\psi : \mathbb{N} \to [0, +\infty)$ be a function. Then*

$$\mu(W'(\psi)) = \begin{cases} 0, & \text{if } \sum_{q=1}^{\infty} \frac{\psi(q)\varphi(q)}{q} < +\infty, \\ 1, & \text{if } \sum_{q=1}^{\infty} \frac{\psi(q)\varphi(q)}{q} = +\infty, \end{cases}$$

where μ is the Lebesgue measure on $[0, 1)$ and $\varphi(x)$ is the Euler phi function, defined to be the number of natural numbers coprime to x that are less than or equal to x.

That is, the convergence (or divergence respectively) of the series

$$\sum_{q=1}^{\infty} \frac{\psi(q)\varphi(q)}{q} \tag{5.7}$$

implies that the Lebesgue measure of $W'(\psi)$ is 0 (or 1 respectively).

Again, the convergence case follows directly from the Borel-Cantelli Lemma, while the divergence case required deeper insight; for the full proof we refer to [26]. We return to results around this theorem later in the thesis.

Khintchine's Theorem tells us about the measure of $W(\psi)$, but to gain a deeper understanding of Khintchine's Theorem is to consider the number of solutions to

(5.1), given that $1 \leq q \leq Q$ for a fixed $Q \in \mathbb{N}$. We add the further restriction that $\psi : \mathbb{N} \to [0, 1/2)$; this ensures that given any integer q, there is maximally one p such that $(p,\ q)$ satisfy (5.1). Thus, counting solution pairs $(p,\ q)$ to (5.1) is equivalent to counting the number of q's for which we can find a p satisfying (5.1), subject to the condition that $1 \leq q \leq Q$. More explicitly, given $x \in [0, 1)$ and $Q \in \mathbb{N}$ we define

$$S(x, Q) := \# \left\{ q \in \mathbb{N} \cap [1, Q] \ : \ \left| x - \frac{p}{q} \right| < \frac{\psi(q)}{q}, \text{ for some } p \in \mathbb{N} \right\};$$

that is, $S(x,\ Q)$ is the number of integer pairs $(p,\ q)$ satisfying (5.1) and $1 \leq q \leq Q$.

In this context, Khintchine's Theorem tells us that if ψ is non-increasing and (5.5) is divergent, then for almost every $x \in [0, 1)$,

$$\lim_{Q \to \infty} S(x, Q) = +\infty.$$

We have seen that there are infinitely many solutions satisfying (5.1), but Khintchine's Theorem does not give a growth rate in the number of solutions, nor does it suggest any quantitative relation between $S(x,\ Q)$ and ψ. We refer to these problems as the quantitative problem of Khintchine's Theorem. Schmidt was able to give asymptotic relations between $S(x,\ Q)$ and ψ; we give the statement as stated in [23]:

Theorem (Schmidt, 1960)**.** *Let $\psi : \mathbb{N} \to [0, 1/2)$. Suppose ψ is non-increasing and* (5.5) *diverges. Then for any $\varepsilon > 0$ and almost every $x \in [0, 1)$, as $Q \to \infty$ we have that*

$$S(x, Q) = 2\Psi(Q) + O_{\psi,\varepsilon,x}\left(\Psi^{1/2}(Q) \log^{2+\varepsilon} \Psi(Q) \right),$$

where for any $Q \in \mathbb{N}$, $\Psi(Q)$ is defined to be

$$\Psi(Q) := \sum_{q=1}^{Q} \psi(q).$$

Roughly speaking, Schmidt's Theorem tells us that given $x \in [0, 1)$, the number of solution pairs (p, q) to (5.1) such that $1 \leq q \leq Q$ is approximately equal to $2\Psi(Q)$ when Q is sufficiently large.

In order for the result to hold for almost all $x \in \mathbb{R}$, we have that the error term is asymptotic; that is, the error term above involves an implicit constant which depends on the point x. These asymptotics make it hard for researchers to apply these theorems to practical experiments; one example of a field where these results may be useful is signal processing, where it is known that Diophantine approximation gives results in this area [4].

5.1.1 Some Asymptotic Results

In many cases, the proofs of statements like Schmidt's Theorem rely on tools such as Lemmas 1.4 and 1.5 of [23], which we state here for completeness.

Lemma (Lemma 1.4 of [23])**.**
Let (X, Σ, μ) be a measure space such that $0 < \mu(X) < \infty$. Let $f_k(x)$, $k \in \mathbb{N}$ be a sequence of non-negative real valued functions, and let f_k, ϕ_k be sequences of real numbers such that

$$0 \leq f_k \leq \phi_k \leq 1,\ k \in \mathbb{N}.$$

Suppose that for every $N \geq 1$ we have that

$$\int_X \left(\sum_{k=1}^{N} (f_k(x) - f_k) \right)^2 d\mu \leq K\Phi(N), \tag{5.8}$$

where K is an absolute constant and $\Phi(N) = \sum_{k=1}^{N} \phi_k$. We further assume that $\Phi(N) \to \infty$ as $N \to \infty$. Then for every $\varepsilon > 0$ and almost all $x \in X$, we have that

$$\sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{N} f_k + O\left(\Phi^{2/3}(N) \log(\Phi(N) + 2)^{1/3+\varepsilon}\right).$$

Under a stronger assumption, we can improve the asymptotics, as we see below in Lemma 1.5 of [23].

Lemma (Lemma 1.5 of [23])**.**
Let (X, Σ, μ) be a measure space such that $0 < \mu(X) < \infty$. Let $f_k(x)$, $k \in \mathbb{N}$ be a sequence of non-negative real valued functions, and let f_k, ϕ_k be sequences of real numbers such that

$$0 \leq f_k \leq \phi_k,\ k \in \mathbb{N}.$$

Let $\Phi(N) = \sum_{k=1}^{N} \phi_k$ and assume that $\Phi(N) \to \infty$ as $N \to \infty$. Suppose that for arbitrary integers m, n, $(1 \leq m < n)$ we have that

$$\int_X \left(\sum_{m \leq k < n} (f_k(x) - f_k) \right)^2 d\mu \leq K \sum_{m \leq k < n} \phi_k, \tag{5.9}$$

where K is an absolute constant. Then for every $\varepsilon > 0$ and almost all $x \in X$, we have that

$$\sum_{k=1}^{\infty} f_k(x) = \sum_{k=1}^{N} f_k + O\left(\Phi^{1/2}(N) \log(\Phi(N) + 2)^{3/2+\varepsilon} + \max_{1 \leq k \leq N} f_k\right).$$

The proof of both of these relies on the Borel-Cantelli Lemmas, which we state here as it will be used throughout the rest of this thesis. We give the Borel-Cantelli Lemmas as stated in [23]. Though we will only use the first Borel-Cantelli Lemma, we state both for completeness.

Lemma (The first Borel-Cantelli Lemma)**.** *Let X be a measure space with measure λ. Let A_j, $j = 1, 2, \ldots$ be a collection of measurable subsets of X. Then, if*

$$\sum_{j=1}^{\infty} \mu\left(A_j\right) < +\infty,$$

then almost all members of X (with respect to λ) belong to only finitely many of the A_j.

This is sometimes referred to as the convergence Borel-Cantelli Lemma. The divergence case is a little trickier; we need some form of independence between the sets.

Lemma (The second Borel-Cantelli Lemma)**.** *Let X be a measure space with measure λ, and suppose that $\lambda\left(X\right) = T < \infty$. Let A_j, $j = 1, 2, \ldots$ be a collection of measurable subsets of X such that*

$$T\lambda\left(A_j \cap A_k\right) = \lambda(A_j)\lambda(A_k)$$

for $j \neq k$. Then, if

$$\sum_{j=1}^{\infty} \lambda(A_j) = \infty,$$

almost all members of X belong to infinitely many of the A_j.

Remark. We note that if $T = 1$, this is the standard definition of independence. We also note that the independence condition can be weakened to an independence condition often referred to as quasi-independence on average: this is the condition where there exists a constant $C > 0$ such that

$$\sum_{s,\, t=1}^{Q} \lambda\left(A_s \cap a_t\right) \leq C\left(\sum_{s=1}^{Q} \lambda\left(A_s\right)\right)^2$$

holds for infinitely many $Q \in \mathbb{N}$. We can then conclude that

$$\lambda\left(\limsup_{q\to\infty} \cap_{t=1}^{\infty} \cup_{q=t}^{\infty} A_q\right) = \frac{1}{C}.$$

For further discussion of this we refer the reader to [50][3].

We finally remark that Lemmas 1.4 and 1.5 of [23], as stated above can be considered stronger versions of the divergence Borel-Cantelli Lemma.

The aim of the next chapter is to make Lemmas 1.4 and 1.5 of [23] effective other than on a subset of size δ, where the choice of δ can depend upon the application of the effective lemma. We will then apply these effective theorems, and some similar, in a variety of contexts which we shall now discuss.

5.2 BACKGROUND FOR APPLICATIONS

In Chapter 7 of this thesis, we will give applications of the theorems proved in Chapter 6 to the following topics:

- Effective Schmidt's Theorem
- Quantitative Koukoulopoulos-Maynard Theorem (see [1])
- Inhomogeneous Diophantine Approximation on M_0-sets (see [43])
- Normal numbers
- Strong Law of Large Numbers

We have discussed Schmidt's Theorem above, so the rest of this subsection will discuss the background for the rest of these topics.

5.2.1 Quantitative Koukoulopoulos-Maynard Theorem

Expanding on the work of Koukoulopoulos and Maynard, an asymptotic relation for the quantitative Duffin-Schaeffer type problem is given by Aistleitner, Borda and Hauke in [1]; that is, the paper gives an asymptotic description of the number of solutions satisfying (5.1) with the extra condition that $\gcd(p, q) = 1$. They proved the following theorem.

Theorem (Aistleitner, Borda, Hauke, 2022)**.** *Let $\psi : \mathbb{N} \to [0, 1/2]$. Suppose* (5.7) *diverges and let $C > 0$ be arbitrary. Then for almost every $x \in [0, 1)$, we have that*

$$S'(x, Q) = \Psi'(Q) \left(1 + O_{\psi,C,x}\left(\frac{1}{(\log(\Psi'(Q)))^C}\right)\right), \tag{5.10}$$

as $Q \to \infty$, where

$$S'(x, Q) := \#\left\{q \in \mathbb{N} \cap [1, Q] : \left|x - \frac{p}{q}\right| < \frac{\psi(q)}{q}, \text{ for some } p \in \mathbb{N} \text{ with } \gcd(p, q) = 1.\right\}, \tag{5.11}$$

and $\Psi'(Q)$ is defined to be

$$\Psi'(Q) := 2\sum_{q=1}^{Q} \frac{\psi(q)\varphi(q)}{q}.$$

The proof of this involves an application of a result akin to Harman's Lemma 1.4 [23], which we give explicitly here. We note the version below is given in a more general setting than in [1].

Lemma 5.2.1.1. *Suppose that (X, Ω, μ) is a measure space such that $0 < \mu(X) < \infty$ and let K be a positive real constant. Let $f_q : X \to [0, K]$ be a sequence of μ-measurable functions and let f_q, ϕ_q be sequences of real numbers such that*

$$0 \leq f_q \leq \phi_q \leq K.$$

Write $\Psi(Q) = \sum_{q=1}^{Q} \phi_q$, and suppose that $\Psi(Q) \to \infty$ and $Q \to \infty$. Suppose that for $C > 4$ and for every $Q \in \mathbb{N}$,

$$\int_X \left(\sum_{1 \leq q \leq Q} (f_q(x) - f_q)\right)^2 d\mu = O\left(\frac{\Psi(Q)^2}{(\log \Psi(Q))^C}\right). \tag{5.12}$$

Then for almost all $x \in X$, as $Q \to \infty$,

$$\sum_{1 \leq q \leq Q} f_q(x) = \sum_{1 \leq q \leq Q} f_q + O\left(\frac{\Psi(Q)}{(\log \Psi(Q))^{\sqrt{C}-1}}\right).$$

This is proved in the necessary case in Section 2 of [1], following the method of proof of Lemma 1.4 of [23]. In Chapter 6, we prove the more general lemma above, before proving an effective version that we shall apply to give a quantitative version of Aistleitner, Borda and Hauke's result.

We note that in this thesis, I will give this effective up to an implicit constant, which we will discuss more in Chapter 7. This constant has been found by my collaborator and will be published separately.

5.2.2 Inhomogeneous Diophantine Approximation on M_0-sets

The effective versions of Lemmas 1.4 and 1.5 of [23] that we will prove can also be applied to give effective results akin to those in [43], which is about inhomogeneous Diophantine approximation on M_0-sets. For background and motivation we refer the reader to the introduction and further discussion in [43]; however, we begin this section by recalling some definitions to make this paper more self contained.

As usual, the Fourier transform of a non-atomic probability measure μ is defined by

$$\hat{\mu}(t) = \int e^{-2\pi i t x} \mathrm{d}\mu(x),\ t \in \mathbb{R}.$$

A closed set $E \subset \mathbb{R}$ is said to be an M_0-set if there exists a probability measure μ on E such that $\hat{\mu}$ vanishes at infinity.

Given an increasing sequence of natural numbers $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$, we call $\mathcal{A}$ lacunary if there exists a constant $K > 1$ such that for all $n \in \mathbb{N}$,

$$\frac{q_{n+1}}{q_n} \geq K. \tag{5.13}$$

Relatedly, let $\alpha \in (0, 1)$. We say that a sequence of increasing natural numbers $\mathcal{A}$ is α-separated if there exists a constant $m_0 \in \mathbb{N}$ such that for any integers $m_0 \leq m < n$, we have that if

$$1 \leq |sq_m - tq_n| < {q_m}^{\alpha}$$

for some s, $t \in \mathbb{N}$, then

$$s > m^{12}.$$

Finally, let $\psi : \mathbb{N} \to \mathbb{R}^+$ be a real, positive function and let $\gamma \in [0, 1]$. We define the counting function

$$R(x, N) = R(x, N; \gamma, \psi, \mathcal{A}) = \#\left\{1 \leq n \leq N \,:\, \|q_n x - \gamma\| \leq \psi(q_n)\right\}, \tag{5.14}$$

where $\|\alpha\| := \min\{|\alpha - m| : m \in \mathbb{Z}\}$ denotes the absolute distance from α to its nearest integer.

We will prove effective versions of Theorems 1 and 4 of [43]; we state these here for full completeness.

Theorem (Theorem 1 of [43]). *Let μ be a probability measure supported on a subset F of $[0, 1]$. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be a lacunary sequence of natural numbers. Let $\gamma \in [0, 1]$*

and $\psi : \mathbb{N} \to [0,1]$ be a real, positive function. Suppose that there exists a constant $A > 2$ such that

$$\hat{\mu}(t) = O\left((\log |t|)^{-A}\right)$$

as $|t| \to \infty$. Then, for any $\varepsilon > 0$, we have that

$$R(x,\, N) = 2\Psi(N) + O\left(\Psi(N)^{2/3}\left(\log(\Psi(N)+2)\right)^{2+\varepsilon}\right)$$

for μ-almost all $x \in F$, where

$$\Psi(N) = \sum_{n=1}^{N} \psi(q_n).$$

We now consider Theorem 4 of [43]

Theorem (Theorem 4 of [43])**.** *Let μ be a probability measure supported on a subset F of $[0,1]$. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be an increasing sequence of natural numbers that is α-separated and satisfies*

$$\log q_n > Cn^{1/B} \tag{5.15}$$

for all $n \geq 2$, and for some constants $B \geq 1$ and $C > 0$.

Let $\gamma \in [0,1]$ and let $\psi : \mathbb{N} \to [0,1]$ be a real, positive function. Suppose there exists a constant $A > 2B$ such that

$$\hat{\mu}(t) = O\left((\log |t|)^{-A}\right).$$

Then, for any $\varepsilon > 0$, we have that

$$R(x, N) = 2\Psi(N) + O\left((\Psi(N) + E(N))^{1/2}\left(\log\left(\Psi(N) + E(N) + 2\right)\right)^{2+\varepsilon}\right),$$

for μ-almost all $x \in F$, where $\Psi(N)$ is given above and

$$E(N) = \sum\sum_{1\leq m<n\leq N} (q_m,\, q_n) \min\left(\frac{\psi(q_m)}{m},\, \frac{\psi(q_n)}{q_n}\right). \tag{5.16}$$

We will make both of these theorems effective other than on a subset of measure less than or equal to an arbitrary $\delta > 0$.

5.2.3 Normal numbers

We begin by recalling some definitions. Set $b \in \mathbb{N}$, $b \geq 2$. Recall that for any real number α there is a unique expansion base b such that

$$\alpha = [\alpha] + \sum_{n=1}^{\infty} a_n b^{-n}, \tag{5.17}$$

where $[\alpha]$ denotes the integer part of α, $0 \leq a_n < b$ and $a_n < b - 1$ infinitely often. Given a fixed α, denote by $A(d,\, b,\, N)$ the number of times d is in the set $\{a_1,\, \ldots,\, a_N\}$. We say that α is simply normal to base b if

$$\lim_{N\to\infty} \frac{A(d,\, b,\, N)}{N} = \frac{1}{b}$$

for all d, $0 \leq d < b$.

We call α entirely normal to base b if it is simply normal base b^n for all $n = 1,\, 2, \ldots$. We say α is absolutely normal if it is entirely normal to all bases $b > 1$.

These definitions are different to those first given by Borel; Theorem 1.2 of [23] shows the definitions are equivalent.

An easy application of Lemma 1.4 of [23] quickly shows that almost all real numbers are simply normal to a base b, which we will give below. Applying the effective version of Lemma 1.4 of [23] in its place allows us to give an upper bound on the number of times a given digit d appears in the base b expansion for almost all real α other than on a subset of measure less than or equal to an arbitrary δ.

To give an example of how we can apply Lemma 1.4 of [23] in the asymptotic form, we will prove that almost all numbers are simply normal to a given base b; we follow the second proof of this given in [23].

We need only prove the result holds in the interval $[0,\, 1)$, as once this is proven, the result holds over all of $\mathbb{R}$. By the kth digit of a real number $x \in [0,\, 1)$ we mean a_k given by (5.17), setting $\alpha = x$. set d to be some integer such that $0 \leq d \leq b - 1$ and write

$$f_k(x) = \begin{cases} 1 \text{ if the } k\text{th digit of } x \text{ is } d, \\ 0 \text{ else.} \end{cases}$$

We further set $f_k = \frac{1}{b}$. We now consider the set

$$\{x \in [0,\, 1) \,:\, \text{the } k\text{th and } j\text{th digits of } x \text{ are both } d,\, j \neq k\}.$$

We note that this set is the union of certain intervals (where we assume $k \geq j$) of the type

$$[db^{-j} + db^{-k} + \alpha,\ db^{-j} + (d+1)b^{-k} + \alpha),$$

where α ranges over all fractions of the form

$$\sum_{\substack{n=1 \\ n \neq j}}^{k-1} a_n b^{-n},\ 0 \leq a_n \leq b-1.$$

We can see there are b^{k-2} intervals, each of length b^{-k}. It follows that, for $j \neq k$,

$$\begin{aligned}\int_0^1 f_k(x) f_j(x) \mathrm{d}x &= \mu\left(\{x \in [0,\, 1) \,:\, \text{the } k\text{th and } j\text{th digits of } x \text{ are both } d,\ j \neq k\}\right) \\ &= b^{-2}.\end{aligned}$$

It then follows that

$$\int_0^1 \left(\sum_{k=1}^{N} \left(f_k(x) - f_k\right) \right)^2 \mathrm{d}x = \sum_{k=1}^{N} b^{-1} \left(1 - b^{-1}\right).$$

thus we can apply Lemma 1.4 of [23] as given above with $\phi_k = b^{-1}$ and $K = 1$, obtaining that (taking $\epsilon = \frac{2}{3}$)

$$A(d,\, b,\, N) = \sum_{k=1}^{N} f_k(x) = \frac{N}{b} + O\left(N^{2/3} \left(\log\left(N+2\right)\right)\right)$$

for almost all $x \in [0,\, 1)$. Thus,

$$\lim_{N \to \infty} \frac{A(d,\, b,\, N)}{N} = \frac{1}{b}$$

for almost all α, proving that almost all numbers are normal.

Using the effective version of Lemma 1.4 of [23] in the above will allow us to give bounds on the number of times a given digit can appear in the base b expansion for all α other than on a subset of measure δ.

5.2.4 Strong Law of Large Numbers

Let $(X,\, \Sigma,\, \mu)$ to be a probability space. For any $k \in \mathbb{N}$, let $(F_k(x))$ be sequence of μ-integrable random variables with mean F and variance $\sigma^2 > 0$ on the probability measure space (X, Ω, μ). The strong law of large numbers says that if all the F_k are independent, then for μ-almost every $x \in X$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} F_k(x) = F.$$

In fact, the assumption that all F_k are independent is stronger than needed for the conclusion to hold. In fact, if it holds that for any $m, n \in \mathbb{N}$, $m < n$ then

$$\int_X \left(\sum_{k=m+1}^{n} (F_k(x) - F) \right)^2 \mathrm{d}\mu \leq \sigma^2 (n - m) \max(1, F). \tag{5.18}$$

We will prove this when we give our Strong Law of Large Numbers. It then follows from Lemma 1.5 of [23] that for almost every $x \in X$, as $N \to \infty$,

$$\frac{1}{N} \sum_{k=1}^{N} F_k(x) = F + O\left(N^{-1/2} \log^2 N \right) \to F.$$

With our effective versions of the lemmas from Harman, we will be able to give effective bounds in the above rather than the given asymptotics.

5.3 GENERAL FORMS OF LEMMAS 1.4 AND 1.5 OF [23]

As noted above, to prove their asymptotic theorem, Aistleitner, Borda and Hauke use a variant on Lemma 1.4 of [23]. This raises the question of how generally can we write these lemmas? As remarked in [23], it is difficult to provide great improvements on the Lemmas as written, but in Chapter 8 we will write these Lemmas in as general a form as possible.

— 6 —

Probabilistic Results and their Proofs

The main tool in proving the asymptotic results above are results of the kind of Lemma 1.4 and 1.5 of [23]. Here we state effective versions of these lemmas which will be use in the applications mentioned in the introduction.

6.1 RESULTS

Theorem 6.1.0.1 (Effective Version of Lemma 1.4 in [23])**.** *Let* (X, Ω, μ) *be a measure space, and suppose that* $0 < \mu(X) < +\infty$*. Let* $f_k(x)$*,* $k \in \mathbb{N}$*, be a sequence of non-negative* μ*-measurable functions, where for all* $k \in \mathbb{N}$*,* $x \in X$*, we have that* $f_k(x) < C$ *for some constant* $C \in \mathbb{R}$*. Let* $\{f_k \in \mathbb{R}\}_{k\in\mathbb{N}}$ *and* $\{\varphi_k \in \mathbb{R}\}_{k\in\mathbb{N}}$ *be sequences of real numbers such that for any* $k \in \mathbb{N}$*,*

$$0 \leq f_k \leq \varphi_k \leq 1. \tag{6.1}$$

For any $N \in \mathbb{N}$*, define*

$$\Phi(N) = \sum_{k=1}^{N} \varphi_k,$$

and suppose that $\lim_{N\to\infty} \Phi(N) = +\infty$*. Further, assume that there exists* $K > 0$ *such that for any* $N \in \mathbb{N}$ *we have that*

$$\int_X \left(\sum_{k=1}^{N} (f_k(x) - f_k) \right)^2 \mathrm{d}\mu(x) \leq K\Phi(N). \tag{6.2}$$

Then for any $\varepsilon > 0$ *and* $\delta > 0$*, there exists* $E_{\varepsilon,\delta} \subset X$ *and* $K_{\varepsilon,\delta} > 0$ *such that* $\mu(E_{\varepsilon,\delta}) < \delta$ *and for any* $x \in X \setminus E_{\varepsilon,\delta}$ *and* $N \in \mathbb{N}$*,*

$$\left| \sum_{k=1}^{N} f_k(x) - \sum_{k=1}^{N} f_k \right| \leq K_{\varepsilon,\delta} \left(\Phi^{2/3}(N) \log^{1/3+\varepsilon} (\Phi(N) + 2) \right),$$

where

$$\begin{aligned}
K_{\varepsilon,\delta} &= \max\{X, Y\},\\
N_{\varepsilon,\delta} &= \min\left\{n \in \mathbb{N} : \varPhi(n) > j_{\varepsilon,\delta}^3 \log^{1+\varepsilon}(j_{\varepsilon,\delta}+2)\right\},\\
j_{\varepsilon,\delta} &= 1 + \left\lceil \exp\left(\frac{1+\log^{-1-\varepsilon} 3}{\varepsilon\delta} K\right)^{1/\varepsilon} \right\rceil,
\end{aligned}$$

where $\varPhi_0 = \min\{\varPhi(n) \in \mathbb{R}^+ : n \in \mathbb{N}\}$ *is the minimal non-zero value* $\varPhi$ *attains and*

$$\begin{aligned}
X &= \frac{C N_{\varepsilon,\delta}}{\max\left(\varPhi_0^{2/3} \log^{1/3+\varepsilon}(\varPhi_0+2), 1\right)},\\
Y &= \frac{4}{\log^{2\varepsilon/3}(\varPhi_0+2)} \left(\frac{\log 4}{\log 3}\right)^{1+\varepsilon} \left(4 + \frac{1+\varepsilon}{\log 3} + \frac{1}{4\log^{1+\varepsilon} 4}\right).
\end{aligned}$$

Similarly, we are able to come up with an effective version of Lemma 1.5 from [23], as follows.

Theorem 6.1.0.2 (Effective Version of Lemma 1.5 in [23])**.** *Let* (X, Ω, μ) *be a measure space, and suppose that* $0 < \mu(X) < +\infty$*. Let* $f_k(x)$*,* $k \in \mathbb{N}$*, be a sequence of non-negative* μ*-measurable functions, where for all* $k \in \mathbb{N}$*,* $x \in X$*, we have that* $f_k(x) < C$ *for some constant* $C \in \mathbb{R}$*. Let* $\{f_k \in \mathbb{R}\}_{k\in\mathbb{N}}$ *and* $\{\varphi_k \in \mathbb{R}\}_{k\in\mathbb{N}}$ *be sequences of real numbers such that, for all* $k \in \mathbb{N}$*,*

$$0 \le f_k \le \varphi_k.$$

For any $N \in \mathbb{N}$*, let*

$$\varPhi(N) = \sum_{k=1}^{N} \varphi_k.$$

and suppose that $\lim_{N\to\infty} \varPhi(N) = +\infty$*. Further, assume that there exists* $K > 0$ *such that for any* $m, n \in \mathbb{N}$*, if* $m < n$ *then*

$$\int_X \left(\sum_{k=m+1}^{n} (f_k(x) - f_k)\right)^2 d\mu(x) \le K\left(\varPhi(n) - \varPhi(m)\right). \tag{6.3}$$

Then for any $\varepsilon > 0$ *and* $\delta > 0$*, there exists* $E_{\varepsilon,\delta} \subset X$ *and* $K_{\varepsilon,\delta} > 0$ *such that* $\mu(E_{\varepsilon,\delta}) < \delta$ *and for any* $x \in X \setminus E_{\varepsilon,\delta}$ *and* $N \in \mathbb{N}$*,*

$$\left|\sum_{k=1}^{N} f_k(x) - \sum_{k=1}^{N} f_k\right| \le K_{\varepsilon,\delta}\left(\varPhi^{1/2}(N) \log^{3/2+\varepsilon} \varPhi(N) + \max_{1\le k\le N} f_k\right),$$

where

$$\begin{aligned} K_{\varepsilon,\delta} &= \max\{X',\, Y'\}, \\ N_{\varepsilon,\delta} &= \max\{n \in \mathbb{N} : \varPhi(n) < r_{\varepsilon,\delta}\}, \\ r_{\varepsilon,\delta} &= \left\lceil \left(\frac{2K}{\varepsilon\delta}\right)^{1/\varepsilon} \right\rceil + 1, \end{aligned}$$

where $\varPhi_0 = \min\{\varPhi(n) \in \mathbb{R}^+ : n \in \mathbb{N}\}$ *is the minimal non-zero value* $\varPhi$ *attains, and*

$$\begin{aligned} X' &= \frac{C N_{\varepsilon,\delta}}{\max\left(\varPhi_0^{1/2} \log^{3/2+\varepsilon}(\varPhi_0 + 2) + f_1, 1\right)}, \\ Y' &= \frac{2}{\log^{3/2+\varepsilon/2} 2} \left(1 + \frac{1}{\sqrt{2}\log^{3/2+\varepsilon} 4}\right) \left(\frac{\log 4}{\log 3}\right)^{3/2+\varepsilon}. \end{aligned}$$

The proof in [1] involves an application of a result akin to Harman's Lemma 1.4 [23], which is given at Lemma 5.2.1.1. Later in this chapter we prove this lemma, before proving the following effective version, which we will use to prove the quantitative Koukoulopoulos-Maynard Theorem referred to in the previous chapter.

Theorem 6.1.0.3. *Suppose that* $(X,\, \Omega,\, \mu)$ *is a measure space such that* $0 < \mu(X) < \infty$*, and let* K *be a positive real constant. Let* $f_q : X \to [0,\, K]$ *be a sequence of* μ*-measurable functions and let* f_q*,* ϕ_q *be sequences of real numbers such that*

$$0 \le f_q \le \phi_q \le K.$$

Write $\Psi(Q) = \sum_{q=1}^{Q} \phi_q$*, and suppose that* $\Psi(Q) \to \infty$ *and* $Q \to \infty$*. Let* $C > 4$ *and* $\mathcal{C} > 0$*, and suppose that for every* $n \in \mathbb{N}$*,*

$$\int_X \left(\sum_{1 \le q \le n} (f_q(x) - f_q) \right)^2 d\mu = \mathcal{C} \left(\frac{\Psi(Q)^2}{(\log \Psi(Q))^C} \right) \tag{6.4}$$

Then for any $\delta > 0$*, there exists an* $E_{C,\delta} \subset X$ *such that* $\mu(E_{C,\delta}) < \delta$ *and for any* $x \in X \setminus E_{C,\delta}$*,*

$$\sum_{q=1}^{Q} f_q(x) \le \sum_{q=1}^{Q} f_q + \max\left\{ k_{C,\delta} K,\, 2\frac{e\Psi(Q) + K}{(\log \Psi(Q))^{\sqrt{C}-1}} + K \right\},$$

where

$$k_{C,\delta} \coloneqq \min\left\{ k \in \mathbb{R} \,:\, \mathcal{C} k^{-\frac{\sqrt{C}}{2}} \left(1 + \frac{2k}{\sqrt{C} - 2}\right) < \delta \right\}. \tag{6.5}$$

The choice of $\mathcal{C}$ depends on the application one has in mind; for our applications we will take $\mathcal{C}$ to be the implicit constant from (5.12), as used in [1]. In this case, a bound for the constant $\mathcal{C}$ (which is also the constant in (6.4)) has been established and will be published separately due to the length of the calculation. This is the remaining implicit constant mentioned in Chapter 5.

We will now give the proof of the above results.

6.2 PROOFS

6.2.1 Proof of Theorem 6.1.0.1

Pick any $\varepsilon > 0$. For any $N \in \mathbb{N}$ and $x \in X$, define

$$\begin{aligned} \Psi(N,x) &= \sum_{k=1}^{N} f_k(x), \\ \Psi(N) &= \sum_{k=1}^{N} f_k, \\ E(N,x) &= \Psi(N,x) - \Psi(N). \end{aligned}$$

For any $j \in \mathbb{N}$, define

$$N_j = \min\left\{n \in \mathbb{N} : \Phi(n) > j^3 \log^{1+\varepsilon}(j+2)\right\}, \tag{6.6}$$

$$A_j = \left\{x \in X : |E(N_j,x)| > j^2 \log^{1+\varepsilon}(j+2)\right\}. \tag{6.7}$$

we note that as $\Psi(N) \to \infty$, N_j is well-defined for all $j \in \mathbb{N}$. Considering (6.2), we see that

$$\begin{aligned} \mu(A_j) \min_{x \in A_j} \left(|E(N_j,\, x)|^2\right) &\leq \int_{A_j} \left(\sum_{k=1}^{N_j} (f_k(x) - f_k)^2\right) \mathrm{d}\mu(x) \\ &\leq \int_{X} \left(\sum_{k=1}^{N_j} (f_k(x) - f_k)^2\right) \mathrm{d}\mu(x) \\ &\leq K\Phi(N_j). \end{aligned} \tag{6.8}$$

It follows immediately from the above and (6.7) that

$$\mu\left(A_j\right) \leq \frac{K\Phi(N_j)}{\min_{x\in A_j}\left(\left|E\left(N_j,\, x\right)\right|^2\right)} \leq \frac{K\Phi(N_j)}{\left(j^2\log^{1+\varepsilon}(j+2)\right)^2} = \frac{K\Phi(N_j)}{\left(j^4\log^{2+2\varepsilon}(j+2)\right)}. \tag{6.9}$$

For any $j \in \mathbb{N}$, $\log^{1+\varepsilon} 3 \leq j^3 \log^{1+\varepsilon}(j+2)$. Further, by (6.1) and (6.6) we see that

$$j^3\log^{1+\varepsilon}(j+2) < \Phi(N_j) \leq j^3\log^{1+\varepsilon}(j+2)+1.$$

By applying the above and (6.7) to (6.9), we obtain that

$$\mu(A_j) \leq \frac{Kj^3\log^{1+\varepsilon}(j+2)+K}{j^4\log^{2+2\varepsilon}(j+2)} \leq \frac{(1+\log^{-1-\varepsilon}3)Kj^3\log^{1+\varepsilon}(j+2)}{j^4\log^{2+2\varepsilon}(j+2)} = \frac{1+\log^{-1-\varepsilon}3}{j\log^{1+\varepsilon}(j+2)}K.$$

We now pick any $\delta > 0$. Let

$$j_{\varepsilon,\delta} = 1 + \left\lceil \exp\left(\frac{1+\log^{-1-\varepsilon}3}{\varepsilon\delta}K\right)^{1/\varepsilon}\right\rceil.$$

It follows that

$$\sum_{j=j_{\varepsilon,\delta}}^{\infty}\mu(A_j) < \int_{j_{\varepsilon,\delta}-1}^{\infty}\frac{(1+\log^{-1-\varepsilon}3)K}{x\log^{1+\varepsilon}x}dx = \frac{1+\log^{-1-\varepsilon}3}{\varepsilon\log^{\varepsilon}(j_{\varepsilon,\delta}-1)}K \leq \delta. \tag{6.10}$$

Let

$$E_{\varepsilon,\delta} = \bigcup_{j=j_{\varepsilon,\delta}}^{\infty} A_j.$$

Notice that, by sub-additivity of measures and (6.10),

$$\mu(E_{\varepsilon,\delta}) = \mu\left(\bigcup_{j=j_{\varepsilon,\delta}}^{\infty} A_j\right) \leq \sum_{j=j_{\varepsilon,\delta}}^{\infty}\mu(A_j) < \delta.$$

After this initial set up, we can now give some lemmas we will use to complete the proof.

Lemma 6.2.1.1. *For any $j \in \mathbb{N}$, $j \geq 2$, we have that*

$$\Psi(N_j) - \Psi(N_{j-1}) \leq \left(3 + \frac{1+\varepsilon}{\log 3} + \frac{1}{4\log^{1+\varepsilon} 4}\right) j^2 \log^{1+\varepsilon}(j+2),$$

where N_j is given at (6.6).

Proof. Define $g : \mathbb{R}^+ \to \mathbb{R}$ by

$$g(j) = j^3 \log^{1+\varepsilon}(j+2).$$

Pick any $j \in \mathbb{N}$ such that $j \geq 2$. By the Mean Value Theorem, there exists $\xi_j \in (j-1, j)$ such that

$$\begin{aligned}
g(j) - g(j-1) &= g'(\xi_j) \\
&= 3\xi_j^2 \log^{1+\varepsilon}(\xi_j+2) + \frac{\xi_j^3}{\xi_j+2}(1+\varepsilon)\log^{\varepsilon}(\xi_j+2) \\
&\leq 3\xi_j^2 \log^{1+\varepsilon}(\xi_j+2) + \frac{\xi_j^2}{\log 3}(1+\varepsilon)\log^{1+\varepsilon}(\xi_j+2) \\
&= \left(3 + \frac{1+\varepsilon}{\log 3}\right)\xi_j^2 \log^{1+\varepsilon}(\xi_j+2) \\
&< \left(3 + \frac{1+\varepsilon}{\log 3}\right) j^2 \log^{1+\varepsilon}(j+2).
\end{aligned}$$

We note that by (6.6), we have that

$$\Phi(N_j - 1) \leq j^3 \log^{1+\varepsilon}(j+2) < \Phi(N_j).$$

From this, (6.1), and the fact that $j \geq 2$, we have that

$$\Phi(N_j) - \Psi(N_j) \geq \Phi(N_{j-1}) - \Psi(N_{j-1}).$$

More explicitly, this is because

$$\Phi(N_j) - \Psi(N_j) = (\Phi(N_{j-1}) - \Psi(N_{j-1})) + \sum_{k=N_{j-1}+1}^{N_j} (\varphi_k - \psi_k),$$

and by (6.1) both terms on the right hand side are greater than equal to zero. Now, from the above and that $j \geq 2$ we see that

$$\begin{aligned}\Psi(N_j) - \Psi(N_{j-1}) &\leq \Phi(N_j) - \Phi(N_{j-1}) \\ &= \Phi(N_j - 1) + \varphi_{N_j} - \Phi(N_{j-1}) \\ &\leq g(j) - g(j-1) + 1 \\ &< \left(3 + \frac{1+\varepsilon}{\log 3}\right) j^2 \log^{1+\varepsilon}(j+2) + \frac{j^2 \log^{1+\varepsilon}(j+2)}{4 \log^{1+\varepsilon} 4} \\ &= \left(3 + \frac{1+\varepsilon}{\log 3} + \frac{1}{4\log^{1+\varepsilon} 4}\right) j^2 \log^{1+\varepsilon}(j+2),\end{aligned}$$

as stated in the lemma. □

Lemma 6.2.1.2. *For any $\varepsilon > 0$ and $j \in \mathbb{N}$, $j \geq 2$, we have that*

$$j^2 \log^{1+\varepsilon}(j+2) \leq \frac{4}{\log^{2\varepsilon/3}(\Phi_0 + 2)} \left(\frac{\log 4}{\log 3}\right)^{1+\varepsilon} \Phi^{2/3}(N_{j-1}) \log^{1/3+\varepsilon}(\Phi(N_{j-1}) + 2).$$

Proof. Since $j \geq 2$,

$$\frac{j^2 \log^{1+\varepsilon}(j+2)}{(j-1)^2 \log^{1+\varepsilon}(j+1)} \leq 4 \left(\frac{\log 4}{\log 3}\right)^{1+\varepsilon}.$$

Thus, considering (6.6), we see that

$$\begin{aligned}j^2 \log^{1+\varepsilon}(j+2) &\leq 4 \left(\frac{\log 4}{\log 3}\right)^{1+\varepsilon} (j-1)^2 \log^{1+\varepsilon}(j+1) \\ &\leq 4 \left(\frac{\log 4}{\log 3}\right)^{1+\varepsilon} \Phi^{2/3}(N_{j-1}) \log^{(1+\varepsilon)/3}(\Phi(N_{j-1}) + 2) \\ &\leq \frac{4}{\log^{2\varepsilon/3}(\Phi_0 + 2)} \left(\frac{\log 4}{\log 3}\right)^{1+\varepsilon} \Phi^{2/3}(N_{j-1}) \log^{1/3+\varepsilon}(\Phi(N_{j-1}) + 2),\end{aligned}$$

as stated in the lemma. □

We are now in a position to complete the proof of Theorem 6.1.0.1. Pick any $x \in X \setminus E_{\varepsilon,\delta}$ and $N \in \mathbb{N}$ such that $N > N_{j_{\varepsilon,\delta}}$. Since by assumption,

$$\lim_{n\to\infty} \Phi(n) = +\infty,$$

there exists $j \in \mathbb{N}$ such that $N_{j-1} \leq N < N_j$ and $j > j_{\varepsilon,\delta}$. Hence $x \notin A_j$. It then follows from (6.7), Lemma 6.2.1.1 and Lemma 6.2.1.2 that

$$\begin{aligned}
\Psi(N,x) - \Psi(N) &\leq \Psi(N_j,x) - \Psi(N_{j-1}) \\
&\leq \Psi(N_j,x) - \Psi(N_j) + \Psi(N_j) - \Psi(N_{j-1}) \\
&\leq |E(N_j,x)| + \Psi(N_j) - \Psi(N_{j-1}) \\
&\leq j^2 \log^{1+\varepsilon}(j+2) + \left(3 + \frac{1+\varepsilon}{\log 3} + \frac{1}{4\log^{1+\varepsilon} 4}\right) j^2 \log^{1+\varepsilon}(j+2) \\
&= \left(4 + \frac{1+\varepsilon}{\log 3} + \frac{1}{4\log^{1+\varepsilon} 4}\right) j^2 \log^{1+\varepsilon}(j+2) \\
&\leq \frac{4}{\log^{2\varepsilon/3}(\varPhi_0+2)} \left(\frac{\log 4}{\log 3}\right)^{1+\varepsilon} \left(4 + \frac{1+\varepsilon}{\log 3} + \frac{1}{4\log^{1+\varepsilon} 4}\right) \cdot \\
&\quad \cdot \varPhi^{2/3}(N_{j-1}) \log^{1/3+\varepsilon}(\varPhi(N_{j-1})+2) \\
&\leq \frac{4}{\log^{2\varepsilon/3}(\varPhi_0+2)} \left(\frac{\log 4}{\log 3}\right)^{1+\varepsilon} \left(4 + \frac{1+\varepsilon}{\log 3} + \frac{1}{4\log^{1+\varepsilon} 4}\right) \cdot \\
&\quad \cdot \varPhi^{2/3}(N) \log^{1/3+\varepsilon}(\varPhi(N)+2). \qquad (6.11)
\end{aligned}$$

We now consider $N \leq N_{\varepsilon,\delta}$. We note that by the definition of $f_k(x)$ and f_k, we have that for all $N \in \mathbb{N}$,

$$-N \leq \sum_{k=1}^{N} (f_k(x) - f_k) \leq CN.$$

It follows that

$$\sum_{k=1}^{N} (f_k(x) - f_k) \leq N(C+1).$$

We note that if $C \geq 1$, then we have that

$$\left|\sum_{k=1}^{N} (f_k(x) - f_k)\right| \leq NC. \qquad (6.12)$$

We now assume that $C \geq 1$; indeed if it is not, then all the functions $f_k(x)$ are also bounded above by 1. Thus, for all $N \leq N_{\varepsilon,\delta}$ inequality (6.12) holds, and for all $N > N_{\varepsilon,\delta}$ we have inequality (6.11). This proves the theorem.

6.2.2 PROOF OF THEOREM 6.1.0.2

We now prove Theorem 6.1.0.2. As commented in [23], we improve the spacing between N_j's to improve the bound obtained in Theorem 6.1.0.1. We initially follow the proof of Lemma 1.5 in [23], before quantifying the remainder term.

For any $j \in \mathbb{N}$, define

$$n_j = \max\{n \in \mathbb{N} : \Phi(n) < j\}.$$

For all $j \in \mathbb{N}$, j can be written in binary as

$$j = \sum_{v=0}^{\lfloor \log_2 j \rfloor} 2^v b(j, v),$$

where $b : \mathbb{N} \times \mathbb{N}_0 \to \{0, 1\}$.

Define $r_j = \lfloor \log_2 j \rfloor$ and let

$$B_j = \left\{ (i, s) \in \mathbb{N}_0 \times \mathbb{N}_0 : i = \sum_{v=s+1}^{r_j} 2^{v-s} b(j, v),\ b(j, s) = 1,\ 0 \le s \le r_j \right\}.$$

Notice that $|B_j| \le r_j + 1$ and

$$(0, n_j] = \bigcup_{(i,s) \in B_j} \left(n_{i2^s}, n_{(i+1)2^s} \right]. \tag{6.13}$$

A concrete example of how this works is given in [23] for $j = 37$.

Returning to the proof, we now define for any $(i, s) \in \mathbb{N}_0 \times \mathbb{N}_0$ and $x \in X$ a function $F : \mathbb{N}_0 \times \mathbb{N}_0 \times X \to \mathbb{R}$ as follows:

$$F(i, s, x) = \sum_{k=n_{i2^s}+1}^{n_{(i+1)2^s}} (f_k(x) - f_k).$$

Then, by (6.13), for any $x \in X$,

$$\sum_{k=1}^{n_j} (f_k(x) - f_k) = \sum_{(i,s) \in B_j} F(i, s, x). \tag{6.14}$$

For any $\varepsilon > 0$, define

$$M_\varepsilon = \frac{\sqrt{2}}{\log^{3/2+\varepsilon/2} 2} > 1.$$

We now give some lemmas we will use in the rest of the proof.

Lemma 6.2.2.1. *For any $\varepsilon > 0$ and $\delta > 0$, there exists $E_{\varepsilon,\delta} \subset X$ and $r_{\varepsilon,\delta} \in \mathbb{N}$ such that $\mu(E_{\varepsilon,\delta}) < \delta$, and for any $x \in X \setminus E_{\varepsilon,\delta}$ and $r \in \mathbb{N}$, if $r > r_{\varepsilon,\delta}$ then*

$$\left| \sum_{k=1}^{n_r} (f_k(x) - f_k) \right| \le M_\varepsilon \left(r^{1/2} \log^{3/2+\varepsilon} (r + 2) \right).$$

Proof. By (6.14), it suffices to prove that there exists $r_{\varepsilon,\delta} \in \mathbb{N}$ such that for any $r \in \mathbb{N}$ such that $r > r_{\varepsilon,\delta}$,

$$\left| \sum_{(i,s)\in B_r} F(i,s,x) \right| \leq \sum_{(i,s)\in B_r} |F(i,s,x)| \leq M_\varepsilon \left(r^{1/2} \log^{3/2+\varepsilon}(r+2) \right).$$

For any $r,\ i,\ s \in \mathbb{N}_0$, $x \in X$, define

$$G(r,x) = \sum_{s=0}^{r} \sum_{i=0}^{2^{r-s}-1} F^2(i,s,x), \tag{6.15}$$

$$\Phi(i,s) = \sum_{k=n_{i2^s}+1}^{n_{(i+1)2^s}} \varphi_k \coloneqq \Phi(n_{(i+1)2^s})$$

$$A_r = \left\{ x \in X : G(r,x) > r^{2+\varepsilon} 2^r \right\}. \tag{6.16}$$

Pick any $r \in \mathbb{N}$. By (6.3),

$$\begin{aligned}
\int_X G(r,x)\, d\mu(x) &\leq K \sum_{s=0}^{r} \sum_{i=0}^{2^{r-s}-1} \Phi(i,s) \\
&= K \sum_{s=0}^{r} \sum_{i=0}^{2^{r-s}-1} \left(\Phi(n_{(i+1)2^s}) - \Phi(n_{i2^s}) \right) \\
&= K \sum_{s=0}^{r} \Phi(n_{2^s}) \\
&\leq K(r+1)\Phi(n_{2^r}) \\
&< 2Kr2^r.
\end{aligned}$$

Since

$$r^{2+\varepsilon} 2^r \mu(A_r) \leq \int_{A_r} G(r,x)\, d\mu(x) < 2rK2^r,$$

we obtain that

$$\mu(A_r) < 2Kr^{-1-\varepsilon}.$$

Pick any $\delta > 0$ and define

$$r_{\varepsilon,\delta} = 1 + \left\lceil \left(\frac{2K}{\varepsilon\delta} \right)^{1/\varepsilon} \right\rceil.$$

It follows that

$$\sum_{r=r_{\varepsilon,\delta}}^{\infty} \mu(A_r) < \int_{r_{\varepsilon,\delta}-1}^{\infty} \frac{2K}{x^{1+\varepsilon}} dx = \frac{2K}{\varepsilon (r_{\varepsilon,\delta}-1)^\varepsilon} \leq \delta.$$

We now take

$$E_{\varepsilon,\delta} = \bigcup_{r=r_{\varepsilon,\delta}}^{\infty} A_r.$$

As in the previous proof, we note that by the sub-additivity of measures,

$$\mu(E_{\varepsilon,\delta}) = \mu\left(\bigcup_{r=r_{\varepsilon,\delta}}^{\infty} A_r\right) \leq \sum_{r=r_{\varepsilon,\delta}}^{\infty} \mu(A_r) < \delta.$$

Pick any $x \in X \setminus E_{\varepsilon,\delta}$. For any $r \in \mathbb{N}$, $r > r_{\varepsilon,\delta}$, we necessarily have that $x \notin A_r$ and

$$G(r,x) \leq r^{2+\varepsilon} 2^r. \tag{6.17}$$

Pick any $j \in \mathbb{N}$ such that $j > 2^{r_{\varepsilon,\delta}}$. Then $\lfloor \log_2 j \rfloor + 1 > r_{\varepsilon,\delta}$. Taking $r = \lfloor \log_2 j \rfloor + 1 > r_{\varepsilon,\delta}$, we find that

$$\sum_{(i,s)\in B_j} |F(i,s,x)| = \sum_{s=0}^{r} \sum_{i=0}^{2^{r-s}-1} |F(i,s,x)| \chi_{B_j}(i,s),$$

where $\chi_{B_j}(i,s)$ is the characteristic function on B_j. Notice that $2^r \leq 2j$. Applying this, the Cauchy-Schwarz inequality, (6.15), (6.16) and (6.17) gives us that

$$\begin{aligned}
\sum_{(i,s)\in B_j} |F(i,s,x)| &\leq \left(\sum_{s=0}^{r} \sum_{i=0}^{2^{r-s}-1} |F(i,s,x)|^2\right)^{1/2} \left(\sum_{s=0}^{r} \sum_{i=0}^{2^{r-s}-1} \chi_{B_j}(i,s)\right)^{1/2} \\
&= G^{1/2}(r,x) |B_j|^{1/2} \\
&\leq r^{1+\varepsilon/2} 2^{r/2} r^{1/2} \\
&= 2^{r/2} r^{3/2+\varepsilon/2} \\
&\leq \sqrt{2} j^{1/2} \log_2^{3/2+\varepsilon/2} j \\
&\leq \frac{\sqrt{2}}{\log^{3/2+\varepsilon/2} 2} j^{1/2} \log^{3/2+\varepsilon/2} j \\
&\leq M_\varepsilon \left(j^{1/2} \log^{3/2+\varepsilon/2} j\right),
\end{aligned}$$

which proves the lemma. □

Lemma 6.2.2.2. *For any $\varepsilon > 0$ and $r \in \mathbb{N}$,*

$$(r+1)^{1/2} \log^{3/2+\varepsilon}(r+3) \leq \sqrt{2} \left(\frac{\log 4}{\log 3}\right)^{3/2+\varepsilon} r^{1/2} \log^{3/2+\varepsilon}(r+2).$$

Proof. This follows from the following bound:

$$\begin{aligned}\frac{(r+1)^{1/2}\log^{3/2+\varepsilon}(r+3)}{r^{1/2}\log^{3/2+\varepsilon}(r+2)} &= \sqrt{\frac{r+1}{r}}\left(\frac{\log(r+3)}{\log(r+2)}\right)^{3/2+\varepsilon}\\ &\leq \sqrt{2}\left(\frac{\log 4}{\log 3}\right)^{3/2+\varepsilon}.\end{aligned}$$

The lemma follows. □

We are now able to finish the proof of Theorem 6.1.0.2. Pick any $\varepsilon > 0$ and $\delta > 0$. By Lemma 6.2.2.1, there exists some μ-measurable $E_{\varepsilon,\delta} \subset X$ and $r_{\varepsilon,\delta} \in \mathbb{N}$ such that $\mu(E_{\varepsilon,\delta}) < \delta$ and for any $x \in X \setminus E_{\varepsilon,\delta}$ and $r \in \mathbb{N}$ with $r > r_{\varepsilon,\delta}$,

$$\left|\sum_{k=1}^{n_r}(f_k(x) - f_k)\right| \leq M_\varepsilon\left(r^{1/2}\log^{3/2+\varepsilon}(r+2)\right).$$

Pick any $n \in \mathbb{N}$ such that $n > n_{r_{\varepsilon,\delta}}$. Since by assumption

$$\lim_{N\to\infty}\Phi(N) = \lim_{r\to\infty} n_r = +\infty,$$

there exists $r \in \mathbb{N}$ such that $n_r < n < n_{r+1}$ and $r > r_{\varepsilon,\delta}$. Notice that for any $x \in X$,

$$\sum_{k=1}^{n_r} f_k(x) \leq \sum_{k=1}^{n} f_k(x) \leq \sum_{k=1}^{n_{r+1}} f_k(x),$$

and for any $x \in X \setminus E_{\varepsilon,\delta}$

$$\left|\sum_{k=1}^{n_r}(f_k(x) - f_k)\right| \leq M_\varepsilon\left(r^{1/2}\log^{3/2+\varepsilon}(r+2)\right), \tag{6.18}$$

and

$$\left|\sum_{k=1}^{n_{r+1}}(f_k(x) - f_k)\right| \leq M_\varepsilon\left((r+1)^{1/2}\log^{3/2+\varepsilon}(r+3)\right). \tag{6.19}$$

We note that $M_\varepsilon > 1$, $\Phi(n_{r+1}) < r+1$ and $r \leq \Phi(n_r+1)$, so

$$\begin{aligned}\sum_{k=n_r+1}^{n_{r+1}} f_k &= f_{n_r+1} + \sum_{k=n_r+2}^{n_{r+1}} f_k\\ &\leq f_{n_r+1} + \sum_{k=n_r+2}^{n_{r+1}} \varphi_k\\ &\leq \max_{1\leq k\leq n} f_k + \Phi(n_{r+1}) - \Phi(n_r+1)\\ &< \max_{1\leq k\leq n} f_k + (r+1) - r\\ &= 1 + \max_{1\leq k\leq n} f_k.\end{aligned}$$

Since

$$\sum_{k=1}^{n_r} f_k(x) - \sum_{k=1}^{n} f_k \le \sum_{k=1}^{n}(f_k(x) - f_k) \le \sum_{k=1}^{n_{r+1}} f_k(x) - \sum_{k=1}^{n} f_k,$$

we get that

$$\begin{aligned}\sum_{k=1}^{n_r}(f_k(x) - f_k) - 1 - \max_{1\le k\le n} f_k &\le \sum_{k=1}^{n}(f_k(x) - f_k)\\ &\le \sum_{k=1}^{n_{r+1}}(f_k(x) - f_k) + 1 + \max_{1\le k\le n} f_k.\end{aligned}$$

Notice that $r^{1/2}\log^{3/2+\varepsilon}(r+2) > 1$. Hence by this, (6.18) and (6.19) we get that

$$\begin{aligned}&- M_\varepsilon \left(1 + \frac{1}{\log^{3/2+\varepsilon} 3}\right)\left(r^{1/2}\log^{3/2+\varepsilon}(r+2) + \max_{1\le k\le n} f_k\right)\\ &\le -M_\varepsilon\left(r^{1/2}\log^{3/2+\varepsilon}(r+2)\right) - 1 - \max_{1\le k\le n} f_k\\ &\le \sum_{k=1}^{n}(f_k(x) - f_k)\\ &\le M_\varepsilon\left((r+1)^{1/2}\log^{3/2+\varepsilon}(r+3)\right) + 1 + \max_{1\le k\le n} f_k\\ &\le M_\varepsilon\left(1 + \frac{1}{\sqrt{2}\log^{3/2+\varepsilon} 4}\right)\left((r+1)^{1/2}\log^{3/2+\varepsilon}(r+3) + \max_{1\le k\le n} f_k\right)\\ &\le \sqrt{2}M_\varepsilon\left(1 + \frac{1}{\sqrt{2}\log^{3/2+\varepsilon} 4}\right)\left(\frac{\log 4}{\log 3}\right)^{3/2+\varepsilon}\left(r^{1/2}\log^{3/2+\varepsilon}(r+2) + \max_{1\le k\le n} f_k\right).\end{aligned} \tag{6.20}$$

Thus, as $r \le \Phi(n)$,

$$\begin{aligned}\left|\sum_{k=1}^{n}(f_k(x) - f_k)\right| &\le \sqrt{2}M_\varepsilon\left(1 + \frac{1}{\sqrt{2}\log^{3/2+\varepsilon} 4}\right)\left(\frac{\log 4}{\log 3}\right)^{3/2+\varepsilon}\cdot\\ &\quad\cdot\left(r^{1/2}\log^{3/2+\varepsilon}(r+2) + \max_{1\le k\le n} f_k\right)\\ &\le K_{\varepsilon,\delta}\left(\Phi^{1/2}(n)\log^{3/2+\varepsilon}(\Phi(n)+2) + \max_{1\le k\le n} f_k\right).\end{aligned} \tag{6.21}$$

We now consider $N \le N_{\varepsilon,\delta}$. We have that for all $n \in \mathbb{N}$,

$$-N \max_{1\le k\le N} f_k \le \sum_{k=1}^{N}(f_k(x) - f_k) \le NC.$$

It follows immediately that

$$\left|\sum_{k=1}^{N}(f_k(x) - f_k)\right| \le N\left(C + \max_{1\le k\le N} f_k\right).$$

For all $N \leq N_{\varepsilon,\delta}$ we have that inequality (6.21) holds, and for all $N > N_{\varepsilon,\delta}$. This concludes the proof of Theorem 6.1.0.2.

6.2.3 Proof of Lemma 5.2.1.1

Proof. We essentially follow the proof given in [1]. Initially assume $C > 4$. Define for any $k \in \mathbb{N}$,

$$Q_k = \min\left\{Q \in \mathbb{N} : \Psi(Q) \geq e^{k^{1/\sqrt{C}}}\right\},$$

and let

$$\mathcal{B}_k = \left\{x \in X \; : \left|\sum_{1\leq q\leq Q_k} \left(f_q(x) - f_q\right)\right| \geq \frac{\Psi\left(Q_k\right)}{\left(\log\Psi\left(Q_k\right)\right)^{C/4}}\right\}.$$

By (5.12) we obtain that

$$\begin{aligned}\mu\left(\mathcal{B}_k\right)\cdot\left(\frac{\Psi\left(Q_k\right)}{\left(\log\Psi\left(Q_k\right)\right)^{C/4}}\right)^2 &\leq \int_{\mathcal{B}_k}\left(\sum_{1\leq q\leq Q_k}\left(f_q(x)-f_q\right)\right)^2 \mathrm{d}\mu\\ &\leq \int_X\left(\sum_{1\leq q\leq Q_k}\left(f_q(x)-f_q\right)\right)^2 \mathrm{d}\mu\\ &= O\left(\frac{\Psi(Q_k)^2}{\left(\log\Psi(Q_k)\right)^C}\right).\end{aligned}$$

It immediately follows that

$$\mu\left(\mathcal{B}_k\right) \leq O\left(\left(\log\Psi\left(Q_k\right)\right)^{-C/2}\right) \leq O\left(k^{-\sqrt{C}/2}\right). \tag{6.22}$$

As $C > 4$ we have that

$$\sum_{k=1}^{\infty}\mu\left(\mathcal{B}_k\right) < +\infty,$$

so applying the Borel-Cantelli Lemma we find that almost all $x \in X$ are contained in at most finitely many of the sets $\mathcal{B}_k$. That is, for almost all x there exists a $k_0(x)$ such that for all $k > k_0(x)$, we have that

$$\left|\sum_{1\leq q\leq Q_k}\left(f_q(x)-f_q\right)\right| \leq \frac{\Psi\left(Q_k\right)}{\left(\log\Psi\left(Q_k\right)\right)^{C/4}}.$$

For any $Q \geq 3$ there is a $k \in \mathbb{N}$ such that $Q_k \leq Q \leq Q_{k+1}$. This means that

$$\sum_{q=1}^{Q_k}\phi_q \leq \sum_{q=1}^{Q}\phi_q \leq \sum_{q=1}^{Q_{k+1}}\phi_q.$$

As by assumption $\phi_q \leq K$ for all $q \in \mathbb{N}$, we have that $\Psi\left(Q_k\right) \in \left[e^{k^{\frac{1}{\sqrt{C}}}}, e^{k^{\frac{1}{\sqrt{C}}}} + K\right]$. It follows that

$$\frac{\Psi(Q_{k+1})}{\Psi(Q_k)} = 1 + O\left(k^{-1+\frac{1}{\sqrt{C}}}\right) = 1 + O\left(\left(\log \Psi(Q_k)\right)^{1-\sqrt{C}}\right).$$

From these formulae and the triangle inequality, it follows that for almost all $x \in X$ there exists a $Q_0 \coloneqq Q_0(x)$ such that for all $Q > Q_0$ we have that

$$\left|\sum_{1\leq q\leq Q} \left(f_q(x) - f_q\right)\right| = O\left(\frac{\Psi(Q)}{\left(\log \Psi(Q)\right)^{\sqrt{C}-1}}\right).$$

This proves the lemma. □

This method of proof is directly taken from [1] and applied to general measure spaces rather than the specific one needed for their paper. The proof of Theorem 1 in [1] is now a special case of the above lemma. We further note that in the proof in [1], they allow $\mathcal{C}$ to be arbitrary, and use this to give a further asymptotic term, which we shall not do as we wish to make this result effective.

6.2.4 Proof of Theorem 6.1.0.3

Proof. First we note that by (6.4), for arbitrary $C > 0$ there exists a $\mathcal{C} \in \mathbb{R}$ such that

$$\int_X \left(\sum_{1\leq q\leq Q} \left(f_q(x) - f_q\right)\right)^2 \mathrm{d}\mu \leq \mathcal{C}\frac{\Psi(Q)^2}{\left(\log \Psi(Q)\right)^C}.$$

As before, for $C > 4$ and any $k \in \mathbb{N}$, let

$$Q_x = \min\left\{Q \in \mathbb{R} \,:\, \Psi(Q) \geq e^{x^{\frac{1}{\sqrt{C}}}}\right\},\ x \geq 1, \tag{6.23}$$

and let

$$\mathcal{B}_k = \left\{x \in X \,:\, \left|\sum_{1\leq q\leq Q_k} \left(f_q(x) - f_q\right)\right| \geq \frac{\Psi\left(Q_k\right)}{\left(\log \Psi\left(Q_k\right)\right)^{C/4}}\right\}.$$

We note that we will generally consider $x \in \mathbb{N}$ when considering Q_x; however, we will need to take an integral so have defined this over the reals.

By (6.4) we obtain that

$$\begin{aligned}\mu\left(\mathcal{B}_k\right)\cdot\left(\frac{\Psi\left(Q_k\right)}{\left(\log\Psi\left(Q_k\right)\right)^{C/4}}\right)^2 &\le \int_{\mathcal{B}_k}\left(\sum_{1\le q\le Q_k}\left(f_q(x)-f_q\right)\right)^2 \mathrm{d}\mu\\ &\le \int_X\left(\sum_{1\le q\le Q_k}\left(f_q(x)-f_q\right)\right)^2 \mathrm{d}\mu\\ &\le \mathcal{C}\frac{\Psi(Q_k)^2}{\left(\log\Psi(Q_k)\right)^C},\end{aligned}$$

so it follows that

$$\mu\left(\mathcal{B}_k\right)\le\mathcal{C}\left(\log\Psi\left(Q_k\right)\right)^{-\frac{C}{2}}\le\mathcal{C}k^{-\frac{\sqrt{C}}{2}}, \tag{6.24}$$

where the second inequality follows from (6.23).

Let

$$k_{C,\delta}:=\min\left\{k\in\mathbb{N}\,:\,\mathcal{C}k^{-\frac{\sqrt{C}}{2}}\left(1+\frac{2k}{\sqrt{C}-2}\right)<\delta\right\}$$

Then, by the integral test for convergence and that $C>4$, it follows that

$$\begin{aligned}\sum_{k=k_{C,\delta}}^{\infty}\mu(\mathcal{B}_k) &\le \mu(\mathcal{B}_{k_{C,\delta}})+\int_{k_{C,\delta}}^{\infty}\mu(\mathcal{B}_x)\mathrm{d}x\\ &\le \mathcal{C}\left(k_{C,\delta}^{-\frac{\sqrt{C}}{2}}+\int_{k_{C,\delta}}^{\infty}x^{-\frac{\sqrt{C}}{2}}\mathrm{d}x\right)\\ &= \mathcal{C}\left(k_{C,\delta}^{-\frac{\sqrt{C}}{2}}+\frac{2}{2-\sqrt{C}}\left[x^{1-\frac{\sqrt{C}}{2}}\right]_{k_{C,\delta}}^{\infty}\right)\\ &= \mathcal{C}k_{C,\delta}^{-\frac{\sqrt{C}}{2}}\left(1+\frac{2k_{C,\delta}}{\sqrt{C}-2}\right)<\delta.\end{aligned}$$

Define

$$E_{C,\delta}=\bigcup_{k=k_{C,\delta}}^{\infty}\mathcal{B}_k.$$

Then by the above, $\mu\left(E_{C,\delta}\right)<\delta$.

We also note as before, as $C>4$ and by considering (6.24) we have that $\sum_{k=1}^{\infty}\mu(\mathcal{B}_k)$ converges, so by the Borel-Cantelli lemma, for almost all $x\in X$, there exists a $k_0=k_0(x)$ such that for all $k>k_0$,

$$\left|\sum_{1\le q\le Q_k}\left(f_q(x)-f_q\right)\right|\le\frac{\Psi\left(Q_k\right)}{\left(\log\Psi\left(Q_k\right)\right)^{C/4}}.$$

For any $Q \in \mathbb{N}$, $Q \geq 3$, as before we have that there exists $k \in \mathbb{N}$ satisfying $Q_k \leq Q \leq Q_{k+1}$ and so

$$\sum_{q=1}^{Q_k} \phi_q \leq \sum_{q=1}^{Q} \phi_q \leq \sum_{q=1}^{Q_{k+1}} \phi_q.$$

By the definition of Q_k and as $\phi_q \leq K$ for all q,

$$\Psi(Q_k) \in \left[e^{k^{\frac{1}{\sqrt{C}}}},\, e^{k^{\frac{1}{\sqrt{C}}}} + K \right].$$

Following methods from earlier in this paper, we prove the following lemma.

Lemma 6.2.4.1. *For any $k \in \mathbb{N}$ we have that*

$$0 \leq \Psi\left(Q_{k+1}\right) - \Psi\left(Q_k\right) \leq \frac{e\Psi\left(Q_k\right)\left(\log\Psi\left(Q_k\right)\right)^{1-\sqrt{C}}}{\sqrt{C}} + K.$$

Proof. We follow the same method we used to prove Lemma 6.2.1.1. Let

$$g(x) = e^{x^{1/\sqrt{C}}}.$$

Then by the mean value theorem, for any $k \in \mathbb{N}$, there exists some $\xi_k \in (k,\, k+1)$ such that

$$g(k+1) - g(k) = g'(\xi_k).$$

Calculating, we see that for any $x \in (k,\, k+1)$,

$$\begin{aligned} g'(x) &= \frac{e^{x^{\frac{1}{\sqrt{C}}}} x^{\frac{1}{\sqrt{C}}-1}}{\sqrt{C}} \\ &\leq \frac{e^{(k+1)^{\frac{1}{\sqrt{C}}}} (k+1)^{\frac{1}{\sqrt{C}}-1}}{\sqrt{C}} \\ &\leq \frac{e \cdot e^{k^{\frac{1}{\sqrt{C}}}} k^{\frac{1}{\sqrt{C}}-1}}{\sqrt{C}} \\ &\leq \frac{e\Psi\left(Q_k\right) k^{\frac{1}{\sqrt{C}}-1}}{\sqrt{C}}, \end{aligned}$$

where the last line follows from the definition of Q_k.

Note that $\Psi\left(Q_{k+1}-1\right)<e^{(k+1)^{\frac{1}{\sqrt{C}}}}$. We now consider

$$\begin{aligned}
\Psi\left(Q_{k+1}\right)-\Psi\left(Q_{k}\right) &= \Psi\left(Q_{k+1}-1\right)+\phi_{Q_{k+1}}-\Psi\left(Q_{k}\right) \\
&\leq e^{(k+1)^{\frac{1}{\sqrt{C}}}}-e^{k^{\frac{1}{\sqrt{C}}}}+K \\
&= g(k+1)-g(k)+K \\
&\leq \frac{e\Psi\left(Q_{k}\right)k^{\frac{1}{\sqrt{C}}-1}}{\sqrt{C}}+K \\
&\leq \frac{e\Psi\left(Q_{k}\right)\left(\log\Psi\left(Q_{k}\right)\right)^{1-\sqrt{C}}}{\sqrt{C}}+K,
\end{aligned}$$

where the second to last line follows from Lemma 6.2.4.1, and the last line from the definition of Q_k given at (6.23). □

We now prove the result. Given $Q\in\mathbb{N}$ such that $Q>k_{C,\delta}$, with $Q_k\leq Q<Q_{k+1}$, we see that for any $x\in X\setminus E_{C,\delta}$,

$$\begin{aligned}
\left|\sum_{q=1}^{Q}\left(f_q(x)-f_q\right)\right| &\leq \left|\sum_{q=1}^{Q_{k+1}}f_q(x)-\sum_{q=1}^{Q_k}f_q\right| \\
&\leq \left|\sum_{q=1}^{Q_{k+1}}f_q(x)-\Psi(Q_k)\right| \\
&= \left|\sum_{q=1}^{Q_{k+1}}f_q(x)-\Psi\left(Q_{k+1}\right)+\Psi\left(Q_{k+1}\right)-\Psi(Q_k)\right| \\
&\leq \left|\sum_{q=1}^{Q_{k+1}}f_q(x)-\Psi\left(Q_{k+1}\right)\right|+\left|\Psi\left(Q_{k+1}\right)-\Psi(Q_k)\right| \\
&\leq \frac{\Psi\left(Q_{k+1}\right)}{\left(\log\Psi\left(Q_{k+1}\right)\right)^{C/4}}+\frac{e\Psi\left(Q_{k}\right)\left(\log\Psi\left(Q_{k}\right)\right)^{1-\sqrt{C}}}{\sqrt{C}}+K, \qquad (6.25)
\end{aligned}$$

where the last line follows as $Q>k_{C,\delta}$ and from Lemma 6.2.4.1.

We note that

$$\begin{aligned}
\frac{\Psi\left(Q_{k+1}\right)}{\left(\log\Psi\left(Q_{k+1}\right)\right)^{C/4}} &\leq \frac{e^{(k+1)^{\frac{1}{\sqrt{C}}}}+K}{\left(\log\Psi\left(Q_{k}\right)\right)^{C/4}} \\
&\leq \frac{e\cdot e^{k^{\frac{1}{\sqrt{C}}}}+K}{\left(\log\Psi\left(Q_{k}\right)\right)^{C/4}} \\
&\leq \frac{e\Psi\left(Q_{k}\right)+K}{\left(\log\Psi\left(Q_{k}\right)\right)^{C/4}}.
\end{aligned}$$

Substituting this into (6.25), we obtain that

$$\begin{aligned}
\left|\sum_{q=1}^{Q}\left(f_q(x)-f_q\right)\right| &\leq \frac{e\Psi\left(Q_k\right)+K}{\left(\log\Psi\left(Q_k\right)\right)^{C/4}} + \frac{e\Psi\left(Q_k\right)\left(\log\Psi\left(Q_k\right)\right)^{1-\sqrt{C}}}{\sqrt{C}} + K \\
&\leq \frac{e\Psi\left(Q_k\right)+K}{\left(\log\Psi\left(Q_k\right)\right)^{\sqrt{C}-1}} + \frac{e\Psi\left(Q_k\right)}{\left(\log\Psi\left(Q_k\right)\right)^{\sqrt{C}-1}} + K \\
&\leq 2\frac{e\Psi\left(Q_k\right)+K}{\left(\log\Psi\left(Q_k\right)\right)^{\sqrt{C}-1}} + K \\
&\leq 2\frac{e\Psi\left(Q\right)+K}{\left(\log\Psi\left(Q\right)\right)^{\sqrt{C}-1}} + K
\end{aligned}$$

It follows that

$$\sum_{q=1}^{Q} f_q(x) \leq \sum_{q=1}^{Q} f_q + 2\frac{e\Psi\left(Q\right)+K}{\left(\log\Psi\left(Q\right)\right)^{\sqrt{C}-1}} + K. \tag{6.26}$$

This inequality holds for $Q > k_{C,\delta}$. We now consider when $Q \leq k_{C,\delta}$. As $f_q(x)$ and f_q are bounded above by K for all q, we trivially have that

$$-k_{C,\delta}K \leq -QK \leq \sum_{q=1}^{Q}\left(f_q(x)-f_q\right) \leq QK \leq k_{C,\delta}K. \tag{6.27}$$

Combining (6.26) and (6.27), we see that for all $Q \in \mathbb{N}$, we have that

$$\left|\sum_{q=1}^{Q} f_q(x) - \sum_{q=1}^{Q} f_q\right| \leq \max\left\{k_{C,\delta}K,\ 2\frac{e\Psi\left(Q\right)+K}{\left(\log\Psi\left(Q\right)\right)^{C}} + K\right\}.$$

$\square$

— 7 —

Applications of the Probabilistic Results

In this chapter we give various applications of the results given in the previous chapter, as lined out in Chapter 5.

7.1 EFFECTIVE SCHMIDT'S THEOREM

In this section we prove the following quantitative version of Schmidt's Theorem.

Theorem 7.1.0.1 (Effective Schmidt's Theorem). *Let $\psi : \mathbb{N} \to [0, 1/2)$ be a non-increasing function. Suppose* (5.5) *diverges. Then for any $\varepsilon > 0$ and $\delta > 0$, there exists some measurable $E_{\varepsilon,\delta} \subset [0, 1)$ such that $\mu(E_{\varepsilon,\delta}) < \delta$ and for any $x \in [0, 1) \setminus E_{\varepsilon,\delta}$ and $Q \in \mathbb{N}$,*

$$|S(x, Q) - 2\Psi(Q)| \leq \max\left(N_{\varepsilon,\delta},\ K_\varepsilon \left(\Psi^{1/2}(Q) \log^{2+\varepsilon}(\Psi(Q) + 1)\right)\right),$$

where

$$\begin{aligned} N_{\varepsilon,\delta} &= 2 \max\left\{ n \in \mathbb{N} : \left\lceil \left(\frac{2K_\varepsilon}{\varepsilon\delta}\right)^{1/\varepsilon} \right\rceil + 1 > A \right\}, \\ A &= 69\Psi(n) \log\left(3\Psi^2(n) + 3\right) \log\left(2 \log\left(3\Psi^2(n) + 3\right)\right), \\ K_\varepsilon &= \frac{28 + 35\,(4)^\varepsilon}{\psi^{1/2}(1) \log^{2+\varepsilon}(\psi(1) + 1)} \cdot 26\,(\varepsilon + 1) \cdot 1.75^\varepsilon. \end{aligned}$$

Roughly speaking, this tells us that other than on a set of Lebesgue measure less than δ we can give an effective double bound for the size of $S(x, Q)$. We further note that if we let $\delta \to 0$ and $Q \to \infty$ we regain Schmidt's Theorem as given above.

We now prove Theorem 7.1.0.1. The proof given in this section is based on Section 4.1 of [23]; the proof given there is an improvement based on Schmidt's original method.

Let $\psi : \mathbb{N} \to (0, 1/2]$ be a non-increasing function. Suppose (5.5) diverges. We begin by making the Lemmas 4.1, 4.2 and 4.3 from [23] effective.

Lemma 7.1.0.2 (Effective Lemma 4.1 in [23])**.** *For any $M, N, k \in \mathbb{N}$, if $M < N$ then*

$$0 \leq N - M + 1 - \sum_{n=M}^{N} \frac{\varphi(k,n)}{n} \leq \frac{N-M}{k} + \log N, \tag{7.1}$$

where

$$\varphi(k,n) := \sum_{\substack{1 \leq m \leq n, \\ \gcd(m,n) \leq k}} 1.$$

Proof. The lower bound follows from the fact that for any $k, n \in \mathbb{N}$, $\varphi(k,n) \leq n$. We now focus on the upper bound.

Notice that for any $k, n \in \mathbb{N}$,

$$\varphi(k,n) \geq n - \sum_{\substack{d \mid n \\ d > k}} \sum_{\substack{m=1 \\ m \equiv 0 \bmod d}}^{n} 1.$$

Thus, for any $M, N \in \mathbb{N}$, if $M < N$ then

$$\begin{aligned}
\sum_{n=M}^{N} \frac{\varphi(k,n)}{n} &\geq \sum_{n=M}^{N} \left(1 - \frac{1}{n} \sum_{\substack{d \mid n \\ d > k}} \frac{n}{d} \right) \\
&= N - M + 1 - \sum_{d=k+1}^{N} \left(\frac{1}{d} \sum_{\substack{n=M \\ n \equiv 0 \bmod d}}^{N} 1 \right) \\
&\geq N - M + 1 - \sum_{d=k+1}^{N} \left(\frac{1}{d} \left(\frac{N-M}{d} + 1 \right) \right) \\
&= N - M + 1 - \sum_{d=k+1}^{N} \frac{N-M}{d^2} - \sum_{d=k+1}^{N} \frac{1}{d} \\
&\geq N - M + 1 - \frac{N-M}{k} - \log N
\end{aligned}$$

$\square$

Lemma 7.1.0.3 (Effective Lemma 4.2 in [23]). *Given an eventually non-increasing function $\psi : \mathbb{N} \to \mathbb{R}$, for any $M, N, k \in \mathbb{N}$, if $M < N$ then*

$$\left(1 - \frac{1}{k}\right) \sum_{n=M}^{N} \psi(n) - \psi(M) \log M - \sum_{n=M}^{N} \frac{\psi(n)}{n} \leq \sum_{n=M}^{N} \frac{\psi(n)\varphi(k,n)}{n} \leq \sum_{n=M}^{N} \psi(n).$$

Proof. Pick any $M, N, k \in \mathbb{N}$ with $M < N$. By partial summation,

$$\sum_{n=M}^{N} \frac{\psi(n)\varphi(k,n)}{n} = \sum_{n=M}^{N-1} \left((\psi(n) - \psi(n+1)) \sum_{m=M}^{n} \frac{\varphi(k,m)}{m} \right) + \psi(N) \sum_{m=M}^{N} \frac{\varphi(k,m)}{m} \tag{7.2}$$

We recall that $\frac{\varphi(k,n)}{n} \leq 1$, and applying this to the above and summing over n we find that

$$\sum_{n=M}^{N-1} \left((\psi(n) - \psi(n+1)) \sum_{m=M}^{n} \frac{\varphi(k,m)}{m} \right) + \psi(N) \sum_{m=M}^{N} \frac{\varphi(k,m)}{m} \leq \sum_{n=M}^{N} \psi(N),$$

which establishes the upper bound of the lemma.

To establish the lower bound, we consider the upper bound of (7.1) akin to the above, and we applying this to (7.2), we obtain that

$$\begin{aligned}
\sum_{n=M}^{N} \frac{\psi(n)\varphi(k,n)}{n} &= \sum_{n=M}^{N-1} \left((\psi(n) - \psi(n+1)) \sum_{m=M}^{n} \frac{\varphi(k,m)}{m} \right) + \psi(N) \sum_{m=M}^{N} \frac{\varphi(k,m)}{m} \\
&\geq \sum_{n=M}^{N-1} \left((\psi(n) - \psi(n+1)) \left(n - M + 1 - \frac{n-M}{k} \log n\right) \right) \\
&= (\psi(M) - \psi(M+1)) (1 - \log M) + \\
&\quad (\psi(M+1) - \psi(M+2)) \left(2 - \frac{1}{k} - \log(M+1)\right) + \cdots \\
&\quad + (\psi(N-1) - \psi(N)) \left((N-M) - \frac{N-M-1}{k} - \log(N-1)\right) \\
&\quad + \psi(N) \left(N - M + 1 - \frac{N-M}{k} - \log N\right) \\
&= \sum_{n=M}^{M} \psi(n) - \frac{1}{k} \sum_{m=M+1}^{N} \psi(m) - \psi(M) \log M + \\
&\quad \sum_{l=M+1}^{N} \psi(l) (\log l - \log(l+1)) \\
&\geq \left(1 - \frac{1}{k}\right) \sum_{n=M}^{N} \psi(n) - \psi(M) \log M - \sum_{n=M}^{N} \left(\frac{\psi(n)}{n}\right),
\end{aligned}$$

where the last line follows as $\log l - \log(l+1) > -\frac{1}{l}$ for all positive $l \in \mathbb{N}$ $\square$

We now give some definitions before stating and proving Lemma 7.1.0.4. For any $N \in \mathbb{N}$, define

$$\begin{aligned}
\Psi(N) &:= \sum_{n=1}^{N} \psi(n), \\
\Gamma(N) &:= \Psi^2(N) + 1 \geq 1, \\
L(N) &:= \log(3\Gamma(N)) \geq \log 3 > 1, \\
L_2(N) &:= \log(2L(N)) \geq \log(2\log 3) > 1/2, \\
\varPhi(N) &:= \sum_{\substack{1 \leq m \leq N, \\ \gcd(m,N) \leq \Gamma(N)}} 1 \leq N.
\end{aligned} \tag{7.3}$$

Lemma 7.1.0.4 (Effective Lemma 4.3 in [23])**.** *For any $N \in \mathbb{N}$,*

$$0 \leq \sum_{n=1}^{N} \psi(n) \left(1 - \frac{\varPhi(n)}{n}\right) \leq 41 L(N) L_2(N).$$

Proof. The first inequality follows directly from (7.3), as for any $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} \frac{\psi(n)\varPhi(n)}{n} \leq \sum_{n=1}^{N} \psi(n).$$

The rest of the proof deals with the second inequality. Pick any $N \in \mathbb{N}$. Define $M_0 := 0$ and let

$$\begin{aligned}
M_j &:= 2^{2^j}, \\
h &:= \min\{j \in \mathbb{N} : \Psi(N) < M_j\} \geq 1, \\
V_r &:= \min\{V \in \mathbb{N} \cup \{0\} : \Psi(V) \geq M_r\}.
\end{aligned}$$

Suppose initially that $h \geq 2$. By the definition of h we have that

$$2^{h-1} \log 2 = \log M_{h-1} \leq \log \Psi(N) < 0.5 L(N).$$

It follows that

$$2^{h+1} \log 2 < 2L(N),$$

so upon taking logs, we obtain that

$$(h+1)\log 2 + \log\log 2 < L_2(N).$$

It follows that

$$\begin{aligned} h &< \frac{L_2(N) - \log\log 2}{\log 2} \\ &< \frac{L_2(N)}{\log 2} \end{aligned}$$

and, from the bounds on $L_2(N)$ above, that

$$3 + h < \left(\frac{3}{\log(2\log 3)} + \frac{1}{\log 2}\right) L_2(N).$$

In the case that $h = 1$, these inequalities are trivially true.

We consider the terms in the right hand side of the above, noting that if $M_r \leq \Psi(n)$ then ${M_r}^2 + 1 \leq \Gamma(n)$ and $\varphi({M_r}^2 + 1, n) \leq \mathit{\Phi}(n)$. Applying these and Lemma 7.1.0.3, it follows that

$$\begin{aligned} \sum_{n=1}^{N} \frac{\psi(n)\mathit{\Phi}(n)}{n} &\geq \sum_{0 \leq r \leq h-1} \sum_{\substack{1 \leq n \leq N \\ M_r \leq \Psi(n) < M_{r+1}}} \frac{\psi(n)\varphi({M_r}^2 + 1, n)}{n} \\ &\geq \sum_{n=1}^{N} \psi(n) - \\ &\quad \left(\sum_{n=1}^{N} \frac{\psi(n)}{n} + \sum_{r=0}^{h-1} \left(\frac{1}{{M_r}^2 + 1} \sum_{\substack{1 \leq n \leq N \\ M_r \leq \Psi(n) < M_{r+1}}} \psi(n) + \psi(V_r) \log V_r \right) \right). \end{aligned}$$

Considering the sums in the final line of the above, we note that

$$\begin{aligned} \sum_{r=0}^{h-1} \left(\frac{1}{{M_r}^2 + 1} \sum_{\substack{1 \leq n \leq N, \\ M_r \leq \Psi(n) < M_{r+1}}} \psi(n) \right) &< \sum_{r=0}^{h-1} \frac{M_{r+1}}{{M_r}^2 + 1} \\ &< 3 + h \\ &< \left(\frac{3}{\log(2\log 3)} + \frac{1}{\log 2} \right) L_2(N), \end{aligned}$$

and the assumption that ψ is non-increasing gives us that

$$\begin{aligned}
\sum_{r=0}^{h-1} \psi(V_r) \log V_r &\leq \sum_{r=0}^{h-1} \left(\psi(V_r) \sum_{n=1}^{V_r} \frac{1}{n} \right) \\
&\leq \sum_{n=1}^{N} \left(\frac{1}{n} \sum_{r=1,\, V_r > n}^{h-1} \psi(V_r) \right) \\
&\leq \sum_{n=1}^{N} \frac{\psi(n)}{n} h \\
&\leq \frac{1}{\log 2} L_2(N) \sum_{n=1}^{N} \frac{\psi(n)}{n}.
\end{aligned}$$

Applying Hölder's inequality, as $1/L(N) < 1$ we obtain that

$$\sum_{n=1}^{N} \frac{\psi(n)}{n} \leq \left(\sum_{n=1}^{N} \psi(n)^{1+L(N)} \right)^{1/(1+L(N))} \left(\sum_{n=1}^{N} \frac{1}{n^{1+1/L(N)}} \right)^{L(N)/(1+L(N))}$$

We recall that for any $\varepsilon < 1$,

$$\sum_{n=1}^{N} n^{-1-\varepsilon} < 2 \int_{1/2}^{\infty} \frac{1}{x^{1+\varepsilon}} \mathrm{d}x < \frac{8}{\varepsilon}.$$

Substituting this into the above, with $\varepsilon = 1/L(N)$, we obtain that

$$\sum_{n=1}^{N} \frac{\psi(n)}{n} \leq 8 \Psi(N)^{1/L(N)} L(N) \leq 8\sqrt{e} L(N).$$

The above follows as, letting $\Psi(N) = x$, we note that $\Psi(N)^{1/L(N)}$ is of the form $x^{1/\log(3x^2+3)}$. By standard methods in calculus, we note that for any $x \geq 0$,

$$x^{1/\log(3x^2+3)} \leq \sqrt{e}.$$

Combining the results above, we get that

$$\begin{aligned}
\sum_{n=1}^{N} \psi(n) \left(1 - \frac{\Phi(n)}{n} \right) &\leq 8\sqrt{e} L(N) + \left(\frac{3}{\log(2\log 3)} + \frac{1}{\log 2} \right) L_2(N) + \\
&\quad \frac{1}{\log 2} L_2(N) 8\sqrt{e} L(N) \\
&\leq A \cdot L(N) L_2(N),
\end{aligned}$$

where

$$A = \left(\frac{8\sqrt{e}}{\log(2\log 3)} + \frac{3}{\log 3 \log(2\log 3)} + \frac{1}{\log(2\log 2)\log 3} + \frac{8\sqrt{e}}{\log(2\log 2)} \right)$$

The result then follows by computing the value

$$A < 41.$$

□

We are now in a position to prove Theorem 7.1.0.1.

Proof of Theorem 7.1.0.1. We are going to apply Theorem 6.1.0.2.

Denote by μ the Lebesgue measure on $[0, 1)$ and let $S^*(\alpha, u, v)$ be the number of solutions $n \in \mathbb{N}$ to

$$|\alpha n - m| < \psi(n), \quad \gcd(m, n) \leq \Gamma(n), \quad u < n \leq v.$$

We now compute

$$\int_{[0,1)} S^*(\alpha, 0, N)\,\mathrm{d}\alpha$$

as follows. For any $u, v \in \mathbb{N}$, if $u < v$ then

$$\int_{[0,1)} S^*(\alpha, u, v)\,\mathrm{d}\alpha = \sum_{n=u+1}^{v} \int_{[0,1)} S^*(\alpha, n-1, n)\,\mathrm{d}\alpha = 2 \sum_{n=u+1}^{v} \psi(n) \frac{\Phi(n)}{n}.$$

We recall that, by the definition of S, we have that,

$$\int_{[0,1)} S(\alpha,\, n)\,\mathrm{d}\alpha = \psi(n).$$

Notice that for any $N \in \mathbb{N}$, $S(\alpha, N) \geq S^*(\alpha, 0, N)$. By Lemma 7.1.0.4, we get that for any $v \in \mathbb{N}$,

$$\int_{[0,1)} \left(S(\alpha, v) - S^*(\alpha, 0, v)\right)\,\mathrm{d}\alpha = 2 \sum_{n=1}^{v} \psi(n) \left(1 - \frac{\Phi(n)}{n}\right) \leq 82 L(v) L_2(v).$$

For any $n \in \mathbb{N}$, define $G : \mathbb{N} \to \mathbb{R}^+$ by $G(n) = L(n)/L_2(n)$. Thus, for any $N \in \mathbb{N}$,

$$\mu\left(\left\{\alpha \in [0, 1) : S(\alpha, N) - S^*(\alpha, 0, N) > L^2(N)\right\}\right) \leq \frac{82}{G(N)}.$$

For any $j \in \mathbb{N} \cup \{0\}$, let $v_j = \min\{n \in \mathbb{N} : L(n) \geq 2^j\}$. Then

$$\sum_{j=0}^{\infty} \frac{1}{G(v_j)} < +\infty.$$

Hence, by the Borel-Cantelli Lemma, for almost every $\alpha \in [0,1)$ and $j > j(\alpha)$,

$$0 < S(\alpha, v_j) - S^*(\alpha, 0, v_j) < L^2(v_j).$$

Note that

$$2^{j-1} \leq L(v_j) \leq L(N) \leq 2^j \leq L(v_j) \leq 2^{j+1}.$$

It follows that $4L(N) \geq 2^{j+1} \geq L(v_j)$, so $16L^2(N) \geq L^2(v_j)$. Thus, for almost every $\alpha \in [0,1)$,

$$0 < S(\alpha, N) - S^*(\alpha, 0, N) \leq 16L^2(N). \tag{7.4}$$

We now study $S^*(\alpha, 0, N)$. We are going to apply Theorem 6.1.0.2 on $S^*(\alpha, 0, N)$. Pick any $u, v \in \mathbb{N}$ with $u < v$. Notice that for any $n \in \mathbb{N}$,

$$1 - \frac{\Phi(n)}{n} \leq 1 \leq 2L(n)L_2(n).$$

Let

$$\begin{aligned} d^*(n) &:= \sum_{d \mid n,\, 1 \leq d \leq \Gamma(n)} 1 \\ \Psi(u,v) &:= \sum_{u<n\leq v} \psi(n). \end{aligned}$$

Then

$$\begin{aligned} \Psi(u,v) \sum_{n=u+1}^{v} \psi(n) \left(1 - \frac{\Phi(n)}{n}\right) &= \sum_{n=u+1}^{n} \psi(n) \left(1 - \frac{\Phi(n)}{n}\right) + \\ &\quad \sum_{r=u+1}^{v} \psi(r) \left(1 - \frac{\Phi(r)}{r}\right) \sum_{m=u+1}^{v} \psi(m) \\ &\leq \sum_{n=u+1}^{v} 2\psi(n)L(n)L_2(n) + \\ &\quad \sum_{r=u+1}^{v} \psi(r) \left(1 - \frac{\Phi(r)}{r}\right) \Psi(r). \end{aligned}$$

Hence, the assumptions of Theorem 6.1.0.2 are satisfied, as by Lemma 4.4 of [23] and inequality (4.2.10) of [23] we have that,

$$\begin{aligned} \int_{[0,1)} \left(S^*(\alpha,u,v) - 2\Psi(u,v)\right)^2 \mathrm{d}\alpha &\leq 4 \sum_{u<n\leq v} d^*(n)\psi(n) + \\ &\quad 4\Psi(u,v) \sum_{u<n\leq v} \psi(n) \left(1 - \frac{\Phi(n)}{n}\right) \\ &\leq 4 \sum_{u<n\leq v} B\psi(n), \end{aligned}$$

where

$$B = d^*(n) + 2L(n)L_2(n) + \left(1 - \frac{\Phi(n)}{n}\right)\Psi(n).$$

By Theorem 6.1.0.2, for any $\varepsilon > 0$ and $\delta > 0$ there exists some measurable $E_{\varepsilon,\delta} \subset [0,1)$ such that $\mu(E_{\varepsilon,\delta}) < \delta$ and for any $x \in [0,1) \setminus E_{\varepsilon,\delta}$ and $N \in \mathbb{N}$,

$$|S^*(\alpha, 0, N) - 2\Psi(N)| \leq K_{\varepsilon,\delta,0}\left({\Psi_1}^{1/2}(N)\log^{3/2+\varepsilon}\Psi_1(N) + \frac{1}{2}\right),$$

where by Lemma 7.1.0.4, with $\psi_0 := \psi(1) > 0$, we have

$$\begin{aligned}
K_{\varepsilon,\delta,0} &= \frac{4}{\log^{2\varepsilon/3}(\psi_0+2)}\left(\frac{\log 4}{\log 3}\right)^{1+\varepsilon}\left(4 + \frac{1+\varepsilon}{\log 3} + \frac{1}{4\log^{1+\varepsilon} 4}\right)\\
&\leq 26(\varepsilon+1)\left(\frac{\log 4}{\log 3 \log^{2/3} 2}\right)^{\varepsilon} \leq 26(\varepsilon+1)1.75^{\varepsilon},\\
\Psi_1(N) &:= \sum_{n=1}^{N}\left(d^*(n) + 2L(n)L_2(n) + \left(1 - \frac{\Phi(n)}{n}\right)\Psi(n)\right)\psi(n)\\
&\leq \sum_{n=1}^{N} d^*(n)\psi(n) + 43\Psi(N)L(N)L_2(N).
\end{aligned}$$

Notice that for any $N \in \mathbb{N}$,

$$\begin{aligned}
\sum_{n=1}^{N} d^*(n)\psi(n) = \sum_{n=1}^{N}\psi(n)\sum_{d\,|\,n,\, d\leq\Gamma(n)} 1 &= \sum_{d=1}^{\Gamma(N)}\sum_{n=1}^{N/d}\psi(kd)\\
&\leq \sum_{d\leq\Gamma(N)}\frac{\Psi(N)}{d}\\
&\leq \Psi(N)\log(3\Gamma(N)) = \Psi(N)L(N)\\
&\leq \frac{1}{\log(2\log 3)}\Psi(N)L(N)L_2(N).
\end{aligned}$$

Thus we find that

$$\Psi_1(N) \leq \left(43 + \frac{1}{\log(2\log 3)}\right)\Psi(N)L(N)L_2(N) < 45\Psi(N)L(N)L_2(N),$$

and taking

$$\psi_0' = \Psi_1(1) \leq 45\psi(1)\log\left(3(\psi^2(1)+1)\right)\log\left(2\log\left(3(\psi^2(1)+1)\right)\right) \leq 30,$$

we find that

$$\begin{aligned}
{\Psi_1}^{1/2}(N)\log^{3/2+\varepsilon}\Psi_1(N) + \frac{1}{2} &\leq \frac{{\psi_0'}^{1/2}\log^{3/2+\varepsilon}\psi_0' + 1/2}{{\psi_0}^{1/2}\log^{2+\varepsilon}(\psi_0+1)}\Psi^{1/2}(N)\log^{2+\varepsilon}(\Psi(N)+1)\\
&\leq \frac{35(4)^{\varepsilon}}{{\psi_0}^{1/2}\log^{2+\varepsilon}(\psi_0+1)}\Psi^{1/2}(N)\log^{2+\varepsilon}(\Psi(N)+1).
\end{aligned}$$

Hence, by the triangle inequality and (7.4),

$$\begin{aligned}
|S(\alpha, N) - 2\Psi(N)| &\leq |S(\alpha, N) - S^*(\alpha, 0, N)| + |S^*(\alpha, 0, N) - 2\Psi(N)| \\
&\leq 16L^2(N) + \frac{35(4)^\varepsilon K_{\varepsilon,\delta,0}}{{\psi_0}^{1/2}\log^{2+\varepsilon}(\psi_0+1)}\Psi^{1/2}(N)\log^{2+\varepsilon}(\Psi(N)+1) \\
&\leq \left(\frac{16(\log(3{\psi_0}^2+3))^2}{{\psi_0}^{1/2}\log^{2+\varepsilon}(\psi_0+1)} + \frac{35(4)^\varepsilon K_{\varepsilon,\delta,0}}{{\psi_0}^{1/2}\log^{2+\varepsilon}(\psi_0+1)}\right) \cdot \\
&\quad \cdot \Psi^{1/2}(N)\log^{2+\varepsilon}\Psi(N) \\
&\leq \frac{28 + 35(4)^\varepsilon K_{\varepsilon,\delta,0}}{{\psi_0}^{1/2}\log^{2+\varepsilon}(\psi_0+1)}\Psi^{1/2}(N)\log^{2+\varepsilon}(\Psi(N)+1) \\
&\leq K_{\varepsilon,}\Psi^{1/2}(N)\log^{2+\varepsilon}(\Psi(N)+1),
\end{aligned}$$

where K_ε is given in the statement of the theorem. This proves the theorem. □

7.2 QUANTITATIVE KOUKOULOPOULOS-MAYNARD THEOREM

Using the probabilistic tools from the previous chapter, we are able to give an explicit version of this theorem.

Theorem 7.2.0.1 (Effective Aistleitner-Borda-Hauke Theorem). *Let $\psi : \mathbb{N} \to [0, 1/2]$ be a function and let $C > 4$. Suppose* (5.7) *diverges. Then there exists some measurable $E_{C,\delta} \subset [0, 1)$ such that $\mu(E_{C,\delta}) < \delta$, and for any $x \in [0, 1) \setminus E_{C,\delta}$ we have that*

$$|S'(x, Q) - \Psi'(Q)| \leq \max\left\{\frac{k_{C,\delta}}{2}, \frac{2e\Psi'(Q)+1}{(\log\Psi'(Q))^{\sqrt{C}-1}} + \frac{1}{2}\right\},$$

up to a constant depending only on C found following the proof of Theorem 2 in [1], where $S'(x, Q)$ and $\Psi'(Q)$ are given above, and $k_{C,\delta}$ is given in (6.5).

We now give the proof of Theorem 7.2.0.1 using Theorem 6.1.0.3.

Proof. The proof follows that of [1], using Theorem 6.1.0.3 in place of Theorem 1 of [1]. First define

$$\mathcal{A}_q := [0,1] \bigcap \bigcup_{\substack{1\leq p\leq q \\ \gcd(p,q)=1}} \left(\frac{p-\psi(q)}{q}, \frac{p+\psi(q)}{q}\right), \quad q \in \mathbb{N}$$

We now note that if μ is the Lebesgue measure on [0, 1], then

$$\mu\left(\mathcal{A}_q\right) = \frac{2\varphi(q)\psi(q)}{q}.$$

In Theorem 6.1.0.3, we set $f_q = \phi_q = \mu\left(\mathcal{A}_q\right)$ and set $f_q(x) = \mathbb{1}_{\mathcal{A}_q}(x)$, where $\mathbb{1}_{\mathcal{A}_q}$ denotes the characteristic function on $\mathcal{A}_q$. We note that this means that $\Psi(Q)$ as defined in Theorem 7.2.0.1 and Theorem 6.1.0.3 are equivalent.

We now note that

$$\int_0^1 \left(\sum_{q=1}^{Q} \left(\mathbb{1}_{\mathcal{A}_q}(x) - \mu\left(\mathcal{A}_q\right)\right)\right)^2 \mathrm{d}x = \sum_{q,\, r\leq Q} \mu\left(\mathcal{A}_q \cap \mathcal{A}_r\right) - \Psi(Q)^2.$$

By Theorem 2 of [1], we have that

$$\sum_{q,\, r\leq Q} \mu\left(\mathcal{A}_q \cap \mathcal{A}_r\right) - \Psi(Q)^2 = O_C\left(\frac{\Psi(Q)^2}{\left(\log \Psi(Q)\right)^C}\right),$$

with an implied constant depending on C. We note that this implied constant is the constant that has been calculated, with the working to be published separately due to length. We can thus apply Theorem 6.1.0.3 and the statement follows. □

7.3 INHOMOGENEOUS DIOPHANTINE APPROXIMATION ON M_0-SETS

We will now prove the following effective versions of Theorems 1 and 4 of [43].

Theorem 7.3.0.1 (Effective Theorem 1 of [43])**.** *Let $F \subset [0,1]$ and μ be an non-atomic probability measure supported on F. Let $\gamma \in [0,1]$ and $\psi : \mathbb{N} \to (0,1]$ be a function. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be a lacunary sequence of positive integers. Suppose there exists $\nu > 0$ and $A > 2$ such that for any $t \in \mathbb{Z}\setminus\{0\}$,*

$$|\hat{\mu}(t)| \leq \frac{\nu}{\left(\log^+ |t|\right)^A}. \tag{7.5}$$

Then for any $\varepsilon > 0$ and $\delta > 0$, there exists μ-measurable $E_{\varepsilon,\delta} \subset F$ such that $\mu(E_{\varepsilon,\delta}) < \delta$ and for any $x \in F \setminus E_{\varepsilon,\delta}$ and $N \in \mathbb{N}$,

$$|R(x,N) - 2\Psi(N)| \leq 2K_{\varepsilon,\delta/2}\left(\Psi(N)^{2/3}\left(\log\Psi(N)+2\right)^{2+\varepsilon}\right) + t_{1,\delta}, \tag{7.6}$$

where $\alpha \in (0,1)$, $\log^+(\beta) := \max(0, \log\beta)$ *and*

$$\Psi(n) = \sum_{k=1}^{n} \psi(k) \tag{7.7}$$

$$\varPhi(N) = \Psi^{4/3}(\log^+ \varPhi(N) + 1) + \Psi(N), \tag{7.8}$$

$$K = 6\max(48, c_1, c_2), \tag{7.9}$$

$$c_1 = \frac{22}{K_0 - 1}, \tag{7.10}$$

$$\begin{aligned} c_2 = {} & 12\left(\frac{3}{2^{2/3}} + 1\right)(1 + \zeta(A-1)) + \frac{8}{K_0 - 1} \\ & + 18\nu C^{-A}\left(\sqrt{2}\zeta\left(A - \frac{1}{2}\right)\left(2 + \alpha^{-A}\right) + 2^{A+1}\zeta\left(\frac{A}{2}\right)\right), \end{aligned} \tag{7.11}$$

$$t_{1,\delta} = \frac{1}{2} + \left(\frac{(1-A)\delta}{2(3 + \nu/C^A)}\right)^{1/(1-A)}, \tag{7.12}$$

where $N_{\varepsilon,\delta}$, $r_{\varepsilon,\delta}$, *and* $K_{\varepsilon,\delta}$ *and* $\varPhi_0$ *are given in Theorem 6.1.0.2.*

Theorem 7.3.0.2 (Effective Theorem 4 of [43]). *Let* $F \subset [0,1]$ *and* μ *be an non-atomic probability measure supported on* F. *Let* $\gamma \in [0,1]$ *and* $\psi : \mathbb{N} \to (0,1]$ *be a function. Let* $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ *be an* α*-separated sequence. Suppose there exists* $\nu > 0$ *and* $A > 2$ *such that for any* $t \in \mathbb{Z} \setminus \{0\}$, (7.5) *is satisfied. Further, assume that* $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ *satisfies the following growth condition:*

$$\log q_n > Cn^{1/B} \tag{7.13}$$

Then for any $\varepsilon > 0$ *and* $\delta > 0$, *there exists* μ*-measurable* $E_{\varepsilon,\delta} \subset F$ *such that* $\mu(E_{\varepsilon,\delta}) < \delta$ *and for any* $x \in F \setminus E_{\varepsilon,\delta}$ *and* $N \in \mathbb{N}$,

$$\begin{aligned} |R(x,N) - 2\Psi(N)| \le K_{\varepsilon,\delta/2} & \left(\left(\Psi(N)\left(\log^+ \Psi(N) + 2\right) + E(N)\right)^{1/2}\right. \\ & \left.\left(\log\left(\Psi(N)\left(\log^+ \Psi(N) + 2\right) + E(N)\right)\right)^{3/2+\varepsilon} + 2\right) + t_{2,\delta}, \end{aligned} \tag{7.14}$$

where $R(x,N)$ *and* $\Psi(N)$ *are given at* (5.14) *and* (7.7) *respectively and* $K_{\varepsilon,\delta}$ *is given in Theorem 6.1.0.2.*

Further,

$$E(N) = \sum\sum_{1\le m<n\le N} (q_m,\, q_n) \min\left(\frac{\psi(q_m)}{q_m},\, \frac{\psi(q_n)}{q_n}\right),$$

$$\Phi(N) = \Psi(N)\left(\log^+ \Psi(N) + 2\right) + E(N), \tag{7.15}$$

$$K = 2\max(48, c_3), \tag{7.16}$$

$$c_3 = 4 + 18\nu C^{-A}\left(\sqrt{2}\zeta\left(\frac{A}{B} - \frac{1}{2}\right)\left(2 + \alpha^{-A}\right) + 2^{A+1}\zeta\left(\frac{A}{2B}\right)\right) + c_2, \tag{7.17}$$

$$t_{2,\delta} = \frac{1}{2} + \left(\frac{(1-\min(9, A/B))\delta}{2(1+\nu/C^A)}\right)^{1/(1-\min(9,A/B)))}, \tag{7.18}$$

where c_2 *is given by* (7.11).

Remark. Lacunary sequence $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ satisfies (7.13) with $B = 1$.

We note that $t_{1,\delta}$ and $t_{2,\delta}$ are needed to account for some assumptions made during the proofs, namely (7.19) and (7.29). In our proofs, we will initially make these assumptions, before proving Lemma 7.3.0.10, which allows us to remove them.

We note that if μ is the Lebesgue measure on $[0, 1]$, then μ is a probability measure and (7.5) is satisfied as the Fourier transform of the Lebesgue measure is the Dirac distribution.

In what follows we will make some results of [43] effective, before going on to prove the theorems above. Specifically, we make Lemmas 5 to 8 and Propositions 1 & 2 in [43] effective, then apply the results to Theorem 1 and 4 in that paper to give the quantitative versions. We begin by giving explicit versions of Lemmas 5 to 8 of [43]; initially we just give the statements of the effective versions of Lemmas 5-8 of [43], and we give the proofs of these results after all the statements.

Lemma 7.3.0.3 (Effective Lemma 5 in [43])**.** *Let* μ *be a non-atomic probability measure supported on* $F \subset [0, 1]$ *and* $(q_n)_{n\in\mathbb{N}}$ *be an increasing sequence of positive integers greater than 4. Suppose there exists* $B \ge 1$ *and* $C > 0$ *such that growth condition* (7.13) *is satisfied. Let* $\gamma \in [0, 1]$ *and* $\psi : \mathbb{N} \to (0, 1]$ *be a function. Suppose there exists* $A > 2B$ *such that* (7.5) *is satisfied.*

Further suppose that for any $\tau > 1$ *and* $n \in \mathbb{N}$*,*

$$\psi(q_n) \ge 3n^{-\tau}. \tag{7.19}$$

It then follows that for arbitrary $a, b \in \mathbb{N}$*, if* $a < b$ *then we have that*

$$\left|\sum_{n=a}^{b} \mu\left(E_{q_n}^{\gamma}\right) - 2\sum_{n=a}^{b} \psi(q_n)\right| \le \min\left(m_1, m_2 \sum_{n=a}^{b} \psi(q_n)\right), \tag{7.20}$$

where for any $q \in \mathbb{N}$,

$$E_q^\gamma = E_q^\gamma(\psi) := \{x \in [0,1] : \|qx - \gamma\| \leq \psi(q)\}, \tag{7.21}$$

$$m_1 = m_1(A,B) = 3 + 3\zeta\Big(\frac{A}{B} - 1\Big), \tag{7.22}$$

$$m_2 = \frac{3}{2^{2/3}} < 2. \tag{7.23}$$

As commented previously, (7.19) leads to another error term. We deal with the error term associated to this in Lemma 7.3.0.10.

We now give a bound for the value

$$S(m,n) := \sum_{k \in \mathbb{Z}\setminus\{0\}} \hat{W}^+_{m,n}(k)\hat{\mu}(-k),$$

where

$$W^+_{m,n}(k) = \left(\left(\sum_{p=0}^{q_m-1} \delta_{\frac{p+\gamma}{q_m}}(k)\right) * \chi^+_{\frac{\psi(q_m)}{q_m},\varepsilon_m}(k)\right)\left(\left(\sum_{r=0}^{q_n-1} \delta_{\frac{r+\gamma}{q_n}}(k)\right) * \chi^+_{\frac{\psi(q_n)}{q_n},\varepsilon_n}(k)\right), \tag{7.24}$$

where $*$ denotes convolution, δ_x is the Dirac delta-function at the point $x \in \mathbb{R}$ and

$$\chi^+_{\delta,\varepsilon}(x) := \begin{cases} 1 & \text{if } ||x|| \leq \delta \\ \frac{1}{\delta\varepsilon}(\delta - ||x||) & \text{if } (1-\varepsilon)\delta < ||x|| \leq \delta \\ 0 & \text{if } ||x|| > \delta, \end{cases}$$

where $||x||$ denotes the distance from $x \in \mathbb{R}$ to the nearest integer; that is, $||x|| := \min\{|x-m| : m \in \mathbb{Z}\}$.

Lemma 7.3.0.4 (Effective Lemma 6 in [43])**.** *Let μ be a probability measure supported on $F \subset [0,1]$. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive integers greater than 4. Suppose there exists $B \geq 1$ and $C > 0$ such that growth condition* (7.13) *is satisfied. Let $\gamma \in [0,1]$, $\psi : \mathbb{N} \to (0,1]$, let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of real numbers in $(0,1]$ and let $\alpha \in (0,1)$. Suppose there exists $A > 2B$ such that* (7.5) *is satisfied. Then for any $m,n \in \mathbb{N}$, if $m < n$ then*

$$|S(m,n)| \leq 9\nu C^{-A}\frac{\psi(q_m)}{n^{A/B}{\varepsilon_n}^{1/2}} + 9\nu C^{-A}\left(1 + \frac{1}{\alpha^A}\right)\frac{\psi(q_n)}{m^{A/B}{\varepsilon_m}^{1/2}} + \tag{7.25}$$

$$\frac{9 \cdot 2^A \nu C^{-A}}{n^{A/B}{\varepsilon_m}^{1/2}{\varepsilon_n}^{1/2}} + |T(m,n)|, \tag{7.26}$$

where T is defined based on the value of α that

$$T(m,n) = T_\alpha(m,n) := \sum_{\substack{s,\,t\in\mathbb{Z}\setminus\{0\} \\ 1\le|sq_m - tq_n|<{q_m}^\alpha}}\sum \hat{W}^+_{q_m,\gamma,\epsilon_m}(sq_m)\hat{W}^+_{q_n,\gamma,\epsilon_n}(tq_n)\hat{\mu}(sq_m - tq_n), \tag{7.27}$$

and $\hat{W}^+_{q_m,\gamma,\epsilon_m}(sq_m)$ is the Fourier transform of (7.24)*.*

The following two lemmas give effective bounds on the size of the quantity $T(m,\,n)$ appearing above. Lemma 7.3.0.5 deals with lacunary sequences and Lemma 7.3.0.6 deals with α-separated sequences.

Lemma 7.3.0.5 (Effective Lemma 7 in [43])**.** *Let μ be a probability measure supported on $F \subset [0,1]$. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be an lacunary sequence of natural numbers, with the constant from the definition at* (5.13) *given by $K = K_0$. Let $\gamma \in [0,1]$, $\alpha \in (0,1)$, $\psi : \mathbb{N} \to (0,1]$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ be a decreasing sequence of real numbers in $(0,1]$. Then for arbitrary $a, b \in \mathbb{N}$ with $a < b$, we have that*

$$\sum_{a\le m<n\le b}\sum |T(m,n)| \le 11K' \sum_{n=a}^{b} \frac{\psi(q_n)}{{\varepsilon_n}^{1/2}},$$

where $T(m,\,n)$ depends on α and it is given by (7.27)*, and K' is given by*

$$K' = \frac{1}{K_0 - 1} > 0. \tag{7.28}$$

Lemma 7.3.0.6 (Effective Lemma 8 in [43])**.** *Let μ be a probability measure supported on $F \subset [0,1]$. Let $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ be an α-separated increasing sequence with $m_0 = 1$. Let $\gamma \in [0,1]$, $\psi : \mathbb{N} \to (0,1]$ and $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of real numbers in $(0,1]$. Suppose that for any $n \in \mathbb{N}$,*

$$\psi(q_n) \ge n^{-9} \tag{7.29}$$

and ${\varepsilon_n}^{-1} \le 2n$. Then for arbitrary $a, b \in \mathbb{N}$, if $a < b$ then

$$\sum_{a\le m<n\le b}\sum |T(m,n)| \le 2\sum_{n=a}^{b} \psi(q_n),$$

where $T(m,\,n)$ is given by (7.27)*.*

Again, the assumption (7.29) leads to another error term; this is dealt with in Lemma 7.3.0.10. For the sake of simplicity, each of the proofs is written as an expansion the original proof in [43]; we refer heavily to the original paper throughout.

Before beginning the proofs, we make a slight remark that, as commented in [43], the equality part for identity (3.2.5 in [23]) does not always hold. In general, we have the following result instead:

Lemma 7.3.0.7. *Let $f : \mathbb{N} \to [0, 1/2]$ be a function, $(a_j)_j$, $(b_j)_j$ be sequences of integers and*

$$\mathcal{E}_j = \{x \in [0, 1] : \|a_j x + b_j\| < f(j)\}.$$

Then for any $j, k \in \mathbb{N}$, the Lebesgue measure λ of $\mathcal{E}_j \cap \mathcal{E}_k$ is upper bounded by

$$\lambda(\mathcal{E}_j \cap \mathcal{E}_k) \leq 4f(j)f(k) + 2\gcd(a_j, a_k) \min\left(\frac{f(j)}{a_j}, \frac{f(k)}{a_k}\right).$$

Proof. Let $\mathcal{F}_i = \{x \in [0, 1] : \|x + b_i\| < f(i)\}$, $c_j = a_j/\gcd(a_j, a_k)$ and $c_k = a_k/\gcd(a_j, a_k)$. Since $\mathcal{F}_k$ is an open interval, for any $y \in \mathbb{R}$, the number of integers in the translation $c_j\mathcal{F}_k - y$ is at most

$$\lambda(c_j\mathcal{F}_k) + 1 \leq c_j\lambda(\mathcal{F}_k) + 1 \leq 2c_j f(k) + 1.$$

By Lemma 3.1 in [23] and the argument contained there,

$$\begin{aligned}\lambda(\mathcal{E}_j \cap \mathcal{E}_k) &\leq \frac{1}{c_j c_k}\lambda(c_j\mathcal{F}_k)\left(\lambda(c_k\mathcal{F}_j) + 1\right) = \frac{1}{c_j c_k}(2c_k f(j))(2c_j f(k) + 1)\\ &= 4f(j)f(k) + 2\gcd(a_j, a_k)\frac{f(j)}{a_j},\end{aligned}$$

and the result follows by interchanging the indices j and k for the last inequality. □

Hence, (3.2.5 in [23]) should be applied in this context by setting $\mathcal{E}_j = E^\gamma_{q_j}$, $a_j = q_j$, $f(j) = \psi(q_j)$, correcting the equals sign to $\leq$, and multiplying the second term by a factor of 2 to obtain

$$\lambda(E^\gamma_{q_m} \cap E^\gamma_{q_n}) \leq 4\psi(q_m)\psi(q_n) + 2\gcd(q_m, q_n) \min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n}\right),$$

Thus, by (105 in [43]), we can modify (88 in [43]) to see that

$$\begin{aligned}\hat{W}^+_{m,n}(0) &\leq 4(1 + \varepsilon_m)(1 + \varepsilon_n)\psi(q_m)\psi(q_n) +\\ &\qquad 2(1 + \varepsilon_m)(1 + \varepsilon_n)\gcd(q_m, q_n) \min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n}\right)\\ &\leq 4\psi(q_m)\psi(q_n) + 12\varepsilon_m\psi(q_m)\psi(q_n) + 8\gcd(q_m, q_n) \min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n}\right),\end{aligned} \tag{7.30}$$

as the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ is bounded above by 1.

We are now in a position to prove the results above.

Proof of Lemma 7.3.0.3. Pick any sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of real numbers in $(0,1]$. By our assumptions, following the original proof in [43], we see that for any $a,b\in\mathbb{N}$, if $a<b$ then

$$P(a,b) := \left|\sum_{n=a}^{b}\mu(E_{q_n}^{\gamma}) - 2\sum_{n=a}^{b}\psi(q_n)\right| \le \sum_{n=a}^{b}\psi(q_n)\varepsilon_n + \sum_{n=a}^{b}\frac{3}{n^{A/B}{\varepsilon_n}^{1/2}}. \tag{7.31}$$

Recall that, by definition, for any $n\in\mathbb{N}$, $\Psi(n)=\sum_{k=1}^{n}\psi(n)$. For any $n\in\mathbb{N}$, let

$$\varepsilon_n = \min\Big(1, (\Psi(n))^{-2}\Big).$$

From (7.31) and Lemma D4 in [43], following the argument of the original proof, we obtain that

$$P(a,b) \le 3 + 3\sum_{n=1}^{\infty}\frac{1}{n^{A/B-1}} = 3 + 3\zeta(A/B-1).$$

To finish the proof, we need to establish the "other" upper bound. Set $\tau = A/B$ in (7.19), and let $\varepsilon_n = 2^{-2/3}\in(0,1)$. Then (7.31) allows us to deduce that

$$P(a,b) \le \sum_{n=a}^{b}\left(\varepsilon_n + \frac{1}{n^{A/(2B)}{\varepsilon_n}^{1/2}}\right)\psi(q_n) \le \sum_{n=a}^{b}\left(\varepsilon_n + \frac{1}{{\varepsilon_n}^{1/2}}\right)\psi(q_n) = \frac{3}{2^{2/3}}\sum_{n=a}^{b}\psi(q_n).$$

We note there is no factor of 3 in the above, as the 3 on the right hand side of (7.31) is cancelled by (7.19).

The upper bound is optimal in this method by the choice of $(\varepsilon_n)_{n\in\mathbb{N}}$; the inequalities are true for any choice of $(\varepsilon_n)_{n\in\mathbb{N}}$, and the function $f_1:\mathbb{R}^+\to\mathbb{R}^+$ defined for $x>0$ by

$$f_1(x) = x + \frac{1}{x^{1/2}},$$

has its global minimum of $3/2^{2/3}$ at $x=\varepsilon_n=2^{-2/3}$. $\square$

Proof of Lemma 7.3.0.4. This proof is done by splitting $S(m,\,n)$ into various parts. Following from the original proof, it suffices to improve some estimates.

By (7.5) and (7.13), for any $n\in\mathbb{N}$ and $t\in\mathbb{Z}\setminus\{0\}$,

$$|\hat{\mu}(-tq_n)| \le \nu C^{-A} n^{-A/B}.$$

By (47 in [43]) and (53 in [43]),

$$|S_1(m,n)| \leq 9\nu C^{-A} \frac{\psi(q_m)}{n^{A/B}{\varepsilon_n}^{1/2}}$$
$$|S_2(m,n)| \leq 9\nu C^{-A} \frac{\psi(q_n)}{m^{A/B}{\varepsilon_m}^{1/2}},$$

where $|S_1(m,n)|$ and $|S_2(m,n)|$ are given at the beginning of the proof of Lemma 6 in [43].

We now find an explicit bound for $S_4(m,\, n)$. Notice that, given the restriction $|sq_m - tq_n| \geq q_n/2$ imposed on $s,\, t \in \mathbb{Z} \setminus \{0\}$, by (7.13) and (7.5) we have that

$$\begin{aligned} |\hat{\mu}(sq_m - tq_n)| &\leq \frac{\nu}{\log^A |sq_m - tq_n|} \leq \frac{\nu}{\log^A (q_n/2)} \\ &\leq \frac{2^A \nu}{\log^A q_n} \leq 2^A \nu C^{-A} n^{-A/B}, \end{aligned}$$

where the second line follows because for any $n \in \mathbb{N}$, $q_n \geq 4$, and from that we deduce that

$$\frac{\log q_n}{\log (q_n/2)} \leq 2.$$

We can now give an explicit upper bound for S_4 as definied in the original proof. We find that

$$|S_4(m,n)| \leq 9(2)^A \nu C^{-A} \frac{1}{n^{A/B}{\varepsilon_m}^{1/2}{\varepsilon_n}^{1/2}}.$$

To bound S_6 as given in the original proof, note that for ${q_n}^\alpha \leq |sq_m - tq_m| < q_n/2$, by (7.13) and (7.5), we have that

$$|\hat{\mu}(sq_m - tq_n)| \leq \frac{\nu}{\alpha^A \log^A q_n} \leq \nu C^{-A} \frac{1}{\alpha^A} m^{-A/B}.$$

It follows from the argument of the original proof that

$$|S_6(m,n)| \leq 9\nu C^{-A} \frac{1}{\alpha^A} \frac{\psi(q_n)}{m^{A/B}{\varepsilon_m}^{1/2}}.$$

Hence the result follows by adding these estimates together, as in the original proof. □

Proof of Lemma 7.3.0.5. Set

$$D = D(m,n) = \frac{q_m}{q_n \psi(q_m){\varepsilon_m}^{1/2}}.$$

Notice that, from the original proof and that as $\varepsilon_n \leq 1$, using (51 of [43]) and (52 of [43]), we have that

$$\begin{aligned}|T(m,n)| &\leq \sum_{t\in\mathbb{N}} \min\left(\frac{4}{\pi^2}\frac{q_m^2}{q_n^2}\frac{1}{t^2\psi(q_m)\varepsilon_m}, (2+\varepsilon_m)\psi(q_m)\right)(2+\varepsilon_n)\psi(q_n) \\ &\leq 9\sum_{t=1}^{\lfloor D\rfloor}(\psi(q_m)\psi(q_n)) + \frac{4}{\pi^2}\frac{{q_m}^2}{{q_n}^2}\frac{\psi(q_n)}{\psi(q_m)\varepsilon_m}\sum_{t=\lfloor D\rfloor+1}^{+\infty}\frac{1}{t^2} \\ &\leq 11\frac{q_m}{q_n}\frac{\psi(q_n)}{{\varepsilon_m}^{1/2}} \leq 11\frac{q_m}{q_n}\frac{\psi(q_n)}{{\varepsilon_n}^{1/2}}.\end{aligned}$$

This follows as $(\varepsilon_n)_{n\in\mathbb{N}}$ is decreasing, and by applying the following estimate:

$$\sum_{t=\lfloor D\rfloor+1}^{+\infty}\frac{1}{t^2} \leq \frac{\pi^2}{6}\int_D^\infty \frac{dt}{t^2} = \frac{\pi^2}{6D} = \frac{\pi^2}{6}\frac{q_n\psi(q_m){\varepsilon_m}^{1/2}}{q_m},$$

Thus, by (5.13), we find that $q_m \leq {K_0}^{m-n}q_n$. By the formula for geometric sums, we find that

$$\sum_{m=1}^{n-1}\frac{q_m}{q_n} \leq \sum_{m=1}^{n-1}\frac{{K_0}^{m-n}q_n}{q_n} < \sum_{i=1}^{\infty}{K_0}^{-i} = \frac{1/K_0}{1-1/K_0} = \frac{1}{K_0-1} = K'.$$

□

Proof of Lemma 7.3.0.6. Notice that, from the original proof, for any $m, n \in \mathbb{N}$,

$$\begin{aligned}|T(m,n)| &\leq \sum_{s>m^3/\psi(q_m)}\frac{1}{\pi^2 s\psi(q_m)\varepsilon_m}(2+\varepsilon_n)\psi(q_n) \\ &= \frac{3}{\pi^2}\frac{\psi(q_n)}{\psi(q_m)\varepsilon_m}\sum_{s>m^3/\psi(q_m)}\frac{1}{s^2} \\ &\leq \frac{1}{2}\frac{\psi(q_n)}{m^3\varepsilon_m} \\ &\leq \frac{\psi(q_n)}{m^2},\end{aligned}$$

by reasoning akin to that in the proof of Lemma 7.3.0.5. Hence for any $a, b \in \mathbb{N}$, if $a < b$ then

$$\sum\sum_{a\leq m<n\leq b}|T(m,n)| \leq \sum_{n=a+1}^{b}\left(\psi(q_n)\sum_{m=a}^{b-1}\frac{1}{m^2}\right) < \frac{\pi^2}{6}\sum_{n=a}^{b}\psi(q_n).$$

We finish the proof by noting that $\pi^2/6 < 2$. □

Our next aim is to establish explicit estimates for Propositions 1 and 2 in [43]. Before we can do this, we first need to find explicit estimates for some inequalities. Take $\delta \in (0,1]$ and for any $n \in \mathbb{N}$,

$$\varepsilon_n := \min\left\{\frac{1}{2^\delta}, \frac{1}{(\sum_{k=a}^n \psi(q_k))^\delta}\right\},$$

notice that

$$\varepsilon_n{}^{-1} \leq \max\left\{2^\delta, n^\delta\right\} < 2n. \tag{7.32}$$

In the case that $\sum_{k=a}^b \psi(q_k) > 2$, inequality (102 in [43]) can be modified to

$$\sum\sum_{a \leq m < n \leq b} \frac{1}{n^{A/B} \varepsilon_m{}^{1/2} \varepsilon_n{}^{1/2}} \leq \zeta\left(\frac{A}{B}\right) \sum_{k=a}^b \psi(q_k).$$

For the case of $\sum_{k=a}^b \psi(q_k) \leq 2$, the inequality (103 in [43]) can be modified into

$$\sum\sum_{a \leq m < n \leq b} \frac{1}{n^{A/B} \varepsilon_m{}^{1/2} \varepsilon_n{}^{1/2}} \leq 2\zeta\left(\frac{A}{2B}\right) \sum_{k=a}^b \psi(q_k).$$

Since the Riemann Zeta function is decreasing on $\mathbb{R}^+$, and

$$\zeta\left(\frac{A}{B}\right) < \zeta\left(\frac{A}{2B}\right) < 2\zeta\left(\frac{A}{2B}\right),$$

in both cases we have that

$$\sum\sum_{a \leq m < n \leq b} \frac{1}{n^{A/B} \varepsilon_m{}^{1/2} \varepsilon_n{}^{1/2}} \leq 2\zeta\left(\frac{A}{2B}\right) \sum_{k=a}^b \psi(q_k). \tag{7.33}$$

By (7.32), we have that

$$\sum_{n=1}^\infty \frac{1}{n^{A/B} \varepsilon_n{}^{1/2}} < \sum_{n=1}^\infty \frac{\sqrt{2n}}{n^{A/B}} = \sqrt{2}\zeta\left(\frac{A}{B} - \frac{1}{2}\right).$$

Thus we can make the inequality before (104 in [43]) explicit by applying Lemma 7.3.0.4 and (7.33) to obtain

$$\sum\sum_{a \leq m < n \leq b} |S(m,n)| \leq 9\nu C^{-A} \left(\sqrt{2}\zeta\left(\frac{A}{B} - \frac{1}{2}\right)\left(2 + \alpha^{-A}\right) + 2^{A+1}\zeta\left(\frac{A}{2B}\right)\right) \sum_{n=a}^b \psi(q_n) +$$
$$\sum\sum_{a \leq m < n \leq b} |T(m,n)|.$$

We now note that

$$\left(\frac{3}{2} + \frac{\log x}{2\log(3/2)}\right) \leq \left(\frac{3}{2\log 2} + \frac{1}{2\log(3/2)}\right) \log x \leq 4 \log x.$$

Thus, we make inequality (108 in [43]) explicit, showing that for any $\delta > 0$ and $a, b \in \mathbb{N}$,

$$\sum\sum_{a \leq m < n \leq b} \varepsilon_m \psi(q_m)\psi(q_n) \leq 4 \left(\sum_{n=a}^{b} \psi(q_n) \right)^{2-\delta} \log \left(\sum_{n=a}^{b} \psi(q_n) \right).$$

Before proceeding, we remark there is a typo on (109 in [43]); the correct version should state:

$$\sum\sum_{a \leq m < n \leq b} W^{+}_{m,n}(0) \leq 2 \left(\sum_{n=a}^{b} \psi(q_n) \right)^{2} + O\left(\sum_{n=a}^{b} \psi(q_n) \right)^{2-\delta} \log \left(\sum_{n=a}^{b} \psi(q_n) \right) \quad (7.34)$$

$$+ O\left(\sum\sum_{a \leq m < n \leq b} \gcd(q_m, q_n) \min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n} \right) \right); \quad (7.35)$$

the coefficient of the main term in [43] is 4 instead of 2. Hence, by (7.30) and bounds given above, (7.34) is explicitly given by

$$\sum\sum_{a \leq m < n \leq b} W^{+}_{m,n}(0) \leq 2 \left(\sum_{n=a}^{b} \psi(q_n) \right)^{2} + 24 \left(\sum_{n=a}^{b} \psi(q_n) \right)^{2-\delta} \log \left(\sum_{n=a}^{b} \psi(q_n) \right)$$
$$+ 4 \sum\sum_{a \leq m < n \leq b} \gcd(q_m, q_n) \min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n} \right).$$

We are now in a position to make Propositions 1 and 2 of [43] effective. By applying Lemma 7.3.0.5 with $\alpha = 1/2$ for the formula for T, the implicit constants of inequality (110 in [43]) are given by

$$\sum\sum_{a \leq m < n \leq b} |T(m,n)| \leq 11K' \sum_{n=a}^{b} \frac{\psi(q_n)}{{\varepsilon_n}^{1/2}} \leq \frac{11K'}{\varepsilon_b^{1/2}} \sum_{n=a}^{b} \psi(q_n) \leq 11K' \left(\sum_{n=a}^{b} \psi(q_n) \right)^{1+\delta/2},$$

where $K' = 1/(K_0 - 1)$, and as the sequence $(q_n)_{n \in \mathbb{N}}$ is lacunary, inequality (111 in [43]) becomes

$$\sum\sum_{a \leq m < n \leq b} \gcd(q_m, q_n) \min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n} \right) \leq K' \sum_{n=a}^{b} \psi(q_n).$$

We are now able to give explicit estimates for the inequalities on page 8620 in [43] . If the assumptions of Proposition 1 in [43] are satisfied, as inequality (104 in [43]) has been made effective, we see that

$$\sum\sum_{a \leq m < n \leq b} \mu(E^{\gamma}_m \cap E^{\gamma}_n) \leq 2 \left(\sum_{n=a}^{b} \psi(q_n) \right)^{2} + 24 \left(\sum_{n=a}^{b} \psi(q_n) \right)^{2-\delta} \log^{+} \left(\sum_{n=a}^{b} \psi(q_n) \right) +$$
$$11K' \left(\sum_{n=a}^{b} \psi(q_n) \right)^{1+\delta/2} + X \sum_{n=a}^{b} \psi(q_n), \quad (7.36)$$

where

$$X = 4K' + 9\nu C^{-A} \left(\sqrt{2}\zeta\Big(\frac{A}{B} - \frac{1}{2}\Big) \left(2 + \alpha^{-A}\right) 2^{A+1}\zeta\Big(\frac{A}{2B}\Big)\right)$$

If instead the assumptions in Proposition 2 in [43] are satisfied, we similarly have that

$$\begin{aligned}\sum\sum_{a\le m<n\le b} \mu(E_m^\gamma \cap E_n^\gamma) \le & 2\left(\sum_{n=a}^{b} \psi(q_n)\right)^2 + 24\left(\sum_{n=a}^{b} \psi(q_n)\right)^{2-\delta} \log^+\left(\sum_{n=a}^{b} \psi(q_n)\right) \\ & + Y\sum_{n=a}^{b} \psi(q_n) + 4\sum\sum_{a\le m<n\le b} \gcd(q_m, q_n) \min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n}\right),\end{aligned} \tag{7.37}$$

where

$$Y = \left(2 + 9\nu C^{-A} \left(\sqrt{2}\zeta\Big(\frac{A}{B} - \frac{1}{2}\Big) \left(2 + \alpha^{-A}\right) + 2^{A+1}\zeta\Big(\frac{A}{2B}\Big)\right)\right)$$

We now obtain explicit versions of Proposition 1 and 2 in [43] by applying Lemma 7.3.0.3, setting $\delta = 2/3$ and $\delta = 1$ to (7.36) and (7.37) respectively. Notice that by Lemma 7.3.0.3 with the stated assumptions,

$$4\left(\sum_{n=a}^{b} \psi(q_n)\right)^2 \le \left(\sum_{n=a}^{b} \mu(E_{q_n}^\gamma)\right)^2 + 4m_1(m_2+1)\sum_{n=a}^{b} \psi(q_n),$$

where m_1 and m_2 are given in (7.22).

Proposition 7.3.0.8 (Effective Proposition 1 in [43]). *Let F, μ, $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$, γ and ψ be as in Theorem 7.3.0.1. Further, assume that ψ satisfies* (7.19)*. Then, for arbitrary a, $b \in \mathbb{N}$, with $a < b$ we have that*

$$\begin{aligned}2\sum\sum_{a\le m<n\le b} \mu(E_{q_m}^\gamma \cap E_{q_n}^\gamma) \le & \left(\sum_{n=a}^{b} \mu(E_{q_n}^\gamma)\right)^2 + 48\left(\sum_{n=a}^{b} \psi(q_n)\right)^{4/3} \log^+\left(\sum_{n=a}^{b} \psi(q_n)\right) \\ & + c_1\left(\sum_{n=a}^{b} \psi(q_n)\right)^{4/3} + c_2\sum_{n=a}^{b} \psi(q_n),\end{aligned}$$

where c_1 and c_2 are given in (7.10) *and* (7.11) *respectively.*

Proposition 7.3.0.9 (Effective Proposition 2 in [43]). *Let F, μ, $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$, γ and ψ be as in Theorem 7.3.0.2. Further, assume ψ satisfies* (7.19)*, $q_1 > 4$ and that $\mathcal{A}$ is α-separated with the implicit constant $m_0 = 1$. Then, for arbitrary a, $b \in \mathbb{N}$*

with $a < b$, we have that

$$2 \sum\sum_{a \le m < n \le b} \mu(E^{\gamma}_{q_m} \cap E^{\gamma}_{q_n}) \le \left(\sum_{n=a}^{b} \mu(E^{\gamma}_{q_n})\right)^2 + 8 \sum\sum_{a \le m < n \le b} \gcd(q_m, q_n) \min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n}\right)$$
$$+48 \left(\sum_{n=a}^{b} \psi(q_n)\right) \log^+ \left(\sum_{n=a}^{b} \psi(q_n)\right) + c_3 \sum_{n=a}^{b} \psi(q_n),$$

where c_3 is given in (7.17).

Finally, we can make use of Theorem 6.1.0.2 to make Theorems 1 and 4 in [43] effective, modulo the error term from assumptions (7.19) and (7.29). For Theorem 1 in [43], we take the following parameter in Theorem 6.1.0.2:

$$\varphi_n = \psi(q_n)\Psi(n)^{1/3}\left(\log^+ \Psi(n) + 1\right) + 2\psi(q_n).$$

Notice that for any $\varepsilon > 0$ and $N \in \mathbb{N}$,

$$\varPhi^{1/2}(N) \log^{3/2+\varepsilon} \varPhi(N) + 2 \le 2\Psi^{2/3}(N)(\log \Psi(N) + 2)^{2+\varepsilon},$$

where $\varPhi$ is as given in the original proof or in (7.8). Upon taking $B = 1$ in the above, we have proven Theorem 7.3.0.1, as in this theorem $\mathcal{A} = (q_n)_{n\in\mathbb{N}}$ is lacunary, and as remarked satisfies (7.13) with $B = 1$.

For Theorem 4 in [43], we set parameters of Theorem 6.1.0.2 as follows:

$$\varphi_n = \psi(q_n)\left(\log^+ \Psi(n) + 2\right) + \sum_{m=1}^{n-1} \gcd(q_m, q_n) \min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n}\right),$$
$$\varPhi(N) = \sum_{n=1}^{N} \varphi_n \le \Psi(N)\left(\log^+ \Psi(N) + 2\right) + E(N),$$

where

$$E(N) = \sum_{m=1}^{N-1} \sum_{n=m+1}^{N} \gcd(q_m, q_n) \min\left(\frac{\psi(q_m)}{q_m}, \frac{\psi(q_n)}{q_n}\right).$$

Taking B as needed to satisfy (7.13), Theorem 7.3.0.2 follows.

We now consider the final, extra term in each each effective theorem, which allow us to remove the two extra assumptions in Lemma 7.3.0.3 and Lemma 7.3.0.6, namely conditions (7.19) and (7.29). In the following lemma, we consider this extra term.

Lemma 7.3.0.10. *Let $\psi : \mathbb{N} \to (0,1]$ be a function and $\mathcal{A} = (q_n)_n$ be an increasing sequence of positive integers. Suppose there exists $\nu > 0$ and $A > 2$ such that for any $t \in \mathbb{R} \setminus [-1,1]$, (7.5) is satisfied. Suppose there exists $B \geq 1$ and $C > 0$ such that $A > 2B$ and (7.13) is satisfied. Suppose $\Psi(n) = \sum_{k=1}^{n} \psi(k)$ is unbounded. Let $\omega : \mathbb{N} \to [0,+\infty)$ be a function. Suppose*

$$\sum_{n=1}^{\infty} \omega(n) < +\infty.$$

Define an auxiliary function $\psi^*(q_n) = \max(\psi(q_n), \omega(n))$. Then for every $\delta > 0$ there exists $F_\delta \subset F$ such that $\mu(F_\delta) < \delta$ and for any $x \in F \setminus F_\delta$ and $N \in \mathbb{N}$,

$$|R(x,N;\gamma,\psi,\mathcal{A}) - R(x,N;\gamma,\psi^*,\mathcal{A})| \leq t_\delta,$$

where

$$t_\delta = \min\left\{ t \in \mathbb{N} : \sum_{n=t}^{\infty} \left(\omega(n) + \frac{\nu}{C^A n^{A/B}} \right) < \frac{\delta}{3} \right\}. \tag{7.38}$$

Proof. By the definition of the counting function (5.14), for any $x \in F$, we have that

$$R(x,N;\gamma,\psi,\mathcal{A}) \leq R(x,N;\gamma,\psi^*,\mathcal{A}) \leq R(x,N;\gamma,\psi,\mathcal{A}) + R(x,N;\gamma,\omega,\mathcal{A}).$$

It follows that, for any $x \in F$,

$$|R(x,N;\gamma,\psi,\mathcal{A}) - R(x,N;\gamma,\psi^*,\mathcal{A})| \leq R(x,N;\gamma,\omega,\mathcal{A}).$$

It suffices to find an upper for $R(x,N;\gamma,\omega,\mathcal{A})$, for any $x \in F \setminus F_\delta$, for some measurable $F_\delta \subset F$ such that $\mu(F_\delta) < \delta$. By Theorem 2 in [43], we see that for almost every $x \in F$, the extra term is exactly given by the counting function

$$R(x,N;\gamma,\omega,\mathcal{A})$$

For any $q \in \mathbb{N}$, define

$$E_q = \{x \in F : \|qx - \gamma\| \leq \omega(q)\}.$$

By Lemma 2 in [43], we know that for any $q \in \mathbb{N}$, if $q \geq 4$ then

$$\mu(E_q) \leq 3\omega(q) + \min\left\{ 3 \max_{s \in \mathbb{Z}} |\hat{\mu}(sq)|, 2\sum_{s=1}^{\infty} \frac{|\hat{\mu}(sq)|}{s} \right\}.$$

By the assumptions of the lemma,

$$\max_{s\in\mathbb{Z}\setminus\{0\}} |\hat{\mu}(sq_n)| \le \frac{\nu}{\log^A q_n} \le \frac{\nu}{C^A n^{A/B}}.$$

Hence, by taking the summation, for any $t \in \mathbb{N}$, we have that

$$\mu\left(\bigcup_{n=t}^{\infty} E_{q_n}\right) \le \sum_{n=t}^{\infty} \mu(E_{q_n}) \le 3\sum_{n=t}^{\infty} w(n) + \frac{3\nu}{C^A}\sum_{n=t}^{\infty}\frac{1}{n^{A/B}} \le 3\sum_{n=t}^{\infty}\left(\omega(n) + \frac{\nu}{C^A n^{A/B}}\right).$$

As the right-most term converges when $t = 1$, we get that for any $\delta > 0$, there exists $t_\delta \in \mathbb{N}$ such that

$$\mu\left(\bigcup_{n=t_\delta}^{\infty} E_{q_n}\right) \le \delta,$$

where t_δ is given by (7.38). By taking $F_\delta = \bigcup_{n=t_\delta} E_{q_n}$, we get that $\mu(F_\delta) < \delta$ and for any $x \in F \setminus F_\delta$,

$$R(x, N; \gamma, \omega, \mathcal{A}) < t_\delta.$$

□

This lemma tells us that it is possible to make two such extra assumptions on ψ and the counting result differs by an additive constant which depends only on ω. In the proof for Theorem 7.3.0.1, we have taken $\tau = A/B$ in (7.19). That is, for Theorem 7.3.0.1, t_δ is given by

$$t_\delta = \min\left\{t \in \mathbb{N} : \sum_{n=t}^{\infty}\left(3n^{-A/B} + \frac{\nu}{C^A n^{A/B}}\right) < \frac{\delta}{3}\right\}.$$

To get a concrete estimate, an upper bound for this t_δ can be obtained by noticing that if

$$\left(3 + \frac{\nu}{C^A}\right)\sum_{n=t}^{\infty}\frac{1}{n^{A/B}} \le \left(3 + \frac{\nu}{C^A}\right)\int_{t-1/2}^{\infty}\frac{dx}{x^{A/B}} = \frac{3+\nu/C^A}{1-A/B}\left(t - \frac{1}{2}\right)^{1-A/B} < \delta,$$

then

$$t_\delta \le \frac{1}{2} + \left(\frac{(1-A/B)\delta}{3+\nu/C^A}\right)^{1/(1-A/B)}.$$

Also, for Theorem 7.3.0.2, t_δ is given by

$$t_\delta = \min\left\{t \in \mathbb{N} : \sum_{n=t}^{\infty}\left(n^{-9} + \frac{\nu}{C^A n^{A/B}}\right) < \frac{\delta}{3}\right\}.$$

To get a concrete estimate, an upper bound for this t_δ can be obtained by noticing that if $d = \min(9, A/B)$,

$$\begin{aligned}\sum_{n=t}^{\infty}\left(n^{-9} + \frac{\nu}{C^A n^{A/B}}\right) &\le \left(1 + \frac{\nu}{C^A}\right)\sum_{n=t}^{\infty}\frac{1}{n^d}\\ &\le \left(1 + \frac{\nu}{C^A}\right)\int_{t-1/2}^{\infty}\frac{dx}{x^d}\\ &= \frac{1+\nu/C^A}{1-d}\left(t - \frac{1}{2}\right)^{1-d} < \delta,\end{aligned}$$

then

$$t_\delta \le \frac{1}{2} + \left(\frac{(1-\min(9, A/B))\delta}{1+\nu/C^A}\right)^{1/(1-\min(9,A/B)))}.$$

Our new respective set $E_{\varepsilon,\delta}$ for Theorems 7.3.0.1 and 7.3.0.2 is given by $E_{\varepsilon,\delta} = E'_{\varepsilon,\delta/2} \cup F_{\delta/2}$, where $E'_{\varepsilon,\delta/2}$ is given by Theorem 6.1.0.2 with the parameters given above, and $F_{\delta/2}$ is given in Lemma 7.3.0.10. This completes the proofs of Theorems 7.3.0.1 and 7.3.0.2.

7.4 NORMAL NUMBERS

An easy application of Lemma 1.4 of [23], as given in Chapter 5, quickly shows that almost all real numbers are simply normal to a base b. Applying Theorem 6.1.0.1 in its place allows us to give an upper bound on the number of times a given digit d appears in the base b expansion for almost all real α other than in a set of measure at most δ.

Theorem 7.4.0.1. *For any $\delta > 0$, there exists a set E_δ of measure at most δ such that for any real number $\alpha \in [0, 1) \setminus E_\delta$, the number of times a given digit d appears in its base b expansion in* (5.17) *up to the N-th digit (that is, $A(d, b, N)$) satisfies*

$$A(d, b, N) \le \min\left\{N, \frac{N}{b} + K_{\varepsilon,\delta}\left(N^{2/3}\log^{1/3+\varepsilon}(N+2)\right)\right\},$$

where $K_{\varepsilon,\delta}$ is given in Theorem 6.1.0.1.

We note that the size of the constant $K_{\varepsilon,\delta}$ impacts the size we need N to be for

$$\frac{N}{b} + K_{\varepsilon,\delta}\left(N^{2/3}\log^{1/3+\varepsilon}(N+2)\right) < N$$

to hold; N is clearly a trivial upper bound for $A(d, b, N)$.

The proof follows the one given in [23] but replaces the use of the ineffective lemma with the effective version given at Theorem 6.1.0.1.

Proof. As the integer part of α has no bearing on whether α is simply normal base b, we can without loss of generality restrict $\alpha \in [0, 1)$. Further, let a_k denote the k-th digit in the base b expansion of α as given at (5.17). Set $d \in \mathbb{Z}$, $0 \leq d < b$.

Let

$$f_k(\alpha) = \begin{cases} 1 \text{ if the } k\text{-th digit of } \alpha \text{ is } d, \\ 0 \text{ otherwise.} \end{cases}$$

Further, let

$$\mathfrak{f}_k = b^{-1}.$$

We note that for $j \neq k$,

$$\int_0^1 f_k(x) f_j(x) \mathrm{d}x = \mu\left(\{x \in [0, 1) : \text{the } k\text{-th and } j\text{-th digits of } x \text{ are both } d\}\right) = b^{-2}.$$

It thus follows that

$$\int_0^1 \left(\sum_{k=1}^{N} (f_k(x) - \mathfrak{f}_k) \right)^2 \mathrm{d}x = \sum_{k=1}^{N} b^{-1} \left(1 - b^{-1}\right).$$

More justification for these equalities can be found in Chapter 5.

It follows that we can apply Theorem 6.1.0.1 with $\varphi_k = b^{-1}$ and $K = 1$. We note that $\Phi_N \leq \frac{N}{b}$, $\sum_{k=1}^{N} \mathfrak{f}_k \leq \frac{N}{b}$ and the result then follows. □

7.5 STRONG LAW OF LARGE NUMBERS

In this section, we will give an effective version of strong law of large numbers.

Let (X, Σ, μ) to be a probability space. For any $k \in \mathbb{N}$, let $(F_k(x))$ be sequence of μ-integrable identically distributed random variables with mean F and variance $\sigma^2 > 0$ on the probability measure space (X, Ω, μ). The strong law of large numbers says that if all the F_k are independent, then for μ-almost every $x \in X$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} F_k(x) = F.$$

In fact, the assumption that all F_k are independent is stronger than needed for the conclusion to hold. In fact, if we have that

$$\int_X \left(\sum_{k=m+1}^{n} (F_k(x) - F) \right)^2 \mathrm{d}\mu \leq \sigma^2 (n-m) \max(1, F), \tag{7.39}$$

for any $m, n \in \mathbb{N}$ with $m < n$, then it follows from Lemma 1.5 of [23] that for almost every $x \in X$, as $N \to \infty$,

$$\frac{1}{N} \sum_{k=1}^{N} F_k(x) = F + O\left(N^{-1/2} \log^2 N\right) \to F.$$

The following lemma shows that assumption (7.39) is indeed weaker than independence. In fact, we further show that the assumption that all the $F_k(x)$ are identical is unnecessary too.

Lemma 7.5.0.1. *Suppose all $F_k(x)$ are independent, with finite means F_k and variances σ_k^2. We assume there is a finite universal bound on the means F_k, and write $\tilde{F}_k = \max\{F_k, 1\}$. Similarly, assume there is a finite universal bound on the σ_k^2 and write $\sigma^2 = \max_k \{\sigma_k^2, 1\}$. Then for any $m, n \in \mathbb{N}$ with $m < n$,*

$$\int_X \left(\sum_{k=m+1}^{n} (F_k(x) - F_k) \right)^2 d\mu \leq K \sum_{k=m+1}^{n} \tilde{F}_k, \tag{7.40}$$

for a constant $K > 0$

Proof. For any $m, n \in \mathbb{N}$, if $m < n$ then, as the $F_k(x)$ are independent, we have that

$$\begin{aligned} \int_X \left(\sum_{k=m+1}^{n} (F_k(x) - F_k) \right)^2 \mathrm{d}\mu &= \sum_{k=m+1}^{n} \int_X (F_k(x) - F_k)^2 \, \mathrm{d}\mu \\ &= \sum_{k=m+1}^{n} \sigma_k^2 \\ &\leq (n-m)\sigma^2 \\ &\leq \sigma^2 \sum_{k=m+1}^{n} \tilde{F}_k. \end{aligned}$$

Thus, the Lemma holds with $K = \sigma^2$. In the case that the $F_k(x)$ are identically distributed, from the final inequality we obtain (7.39). □

Our effective version of strong law of large numbers is as follows:

Theorem 7.5.0.2.
Let (X, Σ, μ) to be a probability space. For any $k \in \mathbb{N}$, let $(F_k(x))$ be sequence of μ-integrable random variables with finite means F_k and finite variances $\sigma_k^2 > 0$ on the probability measure space (X, Ω, μ). We assume there is a finite universal bound for the variances, which we denote by σ^2, and assume there is a finite universal bound for the means F_k. Let $\tilde{F}_k = \max\{F_k, 1\}$, so that $\sum_{k=1}^{\infty} \tilde{F}_k$ diverges.

Suppose that (7.40) holds for any $m, n \in \mathbb{N}$ with $m < n$. Then for any $\varepsilon > 0$ and $\delta > 0$, there exists some μ-measurable $E_{\varepsilon,\delta} \subset X$ such that $\mu(E_{\varepsilon,\delta}) < \delta$ and for any $x \in X \setminus E_{\varepsilon,\delta}$ and $N \in \mathbb{N}$,

$$\left| \frac{1}{N} \sum_{k=1}^{N} (F_k(x) - F_k) \right| \leq K_{\varepsilon,\delta} \left(\frac{\Phi^{1/2}(N) \log^{3/2+\varepsilon}(\Phi(N))}{N} + \frac{\Phi_0}{N} \right), \tag{7.41}$$

where $\Phi(N) = \sum_{k=1}^{N} \tilde{F}_k$ and

$$\begin{aligned} K_{\varepsilon,\delta} &= \max\{\alpha, \beta\}, \\ N_{\varepsilon,\delta} &= \left\lceil \frac{r_{\varepsilon,\delta}}{\Phi_0} - 1 \right\rceil, \\ r_{\varepsilon,\delta} &= \left\lceil \left(\frac{2\sigma^2}{\varepsilon\delta} \right)^{1/\varepsilon} \right\rceil + 1, \end{aligned}$$

with

$$\alpha = \frac{N_{\varepsilon,\delta}}{\max\left(\Phi_0^{1/2} \log^{3/2+\varepsilon}(\Phi_0 + 2) + F, 1\right)},$$

and

$$\beta = \frac{2}{\log^{3/2+\varepsilon/2} 2} \left(1 + \frac{1}{\sqrt{2} \log^{3/2+\varepsilon} 4} \right) \left(\frac{\log 4}{\log 3} \right)^{3/2+\varepsilon},$$

where

$$\Phi_0 = \max_k \left\{ \tilde{F}_k \right\}.$$

We note that we do not need an assumption about the random variables being identically and independently distributed.

Proof. The proof is essentially a direct application of Theorem 6.1.0.2. Take $K = \sigma^2$, $C = 1$ and for any $k \in \mathbb{N}$, $f_k(x) = F_k(x)$, $f_k = F_k$, $\varphi_k = \tilde{F}_k$ and Φ_0 as defined above. We apply Theorem 6.1.0.2, and the results follow by dividing both sides in the inequality by N. Although $C = 1$ may not be a universal bound of $F_k(x)$, it follows from the proof of Theorem 6.1.0.2 that it suffices to assume that $C = 1$. □

If all the $F_k(x)$'s are identically distributed, then $\tilde{F}_k = F$ is the same for all k. Thus, $\Phi(N) = NF$, so substituting this into (7.41), we obtain that

$$\left| \frac{1}{N} \sum_{k=1}^{N} (F_k(x) - F_k) \right| \leq K_{\varepsilon,\delta} \left(\frac{F^{1/2} \log^{3/2+\varepsilon}(NF)}{N^{1/2}} + \frac{F}{N} \right).$$

— 8 —

General Versions of Counting Lemmas

As noted previously, [1] uses a theorem very much like Lemma 1.4 of [23], but not quite the same. We give general versions of Lemmas 1.4 and 1.5 of [23].

Theorem 8.0.0.1. *Let (X, Σ, μ) be a measure space and suppose that $0 < \mu(X) < +\infty$. Let $f_i(x)$, $i \in \mathbb{N}$ be a sequence on non-negative μ-measurable functions, where for all $i \in \mathbb{N}$, $x \in X$ we have that $f_i(x) < K$ for some constant $K \in \mathbb{R}$. Further, let f_i, $\phi_i \in \mathbb{R}$ be sequences of real numbers such that for any $i \in \mathbb{N}$,*

$$0 \leq f_i \leq \phi_i \leq K. \tag{8.1}$$

For any $N \in \mathbb{N}$ define

$$\Phi(N) = \sum_{i=1}^{N} \phi_i,$$

and suppose that $\lim_{N\to\infty} \Phi(N) = +\infty$. Further, assume that for all $N \in \mathbb{N}$ we have that

$$\int_X \left(\sum_{i=1}^{N} (f_i(x) - f_i) \right)^2 d\mu = O(F(\Phi(N))), \tag{8.2}$$

where $F : \mathbb{R} \to \mathbb{R}$ is an eventually strictly increasing function. Further, assume that $G : \mathbb{N} \to \mathbb{R}$ and $H : \mathbb{N} \to \mathbb{R}$ are eventually strictly increasing functions, and that

$$\sum_{k=1}^{\infty} \frac{F(k)}{H(k-1)^2} < +\infty. \tag{8.3}$$

We further define $I : \mathbb{N} \to \mathbb{R}$ such that

$$I(k) \asymp (G(k+1) - G(k)), \tag{8.4}$$

and assume that I is eventually strictly increasing. That is, there exist positive constants c, $C > 0$ such that

$$cI(k) \leq G(k+1) - G(k) \leq CI(k),$$

with I eventually strictly decreasing. Then

$$\sum_{i=1}^{N} f_i(x) = \sum_{i=1}^{N} f_i + O\left(H(G^{-1}(\Phi(N))) + I(G^{-1}(\Phi(N)))\right),$$

where we have assumed that $\Phi(N)$ is sufficiently large so that G is strictly increasing on $(\Phi(N) - \varepsilon, +\infty)$ so the inverse makes sense on the restricted domain.

We note that we are able to write the conclusion in terms of $\Phi(N)$ due to the invertibility of G on the restricted range; indeed, we alternatively could write the conclusion as

$$\sum_{i=1}^{N} f_i(x) = \sum_{i=1}^{N} f_i + O\left(H(k-1) + I(k-1)\right),$$

where k is defined so that

$$N_{k-1} \leq N < N_k,$$

with

$$N_k = \min\left\{N \in \mathbb{N} \,:\, \Phi(N) \geq G(k)\right\}.$$

In some cases, for example Lemma 1.4 of [23], it is easier to argue for asymptotics in terms of $\Phi(N)$ from this result in terms of k; see the examples given below.

We now show the theorem above implies some results we have already seen. We first consider Lemma 1.4 of [23]. We define the functions as follows:

$$\begin{aligned}
F\Phi((N)) &= \sum_{k=1}^{N} \phi_k = \Phi(N), \\
G(k) &= k^3(\log 2k)^{1+\varepsilon}, \\
H(k) &= k^2(\log 2k)^{1+\varepsilon}, \\
I(k) &= k^2(\log 2k)^{1+\varepsilon}.
\end{aligned}$$

We can check these functions satisfy the conditions in the theorem. We thus find that

$$\sum_{i=1}^{N} f_i(x) = \sum_{i=1}^{N} f_i + O\left(k^2(\log 2k)^{1+\varepsilon}\right),$$

where k is defined as above so that $N_{k-1} \leq N < N_k$. Upon noting that

$$k^2(\log 2k)^{1+\varepsilon} = O\left(\Phi(N_{k-1})^{2/3}\log\left(\Phi(N)+2\right)\right)$$

(this is shown explicitly in Lemma 6.2.1.2, or follows from the definitions of G_km and N_K), we attain the result given.

We now consider the case of Lemma 5.2.1.1. Define the functions as follows:

$$\begin{aligned}
F(\Phi(N)) &= \frac{\sum_{k=1}^{N}\phi_k}{\left(\log\sum_{k=1}^{N}\phi_k\right)^C} = \frac{\Phi(N)}{\left(\log\Phi(N)\right)^C},\\
G(k) &= e^{k^{1/\sqrt{C}}},\\
H(k) &= \frac{\Phi(N_k)}{\left(\Phi(N_k)\right)^{C/4}},\\
I(k) &= \left(\log\Phi(N_k)\right)^{-\sqrt{C}+1}.
\end{aligned}$$

We note that N_k is defined to be the smallest N such that $\Phi(N) \geq G(k)$ and thus H and I are functions of k. Following the reasoning given in the proof of Lemma 5.2.1.1 then gives the result.

Proof of Theorem 8.0.0.1. Define

$$N_k = \min\left\{N \,:\, \Phi(N) \geq G(k)\right\}. \tag{8.5}$$

Further, let

$$\mathcal{B}_k = \left\{x \in X \,:\, \left|\sum_{i=1}^{N_k}\left(f_i(x) - f_i\right)\right| \geq H(k-1)\right\}.$$

We find that by (8.2),

$$\begin{aligned}
\mu(\mathcal{B}_k)\left(H(k-1)\right)^2 &\leq \int_{\mathcal{B}_k}\left(\sum_{i=1}^{N_k}\left(f_i(x)-f_i\right)\right)^2 \mathrm{d}\mu\\
&\leq \int_X\left(\sum_{i=1}^{N_k}\left(f_i(x)-f_i\right)\right)^2 \mathrm{d}\mu\\
&= O\left(F(\Phi(N_k))\right) = O\left(F(k)\right)
\end{aligned} \tag{8.6}$$

as N_k is defined in terms of k. It immediately follows that

$$\mu(\mathcal{B}_k) \leq O\left(\frac{F(k)}{H(k-1)^2}\right).$$

By (8.3) we have that

$$\sum_{k=1}^{\infty} \frac{F(k)}{H(k-1)^2}$$

converges, so it follows that

$$\sum_{k=1}^{\infty} \mu(\mathcal{B}_k) \leq O\left(\sum_{k=1}^{\infty} \frac{F(k)}{H(k-1)^2}\right)$$

converges. Applying the Borel-Cantelli Lemma we see that almost all $x \in X$ belong to at most finitely many $\mathcal{B}_k$; that is, for almost all $x \in X$, there exists a $k(x)$ such that for all $k > k(x)$, we have that

$$\left|\sum_{i=1}^{N_k} (f_i(x) - f_i)\right| \leq H(k-1). \tag{8.7}$$

We now need to bound the value of $\Phi(N_k) - \Phi(N_{k-1})$. By (8.5) we note that

$$\begin{aligned}
\Phi(N_k) - \Phi(N_{k-1}) &= \Phi(N_k - 1) + \phi_{N_k} - \Phi(N_{k-1}) \\
&\leq O\left(G(k) + K - G(k-1)\right) \\
&\leq O\left(I(k-1)\right) \qquad (8.8)
\end{aligned}$$

where the last line follows from (8.4).

It now follows that, for $x \in X$ such that $x \notin \bigcup_k \mathcal{B}_k$ isn't in the exceptional set, for sufficiently large N such that $N_{k-1} \leq N \leq N_k$ where $k-1 > k(x)$ we have that

$$\begin{aligned}
\left|\sum_{i=1}^{N} (f_i(x) - f_i)\right| &\leq \left|\sum_{i=1}^{N_k} f_i(x) - \sum_{i=1}^{N_{k-1}} f_i\right| \\
&= \left|\sum_{i=1}^{N_k} f_i(x) - \Phi(k) + \Phi(k) - \Phi(k-1)\right| \\
&\leq \left|\sum_{i=1}^{N_k} f_i(x) - \Phi(k)\right| + |\Phi(k) - \Phi(k-1)| \\
&\leq O\left(H(k-1) + I(k-1)\right). \qquad (8.9)
\end{aligned}$$

We note that this is the result given in the remark above. We now take advantage of the invertibility of G on the restricted domain.

By definition, $\Phi(N_{k-1}) \geq G(k-1)$. We recall that we have

$$\Phi(N_{k-1}) \leq \Phi(N) \leq \Phi(N_k),$$

as $N_{k-1} \leq N \leq N_k$. Assume k is large enough that G is strictly increasing, so the inverse of G exists on the restricted domain we are considering. Then, as G is strictly increasing, so is G^{-1}. Thus

$$G^{-1}(\Phi(N)) \geq G^{-1}(\Phi(N_{k-1})) \geq G^{-1}(G(k-1)) = k - 1.$$

Now, as H and I are also eventually strictly increasing, assuming k is large enough we can substitute the above into (8.9) to find that

$$\left|\sum_{i=1}^{N}(f_i(x) - f_i)\right| \leq O\left(H(G^{-1}(\Phi(N))) + I(G^{-1}(\Phi(N)))\right).$$

□

We now do the same for Lemma 1.5 of [23]

Theorem 8.0.0.2. *Let X be a measure space with measure μ such that $0 < \mu(X) < \infty$. Let $f_k(x)$, $k = 1, 2, \ldots$ be a sequence of non-negative, μ-measurable functions, and let f_k, φ_k be sequences of real numbers such that*

$$0 \leq f_k \leq \varphi_k. \tag{8.10}$$

Write

$$\Phi(N) = \sum_{k=1}^{N} \varphi_k,$$

and assume that $\Phi(N) \to \infty$ as $N \to \infty$.

Suppose that for arbitrary integers m, n, $1 \leq m < n$ we have that

$$\int_X \left(\sum_{m \leq k < n}(f_k(x) - f_k)\right)^2 d\mu = O\left(\tilde{F}\left(\sum_{m \leq k < n}\varphi_k\right)\right), \tag{8.11}$$

where $\tilde{F} : \mathbb{R} \to \mathbb{R}^+$ is an increasing function. Further, let $\tilde{G}$, $\tilde{H}$ and $\tilde{I} : \mathbb{N} \to \mathbb{R}^+$ be increasing functions such that the number

$$n_j = \max\left\{n : \tilde{F}(\Phi(n)) = \tilde{F}\left(\sum_{k=1}^{n}\varphi_k\right) < \tilde{G}(j),\, j \in \mathbb{N}\right\},$$

is well defined,

$$\sum_{r=1}^{\infty}\frac{\tilde{G}(2^r)}{r^{1+\varepsilon}\tilde{I}(2^r)\tilde{H}(r)}$$

converges, and that

$$\tilde{F}(\Phi(n_j) = O\left(\tilde{F}(\Phi(n_{j+1})\right).$$

Then

$$\sum_{k=1}^{N} f_k(x) = \sum_{k=1}^{N} f_k + O\left(AB + \max_{1\le k\le N} f_k\right), \tag{8.12}$$

where

$$\begin{aligned} A &= \left(\log\left(\tilde{G}^{-1}\left(\tilde{F}\left(\Phi\left(N\right)\right)\right)\right) + 2\right)^{3/2+\varepsilon,} \\ B &= \left(\tilde{I}\left(\tilde{G}^{-1}\left(\tilde{F}\left(\Phi\left(N\right)\right)\right)\right)\tilde{H}\left(\log\left(\tilde{G}^{-1}\left(\tilde{F}\left(\Phi\left(N\right)\right)\right)\right)\right)\right)^{1/2}. \end{aligned}$$

We note that, roughly speaking, the purpose of function $\tilde{F}$ is to let you change the growth of the asymptotic bound in (8.11), $\tilde{G}$ is to let you control the rate that n_j grows, and $\tilde{H}$ and $\tilde{I}$ are to let you control the size of the sets

$$\left\{x \in X : G(r, x) > r^{2+\varepsilon}\tilde{I}\left(2^r\right)\tilde{H}(r)\right\},$$

which we will apply the Borel-Cantelli Lemma to in the proof.

We note that Lemma 1.5 follows from the above by setting the functions to the following:

$$\begin{aligned} \tilde{G}(j) &= j \\ \tilde{F}(x) &= x \\ \tilde{H}(j) &= 1 \\ \tilde{I}(j) &= j. \end{aligned}$$

we then note that $\tilde{G}^{-1}(j) = j$, and putting these into (8.12) gives us the result of Lemma 1.5 of [23] as expected.

Proof of Theorem 8.0.0.2. Define the sequence $n_1, n_2, \ldots$ by

$$n_j = \max\left\{n : \tilde{F}(\Phi(n)) = \tilde{F}\left(\sum_{k=1}^{n}\varphi_k\right) < \tilde{G}(j)\right\}. \tag{8.13}$$

We note that the n_j need not be distinct.

Suppose that (8.12) holds for $N = n_j$ for all j. Then, if $n_r < n < n_{r+1}$, we have that

$$\sum_{k=1}^{n_r} f_k(x) \le \sum_{k=1}^{n} f_k(x) \le \sum_{k=1}^{n_{r+1}} f_k(x),$$

while
$$\sum_{k=1}^{n_r} f_k(x) = \sum_{k=1}^{n_r} f_k + O\left(\left(\tilde{I}(r)\tilde{H}(r)\right)^{1/2} (\log(r+2))^{3/2+\varepsilon}\right),$$
and
$$\sum_{k=1}^{n_{r+1}} f_k(x) = \sum_{k=1}^{n_{r+1}} f_k + O\left(\left(\tilde{I}(r+1)\tilde{H}(r+1)\right)^{1/2} (\log(r+3))^{3/2+\varepsilon}\right).$$

Note that by (8.10), we have that

$$\begin{aligned}\sum_{n_r<k\leq n_{r+1}} f_k &\leq \max_{k\leq n_{r+1}} f_k + \Phi\left(n_{r+1}\right) - \Phi\left(n_r+1\right)\\ &\leq 1 + \max_{k\leq n} f_k.\end{aligned}$$

Combining these results then gives us (8.12) under the assumptions mentioned.

It thus remains to establish the result for $N = n_j$. Following the proof of Lemma 1.5 in [23], we express the integer j in binary scale as

$$j = \sum_{0\leq \upsilon\leq \log_2 j} b(j,\, \upsilon)2^{\upsilon}.$$

We then let

$$B(j) = \left\{(i,\, s) : \, i = \sum_{\upsilon=s+1}^{r} b(j,\, \upsilon)2^{\upsilon-s},\, b(j,\, s) = 1,\, 0\leq s\leq r\right\},$$

where $r = r(j) = [\log_2 j]$. We further define

$$F(i,\, s,\, x) = \sum_{u_0<k\leq u_1} \left(f_k(x) - f_k\right),$$

where, for $t\in\{0,\, 1\}$, we define

$$u_t = u_t(i,\, s) = \max\left\{n>0 : \, \Phi(n) < (i+t)2^s\right\}, \tag{8.14}$$

with the convention that $\max\emptyset = 0$. This notation splits up $[1,\, n_j]$ into a suitably small number of blocks; that is,

$$(0,\, n_j] = \bigcup_{(i,\, s)\in B(j)} (u_0,\, u_1],$$

with u_0, u_1 given by (8.14). For further discussion of this, see the discussion and example below (6.13).

To complete the proof and establish the result for $N = n_j$, it remains to demonstrate that

$$\sum_{(i,\,s)} |F(i,\,s,\,x)| = O(([\log_2 j]+1)^{3/2+\varepsilon}\,\tilde{I}\,(j+1)^{1/2}\,\tilde{H}([\log_2 j]+1)^{1/2}.).$$

Set

$$G(r,\,x) = \sum_{\substack{0\le s\le r\\ i<2^{r-s}}} F^2(i,\,s,\,x)$$

and

$$\Phi(i,\,s) = \sum_{u_0<k\le u_1} \varphi_k,$$

with u_t given by (8.14). We now note that by (6.2.4) and (8.13) we have that

$$\begin{aligned}\int_X G(r,\,x)\mathrm{d}\mu &= O\left(\sum_{\substack{0\le s\le r\\ 0\le i<2^{r-s}}} \tilde{F}\left(\Phi\left(i,\,s\right)\right)\right)\\ &\le O\left((r+1)\tilde{F}\left(\Phi(n_{2^r})\right)\right)\\ &\le O\left((r+1)\,\tilde{G}\left(2^r\right)\right).\end{aligned}$$

It follows that

$$\mu\left\{x\in X:\,G(r,\,x) > r^{2+\varepsilon}\tilde{I}\left(2^r\right)\tilde{H}(r)\right\} < O\left(\frac{\tilde{G}\left(2^r\right)}{r^{1+\varepsilon}\tilde{I}(2^r)\tilde{H}(r)}\right).$$

By assumption, $\sum_{r=1}^{\infty}\frac{\tilde{G}(2^r)}{r^{1+\varepsilon}\tilde{I}(2^r)\tilde{H}(r)}$ converges. Thus, by the Borel-Canteli lemma, for almost all $x\in X$ we have that

$$G(r,\,x) < r^{2+\varepsilon}\tilde{I}(2^r)\tilde{H}(r), \tag{8.15}$$

for $r > r(x)$.

We now let $r = [\log_2 j]+1$, and suppose x belongs to the set for which (8.15) holds. Recall that $|B(j)|\le r$. An application of the Cauchy-Schwarz inequality gives us that

$$\begin{aligned}\sum_{(i,\,s)\in B(j)} |F(i,\,s,\,x)| &\le |B(j)|^{1/2}G^{1/2}(r,\,x)\\ &\le r^{1/2}\left(r^{2+\varepsilon}\tilde{I}(2^r)\tilde{H}(r)\right)^{1/2}\\ &\le r^{1/2}r^{1+\varepsilon}\left(\tilde{I}(2^r)\tilde{H}(r)\right)^{1/2}\\ &= r^{3/2+\varepsilon}\left(\tilde{I}(2^r)\tilde{H}(r)\right)^{1/2}\\ &\le ([\log_2 j]+1)^{3/2+\varepsilon}\,\tilde{I}\,(j+1)^{1/2}\,\tilde{H}([\log_2 j]+1)^{1/2}.\end{aligned}$$

Now we note that by (8.13),

$$\tilde{F}\left(\Phi(n_j)\right) < \tilde{G}(j) \leq \tilde{F}\left(\Phi(n_{j+1})\right).$$

As $\tilde{G}(j)$ and $\tilde{F}(j)$ are strictly increasing, it follows that

$$j \leq \tilde{G}^{-1}\left(\tilde{F}\left(\Phi\left(n_{j+1}\right)\right)\right).$$

Now, as by assumption $\tilde{F}(\Phi(n_j) = O\tilde{F}(\Phi(n_{j+1})$, for $N = n_j$, we have established that

$$\sum_{k=1}^{N} f_k(x) = \sum_{k=1}^{N} f_k + O\left(A'B'\right),$$

where

$$A' = \left(\log\left(\tilde{G}^{-1}\left(\tilde{F}\left(\Phi\left(n_j\right)\right)\right)\right) + 2\right)^{3/2+\varepsilon},$$
$$B' = \tilde{I}\left(\tilde{G}^{-1}\left(\tilde{F}\left(\Phi\left(n_j\right)\right)\right)\right)^{1/2} \tilde{H}\left(\log\left(\tilde{G}^{-1}\left(\tilde{F}\left(\Phi\left(n_j\right)\right)\right)\right)\right)^{1/2},$$

and establishing this completes the proof. $\square$

— 9 —

Future Work

To conclude, I will give some general discussions about some potential future work. I will split this into the two sections of my thesis.

9.1 ON THE *abc* CONJECTURE

Other than the obvious open work here, there are a few potential directions the work in this thesis could be took.

The results contained, for example at Theorem 2.1.3.1, involve an implicit constant depending just on the choice of number field K. It would be nice to make this constant as explicit as possible (it will still depend on things like the structure of the unit group) to make applications more explicit when needed.

The main results are also proven over the Hilbert Class Field of the chosen number field K; this is to help deal with factorisation and allow us to apply results in linear forms in logarithms. It would be nice to be able to use these methods over the base field. In the case the base field K has class number one, we are fine as $HCF(K) = K$. It does not seem too tricky to apply these methods in a number field with class number two, as any factorisation into irreducible elements has the same number of elements in the factorisation, so we may be able to, with some work, apply linear forms as we have in this thesis in that case. For fields of class number 3 and above, this problem seems harder as different factorisations can have different numbers of elements [32], so this would require some thought.

Another possible direction forward would be to consider trying to prove a variant of these results for an *abcd*-style conjecture; that is, to consider equations of the form $a_1 + ... + a_{n-1} = a_n$ in coprime integers. This is hard to do with the methods in this thesis, as we only have effective bounds for S-unit equations in two variables [16]. However, for some families of equations like the above, we may be able to use bounds on decomposable form equations. These bounds unfortunately have extra dependencies that their unit equation bounds do not, which means we can't immediately apply them to all equations of the form above. A result in this direction however, would lead to the effective Skolem-Mahler-Lech application being able to be used for families of recurrence relations for larger recurrences.

Finally, as given by Lagarias and Soundararajan in [27], one may wish to generalise results on the smooth *abc* Conjecture to general number fields. As of writing, this has not been possible; the issue comes from the necessary scaling as discussed around the Uniform *abc* Conjecture. In the smooth case, the left hand side of the conjectured inequality can be thought of as additive due to the logarithmic factor, while the right hand side contains no such logarithm and is thus multiplicative. This makes finding a conjecture where both left and right hand sides scale appropriately tricky.

9.2 ON QUANTITATIVE DIOPHANTINE APPROXIMATION

Generally, most future work in this area would be in finding further places to apply these effective theorems to give effective results. One such place could be in target problems for dynamical systems; in this case the results could be considered to tell us the maximal distance the orbit of all points outside an exceptional set could get from a target, under the assumption that the conditions in the theorems hold. Another potential place that these could be applied is in statistics, akin to our Strong Law of Large Numbers. One such potential application that has been suggested is in volatility testing in mathematical finance.

The other obvious work that could be done would be to make our general forms of the Lemmas effective; that is, to give effective versions of the results found in Chapter 8.

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