Definable Sets in Finite Structures

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Abstract

This Thesis is primarily motivated by a conjecture of Anscombe, Macpherson, Steinhorn and Wolf [2]. The conjecture states that, for a homogeneous structure M over a finite relational language, M is elementarily equivalent to the ultraproduct of a 'multidimensional exact class' if and only if M is stable. The right to left statement has already been verified, and so our focus is on the left to right. In this thesis, we confirm the conjecture for certain unstable homogeneous structures such as the universal metrically homogeneous graph of diameter k, the universal homogeneous two-graph and various others, such as the 28 'semi-free' edge-coloured homogeneous graphs described by Cherlin in the appendix of [16]. We also provide some mechanisms for answering the question for other unstable structures.

The core of this thesis is about finite 'n-regular' 3-edge-coloured graphs. For any given n, a classification of sufficiently large n-regular 3-edge-coloured graphs is expected to yield a proof of the 'm.e.c' conjecture in the case of the universal homogeneous 3-coloured graph, and indeed, our results yield some further special cases of the 'm.e.c' conjecture. The main focus is on finite '3-regular' 3-coloured graphs. We classify such structures under certain conditions: when they possess a 'complete neighbourhood', when they are 'monochromatic-triangle-free' and if we increase to '4-regularity' we can classify the imprimitive case as well. In the other scenarios, we employ methods from the theory of association schemes, together with linear algebra, to give a description of the eigenvalues and/or eigenvectors of the neighbourhoods with respect to a base point. We also describe the two known primitive examples of such graphs and prove they are actually homogeneous, which implies n-regularity for each n.

To Rosemary

Contents

Chapter 1

Introduction

This thesis is really the coming together of two different mathematical tales. On the one side we have the notion of *Asymptotic Classes* within the burgeoning topic of pseudofinite model theory, and on the other the classical study of symmetry and regularity conditions in combinatorics. As such, I have written this thesis with the intent that it is accessible to an interested party from either field. You must therefore forgive the author if some of it feels patronisingly basic. It's my aim to include every definition you will need in the Prerequisites 2. However I understand my limitations and so if you want to look to my betters for help, my personal favourite references for background are; General Model Theory: [36], Asymptotic classes: [2], Strongly Regular Graphs:[10], and Association Schemes: [23].

This introduction has some grand aims. Its purpose is to set up the thesis in such a manner as to make you interested in its content, as well as outlining the tale of how this thesis came to be.

1.1 History and Motivation

As previously mentioned, this thesis is borne of the meeting of two different fields of study, and so we will outline both in turn. The over-arching motivations for this thesis perhaps stem more from the model theoretic notions, and certainly that was the initial emphasis. As such, it seems only prudent to start our story there.

As is commonly known, the work of Zoé Chatzidakis, Lou van den Dries and Angus

MacIntyre in 1992 discovered the following Theorem:

Theorem 1.1.1 (Main Theorem, [15]) *Let* $\varphi(\bar{x},\bar{y})$ *be a formula in the language of rings* $\mathcal{L}_{\text{ring}} := \{0,1,+,-,\cdot\}$*, where* $n := l(\bar{x})$ *and* $m := l(\bar{y})$*. Then there exist a constant* $C \in \mathbb{R}^+$ *and a finite set* D *of pairs* $(d,\mu) \in \{0,\ldots,n\} \times \mathbb{Q}^+$ *such that for every finite field* $\mathbb{F}_q$ *and for every* $\bar{a} \in \mathbb{F}_q^m$ *if* $\varphi(\mathbb{F}_q^n,\bar{a}) \neq \emptyset$*, then*

$$\Big||\varphi(\mathbb{F}_q^n,\bar{a})| - \mu q^d\Big| \leq C q^{d-\frac{1}{2}} \tag{1.1.1}$$

for some pair $(d,\mu) \in D$*. Moreover, the parameters are definable; that is, for each* $(d,\mu) \in D$ *there exists an* $\mathcal{L}_{\text{ring}}$*-formula* $\varphi_{(d,\mu)}(\bar{y})$ *such that for every* $\mathbb{F}_q$*,* $\mathbb{F}_q \models \varphi_{(d,\mu)}(\bar{a})$ *if and only if* $\bar{a}$ *satisfies 1.1.1 for* (d,μ)*.*

It is really from this that the notion of an asymptotic class is derived. Indeed, from Macpherson and Steinhorn [35], we can think of a *1-dimensional asymptotic class* as a class of finite structures in some language that satisfies the conclusion of this theorem (with some small modification). Generalising even further, Richard Elwes [21] then develops the idea of an *asymptotic class*, which is in general just a higher (but still finite) dimension version of the same concept.

The idea is then later generalised once more in [2] into the notion of a *multidimensional asymptotic class* of finite structures (Definition 2.5.2). Here we no longer worry about the form of the functions giving approximate cardinalities, and allow different parts of a structure to vary independently, whilst keeping that any uniformly definable family of definable sets has (across the class of structures) a fixed number of (approximate) sizes. We then also have the sister concept of a *multidimensional exact class* or *m.e.c* (of finite structures), which drops the word ‘approximate’, so instead any uniformly definable family of definable sets has a fixed number of exact sizes. It is this idea that is most relevant to this thesis.

We roughly say that an infinite structure is a m.e.c limit if there exists a m.e.c which can roughly approximate it. More formally there will exist an ultraproduct of the m.e.c that is elementarily equivalent to the structure. Asking questions like ’does there exist a m.e.c limit for the Random Graph?’ inherently forces us to consider the finite graphs that would make up such a m.e.c. These questions therefore necessitate us to delve into the study of finite structures.

In particular, we shall be asking these questions of homogeneous structures.

Definition 1.1.2 (Page 10, [11])**.** Let M be a countable $\mathcal{L}$-structure. We say M is *homogeneous* if any isomorphism between finite induced substructures of M can be extended to an automorphism of M.

The motivating question for this thesis was the following conjecture:

Conjecture ([2] Conjecture 4.1.4) *Let M be a homogeneous structure over a finite relational language L. Then there is an m.e.c with ultraproduct elementarily equivalent to M if and only if M is stable.*

The backwards direction was confirmed by Daniel Wolf in [49], and so just the forwards direction remains. In [2] the authors tackle a few particular cases, the unstable homogeneous graphs, certain tournament-free digraphs and the random bipartite graph. The latter has a particularly interesting proof, drawing on results from [25] to show that any sufficiently large member of such a m.e.c. is a perfect matching or complete. We employ similar techniques in this thesis when we discuss the 'imprimitive' cases in Chapter 6.

Another crucial idea in this thesis is how the larger finite structures in a m.e.c. approximating a homogeneous structure will have a high degree of combinatorial regularity. In particular, it is shown in [2] that sufficiently large members of any m.e.c approximating the Random Graph must satisfy strict combintorial regularity conditions, namely they must satisfy n-regularity:

Definition 1.1.3. A finite structure M in a finite relational language $\mathcal{L}$ is said to be n*-regular* if for any n-tuple $\bar{y}$ and formula $\phi(x, \bar{y})$ the size of the set $\phi(M, \bar{y}) = \{x \in M : M \models \phi(x, \bar{y})\}$ is determined only by the isomorphism type of $\bar{y}$.

It is this discovery that propagated into the second half of the thesis. The authors of [2] use the classification of 5-regular graphs [9] to show that such highly regular graphs do not satisfy the properties required for a m.e.c with ultraproduct elementarily equivalent to the Random Graph. The next obvious question is, what about structures that are very similar but with slight adjustments? The initial candidates that were discussed were the universal homogeneous digraph, the universal homogeneous RGB graph and the universal homogeneous 3-hypergraph.

Due to the similarity of the material to [12], the first structure which I opted to tackle was the RGB. More formally the definitions we need are the following

Definition 1.1.4. A graph, G, is *3-edge-coloured* or, in this thesis, simply *3-coloured*, if there exist three binary, symmetric, irreflexive relations (colours), such that every unordered pair of vertices in G satisfies exactly one of these relations.
The universal homogeneous RGB graph is the Fräissé limit (Definition 2.4.14) of the class of all finite 3-coloured graphs.

We can think of this as the countably infinite homogeneous 3-coloured graph that embeds all finite 3-coloured graphs. It quickly became apparent that adapting the methods of [12] (in conjunction with [8]) to the 3-coloured case could only take me so far. That being said, the set up and the ideas that stemmed from the attempt seemed to me to be quite promising. Considering the interaction of the neighbourhoods of a base point (as was done in [12]) led to different cases. Each case came with its own host of conditions that had to be satisfied, some of which appeared to me to be impossibly limiting. It also seemed to me that, on the way to ruling out the exact class case, I may as well attempt a full classification of the finite 3-regular 3-coloured graphs. And so the direction of the thesis shifted towards this goal.

We can roughly think of a graph as a network of points with some edges between them. The graphs of particular interest to this thesis of those with high levels of combinatorial regularity. n-regularity has a natural application to graphs, and in particular 1 and 2-regular graphs have been of great interest to mathematicians. In the literature, a 1-regular graph is generally referred to as a *regular graph* and a 2-regular graph as a *strongly regular graph*.

Strong regularity is a fairly limiting condition, however nowhere near enough to consider a classification of such graphs a likely proposition (at least right now). That being said, when the regularity of the graph is witnessed by its automorphism group, we can actually say a lot. For example, we say a transitive permutation group is *rank 3* if the stabiliser of any point has exactly 3 orbits (Definition 2.3.4). Now by [27], rank 3 permutation groups of even order produce strongly regular graphs. By the work of Liebeck [32] and Liebeck and Saxl [33], the finite rank 3 permutation groups were classified, and so this particular class of strongly regular graphs is completely understood.

Rank three permutation groups actually have implications for graphs with higher levels of combinatorial regularity as well. The work of Smith in [43] gives a description of a particular class of rank 3 permutation groups, which are then shown by Cameron, Goethals and Seidel [Theorem 6.5, [12]] to potentially correspond to graphs that are 'almost 3-regular' (the neighbourhoods of a particular base point are strongly regular). Cameron, Goethals and Seidel further determine that these 'almost 3-regular' graphs are either Smith Graphs (as described in [43]), graphs of pseudo or negative Latin square type, or the Pentagon. The corresponding group theoretic classification of finite 3-homogeneous (a weakening of homogeneity but only for isomorphisms on substructures of size at most 3) graphs was then given by Cameron and Macpherson in Theorem 1.1 in [13], showcasing once again just how useful it is to have the group around.

The story is then picked up with one of Cameron's doctoral students, Buczak. In their thesis [8], Buczak provides a classification of 4-regular graphs, which is then used by Cameron again in [9] to classify 5-regular graphs (and n-regular for $n > 5$ too). We seek to emulate these efforts with 3-regular, 3-coloured graphs.

To do this it will be best to both make use of similar methods and also recent advances in the related field of association schemes. An *association scheme* can be thought of a set of commutative square 01-matrices that sum to give the all 1 matrix and are closed under transposition (Definition 2.2.1). The adjacency matrices of strongly regular graphs, together with the identity matrix, will form an association scheme for instance. In a similar fashion, these schemes can also be used to represent n-coloured graphs that are 2-regular. In 1999, Van Dam looks at symmetric 3-class association schemes (which correspond to 3-coloured 2-regular graphs) in his paper [47]. This work, especially his study of amorphic cases (Definition 4.3.1) was very influential and helpful.

1.2 Literature Review

The current literature surrounding the topics for this thesis is mainly split into three categories. We have the recent work of model theorists on pseudofinite structures, the older papers on combinatorial regularity, and then the more recent work on regular structures and three-class association schemes.

For pseudofinite structures, and in particular the study of m.a.c.s and m.e.c.s, the current state of affairs is best expressed in the manuscript of Anscombe, Macpherson, Steinhorn and Wolf [2].

For work on combinatorial regularity in finite structures there have been some recent movements, but a lot of the crucial work is much older. The work by Delsarte, Goethals and Seidel [20] sets up the notion of spherical 2-distance sets and discusses its consequences. We then get [12] which applies these notions to the 'almost' 3-regular case, followed by the work of Buczak [8] on 4-regular graphs, and Cameron again on 5-regular and 6-transitive graphs in [9].

For more recent work on three-class association schemes and probably most closely related to this thesis is the work of Van Dam [47]. Here he provides stipulations for when a three-class association scheme can exist, and lists all potential examples with under 100 vertices. Other work is more tangential, Jaeger [30] completes work on triply regular association schemes (not necessarily three class) in the context of spin models. Here he gives necessary and sufficient conditions for the existence of triply-regular Bose-Mesner Algebras in the language of spin models. This is followed up by Suda [44] who gives a sufficient condition for a triply regular association schemes, using tight spherical designs. He even shows that every tight $4, 5$ or 7 design gives a triply regular association scheme. Suda also mentions the concept of *real mutually unbiased bases* and *linked systems of symmetric designs*, and shows that these also carry a triply regular association scheme (given certain conditions for the linked system of symmetric designs).

In a very similar area, although using different methods, there has recently been a classification of finite highly regular vertex-coloured graphs by Heinrich, Schneider, and Schweitzer in [25]. The motivations for this paper come from the Weisfeiler-

Leman algorithm [48], a powerful tool for graph isomorphism and automorphism group computation. The interest in this algorithm comes from Babai's very important 'Graph isomorphism in quasi-polynomial time' [3]. The methods in this paper utilise an interplay between local and global symmetry of a graph, and the general principle that a graph with high amount of combinatorial regularity will often have a high degree of symmetry too. It makes sense that combinatorial regularity will play a large part in future work in this field.

1.3 Outline of Thesis

This thesis is very large and I apologise profusely for this. It (hopefully) makes up for the length with some interesting results however.
In Chapter 2 I have put most of the set up, the main introductions to the different fields of study, technical definitions and rudimentary theorems. There is also a breakdown of some of the more influential papers and the theory they introduce.

Chapter 3 is where I have deposited the work I have done on Multidimensional Exact classes. The main result is Theorem 3.2.4, which roughly states that if you can identify a homogeneous structure N with no m.e.c limit within another homogeneous structure M, then M has no m.e.c. limit either. This is used to prove the following theorem:

Theorem 1.3.1 *There does not exist a m.e.c with ultraproduct elementarily equivalent to any of the following structures:*

1. *The universal metrically homogeneous graph of diameter k for any k (Theorem 3.2.7),*
2. *The universal homogeneous two-graph (Theorem 3.2.11),*
3. *Any other unstable reduct of the random graph (Theorem 3.3.9),*
4. *The universal homogeneous n-tournament-free digraph (Theorem 3.4.5),*
5. *The primitive universal homogeneous semi-free 3-edge-coloured graph determined by forbidden triangles (Theorem 3.4.7 and Theorem 11.5.4),*
6. *Any of the known primitive universal homogeneous semi-free, but not free, 4-edge-coloured graphs determined by forbidden triangles (Theorem 3.4.9).*

7. *Any homogeneous unstable imprimitive 3-coloured graph (Theorem 6.1.8)*

After this we move on to work on specifically finite 3-coloured graphs and high levels of regularity. The aim here is to provide results that work towards showing that the universal 3-coloured random graph is not a m.e.c limit. The basic notations and definitions are given in Chapter 4, some examples are given in Chapter 5 and some results on the imprimitive case in 6. This culminates in Theorem 6.1.4, which describes the possible imprimitive examples we might have, and then Theorem 6.1.8, which tells us that they can't form a m.e.c.

Next in Chapter 7, we set the foundations for the mechanisms we will use throughout the thesis. This primarily involves applying linear algebraic and combinatorial results to the multiple scenarios that arise, and discussing their feasibility. The key result from this chapter is Theorem 7.6.20 which rules out certain triangle-free possibilities. However there is a host of very powerful lemmas too.

Chapter 8 is an attempt to generalise the work of [12] to the 3-coloured scenario. Due to the added complexity the extra colour brings this is difficult to do, however we can still get results in certain situations. Theorem 8.2.22 in particular provides very strong information, which we then use in Chapter 9.

In Chapter 9, in particular Theorem 9.3.3, we entirely classify the primitive case in which we have a complete neighbourhood, by showing it can only be the Tricolour Heptagon (Definition 5.2.1). Similar work is done in Chapter 10, where we continue looking at some triangle-free cases. We remove the possibility of any three-coloured triangle being omitted in Theorem 10.1.1, entirely describe the case when we have a single two-coloured triangle omitted in Theorem 10.2.13 and provide a full classification when we have all three monochromatic triangles omitted in Theorem 10.3.1.

The final chapter, Chapter 11 works to provide knowledge about the cases we have left, generally with all triangles present and various other limiting conditions. This culminates in a description of all the eigenvectors of every possible case we could have in Theorem 11.4.2. Some of the implications of this work with respect to m.e.cs are discussed in Section 11.5, and this culminates in Theorem 11.5.4 where we show any unstable universal semi-free 3-edge-coloured graph determined by forbidden triangles can't be 'approximated' by a m.e.c.

You'll be glad to know it doesn't end there, we also have an Appendix. This includes the proof of the uniqueness of the primitive universal semi-free 3-edge-coloured graph. This is relevant to Section 3.4 and is unpublished work I completed as part of a research project before my PhD (although it had to undergo extreme editing).

Chapter 2

Prerequisites

2.1 Graph Theory

This section is based of definitions and basic results from [7] about graphs.

Definition 2.1.1. A *graph* is a set V of vertices equipped with a binary relation E. It is said to be *simple* if E is symmetric and irreflexive.

All the graphs in this thesis will be considered to be simple graphs unless stated otherwise. We also look at variations of graphs with differing languages.

Definition 2.1.2. A *digraph* is a graph where E is anti-symmetric and irreflexive. An n-*coloured graph* is a graph but equipped with n symmetric irreflexive binary relations that partition the set of ordered pairs of distinct elements of V.

Note that in this thesis, we shall always use the term coloured graph to refer to edge-colourings, not vertex-colourings.
We see that a 2-coloured graph is the same thing as a graph, with non-edges being replaced with a colour.

Definition 2.1.3. An m-*hypergraph* is a set of vertices equipped instead with an m-ary symmetric, irreflexive, edge relation E_m, so the edges are m-sets. For any 3-tuple (x, y, z), symmetric here will mean that if (x, y, z) is an edge, then so is any permutation of x, y and z, and irreflexive means that if x, y and z are not distinct then (x, y, z) is not an edge.

We will now define some of the more common terminology I use throughout the thesis.

Definition 2.1.4. A *neighbour* of a vertex x in a graph Γ, is any other vertex adjacent to x in Γ. A *neighbourhood* of a vertex, x, is the induced subgraph on all the neighbours of x.

We will also refer to neighbourhoods defined via colours or directions.

Definition 2.1.5. A graph is *regular* if every vertex is connected to the same number, k, of other vertices, the number being known as the *degree* of the graph. A graph is *strongly regular* if it is regular and the number of common neighbours of any two distinct vertices is entirely determined by whether they are adjacent or not.

In a strongly regular graph, the number of common neighbours of two adjacent vertices is classically referred to as λ and the number of common neighbours of two non-adjacent vertices is referred to as μ.

Lemma 2.1.6 *The complement of a strongly regular graph is also strongly regular.*

A strongly regular graph together with its complement are known as *complementary* strongly regular graphs. We can extend the notion of strong regularity further.

Definition 2.1.7. A graph is n*-regular* if for any subset X of the vertex set, such that $|X|\leq n$, the number of vertices adjacent to every $x \in X$ is a fixed number determined only by the isomorphism type of the induced subgraph on X.

Although this may seem slightly different from the general definition 1.1.3, they work out as the same by inclusion-exclusion principle. It follows that the definition of a strongly regular graph and a 2-regular graph are the same. We can note the lemma.

Lemma 2.1.8 *If a graph is* n*-regular then the neighbourhood and non-neighbourhood of each vertex is* $n-1$*-regular.*

Graphs also can be looked at in terms of linear algebra.

Definition 2.1.9. Enumerate the vertices of a graph Γ with vertex set of size n. Then the *adjacency matrix*, A, of a graph is the $n \times n$ matrix, where if i and j are vertices, then $(A)_{ij} = 1$ if $(i, j) \in E$ and 0 otherwise.

Similarly in an m-coloured graph Γ_m, for an edge relation E_l, the Adjacency matrix A_l of

l in Γ_m, is the $n \times n$ matrix, where if i and j are vertices, then $(A_l)_{ij} = 1$ if $(i, j) \in E_l$ and 0 otherwise.
For digraphs it is a little different. The adjacency matrix, A, of a digraph is the $n \times n$ matrix, where if i and j are vertices, then $(A)_{ij} = 1$ if $(i, j) \in E_+$, $(A)_{ij} = -1$ if $(i, j) \in E_-$ and 0 otherwise.

The condition of strongly regular imposes quite a few conditions on the adjacency matrix. We get that

$$A^2 = kI + \lambda A + \mu(\bar{A})$$

where $\bar{A} = J - A - I$, with J being the all ones matrix.
From [Page 1, [7]] we can note that the all-ones vector u is an eigenvector of A if and only if A represents a regular graph. u will then have eigenvalue k in A (where k is the degree of the graph) and is known as the *principal* eigenvalue.

The other eigenvalues are known as the *non-principal* eigenvalues. In a strongly regular graph with adjacency matrix A, A has two distinct non-principal eigenvalues, r and s, with multiplicities f and g respectively.

Theorem 2.1.10 ([14], Theorem 2.16) *Suppose that G is a strongly regular graph with parameters (n, k, λ, μ). Then it has 2 non-principal eigenvalues r, s with respective multiplicities f, g as follows:*

$$r, s = \frac{1}{2}(\lambda - \mu \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}$$
$$f, g = \frac{1}{2}(v - 1 \pm \frac{(n-1)(\mu - \lambda) - 2k}{\sqrt{(\mu - \lambda)^2 + 4(k - \mu)}})$$

Further f and g must be non-negative integers.

Note that the values of f and g must be non-negative integers. Note that a *trivial* equivalence relation is one where everything is considered equivalent, and a *proper* equivalence relation is one where there exists at lease one equivalence class with size greater than one (i.e. any equivalence relation that isn't just equality).

Definition 2.1.11. A graph, Γ, is called *imprimitive* if either $E\cup =$ (the union of E with equality, denoted $E^=$) or $(\neg E)\cup =$ form a proper nontrivial equivalence relation. It is called *primitive* otherwise.

Another important concept is that of n-extension.

Definition 2.1.12. A graph satisfies the *n-extension axiom* if for any two disjoint sets of size n, say X and Y, there exists a vertex connected to every vertex in X and none in Y.

An example of a graphs which satisfy this axiom for large n would be (sufficiently large relative to n) Paley graphs. These are the graphs defined via the following process: Let p be a prime power such that $p = 1 \bmod 4$. Then let $V = \{1, \ldots, p\}$ and say i is connected to j if and only if $(i - j)$ is a square in the finite field $\mathbb{F}_p$. This forms a strongly regular graph which when we take sufficiently large p (relative to n) satisfies n-extension [5].
Finally, I will define two model theoretic terms in the context of graphs.

Definition 2.1.13. A graph is *homogeneous* if it is a finite or countable graph such that every isomorphism between induced finite subgraphs can be extended to an automorphism. A graph is called *universal* for a family of graphs $\mathcal{F}$, if it contains every graph in $\mathcal{F}$ as an induced subgraph.

An example of this is the Random Graph. The Random Graph is defined to be the countable graph that satisfies the n-extension axiom for all n. It is both homogeneous and universal in that it embeds all countable graphs. Both of these concepts can be extended to include other type of graphs, for instance digraphs or coloured graphs.

We will also need some simple results like the following.

Lemma 2.1.14 *[[7], Section 1.1.3]*

1. *A strongly regular graph is the union of identical disconnected complete graphs if and only if one of its eigenvalues is* -1.
2. *A primitive strongly regular graph with eigenvalue* -1 *is complete.*

Proof. Suppose the strongly regular graph has parameters (n, k, λ, μ). Then we know by Theorem 2.1.10 that the eigenvalues are

$$\frac{1}{2}\left((\lambda - \mu) \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}\right)$$

If we let one of these equal -1 then

$$\begin{aligned}(-2-(\lambda-\mu))^2 &= (\lambda-\mu)^2+4(k-\mu)\\ 4+(\lambda-\mu)^2+4(\lambda-\mu) &= (\lambda-\mu)^2+4(k-\mu)\\ 1+\lambda &= k\end{aligned}$$

which implies that the graph is the disjoint union of complete graphs.
If we start with a union of identical disconnected complete graphs then we know $k=\lambda+1$ and $\mu=0$. So

$$\begin{aligned}r,\ s &= \frac{1}{2}\left(\lambda\pm\sqrt{\lambda^2+4\lambda+4}\right)\\ &= \frac{1}{2}\left(\lambda\pm\lambda+2\right)\end{aligned}$$

So $r=-1$ and $s=k$.
This proves part i), and part ii) follows immediately. □

Hence if a strongly regular graph is primitive and has eigenvalue -1 it must be complete. It is also known that we can determine via the multiplicity of the principal eigenvalue the number of connected components of a regular graph.

Lemma 2.1.15 *[[7], Section 1.1] If G is a regular graph of degree k, then the multiplicity of the eigenvalue k is the number of connected components of G.*

Hence in a complete graph, -1 will have multiplicity $n-1$, and any regular graph where -1 has multiplicity $n-1$ is complete.

Corollary 2.1.16 *If G is a regular graph with degree k, then if the multiplicity of the eigenvalue k is greater than one, it is imprimitive.*

This just follows from the fact that each connected component will form an equivalence class of a non-trivial equivalence relation on the structure.

Lemma 2.1.17 *Suppose the adjacency matrix A of a connected regular graph G has only one eigenvalue, r, aside from k. Then $r=-1$.*

Proof. G is connected and so k has multiplicity 1. Therefore r has multiplicity $n-1$. As trace$(A) = 0$, we know $0 = k + (n-1)r$, implying $r = \frac{-k}{n-1}$. We know r is an algebraic integer as it is the eigenvalue of a 01-matrix, hence $n-1|k$. But $n-1 \geq k$, and therefore $n-1 = k$, and $r = -1$. $\square$

2.2 Association Schemes

The definition of an association scheme is very closely related to that of a strongly regular graph. A lot of this information will be covered in more detail later, as the language of association schemes is the primary one I use when dealing with 3-regular 3-coloured structures, and hence only the basics will be mentioned here. The reference I use is [23].

Definition 2.2.1. An *association scheme* with d classes is a set $\mathcal{A} = \{A_0, \ldots, A_d\}$ of $n \times n$, 01-matrices, for some n, such that:

i) $A_0 = I$.

ii) $\sum_{i=0}^{d} A_i = J$.

iii) $A_i^T \in \mathcal{A}$ for each i.

iv) $A_i A_j = A_j A_i \in \text{span}(\mathcal{A})$

We say that an association scheme is *symmetric* if each of its component matrices are symmetric.

There is a natural translation into strongly regular graphs. A strongly regular graph will form a symmetric association scheme with 2 classes, one being the adjacency matrix A, and the other being $\bar{A}$.

Attached to each association scheme is an algebra known as the *Bose-Mesner algebra*. This is the algebra generated by the matrices in $\mathcal{A}$. Call this algebra $\mathbb{C}[\mathcal{A}]$. The matrices $A_0, \ldots, A_d$ form a basis of this algebra. However we can find a more convenient one. To do this define a partial ordering on idempotent matrices in $\mathbb{C}[\mathcal{A}]$ via $E \leq F$ if $FE = E$. We can create a basis of the minimal idempotents via the result

Theorem 2.2.2 *[23, Theorem 1.5.1] Suppose B is a Bose-Mesner Algebra of an association scheme $\mathcal{A}$. Then B has a basis of matrix idempotents $\{E_0, \ldots, E_d\}$ such that:*

(a) $E_i E_j = \delta_{i,j} E_i$

(b) The columns of E_i are eigenvectors for each matrix in $\mathbb{C}[\mathcal{A}]$

(c) $\sum_{i=0}^{d} E_i = I$

(d) $E_i^* = E_i$

Hence $\{E_0, \ldots, E_d\}$ is the basis of minimal idempotents of our association scheme.

In an association scheme, there exist constants called *intersection numbers*. These are defined to be the p_{jl}^m such that

$$A_j A_l = \sum_{m=0}^{d} p_{jl}^m A_m. \tag{2.2.1}$$

For instance in terms of strongly regular graphs, $d = 2$ and $p_{11}^1 = \lambda$ and $p_{11}^2 = \mu$. We then note that, if $\circ$ is the Hadamard product (entrywise multiplication), then

$$p_{jl}^m A_m = A_m \circ (A_j A_l)$$

We can also define *the eigenvalues of the scheme* as the eigenvalues of the A_i. As these are 01-matrices, Godsil notes [[23], Section 2.1] that the eigenvalues must be algebraic integers.

There also exists a natural dual concept to the intersection numbers, using the idempotent basis of the Bose-Mesner algebra. The *Krein parameters* of the association scheme are the constants q_{jl}^m defined via

$$E_j \circ E_l = \frac{1}{n} \sum_{m=0}^{d} q_{jl}^m E_m. \tag{2.2.2}$$

Alternatively these can represented as follows

$$q_{jl}^m E_m = n E_m (E_j \circ E_l)$$

By bringing in a little linear algebra we can make the eigenvectors much easier to deal with.

Definition 2.2.3. Two matrices A and B are *simultaneously diagonalisable* if there exists a matrix P such that for some two diagonal matrices D_1 and D_2, $P^{-1}AP = D_1$ and $P^{-1}BP = D_2$.

We can see that simultaneously diagonalisable matrices must share a basis of eigenvectors, as these make up the columns of the matrix P.

From [29] we have

Lemma 2.2.4 *Diagonalisable matrices commute if and only if they are simultaneously diagonalisable.*

Therefore by the definition of an association scheme we know that all the constituent matrices are simultaneously diagonalisable, and therefore there exists a basis of eigenvectors common to all of them.

We will now introduce the necessary combinatorial and linear algebraic notions needed to make sense of [12].

Definition 2.2.5 ([29])**.** Let $v_1, \ldots, v_m$ be vectors in an inner product space V with inner product $\langle \cdot, \cdot \rangle$. The *Gram matrix* of the vectors $v_1, \ldots, v_m$ with respect to the inner product $\langle \cdot, \cdot \rangle$ is $G = [\langle v_i, v_j \rangle]_{i,j=1}^{m}$.

Each eigenspace of a graph will form an inner product space, and so we can naturally form a corresponding Gram matrix.

Definition 2.2.6 ([38] and [39])**.** A finite set X in $\mathbb{R}^d$ is called an *s-distance set* if the set of Euclidean distances between any two distinct points of X has size s. If an s-distance set lies in the unit sphere $\mathbf{S}^{d-1}$ then it is known as a *spherical* s-distance set.

In other words, S is a spherical s-distance set if it is a set of unit vectors, and there are s real numbers $\{a_1, \ldots, a_s\}$, with $-1 \leq a_i < 1$ for all i, such that the inner products of distinct vectors of S are one of the a_i.

As we shall see, spherical 2-distance sets have a very natural application for strongly regular graphs. There will exist one distance for adjacent pairs of vertices, and one for non-adjacent pairs.

2.3 Highly Regular Graphs

Because of the large influence on the ideas of this thesis, this subsection is dedicated to the story of n-regular graphs through [12], [9] and [8]. Together, these works provide a classification of all n-regular graphs for $n \geq 4$ and strong information in the case of $n = 3$. This is immensely useful, as demonstrated later, and therefore producing equivalent results for more complex languages would be of great worth too.

Throughout [12] the authors mainly work in the language of association schemes above. However they refer to Γ and Δ as the complementary pair of strongly regular graphs on a vertex set X of cardinality n, with adjacency matrices A and B respectively. For a vertex $x \in X$, let $\Gamma(x)$ and $\Delta(x)$ refer to the sets of vertices in X adjacent to x in Γ and Δ respectively, i.e. the subconstituents. Their aim is to provide a classification of graphs that are very close to 3-regular, where for some $x \in X$, $\Gamma, \Delta, \Gamma(x)$ and $\Delta(x)$ are strongly regular. These graphs are effectively 3-regular with respect to a single point x.

A basic result which they make great use of is the following.

Theorem 2.3.1 ([12], Theorem 2.2) *A strongly regular graph having $k = a - 1,\ n = ma$ is a disjoint union of m complete graphs of size a.*

They then introduce a 'special basis', such that it includes certain vectors involving projections onto the eigenspace for A. The process of creating this basis we mirror in Chapter 8, but with 3-colours. Then using this they find a series of identities involving the transition matrices between the two bases of the Bose-Mesner algebra and the eigenvalues. Next they examine the Krein parameters and quickly show that they must be between 0 and 1 in the case of a strongly regular graph.

They look quite heavily at the neighbourhoods. Let A_1, A_2 be the adjacency matrices of $\Gamma(x), \Delta(x)$ respectively, where these are the neighbourhood and non-neighbourhood of a vertex x.

Theorem 2.3.2 ([12], Theorem 5.1) *Let Γ be a strongly regular graph with eigenvalues k, r, s. Suppose λ is an eigenvalue of $\Gamma(x)$, then, if $\lambda \notin \{r, s\}$, there exists a corresponding eigenvalue, $\mu = r + s - \lambda$, of $\Delta(x)$ such that γ and μ have the same eigenspace.*

The next section is on Smith Graphs, a very important type of graph for the study of n-

regular graphs. The main result regarding them is from [43] but is mentioned in this paper as well. We need to quickly introduce the concept of n-transitive and rank 3 permutation groups.

Definition 2.3.3. A connected simple graph Γ is n-transitive if, for any two ordered n-tuples $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of vertices satisfying $d(x_i, x_j) = d(y_i, y_j)$ for all i (d the distance function of Γ), there is an automorphism of Γ which maps each x_i to y_i.

Definition 2.3.4. For any $k \geq 2$, a transitive group of permutations of Ω, G, is *rank* k if, for $\alpha \in \Omega$, G_α has exactly k orbits.

Let l be the size of $\Delta(x)$ and recall that λ and μ are the classical parameters of the strongly regular graph (Definition 2.1.5). The authors note the following is a reformulation of Theorems E and F from [43].

Theorem 2.3.5 *[12, Theorem 6.1] Let G be a primitive rank 3 permutation group on a finite set X in which, for $x \in X$, the stabilizer G_x with orbits $\{x\}$, $\Gamma(x)$, $\Delta(x)$, has rank 3 or less on both $\Gamma(x)$ and $\Delta(x)$. Assume that $\{k, l\} \neq \{f, g\}$. Then, without loss of generality, the parameters of the graphs Γ and Δ are:*

$$
\begin{aligned}
n &= \frac{2(r-s)^2((2r+1)(r-s)-3r(r+1))}{(r-s)^2-r^2(r+1)^2},\\
k &= \frac{-s((2r+1)(r-s)-r(r+1))}{(r-s)+r(r+1)},\\
l &= \frac{-(s+1)((2r+1)(r-s)-r(r+1))}{(r-s)-r(r+1)},\\
\lambda &= \frac{-r(s+1)((r-s)-r(r+3))}{(r-s)+r(r+1)},\\
\mu &= \frac{-(r+1)s((r-s)-r(r+1))}{(r-s)+r(r+1)},
\end{aligned}
$$

where $r - s \geq r(r+3)$. Here, 'without loss of generality' means that $k > r > s$ denote the eigenvalues of the graph defined by either Γ or Δ.

Any graph with the above parameters, for integer r and s, is known as a *Smith Graph*.

The authors then work out that there are 5 possibilities for the non-principal eigenvalues in their case and manage to classify them in the following theorem, the main result of the paper:

Theorem 2.3.6 *[12, Theorem 6.5] Let* Γ, Δ *be a complementary pair of connected strongly regular graphs on* X*, and suppose there is a vertex* $x \in X$ *for which the subconstituents on* $\Gamma(x)$ *and* $\Delta(x)$ *are both strongly regular. Then one of the following occurs:*

i) Γ *is a pentagon,*

ii) Γ *is of pseudo or negative Latin square type,*

iii) Γ *or* Δ *is a Smith Graph.*

We call a strongly regular graph of *negative Latin square type* if for some r

$$(n, k, \lambda, \mu) = (v^2, r(v+1), -v + r^2 + 3r, r(r+1))$$

and of *pseudo Latin square type* if

$$(n, k, \lambda, \mu) = (v^2, r(v-1), v + r^2 - 3r, r(r-1))$$

We can note that any 3-regular graph must be strongly regular with strongly regular subconstituents and so will be included in the list of Theorem 2.3.6 as well. Recall the Krein parameters q_{jl}^m defined via equation 2.2.2. A final result from this paper that could be useful later is the theorem:

Theorem 2.3.7 *Let* Γ, Δ *be a complementary pair of connected strongly regular graphs. Then* $q_{ii}^i = 0$ *holds for some* $i \in \{1, 2\}$ *if and only if either* Γ *is a pentagon or* Γ *or* Δ *is a Smith Graph.*

Now we move on to Cameron's '6-transitive graphs' [9]. This paper mainly focuses on transitivity in graphs, however the author notes that the main theorem can also be used as a classification of 5-regular graphs when coupled with the results of Theorem 2.3.6.

Cameron starts by noting that a 2-transitive graph of diameter 2 is strongly regular from [40] and then mentions a reformulation of the Theorem 2.3.5.

Theorem 2.3.8 *Let* Γ *be a 3-transitive graph of diameter 2. The one of the following holds:*

i) Γ *is a complete multipartite graph,* *ii)* Γ *is of pseudo or negative Latin square type,*

iii) Γ *or its complement is a Smith Graph.*

He then states the most important theorem of the paper from our perspective:

Theorem 2.3.9 *[9, Theorem 3.2] Let* Γ *be a 5-transitive graph of diameter 2. Then* Γ *is one of the following:*

i) A complete multipartite graph,

ii) a pentagon,

iii) the line graph of $K_{3,3}$.

To prove this they use the fact that if Γ is n-transitive with diameter 2 ($n \geq 3$), then each of the subgraphs $\Gamma(x)$ and $\Delta(x)$ are either a disjoint union of complete graphs or an $(n-1)$-transitive graph of diameter 2, as two points in $\Gamma(x)$ or $\Delta(x)$ are at distance at most 2. They prove this by a series of lemmas that deal with the case by case analysis, basing the arguments mainly on the eigenvalues and their limitations.
Particular results from this analysis that are useful for us are:

Lemma 2.3.10 *If a subconstituent of a Smith graph is a Smith graph, then* $-s = r^2(2r+3)$. *Conversely, if* $r > 1$ *and satisfies* $-s = r^2(2r+3)$*, then both subconstituents are Smith graphs.*

and

Lemma 2.3.11 *No Smith graph has a subconstituent which is the complement of a Smith graph.*

Although theorem 2.3.9 does not directly prove the classification of 5-regular graphs, Cameron notes at the end of the paper that, if you replace the result of Smith with that from [12], then the same argument shows that this is in fact a complete list of the 5-regular graphs as well.
Finally we discuss the thesis of J.M.J Buczak [8]. We start with the definition of a graph which is an extension of the idea that Cameron, Goethals and Seidel talk about in [12]:

Definition 2.3.12 (Definition 0.3, [8])**.** *A graph of type B3* is a strongly regular graph G with strongly regular subconstituents $\Gamma(x), \Delta(x)$ for some $x \in G$, such that for some vertex y in $\Delta(x)$, the subconstituents of $\Delta(x)$ formed by the points joined to y (denoted $\Delta_1(x, y)$) and the points not joined to y ($\Delta_2(x, y)$) are both strongly regular, and similarly, for some vertex z in $\Gamma(x)$, the subconstituents of $\Gamma(x)$ formed by the points joined to z ($\Gamma_1(x, y)$) and the points not joined to z ($\Gamma_2(x, y)$) are both strongly regular too.

This is effectively 4-regularity, but only over specific points. Buczak manages to classify these up to identifying a potential infinite family of B3 graphs indexed by a natural number n, these are the graphs he refers to as $B_3(n)$.
The main result of the paper is then the following classification of 4-regular graphs.

Theorem 2.3.13 (Section 0.5, [8]) *Any finite* 4*-regular graph must be one of the following:*

a) a disjoint union of complete graphs,

b) the pentagon,

c) the lattice graph on 9 vertices,

d) the Schläfli graph on 27 vertices,

e) the Maclaughlin graph on 275 vertices,

f) Any graph of type $B_3(n)$*, for* $n \geq 3$*, if such exists,*

g) The complements of the above.

2.4 Model Theory

Here I will outline a lot of the basic model theory that I use. Most of this section comes from [36] and [45] and is fairly fundamental. As only the faintest explanation is given here, I would advise heading to the texts for more information if you'd like it. Throughout $\mathcal{L}$ is a general language.

Definition 2.4.1. A *finite relational* language is a language which only has relation symbols and also only finitely many relational symbols.

Definition 2.4.2. A complete $\mathcal{L}$-theory has *quantifier elimination* if for every $\mathcal{L}$-formula $\phi(\bar{x})$ there exists a quantifier free formula $\psi(\bar{x})$ such that

$$T \models \forall \bar{x} \left(\phi(\bar{x}) \leftrightarrow \psi(\bar{x}) \right).$$

Definition 2.4.3. A theory, T, that has countable models is ω*-categorical* if for any two models M_1 and M_2, such that $|M_1| = |M_2| = \omega$ and $M_1, M_2 \models T$, then $M_1 \cong M_2$.

Definition 2.4.4 ([45], Definition 5.2.1)**.** Two $\mathcal{L}$-structures, A and B, are *elementarily equivalent* if they have the same theory; that is, for all $\mathcal{L}$-sentences ϕ

$$A \models \phi \Leftrightarrow B \models \phi$$

Definition 2.4.5 ([45], Definition 2.2.6)**.** Let A be a $\mathcal{L}$-structure and $B \subset A$. Then $a \in A$ *realises* a set of $\mathcal{L}(B)$-formulas $\Sigma(x)$, if a satisfies all formulas from $\Sigma(x)$. We write

$$A \models \Sigma(x)$$

We call $\Sigma(x)$ *finitely satisfiable* in A if every finite subset of $\Sigma(x)$ is realised in A.

Definition 2.4.6 ([45], Definition 2.2.8)**.** Let A be an $\mathcal{L}$-structure and B a subset of A. A set $p(x)$ of $\mathcal{L}(B)$-formulas is a *type* over B if $p(x)$ is maximal finitely satisfiable in A. Let $S(B) = S^A(B)$ denote the set of types over B.

Recall Definition 1.1.3:

Definition 2.4.7. A finite structure M in a finite relational language $\mathcal{L}$ is said to be n*-regular* if for any n-tuple $\bar{y}$ and formula $\phi(x, \bar{y})$ the size of the set $\phi(M, \bar{y}) = \{x \in M \ : \ M \models \phi(x, \bar{y})\}$ is determined only by the isomorphism type of $\bar{y}$.

In general in this Thesis we will be looking at homogeneous structures, and because of this (see proof of Lemma 2.5.5), we can note that we will exclusively be looking at n-regularity where ϕ is quantifier free.
For graphs, we will refer to 2-regularity as strongly regularity still.

Definition 2.4.8 ([45], Definition 5.2.1)**.** Let κ be an infinite cardinal. We say that a theory,

T, is *κ-stable* if in each model of T, over every set of parameters of size at most κ, and for each n, there are at most κ many n-types, i.e.

$$|A| \leq \kappa \Rightarrow |S(A)| \leq \kappa.$$

We say a theory is *unstable* if it is not κ-stable for any infinite cardinal κ.

A model is said to be κ-stable if its theory is κ-stable, likewise it is unstable if its theory is unstable.

Definition 2.4.9 ([36])**.** An *Ultrafilter* on a set I is a collection $D \subset \mathcal{P}(I)$ such that:

i) $I \in D, \varnothing \notin D$

ii) if $A, B \in D$ then $A \cap B \in D$

iii) if $A \in D$ and $A \subseteq B \subseteq I$, then $B \in D$

iv) For all $X \subseteq I$ either $X \in D$ or $I \backslash X \in D$

This basically means it provides a notion of being a 'big' subset of I. We say that an ultrafilter is *principal* if for some $i \in I$, $D = \{X \in \mathcal{P}(I) : i \in X\}$. Otherwise it is *non-principal*. It is immediate that every element in a non-principal ultrafilter will be infinite. Now the idea is to use this to create a structure, so we define

Definition 2.4.10 ([36])**.** Suppose we have a class $\mathcal{M}_i$ of $\mathcal{L}$-structures indexed by the infinite set I. Let D be an ultrafilter on I. Then we define the *ultraproduct*, $\mathcal{M} = \prod \mathcal{M}_i/D$, of the $\mathcal{M}_i$'s by defining an equivalence relation $\sim$ on

$$X = \prod_{i \in I} M_i = \left\{ f : I \to \bigcup_{i \in I} M_i : \ f(i) \in M_i \text{ for all } i \right\}$$

where $f \sim g$ if and only if $\{i : f(i) = g(i)\} \in D$.

On the set of equivalence classes we can define an $\mathcal{L}$-structure:

- If c is a constant symbol of $\mathcal{L}$, let $c^{\mathcal{M}}$ be the $\sim$ class of $f_c \in X$ where $f_c(i) = c^{\mathcal{M}_i}$ for all $i \in I$.

- Let f be an n-ary function symbol of $\mathcal{L}$ and suppose that $g_1, ..., g_n \in X$. Then

$f^{\mathcal{M}}(g_1\backslash \sim, \dots g_n\backslash \sim) = f^{\mathcal{M}_i}(g_1(1), \dots, g_n(i))\backslash \sim.$

- If R is a relation symbol in $\mathcal{L}$ then $R^{\mathcal{M}} = \{(g_1\backslash \sim, \dots g_n\backslash \sim) : \{i \in I : (g_1(i), \dots, g_n(i)) \in R^{\mathcal{M}_i}\} \in D\}$.

The next theorem gives us a good understanding of what properties the ultraproduct possesses and also what makes it so useful.

Łoś's Theorem *Let $\phi(x_1, \dots, x_n)$ be an $\mathcal{L}$-formula. Then, $\mathcal{M} \models \phi(g_1/\sim), \dots, g_n/\sim)$ if and only if $\{i \in I : \mathcal{M}_i \models \phi(g_1(i)), \dots, g_n(i))\} \in D$.*

This tells us that a formula holds in the ultraproduct if and only if it holds in 'most' of the models in the class.

We now need to describe another fundamental model theoretic construction.

Definition 2.4.11 ([11])**.** A class $\mathcal{C}$ has the *amalgamation property* if for $A, B_1, B_2 \in \mathcal{C}$ and $\phi_i : A \to B_i$ is an embedding for $i = 1, 2$, then there is a structure $C \in \mathcal{C}$ and embeddings $\psi_i : B_i \to C$ for $i = 1, 2$ so that $\psi_1\phi_1 = \psi_2\phi_2$.

Definition 2.4.12 ([11])**.** A class $\mathcal{C}$ of finite relational structures over $\mathcal{L}$ is a *Fraïssé class* over $\mathcal{L}$ is it satisfies the following four conditions:

(a) $\mathcal{C}$ is closed under isomorphism,

(b) $\mathcal{C}$ is closed under taking induced substructures (this means, take a subset of the domain, and all instances of all relations which are contained within this subset,

(c) $\mathcal{C}$ has only countably many members up to isomorphism,

(d) $\mathcal{C}$ has the amalgamation property.

Definition 2.4.13. Let M be a structure over $\mathcal{L}$. Then the *age of M* is the class of all finite $\mathcal{L}$-structures which are embeddable in M.

Fraïssé's Theorem *A class $\mathcal{C}$ is the age of a countable homogeneous $\mathcal{L}$-structure M if and only if it is a Fraïssé class. If these conditions hold then M is unique up to isomorphism.*

We then get the further definition:

Definition 2.4.14. For a countable $\mathcal{L}$-structure M and a class $\mathcal{C}$ satisfying the conclusion of Fraïssé's Theorem, we refer to M as the *Fraïssé limit* of the class $\mathcal{C}$.

There exists important strengthenings of the amalgamation property:

Definition 2.4.15. [11] A Fraïssé class $\mathcal{C}$ has the *strong amalgamation property* if whenever B_1 and B_2 are structures in $\mathcal{C}$ with a common substructure A, there is an amalgam C of B_1 and B_2 such that the intersection of B_1 and B_2 in C is precisely A.
If as well every instance of a relation in C is contained in either B_1 or B_2, then we say that $\mathcal{C}$ has the *free amalgamation property* also.

We also see that ω-categoricity is generally present when dealing with homogeneous structures.

Lemma 2.4.16 *[11, Remark p.41] A homogeneous structure M which has only finitely many isomorphism types of n-element substructures for each n is ω-categorical. In particular this implies any homogeneous structure in a finite relational language is ω-categorical.*

We can also see that

Lemma 2.4.17 *[11, 2.22] Suppose M is a countable and ω-categorical structure over a relational language. Then M is homogeneous if and only if M has quantifier elimination.*

2.5 Multidimensional Exact classes

For this section $\mathcal{C}$ will refer to a class of finite $\mathcal{L}$-structures and $(\mathcal{C}, \bar{y})$ will denote the set $\{(\mathcal{M}, \bar{a}) : \mathcal{M} \in \mathcal{C}, \bar{a} \in M^{|\bar{y}|}\}$.

Definition 2.5.1. A $\varnothing$*-definable partition* of $(\mathcal{C}, \bar{y})$ is a partition Φ of $(\mathcal{C}, \bar{y})$ into finitely many parts, such that for each $\pi \in \Phi$ there exists an $\mathcal{L}$-formula $\phi_\pi(\bar{y})$ such that

$$\phi_\pi(\mathcal{M}) = \{\bar{b} \in M^{|\bar{y}|} : (\mathcal{M}, \bar{b}) \in \pi\}$$

for each $\mathcal{M} \in \mathcal{C}$.

Further:

Definition 2.5.2. Let R be any set of functions $\mathcal{C} \to \mathbb{R}_{\geq 0}$. A class of finite $\mathcal{L}$-structures is an *R-multidimensional asymptotic class* if for every formula $\phi(x, \bar{y})$ there is a finite $\varnothing$-definable partition Φ of $(\mathcal{C}, \bar{y})$ and an indexed set $H_\phi := \{h_\pi \in R : \pi \in \Phi\}$ such that

$$\Big||\phi(M^{|x|} : \bar{b})| - h_\pi(M)\Big| = o(h_\pi(M))$$

for $(M, \bar{b}) \in \pi$ as $|M| \to \infty$.

When R is understood then we say that C is just a *m.a.c*. If, in the equation, the $o(h_\pi(M))$ can be taken to be zero, then instead $\mathcal{C}$ is known as a *R-multidimensional exact class* or R-m.e.c. This is one of the key definitions of the project. Basically we can consider these finite models to be, in a roughly analytical sense, 'tending' to some structure. Therefore when we take an ultraproduct of the class we approximately get the structure it was tending to.

Definition 2.5.3. We say a structure, M, has a *m.e.c limit* if there exists a m.e.c with ultraproduct elementarily equivalent to M.
Similarly for a given m.e.c $\mathcal{C}$ with ultraproduct elementarily equivalent to M, we would say that M is the *m.e.c limit of* $\mathcal{C}$.

We will make great use of the following fairly simple lemma:

Lemma 2.5.4 *Suppose M is a homogeneous structure over a finite relational language $\mathcal{L}$ such that it is the m.e.c limit of $\mathcal{C}$. Then we can thin out $\mathcal{C}$ such that every sentence true of M holds in cofinitely many members of $\mathcal{C}$.*

Proof. Take I an index of the structures in $\mathcal{C}$, and let D be an ultrafilter of I. We know that M is elementarily equivalent to the ultraproduct of $\mathcal{C}$ via D. Therefore by Łoś's Theorem we know that for any sentence true of M is true in some large infinite subclass of the models in $\mathcal{C}$. We can then thin out $\mathcal{C}$ by removing all the models that do not satisfy any of the sentences. Now for any sentence ϕ true of M say the set of models that satisfy it is indexed by $A \in D$. Now the models that do not satisfy ϕ is the union of the sets $A \setminus A \cap B$ for all B. But $A \setminus A \cap B$ is finite, as $A \cap B$ is infinite by definition of a non-principal ultrafilter, and so the union is finite. We see therefore that A is cofinite. $\square$

You may have noticed the similarity between the definition of a m.e.c and that of n-regularity, and it turns out that these are indeed very similar concepts when it is assumed M is homogeneous. Any suitably large member of a m.e.c with homogeneous limit will indeed be n-regular to some degree. This result is an adaption of a result from Theorem 4.1.6 in [2].

Lemma 2.5.5 *Fix $n \in \mathbb{N}$ and suppose $\mathcal{C}$ is a m.e.c over a finite relational language with ultraproduct elementarily equivalent to a homogeneous structure M. Then there is a function $f : \mathbb{N} \to \mathbb{N}$, such that if $N \in \mathcal{C}$ with $|N|> f(n)$, then N is n-regular.*

Proof. Let $T = \text{Th}(M)$, and suppose that $\mathcal{C}$ is a m.e.c with an ultraproduct $U \models T$. After thinning out $\mathcal{C}$ we may suppose that all non-principal ultraproducts of $\mathcal{C}$ are elementarily equivalent – that is, each element of T holds of cofinitely many $D \in \mathcal{C}$. For any formula $\phi(x, \bar{y})$ there is a finite set E of functions $h : \mathcal{C} \to R$ and some formula $\psi_h(\bar{y})$ for each $h \in E$, such that for any $N \in \mathcal{C}$ and $h \in E$, if $\bar{a} \in M^{|\bar{y}|}$ then

$$N \models \psi_h(\bar{a}) \Rightarrow |\phi(N, \bar{a})|= h(N).$$

Since M is homogeneous, T has quantifier-elimination, and hence there is a quantifier-free formula $\chi_h(\bar{y})$ and $\sigma \in T$ such that $\sigma \models \forall \bar{y}(\psi_h(\bar{y}) \leftrightarrow \chi_h(\bar{y}))$. As σ holds on cofinitely many members of $\mathcal{C}$, provided we work in sufficiently large $N \models T$, we may assume $\psi_h(\bar{y})$ is quantifier-free. Therefore if we take $\bar{y} = (y_1 \ldots, y_n)$ then the size of $\phi(N, \bar{y})$ is determined by the isomorphism type of $\bar{y}$, as if $\bar{y}$ and $\bar{y}'$ are both such that they satisfy ψ_h (i.e have the same isomorphism type), then $|\phi(N, \bar{y})|= |\phi(N, \bar{y}')|= h(N)$. It follows that any sufficiently large $N \in \mathcal{C}$, N is n-regular. □

Chapter 3

Multidimensional Exact Classes

This thesis originated with this chapter. It was the first thing I started working on, and as such much of the motivation for everything in this document stems from it. We've already discussed what a Multidimensional Exact Class, or m.e.c, is in Section 2.5, but in this chapter we aim to explain the current theoretical landscape and obtain some new results.

3.1 The Current Situation

To give some background and motivation for this study in m.e.cs, I will run through some of the current research into m.a.cs and m.e.cs. This shall mainly take the form of running through results and ideas listed in a manuscript by Anscombe, Macpherson, Steinhorn and Wolf [2].

We'll start with some nice examples:

Example 3.1.1 *The Paley graphs P_q form a multidimensional asymptotic class but not a multidimensional exact class. [[2] Example 2.2.5]*

Example 3.1.2 *The collection of all finite abelian groups is a m.e.c.[[2] Theorem 4.2.2]*

Example 3.1.3 *For any $d \in \mathbb{N}$, the class of all finite graphs of degree at most d is a m.e.c. [[2] Theorem 4.3.3]*

Interestingly, we can note that from [Example 3.4, [35]], the class of Paley Graphs is such

that any non-principal ultraproduct is elementarily equivalent to the Random Graph.
A crucial and very useful result for determining whether something is a m.a.c or m.e.c is the following. Let $\langle R\rangle$ denote the ring generated by R under the usual addition and multiplication operations for real-valued functions.

Theorem 3.1.4 *[[2], Theorem 2.4.1] (i) Let $\mathcal{C}$ be a class of L-structures. Suppose that $\mathcal{C}$ satisfies the definition of an R-mac for formulas $\phi(x;\bar{y})$ where x is a singleton. Then $\mathcal{C}$ is an $\langle R\rangle$-mac.*
(ii) [[49], Lemma 2.3.1] The assertion of (i) holds with m.e.c.s in place of m.a.c.s.

This allows us to consider just the formulas $\phi(x,\bar{y})$ where x is a single variable. With the Examples 3.1.1-3.1.3 and other similar motivational examples the authors proposed the conjecture:

Conjecture 3.1.5 ([2] Conjecture 4.1.4)

1. *Let M be a homogeneous structure over a finite relational language L. Then there is an m.e.c with ultraproduct elementarily equivalent to M if and only if M is stable.*
2. *Let M be an unstable homogeneous structure over a finite relational language. Then M is not elementarily equivalent to any structure interpretable in an ultraproduct of a m.e.c.*

Now in his thesis [49], Wolf proves the backwards direction of the first conjecture.

Proposition 3.1.6 ([2] Proposition 4.1.5) *Let M be a stable homogeneous structure over a finite relational language L. Then there is an m.e.c. $\mathcal{C}$ with an infinite ultraproduct which is elementarily equivalent to M.*

This means that the main focus is on the proving the other direction. This has been done for certain unstable homogeneous structures. Let I_n (for $n \geq 3$) denote the digraph consisting of n vertices with no directed edges between them. For each $n \geq 3$, we then define Q_n to be the universal homogeneous I_n-free digraph (these occur in Cherlin's classification of homogeneous digraphs in [16]).

Theorem 3.1.7 *[[2], Theorem 4.1.6.] Let M be any of the following homogeneous* *structures.*

i) Any unstable homogeneous graph.

ii) Any homogeneous tournament.

iii) The digraph Q_n for each $n \geq 3$.

iv) The generic bipartite graph.

Then there is no m.e.c with an ultraproduct elementarily equivalent to M.

The proof of part i) is particularly important for this chapter. The authors note that the proof showing that there doesn't exist a m.e.c limit for the Random Graph, will work for any of the K_n-free graphs (and their complements). And by the Lachlan-Woodrow classification of Unstable homogeneous graphs in [31], this will cover every unstable homogeneous graph.

So suppose there is actually a m.e.c $\mathcal{M}$ with ultraproduct elementarily equivalent to the Random Graph. Then, by Lemma 2.5.5, we know that for any sufficiently large $M \in \mathcal{M}$, M will satisfy 5-regularity. However, by a note added in the proof at the end of [9], we know that any 5-regular graph appears in the list in Theorem 2.3.9. It is therefore either the Pentagon, the line graph $K_{3,3}$, a disjoint union of complete graphs of the same size, or the complement of the latter. However M can be chosen sufficiently large so that it satisfies an extension axiom (Definition 2.1.12) that none of these satisfy.

Because of its frequent use we shall state this as a separate theorem.

Theorem 3.1.8 *[[2], Theorem 4.1.6] There is no m.e.c. with an ultraproduct elementarily equivalent to the Random Graph.*

A large amount of the manuscript [2] is dedicated to the study of generalised measurable structures, defined in [[2] Definition 5.2.1]. The authors show that any m.e.c has a generalised measurable ultraproduct.
For an ordered commutative ring S, there is a concept of $S^{\geq 0}$-measurable, which the authors use to get the following result:

Theorem 3.1.9 ([2] Theorem 5.3.3) *Suppose that $\mathcal{C}$ is an m.e.c, and let $\mathcal{U}$ be an ultrafilter on $\mathcal{C}$ and M the corresponding ultraproduct of $\mathcal{C}$. Then there is an ordered commutative* *ring S (an integral domain) such that M is $S^{\geq 0}$-measurable.*

Definition 3.1.10. [42, Defintion 4.2] A formula $\varphi(\bar{x}, \bar{y})$ has the *strict order property* if there exists an indiscernible sequence $\langle \bar{b}^i : i < \omega \rangle$ such that:

$$[\exists \bar{x} \neg \varphi(\bar{x}, \bar{b}^i) \wedge \varphi(\bar{x}, \bar{b}^j)] \Leftrightarrow i < j$$

A theory T has the *strict order property* if some formula $\varphi(\bar{x}, \bar{y})$ does.

We say that a theory has NSOP if it does not have the strict order property.

Proposition 3.1.11 ([2] Proposition 5.4.1) *If M is a weakly generalised measurable structure then its theory has NSOP.*

Crucially, as generalised measurable structures are also weakly generalised measurable, this means that the ultraproduct of any m.e.c has NSOP.

3.2 Unstable Homogeneous structures with no m.e.c limit

Proving the Random Graph does not have a m.e.c limit turns out to be even more useful than just eliminating any unstable homogeneous graph. This is because we can essentially 'find' it in many other unstable homogeneous structures. It stands to reason that if we have a structure in which we can perfectly define the Random graph (with no extra structure), then we have good reason to believe it also can't be a m.e.c. limit. It may help to think of a motivating example, say the universal metrically homogeneous graph of diameter k. It can be shown (Lemma 3.2.6) that from any base vertex x, the vertices immediately adjacent to x form a graph isomorphic to the Random Graph (without extra structure). Then, because we know the random graph isn't a m.e.c limit, we can use the exact same argument, just with formulas defined over the extra parameter x. We can generalise this concept to get the main result of this section, Theorem 3.2.4. Through this section we think of $\mathcal{L}$ as a finite relational language.

Great thanks are given to Dugald Macpherson who helped me iron out the detailed arguments of this section.

3.2.1 Hide and Seek

There are two main issues we need to address. The first is how do we define the principle of 'finding', and the second is the issue of the structure that we are finding being in a different language. The first is the most easily solved, the notion that we shall require is the following:

Definition 3.2.1. [4, Definition 2.3.1] Let N be an $\mathcal{L}'$-structure and M be an $\mathcal{L}$-structure, such that $N \subseteq M$ and we have $A \subset M$ such that the universe of N is A-definable in M. We say N is *canonically embedded* in M over A if the $\emptyset$-definable sets in N^k are exactly the subsets of N^k (for all positive integers k) which are A-definable in M.

In our study, we will couple this with homogeneity so the choice of A is dependent only on its isomorphism type. A small preliminary lemma we need is the following.

Lemma 3.2.2 *Let M be a homogeneous $\mathcal{L}$-structure.*

1. *If $a_1, \ldots a_k \in M$, then $(M, a_1, \ldots, a_k)$ is homogeneous in a language containing relation symbols for all atomic $\mathcal{L}(\bar{a})$ formulas.*
2. *If N is an infinite $\emptyset$-definable substructure of M then N is homogeneous.*

Proof. Both results follow immediately from the definition of homogeneity. □

As it turns out, in this case we do not have to worry about the nature of A at all, because in a big enough member of our m.e.c, the inclusion of A into our language still gives a m.e.c. More formally, let M be a homogeneous $\mathcal{L}$-structure and $a_1, \ldots, a_k \in M$, and let $\theta(z_1, \ldots, z_k)$ isolate $\mathrm{tp}(a_1, \ldots, a_k)$ (so we may suppose $\theta(\bar{z})$ gives the isomorphism type of $\bar{a}$). Let $\mathcal{C}$ be a m.e.c with all non-principal ultraproducts elementarily equivalent to M. We may suppose that all elements of $\mathcal{C}$ satisfy $\exists \bar{z}\theta(\bar{z})$ by thinning it out to only the members that are large enough. Let $\mathcal{C}(\bar{a})$ be the class of all expansions of members of $\mathcal{C}$ to $\mathcal{L}(\bar{a})$, in which $\theta(a_1, \ldots, a_k)$ holds.

Lemma 3.2.3 *Under the above assumptions, $\mathcal{C}(\bar{a})$ contains a m.e.c all of whose non-principal ultraproducts are elementarily equivalent to $(M, a_1, \ldots, a_k)$.*

Proof. First we need to show that the class $\mathcal{C}(\bar{a})$ is such that all non-principal ultraproducts are elementarily equivalent to $(M, a_1, \ldots, a_k)$. By thinning out $\mathcal{C}(\bar{a})$, we may suppose that each $\mathcal{L}(\bar{a})$-sentence is true of a finite or cofinite subset of $\mathcal{C}(\bar{a})$. Suppose we have a formula $\phi(\bar{z})$ such that $(M, a_1, \ldots, a_k) \models \phi(a_1, \ldots, a_k)$. Then $M \models \forall \bar{z}(\theta(\bar{z}) \to \phi(\bar{z}))$, so this holds in every sufficiently large $P \in \mathcal{C}$, so sufficiently large members of $\mathcal{C}(\bar{a})$ satisfy $\phi(a_1, \ldots, a_k)$. Thus, all non-principal ultraproducts of $\mathcal{C}(\bar{a})$ are elementarily equivalent to $(M, a_1, \ldots, a_k)$.

It remains to show that $\mathcal{C}(\bar{a})$ is also a m.e.c. In order to do this we take an $\mathcal{L}(\bar{a})$ formula $\psi(x_1, \ldots, x_n, y_1, \ldots, y_m, a_1, \ldots, a_k)$. Now, as $\mathcal{C}$ is a m.e.c, for the $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_k)$ there exists an $\emptyset$-definable partition Π of $(\mathcal{C}, \bar{y}\bar{z})$ and a finite set $h_\Pi = \{h_\pi : \pi \in \Pi\}$ of functions $\mathcal{C} \to \mathbb{N}$ such that for each $\pi \in \Pi$, if $(P, \bar{c}\bar{d}) \in \pi$ then $|\psi(P^n, \bar{c}\bar{d})| = h_\pi(P)$. In particular, for $(P, \bar{a}) \in \mathcal{C}(\bar{a})$, we know sets of the form $\psi(P^n, \bar{y}, \bar{a})$ will take only a fixed number of sizes as $\bar{y}$ ranges through P^m, and in sufficiently large P (relative to ψ) the size depends just on the isomorphism type of $\bar{y}\bar{a}$, that is the isomorphism type of $\bar{y}$ over $\bar{a}$. Hence $\mathcal{C}(\bar{a})$ is also a m.e.c. □

We now have enough to prove the desired result.

Theorem 3.2.4 *Let M be a homogeneous $\mathcal{L}$-structure which is a m.e.c limit. Let $a_1, \ldots, a_k \in M$ and let N be an $\bar{a}$-definable subset of M. We also let N' be a homogeneous $\mathcal{L}'$-structure, with universe N, which is canonically embedded in M over $\bar{a}$. Then N' is a m.e.c limit.*

Proof. By Lemmas 3.2.3 and 3.2.2, $(M, a_1, \ldots, a_k)$ is also a homogeneous m.e.c limit (in a language where we add relation symbols for $\bar{a}$-definable atomic relations on M), and so we may drop the constants $\bar{a}$. Let $\mathcal{C}$ be a m.e.c such that all of its non-principal ultraproducts are elementarily equivalent to M. Now by Lemma 2.4.17, we have quantifier-free $\mathcal{L}$-formulas which define N in M and define the $\mathcal{L}'$-relations in N'. Applying these to the members of $\mathcal{C}$ we define finite $\mathcal{L}'$-structures, which can be used to create a class $\mathcal{C}'$. We claim that $\mathcal{C}'$ is a m.e.c with any non-principal ultraproduct elementarily equivalent to N'. We can arrange this class in such a way so that every $\mathcal{L}'$-sentence true of N' is true of cofinitely many members of $\mathcal{C}'$, hence any non-principal ultraproduct will be elementarily equivalent to N'. Now consider an $\mathcal{L}'$ formula $\psi(\bar{x}, \bar{c})$. The size of the set $\psi(P^{|\bar{x}|}, \bar{c})$ is going to be the same as the size of the set defined by some ψ' an $\mathcal{L}$-formula over $\mathcal{C}$ and therefore will range through a fixed set of sizes as $(P, \bar{c})$ range through $(\mathcal{C}', \bar{y})$. Furthermore as N' is canonically embedded in M, we see that the same $\emptyset$-definable partition that exists for $\mathcal{C}$ exists for $\mathcal{C}'$. Hence $\mathcal{C}'$ is a m.e.c with ultraproduct elementarily equivalent to N'. □

We now have a pretty easy method of eliminating certain homogeneous structures. All we have to do is find a canonical embedding of an already eliminated structure. We will now go through some examples of this process.

3.2.2 Metrically Homogeneous graph

Definition 3.2.5 (Section 1, [1])**.** A connected graph is *metrically homogeneous* if it is homogeneous when considered as a metric space in the graph metric, i.e. with binary predicates interpreted by the graph's distance.
The *Universal metrically homogeneous graph of diameter* k, M_k, is the countably infinite metrically homogeneous graph that embeds all finite graphs of diameter k.

We can immediately find that for any vertex x in M_k, the vertices directly adjacent to x are effectively the Random Graph.

Lemma 3.2.6 *Let M_k be the universal metrically homogeneous graph of finite diameter $k > 1$ and let $x \in M$. Set $\Gamma_1(x)$ to be the induced subgraph of all elements of M at distance 1 from x. Then $\Gamma_1(x)$ is isomorphic to the Random graph.*

Proof. To show that a countably infinite graph is isomorphic to the Random graph it is enough to show that it satisfies the n-extension axiom for any n. Fix an n and suppose we have two disjoint sets of size n in $\Gamma_1(x)$, $\{i_1, \dots i_n\}$ and $\{j_1, \dots, j_n\}$. By the universality of M there exists a vertex m at distance 1 from x such that m is connected to i_k and not connected to j_k for all k. This vertex m is therefore in $\Gamma_1(x)$ and hence this means that $\Gamma_1(x)$ satisfies the n-extension axiom. □

We can now see that we have enough to apply Theorem 3.2.4,

Theorem 3.2.7 *For all $k \in \mathbb{N}$, there does not exist a m.e.c with ultraproduct elementarily equivalent to the universal metrically homogeneous graph of diameter k.*

Proof. Let M_k be the universal metrically homogeneous graph of diameter k and let $T = \text{Th}(M)$. Suppose there exists a m.e.c $\mathcal{C}$ with ultraproduct elementarily equivalent to M_k. Now take $x \in M_k$ and consider $\Gamma_1(x)$. By Lemma 3.2.6 this is isomorphic to the Random Graph $\mathcal{R}$.
It is clear that $\mathcal{R}$ is canonically embedded in M_k over x. First for any point $x \in M$ the universe of $\mathcal{R}$ is x-definable. It is also clear, as the language of M_k is binary, that the $\emptyset$-definable sets in $\mathcal{R}$ are exactly the x-definable sets in M_k.
Therefore by Theorem 3.2.4, there cannot be a m.e.c with ultraproduct elementarily equivalent to M_k. □

As you can see this proof didn't use any inherent properties of the metrically homogeneous graph once the Random Graph was found. It therefore stands to reason that such an approach will work for other homogeneous structures where we can 'find' a canonical embedding of the Random Graph over some set.

Remark 3.2.8 *We could use an analogous arguments to show that the metrically homogeneous graph of infinite diameter is also not a m.e.c limit, and indeed that many other metrically homogeneous graphs are not. There exists a catalogue of metrically homogeneous graphs given by Cherlin, which is conjectured to be complete. This is verified for diameter 3 in [1].*

3.2.3 Universal Homogeneous Two-graph

Another example of such a structure is the universal homogeneous two-graph.

Definition 3.2.9 ([34], Example 2.3.1 (4))**.** A *two-graph* is a 3-hypergraph such that any 4-set contains an even number of 3-edges.
The *universal homogeneous two-graph* is the homogeneous countably infinite two-graph that embeds all finite two-graphs.

For any graph, Γ, there exists a two-graph with the same vertex set, whose 3-edges are the

3-sets with an odd number of graph edges, and it can be shown that every two-graph arises in this manner. For example, the universal homogeneous two-graph is the two-graph that can formed in this way from the Random Graph.

As before, we have a copy of the Random Graph in this structure by naming a point, however we have to be very careful that it doesn't possess any extra structure, i.e 3-edges. This is a crucial issue, for example we can find the Random graph in the universal 3-hypergraph, however 3-edges would still remain. This would lead to 3-sets that have the same 2-edge structure, but aren't isomorphic due to the fact that one possesses a 3-edge and the other does not. We could therefore not use the m.e.c condition to dictate that as a graph the structure possessed 5-regularity.

Lemma 3.2.10 *Let M be the universal homogeneous two-graph and take $x \in M$. Consider $\Gamma(x)$ to be the graph on $V(M)\backslash\{x\}$ with y and z connected if and only if $\{x, y, z\}$ is a 3-edge in M. Then $\Gamma(x)$ is isomorphic to the Random graph canonically embedded over x.*

Proof. To show that a countably infinite graph is isomorphic to the Random graph it is enough to show that it satisfies the n-extension axiom for any n. Fix an n and suppose we have two disjoint sets of size n in $\Gamma(x)$, $\{i_1, \ldots i_n\}$ and $\{j_1, \ldots, j_n\}$. We need to find an $m \in M$ such that (i_k, m, x) is a 3-edge and (j_k, m, x) is not a 3-edge for all k. By the universality of M we only need to show that the existence of such an m would be consistent. To do this we just need to show that every 4-set in consideration has an even number of 3-edges. Clearly any 4-set not including m does as this it is already present in M. Now we say if (i_{k_1}, j_{k_2}, x) is not a 3-edge, then (i_{k_1}, j_{k_2}, m) is and vice versa. If (i_{k_1}, i_{k_2}, x) is a 3-edge then (i_{k_1}, i_{k_2}, m) is too and if not then it isn't either. We do the same thing for (j_{k_1}, j_{k_2}, x) and (j_{k_1}, j_{k_2}, m). The above relations ensure all 4-sets including both m and x have an even number of 3-edges. Clearly $(i_{k_1}, i_{k_2}, i_{k_3}, m)$ (and w.l.o.g $(j_{k_1}, j_{k_2}, j_{k_3}, m)$) have this property as they satisfy same relations as $(i_{k_1}, i_{k_2}, i_{k_3}, x)$. Finally, $(i_{k_1}, j_{k_2}, i_{k_3}, m)$ (and likewise $(i_{k_1}, j_{k_2}, i_{k_3}, m)$) have an even number of 3-edges, as it has exactly two 3-edges different from $(i_{k_1}, j_{k_2}, i_{k_3}, x)$, (i_{k_1}, j_{k_2}, m) and (j_{k_2}, i_{k_3}, m). Therefore the existence of m is consistent in M, so m is in M and $\Gamma(x)$ has n-extension. Therefore $\Gamma(x)$ is isomorphic to the Random Graph.

We now need to show that this Random Graph is canonical embedded in M over x. First note that it will carry no extra structure by way of 3-edges, as if (a, b, c) is a 3-edge but $x \notin \{a, b, c\}$ then having a 3-relation on (a, b, c) in $\Gamma(x)$ is equivalent to there being an odd number of edges in (a, b, c) by the definition of the universal homogeneous two-graph, and therefore the 3-edges are determined by the graph structure. If it contains x then we lose the 3-relation as we lose x.
We note that we know have two potential random graphs on $M \setminus \{x\}$, the original one used to determine Γ, which we shall call $\mathcal{R}$ and the graph defined via x, $\mathcal{R}'$. In general these will not be the same graph, however we can see that the graph $\mathcal{R}$ will only provide extra structure to $\mathcal{R}'$ if they determine different two-graphs. We can see fairly quickly that they will determine the same two graph however. If we take points $a, b, c \in M \setminus \{x\}$ then they will be a 3-edge in $T(\mathcal{R}')$ (the two-graph generated by $\mathcal{R}'$) if and only if they have an odd number of edges in $\mathcal{R}'$. This means that there an odd number of xab, xac, xbc are 3-edges in $T(\mathcal{R})$. As in any two-graph, any 4-set must have an even number of 3-edges, we get that abc is a 3-edge in $T(\mathcal{R})$ if and only if it is a 3-edge in $T(\mathcal{R}')$. Hence they generate the same two-graph. □

Then as before we can extend the proof of Theorem 3.1.8 to incorporate this structure.

Theorem 3.2.11 *There does not exist a m.e.c with ultraproduct elementarily equivalent to the universal homogeneous two-graph.*

Proof. This argument follows in much the same way to Theorem 3.2.7. Let M be the universal homogeneous two-graph and let $T = \text{Th}(M)$. Suppose there exists a m.e.c $\mathcal{C}$ with ultraproduct elementarily equivalent to M. Now take $x \in M$ and consider $\Gamma(x)$ to be the graph on $V(M) \setminus \{x\}$ with y and z connected if and only if $\{x, y, z\}$ is a 3-edge in M as before. Then this is isomorphic to the Random graph $\mathcal{R}$ canonically embedded over x by Lemma 3.2.10. □

3.3 Reducts

A good place to look for structures with canonical embeddings is through the study of reducts. To properly define reducts however we will need some more ideas from group theory. Let Ω be a countable set. There exists a natural topology on the symmetric group of Ω, that of *pointwise convergence*. Cameron [11] describes pointwise convergence with the following process: Enumerate $\Omega = \{\alpha_0, \alpha_1, \ldots\}$. Then a sequence (g_n) of permutations tends to the limit g if and only if, for any $k \in \mathbb{N}$, $\alpha_k g_n = \alpha_k g$ and $\alpha_k g_n^{-1} = \alpha_k g^{-1}$ for all sufficiently large n.

Definition 3.3.1. A group is a *topological group* if it carries a topological space such that the group operation and the inverse map are continuous in that space.

Hence in the topology of the symmetric group defined above, we see that any permutation group G on the countable set Ω is a topological group (We have continuity as if $g_n \to g$ and $h_n \to h$ then $g_n h_n \to gh$ and $g_n^{-1} \to g^{-1}$).

A basis for the open sets in this topology is made up of cosets of stabilisers of finite tuples.

Definition 3.3.2. We say a permutation group G on a countable set Ω is a *closed subgroup* of $\mathrm{Sym}(\Omega)$, if it is a subgroup of $\mathrm{Sym}(\Omega)$ and a closed set with regards to the topology of pointwise convergence on $\mathrm{Sym}(\Omega)$.

A result that will make these much easier to deal with in terms of reducts is the following:

Lemma 3.3.3 *[11, 2.6] A subgroup G of $\mathrm{Sym}(\Omega)$ is closed if and only if $G = \mathrm{Aut}(M)$ for some (first-order) structure M on Ω.*

We can now define

Definition 3.3.4 ([46])**.** A *reduct* of an ω-categorical structure M is a permutation group (G, M) such that:

i) Aut$(M) \leq G$

ii) G is a closed subgroup of Sym(M)

Assuming ω-categoricity, this is equivalent to there existing a structure N, for some language $\mathcal{L}$, such that:

iii) N has the same universe as M.

iv) For each relation $R \in \mathcal{L}$, R^N is $\emptyset$-definable in M

v) $G = \text{Aut}(N)$

There are two main methods of proving an unstable reducts of a structure does not have a m.e.c limit. The first is to find a combinatorial representation of the reduct, and then find the original structure as a canonically embedded substructure of it over some finite set. This will then allow the use of Theorem 3.2.4 to eliminate the possibility of a m.e.c limit.
The second is to instead name points and use Lemma 3.2.3. We can demonstrate both using the random graph.

3.3.1 Reducts of the Random Graph

It seems prudent that the first structure we look at through this lens is the Random Graph. In [46], Thomas classifies the possible reducts of the random graph.

Theorem 3.3.5 ([46], Theorem 1) *Let the Random Graph be $\mathcal{R}$. If $(G, \mathcal{R})$ is a reduct of $\mathcal{R}$, then*

$$G \in \{\text{Aut}(\mathcal{R}), D(\mathcal{R}), S(\mathcal{R}), B(\mathcal{R}), \text{Sym}(\mathcal{R})\}$$

where $D(\mathcal{R})$ is the duality group, $S(\mathcal{R})$ is the switching group, and $B(\mathcal{R}) = \langle D(\mathcal{R}), S(\mathcal{R}) \rangle$.

This can be rephrased in the form of combinatorial structures as was done in [Theorem 8, [6]].

Theorem 3.3.6 ([46] or Theorem 8, [6]) *i) $S(\mathcal{R})$ is the automorphism group of the 3-hypergraph whose edges are the 3-element subsets containing an odd number of edges in the Random Graph.*

ii) $D(\mathcal{R})$ is the automorphism group of the 4-hypergraph whose edges are the 4-element subsets containing an odd number of edges in the Random Graph.

iii) $B(\mathcal{R})$ is the automorphism group of the 5-hypergraph whose edges are the 5-element subsets containing an odd number of edges in the Random Graph.

We have already dealt with one of these, for as noted in [41] the 3-hypergraph whose edges are the 3-element subsets containing an odd number of edges in the Random Graph is exactly the universal homogeneous two-graph. We will need the following lemma:

Lemma 3.3.7 *Let $\mathcal{L}_1, \mathcal{L}_2$ be relational languages, and let M_1, M_2 be respectively $\mathcal{L}_1$ and $\mathcal{L}_2$-structures, both homogeneous, with the same domain M and the same automorphism group (so the same $\emptyset$-definable sets). Then M_1 is a m.e.c. limit if and only if M_2 is a m.e.c. limit.*

Proof. Any sentences needed to translate between $\mathcal{L}_1$ and $\mathcal{L}_2$ will hold in sufficiently large members of a m.e.c. Suppose that $\mathcal{C}_1$ is a m.e.c for M_1. For each relation symbol $R(\bar{x})$ of $\mathcal{L}_2$ there is a quantifier-free $\mathcal{L}_1$-formula $\phi_R(\bar{x})$ such that $\{\bar{x} : M_1 \models \phi_R(\bar{x})\} = \{\bar{x} : M_2 \models R(\bar{x})\}$. We translate each member of $\mathcal{C}_1$ into an $\mathcal{L}_2$-structure by interpreting each $\mathcal{L}_2$ relation R by the corresponding $\mathcal{L}_1$-formula ϕ_R. After thinning out $\mathcal{C}_2$ first, we find that all its non-principal ultraproducts are elementarily equivalent to M_2. □

Now, let $\mathcal{R}$ denote the random graph, and D denote the reduct whose automorphism group is the duality group $D(\mathcal{R})$. We may view D as a structure in a language with a single arity 4 relation U which determines an equivalence relation on unordered 2-sets, one class being the edge set of the random graph, the other being the non-edge set. We can view D as a homogeneous structure by expanding the language, and this will not change the $\emptyset$-definable sets.

Similarly we let B denote the reduct whose automorphism group is $B(\mathcal{R})$.

Theorem 3.3.8 *i) There does not exist a m.e.c with ultraproduct elementarily equivalent to D,*

ii) There does not exist a m.e.c with ultraproduct elementarily equivalent to B.

Proof. i) Suppose that $\mathcal{C}$ is a m.e.c all of whose ultraproducts are elementarily equivalent to D. Let a, b be adjacent in $\mathcal{R}$. By Lemma 3.2.3, we can obtain from $\mathcal{C}$ a m.e.c, $\mathcal{C}(a, b)$, all of whose ultraproducts are elementarily equivalent to (D, a, b). Now (in a suitable language) (D, a, b) is a homogeneous structure whose $\emptyset$-definable sets are the same as those of $(\mathcal{R}, a, b)$. It follows by Lemma 3.3.7 that the random graph is a m.e.c. limit, a contradiction.

ii) This is the exact same argument, however we name three points a, b, c.

□

Therefore, using Theorem 3.2.11 and Theorem 3.3.8, all three unstable reducts listed in Theorem 3.3.5 do not have m.e.c limits (Note $\mathrm{Sym}(\mathcal{R})$ is stable). More formally,

Theorem 3.3.9 *There does not exist a m.e.c with ultraproduct elementarily equivalent to any unstable reduct of the random graph.*

This begs the question, is this true in general of unstable reducts of structures with no m.e.c limit? While this question is not answered in this work, we can provide some results to be applied to other cases.

3.4 Other structures

Although we've been using reducts to find universal structures with other universal structures within, we need not do this. We can also do it the other way around, name a universal structure and look to see if it contains some copy of a structure already known not to have a m.e.c limit.

3.4.1 Universal homogeneous n-tournament-free digraph

First we should establish our language for digraphs. For convenience, we shall use three symbols F, B, N (although really just one F would suffice). We say $F(x, y)$ represents an edge going from x to y, $B(x, y) = F(y, x)$ represents an edge going from y to x, and $N(x, y) = \neg F(x, y) \wedge \neg F(y, x)$ represents the absence of an edge between x and y.

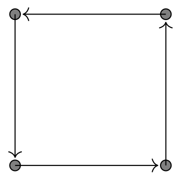
Definition 3.4.1. A *tournament* is a digraph, T, in which for all distinct $x, y \in T$ exactly on of $F(x, y)$ or $B(x, y)$. An *n-tournament* is a tournament of size n.

It is shown by Henson that for any set S of finite tournaments the collection of finite S-free digraphs has the amalgamation property, and hence yields an unstable homogeneous digraph M_S. These are known as *Henson Digraphs*.

Definition 3.4.2. The *universal n-tournament-free digraph* is the Fraïssé limit of the class of all finite digraphs with the omission of any digraph embedding a tournament with n vertices.

We will first consider the 3-tournament-free case.

Theorem 3.4.3 *If M is a finite 2-regular 3-tournament-free digraph then M is isomorphic to the 4-cycle*



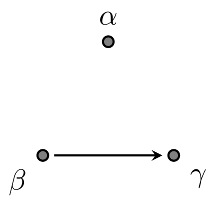
Proof. Suppose M is a finite 2-regular 3-tournament-free digraph. For ease, for any $x \in M$, we will define the sets $F(x) = \{y \in M : F(x, y)\}$, $B(x) = \{y \in M : B(x, y)\}$ and $N(x) = \{y \in M : N(x, y)\}$. By 1-regularity, $|F(x)|= |B(x)|= k$.
Now using 2-regularity, we consider any points $x_1, x_2 \in M$ such that $(x_1, x_2) \in F$. Due to the fact that we know M embeds no tournaments on at least 3 vertices, we can immediately see that $|F(x_1) \cap F(x_2)|= |B(x_1) \cap F(x_2)|= |F(x_1) \cap B(x_2)|= |B(x_1) \cap B(x_2)|= 0$. Also by counting all the arrows to and from x_1 and x_2, we get that $|F(x_1) \cap N(x_2)|= |N(x_1) \cap B(x_2)|= k - 1$ and $|B(x_1) \cap N(x_2)|= |N(x_1) \cap F(x_2)|= k$.
Similarly we consider any points $y_1, y_2 \in M$ such that $N(y_1, y_2)$. As there are no restrictions on this case so we can treat them as we would usual regular digraphs. Suppose $|F(y_1) \cap F(y_2)|= d_1$ and $|B(y_1) \cap B(y_2)|= d_2$. We know that

$$|F(y_1) \cap B(y_2)|= |B(y_1) \cap F(y_2)|= c$$

for some c and

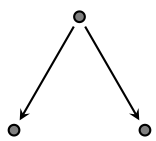
$$|F(y_1) \cap N(y_2)|= |N(y_1) \cap F(y_2)|= |B(y_1) \cap N(y_2)|= |N(y_1) \cap B(y_2)|= e$$

for some e. To see this note that we can count the structure



by starting with α and then comparing the numbers when you pick β next or γ next.

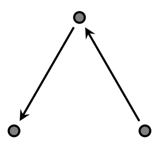
Now we can note that $|F(y_1)| = |(F(y_1) \cap F(y_2))| + |(F(y_1) \cap B(y_2))| + |(F(y_1) \cap N(y_2))| = d_1 + e + c$ and $|B(y_1)| = |(B(y_1) \cap F(y_2))| + |(B(y_1) \cap B(y_2))| + |(B(y_1) \cap N(y_2))| = d_2 + e + c$. But $|F(y_1)| = |B(y_1)| = k$, hence $d_1 = d_2 = d$ and $e = k - d - c$.
Now, let P be the triangle FFN, defined as



We can count P in two different ways either starting with F, giving us $n \cdot k \cdot k - 1$ copies of P, or N, giving us $n \cdot (n - 2k - 1) \cdot d$. This gives us the identity

$$k(k-1) = (n - 2k - 1)d$$

We can do a similar thing for $Q = FBN$.



giving the identity

$$k^2 = (n - 2k - 1)c$$

This leads to the result that

$$\frac{k-1}{k} = \frac{d}{c}$$

As c and d are integers no bigger than k, this implies that $d = k - 1$ and $c = k$. But as $|F(y_1) \cap N(y_2)| = k - d - c \geq 0$ we know $1 - k \geq 0$ and so $k = 1$, as $k - 1 \geq 0$ also. As

$k = c = 1$ we get $n - 2k - 1 = 1$ as well, so $n = 4$. □

These leads immediately on to the following result.

Theorem 3.4.4 *Let* Γ *be the universal homogeneous* 3*-tournament-free digraph. Then there does not exists a m.e.c* $\mathcal{C}$ *such that the ultraproduct of* $\mathcal{C}$ *is elementarily equivalent to* Γ.

Proof. Suppose that the ultraproduct of $\mathcal{C}$ is elementarily equivalent to Γ. Then by Lemma 2.5.5 we know that any sufficiently large member $M \in \mathcal{C}$ will be n-regular. But then by Theorem 3.4.3 the only option is the the 4-cycle. □

We can now generalise this to the n-tournament-free case for a general n, by finding the 3-tournament-free case within it.

Theorem 3.4.5 *If* M *is the universal homogeneous* n*-tournament-free digraph then there does not exist a m.e.c with ultraproduct elementarily equivalent to* M.

Proof. This is simple enough. Let $X = \{x_1, \ldots, x_{n-3}\} \in M$ be such that X carries a tournament isomorphic to a linear ordering from x_1 to x_{n-3} (the choice of tournament actually doesn't matter). Then consider the set of points $Y = \{y \in M \backslash X : F(y, x_i) \text{ for each } i = 1, \ldots, n-3\}$. Then Y is 3-tournament-free and has no other restrictions, hence it is isomorphic to the universal homogeneous 3-tournament-free digraph, and is canonically embedded over X. It follows from Theorem 3.2.4 and Theorem 3.4.4 that there does not exist a m.e.c. with ultraproduct elementarily equivalent to M. □

3.4.2 Amalgamation classes determined by constraints on triangles

Defined by Cherlin in the Appendix of [16], we look at the list of semi-free (but not free) amalgamation classes in binary relational languages with 3 or 4 colours. We show in Theorems 3.4.7 and 3.4.9 that none of the Fraïssé limits of these classes are m.e.c limits.

Definition 3.4.6 ([16]). An amalgamation class of a binary relation language is said to have *semi-free amalgamation* if there is a proper subset of the 2-types of distinct elements which is adequate for the solution of any amalgamation problem.

Let $\mathcal{L} = \{X_1, \ldots, X_k\}$ be a relational language in which every relation is symmetric, binary and irreflexive. Define Δ to be a finite set of $\mathcal{L}$-structures of size 3. Such a structure will be determined by a triple from $\mathcal{L}$, for example $X_1X_1X_2$ determines an $\mathcal{L}$-structure of size 3 in which two pairs satisfy X_1 and the third satisfies X_2. Then define $\mathcal{C}(\Delta)$ to be class of all finite L-structures Γ such that for all $T \in \Delta$, T is not embedded in Γ and the set of unordered pairs is partitioned by the relations $X_1, \ldots X_k$. If $\mathcal{C}(\Delta)$ satisfies the amalgamation property, then we say that $M(\Delta)$ is the Fraïssé limit of $\mathcal{C}(\Delta)$. Cherlin lists the possible Δ such that $\mathcal{C}(\Delta)$ is a primitive semi-free, but not free, amalgamation class in the case where $|\mathcal{L}|= 3$ or $|\mathcal{L}|= 4$, and states that there are $2^{\aleph_0}$ in any case with a larger language. Note he conjectures that these lists are complete but does not prove this. A positive result for the conjecture in the $|\mathcal{L}|= 3$ case is given in the Appendix.

When $\mathcal{L} = \{R, G, B\}$ Cherlin lists just one possibility:

1. $\Delta = \{RBB,\ GGB,\ BBB\}$

When $\mathcal{L} = \{R, G, A, X\}$ we get 27 possibilities:

1. $\Delta_1 = \{RXX,\ GAX,\ AXX\}$
2. $\Delta_2 = \{RXX,\ GAX,\ AXX,\ XXX\}$
3. $\Delta_3 = \{RXX,\ GAX,\ AXX,\ AAX\}$
4. $\Delta_4 = \{RXX,\ GAX,\ AXX,\ AAA\}$
5. $\Delta_5 = \{RXX,\ GAX,\ AXX,\ AAX,\ XXX\}$
6. $\Delta_6 = \{RXX,\ GAX,\ AXX,\ AAA,\ XXX\}$
7. $\Delta_7 = \{RXX,\ GAX,\ AAX,\ AXX,\ AAA\}$
8. $\Delta_8 = \{RXX,\ GAX,\ AAX,\ AXX,\ AAA,\ XXX\}$
9. $\Delta_9 = \{RXX,\ GAX,\ AAX,\ XXX\}$
10. $\Delta_{10} = \{RXX,\ GAX,\ AAX,\ XXX,\ AAA\}$
11. $\Delta_{11} = \{RXX,\ GGX,\ AXX,\ XXX\}$
12. $\Delta_{12} = \{RXX,\ GGX,\ AAX,\ AXX,\ XXX\}$

13. $\Delta_{13} = \{RXX,\ GGX,\ AXX,\ XXX,\ AAA\}$

14. $\Delta_{14} = \{RXX,\ GGX,\ AAX,\ AXX,\ XXX,\ AAA\}$

15. $\Delta_{15} = \{RXX,\ GAX,\ GGX,\ AXX,\ XXX\}$

16. $\Delta_{16} = \{RXX,\ GAX,\ GGX,\ AAX,\ AXX,\ XXX\}$

17. $\Delta_{17} = \{RXX,\ GAX,\ GGX,\ AXX,\ XXX,\ AAA\}$

18. $\Delta_{18} = \{RXX,\ GAX,\ GGX,\ AAX,\ AXX,\ XXX, AAA\}$

19. $\Delta_{19} = \{RXX,\ GAX,\ GGX,\ AAX,\ XXX\}$

20. $\Delta_{20} = \{RXX,\ GAX,\ GGX,\ AAX,\ XXX,\ AAA\}$

21. $\Delta_{21} = \{RAA,\ RXX,\ GAX,\ AAX,\ XXX\}$

22. $\Delta_{22} = \{RAA,\ RXX,\ GAX,\ AAX,\ AXX\}$

23. $\Delta_{23} = \{RAA,\ RXX,\ GAX,\ AAX,\ AXX,\ XXX\}$

24. $\Delta_{24} = \{RAA,\ RXX,\ GAX,\ AXX,\ XXX,\ AAA\}$

25. $\Delta_{25} = \{RAA,\ RXX,\ GAX,\ AAX,\ AXX,\ XXX,\ AAA\}$

26. $\Delta_{26} = \{RRX,\ RAA,\ RXX,\ GAX,\ GXX,\ AAX,\ XXX\}$

27. $\Delta_{27} = \{RRA,\ RRX,\ GAA,\ GAX,\ GXX,\ AAX,\ AXX,\ XXX, AAA\}$

This is a rather dauntingly long list, but we can show that the corresponding Fraïssé limits are not m.e.c limits using earlier results:

Theorem 3.4.7 *Suppose $\mathcal{L} = \{R, G, B\}$ and $\Delta = \{RRB,\ GGB,\ BBB\}$. Then there does not exist a m.e.c with ultraproduct elementarily equivalent to $M(\Delta)$.*

Proof. Define a point x in $M(\Delta)$, then consider the Green neighbourhood $G(x)$ of x. As GGB is a forbidden triangle, there are no Blue edges in this neighbourhood, so it is two coloured. However there is no restriction using only the other two colours. Hence Age$(G(x))$ is the set of all finite two-coloured graphs, and therefore this is isomorphic to the Random Graph.

We can see that this is a canonical embedding as there is no extra structure on the random graph, and so the $\emptyset$-definable sets in $G(x)$ are exactly those that are x-definable in $M(\Delta)$. Hence by Theorem 3.1.8 and Theorem 3.2.4, there does not exist a m.e.c with ultraproduct elementarily equivalent to $M(\Delta)$. □

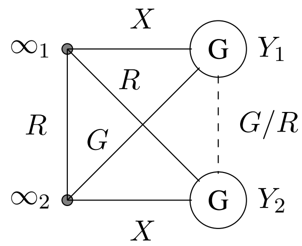
Theorem 3.4.8 *Suppose $\mathcal{L} = \{R, G, A, X\}$. Then if $\Delta \in \{\Delta_1, \ldots, \Delta_{27}\}$, then $M(\Delta)$ has a definable substructure isomorphic to either*

- *The Random Graph*
- *The Random Bipartite graph*

with no extra structure i.e. canonically embedded over suitable parameters.

Proof. First we shall focus on Δ_1. Take a vertex $\infty \in M(\Delta_1)$, and consider $X(\infty)$, the X neighbourhood of ∞. Note that this cannot contain any R or A coloured edges, so is two-coloured. Further we can note that there are no further restrictions within the neighbourhood, as no forbidden triangle solely contains just G and X edges. Hence Age$(X(\infty))$ is the set of all finite two-coloured graphs, and therefore $X(\infty)$ is isomorphic to the Random Graph, canonically embedded over ∞.
We can note that this argument will work for any Δ containing RXX, AXX (or isomorphic conditions) and no further restrictions involving only G and X. Hence, by considering the X-neighbourhood, this argument follows for Δ_3, Δ_4, Δ_7, Δ_9 and Δ_{19} and by considering the A-neighbourhood it follows for Δ_8, Δ_{10}, Δ_{14}, Δ_{18}, Δ_{20}, Δ_{21}, Δ_{22}, Δ_{23} and Δ_{26}. Hence all of these have an induced subgraph isomorphic to the Random graph.
Next we shall look at Δ_2. The only neighbourhood which is not isomorphic to the entire graph is the X-neighbourhood, which is complete in G. Now construct the induced subgraph as follows:
Take vertices ∞_1 and ∞_2 in $M(\Delta_2)$ such that (∞_1, ∞_2) is red. Now consider $Y_1 = \{y \in M(\Delta_2) : X(\infty_1, y) \wedge G(\infty_2, y)\}$ and $Y_2 = \{y \in M(\Delta_2) : X(\infty_2, y) \wedge R(\infty_1, y)\}$. The induced subgraph on both Y_1 and Y_2 is a complete graph in G, however we see that the connections between Y_1 and Y_2 are all of colour G or R (X is removed by XXR, A is removed by GAX). As there are no further restrictions involving just G and R, we can see that the induced subgraph on $Y_1 \cup Y_2$ is indeed isomorphic to the Random bipartite graph,

with G becoming non-edges and R being edges. To illustrate, we have structure



Now as with Δ_1 we can generalise this argument. We can observe that $\Delta_5,\ \Delta_6,\ \Delta_{24}$ and Δ_{25} have all the triangle restrictions of Δ_2, and that their additional forbidden triangles do not affect the above construction of the random bipartite graph. Thus the same argument will apply in these cases.

Although Δ_{12} and Δ_{27} do not follow exactly from this, very similar arguments can be used. For Δ_{12}, with ∞_1 and ∞_2 as above, we instead define $Y_1 = \{y \in M(\Delta_{12}) : X(\infty_1, y) \wedge A(\infty_2, y)\}$ and Y_2 as before. And for Δ_{27}, we use vertices ∞_1 and ∞_2 in $M(\Delta_{27})$ such that (∞_1, ∞_2) is green. Then $Y_1 = \{y \in M(\Delta_{27}) : X(\infty_1, y) \wedge G(\infty_2, y)\}$ and $Y_2 = \{y \in M(\Delta_{27}) : X(\infty_2, y) \wedge G(\infty_1, y)\}$ will give that the induced substructure on $Y_1 \cup Y_2$ is isomorphic to the random bipartite graph.

For Δ_{13}, we take a point ∞ and look at the A-neighbourhood of it. This is 3-coloured as $AAA \in \Delta_{13}$, means it won't have the colour A. Now the remaining forbidden triangles in Δ_{13} without A will hold in this neighbourhood. These are $\Delta' = \{XXX, XGG, XXR\}$. Hence the A-neighbourhood of ∞ is isomorphic to $M'(\Delta')$ (with X, G, R replacing R, G, B). The same this holds for the A-neighbourhood of a point in $M(\Delta_{17})$. However by Theorem 3.4.7, we know these have the random graph as finitely definable substructure.

All we have left are Δ_{11}, Δ_{15} and Δ_{16}. For Δ_{11}, we can find a structure very similar to the random bipartite graph by naming two points ∞_1 and ∞_2 such that (∞_1, ∞_2) is of colour A. Then define $Y_1 = \{y \in M(\Delta_{11}) : X(\infty_1, y) \wedge G(\infty_2, y)\}$ and $Y_2 = \{y \in M(\Delta_{11}) : X(\infty_2, y) \wedge R(\infty_1, y)\}$. Then $Y_1 \cup Y_2$ is two complete graphs in G connected by random edges of colour A and R. We claim that this graph is isomorphic to the Random bipartite graph. Indeed the only difference is that the 'non-edges' internal to the parts are of a different relation to the 'non-edges' between the parts. However as each point in the random bipartite graph knows which part it is in (they are all at distance 2 from each other), we can simply forget the colour R, and recover it if necessary.

For $\Delta = \Delta_{15}$ or Δ_{16} it follows very similarly however we need a slightly different set up. Take ∞_1 and ∞_2 such that (∞_1, ∞_2) is of colour X. Then define $Y_1 = \{y \in M(\Delta) : X(\infty_1, y) \wedge G(\infty_2, y)\}$ and $Y_2 = \{y \in M(\Delta) : X(\infty_2, y) \wedge G(\infty_1, y)\}$. Then $Y_1 \cup Y_2$ is two complete graphs in G connected by random edges of colour X and R, and the same process can be done to show this is isomorphic to the Random bipartite graph. □

Theorem 3.4.9 *Suppose $\mathcal{L} = \{R, G, A, X\}$. Then for $\Delta \in \{\Delta_1, \ldots, \Delta_{27}\}$ there does not exist a m.e.c. with ultraproduct elementarily equivalent to $M(\Delta)$.*

This is just an immediate consequence of Theorem 3.2.4, Theorem 3.4.8 and Theorem 3.1.7. As part of an Undergraduate Research Project (UROP) I did with Prof. David Evans at Imperial College London in 2016, I proved the completeness of the Cherlin's list in the case where $\mathcal{L} = \{R, G, B\}$. More specifically the theorem:

Theorem 3.4.10 *Let $\mathcal{L} = \{R, G, B\}$ be a symmetric, irreflexive, binary, relational language and suppose M is a primitive universal homogeneous L-structure with semi-free, but not free, amalgamation determined by a set of forbidden triangles. Then M is isomorphic to $M(\Delta)$ with*

$$\Delta = \{RBB,\ GGB,\ BBB\}$$

As this was not originally done as part of this PhD research (though it has been heavily revised), I've included the proof in the Appendix.

3.5 Next steps and Open Problems

The next steps to the work are generally to do with ruling out more structures. It seems that once we have certain universal structures, then reducts or other structures defined in similar languages follow fairly quickly. The main next step would be finding a general approach to applying Theorem 3.2.4. At the moment I have been looking at individual structures. However I think there is scope for finding a general theorem that will dictate when a reduct will contain the original structure as a finitely definable substructure.

I would expect that any unstable homogeneous structure that has certain limitations in which types it can use, will probably have a canonical embedding of some other structure with a lesser degree of limitation. This process would continue until you are left with the homogeneous structures with pretty much no limitations, the universal unstable structures. It is my current understanding that these will be the most important pieces of the puzzle. There are immediate examples that come to mind. The universal homogeneous 3-hypergraphs provides a good example to consider. As we saw, the classification of 5-regular graphs resulted from the study of 6-transitive graphs in [9], and their classifications look very similar. In the same paper we find the following theorem.

Theorem 3.5.1 ([9], Theorem 5.1) *Let Δ be a finite k-hypergraph and suppose that any isomorphism between induced sub-hypergraphs on at most $k + 3$ vertices extends to an automorphism of Δ. Then Δ is one of the following:*

i) the complete or null hypergraph,

ii) the hypergraph whose edges are the lines of PG$(2, 2)$, or its complement,

iii) the hypergraph whose edges are the planes of AG$(3, 2)$, or its complement,

iv) the unique regular two-graph on 6 vertices,

v) the unique regular two-graph on 10 points.

Could we use this as a starting point to get a classification of finite n-regular 3-hypergraphs? It seems likely that there may exist some equivalent to the results of Chapter 4 and [12] but utilizing design theory instead of graph theory.

In a similar vein the universal homogeneous digraph would also provide an interesting study. We saw in the case of the 3-tournament free digraph that counting arguments alone were enough to rule it out. One would hope a similar approach would work for this structure.

Another immediate open case is the universal homogeneous 3-coloured graph, which we could also hope to show is not a m.e.c limit from similar methods to the classification of 5-regular graphs. This is what we shall focus on for the rest of the thesis.

Chapter 4

3-coloured Graphs

In this chapter we will outline what we need for the study of highly regular 3-coloured graphs. It will tie together the ideas we need from association schemes and graph theory, and provide strong basic results that will be frequently used in later sections.

4.1 Notation

Throughout the following sections there will be various notation that I will continuously use. This is a continuation of the notation from [12] but extended to meet the needs of this work. Suppose M is a finite 2-regular 3-coloured graph. There is also a table in Appendix B that will summarise the notation given here and throughout the rest of the thesis. The notation we shall use will be as follows:

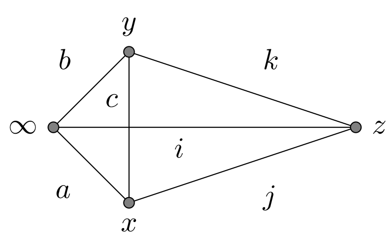
- R, B and G are symmetric irreflexive binary relations representing the 3 colours, and any pair of distinct elements of M satisfy exactly one of them.
- n is the total number of vertices in M.
- A_R, A_B, A_G are the $n \times n$ 01-adjacency matrices of the red, blue and green edges respectively.
- A non-principal eigenvalue is one that does not have eigenvector u (the all 1 vector). Here r_i, s_i and t_i will be the non-principal eigenvalues of A_i for $i \in \{R, G, B\}$. There are three as we are working in a 3-class association scheme. We shall use

the convention later that r_R, r_G and r_B will have the same eigenvectors (shown in Corollary 4.2.8).

- p^i_{jk} is the fixed number of vertices such that if x, y are connected by an edge of type i, then there are p^i_{jk} vertices z which are connected to x in colour j and y in colour k. These are the *double intersection numbers*.
- k_i is the number of edges of type i incident with any vertex.
- J will represent the all one matrix of any dimension, not necessarily square.
- E_0, E_1, E_2, E_3 form the basis of minimal idempotent matrices of the Bose-Mesner algebra of our association scheme. $nE_0 = J$.

We can now extend this to 3-regularity and into the neighbourhoods of some base vertex $\infty \in M$. It should be stated that each neighbourhood of ∞ is itself 2-regular and therefore an association scheme (not necessarily 3-class). We will therefore assume 3-regularity and extend the notation in the following way:

- N^i_{jl} is the $k_l \times k_j$ adjacency matrix of i coloured edges, where for x in the l-neighbourhood of ∞ and y in the j-neighbourhood, of ∞ $(N^i_{jl})_{x,y} = 1$ iff (x, y) is an i-edge and 0 otherwise.
- N^i_{jj} is the $k_j \times k_j$ adjacency matrix of i coloured edges, internal to the j-neighbourhood.
- r_{i_j}, s_{i_j} and t_{i_j} will be the non-principal eigenvalues of N^i_{jj}.
- p^{abc}_{ijk} is the number of vertices z in the following image:



(4.1.1)

These will be called the *triple intersection numbers*.

We should note that triple intersection numbers exist only under the assumption of 3-regularity.

So for illustrative purposes we can see the adjacency matrices fit together as follows

$$A_i = \begin{pmatrix} 0 & \delta_{iR}u_1^T & \delta_{iG}u_2^T & \delta_{iB}u_3^T \\ \delta_{iR}u_1 & N_{RR}^i & N_{RG}^i & N_{RB}^i \\ \delta_{iG}u_2 & N_{GR}^i & N_{GG}^i & N_{GB}^i \\ \delta_{iB}u_3 & N_{BR}^i & N_{BG}^i & N_{BB}^i \end{pmatrix} \tag{4.1.2}$$

with δ_{ij} being the Kronecker delta.

From the definition of our RGB structures we know that $J = I + A_R + A_G + A_B$ and that these (namely I, A_R, A_G and A_B) also form a basis of the Bose-Mesner algebra (see Section 2.2. By Theorem 1.7.1 in [23], we can choose our parameters in such a way that the transition matrices between our two bases are as follows:

	E_0	E_1	E_2	E_3
I	1	1	1	1
A_R	k_R	r_R	s_R	t_R
A_G	k_G	r_G	s_G	t_G
A_B	k_B	r_B	s_B	t_B

(4.1.3)

and

	I	A_R	A_G	A_B
nE_0	1	1	1	1
nE_1	f	$f\alpha_1$	$f\beta_1$	$f\gamma_1$
nE_2	g	$g\alpha_2$	$g\beta_2$	$g\gamma_2$
nE_3	h	$h\alpha_3$	$h\beta_3$	$h\gamma_3$

(4.1.4)

Here A_i has the eigenvalues k_i, r_i, s_i and t_i and f, g and h are the multiplicities for r_i, s_i and t_i respectively (This doesn't depend on i due to 4.2.8).

When using the basis of minimal idempotents, we can think of it as looking at things from the perspective of the eigenspaces instead of the adjacency matrices.

As we will always be dealing with 3-coloured structures we can assume that $k_i \neq 0$ for all i.

We also define a constant D via

$$D := r_G t_R - r_R t_G + s_R t_G - s_G t_R + r_R s_G - r_G s_R \tag{4.1.5}$$

Note that D is the determinant of the $(4, 1)$-minor of the transition matrix in 4.1.3. As such we shall see later in Lemma 4.2.12 that $D \neq 0$. It should also be noted that the choice of colours was immaterial here, by permuting the colours you do not change the form of the equation, however you will multiply by the signature of the permutation. This can be seen by making repeated use of $0 = 1 + r_R + r_G + r_B$ (Lemma 4.2.10) and the corresponding equations for s and t.

Some other shorthand we will use fairly frequently is:

Definition 4.1.1. We say that a 2-regular 3-coloured graph is *strongly regular in* m, if the 2-coloured graph formed by making m the edges and $j \cup l$ the non-edges is strongly regular. We will also here say the adjacency matrix A_m is strongly regular.

It is important to note that a strongly regular adjacency matrix will only have two distinct non-principal eigenvalues as it is exactly the adjacency matrix of a strongly regular graph.

4.2 Preliminary Results

Some initial but important results that come from using this notation are

Lemma 4.2.1 *If i and j are distinct colours and $k_j \neq 0$ then*

$$k_i = \sum_{k \in \{R,G,B\}} p^j_{ik}$$

and if $k_i \neq 0$ then

$$k_i = \sum_{k \in \{R,G,B\}} p^i_{ik} + 1$$

Proof. Fix a vertex x, then fix a vertex y such that (x, y) is of colour j. Then every vertex connected to x by an i coloured edge must also be connected to y in some colour k, hence

if we sum over all possibilities for k, we get the number of i-coloured edges from x.
The proof works exactly the same if (x, y) is of colour i except we have to include the edge (x, y) as well, hence the additional 1 in the sum. □

We can find similar results for the triple intersection numbers. Note that in using triple intersection numbers we are assuming 3-regularity. However first we can note that we can swap around colours inside the intersection number so long as the triangles within are preserved. More formally

Lemma 4.2.2 *For any colours x, y, z, i, j, k, we get*

$$p_{ijk}^{xyz} = p_{jik}^{xzy} = p_{ikj}^{yxz} = p_{kij}^{yzx} = p_{jki}^{zxy} = p_{kji}^{zyx}$$

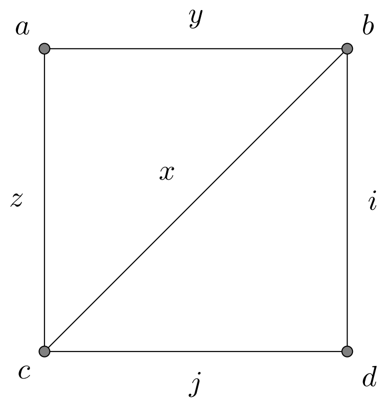
This just comes from inspection of the diagram 4.1.1. They are all valid ways of counting the point z. By using other counting arguments we get the following lemma which we will make great use of:

Lemma 4.2.3 *If x, y, z, i, j are colours and $p_{xz}^{y} \neq 0$, then if $\{i, j\} \neq \{y, z\}$ we get that*

$$p_{ij}^{x} = \sum_{k \in \{R,G,B\}} p_{ijk}^{xyz} = \sum_{k \in \{R,G,B\}} p_{ikj}^{yxz} = \sum_{k \in \{R,G,B\}} p_{kij}^{yzx}$$

If $(i, j) = (y, z)$ then we get the same but with $p_{ij}^{x} - 1$ instead.

Proof. This just comes from counting how many of the following shape exist:



There are multiple ways of doing this, however we shall choose a, then b, then c. This gives a total of $n \cdot k_y \cdot p^y_{xz}$ triangles. Now there are two ways of counting how many options for d there are. Either we can ignore the rest of the structure and note there are p^x_{ij} options (or $p^x_{ij} - 1$ options if $(y,z) = (i,j)$), or we can take the sum of all possible colours of the edge between a and d giving $\sum_{k \in \{R,G,B\}} p^{xyz}_{ijk}$ (or $\sum_{k \in \{R,G,B\}} p^{xyz}_{ijk} - 1$ if $(y,z) = (i,j)$). Hence counting the total number of triangles gives

$$n \cdot k_y \cdot p^y_{xz} \cdot p^x_{ij} = n \cdot k_y \cdot p^y_{xz} \cdot \sum_{k \in \{R,G,B\}} p^{xyz}_{ijk}$$

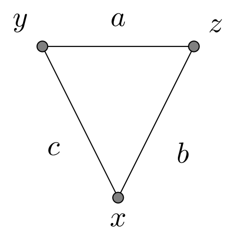
And so if $p^y_{xz} \neq 0$ we get the result. □

Further basic counting arguments give the next few results.

Lemma 4.2.4 *For colours a, b, c, j, l, m we get the following equalities:*

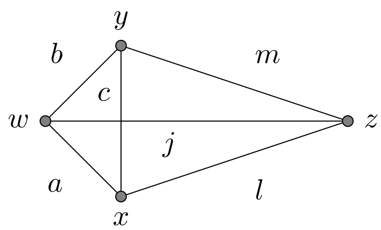
$$k_a p^a_{bc} = k_b p^b_{ac}$$
$$k_a p^a_{bc} p^{abc}_{jlm} = k_b p^b_{jm} p^{bjm}_{acl}$$

Proof. Similar to the above, this comes from counting certain structures in two ways. For the first equation we count the total number of this structure:



We see that if we first fix z, then y, then x we get $nk_a p^a_{bc}$ copies of this triangle. However if instead we fix x then z, then y we get $nk_b p^b_{ac}$ giving the desired equality.

For the second equation instead we look at the structure



We can count this in many ways as well, one way being starting with w, then x, then y, then z giving $nk_a p^a_{bc} p^{abc}_{jlm}$. Another starting with w, then y, then z, then x giving $nk_b p^b_{jm} p^{bjm}_{acl}$ and therefore the result. □

We can see from the two equations that we can arrange it so any triangle and any edge from it are brought out first, which is evident from the argument in the proof.
Following on from this we see what happens when intersection numbers are 0. Note it is convention that if for some colour x if $k_x = 0$, then $p^x_{yz} = 0$ as there can't be any points defined over an edge that doesn't exist. We get the following result.

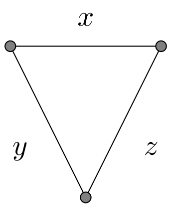
Lemma 4.2.5 *For any colours x, y, z, i, j, k,*

$$p^x_{yz} = 0 \Leftrightarrow p^y_{xz} = 0 \Leftrightarrow p^z_{yz} = 0$$

And

$$p^{xyz}_{ijk} = 0 \Leftrightarrow p^{xij}_{yzk} = 0 \Leftrightarrow p^{iyk}_{xjz} = 0 \Leftrightarrow p^{jkz}_{ixy} = 0$$

Proof. Suppose $p^x_{yz} = 0$, then there does not exist any triangles:



Therefore $p^y_{xz} = p^z_{yx} = 0$ too.
The same idea with 4 points gives the second result. □

An important but simple corollary is the following

Corollary 4.2.6 *Suppose* $p^x_{yz} = 0$. *Then for any* a, b, c, $p^{xyz}_{abc} = p^{xab}_{yzc} = 0$.

Proof. Immediately by Lemma 4.2.3 we see that for any a, b, c, $p^{xab}_{yzc} = 0$. Now we use Lemma 4.2.5 to get $p^{xyz}_{abc} = 0$ too. □

By multiplying out the matrices we get the equations

Lemma 4.2.7 *For distinct colours* j, l, m *we get*

$$A_j^2 = p^j_{jj} A_j + p^m_{jj} A_m + p^l_{jj} A_l + k_j I$$
$$A_j A_m = p^j_{jm} A_j + p^m_{jm} A_m + p^l_{jm} A_l$$

Proof. This follows directly from the definition of an association scheme. □

From this we can see that $A_j A_m = A_m A_j$ and hence from Lemma 2.2.4

Corollary 4.2.8 *There exists a basis of shared eigenvectors for* A_R, A_G *and* A_B.

By multiplying the equations from Lemma 4.2.7 on the left and right by a non-principal eigenvector we get:

Corollary 4.2.9 *For distinct colours* j, l, m *we get*

$$r_j^2 = p^j_{jj} r_j + p^m_{jj} r_m + p^l_{jj} r_l + k_j$$
$$r_j r_m = p^j_{jm} r_j + p^m_{jm} r_m + p^l_{jm} r_l$$

The same results hold for s *and* t*-eigenvalues.*

By the orthogonality of the transition matrix 4.1.3 or by multiplying the equation $J = I + A_R + A_G + A_B$ left and right by a non-principal eigenvector we get:

Lemma 4.2.10 *For eigenvalues* r_R, r_G, r_B

$$0 = 1 + r_R + r_G + r_B$$

As such we will generally remove the third eigenvalue (usually r_B) and keep everything in terms of just two. I will however switch back and forth when it is more helpful to do so. We can also now note:

Corollary 4.2.11

$$(N^x_{mm})^2 + N^x_{mj}N^x_{jm} + N^x_{ml}N^x_{lm} =$$
$$(k_x - p^z_{xx})I + (p^x_{xx} - p^z_{xx})N^x_{mm} + (p^y_{xx} - p^z_{xx})N^y_{mm} + (p^z_{xx} - \delta_{xm})J$$

Proof. This comes from considering the equation from Lemma 4.2.7 with respect to the expansion of A_x in 4.1.2.

$$A_x^2 = \begin{pmatrix} 0 & \delta_{xm}u_1^T & \delta_{xj}u_2^T & \delta_{xl}u_3^T \\ \delta_{xm}u_1 & N^x_{mm} & N^x_{mj} & N^x_{ml} \\ \delta_{xj}u_2 & N^x_{jm} & N^x_{jj} & N^x_{jl} \\ \delta_{xl}u_3 & N^x_{lm} & N^x_{lj} & N^x_{ll} \end{pmatrix} \begin{pmatrix} 0 & \delta_{xm}u_1^T & \delta_{xj}u_2^T & \delta_{xl}u_3^T \\ \delta_{xm}u_1 & N^x_{mm} & N^x_{mj} & N^x_{ml} \\ \delta_{xj}u_2 & N^x_{jm} & N^x_{jj} & N^x_{jl} \\ \delta_{xl}u_3 & N^x_{lm} & N^x_{lj} & N^x_{ll} \end{pmatrix}$$

If we look at just the entry in the second row and second column of A_x^2 we get

$$\delta_{xm}u_1\delta_{xm}u_1^T + (N^x_{mm})^2 + N^x_{mj}N^x_{jm} + N^x_{ml}N^x_{lm}$$

However, from Lemma 4.2.7, we also know that

$$A_x^2 = p^x_{xx}A_x + p^y_{xx}A_y + p^z_{xx}A_z + k_xI$$

Therefore, using $A_z = J - I - A_x - A_y$ and taking just the second row and second column again we get

$$(k_x - p^z_{xx})I + (p^x_{xx} - p^z_{xx})N^x_{mm} + (p^y_{xx} - p^z_{xx})N^y_{mm} + p^z_{xx}J$$

Hence we end up with the equality

$$(N^x_{mm})^2+N^x_{mj}N^x_{jm}+N^x_{ml}N^x_{lm} = (k_x-p^z_{xx})I+(p^x_{xx}-p^z_{xx})N^x_{mm}+(p^y_{xx}-p^z_{xx})N^y_{mm}+(p^z_{xx}-\delta_{xm})J$$

□

Recall the constant D defined via 4.1.5. Utilising Lemma 4.2.10 gives us the following:

Lemma 4.2.12 *In a 2-regular 3-coloured structure $D \neq 0$.*

Proof. Consider the transition matrix from Equation 4.1.3. By adding every row to the bottom row and using Lemma 4.2.10, we see the determinant of the transition matrix is the same as that of the following matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ k_R & r_R & s_R & t_R \\ k_G & r_G & s_G & t_G \\ n & 0 & 0 & 0 \end{pmatrix}$$

The determinant of this matrix is nD and hence the determinant of the transition matrix is $-nD$. The determinant of this matrix must be non-zero and hence D must be. □

We can actually get a general form for any multiplication of adjacency matrices, presuming their dimensions match.

Lemma 4.2.13 *For colours $a_1, a_2, a_3, b_1, b_2, b_3$, if $a_3 = b_2$ and $(a_1, a_2) \neq (b_1, b_3)$ then*

$$N^{a_1}_{a_2a_3}N^{b_1}_{b_2b_3} = p^{a_2b_3R}_{a_3a_1b_1}N^R_{a_2b_3} + p^{a_2b_3G}_{a_3a_1b_1}N^G_{a_2b_3} + p^{a_2b_3B}_{a_3a_1b_1}N^B_{a_2b_3}$$

If $(a_1, a_2) = (b_1, b_3)$ then

$$N^{a_1}_{a_2a_3}N^{b_1}_{b_2b_3} = p^{a_2}_{a_1a_3}I + p^{a_2b_3m}_{a_3a_1b_1}N^m_{a_2b_3} + p^{a_2b_3j}_{a_3a_1b_1}N^j_{a_2b_3} + p^{a_2b_3l}_{a_3a_1b_1}N^l_{a_2b_3}$$

Proof. This follows from direct calculation. We shall first select a row in $N^{a_1}_{a_2a_3}$. This selection corresponds to choosing a vertex in the a_2-neighbourhood of ∞, let's call this x_1.

Each of the non-zero entries in this row represent an edge of colour a_1 going from x_1 into the a_3-neighbourhood. Next we select a column in $N^{b_1}_{b_2 b_3}$. The selection corresponds to a vertex x_2 in the b_3-neighbourhood of ∞, and the non-zero entries in this column represent the edges of colour b_1 going from x_2 to the $b_2 = a_3$ neighbourhood. So the value of the multiplication of x_1's row and x_2's column is the size of the overlap of the two. Note that if $a_2 = b_3$ then we could be selecting the same vertex for both the row and column, i.e. we could have $x_1 = x_2$. In this case they will overlap completely if $a_1 = b_1$ and not at all if $a_1 \neq b_1$. This corresponds to $p^{a_2}_{a_1 a_3}$ or 0 in the positions where $x_1 = x_2$, which is I. Hence if $(a_1, a_2) = (b_1, b_3)$, we will have an extra $p^{a_2}_{a_1 a_3} I$.
Now, supposing $x_1 \neq x_2$, we see that the size of the overlap is determined by the colour of the edge connecting x_1 and x_2. We shall call this c. The size of the overlap is the number of vertices y defined over a a_2, b_3, c triangle with connections a_3 between ∞ and y, a_1 between x_1 and y and b_1 between x_2 and y. Hence it has value $p^{a_2 b_3 c}_{a_3 a_1 b_1}$. The positions where we get this value will correspond to the matrix $N^{c}_{a_2 b_3}$ and therefore, summing over all possible c we get

$$N^{a_1}_{a_2 a_3} N^{b_1}_{b_2 b_3} = p^{a_2 b_3 R}_{a_3 a_1 b_1} N^{R}_{a_2 b_3} + p^{a_2 b_3 G}_{a_3 a_1 b_1} N^{G}_{a_2 b_3} + p^{a_2 b_3 B}_{a_3 a_1 b_1} N^{B}_{a_2 b_3}$$

□

Remark 4.2.14 *We can (and frequently do) use the fact that, for distinct colours m, j and l, $J = I + N^{m}_{mm} + N^{j}_{mm} + N^{l}_{mm}$ and $J = N^{m}_{mj} + N^{j}_{mj} + N^{l}_{mj}$ to make these equations just in terms of two matrices.*

It will also be important to note the following basic fact

Remark 4.2.15 *If for some colour m, $k_m = 0$ then $r_m = s_m = t_m = 0$. Further for some colour c, if $p^{m}_{mc} = 0$ then $r_{c_m} = s_{c_m} = t_{c_m} = 0$.*

This is simply because the adjacency matrix is 0, so all eigenvalues must be.

4.3 Amorphicity

A fairly large set of possible finite 3-coloured 3-regular graphs are those which are such that each colour is strongly regular when considered as a graph in it's own right,

Definition 4.3.1 ([47]). A 3-coloured graph is *Amorphic* if for any distinct colours X, Y, Z, the graph with edges X and non-edges $Y \cup Z$ is strongly regular.

These are troublesome as a lot of our work focuses on using the interactions of neighbourhoods to show that eigenvalues within specific neighbourhoods are equal. However, these graphs are natural examples of such phenomena. We can still say a lot about them though, for example:

Theorem 4.3.2 ([47], Theorem 4.1) *If all three relations of a 3-class association scheme are strongly regular graphs, then they either have parameters* $(n^2, l_i(n-1), n + l_i(l_i - 3), l_i(l_i - 1))$*,* $i = 1, 2, 3$ *or* $(n^2, l_i(n+1), -n + l_i(l_i + 3), l_i(l_i + 1))$*,* $i = 1, 2, 3$.

The proof of this is attributed by Van Dam to Higman [26]. We've also seen these forms before, as they are exactly the Pseudo and Negative Latin square graphs from [12]. In future chapters we can see that the tricolour Clebsch graph (Definition 5.3.2) satisfies the parameters as a negative Latin square graph as well.

We can calculate the eigenvalues of these graphs fairly simply

Lemma 4.3.3 *The eigenvalues of a Pseudo Latin square graph are* $r_i = -l_i$ *with multiplicity* $n^2 - 1 - l_i(n-1)$ *and* $s_i = n - l_i$ *with multiplicity* $l_i(n-1)$.
The eigenvalues of a Negative Latin square graph are $r_i = l_i$ *with multiplicity* $n^2 - 1 - l_i(n+1)$ *and* $s_i = l_i - n$ *with multiplicity* $l_i(n+1)$.

This is immediate from inputting the parameterisations into the equations for the eigenvalues of a strongly regular graph in Lemma 4.2.9.

We can also see that any eigenvector, v of either of these schemes will have eigenvalues of the form (r_1, r_2, s_3), (r_1, s_2, r_3) or (s_1, r_2, r_3). There is also the possibility of the neighbourhoods also being amorphic themselves. Then we can parameterise them in the same way. This leads to the result:

Lemma 4.3.4 *Suppose there exists a m.e.c $\mathcal{C}$ with ultraproduct elementarily equivalent to an unstable homogeneous 3-coloured graph. Then for sufficiently large $M \in \mathcal{C}$, if M is amorphic and x, y are distinct vertices and non-adjacent in some colour R, then $R(x) \cap R(y)$ cannot be amorphic.*

Proof. We know by lemma 2.5.5 that any sufficiently large member of $\mathcal{C}$ will be 4-regular. So therefore suppose for a contradiction that we have $M \in \mathcal{C}$ large enough to be 4-regular and also with x, y distinct vertices in M, non-adjacent in R, and such that $R(x) \cap R(y)$ is amorphic. We know by Theorem 4.3.2 applied to $R(x) \cap R(y)$ (which is 2-regular), that $|R(x) \cap R(y)|$ is a square. However we also know by applying Theorem 4.3.2 to M, that $|R(x) \cap R(y)| = \mu = l_i(l_i + 1)$ or $l_i(l_i - 1)$. As these are products of two consecutive integers they can't possibly be a square, a contradiction. □

This lemma is more powerful than it originally seems. It tells us that if we can show that any finite primitive 3-regular 3-coloured graph is amorphic, then we can answer the m.e.c conjecture.

Chapter 5

Known Examples

In this section I'll present the only two known examples of finite primitive 3-regular, 3-coloured graphs that were found during the course of this work, along with proofs they are indeed 3-regular. Before this we will go over some essential prerequisite ideas that will be used to show those graphs are 3-regular. We will do this by deriving them from the known automorphism groups of finite binary homogeneous structures. These are discussed in Cherlin's [17] and [18].

5.1 Binary permutation groups

This section will discuss literature and results regarding permutation groups. A permutation group G acting on a set X will be denoted by (X, G).

Definition 5.1.1. The *relational complexity* $\rho(X, G)$ of the permutation group G acting on the set X may be defined as the least k for which (X, G) can be viewed as $(\hat{X}, \mathrm{Aut}(\hat{X}))$ with $\hat{X}$ a homogeneous structure whose relations are k-ary.
Alternatively, in terms of permutation groups, *relational complexity* is defined as the least k such that for all $\mathbf{a}, \mathbf{b} \in X^n$ we have

$$\mathbf{a} \sim_k \mathbf{b} \Leftrightarrow \mathbf{a} \sim \mathbf{b}$$

where on the left, $\sim_k$ means that any corresponding k-tuples from **a** and **b** lie in the same

G-orbit, and on the right, $\sim$ means that **a** and **b** lie in the same G-orbit.

As we deal with binary structures, we see therefore that we will be dealing with the case where the relational complexity is 2. These are known as *binary* permutation groups. Other definitions we will need are:

Definition 5.1.2. A permutation group (X, G) is *primitive* if G does not preserve any proper non-trivial equivalence relation on X. A primitive permutation group (X, G) is *affine* if the *socle* of G, the direct product of the minimal normal subgroups of G, is abelian.

If (X, G) is a finite affine primitive permutation group with socle A, then A has the form $(C_p)^k$ for some prime p, written additively. Since A acts regularly on X, we may identify A with X, and G with a semi-direct product AG', where G' is the stabiliser of the elements 0 of A. Here A acts by translation on itself and G' acts by conjugation.

These groups are classified by Cherlin in the following Theorem:

Theorem 5.1.3 (Theorem 1, [18]) *Let (A, AG') be a finite primitive affine binary permutation group. Then either $|G'|\leq 2$ and $|A|\cong C_p$ is cyclic of prime order, or else A can be given the structure of a two-dimensional vector space over a finite field $\mathbb{F}_q$ with $G' = O_2^-(q)$, where A acts by translation and G' acts naturally.*

As Cherlin describes the first case can be thought of as giving a group A or $A\langle\pm 1\rangle$ which is primitive only if A is 1-dimensional over $\mathbb{F}_p$ (AG' is a dihedral group if $|G'|= 2$).

The second case is trickier. It gives a family of examples $VO_2^-(q)$ where V is 2-dimensional over a finite field $\mathbb{F}_q$. More explicitly, Cherlin describes it as using the following process. Identify V with the quadratic extension $\mathbb{F}_{q^2}$ of the base field. Then $O_2^-(q)$ can be thought of as $K\langle\sigma\rangle$ where $\langle\sigma\rangle = \text{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$ and K is the kernel of the norm map from $\mathbb{F}_{q^2}$ to $\mathbb{F}_q$.

By definition of relational complexity, we see that if (A, AG') is a primitive affine binary permutation group, then A is a homogeneous binary relational structure. If a structure is homogeneous, then it is n-regular for any $n \leq |A|$. Therefore to show a 3-coloured structure M is 3-regular (or indeed n-regular) it is enough to show that $(M, \text{Aut}(M))$ is a primitive affine binary permutation group, with 3 orbits on ordered pairs of distinct elements, all symmetric.

This provides us with a strong tool for identifying whether or not a primitive 3-coloured structure can be 3-regular. Indeed, in order to discover the Clebsch graph example (5.3), we considered the implications of this Theorem in the case where the 2-dimensional permutation group would allow for three orbits. However this Theorem alone does not provide sufficient conditions. For this we look to Cherlin's earlier work on the subject in [17].

This paper is on the same subject however uses slightly different terminology. Cherlin describes a group, G as being *strictly linear* and of *dimension* d if it is a subgroup of $\mathrm{AGL}(V)$ for V the d-dimensional translational subgroup of G and $\mathrm{AGL}(V) = V \rtimes \mathrm{GL}(V)$. There is also a slightly broader class of *non-strictly linear* affine groups which live in $\mathrm{A\Gamma L}(V)$ but not $\mathrm{AGL}(V)$, where $\mathrm{A\Gamma L}(V) = V \rtimes \Gamma\mathrm{L}(V)$.

Theorem 5.1.4 (Example 7, [17]) *Let G be a primitive 1-dimensional affine group, not strictly linear. Then the relational complexity of G is strictly less than 5 and G is binary if and only if G has the form* $\mathbb{F}_{q^2} \rtimes (\mu_{q+1} \rtimes \langle\sigma\rangle)$ *with* σ *of order 2.*

Crucially Cherlin then remarks on potential examples of this:

Remark 5.1.5 (Remark Page 14, [17]) *Let* Γ_q *be the binary structure corresponding to the binary 1-dimensional affine group* $\mathbb{F}_{q^2} \rtimes (\mu_{q+1} \rtimes \langle\sigma\rangle)$*. Then* Γ_q *is a symmetric graph with an edge coloring by* $q-1$ *colors.*

He goes on to describe the $q = 4$ case, stating that it produces a graph of order 16 with a 3-edge colouring and no monochromatic triangles, the Tricolour Clebsch Graph. This graph will be discussed further in Section 5.3.

5.2 The Tricolour Heptagon

Definition 5.2.1. The *tricolour heptagon* is the 3-coloured graph on 7 vertices such that each vertex is a point in $\mathbb{F}_7$, and for $a, b \in \mathbb{F}_7$, (a, b) is red if $|a-b|= 1$, green if $|a-b|= 2$ and blue if $|a-b|= 3$.

Visually, this looks like this

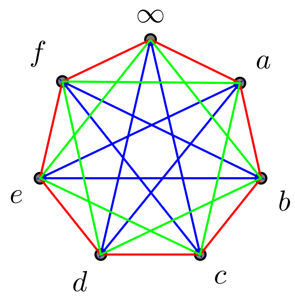


Figure 5.1: Tricolour Heptagon

This is clearly regular as by definition each vertex will have 2 neighbours in each colour. We can prove it is 3-regular by finding and examining its automorphism group.

Lemma 5.2.2 *Let H be the Tricolour Heptagon. Then* $\mathrm{Aut}(H) \cong D_7$.

Proof. We can clearly see that $C_7 \subset \mathrm{Aut}(H)$. Now we can also see that in the stabiliser of the point, say ∞, we get a reflection $\sigma = (ab)(cd)(ef)$. As any automorphism in stab(∞) will preserve coloured neighbourhoods of ∞, we see a could only go to b, c to d and e to f and vice versa. If a was fixed as well, then c must also be fixed, and therefore e too, hence nothing moves. As every point is isomorphic to a (in some colour), the stabiliser of any point is C_2. Hence $D_7 \subset \mathrm{Aut}(H)$ and by the Orbit-stabilizer theorem, we must have the entire automorphism group. □

This fits with Cherlin's description of the 1-dimensional case in Theorem 5.1.3, however we still have to prove it is a binary affine permutation group (we know it's primitive as 7 is prime).

In this context D_7 can be identified with the group generated by the functions $f(x) = x + 1$ and $g(x) = -x$ on $\mathbb{F}_7$.

Theorem 5.2.3 *The Tricolour Heptagon is homogeneous.*

Proof. We will show that any isomorphism between substructures can be extended by one point. This is enough to prove homogeneity by a back and forth argument.

Therefore consider any 2 isomorphic n-tuples in $\mathbb{F}_7$, say $\mathbf{a}_n$ and $\mathbf{b}_n$. Then there exists a function $\sigma(x) = (-1)^l(x + m)$ for some l and m such that $\sigma(\mathbf{a}_n) = \mathbf{b}_n$. We can enumerate

$\mathbf{a}_n$ and $\mathbf{b}_n$ such that $\sigma(a_i) = b_i$ for all i. Now extend $\mathbf{a}_n$ to $\mathbf{a}_{n+1}$ by a new vertex a_{n+1}, and add $(-1)^l(a_{n+1} + m) = b_{n+1}$ to $\mathbf{b}_n$ to get $\mathbf{b}_{n+1}$. We claim that $\mathbf{a}_{n+1}$ and $\mathbf{b}_{n+1}$ are still isomorphic.

First we need to show that b_{n+1} is distinct from all $b_i \in \mathbf{b}_n$. Well suppose for some i, $b_i = b_{n+1}$, then $b_i = (-1)^l(a_{n+1} + m)$, implying $(-1)^l(a_i + m) = (-1)^l(a_{n+1} + m)$, so $a_i = a_{n+1}$ a contradiction. Now we need to show that the new edges are the same colour, i.e for all i, $|a_{n+1} - a_i| = |b_{n+1} - b_i|$. Note that this will be enough, as all other types will be determined by the constituent edges. Well

$$\begin{aligned} |b_{n+1} - b_i| &= |(-1)^l(a_{n+1} + m) - (-1)^l(a_i + m)| \\ &= |(-1)^l(a_{n+1} + m - a_i - m)| \\ &= |a_{n+1} - a_i| \end{aligned}$$

So all the edges match up as well. □

Referring back to the list in the Appendix D of [47], we see that this structure is the smallest possible with all integral eigenvalues.

5.3 The Tricolour Clebsch graph

The other finite primitive 3-regular, 3-coloured graph we find comes from the study of 3-homogeneous structures. I have Dugald Macpherson to thank for the process of finding this example and Gregory Cherlin for the result showing it is 3-regular (and in fact entirely homogeneous by Remark 5.1.5).

We'll start by defining the Clebsch graph in the two-colour case.

Definition 5.3.1 ([14])**.** The *Clebsch graph* is a strongly regular graph with parameters $(16, 5, 0, 2)$. It can be constructed by allowing the vertices to be the even sized subsets of the set $\{1, 2, 3, 4, 5\}$ with two subsets connected if their symmetric difference is of size 4.

It has been shown by Greenwood and Gleason [24, Theorem 4] that there is a three coloured version. This is defined using cubic residues on the field $\mathbb{F}_{16}$. Let x be the generator of the multiplicative group of $\mathbb{F}_{16}$, then the set of cubes in $\mathbb{F}_{16}$ is the following set

$$R := \{x^3,\ x^3+x^2,\ x^3+x,\ x^3+x^2+x+1,\ 1\}$$

We then note that the multiplicative cosets are the following:

$$G := \{x+1,\ x^3+x+1,\ x^2+x+1,\ x^3+x^2+1,\ x\}$$
$$B := \{x^2+x,\ x^2+1,\ x^3+x^2+x,\ x^3+1,\ x^2\}$$

We can now formally define the graph using this

Definition 5.3.2 ([24])**.** The *Tricolour Clebsch Graph* is the 3-coloured graph which takes as its vertex set the elements of $\mathbb{F}_{16}$. For any two distinct vertices $a, b \in \mathbb{F}_{16}$, the edge (a, b) is coloured red if $a - b \in R$, coloured green if $a - b \in G$ and coloured blue if $a - b \in B$.

This can be represented by the following diagram

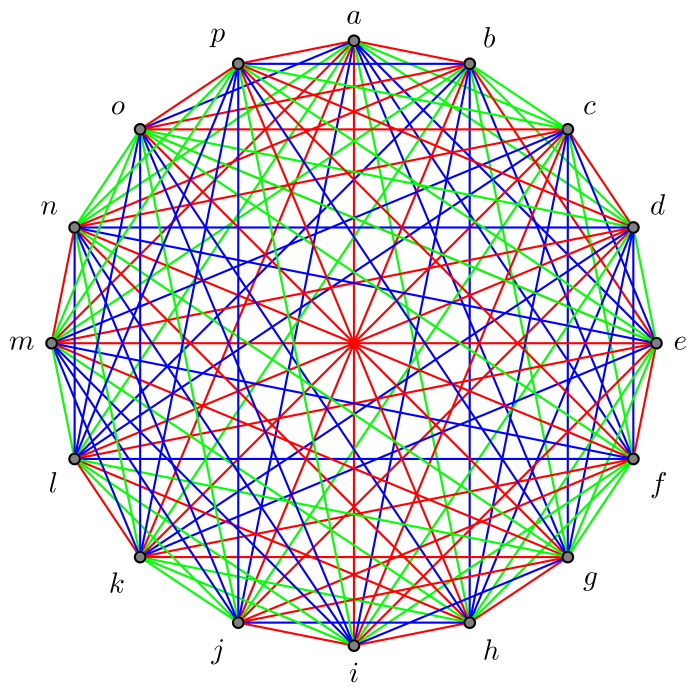


Figure 5.2: Tricolour Clebsch Graph

By the results of Cherlin discussed in Remark 5.1.5, we get:

Theorem 5.3.3 ([17]) *The Tricolour Clebsch Graph is homogeneous.*

And therefore

Corollary 5.3.4 *The Tricolour Clebsch Graph is n-regular for all $n \leq 15$.*

We also have the following small but useful result

Lemma 5.3.5 *[19, Lemma 11] Let Γ be a connected graph with spectrum $\{[5]^1, [1]^f, [-3]^g\}$ for some positive integers f and g. Then Γ is the Clebsch graph.*

This tells us that the Clebsch graph is uniquely determined by its spectrum as a 2-coloured graph, and therefore is uniquely determined by its parameters as a strongly regular graph (as these are determined by the eigenvalues ([12, Lemma 2.1]). This carries over into three colours, as the intersection numbers determine the strongly regular graph parameters for each colour.

Chapter 6

The Imprimitive Case

In the case of 3-regular 3-coloured graphs, the imprimitive case refers to the existence of at least one non-trivial proper equivalence relation on either one or two colours. An equivalence relation on one colour, say R, would take the form of all R-paths being completed only by red edges. Formally the equivalence relation would be $R^=$ where $R^=(x,y)$ if and only if $R(x,y) \vee (x=y)$. Similarly an equivalence relation on two colours B and G would take the form of all B/G-paths being completed by B or G edges. This will be denoted $B^= \cup G^=$.
In terms of intersection numbers, an equivalence relation $R^=$, means that $p^R_{RR} = k_R - 1$ and $p^G_{RR} = p^B_{RR} = 0$, and an equivalence relation $B^= \cup G^=$ means that $p^R_{GG} = p^R_{BG} = p^R_{BB} = 0$.

In this chapter we shall be discussing finite 3-coloured graphs of varying regularity. We shall show that we can completely describe the equivalence relations when we have 4-regularity by Theorem 6.1.4, and that in most cases their form is dictated by a 2-coloured graph. Hence by the earlier work of Buczak [8], we can classify them in Theorem 6.1.6.

We will then discuss some brief ideas on locally imprimitive graphs i.e. when the neighbourhood of a base point is imprimitive.

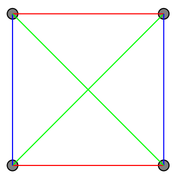
6.1 Globally imprimitive

Lemma 6.1.1 *There does not exist any finite, nontrivial, imprimitive, 2-regular, 3-coloured graphs where the equivalence relations are of the form:*

- *$R^=$, $G^=$ and $R^= \cup G^=$ are equivalence relations*
- *$R^=$ and $G^= \cup B^=$ are equivalence relations*

Proof. Suppose we have a 2-regular 3-coloured graph M with equivalence relations $R^=$, $G^=$ and $R^= \cup G^=$. Then $p^R_{RG} = p^R_{RB} = p^G_{GB} = p^G_{GR} = p^B_{RG} = 0$, implying either $k_G = 0$ or $k_R = p^G_{GR} + p^G_{RR} + p^G_{RB} = 0$ by Lemma 4.2.1, contradicting the fact it is 3-coloured. If instead $R^=$ and $G^= \cup B^=$ are equivalence relations, then $p^R_{RG} = p^R_{RB} = p^R_{GB} = p^R_{GG} = p^R_{BB} = 0$ as any G/B triangle must be completed with G or B. Therefore either $k_R = 0$ or $k_G = p^R_{RG} + p^R_{GG} + p^R_{GB} = 0$ by Lemma 4.2.1, contradicting the fact it is 3-coloured again. □

Lemma 6.1.2 *In a finite nontrivial, imprimitive, 2-regular, 3-coloured graph Γ if $R^=, G^=$ and $B^=$ are equivalence relations, then $k_R = k_B = k_G = 1$ and Γ has the form*



Proof. Suppose M was such a structure. Then $p^R_{RG} = p^R_{RB} = p^G_{GB} = p^G_{GR} = p^B_{BG} = p^B_{BR} = 0$. We can note that for there to exist blue and green edges there must be more than one red clique, so consider two red cliques, and take a point x in one of them. As $p^B_{RB} = 0$, there can only be one blue edge from x into the other clique. Similarly $p^G_{RG} = 0$ implies there can only be one green edge from x into the other clique. There must be a total of two edges from x into the other clique. Hence the red cliques are of size 2 and $k_R = 1$. We can do the same thing with the other colours to get $k_G = k_B = 1$ as well. The only possible way of constructing this graph up to isomorphism is as shown. □

Therefore for an imprimitive 2-regular, 3-coloured graph we have just the cases as follows for possible equivalence relations:

i) $R^=$

ii) $R^=, G^=$

iii) $R^= \cup G^=$

iv) $R^=, R^= \cup G^=$

Lemma 6.1.3 *Suppose M is a finite 4-regular 3-coloured graph, with an equivalence relation on exactly one colour. Then either $p^R_{BG} = 0$ or there exists another equivalence relation on a different colour.*

Proof. Suppose M is a finite 4-regular, 3-regular graph with $G^=$ being an equivalence relation. Then M is made up of complete graphs of size $k_G + 1$, where any two complete graphs are connected by a mixture of red and blue edges. Suppose $p^R_{GB} \neq 0$, i.e. between any two blocks, there exists both red and blue edges.

Now take two blocks C_1 and C_2, and consider the induced subgraph on $C_1 \cup C_2$. We claim this must be 3-regular as well. Indeed, we can see that the definable sets over any given 2-type or 3-type, except those entirely made of G-relations, are either entirely contained within $C_1 \cup C_2$ or entirely not. So all we have to check are the definable sets over a green edge and the green triangle.

So suppose we have we have x_1, x_2, such that (x_1, x_2) is green. Then we know x_1, x_2 are either both in C_1 or C_2, so say C_1. We don't have to worry about definable sets involving more green edges, as these are either entirely in C_1 (for GGG) or empty in M (GGX). Hence for colours $j, l \in \{R, B\}$, we consider the set $Y_{jl} = \{y \in M : j(x_1, y) \wedge l(x_2, y)\}$, and we need to show that the intersection of this and C_2 is the same size for any (x_1, x_2) (we don't have to worry about C_1 as the intersection is empty). This is possible by fixing any point, $x_3 \in C_2$, then a point $y \in Y_{jl}$ is in C_2 if $G(x_3, y)$ or $x_3 = y$ and not if $\neg G(x_3, y)$. Hence $|C_2 \cap Y_{jl}| = |\{y \in M : j(x_1, y) \wedge l(x_2, y) \wedge G(x_3, y)\}|$ ($+1$ if $x_3 \in Y_{jl}$) which is a fixed number by 3-regularity.

Now suppose that z_1, z_2, z_3 form a GGG triangle in C_1. We can do the exact same thing to find the size of the set $C_2 \cap \{y \in M : j(z_1, y) \wedge l(z_2, y) \wedge m(z_3, y)\}$, however this now requires 4-regularity of the graph.

Hence we have that $C_1 \cup C_2$ is 3-regular. Now we can add unary predicates to the language that differentiate C_1 and C_2. These can be thought of as vertex colourings of the parts, and so from [[25], Lemma 4.8], we know that the two parts form a matching in R or B, or are entirely connected in R or B. If we have the latter, then $p^R_{GB} = 0$, so suppose we have a matching for R. This is equivalent to saying that $p^R_{RG} = 0$ inside $C_1 \cup C_2$, but as discussed earlier, the definable sets over non-green 2-types are the same in the full graph, and so $p^R_{RG} = 0$ in M.

Now fix a point ∞ in M, and consider the B-neighbourhood of a point. This will again be made of green blocks with these blocks being the same as before, however missing one point, the red neighbour of ∞ interior to the green block. Suppose $p^R_{RB} \neq 0$ in M, i.e the matchings do not necessarily align between the blocks. Now consider two green blocks in the B-neighbourhood of ∞, C'_1 and C'_2, and the extensions of these blocks in M, C_1 and C_2. So $C'_1 = C_1 \setminus \{y_1\}$ and $C'_2 = C_2 \setminus \{y_2\}$, where y_1 is the red neighbour of ∞ in C_1 and y_2 is the red neighbour of ∞ in C_2. As $p^R_{RB} \neq 0$, we can assume there exists some point x in C'_1 such that x is connected to y_2 by a red edge (as if there does not, we can re-select our block C_1 such that there does). Now we see that between x and C'_2 there cannot exist a red edge, and so $p^{BBB}_{BRG} = 0$. But then for any other vertex in C'_1 there cannot exist a red edge to C'_2 either. However this implies they all must have a red edge with y_2, which can't happen. Hence $p^R_{RB} = 0$.

With $p^R_{RG} = p^R_{RB} = 0$, we see that $R^=$ also forms an equivalence relation. Therefore either $p^R_{GB} = 0$ or $R^=$ is an equivalence relation. □

Theorem 6.1.4 *Any imprimitive 4-regular 3-coloured graph is either*

i) Disconnected X-blocks of size k_1, arranged further into k_3-many XY-blocks, where each XY-block contains k_2-many X-blocks connected entirely by edges of colour Y. Between each XY-block there are edges of colour Z.

ii) Disconnected cliques of size k_1 in colour X which form the 'vertices' of a 4-regular graph in Y and Z.

iii) Multiple isomorphic 4-regular graphs in colours X and Y entirely connected to each other by edges in Z.

iv) A 2-coloured rook graph, with horizontal edges coloured X and vertical edges coloured Y, and the non-edges coloured Z.

Proof. We shall prove this by a case by case analysis of the types of equivalence relations we can encounter.

Firstly we shall work with just the equivalence relation $R^{=}$ (or $B^{=}$ or $G^{=}$). So suppose we have a 4-regular RGB structure, M, where $R^{=}$ is an equivalence relation. In terms of intersection numbers this means $p^G_{RR} = p^B_{RR} = 0$ and $k_R = 1 + p^R_{RR}$. And so we have cliques in R of size $k_R + 1$ with either blue or green edges connecting them. By Lemma 6.1.3, we know in this case $p^R_{BG} = 0$ or $G^{=}$ is also an equivalence relation.

Suppose first $p^R_{BG} = 0$, now cliques R_1 and R_2 are either connected entirely by blue or green. Label all the cliques in M with R_i for $1 \geq i \geq \frac{n}{k_R+1}$. Let Γ be a graph defined on $\frac{n}{k_R+1}$ vertices, with v_i connected to v_j if and only if R_i is connected to R_j by blue edges (basically think of Γ as the quotient on the set of equivalence classes of R). Now Γ is 3-regular as for any triangle $\{v_i, v_j, v_k\}$ the number of points connected to v_i, v_j and v_k is exactly the number of cliques connected by blue edges to R_i, R_j and R_k. Hence we are in scenario ii) from the theorem.

If we have R and G both equivalence relations, then we know $p^R_{RG} = p^R_{RB} = p^G_{GR} = p^G_{GB} = 0$. This implies $p^B_{RG} = 1$, $p^R_{GB} = k_G$, $p^G_{RB} = k_R$ and $k_B = k_R k_G$. Now $n = (k_R + 1)(k_G + 1)$. We see therefore that, if we forget the distinction between red and green and view these only as edges, we have all the necessary and sufficient conditions for a rook graph from [37] and [28].

Now we shall consider the equivalence relation $R^{=} \cup G^{=}$. Here we get that $p^B_{RR} = p^B_{RG} = p^B_{GG} = 0$. Therefore the structure will split into blocks containing only red and green edges, with only blue connecting each block. As we have no restrictions just involving red and green, the blocks can be any 4-regular graph. Hence we are in situation iii).

The final set of equivalence relations is $R^{=}$ and $R^{=} \cup G^{=}$ together. Here we have $p^G_{RR} = p^B_{RR} = p^B_{RR} = p^B_{RG} = p^B_{GG} = 0$. Therefore we have cliques in R which are entirely connected to $\frac{k_B}{k_R+1}$ other cliques by entirely blue edges. These blocks of only blue and red edges are then connected by green edges. Hence we are in case i).

□

We see that the graph described in Lemma 6.1.2 fits into case iv). Notice that case ii) and iii) are defined by primitive 4-regular 2-coloured graphs, as if they were imprimitive they would fall under case i). Hence they are classified by the theorem of Buczak.

Definition 6.1.5. For a 2-coloured graph Γ in R and G, we can form a 3-coloured graph by replacing each vertex x_1 with a set of vertices X_1 of size k_1, where X_1 is complete in a third colour B. This 3-coloured graph shall be known as the *k_1-quotient of G*.
Similarly we can form a graph be introducing k_2 isomorphic copies of Γ and entirely connecting every vertex unconnected vertex by a new colour B. This shall be known as the *k_2-extension of Γ*.

Using this definition we can see that case ii) refers to the k_1-quotients of any 4-regular graph and case iii) refers to the k_2-extension of any 4-regular graph.

Corollary 6.1.6 *Any imprimitive 4-regular 3-coloured graph of case ii) from Theorem 6.1.4 is the k_1-quotient of one of the graphs from Theorem 2.3.13, for any $k_1 > 1$.*
Any imprimitive 4-regular 3-coloured graph of case iii) from Theorem 6.1.4 is the k_1-extension of one of the graphs from Theorem 2.3.13, for any $k_1 > 1$.

Similarly we can use the classification of 5-regular graphs in [12]

Corollary 6.1.7 *Any imprimitive 5-regular 3-coloured graph of case ii) from Theorem 6.1.4 is the k_1-quotient of one of the graphs from Theorem 2.3.9, for any $k_1 > 1$.*
Any imprimitive 5-regular 3-coloured graph of case iii) from Theorem 6.1.4 is the k_1-extension of one of the graphs from Theorem 2.3.9, for any $k_1 > 1$.

Further we now have enough to verify the m.e.c conjecture for the case of imprimitive 3-coloured graphs:

Theorem 6.1.8 *There is no unstable imprimitive homogeneous 3-coloured graph M that is elementarily equivalent to the ultraproduct of a m.e.c.*

Proof. Suppose there exists a m.e.c $\mathcal{C}$ with ultraproduct elementarily equivalent to M. As M is imprimitive, we know that in the theory of M there will exist a sentence saying that the union of some relations (and equality) forms an equivalence relation. After thinning out the m.e.c $\mathcal{C}$, we know that every sentence true of M is true in cofinitely many of the members of $\mathcal{C}$. Combining this with Lemma 2.5.5, we know that any sufficiently large

$\Gamma \in \mathcal{C}$, is imprimitive and 5-regular. Hence it will belong to one of the four types described in Theorem 6.1.4. As the type of equivalence relation is also determined by a sentence in $\mathrm{Th}(M)$, we also know that each $\Gamma \in \mathcal{C}$ large enough to be imprimitive and 5-regular will be of the same type in Theorem 6.1.4. Hence we can effectively just think of m.e.cs of such structures.

We can see that any m.e.c of imprimitive 5-regular graphs of type i) and iv) will have a stable limit and so cannot be M. Now we look at m.e.cs of imprimitive 5-regular graphs of type ii). Here either k_1 goes to infinity or it is fixed. If k_1 is fixed then the m.e.c limit would be the k_1 quotient of an infinite homogeneous graph G_1. Now G_1 will itself be the limit of a m.e.c of graphs, and we know that no such unstable homogeneous graph exists by Theorem 3.1.4. Therefore G_1 must be stable, meaning M is too.
If instead k_1 varies to infinity, every large enough member of $\mathcal{C}$ will be a quotient of the same graph G. Hence the m.e.c limit is the ω-quotient of G, which is stable, and therefore M is stable too.

The same argument works for case iii) but with extensions instead of quotients. □

We can also make the following remark

Remark 6.1.9 *Suppose for any colour m, we have that $k_m = 1$. Then we see that the structure has an equivalence relation in m, with classes of size two. Hence when we assume a structure is finite primitive and 3-coloured, we know that $k_m \geq 2$, for all colours m.*

6.2 Locally Imprimitive

As a lot of the analysis in later sections involves looking at the structure on neighbourhoods we need to think about what happens when the neighbourhoods possess non-trivial equivalence relations. Given an imprimitive neighbourhood, it is not immediate that the full graph is imprimitive. The following is a generalisation of Lemma 3.1 from [22] to the 3-coloured case.

Lemma 6.2.1 *In a 3-regular 3-coloured graph Γ and any vertex $\infty \in \Gamma$, if the R-neighbourhood of ∞ has a non-trivial equivalence relation $B^= \cup G^=$, then the entirety* *of* Γ *has an equivalence relation* $B^= \cup G^=$.

Proof. Let the R,G and B-neighbourhoods of ∞ be $\mathcal{R}$, $\mathcal{G}$ and $\mathcal{B}$ respectively. Consider two vertices u and v in $\mathcal{R}$, such that they are in different $B \cup G$ components. Now note the number of red neighbours of u in $\mathcal{G} \cup \mathcal{B}$ is $p^R_{RB} + p^R_{RG}$, and the same number for v. However the number of common red neighbours of both u and v in $\mathcal{G} \cup \mathcal{B}$ is $p^{RRR}_{RRG} + p^{RRR}_{RRB} = p^R_{RB} + p^R_{RG}$ and must be entirely contained within $\mathcal{G} \cup \mathcal{B}$. Hence these must be the same sets. This means that every vertex in $\mathcal{R}$ has the same red neighbourhood in $\mathcal{G} \cup \mathcal{B}$, and so this must be the entirety of $\mathcal{G} \cup \mathcal{B}$. Now any point $x \in \mathcal{G} \cup \mathcal{B}$ has k_R red neighbours, but must be entirely connected to $\mathcal{R}$ by k_R red edges, meaning that all edges internal to $\mathcal{G} \cup \mathcal{B}$ are blue or green. Hence Γ has a $B^= \cup G^=$ equivalence relation too. □

This will cover neighbourhoods that are imprimitive of type i) and type iii) from Theorem 6.1.4, so long as the colours of the equivalence relations are different to that of the neighbourhood.

Chapter 7

The Eigenspaces of the Neighbourhoods

We will pick up from where we left off in Chapter 4 now, focusing on primitive 3-regular 3-coloured graphs. We start with some results about any 2-regular graph, and then take the neighbourhoods of a 3-regular graph (which are themselves 2-regular) and apply these results to them. We then look at possible ways the eigenspaces of each neighbourhood interact, and the consequences of such interactions. The main results are the splitting of these interactions into three cases using Equation 7.2.2, the characterisation of each of these cases (Sections 7.3, 7.4 and 7.5), and then a host of results about what happens when multiple different interactions occur simultaneously (Section 7.6).

7.1 Eigenspaces of 2-regular graphs

In this subsection we will generally work under the assumption of only 2-regularity. Here we know that the adjacency matrices are simultaneously diagonalisable, and so will have a shared basis of eigenvectors. The aim of this section is to think about how the three different partitions of the shared basis into eigenspaces interact.

Definition 7.1.1. We say the eigenvalue r_m from A_m *corresponds* with the eigenvalue r_j from A_j if they share the same subspace of $\mathbb{R}^n$ as an eigenspace.

The main consequence of eigenvalues corresponding is that they have the same multiplicity. We first will deal with them in terms of the full graph before looking at how they behave in the neighbourhoods.

Lemma 7.1.2 *If $p^j_{mm} \neq p^l_{mm}$ then we can determine r_j given r_m i.e. the eigenspace of r_m is contained within the eigenspace of r_j.*

Proof. From Lemma 4.2.7 and as $J = I + A_m + A_j + A_l$ we know that

$$A_m^2 = (p^m_{mm} - p^l_{mm})A_m + (p^j_{mm} - p^l_{mm})A_j + (k_m - p^l_{mm})I + p^l_{mm}J$$

So consider an eigenvector v of r_m. Now as r_m is non-principal we know $Jv = 0$ and therefore applying v to the above equation gives us

$$\begin{aligned}(p^j_{mm} - p^l_{mm})A_j v =& A_m^2 v - (p^m_{mm} - p^l_{mm})A_m v - (k_m - p^l_{mm})Iv \\ A_j v =& \frac{r_m^2 - (p^m_{mm} - p^l_{mm})r_m - k_m + p^l_{mm}}{p^j_{mm} - p^l_{mm}} v\end{aligned}$$

Hence v is also an eigenvector for A_j if $p^j_{mm} \neq p^l_{mm}$ with eigenvalue

$$r_j = \frac{r_m^2 - (p^m_{mm} - p^l_{mm})r_m - k_m + p^l_{mm}}{p^j_{mm} - p^l_{mm}}$$

Therefore the eigenspace of r_m is contained within the eigenspace of r_j. □

This means that if the graph is not strongly regular in colour m then the eigenspace of r_m is contained within the eigenspace of r_j. If we also have that $p^m_{jj} \neq p^l_{jj}$ then we get that the eigenspace of r_j is also contained within r_m, meaning the eigenspaces of r_j and r_m are equal, so the eigenvectors of r_m and r_j correspond. We can also derive further information.

Lemma 7.1.3 *If $r_m \neq p^j_{mj} - p^l_{mj}$ then we can determine r_j given r_m.*

Proof. This comes from the equations Corollary 4.2.9 and Lemma 4.2.10. These give us

$$r_m r_j = (p^m_{mj} - p^l_{mj})r_m + (p^j_{mj} - p^l_{mj})r_j - p^l_{mj}$$

which we can solve for r_j unless $r_m = p^j_{mj} - p^l_{mj}$. □

Combining these two results we get

Corollary 7.1.4 *Given r_m we can determine r_j unless both $p^j_{mm} = p^l_{mm}$ and $r_m = p^j_{mj} - p^l_{mj}$.*

An interesting consequence of this is the following

Corollary 7.1.5 *Suppose $p^j_{mm} = p^l_{mm}$ and $r_m, s_m, t_m \neq p^j_{mj} - p^l_{mj}$. Then the structure is 2-coloured.*

Proof. As $p^j_{mm} = p^l_{mm}$ the graph is strongly regular in m, and we have two non-principal eigenvalues for A_m. Further, as no eigenvalue of A_m is equal to $p^j_{mj} - p^l_{mj}$, we know from Lemma 7.1.3 that for any eigenvalue r_m of A_m, there exists an eigenvalue r_j of A_j such that the r_m-eigenspace is contained within the r_j-eigenspace. Hence there are a maximum of two distinct non-principal r_j-eigenspaces, and so a maximum of two non-principal eigenvalues of A_j. As we are assuming A_j is not complete, this means there are two distinct non-principal eigenvalues.
Further this implies that they must correspond perfectly, i.e. share eigenspaces, as a surjective function between two sets of the same size is bijective. Now as $0 = 1 + r_m + r_j + r_l$, and the eigenspace of r_m and r_j are the same, we must have only two solutions for r_l as well. In the Bose-Mesner Algebra this is equivalent to there only being two minimal non-principal idempotents in the basis, and if this is the case their must only be two association classes as well. Hence it is two-coloured. □

Suppose that we have both $p^j_{mm} = p^l_{mm}$ and $r_m = p^j_{mj} - p^l_{mj}$. Well then we can note that the structure is strongly regular in m, and so we only have two distinct non-principal eigenvalues for A_m. Further we know that one of them is $p^j_{mj} - p^l_{mj}$. We can then find the other eigenvalue using classical results about the eigenvalues of strongly regular graphs.

Lemma 7.1.6 *Suppose that $p^j_{mm} = p^l_{mm}$ and $r_m = p^j_{mj} - p^l_{mj}$. Then*

$$s_m = p^m_{mm} + p^l_{mj} - p^j_{mm} - p^j_{mj} = p^m_{mm} - p^j_{mm} - r_m$$

And

$$(2p^j_{mj} - 2p^l_{mj} + p^j_{mm} - p^m_{mm})^2 = (p^m_{mm} - p^j_{mm})^2 + 4(k_m - p^j_{mm})$$

Proof. We know from Theorem 2.1.10 that

$$r_m = \frac{1}{2}\left((p^m_{mm} - p^j_{mm}) + \delta(r_m)\sqrt{(p^m_{mm} - p^j_{mm})^2 + 4(k_m - p^j_{mm})}\right)$$

where $\delta(r_m)$ is either 1 or -1.

So then using that $r_m = p^j_{mj} - p^l_{mj}$ we get

$$2p^j_{mj} - 2p^l_{mj} + p^j_{mm} - p^m_{mm} = \delta(r_m)\sqrt{(p^m_{mm} - p^j_{mm})^2 + 4(k_m - p^j_{mm})}$$

Squaring this gives the second equation from the Lemma.

Now inputting this into the equation for the other eigenvalue we get

$$\begin{aligned} s_m =& \frac{1}{2}\left((p^m_{mm} - p^j_{mm}) - \delta(r_m)\sqrt{(p^m_{mm} - p^j_{mm})^2 + 4(k_m - p^j_{mm})}\right) \\ =& \frac{1}{2}\left((p^m_{mm} - p^j_{mm}) - (2p^j_{mj} - 2p^l_{mj} + p^j_{mm} - p^m_{mm})\right) \\ =& p^m_{mm} - p^j_{mm} - p^j_{mj} + p^l_{mj} \end{aligned}$$

□

Applying even more classical results we can also get the multiplicities of our two non-principal eigenvalues in nice terms.

Lemma 7.1.7 *Suppose* $p^j_{mm} = p^l_{mm}$ *and* $r_m = p^j_{mj} - p^l_{mj}$ *and the multiplicity of* r_m *is* f *and the multiplicity of* s_m *is* g*. Then*

$$\begin{aligned} f =& \frac{-(n-1)s_m - k_m}{r_m - s_m} \\ g =& \frac{(n-1)r_m + k_m}{r_m - s_m} \end{aligned}$$

Proof. By Theorem 2.1.10, we have the equation

$$f, g = \frac{1}{2}\left((n-1) \mp \frac{2k_m + (n-1)(p^m_{mm} - p^j_{mm})}{\sqrt{(p^m_{mm} - p^j_{mm})^2 + 4(k_m - p^j_{mm})}}\right)$$

Using results from Lemma 7.1.6 we see this becomes

$$\begin{aligned}
f =& \frac{1}{2}\left((n-1) - \frac{2k_m + (n-1)(p^m_{mm} - p^j_{mm})}{2p^j_{mj} - 2p^l_{mj} - p^m_{mm} + p^j_{mm}}\right) \\
=& \frac{1}{2}\left(\frac{(n-1)(2p^j_{mj} - 2p^l_{mj} - p^m_{mm} + p^j_{mm}) - 2k_m - (n-1)(p^m_{mm} - p^j_{mm})}{2p^j_{mj} - 2p^l_{mj} - p^m_{mm} + p^j_{mm}}\right) \\
=& \frac{(n-1)(p^j_{mj} - p^l_{mj} - p^m_{mm} + p^j_{mm}) - k_m}{2p^j_{mj} - 2p^l_{mj} - p^m_{mm} + p^j_{mm}} \\
=& \frac{-(n-1)s_m - k_m}{r_m - s_m}
\end{aligned}$$

And similarly

$$\begin{aligned}
g =& \frac{1}{2}\left((n-1) + \frac{2k_m + (n-1)(p^m_{mm} - p^j_{mm})}{2p^j_{mj} - 2p^l_{mj} - p^m_{mm} + p^j_{mm}}\right) \\
=& \frac{(n-1)(p^j_{mj} - p^l_{mj}) + k_m}{2p^j_{mj} - 2p^l_{mj} - p^m_{mm} + p^j_{mm}} \\
=& \frac{(n-1)r_m + k_m}{r_m - s_m}
\end{aligned}$$

□

There are also other consequences for the intersection numbers if we are in the case where we have one colour being strongly regular and at least one of the others not.

Lemma 7.1.8 *Suppose that* $p^j_{mm} = p^l_{mm}$ *and* $r_m = p^j_{mj} - p^l_{mj}$*. Then*

$$\begin{aligned}
p^l_{mj} &= (p^m_{mj} - p^l_{mj})(p^j_{mj} - p^l_{mj}) \\
p^j_{ml} &= (p^m_{ml} - p^j_{ml})(p^j_{mj} - p^l_{mj}) \\
p^l_{ml} &= (1 + p^m_{ml} - p^j_{ml})(p^j_{ml} - p^l_{mj}) \\
p^j_{mj} &= (1 + p^m_{mj} - p^l_{mj})(p^j_{ml} - p^l_{mj})
\end{aligned}$$

Proof. The first equation is the result of putting $r_m = p^j_{mj} - p^l_{mj}$ into

$$r_m r_j = (p^m_{mj} - p^l_{mj})r_m + (p^j_{mj} - p^l_{mj})r_j - p^l_{mj}$$

The second comes from doing the same thing with

$$r_m r_l = (p^m_{ml} - p^j_{ml})r_m + (p^l_{ml} - p^m_{ml})r_l - p^j_{ml}$$

Note that as $p^j_{mm} = p^l_{mm}$, $p^j_{mj} - p^l_{mj} = p^l_{ml} - p^j_{ml}$, the term involving r_l still cancels. Next we use

$$\begin{aligned} r_m^2 &= (p^m_{mm} - p^l_{mm})r_m + (p^j_{mm} - p^l_{mm})r_j + k_m - p^l_{mm} \\ 0 &= (p^m_{mm} - p^l_{mm})(p^j_{mj} - p^l_{mj}) + p^l_{mj} + p^l_{ml} - (p^j_{mj} - p^l_{mj})^2 \\ &= (p^j_{mj} - p^l_{mj})(p^j_{mj} - p^l_{mj} + p^j_{mm} - p^m_{mm}) - p^l_{mj} - p^l_{ml} \end{aligned}$$

and substitute in the earlier result for p^l_{mj} to give

$$\begin{aligned} p^l_{ml} &= (p^j_{mj} - p^l_{mj})(p^j_{mj} - p^l_{mj} + p^j_{mm} - p^m_{mm}) - (p^j_{mj} - p^l_{mj})(p^m_{mj} - p^l_{mj}) \\ &= (p^j_{mj} - p^l_{mj})(p^j_{mj} + p^j_{mm} - p^m_{mm} - p^m_{mj}) \\ &= (p^j_{mj} - p^l_{mj})(k_m - p^j_{ml} - k_m + p^m_{ml} + 1) \\ &= (p^j_{mj} - p^l_{mj})(p^m_{ml} - p^j_{ml} + 1) \end{aligned}$$

Now for the final equation, we can do the same thing with

$$r_m^2 = (p^m_{mm} - p^j_{mm})r_m + (p^l_{mm} - p^j_{mm})r_l + k_m - p^j_{mm}$$

however substituting in p^j_{ml} instead. □

Taking stock of these results we get a simple yet fundamentally crucial lemma describing the different partitions of eigenspaces.

Lemma 7.1.9 *In a primitive 2-regular 3-coloured graph, the intersections of the eigenspaces in each colour partition the shared basis of non-principal eigenvectors into*

4 classes.

Proof. We immediately get the first class as the principal eigenvectors. Next in each colour, we get a further partition into 3 classes, and therefore there are 27 possibilities initially. However as $r_l = -1 - r_m - r_j$, we can determine which eigenspace of A_l an eigenvector is in if we know which it is for A_m and A_j. This reduces now to 9 cases.
Suppose first that we have three distinct non-principal eigenvectors of A_m. Then $p^j_{mm} \neq p^l_{mm}$, and hence by Lemma 7.1.2 $E(r_m) \subset E(r_j)$. This must also occur for the other two eigenspace i.e $E(s_m) \subset E(s_j)$ and $E(t_m) \subset E(t_j)$. Therefore if the A_j also has three distinct non-principal eigenvalues, we have equality here. If instead A_j has two distinct eigenvalues, so $s_j = t_j$, then $E(r_m) = E(r_j)$ and $E(s_m) \oplus E(t_m) = E(s_j)$.
Now if we suppose that A_m has only two distinct non-principal eigenvalues then we see we've already covered if A_j has 3, so it just remains to think about if A_j has two distinct non-principal eigenvalues as well. The first possibility is when $E(r_j) \subset E(r_m)$, then we will also have $E(s_m) \subset E(s_j)$ and $E(r_m) \cap E(s_j) \neq \emptyset$. Here the classes that partition V will be $E(r_j), E(s_m)$ and $E(r_m) \cap E(s_j)$.
The final case is when $E(r_m) = E(r_j)$. But then $E(s_m) = E(s_j)$, meaning the graph is simply two-coloured in this scenario. □

It is therefore prudent to think of these classes rather than the individual eigenspaces for the colour.

Definition 7.1.10. We say an *eigenvalue triple* is a tuple of 3 eigenvalues, one from each colour adjacency matrix.
We will say $E(r_m, r_j, r_l)$ to mean $E(r_m) \cap E(r_j) \cap E(r_l)$, or alternatively *the eigenspace for the eigenvalue triple* (r_m, r_j, r_l).

Although it is a slight abuse of notation, I shall occasionally refer to the eigenvalue triple (r_m, r_j, r_l) as the eigenvalue triple 'r_x' when it is in sufficient generality as to not present confusion.

We should note that we could have eigenvalue triples (r_m, r_j, r_l) and (s_m, s_j, s_l) such that $r_m = s_m$ say, however so long as there exists one constituent eigenvalue that is different between them, we still have different eigenvalue triples. This is because they are based on the intersection of the three eigenspaces and this space will still be different.

We will now think about what happens in the imprimitive case. Although we are currently not looking at imprimitive structures, we shall later be applying these results to the neighbourhoods which could themselves be imprimitive, hence it makes sense to deal with it now. The difference here is that the principal eigenvalues can act as 'regular' eigenvalues, i.e. we could have that their eigenspace overlaps with that of a non-principal eigenvalue. Fortunately, we see that if it does, the eigenvector in question will be orthogonal to u still, as it is non-principal in some colour. Therefore we still get the full weight of Lemmas 7.1.2 and 7.1.3.

Lemma 7.1.11 *Suppose for distinct non-principal eigenvalues r_j and s_j, both $E(k_m) \cap E(r_j) \neq \emptyset$ and $E(k_m) \cap E(s_j) \neq \emptyset$. Then $p^l_{mm} = p^j_{mm} = p^l_{mj} = 0$.*

Proof. From Lemma 7.1.2 we get that $p^j_{mm} = p^l_{mm}$ and then from Lemma 7.1.3 we get $k_m = p^j_{mj} - p^l_{mj}$. As $k_m \geq p^j_{mj}$, this implies $p^l_{mj} = 0$ and $k_m = p^j_{mj}$, which further gives $p^j_{mm} = 0$. □

This will mean that the only non-principal eigenvalue of m would be -1, as its complete in m.

We can also note that the multiplicity of a principal eigenvalue k_R will only exceed one if the graphs equivalence relations include $R^=$ or $B^= \cup G^=$. This is because the graph that it is actually an eigenvalue of is A_R, which is effectively just a graph with R-edges and B and G non-edges. Hence we can just go through the list of possibilities and see how many classes they have.

Lemma 7.1.12 *In an imprimitive 2-regular 3-coloured graph, the intersections of the eigenspaces in each colour partition the shared basis of non-principal eigenvectors into 4 classes.*

Proof. As before, we shall group the principal eigenvectors together as the first class and ignore them thereafter. Therefore every other eigenvector we deal with will be orthogonal to u and non-principal. As before we can determine the eigenvalue of A_B by knowing that of A_R and A_G.

Suppose first that we have the equivalence relation $R^=$. Then A_R has two eigenvalues k_R and -1. If $E(k_R)$ has a non-zero intersection with only one eigenspace of A_G, then we know $E(-1)$ has a maximum of two non-zero intersections with eigenspaces of A_G (as they are the solutions of a quadratic). If it only has one then we end up with a two-coloured structure. So it has two, giving three total classes (not including the principal class). If instead $E(k_R)$ has a non-zero intersection with two eigenspaces of A_G, then by Lemma 7.1.11 and 7.1.3, $E(-1)$ must have non-zero intersection with only one eigenspace of A_G, giving three classes again. Note that this argument also works for equivalence classes $R^=, G^=$ and $R^=, R^= \cup G^=$, as we assumed nothing about G.

Finally, If we assume we have the equivalence relation $R^= \cup G^=$, the only non-principal eigenvalue with multiplicity greater than one is k_B. It makes sense to work consider then the eigenvalues of A_B. Suppose first for a contradiction that there are three non-principal eigenvalues of A_B. Then as the multiplicity of k_R is one, one of the non-principal eigenvalues of A_R must have an eigenspace that has a non-zero intersection with two eigenspaces of A_B. But then by Lemma 7.1.2, A_R has only two non-principal eigenvalues, implying they both must double up. However by Lemma 7.1.3 this would mean they are equal, a contradiction. Therefore A_B must have two non-principal eigenvalues and by the previous argument each eigenspace of A_B must not have non-zero intersection with more than one of the non-principal eigenspaces of A_R. Hence these will be the three classes. □

Remark 7.1.13 *We can now look at consequences of the Lemmas 7.1.9 and 7.1.12, and the different 'alignments' of the eigenspaces.*

There are 3 main primitive cases:

i) A_m *has three distinct non-principal eigenvalues* r_m, s_m, t_m *and* A_j *has three distinct non-principal eigenvalues* r_j, s_j, t_j*. They are such that* $E(r_m) = E(r_j)$*,* $E(s_m) = E(s_j)$ *and* $E(t_m) = E(t_j)$*.*

ii) A_m *has two distinct non-principal eigenvalues* r_m, s_m *and* A_j *has three distinct non-principal eigenvalues* r_j, s_j, t_j*. They are such that* $E(r_m) = E(r_j) \oplus E(t_j)$ *and* $E(s_m) = E(s_j)$*. Here* $p^j_{mm} = p^l_{mm}$ *and* $r_m = p^j_{mj} - p^l_{mj}$*.*

iii) A_m *has two distinct non-principal eigenvalues* r_m, s_m *and* A_j *has two distinct non-principal eigenvalues* r_j, s_j*. They are such that* $E(r_m) \cap E(r_j) = E(r_j)$*,* $E(s_j) \cap E(s_m) = E(s_m)$ *and* $E(r_m) \cap E(s_j) \neq 0$*. Here* $p^j_{mm} = p^l_{mm}$*,* $p^m_{jj} = p^l_{jj}$*,* $r_m = p^j_{mj} - p^l_{mj}$ *and* $s_j = p^m_{mj} - p^l_{mj}$*.*

There are a further two breakdowns of the third case. At the moment, from Lemma 4.2.10, we can see the eigenvalues of A_l are $-1 - r_m - r_j$, $-1 - r_m - s_j$ and $-1 - s_m - s_j$. However there is potential for $-1 - s_m - s_j$ to equal $-1 - r_m - r_j$, meaning A_l would only have two distinct non-principal eigenvalues as well. In this case, each colour forms a strongly regular graph. In this case the structure is amorphic (Definition 4.3.1) and it's been very well documented in [47]. We shall come back to this case later.
If however the third case is not Amorphic, then we can entirely describe its eigenvalues anyway

Lemma 7.1.14 *Suppose a primitive regular 3-coloured graph G is strongly regular in both* m *and j, but not strongly regular in* l*. Then we can describe it's eigenvalues as follows:*

$$
\begin{aligned}
&r_m = p^j_{mj} - p^l_{mj},\ s_m = p^m_{mm} - p^j_{mm} - r_m \\
&r_j = p^j_{jj} - p^m_{jj} - s_j,\ s_j = p^m_{mj} - p^l_{mj} \\
&r_l = p^j_{lj} - p^m_{lj},\ s_l = p^m_{ml} - p^j_{ml}\ t_l = 2p^l_{mj} - 1 - p^j_{mj} - p^m_{mj}
\end{aligned}
$$

Proof. We already know $r_m = p^j_{mj} - p^l_{mj}$, $s_m = p^m_{mm} - p^j_{mm} - r_m$ $r_j = p^j_{jj} - p^m_{jj} - s_j$ and $s_j = p^m_{mj} - p^l_{mj}$ by use of Lemma 7.1.6 on both m and j. We also know that $r_l = -1 - r_m - r_j$ so

$$
\begin{aligned}
r_l &= -1 - p^j_{mj} + p^l_{mj} - p^j_{jj} + p^m_{jj} + p^m_{mj} - p^l_{mj} \\
&= p^j_{lj} - p^m_{lj}
\end{aligned}
$$

Similarly $s_l = -1 - s_m - s_j$ so

$$
\begin{aligned}
r_l &= -1 - p^m_{mj} + p^l_{mj} - p^m_{mm} + p^j_{mm} + p^j_{mj} - p^l_{mj} \\
&= p^m_{lm} - p^j_{lm}
\end{aligned}
$$

Finally the third eigenvalue will be determined by $t_l = -1 - r_m - s_j$, so

$$t_l = -1 - p^j_{mj} + p^l_{mj} - p^m_{mj} + p^l_{mj}$$

□

If it was Amorphic we would have the added condition that $r_l = s_l = p^m_{ml} - p^j_{ml}$ and $p^j_{ll} = p^m_{ll}$, which are actually equivalent conditions in this context.

We shall now look at particular combinations of intersection numbers that cannot happen in a 2-regular, 3-coloured structure.

Lemma 7.1.15 *Suppose in a primitive 2-regular, 3-coloured structure that, for distinct colours c and d, $p^m_{cj} = p^m_{dj} = 0$. Then $\{c, d\} = \{m, j\}$.*

Proof. This comes from consideration of the equation

$$r_m r_j = p^m_{mj} r_m + p^j_{mj} r_j + p^l_{mj} r_l$$

Now if we let $p^m_{cj} = p^m_{dj} = 0$, by Lemma 4.2.5, we see this becomes

$$r_m r_j = p^e_{mj} r_e$$

Where e is the colour distinct from c and d. Suppose $e = m$, then either $r_m = 0$ or $r_j = p^m_{mj}$. However the same equations hold for s and t. Now we can't have $r_m = s_m = t_m = 0$ so one of these must be non-zero, so suppose without loss of generality that r_m isn't. Then $r_j = p^m_{mj} = k_j$, implying that the multiplicity of the principal eigenvalue is greater than one. This can only happen when the graph is imprimitive by Corollary 2.1.16 and so $e \neq m$. When $e = j$, we get that either $r_j = 0$ or $r_m = p^j_{mj} = k_m$ and so the same contradiction occurs. Hence $e = l$ and $\{c, d\} = \{m, j\}$. □

Lemma 7.1.16 *In a 2-regular, 3-coloured structure if $p^m_{mm} = p^m_{mj} = p^m_{jj} = p^j_{jj} = 0$ then the structure is imprimitive.*

Proof. For a contradiction, suppose we have a primitive 2-regular, 3-coloured structure with

$p^m_{mm} = p^m_{mj} = p^m_{jj} = p^j_{jj} = 0$. Then by Lemma 7.1.15, Lemma 4.2.9 and Lemma 4.2.5 we get

$$r_m^2 = p^l_{mm} r_l + k_m,\ r_j^2 = p^l_{jj} r_l + k_j,\ r_m r_j = p^l_{mj} r_l$$

And so

$$r_m^2 r_j^2 = (p^l_{mj})^2 r_l^2 = p^l_{mm} p^l_{jj} r_l^2 + (k_m p^l_{jj} + k_j p^l_{mm}) r_l + k_m k_j$$

Now, by Lemma 4.2.1 and Lemma 4.2.5, we have $k_m = p^m_{ml} + 1$, $k_j = p^j_{jl} + 1$ and $k_j = p^m_{jl}$, meaning that $p^l_{mj} = \frac{k_m k_j}{k_l}$, $p^l_{mm} = \frac{k_m}{k_l}(k_m - 1)$ and $p^l_{jj} = \frac{k_j}{k_l}(k_j - 1)$. Combining these with the above equation we have

$$\begin{aligned} 0 &= \frac{k_m k_j}{k_l^2}(1 - k_m - k_j) r_l^2 + \frac{k_m k_j}{k_l}(k_m + k_j - 2) r_l + k_m k_j \\ &= \frac{k_m k_j}{k_l^2}((1 - k_m - k_j) r_l^2 - k_l(1 - k_m - k_j) r_l - k_l r_l + k_l^2) \\ &= \frac{k_m k_j}{k_l^2}((r_l - k_l)((1 - k_m - k_j) r_l - k_l)) \end{aligned}$$

Therefore, as $r_l = k_l$ would imply k_l had multiplicity greater than 1 and the structure would be imprimitive by Corollary 2.1.16, we must have $r_l = \frac{k_l}{1 - k_m - k_j}$. However the same equation applies for s_l and t_l too, implying that $r_l = s_l = t_l$. This would mean the structure was complete in l by Lemma 2.1.17, which wouldn't be 3-coloured. Hence if $p^m_{mm} = p^m_{mj} = p^m_{jj} = p^j_{jj} = 0$ then the structure is imprimitive. □

We can also note other combinations that don't work for similar reasons

Lemma 7.1.17 *In a 2-regular, 3-coloured structure, if* $p^m_{mm} = p^m_{mj} = p^m_{jj} = p^m_{ll} = 0$ *then the structure is imprimitive.*

Proof. We first note by Lemma 4.2.1 and Lemma 4.2.5 that $k_m = p^j_{ml}$ and $k_m - 1 = p^m_{ml}$.

Then

$$\begin{aligned} r_m r_l &= p^m_{ml} r_m + p^j_{ml} r_j \\ -r_m - r_m^2 - r_m r_j &= (k_m - 1) r_m + k_m r_j \\ -r_m(r_m + r_j) &= k_m(r_m + r_j) \end{aligned}$$

Therefore either $r_m + r_j = 0$ or $r_m = -k_m$. The former would imply that $r_l = -1$ meaning it's complete in l or imprimitive, so suppose the latter. But then by Lemma 4.2.9 and Lemma 4.2.5 we get

$$\begin{aligned} (-k_m)^2 &= p^l_{mm} r_l + k_m \\ k_m(k_m - 1) &= p^l_{mm} r_l \end{aligned}$$

But by Lemma 4.2.1, $k_m - 1 = p^m_{ml}$, and by Lemma 4.2.4, $k_l p^l_{mm} = k_m p^m_{ml} = k_m(k_m - 1)$. Hence this implies $r_l = k_l$, meaning the structure is imprimitive by Corollary 2.1.16. □

7.2 Into the Neighbourhoods

Now we also want to consider how this works inside of the neighbourhoods. We shall use freely the notation introduced at the start of Chapter 4. Recall that we fix a point ∞ and look at the neighbourhoods, $R(\infty), G(\infty)$ and $B(\infty)$, respective of that point. In a 3-regular structure, the neighbourhoods are themselves 2-regular, and so all of the above results hold, however k_x becomes p^m_{mx} and the intersection number p^x_{yz} would become p^{mmx}_{myz}. We shall go through this formally now.

Lemma 7.2.1 *For any colour m and distinct colours c, d and e, N^c_{mm} is simultaneously diagonalisable with N^d_{mm} and N^e_{mm}.*

Proof. This is immediate from the fact that the m-neighbourhood is 2-regular and so will form an association scheme. Therefore the adjacency matrices will commute, and are therefore simultaneously diagonalisable by Lemma 2.2.4. □

And so as before there exists a basis of eigenvectors that is shared across all the adjacency matrices within a neighbourhood. For convenience we will group these together.

Definition 7.2.2. We say the *eigenvectors of the m-neighbourhood* are the eigenvectors included in a basis of eigenvectors that is shared between N^R_{mm}, N^G_{mm} and N^B_{mm}.

We will now list and prove the neighbourhood versions of the earlier results. Lemma 7.1.2 becomes:

Lemma 7.2.3 *Suppose x, y, z are distinct colours, m is any colour and r_{x_m} is a non-principal eigenvalue of $N^x_{mm}$with eigenvector v_{r_m}. Then if $p^{mmy}_{mxx} \neq p^{mmz}_{mxx}$, N^y_{mm} has eigenvalue*

$$r_{y_m} = \frac{p^m_{xm} - p^{mmz}_{mxx} + r_{x_m}(p^{mmx}_{mxx} - p^{mmz}_{mxx}) - r^2_{x_m}}{p^{mmz}_{mxx} - p^{mmy}_{mxx}}$$

and the eigenspace of r_{x_m} is contained within the eigenspace of r_{y_m}.

Proof. First note that $Jv_{r_m} = 0$ as v_{r_m} is orthogonal to u. Secondly from Corollary 4.2.11 we get that, for distinct $j, l \neq m$,

$$r^2_{x_m}v_{r_m}+N^x_{mj}N^x_{jm}v_{r_m}+N^x_{ml}N^x_{lm}v_{r_m} = (k_x-p^z_{xx})v_{r_m}+(p^x_{xx}-p^z_{xx})r_{x_m}v_{r_m}+(p^y_{xx}-p^z_{xx})A_{y_m}v_{r_m} \tag{7.2.1}$$

And then using Lemma 4.2.13, we see that

$$N^x_{mj}N^x_{jm}v_{r_m} = (p^m_{xj} - p^{mmz}_{jxx})v_{r_m} + (p^{mmx}_{jxx} - p^{mmz}_{jxx})r_{x_m}v_{r_m} + (p^{mmy}_{jxx} - p^{mmz}_{jxx})A_{y_m}v_{r_m}$$

as well as

$$N^x_{ml}N^x_{lm}v_{r_m} = (p^m_{xl} - p^{mmz}_{lxx})v_{r_m} + (p^{mmx}_{lxx} - p^{mmz}_{lxx})r_{x_m}v_{r_m} + (p^{mmy}_{lxx} - p^{mmz}_{lxx})A_{y_m}v_{r_m}$$

Inputting these values into equation 7.2.1 and rearranging we get

$$\begin{aligned} A_{y_m}v_{r_m} &= \frac{k_x - p^z_{xx} - p^m_{xj} - p^m_{xl} + p^{mmz}_{jxx} + p^{mmz}_{lxx} + r_{x_m}(p^x_{xx} - p^z_{xx} + p^{mmz}_{jxx} + p^{mmz}_{lxx} - p^{mmx}_{jxx} - p^{mmx}_{lxx}) - r^2_{x_m}}{p^z_{xx} - p^{mmz}_{lxx} - p^{mmz}_{jxx} - p^y_{xx} + p^{mmy}_{lxx} + p^{mmy}_{jxx}}v_{r_m} \\ &= \frac{p^m_{mx} - p^{mmz}_{mxx} + r_{x_m}(p^{mmx}_{mxx} - p^{mmz}_{mxx}) - r^2_{x_m}}{p^{mmz}_{mxx} - p^{mmy}_{mxx}}v_{r_m} \end{aligned}$$

and hence v_{r_m} is an eigenvector for A_{y_m}, with the eigenvalue r_{y_m} as defined. □

We shall also note the consequence of $p^{mmy}_{mxx} = p^{mmz}_{mxx}$ as follows

Corollary 7.2.4 *Suppose x, y, z are distinct colours, m is any colour and r_{x_m} is a non-principal eigenvalue of N^x_{mm} with eigenvector v_{r_m}. Then if $p^{mmy}_{mxx} = p^{mmz}_{mxx}$, we have*

$$0 = p^m_{xm} - p^{mmz}_{mxx} + r_{x_m}(p^{mmx}_{mxx} - p^{mmz}_{mxx}) - r^2_{x_m}$$

Proof. We use equation 7.2.1 as in the previous lemma, however we can note that now the coefficient of $A_{y_m}v_{r_m}$ is 0. This gives the result. □

Next we will adapt Lemma 7.1.3:

Lemma 7.2.5 *Suppose x, y, z are distinct colours, m is any colour and r_{x_m} is a non-principal eigenvalue of N^x_{mm} with eigenvector v_{r_m}. Then if $r_{x_m} \neq p^{mmy}_{mxy} - p^{mmz}_{mxy}$, N^y_{mm} has eigenvalue*

$$r_{y_m} = \frac{(p^{mmx}_{mxy} - p^{mmz}_{mxy})r_{x_m} - p^{mmz}_{mxy}}{r_{x_m} + p^{mmz}_{mxy} - p^{mmy}_{myx}}$$

and the eigenspace of r_{x_m} is contained within the eigenspace of r_{y_m}.

Proof. If $r_{x_m} \neq p^{mmy}_{mxy} - p^{mmz}_{mxy}$, then applying Lemma 4.2.9 internal to the m-neighbourhood we get the equation

$$r_{x_m}r_{y_m} = (p^{mmx}_{mxy} - p^{mmz}_{mxy})r_{x_m} + (p^{mmy}_{mxy} - p^{mmz}_{mxy})r_{y_m} - p^{mmz}_{mxy}$$

which we can solve for r_{y_m}, giving the solution. □

We can combine Lemma 7.2.3 and Lemma 7.2.5 again to get the corollaries:

Corollary 7.2.6 *The eigenspace for r_{x_m} is contained within the eigenspace of r_{y_m} unless $p^{mmy}_{mxx} = p^{mmz}_{mxx}$ and $r_{x_m} = p^{mmy}_{mxy} - p^{mmz}_{mxy}$.*

Corollary 7.2.7 *Suppose $p^{mmy}_{mxx} = p^{mmz}_{mxx}$. Then if there does not exist an eigenvalue $r_{x_m} = p^{mmy}_{mxy} - p^{mmz}_{mxy}$ the m-neighbourhood is two-coloured.*

These follow immediately from Corollaries 7.1.4 and 7.1.5 applied to the m-neighbourhood. We will also need a slight but quite important variation on the above result, for which we re-introduce the concept of an eigenvalue triple (Definition 7.1.10):

Lemma 7.2.8 *If we have two eigenvalue triples of the m-neighbourhood, $(r_{x_m}, r_{y_m}, r_{z_m})$ and $(s_{x_m}, s_{y_m}, s_{z_m})$, then if $r_{x_m} = s_{x_m}$, either $r_{x_m} = s_{x_m} = p_{mxy}^{mmy} - p_{mxy}^{mmz}$ or both $r_{y_m} = s_{y_m}$ and $r_{z_m} = s_{z_m}$.*

Proof. Here we follow the eigenspaces of the two eigenvalue triples. Suppose $E_{r_m} = E((r_{x_m}, r_{y_m}, r_{z_m}))$ and $E_{s_m} = E((s_{x_m}, s_{y_m}, s_{z_m}))$. Then as $r_{x_m} = s_{x_m}$, we know that $E_{r_m} \cap E_{s_m}$ is non-empty. Now by Corollary 7.2.6 either $E_{r_m} = E(r_{x_m}) = E(s_{x_m}) = E_{s_m}$ or $r_{x_m} = p_{mxy}^{mmy} - p_{mxy}^{mmz}$ and $p_{mxx}^{mmy} = p_{mxx}^{mmz}$. If $E_{r_m} = E_{s_m}$, then the eigenvalue triples must actually be equal, giving $r_{y_m} = s_{y_m}$ and $r_{z_m} = s_{z_m}$. Hence we get the result. □

We can also note in the case where $r_{x_m} = s_{x_m}$, $r_{y_m} = s_{y_m}$ and $r_{z_m} = s_{z_m}$ that the m-neighbourhood must be two-coloured.

Finally, we get the neighbourhoods variation of Lemma 7.1.9. This is slightly different because the neighbourhoods can indeed be primitive or two-coloured, but also because we have stronger restrictions. Recall V defined via equation 8.1.1.

Lemma 7.2.9 *Suppose we have a primitive 3-regular 3-coloured graph, and we consider the eigenspaces for each colour in a neighbourhood. Then V can be partitioned into a maximum of four classes by the intersections of these eigenspaces.*

Proof. Immediately if we work under the assumption that the neighbourhood is 3-coloured and primitive then we can apply Lemma 7.1.9 to the neighbourhood to get the desired result. If it is 3-coloured and imprimitive we can use Lemma 7.1.12.

Note if it is just one-coloured then $V \setminus \{u\}$ is the eigenspace of -1, and there are no other colours to worry about, so V has just two classes.

If the neighborhood is two-coloured, then it is strongly regular, and the eigenvalues are paired off by $1 + r + s = 0$. Hence we have 3 classes. □

So far we have just been applying results present across any 2-regular structure, however these neighbourhoods have the added benefit of being interlinked with each other as well. From the equations of Lemma 4.2.13 we see that, for an eigenvector v of the m-neighbourhood, we can consider $N^c_{jm}v$ as a potential eigenvector of the j-neighbourhood. As $j \neq m$ Lemma 4.2.13 further tells us that, for distinct colours c, d, e,

$$N^x_{jj}N^c_{jm} = (p^{jmc}_{jxc} - p^{jme}_{jxc})N^c_{jm} + (p^{jmd}_{jxc} - p^{jme}_{jxc})N^d_{jm} + p^{jme}_{jxc}J$$

And therefore for a non-principal eigenvector v of the m-neighbourhood, $Jv = 0$, and we get

$$N^x_{jj}N^c_{jm}v = (p^{jmc}_{jxc} - p^{jme}_{jxc})N^c_{jm}v + (p^{jmd}_{jxc} - p^{jme}_{jxc})N^d_{jm}v \tag{7.2.2}$$

Definition 7.2.10. Now there are a few possibilities here depending on the eigenvector, v chosen. For some distinct colours c and d, either we have:

The 0 case in j: $N^c_{jm}v = N^d_{jm}v = 0$.

The Eigenvector case in j: $N^c_{jm}v \neq 0$ and $N^d_{jm}v = aN^c_{jm}v$ for some constant a. Here $N^c_{jm}v$ will be an eigenvector for the j-neighbourhood.

The Independent case in j: $N^c_{jm}v \neq 0$ and $N^d_{jm}v \neq 0$ but they are linearly independent.

Each case has different implications, but we see at the moment they are tied to a particular eigenvector and not the eigenspace as a whole. It is our aim to tie the cases to eigenvalue conditions instead. This will mean that each eigenvalue triple's eigenspace will be of the same case, which makes sense. This is done with Corollary 7.4.3 and Lemma 7.5.5. We shall now go through each case and note the implications.

7.3 Consequences of Eigenvectors being in the Eigenvector case

As might have been hinted by the name, the main consequence of this case is that for v an eigenvector of the m-neighbourhood, $N^c_{jm}v$ end up being an eigenvector for the j-neighbourhood. Indeed we can state

Lemma 7.3.1 *For v an eigenvector of the m-neighbourhood then if $N^c_{jm}v \neq 0$ and $aN^c_{jm}v = N^d_{jm}v$ then $N^c_{jm}v$ is an eigenvector of the j-neighbourhood.*

Proof. Consider

$$N^x_{jj}N^c_{jm}v = (p^{jmc}_{jxc} - p^{jme}_{jxc})N^c_{jm}v + (p^{jmd}_{jxc} - p^{jme}_{jxc})N^d_{jm}v$$

And so either $a = 0$ and we are done or

$$N^x_{jj}N^c_{jm}v = (p^{jmc}_{jxc} - p^{jme}_{jxc} + \frac{1}{a}p^{jmd}_{jxc} - \frac{1}{a}p^{jme}_{jxc})N^c_{jm}v$$

Therefore as $N^c_{jm}v \neq 0$, it is an eigenvector of N^x_{jj} □

Therefore in this case we see that for an eigenvalue triple of the m-neighbourhood, $(r_{c_m}, r_{d_m}, r_{e_m})$, there will exist an eigenvalue triple $(r_{c_j}, r_{d_j}, r_{e_j})$ of N^c_{jj} such that for some x, $E((r_{c_j}, r_{d_j}, r_{e_j})) = \{N^x_{jm}v : v \in E((r_{c_m}, r_{d_m}, r_{e_m}))\}$, where $E((r_{c_m}, r_{d_m}, r_{e_m})) = E(r_{c_m}) \cap E(r_{d_j}) \cap E(r_{e_j})$. We will say here that the two eigenvalue triples *correspond*.

Lemma 7.3.2 *Suppose $N^c_{jm}v$ is an eigenvector of the j-neighbourhood. Then, for some colour d distinct from c either $N^d_{jm}v$ is 0 or an eigenvector for the j-neighbourhood, with $N^c_{jm}v = aN^d_{jm}v$.*

Proof. This is very straightforward. Note that from Lemma 4.2.13

$$N^x_{jj}N^c_{jm}v = (p^{jmc}_{jxc} - p^{jme}_{jxc})N^c_{jm}v + (p^{jmd}_{jxc} - p^{jme}_{jxc})N^d_{jm}v$$

But $N^x_{jj}N^c_{jm}v = r_{x_j}N^c_{jm}v$ as its an eigenvector. Hence

$$r_{x_j}N^c_{jm}v = (p^{jmc}_{jxc} - p^{jme}_{jxc})N^c_{jm}v + (p^{jmd}_{jxc} - p^{jme}_{jxc})N^d_{jm}v$$
$$(r_{x_j} - p^{jmc}_{jxc} + p^{jme}_{jxc})N^c_{jm}v = (p^{jmd}_{jxc} - p^{jme}_{jxc})N^d_{jm}v$$

So $N^d_{jm}v$ is 0 or a multiple of $N^c_{jm}v$, and therefore also an eigenvector. □

We can find some basic equations for what the constant a is in terms of intersection numbers.

Lemma 7.3.3 *Suppose v is an eigenvector of the m-neighbourhood such that $N^c_{jm}v \neq 0$*

and $aN^c_{jm}v = N^d_{jm}v$ *then* $a = 0$ *or*

$$a = \frac{(p^{mmc}_{jcd} - p^{mme}_{jcd})r_{c_m} + (p^{mmd}_{jcd} - p^{mme}_{jcd})r_{d_m} - p^{mme}_{jcd}}{p^m_{cj} - p^{mme}_{jcc} + (p^{mmc}_{jcc} - p^{mme}_{jcc})r_{c_m} + (p^{mmd}_{jcc} - p^{mme}_{jcc})r_{d_m}}$$

Proof. First assume $a \neq 0$. Now as before we will use the value of the norm squared

$$|N^c_{jm}v|^2 = (p^m_{jc} + p^{mmc}_{jcc}r_{c_m} + p^{mmd}_{jcc}r_{d_m} + p^{mme}_{jcc}r_{e_m})|v|^2$$

and

$$v^T N^d_{mj}N^c_{jm}v = (p^{mmc}_{jcd}r_{c_m} + p^{mmd}_{jcd}r_{d_m} + p^{mme}_{jcd}r_{e_m})|v|^2$$

But also

$$v^T N^d_{mj}N^c_{jm}v = \frac{1}{a}|N^c_{jm}v|^2$$

Hence

$$\frac{1}{a}(p^m_{jc} + p^{mmc}_{jcc}r_{c_m} + p^{mmd}_{jcc}r_{d_m} + p^{mme}_{jcc}r_{e_m}) = (p^{mmc}_{jcd}r_{c_m} + p^{mmd}_{jcd}r_{d_m} + p^{mme}_{jcd}r_{e_m})$$

And so

$$a = \frac{(p^{mmc}_{jcd}r_{c_m} + p^{mmd}_{jcd}r_{d_m} + p^{mme}_{jcd}r_{e_m})}{(p^m_{jc} + p^{mmc}_{jcc}r_{c_m} + p^{mmd}_{jcc}r_{d_m} + p^{mme}_{jcc}r_{e_m})}$$

or, equivalently,

$$a = \frac{(p^{mmc}_{jcd} - p^{mme}_{jcd})r_{c_m} + (p^{mmd}_{jcd} - p^{mme}_{jcd})r_{d_m} - p^{mme}_{jcd}}{p^m_{cj} - p^{mme}_{jcc} + (p^{mmc}_{jcc} - p^{mme}_{jcc})r_{c_m} + (p^{mmd}_{jcc} - p^{mme}_{jcc})r_{d_m}}$$

□

From this we can determine what the corresponding eigenvalues in the j-neighbourhood actually are

Lemma 7.3.4 *Suppose* v *is an eigenvector of the* m*-neighbourhood such that* $N^c_{jm}v \neq 0$

and $N^c_{jm}v = aN^d_{jm}v$ *then if* r_{x_j} *is the eigenvalue of* N^x_{jj} *with eigenvector* $N^c_{jm}v$ *and* $a \neq 0$

$$\begin{aligned} r_{x_j} &= p^{jmc}_{jxc} - p^{jme}_{jxc} + \frac{1}{a}(p^{jmd}_{jxc} - p^{jme}_{jxc}) \\ &= p^{jmc}_{jxc} - p^{jme}_{jxc} + \frac{p^m_{cj} - p^{mme}_{jcc} + (p^{mmc}_{jcc} - p^{mme}_{jcc})r_{c_m} + (p^{mmd}_{jcc} - p^{mme}_{jcc})r_{d_m}}{(p^{mmc}_{jcd} - p^{mme}_{jcd})r_{c_m} + (p^{mmd}_{jcd} - p^{mme}_{jcd})r_{d_m} - p^{mme}_{jcd}}(p^{jmd}_{jxc} - p^{jme}_{jxc}) \end{aligned}$$

If $a = 0$ *then* $r_{x_j} = p^{jmc}_{jxc} - p^{jme}_{jxc}$.

Proof. We know that $N^c_{jm}v$ is an eigenvector, so consider

$$N^x_{jj}N^c_{jm}v = (p^{jmc}_{jxc} - p^{jme}_{jxc})N^c_{jm}v + (p^{jmd}_{jxc} - p^{jme}_{jxc})N^d_{jm}v$$

Hence if $a = 0$, $N^d_{jm}v = 0$ and $r_{x_j} = p^{jmc}_{jxc} - p^{jme}_{jxc}$. If $a \neq 0$ then

$$\begin{aligned} r_{x_j} &= p^{jmc}_{jxc} - p^{jme}_{jxc} + \frac{1}{a}(p^{jmd}_{jxc} - p^{jme}_{jxc}) \\ &= p^{jmc}_{jxc} - p^{jme}_{jxc} + \frac{p^m_{cj} - p^{mme}_{jcc} + (p^{mmc}_{jcc} - p^{mme}_{jcc})r_{c_m} + (p^{mmd}_{jcc} - p^{mme}_{jcc})r_{d_m}}{(p^{mmc}_{jcd} - p^{mme}_{jcd})r_{c_m} + (p^{mmd}_{jcd} - p^{mme}_{jcd})r_{d_m} - p^{mme}_{jcd}}(p^{jmd}_{jxc} - p^{jme}_{jxc}) \end{aligned}$$

□

We can also show that being corresponding is a symmetric relation.

Lemma 7.3.5 *Suppose* v *is an eigenvector of the* m*-neighbourhood such that* $N^c_{jm}v \neq 0$ *and* $N^c_{jm}v = aN^d_{jm}v$*, then* $N^c_{jm}v$ *is in the eigenvector case in* m*.*

Proof. This is fairly trivial, all we have to show is that for some distinct colours x and y, $N^x_{mj}N^c_{jm}v \neq 0$ and $N^x_{mj}N^c_{jm}v = aN^y_{mj}N^c_{jm}v$ for some constant a. Well note that $N^c_{mj}N^c_{jm}v$ is not 0, as otherwise $N^c_{jm}v$ would be. Further

$$N^d_{mj}N^c_{jm}v = (p^{mmc}_{jcd}r_{c_m} + p^{mmd}_{jcd}r_{d_m} + p^{mme}_{jcd}r_{e_m})v$$

which is a multiple of $N^c_{mj}N^c_{jm}v$ as this can also be expressed as a constant multiplied by v. □

As we've seen the $a = 0$ provides a bit of a special case here. As we showed in Lemma

7.3.5, the eigenvector in the j-neighbourhood is also of the eigenvector case, and we can see that if $N^d_{jm}v = 0$ then $N^d_{mj}N^c_{jm}v = 0$. Hence if one direction gets the $a = 0$ results, then so does the other. So in this case we can get a full description of the eigenvalues just in terms of the intersection numbers.

Lemma 7.3.6 *If for v an eigenvector of the m-neighbourhood $N^d_{jm}v = 0$ but $N^c_{jm}v \neq 0$ then for any colour x, $p^{jmc}_{jxd} = p^{jme}_{jxd}$ and $p^{mjc}_{mxd} = p^{mje}_{mxd}$. Further $N^c_{jm}v$ is an eigenvector of the j-neighbourhood with eigenvalues $(r_{x_j}, r_{y_j}, r_{z_j})$ such that*

$$r_{x_j} = p^{jmc}_{jxc} - p^{jme}_{jxc},\ r_{y_j} = p^{jmc}_{jyc} - p^{jme}_{jyc},\ r_{z_j} = p^{jmc}_{jzc} - p^{jme}_{jzc}$$

And if $(r_{x_j}, r_{y_j}, r_{z_j})$ are the eigenvalues of v then

$$r_{x_m} = p^{mjc}_{mxc} - p^{mje}_{mxc},\ r_{y_m} = p^{mjc}_{myc} - p^{mje}_{myc},\ r_{z_m} = p^{mjc}_{mzc} - p^{mje}_{mzc}$$

Proof. All of this is immediate from Lemmas 7.3.4 and 7.3.5, except that for any colour x, $p^{jmc}_{jxd} = p^{jme}_{jxd}$ and $p^{mjc}_{mxd} = p^{mje}_{mxd}$. This comes from the fact that, for any colour x,

$$0 = N^x_{jj}N^d_{jm}v = (p^{jmc}_{jxd} - p^{jme}_{jxd})N^c_{jm}v$$

and

$$0 = N^x_{mm}N^d_{mj}N^c_{jm}v = (p^{mjc}_{mxd} - p^{mje}_{mxd})N^c_{mj}N^c_{jm}v$$

□

7.4 Consequences of Eigenvectors in the 0 case

Lemma 7.4.1 *For distinct colours c, d, e, if v is an eigenvector of the m-neighbourhood with eigenvalues $(r_{c_m}, r_{d_m}, r_{e_m})$ then $N^x_{jm}v = 0$ if and only if*

$$0 = p^m_{xj} + p^{mmc}_{jxx}r_{c_m} + p^{mmd}_{jxx}r_{d_m} + p^{mme}_{jxx}r_{e_m}$$

Proof. This comes from the fact that $N^x_{jm}v = 0$ if and only if $|N^x_{jm}v| = 0$, coupled with the equation from Lemma 4.2.13

$$N^x_{mj}N^x_{jm} = p^m_{xj}I + p^{mmc}_{jxx}N^c_{mm} + p^{mmd}_{jxx}N^d_{mm} + p^{mme}_{jxx}N^e_{mm}$$

Therefore $N^x_{jm}v = 0$ if and only if

$$\begin{aligned}
0 =& |N^x_{jm}v|^2 \\
=& v^T N^x_{mj}N^x_{jm}v \\
=& p^m_{xj}v^T v + p^{mmc}_{jxx}v^T N^c_{mm}v + p^{mmd}_{jxx}v^T N^d_{mm}v + p^{mme}_{jxx}v^T N^e_{mm}v \\
=& (p^m_{xj}I + p^{mmc}_{jxx}r_{c_m} + p^{mmd}_{jxx}r_{d_m} + p^{mme}_{jxx}r_{e_m})|v|^2
\end{aligned}$$

As v is an eigenvector, we know $|v|^2 \neq 0$ and therefore we have the result. □

Remark 7.4.2 *By inputting the equation from Lemma 4.2.10,* $0 = 1 + r_{c_m} + r_{d_m} + r_{e_m}$*, the equation from Lemma 7.4.1 becomes*

$$0 = (p^m_{xj} - p^{mme}_{jxx}) + (p^{mmc}_{jxx} - p^{mme}_{jxx})r_{c_m} + (p^{mmd}_{jxx} - p^{mme}_{jxx})r_{d_m} \qquad (7.4.3)$$

By Lemma 7.4.1 we see that in the 0 case this equation will hold for any colour x*. We can also note that if* $N^x_{jm}v = 0$ *then, for any colour* y *distinct from* x*,* $v^T N^y_{mj}N^x_{jm}v = 0$ *too and so we get a slightly different equation*

$$0 = -p^{mme}_{jxy} + (p^{mmc}_{jxy} - p^{mme}_{jxy})r_{c_m} + (p^{mmd}_{jxy} - p^{mme}_{jxy})r_{d_m} \qquad (7.4.4)$$

It can be fairly easily shown that given Equation 7.4.3 in all colours x*, we get Equation 7.4.4 as a consequence, and so it provides no new information. However, it could be more useful to use this variation on occasion.*

This equation is crucial to a lot of our study, and we will go back to it further on. However for now we shall focus on how this works with corresponding and non-corresponding eigenvalues.

Corollary 7.4.3 *Suppose* $\exists v \in E(r_{c_m}r_{d_m}, r_{e_m})$ *is in the 0 case in* j*. Then the entirety of* $E(r_{c_m}r_{d_m}, r_{e_m})$ *is in the 0 case in* j*.*

Proof. By definition, it is known that for all $v' \in E(r_{c_m} r_{d_m}, r_{e_m})$, $N^c_{mm}v' = r_{c_m}v'$, $N^d_{mm}v' = r_{d_m}v'$, and $N^e_{mm}v' = r_{e_m}v'$. Therefore if there exits v in the 0 case in j, then by Lemma 7.4.1, we have the equation 7.4.3. This will then hold for all $v' \in E(r_{c_m} r_{d_m}, r_{e_m})$, meaning by Lemma 7.4.1, v' is in the 0 case in j. □

We can't necessarily tie the case to an eigenspace of a single colour, $E(r_{c_m})$, in the same way. What happens to the vectors in the other eigenspaces if there exists overlapping eigenspaces is an interesting question. For the most part, they cannot be of case 0 in j, as (r_{c_m}, s_{d_m}) won't generally solve equation 7.4.3 if (r_{c_m}, r_{d_m}) does. However there are certain constraints on the intersection numbers where it is possible that it might. For instance, what if the entire equation 7.4.3 collapses to 0, or the coefficient of $N^d_{mm}v$ is 0? Well it would have to happen for all colours x for that to affect the situation, and that will have some very strong consequences which we shall discuss later.
Now we shall look at the repercussions of just one eigenvalue triple of the m-neighbourhood being the 0 case in j. First note that Equation 7.4.3 actually gives us three linear equations, one for each colour in place of x. Hence we can solve them for r_{c_m} and r_{d_m}, if the discriminant is non-zero. For distinct colours x and y define

$$\mathfrak{D}^{xy}_{mj} := (p^{mme}_{jyy} - p^{mmc}_{jyy})(p^{mme}_{jxx} - p^{mmd}_{jxx}) + (p^{mme}_{jyy} - p^{mmd}_{jyy})(p^{mmc}_{jxx} - p^{mme}_{jxx}) \qquad (7.4.5)$$

This is the discriminant of the Equation 7.4.3 when expressed in x and y.

Lemma 7.4.4 *Suppose $v \in E(r_{c_m}, r_{d_m}, r_{e_m})$ is an eigenvector of the m-neighbourhood, such that v is the 0 case in j, and further suppose for some distinct x, y we have $\mathfrak{D}^{xy}_{mj} \neq 0$ then*

$$r_{c_m} = \frac{(p^m_{yj} - p^{mme}_{jyy})(p^{mme}_{jxx} - p^{mmd}_{jxx}) + (p^m_{xj} - p^{mme}_{jxx})(p^{mmd}_{jyy} - p^{mme}_{jyy})}{(p^{mme}_{jyy} - p^{mmc}_{jyy})(p^{mme}_{jxx} - p^{mmd}_{jxx}) + (p^{mme}_{jyy} - p^{mmd}_{jyy})(p^{mmc}_{jxx} - p^{mme}_{jxx})}$$
$$r_{d_m} = \frac{(p^m_{yj} - p^{mme}_{jyy})(p^{mme}_{jxx} - p^{mmc}_{jxx}) + (p^m_{xj} - p^{mme}_{jxx})(p^{mmc}_{jyy} - p^{mme}_{jyy})}{(p^{mme}_{jyy} - p^{mmc}_{jyy})(p^{mme}_{jxx} - p^{mmd}_{jxx}) + (p^{mme}_{jyy} - p^{mmd}_{jyy})(p^{mmc}_{jxx} - p^{mme}_{jxx})}$$

Proof. From Lemma 7.4.1 we must have that both

$$0 = (p^m_{xj} - p^{mme}_{jxx}) + (p^{mmc}_{jxx} - p^{mme}_{jxx})r_{c_m} + (p^{mmd}_{jxx} - p^{mme}_{jxx})r_{d_m} \qquad (7.4.6)$$

And

$$0 = (p^{m}_{yj} - p^{mme}_{jyy}) + (p^{mmc}_{jyy} - p^{mme}_{jyy})r_{c_m} + (p^{mmd}_{jyy} - p^{mme}_{jyy})r_{d_m} \tag{7.4.7}$$

When considered as a system of linear equations in variables r_{c_m} and r_{d_m}, we see that the determinant is $(p^{mme}_{jyy} - p^{mmc}_{jyy})(p^{mme}_{jxx} - p^{mmd}_{jxx}) + (p^{mme}_{jyy} - p^{mmd}_{jyy})(p^{mmc}_{jxx} - p^{mme}_{jxx})$ which is non-zero. Hence we can solve these equations for r_{c_m} and r_{d_m}, giving the desired results. □

It is important to note here that unless $\mathfrak{D}^{xy}_{mj}$ is 0 for all combinations of distinct x and y, then we can solve these equations for r_{c_m} and r_{d_m}. For this to happen there will be consequences.

Lemma 7.4.5 *For any distinct colours c, d, e, and distinct x and y, $\mathfrak{D}^{xy}_{mj} = 0$ if and only if one of the following occurs:*

i) $p^{mmc}_{jxx} = p^{mme}_{jxx}$ *and* $p^{mmc}_{jyy} = p^{mme}_{jyy}$

ii) $p^{mmd}_{jxx} = p^{mme}_{jxx}$ *and* $p^{mmd}_{jyy} = p^{mme}_{jyy}$

iii) $p^{mmc}_{jxx} = p^{mmd}_{jxx} = p^{mme}_{jxx}$ *or* $p^{mmc}_{jyy} = p^{mmd}_{jyy} = p^{mme}_{jyy}$

iv) For some non-zero constant λ, $p^{mmc}_{jxx} - p^{mme}_{jxx} = \lambda(p^{mmc}_{jyy} - p^{mme}_{jyy})$ *and* $p^{mmd}_{jxx} - p^{mme}_{jxx} = \lambda(p^{mmd}_{jyy} - p^{mme}_{jyy})$

Proof. Clearly if any of i)-iv) hold then $\mathfrak{D}^{xy}_{mj} = 0$, so all that remains to prove is the converse. Well suppose $\mathfrak{D}^{xy}_{mj} = 0$. Then either both the individual terms are 0, giving case i), case ii) or case iii), or neither are and we have

$$(p^{mme}_{jyy} - p^{mmc}_{jyy})(p^{mme}_{jxx} - p^{mmd}_{jxx}) = (p^{mme}_{jyy} - p^{mmd}_{jyy})(p^{mme}_{jxx} - p^{mmc}_{jxx})$$

Hence if we let $\lambda = \frac{p^{mme}_{jxx} - p^{mmd}_{jxx}}{p^{mme}_{jyy} - p^{mmd}_{jyy}}$, we get $\lambda = \frac{p^{mme}_{jxx} - p^{mmc}_{jxx}}{p^{mme}_{jyy} - p^{mmc}_{jyy}}$. Note that denominators must be non-zero as otherwise it would fall into one of the other cases. □

Coupling the equations with the results from the determinant being 0 we can push the consequences even further

Lemma 7.4.6 *Suppose both equations 7.4.6 and 7.4.7 hold. Then* $\mathfrak{D}^{xy}_{mj} = 0$ *if and only if one of the following hold*

i) $p^{mmc}_{jxx} = p^{mme}_{jxx}$, $p^{mmc}_{jyy} = p^{mme}_{jyy}$ *and* $r_{d_m} = \frac{p^m_{xj} - p^{mme}_{jxx}}{p^{mme}_{jxx} - p^{mmd}_{jxx}} = \frac{p^m_{yj} - p^{mme}_{jyy}}{p^{mme}_{jyy} - p^{mmd}_{jyy}}$

ii) $p^{mmd}_{jxx} = p^{mme}_{jxx}$, $p^{mmd}_{jyy} = p^{mme}_{jyy}$ *and* $r_{c_m} = \frac{p^m_{xj} - p^{mme}_{jxx}}{p^{mme}_{jxx} - p^{mmc}_{jxx}} = \frac{p^m_{yj} - p^{mme}_{jyy}}{p^{mme}_{jyy} - p^{mmc}_{jyy}}$

iii) $p^m_{jx} = 0$ *or* $p^m_{jy} = 0$

iv) For some non-zero constant λ, $p^m_{jx} - \lambda p^m_{jy} = p^{mmc}_{jxx} - \lambda p^{mmc}_{jyy} = p^{mmd}_{jxx} - \lambda p^{mmd}_{jyy} = p^{mme}_{jxx} - \lambda p^{mme}_{jyy}$

Proof. Suppose first that $\mathfrak{D}^{xy}_{mj} = 0$. The different cases here align with the cases from Lemma 7.4.5.

Case i) and ii) are immediate from combining either case i) or ii) from Lemma 7.4.5 with the equations 7.4.6 and 7.4.7 and solving for r_{d_m} or r_{c_m} respectively.

For iii) suppose we have $p^{mmc}_{jxx} = p^{mmd}_{jxx} = p^{mme}_{jxx}$ and equation 7.4.6. Then the equation becomes $p^m_{xj} = p^{mme}_{jxx}$, so $p^m_{xj} = p^{mmc}_{jxx} = p^{mmd}_{jxx} = p^{mme}_{jxx}$. Note if $p^m_{ma} = 0$, for any a, then $p^{mma}_{jxx} = 0$ by Lemma 4.2.3. This further implies $p^m_{xj} = 0$ and so we are done.

So suppose $p^m_{ma} \neq 0$ for all colours a. Then, for any colour a, by Lemma 4.2.3, we know that $p^{mma}_{jxy} = p^{mma}_{jxz} = 0$. Now by Lemma 4.2.5, $p^{mzj}_{amx} = 0$ and $p^{myj}_{amx} = 0$ for all a too. But then by Lemma 4.2.3, either $p^j_{mx} = p^{mzj}_{cmx} + p^{mzj}_{dmx} + p^{mzj}_{emx} = 0$ or $p^m_{zj} = 0$. So either we are done, or $p^m_{zj} = 0$. So suppose for a contradiction that $p^m_{zj} = 0$. Applying $p^{myj}_{amx} = 0$ in the same way, we see that either $p^m_{jx} = 0$ or $p^m_{yj} = 0$. Therefore if $p^m_{jx} \neq 0$, both $p^m_{yj} = 0$ and $p^m_{zj} = 0$ and further, as $p^m_{mj} \neq 0$ we know $x = m$. However, now we have $p^m_{jj} = p^m_{jl} = 0$ contradicting Lemma 7.1.15. Hence we must have $p^m_{jx} = 0$.

Finally for case iv), first input $p^{mmc}_{jxx} - p^{mme}_{jxx} = \lambda(p^{mmc}_{jyy} - p^{mme}_{jyy})$ and $p^{mmd}_{jxx} - p^{mme}_{jxx} = \lambda(p^{mmd}_{jyy} - p^{mme}_{jyy})$ into Equations 7.4.6 and 7.4.7. This gives $p^m_{xj} - p^{mme}_{jxx} = \lambda(p^m_{yj} - p^{mme}_{jyy})$. We can then rearrange both equation 7.4.6 and equation 7.4.7 to be equal to $p^{mme}_{jxx} - \lambda p^{mme}_{jyy}$, giving the result.

The other direction is provable from the fact that each of these cases immediately imply their corresponding cases in Lemma 7.4.5. Cases i), ii) and iv) are immediate. In case iii), $p^m_{jx} = 0$ implies $0 = p^{mmc}_{jxx} = p^{mmd}_{jxx} = p^{mme}_{jxx}$ by Lemma 4.2.3, and the similar for $p^m_{jy} = 0$. □

Note that case iii) is the $\lambda = 0$ variation of case iv). Also it is fairly trivial to note that $\mathfrak{D}^{xy}_{mj} = -\mathfrak{D}^{yx}_{mj}$, and so if one is zero, so is the other.
As mentioned earlier, in order to be unable to solve the equations 7.4.3 for r_{c_m} and r_{d_m}, we must have $\mathfrak{D}^{xy}_{mj} = \mathfrak{D}^{yz}_{mj} = \mathfrak{D}^{zx}_{mj} = 0$. Assuming this is the case, we will then have at least one of the outcomes from Lemma 7.4.6 for each pair (x, y), (x, z) and (y, z). We will now discuss the possible combinations of these outcomes.

Remark 7.4.7 *It is important to remark that given the condition $p^m_{ma} = 0$ for some a, although $\mathfrak{D}^{xy}_{mj}$ may be zero, we can still solve Equation 7.4.3 for the eigenvalues as we know by Remark 4.2.15 that $r_{a_m} = 0$. Therefore either the equation is immediately in terms of one eigenvalue, or using $0 = 1 + r_{c_m} + r_{d_m} + r_{e_m}$, it can be.*

Lemma 7.4.8 *Suppose the equation 7.4.3 holds for x, y and z and that $p^m_{jx} = 0$ and $p^m_{ma} \neq 0$ for all colours a. Then $\mathfrak{D}^{xy}_{mj} = \mathfrak{D}^{yz}_{mj} = \mathfrak{D}^{zx}_{mj} = 0$.*

Proof. Suppose $p^m_{jx} = 0$ and both p^m_{jy} and p^m_{jz} are non-zero (as otherwise we are done). Then immediately from Lemma 7.4.6 we get $\mathfrak{D}^{xy}_{mj} = \mathfrak{D}^{xz}_{mj} = 0$, so all the remains to prove is that $\mathfrak{D}^{yz}_{mj} = 0$. Well if $p^m_{jx} = 0$, then for any colour a by Lemma 4.2.3 either $p^m_{ma} = 0$ or both $p^m_{jy} = p^{mma}_{jyy} + p^{mma}_{jyz}$ and $p^m_{jz} = p^{mma}_{jzz} + p^{mma}_{jyz}$. Therefore $p^m_{jy} - p^m_{jz} = p^{mma}_{jyy} - p^{mma}_{jzz}$. Hence we have case iv) from Lemma 7.4.6 with $\lambda = 1$ and $\mathfrak{D}^{yz}_{mj} = 0$. □

We shall isolate the following result from the proof as it will be useful in it's own right later

Corollary 7.4.9 *Suppose that $p^m_{jx} = 0$, $p^m_{ma} \neq 0$ for all colours a and the equation 7.4.3 holds for x, y and z. Then for some distinct y and z, not equal to x, either $p^m_{jy} = 0$, $p^m_{jz} = 0$ or $p^m_{jy} - p^m_{jz} = p^{mmc}_{jyy} - p^{mmc}_{jzz} = p^{mmd}_{jyy} - p^{mmd}_{jzz} = p^{mme}_{jyy} - p^{mme}_{jzz}$.*

We can make some further deductions about other combinations. Suppose we have case i) for x and y and case iii) for y and z. Then it turns out that this implies we actually have case i) for y and z as well. More formally

Lemma 7.4.10 *Suppose that $p^{mmc}_{jyy} = p^{mme}_{jyy}$ and for some non-zero constant λ we have $p^m_{jy} - \lambda p^m_{jz} = p^{mmc}_{jyy} - \lambda p^{mmc}_{jzz} = p^{mmd}_{jyy} - \lambda p^{mmd}_{jzz} = p^{mme}_{jyy} - \lambda p^{mme}_{jzz}$. Then $p^{mmc}_{jzz} = p^{mme}_{jzz}$.*

Proof. We have $p^{mmc}_{jyy} - \lambda p^{mmc}_{jzz} = p^{mme}_{jyy} - \lambda p^{mme}_{jzz}$. Therefore as $p^{mmc}_{jyy} = p^{mme}_{jyy}$, we have $p^{mmc}_{jzz} = p^{mme}_{jzz}$. □

Now we can put all these together to classify the possibilities

Lemma 7.4.11 *Suppose the equation 7.4.3 holds for x, y and z and $p^m_{ma} \neq 0$. Then $\mathfrak{D}^{xy}_{mj} = \mathfrak{D}^{yz}_{mj} = \mathfrak{D}^{zx}_{mj} = 0$ if and only if one of the following occurs:*

1. *Either $p^m_{jx} = 0$, $p^m_{jy} = 0$ or $p^m_{jz} = 0$*
2. *$p^{mmc}_{jxx} = p^{mme}_{jxx}$, $p^{mmc}_{jyy} = p^{mme}_{jyy}$, and $p^{mmc}_{jzz} = p^{mme}_{jzz}$*
3. *$p^{mmd}_{jxx} = p^{mme}_{jxx}$, $p^{mmd}_{jyy} = p^{mme}_{jyy}$, and $p^{mmd}_{jzz} = p^{mme}_{jzz}$*
4. *For some non-zero constants λ and μ and all colours a, $p^m_{jx} - \lambda p^m_{jy} = p^{mma}_{jxx} - \lambda p^{mma}_{jyy}$, $p^m_{jy} - \mu p^m_{jz} = p^{mma}_{jyy} - \mu p^{mma}_{jzz}$ and $p^m_{jx} - \lambda\mu p^m_{jz} = p^{mma}_{jxx} - \lambda\mu p^{mma}_{jzz}$*

Proof. First suppose we have $\mathfrak{D}^{xy}_{mj} = \mathfrak{D}^{yz}_{mj} = \mathfrak{D}^{zx}_{mj} = 0$. Then by Lemma 7.4.6 for each combination of x, y and z we must have one of the cases i)-iv). For convenience in this proof we shall use the shorthand that xy is of case i), for instance.

First suppose that any one of xy, xz or yz is of case iii), then we get case 1) and are done. If any two of xy, xz or yz are case i), then we get case 2) here. Similarly if we assume they any two of xy, xz or yz are case ii), we get case 3) here. We can also see that if any two are of case iv), say w.l.o.g xy and yz, then for all colours a and some non-zero constants λ and μ, $p^m_{jx} - \lambda p^m_{jy} = p^{mma}_{jxx} - \lambda p^{mma}_{jyy}$ and $p^m_{jy} - \mu p^m_{jz} = p^{mma}_{jyy} - \mu p^{mma}_{jzz}$. Combining these two equations gives $p^m_{jx} - \lambda\mu p^m_{jz} = p^{mma}_{jxx} - \lambda\mu p^{mma}_{jzz}$, and so we are in case 4). Therefore the only remaining possibility is a combination involving all three of case i), ii) and iv). However we note for instance xy and xz are case i) and case ii) respectively, then $p^{mmc}_{jxx} = p^{mmd}_{jxx} = p^{mme}_{jxx}$. This, when coupled with Equation 7.4.3, implies $p^m_{xj} = 0$, and so we are actually in case 1).

Now we shall prove the converse. If we suppose we have case 1) then by Lemma 7.4.8, we get $\mathfrak{D}^{xy}_{mj} = \mathfrak{D}^{yz}_{mj} = \mathfrak{D}^{zx}_{mj} = 0$. If we suppose have case 2 or 3, then this immediately implies we have case i) or case ii) respectively from Lemma 7.4.6 for every combination of x, y and z. Similarly, case 4) corresponds to having case iv) from Lemma 7.4.6 for every combination of x, y and z. □

Remark 7.4.12 *It should be noted from the proof that the condition $p^m_{ma} \neq 0$ for all a is only present so we can apply Lemma 7.4.8. Hence Lemma 7.4.11 would therefore be almost exactly the same without this condition, however it would no longer be 'if and only if' as the condition '$p^m_{jx} = 0$, $p^m_{jx} = 0$ or $p^m_{jx} = 0$' would no longer necessarily imply $\mathfrak{D}^{xy}_{mj} = \mathfrak{D}^{yz}_{mj} = \mathfrak{D}^{zx}_{mj} = 0$.*

In case 2), we can note that, as we can solve Equation 7.4.3 for r_{d_m}, if we have $p^{mmc}_{mdd} \neq p^{mme}_{mdd}$ and use Lemma 7.2.3, we can determine r_{c_m}. Similarly we can do the same thing with case 3). Hence in these cases in order to not be able to solve for both r_{c_m} and r_{d_m}, we must have $p^{mmc}_{mdd} = p^{mme}_{mdd}$ in case 2) or $p^{mmc}_{mdd} = p^{mme}_{mdd}$ in case 3) also.
Further from Lemma 7.2.5, we can solve for r_{c_m} given r_{d_m} unless $r_{d_m} = p^{mmc}_{mcd} - p^{mme}_{mcd}$, therefore this condition must be present also.
It should be noted that if $p^m_{ma} = 0$ for any a solving the linear equations is possible, except case 2) when $p^m_{md} = 0$ and the case 3) when $p^m_{mc} = 0$. This shall not be excluded from the following definitions, however it will be noted wherever it affects results.
As we will make frequent reference to these cases we shall define terms for them

Definition 7.4.13. If for some x, $p^m_{jx} = 0$ then we say that the m-neighbourhood is x-*undesirable* with respect to the j-neighbourhood.
If $p^{mmc}_{jxx} = p^{mme}_{jxx}$, $p^{mmc}_{jyy} = p^{mme}_{jyy}$, $p^{mmc}_{jzz} = p^{mme}_{jzz}$, and $p^{mmc}_{mdd} = p^{mme}_{mdd}$, then we say that the m-neighbourhood is d-*semi-undesirable* with respect to the j-neighbourhood.
If for some non-zero constants λ and μ and all colours a, $p^m_{jx} - \lambda p^m_{jy} = p^{mma}_{jxx} - \lambda p^{mma}_{jyy}$, $p^m_{jy} - \mu p^m_{jz} = p^{mma}_{jyy} - \mu p^{mma}_{jzz}$ and $p^m_{jx} - \lambda\mu p^m_{jz} = p^{mma}_{jxx} - \lambda\mu p^{mma}_{jzz}$ then we say that the m-neighbourhood is a *multiple* with respect to the j-neighbourhood.

Remark 7.4.14 *If we have d-semi-undesirability, we can note that we also have $p^{mmc}_{jxy} = p^{mme}_{jxy}$ for all distinct x, y. This is because, by repeated use of Lemma 4.2.3,*

$$\begin{aligned} p^{mmc}_{jxy} &= p^m_{jx} - p^{mmc}_{jxx} - p^{mmc}_{jxz} \\ &= p^m_{jx} - p^{mmc}_{jxx} - p^m_{jz} + p^{mmc}_{jzz} + p^{mmc}_{jyz} \\ &= p^m_{jx} - p^{mmc}_{jxx} - p^m_{jz} + p^{mmc}_{jzz} + p^m_{jy} - p^{mmc}_{jyy} - p^{mmc}_{jxy} \end{aligned}$$

So applying the d-semi-undesirable condition this becomes

$$
\begin{aligned}
2p^{mmc}_{jxy} &= p^m_{jx} - p^{mme}_{jxx} - p^m_{jz} + p^{mme}_{jzz} + p^m_{jy} - p^{mme}_{jyy} \\
&= p^{mme}_{jxy} + p^{mme}_{jxz} - p^{mme}_{jxz} - p^{mme}_{jyz} + p^{mme}_{jxy} + p^{mme}_{jyz} \\
&= 2p^{mme}_{jxy}
\end{aligned}
$$

By combining Lemmas 7.4.4 and 7.4.10 then we can say

Lemma 7.4.15 *Suppose we have an eigenvector v of the m-neighbourhood with eigenvalues $(r_{c_m}, r_{d_m}, r_{e_m})$ which is the 0 case in j. Further suppose we can't find unique solutions for both r_{c_m} and r_{d_m} using the equations 7.4.3 and $0 = 1 + r_{c_m} + r_{d_m} + r_{e_m}$. Then we must have that, for some colour x, either:*

- *The m-neighbourhood is 3-coloured and x-undesirable, x-semi-undesirable or a multiple with respect to j.*
- *The m-neighbourhood is 2-coloured with $p^j_{mj} = p^j_{mm} = 0$.*

Proof. First suppose the m-neighbourhood is not 3-coloured. Well if it is 1-coloured, by Remark 4.2.15, we can solve $0 = 1+r_{c_m}+r_{d_m}+r_{e_m}$ for the eigenvalues. If it is 2-coloured, then one of the eigenvalues is 0, and the other two are for distinct colours a and b. Then they satisfy $0 = 1 + r_{a_m} + r_{b_m}$ and $0 = p^m_{xj} + p^{mma}_{jxx} r_{a_m} + p^{mmb}_{jxx} r_{b_m}$ for all colours x. This becomes

$$0 = p^m_{xj} - p^{mmb}_{jxx} + r_{a_m}(p^{mma}_{jxx} - p^{mmb}_{jxx})$$

Which means we can solve for r_{a_m} and r_{b_m} unless $p^{mma}_{jxx} = p^{mmb}_{jxx}$ for all x. But if this happens we also get $p^m_{xj} = p^{mma}_{jxx} = p^{mmb}_{jxx}$. This implies by Lemma 4.2.3 that $p^{mma}_{jxy} = p^{mma}_{jxz} = 0$ for y, z distinct and not equal to x. But now by Lemma 4.2.4 and Lemma 4.2.3, either $p^m_{jx} = 0$ or both $p^m_{ma} = p^{mjx}_{max}$ and $p^m_{mb} = p^{mjx}_{mbx}$. But again by Lemma 4.2.3, either $p^j_{mx} = 0$ or $p^j_{mx} = p^{mjx}_{max} + p^{mjx}_{mbx} + 1 = p^m_{ma} + p^m_{mb} + 1 = k_m$. The latter implies $p^m_{jy} = p^m_{jz} = 0$ by Lemma 4.2.1. As we could have done the exact same thing for y and z, we see that we must have that two of p^m_{jx}, p^m_{jy} and p^m_{jz} are zero. By Lemma 7.1.15, we must have $p^m_{jm} = p^m_{jj} = 0$.

So now suppose we are 3-coloured. Then we know $p^m_{ma} \neq 0$ for any a and so r_{c_m} and r_{d_m} are non-zero. Hence by Lemma 7.4.4 we can solve the equations 7.4.3 in x, y and z to get unique solutions for r_{c_m} and r_{d_m} if we do not have $\mathfrak{D}^{xy}_{mj} = \mathfrak{D}^{yz}_{mj} = \mathfrak{D}^{zx}_{mj} = 0$. Hence if we cannot do this we must have one of the conclusions from Lemma 7.4.11, which correspond to undesirability, semi-undesirability or a multiple with respect to j. □

At this point we can note that the second option doesn't actually occur, this is due to the later result Theorem 7.6.20.

We can note that there are some strong consequences of the multiple case.

Lemma 7.4.16 *Suppose we have an eigenvalue triple of the m-neighbourhood $(r_{c_m}, r_{d_m}, r_{e_m})$ that is in the 0 case in j. Suppose also that the m-neighbourhood is a multiple with respect to the j-neighbourhood and $p^m_{ma} \neq 0$ for all colours a. Then, for distinct colours x, y, z,*

i) $p^{mmx}_{jcd} p^{mmy}_{jce} = p^{mmx}_{jce} p^{mmy}_{jcd}$,

ii) $p^{mjc}_{mxd} p^{mjc}_{mye} = p^{mjc}_{mxe} p^{mjc}_{myd}$,

iii) $p^{j}_{md} p^{mjc}_{mye} = p^{j}_{me} p^{mjc}_{myd}$,

iv) $p^{mje}_{myc} = p^{mjd}_{myc}$.

Proof. The proof is fairly simple. We know that in the multiple case the equation 7.4.4

$$0 = -p^{mmz}_{jcd} + (p^{mmx}_{jcd} - p^{mmz}_{jcd}) r_{x_m} + (p^{mmy}_{jcd} - p^{mmz}_{jcd}) r_{y_m}$$

must be a linear multiple of the equation 7.4.4

$$0 = -p^{mmz}_{jce} + (p^{mmx}_{jce} - p^{mmz}_{jce}) r_{x_m} + (p^{mmy}_{jce} - p^{mmz}_{jce}) r_{y_m}$$

Hence, for some non-zero constant λ,

$p^{mmz}_{jcd} = \lambda p^{mmz}_{jce}$, $p^{mmx}_{jcd} - p^{mmz}_{jcd} = \lambda (p^{mmx}_{jce} - p^{mmz}_{jce})$ and $p^{mmy}_{jcd} - p^{mmz}_{jcd} = \lambda (p^{mmy}_{jce} - p^{mmz}_{jce})$

The first equation gives us $\lambda = \frac{p^{mmz}_{jcd}}{p^{mmz}_{jce}}$, which we can then substitute into the others giving

$$p^{mmx}_{jcd}p^{mmz}_{jce} = p^{mmx}_{jce}p^{mmz}_{jcd}$$
$$p^{mmy}_{jcd}p^{mmz}_{jce} = p^{mmy}_{jce}p^{mmz}_{jcd}$$

Combining these will give us the third

$$p^{mmx}_{jcd}p^{mmy}_{jce} = p^{mmx}_{jce}p^{mmy}_{jcd}$$

The other results we can find from manipulating these equations. First if we just apply Lemma 4.2.4 to every intersection number we get $p^{mjc}_{mxd}p^{mjc}_{mye} = p^{mjc}_{mxe}p^{mjc}_{myd}$. Now we can use this and Lemma 4.2.3 to note that

$$\begin{aligned} p^{j}_{md}p^{mjc}_{mye} &= (p^{mjc}_{mxd} + p^{mjc}_{myd} + p^{mjc}_{mzd})p^{mjc}_{mye} \\ &= p^{mjc}_{mxe}p^{mjc}_{myd} + p^{mjc}_{myd}p^{mjc}_{mye} + p^{mjc}_{mze}p^{mjc}_{myd} \\ &= p^{mjc}_{myd}(p^{mjc}_{mxe} + p^{mjc}_{mye} + p^{mjc}_{mze}) \\ &= p^{j}_{me}p^{mjc}_{myd} \end{aligned}$$

Finally, making repeated use of Lemma 4.2.4

$$p^{j}_{md}p^{j}_{mc}p^{mjc}_{mye} = p^{j}_{me}p^{j}_{mc}p^{mjc}_{myd}$$
$$p^{j}_{md}p^{j}_{me}p^{mje}_{myc} = p^{j}_{me}p^{j}_{md}p^{mjd}_{myc}$$
$$p^{mje}_{myc} = p^{mjd}_{myc}$$

□

This final equality impacts a lot. Note that m and j are 'swapped' from the previously established convention here as this lemma is from the perspective of assuming that m is a multiple with respect to j. We don't assume this, because the Lemma holds from a single consequence of this, namely the identity $p^{mjd}_{myc} = p^{mje}_{myc}$, and so we don't require the full strength of the condition.

Lemma 7.4.17 *Suppose for distinct colours m, j, any colour y and distinct colours c, d, e we have $p^{mjd}_{myc} = p^{mje}_{myc}$, and also suppose that v' is a non-principal eigenvector of the j-neighbourhood. Then either v' is in the 0 case in m or is in the eigenvector case and corresponds to an eigenvalue triple where $r_{y_m} = p^{mjc}_{myc} - p^{mjd}_{myc}$.*

Proof. Suppose that we are not in the 0 case, so for some colour c, $N^c_{mj}v' \neq 0$. Now we consider the value of $N^y_{mm}N^c_{mj}v'$. From Lemma 4.2.13, and $Jv' = 0$, we get

$$\begin{aligned} N^y_{mm}N^c_{mj}v' &= (p^{mjc}_{myc} - p^{mje}_{myc})N^c_{mj}v' + (p^{mjd}_{myc} - p^{mje}_{myc})N^d_{mj}v' \\ &= (p^{mjc}_{myc} - p^{mje}_{myc})N^c_{mj}v' \end{aligned}$$

Hence $N^c_{mj}v'$ is an eigenvector of the m-neighbourhood, and has eigenvalues $r_{y_m} = p^{mjc}_{myc} - p^{mjd}_{myc}$. □

Further this equality is actually is at odds with the Eigenvector case altogether

Lemma 7.4.18 *Suppose for all b, and all distinct x, y, z, we have $p^{mjy}_{mbz} = p^{mjx}_{mbz}$ and that $p^m_{ja} \neq 0$ for all a. Then we cannot have an eigenvalue triple for the m-neighbourhood in the Eigenvector case in j.*

Proof. From $p^{mjy}_{mbz} = p^{mjx}_{mbz}$ and Lemma 4.2.4 we have for all b that

$$p^m_{jx}p^{mmb}_{jyz} = p^m_{jy}p^{mmb}_{jxz} \tag{7.4.8}$$

Now suppose for a contradiction that we have an eigenvalue triple $(s_{c_m}, s_{d_m}, s_{e_m})$ in the Eigenvector case in j. Note that, for $v \in E(s_{c_m}, s_{d_m}, s_{e_m})$, we know that

$$v^T N^x_{mj}N^z_{jm}v = \left(p^{mmc}_{jxz}s_{c_m} + p^{mmd}_{jxz}s_{d_m} + p^{mme}_{jxz}s_{e_m}\right)|v|^2$$

which by use of Equation 7.4.8 becomes

$$
\begin{aligned}
&= \left(\frac{p^m_{jx}}{p^m_{jy}} p^{mmc}_{jyz} s_{c_m} + \frac{p^m_{jx}}{p^m_{jy}} p^{mmd}_{jyz} s_{d_m} + \frac{p^m_{jx}}{p^m_{jy}} p^{mme}_{jyz} s_{e_m} \right) |v|^2 \\
&= \frac{p^m_{jx}}{p^m_{jy}} v^T N^y_{mj} N^z_{jm} v
\end{aligned}
$$

And so, as we know that for some constant λ, $N^x_{jm}v = \lambda N^y_{jm}v$, we must have that if $N^z_{jm}v \neq 0$,

$$N^x_{jm}v = \frac{p^m_{jx}}{p^m_{jy}} N^y_{jm}v$$

Note that we also get that, if $N^y_{jm}v \neq 0$

$$N^x_{jm}v = \frac{p^m_{jx}}{p^m_{jz}} N^z_{jm}v$$

Now we know $0 = Jv = N^x_{jm}v + N^y_{jm}v + N^z_{jm}v$ and so

$$
\begin{aligned}
0 &= (1 + \frac{p^m_{jx}}{p^m_{jz}} + \frac{p^m_{jx}}{p^m_{jz}}) N^x_{jm}v \\
&= \frac{k_j}{p^m_{jz}} N^x_{jm}v
\end{aligned}
$$

So as $k_j \neq 0$, $N^x_{jm}v = 0$.

But as $p^m_{ja} \neq 0$ for all a, either $N^z_{jm}v = 0$ or $N^y_{jm}v = 0$ as well, implying all three must be 0, a contradiction with the definition of the Eigenvector case. □

Putting this together with Lemma 7.4.16 we get

Corollary 7.4.19 *Suppose we have an eigenvalue triple of the m-neighbourhood $(r_{c_m}, r_{d_m}, r_{e_m})$ that is case 0 in j. Suppose also that the m-neighbourhood is a multiple with respect to the j-neighbourhood. Then if $p^m_{jx} \neq 0$ and $p^m_{mx} \neq 0$ for all x, then no eigenvalue triple of the m-neighbourhood can be in the Eigenvector case in j.*

7.5 Consequences of Eigenvectors in the Independent Case

This is maybe the most oppressive of the three cases, and it turns out we can entirely disprove its existence, which we do in Theorem 9.2.2. However in order to get that far we need to fully explore the limitation and consequences of the Independent cases, which we do here. The main result is Theorem 7.5.2, which both determines the value of all the eigenvalues of an eigenvalue triple in this case, but also determines that the familiar equality $p^{jmb}_{mad} = p^{jmc}_{mad}$ holds. The consequences of this equality are fully explored in this section.
First we can note a rather nice result which seriously limits the prevalence of the Independent case.

Lemma 7.5.1 *Suppose for some c, $p^m_{cj} = 0$. Then, for v an eigenvector for the m-neighbourhood, $N^c_{jm}v = 0$ and $N^d_{jm}v$ is either 0 or an eigenvector of N^m_{jj}, for any $d \neq c$. Hence v cannot be the Independent case in j.*

Proof. We first see that by Lemma 4.2.5 and Lemma 4.2.3,

$$\begin{aligned}|N^c_{jm}v|^2 &= p^m_{cj} + p^{mmm}_{jcc} r_{m_m} + p^{mmj}_{jcc} r_{j_m} + p^{mml}_{jcc} r_{l_m} \\ &= 0\end{aligned}$$

And so $N^c_{jm}v = 0$. Well now, we note that as $J = N^c_{jm} + N^d_{jm} + N^e_{jm}$ and $Jv = 0$, we get $N^e_{jm}v = -N^d_{jm}v$ for $d, e \neq c$. Then by using Lemma 4.2.13 and multiplying by v on the right we see

$$\begin{aligned}N^m_{jj}N^d_{jm}v &= p^{jmc}_{jmd}N^c_{jm}v + p^{jmd}_{jmd}N^d_{jm}v + p^{jme}_{jmd}N^e_{jm}v \\ &= (p^{jmd}_{jmd} - p^{jme}_{jmd})N^d_{jm}v\end{aligned}$$

Hence $N^d_{jm}v$ an eigenvector or it is zero.
Also not that $N^d_{jm}v$ and $N^e_{jm}v$ are not linearly independent so v is not in the Independent case. □

Specifically this means that the combination of some undesirable cases and the Independent case cannot happen which is always good.

Now we look at the general implications of the Independent Case. Note in the following analysis we can safely assume $p_{jc}^{m} \neq 0$ for all c by the previous lemma. We aim to prove the following

Theorem 7.5.2 *For any colours d, e, f, j, m, such that j and m are distinct, suppose v is an eigenvector for N_{mm}^{d}, orthogonal to u, with eigenvalue r_{d_m}, such that $N_{jm}^{e}v$ is non-zero and not an eigenvector of N_{jj}^{f}. Then for all distinct colours a, b, c*

$$\begin{aligned} r_{d_m} &= p_{mad}^{jma} - p_{mad}^{jmb} \\ p_{mad}^{jmb} &= p_{mad}^{jmc} \end{aligned}$$

Proof. For distinct a, b, c, by using Lemma 4.2.13 and multiplying by v on the right we get the equation:

$$\begin{aligned} N_{jm}^{a}N_{mm}^{d}v =& (p_{mad}^{jma} - p_{mad}^{jmb})N_{jm}^{a}v + (p_{mad}^{jmc} - p_{mad}^{jmb})N_{jm}^{c}v \\ 0 =& (p_{mad}^{jma} - p_{mad}^{jmb} - r_{d_m})N_{jm}^{a}v + (p_{mad}^{jmc} - p_{mad}^{jmb})N_{jm}^{c}v \end{aligned}$$

Now we know that $N_{jm}^{a}v$ is non-zero and also that it is linearly independent of $N_{jm}^{c}v$, otherwise it would be an eigenvalue of N_{jj}^{f}. Hence we must have $p_{mad}^{jma} - p_{mad}^{jmb} = r_{d_m}$, and consequently, $p_{mad}^{jmc} = p_{mad}^{jmb}$ too as $N_{jm}^{c}v$ must also be non-zero. ☐

If this situation occurs we can actually determine quite a lot about the rest of the structure.

Lemma 7.5.3 *For colours d, e, e', f and distinct colours j and m, suppose v is an eigenvector for N_{mm}^{d}, orthogonal to u, with eigenvalue r_{d_m}, such that $N_{jm}^{e}v$ is non-zero and not an eigenvector of N_{jj}^{f}. Then, for any eigenvector, v', of N_{jj}^{f}, orthogonal to u, either $N_{mj}^{e'}v' = 0$ or $N_{mj}^{e'}v'$ is an eigenvector for N_{mm}^{d} orthogonal to v but with eigenvalue r_{d_m}.*

Proof. We get immediately from 7.5.2 that $r_{d_m} = p_{mad}^{jma} - p_{mad}^{jmb}$ and $p_{mad}^{jmb} = p_{mad}^{jmc}$ for any distinct a, b, c.

Now fix an eigenvector v' of N^f_{jj} orthogonal to u and consider

$$\begin{aligned} N^d_{mm}N^{e'}_{mj}v' &= (p^{mjR}_{mde'} - p^{mjB}_{mde'})N^R_{mj}v' + (p^{mjG}_{mde'} - p^{mjB}_{mde'})N^G_{mj}v' \\ &= (p^{jmR}_{me'd} - p^{jmB}_{me'd})N^R_{mj}v' + (p^{jmG}_{me'd} - p^{jmB}_{me'd})N^G_{mj}v' \\ &= r_{d_m}N^{e'}_{mj}v' \end{aligned}$$

Therefore either $N^{e'}_{mj}v'$ is an eigenvector of N^d_{mm} with eigenvalue r_{d_m} or it is 0. Note that if $N^{e'}_{mj}v'$ is an eigenvector it must be orthogonal to v, else $N^e_{jm}v$ would be a multiple of v', and therefore an eigenvector of N^f_{jj} as it's non-zero. □

The equality $p^{jmb}_{mad} = p^{jmc}_{mad}$ from Theorem 7.5.2 is the same as the one in Lemma 7.4.18 and, as we know by Lemma 7.5.1 that $p^m_{jx} \neq 0$ for all x, we get the following result.

Corollary 7.5.4 *Suppose we have an eigenvector of the m-neighbourhood with eigenvalues $(r_{c_m}, r_{d_m}, r_{e_m})$ that is the Independent case in j. Then there does not exist an eigenvalue triple of the m-neighbourhood in the Eigenvector case in j.*

Therefore, this coupled with Lemma 7.4.17, means that in this scenario, the j-neighbourhood can only have eigenvalues in the 0-case in m.

Lemma 7.5.5 *Suppose we have an eigenvector of the m-neighbourhood with eigenvalues $(r_{c_m}, r_{d_m}, r_{e_m})$ that is the Independent case in j. Then all eigenvectors of the $E(r_{c_m}, r_{d_m}, r_{e_m})$ are in the independent case.*

Proof. Consider any v' in $E(r_{c_m}, r_{d_m}, r_{e_m})$. v' cannot be in the 0 case as this would imply v was by Corollary 7.4.3. By Corollary 7.5.4, we also cannot have v' be in the Eigenvector case. Hence v' is in the Independent case. □

This allows us to conclude that each eigenspace for an eigenvalue triple is of one consistent case.

We can show further that if we have the Independent case, we can't be in the multiple case either.

Lemma 7.5.6 *Suppose we have an eigenvalue triple of the m-neighbourhood $(r_{c_m}, r_{d_m}, r_{e_m})$ that is the Independent case in j, then the j-neighbourhood can't be a multiple with respect to m.*

Proof. Suppose the j-neighbourhood was a multiple with respect to m. Then by Lemma 7.4.16 we get that for any d, and distinct a, b, c, we have

$$p_{jda}^{jmb} = p_{jda}^{jmc}$$

But now, for some v an eigenvector of the m-neighbourhood belonging to the eigenspace of $(r_{c_m}, r_{d_m}, r_{e_m})$, consider the equation

$$\begin{aligned} N_{jj}^{d} N_{jm}^{a} v &= (p_{jda}^{jma} - p_{jda}^{jmc}) N_{jm}^{a} v + (p_{jda}^{jmb} - p_{jda}^{jmc}) N_{jm}^{b} v \\ &= (p_{jda}^{jma} - p_{jda}^{jmc}) N_{jm}^{a} v \end{aligned}$$

This implies that $N_{jm}^{a} v$ is in fact an eigenvector of the j-neighbourhood. The choice for a was arbitrary so all of them are actually eigenvectors of the j-neighbourhood. Therefore they are not linearly independent, and not in the Independent case. □

7.6 Combining the cases

All of these results have been talking about just a single eigenvalue, but as we know by Lemma 7.2.9, each neighbourhood can have up to three non-principal eigenvalues and so it makes sense to look at how they interact. We will start with multiple 0 cases. Note that in this section we will always assume m, j and l are distinct colours.

Lemma 7.6.1 *There is a maximum of two distinct eigenvalue triples of the m-neighbourhood that are in the 0 case in j.*

Proof. First note that if the m-neighbourhood is not 3-coloured, this is trivially true as we only have a maximum of two eigenvalue triples in this case by Lemma 7.2.9. We also know that there must exist some colour c such that $p_{cj}^{m} \neq 0$ or else both k_m and k_j would be 0. Suppose for a contradiction that every eigenvalue triple was in the 0 case in j and further that for some colour x and distinct colours y and z, we have $p^{mmy}_{mxx} \neq p^{mmz}_{mxx}$. Hence we have by Equation 7.4.3 in c

$$0 = (p^m_{cj} - p^{mmz}_{jcc}) + (p^{mmx}_{jcc} - p^{mmz}_{jcc})r_{x_m} + (p^{mmy}_{jcc} - p^{mmz}_{jcc})r_{y_m}$$

Now as $p^{mmy}_{mxx} \neq p^{mmz}_{mxx}$ we can use Lemma 7.2.3 to get

$$\begin{aligned}
0 =&(p^m_{cj} - p^{mmz}_{jcc}) + (p^{mmx}_{jcc} - p^{mmz}_{jcc})r_{x_m} + (p^{mmy}_{jcc} - p^{mmz}_{jcc})(\frac{p^m_{xm} - p^{mmz}_{mxx} + r_{x_m}(p^{mmx}_{mxx} - p^{mmz}_{mxx}) - r^2_{x_m}}{p^{mmz}_{mxx} - p^{mmy}_{mxx}}) \\
=&\frac{1}{p^{mmz}_{mxx} - p^{mmy}_{mxx}} \Big(r^2_{x_m}(p^{mmz}_{jcc} - p^{mmy}_{jcc}) + \\
&r_{x_m}((p^{mmx}_{mxx} - p^{mmz}_{mxx})(p^{mmy}_{jcc} - p^{mmz}_{jcc}) + (p^{mmz}_{mxx} - p^{mmy}_{mxx})(p^{mmx}_{jcc} - p^{mmz}_{jcc})) + \\
&(p^m_{jc} - p^{mmz}_{jcc})(p^{mmz}_{mxx} - p^{mmy}_{mxx}) + (p^{mmy}_{jcc} - p^{mmz}_{jcc})(p^m_{xm} - p^{mmz}_{mxx})\Big)
\end{aligned}$$

Now we have a quadratic in terms of r_{x_m}. Provided both the coefficients of both r_{x_m} and $r^2_{x_m}$ aren't zero then, this can be solved, giving a maximum of two solutions. So all that remains to be shown is that we can't have both $p^{mmz}_{jcc} = p^{mmy}_{jcc}$ and $(p^{mmx}_{mxx} - p^{mmz}_{mxx})(p^{mmy}_{jcc} - p^{mmz}_{jcc}) = -(p^{mmz}_{mxx} - p^{mmy}_{mxx})(p^{mmx}_{jcc} - p^{mmz}_{jcc})$. Well suppose we did, then we would have $0 = (p^{mmz}_{mxx} - p^{mmy}_{mxx})(p^{mmx}_{jcc} - p^{mmz}_{jcc})$. Now we already know $p^{mmy}_{mxx} \neq p^{mmz}_{mxx}$, and so therefore this would give $p^{mmx}_{jcc} = p^{mmz}_{jcc}$. By Lemma 7.4.6 if $p^{mmx}_{jcc} = p^{mmy}_{jcc} = p^{mmz}_{jcc}$ and we have an eigenvalue triple in the 0 case, then $p^m_{jc} = 0$, a contradiction as we established earlier that $p^m_{jc} \neq 0$. Hence we must have $p^{mmy}_{mxx} = p^{mmz}_{mxx}$ for all distinct x, y and z.

So suppose now that we have for all distinct x, y, z we have $p^{mmy}_{mxx} = p^{mmz}_{mxx}$, and suppose for a contradiction that all three non-principal eigenvalue triples are in the 0 case in j. By the condition that $p^{mmy}_{mxx} = p^{mmz}_{mxx}$ we know that each colour adjacency matrix has only 2 distinct non-principal eigenvalues. The m-neighbourhood must therefore be in case iii) from Remark 7.1.13. We can see then that for each colour x, there exists an eigenvalue of N^x_{mm} that is $p^{mmy}_{mxy} - p^{mmz}_{mxy}$ and this eigenvalue is present in two distinct eigenvalue triples of the m-neighbourhood. Say we now consider just a particular fixed colour x and say w.l.o.g $r_{x_m} = p^{mmy}_{mxy} - p^{mmz}_{mxy}$. From 7.4.3 in x, we get

$$0 = p^m_{xj} - p^{mmz}_{jxx} + (p^{mmx}_{jxx} - p^{mmz}_{jxx})(p^{mmy}_{mxy} - p^{mmz}_{mxy}) + (p^{mmy}_{jxx} - p^{mmz}_{jxx})r_{y_m}$$

So unless the m-neighbourhood is x-semi-undesirable, then we can determine r_{y_m}, contradicting the fact that r_{x_m} is in two distinct eigenvalue triples. Hence the m-neighbourhood is x-semi-undesirable. But then consider the eigenvalue triple without r_{x_m}, and say the x eigenvalue is s_{x_m}. But then by our assumption that this was also in the 0 case in j, by Equation 7.4.3 coupled with x-semi-undesirability, we see

$$0 = p_{xj}^{m} - p_{jxx}^{mmz} + (p_{jxx}^{mmx} - p_{jxx}^{mmz})s_{x_m}$$

Implying that actually s_{x_m} does equal r_{x_m}. But then r_{x_m} is the only eigenvalue of N_{mm}^{x} in the m-neighbourhood, and it will have multiplicity $k_m - 1$, meaning the m-neighbourhood is complete in x. But then it is 1-coloured, a contradiction. □

Therefore a maximum of two of our eigenvalue triples in the m-neighbourhood are in the 0 case in j, however we can reduce this number further depending on their desirability from Definition 7.4.13. This is because if we can solve the Equations 7.4.3 then we determine what the eigenvalue is in terms of intersection numbers, and so any eigenvalue triple in this situation must have the same eigenvalues.

Lemma 7.6.2 *Suppose m is not semi-undesirable, undesirable or a multiple with respect to j, and we have eigenvalue triples $(r_{c_m}, r_{d_m}, r_{e_m})$ and $(s_{c_m}, s_{d_m}, s_{e_m})$ both in the 0 case in j. Then $r_{c_m} = s_{c_m}$, $r_{d_m} = s_{d_m}$ and $r_{e_m} = s_{e_m}$.*

Proof. We know by Lemma 7.4.11 that there must exist some x and y such that $\mathfrak{D}_{mj}^{xy} \neq 0$. Therefore we can apply Lemma 7.4.4 to both $(r_{c_m}, r_{d_m}, r_{e_m})$ and $(s_{c_m}, s_{d_m}, s_{e_m})$. Hence the eigenvalue triples must be equal. □

We can get some similar results surrounding semi-undesirability.

Lemma 7.6.3 *Suppose m is d-semi-undesirable with respect to j, and we have eigenvalue triples*
$(r_{c_m}, r_{d_m}, r_{e_m})$ and $(s_{c_m}, s_{d_m}, s_{e_m})$ both in the 0 case in j. Then $r_{d_m} = s_{d_m}$ and either $r_{c_m} = s_{c_m}$, $r_{e_m} = s_{e_m}$ or $r_{d_m} = p_{mdc}^{mmc} - p_{mdc}^{mme}$.

Proof. Given d-semi-undesirability and Equation 7.4.3 for the eigenvalue triple $(r_{c_m}, r_{d_m}, r_{e_m})$ we get, for all colours x,

$$0 = (p^m_{xj} - p^{mme}_{jxx}) + (p^{mmd}_{jxx} - p^{mme}_{jxx})r_{d_m}$$

Hence we can solve this for r_{d_m}. However we can note that we get the exact same equation for the eigenvalue triple $(s_{c_m}, s_{d_m}, s_{e_m})$, and hence $r_{d_m} = s_{d_m}$.
Now by applying Lemma 7.2.5, we see that either $r_{c_m} = s_{c_m}$ or $r_{d_m} = s_{d_m} = p^{mmc}_{mdc} - p^{mme}_{mdc}$. □

We can consider the Independent case in a similar manner and get strong results.

Lemma 7.6.4 *Suppose we have eigenvalue triples $(r_{c_m}, r_{d_m}, r_{e_m})$ and $(s_{c_m}, s_{d_m}, s_{e_m})$ of m both in the Independent case in j. Then $(r_{c_m}, r_{d_m}, r_{e_m}) = (s_{c_m}, s_{d_m}, s_{e_m})$.*

Proof. Applying Theorem 7.5.2 to the eigenvalue triple $(r_{c_m}, r_{d_m}, r_{e_m})$, we see that we can determine the value of each of the eigenvalues. Doing the same thing to $(s_{c_m}, s_{d_m}, s_{e_m})$ gives the same values, hence eigenvalues of the same colour must be equal. □

Similar, slightly weaker results are available for the eigenvector case too.

Lemma 7.6.5 *Suppose we have eigenvalue triples $(r_{c_m}, r_{d_m}, r_{e_m})$ and $(s_{c_m}, s_{d_m}, s_{e_m})$ of m both in the eigenvector case in j. Further suppose there exists a colour x such that for any eigenvector v of E_{r_m}, $N^x_{jm}v = 0$ and for any eigenvector v' of E_{s_m}, $N^x_{jm}v' = 0$. Then $(r_{c_m}, r_{d_m}, r_{e_m}) = (s_{c_m}, s_{d_m}, s_{e_m})$.*

This is just a straightforward consequence of Lemma 7.3.6.
So far we have only been combining cases going from the m-neighbourhood to the j-neighbourhood. However we can also take into account the consequences of the fact there must also be eigenvectors going between the m and l neighbourhoods. For instance a fairly strong result is the following.

Lemma 7.6.6 *Suppose v is an eigenvector of the m-neighbourhood with eigenvalues $(r_{c_m}, r_{d_m}, r_{e_m})$. Then if for some c, $N^c_{jm}v = N^c_{lm}v = 0$, we have $N^d_{jm}v = N^d_{lm}v = 0$, $N^e_{jm}v = N^e_{lm}v = 0$ and $(r_{c_m}, r_{d_m}, r_{e_m}) = (r_c, r_d, r_e),\ (s_c, s_d, s_e),$ or (t_c, t_d, t_e).*

Proof. Consider the vector $v' = (v^T, 0, 0)^T$. Then

$$A_c v' = \begin{pmatrix} 0 & u^T & 0 & 0 \\ u & N^c_{mm} & N^c_{mj} & N^c_{ml} \\ 0 & N^c_{jm} & N^c_{jj} & N^c_{jl} \\ 0 & N^c_{lm} & N^c_{lj} & N^c_{ll} \end{pmatrix} \begin{pmatrix} 0 \\ v \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ N^c_{mm} v \\ N^c_{jm} v \\ N^c_{lm} v \end{pmatrix} = r_{c_m} \begin{pmatrix} 0 \\ v \\ 0 \\ 0 \end{pmatrix}$$

So v' is a non-principal eigenvector of A_c, and so $r_{c_m} = r_c, s_c$ or t_c. Suppose without loss of generality it is r_c. Then further we also know that v' is a non-principal eigenvector of A_d and A_e with eigenvalue r_d and r_e respectively. Hence

$$r_d v' = A_d v'$$
$$= \begin{pmatrix} 0 & u^T & 0 & 0 \\ u & N^d_{mm} & N^d_{mj} & N^d_{ml} \\ 0 & N^d_{jm} & N^d_{jj} & N^d_{jl} \\ 0 & N^d_{lm} & N^d_{lj} & N^d_{ll} \end{pmatrix} \begin{pmatrix} 0 \\ v \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ N^d_{mm} v \\ N^d_{jm} v \\ N^d_{lm} v \end{pmatrix}$$

And so $N^d_{jm} v = N^d_{lm} v = 0$ and $r_{d_m} = r_d$. The same thing will of course happen for the third colour e, giving the result. □

This basically means that an eigenspace cannot be the $a = 0$ eigenvector case in both j and l. It also says that in the case where, for some eigenvalue triple, we have the 0 case in both j and l, the eigenvalues of the m-neighbourhood are actually eigenvalues of the entire graph.

Corollary 7.6.7 *Suppose there is an eigenvalue triple* $(r_{c_m}, r_{d_m}, r_{e_m})$ *that is the 0 case in both* j *and* l*. Then* $(r_{c_m}, r_{d_m}, r_{e_m}) = (r_c, r_d, r_e),\ (s_c, s_d, s_e),\ or\ (t_c, t_d, t_e)$.

Proof. For any eigenvector $v \in E(r_{c_m}, r_{d_m}, r_{e_m})$, we see $N^c_{jm}v = N^c_{lm}v = 0$. Hence by Lemma 7.6.6, $(r_{c_m}, r_{d_m}, r_{e_m}) = (r_c, r_d, r_e),\ (s_c, s_d, s_e),$ or (t_c, t_d, t_e). □

Furthermore we can find other relationships in this case. This is because we have the equations 7.4.3, but for both j and l. Therefore even if they aren't solvable going from m to j and from m to l, then they may be solvable through the interaction of the two.

Firstly let us just consider the equations 7.4.3 going in both from m to j and from m to l. We have for all colours x,

$$0 = (p^m_{xj} - p^{mme}_{jxx}) + (p^{mmc}_{jxx} - p^{mme}_{jxx})r_{c_m} + (p^{mmd}_{jxx} - p^{mme}_{jxx})r_{d_m} \tag{7.6.9}$$

and

$$0 = (p^m_{xl} - p^{mme}_{lxx}) + (p^{mmc}_{lxx} - p^{mme}_{lxx})r_{c_m} + (p^{mmd}_{lxx} - p^{mme}_{lxx})r_{d_m} \tag{7.6.10}$$

Therefore unless the discriminant of these six linear equations, all in terms of r_{c_m} and r_{d_m}, is 0, we can solve for r_{c_m} and r_{d_m}. We will assume that we cannot solve the set for j and the set for l internally, i.e. we have either undesirability, semi-undesirability or multiples. However each of these scenarios leaves us with at least one non-zero equation still and so we can consider this system.

First we can note that if the m-neighbourhood is c-semi-undesirable with respect to one of either the j or the l-neighbourhood, then either we can solve for the eigenvalues or it must also be c-semi-undesirable with respect to the other.

Lemma 7.6.8 *Suppose* v *is an eigenvector of the* m*-neighbourhood with eigenvalue triple* $(r_{c_m}, r_{d_m}, r_{e_m})$*, and is in the 0 case in both* j *and* l*. Suppose the* m*-neighbourhood is* d*-semi-undesirable with respect to* j*. Then either we can find unique values of* $(r_{c_m}, r_{d_m}, r_{e_m})$ *in terms of intersection numbers (either by Equations 7.6.9 and 7.6.10 or otherwise) or the* m*-neighbourhood is also* d*-semi-undesirable with respect to* l*.*

Further if the m*-neighbourhood is* d*-semi-undesirable with respect to both* j *and* l *and we* *cannot find unique values for* $(r_{c_m}, r_{d_m}, r_{e_m})$ *in terms of intersection numbers, then*

$$r_{d_m} = p^c_{cd} - p^e_{cd},\ \textit{and}\ p^c_{dd} = p^e_{dd}$$

Proof. First suppose the m-neighbourhood is d-semi-undesirable with respect to j and not with respect to l. Then we can solve 7.6.9 for r_{d_m}. Now as the m-neighbourhood is not semi-undesirable with respect to l, there exists some colour x such that 7.6.10 is non-zero (as we can't have $p^m_{jx} = 0$ for all x), and has non-zero coefficient for r_{c_m} (otherwise we'd have d-semi-undesirability). Therefore inputting the value of r_{d_m} into Equation 7.6.10 for this x, can solve for r_{c_m} too.
Now suppose the m-neighbourhood is d-semi-undesirable with respect to j and l and we cannot find a unique solution for r_{c_m} in terms for intersection numbers. Then, for all x, y, we have $p^{mmc}_{jxy} = p^{mme}_{jxy}$ and $p^{mmc}_{lxy} = p^{mme}_{lxy}$ by Remark 7.4.14. Further as we cannot find a unique value of r_{c_m} in terms of intersection numbers, we must have $r_{d_m} = p^{mmc}_{mcd} - p^{mme}_{mcd}$ by Lemma 7.2.5. Well now note that by Lemma 4.2.3,

$$\begin{aligned} p^c_{dd} &= p^{mmc}_{mdd} + p^{mmc}_{ldd} + p^{mmc}_{jdd}(+1 \text{ if } m = d) \\ &= p^{mme}_{mdd} + p^{mme}_{ldd} + p^{mme}_{jdd}(+1) \\ &= p^e_{dd} \end{aligned}$$

Also, by Lemma 4.2.3,

$$\begin{aligned} r_{d_m} &= p^{mmc}_{mcd} - p^{mme}_{mcd} \\ &= p^c_{cd} - p^{mmc}_{lcd} - p^{mmc}_{jcd} - p^e_{cd} + p^{mme}_{lcd} - p^{mme}_{jcd} \\ &= p^c_{cd} - p^e_{cd} \end{aligned}$$

□

We can also note that we did not use the 0 case condition to obtain that $p^c_{dd} = p^e_{dd}$, hence:

Corollary 7.6.9 *Suppose the* m*-neighbourhood is* c*-semi-undesirable with respect to* j *and* l*. Then* $p^c_{dd} = p^e_{dd}$*.*

So in the case where we have that m is d-semi-undesirable in both j and l and an eigenvalue triple of the m-neighbourhood is the 0 case in both j and l, then from combining Corollary 7.6.7 and Lemma 7.6.8 we know that A_d has an eigenvalue $r_d = p^c_{cd} - p^e_{cd}$, and is also strongly regular.

SO far we have only looked at the scenarios involving semi-undesirability, but we could also have combinations including undesirability and multiples. We know that by Lemma 7.4.9 if $p^m_{jx} = 0$ then either we have another $p^m_{jy} = 0$ or the Equation 7.4.3 in y and z are multiples. It seems prudent therefore to discuss the interactions of these conditions.

Lemma 7.6.10 *Suppose the m-neighbourhood is both c and d-undesirable with respect to j. Then $\{c, d\} = \{m, j\}$ and the m-neighbourhood can be undesirable in at most one colour with respect to l.*

Proof. By Lemma 7.1.15 we know that if $p^m_{jc} = p^m_{jd} = 0$ then $\{c, d\} = \{m, j\}$. Hence we get the first statement. Now suppose, for distinct colours c' and d', the m-neighbourhood is both c' and d'-undesirable with respect to l, then we know $p^m_{ml} = p^m_{ll} = 0$ by Lemma 7.1.15. But now we get $k_m = 1 + p^m_{mm}$, a contradiction to primitivity. □

We actually can note the following

Lemma 7.6.11 *Suppose $p^m_{mj} = 0$. Then there cannot exist an eigenvalue triple of the m-neighbourhood that is the 0 case in both j and l.*

Proof. Suppose for a contradiction that there exists an eigenvalue triple $(r_{m_m}, r_{j_m}, r_{l_m})$ that is in the 0 case in both j and l. If $p^m_{mj} = 0$ then Corollary 4.2.9 yields $r_j r_m = p^j_{mj} r_j + p^l_{mj} r_l$. Now $p^m_{mj} = 0$ so $r_{j_m} = 0$ by Remark 4.2.15. But by Corollary 7.6.7, there exists an eigenvalue of A_j equal to r_{j_m}. Suppose this is r_j, hence $r_j = 0$. Therefore $0 = p^l_{mj} r_l$. But if $p^l_{mj} = 0$, then we contradict Lemma 7.1.15, and if $r_l = 0$ then $r_m = -1$, meaning the structure is complete in m or imprimitive by Lemma 2.1.14. Neither of these can happen and so we have a contradiction. □

Corollary 7.6.12 *Suppose that there exists an eigenvalue triple of the m-neighbourhood that is the 0 case in both j and l. Then, for distinct c and d, m cannot be both c and*

d-undesirable with respect to j or l.

Proof. Suppose for a contradiction that the m-neighbourhood is c and d-undesirable with respect to j. This will imply $p^m_{cj} = p^m_{dj} = 0$, which by Lemma 7.1.15, means $p^m_{mj} = p^m_{jj} = 0$. Therefore $p^m_{mj} = 0$ and so we can apply Lemma 7.6.11, to get that we can't have an eigenvalue triple of the m-neighbourhood that is the 0 case in both j and l, a contradiction. The exact same thing will happen if we supposed that the m-neighbourhood was c and d-undesirable with respect to l instead, however with $p^m_{ml} = 0$. □

Hence we don't have to worry about both c and d-undesirability when discussing the double 0 case. We can also think about how the interaction between two of the neighbourhoods affect their interactions with the third. We first look at the Eigenvector case and note the following important definition.

Definition 7.6.13. We shall say that an eigenvector, v of the m-neighbourhood and an eigenvector v' of the j-neighbourhood *correspond* if for some colour c and some constants λ_c and λ'_c, $N^c_{jm}v = \lambda_c v'$ and $N^c_{mj}v' = \lambda'_c v$.

We note that this will occur when we have the Eigenvector case.

Remark 7.6.14 *If we have an eigenvalue triple of the m-neighbourhood $(r_{c_m}, r_{d_m}, r_{e_m})$ that is in the Eigenvector case in j, then there will exist an eigenvalue triple $(r_{c_j}, r_{d_j}, r_{e_j})$ of the j-neighbourhood, such that for every $v \in E(r_{c_m}, r_{d_m}, r_{e_m})$ there exists a corresponding $v' \in E(r_{c_j}, r_{d_j}, r_{e_j})$.*

Therefore we can say that Eigenvalue triples *correspond* if the eigenvectors of their eigenspaces do.

When we have multiple Eigenvector cases we can get quite a few results. We shall see that effectively the Eigenvector case acts as kind of equivalence relation.

Lemma 7.6.15 *For distinct colours j, l and m, if there exist corresponding eigenvectors v_m of the m-neighbourhood and v_j of j-neighbourhood such that v_m is of the independent case in l, then v_j is also of the independent case in l.*

Proof. As v_m and v_j correspond, then for some colour c and non-zero constant A, $N^c_{mj}v_j = Av_m$. Therefore for some colour c', $N^{c'}_{lm}N^c_{mj}v_j = AN^{c'}_{lm}v_m$ must be non-zero and not an eigenvector of the l-neighbourhood, as v_m is in the independent case in l. Now by Lemma 4.2.13

$$N^{c'}_{lm}N^c_{mj}v_j = (p^{ljm}_{mc'c} - p^{ljl}_{mc'c})N^m_{lj}v_j + (p^{ljj}_{mc'c} - p^{ljl}_{mc'c})N^j_{lj}v_j \tag{7.6.11}$$

As this must be non-zero, we can't have that both $N^m_{lj}v_j$ and $N^j_{lj}v_j$ are zero. So the 0 case is not an option or v_j in l. If we suppose that only one of these vectors is 0, say $N^m_{lj}v_j = 0$, then we see the non-zero one, $N^j_{lj}v_j$, will become an eigenvector for the l-neighbourhood. This is because for any colour d, $N^d_{ll}N^j_{lj}v_j = (p^{ljj}_{ldj} - p^{ljl}_{ldj})N^j_{lj}v_j$ by Lemma 4.2.13. However we also see that Equation 7.6.11 becomes

$$AN^{c'}_{lm}v_m = (p^{ljj}_{mc'c} - p^{ljl}_{mc'c})N^j_{lj}v_j$$

And hence this implies that $N^{c'}_{lm}v_m$ is also an eigenvector of the l-neighbourhood, hence we have a contradiction with v_m being the independent case. The same contradiction occurs if we assume instead that $N^j_{lj}v_m$ was zero. Therefore $N^x_{lj}v_j$ is non-zero for all colours x.
If $N^m_{lj}v_j$ and $N^j_{lj}v_j$ are eigenvectors of the l-neighbourhood then by Equation 7.6.11 this would imply that $N^{c'}_{lm}N^c_{mj}v_j$ is an eigenvector of the l-neighbourhood too. This would further imply that $N^{c'}_{lm}v_m$ is, a contradiction to the independent case. Hence we can't have both $N^m_{lj}v_j$ and $N^j_{lj}v_j$ being eigenvectors of the l-neighbourhood. But if only one is, then the other must be as well by Lemma 7.3.2, as they can't be 0. So neither $N^m_{lj}v_j$ or $N^j_{lj}v_j$ are, hence they must be in the independent case, concluding the proof. □

We get a similar result with the 0 case

Lemma 7.6.16 *For distinct colours j, l and m, if there exist corresponding eigenvectors v_m of the m-neighbourhood and v_j of the j-neighbourhood such that v_m is the 0 case in l, then v_j is the 0 case in l as well.*

Proof. As v_m and v_j correspond, then for some colour a and non-zero constants A and A', $N^a_{mj}v_j = Av_m$ and $N^a_{jm}v_m = A'v_j$. We also know by Lemma 7.6.15 that v_j cannot be the Independent case in l, as this would imply v_m was as well. Hence it must be either the Eigenvector case or the 0 case. Well suppose for a contradiction v_j is the Eigenvector case in l. Then for some colour c, $N^c_{lj}v_j$ is an eignevector of the l-neighbourhood and for some non-zero constant B and eigenvector v_l of the l-neighbourhood, $N^c_{lj}v_j = Bv_l$. Hence $\frac{1}{A'}N^c_{lj}N^a_{jm}v_m = Bv_l$. But now, as $N^b_{lm}v_m = 0$ for all b by the 0 case assumption for v_m,

$$\begin{aligned} A'Bv_l &= p^{lmc}_{jca}N^c_{lm}v_m + p^{lmd}_{jca}N^d_{lm}v_m + p^{lme}_{jca}N^e_{lm}v_m \\ &= 0 \end{aligned}$$

However A' and B are both non-zero, implying $v_l = 0$, a contradiction. Hence we cannot have v_j in the Eigenvector case in l, and it must be of the 0 case. □

And finally

Lemma 7.6.17 *For distinct colours j, l and m, if there exists corresponding eigenvectors v_m of the m-neighbourhood and v_j of the j-neighbourhood, such that v_m is of the eigenvector case in l, then v_j is also of the eigenvector case in l.*

Proof. This is just a process of elimination. We know that v_j cannot be the Independent case in l by Lemma 7.6.15 and we know it can't be the 0 case by Lemma 7.6.16. Hence it must be the eigenvector case in l. □

Further to the scenario where we have eigenvalue triples of the m-neighbourhood in the 0 case in both j and l, we can also get results when we have certain other combinations of the cases in j and l.

Lemma 7.6.18 *Suppose $(r_{c_m}, r_{d_m}, r_{e_m})$ is an eigenvalue triple of the m-neighbourhood corresponding to $(r_{c_l}, r_{d_l}, r_{e_l})$ an eigenvalue triple in the l-neighbourhood. Then if $(r_{c_m}, r_{d_m}, r_{e_m})$ is in the 0 case in j, we get, for any x and y either*

1. $p^{lmc}_{jxy} = p^{lmd}_{jxy} = p^{lme}_{jxy}$
2. $N^c_{lm}v = 0$, $p^{lmd}_{jxy} = p^{lme}_{jxy}$ *and* $r_{a_l} = p^{lmd}_{lad} - p^{lme}_{lad}$ *for all* a,
3. $N^d_{lm}v = 0$, $p^{lmc}_{jxy} = p^{lme}_{jxy}$ *and* $r_{a_l} = p^{lmc}_{lac} - p^{lme}_{lac}$ *for all* a,

4. $r_{a_l} = \frac{(p^{lmc}_{lac} - p^{lme}_{lac})(p^{lmd}_{jxy} - p^{lme}_{jxy}) + (p^{lmd}_{lac} - p^{lme}_{lac})(p^{lme}_{jxy} - p^{lmc}_{jxy})}{(p^{lmd}_{jxy} - p^{lme}_{jxy})}$ *for all a.*

Proof. Suppose $(r_{c_m}, r_{d_m}, r_{e_m})$ is in the 0 case in j and has eigenvector v. Then for all y, $N^y_{jm}v = 0$. Now by Lemma 4.2.13, for any x,

$$0 = N^x_{lj}N^y_{jm}v = (p^{lmc}_{jxy} - p^{lme}_{jxy})N^c_{lm}v + (p^{lmd}_{jxy} - p^{lme}_{jxy})N^d_{lm}v$$

As v is in the Eigenvector case in l we know we don't have $N^c_{lm}v = N^d_{lm}v = 0$. Hence either

i) $p^{lmc}_{jxy} = p^{lmd}_{jxy} = p^{lme}_{jxy}$

ii) $N^c_{lm}v = 0$ and $p^{lmd}_{jxy} = p^{lme}_{jxy}$

iii) $N^d_{lm}v = 0$ and $p^{lmc}_{jxy} = p^{lme}_{jxy}$

iv) $N^d_{lm}v = \frac{(p^{lmc}_{jxy} - p^{lme}_{jxy})}{(p^{lme}_{jxy} - p^{lmd}_{jxy})}N^c_{lm}v$

If we suppose i) holds then we get outcome 1) from the Lemma. First suppose its case iii) or iv). Then for any a, using Lemma 4.2.13 we get

$$\begin{aligned} r_{a_l}N^c_{lm}v &= N^a_{ll}N^c_{lm}v \\ &= (p^{lmc}_{lac} - p^{lme}_{lac})N^c_{lm}v + (p^{lmd}_{lac} - p^{lme}_{lac})N^d_{lm}v \end{aligned}$$

Assuming case iii), we input $N^d_{lm}v = 0$ and solve this for r_{a_l} giving $r_{a_l} = p^{lmc}_{lac} - p^{lme}_{lac}$ corresponding to outcome 3) from the lemma.

If we assume case iv) then we see we get the result of outcome 4) from the lemma.

Finally assume we are case ii), then we can do the same thing with $r_{a_l}N^d_{lm}v$ to get outcome 2) from the lemma. □

We can get an equivalent to lemma 5.2 from [12]

Lemma 7.6.19 *Suppose that, for distinct m and j, none of the eigenvectors of the m-neighbourhood are in the 0 case in j, and none of the eigenvectors of the j-neighbourhood are in the 0 case in m. Then $k_m = k_j$.*

Proof. Now suppose for a contradiction that $k_m \neq k_j$. Then for all c we get that N^c_{jm} is not square. This will mean that either $N^c_{mj}N^c_{jm}$ has eigenvalue 0 if $k_m > k_j$ or $N^c_{jm}N^c_{mj}$ has eigenvalue 0 if $k_j > k_m$. Suppose without loss of generality we have the first scenario. Then this is the equivalent of saying that for each c there exists an eigenvector v_c of the m-neighbourhood such that $N^c_{jm}v_c = 0$. If for distinct colours c and d, $v_c = v_d$, then we can note that $N^e_{jm}v_c = -N^c_{jm}v_c - N^d_{jm}v_d = 0$ too, and so this eigenvector is of the 0 case in j, a contradiction. But then each of these eigenvectors must be of the Eigenvector case in j, as the independent case requires $N^x_{jm}v$ to be non-zero for all colours x. Also they must all be from separate eigenvalue triples, meaning that all eigenvalue triples of m are in the Eigenvector case in j. But then each of these eigenvalue triples corresponds to a different eigenvalue triple in j. As the multiplicities of these corresponding pairs of eigenvalue triples are the same, $k_m \leq k_j$, a contradiction. So $k_m = k_j$. □

We conclude this chapter with the following powerful Theorem, that draws from a lot of the results of the section

Theorem 7.6.20 *In a primitive 3-regular 3-coloured structure, for distinct colours m and j, we cannot have both $p^m_{mj} = 0$ and $p^m_{jj} = 0$.*

Proof. Suppose we have $p^m_{mj} = p^m_{jj} = 0$. Then the first thing we want to note is that both $N^m_{mj} = 0$ and $N^j_{mj} = 0$, implying that $N^l_{mj} = J$. Now consider v an eigenvector of the m-neighbourhood and v' an eigenvector of the j-neighbourhood. As $0 = N^m_{mj}v' = N^j_{mj}v' = N^l_{mj}v'$, v' is in the 0 case in m. Similarly $0 = N^m_{mj}v = N^j_{mj}v = N^l_{mj}v$ and so v is in the 0 case in j. Hence there can only be eigenvectors of the 0 case between the m and j-neighbourhoods. Therefore by Lemma 7.6.1 there can only be a maximum two distinct non-principal eigenvalue triples in both the m and the j neighbourhoods.

We now consider the eigenvectors v and v' in terms of l. As $p^m_{mj} = 0$ we know by Lemma 7.6.11 that v can't be the 0 case in l and by Lemma 7.5.1 and $p^m_{mj} = 0$ we know that v can't be the Independent case in l. Hence it must be the Eigenvector case in l. However we get the exact same argument with v' using $p^j_{jm} = 0$. Therefore all eigenvectors of the m and j-neighbourhoods correspond with the eigenvectors of the l-neighbourhood, however are the 0 case between the m and j-neighbourhoods. This contradicts Lemma 7.6.17. Hence we cannot have both $p^m_{mj} = 0$ and $p^m_{jj} = 0$. □

Chapter 8

A More Spectral Approach

In this chapter we follow the ideas of [12] more closely. The aim is to use the concept of a spherical 3-distance set in much the same way that they use a spherical 2-distance set, however it should be noted that the literature on such sets is nowhere near as extensive. The main difference that arises is that in [12], they are guaranteed that eigenvalues correspond by the fact that having one immediately determines the other by $0 = 1 + r + s$. Other such problems also arise during the course of this analysis. However some results can still be gained by using their methods, as we show in this chapter. Ultimately this chapter leads to us gaining a better understanding of the 0 case (Corollary 8.2.13), learning how the idempotents can relate to certain case distributions from Definition 7.2.10 (Lemma 8.2.18) and also being able to determine the eigenvalues in the case where we have the eigenvector case in j and the 0 case in l (Theorem 8.2.22). These methods are useful as they allow us to circumvent the issues of desirability in Definition 7.4.13.

8.1 Initial Results using just 2-regularity

In this section we will be working under just 2-regularity. The initial plan is to find all our parameters in terms of the eigenvalues. Recall the definition of the constant D from Equation 4.1.5 and that it must be non-zero by Lemma 4.2.12.

Definition 8.1.1. We shall also start using the convention that μ_i is f, g or h when $i = 1, 2$ or 3 respectively and $(\lambda_i)_j$ is r_j, s_j or t_j when $i = 1, 2$ or 3 respectively; the subscript i will be modulo 3 here to allow for easier indentation.

It is important to note that the μ_i and λ_i have nothing to do with the classical parameters μ and λ of the strongly regular graph. Recall the constant D, defined via Equation 4.1.5. We can find the following equations:

Lemma 8.1.2 *The multiplicities of the eigenvalues can be expressed in terms of n and the eigenvalues as follows:*

$$Df = (n-1)(s_R t_G - s_G t_R) + k_R(t_G - s_G) + k_G(s_R - t_R)$$
$$Dg = (n-1)(t_R r_G - t_G r_R) + k_R(r_G - t_G) + k_G(t_R - r_R)$$
$$Dh = (n-1)(r_R s_G - r_G s_R) + k_R(s_G - r_G) + k_G(r_R - s_R)$$

Proof. This comes from solving the set of equations that we get from the fact the trace of each adjacency matrix is 0:

$$0 = k_R + fr_R + gs_R + ht_R$$
$$0 = k_G + fr_G + gs_G + ht_G$$
$$n = 1 + f + g + h$$

□

Using this we can also find the constants used in the transition matrix 4.1.4.

Lemma 8.1.3

$$Df\alpha_1 = k_G(s_R - t_R) + (n - k_R)(s_G - t_G) + t_R s_G - t_G s_R$$

$$Dg\alpha_2 = k_G(t_R - r_R) + (n - k_R)(t_G - r_G) + r_R t_G - r_G t_R$$

$$Dh\alpha_3 = k_G(r_R - s_R) + (n - k_R)(r_G - s_G) + s_R r_G - s_G r_R$$

$$Df\beta_1 = k_R(t_G - s_G) + (n - k_G)(t_R - s_R) + s_G t_R - s_R t_G$$

$$Dg\beta_2 = k_R(r_G - t_G) + (n - k_G)(r_R - t_R) + t_G r_R - t_R r_G$$

$$Dh\beta_3 = k_R(s_G - r_G) + (n - k_G)(s_R - r_R) + r_G s_R - r_R s_G$$

$$Df\gamma_1 = k_G(s_R - t_R) - k_R(s_G - t_G) + t_R s_G - t_G s_R$$

$$Dg\gamma_2 = k_G(t_R - r_R) - k_R(t_G - r_G) + r_R t_G - r_G t_R$$

$$Dh\gamma_3 = k_G(r_R - s_R) - k_R(r_G - s_G) + s_R r_G - s_G r_R$$

Proof. Similar to the above, this comes from solving the equations generated by

$$nI = \begin{pmatrix} 1 & 1 & 1 & 1 \\ f & f\alpha_1 & f\beta_1 & f\gamma_1 \\ g & g\alpha_2 & g\beta_2 & g\gamma_2 \\ h & h\alpha_3 & h\beta_3 & h\gamma_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ k_R & r_R & s_R & t_R \\ k_G & r_G & s_G & t_G \\ k_B & r_B & s_B & t_B \end{pmatrix}$$

□

The concept of a spherical 2-distance set (Definition 2.2.6), first introduced in [20], was used extensively in Section 2 of [12]. We can use a similar concept here, except we will have a 3-distance set instead. We shall set up our 2-regular 3-coloured structure in much the same way as they set up strongly regular graphs in Section 2 of [12], however with slight differences where appropriate.

Suppose we have a 2-regular, 3-coloured structure Γ, with vertex set X of cardinality n. As shown in Lemma 7.1.9 and Lemma 7.1.12, the non-zero intersections of the eigenspaces of A_R, A_G, A_B split into 4 different spaces. We shall identify X with an orthonormal basis for the real vector space $V := \mathbb{R}^n$, which is an orthogonal direct sum of these 4 classes, namely:

$$V = V_0 \oplus V_1 \oplus V_2 \oplus V_3,\ V_i = V\pi_i,\ i = 0, 1, 2, 3 \tag{8.1.1}$$

Define X_i to be the set of projections of the elements of X into each V_i, i.e.

$$X_i := \{x\pi_i :\ x \in X\}$$

Now we can see that the idempotent E_i is the Gram matrix (Definition 2.2.5) of the vectors of X_i,

$$E_i = \{\langle x\pi_i, y\pi_i\rangle :\ x, y \in X\}$$

As, by Equation 4.1.4, $E_i = \frac{\mu_i}{n}(I + \alpha_i A_R + \beta_i A_G + \gamma_i A_B)$, we see that the E_i form a spherical 3-distance set in V_i with distances α_i, β_i and γ_i. Hence we may write that

$$\begin{aligned} E_i =& H_i H_i^T \\ I =& H_i^T H_i \end{aligned}$$

where the rows of the H_i are the coordinates of the vectors of X with respect to any orthonormal basis $\mathcal{B}_i$ of V_i. We therefore see that the matrix

$$H := (H_0\ H_1\ H_2\ H_3) \tag{8.1.2}$$

where $n^{\frac{1}{2}} H_0 = u$, is the orthonormal transition matrix from the orthonormal basis X to the orthonormal basis $\mathcal{B} := \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$. For reference, these H_i are the ith characteristic matrices of each eigenspace [Definition 3.4, [20]], and the above form comes from Theorems 3.6 and 5.3 in [20].

We now want to define a specific orthonormal basis $\mathcal{B}$ to form our H_i. We do this so that for a given vertex $x \in X$, the vectors

$$x\pi_0 n^{\frac{1}{2}},\ x\pi_1(n/f)^{\frac{1}{2}},\ x\pi_2(n/g)^{\frac{1}{2}},\ x\pi_3(n/h)^{\frac{1}{2}}$$

belong to $\mathcal{B}_0, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ respectively. This gives us the transition matrix H from X to $\mathcal{B}$ as follows:

$$H = (H_0,\ H_1,\ H_2,\ H_3)$$
$$= \begin{pmatrix} (\frac{1}{n})^{\frac{1}{2}} & (\frac{f}{n})^{\frac{1}{2}} & 0^T & (\frac{g}{n})^{\frac{1}{2}} & 0^T & (\frac{h}{n})^{\frac{1}{2}} & 0^T \\ (\frac{1}{n})^{\frac{1}{2}}u_1 & (\frac{f}{n})^{\frac{1}{2}}\alpha_1 u_1 & (\frac{1}{D})^{\frac{1}{2}}K_1 & (\frac{g}{n})^{\frac{1}{2}}\alpha_2 u_1 & (\frac{1}{D})^{\frac{1}{2}}K_2 & (\frac{h}{n})^{\frac{1}{2}}\alpha_3 u_1 & (\frac{1}{D})^{\frac{1}{2}}K_3 \\ (\frac{1}{n})^{\frac{1}{2}}u_2 & (\frac{f}{n})^{\frac{1}{2}}\beta_1 u_2 & (\frac{1}{D})^{\frac{1}{2}}L_1 & (\frac{g}{n})^{\frac{1}{2}}\beta_2 u_2 & (\frac{1}{D})^{\frac{1}{2}}L_2 & (\frac{h}{n})^{\frac{1}{2}}\beta_3 u_2 & (\frac{1}{D})^{\frac{1}{2}}L_3 \\ (\frac{1}{n})^{\frac{1}{2}}u_3 & (\frac{f}{n})^{\frac{1}{2}}\gamma_1 u_3 & (\frac{1}{D})^{\frac{1}{2}}M_1 & (\frac{g}{n})^{\frac{1}{2}}\gamma_2 u_3 & (\frac{1}{D})^{\frac{1}{2}}M_2 & (\frac{h}{n})^{\frac{1}{2}}\gamma_3 u_3 & (\frac{1}{D})^{\frac{1}{2}}M_3 \end{pmatrix} \tag{8.1.3}$$

Here for each i, K_i, L_i, M_i have width $\mu_i - 1$, and for any i, K_i has height k_R, L_i has height k_G and M_i has height k_B, where width refers to the number of entries in a row, and height the number of entries in a column.

Lemma 8.1.4 *We can express the idempotent basis in terms of the eigenvalues as*

$$\begin{aligned} DE_1 =& (s_Rt_G - s_Gt_R)I + (s_G - t_G)A_R + (t_R - s_R)A_G + \\ &+ \frac{(s_R - t_R)k_G + (t_G - s_G)k_R + t_Rs_G - t_Gs_R}{n}J \\ DE_2 =& (r_Gt_R - r_Rt_G)I + (t_G - r_G)A_R + (r_R - t_R)A_G + \\ &+ \frac{(t_R - r_R)k_G + (r_G - t_G)k_R + r_Rt_G - r_Gt_R}{n}J \\ DE_3 =& (r_Rs_G - r_Gs_R)I + (r_G - s_G)A_R + (s_R - r_R)A_G + \\ &+ \frac{(r_R - s_R)k_G + (s_G - r_G)k_R + s_Rr_G - s_Gr_R}{n}J \end{aligned}$$

Proof. From the expansion of nE_1 into the basis $\{I, A_R, A_G, A_B\}$ that is given by 4.1.4, we get

$$nE_1 = f((1 - \gamma_1)I + (\alpha_1 - \gamma_1)A_R + (\beta_1 - \gamma_1)A_G + \gamma_1 J)$$

And so in terms of the eigenvalues we get

$$DnE_1 = (s_Rt_G - s_Gt_R)nI + (s_G - t_G)nA_R + (t_R - s_R)nA_G + (k_G(s_R - t_R) - k_R(s_G - t_G) + t_Rs_G - t_Gs_R)J$$

The formulas for the other nE_i follow in exactly the same manner. □

Using this we can get the results:

Lemma 8.1.5

$$
\begin{aligned}
K_1K_1^T =&(s_Rt_G - s_Gt_R)I_1 + (s_G - t_G)N_{RR}^R + (t_R - s_R)N_{RR}^G + \frac{Df}{n}(\gamma_1 - \alpha_1^2)J_1\\
L_1L_1^T =&(s_Rt_G - s_Gt_R)I_1 + (s_G - t_G)N_{GG}^R + (t_R - s_R)N_{GG}^G + \frac{Df}{n}(\gamma_1 - \beta_1^2)J_2\\
M_1M_1^T =&(s_Rt_G - s_Gt_R)I_1 + (s_G - t_G)N_{BB}^R + (t_R - s_R)N_{BB}^G + \frac{Df}{n}(\gamma_1 - \gamma_1^2)J_3\\
K_2K_2^T =&(t_Rr_G - t_Gr_R)I_1 + (t_G - r_G)N_{RR}^R + (r_R - t_R)N_{RR}^G + \frac{Dg}{n}(\gamma_2 - \alpha_2^2)J_1\\
L_2L_2^T =&(t_Rr_G - t_Gr_R)I_1 + (t_G - r_G)N_{GG}^R + (r_R - t_R)N_{GG}^G + \frac{Dg}{n}(\gamma_2 - \beta_2^2)J_2\\
M_2M_2^T =&(t_Rr_G - t_Gr_R)I_1 + (t_G - r_G)N_{BB}^R + (r_R - t_R)N_{BB}^G + \frac{Dg}{n}(\gamma_2 - \gamma_2^2)J_3\\
K_3K_3^T =&(r_Rs_G - r_Gs_R)I_1 + (r_G - s_G)N_{RR}^R + (s_R - r_R)N_{RR}^G + \frac{Dh}{n}(\gamma_3 - \alpha_3^2)J_1\\
L_3L_3^T =&(r_Rs_G - r_Gs_R)I_2 + (r_G - s_G)N_{GG}^R + (s_R - r_R)N_{GG}^G + \frac{Dh}{n}(\gamma_3 - \beta_3^2)J_2\\
M_3M_3^T =&(r_Rs_G - r_Gs_R)I_3 + (r_G - s_G)N_{BB}^R + (s_R - r_R)N_{BB}^G + \frac{Dh}{n}(\gamma_3 - \gamma_3^2)J_3
\end{aligned}
$$

Proof. Note that $E_i = H_iH_i^T$ gives us

$$
E_i = \begin{pmatrix}
\frac{\mu_i}{n} & \frac{\mu_i}{n}\alpha_iu_1^T & \frac{\mu_i}{n}\beta_iu_2^T & \frac{\mu_i}{n}\gamma_iu_3^T\\
\frac{\mu_i}{n}\alpha_iu_1 & \frac{\mu_i}{n}\alpha_i^2J + \frac{1}{D}K_iK_i^T & \frac{\mu_i}{n}\alpha_i\beta_iJ + \frac{1}{D}K_iL_i^T & \frac{\mu_i}{n}\alpha_i\gamma_iJ + \frac{1}{D}K_iM_i^T\\
\frac{\mu_i}{n}\beta_iu_2 & \frac{\mu_i}{n}\alpha_i\beta_iJ + \frac{1}{D}L_iK_i^T & \frac{\mu_i}{n}\beta_i^2J + \frac{1}{D}L_iL_i^T & \frac{\mu_i}{n}\beta_i\gamma_1J + \frac{1}{D}L_1M_1^T\\
\frac{\mu_i}{n}\gamma_iu_3 & \frac{\mu_i}{n}\alpha_i\gamma_iJ + \frac{1}{D}M_iK_i^T & \frac{\mu_i}{n}\alpha_i\gamma_iJ + \frac{1}{D}M_iL_i^T & \frac{\mu_i}{n}\gamma_i^2J + \frac{1}{D}M_iM_i^T
\end{pmatrix} \tag{8.1.4}
$$

Comparing this with the equations for E_i in Lemma 8.1.4 and the formulation of A_i from Equation 4.1.2, we get the equation

$$
\begin{aligned}
\frac{\mu_i}{n}\alpha_i^2J + \frac{1}{D}K_iK_i^T =&\frac{1}{D}((\lambda_{i+1})_R(\lambda_{i+2})_G - (\lambda_{i+1})_G(\lambda_{i+2})_R)I + ((\lambda_{i+1})_G - (\lambda_{i+2})_G)N_{RR}^R +\\
&+ ((\lambda_{i+2})_R - (\lambda_{i+1})_R)N_{RR}^G + \frac{D\mu_i\gamma_i}{n}J)
\end{aligned}
$$

So

$$\begin{aligned} K_iK_i^T =&((\lambda_{i+1})_R(\lambda_{i+2})_G - (\lambda_{i+1})_G(\lambda_{i+2})_R)I + ((\lambda_{i+1})_G - (\lambda_{i+2})_G)N_{RR}^R + \\ &+ ((\lambda_{i+2})_R - (\lambda_{i+1})_R)N_{RR}^G + \frac{D\mu_i(\gamma_i - \alpha_i^2)}{n}J) \end{aligned}$$

as required. The same reasoning works for the other equations. □

It is worth noting at this point that the matrices listed in Lemma 8.1.5 are real symmetric and so have real eigenvalues. We can also see that the Equation 8.1.4 can also give us information about other combinations of matrices:

Lemma 8.1.6

$$\begin{aligned} L_iK_i^T &= ((\lambda_{i+1})_G - (\lambda_{i+2})_G)N_{GR}^R + ((\lambda_{i+2})_R - (\lambda_{i+1})_R)N_{GR}^G + \frac{D\mu_i(\gamma_i - \alpha_i\beta_i)}{n}J \\ M_iK_i^T &= ((\lambda_{i+1})_G - (\lambda_{i+2})_G)N_{BR}^R + ((\lambda_{i+2})_R - (\lambda_{i+1})_R)N_{BR}^G + \frac{D\mu_i(\gamma_i - \alpha_i\gamma_i)}{n}J \\ M_iL_i^T &= ((\lambda_{i+1})_G - (\lambda_{i+2})_G)N_{BG}^R + ((\lambda_{i+2})_R - (\lambda_{i+1})_R)N_{BG}^G + \frac{D\mu_i(\gamma_i - \beta_i\gamma_i)}{n}J \end{aligned}$$

Proof. As before, if we compare the matrix 8.1.4 with the equations for E_i in Lemma 8.1.4 and the formulation of A_i from Equation 4.1.2 we get the equation

$$\begin{aligned} \frac{\mu_i}{n}\alpha_i\beta_iJ + \frac{1}{D}L_iK_i^T =&\frac{1}{D}((\lambda_{i+1})_R(\lambda_{i+2})_G - (\lambda_{i+1})_G(\lambda_{i+2})_R)I + ((\lambda_{i+1})_G - (\lambda_{i+2})_G)N_{GR}^R + \\ &+ ((\lambda_{i+2})_R - (\lambda_{i+1})_R)N_{GR}^G + \frac{D\mu_i\gamma_i}{n}J) \end{aligned}$$

So

$$\begin{aligned} L_iK_i^T =&((\lambda_{i+1})_R(\lambda_{i+2})_G - (\lambda_{i+1})_G(\lambda_{i+2})_R)I + ((\lambda_{i+1})_G - (\lambda_{i+2})_G)N_{GR}^R + \\ &+ ((\lambda_{i+2})_R - (\lambda_{i+1})_R)N_{GR}^G + \frac{D\mu_i(\gamma_i - \alpha_i\beta_i)}{n}J) \end{aligned}$$

as required. The same reasoning works for the other equations. □

8.2 Introducing 3-regularity

Recall that, assuming 3-regularity, N^R_{RR}, N^G_{RR} and N^B_{RR} have a shared basis of eigenvectors by Lemma 7.2.1. Hence from this point onward we assume 3-regularity and get the following.

Corollary 8.2.1 *The eigenvectors of $K_iK_i^T$ are exactly the eigenvectors of N^R_{RR}, N^G_{RR} and N^B_{RR}. Likewise for $L_iL_i^T$ and $M_iM_i^T$ but with N^c_{GG} and N^c_{BB} respectively.*

Proof. The fact the eigenvectors of $K_iK_i^T$ are the same as those for N^R_{RR} and N^G_{RR} is immediately evident from the equations, using $Jv = 0$ for any non-principal eigenvector v. □

Continuing this comparison of the two formulations for E_i and A_i we get the following identities

Lemma 8.2.2 *For any colour c,*

$$N^c_{RG} = \frac{1}{n}(k_c + r_c f\alpha_1\beta_1 + s_c g\alpha_2\beta_2 + t_c h\alpha_3\beta_3)J + \frac{1}{D}(r_cK_1L_1^T + s_cK_2L_2^T + t_cK_3L_3^T)$$
$$N^c_{RB} = \frac{1}{n}(k_c + r_c f\alpha_1\gamma_1 + s_c g\alpha_2\gamma_2 + t_c h\alpha_3\gamma_3)J + \frac{1}{D}(r_cK_1M_1^T + s_cK_2M_2^T + t_cK_3M_3^T)$$
$$N^c_{GB} = \frac{1}{n}(k_c + r_c f\beta_1\gamma_1 + s_c g\beta_2\gamma_2 + t_c h\beta_3\gamma_3)J + \frac{1}{D}(r_cL_1M_1^T + s_cL_2M_2^T + t_cL_3M_3^T)$$

Proof. First note that by definition of the E_i, we have

$$A_c = k_cE_0 + r_cE_1 + s_cE_2 + t_cE_3$$

Then using the matrix expansion of E_i from Equation 8.1.4 and comparing it to that of A_i in we get Equation 4.1.2, we get

$$N^c_{RG} = \frac{1}{n}(k_c + r_c f\alpha_1\beta_1 + s_c g\alpha_2\beta_2 + t_c h\alpha_3\beta_3)J + \frac{1}{D}(r_cK_1L_1^T + s_cK_2L_2^T + t_cK_3L_3^T)$$
$$N^c_{RB} = \frac{1}{n}(k_c + r_c f\alpha_1\gamma_1 + s_c g\alpha_2\gamma_2 + t_c h\alpha_3\gamma_3)J + \frac{1}{D}(r_cK_1M_1^T + s_cK_2M_2^T + t_cK_3M_3^T)$$
$$N^c_{GB} = \frac{1}{n}(k_c + r_c f\beta_1\gamma_1 + s_c g\beta_2\gamma_2 + t_c h\beta_3\gamma_3)J + \frac{1}{D}(r_cL_1M_1^T + s_cL_2M_2^T + t_cL_3M_3^T)$$

as required. □

Using these results about $K_iK_i^T$, $L_iL_i^T$ and $M_iM_i^T$, we can start to form some understanding of $K_i^TK_i$, $L_i^TL_i$ and $M_i^TM_i$ as well. We know from the definition that H is an orthogonal matrix, and so using this we can get the following results

Lemma 8.2.3 *For $i = 1, 2$ or 3,*

$$\begin{aligned} DI_{\mu_i - 1} &= K_i^TK_i + L_i^TL_i + M_i^TM_i \\ 1 &= \frac{\mu_i}{n}(1 + \alpha_i^2 + \beta_i^2 + \gamma_i^2) \\ 0 &= \alpha_iK_i^Tu_1 + \beta_iL_i^Tu_2 + \gamma_iM_i^Tu_3 \end{aligned}$$

Proof. We can read these results off from the equation

$$I_{\mu_i} = H_i^TH_i = \begin{pmatrix} \frac{\mu_i}{n}(1 + \alpha_i^2 + \beta_i^2 + \gamma_i^2) & (\frac{\mu_i}{nD})^{\frac{1}{2}}(\alpha_iu_1^TK_i + \beta_iu_2^TL_i + \gamma_iu_3^TM_i) \\ (\frac{\mu_i}{nD})^{\frac{1}{2}}(\alpha_iK_i^Tu_1 + \beta_iL_i^Tu_2 + \gamma_iM_i^Tu_3) & \frac{1}{D}(K_i^TK_i + L_i^TL_i + M_i^TM_i) \end{pmatrix}$$

□

Pushing this idea even further, using the fact that H is an orthogonal matrix, we know $H_i^TH_j = 0$ if $i \neq j$. This leads to

Lemma 8.2.4 *For all $i, j \in \{0, 1, 2, 3\}$ with $i \neq j$ we have*

$$\begin{aligned} 0 &= 1 + \alpha_i\alpha_j + \beta_i\beta_j + \gamma_i\gamma_j \\ 0 &= \alpha_iK_j^Tu_1 + \beta_iL_j^Tu_2 + \gamma_iM_j^Tu_3 \\ 0 &= K_i^TK_j + L_i^TL_j + M_i^TM_j \end{aligned}$$

Proof. As $0 = H_i^TH_j$ we get

$$0 = \begin{pmatrix} \frac{(\mu_i\mu_j)^{\frac{1}{2}}}{n}(1 + \alpha_i\alpha_j + \beta_i\beta_j + \gamma_i\gamma_j) & (\frac{\mu_i}{nD})^{\frac{1}{2}}(\alpha_iu_1^TK_j + \beta_iu_2^TL_j + \gamma_iu_3^TM_j) \\ (\frac{\mu_j}{nD})^{\frac{1}{2}}(\alpha_jK_i^Tu_1 + \beta_jL_i^Tu_2 + \gamma_jM_i^Tu_3) & \frac{1}{D}(K_i^TK_j + L_i^TL_j + M_i^TM_j) \end{pmatrix}$$

□

We now intend to compare the eigenvalues of the matrices $K_iK_i^T$, $L_iL_i^T$ and $M_iM_i^T$ with $K_i^TK_i$, $L_i^TL_i$ and $M_i^TM_i$. We get the following important result.

Lemma 8.2.5 *Any non-zero eigenvalue of $K_iK_i^T$ is an eigenvalue of $K_i^TK_i$ with the same multiplicity as well. The same results hold for L_i and M_i too.*

Proof. We claim that any non-zero eigenvalue of $K_iK_i^T$ is also an eigenvalue of $K_i^TK_i$ as if for some v, $K_iK_i^Tv = \lambda v$, then $K_i^TK_iK_i^Tv = \lambda K_i^Tv$. Now we know $K_i^Tv \neq 0$ as otherwise λ would be zero, so it is an eigenvector for $K_i^TK_i$. We see that λ will have the same multiplicity for both $K_iK_i^T$ and $K_i^TK_i$, as for each eigenvector v of $K_iK_i^T$, there exists a distinct eigenvector K_i^Tv for $K_i^TK_i$. The same argument works the other way around, and so all non-zero eigenvalues are shared, and this works similarly for L_i and M_i. □

Corollary 8.2.6 *If zero is not an eigenvalue of at least one of $K_2K_2^T$ or $L_1L_1^T$ then $k_R = k_G$. Likewise, if zero is not an eigenvalue of $K_3K_3^T$ or $M_1M_1^T$ then $k_R = k_B$ and if zero is not an eigenvalue of $L_3L_3^T$ or $M_2M_2^T$ then $k_G = k_B$.*

Proof. The total multiplicity of the eigenvalues of $K_2K_2^T$ is k_R, however the total multiplicity of the eigenvalues of $K_2^TK_2$ is k_G. If zero is not an eigenvalue then, by the previous lemma, the eigenvalues of these two matrices are the same and have the same multiplicity. Hence $k_R = k_G$. By comparing the total multiplicities of the other matrices in the same way we obtain the other results. □

Lemma 8.2.7 *Let v be an eigenvector of both N_{RR}^R and N_{RR}^G with eigenvalues r_{R_R} and r_{G_R} respectively, and suppose for distinct $m, j, l \in \{1, 2, 3\}$ we have both $K_jK_j^Tv = 0$ and $K_lK_l^Tv = 0$. Then*

$$r_{R_R},\ r_{G_R} = \begin{cases} r_R,\ r_G & m = 1 \\ s_R,\ s_G & m = 2 \\ t_R,\ t_G & m = 3 \end{cases}$$

Proof. We suppose that $m = 1$. However due to the symmetry between the matrices the same argument will work for the others. Now we have that $K_2 K_2^T v = 0$ and $K_3 K_3^T v = 0$, giving by Lemma 8.1.5

$$0 = (t_R r_G - t_G r_R) + (t_G - r_G) r_{R_R} + (r_R - t_R) r_{G_R}$$
$$0 = (r_R s_G - r_G s_R) + (r_G - s_G) r_{R_R} + (s_R - r_R) r_{G_R}$$

Suppose first that $t_G = r_G$. Then either $r_R = t_R$ or $r_{G_R} = r_G$. If both $t_G = r_G$ and $r_R = t_R$ then $D = 0$, a contradiction to Lemma 4.2.12, and so we must have $r_{G_R} = r_G$. Similarly if $t_G = r_G$ and $r_G = s_G$ then $D = 0$ too. Looking at the second equation we see we have two options, either $r_R = s_R$ or not. If we do have $r_R = s_R$, the equation becomes:

$$0 = r_R(s_G - r_G) + (r_G - s_G) r_{R_R}$$

which solves to give $r_{R_R} = r_R$ as required. And if $r_R \neq s_R$, we can use the fact that $r_{G_R} = r_G$ to make the equation:

$$0 = (r_R s_G - r_G s_R) + (r_G - s_G) r_{R_R} + (s_R - r_R) r_G$$
$$0 = r_R(s_G - r_G) + (r_G - s_G) r_{R_R}$$

which also solves to give $r_{R_R} = r_R$ as required.
Hence we now suppose $t_G \neq r_G$. Solving the first equation gives

$$r_{R_R} = \frac{t_R r_G - t_G r_R + (r_R - t_R) r_{G_R}}{(r_G - t_G)}$$

And so

$$0 = (r_R s_G - r_G s_R)(r_G - t_G) + (t_R r_G - t_G r_R)(r_G - s_G) +$$
$$+ r_{G_R}((s_R - r_R)(r_G - t_G) + (r_R - t_R)(r_G - s_G)$$
$$0 = -r_G D + r_{G_R} D$$

So $r_{G_R} = r_G$. Now putting this back into the first equation gives $r_{R_R} = r_R$. □

Lemma 8.2.8 *Suppose v is a non-prinicipal eigenvector of N^R_{RR} and N^G_{RR} such that $K_iK_i^Tv \neq 0$. Then if $N^R_{GR}v = N^G_{GR}v = 0$ then $L_i^TL_iK_i^Tv = 0$, hence K_i^Tv is an eigenvector for both $K_i^TK_i$ and $L_i^TL_i$.*

Proof. Firstly as $K_iK_i^Tv \neq 0$ then we know $K_i^Tv \neq 0$. Now consider $L_i^TL_iK_i^Tv$. By Lemma 8.1.6, we know

$$L_iK_i^T = ((\lambda_{i+1})_G - (\lambda_{i+2})_G)N^R_{GR} + ((\lambda_{i+2})_R - (\lambda_{i+1})_R)N^G_{GR} + \frac{D\mu_i(\gamma_i - \alpha_i\beta_i)}{n}J$$

And therefore $L_iK_i^Tv = 0$ as $N^R_{GR}v = N^G_{GR}v = Jv = 0$. □

Corollary 8.2.9 *Suppose v is an eigenvector of N^R_{RR} and N^G_{RR} such that $K_iK_i^Tv \neq 0$. Then if $N^R_{GR}v = N^G_{GR}v = 0$, K_i^Tv is an eigenvector of $M_i^TM_i$.*

Proof. Combining Lemma 8.2.3 and Lemma 8.2.8 gives us that

$$DK_i^Tv = K_i^TK_iK_i^Tv + M_i^TM_iK_i^Tv$$

And so K_i^Tv is an eigenvector for $M_i^TM_i$. □

Suppose v is a non-principal eigenvector of N^R_{RR} and N^G_{RR} with eigenvalues r_{R_R} and r_{G_R} respectively. Now define the constants $x_j(r), y_j(r), z_j(r)$ as

$$\begin{aligned} x_j(r) &= s_Rt_G - s_Gt_R + (s_G - t_G)r_{R_j} + (t_R - s_R)r_{G_j} \\ y_j(r) &= t_Rr_G - t_Gr_R + (t_G - r_G)r_{R_j} + (r_R - t_R)r_{G_j} \\ z_j(r) &= r_Rs_G - r_Gs_R + (r_G - s_G)r_{R_j} + (s_R - r_R)r_{G_j} \end{aligned} \tag{8.2.5}$$

$x_j(s), y_j(s), z_j(s)$ and $x_j(t), y_j(t), z_j(t)$ are defined similarly for s_{R_j}/s_{G_j} and t_{R_j}/t_{G_j}. We shall see that a consequence of Lemma 8.1.5 and Corollary 8.2.1 is that these are the eigenvalues of $K_iK_i^T, L_iL_i^T$ and $M_iM_i^T$. Let φ_i be such that it is x if $i = 1$, y if $i = 2$ and z if $i = 3$. Then

Lemma 8.2.10 *For $i \in \{1, 2, 3\}$:*

- *The non-principal eigenvalues of* $K_iK_i^T$ *are* $\varphi_{iR}(r), \varphi_{iR}(s)$ *and* $\varphi_{iR}(t)$
- *The non-principal eigenvalues of* $L_iL_i^T$ *are* $\varphi_{iG}(r), \varphi_{iG}(s)$ *and* $\varphi_{iG}(t)$
- *The non-principal eigenvalues of* $M_iM_i^T$ *are* $\varphi_{iB}(r), \varphi_{iB}(s)$ *and* $\varphi_{iB}(t)$

Proof. We shall prove it just for $K_iK_i^T$, however the same method works for the others. We know by Corollary 8.2.1 that a non-principal eigenvector for these is the same as a non-principal eigenvector for the R-neighbourhood. Hence we take the value of $K_iK_i^T$ from Lemma 8.1.5, and then apply a non-principal eigenvector v. This will give us either $\varphi_{iR}(r), \varphi_{iR}(s)$ and $\varphi_{iR}(t)$ depending on the eigenvalue triple to which v belonged. □

Remark 8.2.11 *Note that, for any* j*,*

$$\begin{aligned} x_j(r) + y_j(r) + z_j(r) &= s_Rt_G - s_Gt_R + t_Rr_G - t_Gr_R + r_Rs_G - r_Gs_R \\ &= D \end{aligned}$$

This is true for r, s *or* t.

It is prudent to note that, although the following analysis works with just the R-neighbourhood, the matrices N_{GR}^c and K_i^T, this would all equally work for any other neighbourhood and the corresponding matrices. For example we could have used the G-neighbourhood, the matrices N_{GB}^c and M_i^T. Or the G-neighbourhood, the matrices N_{RG}^c and K_i.

Lemma 8.2.12 *Suppose* v *is a non-principal eigenvector of* N_{RR}^R *and* N_{RR}^G *such that* $K_iK_i^Tv \neq 0$ *for all* i*. Then we can't have* $N_{GR}^Rv = N_{GR}^Gv = 0$.

Proof. Assume for a contradiction that $N_{GR}^Rv = N_{GR}^Gv = 0$. Then we know that $L_iK_i^Tv = 0$ by Lemma 8.1.6. Suppose r_{R_R} is the eigenvalue of N_{RR}^R attached to v. Then by Corollary 8.2.9 we know that K_i^Tv is an eigenvector for $M_i^TM_i$. The corresponding eigenvalue for K_i^Tv is then either 0 or it is an eigenvalue of $M_iM_i^T$ by Lemma 8.2.5. If it is 0 then we get from Lemma 8.2.3 that

$$
\begin{aligned}
DK_1^T v &= K_1^T K_1 K_1^T v + L_1^T L_1 K_1^T v + M_1^T M_1 K_1^T v \\
&= K_1^T K_1 K_1^T v \\
&= x_R(r) K_1^T v
\end{aligned}
$$

so $D = x_R(r)$. But then we can do the same with K_2 and K_3 to get $D = y_R(r) = z_R(r)$. But $D = x_R(r) + y_R(r) + z_R(r) = 3D$, implying $D = 0$ a contradiction.
Therefore $M_i^T M_i K_i^T v$ is non-zero and an eigenvector of $M_i^T M_i$, K_i^T must have eigenvalues $x_B(r), y_B(r)$ or $z_B(r)$ for $i = 1, 2$ or 3 respectively, by Lemma 8.2.5 and Lemma 8.2.10. Using Lemma 8.2.3 again we have

$$
\begin{aligned}
DK_1^T v &= K_1^T K_1 K_1^T v + L_1^T L_1 K_1^T v + M_1^T M_1 K_1^T v \\
&= K_1^T K_1 K_1^T v + M_1^T M_1 K_1^T v \\
&= (x_R(r) + x_B(r)) K_1^T v
\end{aligned}
$$

So $D = x_R(r) + x_B(r)$. Once again we can do the same for the other i giving

$$D = x_R(r) + x_B(r) = y_R(r) + y_B(r) = z_R(r) + z_B(r)$$

But $2D = x_R(r) + x_B(r) + y_R(r) + y_B(r) + z_R(r) + z_B(r) = 3D$, implying $D = 0$, a contradiction. □

From this lemma we can gather that if $N_{GR}^R v = N_{GR}^G v = 0$ then we must have that $K_i K_i^T v = 0$ for some i. In fact the proofs actually go a bit further.

Corollary 8.2.13 *Suppose v is a non-principal eigenvector of the R-neighbourhood such that $N_{GR}^R v = N_{GR}^G v = 0$. Then, for some i, $K_i^T v = 0$.*

This just comes from considering that $v^T K_i K_i^T v = |K_i^T v| = 0$ if and only if $K_i^T v = 0$. Further we get that, for no more than two i, $K_i^T v$ equals 0.

Lemma 8.2.14 *Suppose v is a non-principal eigenvector of the R-neighbourhood. Then* *we cannot have* $K_i^T v = 0$ *for all* i.

Proof. Suppose for a contradiction that $K_1^T v = K_2^T v = K_3^T v = 0$ and say without loss of generality, that v belongs to the eigenspace of the eigenvalue triple $(r_{R_R}, r_{G_R}, r_{B_R})$. Then $K_i K_i^T v = 0$ for all i. But this implies that $x_R(r) = y_R(r) = z_R(r) = 0$, which implies $D = 0$, a contradiction. $\square$

So the only cases to consider are when we have a unique i such that $K_i^T v = 0$ or when we have two distinct i, j such that $K_i^T v = K_j^T v = 0$.

Lemma 8.2.15 *Suppose that* $v \in E(r_{R_R}, r_{G_R}, r_{B_R})$ *is a non-principal eigenvector of the* R*-neighbourhood in the 0 case in G. Then either:*

- *There exists a unique* i *such that* $K_i^T v = 0$ *and* $\varphi_{i_R}(r) = 0$ *and* $\varphi_{i_B}(r) = 0$,
- $K_i^T v = K_j^T v = 0$ *for exactly 2 distinct* i, j *and* $\varphi_{i_R}(r) = 0$, $\varphi_{j_R}(r) = 0$ *and* $\varphi_{l_B}(r) = 0$ *for* l *also distinct from both* i, j.

Proof. We can see that either exactly one or two of the $K_i^T v = 0$ by Corollary 8.2.13 and Lemma 8.2.14. So all that remains to show is the consequences of either option.
Suppose first that there exists a unique i such that $K_i^T v = 0$. We see from Lemma 8.1.5 and multiplying by v on the left and right, that $\varphi_{i_R}(r) = 0$. Now we can use Lemma 8.2.8 and Corollary 8.2.9 for j and l to get that $D = \varphi_{j_R}(r) + \varphi_{j_B}(r) = \varphi_{l_R}(r) + \varphi_{l_B}(r)$. We know also by Remark 8.2.11 that $D = \varphi_{i_R}(r) + \varphi_{j_R}(r) + \varphi_{l_R}(r) = \varphi_{j_R}(r) + \varphi_{l_R}(r)$ and $D = \varphi_{i_B}(r) + \varphi_{j_B}(r) + \varphi_{l_B}(r)$. So

$$\begin{aligned} 2D &= \varphi_{j_R}(r) + \varphi_{j_B}(r) + \varphi_{l_R}(r) + \varphi_{l_B}(r) \\ &= \varphi_{j_R}(r) + \varphi_{l_R}(r) + \varphi_{i_B}(r) + \varphi_{j_B}(r) + \varphi_{l_B}(r) \end{aligned}$$

implying $\varphi_{i_B}(r) = 0$.
Suppose instead we have $K_i^T v = K_j^T v = 0$. Then again, from Lemma 8.1.5 and multiplying by v on the left and right, it follows that $\varphi_{i_R}(r) = \varphi_{j_R}(r) = 0$, so by Remark 8.2.11 we see $D = \varphi_{l_R}(r)$. Using Lemma 8.2.8 and Corollary 8.2.9 we note $D = \varphi_{l_R}(r) + \varphi_{l_B}(r)$. This implies $\varphi_{l_B}(r) = 0$. $\square$

With the second scenario we can go even further.

Lemma 8.2.16 *Suppose $v \in E(r_{R_R}, r_{G_R}, r_{B_R})$ is an eigenvector of the R-neighbourhood in the 0 case in G and that, for distinct i, j, $K_i^T v = K_j^T v = 0$. Then v is in the 0 case in B.*

Proof. As $K_i^T v = K_j^T v = 0$ we know by Lemma 8.2.14 that $K_l^T v \neq 0$ for l distinct from i and j, and by Lemma 8.2.8 and Corollary 8.2.9 it follows that $K_l^T v$ must be an eigenvector of $M_l^T M_l$. By applying $K_l^T v$ to the first equation of Lemma 8.2.3

$$\begin{aligned} DK_l^T v &= K_l^T K_l K_l^T v + M_l^T M_l K_l^T v \\ M_l^T M_l K_l^T v &= (D - \varphi_{lR}(r)) K_l^T v \end{aligned}$$

However by Lemma 8.2.15 and Remark 8.2.11, we know that $D = \varphi_{lR}(r)$, so $M_l^T M_l K_l^T v = 0$. But if $M_l^T M_l K_l^T v = 0$ then $M_l K_l^T v = 0$, and so by applying v to the formulation of N_{BR}^c from Lemma 8.2.2 we get

$$\begin{aligned} N_{BR}^c v &= \frac{1}{D}(r_c M_1 K_1^T v + s_c M_2 K_2^T v + t_c M_3 K_3^T v) \\ &= \frac{(\lambda_l)_c}{D} M_l K_l^T v \\ &= 0 \end{aligned}$$

□

We can actually prove this the other way around as well. However we are going to want some more useful lemmas first that describe the consequences of $K_i^T v$ being 0 in other scenarios. Firstly:

Lemma 8.2.17 *Suppose for some i and some eigenvector $v \in E(r_{R_R}, r_{G_R}, r_{B_R})$ of the R-neighbourhood, we have $K_i^T v = 0$. Then either v is in the 0 case in G or $aN_{GR}^R v = bN_{GR}^G v$ for some a and b, not both 0.*

Proof. Suppose for a contradiction that neither of the outcomes occur. This means that both $N_{GR}^R v$ and $N_{GR}^G v$ are non-zero. Since $K_i^T v = 0$, we get that $L_i K_i^T v = 0$. By Lemma 8.1.6, this means

$$0 = ((\lambda_{i+1})_G - (\lambda_{i+2})_G)N^R_{GR}v + (\lambda_{i+2})_R - (\lambda_{i+1})_R)N^G_{GR}v$$

But as the vectors $N^R_{GR}v$ and $N^G_{GR}v$ are not multiples of each other or zero, we must have that the coefficients are zero. Recall from Definition 8.1.1, that this will imply that $D = 0$, which can't happen by Lemma 4.2.12. Hence either the vectors are multiples or zero. □

In the process of this proof we could also note something about the case when we have $aN^R_{GR}v = bN^G_{GR}v$. This is a variation of Lemma 7.3.3 that we saw earlier, coming from a different angle.

Lemma 8.2.18 *Suppose for some i and some eigenvector v of the R-neighbourhood, we have $K_i^T v = 0$. Then $aN^R_{GR}v = bN^G_{GR}v$ for some a and b and one of:*

i) $N^R_{GR}v = N^G_{GR}v = 0$

ii) $a = 0$, $N^G_{GR}v = 0$ *and* $(\lambda_{i+1})_G = (\lambda_{i+2})_G$,

iii) $b = 0$, $N^R_{GR}v = 0$ *and* $(\lambda_{i+1})_R = (\lambda_{i+2})_R$,

iv) $\frac{a}{b} = \frac{(\lambda_{i+1})_G - (\lambda_{i+2})_G}{(\lambda_{i+1})_R - (\lambda_{i+2})_R}$,

Proof. Suppose for a contradiction that there do not exist a and b such that $aN^R_{GR}v = bN^G_{GR}v$. It follows that both $N^R_{GR}v$ and $N^G_{GR}v$ are non-zero and not multiples of each other. Since $K_i^T v = 0$ we get that $L_i K_i^T v = 0$. By Lemma 8.1.6, this means we have

$$0 = ((\lambda_{i+1})_G - (\lambda_{i+2})_G)N^R_{GR}v + ((\lambda_{i+2})_R - (\lambda_{i+1})_R)N^G_{GR}v \tag{8.2.6}$$

But as the vectors $N^R_{GR}v$ and $N^G_{GR}v$ are not multiples of each other or zero, we must have that the coefficients are zero. However as before, this will imply that $D = 0$. Hence we must have $aN^R_{GR}v = bN^G_{GR}v$ for some a and b.
Suppose that $N^G_{GR}v = 0$ but $N^R_{GR}v \neq 0$. Then $a = 0$ and from equation 8.2.6 $(\lambda_{i+2})_G = (\lambda_{i+1})_G$ giving situation ii). Similarly if $N^R_{GR}v = 0$ but $N^G_{GR}v \neq 0$, then $b = 0$ and from equation 8.2.6 $(\lambda_{i+2})_R = (\lambda_{i+1})_R$ giving situation iii).

Finally if both $N_{GR}^{R}v \neq 0$ and $N_{GR}^{G}v \neq 0$, then from equation 8.2.6 we get

$$\frac{a}{b} = \frac{(\lambda_{i+1})_G - (\lambda_{i+2})_G}{(\lambda_{i+1})_R - (\lambda_{i+2})_R}$$

as required. Note that as both a and b must be non-zero, if either coefficient is zero, then they both are, implying $D = 0$ as before. Hence neither coefficient is zero and this is a well-defined fraction. □

Now if we add also that $K_j^T v = 0$ for some j distinct from i we see that also

Corollary 8.2.19 *Suppose that for some eigenvector v of the R-neighbourhood, $K_i^T v = K_j^T v = 0$ for some distinct i and j. Then either $N_{GR}^{R}v = 0$ or $N_{GR}^{G}v = 0$ or both.*

Proof. Suppose for a contradiction that both $N_{GR}^{R}v \neq 0$ and $N_{GR}^{G}v \neq 0$. As $K_i^T v = 0$ from Lemma 8.2.18 we get $aN_{GR}^{R}v = bN_{GR}^{G}v$ where a and b are such that

$$\frac{a}{b} = \frac{(\lambda_{i+1})_G - (\lambda_{i+2})_G}{(\lambda_{i+1})_R - (\lambda_{i+2})_R}$$

Now as $K_j^T v = 0$ also we get

$$\frac{a}{b} = \frac{(\lambda_{j+1})_G - (\lambda_{j+2})_G}{(\lambda_{j+1})_R - (\lambda_{j+2})_R}$$

Now this is either

$$\frac{a}{b} = \frac{(\lambda_{i+2})_G - (\lambda_{i})_G}{(\lambda_{i+2})_R - (\lambda_{i})_R} \quad \text{or} \quad \frac{a}{b} = \frac{(\lambda_{i})_G - (\lambda_{i+1})_G}{(\lambda_{i})_R - (\lambda_{i+1})_R}$$

Well in the first case, we get that

$$\begin{aligned} 0 &= ((\lambda_{i+2})_R - (\lambda_i)_R)((\lambda_{i+1})_G - (\lambda_{i+2})_G) - ((\lambda_{i+2})_G - (\lambda_i)_G)((\lambda_{i+1})_R - (\lambda_{i+2})_R) \\ &= D \end{aligned}$$

and in the second we also get

$$0 = ((\lambda_i)_R - (\lambda_{i+1})_R)((\lambda_{i+1})_G - (\lambda_{i+2})_G) - ((\lambda_i)_G - (\lambda_{i+1})_G)((\lambda_{i+1})_R - (\lambda_{i+2})_R)$$
$$= D$$

Hence either way we get $D = 0$ a contradiction. □

In fact we can even make the following observation as well.

Corollary 8.2.20 *Suppose that for some eigenvector $v \in E(r_{R_R}, r_{G_R}, r_{B_R})$ of the R-neighbourhood, $K_i^T v = K_j^T v = 0$ for some distinct i and j. Then $N_{GR}^R v = N_{GR}^G v = 0$.*

Proof. By Lemma 8.2.18 and Corollary 8.2.19 we know that if $K_i^T v = 0$ and we don't have $N_{GR}^R v = N_{GR}^G v = 0$, then we must have either $a = 0$, $N_{GR}^G v = 0$ and $(\lambda_{i+1})_G = (\lambda_{i+2})_G$ or $b = 0$, $N_{GR}^R v = 0$ and $(\lambda_{i+1})_R = (\lambda_{i+2})_R$. Further as $K_j^T v = 0$ we must have either $a = 0$, $N_{GR}^G v = 0$ and $(\lambda_{j+1})_G = (\lambda_{j+2})_G$ or $b = 0$, $N_{GR}^R v = 0$ and $(\lambda_{j+1})_R = (\lambda_{j+2})_R$. Suppose for a contradiction that we do not have $N_{GR}^R v = N_{GR}^G v = 0$. Then we must have either $(\lambda_{i+1})_G = (\lambda_{i+2})_G$ and $(\lambda_{j+1})_G = (\lambda_{j+2})_G$ or $(\lambda_{i+1})_R = (\lambda_{i+2})_R$ and $(\lambda_{j+1})_R = (\lambda_{j+2})_R$. However the first would imply $r_G = s_G = t_G$ and the latter would imply $r_R = s_R = t_R$, so either way we get a complete graph in some colour by Lemma 2.1.17, a contradiction. □

The most useful consequence of Lemma 8.2.18 however is the fact that we can fully determine the ratio between the two vectors.

Corollary 8.2.21 *Suppose that for some eigenvector v of the R-neighbourhood, we have $N_{GR}^R v = N_{GR}^G v = 0$. Then $aN_{BR}^R v = bN_{BR}^G v$, for some a and b and one of*

i) $N_{BR}^R v = N_{BR}^G v = 0$

ii) $a = 0$, $N_{BR}^G v = 0$ *and* $(\lambda_{i+1})_G = (\lambda_{i+2})_G$,

iii) $b = 0$, $N_{BR}^R v = 0$ *and* $(\lambda_{i+1})_R = (\lambda_{i+2})_R$,

iv) $\frac{a}{b} = \frac{(\lambda_{i+1})_G - (\lambda_{i+2})_G}{(\lambda_{i+1})_R - (\lambda_{i+2})_R}$, *for some* i.

The proof is just a simple application of Lemma 8.2.12 and then Lemma 8.2.18. However it has very important consequences. For example, if v is an eigenvector as described in Corollary 8.2.21, and we don't have case i), then we can determine what the eigenvalues of the R-neighbourhood are. The case iv) problem is the trickiest but can be found by the following.

Theorem 8.2.22 *Suppose that for some eigenvector* v *of the* R*-neighbourhood,* v *is in the 0 case in G and* $aN^R_{BR}v = bN^G_{BR}v$ *for some non-zero* a *and* b*. Then*

$$r_{c_R} = p^{RBR}_{RcR} - p^{RBB}_{RcR} + \frac{(\lambda_{i+1})_G - (\lambda_{i+2})_G}{(\lambda_{i+1})_R - (\lambda_{i+2})_R}(p^{RBG}_{RcR} - p^{RBB}_{RcR})$$
$$r_{c_B} = p^{BRR}_{BcR} - p^{BRB}_{BcR} + \frac{(\lambda_{i+1})_G - (\lambda_{i+2})_G}{(\lambda_{i+1})_R - (\lambda_{i+2})_R}(p^{BRG}_{BcR} - p^{BRB}_{BcR})$$

for some i.

Proof. First we know that if $aN^R_{BR}v = bN^G_{BR}v$ then $N^R_{BR}v$ is an eigenvector of N^R_{BB} by Lemma 7.3.1. We can calculate the eigenvalue attached to this eigenvector by examining basic equations and using Lemma 4.2.13

$$\begin{aligned}r_{c_B}N^R_{BR}v &= N^c_{BB}N^R_{BR}v \\ &= (p^{BRR}_{BcR} - p^{BRB}_{BcR})N^R_{BR}v + (p^{BRG}_{BcR} - p^{BRB}_{BcR})N^G_{BR}v \\ &= (p^{BRR}_{BcR} - p^{BRB}_{BcR} + \frac{a}{b}(p^{BRG}_{BcR} - p^{BRB}_{BcR}))N^R_{BR}v\end{aligned}$$

But also then we know that this holds in reverse as well, i.e. $av^TN^R_{RB}N^R_{BR}v = bv^TN^G_{RB}N^R_{BR}v$. And so by Lemma 4.2.13

$$\begin{aligned}r_{c_R}v^TN^R_{RB}N^R_{BR}v &= v^TN^c_{RR}N^R_{RB}N^R_{BR}v \\ &= (p^{RBR}_{RcR} - p^{RBB}_{RcR})v^TN^R_{RB}N^R_{BR}v + (p^{RBG}_{RcR} - p^{RBB}_{RcR})v^TN^G_{RB}N^R_{BR}v \\ &= (p^{RBR}_{RcR} - p^{RBB}_{RcR} + \frac{a}{b}(p^{RBG}_{RcR} - p^{RBB}_{RcR}))v^TN^R_{RB}N^R_{BR}v\end{aligned}$$

Now from Lemma 8.2.21 we know that, as a and b are non-zero, we have

$$\frac{a}{b} = \frac{(\lambda_{i+1})_G - (\lambda_{i+2})_G}{(\lambda_{i+1})_R - (\lambda_{i+2})_R}$$

for some fixed i. And so inputting this into the equations, we are done. □

In case ii) and iii) the situation is slightly different but we can still find the eigenvalues.

Lemma 8.2.23 *Suppose that for some eigenvector v of the R-neighbourhood, v is in the 0 case in G. Then if $N_{BR}^G v = 0$ and $N_{BR}^R v \neq 0$, for some i*

$$r_{G_B} = (\lambda_{i+1})_G = (\lambda_{i+2})_G$$

Proof. We know from Lemma 8.2.15 that either $K_i^T v = 0$ for exactly one i, or $K_i^T = K_j^T = 0$ for exactly two distinct i and j. Now by Lemma 8.1.6, $M_i K_i^T v = 0$ will imply that

$$0 = ((\lambda_{i+1})_G - (\lambda_{i+2})_G) N_{BR}^R v + ((\lambda_{i+2})_R - (\lambda_{i+1})_R) N_{BR}^G v$$

meaning $(\lambda_{i+1})_G = (\lambda_{i+2})_G$. We must not have $K_j^T v = 0$ as well, as this would imply $(\lambda_{j+1})_G = (\lambda_{j+2})_G$ too, which would mean that $r_G = s_G = t_G$ and hence the graph was complete in G by Lemma 2.1.17.
Hence we have just $K_i^T v = 0$, but then by Lemma 8.2.15 it follows that $\varphi_{iB}(r) = 0$ too. But this becomes

$$0 = ((\lambda_{i+1})_R(\lambda_{i+2})_G - (\lambda_{i+1})_G(\lambda_{i+2})_R + ((\lambda_{i+1})_G - (\lambda_{i+2})_G))r_{R_B} + ((\lambda_{i+2})_R - (\lambda_{i+1})_R))r_{G_B}$$

$$0 = (\lambda_{i+1})_G((\lambda_{i+1})_R - (\lambda_{i+2})_R) + r_{G_B}((\lambda_{i+2})_G - (\lambda_{i+1})_R)$$

$$0 = ((\lambda_{i+1})_G - r_{G_B})((\lambda_{i+1})_R - (\lambda_{i+2})_R)$$

We note that $(\lambda_{i+1})_R \neq (\lambda_{i+2})_R$, as otherwise D would be 0. Hence $r_{G_B} = (\lambda_{i+1})_G$. □

We note that the exact same result holds but with $N_{GR}^R v = 0$ and $r_{R_B} = (\lambda_{i+1})_R = (\lambda_{i+2})_R$.

Chapter 9

The Complete Neighbourhood case

In this chapter we consider finite primitive 3-regular 3-coloured structures with the extra condition that one neighbourhood of a base point is complete in one of the colours. We obtain a classification (Theorem 9.3.3, showing that the only such example is the Tricolour Heptagon (Definition 5.2.1). I examine the interaction between the complete neighbourhood condition and each of the 0 case, the Eigenvector case and the Independent case from Definition 7.2.10.

In this chapter m, j and l are distinct colours, and every structure is assumed to be finite, primitive, 3-regular and 3-coloured, unless stated. We will choose one neighbourhood, the m-neighbourhood, to be complete, and consider an eigenvector v from its non-principal eigenspace. For most of the arguments in this section, we focus just on two neighbourhoods and the relationship between them, although sometimes we shall have to involve the third neighbourhood.

An important point to note is that, in order to avoid imprimitivity, we can't have the m-neighbourhood being complete in colour m. This is because if it were, then the entire structure would possess an equivalence relation in m. Hence we shall always assume the m-neighbourhood is complete in either j or l, implying $p^m_{mm} = 0$ for the whole of this chapter.

In the following analysis, I keep my colours general. So instead of assigning m to be R without loss of generality, I keep it as just m. This is so that, when we get to the more complicated equalities and identities, it is more easy to spot which positions within the intersection numbers are impacted by each other.

9.1 The 0 case

Lemma 9.1.1 *Suppose, for $x \neq m$, $0 = p^m_{mm} = p^m_{mx}$ and v is a non-principal eigenvector of the m-neighbourhood in the 0 case in j. Then $x = j$.*

Proof. Suppose for a contradiction that $x = l$. Now as $N^c_{jm}v_m = 0$ for all colours c, and therefore $0 = N^c_{mj}N^c_{jm}v$, we note by Lemma 4.2.13

$$0 = p^m_{cj} + p^{mmR}_{jcc}r_{R_m} + p^{mmG}_{jcc}r_{G_m} + p^{mmB}_{jcc}r_{B_m}$$

But, as $p^m_{mm} = p^m_{ml} = 0$, this can be reduced to just

$$0 = p^m_{cj} + p^{mmj}_{jcc}r_{j_m}$$

Now as the m-neighbourhood is complete in j,by Lemma 2.1.17, $r_{j_m} = -1$, giving us the equations

$$p^m_{mj} = p^{mmj}_{jmm},\ p^m_{jj} = p^{mmj}_{jjj},\ p^m_{lj} = p^{mmj}_{jll}$$

According to Lemma 4.2.3 we get

$$p^m_{mj} = p^{mmj}_{jmm} + p^{mmj}_{jml} + p^{mmj}_{jmj}$$
$$p^m_{jj} = p^{mmj}_{jjj} + p^{mmj}_{jjl} + p^{mmj}_{jjm}$$

This gives us the new information that $0 = p^{mmj}_{jml} = p^{mmj}_{jmj} = p^{mmj}_{jjl}$. Now we see that by Lemma 4.2.3 and the fact that $p^m_{ml} = p^m_{mm} = 0$

$$p^j_{ml} = p^{mmj}_{jml} + p^{mmj}_{lml} + p^{mmj}_{mml} = 0$$

and also

$$p^j_{mj} = p^{mmj}_{jmj} + p^{mmj}_{lmj} + p^{mmj}_{mmj} = 0$$

Now as $p^m_{lj} = p^m_{jj} = 0$ by Lemma 4.2.5, we get a contradiction to Lemma 7.1.15. Hence $x \neq l$ and therefore x must equal j. $\square$

Lemma 9.1.2 *Suppose $0 = p^m_{mm} = p^m_{mj}$ and we have some non-principal eigenvector, v of the m-neighbourhood, such that v is in the 0 case in j. Then $p^m_{jj} = 0$.*

Proof. Similarly to the previous lemma, by using $N^c_{jm}v = 0$ for all c and Lemma 4.2.3, we get equations

$$0 = p^{mml}_{jjm} = p^{mml}_{jjl} = p^{mml}_{jlm}$$

Crucially we see that $p^{mml}_{jjl} = 0$ implying $p^{mjj}_{mll} = 0$. Now as $p^{mjj}_{mll} + p^{mjj}_{mml} + p^{mjj}_{mjl} = 0$, either $p^j_{ml} = 0$ or $p^m_{jj} = 0$. By Lemma 7.1.15, we must have $p^m_{jj} = 0$. $\square$

We can now present the classification

Theorem 9.1.3 *Suppose in a primitive 3-regular 3-coloured graph, that for some colour m, the m-neighbourhood is complete. Then for any eigenvector v of the m-neighbourhood, v cannot be in the 0 case in j.*

Proof. Suppose for a contradiction that there exists an eigenvector v of the m-neighbourhood such that v is in the 0 case in j. Then by Lemma 9.1.1 we know that $p^m_{mm} = p^m_{mj} = 0$ and so the m-neighbourhood is complete in l. We also know by Lemma 9.1.2 that $p^m_{jj} = 0$. Hence by Theorem 7.6.20, we have a contradiction. $\square$

9.2 The Independent Case

In this section we will be dealing with the case where we have a complete neighbourhood and this neighbourhood has its only non-principal eigenvalue triple being in the Independent case in another neighbourhood. For the set up suppose the m-neighbourhood is complete and we have the sole eigenvalue triple $(r_{m_m}, r_{j_m}, r_{l_m})$ being in the Independent case in j. Then we definitely have $p^m_{mm} = 0$ and also, either $p^m_{mj} = 0$ or $p^m_{ml} = 0$. However by Lemma 7.5.1, we know that $p^m_{mj} \neq 0$, and hence $p^m_{ml} = 0$. We can also note that $(r_{m_m}, r_{j_m}, r_{l_m}) = (0, -1, 0)$.

By Lemma 7.5.3 we know that the eigenvalue triples of the j-neighbourhood are all either the 0 case or the Eigenvector case in m. However they actually cannot be the Eigenvector case, as there is no eigenvalue triple of the m-neighbourhood for them to correspond with, hence they must all be the 0 case.

Well now we consider how $(r_{m_m}, r_{j_m}, r_{l_m})$ interacts with the l-neighbourhood. By Lemma 7.5.1, we see it can't be in the Independent case in l, and by Theorem 9.1.3 it can't be in the 0 case in l. Hence it must be in the Eigenvector case. From this however we can also force the l neighbourhood to be complete.

At this point, we wish to introduce some notation that allows us to more easily generalise the results of Chapter 8. Recall the K_i, L_i, M_i defined via Equation 8.1.3. We define $(\Pi_m)_i$ as follows:

$$(\Pi_m)_i = \begin{cases} K_i & \text{if } m = R, \\ L_i & \text{if } m = G, \\ M_i & \text{if } m = B \end{cases} \tag{9.2.1}$$

Lemma 9.2.1 *Suppose $p^m_{mm} = p^m_{ml} = 0$, $v \in E(r_{m_m}, r_{j_m}, r_{l_m})$ is an eigenvector of the m-neighbourhood and is in the independent case in j. Then $k_m = k_l$ and the l-neighbourhood must also be complete.*

Proof. Suppose both $k_m \neq k_j$ and $k_m \neq k_l$. Then by Corollary 8.2.6 we must have that two of $(\Pi_m)_1(\Pi_m)_1^T$, $(\Pi_m)_2(\Pi_m)_2^T v$ or $(\Pi_m)_3(\Pi_m)_3^T v$ have an eigenvalue which is 0 (with which two depending on the value of m). As there is only one non-principal eigenvalue triple of the m neighbourhood, every non-principal eigenvector of the m-neighbourhood interacts with the $(\Pi_m)_i$ in the same way. Hence for all non-principal eigenvectors v of the m-neighbourhood, we have two of $(\Pi_m)_1^T v$, $(\Pi_m)_2^T v$, $(\Pi_m)_3^T v$ are equal 0. However if two of these equal 0 then by Corollary 8.2.20, $N^R_{jm} v = N^G_{jm} v = 0$, a contradiction to the fact that v is in the independent case in j. So either $k_m = k_j$ or $k_m = k_l$.

Suppose $k_m = k_j$. Then $k_j = p^m_{mj} + 1$ by Lemma 4.2.3, implying that $p^m_{lj} + p^m_{jj} = 1$ by Lemma 4.2.3 again, and so one of these intersection numbers must be 0. By Lemma 7.1.15 and the fact $p^m_{ml} = 0$, we must have that $p^m_{jj} = 0$. However by Lemma 7.5.1 this would imply that v is either the 0 case or the Eigenvector in j a contradiction.

Hence we have $k_m = k_l$. However by Lemma 7.5.1, we see that $(r_{m_m}, r_{j_m}, r_{l_m})$ can't be in the Independent case in l, and by Theorem 9.1.3 it can't be in the 0 case in l. Hence it must be in the Eigenvector case, and therefore any non-principal eigenvector v corresponds to an eigenvector in the l-neighbourhood. Hence they share the same multiplicity, but v has multiplicity $k_m - 1 = k_l - 1$ and therefore the corresponding eigenvalue triple must be the only eigenvalue triple of the l-neighbourhood. This implies that the l-neighbourhood is also complete. □

We can now see that we have this almost symmetrical situation between the l and the m-neighbourhoods. Both interact with each other in the same way, and must interact with the j-neighbourhood likewise. Hence the l-neighbourhood will also have its sole eigenvalue triple in the Independent case in j. However if we treat the l-neighbourhood as we did the m-neighbourhood, by Lemma 7.5.1 we will get that $p^l_{ll} = p^l_{lm} = 0$, meaning the structure is imprimitive by Lemma 7.1.16. Now combining this and the other results of this section we can state the classification.

Theorem 9.2.2 *Suppose we have a primitive 3-regular, 3-coloured structure with the m-neighbourhood complete. Then there cannot exist an eigenvector $v \in E(r_{m_m}, r_{j_m}, r_{l_m})$ of the m-neighbourhood, with v in the Independent case in j.*

Proof. Suppose for a contradiction that there exists an eigenvector $v \in E(r_{m_m}, r_{j_m}, r_{l_m})$ of the m-neighbourhood, with v in the Independent case in j. We know from Lemma 7.5.1 that $p^m_{mj} \neq 0$, and so we must have $p^m_{mm} = p^m_{ml} = 0$. Now by Lemma 9.2.1 we know that the l-neighbourhood is also complete and we have that the non-principal eigenvalue triples of the m and l-neighbourhoods correspond. By Lemma 7.6.15, this means that the eigenvalue triple of the l-neighbourhood is also in the Independent case in j. So now, applying Lemma 7.5.1 again, we get that $p^l_{lj} \neq 0$, meaning $p^l_{ll} = p^l_{lm} = 0$. However this would mean the structure is imprimitive by Lemma 7.1.16, a contradiction. □

9.3 The Eigenvector Case

The final case we need to consider to get a full classification of finite primitive 3-coloured, 3-regular graphs with complete neighbourhoods is when there exist only corresponding eigenvectors from the complete neighbourhood. We will show there exists only one such structure.

Lemma 9.3.1 *Suppose the m-neighbourhood is complete and is such that its sole eigenvalue triple is in the Eigenvector case in j and l. Then either $k_m = k_j$ and the j-neighbourhood is complete, or $k_m = k_l$, and the l-neighbourhood is complete.*

Proof. Suppose both $k_m \neq k_j$ and $k_m \neq k_l$. Then by Corollary 8.2.6 two of $(\Pi_m)_1^T v$, $(\Pi_m)_2^T v$, $(\Pi_m)_3^T v$ equal 0 (with which two depending on the value of m). However if two of these equal 0 then by Corollary 8.2.20, $N_{jm}^R v = N_{jm}^G v = 0$, a contradiction, and so either $k_m = k_j$ or $k_m = k_j$.

Say without loss of generality, that $k_m = k_j$. As the multiplicity of the v is $k_m - 1$, then we know the multiplicity of $N_{jm}^c v$ is also $k_m - 1 = k_j - 1$, which therefore means it must be the only eigenvector of the j-neighbourhood. Hence the j-neighbourhood is also complete. □

Supposing it is the j-neighbourhood that is also complete, we know that $p_{mm}^m = p_{jj}^j = p_{mx}^m = p_{jy}^j = 0$ for some x and y. By Lemma 7.1.15, we know $(x, y) \neq (j, m)$. Note that this leaves three possible options, namely $(x, y) = (l, l), (l, m)$ or (j, l). However due to the fact that j and m have the exact same conditions placed on them at this point, the options (l, m) and (j, l) will produce the same results, however with j and m swapped.

Theorem 9.3.2 *Let $x \in \{j, l\}$ and $y \in \{m, l\}$. Then there is only one finite primitive 3-regular, 3-coloured structure satisfying $p_{mm}^m = p_{jj}^j = p_{mx}^m = p_{jy}^j = 0$ and $k_m = k_j$, namely the Tricolour Heptagon from Definition 5.2.1.*

Proof. First, note that by Lemma 7.1.15, we know $(x, y) \neq (j, m)$. So suppose $(x, y) = (l, l)$, i.e. $p_{mm}^m = p_{jj}^j = p_{ml}^m = p_{jl}^j = 0$. As $k_m = k_j$, Lemma 4.2.3 yields $p_{jj}^m = p_{jm}^j =$

$k_m - 1$ and $p^j_{mm} = p^m_{mj} = k_m - 1$. Now

$$\begin{aligned} k_m &= p^j_{mm} + p^j_{ml} + p^j_{mj} \\ &= k_m - 1 + p^j_{ml} + k_m - 1 \\ &= 2k_m - 2 + p^j_{ml} \end{aligned}$$

So $p^j_{ml} = 2 - k_m$. As $k_m > 1$, we must have $k_m = 2$, and therefore $p^j_{ml} = 0$. However now $p^l_{mm} = p^l_{jj} = p^l_{mj} = 0$ and so $m^= \cup j^=$ forms an equivalence relation, contradicting our primitivity assumption.

So instead suppose $(x, y) = (j, l)$. Now $k_m - 1 = k_j - 1$ implies $p^m_{ml} = p^j_{jm}$ by Lemma 4.2.3, and $p^{jjm}_{mml} = 0$ by Lemma 4.2.5. Therefore, by Lemma 4.2.3 again

$$p^{jjm}_{jml} + p^{jjm}_{lml} = p^m_{ml} = p^j_{jm} = p^{jjm}_{jml} + 1$$

This implies that $1 = p^{jjm}_{lml}$, which means that $p^j_{lm} = p^{jjm}_{lml} + p^{jjm}_{lmm} + p^{jjm}_{lmj} = 1$. But then by two applications of Lemma 4.2.4,

$$k_j = k_m p^m_{lj} = k_l p^l_{mj}$$

implying first that $p^m_{lj} = 1$ and then that $k_m = k_l p^l_{mj}$. As we know by Theorems 9.1.3 and 9.2.2, the eigenvalue triple of the m-neighbourhood must also be in the Eigenvector case in l and so an eigenvalue triple of the l-neighbourhood corresponds with it. This means the multiplicity of this eigenvalue triple of the l-neighbourhood is $k_m - 1$ and so $k_l \geq k_m$. But as $p^l_{mj} \geq 1$, we must have that $k_l = k_m$ also. So now the l-neighbourhood is also complete as it only has one eigenvalue triple, the one that corresponds to the eigenvalue triple in the m-neighbourhood. Therefore we have three complete neighbourhoods.

We also know $p^m_{lj} = p^j_{lm} = p^l_{jm} = 1$. Comparing the two equations we see:

$$\begin{aligned} p^l_{mm} &= p^{lmj}_{mml} + p^{lmj}_{mmm} + p^{lmj}_{mmj} = p^{lmj}_{mml} \\ p^j_{ml} &= p^{lmj}_{mml} + p^{lmj}_{jml} + p^{lmj}_{lml} = p^{lmj}_{mml} + p^{lmj}_{jml} \end{aligned}$$

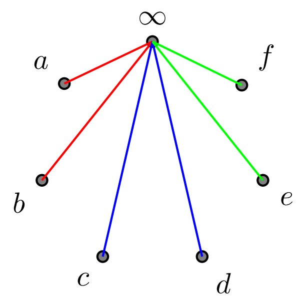
As $p^j_{ml} = 1$, we see that $p^{lmj}_{mml} = 0$ or 1 and hence $p^l_{mm} = 0$ or 1. Of course it must be 1, and so $p^l_{mm} = 1$. Now by Lemma 4.2.4

$$k_m p^m_{ml} = k_l p^l_{mm}$$

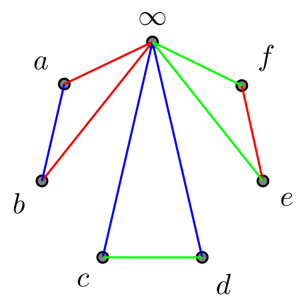
So as $k_m = k_l$ we find that $p^m_{ml} = 1$ too, and therefore $k_m = 1 + p^m_{ml} = 2$ by Lemma 4.2.3. This leads to $k_j = k_l = 2$, and then $p^j_{jm} = 1$ and p^l_{lj} by Lemma 4.2.3. By a further application of Lemma 4.2.4, we get $p^j_{ll} = p^m_{jj} = 1$ as well.
Now suppose this structure actually exists. Then if we assign $m = R$, $j = G$ and $l = B$, we get the following:

$$\begin{aligned}
&k_R = k_B = k_G = 2,\\
&p^R_{RB} = p^R_{GB} = p^R_{GG} = p^B_{BG} = p^B_{RR} = p^B_{RG} = p^G_{GR} = p^G_{BB} = p^G_{RB} = 1\\
&p^R_{RR} = p^R_{RG} = p^G_{GG} = p^G_{GB} = p^B_{BB} = p^B_{BR} = 0
\end{aligned}$$

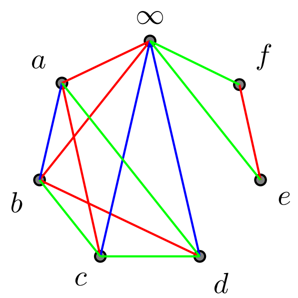
We can start to draw this using $k_R = k_G = k_B = 2$



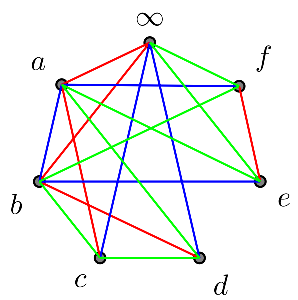
Now we can look at the interior of each neighbourhood. This gets us to



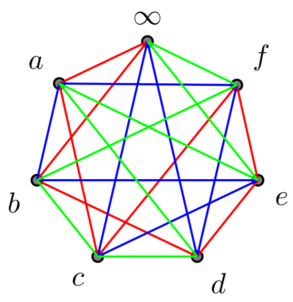
Now we look at the edges going between neighbourhoods. First focus on edges from a, into the blue neighbourhood. We see from the 2-intersection numbers that one edge must be red and one must be green. As c and d are at this point indistinguishable, it doesn't matter which we choose for each. Hence say a to c is red and a to d is green. Then if we focus on b, we see that it must have the opposite composition to match with the 2-intersection numbers. This gives us



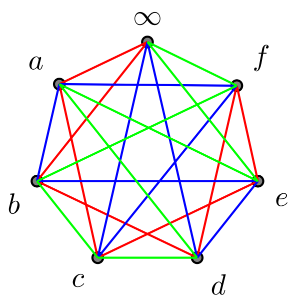
Next we'll look at the edges from the red neighbourhood to the green one, and we find much the same situation. a must have one blue edge and one green edge and it doesn't matter how these are completed. So we pick a to f as blue and a to e as green. b will then have the opposite.



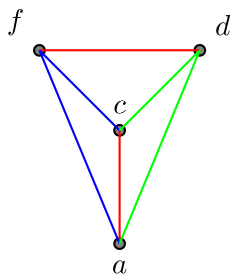
Finally we look at the last neighbourhood interaction. We can see this can be fully determined from the two intersection numbers by choosing the colour of the edge from c to f. First suppose it is a red edge. This gives the structure



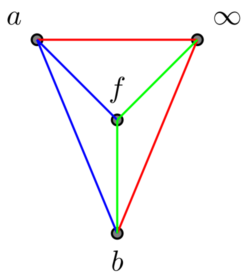
The other configuration is



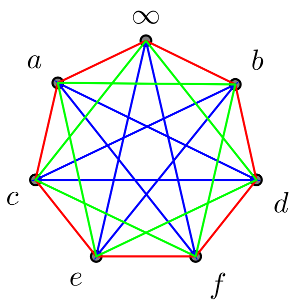
Note that the second configuration is not 3-regular. If we take the triangle (f, d, c) then we get one vertex, a, connected as follows



So p^{RBG}_{BGR} would be 1. However if we look at the triangle (a, ∞, f), which is isomorphic to (f, d, c), there can't possibly be a vertex connected in the same manner as the only other blue edge from a goes to b which forms the following configuration:



Hence we only have one possible configuration, up to isomorphism. This configuration we can redraw as



We can note that it is the Tricolour Heptagon. We know by Theorem 5.2.3 that this is a primitive 3-regular 3-coloured graph and so we have a definite example. Therefore the Tricolour Heptagon is the only possible solution when we have a complete neighbourhood with two corresponding eigenvectors. □

Combining the results of the last few sections we can now provide a full classification of the complete neighbourhood case.

Theorem 9.3.3 *The only finite primitive 3-regular, 3-coloured graph with a complete neighbourhood is the Tricolour Heptagon.*

Proof. Suppose we have a finite primitive 3-regular, 3-coloured graph with the m-neighbourhood complete. Consider the sole eigenvalue triple $(r_{m_m}, r_{j_m}, r_{l_m})$ of the m-neighbourhood. By Theorem 9.1.3 we know $(r_{m_m}, r_{j_m}, r_{l_m})$ is not in the 0 case in j, with j a distinct colour to m. We also know by Theorem 9.2.2 that $(r_{m_m}, r_{j_m}, r_{l_m})$ in not in the Independent case in j, and therefore $(r_{m_m}, r_{j_m}, r_{l_m})$ is in the Eigenvector case in

j. We can do the same thing for l (the third colour distinct from both m and j), to get that $(r_{m_m}, r_{j_m}, r_{l_m})$ is in the Eigenvector case in l as well. Now by Lemma 9.3.1 either the j-neighbourhood is complete with $k_m = k_j$ or the l-neighbourhood is complete with $k_m = k_l$. We suppose without loss of generality that we have the former. However this implies we must have $p_{mm}^{m} = p_{jj}^{j} = p_{mx}^{m} = p_{jy}^{j} = 0$ for some $x \in \{j, l\}$ and $y \in \{m, l\}$. Hence by Theorem 9.3.2, there is only one structure that satisfies this, which is the Tricolour Heptagon. □

Chapter 10

Possibilities when Intersection Numbers are Zero

In this section we will consider what happens when certain double intersection numbers are zero. It will constantly be assumed that no neighbourhood of a base point is complete, as these are already classified. Further assumptions are the usual; finite, primitive, 3-regular and 3-coloured graphs, and m, j and l are distinct colours.

We will attempt to either eliminate such possibilities, or classify them in terms of how the eigenvectors behave in each neighbourhood. We show that $p^R_{GB} = 0$ can't happen in Theorem 10.1.1 and classify the $p^m_{mj} = 0$ case in Theorem 10.2.13. We only partially deal with $p^m_{mm} = 0$, as these intersection numbers being zero doesn't affect the main arguments of this section all that much.

10.1 Some initial results

We can combine arguments similar to those used in Theorem 7.6.20, with earlier ones from Chapter 7, to get the following theorem.

Theorem 10.1.1 *In a primitive 3-regular 3-coloured structure,* $p^R_{GB} \neq 0$.

Proof. Suppose $p^R_{GB} = 0$. Immediately from Lemma 7.5.1 we know that for any two neighbourhoods m and j, every eigenvalue triple of the m-neighbourhood must be the 0 case or the Eigenvector case in j.

Suppose first that for every colour m and j there does not exist an eigenvector of the m-neighbourhood in the 0 case in j. Then we must have $k_R = k_G = k_B$ by Lemma 7.6.19. Now by Lemma 4.2.4, we can note that $p^m_{mj} = p^j_{mm}$. Further as $k_m = k_j$, by Lemma 4.2.3,

$$\begin{aligned} p^j_{ml} + p^j_{mm} + p^j_{mj} &= p^j_{mj} + p^j_{lj} + p^j_{jj} + 1 \\ p^j_{mm} &= p^j_{lj} + p^j_{jj} + 1 \end{aligned}$$

But now $p^R_{GG} = p^R_{BR} + p^R_{RR} + 1$, $p^B_{RR} = p^B_{BG} + p^B_{BB} + 1$ and $p^G_{BB} = p^G_{RG} + p^G_{GG} + 1$. However this, together with the fact $p^j_{mm} = p^m_{mj}$, implies

$$p^R_{GG} = p^R_{GG} + p^R_{RR} + p^G_{GG} + p^B_{BB} + 3$$

a contradiction as all the values are non-negative.

Hence we know that for some distinct m and j, there exists an eigenvector v of the m-neighbourhood in the 0 case in j. So $N^y_{jm}v = 0$ for all colours y. Hence, if l is distinct from j and m, and a is any colour, by Lemma 4.2.13 and Corollary 4.2.6 we see

$$\begin{aligned} 0 &= N^a_{lj} N^y_{jm} v \\ &= p^{lmm}_{jay} N^m_{lm} v + p^{lmj}_{jay} N^j_{lm} v + p^{lml}_{jay} N^l_{lm} v \\ &= (p^{lmm}_{jay} - p^{lml}_{jay}) N^m_{lm} v \end{aligned}$$

Hence either $p^{lmm}_{jay} = p^{lml}_{jay}$ for all a and y or $N^m_{lm}v = 0$. Suppose the former, but then if $a = l$ and $y = j$ we have $p^{lml}_{jlj} = p^{lmm}_{jlj} = 0$ (the latter as $p^m_{lj} = 0$). This means $p^{ljl}_{mlj} = 0$ and we already know $p^{ljl}_{mjj} = 0$, however now by Lemma 4.2.3 either $p^j_{mj} = p^{ljl}_{mlj} + p^{ljl}_{mmj} + p^{ljl}_{mjj} = 0$ or $p^l_{jl} = 0$, both (when combined with the $p^m_{lj} = 0$ condition) contradicting Lemma 7.1.15. Hence we must have $N^m_{lm}v = 0$, and as $N^j_{lm} = 0$, $N^j_{lm}v = 0$ too. Therefore $N^l_{lm}v = Jv = 0$ and v must be in the 0 case in l.

By Lemma 7.6.6 we know that if $(r_{m_m}, r_{j_m}, r_{l_m})$ is the eigenvalue triple attached to v, then for some eigenvalue triple of the entire structure, say (r_m, r_j, r_l), $(r_{m_m}, r_{j_m}, r_{l_m}) =$

(r_m, r_j, r_l). Now by Corollary 4.2.9 and $p^l_{mj} = 0$ we get

$$r_m r_j = p^m_{mj} r_m + p^j_{mj} r_j$$
$$r_m = \frac{p^j_{mj} r_j}{r_j - p^m_{mj}}$$

Note that $r_j \neq p^m_{mj}$ as this would imply $p^j_{mj} p^m_{mj} = 0$, which when combined with $p^m_{lj} = 0$ contradicts Lemma 7.1.15. We can also get the same eigenvalue equation but inside the m-neighbourhood:

$$r_{m_m} r_{j_m} = p^{mmm}_{mmj} r_{m_m} + p^{mmj}_{mmj} r_{j_m}$$

This means

$$r_m r_j = p^{mmm}_{mmj} r_m + p^{mmj}_{mmj} r_j$$

And so

$$p^j_{mj} r_j^2 = p^j_{mj} p^{mmm}_{mmj} r_j + (r_j - p^m_{mj}) p^{mmj}_{mmj} r_j \tag{10.1.1}$$
$$0 = (p^j_{mj} - p^{mmj}_{mmj}) r_j^2 + (p^m_{mj} p^{mmj}_{mmj} - p^j_{mj} p^{mmm}_{mmj}) r_j \tag{10.1.2}$$

We can solve this quadratic as if $p^j_{mj} = p^{mmj}_{mmj}$ and $p^m_{mj} p^{mmj}_{mmj} = p^j_{mj} p^{mmm}_{mmj}$, then $p^m_{mj} = p^{mmm}_{mmj}$ and by Lemma 4.2.3 this would imply $0 = p^{mmm}_{jmj} = p^{mmj}_{mmj} = p^j_{mj}$, a contradiction by Lemma 7.1.15 when combined with $p^m_{jl} = 0$. Therefore $r_j = 0$ or $r_j = \frac{p^j_{mj} p^{mmm}_{mmj} - p^m_{mj} p^{mmj}_{mmj}}{p^{mmj}_{jmj}}$. If $r_j = 0$, then $r_m = 0$ as $p^m_{mj} \neq 0$ by Lemma 7.1.15. Therefore $r_l = -1$ by Lemma 4.2.10 and by Lemma 2.1.14, this implies the graph is complete in l or imprimitive.

Hence we must have

$$r_j = \frac{p^j_{mj} p^{mmm}_{mmj} - p^m_{mj} p^{mmj}_{mmj}}{p^{mmj}_{jmj}}, \; r_m = \frac{p^m_{mj} p^{mmj}_{mmj} - p^j_{mj} p^{mmm}_{mmj}}{p^{mmm}_{jmj}}$$
$$r_l = -1 - \frac{(p^j_{mj} p^{mmm}_{mmj} - p^m_{mj} p^{mmj}_{mmj})(p^{mmm}_{jmj} - p^{mmj}_{jmj})}{p^{mmj}_{jmj} p^{mmm}_{jmj}}$$

This tells us we have a maximum of one eigenvalue triple of the m-neighbourhood in the 0 case in j, as if we had another we could use the exact same deductions and determine that the eigenvalues were all equal.

As in the tricolour heptagon $p^R_{GB} \neq 0$, by Theorem 9.3.3 we know that the neighbourhoods are not complete, so there must exist at least one more eigenvalue triple in each neighbourhood. Consider $S = (s_{m_m}, s_{j_m}, s_{l_m})$ in the m-neighbourhood. This must be in the Eigenvector case in both j and l as it cannot be in the Independent case by Lemma 7.5.1 and cannot be the 0 case by our previous analysis. Suppose v is in the eigenspace of S and v' is the corresponding eigenvector such that $v = N^m_{mj} v'$ in the j-neighbourhood. Then, for any colour y, by Lemma 4.2.13 and Corollary 4.2.6,

$$\begin{aligned} s_{y_m} v &= N^y_{mm} v \\ &= N^y_{mm} N^m_{mj} v' \\ &= (p^{mjm}_{mym} - p^{mjj}_{mym}) N^m_{mj} v' \end{aligned}$$

And so

$$s_{y_m} = p^{mjm}_{mym} - p^{mjj}_{mym} \tag{10.1.3}$$

But as $N^l_{mj} = 0$, we have $N^m_{mj} v = -N^j_{mj} v$ and so

$$\begin{aligned} s_{y_m} v &= N^y_{mm} v \\ &= -N^y_{mm} N^j_{mj} v' \\ &= (p^{mjj}_{myj} - p^{mjm}_{myj}) N^m_{mj} v' \end{aligned}$$

Therefore

$$s_{y_m} = p^{mjm}_{mym} - p^{mjj}_{mym} = p^{mjj}_{myj} - p^{mjm}_{myj} \tag{10.1.4}$$

Further we can do the exact same thing with l giving the equation

$$s_{y_m} = p^{mlm}_{mym} - p^{mll}_{mym} = p^{mll}_{myl} - p^{mlm}_{myl} \tag{10.1.5}$$

From these we can note that by Lemma 4.2.3 and Corollary 4.2.6 we get $s_{l_m} = p^{mjj}_{mlj} = p^m_{ml}$

and $s_{j_m} = p^{mll}_{mjl} = p^m_{mj}$. Now we know that $0 = -1 - s_{m_m} - s_{j_m} - s_{l_m}$ and so $s_{m_m} = -1 - p^m_{ml} - p^m_{mj} = p^m_{mm} - k_m$ by Lemma 4.2.1. However we can also obtain from Equation 10.1.4 that $s_{m_m} = p^{mjm}_{mmm} - p^{mjj}_{mmm}$. Now

$$\begin{aligned} p^m_{mm} - k_m &= p^{mjm}_{mmm} - p^{mjj}_{mmm} \\ &= p^m_{mm} - p^{mjm}_{mmj} - p^{mjj}_{mmm} \end{aligned}$$

Therefore

$$\begin{aligned} k_m &= p^{mjm}_{mmj} + p^{mjj}_{mmm} \\ &= p^j_{mj} + p^j_{mm} - p^{mjm}_{mjj} - p^{mjj}_{mjm} \end{aligned}$$

And so we get $0 = p^{mjm}_{mjj} = p^{mjj}_{mjm}$ as $k_m = p^j_{mj} + p^j_{mm}$ by Lemma 4.2.1 and $p^j_{ml} = 0$. However we have already shown $p^{mjm}_{mjj} = p^{mmj}_{jmj}$ is non-zero by considering the quadratic in Equation 10.1.1. Therefore we have a contradiction, and we know no structure with $p^m_{jl} = 0$ can exist. □

10.2 The $p^m_{mj} = 0$ case

We can apply similar lines of inquiry to get results (namely Theorem 10.2.13) in the case when $p^m_{mj} = 0$. Using Theorem 9.3.3, we shall also work under the assumption that $p^m_{mm} \neq 0$ to avoid making the m-neighbourhood complete.

Lemma 10.2.1 *In a primitive structure, suppose $p^m_{mj} = 0$ and $p^m_{mm} \neq 0$. Then there are two eigenvalue triples in m, one in the 0 case in j and one in the Eigenvector case in j. Further the eigenvalue triple in the Eigenvector case in j is*

$$(p^{mjj}_{mmj} - p^{mjl}_{mmj}, \; p^{mjj}_{mjj} - p^{mjl}_{mjj}, \; p^{mjj}_{mlj} - p^{mjl}_{mlj})$$

Proof. First we immediately know there are only two eigenvalue triples in the m-neighbourhood as it is two coloured, hence it is a strongly regular graph, but not complete.

Now suppose that we have an eigenvalue triple $(r_{c_m}, r_{d_m}, r_{e_m})$ of the m-neighbourhood in the Eigenvector case in j. Then by Remark 7.6.14, there exists an eigenvector of the j-neighbourhood v such that $N^j_{mj}v$ is an eigenvector belonging to the eigenspace of $(r_{c_m}, r_{d_m}, r_{e_m})$. Then for each colour x, by Lemma 4.2.13,

$$\begin{aligned} r_{x_m} N^j_{mj} v &= N^x_{mm} N^j_{mj} v \\ &= p^{mjj}_{mxj} N^j_{mj} v + p^{mjl}_{mxj} N^l_{mj} v \\ &= (p^{mjj}_{mxj} - p^{mjl}_{mxj}) N^j_{mj} v \end{aligned}$$

as $N^m_{mj} = 0$ implies $0 = N^l_{mj}v + N^j_{mj}v$ by Remark 4.2.14. Therefore $r_{x_m} = p^{mjj}_{mxj} - p^{mjl}_{mxj}$ and so there can only be at most one eigenvalue triple in the Eigenvector case in j.

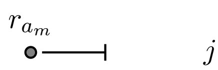
Now suppose that neither eigenvalue triple is in the Eigenvector case in j. Then as $p^m_{mj} = 0$, they must be in the 0 case by Lemma 7.5.1, and further, by Lemma 7.6.11, we know that they must both be in the eigenvector case in l. Then by Lemma 7.6.18 we see that $p^{lmc}_{jxy} = p^{lmd}_{jxy} = p^{lme}_{jxy}$ for all x and y, as otherwise we can determine the eigenvalue, and so $r_{a_l} = s_{a_l}$ for all a. But now $0 = p^{lmm}_{jmj} = p^{lmj}_{jmj} = p^{lml}_{jmj}$, meaning also that $0 = p^{ljm}_{mmj} = p^{ljm}_{mjj} = p^{ljm}_{mlj}$, and so either $p^j_{mj} = 0$, a contradiction to Theorem 7.6.20, or $p^l_{jm} = 0$, a contradiction to Theorem 10.1.1.

And therefore we must have one eigenvalue triple in the 0 case in j and one in the Eigenvector case in j. □

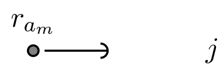
We can represent this in the diagram where nodes represent eigenvalue triples of the labelled neighbourhood and lines represent what case they are in the other neighbourhoods as follows:

Suppose $r_{a_m} = (r_{m_m}, r_{j_m}, r_{l_m})$ and $r_{a_j} = (r_{m_j}, r_{j_j}, r_{l_j})$ are eigenvalue triples. Then, if r_{a_m} is in the eigenvector case in j and it corresponds with r_{a_j} then we would draw the arrow

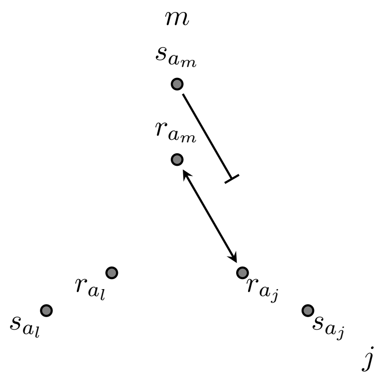
If r_{a_m} is in the 0 case in j then we would draw the arrow



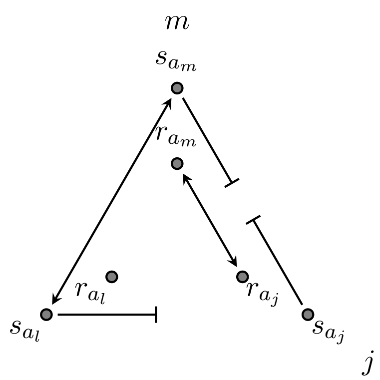
Note that because it doesn't attach to a specific eigenvalue triple, the destination is just the j-neighbourhood in general. And finally if r_{a_m} was in the independent case in j we would draw



Putting this all together for this example we have at the moment



Empty arrows are just where we haven't yet assigned a case to the eigenvalue triple. By Lemma 7.6.11 we know that s_{a_m} can't be in the 0 case in l and therefore it must be in the Eigenvector case. We will say that it corresponds with s_{a_l} without loss of generality. Further by Lemma 7.6.16 this will imply s_{a_l} is the 0 case in j as well. Now by Lemma 7.5.1 we know that s_{a_j} is either in the 0 or Eigenvector case in both m and l. Clearly it must be in the 0 case in m as there is no eigenvalue triple for it to correspond with. Hence the diagram becomes



Now we consider what case r_{a_m} must be in l.

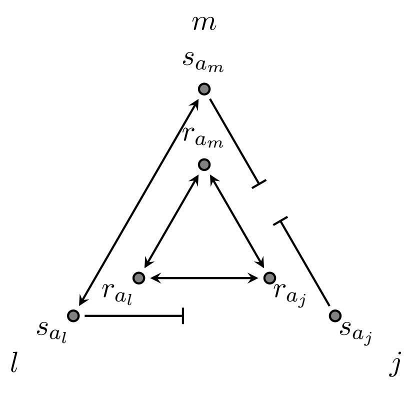
Lemma 10.2.2 *Suppose* $p^m_{mj} = 0$ *and the eigenvalue triples* $(r_{c_m}, r_{d_m}, r_{l_m})$ *and* $(r_{c_j}, r_{d_j}, r_{l_j})$ *correspond. Then both* $(r_{c_m}, r_{d_m}, r_{l_m})$ *and* $(r_{c_j}, r_{d_j}, r_{l_j})$ *are in the Eigenvector case in* l.

Proof. Suppose not. Then $(r_{c_m}, r_{d_m}, r_{l_m})$ is in the 0 case in l by Lemma 7.5.1. Then for v in the eigenspace of $(r_{c_m}, r_{d_m}, r_{l_m})$, we have $N^y_{lm}v = 0$ for all y, and therefore, for any x

$$0 = N^x_{jl}N^y_{lm}v = (p^{jmj}_{lxy} - p^{jml}_{lxy})N^j_{jm}v$$

Now $N^j_{jm}v$ must be non-zero as $(r_{c_m}, r_{d_m}, r_{l_m})$ is in the Eigenvector case in j and we already have $N^m_{jm}v = 0$. Therefore instead we must have that, for all x and y, $p^{jmj}_{lxy} = p^{jml}_{lxy}$. This gives us that $p^{jml}_{lmm} = 0$ as $p^{jmj}_{lmm} = 0$ since $p^m_{mj} = 0$. But now $0 = p^{jml}_{lmm} + p^{jml}_{mmm} + p^{jml}_{jmm}$ and so either $p^j_{ml} = 0$ or $p^l_{mm} = 0$. Both give contradictions, the first by Theorem 10.1.1 and the second as then the m-neighbourhood would be complete in m. Hence we must have that $(r_{c_m}, r_{d_m}, r_{l_m})$ is in the Eigenvector case in l, and therefore by Lemma 7.6.17, $(r_{c_j}, r_{d_j}, r_{l_j})$ must be as well. □

This allows us to complete the diagram a little further



We will now address the issue of how many eigenvalue triples there are in the j-neighbourhood. We have only been using two so far however it turns out that either there are three triples or $p^j_{jj} = 0$.

Lemma 10.2.3 *Suppose* $p^m_{mj} = 0$ *and* $p^m_{mm} \neq 0$. *Then either:*

- *The* j*-neighbourhood has three distinct non-principal eigenvalue triples,*

- $p^j_{jj} = 0$.

Proof. By Lemma 10.2.1, we know that one of the eigenvalue triples of the m-neighbourhood, say $r_{a_m} = (r_{m_m}, r_{j_m}, r_{l_m})$ is in the Eigenvector case in j and the other, $s_{a_m} = (s_{m_m}, s_{j_m}, s_{l_m})$, is in the 0 case in j.

We know that if the j-neighbourhood has only one eigenvalue triple it is complete, but then we would be working with the tricolour heptagon by Theorem 9.3.3. However $p^m_{mm} \neq 0$, so we cannot be.

Suppose that the j-neighbourhood has only two distinct non-principal eigenvalue triples namely $r_{a_j} = (r_{m_j}, r_{j_j}, r_{l_j}$ and $s_{a_j} = (s_{m_j}, s_{j_j}, s_{l_j})$. We know one corresponds to r_{a_m} and we shall say without loss of generality that this is r_{a_j}. However we also know by the fact $p^j_{mm} = 0$ and Lemma 7.5.1, that s_{a_j} is not in the Independent case in m. It also can't be in the Eigenvector case, as there is no eigenvalue triple it could correspond to. Hence s_{a_j} is in the 0 case in l. Now we also know that the j-neighbourhood must be two-coloured as it has only two non-principal eigenvalue triples, so either $p^j_{jj} = 0$, $p^j_{jl} = 0$ or $p^j_{jm} = 0$. We know by Theorem 7.6.20 that $p^j_{jm} \neq 0$.

First suppose $p^j_{jl} = 0$. Well then we see by Lemma 7.6.11, that s_{a_j} can't also be in the 0 case in l. Hence it must be in the Eigenvector case. But then, using Lemma 10.2.2 we have that both eigenvalue triples of the j-neighbourhood are in the eigenvector case in the l-neighbourhood, which contradicts Lemma 10.2.1. Therefore $p^j_{jl} \neq 0$ too.

Therefore if we do not have three distinct non-principal eigenvalue triples, $p^j_{jj} = 0$. □

Now we can further determine the diagram in the $p^j_{jj} = 0$ case using the following small result.

Lemma 10.2.4 *Suppose $p^j_{jj} = p^m_{mj} = 0$. Then no eigenvalue triple of the j-neighbourhood is the 0 case in both m and l.*

Proof. Suppose $p^j_{jj} = p^m_{mj} = 0$ and there exists an eigenvalue triple of the j-neighbourhood that is the 0 case in both m and l. Then by Lemma 7.6.6, we know there exists an eigenvalue r_j of A_j such that $r_j = r_{j_j}$. However as $p^j_{jj} = 0$, we know by Remark 4.2.15 that $r_{j_j} = 0$, and so $r_j = 0$ too. But now by Corollary 4.2.9,

$$\begin{aligned} 0 &= r_m r_j \\ &= p^m_{mj} r_m + p^j_{jm} r_j + p^l_{mj} r_l \\ &= p^l_{mj} r_l \end{aligned}$$

Hence either $r_l = 0$ too, a contradiction, or $p^l_{mj} = 0$, also a contradiction by Theorem 10.1.1. □

Therefore, if $p^j_{jj} = p^m_{mj} = 0$ the diagram looks like this

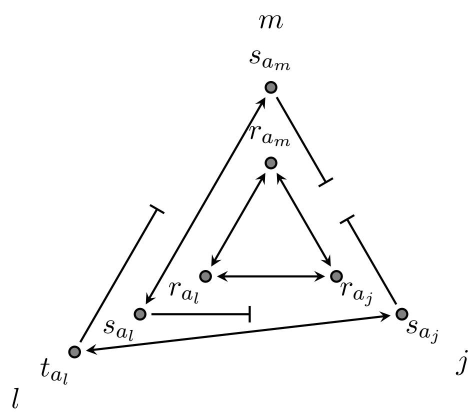


Figure 10.1: $p^j_{jj} = p^m_{mj} = 0$

If $p^j_{jj} \neq 0$ then there is still work to do to fully determine the diagram. We see that the third eigenvalue triple of the j-neighbourhood must be in the 0 case in m as it has no eigenvalue to correspond with.

Claim: Both s_{a_j} and t_{a_j} cannot be in the Eigenvector case in l.
Proof of claim: Suppose for a contradiction both s_{a_j} and t_{a_j} were in the Eigenvector case in l. Then as we know they are both in the 0 case in m we can use Lemma 7.6.18 (with j and m swapped). As we can't have $p^{lmc}_{jxy} = p^{lmd}_{jxy} = p^{lme}_{jxy}$ for all x, y, we must get that they belong in case 2,3 or 4 in Lemma 7.6.18. If s_{a_j} is case 2, then by the condition $p^{lmd}_{jxy} = p^{lme}_{jxy}$, t_{a_j} must also be case 2. The same thing holds for case 3 and 4, and so s_{a_j} and t_{a_j} must be in the same case. But this means they are equal, and so are the same eigenvalue triple, a contradiction to the fact the j-neighbourhood is not 2-coloured. □

Hence either one must be the 0 case in l and one must be the Eigenvector case in l, or both are in the 0 case in l. However:

Lemma 10.2.5 *Suppose $p^m_{mj} = 0$ and an eigenvalue triple $(r_{m_j}, r_{j_j}, r_{l_j})$ of the j-neighbourhood is the 0 case in both m and l. Then $p^{jjl}_{lmm} \neq 0$ and*

$$\begin{aligned} r_m = r_{m_j} =& p^j_{jm} - \frac{p^l_{jm} p^{jjj}_{ljm}}{p^{jjl}_{ljm}}, \\ r_j = r_{j_j} =& \frac{p^j_{jm} p^{jjm}_{lmm} p^{jjl}_{ljm} + p^j_{lm} p^{jjl}_{ljm} - p^l_{jm} p^{jjm}_{lmm} p^{jjj}_{jlm}}{p^{jjj}_{ljm} p^{jjl}_{lmm}}, \\ r_l = r_{l_j} =& \frac{p^l_{jm} p^{jjm}_{lmm} p^{jjj}_{ljm} - p^j_{lm} p^{jjl}_{ljm} - p^j_{jm} p^{jjm}_{lmm} p^{jjl}_{ljm}}{p^{jjl}_{ljm} p^{jjl}_{lmm}} \end{aligned}$$

Proof. Again from Lemma 7.6.6 we know that if we have such an eigenvalue triple, then $r_{a_j} = r_a$ for all a. Therefore we have the two equations by Corollary 4.2.9

$$\begin{aligned} r_j r_m &= p^j_{mj} r_j + p^l_{mj} r_l && (10.2.6) \\ r_j r_m &= p^{jjj}_{jmj} r_j + p^{jjl}_{jmj} r_l \end{aligned}$$

By substituting the bottom equation from the top equation by 4.2.3 and using $p^m_{mj} = 0$, we get

$$0 = p^{jjj}_{lmj} r_j + p^{jjl}_{lmj} r_l \qquad (10.2.7)$$

It is crucial that these coefficients are not both zero and indeed we can show they are.
Claim: p^{jjj}_{lmj} and p^{jjl}_{lmj} are not both zero.
Proof of claim: Suppose for a contradiction $p^{jjj}_{lmj} = p^{jjl}_{lmj} = 0$ then by Lemma 4.2.5, $p^{ljj}_{jmj} = p^{ljj}_{jml} = 0$. But by Lemma 4.2.3 and $p^j_{mm} = 0$, this either means $p^l_{mj} = 0$, which can't happen by Theorem 10.1.1, or $p^l_{jj} = 0$. The latter would mean the j-neighbourhood was two-coloured and hence did not have three distinct eigenvalue triples. However this implies by Lemma 10.2.3 that $p^j_{jj} = 0$ too, and then the j-neighbourhood is complete. So by Theorem 9.3.3, we would have the Tricolour Heptagon, but this doesn't have any eigenvalue triples of any neighbourhood in the 0 case in any other neighbourhood. Hence $p^l_{jj} \neq 0$, a contradiction proving the claim.

We can go further by showing that both r_j and r_l are also non-zero (also implying both p^{jjj}_{lmj} and p^{jjl}_{lmj} are non-zero by Equation 10.2.7). If $r_j = 0$, then by Equation 10.2.6, either $r_l = 0$ or $p^{l}_{mj} = 0$, both of which can't happen, so $r_j \neq 0$.

Suppose now for a contradiction that $r_l = 0$. We know $r_j \neq 0$, so $r_m = p^{j}_{mj}$. Now the eigenvalue triple $(r_{m_j}, r_{j_j}, r_{l_j})$ of the j-neighbourhood is in the 0 case in l. Hence, using Equation 7.4.3 together with the assumption that $r_l = r_{j_l} = 0$ and the fact that $p^{jjj}_{lmm} = 0$ (as $p^{j}_{mm} = 0$), it follows that

$$\begin{aligned} 0 &= p^{j}_{ml} + p^{jjj}_{lmm} r_{j_j} + p^{jjm}_{lmm} r_{m_j} \\ &= p^{j}_{ml} + p^{jjm}_{lmm} p^{j}_{mj} \end{aligned}$$

As these are all non-negative integers, this implies that $p^{j}_{ml} = 0$, a contradicting Theorem 10.1.1. Therefore $r_l \neq 0$.

Hence we can say that

$$r_l = -\frac{p^{jjj}_{lmj}}{p^{jjl}_{lmj}} r_j$$

And therefore

$$r_j r_m = p^{j}_{mj} r_j - \frac{p^{l}_{mj} p^{jjj}_{lmj}}{p^{jjl}_{lmj}} r_j$$

As we already know $r_j \neq 0$, this implies

$$r_m = p^{j}_{mj} - \frac{p^{l}_{mj} p^{jjj}_{lmj}}{p^{jjl}_{lmj}} \tag{10.2.8}$$

We can note the denominator is non-zero as with $r_j \neq 0$ and $r_l \neq 0$, $p^{jjj}_{lmj} = 0$ if and only if $p^{jjl}_{lmj} = 0$, and we've already shown both can't be zero, so neither are.

Now from Equation 7.4.3 and the fact that $p^{jjj}_{lmm} = 0$ again we have that

$$p^{jjm}_{lmm} r_m + p^{jjl}_{lmm} r_l + p^{j}_{lm} = 0$$

Our next step is to show $p^{jjl}_{lmm} \neq 0$ however. Suppose for a contradiction that $p^{jjl}_{lmm} = 0$.

Then we note by Lemma 4.2.5 that $p^{jlm}_{jlm} = 0$ and so by Lemma 4.2.3

$$p^l_{jm} = p^{jlm}_{jmm} + 1 \tag{10.2.9}$$

$$= p^j_{jm} - p^{jlm}_{jml} + 1 \tag{10.2.10}$$

Further we know from Equation 10.2, that $p^{jjm}_{lmm} \neq 0$ (as otherwise $p^j_{lm} = 0$) and

$$r_m = -\frac{p^j_{lm}}{p^{jjm}_{lmm}} = -1 - \frac{p^{jjm}_{lml}}{p^{jjm}_{lmm}}$$

We can now see that $p^{jjm}_{lml} \neq 0$, as if $p^{jjm}_{lml} = 0$ then $r_m = -1$, and so by Lemma 2.1.14, the structure is imprimitive or complete in m, a contradiction. Therefore we know from this that $r_m < -1$ and by Equation 10.2.8 we see

$$p^j_{mj} p^{jjl}_{lmj} < p^l_{mj} p^{jjj}_{lmj}$$

However we can see by Equation 10.2.9 and the fact that $p^{jlm}_{jml} \neq 0$ (as otherwise $p^{jjm}_{lml} = 0$ by Lemma 4.2.5), that $p^l_{mj} \leq p^j_{mj}$. Therefore we can deduce that $p^{jjl}_{lmj} < p^{jjj}_{lmj}$. However applying this to Equation 10.2, we get that

$$-r_l > r_j$$

So we know that $r_m < -1$ and $-r_l > r_j$. However $0 = 1 + r_m + r_j + r_l$ implies $0 < r_j + r_l$. And so $-r_l < r_j$, a contradiction. So $p^{jjl}_{lmm} \neq 0$.

Now using Equations 10.2.8 and 10.2, we can deduce

$$r_l = \frac{p^l_{jm} p^{jjm}_{lmm} p^{jjj}_{ljm} - p^j_{lm} p^{jjl}_{ljm} - p^j_{jm} p^{jjm}_{lmm} p^{jjl}_{ljm}}{p^{jjl}_{ljm} p^{jjl}_{lmm}}$$

Therefore by Equation 10.2, we also get

$$r_j = \frac{p^j_{jm} p^{jjm}_{lmm} p^{jjl}_{ljm} + p^j_{lm} p^{jjl}_{ljm} - p^l_{jm} p^{jjm}_{lmm} p^{jjj}_{jlm}}{p^{jjj}_{ljm} p^{jjl}_{lmm}}$$

□

What this lemma tells us is that there cannot be two distinct eigenvalue triples which are both in the 0 case in m and l. Therefore this means we can complete our diagram as follows

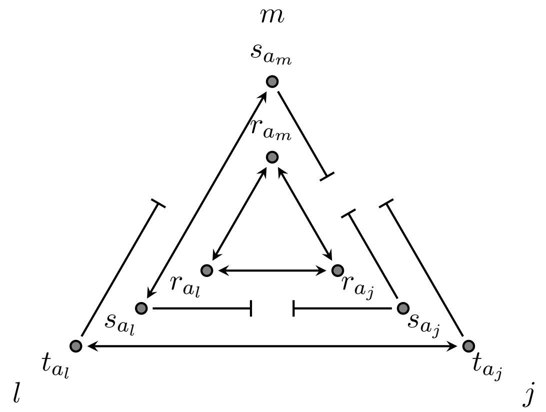


Figure 10.2: $p^m_{mj} = 0,\ p^j_{jj} \neq 0$

Some final deductions we can note from this analysis are as follows:

Corollary 10.2.6 *In a finite primitive 3-regular, 3-coloured structure, suppose* $p^m_{mj} = 0$ *and* $p^m_{mm} \neq 0$*. Then*

i) $p^l_{lx} \neq 0$ *for all* x*,*

ii) $p^m_{lx} \neq 0$ *for all* x*,*

iii) $p^j_{lx} \neq 0$ *for all* x*.*

Proof. For i) if $p^l_{lx} \neq 0$ for any x, then the l-neighbourhood will have only two eigenvalue triples. This doesn't happen in any case as shown in Figure 10.2 and Figure 10.1.

For ii) we immediately know $p^m_{lm} \neq 0$ as $p^m_{lm} = 0$ would make the m-neighbourhood complete in m and $p^m_{lj} \neq 0$ by Theorem 10.1.1. Hence all we have to consider is $p^m_{ll} = 0$, but this is covered by i).

In iii) we get $p^j_{lm} \neq 0$ by Theorem 10.1.1, and $p^j_{ll} \neq 0$ by i). We cannot have $p^j_{jl} = 0$ by Lemma 10.2.3. □

For both of the $p^j_{jj} = 0$ and the $p^j_{jj} \neq 0$ scenarios, if a structure exists we can describe the eigenvalues of the neighbourhoods. We will do this with a swarm of lemmas which will be summarised in Theorem 10.2.13. We label the eigenvalue triples as they are labelled in the Figure 10.1 if $p^j_{jj} = 0$, and Figure 10.2 if not. First we will examine the eigenvalues of the corresponding triangle.

Lemma 10.2.7 *Suppose there exists a finite primitive 3-regular 3-coloured structure with $p^m_{mj} = 0$ and $p^m_{mm} \neq 0$ and $r_{x_m} = (r_{m_m}, r_{j_m}, r_{l_m})$, $r_{x_j} = (r_{m_j}, r_{j_j}, r_{l_j})$ and $r_{x_l} = (r_{m_l}, r_{j_l}, r_{l_l})$ all correspond as eigenvalue triples. Then we have:*

$$\begin{aligned} r_{x_m} &= p^{mjj}_{mxj} - p^{mjl}_{mxj} \\ r_{x_j} &= p^{jmj}_{jxl} - p^{jml}_{jxj} \\ r_{x_l} &= \frac{(p^{lmd}_{lxd} - p^{lmf}_{lxd})(p^{jmj}_{lcd} - p^{jml}_{lcd}) + (p^{lme}_{lxd} - p^{lmf}_{lxd})(p^{jmj}_{lce} - p^{jml}_{lce})}{p^{jmj}_{lcd} - p^{jml}_{lcd}} \\ &= \frac{(p^{ljd'}_{lxd'} - p^{ljf'}_{lxd'})(p^{mjj}_{lc'd'} - p^{mjl}_{lc'd'}) + (p^{lje'}_{lxd'} - p^{ljf'}_{lxd'})(p^{mjj}_{lc'e'} - p^{mjl}_{lc'e'})}{p^{mjj}_{lc'd'} - p^{mjl}_{lc'd'}} \end{aligned}$$

for any colours c, c', distinct colours d, e, f and distinct colours d', e', f'.

Proof. We've already seen in Lemma 10.2.1 that $r_{x_m} = p^{mjj}_{mxj} - p^{mjl}_{mxj}$. We can apply the same idea in reverse to get r_{x_j}. Let v be an eigenvector of the m-neighbourhood in the eigenspace of r_{x_m}. Then $N^j_{jm} v$ is an eigenvector of j-neighbourhood, with eigenvalue r_{x_j}. Hence

$$\begin{aligned} r_{x_j} N^j_{jm} v &= N^x_{jj} N^j_{jm} v \\ &= p^{jmj}_{jxm} N^j_{jm} v + p^{jml}_{jxm} N^l_{jm} v \\ &= (p^{jmj}_{jxj} - p^{jml}_{jxj}) N^j_{jm} v \end{aligned}$$

Hence $r_{x_j} = p^{jmj}_{jxj} - p^{jml}_{jxj}$.

We can also determine r_{x_l} by utilising $N^m_{jm} = 0$. Note that for any c and d,

$$N^c_{jl} N^d_{lm} v = (p^{jmj}_{lcd} - p^{jml}_{lcd}) N^j_{jm} v$$

And therefore, as $\lambda_d N^d_{lm} v = \lambda_e N^e_{lm} v$ for some non-zero constants λ_d and λ_e, we get

$$\lambda_d N^c_{jl} N^d_{lm} v = \lambda_e N^c_{jl} N^e_{lm} v$$
$$\lambda_d (p^{jmj}_{lcd} - p^{jml}_{lcd}) N^j_{jm} v = \lambda_e (p^{jmj}_{lce} - p^{jml}_{lce}) N^j_{jm} v$$

As $N^j_{jm} v \neq 0$, this tells us

$$\frac{\lambda_d}{\lambda_e} = \frac{p^{jmj}_{lce} - p^{jml}_{lce}}{p^{jmj}_{lcd} - p^{jml}_{lcd}}$$

From this we can find r_{x_l} in terms of the intersections numbers as follows. Let d, e, f be distinct colours, then

$$\begin{aligned}
r_{x_l} N^d_{lm} v &= N^x_{ll} N^d_{lm} v \\
&= (p^{lmd}_{lxd} - p^{lmf}_{lxd}) N^d_{lm} v + (p^{lme}_{lxd} - p^{lmf}_{lxd}) N^e_{lm} v \\
&= \left((p^{lmd}_{lxd} - p^{lmf}_{lxd}) + \frac{\lambda_d}{\lambda_e} (p^{lme}_{lxd} - p^{lmf}_{lxd}) \right) N^d_{lm} v \\
&= \left(\frac{(p^{lmd}_{lxd} - p^{lmf}_{lxd})(p^{jmj}_{lcd} - p^{jml}_{lcd}) + (p^{lme}_{lxd} - p^{lmf}_{lxd})(p^{jmj}_{lce} - p^{jml}_{lce})}{(p^{jmj}_{lcd} - p^{jml}_{lcd})} \right) N^d_{lm} v
\end{aligned}$$

We can do the exact same thing with the eigenvectors coming from the j-neighbourhood. So suppose we have an eigenvector v' of the j-neighbourhood that corresponds with v, then $\mu_d N^d_{lj} v' = \mu_e N^e_{lj} v'$ for some non-zero constants μ_d and μ_e. Then as before we use

$$N^c_{ml} N^d_{lj} v' = (p^{mjj}_{lcd} - p^{mjl}_{lcd}) N^j_{mj} v'$$

to give

$$\mu_d N^c_{ml} N^d_{lj} v' = \mu_e N^c_{ml} N^e_{lj} v'$$
$$\mu_d (p^{mjj}_{lcd} - p^{mjl}_{lcd}) N^j_{mj} v' = \mu_e (p^{mjj}_{lce} - p^{mjl}_{lce}) N^j_{mj} v'$$

resulting in

$$\frac{\mu_d}{\mu_e} = \frac{p^{mjj}_{lce} - p^{mjl}_{lce}}{p^{mjj}_{lcd} - p^{mjl}_{lcd}}$$

Therefore for d, e, f distinct colours as before we get that

$$\begin{aligned}
r_{x_l} N^d_{lj} v' &= N^x_{ll} N^d_{lj} v' \\
&= (p^{ljd}_{lxd} - p^{ljf}_{lxd}) N^d_{lj} v' + (p^{lje}_{lxd} - p^{ljf}_{lxd}) N^e_{lm} v \\
&= \left((p^{ljd}_{lxd} - p^{ljf}_{lxd}) + \frac{\mu_d}{\mu_e} (p^{lje}_{lxd} - p^{ljf}_{lxd}) \right) N^e_{lj} v' \\
&= \left(\frac{(p^{ljd}_{lxd} - p^{ljf}_{lxd})(p^{mjj}_{lcd} - p^{mjl}_{lcd}) + (p^{lje}_{lxd} - p^{ljf}_{lxd})(p^{mjj}_{lce} - p^{mjl}_{lce})}{p^{mjj}_{lcd} - p^{mjl}_{lcd}} \right) N^d_{lj} v'
\end{aligned}$$

Therefore, for any c, and any distinct d, e, f,

$$r_{x_l} = \frac{(p^{lmd}_{lxd} - p^{lmf}_{lxd})(p^{jmj}_{lcd} - p^{jml}_{lcd}) + (p^{lme}_{lxd} - p^{lmf}_{lxd})(p^{jmj}_{lce} - p^{jml}_{lce})}{p^{jmj}_{lcd} - p^{jml}_{lcd}}$$

And, for any c', and any distinct d', e', f',

$$r_{x_l} = \frac{(p^{ljd'}_{lxd'} - p^{ljf'}_{lxd'})(p^{mjj}_{lc'd'} - p^{mjl}_{lc'd'}) + (p^{lje'}_{lxd'} - p^{ljf'}_{lxd'})(p^{mjj}_{lc'e'} - p^{mjl}_{lc'e'})}{p^{mjj}_{lc'd'} - p^{mjl}_{lc'd'}}$$

□

We can also determine these in other ways using various previous lemmas, mainly Lemma 7.3.4. At the moment I cannot make much of a conclusion from the different forms of r_{x_l}, except the following identity

$$p^{ljl}_{lmm} p^{mjj}_{lml} = p^{lml}_{lmm} p^{jmj}_{lml}$$

which is currently without much use.

We can also determine some more of the eigenvalues that are not dependent upon p^j_{jj} being zero or not.

Lemma 10.2.8 *Suppose $p^m_{mj} = 0$ and $p^m_{mm} \neq 0$ and the eigenvalue triple $s_{a_m} = (s_{m_m}, s_{j_m}, s_{l_m})$ of the m-neighbourhood is in the 0 case in j and the Eigenvector case in l. Likewise suppose the corresponding eigenvalue triple $s_{a_l} = (s_{m_l}, s_{j_l}, s_{l_l})$ of the l-neighbourhood is in the 0 case in j and the Eigenvector case in m. Then for any colours c*

and d and any distinct colours x, y and z, either $p^{mlm}_{jcd} = p^{mlj}_{jcd} = p^{mll}_{jcd}$ or

$$s_{a_m} = (p^{mlx}_{max} - p^{mlz}_{max}) + \frac{(p^{mlx}_{jcd} - p^{mlz}_{jcd})(p^{mly}_{max} - p^{mlz}_{max})}{(p^{mlz}_{jcd} - p^{mly}_{jcd})}$$

And likewise either $p^{mlm}_{jdc} = p^{mlj}_{jdc} = p^{mll}_{jdc}$ or

$$s_{a_l} = (p^{lmx}_{lax} - p^{lmz}_{lax}) + \frac{(p^{lmx}_{jcd} - p^{lmz}_{jcd})(p^{lmy}_{lax} - p^{lmz}_{lax})}{(p^{lmz}_{jcd} - p^{lmy}_{jcd})}$$

Further there exists values of c and d such that $p^{mlm}_{jcd} = p^{mlj}_{jcd} = p^{mll}_{jcd}$ doesn't hold, and values of c and d such that $p^{mlm}_{jdc} = p^{mlj}_{jdc} = p^{mll}_{jdc}$ doesn't hold.

Proof. In both scenarios, these eigenvalue triples act in exactly the same way, as shown in Figure 10.2. Both correspond with each other and are the 0 case in j. Let v be an eigenvector of s_{a_l}. We know, for any colours c and d, and distinct colours x, y and z,

$$0 = N^c_{mj} N^d_{jl} v = (p^{mlx}_{jcd} - p^{lmz}_{jcd}) N^x_{ml} v + (p^{mly}_{jcd} - p^{mlz}_{jcd}) N^y_{ml} v$$

And therefore

$$(p^{mlx}_{jcd} - p^{mlz}_{jcd}) N^x_{ml} v = (p^{mlz}_{jcd} - p^{mly}_{jcd}) N^y_{ml} v$$

Hence, unless $p^{mlx}_{jcd} = p^{mly}_{jcd} = p^{mlz}_{jcd}$, we have $N^y_{ml} v$ in terms of $N^x_{ml} v$ and we can get

$$\begin{aligned}
s_{a_m} N^x_{ml} v &= N^a_{mm} N^x_{ml} v \\
&= (p^{mlx}_{max} - p^{mlz}_{max}) N^x_{ml} v + (p^{mly}_{max} - p^{mlz}_{max}) N^y_{ml} v \\
&= \left((p^{mlx}_{max} - p^{mlz}_{max}) + \frac{(p^{mlx}_{jcd} - p^{mlz}_{jcd})(p^{mly}_{max} - p^{mlz}_{max})}{(p^{mlz}_{jcd} - p^{mly}_{jcd})} \right) N^x_{ml} v
\end{aligned}$$

And therefore

$$s_{a_m} = (p^{mlx}_{max} - p^{mlz}_{max}) + \frac{(p^{mlx}_{jcd} - p^{mlz}_{jcd})(p^{mly}_{max} - p^{mlz}_{max})}{(p^{mlz}_{jcd} - p^{mly}_{jcd})}$$

Now we can do the exact same thing but with m and l swapped to get s_{a_l}. First for v' an eigenvector of s_{a_l} we have

$$0 = N^c_{lj} N^d_{jm} v' = (p^{lmx}_{jcd} - p^{lmz}_{jcd}) N^x_{lm} v' + (p^{lmy}_{jcd} - p^{lmz}_{jcd}) N^y_{lm} v'$$

And therefore

$$(p^{lmx}_{jcd} - p^{lmz}_{jcd}) N^x_{lm} v' = (p^{lmz}_{jcd} - p^{lmy}_{jcd}) N^y_{lm} v'$$

Hence, unless $p^{lmx}_{jcd} = p^{lmy}_{jcd} = p^{lmz}_{jcd}$, we have $N^y_{lm} v$ in terms of $N^x_{lm} v$ and we can get

$$\begin{aligned}
s_{a_l} N^x_{lm} v &= N^a_{ll} N^x_{lm} v' \\
&= (p^{lmx}_{lax} - p^{lmz}_{lax}) N^x_{lm} v' + (p^{lmy}_{lax} - p^{lmz}_{lax}) N^y_{lm} v' \\
&= \left((p^{lmx}_{lax} - p^{lmz}_{lax}) + \frac{(p^{lmx}_{jcd} - p^{lmz}_{jcd})(p^{lmy}_{lax} - p^{lmz}_{lax})}{(p^{lmz}_{jcd} - p^{lmy}_{jcd})} \right) N^x_{lm} v'
\end{aligned}$$

And therefore

$$s_{a_l} = (p^{lmx}_{lax} - p^{lmz}_{lax}) + \frac{(p^{lmx}_{jcd} - p^{lmz}_{jcd})(p^{lmy}_{lax} - p^{lmz}_{lax})}{(p^{lmz}_{jcd} - p^{lmy}_{jcd})}$$

To prove the final statement we simply need to find colours c and d such that $p^{mlm}_{jcd} = p^{mlj}_{jcd} = p^{mll}_{jcd}$ doesn't hold. The other statement will then not hold when c and d are swapped.
Note that setting $c = m$ and $d = j$ we see that the condition $p^{lmx}_{jcd} = p^{lmy}_{jcd} = p^{lmz}_{jcd}$ cannot hold. This is because if $0 = p^{lmm}_{jmj} = p^{lmj}_{jmj} = p^{lml}_{jmj}$ then $0 = p^{ljm}_{mmj} = p^{ljm}_{mjj} = p^{ljm}_{mlj}$ and hence either $p^l_{jm} = 0$ or $p^j_{mj} = 0$ by Lemma 4.2.3, contradicting either Theorem 7.6.20 or 10.1.1. And so we are done. □

In a very similar manner we now find the third eigenvalue of the l-neighbourhood, and the second eigenvalue of the j-neighbourhood.

Lemma 10.2.9 *Suppose $p^m_{mj} = 0$, and both $t_{a_l} = (t_{m_l}, t_{j_l}, t_{l_l})$ and $t_{a_j} = (t_{m_j}, t_{j_j}, t_{l_j})$ are corresponding eigenvalue triples belonging to the l-neighbourhood and the j-neighbourhood respectively. Then if they are both in the 0 case in m, for some colours*

c and d and any distinct colours x, y and z, either $p^{jlx}_{mcd} = p^{jly}_{mcd} = p^{jlz}_{mcd}$, or we have

$$t_{a_j} = (p^{jlx}_{jax} - p^{jlz}_{jax}) + \frac{(p^{jly}_{jax} - p^{jlz}_{jax})(p^{jlx}_{mcd} - p^{jlz}_{mcd})}{p^{jlz}_{mcd} - p^{jly}_{mcd}}$$

Similarly, either $p^{ljx}_{mcd} = p^{ljy}_{mcd} = p^{ljz}_{mcd}$ or

$$t_{a_l} = (p^{ljx}_{lax} - p^{ljz}_{lax}) + \frac{(p^{ljy}_{lax} - p^{ljz}_{lax})(p^{ljx}_{mcd} - p^{ljz}_{mcd})}{p^{ljz}_{mcd} - p^{ljy}_{mcd}}$$

Further there exist values of c and d such that $p^{jlx}_{mcd} = p^{jly}_{mcd} = p^{jlz}_{mcd}$ doesn't hold, and values of c and d such that $p^{ljx}_{mcd} = p^{ljy}_{mcd} = p^{ljz}_{mcd}$ doesn't hold.

Proof. This is almost exactly the same proof as the last one, however with j instead of m. First if v is an eigenvector of t_{x_l} then

$$\begin{aligned} 0 &= N^c_{jm} N^d_{ml} v \\ &= (p^{jlx}_{mcd} - p^{jlz}_{mcd}) N^x_{jl} v + (p^{jly}_{mcd} - p^{jlz}_{mcd}) N^y_{jl} v \end{aligned}$$

And therefore either $p^{jlx}_{mcd} = p^{jly}_{mcd} = p^{jlz}_{mcd}$ or

$$N^y_{jl} v = \frac{p^{jlx}_{mcd} - p^{jlz}_{mcd}}{p^{jlz}_{mcd} - p^{jly}_{mcd}} N^x_{jl} v$$

Now we know

$$\begin{aligned} t_{a_j} N^x_{jl} v &= N^a_{jj} N^x_{jl} v \\ &= (p^{jlx}_{jax} - p^{jlz}_{jax}) N^x_{jl} v + (p^{jly}_{jax} - p^{jlz}_{jax}) N^y_{jl} v \\ &= (p^{jlx}_{jax} - p^{jlz}_{jax}) N^x_{jl} v + \left(\frac{(p^{jly}_{jax} - p^{jlz}_{jax})(p^{jlx}_{mcd} - p^{jlz}_{mcd})}{p^{jlz}_{mcd} - p^{jly}_{mcd}} \right) N^x_{jl} v \\ &= \left((p^{jlx}_{jax} - p^{jlz}_{jax}) + \frac{(p^{jly}_{jax} - p^{jlz}_{jax})(p^{jlx}_{mcd} - p^{jlz}_{mcd})}{p^{jlz}_{mcd} - p^{jly}_{mcd}} \right) N^x_{jl} v \end{aligned}$$

Therefore either $p^{jlx}_{mcd} = p^{jly}_{mcd} = p^{jlz}_{mcd}$ or

$$t_{a_j} = (p^{jlx}_{jax} - p^{jlz}_{jax}) + \frac{(p^{jly}_{jax} - p^{jlz}_{jax})(p^{jlx}_{mcd} - p^{jlz}_{mcd})}{p^{jlz}_{mcd} - p^{jly}_{mcd}}$$

Similarly if we let v' be an eigenvector of t_{x_j}, then

$$\begin{aligned} 0 &= N^c_{lm} N^d_{mj} v' \\ &= (p^{ljx}_{mcd} - p^{ljz}_{mcd}) N^x_{lj} v' + (p^{ljy}_{mcd} - p^{ljz}_{mcd}) N^y_{lj} v' \end{aligned}$$

Hence either $p^{ljx}_{mcd} = p^{ljy}_{mcd} = p^{ljz}_{mcd}$ or

$$N^y_{lj} v' = \frac{p^{ljx}_{mcd} - p^{ljz}_{mcd}}{p^{ljz}_{mcd} - p^{ljy}_{mcd}} N^x_{jl} v'$$

Now

$$\begin{aligned} t_{a_l} N^x_{lj} v' &= N^a_{ll} N^x_{lj} v' \\ &= (p^{ljx}_{lax} - p^{ljz}_{lax}) N^x_{lj} v' + (p^{ljy}_{lax} - p^{ljz}_{lax}) N^y_{lj} v' \\ &= (p^{ljx}_{lax} - p^{ljz}_{lax}) N^x_{lj} v' + \left(\frac{(p^{ljy}_{lax} - p^{ljz}_{lax})(p^{ljx}_{mcd} - p^{ljz}_{mcd})}{p^{ljz}_{mcd} - p^{ljy}_{mcd}} \right) N^x_{lj} v' \\ &= \left((p^{ljx}_{lax} - p^{ljz}_{lax}) + \frac{(p^{ljy}_{lax} - p^{ljz}_{lax})(p^{ljx}_{mcd} - p^{ljz}_{mcd})}{p^{ljz}_{mcd} - p^{ljy}_{mcd}} \right) N^x_{lj} v' \end{aligned}$$

Therefore either $p^{ljx}_{mcd} = p^{ljy}_{mcd} = p^{ljz}_{mcd}$ or

$$t_{a_l} = (p^{ljx}_{lax} - p^{ljz}_{lax}) + \frac{(p^{ljy}_{lax} - p^{ljz}_{lax})(p^{ljx}_{mcd} - p^{ljz}_{mcd})}{p^{ljz}_{mcd} - p^{ljy}_{mcd}}$$

For the last statement, we note that if $c = j$ and $d = m$ then $p^{jlx}_{mcd} = p^{jly}_{mcd} = p^{jlz}_{mcd}$ doesn't hold. This is because if it did we would have $0 = p^{jlm}_{mjm} = p^{jlj}_{mjm} = p^{jll}_{mjm}$, meaning $0 = p^{jmj}_{lmm} = p^{jmj}_{ljm} = p^{jmj}_{llm}$. This implies either $p^j_{mj} = 0$ or $p^m_{lm} = 0$ by Lemma 4.2.3, however the former contradicts Theorem 7.6.20, and the latter implies the m-neighbourhood is complete in m.

The same contradiction occurs if we have $p^{ljx}_{mcd} = p^{ljy}_{mcd} = p^{ljz}_{mcd}$ and $c = m$ and $d = j$. $\square$

A lot of our work with the triple intersection numbers in this case depends on an equation of the following form not holding: $p^{mlm}_{jcd} = p^{mlj}_{jcd} = p^{mll}_{jcd}$. We have indeed shown that a particular case will always not hold, but we can actually go much further than this. Note that we are dropping the $p^{m}_{mj} = 0$ assumption here.

Lemma 10.2.10 *Suppose in a finite primitive 3-regular 3-coloured graph, for all x and y, and for all distinct c, d, e, we have $p^{mjc}_{lxy} = p^{mjd}_{lxy} = p^{mje}_{lxy}$. Then $p^{m}_{jz} \neq 0$ for all z and $p^{mlc}_{jxy} = p^{mld}_{jxy} = p^{mle}_{jxy}$.*

Proof. Suppose for some z that $p^{m}_{jz} = 0$. Then $p^{mjz}_{lxy} = 0$ for all x and y, by Corollary 4.2.6, and therefore $p^{mjc}_{lxy} = p^{mjd}_{lxy} = p^{mje}_{lxy} = 0$. This implies $p^{mjx}_{lcy} = p^{mjx}_{ldy} = p^{mjx}_{ley} = 0$, hence $p^{m}_{jx} = 0$ or $p^{j}_{ly} = 0$ by Lemma 4.2.3. But this holds for all x and y, meaning it holds for $x = l$ and $y = m$. But here we would have $p^{m}_{jl} = 0$, a contradiction to Theorem 10.1.1. Therefore we know $p^{m}_{jz} \neq 0$ for all z.
Since $p^{mjc}_{lxy} = p^{mjd}_{lxy} = p^{mje}_{lxy}$ then, using Lemma 4.2.4 we see

$$\frac{p^{m}_{lx}}{p^{m}_{jc}} p^{mlx}_{jcy} = \frac{p^{m}_{lx}}{p^{m}_{jd}} p^{mlx}_{jdy} = \frac{p^{m}_{lx}}{p^{m}_{je}} p^{mlx}_{jey}$$

So

$$p^{m}_{jd} p^{m}_{je} p^{mlx}_{jcy} = p^{m}_{je} p^{m}_{jc} p^{mlx}_{jdy} = p^{m}_{jc} p^{m}_{jd} p^{mlx}_{jey} \tag{10.2.11}$$

And crucially

$$p^{mlx}_{jdy} = \frac{p^{m}_{jd}}{p^{m}_{jc}} p^{mlx}_{jcy}, \; p^{mlx}_{jey} = \frac{p^{m}_{je}}{p^{m}_{jc}} p^{mlx}_{jcy}$$

Therefore, combining this with Lemma 4.2.3, we see

$$\begin{aligned} p^l_{jy} &= p^{mlx}_{jcy} + p^{mlx}_{jdy} + p^{mlx}_{jey} \\ &= p^{mlx}_{jcy}\left(1 + \frac{p^m_{jd}}{p^m_{jc}} + \frac{p^m_{je}}{p^m_{jc}}\right) \\ &= p^{mlx}_{jcy}\left(\frac{p^m_{jc} + p^m_{jd} + p^m_{je}}{p^m_{jc}}\right) \\ &= \frac{k_j p^{mlx}_{jcy}}{p^m_{jc}} \end{aligned}$$

Therefore $p^{mlx}_{jcy} = \frac{p^m_{jc} p^l_{jy}}{k_j}$. But this holds for all x, and the value doesn't depend on x at all, hence $p^{mlm}_{jcy} = p^{mlj}_{jcy} = p^{mll}_{jcy}$. Now note that it didn't matter which value we picked for c, and this could range through any colour. □

We see that this goes back the other way by swapping j and l so it is actually an if and only if statement. Further the choice of m was immaterial so we actually get three equivalent conditions.

Corollary 10.2.11 *Suppose in a finite primitive 3-regular 3-coloured graph, for all x and y, and for all distinct c, d, e the following conditions are equivalent*

1. $p^{mjc}_{lxy} = p^{mjd}_{lxy} = p^{mje}_{lxy}$
2. $p^{lmc}_{jxy} = p^{lmd}_{jxy} = p^{lme}_{jxy}$
3. $p^{jlc}_{mxy} = p^{jld}_{mxy} = p^{jle}_{mxy}$
4. $p^{mjc}_{lxy} = p^{mjd}_{lxy} = p^{mje}_{lxy} = \frac{p^j_{ly} p^m_{lx}}{k_l}$
5. $p^{lmc}_{jxy} = p^{lmd}_{jxy} = p^{lme}_{jxy} = \frac{p^m_{jy} p^l_{jx}}{k_j}$
6. $p^{jlc}_{mxy} = p^{jld}_{mxy} = p^{jle}_{mxy} = \frac{p^l_{my} p^j_{mx}}{k_m}$

We can also note the impact of this condition on the eigenvalue interactions.

Lemma 10.2.12 *Suppose for all x and y, and for all distinct c, d, e, $p^{mjc}_{lxy} = p^{mjd}_{lxy} = p^{mje}_{lxy}$. Then there cannot exist an eigenvalue triple of the m-neighbourhood that is in the Eigenvector case in both j and l.*

Proof. Suppose for a contradiction that we have an eigenvalue triple $r_{x_m} = (r_{m_m}, r_{j_m}, r_{l_m})$ of the m-neighbourhood in the Eigenvector case in j and l. Then if v is an eigenvector of r_{x_m} we know, for some colour a, $N^a_{jm}v$ is an eigenvector in the j-neighbourhood. Further by Lemma 7.6.17, we know that for some colour b, $N^b_{lj}N^a_{jm}v$ is an eigenvector for the l-neighbourhood. However

$$N^b_{lj}N^a_{jm}v = (p^{lmc}_{jba} - p^{lme}_{jba})N^c_{lm}v + (p^{lmd}_{jba} - p^{lme}_{jba})N^d_{lm}v$$

and we know by Corollary 10.2.11, $p^{lmc}_{jxy} = p^{lmd}_{jxy} = p^{lme}_{jxy}$, implying $0 = N^b_{lj}N^a_{jm}v$. So $N^b_{lj}N^a_{jm}v$ can't be an eigenvector, a contradiction. □

Hence, as we are guaranteed a triangle of corresponding eigenvalue triples in any situation including $p^m_{mj} = 0$ by Lemma 10.2.1 and Lemma 10.2.2, we can conclude that no variation of $p^{mjc}_{lxy} = p^{mjd}_{lxy} = p^{mje}_{lxy}$ holds.

We then have only one eigenvalue left to determine, which is s_{a_j}. Obviously if $p^j_{jj} = 0$ this doesn't exist, and if $p^j_{jj} \neq 0$ then we know it is case 0 in both m and l. Hence we can determine it using Lemma 10.2.5. Summarising this we get the following theorem:

Theorem 10.2.13 *Suppose for a finite primitive 3-regular, 3-coloured graph, that $p^m_{mj} = 0$. Then all other intersection numbers are non-zero, except perhaps p^j_{jj}. For any colours a, c and c', distinct colours d, e, f and distinct colours d', e' and f', the eigenvalues of the* *neighbourhoods are as follows:*

$$
\begin{aligned}
r_{a_m} &= p_{maj}^{mjj} - p_{maj}^{mjl} \\
r_{a_j} &= p_{jal}^{jmj} - p_{jaj}^{jml} \\
r_{a_l} &= \frac{(p_{lxd}^{lmd} - p_{lxd}^{lmf})(p_{lcd}^{jmj} - p_{lcd}^{jml}) + (p_{lxd}^{lme} - p_{lxd}^{lmf})(p_{lce}^{jmj} - p_{lce}^{jml})}{p_{lcd}^{jmj} - p_{lcd}^{jml}} \\
&= \frac{(p_{lxd'}^{ljd'} - p_{lxd'}^{ljf'})(p_{lc'd'}^{mjj} - p_{lc'd'}^{mjl}) + (p_{lxd'}^{lje'} - p_{lxd'}^{ljf'})(p_{lc'e'}^{mjj} - p_{lc'e'}^{mjl})}{p_{lc'd'}^{mjj} - p_{lc'd'}^{mjl}} \\
s_{a_m} &= (p_{mad}^{mld} - p_{mad}^{mlf}) + \frac{(p_{jcc'}^{mld} - p_{jcc'}^{mlf})(p_{mad}^{mle} - p_{mad}^{mlf})}{(p_{jcc'}^{mlf} - p_{jcc'}^{mle})} \\
s_{a_l} &= (p_{lad}^{lmd} - p_{lad}^{lmf}) + \frac{(p_{jcc'}^{lmd} - p_{jcc'}^{lmf})(p_{lad}^{lme} - p_{lad}^{lmz})}{(p_{jcc'}^{lmf} - p_{jcc'}^{lme})} \\
t_{a_j} &= (p_{jad}^{jld} - p_{jad}^{jlf}) + \frac{(p_{jad}^{jle} - p_{jad}^{jlf})(p_{mcc'}^{jld} - p_{mcc'}^{jlf})}{p_{mcc'}^{jlf} - p_{mcc'}^{jle}} \\
t_{a_l} &= (p_{lad}^{ljd} - p_{lad}^{ljf}) + \frac{(p_{lad}^{lje} - p_{lad}^{ljf})(p_{mcc'}^{ljd} - p_{mcc'}^{ljf})}{p_{mcc'}^{ljf} - p_{mcc'}^{lje}}
\end{aligned}
$$

If $p_{jj}^j = 0$*, then we have only two eigenvalues in the* j*-neighbourhood, so this is all of them.*
If $p_{jj}^j \neq 0$ *then*

$$
\begin{aligned}
r_m = s_{m_j} &= p_{jm}^j - \frac{p_{jm}^l p_{ljm}^{jjj}}{p_{ljm}^{jjl}}, \\
r_j = s_{j_j} &= \frac{p_{jm}^j p_{lmm}^{jjm} p_{ljm}^{jjl} + p_{lm}^j p_{ljm}^{jjl} - p_{jm}^l p_{lmm}^{jjm} p_{jlm}^{jjj}}{p_{ljm}^{jjj} p_{lmm}^{jjl}}, \\
r_l = s_{l_j} &= \frac{p_{jm}^l p_{lmm}^{jjm} p_{ljm}^{jjj} - p_{lm}^j p_{ljm}^{jjl} - p_{jm}^j p_{lmm}^{jjm} p_{ljm}^{jjl}}{p_{ljm}^{jjl} p_{lmm}^{jjl}}
\end{aligned}
$$

The eigenvalues also interact as follows:

If $p_{jj}^j = 0$

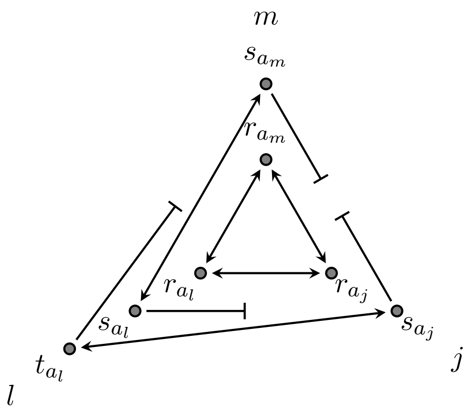


Figure 10.3: Classification of eigenvalue triple cases when $p^m_{mj} = 0, p^j_{jj} = 0$

If $p^j_{jj} \neq 0$

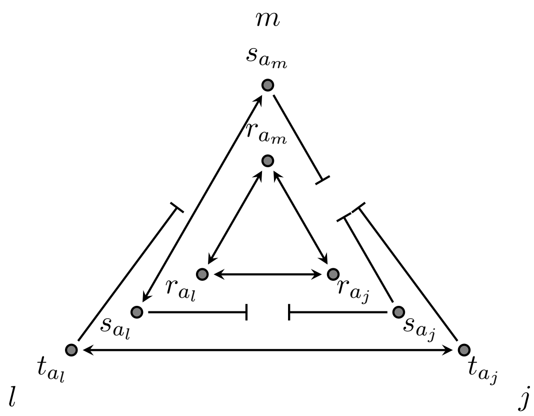


Figure 10.4: Classification of eigenvalue triple cases when $p^m_{mj} = 0, p^j_{jj} \neq 0$

We can summarise all the identities we've found over the course of the classification into one lemma too. These gives us some feasibility equations.

Lemma 10.2.14 *Suppose we have a finite primitive, 3-regular, 3-coloured graph such that* $p^m_{mj} = 0$*. Then it must satisfy the following:*

1. *For all colours* c, c', d, d' *and any distinct colours* x, y, z*:*

$$(p^{lmx}_{jcd} - p^{lmz}_{jcd})(p^{lmz}_{jc'd'} - p^{lmy}_{jc'd'}) = (p^{lmx}_{jc'd'} - p^{lmz}_{jc'd'})(p^{lmz}_{jcd} - p^{lmy}_{jcd})$$

2. *For all colours c, c', d, d' and any distinct colours x, y, z:*

$$(p^{ljx}_{mcd} - p^{ljz}_{mcd})(p^{ljz}_{mc'd'} - p^{ljy}_{mc'd'}) = (p^{ljx}_{mc'd'} - p^{ljz}_{mc'd'})(p^{ljz}_{mcd} - p^{ljy}_{mcd})$$

3. *For all colours c, c' and any distinct colours e and d:*

$$(p^{jmj}_{lce} - p^{jml}_{lce})(p^{jmj}_{lc'd} - p^{jml}_{lc'd}) = (p^{jmj}_{lcd} - p^{jml}_{lcd})(p^{jmj}_{lc'e} - p^{jml}_{lc'e})$$
$$(p^{mjj}_{lce} - p^{mjl}_{lce})(p^{mjj}_{lc'd} - p^{mjl}_{lc'd}) = (p^{mjj}_{lcd} - p^{mjl}_{lcd})(p^{mjj}_{lc'e} - p^{mjl}_{lc'e})$$

We can also note that Theorem 10.2.13 (and Theorem 10.1.1) have some further consequences when applied to undesirability (as defined in 7.4.13)

Lemma 10.2.15 *Suppose the R-neighbourhood is undesirable with respect to G. Then either:*

- $p^R_{RG} = 0$ *and we have either Figure 10.2.13 or 10.2.13 with $m = R$, $j = G$ and $l = B$.*
- $p^R_{GG} = 0$ *and we have either Figure 10.2.13 or 10.2.13 with $m = G$, $j = R$ and $l = B$.*

This follows from the fact that undesirability forces $p^m_{cj} = 0$ for some c.

10.3 Monochromatic-triangle-free

In this section we will discuss the consequences of a 3-coloured 3-regular graph having $p^R_{RR} = p^G_{GG} = p^B_{BB} = 0$ i.e. a 3-coloured, 3-regular structure devoid of any monochromatic triangle. By Ramsey's Theorem, we know that this must be small. In fact the Clebsch Graph from earlier is the largest possible case, as $R(3, 3, 3) = 17$ [24]. But aside from the tricolour Clebsch graph, are there any possibilities? We will now show there are none

Theorem 10.3.1 *Suppose Γ is a primitive, 3-coloured, 3-regular graph with $p^R_{RR} = p^G_{GG} = p^B_{BB} = 0$ but no complete neighbourhoods. Then Γ is isomorphic to the Tricolour Clebsch graph.*

Proof. Firstly we know that no other double intersection numbers are zero as the neighbourhoods are not complete and Theorem 10.1.1 means $p^m_{jl} \neq 0$. This means we can use Lemma 4.2.3 without worrying about double intersection numbers being 0. Focusing just on the m-neighbourhood, we see (using $p^{mmj}_{mjj} = p^{mmj}_{mjm} = 0$) that $p^m_{mj} = p^{mmj}_{mjl} + 1 = p^{mml}_{mjl} + p^{mml}_{mjj}$ and similarly $p^m_{ml} = p^{mml}_{mjl} + 1 = p^{mmj}_{mjl} + p^{mmj}_{mll}$. Now

$$\begin{aligned} k_m =& p^m_{mj} + p^m_{ml} + 1 \\ =& p^{mml}_{mjl} + p^{mmj}_{mjl} + 3 \\ =& p^{mmj}_{mjl} + p^{mmj}_{mll} + p^{mml}_{mjl} + p^{mml}_{mjj} + 1 \end{aligned}$$

Therefore $2 = p^{mmj}_{mll} + p^{mml}_{mjj}$, and as these are non-negative integers we have very few options. Either:

i) $p^{mmj}_{mll} = p^{mml}_{mjj} = 1$

ii) $p^{mmj}_{mll} = 2, p^{mml}_{mjj} = 0$

iii) $p^{mmj}_{mll} = 0, p^{mml}_{mjj} = 2$

We see that ii) and iii) are effectively the same thing, just with j and l swapped, and so we will treat them as one case.

First suppose we have case i), then $p^m_{mj} = 1 + p^{mmj}_{mlj} = 1 + p^{mml}_{mlj}$, so $p^{mmj}_{mlj} = p^{mml}_{mlj}$. But, by repeated us of Lemma 4.2.4,

$$\begin{aligned} p^m_{mj} &= p^m_{mj} p^{mmj}_{mll} \\ &= p^m_{ml} p^{mml}_{mjl} \\ &= p^m_{ml} p^{mml}_{mjj} p^{mmj}_{mlj} \\ &= p^m_{mj} (p^{mmj}_{mlj})^2 \end{aligned}$$

Therefore $p^{mmj}_{mlj} = p^{mml}_{mlj} = 1$. This implies $p^m_{mj} = p^m_{ml} = 2$, and $k_m = 5$.

If instead we suppose we have case ii) (which covers case iii too as previously mentioned), then we see $p^m_{mj} = p^{mml}_{mjl}$. Now as $2p^m_{mj} = p^m_{mj} p^{mmj}_{mll} = p^m_{ml} p^{mml}_{mlj}$, we see that $p^m_{ml} = 2$. We know $p^m_{ml} = 1 + p^{mml}_{mlj} = 1 + p^m_{mj}$, and therefore $p^m_{mj} = 1$ and $k_m = 4$.

Hence we see that for any colour x, $k_x = 4$ or 5. If $k_m = 5$, then we can note that by Lemma 4.2.4, $k_j p^j_{mm} = k_m p^m_{mj} = 10$, hence k_j divides 10 and so it must be 5. The same is true of k_l, hence if we have case i) in one neighbourhood we have it in all the others, meaning $n = 16$ and $k_m = k_j = k_l = 5$. As the Clebsch graph is uniquely determined by its spectrum (Lemma 5.3.5), it is uniquely determined by its intersection numbers, and therefore this must be the tricolour Clebsch graph.

Equally we see that if $k_m = 4$, then $k_j = k_l = 4$. And so if one neighbourhood is case ii) or case iii), then all three are case ii) or case iii). However, if we assume the m-neighbourhood is case ii), then case ii) and case iii) can no longer be treated as the same in either the j or l-neighbourhood. Consider first the j-neighbourhood. We must have either $p^{jjm}_{jll} = 2$ and $p^{jjl}_{jmm} = 0$ or $p^{jjl}_{jmm} = 2$ and $p^{jjm}_{jll} = 0$ as these correspond to case ii) and case iii) in the j-neighbourhood. Suppose first for a contradiction that $p^{jjm}_{jll} = 2$ and $p^{jjl}_{jmm} = 0$. Then $p^j_{jm} = 1$ and $p^j_{jl} = 2$. Crucially, as $k_m = k_j = k_l$ $p^m_{jj} = 1$, $p^l_{mm} = 2$ and $p^m_{jl} = p^l_{jm}$. Now $k_j = p^m_{mj} + p^m_{jj} + p^m_{jl}$ implies $p^m_{jl} = 2$ and therefore $p^l_{jm} = 2$ as well. However $k_m = p^l_{jm} + p^l_{mm} + p^l_{ml}$ now implies $p^l_{ml} = 0$, implying the l-neighbourhood is complete, when we assumed it was not. Hence we must have instead that $p^{jjl}_{jmm} = 2$ and $p^{jjm}_{jll} = 0$ in the j-neighbourhood, and by the same reasoning, $p^{llm}_{ljj} = 2$ and $p^{llj}_{lmm} = 0$ in the l-neighbourhood.

This structure will have the following intersection numbers:

- $n = 13$, $k_R = k_G = k_B = 4$
- $p^m_{mj} = p^j_{jl} = p^l_{lm} = 2$
- $p^m_{ml} = p^j_{jm} = p^l_{lj} = p^m_{jl} = 1$
- $p^{mml}_{mjj} = p^{jjm}_{jll} = p^{llj}_{lmm} = 2$
- $p^{mmj}_{mlj} = p^{mml}_{mlj} = p^{jjl}_{jml} = p^{jjm}_{jml} = p^{llm}_{lmj} = p^{llj}_{ljm} = 1$
- $p^{mmj}_{mll} = p^{jjl}_{jmm} = p^{llm}_{ljj} = 0$

However in trying to fully derive the other triple intersection numbers we will come across an impossible scenario. Namely, as $p^l_{lj} = 1$ and $p^{mml}_{mlj} = 1$, we get that $p^{mml}_{llj} = 0$ by

Lemma 4.2.3. Now, by Lemma 4.2.3 again

$$2 = p_{ll}^{m} = p_{llj}^{mml} + p_{llm}^{mml} + p_{lll}^{mml} = p_{llm}^{mml}$$

But $1 = p_{ml}^{m} = p_{lmm}^{mml} + p_{llm}^{mml} + p_{ljm}^{mml}$ by Lemma 4.2.3, and as $p_{llm}^{mml} = 2$, the RHS is greater than or equal to 2, a contradiction. Hence this cannot be a primitive 3-regular 3-coloured graph. □

Chapter 11

Other Cases

In this chapter we aim to discuss the myriad of other cases not yet investigated in detail, and boil everything down to just a few possibilities as given in Theorem 11.4.2.

First we'll talk about the most awkward of the cases, the independent one. This one crops up annoyingly as a counter example in the broader discussion of non-zero intersection numbers. As such this case is particular relevant to our original motivating problem of finding whether the universal homogeneous 3-coloured graph is a m.e.c. limit. However it provides quite strong conditions itself and using these conditions we will entirely rule out the possibility of it occurring in Theorem 11.1.2.

We then turn our attention to the scenario where we have two eigenvalue triples both in the 0 case in another neighbourhood. In Theorem 11.2.4, we show that this can't occur, except in the case when $p^m_{jj} = 0$ as previously discussed.

To start with we shall only be assuming that we are working in a finite primitive 3-regular 3-coloured graph.

11.1 Independent Case

It is first our intention to eliminate the possibility of the Independent case. With this in mind suppose that we have an eigenvalue triple $(r_{m_m}, r_{j_m}, r_{l_m})$ in m which is in the independent case in j. Now we know that any other eigenvalue triple in the m-neighbourhood can't be in the Eigenvector case in j by Corollary 7.5.4 and can't be in the Independent case in j by Theorem 7.5.2 (as they'd be equal). Further any eigenvalue triples of the j-neighbourhood must be in the 0 case in m. Hence, as we are assuming the j-neighbourhood isn't complete, we must have only two eigenvalue triples in the j-neighbourhood by Lemma 7.6.1, and both must be the 0 case in m. Therefore the j-neighbourhood is two coloured. By Figures 10.2.13 and 10.2.13 from Theorem 10.2.13 (and also by Theorem 10.1.1), we know that no independent case can occur anywhere when $p_{jc}^{m} = 0$ or $p_{jc}^{l} = 0$ for any c. Hence $p_{jj}^{j} = 0$ and the j-neighbourhood is either semi-undesirable or a multiple with respect to m by Lemma 7.6.2 (it can't be undesirable as we've already shown $p_{mc}^{j} \neq 0$ fo all c). Some question remain however. For example what case is $(r_{m_m}, r_{j_m}, r_{l_m})$ in l? How many eigenvalue triples are there in the m-neighbourhood?

First suppose $(r_{m_m}, r_{j_m}, r_{l_m})$ is in the 0 case in l. Then, for some colours c and d, any distinct colours x, y, z and v an eigenvector in the eigenspace of $(r_{m_m}, r_{j_m}, r_{l_m})$, by Lemma 4.2.13,

$$0 = N_{jl}^{c} N_{lm}^{d} v = (p_{lcd}^{jmx} - p_{lcd}^{jmz}) N_{jm}^{x} v + (p_{lcd}^{jmy} - p_{lcd}^{jmz}) N_{jm}^{y} v$$

Therefore we must have $p_{lcd}^{jmx} = p_{lcd}^{jmy} = p_{lcd}^{jmz}$, meaning we have all the results from Corollary 10.2.11. Now if we have that the j-neighbourhood is a multiple with respect to m, we also get $p_{jda}^{jmb} = p_{jda}^{jmc}$ for any d and distinct a, b.

Lemma 11.1.1 *For any distinct colours m, j, l, suppose the m-neighbourhood has exactly two distinct eigenvalue triples, both of which are in the 0 case in j. Then it cannot be d-semi-undesirable for any d.*

Proof. As we have only two distinct eigenvalue triples, we know that the m-neighbourhood is two-coloured, hence for some colour x, $p_{mx}^{m} = 0$. Now, by Theorem 10.2.13, we know that neither $p_{mj}^{m} = 0$ or $p_{ml}^{m} = 0$ as there is no scenario allowing the m-neighbourhood to have two 0 cases in either j or l. Hence we must have $p_{mm}^{m} = 0$.

Suppose that the m-neighbourhood is d-semi-undesirable. Therefore, for colours c and e, distinct from each other and from d, $p_{jxx}^{mmc} = p_{jxx}^{mme}$ for all colours x. If $d \neq m$, then we see by Lemma 7.6.3 that $r_{d_m} = s_{d_m}$. But the m-neighbourhood was two coloured, so these are the only non-principal eigenvalues of N_{mm}^{d} and their multiplicities will sum to $k_m - 1$.

By Lemma 2.1.17, they are therefore equal to -1, meaning that the m-neighbourhood is connected by Lemma 2.1.15, and hence is complete in d.
So we must have $d = m$, and so $r_{d_m} = 0$. However now we get from Equation 7.4.3 that

$$0 = p^{m}_{jx} - p^{mme}_{jxx}$$

And so $p^{m}_{jx} = p^{mmj}_{jxx} = p^{mml}_{jxx}$ as $\{c, e\} = \{j, l\}$. But as p^{m}_{mj} and p^{m}_{ml} are non-zero, this implies by Lemma 4.2.3 that $p^{mmj}_{jxy} = p^{mml}_{jxy} = 0$ for any colour y distinct from x. Hence $p^{myj}_{jmx} = p^{myj}_{lmx} = 0$ by Lemma 4.2.5, meaning either $p^{m}_{yj} = 0$ or $p^{m}_{jx} = p^{myj}_{jmx} + p^{myj}_{lmx} = 0$ by Lemma 4.2.3. But by Theorems 10.2.13 and 10.1.1 this can't happen. $\square$

This was the last remaining possibility that allowed the independent case to occur, and so we can now formally rule it out.

Theorem 11.1.2 *In a 3-regular 3-coloured structure, for any distinct colours m and j there does not exist an eigenvalue triple of the m neighbourhood that is in the Independent case in the j neighbourhood.*

Proof. Suppose there exists an eigenvalue triple $(r_{m_m}, r_{j_m}, r_{l_m})$ of the m-neighbourhood which is in the independent case in j. Now we know by Corollary 7.5.4 that any other eigenvalue triples of the m-neighbourhood can't be in the Eigenvector case in j. We also know that by Theorem 7.5.2 that any other eigenvalue triple of the m-neighbourhood can't be in the Independent case in j, as it would equal $(r_{m_m}, r_{j_m}, r_{l_m})$. Therefore any other eigenvalue triple of the m-neighbourhood must be in the 0 case in j. This further implies that any eigenvalue triple of the j-neighbourhood cannot be in the Eigenvector case in m. We also know by Lemma 7.5.3 that any eigenvalue triple of the j-neighbourhood cannot be the Independent case in m, hence all eigenvalue triples of the j-neighbourhood are in the 0 case in m.

We know by Theorem 9.3.3 that the j-neighbourhood is not complete as the Tricolour Heptagon did not admit an eigenvalue triple in the Independent case anywhere, and we know it can't have three distinct eigenvalues by Lemma 7.6.1. Hence it must have two distinct eigenvalue triples and therefore we know by Lemma 7.6.2 that the j-neighbourhood must be a multiple, undesirable or semi-undesirable with respect to m. By Lemma 7.4.17 we can't have it being a multiple, and by Theorem 10.2.13 it can't be undesirable either. Hence it has to be semi-undesirable, but this is impossible by Lemma 11.1.1. □

11.2 Double 0 cases

Using similar ideas, if we assume $p^m_{jj} \neq 0$ (which is classified in Theorem 10.2.13), we can also rule out the possibility of two of the eigenvalue triples from the m-neighbourhood being the 0 case in j.

Lemma 11.2.1 *In a 3-regular 3-coloured structure, for any distinct colours m and j, suppose $p^m_{jx} \neq 0$ and $p^m_{mx} \neq 0$ for all x and there exists an eigenvalue triple of the m-neighbourhood in the 0 case in j. Then the m-neighbourhood cannot be a multiple with respect to j.*

Proof. Suppose for a contradiction the m-neighbourhood is a multiple with respect to j and that we have an eigenvalue triple $(r_{m_m}, r_{j_m}, r_{l_m})$ of the m-neighbourhood which is in the 0 case in j. Then by Corollary 7.4.19 and Theorem 11.1.2 we know that all the eigenvalue triples of the m-neighbourhood are in the 0 case in j and vice versa. Therefore by Lemma 7.6.1, as neither are complete, we must have that both the m and j-neighbourhoods have two distinct eigenvalue triples. Hence by Lemma 11.1.1, j must also be a multiple with respect to m.
As we have only two distinct eigenvalue triples and we've assumed $p^m_{jx} \neq 0$ for all x, we must have either $p^m_{mm} = 0$ or $p^m_{ml} = 0$. Then either $r_{m_m} = s_{m_m} = 0$ or $r_{l_m} = s_{l_m} = 0$ by Remark 4.2.15. Suppose we have $p^m_{mm} = 0$ and $r_{m_m} = s_{l_m} = 0$, and so by Lemma 4.2.10 we get $0 = r_{j_m} + r_{l_m} + 1$. But this means that we can solve Equation 7.4.3 (note that coefficients are non-zero for at least one colour as part of the multiple assumption), and determine that $r_{j_m} = s_{j_m}$ and $r_{l_m} = s_{l_m}$, meaning we only have one distinct eigenvalue triple in m, a contradiction. The same thing occurs for $p^m_{ml} = 0$, and so we can't have that the m-neighbourhood is a multiple with respect to j. □

Therefore we can ignore this case and we can boil it down to just semi-undesirability when there are double 0 cases.

Corollary 11.2.2 *Suppose that two distinct eigenvalue triples of the m-neighbourhood are in the 0 case in j and $p^m_{jj} \neq 0$. Then the m-neighbourhood is semi-undesirable with respect to j.*

This is the simple combination of Lemma 10.2.15, removing undesirability, and Lemma 11.2.1, removing the multiple case. Note that we can combine this with Lemma 11.1.1 to completely remove this in the 2-coloured case.

Corollary 11.2.3 *Suppose the m-neighbourhood is two-coloured. Then it cannot have two distinct eigenvalue triples both in the* 0 *case in j.*

So now let us delve into the possibilities that occur as a result of this.
Suppose we have that the m-neighbourhood has three distinct eigenvalue triples, two of which, say r_{x_m} and s_{x_m}, are in the 0 case in j. Then m is semi-undesirable with respect to j by Corollary 11.2.2, and also we know by Theorem 11.1.2 that the third eigenvalue triple of the m-neighbourhood, t_{x_m}, is in the Eigenvector case in j. Conveniently we can remove all these cases in one fell swoop.

Theorem 11.2.4 *Suppose that two distinct eigenvalue triples of the m-neighbourhood are in the 0 case in j. Then $p^m_{jj} = 0$.*

Proof. Suppose for a contradiction that $p^m_{jx} \neq 0$ for all x. We know by Corollary 11.2.3 that the m-neighbourhood has three distinct eigenvalue triples, two of which, namely r_{x_m} and s_{x_m}, are both in the 0 case in j. Then we know from Corollary 11.2.2 that the m-neighbourhood must be semi-undesirable with respect to j. We also know that the third eigenvalue triple must be in the Eigenvector case in j as it can't be in the 0 case by Lemma 7.6.1 and can't be independent by Theorem 11.1.2. Say, without loss of generality, that the m-neighbourhood is d-semi-undesirable with respect to j. This means that, for any distinct x and y,

$$\frac{p^m_{jx} - p^{mme}_{jxx}}{p^{mme}_{jxx} - p^{mmd}_{jxx}} = \frac{-p^{mme}_{jxy}}{p^{mme}_{jxy} - p^{mmd}_{jxy}} \tag{11.2.1}$$

Note the denominators are non-zero, as otherwise, when coupled with d-semi-undesirability, they would give $p^{mmc}_{jxx} = p^{mmd}_{jxx} = p^{mme}_{jxx}$ or $p^{mmc}_{jxy} = p^{mmd}_{jxy} = p^{mme}_{jxy}$, which give undesirability, and hence $p^m_{jx} = 0$.

Now, as the third eigenvalue triple t_{x_m} is in the Eigenvector case in j, we know that, if v is its eigenvector,

$$N^x_{jm} v = B_{xy} N^y_{jm} v \tag{11.2.2}$$

where B_{xy} is a constant. Now by Lemma 7.3.3 and the semi-undesirability condition we know that

$$B_{xy} = \frac{(p^{mmd}_{jxx} - p^{mme}_{jxx})t_{d_m} + p^m_{jx} - p^{mme}_{jxx}}{(p^{mmd}_{jxy} - p^{mme}_{jxy})t_{d_m} - p^{mme}_{jxy}}$$

Note that neither the numerator nor the denominator can be zero, as if it was then t_{d_m} would equal r_{d_m} and s_{d_m} meaning the m-neighbourhood was complete in d.

Using Equation 11.2.1 this becomes

$$\begin{aligned} B_{xy} &= \frac{p^m_{jx} - p^{mme}_{jxx}}{-p^{mme}_{jxy}} \\ &= -1 - \frac{p^{mme}_{jxz}}{p^{mme}_{jxy}} \end{aligned}$$

However we know

$$\begin{aligned} 0 &= N^x_{jm} v + N^y_{jm} v + N^z_{jm} v \\ &= N^x_{jm} v \,(1 + B_{yx} + B_{zx}) \\ &= N^x_{jm} v \left(-1 - \frac{p^{mme}_{jyz}}{p^{mme}_{jyx}} - \frac{p^{mme}_{jzy}}{p^{mme}_{jzx}} \right) \end{aligned}$$

And therefore, as all the intersection numbers are positive, we must have $N^x_{jm} v = 0$. However this means that $B_{xy} = 0$ which we've already shown can't happen.

Therefore, as the m-neighbourhood cannot be two coloured by Corollary 11.2.3, we must have $p^m_{jx} = 0$ for some x. Now we know $p^m_{jl} \neq 0$ by Theorem 10.1.1 and $p^m_{mj} \neq 0$ by the fact the m-neighbourhood is three coloured. Hence we know that we must have $p^m_{jj} = 0$. □

We can therefore note that the m-neighbourhood has two eigenvalue triples in the 0 case in j if we have one of the cases from Theorem 10.2.13 (with m and j swapped as in the statement of the theorem). We can therefore see that we must be in the scenario represented by Figure 10.2.13, and so $p^m_{mm} \neq 0$ (as $p^j_{jj} \neq 0$ in the theorem).

11.3 Eigenvector case

The only case for which we have not yet obtained tight restrictions is the Eigenvector case. Many of the possibilities that we have discussed so far involve a neighbourhood with two eigenvalues that are in the eigenvector case with another neighbourhood. So if we could place limits upon this we'd gain a lot of information.

Suppose therefore that the m-neighbourhood is such that it has two distinct eigenvalue triples, $(r_{x_m}, r_{y_m}, r_{z_m})$ and $(s_{x_m}, s_{y_m}, s_{z_m})$ in the Eigenvector case in j. Then we know there exists a system of constants that describe the linear relationships between $N^x_{jm}v$ and $N^y_{jm}v$ for all x, y and an eigenvector v belonging to the eigenspace of $(r_{x_m}, r_{y_m}, r_{z_m})$. We will first assume that $N^x_{jm}v \neq 0$ for all x, however a similar system can be set up when $N^x_{jm}v = 0$. Let v' be an eigenvector belonging to the eigenspace, $(r_{x_j}, r_{y_j}, r_{z_j})$, which corresponds to $(r_{x_m}, r_{y_m}, r_{z_m})$. For any x, y, define constants $B_{xy}, \bar{B}_{xy}$ as

$$N^x_{jm}v = B_{xy}N^y_{jm}v,\ N^x_{mj}v' = \bar{B}_{xy}N^y_{mj}v' \tag{11.3.3}$$

Then we can note

Lemma 11.3.1 *For the system of constants defined in Equation 11.3.3 and any colours x, y:*

- $B_{xy} = \frac{1}{B_{yx}}$
- $0 = 1 + B_{yx} + B_{zx}$
- $B_{xy} = \bar{B}_{xy}$

Proof. The first point is immediate from the definition and the second follows from the fact $0 = N^x_{jm}v + N^y_{jm}v + N^z_{jm}v$. The third comes from the fact that because, by Remark 7.6.14, $N^y_{mj}v'$ is itself an eigenvector of the m-neighbourhood, we know $N^x_{jm}N^y_{mj}v' = B_{xy}N^y_{jm}N^y_{mj}v'$. Further we can say, $N^x_{jm}N^y_{mj}v' = B_{xy}\bar{B}_{yx}N^y_{jm}N^x_{mj}v'$, however

$$\begin{aligned} N^x_{jm}N^y_{mj}v' &= \left(p^{jjx}_{mxy}r_{x_j} + p^{jjy}_{mxy}r_{y_j} + p^{jjz}_{mxy}r_{z_j}\right)v' \\ &= \left(p^{jjx}_{myx}r_{x_j} + p^{jjy}_{myx}r_{y_j} + p^{jjz}_{myx}r_{z_j}\right)v' \\ &= N^y_{jm}N^x_{mj}v' \end{aligned}$$

meaning $B_{xy}\bar{B}_{yx} = 1$, and therefore, $\bar{B}_{yx} = B_{yx}$. □

Now from Lemma 7.3.4 we know that, for any colour a,

$$r_{a_m} = (p^{mjx}_{max} - p^{mjz}_{max}) + B_{yx}(p^{mjy}_{max} - p^{mjz}_{max}) \tag{11.3.4}$$

Therefore if we know B_{yx} then we know r_{a_m} for any a. At the moment we only know B_{yx} in terms of the eigenvalues (from Lemma 7.3.4) and hence it will be unique to each eigenspace. However if we could find it in terms of just the intersection numbers then we could determine that $(r_{x_m}, r_{y_m}, r_{z_m})$ and $(s_{x_m}, s_{y_m}, s_{z_m})$ are actually equal.
We know however that

$$r_{a_m} = (p^{mjy}_{may} - p^{mjz}_{may}) + B_{xy}(p^{mjx}_{may} - p^{mjz}_{may})$$

And so

$$(p^{mjy}_{may} - p^{mjz}_{may}) + B_{xy}(p^{mjx}_{may} - p^{mjz}_{may}) = (p^{mjx}_{max} - p^{mjz}_{max}) + B_{yx}(p^{mjy}_{max} - p^{mjz}_{max})$$

Using the result from Lemma 11.3.1 we can determine that

$$0 = (p^{mjy}_{max} - p^{mjz}_{max})B^2_{yx} + (p^{mjx}_{max} + p^{mjz}_{may} - p^{mjz}_{max} - p^{mjy}_{may})B_{yx} - (p^{mjx}_{may} - p^{mjz}_{may}) \tag{11.3.5}$$

Therefore we have a quadratic for B_{yx}.

Lemma 11.3.2 *In the m-neighbourhood, there do not exist three distinct eigenvalue triples that are all in the eigenvector case in j.*

Proof. So long as the Quadratic 11.3.5 is non-zero there can exist a maximum of 2 different values for B_{yx}. As the same quadratic is formed by each of the eigenvalue triples, we therefore have only two different possible values for the eigenvalue triples. Therefore in order for there to be three distinct eigenvalue triples all in the eigenvector case in j, the quadratic 11.3.5 has coefficients all 0 for every value of a. However this implies for all a, $p^{mjy}_{max} = p^{mjz}_{max}$, meaning $r_{a_m} = s_{a_m} = t_{a_m} = p^{mjx}_{max} - p^{mjz}_{max}$. Hence there cannot be three distinct eigenvalue triples all in the eigenvector case in a. □

We will now consider the case when 11.3.5 is actually a linear equation:

Lemma 11.3.3 *Suppose the m-neighbourhood is such that it has two eigenvalue triples, $(r_{x_m}, r_{y_m}, r_{z_m})$ and $(s_{x_m}, s_{y_m}, s_{z_m})$ in the Eigenvector case in j. Then if there exists a colour a and distinct colours x, y, z such that $p_{max}^{mjy} = p_{max}^{mjz}$, then either $(r_{x_m}, r_{y_m}, r_{z_m}) = (s_{x_m}, s_{y_m}, s_{z_m})$ or $p_{max}^{mjy} = p_{max}^{mjz}$ for all distinct colours x, y, z, and also*

$$r_{a_m} = s_{a_m} = p_{max}^{mjx} - p_{max}^{mjy}$$

Proof. Suppose there exists a colour a and distinct colours x, y, z such that $p_{max}^{mjy} = p_{max}^{mjz}$. Then we know from the quadratic 11.3.5 that

$$0 = (p_{max}^{mjx} + p_{may}^{mjz} - p_{max}^{mjz} - p_{may}^{mjy})B_{yx} - (p_{may}^{mjx} - p_{may}^{mjz})$$

And so either we can solve for B_{yx} in terms of only intersection numbers or both $p_{max}^{mjx} + p_{may}^{mjz} - p_{max}^{mjz} - p_{may}^{mjy}$ and $p_{may}^{mjx} - p_{may}^{mjz}$ are zero. If we had the former, then we are done as both eigenvalue triples would have the same value for B_{yx}, and therefore would be equal. So suppose we have the latter.

Note that now we have $r_{a_m} = s_{a_m} = p_{may}^{mjy} - p_{may}^{mjz} = p_{max}^{mjx} - p_{max}^{mjz}$ by Equation 11.3.4, however it is not only B_{yx} that we can solve for, it could be B_{xz} or B_{yz}, as well. And now we can write

$$r_{a_m} = p_{may}^{mjy} - p_{may}^{mjz} = p_{max}^{mjx} - p_{max}^{mjy} - B_{zx}(p_{max}^{mjz} - p_{max}^{mjy})$$

Again if we can determine B_{zx} in terms of intersection numbers we are done, and so we must have $p_{max}^{mjz} = p_{max}^{mjy}$. We can repeat this same idea with any of the B's leading us to having the equation $p_{mad}^{mje} = p_{mad}^{mjf}$ for all distinct colours d, e, f. Further we must have

$$r_{a_m} = s_{a_m} = p_{mad}^{mjd} - p_{mad}^{mje}$$

□

Assuming we don't have $(r_{x_m}, r_{y_m}, r_{z_m}) = (s_{x_m}, s_{y_m}, s_{z_m})$, we can combine this with

Lemma 7.2.8 to get that

$$r_{a_m} = s_{a_m} = p^{mjx}_{max} - p^{mjy}_{max} = p^{mmb}_{mab} - p^{mmc}_{mab}$$

and that $p^{mmb}_{maa} = p^{mmc}_{maa}$ by Lemma 7.1.2. We can also note that Lemma 11.3.3 tells us that, in the case where Equation 11.3.5 is linear, it actually has all coefficients being 0. We can still form the quadratic in terms of b or c, however we can show these are the same quadratic, so we cannot solve for a unique value of B_{yx}.

We can however form a different quadratic using the eigenvalue equation for $r^2_{b_m}$ from Corollary 4.2.9 applied to the m-neighbourhood. This gives

$$r^2_{b_m} = p^{mma}_{mbb} r_{a_m} + p^{mmb}_{mbb} r_{b_m} + p^{mmc}_{mbb} r_{c_m} + p^{m}_{mb}$$

Using this, and the equivalent one for $r^2_{c_m}$, we can show that this case can't occur.

Lemma 11.3.4 *Suppose the m-neighbourhood is such that it has two eigenvalue triples, $(r_{x_m}, r_{y_m}, r_{z_m})$ and $(s_{x_m}, s_{y_m}, s_{z_m})$ in the Eigenvector case in j. Then if there exists a colour a and distinct colours x, y, z such that $p^{mjy}_{max} = p^{mjz}_{max}$, we must have*

$$p^{mmb}_{mbb} - p^{mmc}_{mbb} = p^{mjx}_{mbx} + p^{mjy}_{mby} - p^{mjz}_{mbx} - p^{mjz}_{mby}$$

Proof. This begins from where Lemma 11.3.3 ends. Suppose we do not have $(r_{x_m}, r_{y_m}, r_{z_m}) = (s_{x_m}, s_{y_m}, s_{z_m})$. Then we know by Lemma 11.3.3 that we actually have $p^{mjy}_{max} = p^{mjz}_{max}$ for all distinct x, y, z. We also know by Lemma 11.3.3 that for all distinct x, y, z we can't have $p^{mjy}_{mbx} = p^{mjz}_{mbx}$ or $p^{mjy}_{mcx} = p^{mjz}_{mcx}$, as then we would get $r_{b_m} = s_{b_m}$ or $r_{c_m} = s_{c_m}$, respectively. So by the equation $0 = 1 + r_{a_m} + r_{b_m} + r_{c_m}$, we could determine that in fact the third eigenvalue was also equal, contradicting $(r_{x_m}, r_{y_m}, r_{z_m}) \neq (s_{x_m}, s_{y_m}, s_{z_m})$.

This means that we can form the quadratic from Equation 11.3.5 in b, i.e.

$$0 = (p^{mjy}_{mbx} - p^{mjz}_{mbx})B^2_{yx} + (p^{mjx}_{mbx} + p^{mjz}_{mby} - p^{mjz}_{mbx} - p^{mjy}_{mby})B_{yx} - (p^{mjx}_{mby} - p^{mjz}_{mby}) \quad (11.3.6)$$

Next we consider the eigenvalue equation for $r_{b_m}^2$.

$$\begin{aligned} r_{b_m}^2 &= p_{mbb}^{mma} r_{a_m} + p_{mbb}^{mmb} r_{b_m} + p_{mbb}^{mmc} r_{c_m} + p_{mb}^{m} \\ &= (p_{mbb}^{mma} - p_{mbb}^{mmc}) r_{a_m} + (p_{mbb}^{mmb} - p_{mbb}^{mmc}) r_{b_m} - p_{mbb}^{mmc} + p_{mb}^{m} \end{aligned}$$

Now inputting the value of the eigenvalues in terms of B_{yx} gives us

$$\begin{aligned} 0 =&((p_{mbx}^{mjx} - p_{mbx}^{mjz}) + B_{yx}(p_{mbx}^{mjy} - p_{mbx}^{mjz}))^2 - \\ &B_{yx}(p_{mbb}^{mmb} - p_{mbb}^{mmc})(p_{mbx}^{mjy} - p_{mbx}^{mjz}) - (p_{mbb}^{mma} - p_{mbb}^{mmc})(p_{max}^{mjx} - p_{max}^{mjy}) - \\ &(p_{mbb}^{mmb} - p_{mbb}^{mmc})(p_{mbx}^{mjx} - p_{mbx}^{mjz}) + p_{mbb}^{mmc} - p_{mb}^{m} \\ 0 =&B_{yx}^2(p_{mbx}^{mjy} - p_{mbx}^{mjz})^2 + \\ &B_{yx}(2(p_{mbx}^{mjy} - p_{mbx}^{mjz})(p_{mbx}^{mjx} - p_{mbx}^{mjz}) - (p_{mbb}^{mmb} - p_{mbb}^{mmc})(p_{mbx}^{mjy} - p_{mbx}^{mjz})) + \\ &(p_{mbx}^{mjx} - p_{mbx}^{mjz})^2 - (p_{mbb}^{mma} - p_{mbb}^{mmc})(p_{max}^{mjx} - p_{max}^{mjy}) - \\ &(p_{mbb}^{mmb} - p_{mbb}^{mmc})(p_{mbx}^{mjx} - p_{mbx}^{mjz}) + p_{mbb}^{mmc} - p_{mb}^{m} \end{aligned}$$

Now we can multiply the quadratic 11.3.6 by $(p_{mbx}^{mjy} - p_{mbx}^{mjz})$ and subtract it to get a linear equation in B_{yx}. The coefficient of B_{yx} in this equation is crucial – if it is non-zero then we can solve for B_{yx} and determine that $(r_{x_m}, r_{y_m}, r_{z_m}) = (s_{x_m}, s_{y_m}, s_{z_m})$, hence it must be zero. This gives us

$$\begin{aligned} &2(p_{mbx}^{mjy} - p_{mbx}^{mjz})(p_{mbx}^{mjx} - p_{mbx}^{mjz}) - (p_{mbb}^{mmb} - p_{mbb}^{mmc})(p_{mbx}^{mjy} - p_{mbx}^{mjz}) \\ &= (p_{mbx}^{mjy} - p_{mbx}^{mjz})(p_{mbx}^{mjx} + p_{mby}^{mjz} - p_{mbx}^{mjz} - p_{mby}^{mjy}) \end{aligned}$$

As we've already determined at the start of the proof that $p_{mbx}^{mjy} \neq p_{mbx}^{mjz}$, we can cancel these to give the equation

$$\begin{aligned} 2p_{mbx}^{mjx} - 2p_{mbx}^{mjz} - p_{mbb}^{mmb} + p_{mbb}^{mmc} &= p_{mbx}^{mjx} + p_{mby}^{mjz} - p_{mbx}^{mjz} - p_{mby}^{mjy} \\ p_{mbx}^{mjx} - p_{mbx}^{mjz} - p_{mbb}^{mmb} + p_{mbb}^{mmc} &= p_{mby}^{mjz} - p_{mby}^{mjy} \\ p_{mbb}^{mmb} - p_{mbb}^{mmc} &= p_{mbx}^{mjx} + p_{mby}^{mjy} - p_{mbx}^{mjz} - p_{mby}^{mjz} \end{aligned}$$

□

11.4 Putting it all together

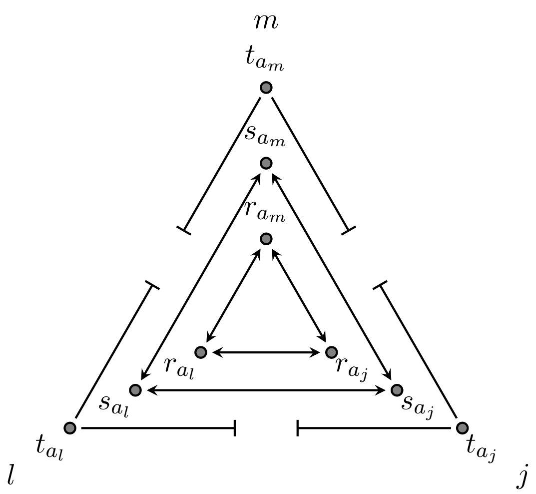
In this section we will work to bring together all that has been discussed over the previous chapters, and conclude with a meaningful theorem about how the eigenvectors of the neighbourhoods must interact. We will also talk about what this means for the intersection numbers.

First coupling together Lemma 11.2.1, Theorem 11.1.2 and Lemma 11.3.2 tells us:

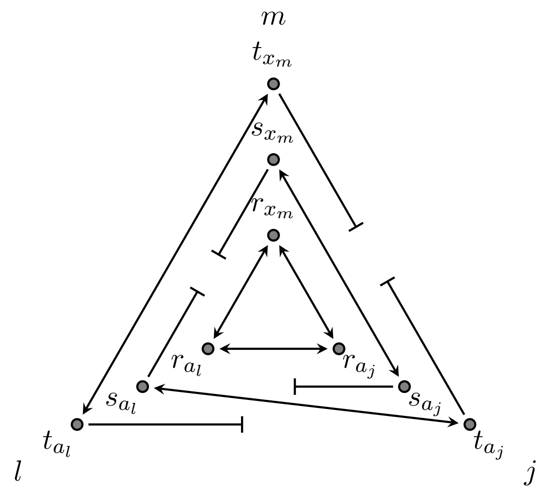
Lemma 11.4.1 *Suppose $p^m_{jj} \neq 0$ and the m-neighbourhood has three distinct eigenvalue triples. Then two eigenvalue triples are in the eigenvector case in j and one is in the 0 case in j.*

We can now list the possible diagrams that arise from supposing each neighbourhood is 3-coloured.

Case 1:



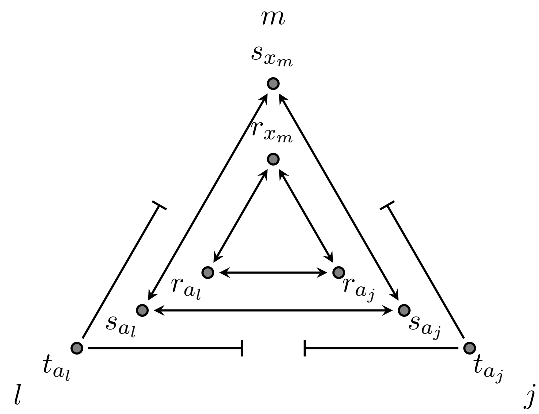
Case 2:



So there are only two possibilities and both contain at least one corresponding eigenvalue triangle.

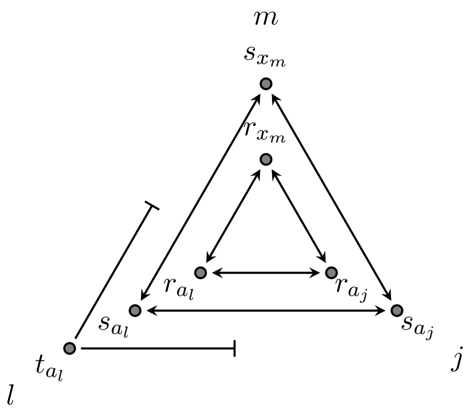
We can also now fully describe when $p^m_{mm} = 0$ and $p^m_{jj} \neq 0$ in just one case.

Case 3:

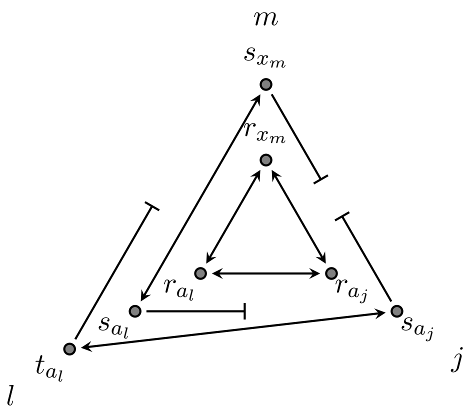


And there are a further two cases when $p^m_{mm} = p^j_{jj} = 0$.

Case 4:

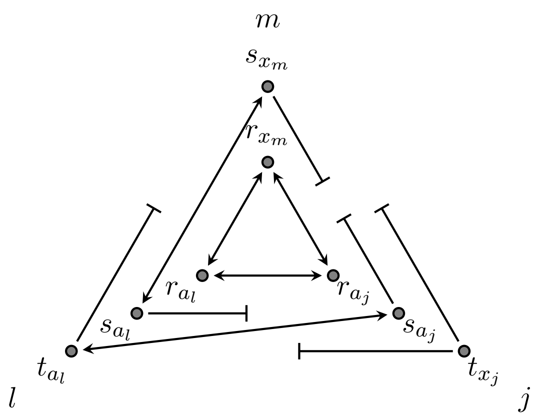


Case 5:



Note that case 5 also covers the $p^j_{jj} = 0$ case from Theorem 10.2.13. There is one more case which also stems from this Theorem and so we'll list it for completeness.

Case 6:



For ease of reference we state this as a Theorem:

Theorem 11.4.2 *Suppose* Γ *is a finite primitive 3-coloured 3-regular graph. Then we have one of the following scenarios:*

- $p^x_{yz} \neq 0$ *for all colours* x, y, z *and the eigenvalue triples of the neighbourhoods are case 1 or case 2.*
- *For some colour* m, $p^m_{mm} = 0$, *for all other combinations of colours* x, y, z, $p^x_{yz} \neq 0$, *and the eigenvalue triples of the neighbourhoods are case 3.*
- *For some distinct colours* m *and* j, $p^m_{mm} = p^j_{jj} = 0$, *for all other combinations of colours* x, y, z, $p^x_{yz} \neq 0$, *and the eigenvalue triples of the neighbourhoods are case 4 or 5.*
- *For some distinct colours* m *and* j, $p^m_{jj} = p^j_{jj} = 0$, *for all other combinations of colours* x, y, z, $p^x_{yz} \neq 0$, *and the eigenvalue triples of the neighbourhoods are case 5.*
- *For some distinct colours* m *and* j, $p^m_{jj} = 0$, *for all other combinations of colours* x, y, z, $p^x_{yz} \neq 0$, *and the eigenvalue triples of the neighbourhoods are case 6.*
- Γ *is the Tricolour Heptagon.*
- Γ *is the Tricolour Clebsch Graph.*

11.5 M.e.c Implications

To bring this thesis around full circle, we look at the implications of Theorem 11.4.2 with regards to m.e.cs.

The most important case and the original motivation was to show that there cannot be a m.e.c with ultraproduct elementarily equivalent to the universal homogeneous 3-coloured graph, which we'll denote by $\mathcal{R}_3$. We can see that if a m.e.c $\mathcal{C}$ has an ultraproduct that was elementarily equivalent to $\mathcal{R}_3$ then for some $n > 3$ any $M \in \mathcal{C}$ sufficiently large with respect to n, will be n-regular and such that $p^x_{yz} \neq 0$ for any colours x, y and z. Hence from Theorem 11.4.2, we know it must be in case 1 or case 2. Further, its neighbourhoods will be $n-1$-regular, and in a large enough member of $\mathcal{C}$ we can assume $p^{mmx}_{myz} \neq 0$ for any m, x, y, z. Therefore the neighbourhoods must also be finite, $n-1$-regular 3-coloured graphs of case 1 or case 2. We can repeat this process indefinitely, as this holds for arbitrarily large n.

It is my belief that issues will arise from the fact that there are double eigenvector cases in each neighbourhood. I have not had the chance to go through all the possible consequences of Equation 11.3.5, but I would guess that, because we have it in distinct colours a, b and c, at least one colour must have the quadratic reduce to 0 i.e. $p^{mjy}_{max} = p^{mjz}_{max}$. This would bring about the conclusion of Lemma 11.3.3. Now in case 1 and case 2, having $r_{a_m} = s_{a_m}$ for some a would cause problems, especially if you had it due to both j and l. In case 2, it forces the neighbourhood to become almost amorphic (strongly regular in two colours), and in both cases we would get some equalities between intersection numbers in j and l.

Other results are slightly easier to apply. We can note an easy application of Theorem 7.6.20.

Theorem 11.5.1 *Suppose $\mathcal{L} = \{R, G, B\}$ with R, G, B colours (binary, symmetric, irreflexive relations), $\{RRG, RGG\} \subseteq \Delta$, and M is a primitive unstable homogeneous Δ-free $\mathcal{L}$-structure. Then there does not exist a m.e.c with ultraproduct elementarily equivalent to M.*

Proof. Suppose for a contradiction there exists a m.e.c $\mathcal{C}$ with ultraproduct elementarily equivalent to M. Well then by Lemma 2.5.4 and Lemma 2.5.5, we know that any sufficiently large member $D \in \mathcal{C}$ is a finite primitive 3-regular 3-coloured graph with no RRG or RGG triangles, i.e. $p^R_{RG} = p^R_{GG} = 0$. But no such structure exists by Theorem 7.6.20. □

We know such structures exists as, recalling the notation from section 3.4, just taking $\Delta = \{RRG, RGG\}$ means the class $\mathcal{C}(\Delta)$ has free amalgamation (by solving the amalgamation problem with B).

We now look at the repercussions of Theorem 9.3.3 in terms of m.e.c which are very similar.

Theorem 11.5.2 *Suppose $\mathcal{L} = \{R, G, B\}$ with R, G, B colours, $\{RRG, RRR\} \subseteq \Delta$, and M is a primitive unstable homogeneous Δ-free $\mathcal{L}$-structure. Then there does not exist a* *m.e.c with ultraproduct elementarily equivalent to M.*

Proof. Suppose for a contradiction there exists a m.e.c $\mathcal{C}$ with ultraproduct elementarily equivalent to M. Well then by Lemma 2.5.4 and Lemma 2.5.5, we know that any sufficiently large member $D \in \mathcal{C}$ is a finite primitive 3-regular 3-coloured graph with no RRG or RRR triangles, i.e. $p^R_{RG} = p^R_{RR} = 0$. But then it has to be the Tricolour Heptagon by Theorem 7.6.20. Hence this will not satisfy the required axioms to be a m.e.c with ultraproduct elementarily equivalent to M. □

Again, we can see such Fraïssé classes exist, as just taking $\Delta = \{RRG, RRR\}$ means $\mathcal{C}(\Delta)$ has free amalgamation (by solving the amalgamation problem with B).

We can also get a similar analogous result for Theorem 10.1.1:

Theorem 11.5.3 *Suppose $\mathcal{L} = \{R, G, B\}$ with R, G, B colours, $\{RGB\} \subseteq \Delta$, and M is a primitive unstable homogeneous Δ-free $\mathcal{L}$-structure. Then there does not exist a m.e.c with ultraproduct elementarily equivalent to M.*

The proof is identical to that of Theorem 11.5.1, with RGB taking the place of RRG and RGG.

Recall Theorem 3.2.4. Using this result, and the work of section 3.4, we now have enough to show that there does not exist a m.e.c limit for any unstable homogeneous graph determined only by forbidden triangles.

Theorem 11.5.4 *Suppose $\mathcal{L} = \{R, G, B\}$ with R, G, B colours, Δ is a non-empty set of triangles such that $\mathcal{C}(\Delta)$ is a Fraïssé class with semi-free amalgamation, and $M(\Delta)$ is the Fraïssé limit of $\mathcal{C}(\Delta)$. Then if $M(\Delta)$ is unstable, there does not exist a m.e.c with ultraproduct elementarily equivalent to $M(\Delta)$.*

Proof. To complete this proof we need to rule out all possible Δ. Suppose for a contradiction that $M(\Delta)$ is unstable and a m.e.c limit. We also know it has to be primitive by Theorem 6.1.8. We can see from Theorems 11.5.1, 11.5.2, and 11.5.3 that we cannot have any sets of the form $\{RRG, RGG\}, \{RRG, RRR\}$ or $\{RGB\}$ in Δ. Further by Theorem 3.4.10 and Theorem 3.4.7, we know that Δ is not of the form $\{BRR, GGR, RRR\}$ and that

Δ is such that $\mathcal{C}(\Delta)$ has free amalgamation. By the fact that the Fraïssé limit is primitive, we can see that any sets of the form $\{RRG, RRB\}$ are not in Δ either (as they form an equivalence relation $R^=$). This leaves with not many options. If we start with $RRG \in \Delta$ then we cannot include a triangle that restricts B in Δ as this would mean we wouldn't have free amalgamation. Hence the only other triangle we can include in Δ is GGG. Similarly if we start with $RRR \in \Delta$ then we can add either RGG or GGG, but no others. Note that $\{RRR, RGG\}$ is the same as $\{RRG, GGG\}$ so we ignore it. This means the only options for Δ are

i) $\Delta_1 = \{RRG\}$

ii) $\Delta_2 = \{RRG, GGG\}$

iii) $\Delta_3 = \{RRR\}$

iv) $\Delta_4 = \{RRR, GGG\}$

We claim that the Fraïssé limits of the classes these define all cannot be m.e.c limits by Theorem 3.2.4.

Indeed, consider $M(\Delta_1)$, then if you name a point $x \in M(\Delta_1)$, the red-neighbourhood of x is a two-coloured graph with no restrictions, and therefore isomorphic to the random graph. Hence the random graph is canonically embedded in $M(\Delta_1)$ over x and therefore $M(\Delta_1)$ cannot be a m.e.c limit by Theorem 3.1.7 and Theorem 3.2.4. We see that $M(\Delta_2)$ and $M(\Delta_3)$, follow by the exact same argument (in $M(\Delta_3)$ the random graph is coloured by G and B but this makes no difference).

$M(\Delta_4)$ is a little different, the same process means that the red-neighbourhood of x, for some $x \in M(\Delta_4)$, is instead the Random triangle-free graph (with G as edges and B as non-edges) canonically embedded in $M(\Delta_4)$ over x. However this still has no m.e.c limit by Theorem 3.1.7, and therefore we can apply Theorem 3.2.4 one last time to show that $M(\Delta_4)$ has no m.e.c limit, and hence no such Δ exists. □

Appendix A

Completeness of Cherlin's List with $\mathcal{L} = \{R, G, B\}$

This appendix is dedicated to the proving of the following theorem.

Theorem 3.4.10 *Let $\mathcal{L} = \{R, G, B\}$ be a symmetric, irreflexive, binary, relational language and suppose M is a primitive universal homogeneous L-structure with semi-free, but not free, amalgamation determined by a set of forbidden triangles. Then M is isomorphic to $M(\Delta)$ with*

$$\Delta = \{RBB,\ GGB,\ BBB\}$$

A.1 Set up for a General Coloured Language

Although the aim is to prove this result for a 3-coloured language, we can set up the notation for a general n-coloured language, to make it easier to generalise. Therefore suppose we are working over a binary relation language $\mathcal{L} = \{A_1, \ldots A_n\}$, where A_i is symmetric and irreflexive. We will be using the notation from Section 3.4.

The aim is to reduce the amalgamation problem for $\mathcal{C}(\Delta)$ to a particular set of finite $\mathcal{L}$-structures. These structures will not necessarily be in $\mathcal{C}(\Delta)$ and in actuality we shall see in Theorem A.1.3 that the problems occur when they are. In a rough sense we set up these

$\mathcal{L}$-structures around two distinct points b and c, in such a way that there cannot be a Δ-free way of connecting b and c. We do this by including points a_i as intermediaries between them, connected in such a way to b and c as to force any edge between b and c to not be a particular colour by some restricted triangle in Δ. If we do this for every colour then (b, c) can't be an edge of any colour. Note we will be able do this for every colour if we do not have free amalgamation.

More formally, fix a set of forbidden triangles Δ. Suppose for a colour $A_i \in \mathcal{L}$, we define the set $\mathcal{A}_i$ as follows. Fix a set of forbidden triangles Δ and consider a triangle $\delta \in \Delta$. If one of the edges in δ is of colour A_i then we remove this edge and add the other two edges into $\mathcal{A}_i$ as an unordered pair i.e. $\mathcal{A}_i = \{\{E_2, E_3\} \ : \ A_1E_2E_3 \in \Delta\}$.

We first construct an $\mathcal{L}$-structure D in the following way. Define a function h that takes as its input an edge and returns the colour of the edge. Start with the vertices b and c and for each colour A_i add a vertex a_i and edges (b, a_i) and (a_i, c) such that $\{h(b, a_i), h(a_i, c)\} \in \mathcal{A}_i$. Now if we have vertices a_j and a_k such that $h(b, a_j) = h(b, a_k)$ and $h(a_j, c) = h(a_k, c)$, we remove the vertex a_k. This leaves us with a finite structure with less than or equal to $n + 2$ points. We leave the colours of the edges between the a_i's unassigned (these will be 'filled in' later). Thus we have not defined an $\mathcal{L}$-structure D, but a family of $\mathcal{L}$-structures.

Definition A.1.1. We define the *Discriminatory class*, $\mathcal{D}(\Delta)$, as the class of all the possible $\mathcal{L}$-structures D as above for a specific Δ. We will call any specific $D \in \mathcal{D}(\Delta)$ a *Discriminatory Structure*.

Now a rather confusing thing to mention, is that what we want is that whether or not $\mathcal{C}(\Delta)$ is a Fraïssé class should hinge on whether or not every element of $\mathcal{D}(\Delta)$ can be completed in a Δ-free way. This leads to the rather paradoxical seeming definition.

Definition A.1.2. We say a discriminatory structure is *completeable in a Δ-free way* if we can assign colours from L to the edges (a_i, a_j) without creating any triangles in Δ.
A Discriminatory Structure is called *flawed* if it cannot be completed in a Δ-free way. $\mathcal{D}(\Delta)$ is called *flawed* if for every $D \in \mathcal{D}(\Delta)$, D is flawed.

We then get the crucial theorem of the classification:

Theorem A.1.3 *Say Δ is a set of n-coloured triangles and $\mathcal{C}(\Delta)$ is the class of all finite n-coloured Δ-free $\mathcal{L}$-structures. Then $\mathcal{C}(\Delta)$ has the strong amalgamation property iff $\mathcal{D}(\Delta)$ is flawed.*

Proof. ($\Rightarrow$) Suppose that $\mathcal{D}(\Delta)$ is not flawed. Then there is a $D \in \mathcal{D}(\Delta)$ such that $D = \{a_1, \ldots, a_j, b, c\}$ is not flawed, for some $j \leq n$. We view D a having been completed, i.e. with colours assigned in Δ-free way to pairs $\{a_i, a_j\}$ for $i \neq j$ (but not to $\{b, c\}$). Therefore take $A = \{a_1, \ldots, a_j\}$ with $B = \{a_1, \ldots, a_j, b\}$ and $C = \{a_1, \ldots, a_j, c\}$, with the embeddings β and γ from A to B and A to C respectively. As D is not flawed, A, B and C are all known to be Δ-free substructures, and hence are structures in $\mathcal{C}(\Delta)$. However, in D, the edge (b, c) cannot be coloured as for all $i \leq j$ the edge pair (b, a_i), (c, a_i) eliminates a set of colours as possibilities, which ultimately partition the set of all colours in $\mathcal{L}$. Therefore no amalgam of B and C over A exists in $\mathcal{C}(\Delta)$, except by possibly amalgamating b and c. Furthermore any strong amalgamation of B and C over A must contain D as $D = B \cup C$. Hence $\mathcal{C}(\Delta)$ does not have the strong amalgamation property.

($\Leftarrow$) Suppose $\mathcal{D}(\Delta)$ is flawed. Consider $A, B_1, B_2 \in \mathcal{C}(\Delta)$ with embeddings $\alpha_1 : A \to B_1$ and $\alpha_2 : A \to B_2$. Then we can identify the image of A in each of these structures such that $\alpha_1(A) = \alpha_2(A)$ and $B_1 \cap B_2 = \alpha_2(A)$. Then define $C = B_1 \cup B_2$. Now all the internal edges of B_1 and B_2 are preserved within C, hence for this to solve the amalgamation problem, all we need to check is the new edges, i.e. the ones between points in $B_1 \backslash \alpha_2(A)$ and $B_2 \backslash \alpha_2(A)$. So take $b_1 \in B_1 \backslash \alpha_2(A)$ and $b_2 \in B_2 \backslash \alpha_2(A)$.

Suppose for a contradiction that the edge (b_1, b_2) cannot be coloured in a Δ-free way. This means that there exist pairs of edges between b_1 and b_2 and, for each $i = 1, \ldots, n$, some a_j for $j \in \{1, \ldots, n\}$ such that $\{h(b_1, a_j), h(b_2, a_j)\} \in \mathcal{A}_i$. Let I be the index set of the j's, then $|I| \leq n$. Therefore, the induced substructure over $\{b_1, b_2, a_1, \ldots, a_{|I|}\}$ is isomorphic to a structure that is in $\mathcal{D}(\Delta)$. However as $\mathcal{D}(\Delta)$ is flawed, this structure cannot exist in Δ-free way. This is a contradiction and therefore such a_i cannot exist. Using this, we can colour (b_1, b_2). Using the same argument, we can assign colour to (b, b_2) for all $b \in B_1 \backslash \alpha_2(A)$. Then, we can treat b_2 as if it were part of $\alpha_2(A)$ and repeat the process for all other elements of B_2. Including all these edges into our C, we get a legitimate Δ-free strong amalgamation of B_1 and B_2 over A. Therefore $\mathcal{C}(\Delta)$ has the strong amalgamation property. $\square$

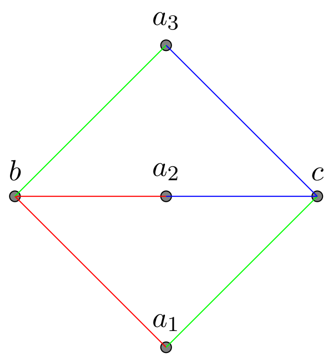
We can note that $\mathcal{C}(\Delta)$ will have the free amalgamation property if and only if for some colour A_i, $\mathcal{A}_i$ is empty.

A.2 Three colours

We shall now assume that $\mathcal{L} = \{R, G, B\}$, with the aim of proving Theorem 3.4.10

Lemma A.2.1 *Over the language $\mathcal{L} = \{R, G, B\}$, if $|\Delta| = 1$, then $\mathcal{C}(\Delta)$ has free amalgamation or $\mathcal{D}(\Delta)$ is not flawed.*

Proof. It is clear that if all the colours are not banned in some way, then we can simply complete the amalgamation by filling in all the edges with that colour, giving us free amalgamation. That leaves us the sole option of $\Delta = \{RGB\}$. However by then examining the discrimination class we can find the discriminatory structure

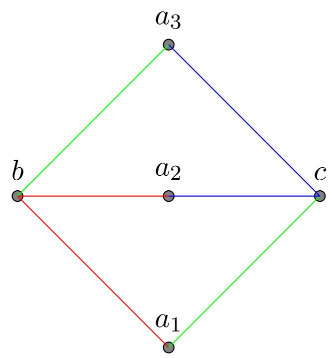


This can be completed in a Δ-free way and hence $\mathcal{D}(\Delta)$ is not flawed. □

We see that this discriminatory structure remains a problem whenever $RGB \in \Delta$ and because it can be completed in multiple different manners, it requires the inclusion of a few more triangles to make it flawed. Recall that the relation $X^=$ is $X \cup =$.

Corollary A.2.2 *Suppose $\mathcal{L} = \{R, G, B\}$ and $RGB \in \Delta$. Then either $\mathcal{C}(\Delta)$ has free amalgamation, or it has an imprimitive Fraïssé limit or $\mathcal{D}(\Delta)$ is not flawed.*

Proof. Suppose $RGB \in \Delta$ and consider the discriminatory structure,



Note that we can also permute the colours to create 2 more similar structures. Let S be the set of all completions of these three structures in an $\{RGB\}$-free way. Then S will have size 9. To make these structures flawed we would need to add triangles into Δ such that every member of S includes at least one of these triangles. We want to find the most efficient way to do this. It is not possible with only 1 triangle as no triangle is common to all completions. If we focus only on triangles of the form XXY we see that every completion contains all but 2 of them, and this pair of absentees is unique to the structure. Therefore to cover all of these structures with two triangles it satisfies to find two of the form XXY that are not an absent pair. As we have 15 options, and only 9 pairs in our completions, this gives us 6 possibilities. However each of these possibilities is of the form $\{XXY, ZZY\}$ or $\{XXY, XXZ\}$; the latter will mean that $X^=$ will form a nontrivial equivalence relation and the former, when combined with the fact RGB is in Δ, will mean $X^= \cup Z^=$ is an equivalence relation. A Fraïssé limit of such a class would therefore be imprimitive.

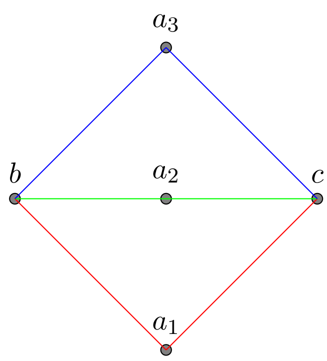
Hence $|\Delta| > 3$. However there is no way to include three triangles of the form XXY and RGB in Δ without forming a definable non-trivial equivalence relation, as before. □

Hence we can now assume that RGB is not in Δ. This immediately leads to the result

Corollary A.2.3 *Suppose $\mathcal{L} = \{R, G, B\}$ and $\{RRR, GGG, BBB\} \subseteq \Delta$. Then either $\mathcal{C}(\Delta)$ has an imprimitive Fraïssé limit or $\mathcal{D}(\Delta)$ is not flawed.*

Proof. First note that we cannot have free amalgamation. Then by Corollary A.2.2, if

$RGB \in \Delta$ we are done, so suppose it is not. In $\mathcal{D}(\Delta)$ we have the structure



Now there exists a completion of this such that (a_1, a_2) is Blue, (a_2, a_3) is Red and (a_3, a_1) is Green. This completion is entirely made up of only the triangle RGB, and so cannot be flawed without $RGB \in \Delta$. $\square$

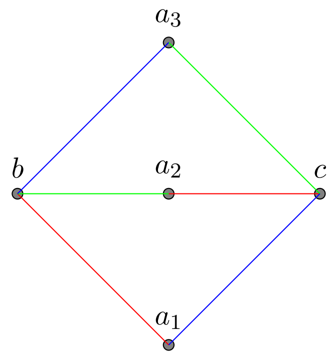
Any Δ cannot contain more than three triangles of the form XXY without necessitating the forming of a quantifier-free definable equivalence relation. We see that there are only two non-isomorphic ways in which three of the XXY triangles might be in Δ without forming a quantifier-free definable equivalence relation. This leads to

Corollary A.2.4
- *Suppose* $\mathcal{L} = \{R, G, B\}$ *and* $|\Delta| > 5$. *Then either* $\mathcal{C}(\Delta)$ *has an imprimitive Fraïssé limit or* $\mathcal{D}(\Delta)$ *is not flawed.*
- *If* $|\Delta| = 5$, *then either* $\Delta = \{RRR, GGG, RRG, GGB, BBR\}$ *or* $\Delta = \{RRR, GGG, RRG, GGR, BBR\}$, *or* $\mathcal{C}(\Delta)$ *has an imprimitive Fraïssé limit or* $\mathcal{D}(\Delta)$ *is not flawed.*

However we can also rule out the only combinations of three triangles of the form XXY that don't form equivalence relations

Lemma A.2.5 *Suppose* $\mathcal{L} = \{R, G, B\}$ *and* $\{RRG, GGB, BBR\} \subseteq \Delta$. *Then either* $\mathcal{C}(\Delta)$ *has an imprimitive Fraïssé limit or* $\mathcal{D}(\Delta)$ *is not flawed.*

Proof. Suppose that $\{RRG, GGB, BBR\} \subseteq \Delta$. Then

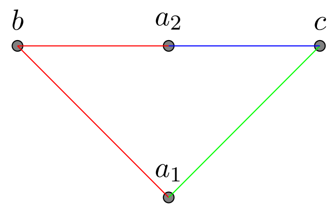


is a discriminatory structure. We can complete it by making (a_1, a_2) green, (a_2, a_3) blue and (a_3, a_1) red. Therefore the only triangles in this completion are RGB, GGR, RRB and BBG, none of which we can include in Δ without inducing a definable non-trivial equivalence relation, or other non-flawed discriminatory structures. □

And

Lemma A.2.6 *Suppose $\mathcal{L} = \{R, G, B\}$ and $\{RRG, GGR, BBR\} \subseteq \Delta$, then either $\mathcal{C}(\Delta)$ has an imprimitive Fraïssé limit or $\mathcal{D}(\Delta)$ is not flawed.*

Proof. Suppose that $\{RRG, GGR, BBR\} \subseteq \Delta$. Then



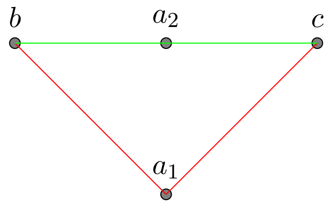
is a discriminatory structure. We can complete it by making (a_1, a_2) blue, and therefore the only triangles in this completion are RRB and GBB, neither of which we can include in Δ without inducing a definable non-trivial equivalence relation. □

Corollary A.2.7 *Suppose $\mathcal{L} = \{R, G, B\}$ and $|\Delta| = 5$, then either $\mathcal{C}(\Delta)$ has an imprimitive Fraïssé limit or $\mathcal{D}(\Delta)$ is not flawed.*

We also know now that if $|\Delta| \geq 3$, then Δ contains a triple XXX and so we focus on possibilities for such Δ.

Lemma A.2.8 *Suppose $\mathcal{L} = \{R, G, B\}$ and $\{RRR, GGG, RRB\} \subseteq \Delta$. Then either $\mathcal{C}(\Delta)$ has an imprimitive Fraïssé limit or $\mathcal{D}(\Delta)$ is not flawed.*

Proof. If $\{RRR, GGG, RRB\} \subseteq \Delta$, we get the following discriminatory structure

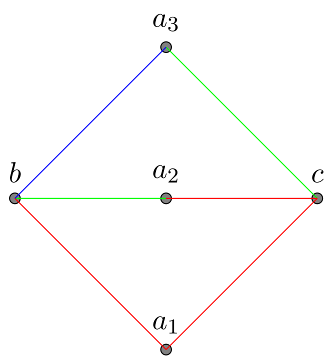


This can be completed by colouring (a_1, a_2) blue, and hence the only way to make this flawed is with $RGB \in \Delta$. But by Corollary A.2.2 this can't happen without $\mathcal{D}(\Delta)$ being flawed or allowing a non-trivial equivalence relation. □

We can similarly rule out another important combination

Lemma A.2.9 *Suppose $\mathcal{L} = \{R, G, B\}$ and $\{RRR, GGR, BBG\} \subseteq \Delta$. Then either $\mathcal{C}(\Delta)$ has an imprimitive Fraïssé limit or $\mathcal{D}(\Delta)$ is not flawed.*

Proof. If $\{RRR, GGR, BBG\} \subseteq \Delta$, we get the following discriminatory structure



This can be completed by colouring (a_1, a_2) blue, (a_2, a_3) red and (a_1, a_3) red. Hence the only way to make this flawed is by having RRB, RRG or RBG in Δ. But we can't have $RRB \in \Delta$ without the consequences of Lemma A.2.5, we can't have RRG without the consequences of Lemma A.2.6, and we can't have RBG without the consequences of Corollary A.2.2. □

Corollary A.2.10 *Suppose* $\mathcal{L} = \{R, G, B\}$ *and* $|\Delta| = 4$. *Then either* $\mathcal{C}(\Delta)$ *has free amalgamation, has an imprimitive Fraïssé limit or* $\mathcal{D}(\Delta)$ *is not flawed.*

Proof. Suppose $|\Delta| = 4$ and $\mathcal{C}(\Delta)$ doesn't have free amalgamation or an imprimitive Fraïssé Limit. Then by Corollary A.2.2, $RGB \notin \Delta$, by Lemmas A.2.5 and A.2.6 we have a maximum of two triangles of the form XXY, and by Corollary A.2.3, we have a maximum of 2 monochrome triangles. Therefore the options for Δ are

i) $\{RRR, GGG, RRB, GGB\}$

ii) $\{RRR, GGG, RRB, BBR\}$

iii) $\{RRR, GGG, RRG, GGB\}$

iv) $\{RRR, GGG, RRG, BBR\}$

However i), ii) and iii) come under the jurisdiction of Lemma A.2.8 and iv) is dealt with by Lemma A.2.9. □

We will now rule out small Δs too

Lemma A.2.11 *Suppose* $\mathcal{L} = \{R, G, B\}$ *and* $|\Delta| = 2$. *Then either* $\mathcal{C}(\Delta)$ *has free amalgamation, has an imprimitive Fraïssé limit or* $\mathcal{D}(\Delta)$ *is not flawed.*

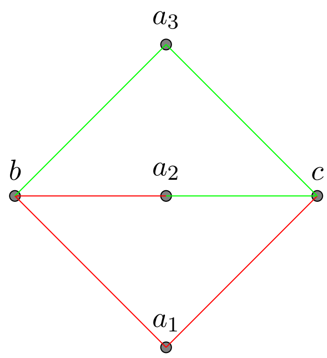
Proof. To avoid free amalgamation each colour must appear in our triangles at least once. To avoid a definable non-trivial equivalence relation we have the following possibilities:

i) $\Delta = \{RRG, GGB\}$

ii) $\Delta = \{RRR, GGB\}$

iii) $\Delta = \{RRG, GBB\}$

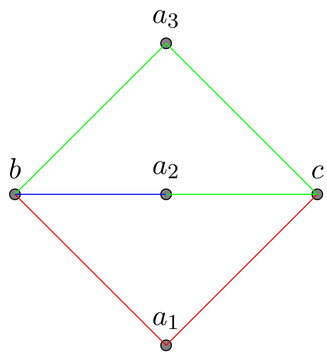
Say we have i). We can see however the discriminatory structure



(A.2.1)

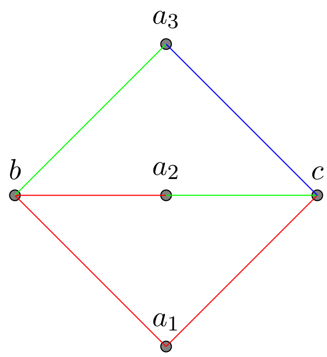
is not flawed in the determining class.

Now suppose we have ii). Then we have a non-flawed discriminatory structure:



(A.2.2)

This leaves just case iii). However we can find the discriminatory structure



(A.2.3)

which is not flawed. Therefore, $|\Delta| \neq 2$ □

As has been the case in most of these proofs, these discriminatory structures can be pushed further and used to rule out certain Δ that embed them.

Corollary A.2.12 *Suppose $\mathcal{L} = \{R, G, B\}$ and $|\Delta| = 3$, and one of the following holds*

1. $\{RRR, GGB\} \subseteq \Delta$
2. $\{RRG, GBB\} \subseteq \Delta$

Then either $\mathcal{C}(\Delta)$ has an imprimitive Fraïssé limit or $\mathcal{D}(\Delta)$ is not flawed.

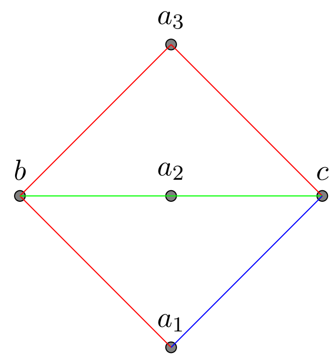
Proof. First start with $\{RRR, GGB\} \subseteq \Delta$ and look at the discriminatory structure A.2.2. This was completed by making (a_1, a_2) green, (a_2, a_3) red and (a_1, a_3) green, hence the only way to make it flawed is to include RGB or GGR in Δ. Now GGR cannot be included without forming a definable non-trivial equivalence relation, and RGB can't by Lemma A.2.2

Now we suppose $\{RRG, GBB\} \subseteq \Delta$ and look at the discriminatory structure A.2.3. This can be completed by making (a_1, a_2) blue, (a_2, a_3) green and (a_1, a_3) green. Now this completion includes triangles GGR, BGG, RRB and RGB. If RRB was in Δ we would end up with a non-trivial equivalence relation. Including GGR in Δ invokes the consequences of Lemma A.2.6, and if BGG were in Δ we would be in the domain of Lemma A.2.5. Then as always Corollary A.2.2 covers the inclusion of RGB. $\square$

We are now at the point where we know Δ has to be of size 3 and has to include at least one monochrome triangle XXX. It also cannot include all monochrome triangles, and if we suppose it has two there is only one possibility if it does not have free amalgamation. We shall consider this next

Lemma A.2.13 *Suppose $\mathcal{L} = \{R, G, B\}$ and $\Delta = \{RRR, GGG, RBB\}$, then $\mathcal{D}(\Delta)$ is not flawed.*

Proof. Simply note we have the discriminatory structure



This is not flawed as it can be completed in a Δ-free way by making (a_1, a_2) blue, (a_2, a_3) blue, and (a_1, a_3) green. □

Hence we have a maximum of one monochrome triangle in Δ.

You will now be relieved to hear we have enough to prove the theorem

Theorem 3.4.10 *Let $\mathcal{L} = \{R, G, B\}$ be a symmetric, irreflexive, binary, relational language and suppose M is a primitive universal homogeneous L-structure with semi-free, but not free, amalgamation determined by a set of forbidden triangles. Then M is isomorphic to $M(\Delta)$ with*

$$\Delta = \{RBB,\ GGB,\ BBB\}$$

Proof. For this we shall just bring together all the previous results. Suppose M is a universal homogeneous $\mathcal{L}$-structure without free amalgamation or a definable equivalence relation. As M does not have free amalgamation and is determined by forbidden triangles, we know that M is isomorphic to $M(\Delta)$ for some set of triangles Δ. Now by Theorem A.1.3 we know that $\mathcal{D}(\Delta)$ is flawed. Hence we can see by Lemmas A.2.1 and A.2.11, and Corollaries A.2.4, A.2.7 and A.2.10, that $|\Delta| = 3$. Now we know $RGB \notin \Delta$ by Corollary A.2.2. Further by Lemmas A.2.5 and A.2.6, Δ must contain a monochrome triangle. However by Lemmas A.2.3, A.2.8 and A.2.13, Δ can only contain a maximum of one monochrome triangle. Hence we have $BBB \in \Delta$ and two triangles of the form XXY. By Corollary A.2.12 part i) neither of these XXY can be without the colour B. To avoid free amalgamation or a non-trivial equivalence relation we have but one option, which is $\Delta = \{RBB,\ GGB,\ BBB\}$ $\square$

Appendix B

3-coloured Graph Notation

This is a list of the more novel and obscure notation that I commonly use throughout the work on 3-coloured graphs.

Notation	Meaning	Where is it defined?
$E^=$	The relation E union with equality	Definition 2.1.11
q^i_{jk}	The Krein Parameter	Equation 2.2.2
R, B, G	Binary, symmetric, irreflexive relations relating to the colours red, blue and green	Section 4.1
n	Number of vertices in the Graph	Section 4.1
A_R, A_B, A_G	Adjacency matrices of the red, blue, green edges	Section 4.1
r_i, s_i, t_i	Non-principal eigenvalues for the A_i	Section 4.1
p^i_{jl}	The double intersection number	Section 4.1
k_i	Number of i coloured edges incident at any vertex	Section 4.1
J	The all one matrix (of any dimensions)	Section 4.1
u	The all one vector (of any dimension)	Section 4.1
I	The identity matrix (of any square dimensions)	Section 4.1
E_i	The minimal idempotents of the Bose-Mesner algebra generated by the A_i	Theorem 2.2.2
N^i_{jl}	Adjacency matrix of i coloured edges from the l-neighbourhood to the j-neighbourhood	Section 4.1
$r_{i_j}, s_{i_j}, t_{i_j}$	Eigenvalues of N^i_{jj}	Section 4.1
p^{abc}_{ijk}	Triple intersection number	Section 4.1
δ_{ij}	Kronecker Delta	Classical
f, g, h	Multiplicities of r_i, s_i, t_i respectively	Equation 4.1.4
α_i, β_i, γ_i	Idempotent constants	Equation 4.1.4
D	A notable structural constant	Equation 4.1.5
r_x or (r_m, r_j, r_l)	Eigenvalue triple of r_m, r_j and r_l	Definition 7.1.10
$E(r_x)$ or $E(r_m, r_j, r_l)$	Eigenspace of the eigenvalue triple	Definition 7.1.10
r_{x_m}	Eigenvalue of N^x_{mm} for any x	Section 7.2
v_{r_m}	Eigenvector of N^x_{mm} with eigenvalue r_{x_m} for any x	Definition 7.2.2
$\mathfrak{D}^{xy}_{mj}$	Discriminant of equation 7.4.3 in x and y	Equation 7.4.5
μ_i	The multiplicity of an eigenvalue, which depends on i	Definition 8.1.1
$(\lambda_i)_j$	An eigenvalue of the graph, which depends on i and j	Definition 8.1.1
H_i	Characteristic matrices of an eigenspace	Equation 8.1.2
K_i, L_i, M_i	Breakdown of the H_i into neighbourhoods	Equation 8.1.3
$x_j(r)$, $y_j(r)$, $z_j(r)$	Constants defined via the eigenvalues of the neighbourhood	Equation 8.2.5
ϕ_i	x, y or z depending on i (in relation to the above constants)	Equation 8.2.5
$(\Pi_m)_i$	One of the K, L or M matrices, depending on m and i	Equation 9.2.1
B_{xy}	Constant used in Eigenvector case	Equation 11.2.2

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