Causality and Measurement in Quantum Field Theory on Fixed Backgrounds

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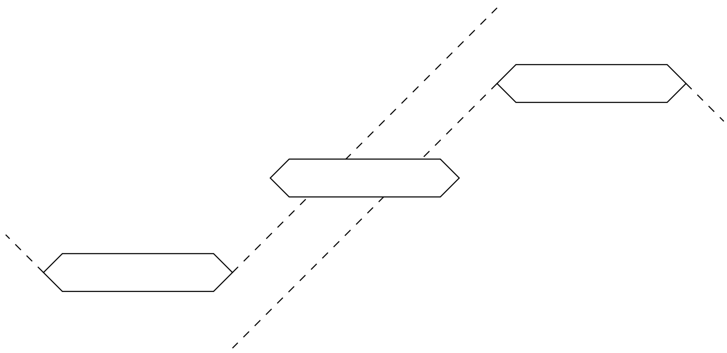

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Abstract

One aspect of causality of quantum field theory on fixed backgrounds, such as globally hyperbolic spacetimes, is that no localised intervention (e.g., a non-selective measurement of a local observable) can affect the outcome of a measurement at spacelike separation. In other words, a description of interventions in QFT needs to respect the causal structure of the background. However, despite the fact that the initial focus of *algebraic* quantum field theory *à la* Haag and Kastler was exactly on interventions (or “operations”), Sorkin’s “Impossible Measurements on Quantum Fields” revealed a major shortcoming of the original description and left QFT without a framework for *causal* interventions and consequently without a framework capable of describing successive measurements.

In this thesis we address this shortcoming. We motivate that the structure of interventions is that of a convex time-orderable pre-factorisation algebra (ctPFA) and, for a given AQFT, we introduce ctPFAs of *causal* quantum channels. Using insight gained from thoroughly analysing scattering off spacetime-compact perturbations of AQFTs, we construct for every AQFT (fulfilling additivity and time-slice) a ctPFA of causal quantum channels, which equips AQFT with a framework for causal interventions.

Although we do not claim that our constructed ctPFAs are *maximal*, we demonstrate that they contain the emergent state-update maps of the AQFT measurement schemes introduced by Fewster and Verch and also generalisations thereof. In particular, for the example of linear real scalar fields, we show that every local observable admits *causal* state-update maps, which finally settles the quest for a *causal* measurement theory for AQFT.

Eventually, through the example of entanglement harvesting we show how causal quantum channels may be used for a fully causal implementation of relativistic quantum information protocols.

Contents

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Author's declaration

I declare that this thesis is a presentation of original work based on my own research carried out under supervision of Christopher J. Fewster and Henning Bostelmann and that I am the sole author. Chapter 5 is based on [1], joint work with Fewster and Bostelmann, and on [2], joint work with Fewster and Jubb. Chapter 6 is based on the single-authored publication [3].

This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

— 1 —

Introduction

The framework of this thesis is that of algebraic quantum field theory on fixed backgrounds and the main theme is that of causality. An important purpose of a fixed background is to supply notions of *localisation* as well as *causality.* The former enables one to ask questions of *when* and *where*, which, for instance, may be answered by referring to a (spatio-temporal) *localisation region* of $3+1$-dimensional Minkowski spacetime. The latter, namely *causality*, is to be understood as follows: a background supplies a relation (between two regions) telling us whether region N_{B} *cannot signal to* some other region N_{A}. If N_{B} cannot signal to N_{A} *and* N_{A} cannot signal to N_{B} we say that the two regions are causally disjoint. An example for that is given by two *spacelike separated* regions of Minkowski spacetime. The heuristic idea is that in that case *no* "intervention" in region N_{A} can affect the outcome of any experiment in region N_{B}. But in order to talk about interventions, there needs to be "something" to intervene on, "something" other than the background (which is supposed to be fixed) for instance a *physical system.*

In the algebraic approach to physics, a physical system is essentially described by a unital $*$-algebra, whose Hermitian elements are considered to be the observables. States are then positive normalised linear functionals and associate *expectation values* to observables. Furthermore, "interventions" are given by quantum channels, i.e., completely positive unit-preserving linear maps, the prime example being a state-update map. The motivation for algebraic quantum field theory is to consistently associate physical systems to the regions of a fixed background in a way that respects the ideas of localisation and causality. In practice, this means that an algebraic quantum field theory associates unital $*$-algebras to regions, with the interpretation that the Hermitian elements are those observables that are accessible by performing an actual experiment in the specified spatio-temporal region.[1] It is then common to

[1]Note that this means that one is most naturally working in the Heisenberg picture, where

assume that algebras associated to causally disjoint regions *commute*.[2] Following this interpretation, one may ask which interventions one is allowed to perform in a given region. The point of view taken in [4] was to allow (at least) every quantum channel that is implemented by the action of an element of the local algebra. To give a concrete example, let U_{A} be a unitary in the algebra associated to a region N_{A}. Then this unitary induces a quantum channel that acts on a state ω as

$$\omega(\cdot) \mapsto \omega(U_{\mathsf{A}}^* \cdot U_{\mathsf{A}}). \tag{1.1}$$

At first sight, this seems like a perfectly fine candidate for a local intervention that respects the causality of the background. Namely, the prediction for the expectation value of a measurement of an observable C in a region N_{C} that is causally disjoint from N_{A} is then given by

$$\omega(U_{\mathsf{A}}^* C U_{\mathsf{A}}) = \omega(C), \tag{1.2}$$

due to the fact that $[U_{\mathsf{A}}, C] = 0$ and $U_{\mathsf{A}}^* U_{\mathsf{A}} = \mathbb{1}$. Hence, the intervention in Eq. (1.1) does not affect the outcome of any experiment performed in the causally disjoint region N_{B}. This fact *seemingly* (but unfortunately *wrongly*) suggested that every such local intervention respects causality and motivated calling the local algebras "algebras of operations" [6]. However, as pointed out by Sorkin in [7], causality imposes stronger restrictions, which becomes apparent when considering more than just two regions, see sketch on title page or also Fig. 3.1. As a consequence, no description of *causal* quantum channels was available and hence neither was a measurement theory due to the lack of criteria that would single out state-update rules that respect causality.[3]

In order to resolve the lack of a description of *causal* interventions we follow a top-down approach in Sec. 3.2 and motivate the structure of *causal* quantum channels for a given algebraic quantum field theory. In particular, for every region there should be a convex set of quantum channels associated to said region. Among others, two important properties of such an association are that

1. the composition of quantum channels should be well-defined (in particular when the order of the localisation regions is not unique, as is the case for causally disjoint regions), and

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states are global and observables are local.

[2]The original motivation for this axiom was to ensure statistical independence of causally disjoint regions, i.e., to ensure that any two states on algebras associated to causally disjoint regions may be considered as the restrictions of a globally defined state, see [4]. However, it turns out that commutativity is logically independent from statistical independence, see for instance [5].

[3]One might consider [8] the first attempt to give such a criterion.

2. quantum channels should not enable any signalling to causally disjoint regions under any circumstances whatsoever.

The obvious question then is whether there are any non-trivial examples of an association of *causal* quantum channels for a given algebraic quantum field theory. Motivated by the results of [1], which is joint work with Bostelmann and Fewster, and the subsequent work of Jubb [9], a tool to answer this question is scattering theory of algebraic quantum field theories on globally hyperbolic spacetimes, which is the topic of Chapter 4. We introduce the notion of K-perturbed variants of algebraic quantum field theories, closely following the presentation in [10] and, motivated by abstract properties of scattering maps, introduce so-called K-maps. In particular, we *prove*

1. that every K-perturbed variant with a compact perturbation zone K given by a disjoint union of two causally orderable sets K_A and K_B decomposes consistently into a K_A- and a K_B-perturbed variant,
2. that for every couple of causally orderable compact perturbation zones, K_A and K_B, and every couple of a K_A- and a K_B-perturbed variant there exists a combined $K_\mathsf{A} \cup K_\mathsf{B}$-perturbed variant, and
3. that every K-map is a scattering map (on some background), which may be regarded as an inverse scattering problem.

These results are important technical achievements of the present work.

At the end of Chapter 4 we then show how K-maps may indeed be used to yield an association of causal quantum channels for any additive algebraic quantum field theory fulfilling time-slice on any globally hyperbolic spacetime.

Equipped with a whole class of causal quantum channels, we return to the question of measurement and state-update maps in Chapter 5. Following [10], we recall the notion of measurement schemes for algebraic quantum field theory on globally hyperbolic spacetimes. We present the results of [1], joint work with Bostelmann and Fewster, in which we showed how (selective as well as non-selective) state-updates *emerge* from measurement schemes and how they respect causality. This demonstrates that a description of measurements with measurement schemes is free of Sorkin's causality issues. A final open question is then whether every observable can be measured using a measurement scheme. For theories of linear real scalar fields fulfilling a normally hyperbolic equation of motion we present the following results.

1. In Sec. 5.3 we show that every local observable can be *exactly* measured with a measurement scheme using a certain K-map. However, this K-map *might* be associated to a somewhat "singular" K-perturbed variant, which is why

2. in Sec. 5.4, based on [2], joint work with Fewster and Jubb, we show that every local observable can be *approximately* measured with a sequence (or rather net) of very well-behaved measurement schemes.

This may be regarded as the final step in closing the issue of impossible measurements on quantum fields [7].

Afterwards, in Chapter 6 we discuss applications to relativistic quantum information. In particular, based on the single-authored work [3], we discuss the protocol of *entanglement harvesting* from a very abstract point of view as well as from a more concrete point of view using *causal* particle detector models used in [3]. These results pave the way for an analysis of truly *causal* quantum information protocols in algebraic quantum field theory.

Finally, in Chapter 7 we give an outlook for future work motivated by the presented results.

— 2 —

Introduction to Algebraic Quantum Theory

Inspirations for the following section are drawn from [11, 12, 13].

2.1 Some motivation for the algebraic approach to physics

In the algebraic approach to physics, the abstract characterisation of a physical system encompasses the specification of two sets, $\mathcal{O}_{\mathbb{R}}$ and $\mathfrak{S}_{\mathbb{R}}$. The elements of $\mathcal{O}_{\mathbb{R}}$ are called *observables* and the elements of $\mathfrak{S}_{\mathbb{R}}$ are called *states*. We also assume that there is a pairing, i.e., a map

$$(\cdot|\cdot) : \mathfrak{S}_{\mathbb{R}} \times \mathcal{O}_{\mathbb{R}} \to \mathbb{R}. \tag{2.1}$$

Generally speaking, one then distinguishes

1. the abstract observable $A \in \mathcal{O}_{\mathbb{R}}$ from
2. the outcomes of individual measurement runs of A when the system is in state ω.

We will proceed by making the assumption that the outcome of (individual) measurement runs are always real *numbers*. In practice, this amounts to the specific choice of a numerical *scale* (which includes a choice of units) of a given measurement device.

On physical grounds[1], we do not require that the abstract characterisation of a physical system allows to predict “the” outcome of a single measurement run of A in a state $\omega \in \mathfrak{S}_{\mathbb{R}}$. Instead, we assume that

1. for every observable $A \in \mathcal{O}_{\mathbb{R}}$ there is a collection of all possible outcomes of individual measurement runs of A called the *measurement spectrum* of A, and

[1] And in order to incorporate *statistical* theories.

2. that for every $A \in \mathcal{O}_{\mathbb{R}}$ and $\omega \in \mathfrak{S}_{\mathbb{R}}$ there is a probability measure $\mu_{A,\omega}$ on $\mathbb{R}$, supported on the measurement spectrum of A, describing measurements of A in state ω.

Importantly, the measures $\mu_{A,\omega}$ are not directly part of our abstract characterisation of a physical system in terms of the triple $(\mathcal{O}_{\mathbb{R}}, \mathfrak{S}_{\mathbb{R}}, (\cdot|\cdot))$ but should ideally be derived from it. A first step in this direction is the assertion that the real number $(\omega|A)$ for $\omega \in \mathfrak{S}_{\mathbb{R}}$ shall have the interpretation of the *expected value* of A in state ω, i.e., it is the first moment of the putative[2] probability measure $\mu_{A,\omega}$.

It should be explicitly emphasised that we have already made several idealisations starting with the assumption that outcomes of individual measurement runs are real *numbers*, while it would be more appropriate to regard them as subsets of $\mathbb{R}$. This is due to the finite *graduation* of any *physical* measurement scale, which only allows for finite precision. Furthermore, we necessarily encounter the standard difficulties in matching (classical) probability theory on the theoretical side with finitely many measurement runs producing actual numerical outcomes (including error bars) on the operational side. We will come back to the first aspect in Chapter 5. As for the second aspect, we will make the standard assumption that an *external* observer in control of the physical system under consideration may use a measurement apparatus to determine outcomes (assumed to be real numbers) of N individual measurement runs of an observable A when the system is in state ω. In each measurement run the system and the measurement apparatus are identical[3] and independently prepared and the outcomes are N numerical values $\mathtt{x}_{A,\omega}[i]$ for $i = 1, ..., N$. The measurement runs can be modelled by a sample of N *independent and identically distributed* (iid) random variables $(\mathtt{X}_{A,\omega,1}, ..., \mathtt{X}_{A,\omega,N})$ consisting of N copies of the putative random variable $\mathtt{X}_{A,\omega}$ with putative distribution $\mu_{A,\omega}$. The *observed* sample mean $\frac{1}{N}\sum_{i=1}^{N} \mathtt{x}_{A,\omega}[i]$ serves as a "good" estimate for the expectation value of the random variable $\mathtt{X}_{A,\omega}$. "Good" here means that the associated random variable, being $\mathtt{X}^{(N)}_{A,\omega} := \frac{1}{N}\sum_{i=1}^{N} \mathtt{X}_{A,\omega,i}$ is an unbiased estimator for the expectation value of $\mathtt{X}_{A,\omega}$, and that furthermore, by Chebyshev's inequality,

$$\mathbb{P}\Big(|\mathtt{X}^{(N)}_{A,\omega} - \mathbb{E}(\mathtt{X}_{A,\omega})| < \epsilon\Big) \geq 1 - \frac{\mathbb{V}\Big(\mathtt{X}^{(N)}_{A,\omega}\Big)}{\epsilon} = 1 - \frac{\mathbb{V}(\mathtt{X}_{A,\omega})}{N\epsilon}, \tag{2.2}$$

where $\mathbb{E}, \mathbb{V}$ denote the expectation value and the variance respectively. Following the usual interpretation of classical probability theory, with a probability of $1 - \frac{\mathbb{V}(\mathtt{X}_{A,\omega})}{N\epsilon}$

[2]Existence and uniqueness of $\mu_{A,\omega}$ will be discussed further below.

[3]In fact, one could conceive that the measurement apparatus used for the individual measurement runs are in fact physically distinct, as long as they can and will be used to measure the *same* observable.

the observed sample mean deviates by at most ϵ from the true expectation value $\mathbb{E}(\mathtt{X}_{A,\omega})$, the first moment of the putative probability measure $\mu_{A,\omega}$, which is predicted to be $(\omega|A)$.[4]

We will now discuss more operational arguments and mathematical idealisations that motivate the additional structure of the triple $(\mathcal{O}_\mathbb{R}, \mathfrak{S}_\mathbb{R}, (\cdot|\cdot))$ in the algebraic approach to physics.

Firstly, we assume that

$$\forall \omega_1, \omega_2 \in \mathfrak{S}_\mathbb{R} : \qquad (\forall A \in \mathcal{O}_\mathbb{R} : (\omega_1|A) = (\omega_2|A)) \implies \omega_1 = \omega_2, \tag{2.3}$$

i.e., two states are identical, if and only if every observable has the same expected value in both of them. Another way of saying this is that the pairing $(\cdot|\cdot)$ induces an *injective* map from $\mathfrak{S}_\mathbb{R}$ to the set of real-valued functions on $\mathcal{O}_\mathbb{R}$. Hence, we will identify any state $\omega \in \mathfrak{S}_\mathbb{R}$ with the function

$$\omega(\cdot) := (\omega|\cdot) : \mathcal{O}_\mathbb{R} \to \mathbb{R}. \tag{2.4}$$

A possible next step, as taken in [12] and [11] is to reciprocally deem two observables $A_1, A_2 \in \mathcal{O}_\mathbb{R}$ identical, if $\forall w \in \mathfrak{S}_\mathbb{R} : \omega(A_1) = \omega(A_2)$, i.e., to assume that the states separate the observables. This means that $A_1 = A_2$ if and only if their expected values agree in every state. In the following however, we will not make this assumption.[5]

Secondly, we assume that there is an observable $\mathbb{1} \in \mathcal{O}_\mathbb{R}$ whose expected value in every state equals 1. This could be seen as a trivial observable that is being measured by an apparatus that displays 1 whenever it is switched on.

Thirdly, for a measurement apparatus for an observable $A \in \mathcal{O}_\mathbb{R}$, we obtain a different measurement apparatus by feeding the outcome to an arbitrary real-valued *polynomial*[6] p, which can be thought of as relabelling the measurement scale. The

[4]As we will see later, $\mathbb{V}(\mathtt{X}_{A,\omega})$ is predicted to be $(\omega|A^2) - (\omega|A)^2$.

[5]It is however certainly always possible to restrict to $\mathcal{O}_\mathbb{R}/_\sim$, where $A \sim B :\iff \forall \omega \in \mathfrak{S}_\mathbb{R} : \omega(A) = \omega(B)$. Then $\mathcal{S}$ separates $\mathcal{O}_\mathbb{R}/_\sim$. Note that every unital C^*-algebra is separated by its normalised positive linear functionals, while this is in general not true for mere unital $*$-algebras: Consider for instance the (Abelian) unital $*$-algebra of all $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$ for $\alpha, \beta \in \mathbb{C}$ equipped with the usual matrix multiplication and *entry-wise* complex conjugation as $*$-operation. There is a single normalised positive linear functional given by $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \mapsto \alpha$, which does not separate the $*$-algebra.

[6]Recall that every measurement apparatus has only finite precision and that every continuous function on a closed interval can be uniformly approximated by polynomials according to Stone-Weierstrass.

abstract observable measured by this "new" device is denoted by $p(A) \in \mathcal{O}_{\mathbb{R}}$, where $B^0 = \mathbb{1}$ for every $B \in \mathcal{O}_{\mathbb{R}}$. Note that in general $(\omega|A^n) \neq (\omega|A)^n$. It is an immediate consequence of the definition of $\sum_{j=1}^{n} \lambda_j A^j$ for $(\lambda_j)_j \subseteq \mathbb{R}$ that for every $\omega \in \mathfrak{S}_{\mathbb{R}}$:

$$\omega\left(\sum_{j=1}^{n} \lambda_j A^j\right) = \sum_{j=1}^{n} \lambda_j \omega(A^j), \tag{2.5}$$

and that in particular

$$\forall A \in \mathcal{O}_{\mathbb{R}} : \ \omega(A^2) \geq 0. \tag{2.6}$$

By construction, for an observable $A \in \mathcal{O}_{\mathbb{R}}$, the real numbers $\omega(A^n)$ for $n \in \mathbb{N}$ should be the higher *moments* of the putative probability measure $\mu_{A,\omega}$. A necessary and sufficient condition for the existence of a (not necessarily unique) Radon probability measure $\mu_{A,\omega}$ on $\mathbb{R}$, see Definition A.1 in [14], such that

$$\omega(A^n) = \int_{\mathbb{R}} x^n \, \mathrm{d}\mu_{A,\omega}(x) \tag{2.7}$$

is given by the requirement that for every real sequence $(\lambda_j)_j \subseteq \mathbb{R}$ with finitely many non-vanishing entries it holds that

$$\sum_{j,k=0}^{\infty} \lambda_j \omega(A^{j+k}) \lambda_k \geq 0. \tag{2.8}$$

This is the solution of the Hamburger moment problem, see Theorem 3.8 in [14].

As a consequence of Eq. (2.6), this requirement is always fulfilled and hence the numbers $\omega(A^n)$ are indeed the moments of a probability measure $\mu_{\omega,A}$. One could then proceed and define the measurement spectrum of A to be the closure of the union of the supports of all $\mu_{A,\omega}$ for varying ω.[7] Note, however, that $\mu_{A,\omega}$, might not be unique. In fact, as argued in [16], it is unclear whether the available information "permits one to fix this choice or somehow reduce the number of possibilities".[8]

Fourthly, given two states $\omega_1, \omega_2 \in \mathfrak{S}_{\mathbb{R}}$ and $\lambda \in [0,1]$, the function $\omega := \lambda\omega_1 + (1-\lambda)\omega_2$ has the operational interpretation of a probabilistic mixture of preparations of the system and, as such, should denote a valid state in $\mathfrak{S}_{\mathbb{R}}$. With this observation $\mathfrak{S}_{\mathbb{R}}$ becomes a convex subset of the $\mathbb{R}$-vector space of real-valued functions on $\mathcal{O}_{\mathbb{R}}$.

Fifthly, we make an assumption "lying beyond the strict operational setting" [12]. We have already seen that $\mathcal{O}_{\mathbb{R}}$ has the structure of a real vector space and that every

[7] See Proposition in Sec. VII.2 on p. 229 in [15].

[8] This refers to the unital $*$-algebra case. For unital C^*-algebras, however, one can associate a unique Radon probability measure, see Proposition 5 (b) in [16].

state in $\mathfrak{S}_{\mathbb{R}}$ is a linear functional. In particular, for every $A, B \in \mathcal{O}_{\mathbb{R}}$, there exists an observable (denoted by $A+B$) such that for every state $\omega \in \mathfrak{S}_{\mathbb{R}}$:

$$\omega(A) + \omega(B) = \omega(A+B). \tag{2.9}$$

Based on this and the previous assumptions we can now define the doubled[9] *Jordan product*

$$\begin{aligned} \{\cdot,\cdot\} &: \mathcal{O}_{\mathbb{R}} \times \mathcal{O}_{\mathbb{R}} \to \mathcal{O}_{\mathbb{R}} \\ \{A, B\} &:= (A+B)^2 - A^2 - B^2, \end{aligned} \tag{2.10}$$

which is clearly always commutative, i.e., $\{A, B\} = \{B, A\}$. We will assume distributivity of $\{\cdot,\cdot\}$, which, while in general not guaranteed, could be derived from the (potentially better motivated) assumption that $\{\cdot,\cdot\}$ is $\mathbb{R}$-homogeneous, see p. 21 in [12].

Finally, we will make another technical assumption with little initial operational motivation other than that it generally seems to be mathematically convenient to view *real* algebraic structures as special subsets of *complex* algebraic structures. Concretely, we will make the assumption that $\mathcal{O}_{\mathbb{R}}$ is given by the Hermitian elements of a complex unital $*$-algebra $\mathcal{O}$ (with an associative but not necessarily commutative product) such that for $A, B \in \mathcal{O}_{\mathbb{R}}$, $\{A, B\}$ is given by the anticommutator of A and B.

Definition 2.1.1 (unital $*$-algebra, Def. 9.1.1 in [17])**.** *A unital $*$-algebra $\mathcal{O}$ is a complex vector space together with an associative, $\mathbb{C}$-bilinear product, a unit $\mathbb{1}$ and a conjugate linear involution ($*$-operation) such that*

$$(AB)^* = B^* A^*. \tag{2.11}$$

An element $A \in \mathcal{O}$ is called

1. *Hermitian, if $A^* = A$,*

2. *positive, if $\exists n \in \mathbb{N}, B_1, ..., B_n \in \mathcal{O} : A = \sum_{j=0}^{n} B_j^* B_j$,*

3. *an effect, if A and $\mathbb{1} - A$ are positive,*

4. *a projection, if $A = A^* = A^2$.*

We write $A \leq B$ if $B - A$ is positive and say $\mathcal{O}$ is ordered *if and only if $\leq$ is an order relation.*

[9] The Jordan product of A and B is given by $\frac{1}{2}\{A, B\}$.

Effects are important observables, which will be considered in more depth in Sec. 2.2.

Lemma 2.1.2 (Remark below Def. 9.4.19 and Proposition 9.1.6 in [17]). *The binary relation $\leq$ is a preorder, i.e., it is reflexive and transitive (but not necessarily antisymmetric). The Hermitian elements together with $\leq$ are a preordered $\mathbb{R}$-vector space, i.e,*

$$(A \leq B) \implies (\forall \text{ Hermitian } C : A + C \leq B + C), \tag{2.12}$$

and for all $\lambda \in \mathbb{R}_0^+$

$$A \leq B \implies \lambda A \leq \lambda B. \tag{2.13}$$

In particular, the set of effects of a unital $*$-algebra is a convex set, i.e., for all effects A, B and for all $\lambda \in [0, 1]$ it holds that

$$\lambda A + (1 - \lambda) B \tag{2.14}$$

is an effect.

Let us now consider states. Having put the focus on the observables, and having identified $\mathcal{O}_\mathbb{R}$ with the Hermitian elements of a unital $*$-algebra $\mathcal{O}$, we now have some freedom which concrete set of states to consider. In the following we will refer to the biggest possible set as the set of states. According to the general reasoning above, a state ω should be an $\mathbb{R}$-linear function from $\mathcal{O}_\mathbb{R}$ to $\mathbb{R}$, that maps $\mathbb{1}$ to 1 and that is positive on positive elements. Observing that

$$A = \frac{1}{2}(A + A^*) + \frac{1}{2\mathrm{i}}(\mathrm{i}A - \mathrm{i}A^*), \tag{2.15}$$

we see that every element is given as a complex linear combination of two Hermitian elements (or four even positive ones, see Eq. (9) in the proof of Proposition 9.1.6 in [17]). Hence, every $\mathbb{R}$-linear function from $\mathcal{O}_\mathbb{R}$ to $\mathbb{R}$ extends uniquely to a $\mathbb{C}$-linear function from $\mathcal{O}$ to $\mathbb{C}$. This motivates the following definition, for which we recall that a point p in a convex set is called *extreme* if and only if for $\lambda \in [0, 1]$

$$p = \lambda x + (1 - \lambda) y \implies (x = y \vee \lambda \in \{0, 1\}). \tag{2.16}$$

Definition 2.1.3 (States). *Let $\mathcal{O}$ be a unital $*$-algebra. A $\mathbb{C}$-linear functional $\omega : \mathcal{O} \to \mathbb{C}$ is called* state, *if it is*

1. *normalised, i.e., $\omega(\mathbb{1}) = 1$, and*

2. *positive, i.e., $\forall A \in \mathcal{O} : \omega(A^* A) \geq 0$.*

The set of states is a convex subset of the algebraic dual space of $\mathcal{O}$. Extreme states are called pure. *A state that is not pure is called mixed.*

A state is called faithful, if

$$\forall A \in \mathcal{O} : \omega(A^*A) = 0 \implies A = 0. \tag{2.17}$$

Remark: This definition differs from Palmer's Def. 9.4.21 in [17], in which he also requires that states are *Hilbert bounded*, see Def. 9.4.2 therein.

Lemma 2.1.4. *States are monotone, i.e.,*

$$A \leq B \implies \omega(A) \leq \omega(B). \tag{2.18}$$

In particular,

$$(0 \leq A \wedge A \leq 0) \implies \omega(A) = 0. \tag{2.19}$$

In summary, in the algebraic approach to physics, a very general description of a physical system is given in terms of a unital $*$-algebra $\mathcal{O}$ together with its set of states $\mathfrak{S}$. $\mathcal{O}$ is spanned by its Hermitian elements, which are the mathematical idealisation of *observables*.

The structure of a unital $*$-algebra is somewhat minimal and generally lacks desirable properties. For instance, it is not guaranteed, that the states separate the algebra, that an observable A and a state ω give rise to a unique random variable $X_{A,\omega}$, that the effects form an effect algebra (see the following section) with projections being precisely the pure effects, or in fact, that there are non-trivial effects at all. In order to improve this situation one may consider unital $*$-algebras with additional structure such as a topology. In Appendix A, we will introduce topological unital $*$-algebras, unital C^*-algebras and unital von Neumann algebras.

2.2 EFFECT ALGEBRAS

Effects are interesting as they are observables with expectation values in $[0, 1]$, which may be interpreted as the "success probability" of observing said "effect". As we will discuss now, the effects of a unital $*$-algebra form a structure called pre-effect algebra, which is of independent interest and may also serve as an axiomatic approach to quantum theory.

Definition 2.2.1 ((pre-)Effect algebra [18])**.** *Let L be a set with two elements $0, \mathbb{1} \in L$ and a partially defined binary operation $\oplus : L \times L \to L$ such that the following conditions hold for all $p, q, r \in L$:*

1. *[Commutative law] If $p \oplus q$ is defined, then $q \oplus p$ is defined and $p \oplus q = q \oplus p$.*

2. *[Associative law] If $q \oplus r$ and $p \oplus (q \oplus r)$ are defined, then $p \oplus q$ and $(p \oplus q) \oplus r$ are defined and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.*

3. *[Orthosupplementation Law] For every $p \in L$ there exists a unique $q \in L$ such that $p \oplus q$ is defined and $p \oplus q = \mathbb{1}$. In this case we write $\mathbb{1} \ominus p := q$.*

Then $(L, \oplus, 0, \mathbb{1})$ is called pre-effect algebra. It is called effect algebra if in addition the following holds:

4. *[Zero-One Law] If $\mathbb{1} \oplus p$ is defined, then $p = 0$.*

We write $p \trianglelefteq q$ if there exists r such that $p \oplus r = q$.

Lemma 2.2.2 ([18]). *Let $(L, \oplus, 0, \mathbb{1})$ be a pre-effect algebra, then $\trianglelefteq$ is a preorder. If $(L, \oplus, 0, \mathbb{1})$ is an effect algebra, then L is partially ordered and $L = \{p \in L | 0 \trianglelefteq p \trianglelefteq \mathbb{1}\}$.*

Definition 2.2.3. *Let $(L, \oplus, 0, \mathbb{1})$ be a pre-effect algebra. We call $p \in L$* sharp, *if the $\trianglelefteq$-infimum of $\{p, \mathbb{1} \ominus p\}$ exists, is unique and equals 0, i.e., if 0 is the unique element such that for every lower bound r of $\{p, \mathbb{1} \ominus p\}$ it holds that $r \trianglelefteq 0$.*

Lemma 2.2.4. *The effects of a unital $*$-algebra, where $\oplus$ is the addition of effects (defined whenever the sum is again an effect), form a pre-effect algebra and for two effects E_1, E_2 it holds that*

$$E_1 \leq E_2 \iff E_1 \trianglelefteq E_2. \tag{2.20}$$

The effects of an ordered *unital $*$-algebra form an effect algebra.*

Remark: Not every non-trivial unital $*$-algebra has non-trivial effects.

Proof. For the first part only the implication "$\implies$" is non-trivial. Suppose $E_1 \leq E_2$ for two effects. Then by definition $0 \leq E_2 - E_1$. Furthermore, since $0 \leq E_1$ and $E_2 \leq \mathbb{1}$, we see that $E_2 - E_1 \leq \mathbb{1}$, hence $E_2 - E_1$ is an effect and $E_1 \trianglelefteq E_2$.

What is left to show is that the "Zero-One Law" holds in an *ordered* unital $*$-algebra. Suppose $\mathbb{1} + P$ is an effect for an effect P. Then, $0 \leq P$ *and* $P \leq 0$, hence $P = 0$. □

We now note the following interesting result.

Lemma 2.2.5 (Lemma 4.4 in [19]). *Let $\mathcal{O}$ be a unital $*$-algebra.*

1. *Extreme effects of $\mathcal{O}$ are sharp.*

2. *If $\mathcal{O}$ is ordered and if for every effect E it holds that $E^2 \leq E$, then sharp effects are projections.*

If $\mathcal{O}$ is a unital C^*-algebra, then E is an extreme effect if and only if E is a projection.

Proof. The first part is Lemma 4.4 in [19], which we reproduce here. Suppose E is an extreme effect. In order to show that 0 is the infimum of $\{E, \mathbb{1} - E\}$ over the effects, we show that any lower bound B that is an effect fulfills $B \leq 0$. The assumptions imply that $E + B$ and $E - B$ are effects, and since

$$E = \frac{1}{2}(E + B) + \frac{1}{2}(E - B), \tag{2.21}$$

extremity of E implies that $E + B = E - B$, hence $B = 0$.

For the second part let E be a sharp effect. Note that $B := E - E^2 = (\mathbb{1} - E) - (\mathbb{1} - E)^2$ is obviously a lower bound for $\{E, \mathbb{1} - E\}$. If now $E^2 \leq E$ for every effect E, it follows that $0 \leq B$, and by sharpness of E it follows that $B \leq 0$, so $B = 0$. Hence E is a projection.

For the last statement see for instance Proposition 7.4.6 in [20] or Proposition 1.6.2 in [21]. □

2.3 BIPARTITE SYSTEMS, FREE COMBINATIONS AND QUANTUM OPERATIONS

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two unital $*$-algebras of two individually well-defined systems, which we may view as subsystems of a bigger bipartite system with unital $*$-algebra $\mathcal{O}$, i.e., $\mathcal{O}_1$ and $\mathcal{O}_2$ are unital $*$-subalgebras of $\mathcal{O}$. Let us now denote by $\mathcal{O}_1 \vee \mathcal{O}_2$ the smallest unital $*$-algebra in $\mathcal{O}$ that contains both $\mathcal{O}_1$ and $\mathcal{O}_2$,[10] which may be considered as the "combination" of the two (sub-)systems. In general, however, $\mathcal{O}_1 \vee \mathcal{O}_2$ does not only depend on $\mathcal{O}_1$ and $\mathcal{O}_2$, but also on "how $\mathcal{O}_1$ and $\mathcal{O}_2$ sit inside $\mathcal{O}$".

It is now an interesting question whether there are physically motivated properties that $\mathcal{O}_1$ and $\mathcal{O}_2$, being *independent* systems, should fulfill and whether they allow to uniquely (up to equivalence) specify their free combination $\mathcal{O}_{1\&2}$ independent from "how they sit inside $\mathcal{O}_1$". Some of such desirable properties are given in the following list.

1. "Statistical independence": For every two states ω_1, ω_2 on $\mathcal{O}_1, \mathcal{O}_2$, there exists a state ω on $\mathcal{O}_{1\&2}$ such that $\omega \restriction \mathcal{O}_j = \omega_j$ for $j = 1, 2$.

2. $\mathcal{O}_1 \cap \mathcal{O}_2 = \mathbb{C}\mathbb{1}$.

[10] In the unital C^*-algebra case it is of course the smallest unital C^*-algebra with that property.

3. "Commutativity" or "Kinematical independence": For every $A_1 \in \mathcal{O}_1$ and $A_2 \in \mathcal{O}_2 : [A_1, A_2] = 0$.

4. "Independence in the product sense": $\mathcal{O}_{1\&2}$ is isomorphic to the algebraic[11] tensor product $\mathcal{O}_1 \otimes \mathcal{O}_2$.

Statistical independence has a very clear operational motivation. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two "independent" subsystems, then the preparation of a state ω_1 on $\mathcal{O}_1$ should not obstruct the preparation of any state ω_2 on $\mathcal{O}_2$.

In the case where the states on $\mathcal{O}_j$ separate the algebra (for instance in the case of C^*-algebras), it is clear that $\mathcal{O}_1 \cap \mathcal{O}_2 = \mathbb{C}\mathbb{1}$ is a necessary condition for statistical independence. It follows that mere commutativity of $\mathcal{O}_1$ and $\mathcal{O}_2$ in $\mathcal{O}$ is not sufficient for statistical independence. It is perhaps surprising that it is also not necessary, as for instance shown on p. 205 in [5] and references therein. While there appear additional conditions in the literature, that together with statistical independence imply kinematical independence (in the case of C^*-algebras), they are considered to be "fairly unmanageable" [22]. A condition based on unitary quantum operations is given in the Proposition in [23]. However, even this argument is not fully satisfactory. [12]

Nevertheless, kinematical independence of $\mathcal{O}_1$ and $\mathcal{O}_2$ will be a standing assumption in what follows.

Among the above properties, independence in the product sense is clearly the strongest, i.e., it implies all the other ones and specifies $\mathcal{O}_{1\&2}$ *by fiat*.

2.3.1 Entanglement

The following discussion in based on the presentation in [3] and the references therein. Following [24] we introduce the notion of product states.

Definition 2.3.1. *Given $\mathcal{A}, \mathcal{B} \subseteq \mathcal{R}$ two commuting, unital $*$-subalgebras of some unital $*$-algebra $\mathcal{R}$, we say that a state ω on $\mathcal{R}$ is*

1. a product state/uncorrelated *on $\mathcal{A} \vee \mathcal{B} \subseteq \mathcal{R}$, if*

$$\forall A \in \mathcal{A},\ \forall B \in \mathcal{B}:\ \omega(AB) = \omega(A)\omega(B), \tag{2.22}$$

[11] In the case of unital C^*-algebras one would of course pick a(n appropriate) C^*-tensor product, which might however not be unique. Similarly for unital von Neumann algebras.

[12] The problem is that generic unitary operations are not necessarily physical as we will discuss in detail in Chapter 3.

2. classically correlated *on* $\mathcal{A} \vee \mathcal{B}$ *if it is a pointwise limit of convex combinations of product states but not a product state itself, and*

3. entangled *on* $\mathcal{A} \vee \mathcal{B}$ *if it is neither of the above.*

Before we discuss the motivation for this definition, let us mention that it in particular encompasses the case where $\mathcal{A} := \mathcal{A}_1 \otimes \mathbb{C}\mathbb{1}$, $\mathcal{B} := \mathbb{C}\mathbb{1} \otimes \mathcal{A}_2$ and $\mathcal{R} = \mathcal{A}_1 \otimes \mathcal{A}_2$. The prime example of product states in this scenario are functionals of the form $\sigma_1 \otimes \sigma_2$, for states σ_1 on $\mathcal{A}$ and σ_2 on $\mathcal{B}$, which are indeed states, see for instance T.7 in Appendix T of [25]. This also holds in the case of unital C^*-algebras, where one may consider $\mathcal{R}$ to be an appropriate C^*-tensor product and $\mathcal{A} \vee \mathcal{B}$ to be the smallest unital C^*-algebra containing both $\mathcal{A}$ and $\mathcal{B}$.

Now the motivation for the definition is as follows. Given two Hermitian elements $A \in \mathcal{A}$, $B \in \mathcal{B}$ and state ω on $\mathcal{A} \vee \mathcal{B}$, the solution of the Hamburger moment problem yields two (possibly non-unique) probability measures $\mu_{A,\omega}$ and $\mu_{B,\omega}$ such that for every $n \in \mathbb{N}$:

$$\omega(A^n) = \int_{\mathbb{R}} x^n \, \mathrm{d}\mu_{A,\omega}(x), \quad \omega(B^n) = \int_{\mathbb{R}} x^n \, \mathrm{d}\mu_{B,\omega}(x), \tag{2.23}$$

i.e., $\mu_{A,\omega}$ and $\mu_{B,\omega}$ are the distributions of random variables $\mathtt{X}_{A,\omega}$ and $\mathtt{X}_{B,\omega}$. On physical grounds, taking also into account that A and B commute, we expect[13] that there exists a vector of random variables $(\mathtt{X}_{A,\omega}, \mathtt{X}_{B,\omega})$ with distribution $\mu_{(A,B),\omega}$ on $\mathbb{R}^2$ such that $\mu_{A,\omega}$ and $\mu_{B,\omega}$ are the respective *marginals*, i.e.,

$$\mu_{A,\omega}(S) = \mu_{(A,B),\omega}(S \times \mathbb{R}), \quad \mu_{B,\omega}(S) = \mu_{(A,B),\omega}(\mathbb{R} \times S), \tag{2.24}$$

for S in the sigma algebra, and such that

$$\omega(A^n B^m) = \int_{\mathbb{R}^n} x^n y^m \, \mathrm{d}\mu_{(A,B),\omega}(x, y). \tag{2.25}$$

If this expectation is true, then it follows immediately that

$$\begin{aligned} \mathrm{Cov}(\mathtt{X}_{A,\omega}, \mathtt{X}_{B,\omega}) &:= \mathbb{E}(\mathtt{X}_{A,\omega}\mathtt{X}_{B,\omega}) - \mathbb{E}(\mathtt{X}_{A,\omega})\mathbb{E}(\mathtt{X}_{B,\omega}) = \omega(AB) - \omega(A)\omega(B), \\ \mathrm{Cor}(\mathtt{X}_{A,\omega}, \mathtt{X}_{B,\omega}) &:= \frac{\mathrm{Cov}(\mathtt{X}_{A,\omega}, \mathtt{X}_{B,\omega})}{\sqrt{\mathbb{V}(\mathtt{X}_{A,\omega})\,\mathbb{V}(\mathtt{X}_{B,\omega})}} = \frac{\omega(AB) - \omega(A)\omega(B)}{\sqrt{(\omega(A^2) - \omega(A)^2)(\omega(B^2) - \omega(B)^2)}}. \end{aligned} \tag{2.26}$$

[13]As mentioned in [16], existence of $\mu_{(A,B),\omega}$ is the multidimensional moment problem on $\mathbb{R}^2$, which is however *not* a straight forward generalisation of the one-dimensional moment problem, see also [14]. As such existence of $\mu_{(A,B),\omega}$ is an assumption for us. In the unital C^*-algebra case, however, existence of $\mu_{(A,B),\omega}$ follows from the spectral theorem for commuting bounded self-adjoint operators.

Notice that, by Cauchy-Schwarz, $-1 \leq \mathrm{Cor}(\mathtt{X}_{A,\omega}, \mathtt{X}_{B,\omega}) \leq 1$ and that $\mathrm{Cov}(\mathtt{X}_{A,\omega}, \mathtt{X}_{B,\omega}) = 0$ if and only if $\mathrm{Cor}(\mathtt{X}_{A,\omega}, \mathtt{X}_{B,\omega}) = 0$.

We now see that ω is a product state on $\mathcal{A} \vee \mathcal{B}$ if and only if for every Hermitian $A \in \mathcal{A}$, $B \in \mathcal{B}$ we have that $\mathrm{Cor}(\mathtt{X}_{A,\omega}, \mathtt{X}_{B,\omega}) = 0$. In other words, ω is a product state if and only if any two observables in $\mathcal{A}$ and $\mathcal{B}$ are uncorrelated.

The motivation for the term "classically correlated" comes from the fact that, in the case of unital C^*-algebras $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{R}$, where $\mathcal{A} := \mathcal{A}_1 \otimes \mathbb{C}\mathbb{1}$, $\mathcal{B} := \mathbb{C}\mathbb{1} \otimes \mathcal{A}_2$ and $\mathcal{R} = \overline{\mathcal{A} \otimes \mathcal{B}}^{\alpha}$, where α refers to an appropriate C^*-norm on the algebraic tensor product, it follows that every state ω on $\mathcal{R}$ is either a product state or classically correlated *if and only if* $\mathcal{A}_1$ or $\mathcal{A}_2$ is Abelian.[14].

Following [24] we now introduce a property whose *failure* is a sufficient condition for entanglement.

Definition 2.3.2 (Verch-Werner ppt property, Definition 3.1 in [24])**.** *For $\mathcal{A}, \mathcal{B} \subseteq \mathcal{R}$ two commuting, unital $*$-subalgebras of some unital $*$-algebra $\mathcal{R}$, we say that a state ω on $\mathcal{A} \vee \mathcal{B} \subseteq \mathcal{R}$ has the* Verch-Werner positive partial transpose (ppt) *property, if and only if for every $N \in \mathbb{N}$ and for all $x_1, ..., x_N \in \mathcal{A}_1$ and $y_1, ..., y_N \in \mathcal{A}_2$:*

$$\sum_{j,k=1}^{N} \omega(x_k x_j^\dagger y_j^\dagger y_k) \geq 0. \tag{2.27}$$

For completeness, we note that this condition may be viewed as a generalisation of the positive partial transpose property of Peres [31] and agrees with it at least for operators on finite-dimensional Hilbert spaces, see Proposition 3.2 in [24].

Lemma 2.3.3 (after Lemma 3.3 in [24])**.** *Every classically correlated state σ has the Verch-Werner ppt property.*

It is then obvious that every state that does *not* have the ppt property is necessarily entangled. The reversed implication, however, does *not* hold in general. There are entangled states between finite-dimensional systems [32, 33], as well as between systems described in Appendix B.5.3, see [34], that *have* the ppt property. Hence a failing of the ppt property is only sufficient for entanglement. Note however Lemma B.5.6, which utilises results from [35].

[14]See Theorem 7 in [26] (based on [27, 28]) and also Proposition 6 in [29] for a direct proof of one implication. See [30] and the comment below Theorem 5.6 in [5] for a slight generalisation in the case of von Neumann algebras.

2.3.2 Quantum operations and quantum channels

Let us now turn to the notion of quantum operations of a system $\mathcal{O}$. One might take the point of view that a quantum operation should act on states, a state-update being the prime example. However, in what follows, we will mostly take the point of view that a quantum operation T is a special map on the algebra under consideration.

A motivation for this point of view is as follows. Let $\mathcal{N}$ be a unital von Neumann algebra with normal states $\mathcal{N}_*$ and let us consider a $\mathbb{C}$-linear map T_* on the set of $\mathbb{C}$-linear functionals of $\mathcal{N}$ that maps positive linear functionals to positive linear functionals. Given that $\mathcal{N}$ has a distinguished set of functionals, it is natural to assume that T_* preserves normal functionals, i.e., $T_*[\mathcal{N}_*] \subseteq \mathcal{N}_*$. Then we can define $T := (T_* \restriction \mathcal{N}_*)^*$ to be the dual map of (the restriction) of T_*, which is a normal positive linear map on $\mathcal{N}$. In fact, every normal positive linear map T arises in this way, see Theorem A.3.4 and also Sec. III. in [36].

This motivates, also in the case of a mere unital $*$- or C^*-algebra $\mathcal{O}$, to consider positive linear maps T on $\mathcal{O}$. Then for every state ω, $T^*\omega = \omega \circ T$ is a positive linear functional on $\mathcal{O}$, i.e., up to normalisation again a state.[15]

There is another natural property that quantum operations should fulfill. Let us consider a system $\mathcal{O}_1$ combined with some system $\mathcal{O}_2$ to form the joint system described by their tensor product $\mathcal{O}_1 \otimes \mathcal{O}_2$. Then it is reasonable to require that for every quantum operation T_1 on $\mathcal{O}_1$ and every quantum operation T_2 on $\mathcal{O}_2$ the combination, $T_1 \otimes T_2$ is a quantum operation on $\mathcal{O}_1 \otimes \mathcal{O}_2$, i.e., the collection of quantum operations should be closed under taking tensor products.

This motivates the following definition.

Definition 2.3.4. *Let $\mathcal{O}_1, \mathcal{O}_2$ be two unital $*$-algebras and let $\mathbb{C}^{n\times n}$ be the nuclear unital C^*-algebra of n by n matrices for $n \in \mathbb{N}$. A $\mathbb{C}$-linear map $T : \mathcal{O}_1 \to \mathcal{O}_2$ is called n-positive, if*

$$T \otimes \mathrm{id}_{\mathbb{C}^{n\times n}} : \mathcal{O}_1 \otimes \mathbb{C}^{n\times n} \to \mathcal{O}_2 \otimes \mathbb{C}^{n\times n} \tag{2.28}$$

is positive, i.e., maps positive elements to positive elements. T is called completely positive, if it is n-positive for every $n \in \mathbb{N}$.

[15]It could be the case that $\omega(T(\mathbb{1})) = 0$, i.e., no normalisation is possible. In the C^*-algebra setting, $\omega(T(\mathbb{1})) = 0$ already implies that $\omega \circ T \equiv 0$ for positive $\omega \circ T$, see for instance Proposition 5 in [29]. Then, the quantum operation can be interpreted as the annihilation of the system.

Remark: We immediately see that every $*$-homomorphism T is completely positive.

Suppose now we have any collection of positive linear maps between unital $*$-algebras that is closed under tensor products and contains the identity maps on $\mathbb{C}^{n\times n}$ for $n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}$. Then this collection is a subset of all completely positive linear maps.

Making the assumption that the identity map id should always be a valid quantum operation on any system, in particular on the finite-dimensional matrix algebras $\mathbb{C}^{n\times n}$, it follows that any set of quantum operations is necessarily a *subset* of all the completely positive maps. In fact, in the case of unital C^*-algebras, it turns out that *every* completely positive map may be regarded as a quantum operation according to the following results.

Lemma 2.3.5 (Theorem 12.3 in [37])**.** *Let $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{B}_1, \mathcal{B}_2$ be four unital C^*-algebras and let $T_\mathcal{A} : \mathcal{A}_1 \to \mathcal{A}_2$ and $T_\mathcal{B} : \mathcal{B}_1 \to \mathcal{B}_2$ be completely positive. Then $T_\mathcal{A} \otimes_\sigma T_\mathcal{B} : \overline{\mathcal{A}_1 \otimes \mathcal{B}_1}^\sigma \to \overline{\mathcal{A}_2 \otimes \mathcal{B}_2}^\sigma$ is completely positive, where σ denotes the minimal C^*-norm on $\mathcal{A}_j \otimes \mathcal{B}_j$ and $T_\mathcal{A} \otimes_\sigma T_\mathcal{B}$ the continuous linear extension of $T_\mathcal{A} \otimes T_\mathcal{B}$.*

This, together with the fact that for every state ω, $\omega(T(\mathbb{1}))$ should have the interpretation of a "success probability" of the quantum operation T, motivates the following definition for the general unital $*$-algebra case.

Definition 2.3.6. *A quantum operation T from a unital $*$-algebra $\mathcal{O}_1$ to a unital $*$-algebra $\mathcal{O}_2$ is a $\mathbb{C}$-linear completely positive map $T : \mathcal{O}_1 \to \mathcal{O}_2$ that is unit-non-increasing, i.e., $T(\mathbb{1}_{\mathcal{O}_1}) \leq \mathbb{1}_{\mathcal{O}_2}$. If T is unit-preserving, i.e., $T(\mathbb{1}_{\mathcal{O}_1}) = \mathbb{1}_{\mathcal{O}_2}$, then T is called a quantum channel.*

Let us note the following property of positive linear maps, which is hence also a property of quantum operations.

Lemma 2.3.7. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two unital $*$-algebras and let $T : \mathcal{O}_1 \to \mathcal{O}_2$ be a positive linear map. Then $T(A^*) = T(A)^*$ for every $A \in \mathcal{O}_1$.*

Proof. By Eq. (9) in the proof of Proposition 9.1.6 in [17] we see that

$$A = \frac{1}{4} \sum_{m=0}^{3} \mathrm{i}^{-m} (\mathbb{1} + \mathrm{i}^m A)^* (\mathbb{1} + \mathrm{i}^m A), \tag{2.29}$$

i.e., A is given as a complex linear combination of 4 positive elements. In particular $T((\mathbb{1}+\mathrm{i}^m A)^*(\mathbb{1}+\mathrm{i}^m A))$ is also positive and hence Hermitian. Then

$$\begin{aligned} T(A^*) &= \frac{1}{4}\sum_{m=0}^{3}(-\mathrm{i})^{-m}T((\mathbb{1}+\mathrm{i}^m A)^*(\mathbb{1}+\mathrm{i}^m A)) \\ &= \frac{1}{4}\sum_{m=0}^{3}(-\mathrm{i})^{-m}T((\mathbb{1}+\mathrm{i}^m A)^*(\mathbb{1}+\mathrm{i}^m A))^* = T(A)^*. \end{aligned} \tag{2.30}$$

□

Lemma 2.3.8. *Suppose $S : \mathcal{O}_2 \to \mathcal{O}_3$ and $T : \mathcal{O}_1 \to \mathcal{O}_2$ are completely positive. Then $S \circ T$ is completely positive. If S, T are quantum operations, then $S \circ T$ is a quantum operation. If S, T are quantum channels, then $S \circ T$ is a quantum channel.*

Furthermore, the set of quantum operations on $\mathcal{O}$ and the set of quantum channels on $\mathcal{O}$ are convex sets.

Proof. For every $n \in \mathbb{N}$, $(S \circ T)\otimes \mathrm{id}_{\mathbb{C}^{n\times n}}$ equals $(S \otimes \mathrm{id}_{\mathbb{C}^{n\times n}})\circ(T \otimes \mathrm{id}_{\mathbb{C}^{n\times n}})$, which, as a composition of positive maps, is positive. It is also immediate that the composition of positive linear unit-non-increasing maps S, T is unit-non-increasing, since

$$\mathbb{1}_{\mathcal{O}_3} - S(T(\mathbb{1}_{\mathcal{O}_1})) = \underbrace{\mathbb{1}_{\mathcal{O}_3} - S(\mathbb{1}_{\mathcal{O}_2})}_{\geq 0} + \underbrace{S(\mathbb{1}_{\mathcal{O}_2} - T(\mathbb{1}_{\mathcal{O}_1}))}_{\geq 0} \geq 0. \tag{2.31}$$

Furthermore, the composition of unit-preserving maps is unit-preserving.

Let now S, T be quantum channels and let $\lambda \in [0,1]$. Then

1. $(\lambda S + (1-\lambda)T)(\mathbb{1}) \leq \lambda\mathbb{1} + (1-\lambda)\mathbb{1} = \mathbb{1}$, so $\lambda S + (1-\lambda)T$ is unit-non-increasing, and

2. for every n

$$(\lambda S + (1-\lambda)T) \otimes \mathrm{id}_{\mathbb{C}^{n\times n}} = \lambda(S \otimes \mathrm{id}_{\mathbb{C}^{n\times n}}) + (1-\lambda)(T \otimes \mathrm{id}_{\mathbb{C}^{n\times n}}), \tag{2.32}$$

which is a convex combination of positive maps and hence positive.

Finally, if S, T are unit-preserving, then $\lambda S + (1-\lambda)T$ is also unit preserving. □

Lemma 2.3.9. *Let $\mathcal{O}$ be a unital $*$-algebra. Let $K_1, ..., K_N \in \mathcal{O}$ be such that $\sum_{j=1}^{N} K_j^* K_j \leq \mathbb{1}$. Then*

$$T(A) := \sum_{j=1}^{N} K_j^* A K_j \tag{2.33}$$

defines a quantum operation (on $\mathcal{O}$) and $K_1, ..., K_N$ are called Kraus operators of T. If $\sum_{j=1}^{N} K_j^* K_j = \mathbb{1}$, then T is a quantum channel.

Proof. We will show that T is completely positive, as the rest is obvious. We will show that for every $n \in \mathbb{N}$ and any $C \in \mathcal{O} \otimes \mathbb{C}^{n \times n}$ we have that

$$(T \otimes \mathrm{id}_{\mathbb{C}^{n\times n}})(C^*C) \geq 0. \tag{2.34}$$

By definition $C = \sum_j C_j \otimes M_j$, where j runs through a finite index set. Then

$$\begin{aligned}(T \otimes \mathrm{id}_{\mathbb{C}^{n\times n}})(C^*C) &= \sum_{j,k} T(C_j^* C_k) \otimes M_j^* M_j = \sum_{j,k} \sum_{l=1}^{N} K_l^* C_j^* C_k K_l \otimes M_j^* M_k \\ &= \sum_{l=1}^{N} \sum_{j,k} (C_j K_l)^* (C_k K_l) \otimes M_j^* M_k = \sum_{l=1}^{N} \left(\sum_j (C_j K_l) \otimes M_j \right)^* \left(\sum_k (C_k K_l) \otimes M_k \right),\end{aligned} \tag{2.35}$$

which is clearly positive. □

By letting the index j run through a countable index set and using continuity, the above proof generalises to the C^*-algebra case. In fact, we can even say more in the case where $\mathcal{O} = BL(\mathcal{H})$ for some (not necessarily separable) complex Hilbert space $\mathcal{H}$.

Lemma 2.3.10 (II.5.5.14 in [38] and Theorem 3.3 in [39])**.** *Let $\mathcal{H}$ be a (not necessarily separable) complex Hilbert space $\mathcal{H}$. Then*

1. *every unit-preserving $*$-automorphism T of $BL(\mathcal{H})$ is inner, i.e., there exists a unitary $U \in BL(\mathcal{H})$ such that $T(A) = U^*AU$ for every $A \in BL(\mathcal{H})$, and*

2. *if $\mathcal{H}$ is separable and T is a normal quantum operation T on $BL(\mathcal{H})$, then there is a set $(K_j)_{j\in J} \subseteq BL(\mathcal{H})$ for a (not necessarily finite or even countable) index set J such that*

   a) $\sum_{j\in J} K_j^* K_j \leq \mathbb{1}$, and

   b) for every $A \in BL(\mathcal{H})$
   $$T(A) = \sum_{j\in J} K_j^* A K_j, \tag{2.36}$$

   where convergence is understood with respect to the topology $\sigma(BL(\mathcal{H}), BL(\mathcal{H})_*)$.

A criterion that guarantees the separability of an underlying Hilbert space for a physical system is for instance given in Proposition 2.17 in [40].

We now introduce an important class of quantum channels.

Lemma 2.3.11. *Let $\mathcal{S}, \mathcal{P}$ be two unital $*$-algebras. Let σ be a state on $\mathcal{P}$. Then we define*

$$\eta_\sigma : \mathcal{S} \otimes \mathcal{P} \to \mathcal{S} \tag{2.37}$$

as the composition of $\mathrm{id}_{\mathcal{S}} \otimes \sigma : \mathcal{S} \otimes \mathcal{P} \to \mathcal{S} \otimes \mathbb{C}$ *with the canonical unit-preserving $*$-isomorphism $\mathcal{S} \otimes \mathbb{C} \cong \mathcal{S}$, which is a quantum channel.*

Furthermore, let $\Theta : \mathcal{S} \otimes \mathcal{P}$ be a unti-preserving $*$-homomorphism. Then for every effect $E \in \mathcal{P}$ it holds that

$$\mathcal{J}^E_{\sigma,\Theta}(A) := \eta_\sigma(\Theta(A \otimes E)) \tag{2.38}$$

is a quantum operation. If $E = \mathbb{1}_{\mathcal{P}}$, then $\mathcal{J}_{\sigma,\Theta} := \mathcal{J}^{\mathbb{1}_{\mathcal{P}}}_{\sigma,\Theta}$ is a quantum channel.

Proof. In Appendix B of [10] it is explicitly shown that $\eta_\sigma : \mathcal{S} \otimes \mathcal{P} \to \mathcal{S}$ is a quantum channel.

Let us now define $\cdot \otimes E : \mathcal{S} \to \mathcal{S} \otimes \mathcal{P}$, which is a quantum operation. It is completely positive since the tensor product of positive elements is positive. By the same fact it follows that $\mathbb{1} \otimes E \leq \mathbb{1} \otimes \mathbb{1}$. We can now write $\mathcal{J}^E_{\sigma,\Theta}$ as a composition of quantum operations $\eta_\sigma \circ \Theta \circ (\cdot \otimes E)$, which is hence itself a quantum operation. Finally, it is easy to see that $\mathcal{J}_{\sigma,\Theta}$ is unit-preserving. □

In summary, given a description of a physical system in terms of a unital $*$-algebra $\mathcal{O}$, "interventions" on the system (by an external observer for instance) are modelled by quantum channels. They are stable under convex combinations, compositions and (at least in some cases) tensor products ("combinations"). Furthermore, the dual of a quantum channel can be thought of as a state-update map.

In the following chapter, we will combine the ideas of the algebraic description of physical systems with the notion of *backgrounds* and then also discuss how quantum channels fit into the picture.

— 3 —

Algebraic Quantum Field Theory, convex time-orderable Pre-Factorisation Algebras and causality

3.1 BACKGROUNDS AND LOCAL PHYSICS

In order to establish a connection between the abstract characterisation of a physical system in terms of a unital $*$-algebra and actual experiments in the real world, we invoked the primitive notions of "external observer" and "measurement apparatus". Another primitive notion of the framework under consideration is that of a *fixed external* background.

The purpose of a background is to provide a reference frame. As such, it primarily enables one to talk about *location* (with respect to the background). Heuristically speaking, it enables one to ask questions about *when* and *where*. Furthermore, a background might come with a notion of *causality*, which tells one from which *region* one can or cannot signal to some other region. Note that this notion of causality already comes together with a notion of *past* and *future*. Finally, a background should also incorporate a notion of causal evolution or dynamics. This motivates the following mathematical structure.

Definition 3.1.1 (Background)**.** *A* background *is a set $\mathfrak{B}$ together with*

1. *a partial order $\subseteq$, for which we say that N_1 is contained in N_2 if and only if $N_1 \subseteq N_2$,*

2. *a relation $\not\lesssim$, for which we say that N_1 cannot signal to N_2 if and only if $N_1 \not\lesssim N_2$, and*

3. *a relation $\sqsubseteq$, for which we say that N_1 is determined by N_2 if and only if $N_1 \sqsubseteq N_2$.*

The elements of $\mathfrak{B}$ *are called* regions*. It is convenient to define a symmetric* causally disjoint *relation* $\perp$ *via*

$$N_1 \perp N_2 :\Longleftrightarrow (N_1 \not\preceq N_2 \wedge N_2 \not\preceq N_1). \tag{3.1}$$

These relations are assumed to have the following properties: $\forall N_1, N_2 \in \mathfrak{B}$

1. $\exists N \in \mathfrak{B} : \ \forall L \in \mathfrak{B} : \ L \sqsubseteq N,$
2. $N_1 \subseteq N_2 \implies N_1 \sqsubseteq N_2,$
3. $N_1 \not\preceq N_2 \implies (\forall L_1 \subseteq N_1, \ \forall L_2 \subseteq N_2 : \ L_1 \not\preceq L_2),$
4. $N_1 \perp N_2 \implies (\forall L_1 \sqsubseteq N_1, \ \forall L_2 \sqsubseteq N_2 : \ L_1 \perp L_2).$

Finally, we call a collection of regions $\{N_j\}_{j\in J}$ causally orderable*, if there exists a (not necessarily unique) linear order* $\leq$ *such that*

$$N_k < N_j \implies N_j \not\preceq N_k. \tag{3.2}$$

Any such $\leq$ *is called a* causal linear order *on* $\{N_j\}_{j\in J}$.

Remark: Recall that a linear order $\leq$ is a reflexive, transitive and antisymmetric relation such that every two elements are comparable. The associated *strict* linear order $<$ is then defined in the following way: $(N_k < N_j) :\Longleftrightarrow (N_k \leq N_j \wedge N_k \neq N_j)$. Other symbols used for (strict) causal linear orders are $\trianglelefteq$ ($\triangleleft$) and $\blacktriangleleft$ with underline ($\blacktriangleleft$).

We now define an important class of backgrounds based on some globally hyperbolic spacetime M that will play a crucial role in later chapters. To that end we recall that J_M^+ denotes the causal future inside M and J_M^- denotes the causal past. Furthermore D_M denotes the domain of dependence or Cauchy development in M. We write $N_1 \perp_M N_2$ if N_1 and N_2 are spacelike separated and $N^{\perp_M} := M \setminus \left(J_M^+(N) \cup J_M^-(N)\right)$, see the beginning of Appendix C.

Definition 3.1.2. *Let* M *be a globally hyperbolic spacetime and let* $K \subseteq M$ *be a compact (possibly empty) set. Then let* $\mathfrak{B}_{(M,K)}$ *be the set of non-empty open and causally convex subsets of* M *that do not intersect* K*. Then we define for all* $N_1, N_2 \in \mathfrak{B}_{(M,K)}$

1. $N_1 \subseteq N_2$ *if and only if* N_1 *is a subset of* N_2*,*
2. $N_1 \not\preceq N_2$ *if and only if* $J_M^+(N_1) \cap N_2 = \emptyset$*, and*

3. $N_1 \sqsubseteq N_2$ if and only if N_1 is a subset of $D_M(N_2)$.

The elements of $\mathfrak{B}_{(M,K)}$ are called K-admissible regions. *For empty K we define $\mathfrak{B}_M := \mathfrak{B}_{(M,\emptyset)}$, whose elements we will simply call regions.*

We will now show that the above backgrounds are indeed backgrounds according to Definition 3.1.1. This should be no surprise, since these backgrounds served as motivation for the abstract Definition 3.1.1. $\mathfrak{B}_M$ is commonly used and hence our Definition 3.1.1 should certainly cover this class, but should also be general enough to cover $\mathfrak{B}_{(M,K)}$. The reason for this, as will become apparent later, is that it allows for a unified and clear presentation of the results of Chapter 4. (There, the natural background for a perturbed variant of a theory on a background $\mathfrak{B}_M$, where the perturbation is only switched on inside K, is given by $\mathfrak{B}_{(M,K)}$.)

Lemma 3.1.3. *Let M be a globally hyperbolic spacetime and let $K \subseteq M$ be a compact (possibly empty) set. Then $\mathfrak{B}_{(M,K)}$ together with $\subseteq, \not\lesssim$ and $\sqsubseteq$ forms a background.*

Furthermore, take $N \in \mathfrak{B}_M$. Then $\mathfrak{B}_{(N,K\cap N)} \subseteq \mathfrak{B}_{(M,K)}$ is a background. Finally, take $K' \subseteq M$ compact such that $K \subseteq K'$. Then $\mathfrak{B}_{(M,K')} \subseteq \mathfrak{B}_{(M,K)}$ is a background.

Proof. The fact that $\subseteq$ is a partial order is obvious.

1. $M_K^{\pm} := M \setminus J_M^{\mp}(K)$ are two K-admissible regions. Furthermore $D_M\left(M_K^{\pm}\right) = M$, see the Appendix in [41]. Then, for every K-admissible region N it follows that N is a subset of $D_M(M_K^{\pm})$, and hence, by definition, $N \sqsubseteq M_K^{\pm}$.

2. For $N_1 \subseteq N_2$, it follows that N_1 is a subset of $D_M(N_1)$, which is a subset of $D_M(N_2)$. But then N_1 is a subset of $D_M(N_2)$, which by definition means that $N_1 \sqsubseteq N_2$.

3. Suppose $N_1 \not\lesssim N_2$, i.e., $J_M^+(N_1) \cap N_2 = \emptyset$. For every $L_1 \subseteq N_1$ and $L_2 \subseteq N_2$ it holds that $J_M^+(L_1)$ is a subset of $J_M^+(N_1)$, hence $J_M^+(L_1) \cap L_2 = \emptyset$, i.e., $L_1 \not\lesssim L_2$.

4. Suppose $N_1 \perp N_2$, i.e., N_1 is spacelike separated from N_2. But then $D_M(N_1)$ is spacelike separated from $D_M(N_2)$. Hence, whenever $L_1 \sqsubseteq N_1$ and $L_2 \sqsubseteq N_2$, it holds that L_1 is spacelike separated from L_2, hence $L_1 \perp L_2$.

Now take $N \in \mathfrak{B}_M$, i.e., N is a causally convex open subset of M. Then, together with the inherited structure, N is a globally hyperbolic spacetime in its own right and $K \cap N \subseteq N$ is a compact subset of N. Hence $\mathfrak{B}_{(N,K\cap N)}$ is a background by the previous argument and $\mathfrak{B}_{(N,K\cap N)} \subseteq \mathfrak{B}_{(M,K)}$. This is due to the fact that an open subset of N is causally convex in N if and only if it is causally convex in M since $J^{\pm}_N(L) = J^{\pm}_M(L) \cap N$ for *every* subset $L \subseteq N$, see Appendix D.

Furthermore, it follows that the restrictions of the relations on $\mathfrak{B}_{(M,K)}$ to $\mathfrak{B}_{(N,K\cap N)}$ agree with the intrinsically defined ones, which can be seen by noticing that for *every* subset $L \subseteq N$ we also have that $D_N(L) = D_M(L) \cap N$, see Appendix D.

Finally, the above argument shows that $\mathfrak{B}_{(M,K')}$ is a background and obviously $\mathfrak{B}_{(M,K')} \subseteq \mathfrak{B}_{(M,K)}$. Furthermore, since the definition of the relations $\subseteq, \not\lesssim$ and $\sqsubseteq$ does not refer to K, the intrinsically defined relations on $\mathfrak{B}_{(M,K')}$ agree with the restrictions of the relations defined on $\mathfrak{B}_{(M,K)}$. □

Let us recall that the prime purpose of a background is to yield a notion of *localisation*. Using this notion, physical systems can then be delineated according to their location. Concretely, this motivates us to associate to every region N of a background $\mathfrak{B}$ a system, i.e., a unital $*$-algebra, whose Hermitian elements are the ones that are observable in the region N. Crucially, this association is assumed to respect some of the structure of the fixed background. To emphasise the role of locality, one may call the resulting framework *local (quantum) physics (LQP)* [6]. A more common name is *algebraic quantum field theory (AQFT)*. However, it is worth mentioning that the physical system under consideration is neither required to be quantum[1] nor do we put special emphasis on the concept of (quantum) *fields*.

Definition 3.1.4. *Let $\mathfrak{B}$ be a background and let $\mathcal{A}^g_{\mathfrak{B}}$ be a unital $*$-algebra. A map $\mathcal{A}_{\mathfrak{B}}$ from $\mathfrak{B}$ to unital $*$-subalgebras of $\mathcal{A}^g_{\mathfrak{B}}$ is called an AQFT if and only if it has the following properties*

1. $\mathcal{A}^g_{\mathfrak{B}} = \bigcup_{N\in\mathfrak{B}} \mathcal{A}_{\mathfrak{B}}(N)$,
2. $N_1 \subseteq N_2 \implies \mathcal{A}_{\mathfrak{B}}(N_1) \subseteq \mathcal{A}_{\mathfrak{B}}(N_2)$, *which is called isotony,*
3. $N_1 \perp N_2 \implies [\mathcal{A}_{\mathfrak{B}}(N_1), \mathcal{A}_{\mathfrak{B}}(N_2)] = \{0\}$, *which is called causally disjoint commutativity.*

A further important and desirable axiom related to causal dynamics (which we, however, do not assume unless explicitly mentioned) called time slice *is the following*

4. $N_1 \sqsubseteq N_2 \implies \mathcal{A}_{\mathfrak{B}}(N_1) \subseteq \mathcal{A}_{\mathfrak{B}}(N_2)$. *In particular, for every region N such that for every other region L we have $L \sqsubseteq N$, it holds that $\mathcal{A}^g_{\mathfrak{B}} = \mathcal{A}_{\mathfrak{B}}(N)$.*

[1] In our algebraic approach, a *quantum* system is one whose corresponding unital $*$-algebra is non-Abelian, whereas a *classical* system is characterised by an Abelian unital $*$-algebra.

The motivation behind these axioms is as follows:

Firstly, $\mathcal{A}^g_{\mathfrak{B}}$ is the (smallest) "global" algebra (hence the superscript g).

Secondly, $\mathcal{A}_{\mathfrak{B}}$ respects the notion of localisation, i.e., for two nested regions $N_1 \subseteq N_2$, every element that is observable in N_1 is also observable in N_2.

Thirdly, local physics on a background $\mathfrak{B}$ is assumed to respect causality. At the current stage, we are satisfied with the heuristic idea that causality is (at least partially) taken care of by the assumption that two spacelike separated observables should commute in order to be independent/commensurable. As emphasised above, commutativity is, however, in fact not necessary for statistical independence. Nevertheless, we adopt this as an assumption here. It can be seen as a prime motivation for the present work to describe in what sense (signalling) causality is *and is not* present in an AQFT $\mathcal{A}_{\mathfrak{B}}$ that fulfills the third property.

Fourthly, $\mathcal{A}_{\mathfrak{B}}$ respects the notion of dynamics. If some region N_1 is fully determined by N_2, then everything that is observable in N_1 is also observable in N_2.

Before we move on let us discuss a notion of *equivalence* of two AQFTs on a common background.

Definition 3.1.5. *Let $\mathfrak{B}$ be a background and let $\mathcal{A}_{\mathfrak{B}}$ and $\mathcal{B}_{\mathfrak{B}}$ be two AQFTs on $\mathfrak{B}$. Then we say that $\mathcal{A}_{\mathfrak{B}}$ and $\mathcal{B}_{\mathfrak{B}}$ are equivalent if and only if there exists a unit-preserving $*$-isomorphism $\chi : \mathcal{A}^g_{\mathfrak{B}} \to \mathcal{B}^g_{\mathfrak{B}}$ such that for every region $N \in \mathfrak{B}$ we have*

$$\chi[\mathcal{A}_{\mathfrak{B}}(N)] = \mathcal{B}_{\mathfrak{B}}(N). \tag{3.3}$$

3.1.1 AQFTs on globally hyperbolic spacetimes

Let us now again consider the special class of backgrounds $\mathfrak{B}_{(M,K)}$ of K-admissible regions of a globally hyperbolic spacetime M. For AQFTs defined over such backgrounds we introduce the following properties (for future reference).

Definition 3.1.6. *Let $\mathfrak{B}_{(M,K)}$ be the background of K-admissible regions of some globally hyperbolic spacetime M and some compact $K \subseteq M$. We say that an AQFT $\mathcal{A}_{\mathfrak{B}_{(M,K)}}$ over $\mathfrak{B}_{(M,K)}$ is*

1. additive*, if for every K-admissible regions $N_1, N_2 \in \mathfrak{B}_{(M,K)}$ the following holds: if $N_1 \cup N_2 \subseteq M$ is a K-admissible region, then*

$$\mathcal{A}_{\mathfrak{B}_{(M,K)}}(N_1 \cup N_2) = \mathcal{A}_{\mathfrak{B}_{(M,K)}}(N_1) \vee \mathcal{A}_{\mathfrak{B}_{(M,K)}}(N_2), \tag{3.4}$$

where the right hand side denotes the smallest unital $*$-algebra in $\mathcal{A}^g_{\mathfrak{B}_{(M,K)}}$ that contains both $\mathcal{A}_{\mathfrak{B}_{(M,K)}}(N_1)$ and $\mathcal{A}_{\mathfrak{B}_{(M,K)}}(N_2)$.

2. *is* outer regular*, if for every K-admissible regions $N_1, N_2 \in \mathfrak{B}_{(M,K)}$ the following holds:*

$$\mathcal{A}_{\mathfrak{B}_{(M,K)}}(N_1 \cap N_2) = \mathcal{A}_{\mathfrak{B}_{(M,K)}}(N_1) \cap \mathcal{A}_{\mathfrak{B}_{(M,K)}}(N_2). \tag{3.5}$$

Let now $\mathfrak{B}_M$ be the background of regions of some globally hyperbolic spacetime M. We say that an AQFT $\mathcal{A}_{\mathfrak{B}_M}$ over $\mathfrak{B}_M$

3. *has the* Haag property*, if the following holds $\forall A \in \mathcal{A}^g_{\mathfrak{B}_M}$:*

$$\begin{aligned}&\Big(\forall B \in \mathcal{A}_{\mathfrak{B}_M}\Big(K^{\perp_M}\Big) :\ [A, B] = 0\Big)\\&\implies (\forall \text{ connected region } L \subseteq M \text{ such that } K \subseteq L : A \in \mathcal{A}_{\mathfrak{B}_M}(L)).\end{aligned} \tag{3.6}$$

The Haag property was introduced in [10], based on the notion of Haag duality [6].

3.1.1.1 RESTRICTIONS AND GLUINGS We have seen above that given a background $\mathfrak{B}_{(M,K)}$ for some compact $K \subseteq M$ we can consider various "sub"-backgrounds such as $\mathfrak{B}_{(N,K\cap N)}$ and $\mathfrak{B}_{(M,K')}$ for some $N \in \mathfrak{B}_M$ and some compact $K' \subseteq M$ that contains K. In particular, given an AQFT $\mathcal{A}_{\mathfrak{B}_{(M,K)}}$ on $\mathfrak{B}_{(M,K)}$ we can then look at the restriction of $\mathcal{A}_{\mathfrak{B}_{(M,K)}}$ to such "sub"-backgrounds.

The following lemma establishes sufficient conditions such that these restrictions are again AQFTs.

Lemma 3.1.7. *Let M be globally hyperbolic and let $K \subseteq K' \subseteq M$ for two compact subsets $K, K' \subseteq M$ and take $N \in \mathfrak{B}_M$. Let furthermore $\mathcal{A}_{\mathfrak{B}_{(M,K)}}$ be an AQFT on $\mathfrak{B}_{(M,K)}$ that fulfills time-slice. Then*

1. $\mathcal{A}_{\mathfrak{B}_{(M,K')}} := \mathcal{A}_{\mathfrak{B}_{(M,K)}} \restriction \mathfrak{B}_{(M,K')}$*, and*
2. $\mathcal{A}_{\mathfrak{B}_{(N,K\cap N)}} := \mathcal{A}_{\mathfrak{B}_{(M,K)}} \restriction \mathfrak{B}_{(N,K\cap N)}$

are AQFTs on their backgrounds fulfilling time-slice with

1. $\mathcal{A}^g_{\mathfrak{B}_{(M,K')}} = \mathcal{A}_{\mathfrak{B}_{(M,K')}}(M \setminus J^-_M(K')) = \mathcal{A}^g_{\mathfrak{B}_{(M,K)}}$*, and*
2. $\mathcal{A}^g_{\mathfrak{B}_{(N,K\cap N)}} = \mathcal{A}_{\mathfrak{B}_{(N,K\cap N)}}(N \setminus J^-_N(K \cap N))$.

If $\mathcal{A}_{\mathfrak{B}_{(M,K)}}$ is additive, so are $\mathcal{A}_{\mathfrak{B}_{(M,K')}}$ and $\mathcal{A}_{\mathfrak{B}_{(N,K\cap N)}}$.

Proof. 1. First we see that for every $L \in \mathfrak{B}_{(M,K')}$ we have that $L \sqsubseteq M \setminus J^-_M(K') \in \mathfrak{B}_{(M,K')}$. Hence, by time-slice of $\mathcal{A}_{\mathfrak{B}_{(M,K)}}$ we see that

$$\bigcup_{L \in \mathfrak{B}_{(M,K')}} \mathcal{A}_{\mathfrak{B}_{(M,K')}}(L) = \mathcal{A}_{\mathfrak{B}_{(M,K')}}(M \setminus J^-_M(K')) = \mathcal{A}^g_{\mathfrak{B}_{(M,K)}}. \tag{3.7}$$

Furthermore, for every $L \in \mathfrak{B}_{(N,K\cap N)}$ we have that $L \sqsubseteq N \setminus J^-_N(K \cap N) \in \mathfrak{B}_{(N,K\cap N)}$. Hence, by time-slice of $\mathcal{A}_{\mathfrak{B}_{(M,K)}}$ we see that

$$\bigcup_{L\in\mathfrak{B}_{(N,K\cap N)}} \mathcal{A}_{\mathfrak{B}_{(N,K\cap N)}}(L) = \mathcal{A}_{\mathfrak{B}_{(N,K\cap N)}}(N \setminus J^-_N(K \cap N)). \tag{3.8}$$

2. This follows from isotony of $\mathcal{A}_{\mathfrak{B}_{(M,K)}}$.

3. If N_1, N_2 in the respective "sub"-backgrounds are causally disjoint, then they are causally disjoint in $\mathfrak{B}_{(M,K)}$, hence causally disjoint commutativity of the two restricted AQFTs follows from causally disjoint commutativity of $\mathcal{A}_{\mathfrak{B}_{(M,K)}}$.

4. Similarly, if N_1 is determined by N_2 in the respective "sub"-backgrounds, then N_1 determined by N_2 in $\mathfrak{B}_{(M,K)}$ and time-slice of the two restricted AQFTs follows from time-slice of $\mathcal{A}_{\mathfrak{B}_{(M,K)}}$.

Finally let $\mathcal{A}_{\mathfrak{B}_{(M,K)}}$ be additive and take $N_1, N_2 \in \mathfrak{B}_{(M,K')}$ such that $N_1 \cup N_2 \in \mathfrak{B}_{(M,K')} \subseteq \mathfrak{B}_{(M,K)}$. But then it follows that

$$\mathcal{A}_{\mathfrak{B}_{(M,K')}}(N_1 \cup N_2) = \mathcal{A}_{\mathfrak{B}_{(M,K)}}(N_1 \cup N_2), \tag{3.9}$$

which, by additivity of $\mathcal{A}_{\mathfrak{B}_{(M,K)}}$, is the smallest unital $*$-algebra of $\mathcal{A}^g_{\mathfrak{B}_{(M,K)}}$ containing $\mathcal{A}_{\mathfrak{B}_{(M,K)}}(N_1)$ and $\mathcal{A}_{\mathfrak{B}_{(M,K)}}(N_2)$. Now, by the first point above, this is the same as the smallest unital $*$-algebra of $\mathcal{A}^g_{\mathfrak{B}_{(M,K')}}$ containing $\mathcal{A}_{\mathfrak{B}_{(M,K')}}(N_1)$ and $\mathcal{A}_{\mathfrak{B}_{(M,K')}}(N_2)$, which shows the result.

Proceeding similarly for $\mathcal{A}_{\mathfrak{B}_{(M,K')}}$ finishes the proof. □

Having discussed restrictions of theories, let us now discuss how we can *glue* theories together. To that end let us consider the situation where $M^\pm \in \mathfrak{B}_M$ are two regions such that $M^+ \cup M^- = M$ and where we have two AQFTs $\mathcal{A}_{\mathfrak{B}_{M^-}}$ and $\mathcal{B}_{\mathfrak{B}_{M^+}}$ whose restrictions to $\mathfrak{B}_{M^+\cap M^-}$ agree. Then it seems reasonable to expect that the two theories may be glued together to form an AQFT $\mathcal{C}_{\mathfrak{B}_M}$. In fact, under certain conditions, it is easy to write down a candidate for $\mathcal{C}_{\mathfrak{B}_M}$, namely $\mathcal{C}_{\mathfrak{B}_M}(N) := \mathcal{A}_{\mathfrak{B}_{M^-}}(N \cap M^-) \vee \mathcal{B}_{\mathfrak{B}_{M^+}}(N \cap M^+)$. The question is then whether $\mathcal{C}_{\mathfrak{B}_M}$ is indeed an AQFT. The answer is given by the following lemma.

Lemma 3.1.8. *Let $M^-, M^+ \subseteq M$ be two regions in a globally hyperbolic spacetime M such that*

1. *$M^- \cup M^+ = M$,*

2. *$M = D_M(M^- \cap M^+)$,*

3. $J^+_M(M^- \cap M^+) \subseteq M^+$ *and* $J^-_M(M^- \cap M^+) \subseteq M^-$*,*

and let $\mathcal{A}_{\mathfrak{B}_{M^-}}$ *and* $\mathcal{B}_{\mathfrak{B}_{M^+}}$ *be two AQFTs fulfilling time-slice and additivity such that*

$$\mathcal{A}_{\mathfrak{B}_{M^- \cap M^+}} = \mathcal{B}_{\mathfrak{B}_{M^- \cap M^+}}. \tag{3.10}$$

Then there exists a unique AQFT $\mathcal{C}_{\mathfrak{B}_M}$ *fulfilling time-slice and additivity such that*

$$\mathcal{C}_{\mathfrak{B}_{M^-}} = \mathcal{A}_{\mathfrak{B}_{M^-}}, \text{ and } \quad \mathcal{C}_{\mathfrak{B}_{M^+}} = \mathcal{B}_{\mathfrak{B}_{M^+}}. \tag{3.11}$$

Concretely, for every region $N \subseteq M$ *we have*

$$\mathcal{C}_{\mathfrak{B}_M}(N) := \mathcal{A}_{\mathfrak{B}_{M^-}}(N \cap M^-) \vee \mathcal{B}_{\mathfrak{B}_{M^+}}(N \cap M^+), \tag{3.12}$$

where the right hand side is the smallest unital $*$*-subalgebra of* $\mathcal{A}^g_{\mathfrak{B}_{M^-}} = \mathcal{B}^g_{\mathfrak{B}_{M^+}}$ *that contains both* $\mathcal{A}_{\mathfrak{B}_{M^-}}(N \cap M^-)$ *and* $\mathcal{B}_{\mathfrak{B}_{M^+}}(N \cap M^+)$*.*

Remark: The following proof relies on two technical auxiliary geometrical lemmas, namely Lemma D.1.2 and Lemma D.1.3. At first sight it might not be obvious why and how these lemmas are useful, however, it simply turns out that these are exactly the results needed for the proof of Lemma 3.1.8.

Proof. By assumption $D_{M^\pm}(M^- \cap M^-) = M^\pm$, so by time-slice of $\mathcal{A}_{\mathfrak{B}_{M^-}}$ and $\mathcal{B}_{\mathfrak{B}_{M^+}}$ we see that $\mathcal{A}^g_{\mathfrak{B}_{M^-}} = \mathcal{B}^g_{\mathfrak{B}_{M^+}}$. Furthermore, for every $N \in \mathfrak{B}_M$, we have that $N \cap M^\pm \in \mathfrak{B}_{M^\pm}$, i.e., $\mathcal{C}_{\mathfrak{B}_M}(N)$ in Eq. (3.12) is well-defined.

We now show that $\mathcal{C}_{\mathfrak{B}_M}$ is an AQFT on $\mathfrak{B}_M$ fulfilling time-slice and additivity.

1. Isotony: Let $L_1 \subseteq L_2$, then $L_1 \cap M^\pm \subseteq L_2 \cap M^\pm$ and $\mathcal{B}_{\mathfrak{B}_{M^+}}(L_1 \cap M^+) \subseteq \mathcal{B}_{\mathfrak{B}_{M^+}}(L_2 \cap M^+)$ and $\mathcal{A}_{\mathfrak{B}_{M^-}}(L_1 \cap M^-) \subseteq \mathcal{A}_{\mathfrak{B}_{M^-}}(L_2 \cap M^-)$ by isotony of $\mathcal{A}_{\mathfrak{B}_{M^-}}$ and $\mathcal{B}_{\mathfrak{B}_{M^+}}$. Hence
$$\begin{aligned}\mathcal{C}_{\mathfrak{B}_M}(L_1) &= \mathcal{B}_{\mathfrak{B}_{M^+}}(L_1 \cap M_1^+) \vee \mathcal{A}_{\mathfrak{B}_{M^-}}(L_1 \cap M_2^-) \\ &\subseteq \mathcal{B}_{\mathfrak{B}_{M^+}}(L_2 \cap M^+) \vee \mathcal{A}_{\mathfrak{B}_{M^-}}(L_2 \cap M^-) = \mathcal{C}_{\mathfrak{B}_M}(L_2).\end{aligned} \tag{3.13}$$

2. Additivity: Let $L_1, L_2, L_1 \cup L_2$ be regions. Let us define $L_j^+ := L_j \cap M^+$ and $L_j^- := L_j \cap M^-$. Then
$$\begin{aligned}\mathcal{C}_{\mathfrak{B}_M}(L_1 \cup L_2) &= \mathcal{B}_{\mathfrak{B}_{M^+}}(L_1^+ \cup L_2^+) \vee \mathcal{A}_{\mathfrak{B}_{M^-}}(L_1^- \cup L_2^-) \\ &= \Big(\mathcal{B}_{\mathfrak{B}_{M^+}}(L_1^+) \vee \mathcal{B}_{\mathfrak{B}_{M^+}}(L_2^+)\Big) \vee \Big(\mathcal{A}_{\mathfrak{B}_{M^-}}(L_1^-) \vee \mathcal{A}_{\mathfrak{B}_{M^-}}(L_2^-)\Big) \\ &= \Big(\mathcal{B}_{\mathfrak{B}_{M^+}}(L_1^+) \vee \mathcal{A}_{\mathfrak{B}_{M^-}}(L_1^-)\Big) \vee \Big(\mathcal{B}_{\mathfrak{B}_{M^+}}(L_2^+) \vee \mathcal{A}_{\mathfrak{B}_{M^-}}(L_2^-)\Big) \\ &= \mathcal{C}_{\mathfrak{B}_M}(L_1) \vee \mathcal{C}_{\mathfrak{B}_M}(L_2),\end{aligned} \tag{3.14}$$
where we used additivtiy of $\mathcal{A}_{\mathfrak{B}_{M^-}}$ and $\mathcal{B}_{\mathfrak{B}_{M^+}}$ and associativity and commutativity of $\vee$.

3. Spacelike commutativity: Let $L_1 \perp_M L_2$ for region $L_1, L_2 \subseteq M$. Then it follows immediately by causally disjoint commutativity of $\mathcal{A}_{\mathfrak{B}_{M^-}}$ and $\mathcal{B}_{\mathfrak{B}_{M^+}}$ that $\mathcal{B}_{\mathfrak{B}_{M^+}}(L_1 \cap M^+)$ and $\mathcal{B}_{\mathfrak{B}_{M^+}}(L_2 \cap M^+)$ commute and that $\mathcal{A}_{\mathfrak{B}_{M^-}}(L_1 \cap M^-)$ and $\mathcal{A}_{\mathfrak{B}_{M^-}}(L_2 \cap M^-)$ commute. What is left to show is that $\mathcal{B}_{\mathfrak{B}_{M^+}}(L_2)$ and $\mathcal{A}_{\mathfrak{B}_{M^-}}(L_1)$ commute for some $L_2 \subseteq M^+$ and $L_1 \subseteq M^-$ such that $L_1 \perp_M L_2$.

   We will now make use of the assumptions point *2* and point *3*. As a result there are $\Sigma_1, \Sigma_2^{\pm} \subseteq M^+ \cap M^-$, three Cauchy surfaces for M, such that there is a causal linear order $\leq$ with $\Sigma_1 < \Sigma_2^- < \Sigma_2^+$ and such that $I_M^+(\Sigma_1) \subseteq M^+$ and $I_M^-(\Sigma_2^+) \subseteq M^-$.

   Let us then decompose L_j into the regions $L_j^{(-,+)} := L_j \cap M^+ \cap M^-$ and $L_1^- := L_1 \cap I_M^-(\Sigma_2^-) \subseteq M^-$ and $L_2^+ := L_2 \cap I_M^+(\Sigma_1) \subseteq M^+$ such that $L_1 = L_1^{(-,+)} \cup L_1^-$ and $L_2 = L_2^{(-,+)} \cup L_2^+$. It then follows by additivity of $\mathcal{A}_{\mathfrak{B}_{M^-}}$ and $\mathcal{B}_{\mathfrak{B}_{M^+}}$, that $\mathcal{A}_{\mathfrak{B}_{M^-}}(L_1) = \mathcal{A}_{\mathfrak{B}_{M^-}}(L_1^{(-,+)}) \vee \mathcal{A}_{\mathfrak{B}_{M^-}}(L_1^-)$ and $\mathcal{B}_{\mathfrak{B}_{M^+}}(L_2) = \mathcal{B}_{\mathfrak{B}_{M^+}}(L_2^{(-,+)}) \vee \mathcal{B}_{\mathfrak{B}_{M^+}}(L_2^+)$. Since by assumption $\mathcal{A}_{\mathfrak{B}_{M^- \cap M^+}} = \mathcal{B}_{\mathfrak{B}_{M^- \cap M^+}}$, what is left to show is that $\mathcal{A}_{\mathfrak{B}_{M^-}}(L_1^-)$ commutes with $\mathcal{B}_{\mathfrak{B}_{M^+}}(L_2^+)$. To that end we apply Lemma D.1.3. By the fact that $L_2^+ \cap (J_M^-(L_1^-) \cup J_M^+(L_1^-)) = \emptyset$ implies $L_2^+ \cap (\overline{J_M^-(L_1^-)} \cup \overline{J_M^+(L_1^-)}) = \emptyset$, and that $L_1^- \subseteq M \setminus J_M^+(\Sigma_2^-)$, we have that

$$
\begin{aligned}
L_2^+ &\subseteq M \setminus (\overline{J_M^-(L_1^-)} \cup \overline{J_M^+(L_1^-)} \cup J_M^-(\Sigma_1)) \\
&\subseteq D_M(M \setminus (\overline{J_M^-(L_1^-)} \cup \overline{J_M^+(L_1^-)} \cup J_M^-(\Sigma_1) \cup J_M^+(\Sigma_2^+))).
\end{aligned} \tag{3.15}
$$

   In particular,

$$
\begin{aligned}
L_2^+ &\subseteq D_M(M \setminus (\overline{J_M^-(L_1^-)} \cup \overline{J_M^+(L_1^-)} \cup J_M^-(\Sigma_1) \cup J_M^+(\Sigma_2^+))) \cap M^+ \\
&= D_{M^+}(\underbrace{M \setminus (\overline{J_M^-(L_1^-)} \cup \overline{J_M^+(L_1^-)} \cup J_M^-(\Sigma_1) \cup J_M^+(\Sigma_2^+))}_{\subseteq M^+}).
\end{aligned} \tag{3.16}
$$

   By local time-slice of $\mathcal{B}_{\mathfrak{B}_{M^+}}$ we then see that

$$
\begin{aligned}
\mathcal{B}_{\mathfrak{B}_{M^+}}(L_2^+) &\subseteq \mathcal{B}_{\mathfrak{B}_{M^+}}(M \setminus (\overline{J^-(L_1^-)} \cup \overline{J^+(L_1^-)} \cup J^-(\Sigma_1) \cup J^+(\Sigma_2^+))) \\
&= \mathcal{A}_{\mathfrak{B}_{M^-}}(M \setminus (\overline{J^-(L_1^-)} \cup \overline{J^+(L_1^-)} \cup J^-(\Sigma_1) \cup J^+(\Sigma_2^+))),
\end{aligned} \tag{3.17}
$$

   which commutes with $\mathcal{A}_{\mathfrak{B}_{M^-}}(L_1^-)$ by causally disjoint commutativity of $\mathcal{A}_{\mathfrak{B}_{M^-}}$.

4. Local time-slice: Let us assume that $L_1 \subseteq D_M(L_2)$ for regions $L_1, L_2 \subseteq M$. Then we want to show that

$$
\begin{aligned}
\mathcal{C}_{\mathfrak{B}_M}(L_1) &= \mathcal{A}_{\mathfrak{B}_{M^-}}(L_1 \cap M^-) \vee \mathcal{B}_{\mathfrak{B}_{M^+}}(L_1 \cap M^+) \\
&\subseteq \mathcal{A}_{\mathfrak{B}_{M^-}}(L_2 \cap M^-) \vee \mathcal{B}_{\mathfrak{B}_{M^+}}(L_2 \cap M^+) = \mathcal{C}_{\mathfrak{B}_M}(L_2).
\end{aligned} \tag{3.18}
$$

   Using the decomposition $L_1 = L_1^- \cup L_1^+$ where $L_1^- := L_1 \cap M^-$ and $L_1^+ := L_1 \cap M^+$ and additivity of $\mathcal{C}_{\mathfrak{B}_M}$, we see that it suffices to show that $\mathcal{C}_{\mathfrak{B}_M}(L_1^{\pm}) \subseteq \mathcal{C}_{\mathfrak{B}_M}(L_2)$.

a) Let us assume that $L_1 \subseteq D_M(L_2)$ for some region $L_2 \subseteq M$ and some region $L_1 \subseteq M^-$. We decompose L_2 accordingly. Then we use Lemma D.1.2, from which it follows that

$$\begin{aligned} L_1 &\subseteq D_M(L_2) \cap M^- \subseteq D_M\Big(D_M(L_2^-) \cup D_M(L_2^+)\Big) \cap M^- \\ &\subseteq D_M\Big(\Big(D_M(L_2^+) \cap M^+ \cup D_M(L_2^-)\Big) \cap M^-\Big) \cap M^- \\ &= D_{M^-}\Big(\Big(D_{M^+}(L_2^+) \cup D_{M^-}(L_2^-)\Big) \cap M^-\Big). \end{aligned} \tag{3.19}$$

By local time-slice and additivity of $\mathcal{A}_{\mathfrak{B}_{M^-}}$ and local time-slice of $\mathcal{B}_{\mathfrak{B}_{M^+}}$ we get

$$\begin{aligned} \mathcal{C}_{\mathfrak{B}_M}(L_1) &= \mathcal{A}_{\mathfrak{B}_{M^-}}(L_1) \subseteq \mathcal{A}_{\mathfrak{B}_{M^-}}\Big(\Big(D_{M^+}(L_2^+) \cup D_{M^-}(L_2^-)\Big) \cap M^-\Big) \\ &= \mathcal{A}_{\mathfrak{B}_{M^-}}\Big(D_{M^+}(L_2^+) \cap M^-\Big) \vee \mathcal{A}_{\mathfrak{B}_{M^-}}(D_{M^-}(L_2^-)) \\ &= \mathcal{A}_{\mathfrak{B}_{M^-}}\Big(D_{M^+}(L_2^+) \cap M^-\Big) \vee \mathcal{A}_{\mathfrak{B}_{M^-}}(L_2^-) \\ &= \mathcal{B}_{\mathfrak{B}_{M^+}}\Big(D_{M^+}(L_2^+) \cap M^-\Big) \vee \mathcal{A}_{\mathfrak{B}_{M^-}}(L_2^-) \\ &\subseteq \mathcal{B}_{\mathfrak{B}_{M^+}}(L_2^+) \vee \mathcal{A}_{\mathfrak{B}_{M^-}}(L_2^-) = \mathcal{C}_{\mathfrak{B}_M}(L_2). \end{aligned} \tag{3.20}$$

b) The case of $L_1 \subseteq M^+$ works essentially mutatis mutandis.

□

Before we finish this section let us make the following remark.

The usage of a fixed (and in particular non-dynamical) background above demonstrates that the framework of AQFT (as used in the present work) somewhat neglects the shift in paradigm initiated by Einstein's general theory of relativity, according to which "the" background *is* dynamical. Nevertheless, at least as a first approximation, we will stick to the above framework of local quantum physics on *fixed* backgrounds. As we will see, even in the presence of fixed background structures questions concerning the causality of physical theories turn out to be not as trivial as one might expect.

3.2 INTERVENTIONS AND LOCALISATION

Having seen how the combination of the algebraic approach to physics together with locality in the form of a background leads to the definition of *local physics* or *AQFT*, let us now further investigate the interplay between the notion of an "intervention"[2] and locality.

[2]An Intervention could be for instance a measurement by an external observer, or a local change of the dynamics etc.

To that end we want to analyse the structure of a collection of abstract interventions without necessarily specifying the physical system on which the interventions act. First of all, such a collection of interventions should naturally be pointed by the identity intervention, i.e., the trivial intervention where nothing happens. Then, due to the fact that we consider *statistical* theories, for any two interventions, any statistical mixture (i.e., convex combination) is also a valid intervention. Hence any set of interventions should have the structure of a pointed convex set. Indeed, the interventions on a physical system given by a unital $*$-algebra $\mathcal{O}$ are the quantum operations on $\mathcal{O}$, and as shown in Lemma 2.3.8, the set of quantum operations forms a pointed convex set with distinguished base point given by the identity id.

Similar to how our discussion of the abstract association of observables to regions led us to AQFT, we now discuss the abstract structure that emerges from combining interventions with *locality* stemming from a background. Incorporating this notion motivates an association of pointed convex sets to every region of a given background. Furthermore, it is reasonable to require that this association respects the structure of the background *at least in parts*. Concretely, we may assume that the association is isotone and base-point-preserving.

An important aspect of interventions that we have not touched upon yet is the idea that successive interventions can be combined to form a new intervention. The *minimal*[3] assumption is that for every finite collection of causally orderable regions $N_1, ..., N_n$ and every collection of interventions $T_1, ..., T_n$ where T_j belongs to the pointed convex set associated to N_j, there exists an intervention T that belongs to the pointed convex set of any region N that contains every N_j. This T has the interpretation of the combination of all the interventions $T_1, ..., T_n$. Clearly, T should depend on each T_j in a convex way, and the combination of base points should yield the base point (being the intervention where nothing happens). The desirable properties of the combination are essentially those of what is known as a *time-ordered product*, which is not very surprising, and the whole structure motivated by this discussion is very close to that of a "time-orderable pre-factorisation algebra (tPFA)" as introduced in [42] based on [43].

[3]A comment to the reader wondering whether the restriction to causally orderable regions is truly necessary: This assumption is indeed a minimal assumption. It is not excluded that certain collections of interventions (naturally associated to certain theories) enjoy more structure that allows to define the combination of interventions whose localisation regions cannot be causally ordered or might even overlap. We will indeed encounter concrete examples below, however, for the abstract discussion we will not require that this more general combination of interventions is *a priori* defined.

Definition 3.2.1 (convex time-orderable pre-factorisation algebra (ctPFA)). *Let $\mathfrak{B}$ be a background. Let $\mathfrak{V}$ be a real pointed vector space with base point $I \neq 0$. A map $\mathfrak{F}_{\mathfrak{B}}$ from $\mathfrak{B}$ to pointed convex subsets of $\mathfrak{V}$ with base point I is called convex time-orderable prefactorization algebra (ctPFA), if*

1. $N_1 \subseteq N_2 \implies \mathfrak{F}_{\mathfrak{B}}(N_1) \subseteq \mathfrak{F}_{\mathfrak{B}}(N_2)$, *and if*
2. *for every $n \in \mathbb{N}$, every n-tuple of* causally orderable *regions $(N_1, ..., N_n)$ and every region N such that $N_j \subseteq N$ for every j, there exists a map $\mathfrak{C}(N_1, ..., N_n; N) : \bigtimes_{j=1}^{n} \mathfrak{F}_{\mathfrak{B}}(N_j) \to \mathfrak{F}_{\mathfrak{B}}(N)$ such that*
   a) *for the empty tuple () we have $\mathfrak{C}((); N) : \{I\} \to \mathfrak{F}_{\mathfrak{B}}(N)$ maps I to I,*
   b) *$\mathfrak{C}(N_1, ..., N_n; N)$ is convex in every slot and base-point preserving,*
   c) *$\mathfrak{C}(N_1; N)$ is the canonical inclusion (in particular $\mathfrak{C}(N_1; N_1)$ is the identity),*
   d) *for every permutation σ we have that the following diagram commutes,*

$$\begin{array}{ccc} \bigtimes_{j=1}^{m} \mathfrak{F}_{\mathfrak{B}}(N_j) & \xrightarrow{\sigma-\text{permute}} & \bigtimes_{j=1}^{m} \mathfrak{F}_{\mathfrak{B}}(N_{\sigma(j)}) \\ & \searrow^{\mathfrak{C}(N_1,...,N_m;N)} & \downarrow{\scriptstyle \mathfrak{C}(N_{\sigma(1)},...,N_{\sigma(m)};N)} \\ & & \mathfrak{F}_{\mathfrak{B}}(N) \end{array}$$

   e) *and for every two collections of causally orderable sets $\bigcup_{j=1}^{m} \bigcup_{k=1}^{n_j} \{L_{jk}\}$ and $\bigcup_{j=1}^{m} \{N_j\}$, with $\bigcup_{k=1}^{n_j} L_{jk} \subseteq N_j$ for every j, the following diagram commutes*

$$\begin{array}{ccc} \bigtimes_{j=1}^{m} \bigtimes_{k=1}^{n_j} \mathfrak{F}_{\mathfrak{B}}(L_{jk}) & \xrightarrow{\times_{j=1}^{m} \mathfrak{C}(L_{j1},...,L_{jn_j};N_j)} & \bigtimes_{j=1}^{m} \mathfrak{F}_{\mathfrak{B}}(N_j) \\ & \searrow^{\mathfrak{C}(L_{11},...,L_{1n_1},...,L_{m1},...,L_{mn_m};N)} & \downarrow{\scriptstyle \mathfrak{C}(N_1,...,N_m;N)} \\ & & \mathfrak{F}_{\mathfrak{B}}(N) \end{array}$$

Remark: Note that what we call *causally orderable* is called *time-orderable* in [42]. Nevertheless, in order to emphasise the connection with [42] we adopt the term convex time-orderable PFA instead of convex causally orderable PFA.

The structure of a ctPFA is very general, and hence it is not surprising that there are immediately many examples available. Let us discuss a concrete one, which straightforwardly generalises and yields a whole class of ctPFAs.

Let $\mathfrak{B}_M$ be the background of regions of a globally hyperbolic manifold M and let $\mathfrak{V}$ be the *free* real vector space over $C_c^\infty(M;\mathbb{R})$. For convenience, let us denote the basis elements[4] by $\mathrm{ad}_S(f)$ for $f \in C_c^\infty(M;\mathbb{R})$ and let us define a distinguished point by $I := \mathrm{ad}_S(0)$.

Let us now define

1. for every region N : $\mathfrak{F}_{\mathfrak{B}_M}(N)$ is the convex hull of $\{\mathrm{ad}_S(f) |\, f \in C_c^\infty(N;\mathbb{R})\}$,
2. for every n-tuple of *causally orderable* regions $(N_1, ..., N_n)$ and every region N such that $N_j \subseteq N$ for every j we define $\mathfrak{C}(N_1, ..., N_n; N) : \bigtimes_{j=1}^{n} \mathfrak{F}_{\mathfrak{B}}(N_1) \to \mathfrak{F}_{\mathfrak{B}}(N)$ via

$$\mathfrak{C}(N_1, ..., N_n; N)(\mathrm{ad}_S(f_1), ..., \mathrm{ad}_S(f_n)) := \mathrm{ad}_S(f_1 + ... + f_n), \tag{3.21}$$

   and convex extension. (Here one could drop the restriction to causally orderable regions.)

In fact, this example is a special case of a more general situation that is discussed in the following lemma.

Lemma 3.2.2. *Let $\mathcal{M}$ be a net of monoids over the open subsets of some globally hyperbolic manifold M, with common unit 0 and composition $+ : \mathcal{M}^g \times \mathcal{M}^g \to \mathcal{M}^g$, where $\mathcal{M}^g := \mathcal{M}(M)$ such that*

1. *$N_1 \subseteq N_2 \implies \mathcal{M}(N_1) \subseteq \mathcal{M}(N_2)$, and*
2. *$f + g \in \mathcal{M}(N_1 \cup N_2)$ for $f \in \mathcal{M}(N_1)$ and $g \in \mathcal{M}(N_2)$.*

Let $\mathfrak{V}$ be the free *real vector space over $\mathcal{M}^g$ with basis denoted by $\mathrm{ad}_S(f)$ for $f \in \mathcal{M}^g$ and let us define a distinguished point by $I := \mathrm{ad}_S(0)$. Then defining*

1. *for every* region *$N \subseteq M$ $\mathfrak{F}^{\mathcal{M}}_{\mathfrak{B}_M}(N)$ to be the convex hull of $\{\mathrm{ad}_S(f) |\, f \in \mathcal{M}(N)\}$, and*

[4]The apparently peculiar choice to denote the basis elements by $\mathrm{ad}_S(f)$ is derived from the fact that $\mathrm{ad}_S(f)$ acts on a certain AQFT via conjugation by a unitary that is usually denoted by $S(f)$, see [44] and [45].

2. *defining for every n-tuple of* causally orderable *regions* $(N_1, ..., N_n)$ *and every region* N *such that* $N_j \subseteq N$ *for every* j, *the map* $\mathfrak{C}(N_1, ..., N_n; N) : \bigtimes_{j=1}^{n} \mathfrak{F}^{\mathcal{M}}_{\mathfrak{B}_M}(N_1) \to \mathfrak{F}^{\mathcal{M}}_{\mathfrak{B}_M}(N)$ *via convex extension of*

$$\mathfrak{C}(N_1, ..., N_n; N)(\mathrm{ad}_S(f_1), ..., \mathrm{ad}_S(f_n)) := \mathrm{ad}_S(f_1 + ... + f_n), \tag{3.22}$$

yields a ctPFA.

Proof. 1. Let N_1, N_2 be regions such that $N_1 \subseteq N_2$. Then, since $N_1 \subseteq N_2 \implies \mathcal{M}(N_1) \subseteq \mathcal{M}(N_2)$, it also follows that $\{\mathrm{ad}_S(f) | \, f \in \mathcal{M}(N_1)\}$ is contained in $\{\mathrm{ad}_S(f) | \, f \in \mathcal{M}(N_2)\}$. But then also $\mathfrak{F}^{\mathcal{M}}_{\mathfrak{B}_M}(N_1) \subseteq \mathfrak{F}^{\mathcal{M}}_{\mathfrak{B}_M}(N_2)$.

2. Let $n \in \mathbb{N}$ and let $(N_1, ..., N_n)$ be an n-tuple of *causally orderable* regions and let N be a region such that $N_j \subseteq N$ for every j. Then

   a) $\mathfrak{C}((); N) : \{I\} \to \mathfrak{F}^{\mathcal{M}}_{\mathfrak{B}_M}(N)$ maps I to I by definition,

   b) $\mathfrak{C}(N_1, ..., N_n; N)$ is convex in every slot by definition,

   c) $\mathfrak{C}(N_1; N)$ obviously maps $\mathrm{ad}_S(f)$ to $\mathrm{ad}_S(f)$, so is the canonical inclusion,

   d) for every permutation σ it follows that

$$\begin{aligned}&\mathfrak{C}(N_1, ..., N_n; N)(\mathrm{ad}_S(f_1), ..., \mathrm{ad}_S(f_n)) = \mathrm{ad}_S(f_1 + ... + f_n)\\ &= \mathrm{ad}_S(f_{\sigma(1)} + ... + f_{\sigma(n)})\\ &= \mathfrak{C}(N_{\sigma(1)}, ..., N_{\sigma(n)}; N)(\mathrm{ad}_S(f_{\sigma(1)}), ..., \mathrm{ad}_S(f_{\sigma(n)})),\end{aligned} \tag{3.23}$$

   from which the rest follows by convex extension.

   e) Finally, for every two collections of causally-orderable sets $\bigcup_{j=1}^{m} \bigcup_{k=1}^{n_j} \{L_{jk}\}$ and $\bigcup_{j=1}^{m} \{N_j\}$ such that for every j it holds that N_j contains L_{jk} for every k, we have that

$$\begin{aligned}&\mathfrak{C}(N_1, ..., N_m; N)\left(\bigtimes_{j=1}^{m} \left(\mathfrak{C}(L_{j1}, ..., L_{jn_j}; N_j)(\mathrm{ad}_S(f_{j1}), ..., \mathrm{ad}_S(f_{jn_j}))\right)\right)\\ &= \mathfrak{C}(N_1, ..., N_m; N)\left(\bigtimes_{j=1}^{m} \left(\mathrm{ad}_S(f_{j1} + ... + f_{jn_j})\right)\right)\\ &= \mathfrak{C}(N_1, ..., N_m; N)(\mathrm{ad}_S(f_{11} + ... + f_{1n_1}), ..., \mathrm{ad}_S(f_{m1} + ... + f_{mn_m}))\\ &= \mathrm{ad}_S(f_{11} + ... + f_{1n_1} + ... + f_{m1} + ... + f_{mn_m})\\ &= \mathfrak{C}(L_{11}, ..., L_{1,n_1}, ..., L_{m1}, ..., L_{mn_m}; N)(\mathrm{ad}_S(f_{11}), ...,\\ &\qquad \mathrm{ad}_S(f_{1n_1}), ..., \mathrm{ad}_S(f_{m1}), ..., \mathrm{ad}_S(f_{mn_m})).\end{aligned} \tag{3.24}$$

□

3.3 ctPFAs of quantum operations

Combining the notion of intervention with that of AQFT, where interventions are given in terms of quantum channels (or even operations), naturally leads to the following definition.

Definition 3.3.1. *A ctPFA $\mathfrak{F}_{\mathfrak{B}}$ over a background $\mathfrak{B}$ is a ctPFA of quantum operations of an AQFT $\mathcal{A}_{\mathfrak{B}}$ if there exists an injective map π from $\bigcup_{N\in\mathfrak{B}} \mathfrak{F}_{\mathfrak{B}}(N)$ to the quantum operations of $\mathcal{A}^g_{\mathfrak{B}}$ such that*

1. $\pi \restriction \mathfrak{F}_{\mathfrak{B}}(N)$ *is a convex map for every $N \in \mathfrak{B}$,*
2. $\pi(I) = \mathrm{id}$*, and*
3. *the combination is a composition, i.e.,*

$$(\pi \circ \mathfrak{C}(N_1, N_2, N))(T_1, T_2) = \begin{cases} \pi(T_1) \circ \pi(T_2) & \text{if } N_2 \not\lesssim N_1, \\ \pi(T_2) \circ \pi(T_1) & \text{if } N_1 \not\lesssim N_2. \end{cases} \tag{3.25}$$

If the image of π is contained in the set of quantum channels, then we say that $\mathfrak{F}_{\mathfrak{B}}$ (together with π) is a ctPFA of quantum channels of $\mathcal{A}_{\mathfrak{B}}$.

The third condition formalises the idea that, in the representation π, $\mathfrak{C}$ is a time-ordered composition of interventions.

A word of caution concerning the order of the composition. If $N_2 \not\lesssim N_1$, then one may say that N_1 is "earlier" than N_2, i.e., the quantum operation $\pi(T_1)$ should be applied "earlier" than $\pi(T_2)$, which is in apparent contradiction with the convention above that the combination is given by $\pi(T_1) \circ \pi(T_2)$. This apparent issue is resolved by the fact that quantum channels act on the global algebra $\mathcal{A}^g_{\mathfrak{B}}$, whereas the *dual* of the composition of the operations, given by

$$(\pi(T_1) \circ \pi(T_2))^* = \pi(T_2)^* \circ \pi(T_1)^*, \tag{3.26}$$

acts on *states* (which are in general not mapped to states because normalisation is not preserved). Here $\pi(T_1)^*$ is indeed applied "earlier" than $\pi(T_2)^*$.

It is now an immediate question whether there are any non-trivial examples of ctPFAs of quantum operations. In fact, there is a well-known example that may be traced back to [4].

Definition 3.3.2. *Let $\mathcal{A}_{\mathfrak{B}}$ be an AQFT over the background $\mathfrak{B}$.*

1. *We say that a quantum operation T of $\mathcal{A}^g_{\mathfrak{B}}$ is $\mathcal{A}_{\mathfrak{B}}(N)$-inner for a region $N \in \mathfrak{B}$ if there is a set $(K_j)_{j\in J} \in \mathcal{A}_{\mathfrak{B}}(N)$ for some finite index set J such that for every $A \in \mathcal{A}^g_{\mathfrak{B}}$*
$$T(A) = \sum_{j\in J} K_j^* A K_j. \tag{3.27}$$

2. *We say that a quantum operation T of $\mathcal{A}^g_{\mathfrak{B}}$ is* localisability-preserving *if for every region $N \in \mathfrak{B}$ we have that*
$$T[\mathcal{A}_{\mathfrak{B}}(N)] \subseteq \mathcal{A}_{\mathfrak{B}}(N). \tag{3.28}$$

We now define for every region $N \in \mathfrak{B}$ $\mathfrak{F}^{\mathrm{HK}}_{\mathfrak{B}}(N)$ to be the (pointed) convex set of all $\mathcal{A}_{\mathfrak{B}}(N)$-inner quantum operations (with base-point given by the identity id*).*

It follows immediately that for causally disjoint N_1, N_2 and every $T_1 \in \mathfrak{F}^{\mathrm{HK}}_{\mathfrak{B}}(N_1)$ and $T_2 \in \mathfrak{F}^{\mathrm{HK}}_{\mathfrak{B}}(N_2)$ we have that $T_1 \circ T_2 = T_2 \circ T_1$. Furthermore, for every $T \in \mathfrak{F}^{\mathrm{HK}}_{\mathfrak{B}}(N)$ and every $A \in \mathcal{A}_{\mathfrak{B}}(L)$ for some L with $L \perp N$ we see that $T(A) = AT(\mathbb{1})$. In particular, if T is a quantum channel, then $T(A) = A$.

We also note the following interesting result.

Lemma 3.3.3. *Let $\mathcal{A}_{\mathfrak{B}}$ be an AQFT over the background $\mathfrak{B}$ such that $\mathcal{A}^g_{\mathfrak{B}} = BL(\mathcal{H})$ for some separable Hilbert space. Then let T be a* normal *quantum operation of $BL(\mathcal{H})$ such that*[5] *for every $A \in \mathcal{A}_{\mathfrak{B}}(N)' \subseteq BL(\mathcal{H})$ we have that $T(A) = T(\mathbb{1})A$. Then T is $\mathcal{A}_{\mathfrak{B}}(N)''$-inner in the sense that there is a subset $(K_j)_{j\in J} \subseteq \mathcal{A}_{\mathfrak{B}}(N)''$ for some index set J such that for every $B \in \mathcal{A}^g_{\mathfrak{B}}$*
$$T(B) = \sum_{j\in J} K_j^* B K_j, \tag{3.29}$$
where convergence is understood in the topology $\sigma(BL(\mathcal{H}), BL(\mathcal{H})_*)$. If $\mathcal{A}_{\mathfrak{B}}(N)'' = \mathcal{A}_{\mathfrak{B}}(N)$, then T is $\mathcal{A}_{\mathfrak{B}}(N)$-inner.

Proof. We observe [47] that for every B
$$\begin{aligned}\sum_{j\in J}[K_j, B]^*[K_j, B] &= \sum_{j\in J} B^* K_j^* K_j B - \sum_{j\in J} K_j^* B^* K_j B - \sum_{j\in J} B^* K_j^* B K_j + \sum_{j\in J} K_j^* B^* B K_j \\ &= B^* T(\mathbb{1}) B - T(B^*) B - B^* T(B) + T(B^* B).\end{aligned} \tag{3.30}$$

[5]The following property is called “weakly localized” in N in [46].

Note that if $T(AB) = AT(B)$ for a positive T and for commuting A, B implies that $T(B)$ commutes with A. To see this note that T is Hermitian, see Proposition 9.9.2 in [17], so $T(AB) = (T(B^*A^*))^*$ and hence

$$[A, T(B)] = T(AB) - (A^*T(B)^*)^* = T(AB) - (T(A^*B^*))^* = 0. \tag{3.31}$$

So, in particular, for $A \in \mathcal{A}_{\mathfrak{B}}(N)' \subseteq BL(\mathcal{H})$. we see, following the assumption and the previous arguments, that

$$\sum_{j\in J} [K_j, A]^*[K_j, A] = 0, \tag{3.32}$$

from which it follows that $\forall j \in J\ [K_j, A] = 0$. Since this holds for every $A \in \mathcal{A}_{\mathfrak{B}}(N)'$, we see that $K_j \in \mathcal{A}_{\mathfrak{B}}(N)''$. □

This result is obviously particularly interesting in the case where the algebras $\mathcal{A}_{\mathfrak{B}}(N)$ are unital von Neumann algebras, i.e., $\mathcal{A}_{\mathfrak{B}}(N)'' = \mathcal{A}_{\mathfrak{B}}(N)$.

We can now show the following theorem.

Theorem 3.3.4. *Let $\mathcal{A}_{\mathfrak{B}}$ be an AQFT over the background $\mathfrak{B}$. Then for every n-tuple of* causally orderable *regions $(N_1, ..., N_n)$ and every region N such that $N_j \subseteq N$ for every j, we define the map $\mathfrak{C}(N_1, ..., N_n; N) : \bigtimes_{j=1}^{n} \mathfrak{F}^{\mathrm{HK}}_{\mathfrak{B}}(N_j) \to \mathfrak{F}^{\mathrm{HK}}_{\mathfrak{B}}(N)$ via*

$$\mathfrak{C}(N_1, ..., N_n; N)(T_1, ..., T_n) := T_{\sigma(1)} \circ ... \circ T_{\sigma(n)}, \tag{3.33}$$

for some permutation σ such that $N_{\sigma(1)} < ... < N_{\sigma(n)}$ for some causal linear order $\leq$. Then $\mathfrak{C}$ is well-defined and $\mathfrak{F}^{\mathrm{HK}}_{\mathfrak{B}}$ together with $\mathfrak{C}$ is a ctPFA of quantum operations.

Before we proceed we show the following useful Lemma.

Lemma 3.3.5. *Let $T_1, ..., T_n$ be endomorphisms of some set, let $\leq$ be the canonical linear order on $\{1, ..., n\}$ and let $\trianglelefteq$ be another order such that for every i, j it holds that*

$$(i \leq j \wedge j \trianglelefteq i) \implies T_i \circ T_j = T_j \circ T_i. \tag{3.34}$$

Then

$$T_1 \circ T_2 \circ ... \circ T_n = T_{\pi(1)} \circ T_{\pi(2)} \circ ...T_{\pi(n)}, \tag{3.35}$$

where $\pi(1) \trianglelefteq \pi(2) \trianglelefteq ... \trianglelefteq \pi(n)$ for a permutation π.

Proof. We proceed by induction. The case $n = 1$ is trivial and the case $n = 2$ follows by assumption, so let us assume that we have shown the claim for $n - 1$ in place of n. Let $\trianglelefteq$ be a linear order on $\{1, ..., n\}$ fulfilling the assumption and let π be the associated permutation. We will show the claim by considering the following cases.

1. Suppose $\pi(n) = n$ upon which we have that $T_{\pi(n)} = T_n$. Then $\trianglelefteq$ (and π) restrict to $\{1, ..., n-1\}$ while still fulfilling the assumptions, and by the induction hypothesis $T_1 \circ \cdots \circ T_{n-1} = T_{\pi(1)} \circ \cdots \circ T_{\pi(n-1)}$, so in particular

$$T_1 \circ ... \circ T_{n-1} \circ T_n = T_{\pi(1)} \circ \circ ... \circ T_{\pi(n-1)} \circ T_{\pi(n)}. \tag{3.36}$$

2. If $\pi(n) \neq n$, then there exists $l \in \mathbb{N}^*$ such that $\pi(l) = n$. In particular $\pi(l) \geq l, l+1, ...n$ but $l \leq l, l+1, ..., n$. Hence, by assumption, T_n commutes with $T_{\pi(l)}, T_{\pi(l+1)}, ..., T_{\pi(n)}$. So we have

$$\begin{aligned} &T_{\pi(1)} \circ ... \circ T_{\pi(l-1)} \circ T_{\pi(l)} \circ T_{\pi(l+1)} \circ ... \circ T_{\pi(n)} \\ &= T_{\pi(1)} \circ ... \circ T_{\pi(l-1)} \circ T_n \circ T_{\pi(l+1)} \circ ... \circ T_{\pi(n)} \\ &= T_{\pi(1)} \circ ... \circ T_{\pi(l-1)} \circ T_{\pi(l+1)} \circ ... \circ T_{\pi(n)} \circ T_n. \end{aligned} \tag{3.37}$$

After appropriate relabelling, we see that we are in the situation of the first case, which finishes the proof.

□

We can now prove Theorem 3.3.4.

Proof of Theorem 3.3.4. 1. Let $N_1 \subseteq N_2$. Then it is easy to see that $\mathfrak{F}^{\mathrm{HK}}_{\mathfrak{B}}(N_1) \subseteq \mathfrak{F}^{\mathrm{HK}}_{\mathfrak{B}}(N_2)$ since $\mathcal{A}_{\mathfrak{B}}(N_1) \subseteq \mathcal{A}_{\mathfrak{B}}(N_2)$.

2. Let $n \in \mathbb{N}$ and let $(N_1, ..., N_n)$ be an n-tuple of *causally orderable* regions and let N be a region such that $N_j \subseteq N$ for every j. Then we show that $\mathfrak{C}$ is well-defined. So let $\leq$ and $\trianglelefteq$ be two causal linear orders on $\{N_1, ...N_n\}$. The note that we have for all i, j that

$$\begin{aligned} &(N_i \leq N_j \wedge N_j \trianglelefteq N_i) \implies N_i \perp N_j \\ &\implies \Big(\forall T_i \mathfrak{F}^{\mathrm{HK}}_{\mathfrak{B}}(N_i), \forall T_j \in \mathfrak{F}^{\mathrm{HK}}_{\mathfrak{B}}(N_j) : T_i \circ T_j = T_j \circ T_i\Big). \end{aligned} \tag{3.38}$$

It then follows from Lemma 3.3.5, isotony of $\mathfrak{F}^{\mathrm{HK}}_{\mathfrak{B}}$ and the fact that a composition of $\mathcal{A}_{\mathfrak{B}}(N)$-inner operations is $\mathcal{A}_{\mathfrak{B}}(N)$-inner, that $\mathfrak{C}$ is well-defined. Furthermore

a) $\mathfrak{C}((); N) : \{I\} \to \mathfrak{F}^{\mathrm{HK}}_{\mathfrak{B}_M}(N)$ maps id to id by definition,

b) $\mathfrak{C}(N_1, ..., N_n; N)$ is convex in every slot,

c) $\mathfrak{C}(N_1; N)$ is the canonical inclusion, and

d) is obvious.

e) Finally, let us consider two collections of causally-orderable sets $\bigcup_{j=1}^{m}\bigcup_{k=1}^{n_j}\{L_{jk}\}$ and $\bigcup_{j=1}^{m}\{N_j\}$ such that for every j it holds that N_j contains L_{jk} for every k. Let

$$\begin{aligned}
&\mathfrak{C}(N_1,...,N_m;N)\left(\bigtimes_{j=1}^{m}\left(\mathfrak{C}(L_{j1},...,L_{jn_j};N_j)(T_{j1},...,T_{jn_j})\right)\right)\\
&=\mathfrak{C}(N_1,...,N_m;N)\left(\bigtimes_{j=1}^{m}\left(T_{j\pi_j(1)}\circ...\circ T_{j\pi_j(n_j)}\right)\right)\\
&=\mathfrak{C}(N_1,...,N_m;N)\left(T_{1\pi_1(1)}\circ...\circ T_{1\pi_1(n_1)},...,T_{m\pi_m(1)}\circ...\circ T_{m\pi_m(n_m)}\right)\\
&=T_{\sigma(1)\pi_{\sigma(1)}(1)}\circ...\circ T_{\sigma(1)\pi_{\sigma(1)}(n_{\sigma(1)})}\circ...\circ T_{\sigma(m)\pi_{\sigma(m)}(1)}\circ...\circ T_{\sigma(m)\pi_{\sigma(m)}(n_{\sigma(m)})},
\end{aligned}\tag{3.39}$$

where π_j are permutations of $\{1,...,n_j\}$ such that $L_{j\pi_j(1)}\blacktriangleleft_j ... \blacktriangleleft_j L_{j\pi_j(n_j)}$ with respect to some causal linear order $\trianglelefteq_j$ and σ is a permutation such that $N_{\sigma(1)}\triangleleft ... \triangleleft N_{\sigma(m)}$ for some causal linear order $\trianglelefteq$. We now show that

$$\underbrace{L_{\sigma(1)\pi_{\sigma(1)}(1)},...,L_{\sigma(1)\pi_{\sigma(1)}(n_{\sigma(1)})}}_{1^{\text{st}}\text{ block}},...,\underbrace{L_{\sigma(m)\pi_{\sigma(m)}(1)},...,L_{\sigma(m)\pi_{\sigma(m)}(n_{\sigma(m)})}}_{m^{\text{th}}\text{ block}},\tag{3.40}$$

is causally ordered upon which the result follows. Let us denote the order by $\leq$ and let us recall that a linear order is called causal if $A<B \implies B \not\lesssim A$. Now suppose $A<B$ and they are both in the *same* block, say the j^{th}, i.e., $A,B\subseteq N_{\sigma(j)}$. Then also $A\blacktriangleleft_{\sigma(j)} B$, and since $\trianglelefteq_{\sigma(j)}$ is a causal linear order, we see that $B\not\lesssim A$. Not suppose that A,B are not in the same block, e.g., $A\in N_{\sigma(j)}$ and $B\in N_{\sigma(k)}$. But then $N_{\sigma(j)}\triangleleft N_{\sigma(k)}$ and since $\trianglelefteq$ is a causal linear order, we have that $N_{\sigma(k)}\not\lesssim N_{\sigma(j)}$. But we see that then also $B\not\lesssim A$, which finishes the proof.

□

The quantum operations in $\mathfrak{F}^{\text{HK}}_{\mathfrak{B}}$ were already discussed by Haag and Kastler in [4]. It seems that the focus was, in fact, more on these quantum operations than on observables. However, as we will see now, this ctPFA of quantum channels does in general not respect causality.

3.4 CTPFAS OF CAUSAL QUANTUM CHANNELS

As we have already discussed, the notion of *causality* in AQFT comes in various shapes. One aspect can be that of statistical independence of spacelike separated regions, which is a notion that does *not* require quantum channels. However, with quantum channels at hand, there is another notion of causality, namely *signalling causality.* In its basic form it states that an observer in control of some region N_1 can in no way *whatsoever* signal to another observer in control of a causally disjoint region N_2. Clearly a necessary requirement for the quantum channels in region N_1 to respect signalling causality is that they act trivially at every N_2 that is causally disjoint, as is the case for the quantum *channels* in $\mathfrak{F}^{\mathrm{HK}}_{\mathfrak{B}}(N_1)$. However, as Sorkin emphasised in [7], this condition is in general *not* sufficient when more than two parties are involved.

3.4.1 Sorkin's superluminal signalling protocol

The ideas of this subsection go back to [7], see also [8].

The essence of Sorkin's signalling protocol does not invoke any "quantumness". In fact, it quite generally hints at the fact that requiring that "interventions" (such as quantum channels) on a (not necessarily quantum) theory act trivially at causally disjoint regions might not be sufficient to rule out superluminal signalling (when the background admits configurations of three regions as illustrated in Fig. 3.1).

To be more specific, let us consider the background of regions of two-dimensional Minkowski spacetime and assume that there are observers Alice, in control of region N_1, and Charlie, in control of region N_3 for spacelike separated N_1 and N_3. Alice will be the sender and Charlie will be the receiver and if there are indeed only Alice and Charlie in the picture, then no matter which local intervention Alice performs, as long as it acts trivially at spacelike separation, Charlie has no chance of receiving a signal. However, the actual requirement is that Alice shall not be able to send a signal to Charlie under any circumstances *whatsoever.* In particular, no signalling should be possible, even if there is another observer Bob in control of some region N_2 as illustrated in Fig. 3.1.

We observe that there is a unique causal linear order on the collection of the three regions, namely $N_1 < N_2 < N_3$. This unique causal linear order results in the following protocol:

1. Alice performs her intervention, then
2. Bob performs his intervention, then, finally,
3. Charlie "checks" for received signals.

Now, in completely abstract terms, we see that Bob's presence could, in principle, have the potential of allowing Alice to send a signal to Charlie. Heuristically speaking,

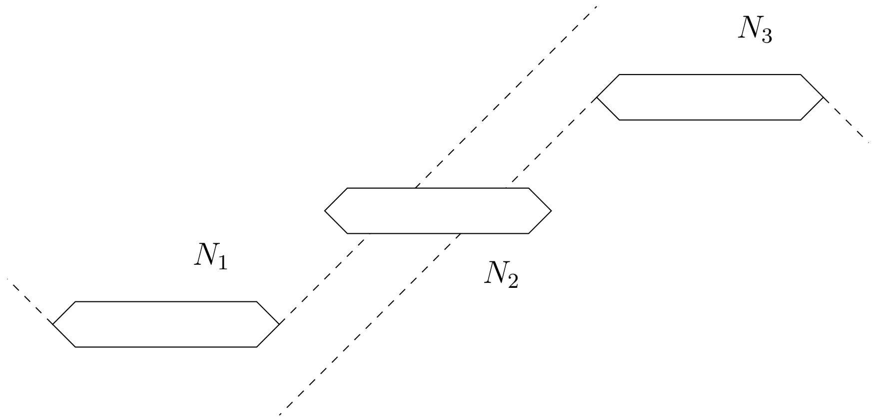


Figure 3.1: Schematic causal configuration of regions N_1, N_2 and N_3.

the fact that Bob's intervention happens between Alice's and Charlie's prevents us from allowing Charlie to directly ignore Alice if the only assumption at hand is that Alice's intervention acts trivially at spacelike separation.

Sorkin's observation suggests that stronger causality conditions are necessary in order for interventions to respect signalling causality.

3.4.2 Causality relations

In order to prepare for the incorporation of *signalling causality* into the framework of ctPFAs of abstract interventions, we introduce a collection of equivalence relations $\bowtie_L$ between certain tuples of causally orderable regions $(N_1, ...N_m)$ and $(R_1, ...R_n)$, one for every region $L \in \mathfrak{B}$.

Definition 3.4.1. *Let $\mathfrak{B}$ be a background and take $m, n \in \mathbb{N}^*$. Let $\{N_1, ..., N_m\}$ and $\{R_1, ..., R_n\}$ be two sets of causally orderable regions. Then we write*

$$(N_1, ..., N_m) \bowtie_L (R_1, ..., R_n) \tag{3.41}$$

if and only if

1. *$L \in \mathfrak{B}$ is a region such that $L \not\lesssim N_j$ and $L \not\lesssim R_k$ for $j = 1, ..., m$ and $k = 1, ..., n$, and*
2. *L is causally disjoint from every region in the symmetric difference of $\{N_1, ..., N_m\}$ and $\{R_1, ..., R_n\}$, i.e., from any region in $\{N_1, ..., N_m\} \setminus \{R_1, ..., R_n\}$ or $\{R_1, ..., R_n\} \setminus \{N_1, ..., N_m\}$.*

The motivation is as follows: Suppose we have a ctPFA of interventions $\mathfrak{F}_{\mathfrak{B}}$ and we have $(N_1, ..., N_m) \bowtie_L (R_1, ..., R_n)$. Then assume we have several agents, one for every (distinct) region in $\{N_1, ..., N_m\} \cup \{R_1, ..., R_n\}$ each of which performs one single intervention. Then consider an observer in L. Now, since $L \not\lesssim N_j$ and $L \not\lesssim R_k$ for $j = 1, ..., m$ and $k = 1, ..., n$, we see that $\{N_1, ..., N_m\} \cup \{L\}$ and $\{R_1, ..., R_n\} \cup \{L\}$ are causally orderable and on each set there is a causal linear order such that L is the greatest or "latest" region. Hence it is reasonable to consider the situation that all agents perform their intervention and *then* the observer in L makes an observation. Based on the principle of signalling causality, we then require that the observer in L should not be able to distinguish the case where only the agents in the regions contained in $\{N_1, ..., N_m\}$ performed their interventions from the case where only the agents in the regions contained in $\{R_1, ..., R_n\}$ performed their interventions. This is precisely because the symmetric difference only comprises agents who are causally disjoint from L and should therefore be irrelevant. This is expressed by the equivalence[6] relations $\bowtie_L$.

In the case of a ctPFA of quantum channels (of an AQFT) it is then natural to speak of *causal* quantum channels if the representation π respects the equivalence relations $\bowtie_L$.

Definition 3.4.2. *Let $\mathfrak{F}_{\mathfrak{B}}$ together with π be a ctPFA of quantum channels of an AQFT $\mathcal{A}_{\mathfrak{B}}$ over a background $\mathfrak{B}$. Then we call it a ctPFA of* causal *quantum channels if $(N_1, ..., N_m) \bowtie_L (N_1, ..., N_{m-1})$ implies that for any region N containing $N_1, ..., N_m$ and any region $\tilde{N}$ containing $N_1, ..., N_{m-1}$ we have*

$$R_{\mathcal{A}_{\mathfrak{B}}(L)} \circ \pi \circ \mathfrak{C}(N_1, ...N_m; N) = R_{\mathcal{A}_{\mathfrak{B}}(L)} \circ \pi \circ \mathfrak{C}(N_1, .., N_{m-1}; \tilde{N}) \circ \mathrm{pr}_{\neg m}, \tag{3.42}$$

where $\mathrm{pr}_{\neg m} : \bigtimes_{j=1}^{m} \mathfrak{F}_{\mathfrak{B}}(N_j) \to \bigtimes_{j=1}^{m-1} \mathfrak{F}_{\mathfrak{B}}(N_j)$ is the canonical projection on all but the m^{th} component and $R_{\mathcal{A}_{\mathfrak{B}}(L)}$ sends a quantum channel T on $\mathcal{A}^g_{\mathfrak{B}}$ to its restriction $T \upharpoonright \mathcal{A}_{\mathfrak{B}}(L)$.

Indeed, this condition captures the requirement that any intervention that happens at a region that is causally disjoint from L may be simply ignored when restricting one's attention to $\mathcal{A}_{\mathfrak{B}}(L)$.

The existence of non-trivial ctPFAs of *causal* quantum channels is established by the following theorem.

[6]To see that $\bowtie_L$ is indeed an equivalence relation suppose that $(N_1, ..., N_m) \bowtie_L (R_1, ..., R_n)$ and $(R_1, ..., R_n) \bowtie_L (P_1, ..., P_q)$. For brevity let us write $A := \{N_1, ..., N_m\}$, $B := \{R_1, ..., R_n\}$ and $C := \{P_1, ..., P_q\}$. Then we use that the symmetric difference of A and C is indeed contained in the symmetric difference of A and B, union the symmetric difference of B and C. By assumption, this set only contains regions that are causally disjoint from L, hence $(N_1, ..., N_m) \bowtie_L (P_1, ..., P_q)$.

Theorem 3.4.3. *Let $\mathcal{A}_{\mathfrak{B}}$ be an AQFT over the background $\mathfrak{B}$. For every region $N \in \mathfrak{B}$ let ${}^{\mathrm{lp}}\mathfrak{F}^{\mathrm{HK}}_{\mathfrak{B}}(N) \subseteq \mathfrak{F}^{\mathrm{HK}}_{\mathfrak{B}}(N)$ be the (pointed) convex set of all* localisability-preserving *$\mathcal{A}_{\mathfrak{B}}(N)$-inner quantum channels (with base-point given by the identity* id*). Then ${}^{\mathrm{lp}}\mathfrak{F}^{\mathrm{HK}}_{\mathfrak{B}}$ and $\mathfrak{C}$, where the latter is defined in Theorem 3.3.4, form a ctPFA of* causal *quantum channels.*

Proof. By observing that the composition of localisability-preserving quantum channels is again a localisability-preseving quantum channel we immediately see that ${}^{\mathrm{lp}}\mathfrak{F}^{\mathrm{HK}}_{\mathfrak{B}}$ and $\mathfrak{C}$ form a ctPFA.

To show that it is causal let us consider appropriate regions $L, N, \tilde{N}$ and $N_1, ..., N_m$ such that $(N_1, ..., N_m) \bowtie_L (N_1, ..., N_{m-1})$. For $j = 1, .., m$ take any $T_j \in {}^{\mathrm{lp}}\mathfrak{F}^{\mathrm{HK}}_{\mathfrak{B}}(N_j)$ such that for some appropriate permutation we have σ we have

$$\mathfrak{C}(N_1, ...N_m; N)(T_1, ..., T_m) = \underbrace{T_{\sigma(1)} \circ ...}_{=:T_{\mathsf{A}}} \circ T_m \circ \underbrace{... \circ T_{\sigma(m)}}_{=:T_{\mathsf{C}}} . \tag{3.43}$$

When evaluated on any $A \in \mathcal{A}_{\mathfrak{B}}(L)$, we see that

$$T_{\mathsf{A}} \circ T_m \circ T_{\mathsf{C}}(A) = T_{\mathsf{A}} \circ T_{\mathsf{C}}(A), \tag{3.44}$$

since $T_{\mathsf{C}}(A) \in \mathcal{A}_{\mathfrak{B}}(L)$ and T_m, being $\mathcal{A}_{\mathfrak{B}}(N)$-inner for $N \perp L$, acts trivially on any element in $\mathcal{A}_{\mathfrak{B}}(L)$. □

Unfortunately, the collection of localisation-preserving and inner quantum channels misses out on a certain class of "interesting" channels, i.e., those that can be derived from local and (spacetime-) compact perturbations of the dynamics of AQFTs, which we will study in the following chapter. Nevertheless, as we will see in Chapter 4, local and (spacetime-) compact perturbations may also be equipped with the structure of a ctPFAs, giving rise to *causal* quantum channels for every *additive* AQFT on the background $\mathfrak{B}_M$ of regions of a globally hyperbolic spacetime M.

— 4 —

Compactly supported perturbations, scattering and causal factorisation

4.1 PERTURBED VARIANTS

Let us fix a globally hyperbolic spacetime M, giving rise to a background $\mathfrak{B}_M$ and let $\mathcal{U}_{\mathfrak{B}_M}$ be an AQFT on $\mathfrak{B}_M$. At the beginning of this chapter we discuss the formalisation of the idea of spacetime-compact *perturbations* of $\mathcal{U}_{\mathfrak{B}_M}$, i.e., perturbations of (the dynamics of) $\mathcal{U}_{\mathfrak{B}_M}$, hence called the *unperturbed* AQFT, that are confined in a compact *perturbation zone* $K \subseteq M$. We follow ideas of Sec. 3.1 in [10].

Guidance for the definition of such compact perturbations may be taken from the "ignorance meta-principle" [48], which may be thought of as underpinning the "generally covariant locality principle" [49]. It states that anything that cannot be known "may be neglected". To an experimenter with control over a region L with no overlap with the perturbation zone K, the accessible local degrees of freedom and their local dynamics are unaffected by a perturbation in K and are hence *equivalent* to the unperturbed AQFT.

Formally speaking, the perturbation of $\mathcal{U}_{\mathfrak{B}_M}$ should itself be an AQFT $\mathcal{P}_{\mathfrak{B}_M}$ on $\mathfrak{B}_M$ such that for every region $L \in \mathfrak{B}_{(M,K)}$, i.e., for every K-admissible region L, the restrictions $\mathcal{U}_{\mathfrak{B}_L}$ and $\mathcal{P}_{\mathfrak{B}_L}$ are equivalent via a unit-preserving $*$-isomorphism $\chi_L : \mathcal{U}^g_{\mathfrak{B}_L} \to \mathcal{P}^g_{\mathfrak{B}_L}$. There is an undesirable ambiguity here, namely it could be that for nested K-admissible regions $L_1 \subseteq L_2$ we have $\chi_{L_1} \neq \chi_{L_2} \restriction \mathcal{U}_{\mathfrak{B}_{L_2}}(L_1)$. We will explicitly exclude this possibility. Finally, since we only care about the perturbed AQFT on K-admissible regions, we will allow perturbed theories to be only defined on $\mathfrak{B}_{(M,K)}$.

Definition 4.1.1 (Perturbed variant)**.** *Let M be a globally hyperbolic spacetime and let $\mathcal{U}_{\mathfrak{B}_M}$ be an AQFT on $\mathfrak{B}_M$. Let $K', K \subseteq M$ be two compact subsets such that $K' \subseteq K$ and let $\mathcal{P}_{\mathfrak{B}_{(M,K')}}$ be an AQFT on $\mathfrak{B}_{(M,K')}$. For every K-admissible region L*

let there be a unit-preserving $*$*-isomorphism* $\chi_L : \mathcal{U}_{\mathfrak{B}_M}(L) \to \mathcal{P}_{\mathfrak{B}_{(M,K')}}(L)$ *such that for every two nested* K*-admissible regions* $L_1 \subseteq L_2$ *the following diagram commutes.*

$$\begin{array}{ccc} \mathcal{U}_{\mathfrak{B}_M}(L_1) & \xrightarrow{\chi_{L_1}} & \mathcal{P}_{\mathfrak{B}_{(M,K')}}(L_1) \\ \downarrow & & \downarrow \\ \mathcal{U}_{\mathfrak{B}_M}(L_2) & \xrightarrow{\chi_{L_2}} & \mathcal{P}_{\mathfrak{B}_{(M,K')}}(L_2) \end{array}$$

Then we call the collection

$$\left(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_{(M,K')}}, \{\chi_L\}_{L\in\mathfrak{B}_{(M,K)}}, K\right) \tag{4.1}$$

a K*-perturbed variant of* $\mathcal{U}_{\mathfrak{B}_M}$ *on* $\mathfrak{B}_{(M,K')}$*. Two* K*-perturbed variants of* $\mathcal{U}_{\mathfrak{B}_M}$ *on* $\mathfrak{B}_{(M,K')}$*, e.g.,*

$$\left(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_{(M,K')}}, \{\chi_L\}_{L\in\mathfrak{B}_{(M,K)}}, K\right) \text{ and } \left(\mathcal{U}_{\mathfrak{B}_M}, \tilde{\mathcal{P}}_{\mathfrak{B}_{(M,K')}}, \{\tilde{\chi}_L\}_{L\in\mathfrak{B}_{(M,K)}}, K\right)$$

are called equivalent*, if and only if there exists a unit-preserving* $*$*-isomorphism* $\alpha : \mathcal{P}^g_{\mathfrak{B}_{(M,K')}} \to \tilde{\mathcal{P}}^g_{\mathfrak{B}_{(M,K')}}$ *such that for every* K*-admissible region* L *we have that*

$$\tilde{\chi}_L = \alpha \circ \chi_L. \tag{4.2}$$

Remarks:

1. We will mainly be interested in the case $K' = \emptyset$ or $K' = K$.
2. Fixing $\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_M}$ and K as above does in general not yield a unique K-perturbed variant. For instance, let each of $\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_M}$ be given by ${}^P\mathcal{W}_{\mathfrak{B}_M}$, i.e., the AQFT of a linear scalar field with Green-hyperbolic equation of motion operator $P : C^\infty(M;\mathbb{R}) \to C^\infty(M;\mathbb{R})$, see Appendix C, and let M and K be such that $K^{\perp_M} = \emptyset$ and K is causally convex.[1] On the one hand let every element of $\{{}^1\chi_L\}_{L\in\mathfrak{B}_{(M,K)}}$ be given by the identity, which is a viable set of identification maps, on the other hand define $\{{}^2\chi_L\}_{L\in\mathfrak{B}_{(M,K)}}$ by

$$^2\chi_N(A) = \begin{cases} A & \text{for } N \cap J^-_M(K) = \emptyset, \\ \sigma(A) & \text{for } N \cap J^+_M(K) = \emptyset, \end{cases} \tag{4.3}$$

where σ is the unique quasi-free unit-preserving $*$-automorphism of ${}^P\mathcal{W}^g_{\mathfrak{B}_M}$ induced by the map $[f]_P \mapsto -[f]_P$, see Appendix B.6.

[1] For instance take M with a compact Cauchy surface Σ and set $K = \Sigma$.

3. Given

$$\left(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_{(M,K')}}, \{\chi_L\}_{L\in\mathfrak{B}_{(M,K)}}, K\right), \tag{4.4}$$

a K-perturbed variant of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_{(M,K')}$ and a compact $\tilde{K} \subseteq M$ such that $K' \subseteq K \subseteq \tilde{K}$ the collection

$$\left(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_{(M,K')}}, \{\chi_L\}_{L\in\mathfrak{B}_{(M,\tilde{K})}}, \tilde{K}\right), \tag{4.5}$$

is a $\tilde{K}$-perturbed variant of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_{(M,K')}$.

4. Finally note that if $\left(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_{(M,K)}}, \{\chi_L\}_{L\in\mathfrak{B}_{(M,K)}}, K\right)$ is a K-perturbed variant, then $\mathcal{U}_{\mathfrak{B}_{(M,K)}}$ and $\mathcal{P}_{\mathfrak{B}_{(M,K)}}$ are not necessarily equivalent. Suppose $\mathcal{U}_{\mathfrak{B}_M}$ and $\mathcal{P}_{\mathfrak{B}_{(M,K)}}$ fulfill time-slice. Then $\chi_{M_K^+}$ and $\chi_{M_K^-}$ are each maps from $\mathcal{U}^g_{\mathfrak{B}_M}$ to $\mathcal{P}^g_{\mathfrak{B}_{(M,K)}}$, but they might be different. In fact, this possibility is precisely what yields the notion of a scattering map, see again Sec. 3.1 in [10].

4.2 SCATTERING

Definition 4.2.1 (Scattering map). *Let $\mathcal{U}_{\mathfrak{B}_M}$ be an AQFT fulfilling time-slice and let $\left(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_{(M,K')}}, \{\chi_L\}_{L\in\mathfrak{B}_{(M,K)}}, K\right)$ be a K-perturbed variant of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_{(M,K')}$ such that $\mathcal{P}_{\mathfrak{B}_{(M,K')}}$ fulfills time-slice. Then we define the scattering map $\Theta : \mathcal{U}^g_{\mathfrak{B}_M} \to \mathcal{U}^g_{\mathfrak{B}_M}$ as*

$$\Theta := \left(\chi_{M_K^-}\right)^{-1} \circ \chi_{M_K^+}, \tag{4.6}$$

which is a unit-preserving $*$-automorphism.

As one would expect, it is indeed true that two equivalent K-perturbed variants have *identical* scattering maps. What is perhaps more surprising is that also the converse holds.

Lemma 4.2.2. *Let $\mathcal{U}_{\mathfrak{B}_M}$ be an AQFT fulfilling time-slice, let $K \subseteq M$ be compact and let*

$$\left(\mathcal{U}_{\mathfrak{B}_M}, {}^1\mathcal{P}_{\mathfrak{B}_{(M,K)}}, \left\{{}^1\chi_L\right\}_{L\in\mathfrak{B}_{(M,K)}}, K\right) \text{ and } \left(\mathcal{U}_{\mathfrak{B}_M}, {}^2\mathcal{P}_{\mathfrak{B}_{(M,K)}}, \left\{{}^2\chi_L\right\}_{L\in\mathfrak{B}_{(M,K)}}, K\right)$$

be two K-perturbed variants of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_{(M,K)}$ with scattering maps ${}^1\Theta$ and ${}^2\Theta$ respectively, such that both ${}^1\mathcal{P}_{\mathfrak{B}_{(M,K)}}$ and ${}^2\mathcal{P}_{\mathfrak{B}_{(M,K)}}$ fulfill time-slice. Then

1. *if the two K-perturbed variants are equivalent, then ${}^1\mathcal{P}_{\mathfrak{B}_{(M,K)}}$ and ${}^2\mathcal{P}_{\mathfrak{B}_{(M,K)}}$ are equivalent,*

2. *if the two K-perturbed variants are equivalent, then ${}^1\Theta = {}^2\Theta$.*

If furthermore $\mathcal{U}_{\mathfrak{B}_M}$ fulfills additivity and K is causally convex, then

3. $^1\Theta = {}^2\Theta \iff$ *the two K-perturbed variants are equivalent.*

Proof. Let the K-perturbed variants be equivalent via a map $\alpha : {}^1\mathcal{P}^g_{\mathfrak{B}_{(M,K)}} \to {}^2\mathcal{P}^g_{\mathfrak{B}_{(M,K)}}$ such that for every K-admissible region L we have that ${}^2\chi_N = \alpha \circ {}^1\chi_N$.

1. Then

$$\begin{aligned} {}^2\Theta &= \left({}^2\chi_{M_K^-}\right)^{-1} \circ {}^2\chi_{M_K^+} = \left(\alpha \circ {}^1\chi_{M^-}\right)^{-1} \circ \alpha \circ {}^1\chi_{M^+} \\ &= \left({}^1\chi_{M_K^-}\right)^{-1} \circ {}^1\chi_{M_K^+} = {}^1\Theta. \end{aligned} \tag{4.7}$$

2. Furthermore, we have for every K-admissible region N that

$$\alpha\left[{}^1\mathcal{P}_{\mathfrak{B}_{(M,K)}}(N)\right] = {}^2\chi_N \circ \left({}^1\chi_N\right)^{-1}\left[{}^1\mathcal{P}_{\mathfrak{B}_{(M,K)}}(N)\right] = {}^2\mathcal{P}_{\mathfrak{B}_{(M,K)}}(N), \tag{4.8}$$

   i.e., ${}^1\mathcal{P}_{\mathfrak{B}_{(M,K)}}$ and ${}^2\mathcal{P}_{\mathfrak{B}_{(M,K)}}$ are equivalent.

3. Let us now assume that $\mathcal{U}_{\mathfrak{B}_M}$ fulfills additivity, that K is causally convex and that ${}^2\Theta = {}^1\Theta$, i.e.,

$$\left({}^2\chi_{M_K^-}\right)^{-1} \circ {}^2\chi_{M_K^+} = \left({}^1\chi_{M_K^-}\right)^{-1} \circ {}^1\chi_{M_K^+}. \tag{4.9}$$

   Then let us define

$$\alpha := {}^2\chi_{M_K^-} \circ \left({}^1\chi_{M_K^-}\right)^{-1} = {}^2\chi_{M_K^+} \circ \left({}^1\chi_{M_K^+}\right)^{-1} : {}^1\mathcal{P}^g_{\mathfrak{B}_{(M,K)}} \to {}^1\mathcal{P}^g_{\mathfrak{B}_{(M,K)}}. \tag{4.10}$$

   The claim is now that for every K-admissible region N we have that ${}^2\chi_N = \alpha \circ {}^1\chi_N$. If N is contained in either M_K^- or M_K^+ this follows by definition. Otherwise note that for general K-admissible region N, for which $N = \left(N \cap M_K^-\right) \cup \left(N \cap M_K^+\right)$ by causal convexity of K, we have that ${}^1\chi_N$ and ${}^2\chi_N$ are uniquely determined by ${}^1\chi_{N\cap M_K^\pm}$ and ${}^2\chi_{N\cap M_K^\pm}$ by additivity of $\mathcal{U}_{\mathfrak{B}_M}$. Then

$$\alpha \circ {}^1\chi_{N\cap M_K^\pm} = {}^2\chi_{M_K^\pm} \circ \left({}^1\chi_{M_K^\pm}\right)^{-1} \circ {}^1\chi_{N\cap M_K^\pm} = {}^2\chi_{M_K^\pm} \restriction \mathcal{U}_{\mathfrak{B}_M}(N \cap M_K^\pm) = {}^2\chi_{N\cap M_K^\pm}. \tag{4.11}$$

□

4.2.1 K-maps

In order to analyse the abstract properties of scattering maps further, we introduce the following class of maps. (The motivation for this definition is that scattering maps are K-maps, which we will show in Lemma 4.2.4).

Definition 4.2.3 (K-homs and K-maps)**.** *Let $\mathcal{A}_{\mathfrak{B}_M}$ be a theory and for some compact $K \subseteq M$ let $T : \mathcal{A}^g_{\mathfrak{B}_M} \to \mathcal{A}^g_{\mathfrak{B}_M}$ be a unit-preserving $*$-homorphism such that*

1. *T acts trivially on $\mathcal{A}_{\mathfrak{B}_M}(L)$ for every region L spacelike separated from K,*
2. *for regions $L^\pm \in \mathfrak{B}_M$ such that $L^\pm \subseteq M^\pm_K$ and $L^+ \sqsubseteq L^-$:*

$$T\Big[\mathcal{A}_{\mathfrak{B}_M}(L^+)\Big] \subseteq \mathcal{A}_{\mathfrak{B}_M}(L^-). \tag{4.12}$$

Then we call T a K-hom. If T is a unit-preserving $*$-automorphism such that in addition also

3. *for regions $L^\pm \in \mathfrak{B}_M$ such that $L^\pm \subseteq M^\pm_K$ and $L^- \sqsubseteq L^+$*

$$T^{-1}\Big[\mathcal{A}_{\mathfrak{B}_M}(L^-)\Big] \subseteq \mathcal{A}_{\mathfrak{B}_M}(L^+), \tag{4.13}$$

4. *and T acts trivially on $\mathcal{A}_{\mathfrak{B}_M}(L \cap M^+_K) \cap \mathcal{A}_{\mathfrak{B}_M}(L \cap M^-_K)$ for every K-admissible region L,*

then we call T a K-map.

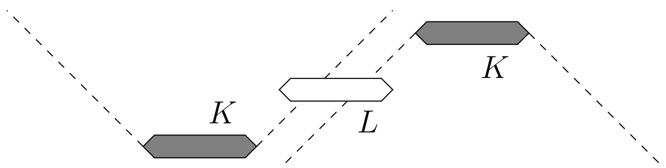


Figure 4.1: Example of a compact K with two connected components and a region L such that $L \not\subseteq M^\pm_K$. In particular, neither of $L \cap M^\pm_K$ is empty and a K-map is required to act trivially on any A in $\mathcal{A}_{\mathfrak{B}_M}(L \cap M^+_K) \cap \mathcal{A}_{\mathfrak{B}_M}(L \cap M^-_K)$.

Remark: Every K-hom/-map is also a K'-hom/-map for every compact $K' \subseteq M$ such that $K' \supseteq K$. Furthermore, if one assumes that $\mathcal{A}_{\mathfrak{B}_M}$ additionally fulfills outer regularity, see Def. 3.1.6, then point *4* follows from point *1*.

The motivation for introducing K-maps is based on the following result, which is a slight extension of Proposition 3.1 in [10].

Lemma 4.2.4 (K-scattering maps are K-maps)**.** *Let $\mathcal{U}_{\mathfrak{B}_M}$ be an AQFT that fulfills time-slice and let*

$$\Big(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_{(M,K')}}, \{\chi_L\}_{L\in\mathfrak{B}_{(M,K)}}, K\Big) \tag{4.14}$$

be a K-perturbed variant of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_{(M,K')}$ such that $\mathcal{P}_{\mathfrak{B}_{(M,K')}}$ fulfills time-slice. Then the resulting scattering map Θ is a K-map.

Proof. Points *1* and *2* in Def. 4.2.3 are covered by Proposition 3.1 in [10] and the proof of point *3* proceeds like the proof of point *2*. For point *4* take A in the intersection $\mathcal{U}_{\mathfrak{B}_M}(L\cap M_K^+)\cap\mathcal{U}_{\mathfrak{B}_M}(L\cap M_K^-)$. Then $\chi_{L\cap M_K^+}(A)=\chi_{M_K^+}(A)$, but also $\chi_{L\cap M_K^+}(A)=\chi_L(A)$, both by the commutative diagram of χ in Def. 4.1.1, and similarly $\chi_{L\cap M_K^-}(A)=\chi_{M_K^-}(A)$ and $\chi_{L\cap M_K^-}(A)=\chi_L(A)$. In particular $\chi_{M_K^+}(A)=\chi_L(A)=\chi_{M_K^-}(A)$ and hence $\Theta(A)=\big((\chi_{M_K^-})^{-1}\circ\chi_{M_K^+}\big)(A)=A$. □

Remark: Motivated by Proposition 3.1 in [10] and the analysis in [1], (completely positive) maps with the property point *2* of Def. 4.2.3 were investigated in [9]. There this specific property was termed "past-support non-increasing (psni)".

4.2.2 Properties of K-homs and K-maps

In this section we show the following results:

1. direct sums and tensor products of K-homs/-maps are K-homs/-maps,
2. every K-hom/-map for arbitrary K inside some region N restricts to a K-hom/-map of the restricted theory on $\mathfrak{B}_N$,
3. the set of all K-homs/-maps for arbitrary K is stable under *strictly causally ordered* compositions, i.e., for $K_1 < K_2$, where $\leq$ is some causal linear order, the composition of K_j-homs/-maps T_j, $T_1\circ T_2$, is a $K_1\cup K_2$-hom/-map, and
4. for spacelike separated K_1, K_2, any two K_j-homs T_j commute, i.e., $T_1\circ T_2 = T_2\circ T_1$.

Lemma 4.2.5. *Let $\mathcal{A}^1_{\mathfrak{B}_M}$ and $\mathcal{A}^2_{\mathfrak{B}_M}$ be AQFTs on $\mathfrak{B}_M$ and, for $j=1,2$ let $T_j:\mathcal{A}^{j,g}_{\mathfrak{B}_M}\to\mathcal{A}^{j,g}_{\mathfrak{B}_M}$ be a K-hom for some compact $K\subseteq M$. Then*

1. *$T_1\oplus T_2:\mathcal{A}^{1,g}_{\mathfrak{B}_M}\oplus\mathcal{A}^{2,g}_{\mathfrak{B}_M}\to\mathcal{A}^{1,g}_{\mathfrak{B}_M}\oplus\mathcal{A}^{2,g}_{\mathfrak{B}_M}$ is a K-hom on $\mathcal{A}^{1,g}_{\mathfrak{B}_M}\oplus\mathcal{A}^{2,g}_{\mathfrak{B}_M}$, and*
2. *$T_1\otimes T_2:\mathcal{A}^{1,g}_{\mathfrak{B}_M}\otimes\mathcal{A}^{2,g}_{\mathfrak{B}_M}\to\mathcal{A}^{1,g}_{\mathfrak{B}_M}\otimes\mathcal{A}^{2,g}_{\mathfrak{B}_M}$ is a K-hom on $\mathcal{A}^{1,g}_{\mathfrak{B}_M}\otimes\mathcal{A}^{2,g}_{\mathfrak{B}_M}$.*

If T_1, T_2 are K-maps, then $T_1\oplus T_2$ and $T_1\otimes T_2$ are K-maps as well.

Proof. Since T_1 and T_2 are unit-preserving $*$-homomorphisms, so are $T_1 \oplus T_2$ and $T_1 \otimes T_2$. Moreover, if T_1 and T_2 are unit-preserving $*$-automorphisms, so are $T_1 \oplus T_2$ and $T_1 \otimes T_2$. It then suffices to check the four conditions of Def. 4.2.3 for elements of the form $v_1 \oplus v_2$ and $v_1 \otimes v_2$ respectively. But since $T_1 \oplus T_2(v_1 \oplus v_2) = T_1 v_1 \oplus T_2 v_2$ and $T_1 \otimes T_2(v_1 \otimes v_2) = T_1 v_1 \otimes T_2 v_2$, the desired properties immediately follow from the properties of T_1 and T_2. □

Lemma 4.2.6. *Let $\mathcal{A}_{\mathfrak{B}_M}$ be an AQFT on $\mathfrak{B}_M$ fulfilling time-slice, let $N \subseteq M$ be a region and let $\mathcal{A}_{\mathfrak{B}_N}$ be the restricted AQFT. Then, for every compact $K \subseteq N$, every K-hom/-map $T : \mathcal{A}^g_{\mathfrak{B}_M} \to \mathcal{A}^g_{\mathfrak{B}_M}$ restricts to a K-hom/-map $T_N : \mathcal{A}^g_{\mathfrak{B}_N} \to \mathcal{A}^g_{\mathfrak{B}_N}$ of $\mathcal{A}_{\mathfrak{B}_N}$.*

Proof. It suffices to show that $T[\mathcal{A}_{\mathfrak{B}_M}(N)] \subseteq \mathcal{A}_{\mathfrak{B}_M}(N)$ in general and that equality holds for a K-map.

Since $K \subseteq N$ is compact, it is easy to see that $N \subseteq D_M(N \setminus J^\pm_M(K))$, hence, by local time-slice and isotony, $\mathcal{A}_{\mathfrak{B}_M}(N) \subseteq \mathcal{A}_{\mathfrak{B}_M}(N \setminus J^\pm_M(K)) \subseteq \mathcal{A}_{\mathfrak{B}_M}(N)$, so $\mathcal{A}_{\mathfrak{B}_M}(N) = \mathcal{A}_{\mathfrak{B}_M}(N \setminus J^\pm_M(K))$. By point *2* in Def. 4.2.3, $T\Big[\mathcal{A}_{\mathfrak{B}_M}(N \setminus J^-_M(K))\Big] \subseteq \mathcal{A}_{\mathfrak{B}_M}(N \setminus J^+_M(K))$.

If T is a K-map, then by point *3* in Def. 4.2.3, $T\Big[\mathcal{A}_{\mathfrak{B}_M}(N \setminus J^-_M(K))\Big] = \mathcal{A}_{\mathfrak{B}_M}(N \setminus J^+_M(K))$. □

Next, we show stability under strictly causally ordered compositions.

Lemma 4.2.7. *Let $K_1, K_2 \subseteq M$ be two causally orderable sets such that $K_1 < K_2$ for an admissible causal linear order $\leq$. Let T_j be a K_j-hom/-map, for $j = 1, 2$, of some AQFT $\mathcal{A}_{\mathfrak{B}_M}$ fulfilling time-slice. Then $T_1 \circ T_2$ is a $K := K_1 \cup K_2$-hom/-map.*

Remark: The case $K_1 = K_2$ is excluded here. In fact, for a K-map T and $n \in \mathbb{Z}$, in general, it cannot be expected that T^n is a K-map unless $n = 0, 1$.

Proof. We need to show that $T_1 \circ T_2$ fulfills the properties of Definition 4.2.3.

1. Let L be a region fully contained in $(K_1 \cup K_2)^{\perp_M} = K_1^{\perp_M} \cap K_2^{\perp_M}$. In particular $T_1 \restriction \mathcal{A}_{\mathfrak{B}_M}(L) = T_2 \restriction \mathcal{A}_{\mathfrak{B}_M}(L) = \mathrm{id}$.
2. Let $L^\pm \subseteq M^\pm_K$ such that $L^+ \subseteq D_M(L^-)$. In particular $D_M(L^-) \setminus J^-_M(K) \supseteq L^+$. By Lemma D.2.1, we have that

$$D_M\Big(D_M(L^-) \setminus (J^-_M(K_1) \cup J^+_M(K_2))\Big) \setminus J^-_M(K) = D_M(L^-) \setminus J^-_M(K), \quad (4.15)$$

hence $L^+ \subseteq D_M(D_M(L^-)\setminus(J_M^-(K_1)\cup J_M^+(K_2)))$. Then, by the properties of T_2, $T_2[\mathcal{A}_{\mathfrak{B}_M}(L^+)] \subseteq \mathcal{A}_{\mathfrak{B}_M}(D_M(L^-)\setminus(J_M^-(K_1)\cup J_M^+(K_2)))$, and by the properties of T_1, we see that

$$T_1\circ T_2\Big[\mathcal{A}_{\mathfrak{B}_M}(L^+)\Big] \subseteq T_1\Big[\mathcal{A}_{\mathfrak{B}_M}(D_M(L^-)\setminus(J_M^-(K_1)\cup J_M^+(K_2)))\Big] \subseteq \mathcal{A}_{\mathfrak{B}_M}(L^-), \tag{4.16}$$

since $L^- \subseteq M_{K_1}^-$, $D_M(L^-)\setminus(J_M^-(K_1)\cup J_M^+(K_2)) \subseteq M_{K_1}^+$ and the latter is obviously contained in $D_M(L^-)$.

If we are dealing with K-maps we need to check two more conditions.

3. Let $L^\pm \subseteq M_K^\pm$ such that $L^- \subseteq D_M(L^+)$. According to Lemma D.2.1, we have

$$D_M\Big(D_M(L^+)\setminus(J_M^-(K_1)\cup J_M^+(K_2))\Big)\setminus J_M^+(K) = D_M(L^+)\setminus J_M^+(K). \tag{4.17}$$

Then, by the properties of T_1, we have that $T_1^{-1}[\mathcal{A}_{\mathfrak{B}_M}(L^-)] \subseteq \mathcal{A}_{\mathfrak{B}_M}(D_M(L^+)\setminus(J_M^-(K_1)\cup J_M^+(K_2)))$, and by the properties of T_2, that

$$\begin{aligned}(T_1\circ T_2)^{-1}\Big[\mathcal{A}_{\mathfrak{B}_M}(L^-)\Big] &\subseteq T_2^{-1}\Big[\mathcal{A}_{\mathfrak{B}_M}(D_M(L^+)\setminus(J_M^-(K_1)\cup J_M^+(K_2)))\Big]\\ &\subseteq \mathcal{A}_{\mathfrak{B}_M}(L^+).\end{aligned} \tag{4.18}$$

4. Let L be a K-admissible region. Then we note that $M_K^\pm \subseteq M_{K_j}^\pm$. In particular, by isotony, $\mathcal{A}_{\mathfrak{B}_M}(L\cap M_K^\pm) \subseteq \mathcal{A}_{\mathfrak{B}_M}(L\cap M_{K_j}^\pm)$, so $\mathcal{A}_{\mathfrak{B}_M}(L\cap M_K^+)\cap\mathcal{A}_{\mathfrak{B}_M}(L\cap M_K^-) \subseteq \mathcal{A}_{\mathfrak{B}_M}(L\cap M_{K_j}^+)\cap\mathcal{A}_{\mathfrak{B}_M}(L\cap M_{K_j}^-)$ for $j=1,2$. But, by the properties of T_j, this means that T_j acts trivially on $\mathcal{A}_{\mathfrak{B}_M}(L\cap M_K^+)\cap\mathcal{A}_{\mathfrak{B}_M}(L\cap M_K^-)$, and hence also $T_1\circ T_2$ acts trivially here.

□

Finally, we show commutativity of K_1- and K_2-homs for $K_1 \perp_M K_2$.

Theorem 4.2.8. *Let $K_1, K_2 \subseteq M$ be two spacelike separated compact sets. Let $\mathcal{A}_{\mathfrak{B}_M}$ be an AQFT fulfilling time-slice and additivity and for $j=1,2$ let T_j be a K_j-hom. Then*

$$T_1\circ T_2 = T_2\circ T_1. \tag{4.19}$$

Proof. We first note that $D_M(K_1^{\perp_M}\cup K_2^{\perp_M}) = M$. By local time-slice and additivity of $\mathcal{A}_{\mathfrak{B}_M}$ we then see that $\mathcal{A}_{\mathfrak{B}_M}^g = \mathcal{A}_{\mathfrak{B}_M}(K_1^{\perp_M})\vee\mathcal{A}_{\mathfrak{B}_M}(K_2^{\perp_M})$. By definition T_j acts trivially on $\mathcal{A}_{\mathfrak{B}_M}(K_j^{\perp_M})$, and by the fact that $K_2 \subseteq K_1^{\perp_M}$ and vice versa, it follows from the first line in the proof of Lemma 4.2.6 and the fact that T_j acts trivially on $\mathcal{A}_{\mathfrak{B}_M}(K_j^{\perp_M})$ that

$$\begin{aligned} T_1 \circ T_2 \restriction \mathcal{A}_{\mathfrak{B}_M}(K_2^{\perp_M}) &= T_1 \restriction \mathcal{A}_{\mathfrak{B}_M}(K_2^{\perp_M}) \quad T_1 \circ T_2 \restriction \mathcal{A}_{\mathfrak{B}_M}(K_1^{\perp_M}) = T_2 \restriction \mathcal{A}_{\mathfrak{B}_M}(K_1^{\perp_M}) \\ T_2 \circ T_1 \restriction \mathcal{A}_{\mathfrak{B}_M}(K_2^{\perp_M}) &= T_1 \restriction \mathcal{A}_{\mathfrak{B}_M}(K_2^{\perp_M}) \quad T_2 \circ T_1 \restriction \mathcal{A}_{\mathfrak{B}_M}(K_1^{\perp_M}) = T_2 \restriction \mathcal{A}_{\mathfrak{B}_M}(K_1^{\perp_M}), \end{aligned} \tag{4.20}$$

so $T_1 \circ T_2$ and $T_2 \circ T_1$ agree on $\mathcal{A}_{\mathfrak{B}_M}(K_1^{\perp_M})$ *and* $\mathcal{A}_{\mathfrak{B}_M}(K_2^{\perp_M})$ and, because they are homomorphisms, they also agree on $\mathcal{A}_{\mathfrak{B}_M}(K_1^{\perp_M}) \vee \mathcal{A}_{\mathfrak{B}_M}(K_2^{\perp_M})$. □

This result generalises as follows.

Theorem 4.2.9. *Let $\mathfrak{K} := \{K_1, ..., K_n\}$ be a set of n distinct compact and causally orderable subsets of M and for every $j = 1, ..., n$ let T_j be a K_j-hom/-map on an AQFT $\mathcal{A}_{\mathfrak{B}_M}$ fulfilling time-slice and additivity. Then there exists a $\left(\bigcup_{j=1}^{n} K_j\right) =: K$-hom/-map $T_{\mathfrak{K}}$ such that for any admissible causal linear order $\trianglelefteq$ and any permutation π with $K_{\pi(1)} \triangleleft K_{\pi(2)} \triangleleft \cdots \triangleleft K_{\pi(n)}$ it holds that*

$$T_{\mathfrak{K}} = T_{\pi(1)} \circ T_{\pi(2)} \circ ... \circ T_{\pi(n)}. \tag{4.21}$$

Proof. After possible relabelling assume that $(K_1, K_2, ..., K_n)$ is strictly causally ordered and set $T_{\mathfrak{K}} := T_1 \circ T_2 \circ \cdots \circ T_n$, which is a K-hom/-map by Lemma 4.2.7. Let $\trianglelefteq$ be a (possibly different) causal linear order. Then note that

$$(i \leq j \wedge K_j \trianglelefteq K_i) \implies K_i \perp K_j \implies T_{K_i} \circ T_{K_j} = T_{K_j} \circ T_{K_i} \tag{4.22}$$

by Theorem 4.2.8. Then the result follows from Lemma 3.3.5. □

4.3 CAUSAL FACTORISATION

Given a K-perturbed variant where K is the disjoint union of two causally orderable sets K_1 and K_2, one would expect that this K-perturbed variant can be decomposed into a K_1- and a K_2-perturbed variant. And similarly, given a K_1- and a K_2-perturbed variant for causally orderable K_1 and K_2, one would expect that they may be combined to a $K := K_1 \cup K_2$-perturbed variant. Moreover, in case either the decomposition or the composition requires a choice of a causal linear order on $\{K_1, K_2\}$, the result needs to be independent of the choice (if there is more than one). And finally, one would expect that either procedure generalises to a collection of finitely many causally orderable perturbation zones.

The purpose of the present section is precisely to establish rigorous results about the decomposition and combination of K-perturbed variants. (For the sake of simplicity, for all K-perturbed variants in this section we will consider the case where, in notation of Definition 4.1.1, $K' = \emptyset$.)

4.3.1 Decomposition

Lemma 4.3.1 (Bipartite decomposition)**.** *Let*

$$\Big(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_M}, \{\chi_L\}_{L\in\mathfrak{B}_{(M,K)}}, K\Big), \tag{4.23}$$

be a K-perturbed variant of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_M$ with scattering map Θ such that both $\mathcal{U}_{\mathfrak{B}_M}$ and $\mathcal{P}_{\mathfrak{B}_M}$ fulfill time-slice and additivity and such that $\mathcal{U}_{\mathfrak{B}_M}$ fulfills outer regularity.

Let $K = K_1 \cup K_2$, where $K_1, K_2 \subseteq M$ are compact, distinct and causally orderable. Then there exist

1. *a K_1-perturbed variant $\Big(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}^1_{\mathfrak{B}_M}, \{\chi^1_L\}_{L\in\mathfrak{B}_{(M,K_1)}}, K_1\Big)$ of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_M$ with scattering map Θ_1, and*
2. *a K_2-perturbed variant $\Big(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}^2_{\mathfrak{B}_M}, \{\chi^2_L\}_{L\in\mathfrak{B}_{(M,K_2)}}, K_2\Big)$ of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_M$ with scattering map Θ_2,*

such that both $\mathcal{P}^1_{\mathfrak{B}_M}$ and $\mathcal{P}^2_{\mathfrak{B}_M}$ fulfill time-slice and additivity. In addition

$$\Theta = \begin{cases} \Theta_1 \circ \Theta_2 & \text{for } K_2 \not\leq K_1, \\ \Theta_2 \circ \Theta_1 & \text{for } K_1 \not\leq K_2. \end{cases} \tag{4.24}$$

Proof. Without loss of generality let us assume that $K_1 \leq K_2$ with respect to a causal linear order $\leq$.

Then we can define the following two couples of associations

1. $\mathcal{P}_{\mathfrak{B}_{M^-_{K_2}}}$ and $\tilde{\mathcal{P}}^1_{\mathfrak{B}_{M_{K_1^+}}} := \chi_{M^+_{K_1}\cap M^-_{K_2}}\Big[\mathcal{U}_{M_{K_1^+}}\Big]$, and
2. $\tilde{\mathcal{P}}^2_{\mathfrak{B}_{M_{K_2^-}}} := \chi_{M^+_{K_1}\cap M^-_{K_2}}\Big[\mathcal{U}_{M_{K_2^-}}\Big]$ and $\mathcal{P}_{\mathfrak{B}_{M^+_{K_1}}}$.

Each of these couples is a couple of two AQFTs $\mathcal{A}_{\mathfrak{B}_{M_1}}$ and $\mathcal{B}_{\mathfrak{B}_{M_2}}$ that coincide on $\mathfrak{B}_{M_1\cap M_2}$ and fulfill time-slice and additivity for appropriately labelled regions $M_1, M_2 \subseteq M$ fulfilling

1. $M_1 \cup M_2 = M$,

2. $M = D_M(M_1 \cap M_2)$,

3. $J^+_M(M_1 \cap M_2) \subseteq M_2$ and $J^-_M(M_1 \cap M_2) \subseteq M_1$.

Then, by Lemma 3.1.8, for each couple there is a glued AQFT, denoted by $\mathcal{P}^1_{\mathfrak{B}_M}$ and $\mathcal{P}^2_{\mathfrak{B}_M}$ respectively, fulfilling time-slice and additivity.

Now we proceed as follows.

1. For $L \in \mathfrak{B}_{(M,K_1)}$ with $L \subseteq M^-_{K_2}$ define

$$\chi^1_L := \chi_L : \mathcal{U}_{\mathfrak{B}_M}(L) \to \mathcal{P}_{\mathfrak{B}_M}(L) = \mathcal{P}^1_{\mathfrak{B}_M}(L), \tag{4.25}$$

and for $L \in \mathfrak{B}_{(M,K_1)}$ with $L \subseteq M^+_{K_1}$ we define

$$\chi^1_L := \chi_{M^+_{K_1} \cap M^-_{K_2}} \restriction \mathcal{U}_{\mathfrak{B}_M}(L) : \mathcal{U}_{\mathfrak{B}_M}(L) \to \chi_{M^+_{K_1} \cap M^-_{K_2}}[\mathcal{U}_{\mathfrak{B}_M}(L)] = \mathcal{P}^1_{\mathfrak{B}_M}(L). \tag{4.26}$$

We first we note that this is well-defined, i.e., for K_1-admissible region $L \subseteq M^+_{K_1} \cap M^-_{K_2}$, we have that $\chi_L = \chi_{M^+_{K_1} \cap M^-_{K_2}} \restriction \mathcal{U}_{\mathfrak{B}_M}(L)$, see the commutative diagram in Def. 4.1.1. Furthermore, it follows from the respective property of χ that for K_1-admissible regions L_1, L_2 with $L_1, L_2 \subseteq M^-_{K_2}$ *or* $L_1, L_2 \subseteq M^+_{K_1}$ the following diagram commutes.

$$\begin{array}{ccc} \mathcal{U}_{\mathfrak{B}_M}(L_1) & \xrightarrow{\chi^1_{L_1}} & \mathcal{P}^1_{\mathfrak{B}_M}(L_1) \\ \big\downarrow & & \big\downarrow \\ \mathcal{U}_{\mathfrak{B}_M}(L_2) & \xrightarrow{\chi^1_{L_2}} & \mathcal{P}^1_{\mathfrak{B}_M}(L_2) \end{array}$$

Then for any K_1-admissible region L it follows that $\chi^1_{L\cap M^+_{K_1}}$ and $\chi^1_{L\cap M^-_{K_2}}$ coincide on $\mathcal{U}_{\mathfrak{B}_M}(L \cap M^+_{K_1} \cap M^-_{K_2})$, which equals $\mathcal{U}_{\mathfrak{B}_M}(L \cap M^+_{K_1}) \cap \mathcal{U}_{\mathfrak{B}_M}(L \cap M^-_{K_2})$ by outer regularity of $\mathcal{U}_{\mathfrak{B}_M}$. Furthermore, note that by additivity of $\mathcal{U}_{\mathfrak{B}_M}$, we have

$$\mathcal{U}_{\mathfrak{B}_M}(L) = \mathcal{U}_{\mathfrak{B}_M}(L \cap M^+_{K_1}) \vee \mathcal{U}_{\mathfrak{B}_M}(L \cap M^-_{K_2}), \tag{4.27}$$

since $M = M^+_{K_1} \cup M^-_{K_2}$. Hence, also using additivity of $\mathcal{P}^1_{\mathfrak{B}_M}$, χ^1_L can be defined in terms of $\chi^1_{L\cap M^+_{K_1}}$ and $\chi^1_{L\cap M^-_{K_2}}$. In summary, $\left(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}^1_{\mathfrak{B}_M}, \{\chi^1_L\}_{L\in\mathfrak{B}_{(M,K_1)}}, K_1\right)$ is a K_1-perturbed variant of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_M$ with scattering map

$$\Theta_1 := \left(\chi^1_{M^-_{K_1}}\right)^{-1} \circ \chi^1_{M^+_{K_1}} = \left(\chi_{M^-_{K_1 \cup K_2}}\right)^{-1} \circ \chi_{M^+_{K_1} \cap M^-_{K_2}}, \tag{4.28}$$

which is a K_1-map according to Lemma 4.2.4.

2. For $L \in \mathfrak{B}_{(M,K_2)}$ with $L \subseteq M^+_{K_1}$ we define

$$\chi^2_L := \chi_L : \mathcal{U}_{\mathfrak{B}_M}(L) \to \mathcal{P}_{\mathfrak{B}_M}(L) = \mathcal{P}^2_{\mathfrak{B}_M}(L), \tag{4.29}$$

and for $L \in \mathfrak{B}_{(M,K_2)}$ with $L \subseteq M^-_{K_2}$ we define

$$\chi^2_L := \chi_{M^+_{K_1} \cap M^-_{K_2}} \restriction \mathcal{U}_{\mathfrak{B}_M}(L) : \mathcal{U}_{\mathfrak{B}_M}(L) \to \chi_{M^+_{K_1} \cap M^-_{K_2}}[\mathcal{U}_{\mathfrak{B}_M}(L)] = \mathcal{P}^2_{\mathfrak{B}_M}(L). \tag{4.30}$$

In complete analogy one then shows that indeed $\left(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}^2_{\mathfrak{B}_M}, \{\chi^2_L\}_{L \in \mathfrak{B}_{(M,K_2)}}, K_2\right)$ is a K_2-perturbed variant of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_M$ with scattering map

$$\Theta_2 := \left(\chi^2_{M^-_{K_2}}\right)^{-1} \circ \chi^2_{M^+_{K_2}} = \left(\chi_{M^+_{K_1} \cap M^-_{K_2}}\right)^{-1} \circ \chi_{M^+_{K_1 \cup K_2}}, \tag{4.31}$$

which is a K_2-map according to Lemma 4.2.4.

We then immediately see that Θ, defined by $\left(\chi_{M^-_K}\right)^{-1} \circ \chi_{M^+_K}$, equals $\Theta_1 \circ \Theta_2$. Then, for $K_1 \perp_M K_2$, it follows from Theorem 4.2.8, that also $\Theta = \Theta_2 \circ \Theta_1$, which finishes the proof. □

The result of Lemma 4.3.1 generalises to the multipartite case as follows.

Theorem 4.3.2 (Multipartite decomposition)**.** *Let*

$$\left(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_M}, \{\chi_L\}_{L \in \mathfrak{B}_{(M,K)}}, K\right), \tag{4.32}$$

be a K-perturbed variant of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_M$ with scattering map Θ such that both $\mathcal{U}_{\mathfrak{B}_M}$ and $\mathcal{P}_{\mathfrak{B}_M}$ fulfill time-slice and additivity and such that $\mathcal{U}_{\mathfrak{B}_M}$ fulfills outer regularity.

Let $\mathfrak{K} := \{K_1, ..., K_n\}$ be a set of n distinct, compact and causally orderable subsets of M such that $K = \bigcup_{j=1}^n K_j$. Then for every j there exist a K_j-perturbed variant

$$\left(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}^j_{\mathfrak{B}_M}, \left\{\chi^j_L\right\}_{L \in \mathfrak{B}_{(M,K_j)}}, K_j\right) \tag{4.33}$$

of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_M$ with scattering map Θ_j, where $\mathcal{P}^j_{\mathfrak{B}_M}$ fulfills time-slice and additivity, such that

$$\Theta = \Theta_{\pi(1)} \circ ... \circ \Theta_{\pi(n)}, \tag{4.34}$$

whenever the sequence $(\pi(1), ..., \pi(n))$ defines an admissible causal linear order on $\mathfrak{K}$.

Proof. We proceed by induction. The case $n = 1$ is trivial, the case $n = 2$ follows from Lemma 4.3.1.

Let us assume that we have shown the claim for $n - 1$ and now we want to show it for n. Let us fix a causal linear order $\leq$ and assume, after possible relabelling, that $K_1 \leq K_2 ... \leq K_n$. Then let us define $\tilde{K}_1 := \bigcup_{j=1}^{n-1} K_j$ and $\tilde{K}_2 := K_n$, which are causally orderable. Then, according to Lemma 4.3.1, there are $\tilde{K}_1$- and $\tilde{K}_2$-perturbed variants with scattering maps $\tilde{\Theta}_1$ and $\tilde{\Theta}_2 = \Theta_n$ such that $\Theta = \tilde{\Theta}_1 \circ \Theta_n$. By our induction hypothesis, there are then K_j-perturbed variants for $j = 1, ..., n-1$ with scattering maps Θ_j such that

$$\Theta = \Theta_1 \circ ... \circ \Theta_{n-1} \circ \Theta_n. \tag{4.35}$$

The result then follows from Theorem4.2.9. □

4.3.2 Composition

Having seen how a single perturbed variant may be decomposed, let us now discuss how a collection of perturbed variants with causally orderable perturbation zones may be combined to a single perturbed variant.

Lemma 4.3.3 (Bipartite combination)**.** *Let $K_1, K_2 \subseteq M$ be two causally orderable compact sets and let there be*

1. *a K_1-perturbed variant $\left(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}^1_{\mathfrak{B}_M}, \{\chi^1_L\}_{L\in\mathfrak{B}_{(M,K_1)}}, K_1\right)$ of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_M$ with scattering map Θ_1, and*
2. *a K_2-perturbed variant $\left(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}^2_{\mathfrak{B}_M}, \{\chi^2_L\}_{L\in\mathfrak{B}_{(M,K_2)}}, K_2\right)$ of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_M$ with scattering map Θ_2,*

such that all three $\mathcal{U}_{\mathfrak{B}_M}$, $\mathcal{P}^1_{\mathfrak{B}_M}$ and $\mathcal{P}^2_{\mathfrak{B}_M}$ fulfill time-slice and additivity. Let us define $K := K_1 \cup K_2$ and let us denote the causal hull of K by $\mathrm{ch}_M(K)$. Then there exists a $\mathrm{ch}_M(K)$-perturbed variant

$$\left(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_M}, \{\chi_L\}_{L\in\mathfrak{B}_{(M,\mathrm{ch}_M(K))}}, \mathrm{ch}_M(K)\right), \tag{4.36}$$

of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_M$ with scattering map Θ such that $\mathcal{P}_{\mathfrak{B}_M}$ fulfills time-slice and additivity and such that

$$\Theta = \begin{cases} \Theta_1 \circ \Theta_2 & \text{for } K_2 \not\leq K_1, \\ \Theta_2 \circ \Theta_1 & \text{for } K_1 \not\leq K_2. \end{cases} \tag{4.37}$$

It additionally $\mathcal{U}_{\mathfrak{B}_M}$ fulfills outer regularity, then there exists a K-perturbed variant

$$\left(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_M}, \{\chi_L\}_{L\in\mathfrak{B}_{(M,K)}}, K\right) \tag{4.38}$$

with scattering map also Θ.

For the proof of Lemma 4.3.3 it is somewhat convenient to work with perturbed variants in standard form, which we define now.

Definition 4.3.4. *A K-perturbed variant $\left(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_M}, \{\chi_L\}_{L\in\mathfrak{B}_{(M,K)}}, K\right)$ with scattering map Θ, where $\mathcal{U}_{\mathfrak{B}_M}$ and $\mathcal{P}_{\mathfrak{B}_M}$ fulfill additivity, is* in standard form, *if*

1. $\mathcal{U}^g_{\mathfrak{B}_M} = \mathcal{P}^g_{\mathfrak{B}_M}$, *and*
2. $\chi_{M^-_K} = \mathrm{id}$ *and* $\chi_{M^+_K} = \Theta$.

The following lemma shows that working with perturbed variants in standard form is not a restriction.

Lemma 4.3.5. *For every K-perturbed variant $\left(\mathcal{U}_{\mathfrak{B}_M}, \tilde{\mathcal{P}}_{\mathfrak{B}_M}, \{\tilde{\chi}_L\}_{L\in\mathfrak{B}_{(M,K)}}, K\right)$, where $\mathcal{U}_{\mathfrak{B}_M}$ and $\tilde{\mathcal{P}}_{\mathfrak{B}_M}$ fulfill additivity, there exists an equivalent K-perturbed variant $\left(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_M}, \{\chi_L\}_{L\in\mathfrak{B}_{(M,K)}}, K\right)$ in standard form.*

Remark: Recall that equivalent K-perturbed variants have identical scattering maps.

Proof. For every region $L \in \mathfrak{B}_M$ let us define $\mathcal{P}_{\mathfrak{B}_M}(L) := \tilde{\chi}^{-1}_{M^-_K}[\mathcal{U}_{\mathfrak{B}_M}(L)]$ and for every K-admissible region N let us define $\chi_N := \tilde{\chi}^{-1}_{M^-_K} \circ \tilde{\chi}_N$. □

We now give the proof of Lemma 4.3.3.

Proof of Lemma 4.3.3. Without loss of generality, let us assume that the two given perturbed variants are in standard form. Furthermore, after possible relabelling, let us assume that $K_1 \leq K_2$ with respect to a causal linear order $\leq$. Then let us look at the restriction $\mathcal{P}^1_{\mathfrak{B}_{M^-_{K_2}}}$ and let us define $\mathcal{P}_{\mathfrak{B}_{M^+_{K_1}}} := \Theta_1\left[\mathcal{P}^2_{\mathfrak{B}_{M^+_{K_1}}}\right]$. Note that their restrictions to $\mathfrak{B}_{M^-_{K_2}\cap M^+_{K_1}}$ are identical, since for every $N \in \mathfrak{B}_{M^-_{K_2}\cap M^+_{K_1}}$ we have that

$$\begin{aligned}\mathcal{P}^1_{\mathfrak{B}_{M^-_{K_2}}}(N) &= \chi^1_{M^+_{K_1}}[\mathcal{U}_{\mathfrak{B}_M}(N)] = \Theta_1[\mathcal{U}_{\mathfrak{B}_M}(N)] = \Theta_1 \circ \left(\chi^2_{M^-_{K_2}}\right)^{-1}\left[\mathcal{P}^2_{\mathfrak{B}_M}(N)\right] \\ &= \Theta_1\left[\mathcal{P}^2_{\mathfrak{B}_{M^+_{K_1}}}(N)\right] = \mathcal{P}_{\mathfrak{B}_{M^+_{K_1}}}(N).\end{aligned} \tag{4.39}$$

Then, by Lemma 3.1.8 (see also the proof of Lemma 4.3.1), there is a glued AQFT $\mathcal{P}_{\mathfrak{B}_M}$ fulfilling time-slice and additivity. Explicitly, we have for every region L that

$$\mathcal{P}_{\mathfrak{B}_M}(L) = \Theta_1\left[\mathcal{P}^2_{\mathfrak{B}_M}(L \cap M^+_{K_1})\right] \vee \mathcal{P}^1_{\mathfrak{B}_M}(L \cap M^-_{K_2}). \tag{4.40}$$

For $L \in \mathfrak{B}_{(M,K)}$ with $L \subseteq M^-_{K_2}$ define

$$\chi_L := \chi^1_L : \mathcal{U}_{\mathfrak{B}_M}(L) \to \mathcal{P}^1_{\mathfrak{B}_M}(L) = \mathcal{P}_{\mathfrak{B}_M}(L), \tag{4.41}$$

and for $L \in \mathfrak{B}_{(M,K)}$ with $L \subseteq M^+_{K_1}$ we define

$$\chi_L := \Theta_1 \circ \chi^2_L : \mathcal{U}_{\mathfrak{B}_M}(L) \to \Theta\Big[\mathcal{P}^2_{\mathfrak{B}_M}(L)\Big] = \mathcal{P}_{\mathfrak{B}_M}(L). \tag{4.42}$$

We first we note that this is well-defined, i.e., for K-admissible region $L \subseteq M^+_{K_1} \cap M^-_{K_2}$, we have that $\chi_L = \chi^1_L = \Theta_1 \restriction \mathcal{U}_{\mathfrak{B}_M}(L) = \Theta_1 \circ \mathrm{id} \restriction \mathcal{U}_{\mathfrak{B}_M}(L) = \Theta_1 \circ \chi^2_L$. Furthermore, it follows from the respective properties of χ^1 and χ^2 that for K-admissible regions L_1, L_2 with $L_1, L_2 \subseteq M^-_{K_2}$ *or* $L_1, L_2 \subseteq M^+_{K_1}$ the following diagram commutes.

$$\begin{array}{ccc} \mathcal{U}_{\mathfrak{B}_M}(L_1) & \xrightarrow{\chi^1_{L_1}} & \mathcal{P}^1_{\mathfrak{B}_M}(L_1) \\ \big\downarrow & & \big\downarrow \\ \mathcal{U}_{\mathfrak{B}_M}(L_2) & \xrightarrow{\chi^1_{L_2}} & \mathcal{P}^1_{\mathfrak{B}_M}(L_2) \end{array}$$

Furthermore, for *any* $\mathrm{ch}_M(K)$-admissible region L it follows that $L = \Big(L \cap M^+_K\Big) \cup (L \cap M^-_K)$. Now $\chi_{L\cap M^+_K}$ and $\chi_{L\cap M^-_K}$ coincide on $\mathcal{U}_{\mathfrak{B}_M}(L \cap M^+_K) \cap \mathcal{U}_{\mathfrak{B}_M}(L \cap M^-_K)$. To see that note that for $A \in \mathcal{U}_{\mathfrak{B}_M}(L \cap M^+_K) \cap \mathcal{U}_{\mathfrak{B}_M}(L \cap M^-_K)$ it holds that $\chi_{L\cap M^-_K}(A) = (A)$ and that $\chi_{L\cap M^+_K}(A) = \Theta_1 \circ \chi^2_{L\cap M^+_K}(A) = \Theta_1 \circ \chi^2_{L\cap M^+_{K_2}}(A) = \Theta_1 \circ \Theta_2(A)$. And by Lemma 4.2.7, $\Theta_1 \circ \Theta_2$ is a $K_1 \cup K_2$-map, so acts trivially on $A \in \mathcal{U}_{\mathfrak{B}_M}(L \cap M^+_K) \cap \mathcal{U}_{\mathfrak{B}_M}(L \cap M^-_K)$.

As a result, for *any* $\mathrm{ch}_M(K)$-admissible region L we can consistently define χ_L on

$$\mathcal{U}_{\mathfrak{B}_M}(L) = \mathcal{U}_{\mathfrak{B}_M}(L \cap M^+_K) \vee \mathcal{U}_{\mathfrak{B}_M}(L \cap M^-_K) \tag{4.43}$$

via $\chi_{L\cap M^+_K}$ and $\chi_{L\cap M^-_K}$, where we used additivity of $\mathcal{U}_{\mathfrak{B}_M}$ and $\mathcal{P}_{\mathfrak{B}_M}$. At this point we see that

$$\Big(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_M}, \{\chi_L\}_{L\in\mathfrak{B}_{(M,\mathrm{ch}_M(K))}}, \mathrm{ch}_M(K)\Big), \tag{4.44}$$

is a $\mathrm{ch}_M(K)$-perturbed variant of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_M$ such that $\mathcal{P}_{\mathfrak{B}_M}$ fulfills time-slice and additivity. The associated scattering map is given by $\Theta_1 \circ \Theta_2$.

Let us now take any K-admissible region L and let us assume that $\mathcal{U}_{\mathfrak{B}_M}$ fulfills outer regularity. Then $L = \Big(L \cap M^+_{K_1}\Big) \cup (L \cap M^-_{K_2})$ and

$$\mathcal{U}_{\mathfrak{B}_M}(L \cap M^+_{K_1}) \cap \mathcal{U}_{\mathfrak{B}_M}(L \cap M^-_{K_2}) = \mathcal{U}_{\mathfrak{B}_M}(L \cap M^+_{K_1} \cap M^-_{K_2}), \tag{4.45}$$

on which $\chi_{L\cap M^+_{K_1}}$ and $\chi_{L\cap M^+_{K_2}}$ agree. Hence we can consistently define χ_L on

$$\mathcal{U}_{\mathfrak{B}_M}(L) = \mathcal{U}_{\mathfrak{B}_M}(L \cap M^+_{K_1}) \vee \mathcal{U}_{\mathfrak{B}_M}(L \cap M^-_{K_2}) \tag{4.46}$$

via $\chi_{L\cap M^+_{K_1}}$ and $\chi_{L\cap M^+_{K_2}}$ using additivity of $\mathcal{U}_{\mathfrak{B}_M}$ and $\mathcal{P}_{\mathfrak{B}_M}$. In summary

$$\Big(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_M}, \{\chi_L\}_{L\in\mathfrak{B}_{(M,K)}}, K\Big) \tag{4.47}$$

is a K-perturbed variant with scattering map also Θ.

A final reference to Theorem 4.2.8 finishes the proof. □

Lemma 4.3.3 generalises to the multipartite case.

Theorem 4.3.6 (Multipartite combination)**.** *Let $\mathfrak{K} := \{K_1, ..., K_n\}$ be a set of n distinct, compact and causally orderable subsets of M and for every j let there be a K_j-perturbed variant*

$$\left(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}^j_{\mathfrak{B}_M}, \left\{\chi^j_L\right\}_{L\in\mathfrak{B}_{(M,K_j)}}, K_j\right) \tag{4.48}$$

of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_M$ with scattering map Θ_j, where $\mathcal{U}_{\mathfrak{B}_M}$ and $\mathcal{P}^j_{\mathfrak{B}_M}$ fulfill time-slice and additivity and where $\mathcal{U}_{\mathfrak{B}_M}$ fulfills outer regularity.

Then there exists a $K := \bigcup_{j=1}^n K_j$-perturbed variant

$$\left(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_M}, \{\chi_L\}_{L\in\mathfrak{B}_{(M,K)}}, K\right), \tag{4.49}$$

of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_M$ with scattering map Θ, where $\mathcal{P}_{\mathfrak{B}_M}$ fulfills time-slice and additivity, such that

$$\Theta = \Theta_{\pi(1)} \circ ... \circ \Theta_{\pi(n)}, \tag{4.50}$$

whenever the sequence $(\pi(1), ..., \pi(n))$ defines an admissible causal linear order on $\mathfrak{K}$.

Remark: Note that we assume outer regularity of $U_{\mathfrak{B}_M}$ from the start. A comparison with Lemma 4.3.3 suggests that this assumption could be dropped possibly resulting in a $\mathrm{ch}_M(K)$-perturbed variant instead of a K-perturbed variant.

Proof. We proceed by induction. The case $n = 1$ is trivial, the case $n = 2$ follows from Lemma 4.3.3.

Let us assume that we have shown the claim for $n - 1$ and now we want to show it for n. Let us fix a causal linear order $\leq$ and assume, after possible relabelling, that $K_1 \leq K_2 ... \leq K_n$. Then $\{K_1, ..., K_{n-1}\}$ is a set of $n - 1$ distinct, compact and causally orderable subsets of M. Hence, by our induction hypothesis, there exists a $\tilde{K}_1 := \bigcup_{j=1}^{n-1} K_j$-perturbed variant

$$\left(\mathcal{U}_{\mathfrak{B}_M}, \tilde{\mathcal{P}}^1_{\mathfrak{B}_M}, \{\chi_L\}_{L\in\mathfrak{B}_{(M,\tilde{K}_1)}}, \tilde{K}_1\right), \tag{4.51}$$

of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_M$ with scattering map $\tilde{\Theta}_1$, where $\tilde{\mathcal{P}}^1_{\mathfrak{B}_M}$ fulfills time-slice and additivity, and where

$$\tilde{\Theta}_1 = \Theta_1 \circ ... \circ \Theta_{n-1}. \tag{4.52}$$

Now note that $\left\{\tilde{K}_1, \tilde{K}_2\right\}$ is a set of two distinct, compact and causally orderable subsets of M. Hence the $\tilde{K}_1$-perturbed variant in Eq. (4.51) can be combined with the $K_n = \tilde{K}_2$-perturbed variant above according to Lemma 4.3.3 yielding a $K := \tilde{K}_1 \cup \tilde{K}_2 = \bigcup_{j=1}^{n}$-perturbed variant

$$\Big(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_M}, \{\chi_L\}_{L\in\mathfrak{B}_{(M,K)}}, K\Big), \tag{4.53}$$

of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_M$ with scattering map Θ, where $\mathcal{P}_{\mathfrak{B}_M}$ fulfills time-slice and additivity, such that

$$\Theta = \tilde{\Theta}_1 \circ \Theta_n = \Theta_1 \circ ... \circ \Theta_{n-1} \circ \Theta_n. \tag{4.54}$$

The result then follows from Theorem 4.2.9. □

4.4 INVERSE SCATTERING

In Definition 4.2.1 we have seen how a K-perturbed variant

$$\Big(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_{(M,K')}}, \{\chi_L\}_{L\in\mathfrak{B}_{(M,K)}}, K\Big) \tag{4.55}$$

on $\mathfrak{B}_{(M,K')}$, where $K' \subseteq K$ and both $\mathcal{U}_{\mathfrak{B}_M}$ and $\mathcal{P}_{\mathfrak{B}_{(M,K')}}$ fulfill time-slice, gives rise to a scattering map Θ on $\mathcal{U}_{\mathfrak{B}_M}$ via

$$\begin{aligned} &\Theta : \mathcal{U}^g_{\mathfrak{B}_M} \to \mathcal{U}^g_{\mathfrak{B}_M} \\ &\Theta := \Big(\chi_{M^-_K}\Big)^{-1} \circ \chi_{M^+_K}, \end{aligned} \tag{4.56}$$

where Θ is in particular a K-map. It is now an interesting question whether this procedure can be inverted, leading us to what may be called *inverse scattering.*

The task is, given an AQFT $\mathcal{U}_{\mathfrak{B}_M}$ fulfilling time-slice and additivity and a K-map T, to construct a $\tilde{K}$-perturbed variant

$$\Big(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}, \{\chi_L\}_{L\in\mathfrak{B}_{(M,\tilde{K})}}, \tilde{K}\Big), \tag{4.57}$$

for some $\tilde{K} \supseteq K$, whose scattering map equals T. That this is indeed possible is shown by the following theorem.

Theorem 4.4.1. *Let $\mathcal{U}_{\mathfrak{B}_M}$ be an AQFT fulfilling time-slice and additivity and let T be a K-map. Then, for every* causally convex *compact $\tilde{K}$ that contains K in its open interior, i.e., $K \subsetneq \mathring{\tilde{K}} \subsetneq \tilde{K}$, there exists a $\tilde{K}$-perturbed variant*

$$\Big(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}, \{\chi_L\}_{L\in\mathfrak{B}_{(M,\tilde{K})}}, \tilde{K}\Big), \tag{4.58}$$

of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_{(M,\tilde{K})}$, where $\mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}$ fulfills time-slice and additivity, whose scattering map equals T. Concretely we have for every $\tilde{K}$-admissible region $L \in \mathfrak{B}_{(M,\tilde{K})}$

$$\mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(L) := T\Big[\mathcal{U}_{\mathfrak{B}_M}(L \cap M^+_{\tilde{K}})\Big] \vee \mathcal{U}_{\mathfrak{B}_M}(L \cap M^-_{\tilde{K}}) \subseteq \mathcal{U}^g_{\mathfrak{B}_M}. \tag{4.59}$$

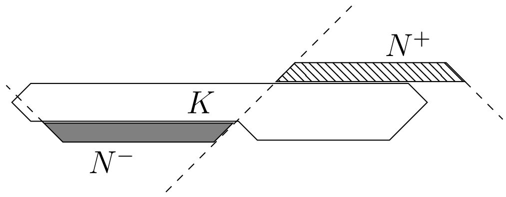


Figure 4.2: Pathological situation in which two spacelike separated regions N^-, N^+ *touch* the perturbation zone K, which motivates to consider an "enlargement" to $\tilde{K} \supsetneq K$.

A few comments are in order.

1. By Lemma 4.2.2 the $\tilde{K}$-perturbed variant above is unique up to equivalence.

2. The slight enlargement of K to $\tilde{K} \supsetneq K$ is necessary in order to show causally disjoint commutativity for (certain) spacelike separated regions (whose closure is) *touching* K, see Fig. 4.2. The existence of such a $\mathring{\tilde{K}}$ with compact closure is guaranteed by compact exhaustion.

3. In a nutshell one could say that by Lemma 4.2.4 every scattering map is a K-map and by the present theorem every K-map is a scattering map. One of the technical details omitted by this simplified statement is the fact that the present theorem "only" yields a perturbed variant on some $\mathfrak{B}_{(M,\tilde{K})}$. In particular, the original perturbation zone of the K-map is not covered by any region in $\mathfrak{B}_{(M,\tilde{K})}$. It is an interesting question whether, and if so, under what conditions the $\tilde{K}$-perturbed variant admits a (unique?) extension to all of $\mathfrak{B}_M$. In Lagrangean approaches such an extension can be achieved using Bogoliubov's formula, see for instance [50] and also [44].

Proof. We start by showing that $\mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}$ as defined in Eq. (4.59) is an AQFT. To that end we will consider T a $\tilde{K}$-map unless for the proof of causally disjoint commutativity.

We note that, due to causal convexity of $\tilde{K}$, for any $\tilde{K}$-admissible region N it follows that $N = \left(N \cap M^-_{\tilde{K}}\right) \cup \left(N \cap M^+_{\tilde{K}}\right)$. If furthermore N is spacelike separated from K, we have that $\mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(N) = \mathcal{U}_{\mathfrak{B}_M}(N)$ by the fact that T acts trivially on $\mathcal{U}_{\mathfrak{B}_M}(N)$.

1. Isotony follows by the same argument as the one in the proof of Lemma 3.1.8, and

2. additivity also follows from the proof of Lemma 3.1.8 by noting that $T[\mathcal{B}_1] \vee T[\mathcal{B}_2] = T[\mathcal{B}_1 \vee \mathcal{B}_2]$.

3. The proof of causally disjoint commutativity proceeds in two steps.

   a) Let us first take $L_1, L_2 \in \mathfrak{B}_{(M,\tilde{K})}$ such that $L_1 \perp_M L_2$ and $L_1 \subseteq M^-_{\tilde{K}}$ and $L_2 \subseteq M^+_{\tilde{K}}$. Then it is a consequence of Corollary D.3.2 that $\mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(L_2) = T[\mathcal{U}_{\mathfrak{B}_M}(L_2)]$ and $\mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(L_1) = \mathcal{U}_{\mathfrak{B}_M}(L_1)$ commute.

   b) To finish the proof of causally disjoint commutativity, let us consider $N_1, N_2 \in \mathfrak{B}_{(M,\tilde{K})}$ such that $N_1 \perp_M N_2$. Then each of $N_1 \cap M^{\pm}_{\tilde{K}}$ is spacelike separated from both $N_2 \cap M^{\pm}_{\tilde{K}}$. In particular $\mathcal{U}_{\mathfrak{B}_M}(N_1 \cap M^-_{\tilde{K}})$ commutes with $\mathcal{U}_{\mathfrak{B}_M}(N_2 \cap M^-_{\tilde{K}})$ and $T\Big[\mathcal{U}_{\mathfrak{B}_M}(N_1 \cap M^+_{\tilde{K}})\Big]$ commutes with $T\Big[\mathcal{U}_{\mathfrak{B}_M}(N_2 \cap M^+_{\tilde{K}})\Big]$. By setting $L_1 := N_1 \cap M^-_{\tilde{K}}$ and $L_2 := N_2 \cap M^+_{\tilde{K}}$ we see from the previous point that $T\Big[\mathcal{U}_{\mathfrak{B}_M}(N_2 \cap M^+_{\tilde{K}})\Big]$ commutes with $\mathcal{U}_{\mathfrak{B}_M}(N_1 \cap M^-_{\tilde{K}})$. Relabelling $N_1 \leftrightarrow N_2$ also shows that $T\Big[\mathcal{U}_{\mathfrak{B}_M}(N_1 \cap M^+_{\tilde{K}})\Big]$ commutes with $\mathcal{U}_{\mathfrak{B}_M}(N_2 \cap M^-_{\tilde{K}})$. Putting everything together finishes the argument.

4. Let us now turn to time-slice. Take $N_1, N_2 \in \mathfrak{B}_{(M,\tilde{K})}$, two $\tilde{K}$-admissible regions, such that and $N_1 \subseteq D_M(N_2)$. What we want to show is that

$$\begin{aligned} \mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(N_1) &= \mathcal{U}_{\mathfrak{B}_M}(N_1 \cap M^-_{\tilde{K}}) \vee T\Big[\mathcal{U}_{\mathfrak{B}_M}(N_1 \cap M^+_{\tilde{K}})\Big] \\ &\subseteq \mathcal{U}_{\mathfrak{B}_M}(N_2 \cap M^-_{\tilde{K}}) \vee T\Big[\mathcal{U}_{\mathfrak{B}_M}(N_2 \cap M^+_{\tilde{K}})\Big] \\ &= \mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(N_2). \end{aligned} \tag{4.60}$$

   Our strategy is to use additivity of $\mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}$. To that end define $N^{\pm}_j := N_j \cap M^{\pm}_{\tilde{K}}$, where clearly $N^{\pm}_1 \subseteq D_M(N_2)$. Then $\mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(N_1) = \mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(N^-_1) \vee \mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(N^+_1)$. In particular, it is sufficient to show that $\mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(N^{\pm}_1) \subseteq \mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(N_2)$. Hence, let us look at the case where $N_1 \subseteq M^-_{\tilde{K}}$ and then at the case where $N_1 \subseteq M^+_{\tilde{K}}$.

   a) Let us assume that $N_1 \subseteq M^-_{\tilde{K}}$. Let us define the $\tilde{K}$-admissible regions $L_{\pm} := D_M(N^{\pm}_2) \setminus J^+_M(\tilde{K})$ and $L := L_+ \cup L_-$. We observe that $N_2 = N^-_2 \cup N^+_2$, so $N_2 \subseteq D_M(N^-_2) \cup D_M(N^+_2)$ and $D_M(N_2) \subseteq D_M(D_M(N^-_2) \cup D_M(N^+_2))$. By assumption

$$N_1 \subseteq D_M(N_2) \setminus J^+_M(\tilde{K}) \subseteq D_M(D_M(N^-_2) \cup D_M(N^+_2)) \setminus J^+_M(\tilde{K}). \tag{4.61}$$

   According to Lemma D.3.3, it then follows that L is a $\tilde{K}$-admissible region contained in $M^-_{\tilde{K}}$ such that $N_1 \subseteq D_M(L)$.

   By definition, $\mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(N_1) = \mathcal{U}_{\mathfrak{B}_M}(N_1)$, and by additivity and time-slice of $\mathcal{U}$ we have

$$\begin{aligned} \mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(N_1) &= \mathcal{U}_{\mathfrak{B}_M}(N_1) \subseteq \mathcal{U}_{\mathfrak{B}_M}(D_M(L)) = \mathcal{U}_{\mathfrak{B}_M}(L) \\ &= \mathcal{U}_{\mathfrak{B}_M}(L_+ \cup L_-) = \mathcal{U}_{\mathfrak{B}_M}(L_-) \vee \mathcal{U}_{\mathfrak{B}_M}(L_+). \end{aligned} \tag{4.62}$$

What is left to show is that

$$\begin{aligned}\mathcal{U}_{\mathfrak{B}_M}(L_-) \vee \mathcal{U}_{\mathfrak{B}_M}(L_+) &\subseteq \mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(N_2) \\ &= \mathcal{U}_{\mathfrak{B}_M}(N_2^-) \vee T\Big[\mathcal{U}_{\mathfrak{B}_M}(N_2^+)\Big].\end{aligned} \tag{4.63}$$

By isotony and time-slice of $\mathcal{U}_{\mathfrak{B}_M}$ we have

$$\begin{aligned}\mathcal{U}_{\mathfrak{B}_M}(L_-) &= \mathcal{U}_{\mathfrak{B}_M}(D_M(N_2^-) \setminus J_M^+(\tilde{K})) \\ &\subseteq \mathcal{U}_{\mathfrak{B}_M}(D_M(N_2^-)) = \mathcal{U}_{\mathfrak{B}_M}(N_2^-),\end{aligned} \tag{4.64}$$

and, by the properties of T, that

$$\mathcal{U}_{\mathfrak{B}_M}(L_+) \subseteq T\Big[\mathcal{U}_{\mathfrak{B}_M}(N_2^+)\Big], \tag{4.65}$$

since $T^{-1}[\mathcal{U}_{\mathfrak{B}_M}(L_+)] \subseteq \mathcal{U}_{\mathfrak{B}_M}(N_2^+)$, as $L_+ \subseteq D_M(N_2^+)$ according to Definition 4.2.3. Hence $\mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(N_1) \subseteq \mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(N_2)$.

b) In complete analogy take $N_1 \subseteq M_{\tilde{K}}^+$ and define the $\tilde{K}$-admissible regions $L_\pm := D_M(N_2^\pm) \setminus J_M^-(\tilde{K})$ and $L := L_+ \cup L_-$. Again, just like before, according to Lemma D.3.3, L is a $\tilde{K}$-admissible region contained in $M_{\tilde{K}}^+$ such that $N_1 \subseteq D_M(L)$. By definition $\mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(N_1) = T[\mathcal{U}_{\mathfrak{B}_M}(N_1)]$ and by additivity and time-slice of $\mathcal{U}_{\mathfrak{B}_M}$, we have

$$\mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(N_1) = \Theta[\mathcal{U}_{\mathfrak{B}_M}(N_1)] \subseteq T[\mathcal{U}_{\mathfrak{B}_M}(L_-)] \vee T[\mathcal{U}_{\mathfrak{B}_M}(L_+)]. \tag{4.66}$$

Again, by time-slice of $\mathcal{U}_{\mathfrak{B}_M}$, we have that $\mathcal{U}_{\mathfrak{B}_M}(L_+) \subseteq \mathcal{U}_{\mathfrak{B}_M}(N_2^+)$, so in particular

$$T[\mathcal{U}_{\mathfrak{B}_M}(L_+)] \subseteq T\Big[\mathcal{U}_{\mathfrak{B}_M}(N_2^+)\Big]. \tag{4.67}$$

Moreover,

$$T[\mathcal{U}_{\mathfrak{B}_M}(L_-)] \subseteq \mathcal{U}_{\mathfrak{B}_M}(N_2^-), \tag{4.68}$$

as $L_- \subseteq D(N_2 \cap M^-)$ according to Definition 4.2.3. Summarised

$$\begin{aligned}\mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(N_1) &\subseteq T[\mathcal{U}_{\mathfrak{B}_M}(L_-)] \vee T[\mathcal{U}_{\mathfrak{B}_M}(L_+)] \\ &\subseteq \mathcal{U}_{\mathfrak{B}_M}(N_2^-) \vee T\Big[\mathcal{U}_{\mathfrak{B}_M}(N_2^+)\Big] = \mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(N_2).\end{aligned} \tag{4.69}$$

This proves the local time-slice property of $\mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}$.

For the following arguments see the proof of Lemma 4.3.1.
For $L \in \mathfrak{B}_{(M,\tilde{K})}$ with $L \subseteq M_{\tilde{K}}^-$ define

$$\chi_L : \mathcal{U}_{\mathfrak{B}_M}(L) \to \mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(L) = \mathcal{U}_{\mathfrak{B}_M}(L), \tag{4.70}$$

as the identity, and for $L \in \mathfrak{B}_{(M,\tilde{K})}$ with $L \subseteq M^+_{\tilde{K}}$ we define

$$\chi_L : \mathcal{U}_{\mathfrak{B}_M}(L) \to \mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}(L) = T[\mathcal{U}_{\mathfrak{B}_M}(L)] \tag{4.71}$$

to be $T \restriction \mathcal{U}_{\mathfrak{B}_M}(L)$. We first note that this is well-defined, i.e., for a $\tilde{K}$-admissible region $L \subseteq M^+_{\tilde{K}} \cap M^-_{\tilde{K}} = K^{\perp_M}$, we have that $T \restriction \mathcal{U}_{\mathfrak{B}_M}(L) = \mathrm{id} \restriction \mathcal{U}_{\mathfrak{B}_M}(L)$.

By additivity of $\mathcal{U}_{\mathfrak{B}_M}$, we now have that for any $\tilde{K}$-admissible region L we have that

$$\mathcal{U}_{\mathfrak{B}_M}(L) = \mathcal{U}_{\mathfrak{B}_M}(L \cap M^-_{\tilde{K}}) \vee \mathcal{U}_{\mathfrak{B}_M}(L \cap M^+_{\tilde{K}}). \tag{4.72}$$

Hence, using additivity of $\mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}$, χ_L may be defined in terms of $\chi_{L\cap M^-_{\tilde{K}}}$ and $\chi_{L\cap M^+_{\tilde{K}}}$, which is consistent, since

$$\chi_{L\cap M^-_{\tilde{K}}} \restriction \mathcal{U}_{\mathfrak{B}_M}(L \cap M^-_{\tilde{K}}) \cap \mathcal{U}_{\mathfrak{B}_M}(L \cap M^+_{\tilde{K}}) = \chi_{L\cap M^+_{\tilde{K}}} \restriction \mathcal{U}_{\mathfrak{B}_M}(L \cap M^-_{\tilde{K}}) \cap \mathcal{U}_{\mathfrak{B}_M}(L \cap M^+_{\tilde{K}}), \tag{4.73}$$

since T acts trivially on any $A \in \mathcal{U}_{\mathfrak{B}_M}(L \cap M^-_{\tilde{K}}) \cap \mathcal{U}_{\mathfrak{B}_M}(L \cap M^+_{\tilde{K}})$, see Definition 4.2.3.

This shows that

$$\left(\mathcal{U}_{\mathfrak{B}_M}, \mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}, \{\chi_L\}_{L\in\mathfrak{B}_{(M,\tilde{K})}}, \tilde{K}\right), \tag{4.74}$$

is a $\tilde{K}$-perturbed variant of $\mathcal{U}_{\mathfrak{B}_M}$ on $\mathfrak{B}_{(M,\tilde{K})}$, where $\mathcal{P}_{\mathfrak{B}_{(M,\tilde{K})}}$ fulfills time-slice and additivity. Finally, the associated scattering map is given by

$$\left(\chi_{M^-_{\tilde{K}}}\right)^{-1} \circ \chi_{M^+_{\tilde{K}}} = \mathrm{id} \circ T = T. \tag{4.75}$$

□

4.5 ctPFAs of causal quantum channels from K-homs

As we have seen in this chapter, K-maps play the important role of scattering automorphisms of AQFTs. As such, they should certainly be regarded as *physical* quantum channels, and hence be a part of a ctPFA of *causal* quantum channels.

In general, K-maps do not need to be *inner* automorphisms and, in particular, they do not need to be localisability-preserving, i.e., they are not contained in the ctPFA of causal quantum channels ${}^{\mathrm{lp}}\mathfrak{F}^{\mathrm{HK}}_{\mathfrak{B}_M}$ introduced earlier, see Theorem 3.4.3. However, K-maps *are* contained in the ctPFA of causal quantum channels we will define now. In fact, it even contains K-homs and quantum channels of the form $\mathcal{J}_{\sigma,T}$, introduced in Lemma 2.3.11, for a K-hom T.[2]

[2]The importance of the fact that the quantum channels $\mathcal{J}_{\sigma,T}$ are contained in a ctPFA of *causal* quantum channels will become clearer once we discuss the physical interpretation of such channels in Chapter 5.

Definition 4.5.1. *Let $\mathcal{S}_{\mathfrak{B}_M}$ be an AQFT that fulfills time-slice and additivity. Then, for every $N \in \mathfrak{B}_M$ let us define the (pointed) set $\mathfrak{F}_{\mathfrak{B}_M}(N)$ (with base-point* id : $\mathcal{S}^g_{\mathfrak{B}_M} \to \mathcal{S}^g_{\mathfrak{B}_M}$*) to comprise all maps*

$$\begin{aligned} &\mathcal{J}_{\sigma,T} : \mathcal{S}^g_{\mathfrak{B}_M} \to \mathcal{S}^g_{\mathfrak{B}_M} \\ &\mathcal{J}_{\sigma,T}(A) := \eta_\sigma\Big(T(A \otimes \mathbb{1}_{\mathcal{P}_{\mathfrak{B}_M}})\Big), \end{aligned} \tag{4.76}$$

where

1. *$\mathcal{P}_{\mathfrak{B}_M}$ is an AQFT fulfilling time-slice and additivity,*
2. *T is a K-hom of the AQFT $\mathcal{S}_{\mathfrak{B}_M} \otimes \mathcal{P}_{\mathfrak{B}_M}$ for some compact $K \subseteq N$, and*
3. *σ is some state on $\mathcal{P}^g_{\mathfrak{B}_M}$.*

It now turns out that, due to the properties of K-homs, $\mathfrak{F}_{\mathfrak{B}_M}$ together with the (appropriate versions of the) maps $\mathfrak{C}$ defined in Eq. (3.33), is a ctPFA of *causal* quantum channels, which is the content of the following theorem.

Theorem 4.5.2. *Let us take $\mathfrak{F}_{\mathfrak{B}_M}$ and $\mathcal{S}_{\mathfrak{B}_M}$ as in Definition 4.5.1. Then for every n-tuple of* causally orderable *regions $(N_1, ..., N_n)$ and every region N such that $N_j \subseteq N$ for every j, we define the map $\mathfrak{C}(N_1, ..., N_n; N) : \bigtimes_{j=1}^{n} \mathfrak{F}_{\mathfrak{B}_M}(N_j) \to \mathfrak{F}_{\mathfrak{B}_M}(N)$ via*

$$\mathfrak{C}(N_1, ..., N_n; N)(\mathcal{J}_{\sigma_1,T_1}, ..., \mathcal{J}_{\sigma_n,T_n}) := \mathcal{J}_{\sigma_{\pi(1)},T_{\pi(1)}} \circ ... \circ \mathcal{J}_{\sigma_{\pi(n)},T_{\pi(n)}}, \tag{4.77}$$

for some permutation π such that $N_{\pi(1)} < ... < N_{\pi(n)}$ for some causal linear order $\leq$. Then $\mathfrak{C}$ is well-defined and $\mathfrak{F}_{\mathfrak{B}_M}$ together with $\mathfrak{C}$ is a ctPFA of causal *quantum channels on $\mathcal{S}_{\mathfrak{B}_M}$.*

Before we turn to the proof we formulate and prove the following auxiliary lemma, see also Theorem 5 in [1].

Lemma 4.5.3. *Let $N_1, ..., N_n$ be a collection of causally ordered regions and let L be a region such that $L \not\lesssim N_j$ for every j. Furthermore, for every j let $K_j \subseteq N_j$ be compact and let T_j be a K_j-hom on some AQFT $\mathcal{A}_{\mathfrak{B}_M}$ that fulfills time-slice and additivity, and, if $N_j \perp L$, then*

$$T_1 \circ ... \circ T_j \circ ... \circ T_n \restriction \mathcal{A}_{\mathfrak{B}_M}(L) = T_1 \circ ... \circ T_{j-1} \circ T_{j+1} \circ ... \circ T_n \restriction \mathcal{A}_{\mathfrak{B}_M}(L). \tag{4.78}$$

Proof. By Theorem 4.2.9, $T_{j+1} \circ .. \circ T_n$ is a $\tilde{K} := K_{j+1} \cup ... \cup K_n$-hom. Moreover, $\tilde{K}$ is contained in $\tilde{N} := \mathrm{ch}(N_{j+1} \cup ... \cup N_n)$ and $N_1, ..., N_j, \tilde{N}$ is clearly causally ordered. Then, by compact exhaustion of $\tilde{N}$, there exists a compact K in $\tilde{N}$ such that $\tilde{K} \subsetneq \mathring{K} \subsetneq K$. Furthermore, it follows that $N_j \subseteq M_K^-$ and $L \subseteq M_K^+$. By Corollary D.3.2, it follows that $(T_{j+1} \circ .. \circ T_n)[\mathcal{A}_{\mathfrak{B}_M}(L)]$ is localisable at spacelike separation from N_j, and hence also at spacelike separation form K_j. Hence $T_j \circ ... \circ T_n \restriction \mathcal{A}_{\mathfrak{B}_M}(L) = T_{j+1} \circ ... \circ T_n \restriction \mathcal{A}_{\mathfrak{B}_M}(L)$ and the assertion follows. □

Let us now turn to the proof of Theorem 4.5.2.

Proof of Theorem 4.5.2. We first note that each $\mathcal{J}_{\sigma,T}$ is a quantum channel according to Lemma 2.3.11. We now show that $\mathfrak{F}_{\mathfrak{B}_M}(N)$ is indeed a convex set.[3] To that end take $\alpha \in [0,1]$ and $\mathcal{J}_{\sigma_j,T_j} \in \mathfrak{F}_{\mathfrak{B}_M}(N)$ for $j = 1,2$. Then we define $\mathcal{P}_{\mathfrak{B}_M} := {}^1\mathcal{P}_{\mathfrak{B}_M} \oplus {}^2\mathcal{P}_{\mathfrak{B}_M}$ and $\sigma := \alpha\sigma_1 \oplus (1-\alpha)\sigma_2$, which is a state on $\mathcal{P}_{\mathfrak{B}_M}$. Then we define $T : \mathcal{S}_{\mathfrak{B}_M} \otimes \mathcal{P}_{\mathfrak{B}_M} \to \mathcal{S}_{\mathfrak{B}_M} \otimes \mathcal{P}_{\mathfrak{B}_M}$, using the identification of $\mathcal{S}_{\mathfrak{B}_M} \otimes \mathcal{P}_{\mathfrak{B}_M}$ with $\left(\mathcal{S}_{\mathfrak{B}_M} \otimes {}^1\mathcal{P}_{\mathfrak{B}_M}\right) \oplus \left(\mathcal{S}_{\mathfrak{B}_M} \otimes {}^2\mathcal{P}_{\mathfrak{B}_M}\right)$, as $T_1 \oplus T_2$, which is a $K_1 \cup K_2$-hom according to Lemma 4.2.5. Then it is easy to check that

$$\alpha\mathcal{J}_{\sigma_1,T_1} + (1-\alpha)\mathcal{J}_{\sigma_2,T_2} = \mathcal{J}_{\sigma,T}. \tag{4.79}$$

We continue with the proof.

1. Let $N_1 \subseteq N_2$. Then it is easy to see that $\mathfrak{F}_{\mathfrak{B}_M}(N_1) \subseteq \mathfrak{F}_{\mathfrak{B}_M}(N_2)$.

2. Let $n \in \mathbb{N}^*$ and let $(N_1, ..., N_n)$ be an n-tuple of *causally orderable* regions and let N be a region such that $N_j \subseteq N$ for every j. Let us then take $\mathcal{J}_{\sigma_j,T_j} \in \mathfrak{F}_{\mathfrak{B}_M}(N_j)$ for $j = 1, ..., n$ and define $\mathcal{P}_{\mathfrak{B}_M} := {}^1\mathcal{P}_{\mathfrak{B}_M} \otimes ... \otimes {}^n\mathcal{P}_{\mathfrak{B}_M}$. Then define

$$\mathbb{F}_j : \mathcal{S}^g_{\mathfrak{B}_M} \otimes \mathcal{P}^g_{\mathfrak{B}_M} \to \mathcal{S}^g_{\mathfrak{B}_M} \otimes {}^j\mathcal{P}^g_{\mathfrak{B}_M} \otimes {}^1\mathcal{P}^g_{\mathfrak{B}_M} \otimes ... \otimes {}^{j-1}\mathcal{P}^g_{\mathfrak{B}_M} \otimes {}^{j+1}\mathcal{P}^g_{\mathfrak{B}_M} \otimes ... \otimes {}^n\mathcal{P}^g_{\mathfrak{B}_M} \tag{4.80}$$

as the canonical isomorphism and set $\sigma := \sigma_1 \otimes ... \otimes \sigma_n$ and

$$\hat{T}_j := (\mathbb{F}_j)^{-1} \circ \left(T_j \otimes \mathrm{id}_{{}^1\mathcal{P}^g_{\mathfrak{B}_M} \otimes ... \otimes {}^{j-1}\mathcal{P}^g_{\mathfrak{B}_M} \otimes {}^{j+1}\mathcal{P}^g_{\mathfrak{B}_M} \otimes ... \otimes {}^n\mathcal{P}^g_{\mathfrak{B}_M}}\right) \circ \mathbb{F}_j. \tag{4.81}$$

It is easy to see that $\hat{T}_j$ is a K_j-hom for every j. Let now π be *any* permutation of $(1, ..., n)$ and set

$$T^\pi := \hat{T}_{\pi(1)} \circ ... \circ \hat{T}_{\pi(n)}. \tag{4.82}$$

[3] I thank R. F. Werner for emphasising this fact.

Then, for effects $E_j \in {}^j\mathcal{P}^g_{\mathfrak{B}_M}$ and $E := E_1 \otimes ... \otimes E_n$ an explicit calculations shows that

$$\mathcal{J}^{E_{\pi(1)}}_{\sigma_{\pi(1)},T_{\pi(1)}} \circ ... \circ \mathcal{J}^{E_{\pi(n)}}_{\sigma_{\pi(n)},T_{\pi(n)}} = \mathcal{J}^{E}_{\sigma,T^\pi}. \tag{4.83}$$

In particular

$$\mathcal{J}_{\sigma_{\pi(1)},T_{\pi(1)}} \circ ... \circ \mathcal{J}_{\sigma_{\pi(n)},T_{\pi(n)}} = \mathcal{J}_{\sigma,T^\pi}. \tag{4.84}$$

As a result, if π induces a causal linear order on $\{N_j\}_{j=1}^n$, then, by Theorem 4.2.9, T^π is a $K_1 \cup ... \cup K_n$-map. The desired properties of $\mathfrak{C}$ then follow in complete analogy to the proof of Theorem 3.3.4.

Finally, we note that this ctPFA is causal according to Definition 3.4.2, which is an immediate consequence of the definition of T^π and Lemma 4.5.3. □

In summary, we have discussed spacetime-compact perturbations of the dynamics of AQFTs and how associated K-maps yield quite generally ctPFAs of causal quantum channels for AQFTs. It is now an interesting question to ask whether this ctPFA is "rich" enough and contains sufficiently many quantum channels that, for instance, may be interpreted as the state-update maps associated to measurements of sufficiently many (ideally all) local observables.

In the next chapter we will see that the ctPFA $\mathfrak{F}_{\mathfrak{B}_M}$ from Def. 4.5.1 in particular contains the non-selective state-update maps associated to measurements described by the measurement schemes for AQFTs introduced in [10].

— 5 —

QMT-observables, instruments and measurement schemes

In this chapter we come back to questions regarding the measurement of observables, see also Sec. 1.2 in [13].

In Chapter 2 we have discussed the notion of an observable and motivated its abstract counterpart to be a Hermitian element A of a unital $*$-algebra. Given a state ω, the theory then determines the moments of a putative distribution $\mu_{A,\omega}$ of a putative random variable $\mathtt{X}_{A,\omega}$. However, as we have emphasised, in general there are several *operationally distinguishable* measures $\mu^{\alpha}_{A,\omega}$, for α in some parameterising index set, all of which are compatible with the predicted moments. A way to avoid this is to shift focus from Hermitian elements A (which we will continue to call *observables*) to *effect-valued measures*, which are the objects that the quantum measurement theory community regards as observables and which we hence call *QMT-observables*.

5.1 QMT-OBSERVABLES AND INSTRUMENTS

The motivation behind QMT-observables is as follows. Suppose we are performing a(n ensemble of) measurement(s). As already mentioned in Chapter 2, it is an overidealisation to say that one *actually* records real numbers. It is much more natural to make the assumption that one in fact records whether the result of a single measurement run lies in a (Borel-)subset $Y \subseteq \mathbb{R}$ of the real numbers. Let then $\mathtt{n}^{(N)}[Y]$ be the number of times a result that falls in the "bin" associated to Y was recorded over N measurement runs. Then,

$$\mathtt{p}^{(N)}[Y] := \frac{\mathtt{n}^{(N)}[Y]}{N}, \tag{5.1}$$

is the *observed* probability that the outcome of the measurement lies in Y. In order to *predict* $\mathtt{p}^{(N)}[Y]$ one now defines a QMT-observable to be an *effect-valued measure (EVM)*.

Definition 5.1.1. *Let $(\mathbb{R}, \mathcal{X})$ be a measurable space and let $\mathcal{A}$ be a unital $*$-algebra. A (normalised) EVM is a map $E : \mathcal{X} \to \mathcal{A}$, valued in the effects of $\mathcal{A}$ such that*

1. $E(\mathbb{R}) = \mathbb{1}$,

2. *for every finite collection of disjoint sets $Y_1, ..., Y_m \in \mathcal{X}$ we have*

$$E\left(\bigcup_{i=1}^{m} Y_i\right) = \sum_{i=1}^{m} E(Y_i). \tag{5.2}$$

Remark: This should be regarded as a *minimal* definition. Further properties might be desirable, see for instance Definition 4.5 (b) in [13].

Now the prediction of the theory is as follows. In the above experiment, there is an underlying random variable $\mathtt{X}_{E,\omega}$ with distribution $\mu_{E,\omega}$ such that for every $Y \in \mathcal{X}$:

$$\mu_{E,\omega}(Y) = \omega(E(Y)). \tag{5.3}$$

We can again look at N iid copies of the random variable $\mathtt{X}_{E,\omega}$ given by $(\mathtt{X}_{E,\omega,1}, ..., \mathtt{X}_{E,\omega,N})$ and for every $Y \in \mathcal{X}$ look at the random variable

$$\mathtt{P}^{(N)}_{Y,E,\omega} := \frac{1}{N} \sum_{i=1}^{N} \chi_Y(\mathtt{X}_{E,\omega,i}), \tag{5.4}$$

where χ_Y is the characteristic function of Y. This is indeed the random variable associated to the numerical value $\mathtt{p}^{(N)}[Y]$. It follows immediately that $\mathtt{P}^{(N)}_{Y,E,\omega}$ is an unbiased estimator for $\mu_{E,\omega}(Y)$. Moreover, it again follows from Chebyshev's inequality that

$$\mathbb{P}\Big(|\mathtt{P}^{(N)}_{Y,E,\omega} - \mathbb{E}(\mathtt{P}^{(N)}_{Y,E,\omega})| < \epsilon\Big) \geq 1 - \frac{\mathbb{V}\Big(\mathtt{P}^{(N)}_{Y,E,\omega}\Big)}{\epsilon} = 1 - \frac{\mathbb{V}(\chi_Y(\mathtt{X}_{E,\omega}))}{N\epsilon}. \tag{5.5}$$

We see again that with a probability of $1 - \frac{\mathbb{V}(\chi_Y(\mathtt{X}_{E,\omega}))}{N\epsilon}$ the observed value $\mathtt{p}^{(N)}[Y]$ deviates by at most ϵ from the true probability $\mu_{E,\omega}(Y)$, which is predicted to be $\omega(E(Y))$.

It is now convenient, and in fact very common, to allow measurable spaces $(\mathbb{R}, \mathcal{X})$ where $\mathcal{X}$ contains "bins" of arbitrarily small size, such as for instance the Borel algebra. This is clearly an idealisation due to the necessarily finite graduation of any measurement scale.

In the next step we introduce so-called *instruments*, first introduced in [51].

Definition 5.1.2. *Let $(\Omega, \mathcal{X})$ be a measurable space and let $\mathcal{A}$ be a unital $*$-algebra. Then we call a map $\mathcal{J}\Big|_{\cdot}$ from $\mathcal{X}$ into the quantum operations on $\mathcal{A}$ an* instrument, *if*

1. $\mathcal{J}\Big|_{\Omega} : \mathcal{A} \to \mathcal{A}$ *is a quantum channel,*
2. *for every finite collection of disjoint sets $Y_1, ..., Y_m \in \mathcal{X}$ we have*

$$\mathcal{J}\Big|_{\bigcup_{i=1}^{m} Y_i} = \sum_{i=1}^{m} \mathcal{J}\Big|_{Y_i}. \tag{5.6}$$

It is easy to see that for every instrument $\mathcal{J}$, the map $\mathcal{J}\Big|_{\cdot}(\mathbb{1}) : \mathcal{X} \to \mathcal{A}$ is an EVM. In fact, given an instrument $\mathcal{J}$ and a state ω on $\mathcal{A}$, one can define

$$\omega'(A) := \frac{\omega(\mathcal{J}|_Y(A))}{\omega(\mathcal{J}|_Y(\mathbb{1}))}, \tag{5.7}$$

provided the denominator does not vanish. In particular $\omega(\mathcal{J}|_Y(\mathbb{1}))$ may be viewed as the *success probability* of performing the quantum operation $\mathcal{J}|_Y$, i.e., it may be interpreted as the probability that the system "survived" the intervention described by the quantum operation $\mathcal{J}|_Y$. This statement needs further clarification, since our current interpretation of $\omega(\mathcal{J}|_Y(\mathbb{1}))$ is the probability that a measurement outcome lies in the set Y. To reconcile these two interpretations we note that the updated state ω' may denote an ensemble of identical systems all of which returned a value in Y for the measurement. It is therefore a *selected* ensemble and the transition $\omega \to \omega'$ may be operationally described as discarding those members of the ensemble that did not return a value in Y. As a result, in the case in which *no* selection happens, i.e., in which $Y = \Omega$ we have

$$\omega'(A) = \omega(\mathcal{J}|_\Omega(A)). \tag{5.8}$$

Now we are confronted with the fact that every instrument defines a unique QMT-observable, but every QMT-observable admits (in general many) different instruments describing a measurement of said QMT-observable. An heuristic way of understanding this ambiguity is that there are many different ways of measuring a QMT-observable, resulting in different quantum operations and different selected ensembles, i.e., different states ω'. The upshot is that an experimentalist telling their friend in the theory department *what* they measured will not allow their friend to determine the right instrument associated to the experiment. The additional input the experimentalist has to provide is *how* they measured.

5.2 MEASUREMENT SCHEMES

The next paragraphs (until Def. 5.2.1) are almost direct quotes from [2].

While currently there seems to be no full explanation of the measurement process in reach, QMT has achieved an understanding of individual steps along the *measurement chain*, i.e., the process by which information about a quantum *system* may be transferred to a *probe*, which might also be assumed to be *quantum*.[1] More concretely, for a given system $\mathcal{S}$ (in AQT given by a unital $*$-algebra) one considers an auxiliary quantum structure called the *probe* $\mathcal{P}$ together with an initial probe state σ, a "pointer" or probe observable B and a "measurement coupling" between system and probe, The idea is that a measurement of B after the coupling has been removed will yield information about a system observable. The collection of $\mathcal{P}$, coupling, probe preparation state σ and probe observable B is known as measurement scheme, see Chapter 10 in [13] and also the precise definition below.

It is important to emphasise that the description of a measurement in terms of measurement schemes neither contains nor requires an explanation of *how* exactly information is extracted from the probe. What needs to be put in is the standard working assumption that information can be observed *somehow*.

In the following we will see how the description of the measurement process in terms of information transfer between a quantum system and a quantum probe can be used to associate well-motivated instruments to system-QMT-observables. Moreover, we will see that these instruments and associated quantum operations very naturally emerge from the framework. Furthermore, in [10] and [52] the concept of measurement schemes was combined with AQFT. We will present and generalise this adaption and show how AQFT-measurement schemes give rise to *causal* instruments and state-updates.

We first start with a general definition in the realm of AQT.

Definition 5.2.1. *Let $\mathcal{S}$ be a unital $*$-algebra and suppose*

1. *$\mathcal{P}$ is a unital $*$-algebra,*

[1] Afterwards, this information can in principle be extracted by a measurement of probe observables. We will *not* be concerned with an analysis of this last step. In particular, we do not touch on the question how classical information can be extracted from the quantum probe, which may be considered part of what is known as "measurement problem".

2. $T : \mathcal{S} \otimes \mathcal{P} \to \mathcal{S} \otimes \mathcal{P}$ *is a unit-preserving* $*$*-homomorphism,*

3. σ *is a state on* $\mathcal{P}$*.*

Then we call the tuple $(\mathcal{P}, T, \sigma)$ *an* AQT-measurement preparation*. Furthermore let* $A \in \mathcal{S}$ *and let there be a* $B \in \mathcal{P}$ *such that*

$$A = \varepsilon^T_\sigma(B) := \eta_\sigma(T(\mathbb{1} \otimes B)). \tag{5.9}$$

Then we call the tuple $(\mathcal{P}, T, \sigma, B)$ *an* AQT-measurement scheme *for* A*. In that case we say* A *is* inducible *(or* induced *by the AQT measurement scheme* $(\mathcal{P}, T, \sigma, B)$*). Similarly for the case of EVMs* A *and* B*, for which in particular*

$$\mathcal{X} \ni Y \mapsto \mathcal{J}^{B(Y)}_{\sigma,T} = \eta_\sigma(T(\cdot \otimes B(Y))), \tag{5.10}$$

see Lemma 2.3.11, is an instrument with associated EVM A*.*

If $B \in \mathcal{P}$ *is Hermitian, we say that* $(\mathcal{P}, T, \sigma, B)$ *is a* Hermitian *AQT-measurement scheme.*

It follows immediately from Eq. (5.9) that if $(\mathcal{P}, T, \sigma, B)$ is an AQT-measurement scheme for A, then for every state ω on $\mathcal{S}$ we have that

$$\omega(A) = \omega \otimes \sigma(T(\mathbb{1} \otimes B)). \tag{5.11}$$

Following our considerations above, the intuitive interpretation associated to the definition is as follows. If $(\mathcal{P}, T, \sigma)$ is a measurement preparation, then we call $\mathcal{P}$ the probe, T the measurement coupling and σ the probe preparation state. If the system is initially prepared in state ω and an observer measures a probe observable B, then, taking the existence of the system $\mathcal{S}$ into account, what one *actually* has to consider is the observable $\mathbb{1} \otimes B$. Furthermore, the probe and the system are somewhat "coupled", which is expressed by T. Taking this coupling into account, the result of the measurement is then given by $\omega \otimes \sigma(T(\mathbb{1} \otimes B))$, which in particular equals $\omega\left(\varepsilon^T_\sigma(B)\right)$. The heuristic idea one should take away from this fact is that by "looking" at the *probe* observable B, one can in fact learn something about the *system* observable $\varepsilon^T_\sigma(B)$.[2]

On the more technical side we note that it follows from (the proof of) Lemma 2.3.11 that $\varepsilon^T_\sigma : \mathcal{P} \to \mathcal{S}$ is a quantum channel, and hence, by Lemma 2.3.7, that $\varepsilon^T_\sigma(B^*) = \varepsilon^T_\sigma(B)^*$.

Let us now particularise to the realm of AQFT, for which, following [10] and [52], we slightly adapt the previous definition.

[2]See [10] for a more sophisticated explanation as to why $\omega \otimes \sigma(T(\mathbb{1} \otimes B))$ is the correct prediction.

Definition 5.2.2. *Let M be a globally hyperbolic spacetime and let $\mathcal{S}_{\mathfrak{B}_M}$ be an AQFT fulfilling time-slice and additivity. Then we call a tuple $(\mathcal{P}_{\mathfrak{B}_M}, T, \sigma)$ a measurement preparation (with coupling zone K) if and only if*

1. *$\mathcal{P}_{\mathfrak{B}_M}$ is an AQFT fulfilling time-slice and additivity and*
2. *$T : \mathcal{S}^g_{\mathfrak{B}_M} \otimes \mathcal{P}^g_{\mathfrak{B}_M} \to \mathcal{S}^g_{\mathfrak{B}_M} \otimes \mathcal{P}^g_{\mathfrak{B}_M}$ is a K-hom, and*
3. *σ is a state on $\mathcal{P}^g_{\mathfrak{B}_M}$.*

In particular $(\mathcal{P}^g_{\mathfrak{B}_M}, T, \sigma)$ is an AQT-measurement preparation.

We say that a measurement preparation $(\mathcal{P}_{\mathfrak{B}_M}, T, \sigma)$ is an FV*-measurement preparation if the K-hom T is the scattering map of a K-perturbed variant of $\mathcal{S}_{\mathfrak{B}_M} \otimes \mathcal{P}_{\mathfrak{B}_M}$ on $\mathfrak{B}_M$.*

We say that a tuple $(\mathcal{P}_{\mathfrak{B}_M}, T, \sigma, B)$ is a measurement scheme for A (with processing region $L \subseteq M$ and coupling zone K) if and only if

1. *$(\mathcal{P}_{\mathfrak{B}_M}, T, \sigma)$ is a measurement preparation (with coupling zone K),*
2. *$B \in \mathcal{P}_{\mathfrak{B}_M}(L)$, and*
3. *$A = \varepsilon^T_\sigma(B)$,*

with the obvious alterations when using the attributes "FV" and "Hermitian".

A few comments are in order:

1. According to Theorem 4.4.1, every K-map is the scattering map of a perturbed variant on some $\mathfrak{B}_{(M,\tilde{K})}$ and it is an open question whether it is also the scattering map of a perturbed variant on $\mathfrak{B}_M$. Hence, the main difference between an FV-measurement scheme and a mere measurement scheme is the potential additional control one *might* have over T, given that T is assumed to be the scattering map of a perturbed variant on $\mathfrak{B}_M$ and not just a K-map.
2. If $\mathcal{S}_{\mathfrak{B}_M}$ fulfills the Haag-property, see Def. 3.1.6, then it can be shown that for *any* $B \in \mathcal{P}_{\mathfrak{B}_M}$, $\varepsilon^T_\sigma(B)$ is contained in $\mathcal{S}_{\mathfrak{B}_M}(L)$ for every *connected* L that contains K, see Theorem 3.3 in [10].
3. If a region L is spacelike separated from K, then for every $B \in \mathcal{P}_{\mathfrak{B}_M}(L)$ we have that $\varepsilon^T_\sigma(B) = \sigma(B)\mathbb{1}$, see Theorem 3.3 in [10].

4. Note that we assume here that the AQFTs under consideration fulfill additivity in order to be able to apply the results from Sec. 4.2 and Sec. 4.3. An alternative would be to proceed as in [10] and [1] and to drop additivity and instead *assume* a version of causal factorisation, see in particular Sec. V. in [1].

5.2.1 Combination of (FV-) measurement schemes and state-updates

The discussion of the following results is based on [1], joint work with Bostelmann and Fewster, see also [10].

Let $(\mathcal{P}^{\mathsf{A}}_{\mathfrak{B}_M}, T_{\mathsf{A}}, \sigma_{\mathsf{A}})$ and $(\mathcal{P}^{\mathsf{B}}_{\mathfrak{B}_M}, T_{\mathsf{B}}, \sigma_{\mathsf{B}})$ be two measurement preparations with coupling zones K_{A} and K_{B} respectively. We now wish to combine these measurement preparations to a new measurement preparation. To that end let us assume that K_{A} and K_{B} are causally orderable, and that, without loss of generality, $K_{\mathsf{B}} \not\lesssim K_{\mathsf{A}}$. In particular, there exists a causal linear order $\leq$, such that $K_{\mathsf{A}} < K_{\mathsf{B}}$. Then we can define the measurement preparation

$$\Big(\mathcal{P}^{\mathsf{A}}_{\mathfrak{B}_M} \otimes \mathcal{P}^{\mathsf{B}}_{\mathfrak{B}_M}, \check{T}_{\mathsf{A}} \circ \check{T}_{\mathsf{B}}, \sigma_{\mathsf{A}} \otimes \sigma_{\mathsf{B}}\Big), \tag{5.12}$$

where $\check{T}_{\mathsf{A}} := T_{\mathsf{A}} \otimes \mathrm{id}_{\mathsf{B}}$ and $\check{T}_{\mathsf{B}} := (\mathbb{F}_2)^{-1} \circ T_{\mathsf{B}} \otimes \mathrm{id}_{\mathsf{A}} \circ \mathbb{F}_2$, where $\mathbb{F}_2 : \mathcal{S}^g_{\mathfrak{B}_M} \otimes \mathcal{P}^{\mathsf{A},g}_{\mathfrak{B}_M} \otimes \mathcal{P}^{\mathsf{B},g}_{\mathfrak{B}_M} \to \mathcal{S}^g_{\mathfrak{B}_M} \otimes \mathcal{P}^{\mathsf{B},g}_{\mathfrak{B}_M} \otimes \mathcal{P}^{\mathsf{A},g}_{\mathfrak{B}_M}$ is the canonical automorphism and where $\mathrm{id}_{\mathsf{J}} : \mathcal{P}^{\mathsf{J},g}_{\mathfrak{B}_M} \to \mathcal{P}^{\mathsf{J},g}_{\mathfrak{B}_M}$ denotes the identity for $\mathsf{J} = \mathsf{A}, \mathsf{B}$. The fact that Eq. (5.12) is indeed a measurement preparation follows since $\check{T}_{\mathsf{A}} \circ \check{T}_{\mathsf{B}}$ is a $K_{\mathsf{A}} \cup K_{\mathsf{B}}$-map according to Lemma 4.2.7.[3] Furthermore, if the two constituents are in fact FV-measurement preparations, then the combination is again an FV-preparation, see Lemma 4.3.3.

As an immediate and important consequence, the combination of measurement preparations yields combinations of measurement schemes, i.e., for any two probe observables $P_{\mathsf{A}} \in \mathcal{P}^{\mathsf{A},g}_{\mathfrak{B}_M}$ and $P_{\mathsf{B}} \in \mathcal{P}^{\mathsf{B},g}_{\mathfrak{B}_M}$ the measurement scheme

$$\Big(\mathcal{P}^{\mathsf{A}}_{\mathfrak{B}_M} \otimes \mathcal{P}^{\mathsf{B}}_{\mathfrak{B}_M}, \check{T}_{\mathsf{A}} \circ \check{T}_{\mathsf{B}}, \sigma_{\mathsf{A}} \otimes \sigma_{\mathsf{B}}, P_{\mathsf{A}} \otimes P_{\mathsf{B}}\Big), \tag{5.13}$$

may be viewed as a combination of the measurement schemes $(\mathcal{P}^{\mathsf{A}}_{\mathfrak{B}_M}, T_{\mathsf{A}}, \sigma_{\mathsf{A}}, P_{\mathsf{A}})$ and $(\mathcal{P}^{\mathsf{B}}_{\mathfrak{B}_M}, T_{\mathsf{B}}, \sigma_{\mathsf{B}}, P_{\mathsf{B}})$.

This rationale generalises to the appropriate multipartite case. For concreteness, we give an in depth analysis of the case of four measurement preparations $(\mathcal{P}^{\mathsf{J}}_{\mathfrak{B}_M}, T_{\mathsf{J}}, \sigma_{\mathsf{J}})$

[3] Here we see that the assumption of causally orderable coupling zones becomes important. Because, unlike in Lagrangean approaches, it is in general unclear how one could combine a K_{A}- and a K_{B}-map to a $K_{\mathsf{A}} \cup K_{\mathsf{B}}$-map in the case where K_{A} and K_{B} are not causally orderable (and potentially even overlapping).

with coupling zone K_J for $\mathsf{J} = \mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$ such that without loss of generality $K_\mathsf{A} < K_\mathsf{B} < K_\mathsf{C} < K_\mathsf{D}$ for some causal linear order $\leq$. The combined measurement preparation is then given by

$$\left(\mathcal{P}^{\mathsf{A}}_{\mathfrak{B}_M} \otimes \mathcal{P}^{\mathsf{B}}_{\mathfrak{B}_M} \otimes \mathcal{P}^{\mathsf{C}}_{\mathfrak{B}_M} \otimes \mathcal{P}^{\mathsf{D}}_{\mathfrak{B}_M}, \hat{T}_\mathsf{A} \circ \hat{T}_\mathsf{B} \circ \hat{T}_\mathsf{C} \circ \hat{T}_\mathsf{D}, \sigma_\mathsf{A} \otimes \sigma_\mathsf{B} \otimes \sigma_\mathsf{C} \otimes \sigma_\mathsf{D}\right), \tag{5.14}$$

where $\hat{T}_\mathsf{J}$ is defined analogously to Eq. (4.81) for $n = 4$ and $\mathsf{A} \equiv 1$, $\mathsf{B} \equiv 2$, etc.

Let us now consider the measurement preparation of Eq. (5.14), which, together with the observable $P := E_\mathsf{A} \otimes E_\mathsf{B} \otimes P_\mathsf{C} \otimes \mathbb{1}_\mathsf{D}$, where $E_\mathsf{A} \in \mathcal{P}^{\mathsf{A},g}_{\mathfrak{B}_M}$, $E_\mathsf{B} \in \mathcal{P}^{\mathsf{B},g}_{\mathfrak{B}_M}$, $P_\mathsf{C} \in \mathcal{P}^{\mathsf{C},g}_{\mathfrak{B}_M}$ and $\mathbb{1}_\mathsf{D}$ denotes the unit in $\mathcal{P}^{\mathsf{D},g}_{\mathfrak{B}_M}$, forms a measurement scheme. It is interesting to compare this measurement scheme to the measurement scheme $(\mathcal{P}^{\mathsf{C}}_{\mathfrak{B}_M}, T_\mathsf{C}, \sigma_\mathsf{C}, P_\mathsf{C})$. To that end we will first state and prove a technical result and provide a detailed discussion of its interpretation later.

In a nutshell: For effects $E_\mathsf{A}, E_\mathsf{B}$, the induced observables of the two measurement schemes differ by a quantum operation, which may be alternatively incorporated in terms of a state-update. Furthermore, these quantum operations respect causality in the sense that they compose and decompose in any admissible causal order, and that there is no retrocausality and no superluminal signalling.

Lemma 5.2.3. *Let $\mathcal{S}_{\mathfrak{B}_M}$ be an AQFT fulfilling additivity and time-slice. Furthermore, for $\mathsf{J} = \mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{D}$, let $(\mathcal{P}^{\mathsf{J}}_{\mathfrak{B}_M}, T_\mathsf{J}, \sigma_\mathsf{J})$ be a measurement preparation with coupling zone K_J such that $K_\mathsf{A} < K_\mathsf{B} < K_\mathsf{C} < K_\mathsf{D}$ for some causal linear order $\leq$. Then the induced observable of the combined measurement scheme*

$$\left(\mathcal{P}^{\mathsf{A}}_{\mathfrak{B}_M} \otimes \mathcal{P}^{\mathsf{B}}_{\mathfrak{B}_M} \otimes \mathcal{P}^{\mathsf{C}}_{\mathfrak{B}_M} \otimes \mathcal{P}^{\mathsf{D}}_{\mathfrak{B}_M}, \hat{T}_\mathsf{A} \circ \hat{T}_\mathsf{B} \circ \hat{T}_\mathsf{C} \circ \hat{T}_\mathsf{D}, \sigma_\mathsf{A} \otimes \sigma_\mathsf{B} \otimes \sigma_\mathsf{C} \otimes \sigma_\mathsf{D}, P\right), \tag{5.15}$$

where $P := E_\mathsf{A} \otimes E_\mathsf{B} \otimes P_\mathsf{C} \otimes \mathbb{1}_\mathsf{D}$ for effects $E_\mathsf{A} \in \mathcal{P}^{\mathsf{A},g}_{\mathfrak{B}_M}$, $E_\mathsf{B} \in \mathcal{P}^{\mathsf{B},g}_{\mathfrak{B}_M}$ and any observable $P_\mathsf{C} \in \mathcal{P}^{\mathsf{C},g}_{\mathfrak{B}_M}$, is given by

$$\varepsilon^{\hat{T}_\mathsf{A} \circ \hat{T}_\mathsf{B} \circ \hat{T}_\mathsf{C} \circ \hat{T}_\mathsf{D}}_{\sigma_\mathsf{A} \otimes \sigma_\mathsf{B} \otimes \sigma_\mathsf{C} \otimes \sigma_\mathsf{D}}(P) = \mathcal{J}^{E_\mathsf{A} \otimes E_\mathsf{B}}_{\sigma_\mathsf{A} \otimes \sigma_\mathsf{B}, \check{T}_\mathsf{A} \circ \check{T}_\mathsf{B}}\left(\varepsilon^{T_\mathsf{C}}_{\sigma_\mathsf{C}}(P_\mathsf{C})\right), \tag{5.16}$$

where $\mathcal{J}^{E_\mathsf{A} \otimes E_\mathsf{B}}_{\sigma_\mathsf{A} \otimes \sigma_\mathsf{B}, \check{T}_\mathsf{A} \circ \check{T}_\mathsf{B}}$ is the quantum operation on $\mathcal{S}^g_{\mathfrak{B}_M}$ introduced in Lemma 2.3.11 and $\varepsilon^{T_\mathsf{C}}_{\sigma_\mathsf{C}}(P_\mathsf{C})$ is the induced observable of the measurement scheme $(\mathcal{P}^{\mathsf{C}}_{\mathfrak{B}_M}, T_\mathsf{C}, \sigma_\mathsf{C}, P_\mathsf{C})$. In particular, for every state ω on $\mathcal{S}^g_{\mathfrak{B}_M}$ it holds, for $\omega\left(\mathcal{J}^{E_\mathsf{A} \otimes E_\mathsf{B}}_{\sigma_\mathsf{A} \otimes \sigma_\mathsf{B}, \check{T}_\mathsf{A} \circ \check{T}_\mathsf{B}}\left(\mathbb{1}_{\mathcal{S}_{\mathfrak{B}_M}}\right)\right) \neq 0$, that

$$\omega\left(\varepsilon^{\hat{T}_\mathsf{A} \circ \hat{T}_\mathsf{B} \circ \hat{T}_\mathsf{C} \circ \hat{T}_\mathsf{D}}_{\sigma_\mathsf{A} \otimes \sigma_\mathsf{B} \otimes \sigma_\mathsf{C} \otimes \sigma_\mathsf{D}}(P)\right) = \omega^{E_\mathsf{A} \otimes E_\mathsf{B}}_{\sigma_\mathsf{A} \otimes \sigma_\mathsf{B}, \check{T}_\mathsf{A} \circ \check{T}_\mathsf{B}}\left(\varepsilon^{T_\mathsf{C}}_{\sigma_\mathsf{C}}(P_\mathsf{C})\right) \omega\left(\mathcal{J}^{E_\mathsf{A} \otimes E_\mathsf{B}}_{\sigma_\mathsf{A} \otimes \sigma_\mathsf{B}, \check{T}_\mathsf{A} \circ \check{T}_\mathsf{B}}\left(\mathbb{1}_{\mathcal{S}_{\mathfrak{B}_M}}\right)\right), \tag{5.17}$$

where we defined the state

$$\omega^{E_\mathsf{A} \otimes E_\mathsf{B}}_{\sigma_\mathsf{A} \otimes \sigma_\mathsf{B}, \check{T}_\mathsf{A} \circ \check{T}_\mathsf{B}} := \frac{\omega \circ \mathcal{J}^{E_\mathsf{A} \otimes E_\mathsf{B}}_{\sigma_\mathsf{A} \otimes \sigma_\mathsf{B}, \check{T}_\mathsf{A} \circ \check{T}_\mathsf{B}}}{\omega\left(\mathcal{J}^{E_\mathsf{A} \otimes E_\mathsf{B}}_{\sigma_\mathsf{A} \otimes \sigma_\mathsf{B}, \check{T}_\mathsf{A} \circ \check{T}_\mathsf{B}}\left(\mathbb{1}_{\mathcal{S}_{\mathfrak{B}_M}}\right)\right)}. \tag{5.18}$$

Furthermore we define

$$\omega_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B},\check{T}_\mathsf{A}\circ\check{T}_\mathsf{B}} := \omega^{1\!\!1_\mathsf{A}\otimes 1\!\!1_\mathsf{B}}_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B},\check{T}_\mathsf{A}\circ\check{T}_\mathsf{B}} = \omega\circ\mathcal{J}_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B},\check{T}_\mathsf{A}\circ\check{T}_\mathsf{B}}. \tag{5.19}$$

In general we have

$$\mathcal{J}^{E_\mathsf{A}\otimes E_\mathsf{B}}_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B},\check{T}_\mathsf{A}\circ\check{T}_\mathsf{B}} = \mathcal{J}^{E_\mathsf{A}}_{\sigma_\mathsf{A},T_\mathsf{A}}\circ\mathcal{J}^{E_\mathsf{B}}_{\sigma_\mathsf{B},T_\mathsf{B}}. \tag{5.20}$$

In particular, if $K_\mathsf{A}\perp_M K_\mathsf{B}$*, then*

$$\mathcal{J}^{E_\mathsf{A}\otimes E_\mathsf{B}}_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B},\check{T}_\mathsf{A}\circ\check{T}_\mathsf{B}} = \mathcal{J}^{E_\mathsf{A}}_{\sigma_\mathsf{A},T_\mathsf{A}}\circ\mathcal{J}^{E_\mathsf{B}}_{\sigma_\mathsf{B},T_\mathsf{B}} = \mathcal{J}^{E_\mathsf{B}}_{\sigma_\mathsf{B},T_\mathsf{B}}\circ\mathcal{J}^{E_\mathsf{A}}_{\sigma_\mathsf{A},T_\mathsf{A}}. \tag{5.21}$$

If in addition $\mathcal{S}_{\mathfrak{B}_M}$ *fulfills the Haag property, then, whenever* $K_\mathsf{A}\perp_M K_\mathsf{C}$ *and* K_C *is connected, we have that*

$$\begin{aligned}\varepsilon^{\hat{T}_\mathsf{A}\circ\hat{T}_\mathsf{B}\circ\hat{T}_\mathsf{C}\circ\hat{T}_\mathsf{D}}_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B}\otimes\sigma_\mathsf{C}\otimes\sigma_\mathsf{D}}(1\!\!1_\mathsf{A}\otimes 1\!\!1_\mathsf{B}\otimes P_\mathsf{C}\otimes 1\!\!1_\mathsf{D}) &= \left(\mathcal{J}_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B},\check{T}_\mathsf{A}\circ\check{T}_\mathsf{B}}\right)\left(\varepsilon^{T_\mathsf{C}}_{\sigma_\mathsf{C}}(P_\mathsf{C})\right)\\ &= \mathcal{J}_{\sigma_\mathsf{B},T_\mathsf{B}}\left(\varepsilon^{T_\mathsf{C}}_{\sigma_\mathsf{C}}(P_\mathsf{C})\right)\end{aligned} \tag{5.22}$$

is independent of T_A *and* σ_A*.*

Proof. Eq. (5.16) follows from a straight forward calculation, and Eq. (5.20) and Eq. (5.21) follow from Eq. (4.83).

For Eq. (5.22) note that $\varepsilon^{T_\mathsf{C}}_{\sigma_\mathsf{C}}(P_\mathsf{C})\in\mathcal{S}_{\mathfrak{B}_M}(N_\mathsf{C})$, whenever N_C is a *connected* region that contains K_C according to Theorem 3.3 in [10]. The claim is now that we can find regions $N_\mathsf{A}, N_\mathsf{B}, N_\mathsf{C}$ such that

1. $K_\mathsf{A}\subseteq N_\mathsf{A}$ and $K_\mathsf{B}\subseteq N_\mathsf{B}$,
2. $N_\mathsf{A} < N_\mathsf{B} < N_\mathsf{C}$ for some causal linear order $\leq$, and
3. $N_\mathsf{A}\perp_M N_\mathsf{C}$ and such that
4. N_C is connected.

Then the desired result follows from Theorem 4.5.2. The existence of appropriate $N_\mathsf{A}, N_\mathsf{B}, N_\mathsf{C}$ is shown in Lemma D.4.1. □

The abstract mathematical results of Lemma 5.2.3 carry some deep meaning and hence their interpretation deserves special attention.

Let us first start with the case where $E_\mathsf{A}\otimes E_\mathsf{B} = 1\!\!1_\mathsf{A}\otimes 1\!\!1_\mathsf{B}$, i.e., where E_A and E_B are the trivial probe observables.

1. The induced observable $\varepsilon^{\hat{T}_\mathsf{A}\circ\hat{T}_\mathsf{B}\circ\hat{T}_\mathsf{C}\circ\hat{T}_\mathsf{D}}_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B}\otimes\sigma_\mathsf{C}\otimes\sigma_\mathsf{D}}(\mathbb{1}_\mathsf{A}\otimes\mathbb{1}_\mathsf{B}\otimes P_\mathsf{C}\otimes\mathbb{1}_\mathsf{D})$ obviously differs, in general, from the induced observable $\varepsilon^{T_\mathsf{C}}_{\sigma_\mathsf{C}}(P_\mathsf{C})$. The latter describes the situation where there is only one observer C (modelled through their measurement scheme), while the former also takes the existence of A, B and D into account.

2. Given a situation where there is an initial system state ω "prepared at early times",[4] the influence of A, B and D on C's measurement scheme can either be taken into account by considering the full measurement scheme in Eq. (5.15), which predicts the expectation value of C's measurement of the probe observable P_C to be $\omega\Big(\varepsilon^{\hat{T}_\mathsf{A}\circ\hat{T}_\mathsf{B}\circ\hat{T}_\mathsf{C}\circ\hat{T}_\mathsf{D}}_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B}\otimes\sigma_\mathsf{C}\otimes\sigma_\mathsf{D}}(\mathbb{1}_\mathsf{A}\otimes\mathbb{1}_\mathsf{B}\otimes P_\mathsf{C}\otimes\mathbb{1}_\mathsf{D})\Big)$, *or*, alternatively, by an *update* or change of the initial system state from ω to $\omega_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B},\check{T}_\mathsf{A}\circ\check{T}_\mathsf{B}}$, upon which the *same* prediction may be expressed as $\omega_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B},\check{T}_\mathsf{A}\circ\check{T}_\mathsf{B}}\Big(\varepsilon^{T_\mathsf{C}}_{\sigma_\mathsf{C}}(P_\mathsf{C})\Big)$, see Eq. (5.16) and Eq. (5.17) and note that $\mathcal{J}^{E_\mathsf{A}\otimes E_\mathsf{B}}_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B},\check{T}_\mathsf{A}\circ\check{T}_\mathsf{B}}$ is unit-preserving for $E_\mathsf{A}\otimes E_\mathsf{B} = \mathbb{1}_\mathsf{A}\otimes\mathbb{1}_\mathsf{B}$. This would correspond to what is known as a *non-selective* state-update. Hence, in the framework of measurement schemes, state-updates may be optionally used as a book-keeping tool instead of working with measurement schemes taking all observers into account. In fact, we see that in any case one would only have to take "earlier" observers into account as D (quite literally) drops out of the equation

$$\omega\Big(\varepsilon^{\hat{T}_\mathsf{A}\circ\hat{T}_\mathsf{B}\circ\hat{T}_\mathsf{C}\circ\hat{T}_\mathsf{D}}_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B}\otimes\sigma_\mathsf{C}\otimes\sigma_\mathsf{D}}(\mathbb{1}_\mathsf{A}\otimes\mathbb{1}_\mathsf{B}\otimes P_\mathsf{C}\otimes\mathbb{1}_\mathsf{D})\Big) = \omega_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B},\check{T}_\mathsf{A}\circ\check{T}_\mathsf{B}}\Big(\varepsilon^{T_\mathsf{C}}_{\sigma_\mathsf{C}}(P_\mathsf{C})\Big). \tag{5.23}$$

Put differently, there is no *retrocausality*.

3. According to Eq. (5.20) and Eq. (5.21), the quantum channel $\mathcal{J}_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B},\check{T}_\mathsf{A}\circ\check{T}_\mathsf{B}}$, whose dual map implements the non-selective state-update, decomposes as a composition of individual quantum channels *in every causally admissible order*, and hence so does the state-update. Also note that this map is *independent of* P_C, i.e., it only depends on the measurement preparation.

4. Finally, it might occur that K_A is in addition *spacelike separated* from K_C. Causality then dictates, that there must not be any difference between the cases where the mere existence of A *is* or *is not* incorporated. Under the technical assumptions stated in the lemma (such as the Haag property and associated appropriate assumptions of conectedness), it then follows from Eq. (5.22) that

$$\varepsilon^{\hat{T}_\mathsf{A}\circ\hat{T}_\mathsf{B}\circ\hat{T}_\mathsf{C}\circ\hat{T}_\mathsf{D}}_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B}\otimes\sigma_\mathsf{C}\otimes\sigma_\mathsf{D}}(\mathbb{1}_\mathsf{A}\otimes\mathbb{1}_\mathsf{B}\otimes P_\mathsf{C}\otimes\mathbb{1}_\mathsf{D}) = \mathcal{J}_{\sigma_\mathsf{B},T_\mathsf{B}}\Big(\varepsilon^{T_\mathsf{C}}_{\sigma_\mathsf{C}}(P_\mathsf{C})\Big). \tag{5.24}$$

Importantly, we see that A may be completely ignored as is required by causality.

[4]This follows Fewster's "slogan" of "prepare early and measure late".

Let us now look at the general case of possibly non-trivial effects $E_\mathsf{A}, E_\mathsf{B}$.

1. Let us first consider the case where P_C is an effect. Then

$$\omega\Big(\varepsilon^{\hat{T}_\mathsf{A}\circ\hat{T}_\mathsf{B}\circ\hat{T}_\mathsf{C}\circ\hat{T}_\mathsf{D}}_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B}\otimes\sigma_\mathsf{C}\otimes\sigma_\mathsf{D}}(E_\mathsf{A}\otimes E_\mathsf{B}\otimes P_\mathsf{C}\otimes 1\!\!1_\mathsf{D})\Big) \tag{5.25}$$

   has the interpretation of the success probability of

   a) A&B observing the effect $E_\mathsf{A}\otimes E_\mathsf{B}$, *and*

   b) C observing the effect P_C, *and*

   c) D observes the trivial observable, i.e., "simply exists".

   Given that, we see from Eq. 5.17 that $\omega^{E_\mathsf{A}\otimes E_\mathsf{B}}_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B},\check{T}_\mathsf{A}\circ\check{T}_\mathsf{B}}\Big(\varepsilon^{T_\mathsf{C}}_{\sigma_\mathsf{C}}(P_\mathsf{C})\Big)$ is the success probability of A&B *and* C observing their effects divided by the success probability of A&B observing their effect. It hence expresses a *conditional expectation* even if P_C is not an effect.

   In other words, suppose the four observers conduct their measurements and in the subsequent joint data analysis C (post-) selects only those experiments that yielded a positive outcome for the probe observable $E_\mathsf{A}\otimes E_\mathsf{B}$ and uses them to calculate an expectation value. Then the prediction for C's result is $\omega^{E_\mathsf{A}\otimes E_\mathsf{B}}_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B},\check{T}_\mathsf{A}\circ\check{T}_\mathsf{B}}\Big(\varepsilon^{T_\mathsf{C}}_{\sigma_\mathsf{C}}(P_\mathsf{C})\Big)$.[5]

   At this point it is also worth emphasising, that the state $\omega^{E_\mathsf{A}\otimes E_\mathsf{B}}_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B},\check{T}_\mathsf{A}\circ\check{T}_\mathsf{B}}$ "[...] cannot, in general, be interpreted conditionally with respect to [...]" the system observable induced by $E_\mathsf{A}\otimes E_\mathsf{B}$, see Sec. 10.2 in [13].

2. The map $\omega\mapsto\omega^{E_\mathsf{A}\otimes E_\mathsf{B}}_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B},\check{T}_\mathsf{A}\circ\check{T}_\mathsf{B}}$ (if defined) corresponds to what is known as a *selective* state-update. Again, it can be written as an iterative state-update *in every causally admissible order*.[6]

3. Finally, note that there is no analogy to A "dropping out" for $K_\mathsf{A}\perp_M K_\mathsf{B}$ and indeed, this is also not expected. C may very well condition on the outcome of an experiment that happens at spacelike separation which could yield a

[5] We intentionally do not consider the case where C conditions on the result of D's outcome (who performs a measurement *after* D). While it is straight forward to write down an expression in this case it does not seem to offer any interesting insight. In particular, there is no reason to expect that D drops out of the equation again. As already noted in [1], "[t]his issue is resolved by reminding oneself that post-selection can only be performed by all observers together in their joint future.".

[6] Note the slightly changed notation in comparison to [1]. Therein the (analogue of) the map $\omega\mapsto\omega^{E_\mathsf{A}\otimes E_\mathsf{B}}_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B},\check{T}_\mathsf{A}\circ\check{T}_\mathsf{B}}$ was denoted by (the analogue of the map) $\mathcal{J}^{E_\mathsf{A}\otimes E_\mathsf{B}}_{\sigma_\mathsf{A}\otimes\sigma_\mathsf{B},\check{T}_\mathsf{A}\circ\check{T}_\mathsf{B}}$.

non-trivial result. This, however, does *not* constitute any form of signalling but is only a sign of *correlation.*

5.3 causal state-updates

An important point of the discussion of measurement schemes above is that a measurement preparation $(\mathcal{P}_{\mathfrak{B}_M}, T, \sigma)$ gives rise to quantum channels $\mathcal{J}_{\sigma,T}$ whose dual map, acting on states on $\mathcal{S}^g_{\mathfrak{B}_M}$, has the interpretation of a non-selective state-update after the measurement of any observable in the image of ε^T_σ. Moreover, $\mathcal{J}_{\sigma,T}$ is *causal* in the sense that it is contained in the ctPFA of causal quantum channels, see Theorem 4.5.2.

In fact, let A be an EVM over $(\mathbb{R}, \mathcal{X})$ of the form $A = \varepsilon^T_\sigma \circ B$ for some EVM B of $\mathcal{P}^g_{\mathfrak{B}_M}$ over $(\mathbb{R}, \mathcal{X})$. Then the instrument $\mathcal{J}^{B(\cdot)}_{\sigma,T}$ is associated to the EVM A, i.e., $\mathcal{J}^{B(\cdot)}_{\sigma,T}(\mathbb{1}) = A(\cdot)$, *and* is *causal* in the sense that $\mathcal{J}_{\sigma,T}$ is causal.

We hence see that measurement preparations yield *causal* update maps for (QMT) observables that are induced by some measurement scheme. The obvious question is now whether every (QMT) observable of a given AQFT $\mathcal{S}_{\mathfrak{B}_M}$ is inducible, which would show that every (QMT) observable admits a *causal* description of its measurement and which would justify the name *observable.*

In the following subsection, we will present the construction of measurement schemes for any observable of any real scalar field AQFT on any globally hyperbolic spacetime fulfilling any normally hyperbolic equation of motion. The crucial idea is to find a K-map that describes an "instantaneous rotation" of certain compactly supported initial data on a Cauchy surface. The (possible) disadvantage of these measurement schemes is, however, that they are not obviously FV.

A somewhat similar idea due to Fewster, presented in Sec. 3 of [2], produces FV measurement schemes for every observable of a Klein-Gordon field, however, only in the case of a globally hyperbolic spacetime with *compact* Cauchy surfaces.

Another option is to use the main result of [2], joint work with Fewster and Jubb, which establishes the existence of *asymptotic* FV measurement schemes for every observable of a real scalar field on a globally hyperbolic spacetime fulfilling a normally hyperbolic equation of motion. We will discuss this approach in more detail in Sec. 5.4.

5.3.1 Measurement schemes for every (QMT-) observable of the linear real scalar field

In the following we use standard results collected in Appendix C.2.1.

Lemma 5.3.1. *Let $S : C^\infty(M;\mathbb{R}) \to C^\infty(M;\mathbb{R})$ and $P : C^\infty(M;\mathbb{R}) \to C^\infty(M;\mathbb{R})$ be normally hyperbolic equation of motion operators and let $\Sigma \subseteq M$ be a spacelike Cauchy surface of M. Let us consider the symplectic space $C^\infty_c(\Sigma;\mathbb{R}^2) \oplus C^\infty_c(\Sigma;\mathbb{R}^2)$, equipped with the symplectic form $\Omega \oplus \Omega$, where*

$$\Omega\Big((\Phi,\Pi),(\tilde{\Phi},\tilde{\Pi})\Big) := \int_\Sigma (\Phi,\Pi)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \tilde{\Phi} \\ \tilde{\Pi} \end{pmatrix} \mathrm{d}V_\Sigma. \tag{5.26}$$

Then, for any $\rho \in C^\infty_c(\Sigma,\mathbb{R})$ we define the invertible linear symplectic map

$$\begin{aligned} \tilde{t}^\Sigma_\rho :& C^\infty_c(\Sigma;\mathbb{R}^2) \oplus C^\infty_c(\Sigma;\mathbb{R}^2) \to C^\infty_c(\Sigma;\mathbb{R}^2) \oplus C^\infty_c(\Sigma;\mathbb{R}^2) \\ &\begin{pmatrix} \Phi_1 \\ \Pi_1 \\ \Phi_2 \\ \Pi_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos\rho \mathbb{1}_{2\times 2} & -\sin\rho \mathbb{1}_{2\times 2} \\ \sin\rho \mathbb{1}_{2\times 2} & \cos\rho \mathbb{1}_{2\times 2} \end{pmatrix}\begin{pmatrix} \Phi_1 \\ \Pi_1 \\ \Phi_2 \\ \Pi_2 \end{pmatrix}, \end{aligned} \tag{5.27}$$

and we denote the associated invertible linear symplectic map on the symplectic space $C^\infty_c(M;\mathbb{R}^2)/(S\oplus P)C^\infty_c(M;\mathbb{R}^2)$ by t^Σ_ρ. Then the associated (appropriate) quasi-free unit-preserving $*$-automorphism denoted by T^Σ_ρ is a K-map on the (appropriate) AQFT ${}^{S\oplus P}\mathcal{W}_{\mathfrak{B}_M}$ or ${}^{S\oplus P}\mathcal{F}_{\mathfrak{B}_M}$, where $K := \mathrm{supp}\rho$ is a compact subset of $\Sigma \subseteq M$.

Proof. $\tilde{t}^\Sigma_\rho$ is obviously linear, invertible and preserves the symplectic form $\Omega\oplus\Omega$. In order to show the result, we want to use Lemma C.3.1, which shows that T^Σ_ρ is a K-map provided that t^Σ_ρ has certain natural properties, which we now prove.

1. Take $[v]_{S\oplus P} = ([v_1]_S,[v_2]_P) \in C^\infty_c(L;\mathbb{R}^2)/(S\oplus P)C^\infty_c(M;\mathbb{R}^2)$ for any region L that is spacelike separated from K and representatives v_1, v_2 with support in L. Then the associated element in $C^\infty_c(\Sigma;\mathbb{R}^2)\oplus C^\infty_c(\Sigma;\mathbb{R}^2)$ (i.e., initial data) is given by $(\Phi_1,\Pi_1,\Phi_2,\Pi_2)$, where $\Phi_1 = ((E^-_S - E^+_S)v_1) \restriction \Sigma$, $\Pi_1 = (\nabla_\mathfrak{n}(E^-_S - E^+_S)v_1) \restriction \Sigma$ and $\Phi_2 = ((E^-_P - E^+_P)v_2) \restriction \Sigma$, $\Pi_2 = (\nabla_\mathfrak{n}(E^-_P - E^+_P)v_2) \restriction \Sigma$. Since $L \perp_M K$, it follows that $\mathrm{supp}(\Phi_1,\Pi_1,\Phi_2,\Pi_2)$ is disjoint from K. In particular $\tilde{t}^\Sigma_\rho$ acts like the identity, which shows that t^Σ_ρ acts like the identity on $[v]_{S\oplus P}$.

2. Consider regions $L^\pm \in \mathfrak{B}_M$ with $L^\pm \subseteq M^\pm_K$ and $L^+ \subseteq D_M(L^-)$ and take $[v]_{S\oplus P} = ([v_1]_S,[v_2]_P) \in C^\infty_c(L^+;\mathbb{R}^2)/(S\oplus P)C^\infty_c(M;\mathbb{R}^2)$. Let the associated

element in $C_c^\infty(\Sigma;\mathbb{R}^2) \oplus C_c^\infty(\Sigma;\mathbb{R}^2)$ be given by $V := (\Phi_1, \Pi_1, \Phi_2, \Pi_2)$. Using a smooth partition of unity, let us then write $V = V^\perp + V^*$, such that $\mathrm{supp}V^\perp$ is disjoint from K and such that $\mathrm{supp}V^* \subseteq D_M^+(L^-) \cap \Sigma$. This is possible since $D_M^+(L^-)$ is open and necessarily contains $K \cap \mathrm{supp}V$ as $L^- \subseteq M_K^-$. It then follows that $\tilde{t}_\rho^\Sigma V = V + \left(\tilde{t}_\rho^\Sigma V^* - V^*\right)$. According to the assumption and time-slice, see Appendix C.3, we see that the element of $C_c^\infty(M;\mathbb{R}^2)/(S \oplus P)C_c^\infty(M;\mathbb{R}^2)$ corresponding to V has a representative with support in L^-, so let us turn to $\left(\tilde{t}_\rho^\Sigma V^* - V^*\right)$. One way of seeing that the associated element has a representative with support in L^- is by noting that $D_M(D_M(L^-)\cap\Sigma) \subseteq D_M(L^-)$ is a region with Cauchy surface $D_M(L^-)\cap\Sigma$ and that $\left(\tilde{t}_\rho^\Sigma V^* - V^*\right)$ has compact support on $D_M(L^-)\cap\Sigma$. Hence it corresponds to an element in $C_c^\infty(D_M(D_M(L^-)\cap\Sigma);\mathbb{R}^n)/PC_c^\infty(D_M(D_M(L^-)\cap\Sigma);\mathbb{R}^n)$, and hence also to an element in $C_c^\infty(L^-;\mathbb{R}^n)/PC_c^\infty(M;\mathbb{R}^n)$. In summary

$$t_\rho^\Sigma\left[C_c^\infty(L^+;\mathbb{R}^n)/PC_c^\infty(L^+;\mathbb{R}^n)\right] \subseteq C_c^\infty(L^-;\mathbb{R}^n)/PC_c^\infty(L^-;\mathbb{R}^n). \tag{5.28}$$

3. Note that $\left(t_\rho^\Sigma\right)^{-1} = t_{-\rho}^\Sigma$. This, together with using the time-reversal symmetry of the previous argument shows that for all regions $L^\pm \in \mathfrak{B}_M$ such that $L^\pm \subseteq M_K^\pm$ and $L^- \subseteq D_M(L^+)$ we have

$$\left(t_\rho^\Sigma\right)^{-1}\left[C_c^\infty(L^-;\mathbb{R}^n)/PC_c^\infty(L^-;\mathbb{R}^n)\right] \subseteq C_c^\infty(L^+;\mathbb{R}^n)/PC_c^\infty(L^+;\mathbb{R}^n). \tag{5.29}$$

□

We can now show the following result.

Theorem 5.3.2. *Let M be a globally hyperbolic spacetime, let $S : C^\infty(M;\mathbb{R}) \to C^\infty(M;\mathbb{R})$ be normally hyperbolic and let ${}^S\mathcal{W}_{\mathfrak{B}_M}$ be the AQFT derived from S. Take any region $N \in \mathfrak{B}_M$ and any region $\tilde{N}$ with compact closure such that $\overline{\tilde{N}} \subseteq N$. Then for any $A \in {}^S\mathcal{W}_{\mathfrak{B}_M}(\tilde{N})$ there exists a measurement scheme*

$$(\mathcal{P}_{\mathfrak{B}_M}, T, \sigma, B) \tag{5.30}$$

for A with compact coupling $K \subseteq N$ and $B \in \mathcal{P}_{\mathfrak{B}_M}(L)$, where $L \subseteq M \setminus J^-(\overline{N})$ is any region such that $N \subseteq D(L)$. In fact, for all regions like N and $\tilde{N}$ above, there exists a measurement preparation

$$({}^S\mathcal{W}_{\mathfrak{B}_M}, T, \sigma) \tag{5.31}$$

such that for every $A \in {}^S\mathcal{W}_{\mathfrak{B}_M}(\tilde{N})$ we have

$$\varepsilon_\sigma^T(A) = A. \tag{5.32}$$

Remark: It is an interesting question whether the measurement schemes used here are FV-measurement schemes.

One of the main implications of this theorem is that *every* EVM A of ${}^S\mathcal{W}_{\mathfrak{B}_M}$ that may be localised in a region $\tilde{N}$ with compact closure admits a *causal* instrument, namely $\mathcal{J}_{\sigma,T}^{A(\cdot)}$. This shows that every observable admits at least one non-selective "state-update-map" that is consistent with causality and hence deserves the name *observable*.

Proof. Take $N \in \mathfrak{B}_M$ with compact closure, let P be normally hyperbolic and let ${}^P\mathcal{W}_{\mathfrak{B}_M}$ be the AQFT derived from P. Let Σ be a spacelike Cauchy surface of N, let $O \subseteq \Sigma$ be open and with compact closure such that $D_M(O) \supseteq \tilde{N}$ and let $\rho \in C_c^\infty(\Sigma; \mathbb{R})$ be such that $\rho \equiv 1$ on O. Let $\nu^{S\to P} : {}^S\mathcal{W}_{\mathfrak{B}_M}{}^g \to {}^P\mathcal{W}_{\mathfrak{B}_M}{}^g$ be the quasi-free unit-preserving $*$-isomorphism that maps $W_S(v)$ to $W_P(\tilde{v})$, where the solutions associated to v and $\tilde{v}$ have the same initial data on Σ. Then

$$({}^P\mathcal{W}_{\mathfrak{B}_M}, T_\rho^\Sigma, \sigma, \nu^{S\to P}(A)) \tag{5.33}$$

is a measurement scheme for A. To see this note that for any $W_S([f]_S)$ with $f \in C_c^\infty(\tilde{N}, \mathbb{R})$ we have that

$$\begin{aligned} T_\rho^\Sigma\Big(\mathbb{1}_{{}^S\mathcal{W}_{\mathfrak{B}_M}^g} \otimes \nu^{S\to P}(W_S([f]_S))\Big) &= T_\rho^\Sigma\Big(W_S(0) \otimes \nu^{S\to P}(W_S([f]_S))\Big) \\ &= W_S([f]_S) \otimes \mathbb{1}_{{}^P\mathcal{W}_{\mathfrak{B}_M}^g}, \end{aligned} \tag{5.34}$$

where we have identified the unique C^*-tensor product of ${}^S\mathcal{W}_{\mathfrak{B}_M}$ and ${}^P\mathcal{W}_{\mathfrak{B}_M}$, see Lemma B.4.2, with ${}^{S\oplus P}\mathcal{W}_{\mathfrak{B}_M}$. To see this note that the element of the symplectic space $C_c^\infty(\Sigma; \mathbb{R}^2) \oplus C_c^\infty(\Sigma; \mathbb{R}^2)$ associated to $W_S(0) \otimes \nu^{S\to P}(W_S(v))$ has support in O, on which $\rho \equiv 1$. There t_ρ^Σ acts as a rotation. By definition of $\nu^{S\to P}$ the claim follows. We use linearity and continuity of T_ρ^Σ to extend the result to arbitrary $A \in {}^S\mathcal{W}_{\mathfrak{B}_M}(\tilde{N})$.

Finally, note that for $P = S$, $\nu^{S\to P}$ is the identity, which finishes the result. □

The main tool of the above theorem is the K-map T_ρ^Σ, where $K = \mathrm{supp}\rho$. By Theorem 4.4.1, this map is associated to a $\tilde{K}$-perturbed variant on $\mathfrak{B}_{(M,\tilde{K})}$. Now it is an open question whether this $\tilde{K}$-perturbed variant admits a unique extension to a $\tilde{K}$-perturbed variant on $\mathfrak{B}_M$, and if it does, it would be very interesting to further investigate and assess how "physical" the associated perturbed AQFT is.

In the following section we will discuss an alternative that relies on perturbed AQFTs derived from linear real scalar fields, which allow to *asymptotically* measure any observable of a linear real scalar field.

5.4 ASYMPTOTIC MEASUREMENT SCHEMES

The following is based on [2], joint work with Fewster and Jubb.

In the following let M be a globally hyperbolic spacetime. Let us introduce the notion of *asymptotic* (FV-) measurement schemes.

Definition 5.4.1. *Let $\mathcal{S}_{\mathfrak{B}_M}$ be an AQFT, τ be a topology on $\mathcal{S}^g_{\mathfrak{B}_M}$ and $A \in \mathcal{S}_{\mathfrak{B}_M}(N)$ for some region N. Then a collection of measurement schemes $({}^\alpha\mathcal{P}_{\mathfrak{B}_M}, T_\alpha, \sigma_\alpha, B_\alpha)$, for α in some index set, such that*

$$\varepsilon^{T_\alpha}_{\sigma_\alpha}(B_\alpha) \to A \tag{5.35}$$

*with respect to τ is called a τ-*asymptotic measurement scheme *for A. In this case we say that A is asymptotically inducible.*

1. *We say that the asymptotic measurement scheme has coupling in $\tilde{N}$, for some (not necessarily compact) subset $\tilde{N} \subseteq M$, if for every α the perturbation zone of T_α is in $\tilde{N}$.*
2. *We say that the asymptotic measurement scheme has processing region $L \in \mathfrak{B}_M$ if for every $\alpha : B_\alpha \in {}^\alpha\mathcal{P}_{\mathfrak{B}_M}(L)$.*
3. *If $({}^\alpha\mathcal{P}_{\mathfrak{B}_M}, T_\alpha, \sigma_\alpha, B_\alpha)$ is Hermitian for every α, then we also call the τ-asymptotic measurement scheme* Hermitian*.*
4. *If $({}^\alpha\mathcal{P}_{\mathfrak{B}_M}, T_\alpha, \sigma_\alpha, B_\alpha)$ is an FV-measurement scheme for every α, then we also call the τ-asymptotic measurement scheme* FV*.*

If a collection of measurement schemes $({}^\alpha\mathcal{P}_{\mathfrak{B}_M}, T_\alpha, \sigma_\alpha, B_\alpha)$ forms a τ-asymptotic measurement scheme for some A, it might not be the case that there exists some $\tilde{\alpha}$ such that $\varepsilon^{T_{\tilde{\alpha}}}_{\sigma_{\tilde{\alpha}}}(B_{\tilde{\alpha}}) = A$, however, for every τ-neighbourhood of A there exists some $\tilde{\alpha}$ such that $\varepsilon^{T_{\tilde{\alpha}}}_{\sigma_{\tilde{\alpha}}}(B_{\tilde{\alpha}})$ is in this neighbourhood. Hence, if the topology τ is physically motivated and if one accepts a certain margin of error for a desired observable A (parametrised by a τ-neighbourhood of A), then an asymptotic measurement scheme allows for a measurement of A within any finite error margin.

In the following we will show that every observable of the AQFT associated to a linear real scalar field fulfilling a normally hyperbolic equation of motion admits an asymptotic FV-measurement scheme.

We first show the following results.

Lemma 5.4.2. *Let $\mathcal{S}_{\mathfrak{B}_M}$ be an AQFT, τ be a topology on $\mathcal{S}^g_{\mathfrak{B}_M}$, $A, A' \in \mathcal{S}_{\mathfrak{B}_M}(N)$ for some region N and let the collection of all $({}^\alpha\mathcal{P}_{\mathfrak{B}_M}, T_\alpha, \sigma_\alpha, B_\alpha)$ for α in some index set be a τ-asymptotic measurement scheme for A.*

1. *If $\left(\mathcal{S}^g_{\mathfrak{B}_M}, \tau\right)$ is a topological vector space and if $({}^\alpha\mathcal{P}_{\mathfrak{B}_M}, T_\alpha, \sigma_\alpha, B'_\alpha)$ is a τ-asymptotic measurement scheme for A', then, for every $c \in \mathbb{C}$, the collection $({}^\alpha\mathcal{P}_{\mathfrak{B}_M}, T_\alpha, \sigma_\alpha, B_\alpha + cB'_\alpha)$ is a τ-asymptotic measurement scheme for $A + cA'$.*

2. *If the topology τ is $*$-compatible, i.e., the $*$-operation is continuous with respect to τ, then the collection $({}^\alpha\mathcal{P}_{\mathfrak{B}_M}, T_\alpha, \sigma_\alpha, B^*_\alpha)$ is a τ-asymptotic measurement scheme for A^*.*

In particular, if a Hermitian A admits a τ-asymptotic measurement scheme for a vector space topology τ that is $*$-compatible, then A admits a Hermitian *τ-asymptotic measurement scheme.*

Proof. 1. This follows immediately from linearity of $\varepsilon^{T_\alpha}_{\sigma_\alpha}$.

2. This follows immediately from the fact that $\varepsilon^{T_\alpha}_{\sigma_\alpha}(B^*_\alpha) = \varepsilon^{T_\alpha}_{\sigma_\alpha}(B_\alpha)^*$.

Finally note that if A is Hermitian and τ as assumed, then

$$({}^\alpha\mathcal{P}_{\mathfrak{B}_M}, T_\alpha, \sigma_\alpha, \frac{1}{2}(B_\alpha + B^*_\alpha)) \tag{5.36}$$

is a Hermitian τ-asymptotic measurement scheme for A. □

We make another elementary observation, which is Lemma 2.2 in [2].

Lemma 5.4.3. *Let $\mathcal{S}_m$ (respectively, $\mathcal{S}_a$) be the set of $A \in \mathcal{S}^g_{\mathfrak{B}_M}$ such that there is a measurement scheme (resp., a τ-asymptotic measurement scheme) for A. Then $\mathcal{S}_a$ is the τ-closure of $\mathcal{S}_m$ in $\mathcal{S}^g_{\mathfrak{B}_M}$. Consequently $\mathcal{S}_a = \mathcal{S}$ if and only if $\mathcal{S}_m$ is τ-dense in $\mathcal{S}^g_{\mathfrak{B}_M}$.*

Proof. Suppose $A \in \mathcal{S}_a$. Then, by definition, there exists a net of elements A_α in $\mathcal{S}_m$ that converges to A, i.e., $\mathcal{S}_a \subseteq \overline{\mathcal{S}_m}$. Conversely, if $A \in \overline{\mathcal{S}_m}$ then $A = \lim_\alpha A_\alpha$ where $(A_\alpha)_\alpha$ is a net of elements in $\mathcal{S}_m$. Accordingly, we may find a measurement scheme H_α for each A_α, whereupon $(H_\alpha)_\alpha$ is a τ-asymptotic measurement scheme for A. Hence $\overline{\mathcal{S}_m} \subseteq \mathcal{S}_a$. □

Let us now turn to a class of FV-measurement preparations for the linear real scalar field that will form the basis of the asymptotic FV-measurement schemes discussed in Sec. 5.4.2.

5.4.1 Quasi-free K-perturbed variants for the linear real scalar field

The following FV-measurement preparations were introduced in [2] and are a generalisation of the ones presented in [10]. We strictly follow the presentation in Sec. 4.3 in [2].

Let S be a formally self-adjoint normally hyperbolic equation of motion operator on $C^\infty(M;\mathbb{R})$ and let P be a formally self-adjoint normally hyperbolic equation of motion operator on $C^\infty(M;\mathbb{R})$ such that $P^{\oplus k}$ is formally self-adjoint and normally hyperbolic on $C^\infty(M;\mathbb{R}^k)$.

Then $S \oplus P^{\oplus k}$ is formally self-adjoint and normally hyperbolic on $C^\infty(M;\mathbb{R}^{k+1})$. For $\lambda \in \mathbb{C}$ we define Q_λ on $C^\infty(M;\mathbb{R}^{k+1}) \simeq C^\infty(M;\mathbb{R}) \oplus C^\infty(M;\mathbb{R}^k)$ conveniently defined in block matrix notation by

$$Q_\lambda := \begin{pmatrix} S & \lambda R^T \\ \lambda R & P^{\oplus k} \end{pmatrix}, \tag{5.37}$$

where $S : C^\infty(M;\mathbb{R}) \to C^\infty(M;\mathbb{R})$ is a 1×1 matrix, $P^{\oplus k} : C^\infty(M;\mathbb{R}^k) \to C^\infty(M;\mathbb{R}^k)$ is a $k \times k$ matrix, and R and R^T are $k \times 1$ and $1 \times k$ matrices so that

$$\begin{aligned} R = \begin{pmatrix} R_1 \\ \vdots \\ R_k \end{pmatrix} : C^\infty(M;\mathbb{R}) \to C^\infty(M;\mathbb{R}^k); \qquad f \mapsto \begin{pmatrix} R_1 f \\ \vdots \\ R_k f \end{pmatrix}, \\ R^T = (R_1, \dots, R_k) : C^\infty(M;\mathbb{R}^k) \to C^\infty(M;\mathbb{R}); \qquad \vec{f} \mapsto \sum_{j=1}^{k} R_j f_j. \end{aligned} \tag{5.38}$$

Here, R_j is the operator of point-wise multiplication with ρ_j, where $\rho_1, ..., \rho_k \in C_c^\infty(M;\mathbb{R})$. For every $\lambda \in \mathbb{R}$, Q_λ is obviously a formally self-adjoint normally hyperbolic equation of motion operator that may be seen as the equation of motion operator derived from the formal Lagrangean density

$$\mathcal{L}_S + \mathcal{L}_{P^{\oplus k}} - \lambda \sum_{j=1}^{k} \rho_j \varphi_S (\varphi_{P^{\oplus k}})_j. \tag{5.39}$$

According to Lemma C.3.2, $S \oplus P^{\oplus k}$ and Q_λ define a K-perturbed variant, where $K \subseteq M$ is a compact subset containing the support of ρ_j for $j = 1, ..., k$. The associated quasi-free scattering map is given in terms of a symplectic map ϑ_λ defined on the combined symplectic spaces. ϑ_λ can be conveniently expressed by its action on representatives in terms of the map $\theta_\lambda : C_c^\infty(M;\mathbb{R}) \oplus C_c^\infty(M;\mathbb{R}^k) \to$

$C_c^\infty(M;\mathbb{R}) \oplus C_c^\infty(M;\mathbb{R}^k)$ defined as

$$\theta_\lambda \begin{pmatrix} F \\ \vec{G} \end{pmatrix} = \begin{pmatrix} F \\ \vec{G} \end{pmatrix} - \begin{pmatrix} 0 & \lambda R^T \\ \lambda R & 0 \end{pmatrix} E^-_{Q_\lambda} \begin{pmatrix} F \\ \vec{G} \end{pmatrix} \tag{5.40}$$

so that

$$\vartheta_\lambda \begin{bmatrix} F \\ \vec{G} \end{bmatrix}_{S\oplus P} = \left[\theta_\lambda \begin{pmatrix} F \\ \vec{G} \end{pmatrix}\right]_{S\oplus P} \tag{5.41}$$

for $(F,\vec{G})^T \in C_c^\infty(M_K^+;\mathbb{R}^{k+1})$.

Let us now make two interesting technical observations, whose importance will become clear in Sec. 5.4.2. The first one follows from Lemma 4.6 in [2].

Lemma 5.4.4. *For $\vec{h} = (h^1,...,h^k)^T \in C_c^\infty(M;\mathbb{R}^k)$ and $\lambda > 0$ define $\vec{h}_\lambda := \vec{h}/\lambda$. Then*

$$\lim_{\lambda\to 0} \mathrm{pr}_1 \left(\theta_\lambda \begin{pmatrix} 0 \\ \vec{h}_\lambda \end{pmatrix}\right) = \lim_{\lambda\to 0} \mathrm{pr}_1 \left(-\begin{pmatrix} 0 & R^T \\ R & 0 \end{pmatrix} E^-_{Q_\lambda} \begin{pmatrix} 0 \\ \vec{h} \end{pmatrix}\right) = -R^T E^-_{P^{\oplus k}} \vec{h} \tag{5.42}$$

in $C_c^\infty(M;\mathbb{R})$.

Proof idea. The proof of Lemma 5.4.4 is given in [2] and is based on (currently) unpublished results [53], according to which the map

$$\mathbb{C} \ni \lambda \mapsto E^-_{Q_\lambda} \tag{5.43}$$

is holomorphic on $\mathbb{C}$ with respect to the topology of bounded convergence of continuous linear maps from the LF space $C_c^\infty(M;\mathbb{C}^{k+1})$ to the Fréchet space $C^\infty(M;\mathbb{C}^{k+1})$. (See [54] for the definition of the topologies involved.) As the linear operator $\begin{pmatrix} 0 & R^T \\ R & 0 \end{pmatrix}$ is continuous from $C^\infty(M;\mathbb{C}^{k+1})$ to $C_c^\infty(M;\mathbb{C}^{k+1})$, the map

$$\lambda \mapsto \begin{pmatrix} 0 & R^T \\ R & 0 \end{pmatrix} E^-_{Q_\lambda} \tag{5.44}$$

is holomorphic on all of $\mathbb{C}$ with respect to the topology of bounded convergence of continuous linear maps from $C_c^\infty(M;\mathbb{C}^{k+1})$ into itself [53]. In particular,

$$\begin{pmatrix} 0 & R^T \\ R & 0 \end{pmatrix} E^-_{Q_\lambda} \begin{pmatrix} 0 \\ \vec{h} \end{pmatrix} \xrightarrow{\lambda\to 0} \begin{pmatrix} 0 & R^T \\ R & 0 \end{pmatrix} \begin{pmatrix} E^-_S & 0^T \\ 0 & E^-_{P^{\oplus k}} \end{pmatrix} \begin{pmatrix} 0 \\ \vec{h} \end{pmatrix} = \begin{pmatrix} R^T E^-_{P^{\oplus k}} \vec{h} \\ \vec{0} \end{pmatrix} \tag{5.45}$$

in $C_c^\infty(M;\mathbb{C}^{k+1})$. □

The second important observation is the following, see Lemma 4.2 in [2].

Lemma 5.4.5. *Let S, P be a normally hyperbolic equation of motion operator on $C^\infty(M;\mathbb{R})$. Then, for every region $N \subseteq M$ and every test function $f \in C_c^\infty(N;\mathbb{R})$ there exists*

1. *a region $\tilde{N} \subseteq N$ with compact closure $\overline{\tilde{N}} \subseteq N$,*
2. *$\tilde{f} \in C_c^\infty(\tilde{N};\mathbb{R})$,*
3. *$\rho \in C_c^\infty(\tilde{N};\mathbb{R})$ and*
4. *$h \in C_c^\infty(M;\mathbb{R})$*

such that $[f]_S = [\tilde{f}]_S$ and

$$\tilde{f} = -RE_P^- h, \tag{5.46}$$

where R is the operator of pointwise multiplication with ρ. In fact, h may be chosen with support in any region $L \subseteq M^+_{\mathrm{supp}\tilde{f}}$ whose domain of dependence contains N.

The proof of Lemma 5.4.5 is given in Appendix B of [2]. We only present the idea of the proof.

Proof idea. The proof may be divided into two parts.

1. Following Lemma B.1 in [2], we pick any Cauchy surface Σ of N and look at the element in $C_c^\infty(\Sigma;\mathbb{R}^2)$ corresponding to $[f]_S$, which has compact support on Σ that is contained in an open set $B \subseteq \Sigma$. Let us now pick initial data with compact support that equals $(1, 0)$ on B. Then there exists a *region* $\tilde{N} \subseteq N$ with compact closure that contains B on which the solution H associated to the picked initial data has no zeros. Then, since $\tilde{N}$ contains B, one can find $\tilde{f} \in C_c^\infty(\tilde{N};\mathbb{R})$ such that $[f]_S = \left[\tilde{f}\right]_S$. Moreover, for any region L that contains N in its domain of dependence, H may be written as $E_P^- h - E_P^+ h$ for some $h \in C_c^\infty(L; R)$.

2. Following Lemma B.2 in [2], let us define $\rho := -\tilde{f}/H$, which is smooth and has compact support in $\tilde{N}$. Let us now choose a region $L \subseteq M^+_{\mathrm{supp}\tilde{f}}$ such that $N \subseteq D_M(L)$. Then

$$\tilde{f} = -RH = -RE_P^- h + RE_P^+ h = -RE_P^- h, \tag{5.47}$$

   for some $h \in C_c^\infty(L;\mathbb{R})$, where we used that $RE_P^+ h \equiv 0$, because $\mathrm{supp}\rho = \mathrm{supp}\tilde{f}$ and hence $J_M^+(L) \cap \mathrm{supp}\rho = \emptyset$.

□

At this point let us note that $\left(C_c^\infty(M;\mathbb{R}^n)/SC_c^\infty(M;\mathbb{R}^n), \tilde{E}_P\right)$, equipped with the quotient topology, is a locally convex nuclear topological vector space, see Appendix C.3. Then we can formulate the following corollary.

Corollary 5.4.6. *For every region $N \subseteq M$ and every collection of test functions $f_1, ..., f_k \in C_c^\infty(N;\mathbb{R})$ there exists*

1. *a region $\tilde{N} \subseteq N$ with compact closure,*

2. *$\rho_1, ..., \rho_k \in C_c^\infty(\tilde{N};\mathbb{R})$, and*

for every region $L \subseteq M^{\pm}_{\tilde{N}}$ whose domain of dependence contains N there exist

3. *$h_1, ..., h_k \in C_c^\infty(L;\mathbb{R})$*

such that

$$\lim_{\lambda\to 0} \mathrm{pr}_1\left(\theta_\lambda\begin{pmatrix} 0 \\ \vec{h}^j/\lambda \end{pmatrix}\right) = -R_j E_P^- h_j \tag{5.48}$$

in $C_c^\infty(M;\mathbb{R})$, where $\vec{h}^j = (0, ..., 0, h_j, 0, ..., 0)^T \in C_c^\infty(M;\mathbb{R}^k)$ and

$$\left[-R_j E_P^- h_j\right]_S = [f_j]_S, \tag{5.49}$$

where θ_λ is defined in Eq. (5.40) *in terms of R, which is defined via $\rho_1, ..., \rho_k$ according to Eq.* (5.38)*. In particular*

$$\lim_{\lambda\to 0}\left[\mathrm{pr}_1\left(\theta_\lambda\begin{pmatrix} 0 \\ \vec{h}^j/\lambda \end{pmatrix}\right)\right]_S = [f_j]_S \tag{5.50}$$

in $C_c^\infty(M;\mathbb{R}^n)/SC_c^\infty(M;\mathbb{R}^n)$.

Proof. For every j let us apply Lemma 5.4.5 to f_j, from which we get regions $\tilde{N}_j$ and functions ρ_j and h_j. Let us now choose $\tilde{N}$ to be the causal hull of $\tilde{N}_1 \cup ... \cup \tilde{N}_k$. Let us now pick any region $L \subseteq M^{\pm}_{\tilde{N}}$ whose domain of dependence contains N. Then according to Lemma 5.4.5, h_j may be chosen in $C_c^\infty(L;\mathbb{R})$.

Then the claim follows from Lemma 5.4.4 and the fact that the quotient map $f \mapsto [f]_S$ is continuous. □

The relevance of this observation will become clear in the next section.

5.4.2 Asymptotic FV-measurement schemes for the linear real scalar field

The discussion of the quasi-free K-perturbed variants of the previous section immediately gives rise to FV-measurement schemes for the field algebra as well as the Weyl algebra defined by S. Furthermore, they may be used to construct asymptotic measurement schemes.

5.4.2.1 THE WEYL ALGEBRA Let us pick a quasi-free state $\sigma^{\otimes k}$ on ${}^{P^{\oplus k}}\mathcal{W}^{g}_{\mathfrak{B}_M}$ and $A \in {}^{S}\mathcal{W}_{\mathfrak{B}_M}(N)$. Let us denote the Weyl generators of ${}^{S}\mathcal{W}^{g}_{\mathfrak{B}_M}$ by W_S, similarly for $P^{\oplus k}$.

Let us assume that $A = \sum_{j=1}^{k} c_j W_S\Big([f_j]_S\Big)$. Using Corollary 5.4.6, let us then define for every $\lambda > 0$ the FV measurement scheme

$$H_\lambda := \left({}^{P^{\oplus k}}\mathcal{W}_{\mathfrak{B}_M}, \Theta_\lambda, \sigma^{\otimes k}, \sum_{j=1}^{k} c_j \frac{W_{P^{\oplus k}}\Big(\big[\vec{h}^j/\lambda\big]_{P^{\oplus k}}\Big)}{\sigma^{\otimes k}\bigg(W_{P^{\oplus k}}\Big(\Big[\mathrm{pr}_{\neg 1}\theta_\lambda\big(0, \vec{h}^j/\lambda\big)^T\Big]_{P^{\oplus k}}\Big)\bigg)}\right), \tag{5.51}$$

for

$$\begin{aligned} &\varepsilon^{\Theta_\lambda}_{\sigma^{\otimes k}}\left(\sum_{j=1}^{k} c_j \frac{W_{P^{\oplus k}}\Big(\big[\vec{h}^j/\lambda\big]_{P^{\oplus k}}\Big)}{\sigma^{\otimes k}\bigg(W_{P^{\oplus k}}\Big(\Big[\mathrm{pr}_{\neg 1}\theta_\lambda\big(0, \vec{h}^j/\lambda\big)^T\Big]_{P^{\oplus k}}\Big)\bigg)}\right) \\ &= \sum_{j=1}^{k} c_j W_S\Big(\Big[\mathrm{pr}_1\theta_\lambda\big(0, \vec{h}^j/\lambda\big)^T\Big]_S\Big), \end{aligned} \tag{5.52}$$

where $\Big[\mathrm{pr}_1\theta_\lambda\big(0, \vec{h}^j/\lambda\big)^T\Big]_S$ converges to $[f_j]_S$ as $\lambda \to 0$ and $\mathrm{pr}_{\neg 1}$ denotes the projection on all but the first component.

What is left in order to show that $(H_\lambda)_{\lambda>0}$ is an asymptotic measurement scheme for A is to find a *reasonable* topology on ${}^{S}\mathcal{W}^{g}_{\mathfrak{B}_M}$ such that $W_S(F_j) \to W_S(F)$ whenever $F_j \to F$ in $C^\infty_c(M;\mathbb{R}^n)/SC^\infty_c(M;\mathbb{R}^n)$. The topology described in Appendix B.4.1, which we will denote by τ^W, fulfills this criterion. In particular, $(H_\lambda)_{\lambda>0}$ is a τ^W-asymptotic FV measurement scheme for A.

Equipped with this insight we can prove the following theorem, which is Theorem 6.4 in [2].

Theorem 5.4.7. *Let τ^W be the topology on ${}^{S}\mathcal{W}^{g}_{\mathfrak{B}_M}$ given in Definition B.4.4. Then, for every region $N \in \mathfrak{B}_M$ with compact closure, every $A \in {}^{S}\mathcal{W}_{\mathfrak{B}_M}(N)$ and every region $L \subseteq M \setminus J^-(\overline{N})$ such that $N \subseteq D(L)$ there exists a τ^W-*asymptotic FV measurement scheme *for A with coupling in N and processing region L.*

If A is Hermitian, then the τ^W-asymptotic FV measurement scheme can be chosen to be Hermitian as well.

Remark: In fact, as noted in [2], for an admissible processing region L, i.e., $L \subseteq M \setminus J^-(\overline{N})$ it holds that $N \subseteq D(L)$ if and only if $N \subseteq D^-(L)$.[7]

Proof. We have seen that the set of all elements in ${}^S\mathcal{W}_{\mathfrak{B}_M}(N)$ possessing asymptotic measurement schemes contains all finite linear combinations of Weyl generators, and hence also their τ^W-closure. Using the fact that τ^W is weaker than the norm topology, see Lemma B.4.7, and that the norm-closure of the set of all finite linear combinations of Weyl generators in ${}^S\mathcal{W}_{\mathfrak{B}_M}(N)$ is all of ${}^S\mathcal{W}_{\mathfrak{B}_M}(N)$, we see that *every $A \in {}^S\mathcal{W}_{\mathfrak{B}_M}(N)$* is asymptotically inducible via FV measurement schemes with coupling in N and processing region L.

Finally, since the $*$-operation is τ^W-continuous, it follows from Lemma 5.4.2 that if A is Hermitian, the associated τ^W-asymptotic FV measurement scheme can be chosen to be Hermitian as well. □

We are now able to prove a simple corollary, which is Corollary 6.5 in [2]. For the definition of $\mathfrak{S}_c$ see Def. B.4.4.

Corollary 5.4.8. *Let N be a region with compact closure. Then, for every $\omega \in \mathfrak{S}_c$ and every element A in the unital von Neumann algebra given by the weak operator topology closure of $\pi_\omega\left[{}^S\mathcal{W}_{\mathfrak{B}_M}(N)\right]$ there exists a net of elements $(A_\alpha)_\alpha \subseteq {}^S\mathcal{W}_{\mathfrak{B}_M}(N)$ and Hermitian FV measurement schemes H_α for each A_α such that $\pi_\omega(A_\alpha) \to A$ in the weak operator topology.*

If A is Hermitian, then every H_α can be chosen to be Hermitian as well.

Proof. We have seen that the set of elements in ${}^S\mathcal{W}_{\mathfrak{B}_M}(N)$ admitting an FV measurement scheme is τ^W-dense in ${}^S\mathcal{W}_{\mathfrak{B}_M}(N)$ and by Lemma B.4.7, the image under π_ω of the set of elements in ${}^S\mathcal{W}_{\mathfrak{B}_M}(N)$ admitting an FV measurement scheme is weak operator topology-dense in the weak operator topology closure of $\pi_\omega\left[{}^S\mathcal{W}_{\mathfrak{B}_M}(N)\right]$, which shows the first part.

For the last sentence note that the $*$-operations is continuous with respect to the weak operator topology (which is a vector space topology), which finishes the proof. □

[7] If $p \in N \cap D^+(L)$, then every past inextendible causal curve through p must intersect L, so $L \cap J^-(\overline{N}) \neq \emptyset$, which is a contradiction.

5.4.2.2 THE FIELD ALGEBRA The main ideas of the previous section may also be used to prove a similar result for the field algebra.

We again closely follow [2] and emphasise, that the general element of ${}^S\mathcal{F}^g_{\mathfrak{B}_M}$ is a complex linear combination of products of generators. Using the canonical commutation relations, every element of ${}^S\mathcal{W}^g_{\mathfrak{B}_M}$ can be expressed as a finite linear combination of symmetrised products of generators, which will be denoted by φ_S. Next, by the multi-linear generalisation of the polarisation identity, see e.g. Eq. (A.4) in [55], every symmetrised n-fold product of $\varphi_S(f_1), ..., \varphi_S(f_n)$ can be written as

$$\begin{aligned}&\sum_{\pi\in S_n}\varphi_S(f_{\pi(1)})...\varphi_S(f_{\pi(n)})\\&=\frac{1}{2^n}\sum_{\epsilon_1,...,\epsilon_n=0}^{1}(-1)^{\epsilon_1+...+\epsilon_n}\left((-1)^{\epsilon_1}\varphi_S(f_1)+...+(-1)^{\epsilon_n}\varphi_S(f_n)\right)^n\\&=\frac{1}{2^n}\sum_{\epsilon_1,...,\epsilon_n=0}^{1}(-1)^{\epsilon_1+...+\epsilon_n}\left(\varphi_S((-1)^{\epsilon_1}f_1+...+(-1)^{\epsilon_n}f_n)\right)^n.\end{aligned} \tag{5.53}$$

Therefore every element of ${}^S\mathcal{F}^g_{\mathfrak{B}_M}$ may be written as a complex linear combination of powers of generators

$$A=\sum_{j=1}^{k}c_j\varphi_S(f_j)^{n_j} \tag{5.54}$$

with real-valued f_j and $c_j\in\mathbb{C}$. Moreover, choosing linearly independent $\varphi_S(f_j)^{n_j}$, the coefficients c_j are real if and only if A is Hermitian.

Using formal power-series with formal (real) parameter x and Corollary 5.4.6, let us now define for $\lambda>0$

$$H_\lambda:=\left({}^{P^{\oplus k}}\mathcal{F}_{\mathfrak{B}_M},\Theta_\lambda,\sigma^{\otimes k},\sum_{j=1}^{k}c_j(-\mathrm{i})^{n_j}\frac{\mathrm{d}^{n_j}}{\mathrm{d}x^{n_j}}\left.\frac{e^{\mathrm{i}x\varphi_{P^{\oplus k}}(\vec{h}^j/\lambda)}}{\sigma^{\otimes k}\left(e^{\mathrm{i}x\varphi_{P^{\oplus k}}\left(\mathrm{pr}_{\neg 1}\theta_\lambda\left(0,\vec{h}^j/\lambda\right)^T\right)}\right)}\right|_{x=0}\right), \tag{5.55}$$

which is an FV-measurement scheme for

$$\begin{aligned}&\sum_{j=1}^{k}c_j(-\mathrm{i})^{n_j}\varepsilon^{\Theta_\lambda}_{\sigma^{\otimes k}}\left(\frac{\mathrm{d}^{n_j}}{\mathrm{d}x^{n_j}}\left.\frac{e^{\mathrm{i}x\varphi_{P^{\oplus k}}(\vec{h}^j/\lambda)}}{\sigma^{\otimes k}\left(e^{\mathrm{i}x\varphi_{P^{\oplus k}}\left(\mathrm{pr}_{\neg 1}\theta_\lambda\left(0,\vec{h}^j/\lambda\right)^T\right)}\right)}\right|_{x=0}\right)\\&=\sum_{j=1}^{k}c_j(-\mathrm{i})^{n_j}\frac{\mathrm{d}^{n_j}}{\mathrm{d}x^{n_j}}\left.e^{\mathrm{i}x\varphi_S\left(\mathrm{pr}_1\theta_\lambda\left(0,\vec{h}^j/\lambda\right)^T\right)}\right|_{x=0}=\sum_{j=1}^{k}c_j\varphi_S\left(\mathrm{pr}_1\theta_\lambda\left(0,\vec{h}^j/\lambda\right)^T\right)^{n_j}.\end{aligned} \tag{5.56}$$

Let us now note that the field algebra ${}^{S}\mathcal{F}^{g}_{\mathfrak{B}_M}$ is a nuclear $*$-algebra, see Appendix C.3. In particular, its topology, which we will denote by τ^{φ}, is a vector space topology for which the $*$-operation is continuous.

One sees immediately from the definition of τ^{φ} and Corollary 5.4.6 that

$$\lim_{\lambda\to 0}\sum_{j=1}^{k} c_j\varphi_S\Big(\mathrm{pr}_1\theta_\lambda\Big(0,\vec{h}^j/\lambda\Big)^T\Big)^{n_j} = \sum_{j=1}^{k} c_j\varphi_S(f_j)^{n_j} = A \tag{5.57}$$

with respect to τ^{φ}.

Finally, we note that for *real* c_j, each measurement scheme H_λ is Hermitian. In summary we have shown the following theorem, which is Theorem 5.1 in [2].

Theorem 5.4.9. *Let* ${}^{S}\mathcal{F}^{g}_{\mathfrak{B}_M}$ *be equipped with* τ^{φ}*. Then, for every region* N *with compact closure, for every* $A \in {}^{S}\mathcal{F}_{\mathfrak{B}_M}(N)$ *and for every region* $L \subseteq M \setminus J^{-}(\overline{N})$ *such that* $N \subseteq D(L)$ *there exists a* τ^{φ}*-asymptotic FV measurement scheme for* A *with coupling in* N *and processing region* L*.*

If A *is Hermitian, then the* τ^{φ}*-asymptotic FV measurement scheme can be chosen to be Hermitian as well.*

In summary, every observable of the AQFT of a linear real scalar field fulfilling a normally hyperbolic equation of motion admits an asymptotic FV measurement scheme.

— 6 —

Applications to Relativistic Quantum Information

Relativistic quantum information (RQI) is the name for the field of research concerned with the study of the quantum information-theoretic aspects of relativistic quantum systems, such as quantum fields.

6.1 Entanglement harvesting

A quantum information-theoretic resource that plays an important role in RQI is entanglement. It is then an interesting question how this resource can, if not efficiently then at least in principle, be accessed and extracted. One way of performing this task is via *entanglement harvesting*.

Before we turn to the formal definitions in the following sections, let us discuss the heuristic idea behind this procedure. Let $\mathcal{S}_1, \mathcal{S}_2$ be two quantum systems and let ω be an entangled state on the free combination $\mathcal{S}_1 \& \mathcal{S}_2$, see Sec. 2.3. Furthermore, let $\mathcal{P}_1, \mathcal{P}_2$ be two quantum probes and let the free combination $\mathcal{P}_1 \& \mathcal{P}_2$ be initialised in an uncorrelated state σ. The goal now is to transfer the entanglement from ω and "entangle" the state σ to get an entangled state σ' on $\mathcal{P}_1 \& \mathcal{P}_2$. If successful, one could say that entanglement has been "harvested", which motivates the name entanglement harvesting. One way of attempting this is to (possibly independently) couple $\mathcal{P}_1$ to $\mathcal{S}_1$ and $\mathcal{P}_2$ to $\mathcal{S}_2$. Then, the resulting probe state σ' may exhibit some correlation or even entanglement. One usually does not explicitly consider $\mathcal{S}_1$ and $\mathcal{S}_2$ but rather a quantum system $\mathcal{S}$ that comprises $\mathcal{S}_1$ and $\mathcal{S}_2$.[1]

Let us now turn to the rigorous formulation of entanglement harvesting in AQT and AQFT.

[1] In fact, in some cases a clear identification of independent subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$ might not always be possible.

6.1.1 The case of AQT

The basic idea behind entanglement harvesting can be explained in the very general framework of AQT as follows.

Let $\mathcal{S}, \mathcal{P}_\mathsf{A}$ and $\mathcal{P}_\mathsf{B}$ be three unital $*$-algebras and let $\left(\mathcal{P}_\mathsf{A} \otimes \mathcal{P}_\mathsf{B}, \Theta_{\{\mathsf{A},\mathsf{B}\}}, \sigma_\mathsf{A} \otimes \sigma_\mathsf{B}\right)$ be a measurement preparation for $\mathcal{S}$. Then for every state ω on $\mathcal{S}$ we can define the state

$$A \mapsto \omega \otimes \sigma_\mathsf{A} \otimes \sigma_\mathsf{B}(\Theta_{\mathsf{A},\mathsf{B}}(\mathbb{1} \otimes A)), \tag{6.1}$$

for $A \in \mathcal{S}$. Defining

$$\mathbb{F} : \mathcal{S} \otimes \mathcal{P}_\mathsf{A} \otimes \mathcal{P}_\mathsf{B} \to \mathcal{P}_\mathsf{A} \otimes \mathcal{P}_\mathsf{B} \otimes \mathcal{S} \tag{6.2}$$

as the canonical isomorphism and setting

$$\Theta^{\mathbb{F}}_{\{\mathsf{A},\mathsf{B}\}} := \mathbb{F} \circ \Theta_{\{\mathsf{A},\mathsf{B}\}} \circ \mathbb{F}^{-1}, \tag{6.3}$$

we can use notation introduced in Lemma 5.2.3 and write

$$\sigma_{\omega,\Theta^{\mathbb{F}}_{\{\mathsf{A},\mathsf{B}\}}}(A) = \sigma_\mathsf{A} \otimes \sigma_\mathsf{B} \otimes \omega\left(\Theta^{\mathbb{F}}_{\mathsf{A},\mathsf{B}}(A \otimes \mathbb{1})\right) = \omega \otimes \sigma_\mathsf{A} \otimes \sigma_\mathsf{B}(\Theta_{\mathsf{A},\mathsf{B}}(\mathbb{1} \otimes A)). \tag{6.4}$$

Hence $\sigma_{\omega,\Theta^{\mathbb{F}}_{\{\mathsf{A},\mathsf{B}\}}}$ is the *effectively* updated state on $\mathcal{P}_\mathsf{A} \otimes \mathcal{P}_\mathsf{B}$ after the perturbation of the *initial* state $\sigma_\mathsf{A} \otimes \sigma_\mathsf{B}$ by the coupling of $\mathcal{P}_\mathsf{A} \otimes \mathcal{P}_\mathsf{B}$ to $\mathcal{S}$ via $\Theta_{\{\mathsf{A},\mathsf{B}\}}$. From the last expression in Eq. (6.4) it follows immediately that

$$\sigma_{\omega,\Theta^{\mathbb{F}}_{\{\mathsf{A},\mathsf{B}\}}} = \omega \circ \varepsilon_{\sigma_\mathsf{A} \otimes \sigma_\mathsf{B}, \Theta_{\{\mathsf{A},\mathsf{B}\}}}. \tag{6.5}$$

Obviously, the *initial* state $\sigma_\mathsf{A} \otimes \sigma_\mathsf{B}$ on $\mathcal{P}_\mathsf{A} \otimes \mathcal{P}_\mathsf{B}$ is a product state and hence shows no correlations between any pair of observables $A \otimes \mathbb{1}_\mathsf{B}$ and $\mathbb{1}_\mathsf{A} \otimes B$. However, this is not necessarily true for the updated state $\sigma_{\omega,\Theta_{\{\mathsf{A},\mathsf{B}\}}}$. We see that the state $\sigma_\mathsf{A} \otimes \sigma_\mathsf{B}$ might have "gotten" correlated or even entangled through the interaction.

Let us now consider the special case where $\Theta_{\mathsf{A},\mathsf{B}}$ decomposes as

$$\Theta_{\{\mathsf{A},\mathsf{B}\}} = \check{\Theta}_\mathsf{A} \circ \check{\Theta}_\mathsf{B}, \tag{6.6}$$

where $\check{\Theta}_\mathsf{J}$ is defined below Eq. (5.12), in which case $\left(\mathcal{P}_\mathsf{A} \otimes \mathcal{P}_\mathsf{B}, \Theta_{\{\mathsf{A},\mathsf{B}\}}, \sigma_\mathsf{A} \otimes \sigma_\mathsf{B}\right)$ may be seen as the combination of the two AQT measurement preparations $(\mathcal{P}_\mathsf{A}, \Theta_\mathsf{A}, \sigma_\mathsf{A})$ and $(\mathcal{P}_\mathsf{B}, \Theta_\mathsf{B}, \sigma_\mathsf{B})$. Then we can show the following useful lemma.

Lemma 6.1.1. *For every $A \otimes B \in \mathcal{P}_\mathsf{A} \otimes \mathcal{P}_\mathsf{B}$ it holds that*

$$\varepsilon_{\sigma_\mathsf{A} \otimes \sigma_\mathsf{B}, \check{\Theta}_\mathsf{A} \circ \check{\Theta}_\mathsf{B}}(A \otimes B) = \eta_{\sigma_\mathsf{A}}(\Theta_\mathsf{A}(\varepsilon_{\sigma_\mathsf{B},\Theta_\mathsf{B}}(B) \otimes A)). \tag{6.7}$$

If in addition $\check{\Theta}_\mathsf{A}$ and $\check{\Theta}_\mathsf{B}$ commute, then

1. $\varepsilon_{\sigma_{\mathsf{A}}\otimes\sigma_{\mathsf{B}},\check{\Theta}_{\mathsf{A}}\circ\check{\Theta}_{\mathsf{B}}}(A\otimes B) = \varepsilon_{\sigma_{\mathsf{A}},\Theta_{\mathsf{A}}}(A)\varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}(B) = \varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}(B)\varepsilon_{\sigma_{\mathsf{A}},\Theta_{\mathsf{A}}}(A)$, *and*
2. *the two subalgebras* $\mathcal{E}_{\mathsf{A}}, \mathcal{E}_{\mathsf{B}} \subseteq \mathcal{S}$, *where* $\mathcal{E}_{\mathsf{J}}$ *is the smallest (closed) subalgebra containing* $\varepsilon_{\sigma_{\mathsf{J}},\Theta_{\mathsf{J}}}[\mathcal{P}_{\mathsf{J}}]$, *commute.*

Proof.

$$\begin{aligned}&\varepsilon_{\sigma_{\mathsf{A}}\otimes\sigma_{\mathsf{B}},\check{\Theta}_{\mathsf{A}}\circ\check{\Theta}_{\mathsf{B}}}(A\otimes B) = \eta_{\sigma_{\mathsf{A}}\otimes\sigma_{\mathsf{B}}}\Big(\check{\Theta}_{\mathsf{A}}\circ\check{\Theta}_{\mathsf{B}}((\mathbb{1}\otimes A\otimes\mathbb{1}_{\mathsf{B}})(\mathbb{1}\otimes\mathbb{1}_{\mathsf{A}}\otimes B))\Big)\\ &= \eta_{\sigma_{\mathsf{A}}\otimes\sigma_{\mathsf{B}}}\Big(\check{\Theta}_{\mathsf{A}}(\mathbb{1}\otimes A\otimes\mathbb{1}_{\mathsf{B}})\check{\Theta}_{\mathsf{A}}\circ\check{\Theta}_{\mathsf{B}}(\mathbb{1}\otimes\mathbb{1}_{\mathsf{A}}\otimes B)\Big).\end{aligned}\tag{6.8}$$

Writing $\Theta_{\mathsf{B}}(\mathbb{1}\otimes B) =: \sum_j S_{B,j}\otimes P_{B,j}$ we see that

$$\begin{aligned}&\varepsilon_{\sigma_{\mathsf{A}}\otimes\sigma_{\mathsf{B}},\check{\Theta}_{\mathsf{A}}\circ\check{\Theta}_{\mathsf{B}}}(A\otimes B) = \sum_j \eta_{\sigma_{\mathsf{A}}\otimes\sigma_{\mathsf{B}}}\Big(\check{\Theta}_{\mathsf{A}}(S_{B,j}\otimes A\otimes P_{B,j})\Big)\\ &= \eta_{\sigma_{\mathsf{A}}}\left(\Theta_{\mathsf{A}}\left(\left(\sum_j S_{B,j}\sigma_{\mathsf{B}}(P_{B,j})\right)\otimes A\right)\right) = \eta_{\sigma_{\mathsf{A}}}(\Theta_{\mathsf{A}}(\varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}(B)\otimes A)).\end{aligned}\tag{6.9}$$

Let us now assume that $\check{\Theta}_{\mathsf{A}}$ and $\check{\Theta}_{\mathsf{B}}$ commute. Then

1. we have that

$$\check{\Theta}_{\mathsf{A}}\circ\check{\Theta}_{\mathsf{B}}(\mathbb{1}\otimes\mathbb{1}_{\mathsf{A}}\otimes B) = \sum_j \Theta_{\mathsf{A}}\Big(S_{B_j}\otimes\mathbb{1}_{\mathsf{A}}\Big)\otimes P_{B_j} = \sum_j \Big(S_{B_j}\otimes\mathbb{1}_{\mathsf{A}}\Big)\otimes P_{B_j}, \tag{6.10}$$

   and hence

$$\begin{aligned}\Theta_{\mathsf{A}}(\varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}(B)\otimes\mathbb{1}_{\mathsf{A}}) &= \sum_j \sigma_{\mathsf{B}}(P_{B_j})\Theta_{\mathsf{A}}\Big(S_{B_j}\otimes\mathbb{1}_{\mathsf{A}}\Big) = \sum_j \sigma_{\mathsf{B}}(P_{B_j})\Big(S_{B_j}\otimes\mathbb{1}_{\mathsf{A}}\Big)\\ &= \varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}(B)\otimes\mathbb{1}_{\mathsf{A}}\,.\end{aligned}\tag{6.11}$$

   Then it is easy to see that

$$\begin{aligned}&\varepsilon_{\sigma_{\mathsf{A}}\otimes\sigma_{\mathsf{B}},\check{\Theta}_{\mathsf{A}}\circ\check{\Theta}_{\mathsf{B}}}(A\otimes B) = \eta_{\sigma_{\mathsf{A}}}(\Theta_{\mathsf{A}}(\varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}(B)\otimes A))\\ &= \eta_{\sigma_{\mathsf{A}}}((\varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}(B)\otimes\mathbb{1}_{\mathsf{A}})\Theta_{\mathsf{A}}(\mathbb{1}\otimes A)) = \varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}(B)\varepsilon_{\sigma_{\mathsf{A}},\Theta_{\mathsf{A}}}(A)\\ &= \eta_{\sigma_{\mathsf{A}}}(\Theta_{\mathsf{A}}(\mathbb{1}\otimes A)(\varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}(B)\otimes\mathbb{1}_{\mathsf{A}})) = \varepsilon_{\sigma_{\mathsf{A}},\Theta_{\mathsf{A}}}(A)\varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}(B).\end{aligned}\tag{6.12}$$

2. When dealing with unital $*$-algebras, this follows from the fact that any element in $\mathcal{E}_{\mathsf{J}}$ is a finite linear combination of finite products of $\varepsilon_{\sigma_{\mathsf{J}},\Theta_{\mathsf{J}}}[\mathcal{P}_{\mathsf{J}}]$.

   (When dealing with unital C^*-algebras, $\mathcal{E}_{\mathsf{J}}$ is the smallest unital C^*-subalgebra containing $\varepsilon_{\sigma_{\mathsf{J}},\Theta_{\mathsf{J}}}[\mathcal{P}_{\mathsf{J}}]$, which is the topological closure of the set of all finite linear combinations of all finite products of elements in $\varepsilon_{\sigma_{\mathsf{J}},\Theta_{\mathsf{J}}}[\mathcal{P}_{\mathsf{J}}]$, which is algebraically closed by continuity of the algebraic operations. Then the commutativity follows also from continuity.)

□

As a direct result we can show the following theorem, see Theorem 8 and Theorem 9 in [3].

Theorem 6.1.2. *Let* $\left(\mathcal{P}_{\mathsf{A}} \otimes \mathcal{P}_{\mathsf{B}}, \check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}, \sigma_{\mathsf{A}} \otimes \sigma_{\mathsf{B}}\right)$ *be a measurement preparation for* $\mathcal{S}$*. Let* ω *be a state on* $\mathcal{S}$*, and for* $\mathsf{J} = \mathsf{A}, \mathsf{B}$ *let* $\tilde{\mathcal{P}}_{\mathsf{J}} \subseteq \mathcal{P}_{\mathsf{J}}$ *be a subalgebra and* $\mathcal{E}_{\mathsf{J}} \subseteq \mathcal{S}$ *be the smallest (closed) subalgebra containing* $\varepsilon_{\sigma_{\mathsf{J}},\Theta_{\mathsf{J}}}\left[\tilde{\mathcal{P}}_{\mathsf{J}}\right]$*. Then*

1. $\sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}$ *is a product state on* $\tilde{\mathcal{P}}_{\mathsf{A}} \otimes \tilde{\mathcal{P}}_{\mathsf{B}}$ *if*
   a) $\mathcal{E}_{\mathsf{B}}$ *is* $\mathbb{C}\mathbb{1}$*, or*
   b) $\mathcal{E}_{\mathsf{A}}$ *is* $\mathbb{C}\mathbb{1}$ and $\Theta_{\mathsf{A}} \restriction \mathcal{E}_{\mathsf{B}} \otimes \mathbb{C}\mathbb{1}_{\mathsf{A}} = \mathrm{id}$.

Let us now assume that $\check{\Theta}_{\mathsf{A}}$ *and* $\check{\Theta}_{\mathsf{B}}$ *commute. Then the following holds.*

2. *If* ω *is a product state on* $(\mathcal{E}_{\mathsf{A}} \vee \mathcal{E}_{\mathsf{B}})$*, then* $\sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}$ *is a product state on* $\tilde{\mathcal{P}}_{\mathsf{A}} \otimes \tilde{\mathcal{P}}_{\mathsf{B}}$*, and*
3. *if* ω *is classically correlated on* $\mathcal{E}_{\mathsf{A}} \vee \mathcal{E}_{\mathsf{B}}$*, then* $\sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}$ *is at most classically correlated on* $\tilde{\mathcal{P}}_{\mathsf{A}} \otimes \tilde{\mathcal{P}}_{\mathsf{B}}$*.*

Remark: Note the asymmetry of the first statement under the exchange $\mathsf{A} \leftrightarrow \mathsf{B}$, which is due to the fact that in general $\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}} \neq \check{\Theta}_{\mathsf{B}} \circ \check{\Theta}_{\mathsf{A}}$.

This result shows that $\sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}$ can only be entangled on $\tilde{\mathcal{P}}_{\mathsf{A}} \otimes \tilde{\mathcal{P}}_{\mathsf{B}}$ under certain circumstances. In particular, in the case where $\check{\Theta}_{\mathsf{A}}$ and $\check{\Theta}_{\mathsf{B}}$ commute, any correlation or entanglement in the state $\sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}$ must have been transferred or "harvested" from the state ω.

Proof. 1. Let us take $A \in \tilde{\mathcal{P}}_{\mathsf{A}}$ and $B \in \tilde{\mathcal{P}}_{\mathsf{B}}$. Then, according to Lemma 6.1.1, we have

$$\sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}(A \otimes B) = \omega\left(\varepsilon_{\sigma_{\mathsf{A}} \otimes \sigma_{\mathsf{B}}, \check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}}(A \otimes B)\right) = \omega \otimes \sigma_{\mathsf{A}}(\Theta_{\mathsf{A}}(\varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}(B) \otimes A)). \tag{6.13}$$

a) If $\mathcal{E}_{\mathsf{B}}$ is $\mathbb{C}\mathbb{1}$, then we can write $\varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}(B) = c_B \mathbb{1} = c_B \varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}(\mathbb{1}_{\mathsf{B}})$, and hence

$$\sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}(A \otimes B) = \sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}(A \otimes \mathbb{1}_{\mathsf{B}})\, c_B. \tag{6.14}$$

Noting that

$$\begin{aligned} c_B &= \omega(c_B \mathbb{1}) = \omega \otimes \sigma_{\mathsf{A}}(\Theta_{\mathsf{A}}(c_B \varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}(\mathbb{1}_{\mathsf{B}}) \otimes \mathbb{1}_{\mathsf{A}})) \\ &= \omega \otimes \sigma_{\mathsf{A}}(\Theta_{\mathsf{A}}(\varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}(B) \otimes \mathbb{1}_{\mathsf{A}})) = \sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}(\mathbb{1}_{\mathsf{A}} \otimes B) \end{aligned} \tag{6.15}$$

shows that $\sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}$ is a product state on $\tilde{\mathcal{P}}_{\mathsf{A}} \otimes \tilde{\mathcal{P}}_{\mathsf{B}}$.

b) Suppose now that $\mathcal{E}_{\mathsf{A}}$ is $\mathbb{C}\mathbb{1}$ *and* $\Theta_{\mathsf{A}} \restriction \mathcal{E}_{\mathsf{B}} \otimes \mathbb{C}\mathbb{1}_{\mathsf{A}} = \mathrm{id}$. Then we can write $\eta_{\sigma_{\mathsf{A}}}(\Theta_{\mathsf{A}}(\varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}(B) \otimes A)) = c_A\, \varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}(B)$ and hence

$$\sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}(A \otimes B) = c_A\, \sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}(\mathbb{1}_{\mathsf{A}} \otimes B). \tag{6.16}$$

Noting that

$$c_A = \omega \otimes \sigma_{\mathsf{A}}(\Theta_{\mathsf{A}}(\varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}(\mathbb{1}_{\mathsf{B}}) \otimes A)) = \sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}(A \otimes \mathbb{1}_{\mathsf{B}}) \tag{6.17}$$

shows that $\sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}$ is a product state on $\tilde{\mathcal{P}}_{\mathsf{A}} \otimes \tilde{\mathcal{P}}_{\mathsf{B}}$.

Let us now assume that $\check{\Theta}_{\mathsf{A}}$ and $\check{\Theta}_{\mathsf{B}}$ commute. Then, by Lemma 6.1.1, we have for every $A \in \tilde{\mathcal{P}}_{\mathsf{A}}$ and every $B \in \tilde{\mathcal{P}}_{\mathsf{B}}$ that

$$\sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}(A \otimes B) = \omega(\varepsilon_{\sigma_{\mathsf{A}},\Theta_{\mathsf{A}}}(A)\varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}(B)) = \omega(\varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}(B)\varepsilon_{\sigma_{\mathsf{A}},\Theta_{\mathsf{A}}}(A)). \tag{6.18}$$

2. Suppose ω is a product state on $(\mathcal{E}_{\mathsf{A}} \vee \mathcal{E}_{\mathsf{B}})$. Then let us take any $A \in \tilde{\mathcal{P}}_{\mathsf{A}}$ and any $B \in \tilde{\mathcal{P}}_{\mathsf{B}}$ and let us write $\tilde{A} = \varepsilon_{\sigma_{\mathsf{A}},\Theta_{\mathsf{A}}}(A) \in \varepsilon_{\sigma_{\mathsf{A}},\Theta_{\mathsf{A}}}\left[\tilde{\mathcal{P}}_{\mathsf{A}}\right]$ and $\tilde{B} = \varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}(B) \in \varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}\left[\tilde{\mathcal{P}}_{\mathsf{B}}\right]$. Then, by assumption,

$$\begin{aligned} \sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}(A \otimes B) &= \omega(\tilde{A}\tilde{B}) = \omega(\tilde{A})\,\omega(\tilde{B}) \\ &= \sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}(A \otimes \mathbb{1}_{\mathsf{B}})\, \sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}(\mathbb{1}_{\mathsf{A}} \otimes B). \end{aligned} \tag{6.19}$$

3. Let us now suppose that ω is classically correlated on $\mathcal{E}_{\mathsf{A}} \vee \mathcal{E}_{\mathsf{B}}$, i.e., ω is a pointwise limit of convex combinations of product states. Then the claim follows immediately by taking convex combinations and pointwise limits of Eq. (6.19).

□

Let us now take the insight that we have gained about entanglement harvesting in the general situation of AQT-measurement preparations and apply them to measurement preparations of AQFTs on globally hyperbolic spacetimes.

6.1.2 The case of AQFTs on globally hyperbolic spacetimes

Let us now particularise the entanglement harvesting protocol to the case of AQFT. To that end let us consider a globally hyperbolic spacetime M and a measurement preparation for an AQFT $\mathcal{S}_{\mathfrak{B}_M}$ (fulfilling additivity and time-slice) of the form $\left(\mathcal{P}^{\mathsf{A}}_{\mathfrak{B}_M} \otimes \mathcal{P}^{\mathsf{B}}_{\mathfrak{B}_M}, \Theta_{\{\mathsf{A},\mathsf{B}\}}, \sigma_{\mathsf{A}} \otimes \sigma_{\mathsf{B}}\right)$, where $\mathcal{P}^{\mathsf{A}}_{\mathfrak{B}_M}$ and $\mathcal{P}^{\mathsf{B}}_{\mathfrak{B}_M}$ are AQFTs (also fulfilling time-slice and additivity) and where $\Theta_{\{\mathsf{A},\mathsf{B}\}}$ is some K-map.

As before, we will consider the situation where $\mathcal{P}^{\mathsf{A}}_{\mathfrak{B}_M}$ and $\mathcal{P}^{\mathsf{B}}_{\mathfrak{B}_M}$ are *independently* coupled to $\mathcal{S}_{\mathfrak{B}_M}$, i.e., where

$$\Theta_{\{\mathsf{A},\mathsf{B}\}} = \check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}} \tag{6.20}$$

for K_{J}-maps Θ_{J} on $\mathcal{S}_{\mathfrak{B}_M} \otimes \mathcal{P}^{\mathsf{J}}_{\mathfrak{B}_M}$ and where we assumed that $K_{\mathsf{B}} \not\precsim K_{\mathsf{A}}$. Then it follows from Lemma 4.2.7 that $\Theta_{\{\mathsf{A},\mathsf{B}\}}$ is a $K_{\mathsf{A}} \dot{\cup} K_{\mathsf{B}}$-map. In this case

$$\left(\mathcal{P}^{\mathsf{A}}_{\mathfrak{B}_M} \otimes \mathcal{P}^{\mathsf{B}}_{\mathfrak{B}_M}, \Theta_{\{\mathsf{A},\mathsf{B}\}}, \sigma_{\mathsf{A}} \otimes \sigma_{\mathsf{B}}\right) \tag{6.21}$$

may be seen as the combination of the two measurement preparations $\left(\mathcal{P}^{\mathsf{A}}_{\mathfrak{B}_M}, \Theta_{\mathsf{A}}, \sigma_{\mathsf{A}}\right)$ and $\left(\mathcal{P}^{\mathsf{B}}_{\mathfrak{B}_M}, \Theta_{\mathsf{B}}, \sigma_{\mathsf{B}}\right)$.

For $\sigma := \sigma_{\mathsf{A}} \otimes \sigma_{\mathsf{B}}$, we are again interested in the state $\sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}$, but possibly not on all of $\mathcal{P}^{\mathsf{A},g}_{\mathfrak{B}_M} \otimes \mathcal{P}^{\mathsf{B},g}_{\mathfrak{B}_M}$ but for instance on $\mathcal{P}^{\mathsf{A}}_{\mathfrak{B}_M}(N_{\mathsf{A}}) \otimes \mathcal{P}^{\mathsf{B}}_{\mathfrak{B}_M}(N_{\mathsf{B}})$ for some regions $N_{\mathsf{J}} \in \mathfrak{B}_M$ such that $N_{\mathsf{J}} \not\precsim K_{\mathsf{J}}$. This describes the situation in which post-procession is only performed in the regions N_{A} and N_{B}. We hence call N_{A} and N_{B} "processing regions".

A simple consequence of Theorem 6.1.2 is the following corollary, see also Theorem 7 and Theorem 8 in [3].[2]

Corollary 6.1.3. *Let* $\left(\mathcal{P}^{\mathsf{A}}_{\mathfrak{B}_M} \otimes \mathcal{P}^{\mathsf{B}}_{\mathfrak{B}_M}, \check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}, \sigma_{\mathsf{A}} \otimes \sigma_{\mathsf{B}}\right)$ *be an AQFT measurement preparation for the AQFT* $\mathcal{S}_{\mathfrak{B}_M}$ *where*

1. $\mathcal{S}_{\mathfrak{B}_M}$, $\mathcal{P}^{\mathsf{A}}_{\mathfrak{B}_M}$ *and* $\mathcal{P}^{\mathsf{B}}_{\mathfrak{B}_M}$ *fulfill additivity and time-slice,*

2. Θ_{J} *is a* K_{J}*-map on* $\mathcal{S}_{\mathfrak{B}_M} \otimes \mathcal{P}^{\mathsf{J}}_{\mathfrak{B}_M}$*, and*

3. $K_{\mathsf{B}} \not\precsim K_{\mathsf{A}}$.

Furthermore, let ω *be a state on* $\mathcal{S}^g_{\mathfrak{B}_M}$ *and consider regions* $N_{\mathsf{J}} \not\precsim K_{\mathsf{J}}$ *for* $\mathsf{J} = \mathsf{A}, \mathsf{B}$*. Then,* $\sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}$ *is a product state on* $\mathcal{P}^{\mathsf{A}}_{\mathfrak{B}_M}(N_{\mathsf{A}}) \otimes \mathcal{P}^{\mathsf{B}}_{\mathfrak{B}_M}(N_{\mathsf{B}})$ *if*

[2] Note, however, that here we already assume that the processing regions fulfill $N_j \not\precsim K_{\mathsf{J}}$ for $\mathsf{J} = \mathsf{A}, \mathsf{B}$.

1. *$N_{\mathsf{B}} \perp_M K_{\mathsf{B}}$, or*

2. *$N_{\mathsf{A}} \perp_M K_{\mathsf{A}}$ and $\overline{N_{\mathsf{B}}}$ is compact and $\overline{N_{\mathsf{B}}} \perp_M K_{\mathsf{A}}$ and $\overline{N_{\mathsf{B}}} \not\precsim K_{\mathsf{B}}$.*

If $K_{\mathsf{A}} \perp_M K_{\mathsf{B}}$, $\mathcal{S}_{\mathfrak{B}_M}$ fulfills the Haag property and $L_{\mathsf{J}} \supseteq K_{\mathsf{J}}$ are connected regions such that $L_{\mathsf{A}} \perp_M L_{\mathsf{B}}$, then

3. *$\sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}}\circ\check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}$ is a product state on $\mathcal{P}^{\mathsf{A}}_{\mathfrak{B}_M}(N_{\mathsf{A}}) \otimes \mathcal{P}^{\mathsf{B}}_{\mathfrak{B}_M}(N_{\mathsf{B}})$ if ω is a product state on $\mathcal{S}_{\mathfrak{B}_M}(L_{\mathsf{A}}) \vee \mathcal{S}_{\mathfrak{B}_M}(L_{\mathsf{B}})$, and*

4. *$\sigma_{\omega,\left(\check{\Theta}_{\mathsf{A}}\circ\check{\Theta}_{\mathsf{B}}\right)^{\mathbb{F}}}$ is at most classically correlated on $\mathcal{P}^{\mathsf{A}}_{\mathfrak{B}_M}(N_{\mathsf{A}}) \otimes \mathcal{P}^{\mathsf{B}}_{\mathfrak{B}_M}(N_{\mathsf{B}})$ if ω is classically correlated on $\mathcal{S}_{\mathfrak{B}_M}(L_{\mathsf{A}}) \vee \mathcal{S}_{\mathfrak{B}_M}(L_{\mathsf{B}})$.*

Before we turn to the proof let us turn to the interpretation of this corollary. An important point is that for certain configurations of the processing regions *no* correlation, in particular no entanglement, can be harvested even if the coupling zones are allowed to be in causal contact. Another important point is that if the coupling zones are assumed to be spacelike separated, then *no* correlation can be harvested unless the initial system state ω already possesses correlation across the degrees of freedom localisable around the coupling zones.

Proof. Let $\mathcal{E}_{\mathsf{J}} \subseteq \mathcal{S}$ be the smallest (closed) subalgebra containing $\varepsilon_{\sigma_{\mathsf{J}},\Theta_{\mathsf{J}}}\left[\mathcal{P}^{\mathsf{J}}_{\mathfrak{B}_M}(N_{\mathsf{J}})\right]$.

1. If $N_{\mathsf{B}} \perp_M K_{\mathsf{B}}$, we have that $\varepsilon_{\sigma_{\mathsf{B}},\Theta_{\mathsf{B}}}\left[\mathcal{P}^{\mathsf{B}}_{\mathfrak{B}_M}(N_{\mathsf{B}})\right] \subseteq \mathbb{C}\mathbb{1}$, see Theorem 3.3 in [10], in particular $\mathcal{E}_{\mathsf{B}} = \mathbb{C}\mathbb{1}$ and by Theorem 6.1.2 the result follows.

2. If $N_{\mathsf{A}} \perp_M K_{\mathsf{A}}$, then $\mathcal{E}_{\mathsf{A}} = \mathbb{C}\mathbb{1}$, see Theorem 3.3 in [10]. For compact $\overline{N_{\mathsf{B}}}$ with $\overline{N_{\mathsf{B}}} \perp_M K_{\mathsf{A}}$ *and* $\overline{N_{\mathsf{B}}} \not\precsim K_{\mathsf{B}}$ we can find a region N_1 in the *region* $M^-_{K_{\mathsf{B}}} \cap \overline{N_{\mathsf{B}}}^{\perp_M}$, such that N_1 has compact closure and contains K_{A}. Furthermore, we can find some compact $\tilde{K}$ that contains K_{B} in its open interior, such that $N_1 \subseteq M^-_{\tilde{K}}$ and $N_{\mathsf{B}} \subseteq M^+_{\tilde{K}}$. Then it follows from Corollary D.3.2 that

$$\check{\Theta}_{\mathsf{A}} \circ \check{\Theta}_{\mathsf{B}}(\mathbb{1}\otimes\mathbb{1}_{\mathsf{A}}\otimes B) = \check{\Theta}_{\mathsf{B}}(\mathbb{1}\otimes\mathbb{1}_{\mathsf{A}}\otimes B). \tag{6.22}$$

   In particular, since Θ_{A} is a homomorphism we have that $\Theta_{\mathsf{A}} \restriction \mathcal{E}_{\mathsf{B}} \otimes \mathbb{C}\mathbb{1}_{\mathsf{A}} = \mathrm{id}$. Then the claim follows from Theorem 6.1.2.

If $K_{\mathsf{A}} \perp_M K_{\mathsf{B}}$, then $\check{\Theta}_{\mathsf{A}}$ and $\check{\Theta}_{\mathsf{B}}$ commute by Theorem 4.2.8. When $\mathcal{S}_{\mathfrak{B}_M}$ fulfills the Haag property and $L_{\mathsf{J}} \supseteq K_{\mathsf{J}}$ are connected regions, then $\mathcal{E}_{\mathsf{J}} \subseteq \mathcal{S}_{\mathfrak{B}_M}(L_{\mathsf{J}})$. If $L_{\mathsf{A}} \perp_M L_{\mathsf{B}}$, then $\mathcal{S}_{\mathfrak{B}_M}(L_{\mathsf{A}})$ and $\mathcal{S}_{\mathfrak{B}_M}(L_{\mathsf{B}})$ commute.

In particular if ω is a product state on $\mathcal{S}_{\mathfrak{B}_M}(L_{\mathsf{A}}) \vee \mathcal{S}_{\mathfrak{B}_M}(L_{\mathsf{B}})$, then ω is a product state on $\mathcal{E}_{\mathsf{A}} \vee \mathcal{E}_{\mathsf{B}}$, similarly if ω is classically correlated. The remaining two points then follow directly from Theorem 6.1.2. $\square$

6.2 PARTICLE DETECTOR MODELS

In the previous section we have performed a model-independent analysis of entanglement harvesting in the general framework of AQT and also in the framework of AQFT on globally hyperbolic spacetimes. In the latter we have utilised measurement preparations mainly due to their good properties when it comes to causality, i.e., the use of K-maps ensures the absence of any unwanted superluminal signalling issues. The probe structures used here were (unspecified) AQFTs as well.

In the RQI literature it is very common to consider very special physical structures as probes, so-called "particle detector models". They go back to [56], in which Unruh introduced (in addition to a relativistic model) a non-relativistic particle in a box as a particle detector. DeWitt then took up this idea and discussed general detector models with discrete internal energy states with a monopole type coupling in [57]. After that, Unruh and Wald discussed a concrete realisation by a two-level system in [58].

A disadvantage of these widely used particle detector models is, however, that the way they are defined either does not reach a desirable level of rigor, or directly violates signalling causality.

6.2.1 *Heuristics of acausal particle detector models*

In the following we will sketch some loose and (partially) heuristic ideas about common particle detector models and motivate an interesting question about them.

The basis for particle detector models widely used in RQI are usually quantum mechanical structures such as a single harmonic oscillator or a single two-level system (i.e., a qubit with two internal energy levels). Due to their internal dynamics it *should* be possible to view them as AQFTs on a background consisting of the open intervals of the real line $\mathbb{R}$. For any globally hyperbolic spacetime M and any inextendible causal curve $\gamma : \mathbb{R} \to M$, such an AQFT on $\mathbb{R}$ *should* then give rise to an AQFT ${}^{\gamma}\mathcal{P}_{\mathfrak{B}_M}$ on $\mathfrak{B}_M$, see for instance Sec. 2.4.3 in [59]. Suppose the system of interest is the AQFT ${}^{P}\mathcal{F}_{\mathfrak{B}_M}$ of a linear real scalar field on M with equation of motion operator P. Then, choosing a state on ${}^{\gamma}\mathcal{P}^{g}_{\mathfrak{B}_M}$, what is left in order to define a measurement preparation for ${}^{P}\mathcal{F}_{\mathfrak{B}_M}$ is a K-map on ${}^{P}\mathcal{F}_{\mathfrak{B}_M} \otimes {}^{\gamma}\mathcal{P}_{\mathfrak{B}_M}$, i.e., a *coupling* between ${}^{P}\mathcal{F}_{\mathfrak{B}_M}$ and ${}^{\gamma}\mathcal{P}_{\mathfrak{B}_M}$ that ideally respects causality.

When it comes to the coupling one can notice (at least) two different approaches in the literature.

1. One approach is to try and find a coupling that is located strictly along the world-line. In our terminology this would (ideally) be a K-map, where K is a compact subset of the image of γ. In practice, one tries to accomplish this by the perturbative quantisation of a singular classical equation of motion,[3] see [57]. It is unclear whether a non-perturbative quantisation of the singular classical equation of motion may be achieved, which could then give rise to the desired scattering map.

2. The second common approach is to perturbatively quantise a *non-local* classical interacting equation of motion, see for instance [58] and [60]. While it seems likely that in fact a consistent non-perturbative quantisation of the interacting theory might be achieved [53], the resulting AQFT would in general *fail* to fulfill the time-slice property. Consequently, the resulting scattering map would *not* be a K-map, opening the door to causality violations. In the perturbative regime, these causality violations are well-known, see [61, 62].

Equipped with the concept of K-maps, we can now ask the following precise question.

Question 6.2.1. *Let* ${}^P\mathcal{F}_{\mathfrak{B}_M}$ *be the AQFT of a linear real scalar field on a globally hyperbolic spacetime* M *of dimension* $1+n$*, where* $n > 0$*, with Green hyperbolic equation of motion operator* P *and let* ${}^\gamma\mathcal{P}_{\mathfrak{B}_M}$ *be an AQFT on* M *stemming from an inextendible causal curve* $\gamma : \mathbb{R} \to M$ *and an AQFT on the real line* $\mathbb{R}$*. For which* n *is there a* K*-map on* ${}^P\mathcal{F}_{\mathfrak{B}_M} \otimes {}^\gamma\mathcal{P}_{\mathfrak{B}_M}$ *that is* not *of the form*

$$\Theta_{{}^P\mathcal{F}_{\mathfrak{B}_M}} \otimes \Theta_{{}^\gamma\mathcal{P}_{\mathfrak{B}_M}}? \tag{6.23}$$

A negative answer to this question would suggest that the particle detector models used in the literature do not fit into the framework of measurement preparations discussed above. However, based on [3] we now introduce a causal particle detector model that *does* have a description in terms of measurement preparations and discuss how it may be applied to investigate entanglement harvesting from a linear real scalar field on a globally hyperbolic spacetime.

6.2.2 A causal particle detector model

Based on ideas in Sec. II. in [63], [64] and Sec. 5.3 in [10], one idea to formulate *causal* particle detector models that was taken up in [3] is to use a linear real scalar

[3] At least for the harmonic oscillator. For the qubit one possibility could be to view the *classical* qubit as a *classical* spinor field.

probe field with equation of motion operator P and couple it to a linear real scalar *system* field with equation of motion operator S in a certain perturbation zone K. The respective AQFTs are denoted by ${}^{P}\mathcal{F}_{\mathfrak{B}_M}$ and ${}^{S}\mathcal{F}_{\mathfrak{B}_M}$. In the next step, one picks a *processing region* N such that $N \not\precsim K$. Then, inside the algebra ${}^{P}\mathcal{F}_{\mathfrak{B}_M}(N)$, one picks a very simple subalgebra which can be viewed as a harmonic oscillator "embedded" in the probe field. It is this "embedded" harmonic oscillator that will then be viewed as a particle detector.

The formal definition is as follows.

Definition 6.2.2. *Let M be a globally hyperbolic spacetime, let $S : C^\infty(M;\mathbb{R}) \to C^\infty(M;\mathbb{R})$ be formally self-adjoint Green hyperbolic and let ${}^{S}\mathcal{F}_{\mathfrak{B}_M}$ be the associated field AQFT. A* causal particle detector *is a tuple $\left({}^{P}\mathcal{F}_{\mathfrak{B}_M}, \Theta_{\lambda\rho}, N, (\varphi_P(h_1), \varphi_P(h_2))\right)$, where*

1. *${}^{P}\mathcal{F}_{\mathfrak{B}_M}$ is the field AQFT of a linear real scalar field with formally self-adjoint Green hyperbolic equation of motion operator $P : C^\infty(M;\mathbb{R}) \to C^\infty(M;\mathbb{R})$, with generators denoted by φ_P,*

2. *$\Theta_{\lambda\rho}$ is the quasi-free K-map on ${}^{S}\mathcal{F}_{\mathfrak{B}_M} \otimes {}^{P}\mathcal{F}_{\mathfrak{B}_M}$ as in Sec. 5.4.1 for some coupling constant λ and some $\rho \in C_c^\infty(M;\mathbb{R})$ with support in K,*

3. *$N \subseteq M$ is a region such that $N \not\precsim K$,*

4. *$h_1, h_2 \in C_c^\infty(N;\mathbb{R})$ such that*

$$[\varphi_P(h_1), \varphi_P(h_2)] = \mathrm{i}\mathbb{1}. \tag{6.24}$$

Remark: In Sec. 5.4.1 we considered *normally* hyperbolic equation of motion operators S, P. In order to apply some results of Sec. 5.4.1 to the present case, we note that for Green-hyperbolic S, P, Q_λ, as defined in Eq. (5.37), is still Green hyperbolic, see [53].

Associated to the choice of $(\varphi_P(h_1), \varphi_P(h_2))$ are annihilation and creation operators, just like for the harmonic oscillator. We define

$$a := \varphi_P^{\mathbb{C}}(h) := \frac{1}{\sqrt{2}}(\varphi_P(h_1) + \mathrm{i}\varphi_P(h_2)), \qquad a^* := \varphi_P^{\mathbb{C}}(\overline{h}) := \frac{1}{\sqrt{2}}(\varphi_P(h_1) - \mathrm{i}\varphi_P(h_2)), \tag{6.25}$$

where $h := \frac{1}{\sqrt{2}}(h_1 + \mathrm{i}h_2) \in C_c^\infty(N;\mathbb{C})$ and where we have defined $\varphi_P^{\mathbb{C}}(h) := \varphi(\mathrm{Re}\, h) + \mathrm{i}\varphi_P(\mathrm{Im}\, h)$. Furthermore, we can define the associated number operator a^*a. It is worth remembering that a, a^* and a^*a do not just depend on the unital $*$-algebra spanned by $\varphi_P(h_1), \varphi_P(h_2)$ and $\mathbb{1}$, but on the ordered pair $(\varphi_P(h_1), \varphi_P(h_2))$.

A situation of interest to which this model may be applied is the one where initial states σ on ${}^P\mathcal{F}_{\mathfrak{B}_M}$ and ω on ${}^S\mathcal{F}_{\mathfrak{B}_M}$ are chosen. It is easily seen that this situation fits into the discussion of measurement preparations and schemes. A causal particle detector $\left({}^P\mathcal{F}_{\mathfrak{B}_M}, \Theta_{\lambda\rho}, N, (\varphi_P(h_1), \varphi_P(h_2))\right)$ together with a state σ determines a measurement scheme

$$\left({}^P\mathcal{F}_{\mathfrak{B}_M}, \Theta_{\lambda\rho}, \sigma, a^*a\right) \tag{6.26}$$

for

$$\varepsilon_{\sigma,\Theta_{\lambda\rho}}(a^*a) = \varphi_S^{\mathbb{C}}(f)\varphi_S^{\mathbb{C}}(\overline{f}) + \sigma\Big(\varphi_P^{\mathbb{C}}(g)\varphi_P^{\mathbb{C}}(\overline{g})\Big), \tag{6.27}$$

where

$$\begin{pmatrix} f_j \\ g_j \end{pmatrix} := \theta_{\lambda\rho}\begin{pmatrix} 0 \\ h_j \end{pmatrix}, \tag{6.28}$$

and $f := \frac{1}{\sqrt{2}}(f_1 + \mathrm{i}f_2)$ and $g := \frac{1}{\sqrt{2}}(g_1 + \mathrm{i}g_2)$ and $\overline{f}$ denotes the complex conjugate of f. Note, however, that $[\varphi_S(f_1), \varphi_S(f_2)] = i(1 - E_P(g_1, g_2))\mathbb{1}_S$.

Following the arguments in Sec. 5.3 in [10], we can perform a perturbative expansion of $\varepsilon_{\sigma,\Theta_{\lambda\rho}}(a^*a)$, upon wich we find

$$\varepsilon_{\sigma,\Theta_{\lambda\rho}}(a^*a) = \sigma(a^*a) + \lambda^2\Big(\sigma(a^*b) + \sigma(b^*a) + \varphi_S^{\mathbb{C}}\Big(RE_P^-\overline{f}\Big)\varphi_S^{\mathbb{C}}\Big(RE_P^- f\Big)\Big) + \mathcal{O}(\lambda^4), \tag{6.29}$$

where $b := \varphi_P^{\mathbb{C}}\Big(RE_S^- RE_P^- h\Big)$, and where we have implicitly extended the domain of R, E_S^- and E_P^- to complex-valued functions. Hence, we see that the system observable induced by the probe number operator a^*a is the expectation value of the number operator in the probe preparation state σ plus higher order terms. Under the assumption that σ is a *ground state* for $(\varphi_P(h_1), \varphi_P(h_2))$, i.e., $\sigma(a^*a) = 0$, it follows, by the Cauchy-Schwarz inequality, that $\sigma(a^*b) = \sigma(b^*a) = 0$, so

$$\varepsilon_{\sigma,\Theta_{\lambda\rho}}(a^*a) = \lambda^2\varphi_S^{\mathbb{C}}\Big(RE_P^-\overline{f}\Big)\varphi_S^{\mathbb{C}}\Big(RE_P^- f\Big) + \mathcal{O}(\lambda^4). \tag{6.30}$$

The interpretation of this result is as follows. Even if the detector is initialised in a *ground state* for $(\varphi_P(h_1), \varphi_P(h_2))$, i.e., in a state where the expected number of particles is initially $\sigma(a^*a) = 0$, the expected number of particles after the interaction, given by

$$\sigma_{\omega,\Theta^{\mathbb{F}}_{\lambda\rho}}(a^*a) = \omega\Big(\varepsilon_{\sigma,\Theta_{\lambda\rho}}(a^*a)\Big) = \lambda^2\omega\Big(\varphi_S^{\mathbb{C}}\Big(RE_P^-\overline{f}\Big)\varphi_S^{\mathbb{C}}\Big(RE_P^- f\Big)\Big) + \mathcal{O}(\lambda^4), \tag{6.31}$$

is in general not zero. This may be understood as particle detection and is what gives the particle detector its name.

Unfortunately, as we will see, every σ in an interesting and commonly used class of probe preparation states has the property that $\sigma(a^*a) \neq 0$, i.e., if one restricts attention to this class of probe preparation states, then particle detectors cannot be initialised in their ground states.

We follow the presentation in [3].

Definition 6.2.3 (Reeh-Schlieder property I)**.** *Let $\mathcal{A}$ be a unital C^*-algebra of operators on a Hilbert space $\mathcal{H}$ and let $\psi \in \mathcal{H}$ be a unit vector. Denote by $\hat{\omega}_\psi$ the state $\hat{\omega}_\psi(\cdot) := \langle\psi| \cdot \psi\rangle$, then we say that $\hat{\omega}_\psi$ has the Reeh-Schlieder property with respect to $\mathcal{A}$ if ψ is a cyclic vector for $\mathcal{A}$, i.e.,*

$$\forall\xi \in \mathcal{H}\ \exists(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A} : \lim_{n\to\infty} A_n\psi = \xi. \tag{6.32}$$

For an abstract unital C^*-subalgebra $\mathcal{A} \subseteq \mathcal{R}$ of a unital C^*-algebra $\mathcal{R}$ with state $\hat{\omega}$ on $\mathcal{R}$ we may look at the GNS representation $\pi_{\hat{\omega}} : \mathcal{R} \to BL(\mathcal{H}_{\hat{\omega}})$, which (is possibly not injective and) maps $\mathcal{A}$ to a unital C^*-subalgebra $\pi_{\hat{\omega}}[\mathcal{A}]$ of the bounded linear operators $BL(\mathcal{H}_{\hat{\omega}})$ on the ($\hat{\omega}$-dependent) complex Hilbert space $(\mathcal{H}_{\hat{\omega}}, \langle\cdot|\cdot\rangle)$. Moreover there exists a unit vector $\Omega_{\hat{\omega}} \in \mathcal{H}_{\hat{\omega}}$ such that $\forall A \in \mathcal{A}$ we have that $\hat{\omega}(A) = \langle\Omega_{\hat{\omega}}|\pi_{\hat{\omega}}(A)\Omega_{\hat{\omega}}\rangle$.

Definition 6.2.4 (Reeh-Schlieder property II)**.** *For an abstract unital C^*-subalgebra $\mathcal{A} \subseteq \mathcal{R}$ of a unital C^*-algebra $\mathcal{R}$ with state $\hat{\omega}$ on $\mathcal{R}$ we say that $\hat{\omega}$ has the Reeh-Schlieder property with respect to $\mathcal{A}$ if $\Omega_{\hat{\omega}} \in \mathcal{H}_{\hat{\omega}}$ is a cyclic vector for $\pi_{\hat{\omega}}[\mathcal{A}]$.*

Definition 6.2.5. *Let M be a globally hyperbolic spacetime, let $P : C^\infty(M;\mathbb{R}^n) \to C^\infty(M;\mathbb{R}^n)$ be Green hyperbolic, let ${}^P\mathcal{F}_{\mathfrak{B}_M}$ be the associated field AQFT and let ${}^P\mathcal{W}_{\mathfrak{B}_M}$ be the associated Weyl AQFT. We say that a quasi-free state σ on ${}^P\mathcal{F}^g_{\mathfrak{B}_M}$ has the* Reeh-Schlieder *for a region $L \subseteq M$ if the corresponding quasi-free state $\hat{\sigma}$ on the unital C^*-algebra ${}^P\mathcal{W}^g_{\mathfrak{B}_M}$ has the Reeh-Schlieder property with respect to ${}^P\mathcal{W}_{\mathfrak{B}_M}(L)$.*

We say that σ is a Reeh-Schlieder state*, if it has the Reeh-Schlieder property for every region $L \subseteq M$.*

Theorem 6.2.6. *Let M be a globally hyperbolic spacetime, let $P : C^\infty(M;\mathbb{R}) \to C^\infty(M;\mathbb{R})$ be Green hyperbolic and let ${}^P\mathcal{F}_{\mathfrak{B}_M}$ be the associated field AQFT. Suppose σ is a quasi-free state on ${}^P\mathcal{F}^g_{\mathfrak{B}_M}$, $N \subseteq M$ is a region such that there is a region $L \subseteq M$ that is spacelike separted from N, and $\left({}^P\mathcal{F}_{\mathfrak{B}_M}, \Theta_{\lambda\rho}, N, (\varphi_P(h_1), \varphi_P(h_2))\right)$ is a causal particle detector model. If σ has the Reeh-Schlieder property for L, then*

$$\sigma(a^*a) > 0. \tag{6.33}$$

Proof. By definition, the state $\hat{\sigma}$ on ${}^P\mathcal{W}^g_{\mathfrak{B}_M}$ has the Reeh-Schlieder property with respect to ${}^P\mathcal{W}_{\mathfrak{B}_M}(L)$. Let now $\hat{\mathcal{B}} \subseteq {}^P\mathcal{W}_{\mathfrak{B}_M}(N)$ be the Weyl algebra over the two-dimensional symplectic space spanned by $[h_1]_P$ and $[h_2]_P$. Then, since ${}^P\mathcal{W}^g_{\mathfrak{B}_M}$ is simple (see Lemma 10 in [29]), the GNS representation $\pi_{\hat{\sigma}}$ is injective (as it is not the zero representation) and $\pi_{\hat{\sigma}}\left[{}^P\mathcal{W}_{\mathfrak{B}_M}(L)\right]$ and $\pi_{\hat{\sigma}}\left[\hat{\mathcal{B}}\right]$ are two commuting, non-Abelian, unital C^*-subalgebras of $BL(\mathcal{H}_\omega)$ and Ω_ω is cyclic for $\pi_{\hat{\sigma}}\left[{}^P\mathcal{W}_{\mathfrak{B}_M}(L)\right]$. Then by Lemma A.2.4, $\hat{\sigma} \restriction \mathcal{B}$ is mixed, and by Lemma B.5.5, $\sigma \restriction \mathcal{B}$ is mixed, where $\mathcal{B} \subseteq {}^P\mathcal{F}_{\mathfrak{B}_M}(N)$ is the smallest unital $*$-algebra containing $\varphi_P(h_1)$ and $\varphi_P(h_2)$. In particular, the associated 2×2 covariance matrix A for the basis $[h_2]_P, [h_2]_P$ fulfills $\det(A) \neq 1$. In fact, $\det(A) > 1$. Together with the fact that $A^\dagger = A$, we then have that $\mathrm{Tr}(A) > 2$, and in particular

$$\sigma(a^\dagger a) = \frac{1}{2}\left(\underbrace{\frac{1}{2}\,\mathrm{Tr}\,(A) - 1}_{>0} + \underbrace{\sigma(\varphi_P(f_1))^2 + \sigma(\varphi_P(f_2))^2}_{\geq 0}\right) > 0. \tag{6.34}$$

□

Remark: The existence of such a region L is stronger than demanding that $N^{\perp_M}$ is non-empty and equivalent to saying that the open interior of $N^{\perp_M}$ is non-empty[4].

The significance of this result is reinforced by the fact that many *physically reasonable* states are Reeh-Schlieder with respect to every region, for example quasi-free states of Klein-Gordon fields on real analytic spacetimes that fulfill the *analytic microlocal spectrum condition* ($a\mu SC$) [66]; see [67] for a general existence result and [68] for the existence of such KMS states on stationary real analytic spacetimes. It turns out that all quasi-free ground and also KMS states on stationary real analytic spacetimes are known to fulfill the $a\mu SC$ [66] and are hence Reeh-Schlieder states. In fact, quasi-free ground and KMS states on general ultrastatic [69] and stationary spacetimes [70] are Reeh-Schlieder. Additionally, the existence of physically reasonable states that have the Reeh-Schlieder property for possibly only a few regions L (which would be sufficient for the above result) on globally hyperbolic spacetimes was analysed in [71], see also Sec. 3.2 in [72]. On flat Minkowski spacetime we even have that every state with bounded energy has the Reeh-Schlieder property for every region, see Sec. II.5.3 in [6].

[4]Take for $N := D_M(\Sigma \setminus \{p\})$ the domain of dependence of a Cauchy surface Σ with one point p removed. Then $N^{\perp_M} = \{p\}$ is non-empty but does not contain a region. However, if the open interior of $N^{\perp_M}$ is non-empty, then it contains a region since M is strongly causal [65].

In summary, a quasi-free Reeh-Schlieder state cannot be the ground state of a causal particle detector model.

6.3 ENTANGLEMENT HARVESTING WITH CAUSAL PARTICLE DETECTOR MODELS

The causal particle detector models introduced in the previous section can be used to revisit entanglement harvesting. To that end let us consider two causal particle detector models

$$\left({}^{P_\mathsf{J}}\mathcal{F}_{\mathfrak{B}_M}, \Theta_{\lambda\rho_\mathsf{J}}, N_\mathsf{J}, \left(\varphi_{P_\mathsf{J}}(h_1^\mathsf{J}), \varphi_{P_\mathsf{J}}(h_2^\mathsf{J})\right)\right) \tag{6.35}$$

for $\mathsf{J} = \mathsf{A}, \mathsf{B}$ with common coupling constant λ and spacelike separated perturbation zones $K_\mathsf{A}, K_\mathsf{B}$, where $\mathrm{supp}\rho_\mathsf{J} \subseteq K_\mathsf{J}$ and $N_\mathsf{J}^{\perp_M}$ has non-empty open interior.

For every J, there is a preferred subalgebra $\mathcal{B}_\mathsf{J} \subseteq {}^{P_\mathsf{J}}\mathcal{F}_{\mathfrak{B}_M}(N_\mathsf{J})$ given by the smallest unital $*$-subalgebra that contains $\left\{\varphi_{P_\mathsf{J}}(h_1^\mathsf{J}), \varphi_{P_\mathsf{J}}(h_2^\mathsf{J})\right\}$.

Let us now pick preparation states σ_J on ${}^{P_\mathsf{J}}\mathcal{F}^g_{\mathfrak{B}_M}$ for $\mathsf{J} = \mathsf{A}, \mathsf{B}$ and set $\sigma := \sigma_\mathsf{A} \otimes \sigma_\mathsf{B}$. Then σ is uncorrelated on $\mathcal{B}_\mathsf{A} \otimes \mathcal{B}_\mathsf{B}$. For the purpose of analysing entanglement harvesting, the question of interest is whether for an initial system state ω on ${}^S\mathcal{F}^g_{\mathfrak{B}_M}$ the state $\sigma_{\omega, \left(\check{\Theta}_{\lambda\rho_\mathsf{A}} \circ \check{\Theta}_{\lambda\rho_\mathsf{B}}\right)^\mathbb{F}}$ is entangled on $\mathcal{B}_\mathsf{A} \otimes \mathcal{B}_\mathsf{B}$. The following result shows that, for a certain class of preparation states ω, σ, this is only possible if λ exceeds some minimal value.

Theorem 6.3.1. *Suppose $\omega, \sigma_\mathsf{A}$ and σ_B are quasi-free with distributional symmetric covariance functions and $\sigma_\mathsf{A} \restriction \mathcal{B}_\mathsf{A}$ and $\sigma_\mathsf{B} \restriction \mathcal{B}_\mathsf{B}$ are mixed, which is in particular the case if σ_A and σ_B are Reeh-Schlieder states. Then there exists $\lambda_{\min} > 0$ such that for all $0 < \lambda < \lambda_{\min}$ the restriction of $\sigma_{\omega, \left(\check{\Theta}_{\lambda\rho_\mathsf{A}} \circ \check{\Theta}_{\lambda\rho_\mathsf{B}}\right)^\mathbb{F}}$ to $\mathcal{B}_\mathsf{A} \otimes \mathcal{B}_\mathsf{B}$ is* not *entangled.*

Proof. For every $\lambda \geq 0$ let us consider $p_S(\lambda)$ defined in Lemma B.5.6 for the state $\sigma_{\omega, \left(\check{\Theta}_{\lambda\rho_\mathsf{A}} \circ \check{\Theta}_{\lambda\rho_\mathsf{B}}\right)^\mathbb{F}}$ on $\mathcal{B}_\mathsf{A} \otimes \mathcal{B}_\mathsf{B}$. By assumption, $\lambda \mapsto p_S(\lambda)$ is continuous and $p_S(0) = -(\det(A) - 1)(\det(B) - 1)$. where A and B are the 2×2 covariance matrices of $\sigma_\mathsf{A} \restriction \mathcal{B}_\mathsf{A}$ and $\sigma_\mathsf{B} \restriction \mathcal{B}_\mathsf{B}$ respectively. From the proof of Theorem 6.2.6, we see that due to the assumed mixedness, $p_S(0) < 0$. Hence, by continuity, there exists $0 < \lambda_{\min}$ such that for all $0 < \lambda < \lambda_{\min}$ $p_S(\lambda) < 0$ and hence, by Lemma B.5.6, the restriction of $\sigma_{\omega, \left(\check{\Theta}_{\lambda\rho_\mathsf{A}} \circ \check{\Theta}_{\lambda\rho_\mathsf{B}}\right)^\mathbb{F}}$ to $\mathcal{B}_\mathsf{A} \otimes \mathcal{B}_\mathsf{B}$ is *not* entangled. □

In summary, a couple of causal particle detector model initialised in quasi-free Reeh-Schlieder states $\sigma_\mathsf{A}, \sigma_\mathsf{B}$ can only harvest entanglement from a quasi-free state ω if the common coupling constant λ exceeds a certain minimal value $\lambda_{\min}$ *that may depend on $\sigma_\mathsf{A}, \sigma_\mathsf{B}$ and ω.*

— 7 —

Summary and Outlook

At the beginning of this thesis we collected some motivation for the algebraic approach to physics, in particular algebraic quantum theory (AQT). The combination of AQT together with a background led us to algebraic quantum field theory (AQFT). We put special emphasis on the background consisting of regions of a globally hyperbolic spacetime M that do not intersect some compact set K, which is denoted by $\mathfrak{B}_{(M,K)}$. In the special case of an AQFT on $\mathfrak{B}_M := \mathfrak{B}_{(M,\emptyset)}$, i.e., where $K = \emptyset$, we recover the standard notion of AQFT on a (possibly curved) globally hyperbolic spacetime.

Our first non-trivial technical result was the "gluing" Lemma 3.1.8, in which we showed, that under appropriate assumptions two AQFTs may be glued together along a region on which they are identical.

Having discussed AQFTs, i.e., the association of unital $*$-algebras to regions of a given background, whose Hermitian elements may be interpreted as locally accessible observables, our aim was to develop a similar framework for *interventions*. In Sec. 3.2 we introduced the structure of a convex time-orderable pre-factorisation algebra (ctPFA), which precisely captured the desired properties of an association of interventions to regions of a given background. In particular, it formalised the idea that one should be able to compose interventions associated to causally orderable regions in any causally admissible order, resulting in another intervention.

Equipped with a framework for interventions, we could then analyse questions of signalling causality. In Sec. 3.4, utilising the insight of Sorkin in [7], we then motivated and formulated a condition describing when a ctPFA of quantum channels of a given AQFT may be called *causal*. Crucially, not every quantum channel of the form $A \mapsto U^*AU$ for some unitary U in a local algebra is causal. This reinforced the point of view that local algebras are algebras of observables (or, more precisely, generated by them) and *not* algebras of physically allowed operations [4].

In order to construct examples of ctPFAs of causal quantum channels inspired by [10] and [1], we first had a closer look at (spacetime-) compact perturbations of AQFTs on globally hyperbolic spacetimes and emergent scattering maps. To that end, following [10], we introduced the notion of *perturbed variants*, which give rise to scattering maps. Motivated by their properties, we introduced what we call K-homs and K-maps. These are homomorphisms and automorphisms of the global algebra of the AQFT under consideration that fulfill certain properties with respect to some compact set K. One of the assumptions in [10] and [1] was that scattering maps (or rather perturbed variants) fulfill "causal factorisation", implying that scattering maps associated to spacelike separated perturbation zones commute. In order to avoid this strong assumption, we first showed, assuming additivity of the underlying AQFT, that two K-homs associated to spacelike separated K_1 and K_2 commute. The fact that scattering maps are K-homs in conjunction with the results of Sec. 4.3 lifted the former *assumption* of causal factorisation to a *theorem*. Not only did we consider this an interesting result in its own right, it also enabled us to elegantly define ctPFAs of quantum channels of AQFTs (fulfilling additivity and time-slice) without having to make the assumption of causal factorisation *by hand*. The last result concerning scattering was Theorem 4.4.1, which showed that not only is every scattering map a K-map, every K-map is a scattering map. An important byproduct of the proof of Theorem 4.4.1 was the auxiliary Corollary D.3.2, which formed an integral part of the proof of one of the most important results, namely Theorem 4.5.2. This theorem showed the existence of ctPFAs of causal quantum channels for AQFTs (fulfilling additivity and time-slice) on globally hyperbolic spacetimes.

In Chapter 5 we discussed and slightly generalised the AQFT measurement schemes of [10]. Most importantly, [10] can be regarded as showing that the quantum channels of the ctPFA of causal quantum channels of Theorem 4.5.2 have an intuitive understanding in terms of state-update maps. (Chronological, [10] of course came first.) In order to answer the remaining open question of whether the ctPFA of causal quantum channels of Theorem 4.5.2 is "rich" enough to describe sufficiently many interventions, we presented two results for the AQFT of a linear real scalar field fulfilling a normally hyperbolic equation of motion. Firstly, we showed, using a specific K-map exchanging initial data, that every observable of the linear real scalar field may be measured by a measurement scheme. In particular, the ctPFA of causal quantum channels of Theorem 4.5.2 contains state-update maps for *every* observable. Secondly, presenting the results of [2], we showed, using measurement schemes of [10], that every observable may be *approximately* measured in a very precise way using *asymptotic measurement schemes*. This demonstrated that the ctPFA of causal quantum channels of Theorem 4.5.2 is indeed "rich" enough to at least describe certain measurements of every observable of a linear real scalar field.

Finally, in Chapter 6 we presented the results of [3], in which we showed with the example of entanglement harvesting, that AQFT measurement schemes may be used to describe relativistic quantum information protocols in a fully causal manner.

7.1 OPEN QUESTIONS

Possibly more important than the questions answered by this thesis are the ones raised by it.

1. *Under what conditions is an AQFT $\mathcal{A}_{\mathfrak{B}_{(M,K)}}$, where M is a globally hyperbolic spacetime and K is a compact subset, the restriction of an AQFT $\mathcal{A}_{\mathfrak{B}_M}$ on the background $\mathfrak{B}_M$ to the background $\mathfrak{B}_{(M,K)}$? When is $\mathcal{A}_{\mathfrak{B}_M}$ unique?*

   The answer to this question is of independent interest, but, in conjunction with Theorem 4.4.1, it is also interesting from an *inverse scattering* point of view. Even an answer for the special case of compact subsets of Cauchy surfaces would be very interesting, as it could help clarify the question whether the measurement schemes of Theorem 5.3.2 are FV or not.

2. *What, if it exists, is the maximal ctPFA of causal quantum channels for a given AQFT $\mathcal{A}_{\mathfrak{B}}$? Is there a dilation-type result for* causal *quantum channels?*

   According to Theorem 4.5.2, $\mathfrak{F}_{\mathfrak{B}_M}$ defined in Definition 4.5.1 for an AQFT $\mathcal{S}_{\mathfrak{B}_M}$ fulfilling additivity and time-slice on a globally hyperbolic spacetime M is a ctPFA of causal quantum channels. As is clear from the definition, every quantum channel in $\mathfrak{F}_{\mathfrak{B}_M}(N)$ comes from a "causal dilation", i.e., from some K-hom on $\mathcal{S}_{\mathfrak{B}_M} \otimes \mathcal{P}_{\mathfrak{B}_M}$ for some $\mathcal{P}_{\mathfrak{B}_M}$. However, it could be the case that there exists a "bigger" ctPFA of causal quantum channels for which $\mathfrak{F}_{\mathfrak{B}_M}$ is a proper sub-ctPFA. In that case, a causal quantum channel strictly contained in the bigger ctPFA would not have a causal dilation of the above described type.

   In the case where $\mathcal{A}^g_{\mathfrak{B}_M}$ is $BL(\mathcal{H})$ for some separable Hilbert space $\mathcal{H}$ and where the local algebras fulfill $\mathcal{A}_{\mathfrak{B}_M}(N)'' = \mathcal{A}_{\mathfrak{B}_M}(N)$, then, following the result of Lemma 3.3.3, a candidate for a reasonable *maximal* ctPFA of quantum channels *could* be the one that associates to every region $N \in \mathfrak{B}_M$ those *normal* quantum channels that

   a) are weakly localized in N, see Lemma 3.3.3, and that

b) fulfill a property analogously to point 2 in Definition 4.2.3,

which is very close to the suggestion in [9]. The advantage of the present suggestion is that it really defines a ctPFA of quantum channels. In particular, two such channels associated to spacelike separated regions are guaranteed to commute. The disadvantage is that this description heavily relies on the fact that normal quantum channels of $BL(\mathcal{H})$ are *inner*, which is in general wrong when working with unital C^*- or even only unital $*$-algebras.

3. *Theorem 4.5.2 proves the existence of a ctPFA of causal quantum channels for every AQFT on a globally spacetime that fulfills time-slice and additivity. Conversely, can one recover an AQFT from a ctPFA? When is a ctPFA a ctPFA of causal quantum channels of some AQFT? When does uniqueness hold?*

A first step in answering this question is to investigate in what way the AQFT defined in [44] and [45] may be viewed as resulting from (an appropriate quotient of) the ctPFA $\mathfrak{F}_{\mathfrak{B}_M}^{C_c^\infty(\cdot;\mathbb{R})}$ defined in and before Lemma 3.2.2. This is motivated by the fact that $\mathfrak{F}_{\mathfrak{B}_M}^{C_c^\infty(\cdot;\mathbb{R})}$ is generated by symbols (namely $\mathrm{ad}_S(\cdot)$) labelled by functions $f \in C_c^\infty(M;\mathbb{R})$, and that the AQFT in [44] and [45] is also generated by symbols (namely $S(\cdot)$) labelled by functions $f \in C_c^\infty(M;\mathbb{R})$ (or even more general functionals) modulo relations. In particular, an element $\mathrm{ad}_S(f)$ of the ctPFA naturally acts by conjugation by $S(f)$, which finally also justifies the notation ad_S. Furthermore, using Bogoliubov's causal factorisation formula for the generators $S(\cdot)$, which is one of the axioms in [44] and [45], it should be possible to show that (an appropriate quotient of) the ctPFA $\mathfrak{F}_{\mathfrak{B}_M}^{C_c^\infty(\cdot;\mathbb{R})}$ is a ctPFA of *causal* quantum channels. This suggests that the AQFT can directly be defined from an appropriate quotient of $\mathfrak{F}_{\mathfrak{B}_M}^{C_c^\infty(\cdot;\mathbb{R})}$, which is then a ctPFA of causal quantum channels. The choice of relations one imposes on $\mathfrak{F}_{\mathfrak{B}_M}^{C_c^\infty(\cdot;\mathbb{R})}$ (which gives rise to the actual ctPFA of interest) is naturally expected to depend on the dynamics of the AQFT one wishes to recover.

The framework of [44] and [45] promise to be an ideal place to start further investigations of the posed questions.

4. *Can the* same *causal quantum channels be reproduced "at a later point in time"?*

Suppose we are given a ctPFA of causal quantum channels $\mathfrak{F}_{\mathfrak{B}_M}$ of some AQFT $\mathcal{A}_{\mathfrak{B}_M}$ on the background of regions of some globally hyperbolic spacetime M

and suppose $T \in \mathfrak{F}_{\mathfrak{B}_M}(N_1)$ for some region N_1. The interpretation is that an agent in control of region N_1 can perform the quantum channel T. Suppose another agent, in control of a region N_2, which we assume to be disjoint[1] from N_1, wishes to perform the very same causal quantum channel. Is this possible? More technically speaking, are there any regions $N_2 \in \mathfrak{B}_M$, disjoint from N_1, such that $T \in \mathfrak{F}_{\mathfrak{B}_M}(N_2)$?

A more concrete question would be the following. Does the ctPFA of Theorem 4.5.2 fulfill time-slice? Again, the framework of [44] and [45] would be an ideal place to start such an investigation. In particular, the proof of time-slice of the AQFT in [45], i.e., Theorem 6.1 therein, suggests that ctPFAs of causal quantum channels possibly do *not* fulfill time-slice. In this case a causal quantum channel could in general *not* be reproduced at a later point in time.

5. *What is the answer to Question 6.2.1?*

   Are there non-trivial K-maps/-homs on the combination of the theory of a linear real scalar field with a physical system on a world-line? How does the situation look in the fully perturbative setting? What is the consequence for common detector models in RQI?

6. *What are possible implications of the discussion of* causal *quantum channels in AQFT for (R)QI?*

   The main lesson is without doubt that not all mathematically conceivable quantum channels on a physical system are physically reasonable and allowed by causality, but also, that there exists a concrete example of a ctPFA of causal quantum channels and that there exists a consistent framework of modelling measurements using causal measurement schemes. This opens the door for an investigation of various (R)QI scenarios in regimes where the quantum field-theoretic nature of the available degrees of freedom becomes relevant.

These five are only some of many interesting questions that the present work not only raises but in fact enables to ask. My hope is that my thesis will prove useful in the continued effort to acquire more insight into the interplay between measurements, causality and quantum field theory.

[1]It is very natural to assume that the two regions are disjoint. Otherwise, the interpretation would be that one agent is partially in the laboratory if the other agent and vice versa.

— A —

More on unital $*$-algebras

We first start with some general observations.

Definition A.0.1. *Let $\mathcal{O}$ be a unital $*$-algebra. A linear subspace $\mathcal{I} \subseteq \mathcal{O}$ such that*

$$\forall A \in \mathcal{O}, \forall B \in \mathcal{I}: \; B^*, AB, BA \in \mathcal{I} \tag{A.1}$$

is called two-sided $*$-ideal. $\mathcal{O}$ is called simple if there are no non-trivial two-sides $*$-ideals.

Definition A.0.2. *Let $\mathcal{O}_1, \mathcal{O}_2$ be two unital $*$-algebras. A $\mathbb{C}$-linear map $\phi : \mathcal{O}_1 \to \mathcal{O}_2$ such that*

1. $\phi(AB) = \phi(A)\phi(B)$,
2. $\phi(A^*) = \phi(A)^*$,

is called a $*$-homomorphism. If in addition

3. $\phi(\mathbb{1}_{\mathcal{O}_1}) = \mathbb{1}_{\mathcal{O}_2}$,

then ϕ is called a unit-preserving $*$-homomorphism. We say that $\mathcal{O}_1$ and $\mathcal{O}_2$ are isomorphic, if there exists a unit-preserving $*$-isomorphism $\phi : \mathcal{O}_1 \to \mathcal{O}_2$.

Definition A.0.3. *Let $\mathcal{O}_2$ be a unital $*$-algebra and let $\mathcal{O}_1$ be a subset of $\mathcal{O}_2$ that is also a unital $*$-algebra. If the canonical inclusion*

$$\iota : \mathcal{O}_1 \hookrightarrow \mathcal{O}_2 \tag{A.2}$$

is a unit-preserving $*$-homomorphism, then $\mathcal{O}_1 \subseteq \mathcal{O}_2$ is called a unital $*$-subalgebra.

Remark: According to this definition, *not* every subset $\mathcal{O}_1 \subseteq \mathcal{O}_2$ that, together with the inherited algebraic structure, forms a unital $*$-algebra is also a unital $*$-subalgebra.[1]

Lemma A.0.4 (Proposition 9.1.3 in[17]). *Let $\mathcal{O}_1, \mathcal{O}_2$ be two unital $*$-algebras. The kernel of a unit-preserving $*$-homomorphism $\phi : \mathcal{O}_1 \to \mathcal{O}_2$ given by*

$$\ker \phi := \{A \in \mathcal{O}_1 | \phi(A) = 0\} \tag{A.3}$$

is a $*$-ideal.

Definition A.0.5 (Def. 9.1.10 and 9.2.1 in [17]). *Let $(\mathcal{H}, \langle \cdot | \cdot \rangle)$ be a (not necessarily separable) Hilbert space and let $\mathcal{D} \subseteq \mathcal{H}$ be a dense linear subspace. Then we define $L_\dagger(\mathcal{D})$ to be the set of linear endomorphisms T of $\mathcal{D}$, such that $T^* := T^\dagger \restriction \mathcal{D}$, where $T^\dagger$ is the adjoint of T with respect to $\langle \cdot | \cdot \rangle$, is an endomorphism of $\mathcal{D}$. Then $L_\dagger(\mathcal{D})$ is a unital $*$-algebra and for every unital $*$-algebra $\mathcal{O}$ a unit-preserving $*$-homomorphism $\phi : \mathcal{O} \to L_\dagger(\mathcal{D})$ is called a $*$-representation of $\mathcal{O}$ on $\mathcal{D}$.*

ϕ is called irreducible if the weak commutant of the representation, i.e., the set of all $C \in BL(\mathcal{H})$ such that

$$\forall x, y \in \mathcal{D} : \langle x | C\pi(A) y \rangle = \langle \pi(A^*) x | C y \rangle \,, \tag{A.4}$$

is trivial.

Lemma A.0.6 (GNS representation, see Sec. 9.4.4 in [17] and Theorem 6.3 in [73]). *Let $\mathcal{O}$ be a unital $*$-algebra and let ω be a state. Then there exists a Hilbert space $\mathcal{H}_\omega$, a dense linear subspace $\mathcal{D}_\omega \subseteq \mathcal{H}_\omega$, a vector $\Omega_\omega \in \mathcal{D}_\omega$ and a $*$-representation $\pi_\omega : \mathcal{O} \to L_\dagger(\mathcal{D}_\omega)$ such that*

1. *Ω_ω is* cyclic *for $\pi_\omega[\mathcal{O}]$, i.e., $\pi_\omega[\mathcal{O}]\Omega_\omega := \{\pi_\omega(A)\Omega_\omega | A \in \mathcal{O}\} = \mathcal{D}_\omega$,*
2. *$\forall A \in \mathcal{O} :\ \omega(A) = \langle \Omega_\omega | A \Omega_\omega \rangle$.*

If ω is faithful, then π_ω is injective. Moreover, ω is pure if and only if π_ω is irreducible.

Remark: In [17], what we call $*$-representation here is referred to as pre-$*$-representation.

[1] Take for instance the set of all 2×2 matrices $\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$ for $\alpha \in \mathbb{C}$ as a subset of all complex 2×2 matrices. With the standard adjoint and matrix multiplication, this forms an Abelian unital $*$-algebra that is isomorphic to $\mathbb{C}$, but that is *not* a unital $*$-subalgebra in our terminology.

A.1 LOCALLY CONVEX TOPOLOGICAL UNITAL $*$-ALGEBRAS

Let us now consider possible additional structures we could equip a unital $*$-algebra with. The first example is that of a locally convex topology giving rise to *locally convex topological* unital $*$-algebras, see Sec. 1.4 in [74].

For future reference, let us recall that a Hausdorff locally convex topological vector space E is called *nuclear* if for every Banach space F the canonical map $E\widehat{\otimes}_\pi B \to E\widehat{\otimes}_\varepsilon B$ is a topological isomorphism, see Theorem 50.1 in [54]. Furthermore, let us note that no infinite dimensional Banach space is nuclear, see Corollary 2 of Proposition 50.2 in [54].

Definition A.1.1. *A locally convex (topological) unital $*$-algebra $\mathcal{O}$ is a unital $*$-algebra $\mathcal{O}$ equipped with a topology τ such that $(\mathcal{O}, \tau)$ is a locally convex topological vector space and such that*

1. *for every $a \in \mathcal{O}$ the left multiplication $l_a : \mathcal{O} \to \mathcal{O}$ given by $b \mapsto ab$ is τ-continuous, and*
2. *the $*$-operation is τ-continuous.*

Furthermore,

3. *if τ is nuclear[2], then $\mathcal{O}$ (together with τ) is called a* nuclear *unital $*$-algebra,*
4. *if the canonical uniformity of $(\mathcal{O}, \tau)$ is complete, then $\mathcal{O}$ is called a* complete *locally convex unital $*$-algebra,*
5. *if the multiplication is jointly continuous, i.e., if $m : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ given by $(a, b) \mapsto ab$ is continuous, then we say that $\mathcal{O}$ is a locally convex unital $*$-algebra with jointly continuous multiplication.*

If $\mathcal{O}_1$ is a subset of $\mathcal{O}$ that is also a locally convex unital $*$-algebra with some topology τ_1, then we say that $\mathcal{O}_1 \subseteq \mathcal{O}_2$ is a locally convex unital $*$-subalgebra, if the canonical inclusion $\iota : \mathcal{O}_1 \to \mathcal{O}$ is a unit-preserving $*$-homomorphism that is a homeomorphism onto its image.

One of the complications that come with topological algebras, which we will also encounter again further below, is the generic ambiguity when defining their tensor product. While for two unital $*$-algebras $\mathcal{O}_1, \mathcal{O}_2$, their *algebraic* tensor product

[2] See Def. 50.1 in [54] and note that then τ is implicitly assumed to be Hausdorff

$\mathcal{O}_1 \otimes \mathcal{O}_2$ carries the canonical structure of a unital $*$-algebra (see Sec. 10.1.30 in [17]), there is in general no canonical locally convex topology on the algebraic tensor product of two locally convex unital $*$-algebras.

Let $\mathcal{O}_1, \mathcal{O}_2$ be two locally convex unital $*$-algebras. Then there are two distinct locally convex topologies on $\mathcal{O}_1 \otimes \mathcal{O}_2$, namely the *projective* one τ_π and the *injective* one τ_ε (see Sec. 43 in [54]). The topology τ_π is stronger than the topology τ_ε, see Proposition 43.5 in [54] i.e.,

$$\mathrm{id} : (\mathcal{O}_1 \otimes \mathcal{O}_2, \tau_\pi) \to (\mathcal{O}_1 \otimes \mathcal{O}_2, \tau_\varepsilon) \tag{A.5}$$

is continuous.

Both $(\mathcal{O}_1 \otimes \mathcal{O}_2, \tau_\pi)$ and $(\mathcal{O}_1 \otimes \mathcal{O}_2, \tau_\varepsilon)$ are locally convex unital $*$-algebras. The fact that the left-multiplications are continuous follows from Proposition 43.6 in [54]. To show that the $*$-operation is continuous, we consider the complex conjugates of $\mathcal{O}_1$ and $\mathcal{O}_2$ and also apply Proposition 43.6 in [54].

If *both* $\mathcal{O}_1, \mathcal{O}_2$ are in addition Hausdorff, so are the topologies τ_π and τ_ε, see below Def. 43.1 and Proposition 43.3 in [54], and we can consider the *unique* Hausdorff completion of $\mathcal{O}_1 \otimes \mathcal{O}_2$ with respect to $\tau_\pi, \tau_\varepsilon$ and denote them by $\overline{\mathcal{O}_1 \otimes \mathcal{O}_2}^{\pi}$ and $\overline{\mathcal{O}_1 \otimes \mathcal{O}_2}^{\varepsilon}$ respectively. In this case the algebraic operations also have continuous extensions and $\overline{\mathcal{O}_1 \otimes \mathcal{O}_2}^{\pi}$ and $\overline{\mathcal{O}_1 \otimes \mathcal{O}_2}^{\varepsilon}$ are complete locally convex unital $*$-algebras.

Let us consider the continuous linear extension of the map in Eq. (A.5)

$$\iota : \overline{\mathcal{O}_1 \otimes \mathcal{O}_2}^{\pi} \to \overline{\mathcal{O}_1 \otimes \mathcal{O}_2}^{\varepsilon}. \tag{A.6}$$

If $\mathcal{O}_1$ is nuclear and $\mathcal{O}_2$ is Hausdorff, then ι is a topological isomorphism and

$$(\mathcal{O}_1 \otimes \mathcal{O}_2, \tau_\pi) = (\mathcal{O}_1 \otimes \mathcal{O}_2, \tau_\varepsilon), \tag{A.7}$$

see Theorem 50.1 in [54]. Furthermore, under these assumptions, the space in Eq. (A.7), as well as its completion, is nuclear, provided *both* $\mathcal{O}_1$ and $\mathcal{O}_2$ are, see Proposition 50.1 in [54].

A.2 C^*-ALGEBRAS

In general, it is still not guaranteed that locally convex topological unital $*$-algebras have non-trivial effects. This imperfection is remedied by considering the following class of *normed* algebras.

Definition A.2.1. *Let $\mathcal{O}$ be a unital $*$-algebra. A norm $\|\cdot\|$ is called C^*-norm if and only if*

1. *for all $A, B \in \mathcal{O} : \|AB\| \leq \|A\|\|B\|$, i.e., $\|\cdot\|$ is* submultiplicative
2. *for all $A \in \mathcal{O} : \|A^*A\| = \|A\|^2$.*

If $\mathcal{O}$ is complete with respect to a C^*-norm $\|\cdot\|$, then $\mathcal{O}$ is called a unital C^*-algebra.

It is an important result that every unital C^*-algebra admits a *unique* C^*-norm that is completely determined by the algebraic relations, see Corollary 2 in [29]. As such, C^*-algebras may be considered special unital $*$-algebras rather than unital $*$-algebras equipped with an additional structure. However, viewing a unital C^*-algebra $\mathcal{O}$ as a unital $*$-algebra and considering a *proper* unital $*$-subalgebra $\mathcal{O}_1 \subsetneq \mathcal{O}$, then $\mathcal{O}_1$ does in general *not* admit a unique C^*-norm (if it admits one at all). This will become important when we discuss tensor products of unital C^*-algebras further below.

The tight connection between the algebraic and the topological structure of unital C^*-algebras is also emphasised by the following lemmas, see Sec. 2.3.1 in [75].

Lemma A.2.2. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two unital C^*-algebras. Every unit-preserving $*$-homomorphism $\phi : \mathcal{O}_1 \to \mathcal{O}_2$ is contractive, i.e.,*

$$\|\phi(\cdot)\|_{\mathcal{O}_2} \leq \|\cdot\|_{\mathcal{O}_1}. \tag{A.8}$$

In particular, ϕ is automatically continuous.

As an immediate consequence, if $\mathcal{O}_2$ is a unital C^*-algebra and $\mathcal{O}_1 \subseteq \mathcal{O}_2$ is a unital $*$-subalgebra, then the canonical inclusion ι is also continuous and in fact $\|\cdot\|_{\mathcal{O}_1}$ and $\|\cdot\|_{\mathcal{O}_2}$ agree on $\mathcal{O}_1$, which is in agreement with the fact that the C^*-norm of a unital C^*-algebra is unique. In particular, $\mathcal{O}_1$ is topologically closed. In summary, we can call $\mathcal{O}_1$ a unital C^*-subalgebra of $\mathcal{O}_2$.

Some further useful properties of unital C^*-algebras are collected in the following lemma, see Sec. 2.3 in [75] and also Proposition 9.7.21 in [17].

Lemma A.2.3. *Let $\mathcal{O}$ be a C^*-algebra. Then*

1. *every state ω on $\mathcal{O}$ is bounded with $\|\omega\| = \sup_{\substack{A\in\mathcal{O}\\ \|A\|=1}} |\omega(A)| = 1$,*

2. $A^*A = 0 \iff A = 0$,[3]

3. the image of a unit-preserving $*$-homomorphism between unital C^*-algebras is a unital C^*-subalgebra of the codomain,

4. the kernel of a unit-preserving $*$-homomorphism ϕ between unital C^*-algebras is a topologically closed $*$-ideal of the domain, in particular, if the domain is simple, then ϕ is injective.

The GNS representation from Lemma A.0.6 can be adapted for unital C^*-algebras, see Theorem 2.3.16 and Theorem 2.3.19 in [75]. In fact, every unital C^*-algebra is $*$-isomorphic to a unital C^*-subalgebra of $BL(\mathcal{H})$ for some complex (not necessarily separable) Hilbert space $\mathcal{H}$, see Theorem 2.1.10 in [75]. Furthermore, by the discussion above, every subset of $BL(\mathcal{H})$ that is a C^*-algebra and that is a unital $*$-subalgebra of $BL(\mathcal{H})$ is topologically closed. As we will see later, the analogue of this statement does *not* hold for unital von Neumann algebras.

Let us now consider the following lemma, which is Lemma 14 in [3].

Lemma A.2.4. *Let $\mathcal{R}$ be a unital C^*-subalgebra of $BL(\mathcal{H})$ for some complex Hilbert space $\mathcal{H}$ and let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{R}$ be two commuting non-Abelian unital C^*-subalgebras. If ψ is a cyclic unit vector for $\mathcal{A}$, i.e., if*

$$\forall \xi \in \mathcal{H}\ \exists (A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A} : \lim_{n\to\infty} A_n\psi = \xi, \tag{A.9}$$

then the state $\omega_\psi \upharpoonright \mathcal{B}$ is mixed, where $\omega_\psi(\cdot) := \langle\psi| \cdot \psi\rangle$.

Proof. We follow an argument in [76]. The proof consists of three parts.

Firstly, we show that there exists a unit vector $y \in \mathcal{H}$ such that $\omega_y \upharpoonright \mathcal{B} \neq \omega_\psi \upharpoonright \mathcal{B}$: we proceed by contradiction. Suppose that for every unit vector $y \in \mathcal{H}$ it holds that $\omega_y \upharpoonright \mathcal{B} = \omega_\psi \upharpoonright \mathcal{B}$. For every self-adjoint $B \in \mathcal{B}$, let us set $X_B := B - \langle\psi|B\psi\rangle\mathbb{1} \in \mathcal{B}$, then for every unit $y \in \mathcal{H} : \langle y|X_B y\rangle = \omega_y(B) - \omega_\psi(B) = 0$. From this and the fact that for self-adjoint X_B, $\|X_B\| = \sup_{\|y\|_\mathcal{H}=1} |\langle y|X_B y\rangle|$ we get that $X_B = 0$. In particular, every self-adjoint element in $\mathcal{B}$ is a multiple of the identity. Since every operator in $\mathcal{B}$ can be written as the sum of two self-adjoint operators, we have that

[3] This does hold for so-called *proper* unital $*$-algebras (see Def. 9.6.18 in [17]), however, not for mere unital $*$-algebras. Consider for instance again the (Abelian) unital $*$-algebra of all $\begin{pmatrix}\alpha & \beta\\ 0 & \alpha\end{pmatrix}$ for $\alpha, \beta \in \mathbb{C}$ equipped with the usual matrix multiplication and *entry-wise* complex conjugation as $*$-operation and set $A = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$

every operator in $\mathcal{B}$ is a multiple of the identity, which contradicts the assumption that $\mathcal{B}$ is non-Abelian.

Secondly, since ψ is cyclic for $\mathcal{A}$, we find $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that $A_n\psi \to y$. Upon restricting to a subsequence and rescaling A_n, the sequence $y_n := A_n\psi$ consists of unit vectors and converges to y, in particular $\omega_{y_n} \to \omega_y$ pointwise. If $\omega_y \restriction \mathcal{B} \neq \omega_\psi \restriction \mathcal{B}$, this shows that there exists $n \in \mathbb{N}$ such that $\omega_{y_n} \restriction \mathcal{B} \neq \omega_\psi \restriction \mathcal{B}$.

Thirdly, we show that there exists $\lambda \in (0,1)$ and a state τ such that $\omega_\psi \restriction \mathcal{B} = \lambda\omega_{y_n} \restriction \mathcal{B} + (1-\lambda)\tau$. We use that for all $B \in \mathcal{B}$

$$\omega_{y_n}(B^\dagger B) = \omega_\psi(A_n^\dagger B^\dagger B A_n) = \omega_\psi(B^\dagger A_n^\dagger A_n B) \leq \|A_n\|^2\omega_\psi(B^\dagger B). \tag{A.10}$$

We set $\lambda := \frac{1}{\|A_n\|^2}$, which (by setting $B = \mathbb{1}$) is in $(0,1]$. To see that $\lambda \neq 1$, note that for two states ω_1 and ω_2 such that $\omega_1(B^\dagger B) \leq \omega_2(B^\dagger B)$, it follows by the Cauchy-Schwarz inequality (see Proposition 5 in [29]) applied to the positive functional $\omega_2 - \omega_1$ (which maps $\mathbb{1}$ to 0), that $\omega_1 = \omega_2$. As a result we see that $\tau := \frac{1}{1-\lambda}(\omega_\psi \restriction \mathcal{B} - \lambda\omega_{y_n} \restriction \mathcal{B})$ is a state, which finishes the proof. □

Having discussed some properties of unital C^*-algebras, let us now once more consider tensor products.

Let $\mathcal{O}_1, \mathcal{O}_2$ be two unital C^*-algebras, which are in particular Banach spaces. In general, the algebraic tensor product $\mathcal{O}_1 \otimes \mathcal{O}_2$ then admits a variety of different *cross norms*, i.e., norms $\|\cdot\|$ such that for all $x_j \in \mathcal{O}_j$:

$$\|x_1 \otimes x_2\| = \|x_1\|_{\mathcal{O}_1}\|x_2\|_{\mathcal{O}_2}. \tag{A.11}$$

Since a C^*-algebra is clearly also a locally convex topological unital $*$-algebra, we can consider the locally convex topologies $\tau_\pi, \tau_\varepsilon$ on $\mathcal{O}_1 \otimes \mathcal{O}_2$, which in fact turn out to be norm-topologies coming from cross-norms

$$\begin{aligned}
\|t\|_\pi &:= \inf\left\{\sum_{j=1}^n \|a_j\|_{\mathcal{O}_1}\|b_j\|_{\mathcal{O}_2} \,|t \in \mathcal{O}_1 \otimes \mathcal{O}_2, t = \sum_{j=1}^n a_j \otimes b_j\right\}, \\
\|t\|_\varepsilon &:= \sup\left\{|(f_1 \otimes f_2)(t)| \,|f_j \in \mathcal{O}_j',\ \|f_j\|_{\mathcal{O}_j'} \leq 1\right\},
\end{aligned} \tag{A.12}$$

where $\mathcal{O}_j'$ is the set of continuous linear functionals on $\mathcal{O}_j$, see Proposition 43.1 and p. 443 in [54], as well as Definition T.3.5 in [25].

The norm $\|\cdot\|_\pi$ is submultiplicative and majorises every cross norm on $\mathcal{O}_1 \otimes \mathcal{O}_2$, see Proposition T.3.6 in [25], and every "resonable" norm $\|\cdot\|$ on the algebraic tensor product fulfills

$$\|\cdot\|_\varepsilon \leq \|\cdot\| \leq \|\cdot\|_\pi, \tag{A.13}$$

see Remark T.3.12 in [25].

However, while $\|\cdot\|_\pi$ is at least submultiplicative, it is in general *not* a C^*-norm,[4] and $\|\cdot\|_\varepsilon$ need not even be submultiplicative, see Remark T.3.7 in [25]. Nevertheless, there *are* C^*-norms on $\mathcal{O}_1 \otimes \mathcal{O}_2$, in particular the *minimal* one $\|\cdot\|_\sigma$ and the maximal one $\|\cdot\|_\mu$, see Remark T.5.2 and Definition T.6.6 in [25]. The names are justified since every C^*-norm $\|\cdot\|$ on $\mathcal{O}_1 \otimes \mathcal{O}_2$ fulfills

$$\|\cdot\|_\varepsilon \leq \|\cdot\|_\sigma \leq \|\cdot\| \leq \|\cdot\|_\mu \leq \|\cdot\|_\pi, \tag{A.14}$$

see Theorem T.6.21 in [25].

The C^*-algebras emerging from the completion of the algebraic tensor product with respect to $\|\cdot\|_\sigma, \|\cdot\|_\mu$ are denoted by $\overline{\mathcal{O}_1 \otimes \mathcal{O}_2}^\sigma$ and $\overline{\mathcal{O}_1 \otimes \mathcal{O}_2}^\mu$ respectively.

There is a canonical $*$-monomorphism

$$\iota : \overline{\mathcal{O}_1 \otimes \mathcal{O}_2}^\mu \to \overline{\mathcal{O}_1 \otimes \mathcal{O}_2}^\sigma, \tag{A.15}$$

which is however not surjective in general. A unital C^*-algebra $\mathcal{O}_1$ is called *nuclear* (as a C^*-algebra) if for every unital C^*-algebra $\mathcal{O}_2$ the map ι is a unit-preserving $*$-isomorphism, i.e., if there exists a unique C^*-norm on $\mathcal{O}_1 \otimes \mathcal{O}_2$.

Remark: A word of caution. Recall that a Hausdorff locally convex topological vector space $\mathcal{O}_1$ is called nuclear, if for every Banach space $\mathcal{O}_2$ the map ι in Eq. (A.6) is a topological isomorphism, see Theorem 50.1 in [54]. Also recall from above that no infinite dimensional Banach space is nuclear. Hence a unital C^*-algebra that is nuclear as a unital C^*-algebra *cannot* be nuclear as a locally convex unital $*$-algebra.

A.3 VON NEUMANN ALGEBRAS

A notion even stronger than that of a unital C^*-algebra is that of a unital von Neumann algebra. One motivation for introducing von Neumann algebras is that a unital C^*-algebra, while always admitting non-trivial effects due to the *continuous functional calculus*, might still not contain any non-trivial *projections*.

Definition A.3.1 (Def. 1.1.2 and Corollary 1.13.3 in [21])**.** *A unital C^*-algebra $\mathcal{N}$ is a unital von Neumann algebra (also called a unital W^*-algebra) if there exists a Banach space $\mathcal{N}_*$ (unique up to isometric isomorphism) whose topological dual space is isometrically isomorphic to $\mathcal{N}$ as a Banach space. $\mathcal{N}_*$ is called the predual of $\mathcal{N}$.*

From the fact that every unit-preserving $*$-isomorphism between unital C^*-algebras is isometric, it follows that if a unital C^*-algebra $\mathcal{N}_2$ is $*$-isomorphic to a

[4] Consequently, Def. 18 in [29] seems to be erroneous.

unital von Neumann algebra $\mathcal{N}_1$, then $\mathcal{N}_2$ is a unital von Neumann algebra as well.

The prime example of a class of unital von Neumann algebras is given by the set of bounded linear operators $BL(\mathcal{H})$ on any (not necessarily separable) Hilbert space $\mathcal{H}$ with predual consisting of trace class operators, see Proposition 2.4.3 in [75] and Theorem 1.15.3 in [21].

Due to the existence of a predual, a von Neumann algebra carries another canonical topology apart from the C^*-norm topology, namely the topology of pointwise convergence denoted by $\sigma(\mathcal{N}, \mathcal{N}_*)$. While every unit-preserving $*$-homomorphism between von Neumann algebras $\mathcal{N}_1$ and $\mathcal{N}_2$ is immediately continuous with respect to the norm topologies, it generally fails to be continuous with respect to $\sigma(\mathcal{N}, \mathcal{N}_*)$.

We consider the following definition.

Definition A.3.2 (III.2.2.1 in [38]). *Let $\mathcal{N}_1, \mathcal{N}_2$ be two von Neumann algebras. A positive map $\varphi : \mathcal{N}_1 \to \mathcal{N}_2$ is called* normal*, if for every increasing net $\{A_\alpha\}_\alpha$ of positive elements in $\mathcal{N}_1$ with upper bound it holds that*

$$\varphi\left(\sup_{\mathcal{N}_1} \{A_\alpha\}_\alpha\right) = \sup_{\mathcal{N}_2} \{\varphi(A_\alpha)\}_\alpha. \tag{A.16}$$

It follows that a unit-preserving $*$-isomorphism between unital von Neumann algebras is normal, see III.2.2 in [38].

Lemma A.3.3 (Proposition III.2.2.2 in [38]). *Let $\mathcal{N}_1, \mathcal{N}_2$ be two von Neumann algebras. A unit-preserving $*$-homomorphism*

$$\phi : (\mathcal{N}_1, \sigma(\mathcal{N}_1, (\mathcal{N}_1)_*)) \to (\mathcal{N}_2, \sigma(\mathcal{N}_2, (\mathcal{N}_2)_*)) \tag{A.17}$$

is continuous if and only if it is normal.

Theorem A.3.4 (Theorem 2.4.21, Theorem 2.4.23, Theorem 2.4.24 in [75]). *Let $\mathcal{N}$ be a unital von Neumann algebra. Then the following holds.*

1. *$\mathcal{N}_*$ embedds canonically into the space of continuous linear functionals on $\mathcal{N}$. A state $\omega : \mathcal{N} \to \mathbb{C}$ is in $\mathcal{N}_*$ if and only if it is normal.*

2. *Unit-preserving $*$-homomorphisms between von Neumann algebras are not necessarily normal.*

3. *A completely positive linear map $\phi : \mathcal{N}_1 \to \mathcal{N}_2$ between unital von Neumann algebras is normal if and only if for every normal state ω on $\mathcal{N}_2$ the function $\omega \circ \phi$ is normal on $\mathcal{N}_1$, see the proof of Proposition III.2.2.2.*

4. *If ω is a normal state, then the corresponding GNS representation $\pi_\omega : \mathcal{N} \to BL(\mathcal{H}_\omega)$ is normal.*

5. *If $\varphi : \mathcal{N} \to BL(\mathcal{H})$ is a normal unit-preserving $*$-homomorphism, then $\varphi[\mathcal{N}]$ is weakly closed, or equivalently, $\varphi[\mathcal{N}]'' = \varphi[\mathcal{N}]$, see Proposition 1.16.2 and Proposition 1.15.1 in [21].*

6. *Every weakly closed unital C^*-subalgebra of $BL(\mathcal{H})$ is a unital von Neumann algebra, see p. 34 in [21].*

7. *If $\varphi : \mathcal{N} \to BL(\mathcal{H})$ is an injective normal unit-preserving $*$-representation, then the normal states on $\mathcal{N}$ are precisely those of the form*

$$A \mapsto \mathrm{Tr}(\rho\pi(A)), \tag{A.18}$$

for a positive trace-class operator ρ with unit trace, see Theorem 2.4.21 [75].

Remark: Let $\mathcal{N}_1$ be a weakly closed unital C^*-subalgebra of $BL(\mathcal{H}_1)$, let $\mathcal{N}_2$ be a unital C^*-subalgebra of $BL(\mathcal{H}_2)$ and let $\varphi : \mathcal{N}_1 \to \mathcal{N}_2$ be a unit-preserving $*$-isomorphism. Then $\mathcal{N}_2$ is a von Neumann algebra and φ is normal. However, unless the canonical extension of φ to a unit-preserving $*$-monomorphism $\tilde{\varphi} : \mathcal{N}_1 \to BL(\mathcal{H}_2)$ is normal, $\mathcal{N}_2$ will in general *not* be a weakly closed C^*-subalgebra of $BL(\mathcal{H}_2)$, see III.2.2.15 in [38].

As we have already mentioned before, there is no analogous situation for C^*-algebras. If $\mathcal{O}$ is a unital $*$-subalgebra of $BL(\mathcal{H})$ with canonical inclusion ι and if $\mathcal{O}$ is a C^*-algebra, then ι is continuous (even isometric) with respect to the norm topologies. If $\mathcal{O}$ is a von Neumann algebra, then ι is *not* necessarily normal, so not necessarily continuous with respect to the topologies $\sigma(\mathcal{O}, \mathcal{O}_*)$ and $\sigma(BL(\mathcal{H}), BL(\mathcal{H})_*)$.

In summary: Every von Neumann algebra is $*$-isomorphic to a weakly closed unital C^*-subalgebra of some $BL(\mathcal{H})$, *but* not every von Neumann algebra that is a $*$-subalgebra of some $BL(\mathcal{H})$ is weakly closed.

Let us finally return to one of our motivations for considering von Neumann algebras, namely that they have sufficiently many projections. Indeed, according to the bounded measurable functional calculus available in a unital von Neumann algebra $\mathcal{N}$, $\mathcal{N}$ admits many projections. In fact, the projections of $\mathcal{N}$ span a norm-dense subspace of $\mathcal{N}$, see p. 71 in [75].

— B —

Canonical Commutation Relation (CCR) algebras

B.1 UNITAL TENSOR $*$-ALGEBRAS

Let V be a $\mathbb{C}$-vector space. Then we define

$$T^{\otimes}(V) := \bigoplus_{n=0}^{\infty} V^{\otimes n}, \tag{B.1}$$

where $V^{\otimes n}$ is the n-fold (algebraic) tensor product of V and $V^{\otimes 0} := \mathbb{C}$. As a subspace of $T^{\otimes}(V)$, $V^{\otimes n}$ is referred to as the n^{th}-sector. The elements of $T(V)$ have finitely many non-vanishing entries. Let us now define the maps $\star_{m,n} : V^{\otimes m} \times V^{\otimes n} \to V^{\otimes m+n}$ through

$$(v_{1,1} \otimes ... \otimes v_{1,m}) \star_{m,n} (v_{2,1} \otimes ... \otimes v_{2,n}) := v_{1,1} \otimes ... \otimes v_{1,m} \otimes v_{2,1} \otimes ... \otimes v_{2,n}, \tag{B.2}$$

where $\star_{0,n}$ and $\star_{n,0}$ is the multiplication by a complex number. These maps are bilinear and give rise to the bilinear map $\star : T^{\otimes}(V) \times T^{\otimes}(V) \to T^{\otimes}(V)$ given by

$$\begin{aligned}(A_0, A_1, ...)\star(B_0, B_1, ...)\\ := \left(\sum_{j+k=0} A_j \star_{j,k} B_k, \sum_{j+k=1} A_j \star_{j,k} B_k, \sum_{j+k=2} A_j \star_{j,k} B_k, ...\right),\end{aligned} \tag{B.3}$$

which turns $T^{\otimes}(V)$ into an associative complex unital algebra called *tensor algebra*.

Let $C_0 : \mathbb{C} \to \mathbb{C}$ be given by the complex conjugation and let $i : V \to V$ be a complex conjugation, i.e., an antilinear involution. Then the complex conjugations $C_n : V^{\otimes n} \to V^{\otimes n}$ given by

$$C_n(v_1 \otimes ... \otimes v_n) := i(v_n) \otimes ... \otimes i(v_1) \tag{B.4}$$

for $n > 0$ define a $*$-operation on $T^{\otimes}(V)$ given by

$$A^* := \left(\bigoplus_{n=0}^{\infty} C_n\right) A, \tag{B.5}$$

i.e., $i : V \to V$ turns $T^{\otimes}(V)$ into a unital $*$-algebra $T^{\otimes}(V, i)$.

Let now V be a nuclear locally convex topological vector space with a continuous complex conjugation i, see Sec. 12.5 in [74]. Then $V^{\otimes n}$ is equipped with a, due to nuclearity *unique*, locally convex topology. Furthermore, $T^{\otimes}(V, i)$ may be equipped with the locally convex direct sum topology, which turns $T^{\otimes}(V, i)$ into a nuclear locally convex topological vector space, see Proposition 50.1 in [54]. We can also consider $T^{\overline{\otimes}}(V, i)$ given by

$$T^{\overline{\otimes}}(V, i) := \bigoplus_{n=0}^{\infty} \overline{V^{\otimes n}}. \tag{B.6}$$

$T^{\otimes}(V, i)$ and also $T^{\overline{\otimes}}(V, i))$ are in fact nuclear unital $*$-algebras. To see this note the continuity of the left-multiplication and the $*$-operation: Let F be a locally convex topological vector space. Since every linear map $T^{\otimes}(V) \to F$ is continuous if and only if its restriction to each sector is continuous, and since the n^{th} sector together with the subset topology is topologically isomorphic to $V^{\otimes n}$ equipped with the tensor product topology, see (iii) on p. 515 in [54], it suffices to show that

1. $\forall n \in \mathbb{N}^* : C_n : V^{\otimes n} \to V^{\otimes n}$ is continuous, and
2. $\forall m, n \in \mathbb{N}^* \; \forall B_n \in V^{\otimes n} : V^{\otimes m} \ni A_m \mapsto A_M \star_{m,n} B_n \in V^{\otimes m+n}$

are continuous, which follows from Proposition 43.6 in [54]. Hence $T^{\otimes}(V, i)$ and (in complete analogy) $T^{\overline{\otimes}}(V, i)$ are nuclear unital $*$-algebras where the latter is in fact complete.

Examples for V, see for instance [77], include various function spaces, such as $C_c^{\infty}(M; \mathbb{C})$, which is a nuclear locally convex topological vector space with a continuous complex conjugation with $\overline{(C_c^{\infty}(M; \mathbb{C}))^{\otimes n}} \cong C_c^{\infty}(M^n; \mathbb{C}^n)$.

B.2 (Pre-)symplectic spaces

Let us now consider a real vector space $V_{\mathbb{R}}$ equipped with a pre-symplectic form $s_{\mathbb{R}}$, i.e., an alternating $\mathbb{R}$-bilinear map $s_{\mathbb{R}} : V_{\mathbb{R}} \times V_{\mathbb{R}} \to \mathbb{R}$. We define

$$\ker s_{\mathbb{R}} := \{u \in V_{\mathbb{R}} | s(u, \cdot) \equiv 0\}. \tag{B.7}$$

If $\ker s_{\mathbb{R}} = \{0\}$, i.e., $s_{\mathbb{R}}$ is weakly nondegenerate, then $s_{\mathbb{R}}$ is called a symplectic form and $(V_{\mathbb{R}}, s_{\mathbb{R}})$ is called a symplectic space.

$s_{\mathbb{R}}$ descends to the quotient $V_{\mathbb{R}}/\ker s_{\mathbb{R}}$ of $V_{\mathbb{R}}$ by the kernel of $s_{\mathbb{R}}$, on which it is weakly nondegenerate. Let now V be the complexification of $V_{\mathbb{R}}$, by $\mathbb{C}$-bilinear extension, $s_{\mathbb{R}}$ extends to an alternating $\mathbb{C}$-bilinear form s, which is weakly nondegenerate if and only if $s_{\mathbb{R}}$ is. Moreover, $V/\ker s$ is isomorphic to the complexification of $V_{\mathbb{R}}/\ker s_{\mathbb{R}}$.

B.3 Field algebra

For what follows see also Sec. 5 in [78].

Given such V and (not necessarily weakly nondegenerate) s, we can define I_{CCR} to be the smallest two-sided $*$-ideal of $T^{\otimes}(V, \bar{\cdot})$ spanned by elements of the form

$$(-\mathrm{i}\hbar s(u,v), 0, u \otimes v - v \otimes u, 0, ...), \tag{B.8}$$

for $u, v \in V$, where $\hbar \in \mathbb{R}$ is some constant. In the case that V carries a locally convex topology and s is separately continuous in each slot, we define I_{CCR} to be the smallest *closed* two-sided $*$-ideal containing the elements above.

The unital $*$-algebra

$$T^{\otimes}(V, \bar{\cdot})/I_s \tag{B.9}$$

is called *off-shell field algebra.* In case V carries a nuclear topology, then $T^{\otimes}(V, \bar{\cdot})/I_s$, together with its topology, is a nuclear unital $*$-algebra, see Proposition 50.1 in [54].

Let us now define I_{shell} to be the smallest (in case of nuclear V *closed*) two-sided $*$-ideal of $T^{\otimes}(V, \bar{\cdot})$ spanned by elements of the form

$$(0, u, ...), \tag{B.10}$$

where $u \in \ker s$. Furthermore, let $I_{\mathrm{shell},s}$ be the smallest (in case of a topology on V *closed*) two-sided $*$-ideal of $T^{\otimes}(V, \bar{\cdot})$ spanned by elements of the form

$$(-\mathrm{i}\hbar s(u,v), 0, u \otimes v - v \otimes u, 0, ...), \qquad (0, w, ...), \tag{B.11}$$

for $u, v \in V$, and $u \in \ker s$, i.e., the smallest (closed) two-sided $*$-ideal containing I_s and I_{shell}. Moreover, I_{shell} descends (under the continuous quotient map) to a (closed) two-sided $*$-ideal of $T^{\otimes}(V, \bar{\cdot})/I_s$ and we have

$$\mathcal{F} := \left(T^{\otimes}(V, \bar{\cdot})/I_s\right)/I_{\mathrm{shell}} \cong T^{\otimes}(V, \bar{\cdot})/I_{\mathrm{shell},s}. \tag{B.12}$$

This is called the (on-shell) field algebra. If V carries a nuclear topology, then so does $\mathcal{F}$.

It is now common to define

$$\varphi_{\mathrm{off}}(f) := [(0, f, 0, ...)]_{I_s} \tag{B.13}$$

and

$$c\mathbb{1} := [(c, 0, ...)]_{I_s}, \tag{B.14}$$

for $f \in V_{\mathbb{R}}$, where $[\cdot]_{I_s} : T^{\otimes}(V, \bar{\cdot}) \to T^{\otimes}(V, \bar{\cdot})/I_s$ is the quotient map. Finite products, linear combinations thereof and their image under the star operation are then defined in the obvious way. In this sense the symbols $\varphi_{\text{off}}(f)$ fulfill

1. $\varphi_{\text{off}}(af + bg) = a\varphi_{\text{off}}(f) + b\varphi_{\text{off}}(g)$ for $a, b \in \mathbb{R}$,
2. $\varphi_{\text{off}}(f)^* = \varphi_{\text{off}}(f)$,
3. $[\varphi_{\text{off}}(f), \varphi_{\text{off}}(g)] = \mathrm{i}\hbar s(f, g)$.

Moreover, $T^{\otimes}(V, \bar{\cdot})/I_s$ is generated by $\mathbb{1}$ and all $\varphi_{\text{off}}(f)$ for $f \in V_{\mathbb{R}}$. The map φ_{off} is called the off-shell field.

In a completely analogous fashion we can define

$$\varphi(f) := [(0, f, 0, ...)]_{I_{\text{shell},s}}, \tag{B.15}$$

which fulfill *in addition* to the three properties above also

$$\forall f \in \ker s_{\mathbb{R}} : \ \varphi(f) = 0, \tag{B.16}$$

and generate $\mathcal{F}$.

Lemma B.3.1. *Let $\mathcal{A}$ be a unital $*$-algebra such that there is a function $\tilde{\varphi} : V \to \mathcal{A}$ that fulfills all the four properties above. Then the smallest unital $*$-subalgebra of $\mathcal{A}$ containing the image of $\tilde{\varphi}$ is isomorphic to $\mathcal{F}$ as a unital $*$-algebra.*

Proof. By Scholium 7.1 in [79], the smallest unital $*$-subalgebra of $\mathcal{A}$ containing the image of φ, which we denote by $\tilde{\mathcal{A}}$, is simple. The map π sending $\tilde{\varphi}(f)$ to $\varphi(f)$ extends to a surjective unit-preserving $*$-homomorphism from $\tilde{\mathcal{A}}$ to $\mathcal{F}$. Injectivity of π follows from simplicity of $\tilde{\mathcal{A}}$. □

Remark: In the following we will set $\hbar = 1$.

B.4 Weyl algebra

Let us now consider a symplectic space $(V_{\mathbb{R}}, s_{\mathbb{R}})$. An $\mathbb{R}$-linear subspace $X \subseteq V_{\mathbb{R}}$ is called symplectic, if $s_{\mathbb{R}} \restriction X$ is a symplectic form on X. For a symplectic subspace, the symplectic complement

$$X^{\perp} := \{u \in V_{\mathbb{R}} | s_{\mathbb{R}}(u, x) = 0 \ \forall x \in X\} \tag{B.17}$$

has trivial intersection with X.

Let us now introduce so-called Weyl generators and the Weyl algebra.

Definition B.4.1. *Let $(V_\mathbb{R}, s_\mathbb{R})$ be a symplectic space, let μ be the counting measure on $V_\mathbb{R}$ and let $L^2(V_\mathbb{R}, \mu)$ be the $\mathbb{C}$-Hilbert space of square-integrable complex valued functions on $V_\mathbb{R}$. Then the Weyl generators $W(v) \in BL(L^2(V_\mathbb{R}, \mu))$ for $v \in V_\mathbb{R}$ are defined as the unitary operators such that*

$$(W(v)F)(x) := e^{\frac{\mathrm{i}}{2}s(v,x)}F(v+x) = \int_V \delta_v(y)e^{\frac{\mathrm{i}}{2}s(y,x)}F(y+x)\,\mathrm{d}\mu(y), \tag{B.18}$$

where $\delta_v(y)$ is the Kronecker delta function centred at v. They fulfill

1. *$W(v)^* = W(-v)$, and*
2. *$W(u)W(v) = e^{-\frac{\mathrm{i}}{2}s(u,v)}W(u+v)$.*

A finite complex linear combination of Weyl generators A is uniquely determined by a complex-valued function z that vanishes everywhere except on at most finitely many points, such that

$$(AF)(x) = \int_V z(y)e^{\frac{\mathrm{i}}{2}s(y,x)}F(y+x)\,\mathrm{d}\mu(y). \tag{B.19}$$

Furthermore, the unital C^*-subalgebra of $BL(L^2(V_\mathbb{R}, \mu))$ generated by all $W(v)$'s for $v \in V$ is called the Weyl algebra.

We collect the following useful results.

Lemma B.4.2. *Let now $\mathcal{A}$ be a unital C^*-algebra and let $W : V_\mathbb{R} \to \mathcal{A}$ be a function such that*

1. *$W(v)^* = W(-v)$, and*
2. *$W(u)W(v) = e^{-\frac{\mathrm{i}}{2}s_\mathbb{R}(u,v)}W(u+v)$.*

Then the smallest unital C^*-subalgebra of $\mathcal{A}$ containing the image of W is isomorphic to the Weyl algebra as a unital C^*-algebra.

Furthermore, the Weyl algebra is a simple nuclear unital C^*-algebra.

Proof. For the first part see Theorem 3 in [29], and Theorem 10.10 in [80] for the nuclearity of the Weyl algebras. See Lemma 10 in [29] for the fact that the Weyl algebra is simple. □

B.4.1 A useful topology

The Weyl algebra over a symplectic space $V_{\mathbb{R}}$ evidently carries the natural norm topology. However, this topology is often too strong for physical purposes, since for instance $\|W(u) - W(v)\| = 2$ for $u \neq v$, see Proposition 7 in [29]. In practice, the underlying symplectic space $V_{\mathbb{R}}$ sometimes carries a vector space topology τ_{symp} for which $s_{\mathbb{R}}$ is separately continuous. This may then be used to define a suitable class of states $\mathfrak{S}_c$, which in turn may be used to define a different *useful* topology on the Weyl algebra. Concretely, the weakest topology containing all strong* operator topologies $\tau^{\omega}_{st^*}$ in GNS representations of states $\omega \in \mathfrak{S}_c$.

The results and proofs of this section are taken from [2], joint work with Fewster and Jubb.

At this point we remind the reader that a topology τ_A on a set X is called weaker (or coarser, or smaller) than topology τ_B on X, if and only if $\tau_A \subseteq \tau_B$. In this case one also says that τ_B is stronger (or finer, or larger) than τ_A. The weakest topology is $\{\emptyset, X\}$, the strongest topology is the power set of X. In particular, every set that is τ_A-open is also τ_B open, but τ_B has (possibly) more open sets, so it is more difficult for a net (Moore-Smith sequence) to converge. Every net $(a_\alpha)_\alpha$ that converges to a point a in the stronger topology τ_B also converges in the weaker topology τ_A, but the converse does not hold in general. If $Z \subseteq X$ is τ_B-dense, then it is also τ_A-dense. Any map $f : X \to Y$ for a topological space Y that is continuous with respect to the topology τ_A is also continuous with respect to the topology τ_B, but the converse does not hold in general. In summary, stronger topologies have more open sets, fewer convergent nets, fewer dense subsets, and more continuous functions into other topological spaces.

Given any state ω on the Weyl algebra $\mathcal{W}$ over the symplectic space $(V_{\mathbb{R}}, s_{\mathbb{R}})$, we can define two interesting topologies, see Definition 6.1 in [2].

Definition B.4.3. *Let ω be a state on $\mathcal{W}$ with GNS representation $\pi_\omega : \mathcal{W} \to BL(\mathcal{H}_\omega)$. Then we define*

1. *the π_ω-weak operator topology τ^{ω}_{w} on $\mathcal{W}$ as the weakest topology such that $\pi_\omega : \mathcal{W} \to (BL(\mathcal{H}_\omega), \tau_w)$ is continuous, where τ_w is the weak operator topology,*

2. *the π_ω-strong operator topology τ^{ω}_{st} on $\mathcal{W}$ as the weakest topology such that $\pi_\omega : \mathcal{W} \to (BL(\mathcal{H}_\omega), \tau_{st})$ is continuous, where τ_{st} is the strong operator topology, and*

3. *the π_ω-strong* operator topology $\tau^\omega_{st^*}$ on $\mathcal{W}$ as the weakest topology such that $\pi_\omega : \mathcal{W} \to (BL(\mathcal{H}_\omega), \tau_{st^*})$ is continuous, where τ_{st^*} is the strong* operator topology.*

Equipped with either τ^ω_w or τ^ω_{st} or $\tau^\omega_{st^*}$, $\mathcal{W}$ is a locally convex topological vector space and

1. *τ^ω_w is generated by the family of seminorms $|\langle x|\pi_\omega(\cdot)y\rangle_\omega|$ for $x, y \in \mathcal{H}_\omega$,*

2. *τ^ω_{st} is generated by the family of seminorms $\|\pi_\omega(\cdot)x\|_\omega$ for $x \in \mathcal{H}_\omega$, and*

3. *$\tau^\omega_{st^*}$ is generated by the family of seminorms $\|\pi_\omega(\cdot)x\|_\omega + \|\pi_\omega(\cdot)^*x\|_\omega$ for $x \in \mathcal{H}_\omega$.*

It holds that

$$\tau^\omega_w \subseteq \tau^\omega_{st} \subseteq \tau^\omega_{st^*} \subseteq \tau_{\|\cdot\|}. \tag{B.20}$$

Given a vector space topology τ_{symp} on $V_\mathbb{R}$ such that $s_\mathbb{R}$ is separately continuous, we can now define the useful topology τ on $\mathcal{W}$, see Definition 6.2 in [2].

Definition B.4.4. *Let $\mathcal{W}$ be the Weyl algebra over a symplectic space $(V_\mathbb{R}, s_\mathbb{R})$ and let τ_{symp} be a vector space topology on $V_\mathbb{R}$ such that $s_\mathbb{R}$ is τ_{symp}-continuous in every slot separately. Then we define*

1. regular *states to be states ω such that for every $u \in V_\mathbb{R}$ the maps $\mathbb{R} \ni t \mapsto \omega(W(tu)) \in \mathbb{C}$ are continuous,*

2. *$\mathfrak{S}_c$ to be the set of all states on $\mathcal{W}$ such that $\omega \circ W : V_\mathbb{R} \to \mathbb{C}$ is continuous with respect to the topology τ_{symp} on $V_\mathbb{R}$, and*

3. *τ to be the locally convex topology generated by the seminorms $\|\pi_\omega(\cdot)x\|_\omega + \|\pi_\omega(\cdot)^*x\|_\omega$ for $x \in \mathcal{H}_\omega$ and $\omega \in \mathfrak{S}_c$, i.e., the weakest topology that contains every $\tau^\omega_{st^*}$ for $\omega \in \mathfrak{S}_c$.*

Remark: $\mathfrak{S}_c$ contains in particular all quasi-free states with jointly τ-continuous symmetric covariance function. Moreover, clearly every state in $\mathfrak{S}_c$ is *regular*.

We now show an auxiliary lemma, which is based on Lemma E.1 in [2].

Lemma B.4.5. *Let $\mathcal{W}$ be the Weyl algebra over $(V_\mathbb{R}, s_\mathbb{R})$ and let ω be a state on $\mathcal{W}$ with GNS representation $\pi_\omega : \mathcal{W} \to BL(\mathcal{H}_\omega)$. Let u_n be a net in $V_\mathbb{R}$ and let $u \in V_\mathbb{R}$ be such that*

1. *$\omega(W(u_n - u)) \longrightarrow 1$ in $\mathbb{C}$, and*

2. *for every $v \in V_{\mathbb{R}}$ it holds that $s_{\mathbb{R}}(u_n - u, v) \longrightarrow 0$ in $\mathbb{C}$.*

Then

$$\pi_\omega(W(u_n)) \longrightarrow \pi_\omega(W(u)) \tag{B.21}$$

with respect to the strong operator topology on $BL(\mathcal{H}_\omega)$.*

Proof. Let us write $W^\omega(x) := \pi_\omega(W(x))$. Using the Weyl relations it is easy to verify that for every $v \in V_{\mathbb{R}}$:

$$(W(u_n) - W(u))W(v) = e^{-\frac{\mathrm{i}}{2}s_{\mathbb{R}}(u_n-u,u+2v)}W(u)W(v)\Big(W(u_n - u) - e^{\frac{\mathrm{i}}{2}s_{\mathbb{R}}(u_n-u,u+2v)}\mathbb{1}\Big), \tag{B.22}$$

and hence, by unitarity of the Weyl generators,

$$\begin{aligned}\|(W^\omega(u_n) - W^\omega(u))W^\omega(v)\Omega_\omega\|_\omega &= \|W(u_n - u)\Omega_\omega - e^{\frac{\mathrm{i}}{2}s_{\mathbb{R}}(u_n-u,u+2v)}\Omega_\omega\|_\omega \\ &\leq \|(W(u_n - u) - \mathbb{1})\Omega_\omega\|_\omega + |e^{\frac{\mathrm{i}}{2}s_{\mathbb{R}}(u_n-u,u+2v)} - 1| \\ &\longrightarrow 0,\end{aligned} \tag{B.23}$$

on noting that $\|(W^\omega(u_n - u) - \mathbb{1})\Omega_\omega\|_\omega^2 = 2 - 2\mathrm{Re}\,\omega(W(u_n - u))$. Taking linear combinations, we have shown that $W^\omega(u_n)\phi \to W^\omega(u)\phi$ for all ϕ in the span of $\{W^\omega(u)\Omega|u \in V_{\mathbb{R}}\}$, which is dense in $\mathcal{H}_\omega$, due to cyclicity and the Weyl relations. This statement extends to all $\phi \in \mathcal{H}_\omega$ because the $W^\omega(u_n)$ are unitary and therefore uniformly bounded.

Finally, if $u_n \to u$, then also $-u_n \to -u$ and the above argument shows that for every $\phi \in \mathcal{H}_\omega$ it holds that

$$\begin{aligned}&\|(W^\pi(u_n) - W^\pi(u))\phi\|_\omega + \|(W^\pi(u_n) - W^\pi(u))^*\phi\|_\omega \\ &= \|(W^\pi(u_n) - W^\pi(u))\phi\|_\omega + \|(W^\pi(-u_n) - W^\pi(-u))\phi\|_\omega \longrightarrow 0.\end{aligned} \tag{B.24}$$

□

A typical application of the above lemma is the following.

Corollary B.4.6. *Let ω be* regular. *Then*

$$\mathbb{R} \ni t \mapsto W^\omega(tu) \in BL(\mathcal{H}_\omega) \tag{B.25}$$

is a strongly continuous one-parameter unitary group.

Let us return to the topology τ. Important properties of this topology are illustrated by the following lemma, which is Lemma 6.3 in [2].

Lemma B.4.7. *Let $\mathcal{W}$ be the Weyl algebra over a symplectic space $(V_{\mathbb{R}}, s_{\mathbb{R}})$ and let τ_{symp} be a vector space topology on $V_{\mathbb{R}}$ such that $s_{\mathbb{R}}$ is separately τ_{symp}-continuous. Then, for every $\omega \in \mathfrak{S}_c$*

$$\tau_w^\omega \subseteq \tau_{st}^\omega \subseteq \tau_{st^*}^\omega \subseteq \tau \subseteq \tau_{\|\cdot\|}, \tag{B.26}$$

and $\omega : (\mathcal{W}, \tau) \to \mathbb{C}$ as well as the GNS representation $\pi_\omega : (\mathcal{W}, \tau) \to BL(\mathcal{H}_\omega)$ is continuous, where $BL(\mathcal{H}_\omega)$ is equipped either with its strong or strong or weak operator topology.*

Furthermore, the star operation is τ-continuous and the map

$$\begin{aligned} W : (V_{\mathbb{R}}, \tau_{\text{symp}}) &\to (\mathcal{W}, \tau) \\ u &\mapsto W(u) \end{aligned} \tag{B.27}$$

is continuous and for every τ-dense $\mathcal{I} \subseteq \mathcal{W}$, $\mathcal{I}$ is τ_w-dense in the von Neumann algebra $(\pi_\omega[\mathcal{W}])''$, i.e.,

$$\overline{\pi_\omega[\mathcal{I}]}^w = \overline{\pi_\omega[\mathcal{W}]}^w = (\pi_\omega[\mathcal{W}])'', \tag{B.28}$$

where $\overline{\cdot}^w$ denotes the closure in $(BL(\mathcal{H}_\omega), \tau_w)$.

Proof. The first sentence follows straightforwardly from Definition B.4.3 and Definition B.4.4.

Furthermore, $W : (V_{\mathbb{R}}, \tau_{\text{symp}}) \to (\mathcal{W}, \tau_{st^*}^\omega)$ is continuous if and only if $\pi_\omega \circ W : (V_{\mathbb{R}}, \tau_{\text{symp}}) \to (BL(\mathcal{H}_\omega), \tau_{st^*})$ is continuous, which is shown in Lemma B.4.5. Hence, by the fact that a function f from some topological space X into $(\mathcal{W}, \tau)$ is continuous if and only if f is continuous into $(\mathcal{W}, \tau_{st^*}^\omega)$ for every $\omega \in \mathfrak{S}_c$, $W : (V_{\mathbb{R}}, \tau_{\text{symp}}) \to (\mathcal{W}, \tau)$ is continuous.

Finally, since τ is stronger than τ_w^ω, $\mathcal{I}$ is τ_w^ω-dense in $\mathcal{W}$ so $\pi_\omega[\mathcal{I}]$ is τ_w-dense in $\pi_\omega[\mathcal{W}]$. Hence $\pi_\omega[\mathcal{I}]$ is also τ_w-dense in $\overline{\pi_\omega[\mathcal{W}]}^w = (\pi_\omega[\mathcal{W}])''$. □

B.4.2 A pre-*Haag-type property*

Let $X \subseteq V_{\mathbb{R}}$ be a symplectic subspace. Then the closed unital C^*-subalgebra of the Weyl algebra over $V_{\mathbb{R}}$ that is generated by all the Weyl generators with indices in X is, according to above, isomorphic to the Weyl algebra of the symplectic vector space $(X, s_{\mathbb{R}} \restriction X)$. We will say that an element in the former is "in the Weyl algebra of $X \subseteq V_{\mathbb{R}}$".

Lemma B.4.8. *Let $(V_{\mathbb{R}}, s_{\mathbb{R}})$ be a symplectic vector space and let X be a symplectic subspace. Let A be an element of the Weyl algebra over $V_{\mathbb{R}}$. Then the following holds*

$$\begin{aligned} &(\forall v \in X : [W(v), A] = 0) \implies \\ &A \text{ is in the Weyl algebra of any symplectic subspace of } V_{\mathbb{R}} \text{ containing } X^\perp. \end{aligned} \tag{B.29}$$

In order to prove this we will utilise the explicit construction of the Weyl algebra above. We will first prove a few auxiliary results summarised in the following lemma.

Lemma B.4.9. *Let $(V_\mathbb{R}, s_\mathbb{R})$ be a symplectic space, let μ be the counting measure on $V_\mathbb{R}$. Then*

1. *For $F \in L^2(V_\mathbb{R}, \mu)$, $F(v) = 0$ except for at most countably many $v \in V_\mathbb{R}$.*
2. *Let X be a symplectic subspace and let ν be the counting measure on X. Then $L^2(X, \nu)$ is a closed subspace and hence $L^2(V_\mathbb{R}, \mu) \simeq L^2(X, \nu) \oplus L^2(X, \nu)^\perp$.*
3. *$D(V_\mathbb{R}) \subseteq L^2(V_\mathbb{R}, \mu)$, defined to be the set of functions that vanish except for at most finitely many $v \in V_\mathbb{R}$ is dense in $L^2(V_\mathbb{R}, \mu)$.*
4. *$G_N \longrightarrow F$ in L^2 implies $G_N \longrightarrow F$ pointwise.*

By definition, for every A in the Weyl algebra over $V_\mathbb{R}$ there is a sequence $(A_N)_{N\in\mathbb{N}}$ such that $A = \lim_{N\to\infty} A_N$ with respect to the norm, where

$$(A_N F)(x) = \int_V z_N(y) e^{\frac{\mathrm{i}}{2} s(y,x)} F(y + x) \,\mathrm{d}\mu(y), \tag{B.30}$$

with $z_N \in D(V)$ for every N. We have

5. *$\exists z_\infty : V \to \mathbb{C}$ such that $\forall y \in V : \lim_{N\to\infty} z_N(y) = z_\infty(y)$,*
6. $$(AF)(x) = \lim_{N\to\infty} \int_V z_\infty(y) e^{\frac{\mathrm{i}}{2} s(y,x)} F_N(y + x) \,\mathrm{d}\mu(y), \tag{B.31}$$
   for every sequence $(F_N)_{N\in\mathbb{N}} \subseteq D(V)$ such that $F_N \to F$ in L^2,
7. *$\|z_\infty\|_{L^2} \leq \|A\| \leq \|z_\infty\|_{L^1}$, where $\|z_\infty\|_{L^1}$ might be ∞.*
8. *Let $V_\infty \subseteq V$ be a symplectic subspace that contains $\operatorname{supp} z_\infty := \{v \in V | z_\infty(v) \neq 0\}$. Then A is in the Weyl algebra of $V_\infty \subseteq V_\mathbb{R}$.*

Remark: In particular, this means that whenever an element of the Weyl algebra can be written as a limit $A = \lim_{N\to\infty} \sum_{v\in V} z_N(v) W(v)$ such that all coefficients of $W(v)$ for $v \in V \setminus \operatorname{supp} z_\infty$ go to zero, it can also be written as a limit $A = \lim_{N\to\infty} \sum_{v\in V} \tilde{z}_N(v) W(v)$ where all coefficients of $W(v)$ for $v \in V \setminus V_\infty$ are the constant zero sequences. It is an interesting fact that we have only been able to prove this for the case that V_∞ is a symplectic space. This suggests that in general coefficient functions that go to zero might not always be replaceable by the constant zero function.

Proof. The first two points are trivial.

3. Let us choose $F \in L^2(V, \mu)$ and let us label its support by the set $\{v_k | k \in \mathbb{N}\}$. Let us define
$$F_n(v_k) := \begin{cases} F(v_k) & \text{for } k \leq n; \\ 0 & \text{else.} \end{cases} \tag{B.32}$$
Then
$$\|F - F_N\|_{L^2}^2 = \sum_{k=0}^{\infty} |F(v_k) - F_N(v_k)|^2 = \sum_{k=n+1}^{\infty} |F(v_k)|^2 \xrightarrow{N\to\infty} 0, \tag{B.33}$$
since $\|F\|_{L^2}^2 = \sum\limits_{k=0}^{\infty} |F(v_k)|^2 < \infty$.

4. Look at
$$|F(x) - G_N(x)|^2 \leq \sum_{k=0}^{\infty} |F(v_k) - G_N(v_k)|^2 = \|F - F_N\|_{L^2}^2. \tag{B.34}$$

5. By assumption $A_N \to A$ in norm, in particular $A_N \to A$ strongly and hence we also have that for every F we have that $A_N F \to AF$ pointwise. In particular
$$(AF)(x) = \lim_{N\to\infty} \int_V z_N(y) e^{\frac{\mathrm{i}}{2}s(y,x)} F(y+x) \,\mathrm{d}\mu(y). \tag{B.35}$$
If we now pick for $F = \delta_0$, we see that $(AF)(-x) = \lim\limits_{N\to\infty} z_N(x) =: z_\infty(x)$.

6. Since A is a bounded operator, we have that $AF_N \to AF$ in L^2 and hence also pointwise. In particular
$$\begin{aligned}(AF)(x) &= \lim_{N\to\infty} \lim_{M\to\infty} \int_V z_M(y) e^{\frac{\mathrm{i}}{2}s(y,x)} F_N(y+x) \,\mathrm{d}\mu(y) \\ &= \lim_{N\to\infty} \int_V z_\infty(y) e^{\frac{\mathrm{i}}{2}s(y,x)} F_N(y+x) \,\mathrm{d}\mu(y),\end{aligned} \tag{B.36}$$
where the second line follows from the fact that for every x and for every N the integrals are finite sums.

7. Let us pick $F = \delta_0$. Then
$$(A\delta_0)(-x) = z_\infty(x), \tag{B.37}$$
in particular
$$\|A\delta_0\|_{L^2} = \|z_\infty\|_{L^2}. \tag{B.38}$$
Note that $\|\delta_0\|_{L^2} = 1$, so $\|A\delta_0\|_{L^2} \leq \|A\|$. For the other inequality we use that A can be written as a limit of A_N. Hence $\|A\| = \lim\limits_{N\to\infty} \|A_N\| \leq \lim\limits_{N\to\infty} \|z_\infty\|_{L^1}$, which might be ∞, where we used unitarity of the Weyl generators.

8. Let us consider $P_\infty A \restriction L^2(V_\infty, \nu)$, which is an operator on $L^2(V_\infty, \nu)$, where $P_\infty \in BL(L^2(V_\mathbb{R}, \mu))$ is the orthogonal projection onto $L^2(V_\infty, \nu) \subseteq L^2(V_\mathbb{R}, \mu)$. Since $A_N \longrightarrow A$ in $BL(L^2(V_\mathbb{R}, \mu))$ we have that $P_\infty A_N \restriction L^2(V_\infty, \nu) \longrightarrow P_\infty A \restriction L^2(V_\infty, \nu)$ in $BL(L^2(V_\infty, \nu))$.

   Let us now define $\tilde{z}_N := z_N \cdot 1\!\!1_{V_\infty}$, where $1\!\!1_{V_\infty}$ is the characteristic function of $V_\infty \subseteq V_\mathbb{R}$, and define $\tilde{A}_N$ via

$$\begin{aligned}(\tilde{A}_N F)(x) :&= \int_V \tilde{z}_N(y) e^{\frac{\mathrm{i}}{2} s(y,x)} F(y+x)\, \mathrm{d}\mu(y) \\ &= \int_V \tilde{z}_N(y-x) e^{\frac{\mathrm{i}}{2} s(y,x)} F(y)\, \mathrm{d}\mu(y),\end{aligned} \tag{B.39}$$

   which is clearly in the Weyl algebra of $V_\infty \subseteq V_\mathbb{R}$.

   Using the fact that V_∞ is a linear space, we can see that $P_\infty \tilde{A}_N \restriction L^2(V_\infty, \nu) = P_\infty A_N \restriction L^2(V_\infty, \nu)$.

   Hence, $P_\infty \tilde{A}_N \restriction L^2(V_\infty, \nu)$ is a sequence of linear combinations of Weyl generators in $BL(L^2(V_\infty, \nu))$ that converges in $BL(L^2(V_\infty, \nu))$. By the uniqueness of the Weyl algebra, the corresponding Weyl generators $\tilde{A}_N$ in $BL(L^2(V_\mathbb{R}, \mu))$ converge as well. Knowing that they converge, we can infer, by exploiting the fact that $\tilde{z}_N \to z_\infty$ pointwise, that the limit needs to be given by A. But then A is in the Weyl algebra of V_∞.

□

We can now prove Lemma B.4.8.

Proof of Lemma B.4.8. Since X is a linear space, we have for $\forall w \in X, \forall t \in \mathbb{R}$: $[W(tw), A] = 0$. Writing A as the norm-limit of A_N's as in Eq. (B.30), let us consider $([W(tw), A_N]F)(x)$. This is easily calculated by recalling that $[W(u), W(v)] = W(u+v)\left(e^{-\frac{\mathrm{i}}{2}s(u,v)} - e^{\frac{\mathrm{i}}{2}s(u,v)}\right) = -2\mathrm{i}\sin\left(\frac{s(u,v)}{2}\right) W(u+v)$. We have explicitly that

$$\begin{aligned}&([W(tw), A_N]F)(x) = e^{\frac{\mathrm{i}}{2}s(tw,x)} \int_V z_N(y) e^{\frac{\mathrm{i}}{2}s(y,x+tw)} F(y+x+tw)\, \mathrm{d}\mu(y) \\ &- \int_V z_N(y) e^{\frac{\mathrm{i}}{2}s(y,x)} F(y+x+tw) e^{\frac{\mathrm{i}}{2}s(tw,x+y)}\, \mathrm{d}\mu(y) \\ &= -2\mathrm{i} \int_V \underbrace{z_N(y-tw) \sin\left(\frac{ts(w,y)}{2}\right)}_{=:c_N(y)} e^{\frac{\mathrm{i}}{2}s(y,x)} F(y+x)\, \mathrm{d}\mu(y).\end{aligned} \tag{B.40}$$

Since $[W(tw), A_N] \longrightarrow 0$ in norm, it follows that $c_N \to 0$ pointwise, i.e.,

$$\forall t\ \forall y:\ z_N(y-tw) \sin\left(\frac{ts(w,y)}{2}\right) \longrightarrow 0, \tag{B.41}$$

in particular (by replacing $y \to y + tw$)

$$\forall t \; \forall y: \; z_N(y) \sin\left(\frac{ts(w,y)}{2}\right) \longrightarrow 0. \tag{B.42}$$

Suppose now that $y_0 \in \mathrm{supp} z_\infty$, i.e., $\lim_{N\to\infty} z_N(y_0) \neq 0$, then $\forall t: \; \sin\left(t\frac{s(w,y_0)}{2}\right) = 0$, which implies that $s(w, y_0) = 0$. Since this holds for every $w \in X$, we see that $\mathrm{supp}\, z_\infty \subseteq X^\perp$. The result then follows by the last point of Lemma B.4.9. □

B.5 STATES

Let $(V_\mathbb{R}, s_\mathbb{R})$ be a symplectic space. Then we may consider the field algebra $\mathcal{F}$ as well as the Weyl algebra $\mathcal{W}$, which are clearly distinct. Nevertheless, certain classes of states on $\mathcal{W}$ are in a one-to-one correspondence with certain classes of state on $\mathcal{F}$. In the following, states on $\mathcal{W}$ will be decorated with a hat, e.g., $\hat{\omega}$.

B.5.1 Analytic and quasi-free states

Definition B.5.1. *A state $\hat{\omega}$ on $\mathcal{W}$ is called* analytic, *if for every $u \in V_\mathbb{R}$ the function*

$$\mathbb{R} \ni t \mapsto \hat{\omega}(W(tu)) \in \mathbb{C} \tag{B.43}$$

is analytic.

Definition B.5.2. *A state $\hat{\omega}$ on $\mathcal{W}$ is called* quasi-free *if for all $v \in \mathbb{R}$*

$$\hat{\omega}(W(v)) = e^{\mathrm{i}\chi(v) - \frac{1}{4}\beta(v,v)}, \tag{B.44}$$

where

1. *χ is an $\mathbb{R}$-linear functional from $V_\mathbb{R}$ to $\mathbb{R}$, and*
2. *$\beta : V_\mathbb{R} \times V_\mathbb{R} \to \mathbb{C}$ is symmetric, $\mathbb{R}$-bilinear and fulfills $\forall u, v \in V_\mathbb{R}$*

$$|s_\mathbb{R}(u,v)|^2 \leq \beta(u,u)\beta(v,v). \tag{B.45}$$

A state ω on $\mathcal{F}$ is called quasi-free if for all $v \in V_\mathbb{R}$

$$\omega(\varphi(v)^n) = (-\mathrm{i})^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{\mathrm{i}x\chi(v) - \frac{1}{4}x^2\beta(v,v)}\Big|_{x=0}, \tag{B.46}$$

where χ and β as above.

We call χ one-point function *and β* symmetric covariance function.

The name for β is motivated by the fact that

$$\begin{aligned}\beta(u,v) =&\omega(\varphi(u)\varphi(v)) - \omega(\varphi(u))\omega(\varphi(v))\\ &+\omega(\varphi(v)\varphi(u)) - \omega(\varphi(v))\omega(\varphi(u)).\end{aligned} \tag{B.47}$$

We clearly see that quasi-free states on $\mathcal{W}$ and on $\mathcal{F}$ are in a one-to-one correspondence.

Definition B.5.3. *Let ω be a quasi-free state on $\mathcal{F}$. Then*

1. *the one-point function is $\omega(\varphi(u)) = \chi(u)$,*
2. *the two-point function is $\omega(\varphi(u)\varphi(v)) = \frac{1}{2}\beta(u,v) + \chi(u)\chi(v) + \frac{1}{2}\mathrm{i}s_{\mathbb{R}}(u,v)$, and*
3. *the symmetrised two-point function is $\omega(\{\varphi(u),\varphi(v)\}) = \beta(u,v) + 2\chi(u)\chi(v)$,*

where $\{\varphi(u),\varphi(v)\} = \varphi(u)\varphi(v) + \varphi(u)\varphi(v)$ is the anti-commutator, and

4. *the* truncated two-point function *is $\omega(\varphi(f)\varphi(g)) - \chi(f)\chi(g) = \frac{1}{2}\beta(f,g) + \frac{1}{2}\mathrm{i}E(f,g)$.*

In particular, $\frac{1}{2}\beta$ is the symmetric part of the truncated two-point function.

Let $(V_{\mathbb{R}}, s_{\mathbb{R}})$ be a symplectic space and let τ_{symp} be a vector space topology on $V_{\mathbb{R}}$ such that $s_{\mathbb{R}}$ is separately τ_{symp}-continuous. Let $\hat{\omega}$ be a quasi-free state on $\mathcal{W}$ with one-point function χ and symmetric covariance function β. Since for every $v \in V_{\mathbb{R}}$ we have that

$$t \mapsto \hat{\omega}(W(tv)) = e^{\mathrm{i}t\chi(v) - \frac{1}{4}t^2\beta(v,v)}, \tag{B.48}$$

we see that $\hat{\omega}$ is *regular*. Let $\pi_{\hat{\omega}}$ denote the GNS representation $\pi_{\hat{\omega}} : \mathcal{W} \to BL(\mathcal{H}_{\hat{\omega}})$. Then, by Corollary B.4.6, we see that for every $v \in V_{\mathbb{R}}$

$$\mathbb{R} \ni t \mapsto \pi_{\hat{\omega}}(W(tv)) \in BL(\mathcal{H}_{\hat{\omega}}) \tag{B.49}$$

is a strongly continuous one-parameter group. Via Stone's theorem, for every $v \in V_{\mathbb{R}}$ there exists a self-adjoint operator $\varphi^{\pi_{\hat{\omega}}}(v)$, such that

$$\pi_{\hat{\omega}}(W(tv)) = e^{\mathrm{i}\varphi^{\pi_{\hat{\omega}}}(v)}. \tag{B.50}$$

One can then show that

$$D^{\varphi}_{\hat{\omega}} := \{\varphi^{\pi_{\hat{\omega}}}(v_1)...\varphi^{\pi_{\hat{\omega}}}(v_n)\Omega_{\hat{\omega}} | n \in \mathbb{N};\ v_1, ..., v_n \in V_{\mathbb{R}}\} \tag{B.51}$$

is dense in $\mathcal{H}_{\hat{\omega}}$ and consist of analytic vectors for every $\varphi^{\pi_{\hat{\omega}}}(v)$, so $D^{\varphi}_{\hat{\omega}}$ is in particular a common core, see Corollary 4.10 in [81]. Moreover, it holds that

$$\langle \Omega_{\hat{\omega}} | \varphi^{\pi_{\hat{\omega}}}(u)\varphi^{\pi_{\hat{\omega}}}(v)\Omega_{\hat{\omega}}\rangle_{\mathcal{H}_{\hat{\omega}}} = \frac{1}{2}\beta(u,v) + \chi(u)\chi(v) + \frac{1}{2}\mathrm{i}s_{\mathbb{R}}(u,v). \tag{B.52}$$

Let us now consider the field algebra $\mathcal{F}$ over $(V_{\mathbb{R}}, s_{\mathbb{R}})$ and let ω be the quasi-free state on $\mathcal{F}$ corresponding to $\hat{\omega}$. Then it turns out, that

$$\varphi(v) \mapsto \varphi^{\pi_{\hat{\omega}}}(v) \upharpoonright D^{\varphi}_{\hat{\omega}} \tag{B.53}$$

uniquely extends to a unit-preserving $*$-representation $\pi^{\varphi}_{\hat{\omega}}$ of $\mathcal{F}$ that is in fact *equivalent* to the GNS representation π_{ω} of $\mathcal{F}$ induced by ω. In particular, ω is pure if and only if $\pi^{\varphi}_{\hat{\omega}}$ is irreducible.

We can now consider the following lemma.

Lemma B.5.4. *Let ω be a quasi-free state on $\mathcal{F}$ and let $\hat{\omega}$ be the corresponding quasi-free state on $\mathcal{W}$. Then ω is pure if and only if $\hat{\omega}$ is pure.*

Proof. Suppose $\hat{\omega}$ be pure, which is the case if and only if the associated GNS representation $\pi_{\hat{\omega}} : \mathcal{W} \to BL(\mathcal{H}_{\hat{\omega}})$ is irreducible, i.e., the commutant of $\pi_{\hat{\omega}}[\mathcal{W}]$ in $BL(\mathcal{H}_{\hat{\omega}})$ is trivial. This is the case if and only if $C \in BL(\mathcal{H}_{\hat{\omega}})$ is trivial whenever for all $u \in V_{\mathbb{R}}$ and for all x, y in a dense subspace it holds that

$$\langle x | C\pi_{\hat{\omega}}(W(u))y\rangle_{\mathcal{H}_{\hat{\omega}}} = \langle \pi_{\hat{\omega}}(W(u)^*)x | Cy\rangle_{\mathcal{H}_{\hat{\omega}}}. \tag{B.54}$$

This follows from the boundedness of $\pi_{\hat{\omega}}(W(u))$. In particular, we can pick $x, y \in D^{\varphi}_{\hat{\omega}}$. By the fact that x, y are analytic vectors and by our arguments above, this equation is equivalent (after replacing u by tu) to

$$\sum_{n=0}^{\infty} t^n \frac{\mathrm{i}^n}{n!} \langle x | C\pi^{\varphi}_{\hat{\omega}}(\varphi(u)^n)y\rangle_{\mathcal{H}_{\hat{\omega}}} = \sum_{n=0}^{\infty} t^n \frac{\mathrm{i}^n}{n!} \langle \pi^{\varphi}_{\hat{\omega}}(\varphi(u)^n)x | Cy\rangle_{\mathcal{H}_{\hat{\omega}}} \tag{B.55}$$

for every $t \in \mathbb{R}$. But this holds if and only if $\forall n \in \mathbb{N}$

$$\langle x | C\pi^{\varphi}_{\hat{\omega}}(\varphi(u)^n)y\rangle_{\mathcal{H}_{\hat{\omega}}} = \langle \pi^{\varphi}_{\hat{\omega}}(\varphi(u)^n)x | Cy\rangle_{\mathcal{H}_{\hat{\omega}}}. \tag{B.56}$$

Then, by the use of multilinear polarisation (as used in Eq. (5.53)), this holds if and only if

$$\langle x | C\pi^{\varphi}_{\hat{\omega}}(A)y\rangle_{\mathcal{H}_{\hat{\omega}}} = \langle \pi^{\varphi}_{\hat{\omega}}(A)x | Cy\rangle_{\mathcal{H}_{\hat{\omega}}} \tag{B.57}$$

for every $x, y \in D^{\varphi}_{\hat{\omega}}$ and for every $A \in \mathcal{F}$. But this holds if and only if ω is pure since $\pi^{\varphi}_{\hat{\omega}}$ is equivalent to the GNS representation π_{ω}. □

B.5.2 Two-dimensional $(V_{\mathbb{R}}, s_{\mathbb{R}})$

Let us now consider a two-dimensional symplectic space $(V_{\mathbb{R}}, s_{\mathbb{R}})$ and let us choose a basis $\{u_1, u_2\}$ such that

$$s_{\mathbb{R}}(u_j, u_k) = s_{jk}, \tag{B.58}$$

where

$$s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{B.59}$$

Such a basis is called a *Darboux* basis. Let now ω be a quasi-free state[1] on the field algebra over $(V_{\mathbb{R}}, s_{\mathbb{R}})$ and define the (2×2) *symmetric covariance matrix* (with respect to the Darboux basis $\{u_1, u_2\}$)

$$A_{jk} := \beta(u_j, u_k). \tag{B.60}$$

Note that A depends on the Darboux basis $\{u_1, u_2\}$. If $\{v_1, v_2\}$ is another Darboux basis, then $\tilde{A}_{jk} := \beta(v_j, v_k)$ is related to A via a symplectic transformation.

Then the following holds, which is Lemma 10 in [3].

Lemma B.5.5. *Let ω be a quasi-free state on the field algebra $\mathcal{F}$ over a two-dimensional symplectic space and let A be the symmetric covariance matrix of ω with respect to a Darboux basis. Then the following are equivalent.*

1. *ω is a pure state on $\mathcal{F}$,*
2. *$\hat{\omega}$ is a pure state on $\mathcal{W}$,*
3. $\det A = 1$,

where $\hat{\omega}$ is the quasi-free state on $\mathcal{W}$ corresponding to ω.

Proof. We show 2. $\iff$ 3. and follow [81]. Let $\hat{\omega}$ be a pure state. Then there is a 2×2 matrix D such that $s = DA$. D has a polar decomposition given by $D = J|D|$, where $J^2 = -\mathbb{1}$, so $s = J|D|A$. It is a well-established fact that the corresponding state is pure if and only if $|D| = \mathbb{1}$, see [81]. We immediately see that $\det(A) = \det(|D|)^{-1}$, so for pure states we have that $\det(A) = 1$. Conversely, the positivity of the state implies that $\||D|\| \leq 1$. Using that the norm of $|D|$ is given by the largest eigenvalue and the fact that the determinant is the product of all the eigenvalues, we can deduce that $\det(A)^{-1} = \det(|D|) = 1$ implies that $|D| = \mathbb{1}$. □

[1] In particular in the case of finite-dimensional $V_{\mathbb{R}}$, quasi-free states are often called (coherent) Gaussian states. See also [82] for many related aspects.

B.5.3 Two two-dimensional spaces

Let us now consider two two-dimensional symplectic spaces $(V_{\mathbb{R}}^{\mathsf{A}}, s_{\mathbb{R}}^{\mathsf{A}})$ and $(V_{\mathbb{R}}^{\mathsf{B}}, s_{\mathbb{R}}^{\mathsf{B}})$ and let us choose a basis $\left\{u_1^{\mathsf{A}}, u_2^{\mathsf{A}}\right\}$ and a basis $\left\{u_1^{\mathsf{B}}, u_2^{\mathsf{B}}\right\}$ such that

$$s_{\mathbb{R}}^{\mathsf{A}}(u_j^{\mathsf{A}}, u_k^{\mathsf{A}}) = s_{\mathbb{R}}^{\mathsf{B}}(u_j^{\mathsf{B}}, u_k^{\mathsf{B}}) = s_{jk}. \tag{B.61}$$

Let us then consider the direct sum $(V_{\mathbb{R}}^{\mathsf{A}} \oplus V_{\mathbb{R}}^{\mathsf{B}}, s_{\mathbb{R}}^{\mathsf{A}} \oplus s_{\mathbb{R}}^{\mathsf{B}})$, which is a four-dimensional symplectic space. Then it holds quite generally, that

$$\begin{aligned} \mathcal{W}_{(V_{\mathbb{R}}^{\mathsf{A}} \oplus V_{\mathbb{R}}^{\mathsf{B}}, s_{\mathbb{R}}^{\mathsf{A}} \oplus s_{\mathbb{R}}^{\mathsf{B}})} &\cong \overline{\mathcal{W}_{(V_{\mathbb{R}}^{\mathsf{A}}, s_{\mathbb{R}}^{\mathsf{A}})} \otimes \mathcal{W}_{(V_{\mathbb{R}}^{\mathsf{B}}, s_{\mathbb{R}}^{\mathsf{B}})}}, \\ \mathcal{F}_{(V_{\mathbb{R}}^{\mathsf{A}} \oplus V_{\mathbb{R}}^{\mathsf{B}}, s_{\mathbb{R}}^{\mathsf{A}} \oplus s_{\mathbb{R}}^{\mathsf{B}})} &\cong \mathcal{F}_{(V_{\mathbb{R}}^{\mathsf{A}}, s_{\mathbb{R}}^{\mathsf{A}})} \otimes \mathcal{F}_{(V_{\mathbb{R}}^{\mathsf{B}}, s_{\mathbb{R}}^{\mathsf{B}})}, \end{aligned} \tag{B.62}$$

see for instance Proposition 18.1-18 in [83] for the first unital C^*-isomorphism. Let now ω be a quasi-free state on $\mathcal{F}_{(V_{\mathbb{R}}^{\mathsf{A}}, s_{\mathbb{R}}^{\mathsf{A}})} \otimes \mathcal{F}_{(V_{\mathbb{R}}^{\mathsf{B}}, s_{\mathbb{R}}^{\mathsf{B}})}$. We define

$$\begin{aligned} A_{jk} &:= \sigma\Big(\Big\{\varphi_{\mathsf{A}}(u_j^{\mathsf{A}}), \varphi_{\mathsf{A}}(u_k^{\mathsf{A}})\Big\}\Big) - 2\sigma'(\varphi_{\mathsf{A}}(u_j^{\mathsf{A}}))\sigma\Big(\varphi_{\mathsf{A}}(u_k^{\mathsf{A}})\Big), \\ B_{jk} &:= \sigma\Big(\Big\{\varphi_{\mathsf{B}}(u_j^{\mathsf{B}}), \varphi_{\mathsf{B}}(u_k^{\mathsf{B}})\Big\}\Big) - 2\sigma'(\varphi_{\mathsf{B}}(u_j^{\mathsf{B}}))\sigma\Big(\varphi_{\mathsf{B}}(u_k^{\mathsf{B}})\Big), \\ C_{jk} &:= \sigma\Big(\Big\{\varphi_{\mathsf{A}}(u_j^{\mathsf{A}}), \varphi_{\mathsf{B}}(u_k^{\mathsf{B}})\Big\}\Big) - 2\sigma\Big(\varphi_{\mathsf{A}}(u_j^{\mathsf{A}})\Big)\sigma'(\varphi_{\mathsf{B}}(u_k^{\mathsf{B}})) \\ &= 2\sigma\Big(\varphi_{\mathsf{A}}(u_j^{\mathsf{A}})\varphi_{\mathsf{B}}(u_k^{\mathsf{B}})\Big) - 2\sigma\Big(\varphi_{\mathsf{A}}(u_j^{\mathsf{A}})\Big)\sigma\Big(\varphi_{\mathsf{B}}(u_k^{\mathsf{B}})\Big), \end{aligned} \tag{B.63}$$

and combine them into the (4×4) symmetric covariance matrix γ, where

$$\gamma_{jk} := \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}_{jk}. \tag{B.64}$$

Furthermore, we define $\Omega := s \oplus s$. Then we have the following result, which is Lemma 16 in [3].

Lemma B.5.6. *The following are equivalent*

1. $\hat{\omega}$ *is* not *Verch-Werner ppt,*

2. *the following inequality*

$$\gamma + \mathrm{i}(s \oplus (-s)) \geq 0. \tag{B.65}$$

   does not *hold,*

3. $p_S > 0$*, where*

$$p_S := -(\det(A) - 1)(\det(B) - 1) + \mathrm{tr}\Big(AsCsBsC^T s\Big) - 2\det(C) - \det(C)^2, \tag{B.66}$$

4. $\hat{\omega}$ is entangled,

5. ω is entangled,

where $\hat{\omega}$ is the corresponding quasi-free state on the Weyl algebra.

We introduce some notation for the proof: We denote by W_{A} and W_{B} the Weyl generators such that $\mathcal{W}_{(V^{\mathsf{A}}_{\mathbb{R}}, s^{\mathsf{A}}_{\mathbb{R}})}$ is generated by $\{W_{\mathsf{A}}(x_1 u_1^{\mathsf{A}} + x_2 u_2^{\mathsf{A}}) | \vec{x} \in \mathbb{R}^2\}$ and $\mathcal{W}_{(V^{\mathsf{B}}_{\mathbb{R}}, s^{\mathsf{B}}_{\mathbb{R}})}$ is generated $\{W_{\mathsf{B}}(y_1 u_1^{\mathsf{B}} + y_2 u_2^{\mathsf{B}}) | \vec{y} \in \mathbb{R}^2\}$. Let us then abuse notation and write $W_{\mathsf{A}}(\vec{x}) \equiv W_{\mathsf{A}}(x_1 u_1^{\mathsf{A}} + x_2 u_2^{\mathsf{A}})$ and $W_{\mathsf{B}}(\vec{y}) \equiv W_{\mathsf{B}}(y_1 u_1^{\mathsf{B}} + y_2 u_2^{\mathsf{B}})$. Similarly, write $\varphi_{\mathsf{A},\mathsf{B}} \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \equiv \varphi_{\mathsf{A}}(x_1 u_1^{\mathsf{A}} + x_2 u_2^{\mathsf{A}}) \otimes \mathbb{1} + \mathbb{1} \otimes \varphi_{\mathsf{B}}(y_1 u_1^{\mathsf{B}} + y_2 u_2^{\mathsf{B}})$.

Let $\hat{\omega}$ be a quasi-free state with 4×4 symmetric covariance matrix γ and one-point function $\chi = (\vec{\chi}_{\mathsf{A}}, \vec{\chi}_{\mathsf{B}})^T \in \mathbb{R}^4$, where we identified $\mathbb{R}^4$ with its dual space via the inner product "$\cdot$".

1. $\iff$ 2.*:* Let us first look at the Verch-Werner ppt condition from Definition 2.3.2. Let $p : \mathbb{R}^2 \to \mathbb{R}^2$ be an $\mathbb{R}$-linear, involutive operator with matrix representation also denoted by p such that $p^T s p = -s$. It gives rise to a $\mathbb{C}$-linear map Π from $\mathcal{W}_{(V^{\mathsf{B}}_{\mathbb{R}}, s^{\mathsf{B}}_{\mathbb{R}})}$ into itself defined on the Weyl generators as $\Pi(W_{\mathsf{B}}(\vec{y})) := W_{\mathsf{B}}(p\vec{y})$. In particular, Π preserves the unit. Then the following lemma holds (see also Proposition 3.2 in [24]).

Lemma B.5.7. *A quasi-free state $\hat{\omega}$ fulfills the Verch-Werner ppt condition if and only if the continuous linear extension of $\underline{\hat{\omega}}(A \otimes B) := \hat{\omega}(A \otimes \Pi(B))$ defines a state.*

Proof. As $\underline{\hat{\omega}}$ is clearly linear and normalised for every state $\hat{\omega}$, we prove the equivalence of the ppt property of $\hat{\omega}$ and the positivity of $\underline{\hat{\omega}}$. First we note that

$$\begin{aligned} \Pi(W_{\mathsf{B}}(\vec{y}_1) W_{\mathsf{B}}(\vec{y}_2)) &= e^{-\frac{\mathrm{i}}{2} \vec{y} \cdot s \vec{y}} \Pi(W_{\mathsf{B}}(\vec{y}_1 + \vec{y}_2)) = e^{\frac{\mathrm{i}}{2} p\vec{y} \cdot s p \vec{y}} W_{\mathsf{B}}(p\vec{y}_1 + p\vec{y}_2) \\ &= W_{\mathsf{B}}(p\vec{y}_2) W_{\mathsf{B}}(p\vec{y}_2) = \Pi(W_{\mathsf{B}}(\vec{y}_2)) \Pi(W_{\mathsf{B}}(\vec{y}_1)). \end{aligned} \tag{B.67}$$

It is also immediate that Π commutes with the star operation. Then (cf. Eq. (2.27))

$$\begin{aligned} &\sum_{\alpha,\beta} \hat{\omega}(A_\beta A_\alpha^* \otimes \Pi(B_\alpha)^* \Pi(B_\beta)) = \sum_{\alpha,\beta} \hat{\omega}(A_\beta A_\alpha^* \otimes \Pi(B_\beta B_\alpha^*)) \\ &= \sum_{\alpha,\beta} \underline{\hat{\omega}}(A_\beta A_\alpha^* \otimes B_\beta B_\alpha^*) = \underline{\hat{\omega}}(X^* X), \end{aligned} \tag{B.68}$$

where $X := \sum_\alpha A_\alpha^* \otimes B_\alpha^*$ and where α, β each run through the same *finite* index set. The fact that every X in the algebraic tensor product can be written in this form together with a density argument finishes the proof. $\square$

An example for p is the map derived from $\tilde{p}(y_1 g_1 + y_2 g_2) := y_1 g_1 - y_2 g_2$, i.e, $p = \operatorname{diag}(1, -1)$. In this case, the condition that $\hat{\omega}$ has the Verch-Werner ppt property is equivalent to

$$\Lambda^T \gamma \Lambda + \mathrm{i}(s \oplus s) \geq 0, \tag{B.69}$$

where $\Lambda = \mathbb{1} \oplus p = \operatorname{diag}(1, 1, 1, -1)$, or likewise

$$\gamma + \mathrm{i}(s \oplus (-s)) \geq 0, \tag{B.70}$$

which is Eq. (B.65). Note in particular that $p^T = p$ and $\Lambda^T = \Lambda$.

2. $\iff$ 3. $\iff$ 4.*:* Simon showed in [35], that Eq. (B.65) is equivalent to $\hat{\omega}$ being classically correlated and that Eq. (B.65) is equivalent to $p_S \leq 0$.

5. $\implies$ 2.*:* The essence of Simon's proof is that if Eq. (B.65) holds, then the quasi-free state ω is "locally related" to a quasi-free state ω_0 with covariance matrix γ_0 which fulfills

$$\gamma_0 - \mathbb{1} \geq 0. \tag{B.71}$$

By "locally related" we mean that there exist unit-preserving $*$-automorphisms Γ_{A} and Γ_{B} acting on $\mathcal{F}_{(V_{\mathbb{R}}^{\mathsf{A}}, s_{\mathbb{R}}^{\mathsf{A}})}$ and $\mathcal{F}_{(V_{\mathbb{R}}^{\mathsf{B}}, s_{\mathbb{R}}^{\mathsf{B}})}$ such that $\omega_0 := \omega \circ (\Gamma_{\mathsf{A}} \otimes \Gamma_{\mathsf{B}})$. In particular ω is entangled if and only if ω_0 is. A similar statement holds for $\hat{\omega}_0 := \hat{\omega} \circ (\hat{\Gamma}_{\mathsf{A}} \otimes \hat{\Gamma}_{\mathsf{B}})$ for unit-preserving C^*-automorphisms $\hat{\Gamma}_{\mathsf{A}}, \hat{\Gamma}_{\mathsf{B}}$. Based on this, 5. $\implies$ 2. follows from the subsequent lemma, which shows that ω_0 is classically correlated.

Lemma B.5.8. *Let $\hat{\omega}_0$ be a quasi-free state with 4×4 symmetric covariance matrix γ_0 that fulfills Eq.* (B.71) *and one-point function $\chi_0 = (\vec{\chi}_{\mathsf{A}}, \vec{\chi}_{\mathsf{B}})^T$, then ω_0 is classically correlated (and so is $\hat{\omega}_0$).*

Remark: The proof of this lemma is based on the *Glauber–Sudarshan P-representation*, see their original publications [84, 85] and also Sec. V. in [86].

Proof. Let us define the *coherent* states

$$\hat{\eta}^{\mathsf{A}}_{\vec{\alpha}}(W_{\mathsf{A}}(\vec{x})) := e^{-\frac{1}{4}\vec{x}\cdot\vec{x} + \mathrm{i}\vec{\alpha}\cdot\vec{x} + \mathrm{i}\vec{\chi}_{\mathsf{A}}\cdot\vec{x}}, \qquad \hat{\eta}^{\mathsf{B}}_{\vec{\beta}}(W_{\mathsf{B}}(\vec{y})) := e^{-\frac{1}{4}\vec{y}\cdot\vec{y} + \mathrm{i}\vec{\beta}\cdot\vec{y} + \mathrm{i}\vec{\chi}_{\mathsf{B}}\cdot\vec{y}}, \tag{B.72}$$

which are pure quasi-free states with non-vanishing one-point function.

We assume that Eq. (B.71) holds, and discuss the case where $\gamma_0 - \mathbb{1}$ is positive definite. Then we can define the non-negative Gaussian function $P(\vec{\alpha}, \vec{\beta})$ for the symmetric, positive definite matrix $(\frac{1}{2}\gamma_0 - \frac{1}{2}\mathbb{1})^{-1}$ via

$$P(\vec{\alpha}, \vec{\beta}) := \frac{1}{\pi^2 \sqrt{\det(\gamma_0 - \mathbb{1})}} e^{-\frac{1}{2}\begin{pmatrix}\vec{\alpha}\\ \vec{\beta}\end{pmatrix}\cdot\left(\frac{1}{2}\gamma_0 - \frac{1}{2}\mathbb{1}\right)^{-1}\begin{pmatrix}\vec{\alpha}\\ \vec{\beta}\end{pmatrix}}. \tag{B.73}$$

It is then easy to see that for all $\vec{x}, \vec{y} \in \mathbb{R}^2$

$$\begin{aligned}
&\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} P(\vec{\alpha},\vec{\beta})\hat{\eta}^{\mathsf{A}}_{\vec{\alpha}}(W_{\mathsf{A}}(\vec{x}))\ \hat{\eta}^{\mathsf{B}}_{\vec{\beta}}(W_{\mathsf{B}}(\vec{y}))\mathrm{d}\vec{\alpha}\,\mathrm{d}\vec{\beta} \\
&= \frac{1}{\pi^2\sqrt{\det(\gamma_0 - \mathbb{1})}} e^{\mathrm{i}\chi_0\cdot\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix} - \frac{1}{4}\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}\cdot\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}} \int_{\mathbb{R}^4} e^{-\frac{1}{2}\xi\cdot\left(\frac{1}{2}\gamma_0 - \frac{1}{2}\mathbb{1}\right)^{-1}\xi + \mathrm{i}\xi\cdot\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}}\mathrm{d}\xi \\
&= \frac{1}{\pi^2\sqrt{\det(\gamma_0 - \mathbb{1})}}\frac{(2\pi)^2}{\sqrt{\det\left((\frac{1}{2}\gamma_0 - \frac{1}{2}\mathbb{1})^{-1}\right)}} e^{\mathrm{i}\chi_0\cdot\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix} - \frac{1}{4}\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}\cdot\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}} e^{-\frac{1}{2}\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}\cdot\left(\frac{1}{2}\gamma_0 - \frac{1}{2}\mathbb{1}\right)\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}} \\
&= e^{\mathrm{i}\chi_0\cdot\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix} - \frac{1}{4}\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}\cdot\gamma_0\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}} = \hat{\omega}_0(W_{\mathsf{A}}(\vec{x})\otimes W_{\mathsf{B}}(\vec{y})),
\end{aligned} \tag{B.74}$$

where we used $\xi := (\vec{\alpha}, \vec{\beta})^T$. Since the integrand is continuous, we can write the integral as a limit of Riemann sums, each of which corresponds to a convex combination (as $P \geq 0$) of coherent states evaluated on a tensor product of Weyl generators. This shows that $\hat{\omega}_0$ is a pointwise limit of convex combinations of products of coherent states and hence classically correlated.

By using the corresponding coherent states on the field algebra, the statement holds for ω_0 as well. To see this, use that

$$\begin{aligned}
&\omega_0\left(\varphi_{\mathsf{A},\mathsf{B}}\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}^n\right) \\
&= (-\mathrm{i})^n\left[\frac{\mathrm{d}^n}{\mathrm{d}a^n} e^{\mathrm{i}a\chi_0\cdot\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix} - a^2\frac{1}{4}\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}\cdot\gamma_0\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}}\right]_{a=0} \\
&= (-\mathrm{i})^n\left[\frac{\mathrm{d}^n}{\mathrm{d}a^n}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} P(\vec{\alpha},\vec{\beta}) e^{-a^2\frac{1}{4}\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}\cdot\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix} + \mathrm{i}a\left(\begin{pmatrix}\vec{\alpha}\\ \vec{\beta}\end{pmatrix} + \begin{pmatrix}\vec{\chi}_{\mathsf{A}}\\ \vec{\chi}_{\mathsf{B}}\end{pmatrix}\right)\cdot\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}}\mathrm{d}\vec{\alpha}\,\mathrm{d}\vec{\beta}\right]_{a=0},
\end{aligned} \tag{B.75}$$

change the order of integration and differentiation and approximate the integral in a similar fashion as before.

The case of a merely positive *semi*-definite matrix $\gamma_0 - \mathbb{1}$, i.e., of rank < 4, can be treated similarly by using a lower-dimensional integral (cf. Sec. V. in [86]). □

2. $\Longrightarrow$ 5.*:* The proof of Lemma B.5.6 is completed by noting that every product or classically correlated state ω fulfills Eq. (B.65).

To see this let us assume that ω is the pointwise limit of ω_n of the form $\omega_n = \sum_j \lambda_j \omega_{\mathsf{A},j} \otimes \omega_{\mathsf{B},j}$, where j runs over some finite index set. Let γ and γ_n be the covariance matrices of ω and ω_n respectively and let $\vec{\chi}_{\mathsf{B},j}$ and B_j be the one-point function and the covariance matrix of (the not necessarily quasi-free) state $\omega_{\mathsf{B},j}$

respectively. Then we note that $p\vec{\chi}_{\mathsf{B},j}$ and $p^T B_j p$ define a quasi-free state $\tilde{\omega}_{\mathsf{B},j}$. Positivity of $\tilde{\omega}_{\mathsf{B},j}$ can be seen as follows: the covariance matrix B_j of *every* (not necessarily quasi-free) state ω_{B} fulfills $B_j + \mathrm{i}s \geq 0$. By explicit computation this is found to be equivalent to $B_j - \mathrm{i}s \geq 0$, which is in turn equivalent to $p^T B_j p + \mathrm{i}s \geq 0$. In particular, $\tilde{\omega}_n := \sum_j \lambda_j \omega_{\mathsf{A},j} \otimes \tilde{\omega}_{\mathsf{B},j}$ is a state with covariance matrix $\tilde{\gamma}_n$ that hence fulfills $\tilde{\gamma}_n + \mathrm{i}\Omega \geq 0$. We note that $\tilde{\gamma}_n = \Lambda^T \gamma_n \Lambda$ and, since $\omega_n \to \omega$ shows that $\gamma_n \to \gamma$, also $\tilde{\gamma}_n \to \Lambda^T \gamma \Lambda$. Finally, as the limit of a convergent sequence of positive semi-definite matrices is positive semi-definite, Eq. (B.69) holds, which is equivalent to Eq. (B.65).

B.6 QUASI-FREE UNIT-PRESERVING $*$-ISOMORPHISMS

Suppose $(V_{\mathbb{R}}^1, s_{\mathbb{R}}^1)$ and $(V_{\mathbb{R}}^2, s_{\mathbb{R}}^2)$ are two symplectic spaces such that there exist a linear isomorphism $t : V_{\mathbb{R}}^1 \to V_{\mathbb{R}}^2$ that is *symplectic*, i.e., such that for every $u, v \in V_{\mathbb{R}}^1$ it holds that

$$s_{\mathbb{R}}^1(u, v) = s_{\mathbb{R}}^2(t(u), t(v)). \tag{B.76}$$

Then the maps

$$W_1(u) \mapsto W_2(t(u)), \text{ and } \varphi_1(u) \mapsto \varphi_2(t(v)) \tag{B.77}$$

define unit-preserving $*$-isomorphisms

$$T^W : \mathcal{W}_{(V_{\mathbb{R}}^1, s_{\mathbb{R}}^1)} \to \mathcal{W}_{(V_{\mathbb{R}}^2, s_{\mathbb{R}}^2)}, \text{ and } T^\varphi : \mathcal{F}_{(V_{\mathbb{R}}^1, s_{\mathbb{R}}^1)} \to \mathcal{F}_{(V_{\mathbb{R}}^2, s_{\mathbb{R}}^2)}. \tag{B.78}$$

It is easy to see that if ω_2 is a quasi-free state on $\mathcal{F}_{(V_{\mathbb{R}}^2, s_{\mathbb{R}}^2)}$, then $\omega_2 \circ T^\varphi$ is a quasi-free state on $\mathcal{F}_{(V_{\mathbb{R}}^1, s_{\mathbb{R}}^1)}$ and $\hat{\omega}_2 \circ T^W$ is a quasi-free state on $\mathcal{W}_{(V_{\mathbb{R}}^1, s_{\mathbb{R}}^1)}$. Hence we call such unit-preserving $*$-isomorphisms *quasi-free*. We will be particularly interested in the case where $(V_{\mathbb{R}}^1, s_{\mathbb{R}}^1) = (V_{\mathbb{R}}^2, s_{\mathbb{R}}^2) = (V_{\mathbb{R}}, s_{\mathbb{R}})$ and t is a linear isomorphism that preserves the symplectic form.

B.7 NETS OF SYMPLECTIC SPACES AND AQFTS

Let $\mathfrak{B}$ be a background and let $Cl_{\mathfrak{B}}$ be an association of symplectic subspaces of some symplectic space $(V_{\mathbb{R}}, s_{\mathbb{R}})$ to regions in $\mathfrak{B}$, such that

1. $V_{\mathbb{R}} = \bigcup_{N \in \mathfrak{B}} Cl_{\mathfrak{B}}(N)$,
2. $N_1 \subseteq N_2 \implies Cl_{\mathfrak{B}}(N_1) \subseteq Cl_{\mathfrak{B}}(N_1)$,
3. $N_1 \perp N_2 \implies s_{\mathbb{R}}(\mathcal{A}_{\mathfrak{B}}(N_1), \mathcal{A}_{\mathfrak{B}}(N_2)) = \{0\}$.

A further desirable axiom (which we do not assume unless explicitly mentioned) is

4. $N_1 \sqsubseteq N_2 \implies Cl_{\mathfrak{B}}(N_1) \subseteq Cl_{\mathfrak{B}}(N_2)$.

We call such a $Cl_{\mathfrak{B}}$ a net of symplectic subspaces. Let us now consider $\mathcal{F}$ and $\mathcal{W}$ to be the field- and the Weyl algebra over $(V_{\mathbb{R}}, s_{\mathbb{R}})$ respectively. Then it is indeed easy to see that the associations $\mathcal{W}_{\mathfrak{B}}$ and $\mathcal{F}_{\mathfrak{B}}$ where

1. $\mathcal{W}_{\mathfrak{B}}(N) \subseteq \mathcal{W}$ is the unital C^*-subalgebra spanned by all $W(u)$ for $u \in Cl_{\mathfrak{B}}(N)$, and

2. $\mathcal{F}_{\mathfrak{B}}(N) \subseteq \mathcal{W}$ is the unital $*$-subalgebra spanned by $\mathbb{1}$ and all $\varphi(u)$ for $u \in Cl_{\mathfrak{B}}(N)$,

are two AQFTs.

— C —

Linear real scalar fields on globally hyperbolic spacetimes

C.1 Elements of Lorentzian geometry

See for instance [87].

In the following let (M, W) be a globally hyperbolic spacetime, i.e., a time-oriented smooth Lorentzian manifold with smooth metric and a Cauchy surface, where M denotes the pointset and W the structure on M. It is common to simply write M for (M, W). Note that, following [88] we do not make assumptions on connectedness.

A Cauchy (hyper-) surface $\Sigma \subseteq M$ is a closed acausal set such that every inextendible causal directed curve intersects Σ exactly once. For a subset $N \subseteq M$, $J_M^{\pm}(N)$ denotes the causal future (+) and past (−), $I_M^{\pm}(N)$ denotes the chronological future and past and $D_M(N)$ denotes the domain of dependence or Cauchy development of N in M. A subset $N \subseteq M$ is called *causally convex* iff it equals its causal hull $\mathrm{ch}_M(N) := J_M^+(N) \cap J_M^-(N)$. We call non-empty, open and causally convex subsets of M *regions*. Every region $N \subseteq M \equiv (M, W)$ together with the inherited structure $W \restriction N$ is a globally hyperbolic spacetime (with possibly countably many connected components) in its own right. If $N \subseteq M$ is a region, so is $D_M(N)$, if $N_1, N_2 \subseteq M$ are regions, so is $N_1 \cap N_2$. For every region $N \subseteq M$, $L \subseteq N$ is a region in N iff it is a region in M. For $K \subseteq M$, $K^{\perp_M} := M \setminus (J_M^+(K) \cup J_M^-(K))$ and $M_K^{\pm} := M \setminus J_M^{\mp}(K)$, which, when K is compact, are regions such that $D_M\left(M_K^{\pm}\right) = M$, see for instance the Appendix of [41]. A finite set $\{K_1, ..., K_n\}$ of compact subsets of M is called causally orderable if there exists a linear order $\leq$ on $\{K_1, ..., K_n\}$ such that

$$K_i < K_j \implies J_M^+(K_j) \cap K_i = \emptyset, \tag{C.1}$$

or equivalently $K_i < K_j$ implies that there exists a Cauchy surface $\Sigma \subseteq M$ separating K_i and K_j such that $K_j \subseteq I_M^+(\Sigma)$ and $K_i \subseteq I_M^-(\Sigma)$, see also Def. 4.1 in [89]. The order does not need to be unique and every such $\leq$ is referred to as an *admissible causal linear order*.

C.2 GREEN HYPERBOLIC (SCALAR) OPERATORS

We follow [88]. A linear differential operator $P : C^\infty(M;\mathbb{R}^n) \to C^\infty(M;\mathbb{R}^n)$ is called *formally self-adjoint*, if $\forall f, g \in C^\infty(M;\mathbb{R}^n)$ such that $\operatorname{supp} f \cap \operatorname{supp} g$ is compact it holds that

$$\int_M (Pf) \cdot g \, \mathrm{d}V_M = \int_M f \cdot (Pg) \, \mathrm{d}V_M, \tag{C.2}$$

where the dot denotes the standard inner product in $\mathbb{R}^n$.

A formally self-adjoint linear differential operator $P : C^\infty(M;\mathbb{R}^n) \to C^\infty(M;\mathbb{R}^n)$ is called *Green* hyperbolic, if there are unique (so-called) advanced $(-)$ and retarded $(+)$ Green's operators $E_P^\pm : C_c^\infty(M;\mathbb{R}^n) \to C^\infty(M;\mathbb{R}^n)$ such that for all $f \in C_c^\infty(M;\mathbb{R}^n)$:

1. $E_P^\pm P f = f$,
2. $P E_P^\pm f = f$, and
3. $\operatorname{supp} E_P^\pm f \subseteq J_M^\pm(\operatorname{supp} f)$.

We may then define the *commutator function* $E_P : C_c^\infty(M;\mathbb{R}^n) \times C_c^\infty(M;\mathbb{R}^n) \to \mathbb{R}$ as

$$E_P(f, g) := \int_M f \cdot ((E_P^- - E_P^+) g) \, \mathrm{d}V_M, \tag{C.3}$$

which is a pre-symplectic form on $C_c^\infty(M;\mathbb{R}^n)$. One can now show that

$$\{0\} \to C_c^\infty(M;\mathbb{R}^n) \xrightarrow{P} C_c^\infty(M;\mathbb{R}^n) \xrightarrow{E_P^- - E_P^+} C_{sc}^\infty(M;\mathbb{R}^n) \xrightarrow{P} C_{sc}^\infty(M;\mathbb{R}^n) \tag{C.4}$$

is exact, where $C_{sc}^\infty(M;\mathbb{R}^n)$ is the set of $f \in C^\infty(M;\mathbb{R}^n)$ such that there exists a compact $K \subseteq M$ such that $\operatorname{supp} f \subseteq J_M^-(K) \cup J_M^+(K)$. In particular, with this the kernel of E_P is given by

$$\ker(E_P^- - E_P^+) = P C_c^\infty(M;\mathbb{R}^n). \tag{C.5}$$

Hence $\left(C_c^\infty(M;\mathbb{R}^n)/P C_c^\infty(M;\mathbb{R}^n), \tilde{E}_P\right)$, where

$$\tilde{E}_P([f]_P, [g]_P) := E_P(f, g), \tag{C.6}$$

and $f \mapsto [f]_P$ is the quotient map, is a symplectic space.

$C_c^\infty(M;\mathbb{R})$, when equipped with its usual topology, is a nuclear topological vector space. Moreover, $E_P^- - E_P^+$ is continuous, see Corollary 3.11 in [88]. We may then equip the quotient space $\left(C_c^\infty(M;\mathbb{R}^n)/PC_c^\infty(M;\mathbb{R}^n), \tilde{E}_P\right)$ with the quotient topology. Since the space $PC_c^\infty(M;\mathbb{R}^n)$, being the kernel of a continuous map $E_P^- - E_P^+$, is closed, it follows that $\left(C_c^\infty(M;\mathbb{R}^n)/PC_c^\infty(M;\mathbb{R}^n), \tilde{E}_P\right)$ is a nuclear topological vector space and $\tilde{E}_P$ is separately continuous.

Finally, for every region $N \subseteq M$, viewed as a globally hyperbolic spacetime on its own, the restriction of P to N, $P_N := P \upharpoonright C^\infty(N;\mathbb{R}) \to C^\infty(N;\mathbb{R})$ is also Green hyperbolic with $E_{P_N}^\pm = E_P^\pm \upharpoonright C_c^\infty(N;\mathbb{R})$, see Sec. 3.2 in [88].

C.2.1 Normally hyperbolic operators

We follow [90]. An important class of formally self-adjoint Green hyperbolic operators are so-called *wave operators* or *normally hyperbolic operators* (which we always implicitly assume to be formally self-adjoint). They are second-order linear differential operators whose principal symbol is given by the Lorentzian metric.

The general form of a formally self-adjoint normally hyperbolic operator P on $C_c^\infty(M;\mathbb{R}^n)$ is

$$Pf = \Box f + V^\alpha \nabla_\alpha f + Wf \tag{C.7}$$

where V^α and W are smooth matrix-valued coefficients, with V^α antisymmetric and $W - W^T = \nabla_\alpha V^\alpha$, see [2].

Normally hyperbolic operators have an important property, see Theorem 3.2.11 in [90]. For every spacelike Cauchy surface $\Sigma \subseteq M$ and for every $u_0, u_1 \in C_c^\infty(M;\mathbb{R}^n)$ there exists a unique $u \in C_{sc}^\infty(M;\mathbb{R}^n)$ such that

$$Pu = 0, \quad u \upharpoonright \Sigma = u_0, \quad \nabla_{\mathfrak{n}} u \upharpoonright \Sigma = u_1, \tag{C.8}$$

where $\mathfrak{n}$ is the future directed timelike unit normal field along Σ. In other words, for every spacelike Σ, the following spaces are equivalent

$$\begin{aligned}&\left(C_c^\infty(M;\mathbb{R}^n)/PC_c^\infty(M;\mathbb{R}^n), \tilde{E}_P\right) \cong \left(\left(E_P^- - E_P^+\right)[C_c^\infty(M;\mathbb{R}^n)], \sigma\right)\\ =&(\ker P \upharpoonright C_{sc}^\infty(M;\mathbb{R}^n), \sigma) \cong (C_c^\infty(\Sigma;\mathbb{R}^n) \oplus C_c^\infty(\Sigma;\mathbb{R}^n), \Omega),\end{aligned} \tag{C.9}$$

where

$$\sigma\Big(\left(E_P^- - E_P^+\right)f, \left(E_P^- - E_P^+\right)g\Big) := E_P(f,g), \tag{C.10}$$

which is well-defined, and

$$\begin{aligned}\Omega\Big((f_1,f_2)^T, (g_1,g_2)^T\Big) &:= \int_\Sigma f_1 \cdot g_2 - f_2 \cdot g_1 \, \mathrm{d}V_\Sigma\\ &= \int_\Sigma (f_1,f_2)^T \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \mathrm{d}V_\Sigma,\end{aligned} \tag{C.11}$$

where we used that

$$E_P(u,v) = \int\limits_{\Sigma} \left(\left(E_P^- - E_P^+\right)u, \nabla_{\mathfrak{n}}\left(E_P^- - E_P^+\right)u\right)^T \begin{pmatrix} 0 & \mathbb{1}_{n\times n} \\ -\mathbb{1}_{n\times n} & 0 \end{pmatrix} \begin{pmatrix} \left(E_P^- - E_P^+\right)v \\ \nabla_{\mathfrak{n}}\left(E_P^- - E_P^+\right)v \end{pmatrix}, \tag{C.12}$$

according to Lemma 4.7.7 in [90].

C.3 AQFTs of linear real scalar fields

A formally self-adjoint Green hyperbolic operator $P : C^\infty(M;\mathbb{R}^n) \to C^\infty(M;\mathbb{R}^n)$ gives rise to net of symplectic subspaces as follows. To every region $N \subseteq M$ we associate $C_c^\infty(N;\mathbb{R}^n)/PC_c^\infty(M;\mathbb{R}^n)$, where we view $C_c^\infty(N;\mathbb{R}^n) \subseteq C_c^\infty(M;\mathbb{R}^n)$. Then this is a net of symplectic subspaces fulfilling all the axioms of Sec. B.7, including time-slice.

Indeed, the first three properties are straightforward. In order to show time-slice we proceed as follows. Let $N_1 \subseteq D_M(N_2)$, then choose two Cauchy surfaces $\Sigma_-, \Sigma_+ \subseteq N_2$ of $D_M(N_2)$, such that $J^+_{D_M(N_2)}(\Sigma_-) \cap \Sigma_+ = \emptyset$. Let us take $f \in C_c^\infty(N_1;\mathbb{R}^n)$. By multiplying with appropriate bump functions, we can write f as sum of three functions in $C_c^\infty(D_M(N_2);\mathbb{R}^n)$, $f = f_- + \tilde{f} + f_+$, such that $\tilde{f} \in C_c^\infty(N_2)$, and supp $f_\pm \subseteq J^\pm_{D_M(N_2)}(\Sigma_\pm)$.

Let us now pick $\chi_\pm \in C^\infty(D_M(N_2);\mathbb{R}^n)$, such that $\chi_\pm$ vanishes on $J_M^\mp(\Sigma^\mp)$ and equals 1 on $J_M^\pm(\Sigma^\pm)$. Then

$$\tilde{f}_\pm := \underbrace{\mp P\chi_\pm\left(E_P^- - E_P^+\right)f_\pm}_{\in C_c^\infty(N_2;\mathbb{R}^n)} = f_\pm - P\underbrace{\left(\chi_\pm E_P^\mp f_\pm\right)}_{\in C_c^\infty(M;\mathbb{R}^n)}. \tag{C.13}$$

As a result $[f]_P = \left[\tilde{f}_- + \tilde{f} + \tilde{f}_+\right]_P$, which is in $C_c^\infty(N_2;\mathbb{R}^n)/PC_c^\infty(M;\mathbb{R}^n)$.

Furthermore, as shown in [10], there is a "symplectic" version of the Haag property. Let $K \subseteq M$ be a compact subset and suppose $f \in C_c^\infty(M;\mathbb{R}^n)$ is such that $\tilde{E}_P([g]_P,[f_P]) = E_P(f,g) = 0$ for every region $N \subseteq K^{\perp_M}$ and every $g \in C_c^\infty(N;\mathbb{R}^n)$, i.e.,

$$\int_M g \cdot ((E_P^- - E_P^+)f)\,\mathrm{d}V_M = 0. \tag{C.14}$$

It then follows that $((E_P^- - E_P^+)f)$ vanishes on $K^{\perp_M}$. Then we quote Lemma 3.1 (i) in [78], which tells us that for any region $L \subseteq M$ with finitely many connected components such that $K \subseteq L$, it holds that $((E_P^- - E_P^+)f) \in (E_P^- - E_P^+)[C_c^\infty(L;\mathbb{R}^n)]$. But this implies that $[f]_P \in C_c^\infty(L;\mathbb{R}^n)/PC_c^\infty(M;\mathbb{R}^n)$.

Now we can follow the ideas sketched in Sec. B.7 and arrive at AQFTs ${}^P\mathcal{W}_{\mathfrak{B}_M}$ and ${}^P\mathcal{F}_{\mathfrak{B}_M}$ that fulfill time-slice. (From our discussion above it follows that the field algebra ${}^P\mathcal{F}_{\mathfrak{B}_M}$ is naturally a nuclear unital $*$-algebra.)

In [10] is was shown that ${}^P\mathcal{F}_{\mathfrak{B}_M}$ has the Haag property. Let us now also show that ${}^P\mathcal{W}_{\mathfrak{B}_M}$ has the Haag property. To that end let $K \subseteq M$ be a compact subset, let $A \in {}^P\mathcal{W}^g_{\mathfrak{B}_M}$ and suppose that for every $B \in {}^P\mathcal{W}_{\mathfrak{B}_M}(K^{\perp_M})$ we have that $[B, A] = 0$. In particular, for every $g \in C_c^\infty(K_M^\perp; \mathbb{R}^n)$ we have that

$$[W([g]_P), A] = 0. \tag{C.15}$$

According to Lemma B.4.8 it then follows that A is in the Weyl algebra over every symplectic space containing the symplectic complement of $C_c^\infty(K_M^\perp; \mathbb{R}^n)/PC_c^\infty(M; \mathbb{R}^n)$. According to above, for every region L with finitely connected components that contains K, $C_c^\infty(L; \mathbb{R}^n)/PC_c^\infty(M; \mathbb{R}^n)$ is such a symplectic subspace, hence

$$A \in {}^P\mathcal{W}_{\mathfrak{B}_M}(L). \tag{C.16}$$

Finally, let us remark that both ${}^P\mathcal{W}_{\mathfrak{B}_M}$ and ${}^P\mathcal{F}_{\mathfrak{B}_M}$ also fulfill additivity and outer regularity. Showing additivity is straight forward by referring to the generators, and so is outer regularity for ${}^P\mathcal{F}_{\mathfrak{B}_M}$. Outer regularity for ${}^P\mathcal{W}_{\mathfrak{B}_M}$ utilises Lemma B.4.9.

C.3.1 Quasi-free K-maps and K-perturbed variants

Lemma C.3.1. *Let $P : C^\infty(M; \mathbb{R}^n) \to C^\infty(M; \mathbb{R}^n)$ be a formally self-adjoint Green hyperbolic operator and let $K \subseteq M$ be a compact set. Let t be a linear isomorphism of $C_c^\infty(M; \mathbb{R}^n)/PC_c^\infty(M; \mathbb{R}^n)$ that preserves the symplectic form such that*

1. *$t(v) = v$ for every $v \in C_c^\infty(L; \mathbb{R}^n)/PC_c^\infty(L; \mathbb{R}^n)$ whenever the region L spacelike separated from K,*

2. *for all regions $L^\pm \in \mathfrak{B}_M$ with $L^\pm \subseteq M_K^\pm$ and $L^+ \subseteq D_M(L^-)$:*

$$t\Big[C_c^\infty(L^+; \mathbb{R}^n)/PC_c^\infty(L^+; \mathbb{R}^n)\Big] \subseteq C_c^\infty(L^-; \mathbb{R}^n)/PC_c^\infty(L^-; \mathbb{R}^n), \tag{C.17}$$

   and

3. *for all regions $L^\pm \in \mathfrak{B}_M$ such that $L^\pm \subseteq M_K^\pm$ and $L^- \subseteq D_M(L^+)$:*

$$t^{-1}\Big[C_c^\infty(L^-; \mathbb{R}^n)/PC_c^\infty(L^-; \mathbb{R}^n)\Big] \subseteq C_c^\infty(L^+; \mathbb{R}^n)/PC_c^\infty(L^+; \mathbb{R}^n). \tag{C.18}$$

Then the emergent quasi-free unit-preserving $*$-automorphisms of the AQFTs ${}^P\mathcal{W}_{\mathfrak{B}_M}$ and ${}^P\mathcal{F}_{\mathfrak{B}_M}$ derived from t are K-maps.

Proof. Let T be the quasi-free unit-preserving $*$-automorphisms of the AQFTs ${}^{P}\mathcal{W}_{\mathfrak{B}_M}$.

1. Take $v \in C_c^\infty(L; \mathbb{R}^n)/PC_c^\infty(L; \mathbb{R}^n)$ for some region L spacelike separated from K. Then
$$TW(v) = W(tv) = W(v). \tag{C.19}$$
By linearity and continuity of T, this extends to every $A \in {}^{P}\mathcal{W}_{\mathfrak{B}_M}(L)$.

2. Consider regions $L^\pm \in \mathfrak{B}_M$ with $L^\pm \subseteq M_K^\pm$ and $L^+ \subseteq D_M(L^-)$ and take $v \in C_c^\infty(L^+; \mathbb{R}^n)/PC_c^\infty(L^+; \mathbb{R}^n)$. Then
$$TW(v) = W(tv) \in {}^{P}\mathcal{W}_{\mathfrak{B}_M}(L^-), \tag{C.20}$$
since $tv \in C_c^\infty(L^-; \mathbb{R}^n)/PC_c^\infty(L^-; \mathbb{R}^n)$. By linearity and continuity of T, this extends to every $A \in {}^{P}\mathcal{W}_{\mathfrak{B}_M}(L^+)$.

3. Consider regions $L^\pm \in \mathfrak{B}_M$ such that $L^\pm \subseteq M_K^\pm$ and $L^- \subseteq D_M(L^+)$ and take $v \in C_c^\infty(L^-; \mathbb{R}^n)/PC_c^\infty(L^-; \mathbb{R}^n)$. Then
$$T^{-1}W(v) = W(t^{-1}v) \in {}^{P}\mathcal{W}_{\mathfrak{B}_M}(L^+), \tag{C.21}$$
because $t^{-1}v \in C_c^\infty(L^+; \mathbb{R}^n)/PC_c^\infty(L^+; \mathbb{R}^n)$. By linearity and continuity of T, this extends to every $A \in {}^{P}\mathcal{W}_{\mathfrak{B}_M}(L^-)$.

4. Note that the AQFTs ${}^{P}\mathcal{W}_{\mathfrak{B}_M}$ and $\mathcal{F}_{\mathfrak{B}_M}$ both fulfill outer regularity.

Finally, the above arguments carry over to the AQFT $\mathcal{F}_{\mathfrak{B}_M}$ by noting that T (and T^{-1}) is a homomorphism and that ever element of any $\mathcal{F}_{\mathfrak{B}_M}(N)$ is given by a finite linear combination of products of elements of the form $\varphi(v)$, on which the (appropriate) quasi-free unit-preserving $*$-autmorphism T acts as $T\varphi(v) = \varphi(tv)$. $\square$

The following lemma concerns quasi-free K-perturbed variants of AQFTs of linear real scalar fields.

Lemma C.3.2. *Let Q and T be two formally self-adjoint Green hyperbolic operators on $C^\infty(M; \mathbb{R}^n)$ such that for some compact $K \subseteq M$ it holds that*
$$Qf = Tf \tag{C.22}$$
for every $f \in C^\infty(M; \mathbb{R}^n)$ whenever $\operatorname{supp} f \cap K = \emptyset$. *Then the following statements are true.*

1. *For every $N \in \mathfrak{B}_{(M,K)}$ there are symplectic linear isomorphisms*

$$\begin{aligned}\chi_N^{\text{symp}} : &\Big(C_c^\infty(N;\mathbb{R}^n)/QC_c^\infty(M;\mathbb{R}^n), \tilde{E}_Q\Big) \to \Big(C_c^\infty(N;\mathbb{R}^n)/TC_c^\infty(M;\mathbb{R}^n), \tilde{E}_T\Big)\\ &[f]_Q \mapsto [f]_T,\end{aligned} \tag{C.23}$$

which are well-defined for representatives $f \in C_c^\infty(N;\mathbb{R}^n)$.

2. $\vartheta := \Big(\chi_{M_K^-}\Big)^{-1} \circ \chi_{M_K^+}$ *is a symplectic linear isomorphism of the symplectic space $\Big(C_c^\infty(M;\mathbb{R}^n)/QC_c^\infty(M;\mathbb{R}^n), \tilde{E}_Q\Big)$. For representatives $f \in C^\infty(M_K^+;\mathbb{R}^n)$ its action is given by*

$$\vartheta[f]_Q = [\theta f]_Q \tag{C.24}$$

where

$$\theta f := f - (T - Q)E_T^- f. \tag{C.25}$$

3. *Let ${}^Q\mathcal{W}_{\mathfrak{B}_M}$ and ${}^T\mathcal{W}_{\mathfrak{B}_M}$ be the AQFTs derived from Q and T and let χ_N be the quasi-free unit-preserving $*$-automorphism associated to χ_N^{symp}. Then*

$$\Big({}^Q\mathcal{W}_{\mathfrak{B}_M}, {}^T\mathcal{W}_{\mathfrak{B}_M}, \{\chi_N\}_{N\in\mathfrak{B}_{(M,K)}}, K\Big) \tag{C.26}$$

is a K-perturbed variant of ${}^Q\mathcal{W}_{\mathfrak{B}_M}$ on $\mathfrak{B}_M$ with scattering map given by Θ, the quasi-free unit-preserving $*$-automorphism associated to ϑ.

4. *Let ${}^Q\mathcal{F}_{\mathfrak{B}_M}$ and ${}^T\mathcal{F}_{\mathfrak{B}_M}$ be the AQFTs derived from Q and T and let χ_N be the quasi-free unit-preserving $*$-automorphism associated to χ_N^{symp}. Then*

$$\Big({}^Q\mathcal{F}_{\mathfrak{B}_M}, {}^T\mathcal{F}_{\mathfrak{B}_M}, \{\chi_N\}_{N\in\mathfrak{B}_{(M,K)}}, K\Big) \tag{C.27}$$

is a K-perturbed variant of ${}^Q\mathcal{F}_{\mathfrak{B}_M}$ on $\mathfrak{B}_M$ with scattering map given by Θ, the quasi-free unit-preserving $*$-automorphism associated to ϑ.

Proof. 1. To see that χ_N^{symp} is well-defined note that two representatives of $[f]_Q$ with support in N differ by some $Qg \in C_c^\infty(N;\mathbb{R}^n)$. Let us now denote the restriction of Q to $C^\infty(N;\mathbb{R}^n)$ by Q_N. Then we see that Qg is in the kernel of $E_Q^- - E_Q^+$, but since $Qg \in C_c^\infty(N;\mathbb{R}^n)$, it is in the kernel of $\Big(E_Q^- - E_Q^+\Big) \restriction C_c^\infty(N;\mathbb{R}^n)$, which is the kernel of $E_{Q_N}^- - E_{Q_N}^+$, see Sec. 3.2 in [88]. But then Qg is in $Q_N C_c^\infty(N;\mathbb{R}^n)$, hence there is a $\tilde{g}$ with support in N such that $Qg = Q\tilde{g}$. Extending by zero, we see that $g - \tilde{g} \in C_c^\infty(M;\mathbb{R}^n)$ is in the kernel of Q and hence must vanish identically. The rest of the claim follows since Q and T are identical on functions with support in N for $N \in \mathfrak{B}_{(M,K)}$.

2. See Appendix D in [10].

The remaining points follow straight forwardly from the fact that $\chi_{N_2}^{\mathrm{symp}} \restriction C_c^\infty(N_1;\mathbb{R}^n)/QC_c^\infty(M;\mathbb{R}^n) = \chi_{N_1}^{\mathrm{symp}}$ for nested K-admissible regions $N_1 \subseteq N_2$. $\square$

— D —

Auxiliary geometrical lemmas

Here we discuss and prove some auxiliary geometrical lemmas. Let M be a globally hyperbolic spacetime. Recall that a region is a non-empty open and causally convex subset of M. In what follows, the concrete definition of causal/timelike future/past-directed curves plays no essential role, apart from the fact that the class should be stable under concatenations (that respect the orientation), see for instance [91]. We may call a causal curve *directed*, if it is either future- or past-directed. For any subset $N \subseteq M$, its chronal and causal future and past are then defined as

$$\begin{aligned} I^{\pm}_M(N) &:= \{q \in M | \exists p \in N \; \exists \text{ timelike future/past-directed curve from } p \text{ to } q\}, \\ J^{\pm}_M(N) &:= \{q \in M | \exists p \in N \; \exists \text{ causal future/past-directed curve from } p \text{ to } q\} \cup N, \end{aligned} \tag{D.1}$$

and its future and past domain of dependence are defined as

$$\begin{aligned} D^{\pm}_M(N) := \{p \in M | &\text{every past/future-inextendible} \\ &\text{causal past/future-directed curve starting at } p \text{ meets } N\}. \end{aligned} \tag{D.2}$$

The domain of dependence of N is then $D_M(N) := D^+_M(N) \cup D^-_M(N)$. In particular

$$\begin{aligned} D_M(N) :=& \{p \in M | \text{every inextendible causal future-directed curve through } p \text{ meets } N\} \\ =& \{p \in M | \text{every inextendible causal past-directed curve through } p \text{ meets } N\} \\ =& \{p \in M | \text{every inextendible causal directed curve through } p \text{ meets } N\}. \end{aligned} \tag{D.3}$$

Useful standard properties are:

1. for every region N, $D_M(N)$ is again a region,

2. $A \subseteq B$ implies that $D_M(A) \subseteq D_M(B)$,

3. $D_M(D_M(A)) = D_M(A)$ and

4. for every Cauchy surface Σ, $J^{\pm}_M(\Sigma)$ is closed.

Furthermore, for every region $N \subseteq M$ and every subset $L \subseteq N$ it holds that

1. $J^{\pm}_N(L) = J^{\pm}_M(L) \cap N$,
2. $D_N(L) = D_M(L) \cap N$ and
3. $\mathrm{ch}_N(L) = \mathrm{ch}_M(L)$,

from which it follows that for every region $N \subseteq M$, $L \subseteq N$ is a region in N iff it is a region in M.

D.1 Lemma 3.1.8

The proof of the gluing Lemma 3.1.8 is based on a couple of auxiliary geometrical lemmas.

Lemma D.1.1. *Let $A \subseteq M$ be* any *subset. Then*

1. $\overline{J^{\pm}_M(A)} = J^{\pm}_M(\overline{J^{\pm}_M(A)})$,
2. *$M \setminus (J^{\pm}_M(A))$ is causally convex.*

Proof. 1. We observe

$$\overline{J^{\pm}_M(A)} \subseteq J^{\pm}_M(\overline{J^{\pm}_M(A)}) \subseteq \overline{I^{\pm}_M(\overline{J^{\pm}_M(A)})} = \overline{I^{\pm}_M(J^{\pm}_M(A))} \subseteq \overline{J^{\pm}_M(J^{\pm}_M(A))} = \overline{J^{\pm}_M(A)}, \tag{D.4}$$

where we used the standard Corollary 2.4.19 in [91] and Proposition 2.11 in [92].

2. See the proof of Lemma A.4 in [41] for related arguments. Let $p_1, p_2 \in M \setminus J^+_M(A)$ be connected by a causal future-directed curve γ starting at p_1. Suppose a point q on γ is outside of $M \setminus J^+_M(A)$, i.e., $q \in J^+_M(A)$. But then $p_1 \in J^+_M(q) \subseteq J^+_M(A)$, which is a contradiction, so $M \setminus J^+_M(A)$ is causally convex.

□

The following lemma is used for proving the local time-slice property of the glued theory.

Lemma D.1.2. *Let $M_1, M_2 \subseteq M$ be two regions in M such that $M_1 \cup M_2 = M$ and such that $M_1 \cap M_2$ contains a Cauchy surface for M. For some region $L \subseteq M$ let us define $L_{1/2} := L \cap M_{1/2}$. Then*

$$D_M(D_M(L_2) \cup D_M(L_1)) \cap M_1 \subseteq D_M((D_M(L_2) \cap M_2 \cup D_M(L_1)) \cap M_1) \cap M_1, \tag{D.5}$$

and $(D_M(L_2) \cap M_2 \cup D_M(L_1)) \cap M_1$ is a region.

Proof. Let $p \in D_M(D_M(L_1) \cup D_M(L_2)) \cap M_1$, i.e., $p \in M_1$ and every inextendible causal future-directed curve, say γ, through p intersects $D_M(L_1) \cup D_M(L_2)$, and we want to show that it also intersects $(D_M(L_2) \cap M_2 \cup D_M(L_1)) \cap M_1$.

1. If γ intersects $D_M(L_1)$, then γ also meets $L_1 \subseteq D_M(L_1) \cap M_1$.

2. So let us assume that γ does not intersect $D_M(L_1)$, so, by assumption, γ then needs to intersects $D_M(L_2)$. Then γ also meets L_2, hence γ meets $D_M(L_2) \cap M_2$. Let us consider $\gamma \cap \partial M_1$. If empty, then γ meets $D_M(L_2) \cap M_2 \cap M_1$.

   So suppose $\gamma \cap \partial_M M_1 \neq \emptyset$ and, in order to derive a contradiction, that γ does not meet $D_M(L_2) \cap M_2 \cap M_1$. Then, by causal convexity of M_1, $\gamma^{-1}[M_1] = (a, b)$, where at least one of a, b is in the domain of γ.

   a) First suppose b is in the domain (if not, skip to point *2b*). Then $\gamma(b) \in M_2 \cap \partial_M M_1$. Since $D_M(L_2)$ is open, by assumption on γ, $\gamma(b) \notin D_M(L_2)$, for otherwise γ would meet $D_M(L_2) \cap M_2 \cap M_1$. But then there exists a future-inextendible causal future-directed curve $\tilde{\gamma}$ starting at $\gamma(b)$ that does not meet $D_M(L_2)$. $\tilde{\gamma}$ cannot meet M_1, because an appropriate concatenation with γ would yield a causal future-directed curve from $p \in M_1$ to another point in M_1, which, by causal convexity of M_1 would have to lie entirely in M_1, which contradicts that $\gamma(b) \notin M_1$. By slight abuse of notation let us now denote the appropriate concatenation of γ and $\tilde{\gamma}$ at $\gamma(b)$ by γ, which is an inextendible causal future-directed curve.

   b) Suppose now that a is in the domain of γ. Then repeat the procedure of point *2a* for $a \leftrightarrow b$ *mutatis mutandis.*

   The above procedure yields an inextendible causal directed curve through $p \in M_1$ that neither intersects $D_M(L_1)$ nor $D_M(L_2)$, which is a contradiction.

Finally let us show that $(D_M(L_2) \cap M_2 \cup D_M(L_1)) \cap M_1$ is a region. It suffices to show that $(D_M(L_2) \cap M_2) \cup D_M(L_1)$ is causally convex. So let us consider a causal future-directed curve γ starting at p_i and ending at p_f for two points $p_i, p_f \in (D_M(L_2) \cap M_2) \cup D_M(L_1)$.

We consider the following cases (the remaining ones are completely analogous).

1. Let $p_i \in D^-_M(L_2) \cap M_2$ and $p_f \in D^-_M(L_1)$ and consider *any* future-inextendible extension $\tilde{\gamma}$ of γ.

   a) Suppose, starting at p_i and following $\tilde{\gamma}$, we reach some point $q \in L_2$ *before* we reach p_f. Then the segment of γ between p_i and q is fully contained in

$D^-_M(L_2) \cap M_2$ (also since $L_2 \subseteq M_2$ and M_2 is causally convex). Following $\tilde{\gamma}$ from $q \in L_2$ we reach p_f and then also some point $r \in L_1$, since $p_f \in D^-_M(L_1)$. But by causal convexity of $L = L_1 \cup L_2$, the segment between q and r is fully contained in L and hence also the segment of γ between q and p_f, which is hence also in $L \subseteq (D_M(L_2) \cap M_2) \cup D_M(L_1)$.

b) Suppose starting at p_i and following $\tilde{\gamma}$, we reach p_f *before* we reach some point $q \in L_2$. But then, since $L_2 \subseteq M_2$, we see that $p_f \in D^-_M(L_2) \cap M_2$ and hence, by causal convexity, that γ is fully contained in $D^-_M(L_2) \cap M_2$.

2. Let $p_i \in D^-_M(L_2) \cap M_2$ and $p_f \in D^+_M(L_1)$. Following γ starting at p_i suppose we reach some $q \in L_2$ before we reach p_f (otherwise $p_f \in D^-_M(L_2) \cap M_2$ see above). Going along γ *backwards* starting at p_f, suppose we reach some $r \in L_1$ *before* we reach p_i. Then the segment between p_i and q is fully in $D^-_M(L_2) \cap M_2$, the segment between q and r is fully in L, and the segment between r and p_f is fully in $D^+_M(L_1)$.

3. Let $p_i \in D^+_M(L_1)$ and $p_f \in D^-_M(L_2) \cap M_2$. Let $\tilde{\gamma}$ be an intextendible extension of γ. Then there is some $q \in L_1$ on $\tilde{\gamma}$ *before* p_i and some $r \in L_2$ on $\tilde{\gamma}$ *after* p_f. By causal convexity of L, we see that $p_i, p_f \in L$.

□

The following is used for causally disjoint commutativity of the glued AQFT.

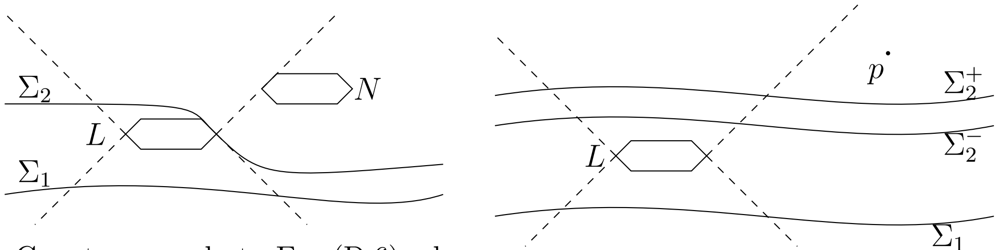
Lemma D.1.3. *Let $\Sigma_1, \Sigma_2^\pm$ be three Cauchy surfaces for M such that there is a causal linear order $\leq$ with $\Sigma_1 < \Sigma_2^- < \Sigma_2^+$. Let L be a region fully contained in $M \setminus J^+_M(\Sigma_2^-)$. Then*

$$M \setminus (\overline{J^-_M(L)} \cup \overline{J^+_M(L)} \cup J^-_M(\Sigma_1)) \subseteq D_M(M \setminus (\overline{J^-_M(L)} \cup \overline{J^+_M(L)} \cup J^-_M(\Sigma_1) \cup J^+(\Sigma_2^+))), \tag{D.6}$$

and $M \setminus (\overline{J^-_M(L)} \cup \overline{J^+_M(L)} \cup J^-_M(\Sigma_1) \cup J^+_M(\Sigma_2^+))$ is a region.

Remark: Using two Cauchy surfaces $\Sigma_2^\pm$ instead of just one is necessary in order to avoid pathological situations as sketched in Fig. D.1a.

Proof. We first note that $J^-_M(L)$ has empty intersection with $J^+_M(\Sigma_2^-)$, and hence also empty intersection with the open set $I^+_M(\Sigma_2^-)$. But then $J^-_M(L) \cap I^+_M(\Sigma_2^-) = \emptyset$ implies that $\overline{J^-_M(L)} \cap I^+_M(\Sigma_2^-) = \emptyset$, and since $J^+_M(\Sigma_2^+) \subseteq I^+_M(\Sigma_2^-)$, we see that also $\overline{J^-_M(L)} \cap J^+_M(\Sigma_2^+) = \emptyset$.



(a) Counterexample to Eq. (D.6) when $\Sigma_2^+ = \Sigma_2^-$ and piece-wise null.

(b) Schematic setting in the proof.

Figure D.1: Illustrations for Lemma D.1.3.

Let us now observe that by Lemma D.1.1 the inclusion we want to show can be phrased as

$$\begin{aligned} &M \setminus \Big(J_M^-(B^-) \cup J_M^+(B^+) \cup J_M^-(\Sigma_1)\Big) \\ &\subseteq D_M\Big(M \setminus \Big(J_M^-(B^-) \cup J_M^+(B^+) \cup J_M^-(\Sigma_1) \cup J_M^+(\Sigma_2^+)\Big)\Big), \end{aligned} \tag{D.7}$$

where $B^+ := \overline{J_M^+(L)}$ and $B^- := \overline{J_M^-(L)}$.

We proceed by contradiction, so let p be contained in the left hand side, but not in the right hand side. Hence there exists an inextendible causal past-directed curve γ through p that does not intersect $M \setminus \Big(J_M^-(B^-) \cup J_M^+(B^+) \cup J_M^-(\Sigma_1) \cup J_M^+(\Sigma_2^+)\Big)$. In particular the trace of γ, and hence also p, needs to fully lie in $J_M^-(B^-) \cup J_M^+(B^+) \cup J_M^-(\Sigma_1) \cup J_M^+(\Sigma_2^+)$. But p cannot lie in $J_M^-(B^-) \cup J_M^+(B^+) \cup J_M^-(\Sigma_1)$, so p needs to lie in $J_M^+(\Sigma_2^+) \setminus J_M^+(B^+)$. Let us divide γ at p into $\gamma^\pm$, such that γ^- is a past-inextendible causal-past-directed curve starting at p, that therefore needs to leave $J_M^+(\Sigma_2^+) \setminus J_M^+(B^+)$ and enter $J^-(\Sigma_1)$. However,

1. $p \notin J_M^+(B^+)$, so γ^- cannot enter $J_M^+(B^+)$,
2. $J_M^+(\Sigma_2^+)$ has empty intersection with $J_M^-(B^-) = \overline{J_M^-(L)}$, so γ^- cannot directly enter $J_M^-(B^-)$ once it leaves $J_M^+(\Sigma_2^+)$, and
3. $J_M^+(\Sigma_2^+)$ has empty intersection with $J_M^-(\Sigma_1)$, so γ^- cannot directly enter $J_M^-(\Sigma_1)$.

We hence see that the trace of γ^- cannot fully lie in $J_M^-(B^-) \cup J_M^+(B^+) \cup J_M^-(\Sigma_1) \cup J_M^+(\Sigma_2^+)$, which is a contradiction.

Finally, what is left to show is that

$$\begin{aligned} &M \setminus (\overline{J_M^-(L)} \cup \overline{J_M^+(L)} \cup J_M^-(\Sigma_1) \cup J_M^+(\Sigma_2^+)) \\ &= M \setminus \Big(J_M^-(B^-) \cup J_M^+(B^+) \cup J_M^-(\Sigma_1) \cup J_M^+(\Sigma_2^+)\Big) \end{aligned} \tag{D.8}$$

is a region. It is clearly open and, by Lemma D.1.1, the intersection of causally convex sets, so causally convex. □

D.2 LEMMA 4.2.7

The following geometrical lemma is used for the proof that the composition of K-maps is again a K-map, i.e., Lemma 4.2.7.

Lemma D.2.1. *Let K_1, K_2 be two causally orderable compact subsets of M, such that $K_1 < K_2$ with respect to some causal linear order $\leq$ and let L be some region in M. Then*

1. $D_M\Big(D_M(L \setminus J_M^+(K_2)) \setminus (J_M^-(K_1) \cup J_M^+(K_2))\Big) \setminus J_M^-(K_1) = D_M(L \setminus J_M^+(K_2)) \setminus J_M^-(K_1)$,
2. $D_M\Big(D_M(L \setminus J_M^-(K_1)) \setminus (J_M^-(K_1) \cup J_M^+(K_2))\Big) \setminus J_M^+(K_2) = D_M(L \setminus J_M^-(K_1)) \setminus J_M^+(K_2)$.

Proof. We only prove the first statement, the second one follows *mutatis mutandis.*

The inclusion "$\subseteq$" is trivial since for arbitrary $A \subseteq B$ we have that $D_M(A) \subseteq D_M(B)$ and $D_M(D_M(A)) = D_M(A)$.

For the inclusion "$\supseteq$" we proceed by contradiction. Let us assume that p is in the right hand side, but not in the left hand side. Then $p \in J_M^+(K_2)$. (Suppose $p \notin J_M^+(K_2)$. Then $p \in D_M(L \setminus J_M^+(K_2)) \setminus (J_M^-(K_1) \cup J_M^+(K_2))$. But then, keeping in mind that p is not in $J_M^-(K_1)$ by assumption, p is in the LHS.) Furthermore, any inextendible causal future/past-directed curve (icdc) through p intersects $L \setminus J_M^+(K_2)$, but at least one icdc, say $\tilde{\gamma}$, does not intersect $D_M(L \setminus J_M^+(K_2)) \setminus (J_M^-(K_1) \cup J_M^+(K_2))$. Hence, $\tilde{\gamma}$ can only intersect $L \setminus J_M^+(K_2)$ in $J_M^-(K_1)$. So let $q \in J_M^-(K_1) \cap (L \setminus J_M^+(K_2))$ be a point on $\tilde{\gamma}$ and γ be the restriction of $\tilde{\gamma}$ that connects p and q. Note that by assumption on p, γ lies fully in $D_M(L \setminus J_M^+(K_2))$, while $\tilde{\gamma}$ does not intersect $D_M(L \setminus J_M^+(K_2)) \setminus (J_M^-(K_1) \cup J_M^+(K_2))$. Hence γ needs to lie in $J_M^-(K_1) \cup J_M^+(K_2)$ connecting $p \in J_M^+(K_2)$ and $q \in J_M^-(K_1)$, which contradicts the fact that the closed sets $J_M^+(K_2), J_M^-(K_1)$ have empty intersection. □

D.3 THEOREM 4.4.1

The following lemma, or rather its corollary, is used to prove causally disjoint commutativity in Theorem 4.4.1. The statement and proof are very similar to Lemma D.1.3.

Lemma D.3.1. *Let $K \subseteq M$ be a compact subset and let $\tilde{K} \subseteq M$ be a compact subset that contains K in its open interior, i.e., such that $K \subsetneq \mathring{\tilde{K}} \subsetneq \tilde{K}$. Let $L \subseteq M$ be a region fully contained in $M^-_{\tilde{K}} = M \setminus J^+_M(\tilde{K})$. Then*

$$M \setminus \Big(\overline{J^+_M(L)} \cup \overline{J^-_M(L)} \cup J^-_M(\tilde{K})\Big) \subseteq D_M\Big(M \setminus \Big(J^+_M(K) \cup \overline{J^+_M(L)} \cup \overline{J^-_M(L)}\Big)\Big), \tag{D.9}$$

and $M \setminus \Big(J^+_M(K) \cup \overline{J^+_M(L)} \cup \overline{J^-_M(L)}\Big)$ is a region fully contained in $M^-_K = M \setminus J^+_M(K)$.

Proof. Let us note that $J^-_M(L)$ has empty intersection with $J^+_M(\tilde{K})$, and hence also empty intersection with $J^+_M(\mathring{\tilde{K}})$, which is open, see Lemma A.8 in [41]. But then $J^-_M(L) \cap J^+_M(\mathring{\tilde{K}}) = \emptyset$ implies that $\overline{J^-_M(L)} \cap J^+_M(\mathring{\tilde{K}}) = \emptyset$, and since $J^+_M(K) \subseteq J^+_M(\mathring{\tilde{K}})$, we see that also $\overline{J^-_M(L)} \cap J^+_M(K) = \emptyset$.

Let us now observe that by Lemma D.1.1 the inclusion we want to show can be phrased as

$$M \setminus \Big(J^+_M(B^+) \cup J^-_M(B^-) \cup J^-_M(K)\Big) \subseteq D_M\Big(M \setminus \Big(J^+_M(K) \cup J^+_M(B^+) \cup J^-_M(B^-)\Big)\Big), \tag{D.10}$$

where $B^+ := \overline{J^+_M(L)}$ and $B^- := \overline{J^-_M(L)}$.

We proceed by contradiction, so let p be contained in the left hand side, but not in the right hand side, so there exists an inextendible causal past-directed curve γ through p that does not intersect $M \setminus \Big(J^+_M(K) \cup J^+_M(B^+) \cup J^-_M(B^-)\Big)$.

In particular the trace of γ, and hence also p, needs to fully lie in $J^+_M(K) \cup J^+_M(B^+) \cup J^-_M(B^-)$. But p cannot lie in $J^-_M(K) \cup J^+_M(B^+) \cup J^-_M(B^-)$, since it is in the left hand side, so p needs to lie in $J^+_M(K) \setminus \Big(J^-_M(K) \cup J^+_M(B^+) \cup J^-_M(B^-)\Big)$.

Let us divide γ at p into $\gamma^\pm$, such that γ^- is a past-inextendible causal past-directed curve starting at p, that therefore needs to leave $J^+_M(K)$. However,

1. $p \notin J^+_M(B^+)$, so γ^- cannot enter $J^+_M(B^+)$,

2. the closed set $J^+_M(K)$ has empty intersection with the closed set $J^-_M(B^-) = \overline{J^-_M(L)}$, so γ^- cannot directly enter $J^-_M(B^-)$.

We hence see that the trace of γ^- cannot fully lie in $J^+_M(K) \cup J^+_M(B^+) \cup J^-_M(B^-)$, which is a contradiction.

Finally, what is left to show is that

$$\begin{aligned} &M \setminus \Big(J^+_M(K) \cup \overline{J^+_M(L)} \cup \overline{J^-_M(L)}\Big) \\ &= M \setminus \Big(J^+_M(K) \cup J^+_M(B^+) \cup J^-_M(B^-)\Big) \end{aligned} \tag{D.11}$$

is a region. It is clearly open, and by Lemma D.1.1, the intersection of causally convex sets, so causally convex. □

The following result is a straightforward consequence of Lemma D.3.1. Together, these results may be seen as a refinement of Theorem 2 and Lemmas 3 and 4 in [1].

Corollary D.3.2. *Let $K \subseteq M$ be a compact subset and let $\tilde{K} \subseteq M$ be a compact subset that contains K in its open interior, i.e., such that $K \subsetneq \mathring{\tilde{K}} \subsetneq \tilde{K}$. Let $L_1 \subseteq M^-_{\tilde{K}}$ and $L_2 \subseteq M^+_{\tilde{K}}$ be two* spacelike separated *regions and let T be a K-hom on an AQFT $\mathcal{A}_{\mathfrak{B}_M}$. Then*

$$T[\mathcal{A}_{\mathfrak{B}_M}(L_2)] \subseteq \mathcal{A}_{\mathfrak{B}_M}\Big(M \setminus \Big(J^+_M(K) \cup \overline{J^+_M(L_1)} \cup \overline{J^-_M(L_1)}\Big)\Big). \tag{D.12}$$

In particular $T[\mathcal{A}_{\mathfrak{B}_M}(L_2)]$ commutes with $\mathcal{A}_{\mathfrak{B}_M}(L_1)$.

Proof. Since L_1 is spacelike separated from L_2, L_2 has empty intersection with $J^+_M(L_1) \cup J^-_M(L_1)$. Since L_2 is open, it also has empty intersection with $\overline{J^+_M(L_1)} \cup \overline{J^-_M(L_1)} = \overline{J^+_M(L_1) \cup J^-_M(L_1)}$, so we have that $L_2 \subseteq M \setminus \Big(\overline{J^+_M(L_1)} \cup \overline{J^-_M(L_1)} \cup J^-_M(\tilde{K})\Big)$, and according to Lemma D.3.1, we have that

$$M \setminus \Big(\overline{J^+_M(L_1)} \cup \overline{J^-_M(L_1)} \cup J^-_M(\tilde{K})\Big) \subseteq D_M\Big(M \setminus \Big(J^+_M(K) \cup \overline{J^+_M(L_1)} \cup \overline{J^-_M(L_1)}\Big)\Big). \tag{D.13}$$

By the properties of the K-hom T we hence have that

$$T[\mathcal{A}_{\mathfrak{B}_M}(L_2)] \subseteq \mathcal{A}_{\mathfrak{B}_M}\Big(M \setminus \Big(J^+_M(K) \cup \overline{J^+_M(L_1)} \cup \overline{J^-_M(L_1)}\Big)\Big). \tag{D.14}$$

By causally disjoint commutativity of $\mathcal{A}_{\mathfrak{B}_M}$ and the fact that $M \setminus (J^+_M(K) \cup \overline{J^+_M(L_1)} \cup \overline{J^-_M(L_1)})$ is spacelike separated from L_1, we see that $T[\mathcal{A}_{\mathfrak{B}_M}(L_2)]$ commutes with $\mathcal{A}_{\mathfrak{B}_M}(L_1)$. □

The following lemma is used to show the local time-slice property in Theorem 4.4.1. The statement and the proof are similar to Lemma D.1.2.

Lemma D.3.3. *Let $K \subseteq M$ be a compact set and let $N^\pm \subseteq M$ be two open sets.*

1. If $N^- \cap J^+_M(K) = \emptyset$, then

$$\begin{aligned} &D_M((D_M(N^+) \cup D_M(N^-)) \setminus J^+_M(K)) \setminus J^+_M(K) \\ &= D_M((D_M(N^+) \cup D_M(N^-))) \setminus J^+_M(K). \end{aligned} \tag{D.15}$$

2. If $N^+ \cap J^-_M(K) = \emptyset$, then

$$\begin{aligned} &D_M((D_M(N^+) \cup D_M(N^-)) \setminus J^-_M(K)) \setminus J^-_M(K) \\ &= D_M((D_M(N^+) \cup D_M(N^-))) \setminus J^-_M(K). \end{aligned} \tag{D.16}$$

Let now N be a $\mathrm{ch}_M(K)$-admissible region and let us define $N^\pm := N \setminus J^\mp_M(K)$. Then

$$(D_M(N^+) \cup D_M(N^-)) \setminus J^+_M(K), \text{ and } (D_M(N^+) \cup D_M(N^-)) \setminus J^-_M(K), \tag{D.17}$$

are causally convex, so in particular they are regions.

Proof. 1. The inclusion "$\subseteq$" is trivial. So take $p \in D_M\Big((D_M(N^+) \cup D_M(N^-))\Big) \setminus J^+_M(K)$, i.e., $p \in M^-_K$ and every inextendible causal future-directed curve, say γ, through p intersects $D_M(N^+) \cup D_M(N^-)$, and we want to show that it also intersects $(D_M(N^+) \cup D_M(N^-)) \setminus J^+_M(K)$.

If γ intersects $D_M(N^-)$, then it also intersects N^-, and since $N^- \cap J^+_M(K) = \emptyset$, it also intersects $(D_M(N^+) \cup D_M(N^-)) \setminus J^+_M(K)$.

Let us assume that γ does *not* intersect $D_M(N^-)$, so it needs to intersect $D_M(N^+)$.

a) Suppose γ meets $D_M(N^+)$ in some point q *before* p. Then $q \notin J^+_M(K)$ (because otherwise, since γ connects q with p, $p \in J^+_M(K)$, which is a contradiction). But then γ intersects $(D_M(N^+) \cup D_M(N^-)) \setminus J^+_M(K)$.

b) Suppose γ meets $D_M(N^+)$ in some point q *after* p.

Let us now consider $\gamma \cap \partial_M M^-_K$. If empty, then γ meets $D_M(N^+) \setminus J^+_M(K)$. Suppose now $\gamma \cap \partial_M M^-_K \neq \emptyset$ and, in order to derive a contradiction, that γ does *not* meet $(D_M(N^+) \cup D_M(N^-)) \setminus J^+_M(K)$. By causal convexity of M^-_K, $\gamma^{-1}\Big[M^-_K\Big] = (a, b)$ where b is in the domain of γ and $\gamma(b) \in \partial_M M^-_K$. Since $D_M(N^+)$ is open, it follows, by assumption on γ, that $\gamma(b) \notin D_M(N^+)$, for otherwise γ would meet $D_M(N^+) \setminus J^+_M(K)$. But then there exists a future-inextendible causal future-directed curve $\tilde\gamma$ starting at $\gamma(b)$ that does not meet $D_M(N^+)$. Clearly, $\tilde\gamma$ lies fully in $J^+_M(K)$, and does not meet $D(N^-)$. Concatenating γ and $\tilde\gamma$ at $\gamma(b)$ yields an inextendible directed curve through p that does not intersect $D_M(N^+)$ or $D_M(N^-)$, which is a contradiction.

2. This follows *mutatis mutandis.*

3. We will show that $D_M(N^+) \cup D_M(N^-)$ is a region. The two respective sets are then simply intersections with the regions $M_K^\pm$, hence also regions. By assumption, $N^\pm$ is open, so $D_M(N^\pm)$ is open as well and hence also the union. So it remains to show that the union is causally convex.

   Note that by the fact that N is $\mathrm{ch}_M(K)$-admissible, we have that $N = N^- \cup N^+$.

   Let now γ be a causal future-directed curve that starts at p_i and ends at p_f for $p_i, p_f \in D_M(N^+) \cup D_M(N^-)$.

   We consider the following cases (the remaining ones are similar).

   a) Let $p_i \in D_M^-(N^-)$ and $p_f \in D_M^-(N^+)$, and let $\tilde{\gamma}$ be a future-inextendible extension of γ.

      i. Suppose, starting at p_i and following $\tilde{\gamma}$, we reach some point $q \in N^-$ *before* we reach p_f. Then the segment between p_i and q is fully contained in $D_M^-(N^-)$ by causal convexity of $D_M^-(N^-)$. Following $q \in N^-$ along $\tilde{\gamma}$ we then reach p_f and then also some point $r \in N^+$ since $p_f \in D_M^-(N^+)$. But by causal convexity of $N = N^- \cup N^+$, the whole segment between q and r is fully contained in N and hence also the segment of γ between q and p_f. As a result, γ is fully contained in $D_M^-(N^-) \cup N \subseteq D_M(N^+) \cup D_M(N^-)$.

      ii. Suppose starting at p_i and following $\tilde{\gamma}$, we reach p_f *before* we reach some point $q \in N^- \subseteq D_M^-(N^-)$. But then, by causal convexity, of $D_M^-(N^-)$, γ is fully contained in $D_M^-(N^-)$.

   b) Let $p_i \in D_M^-(N^-)$ and $p_f \in D_M^+(N^+)$. Following γ starting at p_i suppose we reach some $q \in N^-$ *before* we reach p_f (otherwise $p_f \in D_M^-(N^-)$ by the argument above). Going along γ *backwards*, starting at p_f, suppose we reach some $r \in N^+$ *before* we reach q. Then the segment between p_i and q is fully in $D_M^-(N^-)$, the segment between q and r is fully contained in N and the segment between r and p_f is fully contained in $D_M^+(N^+)$.

□

D.4 LEMMA 5.2.3

The next lemma follows Lemma 6 and the proof of Theorem 5 in [1].

Lemma D.4.1. *Let $K_1, K_2, K_3 \subseteq M$ be three compact sets such that*

1. *$K_1 < K_2 < K_3$ with respect to some causal linear order $\leq$ on $\{K_1, K_2, K_3\}$,*

2. *$K_1 \perp_M K_3$ and such that*

3. *K_3 is connected.*

Then there exist three regions $N_1, N_2, N_3 \subseteq M$ such that

1. *$K_j \subseteq N_j$ for $j = 1, 2, 3$,*

2. *$N_1 \lhd N_2 \lhd N_3$ with respect to some causal linear order $\trianglelefteq$ on $\{N_1, N_2, N_3\}$,*

3. *$N_1 \perp_M N_3$ and such that*

4. *N_3 is connected.*

Proof. By assumption, $K_1 \subseteq M^-_{K_2} \cap K_3^{\perp_M}$. By compact exhaustion of $M^-_{K_2} \cap K_3^{\perp_M}$, we find an open set $G_1 \subseteq M^-_{K_2} \cap K_3^{\perp_M}$ with compact closure such that $K_1 \subseteq G_1$. Let us then set N_1 to be the causal hull of G_1, which is open (see Lemma A.8 in [41]), i.e., a region. Furthermore, N_1 is contained in the causal hull of $\overline{G_1}$, which is compact by Proposition 2.3 in [93], so the region N_1 has compact closure.

It then follows that $K_2 \subseteq M^+_{\overline{N_1}} \cap M^-_{K_3}$. By the same argument, we find a region N_2 with compact closure such that $N_2 \subseteq M^+_{\overline{N_1}} \cap M^-_{K_3}$ and $K_2 \subseteq N_2$.

Finally, let us look at $K_3 \subseteq \overline{N_1}^{\perp_M} \cap M^+_{\overline{N_2}}$. By the same argument we get a region $\tilde{N}_3$ that is fully contained in $\overline{N_1}^{\perp_M} \cap M^+_{\overline{N_2}}$ and that contains K_3. Since K_3 is connected, it is contained in a connected component of $\tilde{N}_3$, which we call N_3, which is also a region. Then N_1, N_2, N_3 fulfill the desired properties. $\square$

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