Gassmann Equivalence and Decompositions of Jacobians

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To my parents

Abstract

This thesis deals with the concept of Gassmann equivalence and its application in obtaining isogenous and isomorphic products of Jacobians of algebraic curves.

We study Gassmann equivalent G-sets with a particular emphasis on rationally, locally integral and integrally Gassmann equivalent G-sets. We develop MAGMA functions that verify the only known example of transitive integral Gassmann equivalent G-sets due to Leonard L. Scott [52] and could potentially be used to obtain new intransitive examples.

Our main results generalize theorems of D. Prasad and C. S. Rajan [47], D. Prasad [46] and D. Arapura et al., [4]. In particular, we show that if C is an algebraic curve, $G \leq Aut(C)$ a finite group and X, Y rationally Gassmann equivalent G-sets then the Jacobians $J(\frac{C \times X}{G})$ and $J(\frac{C \times Y}{G})$ are isogenous. Moreover, if instead the G-sets X, Y are integrally Gassmann equivalent the above isogeny becomes an isomorphism.

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Contents

Chapter 1

Introduction

The main goal of this thesis is to study Gassmann equivalent G-sets for various coefficients and generalize results of [47], [46] and [4] that relate the Jacobians $J(\frac{C\times X}{G})$, $J(\frac{C\times Y}{G})$ arising from Gassmann equivalent G-sets X, Y. We begin with some motivation on the questions that made us interested in studying these topics.

Motivation: L-equivalence

Our study of Gassmann equivalence was motivated by questions regarding the existence of $\mathbb{L}$-equivalent algebraic curves over the complex numbers (check [55], [56] for the definition of $\mathbb{L}$-equivalence). Based on [20], $\mathbb{L}$-equivalent curves have isogenous Jacobians that will actually be isomorphic as soon as the curves are generic. Then, a strategy to construct $\mathbb{L}$-equivalent curves would be to find non-isomorphic curves with isomorphic unpolarized Jacobians. We give a survey of known examples of such curves (see [49],[50],[32],[33],[38],[15],[30],[31]) in Chapter 3. Since most of these examples are not geometric, a more precise motivating question has been the geometric meaning of isomorphism of Jacobians and this thesis aims to contribute in this direction.

While studying the curves with isomorphic Jacobians constructed in [32], we had to understand the table in page 340 of [39] to gain insight on the choice of the equations of curves. In doing this we obtained the reduced and full automorphism groups of hyperelliptic curves defined over the field of complex numbers. This has been studied before in several papers (see [37],[3],[8],[9],[54], [29],[62],[11]). We decided to include the results in Chapter 2, Section 2.2 as they provide a source of examples for applications in Chapter 8.

Most methods that give examples of distinct curves with isomorphic Jacobians only work for curves of small genus (genus 2,3,4) and they are not geometric. The most geometric approach is the one in [46] and allows the construction of non-isomorphic curves of genus at least 203 with isomorphic Jacobians. This lead us to study Gassmann equivalence further.

Gassmann equivalence

Gassmann triples are a group theoretic construction introduced by F. Gassmann in [23]. Let G be a finite group and H_1, H_2 subgroups of G. Then (G, H_1, H_2) is a Gassmann triple if and only if the permutation modules $\mathbb{Q}[G/H_1]$, $\mathbb{Q}[G/H_2]$ are isomorphic. We will call such triples *rational Gassmann triples.* In [46], D. Prasad considers triples such that $\mathbb{Z}[G/H_1]$, $\mathbb{Z}[G/H_2]$ are isomorphic as $\mathbb{Z}[G]$-modules and calls them *refined Gassmann triples.* From now on we will call them *integral* Gassmann triples. In [60], A. V. Sutherland studies various stronger forms of Gassmann equivalence: *local integral*, *integral* and *solvable equivalence.*

Transitive G-sets are isomorphic to G/H for some subgroup H of G. Based on this observation the isomorphism $\mathbb{Q}[G/H_1] \cong \mathbb{Q}[G/H_2]$ can be interpreted as an isomorphism of permutation modules of transitive G-sets $X = G/H_1$, $Y = G/H_2$. In [43], O. Parzanchevski uses this to generalize the notion of Gassmann equivalence to G-sets. G-sets X, Y such that $\mathbb{Q}[X]$, $\mathbb{Q}[Y]$ are isomorphic $\mathbb{Q}[G]$-modules are called *rationally Gassmann equivalent* G-sets. In Chapter [5] we follow [60] and study R-Gassmann equivalence for commutative rings R. In particular we study *rationally*, *p-locally*, *locally integral* and *integrally* Gassmann equivalent G-sets. For the first three cases we use the table of marks of the group G to compute pairs of Gassmann equivalent G-sets. When $R = \mathbb{Q}$ or $\mathbb{Z}_p$ this is essentially the same as finding *Brauer relations* in characteristic zero or in positive characteristic. This has been studied in [5], [6], [7], [51], [61] where explicit relations are given.

Gassmann equivalence has many applications in geometry and number theory and can be used to construct non-isomorphic objects sharing many geometric or arithmetical properties. T. Sunada in [58] used examples of rational Gassmann triples to construct isospectral non-isometric Riemann surfaces. These examples give a negative answer to the question of whether one "can hear the shape of a drum" (see [35],[24]).

On the subject of algebraic curves and their Jacobians, Gassmann triples have been used in [47], [25], [46]. In particular it has been shown that if $f : C \to C'$ is a Galois cover (not necessarily unramified) of algebraic curves with Galois group G and (G, H_1, H_2) a rational (integral) Gassmann triple, then the Jacobians of C/H_1, C/H_2 are isogenous (isomorphic). One can use this result and Scott's example [52] of integral Gassmann triples to construct non-isomorphic curves whose Jacobians are isomorphic unpolarized varieties (see [46]). In [4], D. Arapura et al., prove that if (G, H_1, H_2) is a rational (respectively integral) Gassmann triple and $f : M \to N$ is an unramified Galois cover of algebraic varieties with Galois group G then $H^i(M/H_1, \mathbb{Q})$, $H^i(M/H_2, \mathbb{Q})$ (respectively $H^i(M/H_1, \mathbb{Z})$, $H^i(M/H_2, \mathbb{Z})$) are isomorphic unpolarised Hodge structures.

In geometric applications, O. Parzanchevski [43] uses rationally Gassmann equivalent G-sets instead of traditional Gassmann triples and generalizes the results of T. Sunada [58] by constructing disjoint unions of isospectral Riemann surfaces. Sunada's theorem is then a special case of this for transitive rationally Gassmann equivalent G-sets.

Our goal in this thesis is to deal with the same situation in the context of algebraic curves and their Jacobians. In particular we want to extend the results of [47], [25], [46] for R-Gassmann equivalent G-sets instead of traditional Gassmann triples. We explain our results in the next section.

Main results

Our first result is:

Theorem 1.0.1 (Theorem 7.1.2)**.** *Let C be a projective algebraic curve over the field of the complex numbers with an action of a finite group $G \leq Aut(C)$. Let X, Y be rationally Gassmann equivalent transitive G-sets. Then the Jacobians of $\frac{C\times X}{G}$ and $\frac{C\times Y}{G}$ are isogenous.*

For complex algebraic curves, it extends [47], [25] to the case of rationally Gassmann equivalent G-sets. Our proof is based on [4] and it is sligthly different from those in [47], [25], as it uses the language of Hodge structures.

This proof cannot be generalized to integral Gassmann equivalence and we could not generalize the method of [46] for G-sets.

To deal with this, we used the methods of [4]. Using the exact same approach as in [4] we get:

Theorem 1.0.2 (Theorem 7.1.6)**.** *Let X, Y be R-Gassmann equivalent G-sets, M a smooth projective algebraic variety over a field k with an action of a finite group $G \leq Aut(M)$ and $f : M \to M/G$ a Galois unramified cover. Then:*

1. *If $R = \mathbb{Q}$ we have an isomorphism of rational Hodge structures: $H^k(\frac{M\times X}{G}, \mathbb{Q}) \cong H^k(\frac{M\times Y}{G}, \mathbb{Q})$.*
2. *If $R = \mathbb{Z}$ we have an isomorphism of integral Hodge structures: $H^k(\frac{M\times X}{G}, \mathbb{Z}) \cong H^k(\frac{M\times Y}{G}, \mathbb{Z})$.*
3. *If $R = \mathbb{Z}_p$ we have an isomorphism of etale cohomology groups: $H^k_{et}(\frac{M\times X}{G}, \mathbb{Z}_p) \cong H^k_{et}(\frac{M\times Y}{G}, \mathbb{Z}_p)$.*

Finally, in the case where the varieties are curves, we generalized the theorem of Arapura et al., for ramified covers. In order to do this we used the language of mixed Hodge structures and constructible sheaves to obtain:

Theorem 1.0.3 (Theorem 7.2.4)**.** *Let X, Y be R-Gassmann equivalent G-sets, C a smooth projective algebraic curve over the field of the complex numbers with an action of a finite group $G \leq Aut(C)$ and $f : C \to C/G$ a Galois cover (not necessarily unramified). Then:*

1. *If $R = \mathbb{Q}$ we have an isomorphism of rational Hodge structures: $H^k(\frac{C \times X}{G}, \mathbb{Q}) \cong H^k(\frac{C \times Y}{G}, \mathbb{Q})$.*
2. *If $R = \mathbb{Z}$ we have an isomorphism of integral Hodge structures: $H^k(\frac{C \times X}{G}, \mathbb{Z}) \cong H^k(\frac{C \times Y}{G}, \mathbb{Z})$.*

In particular, this means that if X, Y are rationally (integrally) Gassmann equivalent G-sets then the Jacobians of $\frac{C \times X}{G}$, $\frac{C \times Y}{G}$ are isogenous (isomorphic) (Corollary 7.2.6).

Finally, we notice that there is a connection between the concept of rational Gassmann equivalence and Kani's character equivalence [36]. In particular we prove that:

Theorem 1.0.4 (Theorem 5.2.9)**.** *Let G be a finite group and consider the finite dimensional group algebra $\mathbb{Q}[G]$. Then $X = \bigsqcup G/H_i, Y = \bigsqcup G/K_i$ are rational Gassmann equivalent G-sets if and only if $\epsilon_X = \sum_i \epsilon_{H_i}, \epsilon_Y = \sum_i \epsilon_{K_i}$ are character equivalent (for a subgroup H of G, ϵ_H is defined as $\epsilon_H = \frac{1}{|H|} \sum_{h \in H} h$).*

There is a vast literature on the subject of decomposition of Jacobians up to isogeny and Kani's character equivalence [36] is one of the most commonly used methods. Theorem 5.2.9 shows that these examples can also be obtained using examples of rational Gassmann equivalence and Theorem 7.2.4. Examples of these applications are given in Chapter 8.

Structure of this thesis

This thesis is organized in three main parts.

Part A consists of Chapter 2 and 3.

In Chapter 2 we review background material on algebraic curves and give a survey of known results on the automorphism groups of hyperelliptic curves.

In Chapter 3 we introduce algebraic curves and summarize known results on the existence of distinct curves with isomorphic Jacobians. The aim of these chapters is to provide motivation for the study of Gassmann equivalence and collect results that will be used for applications of our main results in chapter 8.

Part B consists of Chapters 4 and 5.

In Chapter 4 we review the necessary background material from representation theory and set up notation that will be used in the next chapter.

In Chapter 5 we introduce the notion of Gassmann equivalent G-sets. We present various examples and give connections with known results from the literature.

Part C consists of Chapters 6, 7 and 8.

In Chapter 6 we review background material on Hodge structures and constructible sheaves. These are the main tools that will be used in the proofs of our main results.

In Chapter 7 we prove the main results of this thesis which generalize theorems of Prasad and Arapura.

In Chapter 8 we discuss several applications of our main theorem with rational coefficients and obtain decompositions of Jacobians of hyperelliptic curves up to isogeny.

Chapter 2

Algebraic curves and their automorphisms

In this chapter we study algebraic curves and their automorphisms. We start by giving basic definitions of algebraic curves, study properties of algebraic curves with a group action and their quotients, and introduce notation that will be used in the statements of our main results. We review known results on the automorphism groups of algebraic curves and in particular we mention an old theorem of Accola. We continue by focusing on hyperelliptic curves and present known results on their automorphism groups and defining equations.

§ 2.1 Algebraic curves

One of the main objects of interest in this thesis are algebraic curves. By an algebraic curve C we mean a projective, smooth, one-dimensional variety defined over the field of the complex numbers. In this case, the notion of a smooth projective algebraic curve coincides with that of a compact Riemann surface and we will use the two terms interchangeably. To every Riemann surface one can attach an integer called the genus and denoted by g_C, that topologically counts the number of its holes. The results of this section are well known and our main references are [34], [40], [26].

It is well known that a non-constant map f between Riemann surfaces C and C' locally looks like $z \mapsto z^n$ for suitable coordinates around some point p. The integer n is called the *multiplicity* of f at p. Points with non-trivial multiplicity are called *ramification points* and they form a finite subset of C. The image of a ramification point under f is called a *branch point*. Every such map f has a *degree*, denoted by $deg(f)$. At a point

$p' \in C'$ this is defined as the sum of multiplicities of points $p \in C$ that map to p' and one can prove that it is independent of c'.

We now state the following well-known theorem that will be useful for counting the genus of various algebraic curves in Chapter 8.

Theorem 2.1.1 ([40], Theorem 4.16, Hurwitz formula). *Let $f : C \to C'$ be a non-constant holomorphic map between compact Riemann surfaces with genus $g_C, g_{C'}$ respectively. Then*

$$2g_C - 2 = deg(f)(2g_{C'} - 2) + \sum_{p \in C}(mult_p(f) - 1). \tag{2.1}$$

In this thesis we will be studying quotients of curves by the action of a finite group G. We state the definition of a G-action which can be found in any introductory book on algebra:

Definition 2.1.2 (Group action). Let G be a group and X a set. We say that G *acts* on X or that X is a (left) G-set if there is a map from $G \times X$ to X (which we denote by $(g, x) \mapsto g \cdot x$, for $g \in G$ and $x \in X$) that satisfies:

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x \text{ for all } g_1, g_2 \in G, x \in X,$$

$$e_G \cdot x = x \text{ for all } x \in X.$$

If $x \in X$ we define the G-orbit of x in X by

$$Orb_G(x) := \{g \cdot x | g \in G\}$$

and the stabilizer of x in G by

$$Stab_G(x) := \{g \in G | g \cdot x = x\}.$$

In general, when M is a manifold the quotient space M/G has the structure of an orbifold. However, in the case of algebraic curves, the quotient C/G is also an algebraic curve. Moreover, the projection map $p : C \to C/G$ is holomorphic of degree $|G|$ and the multiplicity at a point p is equal to the order of the stabilizer subgroup G_p. By applying 2.1.1 to such projection maps we get the following formula which will be used for calculations in chapter 8:

Proposition 2.1.3 ([40], Corollary 3.7, Riemann-Hurwitz formula). *Let C be an algebraic curve with an effective action by a finite group G. Let $p : C \to C/G$ be the* *projection map. Assume that there are n branch points. Above each branch point p_i' lie $|G|/r_i$ ramification points, where r_i denotes the multiplicity of p at these points. Then:*

$$2g_C - 2 = |G|\{(2g_{C/G} - 2) + \sum_{i=1}^{n}(1 - \frac{1}{r_i})\}. \tag{2.2}$$

In the following remark we introduce some notation that is used in the statement of our main results. The name "Tensor product of G manifolds" was introduced in [43].

Remark 2.1.4 (Products of G-sets with curves with G-action/Tensor product of G manifolds)**.** Later, in Chapter 7 we will consider products of Riemann surfaces with G-sets. Let C be a Riemann surface, G a finite group that acts on C and X a finite G-set. Then we can define an action of G on the product $C \times X$ in a natural way and consider the quotient space $\frac{C \times X}{G}$. It turns out that the quotient space is a disjoint union of quotients of C by subgroups of G. In particular, if $G = \bigsqcup G/H_i$ then $\frac{C \times X}{G} = \bigsqcup C/H_i$.

We will denote by $\mathcal{M}_g$ the moduli space of genus g curves, which classifies isomorphism classes of algebraic curves. While we will not be interested in the geometry of this space, it will be useful to state its dimension, which is $3g - 3$ when $g \geq 2$. We will also study stratifications of this space by automorphisms.

An automorphism of an algebraic curve C is an isomorphism $f : C \to C$. The set of all automorphisms is a group under composition and we denote it by $Aut(C)$. Automorphism groups of algebraic curves of genus $g \geq 2$ are finite and the generic algebraic curve of genus $g \geq 3$ has no automorphisms. The following theorem gives an upper bound on the size of $Aut(C)$.

Theorem 2.1.5 ([40], Theorem 3.9, Hurwitz's theorem)**.** *Let G be a finite group acting on a Riemann surface C of genus $g \geq 2$. Then $|Aut(C)| \leq 84(g - 1)$.*

Now we present an old theorem by Accola that relates the genus g_C of a Riemann surface C with action by a finite group G, with the genus g_{C_i} of quotients of C by various subgroups of G. First, we need a definition:

Definition 2.1.6 (Groups with partition)**.** A finite group G is said to admit a partition if there exists a collection of subgroups H_i, $1 \leq i \leq k$ so that $G = \cup_{i=1}^{k} H_i$ and if $0 < i < j$ then $H_i \cap H_j = \{e_G\}$.

Theorem 2.1.7 ([1], Theorem 5.9)**.** *Let C be a Riemann surface of genus g_C. Suppose that C has a finite group of automorphisms G and that G admits a partition by a* *collection of subgroups* $\{H_i\}$, $1 \leq i \leq k$. *Let* $C_i := C/H_i$ *be the quotients of* C *by the groups of the partition and denote by* g_{C_i} *their genus. Then:*

$$(k-1)g_C + |G|g_{C/G} = \sum_{i=1}^{k} |G_i| g_{C_i} \tag{2.3}$$

In Chapter 8 we will see a generalization of this theorem by Kani, which allows the decompositions of Jacobians of curves with a partition, up to isogeny.

We will now focus on a particular class of algebraic curves, hyperelliptic curves.

Definition 2.1.8 (Hyperelliptic curves, [40])**.** An algebraic curve C is called *hyperelliptic* if and only if there is a holomorphic map $f : C \to \mathbb{P}^1$ which has degree 2.

It is easy to check with Hurwitz's formula 2.1.1 that if C has genus g then the double cover has $2g+2$ ramification points which are called the *Weierstrass points* of C. Another way to think of hyperelliptic curves is as projective completions of smooth affine plane curves given by an equation of the form $y^2 = f(x)$ where $f(x)$ is a polynomial of degree $2g+1$ or $2g+2$ with distinct roots. In that case the roots of the polynomial are the Weierstrass points. Every hyperelliptic curve has an automorphism $a : C \to C$ given by $a(x, y) = (x, -y)$. This automorphism is an involution, called the *hyperelliptic involution*. It is unique and fixes exactly the Weierstrass points.

We can always move 3 of the Weierstrass points to $0, 1, \infty$ by an autmorphism of $\mathbb{P}^1$ so the moduli space of hyperelliptic curves of genus g, which we denote by $\mathcal{H}_g$, has dimension $2g-1$. In the next section we will study the automorphism groups of hyperelliptic curves in more detail.

§ 2.2 Automorphism groups of hyperelliptic curves

Automorphism groups of hyperelliptic curves have been studied by several authors (see [37], [3],[8],[9],[54], [29],[62],[11]). In this section we give an exposition of results appearing in these papers.

From the discussion in the previous section, it is clear that every hyperelliptic curve of genus g can be uniquely associated to an unordered set of $2g+2$ distinct points $(p_1, ..., p_{2g+2}) \in \mathbb{P}^1 \times ... \times \mathbb{P}^1 - \Delta$ where Δ denotes the diagonal. Conversely, to any such set of points we can associate the hyperelliptic curve given by the affine equation:

$$y^2 = (x-p_1)...(x-p_{2g+2}). \tag{2.4}$$

Denote by $(\mathbb{P}^1_n = \mathbb{P}^1 \times ... \times \mathbb{P}^1 - \Delta)/S_n$ the subset of the n-th symmetric product of $\mathbb{P}^1$ with itself consisting of n distinct points. The action of $PGL_2(\mathbb{C})$ on $\mathbb{P}^1$ induces an action on $\mathbb{P}^1_n$. If $P = (p_1, ..., p_n)$ is such a set of n points and $M \in PGL_2(\mathbb{C})$ then the action is given by:

$$MP = (Mp_1, ..., Mp_n)$$

We can now consider the orbit of a set of n points P under the action of $PGL_2(\mathbb{C})$ and thus define the moduli space of sets of n points as:

$$M_n := \mathbb{P}^1_n / PGL_2(\mathbb{C}).$$

We will denote by $[P]$ the image in M_n of a set of n points P.

Theorem 2.2.1. *Let C, C' be two hyperelliptic curves of genus g. Suppose that C is associated to the set of $2g+2$ points $P = (p_1, ..., p_{2g+2})$ and C' is associated to the set of $2g+2$ points $P' = (p'_1, ..., p'_{2g+2})$. Then C and C' are isomorphic if and only if P and P' have the same image in M_{2g+2}.*

Let $P = [p_1, ..., p_n] \in \mathbb{P}^1_n$. The group of automorphisms of P is defined as follows:

$$Aut(P) = \{M \in PGL_2(\mathbb{C}) | [MP] = [P]\}.$$

Proposition 2.2.2. *Let P be a set of n points in $\mathbb{P}^1_n$ and $M \in PGL_2(\mathbb{C})$. Then $Aut(P)$ is a subgroup of the symmetric group of n letters S_n (and hence a finite group) and moreover $Aut(P)$ and $Aut(MP)$ are isomorphic.*

If C is a hyperelliptic curve and i the hyperelliptic involution then we denote by $\overline{Aut(C)} := Aut(C)/\langle i \rangle$ the *reduced automorphism group* of C.

Now, let C be a hyperelliptic curve associated with the set of $2g+2$ points $P = (p_1, ..., p_{2g+2})$. Then the reduced automorphism group of C is isomorphic with $Aut(P)$, the automorphism group of the set of $2g+2$ points P.

Theorem 2.2.3. *Let C be a hyperelliptic curve. The reduced automorphism group $\overline{Aut(C)}$ of C is a finite subgroup of $PGL_2(\mathbb{C})$.*

Finite subgroups of $PGL_2(\mathbb{C})$ are well known and we list them in the following theorem:

Theorem 2.2.4 (18, Section 1.1)**.** *Let G be a finite subgroup of $PGL_2(\mathbb{C})$. Then G is isomorphic to one of the following groups: $\mathbb{Z}_n$ (cyclic group of order n), D_n (dihedral group of order $2n$), A_4, S_4, A_5. Moreover, isomorphic groups are conjugate, and we can choose coordinates such that the action is described as follows:*

1. $\mathbb{Z}_n : \langle z \mapsto \zeta z \rangle$, ζ *is an n-th root of unity.*

2. $D_n : \langle z \mapsto \zeta z, z \mapsto \frac{1}{z} \rangle$, ζ *is an n-th root of unity.*

3. $A_4 : \langle z \mapsto -z, z \mapsto i\frac{z+1}{z-1} \rangle, i^2 = -1.$

4. $S_4 : \langle z \mapsto iz, z \mapsto i\frac{z+1}{z-1} \rangle, i^2 = -1.$

5. $A_5 : \langle z \mapsto \epsilon z, z \mapsto -\frac{x+b}{bx+1} \rangle$, *where* ϵ *is a primitive 5-th root of unity and* $b = -i(\epsilon + \epsilon^4), i^2 = -1.$

We also describe the possible orbits of these actions. These will be used later to find equations of hyperelliptic curves with a given automorphism group.

Theorem 2.2.5 (8, 3 Proposition 4.1)**.** *The possible orbits of the groups in Theorem 2.2.4 are given as follows:*

1. $\mathbb{Z}_n : O_1 = \{1\}$ *which is the non-free orbit of* ∞, $O_2 = \{x\}$ *which is the non-free orbit of* 0 *and finally the free orbit of some point a given by* $O_3 = \{\zeta a, ..., \zeta^n a\}$, ζ *an n-th root of unity. The sizes of these orbits are* $(1, 1, n)$.

2. D_n*: The non-free orbit* $O_1 = \{0, \infty\}$*, the non-free orbit* $O_2 = \{$ *roots of* $x^n - 1\}$*, the non-free orbit* $O_3 = \{$ *roots of* $x^n + 1\}$ *and finally the free orbit* $O_4 = \{$ *roots of* $x^{2n} + ax^n + 1\}$ *with* $a \neq \pm 2$*. The sizes of these orbits are* $(2, n, n, 2n)$.

3. A_4*: The non-free orbit* $O_1 = \{0, \infty, \pm 1, \pm i\}$*, the non-free orbit* $O_2 = \{$ *roots of* $x^4 - 2i\sqrt{3}x^2 + 1\}$*, the non-free orbit* $O_3 = \{$ *roots of* $x^4 + 2i\sqrt{3}x^2 + 1\}$ *and finally the free orbit* $O_4 = \{$ *roots of* $\prod_{i=1}^{3}(x^4 - a_i x^2 + 1)\}$ *with* $a_1 \neq \pm 2, \pm 2i\sqrt{3}$ *and* $a_2 = \frac{2a_1+12}{2-a_1}$, $a_3 = \frac{2a_1-12}{2+a_1}$*. The sizes of these orbits are* $(6, 4, 4, 12)$.

4. S_4*: The non-free orbit* $O_1 = \{0, \infty, \pm 1, \pm i\}$*, the non-free orbit* $O_2 = \{$ *roots of* $x^8 + 14x^4 + 1\}$*, the non-free orbit* $O_3 = \{$ *roots of* $(x^4 + 1)(x^8 - 34x^4) + 1\}$ *and finally the free orbit* $O_4 = \{$ *roots of* $(x^8 + 14x^4 + 1)^3) - a(x^5 - x)^4\}$ *with* $a \neq 108$*. The sizes of these orbits are* $(6, 8, 12, 24)$.

5. A_5*: The non-free orbit* $O_1 = \{0, \infty\} \cup \{$ *roots off*$_1(x) = x(x^{10} + 11ix^5 + 1)\}$*, the non-free orbit* $O_2 = \{$ *roots of* $f_2(x) = x^{20} - 228ix^{15} - 494x^{10} - 228ix^5 + 1\}$*, the non-free orbit* $O_3 = \{$ *roots of* $x^{30} + 522ix^{25} + 10005(x^{20} - x^{10}) - 522ix^5 - 1\}$ *and finally the free orbit* $O_4 = \{$ *roots of* $f_2(x)^3 - af_1(x)^5\}$ *where* $a \neq -1728i$*. The sizes of these orbits are* $(12, 20, 30, 60)$.

Remark 2.2.6. Based on the previous theorem, a hyperelliptic curve will have a non-trivial reduced automorphism group if and only if its $2g+2$ Weierstrass points are a union of the above free and non-free orbits. The equation that needs to be satisfied in each case is:

1. Reduced automorphism group $\mathbb{Z}_n : 2g+2 = i + nd$, i can be equal to 0,1 or 2, d can be any non-negative integer.

2. Reduced automorphism group $D_n : 2g+2 = ni + 2j + 2nd$, i can be equal to 0,1 or 2, j can be either 0 or 1, d can be any non negative integer.

3. Reduced automorphism group $A_4 : 2g+2 = 4i + 6j + 12d$, i can be equal to 0,1 or 2, j can be either 0 or 1, d can be any non negative integer.

4. Reduced automorphism group $S_4 : 2g+2 = 6i + 8j + 12k + 24d$, i, j, k can be either 0 or 1 and d can be any non negative integer.

5. Reduced automorphism group $A_5 : 2g+2 = 12i + 20j + 30k + 60d$, i, j, k can be either 0 or 1 and d can be any non negative integer.

We can now prove that "most" hyperelliptic curves have trivial reduced automorphism group by a dimension argument.

Theorem 2.2.7. *Let C be a generic hyperelliptic curve in $\mathcal{H}_g$ (with $g > 1$), the moduli space of genus g hyperelliptic curves. Then C has trivial reduced automorphism group.*

Proof. We have seen that the dimension of $\mathcal{H}_g$ is $2g-1$ so it suffices to show that hyperelliptic curves with non-trivial reduced automorphism group depend on fewer parameters. If C has reduced automorphism group $\mathbb{Z}_n$ then we can partition the $2g+2$ points into free and non-free orbits. If all orbits are free then the number of parameters is $\frac{2g+2}{n}$. If the free orbit is $\{a, \zeta a, ..., \zeta^{n-1}a\}$ we have to move one of the points to 1 so that we are using the same coordinates. This can be done by applying the transformation $z \mapsto \frac{z}{a}$. So, the actual number of parameters for the hyperelliptic curve C with reduced automorphism group $\mathbb{Z}_n$ is $\frac{2g+2}{n} - 1$ and when n is at least 2 this number is strictly less than $2g-1$. If there are non-free orbits too, the number of parameters is even smaller. If the reduced automorphism group belongs to one of the other 4 cases we have even less parameters and this completes the proof. □

Corollary 2.2.8. *The locus of hyperelliptic curves with non-trivial reduced automorphism group G has dimension less than or equal to g.*

Proof. We just have to consider what is the maximum value of parameters that a hyperelliptic curve with non-trivial reduced automorphism group depends on. Taking into account the proof of the previous theorem we can see that the maximum value occurs for hyperelliptic curves with reduced automorphism group $\mathbb{Z}_2$ and is equal to $\frac{2g+2}{2} - 1 = g$. □

Having found the possible reduced automorphism groups of hyperelliptic curves, we will now study their full automorphism groups. The following basic result can be found in [9],[54].

Lemma 2.2.9. *Let C be a hyperelliptic curve with automorphism group $Aut(C)$ and reduced automorphism group $\overline{Aut(C)}$. Let $g \in Aut(C)$ and denote by $\tilde{g}$ its image in $\overline{Aut(C)}$. Assume that $\tilde{g}$ has order $|\tilde{g}|$. Then the order of g is either equal to $|\tilde{g}|$ in the case that g does not fix any Weierstrass points or equal to $2|\tilde{g}|$ in the case that g fixes some Weierstrass points.*

The discussion that follows can be found in [54]. Let $p : C \to \mathbb{P}^1$ be the double cover of the hyperelliptic curve and denote by W the set of Weierstrass points. Consider the map $f : \mathbb{P}^1 \to \mathbb{P}^1/\overline{Aut(C)}$. The branching indices of f for the 5 different cases of reduced automorphisms groups are (n, n), $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$. Let V be the set consisting of pre-images of branch points of the map f. Then $V = f^{-1}(q_1) \cup f^{-1}(q_2)$ when the reduced automorphism group is $\mathbb{Z}_n$ and $V = f^{-1}(q_1) \cup f^{-1}(q_2) \cup f^{-1}(q_3)$ in the other four cases. When the reduced automorphism group is $\mathbb{Z}_n$ there are four possibilities for the intersection $V \cap W$: $\emptyset$, $f^{-1}(q_1)$, $f^{-1}(q_2)$ and $f^{-1}(q_1) \cup f^{-1}(q_1)$. The equations of the corresponding curves will be given by $y^2 = f_4(x)$, $y^2 = f_1(x)f_4(x)$, $y^2 = f_2(x)f_4(x)$ and $y^2 = f_3(x)f_4(x)$ where $f_1(x), ..., f_4(x)$ are the polynomials appearing in the four orbits of $\mathbb{Z}_n$ in Theorem 2.2.5.

In the other four cases there are eight possibilities for the intersection $V \cap W$ and eight corresponding polynomials. Two of the branching indices are the same for $\mathbb{Z}_n$, D_n and A_4 so we expect to get three equations for $\mathbb{Z}_n$ six equations for D_n, A_4 and eight equations for S_4 and A_5.

We are now ready to find the full automorphism groups of hyperelliptic curves. We will treat the five cases separately.

Case 1: Reduced automorphism group $\mathbb{Z}_n$ (see [9], [54])

We are looking for extensions of $\mathbb{Z}_n$ by $\mathbb{Z}_2$ and these are classified by:

$$H^2(\mathbb{Z}_n, \mathbb{Z}_2) \cong \mathbb{Z}_{(n,2)}. \tag{2.5}$$

When n is odd the only possible extension is $\mathbb{Z}_n \times \mathbb{Z}_2$. When n is even there are two possible extensions, $\mathbb{Z}_n \times \mathbb{Z}_2$ and $\mathbb{Z}_{2n}$.

Based on the discussion above, the three different equations are:

1.
$$y^2 = x^{2g+2} + a_1 x^{n(t-1)} + ... + a_k x^n + 1, t = \frac{2g+2}{n}$$
In this case the intersection $V \cap W$ is empty, elements of order n in $\overline{Aut(C)}$ lift to elements of order n in $Aut(C)$, so the full automorphism group is $\mathbb{Z}_n \times \mathbb{Z}_2$.

2.
$$y^2 = x^{2g+1} + a_1 x^{n(t-1)} + ... + a_k x^n + 1, t = \frac{2g+1}{n}$$
In this case $V \cap W = f^{-1}(q_1)$, elements of order n in $\overline{Aut(C)}$ lift to elements of order $2n$ in $Aut(C)$, so the full automorphism group is $\mathbb{Z}_{2n}$.

3.
$$y^2 = x(x^{nt} + a_1 x^{n(t-1)} + ... + a_k x^n + 1), t = \frac{2g}{n}$$
In this case $V \cap W = f^{-1}(q_1) \cup f^{-1}(q_2)$, elements of order n in $\overline{Aut(C)}$ lift to elements of order $2n$ in $Aut(C)$, so the full automorphism group is $\mathbb{Z}_{2n}$.

Case 2: Reduced automorphism group D_n (see [9], [54])

In this case we have that $\overline{Aut(C)} \cong D_n$ and we are looking for extensions of dihedral groups by $\mathbb{Z}_2$. We have that:

$$H^2(D_2, \mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Also if $n > 1$ and $n \equiv 1 mod 2$ then

$$H^2(D_n, \mathbb{Z}_2) \cong \mathbb{Z}_2$$

and if $n > 2$ and $n \equiv 0 mod 2$ then

$$H^2(D_n, \mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

In the first case the possible extensions are $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, D_4 or the quaternion group Q. In the second case the possible extensions are $\mathbb{Z}_2 \times \mathbb{Z}_n$ or $G_n = \langle x, y | x^{2n}, x^n = y^2, y^{-1}xy = x^{-1} \rangle$. In the last case the possible extensions are $\mathbb{Z}_2 \times D_n$, D_{2n}, G_n, $H_n = \langle x, y | x^2 = y^2, (xy)^n = x^4 = 1 \rangle$, $U_n = \langle x, y | x^{2n} = y^2 = 1, yxy = x^{n-1} \rangle$ and $V_n = \langle x, y | x^4 = y^n = (xy)^2 = (x^{-1}y)^2 = 1 \rangle$.

The general equation of a hyperelliptic curve with reduced automorphism group D_n is:

$$y^2 = x^i (x^n - 1)^j (x^n + 1)^k \prod_{i=1}^{k} (x^{2n} - \lambda_i x^n + 1).$$

When $(i,j,k) = (0,0,0)$ elements of $\overline{Aut(C)}$ lift to elements of orders $(2,2,n)$ in $Aut(C)$ and the full automorphism group is $\mathbb{Z}_2 \times D_n$.

When $(i,j,k) = (1,0,0)$ elements of $\overline{Aut(C)}$ lift to elements of orders $(2,2,2n)$ in $Aut(M)$ and the full automorphism group is D_{2n}.

When $(i,j,k) = (0,1,0)$ or $(i,j,k) = (0,0,1)$ elements of $\overline{Aut(C)}$ lift to elements of orders $(2,4,n)$ in $Aut(C)$ and the full automorphism group is V_n.

When $(i,j,k) = (1,1,0)$ or $(i,j,k) = (1,0,1)$ elements of $\overline{Aut(C)}$ lift to elements of orders $(2,4,2n)$ in $Aut(C)$ and the full automorphism group is U_n.

When $(i,j,k) = (0,1,1)$ elements of $\overline{Aut(C)}$ lift to elements of orders $(4,4,n)$ in $Aut(C)$ and the full automorphism group is H_n.

When $(i,j,k) = (1,1,1)$ elements of $\overline{Aut(C)}$ lift to elements of orders $(4,4,2n)$ in $Aut(C)$ and the full automorphism group is G_n.

Case 3: Reduced automorphism group A_4 (see [9], [54])

In this case we have that $\overline{Aut(C)} \cong A_4$ and we are looking for extensions of A_4 by $\mathbb{Z}_2$. We have that:

$$H^2(A_4, \mathbb{Z}_2) \cong \mathbb{Z}_2$$

The two possible extensions are $\mathbb{Z}_2 \times A_4$ and $SL(2,3)$.

The equations of curves with full automorphism group $\mathbb{Z}_2 \times A_4$ are:

1. $$y^2 = \prod_{i=1}^{3}(x^4 - a_i x^2 + 1)$$

2. $$y^2 = (x^4 - 2i\sqrt{3}x^2 + 1)\prod_{i=1}^{3}(x^4 - a_i x^2 + 1)$$

3. $$y^2 = (x^8 + 14x^4 + 1)\prod_{i=1}^{3}(x^4 - a_i x^2 + 1)$$

The equations of curves with full automorphism group $SL(2,3)$ are:

1. $$y^2 = x(x^4 - 1)\prod_{i=1}^{3}(x^4 - a_i x^2 + 1)$$

2.
$$y^2 = x(x^4-1)(x^4-2i\sqrt{3}x^2+1)\prod_{i=1}^{3}(x^4-a_ix^2+1)$$

3.
$$y^2 = x(x^4-1)(x^8+14x^4+1)\prod_{i=1}^{3}(x^4-a_ix^2+1)$$

Case 4: Reduced automorphism group S_4 (see [9], [54])

In this case we have that $\overline{Aut(C)} \cong S_4$ and we are looking for extensions of S_4 by $\mathbb{Z}_2$. We have that:
$$H^2(S_4, \mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

The four possible extensions are $\mathbb{Z}_2 \times S_4$, $GL(2,3)$, $W_2 = \langle x, y | x^4, y^3, yx^2y^{-1}x^2, (xy)^4 \rangle$ and $W_3 = \langle x, y | x^2, y^3, x^2(xy)^4, (xy)^8 \rangle$.

The equations of curves with full automorphism group $\mathbb{Z}_2 \times S_4$ are:

1.
$$y^2 = ((x^8+14x^4+1)^3 - a(x^5-x)^4)$$

2.
$$y^2 = (x^4+1)(x^8-34x^4+1)((x^8+14x^4+1)^3 - a(x^5-x)^4)$$

The equations of curves with full automorphism group $GL(2,3)$ are:

1.
$$y^2 = (x^8+14x^4+1)((x^8+14x^4+1)^3 - a(x^5-x)^4)$$

2.
$$y^2 = (x^5-x)((x^8+14x^4+1)^3 - a(x^5-x)^4)$$

The equations of curves with full automorphism group W_2 are:

1.
$$y^2 = (x^4+1)(x^8-34x^4+1)(x^8+14x^4+1)((x^8+14x^4+1)^3 - a(x^5-x)^4)$$

2.
$$y^2 = (x^5-x)(x^8+14x^4+1)((x^8+14x^4+1)^3 - a(x^5-x)^4)$$

The equations of curves with full automorphism group W_3 are:

1. $$y^2 = (x^4+1)(x^8-34x^4+1)(x^5-x)((x^8+14x^4+1)^3 - a(x^5-x)^4)$$

2. $$y^2 = (x^8+14x^4+1)(x^4+1)(x^8-34x^4+1)(x^5-x)((x^8+14x^4+1)^3 - a(x^5-x)^4)$$

Case 5: Reduced automorphism group A_5 (see [9], [54])

In this case we have that $\overline{Aut(C)} \cong A_5$ and we are looking for extensions of A_5 by $\mathbb{Z}_2$. We have that:

$$H^2(A_5, \mathbb{Z}_2) \cong \mathbb{Z}_2$$

The two possible extensions are $\mathbb{Z}_2 \times A_4$ and $SL(2,5)$.

Let $f_1(x) = x(x^{10}+11x^5+1)$, $f_2(x) = x^{20} - 228ix^{15} - 494x^{10} - 228ix^5 + 1)$ and $f_3(x) = f_2(x)^3 - af_1(x)^5$.

The equations of the curves with automorphism group $\mathbb{Z}_2 \times A_5$ are:

1. $$y^2 = f_3(x)$$

2. $$y^2 = f_2(x) \cdot f_3(x)$$

3. $$y^2 = f_1(x) \cdot f_3(x)$$

4. $$y^2 = (x^{30} + 522ix^{25} + 10005(x^{20} - x^{10}) - 522ix^5 - 1) \cdot f_3(x)$$

The equations of the curves with automorphism group $SL(2,5)$ are:

1. $$y^2 = f_1(x) \cdot f_2(x) \cdot f_3(x)$$

2. $$y^2 = (x^{30} + 522ix^{25} + 10005(x^{20} - x^{10}) - 522ix^5 - 1) \cdot f_1(x) \cdot f_3(x)$$

3.

$$y^2 = (x^{30} + 522ix^{25} + 10005(x^{20} - x^{10}) - 522ix^5 - 1) \cdot f_2(x) \cdot f_3(x)$$

4.

$$y^2 = (x^{30} + 522ix^{25} + 10005(x^{20} - x^{10}) - 522ix^5 - 1) \cdot f_1(x) \cdot f_2(x) \cdot f_3(x)$$

Chapter 3

Abelian Varieties

In this chapter we review some background material on complex tori and abelian varieties with the goal of discussing some known methods of finding distinct curves with isomorphic Jacobians. In the first section we follow [39] closely. We start with complex tori, introduce polarizations and use them to define abelian varieties. We study the possible endomorphism algebras of abelian varieties and finally we introduce Jacobians of curves and state Torelli's theorem. In the second section we review results from [49], [50],[32],[33],[38],[15],[30],[31].

§ 3.1 Complex tori, Abelian varieties and Jacobians of curves

3.1.1 Complex tori

We will be working over the field of the complex numbers and start our study by defining complex tori:

Definition 3.1.1 (39, Section 1.1)**.** A *complex torus* X is a quotient V/Λ where V is a finite-dimensional complex vector space and Λ a lattice of maximal rank. The dimension of X is the same as that of V.

If we choose bases for V and Λ we can describe a complex torus using its *period matrix*. Let $v_1, ..., v_g$ be a basis of V and $\lambda_!, ..., \lambda_{2g}$ a basis of Λ. We can express the vectors λ_i in terms of the basis of V as $\lambda_i = \sum_{i=1}^{g} z_{ji} v_j$. Then the matrix $\Pi \in M(g \times 2g, \mathbb{C})$:

$$\begin{pmatrix} z_{1,1} & \cdots & z_{1,2g} \\ \vdots & & \vdots \\ z_{g,1} & \cdots & z_{g,2g} \end{pmatrix}$$

is the period matrix of X and a given matrix $\Pi \in M(g \times 2g, \mathbb{C})$ is the period matrix of some complex torus if and only if $\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}$ is nonsingular ([39], Proposition 1.1.2).

Morphisms between complex tori come in two types, translations and homomorphisms. We can restrict our attention to homomorphisms as we can compose an arbitrary morphism with a translation that sends 0 to 0. If f is a morphism between complex tori $X = V/\Lambda$ and $X' = V'/\Lambda'$ then f is induced by a $\mathbb{C}$ linear map $\tilde{f} : V \to V'$ such that $\tilde{f}(\Lambda) \subseteq \Lambda'$ ([39], Proposition 1.2.1). The image of a homomorphism f of complex tori X and X' is a complex subtorus of X' while the connected component of $ker f$ that contains the origin is a subtorus of X ([39], Proposition 1.2.4). This leads to the following important definition.

Definition 3.1.2 (39, page 12)**.** An *isogeny* is a surjective homomorphism of complex tori with finite kernel.

Isogenies define an equivalence relation on the set of complex tori and it is clear that isogenous complex tori have the same dimensions. An example of an isogeny is given by the multiplication by n map. In this case the kernel consinsts of the n-torsion points of the complex torus ([39], Proposition 1.2.5).

3.1.2 Polarizations, Abelian Varieties

In order to define abelian varieties we need the notion of polarization. We state the relevant definitions below:

Definition 3.1.3 (Riemann form)**.** Let $X = V/\Lambda$ be a complex torus. A *Riemann form* on X is a hermitian form $H : V \times V \to \mathbb{C}$ such that ImH is integral on the lattice Λ.

Proposition 3.1.4 (39, Lemma 2.1.7)**.** *There is a 1-1 correspondence between the set of hermitian forms $H : V \times V \to \mathbb{C}$ and the set of alternating forms $E : V \times V \to \mathbb{R}$ such that $E(ix, iy) = E(x, y)$. The correspondence is given by:*

$$E(x, y) = ImH(x, y), H(x, y) = E(ix, y) + iE(x, y).$$

for all $x, y \in V$.

Definition 3.1.5 (Polarization)**.** A *polarization* on a complex torus X is a positive definite Riemann form.

Finally we arrive to the definition of abelian varieties.

Definition 3.1.6. An *abelian variety* is a complex torus that admits a polarization.

Using a theorem from linear algebra by Frobenius, we can find a basis of the lattice such that the alternating form E acquires a special form ([39], Section 3.1). Let $X = V/\Lambda$ be a complex torus which admits a polarization given by a positive definite hermitian form H with associated alternating form E. Then, there is a basis of the lattice Λ with respect to which E is given by the matrix:

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$$

where D is a diagonal matrix with integer entries $d_1, ..., d_g$ such that $d_i | d_{i+1}$ for $i = 1, ..., g-1$. In this case we say that X has a polarization of type $(d_1, ..., d_g)$.

Definition 3.1.7 (Principal polarization)**.** A *principal polarization* is a polarization of type $(1, ..., 1)$. An abelian variety admitting a principal polarization is called a *principally polarized abelian variety.*

Now we will briefly describe where this mysterious alternating form is coming from. This is done in detail in ([39], Chapter 2]) and it follows from ([39], Proposition 2.1.6) that the alternating form E defining a polarization on a complex torus X is coming from the first Chern class $c_1(L)$ of a holomorphic line bundle L on X. The set of equivalence classes of holomorphic line bundles on X is the *Neron-Severi* group of X, denoted by $NS(X)$. One can use an ample line bundle on a complex torus X to construct an embedding of X in a projective space and in that sense abelian varieties are projective complex tori. This gives the connection with the definition of abelian varieties over general fields.

Another way to view polarizations that will be useful when we study the possible endomorphism types of abelian varieties is in terms of maps to the dual abelian variety (see [39], Section 2.4). Let $X = V/L$ be an abelian variety. Then $Pic^0(X)$ can be given the structure of a complex torus which we call the dual complex torus of X and denote it by $\hat{X}$. The map $\phi_L : X \to \hat{X}$ given by $x \mapsto t_x^* L \otimes L^{-1}$ is an isogeny of degree $deg(\phi_L) = det(ImH)$ ([39], Proposition 2.4.9). In particular, for principal polarizations ϕ_L is an isomorphism.

3.1.3 Endomorphism types

Now we are going to study the possible endomorphism algebras of polarized abelian varieties. We will follow ([39], Chapter 5) closely. Let (X, L) be a polarized abelian variety and denote its endomorphism algebra by $End^0(X) := End(X) \otimes \mathbb{Q}$. If $f : X \to X$ is a morphism from X to itself then we also have a dual morphism $\hat{f} : \hat{X} \to \hat{X}$. This extends to a map from $End^0(X)$ to $End^0(\hat{X})$ and we define the *Rosati involution* as $f^t := \phi_L^{-1} \hat{f} \phi_L$.

An element $f \in End^0(X)$ such that $f^t = f$ will be called symmetric and we denote by End^0_{sym} the set of symmetric endomorphisms of $End^0(X)$. The following result will be used in the next section:

Proposition 3.1.8 (39, Proposition 5.2.1)**.** *Let (X, L) be a polarized abelian variety. The map: $NS(X) \otimes \mathbb{Q} \to End^0(X)$, $M \mapsto \phi_L^{-1} \phi_M$ is an isomorphism of $\mathbb{Q}$-vector spaces that restricts to an isomorphism of groups $NS(X) \to End_{sym}(X)$ when the polarization L is principal.*

An abelian variety is called *simple* when its only abelian subvarieties are itself and 0. In general, we have the following:

Theorem 3.1.9 (Poincare's complete reducibility, 39, Theorem 5.3.7)**.** *Let X be an abelian variety. Then there exists an isogeny:*

$$X \to X_1^{n_1} \times ... X_r^{n_r}$$

such that X_i are simple and not isogenous to each other and they are completely determined up to isogeny and permutations.

The following result reduces the classification of endomorphism algebras of abelian varieties to the classification of endomorphism algebras of simple abelian varieties:

Corollary 3.1.10 (39, Corollary 5.3.8)**.** *Let X be an abelian variety and consider the isogeny of the previous theorem $X \to X_1^{n_1} \times ... X_r^{n_r}$. We have that*

$$End^0(X) \cong \bigoplus_{i=1}^{r} Mat_{n_1}(End^0(X_i)).$$

Now when X is a simple abelian variety, $End^0(X)$ is a finite dimensional division algebra over $\mathbb{Q}$ that possesses an involution (the Rosati involution) and such that the trace map $f \mapsto Tr(ff^t)$ is a positive definite quadratic form ([39], Theorem 5.1.8). These algebras have been classified, for the classification see ([39], Section 5.5) or ([27], Section 1.8) or ([17], Theorem 9.6).

As an example, the classification of endomorphism algebras of abelian surfaces is given below:

Example 3.1.11 (27, Example 1.59)**.** Let X be an abelian surface.

1. If X is simple, the possibilities for $End^0(X)$ are:
   (a) $\mathbb{Q}$
   (b) a real quadratic field
   (c) a totally indefinite quaternion algebra over $\mathbb{Q}$
   (d) a CM field
2. If X is isogenous to a product of elliptic curves $X \cong E_1 \times E_2$, the possibilities for $End^0(X)$ are:
   (a) $\mathbb{Q} \times \mathbb{Q}$ when E_1 is not isogenous to E_2
   (b) $Mat_2(\mathbb{Q})$ when E_1, E_2 are isogenous and don't have complex multiplication
   (c) $Mat_2(F)$ when E_1, E_2 are isogenous and have complex multiplication by a CM field F

3.1.4 The moduli space of polarized abelian varieties

The results/notation of this short section can be found in ([39], Chapter 8).

We have seen conditions that ensure when a period matrix defines a complex torus. Similarly, there are conditions which ensure when a period matrix defines a complex torus that is an abelian variety, they are called the *Riemann relations* (see [39], Section 4.2).

Using this, we can define the *Siegel upper half space* of degree g as:

$$\mathcal{A}_g := \{Z \in Mat_g(\mathbb{C}) : Z = Z^t, ImZ > 0\}.$$

Given a point in this space and a diagonal matrix D, we can construct a complex torus with period matrix $(D|Z)$ that has polarization of type D and conversely to any such complex torus we can associate a point in the Siegel upper half space. We say that $\mathcal{A}_g$ is the moduli space parametrizing abelian varieties with polarization of type D. For the dimension of this space we have that $dim\mathcal{A}_g = \frac{g(g+1)}{2}$.

3.1.5 Jacobians of curves and Torelli's theorem

Let C be an algebraic curve. There is a way to associate to C a principally polarized abelian variety of dimension g, the Jacobian $J(C)$ of C. We briefly describe this construction, following ([39], Section 11.1).

Let $\omega_1, ..., \omega_g \in H^0(C, \Omega^1)$ be a basis for the space of holomorphic 1-forms on C and $\delta_1, ..., \delta_{2g}$ a basis of $H_1(C, \mathbb{Z})$. The vectors $\Pi_i = (\int_{\delta_i} \omega_1, ..., \int_{\delta_i} \omega_g)$ are called the periods of C and the $g \times 2g$ matrix Ω with the vectors Π_i as its columns is called the period matrix of C. The $2g$ periods $\Pi_i \in \mathbb{C}^g$ generate a lattice Λ and we define the Jacobian variety $J(C)$ of C to be the complex torus $\mathbb{C}^g/\Lambda$. According to Riemann's bilinear relations, for suitably chosen bases of $H^0(C, \Omega^1)$ and $H_1(C, \mathbb{Z})$ the period matrix Ω of C takes the form $\Omega = (I|Z)$ with $Z = Z^t$ and $ImZ > 0$. This turns the Jacobian $J(C)$ of C into a principally polarized abelian variety. The alternating form E is coming from a suitably chosen *symplectic basis* of $H_1(C, \mathbb{Z})$ and we can construct a line bundle L_E whose first Chern class is E. If θ denotes a global section of L_E we can associate E to a divisor $\Theta = (\theta)$, up to translation. This divisor is called the *Riemann theta divisor*. We can now state Torelli's theorem:

Theorem 3.1.12 (Torelli's theorem)**.** *Let C,C' be smooth projective algebraic curves of genus g. Then C, C' are isomorphic if and only if $(J(C), \Theta)$ and $(J(C'), \Theta')$ are isomorphic as principally polarized abelian varieties.*

We can restate this in terms of the moduli spaces $\mathcal{M}_g$ and $\mathcal{A}_g$ by saying that the *Torelli map* which associates a curve to its polarized Jacobian $(J(C), \Theta)$ is injective. The dimensions of the two spaces coincide for $g \leq 3$ and this will have important implications in the next section. We will also need the following relevant result:

Proposition 3.1.13 (39, Corollary 11.8.2)**.** *a) A principally polarized abelian surface is either the Jacobian of a genus 2 curve or a product of two elliptic curves with the canonical product polarization.*

b) A principally polarized abelian threefold is either the Jacobian of a genus 3 curve, a principally polarized product of an abelian surface with an elliptic curve or a product of three elliptic curves with the canonical product polarization.

While the principally polarized Jacobian of a curve determines the curve completely, one can ask whether the same happens by just considering the Jacobian as a complex torus. In the next section we will see that this doesn't happen as there exist distinct curves whose Jacobians are isomorphic complex tori.

§ 3.2 Distinct curves with isomorphic Jacobians

In this section we will present some known examples of distinct curves with isomorphic Jacobians. The examples will take into consideration the structure of $End^0(J(C))$.

3.2.1 Generic case

We start with the description of $End^0(J(C))$ for a generic genus g curve. The following result is well known:

Theorem 3.2.1 (14, Theorem 3.1)**.** *Let C be a generic curve of genus g. Then, its Jacobian $J(C)$ does not have non-trivial endomorphisms and $End(J(C)) \cong \mathbb{Z}$.*

Then, using 3.1.8 we see that $NS(J(C)) \cong \mathbb{Z}$ which means that the unique principal polarization of $J(C)$ is coming from the theta divisor of C. In particular, generic curves can not share the same unpolarized Jacobian and in order to find such examples we have to look for curves whose Jacobians have non-trivial endomorphisms. The dimension of the subvarieties of $\mathcal{M}_g$ that parametrize curves whose Jacobians have non-trivial endomorphisms have been studied in [16].

It is also known that the number of distinct curves that can share the same unpolarized Jacobian is finite, using a result of Narasimhan and Nori (see [41]).

3.2.2 Real multiplication

The results in this section are from [38]. Let A be a simple abelian variety of dimension g over $\mathbb{C}$ with real multiplication, that is $End^0(A)$ contains a totally real number field K of degree g over $\mathbb{Q}$. Let $\mathfrak{o} \subset K$ denote the maximal order of K and U denote its group of units. If we have an abelian variety A such that $End(A) = \mathfrak{o}$ and $\pi(A)$ denotes the number of isomorphism classes of principal polarizations then $\pi(A) = \#(U^+/U^2)$. Now, using Proposition 3.1.13, a simple abelian variety of dimension 2 or 3 will be the Jacobian of some curve. Using results on fundamental systems of units on totally real quadratic and cubic fields it is possible to construct simple abelian varieties in dimension 2 and 3 with several principal polarizations. These abelian varieties will be Jacobians and the various principal polarizations will be coming from distinct curves that share the same unpolarized Jacobian.

In dimension 4 the dimensions of $\mathcal{M}_g$ and $\mathcal{A}_g$ are no longer the same. One can still use results on fundamental systems of units of totally real quartic fields to construct simple abelian varieties of dimension 4 with several principal polarizations but it now becomes more difficult to tell whether these abelian varieties are Jacobians of some curve. However, using more elaborate results on the Satake compactification of $\mathcal{A}_g$ it is shown in [15] that distinct curves of genus 4 with isomorphic Jacobians do exist. Unfortunately, it is not possible to generalize this method in higher genus and this is actually a common feature of the methods that we study in this section: they only work in low genus.

3.2.3 Quaternionic multiplication

This case has been studied in [50], [49]. In this paper, V. Rotger computes the number of principal polarizations of abelian varieties A such that $End(A)$ is a maximal order in a totally indefinite quaternion algebra ([49], Theorem 1.1).

One can then use this to construct simple abelian varieties with quaternionic multiplications in dimension 2 and 3 with several principal polarizations and as before these abelian varieties will be Jacobians of curves, with the non-isomorphic principal polarizations coming from dinstinct curves.

While the number of principal polarizations of abelian varieties with real multiplcations is bounded, abelian varieties of even dimension with quaternionic multiplication can have arbitrarily many non-isomorphic principal polarizations. This implies that there are arbitrarily large sets of distinct curves $C_1, ..., C_n$ such that $J(C_1) \cong ... \cong J(C_n)$ ([49], Corollary 1.3).

As in the case of real multiplication, the fact that $dim\mathcal{M}_g \neq dim\mathcal{A}_g$ when $g > 3$ makes it difficult to construct such examples in higher genus.

3.2.4 Explicit equations - Products of elliptic curves

Explicit equations of distinct curves with isomorphic unpolarized Jacobians are given in [32] and [33]. In particular, we have the following result:

Theorem 3.2.2 (32, Theorem 1)**.** *Let m be a positive, even, square-free integer with n odd prime divisors. Let $\mathfrak{o}$ denote the ring $\mathbb{Z}[\sqrt{-m}]$ and h its class number. Let S be the set of positive real roots of the polynomial $g(x) = (x+1)^h f(\frac{2^8 x^3}{x+1})$, where f is the polynomial whose roots are the j-invariants of elliptic curves with complex multiplication by $\mathfrak{o}$. Then the set*

$$\{y^2 = x^6 - kx^4 + kx^2 - 1 : k \in S\}$$

contains 2^n non-isomorphic curves with isomorphic unpolarized Jacobian.

As before, using this method we can construct arbitrarily many distinct curves sharing the same unpolarized Jacobian. Even more striking examples are given in ([33], Theorem 1), where it is shown that it is possible for certain hyperelliptic and non-hyperelliptic curves of genus 3 to have isomorphic unpolarized Jacobians.

These results rely on careful considerations regarding the lattice of the Jacobians of the curves and generalizing them to higher genus would require tedious calculations.

Finally, another method to construct curves with isomorphic Jacobians is to count the number of principal polarizations of 2-dimensional abelian varieties that are products of elliptic curves. This has been done in [30],[31],[38] where the number of principal polarizations is associated to certain class numbers of quaternion algebras or hermitian forms.

The number of curves on such an abelian surface is equal to the total number of principal polarizations minus the number of decomposable principal polarizations. It can be shown that in certain cases where $A \cong E_1 \times E_2$ for isogenous elliptic curves with complex multiplication the number of indecomposamble principal polarizations on A is greater than 1.

Generalizing this method in higher dimension again presents difficulties. While in some cases it is still possible to compute the total number of principal polarizations on an abelian variety $A \cong E_1 \times ... \times E_n$, it would be much harder to say which of those polarizations are coming from curves.

A more geometric method for constructing curves with isomorphic Jacobians relies on Gassmann equivalence and Theorem 7.1.3. In the remainder of this thesis we will study Gassmann equivalence in more detail.

Chapter 4

Representation theory

In this chapter we collect basic results from representation theory that will be used in the proofs of Chapter 5. We start with the definitions of representations of groups and their characters, define the operations of restriction and induction and state a property of induced characters. We continue with some definitions on G-sets and study matrices representing homomorphisms between G-sets. Then we meet permutation representations and their characters and finally we define the table of marks of a finite group G and study some of its properties. The table of marks will be used for computations in the next chapter. The results of this chapter can be found in many introductory books on representation theory, for example [21],[63],[42],[53].

§ 4.1 Representation theory of groups

We start with the definition of a representation of a finite group G.

Definition 4.1.1 (Representation of a group G, 63, page 2)**.** Let R be a commutative ring with unity, G a finite group and V an R-module. Denote by $GL(V)$ the group of invertible $R - module$ homomorphisms from V to itself. A *representation* of G is a group homomorphism:

$$\rho : G \to GL(V). \tag{4.1}$$

The simplest example of a representation is the *trivial representation* where we take $\rho(g)$ to be the identity mapping of the R-module V to itself.

In what follows we will treat group representations as $R[G]$-modules ([63], Proposition

1.1.5). Elements of $R[G]$ have the form $\sum_{g\in G} a_g g$ and multiplication is given by

$$(\sum_{g\in G} a_g g)(\sum_{g'\in G} a_{g'} g') = \sum_{k\in G}(\sum_{gg'=k} a_g a_{g'})k. \tag{4.2}$$

Next we define the character of a representation.

Definition 4.1.2 (Character of a representation, 63, page 24)**.** Let ρ be a finite dimensional representation of a group G over a field k. The character of ρ, denoted by χ_ρ is the function $\chi_\rho : G \to k$ defined as:

$$\chi_\rho(g) := tr(\rho(g)). \tag{4.3}$$

Characters only depend on the conjugacy class of the group element g. Such functions are called *class functions*. When $k = \mathbb{C}$ we can define a hermitian inner product on the space of class function of a finite group G as ([63], page 32):

$$\langle \chi, \chi' \rangle_G := \frac{1}{|G|} \sum_{g\in G} \overline{\chi(g)} \chi'(g). \tag{4.4}$$

Now we will define the operations of restriction and induction of representations ([63], Section 4.3).

1. <u>Restriction</u> Let V be a representation of a group G and let H be a subgroup of G. We can get a representation of H just by forgetting about the elements of G that are not in H. We denote this representation by $V{\downarrow}_H^G$.
2. <u>Induction</u> Let W be a representation of H which is a subgroup of a group G. We can view W as an $R[H]$-module and define the $R[G]$-module $W{\uparrow}_H^G := R[G] \otimes_{R[H]} W$.

We will later need an important property of restriction and induction which we state below for reference.

Proposition 4.1.3 (Frobenius reciprocity, 63, Corollary 4.3.8, Corollary 4.3.9)**.** *Let G a finite group and H a subgroup of G. Let V be an $R[H]$-module and W an $R[G]$-module. We have the following isomorphisms:*

$$Hom_{R[G]}(V{\uparrow}_H^G, W) \cong Hom_{R[H]}(V, W{\downarrow}_H^G)$$

$$Hom_{R[G]}(W, V{\uparrow}_H^G) \cong Hom_{R[H]}(W{\downarrow}_H^G, V)$$

These can also be stated in terms of characters. If $\chi{\uparrow}_H^G$ and $\psi{\downarrow}_H^G$ denote the characters of $V{\uparrow}_H^G$ and $W{\downarrow}_H^G$ respectively, than we have:

$$\langle \chi{\uparrow}_H^G, \psi \rangle_G = \langle \chi, \psi{\downarrow}_H^G \rangle_H$$

and

$$\langle \psi, \chi{\uparrow}_H^G \rangle_G = \langle \psi{\downarrow}_H^G, \chi \rangle_H.$$

Finally, one can use the following proposition to compute characters of induced representations.

Proposition 4.1.4 (63, Proposition 4.3.5 and 42, Remark 3.4.1)**.** *Let G be a finite group and H a subgroup of G. Let V be an $R[H]$-module with character χ and R a set of representatives of G/H. The character of $V{\uparrow}_H^G$ is:*

$$\chi{\uparrow}_H^G(g) = \frac{1}{|H|} \sum_{r \in G, r^{-1}gr \in H} \chi(r^{-1}gr) = \sum_{r \in R, r^{-1}gr \in H} \chi(r^{-1}gr). \tag{4.5}$$

In the special case where we have the trivial representation 1_H of H with character χ_{1_H} and the induced representation 1_H^G on G with character $\chi_{1_H^G}$ we get that:

$$\chi_{1_H^G}(g) = \frac{|C_G(g)| \cdot |g^G \cap H|}{|H|} \tag{4.6}$$

where g^G denotes the conjugacy class of g in G.

§ 4.2 G-sets, permutation representations and permutation characters

We met G-sets in Chapter 2. We will now study G-sets and their homomorphisms in more detail. A G-map is a map $f : X \to Y$ between two G-sets X, Y that satisfies:

$$f(g \cdot x) = g \cdot f(x) \text{ for all } g \in G, x \in X.$$

We will denote the set of G-maps between X, Y by $Hom_G(X, Y)$.

A transitive G-set is a G-set which is a G-orbit. Every G-set can be written as a disjoint union of transitive G-sets. Moreover, if X, Y are isomorphic transitive G-sets then $X \cong G/H$ and $Y \cong G/K$ where H, K are conjugate subgroups of G. In general a G-set is isomorphic to $\bigsqcup G/H_i$ for some subgroups H_i of G. Two G-sets $X = \bigsqcup G/H_i$ and $Y = \bigsqcup G/K_i$ are isomorphic if (after possibly reordering) H_i is conjugate to K_i.

G-sets are closely related to a particular class of representations, *permutation representations*. Let X be a G-set. Then, the action of G permutes its elements. Let R be a commutative ring and consider $R[X]$, the free R-module with elements of X as a basis. By extending the action of G on X to an R-linear action of $R[G]$ on $R[X]$ we can consider $R[X]$ as an $R[G]$-module. The corresponding representation is called a *permutation representation* (see [63]).

We will now see a simple example that gives insight on the characters of permutation representations. Let σ be a permutation representation of a G-set X by matrices. For any element $g \in G$ the matrix $M = [\sigma(g)]$ is a permutation matrix and if we use the elements of X to denote its rows and columns, its entries are

$$M_{x_1 x_2} = \begin{cases} 1 & \text{if } x_1 = g \cdot x_2 \\ 0 & \text{otherwise} \end{cases}$$

Example 4.2.1. Consider the symmetric group S_3 acting on a set of three letters $X = \{x_1, x_2, x_3\}$ by $\sigma \cdot x_i = x_{\sigma(i)}$. S_3 is generated by (12) and (23) and the corresponding matrices of this permutation representation are:

$$(12) \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$(23) \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

We see that the trace of these matrices equals the number of points of X that are fixed by (12),(23) respectively.

This holds in general. The permutation matrix corresponding to an element $g \in G$ has entries 0 and 1 and there is a 1 on the diagonal if and only if the corresponding basis element of X is fixed by g. We have that:

$$\chi_{R[X]}(g) = |Fix_X(g)|.$$

Moreover, for a transitive G-set $X = G/H$ we have that the permutation representation $R[G/H]$ coincides with the representation of G induced by the trivial representation on H (see [63], Proposition 4.3.2 and Example 4.3.4). Combining this with 4.1.4, we get that:

Proposition 4.2.2. *Let G be a finite group and $X = \sqcup G/H_i$ a finite G set. Denote by $R[X]$ the permutation representation of X where R is a commutative ring and by $\chi_{1_H^G}$ the character of the induced representation of G by the trivial representation of H. We have the following equalities:*

$$\chi_{R[X]}(g) = |Fix_X(g)| = |C_G(g)| \sum_i \frac{|g^G \cap H_i|}{|H_i|}.$$

For a subgroup K of G, define $Fix_X(K) = \cap_{g \in K} Fix_X(g)$. We will use the following notation: $\chi_{R[X]}(K) = |Fix_X(K)|$. Observe that for a cyclic group $\langle g \rangle$ we have that $\chi_{R[X]}(g) = \chi_{R[X]}(\langle g \rangle)$

If X, Y are G-sets and $\phi : X \to Y$ is a G-map then we can extend it to an R-linear map from $R[X]$ to $R[Y]$. We want to study $R[G]$-homomorphisms between permutation modules. The following proposition classifies $Hom_{R[G]}(R[X], R[Y])$ for transitive G-sets X, Y.

Proposition 4.2.3 (59, Lemma 4.5, page 32)**.** *Let $X \cong G/H$ and $Y \cong G/K$ be transitive G-sets where H, K are subgroups of G. We have an R-module isomorphism*

$$\Phi(H, K) : R^{K \backslash G/H} \to Hom_{R[G]}(R[X], R[Y])$$

defined by:

$$f \mapsto (g_1 H \mapsto \sum_{g_2 K \in G/K} f(K g_2^{-1} g_1 H) g_2 K).$$

For what follows, see the discussion before Proposition 2.6 in [60].

Elements of $Hom_{R[G]}(R[G/H], R[G/K])$ can be identified with matrices $M \in M(m \times n, R)$ where m is the index of K in G and n the index of H in G. If we choose some ordering for the G-sets and ρ_1, ρ_2 denote the permutation representations of G acting on $\{1, ..., n\}$ and $\{1, ..., m\}$ respectively then the entries of M satisfy

$$M_{ij} = M_{\rho_1(g)(i), \rho_2(g)(j)}$$

for all $g \in G$.

For transitive G-sets the following lemma gives a way to calculate the number of variables of the matrix using the hermitian product of induced characters.

Lemma 4.2.4 (42, Lemma 3.4.2)**.** *Let H, K be subgroups of G. Then $\langle 1_H^G, 1_K^G \rangle_G$ is the number of orbits of K on G/H and also equals the number of (K, H)-double cosets of K and H in G.*

For intransitive G-sets $X = \bigsqcup G/H_i$, $Y = \bigsqcup G/H_i$ we can build a similar matrix made up from smaller block matrices corresponding to elements of $Hom_{R[G]}(R[G/H_i], R[G/K_j])$. The number of variables is going to be $\sum_i \sum_j \langle 1_{H_i}^G, 1_{K_j}^G \rangle_G$.

Example 4.2.5. Let G be the symmetric group on 3 letters S_3. It has four conjugacy classes of subgroups: $H_1 = \{e\}, H_2 = \{e, (12)\}, H_3 = \{e, (123)\}$ and G. Consider the G-sets $X \cong G/H_2 \sqcup G/H_2 \sqcup G/H_3$ and $Y \cong G/\{e\} \sqcup G/G \sqcup G/G$. The matrix M corresponding to an element of $Hom_{R[G]}(R[X], R[Y])$ is given below:

$$\begin{pmatrix} x_2 & x_3 & x_1 & x_1 & x_2 & x_3 & x_9 & x_{12} \\ x_1 & x_2 & x_2 & x_3 & x_3 & x_1 & x_9 & x_{12} \\ x_3 & x_1 & x_3 & x_2 & x_1 & x_2 & x_9 & x_{12} \\ x_5 & x_6 & x_4 & x_4 & x_5 & x_6 & x_{10} & x_{13} \\ x_4 & x_5 & x_5 & x_6 & x_6 & x_4 & x_{10} & x_{13} \\ x_6 & x_4 & x_6 & x_5 & x_4 & x_5 & x_{10} & x_{13} \\ x_7 & x_7 & x_8 & x_7 & x_8 & x_8 & x_{11} & x_{14} \\ x_8 & x_8 & x_7 & x_8 & x_7 & x_7 & x_{11} & x_{14} \end{pmatrix}$$

The code for the construction of this matrix is given in Appendix 9.4. In Chapter 5 we will be interested in the determinant of such matrices.

§ 4.3 The table of marks of a finite group

In this section we define the table of marks of a finite group G and provide some examples. The material is standard and taken from ([42], Section 3.5). For more details on the table of marks see [12], [45].

Let $\mathcal{L}(G) = \{H_1, ..., H_n\}$ be a complete list of representatives of conjugacy classes of subgroups of a finite group G. From now on we assume that $|H_i| \leq |H_j|$ for $i \leq j$. Let X be a finite G-set. Consider the function:

$$m_H(X) := |Fix_X(H)|$$

measuring the cardinality of the set of H-fixed points on X. It is called the H-mark on X.

Definition 4.3.1 (42, Definition 3.5.1)**.** The table of marks of G is the square matrix:

$$M(G) = [m_{G/H_i}(H_j)]_{1 \leq i,j \leq n}.$$

Note that conjugate subgroups have the same number of fixed points on any G-set so $M(G)$ is independent of the choice of representatives of $\mathcal{L}(G)$. However it does depend on its ordering.

The table of marks has some interesting properties that are summarized in the following proposition:

Proposition 4.3.2 (42, Lemma 3.5.3, Corollary 3.5.4)**.** *Let $M(G)$ be the table of marks of a finite group G. Then:*

1. *M(G) is invertible.*
2. *$m_{ij} = [N_G(H_i) : H_i] \cdot b_{ij}$ where b_{ij} is the number of subgroups conjugate to H_i. In particular $m_{ii} = [N_G(H_i) : H_i]$.*
3. *The first entry of every row is the index of the corresponding subgroup in G, that is $m_{i1} = [G : H_i]$.*
4. *The number c_{ij} of subgroups of H_i which are conjugate in G to H_j is equal to*

$$c_{ij} = \frac{m_{ij} \cdot m_{j1}}{m_{i1} \cdot m_{jj}} = |H_i| \cdot m_{ij} \cdot |N_G(H_j)|^{-1}.$$

We now give some examples which will be used later in the study of Gassmann equivalence.

Example 4.3.3. Let $G = S_3$ and $\mathcal{L}(G) = \{C_1, C_2, C_3, S_3\}$. The table of marks is:

$$\begin{bmatrix} 6 & & & \\ 3 & 1 & & \\ 2 & 0 & 2 & \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Example 4.3.4 (42, Example 3.5.6)**.** Let $G = A_5$ and $\mathcal{L}(G) = \{C_1, C_2, C_3, V_4, C_5, S_3, D_{10}, A_4, A_5\}$. The table of marks is:

$$\begin{bmatrix} 60 & & & & & & & & \\ 30 & 2 & & & & & & & \\ 20 & 0 & 2 & & & & & & \\ 15 & 3 & 0 & 3 & & & & & \\ 12 & 0 & 0 & 0 & 2 & & & & \\ 10 & 2 & 1 & 0 & 0 & 1 & & & \\ 6 & 2 & 0 & 0 & 1 & 0 & 1 & & \\ 5 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Chapter 5

Gassmann Equivalent G-sets

In this chapter we study the notion of Gassmann equivalence. The structure is based on [60] but instead of Gassmann triples we consider Gassmann equivalent G-sets. We start with some definitions and in the following sections we treat various cases of Gassmann equivalence for different coefficients. We close the chapter with some questions for further research.

§ 5.1 Gassmann equivalent G-sets

Let G be a finite group and X, Y finite G-sets. When X, Y are isomorphic, they give rise to isomorphic permutation representations. However the converse is not true and it is possible for non-isomorphic G-sets to have equivalent permutation representations. This gives rise to the following definition:

Definition 5.1.1 (Gassmann equivalent G-sets)**.** Let R be a commutative ring, G a finite group and X, Y finite G-sets. Then X, Y are *R-Gassmann equivalent* if and only if $R[X]$ and $R[Y]$ are isomorphic as $R[G]$-modules.

We will focus on the following cases: a) $R = \mathbb{Q}$ where we will call X and Y *rationally Gassmann equivalent* b) $R = \mathbb{Z}_p$ (field with p elements) for some prime number p, where we will call X and Y *locally Gassmann equivalent* c) $R = \mathbb{Z}_p$ for all prime numbers p, where we will call X and Y *locally integrally Gassmann equivalent* d) $R = \mathbb{Z}$ where we will call X and Y *integrally Gassmann equivalent.*

Remark 5.1.2. As we have seen, when X and Y are transitive G-sets we can write them as $X \cong G/H$ and $Y \cong G/K$ for some subgroups H, K of G. In this sense, our definition is inspired by the classical examples of Gassmann triples (see [23]). According to our definition these were examples of rationally Gassmann equivalent transitive G-sets. Examples of rationally Gassmann equivalent intransitive G-sets have been studied in [43]. The terminology for cases a)-d) is coming from [60] where the same topic is studied for transitive G-sets.

In each of the cases a) - d) we want to answer the following questions:

- **Question 1** Do examples of R-Gassmann equivalent G-sets exist? Do such examples exist for both transitive and intransitive G-sets?
- **Question 2** Is there an easy computational way to construct such examples?
- **Question 3** Which classes of groups have such examples? What is the order of the smallest group for which such examples exist?
- **Question 4** Is there an explicit description of all R-Gassmann equivalent G-sets for a given group G?

§ 5.2 Rational Gassmann equivalence

The following proposition gives equivalent conditions for G-sets X and Y to be rationally Gassmann equivalent. It is the analogue of (Sutherland [60], Proposition 2.6) and works for either transitive or intransitive G-sets.

Proposition 5.2.1. *Let G be a finite group and X, Y finite G-sets. The following are equivalent:*

1. $\mathbb{Q}[X] \cong \mathbb{Q}[Y]$ *as* $\mathbb{Q}[G]$ *modules;*
2. $\chi_{\mathbb{Q}[X]}(K) = \chi_{\mathbb{Q}[Y]}(K)$ *for every cyclic subgroup K of G;*
3. $X \cong \bigsqcup G/H_i$ *and* $Y \cong \bigsqcup G/K_i$, *where* $H_i, K_i \leq G$ *satisfy*
$$\sum_i \frac{|g^c \cap H_i|}{|H_i|} = \sum_i \frac{|g^c \cap K_i|}{|K_i|}, \forall g \in G. \tag{5.1}$$
4. *Let* $M \in Hom_{\mathbb{Z}[G]}(\mathbb{Z}[X], \mathbb{Z}[Y])$. *Then, there is a choice of the entries of M that make it invertible.*

Proof. (1) clearly implies (2) because equivalent representations have the same character and we saw in Chapter 4 that for cyclic subgroups $K = \langle g \rangle$, $\chi_{R[X]}(K) = \chi_{R[X]}(g)$. Proposition 4.2.2 shows that (2) is equivalent to (3). Clearing the denominators in $Hom_{\mathbb{Q}[G]}(\mathbb{Q}[X], \mathbb{Q}[Y])$ gives the equivalence of (1) and (4). □

Using this definition and the table of marks it is easy to compute examples of rationally Gassmann equivalent G-sets for a given group G. One can start with the table of marks M of G. By deleting rows of the transpose of the table of marks M^t that correspond to non-cyclic subgroups of G we get a new matrix A. Then the nullspace of A gives a basis of rationally Gassmann equivalent G-sets for the group G. This is done using GAP in [43] and we give a MAGMA implementation of the same program in the appendix.

The next theorem characterizes the class of subgroups that have rationally Gassmann equivalent, non-isomorphic G-sets.

Theorem 5.2.2. *Let G be a finite group. Then a pair of non-isomorphic, rationally Gassmann equivalent G-sets X and Y exists if and only if G is non-cyclic. The number of such pairs is equal to the number of conjugacy classes of non-cyclic subgroups of G.*

Proof. We have seen that the table of marks is a lower triangular matrix with non-zero diagonal entries. Also, all subgroups of a cyclic group are cyclic. From the previous discussion and Proposition 5.2.1 it is clear that we will only remove rows from M^t when G is non-cyclic and the nullspace of a A will be non-trivial when the number of rows is less than the number of columns. □

Now we are going to look at some examples.

Example 5.2.3 (43, Section 1.1)**.** Let $G = \{e, a, b, ab\}$ be the non-cyclic group of size four ($G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$). Consider the following subgroups of G: $H_1 = \{e, a\}, H_2 = \{e, b\}, H_3 = \{e, ab\}$. Then the G-sets $X \cong G/H_1 \sqcup G/H_2 \sqcup G/H_3$ and $Y \cong G/e \sqcup G/G \sqcup G/G$ are rationally Gassmann equivalent.

It follows from Theorem 5.2.2 that this is the the group of smallest order for which rationally Gassmann equivalent G-sets exist. Another well known example that we will use for decompositions of Jacobians in Chapter 8 is the following:

Example 5.2.4. Let G be the symmetric group on 3 letters S_3. It has four conjugacy classes of subgroups: $H_1 = \{e\}, H_2 = \{e, (12)\}, H_3 = \{e, (123)\}$ and G. The G-sets $X \cong G/H_2 \sqcup G/H_2 \sqcup G/H_3$ and $Y \cong G/\{e\} \sqcup G/G \sqcup G/G$ are rationally Gassmann equivalent.

Now we are going to present some more examples for general classes of groups. They have been discovered in [43].

Example 5.2.5 (43, Section 4.2)**.** Let $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$ where p is a prime number ($\mathbb{Z}_p$ is the cyclic group of order p). G has $p+1$ subgroups of size (and index) p denoted by $H_i, 1 \leq i \leq p+1$. The G-sets $X = \sqcup_{1 \leq i \leq p+1} G/H_i$ and $Y = G/G \sqcup ...G/G \sqcup G/\{e\}$ (G/G appears p times in Y) are rationally Gassmann equivalent.

Example 5.2.6 (43, Section 4.3)**.** Let $G \cong \mathbb{Z}_p \rtimes \mathbb{Z}_q$ where $p \neq q$ are prime numbers. G has one subgroup Q of size q and q subgroups $P_1, ..., P_q$ of size p which are all conjugate. The G-sets $X = G/P \sqcup ... \sqcup G/P \sqcup G/P$ (G/P appears p times) and $Y = G/G \sqcup ...G/G \sqcup G/\{e\}$ (G/G appears p times in Y) are rationally Gassmann equivalent.

Example 5.2.7 (43, Section 4.3.1)**.** A special case of the previous example (with $p = 2$ and q an odd prime) are the dihedral groups $D_q = \langle \sigma, \tau | \sigma^q, \tau^2, (\sigma\tau)^2 \rangle$ of order $2q$. Then the rationally Gassmann equivalent G-sets that we obtained above are $X = D_q/\langle\tau\rangle \sqcup D_q/\langle\tau\rangle \sqcup D_q/\langle\sigma\rangle$ and $Y = D_q/D_q \sqcup D_q/D_q \sqcup D_q/\{e\}$.

We will now show that in the case of the group algebra $\mathbb{Q}[G]$, rational Gassmann equivalence coincides with the notion of character equivalence introduced by E. Kani in [36]. First we give the definition of character equivalence.

Definition 5.2.8 (Character equivalence,36)**.** Let A be a finite dimensional algebra. Then $a, a' \in A$ are *character equivalent* if and only if:

$$\chi_\rho(a) = \chi_\rho(a')$$

for every representation ρ of A.

Kani applies this in the case where $A = \mathbb{Q}[G]$ and the elements of A are idempotents of the form $\epsilon_H = \frac{1}{|H|} \sum_{h \in H} h \in \mathbb{Q}[G]$. For a G-set $X = \bigsqcup G/H_i$, we define $\epsilon_X := \sum_i \epsilon_{H_i}$. The following result shows that rational Gassmann equivalence coincides with character equivalence in $\mathbb{Q}[G]$. The ingredients for the proof are already there in [36]. What this equivalence shows is that the plethora of examples of decompositions of Jacobians derived using Kani's theorems in [36] can also be derived using rational Gassmann equivalence. We will study this in more detail in Chapter 8.

Theorem 5.2.9 (Equivalence of Kani's character equivalence and rational Gassmann equivalence)**.** *Let G be a finite group and consider the finite dimensional group algebra $\mathbb{Q}[G]$. Then X, Y are rational Gassmann equivalent G-sets if and only if ϵ_X, ϵ_Y are character equivalent.*

Proof. Assume that X, Y can be written as $X = \bigsqcup G/H_i, Y = \bigsqcup G/K_i$. The first thing is to notice that Gassmann equivalence can be written as the following character relation:

$$\sum \chi_{1_{H_i}^G} = \sum \chi_{1_{K_i}^G}. \tag{5.2}$$

Here $\chi_{1_H^G}$ denotes the induced character from the trivial representation on H. If $g \in G$ then $\chi_{1_H^G}(g) = \frac{|C_G(g)||g^G \bigcap H|}{|H|}$, so (5.2) is equivalent to:

$$\sum \frac{|g^G \bigcap H_i|}{|H_i|} = \sum \frac{|g^G \bigcap K_i|}{|K_i|}$$

for every $g \in G$. Now let χ denote any character in the ring of virtual characters of $\mathbb{Q}[G]$. Using Frobenius reciprocity we have that:

$$\chi(\epsilon_X) = \sum \chi(\epsilon_{H_i}) = \sum \langle \chi{\downarrow}_{H_i}^G, 1_{H_i} \rangle_{H_i} = \sum \langle \chi, \chi_{1_{H_i}^G} \rangle_G = \langle \chi, \sum \chi_{1_{H_i}^G} \rangle_G$$

and since $\langle , \rangle_G$ is non-degenerate this completes the proof. □

Finally, regarding Question 3, explicit description of rationally Gassmann equivalent G-sets is given in [5].

§ 5.3 p-Local Gassmann equivalence

For this type of Gassmann equivalence it is useful to look at the class of p-hypoelementary subgroups. The following definition is standard.

Definition 5.3.1. Let p be a prime. A finite group G is said to be p-*hypoelementary* if the quotient of G by $O_p(G)$, the intersection of its p-Sylow subgroups, is cyclic. $O_p(G)$ is a normal p-subgroup of G and G can be written as $O_p(G) \rtimes C_n, (p, n) = 1$.

As in [60], we define

$$d(X, Y) := gcd\{detM : M \in Hom_{\mathbb{Z}[G]}(\mathbb{Z}[X], \mathbb{Z}[Y])\}.$$

The following proposition is the analogue of Proposition 5.2.1 for p-Local Gassmann equivalence. It is just a modified version of (Sutherland [60], Proposition 3.1) that works for both transitive and intransitive G-sets.

Proposition 5.3.2. *Let G be a finite group and X, Y G-sets. The following are equivalent:*

1. $\mathbb{Z}_p[X] \cong \mathbb{Z}_p[Y]$
2. $\chi_{\mathbb{Z}_p[X]}(K) = \chi_{\mathbb{Z}_p[Y]}(K)$ *for every p-hypoelementary subgroup K of G*
3. $p \nmid d(X, Y)$.

As before, it is easy to use the table of marks of G to compute p-locally Gassmann equivalent G-sets. We start with the table of marks M of G and delete rows of M^t that correspond to subgroups of G that are not p-hypoelementary. This way we create a new matrix A and we just have to compute the nullspace of A. This gives the following obvious result:

Theorem 5.3.3. *Let G be a finite group. If G is not p-hypoelementary there exists a pair of non-isomorphic, locally Gassmann equivalent G-sets. The number of such pairs is equal to the number of conjugacy classes of subgroups that are not p-hypoelementary.*

It is possible to explicitely describe examples of p-Local Gassmann equivalent G-sets and this has been done in [51].

§ 5.4 Locally Integral Gassmann equivalence

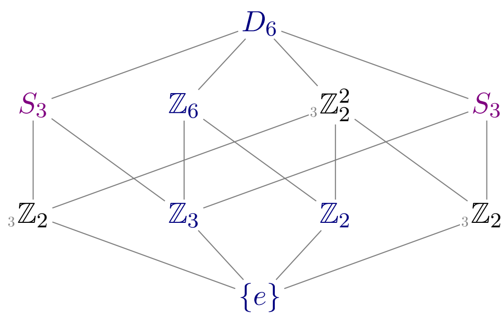
Definition 5.4.1. A finite group G is said to be *hypoelementary* if it is p-hypoelementary for some prime number p.

Definition 5.4.2. Let G be a finite group and X, Y G-sets. If $\mathbb{Z}_p[X] \cong \mathbb{Z}_p[Y]$ for every prime p then X and Y are locally integrally equivalent.

Finding examples of locally integrally Gassmann equivalent G-sets computationally can be done in the same way as before. Starting with the table of marks M we delete rows of M^t that correspond to subgroups of G which are not hypoelementary. This way we get a new matrix A and we compute the nullspace of this matrix. We get the following obvious result:

Theorem 5.4.3. *Let G be a finite group. If G is not hypoelementary there exists a pair of non-isomorphic, locally integrally Gassmann equivalent G-sets.*

Example 5.4.4. The smallest non-hypoelementary group is the dihedral group D_6 of order 12. Using the code in the appendix it is easy to find locally integrally Gasmann equivalent G-sets for this group. The lattice of conjugacy classes of subgroups of D_6 is given below:



Let $X = D_6/D_6 \sqcup D_6/D_6 \sqcup D_6/\mathbb{Z}_3 \sqcup D_6/\mathbb{Z}_2 \sqcup D_6/\mathbb{Z}_2 \sqcup D_6/\mathbb{Z}_2$ and $Y = D_6/S_3 \sqcup D_6/S_3 \sqcup D_6/\mathbb{Z}_6 \sqcup D_6/\mathbb{Z}_2^2 \sqcup D_6/\mathbb{Z}_2^2 \sqcup D_6/\{e\}$. Then X, Y are locally integrallly Gassmann equivalent.

§ 5.5 Integral Gassmann equivalence

The following proposition gives equivalent conditions for integral Gassmann equivalence.

Proposition 5.5.1 (Sutherland [60], Proposition 3.2, Remark 3.3)**.** *Let G be a finite group and X, Y G-sets. The following are equivalent:*

1. $\mathbb{Z}[X] \cong \mathbb{Z}[Y]$
2. *There exists some $M \in Hom_{\mathbb{Z}[G]}(\mathbb{Z}[X], \mathbb{Z}[Y])$ such that $detM = \pm 1$.*

Example 5.5.2 (L.Scott, 52)**.** Let $G = PSL_2(\mathbb{F}_{29})$. G contains two non-conjugate subgroups H_1 and H_2 isomorphic to A_5. It is easy to check that the G-sets $X = G/H_1$ and $Y = G/H_2$ are locally integrally Gassmann equivalent. In order to show that they are integrally Gassmann equivalent we can use Proposition 5.5.1. Using the function OrbitMatrix(G, H_1, H_2) provided in the appendix, a homomorphism from $\mathbb{Z}[X]$ to $\mathbb{Z}[Y]$ can be represented by a 203 x 203 matrix with 8 variables $x_1, ..., x_8 \in \mathbb{Z}$ corresponding to the decomposition of G into 8 double cosets $H_1 g H_2$. Using the function DoubleCosetUnions(G, H_1, H_2), each of these cosets consists of 5,6,10,12,20,30,60,60 right cosets of H_1 respectively. We want to find an assignment of the variables that makes the determinant of this matrix equal to ± 1. In order to guess the right assignment, one can notice that any isomorphism of $\mathbb{Z}[X]$ to $\mathbb{Z}[Y]$ will carry the augmentation map $\mathbb{Z}[X] \to \mathbb{Z}$ into the augmentation map $\mathbb{Z}[Y] \to \mathbb{Z}$ or its negative. This means that we want an assignment such that the following equation holds:

$$5x_1 + 6x_2 + 10x_3 + 12x_4 + 20x_5 + 30x_6 + 60x_7 + 60x_8 = \pm 1. \tag{5.3}$$

An obvious choice would be to set x_1 equal to 1, x_2 equal to -1 and all the other variables equal to 0 and we can see in the appendix 9.3 that this choice actually works.

Examples of intransitive rational, p-local and locally integral Gassmann equivalent G-sets are more abundant compared to transitive ones and one can in general find them in groups of smaller order. As we will see in the next few results, the situation remains the same in the case of integral equivalence. However, as the verification of integral equivalence involves guesswork in the assignment of the variables, it is much harder to actually find explicit examples.

Proposition 5.5.3 (48 Corollary 8.9, 28)**.** *Let Λ be an R-order and M, N a pair of Λ-lattices. Then $N \in \Gamma(M)$ if and only if $M^{\bigoplus k} \cong N^{\bigoplus k}$ for some positive integer k.*

In particular, let $\Lambda = \mathbb{Z}[G]$ and X, Y be locally integral G-sets. Then $\mathbb{Z}[X], \mathbb{Z}[Y]$ are in the same genus and by adding k disjoint copies of X, Y respectively we have that $\mathbb{Z}[X^{\bigoplus k}]$ and $\mathbb{Z}[Y^{\bigoplus k}]$ are integrally Gassmann equivalent.

Proposition 5.5.4. *Let G be a finite group. If G is not hypoelementary then there exists a pair of non-isomorphic, integrally Gassmann equivalent G-sets.*

Proof. The proof is immediate from the above discussion and theorem 5.4.3. $\square$

While existence of integrally Gassmann equivalent intransitive G-sets is guaranteed by the previous proposition, there are still no known explicit examples. In what follows we are going to discuss some computational ways that could be used to search for such examples and the difficulties that are involved in this search.

A strategy to find examples of integral Gassmann equivalence would be to find locally integral Gassmann equivalent G-sets, compute the matrix M of Proposition 5.5.1 using the code in the appendix and try to guess values of the variables that make the determinant of this matrix equal to 1, as we did in Example 5.5.2. In cases where we can compute the determinant symbolically, we can actually solve systems of equations to see if there are values of the variables that accomplish this. This method has been used in ([60], Sections 4.3 and 4.4) for examples of transitive locally integral Gassmann equivalent G-sets. It turns out that these examples are not integrally equivalent.

Example 5.5.5. Here we revisit Example 5.2.4. We computed the corresponding matrix in 4.2.5. It is possible to compute the determinant of this matrix and we get:

$$-9(x_2x_6 - x_3x_6 - x_1x_7 + x_3x_7 + x_1x_8 - x_2x_8)^2(x_{11} - x_{12})(-3x_5x_9x_{11} + 3x_4x_{10}x_{11} - 3x_5x_9x_{12} + 3x_4x_{10}x_{12} + 2x_5x_6x_{13} + 2x_5x_7x_{13} + 2x_5x_8x_{13} - 2x_1x_{10}x_{13} - 2x_2x_{10}x_{13} - 2x_3x_{10}x_{13} - 2x_4x_6x_{14} - 2x_4x_7x_{14} - 2x_4x_8x_{14} + 2x_1x_9x_{14} + 2x_2x_9x_{14} + 2x_3x_9x_{14}).$$

One can check that there are choices of the variables that make this determinant non-zero. However, as expected, the factor of 9 means that the two G-sets are not locally integrally Gassmann equivalent.

The challenge with matrices coming from intransitive locally integral Gassmann equivalent G-sets is that the number of variables grows and symbolic computation of the determinant is much more difficult. One needs to find examples where the number of variables of the matrix is small and use random substitutions of some of the variables. The table in the next page summarizes the results of some calculations regarding the matrix size and number of variables in matrices representing homomorphisms between locally integral Gassmann equivalent G-sets.

In the case of the intransitive example in D_{12} we get a 24×24 matrix with 68 variables. Unfortunately we were not able to compute its determinant symbolically or find choices of the variables that make this determinant equal to 1. Another good source of groups where one could try to find explicit intransitive integrally Gassmann equivalent G-sets are the groups $PSL(2, p)$.

We close this chapter with some questions for further research:

Question A: Are the locally integral intransitive G-sets of Example 5.4.4 integrally Gassmann equivalent?

Question B: Can we find explicit intransitive examples of integrally Gassmann equivalent G-sets in the groups $PSL(2, p)$?

Group	Matrix Size	Number of variables
D_{12}	24	68
D_{24}	48	146
	24	68
	24	38
S_4	40	90
S_5	192	372
	123	162
	222	596
A_5	81	119
$Z_6 \times Z_6$	144	1152
$PSL(2,7)$	212	299
	106	86
	70	60
$PSL(2,11)$	983	1600
	497	430
	309	182
	121	32
$PSL(2,13)$	3188	9806
	1366	1746
$PSL(2,17)$	3470	5146
	1735	1339
	1632	1256
$PSL(2,19)$	5657	2999
	2148	1423
	2661	2175
	399	62
$PSL(2,23)$	25582	116198
	7384	9041
	8190	11325
	16880	48130

Chapter 6

Hodge structures and constructible sheaves

In this section we present some background material on Hodge structures and constructible sheaves that will be necessary for the proof of our results in Chapter 7. In the first section we focus on Grothendieck's formalism of six functors, local systems, Thom-Whitney stratifications and constructible sheaves. Our main references are [19] and [2]. In the second section we review basic definitions on pure and mixed Hodge structures, variations of pure and mixed Hodge structures and explain some connections with constructible sheaves. Our main references are [13] and [57].

§ 6.1 Six functors and Constructible Sheaves

6.1.1 The category Sh(M,R) and operations on sheaves

Let M be a topological space and R a commutative ring. By $Sh(M,R)$ we denote the category of sheaves of $R-$modules on M. Let $f : M \to N$ be a continuous map. In what follows we assume that our topological spaces are locally compact and the maps proper.

We start with a very basic example of a sheaf, which however lies behind many of the cases we treat in Chapter 7.

Example 6.1.1. Let S be an R-module. The *constant sheaf* S on M, denoted by $\underline{S}_M$ is defined by:

$$\underline{S}(U) = \{ \text{ locally constant functions } s : U \to S\}$$

$$res_{V,U}(s) = (s|_U : U \to S).$$

We now define various operations on sheaves (see [2], Chapter 1):

1. **Pull-back or inverse image functor f^*:** Let $\mathcal{F} \in Sh(N, R)$. The pull-back of $\mathcal{F}$ denoted by $f^*\mathcal{F}$ is the sheafification of the presheaf

$$U \mapsto \varinjlim_{\substack{V \subset N \text{ open} \\ V \supset f(U)}} \mathcal{F}(V).$$

2. **Push-forward or direct image functor f_*:** Let $\mathcal{F} \in Sh(M, R)$. Its push-forward is the sheaf $f_*\mathcal{F} \in Sh(N, R)$ given by

$$f_*(V) = \mathcal{F}(f^{-1}(V)).$$

for open $V \subset N$. When N is a point we denote $f_*\mathcal{F}$ by $\Gamma(M, \mathcal{F})$ and call it the set of global sections of $\mathcal{F}$ on M.

3. **Internal Hom functor $\mathcal{H}$om:** Let $\mathcal{F}, \mathcal{G} \in Sh(M, R)$. We define $\mathcal{H}om(\mathcal{F}, \mathcal{G}) \in Sh(M, R)$ as the sheaf defined by:

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = Hom(\mathcal{F}|_U, \mathcal{G}|_U).$$

4. **Tensor product functor $- \otimes -$:** Let $\mathcal{F}, \mathcal{G} \in Sh(M, R)$. We define $\mathcal{F} \otimes \mathcal{G} \in Sh(M, R)$ as the sheaf associated to the following presheaf:

$$U \mapsto \mathcal{F}(U) \otimes \mathcal{G}(U).$$

5. **Proper push-forwrd functor $f_!$:** Let $\mathcal{F} \in Sh(M, R)$. Its proper push-forward is the sheaf $f_!\mathcal{F} \in Sh(N, R)$ given by

$$f_!(U) = \{s \in \mathcal{F}(f^{-1}(U)) : f|_{supps} : supps \to U \text{ is proper }\}.$$

When N is a point we denote $f_!\mathcal{F}$ by $\Gamma_c(M, \mathcal{F})$ and call it the set of global sections with compact support of $\mathcal{F}$ on M.

The following proposition collects various exactness properties of the functors we have just defined (see [2], Chapter 1).

Proposition 6.1.2. *Let $f : M \to N$ be a continuous map of topological spaces. Then:*

1. $f_* : Sh(M,R) \to Sh(N,R)$ *is left exact. This also implies that* Γ *is left exact. If we also assume that* f *is a closed embedding then* f_* *is exact.*

2. $f^* : Sh(N,R) \to Sh(M,R)$ *is exact.*

3. $f_! : Sh(M,R) \to Sh(N,R)$ *is left exact (here we also assume that* M, N *are locally compact). If we also assume that* f *is a locally closed embedding then* $f_!$ *is exact.*

4. $\otimes : Sh(M,R) \times Sh(M,R) \to Sh(M,R)$ *is right exact (exact when* R *is a field).*

5. $\mathcal{H}om : Sh(M,R)^{op} \times Sh(M,R) \to Sh(M,R)$ *is left exact.*

Constant sheaves were introduced in Example 6.1.1. The following definition introduces a generalization of this notion.

Definition 6.1.3 (2, Definition 1.7.1)**.** A *local system* is a sheaf $\mathcal{L}$ on a space M such that there exists an open covering $(U_i)_{i \in I}$ of M so that $\mathcal{L}|_{U_i}$ is a constant sheaf for each $i \in I$. We denote by $Loc(M,R)$ the full subcategory of $Sh(M,R)$ consisting of local systems.

We will mostly be interested in local systems that have finitely generated modules as fibers, these are called local systems of *finite type*.

We also state two propositions that we will use in the proof of Theorem 7.2.4.

Proposition 6.1.4 (2, Section 1.7)**.** *Let* M *be a connected topological space and* x_0 *a basepoint on* M*. Then the category* $Loc(M,R)$ *of local systems on* M *is equivalent with the abelian category* $Rep(\pi_1(M,x_0),R)$ *of representations of* $\pi_1(M,x_0)$ *on* R*-modules over a commutative ring.*

Proposition 6.1.5 (2, Lemma 2.1.22)**.** *Let* M *be a smooth, connected variety,* $U \subset M$ *a Zariski open subset and* $x_0 \in U$ *any point. Then the natural map* $\pi_1(U,x_0) \to \pi_1(M,x_0)$ *is surjective.*

6.1.2 The derived category $\mathcal{D}(Sh(M,R))$ and the functor $f^!$

The category $Sh(M,R)$ is abelian and has enough injectives. As a result, it is possible to derive functors which are left exact on $Sh(M,R)$. We can thus define the derived category $\mathcal{D}(Sh(M,R))$. By $\mathcal{D}^b(Sh(M,R))$, $\mathcal{D}^+(Sh(M,R))$, $\mathcal{D}^-(Sh(M,R))$ we denote the derived categories of complexes of sheaves bounded, bounded below and bounded above respectively.

So, if $f : M \to N$ is a continuous map between topological spaces we have:

- $Rf_* : \mathcal{D}^+(Sh(M,R)) \to \mathcal{D}^+(Sh(N,R))$
- $Rf_! : \mathcal{D}^+(Sh(M,R)) \to \mathcal{D}^+(Sh(N,R))$ (here M, N are locally compact).
- $R\mathcal{H}om : \mathcal{D}^-(Sh(M,R))^{op} \times \mathcal{D}^+(Sh(M,R)) \to \mathcal{D}^+(Sh(M,R))$
- * and $- \otimes -$ which are exact so they already define functors on the derived category of sheaves.

The upshot of working with the derived category is that now we can obtain an adjucntion formula for $Rf_!$.

Definition 6.1.6 (2, Theorem 1.5.4)**.** Let $f : M \to N$ be a continuous map between locally compact topological spaces and R a Noetherian ring with finite global dimension. Assume that $f_!$ has finite cohomological dimension. Then there exists a triangulated functor:

$$f^! : \mathcal{D}^+(Sh(N,R)) \to \mathcal{D}^+(Sh(M,R))$$

such that if $\mathcal{F} \in \mathcal{D}^-(Sh(M,R))$ and $\mathcal{G} \in \mathcal{D}^+(Sh(N,R))$, then we have the following natural transformations:

$$R\mathcal{H}om(Rf_!\mathcal{F}, \mathcal{G}) \cong Rf_* R\mathcal{H}om(\mathcal{F}, f^!\mathcal{G}),$$

$$RHom(Rf_!\mathcal{F}, \mathcal{G}) \cong Rf_* RHom(\mathcal{F}, f^!\mathcal{G}),$$

$$Hom(Rf_!\mathcal{F}, \mathcal{G}) \cong Rf_* Hom(\mathcal{F}, f^!\mathcal{G}).$$

We have now defined **Grothendieck's six operations**. The six functor formalism is behind many properties of cohomology.

For example (see [2], Definition 1.1.17 and Theorem 1.1.18), let M be a topological space and $\mathcal{F} \in \mathcal{D}^+(Sh(M,R))$. The k-th hypercohomology of $\mathcal{F}$, denoted by $\mathbb{H}^k(M,\mathcal{F})$, is the R-module given by

$$\mathbb{H}^k(M,\mathcal{F}) := \mathcal{H}^k(R\Gamma(\mathcal{F})).$$

Similary, the k-th hypercohomology with compact support of $\mathcal{F}$, denoted by $\mathbb{H}^k_c(M,\mathcal{F})$, is the R-module given by

$$\mathbb{H}^k_c(M,\mathcal{F}) := \mathcal{H}^k(R\Gamma_c(\mathcal{F})).$$

In particular, when M is locally contractible and hereditary paracompact ([10], page 21), and R the constant sheaf on M, we have the following natural isomorphisms:

$$\mathbb{H}^k(M,R) \cong H^k_{sing}(M,R)$$

and

$$\mathbb{H}^k_c(M,R) \cong H^k_{sing,c}(M,R).$$

We now consider some useful properties of open-closed decompositions and obtain the sequence of relative cohomology that will be used in Chapter 7. Let M be a topological space with a decomposition $M = U \sqcup Z$ into U open and Z closed. Let $i : Z \hookrightarrow M$ be a closed embedding and let $j : U \hookrightarrow M$ the complementary open embedding. Then (see [2], Chapter 1):

1. We have that $i^* \circ j_! = 0, i^! \circ j_* = 0$ and $j^* \circ i_* = 0$.
2. If $\mathcal{F} \in \mathcal{D}^+(Sh(M,R))$ then we have a natural distinguished triangle:
$$j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to [+1].$$
3. If $\mathcal{F} \in \mathcal{D}^+(Sh(M,R))$ then we have a natural distinguished triangle:
$$i_* i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F} \to [+1].$$
4. Define the **relative cohomology of a closed pair** (M,Z) by
$$\mathbb{H}^k(M,Z,R) = \mathbb{H}^k(M, j_! \underline{R}_{M \setminus Z}).$$
Then there is a natural long exact sequence:
$$... \to \mathbb{H}^k(M,Z,R) \to \mathbb{H}^k(M,R) \to \mathbb{H}^k(Z,R) \to \mathbb{H}^{k+1}(M,Z,R) \to ...$$

Finally we state two results that will play an important role in the proof of Theorem 7.2.4.

Proposition 6.1.7 (Exceptional direct image on a closed subset, [2]**.** *, Chapter 1] Let $i : Z \to X$ be the inclusion of an orientable submanifold into another orientable manifold, d be the real codimension of Z in X and assume that $\mathcal{F}^\cdot \in D^b_c(X)$ has locally constant cohomology on X. Then, $i^! \mathcal{F}^\cdot$ has locally constant cohomology on Z and:*

$$i^! \mathcal{F}^\cdot \cong i^* \mathcal{F}^\cdot [-d].$$

Proposition 6.1.8 (Derived proper base change, 2, Chapter 1)**.** *Consider the following cartesian diagram:*

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{f}} & \tilde{N} \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow{f} & N \end{array}$$

Then, there exist natural isomorphisms:

$$R\tilde{f}_!\tilde{g}^* \cong g^* Rf_!$$

and

$$R\tilde{f}_*\tilde{g}^! \cong g^! R\tilde{f}_*.$$

6.1.3 Stratifications and constructible sheaves

Let $f : M \to N$ be a morphism of complex algebraic varieties and $\mathcal{F}$ a local system on M. Without introducing extra assumptions, the sheaves $R^k f_* \mathcal{F}$ on N are not necessarily locally constant. We will now introduce the notions of stratifications and constructibility and show that the six functors preserve constructibility (see [2], Chapter 2).

Definition 6.1.9. Let M be a variety. A *stratification* of M is a finite collection $(M_i)_{i\in I}$ of disjoint, smooth, connected, locally closed subvarieties such that $M = \cup_{i\in I} M_i$ and such that for any two i, j we have that $\overline{M}_i \cap M_j$ is either empty or M_j. The subvarieties M_i are called the *strata* of the stratification. The set I carries a natural partial order called the closure partial order, given by $j \leq i$ if $M_j \subset \overline{M_i}$.

Example 6.1.10. Let G be a connected algebraic group and M a variety with G-action. Suppose that the action has finitely many orbits. Then the G-orbits constitute a stratification.

Example 6.1.11. Let M be a smooth variety and $Z \subset M$ a divisor with simple normal crossings. Then we can use the irreducible components of Z to define a stratification of M, called the normal crossings stratification.

Definition 6.1.12. Let M be a variety and $(M_i)_{i\in I}$ a stratification. A sheaf $\mathcal{F} \in Sh(M, R)$ is said to be *constructible with respect to* I if each $\mathcal{F}_{M_i}$ is a local system. A sheaf $\mathcal{F}$ is said to be *constructible* if there exists a stratification with respect to which it is constructible. The full subcategory of constructible sheaves is denoted by $Sh_c(M, R)$. An object in $\mathcal{D}^b(Sh(M, R))$ is said to be constructible if there is a stratification with respect to which each cohomology sheaf $H^k(\mathcal{F})$ is constructible. The full subcategory of $\mathcal{D}^b(Sh(M, R))$ consisting of constructible complexes is denoted by $\mathcal{D}^b_c(Sh(M, R))$.

The next result shows that the six functors preserve constructibility.

Proposition 6.1.13 (2, Section 2.7)**.** *Let $f : M \to N$ be a morphism of varieties. Let $\mathcal{F} \in \mathcal{D}^b_c(Sh(M, R))$ and $\mathcal{G} \in \mathcal{D}^b_c(Sh(N, R))$. Then:*

1. $f^*\mathcal{G}, f^!\mathcal{G} \in \mathcal{D}^b_c(Sh(M,R))$
2. $Rf_*\mathcal{F}, Rf_!\mathcal{F} \in \mathcal{D}^b_c(Sh(N,R))$ *if* f *is algebraic or* $f|_{supp}\mathcal{F}$ *is proper.*
3. $R\mathcal{H}om(\mathcal{F},\mathcal{F}'), \mathcal{F}\otimes\mathcal{F}' \in \mathcal{D}^b_c(Sh(M,R))$.

6.1.4 Poincare-Verdier duality

We will now define the Poincare-Verdier duality functor $\mathbb{D}$ and explore its various properties. We follow [2], Section 2.8 closely.

Definition 6.1.14 (The (relative) dualizing complex)**.** Let $f: M \to N$ be a continuous map of finite cohomological dimension so that $f^!$ is well defined. We call $\omega_{M/N} := f^!\mathbb{C}_N \in \mathcal{D}^b_c(Sh(N,R))$ the relative dualizing complex on M over N. When N is a single point we set $\omega_M := \omega_{M/N}$ and call it the dualizing complex on M.

Definition 6.1.15 (Poincare-Verdier duality)**.** The **Poincare-Verdier duality functor** $\mathbb{D} = \mathbb{D}_M : \mathcal{D}^b(Sh(M,R)) \to \mathcal{D}^b(Sh(M,R))$ is defined as

$$\mathcal{F} \to R\mathcal{H}om(\mathcal{F},\omega_M).$$

It is a contravariant endofunctor.

Proposition 6.1.16. *Let* $f: M\to N$ *be a continuous map and* M,N *locally compact. Let* $\mathcal{F}\in\mathcal{D}^b(Sh(M,R))$ *and* $\mathcal{G}\in\mathcal{D}^b(Sh(N,R)$. *The functor* $\mathbb{D}$ *satisfies the following properties:*

1. $f^!\mathbb{D} = \mathbb{D}f^*$.
2. $\mathbb{D}Rf_! = Rf_*\mathbb{D}$.

Proposition 6.1.17. *Let* $f: M\to N$ *be a topological submersion with fiber dimension* d*. Then:*

1. $H^k(\omega_{M/N}) = 0$ *if* $k\neq d$ *and* $H^{-d}(\omega_{M/N})$ *is a local system of rank 1. When* N *is a point it is called the orientation sheaf of* M*.*
2. *If* M,N *are orientable manifolds then* $\omega_{M/N}\cong\mathbb{C}_M[d]$*. Thus in particular* $f^*[d]\cong f^!$.

Example 6.1.18 (Poincare duality)**.** Let M be a topological manifold. For $f: M\to\{pt\}$ we apply $\mathbb{D}Rf_!\cong Rf_*\mathbb{D}$ to $\mathbb{C}_M$ and get $\mathbb{D}Rf_!\mathbb{C}_M\cong Rf_*\mathbb{D}\mathbb{C}_M$ or:

$$\mathbb{D}R\Gamma_c(M,\mathbb{C}_M)\cong R\Gamma(M,\omega_M).$$

By taking cohomology we have that $H^k_c(M, \mathbb{C}_M)^\vee \cong H^{-k}(M, \omega_M)$. Moreover, if M is orientable, then $\omega_M \cong \mathbb{C}_M[d]$ where $d = dim M$. Thus we have:

$$H^k_c(M, \mathbb{C}_M) \cong H^{d-k}(M, \mathbb{C}_M)$$

which is Poincare duality on M.

Example 6.1.19 (Alexander duality)**.** Let M be an orientable manifold, $Z \subset M$ a closed subset and $i : Z \to M$ the closed embedding of Z to M. Then we have:

$$R\Gamma_Z \mathbb{D} \cong Rf_* f^! \mathbb{D} \cong Rf_* \mathbb{D} f^* \cong \mathbb{D} Rf_! f^*$$

Applying this to $\mathbb{C}_M \cong \omega_M[-d]$ we have:

$$R\Gamma_Z \mathbb{C}[d] \cong \mathbb{D}\mathbb{C}_Z.$$

Taking cohomology on both sides, we finally get:

$$H^{d-k}_Z(M, \mathbb{C}_M) \cong H^k_c(Z, \mathbb{C}_Z)^\vee$$

which is Alexander duality.

Unfortunately $\mathbb{D}$ is not a duality on the whole $\mathcal{D}^b(Sh(M, R))$. However, we have the following result:

Proposition 6.1.20. *Let M be a complex analytic space. Then the restriction of $\mathbb{D}$ on $\mathcal{D}^b_c(Sh(M, R)$ is a contravariant endofunctor $\mathbb{D} : \mathcal{D}^b_c(Sh(M, R) \to \mathcal{D}^b_c(Sh(M, R)^{op}$. Furthermore it is an involution, hence it satisfies $\mathbb{D} \circ \mathbb{D} = id$.*

§ 6.2 Hodge Structures

In this section we state basic definitions and results on (mixed) Hodge structures and variations of (mixed) Hodge structures that will be used in the proofs of our main results in Chapter 7.

6.2.1 Pure Hodge Structures

We start with the familiar definition of pure Hodge structures inspired by the Hodge Decomposition theorem on Kähler manifolds (Theorem 6.2.4).

Definition 6.2.1 (Pure Hodge structure, 13, Definition 3.1.1)**.** Let $H_{\mathbb{Z}}$ be a finitely generated free abelian group (lattice) and $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ its complexification. A pure Hodge structure of weight $k \in \mathbb{Z}$ on $H_{\mathbb{C}}$ is a direct sum decomposition $H_{\mathbb{C}} = \oplus_{p+q=k} H^{p,q}$ that satisfies $H^{p,q} = \overline{H^{q,p}}$.

One can also speak about rational (real) Hodge structures by replacing the lattice $H_{\mathbb{Z}}$ with a rational (real) vector space. There is also another way to define Hodge structures using filtrations on $H_{\mathbb{C}}$.

This alternative (but equivalent) definition of pure Hodge structures makes it easier to generalize them to mixed Hodge structures.

Definition 6.2.2 (Pure Hodge Structure - alternative definition)**.** A decreasing filtration

$$H_{\mathbb{C}} = F^0 \supset F^1 \supset \ldots \supset F^k \supset \{0\} \tag{6.1}$$

such that $F^p \cap \overline{F^q} = 0$ whenever $p + q = k + 1$ defines a weight k Hodge structure.

The condition $F^p \cap \overline{F^q} = 0$ whenever $p+q = k+1$ is equivalent to $F^p \oplus \overline{F^{k-p+1}} = H_{\mathbb{C}}$. Moreover, the two definitions are equivalent. Given a decomposition $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ we can define a filtration $\{F^p\}$ as $F^p = \oplus_{r \geq p} H^{r,r-p}$. Conversely, given a filtration $\{F^p\}$ we can define a decomposition of $H_{\mathbb{C}}$ by setting $H^{p,q} = F^p \cap \overline{F^q}$.

Definition 6.2.3 (Morphisms of pure Hodge structures)**.** A morphism f of pure Hodge structures of weight k is a homomorphism of abelian groups $f : H_{\mathbb{Z}} \to H'_{\mathbb{Z}}$ such that $f_{\mathbb{C}} : H_{\mathbb{C}} \to H'_{\mathbb{C}}$ is compatible with the decomposition.

Given some pure Hodge structures, one can use classical linear algebra operations to define new ones.

1. Direct Sum: Given two Hodge structures of weight k we can define their direct sum by taking the underlying lattice to be the direct sum of the two lattices and the (p, q) components to be the direct sum of the (p, q) components of each term. The direct sum is also a Hodge structure of weight k.

2. Dual Hodge Structure: Let $(H_{\mathbb{Z}}, H^{p,q})$ be a Hodge structure of weight k. Letting $H^*_{\mathbb{Z}} := Hom(H_{\mathbb{Z}}, \mathbb{Z})$ and $(H^*)^{p,q} := (H^{-p,-q})^*$ defines a new Hodge structure of weight $-k$.

3. Tensor product: Let $(H_{\mathbb{Z}}, H^{p,q}), (H'_{\mathbb{Z}}, H'^{p,q})$ be two Hodge structures of weights k and k'. Let $H''_{\mathbb{Z}} = H_{\mathbb{Z}} \otimes H'_{\mathbb{Z}}$ and $H''^{p,q} = \oplus_{r+r'=p, s+s'=q} H^{r,s} \otimes H'^{r',s'}$. This defines a new Hodge structure of weight $k + k'$.

One of the most important examples of Hodge structures comes from the cohomology of Kähler manifolds.

Theorem 6.2.4 (Hodge Decomposition)**.** *Let M be a compact Kähler manifold. Then, there exists a decomposition*

$$H^k(M, \mathbb{C}) = \oplus_{p+q=k} H^{p,q}(M) \tag{6.2}$$

where $H^{p,q}(M) = H^q(M, \Omega^p_M)$ and $H^{p,q}(M) = \overline{H^{q,p}}(M)$. If we let $H_{\mathbb{Z}} = H^k(M, \mathbb{Z})/torsion$, then the data $(H_{\mathbb{Z}}, H^{p,q}(M))$ defines a pure Hodge structure.

Example 6.2.5 (Tate Hodge structure)**.** Let $H_{\mathbb{Z}} = 2\pi i\mathbb{Z}$ and set $H_{\mathbb{Z}} = H^{-1,-1}$. This defines a pure Hodge structure of weight -2. We will denote it by $\mathbb{Z}(1)$, as it is the unique 1-dimensional pure Hodge structure of weight -2 up to isomorphism.

Example 6.2.6 (Complex torus associated to a Hodge structure, 22, Remark 1)**.** Let $(H_{\mathbb{Z}}, H^{p,q})$ be a Hodge structure of weight $2k-1$ and consider the corresponding Hodge filtration on $H_{\mathbb{C}}$:

$$H_{\mathbb{C}} = F^0 \supset F^1 \supset ... \supset F^{2k-1} \supset \{0\}. \tag{6.3}$$

Then we can define the following complex torus:

$$J^k(H) = \frac{H_{\mathbb{C}}}{F^k H_{\mathbb{C}} \oplus H_{\mathbb{Z}}}. \tag{6.4}$$

Moreover, morphisms of Hodge structures induce morphisms of the corresponding complex tori. In the particular case where we have a Kähler manifold with the Hodge structure coming from the Hodge decomposition, we get the Intermediate Jacobian.

6.2.2 (Variations of) Mixed Hodge Structures

According to Theorem 6.2.4 every smooth projective algebraic variety over the complex numbers has a pure Hodge structure. Deligne showed that it is possible to extend the definition of pure Hodge structures, in a way that allows considering possibly non-smooth or non-singular varieties, by introducing *mixed Hodge structures*. We are now going to give the definition of this term.

Definition 6.2.7 (Mixed Hodge structure, 13, Definition 3.2.15)**.** A *mixed Hodge structure* is defined by the following data:

1. A lattice $H_{\mathbb{Z}}$.

2. An increasing filtration W of $H_{\mathbb{Z}} \otimes \mathbb{Q}$:

$$... \subset W_0 \subset W_1 \subset W_2 \subset ...$$

This is called the weight filtration.

3. A decreasing filtration F of $H_{\mathbb{C}} \otimes \mathbb{C}$:

$$H_{\mathbb{C}} = F^0 \supset F^1 \supset F^2 \supset ...$$

called the Hodge filtration, which defines a pure Hodge structure of weight k on the graded piece $Gr_k^W H_{\mathbb{Q}} = W^k H_{\mathbb{Q}} / W^{k+1} H_{\mathbb{Q}}$.

This definition depends solely on linear algebra data. It has been shown by Deligne that the cohomology of any complex algebraic variety can be endowed with a mixed Hodge structure ([13], Theorem 3.4.1). The construction of this mixed Hodge structure is described in detail in ([13], Section 3.4).

In the next chapter we will be interested in the mixed Hodge structure of a punctured algebraic curve. The construction of this mixed Hodge structure is described in detail in ([13], Example 3.4.9).

Finally, in the next chapter we will need to describe the mixed Hodge structure on the fibers of morphisms of varieties $f : M \to N$ and for this we need the definition of *variations of mixed Hodge structures.*

Definition 6.2.8 (Variations of mixed Hodge structures, 13, Definition 8.1.13)**.** A *variation of mixed Hodge structures* on an analytic manifold M is defined by the following data:

1. A local system $H_{\mathbb{Z}}$.
2. A finite increasing filtration W of $H_{\mathbb{Z}} \otimes \mathbb{Q}$ by sublocal systems of rational vector spaces.
3. A finite decreasing filtration F by locally free analytic subsheaves of $H_{\mathbb{Z}} \otimes \mathcal{O}_M$ whose sections on M satisfy the Griffiths transversality condition with respect to the connection ∇ defined on $H_{\mathbb{Z}} \otimes \mathcal{O}_M$ by the local system $H_{\mathbb{C}} := H_{\mathbb{Z}} \otimes \mathbb{C}$

$$\nabla(\mathcal{F}^p) \subset \Omega^1_M \otimes_{\mathcal{O}_M} \mathcal{F}^{p-1}$$

(for more details on connections and the Griffiths transversality condition see [13] Chapter 7 and Chapter 8, Section 8.1.2).

4. The filtrations W and F define a mixed Hodge structure on each fiber of the bundle $H_{\mathbb{Z}} \otimes \mathcal{O}_M$ at a point n.

Remark 6.2.9 (13, Corollary 8.1.22)**.** We will be using this in the case where we have a morphism of varieties $f : M \to N$ and the constant sheaf $\underline{\mathbb{Z}_M}$ on M. We will also have a stratification on N. We deduce from the above definitions that for each stratum S and for every i the sheaf $(R^i f_* \underline{\mathbb{Z}_M})|_S$ underlies a variation of mixed Hodge structures.

Chapter 7

Main Results

§ 7.1 Overview of previous results

In this section we are going to discuss some results concerning Gassmann triples and Jacobians of curves. We will deal with the following situation: C is going to be an algebraic curve and $G \leq Aut(C)$ a finite group that acts on C, while X, Y will be transitive R-Gassmann equivalent G-sets, where R is going to be either $\mathbb{Z}$ or $\mathbb{Q}$. We have the following diagram with mappings between various quotients of C:

$$\begin{tikzcd} & C \arrow[dl, "p_1"'] \arrow[dd, "f"] \arrow[dr, "p_2"] & \\ C/H_1 \arrow[dr, "f_1"'] & & C/H_2 \arrow[dl, "f_2"] \\ & C/G & \end{tikzcd}$$

When $R = \mathbb{Q}$ we have the following result:

Theorem 7.1.1 (Prasad-Rajan 47, Gordon-Makover-Webb 25)**.** *Let C be a projective algebraic curve over the field of the complex numbers with an action of a finite group $G \leq Aut(C)$. Let $X = G/H_1, Y = G/H_2$ be rationally Gassmann equivalent transitive G-sets. Then the Jacobians of C/H_1 and C/H_2 are isogenous.*

The following proof uses a variation of the methods of [47], [25] and [4] and the language of Hodge structures.

Proof. The pullbacks p_1^*, p_2^* give isomorphisms of vector spaces:

$$H^1(C/H_1, \mathbb{Q}) \cong H^1(C, \mathbb{Q})^{H_1} \tag{7.1}$$

and

$$H^1(C/H_2, \mathbb{Q}) \cong H^1(C, \mathbb{Q})^{H_2}. \tag{7.2}$$

We also have that ([47], Lemma 1):

$$H^1(C, \mathbb{Q})^{H_1} \cong (H^1(C, \mathbb{Q}) \otimes \mathbb{Q}[G/H_1])^G \tag{7.3}$$

and

$$H^1(C, \mathbb{Q})^{H_2} \cong (H^1(C, \mathbb{Q}) \otimes \mathbb{Q}[G/H_2])^G. \tag{7.4}$$

Since X, Y are rationally Gassmann equivalent we have that $\mathbb{Q}[G/H_1] \cong \mathbb{Q}[G/H_2]$. Combining these together we get an isomorphism of vector spaces: $H^k(C/H_1, \mathbb{Q}) \cong H^k(C/H_2, \mathbb{Q})$. As pullbacks are compatible with Hodge structures, this is actually an isomorphism of Hodge structures. Finally, the isomorphism of rational Hodge structures implies that the Jacobians of C/H_1 and C/H_2 are isogenous (this follows from Example 6.2.6). □

Using the exact same method, we can drop the assumption that the G-sets X, Y are transitive. In what follows we will use the notation of *tensor products of G manifolds.* We get the following result:

Theorem 7.1.2. *Let C be a projective algebraic curve over the field of the complex numbers with an action of a finite group $G \leq Aut(C)$. Let X, Y be rationally Gassmann equivalent G-sets. Then the Jacobians of $\frac{C \times X}{G}$ and $\frac{C \times Y}{G}$ are isogenous.*

Proof. Since X, Y are rationally Gassmann equivalent we have that $\mathbb{Q}[X] \cong \mathbb{Q}[Y]$. We have the following isomorphisms of Hodge structures:

$$H^1(\frac{C \times X}{G}, \mathbb{Q}) \cong H^1(C \times X, \mathbb{Q})^G \cong (H^1(C, \mathbb{Q}) \otimes \mathbb{Q}[X])^G \tag{7.5}$$

and similarly

$$H^1(\frac{C \times Y}{G}, \mathbb{Q}) \cong H^1(C \times Y, \mathbb{Q})^G \cong (H^1(C, \mathbb{Q}) \otimes \mathbb{Q}[Y])^G \tag{7.6}$$

Together, these give the desired isomorphism of rational Hodge structures and the proof is complete. □

This method cannot be used if we replace $\mathbb{Q}$ with $\mathbb{Z}$ because we can't identify $H^k(C/G, \mathbb{Z})$ with $H^k(C, \mathbb{Z})^G$. However, using a different method, Prasad showed in [46] the following:

Theorem 7.1.3 (Prasad, 46). *Let C be a projective algebraic curve over a field k with an action of a finite group $G \leq Aut(C)$ and $f : C \to C/G$ the corresponding Galois cover (not necessarily unramified). Let $X = G/H_1, Y = G/H_2$ be integrally Gassmann equivalent transitive G-sets. Then the Jacobians of C/H_1 and C/H_2 are isomorphic over k.*

The proof is based on the following lemma of Shimura.

Lemma 7.1.4 (46, Lemma 2). *The natural homomorphism (constructed in [46], Lemma 1) from $Hom_G(\mathbb{Z}[G/H_1], \mathbb{Z}[G/H_2])$ to $Hom(J(C/H_1), Jac(C/H_2))$ has the property that for any subgroup H_3 of G with corresponding algebraic curve C/H_3 the natural composition of G-homomorphisms:*

$$\mathbb{Z}[G/H_1] \to \mathbb{Z}[G/H_2] \to \mathbb{Z}[G/H_3] \tag{7.7}$$

corresponds to a composition of the corresponding maps on the Jacobian:

$$Jac(C/H_1) \leftarrow Jac(C/H_2) \leftarrow Jac(C/H_3). \tag{7.8}$$

Proof. (of Theorem 7.1.3) Since X, Y are intergrally Gassmann equivalent we have an isomorphism $\mathbb{Z}[G/H_1] \cong \mathbb{Z}[G/H_2]$ and from the previous lemma this gives an isomorphism $Jac(C/H_1) \cong Jac(C/H_2)$. □

In [4], Arapura et al., consider similar questions regarding the cohomology and Hodge structure of Kähler manifolds. Their method of proof allows more flexibility for various coefficients. They prove the following:

Theorem 7.1.5 (4, Theorem 1.4). *Let $X = G/H_1, Y = G/H_2$ be transitive R-Gassmann equivalent G-sets, M a smooth projective algebraic variety over a field k with an action of a finite group $G \leq Aut(M)$ and $f : M \to M/G$ a Galois unramified cover. Then:*

1. *If $R = \mathbb{Q}$ we have an isomorphism of rational Hodge structures: $H^k(M/H_1, \mathbb{Q}) \cong H^k(M/H_2, \mathbb{Q})$.*

2. *If $R = \mathbb{Z}$ we have an isomorphism of integral Hodge structures: $H^k(M/H_1, \mathbb{Z}) \cong H^k(M/H_2, \mathbb{Z})$.*

3. *If $R = \mathbb{Z}_p$ we have an isomorphism of etale cohomology groups: $H^k_{et}(M/H_1, \mathbb{Z}_p) \cong H^k_{et}(M/H_2, \mathbb{Z}_p)$.*

When f is unramified we can drop the assumption that the G-sets are transitive. Using the exact same method of ([4], page 10) we get the following result:

Theorem 7.1.6. *Let X, Y be R-Gassmann equivalent G-sets, M a smooth projective algebraic variety over a field k with an action of a finite group $G \leq Aut(M)$ and $f : M \to M/G$ a Galois unramified cover. Then:*

1. *If $R = \mathbb{Q}$ we have an isomorphism of rational Hodge structures: $H^k(\frac{M\times X}{G}, \mathbb{Q}) \cong H^k(\frac{M\times Y}{G}, \mathbb{Q})$.*
2. *If $R = \mathbb{Z}$ we have an isomorphism of integral Hodge structures: $H^k(\frac{M\times X}{G}, \mathbb{Z}) \cong H^k(\frac{M\times Y}{G}, \mathbb{Z})$.*
3. *If $R = \mathbb{Z}_p$ we have an isomorphism of etale cohomology groups: $H^k_{et}(\frac{M\times X}{G}, \mathbb{Z}_p) \cong H^k_{et}(\frac{M\times Y}{G}, \mathbb{Z}_p)$.*

Proof. Let $R = \mathbb{Q}$ or $\mathbb{Z}$ and consider the locally constant sheaves $\underline{R}_{\frac{M\times X}{G}}$ and $\underline{R}_{\frac{M\times Y}{G}}$ on $\frac{M\times X}{G}$, $\frac{M\times Y}{G}$ respectively. We have the following diagram:

$$\begin{tikzcd} & M \arrow[d, "f"] & \\ \frac{M\times X}{G} \arrow[r, "f_1"'] & M/G & \frac{M\times Y}{G} \arrow[l, "f_2"] \end{tikzcd}$$

Let $\mathcal{F}_1 = (f_1)_*(\underline{R}_{\frac{M\times X}{G}})$, $\mathcal{F}_2 = (f_2)_*(\underline{R}_{\frac{M\times Y}{G}})$ be the pushforward sheaves on M/G. They are local systems and correspond to representations of the fundamental group of M/G. We also have the monodromy action of $\pi_1(M/G)$ on the fibers of f_1, f_2 which are X and Y respectively. Moreover, the covering map $f : M \to M/G$ gives a surjective group homomorphism $\tilde{f} : \pi_1(M/G) \to G$. It follows that $\mathcal{F}_1$ and $\mathcal{F}_2$ correspond to the $\pi_1(M/G)$ representations $R[X]$ and $R[Y]$ respectively.

Since X and Y are R-Gassmann equivalent, we have that $R[X]$ and $R[Y]$ are isomorphic $R[G]$-modules. It follows that $\mathcal{F}_1$ and $\mathcal{F}_2$ are isomorphic and we have that:

$$H^k(M/G, \mathcal{F}_1) \cong H^k(M/G, \mathcal{F}_2).$$

Since the maps f_1, f_2 are finite sheeted covers, the Leray spectral sequence collapses to give isomorphisms:

$$H^k(\frac{M \times X}{G}, \underline{R}_{\frac{M\times X}{G}}) \cong H^k(M/G, \mathcal{F}_1)$$

and

$$H^k(\frac{M \times Y}{G}, \underline{R}_{\frac{M\times Y}{G}}) \cong H^k(M/G, \mathcal{F}_2).$$

Combining these together, we finally get that:

$$H^k(\frac{M \times X}{G}, \underline{R}_{\frac{M\times X}{G}}) \cong H^k(\frac{M \times Y}{G}, \underline{R}_{\frac{M\times Y}{G}}).$$

This is an isomorphism of cohomology groups. Now we need to notice that the local systems $\mathcal{F}_1$ and $\mathcal{F}_2$ carry Hodge structures and the last isomorphism is compatible with them (see Remark 6.2.9).

The last case when $R = \mathbb{Z}_p$ can be dealt in the same way by using the corresponding etale notions (See [4] for details). $\square$

In the next section we are going to focus to the case where $dimM = 1$. We will show that in that case we can drop the assumption that f is unramified.

§ 7.2 Proof of Main theorem

The goal of this section is to prove the main result of this thesis. We will start by describing the setup and proceed by proving two lemmas before stating and proving the main theorem. In what follows C is a smooth algebraic curve and $G \leq Aut(C)$ is a finite subgroup of the automorphism group of C that acts on it, possibly with ramification.

Consider the map $f : C \to C/G$. Let $Z = \{p_1, ..., p_r\}$ where $p_1, ..., p_r$ are points of C fixed by the action of G. Let $U = C \setminus Z$ be the open complement.

Let $R = \mathbb{Z}$ or $\mathbb{Q}$ and consider the locally constant sheaves $\underline{R}_{\frac{C\times X}{G}}$ and $\underline{R}_{\frac{C\times Y}{G}}$ on $\frac{C\times X}{G}$, $\frac{C\times Y}{G}$ respectively. Let $\mathcal{F}_1 = (f_1)_*(\underline{R}_{\frac{C\times X}{G}})$, $\mathcal{F}_2 = (f_2)_*(\underline{R}_{\frac{C\times Y}{G}})$ be the pushforward sheaves on C/G. Since the covering is possibly ramified, we can't claim that the pushforward sheaves are locally constant. However, $\mathcal{F}_1, \mathcal{F}_2$ are constructible sheaves, locally constant outside of the ramification locus.

First, we are going to prove that the restrictions of the pushforwards sheaves on the unramified locus are isomorphic.

Lemma 7.2.1. *Consider the following diagram:*

$$
\begin{tikzcd}
 & U \arrow[d, "f"] & \\
\frac{U\times X}{G} \arrow[r, "f_1"'] & U/G & \frac{U\times Y}{G} \arrow[l, "f_2"]
\end{tikzcd}
$$

Then, we have that:

$$\mathcal{F}_1 \mid_{U/G} \cong \mathcal{F}_2 \mid_{U/G} .$$

Proof. The covering is unramified over U/G. The restrictions of the pushforward sheaves on U/G, $\mathcal{F}_1 \mid_{U/G}$ and $\mathcal{F}_2 \mid_{U/G}$ are local systems and correspond to representations of the fundamental group of U/G. We also have the monodromy action of $\pi_1(U/G)$ on the fibers of f_1, f_2 which are X and Y respectively. Moreover, the covering map $f : U \to U/G$ gives a surjective group homomorphism $\tilde{f} : \pi_1(U/G) \to G$. It follows that $\mathcal{F}_1 \mid_{U/G}$ and $\mathcal{F}_2 \mid_{U/G}$ correspond to the $\pi_1(U/G)$ representations $R[X]$ and $R[Y]$ respectively.

Since X and Y are R-Gassmann equivalent we have that $R[X]$ and $R[Y]$ are isomorphic $R[G]$-modules. It follows that $\mathcal{F}_1 \mid_{U/G}$ and $\mathcal{F}_2 \mid_{U/G}$ are isomorphic. □

This means that:

$$H^k(U/G, \mathcal{F}_1 \mid_{U/G}) \cong H^k(U/G, \mathcal{F}_2 \mid_{U/G})$$

and since the maps $f_1.f_2$ are finite sheeted covers, the Leray spectral sequences collapse to give isomorphisms:

$$H^k(\frac{U\times X}{G}, \underline{R}_{\frac{U\times X}{G}}) \cong H^k(U/G, \mathcal{F}_1 \mid_{U/G})$$

and

$$H^k(\frac{U\times Y}{G}, \underline{R}_{\frac{U\times Y}{G}}) \cong H^k(U/G, \mathcal{F}_2 \mid_{U/G}).$$

Combining these together, we finally get that:

$$H^k(\frac{U\times X}{G}, \underline{R}_{\frac{U\times X}{G}}) \cong H^k(\frac{U\times Y}{G}, \underline{R}_{\frac{U\times Y}{G}}).$$

Now, let $Z_1 = \frac{Z\times X}{G}, Z_2 = \frac{Z\times Y}{G}$ be the closed complements of $\frac{U\times X}{G}$, $\frac{U\times Y}{G}$ in $\frac{C\times X}{G}$, $\frac{C\times Y}{G}$ respectively. We have the following lemma:

Lemma 7.2.2 (Purity isomorphism). *Let $i_1 : Z_1 \to \frac{C\times X}{G}$ and $i_2 : Z_2 \to \frac{C\times Y}{G}$. Then:*

$$i_1^!(\underline{R}_{\frac{C\times X}{G}}) \cong \underline{R}_{Z_1}[-2]$$

and similarly

$$i_2^!(\underline{R}_{\frac{C\times Y}{G}}) \cong \underline{R}_{Z_2}[-2].$$

Proof. Since Z_1, Z_2 are smooth, this is an immediate application of proposition 6.1.7. □

Next, we need a lemma to show how $\mathcal{F}_i$ are made up from $\mathcal{F}_i\mid_U$ and $\mathcal{F}_i\mid_Z$. Consider the following diagrams of morphisms of open/closed pairs:

$$\begin{tikzcd}
\frac{Z\times X}{G} \arrow[r, "i_1"] \arrow[d, "g_1"] & \frac{C\times X}{G} \arrow[d, "f_1"] & \frac{U\times X}{G} \arrow[l, "j_1"'] \arrow[d, "h_1"] \\
Z/G \arrow[r, "i"'] & C/G & U/G \arrow[l, "j"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\frac{Z\times Y}{G} \arrow[r, "i_2"] \arrow[d, "g_2"] & \frac{C\times Y}{G} \arrow[d, "f_2"] & \frac{U\times Y}{G} \arrow[l, "j_2"'] \arrow[d, "h_2"] \\
Z/G \arrow[r, "i"'] & C/G & U/G \arrow[l, "j"]
\end{tikzcd}$$

.

Lemma 7.2.3. *Consider the distinguished triangle:*

$$i_*i^! \to 1 \to j_*j^* \to [+1].$$

applied to $\mathcal{F}_1, \mathcal{F}_2$. Then we have the following isomorphisms for $Rj_*\mathcal{F}_1\mid_U$ and $Rj_*\mathcal{F}_2\mid_U$:

$$R^0j_*\mathcal{F}_1\mid_U \cong \mathcal{F}_1$$

and

$$R^0j_*\mathcal{F}_2\mid_U \cong \mathcal{F}_2.$$

In particular, as $\mathcal{F}_1\mid_U \cong \mathcal{F}_2\mid_U$ from Lemma 7.2.1, we get that $\mathcal{F}_1 \cong \mathcal{F}_2$.

Proof. Taking the long exact sequence of cohomology sheaves coming from the distinguished triangle and using the purity isomorphism and base change for $i^!$, we get the following:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_1 \arrow[r] & \mathbf{R}^0 j_* \mathcal{F}_1 \mid_U \arrow[dll] \\
0 \arrow[r] & 0 \arrow[r] & \mathbf{R}^1 j_* \mathcal{F}_1 \mid_U \arrow[dll] \\
i_* g_{1*} R_{\frac{C\times X}{G}} \arrow[r] & 0 \arrow[r] & ...
\end{tikzcd}$$

This gives the desired isomorphisms for $\mathcal{F}_1$ and similarly we obtain the desired isomorphisms for $\mathcal{F}_2$. □

Theorem 7.2.4. *Let X, Y be R-Gassmann equivalent G-sets, C a smooth projective algebraic curve over the complex numbers with an action of a finite group $G \leq Aut(C)$ and $f : C \to C/G$ a Galois cover (not necessarily unramified). Then:*

1. *If $R = \mathbb{Q}$ we have an isomorphism of rational Hodge structures: $H^k(\frac{C\times X}{G}, \mathbb{Q}) \cong H^k(\frac{C\times Y}{G}, \mathbb{Q})$.*
2. *If $R = \mathbb{Z}$ we have an isomorphism of integral Hodge structures: $H^k(\frac{C\times X}{G}, \mathbb{Z}) \cong H^k(\frac{C\times Y}{G}, \mathbb{Z})$.*

Proof of main theorem. By Lemma 7.2.1 we see that for each i:

$$H^i(\frac{U\times X}{G}, R_{\frac{U\times X}{G}}) \cong H^i(\frac{U\times Y}{G}, R_{\frac{U\times Y}{G}}). \tag{7.9}$$

Using Lemma 7.2.3:

$$H^2(\frac{Z\times X}{G}, R_{\frac{Z\times X}{G}}) \cong H^2(Z/G, \mathcal{F}_1 \mid_{Z/G}) \cong H^2(Z/G, g_{1*}(R_{\frac{M\times X}{G}})) \cong H^2(M/G, i_* g_{1*}(R_{\frac{M\times X}{G}})) \tag{7.10}$$

and

$$H^2(\frac{Z\times Y}{G}, R_{\frac{Z\times Y}{G}}) \cong H^2(Z/G, \mathcal{F}_2 \mid_{Z/G}) \cong H^2(Z/G, g_{2*}(R_{\frac{M\times Y}{G}})) \cong H^2(M/G, i_* g_{2*}(R_{\frac{M\times Y}{G}})). \tag{7.11}$$

Moreover, the cohomology groups are zero in all other degrees. This means that:

$$H^i(\frac{Z\times Y}{G}, R_{\frac{Z\times Y}{G}}) \cong H^i(\frac{Z\times Y}{G}, R_{\frac{Z\times Y}{G}}) \tag{7.12}$$

for every i. Using Remark 6.2.9, since Z/G is finite and over U/G the covering is unramified, the isomorphisms of (7.9) and (7.12) are isomorphisms of mixed Hodge structures.

The morphisms of open/closed pairs after Lemma 7.2.2 give the following commutative diagram of mixed Hodge structures with exact rows:

$$\begin{tikzcd}
H^i(\frac{C\times X}{G}, R_{\frac{C\times X}{G}}) \arrow[r] \arrow[d, dotted] & H^i(\frac{U\times X}{G}, R_{\frac{U\times X}{G}}) \arrow[r] \arrow[d, "\cong"] & H^{i+1}(\frac{Z\times X}{G}, R_{\frac{Z\times X}{G}}) \arrow[d, "\cong"] \\
H^i(\frac{C\times Y}{G}, R_{\frac{M\times Y}{G}}) \arrow[r] & H^i(\frac{U\times Y}{G}, R_{\frac{U\times Y}{G}}) \arrow[r] & H^{i+1}(\frac{Z\times Y}{G}, R_{\frac{Z\times Y}{G}})
\end{tikzcd}$$

and it follows that $H^i(\frac{C\times X}{G}, R_{\frac{C\times X}{G}}), H^i(\frac{C\times Y}{G}, R_{\frac{C\times Y}{G}})$ are isomorphic Hodge structures, which completes the proof. □

Remark 7.2.5. As in Theorem 7.1.6, the case when $R = \mathbb{Z}_p$ can be dealt in the same way using the corresponding etale notions and one can show that we have an isomorphism of etale cohomology groups: $H^k_{et}(\frac{C\times X}{G}, \mathbb{Z}_p) \cong H^k_{et}(\frac{C\times Y}{G}, \mathbb{Z}_p)$.

Corollary 7.2.6. *Let X, Y be rationally (integrally) Gassmann equivalent G-sets, C a smooth projective algebraic curve over the field of the complex numbers such that $G \leq Aut(C)$ acts on C and $f : C \to C/G$ a Galois cover (not necessarily unramified). Then the Jacobians of $\frac{C\times X}{G}$ and $\frac{C\times Y}{G}$ are isogenous (isomorphic).*

Chapter 8

Applications

In this chapter we provide some applications of our main theorem (Theorem 7.2.4 and Corollary 7.2.6) in the case of rational coefficients and show how it can be used to get decompositions of Jacobians of algebraic curves up to isogeny. Decompositions up to isogeny have been studied extensively (see [44]), often using Kani's theorem (see [36]). We begin by reviewing Kani's theorem. We have seen the relationship between Kani's character equivalence and rationally Gassmann equivalent G-sets in Theorem 5.2.9. Based on this, any decomposition up to isogeny obtained using Kani's theorem, can also be obtained by using examples of rationally Gassmann equivalent G-sets and Corollary 7.2.6 instead. We give some examples and explain a small difference between the two approaches.

§ 8.1 Kani's theorem and rational Gassmann equivalence

We have already discussed Kani's concept of character equivalence and shown that it is equivalent to rational Gassmann equivalence in chapter 5. In [36], it is shown that idempotent relations can be used to get isogeny relations of abelian varieties.

Theorem 8.1.1 (36, Theorem A)**.** *Let A be an abelian variety. Then the idempotent relation*

$$\sum_{i=1}^{n} \epsilon_i \sim \sum_{i=1}^{m} \epsilon_i' \tag{8.1}$$

holds in $End^0(A)$ if and only if we have the following isogeny relation

$$\prod_{i=1}^{n} \epsilon_i(A) \sim \prod_{i=1}^{m} \epsilon_i'(A). \tag{8.2}$$

In [36], relations among idempotents of the form $\epsilon_H = \sum_{h \in H} h \in \mathbb{Q}[G]$ are considered. Let C be a smooth projective curve with $G \leq Aut(C)$ and H a subgroup of G. There is a canonical map of $\mathbb{Q}$-algebras, $\rho : \mathbb{Q}[G] \to End^0 J_C$ (see [39]) and we have that $\rho(\epsilon_H)(J_C)$ is isogenous to $J_{C/H}$ (see [36]). So, if we had a character relation of such idempotents, Theorem 8.1.1 would give us an isogeny relation of Jacobians of quotient curves of C.

In order to apply this theorem, one needs to exhibit such idempotent relations. In this direction, Kani gives the following result which generalizes Theorem 2.1.7.

Theorem 8.1.2 (36, Theorem B)**.** *Let C be an algebraic curve and $G \leq Aut(C)$ a finite group. Let $H_i \leq G$ be subgroups of G such that $G = H_1 \cup ... \cup H_m$ and $H_i \cap H_j = \{e\}$ if $i \neq j$. Then we have the following isogeny relation:*

$$J_C^{m-1} \times J_{C/G}^g \cong J_{C/H_1}^{h_1} \times ... J_{C/H_m}^{h_m}.$$

where $g = |G|$ and $h_i = |H_i|$.

This theorem has been used in several papers to obtain decompositions of Jacobians of curves with non-trivial automorphism groups. One issue with this result is that it is not clear which groups admit partitions and also the subgroups H_i are not explicitly given. As character equivalence is equivalent to rational Gassmann equivalence, instead of using Theorem 8.1.2 one can use our Corollary 7.2.6 and examples of rational Gassmann equivalence to produce the same examples. The advantage of this is that the subgroups H_i can be computed explicitely. Also, the decomposition in Theorem 8.1.2 is often coming from a multiple of the basis for rational Gassmann equivalence and one has to use Poincare reducibility to simplify the decomposition. By using rationally Gassmann equivalent G-sets and Corollary 7.2.6 there is no need to use Poincare reducibility. We illustrate this in the following example.

Example 8.1.3 (44, page 5)**.** Let C be an algebraic curve and assume that $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \langle a, b \rangle$ is contained in the automorphism group of C. Consider the following subgroups of G: $H_1 = \{e, a\}, H_2 = \{e, b\}, H_3 = \{e, ab\}$. These groups give a partition of G and one can apply Theorem 8.1.2 to obtain:

$$J_C^2 \times J_{C/G}^4 \sim J_{C/H_1}^2 \times J_{C/H_2}^2 \times J_{C/H_3}^2.$$

We are interested in relations involving the Jacobian of C so now we have to apply Poincare's reducibility theorem to obtain

$$J_C \times J_{C/G}^2 \sim J_{C/H_1} \times J_{C/H_2} \times J_{C/H_3}. \tag{8.3}$$

The same example can be obtained with our theorem and the rational Gassmann equivalence for $\mathbb{Z}_2 \times \mathbb{Z}_2$. In that case, one obtains the relation 8.3 directly, without having to use Poincare's reducibility theorem. We illustrate it together with other similar examples in the next section.

§ 8.2 Some known examples obtained through rationally Gassmann equivalent G-sets

We start by redoing Example 8.3 using Gassmann equivalence. In all these examples C is a smooth projective algebraic curve and $G \leq Aut(C)$.

Example 8.2.1. Again, let $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and its subgroups $H_1 = \{e, a\}, H_2 = \{e, b\}, H_3 = \{e, ab\}$. Then, we have seen in Example 5.2.3 that the G-sets $X \cong G/H_1 \sqcup G/H_2 \sqcup G/H_3$ and $Y \cong G/\{e\} \sqcup G/G \sqcup G/G$ are rationally Gassmann equivalent. Applying Corollary 7.2.6 we obtain the isogeny relation of Example 8.1.3:

$$J_C \times J_{C/G}^2 \sim J_{C/H_1} \times J_{C/H_2} \times J_{C/H_3}.$$

Similary, using the rationally Gassmann equivalent G-sets in S_3, that we studied in Example 5.2.4, we obtain:

Example 8.2.2. Let G be the symmetric group on 3 letters S_3 and consider its subgroups $H_1 = \{e\}$, $H_2 = \{e, (12)\}$, $H_3 = \{e, (123)\}$ and G. The G-sets $X \cong G/H_2 \sqcup G/H_2 \sqcup G/H_3$ and $Y \cong G/\{e\} \sqcup G/G \sqcup G/G$ are rationally Gassmann equivalent. Applying Corollary 7.2.6 we get:

$$J_{H_2}^2 \times J_{H_3} \cong J_C \times J_{C/G}^2.$$

Now we consider the group $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and the Gassmann equivalent G-sets obtained in 5.2.5.

Example 8.2.3. Applying Corollary 7.2.6 we obtain the following isogeny relation:

$$J_C \times J_{C/G}^p \cong J_{C/H_1} \times ... \times J_{C/H_{p+1}}.$$

This is the same as ([36], Example 1, Section 5).

Finally, an example with the Gassmann equivalent G-sets in the dihedral group D_q, that we obtained in Example 5.2.7.

Example 8.2.4. Applying Corollary 7.2.6 we obtain the following isogeny relation:

$$J_{C/\langle\sigma\rangle} \times J_{C/\langle\tau\rangle}^2 \cong J_{C/D_q}^2 \times J_C.$$

This is the same as ([44], Example 3.1.2).

In the case where we know more details about the generators of the automorphism group of the curve C, we can use the Riemann Hurwitz formula of Chapter 2 to find the genus of the quotient curves and obtain more explicit decompositions. We illustrate this with three examples that can be found in [44].

Example 8.2.5 (44, Theorem 5)**.** Let C be a hyperelliptic curve of the form $y^2 = x^{2g+2}+a_1x^{2g}+a_2x^{2g-2}+...+a_gx^2+1$. Any such hyperelliptic curve has an automorphism group that contains the group $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The three non-trivial automorphisms are: $a : (x, y) \to (-x, y)$, the hyperelliptic involution $b : (x, y) \to (x, -y)$ and $ab : (x, y) \to (-x, -y)$. Example 8.2.1 gives the decomposition:

$$J_{C/\{e\}} \times J_{C/\{G\}} \times J_{C/\{G\}} \cong J_{C/\langle a\rangle} \times J_{C/\langle b\rangle} \times J_{C/\langle ab\rangle}.$$

Disregarding genus 0 curves, we get:

$$J_C \cong J_{C_1} \times J_{C_2}$$

where $C_1 = C/\langle a\rangle$ and $C_2 = C/\langle ab\rangle$.

Using the Riemann Hurwitz formula we get that when g_C is even $g_{C_1} = g_{C_2} = g_C/2$ and when g_C is odd $g_{C_1} = (g_C - 1)/2$ and $g_{C_2} = (g_C + 1)/2$.

Example 8.2.6 (44, Example 3.2.3)**.** Consider the 1-dimensional family of genus 3 curves $C : y^2 = x(x^6 + ax^3 + 1)$. Any such curve has automorphism group D_{12} with generators $r : (x, y) \to (\zeta_3 x, \zeta_6 y)$ and $s : (x, y) \to (1/x, y/x^4)$. This group has a subgroup isomorphic to S_3 generated by r^2 and s. Corollary 7.2.6 and Example 5.2.4 give the following isogeny relation:

$$J_{C/\{e\}} \times J_{C/\{G\}} \times J_{C/\{G\}} \cong J_{C/\langle r^2\rangle} \times J_{C/\langle s\rangle} \times J_{C/\langle sr^2\rangle}.$$

Using the Riemann Hurwitz formula, the three curves on the right have genus 1 and since $\langle r^2\rangle$ and $\langle sr^2\rangle$ are conjugate, the last two Jacobians are isogenous. Disregarding genus 0 curves, we get:

$$J_C \cong E_1 \times E_2^2$$

where E_1, E_2 are elliptic curves.

Example 8.2.7 (44, Example 3.2.4)**.** Consider the genus 3 curve $C : y^2 = x^8 + 14x + 1$. It is the only genus 3 curve with automorphism group $S_4 \times \mathbb{Z}_2$. The generators are $(12) : (x, y) \to (\frac{1-x}{1+x}, -\frac{4y}{(1+x)^4})$, $(1234) : (x, y) \to (ix, y)$, $\tau : (x, y) \to (x, -y)$. This group has a particular subgroup isomorphic to the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and using Corollary 7.2.6 and Example 5.2.3 we get the following isogeny relation:

$$J_{C/\{e\}} \times J_{C/\{G\}} \times J_{C/\{G\}} \cong J_{C/\langle((12)(34),e)\rangle} \times J_{C/\langle((13)(24),e)\rangle} \times J_{C/\langle((14)(23),e)\rangle}.$$

In this case all the subgroups on the right side are conjugate and the corresponding Jacobians of the quotient curves are isogenous. This means that:

$$J_C \sim E^3$$

where E is an elliptic curve.

Similarly, using the results on the full automorphism groups of hyperelliptic curves of Chapter 2 we can obtain most of the other decompositions of Jacobians that are given in [44].

§ 8.3 Conclusions - Future work

While decompositions of Jacobians up to isogeny have been studied extensively, decompositions up to isomorphism are only known in a few special cases. The advantage of Theorem 7.2.4 and Corollary 7.2.6 is that it can give relations of Jacobians up to isomorphism, provided that one knows how to find examples of integrally Gassmann equivalent G-sets. Unfortunately there is currently only one known example (Example 5.5.2) of transitive integrally Gassmann equivalent G-sets. In all the other cases of Gassmann equivalence, such as rational, p-local and locally integral, intransitive examples appear to be more abundant. We expect that it will be possible to find intransitive integrally Gassmann equivalent G-sets and it will be interesting to see what type of isomorphism relations of Jacobians can be obtained through them. For example, if one manages to find such examples in certain groups of type $PSL(2, q)$ it might be possible to decompose Jacobians of Hurwitz curves up to isomorphism.

Chapter 9

Appendix

§ 9.1 Code for Rational Gassmann equivalence

```
function RationalGassmann(G)
    tom:=TableOfMarks(G);
    M:=Transpose(tom);
    Q:=[];
    L:=SubgroupLattice(G);
    for i in [1..#L] do  if IsCyclic(L[i]) then Append(~Q,i);
    end if;
    end for;
    A:=[Q[1]];
    for i in [2..#Q] do
    for j in [1..i-1] do
    if Q[i] ne Q[j] then counter:=0; else counter:=1;
    end if;
    end for;
    if counter ne 1 then Append(~A,Q[i]); end if;
    end for;
    FinalMatrixT:=Matrix([M[1]]);
    for i in [1..#L] do
    if i in A then VerticalJoin(~FinalMatrixT,M[i]);
    end if;
    end for;
    RemoveRow(~FinalMatrixT,1);
```

```
    FinalMatrix:=Transpose(FinalMatrixT);
    Solution:=Nullspace(FinalMatrix);
    return Solution;
end function;

RationalGassmann(SymmetricGroup(3));
RSpace of degree 4, dimension 1 over Integer Ring
Echelonized basis:
( 1 -2 -1  2)
```

§ 9.2 Code for Locally Integral Gassmann Equivalence

```
function LocallyIntegralGassmann(G)
    tom:=TableOfMarks(G);
    M:=Transpose(tom);
    Q:=[];

    K:=PrimeDivisors(Order(G));
    L:=SubgroupLattice(G);
    for i in [1..#L] do for j in [1..#K] do
    if IsCyclic(L[i]/pCore(L[i],K[j])) then Append(~Q,i);
    end if;
    end for;
    end for;
    A:=[Q[1]];
    for i in [2..#Q] do
    for j in [1..i-1] do
    if Q[i] ne Q[j] then counter:=0; else counter:=1;
    end if;
    end for;
    if counter ne 1 then Append(~A,Q[i]); end if;
    end for;
    FinalMatrixT:=Matrix([M[1]]);
    for i in [1..#L] do
    if i in A then VerticalJoin(~FinalMatrixT,M[i]);
    end if;
    end for;
```

```
    RemoveRow(~FinalMatrixT,1);
    FinalMatrix:=Transpose(FinalMatrixT);
    ListSolution:=Nullspace(FinalMatrix);
    return ListSolution;
end function;

LocallyIntegralGassmann(DihedralGroup(6));
RSpace of degree 10, dimension 1 over Integer Ring
Echelonized basis:
( 1 -1 -1 -1 -1  2  1  1  1 -2)
```

§ 9.3 Scott's example of integrally Gassmann equivalent G-sets

```
function OrbitMatrix(G,H1,H2)
     if Type(G) ne GrpPerm then
          pi,G,_:=CosetAction(G,sub<G|>);
          H1:=pi(H1); H2:=pi(H2);
     end if;
     pi1,P1,K1:=CosetAction(G,H1);
     pi2,P2,K2:=CosetAction(G,H2);
     n1:=Degree(P1);
     n2:=Degree(P2);
     X:={<i,j>:i in [1..n1],j in [1..n2]};
     XxG:=CartesianProduct(X,G);
     f:=map<XxG->X|x:-><x[1][1]^pi1(x[2]),x[1][2]^pi2(x[2])>>;
     O:=Orbits(G,GSet(G,X,f));
     A:=AssociativeArray(X);
     for i in [1..#O] do for a in O[i] do A[a]:=i;
     end for; end for;
     M:=Matrix([[A[<i,j>]:i in [1..n1]]:j in [1..n2]]);
     return M,#O,[ExactQuotient(#o,n2):o in O];
end function;

function DoubleCosetUnions(G,H1,H2)
     if Type(G) ne GrpPerm then
          pi,G,_:=CosetAction(G,sub<G|>);
          H1:=pi(H1); H2:=pi(H2);
     end if;
```

```
    pi1,P1,K1:=CosetAction(G,H1);
    pi2,P2,K2:=CosetAction(G,H2);
    n1:=Degree(P1);
    n2:=Degree(P2);
    X:={<i,j>:i in [1..n1],j in [1..n2]};
    XxG:=CartesianProduct(X,G);
    f:=map<XxG->X|x:-><x[1][1]^pi1(x[2]),x[1][2]^pi2(x[2])>>;
    O:=Orbits(G,GSet(G,X,f));
    return [ExactQuotient(#o,n2):o in O];
end function;

G:=PSL(2,29);
S:=[H`subgroup:H in Subgroups(G:IndexEqual:=203)|
#Core(G,H`subgroup) eq 1];
DoubleCosetUnions(G,S[1],S[2]);
M:=OrbitMatrix(G,S[1],S[2]);
for i in [1..203] do
for j in [1..203] do
 if M[i,j] eq 1 then
    M[i,j]:=1;
 elif M[i,j] eq 2 then
    M[i,j]:=-1;
 else
    M[i,j]:=0;
end if;
end for;
end for;
Determinant(M);

[ 5, 6, 10, 12, 20, 30, 60, 60 ]
1
```

§ 9.4 Code for Example 4.2.5

```
G:=SymmetricGroup(3);
tom:=TableOfMarks(G);
M:=Transpose(tom);
Q:=[];
```

```
K:=PrimeDivisors(Order(G));
L:=SubgroupLattice(G);
for i in [1..#L] do  if IsCyclic(L[i]) then Append(~Q,i);
end if; end for;

A:=[Q[1]];
for i in [2..#Q] do
for j in [1..i-1] do
if Q[i] ne Q[j] then counter:=0; else counter:=1;
end if;
end for;
if counter ne 1 then Append(~A,Q[i]); end if;
end for;
FinalMatrixT:=Matrix([M[1]]);
for i in [1..#L] do
if i in A then VerticalJoin(~FinalMatrixT,M[i]);
end if;
end for;
RemoveRow(~FinalMatrixT,1);
FinalMatrix:=Transpose(FinalMatrixT);
Solution:=Nullspace(FinalMatrix);
Solution;

function DoubleCosetUnions(G,H1,H2)
      if Type(G) ne GrpPerm then
           pi,G,_:=CosetAction(G,sub<G|>);
           H1:=pi(H1); H2:=pi(H2);
      end if;
      pi1,P1,K1:=CosetAction(G,H1);
      pi2,P2,K2:=CosetAction(G,H2);
      n1:=Degree(P1);
      n2:=Degree(P2);
      X:={<i,j>:i in [1..n1],j in [1..n2]};
      XxG:=CartesianProduct(X,G);
      f:=map<XxG->X|x:-><x[1][1]^pi1(x[2]),x[1][2]^pi2(x[2])>>;
      O:=Orbits(G,GSet(G,X,f));
      return [ExactQuotient(#o,n2):o in O];
end function;
```

```
Gsetx:=[L[1],L[4],L[4]];
Gsety:=[L[2],L[2],L[3]];

sizes:=[* *];
for i in [1..#Gsetx] do
for j in [1..#Gsety] do
Append(~sizes, DoubleCosetUnions(G, Gsetx[i], Gsety[j]));
end for;
end for;

sizes;

function NumberOfOrbits(G,H1,H2)
      if Type(G) ne GrpPerm then
           pi,G,_:=CosetAction(G,sub<G|>);
           H1:=pi(H1); H2:=pi(H2);
      end if;
      pi1,P1,K1:=CosetAction(G,H1);
      pi2,P2,K2:=CosetAction(G,H2);
      n1:=Degree(P1);
      n2:=Degree(P2);
      X:={<i,j>:i in [1..n1],j in [1..n2]};
      XxG:=CartesianProduct(X,G);
      f:=map<XxG->X|x:-><x[1][1]^pi1(x[2]),x[1][2]^pi2(x[2])>>;
      O:=Orbits(G,GSet(G,X,f));
      return #O;
end function;

function OrbitMatrix(G,H1,H2)
      if Type(G) ne GrpPerm then
           pi,G,_:=CosetAction(G,sub<G|>);
           H1:=pi(H1); H2:=pi(H2);
      end if;
      pi1,P1,K1:=CosetAction(G,H1);
      pi2,P2,K2:=CosetAction(G,H2);
      n1:=Degree(P1);
      n2:=Degree(P2);
      X:={<i,j>:i in [1..n1],j in [1..n2]};
      XxG:=CartesianProduct(X,G);
      f:=map<XxG->X|x:-><x[1][1]^pi1(x[2]),x[1][2]^pi2(x[2])>>;
```

```
      O:=Orbits(G,GSet(G,X,f));
      A:=AssociativeArray(X);
      for i in [1..#O] do for a in O[i] do A[a]:=i;

      end for; end for;
      M:=Matrix([[A[<i,j>]:i in [1..n1]]:j in [1..n2]]);
      return M,#O,[ExactQuotient(#o,n2):o in O];
end function;

M:=[* *];
counter:=0;
for i in [1..#Gsetx] do
for j in [1..#Gsety] do
K:=OrbitMatrix(G, Gsetx[i], Gsety[j]);
rows:=NumberOfRows(K);
columns:=NumberOfColumns(K);
for k in [1..rows] do
for l in [1..columns] do
K[k,l]:=K[k,l]+counter;
end for;
end for;
Append(~M,K);
counter:=counter+NumberOfOrbits(G, Gsetx[i],Gsety[j]);
end for;
end for;

Vertical:=[* *];
for i in [1..#Gsetx] do
Append(~Vertical,M[1+(i-1)*(#Gsetx)]);
end for;
FinalVertical:=[* *];
for i in [1..#Gsetx] do
FinalVertical[i]:=Vertical[i];
for j in [2..#Gsetx] do
FinalVertical[i]:=VerticalJoin(FinalVertical[i],
M[(i-1)*(#Gsetx)+j]);
end for;
end for;
FinalMatrix:=FinalVertical[1];
for i in [2..#Gsetx] do
```

```
FinalMatrix:=HorizontalJoin(FinalMatrix, FinalVertical[i]);
end for;
FinalMatrix;
```

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