Dynamics in anti–de Sitter spacetimes and representations of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$

David Serrano Blanco

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University of York
Mathematics

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Abstract

We present the analysis of the dynamics for a scalar field in the universal covering space of N–dimensional anti–de Sitter spacetime, AdS_N ($N \geq 2$), and for a spinor field satisfying the Dirac equation in the universal covering space of two–dimensional anti-de Sitter spacetime, AdS_2. We apply a prescription for dynamics in static, non–globally hyperbolic spacetimes based on the theory of self–adjoint extensions of operators on Hilbert spaces. This prescription results in a family of field theories with a well–defined initial value problem despite the lack of global–hyperbolicity of the spacetime manifold. Then, we impose the invariance of the associated solution spaces under the infinitesimal action of the isometry group of AdS_N ($\widetilde{\mathrm{SL}}(2,\mathbb{R})$ for $N = 2$ and $\widetilde{\mathrm{SO}}(2, N-1)$ for $N \geq 3$) to determine which among the family of theories obtained by the prescription for dynamics can be used to construct a quantum field theory with a stationary vacuum state.

Contents

List of Tables

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Author's declaration

I declare that the work presented in this thesis, except where otherwise stated, is based on my own research carried out at the University of York and has not been submitted previously for any degree at this or any other university. Sources are acknowledged by explicit references.

The original contributions from the author include the following. Chapter 5, specially Section 5.2, is based on a jointly–authored pre–print [1] with Atsushi Higuchi. Chapter 6, together with the material in Appendices B–E are based on a collaboration with Atsushi Higuchi and Lasse Schmieding which has been accepted for publication [2]. The analysis from the paragraph after Eq. (7.46) in Section 7.2 and onwards until the end of Chapter 7 is an original contribution of the author. Chapter 8 is based on a pre-print [3] by the author under the supervision of Atsushi Higuchi which has been accepted for publication and is currently under production. Appendices F–H contain calculations made by the author under the supervision of Atsushi Higuchi.

— 1 —

Introduction

One of the cornerstones of modern mathematical and theoretical physics is the framework of Quantum Field Theory (QFT). Its predictive power and validation against experimental scrutiny to a remarkably high degree of accuracy has cemented QFT as one of the most successful physical theories of the past century. However, despite its success, QFT has failed to describe a comprehensive and consistent theory of quantum gravity, mostly due to the mathematical inconsistencies that arise when the principles of General Relativity are included in the theory. Nevertheless, attempts to reconcile QFT with the equally successful theory of General Relativity have resulted in a vast number of theories that have broadened the landscape of mathematical physics and have added to our understanding of the possible limitations of the mathematical tools that are used to describe physical phenomena.

Among these theories, Quantum Field Theory in curved spacetime is a framework that aims to describe quantum phenomena in a regime where the influence of the gravitational field on the propagation of the quantum fields is significant, but its description as a quantum field itself can be neglected. Thus, QFT in curved spacetime can be understood as an approximation that takes into account the effects that a gravitational field introduces to the quantum interactions that fall in this particular threshold. Specifically, the gravitational interaction in these theories enters the picture in the form of a background spacetime satisfying Einstein's field equations on which the quantum fields propagate. Several interesting predictions have been obtained by applying this framework to situations in which we believe these considerations do apply, including, but not limited to, the discovery of Hawking radiation [4, 5], the Unruh effect [6, 7, 8], and the Casimir effect [9].

Ever since the effort of QFT in curved spacetime was proven to produce useful results, there has been an ongoing interest in analysing the propagation of quantum fields on a variety of curved backgrounds. A very interesting example is that of anti–de Sitter spacetime, a maximally symmetric vacuum solution to Einstein's field equations of constant negative curvature. This spacetime has two prominent peculiarities. Firstly, it admits the existence of closed timelike curves, a fact that violates the causality of events if taken to be a physical spacetime. Nevertheless, one usually avoids this issue by considering instead the universal covering space, AdS_N, of the N–dimensional anti–de Sitter space as the physical spacetime this solution describes. Secondly, the universal covering space AdS_N is not a globally hyperbolic manifold. Hence, given a set of initial data on a region of spacetime, the equations of motion describing the evolution of the fields may not be able to determine the values of the field throughout the whole spacetime manifold. Despite this drawback, AdS_N has played a prominent role in many areas of mathematical and theoretical physics in the last two decades. The interest to study anti–de Sitter spacetimes in the context of QFT in curved spacetime may be attributed to the outstanding result from string theory known as the AdS/CFT correspondence [10], which conjectures the equivalence between a theory of quantum gravity on AdS_N and a quantum field theory with no gravity defined in its conformal boundary. Hence, the AdS/CFT conjecture may provide a way to describe quantum gravitational effects while circumventing the notorious difficulties encountered when defining a quantum theory of gravity. Interest in classical and quantum theories in anti–de Sitter spacetime has gone well beyond its initial connection to the AdS/CFT correspondence from the viewpoint of string theory and has resulted in the investigation of their properties in a wide variety of different contexts.

Anti–de Sitter spacetime has proven to be a useful model to study QFT's for which the background manifold is not globally hyperbolic. The standard procedure to construct a QFT consists of applying canonical quantisation to the solutions of the classical field equations of motion. For this process to define a consistent quantum theory, the classical field satisfying the equations of motion must have well–defined dynamics. If the spacetime manifold fails to be globally hyperbolic, this last requirement is not guaranteed and, thus, the heavy work for the quantisation of this kind of theories is usually done at the classical level. For the particular case of AdS_N, sensible dynamics for the classical field can be obtained by imposing certain boundary conditions that the field must satisfy at spatial infinity. Some properties of quantum field theories in anti–de Sitter spacetime obtained from these boundary conditions have been studied in the past [11, 12, 13, 14] and more recently [15, 16, 17, 18, 19]. Depending on the context in which such theories are analysed, different arguments may be given for choosing a particular set of boundary conditions over the others but, in general, there is no underlying principle that removes the ambiguity of choice.

In a more general context, if the spacetime manifold admits a global timelike Killing vector field, there exists a prescription that provides a way to obtain well–defined dynamics for the equations of motion. This prescription for dynamics is due to Ishibashi and Wald [20, 21] and, at its core, relies on the theory of self–adjoint extensions of operators on Hilbert spaces. Since AdS_N is maximally symmetric, it admits a global timelike Killing vector and, thus, this prescription can be applied to this case. In particular, Ishibashi and Wald analysed in Ref. [22] the classical scalar, vector and symmetric tensor field theories defined on AdS_N for all $N \geq 3$. The interesting aspect of Ishibashi and Wald's prescription is that it is possible to obtain a family of boundary conditions on the field solutions that result in well–defined dynamics. The usual choices of boundary conditions arise in this prescription as particular cases of the family of admissible boundary conditions. The results of Ishibashi and Wald for AdS_N cover the higher–dimensional cases ($N \geq 3$), all of which behave in a very similar way under their prescription. However, their results do not immediately extend to the two–dimensional case.

The main goal of the research presented in this thesis is to apply the prescription for dynamics of Ishibashi and Wald to free scalar and spinor field theories in the universal covering space of two–dimensional anti–de Sitter spacetime, AdS_2. Additionally, we aim to extend the analysis of Ishibashi and Wald in Ref. [22] by determining which of the boundary conditions resulting from applying their prescription to a scalar field theory in AdS_N for all $N \geq 2$ and to a spinor field in AdS_2 are invariant under the isometry group of the spacetime manifold. For AdS_2, the isometry group is isomorphic to the universal covering group, $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, of $\mathrm{SL}(2,\mathbb{R})$, while for AdS_N, with $N \geq 3$, the isometry group is the connected component of the special indefinite orthogonal group $\mathrm{SO}(2, N-1)$. The invariance of the associated solution spaces under the isometry group of the theory thus provides a criterion to select certain boundary conditions over the others. Furthermore, with the aim of constructing a quantum field theory using the solution spaces that result from the invariant boundary conditions, we determine which among these admit an invariant positive–frequency subspace.

Thus, the general outline of this thesis is the following. The rest of Chapter 1 includes a brief remark on the notational conventions we use in this thesis. In Chapters 2–5 we present the mathematical preliminaries used in our research, including the prescription for dynamics by Ishibashi and Wald, some general properties of N–dimensional anti–de Sitter spacetimes and a review of the representation theory of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. We also present a brief summary of the theory of self–adjoint extensions of operators on Hilbert spaces and include a useful example in which the correspondence with the self–adjoint extensions of a differential operator and a family of boundary conditions is made explicit. In Chapters 6–8, we present the analysis of a minimally coupled, free scalar field in AdS_N ($N \geq 2$), and that of a free spinor field satisfying the Dirac equation in AdS_2. In Chapter 9 we provide a summary of our results. The rest of this thesis consist of Appendices A–H, in which we include some reference material regarding self–adjoint operators on Hilbert spaces as well as some specific calculations used in this thesis.

1.1 NOTATION

We clarify some of the notational conventions we use throughout this work.

Tensors and tensor fields of rank 2 on a smooth manifold are denoted with the use of boldface, *e.g.*, the metric tensor on an N–dimensional pseudo–Riemannian manifold, $\mathcal{M}$ is written as $\boldsymbol{g}$. Local coordinates on $\mathcal{M}$ are represented by the N–tuples $(x^0, x^1, \ldots, x^{N-1})$, collectively written as (x^μ), where $0 \leq \mu \leq N-1$. If the manifold $\mathcal{M}$ is Lorentzian with signature $(-1, 1, \ldots, 1)$, we will denote the spacelike coordinates with Latin superscripts, *e.g.*, $(x^\mu) = (x^0, x^i)$, where $1 \leq i \leq N-1$, and we usually make use of the shorthand $x := (x^i)$. A tensor field $\boldsymbol{g}$ evaluated at a point $p \in \mathcal{M}$ with local coordinates (x^0, x) is denoted by $\boldsymbol{g}_{(x^0,x)}$. The action of vector fields on smooth functions on $\mathcal{M}$ will be denoted by the use of square brackets: For $\xi \in \mathfrak{X}(\mathcal{M})$, and $f \in C^\infty(\mathcal{M})$, we have $\xi[f] \in C^\infty(\mathcal{M})$. We also denote partial differentiation of functions by the symbols

$$\frac{\partial}{\partial x} \quad \text{and} \quad \partial_x\,,$$

interchangeably.

For a Hilbert space $\mathscr{H}$, we adopt the convention in which the inner product $\langle\cdot,\cdot\rangle$ in $\mathscr{H}$ satisfies, for all $x, y \in \mathscr{H}$, and $\lambda_1, \lambda_2 \in \mathbb{C}$,

$$\langle \lambda_1 x, \lambda_2 y\rangle = \overline{\lambda_1}\lambda_2 \langle x, y\rangle\ ,$$

where $\overline{\lambda_1}$ is the complex conjugate of λ_1.

The use of the superscript † will be relatively flexible and will depend on the type of mathematical object it is used on. In general, it denotes the adjoint of a linear map acting on a Hilbert space $\mathscr{H}$. If $T : \mathscr{H} \to \mathscr{H}$ is a linear map, and $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathscr{H}$, then $T^\dagger$ is defined by the relation

$$\langle x, Ty\rangle = \left\langle T^\dagger x, y\right\rangle\ , \tag{1.1}$$

for all $x, y \in \mathscr{H}$ for which the above expression is well–defined (see Definition 2.2.2 in Chapter 2). In particular, if $\mathscr{H} = \mathbb{C}^N$ with its standard inner product, then T is a complex $N \times N$–matrix and $T^\dagger$ is the conjugate transpose of T. If $\mathscr{H}$ is a more general Hilbert space, and T is an abstract operator, then $T^\dagger$ must be understood through Eq. (1.1).

We adopt the convention that defines the set of natural numbers as $\mathbb{N} = \{1, 2, 3, \dots\}$, and we use the symbol $\mathbb{N}_0$ to refer to the set defined by

$$\mathbb{N}_0 = \mathbb{N} \cup \{0\}\,.$$

Finally, we denote the Gaussian hypergeometric function [23, Eq. 15.2.1] by

$$F\left(a, b; c; z\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b_k)}{(c)_k k!} z^k\,, \tag{1.2}$$

for $a, b, c, z \in \mathbb{C}$, where

$$\begin{aligned} (a)_k &= \prod_{l=0}^{k-1} (a + k)\,, \\ &= \frac{\Gamma(a+k)}{\Gamma(a)}\,, \qquad -a \notin \mathbb{N}_0\,, \end{aligned}$$

denotes the Pochhammer symbol [23, Section 5.2(iii)], or rising factorial.

— 2 —

Quantum field theory on static spacetime

The main idea behind quantum field theory (QFT) in curved spacetime is to formulate a theory of a quantum field that propagates in an external, classically describable spacetime generalising the methods and techniques used in usual QFT in Minkowski space as much as possible. The extent to which this can be achieved depends quite substantially on the geometric properties and causal structure of the curved background itself. Quantum fields are first defined as classical solutions of the equations of motion which dictate the dynamics of these fields on the given spacetime. Therefore, to have a deterministic evolution of the system, the equations of motion should define a well–posed initial value problem for the field. This narrows down the types of Lorentzian manifolds that can be treated as the spacetime background for the theory. The most common requirement is that of global hyperbolicity of the spacetime manifold [24]. This condition ensures that, given initial data on a Cauchy surface, the values of the field at any point in spacetime will be completely determined by the Cauchy evolution of the initial data through the equations of motion without violating causality. Globally hyperbolic manifolds are not, however, necessary for the dynamics of a field to be well defined in this sense. If the spacetime allows a one–parameter group of isometries with everywhere timelike orbits which are hypersurface orthogonal, then there is a prescription that ensures a deterministic dynamical evolution of the field [20, 21, 22]. Spacetimes having this property are known as *static* [25, 26, 27], and they usually have the physical interpretation of being solutions of Einstein's equation that represent equilibrium situations.

In this chapter we present the general outline of the canonical quantisation procedure of free scalar and spinor fields on an arbitrary static spacetime. Our main goal is to focus on the construction of the multi–particle Fock space for both cases, specifying the requirements we need for this space to be well defined. Since anti–de Sitter spacetime is a static solution to Einstein's equations in the vacuum, the contents of this chapter will serve as a guide for later calculations and methods that will be applied to the specific case of anti–de Sitter spacetime we are interested in.

2.1 STANDARD STATIC SPACETIMES

The term *spacetime* will refer to any connected N–dimensional pseudo–Riemannian manifold, $(\mathcal{M}, \boldsymbol{g})$, with Lorentzian metric $\boldsymbol{g}$ with signature $(-1, 1, \ldots, 1)$. We recall that a smooth manifold has an associated tangent space $T_p\mathcal{M}$ at every point $p \in \mathcal{M}$, and the union of all tangent spaces of $\mathcal{M}$ forms a vector bundle over $\mathcal{M}$, the tangent bundle $T\mathcal{M}$ [25, 28, 29]. Smooth (local) sections of $T\mathcal{M}$ are called *vector fields*, and the space of vector fields on $\mathcal{M}$ is denoted by $\mathfrak{X}(\mathcal{M})$. A vector field $\xi \in \mathfrak{X}(\mathcal{M})$ is a Killing vector field if the Lie derivative of $\boldsymbol{g}$ along ξ vanishes, *i.e.*, if $\mathcal{L}_\xi \boldsymbol{g} = 0$ [25, 28, 29]. The one–parameter group generated by a Killing vector field ξ is then an isometry of the spacetime. If the spacetime $(\mathcal{M}, \boldsymbol{g})$ admits a global timelike Killing vector field, that is, if $\boldsymbol{g}_p[\xi, \xi] < 0$ for every $p \in \mathcal{M}$, then $(\mathcal{M}, \boldsymbol{g})$ is called *stationary*. On stationary spacetimes it is natural to identify the "time direction" with that of the Killing vector ξ itself, and regard the Killing parameter $t \in \mathbb{R}$ as a time coordinate for some local chart in $\mathcal{M}$. In this sense, the group of isometries generated by ξ expresses the time–translation symmetry of a stationary spacetime. However, stationary spacetimes are not, in general, invariant under time–reversal transformations. The following definition [30], although restrictive, ensures that a stationary spacetime possess this additional invariance:

Definition 2.1.1 *A stationary spacetime $(\mathcal{M}, \boldsymbol{g})$ is called a* ***standard static spacetime*** *if $\mathcal{M}$ is the product manifold $\mathcal{M} = \mathbb{R} \times \Sigma$, endowed with the metric tensor*

$$\boldsymbol{g}_{(t,x)} = -\mathcal{N}(x)^2 \mathrm{d}t \otimes \mathrm{d}t + \boldsymbol{h}_x \,, \tag{2.1}$$

at every $(t, x) \in \mathbb{R} \times \Sigma$, where $(\Sigma, \boldsymbol{h})$ is an $(N-1)$–dimensional Riemannian manifold and $\mathcal{N}^2 = -\boldsymbol{g}[\xi, \xi]$.

In this context, the timelike Killing vector field ξ is referred to as the *static vector field*. From Eq. (2.1) it is clear that the components of the metric tensor are independent of t, and thus, $\boldsymbol{g}$ is invariant under time–translations as well as time–reflections.

To analyse further properties of standard static spacetimes, we explore some useful consequences of choosing the Killing parameter t as one of the coordinates for $\mathcal{M}$. The following analysis can be found in Ref. [25, Section 6.1]. First we choose arbitrary coordinates on Σ, say $\{x^i\}_{i=1}^{N-1}$, so that a point $(t, x) \in \mathbb{R} \times \Sigma$ has the coordinate expression $(t, x^1, \ldots, x^{N-1})$, and the static vector field takes the form $\xi = \partial_t$. Fixing the time coordinate t defines a spacelike hypersurface $\Sigma_t := \{t\} \times \Sigma$ in $\mathcal{M}$ which is referred to as a *static hypersurface* or *static slice*. Since this coordinate system uses the Killing parameter t as a coordinate function, every point of the hypersurface $(t, x) \in \Sigma_t$ lies in a unique integral curve of the static vector field ξ, and from Eq. (2.1) we have $\boldsymbol{g}_{(t,x)}[\xi, \partial_i] = 0$ for all $1 \leq i \leq N-1$. This means that the hypersurface Σ_t is orthogonal to the orbits of the isometries generated by ξ. Letting the coordinate t vary over $\mathbb{R}$ makes it clear that the whole spacetime is foliated into a family of hypersurfaces orthogonal to ξ parametrised by t. Since every $\Sigma_t \subset \mathcal{M}$ is isomorphic to Σ by construction, and the metric $\boldsymbol{h}$ is independent of

t, the geometry of the static slices is constant in time. The manifold Σ is then interpreted as the spatial slice of the spacetime $\mathcal{M}$ which remains constant through time.

A stationary spacetime that admits a global timelike Killing vector field ξ and a hypersurface Σ orthogonal to the orbits of the isometries generated by ξ is said to be *static* [25, 27]. In contrast to standard static spacetimes, static ones allow a more general topology of $\mathcal{M}$ to that of a product manifold. However, if we adopt a coordinate system for a static spacetime for which the timelike coordinate is once again the Killing parameter, t, and identify the hypersurface Σ with all points in $\mathcal{M}$ with the same t–coordinate by a suitable reparametrisation of the Killing orbits if necessary, then, following the same construction as for the standard case above, the orthogonality condition around a neighbourhood of Σ implies that the metric of any static spacetime is locally given by Eq. (2.1). Since in Chapter 3 we will find that anti–de Sitter spacetimes are standard static, we will restrict our analysis to standard static spacetimes from now on.

We address some causal properties of standard static spacetimes. First, we recall some basic notions related to causality for arbitrary spacetimes. Most of the following concepts and definitions are based on Refs. [24, 25, 26].

Let $(\mathcal{M}, \boldsymbol{g})$ be an arbitrary spacetime. For $p \in \mathcal{M}$ it is possible to define the light–cone [25] passing through the origin of $T_p\mathcal{M}$, the tangent space at p, and assign in an arbitrary fashion one half of the cone to be the *future* and the other to be the *past*. If this designation can be made in a continuous fashion for every point in $\mathcal{M}$, then $(\mathcal{M}, \boldsymbol{g})$ is said to be a *time–orientable* spacetime.

A *smooth curve* c on $\mathcal{M}$ is a C^∞ map from $\mathbb{R}$, or an interval of $\mathbb{R}$, into $\mathcal{M}$, $c : \mathbb{R} \to \mathcal{M}$. The number s is referred to as the *parameter* of the curve c. A smooth curve c is said to be a *future–directed timelike curve* if for every $p \in \text{Image}(c)$, the tangent vector $\dot{c}_p$ at p is future–directed *i.e.*, lies in the interior of the future light–cone of p. A smooth curve is said to be *future–directed causal curve* if the tangent vector $\dot{c}_p$ at every p is either a future–directed or a null vector. Past–directed timelike and causal curves are defined analogously. Events on a spacetime, modelled as points in $\mathcal{M}$, connected by (either future– or past–) directed causal curves have the physical interpretation of being *causally related* to each other, *e.g.*, a material or light particle starting at p can only reach the event q if there is a future–directed causal curve starting at p and containing q [25]. It is possible to extend the definitions of future–directed timelike and causal curves to continuous curves on M. A continuous curve c is said to be a future–directed timelike (causal) curve if, for every point $p \in \text{Image}(c)$, there exists a convex normal neighbourhood U of p such that if $c(s_1), c(s_2) \in U$ for some $s_1 < s_2$, then there exists a smooth future–directed timelike (causal) curve connecting $c(s_1)$ and $c(s_2)$. A convex normal neighbourhood of $p \in \mathcal{M}$ is an open neighbourhood U of p such that for every $q, r \in U$, there exists a unique geodesic connecting q and r staying entirely within U.

The future (past) *endpoint* of a future– (past–) directed causal curve c parametrised by $-\infty < s < \infty$, is a point $p \in \mathcal{M}$ such that, for every open neighbourhood U of p, there exists $s_0 \in \mathbb{R}$, such that, for every $s > s_0$, we have $c(s) \in U$. If a curve c does not have a future (past) endpoint, then c is said to be future (past) *inextendible*. Endpoints of causal curves may be interpreted heuristically as "limit points" of the curve as its parameter $s \to \pm\infty$, and thus, inextendible curves are those which do not approach to a point $p \in \mathcal{M}$ as its parameter grows arbitrarily large. With these notions we are ready to define the following two concepts:

Definition 2.1.2 *Let $(\mathcal{M}, \boldsymbol{g})$ be a time–orientable spacetime. A closed subset $\mathcal{S} \subset \mathcal{M}$ is called a* ***Cauchy surface*** *if every inextendible causal curve on $\mathcal{M}$ intersects $\mathcal{S}$ at exactly one point. A spacetime that possesses a Cauchy surface is said to be* ***globally hyperbolic****.*

Cauchy surfaces play an important role when the study of dynamical equations of motion on spacetimes is concerned. The reason has to do with the causal properties that these surfaces have which we shall now explain. A subset $\mathcal{S} \subset \mathcal{M}$ is said to be an *achronal* set if there do not exist $p, q \in \mathcal{S}$ such that p and q are connected by a future– or past–directed timelike curve. For example, the static slices Σ_t of a static spacetime are achronal so no directed timelike curves connect any two points in Σ_t, which follows from the fact that $\mathbb{R}$ is simply connected, and the orthogonality of the static surfaces to the orbits of the Killing field ξ. As any timelike curve connecting any two points on a Cauchy surface $\mathcal{S}$ would necessarily intersect it more than once, from Definition 2.1.2 it follows that any Cauchy surface must be achronal as well. Furthermore, globally hyperbolic spacetimes do not admit closed timelike curves. If this was not the case, then a closed timelike curve intersecting the Cauchy surface would contradict its achronality, while a closed timelike curve not intersecting it would contradict global hyperbolicity [25]. Since no directed causal curves connect any two points on a Cauchy surface $\mathcal{S}$, no events on $\mathcal{S}$ are causally related to each other, and thus, we may interpret $\mathcal{S}$ as representing an instant of time throughout the spacetime.

Events on Cauchy surfaces influence events throughout the whole spacetime in the following sense: For any closed achronal subset $\mathcal{S} \subset \mathcal{M}$, let $D^+(\mathcal{S}) \subset \mathcal{M}$ be the set of all points $p \in \mathcal{M}$ such that any inextendible past–directed causal curve passing through p intersects $\mathcal{S}$. The set $D^+(\mathcal{S})$ is called the *future domain of dependence* of $\mathcal{S}$ and the set $D^-(\mathcal{S})$, defined analogously, is called the *past domain of dependence* of $\mathcal{S}$. The full *domain of dependence* of $\mathcal{S}$ is the set $D(\mathcal{S}) = D^+(\mathcal{S}) \cup D^-(\mathcal{S})$. Definition 2.1.2 implies that if $\mathcal{S}$ is a Cauchy surface, then we have $D(\mathcal{S}) = \mathcal{M}$. If $\mathcal{S}$ is a closed achronal subset, but not a Cauchy surface, the interior of its domain of dependence, $\mathrm{int}[D(\mathcal{S})]$, is said to be a globally hyperbolic region [25, 26], *i.e.*, there is a subset in $\mathrm{int}[D(\mathcal{S})]$ for which $\mathcal{S}$ is a Cauchy surface according to Definition 2.1.2.

Classical fields at any point of a globally hyperbolic spacetime can be predicted or retrodicted from conditions on the instant of time the Cauchy surface $\mathcal{S}$ represents. More specifically, for certain systems of linear partial differential equations called *second order normally hyperbolic systems* on globally hyperbolic spacetimes, unique solutions which depend continuously on initial data defined on a Cauchy surface $\mathcal{S}$ can be found such that sufficiently small perturbations to the initial data corresponds to small changes in the solutions over a fixed compact region of $\mathcal{M}$ and, in addition, changes in the initial data over a closed region Ω of $\mathcal{S}$ should leave unchanged the solutions outside its future domain of dependence $D(\Omega)$ (an extensive discussion regarding hyperbolic systems of equations and their initial value problem can be found in [24, Chapter 7] or [25, Chapter 10]). If solutions of this form can be found, then the differential equations are said to have a *well–posed initial value problem.*

To determine if a given static spacetime is globally hyperbolic or not, we use the following criterion, a proof of which can be found in [30, Proposition 3.5]:

Theorem 2.1.3 *The static slices Σ of a standard static spacetime $\mathcal{M} = \mathbb{R} \times \Sigma$ with metric $\boldsymbol{g}$ given by Eq.* (2.1) *are Cauchy surfaces if and only if the metric $\boldsymbol{g}_R := \boldsymbol{h}/\mathcal{N}^2$ is complete,* i.e.*, if the Riemannian manifold $(\Sigma, \boldsymbol{g}_R)$ is a complete metric space.*

Remark 2.1.4 *The completeness of a Riemannian manifold ensures that the associated metric is geodesically complete, that is, that all inextendible geodesics have infinite proper length [26, Chapter XII]. This result is called the Hopf–Rinow theorem.*

Now, from Definition 2.1.2 it follows that if any of the static slices Σ_t for some $t \in \mathbb{R}$ of a static spacetime is a Cauchy surface, then the spacetime is globally hyperbolic. On the other hand, Theorem 2.1.3 asserts that an arbitrary static spacetime need not be globally hyperbolic.

Definition 2.1.5 *A **time function** on $(\mathcal{M}, \boldsymbol{g})$ is a differentiable function $\tau : \mathcal{M} \to \mathbb{R}$ such that the vector field $\mathrm{d}\tau^\sharp$, is always timelike.*

Remark 2.1.6 *Here, $\mathrm{d}\tau^\sharp \in \mathfrak{X}(\mathcal{M})$ is the gradient of the function τ, defined as the vector field such that, for any $\zeta \in \mathfrak{X}(\mathcal{M})$, we have $\boldsymbol{g}[\mathrm{d}\tau^\sharp, \zeta] = \zeta[\tau]$.*

Time functions play an important role to identify Cauchy surfaces in a manifold. The characterisation is given as a consequence of Theorem 2.1.3 [26, Corollary 11.7]:

Corollary 2.1.7 *On a globally hyperbolic spacetime $(\mathcal{M}, \boldsymbol{g})$, there exists a global time function τ whose level sets are Cauchy surfaces.*

We are interested in the problem of finding solutions to dynamical equations of motion that define a well–posed initial value problem in non–globally hyperbolic standard static spacetimes. As we have explained, the lack of global hyperbolicity implies that for these spacetimes there is no spacelike hypersurface whose domain of dependence is the whole spacetime $\mathcal{M}$, and thus, initial data defined on any spacelike region may fail to predict the dynamical evolution of a system in certain regions of $\mathcal{M}$. However, if the aim is to construct a quantum field theory on a non–globally hyperbolic static spacetime via canonical quantisation of solutions to dynamical equations of motion then, as it turns out, there is one requirement that can be imposed on the theory in order to have a well–defined Fock space of quantum states. In the rest of this chapter, we explain that for scalar and spinor field theories in an arbitrary static spacetime a quantum field theory can be defined if a certain spatial component of the associated classical equations of motion defines a self–adjoint (and additionally, for the case of a scalar field, positive) operator on the Hilbert space of solutions.

2.2 SCALAR AND SPINOR FIELDS ON STANDARD STATIC SPACETIMES

In this section we will consider a standard static spacetime $(\mathcal{M}, \boldsymbol{g})$ with metric tensor given by Eq. (2.1). We will not require the spacetime to be globally hyperbolic. We will obtain the classical field equations for free scalar and spinor theories from a standard variational principle and discuss some of the properties of the associated differential operators when taking into account a decomposition of the equations with respect to the time coordinate given by the Killing parameter.

We first introduce the relevant geometric objects for an N–dimensional static spacetime that will be used throughout this and subsequent chapters. Let us consider the static coordinates introduced in Section 2.1, and denote them by $(x^\mu) = (t, x^1, \ldots, x^{N-1})$, with $0 \le \mu \le N-1$, so that $x^0 = t$ and x^i with $1 \le i \le N-1$ are the local coordinates for Σ as in Definition 2.1.1. Then, the line element associated to the metric (2.1) is given in this coordinate system by

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu \,, \\ &= -\mathcal{N}(x)^2 \mathrm{d}t^2 + h_{ij}(x) \mathrm{d}x^i \mathrm{d}x^j \,, \end{aligned} \tag{2.2}$$

where $g_{\mu\nu} = \boldsymbol{g}[\partial_\mu, \partial_\nu]$ are the components of the metric tensor $\boldsymbol{g}$, and $h_{ij} = \boldsymbol{h}[\partial_i, \partial_j]$, with $1 \le i, j \le N-1$, are the components of the spatial metric tensor of Σ. The components of the Levi–Civita connection are given by ${\Gamma^\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\kappa}(\partial_\lambda g_{\nu\kappa} + \partial_\nu g_{\lambda\kappa} - \partial_\kappa g_{\nu\lambda})$, and from Eq. (2.2) they are found to be given by

$${\Gamma^0}_{i0} = \frac{1}{\mathcal{N}} \partial_i \mathcal{N} \,, \tag{2.3a}$$

$${\Gamma^i}_{00} = h^{ij} \mathcal{N} \partial_j \mathcal{N} \,, \tag{2.3b}$$

$${\Gamma^i}_{jk} = \frac{1}{2} h^{il} \left(\partial_j h_{kl} + \partial_k h_{jl} - \partial_l h_{jk} \right) \,, \tag{2.3c}$$

with h^{ij} denoting the components of the inverse of the metric tensor $\boldsymbol{h}$, and all other components are found to be zero. The action of the covariant derivative associated to the Levi–Civita connection on a vector field $v \in \mathfrak{X}(\mathcal{M})$ is found to be given by

$$\nabla_0 v^\mu = \partial_t v^\mu + \delta^\mu_i h^{ij} \mathcal{N} \partial_j \mathcal{N} v^0 + \delta^\mu_0 \frac{1}{\mathcal{N}} \partial_i \mathcal{N} v^i \,, \tag{2.4a}$$

$$\nabla_i v^\mu = \delta^\mu_0 \frac{1}{\mathcal{N}} \partial_i \mathcal{N} v^0 + \partial_i v^\mu + \delta^\mu_k {\Gamma^k}_{ij} v^j \,. \tag{2.4b}$$

The Lagrangian for the minimally coupled scalar field $\phi(t, x) \in C^\infty(\mathcal{M})$ with mass M in this spacetime [31, 32, 33] is

$$L_{KG} = \int_\Sigma \mathcal{L}_{KG} \left(\phi, \partial_\mu \phi \right) \mathrm{d}x \,, \tag{2.5}$$

with Σ representing any of the static slices Σ_t for some $t \in \mathbb{R}$, with $\mathrm{d}x := \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{N-1}$ and where the Lagrangian density $\mathscr{L}$ is given by

$$\mathscr{L}_{KG}(\phi, \partial_\mu \phi) = \frac{1}{2}\sqrt{h}\left[\mathcal{N}^{-1}(\partial_t \phi)^2 - \mathcal{N}\, h^{ij}(\partial_i \phi)(\partial_j \phi) - \mathcal{N} M^2 \phi^2\right], \tag{2.6}$$

with $h := \det(h_{ij})$. The Euler–Lagrange field equations [27, 34] are derived from the Lagrangian (2.5) and result in

$$-\frac{1}{\mathcal{N}^2}\partial_t^2 \phi + \frac{1}{\sqrt{h}\,\mathcal{N}}\partial_i(\sqrt{h}\, h^{ij} \mathcal{N} \partial_j \phi) - M^2 \phi = 0\,, \tag{2.7}$$

which is the Klein–Gordon equation [25, 27] associated to the static metric (2.1). In fact, we identify the Laplace–Beltrami operator on $(\mathcal{M}, \boldsymbol{g})$ as

$$\begin{aligned} \Box &= \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g} g^{\mu\nu}\partial_\nu\right)\,, \\ &= -\frac{1}{\mathcal{N}^2}\partial_t^2 + \frac{1}{\sqrt{h}\,\mathcal{N}}\partial_i(\sqrt{h}\, h^{ij}\mathcal{N}\partial_j)\,, \end{aligned} \tag{2.8}$$

where $g := |\det(g_{\mu\nu})|$, and $g^{\mu\nu}$ are the components of the inverse of the metric $\boldsymbol{g}$. Now, if the static spacetime $(\mathcal{M}, \boldsymbol{g})$ is not globally hyperbolic, then none of the static slices Σ_t are Cauchy surfaces. In particular, if the initial data are specified on the slice Σ_0 corresponding to $t = 0$, then solutions of Eq. (2.7) will only be defined on the domain of dependence $D(\Sigma_0) \neq \mathcal{M}$ and thus, there may be regions in $\mathcal{M}$ outside $D(\Sigma_0)$ for which no solutions continuously depending on the initial data can be determined. We will briefly describe the general prescription used by Wald [20] to address this particular issue. This will be presented in the form of a condition that a certain differential operator (defined below) must satisfy in order to find solutions of Eq. (2.7) defined throughout all $\mathcal{M}$.

The fact that the second term of Eq. (2.8) does not depend on the t–coordinate allows us to view the linear differential operator A, defined by

$$A := -\frac{\mathcal{N}}{\sqrt{h}}\partial_i \sqrt{h} h^{ij} \mathcal{N} \partial_j + M^2 \mathcal{N}^2\,, \tag{2.9}$$

as an operator on the Hilbert space $\mathscr{H}_{KG} := L^2(\Sigma_0; \mathrm{d}V)$ of square–integrable functions on the static slice Σ_0, where $\mathrm{d}V := \sqrt{h}\mathcal{N}^{-1}\mathrm{d}x$. With respect to this volume measure we can define the inner product between elements Φ_1, Φ_2 of the Hilbert space $\mathscr{H}_{KG}$ by

$$\langle \Phi_1, \Phi_2 \rangle_{KG} := \int_{\Sigma_0} \overline{\Phi_1(x)} \Phi_2(x) \mathrm{d}V\,, \tag{2.10}$$

and thus, the norm of an element $\Phi \in \mathscr{H}_{KG}$ is defined as $\|\Phi\|_{KG} = (\langle \Phi, \Phi \rangle_{KG})^{1/2}$.

To fully describe a linear operator on a Hilbert space, we need to specify the subspace in which it is defined, namely, the *domain* of the operator. In order to define the domain of the operator A in Eq. (2.9), which we denote by $\mathrm{Dom}(A) \subseteq \mathscr{H}_{KG}$, we first recall some important definitions for an arbitrary linear operator T on a Hilbert space $\mathscr{H}$ with inner product $\langle \cdot, \cdot \rangle$ [35, 36, 37]:

Definition 2.2.1 *A linear operator T is said to be* ***densely defined*** *if* $\mathrm{Dom}(T)$ *is dense in $\mathscr{H}$, that is, that every element $f \in \mathscr{H}$ can be obtained as a limit of a sequence $\{f_n\} \in \mathrm{Dom}(T)$. The operator A is said to be* ***closed*** *if, for a sequence $\{f_n\} \in \mathrm{Dom}(T)$ satisfying $f_n \to f$ and $Tf_n \to g$, $f \in \mathrm{Dom}(T)$ and $Tf = g$.*

Equivalently [35], a densely defined operator T is said to be closed if its graph, $G_T := \{(f,g) \in \mathscr{H} \times \mathscr{H} | f \in \mathrm{Dom}(T)\,, g = Tf\}$, is a topologically closed subset of $\mathscr{H} \times \mathscr{H}$ with respect to the topology induced by the norm

$$\langle f, g\rangle_T := \langle f, g\rangle + \langle Tf, Tg\rangle \ . \tag{2.11}$$

Given a densely defined operator T on $\mathscr{H}$, let us define the subspace $\mathcal{D}$ as the space of all $\tilde{f} \in \mathscr{H}$, for which there exists a unique element $F \in \mathscr{H}$ such that

$$\left\langle \tilde{f}, Tf \right\rangle = \langle F, f\rangle \ , \tag{2.12}$$

for every $f \in \mathrm{Dom}(T)$. Relating both functions, $\tilde{f}$ and F, gives rise to the next well–known concept.

Definition 2.2.2 *Let T be a closed, densely defined operator on $\mathscr{H}$, such that (2.12) holds for every $f \in \mathrm{Dom}(T)$. Then, the operator $T^\dagger$, with domain $\mathrm{Dom}(T^\dagger) := \mathcal{D}$, defined by $T^\dagger \tilde{f} := F$, is called the* ***adjoint*** *of the operator A. Equivalently, the adjoint operator $T^\dagger$ of T, is uniquely characterised by*

$$\left\langle \tilde{f}, Tf \right\rangle = \left\langle T^\dagger \tilde{f}, f \right\rangle \ .$$

The reason why we require T to be a densely defined operator is due to the fact that $\mathrm{Dom}(T)$ being dense in $\mathscr{H}$ ensures that F is uniquely determined by the relation (2.12).

Definition 2.2.3 *A densely defined operator T on a Hilbert space $\mathcal{H}$ is called* ***symmetric*** *if $\mathrm{Dom}(T) \subset \mathrm{Dom}(T^\dagger)$, and if $Tf = T^\dagger f$, for all $f \in \mathrm{Dom}(T)$. Equivalently, the operator T is symmetric if and only if*

$$\langle f, Tg\rangle = \langle Tf, g\rangle \ , \tag{2.13}$$

for all $f, g \in \mathrm{Dom}(T)$, that is, if the action of the adjoint $T^\dagger$ on the elements in $\mathrm{Dom}(T)$ is identical to that of T.

The defining relation for a symmetric operator, namely, (2.13), is only valid for elements in the domain of T, *not* for any $f \in \mathscr{H}$. In order to make (2.13) valid for a broader set of functions, we have to consider a more restrictive class of operators.

Definition 2.2.4 *The operator T is called* ***self-adjoint*** *if and only if T is symmetric and $\mathrm{Dom}(T) = \mathrm{Dom}(T^\dagger)$.*

From this definition, it is clear that not every symmetric operator is self–adjoint. Some relevant properties of self–adjoint operators which will be used here and in subsequent chapters can be found in Appendix A.

Returning to the operator A in Eq. (2.9), we will assume that its domain $\mathrm{Dom}(A)$ is specified in such a way that A is a closed, densely defined, self–adjoint operator. We will also require the operator A to be positive [36] with respect to the inner product in Eq. (2.10), *i.e.*, that for any $\Phi \in \mathrm{Dom}(A)$, we have

$$\langle \Phi, A\Phi \rangle_{KG} \geq 0\,. \tag{2.14}$$

This requirement ensures that the spectrum of the operator A is strictly positive and, in turn, that the operator $A^{1/2}$ is well defined. The natural analogue of positive–frequency solutions defined in QFT on Minkowski spacetime should obey the pseudo–differential equation $i\partial_t \phi = A^{1/2}\phi$ [27, Chapter 3]. Thus, if A is strictly positive then the time evolution of the field is unitary (see Eq. (2.16), below). From a more heuristic viewpoint, if A is allowed to have non–positive eigenvalues, the time evolution of the solutions results in field configurations which quickly diverge as the parameter t increases. The assumptions we have imposed on $\mathrm{Dom}(A)$ are not as restrictive as they may seem at first. In fact, if we require the operator A to be only symmetric instead of self–adjoint and positive with respect to the inner product (2.10), then the domain of A can be chosen to be the subspace $C_c^\infty(\Sigma_0)$ of smooth functions of compact support [25, 35] on Σ_0. With this domain, A is a densely defined symmetric operator [20, 21, 35, 36]. In Chapter 5 we will present a general prescription to obtain a positive self–adjoint operator from a symmetric operator, showing that, under certain conditions, the symmetric operator can be suitably "replaced" by a positive self–adjoint operator that correctly encodes the dynamics described by Eq. (2.7).

We can now reformulate the problem of solving Eq. (2.7) into the problem of finding a one–parameter family of vectors $\phi_t \in \mathscr{H}_{KG}$ satisfying the equation

$$A\phi_t = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\phi_t\,. \tag{2.15}$$

As noted by Wald [20], Eqs. (2.7) and (2.15) are not strictly equivalent: The existence of the partial derivative at a fixed spatial position $x \in \Sigma$ of the scalar function $\phi(t,x)$ does not imply the existence of the derivative of the vector $\phi_t \in \mathscr{H}_{KG}$ with respect to the parameter t in the Hilbert space sense, for which convergence is with respect to the L^2–norm of $\mathscr{H}_{KG}$. However, Ishibashi and Wald [20, 21], managed to show that if the operator A is self–adjoint and positive, then Eqs. (2.7) and (2.15) are equivalent inside the domain of dependence $D(\Sigma_0)$ of the initial data surface Σ_0, and that solutions to Eq. (2.15) extend those of the Klein–Gordon equation such that they continuously depend on initial conditions on Σ_0 and can be defined throughout all $\mathcal{M}$. We summarise their results in the following theorem:

Theorem 2.2.5 (Ishibashi-Wald) *Let A be the operator from Eq.* (2.9) *acting on the Hilbert space $\mathscr{H}_{KG}$ with domain* $\mathrm{Dom}(A)$*. If A is a positive and self–adjoint operator on* $\mathrm{Dom}(A)$*, then:*

1. *Given the initial data* $(\phi_0, \dot{\phi}_0) \in \mathrm{Dom}(A) \times \mathrm{Dom}(A)$*, the solution of Eq.* (2.15) *for each* $t \in \mathbb{R}$ *is given by*

$$\phi_t = \cos\left(A^{1/2}t\right)\phi_0 + A^{-1/2}\sin\left(A^{1/2}t\right)\dot{\phi}_0\,, \tag{2.16}$$

where the operators $\cos\left(A^{1/2}t\right)$ *and* $A^{-1/2}\sin\left(A^{1/2}t\right)$*, defined through the functional calculus* [1] *of the self–adjoint operator* A*, are bounded.*

2. *There exists a unique* $\phi \in C^\infty(\mathcal{M})$*, such that, for all* $t \in \mathbb{R}$*,*

$$\phi|_{\Sigma_t} = \phi_t\,, \qquad \xi^\mu \nabla_\mu \phi|_{\Sigma_t} = \frac{\mathrm{d}}{\mathrm{d}t}\phi_t\,, \tag{2.17}$$

and ϕ *satisfies Eq.* (2.7) *throughout* $\mathcal{M}$*.*

Remark 2.2.6 *The original argument in [20, 21] does not require the operator* A *to be self–adjoint, but only positive and symmetric. In their statement, they start with a symmetric operator* A *with* $\mathrm{Dom}(A) = C_c^\infty(\Sigma_0)$ *and then show that any of its positive self–adjoint extensions (see Chapter 5) satisfies points 1 and 2 above. Furthermore, in Ref. [21] they show that their prescription is unique in the sense that any initial value problem for Eq.* (2.7) *must arise from a particular choice of self–adjoint extension of* A*.*

Theorem 2.2.5 ensures that if the operator A is a positive self–adjoint operator, then the solution ϕ to Eq. (2.7) will be defined via initial data not just on the domain of dependence $D(\Sigma_0)$ of the static slice Σ_0 but on all $\mathcal{M}$, even if the spacetime is not globally hyperbolic.

A similar argument can be made for a spinor field satisfying Dirac's equation. In order to define a spinor field on $(\mathcal{M}, \boldsymbol{g})$, we introduce a local orthonormal frame [25, 28, 29] at each tangent space of the manifold $\mathcal{M}$ as follows; let $\{e_a\}_{a=0}^{N-1}$ be a linearly independent set of smooth vector fields and let ${e_a}^\mu$ denote the components of each $e_a \in \mathfrak{X}(\mathcal{M})$ with respect to the coordinates $(x^\mu) = (t, x)$, *i.e.*, $e_a = {e_a}^\mu \partial_\mu$. We define these vector fields such that at each point $p \in \mathcal{M}$ they form an orthonormal basis for the tangent space $T_p\mathcal{M}$. Thus, we require $\boldsymbol{g}_p[e_a, e_b] = \eta_{ab}$ and $\hat{e}^a[e_b] = \delta^a_b$, where η_{ab} are the components of the flat Lorentzian metric $\boldsymbol{\eta} = \mathrm{diag} = (-1, 1, \ldots, 1)$, and $\{\hat{e}^a\}_{a=0}^{N-1}$ is the dual basis for each cotangent space $T_p^*\mathcal{M}$ consisting of the *co–frame fields* with components ${e^a}_\mu$ defined through $\hat{e}^a = {e^a}_\mu \mathrm{d}x^\mu$ [29]. In terms of the components ${e_a}^\mu$, these orthonormality relations read

$$g_{\mu\nu}{e_a}^\mu {e_b}^\nu = \eta_{ab}\,, \qquad {e_a}^\mu {e^b}_\mu = \delta_a^b\,, \qquad {e_a}^\mu {e^a}_\nu = \delta^\mu_\nu\,. \tag{2.18}$$

With respect to this local frame the associated connection 1–form [28, 29] ${\omega^a}_b = {\omega^a}_{b\,\mu}\mathrm{d}x^\mu$, can be defined through its spacetime components as

$${\omega^a}_{b\,\mu} = \left(\partial_\mu {e_b}^\lambda + {\Gamma^\lambda}_{\mu\nu}{e_b}^\nu\right){e^a}_\lambda\,. \tag{2.19}$$

[1] See Definition A.0.7 in Appendix A.

An orthonormal frame compatible with the static metric $\boldsymbol{g}$ in Eq. (2.2) is obtained by letting

$$e_0{}^0 = \frac{1}{\mathcal{N}}\,, \qquad e_0{}^i = 0 = e_i^0\,, \qquad \text{for all } 1 \le i \le N-1\,, \tag{2.20}$$

and by defining $e_i{}^j$ for all $1 \le i,j \le N-1$ to be functions of the spatial coordinates x satisfying Eq. (2.18). Defined in this way, the fields e_a and $\hat{e}^a$ are time–independent. With this choice of orthogonal frame we find that the non–zero components of the connection 1–form in Eq. (2.19) are given by

$$\omega^0{}_{i\,0} = e_i{}^j \partial_j \mathcal{N}(x)\,, \tag{2.21a}$$

$$\omega^i{}_{j\,k} = \left(\partial_k e_j{}^l + \Gamma^l{}_{km} e_j{}^m\right) e^i{}_l\,. \tag{2.21b}$$

Next, we will consider the complex space $\mathbb{C}^{\tilde{N}}$, where $\tilde{N} := 2^{\lfloor N/2 \rfloor}$, with $\lfloor N/2 \rfloor$ denoting the floor function. We will use a representation for the N Dirac gamma matrices [38, 39], γ^a, of dimension $\tilde{N} \times \tilde{N}$ for which $(\gamma^0)^\dagger = -\gamma^0$ and $(\gamma^i)^\dagger = \gamma^i$. Under our signature convention the *anticommutation relations* read

$$\left\{\gamma^a, \gamma^b\right\} = 2\eta^{ab}\mathbb{I}\,, \tag{2.22}$$

where $\{\cdot,\cdot\}$ denotes the matrix anticommutator. We will make use of the quantity Σ^{ab} defined as

$$\Sigma^{ab} := \frac{1}{4}\left[\gamma^a, \gamma^b\right]\,, \tag{2.23}$$

where $[\cdot,\cdot]$ denotes the matrix commutator in $\mathbb{C}^{\tilde{N}}$. We will regard *spinor* fields on the static spacetime $(\mathcal{M}, \boldsymbol{g})$ as elements of the space $C^\infty(\mathcal{M}, \mathbb{C}^{\tilde{N}})$ of smooth maps from $\mathcal{M}$ to $\mathbb{C}^{\tilde{N}}$ [25, 40]. We assume that the spinor bundle is trivial. Given a spinor ψ, we define its *Dirac adjoint* by $\psi^* := \psi^\dagger \gamma^0$. The *spinor covariant derivative* [41, 42] is given by

$$\nabla^{(D)}_\mu = \partial_\mu + \frac{1}{2}\omega_{ab\,\mu}\,\Sigma^{ab}\,, \tag{2.24}$$

where $\omega_{ab\,\mu} = \eta_{ac}\omega^c{}_{b\,\mu}$. The spinor covariant derivatives in the time and spatial directions are obtained using Eq. (2.21), and read

$$\nabla^{(D)}_0 = \partial_t - e_i{}^j \partial_j \mathcal{N} \Sigma^{0i}\,, \tag{2.25a}$$

$$\nabla^{(D)}_i = \partial_i + \frac{1}{2}\omega_{jk\,i}\,\Sigma^{jk}\,, \tag{2.25b}$$

respectively.

The Lagrangian for a free Dirac spinor field $\psi \in C^\infty(\mathcal{M}, \mathbb{C}^{\tilde{N}})$ with mass M is [34]

$$L_D = \int_{\Sigma_0} \mathscr{L}_D(\psi, \psi^*)\mathrm{d}x\,, \tag{2.26}$$

with the Lagrangian density $\mathscr{L}_D$ given by

$$\mathscr{L}_D(\psi, \psi^*) = \sqrt{h}\,\mathcal{N}\psi^*\left(\tilde{\gamma}^\mu \nabla^{(D)}_\mu - M\right)\psi\,, \tag{2.27}$$

where we have defined the spacetime gamma matrices $\tilde{\gamma}^{\mu} = e_a{}^{\mu}\gamma^a$. Variation of the action functional $S_D[\psi, \psi^*] = \int L_D \mathrm{d}t$ with respect to the field ψ^* yields the Euler–Lagrange field equation [27, 34]

$$\tilde{\gamma}^0 \partial_t \psi - \left[\tilde{\gamma}^i \left(\mathcal{N}^2 \nabla_i^{(D)} + \frac{1}{2} \mathcal{N} \partial_i \mathcal{N} \right) - M \mathcal{N}^2 \right] \psi = 0 \,, \tag{2.28}$$

which is the Dirac equation [25, 27] associated to the metric (2.1). Similarly to the case of a scalar field, if the static spacetime is not globally hyperbolic, then Eq. (2.28) will not determine the spinor $\psi(t, x)$ outside the domain of dependence $D(\Sigma_0)$. Nevertheless, the operator inside the square brackets of Eq. (2.28) is time–independent. This allows us to view the differential operator

$$\mathbb{D} := i \mathcal{N}^2 \tilde{\gamma}^0 \left[\tilde{\gamma}^i \left(\nabla_i^{(D)} + \frac{1}{2} \mathcal{N}^{-1} \partial_i \mathcal{N} \right) - M \right] , \tag{2.29}$$

as a linear operator on the Hilbert space $\mathscr{H}_D := L^2(\Sigma_0, \mathbb{C}^{\tilde{N}}; \mathrm{d}V')$ of square–integrable maps $\Sigma_0 \to \mathbb{C}^{\tilde{N}}$, with respect to the measure $\mathrm{d}V' := \sqrt{h}\, \mathrm{d}x$. We will refer to the maps $\Psi : \Sigma_0 \to \mathbb{C}^{\tilde{N}}$ as *spatial spinors*. The inner product between elements $\Psi_1, \Psi_2 \in \mathscr{H}_D$ is defined by

$$\langle \Psi_1, \Psi_2 \rangle_D := \int_{\Sigma_0} \Psi_1(x)^{\dagger} \Psi_2(x) \, \mathrm{d}V' \,, \tag{2.30}$$

and thus, the norm of a spatial spinor $\Psi \in \mathscr{H}_D$ is given by $\|\Psi\|_D := (\langle \Psi, \Psi \rangle_D)^{1/2}$. We note that due to the time–independence of the inner product (2.30), it can be defined as the integral over any of the static slices Σ_t. We will once again impose the crucial assumption that the domain $\mathrm{Dom}(\mathbb{D}) \subset \mathscr{H}_D$ of the operator in Eq. (2.29) is defined such that $\mathbb{D}$ is a densely defined self–adjoint operator with respect to the inner product (2.30) in the sense of Defs. 2.2.1 and 2.2.4.

Thus, following a similar argument as for the scalar field, we will reformulate the problem of finding solutions to Eq. (2.28) into the problem of finding the one–parameter family of vectors $\psi_t \in \mathscr{H}_D$ satisfying the equation

$$\mathbb{D} \psi_t = -i \frac{\mathrm{d}}{\mathrm{d}t} \psi_t \,, \tag{2.31}$$

where once again, the derivative of ψ_t with respect to the parameter t is to be understood in the sense of strong convergence in the Hilbert space norm [35, 43]. We will only concern ourselves with solving Eq. (2.31) and showing that solutions to this equation are also solutions of Eq. (2.28) throughout all $D(\Sigma_0)$. Then, with an additional assumption we will arrive at an analogous result to Theorem 2.2.5 for the operator $\mathbb{D}$, that is, the solution of Eq. (2.31) depends on the set of smooth initial data on Σ_0 and it defines a smooth solution of Eq. (2.28) throughout $\mathcal{M}$.

Proposition 2.2.7 *For any $\psi_0 \in \mathrm{Dom}(\mathbb{D})$, the one–parameter family of vectors*

$$\psi_t = \exp\left(it\mathbb{D}\right) \psi_0 \,, \qquad t \in \mathbb{R} \,, \tag{2.32}$$

where the operator $\exp(it\mathbb{D})$ *is defined by the functional calculus of the self–adjoint operator* $\mathbb{D}$ *in Eq.* (2.29)*, gives a unique solution to Eq.* (2.31)*, with* $\psi_t \in \mathrm{Dom}(\mathbb{D})$ *for all* $t \in \mathbb{R}$*.*

Proof: By point 1 of Proposition A.0.9 in Appendix A and by the self–adjointness of $\mathbb{D}$ we have that, for all $t \in \mathbb{R}$, $u(t) := \exp(it\mathbb{D})$ defines a strongly continuous one–parameter unitary group on $\mathscr{H}_D$ (see Definition A.0.8). Since $\psi_0 \in \mathrm{Dom}(\mathbb{D})$, point 2 of the same proposition implies that

$$\mathbb{D}\psi_0 = \lim_{t\to 0} \frac{\exp(it\mathbb{D})\,\psi_0 - \psi_0}{t}\,, \tag{2.33}$$

where the convergence of the limit is with respect to the norm $\|\cdot\|_D$. Thus, it follows that

$$\begin{aligned}\lim_{s\to 0} \frac{\exp(i(t+s)\mathbb{D})\,\psi_0 - \exp(it\mathbb{D})\,\psi_0}{s} &= \exp(it\mathbb{D}) \lim_{s\to 0} \frac{\exp(is\mathbb{D})\,\psi_0 - \psi_0}{s}\,. \\ &= i\exp(it\mathbb{D})\,\mathbb{D}\psi_0\,. \end{aligned} \tag{2.34}$$

On the other hand, we also have

$$\lim_{s\to 0} \frac{\exp(i(t+s)\mathbb{D})\,\psi_0 - \exp(it\mathbb{D})\,\psi_0}{s} = \lim_{s\to 0} \frac{\exp(is\mathbb{D})\,\psi_t - \psi_t}{s}\,, \tag{2.35}$$

which by Eq. (2.34) exists. Therefore, point 3 of Proposition A.0.9, implies that $\psi_t \in \mathrm{Dom}(\mathbb{D})$ and the right–hand side of Eq. (2.35) is equal to $i\mathbb{D}\psi_t$. Furthermore, the left–hand side of Eq. (2.35) reduces to $\mathrm{d}\psi_t/\mathrm{d}t$. Hence, ψ_t as given by Eq. (2.32) is a solution of Eq. (2.31) for all $t \in \mathbb{R}$. Finally, to prove uniqueness, we consider two solutions ψ_t, ψ'_t of Eq. (2.31), with $\psi_0 = \psi'_0 \in \mathrm{Dom}(\mathbb{D})$. Define $\varphi_t := \psi_t - \psi'_t$ for all $t \in \mathbb{R}$, so that $\varphi_0 = 0$. By the previous argument, $\varphi_t \in \mathrm{Dom}(\mathbb{D})$, thus, $\mathrm{d}\varphi_t/\mathrm{d}t = i\mathbb{D}\varphi_t$, which implies that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \varphi_t, \varphi_t\right\rangle_D &= i\left\langle \varphi_t, \mathbb{D}\varphi_t\right\rangle_D - i\left\langle \mathbb{D}\varphi_t, \varphi_t\right\rangle_D\,, \\ &= 0\,, \end{aligned} \tag{2.36}$$

as $\mathbb{D}$ is a self–adjoint operator. Thus, $\|\varphi_t\|_D^2$ is constant throughout $\mathbb{R}$. Since $\varphi_0 = 0$, we conclude $\psi_t = \psi'_t$ for all $t \in \mathbb{R}$. ■

Now we show that solutions to Eq. (2.31) reproduce the solutions of Eq. (2.28) obtained by Cauchy evolution in the region $D(\Sigma_0)$ by adapting the argument made for the scalar field in [20]. Let φ be a solution obtained through Cauchy evolution of Eq. (2.28) with initial data $\varphi(0,x) = \psi_0$, with $\psi_0 \in C^\infty(\Sigma_0, \mathbb{C}^{\tilde{N}}) \cap \mathrm{Dom}(\mathbb{D})$. For each static slice Σ_t we define $\varphi_t \in \mathscr{H}_D$ by $\varphi_t(x) = \varphi(t,x)$ for all $(t,x) \in \Sigma_t \cap D(\Sigma_0)$. Assume that φ and the solution ψ_t from Eq. (2.32) differ from each other on $D(\Sigma_0)$. Then, there exists a static slice Σ_{t_1} such that $\varphi_{t_1} \neq \psi_{t_1}$, as elements of $\mathscr{H}_D$, on $\Sigma_{t_1} \cap D(\Sigma_0)$. Let $\mathcal{S}$ be a Cauchy surface for $D(\Sigma_0)$ such that $\mathcal{S}$ includes an open neighbourhood within Σ_{t_1} on which $\varphi_{t_1} \neq \psi_{t_1}$. Let $f_{t_1} \in C_c^\infty(\mathcal{S}, \mathbb{C}^{\tilde{N}})$ be a spatial spinor of compact support contained in $\mathcal{S} \cap \Sigma_{t_1}$, with the property that

$$\int_{\mathcal{S}\cap\Sigma_{t_1}} f_{t_1}^\dagger(x)\,(\varphi_{t_1}(x) - \psi_{t_1}(x))\,\mathrm{d}V' \neq 0\,. \tag{2.37}$$

We define f throughout $D(\Sigma_0)$ to be the Cauchy development of the initial data f_{t_1} on $\mathcal{S}$, and outside $D(\Sigma_0)$ in the region between Σ_0 and Σ_{t_1}, we set $f = 0$. Thus, f satisfies Eq. (2.28) throughout the region between Σ_0 and Σ_{t_1}, and for all $0 \leq t \leq t_1$, the restriction $f_t := f|_{\Sigma_t}$ lies in $C_c^\infty(\Sigma_t, \mathbb{C}^{\tilde{N}})$. Consider the quantity

$$c(t) := \int_{\Sigma_{t_1}} f(t,x)^\dagger \left(\varphi(t,x) - \psi_t(x)\right) \mathrm{d}V' \,, \tag{2.38}$$

with t_1 held fixed (we recall that $\mathrm{d}V' = \sqrt{h}\,\mathrm{d}x$ is t–independent). The derivative of c with respect to t is given by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} c(t) &= \int_{\Sigma_{t_1}} \left[f(t,x)^\dagger \left(\partial_t \varphi(t,x) - \frac{\mathrm{d}}{\mathrm{d}t}\psi_t(x) \right) + \partial_t f(t,x)^\dagger \left(\varphi(t,x) - \psi_t(x)\right) \right] \mathrm{d}V' \,, \\ &= \int_{\Sigma_{t_1}} \left[f(t,x)^\dagger \partial_t \varphi(t,x) + \partial_t f(t,x)^\dagger \varphi(t,x) \right] \mathrm{d}V' \\ &\quad - \int_{\Sigma_{t_1}} \left[f(t,x)^\dagger \frac{\mathrm{d}}{\mathrm{d}t}\psi_t(x) + \partial_t f(t,x)^\dagger \psi_t(x) \right] \mathrm{d}V' \,, \end{aligned} \tag{2.39}$$

and, since the spinors f and φ are solutions of Eq. (2.28), we can replace their partial derivatives with respect to t appearing in the first term of the second equality above with $(i\mathbb{D}f)^\dagger$ and $i\mathbb{D}\varphi$, respectively. By integrating this term by parts, and recalling that f_{t_1} is of compact support, we find that it vanishes. Furthermore, we note that for any Σ_t with $0 \leq t \leq t_1$, we have $\partial_t f(t,x) = i(\mathbb{D}f_t)(x)$ by means of the way we have defined f. Thus, using Eq. (2.31) for ψ_t, Eq. (2.39) now reads

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} c(t) =& - \int_{\Sigma_{t_1}} \left[f(t,x)^\dagger (i\mathbb{D}\psi_t)(x) - i(\mathbb{D}f_t)^\dagger(x)\psi_t(x) \right] \mathrm{d}V' \,, \\ =& - i \left\langle f_t, \mathbb{D}\psi_t \right\rangle_D + i \left\langle \mathbb{D}f_t, \psi_t \right\rangle_D \,, \\ =& 0 \,, \end{aligned} \tag{2.40}$$

since $\mathbb{D}$ is a self–adjoint operator. From Eq. (2.38) it is clear that $c(0) = 0$ as $\varphi_0 = \psi_0$. On the other hand, Eq. (2.37) implies that $c(t_1) \neq 0$, and thus, we arrive at a contradiction. Hence, we must have $\varphi_t = \psi_t$ everywhere on $D(\Sigma_0)$.

We have shown that solutions to Eq. (2.28) obtained by Cauchy evolution of the initial data ψ_0 coincide with solutions of Eq. (2.31) on $D(\Sigma_0)$. Now, for any $t \in \mathbb{R}$ and $x \in \Sigma_0$ the solution $\psi_t(x)$ given by Eq. (2.32) is well defined as long as ψ_0 lies in $\mathrm{Dom}(\mathbb{D})$. However, ψ_t and $\mathrm{d}\psi_t/\mathrm{d}t$ might fail to define spatial spinors that are smooth in the spatial variables. The approach taken by Wald in Ref. [20] for the case of the scalar field relies on the fact that one can apply an elliptic regularity theorem [44] to the operator A^k for all $k \in \mathbb{N}$, which cannot be directly adapted to the operator $\mathbb{D}$ in Eq. (2.29). Nevertheless, we will find in Chapter 8 that, for the specific case of interest of two–dimensional anti-de Sitter spacetime, solutions which are smooth on the spatial variable for fixed t can be found when the operator $\mathbb{D}$ is self–adjoint. If the solution ψ_t and its derivative $\mathrm{d}\psi_t/\mathrm{d}t$ for fixed t are smooth with respect to the spatial variables, then spacetime smoothness at an arbitrary point $(t,x) \in \mathcal{M}$ is proven as follows (even for the general case of a static spacetime). Let

Σ_t be the static surface passing through $(t, x) \in \mathcal{M}$. Then on this surface ψ_t and $\mathrm{d}\psi_t/\mathrm{d}t$ are smooth with respect to x. Therefore, the solution φ of Eq. (2.28) with initial data $\varphi(t, x) = \psi_t(x)$ will be smooth throughout $D(\Sigma_t)$. Then, by the same argument leading to Eq. (2.40), the solution $\psi_t(x)$ agrees with $\varphi(t, x)$ on $D(\Sigma_t)$. Since $(t, x) \in D(\Sigma_t)$, this shows that ψ is smooth at (t, x) in the spacetime sense. This in turn also proves that the existence of the derivative $\mathrm{d}/\mathrm{d}t$ of ψ_t in the strong convergence sense implies the existence of the partial derivative ∂_t for the solution φ via spacetime smoothness.

2.3 CANONICAL QUANTISATION OF SCALAR AND SPINOR FIELDS

In this section we will construct the quantum field theories for the scalar and spinor fields satisfying Eqs. (2.7) and (2.28), respectively, through canonical quantisation. We assume that the operators A and $\mathbb{D}$ in Eqs. (2.9) and (2.29) are self–adjoint and in addition, the operator A is also assumed to be positive, *i.e.*, it satisfies Eq. (2.14). We will explain that these conditions are sufficient to obtain the associated multi–particle Fock spaces for the quantum theories. We will follow the usual quantisation procedure which can be found in Refs. [24, 25, 27, 34]. We also show that if the spacetime $(\mathcal{M}, \boldsymbol{g})$ admits an isometry group [25, 29], such that the space of solutions of the dynamical equations obeys a certain invariance condition, then the vacuum state of the theory will be invariant under the action of the isometries.

We consider the Lagrangian density $\mathscr{L}_{KG}(\phi, \partial_\mu\phi)$ of a minimally coupled, free scalar field ϕ on the standard static spacetime $(\mathcal{M}, \boldsymbol{g})$, given by Eq. (2.6). The conjugate momentum density [27, 31, 34] is defined as

$$\begin{aligned}\pi(t, x) &:= \frac{\partial \mathscr{L}_{KG}}{\partial(\partial_t\phi(t, x))} \\ &= \frac{\sqrt{h(x)}}{\mathcal{N}(x)}\partial_t\phi(t, x)\,. \end{aligned} \tag{2.41}$$

Canonical quantisation of the scalar field theory is achieved by regarding the general solution ϕ of the Klein–Gordon equation (2.7) obtained through the Lagrangian density $\mathscr{L}_{KG}$ and the conjugate momentum π in Eq. (2.41), not as functions of the spacetime $(\mathcal{M}, \boldsymbol{g})$ but as *operator–valued distributions* [27, 34] on $\mathcal{M}$. The idea is that for each point $(t, x) \in \mathcal{M}$, the quantity $\phi(t, x)$ defines a linear operator that acts on a Hilbert space $\mathscr{F}_{KG}$ of physical quantum states, which we will define shortly. This is analogous to the way operators are defined in the Heisenberg picture of ordinary (point–particle) Quantum Mechanics [37]. The distributional aspect of ϕ will be explained in more detail after writing the field ϕ in a more explicit form (see Eq. (2.46)). Thus, from now on we will regard ϕ as an operator–valued distribution and refer to it as a *quantum field*. We will require ϕ and π to obey the *equal–time canonical commutation relations* [34] given by

$$[\phi(t, x), \pi(t, x')] = i\delta(x; x')\,, \tag{2.42a}$$

$$[\phi(t, x), \phi(t, x')] = [\pi(t, x), \pi(t, x')] = 0\,, \tag{2.42b}$$

where δ is defined in the distributional sense by

$$\int_{\Sigma_t} \delta(x;x')f(x')\mathrm{d}x = f(x)\,, \tag{2.43}$$

for any smooth compactly supported function f on any of the static slices Σ_t.

Now we will consider solutions of Eq. (2.7) or, more precisely, solutions of Eq. (2.15) which are defined for all $(t,x) \in \mathcal{M}$ by virtue of Theorem 2.2.5. Since $(\mathcal{M}, \boldsymbol{g})$ is static, the orthogonality between the static slices Σ_t and the orbits of the Killing vector ξ [25] allows us to write these solutions[2] in the form

$$\phi_\sigma(t,x) = \frac{1}{\sqrt{2\omega_\sigma}}\Phi_\sigma(x)e^{-i\omega_\sigma t}\,, \tag{2.44}$$

where σ labels the elements of the spectrum $\boldsymbol{\sigma}(A)$ of the operator A in Eq. (2.9), and $\Phi_\sigma \in \mathscr{H}_{KG}$ satisfies the equation

$$A\Phi_\sigma = \omega_\sigma^2 \Phi_\sigma\,. \tag{2.45}$$

Now we will assume that the spectrum of the operator A is purely discrete, and that $\omega_\sigma^2 > 0$ for all σ, *i.e.*, that A is a strictly positive operator. Then, Theorem A.0.3 in Appendix A tells us that the eigenfunctions Φ_σ of the operator A form a complete orthonormal set for $\mathscr{H}_{KG}$, and we set $\langle \Phi_\sigma, \Phi_{\sigma'}\rangle_{KG} = \delta_{\sigma\sigma'}$. This allows us to write the quantum field $\phi(t,x)$ as

$$\phi(t,x) = \sum_\sigma \left[a_\sigma \phi_\sigma(t,x) + a_\sigma^\dagger \overline{\phi_\sigma(t,x)}\right]\,, \tag{2.46}$$

with $\omega_\sigma > 0$ and $\sigma \in \mathbb{N}$, where the mode functions $\phi_\sigma(t,x)$ are defined by Eq. (2.44), and the coefficients $a_\sigma, a_\sigma^\dagger$ are the *annihilation* and *creation operators*, respectively. The operational nature of these quantities will be made clear once the Hilbert space $\mathscr{F}_{KG}$ is properly defined below. The spectrum $\boldsymbol{\sigma}(A)$ is referred to as the *frequency spectrum*, and its elements ω_σ denote the allowed energy frequencies of the quantum field ϕ. We note that Eq. (2.46) is a *mode expansion* in terms of *positive–frequency* solutions.

Now, the infinite sum in Eq. (2.46) may not converge pointwise to an operator on the Hilbert space $\mathscr{F}_{KG}$ [24, 25, 27]. Thus, ϕ must be understood in a distributional sense, *i.e.*, for any smooth compactly supported function f on the spacetime $\mathcal{M}$, the quantity

$$\int_{\mathcal{M}} f(t,x)\phi(t,x)\sqrt{g}\mathrm{d}t\mathrm{d}x\,, \tag{2.47}$$

defines an operator acting on the Hilbert space $\mathscr{F}_{KG}$ and has to be understood formally.

Using the fact that $\{\Phi_\sigma\}_{\sigma\in\mathbb{N}}$ is a complete orthonormal set for $\mathscr{H}_{KG}$, we find that

$$\sum_\sigma \Phi_\sigma(x)\overline{\Phi_\sigma(x')} = \frac{\mathcal{N}(x)}{\sqrt{h(x)}}\delta(x,x')\,. \tag{2.48}$$

[2] Here we are implicitly assuming that the underlying field ϕ is a tempered distribution along the time direction [35].

This completeness relation allows one to show that the equal–time commutation relations (2.42) are equivalent to the commutation relations among the annihilation and creation operators given by

$$[a_\sigma, a^\dagger_{\sigma'}] = \delta_{\sigma\sigma'} \,, \tag{2.49}$$

with all other commutators among a_σ and $a^\dagger_{\sigma'}$ vanishing.

We now introduce the definition of the Hilbert space $\mathscr{F}_{KG}$ of quantum states. The following construction is based on Ref. [25, Section 14.2]. Let us consider the Hilbert space $\mathscr{H}_{KG}$ of solutions of the Klein–Gordon equation and, from it, we construct the n–fold *symmetric* tensor product space denoted by $\otimes^n_S \mathscr{H}_{KG}$. The space $\otimes^n_S \mathscr{H}_{KG}$ consists of continuous n–multilinear maps $\Phi^{(n)} : \mathscr{H}_{KG} \times \cdots \times \mathscr{H}_{KG} \to \mathbb{C}$ which are invariant under permutations of its arguments. This space can be shown to be a Hilbert space with respect to the inner product induced from the tensor product structure in the usual way (see for example Ref. [35, Section II.4, Proposition 1]). The algebraic direct sum of Hilbert spaces defined by

$$\mathscr{F}^{KG}_0 := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} (\otimes^n_S \mathscr{H}_{KG}) \,, \tag{2.50}$$

consists of *terminating* sequences of the form

$$|\Phi\rangle_B = (\Phi^{(0)}, \Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(n)}, 0, \ldots) \,, \tag{2.51}$$

with $\Phi^{(0)} \in \mathbb{C}$ and with each $\Phi^{(j)} \in \otimes^j_S \mathscr{H}_{KG}$. The space $\mathscr{F}^{KG}_0$ is endowed with the norm $\|\cdot\|_{\mathscr{F}_{KG}}$, defined through

$$\|\Phi\|^2_{\mathscr{F}_{KG}} = \langle \Phi \,|\, \Phi\rangle_B := |\Phi^{(0)}|^2 + \sum_{n=0}^{\infty} \|\Phi^{(n)}\|^2_{KG,n} \,, \tag{2.52}$$

for all $|\Phi\rangle_B$, where $\|\cdot\|_{KG,n}$ denotes the norm of $\otimes^n_S \mathscr{H}_{KG}$. The Hilbert space completion $\mathscr{F}_{KG} := \overline{\mathscr{F}^{KG}_0}$ with respect to the norm in Eq. (2.52) is known as the multi–particle *bosonic* Fock space for the Klein–Gordon field, and it represents the Hilbert space of quantum states. Elements in $\mathscr{F}_{KG}$ consist of infinite sequences of finite norm. For each state $\Phi' \in \mathscr{H}_{KG}$, we define the unbounded operators $a(\Phi')$ and $a^\dagger(\Phi')$ on $\mathscr{F}_{KG}$ with dense domain given by $\mathscr{F}^{KG}_0$ as follows. Given $\Phi' \in \mathscr{H}_{KG}$, the action of $a(\Phi')$ on the element $|\Phi\rangle_B \in \mathscr{F}^{KG}_0$ is defined as

$$a(\Phi')\,|\Phi\rangle_B := (\overline{\Phi'} \cdot \Phi^{(1)}, \sqrt{2}\,\overline{\Phi'} \cdot \Phi^{(2)}, \ldots, \sqrt{n}\,\overline{\Phi'} \cdot \Phi^{(n)}, 0, \ldots) \,, \tag{2.53}$$

where $\overline{\Phi'} \cdot \Phi^{(j)} \in \otimes^{j-1}_S \mathscr{H}_{KG}$ denotes the insertion of $\overline{\Phi'}$ into one of the arguments of the functional $\Phi^{(j)}$. Similarly, for each $\Phi' \in \mathscr{H}_{KG}$, the action of the operator $a^\dagger(\Phi')$ on an element in the form of Eq. (2.51) is defined as

$$a^\dagger(\Phi')\,|\Phi\rangle_B := (0, \Phi^{(0)}\tilde{\Phi}', \sqrt{2}\Phi^{(1)} \otimes_S \tilde{\Phi}', \ldots, \sqrt{n+1}\Phi^{(n)} \otimes_S \tilde{\Phi}', 0, \ldots) \,, \tag{2.54}$$

where $\tilde{\Phi}' \in \mathscr{H}_{KG}^*$ is the unique continuous linear functional associated to Φ' (*i.e.*, $\tilde{\Phi}'(\Phi) = \langle \Phi', \Phi \rangle_{KG}$ for all $\Phi \in \mathscr{H}_{KG}$). For any $\Phi_1, \Phi_2 \in \mathscr{H}_{KG}$ we can calculate the commutator between the operators $a(\Phi_1)$ and $a^\dagger(\Phi_2)$. Indeed, using Eqs. (2.53) and (2.54), we obtain

$$
\begin{aligned}
a(\Phi_1)a^\dagger(\Phi_2)\,|\Phi\rangle_B &= \left(\Phi^{(0)}\overline{\Phi_1}\cdot\tilde{\Phi}_2, 2\overline{\Phi_1}\cdot(\Phi^{(1)}\otimes_S\tilde{\Phi}_2), \dots, (n+1)\overline{\Phi_1}\cdot(\Phi^{(n)}\otimes_S\tilde{\Phi}_2), 0, \dots\right)\,, \\
a^\dagger(\Phi_2)a(\Phi_1)\,|\Phi\rangle_B &= \left(0, (\overline{\Phi_1}\cdot\Phi^{(1)})\tilde{\Phi}_2, 2(\overline{\Phi_1}\cdot\Phi^{(2)})\otimes_S\tilde{\Phi}_2 \cdots, n(\overline{\Phi_1}\cdot\Phi^{(n)})\otimes_S\tilde{\Phi}_2, 0, \dots\right)\,,
\end{aligned}
$$

and, since for each $1 \leq j \leq n$, we have[3]

$$
\begin{aligned}
(j+1)\overline{\Phi_1}\cdot(\Phi^{(j)}\otimes_S\tilde{\Phi}_2) - j(\overline{\Phi_1}\cdot\Phi^{(j)})\otimes_S\tilde{\Phi}_2 &= (\overline{\Phi_1}\cdot\tilde{\Phi}_2)\Phi^{(j)}\,, \\
&= \left\langle \Phi_1, \overline{\Phi_2}\right\rangle_{KG}\Phi^{(j)}\,,
\end{aligned}
\tag{2.55}
$$

then it follows that

$$
\left[a(\Phi_1), a^\dagger(\Phi_2)\right]|\Phi\rangle_B = \left\langle \Phi_1, \overline{\Phi_2}\right\rangle_{KG}|\Phi\rangle_B\,. \tag{2.56}
$$

Similar calculations show that all other commutators between the operators $a(\Phi)$ and $a^\dagger(\Phi)$ vanish. Hence, if we consider the orthonormal basis $\{\Phi_\sigma\}_{\sigma\in\boldsymbol{\sigma}(A)}$ of $\mathscr{H}_{KG}$, we can define the operators $a_\sigma := a(\Phi_\sigma)$ and $a_\sigma^\dagger := a^\dagger(\overline{\Phi_\sigma})$ for each basis element Φ_σ. Then, Eq. (2.56) reduces to Eq. (2.49) and the operators a_σ and $a_\sigma^\dagger$ are precisely the operators appearing in the mode expansion of Eq. (2.46).

The definition of the action of the annihilation operators a_σ through Eq. (2.53) implies that the element

$$
|0\rangle_B := (1, 0, 0, \dots) \in \mathscr{F}_{KG}\,, \tag{2.57}
$$

known as the *bosonic vacuum state*, is uniquely characterised, up to a phase factor, by the condition

$$
a_\sigma\,|0\rangle_B = 0\,, \tag{2.58}
$$

for all $\sigma \in \boldsymbol{\sigma}(A)$. Note that Eq. (2.52) we obtain $\|0\|^2_{\mathscr{F}_{KG}} = \langle 0|0\rangle_B := 1$.

A basis for the space $\mathscr{F}_{KG}$ can be constructed from the action of the creation operators on the vacuum state. Consider the orthonormal basis $\{\Phi_\sigma\}_{\sigma\in\boldsymbol{\sigma}(A)}$ of $\mathscr{H}_{KG}$ of eigenfunctions of the operator A. Then, since the spectrum of A is assumed to be purely discrete, the elements of the form $\tilde{\Phi}_{\sigma_1}\otimes_S\cdots\otimes_S\tilde{\Phi}_{\sigma_n}$ form a basis for the Hilbert spaces $\otimes_S^n\mathscr{H}_{KG}$, and varying $n\in\mathbb{N}$, we obtain a basis for $\mathscr{F}_{KG}$ [27]. Now, we consider the elements in $\mathscr{F}_{KG}$ of the form

$$
\left|n_{1\,\sigma_1}, n_{2\,\sigma_2}, \dots, n_{k\,\sigma_k}\right\rangle_B := \frac{1}{\sqrt{n_1!n_2!\cdots n_k!}}\left(a_{\sigma_1}^\dagger\right)^{n_1}\left(a_{\sigma_2}^\dagger\right)^{n_2}\cdots\left(a_{\sigma_k}^\dagger\right)^{n_k}|0\rangle_B\,, \tag{2.59}
$$

for some $k\in\mathbb{N}$ and $n_1,\dots,n_k\in\mathbb{N}$. A straightforward calculation using Eqs. (2.54) and (2.57) shows that for each $k\in\mathbb{N}$ and $n_1,\dots n_k\in\mathbb{N}$, the element in Eq. (2.59) is

[3]This follows immediately after evaluating the left–hand side on $(\phi_1, \phi_2, \dots, \phi_j)$ and using the symmetry of the functionals with respect to permutations of their arguments.

proportional to the basis element $\tilde{\Phi}_{\sigma_1}^{\otimes n_1} \otimes_S \cdots \otimes_S \tilde{\Phi}_{\sigma_k}^{\otimes n_k}$. Hence, the states of the form of Eq. (2.59), form a basis of $\mathscr{F}_{KG}$. The normalisation of these basis elements is obtained through

$$\begin{aligned} &\left\langle n_{1\,\sigma_1}, \ldots, n_{j\,\sigma_j} \middle| m_{1\,\sigma'_1}, \ldots, m_{k\,\sigma'_k} \right\rangle_B \\ &:= \delta_{jk} \sum_{\alpha} \delta_{n_1,m_{\alpha(1)}} \cdots \delta_{n_j,m_{\alpha(k)}} \delta_{\sigma_1,\sigma'_{\alpha(1)}} \ldots \delta_{\sigma_j,\sigma'_{\alpha(k)}}\,, \end{aligned} \tag{2.60}$$

where the sum is over all permutations α of the set of integers $\{1, 2, \ldots, k\}$.

With respect to the inner product on $\mathscr{F}_{KG}$ obtained by linear extension of Eq. (2.60), it can be shown that the operators $a_\sigma^\dagger$ are in fact the conjugate transpose of the operators a_σ for every σ. The basis we have chosen is known as the *occupancy number basis*. Elements of the basis are called *Fock states*. We may interpret the Fock states in Eq. (2.59) as indicating the number of particles in a given quantum state. For example, the Fock state $|n_\sigma\rangle_B = (n!)^{-1/2}(a_\sigma^\dagger)^n\,|0\rangle_B$ indicates that there are n indistinguishable particles in the state Φ_σ with associated energy frequency ω_σ. From Eqs. (2.53) and (2.54) it follows that the actions of the operators $a_\sigma^\dagger$ and a_σ on this element are given by

$$a_\sigma^\dagger\,|n_\sigma\rangle_B = \sqrt{n+1}\,|(n+1)_\sigma\rangle_B\,, \tag{2.61a}$$

$$a_\sigma\,|n_\sigma\rangle_B = \sqrt{n}\,|(n-1)_\sigma\rangle_B\,, \tag{2.61b}$$

and thus, $a_\sigma^\dagger$ increases the number of particles in the state Φ_σ by one and a_σ reduces the number of particles by one, hence the naming convention of creation and annihilation operators, respectively. The actions of these operators on the element (2.59) are carried out using the commutation relations (2.49) and Eq. (2.61).

We note that the completeness and orthonormality of the set $\{\Phi_\sigma\}_{\sigma\in\mathbb{N}}$, which was directly implied from the self–adjointness of the operator A in Eq. (2.9), played a fundamental role in the construction of the Fock space $\mathscr{F}_{KG}$.

The analogous Fock space $\mathscr{F}_D$ of quantum states of a spinor field satisfying the Dirac equation is constructed in a similar way. Considering the Lagrangian density $\mathscr{L}_D$ given by Eq. (2.27), the conjugate momentum density is found to be

$$\begin{aligned} \Pi(t,x) &= \frac{\partial \mathscr{L}}{\partial(\partial_0 \psi(t,x))} \\ &= i\sqrt{h(x)}\,\psi^\dagger(t,x)\,, \end{aligned} \tag{2.62}$$

where the first equality is obtained through the left–functional derivative of the action S_D. As with the scalar field, canonical quantisation of the spinor field theory is achieved regarding the solution ψ of Eq. (2.28) and the conjugate momentum Π in Eq. (2.62) as operator–valued distributions for the Hilbert space $\mathscr{F}_D$ of quantum states instead of a spinor field. Thus, the field ψ is now regarded as a fermionic quantum field.

Now, denoting the $\tilde{N}$ component functions of the spinor ψ by $\psi_{\dot{a}}$ with $\dot{a} = 1, \ldots, \tilde{N}$, we impose the equal–time canonical anticommutation relations [27, 34] given by

$$\{\psi_{\dot{a}}(t,x), \Pi_{\dot{b}}(t,y)\} = \delta_{\dot{a}\dot{b}}\delta(x,y)\,, \tag{2.63a}$$

$$\{\psi_{\dot{a}}(t,x), \psi_{\dot{b}}(t,y)\} = 0 = \{\Pi_{\dot{a}}(t,x), \Pi_{\dot{b}}(t,y)\}\ , \tag{2.63b}$$

where δ is once again defined through Eq. (2.43).

Solutions to Eq. (2.31) which are defined for all $(t,x) \in \mathcal{M}$ can be found in the form

$$\psi_{\tilde{\sigma}}(t,x) = \Psi_{\tilde{\sigma}}(x)e^{-i\omega_{\tilde{\sigma}}t}\,, \tag{2.64}$$

where $\tilde{\sigma}$ labels the elements of the spectrum $\boldsymbol{\sigma}(\mathbb{D})$ in Eq. (2.29), and the spatial spinor $\Psi_{\tilde{\sigma}} \in \mathscr{H}_D$ satisfies the equation

$$\mathbb{D}\Psi_{\tilde{\sigma}} = \omega_{\tilde{\sigma}}\Psi_{\tilde{\sigma}}\,. \tag{2.65}$$

Now, suppose that the spectrum of the operator $\mathbb{D}$ is purely discrete. Since $\mathbb{D}$ is assumed to be self–adjoint, then Theorem A.0.3 implies that the set of spatial spinors satisfying Eq. (2.65) forms a complete orthonormal set for the Hilbert space $\mathscr{H}_D$. Furthermore, we will assume that $\omega_{\tilde{\sigma}} \neq 0$ for all $\tilde{\sigma}$. Thus, we set $\langle \Psi_{\tilde{\sigma}}, \Psi_{\tilde{\sigma}'}\rangle_D = \delta_{\tilde{\sigma}\tilde{\sigma}'}$, and write the quantum field $\psi(t,x)$ as [27, 34]

$$\psi(t,x) = \sum_{\tilde{\sigma}} \left[a_{\tilde{\sigma}}\Psi_{\tilde{\sigma}}(x)e^{-i\omega_{\tilde{\sigma}}t} + b^\dagger_{\tilde{\sigma}}\left(\Psi_{\tilde{\sigma}}(x)\right)^c e^{i\omega_{\tilde{\sigma}}t}\right]\,. \tag{2.66}$$

Here, the coefficients $a_{\tilde{\sigma}}$ and $b^\dagger_{\tilde{\sigma}}$ are operators acting on the space $\mathscr{F}_D$. The spatial spinor $(\Psi_{\tilde{\sigma}})^c$ in Eq. (2.66) is the *charge conjugate spinor*, defined by

$$(\Psi_{\tilde{\sigma}})^c = C\left(\Psi^*_{\tilde{\sigma}}\right)^T\ , \tag{2.67}$$

where the *charge conjugation matrix* C is defined through $C^{-1}\gamma^a C = -(\gamma^a)^T$, and required to satisfy $C^{-1} = C^\dagger$. The reason to consider the charge conjugate spinor in the expansion of the quantum field (2.66) is that if $\Psi_{\tilde{\sigma}}$ is a solution of Eq. (2.65) with $\omega_{\tilde{\sigma}} > 0$, then a solution of this equation with $\omega_{\tilde{\sigma}} \to -\omega_{\tilde{\sigma}}$ is given by $\Psi^c_{\tilde{\sigma}}$. Therefore, the sum in Eq. (2.66) is over all σ such that $\omega_{\tilde{\sigma}} > 0$, and in this way all positive– and negative–frequency solutions are used for the expansion of the quantum field ψ.

The sum in Eq. (2.66) is again to be understood in the distributional sense: For any smooth, compactly supported map $F : \mathcal{M} \to \mathbb{C}^{\tilde{N}}$, the quantity

$$\int_{\mathcal{M}} F(t,x)^\dagger\psi(t,x)\sqrt{g}\mathrm{d}t\mathrm{d}x\,, \tag{2.68}$$

defines an operator on the space $\mathscr{F}_D$.

Since the set $\{\Psi_{\tilde{\sigma}}\}_{\tilde{\sigma}\in\mathbb{N}}$ is a complete orthonormal set for $\mathscr{H}_D$, we find that

$$\sum_{\tilde{\sigma}} \Psi_{\tilde{\sigma}}(x')^\dagger\Psi_{\tilde{\sigma}}(x) = \frac{1}{\sqrt{h(x)}}\delta(x,x')\,. \tag{2.69}$$

This relation is used to show that the anticommutation relations between the fields ψ and $\psi^\dagger$ in Eq. (2.63) are equivalent to the equal–time anticommutation relations between the creation and annihilation operators given by

$$\left\{a_{\tilde{\sigma}}, a^\dagger_{\tilde{\sigma}'}\right\} = \delta_{\tilde{\sigma}\tilde{\sigma}'} = \left\{b_{\tilde{\sigma}}, b^\dagger_{\tilde{\sigma}'}\right\} , \tag{2.70}$$

with all the other anticommutators vanishing.

The Hilbert space $\mathscr{F}_D$ of fermionic quantum states is then defined in a way slightly different to the case of a scalar field. From the space of solutions $\mathscr{H}_D$ we construct the n–fold *antisymmetric* tensor product space $\otimes^n_{AS}\mathscr{H}_D$, whose elements are the continuous alternating n–multilinear maps $\Psi^{(n)} : \mathscr{H}_D \times \cdots \times \mathscr{H}_D \to \mathbb{C}$. For each $n \in \mathbb{N}$, the space $\otimes^n_{AS}\mathscr{H}_D$ is a Hilbert space with respect to the inner product induced from the tensor product structure [35, Section II.4]. We define the algebraic direct sum of Hilbert spaces in analogy with Eq. (2.50) by

$$\mathscr{F}^D_0 := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} (\otimes^n_{AS}\mathscr{H}_D) \,. \tag{2.71}$$

Elements in $\mathscr{F}^D_0$ are finite sequences of the form

$$|\Psi\rangle_F := (\Psi^{(0)}, \Psi^{(1)}, \dots, \Psi^{(n)}, 0, \dots) \,, \tag{2.72}$$

where $\Psi^{(0)} \in \mathbb{C}$, and $\Psi^{(j)} \in \otimes^j_{AS}\mathscr{H}_D$. The norm in $\mathscr{F}^D_0$ is given by

$$\|\Psi\|^2_{\mathscr{F}_D} = \langle \Psi \,|\, \Psi\rangle_F := |\Psi^{(0)}|^2 + \sum_{n=0}^{\infty} \|\Psi^{(n)}\|^2_{D,n} \,, \tag{2.73}$$

for all $|\Psi\rangle_F$, with $\|\cdot\|_{D,n}$ the norm in $\otimes^n_{AS}\mathscr{H}_D$. The Hilbert space of quantum states for the Dirac field, $\mathscr{F}_D$, is defined as the completion of $\mathscr{F}^D_0$ with respect to the norm in Eq. (2.73), *i.e.*, $\mathscr{F}_D = \overline{\mathscr{F}^D_0}$. For each $\Psi' \in \mathscr{H}_D$, we define the operators $a(\Psi')$ and $b(\Psi')$ with domain $\mathscr{F}^D_0$, whose action on elements of the form of Eq. (2.72) is given by

$$a(\Psi')\,|\Psi\rangle_F := (\Psi'^c \cdot \Psi^{(1)}, \sqrt{2}\Psi'^c \cdot \Psi^{(2)}, \dots, \sqrt{n}\Psi'^c \cdot \Psi^{(n)}, 0, \dots) \,, \tag{2.74a}$$
$$b(\Psi')\,|\Psi\rangle_F := (\Psi' \cdot \Psi^{(1)}, \sqrt{2}\Psi' \cdot \Psi^{(2)}, \dots, \sqrt{n}\Psi' \cdot \Psi^{(n)}, 0, \dots) \,, \tag{2.74b}$$

where $\Psi' \cdot \Psi^{(j)}, \Psi'^c \cdot \Psi^{(j)} \in \otimes^{j-1}_{AS}\mathscr{H}_D$ denote the skew–symmetric insertions of Ψ' and Ψ'^c, respectively, into one of the arguments of the functional $\Psi^{(j)}$. Similarly, for each $\Psi' \in \mathscr{H}_D$, we define the operators $a^\dagger(\Psi')$ and $b^\dagger(\Psi')$ with dense domain $\mathscr{F}^D_0$ by the action on the elements in Eq. (2.72) given by

$$a^\dagger(\Psi')\,|\Psi\rangle_F := (0, \Psi^{(0)}\tilde{\Psi}', \sqrt{2}\tilde{\Psi}' \otimes_{AS} \Psi^{(1)}, \dots, \sqrt{n+1}\tilde{\Psi}' \otimes_{AS} \Psi^{(n)}, 0, \dots) \,, \tag{2.75a}$$
$$b^\dagger(\Psi')\,|\Psi\rangle_F := (0, \Psi^{(0)}\tilde{\Psi}'^c, \sqrt{2}\tilde{\Psi}'^c \otimes_{AS} \Psi^{(1)}, \dots, \sqrt{n+1}\tilde{\Psi}'^c \otimes_{AS} \Psi^{(n)}, 0, \dots) \,, \tag{2.75b}$$

where $\tilde{\Psi}'$ and $\tilde{\Psi}'^c$ denote the continuous functionals associated to the elements Ψ' and Ψ'^c, respectively. We note that a direct consequence of definitions (2.74) and (2.75) is that, for any $\Psi' \in \mathscr{H}_D$, we have $a(\Psi')^2 = a^\dagger(\Psi')^2 = b(\Psi')^2 = b^\dagger(\Psi')^2 = 0$. We calculate the anitcommutator between the operators $a(\Psi_1)$ and $a^\dagger(\Psi_2)$ for any two $\Psi_1, \Psi_2 \in \mathscr{H}_D$. Using Eqs. (2.74) and (2.75), we obtain

$$a(\Psi_1)a^\dagger(\Psi_2)\left|\Psi\right\rangle_F = \left(\Psi^{(0)}\Psi_1^c\cdot\tilde{\Psi}_2, 2\Psi_1^c\cdot(\tilde{\Psi}_2\otimes_{AS}\Psi^{(1)}), \ldots, (n+1)\Psi_1^c\cdot(\tilde{\Psi}_2\otimes_{AS}\Psi^{(n)}), 0, \ldots\right),$$
$$a^\dagger(\Psi_2)a(\Psi_1)\left|\Psi\right\rangle_F = \left(0, (\Psi_1^c\cdot\Psi^{(1)})\tilde{\Psi}_2, 2\tilde{\Psi}_2\otimes_{AS}(\Psi_1^c\cdot\Psi^{(2)}), \ldots, n\tilde{\Psi}_2\otimes_{AS}(\Psi_1^c\cdot\Psi^{(n)}), 0, \ldots\right),$$

and, since for all $1 \leq j \leq n$ we have

$$\begin{aligned}(j+1)\Psi_1^c\cdot(\tilde{\Psi}_2\otimes_{AS}\Psi^{(j)}) + j\tilde{\Psi}_2\otimes_{AS}(\Psi_1^c\cdot\Psi^{(j)}) &= \tilde{\Psi}_2(\Psi_1^c)\Psi^{(j)}\,,\\ &= \left\langle\Psi_2, \Psi_1^c\right\rangle_D\Psi^{(j)}\,,\end{aligned} \tag{2.76}$$

then it follows that

$$\left\{a(\Psi_1), a^\dagger(\Psi_2)\right\}\left|\Psi\right\rangle_F = \left\langle\Psi_2, \Psi_1^c\right\rangle_D\left|\Psi\right\rangle_F\,. \tag{2.77}$$

An analogous calculation can be done for the anticommutator between the operators $b(\Psi_1)$ and $b^\dagger(\Psi_2)$, leading to

$$\left\{b(\Psi_1), b^\dagger(\Psi_2)\right\}\left|\Psi\right\rangle_F = \left\langle\Psi_2^c, \Psi_1\right\rangle_D\left|\Psi\right\rangle_F\,, \tag{2.78}$$

and using Eqs. (2.74) and (2.75) it can readily be verified that all other anticommutators vanish. Finally, let us consider the orthonormal basis $\{\Psi_{\tilde{\sigma}}\}_{\tilde{\sigma}\in\boldsymbol{\sigma}(\mathbb{D})}$ and, for each positive–frequency basis element $\Psi_{\tilde{\sigma}}$ let us define the operators $a_{\tilde{\sigma}} := a(\Psi_{\tilde{\sigma}})$, $b_{\tilde{\sigma}} := b(\Psi_{\tilde{\sigma}})$, $a_{\tilde{\sigma}}^\dagger := a^\dagger(\Psi_{\tilde{\sigma}}^c)$, and $b_{\tilde{\sigma}}^\dagger := b^\dagger(\Psi_{\tilde{\sigma}}^c)$. Then, Eqs. (2.77) and (2.78) with $\Psi_1 = \Psi_{\tilde{\sigma}}$ and with $\Psi_2 = \Psi_{\tilde{\sigma}'}^c$, reduce[4] to Eq. (2.70). Hence, the operators $a_{\tilde{\sigma}}$ and $b_{\tilde{\sigma}}^\dagger$ are precisely the ones appearing in the mode expansion of the quantum field in Eq. (2.66).

Similarly to the case of a scalar field, Eq. (2.53) implies that the element

$$\left|0\right\rangle_F := (1, 0, 0, \ldots) \in \mathscr{F}_D\,, \tag{2.79}$$

is uniquely characterised by the condition

$$a_{\tilde{\sigma}}\left|0\right\rangle_F = 0 = b_{\tilde{\sigma}}\left|0\right\rangle_F\,, \tag{2.80}$$

for all $\tilde{\sigma}$. The state $\left|0\right\rangle_F$ is known as the *fermionic vacuum state*. By means of Eq. (2.73), the norm of the vacuum state is $\|0\|^2_{\mathscr{F}_D} = \left\langle 0\middle|0\right\rangle_F = 1$.

The occupancy number basis for the fermionic Fock space is constructed in a similar fashion to the bosonic case. Let $\{\Psi_{\tilde{\sigma}}\}_{\tilde{\sigma}\in\boldsymbol{\sigma}(\mathbb{D})}$ be the orthonormal basis of $\mathscr{H}_D$ of eigenvectors of $\mathbb{D}$. Note that the elements $\Psi_{\tilde{\sigma}}^c$ are included in the basis since they correspond to the eigenvalues $-\omega_{\tilde{\sigma}}$. The elements of the form $\tilde{\Psi}_{\tilde{\sigma}_1}\otimes_{AS}\cdots\otimes_{AS}\tilde{\Psi}_{\tilde{\sigma}_n}$, with $\tilde{\sigma}_1 < \cdots < \tilde{\sigma}_n$ form a basis for the space $\otimes^n_{AS}\mathscr{H}_D$. Varying $n \in \mathbb{N}$ we obtain a basis for $\mathscr{F}_D$. Using Eqs. (2.75) and (2.79), it can be shown that, for each $j, k \in \mathbb{N}$, the element in $\mathscr{F}_D$ defined by

$$\left|1_{\tilde{\sigma}_1}, \ldots, 1_{\tilde{\sigma}_j}; 1_{\tilde{\sigma}_{j+1}}, \ldots, 1_{\tilde{\sigma}_{j+k}}\right\rangle_F := a_{\tilde{\sigma}_1}^\dagger\cdots a_{\tilde{\sigma}_j}^\dagger b_{\tilde{\sigma}_{j+1}}^\dagger\cdots b_{\tilde{\sigma}_{j+k}}^\dagger\left|0\right\rangle_F\,. \tag{2.81}$$

[4]We use the fact that the inner product in $\mathscr{H}_D$ satisfies $\left\langle\Psi_1^c, \Psi_2^c\right\rangle_D = \left\langle\Psi_2, \Psi_1\right\rangle_D$.

where $\tilde{\sigma}_1 < \cdots < \tilde{\sigma}_j$, and $\tilde{\sigma}_{j+1} < \cdots < \tilde{\sigma}_{j+k}$, is proportional to the basis element

$$\tilde{\Psi}_{\tilde{\sigma}_1} \otimes_{AS} \cdots \tilde{\Psi}_{\tilde{\sigma}_j} \otimes_{AS} \tilde{\Psi}^c_{\tilde{\sigma}_{j+1}} \otimes_{AS} \cdots \otimes_{AS} \tilde{\Psi}^c_{\tilde{\sigma}_{j+k}} \,,$$

and, thus, the elements of the form of Eq. (2.81) provide a basis for the fermionic Fock space. The normalisation of these elements is obtained using Eq. (2.73), and we obtain

$$\begin{aligned} &\left\langle 1_{\tilde{\sigma}_1}, \ldots, 1_{\tilde{\sigma}_j}; 1_{\tilde{\sigma}_{j+1}}, \ldots, 1_{\tilde{\sigma}_{j+k}} \middle| 1_{\tilde{\sigma}'_1}, \ldots, 1_{\tilde{\sigma}'_l}; 1_{\tilde{\sigma}'_{l+1}}, \ldots, 1_{\tilde{\sigma}'_{l+m}} \right\rangle_F \\ &= \delta_{jl}\delta_{km} \sum_{\alpha,\beta} \mathrm{sgn}(\alpha)\mathrm{sgn}(\beta) \delta_{\tilde{\sigma}_1 \tilde{\sigma}_{\alpha(1)}} \cdots \delta_{\tilde{\sigma}_j \tilde{\sigma}_{\alpha(j)}} \delta_{\tilde{\sigma}_{j+1} \tilde{\sigma}_{\beta(j+1)}} \cdots \delta_{\tilde{\sigma}_{j+k} \tilde{\sigma}_{\beta(j+k)}} \,, \end{aligned} \tag{2.82}$$

where the sum runs over all permutations α of the set $\{1, \ldots, j\}$ and all permutations β of the set $\{j+1, \ldots, j+k\}$. In analogy with the particle interpretation for the bosonic Fock space, the Fock state defined by $|1_{\tilde{\sigma}}; 0\rangle_F := a^\dagger_{\tilde{\sigma}} |0\rangle_F$ represents a particle occupying the quantum state $\psi_{\tilde{\sigma}}$ with energy frequency $\omega_{\tilde{\sigma}}$. However, since $(a^\dagger_{\tilde{\sigma}})^2 = 0$, we have $a^\dagger_{\tilde{\sigma}} |1_{\tilde{\sigma}}; 0\rangle_F = 0$. The same is true for the Fock state defined by $|0; 1_{\tilde{\sigma}}\rangle_F := b^\dagger_{\tilde{\sigma}} |0\rangle_F$ representing a particle in the quantum state $\psi^c_{\tilde{\sigma}}$ with energy frequency $-\omega_{\tilde{\sigma}}$, *i.e.*, $b^\dagger_{\tilde{\sigma}} |0; 1_{\tilde{\sigma}}\rangle_F = 0$. This behaviour reflects the fact that fermionic fields obey the Pauli exclusion principle [37], so that only one particle can occupy a given quantum state at a time.

As for the case of a scalar field, a key requirement for the construction of the space of quantum states for the spinor field theory is the self–adjointness of the operator $\mathbb{D}$ in Eq. (2.29) as well as the absence of zero–modes.

2.3.1 Invariance of the vacuum states under the isometry group

Now that we have constructed the quantum field theories associated to free scalar and spinor fields on a general standard static spacetime, we will explore additional properties that the spaces of quantum states have if we further assume that the spacetime $(\mathcal{M}, \boldsymbol{g})$ admits a larger group of isometries and if we require the spaces of solutions to the dynamical equations to be invariant under these transformations. The isometries will be generated by a set of linearly independent Killing vector fields which includes the static vector field ξ.

Let ξ' be a Killing vector of the spacetime $(\mathcal{M}, \boldsymbol{g})$. The action of ξ' on the quantum scalar field ϕ is realised through the Lie derivative $\mathcal{L}_{\xi'}$ in the direction of ξ'. Since the Laplace–Beltrami operator $\Box$ in Eq. (2.8) commutes with $\mathcal{L}_{\xi'}$, the space of solutions of the Klein–Gordon equation (2.7) is invariant under the isometries of $(\mathcal{M}, \boldsymbol{g})$. Hence, for all σ we obtain

$$\mathcal{L}_{\xi'} \phi_\sigma(t, x) = \sum_{\sigma'} \Lambda_{\sigma\sigma'} \phi_{\sigma'}(t, x) + \sum_{\sigma'} \tilde{\Lambda}_{\sigma\sigma'} \overline{\phi_{\sigma'}(t, x)} \,, \tag{2.83}$$

with $\Lambda_{\sigma\sigma'}, \tilde{\Lambda}_{\sigma\sigma'} \in \mathbb{C}$. Substituting this expression into Eq. (2.46), we have

$$\mathcal{L}_{\xi'} \phi(t, x) = \sum_{\sigma} \sum_{\sigma'} \left[(a_{\sigma'} \Lambda_{\sigma'\sigma} + a^\dagger_{\sigma'} \overline{\tilde{\Lambda}_{\sigma'\sigma}}) \phi_\sigma(t, x) + (a^\dagger_{\sigma'} \overline{\Lambda_{\sigma'\sigma}} + a_{\sigma'} \tilde{\Lambda}_{\sigma'\sigma}) \overline{\phi_\sigma(t, x)} \right] . \tag{2.84}$$

Thus, the infinitesimal transformation of the annihilation operators a_σ corresponding to the symmetry transformation generated by ξ' is given by

$$\delta_{\xi'} a_\sigma = \sum_{\sigma'} \left(a_{\sigma'} \Lambda_{\sigma'\sigma} + a^\dagger_{\sigma'} \overline{\tilde{\Lambda}_{\sigma'\sigma}} \right) . \tag{2.85}$$

For the bosonic vacuum state $|0\rangle_B$ defined by Eq. (2.58) to be invariant under the spacetime symmetry transformation corresponding to the Killing vector ξ', we need to have $\tilde{\Lambda}_{\sigma\sigma'} = 0$. That is,

$$\mathcal{L}_{\xi'} \phi_\sigma(t, x) = \sum_{\sigma'} \Lambda_{\sigma\sigma'} \phi_{\sigma'}(t, x) . \tag{2.86}$$

In other words, for $|0\rangle_B$ to be invariant under this symmetry transformation, the positive–frequency solutions $\phi_\sigma(t, x)$ given by Eq. (2.44) with $\omega_\sigma > 0$ must transform among themselves without any component of negative–frequency solutions. Using the Lie derivative, we can also see that $|0\rangle_B$ is stationary, *i.e.* invariant under time–translation symmetry induced by the static vector field $\xi = \partial_t$. From the form of the solutions ϕ_σ in Eq. (2.44), the Lie derivative in the direction of the static vector field reads $\mathcal{L}_\xi \phi_\sigma = -i\omega_\sigma \phi_\sigma$. This implies that

$$\mathcal{L}_\xi \phi = -i \sum_\sigma \omega_\sigma \left(a_\sigma \phi_\sigma(t, x) - a^\dagger_\sigma \overline{\phi_\sigma(t, x)} \right) , \tag{2.87}$$

and thus, the transformed coefficients $\delta_\xi a_\sigma = -i\omega_\sigma a_\sigma$ and $\delta_\xi b^\dagger_\sigma = i\omega_\sigma b^\dagger_\sigma$ define the same vacuum state $|0\rangle_B$.

For the case of a spinor field, the action of the Killing vector ξ' is realised through the *spinorial Lie derivative* in the direction of ξ', defined by [42, 41]

$$\mathcal{L}^{(D)}_{\xi'} \psi = \xi'^\mu \nabla_\mu \psi + \frac{1}{4} (\nabla_\mu \xi'_\nu) \gamma^\mu \gamma^\nu \psi . \tag{2.88}$$

As the Dirac operator $\tilde{\gamma}^\mu \nabla^{(D)}_\mu$ commutes with the Lie derivative $\mathcal{L}^{(D)}_{\xi'}$, space of solutions of Eq. (2.28) must be invariant under the infinitesimal transformation induced by ξ'. Hence,

$$\mathcal{L}^{(D)}_{\xi'} \psi_{\tilde{\sigma}}(t, x) = \sum_{\tilde{\sigma}'} \left(\Lambda^{(D)}_{\tilde{\sigma}\tilde{\sigma}'} \psi_{\sigma'}(t, x) + \tilde{\Lambda}^{(D)}_{\tilde{\sigma}\tilde{\sigma}'} \psi^c_{\tilde{\sigma}'}(t, x) \right) , \tag{2.89}$$

where $\Lambda^{(D)}_{\tilde{\sigma}\tilde{\sigma}'}, \tilde{\Lambda}^{(D)}_{\tilde{\sigma}\tilde{\sigma}'} \in \mathbb{C}$. After substituting this expression into Eq. (2.64), we obtain

$$\mathcal{L}^{(D)}_{\xi'} \psi(t, x) = \sum_{\tilde{\sigma}'} \sum_{\tilde{\sigma}} \left[\left(a_{\tilde{\sigma}} \Lambda^{(D)}_{\tilde{\sigma}\tilde{\sigma}'} + b^\dagger_{\tilde{\sigma}} \overline{\tilde{\Lambda}^{(D)}_{\tilde{\sigma}\tilde{\sigma}'}} \right) \psi_{\tilde{\sigma}'}(t, x) + \left(a_\sigma \tilde{\Lambda}^{(D)}_{\tilde{\sigma}\tilde{\sigma}'} + b^\dagger_{\tilde{\sigma}} \overline{\Lambda^{(D)}_{\tilde{\sigma}\tilde{\sigma}'}} \right) \psi^c_{\tilde{\sigma}'}(t, x) \right] , \tag{2.90}$$

where we have used the fact that $\left(\mathcal{L}^{(D)}_{\xi'} \psi \right)^c = \mathcal{L}^{(D)}_{\xi'} \psi^c$. Thus, we see that the Lie derivative on ψ induces the transformation on the operators

$$a_{\tilde{\sigma}} \mapsto \sum_{\tilde{\sigma}'} \left(a_{\tilde{\sigma}'} \Lambda^{(D)}_{\tilde{\sigma}'\tilde{\sigma}} + b^\dagger_{\tilde{\sigma}'} \overline{\tilde{\Lambda}^{(D)}_{\tilde{\sigma}'\tilde{\sigma}}} \right) , \tag{2.91a}$$

$$b^\dagger_{\tilde{\sigma}} \mapsto \sum_{\tilde{\sigma}'} \left(a_{\sigma'} \tilde{\Lambda}^{(D)}_{\tilde{\sigma}'\tilde{\sigma}} + b^\dagger_{\tilde{\sigma}'} \overline{\Lambda^{(D)}_{\tilde{\sigma}'\tilde{\sigma}}} \right) . \tag{2.91b}$$

Hence, for the fermionic vacuum state $|0\rangle_F$ defined by Eq. (2.80) to be invariant under the spacetime symmetry transformation corresponding to the Killing vector ξ', we need to have $\tilde{\Lambda}^{(D)}_{\tilde{\sigma}\tilde{\sigma}'} = 0$. That is,

$$\mathcal{L}^{(D)}_{\xi'}\psi_{\tilde{\sigma}}(t,x) = \sum_{\tilde{\sigma}'} \Lambda^{(D)}_{\tilde{\sigma}\tilde{\sigma}'}\psi_{\tilde{\sigma}'}(t,x)\,. \tag{2.92}$$

Once again, this means for the vacuum state $|0\rangle_F$ to be invariant under this symmetry transformation, the positive–frequency solutions must transform among themselves without any of negative–frequency solutions. Finally, we note that $|0\rangle_F$ is also stationary: The action of the spinorial Lie derivative (2.88) in the direction of $\xi = \partial_t$ on a spinor in the form of Eq. (2.64) reads $\mathcal{L}^{(D)}_{\xi}\psi_{\tilde{\sigma}} = \partial_t\psi_{\tilde{\sigma}} = -i\omega_{\tilde{\sigma}}\psi_{\tilde{\sigma}}$. Therefore, we have

$$\mathcal{L}^{(D)}_{\xi}\psi(t,x) = -i\sum_{\tilde{\sigma}} \omega_{\tilde{\sigma}}\left(a_{\tilde{\sigma}}\psi_{\tilde{\sigma}}(t,x) - b^{\dagger}_{\tilde{\sigma}}\psi^{c}_{\tilde{\sigma}}(t,x)\right)\,, \tag{2.93}$$

and thus, the transformed operators $\delta_\xi a_{\tilde{\sigma}} = -i\omega_{\tilde{\sigma}}a_{\tilde{\sigma}}$ and $\delta_\xi b^{\dagger}_{\tilde{\sigma}} = i\omega_{\tilde{\sigma}}b^{\dagger}_{\tilde{\sigma}}$ define the same vacuum state $|0\rangle_F$.

— 3 —

The geometry of anti–de Sitter spacetimes

Einstein's field equations, which relate the geometry of spacetime with the distribution of matter and energy in the universe, are a system of coupled non–linear second order partial differential equations for the components of the metric tensor [25, 45]. Due to the complexity of Einstein's equations, finding exact solutions is a highly non–trivial task. Nevertheless, exact solutions were found not long after Einstein himself published his seminal paper containing the field equations, albeit for a highly symmetric case in the absence of matter or energy. Solutions to the Einstein's field equations where no sources of matter or energy are present are called *vacuum solutions*, and even if they do not fully represent the "reality" of physical spacetimes, they provide an idea of the qualitative properties that can arise in General Relativity and, thus, of the possible behaviour of realistic solutions.

Amongst the vacuum solutions, those of constant scalar curvature [28, 29] represent the simplest exact solutions to Einstein's field equations. These solutions are characterised by the sign of the scalar curvature, the trivial case of Minkowski spacetime, $\mathbb{R}^{1,3}$, corresponding to the value of zero. The four–dimensional anti–de Sitter manifold is a static vacuum solution with negative constant scalar curvature [24, 26, 27]. This solution, as well as its N–dimensional analogues, are maximally symmetric spaces [25], meaning that the number of linearly independent Killing vector fields that the metric admits is $N(N+1)/2$. In particular, this means that the isometry group generated by the Killing vectors acts transitively on the anti de–Sitter manifold.

Even though anti–de Sitter manifolds are highly symmetric solutions to Einstein's field equations they are deemed as unphysical mainly due to a causal property they possess: Closed timelike curves [24, 26]. This issue may be remedied by considering, not the anti de–Sitter manifold as the physical spacetime, but instead its universal covering space [29]. This is achieved by "unrolling" the time coordinate of the original anti–de Sitter manifold[1] to obtain an N–dimensional, simply connected Lorentzian manifold with no closed timelike curves. We will refer to this universal covering space as the *anti–de Sitter spacetime*, AdS_N. Another issue arises when considering the universal covering space, that is, the lack of global hyperbolicity of AdS_N. However, as our main point of focus is the construction

[1]The concept of unrolling the time coordinate will be properly defined in Section 3.1.

of scalar and spinor field theories in anti–de Sitter spacetimes, the machinery developed by Wald described in Chapter 2 will help us circumvent the consequences that this issue might produce for our specific goal.

In this chapter we will describe the geometric and causal properties of AdS_N, introducing a global coordinate system that will be used throughout this and subsequent chapters. We will explain how the universal covering space is defined and show that the anti–de Sitter spacetime is not globally hyperbolic. We will also pay special attention to the case $N = 2$ for which the geometry of the spacetime is slightly different from the higher–dimensional cases. Since AdS_N is a standard static spacetime in the sense of Definition 2.1.1, we will be using most of the concepts and results introduced in Chapter 2.

3.1 THE ANTI–DE SITTER MANIFOLD AND ITS UNIVERSAL COVERING SPACE

The N–dimensional anti–de Sitter manifold can be defined [46] as a one–sheeted hyperboloid, Hyp_N, embedded in the $(N+1)$–dimensional flat Lorentzian manifold, $\mathbb{R}^{2,N-1}$, endowed with the metric $\boldsymbol{\eta}_{N+1} = \mathrm{diag}(-1,-1,1,\ldots,1)$. Denoting the standard rectangular coordinates in the *ambient space* $\mathbb{R}^{2,N-1}$ by $(y^1, y^2, \ldots, y^{N+1})$, we can describe this hyperboloid by the points satisfying the constraint

$$-\left(y^1\right)^2 - \left(y^2\right)^2 + \sum_{i=3}^{N+1} \left(y^i\right)^2 = -R^2\,, \tag{3.1}$$

where the parameter $R > 0$ is known as the *radius of curvature*[2]. Here, and hereafter we set $R = 1$, and choose the units $c = \hbar = 1$, with c the speed of light and $\hbar$ the reduced Planck's constant.

The metric for the anti–de Sitter manifold is induced from the ambient metric $\boldsymbol{\eta}_{N+1}$ of $\mathbb{R}^{2,N-1}$. The line element in the ambient space is given in terms of the rectangular coordinates by

$$\mathrm{d}s^2_{N+1} = -\left(\mathrm{d}y^1\right)^2 - \left(\mathrm{d}y^2\right)^2 + \sum_{i=3}^{N+1} \left(\mathrm{d}y^i\right)^2\,. \tag{3.2}$$

A change of coordinates in $\mathbb{R}^{2,N-1}$ from the rectangular system to hyperspherical–like coordinates given by

$$y^1 = r\sec\rho\,\cos\tilde{t}\,, \tag{3.3a}$$

$$y^2 = r\sec\rho\,\sin\tilde{t}\,, \tag{3.3b}$$

$$y^3 = r\tan\rho\,\cos\theta_1\,, \tag{3.3c}$$

$$y^i = r\tan\rho\left(\prod_{j=1}^{i-3}\sin\theta_j\right)\cos\theta_{i-2}\,, \qquad 3 < i \le N\,, \tag{3.3d}$$

$$y^{N+1} = r\tan\rho\left(\prod_{j=1}^{N-2}\sin\theta_j\right)\,, \tag{3.3e}$$

[2]Viewed as a solution to Einstein's field equations, the radius of curvature of the N–dimensional anti–de Sitter manifold encodes the value of the (negative) cosmological constant via $R^2 = -(N-1)(N-2)/(2\Lambda)$

with $r \in [0, \infty)$ and with

$$\tilde{t} \in [-\pi, \pi)\,, \tag{3.4a}$$

$$\rho \in \left[0, \frac{\pi}{2}\right)\,, \tag{3.4b}$$

$$\theta_j \in [0, \pi]\,, \quad 1 \leq j \leq N-3\,, \tag{3.4c}$$

$$\theta_{N-2} \in [0, 2\pi)\,, \tag{3.4d}$$

reduces the hyperboloid equation (3.1) to $r^2 = 1$.

At this point it is worth pointing out that the parametrisation of $\mathbb{R}^{2,N-1}$ given by Eqs. (3.3) and (3.4) does not provide a complete coordinate system for the case $N = 2$. For this case, there are no angular coordinates θ_k, and thus, the variables in Eqs. (3.3d) and (3.3e) are omitted, and we have $y^3 = r \tan \rho$. However, $\rho \in [0, \pi/2)$ covers only the half–space of $\mathbb{R}^{2,1}$ given by $y^3 \geq 0$. Hence, for the particular case $N = 2$ we will always understand the variable ρ to be defined by $\rho \in (-\pi/2, \pi/2)$ instead of Eq. (3.4b). The fact that the $N = 2$ case allows this parametrisation will result in a very important and non–trivial distinction between the behaviour of the scalar field theory in dimension $N = 2$ and those in in dimension $N \geq 3$.

We can now define global coordinates for the anti–de Sitter manifold Hyp_N as

$$\left(\tilde{x}^0, x^1, x^2 \ldots, x^{N-1}\right) := \left(\tilde{t}, \rho, \theta_1, \ldots, \theta_{N-2}\right)\,. \tag{3.5}$$

With respect to these coordinates the line element induced from (3.2) takes the form

$$\mathrm{d}s^2_{\mathrm{Hyp}_N} = \sec^2 \rho \left(-\mathrm{d}\tilde{t}^2 + \mathrm{d}\rho^2 + \sin^2 \rho \, \mathrm{d}\Omega^2_{N-2}\right)\,, \tag{3.6}$$

where $\mathrm{d}\Omega^2_{N-2}$ is the line element of the $(N-2)$–sphere [28, 29] given by

$$\mathrm{d}\Omega^2_{N-2} = \mathrm{d}\theta_1^2 + \sum_{i=3}^{N-1} \left(\prod_{l=1}^{i-2} \sin^2 \theta_l\right) \mathrm{d}\theta^2_{i-1}\,. \tag{3.7}$$

The parametrisation of the coordinates (3.5) in terms of the ranges of the parameters defined in Eq. (3.4) makes clear that the anti–de Sitter manifold has the topology [24, 46] of the product $S^1 \times \mathbb{R}^{N-1}$, where S^1 is the unit circle and $\mathbb{R}^{N-1}$ corresponds to the hypersurfaces of constant $\tilde{t}$. Due to this fact it is clear that any curve parametrised by $\tilde{t}$ whose image consists of points with constant spatial coordinates $(\rho, \theta_1, \ldots, \theta_{N_2})$ is a closed timelike curve. However, these curves are not contractible [24] and thus, the anti–de Sitter manifold is not simply connected.

The universal covering space, AdS_N, of Hyp_N is realised by redefining the time coordinate $\tilde{t} \to t \in \mathbb{R}$, which can be interpreted as the “unrolling” of the unit circle S^1. More specifically, by considering the equivalence relation $t \sim t + 2\pi$ for $t \in \mathbb{R}$, we can interpret the coordinate $\tilde{t}$ as the map $\tilde{t} : \mathbb{R} \to S^1$ that sends t to its equivalence class, and thus, $\tilde{t} : t \mapsto \tilde{t}(t)$ defines the covering map which is extended to $\mathrm{AdS}_N \to \mathrm{Hyp}_N$. We will refer to the universal covering space AdS_N as the N–dimensional **anti–de Sitter spacetime**. The (global) coordinate system for AdS_N is thus given by

$$\left(x^0, x^1, x^2 \ldots, x^{N-1}\right) := (t, \rho, \theta_1, \ldots, \theta_{N-2}) \,, \tag{3.8}$$

with $t \in \mathbb{R}$ and with the range of the spatial variables $x := (\rho, \theta_1, \ldots, \theta_{N-2})$ given by Eqs. (3.4b)–(3.4d). By construction, AdS_N is the product manifold $\mathbb{R} \times \mathbb{R}^{N-1}$. Using Eq. (3.6) we see that the metric tensor $\boldsymbol{g}$ of AdS_N is given with respect to the coordinates (t, x) by

$$\boldsymbol{g}_{(t,x)} = -\sec^2\!\rho \, \mathrm{d}t \otimes \mathrm{d}t + \sec^2\!\rho \, \mathrm{d}\rho \otimes \mathrm{d}\rho + \tan^2\!\rho \, \mathrm{d}\Omega^2_{N-2} \,, \tag{3.9}$$

and thus, identifying

$$\mathcal{N}(x)^2 = \sec^2\!\rho \,, \tag{3.10a}$$

$$\boldsymbol{h}_x = \sec^2\!\rho \, \mathrm{d}\rho \otimes \mathrm{d}\rho + \tan^2\!\rho \, \mathrm{d}\Omega^2_{N-2} \,, \tag{3.10b}$$

and $\Sigma := \mathbb{R}^{N-1}$, we see that $(\mathrm{AdS}_N, \boldsymbol{g})$ is a standard static spacetime in the sense of Definition 2.1.1. The coordinates (t, x) precisely correspond to a choice of the static coordinates introduced in Chapter 2 and the static Killing vector field is given by $\xi_0 := \partial_t$, as expected.

The Levi–Civita connection compatible with the metric tensor in Eq. (3.9) is calculated using Eq. (2.3). We find that the non–zero components ${\Gamma^\mu}_{\nu\lambda}$ of the connection in terms of the static coordinates (3.8) are found to be given by

$${\Gamma^0}_{01} = {\Gamma^1}_{00} = {\Gamma^1}_{11} = -{\Gamma^1}_{22} = \tan\rho \,, \tag{3.11a}$$

$${\Gamma^1}_{ii} = -\tan\rho \prod_{l=1}^{i-2} \sin^2\theta_l \,, \tag{3.11b}$$

$${\Gamma^2}_{12} = {\Gamma^i}_{1i} = \csc\rho \sec\rho \,, \tag{3.11c}$$

$${\Gamma^i}_{ji} = \cot\theta_{j-1} \,, \tag{3.11d}$$

$${\Gamma^j}_{ii} = -\cot\theta_{j-1} \prod_{l=j-1}^{i-2} \sin^2\theta_l \,, \tag{3.11e}$$

for $3 \le i \le N-1$ and $2 \le j \le i-1$.

A t–independent orthogonal frame in the form of Eq. (2.20) can be chosen by defining the vector fields $e_a \in \mathfrak{X}(\mathrm{AdS}_N)$, with $a = 0, 1, \ldots, N-1$ given by

$$e_0 = \cos\rho \, \frac{\partial}{\partial t} \,, \tag{3.12a}$$

$$e_1 = \cos\rho \, \frac{\partial}{\partial \rho} \,, \tag{3.12b}$$

$$e_2 = \cot\rho \, \frac{\partial}{\partial \theta_1} \,, \tag{3.12c}$$

$$e_i = \cot\rho \left(\prod_{l=1}^{i-2} \csc\theta_l \right) \frac{\partial}{\partial \theta_{i-1}} \,, \qquad 3 \le i \le N-1 \,. \tag{3.12d}$$

The components e_a^μ of these vector fields are found to satisfy Eqs. (2.18), and thus, define an orthogonal frame at each $(t,x) \in \mathrm{AdS}_N$ except at the points for which any of the angular coordinates $\theta_1,\ldots,\theta_{N-2}$ vanish, as the vector fields e_i in Eq. (3.12d) become singular[3]. With respect to the frame $\{e_a\}_{a=0}^{N-1}$, the non–zero components ${\omega^a}_{b\,\mu}$ of the connection 1–form are obtained using Eq. (2.21) and read

$$
\begin{aligned}
{\omega^0}_{1\,0} &= \tan\rho\,, &(3.13a)\\
{\omega^2}_{1\,2} &= \sec\rho\,, &(3.13b)\\
{\omega^i}_{1\,i} &= \sec\rho\prod_{l=1}^{i-2}\sin\theta_l\,, &(3.13c)\\
{\omega^i}_{j\,i} &= \cot\theta_{j-1}\prod_{l=j-1}^{i-2}\sin\theta_l\,, &(3.13d)
\end{aligned}
$$

with $3 \le i \le N-1$ and $2 \le j \le i-1$. Using either Eqs. (3.9) and (3.11), or Eqs. (2.8) and (3.10), we can calculate the Laplace–Beltrami operator [25] associated to the anti–de Sitter spacetime. This is given by

$$
\Box_{\mathrm{AdS}_N} = \cot^2\rho\left(-\sin^2\rho\,\frac{\partial^2}{\partial t^2} + \sin^2\rho\,\frac{\partial^2}{\partial\rho^2} + (N-2)\tan\rho\,\frac{\partial}{\partial\rho} + \Delta_{N-2}\right) \tag{3.14}
$$

where Δ_{N-2} is the Laplacian of the $(N-2)$–sphere [28, 29].

We conclude this section with a discussion regarding the symmetries of anti–de Sitter spacetime by analysing the properties of the isometry group of AdS_N. We start by considering the symmetry group of transformations of $\mathbb{R}^{N+1}$ preserving the metric $\boldsymbol{\eta}_{N+1}$ in Eq. (3.2), that is, $\mathrm{O}(2,N-1)$, the indefinite orthogonal group [47]. In full analogy with the Lorentz group [29, 47, 48] we will consider the identity component of $\mathrm{O}(2,N-1)$ preserving time (y^1 and y^2 coordinates) and space ($y^3\ \ldots, y^{N-1}$ coordinates) orientations, which we will denote[4] by $\mathrm{SO}(2,N-1)$.

An element G of $\mathrm{SO}(2,N-1)$ is characterised, in the fundamental (matrix) representation acting on $\mathbb{R}^{2,N-1}$, by the condition $G^T\boldsymbol{\eta}_{_{N+1}}G = \boldsymbol{\eta}_{_{N+1}}$, where G^T denotes the transpose of the $(N+1)\times(N+1)$–matrix G, and by the requirement $\det(A) = 1 = \det(D)$, where A and D are 2×2– and $(N-1)\times(N-1)$–matrices, respectively, defined by

$$
G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\,, \tag{3.15}
$$

with B a $2\times(N-1)$–matrix and C a $(N-1)\times 2$–matrix.

By denoting elements in $\mathbb{R}^{2,N-1}$ as column vectors $y = (y^1,\ldots,y^{N+1})^T$, we can rewrite the hyperboloid equation (3.1) that defines the anti–de Sitter manifold, Hyp_N, as $y^T\boldsymbol{\eta}_{_{N+1}}y =$

[3] If necessary we may choose a different chart in AdS_N for which the point $(t,\rho,\theta_1,\ldots,\theta_{N-2})$ has none of its angular coordinates equal to zero.

[4] The correct notation for this group should be $\mathrm{SO}_0^+(2,N-1)$, whereas $\mathrm{SO}(2,N-1)$ usually stands for elements in $\mathrm{O}(2,N-1)$ with unit determinant and no further constraints. As we will not deal with the latter, we drop the cumbersome indices 0 and + for the sake of simplicity.

-1. By applying the transformation $y \mapsto Gy$, we see that $(Gy)^T \boldsymbol{\eta}_{N+1} Gy = y^T \boldsymbol{\eta}_{N+1} y$, and thus, for any $y \in \mathrm{Hyp}_N$, we have $Gy \in \mathrm{Hyp}_N$ for all $G \in \mathrm{SO}(2, N-1)$. Since the metric for the anti–de Sitter manifold can be obtained by the restriction of $\boldsymbol{\eta}_{N+1}$ to the hyperboloid Hyp_N, $\mathrm{SO}(2, N-1)$ is a symmetry group of this space preserving the metric structure. Hence, it is the isometry group of the anti–de Sitter manifold.

Now, since anti de–Sitter spacetime, AdS_N, is defined as the universal covering space of Hyp_N, the isometry group of AdS_N is $\widetilde{\mathrm{SO}}(2, N-1)$, the *universal covering group*[5] of $\mathrm{SO}(2, N-1)$ [47, 49]. We will not concern ourselves with the full description of the isometry group, but instead we will take the infinitesimal approach (which will prove to be of greater relevance when we analyse the scalar and spinor fields on AdS_N in Chapters 6, 7 and 8). Both Lie groups $\mathrm{SO}(2, N-1)$ and $\widetilde{\mathrm{SO}}(2, N-1)$ share the Lie algebra $\mathfrak{so}(2, N-1)$ consisting of real traceless matrices of the form [47]

$$X = \begin{pmatrix} 0 & a & b_{11} & b_{12} & \cdots & b_{1,N-1} \\ -a & 0 & b_{21} & b_{22} & \cdots & b_{2,N-1} \\ b_{11} & b_{21} & 0 & c_{12} & \cdots & c_{1,N-1} \\ b_{12} & b_{22} & -c_{12} & 0 & \cdots & c_{2,N-1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{1,N-1} & b_{2,N-1} & -c_{1,N-1} & -c_{2,N-1} & \cdots & 0 \end{pmatrix} . \tag{3.16}$$

A basis for the Lie algebra $\mathfrak{so}(2, N-1)$ is constructed as follows. Let E_{ij}, with $1 \le i, j \le N+1$ denote the $(N+1) \times (N+1)$ matrix with a 1 in the position (i, j) and zero elsewhere. We define the $N(N+1)/2$ matrices $O_{ij} := \eta_{jk} E_{ki} - \eta_{ik} E_{kj}$ where η_{ij} denotes the (i, j) element of $\boldsymbol{\eta}_{N+1}$. The matrices O_{ij} can be renamed for specific values of i, j as follows:

$$I_{12} := E_{12} - E_{21} \,, \tag{3.17a}$$
$$J_{ij} := E_{ij} - E_{ji} \,, \qquad 3 \le i < j \le N+1 \,, \tag{3.17b}$$
$$K_{ij} := E_{ij} + E_{ji} \,, \qquad i = 1, 2, \quad 3 \le j \le N+1 \,, \tag{3.17c}$$

Then, the matrix $X \in \mathfrak{so}(2, N-1)$ in Eq. (3.16) can be written as

$$X = a I_{12} + \sum_{i=1}^{2} \sum_{j=3}^{N+1} b_{i,j-2} K_{ij} + \sum_{i<j}^{N+1} c_{i-2,j-2} J_{ij} \,. \tag{3.18}$$

The matrices O_{ij} form a basis for the Lie algebra $\mathfrak{so}(2, N-1)$ and obey the commutation relations

$$[O_{ij}, O_{kl}] = \eta_{ik} O_{jl} - \eta_{il} O_{jk} - \eta_{jk} O_{il} + \eta_{jl} O_{ik} \,, \tag{3.19}$$

and thus, they are the infinitesimal generators of $\mathrm{SO}(2, N-1)$ and $\widetilde{\mathrm{SO}}(2, N-1)$. Using Eqs. (3.17) and (3.19), we can determine the commutation relations between the individual sectors of the $\mathfrak{so}(2, N-1)$ algebra. Indeed, the one–dimensional subalgebra generated

[5] We will give a rigorous characterisation of a universal covering group in Chapter 4, more specifically in Proposition 4.1.3.

by I_{12} is abelian and isomorphic to SO(2). All other non–zero commutation relations involving I_{12} are given by

$$[I_{ij}, K_{kl}] = \delta_{jk} K_{il} - \delta_{ik} K_{jl} \,, \tag{3.20}$$

for all $i, j, k = 1, 2$ and $3 \leq l \leq N + 1$. We also have, for $i, j = 1, 2$ and $k, l, m, n = 3, \ldots, N + 1$, the following

$$[J_{kl}, J_{mn}] = \delta_{km} J_{ln} - \delta_{kn} J_{lm} - \delta_{lm} J_{kn} + \delta_{ln} J_{km} \,, \tag{3.21a}$$

$$[K_{ik}, K_{jl}] = \delta_{kl} I_{ij} - \delta_{ij} J_{kl} \,, \tag{3.21b}$$

$$[K_{ik}, J_{lm}] = \delta_{kl} K_{im} + \delta_{km} K_{il} \,. \tag{3.21c}$$

From these relations we see that the elements J_{kl} defined in Eq. (3.17b) span a subalgebra isomorphic to $\mathfrak{so}(N-1)$.

For the special case of $N = 2$, the Lie algebra $\mathfrak{so}(2,1)$ has the three generators I_{12}, K_{13} and K_{23} which satisfy the commutation relations

$$[I_{12}, K_{23}] = K_{13} \,, \quad [I_{12}, K_{13}] = -K_{23} \,, \quad [K_{13}, K_{23}] = I_{12} \,. \tag{3.22}$$

These are the commutation relations of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, the special linear group of real 2×2 matrices [47, 49]. This in turn implies that the isometry group for the two–dimensional anti–de Sitter spacetime AdS_2 is the universal covering group, $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, of $\mathrm{SL}(2,R)$. We will discuss some technical aspects of this group and its unitary irreducible representations in Chapter 4. For now we will continue to treat all cases for arbitrary N in the same footing.

The action of $\widetilde{\mathrm{SO}}(2, N-1)$ on AdS_N induces an action of $\mathfrak{so}(2, N-1)$ on smooth functions of the spacetime realised in the algebra of vector fields $\mathfrak{X}(\mathrm{AdS}_N)$ [28, 29]. Since $\widetilde{\mathrm{SO}}(2, N-1)$ is the isometry group of AdS_N, for each generator $O_{ij} \in \mathfrak{so}(2, N-1)$ there is an associated Killing vector field. Given $f \in C^\infty(\mathbb{R}^{2,N-1})$ and $X \in \mathfrak{so}(2, N-1)$, we calculate the action of the vector field $\xi(X)$ on f as a function in $\mathbb{R}^{2,N-1}$, given by

$$\xi(X)[f] := \frac{\mathrm{d}}{\mathrm{d}\epsilon} f\left((\mathbb{I} + \epsilon X) y\right)\Big|_{\epsilon=0} \,, \tag{3.23}$$

and then obtain the associated action on $C^\infty(\mathrm{AdS}_N)$ by restriction to the hyperboloid Hyp_N. Using Eqs. (3.17) and (3.23) we obtain the following vector fields

$$\xi_0 := \xi(I_{12}) = y^2 \frac{\partial}{\partial y^1} - y^1 \frac{\partial}{\partial y^2} \,, \tag{3.24a}$$

$$\boldsymbol{K}_j := \xi(K_{1j}) = y^1 \frac{\partial}{\partial y^j} + y^j \frac{\partial}{\partial y^1} \,, \quad 3 \leq j \leq N+1 \,, \tag{3.24b}$$

$$\boldsymbol{B}_j := \xi(K_{2,j}) = y^2 \frac{\partial}{\partial y^j} + y^j \frac{\partial}{\partial y^2} \,, \quad 3 \leq j \leq N+1 \,, \tag{3.24c}$$

$$\boldsymbol{J}_{ij} := \xi(J_{ij}) = y^j \frac{\partial}{\partial y^i} - y^i \frac{\partial}{\partial y^j} \,, \quad 3 \leq i < j \leq N+1 \,. \tag{3.24d}$$

From these expressions it is clear that ξ_0 describes an infinitesimal rotation of the y^1y^2–plane in $\mathbb{R}^{2,N-1}$, and the vector fields $\boldsymbol{J}_{ij}$ correspond to purely spatial infinitesimal rotations [28, 29]. The vector fields $\boldsymbol{K}_j$ and $\boldsymbol{B}_j$ can be understood as generalised *boosts*, in analogy with the Lorentz algebra structure [48]. (See Eq. (3.21).) Finally, to obtain the action of the Killing vector fields (3.24) on AdS$_N$ we write the partial derivatives $\partial/\partial y^i$ in terms of the static coordinates (3.4) using the coordinate transformation in Eq. (3.3) setting $r = 1$. We obtain

$$\frac{\partial}{\partial y^1} = -\cos\rho\,\sin t\,\frac{\partial}{\partial t} + \cot\rho\,\cos\rho\,\cos t\,\frac{\partial}{\partial\rho}\,, \tag{3.25a}$$

$$\frac{\partial}{\partial y^2} = \cos\rho\,\cos t\,\frac{\partial}{\partial t} + \cot\rho\,\cos\rho\,\sin t\,\frac{\partial}{\partial\rho}\,, \tag{3.25b}$$

$$\frac{\partial}{\partial y^3} = -\cot\rho\,\sin\theta_1\frac{\partial}{\partial\theta_1}\,, \tag{3.25c}$$

$$\frac{\partial}{\partial y^i} = \cot\rho\left(\prod_{k=1}^{i-3}\csc\theta_k\right)\left[\cos\theta_{i-2}\sum_{j=1}^{i-3}\cot\theta_j\left(\prod_{k=j}^{i-3}\sin^2\theta_k\right)\frac{\partial}{\partial\theta_j} - \sin\theta_{i-2}\frac{\partial}{\partial\theta_{i-2}}\right]\,, \tag{3.25d}$$

$$\frac{\partial}{\partial y^{N+1}} = \cot\rho\left(\prod_{k=1}^{N-3}\csc\theta_k\right)\left[\sum_{j=1}^{N-2}\cot\theta_j\left(\prod_{k=j}^{N-2}\sin^2\theta_k\right)\frac{\partial}{\partial\theta_j} + \cos\theta_{N-2}\frac{\partial}{\partial\theta_{N-2}}\right]\,, \tag{3.25e}$$

where $4 \le i \le N$. Then, the Killing vector fields in Eq. (3.24) may be written in terms of (t, x). In particular, Eqs. (3.24a), (3.25a) and (3.25b) imply that $\xi_0 = \partial/\partial t$, and thus, ξ_0 corresponds to the static Killing vector field of AdS$_N$. It is also worth pointing out that Eqs. (3.3), (3.24d) and (3.25) imply that the Killing vectors $\boldsymbol{J}_{ij}$ only depend on the angular coordinates θ. Similarly, the two boost–like Killing vectors $\boldsymbol{K}_3$ and $\boldsymbol{B}_3$ are given by

$$\boldsymbol{K}_3 = -\cos\theta_1\left(\sin t\,\sin\rho\,\frac{\partial}{\partial t} - \cos t\,\cos\rho\,\frac{\partial}{\partial\rho}\right) - \frac{\cos t\,\sin\theta_1}{\sin\rho}\frac{\partial}{\partial\theta_1}\,, \tag{3.26a}$$

$$\boldsymbol{B}_3 = \cos\theta_1\left(\cos t\,\sin\rho\,\frac{\partial}{\partial t} + \sin t\,\cos\rho\,\frac{\partial}{\partial\rho}\right) - \frac{\sin t\,\sin\theta_1}{\sin\rho}\frac{\partial}{\partial\theta_1}\,. \tag{3.26b}$$

For the case $N = 2$, the only Killing vector fields are given by ξ_0, $\boldsymbol{K}_3$ and $\boldsymbol{B}_3$, the latter are obtained by setting $\theta_1 = 0$ in Eq. (3.26).

An N–dimensional spacetime that admits $N(N+1)/2$ Killing vector fields is said to be *maximally symmetric* [25, 45]. This is clearly the case for anti–de Sitter spacetime as the dimension of $\mathfrak{so}(2, N-1)$ is the number of generators in Eq. (3.17). Since AdS$_N$ is a maximally symmetric spacetimes of constant curvature, the components of the Riemann curvature tensor can be written as [50]

$$R_{\mu\nu\kappa\lambda} = -(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\lambda}g_{\nu\kappa})\,, \tag{3.27}$$

(we have taken into account our initial choice $R = 1$ in Eq. (3.1)) and the Ricci curvature $\boldsymbol{R}$ is then proportional to the metric tensor, *i.e.*, $\boldsymbol{R} = -(N-1)\boldsymbol{g}$. Hence, the Ricci scalar, defined as the trace of the Ricci curvature [25, 28, 29] with respect to the metric tensor,

$\mathcal{R} := \mathrm{Tr}(\boldsymbol{R})$ is given by $\mathcal{R} = -N(N-1)$. By considering these facts, we see that the metric $\boldsymbol{g}$ for AdS_N satisfies

$$\boldsymbol{R} - \frac{1}{2}\mathcal{R}\boldsymbol{g} = \frac{1}{2}(N-1)(N-2)\boldsymbol{g}\,, \tag{3.28}$$

which is precisely the vacuum Einstein equation [25, 45] with cosmological constant $\Lambda = -(N-1)(N-2)/2$.

3.2 LACK OF GLOBAL HYPERBOLICITY

Now that we have reviewed some of the relevant geometric properties of AdS_N, we briefly explain why this spacetime is not globally hyperbolic. This fact is well known and has been analysed thoroughly, thus, we will limit ourselves to give an intuitive explanation for why this is the case. Several arguments for the lack of global hyperbolicity of AdS_N can be found throughout the literature, for example in [24, 26, 27, 30, 34, 51, 52].

If we consider the static coordinates $(t, \rho, \theta_1, \ldots, \theta_{N-2})$, it follows that spatial infinity is described by points in AdS_N for which $\rho \to \pi/2$, with all other coordinates fixed. From Eq. (3.9) we see that the function $\mathcal{N}$ diverges as we take the limit to spatial infinity, and thus, the metric is not defined for these points. We can analyse the causal properties of AdS_N by considering the *conformally related* metric $\boldsymbol{g}^C$ satisfying $\boldsymbol{g} = \mathcal{N}^2\boldsymbol{g}^C$, where $\boldsymbol{g}$ is given by Eq. (3.9). The transformation given by $\boldsymbol{g} \mapsto \boldsymbol{g}^C$ is known as a *conformal transformation* [24, 25, 28, 29, 45], and it has the property that it preserves null hypersurfaces. The non–physical metric tensor $\boldsymbol{g}^C$ describes a pseudo–Riemannian manifold $\widetilde{\mathrm{AdS}_N}$ in which the "points at infinity" of AdS_N are represented as the timelike boundary $\rho = \pi/2$, and the metric $\boldsymbol{g}^C$ is well defined at this boundary. Since null hypersurfaces and time orientation are preserved under the conformal transformation [24], the causal structure of $\widetilde{\mathrm{AdS}_N}$ is the same as for anti–de Sitter spacetime.

The existence of this timelike boundary in $\widetilde{\mathrm{AdS}_N}$ is precisely what prevents AdS_N to be a globally hyperbolic spacetime. The line element associated to the metric $\boldsymbol{g}^C$ is given by

$$\mathrm{d}s_C^2 = -\mathrm{d}t^2 + \mathrm{d}\rho^2 + \sin^2\!\rho\,\mathrm{d}\Omega_{N-2}^2\,. \tag{3.29}$$

If we then consider a null geodesic $c : I \subset \mathbb{R} \to \widetilde{\mathrm{AdS}_N}$ emanating from the origin with respect to the coordinates (t, x), and moving radially outwards, then we must have $\mathrm{d}s_C^2 = 0$, with $\mathrm{d}\Omega_{N-2}^2 = 0$. The equation describing this geodesic parametrised by t_0 is simply given by $\mathrm{d}t = \pm\mathrm{d}\rho$, so that $\rho(t_0) = t_0$ and $\rho(t_0) = -t_0$ describe the geodesics starting at $p = (0, 0, \theta_0)$ with the angular coordinates $\theta_0 \in S^{N-2}$ held fixed. This means that a null geodesic from the origin intersects the timelike boundary $\rho = \pi/2$ after a finite coordinate time, $\Delta t_0 = \pi/2$. Note that this argument is easily extended to any point $p \in \widetilde{\mathrm{AdS}_N}$ other than the origin. In terms of AdS_N, this implies that a light signal emanating from p will escape to spatial infinity after a finite amount of time has elapsed. To see why this fact implies the lack of global hyperbolicity, let us assume that AdS_N is globally hyperbolic, and that there exists a Cauchy surface $\mathcal{S}$ containing $p \in \mathrm{AdS}_N$. Now, the null geodesic described above is an inextendible future–directed causal curve in AdS_N (points with $\rho = \pi/2$ are only defined in $\widetilde{\mathrm{AdS}_N}$, so the geodesic has no future endpoint). By Corollary 2.1.7, there exists a global time function $\tau : \mathrm{AdS}_N \to \mathbb{R}$, and the Cauchy surface $\mathcal{S}$ can be described as the level curve $\tau = \tau_1$ for some $\tau_1 \in \mathbb{R}$. The time function τ is clearly bounded along the null geodesic c starting at p and escaping to infinity at t_{max}, the upper bound given by its value at $c(t_{max})$, say $\tau_2 \in \mathbb{R}$. For any $\varepsilon > 0$, the level curve $\tau = \tau_2 + \varepsilon$ is, by Corollary 2.1.7, a Cauchy surface, $\mathcal{S}_{\tau_2+\varepsilon}$. However, as any point lying in the null geodesic c satisfies $\tau(c(t)) \leq \tau_2$, the surface $\mathcal{S}_{\tau_2+\varepsilon}$ does not intersect $c(t)$ for any t, and thus, cannot be a Cauchy surface by Definition 2.1.2. Since $\mathcal{S}_{\tau_2+\varepsilon}$ was obtained by Cauchy development of $\mathcal{S}$ through τ, $\mathcal{S}$ cannot be a Cauchy surface, so we arrive at a contradiction. As $\mathcal{S}$ was chosen arbitrarily this argument implies that no Cauchy surfaces can exist in AdS_N, hence, it is not a globally hyperbolic spacetime.

— 4 —

Representation theory of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$

One of the main aspects of QFT in Minkowski spacetime, $\mathbb{R}^{1,3}$, is the role that the unitary representations of the Poincaré group [48], the isometry group of $\mathbb{R}^{1,3}$, have in defining physically acceptable quantum theories. Trying to reconcile Quantum Mechanics and Special Relativity was what led to the framework of QFT in the first place, and this was achieved, in broad terms, by requiring the fields that represent quantum observables to form a unitary irreducible representation (UIR) of the Poincaré group [48, 53, 54].

Extending the methods and techniques of QFT in Minkowski spacetime to a QFT defined in a more general curved background suggests, in a certain way, that requiring the fields to be invariant under the isometry group (if any) of the curved spacetime may be a sensible requirement for physically acceptable theories.

As we mentioned in Chapter 3, the isometry group of anti de–Sitter spacetime, AdS_N, is the universal covering group of $\mathrm{SO}(2, N-1)$, which in the two–dimensional case reduces to $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, the universal covering group of $\mathrm{SL}(2,\mathbb{R})$. The classification of all possible UIRs of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ up to isomorphism was mainly due to Pukanszky [55] who followed an approach similar to that of Bargmann [56] for the case of $\mathrm{SL}(2,\mathbb{R})$. The classification of UIRs of $\mathrm{SO}(p,q)$, for $p, q \in \mathbb{N}$, can be found in a series of papers by Limić, Niederle and Rączka [57, 58, 59], while the UIRs for the universal covering group $\widetilde{\mathrm{SO}}(2,3)$ were classified by Ehrman [60], and the generalisation to any p and q might be carried out in a very similar way.

In this chapter we will briefly review some of the general concepts in the representation theory of Lie groups and Lie algebras. We will present some fundamental definitions and results for general Lie groups and then specialise to the properties of the groups of $\mathrm{SL}(2,\mathbb{R})$ and $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ and the representations of their Lie algebra. Then we will explain how the UIRs of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ can be obtained via the representations of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, and we will arrive at the classification given by Pukanszky. We will adopt the notation and conventions of Ref. [61] which are more closely related to the way we will apply the theory to the particular case of AdS_2.

4.1 Lie groups, Lie algebras and their representations

In this section we give some elementary definitions and results regarding Lie groups and Lie algebras. Most of the material presented here is well known and can be found in standard textbooks, but we will base our notation, definitions, and results mainly on Refs. [47, 49, 62].

We will refer to a set S as a separable topological space if its underlying topology is that of a separable metric space. A **Lie group** G is a separable topological group [47] with the structure of a smooth manifold compatible with the given topology in such a way that the group multiplication $G \times G \to G$ and inversion are smooth maps. An *analytic* group is a connected Lie group. If the underlying manifold defining the Lie group G is compact, then G is said to be a compact group. A *matrix Lie group*, or closed linear Lie group, is a topologically closed subgroup of $\mathrm{GL}(n,\mathbb{C})$, where $\mathrm{GL}(n,\mathbb{C})$ with $n \in \mathbb{N}$ is the group of invertible $n \times n$ matrices with complex entries [49].

Let $\Phi : G \to H$ be a smooth map between the Lie groups G and H. If Φ satisfies $\Phi(g_1 g_2) = \Phi(g_1)\Phi(g_2)$ for all $g_1, g_2 \in G$, then Φ is said to be a *group homomorphism*. A *local homomorphism* between analytic Lie groups G and H, is a pair (Φ, U), where $U \subset G$ is an open connected neighbourhood of $e \in G$ and $\Phi : G \to H$ is a smooth map such that $\Phi(g_1, g_2) = \Phi(g_1)\Phi(g_2)$ whenever g_1, g_2 and $g_1 g_2$ lie in U.

A finite dimensional **Lie algebra** $\mathfrak{g}$ is a vector space over a field $\mathbb{F}$ with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket*, satisfying $[X,Y] = -[Y,X]$, and the Jacobi identity

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]] \,, \tag{4.1}$$

for all $X, Y, Z \in \mathfrak{g}$. Given a basis $\{X_i\}_{i=1}^{D}$ of a D–dimensional Lie algebra $\mathfrak{g}$, the commutator between the vectors X_i must satisfy

$$[X_i, X_j] = c^k{}_{ij} X_k \,, \tag{4.2}$$

for some $c^k{}_{ij} \in \mathbb{F}$, with $1 \le i,j,k \le D$ [47]. The quantities $c^k{}_{ij} \in \mathbb{F}$ are the *structure constants* of the Lie algebra $\mathfrak{g}$, and they are basis–dependent. The structure constants, by linearity, determine the Lie brackets of all elements of $\mathfrak{g}$. If all the structure constants vanish in a given basis, then the Lie algebra $\mathfrak{g}$ is said to be commutative or **abelian**. A subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is said to be a Lie subalgebra if, for any $X, Y \in \mathfrak{h}$, $[X,Y] \in \mathfrak{h}$. A linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ between two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, satisfying $\phi\left([X,Y]_{\mathfrak{g}}\right) = [\phi(X), \phi(Y)]_{\mathfrak{h}}$ for all $X, Y \in \mathfrak{g}$ is called a *Lie algebra homomorphism*.

Several important types of Lie algebras are found by analysing how certain subalgebras behave under operations involving Lie brackets; we now review the definitions of some of these that will be relevant for our purposes [47, Chapter 1]. Let $\mathfrak{g}$ be a Lie algebra, and for any two subspaces $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{g}$, write $[\mathfrak{a}, \mathfrak{b}]$ to denote the linear span of elements of the form $[X,Y]$ with $X \in \mathfrak{a}$ and $Y \in \mathfrak{b}$. An *ideal* $\mathfrak{a}$ in $\mathfrak{g}$ is a subspace satisfying $[\mathfrak{a}, \mathfrak{g}] \subseteq \mathfrak{a}$. The **centre** of $\mathfrak{g}$ is the subspace $Z_{\mathfrak{g}}$ consisting of all $X \in \mathfrak{g}$ such that $[X,Y] = 0$ for all $Y \in \mathfrak{g}$. Now, consider the following subspaces of $\mathfrak{g}$: The subspaces $\mathfrak{g}^k$ defined recursively by $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^1 = [\mathfrak{g},\mathfrak{g}]$, and $\mathfrak{g}^{k+1} = [\mathfrak{g}^k,\mathfrak{g}^k]$, and the subspaces $\mathfrak{g}_k$ defined recursively by $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_1 = [\mathfrak{g},\mathfrak{g}]$, and $\mathfrak{g}_{k+1} = [\mathfrak{g},\mathfrak{g}_k]$. A lie algebra $\mathfrak{g}$ is said to be *solvable* if $\mathfrak{g}^k = 0$ for some k. A Lie algebra $\mathfrak{g}$ satisfying $\mathfrak{g}_k = 0$ for some k is said to be **nilpotent**. If $\mathfrak{g}$ has no non–zero solvable ideals, then $\mathfrak{g}$ is called a **semisimple** Lie algebra.

Now we briefly review the definition of a universal enveloping algebra, and refer to [47, Chapter 3] for further details. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$[1]. Since $\mathfrak{g}$ is a vector space, we can consider the *tensor algebra* of $\mathfrak{g}$ defined by [47, 63]

$$T(\mathfrak{g}) := \bigoplus_{k=0}^{\infty} \left(\otimes^k \mathfrak{g}\right) , \tag{4.3}$$

where $\otimes^k\mathfrak{g}$ is the k–fold tensor product of the vector space $\mathfrak{g}$, and $\otimes^0\mathfrak{g} := \mathbb{C}$. $T(\mathfrak{g})$ is an associative algebra with identity $\mathbf{1}$ in $\mathbb{C}$. Consider the two–sided (left– and right–) ideal generated by all $(X \otimes Y - Y \otimes X - [X,Y]) \in T(\mathfrak{g})$, with $X, Y \in \mathfrak{g}$. The quotient of $T(\mathfrak{g})$ by this two–sided ideal, denoted by $U(\mathfrak{g})$, is known as the **universal enveloping algebra** of $\mathfrak{g}$ [47, 49, 63]. The universal enveloping algebra is also associative and unital (with identity $\mathbf{1} \in \mathbb{C}$), and has a universal mapping property [63, Proposition 3.1]. The Lie algebra $\mathfrak{g}$ naturally embeds in $U(\mathfrak{g})$ by means of Eq. (4.3). The centre of $U(\mathfrak{g})$ is the set $Z(\mathfrak{g})$ of elements $\widetilde{X} \in U(\mathfrak{g})$ such that $\widetilde{X}Y = Y\widetilde{X}$ for all $Y \in \mathfrak{g}$ [47, Proposition 5.22]. We will return to universal enveloping algebras once we introduce the concept of representations.

The link between Lie groups and Lie algebras arises from the smooth manifold structure of the Lie group G [47, Chapter 1]. For any $g \in G$, let $L_g : G \to G$ be the *left–translation* map by g, defined by $L_g(g') = gg'$ for all $g' \in G$. Now, consider a smooth vector field $X \in \mathfrak{X}(G)$. A vector field X is said to be a *left–invariant vector field* if, for any $g, g' \in G$, we have $X_{gg'} = (\mathrm{d}L_g)_{g'}(X_{g'})$, where $(\mathrm{d}L_g)_{g'} : T_{g'}G \to T_{gg'}G$ denotes the differential or pushforward [28, 29] of the map L_g at g'. The subspace $\mathfrak{g}$ of left–invariant vector fields on G forms a Lie subalgebra of $\mathfrak{X}(G)$, and $\mathfrak{g}$ is said to be the Lie algebra of the Lie group G. The map $X \mapsto X_e$, with $e \in G$ denoting the identity element of G, is a vector space isomorphism $\mathfrak{g} \to T_eG$ onto the tangent space of G at the identity [47], and thus, we may identify the Lie algebra of G with T_eG. Hence, every Lie group G has a unique associated Lie algebra $\mathfrak{g}$, up to vector space isomorphism canonically identified with its tangent space at the identity.

This relation between Lie groups and Lie algebras also extends to homomorphisms [47, Chaper I, Section 10]: Let $\Phi : G \to H$ be a smooth homomorphism between the Lie groups G and H with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Let $(\mathrm{d}\Phi)_g : T_gG \to T_{\Phi(g)}H$ be the differential of Φ at $g \in G$. Then, the map $(\mathrm{d}\Phi)_e : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism. This means that for any Lie group homomorphism there is an associated Lie algebra homomorphism. The passage in the reverse direction, in general, only works locally. The result is that, given two analytic groups G and H and a homomorphism ϕ between their Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively, then there exists a *local* homomorphism $\Phi : G \to H$, such that $\mathrm{d}\Phi = \phi$.

[1] If $\mathfrak{g}$ is a real Lie algebra, then we shall take the *complexification* of $\mathfrak{g}$ instead, given by $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus i\mathfrak{g}$ [49].

The lifting of homomorphisms of Lie algebras to homomorphisms of Lie groups described above can be used to define the exponential map of a Lie algebra. We briefly review the well–known definition and refer to Ref. [47, Chapter I, Section 10] for further details. Let $\mathbb{R}$ the real additive group and denote by $\mathfrak{r}$ its one–dimensional Lie algebra spanned by $T := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}$. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Given $X \in \mathfrak{g}$, the map $T \mapsto X$ defines a Lie algebra homomorphism $\mathfrak{r} \to \mathfrak{g}$, and it lifts to a smooth (local) group homomorphism $\Phi_X : \mathbb{R} \to G$. Now, the map Φ_X defines a smooth curve on G with $\Phi_X(0) = e$, where $e \in G$ is the identity element, whose tangent vector at e is X. Hence, Φ_X is an integral curve of the left–invariant vector field $\tilde{X}$ associated to X by $\tilde{X}_e = X$. The **exponential map** of the Lie algebra $\mathfrak{g}$, denoted by $\exp : \mathfrak{g} \to G$ is then defined by $\exp(X) := \Phi_X(1)$.

The exponential map, $\exp : \mathfrak{g} \to G$, is a diffeomorphism for any sufficiently small neighbourhood of $0 \in \mathfrak{g}$ onto an open neighbourhood of $e \in G$ [63]. Using the exponential map two local coordinate systems for G can be constructed [47, 63]. Given a basis $\{X_i\}_{i=1}^{D}$ of the Lie algebra $\mathfrak{g}$, the map

$$(x^1, \ldots, x^D) \mapsto g\, \exp(x^1 X_1 + \cdots x^D X_D)\,, \tag{4.4}$$

carries a sufficiently small neighbourhood around $0 \in \mathbb{R}^D$ diffeomorphically onto an open neighbourhood of g in G. Hence, the inverse map defines a compatible chart about g, and defines *canonical coordinates of the first kind*. Similarly, if the Lie algebra $\mathfrak{g}$ of an analytic Lie group G is the direct sum of Lie algebras of one–dimensional subspaces, $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_D$, and if U_k is a sufficiently small open neighbourhood of $0 \in \mathfrak{g}_k$ for $1 \leq k \leq D$, then the map

$$(X_1, \ldots, X_D) \mapsto g\, \exp(X_1) \cdots \exp(X_D)\,, \tag{4.5}$$

is a diffeomorphism of $U_1 \times \cdots \times U_D$ onto an open neighbourhood of $g \in G$. The local coordinates given by the inverse map of Eq. (4.5) are known as *canonical coordinates of the second kind*.

The exponential map allows to map elements of a Lie algebra to the associated analytic group but, in the general case, the surjectivity of exp only holds locally. This implies that for a general analytic group G, it may not be possible to map the Lie algebra $\mathfrak{g}$ onto the entire Lie group through a single mapping of the form $\exp(X)$ with $X \in \mathfrak{g}$. In other words, there may exist elements $g \in G$ such that no $X \in \mathfrak{g}$ satisfies $g = \exp(X)$ [49, 64]. However, Eq. (4.5) implies that if the Lie group G is connected, then any element $g \in G$ can be written as a product of exponentials of elements of the Lie algebra $\mathfrak{g}$. The algebraic properties that the Lie algebra $\mathfrak{g}$ has can thus be used to characterise the associated Lie group via this correspondence. For example: A Lie group G with Lie algebra $\mathfrak{g}$ is said to be a **nilpotent group** if its Lie algebra is nilpotent and G is said to be a **semisimple group** if $\mathfrak{g}$ is a semisimple Lie algebra.

The local nature of the exponential map shows the reason why a Lie algebra $\mathfrak{g}$ may have two or more non–isomorphic analytic groups associated to it [49]; the exponential map does not provide information about the global nature of the group except in very few specific cases [47]. However, there is a canonical way to assign a unique Lie group (up to isomorphism) to a given Lie algebra, given by the following theorem [65, Corollary 3.43]:

Theorem 4.1.1 *For any real or complex Lie algebra $\mathfrak{g}$, there is a unique, up to isomorphism, simply connected analytic group G whose Lie algebra is $\mathfrak{g}$. Any other analytic group G' with Lie algebra $\mathfrak{g}$ must be of the form $G' = G/H$, for some discrete central subgroup $H \subset G$.*

Remark 4.1.2 *The key property of this unique Lie group associated to $\mathfrak{g}$ is its simple connectedness. A pathwise connected topological space is said to be simply connected if every loop based at a point can be continuously deformed to the given point with the point itself held fixed [47, 49]. A central subgroup $H \subset G$ is a subgroup contained in the centre Z of G, that is, the set of elements that commute with every element of G. A subgroup H of G is discrete if the subspace topology induced from G is discrete.*

Theorem 4.1.1 gives the form of all possible analytic groups with Lie algebra $\mathfrak{g}$. If one starts with an analytic group G which is not simply connected we may use this result to find an analytic group $\widetilde{G}$ which is simply connected and that has the same Lie algebra as G. If G is an analytic group, then it is pathwise connected, and *locally* simply connected [47]. Hence, G, as a separable topological space, admits a universal covering space, $\widetilde{G}$. By definition [29, 47], the universal covering space $\widetilde{G}$ is the unique (up to isomorphism) simply connected covering of G, and the following result ensures that $\widetilde{G}$ is an analytic group [47, Propositions 1.97 and 1.99]:

Proposition 4.1.3 *Let G be an analytic group and let $\widetilde{G}$ be its universal covering space. Then there exists a unique multiplication on $\widetilde{G}$ that makes $\widetilde{G}$ into an analytic group such that the covering map is a group homomorphism.*

The group $\widetilde{G}$ obtained in this proposition is known as the **universal covering group** of G. If we apply Theorem 4.1.1 to $\widetilde{G}$, then it follows that both $\widetilde{G}$ and G have the same Lie algebra and that G is isomorphic to $\widetilde{G}/H$, with H a discrete subgroup of the centre of $\widetilde{G}$. In Section 4.2 we will explain how these results translate to the specific case of the group $\mathrm{SL}(2,\mathbb{R})$.

Next, we review certain concepts regarding representations of Lie algebras and their relation with representations of the associated Lie group. We begin by giving the general definition [63]:

Definition 4.1.4 *A **representation of a Lie group** G on a complex Hilbert space $\mathscr{H}$ is a homomorphism Π of G into the group of bounded linear operators on $\mathscr{H}$ with bounded inverses, such that the resulting map $G \times \mathscr{H} \to \mathscr{H}$ is continuous. A **representation of a Lie algebra** $\mathfrak{g}$ on a complex Hilbert space $\mathscr{H}$ is a homomorphism π from the Lie algebra $\mathfrak{g}$ to the Lie algebra of all linear transformations of $\mathscr{H}$ into itself.*

An *invariant subspace* for a representation Π of G is a vector subspace $\mathscr{I} \subseteq \mathscr{H}$ such that $\Pi(g)\mathscr{I} \subseteq \mathscr{I}$ for all $g \in G$. A representation Π is said to be **irreducible** if it has no closed invariant subspaces other than $\{0\}$ and $\mathscr{H}$. For $g \in G$, let $\Pi(g)^\dagger$ denote the adjoint operator of $\Pi(g)$ with respect to the inner product of $\mathscr{H}$ as given by Definition 2.2.2. If $\Pi(g)\Pi(g)^\dagger = \Pi(g)^\dagger\Pi(g) = \mathbb{I}$ for all $g \in G$, with $\mathbb{I}$ the identity operator on $\mathscr{H}$, then the representation Π is called **unitary**. Two representations Π on $\mathscr{H}$ and Π' on $\mathscr{H}'$ of a Lie group G are *unitarily equivalent* if there exists a bounded linear unitary map $E : \mathscr{H} \to \mathscr{H}'$ with a bounded inverse such that $\Pi'(g)E = E\Pi(g)$ for all $g \in G$. An important result concerning unitary irreducible representations of Lie groups is given by Schur's lemma [63, Proposition 1.5]

Theorem 4.1.5 (Schur's lemma) *Let G be a topological group. A unitary representation Π of G on a Hilbert space $\mathscr{H}$ is irreducible if and only if the only bounded linear operators on $\mathscr{H}$ commuting with all $\Pi(g)$, $g \in G$, are the scalar operators.*

Invariant subspaces and irreducible representations of Lie algebras are defined analogously to those corresponding to Lie group representations. If π is a representation of a Lie algebra $\mathfrak{g}$ on a Hilbert space $\mathscr{H}$, then there exists a unique algebra homomorphism $\tilde{\pi}$ from the universal enveloping algebra $U(\mathfrak{g})$ to the algebra of all linear transformations of $\mathscr{H}$ into itself such that $\tilde{\pi}(\mathbf{1}) = \mathbb{I}$ and $\tilde{\pi}(X) = \pi(X)$ for all $X \in \mathfrak{g} \subset U(\mathfrak{g})$ [49, Proposition 9.9]. The homomorphism $\tilde{\pi}$ is a representation of the algebra $U(\mathfrak{g})$, and in this sense it extends the representation π of $\mathfrak{g}$ to $U(\mathfrak{g})$. An analogue of Theorem 4.1.5 for irreducible representations of the universal enveloping algebra $U(\mathfrak{g})$ is given by the following result [47, Corollary 3.6 and Proposition 5.19].

Theorem 4.1.6 (Dixmier) *Let $\mathfrak{g}$ be a complex Lie algebra, and let $\tilde{\pi}$ be an irreducible representation of $U(\mathfrak{g})$ on $\mathscr{H}$. Then, the only $U(\mathfrak{g})$–linear maps $\mathscr{H} \to \mathscr{H}$ are the scalar multiples of the identity.*

Remark 4.1.7 *A $U(\mathfrak{g})$–linear map is an element of $\mathrm{End}_{U(\mathfrak{g})}(\mathscr{H}, \mathscr{H})$, with $\mathscr{H}$ identified with a left $U(\mathfrak{g})$–module [47, 63]. Explicitly, given $f_1, f_2 \in \mathscr{H}$, a map $L : \mathscr{H} \to \mathscr{H}$ is $U(\mathfrak{g})$–linear if $L(\tilde{\pi}(\widetilde{X})f_1 + \tilde{\pi}(\widetilde{Y})f_2) = \tilde{\pi}(\widetilde{X})L(f_1) + \tilde{\pi}(\widetilde{Y})L(f_2)$ for all $\widetilde{X}, \widetilde{Y} \in U(\mathfrak{g})$. If $L = \tilde{\pi}(\widetilde{Z})$ for some $\widetilde{Z} \in U(\mathfrak{g})$, this means that $\widetilde{Z}$ is an element of $Z(\mathfrak{g})$. Thus, any element of $Z(\mathfrak{g})$ must be mapped to a multiple of the identity operator in $\mathscr{H}$ if $\tilde{\pi}$ is irreducible.*

The result of this theorem gives a useful criterion to determine if a given representation of a Lie algebra is irreducible: Any element of the centre $Z(\mathfrak{g})$ must act as a scalar operator on a Hilbert space $\mathscr{H}$ on which a representation of $U(\mathfrak{g})$ and thus, a representation of $\mathfrak{g}$ acts. We will use this fact in Section 4.3 as a tool to classify irreducible representations of $\mathfrak{sl}(2, \mathbb{R})$.

Now we review the relation between representations of a Lie group and representations of its Lie algebra. Given a Lie group with Lie algebra $\mathfrak{g}$ and a representation Π of G on a finite–dimensional Hilbert space $\mathscr{H}$, the differential $(\mathrm{d}\Pi)_e$ of Π at the identity is a Lie algebra representation of $\mathfrak{g}$ on $\mathscr{H}$, and $\mathrm{d}\Pi$ uniquely determines Π. Conversely, if π is a representation of the Lie algebra $\mathfrak{g}$ on a finite–dimensional Hilbert space $\mathscr{H}$ and if G is a simply connected analytic Lie group with Lie algebra $\mathfrak{g}$, then there exists a representation Π of G with $\mathrm{d}\Pi = \pi$ [63, Chapter I, Section 3]. The relation between finite–dimensional representations of a Lie group and finite–dimensional representations of the associated Lie algebra gives a correspondence between invariant subspaces and irreducibility: If a finite–dimensional representation of a Lie group admits an invariant subspace $\mathscr{I}$, then $\mathscr{I}$ is also an invariant subspace for the representation of the Lie algebra. Similarly, if the representation Π of G on $\mathscr{H}$ is unitary, then differentiation at the identity implies that $\mathrm{d}\Pi = \pi$ maps elements of $\mathfrak{g}$ to skew–Hermitian operators on $\mathscr{H}$, *i.e.*, $\pi(X)^\dagger = -\pi(X)$ for all $X \in \mathfrak{g}$.

The correspondence between representations of Lie groups and representations of Lie algebras becomes more complicated when the underlying Hilbert space $\mathscr{H}$ is infinite–dimensional. We will present a brief summary of the process needed to associate a representation of the Lie algebra $\mathfrak{g}$ of G to a representation Π of G on an infinite–dimensional Hilbert space. We omit the more technical details and refer to [62, Chapter VI, Section 1] and in [63, Chapter 3, Sections 3 and 4] for a complete discussion.

Let G be an analytic group and Π a representation of G on an infinite–dimensional Hilbert space $\mathscr{H}$. An element $f \in \mathscr{H}$ is said to be a C^∞–vector if the map $g \mapsto \Pi(g)f$ is of class C^∞. The C^∞–vectors form a dense subspace of $\mathscr{H}$, and we denote this subspace by $C^\infty(\Pi)$. Given $f \in C^\infty(\Pi)$, and $X \in \mathfrak{g}$, we define the linear mapping $C^\infty(\Pi) \to C^\infty(\Pi)$ by

$$\pi(X)f := \lim_{t\to 0} \frac{\Pi(\exp(tX))f - f}{t}\,. \tag{4.6}$$

For every $X \in \mathfrak{g}$, the map $\pi(X)$ satisfies $\pi(x)(C^\infty(\Pi)) \subseteq C^\infty(\Pi)$, and it defines a Lie algebra representation of $\mathfrak{g}$ on $C^\infty(\Pi)$ [63, Proposition 3.9]. The representation Π of G on $\mathscr{H}$ leaves $C^\infty(\Pi)$ stable and thus, Eq. (4.6) is an analogue to $\mathrm{d}\Pi = \pi$ for the finite–dimensional case. The important fact about the subspace $C^\infty(\Pi)$ is that it is dense in $\mathscr{H}$ [63, Theorem 3.15]. Therefore, any $f \in \mathscr{H}$ can be approximated by a sequence of elements in $C^\infty(\Pi)$ and in particular, of elements in $\pi(C^\infty(\Pi))$.

The reason to focus our attention to infinite dimensional representations of Lie algebras and Lie groups is that for non–compact semisimple groups such as $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SO}(2,N-1)$ [47] UIRs are, in general, infinite–dimensional. The relation between infinite–dimensional representations at the Lie group and the Lie algebra level allows us to work with representations of the Lie algebra to classify all UIRs which is a far simpler task than dealing with the group representations.

4.2 General properties of $\mathrm{SL}(2,\mathbb{R})$, $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ and $\mathfrak{sl}(2,\mathbb{R})$

In this section we will present some properties of the Lie group $\mathrm{SL}(2,\mathbb{R})$, the realisation of its universal covering group and some aspects of the structure of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. We will briefly describe the construction of all the finite–dimensional irreducible representations of the Lie algebra and leave the classification of all possible UIRs of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ for the next section.

The real special linear group $\mathrm{SL}(2,\mathbb{R})$ is the matrix Lie group which in its linear representation is given by the set of real 2×2 invertible matrices of determinant 1, *i.e.*,

$$\mathrm{SL}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a,b,c,d \in \mathbb{R}\,, ac-bd=1 \right\} . \tag{4.7}$$

Any element $g \in \mathrm{SL}(2,\mathbb{R})$, can be uniquely written in terms of the product of matrices given by

$$\begin{aligned} g(\theta,s,y) &= k(\theta)a(s)n(y)\,, \\ &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\,, \end{aligned} \tag{4.8}$$

with $\theta \in [0,2\pi)$ and $s,y\in\mathbb{R}$ [47, Theorem 6.46]. The parametrisation of elements $g\in\mathrm{SL}(2,\mathbb{R})$ of the form (4.8) provides a global description of $\mathrm{SL}(2,\mathbb{R})$ with the identity element obtained by setting $e=g(0,0,0)$. Using this parametrisation it is also possible to show [49] that the centre Z of $\mathrm{SL}(2,\mathbb{R})$ consists of the two elements, e and $-e=g(\pi,0,0)$. From this decomposition it follows that the topology of $\mathrm{SL}(2,\mathbb{R})$ is that of $S^1\times\mathbb{R}^2$, and thus, $\mathrm{SL}(2,\mathbb{R})$ is a non–compact analytic group which is not simply connected.

Equation (4.8) also gives a decomposition of $\mathrm{SL}(2,\mathbb{R})$ in terms of the subgroups $K:=\{k(\theta)|\,\theta\in[0,2\pi)\}$, $A:=\{a(s)|\,s\in\mathbb{R}\}$ and $N:=\{n(y)|\,y\in\mathbb{R}\}$, where $K\simeq\mathrm{SO}(2)$ is compact, A is abelian and N is nilpotent. This is the *Iwasawa decomposition* [47] of $\mathrm{SL}(2,\mathbb{R})$. The multiplication map $K\times A\times N\to\mathrm{SL}(2,\mathbb{R})$ given by $(k,a,n)\mapsto kan$ defines a surjective diffeomorphism [63, Theorem 5.12], but not a group homomorphism.

The group $\mathrm{SL}(2,\mathbb{R})$ is not simply connected [64, Section 7.2], but it admits a universal covering group, $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. From its Iwasawa decomposition (4.8) it is clear that the subgroup $K=\mathrm{SO}(2)$, topologically equivalent to S^1, has the real additive group $\mathbb{R}$ as its universal covering group, and the universal covering group of $\mathrm{SL}(2,\mathbb{R})$ can be understood as associating this "unrolling" of S^1 to the whole group. Instead of describing the group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ via a covering map on $\mathrm{SL}(2,\mathbb{R})$, we will use Theorem 4.1.1 to define the universal covering group through the Lie algebra of $\mathrm{SL}(2,\mathbb{R})$, which we now review.

The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ consists of the space of traceless 2×2 matrices over the reals. This is easily seen from the fact that any smooth curve $c:\mathbb{R}\to\mathrm{SL}(2,\mathbb{R})$ passing through the identity at $t=0$ satisfies $\det(c(t))=1$ and, thus, $\mathrm{Tr}(\dot c(0))=0$ [49]. We will take advantage of the Iwasawa decomposition (4.8) to obtain the generators of the subgroups K, A and N by defining the smooth curves on $\mathrm{SL}(2,\mathbb{R})$ given by $c_1(t)=g(t,0,0)$, $c_2(t)=g(0,t,0)$ and $c_3(t)=g(0,0,t)$. We have $c_i(0)=e$ for all $i=1,2,3$. The corresponding tangent vectors in $\mathfrak{sl}(2,\mathbb{R})$ are given by $X_i:=\dot c_i(0)$, and we find

$$X_1=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \quad X_2=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,, \quad X_3=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,. \tag{4.9}$$

The tangent vectors in Eq. (4.9) form a basis of $\mathfrak{sl}(2,\mathbb{R})$, and are related to the more familiar basis [49],

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{4.10}$$

by $X_1 = E - F$, $X_2 = H$ and $X_3 = E$. The Lie brackets between the elements (4.10) are given by

$$[H, E] = 2E\,, \quad [H, F] = -2F\,, \quad [E, F] = H\,. \tag{4.11}$$

From these relations, it follows that $\mathfrak{sl}(2,\mathbb{R})$ is non–abelian and the only ideal in $\mathfrak{sl}(2,\mathbb{R})$ is $\{0\}$, which means that it is a semisimple Lie algebra. This also implies that the centre $Z_{\mathfrak{sl}(2,\mathbb{R})}$ of the Lie algebra is trivial.

A more convenient basis of $\mathfrak{sl}(2,\mathbb{R})$ can be found as follows. Consider the *adjoint* map $\mathrm{ad} : \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{End}_{\mathbb{R}}(\mathfrak{sl}(2,\mathbb{R}))$, defined as $\mathrm{ad}_X(Y) := [X, Y]$ for all $X, Y \in \mathfrak{sl}(2,\mathbb{R})$. The map ad defines a three–dimensional representation of $\mathfrak{sl}(2,\mathbb{R})$ on itself [49]. For this representation, each $X \in \mathfrak{sl}(2,\mathbb{R})$ is mapped to a linear operator on the Lie algebra given by a 3×3 matrix. Thus, we can define a bilinear form $B_K : \mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{sl}(2,\mathbb{R})$, given by

$$B_K(X, Y) := \mathrm{Tr}(\mathrm{ad}_X \mathrm{ad}_Y)\,, \tag{4.12}$$

for $X, Y \in \mathfrak{sl}(2,\mathbb{R})$. The bilinear map B_K is known as the *Killing form*. A straightforward calculation shows that $B_K(\mathrm{ad}_X(Y), Z) = -B_K(Y, \mathrm{ad}_X(Z))$ for all $X, Y, Z \in \mathfrak{sl}(2,\mathbb{R})$. A very important result [47, Theorem 1.45], known as Cartan's criterion for semisimplicity, states that on any semisimple Lie algebra, B_K is non–degenerate. Since $\mathfrak{sl}(2,\mathbb{R})$ is semisimple, the Killing form defines a (pseudo–) metric on the Lie algebra. This is directly seen by choosing a basis for $\mathfrak{sl}(2,\mathbb{R})$, say $\{H, E, F\}$ in Eq. (4.10) and computing the matrix representation B of B_K in that basis. Using Eqs. (4.11) and (4.12) it can be shown that B_K has signature $(-,+,+)$. The basis that diagonalises the Killing form is given by

$$\Lambda_0 = \frac{1}{2}(E - F)\,, \quad \Lambda_1 = \frac{1}{2}(E + F)\,, \quad \Lambda_2 = \frac{1}{2}H\,, \tag{4.13}$$

and Eq. (4.11) implies that the commutation relations for these elements are given by

$$[\Lambda_0, \Lambda_1] = \Lambda_2\,, \quad [\Lambda_0, \Lambda_2] = -\Lambda_1\,, \quad [\Lambda_1, \Lambda_2] = -\Lambda_0\,. \tag{4.14}$$

Note that these commutation relations correspond to those given by Eq. (3.22) for $\mathfrak{so}(2,1)$ by identifying $\Lambda_0 \to I_{12}$, $\Lambda_1 \to K_{23}$ and $\Lambda_2 \to K_{13}$. Hence, it is now clear that $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(2,1)$ are indeed isomorphic.

Using the non–degeneracy of the Killing form we can find a basis for the dual vector space $\mathfrak{sl}(2,\mathbb{R})^*$. The dual basis $\{\Lambda'_0, \Lambda'_1, \Lambda'_2\}$ is defined by requiring $B_K(\Lambda'_i, \Lambda_j) = \eta_{ij}$. This results in $\Lambda'_i = \Lambda_i/2$ for $i = 0, 1, 2$. With this result, we can now consider the universal enveloping algebra $U(\mathfrak{g})$, with $\mathfrak{g} := \mathfrak{sl}(2,\mathbb{C})$, the complexification of $\mathfrak{sl}(2,\mathbb{R})$ [49, Proposition

3.38, Eq. 3.17]. Let us define the element $Q := \sum_i \widetilde{X}_i' \widetilde{X}_i \in U(\mathfrak{g})$ for any choice of basis $\{\widetilde{X}_i\}_{i=1}^3$ of $\mathfrak{g} \subset U(\mathfrak{g})$. The element $Q \in U(\mathfrak{g})$ is known as the **Casimir element** and, in terms of the basis elements $\{\Lambda_0, \Lambda_1, \Lambda_2\}$, it is given by

$$Q = \frac{1}{2}\left(-\widetilde{\Lambda}_0^2 + \widetilde{\Lambda}_1^2 + \widetilde{\Lambda}_2^2\right), \tag{4.15}$$

where we have used Eq. (4.14). For any complex semisimple Lie algebra $\mathfrak{g}$ (not just $\mathfrak{sl}(2,\mathbb{C})$) the element Q commutes in $U(\mathfrak{g})$ with all $\widetilde{X} \in \mathfrak{g}$ which means that $Q \in Z(\mathfrak{g})$ [47, Proposition 5.24]. The element Q in Eq. (4.15) will play a central role once we study irreducible representations of $\mathfrak{sl}(2,\mathbb{R})$ at the end of this section.

Now, if we consider the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, then exponentiation of this Lie algebra canonically defines the universal covering group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{R})$ [61]. It is well known that the Lie group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is not a matrix Lie group [49, Proposition 5.16] and, thus, it is not possible to define convergence of $\exp(X)$ for $X \in \mathfrak{sl}(2,\mathbb{R})$ with respect to the topology of 2×2 matrices. However, exponentiation of $\mathfrak{sl}(2,\mathbb{R})$ can be defined abstractly using the canonical coordinates of the second kind given by Eq. (4.5) which, for this case, read

$$(\tilde{\theta}\Lambda_0, a\Lambda_1, b\Lambda_2) \mapsto \tilde{g}\exp(\tilde{\theta}\Lambda_0)\exp(a\Lambda_1)\exp(b\Lambda_2), \tag{4.16}$$

for $\widetilde{\theta}, a, b \in \mathbb{R}$ and $\tilde{g} \in \widetilde{\mathrm{SL}}(2,\mathbb{R})$. Furthermore, the canonical coordinates of the first kind (4.4) define the inverse of the coordinate chart around $\tilde{g} \in \widetilde{\mathrm{SL}}(2,\mathbb{R})$, given by

$$(\widetilde{\theta}, a, b) \mapsto \tilde{g}\exp\left(\widetilde{\theta}\Lambda_0 + a\Lambda_1 + b\Lambda_2\right). \tag{4.17}$$

To relate $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ with the result of Theorem 4.1.1, we need to identify a discrete central subgroup $H \subset \widetilde{\mathrm{SL}}(2,\mathbb{R})$ mentioned in this statement. As it turns out, the centre of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is the discrete subgroup $\widetilde{Z} := \{\exp(2\pi n\Lambda_0) | \, n \in \mathbb{Z}\}$. Applying Theorem 4.1.1, one finds that $\widetilde{\mathrm{SL}}(2,\mathbb{R})/\widetilde{Z} = \mathrm{SO}_0(2,1)$ [55, 56, 61]. Now, the group $\mathrm{SL}(2,\mathbb{R})$ is a double cover of the group $\mathrm{SO}_0(2,1)$, and we have $\mathrm{SL}(2,\mathbb{R})/\mathbb{Z}_2 \simeq \mathrm{SO}_0(2,1)$ [55] with $\mathbb{Z}_2$ the cyclic group of order 2. Thus, Eq. (4.16) gives a simple way to understand the relation between the groups $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, $\mathrm{SL}(2,\mathbb{R})$ and $\mathrm{SO}_0(2,1)$, at least near their identity elements. The map in Eq. (4.16) with $\tilde{g} = \tilde{e}$, where $\tilde{e}$ denotes the identity in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, takes the triple $(\widetilde{\theta}\Lambda_0, a\Lambda_1, b\Lambda_2)$ and maps it to $\exp(\tilde{\theta}\Lambda_0)\exp(a\Lambda_1)\exp(b\Lambda_2)$. Now, we write $\widetilde{\theta} = \theta + 4\pi n$, for some $\theta \in [0, 4\pi)$ and $n \in \mathbb{Z}$. Thus, we have

$$\exp(\widetilde{\theta}\Lambda_0)\exp(a\Lambda_1)\exp(b\Lambda_2) = \exp(\theta\Lambda_0)\exp(a\Lambda_1)\exp(b\Lambda_2)\exp(2\pi n\Lambda_0), \tag{4.18}$$

where we have used the fact that $\exp(2\pi n\Lambda_0) \in \widetilde{Z}$. This implies that, for elements near the identity $\tilde{e}$ of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, the quotient map $q : \widetilde{\mathrm{SL}}(2,\mathbb{R}) \to \mathrm{SO}_0(2,1)$ is given by

$$\exp(\widetilde{\theta}\Lambda_0)\exp(a\Lambda_1)\exp(b\Lambda_2) \mapsto \exp(\theta\Lambda_0)\exp(a\Lambda_1)\exp(b\Lambda_2), \tag{4.19}$$

and may be interpreted as the map induced by $\tilde{\theta} = \theta + 4\pi n \mapsto \theta$. For the particular purposes of our analysis, a full description of the Lie group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ will not be necessary as the classification of the UIRs of this group that will be used to analyse scalar and spinor fields in AdS_2 only depends on the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ and the element $\exp(2\pi\Lambda_0) \in \widetilde{Z}$ [55, 61].

To conclude this section we review a very well–known result that characterises all the possible finite–dimensional irreducible representations of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$, as the structure of these representations resembles the that of some of the infinite–dimensional representations that we will review in Section 4.3. We will note that none of the finite–dimensional irreducible representations except for the trivial representation can be unitary, which is precisely the reason why we will need to consider the infinite–dimensional case if we require the representations to be unitary. The following construction is based on Ref. [49, Chapter 4, Section 4.6]. A more exhaustive analysis can be found in Ref. [62, Chapter 6, Section 2].

Let us consider the basis of $\mathfrak{sl}(2,\mathbb{R})$ given by the elements $\{\Lambda_i\}$ in Eq. (4.13). For any representation π of $\mathfrak{sl}(2,\mathbb{R})$, we define the *ladder* operators given by

$$L_0 := i\pi(\Lambda_0)\,, \tag{4.20a}$$
$$L_\pm := \pi(\Lambda_1) \pm i\pi(\Lambda_2)\,. \tag{4.20b}$$

which, by means of Eq. (4.14), are found to satisfy the commutation relations

$$[L_0, L_+] = L_+\,, \qquad [L_0, L_-] = -L_-\,, \qquad [L_+, L_-] = 2L_0\,. \tag{4.21}$$

Now, let V_n be a complex vector space with $\dim(V) = n+1$ and let $\{v_k\}_{k=0}^{n}$ be a basis of V_n. Consider the representation $\pi_n : \mathfrak{sl}(2,\mathbb{R}) \to \mathrm{End}_{\mathbb{C}}(V_n)$, defined by the formulae

$$L_0 v_k := \left(\frac{n}{2} - k\right) v_k\,, \tag{4.22a}$$
$$L_+ v_k := \begin{cases} k(n-k+1)v_{k-1}\,, & k > 0\,,\\ 0\,, & k = 0\,,\end{cases} \tag{4.22b}$$
$$L_- v_k := \begin{cases} v_{k+1}\,, & k < n\,,\\ 0\,, & k = n\,,\end{cases} \tag{4.22c}$$

with L_0 and $L_\pm$ defined by Eq. (4.20) with π_n instead of π. Then, the representation π_n is irreducible and, every finite–dimensional irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$ is isomorphic to π_n for some $n \in \mathbb{N}_0$ [49, Theorem 4.32]. Thus, the finite–dimensional irreducible representations of $\mathfrak{sl}(2,\mathbb{R})$ are labelled by the non–negative integers $n \geq 0$. The representation labelled by $n = 0$ corresponding to the trivial representation for which $\pi_0(X) = \mathbb{I}$ for all $X \in \mathfrak{sl}(2,\mathbb{R})$. Since the representations labelled by n are irreducible, Theorem 4.1.6 tells us that the Casimir element (4.15) acts as a scalar multiple of the identity. Indeed, in terms of the operators in Eq. (4.20) the Casimir element Q, up to a constant, is mapped to the operator

$$Q_n = L_0^2 + \frac{1}{2}\left(L_+L_- + L_-L_+\right)\,, \tag{4.23}$$

and, using Eq. (4.22), it can readily be verified that $Q_n = (n^2/4)\mathbb{I}$. Thus, the Casimir element in the representation is completely determined by its eigenvalue $q = n^2/4$ which depends on the label n. A similar situation will be encountered in the infinite dimensional case.

None of the representations π_n, for $n \neq 0$ are unitary. Indeed, let π_n be the irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$ on the vector space V_n corresponding to the integer $n > 0$, with $\dim(V_n) = n+1$ given by (4.22). Then for π_n to be unitary we must have $\pi_n(X)^\dagger = -\pi_n(X)$ for all $X \in \mathfrak{sl}(2,\mathbb{R})$, where $\pi_n(X)^\dagger$ denotes conjugate transposition with respect to the Hermitian inner product $\langle\cdot,\cdot\rangle$ in $V_n \simeq \mathbb{C}^{n+1}$. Thus, Eq. (4.20) implies that $L_0^\dagger = L_0$ and $L_\pm^\dagger = -L_\mp$. If $v_0 \in V_n$ is normalised such that $\|v_0\|^2 = \langle v_0, v_0\rangle = 1$, then, for all $k \leq n$, we have

$$\|v_k\|^2 = \langle L_- v_{k-1}, v_k\rangle \ , \tag{4.24}$$

$$= -\langle v_{k-1}, L_+ v_k\rangle \ , \tag{4.25}$$

$$= -k(n-k+1)\,\|v_{k-1}\|^2 \ , \tag{4.26}$$

where we have used Eq. (4.22b). However, $n-k+1 \geq 1$ for all $0 \leq k \leq n$. In particular, $\|v_1\|^2 = -n\,\|v_0\|^2 = -n$, and we arrive at an inconsistent result. Hence, π_n cannot be unitary. Clearly the trivial representation $n = 0$ does not present this problem, and it is the only finite–dimensional UIR of $\mathfrak{sl}(2,\mathbb{R})$.

4.3 UNITARY IRREDUCIBLE REPRESENTATIONS OF $\widetilde{\mathrm{SL}}(2,\mathbb{R})$

Now that we have identified all finite–dimensional irreducible representations of $\mathfrak{sl}(2,\mathbb{R})$ and verified that none of them, except for the trivial representation, are unitary, we will briefly describe the classification of all the possible UIRs of $\mathfrak{sl}(2,\mathbb{R})$ that arise from the group representations of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. We will use the notation and conventions similar to those of Ref. [61] which, in turn, are based on the original results by Pukanszky [55].

The construction of finite–dimensional representations of $\mathfrak{sl}(2,\mathbb{R})$ in Section 4.2 relied on the diagonalisation of the operator L_0, corresponding to the Lie algebra element $i\Lambda_0$ (see Eq. (4.22a)). The basis for the vector space on which the representations act is the eigenbasis of L_0, and the operators $L_\pm$ act as ladder operators between the consecutive one–dimensional eigenspaces. We will adopt the same approach to construct representations of $\mathfrak{sl}(2,\mathbb{R})$ that are related to the group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. The main difference between representations associated to $\mathrm{SL}(2,\mathbb{R})$ and those associated to $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ will arise in the admissible eigenvalues of the operator L_0.

Let $\mathscr{H}$ be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and let π be a unitary irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$ on the subspace of C^∞–vectors of $\mathscr{H}$ (as discussed in Section 4.1). We define the operators L_0 and $L_\pm$ acting on $\mathscr{H}$ by the relations (4.20). If the representation π comes from a unitary irreducible representation Π of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ on $\mathscr{H}$ defined through Eq. (4.6), then Theorem 4.1.5 implies that any element in the centre $\widetilde{Z}$ of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is mapped by Π to a bounded operator that acts as multiplication by a scalar on $\mathscr{H}$. In particular, the element $\exp(2\pi\Lambda_0) \in \widetilde{Z}$ satisfies $\Pi(\exp(2\pi\Lambda_0)) = e^{-2\pi i\mu}\mathbb{I}$, for some $\mu \in \mathbb{R}/\mathbb{Z}$ [61]. From this fact we note that representations of $\mathrm{SL}(2,\mathbb{R})$ correspond to the values $e^{-2\pi i\mu} = 1$. Now, let ϕ_ω be an eigenvector of L_0 with eigenvalue ω. Since $L_0 = i\pi(\Lambda_0)$ and $\Pi(\exp(2\pi\Lambda_0))\phi_\omega = e^{-2\pi i\mu}\phi_\omega$, the eigenvalue ω must satisfy $\omega = \mu + k$, for $k \in \mathbb{Z}$.

Similarly, Theorem 4.1.6 tells us that the Casimir element is constant over an irreducible representation. With a slight abuse of notation, we denote the operator $\pi(2Q)$ on $\mathscr{H}$ by Q. In terms of the operators L_0 and $L_\pm$, the operator Q will be given by the right–hand side of Eq. (4.23). Let $q := \lambda(\lambda - 1)$ be the eigenvalue of Q in this representation. Since the representation π is unitary, the adjoint operators of L_0 and $L_\pm$ with respect to the inner product in $\mathscr{H}$ must satisfy $L_0^\dagger = L_0$ and $L_\pm^\dagger = -L_\mp$. Hence, the eigenvalue q of Q is real. This allows us to restrict the values of the parameter λ as

$$\lambda \in \frac{1}{2} + i\mathbb{R}^+ \qquad \text{or} \qquad \lambda \in \mathbb{R} \text{ and } \lambda \geq \frac{1}{2}\,, \tag{4.27}$$

for which we have $q < -1/4$ and $q \geq -1/4$, respectively. We have taken into account the fact that q is invariant under the transformation $\lambda \mapsto 1 - \lambda$ in restricting the values for λ in Eq. (4.27).

Now, one finds from the commutation relations (4.21) that $L_0 L_\pm \phi_\omega = (\omega \pm 1)L_\pm\phi_\omega$. We also find from Eqs. (4.21) and (4.23) that

$$\langle L_-\phi_\omega, L_-\phi_\omega\rangle = -\langle \phi_\omega, L_+L_-\phi_\omega\rangle = -q + \omega^2 - \omega\,, \tag{4.28a}$$

$$\langle L_+\phi_\omega, L_+\phi_\omega\rangle = -\langle \phi_\omega, L_-L_+\phi_\omega\rangle = -q + \omega^2 + \omega\,. \tag{4.28b}$$

If the representation is unitary, then the right–hand side of Eq. (4.28) must be non–negative. Recalling $\omega = \mu + k$, $k \in \mathbb{Z}$ and $q = \lambda(\lambda - 1)$, we can write this requirement as

$$(k + \mu - \lambda)(k + \mu + \lambda - 1) \geq 0\,, \tag{4.29a}$$

$$(k + \mu + \lambda)(k + \mu - \lambda + 1) \geq 0\,, \tag{4.29b}$$

for the value of k for every eigenvector ϕ_ω in the given representation. Notice that Eq. (4.29b) is obtained from Eq. (4.29a) by letting $(k,\mu) \mapsto (-k,-\mu)$.

The non–trivial UIRs are thus labelled by the pair (λ,μ), and they are classified depending on the possible values that k and μ can take for a given λ in order to satisfy the system of inequalities given by Eq. (4.29). Two of the types of representations are found by requiring that Eqs. (4.29) are satisfied by all $k \in \mathbb{Z}$, one corresponding to $\lambda = 1/2 + is$ and the other to $1/2 < \lambda < 1$. Others are found by requiring that both Eqs. (4.29a) and (4.29b) are satisfied for $k \in \mathbb{N}$ and that the equality in Eq. (4.29a) is satisfied by $k = 0$, or that Eqs. (4.29a) and (4.29b) are satisfied for $-k \in \mathbb{N}$ and that the equality in Eq. (4.29b) is satisfied by $k = 0$. In this manner, one finds the following UIRs, which exhaust all non–trivial UIRs up to isomorphisms [55, 61]:

1. **Principal series representations:** $\mathscr{P}^\mu_{is}$ for $\lambda = 1/2 + is$, with $s \in \mathbb{R}^+$, $-1/2 < \mu \leq 1/2$, and $\omega = \mu + k$, where $k \in \mathbb{Z}$. The Casimir eigenvalue satisfies $q < -1/4$.

2. **Complementary series representations:** $\mathscr{C}^\mu_\lambda$ for $0 < \lambda < 1/2$, with $|\mu| < \lambda$ and $\omega = \mu + k$, where $k \in \mathbb{Z}$. The Casimir eigenvalue satisfies $-1/4 < q < 0$.

3. **Discrete series representations:**

   - $\mathscr{D}^{\pm}_{\lambda}$ for $\lambda > 1/2$, with $\mu = \pm\lambda$, and $\omega = \pm(\lambda + k)$, respectively, where $k \in \mathbb{N}_0$.
   - $\mathscr{D}^{\pm}_{1-\lambda}$ for $1/2 < \lambda < 1$, with $\mu = \pm(1-\lambda)$ and $\omega = \pm(1-\lambda+k)$, respectively, where $k \in \mathbb{N}_0$.

   The Casimir eigenvalue satisfies $q > -1/4$.

4. **Mock–discrete series representations:** $\mathscr{D}^{\pm}_{1/2}$ for $\mu = \lambda = 1/2$, and $\omega = \pm(1/2+k)$, respectively, with $k \in \mathbb{N}_0$. The Casimir eigenvalue is $q = -1/4$.

The principal and complementary series representations are collectively referred to as *continuous* series representations. For these representations, the spectrum of the operator L_0 consist of positive and negative eigenvalues $\omega = \mu + k$ extending to infinity in both directions separated by integer steps. The positive discrete series representations $\mathscr{D}^{+}_{\lambda}$ and $\mathscr{D}^{+}_{1-\lambda}$ on the other hand are *lowest–weight modules* [49] with lowest weights λ and $1-\lambda$, respectively. Similarly, the negative discrete series $\mathscr{D}^{-}_{\lambda}$ and $\mathscr{D}^{-}_{1-\lambda}$ are *highest–weight modules* with highest weights $-\lambda$ and $\lambda - 1$, respectively. Mock–discrete series representations $\mathscr{D}^{\pm}_{1/2}$, are related to the limit of the reducible representation $\mathscr{P}^{1/2}_{0}$ by $\mathscr{P}^{1/2}_{0} \simeq \mathscr{D}^{+}_{1/2} \oplus \mathscr{D}^{-}_{1/2}$ [55, 61, 63].

— 5 —

Self–adjoint extensions of operators on Hilbert spaces

The formulation of Quantum Mechanics and QFT, heavily relies on the theory of linear self–adjoint operators on a given Hilbert space. Indeed, quantum observables are defined to be self–adjoint operators acting on the space of states. Having a well–defined set of self–adjoint operators is essential in describing a mathematically rigorous and physically coherent quantum theory. In Chapter 2 we found that the self–adjointness of the differential operators A and $\mathbb{D}$ defined in Eqs. (2.9) and (2.29), respectively, was a key assumption that allows to have a well–posed initial value problem for scalar and spinor field theories on a standard static spacetime which is not globally hyperbolic, like anti–de Sitter spacetimes. However, for most of the relevant systems in quantum theories, we do not start with a self–adjoint operator, but only with a symmetric one. The task is then to modify the given symmetric operator in such a way that we end up with a self–adjoint operator which is related to the one we started with.

From a mathematical point of view, an operator acting on a Hilbert space is defined via its action on the states and the domain on which it is allowed to act. One must make a clear distinction between an *operation*, i.e., the action on states, and an *operator*, i.e., the operation together with its domain. One must distinguish between a merely symmetric (or Hermitian) operator and a self–adjoint one having the same operation but defined on different domains. Constructing a self–adjoint operator from a symmetric operator consists in extending the original domain in a specific way.

This process is known as finding a *self–adjoint extension* of an operator. It was originally introduced by Weyl [66] in the context of differential operators, and then generalized by von Neumann [67] for general linear operators defined on a Hilbert space. The main result, von Neumann's theorem, states that every admissible self–adjoint extension is in one–to–one correspondence with the parameters of a unitary map on a certain subspace of the domain of the adjoint operator. The literature describing this procedure is extensive from a mathematical point of view [35, 36, 43, 68, 69], and it has only recently been emphasised in physics related literature [37, 70, 71, 72, 73].

Even though the prescription given by von Neumann is elegant and self–contained, the explicit construction of self–adjoint extensions of symmetric operators is usually presented in the literature for specific examples. The explicit correspondence between the domain of the resulting self–adjoint operator and the set of boundary conditions associated with them is most commonly introduced as a convenient way to describe the domain instead of a consequence of the self–adjoint extension prescribed by von Neumann's theorem.

In this chapter we will review the process of finding all the possible self–adjoint extensions associated to a symmetric operator acting on a Hilbert space based on von Neumann's theorem. In Chapters 6 and 8 we will show that the operators A and $\mathbb{D}$ that arise from the Klein–Gordon and Dirac equations in AdS_2, respectively, can be reduced to a one–dimensional Schrödinger operator with a symmetric potential acting on a certain Hilbert space. Thus, after introducing the theory behind the self–adjoint extensions, we will apply the resulting machinery to this particular operator. Furthermore, we will show an explicit way in which the domains of all the possible self–adjoint extensions of this operator can be associated in a one–to–one fashion to a family of boundary conditions. We will make use of the definitions introduced in Chapter 2, Section 2.2 involving general linear operators on a Hilbert space.

5.1 THE THEORY OF SELF–ADJOINT EXTENSIONS

In this section we will review the theory behind the self–adjoint extensions of symmetric operators on a Hilbert space. We will consider an arbitrary symmetric operator T in the sense of Definition 2.2.3 acting on a separable Hilbert space $\mathscr{H}$. Then we will describe the requirements that T must satisfy in order to admit closed extensions that are self–adjoint according to Definition 2.2.4. The admissible self–adjoint extensions, as given by von Neumann's theorem, are described in terms of their domains, which in turn are parametrised by certain unitary maps. We will show that, if T is a second order differential operator on a Hilbert space of functions, then this family of domains can be put into a one–to–one correspondence with a family of boundary conditions that the functions in these domains must satisfy. The material covered in this section is mostly based on Refs. [35, 36, 37, 43].

We start with a simple criterion to determine if a symmetric operator is indeed self–adjoint. Assume that T is a self–adjoint operator on $\mathscr{H}$. From Definitions 2.2.2 and 2.2.4, we have $T^\dagger = T$ and $\mathrm{Dom}(T)^\dagger = \mathrm{Dom}(T)$. Now, we further assume that there is an element $g \in \mathrm{Dom}(T) = \mathrm{Dom}(T^\dagger)$ such that $T^\dagger g = \pm i\lambda g$, for some real $\lambda > 0$. Then, using the fact that $Tg = \pm i\lambda g$, it follows that

$$\mp i\lambda \langle g, g\rangle = \langle \pm i\lambda g, g\rangle = \langle Tg, g\rangle = \left\langle g, T^\dagger g\right\rangle = \langle g, \pm i\lambda g\rangle = \pm i\lambda \langle g, g\rangle \ , \tag{5.1}$$

and, thus, $g = 0$. This means that if T is self–adjoint, then the equations $T^\dagger g = \pm i\lambda g$ cannot have non–trivial normalisable solutions. Note that the requirement $\lambda > 0$ is imposed in order to satisfy (5.1), however this relation is valid even if we generalise to $Tg = \pm z_\pm g$, with $z_+ \in \mathbb{C}$ in the upper half–plane and $z_- \in \mathbb{C}$ in the lower half–plane. We will restrict ourselves to the imaginary axis, *i.e.*, to $\pm i\lambda$, in order to simplify calculations. This result is part of the proof of the next statement.

Theorem 5.1.1 (Basic criterion for self–adjointness) *Let T be a densely defined, symmetric operator on $\mathscr{H}$. Then, the following statements are equivalent:*

1. *T is self-adjoint.*
2. *T is closed and* $\mathrm{Ker}(T^{\dagger} \pm i\lambda) = \{0\}$.
3. $\mathrm{Range}(T \pm i\lambda) = \mathscr{H}$.

The implication 1$\Longrightarrow$2 follows from the argument used in Eq. (5.1) and $T^{\dagger}$ is closed if T is symmetric. For a detailed proof of the other implications, see for example [36, Theorem VIII.3].

From this criterion, it is clear that, for a given symmetric operator T, the spaces

$$\mathscr{K}_{+} := \mathrm{Ker}(T^{\dagger} - i\lambda)\,, \qquad \mathscr{K}_{-} := \mathrm{Ker}(T^{\dagger} + i\lambda)\,, \tag{5.2}$$

play an important role in determining whether or not T is self-adjoint. These subspaces of $\mathscr{H}$ are called **deficiency subspaces** associated to T, while the numbers $n_{\pm} := \dim \mathscr{K}_{\pm}$, are called **deficiency indices**.

Now, the problem we are interested in is the following. Given a densely defined symmetric operator T which is not self–adjoint, we may ask if there is a way to construct a self–adjoint operator whose action is identical to that of the original. In other words, the question is: Is there a way to modify a symmetric operator T to make it self–adjoint? To answer this question, we first introduce the concept of the extension of an operator.

Definition 5.1.2 *Let T and T_U be linear operators on $\mathscr{H}$. The operator T_U is said to be an **extension** of T, if and only if* $\mathrm{Dom}(T) \subset \mathrm{Dom}(T_U)$, *and* $T_U f = Tf$, *for all* $f \in \mathrm{Dom}\,(T)$.

A relevant example of an extension is that of the **closure** of an operator. An operator A is said to be **closable** if there exists a closed operator in the sense of Definition 2.2.1 whose domain contains $\mathrm{Dom}(T)$ and it has the same action of T, *i.e.*, if T admits a closed extension. The smallest closed extension of T, denoted by $\bar{T}$ is called the closure of T. Similarly, Definition (2.2.2) implies that the adjoint $T^{\dagger}$ of a symmetric operator T is an extension of T and we have $\mathrm{Dom}(T) \subseteq \mathrm{Dom}(T^{\dagger})$.

For a densely defined symmetric operator T, we will aim to find an extension T_U which is self–adjoint. Before introducing the formal result, we will first give a description of the machinery involved in finding such extensions.

The prescription: Let us begin with a closed, densely defined symmetric operator T, such that $\mathscr{K}_{\pm} \neq \{0\}$. Hence, according to Theorem 5.1.1, T fails to be self–adjoint. This means that at least one of the equations

$$T^{\dagger} g_{\pm} = \pm i\lambda g_{\pm}\,, \qquad \lambda > 0\,, \tag{5.3}$$

has non–trivial solutions in $\mathrm{Dom}(T^{\dagger})$.

For any closed, densely defined symmetric operator, T, the domain of the adjoint operator $T^\dagger$ is given by [36, Chapter X.I]

$$\mathrm{Dom}(T^\dagger) = \mathrm{Dom}(T) \oplus_T \mathscr{K}_+ \oplus_T \mathscr{K}_- \,, \tag{5.4}$$

where the sum of vector spaces is orthogonal with respect to the inner product

$$\langle f_1, f_2 \rangle_T := \langle f_1, f_2 \rangle + \frac{1}{\lambda^2} \left\langle T^\dagger f_1, T^\dagger f_2 \right\rangle \,, \tag{5.5}$$

for all $f_1, f_2 \in \mathrm{Dom}(T^\dagger)$.

Now, consider a closed symmetric extension T_U of the operator T in the sense of Definition 5.1.2. If $T_U^\dagger$ denotes the adjoint operator of T_U, then we have $\mathrm{Dom}(T_U^\dagger) \subseteq \mathrm{Dom}(T^\dagger)$. Hence, by Eq. (5.4) and the fact that $\mathrm{Dom}(T_U) \subseteq \mathrm{Dom}(T_U^\dagger)$, we must have

$$\mathrm{Dom}(T_U) \subseteq \mathrm{Dom}(T) \oplus_T \mathscr{K}_+ \oplus_T \mathscr{K}_- \,. \tag{5.6}$$

The explicit form of the domain of T_U can be found from this inclusion in the following way. First, Definition 5.1.2 states that $\mathrm{Dom}(T) \subseteq \mathrm{Dom}(T_U)$ and, thus, Eq. (5.6) implies that there must exist a subspace $\mathscr{S} \subseteq \mathscr{K}_+ \oplus_T \mathscr{K}_-$, such that $\mathrm{Dom}(T_U) = \mathrm{Dom}(T) \oplus_T \mathscr{S}$. This means that any element $f \in \mathrm{Dom}(T_U)$ is of the form $f = f_0 + g_+ + g_-$, where $f_0 \in \mathrm{Dom}(T)$, and $g_\pm \in \mathscr{S}_\pm$, with $\mathscr{S}_\pm$ subspaces of $\mathscr{K}_\pm$ such that $\mathscr{S} = \mathscr{S} \oplus_T \mathscr{S}_-$. The fact that T_U is a symmetric operator by definition implies that for $f \in \mathrm{Dom}(T_U)$ we have

$$\langle f_0 + g_- + g_-, T_U(f_0 + g_- + g_-) \rangle = \langle T_U(f_0 + g_- + g_-), f_0 + g_- + g_- \rangle \,. \tag{5.7}$$

Using the fact that the action of the operators $T^\dagger$, $T_U^\dagger$ and T_U is equal to the action of T on $f_0 \in \mathrm{Dom}(T)$, we obtain

$$\left\langle T_U^\dagger(g_+ + g_-), g_+ + g_- \right\rangle = \left\langle g_+ + g_-, T_U^\dagger(g_+ + g_-) \right\rangle \,, \tag{5.8}$$

and since $T_U^\dagger(g_+ + g_-) = T^\dagger(g_+ + g_-)$, Eq. (5.3) implies that

$$-i\lambda \langle g_+ - g_-, g_+ + g_- \rangle = i\lambda \langle g_+ + g_-, g_+ - g_- \rangle \,. \tag{5.9}$$

Hence, $g_+ + g_- \in \mathscr{S}$ must satisfy $\|g_+\| = \|g_-\|$. Now, consider an orthonormal basis for the space $\mathscr{S}$ with respect to the inner product in Eq. (5.5) given by $\{s_i^+ + s_i^-\}_{i=1}^k$, with $k = \dim(\mathscr{S})$ and with $s_i^\pm \in \mathscr{K}_\pm$. Using the fact that the deficiency spaces are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_T$, and that, for any $g_+ + g_- \in \mathscr{S}$ we must have $\|g_+\| = \|g_-\|$, it can be shown that the vectors $s_i^\pm$ satisfy

$$\left\langle s_i^+, s_j^+ \right\rangle = \left\langle s_i^-, s_j^- \right\rangle = \frac{1}{4}\delta_{ij} \,, \tag{5.10}$$

for all $1 \leq i, j \leq k$, with respect to the inner product in $\mathscr{H}$. Thus, the subspace $\mathscr{S}_+$ is spanned by the elements $\{s_i^+\}_{i=1}^k$, and the condition $\|g_+\| = \|g_-\|$ for all $g_+ + g_- \in \mathscr{S}$ tells us that any element in $\mathscr{S}$ is of the form $g + Ug$, with $g \in \mathscr{S}_+$, and where $U : \mathscr{S}_+ \to \mathscr{K}_-$

is a partial isometry from the subspace $\mathscr{S}_+$ into $\mathscr{K}_-$ with respect to the inner product in $\mathscr{H}$. Hence, the domain of any symmetric extension T_U of T must be given by

$$\mathrm{Dom}(T_U) = \mathrm{Dom}(T) \oplus_T \{g_+ + g_- \in \mathscr{K}_+ \oplus_T \mathscr{K}_- \,|\, g_+ \in \mathscr{S}_+ \,,\, g_- = Ug_+\} \,, \tag{5.11}$$

for some partial isometry $U : \mathscr{S}_+ \to \mathscr{K}_-$. We note that the choice of map U determines the subspace $\mathrm{Dom}(T_U)$, and this correspondence is one–to–one [36, Chapter X.I, Lemma].

For any symmetric extension T_U of T, we can find the domain of the adjoint operator $T_U^\dagger$. Let $f_1 \in \mathrm{Dom}(T^\dagger)$ and $f_2 \in \mathrm{Dom}(T_U)$. By Eq. (5.4), we can write $f_1 = f_{1,0} + h_+ + h_-$, with $f_{1,0} \in \mathrm{Dom}(T)$, and $h_\pm \in \mathrm{Dom}(\mathscr{K}_\pm)$. Similarly, by Eq. (5.11) f_2 can be written in the form $f_2 = f_{2,0} + g_+ + Ug_+$, with $f_{2,0} \in \mathrm{Dom}(T)$ and $g_+ \in \mathscr{S}_+$. Since T_U is symmetric, the requirement for f_1 to be in $\mathrm{Dom}(T_U)^\dagger$ is equivalent to

$$\langle f_1, T_U f_2\rangle - \left\langle T^\dagger f_1, f_2 \right\rangle = 0 \,, \tag{5.12}$$

for all $f_2 \in \mathrm{Dom}(T_U)$. By expanding this expression and by using Eq. (5.3) we find that Eq. (5.12) is equivalent to the requirement

$$\langle h_+, g_+\rangle - \langle h_-, Ug_+\rangle = 0 \,, \tag{5.13}$$

for all $g_+ \in \mathscr{S}_+$. Let $h_+ = s_+ + s_+^\perp$, with $s_+ \in \mathscr{S}_+$ and $s_+^\perp \in (\mathscr{S}_+)^\perp$, where $(\mathscr{S}_+)^\perp$ denotes the orthogonal complement of $\mathscr{S}_+$ in $\mathscr{K}_+$ with respect to the inner product in $\mathscr{H}$. Similarly, we let $h_- = s_- + s_-^\perp$, where $s_- \in U(\mathscr{S}_+)$ and $s_-^\perp \in [U(\mathscr{S}_+)]^\perp$ with $[U(\mathscr{S}_+)]^\perp$ denoting the orthogonal complement of $U(\mathscr{S}_+)$ in $\mathscr{K}_-$. Then, Eq. (5.13) reduces to the condition

$$\langle s_+, g_+\rangle = \langle s_-, Ug_+\rangle \,, \tag{5.14}$$

for all g_+, and no further restrictions on $s_+^\perp$ or $s_-^\perp$. This is satisfied if $s_- = Us_+$, that is, if $s_+ + s_- \in \mathscr{S}$. Hence, for f_1 to be in $\mathrm{Dom}(T_U^\dagger)$ we must have

$$f_1 = f_{1,0} + (s_+ + Us_+) + s_+^\perp + s_-^\perp \,. \tag{5.15}$$

Thus, the domain of $T_U^\dagger$ for a given symmetric extension T_U is given by

$$\mathrm{Dom}(T_U^\dagger) = \mathrm{Dom}(T_U) \oplus_T (\mathscr{S}_+)^\perp \oplus_T [U(\mathscr{S}_+)]^\perp \,, \tag{5.16}$$

where the sum of vector spaces is orthonormal due to the fact that if $S_1 \oplus S_2$ is an orthonormal sum, then so is $S_1^\perp \oplus S_2^\perp$ [35].

Thus far, we have shown that if we are given a closed, densely defined symmetric operator T on a Hilbert space $\mathscr{H}$ with deficiency subspaces $\mathscr{K}_\pm$, then the symmetric extensions T_U of T are parametrised by all the possible subspaces $\mathscr{S}_+$. Furthermore, the domains of T_U and $T_U^\dagger$ are given by Eqs. (5.11) and (5.16), respectively.

The task of finding, amongst the possible symmetric extensions T_U, a self–adjoint extension of the operator T, thus reduces to determining the conditions needed for T_U

to satisfy $\mathrm{Dom}(T_U) = \mathrm{Dom}(T_U^\dagger)$. By considering the description of $\mathrm{Dom}(T_U^\dagger)$ given in Eq. (5.16), we see that, whenever $(\mathscr{S}_+)^\perp = [U(\mathscr{S}_+)]^\perp = \{0\}$, then this condition is satisfied. Hence, if $\mathscr{S}_+ = \mathscr{K}_+$, the operator T_U is a self–adjoint extension of the operator T. Furthermore, we note that $[U(\mathscr{K}_+)]^\perp = \{0\}$ implies that the deficiency indices satisfy $n_+ = n_-$. If this is the case, Eq. (5.11) implies that the domains of the self–adjoint extensions T_U of the operator T are given by

$$\mathrm{Dom}(T_U) = \{f_0 + g + Ug \mid f_0 \in \mathrm{Dom}(T),\, g \in \mathscr{K}_+\}\ , \tag{5.17}$$

for each unitary map $U : \mathscr{K}_+ \to \mathscr{K}_-$. The action of T_U on this domain is given by

$$T_U(f_0 + g + Ug) := T^\dagger f_0 + i\lambda g - i\lambda U g\,. \tag{5.18}$$

Equivalently, if $\mathscr{S}_+ = \mathscr{K}_+$ then the calculation leading to Eq. (5.11) implies that the subspace $\mathscr{S}$ corresponds to a maximal subspace of $\mathrm{Dom}(T^\dagger)$ on which the operator $T^\dagger$ is symmetric, and we have [36, 68]

$$\mathrm{Dom}(T_U) = \mathrm{Dom}(T) \oplus \mathscr{S}\,, \tag{5.19}$$

.

The prescription described above constitutes most of the proof of a well–known theorem, originally proposed by Weyl [66], and then generalised by von Neumann [67]. In general terms, the aforementioned theorem reads:

Theorem 5.1.3 *Let T be a closed, densely defined symmetric operator on a Hilbert space $\mathscr{H}$, with deficiency indices n_+, n_-. Then,*

1. *T is self-adjoint if and only if $n_+ = 0 = n_-$.*

2. *T has self-adjoint extensions if and only if $n_+ = n_-$. The self-adjoint extensions of T are in one–to–one correspondence with the unitary maps from $\mathscr{K}_+$ to $\mathscr{K}_-$.*

3. *If either $n_+ = 0 \neq n_-$, or $n_- = 0 \neq n_+$, then T has no non-trivial self-adjoint extensions.*

A detalied proof of this theorem can be found in standard literature [36, Theorem X.2]. Using this result, we can give a concise set of steps to find the self–adjoint extensions of a symmetric operator in the following way. Let us assume T is a densely defined, symmetric operator on $\mathscr{H}$ with domain $\mathrm{Dom}(T)$. Then,

1. We verify that T is a closed operator in the sense of Definition 2.2.1. If not, we will then consider the closure $\bar{T}$ of the operator T. By Definition 2.2.1, we achieve this by extending the original domain of T such that it is closed under the inner product in Eq. (2.11) and T is still symmetric.

2. Then, we calculate the deficiency indices for $\bar{T}$. In other words, we verify if the deficiency equations (5.3) have normalisable solutions in the domain of $\bar{T}^\dagger$. If the deficiency indices are not equal, then Theorem (5.1.3) implies that there are no non–trivial self–adjoint extensions of the operator T. If the deficiency indices are equal, say, $n := n_+ = n_-$, then T admits non–trivial self–adjoint extensions. For the sake of simplicity, we shall assume that n is finite. The case in which n is not finite can be treated in a similar manner to the finite case, see for example the discussion in [69, Chapter 3]. Therefore, we denote the set of n linearly independent (normalised) solutions of (5.3) by $\left(g_+^j\right)$ and $\left(g_-^j\right)$, $1 \leq j \leq n$, respectively.

3. If the deficiency indices both equal to zero, then the only self–adjoint extension of T, is its closure, $\bar{T}$, otherwise, the self–adjoint extensions of T are parametrised by the unitary maps $U : \mathscr{K}_+ \to \mathscr{K}_-$. For finite n, these unitary maps admit matrix representations given by unitary $n \times n$ matrices U_M. Thus, we consider the extension T_U of T defined via its domain by Eq. (5.17) and its action on this domain in Eq. (5.18).

4. Finally, we may explicitly describe the domain of the self–adjoint extension T_U in terms of the solutions of the deficiency equations (5.3) and the possible choices of $n \times n$ unitary matrices U_M. Considering the bases $\{g_\pm^{(i)}\}_{i=1}^n$ of $\mathscr{K}_\pm$, we can write the action of $U : \mathscr{K}_+ \to \mathscr{K}_-$ as

$$U\left(\sum_{i=1}^n \alpha_i g_+^{(i)}\right) = \sum_{i=1}^n \beta_i g_-^{(i)}\,, \tag{5.20}$$

for $\alpha_i \in \mathbb{C}$, and where the coefficients β_i are given by

$$\beta_i = \sum_{j=1}^n u_{ij}\alpha_j\,, \tag{5.21}$$

where the numbers $u_{ij} \in \mathbb{C}$ are the elements of the $n \times n$ unitary matrix $U_M = (u_{ij})$ representing the map U. Hence, Eq. (5.17) implies that any $f \in \mathrm{Dom}(T_U)$ is given by

$$f = f_0 + \sum_{i=1}^n \alpha_i g_+^{(i)} + \sum_{i=1}^n \left(\sum_{j=1}^n u_{ij}\alpha_j\right) g_-^{(i)}\,, \tag{5.22}$$

where $f_0 \in \mathrm{Dom}(\overline{T})$.

The prescription presented above applies for an arbitrary densely defined symmetric operator T on a Hilbert space $\mathscr{H}$. However, if the Hilbert space $\mathscr{H}$ is a space of square–integrable functions, and if T is a differential operator, then it is possible to obtain a description of the admissible self–adjoint extensions, T_U, of the operator T in terms of *boundary conditions* that the elements in $\mathrm{Dom}(T_U)$ satisfy. This is seen as follows. Let T_U be a self–adjoint extension of T with domain given by Eq. (5.17). If $\langle\cdot,\cdot\rangle$ is the L^2–inner product, then Eqs. (5.4) and (5.19) imply that restricting the elements $f \in \mathrm{Dom}(T^\dagger)$ to satisfy the condition

$$0 = \left\langle f, T_U \tilde{f} \right\rangle - \left\langle T^\dagger f, \tilde{f} \right\rangle , \tag{5.23}$$

for all $\tilde{f} \in \mathrm{Dom}(T_U)$, is equivalent to finding the maximal subspace $\mathscr{S} \subseteq \mathscr{K}_+ \oplus_T \mathscr{K}_-$ that characterises the self–adjoint extension T_U. However, if the domain of $T^\dagger$ consists of sufficiently differentiable functions[1], then the right–hand side of Eq. (5.23) can be written in terms of the boundary values of the functions f and $\tilde{f}$ using, for example, integration by parts. Now, the functions $\tilde{f}$ are given by Eq. (5.22) and, thus, the values of $\tilde{f}$ at the boundary are known and they depend on the entries of the unitary matrix U_M. Hence, Eq. (5.23) reduces to a boundary condition on the function $f \in \mathrm{Dom}(T^\dagger)$ that depends on the matrix U_M. Thus, for a given self–adjoint extension T_U we end up with a set of boundary conditions that determines $\mathrm{Dom}(T_U)$. We will refer to the boundary conditions obtained in this way as **self–adjoint boundary conditions**.

Suppose that, for a given densely defined symmetric differential operator T with domain $\mathrm{Dom}(T)$ and deficiency indices satisfying $n_+ = n_-$, the domain of the adjoint operator $T^\dagger$ is known *a priori*. Then the existence of the self–adjoint extensions provided by Theorem 5.1.3 and the definition of the subspace $\mathscr{S}$ as a maximal subspace of $\mathrm{Dom}(T^\dagger)$ for which $T^\dagger$ is a symmetric operator, imply that we may find the self–adjoint boundary conditions directly by imposing

$$\left\langle f, T^\dagger f \right\rangle = \left\langle T^\dagger f, f \right\rangle , \tag{5.24}$$

for $f \in \mathrm{Dom}(T^\dagger)$. Thus, imposing the symmetry condition (5.24) on $\mathrm{Dom}(T^\dagger)$, is usually preferred for calculations whenever the domain of $T^\dagger$ is known explicitly [71, 72, 73]. However, in practice, the domain $\mathrm{Dom}(T^\dagger)$ of $T^\dagger$ does not always have a simple or straightforward description as a subspace of square–integrable functions.

Having this in mind, we will now apply this machinery to the particular case of a one–dimensional Schrödinger operator with symmetric potential on a finite interval to show how the self–adjoint extensions of this operator can be given in terms of self–adjoint boundary conditions as described above. Our aim is to classify all the possible self–adjoint extensions for the associated operator with respect to different choices of unitary matrices and find the explicit self–adjoint boundary conditions for each case.

5.2 SELF–ADJOINT EXTENSIONS OF THE SCHRÖDINGER OPERATOR

In this section we illustrate how the theory of self–adjoint extensions is applied to the particular case of a differential operator T acting on a space of functions. The operator T we are focusing on is a one–dimensional Schrödinger operator with a symmetric (even) potential term acting on the Hilbert space of square–integrable functions on a finite interval.

[1]This condition on $\mathrm{Dom}(T^\dagger)$ is related to the regularity theorem for weak solutions of partial differential equations. For further details we refer to [35, Chapter V] and [36, Chapter IX].

The reason we are choosing this operator as an example is due to the fact that the problem of finding well–defined dynamics of a scalar field theory in AdS_2 and in AdS_N, $N \geq 3$, will turn out to be equivalent to finding the self–adjoint extensions of Schrödinger operator of this form associated to the Klein–Gordon equation for these cases (see Chapters 6 and 7).

Thus, we consider the operator defined on $\mathscr{H} = L^2[-a, a]$, with a, a finite positive real number, by

$$T := -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)\,, \tag{5.25}$$

where the real–valued piecewise–continuous potential satisfies $V(-x) = V(x)$, and such that $V \in L^2_{\mathrm{loc}}(-a, a)$, *i.e.*, that if $f \in C_0^\infty(-a, a)$, then $Vf \in L^2[-a, a]$ [37, 43]. The domain of the operator is given by

$$\mathrm{Dom}(T) = \left\{ f \in AC^2[-a, a] \mid f(a) = f(-a) = f'(a) = f'(-a) = 0 \right\}\,, \tag{5.26}$$

where $AC^2[-a, a]$ denotes the set of functions $f \in \mathscr{H}$ whose weak derivatives up to second order are in $AC[-a, a]$, in particular, they are continuously differentiable [36, Chapter X, Example 2]. We choose the domain in this manner to ensure that T is a closed, densely defined symmetric operator [35, 68, 74]. It is clear that T is not a self–adjoint operator because [35, 37]

$$\mathrm{Dom}(T^\dagger) = \left\{ f \in AC^2[-a, a] \right\}\,, \tag{5.27}$$

that is, even though the formal expressions for T and $T^\dagger$ as differential operators are identical, their domains are not the same.

We will now apply the prescription described in the Section 5.1 to find the self–adjoint extensions of T, and we start by finding its deficiency indices. Thus, we look for normalisable solutions of Eq. (5.3) with $\lambda = 1$. By the assumption that $V(x)$ is an even potential, we may choose the functions $g_+, g_- \in AC^2[-a, a]$ as the normalised even and odd eigenfunctions of the equation

$$T^\dagger\, g_\pm(x) = i g_\pm(x)\,. \tag{5.28}$$

We note that since $g_+, g_- \in AC^2[-a, a]$, all weak solutions of Eq. (5.28) are continuously differentiable, and, thus, we can think of this equation as an ordinary differential equation, *i.e.*, we rule out distributional solutions. Thus, $\mathscr{K}_+ = \mathrm{span}\{g_+, g_-\}$, where $\mathscr{K}_+$ is defined by Eq. (5.2) and it immediately follows that $\mathscr{K}_- = \mathrm{span}\{\overline{g_+}, \overline{g_-}\}$. Thus, we have $n_+ = n_- = 2$, and, hence, by Theorem 5.1.3, the self–adjoint extensions of T are parametrised by a 2×2 unitary matrix.

Let T_U denote the self–adjoint extension of T. Then, by Eqs. (5.17) and (5.22), we have that an element $f \in \mathrm{Dom}(T_U)$ must be of the form

$$f = f_0 + c_1 f_+ + c_2 f_- \in AC^2[-a, a]\,, \tag{5.29}$$

for some $c_1, c_2 \in \mathbb{C}$, where $f_0 \in \mathrm{Dom}(T)$ and where we have defined the functions

$$f_+(x) = g_+(x) + u_{11}\overline{g_+(x)} + u_{12}\overline{g_-(x)}\,, \tag{5.30a}$$

$$f_-(x) = g_-(x) + u_{21}\overline{g_+(x)} + u_{22}\overline{g_-(x)}\,, \tag{5.30b}$$

with u_{ij} denoting the ij–elements of a 2×2 unitary matrix U_M. Note that Eq. (5.30) implies that $c_1 f_+ + c_2 f_-$ belongs to a maximal subspace $\mathscr{S} \subset \mathrm{Dom}(T^\dagger)$ for which $T^\dagger$ is symmetric.

Even though Eqs. (5.29) and (5.30) specify the domain of the self–adjoint extension T_U of T completely, it is not written in a form suitable for finding the spectrum of T_U. It is more convenient to specify the self–adjoint extension as a set of boundary conditions of functions in $AC^2[-a, a]$ at $x = \pm a$ as discussed at the end of Section 5.1. This is achieved by restricting an arbitrary function $f \in \mathrm{Dom}(T^\dagger)$ to satisfy Eq. (5.23) for all $\tilde{f} \in \mathrm{Dom}(T_U)$ of the form of Eq. (5.29) above. This restriction leads to a set of boundary conditions for the function f given by

$$\begin{pmatrix} f'(a) - i f(a) \\ f'(-a) + i f(-a) \end{pmatrix} = \mathcal{U} \begin{pmatrix} f'(a) + i f(a) \\ f'(-a) - i f(-a) \end{pmatrix}\,, \tag{5.31}$$

where the unitary matrix $\mathcal{U}$ is in one–to–one correspondence with the matrix U_M of Eq. (5.30). In Appendix B we present the explicit calculation that shows the correspondence between the domain of T_U described by Eqs. (5.29) and (5.30) and the boundary condition (5.31). The proof given in Appendix B concerns the more general case of the Schrödinger operator associated to the Klein–Gordon equation in AdS_2, but the calculations follow analogously for the operator T in Eq. (5.25). We also note that the set of boundary conditions (5.31) is the same as those inferred using other methods for free quantum particle in a box [71].

To conclude this chapter, we will now write the self–adjoint boundary conditions in Eq. (5.31) in a more familiar form. We will point out the types of boundary conditions which are most frequently used as special cases.

We first write the boundary conditions (5.31) as follows:

$$(\mathbb{I} - \mathcal{U}) \begin{pmatrix} f'(a) \\ f'(-a) \end{pmatrix} = i(\mathbb{I} + \mathcal{U}) \begin{pmatrix} f(a) \\ -f(-a) \end{pmatrix}. \tag{5.32}$$

To rewrite these boundary conditions in a more familiar form, it is useful to classify them according to whether or not the matrices $\mathbb{I} - \mathcal{U}$ or $\mathbb{I} + \mathcal{U}$ are singular.

Case I: both $\mathbb{I} - \mathcal{U}$ and $\mathbb{I} + \mathcal{U}$ are regular. In this case we can write Eq. (5.32) as

$$\begin{pmatrix} f'(a) \\ f'(-a) \end{pmatrix} = H \begin{pmatrix} f(a) \\ -f(-a) \end{pmatrix}, \tag{5.33}$$

where

$$H := i(\mathbb{I} - \mathcal{U})^{-1}(\mathbb{I} + \mathcal{U})\,. \tag{5.34}$$

We note that the matrix H is the Cayley transform of the unitary matrix $\mathcal{U}$ [35, 43]. One can readily show that the 2×2 matrix H is Hermitian and invertible. Furthermore, the matrices H and $\mathcal{U}$ commute and $\mathcal{U} = (H - i\mathbb{I})^{-1}(H + i\mathbb{I})$. Thus, the invertible 2×2 Hermitian matrix H and the 2×2 unitary matrix $\mathcal{U}$, such that $\mathbb{I}\pm\mathcal{U}$ are non–singular, are in one–to–one correspondence through the relation (5.34).

By writing

$$H = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & -\gamma \end{pmatrix}, \tag{5.35}$$

where $\alpha,\gamma\in\mathbb{R}$, $\beta\in\mathbb{C}$ and $\alpha\gamma+|\beta|^2\neq 0$, we find that Eq. (5.33) becomes

$$f'(a) = \alpha f(a) - \beta f(-a)\,, \tag{5.36a}$$
$$f'(-a) = \overline{\beta} f(a) + \gamma f(-a)\,. \tag{5.36b}$$

Since the Hermitian matrix H is invertible, one may also express the boundary conditions here by writing $f(a)$ and $f(-a)$ as linear combinations of $f'(a)$ and $f'(-a)$. Notice that if H is diagonal, which implies by virtue of Eq. (5.34) that $\mathcal{U}$ is also diagonal, then Eqs. (5.36a) and (5.36b) become $f'(a) = \alpha f(a)$ and $f'(-a) = \gamma f(-a)$, with $\alpha\neq 0$ and $\gamma\neq 0$. These boundary conditions are called the **Robin boundary conditions**.

Case II. $\mathbb{I}+\mathcal{U}$ is singular and $\mathbb{I}-\mathcal{U}$ is regular. This case is similar to case I and the boundary conditions are given by Eq. (5.36) except that the Hermitian matrix H is not invertible. Thus, $f'(a)$ and $f'(-a)$ are proportional to each other as linear combinations of $f(a)$ and $f(-a)$. For the special case with $\mathcal{U} = -\mathbb{I}$ we have $H = 0$, and conditions (5.36) reduce to $f'(a) = f'(-a) = 0$, *i.e.*, the **Neumann boundary conditions** at both $\pm a$.

Case III. $\mathbb{I}-\mathcal{U}$ is singular and $\mathbb{I}+\mathcal{U}$ is regular. In this case the matrix H in Eq. (5.36) is not defined. Instead, we can write Eq. (5.32) as

$$\begin{pmatrix} f(a) \\ -f(-a) \end{pmatrix} = H' \begin{pmatrix} f'(a) \\ f'(-a) \end{pmatrix}, \tag{5.37}$$

where

$$H' := -i(\mathbb{I}+\mathcal{U})^{-1}(\mathbb{I}-\mathcal{U})\,, \tag{5.38}$$

which is Hermitian but not invertible. By letting

$$H' = \begin{pmatrix} \alpha' & -\beta' \\ -\overline{\beta'} & -\gamma' \end{pmatrix}, \tag{5.39}$$

we find

$$f(a) = \alpha' f'(a) - \beta' f'(-a)\,, \tag{5.40a}$$
$$f(-a) = \overline{\beta}' f'(a) + \gamma' f'(-a)\,, \tag{5.40b}$$

with $\alpha', \gamma' \in \mathbb{R}$, and $\beta' \in \mathbb{C}$. Since the matrix H' is singular, $f(a)$ and $f(-a)$ are proportional to each other as linear combinations of $f'(a)$ and $f'(-a)$. For the special case with $\mathcal{U} = \mathbb{I}$, that is, for $H' = 0$, the boundary conditions reduce to $f(a) = f(-a) = 0$, namely, the **Dirichlet boundary conditions** at both $\pm a$.

Case IV. Both $\mathbb{I} \pm \mathcal{U}$ are singular. In this case $\mathcal{U}$ has 1 and -1 as eigenvalues. Then, $\mathcal{U}$ can be given as

$$\mathcal{U} = \begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix}, \tag{5.41}$$

where $\theta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$ are the polar and azimuthal angles, respectively, in the standard spherical polar coordinates. (Thus, the matrix $\mathcal{U}$ is a Pauli spin matrix [29] in the direction specified by the angles θ and φ.) By substituting this equation into Eq. (5.32) we obtain

$$\sin\frac{\theta}{2}\left[f'(a)\sin\frac{\theta}{2} - f'(-a)e^{-i\varphi}\cos\frac{\theta}{2}\right] = i\cos\frac{\theta}{2}\left[f(a)\cos\frac{\theta}{2} - f(-a)e^{-i\varphi}\sin\frac{\theta}{2}\right], \tag{5.42a}$$

$$-\cos\frac{\theta}{2}\left[f'(a)e^{i\varphi}\sin\frac{\theta}{2} - f'(-a)\cos\frac{\theta}{2}\right] = i\sin\frac{\theta}{2}\left[f(a)e^{i\varphi}\cos\frac{\theta}{2} - f(-a)\sin\frac{\theta}{2}\right]. \tag{5.42b}$$

These equations are equivalent to

$$f'(-a)\cos\frac{\theta}{2} = f'(a)e^{i\varphi}\sin\frac{\theta}{2}, \tag{5.43a}$$

$$f(-a)\sin\frac{\theta}{2} = f(a)e^{i\varphi}\cos\frac{\theta}{2}. \tag{5.43b}$$

If $\theta \in (0, \pi)$, then we can write these boundary conditions as

$$f(-a) = Kf(a), \tag{5.44a}$$

$$f'(-a) = \frac{1}{\overline{K}}f'(a), \tag{5.44b}$$

where $K = e^{i\varphi}\cot(\theta/2)$ is any non–zero complex number. For $\theta = \pi/2$ ($|K| = 1$) we have $f(-a) = e^{i\varphi}f(a)$ and $f'(-a) = e^{i\varphi}f'(a)$. The boundary condition given by this set of equations is often called an **automorphic boundary condition**. In particular, if $\varphi = 0$ ($K = 1$), we have the **periodic boundary condition**, whereas if $\varphi = \pi$, we have the **anti–periodic boundary condition**.

If $\theta = 0$, then Eqs. (5.43a) and (5.43b) become $f(a) = f'(-a) = 0$. Thus, we have the Dirichlet boundary condition at a and the Neumann boundary condition at $-a$. On the other hand, if $\theta = \pi$, then they become $f(-a) = f'(a) = 0$. Thus, we have the Dirichlet boundary condition at $-a$ and the Neumann boundary condition at a. These boundary conditions will be collectively referred to as **mixed boundary conditions**.

— 6 —

Scalar field theory in AdS_2

The main goal of this chapter is to study the dynamics of a free scalar field theory in two–dimensional anti–de Sitter space, AdS_2, using Theorem 2.2.5 to obtain a well–posed initial value problem. We will note that the spatial component of the Klein–Gordon operator in AdS_2 is not a self–adjoint operator. Thus, in order to use Theorem 2.2.5, we will first apply the machinery described in Chapter 5 to this operator to obtain the self–adjoint extensions associated to it. As mentioned in Chapter 3, in this spacetime there are two disjoint spatial boundaries unlike in the higher–dimensional case studied by Ishibashi and Wald in Ref. [22]. Due to this fact the self–adjoint extensions for the spatial component of the Klein–Gordon operator in AdS_2 are richer than in the higher–dimensional case. The results of Ishibashi and Wald give some special self–adjoint extensions in two dimensions but not all of them, as we will show in Chapter 7. This is because, for low values of the squared mass of the field, the self–adjoint extensions in AdS_N with $N \geq 3$ are parametrised by one real number whereas the extensions for the two–dimensional case are parametrised by a 2×2 unitary matrix.

We also note that not all of the consistent theories obtained through the self–adjoint extensions associated to the Klein–Gordon operator may be of physical interest. Depending on the context in which such theories are analysed, different arguments may be given for choosing a particular theory over the others. One possibility is to require the invariance under the isometry group of the spacetime. Applying this requirement to scalar field theory in AdS_2, among the family of different consistent theories arising from self–adjoint boundary conditions we may choose those whose positive–frequency solutions form a UIR of the symmetry group of AdS_2, *i.e.* $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. We will refer to the classification of the UIRs of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ given in Chapter 4 in order to identify the self–adjoint boundary conditions that are invariant under $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. Then, we identify the boundary conditions among these that lead to an $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–invariant positive–frequency subspace, which corresponds to an $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–invariant vacuum state as constructed for a general static spacetime in Chapter 2. We also study the cases where the boundary conditions are invariant but the vacuum state is not and identify the UIR to which the vacuum state belongs.

Free scalar and spinor field theories have previously been studied, under the lens of supersymmetry by Sakai and Tanii in Ref. [14]. In their analysis the boundary conditions for the mode functions are determined by imposing the vanishing of energy flux at the conformal boundaries. As we shall show, the boundary conditions stemming from the energy flux condition coincide with the self–adjoint extensions corresponding to the boundary conditions invariant under the action of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$.

Thus, we analyse a scalar field of mass M obeying the Klein–Gordon equation in AdS_2. More specifically, we study the self–adjoint boundary conditions for this equation and find those such that the positive–frequency solutions allow to define a UIR of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. The type of self–adjoint boundary conditions depends on the value of the mass of the field. If the mass is sufficiently large, the boundary conditions are uniquely determined by requiring the solutions to the Klein–Gordon equation to be normalisable with respect to the Klein–Gordon inner product, while in a certain range of low mass parameter the boundary conditions need to be specified. The theory of self–adjoint extensions will be used to obtain such boundary conditions. Then we determine for which self–adjoint extensions the corresponding positive–frequency solutions form UIRs. It will be found that only two types of boundary conditions preserve the symmetry of anti–de Sitter spacetime for the positive–frequency subspace of the solution space.

6.1 Solutions of the Klein–Gordon equation in AdS_2

Let us consider the two–dimensional anti–de Sitter spacetime, AdS_2. The static coordinate system in Eq. (3.8) for $N = 2$ reduces to (t, ρ), with $t \in \mathbb{R}$ and $\rho \in (-\pi/2, \pi/2)$. The static slices Σ_t for $t \in \mathbb{R}$ then correspond to the lines of constant time, isomorphic to the interval $(-\pi/2, \pi/2)$. By Eq. (3.6), the line element for AdS_2 written in these coordinates thus reads

$$\mathrm{d}s^2 = \sec^2\rho\,\left(-\mathrm{d}t^2 + \mathrm{d}\rho^2\right)\,. \tag{6.1}$$

As stated in Chapter 3, the three Killing vector fields of AdS_2 are given by the static vector field $\xi_0 = \partial/\partial t$, and the boost–like vector fields $\boldsymbol{K}_3$ and $\boldsymbol{B}_3$ given by Eq. (3.26) with $\theta_1 = 0$. Since these vector fields leave the metric in Eq. (6.1) invariant, they correspond to the generators of the group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. Thus, we can identify these vector fields with the basis $\{\Lambda_i\}_{i=0}^{2}$ of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ given by Eq. (4.13). We rename the boost–like Killing vectors as $\xi_1 := \boldsymbol{K}_3$, and $\xi_2 := \boldsymbol{B}_3$, and we have

$$\xi_0 = \frac{\partial}{\partial t}\,, \tag{6.2a}$$

$$\xi_1 = -\sin t\,\sin\rho\,\frac{\partial}{\partial t} + \cos t\,\cos\rho\,\frac{\partial}{\partial\rho}\,, \tag{6.2b}$$

$$\xi_2 = \cos t\,\sin\rho\,\frac{\partial}{\partial t} + \sin t\,\cos\rho\,\frac{\partial}{\partial\rho}\,. \tag{6.2c}$$

Hence, the map $\Lambda_i \mapsto \xi_i$, for $i = 0, 1, 2$, corresponds to the induced action of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ on $C^\infty(\mathrm{AdS}_2)$.

Now, using Eq. (3.14), we see that the Laplace–Beltrami operator, $\Box_{\mathrm{AdS}_2}$, for AdS_2 takes the form

$$\Box_{\mathrm{AdS}_2} = \cos^2\rho \left(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \rho^2}\right). \tag{6.3}$$

On the other hand, the action of the Casimir element Q of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ given by Eq. (4.15) on a function $f \in C^\infty(\mathrm{AdS}_2)$ is given, up to a constant factor, by $Qf = -\xi_0[\xi_0 f] + \xi_1[\xi_1 f] + \xi_2[\xi_2 f]$. A simple calculation using Eq. (6.2) shows that the Laplace–Beltrami operator (6.3) and the action of the Casimir element on AdS_2 coincide, *i.e.*, $Q = \Box_{\mathrm{AdS}_2}$.

The Klein–Gordon equation for a scalar field ϕ of mass M in AdS_2 is given by $(\Box_{\mathrm{AdS}_2} - M^2)\phi = 0$, which in static coordinates (t,ρ) reads

$$\cos^2\rho \left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial \rho^2} - \frac{\lambda(\lambda-1)}{\cos^2\rho}\right]\phi(t,\rho) = 0\,. \tag{6.4}$$

Here we have let $M^2 = \lambda(\lambda-1)$ so that it is identified with the parametrisation of the eigenvalue q of the Casimir element as given in Section 4.3 of Chapter 4.

We first describe the solutions to Eq. (6.4) of the form of Eq. (2.44), so that

$$\phi(t,\rho) = \Phi_\omega(\rho)e^{-i\omega t}\,, \qquad \omega > 0\,. \tag{6.5}$$

Thus, the spatial component Φ_ω satisfies Eq. (2.45), *i.e.*, $A\Phi_\omega = \omega^2\Phi_\omega$, where, for this particular case, the differential operator A given in Eq. (2.9) reduces to

$$A = -\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{\lambda(\lambda-1)}{\cos^2\rho}\,. \tag{6.6}$$

By following the prescription for a scalar field in an arbitrary static spacetime introduced in Chapter 2, we see that the operator A in Eq. (6.6) is defined on the Hilbert space $\mathscr{H}_{KG}$ of square–integrable functions on the static slice $\Sigma_0 = (-\pi/2, \pi/2)$ with respect to the measure $\mathrm{d}V = \sqrt{h}\mathcal{N}^{-1}\mathrm{d}\rho$. From Eq. (3.10), it follows that for AdS_2 we have $\sqrt{h} = \mathcal{N} = \sec\rho$. Hence, $\mathrm{d}V = \mathrm{d}\rho$ and, thus, the inner product in Eq. (2.10) between two elements $\Phi_1, \Phi_2 \in \mathscr{H}_{KG}$ is given by

$$\langle \Phi_1, \Phi_2\rangle_{KG} = \int_{-\pi/2}^{\pi/2} \overline{\Phi_1(\rho)}\,\Phi_2(\rho)\,\mathrm{d}\rho\,. \tag{6.7}$$

The inner product in $\mathscr{H}_{KG}$ induces an inner product on the solutions ϕ of the form (6.5). Indeed, if ϕ_1 and ϕ_2 are solutions of Eq. (6.4) with $\omega_1, \omega_2 > 0$ and $\omega_1 \neq \omega_2$, then the pairing

$$\begin{aligned}(\phi_1,\phi_2)_{KG} &:= i\int_{-\pi/2}^{\pi/2}\left(\overline{\phi_1(t,\rho)}\frac{\partial\phi_2(t,\rho)}{\partial t} - \frac{\partial\overline{\phi_1(t,\rho)}}{\partial t}\phi_2(t,\rho)\right)\mathrm{d}\rho\,,\\ &= (\omega_2-\omega_1)e^{-i(\omega_2-\omega_1)t}\,\langle\Phi_{\omega_1},\Phi_{\omega_2}\rangle_{KG}\,,\end{aligned} \tag{6.8}$$

where in the last line we have used Eq. (6.4), is time–independent and defines a non–degenerate bilinear form for each $t \in \mathbb{R}$.

In order to completely define the operator A in Eq. (6.6), we will consider the "natural" domain [20, 22, 35, 43] given by $\mathrm{Dom}(A) = C_c^\infty(-\pi/2, \pi/2)$, where $C_c^\infty(-\pi/2, \pi/2)$ stands for the set of compactly supported smooth functions with support away from the boundary $\rho = \pm\pi/2$. On this domain, the operator A is symmetric in the sense of Definition 2.2.3 with respect to the inner product (6.7), that is, it satisfies $\langle A\Phi_1, \Phi_2\rangle_{KG} = \langle \Phi_1, A\Phi_2\rangle_{KG}$ for all $\Phi_1, \Phi_2 \in \mathrm{Dom}(A)$. This follows by a simple integration by parts. Furthermore, the set $C_c^\infty(-\pi/2, \pi/2)$ is dense in $\mathscr{H}_{KG}$ [35] and, thus, A is densely defined on this domain.

Now, we consider the adjoint operator, $A^\dagger$, of the operator A. From Definition 2.2.2, we have that if $\Phi' \in \mathrm{Dom}(A^\dagger)$, then

$$\langle \Phi', A\Phi\rangle_{KG} = \left\langle A^\dagger\Phi', \Phi\right\rangle_{KG}, \tag{6.9}$$

for all $\Phi \in \mathrm{Dom}(A) = C_c^\infty(-\pi/2, \pi/2)$. It is known that, if $\Phi \in \mathrm{Dom}(A^\dagger)$, then the derivative $\mathrm{d}\Phi/\mathrm{d}\rho$ exists in $L^2(-\pi/2, \pi/2)$ and is absolutely continuous [36]. An important consequence of this fact is that the operator $A^\dagger$ is the same differential operator as A on $\Phi \in \mathrm{Dom}(A^\dagger)$ except on a measure–zero set, where Φ may not be twice differentiable, and that, if $\Phi_1 \in \mathrm{Dom}(A^\dagger)$ and $\Phi_2 \in \mathrm{Dom}(A)$, then the following equality from integration by parts holds:

$$\left\langle A^\dagger\Phi_1, \Phi_2\right\rangle_{KG} - \langle \Phi_1, A\Phi_2\rangle_{KG} = \left[\overline{\Phi_1(\rho)}\frac{\mathrm{d}\Phi_2(\rho)}{\mathrm{d}\rho} - \frac{\mathrm{d}\overline{\Phi_1(\rho)}}{\mathrm{d}\rho}\Phi_2(\rho)\right]_{\rho\to-\pi/2}^{\rho\to\pi/2}. \tag{6.10}$$

We note also that $A^\dagger\Phi = A\Phi$ if $\Phi \in \mathrm{Dom}(A) = C_c^\infty(-\pi/2, \pi/2)$.

The operator A is not self–adjoint because $\mathrm{Dom}(A) \neq \mathrm{Dom}(A^\dagger)$, the latter being larger [36, 43]. To define a quantum theory of this scalar field as described in Chapter 2, Section 2.3, we need to find a self–adjoint operator A_U with its domain satisfying $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(A_U) \subseteq \mathrm{Dom}(A^\dagger)$, such that $A_U\Phi = A^\dagger\Phi$ if $\Phi \in \mathrm{Dom}(A_U)$. Hence, we will apply the theory of self–adjoint extensions from Chapter 5 to the operator A. Since the operator A defined by Eq. (6.6) is of the form of Eq. (5.25), that is, a one–dimensional Schrödinger differential operator with potential term given by

$$V(\rho) = \frac{\lambda(\lambda-1)}{\cos^2\rho}, \tag{6.11}$$

and with $V\Phi \in L^2[-\pi/2, \pi/2]$ for all $\Phi \in C_0^\infty(-\pi/2, \pi/2)$, the analysis of the self–adjoint extensions of A will follow closely the analysis developed for the operator T.

Now, from the prescription presented in Chapter 2, Section 2.3, we have that, given a self–adjoint operator A_U with positive spectrum, one can define a quantum theory with a stationary vacuum state for this scalar field. Hence, we will only consider the *positive* self–adjoint extensions of the operator A. Now the operator A is positive for $\lambda \in \mathbb{R}$ because

$$A = \left(-\frac{\mathrm{d}}{\mathrm{d}\rho} + \lambda\tan\rho\right)\left(\frac{\mathrm{d}}{\mathrm{d}\rho} + \lambda\tan\rho\right) + \lambda^2. \tag{6.12}$$

It is shown in Appendix C that the operator A is unbounded from below if $M^2 = \lambda(\lambda-1) < -1/4$, *i.e.*, if λ is imaginary. (The method for the proof is similar to the higher–dimensional case [22].) For this reason, we assume that $M^2 \geq -1/4$ and, hence, that $\lambda \in \mathbb{R}$ and $\lambda \geq 1/2$ from now on. This constraint on the mass values is analogous to the Breitenlohner–Freedman (B-F) bound that occurs in the four–dimensional case [13]. In Chapter 7 we will find an analogous result for the N–dimensional cases.

We note that since A is a positive symmetric operator, then at least one positive self–adjoint extension exists, namely, the *Friedrichs extension* [43, Section 10.4]. We will briefly comment on the relation between this particular self–adjoint extension and the ones obtained by applying von Neumann's theorem in Section 6.2.

Having these facts in mind, we will proceed with the analysis of the self–adjoint extensions of the operator A. Since A is not self–adjoint, we will begin by finding solutions in $\mathscr{H}_{KG}$ of the equation $A^\dagger \Phi_\omega = \omega^2 \Phi_\omega$, which reads

$$-\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}\Phi_\omega(\rho) + \left[\frac{\lambda(\lambda-1)}{\cos^2\rho} - \omega^2\right]\Phi_\omega(\rho) = 0\,. \tag{6.13}$$

Then, we will use the machinery developed in Chapter 5 to obtain a family of boundary conditions for these solutions that correspond to the family of self–adjoint extensions A_U of the operator A, effectively restricting the domain of $A^\dagger$ in such a way that it coincides with $\mathrm{Dom}(A_U)$ given by Theorem 5.1.3.

Two independent solutions of Eq. (6.13) are given in terms of the Gaussian hypergeometric functions [23, Chapter 15] and read

$$\Phi_\omega^{(1)}(\rho) = (\cos\rho)^\lambda\, F\left(\frac{\lambda+\omega}{2}, \frac{\lambda-\omega}{2}; \frac{1}{2}; \sin^2\rho\right)\,, \tag{6.14a}$$

$$\Phi_\omega^{(2)}(\rho) = \sin\rho\,(\cos\rho)^\lambda\, F\left(\frac{1+\lambda+\omega}{2}, \frac{1+\lambda-\omega}{2}; \frac{3}{2}; \sin^2\rho\right)\,. \tag{6.14b}$$

To find self–adjoint extensions of the operator A we need to analyse the behaviour of the solutions (6.14) at the boundaries $\rho = \pm\pi/2$. We first find for which values of λ these solutions are square–integrable by examining their asymptotic behaviour at the boundaries. It turns out that it is convenient to analyse them in the following three cases separately:

1. $\lambda > 3/2$ with $\lambda \neq k + 1/2$ for any $k \in \mathbb{N}$;
2. $\lambda = 1/2$ and $\lambda = k + 1/2$ for $k \in \mathbb{N}$;
3. $1/2 < \lambda < 3/2$.

For the cases 1 and 3, we use the following transformation formulas for the hypergeometric function, which are valid for $\lambda \neq k + \frac{1}{2}$, $k \in \mathbb{Z}$ [23, Eq. 15.8.4]:

$$\begin{aligned}\Phi_\omega^{(1)}(\rho) =& (\cos\rho)^\lambda\, A_1(\omega) F\left(\frac{\lambda+\omega}{2}, \frac{\lambda-\omega}{2}; \frac{1}{2}+\lambda; \cos^2\rho\right)\,,\\ &+ (\cos\rho)^{1-\lambda}\, B_1(\omega)\, F\left(\frac{1-\lambda+\omega}{2}, \frac{1-\lambda-\omega}{2}; \frac{3}{2}-\lambda; \cos^2\rho\right)\,,\end{aligned} \tag{6.15a}$$

$$\begin{aligned}\Phi_\omega^{(2)}(\rho) =& \sin\rho\,\Bigg[(\cos\rho)^\lambda\, A_2(\omega)\, F\left(\frac{1+\lambda+\omega}{2}, \frac{1+\lambda-\omega}{2}; \frac{1}{2}+\lambda; \cos^2\rho\right)\,,\\ &+ (\cos\rho)^{1-\lambda}\, B_2(\omega)\, F\left(\frac{2-\lambda+\omega}{2}, \frac{2-\lambda-\omega}{2}; \frac{3}{2}-\lambda; \cos^2\rho\right)\Bigg]\,,\end{aligned} \tag{6.15b}$$

where

$$A_1(\omega) := \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}-\lambda\right)}{\Gamma\left(\frac{1-\lambda+\omega}{2}\right)\Gamma\left(\frac{1-\lambda-\omega}{2}\right)}\,, \quad B_1(\omega) := \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\lambda-\frac{1}{2}\right)}{\Gamma\left(\frac{\lambda+\omega}{2}\right)\Gamma\left(\frac{\lambda-\omega}{2}\right)}\,, \tag{6.16a}$$

$$A_2(\omega) := \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}-\lambda\right)}{\Gamma\left(\frac{2-\lambda+\omega}{2}\right)\Gamma\left(\frac{2-\lambda-\omega}{2}\right)}\,, \quad B_2(\omega) := \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\lambda-\frac{1}{2}\right)}{\Gamma\left(\frac{1+\lambda+\omega}{2}\right)\Gamma\left(\frac{1+\lambda-\omega}{2}\right)}\,. \tag{6.16b}$$

We define the variable $\widetilde{\rho} := \pi/2 - |\rho|$. Near the spatial boundaries, $\rho \to \pm\pi/2$, we have $\widetilde{\rho} \to 0$, and $\cos\rho \approx \widetilde{\rho}$. Since $F(a,b;c;x) = 1 + O(x)$ for $|x| \ll 1$, the behaviour of the general solution $\Phi(\rho) = C_1\Phi_\omega^{(1)}(\rho) + C_2\Phi_\omega^{(2)}(\rho)$, with $C_1, C_2 \in \mathbb{C}$, of Eq. (6.13) for the cases 1 and 3 around $\rho = \pm\pi/2$ is given by

$$\Phi(\rho) \sim \widetilde{\rho}^{\lambda}(C_1A_1(\omega) \pm C_2A_2(\omega) + O(\widetilde{\rho}^2)) + \widetilde{\rho}^{1-\lambda}(C_1B_1(\omega) \pm C_2B_2(\omega) + O(\widetilde{\rho}^2))\,. \tag{6.17}$$

For the solutions with $\lambda = k + 1/2$ in case 2 the transformation formulas given by Eqs. (6.15) and (6.16) are ill–defined. For this case the following transformation formulas are used instead [23, Eq. 15.8.9]

$$\begin{aligned}\Phi_\omega^{(1)}(\rho) =& H_1(\omega)(\cos\rho)^{k+\frac{1}{2}}\sum_{j=0}^{\infty}\frac{\left(\frac{k+\omega}{2}+\frac{1}{4}\right)_j\left(\frac{k-\omega}{2}+\frac{1}{4}\right)_j}{j!(j+k)!}(\cos\rho)^{2j}\left(\ln(\cos^2\rho)+h_1(j)\right)\,,\\ &+ B_1(\omega)(\cos\rho)^{-k+\frac{1}{2}}\sum_{j=0}^{k-1}\frac{\left(\frac{-k+\omega}{2}+\frac{1}{4}\right)_j\left(\frac{-k-\omega}{2}+\frac{1}{4}\right)_j}{j!(1-k)_j}(\cos\rho)^{2j}\,,\end{aligned} \tag{6.18a}$$

$$\begin{aligned}\Phi_\omega^{(2)}(\rho) =& H_2(\omega)\sin\rho(\cos\rho)^{k+\frac{1}{2}}\sum_{j=0}^{\infty}\frac{\left(\frac{k+\omega}{2}+\frac{3}{4}\right)_j\left(\frac{k-\omega}{2}+\frac{3}{4}\right)_j}{j!(j+k)!}(\cos\rho)^{2j}\left(\ln(\cos^2\rho)+h_2(j)\right)\,,\\ &+ B_2(\omega)\sin\rho(\cos\rho)^{-k+\frac{1}{2}}\sum_{j=0}^{k-1}\frac{\left(\frac{-k+\omega}{2}+\frac{3}{4}\right)_j\left(\frac{-k-\omega}{2}+\frac{3}{4}\right)_j}{j!(1-k)_j}(\cos\rho)^{2j}\,,\end{aligned} \tag{6.18b}$$

where we have defined

$$H_1(\omega) = \frac{(-1)^{k+1}\Gamma(1/2)}{\Gamma\left(\frac{-k+\omega}{2}+\frac{1}{4}\right)\Gamma\left(\frac{-k-\omega}{2}+\frac{1}{4}\right)}\,, \quad B_1(\omega) = \frac{\Gamma(k)\Gamma(1/2)}{\Gamma\left(\frac{k+\omega}{2}+\frac{1}{4}\right)\Gamma\left(\frac{k-\omega}{2}+\frac{1}{4}\right)}\,, \tag{6.19a}$$

$$H_2(\omega) = \frac{(-1)^{k+1}\Gamma(3/2)}{\Gamma\left(\frac{-k+\omega}{2}+\frac{3}{4}\right)\Gamma\left(\frac{-k-\omega}{2}+\frac{3}{4}\right)}\,, \quad B_2(\omega) = \frac{\Gamma(k)\Gamma(3/2)}{\Gamma\left(\frac{k+\omega}{2}+\frac{3}{4}\right)\Gamma\left(\frac{k-\omega}{2}+\frac{3}{4}\right)}\,, \tag{6.19b}$$

and the constants $h_1(j)$ and $h_2(j)$ are given by

$$h_1(j) = \psi\left(\frac{k+\omega}{2}+\frac{1}{4}+j\right)+\psi\left(\frac{k-\omega}{2}+\frac{1}{4}+j\right)-\psi(j+1)-\psi(j+k+1)\,, \tag{6.20a}$$

$$h_2(j) = \psi\left(\frac{k+\omega}{2}+\frac{3}{4}+j\right)+\psi\left(\frac{k-\omega}{2}+\frac{3}{4}+j\right)-\psi(j+1)-\psi(j+k+1)\,, \tag{6.20b}$$

and $\psi(x)$ is the digamma function [23, Eq. 5.2.2]. Note first that the leading behaviour of these functions is the same as in case 1 if $k \geq 1$ ($\lambda \geq 3/2$). For $k = 0$ ($\lambda = 1/2$) the leading behaviour for $\Phi(\rho) = C_1 \Phi_\omega^{(1)}(\rho) + C_2 \Phi_\omega^{(2)}(\rho)$ is found as

$$\begin{aligned}\Phi(\rho) \sim \widetilde{\rho}^{\,\frac{1}{2}} &\left[\ln(\widetilde{\rho}^2)(C_1 H_1(\omega) \pm C_2 H_2(\omega)) + (C_1 H_1(\omega) h_1(0) \pm C_2 H_2(\omega) h_2(0))\right] \\ &+ O(\widetilde{\rho}^{\frac{3}{2}} \ln(\widetilde{\rho}^2)) \,. \end{aligned} \tag{6.21}$$

Using Eqs. (6.17) and (6.21) we can determine when we have square–integrable solutions for each value of λ in cases 1–3 as follows.

1. For this case we have $2\lambda > 3$. Hence, the first term in Eq. (6.17) is square–integrable. However, because $2 - 2\lambda < -1$, the second term is not square–integrable unless it vanishes. Hence, the solution $\Phi(\rho)$ is square–integrable if and only if $C_1 B_1(\omega) \pm C_2 B_2(\omega) = 0$. This can be achieved for both $\rho = \pm\pi/2$ if and only if $B_1(\omega) = 0$ and $C_2 = 0$, or $B_2(\omega) = 0$ and $C_1 = 0$. From Eq. (6.16a) we find that the conditions $B_1(\omega) = 0$ and $C_2 = 0$ give the even solution $\Phi_\omega^{(1)}(\rho)$ with $\omega = \lambda + 2\ell$, $\ell \in \mathbb{N}_0$ while the conditions $B_2(\omega) = 0$ and $C_1 = 0$ give the odd solution $\Phi_\omega^{(2)}(\rho)$ with $\omega = \lambda + 2\ell + 1$, $\ell \in \mathbb{N}_0$. These two cases can be combined to give the positive–frequency functions as [23]

$$\phi_n^{\mathrm{I}}(t,\rho) = N_n^{\mathrm{I}} (\cos\rho)^\lambda \, P_n^{(\lambda-1/2,\lambda-1/2)}(\sin\rho) e^{-i\omega_n^{\mathrm{I}} t} \,, \quad \omega^{\mathrm{I}} = \lambda + n \,. \tag{6.22}$$

where $P_n^{(a,b)}(x)$, $n \in \mathbb{N}_0$, are the Jacobi polynomials defined by [23, Eq. 18.5.7]

$$P_n^{(a,b)}(x) := \frac{\Gamma(n+a+1)}{n!\Gamma(a+1)} F\left(n+a+b+1, -n; a+1; \frac{1-x}{2}\right) \,, \tag{6.23}$$

and N_n^{I} are normalisation constants such that the mode functions $\phi_n^{\mathrm{I}}(t,\rho)$ are normalised by the Klein–Gordon inner product (6.8),

$$(\phi_m^{\mathrm{I}}, \phi_n^{\mathrm{I}})_{KG} = 2\omega_n \left\langle \Phi_m^{\mathrm{I}}, \Phi_n^{\mathrm{I}} \right\rangle_{KG} = \delta_{mn} \,, \tag{6.24}$$

if we write $\phi_n^{\mathrm{I}}(t,\rho) = \Phi_n^{\mathrm{I}}(\rho) e^{-i\omega_n^{\mathrm{I}} t}$. These constants are found by using the standard normalisation integral for the Jacobi polynomials (see e.g., [23, Table 18.3.1]),

$$\begin{aligned} &\int_{-1}^{1} (1-x)^a (1+x)^b P_n^{(a,b)}(x) P_m^{(a,b)}(x) \, dx \\ &= \frac{2^{a+b+1}\Gamma(a+n+1)\Gamma(b+n+1)}{n!(a+b+1+2n)\Gamma(a+b+n+1)} \delta_{nm} \,, \quad a, b > -1 \,. \end{aligned} \tag{6.25}$$

This gives the normalisation constant

$$N_n^{\mathrm{I}} = \frac{\sqrt{n!\Gamma(2\lambda+n)}}{2^\lambda \Gamma(\lambda+n+1/2)} \,. \tag{6.26}$$

2. As we stated before, if $k \geq 1$, then the leading terms for $\rho \to \pm\pi/2$ are identical with those in case 1. Hence, the only square–integrable functions (up to a normalisation

factor) are given again by $\Phi^{(1)}_\omega(\rho)$ with $\omega = \lambda + 2\ell$, $\ell \in \mathbb{N}_0$, in Eq. (6.18a) and $\Psi^{(2)}_\omega(\rho)$ with $\omega = \lambda + 2\ell + 1$, $\ell \in \mathbb{N}_0$ in Eq. (6.18b), where $\lambda = k + 1/2$. Equations (6.18a) and (6.18b) are ill–defined for these values of ω as they stand, but by observing that, for $j \leq \ell$,

$$\lim_{\omega\to\lambda+2\ell} H_1(\omega)h_1(j) = \frac{(-1)^{\ell+1}\Gamma(\frac{1}{2})\Gamma(1+k+\ell)}{\Gamma(\ell+\frac{1}{2})}\,, \tag{6.27a}$$

$$\lim_{\omega\to\lambda+2\ell+1} H_2(\omega)h_2(j) = \frac{(-1)^{\ell+1}\Gamma(\frac{3}{2})\Gamma(1+k+\ell)}{\Gamma(\ell+\frac{3}{2})}\,, \tag{6.27b}$$

with these limits vanishing if $j \geq \ell + 1$, we find that $\Phi^{(1)}_\omega(\rho)$ and $\Phi^{(2)}_\omega(\rho)$ behave like $(\cos\rho)^\lambda$ as $\rho \to \pm\pi/2$ for $\omega = \lambda + 2\ell$ and $\omega = \lambda + 2\ell + 1$, respectively. Thus, also in these cases, the Klein–Gordon normalised positive–frequency mode functions are given by Eq. (6.22). For the case $\lambda = 1/2$, the function $\Phi(\rho)$ in Eq. (6.21) is square integrable for all C_1, C_2 and ω. To treat this case, we need to analyse the boundary conditions which give self–adjoint extensions of the operator A given by Eq. (6.6). This analysis will be given in Sec. 6.3.1.

3. Here, we have $-1 < 2 - 2\lambda < 1$, so the function $\Phi(\rho)$ in Eq. (6.17) is square integrable for all values of C_1, C_2 and ω. Therefore, we need to determine the boundary conditions which give self–adjoint extensions of the operator A. This task will be carried out in Sec. 6.3.1.

In this section we have seen that the leading behaviour of the solutions to the eigenvalue equation for the adjoint $A^\dagger$ is uniquely determined if $\lambda \geq 3/2$. In these cases, the adjoint $A^\dagger$ turns out to be self–adjoint, as we discuss below. On the other hand, for $1/2 \leq \lambda < 3/2$ all solutions are square integrable. For these cases, we need to find suitable boundary conditions at $\rho = \pm\pi/2$ which give self–adjoint extensions of the operator A in Eq. (6.6). This task will be carried out next.

6.2 SELF–ADJOINT EXTENSIONS OF THE OPERATOR A

Now we discuss the self–adjoint extensions of the operator A defined in Eq. (6.6) on the domain $\mathrm{Dom}\,(A) = C^\infty_c(-\pi/2, \pi/2)$.

Following a similar prescription to that of the operator T presented in Chapter 5, we start by finding the deficiency subspaces $\mathscr{K}_\pm$ defined by Eq. (5.2) of the symmetric operator A, which are defined as the linear span of the (normalisable) solutions to the equations

$$A^\dagger\Phi_\pm(\rho) = \pm 2i\Phi_\pm(\rho)\,. \tag{6.28}$$

Notice that this equation is Eq. (6.13) with $\omega^2 = \pm 2i$. Thus, each equation has two linearly independent solutions. The solutions in $\mathscr{K}_+$, if they are square–integrable, are given by

Eq. (6.14) with $\omega = 1 + i$ as

$$\Phi^{(1)}(\rho) = c_1 (\cos\rho)^{\lambda}\, F\left(\frac{\lambda+1+i}{2}, \frac{\lambda-1-i}{2}; \frac{1}{2}; \sin^2\rho\right), \tag{6.29a}$$

$$\Phi^{(2)}(\rho) = c_2 \sin\rho\,(\cos\rho)^{\lambda}\, F\left(\frac{\lambda+2+i}{2}, \frac{\lambda-i}{2}; \frac{3}{2}; \sin^2\rho\right), \tag{6.29b}$$

where the normalisation constants $c_j \in \mathbb{R}$, $j = 1, 2$, are chosen so that

$$\left\langle \Phi^{(1)}, \Phi^{(1)} \right\rangle_{KG} = \left\langle \Phi^{(2)}, \Phi^{(2)} \right\rangle_{KG} = 1\,. \tag{6.30}$$

Note that $\langle \Phi^{(1)}, \Phi^{(2)} \rangle = 0$ because $\Phi^{(1)}$ and $\Phi^{(2)}$ are even and odd functions, respectively. We have $\Phi^{(1)}, \Phi^{(2)} \in \mathscr{K}_+$ and $\overline{\Phi^{(1)}}, \overline{\Phi^{(2)}} \in \mathscr{K}_-$ if $\Phi^{(1)}$ and $\Phi^{(2)}$ are square–integrable.

From the analysis preceding the solutions given by Eq. (6.22), we know that if $\lambda \geq 3/2$, then Eq. (6.13) has square–integrable solutions only if $\omega = \omega_n^{\mathrm{I}} = \lambda + n$, with $n \in \mathbb{N}_0$. This means that there are no square–integrable solutions to Eq. (6.28) for $\omega = 1 \pm i$. Then it follows that the spaces $\mathscr{K}_\pm$ defined by Eq. (5.2) are both zero–dimensional, *i.e.* $n_\pm = 0$, and so there is only one self–adjoint extension for A, namely, its closure $\bar{A} (= A^\dagger)$. In this case the spectrum of $\bar{A}$ is discrete. That is, the eigenfunctions Φ_n^{I} form an orthonormal basis for $L^2[-\pi/2, \pi/2]$. Note also that the eigenvalues, $(\omega_n^{\mathrm{I}})^2$, are all positive. Since the self–adjoint extension $\overline{A}$ is unique and, as mentioned in Section 6.1, A admits a Friedrichs extension A_F [36, Theorem X.26], we conclude that $A_F = \overline{A}$. With all these facts put together, we conclude that, the quantum field theory and the vacuum state can be constructed using the mode functions $\phi_n^{\mathrm{I}}(t, \rho)$ following the general procedure outlined in Chapter 2.

For the cases $1/2 \leq \lambda < 3/2$ both solutions in Eq. (6.29) are square–integrable and, hence, in $\mathrm{Dom}(A^\dagger)$. This follows from the fact that these functions have the same asymptotic behaviour as that of Eq. (6.17) if $1/2 \leq \lambda < 3/2$, and of Eq. (6.21) if $\lambda = 1/2$, which are both square–integrable for any value of ω, in particular, for $\omega = 1 \pm i$. Hence, we have $n_+ = n_- = 2$. Thus, by Eq. (5.19), the self–adjoint extensions of A are characterised by the subspaces $\mathscr{S}$ of $\mathscr{K}_+ \oplus \mathscr{K}_-$ on which the operator $A^\dagger$ is symmetric.

Let $\Phi^{(i)} = \Phi_+^{(i)} + \Phi_-^{(i)}$, $i = 1, 2$, where $\Phi_+^{(i)} \in \mathscr{K}_+$ and $\Phi_-^{(i)} \in \mathscr{K}_-$. Then, the condition $\langle \Phi^{(i)}, A^\dagger \Phi^{(j)} \rangle = \langle A^\dagger \Phi^{(i)}, \Phi^{(j)} \rangle$ implies $\langle \Phi_+^{(i)}, \Phi_+^{(j)} \rangle = \langle \Phi_-^{(i)}, \Phi_-^{(j)} \rangle$. Thus, if $\Phi^{(i)}$, $i = 1, 2$ is in $\mathscr{S}$, then $\Phi_-^{(i)} = U \Phi_+^{(i)}$, where $U : \mathscr{K}_+ \to \mathscr{K}_-$ is a unitary map. Hence, the self–adjoint extensions are characterised by the following 2×2 unitary map U:

$$U\Phi_+ := U \begin{pmatrix} \Phi^{(1)} \\ \Phi^{(2)} \end{pmatrix} = \begin{pmatrix} u_{11}\overline{\Phi^{(1)}} + u_{12}\overline{\Phi^{(2)}} \\ u_{21}\overline{\Phi^{(1)}} + u_{22}\overline{\Phi^{(2)}} \end{pmatrix}, \tag{6.31}$$

where the 2×2 matrix,

$$U_{\mathrm{M}} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}, \tag{6.32}$$

is unitary. By Eqs. (5.17) and (5.18), the domain of the self–adjoint extension A_U is given by

$$\mathrm{Dom}\,(A_U) := \left\{ \Phi + \Phi_+ + U\Phi_+ \;\middle|\; \Phi \in \mathrm{Dom}(\bar{A}),\ \Phi_+ \in \mathscr{K}_+ \right\}, \tag{6.33}$$

where the operator A_U acts on this domain as

$$A_U(\Phi + \Phi_+ + U\Phi_+) = A^\dagger\Phi + 2i\Phi_+ - 2iU\Phi_+ \,. \tag{6.34}$$

Although Eqs. (6.31), (6.33) and (6.34) give all self–adjoint extensions of the operator A, it will be more convenient to describe them in terms of boundary values of the functions in the domain of A_U as we did for the operator T in Chapter 5. Since the functions in the deficiency subspace $\mathscr{K}_+ \oplus \mathscr{K}_-$ behave either like $(\cos\rho)^{1-\lambda}$ or $(\cos\rho)^\lambda$ for $1/2 < \lambda < 3/2$ and either like $(\cos\rho)^{1/2}$ or $(\cos\rho)^{1/2}\ln(\cos^2\rho)$ for $\lambda = 1/2$ as $\rho \to \pm\pi/2$, we define the following quantities in order to extract the boundary behaviour of these functions:

$$\widetilde{\Phi}^{(\lambda)}(\rho) := (\cos\rho)^{\lambda-1}\Phi(\rho)\,, \tag{6.35a}$$

$$D\widetilde{\Phi}^{(\lambda)}(\rho) := (\cos\rho)^{2-2\lambda}\frac{\mathrm{d}}{\mathrm{d}\rho}\left((\cos\rho)^{\lambda-1}\Phi(\rho)\right)\,, \tag{6.35b}$$

for the case $1/2 < \lambda < 3/2$, and

$$\widetilde{\Phi}^{(1/2)}(\rho) := \frac{(\cos\rho)^{-1/2}}{\ln(\cos^2\rho) - 1}\Phi(\rho)\,, \tag{6.36a}$$

$$D\widetilde{\Phi}^{(1/2)}(\rho) := (\cos\rho)[\ln(\cos^2\rho) - 1]^2\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{(\cos\rho)^{-1/2}}{\ln(\cos^2\rho) - 1}\Phi(\rho)\right)\,, \tag{6.36b}$$

if $\lambda = 1/2$. At least one of the boundary values $\widetilde{\Phi}^{(\lambda)}(\pm\pi/2)$ and $D\widetilde{\Phi}^{(\lambda)}(\pm\pi/2)$ is non–zero for $\Phi \in \mathscr{K}_+ \oplus \mathscr{K}_-$ (except for the zero function), and they vanish if $\Phi \in \mathrm{Dom}(\bar{A})$ as shown in Appendix D. Note also that for any given set of four values $\widetilde{\Phi}^{(\lambda)}(\pm\pi/2)$ and $D\widetilde{\Phi}^{(\lambda)}(\pm\pi/2)$, one can find a function $\Phi \in \mathrm{Dom}(A^\dagger)$ which has these boundary values. (For example, for $1/2 < \lambda < 3/2$ a smooth function Φ satisfying $\Phi(\rho) = (\cos\rho)^{1-\lambda}$ for $\pi/4 \le \rho < \pi/2$ and $\Phi(\rho) = 0$ for $-\pi/2 \le \rho \le 0$ is in $\mathrm{Dom}(A^\dagger)$ and has $\widetilde{\Phi}^{(\lambda)}(\pi/2) = 1$ and $\widetilde{\Phi}^{(\lambda)}(-\pi/2) = D\widetilde{\Phi}^{(\lambda)}(\pm\pi/2) = 0$.) These facts imply that the four–dimensional vector $(\widetilde{\Phi}^{(\lambda)}(-\pi/2), \widetilde{\Phi}^{(\lambda)}(\pi/2), D\widetilde{\Phi}^{(\lambda)}(-\pi/2), D\widetilde{\Phi}^{(\lambda)}(\pi/2))$ uniquely determines a function $\Phi \in \mathscr{K}_+ \oplus \mathscr{K}_-$. Thus, a self–adjoint extension of A can be characterised by a two–dimensional subspace of the four–dimensional vector space consisting of these vectors which characterises a subspace $\mathscr{S} \subset \mathscr{K}_+ \oplus \mathscr{K}_-$, for which the operator $A^\dagger$ is symmetric. We now find such subspaces of this four–dimensional vector space.

For either $\lambda = 1/2$ or $1/2 < \lambda < 3/2$, if $\Phi_1, \Phi_2 \in \mathscr{K}_+ \oplus \mathscr{K}_-$, we have from Eq. (6.10)

$$\begin{aligned}
&\left\langle A^\dagger\Phi_1, \Phi_2\right\rangle_{KG} - \left\langle \Phi_1, A^\dagger\Phi_2\right\rangle_{KG} \\
&= \overline{\widetilde{\Phi}_1^{(\lambda)}(\pi/2)}D\widetilde{\Phi}_2^{(\lambda)}(\pi/2) - \overline{D\widetilde{\Phi}_1^{(\lambda)}(\pi/2)}\widetilde{\Phi}_2^{(\lambda)}(\pi/2) \\
&\quad - \left[\overline{\widetilde{\Phi}_1^{(\lambda)}(-\pi/2)}D\widetilde{\Phi}_2^{(\lambda)}(-\pi/2) - \overline{D\widetilde{\Phi}_1^{(\lambda)}(-\pi/2)}\widetilde{\Phi}_2^{(\lambda)}(-\pi/2)\right]\,.
\end{aligned} \tag{6.37}$$

Thus, the condition $\left\langle A^\dagger\Phi_1, \Phi_2\right\rangle_{KG} - \left\langle \Phi_1, A^\dagger\Phi_2\right\rangle_{KG} = 0$ can be written as

$$\begin{aligned}
&\overline{\left[D\widetilde{\Phi}_1^{(\lambda)}(\pi/2) - i\widetilde{\Phi}_1^{(\lambda)}(\pi/2)\right]}\left[D\widetilde{\Phi}_2^{(\lambda)}(\pi/2) - i\widetilde{\Phi}_2^{(\lambda)}(\pi/2)\right] \\
&+ \overline{\left[D\widetilde{\Phi}_1^{(\lambda)}(-\pi/2) + i\widetilde{\Phi}_1^{(\lambda)}(-\pi/2)\right]}\left[D\widetilde{\Phi}_2^{(\lambda)}(-\pi/2) + i\widetilde{\Phi}_2^{(\lambda)}(-\pi/2)\right] \\
&= \overline{\left[D\widetilde{\Phi}_1^{(\lambda)}(\pi/2) + i\widetilde{\Phi}_1^{(\lambda)}(\pi/2)\right]}\left[D\widetilde{\Phi}_2^{(\lambda)}(\pi/2) + i\widetilde{\Phi}_2^{(\lambda)}(\pi/2)\right] \\
&\quad + \overline{\left[D\widetilde{\Phi}_1^{(\lambda)}(-\pi/2) - i\widetilde{\Phi}_1^{(\lambda)}(-\pi/2)\right]}\left[D\widetilde{\Phi}_2^{(\lambda)}(-\pi/2) - i\widetilde{\Phi}_2^{(\lambda)}(-\pi/2)\right]. \qquad (6.38)
\end{aligned}$$

This relation is equivalent to

$$\begin{pmatrix} D\widetilde{\Phi}^{(\lambda)}(\pi/2) - i\widetilde{\Phi}^{(\lambda)}(\pi/2) \\ D\widetilde{\Phi}^{(\lambda)}(-\pi/2) + i\widetilde{\Phi}^{(\lambda)}(-\pi/2) \end{pmatrix} = \mathcal{U}\begin{pmatrix} D\widetilde{\Phi}^{(\lambda)}(\pi/2) + i\widetilde{\Phi}^{(\lambda)}(\pi/2) \\ D\widetilde{\Phi}^{(\lambda)}(-\pi/2) - i\widetilde{\Phi}^{(\lambda)}(-\pi/2) \end{pmatrix}, \qquad (6.39)$$

for all $\Phi \in \mathscr{S}$ for a fixed 2×2 unitary matrix $\mathcal{U}$. (This relation specifying the two–dimensional subspace $\mathscr{S}$ of $\mathscr{K}_+ \oplus \mathscr{K}_-$ is analogous to that for a free quantum particle in a box.) Similarly to the case of the operator T in Chapter 2, the unitary matrices $\mathcal{U}$ above and U_M in Eq. (6.32) both characterise a subspace $\mathscr{S}$. We find the explicit map $U_\mathrm{M} \mapsto \mathcal{U}$ which identifies the unitary matrix $\mathcal{U}$ that specifies the same subspace $\mathscr{S}$ as the unitary matrix U_M in Appendix B.

Note that the condition (6.39) can also be expressed in a similar way to the self–adjoint boundary conditions of the Schrödinger operator T given by Eq. (5.32) as

$$(\mathbb{I} - \mathcal{U})\begin{pmatrix} D\widetilde{\Phi}^{(\lambda)}(\pi/2) \\ D\widetilde{\Phi}^{(\lambda)}(-\pi/2) \end{pmatrix} = i(\mathbb{I} + \mathcal{U})\begin{pmatrix} \widetilde{\Phi}^{(\lambda)}(\pi/2) \\ -\widetilde{\Phi}^{(\lambda)}(-\pi/2) \end{pmatrix}. \qquad (6.40)$$

Thus, we have a family of self–adjoint operators A_U parametrised by the four entries of the unitary matrix $\mathcal{U}$. The quantities $\widetilde{\Phi}^{(\lambda)}(\pm\pi/2)$ and $D\widetilde{\Phi}^{(\lambda)}(\pm\pi/2)$ for the general eigenfunctions $C_1\Phi_\omega^{(1)}(\rho) + C_2\Phi_\omega^{(2)}(\rho)$, where $\Phi_\omega^{(1)}(\rho)$ and $\Phi_\omega^{(2)}(\rho)$ are given by Eq. (6.15), are linear in C_1 and C_2 with their coefficients being functions of the frequency ω. Hence, Eq. (6.40) gives two equations linear in C_1 and C_2. The condition for the existence of non–trivial solutions for C_1 and C_2 gives the spectrum of ω^2 for each unitary matrix $\mathcal{U}$. It is known that the spectrum of the self–adjoint extension of a second–order differential operator with deficiency indices $n_\pm = 2$ is discrete [71, 75]. Therefore, the eigenfunctions of the operator A_U satisfying the boundary conditions (6.40) form a basis for the Hilbert space $L^2[-\pi/2, \pi/2]$. If all eigenvalues ω^2 are positive, then one can follow the standard procedure to quantise this theory with a stationary vacuum state as outlined in Section 2.3 of Chapter 2. We provide an example of boundary conditions with negative eigenvalues for ω^2 in Appendix E.

Next, let us write the boundary conditions (6.40) in a more familiar form following the same classification we found in Section 5.2 of Chapter 5. First let us consider the case for which the matrix $\mathbb{I} - \mathcal{U}$ is regular. In this case the matrix $i(\mathbb{I} - \mathcal{U})^{-1}(\mathbb{I} + \mathcal{U})$ is Hermitian (similarly to the matrix H in Eq. (5.34)). Hence, Eq. (6.40) can be written as

$$D\widetilde{\Phi}^{(\lambda)}(\pi/2) = \alpha\,\widetilde{\Phi}^{(\lambda)}(\pi/2) - \beta\,\widetilde{\Phi}^{(\lambda)}(-\pi/2)\,, \tag{6.41a}$$

$$D\widetilde{\Phi}^{(\lambda)}(-\pi/2) = \overline{\beta}\,\widetilde{\Phi}^{(\lambda)}(\pi/2) + \gamma\,\widetilde{\Phi}^{(\lambda)}(-\pi/2)\,, \tag{6.41b}$$

where $\alpha, \gamma \in \mathbb{R}$ and $\beta \in \mathbb{C}$. Notice that if $\mathcal{U}$ is a diagonal matrix, then $\beta = 0$ and Eq. (6.41) reduces to what can be called a **generalised Robin** boundary condition, in a similar way to the boundary condition of Case I in Section 5.2. A special case of Eq. (6.41) is given by $\alpha = \beta = \gamma = 0$ (corresponding to $\mathcal{U} = -\mathbb{I}$) as

$$D\widetilde{\Phi}^{(\lambda)}(\pi/2) = D\widetilde{\Phi}^{(\lambda)}(-\pi/2) = 0\,. \tag{6.42}$$

We note that Eq. (6.42) corresponds to the Case II in Section 5.2 for which the resulting boundary condition is a Neumann boundary condition. Thus, we call Eq. (6.42) the **generalised Neumann** boundary condition. If $\mathbb{I} + \mathcal{U}$ is invertible, then the matrix $i(\mathbb{I}+\mathcal{U})^{-1}(\mathbb{I}-\mathcal{U})$ is Hermitian, and Eq. (6.40) can be given as

$$\widetilde{\Phi}^{(\lambda)}(\pi/2) = a\,D\widetilde{\Phi}^{(\lambda)}(\pi/2) - b\,D\widetilde{\Phi}^{(\lambda)}(-\pi/2)\,, \tag{6.43a}$$

$$\widetilde{\Phi}^{(\lambda)}(-\pi/2) = \overline{b}\,D\widetilde{\Phi}^{(\lambda)}(\pi/2) + c\,D\widetilde{\Phi}^{(\lambda)}(-\pi/2)\,, \tag{6.43b}$$

where $a, c \in \mathbb{R}$ and $b \in \mathbb{C}$. The special case with $a = b = c = 0$ (corresponding to $\mathcal{U} = \mathbb{I}$) is

$$\widetilde{\Phi}^{(\lambda)}(\pi/2) = \widetilde{\Phi}^{(\lambda)}(-\pi/2) = 0\,, \tag{6.44}$$

which we call the **generalised Dirichlet** boundary condition in accordance with the boundary condition we found for Case III in Section 5.2.

Similarly to Case IV in Section 5.2, if the matrices $\mathbb{I} \pm \mathcal{U}$ are both singular, *i.e.*, if the eigenvalues of $\mathcal{U}$ are ± 1, then it is a Pauli spin matrix and can be parametrised as

$$\mathcal{U} = \begin{pmatrix} \cos 2\theta & e^{i\varphi}\sin 2\theta \\ e^{-i\varphi}\sin 2\theta & -\cos 2\theta \end{pmatrix}\,, \tag{6.45}$$

where $\theta, \varphi \in \mathbb{R}$. Then Eq. (6.40) reduces to

$$\widetilde{\Phi}^{(\lambda)}(\pi/2)\cos\theta = \widetilde{\Phi}^{(\lambda)}(-\pi/2)\,e^{i\varphi}\sin\theta\,, \tag{6.46a}$$

$$D\widetilde{\Phi}^{(\lambda)}(\pi/2)\sin\theta = D\widetilde{\Phi}^{(\lambda)}(-\pi/2)\,e^{i\varphi}\cos\theta\,. \tag{6.46b}$$

The special cases for which $\theta = \pi/4$ give a set of boundary conditions that will be referred to as the **generalised automorphic** boundary conditions. If we set $\varphi = 0$ as well, then we have the periodic boundary condition. The special case with $\theta = \pi/2$ reads

$$\widetilde{\Phi}^{(\lambda)}(\pi/2) = D\widetilde{\Phi}^{(\lambda)}(-\pi/2) = 0\,, \tag{6.47}$$

whereas the case with $\theta = 0$ reads

$$\widetilde{\Phi}^{(\lambda)}(-\pi/2) = D\widetilde{\Phi}^{(\lambda)}(\pi/2) = 0\,. \tag{6.48}$$

These boundary conditions will be called the **mixed** boundary conditions.

Most of the boundary conditions given by Eq. (6.40) will result in a rather complicated spectrum of the frequency ω. This is related to the fact that they are not invariant under the symmetry group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ of AdS_2. Next, we identify all boundary conditions among those given by Eq. (6.40) that are invariant under this symmetry group.

6.3 The invariant self–adjoint boundary conditions

In order to determine which of the (positive) self–adjoint extensions of the operator A result in a representation of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, we first find the boundary conditions which are invariant under the infinitesimal action of the group. To do so, we consider the Killing vector fields from Eq. (6.2). For each $\Lambda_i \in \mathfrak{sl}(2,\mathbb{R})$, $i = 0,1,2$, the map $\pi : \Lambda_i \mapsto \xi_i$ defines a representation of $\mathfrak{sl}(2,\mathbb{R})$ on smooth functions on AdS_2. Thus, the action of the ladder operators L_0, $L_\pm$ defined by Eq. (4.20) on $C^\infty(\mathrm{AdS}_2)$ is given by

$$L_0 := i\xi_0 = \frac{\partial}{\partial t}\,, \tag{6.49a}$$

$$L_\pm := \xi_1 \pm i\xi_2 = e^{\pm it}\left(\cos\rho\,\frac{\partial}{\partial\rho} \pm i\sin\rho\,\frac{\partial}{\partial t}\right). \tag{6.49b}$$

The action of these operators on the mode functions of the form $\phi(t,\rho) = \Phi_\omega(\rho)e^{-i\omega t}$ in our local coordinates are found to be given by

$$\mp iL_\pm\phi(t,\rho) = e^{-i(\omega\pm1)t}\left(\cos\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\Phi_\omega(\rho) \mp \omega\sin\rho\,\Phi_\omega(\rho)\right). \tag{6.50}$$

Thus, at $t = 0$ the function $\Phi_\omega(\rho)$ and its derivative transform under the action of $\mp iL_\pm$ as follows:

$$\delta_\pm\Phi_\omega(\rho) = \cos\rho\,\frac{\mathrm{d}}{\mathrm{d}\rho}\Phi_\omega(\rho) \mp \omega\sin\rho\,\Phi_\omega(\rho)\,, \tag{6.51a}$$

$$\begin{aligned}\delta_\pm\left(\frac{\mathrm{d}}{\mathrm{d}\rho}\Phi_\omega(\rho)\right) &= (-1\mp\omega)\sin\rho\,\frac{\mathrm{d}}{\mathrm{d}\rho}\Phi_\omega(\rho) + (-\omega^2\mp\omega)\cos\rho\,\Phi_\omega(\rho)\\ &\quad - \frac{\lambda(1-\lambda)}{\cos\rho}\Phi_\omega(\rho)\,,\end{aligned} \tag{6.51b}$$

where we have used the Klein–Gordon equation (6.4) to find Eq. (6.51b).

First, we examine the cases with $1/2 < \lambda < 3/2$. Using the definitions of $\widetilde{\Phi}^{(\lambda)}_\omega$ and $D\widetilde{\Phi}^{(\lambda)}_\omega$ in Eq. (6.35), we find from Eq. (6.51)

$$\delta_-\widetilde{\Phi}^{(\lambda)}_\omega(\rho) = (\omega-1+\lambda)\sin\rho\,\widetilde{\Phi}^{(\lambda)}(\rho) + (\cos\rho)^{2\lambda-1}D\widetilde{\Phi}^{(\lambda)}_\omega(\rho)\,, \tag{6.52a}$$

$$\begin{aligned}\delta_-D\widetilde{\Phi}^{(\lambda)}_\omega(\rho) &= (\omega-\lambda)\sin\rho\,D\widetilde{\Phi}^{(\lambda)}_\omega(\rho)\\ &\quad + \left[\lambda(\lambda-1)-\omega(\omega-1)\right](\cos\rho)^{3-2\lambda}\widetilde{\Phi}^{(\lambda)}_\omega(\rho)\,.\end{aligned} \tag{6.52b}$$

The formulas for δ_+ are obtained from these by letting $\omega \mapsto -\omega$. Then

$$\delta_-\widetilde{\Phi}^{(\lambda)}_\omega(\pm\pi/2) = \pm(\omega-1+\lambda)\widetilde{\Phi}^{(\lambda)}_\omega(\pm\pi/2)\,, \tag{6.53a}$$

$$\delta_-D\widetilde{\Phi}^{(\lambda)}_\omega(\pm\pi/2) = \pm(\omega-\lambda)D\widetilde{\Phi}^{(\lambda)}_\omega(\pm\pi/2)\,. \tag{6.53b}$$

Now, if Φ_{ω_1} and Φ_{ω_2} are eigenfunctions with the same boundary condition with $\omega_1, \omega_2 \in \mathbb{R}$, then

$$\begin{aligned}
&\langle A_U \Phi_{\omega_1}, \Phi_{\omega_2}\rangle_{KG} - \langle \Phi_{\omega_1}, A_U \Phi_{\omega_2}\rangle_{KG} \\
&= \left[\overline{\widetilde{\Phi}^{(\lambda)}_{\omega_1}(\pi/2)} D\widetilde{\Phi}^{(\lambda)}_{\omega_2}(\pi/2) - \overline{D\widetilde{\Phi}^{(\lambda)}_{\omega_1}(\pi/2)}\widetilde{\Phi}^{(\lambda)}_{\omega_2}(\pi/2)\right] \\
&\quad - \left[\overline{\widetilde{\Phi}^{(\lambda)}_{\omega_1}(-\pi/2)} D\widetilde{\Phi}^{(\lambda)}_{\omega_2}(-\pi/2) - \overline{D\widetilde{\Phi}^{(\lambda)}_{\omega_1}(-\pi/2)}\widetilde{\Phi}^{(\lambda)}_{\omega_2}(-\pi/2)\right] \\
&= 0\,.
\end{aligned} \tag{6.54}$$

If this boundary condition is invariant under the transformation $\delta_-^{(\lambda)}$, then $\delta_-\Phi_\omega$ must satisfy the same boundary condition as Φ_ω. This implies

$$\langle A_U \delta_-\Phi_{\omega_1}, \Phi_{\omega_2}\rangle_{KG} - \langle \delta_-\Phi_{\omega_1}, A_U \Phi_{\omega_2}\rangle_{KG} = 0\,. \tag{6.55}$$

Then,

$$\begin{aligned}
&[\langle A_U \delta_-\Phi_{\omega_1}, \Phi_{\omega_2}\rangle_{KG} - \langle \delta_-\Phi_{\omega_1}, A_U \Phi_{\omega_2}\rangle_{KG}] + [\langle A_U \Phi_{\omega_1}, \delta_-\Phi_{\omega_2}\rangle_{KG} - \langle \Phi_{\omega_1}, A_U \delta_-\Phi_{\omega_2}\rangle_{KG}] \\
&= (\omega_1+\omega_2-1)\left[\overline{\widetilde{\Phi}^{(\lambda)}_{\omega_1}(\pi/2)} D\widetilde{\Phi}^{(\lambda)}_{\omega_2}(\pi/2) - \overline{D\widetilde{\Phi}^{(\lambda)}_{\omega_1}(\pi/2)}\widetilde{\Phi}^{(\lambda)}_{\omega_2}(\pi/2)\right] \\
&\quad + (\omega_1+\omega_2-1)\left[\overline{\widetilde{\Phi}^{(\lambda)}_{\omega_1}(-\pi/2)} D\widetilde{\Phi}^{(\lambda)}_{\omega_2}(-\pi/2) - \overline{D\widetilde{\Phi}^{(\lambda)}_{\omega_1}(-\pi/2)}\widetilde{\Phi}^{(\lambda)}_{\omega_2}(-\pi/2)\right] \\
&= 0\,.
\end{aligned} \tag{6.56}$$

Equations (6.54) and (6.56) are compatible with each other if and only if

$$\begin{aligned}
&\overline{\widetilde{\Phi}^{(\lambda)}_{\omega_1}(\pi/2)} D\widetilde{\Phi}^{(\lambda)}_{\omega_2}(\pi/2) - \overline{D\widetilde{\Phi}^{(\lambda)}_{\omega_1}(\pi/2)}\widetilde{\Phi}^{(\lambda)}_{\omega_2}(\pi/2) \\
&= \overline{\widetilde{\Phi}^{(\lambda)}_{\omega_1}(-\pi/2)} D\widetilde{\Phi}^{(\lambda)}_{\omega_2}(-\pi/2) - \overline{D\widetilde{\Phi}^{(\lambda)}_{\omega_1}(-\pi/2)}\widetilde{\Phi}^{(\lambda)}_{\omega_2}(-\pi/2) \\
&= 0\,,
\end{aligned} \tag{6.57}$$

for all pairs $\{\Phi_{\omega_1}, \Phi_{\omega_2}\}$ such that $\omega_1+\omega_2 \neq 1$ (and there are infinitely many such pairs). This implies that the unitary matrix $\mathcal{U}$ must be diagonal for the boundary condition to be invariant under the $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ transformations. That is,

$$(1-e^{i\alpha_\pm}) D\widetilde{\Phi}^{(\lambda)}_{\omega}(\pm\pi/2) = \pm i(1+e^{i\alpha_\pm})\widetilde{\Phi}^{(\lambda)}_{\omega}(\pm\pi/2)\;, \tag{6.58}$$

where $\alpha_\pm \in \mathbb{R}$. Applying δ_- to these equations gives

$$\pm(\omega-\lambda)(1-e^{i\alpha_\pm}) D\widetilde{\Phi}^{(\lambda)}_{\omega}(\pm\pi/2) = i(\omega-1+\lambda)(1+e^{i\alpha_\pm})\widetilde{\Phi}^{(\lambda)}_{\omega}(\pm\pi/2)\;. \tag{6.59}$$

Equations (6.58) and (6.59) are compatible with each other if and only if $e^{i\alpha_\pm} = \pm 1$, *i.e.*, if and only if $\widetilde{\Phi}^{(\lambda)}(\pi/2) = 0$ or $D\widetilde{\Phi}^{(\lambda)}(\pi/2) = 0$, and $\widetilde{\Phi}^{(\lambda)}(-\pi/2) = 0$ or $D\widetilde{\Phi}^{(\lambda)}(-\pi/2) = 0$. Thus, the only $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–invariant boundary conditions are the generalised Dirichlet and Neumann boundary conditions, given by Eqs. (6.44) and (6.42), respectively, and the mixed boundary conditions given by Eqs, (6.47) and (6.48). It can readily be seen that these boundary conditions are also invariant under the δ_+–transformation, and hence under all $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ transformations.

Next, we turn to the case $\lambda = 1/2$. The transformation of Φ_ω and its derivative under the infinitesimal group action is given by Eq. (6.51) with $\lambda = 1/2$. From the definitions of $\widetilde{\Phi}^{(1/2)}$ and $D\widetilde{\Phi}^{(1/2)}$ given by Eq. (6.36), we find

$$\begin{aligned}\delta_- \widetilde{\Phi}_\omega^{(1/2)}(\rho) &= \left(\omega - \frac{1}{2} - \frac{2}{\ln(\cos^2\rho) - 1}\right) \sin\rho\, \widetilde{\Phi}_\omega^{(1/2)}(\rho) \\ &\quad + \frac{1}{\left[\ln(\cos^2\rho) - 1\right]^2} D\widetilde{\Phi}_\omega^{(1/2)}(\rho)\,, \qquad (6.60a)\\ \delta_- D\widetilde{\Phi}_\omega^{(1/2)}(\rho) &= \left\{-4 + \cos^2\rho \left[\left(-\omega^2 + \omega - \frac{1}{4}\right)\left[\ln(\cos^2\rho) - 1\right]^2 + 4\right]\right\} \widetilde{\Phi}_\omega^{(1/2)}(\rho) \\ &\quad + \left(\omega - \frac{1}{2} + \frac{2}{\ln(\cos^2\rho) - 1}\right) \sin\rho\, D\widetilde{\Phi}_\omega^{(1/2)}(\rho)\,. \qquad (6.60b)\end{aligned}$$

Thus, we obtain

$$\delta_- \widetilde{\Phi}_\omega^{(1/2)}(\pm\pi/2) = \pm(\omega - 1/2)\, \widetilde{\Phi}_\omega^{(1/2)}(\pm\pi/2)\,, \qquad (6.61a)$$

$$\delta_- D\widetilde{\Phi}_\omega^{(1/2)}(\pm\pi/2) = \pm(\omega - 1/2)\, D\widetilde{\Phi}_\omega^{(1/2)}(\pm\pi/2) - 4\widetilde{\Phi}_\omega^{(1/2)}(\pm\pi/2)\,. \qquad (6.61b)$$

It can be shown as in the cases with $1/2 < \lambda < 3/2$ that, for the boundary condition to be invariant under the δ_-–transformation, the matrix $\mathcal{U}$ must be diagonal. Thus, we have boundary conditions of the form given by Eq. (6.58) in this case as well. Then the δ_-–transformation gives

$$\begin{aligned}&\pm(\omega - 1/2)(1 - e^{i\alpha_\pm}) D\widetilde{\Phi}_\omega^{(1/2)}(\pm\pi/2) \\ &= \left[i(\omega - 1/2)(1 + e^{i\alpha_\pm})\widetilde{\Phi}_\omega^{(\lambda)}(\pm\pi/2) + 4(1 - e^{i\alpha_\pm})\right] \widetilde{\Phi}_\omega^{(1/2)}(\pm\pi/2)\,. \qquad (6.62)\end{aligned}$$

This equation is compatible with Eq. (6.58) if and only if $e^{i\alpha_\pm} = 1$. Thus, the only $\widetilde{\mathrm{SL}}(2,\mathbb{R})$-invariant boundary condition is the generalised Dirichlet boundary condition in Eq. (6.44). This was expected because all invariant boundary conditions for $1/2 < \lambda < 3/2$, the generalised Dirichlet, Neumann and mixed boundary conditions, tend to the generalised Dirichlet boundary condition in the limit $\lambda \to 1/2$.

6.3.1 Mode functions resulting from the invariant self–adjoint boundary conditions

Now that we have found the boundary conditions which are invariant under the group action $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, we shall find the frequency spectrum and the corresponding mode functions for these boundary conditions. Since all invariant boundary conditions are either the generalised Dirichlet or Neumann boundary condition at each boundary, it is convenient to use the solutions to Eq. (6.4) satisfying one of these conditions at $\rho = \pi/2$.

Let us start with the cases with $1/2 < \lambda < 3/2$. The solutions which satisfy the generalised Dirichlet or Neumann boundary condition at $\rho = \pi/2$ are

$$\Phi_\omega^{(\mathrm{D},\lambda)}(\rho) = (\cos\rho)^\lambda F\left(\lambda + \omega, \lambda - \omega; \lambda + \frac{1}{2}; \frac{1 - \sin\rho}{2}\right)\,, \qquad (6.63a)$$

$$\Phi_\omega^{(\mathrm{N},\lambda)}(\rho) = (\cos\rho)^{1-\lambda} F\left(1 - \lambda + \omega, 1 - \lambda - \omega; \frac{3}{2} - \lambda; \frac{1 - \sin\rho}{2}\right)\,, \qquad (6.63b)$$

respectively. Note that the function $\Phi_\omega^{(\mathrm{N},\lambda)}(\rho)$ is obtained from $\Phi_\omega^{(\mathrm{D},\lambda)}(\rho)$ by letting $\lambda \mapsto 1-\lambda$. Using Eqs. (6.16) and (6.17), we obtain the behaviour of these functions for $\rho \to -\pi/2$, which is given by

$$\Phi_\omega^{(\mathrm{D},\lambda)}(\rho) \approx \frac{\cos\pi\omega}{\cos\pi\lambda}(\cos\rho)^{\lambda} + \frac{\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma\left(\lambda-\frac{1}{2}\right)}{2^{1-2\lambda}\Gamma\left(\lambda+\omega\right)\Gamma\left(\lambda-\omega\right)}(\cos\rho)^{1-\lambda}\,, \tag{6.64a}$$

$$\Phi_\omega^{(\mathrm{N},\lambda)}(\rho) \approx -\frac{\cos\pi\omega}{\cos\pi\lambda}(\cos\rho)^{1-\lambda} + \frac{\Gamma\left(\frac{3}{2}-\lambda\right)\Gamma\left(\frac{1}{2}-\lambda\right)}{2^{2\lambda-1}\Gamma\left(1-\lambda+\omega\right)\Gamma\left(1-\lambda-\omega\right)}(\cos\rho)^{\lambda}\,. \tag{6.64b}$$

Dirichlet boundary condition. The function $\Phi_\omega^{(\mathrm{D},\lambda)}(\rho)$ given by Eq. (6.64a) satisfies the generalised Dirichlet boundary condition also at $\rho = -\pi/2$ (with $\omega > 0$) if and only if $\Gamma(\lambda-\omega) = \infty$, *i.e.*, if and only if $\omega = \omega_n^{\mathrm{I}} := \lambda + n$, $n \in \mathbb{N}_0$. The mode functions are given by Eq. (6.22).

Neumann boundary condition. We first discuss the cases with $\lambda \neq 1$. The function $\Phi_\omega^{(\mathrm{N},\lambda)}(\rho)$ given by Eq. (6.64b) satisfies the generalised Neumann boundary condition also at $\rho = -\pi/2$ (with $\omega > 0$) if and only if $\Gamma(1-\lambda-\omega) = \infty$ or $\Gamma(1-\lambda+\omega) = \infty$, *i.e.*, if and only if $\omega = 1-\lambda+n$, $n \in \mathbb{N}$, or $\omega = |1-\lambda|$. For either case the positive–frequency mode functions are obtained by substituting these values of ω into Eq. (6.64b) and then by using the definition of the Jacobi polynomials in Eq. (6.23). We obtain

$$\phi_n^{\mathrm{II}}(t,\rho) = N_n^{\mathrm{II}}(\cos\rho)^{1-\lambda}P_n^{(1/2-\lambda,1/2-\lambda)}(\sin\rho)e^{-i\omega_n^{\mathrm{II}}t}\,, \tag{6.65}$$

where $\omega_n^{\mathrm{II}} = 1-\lambda+n$, $n \in \mathbb{N}$, and $\omega_0^{\mathrm{II}} = |1-\lambda|$. The normalisation constant such that $(\phi_m^{\mathrm{II}}, \phi_n^{\mathrm{II}})_{KG} = \delta_{mn}$ can be found by using Eq. (6.25) as

$$N_n^{\mathrm{II}} = \frac{\sqrt{n!|\Gamma(2-2\lambda+n)|}}{2^{1-\lambda}\Gamma(3/2-\lambda+n)}\,. \tag{6.66}$$

Notice that $\omega_1 - \omega_0 = 3-2\lambda \neq 1$ if $1 < \lambda < 3/2$.

For $\lambda = 1$, we can write the positive–frequency solutions with the Neumann boundary condition as

$$\phi_n^{(\mathrm{II},\lambda=1)}(t,\rho) = \begin{cases} \frac{1}{\sqrt{\pi n}}\sin n\rho\, e^{-int}\,, & n+1 \in 2\mathbb{N}\,, \\ \frac{1}{\sqrt{\pi n}}\cos n\rho\, e^{-int}\,, & n \in 2\mathbb{N}\,. \end{cases} \tag{6.67}$$

We have $(\phi_m^{(\mathrm{II},\lambda=1)}, \phi_n^{(\mathrm{II},\lambda=1)})_{KG} = \delta_{mn}$ for $m, n \in \mathbb{N}$. Note that there are solutions with $\omega = 0$:

$$\phi_0^{(\mathrm{II},\lambda=1)}(t,\rho) = At + B\,, \tag{6.68}$$

where A and B are constants.

Mixed boundary conditions. The function $\Phi_\omega^{(\mathrm{D},\lambda)}(\rho)$ in Eq. (6.64a), which satisfies the generalised Dirichlet boundary condition at $\rho = \pi/2$, will satisfy the generalised Neumann boundary condition at $\rho = -\pi/2$ if and only if $\cos \pi\omega = 0$, *i.e.*, $\omega = n + 1/2$, $n \in \mathbb{N}_0$ (for $\omega > 0$). Then

$$\begin{aligned}\Phi_\omega^{(\mathrm{D},\lambda)}(\rho)|_{\omega=n+1/2} &= (\cos\rho)^\lambda F\left(\lambda+n+\frac{1}{2}, \lambda-n-\frac{1}{2}; \lambda+\frac{1}{2}; \frac{1-\sin\rho}{2}\right) \\ &= (\cos\rho)^\lambda \left(\frac{1+\sin\rho}{2}\right)^{1/2-\lambda} F\left(1+n, -n; \lambda+\frac{1}{2}; \frac{1-\sin\rho}{2}\right) . \end{aligned} \tag{6.69}$$

The positive–frequency mode functions corresponding to this function are

$$\phi_n^{\mathrm{III}}(t,\rho) = N_n^{\mathrm{III}} (\cos\rho)^\lambda (1+\sin\rho)^{1/2-\lambda} P_n^{(\lambda-1/2,-\lambda+1/2)}(\sin\rho) e^{-i\omega_n^{\mathrm{III}} t} , \tag{6.70}$$

where $\omega_n^{\mathrm{III}} = n + 1/2$ and where the normalisation constant such that $(\phi_m^{\mathrm{III}}, \phi_n^{\mathrm{III}})_{KG} = \delta_{mn}$, found using Eq. (6.25), is

$$N_n^{\mathrm{III}} = \frac{n!}{\sqrt{2\Gamma(\lambda+n+1/2)\Gamma(3/2-\lambda+n)}} . \tag{6.71}$$

The positive–frequency mode functions satisfying the generalised Dirichlet and Neumann boundary conditions at $\rho = -\pi/2$ and $\rho = \pi/2$, respectively, are

$$\phi_n^{\mathrm{IV}}(t,\rho) = N_n^{\mathrm{III}} (\cos\rho)^\lambda (1-\sin\rho)^{1/2-\lambda} P_n^{(-\lambda+1/2,\lambda-1/2)}(\sin\rho) e^{-i\omega_n^{\mathrm{III}} t} . \tag{6.72}$$

Since $P_n^{(\alpha,\beta)}(-x) = P_n^{(\beta,\alpha)}(x)$, we have

$$\phi_n^{\mathrm{IV}}(t,\rho) = \phi_n^{\mathrm{III}}(t,-\rho) , \tag{6.73}$$

as expected.

Next let us discuss the case $\lambda = 1/2$. As we saw, the only invariant boundary condition in this case is the generalised Dirichlet boundary condition. We have

$$\begin{aligned}\Phi_\omega^{(\mathrm{D},1/2)}(\rho) &= (\cos\rho)^{1/2} F\left(\frac{1}{2}+\omega, \frac{1}{2}-\omega; 1; \frac{1-\sin\rho}{2}\right) \\ &= (\cos\rho)^{1/2} \mathrm{P}_{\omega-1/2}(\sin\rho) , \end{aligned} \tag{6.74}$$

where $\mathrm{P}_\nu(x)$ is the Legendre function of the first kind. Then, since [23, Eq. 14.9.10]

$$\mathrm{P}_{\omega-1/2}(-x) = \frac{2}{\pi}\cos\pi\omega\, \mathrm{Q}_{\omega-1/2}(x) + \sin\pi\omega \mathrm{P}_{\omega-1/2}(x) , \tag{6.75}$$

where $\mathrm{Q}_\nu(x)$ is the Legendre function of the second kind with $\mathrm{Q}_\nu(x) \approx -\ln(1-x)/2$ as $x \to 1$ [23, Eq. 14.8.3], we must have $\cos\pi\omega = 0$ to have the generalised Dirichlet boundary condition at $\rho = -\pi/2$ as well. Thus, we obtain the positive–frequency mode functions in this case as

$$\phi_n^{\mathrm{V}}(t,\rho) = \frac{1}{\sqrt{2}} (\cos\rho)^{1/2} \mathrm{P}_n(\sin\rho) e^{-i\omega_n^{\mathrm{V}} t} , \tag{6.76}$$

where $\omega_n^{\mathrm{V}} = n + 1/2$. Note that $\mathrm{P}_n(x) = P_n^{(0,0)}(x)$. These mode functions are normalised so that $(\phi_m^{\mathrm{V}}, \phi_n^{\mathrm{V}})_{KG} = \delta_{mn}$.

6.3.2 Boundary conditions from vanishing energy flux at the boundaries

The $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–invariant boundary conditions and the solutions to Eq. (6.4) found in Section 6.3.1 are identical with the results of Sakai and Tanii [14] based on the requirement that the energy flux should vanish at the endpoints $\rho = \pm\pi/2$. (This requirement is analogous to that used in the higher–dimensional case [11, 12, 76].) We briefly review this derivation.

The stress–energy tensor with the conformal coupling constant β is given by [14]

$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\kappa\sigma}\partial_\kappa\phi\,\partial_\sigma\phi - \frac{1}{2}g_{\mu\nu}M^2\phi^2 + \frac{1}{2}\beta(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + R_{\mu\nu})\phi^2\,, \tag{6.77}$$

where $R_{\mu\nu} = g_{\mu\nu}$ is the Ricci curvature, and ∇_μ is the covariant derivative given by Eq. (2.4) with $N = 2$ and $\sqrt{h} = \mathcal{N} = \sec\rho$. The requirement of vanishing energy flux reads

$$\sqrt{-\det g}\, g^{1\mu}T_{\mu\nu}\xi_0^\nu\Big|_{\rho=\pm\pi/2} = 0\,, \tag{6.78}$$

where $\xi_0^\mu = \delta_0^\mu$ are the components of the Killing vector field ξ_0 in Eq. (6.2) in global coordinates (t,ρ). From Eq. (6.1), we have $\boldsymbol{g} = \sec^2\rho\,\mathrm{diag}(-1,1)$, and therefore by Eq. (2.3), the only non–vanishing connection components are $\Gamma^0_{01} = \Gamma^1_{00} = \Gamma^1_{11} = \tan\rho$. By substituting the mode functions given as in Eq. (6.5) into Eq. (6.78), we find

$$\left((1-2\beta)\frac{\mathrm{d}\Phi_\omega(\rho)}{\mathrm{d}\rho} + \beta\tan\rho\,\Phi_\omega(\rho)\right)\Phi_\omega(\rho)\Big|_{\rho=\pm\pi/2} = 0\,. \tag{6.79}$$

For $\lambda \geq 3/2$ we have $\Phi_\omega(\rho) \sim (\cos\rho)^\lambda$, and hence this condition is satisfied. For $1/2 < \lambda < 3/2$, the condition (6.79) becomes

$$\begin{aligned}&\Big\{(1-2\beta)D\widetilde{\Phi}^{(\lambda)}_\omega(\rho)\\ &+ \left[(3-2\lambda)\beta - (1-\lambda)\right](\cos\rho)^{1-2\lambda}\sin\rho\,\widetilde{\Phi}^{(\lambda)}_\omega(\rho)\Big\}\,\widetilde{\Phi}^{(\lambda)}_\omega(\rho)\Big|_{\rho=\pm\pi/2} = 0\,,\end{aligned} \tag{6.80}$$

where $\widetilde{\Phi}^{(\lambda)}_\omega(\rho)$ and $D\widetilde{\Phi}^{(\lambda)}_\omega(\rho)$ are defined by Eq. (6.35). This condition is satisfied by the Dirichlet boundary condition $\widetilde{\Phi}^{(\lambda)}(\pm\pi/2) = 0$ for all β. If we choose

$$\beta = \frac{1-\lambda}{3-2\lambda}\,, \tag{6.81}$$

then the Neumann boundary condition $D\widetilde{\Phi}^{(\lambda)}(\pm\pi/2) = 0$ and the mixed boundary conditions $D\widetilde{\Phi}^{(\lambda)}(\pi/2) = \widetilde{\Phi}^{(\lambda)}(-\pi/2) = 0$ or $D\widetilde{\Phi}^{(\lambda)}(-\pi/2) = \widetilde{\Phi}^{(\lambda)}(\pi/2) = 0$ satisfy the condition (6.80).

For $\lambda = 1/2$, Eq. (6.79) reads

$$\begin{aligned}&\Bigg\{(1-2\beta)D\widetilde{\Phi}^{(\lambda)}(\rho) + \Bigg[\left(2\beta - \frac{1}{2}\right)\left[\ln(\cos^2\rho) - 1\right]^2\\ &\qquad -2(1-2\beta)\left[\ln(\cos^2\rho) - 1\right]\Bigg]\tan\rho\,\widetilde{\Phi}^{(\lambda)}(\rho)\Bigg\}\,\widetilde{\Phi}^{(\lambda)}(\rho)\Big|_{\rho=\pm\pi/2} = 0\,.\end{aligned} \tag{6.82}$$

This is satisfied only by the Dirichlet boundary condition $\widetilde{\Phi}^{(\lambda)}(\pm\pi/2) = 0$. Thus, for all values of λ the energy fluxes at $\rho = \pm\pi/2$ vanish for some values of β if and only if $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–invariant boundary conditions are imposed.

The above analysis shows that, among all the allowed self–adjoint boundary conditions, only those which are invariant under $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ result in solution spaces satisfying the energy flux condition at the endpoints $\rho = \pm\pi/2$. The physical interpretation of the vanishing of the energy flux at the boundary (not necessarily the vanishing of the energy flux at each endpoint), is that the energy of the system is conserved and, thus, the model describes a closed system. On the other hand, the self–adjointness of the operators A_U also entails that we are dealing with an isolated system. Thus, one might think that there is a contradiction in place since not all self–adjoint boundary conditions satisfy the condition (6.78). This is not, however, entirely the case: We note that the energy condition in Eq. (6.78) is stronger than the similar but more general requirement

$$\sqrt{-\det g}\, g^{1\mu} T_{\mu\nu} \xi_0^\nu \Big|_{-\pi/2}^{\pi/2} = 0\,, \tag{6.83}$$

for which conservation of energy is also guaranteed. It might be the case that certain non–invariant self–adjoint boundary conditions do correspond to solutions satisfying Eq. (6.83). Further analysis may be necessary to verify if this situation holds true. Another possibility is that the non–invariant self–adjoint boundary conditions lead to solutions that satisfy a different energy condition: One in which part of the energy flows along the boundary, similar to the cases analysed in Ref. [77]. Once again, further analysis is needed to verify these claims.

6.3.3 *Boundary conditions leading to unitary irreducible representations*

In this subsection we discuss the mode functions found in Section 6.3.1 with reference to the UIRs of the group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ listed in Section 4.3. Let us start with the cases with $\lambda \geq 3/2$. In each of these cases the positive–frequency mode functions were found to have spatial components given by Eq. (6.22) and frequencies $\omega_n = \lambda + n$, with $n \in \mathbb{N}_0$. By comparing the ranges for λ and the frequencies $\omega = \mu + k$ with the classification given in Section 4.3, we find that the eigenvectors of the Casimir operator, *i.e.*, $\Box$ in Eq. (6.3), form a discrete series representation $\mathscr{D}^+_\lambda$ labelled by the pair (λ, λ).

Next, let us discuss the cases with $1/2 < \lambda < 3/2$. For each of these cases there are four possible $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–invariant self–adjoint extensions of the operator A. Correspondingly, there are four sets of positive–frequency solutions to the Klein–Gordon equation. The positive–frequency mode functions with the generalised Dirichlet boundary condition, $\phi^{\mathrm{I}}_n(t,\rho)$ in Eq. (6.22), form the discrete series representation $\mathscr{D}^+_\lambda$ as in the cases $\lambda \geq 3/2$.

For the positive–frequency mode functions with the generalised Neumann boundary condition, $\phi^{\mathrm{II}}_n(t,\rho)$ in Eq. (6.65), we need to discuss the cases $1/2 < \lambda < 1$, $\lambda = 1$ and $1 < \lambda < 3/2$ separately. If $1/2 < \lambda < 1$, then these mode functions form a discrete series representation $\mathscr{D}^+_{1-\lambda}$ with the positive frequencies $\omega_n = (1-\lambda) + n$, $n \in \mathbb{N}_0$.

If $\lambda = 1$, then the mode function $\phi_0^{\mathrm{II}}(t,\rho)$ has zero frequency. There is no $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–invariant vacuum state because of this mode function. This situation is analogous to the absence of de Sitter–invariant vacuum state for the massless minimally–coupled scalar field in de Sitter space [78]. Now, if we let the ladder operator L_- act on $\phi_1^{\mathrm{II}}(t,\rho) \propto \sin\rho\, e^{-it}$ we have by Eq. (6.50)

$$iL_-\phi_1^{\mathrm{II}}(t,\rho) = \frac{1}{\sqrt{\pi}}\,. \tag{6.84}$$

Since a constant function is orthogonal to all mode functions including itself with respect to the Klein–Gordon inner product (6.7), we can identify it with 0. This amounts to identifying $\phi_n^{\mathrm{II}}(t,\rho) + \text{constant}$ with $\phi_n^{\mathrm{II}}(t,\rho)$ for $n \geq 1$. With this identification, the mode functions $\phi_n^{\mathrm{II}}(t,\rho)$ form the UIR $\mathscr{D}_1^+$. Thus, it is possible to define an $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–invariant vacuum state if we "quotient out" the zero–frequency sector.(A de Sitter invariant vacuum state can be constructed also for the massless minimally–coupled scalar field in de Sitter space in this manner [79, 80].)

If $1 < \lambda < 3/2$, then the positive–frequency mode function with $n = 1$ satisfies

$$iL_-\phi_1^{\mathrm{II}}(t,\rho) = \sqrt{2(\lambda-1)}\;\overline{\phi_0^{\mathrm{II}}(t,\rho)}\,, \tag{6.85}$$

where Eq. (6.50) has been used. That is, the infinitesimal transformation of the positive–frequency mode $\phi_1^{\mathrm{II}}(t,\rho)$ leads to a negative–frequency mode $\overline{\phi_0^{\mathrm{II}}(t,\rho)}$. Thus, the positive–frequency subspace of the solutions to the Klein–Gordon equation and the vacuum state are not invariant under $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ transformations (see Section 2.3). In this case, the positive–frequency mode functions $\phi_n^{\mathrm{II}}(t,\rho)$, $n \in \mathbb{N}$, together with the negative–frequency mode function $\overline{\phi_0^{\mathrm{II}}(t,\rho)}$ form a representation in the *non–unitary discrete series* [61] denoted by $\mathscr{F}^{\pm}_{1-\lambda}$.

Next, let us discuss the solutions obeying the mixed boundary conditions for $1/2 < \lambda < 3/2$. Also in this case, the space of positive–frequency solutions and the vacuum state are non–invariant because positive–frequency mode functions and negative–frequency mode functions mix under the $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ transformations. This can be seen by using Eq. (6.50) as

$$iL_-\phi_0^{\mathrm{III}}(t,\rho) = \left(\frac{1}{2}-\lambda\right)\overline{\phi_0^{\mathrm{III}}(t,\rho)}\,, \tag{6.86a}$$

$$iL_-\phi_0^{\mathrm{IV}}(t,\rho) = -\left(\frac{1}{2}-\lambda\right)\overline{\phi_0^{\mathrm{IV}}(t,\rho)}\,. \tag{6.86b}$$

For either of the mixed boundary conditions, the positive– and negative–frequency mode functions together form a non–unitary representation of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$.

For $\lambda = 1/2$, the positive–frequency mode functions with the generalised Dirichlet boundary conditions, $\phi_n^{\mathrm{V}}(t,\rho)$, $n \in \mathbb{N}_0$, form a unitary representation because in this case we have $L_-\phi_0^{\mathrm{V}}(t,\rho) = 0$. This representation is $\mathscr{D}^+_{1/2}$, which is a representation in the mock–discrete series, labelled by the pair $(1/2, 1/2)$.

6.4 INVARIANT THEORIES WITH NO INVARIANT POSITIVE–FREQUENCY SUBSPACE

In Section 6.3.3 we found that some boundary conditions lead to an $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–invariant theory with non–invariant vacuum state if $1/2 < \lambda < 3/2$. The Klein–Gordon inner product (6.8) is $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–invariant with any of these boundary conditions. This implies that the $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ transformations on the quantum field are Bogoliubov transformations [24, 27, 34], which mix annihilation and creation operators. Since a theory with any of these boundary conditions is $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–invariant but the vacuum state is not, it must be possible to find a UIR of this group which the vacuum state belongs to. We may associate to these cases a type of broken symmetry. In this section we identify this representation.

We start with the cases with the mixed boundary condition in Eq. (6.47) with $1/2 < \lambda < 3/2$. (The cases satisfying the boundary condition (6.48) are similar.) We expand the quantum field operator $\phi(t,\rho)$ as in Eq. (2.46) so that

$$\phi(t,\rho) = \sum_{n=0}^{\infty}\left[a_n\phi_n^{\mathrm{III}}(t,\rho) + a_n^{\dagger}\overline{\phi_n^{\mathrm{III}}(t,\rho)}\right]. \tag{6.87}$$

Then, the conserved quantum charge for the symmetry generated by $L_\pm$ is

$$\begin{aligned}\hat{L}_\pm &= \frac{1}{2}(\phi, L_\pm\phi)_{KG} \\ &= i\int_{-\pi/2}^{\pi/2}\left[\overline{\phi(t,\rho)}\frac{\partial}{\partial t}L_\pm\phi(t,\rho) - \frac{\partial\overline{\phi(t,\rho)}}{\partial t}L_\pm\phi(t,\rho)\right]\mathrm{d}\rho\,. \end{aligned}\tag{6.88}$$

We use the ladder operators for the Jacobi polynomials [23, Eqs. 18.9.17, 18.9.18]

$$\begin{aligned}&(2n+\alpha+\beta)(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(\alpha,\beta)}(x) \\ &= n(\alpha-\beta-(2n+\alpha+\beta)x)P_n^{(\alpha,\beta)}(x) + 2(n+\alpha)(n+\beta)P_{n-1}^{(\alpha,\beta)}(x)\,, \end{aligned}\tag{6.89a}$$

$$\begin{aligned}&(2n+\alpha+\beta+2)(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(\alpha,\beta)}(x) \\ &= (n+\alpha+\beta+1)(\alpha-\beta+(2n+\alpha+\beta+2)x)P_n^{(\alpha,\beta)}(x) \\ &\quad - 2(n+1)(n+\alpha+\beta+1)P_{n+1}^{(\alpha,\beta)}(x)\,, \end{aligned}\tag{6.89b}$$

to find $L_+\phi_n^{\mathrm{III}} = -ik_n\phi_{n+1}^{\mathrm{III}}$ $(n\geq 0)$, $L_-\phi_n^{\mathrm{III}} = -ik_{n-1}\phi_{n-1}^{\mathrm{III}}$ $(n\geq 1)$, $L_-\overline{\phi_n^{\mathrm{III}}} = ik_n\overline{\phi_{n+1}^{\mathrm{III}}}$ $(n\geq 0)$, $L_+\overline{\phi_n^{\mathrm{III}}} = ik_{n-1}\overline{\phi_{n-1}^{\mathrm{III}}}$ $(n\geq 1)$, where

$$k_n = \left[\left(\lambda+n+\frac{1}{2}\right)\left(n+\frac{3}{2}-\lambda\right)\right]^{1/2}. \tag{6.90}$$

On the other hand, we have $L_-\phi_0^{\mathrm{III}} = -i(1/2-\lambda)\overline{\phi_0^{\mathrm{III}}}$ (see Eq. (6.86a)) and $L_+\overline{\phi_0^{\mathrm{III}}} = i(1/2-\lambda)\phi_0^{\mathrm{III}}$. By using these formulas and the orthonormality relations $(\phi_m^{\mathrm{III}},\phi_n^{\mathrm{III}})_{KG} = -(\overline{\phi_m^{\mathrm{III}}},\overline{\phi_n^{\mathrm{III}}})_{KG} = \delta_{mn}$ in Eq. (6.88), we find

$$i\hat{L}_+ = \sum_{n=0}^{\infty}k_n a_{n+1}^{\dagger}a_n - \frac{1}{2}\left(\lambda-\frac{1}{2}\right)(a_0^{\dagger})^2\,, \tag{6.91}$$

$$i\hat{L}_- = \sum_{n=0}^{\infty}k_n a_n^{\dagger}a_{n+1} - \frac{1}{2}\left(\lambda-\frac{1}{2}\right)(a_0)^2\,. \tag{6.92}$$

Then,

$$[\hat{L}_+, \hat{L}_-] = \sum_{n=1}^{\infty} k_n^2 (a_n^\dagger a_n - a_{n+1}^\dagger a_{n+1}) + \frac{1}{2}\left(\lambda - \frac{1}{2}\right)^2 (a_0^\dagger a_0 + a_0 a_0^\dagger)\,. \tag{6.93}$$

We note that the first term above should annihilate the bosonic vacuum state $|0\rangle_B$, thus, the manipulation of this sum must be understood formally, keeping the operators normal–ordered. Thus, we obtain

$$[\hat{L}_+, \hat{L}_-] = 2\hat{L}_0\,, \tag{6.94}$$

where

$$\hat{L}_0 = \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) a_n^\dagger a_n + \frac{1}{4}\left(\lambda - \frac{1}{2}\right)^2 \mathbb{I}\,. \tag{6.95}$$

(If we substituted the equality $a_n^\dagger a_n - a_{n+1}^\dagger a_{n+1} = a_n a_n^\dagger - a_{n+1} a_{n+1}^\dagger$ into Eq. (6.93) and manipulated the summation formally, we would find an infinite constant added to $\hat{L}_0$.)

From Eq. (6.49), we know that the operator $L_0 = [L_+, L_-]/2$ corresponds to the action of the timelike Killing vector ξ_0 on the solution space. Thus, by comparing the commutation relation (6.94) with Eq. (4.21) we conclude that the operator $\hat{L}_0$ should be identified as the time–translation generator, *i.e.*, the energy operator. The vacuum state $|0\rangle_B$ annihilated by all a_n, $n = 0, 1, 2, \ldots$, satisfies $\hat{L}_- |0\rangle_B = 0$ and $\hat{L}_0 |0\rangle_B = [(\lambda - 1/2)^2/4]\,|0\rangle_B$. Hence, the state $|0\rangle_B$ belongs to the representation $\mathscr{D}^+_{(\lambda-1/2)^2/4}$. The other states in this representation can be found by applying $\hat{L}^+$ repeatedly on $|0\rangle_B$.

The theory with $1 < \lambda < 3/2$ obtained by imposing the Neumann boundary condition (6.42) can be studied in the same manner. For these cases, we find $L_+\phi_n^{\mathrm{II}} = -iq_n\phi_{n+1}^{\mathrm{II}}$ $(n \geq 1)$, $L_-\phi_n^{\mathrm{II}} = iq_{n-1}\phi_{n-1}^{\mathrm{II}}$ $(n \geq 2)$, $L_-\overline{\phi_n^{\mathrm{II}}} = iq_n\overline{\phi_{n+1}^{\mathrm{II}}}$ $(n \geq 1)$ and $L_+\overline{\phi_n^{\mathrm{II}}} = -iq_{n-1}\overline{\phi_{n-1}^{\mathrm{II}}}$ $(n \geq 2)$, where

$$q_n = \sqrt{(n+1)(2 - 2\lambda + n)}\,. \tag{6.96}$$

On the other hand, we have $L_+\overline{\phi_0^{\mathrm{II}}} = i\sqrt{2(\lambda-1)}\phi_1^{\mathrm{II}}$, $L_-\phi_1^{\mathrm{II}} = i\sqrt{2(\lambda-1)}\,\overline{\phi_0^{\mathrm{II}}}$, $L_-\phi_0^{\mathrm{II}} = -i\sqrt{2(\lambda-1)}\,\overline{\phi_1^{\mathrm{II}}}$, $L_+\overline{\phi_1^{\mathrm{II}}} = -i\sqrt{2(\lambda-1)}\phi_0^{\mathrm{II}}$ and $L_-\overline{\phi_0^{\mathrm{II}}} = L_+\phi_0^{\mathrm{II}} = 0$. Then we find

$$i\hat{L}_+ = \sum_{n=1}^{\infty} q_n a_{n+1}^\dagger a_n - \sqrt{2(\lambda-1)}a_1^\dagger a_0^\dagger\,, \tag{6.97}$$

$$i\hat{L}_- = \sum_{n=1}^{\infty} q_n a_n^\dagger a_{n+1} - \sqrt{2(\lambda-1)}a_1 a_0\,. \tag{6.98}$$

Then, in the same way as in the cases with the mixed boundary condition, we find

$$[\hat{L}_+, \hat{L}_-] = 2\hat{L}_0\,, \tag{6.99}$$

where

$$\hat{L}_0 = \sum_{n=0}^{\infty} \omega_n^{\mathrm{II}} a_n^\dagger a_n + (\lambda - 1)\mathbb{I}\,. \tag{6.100}$$

Thus, the vacuum state in this theory belongs to the representation $\mathscr{D}^+_{\lambda-1}$.

We conclude this chapter summarising our main results for the analysis of a scalar field in AdS_2 in Table 6.1. Here, the symbols $\overrightarrow{\Phi}^{(\lambda)}$ and $\overrightarrow{D\Phi}^{(\lambda)}$ stand for the column vectors $(\tilde{\Phi}^{(\lambda)}(\pi/2), -\tilde{\Phi}^{(\lambda)}(-\pi/2))^T$ and $(D\tilde{\Phi}^{(\lambda)}(\pi/2), D\tilde{\Phi}^{(\lambda)}(-\pi/2))^T$, respectively. We also define the matrices $U_+ := i(\mathbb{I} + \mathcal{U})$ and $U_- := \mathbb{I} - \mathcal{U}$.

Table 6.1: Self–adjoint boundary conditions for scalar field in AdS_2

$\boldsymbol{M}$ $(\boldsymbol{\lambda})$	**SABCs**	**Inv. SABCs**	**Spectrum** $(\boldsymbol{\omega})$	**Inv. P-F solutions**	**UIR**
$M^2 \geq \frac{3}{4}$ $\left(\lambda \geq \frac{3}{2}\right)$	Dirichlet	Dirichlet	$\omega = \pm\omega_n^{(\lambda,\mathrm{D})}$ $\omega_n^{(\lambda,\mathrm{D})} = \lambda + n$	ϕ_n^{I}, $\omega_n^{(\lambda,\mathrm{D})}$	$\mathscr{D}_\lambda^+$
$0 < M^2 < \frac{3}{4}$ $\left(1 < \lambda < \frac{3}{2}\right)$	$U_- \overrightarrow{D\Phi}^{(\lambda)} = U_+ \overrightarrow{\Phi}^{(\lambda)}$	Dirichlet	$\omega = \pm\omega_n^{(\lambda,\mathrm{D})}$ $\omega_n^{(\lambda,\mathrm{D})} = \lambda + n$	ϕ_n^{I}, $\omega_n^{(\lambda,\mathrm{D})}$	$\mathscr{D}_\lambda^+$
		Neumann	$\omega = \pm\omega_n^{(\lambda,\mathrm{N})}$ $\omega_n^{(\lambda,\mathrm{N})} = 1 - \lambda + n$	–	–
		Mixed I	$\omega_j^{(\mathrm{M})} = j + \frac{1}{2}$ $j \in \mathbb{Z}$	–	–
		Mixed II		–	–
$M^2 = 0$ $(\lambda = 1)$	$U_- \overrightarrow{D\Phi}^{(1)} = U_+ \overrightarrow{\Phi}^{(1)}$	Dirichlet	$\omega = \pm\omega_n^{(1,\mathrm{D})}$, $\omega_n^{(1,\mathrm{D})} = 1 + n$	ϕ_n^{I}, $\omega_n^{(1,\mathrm{D})}$	$\mathscr{D}_1^+$
		Neumann	$\omega_j^{(1,\mathrm{N})} = j$ $j \in \mathbb{Z}$	$\left[\phi_{n+1}^{\mathrm{II}}\right]$, $\omega_{n+1}^{(1,\mathrm{N})}$	
		Mixed I	$\omega_j^{(\mathrm{M})} = j + \frac{1}{2}$ $j \in \mathbb{Z}$	–	–
		Mixed II		–	–
$-\frac{1}{4} < M^2 < 0$ $\left(\frac{1}{2} < \lambda < 1\right)$	$U_- \overrightarrow{D\Phi}^{(\lambda)} = U_+ \overrightarrow{\Phi}^{(\lambda)}$	Dirichlet	$\omega = \pm\omega_n^{(\lambda,\mathrm{D})}$, $\omega_n^{(\lambda,\mathrm{D})} = \lambda + n$	ϕ_n^{I}, $\omega_n^{(\lambda,\mathrm{D})}$	$\mathscr{D}_\lambda^+$
		Neumann	$\omega = \pm\omega_n^{(\lambda,\mathrm{N})}$, $\omega_n^{(\lambda,\mathrm{N})} = 1 - \lambda + n$	ϕ_n^{II}, $\omega_n^{(\lambda,\mathrm{N})}$	$\mathscr{D}_{1-\lambda}^+$
		Mixed I	$\omega_j^{(\mathrm{M})} = j + \frac{1}{2}$, $j \in \mathbb{Z}$	–	–
		Mixed II		–	–
$M^2 = -\frac{1}{4}$ $\left(\lambda = \frac{1}{2}\right)$	$U_- \overrightarrow{D\Phi}^{(\frac{1}{2})} = U_+ \overrightarrow{\Phi}^{(\frac{1}{2})}$	Dirichlet	$\omega_j^{(1/2,\mathrm{D})} = j + \frac{1}{2}$, $j \in \mathbb{Z}$	ϕ_n^{V}, $\omega_n^{(1/2,\mathrm{D})}$	$\mathscr{D}_{1/2}^+$

Here n is assumed to be in $\mathbb{N}_0$.
The symbol $\left[\phi_{n+1}^{\mathrm{II}}\right]$ stands for the equivalence class of modes modulo a constant term.

— 7 —

Scalar field theory in AdS_N

Now that we have analysed a scalar field theory in the two–dimensional anti–de Sitter space, we shall extend our analysis to the higher dimensional cases. We shall consider a scalar field on an N–dimensional anti–de Sitter spacetime, AdS_N, with $N \geq 3$. We will proceed in a very similar way to that of Chapter 6, first obtaining general solutions to the Klein–Gordon equation in AdS_N, then finding all possible self–adjoint extensions of a certain one–dimensional operator in the form of boundary conditions and finally finding which, among these, are invariant under the isometry group of the spacetime.

The analysis of a scalar field in AdS_N using the theory of self–adjoint extensions has been done by Ishibashi and Wald [22]. The main objective in their analysis was to prove that a scalar field theory in AdS_N can be constructed in a way such that the dynamics for the classical field is well defined. They proceeded to show that the radial component of the Klein–Gordon equation defines a densely defined symmetric operator on a certain Hilbert space that admits a family of positive self–adjoint extensions. Then, they applied von Neumann's theorem (Theorem 5.1.3) to determine the domains of the self–adjoint extensions using the method presented in Chapter 5. Following this, they found that the positive self–adjoint extensions are parametrised by a real number.

In this chapter we will rework the analysis by Ishibashi and Wald for the scalar field in AdS_N, and, after replicating their results, we will extend their analysis by imposing invariance under the isometry group of AdS_N to determine which among these theories result in invariant vacuum states. We will follow a similar method to the one used in Ref. [21] (which is the same method used in Chapter 6 for a scalar field in AdS_2). Thus, we present the general positive–frequency solutions to the Klein–Gordon equation in AdS_N and analyse the behaviour of the radial component of the solutions for different ranges of the mass of the field. For certain values of the mass parameter the associated radial operator admits a family of self–adjoint extensions prescribed by von Neumann's theorem. After finding the domains of the admissible self–adjoint extensions of the radial operator, we show that these domains can be put in a one–to–one correspondence with a family of boundary conditions. Additionally, after we find the boundary conditions characterising all possible self–adjoint extensions we will impose invariance under the isometry group, $\mathrm{SO}(2, N-1)$, of AdS_N as a criterion to determine which among these boundary conditions are of physical interest. Finally, we will consider the action of one of the boost–like Killing vector fields of AdS_N on the mode solutions arising from the different boundary conditions that characterise the admissible self–adjoint extensions of the radial operator.

7.1 SOLUTIONS OF THE KLEIN–GORDON EQUATION IN AdS_N

Let us consider the N–dimensional anti–de Sitter spacetime, AdS_N, with $N \geq 3$. We use the static coordinate system given by Eq. (3.8), where the ranges of the coordinates $(t, \rho, \theta_1, \ldots, \theta_{N-2})$ are given by Eq. (3.4). The line element for AdS_N in these coordinates is given by Eq. (3.6). Using Eqs. (3.7) and (3.10), we also see that for AdS_N we have

$$\mathcal{N}(\rho, \theta) = \sec\rho \,, \tag{7.1a}$$

$$\sqrt{h(\rho,\theta)} = \sec\rho\,(\tan\rho)^{N-2} \prod_{k=1}^{N-3} (\sin\theta_k)^{N-k-2} \,, \tag{7.1b}$$

where we have denoted $\theta = (\theta_1, \ldots, \theta_{N-2})$.

A scalar field ϕ of mass M satisfies the Klein–Gordon equation $\left(\Box_{\mathrm{AdS}_N} - M^2\right)\phi = 0$, where $\Box_{\mathrm{AdS}_N}$ is the Laplace–Beltrami operator on AdS_N given by Eq. (3.14). Similarly to the two–dimensional case in Chapter 6, we are interested in positive–frequency solutions of the form

$$\phi(t, \rho, \theta) = \Phi_\omega(\rho, \theta) e^{-i\omega t} \,, \qquad \omega > 0 \,. \tag{7.2}$$

Expressing the solutions of the Klein–Gordon equation in this way, we can identify the spatial component functions Φ_ω as solutions to the equation

$$A\Phi_\omega = \omega^2 \Phi_\omega \,, \tag{7.3}$$

where the operator A, defined for a general static spacetime in Eq. (2.9), is obtained for AdS_N using Eq. (3.14), and reads

$$A := -\frac{\partial^2}{\partial\rho^2} - \frac{N-2}{\sin\rho\cos\rho}\frac{\partial}{\partial\rho} + \frac{M^2}{\cos^2\rho} - \frac{1}{\sin^2\rho}\Delta_{N-2} \,. \tag{7.4}$$

The space of solutions of Eq. (7.3) thus forms the Hilbert space $\mathscr{H}_{KG} := L^2(\Sigma, \mathrm{d}V)$, where Σ corresponds to any of the hypersurfaces Σ_t of constant t–coordinate, and

$$\begin{aligned} \mathrm{d}V &= \sqrt{h}\mathcal{N}^{-1} \mathrm{d}\rho\, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{N-2} \,, \\ &= (\tan\rho)^{N-2} \left(\prod_{k=1}^{N-3} (\sin\theta_k)^{N-k-2} \right) \mathrm{d}\rho\, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{N-2} \,, \end{aligned} \tag{7.5}$$

where we have used Eq. (7.1). Following the theory presented in Chapter 2, the inner product of $\mathscr{H}_{KG}$ is given by Eq. (2.10), and for $\Phi_1, \Phi_2 \in \mathscr{H}_{KG}$ it reads

$$\langle \Phi_1, \Phi_2 \rangle_{KG} = \int_\Sigma \overline{\Phi_1(\rho,\theta)} \Phi_2(\rho,\theta) (\tan\rho)^{N-2} \mathrm{d}\rho\, \mathrm{d}\Omega_{N-2} \,, \tag{7.6}$$

where we have denoted the volume element of the unit $(N-2)$–sphere as

$$\mathrm{d}\Omega_{N-2} := \left(\prod_{k=1}^{N-3} (\sin\theta_k)^{N-k-2}\right) \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{N-2}\,. \tag{7.7}$$

The inner product (7.6) induces an inner product on the space of positive–frequency solutions of the Klein–Gordon equation. Indeed, for any two solutions ϕ_1, ϕ_2, of the form of Eq. (7.2) with $\omega_1, \omega_2 > 0$ and $\omega_1 \neq \omega_2$, the inner product is given by

$$\begin{aligned}(\phi_1, \phi_2)_{KG} &:= i \int_\Sigma \left(\overline{\phi_1(t,\rho,\theta)} \frac{\partial \phi_2(t,\rho,\theta)}{\partial t} - \frac{\partial \overline{\phi_1(t,\rho,\theta)}}{\partial t} \phi_2(t,\rho,\theta)\right) (\tan\rho)^{N-2} \mathrm{d}\rho\, \mathrm{d}\Omega_{N-2}\,, \\ &= (\omega_2 + \omega_1) e^{-i(\omega_2-\omega_1)t} \left\langle \Phi_{\omega_1}, \Phi_{\omega_2} \right\rangle_{KG}\,. \end{aligned} \tag{7.8}$$

With these facts in mind, we now look for solutions of Eq. (7.3) which are normalisable with respect to the inner product (7.6). We begin by finding the general solutions for different values of the mass M of the field. Let us consider the separation of variables for the function Φ_ω of the form

$$\Phi_\omega(\rho, \theta) = R_\omega(\rho) Y(\theta)\,. \tag{7.9}$$

Substituting this expression into Eq. (7.3) and rearranging we obtain

$$\frac{\sin^2\rho}{R_\omega(\rho)} \left[-\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{N-2}{\sin\rho\cos\rho} \frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{M^2}{\cos^2\rho} - \omega^2\right] R_\omega(\rho) = \frac{1}{Y(\theta)} \Delta_{N-2} Y(\theta)\,. \tag{7.10}$$

Since the right–hand side of this expression equals a constant whenever $Y(\theta)$ is an eigenfunction of the Laplace operator on the $(N-2)$–sphere, we can consider these functions to be solutions of the equation

$$\Delta_{N-2} Y_{\boldsymbol{l}}(\theta) = -l_1(l_1 + N - 3) Y_{\boldsymbol{l}}(\theta)\,, \tag{7.11}$$

where $l_1 \in \mathbb{N}_0$, and $\boldsymbol{l}$ labels the full spectrum of Δ_{N-2}. Solutions to Eq. (7.11) are given by the scalar hyperspherical harmonics [81, 82, 83]. Adopting a convention similar to that of Ref. [84], the normalised solutions to Eq. (7.11) are given by

$$Y_{\boldsymbol{l}}(\theta) = \frac{1}{\sqrt{2\pi}} \left(\prod_{k=1}^{N-3} \overline{P}^{l_2}_{k,l_k}(\theta_k)\right) e^{i l_{N-2} \theta_{N-2}}\,, \tag{7.12}$$

where the integers $l_1, l_2, \ldots, l_{N-2}$ satisfying

$$l_1 \geq l_2 \geq \cdots l_{N-3} \geq |l_{N-2}|\,, \tag{7.13}$$

comprise the spectral label $\boldsymbol{l} := (l_1, \ldots, l_{N-2})$. The functions $\overline{P}^{l}_{k,L}$ in Eq. (7.12) depend on the associated Legendre functions of the first kind [23, Eq. 14.3.9],

$$P_\nu^{-\mu}(x) = \frac{1}{\Gamma(1+\mu)} \left(\frac{x-1}{x+1}\right)^{\mu/2} F\left(\nu+1, -\nu; 1+\mu; \frac{1-x}{2}\right)\,, \tag{7.14}$$

via the definition

$$\overline{P}^{l}_{k,L}(\theta) := c^{l}_{k,L}(\sin\theta)^{-(N-k-3)/2} P^{-l-(N-k-3)/2}_{L+(N-k-3)/2}(\cos\theta)\,. \tag{7.15}$$

The constants $c^{l}_{k,L}$ are given by [84]

$$c^{l}_{k,L} = \left(\frac{2L+N-k-2}{2}\frac{(L+l+N-k-3)!}{(L-l)!}\right)^{1/2}, \tag{7.16}$$

and normalise the functions $Y_{\boldsymbol{l}}$ with respect to the angular component of the inner product in Eq. (7.6), that is,

$$\int_{S^{N-2}} \overline{Y_{\boldsymbol{l}}(\theta)}\, Y_{\boldsymbol{l}'}(\theta) \mathrm{d}\Omega_{N-2} = \delta_{\boldsymbol{l},\boldsymbol{l}'}\,, \tag{7.17}$$

where $\delta_{\boldsymbol{l},\boldsymbol{l}'} = \delta_{l_1,l'_1}\cdots\delta_{l_{N-2},l'_{N-2}}$.

Substituting Eq. (7.11) into Eq. (7.10), we obtain a differential equation for the radial function R_{ω,l_1} given by

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{N-2}{\sin\rho\,\cos\rho}\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{M^2}{\cos^2\rho} + \frac{l_1(l_1+N-3)}{\sin^2\rho}\right] R_{\omega,l_1}(\rho) = \omega^2 R_{\omega,l_1}(\rho)\,. \tag{7.18}$$

In order to simplify this equation, let us consider the function r_{ω,l_1} defined through the relation

$$R_{\omega,l_1}(\rho) = (\cot\rho)^{\frac{N-2}{2}}\, r_{\omega,l_1}(\rho)\,. \tag{7.19}$$

The radial equation (7.18) reduces to a differential equation for the function r_{ω,l_1} given by

$$A_{\text{Rad}} r_{\omega,l_1} = \omega^2\, r_{\omega,l_1}\,, \tag{7.20}$$

where the differential operator A_{Rad} is given by

$$A_{\text{Rad}} := -\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \left(\frac{\nu^2-1/4}{\cos^2\rho} + \frac{\sigma^2-1/4}{\sin^2\rho}\right)\,. \tag{7.21}$$

with the constants σ and ν defined as

$$\sigma := l_1 + \frac{N-3}{2}\,, \qquad \nu := \sqrt{M^2 + \left(\frac{N-1}{2}\right)^2}\,. \tag{7.22}$$

The operator A_{Rad} is of the form of the Schrödinger operator T in Eq. (5.25) with potential term given by

$$V(\rho) = \frac{\frac{N}{2}\left(\frac{N-2}{2}\right) + M^2}{\cos^2\rho} + \frac{\left(\frac{N-2}{2}\right)^2 - \frac{N-2}{2} + l_1(l_1+N-3)}{\sin^2\rho}\,. \tag{7.23}$$

Hence, we are to investigate the properties of the radial operator in Eq. (7.21) in a way similar to the analysis of the operator A for the two–dimensional case in Chapter 6.

We first note that the inner product of elements in the Hilbert space $\mathscr{H}_{KG}$ of square–integrable functions on the static slice Σ, given by Eq. (7.6) induces an inner product on the (scaled) radial component functions, r_{ω,l_1}, by means of the decomposition in Eq. (7.2) and the definition in Eq. (7.19). Given two elements $\Phi_1, \Phi_2 \in \mathscr{H}_{KG}$ such that $\Phi_1(\rho,\theta) = R_{\omega_1,l_1}(\rho)Y_{\boldsymbol{l}}(\theta)$ and $\Phi_2(\rho,\theta) = R_{\omega_2,l_1'}(\rho)Y_{\boldsymbol{l}'}(\theta)$, Eq. (7.6) together with Eq. (7.17) imply that

$$\langle \Phi_1, \Phi_2 \rangle_{KG} = \delta_{\boldsymbol{l},\boldsymbol{l}'} \int_0^{\pi/2} \overline{r_{\omega_1,l_1}(\rho)} r_{\omega_2,l_1}(\rho) \mathrm{d}\rho\,. \tag{7.24}$$

Hence, we will consider solutions to Eq. (7.20) lying in the Hilbert space $L^2[0,\pi/2]$, with inner product given by

$$\langle r_{\omega_1,l_1}, r_{\omega_2,l_1} \rangle_{\text{Rad}} = \int_0^{\pi/2} \overline{r_{\omega_1,l_1}(\rho)} r_{\omega_2,l_1}(\rho) \mathrm{d}\rho\,. \tag{7.25}$$

To completely specify the operator A_{Rad} we will consider its domain to be defined by $\mathrm{Dom}(A_{\text{Rad}}) := C_c^\infty(0,\pi/2)$, the space of compactly supported smooth functions on the interval $(0,\pi/2)$ with support away from the boundary at $\rho = \pi/2$ [22, 35]. On this domain, the operator A_{Rad} is densely defined in the sense of Definition 2.2.1 and it is symmetric with respect to the inner product of Eq. (7.25).

Since the operator A_{Rad} is symmetric and densely defined, the adjoint operator $A_{\text{Rad}}^\dagger$ exists and it is given by Definition 2.2.2. For any $r' \in \mathrm{Dom}(A_{\text{Rad}}^\dagger)$, we have

$$\langle r', A_{\text{Rad}} r \rangle_{\text{Rad}} = \left\langle A_{\text{Rad}}^\dagger r', r \right\rangle_{\text{Rad}}\,, \tag{7.26}$$

for all $r \in \mathrm{Dom}(A_{\text{Rad}})$.

The operator A_{Rad} with domain $C_c^\infty(0,\pi/2)$ shares several properties with the operator A in Eq. (6.6) from Chapter 6. In fact, for $N = 2$ and $l_1 = 0$, Eq. (7.22) implies that $\sigma^2 = 1/4$ and $\nu^2 = M^2 + 1/4$. Thus, Eq. (7.21) reduces to Eq. (6.6) as a differential operator if we identify $\nu^2 - 1/2$ with $\lambda(\lambda - 1)$ in Eq. (6.11). In this "limiting" case, both operators A_{Rad} and A are defined on compactly supported smooth functions on a finite interval. Hence, similarly to the case of the operator A, if $r \in \mathrm{Dom}(A_{\text{Rad}}^\dagger)$, then the derivative $\mathrm{d}r/\mathrm{d}\rho$ exists in $L^2(0,\pi/2)$ and it is absolutely continuous [36]. Therefore, the operator $A_{\text{Rad}}^\dagger$ is the same differential operator as A_{Rad} on $r \in \mathrm{Dom}(A_{\text{Rad}}^\dagger)$ except on a measure–zero set where r may not be twice differentiable. Similarly, if $r_1, r_2 \in \mathrm{Dom}(A_{\text{Rad}}^\dagger)$, then we have

$$\left\langle A_{\text{Rad}}^\dagger r_1, r_2 \right\rangle_{\text{Rad}} - \langle r_1, A_{\text{Rad}} r_2 \rangle_{\text{Rad}} = \left[\overline{r_1(\rho)} \frac{\mathrm{d}r_2(\rho)}{\mathrm{d}\rho} - \frac{\mathrm{d}\overline{r_1(\rho)}}{\mathrm{d}\rho} r_2(\rho) \right]_{\rho\to 0}^{\rho\to\pi/2}\,, \tag{7.27}$$

and, if $r \in \mathrm{Dom}(A_{\text{Rad}})$, then $A_{\text{Rad}}^\dagger r = A_{\text{Rad}} r$.

The operator A_{Rad} is not self–adjoint since $\mathrm{Dom}(A_{\text{Rad}}) \neq \mathrm{Dom}(A_{\text{Rad}}^\dagger)$, the latter being a larger subspace of $L^2[0,\pi/2]$ [36, 43]. We will thus apply the theory of self–adjoint extensions of Chapter 5 to the operator A_{Rad} to find a family of operators $(A_{\text{Rad}})_U$ satisfying $\mathrm{Dom}(A_{\text{Rad}}) \subseteq \mathrm{Dom}((A_{\text{Rad}})_U) \subseteq \mathrm{Dom}(A_{\text{Rad}}^\dagger)$, and $(A_{\text{Rad}})_U r = A_{\text{Rad}} r$ for all $r \in \mathrm{Dom}(A_{\text{Rad}})$.

Since our goal is to define a quantum theory with a stationary vacuum state for the scalar field ϕ following the prescription of Chapter 2, we will only consider *positive* self–adjoint extensions of the operator A_{Rad}. Then, each $(A_{\text{Rad}})_U$ will be a self–adjoint operator with positive spectrum. Now, if $\nu^2 \geq 0$, then the operator A_{Rad} in Eq. (7.21) can be written as

$$A_{\text{Rad}} = \left(-\frac{\mathrm{d}}{\mathrm{d}\rho} + F(\rho)\right)\left(\frac{\mathrm{d}}{\mathrm{d}\rho} + F(\rho)\right) + (\nu - \sigma - 1)^2\,, \tag{7.28}$$

where we have defined

$$F(\rho) = -\left(\nu - \frac{1}{2}\right)\tan\rho - \left(\sigma + \frac{1}{2}\right)\cot\rho\,. \tag{7.29}$$

Then, for arbitrary $r \in \mathrm{Dom}(A_{\text{Rad}}) = C_c^\infty(0, \pi/2)$, we have [22]

$$\langle r, A_{\text{Rad}} r\rangle_{\text{Rad}} = (\nu - \sigma - 1)^2 \int_0^{\pi/2} |r(\rho)|^2\,\mathrm{d}\rho + \int_0^{\pi/2} \left|\frac{\mathrm{d}r(\rho)}{\mathrm{d}\rho} + F(\rho)r(\rho)\right|^2 \mathrm{d}\rho\,, \tag{7.30}$$

and, thus, the operator A_{Rad} is positive for $\nu^2 \geq 0$. If, on the other hand, $\nu^2 < 0$, an argument similar to that for the operator A presented in Appendix C shows that the operator A_{Rad} is unbounded below, and thus, it is not a positive semi–definite operator [22, Proposition 3.1]. Hence, we will only consider the values for ν such that $\nu^2 \geq 0$ which, by Eq. (7.22) corresponds to the mass of the scalar field ϕ taking on the values $M^2 \geq -(N-1)^2/4$. We note that this value corresponds to the generalisation of the B-F bound [12, 22] for the higher–dimensional cases.

To find the self–adjoint extensions of the operator A_{Rad} we begin by finding square–integrable solutions of the equation $A^\dagger_{\text{Rad}} r_{\omega,l_1} = \omega^2 r_{\omega_l 1}$, that is,

$$-\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} r_{\omega,l_1}(\rho) + \left[\frac{\nu^2 - 1/4}{\cos^2\rho} + \frac{\sigma^2 - 1/4}{\sin^2\rho} - \omega^2\right] r_{\omega,l_1}(\rho) = 0\,. \tag{7.31}$$

We note that the functional form of the general solutions of Eq. (7.31) depends on the values of the constants σ and ν due to their roles as coefficients of the singular terms in the differential equation. In particular, Eq. (7.31) can be taken to the form of a hypergeometric equation whose second linearly independent solution depends on whether or not the constant σ is an integer or a *half–integer* [23, Eqs. 15.10.6–15.10.10]. Thus, we follow the analysis of Ref. [22, Section 3.2] to write the general solution of the differential equation in terms of the hypergeometric functions as

$$\begin{aligned} r_{\omega,l_1}(\rho) =& C_1(\sin\rho)^{\sigma+\frac{1}{2}}(\cos\rho)^{\nu+\frac{1}{2}} F\left(\zeta^\omega_{\nu,\sigma}, \zeta^{-\omega}_{\nu,\sigma}; 1+\sigma; \sin^2\rho\right) \\ &+ C_2(\sin\rho)^{\frac{1}{2}-\sigma}(\cos\rho)^{\nu+\frac{1}{2}} \mathcal{F}_{\nu,\sigma}(\rho)\,, \end{aligned} \tag{7.32}$$

where the quantities $\zeta^\omega_{\nu,\sigma}$ are defined by

$$\zeta^\omega_{\nu,\sigma} := \frac{1}{2}(1 + \nu + \sigma + \omega)\,, \tag{7.33}$$

and where $C_1, C_2 \in \mathbb{C}$ are arbitrary constants. The function $\mathcal{F}_{\nu,\sigma}$ is given by the hypergeometric function

$$\mathcal{F}_{\nu,\sigma}(\rho) = F\left(\zeta^\omega_{\nu+1,-\sigma}, \zeta^{-\omega}_{\nu+1,-\sigma}; 1-\sigma; \sin^2\rho\right)\,, \tag{7.34}$$

if $\sigma \notin \mathbb{N}_0$, and by [23, Eq. 15.10.7b]

$$\begin{aligned}\mathcal{F}_{\nu,\sigma}(\rho) = \ln\left(\sin^2\rho\right) F\left(\zeta^{\omega}_{\nu,\sigma}, \zeta^{-\omega}_{\nu,\sigma}; 1+\sigma; \sin^2\rho\right) + \sum_{k=0}^{\infty} \frac{(\zeta^{\omega}_{\nu,\sigma})_k (\zeta^{-\omega}_{\nu,\sigma})_k}{(1+\sigma)_k k!} (\sin\rho)^{2k} \Big[\psi(\zeta^{\omega}_{\nu,\sigma}+k) \\ + \psi(\zeta^{-\omega}_{\nu,\sigma}+k) - \psi(\sigma+k+1) - \psi(k+1)\Big] \\ - \sum_{k=1}^{\sigma} \frac{(-1)^k \sigma!(k-1)!}{(\sigma-k)!(1-\zeta^{\omega}_{\nu,\sigma})_k (1-\zeta^{-\omega}_{\nu,\sigma})_k} (\sin\rho)^{-2k}\,, \end{aligned} \tag{7.35}$$

if $\sigma \in \mathbb{N}$. For $\sigma = 0$ the function $\mathcal{F}_{\nu,0}$ is given by Eq. (7.35) with the last sum understood to be zero.

So far, the analysis of the operator A_{Rad} has been analogous to that of the operators T of Chapter 5 and A of Chapter 6, including the differential equations for their respective eigenvalue problems and their general solutions. However, for the particular case of AdS$_N$ with $N \geq 3$, the general solution in Eq. (7.32) needs to be restricted at this point in order to be a physically acceptable radial component of the Klein–Gordon field. Indeed, before proceeding any further we must verify that Eq. (7.32) defines a square–integrable function at the coordinate origin of the static slice Σ which corresponds to the value $\rho = 0$. It is worth pointing out that this regularity condition at $\rho = 0$ is already a restriction on the domain of the operator A_{Rad}. We restrict the domain of A_{Rad} before we analyse its self–adjoint extensions for the following reason. If we apply the theory of self–adjoint extensions to the operator A_{Rad} with natural domain $C_0^{\infty}(0, \pi/2)$ without any restrictions at $\rho = 0$, then Eq. (7.27) implies that the self–adjoint extensions of A_{Rad} can be put in correspondence with the boundary data at $\rho = 0$ and $\rho = \pi/2$, just as the two–dimensional case. However, that would imply that we are regarding $\rho = 0$ as an endpoint of the boundary. Since the coordinate origin $\rho = 0$ has no special bearing for the theory, simply being a consequence of the chart we work with, there is no (physical) reason to consider the SABCs of the operator A_{Rad} which depend on the data at $\rho = 0$.

By direct inspection of Eqs. (7.32), (7.34) and (7.35), it is clear that the function with coefficient C_2 is not square–integrable at $\rho = 0$ for $\sigma \geq 1$ due to the prefactor $(\sin\rho)^{1/2-\sigma}$. Hence, for these values of σ we must impose the condition $C_2 = 0$ on the general solution (7.32). (Note that this issue was not present in the case of the spatial component of the scalar field in AdS$_2$ since none of the functions in Eq. (6.14) were ill–defined at $\rho = 0$.) The cases with $\sigma = 0$ (occurring only for a scalar field in AdS$_3$ with $l_1 = 0$) and $\sigma = 1/2$ (occurring only for a scalar field in AdS$_4$ with $l_1 = 0$) do not yield physically acceptable solutions unless $C_2 = 0$, as pointed out by Ishibashi and Wald [22, Section 3.2]. Therefore, from this point onwards we will set $C_2 = 0$ for all $N \geq 3$ and all values of σ, and consider the general solution of Eq. (7.31) given by

$$r_{\omega,l_1}(\rho) = N_{l_1,\omega} (\sin\rho)^{\sigma+\frac{1}{2}} (\cos\rho)^{\nu+\frac{1}{2}} F\left(\zeta^{\omega}_{\nu,\sigma}, \zeta^{-\omega}_{\nu,\sigma}; 1+\sigma; \sin^2\rho\right)\,, \tag{7.36}$$

where $N_{l_1,\omega}$ is a normalisation constant.

7.2 Self-adjoint extensions of the radial operator

Now that we have found the general solution of the radial component of the Klein–Gordon equation, we will apply the theory of Chapert 5 to find the positive self–adjoint extensions of the operator A_{Rad} defined in Eq. (7.21) with domain given by $\mathrm{Dom}(A_{\text{Rad}}) = C_c^\infty(0,\pi/2)$. We will take a similar approach to that of the scalar field on two–dimensional anti–de Sitter spacetime presented in Chapter 6. Thus, we begin by finding solutions in $L^2[0,\pi/2]$ of the equation $A_{\text{Rad}}^\dagger r_\pm = \pm 2ir_\pm$ which defines the deficiency spaces $\mathscr{K}_\pm$ of the operator A_{Rad} as given by Eq. (5.2) with $\lambda = 2$. The number of linearly independent solutions to these equations gives the deficiency indices $n_\pm$ which in turn determine whether or not the operator A_{Rad} admits any self–adjoint extensions by means of Theorem 5.1.3. The following calculations, up to and including Eq. (7.46) are adapted from the analysis in Ref. [22, Section 3.3].

Thus, let us consider the equations

$$-\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} r_\pm(\rho) + \left[\frac{\nu^2-1/4}{\cos^2\rho} + \frac{\sigma^2-1/4}{\sin^2\rho}\right] r_\pm(\rho) = \pm 2ir_\pm(\rho)\,, \tag{7.37}$$

for $r_\pm \in L^2(0,\pi/2)$ and $\nu \geq 0$. Since $\sigma,\nu\in\mathbb{R}$, it follows that if r_+ is a solution to the positive eigenvalue equation, then the function $r_- := \overline{r_+}$ is a solution to the negative eigenvalue equation. Now, Eq. (7.37) for r_+ is of the same form of Eq. (7.31) for r_{ω,l_1} with $\omega^2 = 2i$, that is, $\omega = 1+i$. Hence, the general solutions to Eq. (7.37) which are square–integrable at $\rho = 0$ are given by

$$r_\pm(\rho) = N_\pm (\sin\rho)^{\sigma+1/2}(\cos\rho)^{\nu+1/2} F\left(\zeta_{\nu,\sigma}^{1\pm i}, \zeta_{\nu,\sigma}^{-1\mp i}; 1+\sigma; \sin^2\rho\right)\,, \tag{7.38}$$

where we have used Eq. (7.33), and $N_\pm \in \mathbb{C}$ are normalisation constants satisfying $\overline{N_+} = N_-$.

We now determine for which values of the mass parameter $\nu\in\mathbb{R}$ the functions in Eq. (7.38) are square–integrable at $\rho = \pi/2$. To do this, we analyse the asymptotic behaviour of $r_\pm$ as $\rho\to\pi/2$ by transforming the argument $\sin^2\rho$ of the hypergeometric function in Eq. (7.38) to $\cos^2\rho$. Since the functions $r_\pm$ in Eq. (7.38) can be obtained by taking $\omega = 1\pm i$ in Eq. (7.32), we can analyse the asymptotic expansions of r_{ω,l_1} for arbitrary $\omega\in\mathbb{C}$ and then specialise to the cases of interest [22, Section 3.3].

If $\nu\notin\mathbb{N}_0$, then we use the transformation of the hypergeometric function [23, Eq. 15.8.4] to write r_{ω,l_1} as

$$\begin{aligned} r_{\omega,l_1}(\rho) =& N_{\omega,l_1}(\sin\rho)^{\sigma+\frac{1}{2}}\Big[A_{-\nu}^{\omega}(\cos\rho)^{\nu+\frac{1}{2}} F\left(\zeta_{\nu,\sigma}^{\omega},\zeta_{\nu,\sigma}^{-\omega};1+\nu;\cos^2\rho\right) \\ &+ A_{\nu}^{\omega}(\cos\rho)^{-\nu+\frac{1}{2}} F\left(\zeta_{-\nu,\sigma}^{\omega},\zeta_{-\nu,\sigma}^{-\omega};1-\nu;\cos^2\rho\right)\Big]\,, \end{aligned} \tag{7.39}$$

where we have defined

$$A_\nu^\omega := \frac{\Gamma(1+\sigma)\Gamma(\nu)}{\Gamma(\zeta_{\nu,\sigma}^{\omega})\Gamma(\zeta_{\nu,\sigma}^{-\omega})}\,. \tag{7.40}$$

For $\nu = m \in \mathbb{N}_0$, the transformation of the hypergeometric function leading to Eq. (7.39) is not well defined. Instead, we can use the limiting case of the transformation [23, Eqs. 15.8.10 and 15.8.12] to obtain

$$\begin{aligned} r_{\omega,l_1}(\rho) =& N_{\omega,l_1}(\sin\rho)^{\sigma+\frac{1}{2}}\left[H^{\omega}_{m}(\cos\rho)^{m+\frac{1}{2}}\sum_{k=0}^{\infty}\frac{(\zeta^{\omega}_{m,\sigma})_k(\zeta^{-\omega}_{m,\sigma})_k}{k!(m+k)!}\left[\ln(\cos^2\rho)+h_{\omega}(k)\right](\cos\rho)^{2k}\right. \\ &\left. +(-1)^{m-1}H^{\omega}_{-m}(\cos\rho)^{-m+\frac{1}{2}}\sum_{k=0}^{m-1}\frac{(\zeta^{\omega}_{-m,\sigma})_k(\zeta^{-\omega}_{-m,\sigma})_k}{k!(1-m)_k}(\cos\rho)^{2k}\right], \end{aligned} \tag{7.41}$$

where we have defined the quantities

$$H^{\omega}_{m} := \frac{(-1)^{m+1}\Gamma(1+\sigma)}{\Gamma(\zeta^{\omega}_{-m,\sigma})\Gamma(\zeta^{-\omega}_{-m,\sigma})}, \tag{7.42}$$

and

$$h_{\omega}(k) := \psi(\zeta^{\omega}_{m,\sigma}+k)+\psi(\zeta^{-\omega}_{m,\sigma}+k)-\psi(k+m+1)-\psi(k+1). \tag{7.43}$$

Defining the variable $\widetilde{\rho} = \pi/2-\rho$, we see that $\cos\rho \sim \widetilde{\rho}$ as $\widetilde{\rho} \to 0$. Let us first consider the case for which $\nu \notin \mathbb{N}_0$. The asymptotic expansion of Eq. (7.39) as $\widetilde{\rho} \to 0$, is given by

$$r_{\omega,l_1}(\rho) \sim N_{\omega,l_1}\left[A^{\omega}_{-\nu}\widetilde{\rho}^{\,\nu+\frac{1}{2}}+A^{\omega}_{\nu}\widetilde{\rho}^{\,-\nu+\frac{1}{2}}\right]\left[1+O\left(\widetilde{\rho}^{\,2}\right)\right]. \tag{7.44}$$

Now, if $\nu > 1$, the second term in Eq. (7.44) is not square–integrable near the spatial boundary $\rho = \pi/2$ since the exponent of $\tilde{\rho}$ satisfies $-\nu+1/2 < -1/2$. Hence, for $\omega = 1 \pm i$ the only acceptable solution is obtained by setting $N_{1\pm i,l_1} = 0$. This means that, for this range of ν, the positive deficiency subspace $\mathscr{K}_+$ (and thus, $\mathscr{K}_-$ as well) is zero–dimensional and we have $n_\pm = 0$. Hence, as a consequence of Theorem 5.1.3, the only self–adjoint extension of A_{Rad} is its closure, $\overline{A_{\text{Rad}}}$. On the other hand, if $0 < \nu < 1$, Eq. (7.44) implies that r_{ω,l_1} is a square–integrable solution near the boundary. Therefore, we have $n_\pm = 1$ and, thus, Theorem 5.1.3 implies the existence of a family of self–adjoint extensions $(A_{\text{Rad}})_U$ of A_{Rad} parametrised by a unitary map $U : \mathscr{K}_+ \to \mathscr{K}_-$. Since both deficiency spaces are one–dimensional, Eq. (5.20) implies that we can represent the map U as

$$U : r_+ \mapsto Ur_+ := e^{i\alpha}r_-, \tag{7.45}$$

with $\alpha \in [-\pi,\pi)$, and $r_- \in \mathscr{K}_-$.

Similarly, for the case with $\nu = m \in \mathbb{N}_0$, the asymptotic behaviour of r_{ω,l_1} in Eq. (7.41) as $\widetilde{\rho} \to 0$ is given by

$$r_{\omega,l_1}(\rho) \sim N_{\omega,l_1}\left[H^{\omega}_{m}\widetilde{\rho}^{\,m+\frac{1}{2}}\ln(\widetilde{\rho}^{\,2})+(-1)^{m-1}H^{\omega}_{-m}\widetilde{\rho}^{\,-m+\frac{1}{2}}\right]\left[1+O\left(\widetilde{\rho}^{\,2}\right)\right]. \tag{7.46}$$

From this expression it follows that if $m \geq 1$, then the second term in Eq. (7.46) is not square–integrable for any $\omega \in \mathbb{C}$, since the exponent of $\tilde{\rho}$ satisfies $-m+1/2 < -1/2$. Hence, the deficiency spaces are again zero-dimensional, and we have $n_\pm = 0$. If $m = 0$, the second term is absent, and the term proportional to $\widetilde{\rho}^{\,1/2}\ln(\widetilde{\rho}^{\,2})$ goes to zero as $\widetilde{\rho} \to 0$. This means that for $m = 0$, the solution in Eq. (7.46) is square–integrable, and the deficiency spaces are both one–dimensional. By Theorem (5.1.3), the only self–adjoint extension of A_{Rad} with $\nu = m \geq 1$ is its closure, $\overline{A_{\text{Rad}}}$, while if $m = 0$, there exists a family of self–adjoint extensions of A_{Rad} parametrised by $\alpha \in (-\pi, \pi]$ through Eq. (7.45).

From the analysis above, we have found that the admissible self–adjoint extensions of the operator A_{Rad} depend on the value of the mass parameter ν. Therefore, it will be convenient to separate the rest of the analysis of the self–adjoint extensions of A_{Rad} into the following three cases:

1. $\nu \geq 1$,
2. $0 < \nu < 1$,
3. $\nu = 0$.

For Case 1 we have $n_\pm = 0$. Hence, Theorem 5.1.3 implies that the unique self–adjoint extension of A_{Rad} is its closure $\overline{A_{\text{Rad}}}$. A simple argument using the fact that $\overline{A_{\text{Rad}}} = (A^\dagger_{\text{Rad}})^\dagger$ [35] shows that if $r \in \text{Dom}(\overline{A_{\text{Rad}}})$, then r must satisfy the boundary condition

$$\left[(\cos\rho)^{\nu-1/2} r(\rho)\right]\Big|_{\rho=\frac{\pi}{2}} = 0\,. \tag{7.47}$$

We leave the proof of this result in Appendix F. Thus, the boundary condition in Eq. (7.47) uniquely determines the self–adjoint extension of A_{Rad} for this particular case.

For Case 2 we have $n_\pm = 1$, and thus, the self–adjoint extensions of A_{Rad} are parametrised by the unitary maps in Eq. (7.45). Theorem 5.1.3 together with Eqs. (5.17) and (7.45) provide the description of the self–adjoint extensions $(A_{\text{Rad}})_\alpha$ in terms of their domains given by

$$\text{Dom}\left((A_{\text{Rad}})_\alpha\right) = \left\{ r_0 + r_+ + e^{i\alpha}\overline{r_+} \,\middle|\, r_0 \in \text{Dom}(\overline{A_{\text{Rad}}}), r_+ \in \mathscr{K}_+ \right\}\,, \tag{7.48}$$

where the action of $(A_{\text{Rad}})_\alpha$ on $\text{Dom}((A_{\text{Rad}})_\alpha)$ is given by

$$(A_{\text{Rad}})_\alpha(r_0 + r_+ + e^{i\alpha}\overline{r_+}) = A^\dagger_{\text{Rad}} r_0 + 2ir_+ - 2ie^{i\alpha}\overline{r_+}\,. \tag{7.49}$$

Using the prescription presented in Section 5.1, we find the boundary conditions that elements in $\text{Dom}\left((A_{\text{Rad}})_\alpha\right)$ satisfy by identifying the corresponding maximal subspace $\mathscr{S} \subseteq \text{Dom}(A^\dagger_{\text{Rad}})$ on which $A^\dagger_{\text{Rad}}$ is symmetric for all elements in $\text{Dom}((A_{\text{Rad}})_\alpha)$. This is achieved by restricting the functions in $\text{Dom}(A^\dagger_{\text{Rad}})$ via Eq. (5.23). We write this condition for $r \in \text{Dom}(A^\dagger_{\text{Rad}})$ as

$$\langle r, (A_{\text{Rad}})_\alpha s\rangle_{\text{Rad}} - \left\langle A^\dagger_{\text{Rad}} r, s\right\rangle_{\text{Rad}} = 0 \tag{7.50}$$

for all $s \in \text{Dom}((A_{\text{Rad}})_\alpha)$. Equation (7.48) allows us to write $s = s_0 + s_+ + e^{-i\alpha}s_+$ with $s_0 \in \text{Dom}(\overline{A_{\text{Rad}}})$ and $s_+ \in \mathscr{K}_+$. Now we use Eq. (7.27), to rewrite this condition as

$$r(\pi/2)\left[\overline{s'_+(\pi/2)} + e^{-i\alpha}s'_+(\pi/2)\right] - r'(\pi/2)\left[\overline{s_+(\pi/2)} + e^{-i\alpha}s_+(\pi/2)\right] = 0\,, \tag{7.51}$$

where we have used Eq. (7.49) and the fact that

$$\langle r, (A_{\text{Rad}})_\alpha s_0 \rangle_{\text{Rad}} - \left\langle A^\dagger_{\text{Rad}} r, s_0 \right\rangle_{\text{Rad}} = 0\,, \tag{7.52}$$

by the symmetry of $A^\dagger_{\text{Rad}}$. A straightforward calculation shows that Eq. (7.51) is equivalent to

$$\begin{aligned} 0 =& \widetilde{r}^{(\nu)}(\pi/2)\left[\overline{D\widetilde{s_+}^{(\nu)}(\pi/2)} + e^{-i\alpha} D\widetilde{s_+}^{(\nu)}(\pi/2)\right] \\ &- D\widetilde{r}^{(\nu)}(\pi/2)\left[\overline{\widetilde{s_+}^{(\nu)}(\pi/2)} + e^{-i\alpha}\widetilde{s_+}^{(\nu)}(\pi/2)\right]\,, \end{aligned} \tag{7.53}$$

where the functions $\widetilde{r}^{(\nu)}$, $D\widetilde{r}^{(\nu)}$, are defined by

$$\widetilde{r}^{(\nu)}(\rho) := (\cos\rho)^{\nu-\frac{1}{2}} r(\rho)\,, \tag{7.54a}$$

$$D\widetilde{r}^{(\nu)}(\rho) := (\cos\rho)^{1-2\nu}\frac{\mathrm{d}}{\mathrm{d}\rho}\widetilde{r}^{(\nu)}(\rho)\,, \tag{7.54b}$$

for $0 < \nu < 1$. Similarly to the case of a scalar field in AdS$_2$, the functions in Eqs. (7.54) are defined in order to extract the boundary behaviour of the solutions at $\rho = \pi/2$. (Compare with Eqs. (6.35) and (6.36).) The functions $\widetilde{s_+}^{(\nu)}$ and $D\widetilde{s_+}^{(\nu)}$ are defined analogously using Eq. (7.54). Since the function s_+ admits the transformation given by Eq. (7.39), the definitions in Eq. (7.54) imply that

$$\widetilde{s_+}^{(\nu)}(\pi/2) = A^{1+i}_{\nu}\,, \qquad D\widetilde{s_+}^{(\nu)}(\pi/2) = -2\nu A^{1+i}_{-\nu}\,. \tag{7.55}$$

Hence, by substituting this into Eq. (7.53) we obtain

$$2\nu\widetilde{r}^{(\nu)}(\pi/2)\left[\overline{A^{1+i}_{-\nu}} + e^{-i\alpha}A^{1+i}_{-\nu}\right] + D\widetilde{r}^{(\nu)}(\pi/2)\left[\overline{A^{1+i}_{\nu}} + e^{-i\alpha}A^{1+i}_{\nu}\right] = 0\,. \tag{7.56}$$

A simple rearrangement shows that Eq. (7.56) can be rewritten as

$$\begin{aligned} &\left[\overline{A^{1+i}_{\nu}} + 2i\nu\overline{A^{1+i}_{-\nu}} + e^{-i\alpha}(A^{1+i}_{\nu} + 2i\nu A^{1+i}_{-\nu})\right]\left(D\widetilde{r}^{(\nu)}(\pi/2) - i\widetilde{r}^{(\nu)}(\pi/2)\right) \\ &= -\left[\overline{A^{1+i}_{\nu}} - 2i\nu\overline{A^{1+i}_{-\nu}} + e^{-i\alpha}(A^{1+i}_{\nu} - 2i\nu A^{1+i}_{-\nu})\right]\left(D\widetilde{r}^{(\nu)}(\pi/2) + i\widetilde{r}^{(\nu)}(\pi/2)\right)\,. \end{aligned} \tag{7.57}$$

Now, from the definition of A^{ω}_{ν} in Eq. (7.40) it follows that the expressions inside square brackets in Eq. (7.57) are never zero. Hence, we can write Eq. (7.57) as

$$\left(D\widetilde{r}^{(\nu)}(\pi/2) - i\widetilde{r}^{(\nu)}(\pi/2)\right) = \mathcal{U}_\alpha\left(D\widetilde{r}^{(\nu)}(\pi/2) + i\widetilde{r}^{(\nu)}(\pi/2)\right)\,, \tag{7.58}$$

where we have defined

$$\mathcal{U}_\alpha := -\frac{\overline{A^{1+i}_{\nu}} - 2i\nu\overline{A^{1+i}_{-\nu}} + e^{-i\alpha}(A^{1+i}_{\nu} - 2i\nu A^{1+i}_{-\nu})}{\overline{A^{1+i}_{\nu}} + 2i\nu\overline{A^{1+i}_{-\nu}} + e^{-i\alpha}(A^{1+i}_{\nu} + 2i\nu A^{1+i}_{-\nu})}\,. \tag{7.59}$$

Substituting Eq. (7.40) into Eq. (7.59), and applying the identity $\Gamma(x+1) = x\Gamma(x)$ for the gamma function, we obtain a simpler expression for $\mathcal{U}_\alpha$, given by

$$\mathcal{U}_\alpha = -\frac{C_{\nu,\sigma}e^{i\alpha} + \overline{D_{\nu,\sigma}}}{D_{\nu,\sigma}e^{i\alpha} + \overline{C_{\nu,\sigma}}} \tag{7.60}$$

where we have defined

$$C_{\nu,\sigma} = (\sigma+\nu+i)\Gamma(\nu)\left|\Gamma(\zeta^{1+i}_{-\nu,\sigma})\right|^2 + 2i(\sigma-\nu+i)\Gamma(1-\nu)\left|\Gamma(\zeta^{1+i}_{\nu,\sigma})\right|^2, \tag{7.61a}$$

$$D_{\nu,\sigma} = (\sigma+\nu-i)\Gamma(\nu)\left|\Gamma(\zeta^{1+i}_{-\nu,\sigma})\right|^2 + 2i(\sigma-\nu-i)\Gamma(1-\nu)\left|\Gamma(\zeta^{1+i}_{\nu,\sigma})\right|^2. \tag{7.61b}$$

Since σ, ν and α are real parameters, Eq. (7.60) implies that $|\mathcal{U}_\alpha| = 1$, and we write $\mathcal{U}_\alpha = e^{iu}$ for some $u \in (-\pi,\pi]$. Now, it is clear from Eq. (7.60) and from the fact that $|C_{\nu,\sigma}|^2 \neq |D_{\nu,\sigma}|^2$ that, given $\alpha_1,\alpha_2 \in (-\pi,\pi]$ with $\alpha_1 \neq \alpha_2$, we have $\mathcal{U}_{\alpha_1} \neq \mathcal{U}_{\alpha_2}$. Hence, given $\alpha \in (-\pi,\pi]$ there exists a unique $u \in (-\pi,\pi]$ such that $\mathcal{U}_\alpha = e^{iu}$. Similarly, given $u \in (-\pi,\pi]$ we can find

$$e^{i\alpha} = -\frac{D_{\nu,\sigma} + \overline{C_{\nu,\sigma}}e^{iu}}{\overline{D_{\nu,\sigma}}e^{iu} + C_{\nu,\sigma}}. \tag{7.62}$$

Thus, given $u_1,u_2 \in (-\pi,\pi]$, with $u_1 \neq u_2$, we have $e^{i\alpha_1} \neq e^{i\alpha_2}$ since $|C_{\nu,\sigma}|^2 \neq |D_{\nu,\sigma}|^2$. Hence, the correspondence $\alpha \mapsto \mathcal{U}_\alpha$ is one–to–one. This means that Eq. (7.58) is a one–dimensional analogue of Eq. (5.31), and thus, can be rearranged into

$$\left(1-e^{iu}\right) D\widetilde{r}^{\,(\nu)}(\pi/2) = i\left(1+e^{iu}\right)\widetilde{r}^{\,(\nu)}(\pi/2). \tag{7.63}$$

The correspondence between α and u implies that Eq. (7.63) is, in the terminology introduced in Chapter 5, a family of self–adjoint boundary conditions which completely determines the domains of the self–adjoint extensions of the operator A_{Rad}. We note that if $u=0$, Eq. (7.63) describes a **generalised Dirichlet boundary condition**, that is,

$$\left[(\cos\rho)^{\nu-\frac{1}{2}}r(\rho)\right]\Big|_{\rho=\frac{\pi}{2}} = 0. \tag{7.64}$$

If $u=\pi$ then we have a **generalised Neumann boundary condition**, given by

$$\left[(\cos\rho)^{1-2\nu}\frac{\mathrm{d}}{\mathrm{d}\rho}\left((\cos\rho)^{\nu-\frac{1}{2}}r(\rho)\right)\right]\Big|_{\rho=\frac{\pi}{2}} = 0. \tag{7.65}$$

All other values of u give **generalised Robin boundary conditions**, which we write as

$$\sin\frac{u}{2}\left[(\cos\rho)^{1-2\nu}\frac{\mathrm{d}}{\mathrm{d}\rho}\left((\cos\rho)^{\nu-\frac{1}{2}}r(\rho)\right)\right]\Big|_{\rho=\frac{\pi}{2}} + \cos\frac{u}{2}\left[(\cos\rho)^{\nu-\frac{1}{2}}r(\rho)\right]\Big|_{\rho=\frac{\pi}{2}} = 0. \tag{7.66}$$

Now we perform a similar analysis for Case 3, that is, for $\nu=0$. For this value we have $n_\pm = 1$ and, thus, by Theorem 5.1.3, we have that the self–adjoint extensions of A_{Rad} are also parametrised by a real number $\alpha \in (-\pi,\pi]$. The domain of the self–adjoint extension $(A_{\text{Rad}})_\alpha$ is given by Eq. (7.48), but with the solutions of the deficiency spaces, $r_\pm$ given instead by Eq. (7.41) with $\nu = m = 0$. Thus, the problem of finding the maximal subspace $\mathscr{S}$, on which $A^\dagger_{\text{Rad}}$ is symmetric, characterising the domain of $(A_{\text{Rad}})_\alpha$ follows analogously to the treatment of Case 2 above. Indeed, the condition for $r \in \mathrm{Dom}(A^\dagger_{\text{Rad}})$ given by Eq. (7.50) is equivalent to finding the subspace $\mathscr{S}$ for which $A^\dagger_{\text{Rad}}$ is a symmetric operator for all elements $\mathrm{Dom}((A_{\text{Rad}})_\alpha)$. However, for this case, Eq. (7.50) reduces instead to

$$\begin{aligned} 0 =& \widetilde{r}^{\,(0)}(\pi/2)\left[\overline{D\widetilde{s_+}^{\,(0)}(\pi/2)} + e^{-i\alpha} D\widetilde{s_+}^{\,(0)}(\pi/2)\right] \\ &- D\widetilde{r}^{\,(0)}(\pi/2)\left[\overline{\widetilde{s_+}^{\,(0)}(\pi/2)} + e^{-i\alpha}\widetilde{s_+}^{\,(0)}(\pi/2)\right], \end{aligned} \tag{7.67}$$

where the functions $\widetilde{r}^{\,(0)}$, $D\widetilde{r}^{\,(0)}$ are defined in a way such that the asymptotic behaviour at $\rho = \pi/2$ is finite. These are given by

$$\widetilde{r}^{\,(0)}(\rho) := \frac{(\cos\rho)^{-1/2}}{\ln(\cos^2\rho)} r(\rho)\,, \tag{7.68a}$$

$$D\widetilde{r}^{\,(0)}(\rho) := \cos\rho\left[\ln(\cos^2\rho)\right]^2 \frac{\mathrm{d}}{\mathrm{d}\rho}\widetilde{r}^{\,(0)}(\rho)\,, \tag{7.68b}$$

and the functions $\widetilde{s_+}^{\,(0)}$ and $D\widetilde{s_+}^{\,(0)}$ are defined analogously. (Compare with functions defined for Case 1 by Eq. (7.54)) We take the function s_+ given in the form of Eq. (7.41) and use Eq. (7.68) to obtain

$$\widetilde{s_+}^{\,(0)}(\pi/2) = H_0^{1+i}\,, \qquad D\widetilde{s_+}^{\,(0)}(\pi/2) = -2H_0^{1+i} h_{1+i}(0)\,, \tag{7.69}$$

where H_0^{1+i} and $h_{1+i}(0)$ are given by Eqs. (7.42) and (7.43), respectively.

We substitute Eq. (7.69) into Eq. (7.67) to obtain

$$\begin{aligned} 0 =& 2\widetilde{r}^{\,(0)}(\pi/2)\left[\overline{H_0^{1+i} h_{1+i}(0)} + e^{-i\alpha} H_0^{1+i} h_{1+i}(0)\right] \\ &+ D\widetilde{r}^{\,(0)}(\pi/2)\left[\overline{H_0^{1+i}} + e^{-i\alpha} H_0^{1+i}\right], \end{aligned} \tag{7.70}$$

which, after rearranging appropriately can be taken to the form

$$\begin{aligned} &-\left[\overline{H_0^{1+i}}\left(1 + 2i\overline{h_{1+i}(0)}\right) + e^{-i\alpha} H_0^{1+i}\left(1 + 2ih_{1+i}(0)\right)\right]\left(D\widetilde{r}^{\,(0)}(\pi/2) - i\widetilde{r}^{\,(0)}(\pi/2)\right) = \\ &\left[\overline{H_0^{1+i}}\left(1 - 2i\overline{h_{1+i}(0)}\right) + e^{-i\alpha} H_0^{1+i}\left(1 - 2ih_{1+i}(0)\right)\right]\left(D\widetilde{r}^{\,(0)}(\pi/2) + i\widetilde{r}^{\,(0)}(\pi/2)\right). \end{aligned} \tag{7.71}$$

The definitions in Eqs. (7.42) and (7.43) of the quantities H_0^{1+i} and $h_{1+i}(0)$ imply that the expressions in square brackets in Eq. (7.71) are never zero. Thus, we may write this equation as Eq. (7.58) with $\nu = 0$, and $\mathcal{U}_\alpha$ given instead by

$$\mathcal{U}_\alpha = -\frac{\overline{H_0^{1+i}}\left(1 + 2i\overline{h_{1+i}(0)}\right) + e^{-i\alpha} H_0^{1+i}\left(1 + 2ih_{1+i}(0)\right)}{\overline{H_0^{1+i}}\left(1 - 2i\overline{h_{1+i}(0)}\right) + e^{-i\alpha} H_0^{1+i}\left(1 - 2ih_{1+i}(0)\right)}\,. \tag{7.72}$$

From this expression it is clear that the numerator is the complex conjugate of the denominator and, thus, we have $|\mathcal{U}_\alpha| = 1$. Finally, a straightforward calculation similar to that involving Eq. (7.59) shows that the correspondence $\alpha \mapsto \mathcal{U}_\alpha$ is one–to–one. Hence, given $\alpha \in (-\pi, \pi]$ there is a unique $u \in (-\pi, \pi]$ such that $\mathcal{U}_\alpha = e^{iu}$. Therefore, the condition in Eq. (7.71) for the function $r \in \mathrm{Dom}(A^\dagger_{\text{Rad}})$ completely characterises $\mathrm{Dom}((A_{\text{Rad}})_\alpha)$. Rearranging this equation, we obtain the self–adjoint boundary condition

$$\left(1 - e^{iu}\right) D\widetilde{r}^{\,(0)}(\pi/2) = i\left(1 + e^{iu}\right)\widetilde{r}^{\,(0)}(\pi/2)\,, \tag{7.73}$$

parametrised by u. Since Eq. (7.73) is of the same form of Eq. (7.63), we have the same types of boundary conditions as for Case 2, that is, a generalised Dirichlet boundary condition,

$$\left[\frac{(\cos\rho)^{-1/2}}{\ln(\cos^2\rho)}r(\rho)\right]\Bigg|_{\rho=\frac{\pi}{2}} = 0\,, \tag{7.74}$$

for $u = 0$, a generalised Neumann boundary condition,

$$\left[\cos\rho\left[\ln(\cos^2\rho)\right]^2\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{(\cos\rho)^{-1/2}}{\ln(\cos^2\rho)}r(\rho)\right)\right]\Bigg|_{\rho=\frac{\pi}{2}} = 0\,, \tag{7.75}$$

for $u = \pi$, and a family of generalised Robin boundary conditions for all other values of u given by

$$\begin{aligned}&\sin\frac{u}{2}\left[\cos\rho\left[\ln(\cos^2\rho)\right]^2\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{(\cos\rho)^{-1/2}}{\ln(\cos^2\rho)}r(\rho)\right)\right]\Bigg|_{\rho=\frac{\pi}{2}}\\ &+\cos\frac{u}{2}\left[\frac{(\cos\rho)^{-1/2}}{\ln(\cos^2\rho)}r(\rho)\right]\Bigg|_{\rho=\frac{\pi}{2}} = 0\,.\end{aligned} \tag{7.76}$$

7.3 INVARIANT SELF–ADJOINT BOUNDARY CONDITIONS

Now that we have found the family of self–adjoint boundary conditions parametrised by $u \in (-\pi, \pi]$, we will find for which values of u Eqs. (7.63) and (7.73) result in mode solutions of the Klein–Gordon equation which are invariant under the isometry group, $\mathrm{SO}(2, N-1)$. Similarly to the analysis in Chapter 6, we will take an infinitesimal approach and, thus, we will analyse the action of the Killing vector fields of AdS_N on the mode functions resulting from the self–adjoint boundary conditions obtained in the previous section.

Before we proceed with the analysis of the invariance of the self–adjoint boundary conditions, we point out some useful remarks regarding the infinitesimal action of $\mathrm{SO}(2, N-1)$ on the space of solutions of the Klein–Gordon equation given in terms of the decomposition in Eq. (7.2). This will be useful to simplify the task we are concerned with. First, we recall that the Killing vector fields of AdS_N, introduced in Chapter 3, are given by Eqs. (3.24) and (3.25). The decomposition into positive–frequency solutions for the Klein–Gordon equation given in Eq. (7.2) that we are considering corresponds to finding simultaneous eigenfunctions of the operator $\Box_{\mathrm{AdS}_N}$ and the static Killing vector field $\xi_0 = \partial_t$ in Eq. (3.24a). Hence, the space of solutions ϕ_{ω,l_1} of this particular form we are considering is invariant under ξ_0 and we clearly have $\xi_0\phi_{\omega,l_1} = -i\omega\phi_{\omega,l_1}$.

Now, the commutation relations in Eq. (3.21) for the generators of $\mathfrak{so}(2, N-1)$ imply that the Killing vector fields $\boldsymbol{J}_{ij}$, with $3 \le i, j \le N+1$ given by Eq. (3.24d) correspond to spatial rotations of the static slices Σ_t. Since we have considered solutions $\phi(t, \rho, \theta)$ of the form

$$\phi_{\omega,l_1}(t,\rho,\theta) = (\cot\rho)^{\frac{N-2}{2}}r_{\omega,l_1}(\rho)Y_l(\theta)e^{-i\omega t}\,, \tag{7.77}$$

we have that the Killing vector $\boldsymbol{J}_{ij}$ for any $3 \le i, j \le N+1$ acts on ϕ by

$$(\boldsymbol{J}_{ij}\phi_{\omega,l_1})(t,\rho,\theta) = (\cot\rho)^{\frac{N-2}{2}} r_{\omega,l_1}(\rho)\,[\boldsymbol{J}_{ij}Y_l]\,(\theta)e^{-i\omega t}\,, \tag{7.78}$$

where we have used the fact that the Killing vectors $\boldsymbol{J}_{ij}$ do not depend on the variables t and ρ, which follows from Eqs. (3.24) and (3.25). Now, since the hyperspherical harmonics are eigenfunctions of the Laplacian operator Δ_{N-2} of the $(N-2)$–sphere and this operator commutes with all $\boldsymbol{J}_{ij}$, the function $\boldsymbol{J}_{ij}Y_l$ is a linear combination of hyperspherical harmonics with the same label l_1 [85, Chapter 9]. (Indeed, the space of functions $Y_{l_1,\ldots,l_{N-2}}$ with fixed l_1 forms an irreducible representation of the group $\mathrm{SO}(N-1)$ whose infinitesimal generators are given by $\boldsymbol{J}_{ij}$, and the Casimir element corresponds to the Laplacian Δ_{N-2}.) From this result it follows that any function ϕ of the form of Eq. (7.2) is invariant under the action of the Killing vectors $\boldsymbol{J}_{ij}$, regardless of the boundary condition (7.63) the radial function r_{ω,l_1} defined by Eq. (7.19) obeys.

Finally, let us assume that the space of solutions ϕ_{ω,l_1} is invariant under the action of at least one of the two boost–like Killing vector fields, $\boldsymbol{K}_k$ in Eq. (3.24b) or $\boldsymbol{B}_k$ in Eq. (3.24c). Let us denote these Killing vectors collectively as $\boldsymbol{K}_{ik}$, with $i \in \{1,2\}$ and $k \in \{3,\ldots,N+1\}$, so that $\boldsymbol{K}_{1k} = \boldsymbol{K}_k$ and $\boldsymbol{K}_{2k} = \boldsymbol{B}_k$. This implies that we must have

$$[\boldsymbol{K}_{ik}\phi_{\omega,l_1}](t,\rho,\theta) = \sum_{\omega',l_1'} \mathcal{C}(\omega,\omega';l_1,l_1')R_{\omega',l_1'}(\rho)Y_{l'}(\theta)e^{-i\omega' t}\,, \tag{7.79}$$

for some coefficients $\mathcal{C}(\omega,\omega';l_1,l_1')$, and where the sum runs through the allowed eigenvalues ω and l_1. Now, the commutation relations of the elements in $\mathfrak{so}(2,N-1)$ given in Eq. (3.21) imply that the Killing vectors must satisfy

$$[\boldsymbol{K}_{ik},\boldsymbol{J}_{kl}] = \boldsymbol{K}_{il}\,. \tag{7.80}$$

Since the Killing vector $\boldsymbol{J}_{kl}$ leaves the space of solutions ϕ_{ω,l_1} invariant, and we have assumed that Eq. (7.79) holds, the function $\boldsymbol{J}_{kl}\circ\boldsymbol{K}_{ik}\phi_{\omega,l_1}$ must also be a linear combination of solutions $\phi_{\omega',l_1'}$. On the other hand, we have, by Eq. (7.80)

$$\begin{aligned}\boldsymbol{J}_{kl}\circ\boldsymbol{K}_{ik}\phi_{\omega,l_1} &= [\boldsymbol{J}_{kl},\boldsymbol{K}_{ik}]\,\phi_{\omega,l_1} + \boldsymbol{K}_{ik}\circ\boldsymbol{J}_{kl}\phi_{\omega,l_1}\,,\\ &= -\boldsymbol{K}_{il}\phi_{\omega,l_1} + \boldsymbol{K}_{ik}\circ\boldsymbol{J}_{kl}\phi_{\omega,l_1}\,.\end{aligned} \tag{7.81}$$

and by the invariance of the space of solutions under $\boldsymbol{J}_{kl}$, the last term is a linear combination of solutions as well. Hence, the function $\boldsymbol{K}_{il}\phi_{\omega,l_1}$ must be of the form of Eq. (7.79) and, thus, we conclude that if the space of solutions of the Klein–Gordon equation is invariant under the action of any of the boost–like Killing vector fields $\boldsymbol{K}_{ik}$, then it must also be invariant under the action of all Killing vector fields, that is, solutions are invariant under the infinitesimal action of $\mathrm{SO}(2,N-1)$.

Thus, we will determine which of the self–adjoint boundary conditions in Eqs. (7.63) and (7.73) result in invariant solutions spaces under the action of the Killing vector field $\boldsymbol{K}_3$ given by Eq. (3.26a), that is,

$$\boldsymbol{K}_3 = -\cos\theta_1\left(\sin t\,\sin\rho\,\frac{\partial}{\partial t} - \cos t\,\cos\rho\,\frac{\partial}{\partial\rho}\right) - \frac{\cos t\,\sin\theta_1}{\sin\rho}\frac{\partial}{\partial\theta_1}\,. \tag{7.82}$$

By the argument presented above, this requirement is sufficient to prove invariance of the resulting solution spaces under the infinitesimal action of SO(2, , $N-1$).

Let us consider the solution ϕ_{ω,l_1} given by Eq. (7.77) and calculate the function that results after applying $\boldsymbol{K}_3$ to this solution. First we note that the last term of Eq. (7.82) acts only on the hyperspherical harmonic $Y_{\boldsymbol{l}}(\theta)$. For the sake of clarity, we change the notation of the label $\boldsymbol{l} = (l_1, \ldots, l_{N-2})$ of the hyperspherical harmonics to $\boldsymbol{l} = (l_1, \tilde{l}\,)$. Thus, we begin by differentiating the function $\overline{P}^{l_2}_{1,l_1}(\theta_1)$ defined in Eq. (7.15) with respect to θ_1, to obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta_1}\overline{P}^{l_2}_{1,l_1}(\theta_1) =& c^{l_2}_{1,l_1}(\sin\theta_1)^{-\frac{N-4}{2}}\left[\frac{\mathrm{d}}{\mathrm{d}\theta_1}P^{-(l_2+(N-4)/2)}_{l_1+(N-4)/2}(\cos\theta_1)\right. \\
&\left.-\frac{N-4}{2}\cot\theta_1 P^{-(l_2+(N-4)/2)}_{l_1+(N-4)/2}(\cos\theta_1)\right], \\
=&\frac{c^{l_2}_{1,l_1}}{\sin\theta_1}\left[\frac{l_1(l_1+l_2+N-3)}{c^{l_2}_{1,l_1+1}(2l_1+N-3)}\overline{P}^{l_2}_{1,l_1+1}(\theta_1)\right. \\
&\left.-\frac{(l_1-l_2)(l_1+N-3)}{c^{l_2}_{1,l_1-1}(2l_1+N-3)}\overline{P}^{l_2}_{l,l_1-1}(\theta_1)\right],
\end{aligned}
\tag{7.83}
$$

where we have used the identities for the associated Legendre functions [23, Eqs. 14.10.3 and 14.10.5]

$$
x\,P^{-\mu}_{\nu} = \frac{\nu+\mu+1}{2\nu+1}P^{-\mu}_{\nu+1}(x) + \frac{\nu-\mu}{2\nu+1}P^{-\mu}_{\nu-1}(x)\,, \tag{7.84a}
$$

$$
\sqrt{1-x^2}\frac{\mathrm{d}}{\mathrm{d}x}P^{-\mu}_{\nu}(x) = -\frac{1}{\sqrt{1-x^2}}\left(\nu x\,P^{-\mu}_{\nu}(x) - (\nu-\mu)P^{-\mu}_{\nu-1}(x)\right), \tag{7.84b}
$$

Thus, using Eq. (7.83) and the definition of the functions $Y_{l_1,\tilde{l}}(\theta)$ in Eq. (7.12) we obtain the following:

$$
\sin\theta_1\frac{\partial}{\partial\theta_1}Y_{l_1,\tilde{l}}(\theta) = l_1 c_1 Y_{l_1+1,\tilde{l}}(\theta) - (l_1+N-3)c_2 Y_{l_1-1,\tilde{l}}(\theta)\,, \tag{7.85}
$$

where we have defined the constants

$$
\begin{aligned}
c_1 &:= \left(\frac{(l_1-l_2+1)(l_1+l_2+N-3)}{(2l_1+N-1)(2l_1+N-3)}\right)^{1/2}, \\
c_2 &:= \left(\frac{(l_1-l_2)(l_1+l_2+N-4)}{(2l_1+N-3)(2l_1+N-5)}\right)^{1/2}.
\end{aligned}
\tag{7.86}
$$

The action of $\boldsymbol{K}_3$ on the solution ϕ_{ω,l_1} in Eq. (7.77) is thus obtained from Eqs. (7.82) and (7.85), and reads

$$
\begin{aligned}
&[\boldsymbol{K}_3\phi_{\omega,l_1}](t,\rho,\theta) \\
&= (\cot\rho)^{\frac{N-2}{2}}\left[\cos\theta_1 Y_{l_1,\tilde{l}}(\theta)\left(\cos t\cos\rho\frac{\mathrm{d}}{\mathrm{d}\rho} + i\omega\sin t\sin\rho - \frac{N-2}{2}\frac{\cos t}{\sin\rho}\right)r_{\omega,l_1}(\rho)\right. \\
&\left.-\frac{\cos t}{\sin\rho}\left(l_1 c_1 Y_{l_1+1,\tilde{l}}(\theta) - (l_1+N-3)c_2 Y_{l_1-1,\tilde{l}}(\theta)\right)r_{\omega,l_1}(\rho)\right]e^{-i\omega t}\,.
\end{aligned}
\tag{7.87}
$$

Using the identity (7.84) for the Legendre functions, we obtain the relation between the hyperspherical harmonics given by

$$\cos\theta_1 Y_{l_1,\tilde{l}}(\theta) = c_1 Y_{l_1+1,\tilde{l}}(\theta) + c_2 Y_{l_1-1,\tilde{l}}(\theta)\,, \tag{7.88}$$

where the constants c_1 and c_2 are given by Eq. (7.86). Thus, we substitute Eq. (7.88) into Eq. (7.87) and we write the functions $\sin t$ and $\cos t$ in terms of exponentials to obtain

$$\begin{aligned}
&[\boldsymbol{K}_3\phi_{\omega,l_1}](t,\rho,\theta) \\
&= \frac{1}{2}(\cot\rho)^{\frac{N-2}{2}}\left\{\left[\left(c_1 Y_{l_1+1,\tilde{l}}(\theta) + c_2 Y_{l_1-1,\tilde{l}}(\theta)\right)\left(\cos\rho\frac{\mathrm{d}}{\mathrm{d}\rho} - \frac{N-2}{2\sin\rho} + \omega\sin\rho\right) r_{\omega,l_1}(\rho)\right.\right. \\
&- \left.\frac{1}{\sin\rho}\left(l_1 c_1 Y_{l_1+1,\tilde{l}}(\theta) - (l_1+N-3)c_2 Y_{l_1-1,\tilde{l}}(\theta)\right) r_{\omega,l_1}(\rho)\right] e^{-i(\omega-1)t} \\
&+ \left[\left(c_1 Y_{l_1+1,\tilde{l}}(\theta) + c_2 Y_{l_1-1,\tilde{l}}(\theta)\right)\left(\cos\rho\frac{\mathrm{d}}{\mathrm{d}\rho} - \frac{N-2}{2\sin\rho} - \omega\sin\rho\right) r_{\omega,l_1}(\rho)\right. \\
&- \left.\left.\frac{1}{\sin\rho}\left(l_1 c_1 Y_{l_1+1,\tilde{l}}(\theta) - (l_1+N-3)c_2 Y_{l_1-1\tilde{l}}(\theta)\right) r_{\omega,l_1}(\rho)\right] e^{-i(\omega+1)t}\right\}\,.
\end{aligned} \tag{7.89}$$

Finally, after grouping the terms with the same hyperspherical harmonic factor together, we get

$$\begin{aligned}
&[\boldsymbol{K}_3\phi_{\omega,l_1}](t,\rho,\theta) \\
&= \frac{1}{2}(\cot\rho)^{\frac{N-2}{2}}\left\{c_1 Y_{l_1+1,\tilde{l}}(\theta)e^{-i(\omega-1)t}\left[\cos\rho\frac{\mathrm{d}}{\mathrm{d}\rho} - \frac{\sigma+1/2}{\sin\rho} + \omega\sin\rho\right] r_{\omega,l_1}(\rho)\right. \\
&+ c_2 Y_{l_1-1,\tilde{l}}(\theta)e^{-i(\omega-1)t}\left[\cos\rho\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{\sigma-1/2}{\sin\rho} + \omega\sin\rho\right] r_{\omega,l_1}(\rho) \\
&+ c_1 Y_{l_1+1,\tilde{l}}(\theta)e^{-i(\omega+1)t}\left[\cos\rho\frac{\mathrm{d}}{\mathrm{d}\rho} - \frac{\sigma+1/2}{\sin\rho} - \omega\sin\rho\right] r_{\omega,l_1}(\rho) \\
&+ \left. c_2 Y_{l_1-1,\tilde{l}}(\theta)e^{-i(\omega+1)t}\left[\cos\rho\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{\sigma-1/2}{\sin\rho} - \omega\sin\rho\right] r_{\omega,l_1}(\rho)\right\}\,.
\end{aligned} \tag{7.90}$$

where we have used the definition of the label σ from Eq. (7.22) to identify the quantities $l_1 + (N-2)/2 = \sigma + 1/2$ and $l_1 + (N-4)/2 = \sigma - 1/2$.

Now, we define the following operators acting on the radial function $r_{\omega,l_1}(\rho)$ by

$$\left[L_{+\pm}r_{\omega,l_1}\right](\rho) := \left(\cos\rho\frac{\mathrm{d}}{\mathrm{d}\rho} - \frac{\sigma+\frac{1}{2}}{\sin\rho} \mp \omega\sin\rho\right) r_{\omega,l_1}(\rho)\,, \tag{7.91a}$$

$$\left[L_{-\pm}r_{\omega,l_1}\right](\rho) := \left(\cos\rho\frac{\mathrm{d}}{\mathrm{d}\rho} + \frac{\sigma-\frac{1}{2}}{\sin\rho} \mp \omega\sin\rho\right) r_{\omega,l_1}(\rho)\,, \tag{7.91b}$$

which correspond to the expressions in square brackets in Eq. (7.90). Since $\boldsymbol{K}_3$ commutes with the Casimir operator, *i.e.*, the Laplace–Beltrami operator in Eq. (3.14), direct inspection of Eq. (7.90) shows that the functions resulting from the above expressions must be proportional to solutions with the appropriate labels (ω, l_1), that is, that $L_{\pm,\pm}r_{\omega,l_1} \propto r_{\omega\pm1,l_1\pm1}$ and $L_{\pm,\mp}r_{\omega,l_1} \propto r_{\omega\pm1,l_1\mp1}$. We leave the proof showing that these transformations are indeed satisfied in Appendix G, where we obtain that the operator $\boldsymbol{K}_3$ acting on ϕ_{ω,l_1}

results in a linear combination of solutions. Hence, Eq. (7.90) reads

$$\begin{aligned}&[\boldsymbol{K}_3\phi_{\omega,l_1}](t,\rho,\theta)\\&= N_{\omega,l_1}\left[\frac{c_1\zeta_{\nu,\sigma}^{-\omega}(\zeta_{\nu,\sigma}^{\omega}-\sigma-1)}{N_{\omega-1,l_1+1}(1+\sigma)}\phi_{\omega-1,l_1+1}(t,\rho,\theta)+\frac{c_2\sigma}{N_{\omega-1,l_1-1}}\phi_{\omega-1,l_1-1}(t,\rho,\theta)\right.\\&\left.+\frac{c_1\zeta_{\nu,\sigma}^{\omega}(\zeta_{\nu,\sigma}^{-\omega}-\sigma-1)}{N_{\omega+1,l_1+1}(1+\sigma)}\phi_{\omega+1,l_1+1}(t,\rho,\theta)+\frac{c_2\sigma}{N_{\omega+1,l_1-1}}\phi_{\omega+1,l_1-1}(t,\rho,\theta)\right],\end{aligned} \tag{7.92}$$

where N_{ω,l_1} is the normalisation constant of r_{ω,l_1} with respect to the inner product in Eq. (7.25).

We now find for which values of $u\in(-\pi,\pi]$ the space of solutions of Eq. (7.31) satisfying the self–adjoint boundary conditions are invariant under the action of the vector field $\boldsymbol{K}_3$. Once again, we divide the analysis into three cases.

For Case 1 we have $1\le\nu$, and the only self–adjoint boundary condition is given by Eq. (7.47). Let r_{ω,l_1} be be a solution of Eq. (7.31) satisfying the boundary condition in Eq. (7.47). Now, let us consider the function $\hat{r}_{+-}:=L_{+-}r_{\omega,l_1}$ and calculate the quantities $\widetilde{\hat{r}_{+-}}^{(\nu)}$ and $D\widetilde{\hat{r}_{+-}}^{(\nu)}$ defined through Eq. (7.54). We obtain

$$\widetilde{\hat{r}_{+-}}^{(\nu)}(\rho) = (\cos\rho)^{2\nu}D\widetilde{r_{\omega,l_1}}^{(\nu)}(\rho)+\left[\left(\omega+\nu-\frac{1}{2}\right)\sin\rho-\frac{\sigma+\frac{1}{2}}{\sin\rho}\right]\widetilde{r_{\omega,l_1}}^{(\nu)}(\rho)\,, \tag{7.93a}$$

$$\begin{aligned}D\widetilde{\hat{r}_{+-}}^{(\nu)}(\rho) &= \left[\left(\omega-\nu-\frac{1}{2}\right)\sin\rho-\frac{\sigma+\frac{1}{2}}{\sin\rho}\right]D\widetilde{r_{\omega,l_1}}^{(\nu)}(\rho)\\&+\left[\nu^2-\frac{1}{4}+\omega(1-\omega)+\frac{\left(\sigma+\frac{1}{2}\right)^2}{\sin^2\rho}\right](\cos\rho)^{2-2\nu}\widetilde{r_{\omega,l_1}}^{(\nu)}(\rho)\,,\end{aligned} \tag{7.93b}$$

where we have used Eq. (7.31) to obtain Eq. (7.93b). We note that the function $D\widetilde{r_{\omega,l_1}}^{(\nu)}$ takes on a finite value at $\rho=\pi/2$. This is seen from Eqs. (7.44) and (7.46) and the definition of $D\widetilde{r_{\omega,l_1}}^{(\nu)}$ in Eq. (7.54).

Now, since the function r_{ω,l_1} satisfies Eq. (7.47) and since $D\widetilde{r_{\omega,l_1}}^{(\nu)}$ is finite at $\rho=\pi/2$, Eq. (7.93) implies that $\widetilde{\hat{r}_{+-}}^{(\nu)}(\pi/2)=0$ and $D\widetilde{\hat{r}_{+-}}^{(\nu)}(\pi/2)$ remains finite. A similar argument shows that the analogously defined functions $\hat{r}_{\pm\pm}$ and $\hat{r}_{-+}$ also satisfy Eq. (7.47). Hence, this boundary condition results in a space of solutions which is invariant under $\boldsymbol{K}_3$ and, thus, under the infinitesimal action of $\mathrm{SO}(2,N-1)$.

For Case 2 we have $0<\nu<1$, and the self-adjoint boundary conditions determining $\mathrm{Dom}((A_{\mathrm{Rad}})_\alpha)$ are given by Eq. (7.63), which we rewrite as

$$\sin\frac{u}{2}D\widetilde{r}^{(\nu)}(\pi/2)+\cos\frac{u}{2}\widetilde{r}^{(\nu)}(\pi/2)=0\,, \tag{7.94}$$

for all $r\in\mathrm{Dom}((A_{\mathrm{Rad}})_\alpha)$. Assume r_{ω,l_1} is a solution of Eq. (7.31) satisfying Eq. (7.94), and let $\hat{r}_{+-}:=L_{+-}r_{\omega,l_1}$. The functions $\widetilde{\hat{r}_{+-}}^{(\nu)}$ and $D\widetilde{\hat{r}_{+-}}^{(\nu)}$ are calculated using Eqs. (7.54) and (7.91a), and they are found to be given by Eq. (7.93) with the appropriate range of the parameter ν. Thus, we evaluate Eq. (7.93) at $\rho = \pi/2$ to obtain

$$\widetilde{\hat{r}_{+-}}^{(\nu)}(\pi/2) = (\omega + \nu - \sigma - 1)\, \widetilde{r_{\omega,l_1}}^{(\nu)}(\pi/2)\,, \tag{7.95a}$$

$$D\widetilde{\hat{r}_{+-}}^{(\nu)}(\pi/2) = (\omega - \nu - \sigma - 1)\, D\widetilde{r_{\omega,l_1}}^{(\nu)}(\pi/2)\,. \tag{7.95b}$$

Hence, we have

$$\begin{aligned}&\sin\frac{u}{2} D\widetilde{\hat{r}_{+-}}^{(\nu)}(\pi/2) + \cos\frac{u}{2}\widetilde{\hat{r}_{+-}}^{(\nu)}(\pi/2)\\ &= (\omega - \nu - \sigma - 1)\sin\frac{u}{2} D\widetilde{r_{\omega,l_1}}^{(\nu)}(\pi/2) + (\omega + \nu - \sigma - 1)\cos\frac{u}{2}\widetilde{r_{\omega,l_1}}^{(\nu)}(\pi/2)\,. \end{aligned}\tag{7.96}$$

Now, let $u \neq 0$ and $u \neq \pi$, so that Eq. (7.94) corresponds to a generalised Robin boundary condition. Then, since r_{ω,l_1} satisfies the boundary condition in Eq. (7.94), we can write $D\widetilde{\hat{r}_{+-}}^{(\nu)}(\pi/2) = -\cot(u/2)\widetilde{\hat{r}_{+-}}^{(\nu)}(\pi/2)$, and by substituting this into Eq. (7.96) we obtain

$$\sin\frac{u}{2} D\widetilde{\hat{r}_{+-}}^{(\nu)}(\pi/2) + \cos\frac{u}{2}\widetilde{\hat{r}_{+-}}^{(\nu)}(\pi/2) = 2\nu\widetilde{r_{\omega,l_1}}^{(\nu)}(\pi/2)\,. \tag{7.97}$$

Since $\nu \neq 0$, the right–hand side of this expression is never zero and, thus, the functions $\widetilde{\hat{r}_{+-}}^{(\nu)}$ and $D\widetilde{\hat{r}_{+-}}^{(\nu)}$ do not satisfy Eq. (7.94). Hence, the generalised Robin boundary conditions (7.66) do not result in an invariant space of solutions.

Now if $u = 0$, *i.e.*, if we have a generalised Dirichlet boundary condition, then Eq. (7.94) reduces to $\widetilde{r_{\omega,l_1}}^{(\nu)}(\pi/2) = 0$. However, Eq. (7.95a) directly implies that if r_{ω,l_1} satisfies this condition, then so does $\hat{r}_{+-}$. A similar argument using Eq. (7.91) shows that this is also the case for the functions. Thus, the Dirichlet boundary condition (7.64) results in an invariant space of solutions.

Similarly, if $u = \pi$, then Eq. (7.94) reduces to $D\widetilde{r_{\omega,l_1}}^{(\nu)}(\pi/2) = 0$, that is, a generalised Neumann boundary condition. If r_{ω,l_1} satisfies this condition, then Eq. (7.95b) implies that $D\widetilde{\hat{r}_{+-}}^{(\nu)}(\pi/2) = 0$ and, once again, the same holds true for the functions $L_{\pm\pm}r_{\omega,l_1}$ and $L_{-+}r_{\omega,l_1}$. Hence, the generalised Neumann boundary condition (7.65) yields an invariant space of solutions.

Finally, we consider the Case 3. The self–adjoint boundary conditions in Eq. (7.73) are written as

$$\sin\frac{u}{2} D\tilde{r}^{(0)}(\pi/2) + \cos\frac{u}{2}\tilde{r}^{(0)}(\pi/2) = 0\,, \tag{7.98}$$

for all $r \in \mathrm{Dom}((A_{\mathrm{Rad}})_\alpha)$. Equations (7.68) and (7.91) imply that if r_{ω,l_1} is a solution of Eq. (7.31) and $\hat{r}_{+-} = L_{+-}r_{\omega,l_1}$, then the functions $\widetilde{\hat{r}_{+-}}^{(0)}$ and $D\widetilde{\hat{r}_{+-}}^{(0)}$, are given by

$$\begin{aligned}\widetilde{\hat{r}_{+-}}^{(0)}(\rho) = &\left[\omega\sin\rho - \frac{\sigma + \frac{1}{2}}{\sin\rho} - \frac{[\ln(\cos^2\rho) + 4]}{2\ln(\cos^2\rho)}\right]\widetilde{r_{\omega,l_1}}^{(0)}(\rho)\\ &+ \frac{1}{[\ln(\cos^2\rho)]^2} D\widetilde{r_{\omega,l_1}}^{(0)}(\rho)\,,\end{aligned}\tag{7.99a}$$

$$\begin{aligned}D\widetilde{\hat{r}_{+-}}^{(0)}(\rho) = &\left[\cos^2\rho[\ln(\cos^2\rho)]^2\left(\omega(1-\omega) - \frac{1}{4} + \frac{\left(\sigma + \frac{1}{2}\right)^2}{\sin^2\rho}\right) - 4\sin^2\rho\right]\widetilde{r_{\omega,l_1}}^{(0)}(\rho)\\ &+ \left[\left(\omega - \frac{1}{2}\right)\sin\rho - \frac{\sigma + \frac{1}{2}}{\sin\rho} + \frac{2\sin\rho}{\ln(\cos^2\rho)}\right] D\widetilde{r_{\omega,l_1}}^{(0)}(\rho)\,.\end{aligned}\tag{7.99b}$$

We evaluate these functions at $\rho = \pi/2$ to obtain

$$\widetilde{\hat{r}_{+-}}^{(0)}(\pi/2) = (\omega - \sigma - 1)\widetilde{r_{\omega,l_1}}^{(0)}(\pi/2)\,, \tag{7.100a}$$

$$D\widetilde{\hat{r}_{+-}}^{(0)}(\pi/2) = (\omega - \sigma - 1)D\widetilde{r_{\omega,l_1}}^{(0)}(\pi/2) - 4\widetilde{r_{\omega,l_1}}^{(0)}(\pi/2)\,. \tag{7.100b}$$

Let us assume that r_{ω,l_1} satisfies Eq. (7.98) with $u \neq 0$ and $u \neq \pi$. Then, we calculate

$$\begin{aligned}
&\sin\frac{u}{2}D\widetilde{\hat{r}_{+-}}^{(0)}(\pi/2) + \cos\frac{u}{2}\widetilde{\hat{r}_{+-}}^{(0)}(\pi/2) \\
&= (\omega - \sigma - 1)\left[\sin\frac{u}{2}D\widetilde{r_{\omega,l_1}}^{(0)}(\pi/2) + \cos\frac{u}{2}\widetilde{r_{\omega,l_1}}^{(0)}(\pi/2)\right] \\
&- 4\sin\frac{u}{2}\widetilde{r_{\omega,l_1}}^{(0)}(\pi/2)\,.
\end{aligned} \tag{7.101}$$

Then, since r_{ω,l_1} satisfies the boundary condition, Eq. (7.98) implies that the terms in square brackets above vanish. Thus, Eq. (7.101) reduces to

$$\sin\frac{u}{2}D\widetilde{r_{\omega,l_1}}^{(0)}(\pi/2) + \cos\frac{u}{2}\widetilde{\hat{r}_{+-}}^{(0)}(\pi/2) = -4\sin\frac{u}{2}\widetilde{r_{\omega,l_1}}^{(0)}(\pi/2)\,, \tag{7.102}$$

and the right–hand side is never zero. Hence, the function $\hat{r}_{+-}$ does not satisfy the self–adjoint boundary condition (7.98). This implies that the generalised Robin boundary conditions (7.76) do not result in an invariant space of solutions.

Now, if r_{ω,l_1} satisfies Eq. (7.98) with $u = 0$, that is, the generalised Dirichlet boundary condition $\widetilde{r_{\omega,l_1}}^{(0)}(\pi/2) = 0$, then Eq. (7.100a) directly implies that $\widetilde{\hat{r}_{+-}}^{(0)}(\pi/2) = 0$. A similar argument shows that all other functions obtained by applying the operators in Eq. (7.91) to the Dirichlet function r_{ω,l_1} satisfy the boundary condition as well. Thus, the generalised Dirichlet boundary condition (7.74) yields an invariant space of solutions.

Finally, let us assume that r_{ω,l_1} satisfies the boundary condition in Eq. (7.98) with $u = \pi$, and we have $D\widetilde{r_{\omega,l_1}}^{(0)}(\pi/2) = 0$. This time, Eq. (7.100a) implies that

$$D\widetilde{\hat{r}_{+-}}^{(0)}(\pi/2) = -4\widetilde{r_{\omega,l_1}}^{(0)}(\pi/2)\,, \tag{7.103}$$

and, thus, the function $\hat{r}_{+-}$ does not satisfy the same boundary condition that r_{ω,l_1} does. Hence, the generalised Neumann boundary condition (7.75) does not result in an invariant space of solutions.

7.4 MODE FUNCTIONS SATISFYING THE INVARIANT BOUNDARY CONDITIONS

Now that we have determined which amongst all the possible self–adjoint boundary conditions result in solution spaces which are invariant under SO$(2, N-1)$, we can give the explicit form of the associated mode solutions for each of these cases. The frequency spectrum for the Dirichlet, Neumann and Robin mode solutions can be found, for example, in Ref. [16]. We note that the values of the frequencies we obtain for the Dirichlet and Neumann modes correspond to these results.

Let us begin by considering all cases for which $\nu \notin \mathbb{N}_0$. Thus, we can consider the general solution of Eq. (7.31) in the form given by Eq. (7.39). We use the definition given in Eq. (7.54a), to obtain the function $\widetilde{r_{\omega,l_1}}^{(\nu)}$, namely,

$$\begin{aligned}\widetilde{r_{\omega,l_1}}^{(\nu)}(\rho) =& N_{\omega,l_1}(\sin\rho)^{\sigma+\frac{1}{2}}\left[A^{\omega}_{-\nu}(\cos\rho)^{2\nu}F\left(\zeta^{\omega}_{\nu,\sigma},\zeta^{-\omega}_{\nu,\sigma};1+\nu;\cos^2\rho\right)\right.\\ &\left.+\ A^{\omega}_{\nu}F\left(\zeta^{\omega}_{-\nu,\sigma},\zeta^{-\omega}_{-\nu,\sigma};1-\nu;\cos^2\rho\right)\right], \end{aligned} \tag{7.104}$$

where we have extended the definition in Eq. (7.54a) to encompass the cases where $\nu > 1$ and $\nu \notin \mathbb{N}$. Using the definition in Eq. (7.54b), and the identity for the hypergeometric function [23, Eq. 15.5.1]

$$\frac{\mathrm{d}}{\mathrm{d}x}F(a,b;c;x) = \frac{ab}{c}F(a+1,b+1;c+1;x)\,, \tag{7.105}$$

we calculate the function $D\widetilde{r_{\omega,l_1}}^{(\nu)}$, which reads

$$\begin{aligned}D\widetilde{r_{\omega,l_1}}^{(\nu)}(\rho) =& N_{\omega,l_1}(\sin\rho)^{\sigma+\frac{3}{2}}\left[A^{\omega}_{-\nu}\cos^2\rho\left(\frac{\sigma+1/2}{\sin\rho}F_1(\rho)-2\frac{\zeta^{\omega}_{\nu,\sigma}\zeta^{-\omega}_{\nu,\sigma}}{1+\nu}\tilde{F}_1(\rho)\right)\right.\\ &\left.+\ A^{\omega}_{\nu}(\cos\rho)^{2-2\nu}\left(\frac{\sigma+1/2}{\sin\rho}-2\frac{\zeta^{\omega}_{-\nu,\sigma}\zeta^{-\omega}_{-\nu,\sigma}}{1-\nu}\tilde{F}_2(\rho)\right)-2\nu A^{\omega}_{-\nu}F_1(\rho)\right], \end{aligned} \tag{7.106}$$

where we have defined the functions

$$F_1(\rho) := F\left(\zeta^{\omega}_{\nu,\sigma},\zeta^{-\omega}_{\nu,\sigma};1+\nu;\cos^2\rho\right), \tag{7.107a}$$
$$F_2(\rho) := F\left(\zeta^{\omega}_{-\nu,\sigma},\zeta^{-\omega}_{-\nu,\sigma};1-\nu;\cos^2\rho\right), \tag{7.107b}$$
$$\tilde{F}_1(\rho) := F\left(1+\zeta^{\omega}_{\nu,\sigma},1+\zeta^{-\omega}_{\nu,\sigma};2+\nu;\cos^2\rho\right), \tag{7.107c}$$
$$\tilde{F}_2(\rho) := F\left(1+\zeta^{\omega}_{-\nu,\sigma},1+\zeta^{-\omega}_{-\nu,\sigma};2-\nu;\cos^2\rho\right). \tag{7.107d}$$

Let us then apply the generalised Dirichlet boundary condition (7.64) to the solution r_{ω,l_1}, that is, $\widetilde{r_{\omega,l_1}}^{(\nu)}(\pi/2) = 0$. (We note that the unique self–adjoint boundary condition for $1 < \nu$ given by Eq. (7.47) is also a Dirichlet boundary condition of this type.) Hence, we evaluate Eq. (7.104) at $\rho = \pi/2$. Since $F(a,b;c;0) = 1$ for all $a,b,c \in \mathbb{C}$, we have that $\widetilde{r_{\omega,l_1}}^{(\nu)}(\pi/2) = N_{\omega,l_1}A^{\omega}_{\nu}$. Thus, the function r_{ω,l_1} satisfies the Dirichlet boundary condition if $A^{\omega}_{\nu} = 0$. The definition of A^{ω}_{ν} in Eq. (7.40) implies that this is satisfied whenever $\Gamma(\zeta^{\pm\omega}_{\nu,\sigma})^{-1} = 0$, that is, whenever

$$\frac{1+\sigma+\nu\pm\omega}{2} = -n\,, \tag{7.108}$$

for any $n \in \mathbb{N}_0$. Hence, the Dirichlet boundary condition is satisfied if we constrain the frequency values to $\omega = \pm\omega^{(\nu,\mathrm{D})}_n$, where we have defined

$$\omega^{(\nu,\mathrm{D})}_n := 2n+1+\sigma+\nu\,, \tag{7.109}$$

for $n \in \mathbb{N}_0$. Since we are interested in positive–frequency solutions, we choose the positive frequency parameter in Eq. (7.109). We substitute $\omega = \omega^{(\nu,\mathrm{D})}_n$ into Eq. (7.39) to obtain the Dirichlet solution $r^{(\nu,\mathrm{N})}_{n,l_1} := r_{\omega_n^{(\nu,\mathrm{D})},l_1}$. We find

$$\begin{aligned} r^{(\nu,\mathrm{D})}_{n,l_1}(\rho) &= N_{\omega_n^{(\nu,\mathrm{D})},l_1} A^{\omega_n^{(\nu,\mathrm{D})}}_{-\nu} (\sin\rho)^{\sigma+\frac{1}{2}} (\cos\rho)^{\nu+\frac{1}{2}} F\left(n+\nu+\sigma+1, -n; 1+\nu; \cos^2\rho\right) , \\ &= N^{(\nu,\mathrm{D})}_{n,l_1} (\sin\rho)^{\sigma+\frac{1}{2}} (\cos\rho)^{\nu+\frac{1}{2}} P^{(\sigma,\nu)}_n(\cos 2\rho) , \end{aligned} \tag{7.110}$$

where $P^{(a,b)}_n(x)$ is a Jacobi polynomial given by Eq. (6.23) and $N^{(\nu,\mathrm{D})}_{n,l_1}$ is a normalisation constant.

Now, for special case for which $\nu \in \mathbb{N}$ the function r_{ω,l_1} given by Eq. (7.39) is ill–defined. Hence, we consider the transformation given Eq. (7.41) instead. Using this definition of the solution r_{ω,l_1}, we calculate

$$\begin{aligned} (\cos\rho)^{m-\frac{1}{2}} r_{\omega,l_1}(\rho) =& N_{\omega,l_1} (\sin\rho)^{\sigma+\frac{1}{2}} \Bigg[H^{\omega}_m (\cos\rho)^{2m+1} \sum_{k=0}^{\infty} C_k \left[\ln(\cos^2\rho) + h_\omega(k)\right] (\cos\rho)^{2k} \\ &+ (-1)^{m-1} H^{\omega}_{-m} \sum_{k=0}^{m-1} D_k (\cos\rho)^{2k} \Bigg] , \end{aligned} \tag{7.111}$$

where we have defined the coefficients C_k and D_k as

$$C_k := \frac{(\zeta^{\omega}_{m,\sigma})_k (\zeta^{-\omega}_{m,\sigma})_k}{k!(m+k)!} , \qquad D_k := \frac{(\zeta^{\omega}_{-m,\sigma})_k (\zeta^{-\omega}_{-m,\sigma})_k}{k!(1-m)_k} . \tag{7.112}$$

We impose the boundary condition (7.47) on the function r_{ω,l_1} by evaluating Eq. (7.111) at $\rho = \pi/2$ and equating the result to zero. Since $(\cos\rho)^{2m+1}\ln(\cos\rho) \to 0$ as $\rho \to \pi/2$, Eq. (7.111) implies that

$$\left[(\cos\rho)^{m-\frac{1}{2}} r_{\omega,l_1}(\rho)\right]\Big|_{\rho=0\pi/2} = N_{\omega,l_1} (-1)^{m-1} H^{\omega}_{-m} . \tag{7.113}$$

Hence, the function r_{ω,l_1} satisfies the Dirichlet boundary condition if $H^{\omega}_{-m} = 0$. Using the definition of the constant H^{ω}_{-m} given by Eq. (7.42), we see that this condition is equivalent to $\Gamma(\zeta^{\pm\omega}_{m,\sigma})^{-1} = 0$, which is satisfied if and only if $\omega = \pm\omega^{(m,\mathrm{D})}_n$, that is, if the frequency spectrum is given by Eq. (7.109) with $\nu = m$. Once again, since we are considering positive–frequency solutions we take $\omega = \omega^{(m,\mathrm{D})}_n$. The substitution of this value of ω into Eq. (7.41) is a bit more subtle compared to the case where $\nu \notin \mathbb{N}$. If we substitute $\omega^{(m,\mathrm{D})}_n$ into H^{ω}_m, Eq. (7.42) implies that

$$H^{\omega_n^{(m,\mathrm{D})}}_m = \frac{(-1)^{m+1}\Gamma(1+\sigma)}{\Gamma(2n+1+\sigma)\Gamma(-m)} = 0 . \tag{7.114}$$

Thus, it would appear that the Dirichlet solution $r_{\omega_n^{(m,\mathrm{D})},l_1}$ is trivial. However, we note that not all the combinations $H^{\omega_n^{(m,\mathrm{D})}}_m h_{\omega_n^{(m,\mathrm{D})}}(k)$ in Eq. (7.41) vanish. From the definition of the constants h_ω in Eq. (7.43) and the result in Eq. (7.114), we see that

$$\begin{aligned} H_m^{\omega_n^{(m,\mathrm{D})}} h_{\omega_n^{(m,\mathrm{D})}}(k) &= \lim_{\epsilon\to 0} \frac{(-1)^{m+1}\Gamma(1+\sigma)}{\Gamma(2n+1+\sigma)\Gamma(-m+\epsilon)}\psi(k-m+\epsilon)\,, \\ &= \frac{(-1)^{m+1}\Gamma(1+\sigma)}{\Gamma(2n+1+\sigma)} \lim_{\epsilon\to 0}\frac{\psi(k-m+\epsilon)}{\Gamma(-m+\epsilon)}\,, \\ &= \frac{(-1)^{m+1}\Gamma(1+\sigma)}{\Gamma(2n+1+\sigma)}(-1)^m m! \lim_{\epsilon\to 0}\frac{\psi(\epsilon)}{\Gamma(\epsilon)}\,, \\ &= \frac{\Gamma(1+\sigma)m!}{\Gamma(2n+1+\sigma)}\,, \end{aligned} \tag{7.115}$$

where we have used the fact that $\psi(x)/\Gamma(x) \to -1$ as $x \to 0$. Thus, a straightforward calculation shows that, after substituting $\omega = \omega_n^{(m,\mathrm{D})}$ into Eq. (7.41) and applying the result of Eq. (7.115), the solution $r_{\omega_n^{(m,\mathrm{D})},l_1}$ reduces to Eq. (7.110) with $\nu = m$. Thus, by this analysis we conclude that the generalised Dirichlet boundary conditions result in the mode functions given by Eq. (7.110) for all $1 \le \nu$ and $0 < \nu < 1$.

Now let us consider $0 < \nu < 1$. For these values of the parameter ν, the general solution is also given by Eq. (7.39) and the functions $\widetilde{r_{\omega,l_1}}^{(\nu)}$ and $D\widetilde{r_{\omega,l_1}}^{(\nu)}$ are given by Eqs. (7.104) and (7.106). We evaluate Eq. (7.106) at $\rho = \pi/2$ and we obtain

$$D\widetilde{r_{\omega,l_1}}^{(\nu)}(\pi/2) = -2\nu A^{\omega}_{-\nu}\,. \tag{7.116}$$

Now, let r_{ω,l_1} be a solution satisfying the generalised Neumann boundary condition in Eq (7.65). Hence, we must have $D\widetilde{r_{\omega,l_1}}^{(\nu)}(\pi/2) = 0$, which is satisfied if $A^{\omega}_{-\nu} = 0$. The definition in Eq. (7.40) implies that this is satisfied whenever $\Gamma(\zeta^{\pm\omega}_{-\nu,\sigma})^{-1} = 0$, that is,

$$\frac{1+\sigma-\nu\pm\omega}{2} = -n\,, \tag{7.117}$$

with $n \in \mathbb{N}_0$. Hence, the frequency spectrum is constrained by $\omega = \pm\omega_n^{(\nu,\mathrm{N})}$, where

$$\omega_n^{(\nu,\mathrm{N})} := 2n+1+\sigma-\nu\,. \tag{7.118}$$

We choose the positive frequencies and substitute $\omega = \omega_n^{(\nu,\mathrm{N})}$ into Eq. (7.39) to obtain the Neumann modes $r_{n,l_1}^{(\nu,\mathrm{N})} := r_{\omega_n^{(\nu,\mathrm{N})},l_1}$. We find

$$\begin{aligned} r_{n,l_1}^{(\nu,\mathrm{N})}(\rho) &= N_{\omega_n^{(\nu,\mathrm{N})},l_1} A_\nu^{\omega_n^{(\nu,\mathrm{N})}} (\sin\rho)^{\sigma+\frac{1}{2}}(\cos\rho)^{\frac{1}{2}-\nu} F\left(n-\nu+\sigma+1, -n; 1-\nu; \cos^2\rho\right)\,, \\ &= N_n^{(\nu,\mathrm{N})}(\sin\rho)^{\sigma+\frac{1}{2}}(\cos\rho)^{\frac{1}{2}-\nu} P_n^{(\sigma,-\nu)}(\cos 2\rho)\,, \end{aligned} \tag{7.119}$$

where we have used the definition of the Jacobi polynomials in Eq. (6.23) and $N_n^{(\nu,\mathrm{N})}$ is a normalisation constant.

Finally, we consider the case $\nu = 0$. The general solution r_{ω,l_1} is given by Eq. (7.41) with $m = 0$ (the last term omitted). Now, Eq. (7.68) implies that the function $\widetilde{r_{\omega,l_1}}^{(0)}$ is given by

$$\widetilde{r_{\omega,l_1}}^{(0)}(\rho) = N_{\omega,l_1}(\sin\rho)^{\sigma+\frac{1}{2}}\left[H_0^\omega \sum_{k=0}^{\infty}\frac{(\zeta^{\omega}_{0,\sigma})_k(\zeta^{-\omega}_{0,\sigma})_k}{(k!)^2}\left(1+\frac{h_\omega(k)}{\ln(\cos^2\rho)}\right)(\cos\rho)^{2k}\right]\,. \tag{7.120}$$

The only invariant self–adjoint boundary condition in this case corresponds to the generalised Dirichlet boundary condition in Eq. (7.74), that is, $\widetilde{r_{\omega,l_1}}^{(0)}(\pi/2) = 0$. From Eq. (7.120) we have $\widetilde{r_{\omega,l_1}}^{(0)}(\pi/2) = N_{\omega,l_1} H_0^{\omega}$. Hence, the solution r_{ω,l_1} satisfies the Dirichlet condition if $H_0^{\omega} = 0$. The definition in Eq. (7.42) implies that this is the case whenever $\Gamma(\zeta_{0,\sigma}^{\pm\omega})^{-1} = 0$, that is, if

$$\frac{1+\sigma\pm\omega}{2} = -n\,, \tag{7.121}$$

with $n \in \mathbb{N}_0$. This constrains the allowed frequencies to satisfy $\omega = \pm\omega_n^{(0,\mathrm{D})}$, where $\omega_n^{(0,\mathrm{D})}$ is defined by taking $\nu = 0$ in either Eq. (7.109) or (7.118). We take the positive frequency spectrum and set $\omega = \omega_n^{(0,\mathrm{D})}$. We substitute this value of ω into Eq. (7.41) in a very similar way to the case $\nu \in \mathbb{N}$ above. Thus, using Eq. (7.114), we obtain the Dirichlet solution $r_{n,l_1}^{(0,\mathrm{D})} := r_{\omega_{n,l_1}^{(0,\mathrm{D})}}$, given by

$$\begin{aligned} r_{n,l_1}^{(0,\mathrm{D})}(\rho) &= N_{\omega_n^{(0,\mathrm{D})},l_1} (\sin\rho)^{\sigma+\frac{1}{2}} (\cos\rho)^{\frac{1}{2}} F\left(-n, n+\sigma+1; 1+\sigma; \sin^2\rho\right)\,, \\ &= N_{n,l_1}^{(0,\mathrm{D})} (\sin\rho)^{\sigma+\frac{1}{2}} (\cos\rho)^{\frac{1}{2}} P_n^{(\sigma,0)}(\cos 2\rho)\,, \end{aligned} \tag{7.122}$$

where we have used Eq. (6.23), and $N_{n,l_1}^{(0,\mathrm{D})}$ is a normalisation constant.

To conclude this section we present the full mode solutions of the Klein–Gordon equation given by Eq. (7.2) associated to all the $\mathrm{SO}(2, N-1)$–invariant self–adjoint boundary conditions for the different values of the mass parameter ν. The normalisation constants of the radial components are obtained by imposing the condition

$$\langle r_{n,l_1}, r_{n',l_1}\rangle_{\mathrm{Rad}} = \delta_{nn'}\,, \tag{7.123}$$

with the inner product defined in Eq. (7.25). We have made use of the orthonormality of the Jacobi polynomials given in Eq. (6.25). We adopt our original notation used in Section 7.1 and denote the hyperspherical harmonics defined by Eqs. (7.12) and (7.15) by $Y_{\boldsymbol{l}}(\theta)$ where $\boldsymbol{l} = (l_1, \ldots, l_{N-2})$. We also use the definition of σ in Eq. (7.22) to write the mode functions in terms of the angular number l_1.

Case 1: $\nu \geq 1$ $(M^2 \geq (N-3)(N+1)/4)$. The only self–adjoint boundary condition for the radial component is the generalised Dirichlet boundary condition given by Eq. (7.47). The normalised positive–frequency mode solutions are given by

$$\phi_n^{(\nu,\mathrm{D})}(t,\rho,\theta) = N_{n,l_1}^{(\nu,\mathrm{D})} (\sin\rho)^{l_1} (\cos\rho)^{\nu+\frac{N-1}{2}} P_n^{\left(l_1+\frac{N-3}{2},\nu\right)}(\cos 2\rho) Y_{\boldsymbol{l}}(\theta) e^{-i\omega_{n,l_1}^{(\nu,\mathrm{D})} t}\,, \tag{7.124}$$

with frequency spectrum given by

$$\omega_{n,l_1}^{(\nu,\mathrm{D})} = 2n+\nu+l_1+\frac{N-1}{2}\,, \qquad n \in \mathbb{N}_0\,, \tag{7.125}$$

and normalisation constant given by

$$N_{n,l_1}^{(\nu,\mathrm{D})} = \left[\frac{2\omega_{n,l_1}^{(\nu,\mathrm{D})} n!\,\Gamma\left(n+\nu+l_1+\frac{N-1}{2}\right)}{\Gamma(n+\nu+1)\Gamma\left(n+l_1+\frac{N-1}{2}\right)}\right]^{\frac{1}{2}}\,. \tag{7.126}$$

Case 2: $0 < \nu < 1$ $(-(N-1)^2/4 < M^2 < (N-3)(N+1)/4)$. The invariant self–adjoint boundary conditions are the generalised Dirichlet (7.64) and Neumann (7.65) boundary conditions. The Dirichlet modes are given by Eq. (7.124). The normalised positive–frequency Neumann modes are given by

$$\phi_n^{(\nu,\mathrm{N})}(t,\rho,\theta) = N_{n,l_1}^{(\nu,\mathrm{N})}(\sin\rho)^{l_1}(\cos\rho)^{\frac{N-1}{2}-\nu}P_n^{\left(l_1+\frac{N-3}{2},-\nu\right)}(\cos 2\rho)Y_{\boldsymbol{l}}(\theta)e^{-i\omega_{n,l_1}^{(\nu,\mathrm{N})}t}\,, \tag{7.127}$$

with frequency spectrum given by

$$\omega_{n,l_1}^{(\nu,\mathrm{N})} = 2n - \nu + l_1 + \frac{N-1}{2}\,, \qquad n \in \mathbb{N}_0\,, \tag{7.128}$$

and normalisation constant given by

$$N_{n,l_1}^{(\nu,\mathrm{N})} = \left[\frac{2\omega_{n,l_1}^{(\nu,\mathrm{N})}n!\Gamma\left(n-\nu+l_1+\frac{N-1}{2}\right)}{\Gamma(n-\nu+1)\Gamma\left(n+l_1+\frac{N-1}{2}\right)}\right]^{\frac{1}{2}}\,, \tag{7.129}$$

We note that, for the case $N = 4$, that is, for a scalar field in AdS$_4$, the Dirchlet and Neumann mode functions given by Eq. (7.124) and (7.127), respectively, reduce to those found by Breitenlöhner and Freedman in Refs. [12, 13]. In their analysis, they showed that for $-9/4 < M^2 < -5/4$ two types of mode solutions, corresponding to Dirichlet and Neumann boundary conditions, make the energy flux at the boundaries to be zero. The freedom of choice between these two sets of mode functions can then be explained using the fact that only these two solution spaces coming from the self–adjoint boundary conditions are invariant under $\widetilde{\mathrm{SO}}(2, N-1)$.

Case 3: $\nu = 0$ $(M^2 = -(N-1)^2/4)$. The only invariant self–adjoint boundary condition for this case is the generalised Dirichlet boundary condition (7.74). The normalised positive–frequency Dirichlet modes are given by

$$\phi_n^{(0,\mathrm{D})}(t,\rho,\theta) = N_{n,l_1}^{(0,\mathrm{D})}(\sin\rho)^{l_1}(\cos\rho)^{\frac{N-1}{2}}P_n^{\left(l_1+\frac{N-3}{2},0\right)}(\cos 2\rho)Y_{\boldsymbol{l}}(\theta)e^{-i\omega_{n,l_1}^{(\nu,\mathrm{D})}t}\,, \tag{7.130}$$

with frequency spectrum given by

$$\omega_{n,l_1}^{(0,\mathrm{D})} = 2n + l_1 + \frac{N-1}{2}\,, \qquad n \in \mathbb{N}_0\,, \tag{7.131}$$

and normalisation constant given by

$$N_{n,l_1}^{(\nu,\mathrm{D})} = \left[2\omega_{n,l_1}^{(0,\mathrm{D})}\right]^{\frac{1}{2}}\,. \tag{7.132}$$

We note that these mode solutions correspond to Eqs. (7.124) and (7.127) with $\nu = 0$. Thus, the Dirichlet and Neumann mode solutions in Eqs. (7.124) and (7.127) for $0 < \nu < 1$ approach the Dirichlet modes in Eq. (7.130) in the limit as the mass squared approaches to the critical value $-(N-1)^2/4$. We also note that the Dirichlet modes in Eq. (7.130) for $N = 4$ coincide with those found by imposing the vanishing of the energy flux at the spatial boundary in Refs. [11, 12].

7.5 Invariant positive–frequency subspaces

In Section 7.3 we have shown that if the space of solutions to Klein–Gordon equation is invariant under the action of the Killing vector field $\boldsymbol{K}_3$, then it is invariant under the infinitesimal action of $\mathrm{SO}(2, N-1)$. We found in Section 7.4 that only the Dirichlet and Neumann modes form invariant solution spaces. We now turn to the task of determining whether or not the space of solutions for these cases, namely, the ones spanned by the mode functions in Eqs. (7.124), (7.127) and (7.130), respectively, results in invariant positive–frequency spaces. We will first show that for any of these sets of mode solutions, the action of the boost–like Killing vector $\boldsymbol{K}_3$ in Eq. (7.82) on any positive–frequency mode is a linear combination of positive–frequency modes. Then, using the Killing algebra, we will show that this fact implies that any other Killing vector field preserves positive–frequency solutions.

Let ϕ_{ω_n,l_1} be any of the three mode solutions of Eqs. (7.124), (7.127), or (7.130) such that the associated frequency satisfies $\omega_n > 0$. Then, the action of the vector field $\boldsymbol{K}_3$ on ϕ_{ω_n,l_1} is given by Eq. (7.92). From this expression, it follows that $\boldsymbol{K}_3\phi_{\omega_n,l_1}$ is a linear combination of the four mode solutions $\phi_{\omega_n\pm1,l_1\pm1}$, and $\phi_{\omega_n\pm1,l_1\mp1}$. Also, if $n = 0 = l_1$, then Eqs. (7.86) and (7.92) imply that $\boldsymbol{K}_3\phi_{\omega_0,0} \propto \phi_{\omega_0+1,l_1+1}$. Now, since $\omega_n > 0$ for all $n \in \mathbb{N}_0$, the two modes, $\phi_{\omega_n+1,l_1\pm1}$, have positive frequencies. The other two mode functions have frequencies $\omega_n - 1$, with $l_1 - 1$ and $l_1 + 1$, respectively. Thus, the frequencies $\omega_n - 1$ will be positive unless $\omega_n < 1$ for some n. The explicit form of the frequency spectrum for the invariant spaces is given by $\omega_n = 2n + \varepsilon\nu + l_1 + (N-1)/2$, where $\varepsilon = \pm1$, so that $\varepsilon = 1$ corresponds to the Dirichlet modes and $\varepsilon = -1$ corresponds to the Neumann modes. If $\nu \geq 1$, then only the Dirichlet modes ($\varepsilon = 1$) are invariant, and for these cases it follows that $\omega_n \geq 2$ for all $n, l_1 \in \mathbb{N}_0$, since $(N-1)/2 \geq 1$ for all $N \geq 3$. Hence, $\boldsymbol{K}_3\phi_{\omega_n,l_1}$ is a linear combination of positive frequency modes for all $n \in \mathbb{N}_0$. Now, for the Dirichlet modes ($\varepsilon = 1$) with $0 \leq \nu < 1$, we have $\omega_n \geq 1$ for all $n, l_1 \in \mathbb{N}_0$ and, thus, $\boldsymbol{K}_3\phi_{\omega_n,l_1}$ is again positive–frequency. Finally, for the Neumann modes ($\varepsilon = -1$) which only appear for $0 < \nu < 1$, we have $\omega_n \geq 1$ for all $n, l_1 \in \mathbb{N}$, since the frequency ω_0 with $l_1 = 0$ does satisfy $0 < \omega_0 < 1$ (since if $N = 3$, we have $\omega_0 = -\nu + (N-1)/2 < 1$). However, as mentioned above, we have $\boldsymbol{K}_3\phi_{\omega_0,0} \propto \phi_{\omega_0+1,l_1+1}$, so no negative frequency solution appears for this particular case either. Hence, $\boldsymbol{K}_3\phi_{\omega_n,l_1}$ is a linear combination of positive–frequency solutions for all $n, l_1 \in \mathbb{N}_0$ for the Neumann modes as well. Therefore, we have shown that for any of the invariant mode solution spaces, the positive–frequency subspaces are invariant under the action of the Killing vector $\boldsymbol{K}_3$.

Now, let ϕ_{n,l_1} be any of the positive–frequency mode solutions with $\omega_{n,l_1} > 0$ coming from the Dirichlet or Neumann boundary conditions. The first thing to note is that, since $[\xi_0, \boldsymbol{J}_{kl}] = 0$ for all $3 \leq k < l < N-2$, where ξ_0 is the time–like Killing vector field and $\boldsymbol{J}_{kl}$ is defined in Eq. (3.24d), the action of $\boldsymbol{J}_{kl}$ leaves the frequency invariant. Thus, $\boldsymbol{J}_{kl}\phi_{n,l_1}$ must be a linear combination of modes with the same frequency ω_{n,l_1}. Similarly, by our previous result, we can write $\boldsymbol{K}_3\phi_{n,l_1} = \sum C_{n,n';l_1,l_1'}\phi_{n',l_1'}$, with each $\phi_{n',l_1'}$

a positive–frequency mode. Since Eq. (7.80) implies that

$$\boldsymbol{K}_j = [\boldsymbol{K}_3, \boldsymbol{J}_{3j}] \ , \tag{7.133}$$

for all $j > 3$, where $\boldsymbol{K}_j$ is given by Eq. (3.24b), the action of $\boldsymbol{K}_j$ on the mode function ϕ_{n,l_1} reduces to

$$\boldsymbol{K}_j \phi_{n,l_1} = (\boldsymbol{K}_3 \circ \boldsymbol{J}_{3j})\, \phi_{n,l_1} - \sum_{n,l_1'} C_{n,n';l_1,l_1'} \boldsymbol{J}_{3j} \phi_{n',l_1'} \, . \tag{7.134}$$

We immediately note that the second term is a linear combination of positive–frequency modes due to the invariance of the frequency by the action of $\boldsymbol{J}_{3j}$. Similarly, the function $\boldsymbol{J}_{3j}\phi_{n,l_1}$ is a linear combination of positive–frequency modes and the action of $\boldsymbol{K}_3$ on these functions preserves the positive–frequency subspace. Hence, Eq. (7.134) implies that the mode function $\boldsymbol{K}_j \phi_{n,l_1}$ is a linear combination of positive–frequency modes for all $j > 3$. Finally, we note that Eq. (3.20) implies that $\boldsymbol{B}_j = [\boldsymbol{K}_j, \xi_0]$ for all $j \geq 3$, where the boost–like Killing vector $\boldsymbol{B}_j$ is given by Eq. (3.24c). Thus, we calculate the action of $\boldsymbol{B}_j$ on the mode ϕ_{n,l_1} to obtain

$$\boldsymbol{B}_j \phi_{n,l_1} = -i\omega_{n,l_1} \boldsymbol{K}_j \phi_{n,l_1} - \xi_0 \boldsymbol{K}_j \phi_{n,l_1} \, , \tag{7.135}$$

we have used $\xi_0 \phi_{n,l_1} = -i\omega_{n,l_1}\phi_{n,l_1}$. Since we previously found that $\boldsymbol{K}_j\phi_{n,l_1}$ is a linear combination of positive–frequency mode solutions for all $j \geq 3$, Eq. (7.135) implies that $\boldsymbol{B}_j\phi_{n,l_1}$ must also be a linear combination of positive–frequency modes for all $j \geq 3$. Hence, we can conclude that the subspace of positive–frequency solutions for the Dirichlet and Neumann boundary conditions is invariant under the action of the Killing vectors $\boldsymbol{K}_j$, $\boldsymbol{B}_j$ and $\boldsymbol{J}_{kl}$ for all j, k, l.

In conclusion, we have shown that the Dirichlet modes $\phi^{(\nu,\mathrm{D})}_{n,l_1}$ for all $0 \leq \nu$, and the Neumann modes $\phi^{(\nu,\mathrm{N})}_{n,l_1}$ for $0 < \nu < 1$, form invariant positive–frequency subspaces. Thus, it is possible to construct quantum field theories with a stationary vacuum state for these solution spaces by following the general prescription presented in Section 2.3 of Chapter 2.

We conclude this chapter summarising our main results for the analysis of a scalar field in AdS$_N$ in Table 7.1. In this table, the symbols $\widetilde{r}^{(\nu)}$ and $D\widetilde{r}^{(\nu)}$ represent the evaluations $\widetilde{r}^{(\nu)}(\pi/2)$ and $D\widetilde{r}^{(\nu)}(\pi/2)$, respectively. We have also defined the numbers $u_- := 1 - e^{iu}$ and $u_+ := i(1 + e^{iu})$.

Table 7.1: Self–adjoint boundary conditions for scalar field in AdS_N

M (ν)	SABCs	Inv. SABCs	Spectrum (ω)	Inv. P-F sol.
$M^2 \geq M_{BF}$ ($\nu \geq 1$)	Dirichlet	Dirichlet	$\omega = \pm\omega_{n,l_1}^{(\nu,\mathrm{D})}$, $\omega_{n,l_1}^{(\nu,\mathrm{D})} = 2n + \nu + \sigma + 1$	$\phi_n^{(\nu,\mathrm{D})}$, $\omega_{n,l_1}^{(\nu,\mathrm{D})}$
$-1 - M_{BF} < M^2 < M_{BF}$ ($0 < \nu < 1$)	$u_- D\widetilde{r}^{(\nu)} = u_+\widetilde{r}^{(\nu)}$	Dirichlet	$\omega = \pm\omega_{n,l_1}^{(\nu,\mathrm{D})}$, $\omega_{n,l_1}^{(\nu,\mathrm{D})} = 2n + \nu + \sigma + 1$	$\phi_n^{(\nu,\mathrm{D})}$, $\omega_{n,l_1}^{(\nu,\mathrm{D})}$
		Neumann	$\omega = \pm\omega_{n,l_1}^{(\nu,\mathrm{N})}$, $\omega_{n,l_1}^{(\nu,\mathrm{N})} = 2n - \nu + \sigma + 1$	$\phi_n^{(\nu,\mathrm{N})}$, $\omega_{n,l_1}^{(\nu,\mathrm{N})}$
$M^2 = -1 - M_{BF}$ ($\nu = 0$)	$u_- D\widetilde{r}^{(0)} = u_+\widetilde{r}^{(0)}$	Dirichlet	$\omega = \pm\omega_{n,l_1}^{(0,\mathrm{D})}$, $\omega_{n,l_1}^{(0,\mathrm{D})} = 2n + \sigma + 1$	$\phi_n^{(0,\mathrm{D})}$, $\omega_{n,l_1}^{(0,\mathrm{D})}$

The number $M_{BF} := (N-3)(N+1)/4$ is the B-F bound.
Here n is always assumed to be in $\mathbb{N}_0$.
We have taken the definition $\sigma = l_1 + (N-3)/2$

— 8 —

Dirac spinors in AdS_2

In this chapter we analyse a spinor field of mass $0 \leq M$ obeying the Dirac equation in AdS_2. We will apply the prescription introduced in Chapter 2 for spinor fields. Following the approach taken for the scalar field case in Chapter 6, we will first obtain the general solutions of the Dirac equation via separation of variables, decomposing the solution with respect to the frequency spectrum. Since our goal is to prescribe well–defined dynamics for the classical field in order to obtain a quantum field theory via canonical quantisation (as in Section 2.3), we consider only positive–frequency solutions. The spatial component of the Dirac operator, defined on a suitable Hilbert space of functions, is a densely defined symmetric operator which fails to be self–adjoint. Hence, we will apply the theory of self–adjoint extensions presented in Chapter 5 to this operator. The theory of self–adjoint extensions has been applied to a Dirac field in AdS_4 in a slightly different way to the one we have used for the scalar field in AdS_N by Bachelot [86], where certain boundary conditions are highlighted.

The admissible self–adjoint extensions provided by von Neumann's theorem 5.1.3 are given in terms of their domains. Similarly to the analysis of scalar field theory in AdS_2 and AdS_N, $N \geq 3$, we associate to each self–adjoint extension a self–adjoint boundary condition that the elements in its domain must satisfy. However, we will use a different method to obtain the associated self–adjoint boundary conditions to the one we applied in Chapters 6 and 7 for the case of scalar field theories. This equivalent approach is based on the analysis in Ref. [71]. The type of self–adjoint boundary condition depends on the absolute value of the mass of the field. If $|M|$ is sufficiently large, the boundary conditions are uniquely determined by requiring the solutions to the Dirac equation to be normalisable with respect to the Dirac inner product, which is equivalent to the fact that the spatial Dirac operator for this mass range is essentially self–adjoint and thus, has a unique self–adjoint extension. On the other hand, in a certain range of low mass parameter, similarly to the case of a scalar field, the self–adjoint extensions will be parametrised by a 2×2 unitary matrix.

We will then determine which of the self–adjoint boundary conditions result in invariant mode solutions under the infinitesimal action of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, which can be realised through a certain Lie derivative operator defined on spinor solutions. We will then find which of the resulting invariant mode solution spaces admit an invariant positive– or negative–frequency subspace and hence, result in a vacuum state invariant under the $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ action following the prescription of Section 2.3. Finally, we analyse the cases for which the invariant boundary conditions result in modes which do not admit this frequency spectrum splitting and thus, describe quantum theories with non–invariant vacuum sectors.

8.1 SOLUTIONS OF THE DIRAC EQUATION IN AdS_2

Let us consider the two–dimensional anti–de Sitter spacetime, AdS_2. We use the static coordinate system defined in Eq. (3.8) which, for $N = 2$, reduces to $(x^0, x^1) = (t, \rho)$, with $t \in \mathbb{R}$ and $\rho \in (-\pi/2, \pi/2)$. The line element for AdS_2 is given by Eq. (6.1). With respect to the metric tensor, the non–zero components of the Levi–Civita connection are given by Eq. (3.11), and for $N = 2$, they reduce to ${\Gamma^0}_{01} = {\Gamma^1}_{00} = {\Gamma^1}_{11} = \tan\rho$. We adopt the local orthonormal frame $\{e_a\}_{a=0,1}$ given by Eq. (3.12). For the two–dimensional case, the non–zero components ${e_a}^\mu$ of the frame fields are given by

$$ {e_0}^0 = \cos\rho = {e_1}^1 \,. \tag{8.1} $$

With respect to the orthonormal frame, the connection 1–form ${\omega^a}_b$, defined in Eq. (2.19), has non–zero components ${\omega^a}_{b\,\mu}$ given by Eq. (3.13), which read ${\omega^0}_{1\,0} = {\omega^1}_{0\,0} = \tan\rho$

Following a convention similar to that of Ref. [14], we will use the 2–dimensional representation of the 2×2–gamma matrices γ^a given by

$$ \gamma^0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \,, \qquad \gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \,, \tag{8.2} $$

which satisfy the anticommutation relation in Eq. (2.22), that is, $\{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbb{I}$, with $\eta^{ab} = \mathrm{diag}(-1, 1)$. For this choice of gamma matrices, we have $(\gamma^0)^\dagger = -\gamma^0$ and $(\gamma^1)^\dagger = \gamma^1$. From Eq. (2.23), we see that the quantities Σ^{ab} for the gamma matrices in Eq.(8.2) are given by

$$ \Sigma^{01} = \frac{1}{2} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \,, \tag{8.3} $$

and by $\Sigma^{10} = -\Sigma^{01}$. In this representation the charge conjugation matrix C is given by $C = 2z\Sigma^{01}$ for any $z \in \mathbb{C}$ on the unit circle. We choose $z = -1$, and write

$$ C := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \,, \tag{8.4} $$

and, from Eq. (8.2), it follows that we have $C = -\gamma^0\gamma^1$.

As discussed in Chapter 2, spinor fields in AdS_2 will be regarded as elements of the space $C^\infty(\mathrm{AdS}_2, \mathbb{C}^2)$, of $\mathbb{C}^2$–valued smooth functions on anti–de Sitter spacetime. The spin covariant derivative acting on a spinor $\psi(t, \rho)$ in AdS_2 is defined using Eqs. (2.24) and (2.25). With respect to our choice of orthonormal frame (8.1), the spinor covariant derivatives in the time and spatial directions are found to be given by

$$ \nabla^{(D)}_0 = \partial_t - \frac{1}{2}\tan\rho\,\Sigma^{01} \,, \tag{8.5a} $$

$$ \nabla^{(D)}_1 = \partial_\rho \,, \tag{8.5b} $$

respectively. We recall that the spacetime gamma matrices are defined by the relation $\tilde{\gamma}^\mu := {e_a}^\mu \gamma^a$. Hence, the Dirac equation defined for a standard static spacetime in Eq. (2.28) reduces, for the case of AdS_2 in global coordinates (t, ρ), to

$$\cos\rho \left[\gamma^0 \partial_t + \gamma^1 \left(\partial_\rho + \frac{1}{2}\tan\rho\right)\right] \psi(t,\rho) = M\psi(t,\rho)\,, \tag{8.6}$$

where $\psi \in C^\infty\left(\mathrm{AdS}_2; \mathbb{C}^2\right)$, and $M \in \mathbb{R}$ denotes the mass of the field.

It will be convenient to define the two–component spinor $\tilde{\psi}$ via the relation

$$\psi(t,\rho) = (\cos\rho)^{\frac{1}{2}} \tilde{\psi}(t,\rho)\,, \tag{8.7}$$

so that Eq. (8.6) is equivalent to the equation for the spinor $\tilde{\psi}$ given by

$$\left(\gamma^0 \partial_t + \gamma^1 \partial_\rho\right) \tilde{\psi}(t,\rho) = M \sec\rho\, \tilde{\psi}(t,\rho)\,. \tag{8.8}$$

A simple substitution shows that if $\tilde{\psi}$ is a solution of this equation with mass M, then $i\Sigma^{01}\tilde{\psi}$ is a solution with mass $-M$. Then, without loss of generality, we will only consider solutions to Eq. (8.8) with $M \geq 0$.

Since we are interested in the description of the solutions of Eq. (8.6) in terms of positive– and negative–frequency mode spinors, we will consider solutions $\tilde{\psi}$ of the form

$$\tilde{\psi}(t,\rho) = \Psi_\omega(\rho) e^{-i\omega t}\,, \qquad \omega \neq 0\,, \tag{8.9}$$

where $\Psi_\omega \in C^\infty((-\pi/2, \pi/2), \mathbb{C}^2)$. Following the description of spinors in standard static spacetimes given in Chapter 2, we identify the spatial spinor associated to ψ with $(\cos\rho)^{1/2}\Psi_\omega$. We recall that the Hilbert space $\mathscr{H}_D$ was defined as $L^2\left([-\pi/2, \pi/2]; \mathbb{C}^2\right)$, the space of square–integrable spatial spinors with respect to the measure $\mathrm{d}V' = \sqrt{h}\,\mathrm{d}\rho = \sec\rho\,\mathrm{d}\rho$ (which follows by Eq. (7.1)). Considering the scaling of the spatial spinor defined in Eq. (8.7), the scaled spatial spinors Ψ_ω can be identified with the elements of the Hilbert space $L^2\left([-\pi/2, \pi/2]; \mathbb{C}^2\right)$ with respect to the measure $\mathrm{d}\rho$. Since the multiplication map by the factor $(\cos\rho)^{1/2}$ is a unitary isomorphism between these two Hilbert spaces, we will refer to the latter by the same symbol, $\mathscr{H}_D$.

With these considerations in mind, we see that Eq. (2.30) implies that the inner product between the elements $\Psi_1, \Psi_2 \in \mathscr{H}_D$, is given by

$$\langle \Psi_1, \Psi_2 \rangle_D = \int_{-\pi/2}^{\pi/2} \Psi_1(\rho)^\dagger \Psi_2(\rho) \mathrm{d}\rho\,, \tag{8.10}$$

where $\Psi^\dagger$ denotes the conjugate transpose of $\Psi(\rho) \in \mathbb{C}^2$. The inner product of $\mathscr{H}_D$ induces an inner product between the spinors $\psi \in L^2\left(\mathrm{AdS}_2; \mathbb{C}^2\right)$. Given two solutions, ψ_1 and ψ_2 of Eq. (8.6), of the form $\psi_j(t,\rho) = (\cos\rho)^{1/2}\Psi_{\omega_j}(\rho) e^{-i\omega_j t}$, $j = 1, 2$, with non–zero frequencies $\omega_1 \neq \omega_2$, the pairing

$$\begin{aligned}(\psi_1, \psi_2)_D &:= \int_{-\pi/2}^{\pi/2} \psi_1(t,\rho)^\dagger \psi_2(t,\rho) \frac{\mathrm{d}\rho}{\cos\rho}\,,\\ &= e^{-i(\omega_2 - \omega_1)t} \left\langle \Psi_{\omega_1}, \Psi_{\omega_2} \right\rangle_D\,,\end{aligned} \tag{8.11}$$

defines a non–degenerate bilinear form for a fixed $t \in \mathbb{R}$.

From Eqs. (8.6) and (8.8), it follows that the spatial spinor Ψ_ω satisfies the equation

$$\mathbb{D}\,\Psi_\omega(\rho) = \omega \Psi_\omega(\rho)\,, \tag{8.12}$$

where we have defined the operator[1]

$$\mathbb{D} := i\gamma^0\gamma^1 \frac{\mathrm{d}}{\mathrm{d}\rho} - i\gamma^0 M \sec\rho\,. \tag{8.13}$$

Analogously to the case of a scalar field in AdS_2, we will consider the operator $\mathbb{D}$ acting on the natural domain $\mathrm{Dom}(\mathbb{D}) = C_c^\infty\left((-\pi/2, \pi/2), \mathbb{C}^2\right)$, that is, the subspace of compactly supported smooth maps $(-\pi/2, \pi/2) \to \mathbb{C}^2$ with support away from the boundary. We note that, since any element in $\mathscr{H}_D$ can be approximated arbitrarily well by elements in $C_c^\infty\left((-\pi/2, \pi/2); \mathbb{C}^2\right)$, we have $\overline{\mathrm{Dom}(\mathbb{D})} = \mathscr{H}_D$. Hence, $\mathbb{D}$ is a densely defined operator. Furthermore, the operator $\mathbb{D}$ is a symmetric operator with respect to the inner product given by Eq. (8.10), that is, for all $\Psi_1, \Psi_2 \in \mathrm{Dom}(\mathbb{D})$, we have

$$\langle \Psi_1, \mathbb{D}\Psi_2 \rangle_D = \langle \mathbb{D}\Psi_1, \Psi_2 \rangle_D\,, \tag{8.14}$$

which follows by using Eq. (8.10) and integrating by parts.

Now, we consider the adjoint operator $\mathbb{D}^\dagger$ of $\mathbb{D}$. From Definition 2.2.2 we have that if $\Psi' \in \mathrm{Dom}(\mathbb{D}^\dagger)$, then

$$\langle \Psi', \mathbb{D}\Psi \rangle_D = \left\langle \mathbb{D}^\dagger \Psi', \Psi \right\rangle_D\,, \tag{8.15}$$

for all $\Psi \in \mathrm{Dom}(\mathbb{D})$. In general, $\mathrm{Dom}(\mathbb{D}) \neq \mathrm{Dom}(\mathbb{D}^\dagger)$, and thus, the operator $\mathbb{D}$ is not self–adjoint[2]. In fact, the domain of the adjoint operator is found to satisfy

$$\mathrm{Dom}(\mathbb{D}) \subseteq \mathrm{Dom}(\mathbb{D}^\dagger)\,, \tag{8.16}$$

but we can easily find elements in $\mathscr{H}_D$ not contained in the domain of $\mathbb{D}$ satisfying Eq. (8.15) (*e.g.*, spatial spinors Ψ' for which the component functions $\Psi'^{(1)}, \Psi'^{(2)}$ are absolutely continuous functions on $(-\pi/2, \pi/2)$ do satisfy Eq. (8.15) and are not contained in $\mathrm{Dom}(\mathbb{D})$).

Thus, we will apply the theory of self–adjoint extensions presented in Chapter 5 to the operator $\mathbb{D}$ to find a family of self–adjoint operators $\mathbb{D}_U$. Following the general prescription of Section 5.1, we will first look for solutions in $\mathscr{H}_D$ satisfying the equations $\mathbb{D}^\dagger \Psi_\pm = \pm i \Psi_\pm$ in order to find the deficiency indices $n_\pm$ of the operator $\mathbb{D}$. Then, we will apply von Neumann's theorem to identify the domains of the admissible self–adjoint extensions of $\mathbb{D}$.

[1] The operator $\mathbb{D}$ above and the one defined in Eq. (2.29) are not the same for $N = 2$, but instead related by the intertwiner defined as multiplication by $(\cos\rho)^{1/2}$.

[2] We will explicitly prove the lack of self–adjointness of $\mathbb{D}$ shortly by calculating its deficiency indices and applying von Neumann's theorem.

Similarly to the case of a scalar field, we determine if normalisable solutions of $\mathbb{D}^\dagger\Psi_\omega = \omega\Psi$, that is,

$$\left[i\gamma^0\gamma^1\frac{\mathrm{d}}{\mathrm{d}\rho} - i\gamma^0 M\sec\rho\right]\Psi_\omega(\rho) = \omega\Psi_\omega(\rho)\,, \tag{8.17}$$

exist for $\omega \in \mathbb{C}$. Thus, by specialising to the particular values of $\omega = \pm i$, solving Eq. (8.17) is equivalent to determining the deficiency subspaces $\mathscr{K}_\pm$ of the operator $\mathbb{D}$. The number of linearly independent solutions will thus provide the deficiency indices of $\mathbb{D}$.

In order to find the general solutions, we project Eq. (8.17) onto the components of the spinor Ψ_ω with respect to the gamma matrix representation we have chosen. Let us define

$$\Psi_\omega = \begin{pmatrix}\Psi_\omega^{(1)}\\ \Psi_\omega^{(2)}\end{pmatrix}\,, \tag{8.18}$$

with $\Psi_\omega^{(1)}, \Psi_\omega^{(2)}$ complex–valued functions on the interval $(-\pi/2, \pi/2)$. Hence, Eq. (8.17) is equivalent to the coupled system of equations given by

$$\frac{\mathrm{d}}{\mathrm{d}\rho}\Psi_\omega^{(1)}(\rho) + M\sec\rho\,\Psi_\omega^{(1)}(\rho) = \omega\Psi_\omega^{(2)}(\rho)\,, \tag{8.19a}$$

$$-\frac{\mathrm{d}}{\mathrm{d}\rho}\Psi_\omega^{(2)}(\rho) + M\sec\rho\,\Psi_\omega^{(2)}(\rho) = \omega\Psi_\omega^{(1)}(\rho)\,. \tag{8.19b}$$

We note that if Ψ_ω is a solution of Eq. (8.12) with $\omega \in \mathbb{R}$ and $\omega > 0$, then the charge conjugate spinor $\Psi_\omega^c := C(\gamma^0)^T\overline{\Psi_\omega}$, where C is given by Eq. (8.4), is a solution of Eq. (8.12) with $-\omega$.

Now, by eliminating $\Psi_\omega^{(2)}$ in Eq. (8.19) we obtain the second order equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}\Psi_\omega^{(1)}(\rho) + \left[\omega^2 + M\sec\rho\,\tan\rho - M^2\sec^2\rho\right]\Psi_\omega^{(1)}(\rho) = 0\,. \tag{8.20}$$

A general solution to this equation when $M - 1/2 \notin \mathbb{N}_0$ is given in terms of the Gaussian hypergeometric functions [23], and reads

$$\begin{aligned}\Psi_\omega^{(1)}(\rho) =& (2M+1)C_1\,\sigma(\rho)^M F\left(\omega, -\omega; \frac{1}{2}+M; \frac{1-\sin\rho}{2}\right)\\ &+ \omega\,C_2\,\cos\rho\,\sigma(\rho)^{-M} F\left(1+\omega, 1-\omega; \frac{3}{2}-M; \frac{1-\sin\rho}{2}\right)\,,\end{aligned} \tag{8.21}$$

where we have defined

$$\sigma(\rho) := \left(\frac{1-\sin\rho}{1+\sin\rho}\right)^{\frac{1}{2}}\,, \tag{8.22}$$

and with $C_1, C_2 \in \mathbb{C}$ arbitrary constants. Using this solution, we now define the second spinor component $\Psi_\omega^{(2)}$ through Eq. (8.19a). By using Eq. (7.105) it can readily be verified that the second component $\Psi_\omega^{(2)}$ is given by

$$\begin{aligned}\Psi_\omega^{(2)}(\rho) =& \omega\,C_1\cos\rho\,\sigma(\rho)^M F\left(1+\omega, 1-\omega; \frac{3}{2}+M; \frac{1-\sin\rho}{2}\right)\\ &+ (2M-1)C_2\,\sigma(\rho)^{-M} F\left(\omega, -\omega; \frac{1}{2}-M; \frac{1-\sin\rho}{2}\right)\,.\end{aligned} \tag{8.23}$$

If $M = 1/2 + k$, with $k \in \mathbb{N}_0$, then it can be shown that the general solutions are instead given by

$$\begin{aligned}\Psi^{(1)}_\omega(\rho) =&\sigma(\rho)^{\frac{1}{2}}\left[C_1\left(\mathsf{P}^{-k}_\omega(\sin\rho) + \mathsf{P}^{-k}_{\omega-1}(\sin\rho)\right)\right.\\ &\left.+ C_2\left(\mathsf{Q}^{-k}_\omega(\sin\rho) + \mathsf{Q}^{-k}_{\omega-1}(\sin\rho)\right)\right]\,, \end{aligned}\tag{8.24a}$$

$$\begin{aligned}\Psi^{(2)}_\omega(\rho) =&\sigma(\rho)^{-\frac{1}{2}}\left[C_1\left(\mathsf{P}^{-k}_{\omega-1}(\sin\rho) - \mathsf{P}^{-k}_\omega(\sin\rho)\right)\right.\\ &\left.+ C_2\left(\mathsf{Q}^{-k}_{\omega-1}(\sin\rho) - \mathsf{Q}^{-k}_\omega(\sin\rho)\right)\right]\,, \end{aligned}\tag{8.24b}$$

where P^μ_ν and Q^μ_ν are Ferrers functions of the first and second kind, respectively. These are given by [23, Eqs. 14.3.1, 14.3.2]

$$\mathsf{P}^\mu_\nu(x) = \frac{1}{\Gamma(1-\mu)}\left(\frac{1+x}{1-x}\right)^{\mu/2} F\left(\nu+1, -\nu; 1-\mu; \frac{1-x}{2}\right)\,, \tag{8.25a}$$

$$\mathsf{Q}^\mu_\nu(x) = \frac{\pi}{\sin\mu\pi}\left(\cos\mu\pi\,\mathsf{P}^\mu_\nu(x) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}\mathsf{P}^{-\mu}_\nu(x)\right)\,. \tag{8.25b}$$

The functions Q^{-k}_ν with $k \in \mathbb{N}_0$ can be defined by substituting Eq. 14.9.3 in Ref. [23] into Eq. (8.25b) and taking the limit $\mu \to -k$. Solutions for $M = 1/2$ reduce to Legendre functions by means of the relation $\mathsf{P}^0_\omega(x) = \mathsf{P}_\omega(x)$. We also note that the solutions for the massless spinor field can be directly obtained from Eqs. (8.19a) and (8.20) by setting $M = 0$, in which case the components are simply given by

$$\Psi^{(1)}_{\omega,M=0}(\rho) = \tilde{C}_1\cos\omega\rho + \tilde{C}_2\sin\omega\rho\,, \tag{8.26a}$$

$$\Psi^{(2)}_{\omega,M=0}(\rho) = -\tilde{C}_1\sin\omega\rho + \tilde{C}_2\cos\omega\rho\,, \tag{8.26b}$$

for some $\tilde{C}_1, \tilde{C}_2 \in \mathbb{C}$.

In order to determine if square–integrable solutions exist, we need to find for which values of M the functions in Eqs. (8.21) and (8.23) are square–integrable. This can be done by analysing the asymptotic behaviour of these solutions at the boundary. The leading behaviour of the hypergeometric functions appearing in these solutions at $\rho = \pm\pi/2$ is different for different values of the mass of the spinor field M, thus, it will be convenient to perform this analysis separately for the following cases:

1. $0 \le M < 1/2$.

2. $M > 1/2$, with $M - 1/2 \notin \mathbb{N}$.

3. $M = 1/2 + k$, with $k \in \mathbb{N}_0$.

We analyse cases 1 and 2 first. If $M = 0$, then it is clear from Eq. (8.26) that both solutions are square–integrable for any $\tilde{C}_1, \tilde{C}_2, \omega \in \mathbb{C}$. Therefore, the deficiency indices for the massless case are given by $n_\pm = 2$. Now, for the non–zero values of M falling on these ranges, we evaluate the functions (8.21) and (8.23) at $\rho = \pi/2 - \epsilon$ for sufficiently small

$\epsilon > 0$. Then, using the fact that $F(a,b;c;\epsilon) = 1 + O(\epsilon)$ as $\epsilon \to 0$, it follows that

$$\Psi^{(1)}_\omega\left(\frac{\pi}{2}-\epsilon\right) = \left[(2M+1)C_1 + O\left(\epsilon^2\right)\right]\epsilon^M + \left[\omega\, C_2 + O\left(\epsilon^2\right)\right]\epsilon^{1-M}\,, \tag{8.27a}$$

$$\Psi^{(2)}_\omega\left(\frac{\pi}{2}-\epsilon\right) = \left[\omega\, C_1 + O\left(\epsilon^2\right)\right]\epsilon^{1+M} + \left[(2M-1)C_2 + O\left(\epsilon^2\right)\right]\epsilon^{-M}\,. \tag{8.27b}$$

To evaluate the component functions near $\rho = -\pi/2$, we use the transformation formula for the hypergeometric function [23, Eq. 15.8.4],

$$\begin{aligned} F(a,b;c;x) =& \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-x)^{c-a-b}F(c-a,c-b;c-a-b+1;1-x) \\ &+ \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b;a+b-c+1;1-x)\,, \end{aligned} \tag{8.28}$$

so that we can write Eqs. (8.21) and (8.23) as

$$\begin{aligned} \Psi^{(1)}_\omega(\rho) =& (2M+1)C_1\sigma(\rho)^M\left[A_1^{(M)}F_1^M(\rho) + A_2^{(M)}\left(1+\sin\rho\right)^{M+\frac{1}{2}}F_2^M(\rho)\right] \\ &+ \omega C_2\cos\rho\,\sigma(\rho)^{-M}\left[B_1^{(M)}F_3^M(\rho) + B_2^{(M)}(1+\sin\rho)^{-M-\frac{1}{2}}F_4^M(\rho)\right]\,, \end{aligned} \tag{8.29}$$

and

$$\begin{aligned} \Psi^{(2)}_\omega(\rho) =& \omega\, C_1\cos\rho\,\sigma(\rho)^{-M}\left[B_1^{(-M)}F_3^{-M}(\rho) + B_2^{(-M)}\left(1+\sin\rho\right)^{M-\frac{1}{2}}F_4^{-M}(\rho)\right] \\ &+ (2M-1)C_2\sigma(\rho)^{-M}\left[A_1^{(-M)}F_1^{-M}(\rho) + A_2^{(-M)}\left(1+\sin\rho\right)^{\frac{1}{2}-M}F_2^{-M}(\rho)\right]\,, \end{aligned} \tag{8.30}$$

respectively, where we have defined the quantities

$$A_1^{(M)} = \frac{\Gamma\left(\frac{1}{2}+M\right)^2}{\Gamma\left(\frac{1}{2}+M+\omega\right)\Gamma\left(\frac{1}{2}+M-\omega\right)}\,,\quad A_2^{(M)} = \frac{\Gamma\left(\frac{1}{2}+M\right)\Gamma\left(-\frac{1}{2}-M\right)}{\Gamma\left(\omega\right)\Gamma\left(-\omega\right)}\,, \tag{8.31a}$$

$$B_1^{(M)} = \frac{\Gamma\left(\frac{3}{2}-M\right)\Gamma\left(-M-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-M+\omega\right)\Gamma\left(\frac{1}{2}-M-\omega\right)}\,,\quad B_2^{(M)} = \frac{\Gamma\left(\frac{3}{2}-M\right)\Gamma\left(\frac{1}{2}+M\right)}{\Gamma\left(1+\omega\right)\Gamma\left(1-\omega\right)}\,. \tag{8.31b}$$

and the functions $F_j^M(\rho)$, $j = 1,\ldots 4$, are the hypergeometric functions of argument $(1+\sin\rho)/2$ that result from the transformations in Eq. (8.28) and satisfy $F_j^M(\rho) = 1 + O(1+\sin\rho)$ as $x \to -\pi/2$. We are now able to evaluate these functions at $\rho = \epsilon - \pi/2$, for the same small parameter $\epsilon > 0$ above. This results in

$$\begin{aligned} \Psi^{(1)}_\omega\left(\epsilon-\frac{\pi}{2}\right) =& \left[(2M+1)C_1A_1^{(M)} + C_2\omega\, B_2^{(M)} + O\left(\epsilon^2\right)\right]\epsilon^{-M} \\ &+ \left[(2M+1)C_1A_2^{(M)} + C_2\omega\, B_1^{(M)} + O\left(\epsilon^2\right)\right]\epsilon^{M+1}\,, \end{aligned} \tag{8.32a}$$

$$\begin{aligned} \Psi^{(2)}_\omega\left(\epsilon-\frac{\pi}{2}\right) =& \left[C_1\,\omega\, B_1^{(-M)} + (2M-1)C_2A_2^{(-M)} + O\left(\epsilon^2\right)\right]\epsilon^{1-M}\,, \\ &+ \left[C_1\,\omega\, B_2^{(-M)} + (2M-1)C_2A_1^{(-M)}O\left(\epsilon^2\right)\right]\epsilon^{M}\,. \end{aligned} \tag{8.32b}$$

Since the inner product in Eq. (8.10) is written in terms of the components in Eq. (8.18) as

$$\langle\Psi_1,\Psi_2\rangle_D = \int_{-\pi/2}^{\pi/2}\left(\overline{\Psi_1^{(1)}(\rho)}\Psi_2^{(1)}(\rho) + \overline{\Psi_1^{(2)}(\rho)}\Psi_2^{(2)}(\rho)\right)\mathrm{d}\rho\,, \tag{8.33}$$

it follows that the spinor Ψ_ω will be normalisable if the asymptotic behaviour of the function $|\Psi_\omega^{(1)}(\rho)|^2 + |\Psi_\omega^{(2)}(\rho)|^2$ as $\rho \to \pm\pi/2$ is given by $(\pi/2 - |\rho|)^r$ with $r > -1$. Using Eq. (8.27) and the fact that $M \geq 0$ for the cases we are considering, we have that

$$\left|\Psi_\omega^{(1)}\left(\frac{\pi}{2} - \epsilon\right)\right|^2 + \left|\Psi_\omega^{(2)}\left(\frac{\pi}{2} - \epsilon\right)\right|^2 \sim |C_2|^2 \left(|\omega|^2 \epsilon^{2-2M} + (2M-1)^2 \epsilon^{-2M}\right), \tag{8.34}$$

and,

$$\begin{aligned}\left|\Psi_\omega^{(1)}\left(\epsilon - \frac{\pi}{2}\right)\right|^2 + \left|\Psi_\omega^{(2)}\left(\epsilon - \frac{\pi}{2}\right)\right|^2 \sim &\left|C_1\, \omega\, B_1^{(-M)} + (2M-1) C_2 A_2^{(-M)}\right|^2 \epsilon^{2-2M} \\ &+ \left|(2M+1) C_1 A_1^{(M)} + C_2\, \omega\, B_2^{(M)}\right|^2 \epsilon^{-2M}, \end{aligned} \tag{8.35}$$

as $\epsilon \to 0$. Thus, it follows that if $0 < M < 1/2$, then the leading term of these expressions is proportional to ϵ^r, with $r > -1$ at both endpoints. Therefore, the two component functions $\Psi_\omega^{(1)}$ and $\Psi_\omega^{(2)}$ are square–integrable for any $C_1, C_2, \omega \in \mathbb{C}$, in particular, for $\omega = \pm i$, and thus, the deficiency subspaces of $\mathbb{D}$ for this mass range have dimension $n_\pm = 2$. By von Neumann's theorem 5.1.3, the operator $\mathbb{D}$ admits a family of self–adjoint extensions parametrised by the isometries from $\mathscr{K}_+$ to $\mathscr{K}_-$ which, due to finite–dimensionality, can be realised as 2×2–unitary matrices. The self–adjoint extensions of the operator $\mathbb{D}$ for this case will be obtained in Sec. 8.2.

On the other hand, for Case 2, if $1/2 < M < 3/2$, the terms with ϵ^{2-2M} decay faster than ϵ^{-1}, so the singular behaviour comes from the terms proportional to ϵ^{-2M}. For the spinor Ψ_ω to be square–integrable at both endpoints, we must have

$$C_2 = 0\,, \quad \text{and} \quad A_1^{(M)} = 0\,. \tag{8.36}$$

If $M > 3/2$ and $M - 1/2 \notin \mathbb{N}$, then terms proportional to ϵ^{2-2M} are also singular, so square–integrable solutions for this range need to satisfy Eq. (8.36) as well as the additional condition

$$\omega\, B_1^{(-M)} = 0\,. \tag{8.37}$$

Using the definitions of these quantities in Eq. (8.31) we note that $B_1^{(-M)}$ is proportional to $A_1^{(M)}$ as a function of ω, so the only positive values of ω for which $A_1^{(M)}$, and therefore $B_1^{(-M)}$, vanish are given by $\omega = \omega_n^I$, where

$$\omega_n^I := \frac{1}{2} + M + n\,, \qquad n \in \mathbb{N}_0\,. \tag{8.38}$$

This implies that no square–integrable solutions exist for $\omega = \pm i$, and therefore, the deficiency spaces are both zero–dimensional and thus, by von Neumann's theorem, the operator $\mathbb{D}$ is essentially self–adjoint. This means that the unique self–adjoint extension for the operator $\mathbb{D}$ is its closure, $\overline{\mathbb{D}}$.

A similar conclusion holds for Case 3. First we note that for the values of M we are considering, the general solution to Eq. (8.19a) with $\omega = 0$ is given by $\Psi(\rho) =$

$C_1(\sigma(\rho)^M, 0)^T + C_2(0, \sigma(\rho)^{-M})^T$, where $\sigma(\rho)$ is defined by Eq. (8.22). This solution is not square–integrable for any $M \geq 1/2$, so we will continue the analysis for these values of M assuming $\omega \neq 0$. Let us now consider the functions in Eq. (8.24) with $k > 0$. As shown in Appendix H, the asymptotic behaviour of the these component functions at $\rho = \pi/2 - \epsilon$, for sufficiently small $\epsilon > 0$ is given by

$$\Psi_\omega^{(1)}\left(\frac{\pi}{2} - \epsilon\right) = \left[C_2 \omega A_3^{(k)} + O\left(\epsilon^2\right)\right] \epsilon^{-k+\frac{1}{2}}\,, \tag{8.39a}$$

$$\Psi_\omega^{(2)}\left(\frac{\pi}{2} - \epsilon\right) = \left[C_2 k A_3^{(k)} + O\left(\epsilon^2\right)\right] \epsilon^{-k-\frac{1}{2}}\,, \tag{8.39b}$$

and similarly, at $\rho = \epsilon - \pi/2$, we have

$$\Psi_\omega^{(1)}\left(\epsilon - \frac{\pi}{2}\right) = k\, A_3^{(k)} \left[C_1 \frac{2}{\pi} \sin \pi(\omega - k) + C_2 \cos \pi(\omega - k) + O\left(\epsilon^2\right)\right] \epsilon^{-k-\frac{1}{2}}\,, \tag{8.40a}$$

$$\Psi_\omega^{(2)}\left(\epsilon - \frac{\pi}{2}\right) = \omega\, A_3^{(k)} \left[C_1 \frac{2}{\pi} \sin \pi(\omega - k) + C_2 \cos \pi(\omega - k) + O\left(\epsilon^2\right)\right] \epsilon^{-k+\frac{1}{2}}\,, \tag{8.40b}$$

where we have defined

$$A_3^{(k)} := \frac{2^k \Gamma(k) \Gamma(\omega - k)}{\Gamma(\omega + k + 1)}\,. \tag{8.41}$$

The behaviour of the modulus squared of the spinor at the boundary is obtained using Eqs. (8.39) and (8.40) and it can readily be verified that it is given by Eq. (8.34) at $\rho = \pi/2$ and by Eq. (8.35) at $\rho = -\pi/2$, with $M = k + 1/2$. From these approximations it is clear that the leading terms at both endpoints are of the form ϵ^r with $r < -1$, so in order to obtain square–integrable solutions the expressions on the left–hand side must vanish simultaneously. This occurs only when $|A_3^{(k)}|^2 = 0$, or when $C_2 = 0$ and $\sin \pi(\omega - k) = 0$. From Eq. (8.41), the former case only happens when $\omega = -n - k - 1$ with $n \in \mathbb{N}_0$, and the latter only happens when $\omega = n + k + 1$. Therefore, no square–integrable solutions for $\omega = \pm i$ exist for this case either, thus, the deficiency indices are once again $n_\pm = 0$, hence, the unique self–adjoint extension is given by the closure $\overline{\mathbb{D}}$.

If $k = 0$, then the solutions in Eq. (8.24) are given in terms of Legendre functions which have a different asymptotic expansion at the endpoints of the boundary. From the analysis in Appendix H we find that

$$\left|\Psi_\omega^{(1)}\left(\frac{\pi}{2} - \epsilon\right)\right|^2 + \left|\Psi_\omega^{(2)}\left(\frac{\pi}{2} - \epsilon\right)\right|^2 \sim \frac{|C_2|^2}{|\omega|^2}\, \epsilon^{-1}\,, \tag{8.42a}$$

$$\left|\Psi_\omega^{(1)}\left(\epsilon - \frac{\pi}{2}\right)\right|^2 + \left|\Psi_\omega^{(2)}\left(\epsilon - \frac{\pi}{2}\right)\right|^2 \sim \left|C_1 \frac{2}{\pi\omega} \sin \pi\omega + C_2 \frac{1}{\omega} \cos \pi\omega\right|^2 \epsilon^{-1}\,, \tag{8.42b}$$

as $\epsilon \to 0$. Once again, the spinor solution will be square–integrable if the above expressions on the left–hand side vanish. This only happens if $C_2 = 0$ and $\sin \pi\omega = 0$, the latter condition restricting the values of ω to be $\omega \in \mathbb{Z}$, and we once again note that these values are also of the form $\omega_n^{(I)}$ in Eq. (8.38) with $M = 1/2$. Thus, by the same argument as for the case $k > 0$, the deficiency spaces are zero–dimensional, and thus we will treat this case in a way similar to the Case 2.

Let us summarise the results of this section. Looking at the asymptotic behaviour of the solutions at the boundary we have found that the deficiency indices of the operator $\mathbb{D}$ are $n_\pm = 2$ when $0 \leq M < 1/2$ (Case 1 above) and $n_\pm = 0$ when the mass of the spinor field satisfies $M \geq 1/2$ (Cases 2 and 3). For the latter case, square–integrable solutions exist only for the frequencies given by Eq. (8.38). Now that we have found the deficiency indices of the operator $\mathbb{D}$, we will proceed to obtain the associated self–adjoint extensions $\mathbb{D}_U$.

8.2 SELF-ADJOINT EXTENSIONS OF THE OPERATOR $\mathbb{D}$

In this section we will find the self–adjoint extensions of the operator $\mathbb{D}$. We start by analysing the case with $0 \leq M < 1/2$ for which we have found that $n_\pm = 2$. Since the operator $\mathbb{D}$ is densely defined and symmetric, the domain of its adjoint operator is given by Eq. (5.4). Hence, based on the theory presented in Chapter 5, we know that the domain of the admissible self–adjoint extensions $\mathbb{D}_U$ of $\mathbb{D}$ must be of the form of Eq. (5.19), that is,

$$\mathrm{Dom}(\mathbb{D}_U) = \mathrm{Dom}(\overline{\mathbb{D}}) \oplus_{\mathbb{D}} \mathscr{S}\,, \tag{8.43}$$

where $\mathscr{S} \subseteq \mathscr{K}_+ \oplus_{\mathbb{D}} \mathscr{K}_-$ is a maximal subspace on which the operator $\mathbb{D}^\dagger$ is symmetric. This fact makes possible the description of all the self–adjoint extensions of the operator $\mathbb{D}$ in terms of boundary conditions which we will impose on the solutions of Eq. (8.17). These boundary conditions can then be obtained by finding the conditions that elements of $\mathscr{S}$ satisfy. This equivalent approach to finding the self–adjoint extensions of $\mathbb{D}$ was applied to several one–dimensional differential operators in Ref. [71]. We expand that analysis and apply it to the present case.

For the analysis that follows, we will need to manipulate the boundary values of the solutions with $0 \leq M < 1/2$ which, by the analysis in Section 8.1, were found to be given only in terms of the asymptotic behaviour of the solutions at the boundary. Therefore, it will be convenient to define the component functions $\widetilde{\Psi^{(1)}}$ and $\widetilde{\Psi^{(2)}}$ that contain the same leading behaviour as the components $\Psi^{(1)}$ and $\Psi^{(2)}$ but take on finite values when evaluated at $\rho = \pm\pi/2$. Hence, given $\Phi \in \mathrm{Dom}(\mathbb{D}^\dagger)$ and $0 < M < 1/2$, we define

$$\widetilde{\Psi^{(1)}}(\rho) := \sigma(\rho)^{-M}\Psi^{(1)}(\rho)\,, \tag{8.44a}$$

$$\widetilde{\Psi^{(2)}}(\rho) := \sigma(\rho)^{M}\Psi^{(2)}(\rho)\,. \tag{8.44b}$$

For the case $M = 0$ the functions on the left–hand side are defined trivially by $\widetilde{\Psi^{(1)}} = \Psi^{(1)}$ and $\widetilde{\Psi^{(2)}} = \Psi^{(2)}$. Thus, if Φ_ω is any solution of Eq. (8.17) with $0 < M < 1/2$ and fixed $\omega \in \mathbb{C}$ then, with Eqs. (8.27) and (8.32), a straightforward calculation shows that the component functions defined through Eq. (8.44) satisfy

$$\begin{pmatrix} \widetilde{\Psi_\omega^{(1)}}\left(\frac{\pi}{2}\right) \\ \widetilde{\Psi_\omega^{(2)}}\left(\frac{\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} (2M+1)C_1 \\ \\ (2M-1)C_2 \end{pmatrix}, \tag{8.45a}$$

$$\begin{pmatrix} \widetilde{\Psi_\omega^{(1)}}\left(-\frac{\pi}{2}\right) \\ \widetilde{\Psi_\omega^{(2)}}\left(-\frac{\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} (2M+1)C_1 A_1^{(M)} + 2^{M+\frac{1}{2}} C_2\, \omega\, B_2^{(M)} \\ \\ 2^{M-\frac{1}{2}} C_1\, \omega\, B_2^{(-M)} + (2M-1)C_2\, A_1^{(-M)} \end{pmatrix}, \tag{8.45b}$$

and from Eq. (8.26) it is clear that solutions for $M = 0$ satisfy the simpler relations

$$\begin{pmatrix} \widetilde{\Psi_\omega^{(1)}}\left(\pm\frac{\pi}{2}\right) \\ \widetilde{\Psi_\omega^{(2)}}\left(\pm\frac{\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} \tilde{C}_1 \cos\frac{\omega\pi}{2} \pm \tilde{C}_2 \sin\frac{\omega\pi}{2} \\ \mp\tilde{C}_1 \sin\frac{\omega\pi}{2} + \tilde{C}_2 \cos\frac{\omega\pi}{2} \end{pmatrix}. \tag{8.46}$$

Let us recall that the elements of deficiency subspaces $\mathscr{K}_+$ and $\mathscr{K}_-$ are linear combinations of the solutions Ψ_ω of Eq. (8.12) with $\omega = \pm i$, respectively. If $M = 0$, a solution $\Psi_{\pm i}$ of this equation is given by Eq. (8.26) and, if $M \neq 0$ it is given by Eqs. (8.21) and (8.23) instead. Since $\mathscr{K}_\pm$ is two–dimensional, its elements are characterised by the two coefficients of the solution $\Psi_{\pm i}$ which we can denote by $C_1^\pm$ and $C_2^\pm$. Equations (8.45) and (8.46), with $C_1, C_2, \tilde{C}_1, \tilde{C}_2$ replaced by $C_1^\pm$ and $C_2^\pm$, imply that these coefficients are completely determined by the values of the functions $\widetilde{\Psi_{\pm i}^{(1)}}$ and $\widetilde{\Psi_{\pm i}^{(2)}}$ at the boundary. Therefore, an element $\Psi = \Psi_i + \Psi_{-i} \in \mathscr{K}_+ \oplus_{\mathbb{D}} \mathscr{K}_-$ is in one–to–one correspondence with the vector

$$\left(\widetilde{\Psi^{(1)}}\left(\frac{\pi}{2}\right), \widetilde{\Psi^{(2)}}\left(\frac{\pi}{2}\right), \widetilde{\Psi^{(1)}}\left(-\frac{\pi}{2}\right), \widetilde{\Psi^{(2)}}\left(-\frac{\pi}{2}\right)\right)^T \in \mathbb{C}^4, \tag{8.47}$$

of boundary data. This means that finding the two–dimensional subspace $\mathscr{S}$ that characterises a self–adjoint extension of $\mathbb{D}$ is equivalent to finding the two–dimensional subspace of $\mathbb{C}^4$ of boundary data on which $\mathbb{D}^\dagger$ is symmetric. Now, from the definition of the subspace $\mathscr{S}$, any element $\Psi \in \mathscr{S}$ must satisfy

$$0 = \left\langle \mathbb{D}^\dagger \Psi, \Psi \right\rangle_D - \left\langle \Psi, \mathbb{D}^\dagger \Psi \right\rangle_D . \tag{8.48}$$

Using the inner product from Eq. (8.33), expanding in terms of the components $\Psi^{(1)}$ and $\Psi^{(2)}$ of the spinor Ψ and integrating by parts, we find that this condition becomes

$$\begin{aligned} 0 &= \left[\overline{\Psi^{(1)}(\rho)}\Psi^{(2)}(\rho) - \overline{\Psi^{(2)}(\rho)}\Psi^{(1)}(\rho)\right]\Big|_{-\pi/2}^{\pi/2}, \\ &= \left[\overline{\widetilde{\Psi^{(1)}}(\rho)}\widetilde{\Psi^{(2)}}(\rho) - \overline{\widetilde{\Psi^{(2)}}(\rho)}\widetilde{\Psi^{(1)}}(\rho)\right]\Big|_{-\pi/2}^{\pi/2}, \end{aligned} \tag{8.49}$$

where the second equality is obtained using the definition in Eq. (8.44). We can rewrite Eq. (8.49) as

$$\begin{aligned} 0 = &\left|\widetilde{\Psi^{(2)}}\left(\frac{\pi}{2}\right) + i\widetilde{\Psi^{(1)}}\left(\frac{\pi}{2}\right)\right|^2 - \left|\widetilde{\Psi^{(2)}}\left(\frac{\pi}{2}\right) - i\widetilde{\Psi^{(1)}}\left(\frac{\pi}{2}\right)\right|^2 \\ &+ \left|\widetilde{\Psi^{(2)}}\left(-\frac{\pi}{2}\right) - i\widetilde{\Psi^{(1)}}\left(-\frac{\pi}{2}\right)\right|^2 - \left|\widetilde{\Psi^{(2)}}\left(-\frac{\pi}{2}\right) + i\widetilde{\Psi^{(1)}}\left(-\frac{\pi}{2}\right)\right|^2 . \end{aligned} \tag{8.50}$$

This expression implies that the components of the vectors of the form of Eq. (8.47) that belong to the symmetric subspace $\mathscr{S}$ must satisfy

$$\left| \begin{pmatrix} \widetilde{\Psi^{(2)}}\left(\frac{\pi}{2}\right) + i\widetilde{\Psi^{(1)}}\left(\frac{\pi}{2}\right) \\ \widetilde{\Psi^{(2)}}\left(-\frac{\pi}{2}\right) - i\widetilde{\Psi^{(1)}}\left(-\frac{\pi}{2}\right) \end{pmatrix} \right|^2 = \left| \begin{pmatrix} \widetilde{\Psi^{(2)}}\left(\frac{\pi}{2}\right) - i\widetilde{\Psi^{(1)}}\left(\frac{\pi}{2}\right) \\ \widetilde{\Psi^{(2)}}\left(-\frac{\pi}{2}\right) + i\widetilde{\Psi^{(1)}}\left(-\frac{\pi}{2}\right) \end{pmatrix} \right|^2 . \tag{8.51}$$

By writing the element from Eq. (8.47) as $(\mathbf{\Psi}_1, \mathbf{\Psi}_2)^T$, where

$$\mathbf{\Psi}_1 = \begin{pmatrix} \widetilde{\Psi^{(2)}}\left(\frac{\pi}{2}\right) + i\widetilde{\Psi^{(1)}}\left(\frac{\pi}{2}\right) \\ \widetilde{\Psi^{(2)}}\left(-\frac{\pi}{2}\right) - i\widetilde{\Psi^{(1)}}\left(-\frac{\pi}{2}\right) \end{pmatrix}, \quad \mathbf{\Psi}_2 = \begin{pmatrix} \widetilde{\Psi^{(2)}}\left(\frac{\pi}{2}\right) - i\widetilde{\Psi^{(1)}}\left(\frac{\pi}{2}\right) \\ \widetilde{\Psi^{(2)}}\left(-\frac{\pi}{2}\right) + i\widetilde{\Psi^{(1)}}\left(-\frac{\pi}{2}\right) \end{pmatrix}, \tag{8.52}$$

we find that Eq. (8.51) and the linearity of $\mathscr{S}$, imply that if $(\mathbf{0}, \mathbf{\Psi}_2)^T \in \mathscr{S}$, then $\mathbf{\Psi}_2 = \mathbf{0}$, and thus, if $(\mathbf{\Psi}_1, \mathbf{\Psi}_2)^T, (\mathbf{\Psi}_1, \mathbf{\Psi'}_2)^T \in \mathscr{S}$, then $\mathbf{\Psi}_2 = \mathbf{\Psi'}_2$. Therefore, any $(\mathbf{\Psi}_1, \mathbf{\Psi}_2)^T \in \mathscr{S}$ must satisfy $\mathbf{\Psi_2} = U(\mathbf{\Psi_1})$ for some linear function U. Hence, for any $\Psi \in \mathscr{S}$, we must have

$$U \begin{pmatrix} \widetilde{\Psi^{(2)}}\left(\frac{\pi}{2}\right) + i\widetilde{\Psi^{(1)}}\left(\frac{\pi}{2}\right) \\ \widetilde{\Psi^{(2)}}\left(-\frac{\pi}{2}\right) - i\widetilde{\Psi^{(1)}}\left(-\frac{\pi}{2}\right) \end{pmatrix} = \begin{pmatrix} \widetilde{\Psi^{(2)}}\left(\frac{\pi}{2}\right) - i\widetilde{\Psi^{(1)}}\left(\frac{\pi}{2}\right) \\ \widetilde{\Psi^{(2)}}\left(-\frac{\pi}{2}\right) + i\widetilde{\Psi^{(1)}}\left(-\frac{\pi}{2}\right) \end{pmatrix}, \tag{8.53}$$

where U is a 2×2 matrix. For the boundary values of $\Psi \in \mathscr{S}$ to span a two–dimensional space under this condition, the vector $\mathbf{\Psi}_1$ of Eq. (8.52) must take on all possible values. This fact, together with Eq. (8.51), implies that U must be a unitary matrix. Conversely, for any two $\Psi, \Psi' \in \mathscr{K}_+ \oplus_{\mathbb{D}} \mathscr{K}_-$ satisfying Eq. (8.53) for a fixed unitary matrix U, we have $\left\langle \mathbb{D}^\dagger \Psi, \Psi' \right\rangle = \left\langle \Psi, \mathbb{D}^\dagger \Psi' \right\rangle$. Thus, we conclude that the unitary matrix U specifies a self–adjoint extension $\mathbb{D}_U$ of the operator $\mathbb{D}$ through the boundary data of the element Ψ. In this way, every self–adjoint extension $\mathbb{D}_U$ is characterised by the boundary condition (8.53) which we rewrite as

$$(\mathbb{I} - U) \begin{pmatrix} \widetilde{\Psi^{(2)}}\left(\frac{\pi}{2}\right) \\ \widetilde{\Psi^{(2)}}\left(-\frac{\pi}{2}\right) \end{pmatrix} = i(\mathbb{I} + U) \begin{pmatrix} \widetilde{\Psi^{(1)}}\left(\frac{\pi}{2}\right) \\ -\widetilde{\Psi^{(1)}}\left(-\frac{\pi}{2}\right) \end{pmatrix}. \tag{8.54}$$

With this, we are now able to solve the eigenvalue problem of Eq. (8.17). We will solve the equation $\mathbb{D}^\dagger \Phi_\omega = \omega \Psi_\omega$ now with $\omega \in \mathbb{R}$ to be determined by requiring that the solutions as given by Eqs. (8.21) and (8.23) are elements of $\mathrm{Dom}(\mathbb{D}_U)$. Based on the analysis carried out in this section, we note that this is equivalent to finding solutions to this equation satisfying the boundary condition in Eq. (8.54) for each matrix U.

Finally, for the solutions with mass satisfying $M \geq 1/2$, as shown in Section 8.1, the deficiency indices satisfy $n_\pm = 0$ and thus, the only self–adjoint extension of the operator $\mathbb{D}$ is its closure $\overline{\mathbb{D}}$. From Eq. (5.4), it follows that $\mathrm{Dom}(\overline{\mathbb{D}}) = \mathrm{Dom}(\mathbb{D}^\dagger)$, so finding solutions in $\mathrm{Dom}(\overline{\mathbb{D}})$ is equivalent to finding square–integrable solutions of $\mathbb{D}^\dagger \Psi_\omega = \omega \Psi_\omega$. We recall that, from the analysis of the deficiency spaces in Section 8.1, square–integrable solutions for this equation exist only when the frequency ω is restricted to be of the form ω_n^I as given in Eq. (8.38). We can rephrase this fact in a way more similar to that of the cases with $0 \leq M < /2$, that is, in terms of a boundary condition as follows: From the definitions of the component functions $\widetilde{\Psi_\omega^{(1)}}$ and $\widetilde{\Psi_\omega^{(2)}}$ in Eq. (8.44), it can be directly verified that imposing the boundary condition, which we will refer to as **Dirichlet type I** boundary condition, given by

$$\widetilde{\Psi_\omega^{(2)}}\left(\frac{\pi}{2}\right) = 0 = \widetilde{\Psi_\omega^{(1)}}\left(-\frac{\pi}{2}\right) , \tag{8.55}$$

on the general solutions in Eqs. (8.21) and (8.23) if $M - 1/2 \notin \mathbb{N}_0$ and, in Eq. (8.24) if $M - 1/2 \in \mathbb{N}_0$, results in the same frequency spectrum ω_n^I and restriction $C_2 = 0$ that were obtained in Section 8.1 by requiring square–integrability of the spatial spinor Ψ_ω. Hence, we can conclude that for all $M \geq 1/2$, the unique self–adjoint extension of $\mathbb{D}$ is characterised by the Dirichlet type I boundary condition. This fact will be used in Sec. 8.3 when analysing the invariance of the solutions under the action of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$.

8.3 Invariant self–adjoint boundary conditions

Instead of finding the spectrum of the operator $\mathbb{D}_U$ for every unitary matrix U, we will only focus on those matrices which, via Eq. (8.54), result in boundary conditions that remain invariant under infinitesimal $\widetilde{\mathrm{SL}}(2, \mathbb{R})$–transformations. In order to do this, we need to consider the infinitesimal action of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ on the space of spinors, $C^\infty(\mathrm{AdS}_2; \mathbb{C}^2)$.

We recall that, for a spinor field $\psi \in C^\infty(\mathrm{AdS}_2, \mathbb{C}^2)$, the spinorial Lie derivative in the direction of an arbitrary vector field $\xi \in \mathfrak{X}(\mathrm{AdS}_2)$, is given by Eq. (2.88). Thus, the spinorial Lie derivatives in the direction of the Killing vector fields of AdS_2 from Eq. (6.2) define an infinitesimal action of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ on the space of spinor fields. These read

$$\mathcal{L}_{\xi_0} = \partial_t \, , \tag{8.56a}$$

$$\mathcal{L}_{\xi_1} = \cos t \, \sin\rho \, \partial_t + \sin t \, \cos\rho \, \partial_\rho + \cos t \, \cos\rho \, \Sigma^{01} \, , \tag{8.56b}$$

$$\mathcal{L}_{\xi_2} = - \sin t \, \sin\rho \, \partial_t + \cos t \, \cos\rho \, \partial_\rho - \sin t \, \cos\rho \, \Sigma^{01} \, . \tag{8.56c}$$

From these operators we can construct the associated time–translation operator $\mathcal{L}_0 := i\mathcal{L}_{\xi_0}$ and the ladder operators

$$\begin{aligned} \mathcal{L}_\pm &:= \mathcal{L}_{\xi_1} \pm i\mathcal{L}_{\xi_2} \, , \\ &= e^{\mp it}\left(\pm i \cos\rho \, \partial_\rho + \sin\rho \, \partial_t + \cos\rho \, \Sigma^{01}\right) , \end{aligned} \tag{8.57}$$

which will be more convenient to use once we obtain a mode decomposition of the solutions. We note that the commutation relations between these operators are given by

$$[\mathcal{L}_0, \mathcal{L}_\pm] = \pm\mathcal{L}_\pm \, , \qquad [\mathcal{L}_+, \mathcal{L}_-] = 2\mathcal{L}_0 \, . \tag{8.58}$$

On the representation of $\mathfrak{sl}(2, \mathbb{R})$ on the space of spinors, the Casimir element Q defined by Eq. (4.15) acts by

$$Q = \mathcal{L}_0^2 + \frac{1}{2}\left(\mathcal{L}_+\mathcal{L}_- + \mathcal{L}_-\mathcal{L}_+\right) . \tag{8.59}$$

It can readily be verified using Eqs. (8.56) and (8.57), that the Dirac operator $\tilde{\gamma}^\mu \nabla^{(D)}_\mu$ on AdS_2, given by the left–hand side of Eq. (8.6) is related to the Casimir operator Q in Eq. (8.59) by

$$\left(\tilde{\gamma}^\mu \nabla^{(D)}_\mu\right)^2 = Q + \frac{1}{4}\mathbb{I}\,. \tag{8.60}$$

Hence, the eigenvalue q of the Casimir operator for the representation on $C^\infty\left(\mathrm{AdS}_2, \mathbb{C}^2\right)$ satisfies, by Eq. (8.6), the relation

$$q = M^2 - \frac{1}{4}\,. \tag{8.61}$$

We recall the fact that we have been considering spinor solutions of the form given by Eq. (8.7). Hence, we will denote by $L_\pm$ the associated ladder operators acting on the spinor $\tilde{\psi} = (\cos\rho)^{-1/2}\psi$. From the decomposition into mode solutions in Eq. (8.9), it follows that the action of the ladder operators on spinors of the form $\tilde{\psi}(t,\rho) = \Psi_\omega(\rho)e^{-i\omega t}$ is given by

$$L_\pm\left[\Psi_\omega(\rho)e^{-i\omega t}\right] = \pm i\left[\cos\rho\,\frac{\mathrm{d}}{\mathrm{d}\rho} - \left(\frac{1}{2}\pm\omega\right)\sin\rho \mp i\cos\rho\,\Sigma^{01}\right]\Psi_\omega(\rho)e^{-i(\omega\pm 1)t}\,. \tag{8.62}$$

From this expression we see that at $t=0$, the component functions $\Psi^{(1)}_\omega$ and $\Psi^{(2)}_\omega$ of the spinor Ψ_ω transform under the action of $\mp iL_\pm$ as

$$(\delta_\pm\Psi_\omega)^{(1)}(\rho) := -\left[M + \left(\frac{1}{2}\pm\omega\right)\sin\rho\right]\Psi^{(1)}_\omega(\rho) + \left(\omega\pm\frac{1}{2}\right)\cos\rho\,\Psi^{(2)}_\omega(\rho)\,, \tag{8.63a}$$
$$(\delta_\pm\Psi_\omega)^{(2)}(\rho) := -\left(\omega\pm\frac{1}{2}\right)\cos\rho\,\Psi^{(1)}_\omega(\rho) + \left[M - \left(\frac{1}{2}\pm\omega\right)\sin\rho\right]\Psi^{(2)}_\omega(\rho)\,. \tag{8.63b}$$

where we have used Eq. (8.19a) to eliminate the derivative terms. Using the definitions of the components $\widetilde{\Psi^{(1)}_\omega}$ and $\widetilde{\Psi^{(2)}_\omega}$ in Eq. (8.44), we find that

$$\begin{aligned}\widetilde{(\delta_\pm\Psi_\omega)}^{(1)}(\rho) = &-\left[M + \left(\frac{1}{2}\pm\omega\right)\sin\rho\right]\widetilde{\Psi^{(1)}_\omega}(\rho)\\ &+\left(\omega\pm\frac{1}{2}\right)\cos\rho\,\sigma(\rho)^{-2M}\widetilde{\Psi^{(2)}_\omega}(\rho)\,,\end{aligned} \tag{8.64a}$$
$$\begin{aligned}\widetilde{(\delta_\pm\Psi_\omega)}^{(2)}(\rho) = &-\left(\omega\pm\frac{1}{2}\right)\cos\rho\,\sigma(\rho)^{2M}\widetilde{\Psi^{(1)}_\omega}(\rho)\\ &+\left[M - \left(\frac{1}{2}\pm\omega\right)\sin\rho\right]\widetilde{\Psi^{(2)}_\omega}(\rho)\,.\end{aligned} \tag{8.64b}$$

In particular, for the δ_-–transformation, we have

$$\widetilde{(\delta_-\Psi_\omega)}^{(1)}\left(\pm\frac{\pi}{2}\right) = -\left[M \pm \left(\frac{1}{2}-\omega\right)\right]\widetilde{\Psi^{(1)}_\omega}\left(\pm\frac{\pi}{2}\right)\,, \tag{8.65a}$$
$$\widetilde{(\delta_-\Psi_\omega)}^{(2)}\left(\pm\frac{\pi}{2}\right) = \left[M \mp \left(\frac{1}{2}-\omega\right)\right]\widetilde{\Psi^{(2)}_\omega}\left(\pm\frac{\pi}{2}\right)\,. \tag{8.65b}$$

and the values for δ_+ are obtained by replacing $\omega\mapsto-\omega$. We also note that these boundary values are valid for the massless case by setting $M=0$.

Now, if Ψ_{ω_1} and Ψ_{ω_2} are two solutions of the equation $\mathbb{D}_U\Psi = \omega\Psi$, with $\omega_1, \omega_2 \in \mathbb{R}$, satisfying the same boundary condition (8.54) for a fixed matrix U, then we must have $0 = \langle \mathbb{D}_U \Psi_{\omega_1}, \Psi_{\omega_2}\rangle_D - \langle \Psi_{\omega_1}, \mathbb{D}_U \Psi_{\omega_2}\rangle_D$, that is,

$$\begin{aligned} 0 = &\left[\overline{\widetilde{\Psi_{\omega_1}^{(1)}}\left(\frac{\pi}{2}\right)}\widetilde{\Psi_{\omega_2}^{(2)}}\left(\frac{\pi}{2}\right) - \overline{\widetilde{\Psi_{\omega_1}^{(2)}}\left(\frac{\pi}{2}\right)}\widetilde{\Psi_{\omega_2}^{(1)}}\left(\frac{\pi}{2}\right)\right] \\ &- \left[\overline{\widetilde{\Psi_{\omega_1}^{(1)}}\left(-\frac{\pi}{2}\right)}\widetilde{\Psi_{\omega_2}^{(2)}}\left(-\frac{\pi}{2}\right) - \overline{\widetilde{\Psi_{\omega_1}^{(2)}}\left(-\frac{\pi}{2}\right)}\widetilde{\Psi_{\omega_2}^{(1)}}\left(-\frac{\pi}{2}\right)\right] . \end{aligned} \tag{8.66}$$

From Eq. (8.65), it follows that if the boundary condition is invariant under the transformation induced by $L_\pm$, then the relation

$$\begin{aligned} 0 &= \left[\langle \mathbb{D}_U \delta_- \Psi_{\omega_1}, \Psi_{\omega_2}\rangle - \langle \delta_- \Psi_{\omega_1}, \mathbb{D}_U \Psi_{\omega_2}\rangle\right] , \\ &= -(1-\omega_1-\omega_2)\left[\overline{\widetilde{\Psi_{\omega_1}^{(1)}}\left(\frac{\pi}{2}\right)}\widetilde{\Psi_{\omega_2}^{(2)}}\left(\frac{\pi}{2}\right) - \overline{\widetilde{\Psi_{\omega_1}^{(2)}}\left(\frac{\pi}{2}\right)}\widetilde{\Psi_{\omega_2}^{(1)}}\left(\frac{\pi}{2}\right)\right] \\ &\quad -(1-\omega_1-\omega_2)\left[\overline{\widetilde{\Psi_{\omega_1}^{(1)}}\left(-\frac{\pi}{2}\right)}\widetilde{\Psi_{\omega_2}^{(2)}}\left(-\frac{\pi}{2}\right) - \overline{\widetilde{\Psi_{\omega_1}^{(2)}}\left(-\frac{\pi}{2}\right)}\widetilde{\Psi_{\omega_2}^{(1)}}\left(-\frac{\pi}{2}\right)\right] , \end{aligned} \tag{8.67}$$

(here, the operator $\mathbb{D}_U$ is not transformed) must also be satisfied. Eqs. (8.66) and (8.67) are compatible with each other if and only if the two equations

$$\left[\overline{\widetilde{\Psi_{\omega_1}^{(1)}}\left(\pm\frac{\pi}{2}\right)}\widetilde{\Psi_{\omega_2}^{(2)}}\left(\pm\frac{\pi}{2}\right) - \overline{\widetilde{\Psi_{\omega_1}^{(2)}}\left(\pm\frac{\pi}{2}\right)}\widetilde{\Psi_{\omega_2}^{(1)}}\left(\pm\frac{\pi}{2}\right)\right] = 0 , \tag{8.68}$$

are simultaneously satisfied. For this to hold for all pairs $\{\omega_1, \omega_2\}$, we must have

$$\widetilde{\Psi_{\omega}^{(2)}}\left(\pm\frac{\pi}{2}\right) \propto \widetilde{\Psi_{\omega}^{(1)}}\left(\pm\frac{\pi}{2}\right) , \tag{8.69}$$

for both ω_1 and ω_2, with the proportionality constant being the same real number, or $\widetilde{\Psi_{\omega}^{(1)}} = 0$ at each endpoint. This is only true if the unitary matrix U in Eq. (8.54) is diagonal, that is, if

$$(1 - e^{i\alpha_\pm})\widetilde{\Psi_{\omega}^{(2)}}\left(\pm\frac{\pi}{2}\right) = \pm i(1 + e^{i\alpha_\pm})\widetilde{\Psi_{\omega}^{(1)}}\left(\pm\frac{\pi}{2}\right) , \tag{8.70}$$

for some $\alpha_\pm \in \mathbb{R}$.

Now, using Eq. (8.64), we see that that if these boundary conditions are invariant, then the components $(\delta_-\Psi_\omega)^{(1)}$ and $(\delta_-\Psi_\omega)^{(2)}$ must satisfy

$$\left(\frac{1}{2} - \omega \mp M\right)(1 - e^{i\alpha_\pm})\widetilde{\Psi_{\omega}^{(2)}}\left(\pm\frac{\pi}{2}\right) = \pm i\left(\frac{1}{2} - \omega \pm M\right)(1 + e^{i\alpha_\pm})\widetilde{\Psi_{\omega}^{(1)}}\left(\pm\frac{\pi}{2}\right) . \tag{8.71}$$

From these relations, it is clear that if $M = 0$, then Eq. (8.71) reduces to Eq. (8.70), thus, the self–adjoint extensions parametrised by a diagonal matrix U are all invariant under both ladder operators. In contrast, if $0 < M < 1/2$, then for Eq. (8.70) to be consistent with Eq. (8.71), there are only 4 different cases for the values that the numbers $e^{i\alpha_\pm}$ can take:

1. $e^{i\alpha_\pm} = \mp 1$, *i.e.*, $U = \mathrm{diag}(-1, 1)$. Then the invariant boundary condition is the Dirichlet type I condition that was also found for the case $M \geq 1/2$ in Eq. (8.55), namely,

$$\widetilde{\Psi_\omega^{(2)}}\left(\frac{\pi}{2}\right) = 0 = \widetilde{\Psi_\omega^{(1)}}\left(-\frac{\pi}{2}\right). \tag{8.72}$$

2. $e^{i\alpha_\pm} = \pm 1$, *i.e.*, $U = \mathrm{diag}(1, -1)$. Then, Eq. (8.54) reduces to

$$\widetilde{\Psi_\omega^{(1)}}\left(\frac{\pi}{2}\right) = 0 = \widetilde{\Psi_\omega^{(2)}}\left(-\frac{\pi}{2}\right), \tag{8.73}$$

   which we will refer to as the **Dirichlet type II** condition.

3. $e^{i\alpha_\pm} = 1$, *i.e.*, $U = \mathbb{I}$. This self–adjoint boundary condition then takes the form of a Dirichlet boundary condition for the first weighted component of the spinor, namely,

$$\widetilde{\Psi_\omega^{(1)}}\left(\frac{\pi}{2}\right) = 0 = \widetilde{\Psi_\omega^{(1)}}\left(-\frac{\pi}{2}\right). \tag{8.74}$$

   We will call this boundary condition **Dirichlet type III**.

4. $e^{i\alpha_\pm} = -1$, *i.e.*, $U = -\mathbb{I}$. In this case, we have a Dirichlet boundary condition for the second weighted component of the spinor, namely,

$$\widetilde{\Psi_\omega^{(2)}}\left(\frac{\pi}{2}\right) = 0 = \widetilde{\Psi_\omega^{(2)}}\left(-\frac{\pi}{2}\right). \tag{8.75}$$

   This boundary condition will be referred to as **Dirichlet type IV**.

For the case $M \geq 1/2$ we recall that the unique self–adjoint extension of the operator $\mathbb{D}$ is its closure $\overline{\mathbb{D}}$. As pointed out at the end of Sec. 8.2, square–integrable solutions of $\mathbb{D}^\dagger \Psi_\omega = \omega \Psi_\omega$ can be characterised by solutions satisfying the Dirichlet type I boundary condition in Eq. (8.55). This boundary condition is the same as those found in Eq. (8.73) above, and thus, also invariant under the infinitesimal action of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$.

8.4 MODE SOLUTIONS SATISFYING THE INVARIANT BOUNDARY CONDITIONS

Now that we have found which of the boundary conditions that characterise the admissible self–adjoint extensions of the operator $\mathbb{D}$ are invariant under $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, we shall find the frequency spectrum and the corresponding mode solutions for each of these boundary conditions. It will be convenient to analyse the massless and massive cases separately.

8.4.1 Massless field

We know from the analysis in Sec. 8.3 that any boundary condition of the form given by Eq. (8.54) with a diagonal matrix U will result in an invariant self–adjoint extension of $\mathbb{D}$.

Let us reparametrise this matrix as $U = \mathrm{diag}(e^{2i\beta_+}, e^{2i\beta_-})$, with $0 \leq \beta_\pm \leq \pi$, so that the boundary condition for this case is now written as

$$\cos\beta_\pm \Psi_\omega^{(1)}\left(\pm\frac{\pi}{2}\right) = \mp \sin\beta_\pm \Psi_\omega^{(2)}\left(\pm\frac{\pi}{2}\right), \tag{8.76}$$

which, by Eq. (8.46), read

$$\cos\left(\frac{\omega\pi}{2} + \beta_\pm\right)\tilde{C}_1 \pm \sin\left(\frac{\omega\pi}{2} + \beta_\pm\right)\tilde{C}_2 = 0\,. \tag{8.77}$$

To have non–trivial solutions for $\tilde{C}_1$ and $\tilde{C}_2$, the determinant of the associated linear system of equations should vanish, that is, $\sin(\omega\pi + \beta_+ + \beta_-) = 0$, which means that the frequency ω is restricted by this condition to be of the form

$$\omega_j = -\frac{1}{\pi}(\beta_+ + \beta_-) + j\,, \qquad j \in \mathbb{Z}\,. \tag{8.78}$$

In order to substitute these values back into Eq. (8.77), we need to treat the cases for which j is even and odd separately. By a direct calculation it can readily be verified that the constants $\tilde{C}_1$ and $\tilde{C}_2$ must then satisfy

$$\begin{aligned}
&\cos\left(\frac{\beta_+ - \beta_-}{2}\right)\tilde{C}_1 + \sin\left(\frac{\beta_+ - \beta_-}{2}\right)\tilde{C}_2 = 0\,, \quad \text{if } \; j = 2m\,,\\
&\sin\left(\frac{\beta_+ - \beta_-}{2}\right)\tilde{C}_1 - \cos\left(\frac{\beta_+ - \beta_-}{2}\right)\tilde{C}_2 = 0\,, \quad \text{if } \; j = 2m+1\,,
\end{aligned} \tag{8.79}$$

for $m \in \mathbb{Z}$. Substituting the values of $\tilde{C}_1$ and $\tilde{C}_2$ into Eq. (8.77), and relabelling the index m to $m+1 \in \mathbb{Z}$ for later convenience, we find that the mode solutions of the form of Eq. (8.9) will be given by

$$\tilde{\psi}_{2m,0}(t,\rho) = N_{2m}\begin{pmatrix} \cos\left[(2m+1-\beta)\,\rho - B\right] \\ -\sin\left[(2m+1-\beta)\,\rho - B\right]\end{pmatrix} e^{-i(2m+1-\beta)t}\,, \tag{8.80a}$$

$$\tilde{\psi}_{2m+1,0}(t,\rho) = N_{2m+1}\begin{pmatrix} \sin\left[(2m+2-\beta)\,\rho - B\right] \\ \cos\left[(2m+2-\beta)\,\rho - B\right]\end{pmatrix} e^{-i(2m+2-\beta)t}\,, \tag{8.80b}$$

where we have used Eq. (8.9) and where we have defined $\beta := (\beta_+ + \beta_-)/\pi$ and $B := (\beta_+ - \beta_-)/2$. Using the inner product in Eq. (8.10), we find that $N_j = \pi^{-1/2}$ for all $j \in \mathbb{Z}$.

8.4.2 Massive field

For the massive spinor field with mass satisfying $0 < M < 1/2$, we only have four unitary matrices U leading to invariant self–adjoint extensions as listed at the end of Sec. 8.3. In each of these cases we impose the associated boundary conditions to the general solutions Ψ_ω given by Eqs. (8.21) and (8.23) and then we substitute the resulting spatial components into the scaled Dirac spinors $\tilde{\psi}$ via Eq. (8.9). For all the mode functions obtained below, we use the definition of the Jacobi polynomials given by Eq. (6.23).

1. $U = \mathrm{diag}(-1, 1)$: The Dirichlet type I boundary condition in Eq. (8.72) applied to Eq. (8.45) reduces to the requirement $C_2 = 0$ and $A_1^{(M)} = 0$, which restricts the values of ω to be of the form $\pm\omega_n^I = \pm(1/2 + M + n)$, with $n \in \mathbb{N}_0$. Then the solutions take the form

$$\tilde{\psi}^I_{n,M}(t,\rho) = N^I_{n,M}(\cos\rho)^M \begin{pmatrix} (1+\sin\rho)^{\frac{1}{2}} P_n^{\left(-\frac{1}{2}+M, \frac{1}{2}+M\right)}(\sin\rho) \\ (1-\sin\rho)^{\frac{1}{2}} P_n^{\left(\frac{1}{2}+M, -\frac{1}{2}+M\right)}(\sin\rho) \end{pmatrix} e^{-i\omega_n^I t}\,, \tag{8.81a}$$

$$\tilde{\psi}^I_{-n,M}(t,\rho) = N^I_{n,M}(\cos\rho)^M \begin{pmatrix} (1+\sin\rho)^{\frac{1}{2}} P_n^{\left(-\frac{1}{2}+M, \frac{1}{2}+M\right)}(\sin\rho) \\ -(1-\sin\rho)^{\frac{1}{2}} P_n^{\left(\frac{1}{2}+M, -\frac{1}{2}+M\right)}(\sin\rho) \end{pmatrix} e^{i\omega_n^I t}\,, \tag{8.81b}$$

with the normalisation constant given by

$$N^I_{n,M} = \frac{\sqrt{n!\Gamma(n+2M+1)}}{2^{M+\frac{1}{2}}\Gamma(1/2+M+n)}\,. \tag{8.82}$$

2. $U = \mathrm{diag}(1, -1)$: The Dirichlet type II boundary condition given in Eq. (8.73) reduces to $C_1 = 0$ and $A_1^{(-M)} = 0$, which restricts the values of ω to be of the form $\omega = \omega_n^{II} := n - M + 1/2$, with $n \in \mathbb{N}_0$ for the positive–frequency modes, and $\omega = -\omega_n^{II}$ for the negative–frequency modes. Then the solutions are found to be given by

$$\tilde{\psi}^{II}_{n,M}(t,\rho) = N^{II}_{n,M}(\cos\rho)^{-M} \begin{pmatrix} (1-\sin\rho)^{\frac{1}{2}} P_n^{\left(\frac{1}{2}-M, -\frac{1}{2}-M\right)}(\sin\rho) \\ -(1+\sin\rho)^{\frac{1}{2}} P_n^{\left(-\frac{1}{2}-M, \frac{1}{2}-M\right)}(\sin\rho) \end{pmatrix} e^{-i\omega_n^{II} t}\,, \tag{8.83a}$$

$$\tilde{\psi}^{II}_{-n,M}(t,\rho) = N^{II}_{n,M}(\cos\rho)^{-M} \begin{pmatrix} (1-\sin\rho)^{\frac{1}{2}} P_n^{\left(\frac{1}{2}-M, -\frac{1}{2}-M\right)}(\sin\rho) \\ (1+\sin\rho)^{\frac{1}{2}} P_n^{\left(-\frac{1}{2}-M, \frac{1}{2}-M\right)}(\sin\rho) \end{pmatrix} e^{i\omega_n^{II} t}\,, \tag{8.83b}$$

with

$$N^{II}_{n,M} = \frac{\sqrt{n!\Gamma(n-2M+1)}}{2^{\frac{1}{2}-M}\Gamma(1/2-M+n)}\,. \tag{8.84}$$

3. $U = \mathbb{I}$: The Dirichlet type III boundary condition for the first component in Eq. (8.74) implies that $C_1 = 0$ and $B_2^{(M)} = 0$. Using the definitions in Eq. (8.31), we find that this boundary condition restricts the value of ω to be either zero or of the form $\omega = \omega_n^{III} := n$, with $n \in \mathbb{N}$ for the positive–frequency modes and $\omega = -\omega_n^{III}$ for the negative–frequency modes. By substituting these values into Eqs. (8.21) and (8.23), we find that the mode solutions $\tilde{\psi}$ reduce to the forms

$$\tilde{\psi}^{III}_{n,M}(t,\rho) = N^{III}_{n,M}\left(\frac{1+\sin\rho}{1-\sin\rho}\right)^{\frac{M}{2}} \begin{pmatrix} \cos\rho\, P_{n-1}^{\left(\frac{1}{2}-M, \frac{1}{2}+M\right)}(\sin\rho) \\ -2P_n^{\left(-\frac{1}{2}-M, -\frac{1}{2}+M\right)}(\sin\rho) \end{pmatrix} e^{-i\omega_n^{III} t}\,, \tag{8.85a}$$

$$\tilde{\psi}^{III}_{0,M}(t,\rho) = N^{III}_{0,M}\left(\frac{1+\sin\rho}{1-\sin\rho}\right)^{\frac{M}{2}} \begin{pmatrix} 0 \\ -2 \end{pmatrix}\,, \tag{8.85b}$$

$$\tilde{\psi}^{III}_{-n,M}(t,\rho) = N^{III}_{n,M}\left(\frac{1+\sin\rho}{1-\sin\rho}\right)^{\frac{M}{2}} \begin{pmatrix} \cos\rho\, P_{n-1}^{\left(\frac{1}{2}-M, \frac{1}{2}+M\right)}(\sin\rho) \\ 2P_n^{\left(-\frac{1}{2}-M, -\frac{1}{2}+M\right)}(\sin\rho) \end{pmatrix} e^{i\omega_n^{III} t}\,, \tag{8.85c}$$

where

$$N^{III}_{n,M} = \frac{n!}{2\sqrt{\Gamma(1/2+M+n)\Gamma(1/2-M+n)}} \,. \tag{8.86}$$

4. $U = -\mathbb{I}$: The Dirichlet type IV boundary condition in Eq. (8.75) implies that $C_2 = 0$ and $\omega\, B_2^{(-M)} = 0$. Once again, the definitions in Eq. (8.31) imply that these conditions restrict the value of ω to be either zero or once again of the form $\omega = \omega_n^{III} = n$, with $n \in \mathbb{N}$ for the positive–frequency modes and $\omega = -\omega_n^{III}$ for the negative–frequency modes. Thus, the associated mode solutions $\tilde{\psi}$ reduce to the form

$$\tilde{\psi}^{IV}_{n,M}(t,\rho) = N^{IV}_{n,M}\left(\frac{1-\sin\rho}{1+\sin\rho}\right)^{\frac{M}{2}} \begin{pmatrix} 2P_n^{\left(-\frac{1}{2}+M,-\frac{1}{2}-M\right)}(\sin\rho) \\ \cos\rho\, P_{n-1}^{\left(\frac{1}{2}+M,\frac{1}{2}-M\right)}(\sin\rho) \end{pmatrix} e^{-i\omega_n^{III} t} \,, \tag{8.87a}$$

$$\tilde{\psi}^{IV}_{0,M}(t,\rho) = N^{IV}_{0,M}\left(\frac{1-\sin\rho}{1+\sin\rho}\right)^{\frac{M}{2}} \begin{pmatrix} 2 \\ 0 \end{pmatrix} , \tag{8.87b}$$

$$\tilde{\psi}^{IV}_{-n,M}(t,\rho) = N^{IV}_{n,M}\left(\frac{1-\sin\rho}{1+\sin\rho}\right)^{\frac{M}{2}} \begin{pmatrix} 2P_n^{\left(-\frac{1}{2}+M,-\frac{1}{2}-M\right)}(\sin\rho) \\ -\cos\rho\, P_{n-1}^{\left(\frac{1}{2}+M,\frac{1}{2}-M\right)}(\sin\rho) \end{pmatrix} e^{i\omega_n^{III} t} \,, \tag{8.87c}$$

with $N^{IV}_{n,M} = N^{III}_{n,M}$ as given in Eq. (8.86).

From the remarks at the end of Section 8.3, all square–integrable solutions with $M \geq 1/2$ are invariant under $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ and satisfy the boundary condition in Eq. (8.55). As shown in Sec. 8.1, imposing this boundary condition (or equivalently, requiring square–integrable solutions) restricts the values of the frequencies to be ω_n^I and requires the second linearly independent solution to vanish. We substitute these conditions into the general solutions found for $M \geq 1/2$ as follows:

1. For $M - 1/2 \notin \mathbb{N}_0$: We substitute $C_2 = 0$ and $\omega_n^I = 1/2 + M + n$, $n \in \mathbb{N}_0$ into the general solutions given by Eq. (8.21) and (8.23). This results in the mode solutions $\tilde{\psi}^I_{n,M}$ in Eq. (8.81a). This was indeed expected as both sets satisfy the same boundary condition (8.55), the only difference being the values of the mass M in each in case.

2. For $M = k + 1/2$, with $k \in \mathbb{N}$: We substitute $C_2 = 0$ and $\omega_n^I = k+n+1$, $n \in \mathbb{N}_0$ into the general solutions given by Eq. (8.24). This results in the spatial components

$$\Psi^{(1)}_{\omega_n^I}(\rho) = C_1 \sigma(\rho)^{\frac{1}{2}} \left(\mathsf{P}^{-k}_{k+n+1}(\sin\rho) + \mathsf{P}^{-k}_{k+n}(\sin\rho)\right) , \tag{8.88a}$$

$$\Psi^{(2)}_{\omega_n^I}(\rho) = C_1 \sigma(\rho)^{-\frac{1}{2}} \left(\mathsf{P}^{-k}_{k+n}(\sin\rho) - \mathsf{P}^{-k}_{k+n+1}(\sin\rho)\right) . \tag{8.88b}$$

After writing the Ferrers functions above in terms of Gaussian hypergeometric functions using Eq. (8.25), we find that the components above reduce, via Eq. (6.23), to Jacobi polynomials, and thus, the mode solutions are found to be given by

$$\tilde{\psi}^{V}_{n,k}(t,\rho) = N^{V}_{n,k}(\cos\rho)^{k+\frac{1}{2}} \begin{pmatrix} (1+\sin\rho)^{\frac{1}{2}} P_n^{(k,k+1)}(\sin\rho) \\ (1-\sin\rho)^{\frac{1}{2}} P_n^{(k+1,k)}(\sin\rho) \end{pmatrix} e^{-i(k+n+1)t} \,, \tag{8.89}$$

with

$$N^V_{n,k} = \frac{\sqrt{n!(2k+n+1)!}}{2^{k+1}(n+k)!} \,. \tag{8.90}$$

From these expressions it is clear that the mode solutions $\tilde{\psi}^V_{n,k}$ are of the same form as $\tilde{\psi}^I_{n,M}$ in Eq. (8.81a) with $M = k + 1/2$.

3. For $M = 1/2$: Substituting $C_2 = 0$ and $\omega^I_n = m$, where $m \in \mathbb{N}$ into Eq. (8.24) with $k = 0$, we find the spatial components of the spinor solutions as

$$\Psi^{(1)}_{\omega^I_m} = C_1 \sigma(\rho)^{\frac{1}{2}} \left(\mathsf{P}_{m+1}(\sin\rho) + \mathsf{P}_m(\sin\rho)\right) \,, \tag{8.91a}$$
$$\Psi^{(2)}_{\omega^I_m} = C_1 \sigma(\rho)^{-\frac{1}{2}} \left(\mathsf{P}_m(\sin\rho) - \mathsf{P}_{m+1}(\sin\rho)\right) \,. \tag{8.91b}$$

To match the functional form of these component functions to the previous cases, we use the fact that Legendre polynomials are related to Jacobi polynomials by $\mathsf{P}_m(x) = P^{(0,0)}_m(x)$. By applying recursion relations for the combinations above, it can readily be verified that the resulting mode solutions reduce to

$$\tilde{\psi}^{VI}_{n,1/2}(t,\rho) = N^{VI}_n (\cos\rho)^{\frac{1}{2}} \begin{pmatrix} (1+\sin\rho)^{\frac{1}{2}} P^{(0,1)}_n(\sin\rho) \\ (1-\sin\rho)^{\frac{1}{2}} P^{(1,0)}_n(\sin\rho) \end{pmatrix} e^{-i(n+1)t} \,, \tag{8.92}$$

where $n \in \mathbb{N}_0$, and $N^{VI}_n = \sqrt{n+1/2}$. From these expressions it follows that these modes are of the form of $\tilde{\psi}^I_{n,M}$ (and thus, of $\tilde{\psi}^V_{n,k}$) with $M = 1/2$ ($k = 0$, respectively).

Thus, the mode solutions found for all possible values of $M \geq 1/2$ reduce to the form of the spinors $\tilde{\psi}^I_{n,M}$ as given by Eq. (8.81a).

8.5 MODE SOLUTIONS LEADING TO INVARIANT POSITIVE–FREQUENCY SUBSPACES

We will now determine which of the solution spaces that result from the $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–invariant self–adjoint boundary conditions found in the previous section split into invariant positive– and negative–frequency subspaces and thus, lead to an invariant vacuum state via the fermionic Fock space construction outlined in chapter 2, Section 2.3. It is clear that all the sets of mode solutions listed in Section 8.4 form a unitary representation of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ with respect to the inner product in Eq. (8.10), as from Eq. (8.62) we have $L^\dagger_\pm = -L_\mp$, and $\langle \Psi_{\omega,M}, \Psi_{\omega,M} \rangle_D > 0$ for all values of M and all ω in the frequency spectrum. In order to determine if any of these representations admits a splitting into invariant positive– or negative–frequency subspaces, we will use the action of the operators $L_\pm$ from Eqs. (8.62) and (8.63) on each of the sets of mode solutions to determine if a particular mode is annihilated by any of these operators and thus, defines a highest– or lowest–weight vector of an invariant subspace. Once again, it will be more convenient to treat the massless and massive cases separately.

8.5.1 Massless spinor

We consider the modes in Eq. (8.80) collectively written as $\tilde{\psi}_{j,0}$ for $j \in \mathbb{Z}$. By applying the operators $L_\pm$ on both $\tilde{\psi}_{2n,0}$ and $\tilde{\psi}_{2n+1,0}$, it can readily be verified that

$$(L_\pm \tilde{\psi}_{j,0})(t,\rho) = i(-1)^{j+1}\left(\frac{1}{2} \pm \omega_j\right)\tilde{\psi}_{j\pm 1,0}(t,\rho)\,. \tag{8.93}$$

We recall that the frequencies are given by $\omega_j = j + 1 - \beta$, with $\beta = (\beta_+ + \beta_-)/\pi$ so that $0 \le \beta < 2$. This implies that the right–hand side of the expression above vanishes only for $\beta = 1/2$ or $\beta = 3/2$. Thus, for any other possible value of β and $j \in \mathbb{Z}$ all elements $\tilde{\psi}_{j,0}$ can be reached by applying the operators $L_\pm$. Hence, the representation is irreducible for $\beta \ne 1/2, 3/2$. In order to relate this irreducible representation to the UIRs of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ given by the classification at the end of Section 4.3, we need only note that Eq. (8.61) implies that the mass of the spinor field M relates to the eigenvalue parameter λ (where $q = \lambda(\lambda-1)$), by $\lambda = M + 1/2$. Hence, we identify these irreducible representations with the unitary principal series of the form $\mathscr{P}_0^\mu$, with the values of μ given by

$$\mu = \begin{cases} -\beta, & 0 \le \beta < \frac{1}{2} \\ 1-\beta, & \frac{1}{2} < \beta < \frac{3}{2}\,, \\ 2-\beta, & \frac{3}{2} < \beta < 2\,. \end{cases} \tag{8.94}$$

Now we turn to the specific cases in which the resulting representations are reducible. When $\beta = 1/2$, we see from Eq. (8.93) that $L_+\tilde{\psi}_{-1,0}$ and $L_-\tilde{\psi}_{0,0}$, with frequencies $\omega_{-1} = -1/2$ and $\omega_0 = 1/2$ respectively, vanish. Therefore, the representation splits into the two invariant subspaces spanned by the positive–frequency modes $\left\{\tilde{\psi}_{n,0}\right\}_{n\in\mathbb{N}_0}$, and the negative–frequency modes $\left\{\tilde{\psi}_{-n,0}\right\}_{n\in\mathbb{N}}$, respectively. The explicit form of the mode solutions is obtained by writing Eq. (8.80) in terms of one of the parameters, say β_+, so that $\beta_- = \pi/2 - \beta_+ \ge 0$ and $B = \beta_+ - \pi/4$. By doing this, we find that, for $\beta_+ \in [0, \pi/2)$ and $n \in \mathbb{N}_0$, the invariant positive–frequency subspace is spanned by the modes

$$\tilde{\psi}^{\beta_+}_{2n,0}(t,\rho) = \frac{1}{\sqrt{\pi}} R\left(\frac{\pi}{4} - \beta_+\right)\begin{pmatrix} \cos\left(2n+\frac{1}{2}\right)\rho \\ -\sin\left(2n+\frac{1}{2}\right)\rho \end{pmatrix} e^{-i\left(2n+\frac{1}{2}\right)t}\,, \tag{8.95a}$$

$$\tilde{\psi}^{\beta_+}_{2n+1,0}(t,\rho) = \frac{1}{\sqrt{\pi}} R\left(\frac{\pi}{4} - \beta_+\right)\begin{pmatrix} \sin\left(2n+1+\frac{1}{2}\right)\rho \\ \cos\left(2n+1+\frac{1}{2}\right)\rho \end{pmatrix} e^{-i\left(2n+1+\frac{1}{2}\right)t}\,, \tag{8.95b}$$

where we have defined $R(\theta)$ as the 2×2 rotation matrix parametrised by the angle θ. The negative–frequency subspace is obtained by taking $-n \in \mathbb{N}$.

Similarly, when $\beta = 3/2$, the transformed modes $L_+\tilde{\psi}_{0,0}$ and $L_-\tilde{\psi}_{1,0}$, with frequencies $\omega_0 = -1/2$ and $\omega_1 = 1/2$ respectively, are the only ones vanishing, and thus, the representation once again splits into the two invariant subspaces spanned by $\left\{\tilde{\psi}_{n,0}\right\}_{n\in\mathbb{N}}$ and $\left\{\tilde{\psi}_{-n,0}\right\}_{n\in\mathbb{N}_0}$. The explicit form of the mode solutions is once again obtained by writing

Eq. (8.80) in terms of β_+, so that $\beta_- = 3\pi/2 - \beta_+ < \pi$ and $B = \beta_+ - 3\pi/4$. Also, after shifting the labels of the spinors so that the lowest positive–frequency mode is $\Psi_{0,0}$, we have that, for $\beta_+ > \pi/2$ and $n \in \mathbb{N}_0$, the positive–frequency subspace is spanned by

$$\tilde{\psi}^{\beta_+}_{2n,0}(t,\rho) = \frac{1}{\sqrt{\pi}} R\left(\frac{3\pi}{4} - \beta_+\right) \begin{pmatrix} \sin\left(2n+\frac{1}{2}\right)\rho \\ \cos\left(2n+\frac{1}{2}\right)\rho \end{pmatrix} e^{-i\left(2n+\frac{1}{2}\right)t}\,, \tag{8.96a}$$

$$\tilde{\psi}^{\beta_+}_{2n+1,0}(t,\rho) = \frac{1}{\sqrt{\pi}} R\left(\frac{3\pi}{4} - \beta_+\right) \begin{pmatrix} \cos\left(2n+1+\frac{1}{2}\right)\rho \\ -\sin\left(2n+1+\frac{1}{2}\right)\rho \end{pmatrix} e^{-i\left(2n+1+\frac{1}{2}\right)t}\,, \tag{8.96b}$$

and the negative–frequency modes are obtained by considering instead $-n \in \mathbb{N}$. However, by writing $R(3\pi/4 - \beta_+) = R(\pi/4 - \beta_+)R(\pi/2)$ above, and noting that the matrix $R(\pi/2)$ maps a two component spinor $(a,b)^T$ to $(b,-a)^T$, we note that the modes in Eq. (8.96) in fact reduce to the same form of the modes in Eq. (8.95) (up to a minus sign for the odd modes). Therefore, regardless of the value of $\beta_+ \in [0,\pi)$, the invariant subspaces the representation splits into are given by the linear span of the modes appearing in Eq. (8.95).

Before we identify the resulting subspaces with the known UIR's, we write the mode solutions in terms of Jacobi polynomials so that we can match the functional form of the spinors with $M \neq 0$ found in Section 8.4.2. If we use the fact that[3]

$$\begin{aligned}\cos\left(n+\frac{1}{2}\right)\rho =& C_n\sqrt{\frac{2}{\pi}}\left((-1)^n(1+\sin\rho)^{\frac{1}{2}}P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}(\sin\rho)\right.\\ &\left.+(1-\sin\rho)^{\frac{1}{2}}P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(\sin\rho)\right)\,,\end{aligned} \tag{8.97a}$$

$$\begin{aligned}\sin\left(n+\frac{1}{2}\right)\rho =& C_n\sqrt{\frac{2}{\pi}}\left((1+\sin\rho)^{\frac{1}{2}}P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}(\sin\rho)\right.\\ &\left.+(-1)^n(1-\sin\rho)^{\frac{1}{2}}P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(\sin\rho)\right)\,,\end{aligned} \tag{8.97b}$$

where we have defined the constants

$$C_n := \frac{(-1)^{\lfloor\frac{n}{2}\rfloor}n!}{\sqrt{2}\Gamma\left(n+\frac{1}{2}\right)}\,, \tag{8.98}$$

then a straightforward calculation shows that the modes in Eq. (8.95) can be collectively written as

$$\tilde{\psi}^{\beta_+}_{n,0}(t,\rho) = C_n R\left(\frac{\pi}{2} - \beta_+\right) \begin{pmatrix} (1+\sin\rho)^{\frac{1}{2}}P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}(\sin\rho) \\ (1-\sin\rho)^{\frac{1}{2}}P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(\sin\rho) \end{pmatrix} e^{-i\left(\frac{1}{2}+n\right)t}\,, \tag{8.99a}$$

$$\tilde{\psi}^{\beta_+}_{-n,0}(t,\rho) = C_n R\left(\frac{\pi}{2} - \beta_+\right) \begin{pmatrix} (1+\sin\rho)^{\frac{1}{2}}P_n^{\left(-\frac{1}{2},\frac{1}{2}\right)}(\sin\rho) \\ -(1-\sin\rho)^{\frac{1}{2}}P_n^{\left(\frac{1}{2},-\frac{1}{2}\right)}(\sin\rho) \end{pmatrix} e^{i\left(\frac{1}{2}+n\right)t}\,, \tag{8.99b}$$

[3]These identities can be obtained using the power series expansions for the Jacobi polynomials [23, Eqs. 18.5.7, 18.5.8]. For this particular case the author found it more convenient to prove these formulas via induction.

for all $\beta_+ \in [0, \pi)$ and $n \in \mathbb{N}_0$. We note that the matrix $R(\theta)$ can be written as $R(\theta) = \exp(-2i\theta\Sigma^{01})$ by means of Eq. (8.3). Now, it is a well–known fact that the massless Dirac equation has a global internal chiral symmetry [42]. In a two–dimensional spacetime, chirality corresponds to the spinor components of the solutions to Eq. (8.6) with $M = 0$ being left– or right–moving plane waves. In terms of the gamma matrix representation in Eq. (8.2), the chiral transformation is given by $\psi \mapsto \exp(2i\theta\Sigma^{01})\psi$ and two massless Dirac spinors differing by a chiral transformation are taken to be equivalent under this symmetry. Then, the action of the rotation matrix $R(\pi/2 - \beta)$ in Eq. (8.99) is in fact a chiral transformation on the modes $\tilde{\psi}^{\beta_+}_{\pm n,0}$. Therefore, the sets of mode functions with different values of β_+ define a unique representation up to chiral equivalence. If we refer to the classification of UIR's in Section 4.3, we can directly identify the linear span of the positive–frequency mode solutions in Eq. (8.99a) with the positive mock–discrete series representation $\mathscr{D}^+_{1/2}$. Similarly, the negative–frequency subspace spanned by the modes in Eq. (8.99b) is identified with the negative mock–discrete series $\mathscr{D}^-_{1/2}$. The fact that the unitary representation spanned by both positive– and negative–frequency subspaces splits into the two invariant subspaces is consistent with the representation theory of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. This follows from the fact that the reducible representation spanned by both positive– and negative–frequency modes corresponds to the unitary principal series $\mathscr{P}^{1/2}_0$ which, as discussed in Section 4.3, is known to have the decomposition into irreducible subspaces $\mathscr{P}^{1/2}_0 \simeq \mathscr{D}^+_{1/2} \oplus \mathscr{D}^-_{1/2}$.

We also note that the invariant sets of mode functions corresponding to the Dirichlet types I–IV boundary conditions in Eqs. (8.81)–(8.87) reduce to certain massless sets of modes in the limit $M \to 0$. The Dirichlet type I and type II modes with $M = 0$ reduce to Eq. (8.99) with $\beta_+ = \pi/2$ and $\beta_+ = 0$, respectively. To see the correspondence with the Dirichlet type III and type IV modes, we consider the invariant massless mode functions in Eq. (8.80) forming the principal series $\mathscr{P}^\mu_0$, with $\mu = 0$. From Eq. (8.94) we note that this restricts the values of $\beta_\pm$ to satisfy $\beta = 1$. Thus, by setting $\beta_- = \pi - \beta_+$ in Eq. (8.80), the invariant massless modes reduce to

$$\tilde{\psi}_{2m,0}(t, \rho) = \frac{1}{\sqrt{\pi}} R(-\beta_+) \begin{pmatrix} -\sin\, 2m\rho \\ -\cos\, 2m\rho \end{pmatrix} e^{-2imt}\,, \tag{8.100a}$$

$$\tilde{\psi}_{2m+1,0}(t, \rho) = \frac{1}{\sqrt{\pi}} R(-\beta_+) \begin{pmatrix} \cos(2m+1)\rho \\ -\sin(2m+1)\rho \end{pmatrix} e^{-i(2m+1)t}\,, \tag{8.100b}$$

where $R(\beta_+)$ is a rotation matrix by the angle β_+. We now write these modes in a form more readily recognisable as the massless limit of the massive cases. Consider the identities [23, Eqs. 18.5.1, 18.5.2] $\cos\, n\theta = T_n(\cos\theta)$, and $\sin\, n\theta = \sin\theta U_{n-1}(\cos\theta)$, where $n \in \mathbb{N}_0$ and T_n, U_n are Chebyshev polynomials of the first and second kind, respectively, and the relations [87, Eq. 8.962.3]

$$T_n(x) = \frac{n!\sqrt{\pi}}{\Gamma(n + 1/2)} P^{(-1/2,-1/2)}_n(x)\,, \qquad U_n(x) = \frac{(n+1)!\sqrt{\pi}}{2\Gamma(n+3/2)} P^{(1/2,1/2)}_n(x)\,. \tag{8.101}$$

Hence, Eq. (8.100) can be written, for $n \in \mathbb{N}$, as

$$\tilde{\psi}_{n,0}(t,\rho) = (-1)^n N^{III}_{n,0} R(-\beta_+) \begin{pmatrix} \cos\rho \, P^{(1/2,1/2)}_{n-1}(\sin\rho) \\ -2P^{(-1/2,-1/2)}_{n}(\sin\rho) \end{pmatrix} e^{-int}\,, \tag{8.102a}$$

$$\tilde{\psi}_{0,0}(t,\rho) = N^{III}_{0,0} R(-\beta_+) \begin{pmatrix} 0 \\ -2 \end{pmatrix}\,, \tag{8.102b}$$

$$\tilde{\psi}_{-n,0}(t,\rho) = (-1)^n N^{III}_{n,0} R(-\beta_+) \begin{pmatrix} -\cos\rho \, P^{(1/2,1/2)}_{n-1}(\sin\rho) \\ -2P^{(-1/2,-1/2)}_{n}(\sin\rho) \end{pmatrix} e^{int}\,. \tag{8.102c}$$

From these expressions it follows that if $\beta_+ = 0$, then the modes above are the Dirichlet type III mode solutions in Eq. (8.85) with $M = 0$, and if $\beta_+ = \pi/2$, then we obtain the Dirichlet type IV mode solutions in Eq. (8.87) with $M = 0$. Similarly to the previous situation, all other values of β_+ are equivalent to either of these two mode solutions up to a chiral transformation.

8.5.2 Massive spinor

Now we analyse the massive mode solutions that resulted from imposing the self–adjoint boundary conditions which are invariant under $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. Thus, we will determine which of the sets of mode solutions appearing in Eqs. (8.81)–(8.87) span a solution space with an invariant positive– or negative–frequency subspace by applying the ladder operators $L_\pm$ on the lowest positive– and highest negative–frequency modes and determine for which cases these modes are annihilated.

We first consider the two sets of mode solutions $\tilde{\psi}^{I}_{n,M}$ for all $M > 0$ and $\tilde{\psi}^{II}_{n,M}$ for $0 < M < 1/2$, given by Eqs. (8.81) and (8.83). Using Eq. (8.62) and standard recurrence relations for the Jacobi polynomials [23, Secs. 18.9(i), 18.9(iii)] we find that the set of Dirichlet type I modes transform under the ladder operators $L_\pm$ as

$$L_\pm \tilde{\psi}^{I}_{n,M} = -i\sqrt{(n+1/2 \pm 1/2)(n+2M+1/2 \pm 1/2)}\,\tilde{\psi}^{I}_{n\pm1,M}\,, \tag{8.103a}$$

$$L_\pm \tilde{\psi}^{I}_{-n,M} = i\sqrt{(n+1/2 \mp 1/2)(n+2M+1/2 \mp 1/2)}\,\tilde{\psi}^{I}_{-n\pm1,M}\,, \tag{8.103b}$$

for all $n \geq 0$ (here, we use the notation $\tilde{\psi}^{I}_{-0,M}$ to denote the highest negative–frequency mode). Similarly, the Dirichlet type II mode solutions transform as in Eq. (8.103) with M replaced by $-M$. From these expressions we see that $L_-\tilde{\psi}_{0,M} = 0 = L_+\tilde{\psi}_{-0,M}$, and thus, both the positive– and negative–frequency subspaces are invariant for these two sets of mode solutions. Furthermore, using the classification of UIR's at the end of Sec. 8.1, we can identify these subspaces with the discrete series representations: The mode solutions $\tilde{\psi}^{I}_{\pm n,M}$ form the representation $\mathscr{D}^{\pm}_{1/2+M}$, while the mode solutions $\tilde{\psi}^{II}_{\pm n,M}$ form the representation $\mathscr{D}^{\pm}_{1/2-M}$.

On the other hand, Dirichlet type III and type IV mode solutions in Eqs. (8.85) and (8.87), respectively, form an irreducible representation and thus, do not split into invariant positive– and negative–frequency subspaces. This can be seen as follows: Using Eq. (8.62) we find that both sets of mode solutions, $\tilde{\psi}_n^{III}$ and $\tilde{\psi}_n^{IV}$, transform under the action of the ladder operators $L_\pm$ as

$$L_\pm \tilde{\psi}_n = -i\sqrt{(n+M\pm 1/2)(n-M\pm 1/2)}\,\tilde{\psi}_{n\pm 1}\,, \tag{8.104a}$$

$$L_\pm \tilde{\psi}_{-n} = i\sqrt{(n+M\mp 1/2)(n-M\mp 1/2)}\,\tilde{\psi}_{-n\pm 1}\,, \tag{8.104b}$$

for all $n \geq 1$, and

$$L_\pm \tilde{\psi}_0 = \mp i\sqrt{1/4 - M^2}\,\tilde{\psi}_{\pm 1}\,. \tag{8.105}$$

Since these solutions are valid only for $0 < M < 1/2$, the right–hand sides of Eqs. (8.104) and (8.105) are never zero for any $n \in \mathbb{N}_0$, and thus, all consecutive modes appearing in Eqs. (8.85) and (8.87) can be reached by applying ladder operators $L_\pm$, so the associated spaces spanned by these modes are irreducible under the action of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. In fact, using the classification of UIRs, we can identify both of these solution spaces with the complementary series representations $\mathscr{C}^0_{1/2+M}$.

We have found that only the mode solutions stemming from the invariant self–adjoint boundary conditions that span a solution space with invariant positive– and negative–frequency subspaces are those coming from Dirichlet type I boundary condition given by Eq. (8.81) and from the Dirichlet type II boundary condition given by Eq. (8.83), which correspond to the self–adjoint extensions of the operator $\mathbb{D}$ labelled by the matrices $U = \mathrm{diag}(\mp 1, \pm 1)$, respectively.

It is also worth noting that the massless and massive mode solutions that result from imposing the Dirichlet type I–IV boundary conditions are invariant under charge conjugation, $\tilde{\psi} \mapsto \tilde{\psi}^c = C(\gamma^0)^T\overline{\tilde{\psi}}$ with $C = -2\Sigma^{01}$. This fact follows immediately by noting that all of the negative–frequency modes from these sets of mode solutions satisfy $\tilde{\psi}_{-n} = -\gamma^1\overline{\tilde{\psi}_n}$ for all $n \geq 0$. From Eqs. (8.2) and (8.3), we have $C(\gamma^0)^T = -\gamma^1$ and thus, all Dirichlet type I–IV modes satisfy $\tilde{\psi}_n^c = \tilde{\psi}_{-n}$. Furthermore, the only self–adjoint boundary conditions that are invariant under both, charge conjugation and $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–transformations, are those corresponding to the four unitary matrices $U = \mathrm{diag}(\pm 1, \mp 1)$ and $U = \pm\mathbb{I}$. A simple calculation shows that if $\tilde{\psi}_\omega$ satisfies the general self–adjoint boundary condition of Eq. (8.54), then the charge conjugate $\tilde{\psi}^c_\omega = -\gamma^1\tilde{\psi}_\omega$ satisfies the same boundary condition if and only if the matrix U satisfies $\overline{U} = U$. We can then determine which of these charge conjugation–invariant boundary conditions are also invariant under $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–transformations using the analysis of Sec. 8.3 but assuming $\overline{U} = U$. We find that if $M = 0$ the matrix U must also be diagonal and the only unitary matrices satisfying both of these requirements are $U = \mathrm{diag}(\pm 1, \mp 1)$ and $U = \pm\mathbb{I}$. If $M \neq 0$, we find the same four unitary matrices.

8.6 INVARIANT THEORIES WITH NO INVARIANT POSITIVE–FREQUENCY SUBSPACES

In Section 8.5 we have found that only certain $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–invariant self–adjoint boundary conditions result in invariant positive–frequency subspaces, and thus, in an invariant vacuum state once the fermionic Fock space construction, as outlined in Section 2.3, is performed. We also noted that the rest of the $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–invariant self–adjoint boundary conditions result in unitary representations that do not split into positive– or negative–frequency subspaces and thus, no invariant vacuum state can be found. Instead, since the inner product (8.10) is $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–invariant for any of these boundary conditions, the ladder operators $L_\pm$ acting on the quantum field are Bogoliubov transformations [24, 27, 25] that mix the creation and annihilation operators. This implies that for these theories there must be UIRs of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ to which the associated vacuum states belong. In this section we find these representations.

We will start by considering the massless modes in Eq. (8.80). Without loss of generality, we will choose the parameters $\beta_\pm$ of the unitary matrix U such that $\omega_0 = \mu > 0$ is the lowest positive frequency. The analysis for the other representations labelled by μ in Eq. (8.94) can be carried out by appropriately relabelling the frequency index. The case $\mu = 0$ will be analysed separately.

We recall that the frequency spectrum is given by $\omega_j = j + \mu$, $j \in \mathbb{Z}$, and for the sake of simplicity, we will denote the associated mode solutions by $\tilde{\psi}_j$ instead of $\tilde{\psi}_{j,0}$. The negative–frequency modes are given by $\tilde{\psi}_{-j}$ for $j > 0$ and, thus, the quantum field $\tilde{\psi}$ is expanded in terms of the complete set of mode solutions as

$$\tilde{\psi} = \sum_{j\geq 0}\left(a_j\tilde{\psi}_j + b_j^\dagger\tilde{\psi}_{-j-1}\right), \tag{8.106}$$

where the operators a_j, b_k satisfy

$$\left\{a_j, a_k^\dagger\right\} = \delta_{jk}\mathbb{I} = \left\{b_j, b_k^\dagger\right\}, \tag{8.107}$$

and all other anticommutators vanish. Using the action of the ladder operators on the modes $\tilde{\psi}_j$ given by Eq. (8.93), we find that

$$\begin{aligned} L_+\tilde{\psi} =& i\sum_{j\geq 0}(-1)^{j+1}\left(\left(\omega_j + \frac{1}{2}\right)a_j\tilde{\psi}_{j+1} + \left(\omega_{-j-1} - \frac{1}{2}\right)b_{j+1}^\dagger\tilde{\psi}_{-j-1}\right)\\ &+ i\left(\mu - \frac{1}{2}\right)b_0^\dagger\tilde{\psi}_0\,, \end{aligned} \tag{8.108a}$$

$$\begin{aligned} L_-\tilde{\psi} =& i\sum_{j\geq 0}(-1)^{j+1}\left(\left(\omega_j + \frac{1}{2}\right)a_{j+1}\tilde{\psi}_j + \left(\omega_{-j-1} - \frac{1}{2}\right)b_j^\dagger\tilde{\psi}_{-j-2}\right)\\ &+ i\left(\mu - \frac{1}{2}\right)a_0\tilde{\psi}_{-1}\,. \end{aligned} \tag{8.108b}$$

Using the inner product in Eq. (8.11) in terms of the field $\tilde{\psi} = (\cos\rho)^{-1/2}\psi$, for which the mode solutions $\tilde{\psi}_j$ satisfy $\left(\tilde{\psi}_j, \tilde{\psi}_k\right)_D = \delta_{jk}$, we define the conserved quantum charges for the symmetry generated by $L_\pm$ by

$$\hat{L}_\pm := \left(\tilde{\psi}, L_\pm\tilde{\psi}\right)_D, \tag{8.109}$$

and from Eq. (8.108), we find that these can be written in terms of the annihilation and creation operators as

$$\hat{L}_+ = i\sum_{j\geq 0}(-1)^{j+1}\Big(\Big(\omega_j+\frac{1}{2}\Big)\,a^\dagger_{j+1}a_j+\Big(\frac{1}{2}-\omega_{-j-1}\Big)\,b^\dagger_{j+1}b_j\Big)+i\left(\mu-\frac{1}{2}\right)a^\dagger_0b^\dagger_0\,, \tag{8.110a}$$

$$\hat{L}_- = i\sum_{j\geq 0}(-1)^{j+1}\Big(\Big(\omega_j+\frac{1}{2}\Big)\,a^\dagger_{j}a_{j+1}+\Big(\frac{1}{2}-\omega_{-j-1}\Big)\,b^\dagger_{j}b_{j+1}\Big)-i\left(\mu-\frac{1}{2}\right)a_0b_0\,. \tag{8.110b}$$

Next, we calculate the commutators between these charges. Using the anticommutation relations in Eq. (8.107) and the fact that $\omega_j = j+\mu$ for $j\in\mathbb{Z}$, it can readily be verified that

$$\left[\hat{L}_+,\hat{L}_-\right] = 2\sum_{j\geq 0}\left(\omega_j a^\dagger_j a_j - \omega_{-j-1}b^\dagger_j b_j\right)+\left(\mu-\frac{1}{2}\right)^2\mathbb{I}\,. \tag{8.111}$$

We then define the operator

$$\hat{L}_0 := \sum_{j\geq 0}\left(\omega_j a^\dagger_j a_j - \omega_{-j-1}b^\dagger_j b_j\right)+2\kappa\mathbb{I}\,, \tag{8.112}$$

with $\kappa = (\mu-1/2)^2/2$. If we then compare Eq. (8.111) with Eq. (8.58), we can identify $\hat{L}_0$ with the time–translation charge induced from $\mathcal{L}_0$. By applying the general prescription in Section 2.3, the fermionic vacuum state $|0\rangle_F$ is defined by the requirement that for all $j\geq 0$, $a_j\,|0\rangle_F = 0 = b_j\,|0\rangle_F$, and we see that

$$\hat{L}_0\,|0\rangle_F = \lambda\,|0\rangle_F\;,\qquad \hat{L}_-\,|0\rangle_F = 0\,, \tag{8.113}$$

the latter resulting directly from Eq. (8.110b). The fermionic Fock space is thus a weight–module with lowest weight κ and lowest–weight vector $|0\rangle_F$. From the classification of UIRs at the end of Section 4.3 we see that this representation is isomorphic to the discrete series $\mathscr{D}^+_\kappa$.

We now turn to the analysis of the massless modes with $\mu = 0$ and the massive modes with $0 < M < 1/2$ satisfying the self–adjoint boundary condition with $U = \mathbb{I}$, given by Eq. (8.85). We recall the fact that the massless modes with $\mu = 0$ can be written as in Eq. (8.102) and thus, are equivalent to the Dirichlet type III modes with $M = 0$ up to a chiral transformation, therefore, this analysis includes the case for which $M = 0$ and $\mu = 0$. Furthermore, the massive Dirichlet type IV modes in Eq. (8.87) are related to the Dirichlet type III modes by $\tilde{\psi}^{IV}_n = (-1)^n\mathbb{P}\tilde{\psi}^{III}_n$, where $\mathbb{P}\tilde{\psi}(t,\rho) = i\gamma^0\tilde{\psi}(t,-\rho)$ is the parity transformation acting on the spinor $\tilde{\psi}$. From the fact that $\mathbb{P}L_\pm\mathbb{P} = -L_\pm$, it follows that the ladder operators take the same form for the Dirichlet type IV modes as for the Dirichlet type III modes. Thus, without loss of generality, we will consider the mode solutions $\tilde{\psi}^{III}_{\pm n}$ with $0\leq M < 1/2$ to include all remaining cases. For these theories, the quantum field is expanded as

$$\tilde{\psi}^{III} = \sum_{n=1}^{\infty}\left(a_n\tilde{\psi}^{III}_n + b^\dagger_n\tilde{\psi}^{III}_{-n}\right)+a_0\tilde{\psi}^{III}_0\,, \tag{8.114}$$

with the annihilation and creation operators satisfying Eq. (8.107). Using the transformation in Eq. (8.104), we find that the action of the ladder operators on the quantum field is given by

$$L_+\tilde{\psi}^{III} = -\,i\sum_{n=1}^{\infty} C_n\left(a_n\tilde{\psi}^{III}_{n+1} - b^\dagger_{n+1}\tilde{\psi}^{III}_{-n}\right) + iC_0\left(b^\dagger_1\tilde{\psi}^{III}_0 - a_0\Psi^{III}_1\right)\,, \tag{8.115a}$$

$$L_-\tilde{\psi}^{III} = -\,i\sum_{n=1}^{\infty} C_n\left(a_{n+1}\tilde{\psi}^{III}_{n} - b^\dagger_{n}\tilde{\psi}^{III}_{-n-1}\right) + iC_0\left(a_0\tilde{\psi}^{III}_{-1} - a_1\tilde{\psi}^{III}_0\right)\,, \tag{8.115b}$$

where we have defined the constants $C_n = \sqrt{(n+M+1/2)(n-M+1/2)}$.

Once again, we define the conserved quantum charges $\hat{L}_\pm$ by Eq. (8.109), and considering the fact that the mode solutions $\tilde{\psi}^{III}_n$ are also orthonormal with respect to the inner product (8.11), we find the quantum operators $\hat{L}_\pm$ to be given by

$$\hat{L}_+ = -i\sum_{n=1}^{\infty} C_n\left(a^\dagger_{n+1}a_n + b^\dagger_{n+1}b_n\right) + iC_0\left(a^\dagger_0 b^\dagger_1 - a^\dagger_1 a_0\right)\,, \tag{8.116a}$$

$$\hat{L}_- = -i\sum_{n=1}^{\infty} C_n\left(a^\dagger_{n}a_{n+1} + b^\dagger_{n}b_{n+1}\right) + iC_0\left(b_1 a_0 - a^\dagger_0 a_1\right)\,. \tag{8.116b}$$

The commutator between these charges is calculated using the anticommutation relations in Eq. (8.107) and the fact that $C_n^2 - C_{n-1}^2 = 2n$. This is found to be given by

$$\left[\hat{L}_+, \hat{L}_-\right] = 2\sum_{n=1}^{\infty} n\left(a^\dagger_n a_n + b^\dagger_n b_n\right) + C_0^2\mathbb{I}\,. \tag{8.117}$$

By comparing this with Eq. (8.58), we identify the right–hand side with $2\hat{L}_0$, where

$$\hat{L}_0 = \sum_{n=1}^{\infty} n\left(a^\dagger_n a_n + b^\dagger_n b_n\right) + \frac{1}{2}\left(\frac{1}{4} - M^2\right)\mathbb{I}\,, \tag{8.118}$$

is the time–translation charge operator. The canonical vacuum state $|0\rangle_F$ satisfies

$$\hat{L}_0\,|0\rangle_F = \frac{1}{2}\left(\frac{1}{4} - M^2\right)|0\rangle_F\,, \qquad \hat{L}_-\,|0\rangle_F = 0\,. \tag{8.119}$$

However, in contrast to the previous case, the vacuum sector with energy $(1/4 - M^2)/2$ has a double degeneracy: The state $a^\dagger_0\,|0\rangle_F$ also satisfies

$$\hat{L}_0 a^\dagger_0\,|0\rangle_F = \left((1/4 - M^2)/2\right) a^\dagger_0\,|0\rangle_F\,, \qquad \hat{L}_- a^\dagger_0\,|0\rangle_F = 0 \tag{8.120}$$

This was indeed expected from the fact that there is a zero–frequency mode $\tilde{\psi}^{III}_0$ in the solution space. We therefore have a two–parameter family of (normalised) vacuum states, given by

$$|0;\alpha\rangle_F := \alpha\,|0\rangle_F + \left(1 - |\alpha|^2\right)^{\frac{1}{2}} a^\dagger_0\,|0\rangle_F\,, \qquad \alpha \in \mathbb{C}\,, \tag{8.121}$$

which implies that for every $\alpha \in \mathbb{C}$, the vacuum sector for these theories generates the lowest–weight module isomorphic to the discrete series representation $\mathscr{D}^+_{(1/4-M^2)/2}$.

We conclude this chapter summarising our main results on the analysis of a Dirac field in AdS$_2$ in Table 8.1. The symbols $\overrightarrow{\Psi}^{(1)}$ and $\overrightarrow{\Psi}^{(2)}$ stand for the column vectors $(\widetilde{\Psi^{(1)}}(\pi/2), -\widetilde{\Psi^{(1)}}(-\pi/2))^T$ and $(\widetilde{\Psi^{(2)}}(\pi/2), \widetilde{\Psi^{(2)}}(-\pi/2))^T$, respectively. We have also defined the matrices $\mathrm{U}_+ := i(\mathbb{I} + U)$, $\mathrm{U}_- := \mathbb{I} - U$ and $H := -\mathrm{diag}(\tan\beta_+, \tan\beta_-)$.

Table 8.1: Self–adjoint boundary conditions for a Dirac field in AdS$_2$

M	**SABCs**	**Inv. SABCs**	**Spectrum (ω)**	**Inv. P-F sol.**	**Rep.**
$M \geq \frac{1}{2}$	Dirichlet I	Dirichlet I	$\omega = \pm\omega_n^I$, $\omega_n^I = \frac{1}{2} + M + n$	$\tilde{\psi}_{n,M}^I$, ω_n^I	$\mathscr{D}_{\frac{1}{2}+M}^+ \oplus \mathscr{D}_{\frac{1}{2}+M}^-$
$0 < M < \frac{1}{2}$	$\mathrm{U}_- \overrightarrow{\Psi}^{(2)} = \mathrm{U}_+ \overrightarrow{\Psi}^{(1)}$	Dirichlet I	$\omega = \pm\omega_n^I$, $\omega_n^I = \frac{1}{2} + M + n$	$\tilde{\psi}_{n,M}^I$, ω_n^I	$\mathscr{D}_{\frac{1}{2}+M}^+ \oplus \mathscr{D}_{\frac{1}{2}+M}^-$
		Dirichlet II	$\omega = \pm\omega_n^{II}$, $\omega_n^{II} = \frac{1}{2} - M + n$	$\tilde{\psi}_{n,M}^{II}$, ω_n^{II}	$\mathscr{D}_{\frac{1}{2}-M}^+ \oplus \mathscr{D}_{\frac{1}{2}-M}^-$
		Dirichlet III	$\omega_j^{III} = j \in \mathbb{Z}$	–	$\mathscr{C}_{\frac{1}{2}+M}^0$
		Dirichlet IV		–	$\mathscr{C}_{\frac{1}{2}+M}^0$
$M = 0$	$\mathrm{U}_- \overrightarrow{\Psi}^{(2)} = \mathrm{U}_+ \overrightarrow{\Psi}^{(1)}$	$\overrightarrow{\Psi}^{(1)} = H\overrightarrow{\Psi}^{(2)}$	$\omega_j = \frac{1}{2} + j$, $j \in \mathbb{Z}$	$\left[\tilde{\psi}_{n,0}^I\right], \omega_n$ (Chir. Equiv.)	$\mathscr{D}_{\frac{1}{2}}^+ \oplus \mathscr{D}_{\frac{1}{2}}^-$ $(\simeq \mathscr{P}_0^{\frac{1}{2}})$
			$\omega_j = j - \beta$, $j \in \mathbb{Z}, \beta \in \left[0, \frac{1}{2}\right)$	–	$\mathscr{P}_0^{\beta}$
			$\omega_j = j - \beta$, $j \in \mathbb{Z}, \beta \in \left(\frac{1}{2}, \frac{3}{2}\right)$	–	$\mathscr{P}_0^{1-\beta}$
			$\omega_j = j - \beta$, $j \in \mathbb{Z}, \beta \in \left(\frac{3}{2}, 2\right)$	–	$\mathscr{P}_0^{2-\beta}$

Here n is always assumed to be in $\mathbb{N}_0$. We also write $\beta = (\beta_+ + \beta_-)/\pi$.
The symbol $\left[\tilde{\psi}_{n,0}^I\right]$ denotes the equivalence class of $\tilde{\psi}_{n,0}^I$ w.r.t. chiral equivalence.

— 9 —

Conclusions

In this thesis we have presented the analysis of the dynamics for a scalar field in N–dimensional anti–de Sitter spacetime AdS_N ($N \geq 2$), and for a spinor field satisfying the Dirac equation in two–dimensional anti-de Sitter spacetime, AdS_2. The case of a scalar field in AdS_N for $N \geq 3$ was based on the original results by Ishibashi and Wald in Ref. [22], which we reworked to more closely resemble the way we approached the two–dimensional case. By applying the prescription for dynamics in static, non–globally hyperbolic spacetimes of Ishibashi and Wald, we have found a family of field theories with a well–defined initial value problem despite the lack of global–hyperbolicity of the spacetime manifold. Additionally, we studied the invariance of the associated solution spaces under the infinitesimal action of the isometry group of the spacetime ($\widetilde{\mathrm{SL}}(2, \mathbb{R})$ for AdS_2 and $\widetilde{\mathrm{SO}}(2, N-1)$ for AdS_N). We determined which, among the family of theories obtained by the prescription for dynamics, can be used to construct a quantum field theory with a stationary vacuum state.

We summarise our results for each of the cases we analysed as follows:

- For a free scalar field of mass M in AdS_2, the prescription for dynamics reduces to determining the admissible positive self–adjoint extensions of the spatial component, A, of the Klein–Gordon equation. We associated a unique boundary condition for the spatial solutions at infinity ($\rho = \pm\pi/2$) to each self–adjoint extension of the operator A. The number of admissible self–adjoint extensions and, thus, of boundary conditions, is given by von Neumann's theorem, and we found that it depends on the value of the mass parameter λ, defined by $M^2 = \lambda(\lambda - 1)$. If the eigenvalues ω^2 for the self-adjoint extension A_U of the operator A are positive, then one can define a quantum field theory with a stationary vacuum state following the standard procedure. We noted that λ has to be real for A to be a positive operator and, since M^2 remains unchanged under $\lambda \leftrightarrow 1 - \lambda$, we were able to restrict our analysis to $\lambda \geq 1/2$.

  For $\lambda \geq 3/2$ the self–adjoint extension of the operator A is unique and determined to correspond to the generalised Dirichlet boundary condition. For $1/2 \leq \lambda < 3/2$ the self–adjoint extensions of A are labelled by a 2×2 unitary matrix $\mathcal{U}$, which parametrises the boundary conditions. These boundary conditions are analogous to the self–adjoint boundary conditions for the free quantum particle in a box [71].

Next, we determined the self–adjoint boundary conditions which are invariant under the action of the group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. The generalised Dirichlet boundary condition is invariant for all $\lambda \geq 1/2$. For $1/2 < \lambda < 3/2$, there are additional invariant boundary conditions, which are the generalised Neumann boundary condition and the mixed boundary conditions consisting of the generalised Dirichlet boundary condition at one end and the generalised Neumann boundary condition at the other. We also noted that the mode solutions obtained from these boundary conditions are identical to those obtained by Sakai and Tanii [14] by requiring the vanishing of the energy flux at each boundary.

The set of solutions to the Klein–Gordon equation satisfying an invariant boundary condition forms a representation of the group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, but this representation may not be unitary. For the stationary vacuum state to be invariant under the $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ symmetry, the positive–frequency ($\omega > 0$) subset of the solutions must form a unitary representation. We found that the positive–frequency solutions form a unitary representation for the generalized Dirichlet boundary condition for all $\lambda \geq 1/2$ and for the generalised Neumann boundary condition for $1/2 < \lambda < 1$. The generalised Neumann boundary condition does not lead to a unitary representation for $1 < \lambda < 3/2$, and the mixed boundary conditions do not lead to a unitary representation for any value of λ. For $\lambda = 1$ ($M^2 = 0$), the Neumann boundary condition allows spatially constant solutions. For this case, the spatially non–constant positive–frequency mode functions form a unitary representation.

Finally, we studied the cases where the boundary condition is $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–invariant but the positive–frequency subspace is not. This situation occurs for the cases with the mixed boundary conditions with $1/2 < \lambda < 3/2$ and those with the Neumann boundary condition with $1 < \lambda < 3/2$. In particular, we found that the vacuum state, which is not invariant under the $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ transformations, belongs to a UIR in each case: For the former, the vacuum belongs to the discrete series $\mathscr{D}^{+}_{(\lambda-1/2)^2/4}$, and for the latter, the vacuum state belongs to $\mathscr{D}^{+}_{1-\lambda}$.

- For a free scalar field in AdS_N with $N \geq 3$, the prescription for dynamics reduces to determining the positive self–adjoint extensions of the radial component A_{Rad} of the Klein–Gordon operator. The analysis originally done by Ishibashi and Wald in Ref. [22] for this case concluded that the number of self–adjoint extensions of A_{Rad} depends on the mass of the field M, and the dimension, N, of the spacetime. In particular, they showed that the mass of the field has to satisfy $M^2 \geq -(N-1)^2/4$ for A_{Rad} to be a positive operator and, thus, to have positive self–adjoint extensions. Similarly, they found that if $M^2 \geq (N-3)(N+1)/4$, then the self–adjoint extension of A_{Rad} is unique, and determined to be the closure of A_{Rad}. For $-(N-1)^2/4 \leq M^2 < (N-3)(N+1)/4$, the self–adjoint extensions of A_{Rad} are parametrised by a real number $\alpha \in (-\pi, \pi]$. This situation differs from the two–dimensional case (scalar field in AdS_2) since the radial coordinate characterises spatial infinity in AdS_N as a single point, $\rho = \pi/2$, instead of a pair of endpoints, $\rho = \pm\pi/2$, which characterises spatial infinity in AdS_2.

Using a slightly more explicit approach to that of Ishibashi and Wald, we associated a unique boundary condition for the radial solutions to each of the admissible positive self–adjoint extensions of the operator A_{Rad}. For the case with $M^2 \geq (N-3)(N+1)/4$ the unique self–adjoint extension of A_{Rad} was determined to correspond to a generalised Dirichlet boundary condition at $\rho = \pi/2$. When the mass satisfies $-(N-1)^2/4 \leq M^2 < (N-3)(N+1)/4$, the self–adjoint extensions correspond to a family of generalised Robin boundary conditions parametrised by $u \in (-\pi, \pi]$, with the generalised Dirichlet and Neumann boundary conditions arising as special cases ($u = 0$ and $u = \pi$, respectively).

Next, we imposed invariance under the infinitesimal action of $\widetilde{\mathrm{SO}}(2, N-1)$ on the sets of solutions corresponding to the admissible self–adjoint boundary conditions. We showed that it is sufficient to verify if a given set of mode solutions is invariant under the action of any of the boost–like Killing vector fields of AdS_N in order for this space to be invariant under all infinitesimal $\widetilde{\mathrm{SO}}(2, N-1)$–transformations. Considering this, the generalised Dirichlet boundary condition was found to be invariant for all $M^2 \geq -(N-1)^2/4$. For $-(N-1)^2/4 < M^2 < (N-3)(N+1)/4$, the only other type of boundary condition that results in invariant mode solutions is the generalised Neumann boundary condition. We noted that for $N = 4$, these two sets of mode solutions correspond to the ones found in Refs. [11, 12, 13].

Finally, we explored the behaviour of the sets of mode solutions resulting from the invariant self–adjoint boundary conditions under the action of the boost–like Killing vector field $\boldsymbol{K}_3$ in Eq. (7.82). In particular, we considered the positive–frequency modes for the generalised Dirichlet and Neumann boundary conditions. We found that these mode functions are mapped to positive–frequency modes by action of the Killing vector field $\boldsymbol{K}_3$. By means of the Lie algebra structure of the Killing vector fields of AdS_N, this result implies that all the sets of mode solutions resulting from the invariant self–adjoint boundary conditions form invariant positive–frequency subspaces and, thus, a unitary representation of $\widetilde{\mathrm{SO}}(2, N-1)$. Hence, for each of these cases a quantum field theory with an invariant vacuum state can be constructed by following the general prescription of Section 2.3.

- For a free spinor field of mass M satisfying the Dirac equation in AdS_2, the prescription of dynamics is equivalent to finding the admissible self–adjoint extensions of the spatial component $\mathbb{D}$ of the Dirac operator. Similarly to the scalar field case, the number of self–adjoint boundary conditions depends on the mass of the field M. Since solutions of the spatial Dirac equation with mass $-M$ can be obtained from the solutions with mass M, we restricted our analysis to $M \geq 0$.

  For $M \geq 1/2$ the self–adjoint extension of the operator $\mathbb{D}$ is unique and determined to correspond to the Dirichlet type I boundary condition for the spatial component of the spinor solutions at the endpoints $\rho = \pm\pi/2$. For $0 \le M < 1/2$ the self-adjoint extensions of $\mathbb{D}$ are labelled by a 2×2 unitary matrix U, which parametrises the boundary conditions.

Next, we determined the self–adjoint boundary conditions which are invariant under the action of the group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. For $0 < M < 1/2$, we found that the only unitary matrices U parametrising the boundary conditions which result in invariant mode solutions are given by $U = \mathrm{diag}(\mp 1, \pm 1)$ and $U = \pm\mathbb{I}$. These matrices correspond to the Dirichlet boundary conditions of type I, II, III and IV, respectively, defined in Eqs. (8.72)–(8.75). For the massless case $M = 0$, we found that any diagonal unitary matrix U gives a set of boundary conditions that result in invariant mode solutions. We also noted that the Dirichlet type I–IV boundary conditions for all $0 \le M < 1/2$ and the Dirichlet type I boundary condition for all $M \ge 0$ are invariant under charge conjugation.

The set of solutions to the Dirac equation satisfying an invariant boundary condition forms a unitary representation of the group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, but this representation may not split into invariant positive– and negative–frequency subspaces, which is necessary for the vacuum state of the quantised theory to be isometry–invariant. We found that the positive–frequency solutions span invariant subspaces for the Dirichlet type I boundary condition for all $M \ge 0$. For $0 \le M < 1/2$ there is an additional invariant boundary condition that leads to an invariant positive–frequency subspace, which is the Dirichlet type II boundary condition. The mode functions resulting from these boundary conditions were identified with the sum of discrete series representations, $\mathscr{D}^+_{1/2+M} \oplus \mathscr{D}^-_{1/2+M}$ for the Dirichlet type I modes and $\mathscr{D}^+_{1/2-M} \oplus \mathscr{D}^-_{1/2-M}$ for the Dirichlet type II modes. Both Dirichlet types III and IV mode functions are identified with the complementary series representations $\mathscr{C}^0_{1/2+M}$ which are already irreducible and do not split into invariant positive– and negative–frequency subspaces. For the massless case we found that the only diagonal unitary matrices corresponding to boundary conditions that result in invariant positive– and negative–frequency subspaces are of the form $U = \mathrm{diag}(e^{2i\beta_+}, -e^{-2i\beta_+})$, with $\beta_+ \in [0, \pi)$. The particular cases for $\beta_+ = \pi/2$ and $\beta_+ = 0$ correspond to the massless Dirichlet conditions of type I and II, respectively. We noted that all the other massless mode solutions that form invariant positive–frequency subspaces are in fact related to the Dirichlet type I and II mode solutions by a chiral transformation realised as the action of the rotation by the angle $\pi/2 - \beta_+$ on the spatial components of these modes. Since the massless Dirac equation is invariant under chiral transformations, the solutions parametrised by β_+ are taken to be equivalent, and thus, can be identified with the Dirichlet type I (or type II) mode solutions. These mode solutions, up to a chiral transformation, are identified with the sum of mock–discrete series representations $\mathscr{D}^+_{1/2} \oplus \mathscr{D}^-_{1/2}$. For all other diagonal matrices, the associated self–adjoint boundary condition results in mode functions forming the principal series representation $\mathscr{P}^\mu_0$, where μ depends on the parameters $\beta_\pm$ via Eq. (8.94). It is worth pointing out that, besides the massless mode solutions forming the principal series

$\mathscr{P}_0^\mu$, all other mode solutions that we obtained from the $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–invariant self–adjoint extensions of the operator $\mathbb{D}$, *i.e.*, the Dirichlet type I modes for all $M \geq 0$, the Dirichlet types I–IV for $0 < M < 1/2$, and the massless modes parametrised by β_+, up to a chiral equivalence, also correspond to the modes found by Sakai and Tanii [14].

Finally, we examined the cases for which the self–adjoint boundary conditions are $\widetilde{\mathrm{SL}}(2,\mathbb{R})$–invariant but the solution spaces do not split into invariant positive– and negative–frequency subspaces, *i.e.*, the massless solution spaces satisfying the boundary conditions with $\beta_+ + \beta_- \neq \pi/2, 3\pi/2$, and the massive solution spaces satisfying the Dirichlet type III and IV boundary conditions. Due to the lack of an invariant positive–frequency subspace, the vacuum state associated to these theories is not invariant, but instead belongs to a UIR of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$. For the massless theories, we found that the vacuum state belongs to the discrete series representation $\mathscr{D}_\kappa^+$, with $\kappa = (\mu - 1/2)^2/2$, and μ given by Eq. (8.94). The massless theory with $\mu = 0$, and the massive theories corresponding to the Dirichlet type III and type IV mode solutions resulted in a doubly degenerate vacuum sector. The UIR to which the vacuum state $|0;\alpha\rangle$, for $\alpha \in \mathbb{C}$ belongs is isomorphic to the discrete series $\mathscr{D}^+_{(1/4-M^2)/2}$.

As pointed out above, the sets of mode solutions that we obtained from the invariant self–adjoint boundary conditions for the scalar and spinor fields in AdS_2 correspond to the mode solutions obtained by Sakai and Tanii by a different argument. The latter were obtained by requiring the energy flux of the fields to vanish separately at each endpoint of the boundary. Thus, it will be interesting to investigate deeper connections, if any, between these two requirements.

From the analysis of a scalar field in AdS_N for $N \geq 3$ we concluded that only the generalised Dirichlet and Neumann boundary conditions result in $\widetilde{\mathrm{SO}}(2, N-1)$–invariant sets of mode solutions. We examined the action of the Killing vector $\boldsymbol{K}_3$ on the lowest positive–frequency modes of each of these sets and found that these are mapped to positive–frequency modes solutions, implying that these solution spaces admit invariant positive–frequency subspaces and, thus, form a unitary representation of $\widetilde{\mathrm{SO}}(2, N-1)$ resulting in a stationary vacuum sate. Thus, contrary to the $N = 2$ case, there are no invariant theories with a non-invariant vacuum state for the higher–dimensional case. It will be interesting to explore if an analogous situation is also present for spinor field theories in AdS_N with $N \geq 3$.

Our results also show that there is a fundamental difference between the scalar field theories in AdS_2 and the scalar field theories in AdS_N for $N \geq 3$. For both of these cases the self–adjoint extensions are parametrised by non–trivial unitary maps only when the mass of the field lies within a particular range of low values, and if the mass is sufficiently large, then only one self–adjoint boundary condition exists. However, for the low mass range, the unitary map parametrising the self–adjoint boundary conditions is a 2×2 matrix for AdS_2, while for AdS_N the unitary map is given by a phase, e^{iu}. As previously discussed, this is due to the fact that spatial infinity consists of the endpoints $\{-\pi/2, \pi/2\}$

for AdS_2 and of the single point $\{\pi/2\}$ for AdS_N with $N \geq 3$. Since the spinor field in AdS_2 also admits a family of boundary conditions parametrised by a 2×2 unitary matrix for the low–mass range, it will be interesting to confirm if the boundary condition for self–adjointness is parametrised by e^{iu} for the low–mass range in the higher–dimensional case.

We have analysed the dynamics of free scalar and spinor fields in AdS_N and AdS_2, respectively. Our results provide certain choices for sensible dynamics in each case and thus allow to construct the associated free quantum field theories. This in turn opens the possibility for the analysis of interacting field theories, the first step to achieve this being constructing the associated propagators for these cases. An interesting question from this perspective is the compatibility between the isometry invariance condition which we have focused on and the Hadamard condition on the two–point functions associated to the theories. A similar analysis has been done in the case of automorphic fields in two–dimensional de Sitter space [88], where invariant Hadamard and non–Hadamard states have been found. It would be interesting to see if any similar conclusions can be found for the anti–de Sitter case.

— A —

Some properties of Self–adjoint operators

In this chapter we present relevant concepts and standard results related to self–adjoint operators on a separable Hilbert space $\mathscr{H}$. All of the statements and proofs are quite standard and can be found thoguhout the literature, for example in [35, 36, 37, 43].

We will denote elements of $\mathscr{H}$ by $f, g, h, \ldots$ etc. The inner product between two elements $f, g \in \mathscr{H}$ is denoted by $\langle f, g\rangle$, and the norm of f is defined by $\|f\| := (\langle f, f\rangle)^{1/2}$. An operator T is a linear map $\mathrm{Dom}(T) \to \mathscr{H}$, where $\mathrm{Dom}(T) \subseteq \mathscr{H}$ denotes the domain of T.

Definition A.0.1 *An operator $T : \mathscr{H} \to \mathscr{H}$ is said to be* ***bounded*** *if the operator norm of T, given by*

$$\|T\|_{OP} := \sup_{\|f\|=1} \|Tf\| \, , \tag{A.1}$$

is finite. An ***unbounded*** *operator T is a linear map from a dense subspace $\mathrm{Dom}(T) \subseteq \mathscr{H}$ into $\mathscr{H}$.*

It is worth pointing out that from this definition an unbounded operator is *not necessarily bounded*.

Definition A.0.2 *Let T be an unbounded operator on $\mathscr{H}$. A number $\omega \in \mathbb{C}$ belongs to the* ***resolvent set*** *of T if there exists a bounded operator B such that:*

1. *For all $f \in \mathscr{H}$, Bf belongs to $\mathrm{Dom}(T)$, and $(T - \omega\mathbb{I})Bf = f$.*
2. *For all $f \in \mathrm{Dom}(T)$ we have $B(T - \omega\mathbb{I})f = f$.*

If no such bounded operator B exists, then ω belongs to the ***spectrum*** *$\boldsymbol{\sigma}(T)$.*

Theorem A.0.3 *Let T be a self–adjoint operator on an infinite–dimensional Hilbert space $\mathscr{H}$. Then, the following are equivalent:*

1. *There exists a real sequence $(\omega_n)_{n\in\mathbb{N}}$ and an orthonormal basis $\{f_n\}_{n\in\mathbb{N}}$ on $\mathscr{H}$ such that $|\omega_n| \to \infty$ as $n \to \infty$, and $Tf_n = \omega_n f_n$ for all $n \in \mathbb{N}$.*
2. *T has a purely discrete spectrum.*

Remark A.0.4 *An operator T has a **purely discrete spectrum** if $\boldsymbol{\sigma}(T)$ consists only of eigenvalues of finite multiplicities which have no finite accumulation point.*

Proof: See Ref. [43, Proposition 5.12].

Definition A.0.5 *A **spectral measure** on the σ–algebra $\mathfrak{B}(\Omega)$ of subsets of a set Ω is a mapping μ of $\mathfrak{B}(\Omega)$ into the orthogonal projections on a Hilbert space $\mathscr{H}$ such that:*

1. $\mu(\emptyset) = 0$.
2. $\mu(\Omega) = \mathbb{I}$.
3. $\mu(\Omega_1 \cap \Omega_2) = \mu(\Omega_1)\mu(\Omega_2)$.
4. $\mu\left(\bigcup_{n\in\mathbb{N}} M_n\right) = \sum_{n\in\mathbb{N}} \mu(M_n)$ *for any sequence $(M_n)_{n\in\mathbb{N}}$ of pairwise disjoint sets from $\mathfrak{B}(\Omega)$ whose union is also in $\mathfrak{B}(\Omega)$.*

Theorem A.0.6 (Spectral theorem for self–adjoint operators) *Let T be a possibly unbounded self–adjoint operator on $\mathscr{H}$. Then, there exists a unique spectral measure μ_T on the Borel σ–algebra $\mathfrak{B}(\mathbb{R})$ such that*

$$T = \int_{\mathbb{R}} \omega \, \mathrm{d}\mu_T(\omega) \,. \tag{A.2}$$

Proof: See Ref. [43, Theorem 5.7].

Definition A.0.7 *Let T be a self–adjoint operator with spectral measure μ_T. For any measurable function f on $\mathfrak{B}(T)$, define the possibly unbounded operator $f(T)$ by*

$$f(T) := \int_{\mathbb{R}} f(\omega) \mathrm{d}\mu_T(\omega) \,, \tag{A.3}$$

defined on the domain

$$\mathrm{Dom}(f(T)) := \left\{ g \in \mathscr{H} \,\middle|\, \int_{\mathbb{R}} |f(\omega)|^2 \mathrm{d} \left\langle \mu_T(\omega) g, g \right\rangle < \infty \right\} \,. \tag{A.4}$$

*An operator $f(T)$ defined by Eq. (A.3) is said to be obtained though the **functional calculus** of T.*

Definition A.0.8 *A **one–parameter unitary group** on $\mathscr{H}$ is a family $u(t)$, $t \in \mathbb{R}$, of unitary operators such that $U(0) = \mathbb{I}$, and $u(t+s) = u(t)u(s)$ for all $t, s \in \mathbb{R}$. A one–parameter unitary group is said to be **strongly continuous** if*

$$\lim_{s\to t} \|u(t)f - u(s)f\| = 0 \,, \tag{A.5}$$

for all $f \in \mathscr{H}$ and all $t \in \mathbb{R}$.

Proposition A.0.9 *Suppose T is a self–adjoint operator on $\mathscr{H}$ and for $t \in \mathbb{R}$, let $u(t)$ be defined by*

$$u(t) = \exp\left(itT\right) , \tag{A.6}$$

where the operator on the right–hand side is defined through the functional calculus of T as in Definition A.0.7. Then, the following hold:

1. *u is a strongly continuous one–parameter unitary group.*
2. *For all $f \in \mathrm{Dom}(T)$, we have*

$$Tf = \lim_{t\to 0} \frac{1}{i} \frac{u(t)f - f}{t} , \tag{A.7}$$

   where the limit is in the norm topology of $\mathscr{H}$.

3. *For all $f \in \mathscr{H}$, if the limit*

$$\lim_{t\to 0} \frac{1}{i} \frac{u(t)f - f}{t} , \tag{A.8}$$

   exists, then $f \in \mathrm{Dom}(T)$ and the limit is equal to Tf.

Proof: See Ref. [37, Proposition 10.14].

— B —

Relation between the two descriptions of self–adjoint extensions

In this chapter we find a one–to–one map between the unitary matrix U_{M} characterising the map $U : \mathscr{K}_+ \to \mathscr{K}_-$ defined in Eq. (5.20) and the boundary conditions obtained by imposing the restriction given by Eq. (5.23). The calculations presented below are given in terms of the operator A defined in Eq. (6.6). Hence, we show the correspondence between the unitary matrix U_{M} in Eq. (6.31) and the unitary matrix $\mathcal{U}$ defined by Eq. (6.40). These two matrices characterise the self–adjoint extensions of the operator A in two different ways. However, the following procedure also applies to the Schrödinger operator T of Chapter 5 given by Eq. (5.25). For this operator the calculations in this chapter show that Eqs. (5.29) and (5.30) which describe $\mathrm{Dom}(T_U)$ are equivalent to the boundary condition given by Eq. (5.31), and the matrix U_M is in one–to–one correspondence to the matrix $\mathcal{U}$.

We write the matrix U_{M} as U in this appendix for simplicity.

Let $\Phi \in \mathrm{Dom}(A_U)$, where A_U is a self–adjoint extension of A, and let $f \in \mathscr{S} \subset \mathscr{K}_+ \oplus \mathscr{K}_-$. (Recall that $\mathrm{Dom}(A_U) = \mathrm{Dom}(\bar{A}) \oplus \mathscr{S}$.) Let $\{g_1, g_2\}$ be an orthonormal basis of $\mathscr{K}_+$. Hence, $\{\overline{g_1}, \overline{g_2}\}$ is an orthonormal basis of $\mathscr{K}_-$. Then, from Eqs. (6.31) and (6.33), the element $f \in \mathscr{S}$ is written as

$$f = c_1 \frac{1}{2}(g_1 + u_{11}\overline{g_1} + u_{21}\overline{g_2}) + c_2 \frac{1}{2}(g_2 + u_{12}\overline{g_1} + u_{22}\overline{g_2})\,, \tag{B.1}$$

for some $c_1, c_2 \in \mathbb{C}$. The elements defined by

$$G_j = \frac{1}{2}\left(g_j + u_{j1}\overline{g_1} + u_{j2}\overline{g_2}\right)\,, \qquad j = 1, 2\,. \tag{B.2}$$

define an orthonormal basis of $\mathscr{S}$ with respect to the inner product $\langle \cdot, \cdot \rangle_A$ (see the construction in Section 5.1). Since the operator A_U is symmetric, it follows that

$$\begin{aligned} 0 &= \langle \Phi, A_U f \rangle_{KG} - \langle A_U \Phi, f \rangle_{KG}\,, \\ &= \langle \Phi, A_U (c_1 G_1 + c_2 G_2) \rangle_{KG} - \langle A_U \Phi, c_1 G_1 + c_2 G_2 \rangle_{KG}\,. \end{aligned} \tag{B.3}$$

Let us consider the two cases: $c_1 \neq 0$, $c_2 = 0$ and $c_1 = 0$, $c_2 \neq 0$. By integration by parts, the resulting equations can be written in terms of the boundary values as

$$\Phi(\rho)\,\overline{G_j'(\rho)}\Big|_{\pi/2} - \Phi'(\rho)\,\overline{G_j(\rho)}\Big|_{\pi/2} - \Phi(\rho)\,\overline{G_j'(\rho)}\Big|_{-\pi/2} + \Phi'(\rho)\,\overline{G_j(\rho)}\Big|_{-\pi/2} = 0\,. \tag{B.4}$$

From Eq. (6.29) we know that the solutions in the deficiency spaces $\mathscr{K}_+ \oplus \mathscr{K}_-$ behave at the boundaries in a way similar to the solutions of the original eigenvalue problem. Thus, we can write Eq. (B.4) as follows:

$$\begin{aligned}\widetilde{\Phi}^{(\lambda)}\,(\pi/2)\,\overline{D\widetilde{G}_j^{(\lambda)}\,(\pi/2)} - D\widetilde{\Phi}^{(\lambda)}\,(\pi/2)\,\overline{\widetilde{G}_j^{(\lambda)}\,(\pi/2)} \\ - \widetilde{\Phi}^{(\lambda)}\,(-\pi/2)\,\overline{D\widetilde{G}_j^{(\lambda)}\,(-\pi/2)} + D\widetilde{\Phi}^{(\lambda)}\,(-\pi/2)\,\overline{\widetilde{G}_j^{(\lambda)}\,(-\pi/2)} = 0\,,\end{aligned} \tag{B.5}$$

where $\widetilde{G}_j^{(\lambda)}(\rho)$ and $D\widetilde{G}_j^{(\lambda)}(\rho)$ are defined similarly to $\widetilde{\Phi}^{(\lambda)}(\rho)$ and $D\widetilde{\Phi}^{(\lambda)}(\rho)$ in Eqs. (6.35) and (6.36). Next, we define

$$\mathcal{A} := \begin{pmatrix} D\widetilde{g_1}^{(\lambda)}(\pi/2) & 0 \\ 0 & D\widetilde{g_2}^{(\lambda)}(\pi/2) \end{pmatrix} = \begin{pmatrix} -D\widetilde{g_1}^{(\lambda)}(-\pi/2) & 0 \\ 0 & D\widetilde{g_2}^{(\lambda)}(-\pi/2) \end{pmatrix}, \tag{B.6a}$$

$$\mathcal{B} := \begin{pmatrix} \widetilde{g_1}^{(\lambda)}(\pi/2) & 0 \\ 0 & \widetilde{g_2}^{(\lambda)}(\pi/2) \end{pmatrix} = \begin{pmatrix} \widetilde{g_1}^{(\lambda)}(-\pi/2) & 0 \\ 0 & -\widetilde{g_2}^{(\lambda)}(-\pi/2) \end{pmatrix}, \tag{B.6b}$$

where $\widetilde{g_j}^{(\lambda)}$ and $D\widetilde{g_j}^{(\lambda)}$ are defined from g_j, $j = 1, 2$, in the same way as $\widetilde{\Phi}^{(\lambda)}$ and $D\widetilde{\Phi}^{(\lambda)}$ are defined from Φ. The second equalities in Eqs. (B.6a) and (B.6b) follow from the fact that g_1 and g_2 are even and odd, respectively. Then, Eq. (B.5) can be written in a matrix form as

$$\left(\overline{\mathcal{A}} + \overline{U}\mathcal{A}\right)\overrightarrow{\Phi} = \left(\overline{\mathcal{B}} + \overline{U}\mathcal{B}\right)\overrightarrow{D\Phi}\,, \tag{B.7}$$

where

$$\overrightarrow{\Phi} := \begin{pmatrix} \widetilde{\Phi}^{(\lambda)}\,(\pi/2) + \widetilde{\Phi}^{(\lambda)}\,(-\pi/2) \\ \widetilde{\Phi}^{(\lambda)}\,(\pi/2) - \widetilde{\Phi}^{(\lambda)}\,(-\pi/2) \end{pmatrix}, \qquad \overrightarrow{D\Phi} := \begin{pmatrix} D\widetilde{\Phi}^{(\lambda)}\,(\pi/2) - D\widetilde{\Phi}^{(\lambda)}\,(-\pi/2) \\ D\widetilde{\Phi}^{(\lambda)}\,(\pi/2) + D\widetilde{\Phi}^{(\lambda)}\,(-\pi/2) \end{pmatrix}. \tag{B.8}$$

It is useful to note that, by expressing the relation

$$\langle g_j, A_U g_j\rangle_{KG} - \langle A_U g_j, g_j\rangle_{KG} = 4i \tag{B.9}$$

with $j = 1, 2$, in terms of the boundary values $\widetilde{g_j}^{(\lambda)}(\pm\pi/2)$ and $D\widetilde{g_j}^{(\lambda)}(\pm\pi/2)$, one finds

$$\mathcal{B}\overline{\mathcal{A}} - \mathcal{A}\overline{\mathcal{B}} = 2i\mathbb{I}\,. \tag{B.10}$$

We rearrange Eq. (B.7) as

$$\left[\overline{\mathcal{B}} - i\overline{\mathcal{A}} + \overline{U}(\mathcal{B} - i\mathcal{A})\right]\left(\overrightarrow{D\Phi} - i\overrightarrow{\Phi}\right) = -\left[\overline{\mathcal{B}} + i\overline{\mathcal{A}} + \overline{U}(\mathcal{B} + i\mathcal{A})\right]\left(\overrightarrow{D\Phi} + i\overrightarrow{\Phi}\right). \tag{B.11}$$

The matrices $\overline{\mathcal{B}} \pm i\overline{\mathcal{A}} + \overline{U}(\mathcal{B} \pm i\mathcal{A})$ are invertible because the relation (B.10) implies that there are no non.-trivial solutions $\vec{a}$ to either of the equations $\|(\overline{\mathcal{B}} \pm i\overline{\mathcal{A}})\vec{a}\|^2 = \|(\mathcal{B} \pm i\mathcal{A})\vec{a}\|^2$. Then, the matrix $\widetilde{\mathcal{U}}$ defined by

$$\widetilde{\mathcal{U}} := -\left[\overline{\mathcal{B}} - i\overline{\mathcal{A}} + \overline{U}(\mathcal{B} - i\mathcal{A})\right]^{-1}\left[\overline{\mathcal{B}} + i\overline{\mathcal{A}} + \overline{U}(\mathcal{B} + i\mathcal{A})\right], \tag{B.12}$$

is unitary and the map $U \mapsto \widetilde{\mathcal{U}}$ is a bijection as we show below. Thus, the self–adjoint extensions characterised by the unitary matrix U is indeed equivalently characterised by another unitary matrix $\widetilde{\mathcal{U}}$ which specifies the boundary conditions.

The unitarity of $\widetilde{\mathcal{U}}$ follows from

$$V_1 V_1^\dagger - V_2 V_2^\dagger = 4(\mathbb{I} - \overline{U}\,\overline{U}^\dagger)\,, \tag{B.13}$$

where

$$V_1 := \overline{\mathcal{B}} - i\overline{\mathcal{A}} + \overline{U}(\mathcal{B} - i\mathcal{A})\,, \tag{B.14a}$$

$$V_2 := \overline{\mathcal{B}} + i\overline{\mathcal{A}} + \overline{U}(\mathcal{B} + i\mathcal{A})\,, \tag{B.14b}$$

since $\widetilde{\mathcal{U}} = -V_1^{-1}V_2$ and $\overline{U}\,\overline{U}^\dagger = \mathbb{I}$. Equation (B.13) results from Eq. (B.10). Next, we show that the map $U \mapsto \widetilde{\mathcal{U}}$ is a bijection by demonstrating that the matrix U satisfying $V_1\widetilde{\mathcal{U}} = -V_2$ for a given unitary matrix $\widetilde{\mathcal{U}}$ exists and is unique. This equation is solved uniquely for $\overline{U}$ if and only if the homogeneous equation

$$\overline{U}(\mathcal{B} - i\mathcal{A})\widetilde{\mathcal{U}} = -\overline{U}(\mathcal{B} + i\mathcal{A})\,, \tag{B.15}$$

admits only the trivial solution $\overline{U} = 0$. Indeed, if Eq. (B.15) is satisfied, then

$$\overline{U}(\mathcal{B} - i\mathcal{A})(\overline{\mathcal{B}} + i\overline{\mathcal{A}})\overline{U}^\dagger = \overline{U}(\mathcal{B} + i\mathcal{A})(\overline{\mathcal{B}} - i\overline{\mathcal{A}})\overline{U}^\dagger\,. \tag{B.16}$$

Then, by Eq. (B.10) we find $\overline{U}\,\overline{U}^\dagger = 0$, which implies $\overline{U} = 0$.

Thus, we can write Eq. (B.11) as

$$(\mathbb{I} - \widetilde{\mathcal{U}})\overrightarrow{D\Phi} = i(\mathbb{I} + \widetilde{\mathcal{U}})\overrightarrow{\Phi}\,, \tag{B.17}$$

where $\widetilde{\mathcal{U}}$ is unitary. Then, by defining

$$\mathcal{U} := \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\widetilde{\mathcal{U}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\,, \tag{B.18}$$

we arrive at Eq. (6.40). It is clear that the map $U \mapsto \mathcal{U}$ is a bijection because the map $U \mapsto \widetilde{\mathcal{U}}$ is.

— C —

The operator A with $M^2 < -1/4$

In this appendix we demonstrate that the operator A with $M^2 < -1/4$ is unbounded from below. In this case we have

$$A = -\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{1/4 + a}{\cos^2 \rho}\,, \tag{C.1}$$

with $a > 0$. We first observe

$$\int_{-\eta}^{\eta} (\cos\rho)^{1/2} A (\cos\rho)^{1/2} \mathrm{d}\rho = -2a \ln(\sec\eta + \tan\eta) + \frac{1}{2}\sin\eta\,, \tag{C.2}$$

for $0 < \eta < \pi/2$. Notice that this integral diverges to $-\infty$ as $\eta \to \pi/2$. Let $\pi/6 < \eta < \pi/2$ and $\epsilon = (\pi/2 - \eta)/2$. Then $0 < \eta - \epsilon < \eta + \epsilon < \pi/2$.

Let $f \in \mathrm{Dom}(A)$ be defined by

$$f(\rho) := \begin{cases} (\cos\rho)^{1/2} & \text{if} \quad |\rho| \le \eta - \epsilon\,, \\ (\cos\rho)^{1/2}\chi((|\rho| - \eta)/\epsilon) & \text{if} \quad |\rho| \ge \eta - \epsilon\,, \end{cases} \tag{C.3}$$

where χ is a smooth monotonically–decreasing function satisfying the condition that $\chi(x) = 1$ if $x \le -1$ and $\chi(x) = 0$ if $x \ge 1$. We have $f \in \mathrm{Dom}\,(A)$ because $f(\rho) = 0$ if $\eta + \epsilon \le |\rho| < \pi/2$. We have

$$\int_{-\pi/2}^{\pi/2} |f(\rho)|^2 \mathrm{d}\rho \le 2\,. \tag{C.4}$$

and

$$\int_{-\pi/2}^{\pi/2} \overline{f(\rho)} A f(\rho) \mathrm{d}\rho = \int_{-\eta+\epsilon}^{\eta-\epsilon} \overline{f(\rho)} A f(\rho) \mathrm{d}\rho + 2\int_{\eta-\epsilon}^{\eta+\epsilon} \overline{f(\rho)} A f(\rho) \mathrm{d}\rho\,. \tag{C.5}$$

Since the first integral diverges to $-\infty$ as $\eta - \epsilon \to \pi/2$ by Eq. (C.2), if the second integral is bounded in this limit, then the operator A is unbounded from below.

For $\eta - \epsilon < \rho < \eta + \epsilon$ we find

$$\begin{aligned} f(\rho) A f(\rho) = \chi((\rho - \eta)/\epsilon) \Big[& -\frac{1}{\epsilon^2}\cos\rho\, \chi''((\rho-\eta)/\epsilon) + \frac{1}{\epsilon}\sin\rho\, \chi'((\rho-\eta)/\epsilon) \\ & + \left(-\frac{a}{\cos\rho} + \frac{1}{4}\cos\rho\right)\chi((\rho-\eta)/\epsilon)\Big]\,. \end{aligned} \tag{C.6}$$

Let $|\chi''(x)| \leq C_2$, $|\chi'(x)| \leq C_1$ and recall $|\chi(x)| \leq 1$. Then,

$$|f(\rho)Af(\rho)| \leq \frac{C_2}{\epsilon^2}\sin 3\epsilon + \frac{C_1}{\epsilon} + \frac{a}{\sin\epsilon} + \frac{1}{4}\,. \tag{C.7}$$

Then,

$$\begin{aligned}\left|\int_{\eta-\epsilon}^{\eta+\epsilon} \overline{f(\rho)} A f(\rho) \mathrm{d}\rho\right| &\leq \int_{\eta-\epsilon}^{\eta+\epsilon} |\overline{f(\rho)} A f(\rho)| \mathrm{d}\rho \\ &\leq \frac{2C_2}{\epsilon}\sin 3\epsilon + 2C_1 + \frac{2a\epsilon}{\sin\epsilon} + \frac{\epsilon}{2} \\ &\to 6C_2 + 2C_1 + 2a\,,\end{aligned} \tag{C.8}$$

as $\epsilon = (\pi/2 - \eta)/2 \to 0$. Hence, the second term in Eq. (C.5) is indeed bounded for η and ϵ in the range considered. Therefore, we can take $\eta - \epsilon \to \pi/2$, and the operator A is unbounded from below.

— D —

The closure of the operator A

In this appendix we demonstrate that, if $\Phi \in \mathrm{Dom}(\bar{A})$, then

$$\widetilde{\Phi}^{(\lambda)}(\pm\pi/2) = D\widetilde{\Phi}^{(\lambda)}(\pm\pi/2) = 0\,, \tag{D.1}$$

where $\bar{A}$ denotes the closure of the operator A in Eq. (6.6) and where $\widetilde{\Phi}^{(\lambda)}$ and $D\widetilde{\Phi}^{(\lambda)}$ are defined by Eqs. (6.35) and (6.36). First we examine the case with $1/2 < \lambda < 3/2$. Let $f_1(\rho)$ and $f_2(\rho)$ be smooth functions whose support is in $[0, \pi/2]$ and which take the values $(\cos\rho)^\lambda$ and $(\cos\rho)^{1-\lambda}$, respectively, for $\rho \in [\pi/4, \pi/2)$. Then since λ and $1-\lambda$ are both larger than $-1/2$ so that $f_1, f_2 \in L^2[-\pi/2, \pi/2]$ and since, for $\rho \in [\pi/4, \pi/2)$,

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{\lambda(\lambda-1)}{\cos^2\rho}\right) f_1(\rho) = \lambda^2(\cos\rho)^\lambda, \tag{D.2a}$$

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{\lambda(\lambda-1)}{\cos^2\rho}\right) f_2(\rho) = (1-\lambda)^2(\cos\rho)^{1-\lambda}\,, \tag{D.2b}$$

we have $A^\dagger f_1, A^\dagger f_2 \in L^2[-\pi/2, \pi/2]$ and therefore $f_1, f_2 \in \mathrm{Dom}(A^\dagger)$.

Now, suppose $\Phi \in \mathrm{Dom}\left(\bar{A}\right)$. Since $\bar{A} = (A^\dagger)^\dagger$ (see Reference [35]), we have by definition

$$\langle c_1 f_1 + c_2 f_2, \bar{A}\Phi\rangle - \langle c_1 A^\dagger f_1 + c_2 A^\dagger f_2, \Phi\rangle = 0\,, \tag{D.3}$$

where $c_1, c_2 \in \mathbb{C}$. This can be written as

$$\lim_{a\to\pi/2}\int_0^a \left\{\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}\left[\overline{c_1}f_1(\rho) + \overline{c_2}f_2(\rho)\right]\Phi(\rho) - \left[\overline{c_1}f_1(\rho) + \overline{c_2}f_2(\rho)\right]\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}\Phi(\rho)\right\}\mathrm{d}\rho = 0. \tag{D.4}$$

Then, by integration by parts we have

$$\begin{aligned}\lim_{\rho\to\pi/2}\Big\{&(1-2\lambda)\overline{c_1}(\cos\rho)^{\lambda-1}\sin\rho\Phi(\rho)\\ &-\left[\overline{c_1}(\cos\rho)^{2\lambda-1} + \overline{c_2}\right](\cos\rho)^{2-2\lambda}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[(\cos\rho)^{\lambda-1}\Phi(\rho)\right]\Big\} = 0.\end{aligned} \tag{D.5}$$

Let $c_1 = 0$ and $c_2 = 1$. Then,

$$\lim_{\rho\to\pi/2}(\cos\rho)^{2-2\lambda}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[(\cos\rho)^{\lambda-1}\Phi(\rho)\right] = 0. \tag{D.6}$$

That is, $D\widetilde{\Phi}^{(\lambda)}(\pi/2) = 0$. Next, let $c_1 = 1$ and $c_2 = 0$. Then since $(\cos\rho)^{2\lambda-1} \to 0$ and $\sin\rho \to 1$ as $\rho \to \pi/2$, we find

$$\lim_{\rho\to\pi/2} (\cos\rho)^{\lambda-1}\Phi(\rho) = 0. \tag{D.7}$$

That is, $\widetilde{\Phi}^{(\lambda)}(\pi/2) = 0$. We can construct a similar argument to show that we also have $D\widetilde{\Phi}^{(\lambda)}(-\pi/2) = \widetilde{\Phi}^{(\lambda)}(-\pi/2) = 0$.

For $\lambda = 1/2$ we can let $f_1(\rho) = (\cos\rho)^{1/2}$ and $f_2(\rho) = (\cos\rho)^{1/2}[\ln(\cos^2\rho) - 1]$ for $\rho \in [\pi/4, \pi/2)$ and let them vanish for $\rho \in [-\pi/2, 0]$. We find that f_2 is also in $\mathrm{Dom}(A^\dagger)$ because

$$\begin{aligned}\left(-\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{1}{4\cos^2\rho}\right)(\cos\rho)^{1/2}\left[\ln(\cos^2\rho) - 1\right] =& \frac{1}{4}(\cos\rho)^{1/2}\left[\ln(\cos^2\rho) - 1\right] \\ &+ 2(\cos\rho)^{1/2}.\end{aligned} \tag{D.8}$$

Proceeding in the same way as before, if $\Phi \in \mathrm{Dom}(\bar{A})$, then we find, instead of Eq. (D.5),

$$\begin{aligned}&\lim_{\rho\to\pi/2}\left\{\frac{2\overline{c_1}\sin\rho}{(\cos\rho)^{1/2}\left[\ln(\cos^2\rho) - 1\right]}\Phi(\rho)\right.\\ &\left.-\left[\frac{\overline{c_1}}{\left[\ln(\cos^2\rho) - 1\right]} + \overline{c_2}\right](\cos\rho)\left[\ln(\cos^2\rho) - 1\right]^2\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{\Phi(\rho)}{(\cos\rho)^{1/2}\left[\ln(\cos^2\rho) - 1\right]}\right)\right\}\\ &= 0.\end{aligned} \tag{D.9}$$

By choosing $c_1 = 0$ and $c_2 = 1$, we find

$$\lim_{\rho\to\pi/2}(\cos\rho)\left[\ln(\cos^2\rho) - 1\right]^2\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\frac{\Phi(\rho)}{(\cos\rho)^{1/2}\left[\ln(\cos^2\rho) - 1\right]}\right) = 0. \tag{D.10}$$

That is, $D\widetilde{\Phi}^{(1/2)}(\pi/2) = 0$. Next we choose $c_1 = 1$ and $c_2 = 0$ and we find, since $\sin\rho \to 1$ and $\ln(\cos^2\rho) \to -\infty$ as $\rho \to \pi/2$,

$$\lim_{\rho\to\pi/2}\frac{\Phi(\rho)}{(\cos\rho)^{1/2}\left[\ln(\cos^2\rho) - 1\right]} = 0. \tag{D.11}$$

That is, $\widetilde{\Phi}^{(1/2)}(\pi/2) = 0$. We can argue in a similar manner to conclude $D\widetilde{\Phi}^{(1/2)}(-\pi/2) = \widetilde{\Phi}^{(1/2)}(-\pi/2) = 0$. In fact, it is possible to show that if $\Phi \in \mathrm{Dom}(\bar{A})$, then

$$\lim_{\rho\to\pm\pi/2}(\cos\rho)^{-3/2}\Phi(\rho) = 0\,, \tag{D.12}$$

if $1/2 \le \lambda < 3/2$, which is stronger than one of the results, $\widetilde{\Phi}^{(\lambda)}(\pm\pi/2) = 0$.

— E —

Boundary conditions with negative eigenvalues of A_U

In this example we let $\lambda = 1$ so that the eigenvalue problem is given by

$$-\frac{\mathrm{d}^2}{\mathrm{d}\rho^2}\Phi(\rho) = \omega^2\Phi(\rho)\,. \tag{E.1}$$

Choosing the unitary matrix in Eq. (6.40) to be diagonal, we find that the following boundary conditions are possible:

$$\Phi\left(\pm\pi/2\right) = \pm\alpha\Phi'\left(\pm\pi/2\right)\,, \tag{E.2}$$

where we choose $\alpha > 0$. Two independent solutions to Eq. (E.1) with $\omega^2 = -\nu^2 < 0$ are

$$\Phi_\nu^{(1)}(\rho) = \cosh(\nu\rho)\,, \tag{E.3a}$$
$$\Phi_\nu^{(2)}(\rho) = \sinh(\nu\rho)\,. \tag{E.3b}$$

The functions $\Phi_\nu^{(1)}$ and $\Phi_\nu^{(2)}$ satisfy the boundary conditions (E.2) if

$$\coth\left(\tfrac{\nu\pi}{2}\right) = \alpha\nu\,, \tag{E.4a}$$
$$\tanh\left(\tfrac{\nu\pi}{2}\right) = \alpha\nu\,, \tag{E.4b}$$

respectively. Equation (E.4a) has a solution for all $\alpha > 0$ whereas Eq. (E.4b) has a solution if $0 < \alpha < \pi/2$. It is interesting that in the limit $\alpha \to 0$ (the Dirichlet limit) we have $\nu \to \infty$ and hence $\omega^2 \to -\infty$.

— F —

The closure of the operator A_{Rad}

In this appendix we show that any element r in $\mathrm{Dom}(\overline{A_{\text{Rad}}})$ where A_{Rad} is the radial operator defined in Eq. (7.21), with the mass parameter satisfying $\nu \geq 1$, satisfies the boundary condition

$$\left[(\cos\rho)^{\nu-\frac{1}{2}} r(\rho)\right]\Big|_{\rho=\pi/2} = 0\,. \tag{F.1}$$

The proof follows a similar argument to the one used in Appendix D.

Let s be a smooth function with support in $[0, \pi/2]$. Assume that the function s is supported away from $\rho = 0$, and that, for $\rho \in [\pi/4, \pi/2)$, the function s takes the values $(\sin\rho)^{\sigma+1/2}(\cos\rho)^{\nu+1/2}$, with $\sigma = 0, 1/2, 1, \ldots$, and $\nu \geq 1$. This implies that, $s \in L^2[0, \pi/2]$. Since, for $\rho \in [\pi/4, \pi/2)$ the function s satisfies

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{\nu^2 - 1/4}{\cos^2\rho} + \frac{\sigma^2 - 1/4}{\sin^2\rho}\right) s(\rho) = (\sigma + \nu + 1)^2 s(\rho)\,, \tag{F.2}$$

we have $A^\dagger_{\text{Rad}} s \in L^2[0, \pi/2]$. Hence, $s \in \mathrm{Dom}(A^\dagger_{\text{Rad}})$.

Let $r \in \mathrm{Dom}(\overline{A_{\text{Rad}}})$. Now, the operator A_{Rad} is densely defined and symmetric with respect to the inner product in Eq. (7.25), which implies that $\overline{A_{\text{Rad}}} = (A^\dagger_{\text{Rad}})^\dagger$. Then, we must have

$$0 = \left\langle A^\dagger_{\text{Rad}} s, r\right\rangle_{\text{Rad}} - \left\langle s, \overline{A_{\text{Rad}}} r\right\rangle_{\text{Rad}}\,. \tag{F.3}$$

Using Eqs. (7.21) and (7.25) we can write Eq. (F.3) as

$$0 = \lim_{a\to\pi/2} \int_0^a \left[\frac{\mathrm{d}^2\overline{s(\rho)}}{\mathrm{d}\rho^2} r(\rho) - \overline{s(\rho)}\frac{\mathrm{d}^2 r(\rho)}{\mathrm{d}\rho^2}\right] \mathrm{d}\rho\,, \tag{F.4}$$

and, after integrating by parts we obtain

$$\begin{aligned}
0 &= \lim_{\rho\to\pi/2} \left[\frac{\mathrm{d}\overline{s(\rho)}}{\mathrm{d}\rho} r(\rho) - \overline{s(\rho)}\frac{\mathrm{d}r(\rho)}{\mathrm{d}\rho}\right]\,,\\
&= \lim_{\rho\to\pi/2} \left\{\left[\left(\sigma + \frac{1}{2}\right)(\sin\rho)^{\sigma-\frac{1}{2}}(\cos\rho)^{\nu+\frac{3}{2}} - \left(\nu + \frac{1}{2}\right)(\sin\rho)^{\sigma+\frac{3}{2}}(\cos\rho)^{\nu-\frac{1}{2}}\right] r(\rho)\right.\\
&\quad \left. -(\sin\rho)^{\sigma+\frac{1}{2}}(\cos\rho)^{\nu+\frac{1}{2}}\frac{\mathrm{d}r(\rho)}{\mathrm{d}\rho}\right\}\,.
\end{aligned} \tag{F.5}$$

We note that

$$\begin{aligned}(\sin\rho)^{\sigma+\frac{1}{2}}(\cos\rho)^{\nu+\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}\rho}r(\rho) =&(\sin\rho)^{\sigma+\frac{1}{2}}\cos\rho\frac{\mathrm{d}}{\mathrm{d}\rho}\left[(\cos\rho)^{\nu-\frac{1}{2}}r(\rho)\right]\\ &+\left(\nu-\frac{1}{2}\right)(\sin\rho)^{\sigma+\frac{3}{2}}\left[(\cos\rho)^{\nu-\frac{1}{2}}r(\rho)\right]\,,\end{aligned} \tag{F.6}$$

We substitute Eq. (F.6) into Eq. (F.5) to obtain

$$\begin{aligned}0=\lim_{\rho\to\pi/2}\Bigg\{&\left(\sigma+\frac{1}{2}\right)(\sin\rho)^{\sigma-\frac{1}{2}}\cos^2\rho\left[(\cos\rho)^{\nu-\frac{1}{2}}r(\rho)\right]-2\nu(\sin\rho)^{\sigma+\frac{3}{2}}\left[(\cos\rho)^{\nu-\frac{1}{2}}r(\rho)\right]\\ &-(\sin\rho)^{\sigma+\frac{3}{2}}(\cos\rho)^{2\nu}(\cos\rho)^{1-2\nu}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[(\cos\rho)^{\nu-\frac{1}{2}}r(\rho)\right]\Bigg\}\,.\end{aligned} \tag{F.7}$$

Since $\sin\rho\to 1$ as $\rho\to\pi/2$, we have

$$0=\lim_{\rho\to\pi/2}\left\{-2\nu\left[(\cos\rho)^{\nu-\frac{1}{2}}r(\rho)\right]-(\cos\rho)^{2\nu}(\cos\rho)^{1-2\nu}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[(\cos\rho)^{\nu-\frac{1}{2}}r(\rho)\right]\right\}\,. \tag{F.8}$$

Finally, we note that $(\cos\rho)^{2\nu}\to 0$ as $\rho\to\pi/2$. Thus, for r to be in $\mathrm{Dom}(\overline{A_{\mathrm{Rad}}})$, we must have

$$-2\nu\left[(\cos\rho)^{\nu-\frac{1}{2}}r(\rho)\right]\Big|_{\rho\to\pi/2}=0\,, \tag{F.9}$$

and the factor $(\cos\rho)^{1-2\nu}\frac{\mathrm{d}}{\mathrm{d}\rho}[(\cos\rho)^{\nu-1/2}r(\rho)]$ must remain finite as $\rho\to\pi/2$. A simple power–counting argument shows that if Eq. (F.9) is satisfied, then this factor is indeed regular at $\rho=\pi/2$ and, thus, the requirement reduces to Eq. (F.1).

— G —

Infinitesimal transformations of the functions r_{ω,l_1}

In this appendix we verify that the expression in Eq. (7.92) holds for the solutions r_{ω,l_1} of Eq. (7.31).

Let us consider the general solution, r_{ω,l_1}, given by Eq. (7.36). For notational simplicity, we write this function as

$$r_{\omega,l_1}(\rho) = N_{\omega,l_1}(\sin\rho)^{\sigma+\frac{1}{2}}(\cos\rho)^{\nu+\frac{1}{2}}F^{\omega}_{\nu,\sigma}(\rho)\,, \tag{G.1}$$

where we have defined

$$F^{\omega}_{\nu,\sigma}(\rho) := F\left(\zeta^{\omega}_{\nu,\sigma},\zeta^{-\omega}_{\nu,\sigma};1+\sigma;\sin^2\rho\right)\,. \tag{G.2}$$

We note that, from the definitions of the quantities $\zeta^{\omega}_{\nu,\sigma}$ in Eq. (7.33), it follows that

$$\zeta^{\omega}_{\nu,\sigma}+1 = \zeta^{\omega+1}_{\nu,\sigma+1}\,, \qquad \zeta^{-\omega}_{\nu,\sigma}+1 = \zeta^{-\omega}_{\nu+1,\sigma+1} = \zeta^{-(\omega-1)}_{\nu,\sigma+1}\,. \tag{G.3}$$

Considering these definitions, we will now apply the operators $L_{\pm\pm}$, $L_{\pm\mp}$ defined through Eq. (7.91) to the function in Eq. (G.1). We begin by calculating $L_{+-}r_{\omega,l_1}$. Since the derivative of the hypergeometric functions is given by Eq. (7.105), we have, by means of definition (G.2),

$$\frac{\mathrm{d}}{\mathrm{d}\rho}F^{\omega}_{\nu,\sigma}(\rho) = 2\frac{\zeta^{\omega}_{\nu,\sigma}\zeta^{-\omega}_{\nu,\sigma}}{1+\sigma}F^{\omega}_{\nu+1,\sigma+1}(\rho)\,. \tag{G.4}$$

Thus, using this identity we obtain

$$\begin{aligned}\cos\rho\frac{\mathrm{d}}{\mathrm{d}\rho}r_{\omega,l_1}(\rho) =&(\sin\rho)^{\sigma+\frac{1}{2}}(\cos\rho)^{\nu+\frac{1}{2}}\left[\frac{2\zeta^{\omega}_{\nu,\sigma}\zeta^{-\omega}_{\nu,\sigma}}{1+\sigma}\sin\rho\,\cos^2\rho\,F^{\omega}_{\nu+1,\sigma+1}(\rho)\right.\\ &\left.+\left(\frac{\sigma+\frac{1}{2}}{\sin\rho}\cos^2\rho-\left(\nu+\frac{1}{2}\right)\sin\rho\right)F^{\omega}_{\nu,\sigma}\right]\,.\end{aligned} \tag{G.5}$$

We substitute Eq. (G.5) into Eq. (7.91a) to obtain

$$\begin{aligned}[L_{+-}r_{\omega,l_1}](\rho) =&2(\sin\rho)^{\sigma+\frac{3}{2}}(\cos\rho)^{\nu+\frac{1}{2}}\left[\frac{\zeta^{\omega}_{\nu,\sigma}\zeta^{-\omega}_{\nu,\sigma}}{1+\sigma}(1-\sin^2\rho)F^{\omega}_{\nu+1,\sigma+1}(\rho)\right.\\ &\left.-\zeta^{-\omega}_{\nu,\sigma}F^{\omega}_{\nu,\sigma}(\rho)\right]\,.\end{aligned} \tag{G.6}$$

Now, using the identity for the hypergeometric functions [23, Eq. 15.5.13], given by

$$\frac{ab}{c}(1-z)F(a+1,b+1;c+1;z) = \frac{b(a-c)}{c}F(a,b+1;c+1;z) + bF(a,b;c;z)\,, \tag{G.7}$$

we obtain, by means of Eq. (G.3), the identity

$$\frac{\zeta^{\omega}_{\nu,\sigma}\zeta^{-\omega}_{\nu,\sigma}}{1+\sigma}(1-\sin^2\rho)F^{\omega}_{\nu+1,\sigma+1}(\rho) = \frac{\zeta^{-\omega}_{\nu,\sigma}(\zeta^{\omega}_{\nu,\sigma}-1-\sigma)}{1+\sigma}F^{\omega-1}_{\nu,\sigma+1}(\rho) + \zeta^{-\omega}_{\nu,\sigma}F^{\omega}_{\nu,\sigma}(\rho)\,. \tag{G.8}$$

Hence, after substituting Eq. (G.8) into Eq. (G.6), we obtain

$$\begin{aligned}[L_{+-}r_{\omega,l_1}](\rho) &= 2\frac{\zeta^{-\omega}_{\nu,\sigma}(\zeta^{\omega}_{\nu,\sigma}-\sigma-1)}{1+\sigma}(\sin\rho)^{\sigma+\frac{3}{2}}(\cos\rho)^{\nu+\frac{1}{2}}F^{\omega-1}_{\nu,\sigma+1}(\rho)\,,\\ &= 2\frac{\zeta^{-\omega}_{\nu,\sigma}(\zeta^{\omega}_{\nu,\sigma}-\sigma-1)}{1+\sigma}r_{\omega-1,l_1+1}(\rho)\,,\end{aligned} \tag{G.9}$$

where in the last line we have used Eq. (G.1) and the fact that $\sigma \mapsto \sigma \pm 1$ corresponds to $l_1 \mapsto l_1 \pm 1$ by means of Eq. (7.22).

Next, we substitute Eq. (G.5) into Eq. (7.91b) to obtain

$$\begin{aligned}[L_{--}r_{\omega,l_1}](\rho) =& 2(\sin\rho)^{\sigma-\frac{1}{2}}(\cos\rho)^{\nu+\frac{1}{2}}\left[\frac{\zeta^{\omega}_{\nu,\sigma}\zeta^{-\omega}_{\nu,\sigma}}{1+\sigma}\sin^2\rho(1-\sin^2\rho)F^{\omega}_{\nu+1,\sigma+1}(\rho)\right.\\ &\left.+\left(\sigma-\zeta^{-\omega}_{\nu,\sigma}\sin^2\rho\right)F^{\omega}_{\nu,\sigma}(\rho)\right]\,.\end{aligned} \tag{G.10}$$

Now we use the identities [23, Eqs. 15.5.19, 15.5.12]

$$\begin{aligned}\frac{ab}{c}z(1-z)F(a+1,b+1;c+1;z) &= (c-a)F(a-1,b;c;z)\\ &\quad -(c-a-bz)F(a,b;c;z)\,,\end{aligned} \tag{G.11a}$$

$$(c-1)F(a-1,b;c-1;z) = (c-a)F(a-1,b;c;z) + (a-1)F(a,b;c;z)\,, \tag{G.11b}$$

to obtain

$$\frac{\zeta^{\omega}_{\nu,\sigma}\zeta^{-\omega}_{\nu,\sigma}}{1+\sigma}\sin^2\rho(1-\sin^2\rho)F^{\omega}_{\nu+1,\sigma+1}(\rho) = \sigma F^{\omega-1}_{\nu,\sigma-1}(\rho) - (\sigma-\zeta^{-\omega}_{\nu,\sigma}\sin^2\rho)F^{\omega}_{\nu,\sigma}(\rho)\,, \tag{G.12}$$

where we have used Eq. (G.3) to infer that $\zeta^{\omega}_{\nu,\sigma}-1 = \zeta^{\omega-1}_{\nu,\sigma-1}$ and $\zeta^{-\omega}_{\nu,\sigma} = \zeta^{-\omega+1}_{\nu,\sigma-1}$. We substitute Eq. (G.12) into Eq. (G.10) to obtain

$$\begin{aligned}[L_{--}r_{\omega,l_1}](\rho) =& 2\sigma(\sin\rho)^{\sigma-\frac{1}{2}}(\cos\rho)^{\nu+\frac{1}{2}}F^{\omega-1}_{\nu,\sigma-1}(\rho)\,,\\ =& 2\sigma r_{\omega-1,l_1-1}(\rho)\,.\end{aligned} \tag{G.13}$$

Next we substitute Eq. (G.5) into Eq. (7.91a), so that we have

$$\begin{aligned}[L_{++}r_{\omega,l_1}](\rho) =& 2(\sin\rho)^{\sigma+\frac{3}{2}}(\cos\rho)^{\nu+\frac{1}{2}}\left[\frac{\zeta^{\omega}_{\nu,\sigma}\zeta^{-\omega}_{\nu,\sigma}}{1+\sigma}(1-\sin^2\rho)F^{\omega}_{\nu+1,\sigma+1}(\rho)\right.\\ &\left.-\zeta^{\omega}_{\nu,\sigma}F^{\omega}_{\nu,\sigma}(\rho)\right]\,.\end{aligned} \tag{G.14}$$

We consider the identity in Eq. (G.7) with $b \leftrightarrow a$, that is

$$\begin{aligned}\frac{ab}{c}(1-z)F(a+1,b+1;c+1;z) =& \frac{a(b-c)}{c}F(a,b+1;c+1;z)\\ &+aF(a,b;c;z)\,.\end{aligned} \tag{G.15}$$

This implies that we have

$$\frac{\zeta^{\omega}_{\nu,\sigma}\zeta^{-\omega}_{\nu,\sigma}}{1+\sigma}(1-\sin^2\rho)F^{\omega}_{\nu+1,\sigma+1}(\rho) = \frac{\zeta^{\omega}_{\nu,\sigma}(\zeta^{-\omega}_{\nu,\sigma}-1-\sigma)}{1+\sigma}F^{\omega+1}_{\nu,\sigma+1}(\rho)+\zeta^{\omega}_{\nu,\sigma}F^{\omega}_{\nu,\sigma}(\rho)\,. \tag{G.16}$$

Hence, substituting this identity into Eq. (G.14) we obtain

$$\begin{aligned}[L_{++}r_{\omega,l_1}](\rho) &= 2\frac{\zeta^{\omega}_{\nu,\sigma}(\zeta^{-\omega}_{\nu,\sigma}-\sigma-1)}{1+\sigma}(\sin\rho)^{\sigma+\frac{3}{2}}(\cos\rho)^{\nu+\frac{1}{2}}F^{\omega+1}_{\nu,\sigma+1}(\rho)\,,\\ &= 2\frac{\zeta^{\omega}_{\nu,\sigma}(\zeta^{-\omega}_{\nu,\sigma}-\sigma-1)}{1+\sigma}r_{\omega+1,l_1+1}(\rho)\,.\end{aligned} \tag{G.17}$$

Finally, we substitute Eq. (G.5) into Eq. (7.91). This results in

$$\begin{aligned}[L_{-+}r_{\omega,l_1}](\rho) =& 2(\sin\rho)^{\sigma-\frac{1}{2}}(\cos\rho)^{\nu+\frac{1}{2}}\left[\frac{\zeta^{\omega}_{\nu,\sigma}\zeta^{-\omega}_{\nu,\sigma}}{1+\sigma}\sin^2\rho(1-\sin^2\rho)F^{\omega}_{\nu+1,\sigma+1}(\rho)\right.\\ &\left.+\left(\sigma-\zeta^{\omega}_{\nu,\sigma}\sin^2\rho\right)F^{\omega}_{\nu,\sigma}(\rho)\right]\,.\end{aligned} \tag{G.18}$$

Now, we consider the identities in Eq.(G.11) with $b \leftrightarrow a$, that is,

$$\begin{aligned}\frac{ab}{c}z(1-z)F(a+1,b+1;c+1;z) =& (c-b)F(a,b-1;c;z)\\ &-(c-b-az)F(a,b;c;z)\,,\end{aligned} \tag{G.19a}$$

$$(c-1)F(a,b-1;c-1;z) = (c-b)F(a,b-1;c;z)+(b-1)F(a,b;c;z)\,. \tag{G.19b}$$

Thus, we have

$$\frac{\zeta^{\omega}_{\nu,\sigma}\zeta^{-\omega}_{\nu,\sigma}}{1+\sigma}\sin^2\rho(1-\sin^2\rho)F^{\omega}_{\nu+1,\sigma+1}(\rho) = \sigma F^{\omega+1}_{\nu,\sigma-1}(\rho)-(\sigma-\zeta^{\rho}_{\nu,\sigma}\sin^2\rho)F^{\omega}_{\nu,\sigma}(\rho)\,. \tag{G.20}$$

After substituting this expression into Eq. (G.18), we obtain

$$\begin{aligned}[L_{-+}r_{\omega,l_1}](\rho) &= 2\sigma(\sin\rho)^{\sigma-\frac{1}{2}}(\cos\rho)^{\nu+\frac{1}{2}}F^{\omega+1}_{\nu,\sigma-1}(\rho)\,,\\ &= 2\sigma r_{\omega+1,l_1-1}(\rho)\,.\end{aligned} \tag{G.21}$$

Hence, if we substitute Eqs. (G.9), (G.13), (G.17) and (G.21) into Eq. (7.90), we obtain Eq. (7.92).

— H —

Asymptotic behaviour of Ferrers functions

In this section we show that the asymptotic behaviour at the spatial boundary of the component functions in Eq. (8.24) corresponding to Dirac spinors of mass $M = k + 1/2$ ($k \in \mathbb{N}_0$), is given by Eqs. (8.39) and (8.40) if $k > 0$, and we show Eqs. (8.42) for the case $k = 0$.

We begin by analysing the case $k > 0$. The behaviour of the Ferrers functions $\mathsf{P}_\omega^{-k}(x)$ and Q_ω^{-k}, for $\omega \in \mathbb{C}$ with $\omega \neq 0$, at the singular point $x = 1$ is given by [23, Eqs. 14.8.1, 14.8.4]

$$\mathsf{P}_\omega^{-k}(x) \approx \frac{1}{\Gamma(1+k)} \left(\frac{1-x}{2}\right)^{\frac{k}{2}} , \tag{H.1a}$$

$$\mathsf{Q}_\omega^{-k}(x) \approx \frac{\Gamma(k)\Gamma(\omega-k+1)}{2\Gamma(\omega+k+1)} \left(\frac{2}{1-x}\right)^{\frac{k}{2}} , \tag{H.1b}$$

where $f(x) \approx g(x)$ if and only if $f(x)/g(x) \to 1$ as $x \to c$. For sufficiently small $\epsilon > 0$, we have that if $\rho = \pi/2 - \epsilon$, then $\sin\rho = \cos\epsilon$, thus, Eq. (H.1) with $x = \cos\epsilon$ implies that

$$\mathsf{P}_\omega^{-k}(\cos\epsilon) \pm \mathsf{P}_{\omega-1}^{-k}(\cos\epsilon) \approx 0 , \tag{H.2a}$$

$$\mathsf{Q}_\omega^{-k}(\cos\epsilon) + \mathsf{Q}_{\omega-1}^{-k}(\cos\epsilon) \approx \frac{2^k\omega\Gamma(\omega-k)\Gamma(k)}{\Gamma(\omega+k+1)}\epsilon^{-k} , \tag{H.2b}$$

$$\mathsf{Q}_\omega^{-k}(\cos\epsilon) - \mathsf{Q}_{\omega-1}^{-k}(\cos\epsilon) \approx \frac{2^k k\Gamma(\omega-k)\Gamma(k)}{\Gamma(\omega+k+1)}\epsilon^{-k} , \tag{H.2c}$$

where we have used the estimate $\cos\epsilon \sim 1 - \epsilon^2/2$ so that $(1-\cos\epsilon)^{-k/2} \sim \epsilon^{-k}$. We note that for the same value of ρ, we have

$$\left(\frac{1-\sin\rho}{1+\sin\rho}\right)^{\frac{1}{4}} \sim \epsilon^{\frac{1}{2}} . \tag{H.3}$$

Then, it follows that the component functions in Eq. (8.24) evaluated at $\rho = \pi/2 - \epsilon$ behave as

$$\Psi_\omega^{(1)}\left(\frac{\pi}{2} - \epsilon\right) \sim C_2 \frac{2^k\omega\Gamma(\omega-k)\Gamma(k)}{\Gamma(\omega+k+1)}\epsilon^{-k+\frac{1}{2}} , \tag{H.4a}$$

$$\Psi_\omega^{(2)}\left(\frac{\pi}{2} - \epsilon\right) \sim C_2 \frac{2^k k\Gamma(\omega-k)\Gamma(k)}{\Gamma(\omega+k+1)}\epsilon^{-k-\frac{1}{2}} , \tag{H.4b}$$

which are precisely the expressions in Eq. (8.39).

To analyse the behaviour of the component functions $\Psi_\omega^{(1)}$ and $\Psi_\omega^{(2)}$ at the other endpoint $\rho = -\pi/2$, we use the connection formulas for the Ferrers functions, [23, Eqs. 14.9.10, 14.9.8] namely

$$\begin{aligned}
\mathsf{P}_\omega^{-k}(-x) &= \cos\pi(\omega-k)\mathsf{P}_\omega^{-k}(x) - \frac{2}{\pi}\sin\pi(\omega-k)\mathsf{Q}_\omega^{-k}(x)\,,\\
\mathsf{Q}_\omega^{-k}(-x) &= -\cos\pi(\omega-k)\mathsf{Q}_\omega^{-k}(x) - \frac{\pi}{2}\sin\pi(\omega-k)\mathsf{P}_\omega^{-k}(x)\,,
\end{aligned} \tag{H.5}$$

from which it follows that

$$\begin{aligned}
\mathsf{P}_\omega^{-k}(-x) \pm \mathsf{P}_{\omega-1}^{-k}(-x) &= \cos\pi(\omega-k)\left(\mathsf{P}_\omega^{-k}(x) \mp \mathsf{P}_{\omega-1}^{-k}(x)\right)\\
&\quad - \frac{2}{\pi}\sin\pi(\omega-k)\left(\mathsf{Q}_\omega^{-k}(x) \mp \mathsf{Q}_{\omega-1}^{-k}(x)\right)\,,
\end{aligned} \tag{H.6a}$$

$$\begin{aligned}
\mathsf{Q}_\omega^{-k}(-x) \pm \mathsf{Q}_{\omega-1}^{-k}(-x) &= -\cos\pi(\omega-k)\left(\mathsf{Q}_\omega^{-k}(x) \mp \mathsf{Q}_{\omega-1}^{-k}(x)\right)\\
&\quad - \frac{\pi}{2}\sin\pi(\omega-k)\left(\mathsf{P}_\omega^{-k}(x) \mp \mathsf{P}_{\omega-1}^{-k}(x)\right)\,.
\end{aligned} \tag{H.6b}$$

Taking the plus sign above and setting $\rho = \epsilon - \pi/2$, so that $-x = \sin\rho = -\cos\epsilon$, and substituting Eqs. (H.2a) and (H.2c) we obtain

$$\mathsf{P}_\omega^{-k}(-\cos\epsilon) + \mathsf{P}_{\omega-1}^{-k}(-\cos\epsilon) \approx \sin\pi(\omega-k)\frac{2^{k+1}k\Gamma(\omega-k)\Gamma(k)}{\pi\Gamma(\omega+k+1)}\epsilon^{-k}\,, \tag{H.7a}$$

$$\mathsf{Q}_\omega^{-k}(-\cos\epsilon) + \mathsf{Q}_{\omega-1}^{-k}(-\cos\epsilon) \approx \cos\pi(\omega-k)\frac{2^{k}k\Gamma(\omega-k)\Gamma(k)}{\Gamma(\omega+k+1)}\epsilon^{-k}\,, \tag{H.7b}$$

therefore, using the analogous estimate appearing in Eq. (H.3) this time with $\rho \to -\rho$, it follows that the behavior of the component function $\Psi_\omega^{(1)}$ is found to be

$$\Psi_\omega^{(1)}\left(\epsilon - \frac{\pi}{2}\right) \approx \frac{2^k k\Gamma(\omega-k)\Gamma(k)}{\Gamma(\omega+k+1)}\left[C_1\frac{2}{\pi}\sin\pi(\omega-k) + C_2\cos\pi(\omega-k)\right]\epsilon^{-k-\frac{1}{2}}\,. \tag{H.8}$$

Similarly, taking the minus sign in Eq. (H.7) and $\rho = \epsilon - \pi/2$ we obtain

$$\mathsf{P}_\omega^{-k}(-\cos\epsilon) - \mathsf{P}_{\omega-1}^{-k}(-\cos\epsilon) \approx \sin\pi(\omega-k)\frac{2^{k+1}\omega\Gamma(\omega-k)\Gamma(k)}{\pi\Gamma(\omega+k+1)}\epsilon^{-k}\,, \tag{H.9a}$$

$$\mathsf{Q}_\omega^{-k}(-\cos\epsilon) - \mathsf{Q}_{\omega-1}^{-k}(-\cos\epsilon) \approx \cos\pi(\omega-k)\frac{2^{k}\omega\Gamma(\omega-k)\Gamma(k)}{\Gamma(\omega+k+1)}\epsilon^{-k}\,, \tag{H.9b}$$

and thus, the second component $\Psi_\omega^{(2)}$ satisfies

$$\Psi_\omega^{(2)}\left(\epsilon - \frac{\pi}{2}\right) \approx \frac{2^k \omega\Gamma(\omega-k)\Gamma(k)}{\Gamma(\omega+k+1)}\left[\frac{2}{\pi}C_1\sin\pi(\omega-k) + C_2\cos\pi(\omega-k)\right]\epsilon^{-k+\frac{1}{2}}\,. \tag{H.10}$$

Equations (H.8) and (H.10) are Eq. (8.40).

For the case when $k = 0$, the Ferrers functions in the component functions of Eq. (8.24) reduce to Legendre functions of the form $\mathsf{P}_\omega(x)$ and $\mathsf{Q}_\omega(x)$. At $x = 1$ we have $\mathsf{P}_\omega(1) = 1$ for the functions of the first kind, and for the functions of the second kind we have, as $x \to 1$,

$$\mathsf{Q}_\omega(x) \approx \mathsf{P}_\omega(x)\left[\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) - \gamma - \psi(\omega+1)\right]\,, \tag{H.11}$$

where ψ denotes the digamma function, and γ the Euler constant $\psi(1)$. Letting $\rho = \pi/2 - \epsilon$, for sufficiently small values of $\epsilon > 0$ we use Eq. (H.11) with $x = \cos\epsilon$ to obtain

$$\mathsf{Q}_\omega(\cos\epsilon) + \mathsf{Q}_{\omega-1}(\cos\epsilon) \approx \ln 2 - 2\ln\epsilon - 2\gamma - 2\psi(\omega) - \frac{1}{\omega}\,, \tag{H.12a}$$

$$\mathsf{Q}_\omega(\cos\epsilon) - \mathsf{Q}_{\omega-1}(\cos\epsilon) \approx -\frac{1}{\omega}\,, \tag{H.12b}$$

where we have used the estimate $1 - \cos\epsilon \sim \epsilon^2/2$ and the identity $\psi(\omega+1) = \psi(\omega) + 1/\omega$. Using Eq. (H.3) we take into account the prefactors of the component functions in Eq. (8.24), and thus we have

$$\Psi^{(1)}_\omega\left(\frac{\pi}{2} - \epsilon\right) \approx -2\, C_2\, \epsilon^{\frac{1}{2}} \ln\epsilon\,, \tag{H.13a}$$

$$\Psi^{(2)}_\omega\left(\frac{\pi}{2} - \epsilon\right) \approx \frac{1}{\omega} C_2 \epsilon^{-\frac{1}{2}}\,, \tag{H.13b}$$

hence, considering that $\epsilon^{1/2}\ln\epsilon \to 0$ as $\epsilon \to 0$, it is clear that Eq. (8.42a) immediately follows.

To evaluate at the other endpoint of the spatial boundary, we use the connection formulas for the Legendre functions which are obtained by setting $k = 0$ in Eq. (H.5). It then follows that the identities in Eq. (H.6) with $k = 0$ are valid for the Legendre functions. By setting $-x = \sin\rho$ with $\rho = \epsilon - \pi/2$, and using Eq. (H.12) we have that

$$\mathsf{P}_\omega(-\cos\epsilon) + \mathsf{P}_{\omega-1}(-\cos\epsilon) \approx \frac{2}{\pi\omega}\sin\pi\omega\,, \tag{H.14a}$$

$$\mathsf{Q}_\omega(-\cos\epsilon) + \mathsf{Q}_{\omega-1}(-\cos\epsilon) \approx \frac{1}{\omega}\cos\pi\omega\,, \tag{H.14b}$$

for the plus sign, and

$$\begin{aligned}\mathsf{P}_\omega(-\cos\epsilon) - \mathsf{P}_{\omega-1}(-\cos\epsilon) \approx{}& -\frac{2}{\pi}\sin\pi\omega\left(2\ln\epsilon - \ln 2 - 2\gamma - 2\psi(\omega) - \frac{1}{\omega}\right)\\ &+ 2\cos\pi\omega\,,\end{aligned} \tag{H.15a}$$

$$\begin{aligned}\mathsf{Q}_\omega(-\cos\epsilon) - \mathsf{Q}_{\omega-1}(-\cos\epsilon) \approx{}& -\cos\pi\omega\left(2\ln\epsilon - \ln 2 - 2\gamma - 2\psi(\omega) - \frac{1}{\omega}\right)\\ &- \frac{\pi}{2}\sin\pi\omega\,,\end{aligned} \tag{H.15b}$$

for the minus sign. Once again, using Eq. (H.3) the expressions above imply that

$$\Psi^{(1)}_\omega\left(\epsilon - \frac{\pi}{2}\right) \approx \left[C_1 \frac{2}{\pi\omega}\sin\pi\omega + C_2 \frac{1}{\omega}\cos\pi\omega\right]\epsilon^{-\frac{1}{2}}\,, \tag{H.16a}$$

$$\Psi^{(2)}_\omega\left(\epsilon - \frac{\pi}{2}\right) \approx -\left[C_1 \frac{2}{\pi}\sin\pi\omega + \cos\pi\omega\right]\epsilon^{\frac{1}{2}}\ln\,\epsilon\,. \tag{H.16b}$$

Using these approximations and the fact that $\epsilon^{1/2}\ln\epsilon \to 0$ as $\epsilon \to 0$, we have the asymptotic expansion given in Eq. (8.42b) immediately.

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