CONGRUENCES RELATED TO HILBERT MODULAR FORMS OF INTEGER AND HALF-INTEGER WEIGHTS.

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In memory of my loving grandparents.

Abstract

Let K be a totally real quadratic field of narrow class number 1. In this thesis, we investigate congruences between Fourier coefficients of classical modular forms and then generalise these congruences to Hilbert modular forms of parallel weight over K. Given an odd prime p, we first prove mod p congruences between Fourier coefficients of integer weight ordinary Hilbert eigenforms that are at the same level but whose weights differ by an odd multiple of $(p-1)$. These eigenforms belong to *Hida's* p-adic family of eigenforms. We are then able to lift these congruences to congruences between Fourier coefficients of half-integer weight ordinary Hilbert eigenforms that are at the same level but whose weights differ by an odd multiple of $\frac{p-1}{2}$. As an example, we briefly work with the real quadratic field $K = \mathbb{Q}(\sqrt{5})$ and see the significance of Fourier coefficients of half-integer weight Hilbert modular forms and the related congruences.

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"Modular forms are functions on the complex plane that are inordinately symmetric.
They satisfy so many internal symmetries that their mere existence seem like accidents.
But they do exist."——Barry Mazur.

Contents

Introduction

One of the biggest breakthrough's of the 20th century in number theory was the proof of the complete *Shimura-Taniyama* conjecture (*Taylor, Wiles, Breuil, Conrad, Diamond, et al.*), now known as the Modularity theorem. The Modularity theorem asserts that:

Every elliptic curve over $\mathbb{Q}$ is modular.

Loosely speaking, if we take the solution set of the equation $y^2 = x^3 + ax + b$ where $a, b \in \mathbb{Q}$ and throw an additional point in this set, called the point at infinity, what we end up with is an elliptic curve over $\mathbb{Q}$, denoted $E/\mathbb{Q}$. For each prime p, we associate an arithmetic quantity $a_p(E)$ to $E/\mathbb{Q}$. This arithmetic data of $a_p(E)$'s is stored in an *Euler* product which determines the L-function $L(E,s)$ of an elliptic curve. The idea of modularity of elliptic curves involves linking arithmetic data in $L(E,s)$ to some set of bizarre complex functions showing a lot of symmetry. These complex functions are called modular forms. Vaguely speaking, modular forms are analytic functions that satisfy a certain type of functional equation and a growth condition. A functional equation is a type of relation which relates the value of a function at one point to another while the growth condition examines the bounds on the function as the input grows very large. The functional equation of modular forms relates their values at a complex point z to their values at some linear fractional transformations $\frac{az+b}{cz+d}$ for any integers a, b, c, d satisfying $(ad - bc) = 1$. This gives rise to a lot of symmetries! The functional equation of the modular forms makes them periodic in nature and hence they have a nice Fourier expansion. Their Fourier coefficients encode interesting arithmetic information and are highly intriguing to number theorists. As we will see, this thesis is essentially all about these Fourier coefficients and their mod p congruences. There are two fixed parameters, weight k and level N associated to each modular form which are precisely defined in Chapter 1 of this thesis.

We can now explain modularity of elliptic curves over $\mathbb{Q}$ a bit more precisely. Let $E/\mathbb{Q}$ be an elliptic curve of conductor N_E. The conductor is an invariant associated to $E/\mathbb{Q}$ that encodes informations about mod p reductions of $E/\mathbb{Q}$. We say $E/\mathbb{Q}$ is modular if there exists a newform[1] $f(z)$ of weight 2 and level N_E with Fourier coefficients $a_n(f)$ such that $a_p(E) = a_p(f)$ for all primes p. In other words, if we define the L-function of $f(z)$ using its Fourier coefficients as $L(f,s) = \sum a_n(f)n^{-s}$, then modularity of $E/\mathbb{Q}$ implies

$$L(E,s) = L(f,s) \text{ for } s \in \mathbb{C}.$$

This gives us a functional equation for $L(E,s)$ and extends it to entire complex plane. The functional equation of $L(E,s)$ relates its values at s with those at $(2-s)$. The value $s = 1$ is the only integer point lying on its critical strip $0 < \text{Re}(s) < 2$ and is called the critical value of $L(E,s)$. On the other hand $s = k/2$ which is equal to 1 in our case is the central critical value of $L(f,s)$. So modularity of $E/\mathbb{Q}$ links critical values of L-functions of elliptic curves with central critical values of L-functions of modular forms. But why do we care about this?

[1] A "newform" is a special modular form that satisfies some additional properties, as we will see in Chapter 1.

Well, we will see it in a bit.

A natural question that arises concerning elliptic curves is to describe, and find if possible, all the rational solutions to an elliptic curve $E/\mathbb{Q}$. We denote this rational solution set by $E(\mathbb{Q})$. The number of independent rational points of infinite order on $E/\mathbb{Q}$ is called the rank$(E/\mathbb{Q})$. It turns out to be a rather hard problem to find an algorithm to determine the rank$(E/\mathbb{Q})$. The famous conjecture of *Birch* and *Swinnerton-Dyer* (BSD) predicts the formula for the rank$(E/\mathbb{Q})$. More precisely, the weak form of BSD asserts

$$\mathrm{ord}_{s=1}(L(E,s)) = \mathrm{rank}(E/\mathbb{Q}).$$

Here $\mathrm{ord}_{s=1}(L(E,s))$ is the order of vanishing of the $L(E,s)$ at $s=1$. Using modularity of elliptic curves over $\mathbb{Q}$, this is equal to $\mathrm{ord}_{s=1}(L(f,s))$ where $f(z)$ is the modular form attached to E and $s=1$ is the central critical value of $L(f,s)$. This is one of the many instances where the central critical values of L-functions of modular forms hold great importance in number theory.

In 1973, *Goro Shimura* [Shi73] enriched the theory of half-integer weight modular forms by giving a correspondence which sends modular forms of weight $k+\frac{1}{2}$ to modular forms of even integer weight $2k$ where k is a positive integer. Note that *Shimura's* correspondence does not give us an isomorphism. In 1980, *Kohnen* [Koh82] discovered that it was possible to impose some conditions on Fourier coefficients of half-integer weight modular forms and hence obtain an isomorphism via *Shimura's* map. Around the same time, *Waldspurger* [Wal81] established a close link between Fourier coefficients of half-integer weight modular forms and central critical values of L-functions of the twists of the corresponding integer weight modular forms associated under *Shimura's* map. This shifts our attention to the theory of half-integer weight modular forms and their Fourier coefficients. We explain this a bit further.

Let D be the fundamental discriminant of some quadratic field with an associated quadratic character $\chi_D = \left(\frac{D}{\bullet}\right)$. Then the twist of modular form $f(z)=\sum a_n(f)q^n$ by χ_D is given by $(f\otimes\chi_D)(z)=\sum \chi_D(n)a_n(f)q^n$. The modular form $f(z)$ attached to $E/\mathbb{Q}$ is a newform of weight 2. Then by *Kohnen's* isomorphism [Koh82] there exists a unique form $g(z)=\sum b_n q^n$ of weight $\frac{3}{2}$ attached to $f(z)$. *Waldspurger's* theorem implies

$$L(f\otimes\chi_D,1) = * \,.\, b_{|D|}^2$$

where $*$ is a constant that is well understood. Moreover, if the choice of the quadratic twist D results in the sign of the functional equation of $L(f\otimes\chi_D,s)$ to be positive, then $L(f\otimes\chi_D,s)$ vanishes at $s=1$ to an even order. In this case, if $b_{|D|}=0$, then *Waldspurger's* theorem implies that $L(f\otimes\chi_D,s)$ vanishes at $s=1$ to order at least two. On the other hand, the conjecture of *Birch* and *Swinnerton-Dyer* predicts that rank$(E_D/\mathbb{Q})$ in this case must be at least two. If we can somehow check that the rank$(E_D/\mathbb{Q})$ is at least two, then this would provide evidence in support of the conjecture of *Birch* and *Swinnerton-Dyer*. In order to get an overview of the whole situation, see figure 0.0.1.

The *Waldspurger's* theorem can be further combined with congruences between modular forms to provide concrete evidence for the *Bloch-Kato* conjecture. The Tamagawa number conjecture of *Bloch* and *Kato* generalises the BSD conjecture to any arbitrary motive. It describes the behavior at integers of the L-function associated to a motive over $\mathbb{Q}$ by relating it to the order of certain *Bloch-Kato* Selmer groups. In general, very little is known about *Bloch-Kato* Selmer groups. The congruences between modular forms modulo a prime p can help us to construct non-trivial elements of order p in these *Bloch-Kato* Selmer groups.

In order to get a flavour of the study of congruences between modular forms, see the work of *Neil Dummigan* in [Dum01] where he works over mod 11 congruence between modular form $f(z)$ attached to the elliptic curve of conductor 11 and the well known discriminant function $\Delta(z)$ to provide evidence in support of the *Bloch-Kato* conjecture, see figure 0.0.2.

Figure 0.0.1.

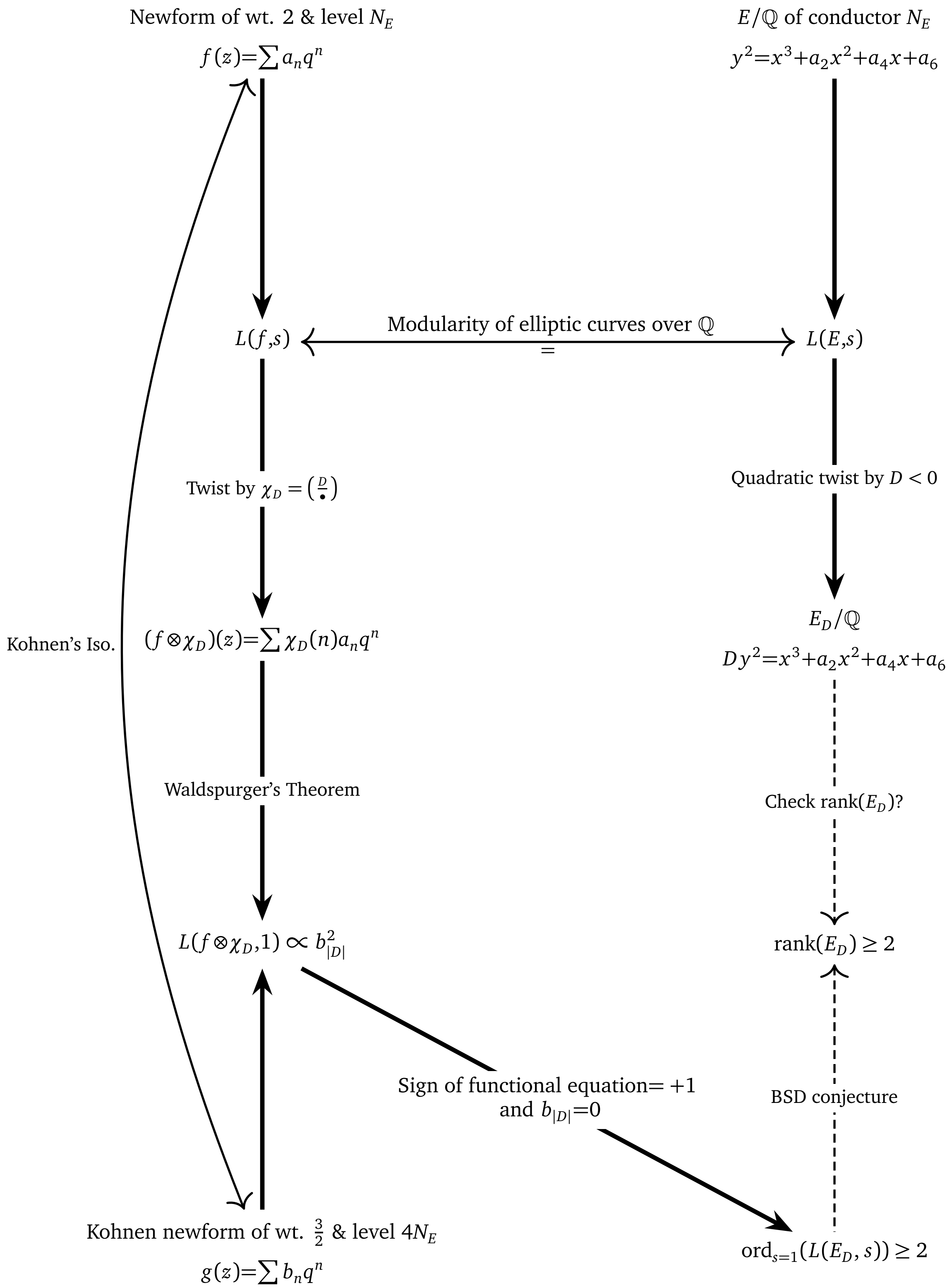
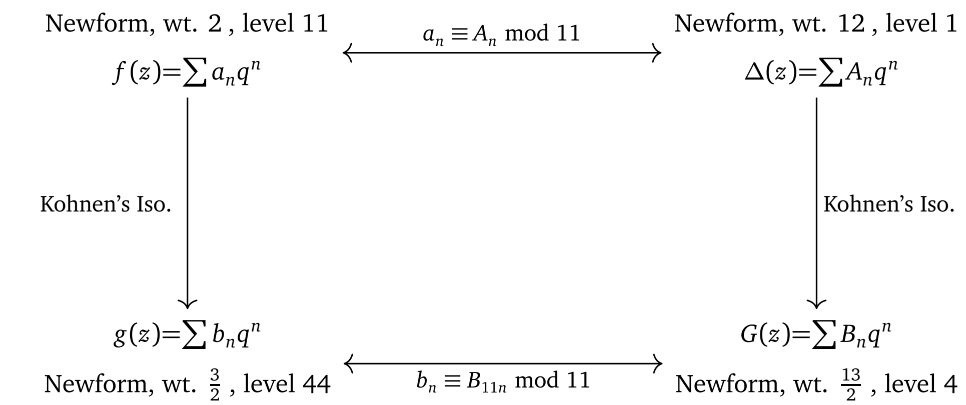
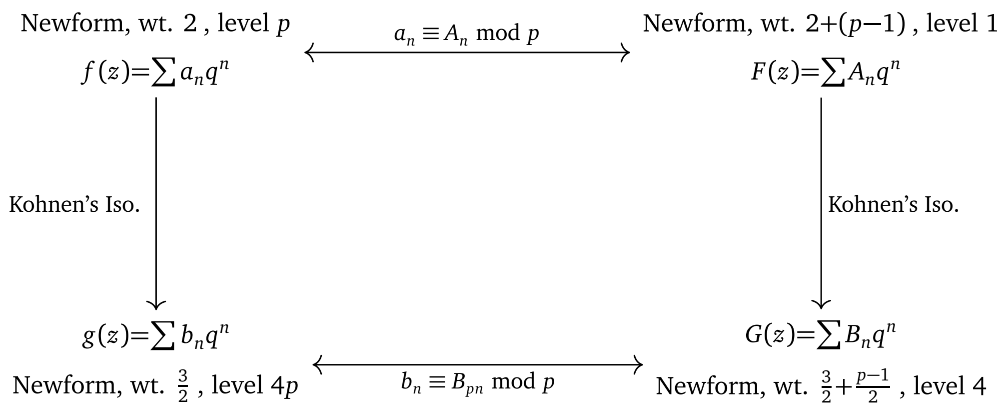

Newform of wt. 2 & level N_E
$f(z)=\sum a_n q^n$
$E/\mathbb{Q}$ of conductor N_E
$y^2=x^3+a_2x^2+a_4x+a_6$
$L(f,s)$
Modularity of elliptic curves over $\mathbb{Q}$
$=$
$L(E,s)$
Twist by $\chi_D = \left(\frac{D}{\bullet}\right)$
Quadratic twist by $D<0$
Kohnen's Iso.
$(f\otimes\chi_D)(z)=\sum \chi_D(n)a_n q^n$
$E_D/\mathbb{Q}$
$Dy^2=x^3+a_2x^2+a_4x+a_6$
Waldspurger's Theorem
Check rank(E_D)?
$L(f\otimes\chi_D,1) \propto b_{|D|}^2$
rank(E_D) ≥ 2
Sign of functional equation= +1
and $b_{|D|}=0$
BSD conjecture
Kohnen newform of wt. $\frac{3}{2}$ & level $4N_E$
$g(z)=\sum b_n q^n$
$\text{ord}_{s=1}(L(E_D,s)) \geq 2$

Figure 0.0.2. .



Building up on the aforementioned example of mod 11 congruence by *Dummigan*, *McGraw* and *Ono* [MO03, Theorem 2] showed that the congruences between modular forms modulo an odd prime p lift via *Kohnen's* isomorphism to fairly similar mod p congruences between half-integer weight modular forms just with a slight shift in Fourier coefficients, see figure 0.0.3.

Figure 0.0.3. Given odd prime p.



McGraw and *Ono's* recipe involves use of ingredients like multiplying $f(z)$ in figure 0.0.3 by a suitable Eisenstein series to add up the weight $2 \to 2+(p-1)$ and a level lowering operator called Trace operator to lower the level $p \to 1$. They then lift to an eigenform $F(z)$ using *Deligne* and *Serre's* lifting lemma and pair it with a uniqueness assumption to imply $F(z)$ is unique. They then pass on to half-integer weight modular forms via *Kohnen's* map and work in a similar way. We avoid going into details of *McGraw* and *Ono's* approach here but readers can have a look at [MO03] for details.

McGraw and *Ono's* work as shown in figure 0.0.3 implies

$$b_{|D|} \equiv B_{|pD|} \text{ mod } p. \tag{1}$$

where we choose our fundamental discriminants D such that the sign of the functional equation of $L(f \otimes \chi_D, 1)$ is positive.

The *Waldspurger's* Theorem connects the Fourier coefficients $b_{|D|}$ and $B_{|pD|}$ on either side of the congruence (1) to the central critical L-values of the twisted L-functions of $f(z)$ and $F(z)$ respectively. More precisely,

$$L(f \otimes \chi_D, 1) = * \,.\, b_{|D|}^2 \quad \text{and} \quad L\left(F \otimes \chi_{D(p)}, \tfrac{p+1}{2}\right) = *' \,.\, B_{|pD|}^2$$

where $D(p) = (-1)^{\frac{p-1}{2}} Dp$ and $*$, $*'$ are well understood constant terms.

We now take two fundamental discriminants D and D_0 such that sign of the functional equation of both $L(f \otimes \chi_D, 1)$ and $L(f \otimes \chi_{D_0}, 1)$ is positive due to certain criteria by $\left(\frac{D}{p}\right)$ and $\left(\frac{D_0}{p}\right)$. Suppose $b_{|D_0|} \not\equiv 0 \bmod p$. This choice is possible as we can normalise newform $g(z)$ such that there exists a D_0 for which the Fourier coefficient $b_{|D_0|} \not\equiv 0 \bmod p$. Then $B_{|pD_0|} \not\equiv 0 \bmod p$. Then an immediate implication [Dum01, Corollary 4.2] of the *Waldspurger's* theorem is

$$\left(\frac{D}{D_0}\right)^{\frac{1}{2}} \cdot \frac{L\left(f \otimes \chi_D, 1\right)}{L\left(f \otimes \chi_{D_0}, 1\right)} = \frac{b_{|D|}^2}{b_{|D_0|}^2} \quad \text{and} \quad \left(\frac{D(p)}{D_0(p)}\right)^{\frac{p}{2}} \cdot \frac{L\left(F \otimes \chi_{D(p)}, \frac{p+1}{2}\right)}{L\left(F \otimes \chi_{D_0(p)}, \frac{p+1}{2}\right)} = \frac{B_{|pD|}^2}{B_{|pD_0|}^2}.$$

In the case when $b_D = 0$, congruence (1) implies $B_{|pD|} \equiv 0 \bmod p$. Then

$$\left(\frac{D}{D_0}\right)^{\frac{p}{2}} \cdot \frac{L\left(F \otimes \chi_{D(p)}, \frac{p+1}{2}\right)}{L\left(F \otimes \chi_{D_0(p)}, \frac{p+1}{2}\right)}$$

is a rational number divisible by p. If this rational number is non-zero, then the *Bloch-Kato* conjecture, applied to these critical values, predicts that some Shafarevich-Tate group for $(F \otimes \chi_{D(p)})(z)$ has order divisible by p, even by p^2. As an example of divisibility of some Shafarevich-Tate group for $(\Delta \otimes \chi_{D11})(z)$ by $p = 11$, see [Dum01]. Thus, the congruences between Fourier coefficients of half-integer weight modular forms have strong implications and we are motivated to extend these congruences to half-integer weight Hilbert modular forms.

Part I of this thesis provides an alternative approach to prove such congruences between modular forms modulo powers of prime p and also lift to almost similar congruences between modular forms of half-integer weight with a slight shift in Fourier coefficients. The main aim to do so is to set up a mechanism that can be easily generalised to Hilbert modular forms of integer and half-integer weight associated to totally real quadratic fields of narrow class number 1, which is the primary objective of the remaining **Part II** of this thesis.

Over a century ago, *Ludwig Otto Blumenthal* took the first initiative to work on sketches of *David Hilbert* with the aim of creating a theory of modular functions of several complex variables. These are now called Hilbert modular forms. In this thesis, we study Hilbert modular forms over totally real quadratic fields of narrow class number 1 that involve two complex variables. We are motivated to study congruences related to Hilbert modular forms due to the remarkable progress that has been made in this field in the 21^{th} century. In 2014, the modularity over any real quadratic field was proved by *Freitas, Le Hung and Siksek* in [FHS15, Theorem 1]. This means starting with an elliptic curve E/K where K is a totally real quadratic field of narrow class number 1, we can associate a Hilbert modular form of parallel weight 2 and level $\mathfrak{n}_E$, an integral ideal equal to conductor of E/K. Using combined work of *Hiraga* and *Ikeda* [HI13] and *Ren He-Su* [Su18] on generalisation of *Kohnen's isomorphism* to Hilbert modular forms, we then pass to Hilbert modular forms of half-integer weight. The Fourier coefficients of these half-integer weight Hilbert modular forms again connect to central critical values of the L-function of twists of Hilbert modular forms. This is work (2003) of *Baruch* and *Mao* [BM03] who generalised the *Waldspurger*'s Theorem to Hilbert modular forms over totally real fields. Quite recently (2021), *Sirolli* and *Tornaría* [ST21] have derived a more explicit formula to compute central critical values of the L-function of twists of Hilbert modular forms. This provides us a smooth layout to test and give evidence for the BSD conjecture for elliptic curves over totally real quadratic fields of narrow class number 1. In this thesis, we take a step further and prove congruences between Hilbert modular forms of integer and half-integer weight. These congruences leave open scope to work in the direction of visualising elements in certain *Bloch-Kato* Selmer groups. In this thesis, we do not include details about this application but rather use it as our motivation for research.

Part I of this thesis is about congruences related to classical modular forms of integer weight and half-integer weight.

Chapter 1 contains a brief overview of background classical modular forms of integer weight. It includes basic definitions, Hecke-theory of modular forms and statements of some classical results involving old/new spaces. We include details of some further operators that will be seen in the later chapters.

Chapter 2 develops the theory of classical half-integer weight modular forms closely analogous to chapter 1. We then move on to fundamental theorems of *Shimura's* correspondence and *Kohnen's isomorphism*. These connect our congruences between integer weight modular forms to those of half-integer weight modular forms.

Chapter 3 introduces the basic facts about Eisenstein series and the related mod p congruences. These congruences arise from the *von-Staudt Clausen* congruences involving denominators of Bernoulli numbers. We then move to generalised Eisenstein series with a character and prove similar congruences using known facts about generalised Bernoulli numbers.

Chapter 4 introduces the theory of ordinary modular forms. This will be used while proving the main result where we project our spaces onto their ordinary parts and work within these. The reason to do so is to invoke *Hida's* control theorem about constancy of dimensions of ordinary spaces of modular forms. Next, we go over p-stabilisation of modular forms to see how we can force a prime p in the level even if p initially does not divide the level of the space of modular forms. We then develop an analogous theory of ordinary half-integer weight modular forms and their p-stabilisation. This chapter heavily uses the knowledge of further operators developed in chapter 1 and 2.

Chapter 5 will present the main results in **Part I** of our thesis including a uniqueness assumption that is crucial for the main result to follow. We then prove a series of propositions building up to the main congruence (Theorem 5.2.4) between integer weight modular forms. Given an odd rational prime p, the main congruence relates Fourier coefficients of modular forms modulo fixed powers of prime p where the modular forms are of varying even weights, $2k$ and $2k'$ with $2k \equiv 2k' \bmod (p-1)$. These spaces of modular forms are then projected to their respective ordinary parts. We consider our spaces on both sides of the congruences to be at level Np such that they are definitely new at N but could be possibly old at p. This means they could have been attained using p-stabilisation of ordinary modular forms.
The same thing is then repeated for half-integer weight modular forms that are related to the integer weight congruence using *Kohnen's* map (Theorem 5.3.3). Thus, we lift to congruences between half-integer weight modular forms modulo same fixed powers of prime p.

Part II of this thesis generalises the congruences related to modular forms to Hilbert modular forms associated to totally real quadratic field of class number 1.

Chapter 6 introduces notation and terminology required with working with totally real quadratic fields of narrow class number 1. We then develop significant features of Hilbert modular forms. Here we observe some fair differences with the classical modular forms. For instance, the *Koecher's* principle which makes Hilbert modular forms automatically holomorphic at the cusps. We introduce theory of Hecke operators and old/new spaces of Hilbert modular forms and also give statements of well known theorems.

Chapter 7 develops the theory of half-integer weight Hilbert modular forms analogous to chapter 6. Then we shift our focus on generalisations of fundamental theorems of *Shimura's* correspondence and *Kohnen's* isomorphism to Hilbert modular forms. Generalisation of *Kohnen's* isomorphism uses more of a representation theoretic approach. We refrain from dwelling deep into the background for this and give explanation only as per requirement of the thesis.

Chapter 8 gives an overview of Dedekind zeta function and Hecke L-function. We then introduce Hilbert Eisenstein series and related mod p congruences. We will observe differences in absolute convergence of Hilbert Eisenstein series and Classical Eisenstein series introduced in chapter 3. We then move to details about generalised Hilbert Eisenstein series with a character and prove similar mod p congruences for them.

Chapter 9 generalises the theory of ordinary modular forms introduced in chapter 4 to the Hilbert case. We also state the existing *Hida's* control theorem about constancy of dimensions of ordinary spaces of Hilbert modular forms. We introduce p-stabilised Hilbert modular forms for primes p that do not ramify in our real quadratic field K. We then work our way examining ordinary Hilbert modular forms of half-integer weight and their p-stabilisation.

Chapter 10 will present the main results in **Part II** of our thesis (Theorems 10.2.5 and 10.3.3). After building the background for Hilbert modular forms from chapter $6-9$, and setting up a plan in chapter 5 for proving congruences related to classical modular forms, we are ready to present an analogous proof of congruences related to Hilbert modular forms modulo powers of prime p. We again use two assumptions here. The first assumption is quite analogous to the one introduced in Chapter 5 while the other involves divisibility of numerators of generalised Bernoulli numbers. In this chapter, we basically generalise all our results in Chapter 5 for classical modular forms to Hilbert modular forms over totally real quadratic field of narrow class number 1.

Chapter 11 will briefly overview elliptic curves over an arbitrary number field K and the generalisation of *Waldspurger's* theroem to Hilbert modular forms. We then present an example of an elliptic curve over $\mathbb{Q}(\sqrt{5})$ to give evidence in the direction of BSD conjecture for elliptic curves over totally real quadratic fields of narrow class number 1. The main theorems proved in Chapter 10, Theorem 10.2.5 and Theorem 10.3.3, leave scope to further provide evidence for *Bloch-Kato* conjecture in future by constructing elements of prime order p in the *Bloch-Kato* Selmer group.

Part I

Congruences related to classical modular forms

CHAPTER 1

Modular forms of integer weight

1.1 Introduction

Let $\mathscr{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ denote the upper half plane.
Then the group of matrices

$$GL_2^+(\mathbb{R}) := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \text{ and } (ad - bc) > 0 \right\}$$

acts on $\mathscr{H}$ via *möbius* transformations, that is

$$(\gamma, z) \mapsto \gamma z = \frac{az + b}{cz + d}.$$

We now define the automorphy factor $j(\gamma, z)$.

Definition 1.1.1 (Automorphy factor)**.** *The automorphy factor is a function*

$$j : GL_2^+(\mathbb{R}) \times \mathscr{H} \to \mathbb{C} \quad \textit{such that} \quad j(\gamma, z) = (cz + d) \quad \textit{for} \quad \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).$$

One can observe that for every $\gamma_1, \gamma_2 \in GL_2^+(\mathbb{R})$ and for every $z \in \mathbb{H}$, we have

$$j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2 z) j(\gamma_2, z).$$

This is called the *cocycle* relation of the automorphy factor $j(\gamma, z)$.

Let $k \in \mathbb{Z}_{>0}$. We can then define an operator called the weight k-slash operator that acts on complex functions $f : \mathscr{H} \to \mathbb{C}$ defined on the upper half plane.

Definition 1.1.2 (k-slash operator)**.** *Let $k \in \mathbb{Z}_{>0}$. Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{R})$ and $f : \mathscr{H} \to \mathbb{C}$ be a complex function defined on the upper half-plane. Then define the weight k-slash operator as follows*

$$f(z)|_k \gamma = (det\ \gamma)^{\frac{k}{2}} (cz + d)^{-k} f(\gamma z).$$

The cocycle property and the multiplicativity of the determinant implies that if $f(z)$ is a complex function on $\mathscr{H}$, then:

$$f(z)|_k(\gamma_1\gamma_2) = (f(z)|_k\gamma_1)|_k\gamma_2 \text{ for all } \gamma_1, \gamma_2 \in GL_2^+(\mathbb{R}).$$

That is, for each k, the weight-k slash operator defines an action of $GL_2^+(\mathbb{R})$ on functions on the upper-half plane $\mathscr{H}$.

Let $SL_2(\mathbb{Z})$ denote the subgroup of $GL_2^+(\mathbb{R})$ consisting all matrices with integer entries and determinant 1. This subgroup $SL_2(\mathbb{Z})$ is called the *modular group*.

Let $N \in \mathbb{Z}_{>0}$. We will now a define special subgroup of the modular group called the *congruence subgroup* of level N.

Definition 1.1.3 (Congruence subgroup)**.** *Let $N \in \mathbb{Z}_{>0}$. We define the principal congruence subgroup of level N as*

$$\begin{aligned}\Gamma(N) &= ker(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/N\mathbb{Z})) \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\Big|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \; mod\; N \right\}.\end{aligned}$$

Then a congruence subgroup Γ is defined as any subgroup of $SL_2(\mathbb{Z})$ that contains a principal congruence subgroup $\Gamma(N)$ for some N.

We now fix our congruence subgroup $\Gamma = \Gamma_0(N)$ that is defined as

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\Big|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix} \text{ mod } N \right\}.$$

Before we define modular forms, we will define cusps of congruence subgroups. Let $i\infty$ denote the infinity along the imaginary axis. In general, the points at infinity of the upper half plane $\mathscr{H}$ are $\mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{i\infty\}$. The modular group $SL_2(\mathbb{Z})$ acts transitively on $\mathbb{P}^1(\mathbb{Q})$ as follows:

$$\begin{aligned}\begin{pmatrix} a & b \\ c & d \end{pmatrix} . \frac{r}{s} &= \frac{a\frac{r}{s} + b}{c\frac{r}{s} + d} \\ &= \frac{ar + bs}{cr + ds}\end{aligned}$$

and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} . \infty = \frac{a}{c}.$$

Thus, there is only one orbit of $\mathbb{P}^1(\mathbb{Q})$ under the action of the modular group $SL_2(\mathbb{Z})$. In general, for a congruence subgroup Γ, the number of orbits of $\mathbb{P}^1(\mathbb{Q})$ under the action of Γ may be greater than one.

Definition 1.1.4 (Cusps)**.** *The orbits of $\mathbb{P}^1(\mathbb{Q})$ under the action of a congruence group Γ are called the cusps of Γ.*

Note 1.1.5. The number of cusps is always finite [DS05, Lemma 2.4.1]. At level 1, we have only one cusp and that is at $i\infty$.

Definition 1.1.6 (Modular form)**.** *Let $k, N \in \mathbb{Z}_{>0}$. A modular form of weight k and level N is a function $f : \mathscr{H} \to \mathbb{C}$ such that*

- *$f(z)$ is holomorphic on $\mathscr{H}$;*
- *$f(z)$ is invariant under the k-slash action of $\Gamma_0(N)$. In other words,*

$$f(\gamma z) = (cz + d)^k f(z) \text{ where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N);$$

- *$f(z)$ is holomorphic at all the cusps.*

The space of all modular forms of weight k and level N form a finite dimensional complex vector space [DS05, Chapter 3] which is denoted by $M_k(\Gamma_0(N))$.

Note 1.1.7. We haven't precisely explained what holomorphicity at all the cusps means. The fact that $f(z)$ is holomorphic at the cusp $i\infty$ means that $|f(z)|$ remains bounded as $\mathrm{Im}(z) \to \infty$. The condition that $f(z)$ is holomorphic at all cusps is then equivalent to $f(z)|_k\gamma$ being holomorphic at $i\infty$ for all $\gamma \in \Gamma_0(1)$. For details, we refer to [RS11, Section 3.1 and Section 8.1].

The congruence subgroup $\Gamma_0(N)$ contains the matrix $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. If $f(z) \in M_k(\Gamma_0(N))$, $f(z)$ is invariant under the k-slash action of $\Gamma_0(N)$. In particular, we have

$$f(z)|_k\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = f(z)$$

or

$$f(z+1) = f(z).$$

Thus, $f(z)$ is periodic and admits the following Fourier expansion at the cusp at $i\infty$,

$$f(z) = \sum_{n=0}^{\infty} a_n q^n \text{ where } q = e^{2\pi i z},\ a_n \in \mathbb{C}. \tag{1.1}$$

We also refer to this as the q-expansion of the modular form. This condition of holomorphicity at all the cusps is equivalent to saying there is no occurence of negative powers of q in the Fourier expansion 1.1 of $f(z)$ at $i\infty$.

Definition 1.1.8 (Cusp form)**.** *A cusp form $f(z)$ of weight k and level N is a modular form that vanishes at all cusps. This happens if a_0 vanishes in the Fourier expansion of $f(z)|_k\gamma$ for all $\gamma \in \Gamma_0(1)$.*
The space of all cusp forms of weight k and level N form a subspace of $M_k(\Gamma_0(N))$ and is denoted by $S_k(\Gamma_0(N))$.

The spaces $M_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$ are finite dimensional complex vector spaces [DS05, pg. 4].

Examples of modular forms:

1. The zero function on $\mathscr{H}$, $f(z) = 0$ for all z, is a modular form of every weight.
   Every constant function on $\mathscr{H}$, $f(z) = c$ for all z where $c \in \mathbb{C}$, is a modular form of weight 0.

2. Eisenstein series: We now give a non-trivial example of a modular form.
   Let $k > 2$ be an even integer. Then we define the Eisenstein series of weight k to be the two-dimensional analog of the Riemann zeta function $\zeta(k) = \sum_{n=1}^{\infty} n^{-k}$,

   $$G_k(z) := \sum_{(m,n)}{}' \frac{1}{(mz+n)^k}, \quad z \in \mathscr{H},$$

   where $\sum'$ means to sum over nonzero integer pairs $(m,n) \in \mathbb{Z}^2 \backslash \{(0,0)\}$.
   Then $G_k(z)$ is a modular form of weight k, see section 3.1 for further details.

   The Fourier expansions of the first few normalised Eisenstein series $E_k(z) = \frac{G_k(z)}{2\zeta(k)}$ are:

   $$E_2(z) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n;$$
   $$E_4(z) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n;$$
   $$E_6(z) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n;$$

$$E_8(z) = 1 + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n;$$
$$E_{10}(z) = 1 - 264\sum_{n=1}^{\infty}\sigma_9(n)q^n,$$
where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ is the sum of divisors of n and $q = e^{2\pi iz}$.
Note that $E_k(z) = \sum_{n=0}^{\infty} a_n q^n$ is not a cusp form as the Fourier coefficient a_0 is equal to 1 in each case.

3. Cusp forms: We now look at examples of some cusp forms.

Let

$$\begin{aligned} f(z) &:= q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2 \text{ where } q = e^{2\pi iz} \\ &= q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 + O(q^{10}). \end{aligned}$$

Then $f(z)$ is a cusp form of weight 2 and level 11. It's the modular form attached to the elliptic curve $E/\mathbb{Q}$ of conductor 11 defined by the Weierstrass equation $y^2 + y = x^3 - x^2 - 10x - 20$. For more properties about this cusp form and its related objects, see [LMF22, Classical, Label: 11.a2].

Consider another example of a cusp form defined as:

$$\begin{aligned} \Delta(z) &:= q\prod_{n=1}^{\infty}(1-q^n)^{24} \\ &= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 \\ &\quad - 6048q^6 - 16744q^7 + 84480q^8 - 113643q^9 + O(q^{10}) \end{aligned}$$

where $q = e^{2\pi iz}$.

Then $\Delta(z)$ is a cusp form of weight 12 and level 1. It's also called the Discriminant function. For more properties about this cusp form and its related objects, see [LMF22, Classical, Label: 1.12.a].

1.1.1 Modular forms with a character

Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\star \to \mathbb{C}^\star$ be a Dirichlet character modulo N. Then we say $f(z)$ is a modular form of weight k, level N and character χ mod N if the second condition in definition 1.1.6 is replaced by

$$f(\gamma z) = \chi(d)(cz+d)^k f(z) \text{ where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

We denote the space of all modular forms of weight k and level N and character χ mod N by $M_k(\Gamma_0(N), \chi)$.

1.2 Hecke operators

Given two congruences subgroups, Γ_1 and Γ_2 and a fixed element $\alpha \in GL_2^+(\mathbb{Q})$, we can define the following double coset

$$\Gamma_1\alpha\Gamma_2 = \{\gamma_1\alpha\gamma_2 \mid \gamma_1 \in \Gamma_1,\ \gamma_2 \in \Gamma_2\}.$$

These double cosets act via k-slash operator on our space of modular forms $M_k(\Gamma_1)$ and map them to modular forms in the space $M_k(\Gamma_2)$, see [DS05, Section 5.1]. This action is denoted as $|_k[\Gamma_1\alpha\Gamma_2]$.

We are interested in the case when $\Gamma_1 = \Gamma_2 = \Gamma_0(N)$ and $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$ where p is a prime. We can then define linear, commuting endomorphism maps on the complex vector space $M_k(\Gamma_0(N))$ and its subspace $S_k(\Gamma_0(N))$ that act on these spaces via the double coset $|_k[\Gamma_0(N)\alpha\Gamma_0(N)]$. We call these maps *Hecke operators*, denoted as T_p for each prime p. In this section, we will go over some properties of Hecke operators and their action on the q-expansions of modular forms in $M_k(\Gamma_0(N))$. For detailed background on double coset operators and Hecke operators, refer to [DS05, Chapter 5].

Definition 1.2.1. *Let $f(z) \in M_k(\Gamma_0(N))$. Then for each prime p, we define the Hecke operator T_p in terms of the k-slash operator on $M_k(\Gamma_0(N))$ as follows*

$$f(z)|_k T_p = p^{\frac{k}{2}-1} \begin{cases} \sum_{j=0}^{p-1} f(z)|_k \beta_j & p \mid N \\ \sum_{j=0}^{p-1} f(z)|_k \beta_j \; + \; f(z)|_k \beta_\infty & p \nmid N \end{cases}$$

where $\beta_j = \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}$ *and* $\beta_\infty = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ *for* $j \in \mathbb{Z}/p\mathbb{Z}$.

Note 1.2.2. Note that we have a different normalisation here as our k-slash operator is defined in a slightly different way than in [DS05, Proposition 5.2.1].

The Hecke operators are commutative, that is, $T_p T_q = T_q T_p$ for distinct primes p, q [DS05, Proposition 5.2.4]. To define T_n, set $T_1 = 1$ (the identity operator); T_p is already defined for primes p. For powers of primes, T_{p^r} is defined using recursion formula

$$T_{p^r} = T_{p^{r-1}} T_p - p^{k-1} T_{p^{r-2}} \text{ where } r \geq 2$$

given in [DS05, Section 5.3, pg. 178] and hence is a polynomial with integer coefficients in T_p. Finally, we extend our definition to T_n for any integer $n \in \mathbb{N}$ by using multiplicativity, $T_n = \prod_i T_{p_i^{r_i}}$ where $n = \prod_i p_i^{r_i}$.

We now state the action of Hecke operators T_p on the Fourier coefficients of the modular form $f(z) \in M_k(\Gamma_0(N))$ [DS05, Proposition 5.2.2].

Proposition 1.2.3 (Hecke operators on q-expansions)**.** *Let $f(z) \in M_k(\Gamma_0(N))$ with Fourier expansion $f(z) = \sum_{n=0}^{\infty} a_n q^n$. Then $f(z)|_k T_p \in M_k(\Gamma_0(N))$ with Fourier expansion $f(z)|_k T_p = \sum_{n=0}^{\infty} b_n q^n$ such that*

$$b_n = \begin{cases} a_{pn} & p \mid N; \\ a_{pn} + p^{k-1} a_{n/p} & p \nmid N. \end{cases}$$

Here $a_{n/p} = 0$ when n/p is not an integer.

Definition 1.2.4 (Hecke algebra)**.** *The Hecke algebra of weight k and level N acting on $M_k(\Gamma_0(N))$ is the commutative $\mathbb{C}$-sub-algebra of $End(M_k(\Gamma_0(N))$ generated by Hecke operators T_p over all primes p in $\mathbb{Z}_{>0}$.*
We denote the Hecke algebra of weight k and level N by $\mathbb{T}_k(N)$.

Definition 1.2.5 (Hecke eigenform)**.** *We say $f(z) \in M_k(\Gamma_0(N))$ is a Hecke eigenform if it's a simultaneous eigenvector all Hecke operators in $\mathbb{T}_k(N)$.*

Note 1.2.6. We will specify whenever we refer to a Hecke eigenform for all Hecke operators T_p for p not dividing the level N. In general, our definition of a Hecke eigenform refers to $f(z)$ being an eigenvector for all operators T_n for $n \in \mathbb{N}$.

Definition 1.2.7 (Normalised eigenform)**.** *Let $f(z) \in S_k(\Gamma_0(N))$ be a Hecke eigenform with the following Fourier expansion*

$$f(z) = \sum_{n=1}^{\infty} a_n q^n.$$

We say $f(z)$ is normalised if $a_1 = 1$.

1.3 Further operators

In this section, we will introduce more operators on the space $M_k(\Gamma_0(N))$.

1.3.1 Operator V_1

Let $f(z) \in M_k(\Gamma_0(N))$. Let $d \in \mathbb{Z}_{>0}$ such that $(d,N)=1$. Note that $M_k(\Gamma_0(N)) \subset M_k(\Gamma_0(Nd))$. Hence, $f(z)$ can always be viewed as a modular form in $M_k(\Gamma_0(Nd))$. More formally, define an operator

$$|_k V_1 : M_k(\Gamma_0(N)) \to M_k(\Gamma_0(Nd)) \text{ such that } f(z) \mapsto f(z).$$

1.3.2 Operator V_d

Let $f(z) \in M_k(\Gamma_0(N))$. Let $d \in \mathbb{Z}_{>0}$ such that $(d,N)=1$. Define the operator V_d on $f(z)$ in terms of k-slash action as

$$\begin{aligned} f(z)|_k V_d := d^{-\frac{k}{2}} f(z)\Big|_k \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}. &= d^{-\frac{k}{2}} f(z)|_k \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \\ &= d^{-\frac{k}{2}} d^{\frac{k}{2}} f(dz) \\ &= f(dz). \end{aligned}$$

Thus, if $f(z) = \sum_{n=0}^{\infty} a_n q^n$, then $f(z)|_k V_d = \sum_{n=0}^{\infty} a_n q^{dn}$.

Proposition 1.3.1. *Let $f(z) \in M_k(\Gamma_0(N))$. Let $d \in \mathbb{Z}_{>0}$ such that $(d,N)=1$. Then $f(dz) \in M_k(\Gamma_0(Nd))$.*

Proof. See [DS05, Ex. 1.2.11]. ☐

Thus, we have a map

$$|_k V_d : M_k(\Gamma_0(N)) \to M_k(\Gamma_0(Nd)) \text{ such that } f(z) \mapsto f(dz).$$

1.3.3 Operator U_d

Let $d \in \mathbb{Z}_{>0}$ and let $f(z) \in M_k(\Gamma_0(N))$ with Fourier expansion $f(z) = \sum_{n=0}^{\infty} a_n q^n$. Define the action of U_d operator on Fourier coefficients of $f(z)$ in the following way;

$$f(z)|_k U_d := \sum_{n=0}^{\infty} a_{dn} q^n.$$

Proposition 1.3.2. *Let $d \in \mathbb{Z}_{>0}$ be a divisor of N. Then $|_k U_d$ maps the space $M_k(\Gamma_0(N))$ to itself.*

Proof. This follows from the observation that the action of U_d on the Fourier expansion of a modular form $f(z)$ in $M_k(\Gamma_0(N))$ is equivalent to the action of the Hecke operator T_d on $f(z)$, see proposition 1.2.3. ☐

Now let ℓ be a prime such that $\ell \nmid N$, then again from proposition 1.2.3, we have

$$f(z)|_k U_\ell = f(z)|_k T_\ell - \ell^{k-1} f(z)|_k V_\ell.$$

In this case, we observe that the submodule generated by the action of $|_k U_\ell$ operator on $f(z)$ lies in the span of the set $\{f(z)|_k T_\ell, f(z)|_k V_\ell\}$ which has at most level $N\ell$. We state this as a remark for its use in later chapters.

Remark 1.3.3. If prime $\ell \nmid N$ and $f(z) \in M_k(\Gamma_0(N))$, then $f(z)|_k U_\ell$ can have at most level $N\ell$.

1.4 Old and new spaces

Let $M \in \mathbb{Z}_{>0}$. Let ℓ be an odd rational prime that is co-prime to M. Let $f(z) \in S_k(\Gamma_0(M))$ be a cusp form of weight k and level M. Now ℓ is not in the level but using operators $|_k V_1$ and $|_k V_\ell$ defined in section 1.3, we can force ℓ in our level. Recall that

$$|_k V_1 : S_k(\Gamma_0(M)) \to S_k(\Gamma_0(M\ell)) \text{ such that } f(z) \mapsto f(z)$$

and

$$|_k V_\ell : S_k(\Gamma_0(M)) \to S_k(\Gamma_0(M\ell)) \text{ such that } f(z) \mapsto f(\ell z).$$

Let $N = M\ell$. We can now define the space of ℓ-old forms of $S_k(\Gamma_0(N))$.

Definition 1.4.1 (ℓ-old forms at level N). *We define the space of ℓ-old forms of $S_k(\Gamma_0(N))$, denoted as $S_k^{\ell\text{-}old}(\Gamma_0(N))$ as the subspace of $S_k(\Gamma_0(N))$ generated by spanning set $\{f_1(z)|_k V_1, f_2(z)|_k V_\ell\}$ where $f_1, f_2 \in S_k(\Gamma_0(N/\ell))$.*

$$S_k^{\ell\text{-}old}(\Gamma_0(N)) = S_k(\Gamma_0(N/\ell))|_k V_1 \oplus S_k(\Gamma_0(N/\ell))|_k V_\ell.$$

We can now do this for every proper divisor d of N and hence define the space of all old forms at level N.

Definition 1.4.2 (Old subspace). *Let $N \in \mathbb{Z}_{>0}$ be an integer. We define the old subspace of $S_k(\Gamma_0(N))$ as*

$$S_k^{old}(\Gamma_0(N)) := \bigoplus_{d|N,\, d\neq 1} S_k^{d\text{-}old}(\Gamma_0(N)).$$

We now define the hyperbolic measure $d\mu$ on $\mathscr{H}$ to define an inner product on our space $S_k(\Gamma_0(N))$ [DS05, Section 5.4].

$$d\mu(z) = \frac{dx\,dy}{y^2} \text{ where } z = x + iy \in \mathscr{H}.$$

Definition 1.4.3 (Petersson inner product). *Let $f, g \in S_k(\Gamma_0(N))$. We define the Petersson inner product of f and g by,*

$$\langle f, g\rangle := \int_{\Gamma_0(N)\backslash\mathscr{H}} f(z)\overline{g(z)}y^k d\mu(z).$$

where $z = x + iy \in \mathscr{H}$.

Definition 1.4.4 (New subspace). *We define the new subspace of $S_k(\Gamma_0(N))$ as the orthogonal complement of the old subspace with respect to the Petersson inner product.*

$$S_k^{new}(\Gamma_0(N)) := S_k^{old}(\Gamma_0(N))^\perp.$$

Hence, our space of cusp forms at level N has the following direct decomposition.

$$S_k(\Gamma_0(N)) = S_k^{\text{new}}(\Gamma_0(N)) \oplus S_k^{\text{old}}(\Gamma_0(N))$$

or

$$S_k(\Gamma_0(N)) = \bigoplus_{\substack{M|N \\ d|NM^{-1}}} S_k^{\text{new}}(\Gamma_0(M))|_k V_d. \tag{1.2}$$

Definition 1.4.5 (Newform). *A newform $f(z) = \sum_{n=1}^\infty a_n q^n \in S_k^{new}(\Gamma_0(N))$ is normalised such that $a_1 = 1$ and is an eigenform for all Hecke operators T_n for $n \in \mathbb{Z}_{>0}$.*

We now state one of the main results in this section, see [DS05, Theorem 5.8.2].

Proposition 1.4.6. *The space $S_k^{new}(\Gamma_0(N))$ has an orthogonal basis of newforms.*

From equation 1.2 and proposition 1.4.6, it follows that the space of cusp forms $S_k(\Gamma_0(N))$ has a basis of Hecke eigenforms that are eigenforms for all Hecke operators T_n for $n \in \mathbb{Z}_{>0}$.

The Fourier coefficients of a newform can be recovered from its Hecke eigenvalues, see same result [DS05, Theorem 5.8.2]. More formally, if $f(z) \in S_k^{\text{new}}(\Gamma_0(N))$ is a newform with Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} a_n q^n$$

and has Hecke eigenvalues $\lambda_{T_n}(f)$ corresponding to action of Hecke operator T_n for $n \in \mathbb{Z}_{>0}$, then

$$\lambda_{T_n}(f) = a_n.$$

We now state a crucial result in this section that forces the Fourier coefficients of newforms to lie in a ring of integers.

Proposition 1.4.7. *Let*

$$f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_{2k}^{new}(\Gamma_0(N))$$

be a newform. Then there exists a fixed number field L_f with ring of integers $\mathcal{O}_f$ such that for all $n \in \mathbb{N}$, the Fourier coefficients $a_n \in \mathcal{O}_f$.

Proof. See [Shi94, Theorem 3.52]. □

CHAPTER 2

Modular forms of half-integer weight

2.1 Introduction

We now shift our attention to modular forms of weight $k + \frac{1}{2}$, $k \in \mathbb{Z}$, which is a non-integral weight and lies midway between two integers. It might seem at first that we can easily generalise the definitions and transformation laws of classical integral modular forms by replacing our weight k by $k + \frac{1}{2}$ in chapter 1. Roughly speaking, we expect a half-integer weight modular forms $g(z)$ to satisfy a functional equation of the type

$$g(\gamma z) = (cz + d)^{k+\frac{1}{2}} g(z) \text{ for } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$$

where Γ is some congruence subgroup of $SL_2(\mathbb{Z})$. However, one needs to be careful as we will see there is more than one possible choice of the branch of square root involved in the automorphy factor. In order to eliminate inconsistencies, we must introduce a quadratic character corresponding to some quadratic extension of $\mathbb{Q}$ and shift to a bigger group than just $GL_2^+(\mathbb{R})$ called the *metaplectic group*. We begin by looking at our first motivating example of a half-integer weight modular form called the *Theta Function* and then define half-integer weight modular forms in a more general way. For details, refer to [Kob93].

Definition 2.1.1 (Jacobi's Theta function)**.** *The Theta function is defined as*

$$\theta : \mathscr{H} \to \mathbb{C} \text{ such that } \theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \ldots$$

where $q = e^{2\pi i z}$.

Proposition 2.1.2. *$\theta^2(z)$ is a modular form of weight 1 and level 4 with a Dirichlet character χ_4,*

$$\theta^2(z) \in M_1(\Gamma_0(4), \chi_4).$$

Here χ_4 denotes the unique non-trivial character modulo 4 defined as

$$\chi_4(d) = \left(\frac{-1}{d}\right) = \begin{cases} 1 & \text{if } d \equiv 1 \mod 4; \\ -1 & \text{if } d \equiv 3 \mod 4. \end{cases}$$

The transformation law for $\theta^2(z)$ is given by

$$\theta^2(\gamma z) = \left(\frac{-1}{d}\right)(cz + d)\theta^2(z) \text{ where } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4).$$

Proof. See [Kob93, Proposition 30, pg. 138]. ☐

Definition 2.1.3 (Quadratic residue symbol $\left(\frac{c}{d}\right)$ with $c, d \in \mathbb{Z},\ d \neq 0$)**.** *We now introduce a quadratic residue symbol $\left(\frac{c}{d}\right)$ with $c, d \in \mathbb{Z},\ d \neq 0$, as defined in [Shi73, pg. 442] and is characterised by the following properties:*

(i) $\left(\frac{c}{d}\right) = 0$ *if* $(c, d) \neq 1$.

(ii) If d is an odd prime, $\left(\frac{c}{d}\right)$ coincides with the ordinary quadratic residue symbol, that is, it is one less than the number of solutions of $x^2 \equiv c \bmod d$.

(iii) If $d > 0$, the map $c \mapsto \left(\frac{c}{d}\right)$ defines a character modulo d.

(iv) If $c \neq 0$, the map $d \mapsto \left(\frac{c}{d}\right)$ defines a character modulo a divisor of $4c$, whose conductor is the conductor of $\mathbb{Q}(\sqrt{c})$ over $\mathbb{Q}$.

(v) $\left(\frac{c}{-1}\right) = 1$ *or* -1 *according as* $c > 0$ *or* $c < 0$.

(vi) $\left(\frac{0}{\pm 1}\right) = 1$.

Note 2.1.4. It should be noted that our notation does not agree with the traditional symbol with the property $\left(\frac{c}{d}\right) = \left(\frac{c}{|d|}\right)$. In fact we have

$$\left(\frac{c}{d}\right) = \eta . \left(\frac{c}{|d|}\right) \text{ with } \eta = \begin{cases} -1 & \text{if } c < 0 \text{ and } d < 0, \\ 1 & \text{if } c > 0 \text{ and } d > 0, \end{cases}$$

especially

$$\left(\frac{-1}{d}\right) = (-1)^{\frac{d-1}{2}}$$

for all positive or negative odd integers d.

Before we state the transformation law for the Theta function, we must note the convention for taking square roots.

Note 2.1.5 (Convention for taking square roots). We will always take the branch of the square root having argument in $(-\pi/2, \pi/2]$. For any integer k, we take $z^{k/2} = (\sqrt{z})^k$.

Theorem 2.1.6 (Hecke)**.** *The Theta function, denoted $\theta(z)$, is a modular form of weight $1/2$ and level 4 which transforms as*

$$\theta(\gamma z) = j(\gamma, z)\theta(z) \text{ where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$$

where the automorphy factor $j(\gamma, z)$ involves the quadratic residue symbol $\left(\frac{c}{d}\right)$ with $c, d \in \mathbb{Z},\ d \neq 0$ defined in 2.1.3, a fourth root of unity which depends on d mod 4 (boxed below), and an explicit branch of the square root function.

$$\begin{aligned} j(\gamma, z) &= \boxed{\sqrt{\left(\frac{-1}{d}\right)}^{\,-1}} \left(\frac{c}{d}\right) \sqrt{cz + d} \\ &= \epsilon_d^{-1} \left(\frac{c}{d}\right) \sqrt{cz + d} \end{aligned}$$

where

$$\epsilon_d = \sqrt{\left(\frac{-1}{d}\right)} = \begin{cases} 1 & \textit{if } d \equiv 1 \mod 4; \\ i & \textit{if } d \equiv 3 \mod 4. \end{cases}$$

for odd d.

Proof. See, [Kob93, Theorem 4.3, pg. 148]. □

We now introduce the *metaplectic Group*.

Definition 2.1.7 (Metaplectic Group). *Let $T = \{\omega \in \mathbb{C} \mid |\omega| = 1\}$ and let $\mathfrak{G}$ denote the set of all couples $(\alpha, \phi(z))$ such that $\alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(\mathbb{R})$ and $\phi(z)$ is a holomorphic function on $\mathscr{H}$ such that*

$$\phi(z)^2 = t \cdot \det(\alpha)^{-1/2}(cz + d)$$

where $t \in T$.
We define a multiplication law $\star$ on $\mathfrak{G}$ as

$$(\alpha, \phi(z)) \star (\beta, \psi(z)) := (\alpha\beta, \phi(\beta z)\psi(z))$$

This forms a group $(\mathfrak{G}, \star)$ called the Metaplectic Group of $GL_2^+(\mathbb{R})$, see [Kob93, Proposition 1, pg. 179].

Remark 2.1.8. Note that $t \in T$ in Definition 2.1.7 does not depend on z in the sense that it is not a function of z. In fact, in our case it's sufficient to restrict $T = \{\pm 1, \pm i\}$ to allow square roots of $\left(\frac{-1}{d}\right)(cz + d)$ to define the four sheeted covering of $GL_2^+(\mathbb{Z})$. In this case, $\phi(z)^2 = \left(\frac{-1}{d}\right)(cz + d)$ and $t = \{\pm 1\}$.
Let $\mathfrak{G}_1$ be the subgroup of $\mathfrak{G}$ defined as

$$\mathfrak{G}_1 := \{(\alpha, \phi(z)) \in \mathfrak{G} \mid \det(\alpha) = 1\}.$$

Let N be an odd, square-free integer. There are infinitely many ways to lift an element of $GL_2^+(\mathbb{R})$ to its metaplectic cover $\mathfrak{G}$ depending on the choice of $t \in T$. We fix the choice of $\phi(z)$ to establish an isomorphism between $\Gamma_0(4N)$ and a subgroup of $\mathfrak{G}_1$ with integer entries denoted by $\widetilde{\Gamma}_0(4N)$

$$\alpha \mapsto \alpha^\star = (\alpha, \phi(z))$$

where

$$\phi(z) = \left(\frac{-1}{d}\right)^{-1/2} \left(\frac{c}{d}\right)(cz + d)^{1/2}.$$

Here the symbols $\left(\frac{-1}{d}\right)^{-1/2}$ and $\left(\frac{c}{d}\right)$ have the same meaning as in Theorem 2.1.6.
Remark 2.1.9. When we refer to congruence subgroup of level $4N$ in the case of half-integer weight modular forms, we will always mean $\widetilde{\Gamma}_0(4N)$.

Definition 2.1.10 (($k+\frac{1}{2}$)-slash operator). *Let $\xi = (\alpha, \phi(z)) \in \mathfrak{G}$ where $\alpha \in GL_2^+(\mathbb{Z})$ and $\phi(z)$ is a holomorphic function on $\mathscr{H}$ defined as $\phi(z) := \left(\frac{-1}{d}\right)^{-1/2}\left(\frac{c}{d}\right) det(\alpha)^{-1/4}(cz + d)^{1/2}$. For a complex valued function $g(z)$ on the upper half plane, we define an operator $|_{k+\frac{1}{2}}\xi$ as*

$$g(z)|_{k+\frac{1}{2}}\xi := \phi(z)^{-(2k+1)} g(\alpha z).$$

This is the $(k + \frac{1}{2})$-slash operator on half-integer weight modular forms.

Definition 2.1.11 (Half-integer weight modular form). *Let $k, N \in \mathbb{Z}_{>0}$. A function $g : \mathscr{H} \to \mathbb{C}$ is said to be a modular form of weight $k + \frac{1}{2}$ and level $4N$ if*

- *$g(z)$ is holomorphic on $\mathscr{H}$;*
- *$g(z)$ is invariant under the action of $|_{k+\frac{1}{2}}\widetilde{\Gamma}_0(4N)$ defined in 2.1.10. In other words*

$$g(z)|_{k+\frac{1}{2}}\alpha^\star = g(z) \quad \textit{for all} \quad \alpha^\star \in \widetilde{\Gamma}_0(4N).$$

- *$g(z)$ is holomorphic at all cusps of $\Gamma_0(4N)$.*

We denote the space of all half integer weight modular forms of weight $k+\frac{1}{2}$ *and level* $4N$ *as* $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N))$.

The congruence subgroup $\widetilde{\Gamma}_0(4N)$ contains the element $\left(\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right),1\right)$. If $g(z)\in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N))$, then $g(z)$ is invariant under the $(k+\frac{1}{2})$-slash action of $\widetilde{\Gamma}_0(4N)$. In particular, we have

$$g(z)|_{k+\frac{1}{2}}\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},1\right)=g(z)$$

or

$$g(z+1)=g(z).$$

Thus, $g(z)$ is periodic and admits the following Fourier expansion (q-expansion) at $i\infty$,

$$g(z)=\sum_{n=0}^{\infty}b_nq^n \text{ where } q=e^{2\pi iz},\ b_n\in\mathbb{C}. \tag{2.1}$$

In the same way as in the integer case (See note 1.1.7), $g(z)$ is holomorphic at the cusp $i\infty$ of $\Gamma_0(4N)$ means that $|g(z)|$ is bounded as $\mathrm{Im}(z)\to\infty$. This condition is equivalent to saying there is no occurrence of negative powers of q in the Fourier expansion of $g(z)$ at $i\infty$. For further details about the meaning of *holomorphicity at a cusp*, refer to [Shi73, pg. 144] and [Kob93, pgs. 180-182].

Definition 2.1.12 (Half-integer weight cusp form)**.** *A half-integer weight cusp form* $g(z)$ *of weight* $k+\frac{1}{2}$ *and level* $4N$ *is a half-integer weight modular form that vanishes at all cusps of* $\Gamma_0(4N)$*. This happens if* b_0 *(the* 0^{th} *Fourier coefficient) vanishes in the Fourier expansion of* $g(z)$ *at each cusp of* $\Gamma_0(4N)$*.*
The space of all half-integer weight cusp forms of weight $k+\frac{1}{2}$ *and level* $4N$ *forms a subspace of* $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N))$ *and is denoted by* $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N))$.

The spaces $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N))$ and $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N))$ are finite dimensional complex vector spaces. The explicit dimension formulas for these spaces can be found in [CO77].

2.1.1 Half-integer weight modular forms with a character

Let χ_N be a Dirchlet character modulo N and we will assume that χ_N is quadratic. We now define

$$\chi_{4N}:=\left(\frac{4\chi_N(-1)}{\bullet}\right)\chi_N.$$

We say that $g(z)$ is a half-integer weight modular form of weight $k+\frac{1}{2}$, level $4N$ and charcter χ_{4N} if the second condition of definition 2.1.11 is replaced by

$$g(z)|_{k+\frac{1}{2}}\alpha^\star=\chi_{4N}(d)g(z)\quad \text{for all}\quad \alpha^\star\in\widetilde{\Gamma}_0(4N).$$

Note 2.1.13. For simplicity, we will denote χ_{4N} by χ.
We denote the space of all half-integer weight modular forms of weight $k+\frac{1}{2}$, level $4N$ and character χ by $M_{k+\frac{1}{2}}(\Gamma_0(4N),\chi)$.

2.2 Hecke operators

As in the case of integer weight modular forms, there are Hecke operators which act on spaces of half-integer weight modular forms.

Let p be a prime. In this case, we are interested in the double coset is $\left(\widetilde{\Gamma}_0(4N)\ \xi\ \widetilde{\Gamma}_0(4N)\right)$ where $\xi=\left(\left(\begin{smallmatrix}1&0\\0&p^2\end{smallmatrix}\right),p^{1/2}\right)$. We can then define linear, commuting endomorphism maps on the complex vector space $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N))$ and its subspace $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(N))$ via the operator $|_{k+\frac{1}{2}}[\widetilde{\Gamma}_0(4N)\ \xi\ \widetilde{\Gamma}_0(4N)]$. For details on Hecke operators on half-integer weight modular forms, refer to [Kob93] and [Shi73].

Definition 2.2.1. *Let $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N), \chi)$. Then for each prime p, we define the Hecke operator T_{p^2} in terms of $(k+\frac{1}{2})$-slash operator on $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N), \chi)$ as follows*

$$g(z)|_{k+\frac{1}{2}} T_{p^2} = p^{k-\frac{3}{2}} \begin{cases} \sum_{j=0}^{p^2-1} g(z)|_{k+\frac{1}{2}} \gamma_j^\star + \chi(p) \sum_{j=1}^{p-1} g(z)|_{k+\frac{1}{2}} \beta_j^\star + \chi(p^2) g(z)|_{k+\frac{1}{2}} \alpha^\star & \text{if } p \nmid N \\ \sum_{j=0}^{p^2-1} g(z)|_{k+\frac{1}{2}} \gamma_j^\star & \text{if } p \mid N \end{cases}$$

where

$$\gamma_j^\star = \left(\begin{pmatrix} 1 & j \\ 0 & p^2 \end{pmatrix}, p^{\frac{1}{2}} \right);$$

$$\beta_j^\star = \left(\begin{pmatrix} p & j \\ 0 & p \end{pmatrix}, \varepsilon_p^{-1} \left(\frac{-j}{p} \right) \right) \text{ where } \varepsilon_p = 1 \text{ or } i \text{ depending on } p \equiv 1 \text{ or } 3 \bmod 4;$$

$$\alpha^\star = \left(\begin{pmatrix} p^2 & 0 \\ 0 & 1 \end{pmatrix}, p^{-\frac{1}{2}} \right).$$

Note 2.2.2. Here χ is a Dirichlet character modulo $4N$ and we assume it is always quadratic in our case, see section 2.1.1.

The Hecke operators are commutative, that is, $T_{p^2} T_{q^2} = T_{q^2} T_{p^2}$ for distinct primes p, q. See proof of [Shi73, Proposition 1.6]. In order to define T_{n^2} for any $n \in \mathbb{Z}_{>0}$, set $T_1 = 1$ (identity operator); T_{p^2} is already defined for primes p. For powers of squares of prime, define $T_{p^{2\nu}}$ using recursion formula [Pur14, Section 4]. Finally, we extend our definition to T_{n^2} for any $n \in \mathbb{Z}_{>0}$ by using mutiplicativity, $T_{n^2} = \prod_i T_{p_i^{2\nu_i}}$ where $n = p_i^{\nu_i}$.

Remark 2.2.3. We have only defined Hecke operators for squares of integers $n \in \mathbb{Z}_{>0}$. This is because for the double coset operator $|_{k+\frac{1}{2}}[\widetilde{\Gamma}_0(4N)\ \xi\ \widetilde{\Gamma}_0(4N)]$ where $\xi = \left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right), p^{1/4} \right)$ acts as zero operator on the spaces $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N), \chi)$, see [Shi73, pg. 450].

We now state the action of Hecke operator T_{p^2} on the Fourier coefficients of the half-integer weight modular form $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N), \chi)$ which is result by *Shimura*, see [Shi73, Theorem 1.7].

Proposition 2.2.4 (Hecke operators on q-expansion)**.** *Let $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N), \chi)$ with Fourier expansion $g(z) = \sum_{n=0}^{\infty} b_n q^n$. Then $g(z)|_k T_{p^2} \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N), \chi)$ with Fourier expansion $g(z)|_{k+\frac{1}{2}} T_{p^2} = \sum_{n=0}^{\infty} c_n q^n$ such that*

$$c_n = \begin{cases} b_{p^2 n} + \chi^\star(p) p^{k-1} \left(\frac{n}{p}\right) b_n + \chi^\star(p^2) p^{2k-1} b_{n/p^2} & \text{if } p \nmid N; \\ b_{p^2 n} & \text{if } p \mid N. \end{cases}$$

where $\chi^\star(\bullet) = \left(\frac{-1}{\bullet}\right)^k \chi(\bullet)$.
Here $b_{n/p^2} = 0$ when n/p^2 is not an integer

The fact that $g(z)|_{k+\frac{1}{2}} T_{p^2}$ lies in $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N), \chi)$ follows from [Shi73, pg. 450].

Definition 2.2.5 (Hecke algebra)**.** *The Hecke algebra of weight $k+\frac{1}{2}$, level $4N$ and character χ acting on $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N), \chi)$ is the commutative $\mathbb{C}$-subalgebra of End($M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N), \chi)$) generated by the Hecke operators T_{p^2} for all primes $p \in \mathbb{Z}_{>0}$.*
We denote the Hecke Algebra of weight $k+\frac{1}{2}$, level $4N$ and character χ by $\widetilde{\mathbb{T}}_{k+\frac{1}{2}}(4N, \chi)$.

Definition 2.2.6 (Hecke eigenform)**.** *We say $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N), \chi)$ is a Hecke eigenform of half-integer weight if it's a simultaneous eigenvector for all Hecke operators in $\widetilde{\mathbb{T}}_{k+\frac{1}{2}}(4N, \chi)$.*

Note 2.2.7. We will specify whenever we refer to a Hecke eigenform for all Hecke operators T_{p^2} for p not dividing the level. In general, our definition of a Hecke eigenform refers to $g(z)$ being an eigenvector for all operators T_{n^2} for $n \in \mathbb{Z}_{>0}$.

2.3 Further operators

In this section, we will introduce some operators on the space $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N),\chi)$ that will often show up later in this thesis.

2.3.1 Operator V_1

Let $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N),\chi)$. Let $d \in \mathbb{Z}_{>0}$ such that $(d,N)=1$. Note that $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N),\chi) \subset M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Nd),\chi)$. Hence, any half-integer weight modular form $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N),\chi)$ can always be viewed as a half-integer weight form in $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Nd),\chi)$. More formally, define an operator

$$|_{k+\frac{1}{2}}V_1 : M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N),\chi) \to M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Nd),\chi) \text{ such that } g(z) \mapsto g(z).$$

2.3.2 Operator V_d

Let $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N),\chi)$. Let $d \in \mathbb{Z}_{>0}$ such that $(d,N)=1$. Define the action of operator V_d on $g(z)$ in terms of the $\left(k+\frac{1}{2}\right)$-action in the following way,

$$\begin{aligned} g(z)|_{k+\frac{1}{2}}V_d &:= d^{-\frac{(2k+1)}{4}} g(z)|_{k+\frac{1}{2}}\left(\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}, d^{-\frac{1}{4}}\right) \\ &= d^{-\frac{(2k+1)}{4}} d^{\frac{(2k+1)}{4}} g(dz) \\ &= g(dz). \end{aligned}$$

Thus, if $g(z)=\sum_{n=0}^{\infty} b_n q^n$, then $g(z)|_{k+\frac{1}{2}}V_d = \sum_{n=0}^{\infty} b_n q^{dn}$.

Proposition 2.3.1. *Let $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N),\chi)$. Let $d \in \mathbb{Z}_{>0}$ such that $(d,N)=1$. Then $g(dz) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Nd),\chi')$ where $\chi'(\bullet)=\chi(\bullet)\left(\frac{d}{\bullet}\right)$.*

Proof. See [Shi73, Proposition 1.3]. ☐

Thus, we have a map

$$|_{k+\frac{1}{2}}V_d : M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N),\chi) \to M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Nd),\chi') \text{ such that } g(z) \mapsto g(dz).$$

2.3.3 Operator U_d

Let $d \in \mathbb{Z}_{>0}$ and let $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N),\chi)$ with Fourier expansion $g(z)=\sum_{n=0}^{\infty} b_n q^n$. Define the action of operator U_d on the Fourier coefficients of $g(z)$ in the following way,

$$g(z)|_{k+\frac{1}{2}}U_d := \sum_{n=0}^{\infty} b_{dn} q^n.$$

Proposition 2.3.2. *Let $d \in \mathbb{Z}_{>0}$ be a divisor of N. Then $|_{k+\frac{1}{2}}U_d$ maps the space $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N),\chi)$ to $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N),\chi')$ where $\chi'(\bullet)=\chi(\bullet)\left(\frac{d}{\bullet}\right)$.*

Proof. [Shi73, Proposition 1.5] ☐

Let $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N),\chi)$. Suppose ℓ be a prime such that $\ell \nmid N$. Then $g(z)|_{k+\frac{1}{2}}U_\ell$ can have at most level $4N\ell$. This can be observed by viewing $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N),\chi)$ as $g(z)|_{k+\frac{1}{2}}V_1 \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N\ell),\chi)$. Using proposition 1.3.2, we get $\left(g(z)|_{k+\frac{1}{2}}V_1\right)|_{k+\frac{1}{2}}U_\ell \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N\ell),\chi')$ where χ' where $\chi'(\bullet)=\chi(\bullet)\left(\frac{\ell}{\bullet}\right)$.

Again, let $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N), \chi)$. Suppose ℓ is a prime such that $\ell \nmid N$. Then $g(z)|_{k+\frac{1}{2}} U_{\ell^2}$ can have at most level $4N\ell$. This can be again observed by viewing $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N), \chi)$ as $g(z)|_{k+\frac{1}{2}} V_1 \subset M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N\ell), \chi)$. Using proposition 2.2.4, we get $\left(g(z)|_{k+\frac{1}{2}} V_1\right)|_{k+\frac{1}{2}} U_{\ell^2} \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N\ell), \chi)$.

Remark 2.3.3. We now make two remarks that will be useful later.

1. If prime $\ell \nmid N$ and $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N), \chi)$, then $g(z)|_{k+\frac{1}{2}} U_\ell$ can have at most level $4N\ell$ and the character of $g(z)|_{k+\frac{1}{2}} U_\ell$ changes to χ' where $\chi'(\bullet) = \chi(\bullet)\left(\frac{\ell}{\bullet}\right)$.

2. If prime $\ell \nmid N$ and $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N), \chi)$, then $g(z)|_{k+\frac{1}{2}} U_{\ell^2}$ can have at most level $4N\ell$ while the character of $g(z)|_{k+\frac{1}{2}} U_{\ell^2}$ remains same as χ.

2.4 *Shimura's* correspondence

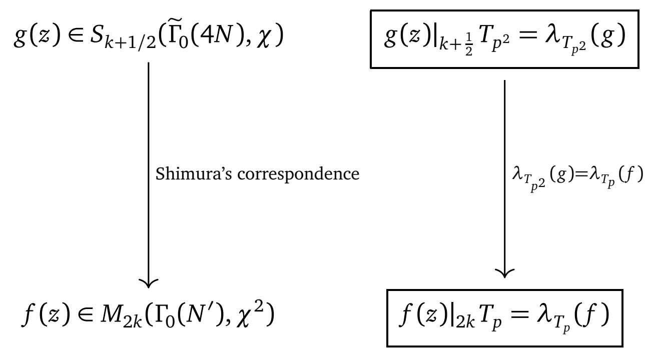
In 1973, *Goro Shimura* developed a beautiful connection between half-integer weight modular forms of weight $k + 1/2$ and integer weight modular forms of even weight $2k$ where $k \in \mathbb{Z}_{>0}$. This connection is known as the *Shimura's correspondence* [Shi73, Main Theorem, pg. 458].

Theorem 2.4.1 (*Shimura's* correspondence)**.** *Let $k, N \in \mathbb{Z}_{>0}$ and let χ be a Dirichlet character modulo $4N$. Suppose, we are given a non-zero half-integer weight modular form $g(z) \in S_{k+1/2}(\widetilde{\Gamma}_0(4N), \chi)$ which is an eigenform for the T_{p^2} operator over all primes p with corresponding eigenvalue $\lambda_{T_{p^2}}(g) \in \mathbb{C}$.*
Then there exists an even integer weight modular form $f(z) \in M_{2k}(\Gamma_0(N'), \chi^2)$ of weight $2k$, level N' dividing $4N$ and character χ^2 such that $f(z)$ is an eigenform for the T_p operator for all primes p with corresponding eigenvalue $\lambda_{T_p}(f) \in \mathbb{C}$ and for each p satisfies

$$\lambda_{T_p}(f) = \lambda_{T_{p^2}}(g).$$

Remark 2.4.2. Note that for $k \geq 2$, $f(z)$ will be a cusp form.

Figure 2.4.3.



In other words, if *Shi*( ) denotes the *Shimura's* map from $S_{k+1/2}(\widetilde{\Gamma}_0(4N), \chi) \to M_{2k}(\Gamma_0(N'), \chi^2)$, then for each prime $p \nmid N$

$$\mathrm{Shi}\left(g(z)|_{k+\frac{1}{2}} T_{p^2}\right) = \mathrm{Shi}(g(z))|_{2k} T_p.$$

For explicit relation between L-functions of $g(z)$ and $f(z)$ as originally stated in *Shimura's* Theorem, refer to [Shi73, Main Theorem, pg. 458].

Remark 2.4.4. We make a few remarks here that have been taken from taken from [Kob93, pg. 213].

1. *Shimura* proved that the level of N' in Theorem 2.4.1 divides $4N$. In 1975, *Niwa* [Niw75] showed one could always take N' as $2N$ but other choices are also possible.
2. When we refer to the image of a single eigenform $g(z)$ under the *Shimura's* map, we shall always mean normalised eigenform $f(z)$ in Theorem 2.4.1. But if we have a fixed basis $g_i(z)$ of eigenforms with $Shi(g_i(z)) = f_i(z)$ for each i, then we can extend the *Shimura's* map by linearity as $Shi(\sum_i a_i g_i(z)) = \sum_i a_i f_i(z)$ which is not necessarily a normalised eigenform.
3. The *Shimura's* map is not necessarily a one to one map. This mean we could land on the same $f(z)$ starting with two distinct $g(z)$'s in $S_{k+1/2}(\widetilde{\Gamma}_0(4N)), \chi)$ using Theorem 2.4.1.

2.5 *Kohnen's* isomorphism

Kohnen defined suitable subspaces of cusp forms of half-integer weight modular forms under which *Shimura's* correspondence gives an isomorphism with the corresponding space of newforms of integer weight. To begin with, we define the a subspace of space of half-integer weight cusp forms by imposing some conditions on the vanishing of Fourier coefficients. We call this subspace *Kohnen's plus space* and denote it by $S^+_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N), \chi)$.

Definition 2.5.1 (*Kohnen's* plus space)**.** *The Kohnen's plus space of $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N), \chi)$ is defined as*

$$S^+_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N), \chi) = \left\{ g(z) = \sum_{n=1}^{\infty} b_n q^n \in S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N), \chi) \mid b_n = 0 \text{ unless } \varepsilon(-1)^k n \equiv 1 \text{ or } 0 \bmod 4 \right\}$$

where $q = e^{2\pi i z}$.
Note here $\chi = \left(\frac{4\varepsilon}{\bullet}\right)\chi_N$ and $\varepsilon = \chi_N(-1)$ where χ_N is a primitive Dirichlet character modulo N which is assumed to be a quadratic character unless specified otherwise.

We now define the old and new subspaces of half-integer weight modular forms inside the *Kohnen's plus space*. In order to define half-integer weight oldforms, *Kohnen* used half-integer weight cusp form in $S^+_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N), \chi)$ and acted on them by U_{ℓ^2} operator for all primes $\ell \nmid N$. Recall that U_{ℓ^2} operator replaces the n^{th} Fourier coefficient of a half-integer weight modular form by $(\ell^2 n)^{\text{th}}$ coefficient. *Kohnen* makes the choice of U_{ℓ^2} operator over V_ℓ as the action of U_{ℓ^2} operator commutes with the *Shimura's* map.

Note 2.5.2. For simplicity, we will often avoid writing the character χ and take it to be trivial but all the definitions can be stated for a space with a non-trivial character as well.

We next define the old and new subspaces of the *Kohnen's* plus space.

Definition 2.5.3 (Old space of the *Kohnen's* plus space)**.**

$$S^{old}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N)) := \sum_{\substack{M|N \\ M<N}} \left(S^+_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4M)) + S^+_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4M))|_{k+\frac{1}{2}} U_{(N/M)^2} \right).$$

Definition 2.5.4 (New space of the *Kohnen's* plus space)**.** *The new space of half-integer weight cusp forms in the plus space is defined as the orthogonal complement of the old space defined in 2.5.3.*

$$S^{new}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N)) = \left(S^{old}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N)) \right)^{\perp}$$

As such, the *Kohnen's* plus space decomposes into old and new subspaces.

$$S^+_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N)) = S^{\text{old}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N)) \oplus S^{\text{new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N)).$$

We will now state our main results in this section by *Kohnen* which gives an isomorphism between eigenforms in $S^{\text{new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N))$ and integer weight eigenforms in $S^{\text{new}}_{2k}(\Gamma_0(N))$. For details and proofs, refer to [Koh82, Section 5, Theorem 2].

Theorem 2.5.5. *Let $N > 0$ be an odd, square-free integer.*

(i) $S^{+}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N)) = \bigoplus_{\substack{dM|N\\ d,M\geq 1}} S^{new}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4M))|_{k+\frac{1}{2}} U_{d^2}$.

(ii) *The space $S^{new}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N))$ has an orthogonal basis of common eigenforms for all operators T_{p^2} for prime $p \nmid N$, uniquely determined up to scalar multiplication with non-zero complex numbers. These are also eigenforms for the $T_{p^2}(= U_{p^2})$ operators for prime $p \mid N$ with eigenvalue p^{k-1}.*

(iii) *If $g(z) \in S^{new}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N))$ is an eigenform for the T_{p^2} operator for all primes p with eigenvalue $\lambda_{T_{p^2}}(g)$, then there exists an eigenform $f(z) \in S^{new}_{2k}(\Gamma_0(N))$ for the T_p operator for all primes p, uniquely determined up to multiplication with a non-zero complex number, which satisfies $f(z)|_{2k}T_p = \lambda_{T_{p^2}}(g)f(z)$.*
The same happens if we work the other way round, starting with an integer weight eigenform in the new space and passing on to half-integer weight eigenform in the new space.

Remark 2.5.6. From Theorem 1.4.7, if $f(z) = \sum_{n=1}^{\infty} a_n q^n \in S^{\text{new}}_{2k}(\Gamma_0(N))$ is normalised such that $a_1 = 1$, then there exists a fixed number field L_f such that for each n, $a_n \in \mathcal{O}_f$, where $\mathcal{O}_f$ is the ring of integers of L_f. Recall that we refer to such forms as *newforms* of integer weight. Let $g(z) = \sum_{n=1}^{\infty} b_n q^n \in S^{\text{new}}_{2k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N))$ be a half-integer weight eigenform that corresponds to $f(z)$ via *Kohnen's* isomorphism. It could be possible that $b_1 = 0$, and we cannot normalise $g(z)$ at all by dividing every Fourier coefficient by b_1. If $b_1 \neq 0$, then we try to normalise $g(z)$ such that $b_1 = 1$. However, this normalisation does not guarantee that the Fourier coefficients of $g(z)$ are algebraic integers. But there exists a normalisation of eigenform $g(z)$ such that its Fourier coefficient's lie in the same ring of integers $\mathcal{O}_f$ as it's corresponding integer weight eigenform $f(z)$, see [Ste94, Proposition 2.3.1]. We refer to such normalised half-integer weight eigenforms as *Kohnen newforms*.

Definition 2.5.7 (*Kohnen* Newforms). *Let $g(z) \in S^{new}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N))$ be an eigenform and the newform $f(z) \in S^{new}_{2k}(\Gamma_0(N))$ be its image under Kohnen's isomorphism. We refer to $g(z)$ as a Kohnen newform if its Fourier coefficients are algebraic integers in the same ring of integers that contains Fourier coefficients of newform $f(z)$.*

CHAPTER 3

Eisenstein series and related congruences

3.1 Classical Eisenstein Series

Let $k > 2$ be an even integer. For $z \in \mathcal{H}$, define the series $G_k(z)$ as,

$$G_k(z) := \sum_{m,n}{}' \frac{1}{(mz+n)^k} \tag{3.1}$$

where the summation runs over pairs of integers $(m,n) \neq (0,0)$.

Proposition 3.1.1. $G_k(z) \in M_k(\Gamma_0(1))$.

Proof. See [Kob93, III.2, Proposition 5, pg. 110]. ☐

Note 3.1.2. We have chosen k to be an even integer strictly greater than two. This is because for $k = 2$, $G_2(z)$ fails to converge absolutely and hence is not a modular form, see [Kob93, III.2, pg. 112].

Next, we give the Fourier expansion of $G_k(z)$.

Proposition 3.1.3. *Let $k > 2$ be an even integer and let $z \in \mathcal{H}$. Then the modular form defined in 3.1 has the following Fourier expansion*

$$G_k(z) = 2\zeta(k)\left(1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right) \text{ where } q = e^{2\pi i z}.$$

Here $\zeta(k) = \sum_{n=1}^{\infty}\frac{1}{n^k}$ is the Riemann-zeta function, B_k is the k^{th} Bernoulli number — which is a rational number defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}$$

and σ_k is the $(k-1)^{th}$ divisor sum of n defined as

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}.$$

Proof. See [Kob93, III.2, Proposition 6, pgs. 110-111]. ☐

Definition 3.1.4 (Normalised Eisenstein Series). *The normalised Eisenstein series $E_k(z)$ is obtained by dividing $G_k(z)$ by the constant $2\zeta(k)$, that is,*

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n \text{ where } q = e^{2\pi i z}.$$

Remark 3.1.5. It is clear from Definition 3.1.4 that $E_k(z)$ has rational Fourier coefficients.

3.2 Eisenstein series modulo p

Let p be an odd rational prime. We will now define a new series $\mathscr{E}_p(z)$ as follows.

$$\mathscr{E}_p(z) := \begin{cases} E_{p-1}(z) - p^{(p-1)}E_{p-1}(pz) & \text{if } p \geq 5; \\ E_2(z) - 3E_2(3z) & \text{if } p = 3. \end{cases}$$

We will now state [Ste07, Theorem 5.8, pg. 86] that implies $\mathscr{E}_p(z) \in M_{p-1}(\Gamma_0(p))$. This is originally proved by *Miyake* in [Miy89, Chapter 7].

Theorem 3.2.1. *Let $k \geq 4$ be an even integer and let be t be a positive integer. Then the power series $E_k(tz)$ defines an element of $M_k(\Gamma_0(t))$. Moreover, in the case of $k = 2$, $E_2(z) - tE_2(tz)$ is a modular form in $M_2(\Gamma_0(t))$.*

Thus, it follows from Theorem 3.2.1 that for $p \geq 5$, $\mathscr{E}_p(z)$ is a modular form in $M_{p-1}(\Gamma_0(p))$ and for the case $p = 3$, $\mathscr{E}_3(z)$ is a modular form in $M_2(\Gamma_0(3))$.

We next try to prove a congruence modulo p satisfied by $\mathscr{E}_p(z)$. In order to do so, we will need the von-Staudt-Clausen theorem about the denominators of Bernoulli numbers. This theorem was first proved Karl von Staudt [Sta40] and also by Thomas Clausen [Cla40] independently in 1840.

Theorem 3.2.2 (von-Staudt-Clausen). *Let $n \in \mathbb{N}$. Then*

$$B_{2n} + \sum_{(q-1)|2n} \frac{1}{q}$$

is an integer. In particular, the denominator of B_{2n} is exactly the product of primes q for which $(q-1)|2n$.

Now since p is an odd prime, $p - 1$ is even. It follows from Theorem 3.2.2 that

$$\text{denominator } B_{p-1} = \prod_{\substack{q \text{ prime} \\ (q-1)|(p-1)}} q$$

is divisible by p. In other words,

$$B_{p-1}^{-1} \equiv 0 \bmod p \tag{3.2}$$

Lemma 3.2.3. *Let p be an odd prime and $\mathscr{E}_p(z) \in M_{p-1}(\Gamma_0(p))$ be the modular defined as*

$$\mathscr{E}_p(z) = \begin{cases} E_{p-1}(z) - p^{(p-1)}E_{p-1}(pz) & \text{if } p \geq 5; \\ E_2(z) - 3E_2(3z) & \text{if } p = 3. \end{cases}$$

Then

$$\mathscr{E}_p(z) \equiv 1 \text{ mod } p.$$

Proof. We first observe that the Fourier coefficients of $\mathscr{E}_p(z)$ are p-integral. In other words, the Fourier coefficients of $\mathscr{E}_p(z)$ lie in $\mathbb{Z}_{(p)}$, where $\mathbb{Z}_{(p)} = \{(x/y) \in \mathbb{Q} \mid x, y \in \mathbb{Z} \text{ and } p \nmid y\}$. From definition 3.1.4, the Fourier expansion of $E_{p-1}(z)$ is

$$E_{p-1}(z) = 1 + 2(p-1)B_{p-1}^{-1}\sum_{n=1}^{\infty}\sigma_{p-2}(n)q^n \quad \text{where } q = e^{2\pi i z}.$$

Now Theorem 3.2.2 holds true only if B_{p-1}^{-1} is p-integral. Thus, we get that the Fourier coefficients of $E_{p-1}(z)$ lie in $\mathbb{Z}_{(p)}$ for $p \geq 5$.
Also, note that the Eisenstein series $E_2(z)$ is given by the following Fourier expansion

$$E_2(z) = 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n.$$

Hence, it is easy to see that the Fourier coefficients of $E_2(z)$ also lie in $\mathbb{Z}_{(3)}$.
Together, we can conclude using the definition of $\mathscr{E}_p(z)$ that its Fourier coefficients lie in $\mathbb{Z}_{(p)}$.

Now we show $\mathscr{E}_p(z) \equiv 1 \bmod p$. The von-Staudt Clausen Theorem 3.2.2 states that

$$B_{p-1}^{-1} \equiv 0 \bmod p.$$

Using this fact in the Fourier expansion of $E_{p-1}(z)$ in Definition 3.1.4, it follows,

$$E_{p-1}(z) \equiv 1 \bmod p$$

Hence, we get

$$\begin{aligned}\mathscr{E}_p(z) &\equiv E_{p-1}(z) \bmod p\\ &\equiv 1 \bmod p.\end{aligned}$$

□

Corollary 3.2.4. *Let p be an odd prime and $\mathscr{E}_p(z) \in M_{p-1}(\Gamma_0(p))$ be the modular defined as*

$$\mathscr{E}_p(z) = \begin{cases} E_{p-1}(z) - p^{(p-1)}E_{p-1}(pz) & \text{if } p \geq 5;\\ E_2(z) - 3E_2(3z) & \text{if } p = 3.\end{cases}$$

Let $j \in \mathbb{Z}_{\geq 0}$ be a fixed integer. Then

$$\mathscr{E}_p(z)^{p^j} \equiv 1 \text{ mod } p^j.$$

Proof. If $j = 0$, then result directly follows from lemma 3.2.3. So assume $j \in \mathbb{Z}_{>0}$.

Again using lemma 3.2.3, we have $\mathscr{E}_p(z) \equiv 1 \bmod p$. We can thus write $\mathscr{E}_p(z) = 1 + \mathscr{P}p$ where $\mathscr{P}$ is a power series in q whose coefficients are rational numbers and are integral at p. This implies

$$\mathscr{E}_p(z)^{p^j} = (1 + \mathscr{P}p)^{p^j}.$$

Using Binomial theorem, we can write

$$\begin{aligned}\mathscr{E}_p(z)^{p^j} &= \sum_{k=0}^{p^j}\binom{p^j}{k}(\mathscr{P}p)^k.\\ &= 1 + \sum_{k=1}^{p^j}\binom{p^j}{k}(\mathscr{P}p)^k.\end{aligned}$$

Let $k = r.p^i$ where i is a non-negative integer and r is a positive integer such that $p \nmid r$.

Let v_p be the standard p-adic valuation on $\mathbb{Q}$: For any integer x, define $v_p(x)$ as the the highest exponent of p in the prime factorisation of x. For an arbitrary rational number $\frac{x}{y}$, define $v_p\left(\frac{x}{y}\right) = v_p(x) - v_p(y)$. Then an immediate consequence of *Kummer's* Theorem [Kum52] is the fact that

$$v_p\left(\binom{p^j}{rp^i}\right) = (j-i)$$

which implies

$$v_p\left(\binom{p^j}{rp^i}(\mathscr{P}p)^{rp^i}\right) = (j-i) + rp^i.$$

If $i = 0$, then

$$\begin{aligned} v_p\left(\binom{p^j}{r}(\mathscr{P}p)^r\right) &= j + r \\ &\geq j+1. \end{aligned} \tag{3.3}$$

If $i \neq 0$, then

$$\begin{aligned} v_p\left(\binom{p^j}{k}(\mathscr{P}p)^{rp^i}\right) &= (j-i) + rp^i \\ &\geq (j-i) + p^i. \end{aligned}$$

Next, using induction, it is easy to see that $p^i > i$ for any positive integer i. Hence, for $i \neq 0$, we have $(j + p^i - i) > j$ or

$$v_p\left(\binom{p^j}{k}(\mathscr{P}p)^{rp^i}\right) \geq j+1. \tag{3.4}$$

From 3.3 and 3.4, it follows that p^{j+1} divides every binomial coefficient in the sum $\sum_{k=1}^{p^j}\binom{p^j}{k}(\mathscr{P}p)^k$.
Thus, we conclude

$$\mathscr{E}_p(z)^{p^j} \equiv 1 \bmod p^{j+1}.$$

□

3.3 Generalised Eisenstein Series

Let $k,\ N \in \mathbb{Z}_{>0}$. Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ be a non-trivial primitive Dirichlet character. We now introduce a new series $G_{k,\chi}(z)$ with a character χ, called the generalised Eisenstein series.

$$G_{k,\chi}(z) := \sum_{(m,n)}{}' \chi^{-1}(n)(mNz+n)^{-k} \tag{3.5}$$

where $\sum'$ indicates that the sum is taken over all ordered pairs of integers excluding $(0,0)$.

Proposition 3.3.1. $G_{k,\chi}(z) \in M_k(\Gamma_0(N), \chi)$.

Proof. For proof, see [Miy89, Chapter 7]. □

Note 3.3.2. In the definition above, k is allowed to be any positive integer including $k = 1$ and $k = 2$, see [Miy89, Section 7.2]. This is allowed as long as character χ is non-trivial. In general, $G_{k,\chi}(z)$ is always a modular form in $M_k(\Gamma_0(N), \chi)$ unless both $k = 2$ and character χ is trivial.

Next, we give the Fourier expansion of $G_{k,\chi}(z)$.

Proposition 3.3.3. *The modular form defined in 3.5 has the Fourier expansion*

$$G_{k,\chi}(z) = 2N^{-k}\frac{\chi(-1)N}{\mathfrak{g}(\chi)}\frac{(-2\pi\sqrt{-1})^k}{(k-1)!}\left(\frac{-B_{k,\chi}}{2k}\right)\left(1-\frac{2k}{B_{k,\chi}}\sum_{n=1}^{\infty}\sigma_{k-1,\chi}(n)q^n\right) \tag{3.6}$$

where $\mathfrak{g}(\chi)$ *is the Gauss sum given by* $\sum_{j=1}^{N}\chi(j)e^{2\pi i n j/N}$, $B_{k,\chi}$ *is the kth generalised Bernoulli number, that is a rational number defined by the generating function*

$$\sum_{j=1}^{N}\chi(j)\frac{xe^{jx}}{e^{Nx}-1} = \sum_{k=0}^{\infty}B_{k,\chi}\frac{x^k}{k!}$$

and $\sigma_{k-1,\chi}$ *is the* $(k-1)^{th}$ *generalised divisor sum defined as* $\sigma_{k-1,\chi}(n) = \sum_{d|n}\chi(d)d^{k-1}$.

Proof. See [Hid93, pgs. 127-28]. □

Definition 3.3.4 (Normalised generalised Eisenstein series). *We define the normalised generalised Eisenstein series as*

$$\begin{aligned} E_{k,\chi}(z) &:= \left[2N^{-k}\frac{\chi(-1)N}{\mathfrak{g}(\chi)}\frac{(-2\pi\sqrt{-1})^k}{(k-1)!}\left(\frac{-B_{k,\chi}}{2k}\right)\right]^{-1} G_{k,\chi}(z) \\ &= 1-\frac{2k}{B_{k,\chi}}\sum_{n=1}^{\infty}\sigma_{k-1,\chi}(n)q^n \end{aligned}$$

where $q = e^{2\pi i z}$.

Remark 3.3.5. It is clear from Definition 3.3.4 that $E_{k,\chi}(z)$ has rational Fourier coefficients.

3.4 Generalised Eisenstein series modulo p

Let p be an odd rational prime. Let $\chi_p = \left(\frac{\bullet}{p}\right)$ be the Kronecker symbol which is clearly non-trivial. We then define $\widetilde{\mathscr{E}}_p(z)$ to be the generalised Eisenstein series of weight $\frac{p-1}{2}$, level p and character χ_p. More precisely,

$$\widetilde{\mathscr{E}}_p(z) := E_{\frac{p-1}{2},\chi_p}(z).$$

We next try to prove a congruence modulo p satisfied by $\widetilde{\mathscr{E}}_p(z)$. In order to do so, we will need an analogue of von-Staudt-Clausen Theorem 3.2 for generalised Bernoulli numbers. This is stated in [Ste07, Theorem 5.7] and identifies the denominator of generalised Bernoulli numbers.

Theorem 3.4.1. *Let p be an odd rational prime and* $\chi_p = \left(\frac{\bullet}{p}\right)$ *be the Krocnecker symbol. Then*

$$\left(B_{\frac{p-1}{2},\chi_p}\right)^{-1} \equiv 0 \ mod\ p.$$

We omit details here but for proof, see [Car59a] and [Car59b].

Lemma 3.4.2. *Let p be an odd rational prime and* $\chi_p = \left(\frac{\bullet}{p}\right)$ *be the Kronecker symbol. Let* $\widetilde{\mathscr{E}}_p(z) \in M_{\frac{p-1}{2}}(\Gamma_0(p), \chi_p)$ *be the modular form defined as*

$$\widetilde{\mathscr{E}}_p(z) = E_{\frac{p-1}{2},\chi_p}(z)$$

Then

$$\widetilde{\mathscr{E}}_p(z) \equiv 1 \ mod \ p.$$

Proof. From the definition 3.3.4, the Fourier expansion of $E_{\frac{p-1}{2},\left(\frac{\bullet}{p}\right)}(z)$ is

$$E_{\frac{p-1}{2},\left(\frac{\bullet}{p}\right)}(z) = 1 + (p-1)\left(B_{\frac{p-1}{2},\left(\frac{\bullet}{p}\right)}\right)^{-1} \sum_{n=1}^{\infty} \sigma_{\frac{p-3}{2}}(n) q^n; \qquad q = e^{2\pi i z}.$$

Note that Theorem 3.4.1 holds true and this means that the Fourier coefficients of $E_{\frac{p-1}{2},\left(\frac{\bullet}{p}\right)}(z)$ must lie in $\mathbb{Z}_{(p)}$. Thus, we can use the definition of $\widetilde{\mathscr{E}}_p(4z)$ to conclude that its Fourier coefficients lie in $\mathbb{Z}_{(p)}$.

From Theorem 3.4.1, we have

$$\left(B_{\frac{p-1}{2},\chi_p}\right)^{-1} \equiv 0 \bmod p.$$

Using this fact in the Fourier expansion of $E_{\frac{p-1}{2},\chi_p}(z)$ in definition 3.3.4, it follows

$$E_{\frac{p-1}{2},\chi_p}(z) \equiv 1 \bmod p$$

or

$$\widetilde{\mathscr{E}}_p(z) \equiv 1 \bmod p.$$

□

CHAPTER 4

Ordinary modular forms and p-stabilisation

4.1 Introduction

Let $k \in \mathbb{Z}_{>0}$ and let $N \in \mathbb{Z}_{>0}$ be an odd, square-free integer. Also, let p be an odd rational prime.

Let $f(z) \in S_{2k}^{new}(\Gamma_0(N))$ be a newform. Then by Theorem 1.4.7, the Fourier coefficients of $f(z)$ lie in a ring of integers $\mathcal{O}_f$ of some fixed number field L_f. Let $g(z) \in S_{k+\frac{1}{2}}^{\text{new}}(\widetilde{\Gamma_0}(4N))$ be the image of $f(z)$ under *Kohnen's* isomorphism that is normalised such that its Fourier coefficients lie in the same ring of integers as that of $f(z)$, that is its corresponding integer weight newform under *Kohnen's* map. This normalisation exists, see remark 2.5.6. Now, let us take a big enough number field $L/\mathbb{Q}$ containing number fields L_f for all newforms $f(z)$ in $S_{2k}^{new}(\Gamma_0(N))$.

Let $\mathcal{O}_L$ be the ring of integers of L. In general, any prime ideal $\mathfrak{p}$ of $\mathcal{O}_L$ determines a valuation $v_{\mathfrak{p}}$ of L, up to a scalar.

Definition 4.1.1 ($\mathfrak{p}$-adic valuation). *For $x \in \mathcal{O}_L$, we define the $\mathfrak{p}$-adic valuation $v_{\mathfrak{p}}(x) = n$ where n is the highest integer such that $x \in \mathfrak{p}^n$. For any $\alpha = \frac{x}{y} \in L$, we define $v_{\mathfrak{p}}(\alpha) = v_{\mathfrak{p}}(x) - v_{\mathfrak{p}}(y)$.*

Remark 4.1.2. If the prime ideal $\mathfrak{p}$ lies above p, then it extends the p-adic valuation. It is possible to normalise the valuation $v_{\mathfrak{p}}$ differently by choosing any integer $t \in \mathbb{N}$ and defining $v_{\mathfrak{p}}(x) = \frac{n}{t}$. If we choose $t = 1$, then the valuation is normalised to have values in $\mathbb{Z}$. However, this does not guarantee that its restriction to $\mathbb{Q}$ will have image in $\mathbb{Z}$. For this reason, we now fix prime ideal $\mathfrak{p}$ above p throught out this chapter and normalise $v_{\mathfrak{p}}$ suitably so that $v_{\mathfrak{p}}(p) = v_p(p) = 1$, where v_p is the standard p-adic valuation on $\mathbb{Q}$. This normalisation is achieved by taking $t = e_{\mathfrak{p}}$ where $e_{\mathfrak{p}}$ is the ramification index of $\mathfrak{p}$ in factorisation of $p\mathcal{O}_L$.
For a power series $f(z) = \sum_{n=1}^{\infty} a_n q^n$, we define

$$v_p(f(z)) := \inf(v_p(a_n)).$$

Let $F^{\mathfrak{p}}$ be a finite extension of $\mathbb{Q}_p$ containing L that extends v_p to $v_{\mathfrak{p}}$. Let $\mathcal{O}_{F^{\mathfrak{p}}}$ be its corresponding ring of integers. We can then embed $L \hookrightarrow \overline{\mathbb{Q}_p}$ or embed $L \hookrightarrow \mathbb{C}$. It therefore makes sense to view Fourier coefficients $\mathfrak{p}$-adically embedded in $\mathcal{O}_{F^{\mathfrak{p}}}$.

Let $S_{2k}(\Gamma_0(N); \mathcal{O}_L)$ be the $\mathcal{O}_L$-submodule of $S_{2k}(\Gamma_0(N); L)$ containing cusp forms in $S_{2k}(\Gamma_0(N); L)$ that have Fourier coefficients in $\mathcal{O}_L$. Now define

$$S_{2k}(\Gamma_0(N); \mathcal{O}_{F^{\mathfrak{p}}}) := S_{2k}(\Gamma_0(N); \mathcal{O}_L) \otimes_{\mathcal{O}_L} \mathcal{O}_{F^{\mathfrak{p}}}.$$

Let $\mathbb{T}_{2k}(N;\mathcal{O}_L)$ be the commutative $\mathcal{O}_L$-subalgebra of the $\mathrm{End}(S_{2k}(\Gamma_0(N);L)$ generated by T_m where $m \in \mathbb{Z}_{>0}$. Define

$$\mathbb{T}_{2k}(N;\mathcal{O}_{F^{\mathfrak{p}}}) := \mathbb{T}_{2k}(N;\mathcal{O}_L) \otimes_{\mathcal{O}_L} \mathcal{O}_{F^{\mathfrak{p}}}.$$

Similarly, let $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N);\mathcal{O}_L)$ be the $\mathcal{O}_L$-submodule of $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N);L)$ containing cusp forms in $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N);L)$ that have Fourier coefficients in $\mathcal{O}_L$. Now define

$$S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N);\mathcal{O}_{F^{\mathfrak{p}}}) := S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N);\mathcal{O}_L) \otimes_{\mathcal{O}_L} \mathcal{O}_{F^{\mathfrak{p}}}.$$

Let $\widetilde{\mathbb{T}}_{k+\frac{1}{2}}(4N;\mathcal{O}_L)$ be the commutative $\mathcal{O}_L$-subalgebra of the $\mathrm{End}(S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N);L))$ generated by T_{m^2} where $m \in \mathbb{Z}_{>0}$. Define

$$\widetilde{\mathbb{T}}_{k+\frac{1}{2}}(4N;\mathcal{O}_{F^{\mathfrak{p}}}) := \widetilde{\mathbb{T}}_{k+\frac{1}{2}}(4N;\mathcal{O}_L) \otimes_{\mathcal{O}_L} \mathcal{O}_{F^{\mathfrak{p}}}.$$

4.2 Ordinary modular forms of integer weight

Let $f(z) \in S_{2k}(\Gamma_0(N);\mathcal{O}_L)$ be a cusp form. We intend to define an idempotent element in $\mathbb{T}_{2k}(N;\mathcal{O}_{F^{\mathfrak{p}}})$.

Definition 4.2.1 (p-ordinary projector). *Let p be an odd rational prime. Then define the $\mathfrak{p}$-adic limit*

$$e := \lim_{n\to\infty} T_p^{n!}.$$

The limit $e \in \mathbb{T}_{2k}(N;\mathcal{O}_{F^{\mathfrak{p}}})$ exists and e satisfies $e^2 = e$ [Hid93, Lemma 1, pg. 201].

Note 4.2.2. Note that for $p \mid N$, $T_p = U_p$ and we can alternatively write $e = \lim_{n\to\infty} U_p^{n!}$.

We will now define p-ordinary cusp forms.

Definition 4.2.3 (Ordinary modular forms). *Let $f(z) \in S_{2k}(\Gamma_0(N);\mathcal{O}_L)$ be a cusp form. Then $f(z)$ is p-ordinary if*

$$f(z)|_{2k}e = f(z).$$

The image of $S_{2k}(\Gamma_0(N);\mathcal{O}_L)$ under the ordinary projection by $|_{2k}e$ is called the space of ordinary cusp forms. We denote the subspace of ordinary cusp forms in $S_{2k}(\Gamma_0(N);\mathcal{O}_L)$ by $S_{2k}^{ord}(\Gamma_0(N);\mathcal{O}_L)$.

We now make a few observations about p-ordinary projection of eigenforms in $S_{2k}(\Gamma_0(N);\mathcal{O}_L)$. Let $f(z) \in S_{2k}(\Gamma_0(N);\mathcal{O}_L)$ be a T_p eigenform and let its corresponding eigenvalue in $\mathcal{O}_L$ be $\lambda_{T_p}(f)$. Then we have

$$f(z)|_{2k}e = \begin{cases} f(z) & \text{if } |\lambda_{T_p}(f)|_{\mathfrak{p}} = 1; \\ 0 & \text{if } |\lambda_{T_p}(f)|_{\mathfrak{p}} < 1. \end{cases}$$

Thus, the T_p eigenform $f(z)$ is said to be p-ordinary if its T_p eigenvalue $\lambda_{T_p}(f)$ is a $\mathfrak{p}$-adic unit for the fixed prime ideal $\mathfrak{p} \subset \mathcal{O}_L$ lying above p.

4.3 Control Theorem

We consider the level of cuspidal space to be N as before. Let p be an odd prime such that $p \mid N$ but $p^2 \nmid N$. Suppose our cuspidal space $S_{2k}(\Gamma_0(N),\chi;\mathcal{O}_L)$ has a fixed Dirichlet character χ mod N and let $\mathcal{O}_L[\chi]$ contain all values of χ. We now state the *Hida's* Theorem of constancy of p-ordinary rank. For this, we first need to introduce the Teichmüller character.

Definition 4.3.1 (Teichmüller character)**.** *The Teichmüller character is a homomorphism between multiplicative groups*

$$\omega : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{Z}_p^\times$$

such that

- $\omega(x)$ *is a unique* $(p-1)^{th}$ *root of unity in* $\mathbb{Z}_p$*;*
- $\omega(x) \equiv x \bmod p$*.*

Theorem 4.3.2. *Let* $N \in \mathbb{Z}_{>0}$ *and let* p *be an odd rational prime such that* $p \mid N$ *but* $p^2 \nmid N$*. Let* ω *be the Teichmüller character defined in 4.3.1. Then the rank of* $\left(S_{2k}^{ord}(\Gamma_0(N), \chi\omega^{-2k}; \mathcal{O}_L[\chi])\right)$ *is constant.*

This theorem is proved and originally stated by *Hida* in [Hid93, Theorem 3, pg.215].

Note 4.3.3. We have taken our weight to be an even integer $2k$ so that we can later have a smooth transition to half-integer weight modular forms of weight $k+\frac{1}{2}$ under *Kohnen's* map. However, the original theorem is valid for all integer weight strictly greater than 1.

We can choose χ to be a suitable power of the Teichmüller map ω and obtain the following corollary to Theorem 4.3.2.

Corollary 4.3.4. *Let* p *be an odd rational prime and let* k*,* $k' \in \mathbb{Z}_{>0}$ *such that* $2k \equiv 2k' \bmod (p-1)$*. Let* N *be a positive integer such that* $p \mid N$ *but* $p^2 \nmid N$*. Then*

$$rank\left(S_{2k}^{ord}(\Gamma_0(N); \mathcal{O}_L)\right) = \ rank\ \left(S_{2k'}^{ord}(\Gamma_0(N); \mathcal{O}_L)\right).$$

4.4 p-stabilisation of modular forms of integer weight

Let $k \in \mathbb{Z}_{>0}$. Let N be an odd, positive and square-free integer. Let p be an odd prime that is co-prime to N. Let

$$f(z) = \sum_{n=1}^{\infty} a_n q^n \in S_{2k}^{new}(\Gamma_0(N); \mathcal{O}_L)$$

be a newform. Even though p does not divide the level N of $f(z)$, it is possible to force p in the level by passing to a p-oldform.

Note that $f(z)$ is a newform and hence is an eigenvector for the T_p operator. Let the eigenvalue of $f(z)$ for the T_p operator be denoted by $\lambda_{T_p}(f)$. Then

$$f(z)|_{2k}T_p = \lambda_{T_p}(f)f(z). \tag{4.1}$$

Let V_1 and V_p be operators defined in section 1.3 that map the space $S_{2k}(\Gamma_0(N); \mathcal{O}_L)$ to $S_{2k}(\Gamma_0(Np); \mathcal{O}_L)$ and act as

$$|_{2k}V_1 : f(z) \mapsto f(z) \quad \text{ and } \quad |_{2k}V_p : f(z) \mapsto f(pz).$$

From Theorem 1.2.3, T_p acts on $f(z)$ as follows

$$f(z)|_{2k}T_p = f(z)|_{2k}U_p + p^{2k-1}f(pz).$$

Using 4.1 and rearranging the terms, we can write

$$f(z)|_{2k}U_p = \lambda_{T_p}(f)f(z) - p^{2k-1}f(pz) \tag{4.2}$$

or

$$f(z)|_{2k}U_p = \lambda_{T_p}(f)(f(z)|_{2k}V_1) - p^{2k-1}(f(z)|_{2k}V_p).$$

Next, we act 4.2 by U_p again and get

$$\begin{aligned} f(z)|_{2k}U_p^2 &= \lambda_{T_p}(f)f(z)|_{2k}U_p - p^{2k-1}f(pz)|_{2k}U_p \\ &= \lambda_{T_p}(f)f(z)|_{2k}U_p - p^{2k-1}f(z). \end{aligned}$$

This relation rewrites as

$$f(z)|_{2k}(U_p^2 - \lambda_{T_p}(f)U_p + p^{2k-1}) = 0.$$

Thus U_p satisfies the quadratic polynomial $x^2 - \lambda_{T_p}(f)x + p^{2k-1}$ on the two-dimensional subspace spanned by $\{f(z), f(z)|_{2k}U_p\}$ that has level at most Np, see remark 1.3.3. We may factor this quadratic polynomial as $(x-\alpha_p)(x-\beta_p)$ where α_p and β_p are algebraic integers satisfying $\alpha_p + \beta_p = \lambda_{T_p}(f)$ and $\alpha_p\beta_p = p^{2k-1}$.

We now define two U_p eigenforms in $S_{2k}^{p\text{-old}}(\Gamma_0(Np);\mathcal{O}_L)$ below:

$$f_{\alpha_p}(z) := f(z)|_{2k}(U_p - \beta_p) \quad \text{such that} \quad f_{\alpha_p}(z)|_{2k}U_p = \alpha_p f_{\alpha_p}(z)$$

and

$$f_{\beta_p}(z) := f(z)|_{2k}(U_p - \alpha_p) \quad \text{such that} \quad f_{\beta_p}(z)|_{2k}U_p = \beta_p f_{\beta_p}(z).$$

We call $f_{\alpha_p}(z)$ and $f_{\beta_p}(z)$, the p-stabilised forms at level Np associated to the newform $f(z)$ at level N.

Remark 4.4.1. Now if $f(z) \in S_{2k}^{\text{new, ord}}(\Gamma_0(N);\mathcal{O}_L)$ is p-ordinary, then its T_p eigenvalue must be a $\mathfrak{p}$-adic unit, where $\mathfrak{p}$ is the fixed prime ideal above p. In other words, $|\lambda_{T_p}(f)|_{\mathfrak{p}} = 1$. We can therefore choose α_p to be a $\mathfrak{p}$-adic unit. This then fixes a unique ordinary p-stabilised form $f_{\alpha_p}(z) \in S_{2k}^{\text{ord}}(\Gamma_0(Np);\mathcal{O}_L)$ with U_p eigenvalue being α_p, a $\mathfrak{p}$-adic unit.

4.5 Ordinary modular forms of half-integer weight

Let $g(z) \in S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N);\mathcal{O}_L)$ be a half-integer weight cusp form visualised to lie in $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N);\mathcal{O}_F)$. We intend to define an idempotent in $\widetilde{\mathbb{T}}_{k+\frac{1}{2}}(4N;\mathcal{O}_{F^{\mathfrak{p}}})$.

Definition 4.5.1 (p-ordinary projector)**.** *Let p be an odd rational prime. Then for each prime p, define the $\mathfrak{p}$-adic limit*

$$\widetilde{e} := \lim_{n\to\infty} T_{p^2}^{n!}$$

The limit $\widetilde{e} \in \widetilde{\mathbb{T}}_{k+\frac{1}{2}}(4N;\mathcal{O}_{F^{\mathfrak{p}}})$ exists and $\widetilde{e}$ satisfies $\widetilde{e}^2 = \widetilde{e}$ [Hid93, Lemma 1, pg. 201].

Note 4.5.2. Note that for $p \mid N$, $T_{p^2} = U_{p^2}$ and we can alternatively write $\widetilde{e} = \lim_{n\to\infty} U_{p^2}^{n!}$.

We will now define p-ordinary cusp forms of half-integer weight.

Definition 4.5.3. *Let $g(z) \in S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N);\mathcal{O}_L)$ be a half-integer weight cusp form. Then $g(z)$ is said to be p-ordinary if*

$$g(z)|_{k+\frac{1}{2}}\widetilde{e} = g(z).$$

The image of $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N);\mathcal{O}_L)$ under the ordinary projection by $|_{k+\frac{1}{2}}\widetilde{e}$ is called the space of half-integer weight ordinary cusp forms. We denote the space of half-integer weight ordinary cusp forms in $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N);\mathcal{O}_L)$ by $S_{k+\frac{1}{2}}^{ord}(\widetilde{\Gamma}_0(4N);\mathcal{O}_L)$.

Let $g(z) \in S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N);\mathscr{O}_L)$ be a T_{p^2} eigenform for some prime $p \mid N$ (resp. $p \nmid N$). Let $\lambda_{T_{p^2}}(g)$ denote the Hecke-eigenvalue of $g(z)$ under the action of T_{p^2} operator, that is,

$$g(z)|_{k+\frac{1}{2}} T_{p^2} = \lambda_{T_{p^2}}(g) g(z).$$

Then we have

$$g(z)|_{k+\frac{1}{2}} \widetilde{e} = \begin{cases} g(z) & \text{if } |\lambda_{T_{p^2}}(g)|_{\mathfrak{p}} = 1; \\ 0 & \text{if } |\lambda_{T_{p^2}}(g)|_{\mathfrak{p}} < 1. \end{cases}$$

Thus, T_{p^2} eigenform $g(z)$ for all primes $p \mid N$ (resp. $p \nmid N$) is said to be p-ordinary if its T_{p^2} eigenvalue $\lambda_{T_{p^2}}(g)$ is a $\mathfrak{p}$-adic unit for the fixed prime ideal $\mathfrak{p} \subset \mathscr{O}_L$ lying above p.

4.6 p-stabilisation of modular forms of half-integer weight

Let $k \in \mathbb{Z}_{>0}$. Let N be an odd, positive and square-free integer. Let p be an odd prime that is co-prime to N.

Let

$$g(z) = \sum_{n=1}^{\infty} b_n q^n \in S^{\text{new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N);\mathscr{O}_L)$$

be a *Kohnen* newform. Even though p does not divide the level $4N$ of $g(z)$, it is possible to force p in the level by passing to an oldform of half-integer weight.

Note that $g(z)$ is a newform of half-integer weight and hence is an eigenvector for the T_{p^2} operator for $p \nmid N$. Let the eigenvalue of $g(z)$ for the T_{p^2} operator be denoted by $\lambda_{T_{p^2}}(g)$.

$$g(z)|_{k+\frac{1}{2}} T_{p^2} = \lambda_{T_{p^2}}(g) g(z). \tag{4.3}$$

Let V_1 be a map from space $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N);L)$ to $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np);L)$ and let V_{p^2} and $V_{\left(\frac{\bullet}{p}\right)}$ be maps defined from the space $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N);L)$ to $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np^2);L)$ given by

$$|_{k+\frac{1}{2}} V_1 : g(z) \mapsto g(z), \quad |_{k+\frac{1}{2}} V_{p^2} : g(z) \mapsto g(pz) \quad \text{and} \quad |_{k+\frac{1}{2}} V_{\left(\frac{\bullet}{p}\right)} : g(z) \mapsto g_{\left(\frac{\bullet}{p}\right)}(z)$$

respectively.

Note 4.6.1. V_1 and V_p have been defined explicitly before in section 6.3. Here $g_{\left(\frac{\bullet}{p}\right)}(z) = \sum_{n=1}^{\infty} \left(\frac{n}{p}\right) b_n q^n$ is the twist of $g(z)$ by the Legendre symbol $\left(\frac{\bullet}{p}\right)$. The level of $g_{\left(\frac{\bullet}{p}\right)}(z)$ is irrelevant to our result as we will see next that it's killed under the action of the U_{p^2} operator. So we skip the details regarding the level of $g_{\left(\frac{\bullet}{p}\right)}(z)$.

From Theorem 2.2.4, T_{p^2} operator acts on the Fourier coefficients of $g(z)$ as follows,

$$g(z)|_{k+\frac{1}{2}} T_{p^2} = g(z)|_{k+\frac{1}{2}} U_{p^2} + \left(\frac{-1}{p}\right)^k p^{k-1} g_{\left(\frac{\bullet}{p}\right)}(z) + p^{2k-1} g(p^2 z).$$

Using equation 4.3 and rearranging the terms, we can write

$$g(z)|_{k+\frac{1}{2}} U_{p^2} = \lambda_{T_{p^2}}(g) g(z) - \left(\frac{-1}{p}\right)^k p^{k-1} g_{\left(\frac{\bullet}{p}\right)}(z) - p^{2k-1} g(p^2 z) \tag{4.4}$$

or

$$g(z)|_{k+\frac{1}{2}} U_{p^2} = \lambda_{T_{p^2}}(g) \left(g(z)|_{k+\frac{1}{2}} V_1\right) - \left(\frac{-1}{p}\right)^k p^{k-1} \left(g(z)|_{k+\frac{1}{2}} V_{\left(\frac{\bullet}{p}\right)}\right) - p^{2k-1} \left(g(z)|_{k+\frac{1}{2}} V_{p^2}\right).$$

Next we act 4.4 by U_{p^2} again and get

$$g(z)|_{k+\frac{1}{2}}U_{p^2}^2 = \lambda_{T_{p^2}}(g)g(z)|_{k+\frac{1}{2}}U_{p^2} - \left(\frac{-1}{p}\right)^k p^{k-1} g_{\left(\frac{\bullet}{p}\right)_2}(z)|_{k+\frac{1}{2}}U_{p^2} - p^{2k-1}g(p^2z)|_{k+\frac{1}{2}}U_{p^2}.$$

From the definition of the Legendre symbol, $\left(\frac{p^2n}{p}\right) = 0$. Thus,

$$\begin{aligned} g_{\left(\frac{\bullet}{p}\right)}(z)|_{k+\frac{1}{2}}U_{p^2} &= \sum_{n=1}^{\infty}\left(\frac{p^2n}{p}\right) b_{p^2n}q^n \\ &= 0. \end{aligned}$$

Hence, we conclude $g_{\left(\frac{\bullet}{p}\right)}(z)$ lies in the kernel of U_{p^2}. It follows,

$$g(z)|_{k+\frac{1}{2}}U_{p^2}^2 = \lambda_{T_{p^2}}(g)g(z)|_{k+\frac{1}{2}}U_{p^2} - p^{2k-1}g(z).$$

This relation rewrites as

$$g(z)|_{k+\frac{1}{2}}\left(U_{p^2}^2 - \lambda_{T_{p^2}}(g)U_{\mathfrak{p}^2} + p^{2k-1}\right) = 0.$$

Thus, U_{p^2} satisfies the quadratic polynomial $x^2 - \lambda_{T_{p^2}}(g)x + p^{2k-1}$ on the two dimensional space spanned by $\{g(z), g(z)|_{k+\frac{1}{2}}U_{p^2}\}$ that has level at most $4Np$, see remark 2.3.3. We may factor this quadratic polynomial as $(x-\alpha_p)(x-\beta_p)$ where α_p and β_p are algebraic integers satisfying $\alpha_p + \beta_p = \lambda_{T_{p^2}}(g)$ and $\alpha_p\beta_p = p^{2k-1}$.

We now define two U_{p^2} eigenforms at level $4Np$,

$$g_{\alpha_p}(z) := g(z)|_{k+\frac{1}{2}}(U_{p^2} - \beta_p) \quad \text{such that} \quad g_{\alpha_p}(z)|_{k+\frac{1}{2}}U_{p^2} = \alpha_p g_{\alpha_p}(z)$$

and

$$g_{\beta_p}(z) := g(z)|_{k+\frac{1}{2}}(U_{p^2} - \alpha_p) \quad \text{such that} \quad g_{\beta_p}(z)|_{k+\frac{1}{2}}U_{p^2} = \beta_p g_{\beta_p}(z).$$

We call $g_{\alpha_p}(z)$ and $g_{\beta_p}(z)$, the p-stabilised forms at level $4Np$ associated to the half-integer weight newform $g(z)$ at level $4N$.

Remark 4.6.2. Now if $g(z) \in S_{k+\frac{1}{2}}^{\text{new, ord}}(\widetilde{\Gamma}_0(4N); \mathcal{O}_L)$ is p-ordinary, then its T_{p^2} eigenvalue must be a $\mathfrak{p}$-adic unit, where $\mathfrak{p}$ is the fixed prime ideal above p. In other words, $|\lambda_{T_{p^2}}(g)|_{\mathfrak{p}} = 1$. We can therefore choose α_p to be a $\mathfrak{p}$-adic unit. This then fixes a unique ordinary p-stabilised form $g_{\alpha_p}(z) \in S_{k+\frac{1}{2}}^{\text{ord}}(\widetilde{\Gamma}_0(4Np); \mathcal{O}_L)$ with U_{p^2} eigenvalue being α_p, a $\mathfrak{p}$-adic unit.

CHAPTER 5

Congruences related to modular forms of integer & half-integer weight

5.1 Notation

Let $k \in \mathbb{Z}_{>0}$ and let N be a positive, odd, square-free integer. Fix an odd rational prime p that is coprime to N.

Let M be a positive integer such that $M \mid Np$. Let $f(z) \in S_{2k}^{\text{new}}(\Gamma_0(M))$ be a newform. Then by Theorem 1.4.7, for any normalised integer weight eigenform, in particular for $f(z) = \sum_{n=1}^{\infty} a_n q^n$, there exists a fixed number field L_f with ring of integers $\mathcal{O}_f$ such that for each $n \in \mathbb{Z}_{>0}$, $a_n \in \mathcal{O}_f$.

Let $g(z) \in S_{k+\frac{1}{2}}^{\text{new}}(\widetilde{\Gamma}_0(4M))$ be a *Kohnen* newform associated to $f(z)$ via *Kohnen's* isomorphism that is normalised in a way that its Fourier coefficients lie in the same ring of integers as that of $f(z)$, see remark 2.5.6. More precisely, if $g(z) = \sum_{n=1}^{\infty} b_n q^n$, then for each $n \in \mathbb{Z}_{>0}$, $b_n \in \mathcal{O}_f$.

Let $j \in \mathbb{Z}_{>0}$ be a fixed integer and define $k' := k + \frac{p^j(p-1)}{2}$. Now let us take a big enough number field $L/\mathbb{Q}$ containing number fields L_f and $L_{\mathscr{F}}$ for all newforms f and $\mathscr{F}$ in $S_{2k}^{new}(\Gamma_0(M))$ and $S_{2k'}^{new}(\Gamma_0(M))$ respectively over all divisors M of Np.

Let $\mathcal{O}_L$ be the ring of integers of L. We now fix a prime ideal $\mathfrak{p}$ above p such that the $\mathfrak{p}$-adic valuation $v_{\mathfrak{p}}$ is normalised suitably so that $v_{\mathfrak{p}}(p) = v_p(p) = 1$, where v_p is the standard p-adic valuation on $\mathbb{Q}$. This normalisation is achieved by defining for any $x \in \mathcal{O}_L$, $v_{\mathfrak{p}}(x) = n/e_{\mathfrak{p}}$ where n is the highest integer such that $x \in \mathfrak{p}^n$ and $e_{\mathfrak{p}}$ is the ramification index of $\mathfrak{p}$ in factorisation of $p\mathcal{O}_L$. This normalisation guarantees that $v_{\mathfrak{p}}$ when restricted to $\mathbb{Q}$ will have image in $\mathbb{Z}$.

For a power series $f(z) = \sum_{n=1}^{\infty} a_n q^n$, we define

$$v_p(f(z)) = \inf(v_p(a_n)).$$

Denote the set of $\mathfrak{p}$-integral elements in L by $\mathcal{O}_{(\mathfrak{p})}$, that is

$$\mathcal{O}_{(\mathfrak{p})} = \left\{ \frac{x}{y} \mid x, y \in \mathcal{O}_L,\ y \notin \mathfrak{p} \right\}.$$

Let $F^{\mathfrak{p}}$ be a finite extension of $\mathbb{Q}_p$ containing L that extends the valuation v_p to $v_{\mathfrak{p}}$. Let $\mathcal{O}_{F^{\mathfrak{p}}}$ be its corresponding ring of integers. We can then embed $L \hookrightarrow \overline{\mathbb{Q}_p}$ or embed $L \hookrightarrow \mathbb{C}$. It therefore makes sense to view Fourier coefficients $\mathfrak{p}$-adically embedded in $\mathcal{O}_{F^{\mathfrak{p}}}$.

Let $S_{2k}(\Gamma_0(M);\mathcal{O}_L)$ be the $\mathcal{O}_L$ submodule of $S_{2k}(\Gamma_0(M);L)$ containing all cusp forms in $S_{2k}(\Gamma_0(M);L)$ that have Fourier coefficients in $\mathcal{O}_L$. Now define

$$S_{2k}(\Gamma_0(M);\mathcal{O}_{F^{\mathfrak{p}}}) := S_{2k}(\Gamma_0(M);\mathcal{O}_L) \otimes_{\mathcal{O}_L} \mathcal{O}_{F^{\mathfrak{p}}}.$$

Similarly, let $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4M);\mathcal{O}_L)$ be the $\mathcal{O}_L$ submodule of $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4M);L)$ containing all half-integer weight cusp forms in $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4M);L)$ that have Fourier coefficients in $\mathcal{O}_L$. Also, define

$$S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4M);\mathcal{O}_{F^{\mathfrak{p}}}) := S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4M);\mathcal{O}_L) \otimes_{\mathcal{O}_L} \mathcal{O}_{F^{\mathfrak{p}}}.$$

Let $\mathbb{T}_{2k}(M;\mathcal{O}_L)$ be the commutative $\mathcal{O}_L$-subalgebra of the $\mathrm{End}(S_{2k}(\Gamma_0(M);L))$ generated by T_ℓ over all primes ℓ. Define

$$\mathbb{T}_{2k}(M;\mathcal{O}_{F^{\mathfrak{p}}}) := \mathbb{T}_{2k}(M;\mathcal{O}_L) \otimes_{\mathcal{O}_L} \mathcal{O}_{F^{\mathfrak{p}}}.$$

Similarly, let $\widetilde{\mathbb{T}}_{k+\frac{1}{2}}(4M;\mathcal{O}_L)$ be the commutative $\mathcal{O}_L$-subalgebra of the $\mathrm{End}(S_{k+\frac{1}{2}}(\Gamma_0(4M);L))$ generated by T_{ℓ^2} over all primes ℓ. Define

$$\widetilde{\mathbb{T}}_{k+\frac{1}{2}}(4M;\mathcal{O}_{F^{\mathfrak{p}}}) := \widetilde{\mathbb{T}}_{k+\frac{1}{2}}(4M;\mathcal{O}_L) \otimes_{\mathcal{O}_L} \mathcal{O}_{F^{\mathfrak{p}}}.$$

5.2 Congruences related to modular forms- Integer weight case

In this section, we prove a series of results to establish a mod p congruence between Fourier coefficients of integer weight N-new eigenforms with varying weights. Before we do that, we briefly set up the key ingredients required in this section.

From equation 1.2, we have the following decomposition of complex vector spaces of cusp forms,

$$S_{2k}(\Gamma_0(Np);L) = \bigoplus_{\substack{M\mid Np \\ d\mid NpM^{-1}}} S_{2k}^{\mathrm{new}}(\Gamma_0(M);L)|_{2k}V_d \tag{5.1}$$

where d runs over all positive divisors of NpM^{-1}.

Remark 5.2.1. We can replace the V_d operator in the above definition by the U_d operator for d not dividing the level. Recall that the U_d operator replaces every n^{th} Fourier coefficient for $n \in \mathbb{Z}_{>0}$ in the q expansion of a cusp form with the $(dn)^{\text{th}}$ Fourier coefficient. See remark 1.3.3 and [Koh82, pg. 68].

We now give the explicit definition of the space of N-new forms at level Np.

$$S_{2k}^{N\text{-new}}(\Gamma_0(Np);L) = S_{2k}^{\mathrm{new}}(\Gamma_0(Np);L) \oplus \left(S_{2k}^{\mathrm{new}}(\Gamma_0(N);L)|_{2k}U_p\right) \oplus S_{2k}^{\mathrm{new}}(\Gamma_0(N);L). \tag{5.2}$$

Let I denote a finite index set such that $|\,I\,| = \dim(S_{2k}^{N\text{-new}}(\Gamma_0(Np);L))$. Recall from Proposition 1.4.6 that the newspace has an orthogonal basis of newforms with Fourier coefficients in $\mathcal{O}_L$. This can be applied to each newspace in the direct sum in 5.2. Hence, we can take a basis $\{f_i^N(z)\}_{i\in I} \subseteq S_{2k}^{N\text{-new}}(\Gamma_0(Np);\mathcal{O}_L)$ to be a basis of the space $S_{2k}^{N\text{-new}}(\Gamma_0(Np);L)$ consisting of N-new Hecke eigenforms for all Hecke operators T_ℓ over all primes ℓ (including $T_p = U_p$).

Note 5.2.2. We have used the superscript N on the top of the cusp form $f(z)$ in order to clarify that the cusp form is new at level N.

Let us consider the p-ordinary projection operator $e = \lim_{n\to\infty} U_p^{n!}$, defined in 4.2.1. We apply this projection operator on each element in our basis $\{f_i^N(z)\}_{i\in I}$ as follows

$$\{f_i^N(z)|_{2k}e\}_{i\in I} = \{f_i^N(z)\}_{i=1}^{m_N} \cup \{0\}.$$

Here $m_N = \dim(S_{2k}^{N\text{-new, ord}}(\Gamma_0(Np);L))$ and $\{f_i^N(z)\}_{i=1}^{m_N}$ include p-stabilised U_p eigenforms obtained from newforms in decomposition 5.2. Hence, the set $\{f_i^N(z)\}_{i=1}^{m_N} \subseteq S_{2k}^{N\text{-new, ord}}(\Gamma_0(Np);\mathcal{O}_L)$ consists of m_N distinct N-new ordinary eigenforms that form a basis for $S_{2k}^{N\text{-new, ord}}(\Gamma_0(Np);L)$.

5.2.1 Main assumption

We now state the main assumption that is a key ingredient in building the proof of the main theorem in this chapter. Note that by Hecke eigenforms in assumption 1, we mean eigenforms for all Hecke operators T_ℓ for all primes ℓ.

Assumption 1. *Let*

$$\{f_i^N(z)\}_{i=1}^{m_N} \subseteq S_{2k}^{N\text{-new,ord}}(\Gamma_0(Np); \mathcal{O}_L)$$

be a basis of eigenforms for $S_{2k}^{N\text{-new,ord}}(\Gamma_0(Np); L)$ consisting of N-new eigenforms with $\mathcal{O}_L$-integral Fourier coefficients and scaled in a way such that every element in the basis has at least one Fourier coefficient that is not divisible by $\mathfrak{p}$.

Let $N' \in \mathbb{Z}_{>0}$ be a divisor of N and let

$$f^{N'}(z) \in S_{2k}^{N'\text{-new,ord}}(\Gamma_0(N'p); \mathcal{O}_L)$$

also be an eigenform with $\mathcal{O}_L$-integral Fourier coefficients.

Suppose for all primes $\ell \nmid Np$,

$$\lambda_{T_\ell}(f_i^N) \equiv \lambda_{T_\ell}(f^{N'}) \text{ mod } \mathfrak{p}$$

where $\lambda_{T_\ell}(f_i^N)$ and $\lambda_{T_\ell}(f^{N'})$ denote the Hecke-eigenvalues of $f_i^N(z)$ and $f^{N'}(z)$ for T_ℓ operator respectively. Then

$$f^{N'}(z) = \alpha f_i^N(z) \text{ for some } \alpha \in \mathcal{O}_L.$$

Remark 5.2.3. In our thesis, we are mostly working with one dimensional new spaces of cusp forms, so our assumption 1 holds trivially. However, generally it's quite difficult to argue why one can always find forms for which assumption 1 holds. As such, assumption 1 might appear quite strong to the reader and not amenable to computation as the dimension of the spaces of cusp forms increases. However, there are conjectures for spaces of cusp forms of weight 2 and prime level p that increase the likelihood of existence of newforms for which assumption 1 will be true. In [CS04], *Calegari* and *Stein* conjecture that for any prime p, there are no mod p congruences between two distinct newforms in $S_2(\Gamma_0(p); \mathcal{O}_{F_\mathfrak{p}})$ but there are almost always many such congruences with newforms in spaces with weight greater than 2. They also show that the only prime $p < 50923$ for which there is a congruence between two weight 2 newforms is $p = 389$. In theory, congruences between newforms arise in two ways: from the failure of the Hecke ring $\mathbb{T}_{2k}(p; \mathcal{O}_{F_\mathfrak{p}})$ to be integrally closed or from ramification in the coefficient fields of the newforms. So, the conjecture of *Calegari* and *Stein* asserts that the Hecke algebra $\mathbb{T}_2(p; \mathcal{O}_{F_\mathfrak{p}})$ is integrally closed. In [AR11], *Ahlgre* and *Rouse* extend the work of *Calegari* and *Stein* and work with weights 4 and 6. Thus, in the light of these conjectures which have been tested for considerably many primes p, we expect to find N-new ordinary forms that should satisfy assumption 1. With this expectation, we will also make a similar assumption 2 when we move to congruences between Hilbert modular forms in Chapter 10. The task to find explicit examples where we illustrate how this assumption is tested remains a part of future research work and cannot be included in the current thesis due to time constraints.

5.2.2 Main Theorem

We now state our main result. Again, note that by Hecke eigenforms in Theorem 5.2.4, we mean eigenforms for all Hecke operators T_ℓ for all primes ℓ.

Theorem 5.2.4 (Main Theorem). *Let $m_N = \dim\left(S_{2k}^{N\text{-new, ord}}(\Gamma_0(Np); L)\right)$ and let*

$$\{f_i^N(z)\}_{i=1}^{m_N} \subseteq S_{2k}^{N\text{-new,ord}}(\Gamma_0(Np); \mathcal{O}_L)$$

be a basis for $S_{2k}^{N\text{-new, ord}}(\Gamma_0(Np); L)$ consisting of of N-new ordinary eigenforms with $\mathcal{O}_L$-integral Fourier coefficients and scaled in a way such that every element in the basis has at least one Fourier coefficient that is not divisible by $\mathfrak{p}$.
Let $j \in \mathbb{Z}_{>0}$ be a fixed integer and define $k' := k + \frac{p^j(p-1)}{2}$. Suppose assumption 1 holds for the basis $\{f_i^N(z)\}_{i=1}^{m_N}$. Then for each integer i such that $1 \leq i \leq m_N$, there exists an N-new ordinary eigenform

$$\mathfrak{F}_i^N(z) \in S_{2k'}^{N\text{-new,ord}}(\Gamma_0(Np); \mathcal{O}_{(\mathfrak{p})}),$$

unique up to scalar multiplication in $\mathcal{O}_{(\mathfrak{p})}$ such that we have the following congruence of Fourier coefficients:

$$f_i^N(z) \equiv \mathfrak{F}_i^N(z) \bmod p^{j+1}.$$

Figure 5.2.5.

$$f_i^N \in S_{2k}^{N\text{-new, ord}}(\Gamma_0(Np); \mathcal{O}_L) \xrightarrow{\equiv \text{ mod } p^{j+1} \text{ (Fourier coefficients)}} \mathfrak{F}_i^N \in S_{2k'}^{N\text{-new, ord}}(\Gamma_0(Np); \mathcal{O}_{(\mathfrak{p})})$$

$\downarrow$ N-new, ordinary, eigenform of weight $2k$

$\downarrow$ N-new, ordinary, eigenform of weight $2k'$

Remark 5.2.6. We note that Theorem 5.2.4 also holds true for all weights $k' = k + t\frac{(p-1)}{2}$ where t is a positive integer.

In order to give a nice structure to the proof of Theorem 5.2.4, we break it down into a series of propositions which when combined will eventually imply the result.

Proposition 5.2.7. *Let*

$$f(z) \in S_{2k}(\Gamma_0(Np); \mathcal{O}_L)$$

be a cusp form with $\mathcal{O}_L$-integral Fourier coefficients.
Let $j \in \mathbb{Z}_{>0}$ be a fixed integer and define $k' := k + \frac{p^j(p-1)}{2}$. Then there exists a cusp form

$$F(z) \in S_{2k'}(\Gamma_0(Np); \mathcal{O}_{(\mathfrak{p})})$$

such that we have the following congruence of Fourier coefficients:

$$f(z) \equiv F(z) \bmod p^{j+1}.$$

Figure 5.2.8.

$$f \in S_{2k}(\Gamma_0(Np); \mathcal{O}_L) \xrightarrow{\equiv \text{ mod } p^{j+1} \text{ (Fourier coefficients)}} F \in S_{2k'}(\Gamma_0(Np); \mathcal{O}_{(\mathfrak{p})})$$

$\downarrow$ Cusp form of weight $2k$

$\downarrow$ Cusp form of weight $2k'$

Proof. Define

$$F(z) := f(z)\mathscr{E}_p(z)^{p^j}$$

where $\mathscr{E}_p(z) \in M_{p-1}(\Gamma_0(p))$ is defined in terms of Eisenstein series $E_{p-1}(z)$ as

$$\mathscr{E}_p(z) = \begin{cases} E_{p-1}(z) - p^{p-1}E_{p-1}(pz) & \text{if } p \geq 5; \\ E_2(z) - 3E_2(3z) & \text{if } p = 3. \end{cases}$$

Recall that $\mathscr{E}_p(z)$ has p-integral Fourier coefficients (see proof of Lemma 3.2.3) and note that it's given that the Fourier coefficients of $f(z)$ lie in $\mathscr{O}_L$. This implies that the Fourier coefficients of $F(z)$ lie in $\mathscr{O}_{(\mathrm{p})}$. Further by corollary 3.2.4, we have

$$\mathscr{E}_p(z)^{p^j} \equiv 1 \bmod p^{j+1}.$$

Hence, we conclude,

$$f(z) \equiv F(z) \bmod p^{j+1}.$$

□

We next try to show that the action of Hecke operators $|_{2k}T_\ell$ for primes ℓ on $f(z)$ mod p^{j+1} is the same as the action of Hecke operators $|_{2k'}T_\ell$ on $F(z)$ mod p^{j+1} where $f(z)$ and $F(z)$ are defined in proposition 5.2.7.

Corollary 5.2.9. *Let $f(z) \in S_{2k}(\Gamma_0(Np); \mathscr{O}_L)$ and $F(z) \in S_{2k'}(\Gamma_0(Np); \mathscr{O}_{(\mathrm{p})})$ be cusp forms as defined in proposition 5.2.7 such that*

$$f(z) \equiv F(z) \textit{ mod } p^{j+1}.$$

Then we have

$$f(z)|_{2k}T_\ell \equiv F(z)|_{2k'}T_\ell \textit{ mod } p^{j+1}$$

for all primes ℓ.

Proof. Let $f(z) = \sum_{n=1}^{\infty} a_n q^n$ and $F(z) = \sum_{n=1}^{\infty} A_n q^n$ be the Fourier expansions of $f(z)$ and $F(z)$ defined in proposition 5.2.7. Then we have

$$a_n \equiv A_n \bmod p^{j+1}. \tag{5.3}$$

Case 1: $\ell \nmid Np$.

From proposition 1.2.3, we can write for all prime $\ell \nmid Np$,

$$F(z)|_{2k'}T_\ell = \sum_{n=1}^{\infty}(A_{\ell n} + \ell^{2k'-1}A_{n/\ell})q^n. \tag{5.4}$$

From 5.3 and 5.4, we have

$$\begin{aligned} F(z)|_{2k'}T_\ell &\equiv \sum_{n=1}^{\infty}(a_{\ell n} + \ell^{2\left(k+\frac{p^j(p-1)}{2}\right)-1}a_{n/\ell})q^n \bmod p^{j+1} \\ &= \sum_{n=1}^{\infty}(a_{\ell n} + \ell^{2k-1}\left(\ell^{p-1}\right)^{p^j}a_{n/\ell})q^n \bmod p^{j+1}. \end{aligned}$$

Now by *Fermat's Little Theorem*, we have $\ell^{p-1} \equiv 1 \bmod p$. Then by a similar argument as in the proof of corollary 3.2.4, it follows $\left(\ell^{p-1}\right)^{p^j} \equiv 1 \bmod p^{j+1}$.

Thus,

$$\begin{aligned} F(z)|_{2k'} T_\ell &\equiv \sum_{n=1}^{\infty} (a_{\ell n} + \ell^{2k-1} a_{n/\ell}) q^n \bmod p^{j+1} \\ &\equiv f(z)|_{2k} T_\ell \bmod p^{j+1}. \end{aligned}$$

Case 2: $\ell \mid Np$.

Now, for all primes $\ell \mid Np$, we can write

$$f(z)|_{2k} T_\ell = \sum_{n=1}^{\infty} a_{\ell n} q^n \quad \text{and} \quad F(z)|_{2k'} T_\ell = \sum_{n=1}^{\infty} A_{\ell n} q^n. \tag{5.5}$$

From equations 5.3 and 5.5, it follows

$$a_{\ell n} \equiv A_{\ell n} \bmod p^{j+1}$$

or

$$f(z)|_{2k} T_\ell \equiv F(z)|_{2k'} T_\ell \bmod p^{j+1}.$$

□

We next show that if we are given that $f(z)$ in proposition 5.2.7 is an ordinary cusp form of weight $2k$, then $F(z)$ in proposition 5.2.7 is also an ordinary cusp form of weight $2k'$.

Proposition 5.2.10. *Let*

$$f(z) \in S_{2k}^{ord}(\Gamma_0(Np); \mathcal{O}_L)$$

be an ordinary cusp form with $\mathcal{O}_L$-integral Fourier coefficients.
Let $j \in \mathbb{Z}_{>0}$ be a fixed integer and define $k' := k + \frac{p^j(p-1)}{2}$. Then there exists an ordinary cusp form

$$F^\circ(z) \in S_{2k'}^{ord}(\Gamma_0(Np); \mathcal{O}_{(\mathfrak{p})})$$

such that we have the following congruence of Fourier coefficients:

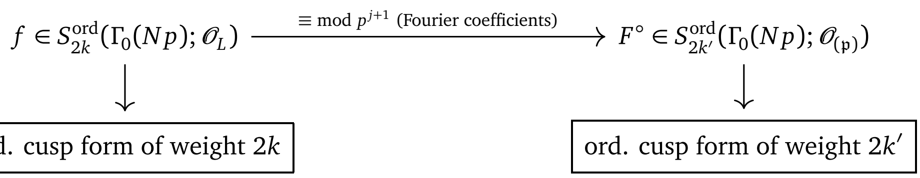
$$f(z) \equiv F^\circ(z) \bmod p^{j+1}.$$

Moreover,

$$f(z)|_{2k} T_\ell \equiv F^\circ(z)|_{2k'} T_\ell \bmod p^{j+1}$$

for all primes ℓ.

Figure 5.2.11.



Proof. Define

$$F^{\circ}(z) := F(z)|_{2k'}e$$

where $e = \lim_{n\to\infty} U_p^{n!}$ is the p-ordinary projector and $F(z) \in S_{2k'}(\Gamma_0(Np); \mathcal{O}_{(\mathfrak{p})})$ is the cusp of weight form obtained in proposition 5.2.7.
Note e is an idempotent, that is $e^2 = e$. This implies

$$\begin{aligned} F^{\circ}(z)|_{2k'}e &= F(z)|_{2k'}e^2 \\ &= F(z)|_{2k'}e \\ &= F^{\circ}(z). \end{aligned}$$

Hence $F^{\circ}(z)$ is ordinary at p and lies in $S^{\text{ord}}_{2k'}(\Gamma_0(Np); \mathcal{O}_{(\mathfrak{p})})$.

Next, we use corollary 5.2.9 and write

$$\begin{aligned} F^{\circ}(z) &= F(z)|_{2k'}e \\ &\equiv f(z)|_{2k}e \bmod p^{j+1} \end{aligned}$$

Since $f(z)$ is given to be ordinary, then $f(z)|_{2k}e = f(z)$. So, we can write

$$F^{\circ}(z) \equiv f(z) \bmod p^{j+1}.$$

Then the mod p^{j+1} equivalence of Hecke action follows directly from corollary 5.2.9. □

We next show that if we are given that $f(z)$ in proposition 5.2.10 is an N-new ordinary eigenform of weight $2k$ for which assumption 1 holds true, then there exists $\mathscr{F}(z)$, a unique ordinary eigenform of weight $2k'$ (unique up to scalar multiplication in $\mathcal{O}_L$) which satisfies a mod $\mathfrak{p}$ congruence of Hecke-eigenvalues with $f(z)$ for all Hecke operators T_ℓ over all primes $\ell \nmid Np$.

Proposition 5.2.12. *Let $m_N = dim\left(S^{N\text{-new, ord}}_{2k}(\Gamma_0(Np); L)\right)$ and let*

$$\{f_i^N(z)\}_{i=1}^{m_N} \subseteq S^{N\text{-new, ord}}_{2k}(\Gamma_0(Np); \mathcal{O}_L)$$

be a basis for $S^{N\text{-new, ord}}_{2k}(\Gamma_0(Np); L)$ consisting of N-new ordinary eigenforms with $\mathcal{O}_L$-integral Fourier coefficients and let each $f_i^N(z)$ be scaled in a way such that it has at least one Fourier coefficient that is not divisible by $\mathfrak{p}$. Also, assume that assumption 1 holds true for the basis $\{f_i^N(z)\}_{i=1}^{m_N}$.
Let $j \in \mathbb{Z}_{>0}$ be a fixed integer and define $k' := k + \frac{p^j(p-1)}{2}$. Then for each i such that $1 \le i \le m_N$, there exists an ordinary eigenform

$$\mathscr{F}_i(z) \in S^{ord}_{2k'}(\Gamma_0(Np); \mathcal{O}_L)$$

that is unique up to scalar multiplication in $\mathcal{O}_L$ and for all primes $\ell \nmid Np$ satisfies the congruence

$$\lambda_{T_\ell}(f_i^N) \equiv \lambda_{T_\ell}(\mathscr{F}_i) \ mod\ \mathfrak{p}.$$

Figure 5.2.13.

$$f_i^N \in S^{N\text{-new, ord}}_{2k}(\Gamma_0(Np); \mathcal{O}_L) \xrightarrow{\equiv \bmod \mathfrak{p} \text{ (Hecke eigenvalues)}} \mathscr{F}_i \in S^{\text{ord}}_{2k'}(\Gamma_0(Np); \mathcal{O}_L)$$

↓ N-new ord. eigenform of weight $2k$

↓ ord. eigenform of weight $2k'$

Proof. For simplicity, let us fix the integer i, say $i = 1$. Then by proposition 5.2.10, for an ordinary cusp form $f_1^N(z) \in S_{2k}^{N\text{-new, ord}}(\Gamma_0(Np); \mathscr{O}_L)$, there exists an ordinary cusp form $F_1^\circ(z) \in S_{2k'}^{\text{ord}}(\Gamma_0(Np); \mathscr{O}_{(\mathfrak{p})})$ such that

$$f_1^N(z) \equiv F_1^\circ(z) \bmod p^{j+1}.$$

Just as in equation 5.1, the space $S_{2k'}(\Gamma_0(Np); L)$ also decomposes as

$$S_{2k'}(\Gamma_0(Np); L) = \bigoplus_{\substack{M|Np \\ d|NpM^{-1}}} S_{2k'}^{\text{new}}(\Gamma_0(M); L)|_{2k'} U_d \tag{5.6}$$

where d runs over positive divisors of NpM^{-1}.

For each M, we have an orthogonal basis of newforms for $S_{2k'}^{\text{new}}(\Gamma_0(M); L)$, see Proposition 1.4.6. These newforms have Fourier coefficients in $\mathscr{O}_L$. Let S be a finite index set with cardinality equal to $\dim(S_{2k'}(\Gamma_0(Np); L)$. Thus, we can take a basis $\{\mathscr{F}_s(z)\}_{s\in S} \subseteq S_{2k'}(\Gamma_0(Np); \mathscr{O}_L)$ with $\mathscr{O}_L$-integral Fourier coefficients for $S_{2k'}(\Gamma_0(Np); L)$ that consists of eigenforms for Hecke-operators T_ℓ for all primes ℓ (including $T_p = U_p$).
Then we can write $F_1^\circ(z) \in S_{2k'}^{\text{ord}}(\Gamma_0(Np); \mathscr{O}_{(\mathfrak{p})})$ as a linear combination of this basis of eigenforms:

$$F_1^\circ(z) = \sum_{s\in S} \alpha_s \mathscr{F}_s(z) \text{ where } \alpha_s \in L. \tag{5.7}$$

We next apply the p-ordinary projector $e_p = \lim_{n\to\infty} U_p^{n!}$ on either side on equation 5.7.

$$\begin{aligned} F_1^\circ(z) &= \sum_{s\in S} \alpha_s \mathscr{F}_s(z)|_{2k'} e \\ &= \sum_{s\in S} \alpha_s \left(\mathscr{F}_s(z)|_{2k'} e\right) \\ &= \sum_{s=1}^{t} \alpha_s \mathscr{F}_s(z) \end{aligned}$$

where $t = \dim(S_{2k'}^{\text{ord}}(\Gamma_0(Np); L))$ and $\{\mathscr{F}_s(z)\}_{s=1}^t$ are p-stabilised U_p eigenforms obtained from newforms in decomposition 5.6.

Claim: We claim that there exists an integer s such that $1 \le s \le t$, say $s = 1$, such that for all primes $\ell \nmid Np$, we have

$$\lambda_{T_\ell}(f_1^N) \equiv \lambda_{T_\ell}(\mathscr{F}_s) \bmod \mathfrak{p}.$$

We assume the contrary and try to reach a contradiction.

Suppose for every integer s such that $1 \le s \le t$, there exists some prime ℓ_s such that

$$\ell_s \nmid Np \quad \text{and} \quad \mathfrak{p} \nmid \left(\lambda_{T_{\ell_s}}(f_1^N) - \lambda_{T_{\ell_s}}(\mathscr{F}_s)\right). \tag{5.8}$$

Now let us operate equation 5.7 by the operator $\big|_{2k'} \prod_{s=1}^t \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s)\right)$ on either side. We will see that this product operator kills every term in the sum on the right hand side of this equation.

$$\begin{aligned} F_1^\circ(z)\Big|_{2k'} \prod_{s=1}^{t}\left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s)\right) &= \left(\sum_{s=1}^{t} \alpha_s \mathscr{F}_s(z)\right)\Bigg|_{2k'} \prod_{s=1}^{t}\left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s)\right) \\ &= \left(\sum_{s=1}^{t} \alpha_s \mathscr{F}_s(z)\right)\Bigg|_{2k'} \left(T_{\ell_1} - \lambda_{T_{\ell_1}}(\mathscr{F}_1)\right) \prod_{s=2}^{t}\left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s)\right) \end{aligned}$$

$$\begin{aligned}
&= \left(\sum_{s=1}^{t} \alpha_s (\mathscr{F}_s(z)|_{2k'} T_{\ell_1}) - \lambda_{T_{\ell_1}}(\mathscr{F}_1) \sum_{s=1}^{t} \alpha_s \mathscr{F}_s(z) \right) \Bigg|_{2k'} \prod_{s=2}^{t} \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s) \right) \\
&= \left(\sum_{s=1}^{t} \alpha_s \lambda_{T_{\ell_1}}(\mathscr{F}_s)\mathscr{F}_s(z) - \sum_{s=1}^{t} \alpha_s \lambda_{T_{\ell_1}}(\mathscr{F}_1)\mathscr{F}_s(z) \right) \Bigg|_{2k'} \prod_{s=2}^{t} \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s) \right) \\
&= \left(\sum_{s=1}^{t} \alpha_s \left(\lambda_{T_{\ell_1}}(\mathscr{F}_s) - \lambda_{T_{\ell_1}}(\mathscr{F}_1) \right) \mathscr{F}_s(z) \right) \Bigg|_{2k'} \prod_{s=2}^{t} \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s) \right). \\
&= \sum_{s=1}^{t} \alpha_s \prod_{w=1}^{t} \left(\lambda_{T_{\ell_w}}(\mathscr{F}_s) - \lambda_{T_{\ell_w}}(\mathscr{F}_w) \right) \mathscr{F}_s(z). \\
&= 0.
\end{aligned} \tag{5.9}$$

Now replacing $F_1^{\circ}(z)$ modulo p^{j+1} by $f_1^N(z)$, we see that the same product operator $\big|_{2k'} \prod_{s=1}^{t} \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s) \right)$ modulo p^{j+1} does not act as a zero operator on $F_1^{\circ}(z)$.

$$\begin{aligned}
F_1^{\circ}(z) \Bigg|_{2k'} \prod_{s=1}^{t} \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s) \right) &\equiv f_1^N(z) \Bigg|_{2k} \prod_{s=1}^{t} \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s) \right) \bmod p^{j+1} \\
&= f_1^N(z) \Bigg|_{2k} (T_{\ell_2} - \lambda_{T_{\ell_2}}(\mathscr{F}_2)) \prod_{\substack{s=1 \\ s\neq 2}}^{t} \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s) \right) \bmod p^{j+1} \\
&= \left(f_1^N(z)|_{2k} T_{\ell_2} - \lambda_{T_{\ell_2}}(\mathscr{F}_2) f_1^N(z) \right) \Bigg|_{2k} \prod_{\substack{s=1 \\ s\neq 2}}^{t} \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s) \right) \bmod p^{j+1} \\
&= \left(\lambda_{T_{\ell_2}}(f_1^N) f_1^N(z) - \lambda_{T_{\ell_2}}(\mathscr{F}_2) f_1^N(z) \right) \Bigg|_{2k} \prod_{\substack{s=1 \\ s\neq 2}}^{t} \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s) \right) \bmod p^{j+1} \\
&= \left(\lambda_{T_{\ell_2}}(f_1^N) - \lambda_{T_{\ell_2}}(\mathscr{F}_2) \right) f_1^N(z) \Bigg|_{2k} \prod_{s=2}^{t} \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s) \right) \bmod p^{j+1} \\
&= \prod_{s=1}^{t} \left(\lambda_{T_{\ell_s}}(f_1^N) - \lambda_{T_{\ell_s}}(\mathscr{F}_s) \right) f_1^N(z) \bmod p^{j+1}.
\end{aligned}$$

First note that $f_1^N(z)$ modulo $\mathfrak{p}$ is non-zero. This is because we are a suitable scaling under which at least one Fourier coefficient of $f_1^N(z)$ is not divisible by $\mathfrak{p}$. Also, assumption 5.8 implies $\mathfrak{p}$ does not divide the product $\prod_{s=1}^{t} \left(\lambda_{T_{\ell_s}}(f_1^N) - \lambda_{T_{\ell_s}}(\mathscr{F}_s) \right)$. Thus, we conclude

$$F_1^{\circ}(z) \Bigg|_{2k'} \prod_{s=1}^{t} \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s) \right) \not\equiv 0 \bmod \mathfrak{p}. \tag{5.10}$$

From 5.9 and 5.10, we have reached a contradiction. Hence, our assumption 5.8 is false. Therefore, there exists an integer s such that $1 \le s \le t$ (say $s = 1$) for which for all primes $\ell \nmid Np$,

$$\lambda_{T_\ell}(f_1^N) \equiv \lambda_{T_\ell}(\mathscr{F}_1) \bmod \mathfrak{p}.$$

We can repeat the above proof for every i such that $2 \le i \le m_N$. Therefore, for each $1 \le i \le m_N$, there exists an integer s_i where $1 \le s_i \le t$ such that for all primes $\ell \nmid Np$, we have

$$\lambda_{T_\ell}(f_i^N) \equiv \lambda_{T_\ell}(\mathscr{F}_{s_i}) \bmod \mathfrak{p}. \tag{5.11}$$

Uniqueness: Let $s_2 = 2$. Then $\mathscr{F}_2(z)$ is the ordinary eigenform that satisfies the congruence 5.11 with $f_2^N(z)$.

Suppose $\mathscr{F}_2(z)$ is distinct from $\mathscr{F}_1(z)$, that is, there does not exist any $\mathscr{B} \in \mathscr{O}_L$ for which $\mathscr{F}_2(z) = \mathscr{B}\mathscr{F}_1(z)$ but if possible, let $\mathscr{F}_2(z)$ also satisfy the following congruence with $f_1^N(z)$:

$$\lambda_{T_\ell}(f_1^N) \equiv \lambda_{T_\ell}(\mathscr{F}_2) \bmod \mathfrak{p}.$$

Then it follows

$$\lambda_{T_\ell}(f_1^N) \equiv \lambda_{T_\ell}(f_2^N) \bmod \mathfrak{p}.$$

Then by assumption 1, $f_2^N(z) = \mathscr{C} f_1^N(z)$ for some $\mathscr{C} \in \mathscr{O}_L$. This is not possible as the set $\{f_i^N(z)\}_{i=1}^{m_N}$ forms a basis of N-new ordinary eigenforms for the space $S_{2k}^{N\text{-new,ord}}(\Gamma_0(Np); \mathscr{O}_L)$ and hence consists of m_N distinct elements.

Hence, for each $1 \le i \le m_N$, $\mathscr{F}_{s_i}(z) \in S_{2k'}^{\text{ord}}(\Gamma_0(Np); \mathscr{O}_L)$ is the unique ordinary eigenform up to scalar multiplication in $\mathscr{O}_L$ which satisfies congruence 5.11 with $f_i^N(z)$. For simplicity, we choose $s_i = i$. ☐

Proposition 5.2.12 gives us a set of ordinary eigenforms $\{\mathscr{F}_i(z)\}_{i=1}^{m_N}$ in $S_{2k'}^{\text{ord}}(\Gamma_0(Np); \mathscr{O}_L)$, with each element in the set being unique up to scalar multiplication in $\mathscr{O}_L$ such that for all primes $\ell \nmid Np$

$$\lambda_{T_\ell}(f_i^N) \equiv \lambda_{T_\ell}(\mathscr{F}_i) \bmod \mathfrak{p}.$$

We next want to show that the elements in the set $\{\mathscr{F}_i(z)\}_{i=1}^{m_N}$ are N-new and forms a basis for the space $S_{2k'}^{N\text{-new, ord}}(\Gamma_0(Np); L)$. For this, we use induction.

Proposition 5.2.14. *Let* $m_N = \dim\left(S_{2k}^{N\text{-new, ord}}(\Gamma_0(Np); L)\right)$ *and let*

$$\{f_i^N(z)\}_{i=1}^{m_N} \subset S_{2k}^{N\text{-new, ord}}(\Gamma_0(Np); \mathscr{O}_L)$$

be a basis for $S_{2k}^{N\text{-new, ord}}(\Gamma_0(Np); L)$ *consisting of N-new ordinary eigenforms with $\mathscr{O}_L$-integral Fourier coefficients and let each $f_i^N(z)$ be scaled in a way such that it has at least one Fourier coefficient that is not divisible by $\mathfrak{p}$. Also, assume that assumption 1 holds true for the basis* $\{f_i^N(z)\}_{i=1}^{m_N}$.
Let $j \in \mathbb{Z}_{>0}$ be a fixed integer and define $k' := k + \frac{p^j(p-1)}{2}$. Then there exists a set of N-new ordinary eigenforms

$$\{\mathscr{F}_i^N(z)\}_{i=1}^{m_N} \subset S_{2k'}^{N\text{-new, ord}}(\Gamma_0(Np); \mathscr{O}_L)$$

with $\mathscr{O}_L$-integral Fourier coefficients such that it forms a basis for the space $S_{2k'}^{N\text{-new, ord}}(\Gamma_0(Np); L)$ and for all primes $\ell \nmid Np$ satisfies the congruence

$$\lambda_{T_\ell}(f_i^N) \equiv \lambda_{T_\ell}(\mathscr{F}_i^N) \bmod \mathfrak{p}.$$

Figure 5.2.15.

$$f_i^N \in S_{2k}^{N\text{-new, ord}}(\Gamma_0(Np); \mathscr{O}_L) \xrightarrow{\equiv \bmod \mathfrak{p} \text{ (Hecke eigenvalues)}} \mathscr{F}_i^N \in S_{2k'}^{N\text{-new, ord}}(\Gamma_0(Np); \mathscr{O}_L)$$

$\downarrow$ N-new ord. eigenform of weight $2k$

$\downarrow$ N-new ord. eigenform of weight $2k'$

Proof. We will use induction on the level to prove this proposition.

- *Base Step:* Let $N = 1$.
  Let $\{f_i^1(z)\}_{i=1}^{m_1} \subseteq S_{2k}^{\text{1-new,ord}}(\Gamma_0(p); \mathcal{O}_L)$ be a basis of $S_{2k}^{\text{1-new,ord}}(\Gamma_0(p); L)$ consisting of 1-new ordinary eigenforms where $m_1 = \dim\, (S_{2k}^{\text{1-new,ord}}(\Gamma_0(p); L)$. Then using proposition 5.2.12, we have a set of ordinary eigenforms $\{\mathscr{F}_i^1(z)\}_{i=1}^{m_1} \subset S_{2k'}^{\text{ord}}(\Gamma_0(p); \mathcal{O}_L)$ with each element unique up to scalar multiplication in $\mathcal{O}_L$ such that for all primes $\ell \nmid p$,

  $$\lambda_{T_\ell}(f_i^1) \equiv \lambda_{T_\ell}(\mathscr{F}_i^1) \bmod \mathfrak{p}.$$

  Now observe that every form in $S_{2k'}(\Gamma_0(p))$ is trivially 1-new. So, $S_{2k'}(\Gamma_0(p)) = S_{2k'}^{\text{1-new}}(\Gamma_0(p))$. In particular,

  $$\{\mathscr{F}_i^1(z)\}_{i=1}^{m_1} \subseteq S_{2k'}^{\text{ord}}(\Gamma_0(p); \mathcal{O}_L) = S_{2k'}^{\text{1-new,ord}}(\Gamma_0(p); \mathcal{O}_L).$$

  Next, we want to show $\dim S_{2k'}^{\text{ord}}(\Gamma_0(p); L) = m_1$. Since $2k' \equiv 2k \bmod p-1$. Then we can apply *Hida's Control Theorem*, see corollary 4.3.4, and get

  $$\begin{aligned} \dim(S_{2k'}^{\text{ord}}(\Gamma_0(p); L)) &= \dim(S_{2k}^{\text{ord}}(\Gamma_0(p); L)) \\ &= m_1. \end{aligned} \tag{5.12}$$

  Thus, the set $\{\mathscr{F}_i^1(z)\}_{i=1}^{m_1}$ contains m_1 distinct elements which form a basis of ordinary eigenforms for the space $S_{2k'}^{\text{ord}}(\Gamma_0(p); L)$.

- *Induction Hypothesis*: Let us assume that proposition 5.2.14 holds for every integer $N' \in \mathbb{Z}_{>0}$ such that $N' \mid N$ but $N' \neq N$.

- *Induction Step*: We are given that $\{f_i^N(z)\}_{i=1}^{m_N} \subset S_{2k}^{N\text{-new, ord}}(\Gamma_0(Np); \mathcal{O}_L)$ is a basis of N-new ordinary eigenforms for the space $S_{2k}^{N\text{-new,ord}}(\Gamma_0(Np); L)$ where $m_N = \dim\, (S_{2k}^{N\text{-new, ord}}(\Gamma_0(Np); L))$. Then by proposition 5.2.12, we have a set of ordinary eigenforms, say $\{\mathscr{F}_i^N(z)\}_{i=1}^{m_N} \subset S_{2k'}^{\text{ord}}(\Gamma_0(Np); \mathcal{O}_L)$ with each element unique up to scalar multiplication in $\mathcal{O}_L$ such that for all primes $\ell \nmid Np$, we have

  $$\lambda_{T_\ell}(f_i^N) \equiv \lambda_{T_\ell}(\mathscr{F}_i^N) \bmod \mathfrak{p}.$$

  Note 5.2.16. Note that we have made an abuse of notation by writing the superscript N for each element in the set $\{\mathscr{F}_i^N(z)\}_{i=1}^{m_N}$ of ordinary eigenforms above. However, it hasn't yet been shown that the elements are N-new but that's our goal. We make this abuse of notation to distinguish that this set is the nominated set that we need.

  As before, let $N' \in \mathbb{Z}_{>0}$ such that $N' \mid N$ but $N' \neq N$. For simplicity, let us consider $i = 1$. Suppose there exists an N'-new eigenform $\mathscr{F}^{N'}(z) \in S_{2k'}^{N'\text{-new, ord}}(\Gamma_0(N'p); \mathcal{O}_L)$ for some proper divisor N' of N such that for all primes $\ell \nmid Np$, we have

  $$\lambda_{T_\ell}(\mathscr{F}^{N'}) = \lambda_{T_\ell}(\mathscr{F}_1^N)$$

  and

  $$\lambda_{T_\ell}(f_1^N) \equiv \lambda_{T_\ell}(\mathscr{F}^{N'}) \bmod \mathfrak{p}. \tag{5.13}$$

  However, induction hypothesis implies that $\mathscr{F}^{N'}(z)$ is a unique form up to scalar multiplication in $\mathcal{O}_L$ that satisfies the congruence 5.13 with an N'-new ordinary eigenform $f^{N'}(z) \in S_{2k}^{N'\text{-new,ord}}(\Gamma_0(Np); \mathcal{O}_L)$. That is, for all primes, $\ell \nmid Np$, we have

  $$\lambda_{T_\ell}(f^{N'}) \equiv \lambda_{T_\ell}(\mathscr{F}^{N'}) \bmod \mathfrak{p}. \tag{5.14}$$

  From congruences 5.13 and 5.14, we get for all primes $\ell \nmid Np$, we have

  $$\lambda_{T_\ell}(f^{N'}) \equiv \lambda_{T_\ell}(f_1^N) \bmod \mathfrak{p}$$

which is contradiction to our assumption 1. Thus, $\mathscr{F}_1^N(z)$ must be N-new.

Now we want to show that $\dim(S_{2k'}^{N\text{-new,ord}}(\Gamma_0(Np);L)) = m_N$. From induction hypothesis,

$$\dim(S_{2k'}^{N'\text{-new,ord}}(\Gamma_0(N'p);L)) = \dim(S_{2k}^{N'\text{-new,ord}}(\Gamma_0(N'p);L)). \tag{5.15}$$

Since $2k' \equiv 2k \bmod p-1$, then by *Hida's Control Theorem*, see corollary 4.3.4, we have

$$\dim(S_{2k'}^{\text{ord}}(\Gamma_0(Np);L)) = \dim(S_{2k}^{\text{ord}}(\Gamma_0(Np);L)). \tag{5.16}$$

Let $\sigma_0(n)$ denote the divisor function that counts the number of positive divisors of an integer $n \in \mathbb{Z}_{>0}$. Using equations 5.15 and 5.16, we get

$$\begin{aligned}
&\dim(S_{2k'}^{N\text{-new,ord}}(\Gamma_0(Np);L)) \\
&= \dim(S_{2k'}^{\text{ord}}(\Gamma_0(Np);L)) - \sum_{\substack{N' \mid N \\ N' \neq N}} \sigma_0(NN'^{-1}) \dim(S_{2k'}^{N'\text{-new,ord}}(\Gamma_0(N'p);L)) \\
&= \dim(S_{2k}^{\text{ord}}(\Gamma_0(Np);L)) - \sum_{\substack{N' \mid N \\ N' \neq N}} \sigma_0(NN'^{-1}) \dim(S_{2k}^{N'\text{-new,ord}}(\Gamma_0(N'p);L)) \\
&= \dim(S_{2k}^{N\text{-new,ord}}(\Gamma_0(Np);L)) \\
&= m_N.
\end{aligned}$$

Hence $\{\mathscr{F}_i^N(z)\}_{i=1}^{m_N}$ is a basis of N-new ordinary eigenforms for the space $S_{2k'}^{N\text{-new,ord}}(\Gamma_0(Np);L)$.

□

5.2.3 Proof of Main Theorem

We now give the proof of the main Theorem 5.2.4.

Figure 5.2.17.

$$\begin{array}{ccc}
f_i^N \in S_{2k}^{N\text{-new, ord}}(\Gamma_0(Np);\mathscr{O}_L) & \xrightarrow{\equiv \bmod p^{j+1} \text{ (Fourier coefficients)}} & \mathscr{F}_i^N \in S_{2k'}^{N\text{-new, ord}}(\Gamma_0(Np);\mathscr{O}_{(\mathfrak{p})}) \\
\downarrow & & \downarrow \\
\boxed{N\text{-new ord. eigenform of wt. } 2k} & & \boxed{N\text{-new ord. eigenform of wt. } 2k'}
\end{array}$$

Proof. For simplicity, let us fix an integer i such that $1 \le i \le m_N$, say $i = 1$.

By proposition 5.2.12, for N-new ordinary eigenform $f_1^N(z) \in S_{2k}^{N\text{-new, ord}}(\Gamma_0(Np);\mathscr{O}_L)$, there exists an ordinary cusp form in $F_1^\circ(z) \in S_{2k'}^{\text{ord}}(\Gamma_0(Np);\mathscr{O}_{(\mathfrak{p})})$ such that we have the following congruence of Fourier coefficients

$$f_1^N(z) \equiv F_1^\circ(z) \bmod p^{j+1}.$$

Let $t = \dim(S_{2k'}^{\text{ord}}(\Gamma_0(Np);L))$ and $\{\mathscr{F}_s^N(z)\}_{s=1}^t \subset S_{2k'}^{\text{ord}}(\Gamma_0(Np);\mathscr{O}_L)$ be a basis for $S_{2k'}^{\text{ord}}(\Gamma_0(Np);L)$ consisting of ordinary eigenforms.

Note 5.2.18. We have a slight abuse of notation when we write the superscript N for each ordinary eigenform in the set $\{\mathscr{F}_s^N(z)\}_{s=1}^t$. These ordinary eigenforms are not necessarily N-new but include the set of N-new ordinary eigenforms as well. Note that $t \ge m_N$.

So, we can write $F_1^\circ(z)$ as a linear combination of elements of the basis $\{\mathscr{F}_s^N(z)\}_{s=1}^t$.

$$F_1^\circ(z) = \sum_{s=1}^t \alpha_s \mathscr{F}_s^N(z) \text{ where } \{\alpha_s\}_{s=1}^t \subset L.$$

By proposition 5.2.14, for N-new ordinary eigenform $f_1^N(z) \in S_{2k}^{N\text{-new,ord}}(\Gamma_0(Np); \mathscr{O}_L)$, there exists an N-new ordinary eigenform in $S_{2k'}^{N\text{-new, ord}}(\Gamma_0(Np); \mathscr{O}_L)$, say $\mathscr{F}_1^N(z)$, unique up to scalar multiplication in $\mathscr{O}_L$ such that for all primes $\ell \nmid Np$, we have

$$\lambda_{T_\ell}(f_1^N) \equiv \lambda_{T_\ell}(\mathscr{F}_1^N) \bmod \mathfrak{p}. \tag{5.17}$$

This uniqueness implies that for every integer s such that $2 \le s \le t$ but $s \neq 1$, there exists some prime $\ell_s \nmid Np$ such that

$$\mathfrak{p} \nmid \left(\lambda_{T_{\ell_s}}(f_1^N) - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N)\right).$$

So, we have

$$\mathfrak{p} \nmid \prod_{s=2}^t \left(\lambda_{T_{\ell_s}}(f_1^N) - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N)\right).$$

In other words,

$$\frac{1}{\prod_{s=2}^t \left(\lambda_{T_{\ell_s}}(f_1^N) - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N)\right)} \in \mathscr{O}_{(\mathfrak{p})}. \tag{5.18}$$

Define

$$\mathfrak{F}_1^N(z) := \frac{1}{\prod_{s=2}^t \left(\lambda_{T_{\ell_s}}(f_1^N) - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N)\right)} F_1^\circ(z)\Big|_{2k'} \prod_{s=2}^t \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N)\right). \tag{5.19}$$

Claim: We claim that $\mathfrak{F}_1^N(z) = \mathscr{B}\mathscr{F}_1^N(z)$ for some $\mathscr{B} \in \mathscr{O}_{(\mathfrak{p})}$ and $\mathfrak{F}_1^N(z) \equiv f_1^N(z) \bmod p^{j+1}$.

We will first show that $\mathfrak{F}_1^N(z) = \mathscr{B}\mathscr{F}_1^N(z)$ for some $\mathscr{B} \in \mathscr{O}_{(\mathfrak{p})}$.

$$\begin{aligned}
&F_1^\circ(z)\Big|_{2k'} \prod_{s=2}^t \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N)\right) \\
&= \left(\sum_{s=1}^t \alpha_s \mathscr{F}_s^N(z)\right)\Bigg|_{2k'} \prod_{s=2}^t \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N)\right) \\
&= \left(\sum_{s=1}^t \alpha_s \mathscr{F}_s^N(z)\right)\Bigg|_{2k'} \left(T_{\ell_2} - \lambda_{T_{\ell_2}}(\mathscr{F}_2^N)\right) \prod_{s=3}^t \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N)\right) \\
&= \left(\sum_{s=1}^t \alpha_s (\mathscr{F}_s^N(z)|_{2k'} T_{\ell_2}) - \lambda_{T_{\ell_2}}(\mathscr{F}_2^N) \sum_{s=1}^t \alpha_s \mathscr{F}_s^N(z)\right)\Bigg|_{2k'} \prod_{s=3}^t \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N)\right) \\
&= \left(\sum_{s=1}^t \alpha_s \lambda_{T_{\ell_2}}(\mathscr{F}_s^N) \mathscr{F}_s^N(z) - \sum_{s=1}^t \alpha_s \lambda_{T_{\ell_2}}(\mathscr{F}_2^N) \mathscr{F}_s^N(z)\right)\Bigg|_{2k'} \prod_{s=3}^t \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N)\right) \\
&= \left(\sum_{s=1}^t \alpha_s \left(\lambda_{T_{\ell_2}}(\mathscr{F}_s^N) - \lambda_{T_{\ell_2}}(\mathscr{F}_2^N)\right) \mathscr{F}_s^N(z)\right)\Bigg|_{2k'} \prod_{s=3}^t \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N)\right). \\
&= \sum_{s=1}^t \alpha_s \prod_{w=2}^t \left(\lambda_{T_{\ell_w}}(\mathscr{F}_s^N) - \lambda_{T_{\ell_w}}(\mathscr{F}_w^N)\right) \mathscr{F}_s^N(z).
\end{aligned}$$

$$= \alpha_1 \mathscr{F}_1^N(z) \prod_{w=2}^{t} \left(\lambda_{T_{\ell_w}}(\mathscr{F}_1^N) - \lambda_{T_{\ell_w}}(\mathscr{F}_w^N) \right)$$

or

$$\alpha_1 = \frac{F_1^\circ(z)|_{2k'} \prod_{s=2}^{t} \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N) \right)}{\mathscr{F}_1^N(z) \prod_{w=2}^{t} \left(\lambda_{T_{\ell_w}}(\mathscr{F}_1^N) - \lambda_{T_{\ell_w}}(\mathscr{F}_w^N) \right)}.$$

We want to show that $\alpha_1 \in \mathscr{O}_{(\mathfrak{p})}$.

Note that $F_1^\circ(z)|_{2k'} \prod_{s=2}^{t}(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N))$ has Fourier coefficients in $\mathscr{O}_{(\mathfrak{p})}$. This is because $\lambda_{T_{\ell_s}}(\mathscr{F}_s^N) \in \mathscr{O}_L$ and the Hecke-algebra $\mathbb{T}_{2k'}(Np; \mathscr{O}_{(\mathfrak{p})}) \subseteq \operatorname{End}(S_{2k'}(\Gamma_0(Np); \mathscr{O}_{(\mathfrak{p})}))$.

Next, $\mathscr{F}_1^N(z)$ is unique up to scalar multiplication in $\mathscr{O}_L$ and we can scale it in a way such that at least one of its Fourier coefficient is not divisible by $\mathfrak{p}$. Thus, $\mathfrak{p} \nmid \mathscr{F}_1^N(z)$.
Now, suppose there exists some w such that $2 \leq w \leq t$ for which $\mathfrak{p} \mid \left(\lambda_{T_{\ell_w}}(\mathscr{F}_1^N) - \lambda_{T_{\ell_w}}(\mathscr{F}_w^N) \right)$, then

$$\mathfrak{p} \mid \left(\lambda_{T_{\ell_w}}(\mathscr{F}_1^N) - \lambda_{T_{\ell_w}}(f_1^N) + \lambda_{T_{\ell_w}}(f_1^N) - \lambda_{T_{\ell_w}}(\mathscr{F}_w^N) \right).$$

Since $\mathfrak{p} \mid (\lambda_{T_{\ell_w}}(\mathscr{F}_1^N) - \lambda_{T_{\ell_w}}(f_1^N))$ by congruence 5.17, it follows,

$$\mathfrak{p} \mid \left(\lambda_{T_{\ell_w}}(f_1^N) - \lambda_{T_{\ell_w}}(\mathscr{F}_w^N) \right).$$

This is a contradiction to 5.18. Thus,

$$\mathfrak{p} \nmid \prod_{w=2}^{t} \left(\lambda_{T_{\ell_w}}(\mathscr{F}_1^N) - \lambda_{T_{\ell_w}}(\mathscr{F}_w^N) \right). \tag{5.20}$$

We can hence conclude $\alpha_1 \in \mathscr{O}_{(\mathfrak{p})}$.

We thus have

$$\begin{aligned}
\mathfrak{F}_1^N(z) &= \frac{1}{\prod_{s=2}^{t} \left(\lambda_{T_{\ell_s}}(f_1^N) - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N) \right)} F_1^\circ(z) \Big|_{2k'} \prod_{s=2}^{t} \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N) \right) \\
&= \alpha_1 \frac{\prod_{w=2}^{t} \left(\lambda_{T_{\ell_w}}(\mathscr{F}_1^N) - \lambda_{T_{\ell_w}}(\mathscr{F}_w^N) \right)}{\prod_{s=2}^{t} \left(\lambda_{T_{\ell_s}}(f_1^N) - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N) \right)} \mathscr{F}_1^N(z) \\
&= \mathscr{B} \mathscr{F}_1^N(z)
\end{aligned}$$

where the fact that $\alpha_1 \in \mathscr{O}_{(\mathfrak{p})}$ along with 5.18 implies

$$\mathscr{B} = \alpha_1 \frac{\prod_{w=2}^{t} \left(\lambda_{T_{\ell_w}}(\mathscr{F}_1^N) - \lambda_{T_{\ell_w}}(\mathscr{F}_w^N) \right)}{\prod_{s=2}^{t} \left(\lambda_{T_{\ell_s}}(f_1^N) - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N) \right)} \in \mathscr{O}_{(\mathfrak{p})}.$$

We will now show $\mathfrak{F}_1^N(z) \equiv f_1^N(z) \bmod p^{j+1}$.
In order to do so, we will again look at action of $\prod_{s=2}^{t}(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N))$ on $F_1^\circ(z)$ modulo p^{j+1}.

$$F_1^\circ(z) \Big|_{2k'} \prod_{s=2}^{t} \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N) \right) \equiv f_1^N(z) \Big|_{2k} \prod_{s=2}^{t} \left(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N) \right) \bmod p^{j+1}$$

We see how one of the terms out the product $\prod_{s=2}^{t}(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N))$ acts on $f_1^N(z)$ to observe the pattern.

$$\begin{aligned}
F_1^{\circ}(z)\Big|_{2k'} \prod_{s=2}^{t}\Big(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N)\Big) &\equiv f_1^N(z)\Big|_{2k}(T_{\ell_2} - \lambda_{T_{\ell_2}}(\mathscr{F}_2^N))\prod_{s=3}^{t}\Big(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N)\Big) \bmod p^{j+1}\\
&= \Big(f_1^N(z)|_{2k}T_{\ell_2} - \lambda_{T_{\ell_2}}(\mathscr{F}_2^N)f_1^N(z)\Big)\Big|_{2k}\prod_{s=3}^{t}\Big(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N)\Big) \bmod p^{j+1}\\
&= \Big(\lambda_{T_{\ell_2}}(f_1^N)f_1^{\mathrm{n}}(z) - \lambda_{T_{\ell_2}}(\mathscr{F}_2^N)f_1^N(z)\Big)\Big|_{2k}\prod_{s=3}^{t}\Big(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N)\Big) \bmod p^{j+1}\\
&= \Big(\lambda_{T_{\ell_2}}(f_1^N) - \lambda_{T_{\ell_2}}(\mathscr{F}_2^N)\Big)f_1^N(z)\Big|_{2k}\prod_{s=3}^{t}\Big(T_{\ell_s} - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N)\Big) \bmod p^{j+1}\\
&= \prod_{s=2}^{t}\Big(\lambda_{T_{\ell_s}}(f_1^N) - \lambda_{T_{\ell_s}}(\mathscr{F}_s^N)\Big)f_1^N(z) \bmod p^{j+1}. \qquad (5.21)
\end{aligned}$$

Using definition 5.19 and 5.21, we get

$$\mathfrak{F}_1^N(z) \equiv f_1^N(z) \bmod p^{j+1}. \qquad (5.22)$$

Thus, we have shown that for each integer i such that $1 \le i \le m_N$, there exists an N-new ordinary eigenform $\mathfrak{F}_i^N(z) \in S_{2k'}^{N\text{-new, ord}}(\Gamma_0(Np); \mathscr{O}_{(\mathfrak{p})})$ with $\mathfrak{p}$-integral Fourier coefficients such that we have the following congruence of Fourier coefficients:

$$\mathfrak{F}_i^N(z) \equiv f_i^N(z) \bmod p^{j+1}. \qquad (5.23)$$

Uniqueness: Let $\mathfrak{F}_2^N(z) \in S_{2k}^{N\text{-new, ord}}(\Gamma_0(Np); \mathscr{O}_{(\mathfrak{p})})$ be the N-new ordinary eigenform that satisfies congruence 5.23 with $f_2^N(z)$.

Suppose there does not exist any $\mathscr{B}' \in \mathscr{O}_{(\mathfrak{p})}$ for which $\mathfrak{F}_2^N(z) = \mathscr{B}'\mathscr{F}_1^N(z)$ but if possible, let $\mathfrak{F}_2^N(z)$ also satisfy the congruence

$$\mathfrak{F}_2^N(z) \equiv f_1^N(z) \bmod p^{j+1}.$$

Then we can write

$$f_2^N(z) \equiv f_1^N(z) \bmod p^{j+1}$$

which implies for all primes ℓ,

$$\big(\lambda_{T_\ell}(f_2^N) - \lambda_{T_\ell}(f_2^N)\big)f_1^N(z) \equiv 0 \bmod p^{j+1}.$$

Since, $\mathfrak{p} \nmid f_1^N(z)$ due to our choice of scaling, we have $\mathfrak{p} \mid \big(\lambda_{T_\ell}(f_2^N) - \lambda_{T_\ell}(f_2^N)\big)$. This is a contradiction to the assumption 1. Thus, $\mathfrak{F}_i^N(z)$ is the unique N-new ordinary eigenform up to scalar multiplication in $\mathscr{O}_{(\mathfrak{p})}$ that satisfies the congruence 5.23 with $f_i^N(z)$. □

5.3 Congruences related to modular forms-Half-integer weight case

From Theorem 2.5.5 (i), we have the following direct decomposition of the *Kohnen* plus space,

$$S^{+}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); L) = \bigoplus_{\substack{M|Np\\ d|NpM^{-1}}} S^{\text{new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4M); L)|_{k+\frac{1}{2}}U_{d^2}$$

where d runs over all divisors of NpM^{-1}.

We now give the explicit definition of the space of half-integer weight $4N$-new forms at level $4Np$.

$$S^{4N\text{-new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); L) := S^{\text{new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); L) \oplus \left(S^{\text{new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N); L)|_{k+\frac{1}{2}} U_{p^2} \right) \oplus S^{\text{new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N); L). \tag{5.24}$$

From the definition of N-new forms of integer weight at level Np in 5.2 and the definition of $4N$-new forms of half-integer weight at level $4Np$ in 5.24, we obtain the following relationship.

Figure 5.3.1.

$$\begin{array}{ccccccc}
S^{\text{new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); L) & \bigoplus & S^{\text{new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4N); L) & |_{k+\frac{1}{2}} U_{p^2} & \bigoplus & S^{\text{new}}_{k+\frac{1}{2}}(\Gamma_0(4N); L) \\
\updownarrow \text{Kohnen's Iso.} & & \updownarrow \text{Kohnen's Iso.} & \updownarrow & & \updownarrow \text{Kohnen's Iso.} \\
S^{\text{new}}_{2k}(\Gamma_0(Np); L) & \bigoplus & S^{\text{new}}_{2k}(\Gamma_0(N); L) & |_{2k} U_p & \bigoplus & S^{\text{new}}_{2k}(\Gamma_0(Np); L)
\end{array}$$

Using *Kohnen's* isomorphism along with the fact that $|_{k+\frac{1}{2}} U_{p^2}$ operator in half-integer weight case corresponds to $|_{2k} U_p$ operator in the integer weight case, we conclude that the spaces $S^{4N\text{-new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); L)$ and $S^{N\text{-new}}_{2k}(\Gamma_0(Np); L)$ are isomorphic as Hecke modules and so are there respective ordinary projections.

Figure 5.3.2.

$$\begin{array}{cccc}
S^{4N\text{-new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); L) & \Big|_{k+\frac{1}{2}} \lim_{n\to\infty} U^{n!}_{p^2} & = & S^{4N\text{-new, ord}}_{k+\frac{1}{2}}(\Gamma_0(4N); L) \\
\updownarrow \text{Iso.} & \updownarrow & & \updownarrow \text{Iso.} \\
S^{N\text{-new}}_{2k}(\Gamma_0(Np); L) & \Big|_{2k} \lim_{n\to\infty} U^{n!}_{p} & = & S^{N\text{-new, ord}}_{2k}(\Gamma_0(Np); L)
\end{array}$$

Let

$$\{g^N_i(z)\}^{m_N}_{i=1} \subseteq S^{4N\text{-new, ord}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); \mathcal{O}_L)$$

be a basis for $S^{4N\text{-new, ord}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); L)$ consisting of $4N$-new ordinary eigenforms of half-integer weight with $\mathcal{O}_L$-integral Fourier coefficients (see remark 2.5.6) that is obtained from the basis

$$\{f^N_i(z)\}^{m_N}_{i=1} \subseteq S^{N\text{-new, ord}}_{2k}(\Gamma_0(Np); \mathcal{O}_L)$$

for $S^{N\text{-new, ord}}_{2k}(\Gamma_0(Np); L)$ consisting of N-new ordinary eigenforms of integer weight with $\mathcal{O}_L$-integral Fourier coefficients via the isomorphism in figure 5.3.1.

5.3.1 Main Theorem

We now state our main result about congruences between half-integer weight eigenforms of varying weights. Note that by Hecke eigenforms in Theorem 5.3.3, we mean eigenforms for all Hecke operators T_{ℓ^2} for all prime ℓ.

Theorem 5.3.3 (Main Theorem). *Let* $m_N = \dim\left(S_{k+\frac{1}{2}}^{4N\text{-new, ord}}(\widetilde{\Gamma}_0(4Np);L)\right)$ *and let*

$$\{g_i^N(z)\}_{i=1}^{m_N} \subseteq S_{k+\frac{1}{2}}^{4N\text{-new,ord}}(\widetilde{\Gamma}_0(4Np);\mathcal{O}_L)$$

be a basis for $S_{k+\frac{1}{2}}^{4N\text{-new, ord}}(\widetilde{\Gamma}_0(4Np);L)$ *consisting of* $4N$*-new ordinary half-integer weight eigenforms with* $\mathcal{O}_L$*-integral Fourier coefficients that is obtained from the basis*

$$\{f_i^N(z)\}_{i=1}^{m_N} \subseteq S_{2k}^{N\text{-new, ord}}(\Gamma_0(Np);\mathcal{O}_L)$$

for $S_{2k}^{N\text{-new, ord}}(\Gamma_0(Np);L)$ *consisting of* N*-new ordinary eigenforms of integer weight with* $\mathcal{O}_L$*-integral Fourier coefficients via the isomorphism in figure 5.3.1. For each* i *where* $1 \leq i \leq m_N$*, let* $g_i^N(z)$ *and* $f_i^N(z)$ *be scaled in a way such that they have at least one Fourier coefficient in the* q *expansion that is not divisible by* $\mathfrak{p}$.
Let $j \in \mathbb{Z}_{>0}$ *be a fixed integer and define* $k' = k+\frac{p^j(p-1)}{2}$*. Suppose assumption 1 holds true for the basis* $\{f_i^N(z)\}_{i=1}^{m_N}$*. Then for each integer* i *such that* $1 \leq i \leq m_N$*, there exists a* $4N$*-new ordinary half-integer weight eigenform*

$$\mathfrak{G}_i^N(z) \in S_{k'+\frac{1}{2}}^{4N\text{-new,ord}}(\widetilde{\Gamma}_0(4Np);\mathcal{O}_{(\mathfrak{p})})$$

unique up to scalar multiplication in $\mathcal{O}_{(\mathfrak{p})}$ *such that we have the following congruence of Fourier coefficients:*

$$g_i^N(z) \equiv \mathfrak{G}_i^N(z)|_{k'+\frac{1}{2}}U_p \ mod\ p^{j+1}.$$

Figure 5.3.4.

$$g_i^N \in S_{k+\frac{1}{2}}^{4N\text{-new,ord}}(\widetilde{\Gamma}_0(4Np);\mathcal{O}_L) \xrightarrow{\equiv \text{ mod } p^{j+1} \text{ (Shifted } |U_p)} \mathfrak{G}_i^N \in S_{k'+\frac{1}{2}}^{4N\text{-new,ord}}(\widetilde{\Gamma}_0(4Np);\mathcal{O}_{(\mathfrak{p})})$$

$\downarrow$

$4N$-new ord. eigenform of weight $k+\frac{1}{2}$		$4N$-new ord. eigenform of weight $k'+\frac{1}{2}$

Remark 5.3.5. We note that Theorem 5.3.3 also holds true for all weights $k' = k + t\frac{(p-1)}{2}$ where t is an odd positive integer.

In order to give a nice structure to the proof of Theorem 5.3.3, we break it down into a series of propositions which when combined will eventually imply the result.

Proposition 5.3.6. *Let*

$$g(z) \in S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np);\mathcal{O}_L)$$

be a cusp form of half-integer weight with $\mathcal{O}_L$*-integral Fourier coefficients.*
Let $j \in \mathbb{Z}_{>0}$ *be a fixed integer and define* $k' = k + \frac{p^j(p-1)}{2}$*. Then there exists a cusp form of half-integer weight*

$$G(z) \in S_{k'+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np);\mathcal{O}_{(\mathfrak{p})})$$

such that we have the following congruence of Fourier coefficients:

$$g(z)|_{k+\frac{1}{2}}U_p \equiv G(z) \ mod\ p^{j+1}.$$

Figure 5.3.7.

$$g \in S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np);\mathscr{O}_L) \xrightarrow{(\text{Shifted } |U_p) \equiv \text{mod } p^{j+1}} G \in S_{k'+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np);\mathscr{O}_{(\mathfrak{p})})$$

$$\downarrow$$

Cusp form of weight $k+\frac{1}{2}$ | Cusp form of weight $k'+\frac{1}{2}$

Before we give the proof of proposition 5.3.6, we will state a prerequisite lemma given in [Kob93, Proposition 3, pg. 183].

Lemma 5.3.8. *Let p be an odd rational prime. Then*

$$M_{\frac{p-1}{2}}\left((\Gamma_0(4N),\left(\frac{\bullet}{p}\right)\right) = M_{\frac{p-1}{2}}\left(\widetilde{\Gamma}_0(4N),\left(\frac{-1}{\bullet}\right)^{\frac{p-1}{2}}\left(\frac{\bullet}{p}\right)\right).$$

Proof of proposition 5.3.6.

Define

$$G(z) := \left(g(z)|_{k+\frac{1}{2}}U_p\right)\widetilde{\mathscr{E}}_p(4z)^{p^j}$$

where $\widetilde{\mathscr{E}}_p(4z) \in M_{\frac{p-1}{2}}\left(\Gamma_0(4p),\left(\frac{\bullet}{p}\right)\right)$ is defined to be the generalised Eisenstein series $E_{\frac{p-1}{2},\left(\frac{\bullet}{p}\right)}(4z)$, see section 3.4.

It's given that the Fourier coefficients of $g(z)$ lie in $\mathscr{O}_L$. From proposition 2.3.2, we note that the action of U_p on $g(z) \in S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np);\mathscr{O}_L)$ twists the character of this space by $\left(\frac{p}{\bullet}\right)$, that is,

$$g(z)|_{k+\frac{1}{2}}U_p \in S_{k+\frac{1}{2}}\left(\widetilde{\Gamma}_0(4Np),\left(\frac{p}{\bullet}\right);\mathscr{O}_L\right). \tag{5.25}$$

Recall that $\widetilde{\mathscr{E}}_p(z)$ has p-integral Fourier coefficients (see proof of lemma 3.4.2). Also, by the lemma 5.3.8, we have

$$\widetilde{\mathscr{E}}_p(4z) \in M_{\frac{p-1}{2}}\left(\widetilde{\Gamma}_0(4p),\left(\frac{-1}{\bullet}\right)^{(p-1)/2}\left(\frac{\bullet}{p}\right);\mathbb{Z}_{(p)}\right) \tag{5.26}$$

It's clear from 5.25 and 5.26 that

$$G(z) \in S_{k'+\frac{1}{2}}\left(\widetilde{\Gamma}_0(4Np),\left(\frac{-1}{\bullet}\right)^{(p-1)/2}\left(\frac{\bullet}{p}\right)\left(\frac{p}{\bullet}\right);\mathscr{O}_{(\mathfrak{p})}\right).$$

Since quadratic reciprocity implies $\left(\frac{-1}{\bullet}\right)^{(p-1)/2}\left(\frac{\bullet}{p}\right)\left(\frac{p}{\bullet}\right) = 1$, it follows

$$G(z) \in S_{k'+\frac{1}{2}}\left(\widetilde{\Gamma}_0(4Np);\mathscr{O}_{(\mathfrak{p})}\right).$$

Further by lemma 3.4.2, we have

$$\widetilde{\mathscr{E}}_p(4z) \equiv 1 \text{ mod } p.$$

Now, using a similar argument as in the proof of corollary 3.2.4, we deduce that

$$\widetilde{\mathscr{E}}_p(4z)^{p^j} \equiv 1 \text{ mod } p.$$

Hence, we conclude,

$$g(z)|_{k+\frac{1}{2}}U_p \equiv G(z) \text{ mod } p^{j+1}.$$

□

We next show that the action of Hecke operators $|_{k+\frac{1}{2}}T_{\ell^2}$ for all primes ℓ on $g(z)|_{k+\frac{1}{2}}U_p \bmod p^{j+1}$ is the same as the action of the Hecke operators $|_{k'+\frac{1}{2}}T_{\ell^2}$ for all primes ℓ on $G(z) \bmod p^{j+1}$ where $g(z)$ and $G(z)$ are defined as in proposition 5.3.6.

Corollary 5.3.9. *Let $g(z) \in S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); \mathcal{O}_L)$ and $G(z) \in S_{k'+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); \mathcal{O}_{(\mathrm{p})})$ be a cusp forms of half-integer weight as in proposition 5.3.6 such that*

$$g(z)|_{k+\frac{1}{2}}U_p \equiv G(z) \bmod p^{j+1}.$$

Then we have

$$\left(g(z)|_{k+\frac{1}{2}}U_p\right)|_{k+\frac{1}{2}}T_{\ell^2} \equiv G(z)|_{k'+\frac{1}{2}}T_{\ell^2} \bmod p^{j+1}$$

for all primes ℓ.

Proof. Let $g(z) = \sum_{n=1}^{\infty} b_n q^n$ and $G(z) = \sum_{n=1}^{\infty} B_n q^n$ be the Fourier expansions of $g(z)$ and $G(z)$ defined in proposition 5.3.6. Then we have

$$b_{pn} \equiv B_n \bmod p^{j+1}. \tag{5.27}$$

Case 1: $\ell \nmid 4Np$.

From proposition 2.2.4, we can write for all primes $\ell \nmid 4Np$,

$$G(z)|_{k'+\frac{1}{2}}T_{\ell^2} = \sum_{n=1}^{\infty}\left(B_{\ell^2 n} + \ell^{k'-1}\left(\frac{-1}{\ell}\right)^{k'}\left(\frac{n}{\ell}\right)B_n + \ell^{2k'-1}B_{n\ell^{-2}}\right)q^n. \tag{5.28}$$

From 5.27 and 5.28, we have

$$G(z)|_{k'+\frac{1}{2}}T_{\ell^2} \equiv \sum_{n=1}^{\infty}\left(b_{p\ell^2 n} + \left(\frac{-1}{\ell}\right)^{k'}\ell^{k'-1}\left(\frac{n}{\ell}\right)b_{pn} + \ell^{2k'-1}b_{pn\ell^{-2}}\right)q^n \bmod p^{j+1}. \tag{5.29}$$

The term inside the summand on the right hand side of the congruence 5.29 can be rewritten as

$$\begin{aligned}
&b_{\ell^2 pn} + \left(\frac{-1}{\ell}\right)^{k+\frac{p^j(p-1)}{2}}\ell^{k+\frac{p^j(p-1)}{2}-1}\left(\frac{n}{\ell}\right)b_{pn} + \ell^{2\left(k+\frac{p^j(p-1)}{2}\right)-1}b_{pn\ell^{-2}}\\
=&b_{\ell^2 pn} + \left(\frac{-1}{\ell}\right)^{k+\frac{p^j(p-1)}{2}}\ell^{k-1}\left(\ell^{\frac{(p-1)}{2}}\right)^{p^j}\left(\frac{p}{\ell}\right)^2\left(\frac{n}{\ell}\right)b_{pn} + \ell^{2k-1}\left(\ell^{(p-1)}\right)^{p^j}b_{pn\ell^{-2}}\\
=&b_{\ell^2 pn} + \left(\frac{-1}{\ell}\right)^{k}\ell^{k-1}\left(\left(\frac{-1}{\ell}\right)^{\frac{(p-1)}{2}}\ell^{\frac{(p-1)}{2}}\right)^{p^j}\left(\frac{p}{\ell}\right)\left(\frac{pn}{\ell}\right)b_{pn} + \ell^{2k-1}\left(\ell^{(p-1)}\right)^{p^j}b_{pn\ell^{-2}}
\end{aligned} \tag{5.30}$$

We now look at the terms $\left(\ell^{(p-1)}\right)^{p^j}$ and $\left(\left(\frac{-1}{\ell}\right)^{\frac{(p-1)}{2}}\ell^{\frac{(p-1)}{2}}\right)^{p^j}$.
Since ℓ is an integer, by *Fermat's Little Theorem*, we have $\ell^{p-1} \equiv 1 \bmod p$. Then using a similar argument as in the proof of corollary 3.2.4, we get

$$\left(\ell^{p-1}\right)^{p^j} \equiv 1 \bmod p^{j+1}. \tag{5.31}$$

Again since ℓ is an integer, by *Euler's Criterion*, we have $\ell^{\frac{(p-1)}{2}} \equiv \left(\frac{\ell}{p}\right) \bmod p$. Then it follows,

$$\begin{aligned}
\left(\frac{-1}{\ell}\right)^{\frac{(p-1)}{2}}\ell^{\frac{(p-1)}{2}}\left(\frac{p}{\ell}\right) &\equiv \left(\frac{-1}{\mathscr{L}}\right)^{\frac{(p-1)}{2}}\left(\frac{\ell}{p}\right)\left(\frac{p}{\ell}\right) \bmod p\\
&= 1 \bmod p.
\end{aligned}$$

Again, using a similar argument as in the proof of corollary 3.2.4, we get

$$\left(\left(\frac{-1}{\ell}\right)^{\frac{(p-1)}{2}} \ell^{\frac{(p-1)}{2}} \left(\frac{p}{\ell}\right)\right)^{p^j} \equiv 1 \bmod p^{j+1}$$

or

$$\left(\frac{-1}{\ell}\right)^{\frac{p^j(p-1)}{2}} \ell^{\frac{p^j(p-1)}{2}} \left(\frac{p}{\ell}\right) \equiv 1 \bmod p^{j+1}. \tag{5.32}$$

Using 5.31 and 5.32 together in 5.30, we get

$$\begin{aligned} G(z)|_{k'+\frac{1}{2}} T_{\ell^2} &\equiv \sum_{n=1}^{\infty} \left(b_{p\ell^2 n} + \left(\frac{-1}{\ell}\right)^k \ell^{k-1} \left(\frac{pn}{\ell}\right) b_{pn} + \ell^{2k-1} b_{pn\ell^{-2}} \right) q^n \bmod p^{j+1} \\ &= \left(g(z)|_{k+\frac{1}{2}} U_p\right)|_{k+\frac{1}{2}} T_{\ell^2} \bmod p^{j+1}. \end{aligned}$$

Case 2: $\ell \mid 4Np$.

Again using proposition 2.2.4, we can write for all primes $\ell \mid 4Np$,

$$\left(g(z)|_{k+\frac{1}{2}} U_p\right)|_{k+\frac{1}{2}} T_{\ell^2} = \sum_{n=1}^{\infty} b_{\ell^2 pn} q^n \quad \text{and} \quad G(z)|_{k'+\frac{1}{2}} T_{\ell^2} = \sum_{n=1}^{\infty} B_{\ell^2 n} q^n. \tag{5.33}$$

From 5.27 and 5.33, it follows,

$$b_{p(\ell^2 n)} \equiv B_{(\ell^2 n)} \bmod p^{j+1}$$

which implies

$$\left(g(z)|_{k+\frac{1}{2}} U_p\right)|_{k+\frac{1}{2}} T_{\ell^2} \equiv G(z)|_{k'+\frac{1}{2}} T_{\ell^2} \bmod p^{j+1}.$$

□

We will next try to replace the half-integer weight cusp form $G(z)$ defined in proposition 5.3.6 by an ordinary half-integer weight cusp form of the same half-integer weight $k' + \frac{1}{2}$, provided that we now take $g(z)$ to be ordinary.

Proposition 5.3.10. *Let*

$$g(z) \in S^{ord}_{k+\frac{1}{2}}(\widetilde{\Gamma_0}(4Np); \mathcal{O}_L)$$

be an ordinary cusp form of half-integer weight with $\mathcal{O}_L$-integral Fourier coefficients.
Let $j \in \mathbb{Z}_{>0}$ be a fixed integer and define $k' = k + \frac{p^j(p-1)}{2}$. Then there exists an ordinary cusp form of half-integer weight

$$G^{\circ}(z) \in S^{ord}_{k'+\frac{1}{2}}(\widetilde{\Gamma_0}(4Np); \mathcal{O}_{(\mathfrak{p})})$$

such that we have the following congruence of Fourier coefficients:

$$g(z)|_{k+\frac{1}{2}} U_p \equiv G^{\circ}(z) \bmod p^{j+1}.$$

Moreover,

$$\left(g(z)|_{k+\frac{1}{2}} U_p\right)|_{k+\frac{1}{2}} T_{\ell^2} \equiv G^{\circ}(z)|_{k'+\frac{1}{2}} T_{\ell^2} \bmod p^{j+1}$$

for all primes ℓ.

Figure 5.3.11.

$$g \in S^{\mathrm{ord}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); \mathscr{O}_L) \xrightarrow{(\text{Shifted } |U_p) \equiv \bmod p^{j+1}} G^{\circ} \in S^{\mathrm{ord}}_{k'+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); \mathscr{O}_{(\mathfrak{p})})$$

$\downarrow$

Ord. cusp form of weight $k + \dfrac{1}{2}$

Ord. cusp form of weight $k' + \dfrac{1}{2}$

Proof. Define

$$G^{\circ}(z) := G(z)|_{k'+\frac{1}{2}}\widetilde{e}.$$

where $\widetilde{e}$ is the p-ordinary projection operator defined in section 4.5 and $G(z) \in S_{k'+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); \mathscr{O}_{(\mathfrak{p})})$ is the half-integer weight cusp form defined in proposition 5.3.6.

Now $\widetilde{e}$ is an idempotent operator, that is, $\widetilde{e}^2 = \widetilde{e}$. This implies

$$\begin{aligned} G^{\circ}(z)|_{k'+\frac{1}{2}}\widetilde{e} &= G(z)|_{k'+\frac{1}{2}}\widetilde{e}^2 \\ &= G(z)|_{k'+\frac{1}{2}}\widetilde{e} \\ &= G^{\circ}(z). \end{aligned}$$

Hence $G^{\circ}(z)$ is ordinary at p and lies in $S^{\mathrm{ord}}_{k'+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); \mathscr{O}_{(\mathfrak{p})})$.

Claim: We now claim that $G^{\circ}(z) \equiv g(z)|_{k+\frac{1}{2}}U_p \bmod p^{j+1}$.

It is given that $g(z)$ is ordinary at p. So, $g(z)|_{k+\frac{1}{2}}\widetilde{e} = g(z)$. Also, note that U_{p^2} acts in the same way on $g(z)$ as U_p^2. Then by Corollary 5.3.9, we can write

$$\begin{aligned} G(z)|_{k'+\frac{1}{2}}\widetilde{e} &\equiv \left(g(z)|_{k+\frac{1}{2}}U_p\right)|_{k+\frac{1}{2}}\widetilde{e} \bmod p^{j+1} \\ &= \left(g(z)|_{k+\frac{1}{2}}U_p\right)|_{k+\frac{1}{2}} \lim_{n\to\infty} U_{p^2}^{n!} \bmod p^{j+1} \\ &= \left(g(z)|_{k+\frac{1}{2}}U_p\right)|_{k+\frac{1}{2}} \lim_{n\to\infty} U_p^{2(n!)} \bmod p^{j+1} \\ &= g(z)|_{k+\frac{1}{2}} \lim_{n\to\infty} U_p^{2(n!)+1} \bmod p^{j+1} \\ &= \left(g(z)|_{k+\frac{1}{2}} \lim_{n\to\infty} U_p^{2(n!)}\right)|_{k+\frac{1}{2}}U_p \bmod p^{j+1} \\ &= \left(g(z)|_{k+\frac{1}{2}} \lim_{n\to\infty} U_{p^2}^{n!}\right)|_{k+\frac{1}{2}}U_p \bmod p^{j+1} \\ &= \left(g(z)|_{k+\frac{1}{2}}\widetilde{e}\right)|_{k+\frac{1}{2}}U_p \bmod p^{j+1} \\ &= g(z)|_{k+\frac{1}{2}}U_p \bmod p^{j+1}. \end{aligned}$$

Thus, we have shown that

$$G^{\circ}(z) \equiv g(z)|_{k+\frac{1}{2}}U_p \bmod p^{j+1}.$$

The equivalence of action of Hecke operators modulo p^{j+1} now follows in the same way as in corollary 5.3.9. $\square$

We next show that if we are given that the ordinary half-integer weight cusp form $g(z) \in S^{\text{ord}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); \mathscr{O}_L)$ in proposition 5.3.10 is a $4N$-new ordinary eigenform that is obtained from the N-new ordinary eigenform of integer weight, say $f^N(z) \in S^{N\text{-new, ord}}_{2k}(\Gamma_0(Np); \mathscr{O}_L)$ via the isomorphism in figure 5.3.1, then there exists $\mathscr{G}(z) \in S^{\text{ord}}_{k'+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); \mathscr{O}_L)$, a unique $4N$-new ordinary eigenform of half-integer weight (unique up to scalar multiplication in $\mathscr{O}_L$) which satisfies a mod $\mathfrak{p}$ congruence of Hecke eigenvalues with $g(z)$ for all Hecke operators T_{ℓ^2} over all primes $\ell \nmid 4Np$.
Note that by Hecke eigenforms in proposition 5.3.12, we mean eigenforms for all Hecke operators T_{ℓ^2} for all primes ℓ.

Proposition 5.3.12. *Let* $m_N = \dim\left(S^{4N\text{-new, ord}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); L)\right)$ *and let*

$$\{g_i^N(z)\}_{i=1}^{m_N} \subseteq S^{4N\text{-new, ord}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); \mathscr{O}_L)$$

be a basis for $S^{4N\text{-new, ord}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); L)$ *consisting of* $4N$*-new ordinary eigenforms of half-integer weight with* $\mathscr{O}_L$*-integral Fourier coefficients that is obtained from the basis*

$$\{f_i^N(z)\}_{i=1}^{m_N} \subseteq S^{N\text{-new, ord}}_{2k}(\Gamma_0(Np); \mathscr{O}_L) \tag{5.34}$$

for $S^{N\text{-new, ord}}_{2k}(\Gamma_0(Np); L)$ *consisting of* N*-new ordinary eigenforms of integer weight with* $\mathscr{O}_L$*-integral Fourier coefficients via the isomorphism given in figure 5.3.1. For each* i *where* $1 \leq i \leq m_N$*, let* $g_i^N(z)$ *and* $f_i^N(z)$ *be scaled in a way such that they have at least one Fourier coefficient in the* q *expansion that is not divisible by* $\mathfrak{p}$*. Suppose assumption* 1 *holds true for the basis* $\{f_i^N(z)\}_{i=1}^{m_N}$*.*
Let $j \in \mathbb{Z}_{>0}$ *be a fixed integer and define* $k' = k + \frac{p^j(p-1)}{2}$*. Then for each integer* i *such that* $1 \leq i \leq m_N$*, there exists a* $4N$*-new ordinary eigenform form of half-integer weight*

$$\mathscr{G}_i^N(z) \in S^{ord}_{k'+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); \mathscr{O}_L)$$

unique up to scalar multiplication in $\mathscr{O}_L$ *such that for all primes* $\ell \nmid 4Np$*, we have*

$$\lambda_{T_{\ell^2}}(g_i^N) \equiv \lambda_{T_{\ell^2}}(\mathscr{G}_i^N) \text{ mod } \mathfrak{p}.$$

Figure 5.3.13.

$$g_i^N \in S^{4N\text{-new, ord}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); \mathscr{O}_L) \xrightarrow{\text{Hecke eigenvalues} \equiv \text{mod } \mathfrak{p}} \mathscr{G}_i^N \in S^{4N-\text{new, ord}}_{k'+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); \mathscr{O}_L)$$

$\downarrow$ $4N$-new ord. eigenform of weight $k + \frac{1}{2}$

$\downarrow$ $4N$-new ord. eigenform of weight $k' + \frac{1}{2}$

Proof. From the isomorphism given in figure 5.3.1, for all primes $\ell \nmid 4Np$, we have

$$\lambda_{T_\ell}(f_i^N) = \lambda_{T_{\ell^2}}(g_i^N). \tag{5.35}$$

Then by proposition 5.2.12 and proposition 5.2.14, there exists a basis $\{\mathscr{F}_i^N(z)\}_{i=1}^{m_N} \subset S^{N\text{-new, ord}}_{2k'}(\Gamma_0(Np); \mathscr{O}_L)$ of N-new ordinary eigenforms for the space $S^{N\text{-new, ord}}_{2k'}(\Gamma_0(Np); L)$ such that each element in the basis is unique up to scalar multiplication in $\mathscr{O}_L$ and for all primes $\ell \nmid Np$ satisfies

$$\lambda_{T_\ell}(f_i^N) \equiv \lambda_{T_\ell}(\mathscr{F}_i^N) \text{ mod } \mathfrak{p} \tag{5.36}$$

In a similar way as in figure 5.3.1, we conclude that the space $S_{k'+\frac{1}{2}}^{4N\text{-new}}(\widetilde{\Gamma}_0(4Np);L)$ is mapped isomorphically onto $S_{2k'}^{N\text{-new}}(\Gamma_0(Np);L)$ and these spaces have isomorphic ordinary projections. We can hence take

$$\{\mathscr{G}_i^N(z)\}_{i=1}^{m_N} \subseteq S_{k'+\frac{1}{2}}^{4N\text{-new, ord}}(\widetilde{\Gamma}_0(4Np);\mathscr{O}_L)$$

to be a basis of $4N$-new ordinary eigenforms of half-integer weight obtained from the basis

$$\{\mathscr{F}_i^N(z)\}_{i=1}^{m_N} \subseteq S_{2k'}^{N\text{-new, ord}}(\Gamma_0(Np);\mathscr{O}_L)$$

of N-new ordinary eigenforms of integer weight for $S_{2k'}^{N\text{-new, ord}}(\Gamma_0(Np);L)$.

Then for all primes $\ell \nmid 4Np$, we have

$$\lambda_{T_\ell}(\mathscr{F}_i^N) = \lambda_{T_{\ell^2}}(\mathscr{G}_i^N). \tag{5.37}$$

Hence, it follows from 5.35, 5.36 and 5.37 that

$$\lambda_{T_{\ell^2}}(g_i^N) \equiv \lambda_{T_{\ell^2}}(\mathscr{G}_i^N) \bmod \mathfrak{p}. \tag{5.38}$$

Uniqueness: Let $\mathscr{G}_1^N(z), \mathscr{G}_2^N(z) \in S_{k'+\frac{1}{2}}^{4N\text{-new, ord}}(\widetilde{\Gamma}_0(4Np);\mathscr{O}_L)$ be two $4N$-new ordinary eigenforms of half-integer weight in the basis $\{\mathscr{G}_i^N(z)\}_{i=1}^{m_N}$. Then using congruence 5.38, they satisfy the respective congruences,

$$\lambda_{T_{\ell^2}}(g_1^N) \equiv \lambda_{T_{\ell^2}}(\mathscr{G}_1^N) \bmod \mathfrak{p}$$

and

$$\lambda_{T_{\ell^2}}(g_2^N) \equiv \lambda_{T_{\ell^2}}(\mathscr{G}_2^N) \bmod \mathfrak{p}$$

where $g_1^N(z), g_2^N(z) \in S_{k+\frac{1}{2}}^{4N\text{-new, ord}}(\widetilde{\Gamma}_0(4Np);\mathscr{O}_L)$ are $4N$-new ordinary eigenforms of half-integer weight in the basis $\{g_i^N(z)\}_{i=1}^{m_N}$.
Suppose there does not exist any $\mathscr{B} \in \mathscr{O}_L$ such that $\mathscr{G}_2^N(z) = \mathscr{B}\mathscr{G}_1^N(z)$ but for all primes $\ell \nmid 4Np$, $\mathscr{G}_2^N(z)$ also satisfies the following congruence

$$\lambda_{T_{\ell^2}}(g_1^N) \equiv \lambda_{T_{\ell^2}}(\mathscr{G}_2^N) \bmod \mathfrak{p}.$$

Then, using equation 5.35, we have $\lambda_{T_\ell}(f_1^N) = \lambda_{T_{\ell^2}}(g_1^N)$ where $f_1^N(z) \in S_{2k}^{N\text{-new, ord}}(\Gamma_0(Np);\mathscr{O}_L)$ is an N-new ordinary eigenform in the basis $\{f_i^N(z)\}_{i=1}^{m_N}$. Similarly, from equation 5.37, we have $\lambda_{T_\ell}(\mathscr{F}_2^N) = \lambda_{T_{\ell^2}}(\mathscr{G}^N)$. It then follows:

$$\begin{aligned} \lambda_{T_\ell}(f_1^N) &= \lambda_{T_{\ell^2}}(g_1^N) \\ &\equiv \lambda_{T_{\ell^2}}(\mathscr{G}_2^N) \bmod \mathfrak{p} \\ &= \lambda_{T_\ell}(\mathscr{F}_2^N) \bmod \mathfrak{p}. \\ &\equiv \lambda_{T_\ell}(f_2^N) \bmod \mathfrak{p}. \end{aligned}$$

This is a contradiction to our assumption 1 as $f_1^N(z)$ and $f_2^N(z)$ are two distinct basis elements. Hence $\mathscr{G}_1^N(z)$ is unique up to scalar multiplication in $\mathscr{O}_L$. ☐

5.3.2 Proof of Main Theorem

We now give the proof of the main Theorem 5.3.3. Before we do that, we will prove the following lemma which will be required to complete the proof of our Main Theorem.

Lemma 5.3.14. *Let*

$$g(z) \in S_{k+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); \mathcal{O}_L)$$

be a half-integer weight cusp form. Then for all primes $\ell \nmid 4Np$, the action of U_p and T_{ℓ^2} on $g(z)$ is commutative, that is,

$$\left(g(z)|_{k+\frac{1}{2}} U_p\right)_{k+\frac{1}{2}} T_{\ell^2} = \left(g(z)|_{k+\frac{1}{2}} T_{\ell^2}\right)_{k+\frac{1}{2}} U_p.$$

Proof. Let $g(z) = \sum_{n=1}^{\infty} b_n q^n$. Then

$$g(z)|_{k+\frac{1}{2}} U_p = \sum_{n=1}^{\infty} b_{pn} q^n \in S_{k+\frac{1}{2}}\big(\widetilde{\Gamma}_0(4Np), \big(\tfrac{p}{\bullet}\big); \mathcal{O}_L\big).$$

Then

$$\begin{aligned}\left(g(z)|_{k+\frac{1}{2}} U_p\right)|_{k+\frac{1}{2}} T_{\ell^2} &= \left(\sum_{n=1}^{\infty} b_{pn} q^n\right)\Big|_{k+\frac{1}{2}} T_{\ell^2} \\ &= \sum_{n=1}^{\infty}\left(b_{\ell^2 pn} + \chi^\star(\ell)\left(\frac{n}{\ell}\right)\ell^{k-1} b_{\ell pn} + \chi^\star(\ell^2)\ell^{2k-1} b_{pn/\ell^2}\right) q^n\end{aligned}$$

where $\chi^\star(\bullet) = \left(\frac{-1}{\bullet}\right)^k \chi(\bullet)$ and in our case $\chi(\bullet) = \left(\frac{p}{\bullet}\right)$.

Thus,

$$\begin{aligned}\left(g(z)|_{k+\frac{1}{2}} U_p\right)|_{k+\frac{1}{2}} T_{\ell^2} &= \sum_{n=1}^{\infty}\left(b_{\ell^2 pn} + \left(\frac{-1}{\ell}\right)^k \left(\frac{p}{\ell}\right)\left(\frac{n}{\ell}\right)\ell^{k-1} b_{\ell pn} + \ell^{2k-1} b_{pn/\ell^2}\right) q^n \\ &= \sum_{n=1}^{\infty}\left(b_{p\ell^2 n} + \left(\frac{-1}{\ell}\right)^k \left(\frac{pn}{\ell}\right)\ell^{k-1} b_{p\ell n} + \ell^{2k-1} b_{pn/\ell^2}\right) q^n \\ &= \left(\sum_{n=1}^{\infty}\left(b_{\ell^2 n} + \left(\frac{-1}{\ell}\right)^k \left(\frac{n}{\ell}\right)\ell^{k-1} b_{\ell n} + \ell^{2k-1} b_{n/\ell^2}\right) q^n\right)\Bigg|_{k+\frac{1}{2}} U_p \\ &= \left(g(z)|_{k+\frac{1}{2}} T_{\ell^2}\right)|_{k+\frac{1}{2}} U_p.\end{aligned}$$

□

We will now give the proof the main theorem 5.3.3 of this section.

Figure 5.3.15.

$$g_i^N \in S_{k+\frac{1}{2}}^{4N\text{-new,ord}}(\widetilde{\Gamma}_0(4Np); \mathcal{O}_L) \xrightarrow{\equiv \bmod p^{j+1} \text{ (Shifted } |U_p)} \mathfrak{G}_i^N \in S_{k'+\frac{1}{2}}^{4N\text{-new,ord}}(\widetilde{\Gamma}_0(4Np); \mathcal{O}_{(\mathfrak{p})})$$

$$\downarrow$$

$$\boxed{4N\text{-new ord. eigenform of weight } k + \frac{1}{2}} \qquad \boxed{4N\text{-new ord. eigenform of weight } k' + \frac{1}{2}}$$

Proof of Theorem 5.3.3. For simplicity, let us fix the integer i such that $1 \le i \le m_N$, say $i = 1$.

By proposition 5.3.10, for a $4N$-new ordinary eigenform $g_1^N(z) \in S_{k+\frac{1}{2}}^{4N\text{-new,ord}}(\widetilde{\Gamma}_0(4Np); \mathscr{O}_L)$, there exists an ordinary half-integer weight cusp form $G_1^{\circ}(z) \in S_{k'+\frac{1}{2}}^{\text{ord}}(\widetilde{\Gamma}_0(4Np); \mathscr{O}_{(\mathfrak{p})})$ such that we have the following congruence of Fourier coefficients:

$$g_1^N(z)|_{k+\frac{1}{2}} U_p \equiv G_1^{\circ}(z) \bmod p^{j+1}.$$

Let $t = \dim(S_{k'+\frac{1}{2}}^{\text{ord}}(\widetilde{\Gamma}_0(4Np); L))$ and $\{\mathscr{G}_s^N(z)\}_{s=1}^t \subseteq S_{k'+\frac{1}{2}}^{\text{ord}}(\widetilde{\Gamma}_0(4Np); \mathscr{O}_L)$ be a basis of $S_{k'+\frac{1}{2}}^{\text{ord}}(\widetilde{\Gamma}_0(4Np); L)$ consisting of ordinary half-integer weight eigenforms.

Note 5.3.16. We have made a slight abuse of notation here when we wrote the superscript N for each element in the set $\{\mathscr{G}_s^N(z)\}_{s=1}^t$. These ordinary half-integer weight eigenforms are not necessarily $4N$-new but include the set of $4N$-new ordinary half-integer weight eigenforms as well. Note that $t \ge m_N$.

So, we can write $G_1^{\circ}(z)$ as a linear combination of elements of the basis $\{\mathscr{G}_s^N(z)\}_{s=1}^t$.

$$G_1^{\circ}(z) = \sum_{s=1}^{t} \beta_s \mathscr{G}_s^N(z) \text{ where } \{\beta_s\}_{s=1}^t \subset L.$$

By proposition 5.3.12, for a $4N$-new ordinary eigenform $g_1^N(z) \in S_{k+\frac{1}{2}}^{4N\text{-new,ord}}(\widetilde{\Gamma}_0(4Np); \mathscr{O}_L)$, there exists a $4N$-new ordinary half-integer weight eigenform in $S_{k'+\frac{1}{2}}^{4N\text{-new, ord}}(\widetilde{\Gamma}_0(4Np); \mathscr{O}_L)$, say $\mathscr{G}_1^N(z)$, unique up to scalar multiplication by elements in $\mathscr{O}_L$ such that for all primes $\ell \nmid 4Np$, we have

$$\lambda_{T_{\ell^2}}(g_1^N) \equiv \lambda_{T_{\ell^2}}(\mathscr{G}_1^N) \bmod \mathfrak{p}. \tag{5.39}$$

This uniqueness implies that for every integer s such that $2 \le s \le t$ but $s \ne 1$, there exists some prime $\ell_s \nmid 4Np$ such that

$$\mathfrak{p} \nmid \left(\lambda_{T_{\ell_s^2}}(g_1^N) - \lambda_{T_{\ell_s^2}}(\mathscr{G}_s^N)\right).$$

So, we have

$$\mathfrak{p} \nmid \prod_{s=2}^{t} \left(\lambda_{T_{\ell_s^2}}(g_1^N) - \lambda_{T_{\ell_s^2}}(\mathscr{G}_s^N)\right). \tag{5.40}$$

In other words,

$$\frac{1}{\prod_{s=2}^{t} \left(\lambda_{T_{\ell_s^2}}(g_1^N) - \lambda_{T_{\ell_s^2}}(\mathscr{G}_s^N)\right)} \in \mathscr{O}_{(\mathfrak{p})}. \tag{5.41}$$

Define

$$\mathfrak{G}_1^N(z) := \frac{\lambda_{U_{p^2}}(g_1^N)^{-1}}{\prod_{s=2}^{t} \left(\lambda_{T_{\ell_s^2}}(g_1^N) - \lambda_{T_{\ell_s^2}}(\mathscr{G}_s^N)\right)} G_1^{\circ}(z)\Bigg|_{k'+\frac{1}{2}} \prod_{s=2}^{t} \left(T_{\ell_s^2} - \lambda_{T_{\ell_s^2}}(\mathscr{G}_s^N)\right). \tag{5.42}$$

Claim 1: $\mathfrak{G}_1^N(z) = \lambda_{U_{p^2}}(g_1^N)^{-1} \mathscr{C} \mathscr{G}_1^N(z)$ for some $\mathscr{C} \in \mathscr{O}_{(\mathfrak{p})}$.

Note 5.3.17. The eigenvalue $\lambda_{U_{p^2}}(g_1^N)^{-1} \in \mathscr{O}_{(\mathfrak{p})}$. This follows from the fact that $g_1^N(z)$ is an ordinary eigenform of half-integer weight and hence its U_{p^2} eigenvalue $\lambda_{U_{p^2}}(g_1^N)$ is a $\mathfrak{p}$-adic unit, that is $\mathfrak{p} \nmid \lambda_{U_{p^2}}(g_1^N)$.

Now we want to show that in definition 5.42, $G_1^{\circ}(z)|_{k'+\frac{1}{2}} \prod_{s=2}^{t}(T_{\ell_s^2} - \lambda_{T_{\ell_s^2}}(\mathscr{G}_s^N))$ has Fourier coefficients in $\mathscr{O}_{(\mathfrak{p})}$.

$$\begin{aligned}
&G_1^{\circ}(z)\bigg|_{k'+\frac{1}{2}} \prod_{s=2}^{t}\Big(T_{\ell_s^2} - \lambda_{T_{\ell_s^2}}(\mathscr{G}_s^N)\Big)\\
&= \left(\sum_{s=1}^{t}\beta_s \mathscr{G}_s^N(z)\right)\bigg|_{k'+\frac{1}{2}} \prod_{s=2}^{t}\Big(T_{\ell_s^2} - \lambda_{T_{\ell_s^2}}(\mathscr{G}_s^N)\Big)\\
&= \left(\sum_{s=1}^{t}\beta_s \mathscr{G}_s^N(z)\right)\bigg|_{k'+\frac{1}{2}} \Big(T_{\ell_2^2} - \lambda_{T_{\ell_2^2}}(\mathscr{G}_2^N)\Big)\prod_{s=3}^{t}\Big(T_{\ell_s^2} - \lambda_{T_{\ell_s^2}}(\mathscr{G}_s^N)\Big)\\
&= \left(\sum_{s=1}^{t}\beta_s (\mathscr{G}_s^N(z)|_{k'+\frac{1}{2}} T_{\ell_2^2}) - \lambda_{T_{\ell_2^2}}(\mathscr{G}_2^N)\sum_{s=1}^{t}\beta_s \mathscr{G}_s^N(z)\right)\bigg|_{k'+\frac{1}{2}} \prod_{s=3}^{t}\Big(T_{\ell_s^2} - \lambda_{T_{\ell_s^2}}(\mathscr{G}_s^N)\Big)\\
&= \left(\sum_{s=1}^{t}\beta_s \lambda_{T_{\ell_2^2}}(\mathscr{G}_s^N)\mathscr{G}_s^N(z) - \sum_{s=1}^{t}\beta_s \lambda_{T_{\ell_2^2}}(\mathscr{G}_2^N)\mathscr{G}_s^N(z)\right)\bigg|_{k'+\frac{1}{2}} \prod_{s=3}^{t}\Big(T_{\ell_s^2} - \lambda_{T_{\ell_s^2}}(\mathscr{G}_s^N)\Big)\\
&= \left(\sum_{s=1}^{t}\beta_s \Big(\lambda_{T_{\ell_2^2}}(\mathscr{G}_s^N) - \lambda_{T_{\ell_2^2}}(\mathscr{G}_2^N)\Big)\mathscr{G}_s^N(z)\right)\bigg|_{k'+\frac{1}{2}} \prod_{s=3}^{t}\Big(T_{\ell_s^2} - \lambda_{T_{\ell_s^2}}(\mathscr{G}_s^N)\Big).\\
&= \sum_{s=1}^{t}\beta_s \prod_{w=2}^{t}\Big(\lambda_{T_{\ell_w^2}}(\mathscr{G}_s^N) - \lambda_{T_{\ell_w^2}}(\mathscr{G}_w^N)\Big)\mathscr{G}_s^N(z).\\
&= \beta_1 \mathscr{G}_1^N(z)\prod_{w=2}^{t}\Big(\lambda_{T_{\ell_w^2}}(\mathscr{G}_1^N) - \lambda_{T_{\ell_w^2}}(\mathscr{G}_w^N)\Big) \qquad (5.43)
\end{aligned}$$

or

$$\beta_1 = \frac{G_1^{\circ}(z)|_{k'+\frac{1}{2}} \prod_{s=2}^{t}\Big(T_{\ell_s^2} - \lambda_{T_{\ell_s^2}}(\mathscr{G}_s^N)\Big)}{\mathscr{G}_1^N(z)\prod_{w=2}^{t}\Big(\lambda_{T_{\ell_w^2}}(\mathscr{G}_1^N) - \lambda_{T_{\ell_w^2}}(\mathscr{G}_w^N)\Big)}.$$

We want to show that $\beta_1 \in \mathscr{O}_{(\mathfrak{p})}$.

Note that $G_1^{\circ}(z)|_{k'+\frac{1}{2}} \prod_{s=2}^{t}(T_{\ell_s^2} - \lambda_{T_{\ell_s^2}}(\mathscr{G}_s^N))$ has Fourier coefficients in $\mathscr{O}_{(\mathfrak{p})}$. This is because $\lambda_{T_{\ell_s^2}}(\mathscr{G}_s^N) \in \mathscr{O}_L$ and the Hecke-algebra $\widetilde{\mathbb{T}}_{k'+\frac{1}{2}}(4Np; \mathscr{O}_{(\mathfrak{p})}) \subseteq \operatorname{End}(S_{k'+\frac{1}{2}}(\widetilde{\Gamma}_0(4Np); \mathscr{O}_{(\mathfrak{p})}))$.

Next, we are given that $\mathscr{G}_1^N(z) \in S_{k'+\frac{1}{2}}^{4N\text{-new, ord}}(\widetilde{\Gamma}_0(4Np); \mathscr{O}_L)$ is unique up to scalar multiplication in $\mathscr{O}_L$. We can therefore assume it is scaled in a way such that at least one of it's Fourier coefficient is not divisible by $\mathfrak{p}$. Now, suppose there exists some w such that $2 \le w \le t$ for which $\mathfrak{p} \mid \Big(\lambda_{T_{\ell_w^2}}(\mathscr{G}_1^N) - \lambda_{T_{\ell_w^2}}(\mathscr{G}_w^N)\Big)$, then

$$\mathfrak{p} \mid \Big(\lambda_{T_{\ell_w^2}}(\mathscr{G}_1^N) - \lambda_{T_{\ell_w^2}}(g_1^N) + \lambda_{T_{\ell_w^2}}(g_1^N) - \lambda_{T_{\ell_w^2}}(\mathscr{G}_w^N)\Big).$$

Since $\mathfrak{p} \mid (\lambda_{T_{\ell_w^2}}(\mathscr{G}_1^N) - \lambda_{T_{\ell_w^2}}(g_1^N))$ by 5.39, it follows,

$$\mathfrak{p} \mid \Big(\lambda_{T_{\ell_w^2}}(g_1^N) - \lambda_{T_{\ell_w^2}}(\mathscr{G}_w^N)\Big).$$

This is a contradiction to uniqueness of $\mathscr{G}_1^N(z)$ in 5.40. Thus,

$$\mathfrak{p} \nmid \prod_{w=2}^{t}\Big(\lambda_{T_{\ell_w^2}}(\mathscr{G}_1^N) - \lambda_{T_{\ell_w^2}}(\mathscr{G}_w^N)\Big). \qquad (5.44)$$

We can hence conclude

$$\beta_1 \in \mathcal{O}_{(\mathfrak{p})}. \tag{5.45}$$

We thus have

$$\begin{aligned}
\mathfrak{G}_1^N(z) &= \frac{\lambda_{U_{p^2}}(g_1^N)^{-1}}{\prod_{s=2}^t \left(\lambda_{T_{\ell_s^2}}(g_1^N) - \lambda_{T_{\ell_s^2}}(\mathcal{G}_s^N)\right)} G_1^\circ(z)\bigg|_{k'+\frac{1}{2}} \prod_{s=2}^t \left(T_{\ell_s^2} - \lambda_{T_{\ell_s^2}}(\mathcal{G}_s^N)\right) \\
&= \lambda_{U_{p^2}}(g_1^N)^{-1} \beta_1 \frac{\prod_{w=2}^t \left(\lambda_{T_{\ell_w^2}}(\mathcal{G}_1^N) - \lambda_{T_{\ell_w^2}}(\mathcal{G}_w^N)\right)}{\prod_{s=2}^t \left(\lambda_{T_{\ell_s^2}}(g_1^N) - \lambda_{T_{\ell_s^2}}(\mathcal{G}_s^N)\right)} \mathcal{G}_1^N(z) \\
&= \lambda_{U_{p^2}}(g_1^N)^{-1} \mathscr{C} \mathcal{G}_1^N(z)
\end{aligned}$$

where 5.41, and 5.45 together imply

$$\mathscr{C} = \beta_1 \frac{\prod_{w=2}^t \left(\lambda_{T_{\ell_w^2}}(\mathcal{G}_1^N) - \lambda_{T_{\ell_w^2}}(\mathcal{G}_w^N)\right)}{\prod_{s=2}^t \left(\lambda_{T_{\ell_s^2}}(g_1^N) - \lambda_{T_{\ell_s^2}}(\mathcal{G}_s^N)\right)} \in \mathcal{O}_{(\mathfrak{p})}.$$

Claim 2: $\mathfrak{G}_1^N(z)|_{k'+\frac{1}{2}} U_p \equiv g_1^N(z) \bmod p^{j+1}$.

We will now again look at action of $\prod_{s=2}^t (T_{\ell_s^2} - \lambda_{T_{\ell_s^2}}(\mathcal{G}_s^N))$ on $G_1^\circ(z)$ modulo p^{j+1}.

$$G_1^\circ(z)\bigg|_{k'+\frac{1}{2}} \prod_{s=2}^t \left(T_{\ell_s^2} - \lambda_{T_{\ell_s^2}}(\mathcal{G}_s^N)\right) \equiv \left(g_1^N(z)|_{k+\frac{1}{2}} U_p\right)\bigg|_{k+\frac{1}{2}} \prod_{s=2}^t \left(T_{\ell_s^2} - \lambda_{T_{\ell_s^2}}(\mathcal{G}_s^N)\right) \bmod p^{j+1}$$

We see how one of the terms out the product $\prod_{s=2}^t (T_{\ell_s^2} - \lambda_{T_{\ell_s^2}}(\mathcal{G}_s^N))$ acts on $g_1^N(z)|_{k+\frac{1}{2}} U_p$ to observe the pattern.

$$\begin{aligned}
&G_1^\circ(z)\bigg|_{k'+\frac{1}{2}} \prod_{s=2}^t \left(T_{\ell_s^2} - \lambda_{T_{\ell_s^2}}(\mathcal{G}_s^N)\right) \\
&\equiv \left(g_1^N(z)|_{k+\frac{1}{2}} U_p\right)\bigg|_{k+\frac{1}{2}} (T_{\ell_2^2} - \lambda_{T_{\ell_2^2}}(\mathcal{G}_2^N)) \prod_{s=3}^t \left(T_{\ell_s^2} - \lambda_{T_{\ell_s^2}}(\mathcal{G}_s^N)\right) \bmod p^{j+1} \\
&= \left(g_1^N(z)|_{k+\frac{1}{2}} U_p T_{\ell_2^2} - \lambda_{T_{\ell_2^2}}(\mathcal{G}_2^N)\, g_1^N(z)|_{k+\frac{1}{2}} U_p\right)\bigg|_{k+\frac{1}{2}} \prod_{s=3}^t \left(T_{\ell_s^2} - \lambda_{T_{\ell_s^2}}(\mathcal{G}_s^N)\right) \bmod p^{j+1} \\
&= \left(g_1^N(z)|_{k+\frac{1}{2}} T_{\ell_2^2} U_p - \lambda_{T_{\ell_2^2}}(\mathcal{G}_2^N)\, g_1^N(z)|_{k+\frac{1}{2}} U_p\right)\bigg|_{k+\frac{1}{2}} \prod_{s=3}^t \left(T_{\ell_s^2} - \lambda_{T_{\ell_s^2}}(\mathcal{G}_s^N)\right) \bmod p^{j+1} \\
&= \left(\lambda_{T_{\ell_2^2}}(g_1^N)\, g_1^N(z)|_{k+\frac{1}{2}} U_p - \lambda_{T_{\ell_2^2}}(\mathcal{G}_2^N)\, g_1^N(z)|_{k+\frac{1}{2}} U_p\right)\bigg|_{k+\frac{1}{2}} \prod_{s=3}^t \left(T_{\ell_s^2} - \lambda_{T_{\ell_s^2}}(\mathcal{G}_s^N)\right) \bmod p^{j+1} \\
&= \left(\lambda_{T_{\ell_2^2}}(g_1^N) - \lambda_{T_{\ell_2^2}}(\mathcal{G}_2^N)\right)\left(g_1^N(z)|_{k+\frac{1}{2}} U_p\right)\bigg|_{k+\frac{1}{2}} \prod_{s=3}^t \left(T_{\ell_s^2} - \lambda_{T_{\ell_s^2}}(\mathcal{G}_s^N)\right) \bmod p^{j+1} \\
&= \prod_{s=2}^t \left(\lambda_{T_{\ell_s^2}}(g_1^N) - \lambda_{T_{\ell_s^2}}(\mathcal{G}_s^N)\right)\left(g_1^N(z)|_{k+\frac{1}{2}} U_p\right) \bmod p^{j+1}. \qquad (5.46)
\end{aligned}$$

Using 5.42 and 5.46, we get

$$\mathfrak{G}_1^N(z) \equiv \lambda_{U_{p^2}}(g_1^N)^{-1} g_1^N(z)|_{k+\frac{1}{2}} U_p \bmod p^{j+1}. \tag{5.47}$$

We now act on both sides in the congruence 5.47 by U_p operator and get

$$\begin{aligned} \mathfrak{G}_1^N(z)|_{k'+\frac{1}{2}} U_p &\equiv \lambda_{U_{p^2}}(g_1^N)^{-1} g_1^N(z)|_{k+\frac{1}{2}} U_p^2 \bmod p^{j+1} \\ &= \lambda_{U_{p^2}}(g_1^N)^{-1} \lambda_{U_{p^2}}(g_1^N) g_1^N(z) \bmod p^{j+1} \\ &= g_1^N(z) \bmod p^{j+1}. \end{aligned} \tag{5.48}$$

Uniqueness: Let $\mathfrak{G}_2^N(z) \in S_{k'+\frac{1}{2}}^{4N\text{-new, ord}}(\widetilde{\Gamma}_0(4Np); \mathcal{O}_{(\mathfrak{p})})$ be a $4N$-new ordinary half-integer weight eigenform for all operators T_{ℓ^2} over all primes ℓ such that it satisfies congruence of the type 5.48 with $g_2^N(z)$:

$$\mathfrak{G}_2^N(z)|_{k'+\frac{1}{2}} U_p \equiv g_2^N(z) \bmod p^{j+1}$$

Suppose there does not exist any $\mathscr{D} \in \mathcal{O}_{(\mathfrak{p})}$ such that $\mathfrak{G}_2^N(z) = \mathscr{D}\mathfrak{G}_1^N(z)$ but $\mathfrak{G}_2^N(z)$ also satisfies congruence 5.48 with $g_1^N(z)$:

$$\mathfrak{G}_2^N(z)|_{k'+\frac{1}{2}} U_p \equiv g_1^N(z) \bmod p^{j+1}.$$

It the follows

$$g_1^N(z) \equiv g_2^N(z) \bmod p^{j+1}$$

which implies

$$g_1^N(z)|_{k+\frac{1}{2}} T_{\ell^2} \equiv g_2^N(z)|_{k'+\frac{1}{2}} T_{\ell^2} \bmod p^{j+1}.$$

So, we have for all primes ℓ,

$$\lambda_{T_{\ell^2}}(g_2^N) g_2^N(z) \equiv \lambda_{T_{\ell^2}}(g_1^N) g_1^N(z) \bmod p^{j+1}.$$

We can then write

$$\lambda_{T_\ell}(f_2^N) g_2^N(z) \equiv \lambda_{T_\ell}(f_1^N) g_1^N(z) \bmod p^{j+1}$$

which implies

$$\left(\lambda_{T_\ell}(f_2^N) - \lambda_{T_\ell}(f_1^N)\right) g_1^N(z) \equiv 0 \bmod p^{j+1}.$$

Now $\mathfrak{p} \nmid g_1^N(z)$ due to choice of scaling of $g_1^N(z)$. So we get for all primes $\ell \nmid Np$, $\mathfrak{p} \mid (\lambda_{T_\ell}(f_2^N) - \lambda_{T_\ell}(f_1^N))$. This is a contradiction to assumption 1 as $f_1^N(z)$ and $f_2^N(z)$ are distinct basis elements. Thus, $\mathfrak{G}_1^N(z)$ is the unique $4N$-new half-integer weight ordinary eigenform up to scalar multiplication in $\mathcal{O}_{(\mathfrak{p})}$ that satisfies the congruence 5.48 with $g_1^N(z)$. □

Part II

Congruences related to Hilbert modular forms

CHAPTER 6

Hilbert modular forms of integer weight

6.1 Introduction

Let $K/\mathbb{Q}$ be a *totally real number field*. By a totally real number field, we mean that for every embedding of K in $\mathbb{C}$, the image lies inside $\mathbb{R}$. In this thesis, we consider K to be a real quadratic number field. Let $\sigma_i : K \hookrightarrow \mathbb{R}$, $i = 1, 2$ be the two real embeddings of K in $\mathbb{R}$. For any $\xi \in K$, we let $\sigma_i(\xi) = \xi_i$, $i = 1, 2$.

We recall some preliminary definitions and use [BHZ08] as our reference.

Let $\mathcal{O}_K$ be the ring of integers of K. A fractional ideal of $\mathcal{O}_K$ is a finitely generated $\mathcal{O}_K$ submodule of K. Fractional ideals form a group under ideal multiplication with $\mathcal{O}_K$ as the identity element. The inverse of a fractional ideal $\mathfrak{f} \subset K$ is

$$\mathfrak{f}^{-1} = \{x \in K \mid x\mathfrak{f} \subset \mathcal{O}_K\}.$$

For the quadratic field K, we have the formula $\mathfrak{f}^{-1} = \frac{1}{N(\mathfrak{f})}\mathfrak{f}'$ where $\mathfrak{f}'$ is the conjugate of $\mathfrak{f}$ and $N(\mathfrak{f})$ is the ideal norm defined as the lattice index $[\mathcal{O}_K : \mathfrak{f}] \in \mathbb{Q}_{\geq 0}$.

Two fractional ideals $\mathfrak{f}$ and $\mathfrak{g}$ are said to be equivalent if there exists $r \in K$ such that $\mathfrak{f} = r\mathfrak{g}$. The group of these equivalence classes is a finite abelian group called the *ideal class group* of K and is denoted by $Cl(K)$. We will next define the *narrow ideal class group*.

Definition 6.1.1 (Totally positive element). *An element $\xi \in K$ is called totally positive if $\xi_i > 0$ for $i = 1, 2$. We often denote an element ξ is totally positive by writing $\xi \gg 0$.*

Definition 6.1.2 (Narrow ideal class group). *Two fractional ideals $\mathfrak{f}$ and $\mathfrak{g}$ are said to be equivalent in the narrow sense if there exists $r \in K$ with $r \gg 0$ such that $\mathfrak{f} = r\mathfrak{g}$. The group of these equivalence classes is called the narrow ideal class group of K and is denoted by $Cl^+(K)$.*

The (narrow) class number of K is the order of the (narrow) ideal class group. It measures how far $\mathcal{O}_K$ is from being a principal ideal domain. From now onwards, we assume that K has *narrow class number* 1, this means every ideal is generated by a single element.

Definition 6.1.3 (Inverse different ideal). *The dual of $\mathcal{O}_K$ is a fractional ideal that is the set of all elements in K with integral trace. It's called the Inverse Different and is denoted by $\mathfrak{d}^{-1}$.*

$$\mathfrak{d}^{-1} := \mathcal{O}_K^V = \{x \in K \mid Tr(x\mathcal{O}_K) \subset \mathbb{Z}\}.$$

We fix δ^{-1} *to be a generator of* $\mathfrak{d}^{-1}$.

Definition 6.1.4 (Different ideal). *The Different ideal* $\mathfrak{d}$ *is an integral ideal of* K *defined as*

$$\mathfrak{d} = \{x \in K \mid x\mathcal{O}_K^V \subset \mathcal{O}_K\}.$$

For any integral ideal $\mathfrak{m}$ in $\mathcal{O}_K$, the ideal norm $N(\mathfrak{m}) = \#(\mathcal{O}_K/\mathfrak{m})$. Next, we briefly recall the factorisation of ideals in quadratic field K. If p is a rational prime, then the ideal (p) generated by p in K has one of the following three factorisations.

Inert: $(p) = \mathfrak{p}$ where $\mathfrak{p}$ in a prime ideal in $\mathcal{O}_K$ and $N(\mathfrak{p}) = p^2$.

Split: $(p) = \mathfrak{p}_1\mathfrak{p}_2$ where $\mathfrak{p}_1$, $\mathfrak{p}_2$ are prime ideals in $\mathcal{O}_K$ and $N(\mathfrak{p}_1) = N(\mathfrak{p}_2) = p$.

Ramified: $(p) = \mathfrak{p}^2$ where $\mathfrak{p}$ is a prime ideal in $\mathcal{O}_K$ and $N(\mathfrak{p}) = p$.

6.1.1 Hilbert modular forms

Let

$$GL_2^+(K) = \{\gamma \in GL_2(K) \mid \det\gamma \in K \text{ such that } \det\gamma \gg 0\}.$$

There is a natural embedding of $GL_2^+(K)$ in $\left(GL_2^+(\mathbb{R})\right)^2$:

$$GL_2^+(K) \hookrightarrow \left(GL_2^+(\mathbb{R})\right)^2 \text{ with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (\gamma_1, \gamma_2) = \left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right)$$

Let $\mathcal{H}^2$ denote two copies of the complex upper half plane. Throughout our thesis, we will use $z = (z_1, z_2)$ as a standard variable on $\mathcal{H}^2$. The group $GL_2^+(K)$ acts on $\mathcal{H}^2$ via fractional linear transformations,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} . z = \left(\frac{a_1z_1 + b_1}{c_1z_1 + d_1}, \frac{a_2z_2 + b_2}{c_2z_2 + d_2}\right).$$

Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2^+(K)$, then $\gamma . z \in \mathcal{H}^2$ [BHZ08, Lemma 1.2, p. 107].

Let $k \in \mathbb{Z}_{>0}$ and let $f : \mathcal{H}^2 \to \mathbb{C}$ be a complex function. Then we can define the k-slash operator on $f(z)$.

Definition 6.1.5 (k-slash operator). *Let* $k \in \mathbb{Z}_{>0}$. *Let* $\gamma \in GL_2^+(K)$ *and let* $f : \mathcal{H}^2 \to \mathbb{C}$ *be a complex function. Then we define the k-slash operator as follows*

$$f(z)|_k\gamma = \det\gamma^{k/2} \prod_{i=1,2} (c_iz_i + d_i)^{-k} f(\gamma z)$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(K)$, $z = (z_1, z_2) \in \mathcal{H}^2$ *and* $N_{K/\mathbb{Q}}$ *is the field norm.*

Let

$$SL_2(K) = \{\gamma \in GL_2^+(K) \mid \det\gamma = 1\}.$$

In general, a subgroup Γ of $SL_2(K)$ is called a congruence subgroup if it is commensurable with $SL_2(K)$, that is, $\Gamma \cap SL_2(K)$ is of finite index in Γ as well as in $SL_2(K)$.
Let $\mathfrak{n} \subset \mathcal{O}_K$ be an integral ideal, then the congruence subgroup $\Gamma[2\mathfrak{d}^{-1}, 2^{-1}\mathfrak{d}\mathfrak{n}]$ of $SL_2(K)$ of level $\mathfrak{n}$ is defined as

$$\Gamma[2\mathfrak{d}^{-1}, 2^{-1}\mathfrak{d}\mathfrak{n}] := \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(K) \mid a, d \in \mathcal{O}_K,\ b \in 2\mathfrak{d}^{-1},\ c \in 2^{-1}\mathfrak{d}\mathfrak{n}\right\}.$$

Note 6.1.6. For simplicity, we will denote $\Gamma[2\mathfrak{d}^{-1}, 2^{-1}\mathfrak{d}\mathfrak{n}]$ by $\Gamma_{\mathfrak{n}}$.

The group $SL_2(K)$ also acts on $\mathbb{P}^1(K) = K \cup \{\infty\}$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} . \frac{\alpha}{\beta} = \frac{a\frac{\alpha}{\beta} + b}{c\frac{\alpha}{\beta} + d} = \frac{a\alpha + b\beta}{c\alpha + d\beta}.$$

Note that this action is transitive by observing $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).\infty = \frac{a}{c}$.

Definition 6.1.7 (Cusps). *Let Γ be a congruence subgroup of $SL_2(K)$. Then the Γ-classes of $\mathbb{P}^1(K)$ are called the cusps of Γ.*

Definition 6.1.8 (Hilbert modular form). *Let $k \in \mathbb{Z}_{>0}$ and let $\mathfrak{n} \subset \mathcal{O}_K$ be an integral ideal of K. A Hilbert modular form of parallel weight k and level $\mathfrak{n}$ is defined as a holomorphic function $f : \mathcal{H}^2 \to \mathbb{C}$ such that*

$$f(z)|_k\gamma = f(z) \quad \text{for all } \gamma \in \Gamma_{\mathfrak{n}}.$$

The space of all Hilbert modular forms of parallel weight k and level $\mathfrak{n}$ forms a finite dimensional complex vector space [BHZ08, pg.118-119] which is denoted by $M_k(\Gamma_{\mathfrak{n}})$.

Remark 6.1.9. In contrast to the classical modular forms, a holomorphic Hilbert modular form is automatically holomorphic at all the cusps including infinity. This is due to *Götzky-Koecher principle* [BHZ08, Theorem 1.20, pg. 114].

As in the case of classical modular forms, Hilbert modular forms also admit a Fourier expansion. In order to give the Fourier expansion, we observe periodicity of a Hilbert modular form. The congruence subgroup $\Gamma_{\mathfrak{n}}$ contains elements of the form $\left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right)$ where $\mu \in 2\mathfrak{d}^{-1}$. If $f(z) \in \Gamma_{\mathfrak{n}}$, then $f(z)$ is invariant under the action of $\Gamma_{\mathfrak{n}}$, that is,

$$f(z)\Big|_{k+\frac{1}{2}} \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} = f(z)$$

or

$$f(z+\mu) = f(z).$$

Thus, $f(z)$ is periodic and admits the following Fourier expansion

$$f(z) = \sum_{\xi \in ((\delta^{-1})^{\vee})^+ \cup \{0\}} a_\xi e^{2\pi i \mathrm{Tr}\left(\frac{\xi}{2} z\right)}$$

or

$$f(z) = \sum_{\xi \in \mathcal{O}_K^+ \cup \{0\}} a_\xi e^{2\pi i \mathrm{Tr}\left(\frac{\xi}{2} z\right)}.$$

where $\mathrm{Tr}\left(\frac{\xi}{2} z\right) = \frac{\xi_1}{2} z_1 + \frac{\xi_2}{2} z_2$ and $a_\xi \in \mathbb{C}$.

Our Fourier expansion is indexed in this way due to our choice of congruence subgroup which is similar to that of *N. Sirolli* in [Sir14, pg. 28].

Definition 6.1.10 (Hilbert cusp form). *A Hilbert cusp form $f(z)$ of weight k and level $\mathfrak{n}$ is a Hilbert modular form that vanishes at all the cusps. This happens if a_0 vanishes in the Fourier expansion of $f(z)|_k\gamma$ for all $\gamma \in \Gamma_{(1)}$. The space of all Hilbert cusp forms of weight k and level $\mathfrak{n}$ forms a subspace of $M_k(\Gamma_{\mathfrak{n}})$ and is denoted by $S_k(\Gamma_{\mathfrak{n}})$.*

6.1.2 Hilbert modular form with a character

We now introduce Hilbert modular form with an associated Hecke character ψ of K. We denote the adelization of K by $K_{\mathbb{A}}$ and the ideles by $K_{\mathbb{A}}^{\times}$. A Hecke character of K is a character of $K_{\mathbb{A}}^{\times}$ which is trivial on $K^{\times}$ and takes values in the unit circle.
For any place v, let ψ_v denote the restriction of ψ to K_v. Let $\psi_{\mathfrak{n}} = \prod_{v|\mathfrak{n}} \psi_v$ denote the finite part of ψ of conductor $\mathfrak{n} \subset \mathcal{O}_K$ and let $\psi_\infty = \prod_{v\in\infty} \psi_v$ denote its infinity type. By conductor of ψ, we always mean the conductor of its finite part. Throughout this text, by a Hecke character, we mean a finite order Hecke character, that is, our infinity type is trivial.

We can now say that $f(z)$ is a Hilbert modular form of parallel weight k, level $\mathfrak{n}$ and character ψ if defining property of Hilbert modular forms in 6.1.8 is replaced by

$$f(z)|_k\gamma = \psi_{\mathfrak{n}}(a_\gamma)f(z) \text{ where } \gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \Gamma_{\mathfrak{n}} \tag{6.1}$$

We denote the space of all Hilbert modular forms of parallel weight k, level $\mathfrak{n}$ and character ψ by $M_k(\Gamma_{\mathfrak{n}}, \psi)$.

6.2 Hecke operators

We now introduce theory of Hecke operators for Hilbert modular forms analogous to that developed for classical modular forms. The Hecke operators in this case are indexed by ideals rather than by integers. For details about Hecke operators, one can refer to *Shimura's* paper [Shi81] as well as [Oza17, Section 2.4].

Definition 6.2.1. *Let $f(z) \in M_k(\Gamma_{\mathfrak{n}})$. Let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime ideal generated by $\wp \gg 0$. Then for each $\mathfrak{p}$, we define the Hecke operator $T_{\mathfrak{p}}$ in terms of the k-slash operator on $M_k(\Gamma_{\mathfrak{n}})$ as follows*

$$f(z)|_k T_{\mathfrak{p}} = N(\wp)^{\frac{k}{2}-1} \begin{cases} \sum_{\alpha\in\mathcal{O}_K/\mathfrak{p}} f(z)|_k\gamma_\alpha & \text{if } \mathfrak{p} \mid \mathfrak{n}; \\ \sum_{\alpha\in\mathcal{O}_K/\mathfrak{p}} f(z)|_k\gamma_\alpha \; + \; f(z)|_k\gamma_\infty & \text{if } \mathfrak{p} \nmid \mathfrak{n} \end{cases}$$

where $\gamma_\alpha = \begin{pmatrix} 1 & 2\delta^{-1}\alpha \\ 0 & \wp \end{pmatrix}$ *and* $\gamma_\infty = \begin{pmatrix} \wp & 0 \\ 0 & 1 \end{pmatrix}$ *for* $\alpha \in \mathcal{O}_K/\mathfrak{p}$.

The Hecke operators are commutative, that is, $T_{\mathfrak{p}}T_{\mathfrak{q}} = T_{\mathfrak{q}}T_{\mathfrak{p}}$, where $\mathfrak{p}$, $\mathfrak{q}$ are distinct prime ideals in $\mathcal{O}_K$. This can be shown as in the case of classical modular forms as in [DS05, Proposition 5.2.4, pg. 173]. It involves applying Hecke action on Fourier coefficients twice and observing that the argument is symmetric in $\mathfrak{p}$ and $\mathfrak{q}$.

To define $T_{\mathfrak{m}}$ for any integral ideal $\mathfrak{m} \subset \mathcal{O}_K$, set $T_{(1)} = T_{\mathcal{O}_K}$ as the identity operator; $T_{\mathfrak{p}}$ is already defined for prime ideals in $\mathcal{O}_K$. For powers of primes ideals, $T_{\mathfrak{p}^r}$ is defined using recursion formula

$$T_{\mathfrak{p}^r} = T_{\mathfrak{p}^{r-1}} T_{\mathfrak{p}} - N(\mathfrak{p})^{k-1} T_{\mathfrak{p}^{r-2}} \text{ where } r \geq 2$$

given in [Oza17, pgs. 19-20] and hence is a polynomial with integer coefficients in $T_{\mathfrak{p}}$. Finally, we extend our definition to $T_{\mathfrak{m}}$ for any integral ideal $\mathfrak{m} \subset \mathcal{O}_K$ by using multiplicativity, $T_{\mathfrak{n}} = \prod_i T_{\mathfrak{p}_i^{r_i}}$ where $\mathfrak{m} = \prod_i \mathfrak{p}_i^{r_i}$.

We now state the action of Hecke operator $T_{\mathfrak{p}}$ on the Fourier coefficients of the modular form $f(z) \in M_k(\Gamma_{\mathfrak{n}})$, see [Oza17, Remark 2.4.6].

Proposition 6.2.2 (Hecke operator on Fourier expansions). *Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_K$ generated by $\wp \gg 0$. Let $f(z) \in M_k(\Gamma_{\mathfrak{n}})$ with Fourier expansion*

$$f(z) = \sum_{\xi\in\mathcal{O}_K^+\cup\{0\}} a_\xi e^{2\pi i Tr(\frac{\xi}{2}z)}.$$

Then $f(z)|_k T_{\mathfrak{p}} \in M_k(\Gamma_{\mathfrak{n}})$ *with Fourier expansion*

$$f(z)|_k T_{\mathfrak{p}} = \sum_{\xi \in \mathcal{O}_K^+ \cup \{0\}} b_\xi e^{2\pi i Tr\left(\frac{\xi}{2} z\right)}$$

such that

$$b_\xi = \begin{cases} a_{\wp\xi} & \text{if } \wp \mid \mathfrak{n}; \\ a_{\wp\xi} + N(\wp)^{k-1} a_{\xi\wp^{-1}} & \text{if } \wp \nmid \mathfrak{n}. \end{cases}$$

Here $a_{\xi\wp^{-1}} = 0$ *when* $\xi\wp^{-1} \notin \mathcal{O}_K$.

The fact that $f(z)|_k T_{\mathfrak{p}} \in M_k(\Gamma_{\mathfrak{n}})$ follows from [Oza17, Proposition 2.4.2]. The Fourier expansion of $f(z)|_k T_{\mathfrak{p}}$ can be obtained by a direct computation once we have proved the following lemma.

Lemma 6.2.3. *Let* $\xi \in \mathcal{O}_K^+ \cup \{0\}$ *and let* $\mathfrak{p} \subset \mathcal{O}_K$ *be a prime ideal generated by* $\wp \gg 0$. *Then*

$$\sum_{\alpha \in (\mathcal{O}_K/\wp\mathcal{O}_K)} e^{2\pi i Tr\left(\frac{\xi\delta^{-1}\alpha}{\wp}\right)} = \begin{cases} N(\wp) & \text{if } \wp \mid \xi; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We will divide our proof into two cases.

Case 1: Let us first assume that $\wp \mid \xi$. Then $(\xi/\wp) \in \mathcal{O}_K$. Also, $\alpha \in \mathcal{O}_K$. Recall that the inverse different ideal is defined as

$$\mathfrak{d}^{-1} = \{x \in K \mid \mathrm{Tr}(yx) \in \mathbb{Z} \quad \forall\, y \in \mathcal{O}_K\}$$

Since δ^{-1} is the generator of $\mathfrak{d}^{-1}$, it implies $\mathrm{Tr}\left(\frac{\xi\delta^{-1}\alpha}{\wp}\right)$ must be an integer. It follows that

$$e^{2\pi i\, \mathrm{Tr}\left(\frac{\xi\delta^{-1}\alpha}{\wp}\right)} = 1.$$

Thus, we have

$$\sum_{\alpha \in (\mathcal{O}_K/\wp\mathcal{O}_K)} e^{2\pi i \mathrm{Tr}\left(\frac{\xi\delta^{-1}\alpha}{\wp}\right)} = N(\wp)$$

where $N(\wp) = |\mathcal{O}_K/\wp\mathcal{O}_K|$.

Case 2: Now let us assume that $\wp \nmid \xi$. Then $(\xi\delta^{-1}/\wp) \notin \mathfrak{d}^{-1}$. This is because if $(\xi\delta^{-1}/\wp) \in \mathfrak{d}^{-1}$, then $(\xi/\wp) \in \mathcal{O}_K$ and that contradicts our assumption.
Now since, $(\xi/\wp) \notin \mathcal{O}_K$, then there exists an algebraic integer $\gamma \in \mathcal{O}_K$ such that $\mathrm{Tr}\left(\frac{\xi\delta^{-1}\gamma}{\wp}\right) \notin \mathbb{Z}$. This implies $e^{2\pi\, \mathrm{Tr}\left(\frac{\xi\delta^{-1}\gamma}{\wp}\right)} \neq 1$. We can then write

$$\begin{aligned} \sum_{\alpha \in (\mathcal{O}_K/\wp\mathcal{O}_K)} e^{2\pi i \mathrm{Tr}\left(\frac{\xi\delta^{-1}\alpha}{\wp}\right)} &= \sum_{\alpha \in (\mathcal{O}_K/\wp\mathcal{O}_K)} e^{2\pi i \mathrm{Tr}\left(\frac{\xi\delta^{-1}(\alpha+\gamma)}{\wp}\right)} \\ &= \sum_{\alpha \in (\mathcal{O}_K/\wp\mathcal{O}_K)} e^{2\pi i \mathrm{Tr}\left(\frac{\xi\delta^{-1}\alpha}{\wp}\right)} e^{2\pi i \mathrm{Tr}\left(\frac{\xi\delta^{-1}\gamma}{\wp}\right)} \\ &= e^{2\pi i \mathrm{Tr}\left(\frac{\xi\delta^{-1}\gamma}{\wp}\right)} \sum_{\alpha \in (\mathcal{O}_K/\wp\mathcal{O}_K)} e^{2\pi i \mathrm{Tr}\left(\frac{\xi\delta^{-1}\alpha}{\wp}\right)} \end{aligned}$$

or

$$\left(\sum_{\alpha \in (\mathcal{O}_K/\wp\mathcal{O}_K)} e^{2\pi i \mathrm{Tr}\left(\frac{\xi\delta^{-1}\alpha}{\wp}\right)}\right) \cdot \left(1 - e^{2\pi i \mathrm{Tr}\left(\frac{\xi\delta^{-1}\gamma}{\wp}\right)}\right) = 0.$$

Since $e^{2\pi i \mathrm{Tr}\left(\frac{\xi\delta^{-1}\gamma}{\wp}\right)} \neq 1$, we get $\sum_{\alpha \in (\mathcal{O}_K/\wp\mathcal{O}_K)} e^{2\pi i \mathrm{Tr}\left(\frac{\xi\delta^{-1}\alpha}{\wp}\right)} = 0$.

□

Definition 6.2.4 (Hecke algebra). *The Hecke algebra of weight k and level $\mathfrak{n}$ acting on $M_k(\Gamma_\mathfrak{n})$ is the commutative $\mathbb{C}$-subalgebra of End($M_k(\Gamma_\mathfrak{n})$) generated by Hecke operators $T_\mathfrak{p}$ over all prime ideals $\mathfrak{p}$ in $\mathcal{O}_K$.*
We will denote the Hecke algebra of weight k and level $\mathfrak{n}$ by $\mathbb{T}_k(\mathfrak{n})$.

Definition 6.2.5 (Hilbert Hecke eigenform). *We say $f(z) \in M_k(\Gamma_\mathfrak{n})$ is a Hilbert Hecke eigenform if it is a simultaneous eigenvector for all Hecke operators in $\mathbb{T}_k(\mathfrak{n})$.*

Note 6.2.6. We will specify whenever we refer to a Hecke eigenform for all Hecke operators $T_\mathfrak{p}$ for $\mathfrak{p}$ not dividing the level $\mathfrak{n}$. In general, our definition of a Hecke eigenform refers to $f(z)$ being an eigenvector for all operators $T_\mathfrak{m}$ over all integral ideals $\mathfrak{m}$ in $\mathcal{O}_K$.

Definition 6.2.7 (Normalised Hilbert eigenform). *Let $f(z) \in S_k(\Gamma_\mathfrak{n})$ be a Hilbert Hecke eigenform with the following Fourier expansion*

$$f(z) = \sum_{\xi \in \mathcal{O}_K^+} a_\xi e^{2\pi i \, Tr\left(\frac{\xi}{2} z\right)}.$$

We say $f(z)$ is normalised if $a_{(1)} = 1$.

6.3 Further operators

In this section, we will introduce more operators on $M_k(\Gamma_\mathfrak{n})$.

6.3.1 Operator $V_{(1)}$

Let $f(z) \in M_k(\Gamma_\mathfrak{n})$. Let $\mathfrak{m} \subset \mathcal{O}_K$ be an integral ideal such that $\mathfrak{m}$ is coprime to level $\mathfrak{n}$. Note that $M_k(\Gamma_\mathfrak{n}) \subset M_k(\Gamma_{\mathfrak{nm}})$. Hence, $f(z)$ can always be viewed as a modular form in $M_k(\Gamma_{\mathfrak{nm}})$. More formally, define an operator

$$|_k V_{(1)} : M_k(\Gamma_\mathfrak{n}) \to M_k(\Gamma_{\mathfrak{nm}}) \text{ such that } f(z) \mapsto f(z).$$

6.3.2 Operator $V_\mathfrak{m}$

Let $f(z) \in M_k(\Gamma_\mathfrak{n})$. Let $\mathfrak{m} \subset \mathcal{O}_K$ be an integral ideal such that $\mathfrak{m}$ is coprime to level $\mathfrak{n}$. Let $\mathscr{M}, \mathscr{N} \in \mathcal{O}_K^+$ be totally positive generators of ideals $\mathfrak{m}$ and $\mathfrak{n}$ respectively. Define the operator $V_\mathfrak{m}$ on $f(z)$ in terms of k-slash action as

$$\begin{aligned} f(z)|_k V_\mathfrak{m} &:= N(\mathscr{M})^{-\frac{k}{2}} f(z)|_k \begin{pmatrix} \mathscr{M} & 0 \\ 0 & 1 \end{pmatrix} \\ &= N(\mathscr{M})^{-\frac{k}{2}} N(\mathscr{M})^{\frac{k}{2}} f(\mathscr{M} z) \\ &= f(\mathscr{M} z) \end{aligned}$$

Thus, if $f(z) = \sum_{\xi \in \mathcal{O}_K^+} a_\xi e^{2\pi i \mathrm{Tr}\left(\frac{\xi}{2} z\right)}$, then $f(z)|_k V_\mathfrak{m} = \sum_{\xi \in \mathcal{O}_K^+} a_\xi e^{2\pi i \mathrm{Tr}\left(\frac{\xi}{2} \mathscr{M} z\right)}$.

Proposition 6.3.1. *Let $f(z) \in M_k(\Gamma_\mathfrak{n})$. Let $\mathfrak{m} \subset \mathcal{O}_K$ be an integral ideal that is co-prime to our level $\mathfrak{n}$. Let $\mathscr{M}, \mathscr{N} \in \mathcal{O}_K^+$ be totally positive generators of ideals $\mathfrak{m}$ and $\mathfrak{n}$ respectively. Then $f(\mathscr{M} z) \in M_k(\Gamma_{\mathfrak{nm}})$.*

Proof. This can be observed by showing $f(\mathscr{M} z)$ is invariant under k-slash action by any arbitrary element in $\Gamma_{\mathfrak{nm}}$.

$$\Gamma_{\mathfrak{nm}} = \left\{ \left(\begin{smallmatrix} a & b \\ \mathscr{M} c & d \end{smallmatrix} \right) \in SL_2(K) \mid a, d \in \mathcal{O}_K,\ c \in 2\mathscr{N} \mathfrak{d}^{-1} \text{ and } d \in 2^{-1} \mathfrak{d} \right\}.$$

Then

$$\begin{aligned} f(\mathscr{M}z)\big|_k \begin{pmatrix} a & b \\ \mathscr{M}c & d \end{pmatrix} &= \left(N(\mathscr{M})^{-\frac{k}{2}} f(z)\big|_k \begin{pmatrix} \mathscr{M} & 0 \\ 0 & 1 \end{pmatrix} \right) \Big|_k \begin{pmatrix} a & b \\ \mathscr{M}c & d \end{pmatrix} \\ &= N(\mathscr{M})^{-\frac{k}{2}} f(z)\big|_k \begin{pmatrix} \mathscr{M}a & \mathscr{M}b \\ \mathscr{M}c & d \end{pmatrix} \\ &= N(\mathscr{M})^{-\frac{k}{2}} f(z)\big|_k \begin{pmatrix} a & \mathscr{M}b \\ c & d \end{pmatrix} \begin{pmatrix} \mathscr{M} & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

Now $f(z) \in M_k(\Gamma_{\mathfrak{n}})$. So, for $\left(\begin{smallmatrix} a & \mathscr{M}b \\ c & d \end{smallmatrix}\right) \in \Gamma_{\mathfrak{n}}$, we have $f(z)|_k \left(\begin{smallmatrix} a & \mathscr{M}b \\ c & d \end{smallmatrix}\right) = f(z)$. It follows

$$\begin{aligned} f(\mathscr{M}z)\big|_k \begin{pmatrix} a & b \\ \mathscr{M}c & d \end{pmatrix} &= N(\mathscr{M})^{-\frac{k}{2}} f(z)\big|_k \begin{pmatrix} \mathscr{M} & 0 \\ 0 & 1 \end{pmatrix} \\ &= N(\mathscr{M})^{-\frac{k}{2}} N(\mathscr{M})^{\frac{k}{2}} f(\mathscr{M}z) \\ &= f(\mathscr{M}z). \end{aligned}$$

□

Thus, we have a map

$$|_k V_{\mathfrak{m}} : M_k(\Gamma_{\mathfrak{n}}) \to M_k(\Gamma_{\mathfrak{nm}}) \text{ such that } f(z) \mapsto f(\mathscr{M}z).$$

6.3.3 Operator $U_{\mathfrak{m}}$

Let $\mathfrak{m} \subset \mathscr{O}_K$ be an integral ideal and let $f(z) \in M_k(\Gamma_{\mathfrak{n}})$ with Fourier expansion $f(z) = \sum_{\xi \in \mathscr{O}_K^+} a_\xi e^{2\pi i \mathrm{Tr}\left(\frac{\xi}{2} z\right)}$. Let $\mathscr{M}, \mathscr{N} \in \mathscr{O}_K^+$ be totally positive generators of ideals $\mathfrak{m}$ and $\mathfrak{n}$ respectively. Define the $U_{\mathfrak{m}}$ operator on the Fourier coefficients of $f(z)$ in the following way,

$$f(z)|_k U_{\mathfrak{m}} := \sum_{\xi \in \mathscr{O}_K^+} a_{\mathscr{M}\xi} e^{2\pi i \mathrm{Tr}\left(\frac{\xi}{2} z\right)}.$$

Proposition 6.3.2. *Let $\mathfrak{m} \subset \mathscr{O}_K$ be an integral ideal such that $\mathfrak{m} \mid \mathfrak{n}$. Then $|_k U_{\mathfrak{m}}$ takes $M_k(\Gamma_{\mathfrak{n}})$ to itself.*

Proof. This follows from proposition 6.2.2 once we observe in the case $\mathfrak{m} \mid \mathfrak{n}$, we have $U_{\mathfrak{m}} = T_{\mathfrak{m}}$. □

Now let $\mathfrak{l} \subset \mathscr{O}_K$ be a prime ideal such that $\mathfrak{l} \nmid \mathfrak{n}$, then again from proposition 6.2.2, we have

$$f(z)|_k U_{\mathfrak{l}} = f(z)|_k T_{\mathfrak{l}} - N(\mathfrak{l})^{k-1} f(z)|_k V_{\mathfrak{l}}.$$

In this case, we observe that the submodule generated by the action of $|_k U_{\mathfrak{l}}$ operator on $f(z)$ lies in the span of the set $\{f(z)|_k T_{\mathfrak{l}}, f(z)|_k V_{\mathfrak{l}}\}$ which has at most level $\mathfrak{nl}$. We state this as a remark for its use in later chapters.

Remark 6.3.3. If $\mathfrak{l} \nmid \mathfrak{n}$ and $f(z) \in M_k(\Gamma_{\mathfrak{n}})$, then $f(z)|_k U_{\mathfrak{l}} \in M_k(\Gamma_{\mathfrak{nl}})$.

6.4 Old and New spaces

Let $\mathfrak{m} \subset \mathscr{O}_K$ be an integral ideal. Let $\mathfrak{p} \subset \mathscr{O}_K$ be a prime ideal generated by $\wp \gg 0$ such that it is co-prime to $\mathfrak{m}$. Consider a cusp form $f(z) \in S_k(\Gamma_{\mathfrak{m}})$ such that the level is not divisible by the prime $\mathfrak{p}$. However, it is possible to raise the level by using operators $|_k V_{(1)}$ and $|_k V_{\mathfrak{p}}$ defined in section 6.3. Recall that

$$|_k V_{(1)} : S_k(\Gamma_{\mathfrak{m}}) \to S_k(\Gamma_{\mathfrak{mp}}) \text{ such that } f(z) \mapsto f(z);$$

and

$$|_k V_{\mathfrak{p}} : S_k(\Gamma_{\mathfrak{m}}) \to S_k(\Gamma_{\mathfrak{mp}}) \text{ such that } f(z) \mapsto f(\wp z).$$

Let $\mathfrak{n} = \mathfrak{mp}$. We can now define the space of $\mathfrak{p}$-old Hilbert cusp forms of $S_k(\Gamma_{\mathfrak{n}})$.

Definition 6.4.1 ($\mathfrak{p}$-old forms at level $\mathfrak{n}$)**.** *We define the space of $\mathfrak{p}$-old forms of $S_k(\Gamma_\mathfrak{n})$, denoted as $S_k^{\mathfrak{p}\text{-}old}(\Gamma_\mathfrak{n})$ as the subspace of $S_k(\Gamma_\mathfrak{n})$ generated by $\{f_1(z)|_k V_{(1)}, f_2(z)|_k V_\mathfrak{p}\}$ where $f_1, f_2 \in S_k(\Gamma_{\mathfrak{n}\mathfrak{p}^{-1}})$.*

$$S_k^{\mathfrak{p}\text{-}old}(\Gamma_\mathfrak{n}) = S_k(\Gamma_{\mathfrak{n}\mathfrak{p}^{-1}})|_k V_{(1)} \oplus S_k(\Gamma_{\mathfrak{n}\mathfrak{p}^{-1}})|_k V_\mathfrak{p}.$$

We can now do this for every ideal $\mathfrak{b} \subset \mathcal{O}_K$, $\mathfrak{b} \neq (1)$ that divides $\mathfrak{n}$ and hence define the space of all Hilbert old forms at level $\mathfrak{n}$.

Definition 6.4.2 (Hilbert old subspace)**.** *Let $\mathfrak{n} \subset \mathcal{O}_K$ be an integral ideal. We define the Hilbert old subspace of $S_k(\Gamma_\mathfrak{n})$ as*

$$S_k^{old}(\Gamma_\mathfrak{n}) := \bigoplus_{\mathfrak{b}|\mathfrak{n}, \mathfrak{b}\neq(1)} \left(S_k(\Gamma_{\mathfrak{n}\mathfrak{b}^{-1}})|_k V_{(1)} \oplus S_k(\Gamma_{\mathfrak{n}\mathfrak{b}^{-1}})|_k V_\mathfrak{n} \right).$$

We now define the hyperbolic measure $d\mu$ on $\mathcal{H}^2$ to define an inner product on our space $S_k(\Gamma_\mathfrak{n})$ [Shi81, pg. 651].

$$d\mu(z) = \frac{dx_1 dy_1}{y_1^2} \frac{dx_2 dy_2}{y_2^2} \text{ where } z = (x_1 + iy_1, x_2 + iy_2) \in \mathcal{H}^2.$$

Definition 6.4.3 (Petersson inner product)**.** *Let $f, g \in S_k(\Gamma_\mathfrak{n})$. We define the Petersson inner product of f and g by,*

$$\langle f, g \rangle := \int_{\Gamma_\mathfrak{n} \backslash \mathcal{H}^2} f(z)\overline{g(z)}(y_1 y_2)^k d\mu(z)$$

where $z = (x_1 + iy_1, x_2 + iy_2) \in \mathcal{H}^2$.

Definition 6.4.4 (Hilbert new subspace)**.** *We define the Hilbert new subspace of $S_k(\Gamma_\mathfrak{n})$ as the orthogonal complement of the Hilbert old subspace with respect to the Petersson inner product,*

$$S_k^{new}(\Gamma_\mathfrak{n}) := S_k^{old}(\Gamma_\mathfrak{n})^\perp.$$

Hence, our space of Hilbert cusp forms at level $\mathfrak{n}$ has the following direct decomposition.

$$S_k(\Gamma_\mathfrak{n}) = S_k^{\text{new}}(\Gamma_\mathfrak{n}) \oplus S_k^{\text{old}}(\Gamma_\mathfrak{n})$$

or

$$S_k(\Gamma_\mathfrak{n}) = \bigoplus_{\substack{\mathfrak{m}|\mathfrak{n} \\ \mathfrak{b}\subset\mathcal{O}_K,\ \mathfrak{b}|\mathfrak{n}\mathfrak{m}^{-1}}} S_k^{\text{new}}(\Gamma_\mathfrak{m})|_k V_\mathfrak{b}. \tag{6.2}$$

Definition 6.4.5 (Hilbert newform)**.** *A Hilbert newform $f(z) = \sum_{\xi\in\mathcal{O}_K^+} a_\xi e^{2\pi i \operatorname{Tr}\left(\frac{\xi}{2} z\right)} \in S_k^{new}(\Gamma_\mathfrak{n})$ is normalised such that $a_{(1)} = 1$ and is an eigenform for all Hecke operators $T_\mathfrak{m}$ over all integral ideals in $\mathcal{O}_K$.*

We now state one of the main results in this section which follows from [Shi81, Proposition 2.4]. Also, see [SW93, pg. 7] for further explanation on the same.

Proposition 6.4.6. *The space $S_k^{new}(\Gamma_\mathfrak{n})$ has an orthogonal basis of Hilbert newforms.*

From equation 6.2 and proposition 6.4.6, it follows that the space of Hilbert cusp forms $S_k(\Gamma_\mathfrak{n})$ has a basis of Hilbert eigenforms that are eigenforms for all Hecke operators $T_\mathfrak{m}$ over all integral ideals $\mathfrak{m}$ in $\mathcal{O}_K$.

Similar to the classical case, the Fourier coefficients of a Hilbert newform can be retained from its Hecke eigenvalues, see [Shi81, Section 2]. More formally, if $f(z) \in S_k(\Gamma_\mathfrak{n})$ is a Hilbert newform with Fourier expansion

$$f(z) = \sum_{\xi\in\mathcal{O}_K^+} a_\xi e^{2\pi i \operatorname{Tr}\left(\frac{\xi}{2} z\right)}.$$

and has Hecke eigenvalues $\lambda_{T_{\mathfrak{m}}}(f)$ corresponding to action of Hecke operator $T_{\mathfrak{m}}$ over integral ideals $\mathfrak{m}$ in $\mathcal{O}_K$, then

$$\lambda_{T_{\mathfrak{m}}}(f) = a_{\mathscr{M}} \tag{6.3}$$

where $\mathscr{M}$ is a totally positive generator of integral ideal $\mathfrak{m}$.

We now state a crucial result in this section that forces the Fourier coefficients of Hilbert newforms to lie in a ring of integers. It immediately follows from the following proposition.

Proposition 6.4.7. *Let $f(z) \in S_k(\Gamma_{\mathfrak{n}})$ be a Hilbert eigenform such that*

$$f(z)|_k T_{\mathfrak{m}} = \lambda_{T_{\mathfrak{m}}}(f)\, f(z)$$

for all integral ideals $\mathfrak{m}$ in $\mathcal{O}_K$. Then the Hecke eigenvalues $\lambda_{T_{\mathfrak{m}}}(f)$ are algebraic integers.

Proof. See [Shi81, Proposition 2.2]. ☐

Corollary 6.4.8. *Let $f(z) \in S_k(\Gamma_{\mathfrak{n}})$ be a Hilbert newform with Fourier expansion*

$$f(z) = \sum_{\xi \in \mathcal{O}_K^+} a_\xi e^{2\pi i\, Tr\left(\frac{\xi}{2} z\right)}.$$

Then there exists some fixed number field L_f with ring of integers $\mathcal{O}_f$ such that for all $\xi \in \mathcal{O}_K^+$, the Fourier coefficients $a_\xi \in \mathcal{O}_f$.

Proof. See equation 6.3 combined with proposition 6.4.7. ☐

CHAPTER 7

Hilbert modular forms of half-integer weight

7.1 Introduction

We now shift our attention to Hilbert modular forms of parallel weight $k+\frac{1}{2},\ k \in \mathbb{Z}_{>0}$, which lies midway between two integers. We will closely follow chapter 2 to build up the theory of half-integer weight Hilbert modular forms. For more details, refer to [Shi87]. We begin by looking at our first motivating example of a half-integer Hilbert weight modular form called the Hecke Theta Function which generalises the Jacobi Theta function over number fields. Refer to section 6.1 to refresh the background notation.

Definition 7.1.1 (Hecke Theta function). *The Hecke Theta function is defined as*

$$\Theta : \mathscr{H}^2 \to \mathbb{C} \text{ such that } \Theta(z) = \sum_{\xi \in \mathcal{O}_K} e^{2\pi i \, Tr\left(\frac{\xi^2}{2} z\right)}$$

where $z = (z_1, z_2) \in \mathscr{H}^2$ *and* $Tr\left(\frac{\xi^2}{2} z\right) = \frac{\xi_1^2}{2} z_1 + \frac{\xi_2^2}{2} z_2$ *given two totally positive real embeddings* ξ_1 *and* ξ_2 *of* ξ *in* $\mathbb{R}^2$.

We now set some notation to state the transformation law of the Hecke-Theta function.

Let D_K be the absolute discriminant of totally real quadratic field K.

Let $\gamma = \left(\begin{smallmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{smallmatrix}\right) \in \Gamma_{(4)}$ where $\Gamma_{(4)} = \Gamma[2\mathfrak{d}^{-1}, 2^{-1}\mathfrak{d}4]$.

- For $\alpha \neq 0$, define

$$\varepsilon(\alpha) := (i \ \mathrm{sgn}(\alpha))^{\frac{1}{2}} \, 2^{-1} D_K^{-\frac{1}{2}} \sum_{j \in (\delta/2\mathcal{O}_K)} e^{2\pi i \ \mathrm{Tr}(-j^2\alpha/4)}$$

  where sgn denotes the sign function and $\mathrm{sgn}(\alpha) \in \{(\pm 1, \pm 1)\}$. Note that $(i \ \mathrm{sgn}(\alpha))^{\frac{1}{2}}$ is meant in the multi-index notation.

- Let s be the number of negative embeddings of α. Then define

$$\tilde{\varepsilon}(\alpha) := i^s.$$

Next, we introduce a generalised quadratic symbol for the totally real quadratic field K with narrow class number 1.

Definition 7.1.2 (Quadratic symbol $\left(\frac{\bullet}{\bullet}\right)_2$ over K). *For a prime ideal $\mathfrak{p} \subset \mathcal{O}_K$ and $\alpha \in \mathcal{O}_K$, we define a quadratic symbol $\left(\frac{\alpha}{\mathfrak{p}}\right)_2$ by*

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_2 = \begin{cases} 1 & \text{if } \alpha \text{ is a square in } (\mathcal{O}_K/\mathfrak{p})^* \\ -1 & \text{if } \alpha \text{ is not a square in } (\mathcal{O}_K/\mathfrak{p})^* \\ 0 & \text{if } \alpha \in \mathfrak{p} \end{cases}.$$

We extend it by multiplicativity to all nonzero ideals of $\mathcal{O}_K$. For $0 \neq \beta \in \mathcal{O}_K$, write $\left(\frac{\alpha}{\beta}\right)_2 = \left(\frac{\alpha}{\beta\mathcal{O}_K}\right)_2$.

We now give the transformation law of Hecke Theta function. For details and proof of these results, refer to [Gar90, pg.142-143].

Proposition 7.1.3. *The Hecke Theta function is a Hilbert modular form of weight $\frac{1}{2}$ and level $4\mathcal{O}_K$ with a certain Hecke character and transforms as,*

$$\Theta(\gamma z) = \varepsilon(a_\gamma)\tilde{\varepsilon}(a_\gamma)\left(\frac{\frac{-1}{2}c_\gamma}{a_\gamma}\right)_2 N(c_\gamma z + d_\gamma)^{\frac{1}{2}}\Theta(z).$$

Also, we have $\Theta^2(z)$ is a Hilbert modular form of weight 1 and level $4\mathcal{O}_K$ with a Hecke character ψ that is defined modulo $4\mathcal{O}_K$ satisfying

$$\Theta^2(\gamma z) = \psi(a_\gamma)N(c_\gamma z + d_\gamma)\,\Theta^2(z)$$

where $N(c_\gamma z + d_\gamma) = \prod_{i=1,2}\left(c_{\gamma_i}z_i + d_{\gamma_i}\right)$ and $\psi(a_\gamma) = \varepsilon(a_\gamma)^2\tilde{\varepsilon}(a_\gamma)^2 = sgn(a_\gamma)^1\left(\frac{-1}{a_\gamma}\right)_2$.

Note 7.1.4 (Convention for taking square roots). The convention for taking square roots remains same as fixed in chapter 2. We shall always take the branch of the square root having argument in $(-\pi/2, \pi/2]$.

We now introduce the *metaplectic Group* of $GL_2^+(K)$.

Definition 7.1.5 (Metaplectic Group of $GL_2^+(K)$). *Let $T = \{z \in \mathbb{C} \mid |z| = 1\}$ and let $\mathfrak{G}^K$ denote all couples $(\alpha, \Phi(z))$ such that $\alpha = \left(\begin{smallmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{smallmatrix}\right) \in GL_2^+(K)$ and $\Phi(z)$ is a holomorphic function on $\mathcal{H}^2$ defined as*

$$\Phi(z)^2 = t \cdot N(\det(\alpha))^{-1/2}N(c_\alpha z + d_\alpha) \text{ where } t \in T.$$

We define a multiplication law $\star$ on $\mathfrak{G}^K$ as

$$(\alpha, \Phi(z)) \star (\beta, \Psi(z)) := (\alpha\beta, \Phi(\beta z)\Psi(z)).$$

This forms a group $(\mathfrak{G}^K, \star)$ called the Metaplectic Group of $GL_2^+(K)$.

Note 7.1.6. In order to verify $(\mathfrak{G}^K, \star)$ is a group, use multiplicativity of norm and follow proof of [Kob93, Proposition 1, pg. 179]

Let $\mathfrak{G}_1^K$ be the subgroup of $\mathfrak{G}^K$ defined as

$$\mathfrak{G}_1^K := \{(\alpha, \Phi(z)) \in \mathfrak{G}^K \mid N(\det(\alpha)) = 1\}.$$

Let $\mathfrak{n} \subset \mathcal{O}_K$ be a square-free integral ideal of odd norm. There are infinitely many ways to lift an element of $GL_2^+(K)$ to its metaplectic group $\mathfrak{G}^K$ depending on the choice of $t \in T$. We fix our choice of $\Phi(z)$ to establish an isomorphism between $\Gamma_{4\mathfrak{n}}$ and a subgroup of $\mathfrak{G}_1^K$ denoted by $\widetilde{\Gamma}_{4\mathfrak{n}}$,

$$\alpha \mapsto \alpha^\star = (\alpha, \Phi(z))$$

where

$$\Phi(z) = \varepsilon(a_\gamma)\tilde{\varepsilon}(a_\gamma)\left(\frac{\frac{-1}{2}c_\gamma}{a_\gamma}\right)_2 N(\det(\alpha))^{-1/4}N(c_\alpha z + d_\alpha)^{1/2}. \tag{7.1}$$

Here the symbols $\varepsilon(a_\gamma)$, $\tilde{\varepsilon}(a_\gamma)$ and $\left(\frac{\frac{-1}{2}c_\gamma}{a_\gamma}\right)_2$ have the same meaning as in Theorem 7.1.3.

Remark 7.1.7. When we refer to congruence subgroup of level $4\mathfrak{n}$ in the case of half-integer weight Hilbert modular forms, we will always mean $\widetilde{\Gamma}_{4\mathfrak{n}}$.

Definition 7.1.8 (($k+\frac{1}{2}$)-slash operator). *Let $\alpha^\star = (\alpha, \Phi(z)) \in \mathfrak{G}^K$ where $\alpha \in GL_2^+(\mathbb{R}^2)$ and $\Phi(z)$ is holomorphic function on $\mathscr{H}^2$ defined as $\Phi(z) := \varepsilon(a_\gamma)\tilde{\varepsilon}(a_\gamma)\left(\frac{\frac{-1}{2}c_\gamma}{a_\gamma}\right)_2 N(\det(\alpha))^{-1/4}N(c_\alpha z + d_\alpha)^{1/2}$. For a complex valued function $g(z)$ on $\mathscr{H}^2$, we define an operator $|_{k+\frac{1}{2}}\alpha^\star$ as*

$$g(z)|_{k+\frac{1}{2}}\alpha^\star := \Phi(z)^{-(2k+1)}g(\alpha z). \tag{7.2}$$

This is the $(k+\frac{1}{2})$-slash action in the case of half-integer weight Hilbert modular forms.

Definition 7.1.9 (Half-integer weight Hilbert modular form). *Let $k \in \mathbb{Z}_{>0}$ and let $4\mathfrak{n} \subset \mathscr{O}_K$ be an integral ideal in K. A half-integer weight Hilbert modular form of parallel weight $k+\frac{1}{2}$ and level $4\mathfrak{n}$ is defined as a holomorphic function $g : \mathscr{H}^2 \to \mathbb{C}$ such that*

$$g(z)|_{k+\frac{1}{2}}\alpha^\star = g(z) \quad \textit{for all } \alpha^\star \in \widetilde{\Gamma}_{4\mathfrak{n}}.$$

We denote the space of all half-integer weight Hilbert modular forms of parallel weight $k+\frac{1}{2}$ and level $4\mathfrak{n}$ by $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}})$.

Remark 7.1.10. Similar to the case of integer weight Hilbert modular forms, $g(z)$ is automatically holomorphic at all the cusps including infinity. This is again due to *Götzky-Koecher principle* [BHZ08, Theorem 1.20, pg. 114].

From [Shi87, Proposition 3.1], $g(z)$ admits the following Fourier expansion corresponding to ideal $\mathscr{O}_K$

$$g(z) = \sum_{\xi \in \mathscr{O}_K^+ \cup \{0\}} b_{\xi,\mathscr{O}_K} e^{2\pi i \,\mathrm{Tr}\left(\frac{\xi}{2}z\right)} \tag{7.3}$$

where $\mathrm{Tr}\left(\frac{\xi}{2}z\right) = \frac{\xi_1}{2}z_1 + \frac{\xi}{2}2z_2$ and $b_{\xi,\mathscr{O}_K} \in \mathbb{C}$.

Further, if $\mathfrak{b}$ is a fractional ideal in K generated by $\mathscr{B}$,then the Fourier coefficients satisfy the following property

$$b_{\mathscr{B}^2\xi,\mathscr{O}_K} = N(\mathscr{B})^k b_{\xi,\mathfrak{b}}$$

for all $\mathscr{B} \in K^\star$.

Note 7.1.11. For simplicity, we will write b_ξ instead of $b_{\xi,\mathscr{O}_K}$ wherever possible.

Definition 7.1.12 (Half-integer weight Hilbert cusp form). *A half-integer weight Hilbert cusp-form $g(z)$ of weight $k+\frac{1}{2}$ and level $4\mathfrak{n}$ is a half-integer weight Hilbert modular form that vanishes at all the cusps of $\Gamma_{4\mathfrak{n}}$. This happens if $b_{(0)}$ (the 0^{th} Fourier coefficient) vanishes in the Fourier expansion of $g(z)$ at each cusp of $\Gamma_{4\mathfrak{n}}$.*

The space of all half-integer weight Hilbert cusp forms of weight $k+\frac{1}{2}$ and level $4N$ forms a subspace of $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}})$ and is denoted by $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}})$.

7.1.1 Half-integer weight Hilbert modular forms with a character

Given a square-free integral ideal $\mathfrak{n} \subset \mathcal{O}_K$ of odd norm, let $\psi_{\mathfrak{n}}$ be a finite order Hecke character of K of conductor $\mathfrak{n}$ defined in section 6.1. We fix $\psi_{\mathfrak{n}}$ to be the generalised quadratic character in K which will be given in terms of the quadratic residue symbol defined in 7.1.2. We now define

$$\psi_{4\mathfrak{n}} := \left(\frac{4\psi_{\mathfrak{n}}(-1)}{\bullet}\right)_2 \psi_{\mathfrak{n}}.$$

We say that $g(z)$ is a half-integer weight Hilbert modular form of weight $k+\frac{1}{2}$, level $4\mathfrak{n}$ and character $\psi_{4\mathfrak{n}}$ if the invariance condition in definition 7.1.9 is replaced by

$$g(z)|_{k+\frac{1}{2}}\alpha^\star = \psi_{4\mathfrak{n}}(a_\alpha)\, g(z) \quad \text{for all} \quad \alpha^\star = \left(\begin{pmatrix} a_\alpha & b_\alpha \\ c_\alpha & d_\alpha \end{pmatrix}, \Phi(z)\right) \in \widetilde{\Gamma}_{4\mathfrak{n}}.$$

Note 7.1.13. For simplicity, we will denote $\psi_{4\mathfrak{n}}$ by ψ.

We denote the space of all half-integer weight Hilbert modular forms of weight $k+\frac{1}{2}$, level $4\mathfrak{n}$ and character ψ by $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}, \psi)$.

7.2 Hecke operators

We will now define Hecke operators acting on spaces of half-integer weight Hilbert modular forms. For details about Hecke operators on half-integer weight Hilbert modular forms, one can refer to *Shimura's Paper* [Shi87, Section 5].

Definition 7.2.1. *Let $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}, \psi)$. Let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime ideal generated by $\wp \gg 0$. Then for each prime ideal $\mathfrak{p}$, we define the Hecke operator $T_{\mathfrak{p}^2}$ in terms of $\left(k+\frac{1}{2}\right)$-slash operator on $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}, \psi)$ as follows*

$$g(z)|_{k+\frac{1}{2}} T_{\mathfrak{p}^2} = N(\wp)^{k-\frac{3}{2}} \psi_\infty(\wp) \begin{cases} \displaystyle\sum_{h \in (\mathcal{O}_K/\wp^2\mathcal{O}_K)} g(z)|\gamma_h^\star + \psi(\wp) \sum_{h \in (\mathcal{O}_K/\wp\mathcal{O}_K)^*} g(z)|\beta_h^\star + \psi(\wp^2) g(z)|\alpha^\star & \text{if } \mathfrak{p} \nmid \mathfrak{n} \\ \displaystyle\sum_{h \in (\mathcal{O}_K/\wp^2\mathcal{O}_K)} g(z)|\gamma_h^\star & \text{if } \mathfrak{p} \mid \mathfrak{n} \end{cases}$$

where $\gamma_h = \begin{pmatrix} 1 & 2\delta^{-1}h \\ 0 & \wp^2 \end{pmatrix}$, $\beta_h = \begin{pmatrix} \wp & 2\delta^{-1}h \\ 0 & \wp \end{pmatrix}$ *and* $\alpha = \begin{pmatrix} \wp & 0 \\ 0 & \wp \end{pmatrix}$.
The explicit metaplectic lifts of γ_h, β_h and α can be found in [Shi87, Proposition 5.3].

Note 7.2.2. Here ψ is the finite order Hecke character modulo $4\mathfrak{n}$ which in our case has been assumed to be given in terms of generalised quadratic residue symbols, see 7.1.1.

The Hecke operators are commutative, that is, $T_{\mathfrak{p}^2}T_{\mathfrak{q}^2} = T_{\mathfrak{q}^2}T_{\mathfrak{p}^2}$ where $\mathfrak{p}$ and $\mathfrak{q}$ are distinct prime ideals in $\mathcal{O}_K$, see [Shi87, Proposition 5.2]. In contrast to the classical case, *Shimura* in [Shi87] extends the definition of Hecke operators only to square free integral ideals $\mathfrak{m}$ in $\mathcal{O}_K$ via mutiplicativity, that is, $T_{\mathfrak{m}^2} = \prod_i T_{\mathfrak{p}_i^2}$ where $\mathfrak{m} = \prod_i \mathfrak{p}_i$.

We now state the action of Hecke operator $T_{\mathfrak{p}^2}$ on the Fourier coefficients of the half-integer weight Hilbert modular form $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}, \psi)$ which is a result by *Shimura*, see [Shi87, Proposition 5.3].

Proposition 7.2.3 (Hecke operators on Fourier expansions)**.** *Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}_K$ generated by $\wp \gg 0$. Let $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}, \psi)$ with Fourier expansion*

$$g(z) = \sum_{\xi \in \mathcal{O}_K^+} b_\xi e^{2\pi i \, Tr\left(\frac{\xi}{2} z\right)}.$$

Then $g(z)|_{k+\frac{1}{2}}T_{\mathfrak{p}^2} \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}, \psi)$ *with Fourier expansion*

$$g(z)|_{k+\frac{1}{2}}T_{\mathfrak{p}^2} = \sum_{\xi \in \mathscr{O}_K^+} c_\xi e^{2\pi i \, Tr\left(\frac{\xi}{2}z\right)}$$

such that

$$\psi_\infty(\wp)c_\xi = \begin{cases} b_{\wp^2\xi} + \psi^\star(\wp)N(\wp)^{k-1}\left(\frac{\xi}{\wp}\right)_2 b_\xi + \psi^\star(\wp^2)N(\wp)^{2k-1}b_{\xi\wp^{-2}} & \text{if } \mathfrak{p} \nmid \mathfrak{n}; \\ b_{\wp^2\xi} & \text{if } \mathfrak{p} \mid \mathfrak{n}. \end{cases}.$$

where $\psi^\star(\bullet) = \left(\frac{-1}{\bullet}\right)_2^k \psi(\bullet)$.

Here $b_{\xi\wp^{-2}} = 0$ *when* $\wp^2 \nmid \xi$.

The fact that $g(z)|_{k+\frac{1}{2}}T_{\mathfrak{p}^2} \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}, \psi)$ follows from [Shi87, pg. 787].

Note 7.2.4. Since we have taken our infinity type ψ_∞ to be trivial in section 7.1.1, we can ignore it in the above formula in definition 7.2.3.

Definition 7.2.5 (Hecke algebra). *The Hecke algebra of weight* $k+\frac{1}{2}$*, level* $4\mathfrak{n}$ *and Hecke character* ψ *acting on* $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}, \psi)$ *is the commutative* $\mathbb{C}$*-subalgebra of* $End\Big(M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}, \psi)\Big)$ *generated by the Hecke operators* $T_{\mathfrak{m}^2}$ *for all square-free integral ideals* $\mathfrak{m}$ *in* $\mathscr{O}_K$*.*
We denote the Hecke algebra of weight $k+\frac{1}{2}$*, level* $4\mathfrak{n}$ *and character* ψ *by* $\widetilde{\mathbb{T}}_{k+\frac{1}{2}}(4\mathfrak{n}, \psi)$*.*

Definition 7.2.6 (Hecke eigenform). *We say* $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}, \psi)$ *is a Hilbert Hecke eigenform of half-integer weight if it's a simultaneous eigenvector for all Hecke operators in* $\widetilde{\mathbb{T}}_{k+\frac{1}{2}}(4\mathfrak{n}, \psi)$*.*

Note 7.2.7. We will specify whenever we refer to a Hecke eigenform for all Hecke operators $T_{\mathfrak{p}^2}$ for $\mathfrak{p}$ not dividing the level $4\mathfrak{n}$. In general, our definition of a Hecke eigenform refers to $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}, \psi)$ being an eigenvector for all Hecke operators $T_{\mathfrak{m}^2}$ over all square-free integral ideals $\mathfrak{m}$ in $\mathscr{O}_K$.

7.3 Further operators

We now introduce some more operators acting on $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}, \psi)$.

7.3.1 Operator $V_{(1)}$

Let $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}, \psi)$. Let $\mathfrak{m} \subset \mathscr{O}_K$ be an integral ideal such that $\mathfrak{m}$ is co-prime to level $\mathfrak{n}$. Note that $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}, \psi) \subset M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}\mathfrak{m}}, \psi)$. Hence, $g(z)$ can always be viewed as a half-integer Hilbert modular form in $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}\mathfrak{m}}, \psi)$. More formally, define an operator

$$|_{k+\frac{1}{2}}V_{(1)} : M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}, \psi) \to M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}\mathfrak{m}}, \psi) \text{ such that } g(z) \mapsto g(z).$$

7.3.2 Operator $V_{\mathfrak{m}}$

Let $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}, \psi)$. Let $\mathfrak{m} \subset \mathscr{O}_K$ be an integral ideal such that $\mathfrak{m}$ is co-prime to level $\mathfrak{n}$. Let $\mathscr{M}, \mathscr{N} \in \mathscr{O}_K^+$ be totally positive generators of ideals $\mathfrak{m}$ and $\mathfrak{n}$ respectively. Define the operator $V_{\mathfrak{m}}$ on $g(z)$ in terms of $\left(k+\frac{1}{2}\right)$-slash action as

$$\begin{aligned} g(z)|_{k+\frac{1}{2}}V_{\mathfrak{m}} &:= N(\mathscr{M})^{-\frac{(2k+1)}{4}} g(z)|_{k+\frac{1}{2}}\left(\begin{pmatrix} \mathscr{M} & 0 \\ 0 & 1 \end{pmatrix}, N(\mathscr{M})^{-\frac{1}{4}}\right) \\ &= N(\mathscr{M})^{-\frac{(2k+1)}{4}} N(\mathscr{M})^{\frac{(2k+1)}{4}} g(\mathscr{M}z) \\ &= g(\mathscr{M}z). \end{aligned}$$

Thus, if $g(z)=\sum_{\xi\in\mathscr{O}_K^+} b_\xi e^{2\pi i\,\mathrm{Tr}\left(\frac{\xi}{2}z\right)}$, then $g(z)|_{k+\frac{1}{2}}V_\mathfrak{m}=\sum_{\xi\in\mathscr{O}_K^+} c_\xi e^{2\pi i\,\mathrm{Tr}\left(\frac{\xi}{2}\mathscr{M}z\right)}$.

Proposition 7.3.1. *Let $g(z)\in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}},\psi)$. Let $\mathfrak{m}\subset\mathscr{O}_K$ be an integral ideal such that $\mathfrak{m}$ is co-prime to level $\mathfrak{n}$. Let $\mathscr{M},\ \mathscr{N}\in\mathscr{O}_K^+$ be totally positive generators of ideals $\mathfrak{m}$ and $\mathfrak{n}$ respectively. Then $g(\mathscr{M}z)\in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}\mathfrak{m}},\psi')$ where $\psi'(\bullet)=\psi(\bullet)\left(\frac{\mathscr{M}}{\bullet}\right)_2$.*

Proof. See [Shi87, Proposition 3.2]. ☐

Thus, we have a map

$$|_{k+\frac{1}{2}}V_\mathfrak{m}: M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}},\psi)\to M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}\mathfrak{m}},\psi') \text{ such that } g(z)\mapsto g(\mathscr{M}z).$$

7.3.3 Operator $U_\mathfrak{m}$

Let $\mathfrak{m}\subset\mathscr{O}_K$ be an integral ideal and let $g(z)\in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}},\psi)$ with Fourier expansion $g(z)=\sum_{\xi\in\mathscr{O}_K^+} b_\xi e^{2\pi i\,\mathrm{Tr}\left(\frac{\xi}{2}z\right)}$. Let $\mathscr{M},\ \mathscr{N}\in\mathscr{O}_K^+$ be totally positive generators of ideals $\mathfrak{m}$ and $\mathfrak{n}$ respectively. Define the operator $U_\mathfrak{m}$ on the Fourier coefficients of $g(z)$ in the following way:

$$g(z)|_{k+\frac{1}{2}}U_\mathfrak{m}:=\sum_{\xi\in\mathscr{O}_K^+} b_{\mathscr{M}\xi}e^{2\pi i\,\mathrm{Tr}\left(\frac{\xi}{2}z\right)}.$$

We next want to show how the action of $U_\mathfrak{m}$ operator on $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}},\psi)$ affects the character of this space. This has been already proved for Hilbert modular forms of weight $\frac{1}{2}$ in [AS08, Lemma 4.3 and 4.4]. We now use the same proof to generalise these results for any half-integral weight $k+\frac{1}{2}$, including more details and explanation wherever possible.

Lemma 7.3.2. *Let $g(z)\in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}},\psi)$ and let $\mathfrak{m}\subset\mathscr{O}_K$ be an integral ideal such that $\mathfrak{m}\mid\mathfrak{n}$. Let $\mathscr{M},\ \mathscr{N}\in\mathscr{O}_K^+$ be totally positive generators of ideals $\mathfrak{m}$ and $\mathfrak{n}$ respectively. Set $h(z)=g(\mathscr{M}^{-1}z)$. Then*

$$h(z)|_{k+\frac{1}{2}}\gamma^\star=\psi(a_\gamma)\left(\frac{\mathscr{M}}{a_\gamma}\right)_2 h(z)$$

for any $\gamma^\star=(\gamma,\Phi_\gamma(z))$ where

$$\gamma=\begin{pmatrix}a_\gamma & b_\gamma\\ c_\gamma & d_\gamma\end{pmatrix}\in\Gamma_{4\mathfrak{n},\mathfrak{m}}.$$

Here $\Gamma[2\mathfrak{d}^{-1}\mathscr{M},2^{-1}\mathfrak{d}4\mathscr{N}]=\Gamma_{4\mathfrak{n},\mathfrak{m}}$.

Proof. Let $M^\star=(M,N(\mathscr{M})^{\frac{1}{4}})$ where $M=\begin{pmatrix}1&0\\0&\mathscr{M}\end{pmatrix}$.
We define $\gamma':=M\gamma M^{-1}$. Then

$$\begin{aligned}\gamma'&=\begin{pmatrix}1&0\\0&\mathscr{M}\end{pmatrix}\begin{pmatrix}a_\gamma&b_\gamma\\c_\gamma&d_\gamma\end{pmatrix}\begin{pmatrix}1&0\\0&\mathscr{M}^{-1}\end{pmatrix}\\&=\begin{pmatrix}a_\gamma&b_\gamma\\\mathscr{M}c_\gamma&\mathscr{M}d_\gamma\end{pmatrix}\begin{pmatrix}1&0\\0&\mathscr{M}^{-1}\end{pmatrix}\\&=\begin{pmatrix}a_\gamma&b_\gamma\mathscr{M}^{-1}\\\mathscr{M}c_\gamma&d_\gamma\end{pmatrix}\\&\in\begin{pmatrix}\mathscr{O}_K&2\mathfrak{d}^{-1}\mathscr{M}\mathscr{M}^{-1}\\\mathscr{M}2^{-1}\mathfrak{d}4\mathscr{N}&\mathscr{O}_K\end{pmatrix}\end{aligned}$$

Thus, $\gamma' \in \Gamma_{4\mathfrak{n}}$. So, we can lift γ' to $\gamma'^{\star} \in \widetilde{\Gamma}_{4\mathfrak{n}}$ using $\Phi_{\gamma'}(z)$ defined in 7.1, that is,

$$\Phi_{\gamma'}(z) = \varepsilon(a_\gamma)\tilde{\varepsilon}(a_\gamma)\left(\frac{\frac{-1}{2}\mathscr{M}c_\gamma}{a_\gamma}\right)_2 N(\mathscr{M}c_\gamma z + d_\gamma)^{\frac{1}{2}}. \tag{7.4}$$

Also, it is given that $\gamma^\star \in \widetilde{\Gamma}_{4\mathfrak{n}}$.

Next, we evaluate $\Phi_M(z)\Phi_\gamma(z)\Phi_{M^{-1}}(z)$.

$$\begin{aligned}
M^\star\gamma^\star M^{-1\star} &= \left(\begin{pmatrix}1 & 0\\ 0 & \mathscr{M}\end{pmatrix}, N(\mathscr{M})^{\frac{1}{4}}\right)\left(\gamma, \Phi_\gamma(z)\right)\left(\begin{pmatrix}1 & 0\\ 0 & \mathscr{M}^{-1}\end{pmatrix}, N(\mathscr{M})^{-\frac{1}{4}}\right)\\
&= \left(\begin{pmatrix}1 & 0\\ 0 & \mathscr{M}\end{pmatrix}, N(\mathscr{M})^{\frac{1}{4}}\right)\left(\gamma M^{-1}, \Phi_\gamma(M^{-1}z)N(\mathscr{M})^{-\frac{1}{4}}\right)\\
&= \left(M\gamma M^{-1}, N(\mathscr{M})^{\frac{1}{4}}\Phi_\gamma(M^{-1}z)N(\mathscr{M})^{-\frac{1}{4}}\right)\\
&= \left(M\gamma M^{-1}, \Phi_\gamma(\mathscr{M}z)\right)
\end{aligned}$$

where

$$\Phi_\gamma(\mathscr{M}z) = \varepsilon(a_\gamma)\tilde{\varepsilon}(a_\gamma)\left(\frac{\frac{-1}{2}c_\gamma}{a_\gamma}\right)_2 N(\mathscr{M}c_\gamma z + d_\gamma)^{\frac{1}{2}} \tag{7.5}$$

Thus, we can conclude from 7.4 and 7.5 that for any $\gamma \in \Gamma_{4\mathfrak{n},\mathfrak{m}}$

$$(M\gamma M^{-1})^\star = \left(I, \left(\frac{\mathscr{M}}{a_\gamma}\right)_2\right)(M^\star\gamma^\star M^{-1\star}) \text{ where } I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

We can now rewrite the above expression for the value of $\gamma^\star$ below

$$\gamma^\star = M^{-1\star}\left(I, \left(\frac{\mathscr{M}}{a_\gamma}\right)_2\right)(M\gamma M^{-1})^\star M^\star.$$

Now we act $h(z)$ by $\gamma^\star \in \widetilde{\Gamma}_{4\mathfrak{n},\mathfrak{m}}$ and we will see the required change in the character.

$$\begin{aligned}
h(z)|_{k+\frac{1}{2}}\gamma^\star &= h(z)|_{k+\frac{1}{2}}M^{-1\star}\left(I, \left(\frac{\mathscr{M}}{a_\gamma}\right)_2\right)(M\gamma M^{-1})^\star M^\star\\
&= N(\mathscr{M})^{\frac{1}{4}(2k+1)}h(M^{-1}z)|_{k+\frac{1}{2}}\left(I, \left(\frac{\mathscr{M}}{a_\gamma}\right)_2\right)(M\gamma M^{-1})^\star M^\star\\
&= N(\mathscr{M})^{\frac{1}{4}(2k+1)}h(\mathscr{M}z)|_{k+\frac{1}{2}}\left(I, \left(\frac{\mathscr{M}}{a_\gamma}\right)_2\right)(M\gamma M^{-1})^\star M^\star\\
&= N(\mathscr{M})^{\frac{1}{4}(2k+1)}g(z)|_{k+\frac{1}{2}}\left(I, \left(\frac{\mathscr{M}}{a_\gamma}\right)_2\right)(M\gamma M^{-1})^\star M^\star\\
&= N(\mathscr{M})^{\frac{1}{4}(2k+1)}\left(\frac{\mathscr{M}}{a_\gamma}\right)_2\psi(a_{\gamma'})g(z)|_{k+\frac{1}{2}}M^\star\\
&= N(\mathscr{M})^{\frac{1}{4}(2k+1)}N(\mathscr{M})^{-\frac{1}{4}(2k+1)}\left(\frac{\mathscr{M}}{a_\gamma}\right)_2\psi(a_{\gamma'})g\left(\mathscr{M}^{-1}z\right)\\
&= \left(\frac{\mathscr{M}}{a_\gamma}\right)_2\psi(a_{\gamma'})h(z).
\end{aligned}$$

Since $a_{\gamma'} = a_\gamma$, we have

$$h(z)|_{k+\frac{1}{2}}\gamma^\star = \left(\frac{\mathscr{M}}{a_\gamma}\right)_2 \psi(a_\gamma)\, h(z) \text{ for any } \gamma^\star \in \widetilde{\Gamma}_{4\mathfrak{n},\mathfrak{m}}.$$

□

Proposition 7.3.3. *Let $\mathfrak{m} \subset \mathcal{O}_K$ be an integral ideal such that $\mathfrak{m}$ divides the level $\mathfrak{n}$. Let $\mathscr{M}$, $\mathscr{N} \in \mathcal{O}_K^+$ be totally positive generators of ideals $\mathfrak{m}$ and $\mathfrak{n}$ respectively. Then the action of operator $U_\mathfrak{m}$ maps the space $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}, \psi)$ to $M_{k+\frac{1}{2}}\left(\widetilde{\Gamma}_{4\mathfrak{n}}, \psi\left(\frac{\mathscr{M}}{\bullet}\right)_2\right)$.*

Proof. For a fixed $\alpha \in (\mathcal{O}_K/\mathfrak{m})$, consider the matrix

$$\begin{pmatrix} 1 & 2\alpha\delta^{-1} \\ 0 & \mathscr{M} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \mathscr{M} \end{pmatrix}\begin{pmatrix} 1 & 2\alpha\delta^{-1} \\ 0 & 1 \end{pmatrix}.$$

Let $M^\star := (M, N(\mathscr{M})^{\frac{1}{4}})$ where $M = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathscr{M} \end{smallmatrix}\right)$ and $B_\alpha^\star := (B_\alpha, 1)$ where $B_\alpha = \left(\begin{smallmatrix} 1 & 2\alpha\delta^{-1} \\ 0 & 1 \end{smallmatrix}\right)$.
Note that $M^\star \notin \widetilde{\Gamma}_{4\mathfrak{n}}$ but $B_\alpha^\star \in \widetilde{\Gamma}_{4\mathfrak{n}}$.

Let $h(z) = g(\mathscr{M}^{-1}z)$ where $g(z) \in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}, \psi)$. Then

$$h(z) = N(\mathscr{M})^{\frac{1}{4}(2k+1)} g(z)|_{k+\frac{1}{2}}\left(\begin{pmatrix} 1 & 0 \\ 0 & \mathscr{M} \end{pmatrix}, N(\mathscr{M})^{\frac{1}{4}}\right)$$

which implies

$$h(z)|_{k+\frac{1}{2}}B_\alpha^\star = N(\mathscr{M})^{\frac{1}{4}(2k+1)} g(z)|_{k+\frac{1}{2}}\left(\begin{pmatrix} 1 & 2\alpha\delta^{-1} \\ 0 & \mathscr{M} \end{pmatrix}, N(\mathscr{M})^{\frac{1}{4}}\right).$$

Let $\gamma \in \Gamma_{4\mathfrak{n}}$. Then

$$\begin{aligned} \left(g(z)|_{k+\frac{1}{2}}U_\mathfrak{m}\right)|_{k+\frac{1}{2}}\gamma^\star &= N(\mathscr{M})^{\frac{k}{2}-\frac{3}{4}} \sum_{\alpha\in\mathcal{O}_K/\mathfrak{m}} \left(g(z)|_{k+\frac{1}{2}}\left(\begin{pmatrix} 1 & 2\alpha\delta^{-1} \\ 0 & \mathscr{M} \end{pmatrix}, N(\mathscr{M})^{\frac{1}{4}}\right)\right)|_{k+\frac{1}{2}}\gamma^\star \\ &= N(\mathscr{M})^{\frac{k}{2}-\frac{3}{4}-\frac{k}{2}-\frac{1}{4}} \sum_{\alpha\in\mathcal{O}_K/\mathfrak{m}} h(z)|_{k+\frac{1}{2}}B_\alpha^\star\gamma^\star \\ &= N(\mathscr{M})^{-1} \sum_{\alpha\in\mathcal{O}_K/\mathfrak{m}} h(z)|_{k+\frac{1}{2}}B_\alpha^\star\gamma^\star. \end{aligned} \tag{7.6}$$

We now try to observe how $B_\alpha^\star\gamma^\star$ acts on $h(z) = g(\mathscr{M}^{-1}z)$.
Since B_α, $\gamma \in \Gamma_{4\mathfrak{n}}$, then $B_\alpha\gamma \in \Gamma_{4\mathfrak{n}}$.

Now let us look at the set

$$\{\Gamma_{4\mathfrak{n},\mathfrak{m}}B_{\alpha_1},\ \Gamma_{4\mathfrak{n},\mathfrak{m}}B_{\alpha_2},\ \Gamma_{4\mathfrak{n},\mathfrak{m}}B_{\alpha_3}, \ldots,\ \Gamma_{4\mathfrak{n},\mathfrak{m}}B_{\alpha_n}\} \text{ where } n = |\mathcal{O}_k/\mathfrak{m}|.$$

Then the right coset decomposition of $\Gamma_{4\mathfrak{n},\mathfrak{m}}$ in $\Gamma_{4\mathfrak{n}}$ using B_α is given by

$$\Gamma_{4\mathfrak{n}} = \bigcup_{i=1}^{n} \Gamma_{4\mathfrak{n},\mathfrak{m}}B_{\alpha_i}.$$

Now for a fixed $\alpha \in (\mathcal{O}_K/\mathfrak{m})$, since $B_\alpha\gamma \in \Gamma_{4\mathfrak{n}}$, it must belong to one these cosets $\{\Gamma_{4\mathfrak{n},\mathfrak{m}}B_{\alpha_i}\}_{i=1}^n$ for some fixed i, that is, $B_\alpha\gamma = \gamma_i B_{\alpha_i}$ where $\gamma_i \in \Gamma_{4\mathfrak{n},\mathfrak{m}}$. However, as we vary α, different choices of α for B_α will give different choices of such cosets with varying i's. Now since all γ, γ_i, B_α, $B_{\alpha_i} \in \Gamma_{4\mathfrak{n}}$, we have

$$(B_\alpha\gamma)^\star = (\gamma_i B_{\alpha_i})^\star \quad \text{if and only if} \quad B_\alpha^\star\gamma^\star = \gamma_i^\star B_{\alpha_i}^\star.$$

Replacing $B_\alpha^\star\gamma^\star$ in equation 7.6 by $\gamma_i^\star B_{\alpha_i}^\star$ and then using lemma 7.3.2, we get

$$\begin{aligned}\left(g(z)|_{k+\frac{1}{2}}U_{\mathfrak{m}}\right)|_{k+\frac{1}{2}}\gamma^\star &= N(\mathscr{M})^{-1}\sum_{i=1}^{n}h(z)|_{k+\frac{1}{2}}\gamma_i^\star B_{\alpha_i}^\star\\ &= N(\mathscr{M})^{-1}\Psi(a_{\gamma_i})\sum_{i=1}^{m}h(z)|_{k+\frac{1}{2}}B_{\alpha_i}^\star\end{aligned}$$

where $\Psi(\bullet)=\psi(\bullet)\left(\frac{\mathscr{M}}{\bullet}\right)_2$.

It follows

$$\begin{aligned}\left(g(z)|_{k+\frac{1}{2}}U_{\mathfrak{m}}\right)|_{k+\frac{1}{2}}\gamma^\star &= N(\mathscr{M})^{-1}N(\mathscr{M})^{\frac{1}{4}(2k+1)}\Psi(a_{\gamma_i})\sum_{i=1}^{n}g(z)|_{k+\frac{1}{2}}\left(\begin{pmatrix}1 & 2\alpha_i\delta^{-1}\\ 0 & \mathscr{M}\end{pmatrix}, N(\alpha)^{\frac{1}{4}}\right)\\ &= N(\mathscr{M})^{-1}N(\mathscr{M})^{\frac{1}{4}(2k+1)}\Psi(a_{\gamma_i})N(\mathscr{M})^{-\frac{1}{4}(2k+1)}\sum_{i=1}^{n}g\left(\frac{z+2\alpha_i\delta^{-1}}{\mathscr{M}}\right)\\ &= \Psi(a_{\gamma_i})g(z)|_{k+\frac{1}{2}}U_{\mathfrak{m}}.\end{aligned}$$

Since $\gamma_i=B_\alpha\gamma B_{\alpha_i}^{-1}$, we can compare entries on either side and it easily follows that

$$a_{\gamma_i}\equiv a_\gamma \bmod 4\mathfrak{n}.$$

It hence follows that

$$\begin{aligned}\left(g(z)|_{k+\frac{1}{2}}U_{\mathfrak{m}}\right)|_{k+\frac{1}{2}}\gamma^\star &= \Psi(a_\gamma)g(z)|_{k+\frac{1}{2}}U_{\mathfrak{m}}\\ &= \psi(a_\gamma)\left(\frac{\mathscr{M}}{a_\gamma}\right)_2 g(z)|_{k+\frac{1}{2}}U_{\mathfrak{m}}.\end{aligned}$$

Since, $\gamma\in\Gamma_{4\mathfrak{n}}$ was arbitrary, we can say that the action of $U_{\mathfrak{m}}$ operator on $M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}},\psi)$ adds a character $\left(\frac{\mathscr{M}}{\bullet}\right)_2$ to the space. ☐

Let $g(z)\in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}},\psi)$. Suppose $\mathfrak{l}\subset\mathscr{O}_K$ is a prime ideal such that $\mathfrak{l}\nmid\mathfrak{n}$. Then $g(z)|_{k+\frac{1}{2}}U_{\mathfrak{l}}$ can have at most level $4\mathfrak{n}\mathfrak{l}$. This can be observed by viewing $g(z)\in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}},\psi)$ as $g(z)|_{k+\frac{1}{2}}V_{(1)}\in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}\mathfrak{l}},\psi)$. Using proposition 7.3.3, we get $\left(g(z)|_{k+\frac{1}{2}}V_{(1)}\right)|_{k+\frac{1}{2}}U_{\mathfrak{l}}\in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}\mathfrak{l}},\Psi)$ where $\Psi(\bullet)=\psi(\bullet)\left(\frac{\mathscr{L}}{\bullet}\right)_2$.

Again, let $g(z)\in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}},\psi)$. Suppose $\mathfrak{l}\subset\mathscr{O}_K$ is a prime ideal such that $\mathfrak{l}\nmid\mathfrak{n}$. Then $g(z)|_{k+\frac{1}{2}}U_{\mathfrak{l}^2}$ can have at most level $4\mathfrak{n}\mathfrak{l}$. This can again be observed by viewing $g(z)\in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}},\psi)$ as $g(z)|_{k+\frac{1}{2}}V_{(1)}\in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}\mathfrak{l}},\psi)$. Using proposition 7.2.3, we have $\left(g(z)|_{k+\frac{1}{2}}V_{(1)}\right)|_{k+\frac{1}{2}}U_{\mathfrak{l}^2}\in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}\mathfrak{l}},\psi)$.

Remark 7.3.4. We now make two remarks that will be useful later.

1. If prime ideal $\mathfrak{l}\subset\mathscr{O}_K$ such that $\mathfrak{l}\nmid\mathfrak{n}$ and $g(z)\in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}},\psi)$, then $g(z)|_{k+\frac{1}{2}}U_{\mathfrak{l}}$ can have at most level $4\mathfrak{n}\mathfrak{l}$ and the character of $g(z)|_{k+\frac{1}{2}}U_{\mathfrak{l}}$ changes to Ψ where $\Psi(\bullet)=\psi(\bullet)\left(\frac{\mathscr{L}}{\bullet}\right)$.
2. If prime ideal $\mathfrak{l}\subset\mathscr{O}_K$ such that $\mathfrak{l}\nmid\mathfrak{n}$ and $g(z)\in M_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}},\psi)$, then $g(z)|_{k+\frac{1}{2}}U_{\mathfrak{l}^2}$ can have at most level $4\mathfrak{n}\mathfrak{l}$ and the character of $g(z)|_{k+\frac{1}{2}}U_{\mathfrak{l}^2}$ remains same as ψ.

7.4 Generalisation of *Shimura's* correspondence to Hilbert modular forms

In section 2.4, we introduced *Shimura's correspondence* which is a Hecke-Linear map between half-integer weight modular forms and integer weight modular form. In this section, we will state the generalisation of the same results to Hilbert modular forms which is again work of *Shimura* and follows from [Shi87, Theorem 6.1 and Theorem 6.2].

Theorem 7.4.1 (*Shimura's* correspondence for Hilbert modular forms).
Let $k \in \mathbb{Z}_{>0}$ and let $\mathfrak{n} \subset \mathcal{O}_K$ be an integral ideal. Let ψ be a finite order Hecke character modulo $4\mathfrak{n}$ which in our case is quadratic in the sense of being given in terms of generalised quadratic residue symbol, see section 7.1.1. Suppose we are given a non-zero half integer weight Hilbert eigenform $g(z) \in S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}, \psi)$ for all Hecke operators $T_{\mathfrak{p}^2}$ over all prime ideals $\mathfrak{p} \subset \mathcal{O}_K$ with corresponding eigenvalues $\lambda_{T_{\mathfrak{p}^2}}(g) \in \mathbb{C}$, that is,

$$g(z)_{k+\frac{1}{2}} T_{\mathfrak{p}^2} = \lambda_{T_{\mathfrak{p}^2}}(g) g(z) \quad \textit{for all prime ideals } \mathfrak{p}.$$

Then there exists an integer weight Hilbert modular newform $f(z) \in M_{2k}(\Gamma_{2\mathfrak{n}}, \psi^2)$ for all Hecke operators $T_{\mathfrak{p}}$ over all prime ideals $\mathfrak{p} \subset \mathcal{O}_K$ with corresponding eigenvalues $\lambda_{T_{\mathfrak{p}}}(f) \in \mathbb{C}$, that is,

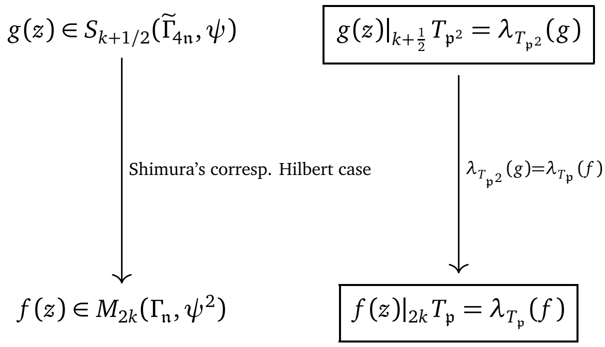
$$f(z)|_{2k} T_{\mathfrak{p}} = \lambda_{T_{\mathfrak{p}}}(f) f(z) \quad \textit{for all prime ideals } \mathfrak{p}.$$

such that

$$\lambda_{T_{\mathfrak{p}^2}}(g) = \lambda_{T_{\mathfrak{p}}}(f).$$

Note 7.4.2. Since ψ is fixed as a quadratic symbol for us, we conclude ψ^2 is trivial character.

Figure 7.4.3.



7.5 Generalisation of *Kohnen's* Isomorphism to Hilbert modular forms

Let K be a real quadratic real field with narrow class number 1 and let $\mathfrak{n} \subset \mathcal{O}_K$ be a square-free integral ideal of odd norm. Let $K_{\mathbb{A}}$ denote the adelization of K. The strong approximation theorem enables us to view Hilbert modular forms of integer weight as automorphic forms on $PGL_2(K_{\mathbb{A}})$. Similarly, let $Mp_2(K_{\mathbb{A}})$ be the metaplectic double cover of $PGL_2(K_{\mathbb{A}})$. Again, Hilbert modular forms of half-integer weight are automorphic forms on $Mp_2(K_{\mathbb{A}})$. We will now use the passage of automorphic representations associated to automorphic forms to connect spaces of integer weight Hilbert modular forms with half-integer weight Hilbert modular forms which will then enable us to generalise *Kohnen's isomorphism* to Hilbert case. We begin by providing an outline of this approach, see figure 7.5.1 and we then closely follow [Su18] for definitions and details.

Let $A_{2k}^{\text{cusp}}(\mathfrak{n})$ be the space of automorphic cusp forms of weight $2k$ and level $\mathfrak{n}$ where $\mathfrak{n} \subset \mathcal{O}_K$ is an integral ideal. For a finite place v associated to prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$, define

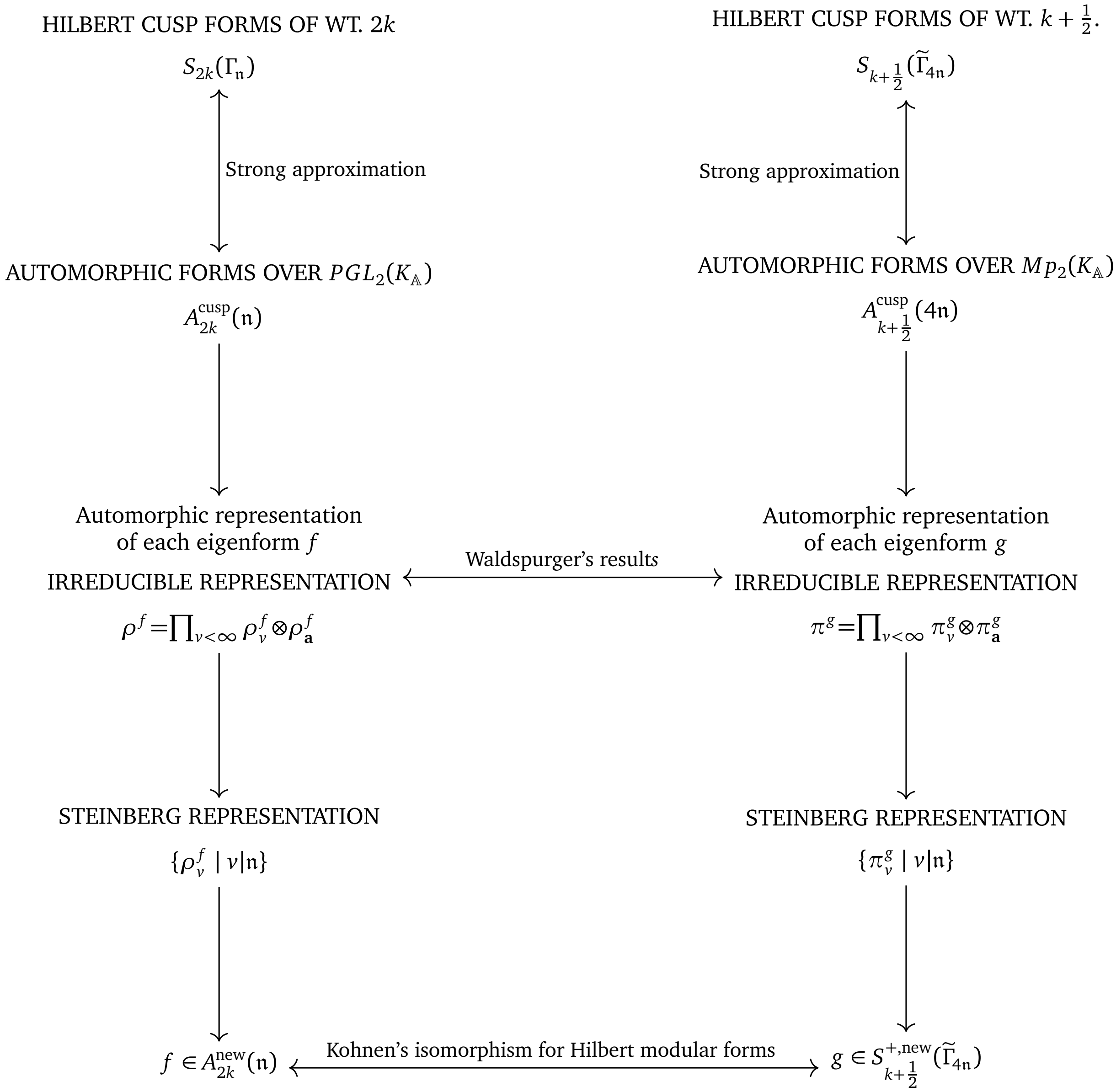
$$(\Gamma_{\mathfrak{n}})_v := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL_2(K_v) \middle| \, a, d \in \mathcal{O}_{K_v}, \; b \in 2\mathfrak{d}_v^{-1} \text{ and } c \in 2^{-1}\mathfrak{d}_v \mathfrak{n}_v \text{ such that } ad - bc = 1 \right\}.$$

Each eigenform $f \in A_{2k}^{\text{cusp}}(\mathfrak{n})$ has an associated irreducible representation ρ^f.

$$\text{Eigenform } f \in A_{2k}^{\text{cusp}}(\mathfrak{n}) \longleftrightarrow \rho^f = \prod_{v<\infty} \rho_v^f \otimes \rho_{\mathbf{a}}^f.$$

Here ρ_v^f is the local representation of $PGL_2(K_v)$ at a finite place v and $\rho_{\mathbf{a}}^f$ is the product of irreducible representation at the archimedean places.

Figure 7.5.1.



We can have the following three cases.

Case 1: $v \mid \mathfrak{n}$ **and** ρ_v^f **is steinberg.**

In this case, $(\Gamma_{\mathfrak{n}})_v$ is *Iwahori*[1] at the finite places $v \mid \mathfrak{n}$ and we have a unique fixed vector under right action of $(\Gamma_{\mathfrak{n}})_v$. The associated representation ρ_v^f is referred to as *steinberg representation*. We then say that $f(z)$ lies in the *new space*.

Case 2: $v \mid \mathfrak{n}$ and ρ_v^f is spherical.
In this case, $(\Gamma_{\mathfrak{n}})_v$ is *Iwahori* at the finite places $v \mid \mathfrak{n}$ and we have a fixed vector under right action of $(\Gamma_{\mathfrak{n}})_v$ but it need not be unique. The associated representation ρ_v^f is referred to as *spherical representation*. We then say that $f(z)$ lies in the *old space*.

Case 3: $v \nmid \mathfrak{n}$ and ρ_v^f is spherical.
In this case, $(\Gamma_{\mathfrak{n}})_v$ is maximal compact at the finite places $v \nmid \mathfrak{n}$ and we have a unique fixed vector under right action of $(\Gamma_{\mathfrak{n}})_v$. This is called the *spherical vector*.

We are interested in **Case** 1. We will denote the new space of automorphic forms by $A_{2k}^{new}(\mathfrak{n})$.

$$\text{Eigenform } f \in A_{2k}^{\text{new}}(\mathfrak{n}) \longleftrightarrow \rho_v^f \text{ is steinberg at the finite places } v \mid \mathfrak{n}.$$

Before we move to the right hand side of figure 7.5.1 about Hilbert modular forms of half-integer weight, we go over a few definitions.

Let $Mp_2(K_{\mathbb{A}})$ be the metaplectic double cover of $GL_2^+(K_{\mathbb{A}})$. Then half-integer Hilbert cusp forms in $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}})$ are essentially automorphic forms over $Mp_2(K_{\mathbb{A}})$.

Definition 7.5.2 (Generalised *Kohnen* plus space). *We define the Kohnen plus space $S^+_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}})$ as the subspace of $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}})$ that contains half-integer weight Hilbert cusp forms whose ξ^{th} coefficient vanishes unless $(-1)^k\xi \equiv \square \ mod\ 4$.*

$$S^+_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}) = \left\{ g \in S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}) \;\middle|\; g(z) = \sum_{\substack{\xi \in \mathcal{O}_K^+ \\ (-1)^k\xi \equiv \square \ mod\ 4}} b_\xi e^{2\pi i \, Tr\left(\frac{\xi}{2} z\right)} \right\}.$$

From [Su18, pg.170], the generalised *Kohnen* plus space decomposes into old and new subspaces as in the classical case of half-integer weight modular forms, that is,

$$S^+_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}) = S^{\text{new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}) \oplus S^{\text{old}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}) \tag{7.7}$$

where the definitions old subspace and the new subspace of the generalised *Kohnen* plus space are defined by *Ren-He Su* in [Su18] as follows.

Definition 7.5.3 (Old subspace of the generalised *Kohnen* plus space). *The old space of the generalised Kohnen plus space is defined as*

$$S^{old}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}) := \sum_{\substack{\mathfrak{m} \mid \mathfrak{n} \\ \mathfrak{m} \neq \mathfrak{n}}} \left(S^+_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{m}}) + S^+_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{m}})|_{k+\frac{1}{2}} U_{(\mathfrak{n}\mathfrak{m}^{-1})^2} \right).$$

Definition 7.5.4 (New subspace of generalised *Kohnen* plus space).
The new subspace of generalised Kohnen plus space is defined as the orthogonal complement of the old space defined in 7.5.3.

$$S^{new}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}) := \left(S^{old}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}) \right)^{\perp}$$

[1] *Iwahori* subgroup:= It is a subgroup of a reductive algebraic subgroup over a non-archimedean local field that is analogous to the *Borel* subgroup of an algebraic group. For $GL_n(\mathbb{R})$, the *Borel* subgroup is given by the set of upper triangular matrices.

Using definition 7.5.3 and decomposition 7.7, we have the following decomposition for the generalised *Kohnen* plus space

$$S^{+}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}) = \oplus_{\mathfrak{m}|\mathfrak{n}} \oplus_{\mathfrak{b}|\mathfrak{m}^{-1}\mathfrak{n}} S^{\text{new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{m}})|_{k+\frac{1}{2}} U_{\mathfrak{b}^2}. \tag{7.8}$$

where $\mathfrak{b}$ runs over integral ideal in $\mathcal{O}_K$ that divide $\mathfrak{m}^{-1}\mathfrak{n}$.

We now state a result by *Hiraga and Ikeda* [HI13] about the new subspace of generalised *Kohnen* plus space being generated by a basis of eigenforms. The version of our statement appears in *Re-He Su's* short article [Su, Theorem 3(ii)].

Theorem 7.5.5. *The space generalised Kohnen plus space $S^{+}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}})$ has a basis consisting of Hilbert Hecke eigenforms of half-integer weight over $\mathbb{C}$.*

Next, we want to introduce the concept of newforms for half-integer weight Hilbert modular forms. We note that each Hecke eigenform $g \in S^{+}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}})$ has an associated irreducible representation π^g of $Mp_2(K_{\mathbb{A}})$ which we can write as a product of local representations.

$$\text{Eigenform } g \in S^{+}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}) \longleftrightarrow \pi^g = \prod_{v<\infty} \pi^g_v \otimes \pi^g_{\mathbf{a}}$$

where π^g_v is a local irreducible representations of $Mp_2(K_v)$ at a finite place v and $\pi^g_{\mathbf{a}}$ is the irreducible representation at the archimedean places.

Definition 7.5.6 (*Kohnen* newform)**.** *A Hecke eigenform $g \in S^{+}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}})$ is called a Kohnen newform if for any finite place $v|\mathfrak{n}$, the local irreducible representation π^g_v of $Mp_2(K_v)$ is equivalent to a steinberg representation.*

For further details about the representation theoretic method of defining $S^{new}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}})$, refer to [Su18].

The main link between irreducible representations π_g of $Mp_2(K_{\mathbb{A}})$ with irreducible cuspidal automorphic representation ρ_f of $PGL_2(K_{\mathbb{A}})$ is given by the *Waldspurger's* results in [GL18].

$$\pi_g \text{ of } Mp_2(K_{\mathbb{A}}) \longleftrightarrow \rho_f \text{ of } PGL_2(K_{\mathbb{A}}).$$

We can now state our main result by *Ren-He, Su* [Su18, Theorem 1.3].

Theorem 7.5.7 (Generalisation of *Kohnen's* isomorphism to Hilbert modular forms)**.** *There exists a Hecke isomorphism between the spaces $S^{new}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}})$ and $A^{new}_{2k}(\mathfrak{n})$.*
For level $\mathfrak{n} = (1)$, this theorem was initially proved by Hiraga and Ikeda, see [HI13].

CHAPTER 8

Hilbert Eisenstein series and related congruences

Let K be a totally real quadratic field with narrow class number 1 and let $\mathcal{O}_K$ be its ring of integers. Given discriminant D of K, let χ_D be the associated quadratic character of K given by the Kronecker symbol $\left(\frac{D}{\bullet}\right)$. Let $\mathfrak{d}_K^{-1} \subset K$ and $\mathfrak{d}_K \subset \mathcal{O}_K$ be the inverse different and the different ideal respectively. Let $\delta \in \mathcal{O}_K$ be a totally positive generator of $\mathfrak{d}$. Let $\mathcal{O}_K^\star$ denote the group of units in K, that is $\mathcal{O}_K^\star = \{\xi \in \mathcal{O}_K \mid N(\xi) \in \{\pm 1\}\}$.

8.1 Dedekind zeta function and some formulae

We briefly overview the Dedekind zeta function and its related functional equation that will be essential to introduce *Hilbert Eisenstein series* in section 8.2.

Definition 8.1.1. *We define the Dedekind zeta function of K for complex numbers s with real part* $\Re(s) > 1$ *as*

$$\zeta_K(s) := \sum_{\mathfrak{n} \subset \mathcal{O}_K} N(\mathfrak{n})^{-s}.$$

Here the sum is taken over all non-zero integral ideals $\mathfrak{n} \subset \mathcal{O}_K$ *and* $N(\mathfrak{n}) = [\mathcal{O}_K : \mathfrak{n}]$ *denotes the absolute norm of* $\mathfrak{n}$.

We now state a few facts about the completed Dedekind zeta function and associated functional equation. For details, refer to [Neu99, Chapter VII.5].

The Dedekind zeta function $\zeta_K(s)$ has a meromorphic continuation to the full complex plane given by the completed zeta function $\Lambda_K(s)$,

$$\Lambda_K(s) = D^{s/2}\pi^{-s}\Gamma\left(\frac{s}{2}\right)^2 \zeta_K(s) \tag{8.1}$$

where $\Gamma(s)$ is the Gamma function. It satisfies the functional equation

$$\Lambda_K(s) = \Lambda_K(1-s). \tag{8.2}$$

Let $k \geq 2$ be an even integer. Using 8.1 and 8.2, we can write

$$D^{k/2}\pi^{-k}\Gamma\left(\frac{k}{2}\right)^2 \zeta_K(k) = D^{(1-k)/2}\pi^{-(1-k)}\Gamma\left(\frac{1-k}{2}\right)^2 \zeta_K(1-k)$$

or

$$\zeta_K(k) = D^{1/2-k}\pi^{2k-1}\left(\frac{\Gamma\left(\frac{1-k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2 \zeta_K(1-k) \tag{8.3}$$

We now try to further simplify the terms on the right hand side of equation 8.3. In order to do this, we give the factorisation of $\zeta_K(1-k)$.

Proposition 8.1.2. *Let K be a totally real quadratic field of narrow class number 1. Let D be the discriminant of K and let χ_D be the associated quadratic character of K given by the Kronecker symbol $\left(\frac{D}{\bullet}\right)$. Let $k \geq 2$ be an even integer. Then the Dedekind zeta function of K factors as*

$$\zeta_K(1-k) = \zeta(1-k)\, L\,(1-k, \chi_D)\,.$$

Here

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is the Riemann Zeta function defined over complex variable s for which $\Re(s) > 1$ and

$$L\,(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

is the Dirichlet L function defined over complex variable s for which $\Re(s) > 1$ and has Dirichlet character χ.

Proof. See [Was97, Theorem 4.3] and [LR10, Theorem 7]. □

We now relate $\zeta(1-k)$ with Bernoulli numbers and $L\,(1-k, \chi_D)$ with generalised Bernoulli numbers.

Proposition 8.1.3. *Let K be a totally real quadratic field of narrow class number 1. Let D be the discriminant of K and let χ_D be the associated quadratic character of K given by the Kronecker symbol $\left(\frac{D}{\bullet}\right)$. Let $k \geq 2$ be an even integer. Also, let B_k, B_{k,χ_D} denote the k^{th} Bernoulli and the k^{th} generalised Bernoulli numbers respectively. Then, we have*

$$\zeta(1-k) = -\frac{B_k}{k};$$
$$L\,(1-k, \chi_D) = -\frac{B_{k,\chi_D}}{k}.$$

Proof. See [Was97, Theorem 4.2] and [LR10, Theorem 3]. □

From propositions 8.1.2 and 8.1.3, we have

$$\zeta_K(1-k) = \frac{B_k B_{k,\chi_D}}{k^2} \in \mathbb{Q}. \tag{8.4}$$

Now, we simplify the other factor $\left(\frac{\Gamma\left(\frac{1-k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2$ in equation 8.3.

Since, $k \geq 2$ is an even integer, $k/2$ is also an integer.
We now quote a formula from [Old09, Section 43:4, pg. 437] satisfied by the Gamma function for any integer n, in particular for $n = k/2$,

$$\Gamma\left(\frac{1}{2} - n\right) = \frac{(-1)^n 2^n \sqrt{\pi}}{(2n-1)!!} \text{ for } n \in \mathbb{N} \cup \{0\} \tag{8.5}$$

where !! denotes *multiple factorial* and from [Old09, Section 2:13:4, pg. 25], its relation with normal factorial is

$$(2n+1)!! = \frac{(2n+1)!}{2^n n!}. \tag{8.6}$$

Also, it is well known that

$$\Gamma(n) = (n-1)! \text{ for } n \in \mathbb{N} \tag{8.7}$$

Using formulae 8.5 and 8.7, we can write

$$\begin{aligned}
\left(\frac{\Gamma\left(\frac{1-k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2 &= \left(\frac{\Gamma\left(\frac{1}{2}-n\right)}{\Gamma(n)}\right)^2 \\
&= \left(\frac{(-1)^n 2^n \sqrt{\pi}}{(2n-1)!!}\right)^2 \left(\frac{1}{(n-1)!}\right)^2 \\
&= \left(\frac{(-1)^n 2^n \sqrt{\pi} 2^{n-1}}{(2(n-1)+1)!!(n-1)!2^{n-1}}\right)^2.
\end{aligned}$$

Using identity 8.6 for $(n-1)$, we can write $(2(n-1)+1)!!(n-1)!2^{n-1} = (2(n-1)+1)!$. Then

$$\begin{aligned}
\left(\frac{\Gamma\left(\frac{1-k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2 &= \left(\frac{(-1)^n 2^{2n-1} \sqrt{\pi}}{(2(n-1)+1)!}\right)^2 \\
&= \left(\frac{(-1)^n 2^{2n-1} \sqrt{\pi}}{(2n-1)!}\right)^2 \\
&= \left(\frac{(-1)^{k/2} 2^{k-1} \sqrt{\pi}}{(k-1)!}\right)^2 \\
&= \frac{(-1)^k 2^{2k} \pi}{4(k-1)!^2} \\
&= \frac{2^{2k} \pi}{4(k-1)!^2}, \ k \text{ even.}
\end{aligned} \tag{8.8}$$

We use 8.4 and 8.8 to simplify equation 8.3.

$$\begin{aligned}
\zeta_K(k) &= D^{1/2-k} \pi^{2k-1} \frac{2^{2k}\pi}{4(k-1)!^2} \frac{B_k B_{k,\chi_D}}{k^2} \\
&= \frac{D^{1/2-k}(2\pi)^{2k} B_k B_{k,\chi_D}}{4(k-1)!^2 k^2}.
\end{aligned} \tag{8.9}$$

8.2 Hilbert Eisenstein Series

Definition 8.2.1 (Hilbert Eisenstein Series). *Let $k > 2$ be an even integer. We define the Hilbert Eisenstein Series $G_k(z)$ as*

$$G_k(z) := \sum_{(x,y)}{}' N(xz+y)^{-k} \tag{8.10}$$

where the summation $\sum'$ runs over representatives $(x,y) \in \left((2^{-1}\delta\mathcal{O}_K \times \mathcal{O}_K) - \{(0,0)\}\right)/\mathcal{O}_K^\star$.

Note 8.2.2. Note that the representatives (x,y) in Definition 8.2.1 run over $\left((2^{-1}\delta\mathcal{O}_K \times \mathcal{O}_K) - \{(0,0)\}\right)/\mathcal{O}_K^{\star}$ in contrast to $((\mathcal{O}_K \times \mathcal{O}_K) - \{(0,0)\})/\mathcal{O}_K^{\star}$ in [Gar90]. We have an additional term $2^{-1}\delta$ which is due to the choice our congruence subgroup of level 1 defined as $\Gamma[2\mathfrak{d}^{-1}, 2^{-1}\mathfrak{d}]$ in contrast to congruence group $SL_2(\mathcal{O}_K)$ as in [Gar90]. However, there are references which explicitly consider Eisenstein series in the form as in our definition 8.2.1 and even more broadly, see [vdG88, pg. 19].

Proposition 8.2.3. *Let $k > 2$ be an even integer. Then $G_k(z) \in M_k(\Gamma_{(1)})$.*

Proof. See [Gar90, Section 1.5]. ☐

Remark 8.2.4. We can also define Hilbert Eisenstein series for weight $k = 2$ by analytic continuation using *Hecke-Trick*. Hence, it turns out that Hilbert Eisenstein series of weight $k = 2$ is holomorphic, see [BHZ08, Remark 1.36, pg.121]. Thus, Hilbert Eisenstein series $G_2(z) \in M_2(\Gamma_{(1)})$.

Next, we give the Fourier expansion of $G_k(z)$ at infinity. We will work out some steps explicitly in the proof so that it's easy to follow the proof of Fourier expansion of Generalised Hilbert Eisenstein series in section 8.5.

Proposition 8.2.5. *Let $k \geq 2$ be an even integer. Let D be the discriminant of the number field K. Then $G_k(z)$ has the following Fourier expansion at infinity*

$$G_k(z) = \zeta_K(k) + \frac{(2\pi i)^{2k}}{(k-1)!^2} D^{\frac{1}{2}-k} \sum_{\xi \in \mathcal{O}_K^+} \sigma_{k-1}(\xi) e^{2\pi i \, Tr\left(\frac{\xi}{2} z\right)}.$$

where

$$\zeta_K(k) = \sum_{\mathfrak{n} \subset \mathcal{O}_K} N(\mathfrak{n})^{-k}$$

is the Dedekind zeta function of K at k and $\sigma_{k-1}(\xi)$ is the $(k-1)^{th}$ divisor sum of norm of all integral ideals dividing (ξ) given by

$$\sigma_{k-1}(\xi) = \sum_{\substack{\mathfrak{r}' \subseteq \mathcal{O}_K \\ \mathfrak{r}' | (\xi)}} N(\mathfrak{r}')^{k-1}.$$

Proof. Using Definition 8.10, we can write

$$G_k(z) = \sum_{(x,y)}{}' N(xz+y)^{-k}$$

where the summation $\sum'$ runs over representatives $(x,y) \in \left((2^{-1}\delta\mathcal{O}_K \times \mathcal{O}_K) - \{(0,0)\}\right)/\mathcal{O}_K^{\star}$

Now we split the sum further into two sums, one with $x = 0$ and other with $x \neq 0$.

$$\begin{aligned} G_k(z) &= \sum_{\substack{y \in \mathcal{O}_K^+ \\ x=0}} N(y)^{-k} + \sum_{x \in 2^{-1}\delta\mathcal{O}_K^+} \left(\sum_{y \in \mathcal{O}_K^+ \cup \{0\}} N(xz+y)^{-k} \right) \\ &= \zeta_k(k) + \sum_{x \in 2^{-1}\delta\mathcal{O}_K^+} \left(\sum_{y \in \mathcal{O}_K^+ \cup \{0\}} N(xz+y)^{-k} \right) \end{aligned} \tag{8.10}$$

where $\zeta_k(k)$ is the Dedekind zeta function, see definition 8.1.1.
We now use the formula [vdG88, pg. 19] obtained using Poisson summation formula in [SW71, pg. 252] that is given below,

$$\sum_{y \in \mathcal{O}_K^+ \cup \{0\}} N(xz+y)^{-k} = \frac{(2\pi i)^{2k}}{(k-1)!^2} D^{\frac{1}{2}-k} \sum_{\substack{\lambda \in \mathfrak{d}^{-1} \\ x\lambda \gg 0}} N(\lambda\delta)^{k-1} e^{2\pi i \, \mathrm{Tr}(x\lambda z)}.$$

Then the formula for $G_k(z)$ above in 8.10 simplifies to

$$\begin{aligned} G_k(z) &= \zeta_k(k) + \frac{(2\pi i)^{2k}}{(k-1)!^2} D^{\frac{1}{2}-k} \sum_{x\in 2^{-1}\delta\mathcal{O}_K^+} \sum_{\substack{\lambda\in\mathfrak{d}^{-1} \\ x\lambda\gg 0}} N(\lambda\delta)^{k-1} e^{2\pi i \operatorname{Tr}(x\lambda z)} \\ &= \zeta_k(k) + \frac{(2\pi i)^{2k}}{(k-1)!^2} D^{\frac{1}{2}-k} \sum_{x\in 2^{-1}\delta\mathcal{O}_K^+} \sum_{x\lambda\in 2^{-1}\mathcal{O}_K^+} N(x^{-1}x\lambda\delta)^{k-1} e^{2\pi i \operatorname{Tr}(x\lambda z)}. \end{aligned}$$

For simplicity, let $\nu = x\lambda$. Then

$$\begin{aligned} G_k(z) &= \zeta_k(k) + \frac{(2\pi i)^{2k}}{(k-1)!^2} D^{\frac{1}{2}-k} \sum_{x\in 2^{-1}\delta\mathcal{O}_K^+} \sum_{\nu\in 2^{-1}\mathcal{O}_K^+} N(x^{-1}\nu\delta)^{k-1} e^{2\pi i \operatorname{Tr}(\nu z)} \\ &= \zeta_k(k) + \frac{(2\pi i)^{2k}}{(k-1)!^2} D^{\frac{1}{2}-k} \sum_{\nu\in 2^{-1}\mathcal{O}_K^+} \sum_{2x\delta^{-1}\in\mathcal{O}_K^+} N((2x\delta^{-1})^{-1}2\nu)^{k-1} e^{2\pi i \operatorname{Tr}(\nu z)}. \end{aligned}$$

Again, for simplicity, put $r = 2x\delta^{-1}$. Then

$$G_k(z) = \zeta_k(k) + \frac{(2\pi i)^{2k}}{(k-1)!^2} D^{\frac{1}{2}-k} \sum_{\nu\in 2^{-1}\mathcal{O}_K^+} \sum_{r\in\mathcal{O}_K^+} N(2r^{-1}\nu)^{k-1} e^{2\pi i \operatorname{Tr}(\nu z)}.$$

Let $\xi = 2\nu$. Then

$$G_k(z) = \zeta_k(k) + \frac{(2\pi i)^{2k}}{(k-1)!^2} D^{\frac{1}{2}-k} \sum_{\xi\in\mathcal{O}_K^+} \sum_{r\in\mathcal{O}_K^+} N(\xi r^{-1})^{k-1} e^{2\pi i \operatorname{Tr}\left(\frac{\xi}{2}z\right)}.$$

Let $rr' = \xi$ where $r' \gg 0$ generates the integral ideal $\mathfrak{r}'$. Then $\mathfrak{r}' \mid (\xi)$. Hence, it follows,

$$\begin{aligned} G_k(z) &= \zeta_k(k) + \frac{(2\pi i)^{2k}}{(k-1)!^2} D^{\frac{1}{2}-k} \sum_{\xi\in\mathcal{O}_K^+} \left(\sum_{\substack{\mathfrak{r}'\subseteq\mathcal{O}_K^+ \\ \mathfrak{r}'\mid(\xi)}} N(\mathfrak{r}')^{k-1} \right) e^{2\pi i \operatorname{Tr}\left(\frac{\xi}{2}z\right)} \\ &= \zeta_k(k) + \frac{(2\pi i)^{2k}}{(k-1)!^2} D^{\frac{1}{2}-k} \sum_{\xi\in\mathcal{O}_K^+} \sigma_{k-1}(\xi) e^{2\pi i \operatorname{Tr}\left(\frac{\xi}{2}z\right)} \end{aligned}$$

where

$$\sigma_{k-1}(\xi) = \sum_{\substack{\mathfrak{r}'\subseteq\mathcal{O}_K^+ \\ \mathfrak{r}'\mid(\xi)}} N(\mathfrak{r}')^{k-1}$$

is the $(k-1)^{\text{th}}$ divisor sum of norm of all integral ideals dividing (ξ). □

Definition 8.2.6 (Normalised Hilbert Eisenstein Series)**.** *The normalised Hilbert Eisenstein series is defined as*

$$E_k(z) := \zeta_K(k)^{-1} G_k(z)$$

where $\zeta_K(s)$ is the Dedekind zeta function of K.

Then $E_k(z)$ has the following Fourier expansion at infinity

$$E_k(z) = 1 + C_k \sum_{\xi\in\mathcal{O}_K^+} \sigma_{k-1}(\xi) e^{2\pi i \operatorname{Tr}\left(\frac{\xi z}{2}\right)} \tag{8.11}$$

where

$$C_k = \frac{(2\pi)^{2k} D^{1/2-k}}{\zeta_K(k)(k-1)!^2}.$$

We next want to show that $E_k(z)$ has rational coefficients. The coefficients of $E_k(z)$ involve two terms, C_k and $\sigma_{k-1}(\xi)$, defined in Theorem 8.12. Here $\sigma_{k-1}(\xi)$ is just a sum of norms of integral ideals and is thus rational by definition. So, we now show that the other term C_k is rational as well. For this, we will use formula obtained in equation 8.9 which implies

$$\zeta_K(k) = \frac{D^{1/2-k}(2\pi)^{2k}}{4(k-1)!^2} \cdot \frac{B_k B_{k,\chi_D}}{k^2}$$

where B_k and B_{k,χ_D} denote the k^{th} Bernoulli and generalised Bernoulli number respectively.
Then the value of the constant in our coefficient C_k in the Fourier expansion of the normalised Hilbert Eisenstein series given in 8.11, simplifies to

$$\begin{aligned} C_k &= \frac{(2\pi)^{2k} D^{1/2-k}}{(k-1)!^2} \cdot \frac{1}{\zeta_K(k)} \\ &= \frac{(2\pi)^{2k} D^{1/2-k}}{(k-1)^2} \cdot \frac{4(k-1)!^2 k^2}{D^{1/2-k}(2\pi)^{2k} B_k B_{k,\chi_D}} \\ &= \frac{(2k)^2}{B_k B_{k,\chi_D}}. \end{aligned}$$

Hence, we can say that the Fourier expansion of $E_k(z)$ is given by

$$E_k(z) = 1 + \frac{(2k)^2}{B_k B_{k,\chi_D}} \sum_{\xi \in \mathcal{O}_K^+} \sigma_{k-1}(\xi) e^{2\pi i \operatorname{Tr}\left(\frac{\xi z}{2}\right)} \tag{8.12}$$

where the Fourier coefficients of $E_k(z)$ lie in $\mathbb{Q}$.

8.3 Hilbert Eisenstein series modulo p

Let $k \geq 2$ be an even integer. In the previous section, we obtained the following Fourier expansion of normalised Hilbert Eisenstein series $E_k(z) \in M_k(\Gamma_{(1)})$ at level 1,

$$E_k(z) = 1 - \frac{(2k)^2}{B_k B_{k,\chi_D}} \sum_{\xi \in \mathcal{O}_K^+} \sigma_{k-1}(\xi) e^{2\pi i \operatorname{Tr}\left(\frac{\xi z}{2}\right)}.$$

Let p be an odd rational prime that is unramified in K. We now define a series $\mathcal{E}_p(z)$ as follows,

$$\mathcal{E}_p(z) := E_{p-1}(z) - N(p\mathcal{O}_K)^{\frac{p-1}{2}} E_{p-1}(pz).$$

Since $E_{p-1}(z),\ E_{p-1}(pz) \in M_{p-1}(\Gamma_{(p)})$, this implies $\mathcal{E}_p(z) \in M_{p-1}(\Gamma_{(p)})$.

We will next try to prove a congruence modulo p satisfies by $\mathcal{E}_p(z)$ that will be an essential lemma required to prove the main theorem in chapter 10.

Lemma 8.3.1. *Let K be a totally real quadratic field of narrow class number 1. Let D be the discriminant of K and let χ_D be the associated quadratic character of K given by the Kronecker symbol $\left(\frac{D}{\bullet}\right)$. Let p be an odd rational prime that is unramified in K. Also, let B_{p-1}, B_{p-1,χ_D} denote the $(p-1)^{th}$ Bernoulli and the $(p-1)^{th}$ generalised Bernoulli number respectively and suppose that $p \nmid B_{p-1,\chi_D}$. Then we have*

$$\mathcal{E}_p(z) \equiv 1 \ mod\ p.$$

Proof. From equation 8.12, the Fourier expansion of $E_{p-1}(z)$ is

$$E_{p-1}(z) = 1 + 4(p-1)^2 B_{p-1}^{-1} \frac{1}{B_{p-1,\chi_D}} \sum_{\xi \in \mathcal{O}_K^+} \sigma_{p-2}(\xi) e^{2\pi i \operatorname{Tr}\left(\frac{\xi z}{2}\right)}.$$

Then using von-Staudt Clausen theorem 3.2.2 and the assumption $p \nmid B_{p-1,\ \chi_D}$ considered in our hypothesis, we get that the Fourier coefficients of $E_{p-1}(z)$ lie in $\mathbb{Z}_{(p)}$. Then it's clear from the definition of $\mathscr{E}_p(z)$ that its Fourier coefficients lie in $\mathbb{Z}_{(p)}$. Further combining these facts with Fourier expansion of $E_{p-1}(z)$ obtained, we get the congruence

$$E_{p-1}(z) \equiv 1 \bmod p$$

and

$$E_{p-1}(pz) \equiv 1 \bmod p.$$

It follows

$$\begin{aligned} \mathscr{E}_p(z) &= E_{p-1}(z) - N(p\mathcal{O}_K)^{\frac{p-1}{2}} E_{p-1}(pz) \\ &= E_{p-1}(z) - p^{2\frac{p-1}{2}} E_{p-1}(pz) \bmod p \\ &\equiv E_{p-1}(z) \bmod p \\ &\equiv 1 \bmod p. \end{aligned}$$

□

8.4 Hecke L-function and some formulae

We briefly overview the Hecke L-function and its related functional equation that will be essential to introduce *Generalised Hilbert Eisenstein series* in section 8.5.

Let p be a rational prime such that is unramified in K. Let ψ be a fixed Hecke character of K defined as

$$\psi := \chi_p \circ N_{K/\mathbb{Q}}. \tag{8.13}$$

Here $\chi_p = \left(\frac{\bullet}{p}\right)$ denotes the Kronecker symbol and $N_{K/\mathbb{Q}}$ denotes the absolute norm.

Definition 8.4.1. *We define the Hecke L-function of K for complex numbers s with real part $\Re(s) > 1$ and Hecke character ψ as*

$$L_K(s,\psi) = \sum_{\mathfrak{n} \subset \mathcal{O}_K} \psi(\mathfrak{n}) N(\mathfrak{n})^{-s}.$$

Here the sum is taken over all non-zero integral ideals $\mathfrak{n} \subset \mathcal{O}_K$ and $N(\mathfrak{n}) = [\mathcal{O}_K : \mathfrak{n}]$ denotes the absolute norm of $\mathfrak{n}$.

We now state a few facts about the completed Hecke L-function and associated functional equation that has been taken from [Zam16, Section 2].

The Hecke L-function admits an analytic continuation on the full complex plane and is given by the completed Hecke L-function $\Lambda_K(s,\psi)$,

$$\Lambda_K(s,\psi) = (D\, N(p\mathcal{O}_K))^{\frac{s}{2}} \left(\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)\right)^{r_1} L_K(s,\psi) \tag{8.14}$$

where Γ is the Gamma function and r_1 denotes the number of real embeddings of K.
It satisfies the following functional equation

$$\Lambda_K(s,\psi) = w(\psi)\Lambda_K(1-s,\psi^{-1}). \tag{8.15}$$

Here $w(\psi) \in \mathbb{C}$ is the root number and $|w(\psi)| = 1$. It satisfies the following relation

$$w(\psi) = \tau(\psi)N(p\mathcal{O}_K)^{\frac{1}{2}} \tag{8.16}$$

where

$$\tau(\psi) = \sum_{j\in(\mathcal{O}_K/p\mathcal{O}_K)^*} \psi(j)e^{2\pi i\ \mathrm{Tr}\left(\frac{j}{p}\delta^{-1}\right)}$$

is a Gauss Sum.

Let $k \in \mathbb{Z}_{>0}$. Using equation 8.15, we can write

$$\Lambda_K(k,\psi^{-1}) = w(\psi^{-1})\Lambda_K(1-k,\psi).$$

Then from 8.14, it follows

$$(D\,N(p\mathcal{O}_K))^{\frac{k}{2}}\left(\pi^{-\frac{k}{2}}\Gamma\left(\tfrac{k}{2}\right)\right)^{r_1} L_K(k,\psi^{-1}) = w(\psi^{-1})(D\,N(p\mathcal{O}_K))^{\frac{1-k}{2}}\left(\pi^{-\frac{1-k}{2}}\Gamma\left(\tfrac{1-k}{2}\right)\right)^{r_1} L_K(1-k,\psi).$$

This simplifies to

$$L_K(k,\psi^{-1}) = w(\psi^{-1})\,(D\,N(p\mathcal{O}_K))^{\frac{1}{2}-k}\left(\pi^{k-\frac{1}{2}}\frac{\Gamma\left(1-\frac{k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2 L_K(1-k,\psi)$$

Using relation 8.16, we get

$$L_K(k,\psi^{-1}) = \left(N(p\mathcal{O}_K)^{-\frac{1}{2}}\tau(\psi^{-1})\right)D^{\frac{1}{2}-k}N(p\mathcal{O}_K)^{\frac{1}{2}-k}\pi^{2k-1}\left(\frac{\Gamma\left(1-\frac{k}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}\right)^2 L_K(1-k,\psi).$$

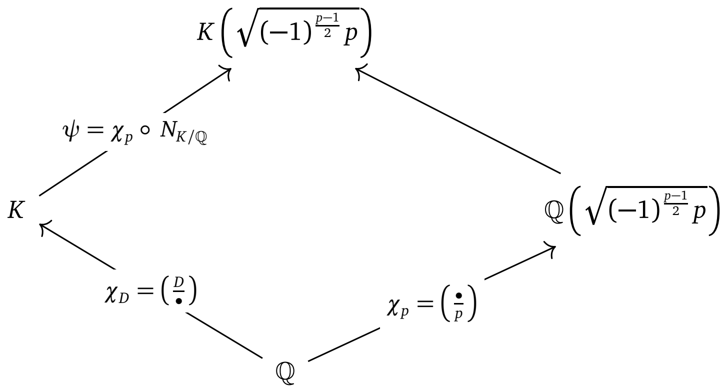
We now use the formula 8.8 to get

$$L_K(k,\psi^{-1}) = \tau(\psi^{-1})D^{\frac{1}{2}-k}N(p\mathcal{O}_K)^{-k}\frac{(2\pi)^{2k}}{2^2(k-1)!^2}L_K(1-k,\psi). \tag{8.17}$$

We now try to further simplify the term on the right hand side of equation 8.17. In order to this, we give factorisation of $L_K(1-k,\psi)$.

We have the following bi-quadratic diagram of field extensions with their respective characters.

Figure 8.4.2.



Now the bi-quadratic extension $K' = \mathbb{Q}\left(\sqrt{(-1)^{\frac{p-1}{2}}p}, \sqrt{D}\right)$ is an abelian extenstion of $\mathbb{Q}$. For abelian extensions over $\mathbb{Q}$, there is a factorisation of the Dedekind zeta function into a product of Dirichlet L-functions for character of the Galois group viewed as a quotient group of some $(\mathbb{Z}/m\mathbb{Z})^\star$ where the abelian extension is inside the m^{th} cyclotomic field (*Kronecker-Weber* theorem), see [Ove14, pg. 301] for further explanation.

Thus, for $K'/\mathbb{Q}$, we have

$$\zeta_{K'}(s) = \zeta_{\mathbb{Q}}(s) \times L\left(s, \chi_p\right) \times L\left(s, \chi_D\right) \times L\left(s, \chi_p\chi_D\right). \tag{8.18}$$

Also, we have in general

$$\begin{aligned}\zeta_{K'}(s) &= \zeta_K(s) \times L_K\left(s, \chi_p \circ N_{K/\mathbb{Q}}\right)\\ &= \zeta_{\mathbb{Q}}(s) \times L\left(s, \chi_D\right) \times L_K\left(s, \chi_p \circ N_{K/\mathbb{Q}}\right).\end{aligned} \tag{8.19}$$

From 8.18 and 8.19, we get

$$L_K\left(1-k,\ \chi_p \circ N_{K/\mathbb{Q}}\right) = L\left(1-k, \chi_p\right) L\left(1-k, \chi_p\chi_D\right).$$

Using proposition 8.1.3, we get

$$L_K\left(1-k,\ \chi_p \circ N_{K/\mathbb{Q}}\right) = \frac{B_{k,\chi_p} B_{k,\chi_p\chi_D}}{k^2}. \tag{8.20}$$

8.5 Generalised Hilbert Eisenstein Series

We now introduce a new series $G_{k,\psi}(z)$ with the character ψ called the generalised Hilbert Eisenstein series

$$G_{k,\psi}(z) := \sum_{(x,y)}{}' \psi^{-1}(y) N(xpz + y)^{-k} \tag{8.21}$$

where the summation $\sum'$ runs over representatives $(x, y) \in \left(\left(2^{-1}\delta\mathcal{O}_K \times \mathcal{O}_K\right) - \{(0,0)\}\right)/\mathcal{O}_K^\star$ and p is a rational odd prime that is unramified in K.

Note 8.5.1. The character $\psi = \chi_p \circ N_{K/\mathbb{Q}}$ remains the same as fixed in the previous section.

We next want to show that the series $G_{k,\psi}(z)$ is a Hilbert modular form of weight k, level (p) and character ψ. Before that, we state a lemma that gives equivalence of two characters [Lem00, Proposition 4.2(iii)].

Lemma 8.5.2. *Let $K/\mathbb{Q}$ be a totally real quadratic field of narrow class number* 1 *with ring of integers $\mathcal{O}_K$. Let p be an odd rational prime and let $\alpha \in \mathcal{O}_K$. Then we have*

$$\left(\frac{\alpha}{p\mathcal{O}_K}\right)_{2,K} = \left(\frac{N(\alpha)}{p}\right)_{\mathbb{Q}}$$

where $\left(\frac{\bullet}{p\mathcal{O}_K}\right)_{2,K}$ denotes the power residue symbol defined in 7.1.2 while $\left(\frac{N(\bullet)}{p}\right)_{\mathbb{Q}}$ denotes the normal Kronecker symbol with $N(\alpha)$ being the $\mathrm{Norm}_{K/\mathbb{Q}}$ of α.

Remark 8.5.3. Now we can write $\psi = \left(\frac{N(\bullet)}{p}\right)$ also as $\psi = \left(\frac{\bullet}{p\mathcal{O}_K}\right)_{2,K}$. For simplicity, we will often denote $\left(\frac{\bullet}{p\mathcal{O}_K}\right)_{2,K}$ by $\left(\frac{\bullet}{p}\right)_2$.

Proposition 8.5.4. $G_{k,\psi}(z) \in M_k(\Gamma_{(p)}, \psi)$.

Proof. In order to prove $G_{k,\psi}(z)$ a Hilbert modular form in $M_k(\Gamma_{(p)},\psi)$, we need to show it satisfies the following two conditions.

Condition 1: *The series $G_{k,\psi}(z)$ is invariant under the action of $\Gamma_{(p)}$.*
Let $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \Gamma_{(p)}$. This means $a_\gamma, d_\gamma \in \mathscr{O}_K$, $b_\gamma \in (2\delta^{-1})$ and $c_\gamma \in (2^{-1}\delta p)$.
Then

$$\begin{aligned}
G_{k,\psi}(\gamma z) &= {\sum_{(x,y)}}' \psi^{-1}(y) N\left(xp\left(\frac{a_\gamma z + b_\gamma}{c_\gamma z + d_\gamma}\right) + y\right)^{-k} \\
&= {\sum_{(x,y)}}' \psi^{-1}(y) N\left(xp(a_\gamma z + b_\gamma) + y(c_\gamma z + d_\gamma)\right)^{-k} N(c_\gamma z + d_\gamma)^k \\
&= N(c_\gamma z + d_\gamma)^k {\sum_{(x,y)}}' \psi^{-1}(y) N\left((xp, y)\begin{pmatrix} a_\gamma z + b_\gamma \\ c_\gamma z + d_\gamma \end{pmatrix}\right)^{-k} \\
&= N(c_\gamma z + d_\gamma)^k {\sum_{(x,y)}}' \psi^{-1}(y) N\left((xp, y)\begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix}\right)^{-k} \\
&= N(c_\gamma z + d_\gamma)^k {\sum_{(x,y)}}' \psi^{-1}(y) N\left((xpa_\gamma + yc_\gamma, xpb_\gamma + yd_\gamma)\begin{pmatrix} z \\ 1 \end{pmatrix}\right)^{-k}. \qquad (8.22)
\end{aligned}$$

Let $x' = \frac{1}{p}(xpa_\gamma + yc_\gamma)$ and $y' = xpb_\gamma + yd_\gamma$. Then $(x', y') \in \left((2^{-1}\delta\mathscr{O}_K \times \mathscr{O}_K) - \{(0,0)\}\right)/\mathscr{O}_K^\star$.

(8.23)

We can now write $\psi^{-1}(y') = \left(\frac{xpb_\gamma + yd_\gamma}{p}\right)_2^{-1} = \left(\frac{yd_\gamma}{p}\right)_2^{-1} = \left(\frac{y}{p}\right)_2^{-1}\left(\frac{d_\gamma}{p}\right)_2^{-1}$. In other words,

$$\left(\frac{d_\gamma}{p}\right)_2 \psi^{-1}(y') = \psi^{-1}(y). \qquad (8.24)$$

Also, we know that $N(a_\gamma d_\gamma - c_\gamma b_\gamma) = 1$, that is, $\left(\frac{a_\gamma d_\gamma - c_\gamma b_\gamma}{p}\right)_2 = 1$. This implies $\left(\frac{a_\gamma d_\gamma}{p}\right)_2 = 1$ or

$$\left(\frac{a_\gamma}{p}\right)_2 = \left(\frac{d_\gamma}{p}\right)_2. \qquad (8.25)$$

Using 8.24 and 8.25 can now further simplify 8.22 and write

$$\begin{aligned}
G_{k,\psi}(\gamma z) &= N(c_\gamma z + d_\gamma)^k {\sum_{(x',y')}}' \psi(a_\gamma)\psi^{-1}(y') N\left((x'p, y')\begin{pmatrix} z \\ 1 \end{pmatrix}\right)^{-k} \\
&= \psi(a_\gamma) N(c_\gamma z + d_\gamma)^k {\sum_{(x',y')}}' \psi^{-1}(y') N\left(x'pz + y'\right)^{-k} \\
&= \psi(a_\gamma) N(c_\gamma z + d_\gamma)^k G_{k,\psi}(z).
\end{aligned}$$

Condition 2: *The series $G_{k,\psi}(z)$ is uniformly convergent for z in compact subsets of $\mathscr{H}^2$ for all $k \in \mathbb{Z}_{>0}$.*

Since ψ is non-trivial in our case, the series $G_{k,\psi}(z)$ is uniformly convergent for z in compact subsets of $\mathscr{H}^2$ for all $k \in \mathbb{Z}_{>0}$. This follows from [Gar90, Section 4.7]. The only case it wouldn't converge is when ψ is trivial and $K = \mathbb{Q}$ with $k = 2$.

Thus, we conclude that $G_{k,\psi}(z) \in M_k(\Gamma_{(p)}, \psi)$, see [Gar90, Section 1.5]. □

Next, we give the Fourier expansion of $G_{k,\psi}(z)$ at infinity. In order to do this, we will need the following lemma.

Lemma 8.5.5. *Let $k \in \mathbb{Z}_{>0}$ and let p be an odd rational prime. Let $\mathfrak{d}^{-1}$ be the inverse different ideal of K and let δ^{-1} be its generator. Let $\psi = \chi_p \circ N_{K/\mathbb{Q}}$ be the Hecke character defined in* 8.13. *Then for $\lambda \in \mathfrak{d}^{-1}$, we have*

$$\sum_{j \in (\mathcal{O}_K/p\mathcal{O}_K)^\star} \psi^{-1}(j) e^{2\pi i\, Tr\left(\lambda \frac{j}{p}\right)} = \psi(\lambda\delta)\tau(\psi^{-1})$$

where

$$\tau(\psi^{-1}) = \sum_{j \in (\mathcal{O}_K/p\mathcal{O}_K)^\star} \psi^{-1}(j) e^{2\pi i\, Tr\left(\frac{j}{p}\delta^{-1}\right)}$$

is a Gauss sum.

Proof.

$$\sum_{j \in (\mathcal{O}_K/p\mathcal{O}_K)^\star} \psi^{-1}(j) e^{2\pi i\, \mathrm{Tr}\left(\lambda \frac{j}{p}\right)} = \sum_{j \in (\mathcal{O}_K/p\mathcal{O}_K)^\star} \left(\frac{j}{p}\right)_2^{-1} e^{2\pi i\, \mathrm{Tr}\left(\frac{\lambda\delta j}{p}\delta^{-1}\right)}$$

We now consider two cases, $\lambda\delta \notin (p)$ and $\lambda\delta \in (p)$.

Now if $\lambda\delta \notin (p)$, then $\lambda\delta j \in (\mathcal{O}_K/p\mathcal{O}_K)^\star$. Thus, $j \to \lambda\delta j$ and we have

$$\begin{aligned}
\sum_{j \in (\mathcal{O}_K/p\mathcal{O}_K)^\star} \left(\frac{j}{p}\right)_2^{-1} e^{2\pi i\, \mathrm{Tr}\left(\frac{\lambda\delta j}{p}\delta^{-1}\right)} &= \sum_{\lambda\delta j \in (\mathcal{O}_K/p\mathcal{O}_K)^\star} \left(\frac{\lambda\delta j}{p}\right)_2^{-1} \left(\frac{\lambda\delta}{p}\right)_2 e^{2\pi i\, \mathrm{Tr}\left(\frac{\lambda\delta j}{p}\delta^{-1}\right)} \\
&= \left(\frac{\lambda\delta}{p}\right)_2 \sum_{j \in (\mathcal{O}_K/p\mathcal{O}_K)^\star} \left(\frac{j}{p}\right)_2^{-1} e^{2\pi i\, \mathrm{Tr}\left(\frac{j}{p}\delta^{-1}\right)} \\
&= \psi(\lambda\delta)\tau(\psi^{-1}).
\end{aligned}$$

Now if $\lambda\delta \in (p)$, then $\lambda\delta j p^{-1} \in \mathcal{O}_K$. Then by definition of inverse different ideal, $\mathrm{Tr}(\lambda\delta j p^{-1}\delta^{-1}) \in \mathbb{Z}$.

This implies

$$\sum_{j \in (\mathcal{O}_K/p\mathcal{O}_K)^\star} \left(\frac{j}{p}\right)_2^{-1} e^{2\pi i\, \mathrm{Tr}\left(\frac{\lambda\delta j}{p}\delta^{-1}\right)} = \sum_{j \in (\mathcal{O}_K/p\mathcal{O}_K)^\star} \left(\frac{j}{p}\right)_2.$$

We claim that the sum $\sum_{j \in (\mathcal{O}_K/p\mathcal{O}_K)^\star} \left(\frac{j}{p}\right)_2 = 0$.

Let $S = \sum_{j \in (\mathcal{O}_K/p\mathcal{O}_K)^\star} \left(\frac{j}{p}\right)_2$. Since $\left(\frac{\bullet}{p}\right)_2$ finite order non-trivial character, there exists $a \in (\mathcal{O}_K/p\mathcal{O}_K)^\star$ such that $\left(\frac{a}{p}\right)_2 \neq 1$. Then we have

$$\begin{aligned}
S \cdot \left(\frac{a}{p}\right)_2 &= \sum_{j \in (\mathcal{O}_K/p\mathcal{O}_K)^\star} \left(\frac{j}{p}\right)_2 \left(\frac{a}{p}\right)_2 \\
&= \sum_{(ja) \in (\mathcal{O}_K/p\mathcal{O}_K)^\star} \left(\frac{ja}{p}\right)_2 \\
&= S.
\end{aligned}$$

It follows that $S\left(\left(\frac{a}{p}\right)_2 - 1\right) = 0$ or $S = 0$. In other words, we can say that $S = \psi(\lambda\delta)\tau(\psi^{-1})$ as $\left(\frac{\lambda\delta}{p}\right)_2 = 0$.

Thus, we have shown

$$\sum_{j\in(\mathcal{O}_K/p\mathcal{O}_K)^\star} \psi^{-1}(j)e^{2\pi i\,\mathrm{Tr}\left(\lambda\frac{j}{p}\right)} = \psi(\lambda\mathfrak{d})\tau(\psi^{-1}).$$

□

Proposition 8.5.6. *Let $k \in \mathbb{Z}_{>0}$ and let p be an odd rational prime that is unramified in K. Let $\psi = \chi_p \circ N_{K/\mathbb{Q}}$ be the Hecke character defined in 8.13. Then $G_{k,\psi}(z)$ has the following Fourier expansion at infinity,*

$$G_{k,\psi}(z) = L_K\left(k,\psi^{-1}\right) + N(p\mathcal{O}_K)^{-k}\frac{(2\pi i)^{2k}}{(k-1)!^2}D^{\frac{1}{2}-k}\tau(\psi^{-1})\sum_{\xi\in\mathcal{O}_K^+}\sigma_{k-1,\psi}(\xi)e^{2\pi i Tr\left(\frac{\xi}{2}z\right)}$$

where D is the discriminant of K,

$$L_K\left(k,\psi^{-1}\right) = \sum_{\mathfrak{n}\subset\mathcal{O}_K}\psi^{-1}(\mathfrak{n})N(\mathfrak{n})^{-k}$$

is the Hecke L-function defined in 8.4.1,

$$\sigma_{k-1,\psi}(\xi) = \sum_{\substack{\mathfrak{r}'\subset\mathcal{O}_K\\ \mathfrak{r}'|(\xi)}}\psi(r')N(r')^{k-1}$$

is the $(k-1)^{th}$ generalised divisor sum of twisted norm of integral ideals dividing (ξ) and

$$\tau(\psi^{-1}) = \sum_{j\in(\mathcal{O}_K/p\mathcal{O}_K)^\star}\psi^{-1}(j)e^{2\pi i\,Tr\left(\frac{j}{p}\delta^{-1}\right)}$$

is a Gauss sum.

Proof. Using definition 8.21, we can write

$$G_{k,\psi}(z) = {\sum_{(x,y)}}'\psi^{-1}(y)N(xpz+y)^{-k}$$

where the summation $\sum'$ runs over representatives $(x,y)\in\left(\left(2^{-1}\delta\mathcal{O}_K\times\mathcal{O}_K\right)-\{(0,0)\}\right)/\mathcal{O}_K^\star$.

Now we split the sum further into two sums, one with $x=0$ and other with $x\neq 0$.

$$\begin{aligned}G_{k,\psi}(z) &= \sum_{\substack{y\in\mathcal{O}_K^+\\ x=0}}\psi^{-1}(y)N(y)^{-k} + \sum_{x\in 2^{-1}\delta\mathcal{O}_K^+}\sum_{y\in\mathcal{O}_K^+\cup\{0\}}\psi^{-1}(y)N(xpz+y)^{-k}\\ &= L_K(k,\psi^{-1}) + \sum_{x\in 2^{-1}\delta\mathcal{O}_K^+}\sum_{y\in\mathcal{O}_K^+\cup\{0\}}\psi^{-1}(y)N(xpz+y)^{-k}\end{aligned}$$

where $L_K(k,\psi^{-1})$ is the Hecke L-function, see definition 8.4.1.

Now let $y = y'p + j$ where $j\in(\mathcal{O}_K/p\mathcal{O}_K)^\star$. Then

$$\begin{aligned}G_{k,\psi}(z) &= L_K(k,\psi^{-1}) + \sum_{x\in 2^{-1}\delta\mathcal{O}_K^+}\sum_{y'\in\mathcal{O}_K^+\cup\{0\}}\sum_{j\in(\mathcal{O}_K/p\mathcal{O}_K)^\star}\psi^{-1}(j)N(xpz+y'p+j)^{-k}\\ &= L_K(k,\psi^{-1}) + N(p\mathcal{O}_K)^{-k}\sum_{x\in 2^{-1}\delta\mathcal{O}_K^+}\sum_{y'\in\mathcal{O}_K^+\cup\{0\}}\sum_{j\in(\mathcal{O}_K/p\mathcal{O}_K)^\star}\psi^{-1}(j)N\left(\left(xz+\frac{j}{p}\right)+y'\right)^{-k}\end{aligned}$$

$$= L_K(k,\psi^{-1}) + N(p\mathcal{O}_K)^{-k} \sum_{x\in 2^{-1}\delta\mathcal{O}_K^+} \sum_{j\in(\mathcal{O}_K/p\mathcal{O}_K)^*} \psi^{-1}(j)\left(\sum_{y'\in\mathcal{O}_K^+\cup\{0\}} N\left(\left(xz+\frac{j}{p}\right)+y'\right)^{-k}\right) \tag{8.26}$$

We will use the formula [vdG88, pg. 19] given below

$$\sum_{y'\in\mathcal{O}_K^+\cup\{0\}} N\left(\left(xz+\frac{j}{p}\right)+y'\right)^{-k} = \frac{(2\pi i)^{2k}}{(k-1)!^2} D^{\frac{1}{2}-k} \sum_{\substack{\lambda\in\mathfrak{d}^{-1}\\ x\lambda\gg 0}} N(\lambda\delta)^{k-1} e^{2\pi i\,\mathrm{Tr}\left(\lambda\left(xz+\frac{j}{p}\right)\right)}$$

where $x\neq 0$. This formula is obtained using Poisson summation formula, see [SW71, pg. 252].

Then formula for $G_{k,\psi}(z)$ in equation 8.26 above simplifies to

$$L_K(k,\psi^{-1}) + N(p\mathcal{O}_K)^{-k}\frac{(2\pi i)^{2k}}{(k-1)!^2} D^{\frac{1}{2}-k} \sum_{x\in 2^{-1}\delta\mathcal{O}_K^+} \sum_{j\in(\mathcal{O}_K/p\mathcal{O}_K)^*} \psi^{-1}(j) \sum_{\substack{\lambda\in\mathfrak{d}^{-1}\\ x\lambda\gg 0}} N(\lambda\delta)^{k-1} e^{2\pi i\,\mathrm{Tr}\left(\lambda\left(xz+\frac{j}{p}\right)\right)}$$

$$=L_K(k,\psi^{-1}) + N(p\mathcal{O}_K)^{-k}\frac{(2\pi i)^{2k}}{(k-1)!^2} D^{\frac{1}{2}-k} \sum_{x\in 2^{-1}\delta\mathcal{O}_K^+} \sum_{\substack{\lambda\in\mathfrak{d}^{-1}\\ x\lambda\gg 0}} N(\lambda\delta)^{k-1} \sum_{j\in(\mathcal{O}_K/p\mathcal{O}_K)^*} \psi^{-1}(j) e^{2\pi i\,\mathrm{Tr}\left(\lambda\frac{j}{p}\right)} e^{2\pi i\,\mathrm{Tr}(\lambda xz)}.$$

Using lemma 8.5.5, $G_{k,\psi}(z)$ further simplifies to

$$L_K(k,\psi^{-1}) + N(p\mathcal{O}_K)^{-k}\frac{(2\pi i)^{2k}}{(k-1)!^2} D^{\frac{1}{2}-k}\tau(\psi^{-1}) \sum_{x\in 2^{-1}\delta\mathcal{O}_K^+} \sum_{\substack{\lambda\in\mathfrak{d}^{-1}\\ x\lambda\gg 0}} \psi(\lambda\delta) N(\lambda\delta)^{k-1} e^{2\pi i\,\mathrm{Tr}(\lambda xz)}$$

$$=L_K(k,\psi^{-1}) + N(p\mathcal{O}_K)^{-k}\frac{(2\pi i)^{2k}}{(k-1)!^2} D^{\frac{1}{2}-k}\tau(\psi^{-1}) \sum_{x\in 2^{-1}\delta\mathcal{O}_K^+} \sum_{x\lambda\in 2^{-1}\mathcal{O}_K^+} \psi(x^{-1}x\lambda\delta) N(x^{-1}x\lambda\delta)^{k-1} e^{2\pi i\,\mathrm{Tr}(\lambda xz)}.$$

For simplicity, let $\nu=\lambda x$. Then our formula for $G_{k,\psi}(z)$ rewrites as

$$L_K(k,\psi^{-1}) + N(p\mathcal{O}_K)^{-k}\frac{(2\pi i)^{2k}}{(k-1)!^2} D^{\frac{1}{2}-k}\tau(\psi^{-1}) \sum_{x\in 2^{-1}\delta\mathcal{O}_K^+} \sum_{\nu\in 2^{-1}\mathcal{O}_K^+} \psi(x^{-1}\nu\delta) N(x^{-1}\nu\delta)^{k-1} e^{2\pi i\,\mathrm{Tr}(\nu z)}$$

$$=L_K(k,\psi^{-1}) + N(p\mathcal{O}_K)^{-k}\frac{(2\pi i)^{2k}}{(k-1)!^2} D^{\frac{1}{2}-k}\tau(\psi^{-1}) \sum_{\nu\in 2^{-1}\mathcal{O}_K^+} \sum_{2\delta^{-1}x\in\mathcal{O}_K^+} \psi((\delta^{-1}x)^{-1}\nu) N((\delta^{-1}x)^{-1}\nu)^{k-1} e^{2\pi i\,\mathrm{Tr}(\nu z)}.$$

For simplicity, let $r=2\delta^{-1}x$. Then our formula for $G_{k,\psi}(z)$ is given by

$$L_K(k,\psi^{-1}) + N(p\mathcal{O}_K)^{-k}\frac{(2\pi i)^{2k}}{(k-1)!^2} D^{\frac{1}{2}-k}\tau(\psi^{-1}) \sum_{2\nu\in\mathcal{O}_K^+} \sum_{r\in\mathcal{O}_K^+} \psi(2r^{-1}\nu) N(2r^{-1}\nu)^{k-1} e^{2\pi i\,\mathrm{Tr}(\nu z)}.$$

Let $\xi=2\nu$. Then

$$G_{k,\psi}(z) = L_K(k,\psi^{-1}) + N(p\mathcal{O}_K)^{-k}\frac{(2\pi i)^{2k}}{(k-1)!^2} D^{\frac{1}{2}-k}\tau(\psi^{-1}) \sum_{\xi\in\mathcal{O}_K^+} \sum_{r\in\mathcal{O}_K^+} \psi(r^{-1}\xi) N(r^{-1}\xi)^{k-1} e^{2\pi i\,\mathrm{Tr}\left(\frac{\xi}{2}z\right)}.$$

Now $rr'=\xi$. Then $\mathfrak{r}'\mid(\xi)$ where $\mathfrak{r}'\subset\mathcal{O}_K$. Hence, it follows,

$$G_{k,\psi}(z) = L_K(k,\psi^{-1}) + N(p\mathcal{O}_K)^{-k}\frac{(2\pi i)^{2k}}{(k-1)!^2} D^{\frac{1}{2}-k}\tau(\psi^{-1}) \sum_{\xi\in\mathcal{O}_K^+} \left(\sum_{\substack{\mathfrak{r}'\subset\mathcal{O}_K\\ \mathfrak{r}'\mid(\xi)}} \psi(r') N(r')^{k-1}\right) e^{2\pi i\,\mathrm{Tr}\left(\frac{\xi}{2}z\right)}$$

or

$$G_{k,\psi}(z) = L_K(k,\psi^{-1}) + N(p\mathcal{O}_K)^{-k}\frac{(2\pi i)^{2k}}{(k-1)!^2}D^{\frac{1}{2}-k}\tau(\psi^{-1})\sum_{\xi\in\mathcal{O}_K^+}\sigma_{k-1,\psi}(\xi)e^{2\pi i\,\mathrm{Tr}\left(\frac{\xi}{2}z\right)}$$

where

$$\sigma_{k-1,\psi}(\xi) = \sum_{\substack{\mathfrak{r}'\subset\mathcal{O}_K\\ \mathfrak{r}'|(\xi)}}\psi(r')N(r')^{k-1}$$

is the twisted $(k-1)^{\text{th}}$ divisor sum of norm of integral ideals dividing (ξ). ☐

Definition 8.5.7 (Normalised generalised Hilbert Eisenstein series)**.** *The normalised generalised Hilbert Eisenstein series is defined as*

$$E_{k,\psi}(z) := \frac{1}{L_K(k,\psi^{-1})}G_{k,\psi}(z)$$

where $L_K(k,\psi^{-1})$ is the Hecke L-function.

Then $E_{k,\psi}(z)$ has the following Fourier expansion at infinity

$$E_{k,\psi}(z) = 1 + C_k\sum_{\xi\in\mathcal{O}_K^+}\sigma_{k-1,\psi}(\xi)e^{2\pi i\,\mathrm{Tr}\left(\frac{\xi}{2}z\right)} \tag{8.27}$$

where

$$C_k = \frac{1}{L_K(k,\psi^{-1})}\frac{(2\pi i)^{2k}}{(k-1)!^2}D^{\frac{1}{2}-k}N(p\mathcal{O}_K)^{-k}\tau(\psi^{-1}).$$

We next want to show that $E_{k,\psi}(z)$ has rational coefficients. The coefficients of $E_{k,\psi}(z)$ involves two main C_k and $\sigma_{k-1,\psi}(\xi)$, see equation 8.27 above. Here $\sigma_{k-1,\psi}(\xi)$ is the sum of norms of integral ideals along with a twist by ψ which takes values in $\{-1,0,1\}$. Thus, the sum $\sigma_{k-1,\psi}(\xi)$ takes rational values. So, we now need to show that the other term C_k is rational as well. For this, we will first use the functional equation of Hecke L-function given in equation 8.17, that is,

$$L_K(k,\psi^{-1}) = \tau(\psi^{-1})D^{\frac{1}{2}-k}N(p\mathcal{O}_K)^{-k}\frac{(2\pi)^{2k}}{2^2(k-1)!^2}L_K(1-k,\psi).$$

Further formula 8.20 gives factorisation of Hecke L-function $L_K(1-k,\psi)$ as $(B_{k,\chi_p}B_{k,\chi_p\chi_D}/k^2)$. Recall that here $\psi=\chi_p\circ N_{K/\mathbb{Q}}$ while $\chi_p=\left(\frac{\bullet}{p}\right)$ and $\chi_D=\left(\frac{D}{\bullet}\right)$.

This implies

$$L_K(k,\psi^{-1}) = \tau(\psi^{-1})D^{\frac{1}{2}-k}N(p\mathcal{O}_K)^{-k}\frac{(2\pi)^{2k}}{2^2(k-1)!^2}\frac{B_{k,\chi_p}B_{k,\chi_p\chi_D}}{k^2}.$$

Then the term C_k simplifies to

$$C_k = \frac{(2k)^2}{B_{k,\chi_p}B_{k,\chi_p\chi_D}}.$$

This implies that C_k is rational as Bernoulli numbers and generalised Bernoulli numbers are rational.

Hence, we can say that the Fourier expansion of $E_{k,\psi}(z)$ is given by

$$E_{k,\psi}(z) = 1 + \frac{(2k)^2}{B_{k,\chi_p}B_{k,\chi_p\chi_D}}\sum_{\xi\in\mathcal{O}_K^+}\sigma_{k-1,\psi}(\xi)e^{2\pi i\,\mathrm{Tr}\left(\frac{\xi}{2}z\right)} \tag{8.28}$$

where the Fourier coefficients of $E_{k,\psi}(z)$ take rational values.

8.6 Generalised Hilbert Eisenstein series modulo p

Let p be an odd rational prime that is unramified in K. Let $\psi = \chi_p \circ N_{K/\mathbb{Q}}$ be the Hecke character which is clearly non-trivial. We then define $\widetilde{\mathscr{E}}_p(z)$, the generalised Hilbert Eisenstein series of weight $\frac{p-1}{2}$, level (p) and character ψ. That is,

$$\widetilde{\mathscr{E}}_p(z) := E_{\frac{p-1}{2},\psi}(z).$$

We next try to prove a congruence modulo p satisfied by $\widetilde{\mathscr{E}}_p(z)$.

Lemma 8.6.1. *Let K be a totally real quadratic field of narrow class number 1. Let D be the discriminant of K and let χ_D be the associated quadratic character of K given by the Kronecker symbol $\left(\frac{D}{\bullet}\right)$. Let p be an odd rational prime that is unramified in K. Let $\chi_p = \left(\frac{\bullet}{p}\right)$ be the Krocnecker symbol and let $B_{\frac{p-1}{2},\chi_p}$ and $B_{\frac{p-1}{2},\chi_p\chi_D}$ denote the $(p-1/2)^{th}$ generalised Bernoulli numbers with character χ_p and $\chi_p\chi_D$ respectively and suppose that $p \nmid B_{\frac{p-1}{2},\chi_p\chi_D}$. For, $\psi = \chi_p \circ N_{K/\mathbb{Q}}$, let $\widetilde{\mathscr{E}}_p(z) \in M_{\frac{p-1}{2}}(\Gamma_{(p)}, \psi)$ be the modular form defined as*

$$\widetilde{\mathscr{E}}_p(z) = E_{\frac{p-1}{2},\psi}(z).$$

Then

$$\widetilde{\mathscr{E}}_p(z) \equiv 1 \ mod\ p.$$

Proof. From Theorem 3.4.1, we have $p \mid B^{-1}_{\frac{p-1}{2},\chi_p}$ and we have been given that $p \nmid B_{\frac{p-1}{2},\chi_p\chi_D}$. Combining these facts with the Fourier expansion of $E_{\frac{p-1}{2},\psi}(z)$ obtained in equation 8.28, it follows that

$$E_{\frac{p-1}{2},\chi_p}(z) \equiv 1 \bmod p$$

or

$$\widetilde{\mathscr{E}}_p(z) \equiv 1 \bmod p.$$

□

CHAPTER 9

Ordinary Hilbert modular forms and p-stabilisation

9.1 Introduction

Let $k \in \mathbb{Z}_{>0}$. Let K be a totally real quadratic field with narrow class number 1 and let $\mathcal{O}_K$ be it's ring of integers. Let $\mathfrak{n}$ be a square-free integral ideal of odd norm in $\mathcal{O}_K$. Let p be an odd rational prime that is unramified in K, that is, its unramified in K. Let $v_{\mathfrak{p}}$ be the valuation determined by the prime ideal $\mathfrak{p}$ above p that extends the standard valuation v_p on $\mathbb{Q}$. Since p does not ramify in K, the ramification index of $\mathfrak{p}$ in p is $e_{\mathfrak{p}} = 1$. Thus if we normalise $v_{\mathfrak{p}}$ such that for $x \in \mathcal{O}_K$, $v_{\mathfrak{p}}(x) = n/e_{\mathfrak{p}}$ where $x \in \mathfrak{p}^n$ and $e_{\mathfrak{p}} = 1$, then the image of both v_p and $v_{\mathfrak{p}}$ lies in $\mathbb{Z}$ as the ramification index of $\mathfrak{p}$ in p is 1.

Let $f(z) \in S_{2k}(\Gamma_{\mathfrak{n}})$ be a Hilbert newform. Then by [Shi81, Section 2, pg. 650], the Fourier coefficients of $f(z)$ lie in a ring of integers $\mathcal{O}_f$ of a fixed number field L_f. Let $g(z) \in S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}})$ be the image of $f(z)$ under *Kohnen's* isomorphism. Then by [Shi87, Lemma 8.8] and [Shi87, Proposition 8.9], the Fourier coefficients of $g(z)$ are algebraic numbers. Further, we will see from Theorem 10.1.1 in chapter 10 that there exists a normalisation that would make these Fourier coefficients lie in a ring of integers $\mathcal{O}_g$ of a fixed number field L_g. Now, let us take a big enough number field L/K containing number fields L_f and L_g for all Hilbert newforms $f(z) \in S_{2k}(\Gamma_{\mathfrak{n}})$ and their associated Hilbert newforms of half-integer weight $g(z)$ respectively.

Let $\mathcal{O}_L$ be the ring of integers of L. We now fix a prime ideal $\mathfrak{P} \subset \mathcal{O}_L$ above $\mathfrak{p}$ such that the valuation $v_{\mathfrak{P}}$ determined by $\mathfrak{P}$ is normalised as: For $x \in \mathcal{O}_L$, $v_{\mathfrak{P}}(x) = \frac{n}{e_{\mathfrak{P}}}$ where $x \in \mathfrak{P}^n$ and $e_{\mathfrak{P}}$ is the ramification index of $\mathfrak{P}$ in the factorisation of $\mathfrak{p}\mathcal{O}_L$. In other words, we have $v_{\mathfrak{P}}(\mathfrak{p}) = v_{\mathfrak{p}}(\mathfrak{p}) = 1$.

For a power series $f(z) = \sum_{\xi \in \mathcal{O}_K^+} a_\xi e^{2\pi i \mathrm{Tr}\left(\frac{\xi}{2} z\right)}$, we define

$$v_p(f(z)) := \inf(v_p(a_\xi)).$$

Let $F^{\mathfrak{P}}$ be a finite extension of $\mathbb{Q}_p$ containing L that extends v_p to $v_{\mathfrak{P}}$. Let $\mathcal{O}_{F^{\mathfrak{P}}}$ be its corresponding ring of integers. We can then embed $L \hookrightarrow \overline{\mathbb{Q}_p}$ or embed $L \hookrightarrow \mathbb{C}$. It therefore makes sense to view the Fourier coefficients of $f(z)$ $\mathfrak{P}$-adically embedded in $\mathcal{O}_{F^{\mathfrak{P}}}$.

Let $S_{2k}(\Gamma_{\mathfrak{n}}; \mathcal{O}_L)$ be the $\mathcal{O}_L$-submodule of $S_{2k}(\Gamma_{\mathfrak{n}}; L)$ containing Hilbert cusp forms in $S_{2k}(\Gamma_{\mathfrak{n}}; L)$ that have Fourier coefficients in $\mathcal{O}_L$. Now define

$$S_{2k}(\Gamma_{\mathfrak{n}}; \mathcal{O}_{F^{\mathfrak{P}}}) := S_{2k}(\Gamma_{\mathfrak{n}}; \mathcal{O}_L) \otimes_{\mathcal{O}_L} \mathcal{O}_{F^{\mathfrak{P}}}.$$

Let $\mathbb{T}_{2k}(\mathfrak{n};\mathcal{O}_L)$ be the commutative $\mathcal{O}_L$-subalgebra of the $\mathrm{End}(S_{2k}(\Gamma_\mathfrak{n};L))$ generated by $T_\mathfrak{m}$ where $\mathfrak{m}$ runs over all integral ideals in $\mathcal{O}_K$. Define

$$\mathbb{T}_{2k}(\mathfrak{n};\mathcal{O}_{F^\mathfrak{P}}) := \mathbb{T}_{2k}(\mathfrak{n};\mathcal{O}_L)\otimes_{\mathcal{O}_L}\mathcal{O}_{F^\mathfrak{P}}.$$

Similarly, let $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}};\mathcal{O}_L)$ be the $\mathcal{O}_L$-submodule of $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}};L)$ containing half-integer weight Hilbert cusp forms in $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}};L)$ that have Fourier coefficients in $\mathcal{O}_L$. Now define

$$S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}};\mathcal{O}_{F^\mathfrak{P}}) := S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}};\mathcal{O}_L)\otimes_{\mathcal{O}_L}\mathcal{O}_{F^\mathfrak{P}}.$$

Let $\widetilde{\mathbb{T}}_{k+\frac{1}{2}}(4\mathfrak{n};\mathcal{O}_L)$ be the commutative $\mathcal{O}_L$-subalgebra of the $\mathrm{End}(S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}};L))$ generated by $T_{\mathfrak{m}^2}$ where $\mathfrak{m}$ runs over all square-free integral ideals in $\mathcal{O}_K$. Define

$$\widetilde{\mathbb{T}}_{k+\frac{1}{2}}(4\mathfrak{n};\mathcal{O}_{F^\mathfrak{P}}) := \widetilde{\mathbb{T}}_{k+\frac{1}{2}}(4\mathfrak{n};\mathcal{O}_L)\otimes_{\mathcal{O}_L}\mathcal{O}_{F^\mathfrak{P}}.$$

9.2 Ordinary Hilbert modular forms of integer weight

Let $f(z)\in S_{2k}(\Gamma_\mathfrak{n};\mathcal{O}_L)$ be a Hilbert cusp form. We intend to define an idempotent element in $\mathbb{T}_{2k}(\mathfrak{n};\mathcal{O}_{F^\mathfrak{P}})$.

Definition 9.2.1 ($\mathfrak{p}$-ordinary projector). *Let p be an odd rational prime. Then for each prime ideal $\mathfrak{p}\subset\mathcal{O}_K$ lying above p, define the $\mathfrak{P}$-adic limit*

$$\mathfrak{e}_\mathfrak{p} := \lim_{n\to\infty} T_\mathfrak{p}^{n!}.$$

The limit $\mathfrak{e}_\mathfrak{p}\in\mathbb{T}_{2k}(\mathfrak{n};\mathcal{O}_{F^\mathfrak{P}})$ exists and $\mathfrak{e}_\mathfrak{p}$ satisfies $\mathfrak{e}_\mathfrak{p}^2=\mathfrak{e}_\mathfrak{p}$ [Hid93, Lemma 1, pg. 201].

Note 9.2.2. Note that for $\mathfrak{p}\mid\mathfrak{n}$, $T_\mathfrak{p}=U_\mathfrak{p}$ and we can alternatively write $\mathfrak{e}_\mathfrak{p}=\lim_{n\to\infty}U_\mathfrak{p}^{n!}$.
We will now define p-ordinary Hilbert cuspforms.

Definition 9.2.3 (p-ordinary Hilbert cusp form). *Let $f(z)\in S_{2k}(\Gamma_\mathfrak{n};\mathcal{O}_L)$ be a Hilbert cusp form. Then $f(z)$ is $\mathfrak{p}$-ordinary if*

$$f(z)|_{2k}\mathfrak{e}_\mathfrak{p} = f(z).$$

Further, we say that $f(z)$ is p-ordinary if $f(z)$ is $\mathfrak{p}$-ordinary for every prime ideal $\mathfrak{p}\subset\mathcal{O}_K$ lying above p. In other words, $f(z)$ is p-ordinary if

$$f(z)|_{2k}\prod_{\substack{\mathfrak{p}\ prime\\ \mathfrak{p}|(p)}}\mathfrak{e}_\mathfrak{p} = f(z).$$

The image of $S_{2k}(\Gamma_\mathfrak{n};\mathcal{O}_L)$ under the ordinary projection by $|_{2k}\prod_{\substack{\mathfrak{p}\ prime\\ \mathfrak{p}|(p)}}\mathfrak{e}_\mathfrak{p}$ is called the space of ordinary Hilbert cusp forms. We denote the subspace of ordinary Hilbert cusp forms in $S_{2k}(\Gamma_\mathfrak{n};\mathcal{O}_L)$ by $S_{2k}^{ord}(\Gamma_\mathfrak{n};\mathcal{O}_L)$.

We now make a few observations about p-ordinary projection of Hilbert eigenforms in $S_{2k}(\Gamma_\mathfrak{n};\mathcal{O}_L)$. Let $f(z)\in S_{2k}(\Gamma_\mathfrak{n};\mathcal{O}_L)$ be a $T_\mathfrak{p}$ eigenform for each prime ideal $\mathfrak{p}\subset\mathcal{O}_K$ lying above p. Let $\lambda_{T_\mathfrak{p}}(f)$ denote the Hecke-eigenvalue of the $f(z)$ under the action of $T_\mathfrak{p}$ operator, that is,

$$f(z)|_{2k}T_\mathfrak{p} = \lambda_{T_\mathfrak{p}}(f)f(z).$$

Then we have

$$f(z)|_{2k}\mathfrak{e}_\mathfrak{p} = \begin{cases} f(z) & \text{if } |\lambda_{T_\mathfrak{p}}(f)|_\mathfrak{P}=1;\\ 0 & \text{if } |\lambda_{T_\mathfrak{p}}(f)|_\mathfrak{P}<1.\end{cases}$$

Thus, $T_\mathfrak{p}$ eigenform $f(z)$ is said to be $\mathfrak{p}$-ordinary if its $T_\mathfrak{p}$ eigenvalue $\lambda_{T_\mathfrak{p}}(f)$ is a $\mathfrak{P}$-adic unit for the fixed prime ideal $\mathfrak{P}\subset\mathcal{O}_L$ lying above $\mathfrak{p}$.

9.3 Control Theorem

We now consider our level of the Hilbert cuspidal space to be $\mathfrak{n}$. Let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime ideal above p such that $\mathfrak{p} \mid \mathfrak{n}$ but $\mathfrak{p}^2 \nmid \mathfrak{n}$. Suppose our Hilbert cuspidal space has a fixed finite order Hecke character ψ and let $\mathcal{O}_L[\psi]$ contain all values of ψ. We will now formally state the generalisation of Theorem 4.3.4 for Hilbert modular forms. The statement of this Theorem can be found in [Oza17, Theorem 3.5.1, pg. 26] and involves theory of Λ-adic Hilbert modular forms. For background refer to [Oza17, Chapter 3].

Theorem 9.3.1. *Let $k \in \mathbb{Z}_{>0}$. Let $\mathfrak{n}$ be a square-free integral ideal of odd norm in $\mathcal{O}_K$. Let p be an odd rational prime that is unramified in K and let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime ideal above p such that $\mathfrak{p} \mid \mathfrak{n}$ but $\mathfrak{p}^2 \nmid \mathfrak{n}$. Let ω be the Teichmüller character defined in 4.3.1. Then the rank of $S_{2k}^{ord}(\Gamma_\mathfrak{n}, \psi\omega^{1-2k}; \mathcal{O}_L[\psi])$ is constant.*

This theorem was actually proved and originally stated by *Hida* in [Hid91, Theorem 3.4] for Hecke-algebras but can be translated using duality between the aforementioned Hecke algebra and the space of ordinary Λ-adic cusp forms [Hid91, Theorem 5.6].

Note 9.3.2. We have taken our weight to be an even integer $2k$ so that we can later have a smooth transition to half-integer weight Hilbert modular forms of parallel weight $k + \frac{1}{2}$ under generalised *Kohnen's* map in Hilbert case stated in Theorem 7.5.7. However, Theorem 9.3.1 is valid for all integer weights strictly greater than 1.

We can choose ψ to be a suitable power of the Teichmüller map ω and obtain the following corollary to Theorem 9.3.1.

Corollary 9.3.3. *Let p be an odd rational prime that is unramified in K and let $k, k' \in \mathbb{Z}_{>0}$ such that $2k \equiv 2k' \bmod (p-1)$. Let $\mathfrak{n}$ be a square-free integral ideal of odd norm in $\mathcal{O}_K$ and $\mathfrak{p} \subset \mathcal{O}_K$ be a prime ideal above p such that $\mathfrak{p} \mid \mathfrak{n}$ but $\mathfrak{p}^2 \nmid \mathfrak{n}$. Then*

$$rank\ \left(S_{2k}^{ord}(\Gamma_\mathfrak{n}; \mathcal{O}_L)\right) = \ rank\left(S_{2k'}^{ord}(\Gamma_\mathfrak{n}; \mathcal{O}_L)\right).$$

9.4 p-stabilisation of Hilbert modular forms of integer weight

We now develop an analogous theory of p-stabilised form as in the integer case, see section 4.4.
Let K be a totally real quadratic field with narrow class number 1 and let $\mathcal{O}_K$ be its corresponding ring of integers. Let $k \in \mathbb{Z}_{>0}$ and let $\mathfrak{n}$ be a square-free integral ideal of odd norm in $\mathcal{O}_K$. Let p be an odd rational prime that is unramified in K and let the ideal $p\mathcal{O}_K$ be co-prime to $\mathfrak{n}$.

9.4.1 p is inert in K

Let

$$f(z) = \sum_{\xi \in \mathcal{O}_K^+} a_\xi e^{2\pi i \mathrm{Tr}\left(\frac{\xi}{2} z\right)} \in S_{2k}^{\mathrm{new}}(\Gamma_\mathfrak{n}; \mathcal{O}_L)$$

be a Hilbert newform. Since p is inert in K, $(p) = \mathfrak{p}$ is a prime ideal in $\mathcal{O}_K$. Even though prime $\mathfrak{p}$ does not divide the level $\mathfrak{n}$, we can force $\mathfrak{p}$ in the level by passing to a $\mathfrak{p}$-old Hilbert modular form.

Note that $f(z)$ is a newform and hence is an eigenvector for the $T_\mathfrak{p}$ operator for $\mathfrak{p} \nmid \mathfrak{n}$. Let the eigenvalue of $f(z)$ for the $T_\mathfrak{p}$ operator be denoted by $\lambda_{T_\mathfrak{p}}(f)$. Then

$$f(z)|_{2k} T_\mathfrak{p} = \lambda_{T_\mathfrak{p}}(f) f(z). \tag{9.1}$$

Let $V_{(1)}$ and $V_\mathfrak{p}$ be maps defined in section 6.3 that maps the space $S_{2k}(\Gamma_\mathfrak{n}; \mathcal{O}_L)$ to $S_{2k}(\Gamma_{\mathfrak{n}\mathfrak{p}}; \mathcal{O}_L)$ and are given by

$$|_{2k} V_{(1)} : f(z) \mapsto f(z) \quad \text{and} \quad |_{2k} V_\mathfrak{p} : f(z) \mapsto f(pz).$$

From Theorem 6.2.2, $T_\mathfrak{p}$ acts on $f(z)$ as follows

$$f(z)|_{2k}T_\mathfrak{p} = f(z)|_{2k}U_\mathfrak{p} + N(p)^{2k-1}f(pz).$$

Using 9.1 and rearranging the terms, we can write

$$f(z)|_{2k}U_\mathfrak{p} = \lambda_{T_\mathfrak{p}}(f)f(z) - N(p)^{2k-1}f(pz) \tag{9.2}$$

or

$$f(z)|_{2k}U_\mathfrak{p} = \lambda_{T_\mathfrak{p}}(f)(f(z)|_{2k}V_{(1)}) - N(p)^{2k-1}(f(z)|_{2k}V_\mathfrak{p}).$$

Next, we act 9.2 by $U_\mathfrak{p}$ again and get

$$\begin{aligned} f(z)|_{2k}U_\mathfrak{p}^2 &= \lambda_{T_\mathfrak{p}}(f)f(z)|_{2k}U_\mathfrak{p} - N(p)^{2k-1}f(pz)|_{2k}U_\mathfrak{p} \\ &= \lambda_{T_\mathfrak{p}}(f)f(z)|_{2k}U_\mathfrak{p} - N(p)^{2k-1}f(z). \end{aligned}$$

This relation rewrites as

$$f(z)|_{2k}(U_\mathfrak{p}^2 - \lambda_{T_\mathfrak{p}}(f)U_\mathfrak{p} + N(p)^{2k-1}) = 0.$$

Thus $U_\mathfrak{p}$ satisfies the quadratic polynomial $x^2 - \lambda_{T_\mathfrak{p}}(f)x + N(p)^{2k-1}$ on the two-dimensional space spanned by $\{f(z), f(z)|_{2k}U_\mathfrak{p}\}$ that has level at most $\mathfrak{np}$, see remark 6.3.3. We may factor this quadratic polynomial as $(x-\alpha_\mathfrak{p})(x-\beta_\mathfrak{p})$ where $\alpha_\mathfrak{p}$ and $\beta_\mathfrak{p}$ are algebraic integers satisfying $\alpha_\mathfrak{p} + \beta_\mathfrak{p} = \lambda_{T_\mathfrak{p}}(f)$ and $\alpha_\mathfrak{p}\beta_\mathfrak{p} = N(p)^{2k-1}$.

We now define two $U_\mathfrak{p}$ eigenforms in $S_{2k}^{\mathfrak{p}\text{-old}}(\Gamma_{\mathfrak{np}}; \mathcal{O}_L)$ below:

$$f_{\alpha_\mathfrak{p}}(z) := f(z)|_{2k}(U_\mathfrak{p} - \beta_\mathfrak{p}) \text{ such that } f_{\alpha_\mathfrak{p}}(z)|_{2k}U_\mathfrak{p} = \alpha_\mathfrak{p} f_{\alpha_\mathfrak{p}}(z)$$

and

$$f_{\beta_\mathfrak{p}}(z) := f(z)|_{2k}(U_\mathfrak{p} - \alpha_\mathfrak{p}) \text{ such that } f_{\beta_\mathfrak{p}}(z)|_{2k}U_\mathfrak{p} = \beta_\mathfrak{p} f_{\beta_\mathfrak{p}}(z).$$

We call $f_{\alpha_\mathfrak{p}}(z)$ and $f_{\beta_\mathfrak{p}}(z)$, the Hilbert $\mathfrak{p}$-stabilised forms at level $\mathfrak{np}$ associated to the Hilbert newform $f(z)$ at level $\mathfrak{n}$.

Remark 9.4.1. Now if $f(z) \in S_{2k}^{\text{new, ord}}(\Gamma_\mathfrak{n}; \mathcal{O}_L)$ is p-ordinary, then its $T_\mathfrak{p}$ eigenvalue must be a $\mathfrak{P}$-adic unit, where $\mathfrak{P} \subset \mathcal{O}_L$ is the fixed prime ideal above $\mathfrak{p}$. In other words, $|\lambda_{T_\mathfrak{p}}(f)|_\mathfrak{P} = 1$. We can therefore choose $\alpha_\mathfrak{p}$ to be a $\mathfrak{P}$-adic unit. This then fixes a unique oridnary $\mathfrak{p}$-stabilised form $f_{\alpha_\mathfrak{p}}(z) \in S_{2k}^{\text{ord}}(\Gamma_{\mathfrak{np}}; \mathcal{O}_L)$ with $U_\mathfrak{p}$ eigenvalue being $\alpha_\mathfrak{p}$, a $\mathfrak{P}$-adic unit.

9.4.2 p is split in K

Let

$$f(z) = \sum_{\xi \in \mathcal{O}_K^+} a_\xi e^{2\pi i \mathrm{Tr}(\frac{\xi}{2}z)} \in S_{2k}^{\text{new}}(\Gamma_\mathfrak{n}; \mathcal{O}_L)$$

be a Hilbert newform. Since p splits in K, $(p) = \mathfrak{p}_1\mathfrak{p}_2$ where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime ideals in $\mathcal{O}_K$. Even though (p) does not divide the level $\mathfrak{n}$, we can force (p) in the level by passing to a (p)-old Hilbert modular form.

Note that $f(z)$ is a newform and hence is an eigenvector for the $T_{\mathfrak{p}_1}$ operator for $\mathfrak{p}_1 \nmid \mathfrak{n}$. Let the eigenvalue of $f(z)$ for the $T_{\mathfrak{p}_1}$ operator be denoted by $\lambda_{T_{\mathfrak{p}_1}}(f)$. Then

$$f(z)|_{2k}T_{\mathfrak{p}_1} = \lambda_{T_{\mathfrak{p}_1}}(f)f(z). \tag{9.3}$$

Let $\wp_1 \in \mathscr{O}_K^+$ be a totally positive generator of $\mathfrak{p}_1$. Let $V_{(1)}$ and $V_{\mathfrak{p}_1}$ be maps defined from the space $S_{2k}(\Gamma_{\mathfrak{n}};\mathscr{O}_L)$ to $S_{2k}(\Gamma_{\mathfrak{n}\mathfrak{p}_1};\mathscr{O}_L)$ that are given by

$$|_{2k}V_{(1)} : f(z) \mapsto f(z) \quad \text{and} \quad |_{2k}V_{\mathfrak{p}_1} : f(z) \mapsto f(\wp_1 z).$$

From Theorem 6.2.2, $T_{\mathfrak{p}_1}$ acts on $f(z)$ as follows,

$$f(z)|_{2k}T_{\mathfrak{p}_1} = f(z)|_{2k}U_{\mathfrak{p}_1} + N(\wp_1)^{2k-1}f(\wp_1 z).$$

Using 9.3 and rearranging the terms, we can write

$$f(z)|_{2k}U_{\mathfrak{p}_1} = \lambda_{T_{\mathfrak{p}_1}}(f)f(z) - N(\wp_1)^{2k-1}f(\wp_1 z) \tag{9.4}$$

or

$$f(z)|_{2k}U_{\mathfrak{p}_1} = \lambda_{T_{\mathfrak{p}_1}}(f)(f(z)|_{2k}V_{(1)}) - N(\wp_1)^{2k-1}(f(z)|_{2k}V_{\mathfrak{p}_1}).$$

Next, we act 9.4 by $U_{\mathfrak{p}_1}$ again and get

$$\begin{aligned} f(z)|_{2k}U_{\mathfrak{p}_1}^2 &= \lambda_{T_{\mathfrak{p}_1}}(f)f(z)|_{2k}U_{\mathfrak{p}_1} - N(\wp_1)^{2k-1}f(\wp_1 z)|_{2k}U_{\mathfrak{p}_1} \\ &= \lambda_{T_{\mathfrak{p}_1}}(f)f(z)|_{2k}U_{\mathfrak{p}_1} - N(\wp_1)^{2k-1}f(z). \end{aligned}$$

This relation rewrites as

$$f(z)|_{2k}(U_{\mathfrak{p}_1}^2 - \lambda_{T_{\mathfrak{p}_1}}(f)U_{\mathfrak{p}_1} + N(\wp_1)^{2k-1}) = 0.$$

Thus $U_{\mathfrak{p}_1}$ satisfies the quadratic polynomial $x^2 - \lambda_{T_{\mathfrak{p}_1}}(f)x + N(\wp_1)^{2k-1}$ on the two-dimensional space spanned $\{f(z), f(z)|_{2k}U_{\mathfrak{p}_1}\}$ that has level at most $\mathfrak{n}\mathfrak{p}_1$, see remark 6.3.3. We may factor this quadratic polynomial as $(x-\alpha_{\mathfrak{p}_1})(x-\beta_{\mathfrak{p}_1})$ where $\alpha_{\mathfrak{p}_1}$ and $\beta_{\mathfrak{p}_1}$ are algebraic integers satisfying $\alpha_{\mathfrak{p}_1} + \beta_{\mathfrak{p}_1} = \lambda_{T_{\mathfrak{p}_1}}(f)$ and $\alpha_{\mathfrak{p}_1}\beta_{\mathfrak{p}_1} = N(\wp_1)^{2k-1}$.

We now define two $U_{\mathfrak{p}_1}$ eigenforms at level $\mathfrak{n}\mathfrak{p}_1$,

$$f_{\alpha_{\mathfrak{p}_1}}(z) := f(z)|_{2k}(U_{\mathfrak{p}_1} - \beta_{\mathfrak{p}_1}) \text{ such that } f_{\alpha_{\mathfrak{p}_1}}(z)|_{2k}U_{\mathfrak{p}_1} = \alpha_{\mathfrak{p}_1}f_{\alpha_{\mathfrak{p}_1}}(z) \tag{9.5}$$

and

$$f_{\beta_{\mathfrak{p}_1}}(z) := f(z)|_{2k}(U_{\mathfrak{p}_1} - \alpha_{\mathfrak{p}_1}) \text{ such that } f_{\beta_{\mathfrak{p}_1}}(z)|_{2k}U_{\mathfrak{p}_1} = \beta_{\mathfrak{p}_1}f_{\beta_{\mathfrak{p}_1}}(z).$$

We call $f_{\alpha_{\mathfrak{p}_1}}(z)$ and $f_{\beta_{\mathfrak{p}_1}}(z)$, Hilbert $\mathfrak{p}_1$-stabilised forms at level $\mathfrak{n}\mathfrak{p}_1$ associated to the Hilbert newform $f(z)$ at level $\mathfrak{n}$.

Again, $f(z)$ is a newform and hence an eigenvector for $T_{\mathfrak{p}_2}$ operator for $\mathfrak{p}_2 \nmid \mathfrak{n}$. Let the eigenvalue of $f(z)$ for the $T_{\mathfrak{p}_2}$ operator be denoted by $\lambda_{T_{\mathfrak{p}_2}}(f)$. We now apply $T_{\mathfrak{p}_2}$ operator on $f_{\alpha_{\mathfrak{p}_1}}(z)$.

$$\begin{aligned} f_{\alpha_{\mathfrak{p}_1}}(z)|_{2k}T_{\mathfrak{p}_2} &= f(z)|_{2k}(U_{\mathfrak{p}_1} - \beta_{\mathfrak{p}_1})T_{\mathfrak{p}_2} \\ &= f(z)|_{2k}U_{\mathfrak{p}_1}T_{\mathfrak{p}_2} - \beta_{\mathfrak{p}_1}f(z)|_{2k}T_{\mathfrak{p}_2}. \end{aligned}$$

Since $U_{\mathfrak{p}_1}$ and $T_{\mathfrak{p}_2}$ commute, we can write

$$\begin{aligned} f_{\alpha_{\mathfrak{p}_1}}(z)|_{2k}T_{\mathfrak{p}_2} &= \left(f(z)|_{2k}T_{\mathfrak{p}_2}\right)U_{\mathfrak{p}_1} - \beta_{\mathfrak{p}_1}\left(f(z)|_{2k}T_{\mathfrak{p}_2}\right) \\ &= \left(f(z)|_{2k}T_{\mathfrak{p}_2}\right)|_{2k}(U_{\mathfrak{p}_1} - \beta_{\mathfrak{p}_1}) \\ &= \lambda_{T_{\mathfrak{p}_2}}(f)f(z)(U_{\mathfrak{p}_1} - \beta_{\mathfrak{p}_1}) \\ &= \lambda_{T_{\mathfrak{p}_2}}(f)f_{\alpha_{\mathfrak{p}_1}}(z). \end{aligned} \tag{9.6}$$

Thus $f_{\alpha_{\mathfrak{p}_1}}(z)$ is a $T_{\mathfrak{p}_2}$ eigenform.

Let $\wp_2 \in \mathcal{O}_K^+$ be a totally positive generator of $\mathfrak{p}_2$. Let $V'_{(1)}$ and $V_{\mathfrak{p}_2}$ be maps defined from the space $S_{2k}(\Gamma_{\mathfrak{n}\mathfrak{p}_1};\mathcal{O}_L)$ to $S_{2k}(\Gamma_{\mathfrak{n}\mathfrak{p}_1\mathfrak{p}_2};\mathcal{O}_L)$ given by

$$V'_{(1)}: f(z)\mapsto f(z) \text{ and } V_{\mathfrak{p}_2}: f(z)\mapsto f(\wp_2 z).$$

From Theorem 6.2.2, $T_{\mathfrak{p}_2}$ acts on $f_{\alpha_{\mathfrak{p}_1}}(z)$ as follows

$$f_{\alpha_{\mathfrak{p}_1}}(z)|_{2k}T_{\mathfrak{p}_2} = f_{\alpha_{\mathfrak{p}_1}}(z)|_{2k}U_{\mathfrak{p}_2} + N(\wp_2)^{2k-1} f_{\alpha_{\mathfrak{p}_1}}(\wp_2 z).$$

Using 9.6 and rearranging the terms, we can write

$$f_{\alpha_{\mathfrak{p}_1}}(z)|_{2k}U_{\mathfrak{p}_2} = \lambda_{T_{\mathfrak{p}_2}}(f) f_{\alpha_{\mathfrak{p}_1}}(z) - N(\wp_2)^{2k-1} f_{\alpha_{\mathfrak{p}_1}}(\wp_2 z) \tag{9.7}$$

or

$$f_{\alpha_{\mathfrak{p}_1}}(z)|_{2k}U_{\mathfrak{p}_2} = \lambda_{T_{\mathfrak{p}_2}}(f)\left(f_{\alpha_{\mathfrak{p}_1}}(z)|_{2k}V'_{(1)}\right) - N(\wp_2)^{2k-1}\left(f_{\alpha_{\mathfrak{p}_1}}(z)|_{2k}V_{\mathfrak{p}_2}\right).$$

Next, we act 9.7 by $U_{\mathfrak{p}_2}$ again and get

$$\begin{aligned} f_{\alpha_{\mathfrak{p}_1}}(z)|_{2k}U^2_{\mathfrak{p}_2} &= \lambda_{T_{\mathfrak{p}_2}}(f) f_{\alpha_{\mathfrak{p}_1}}(z)|_{2k}U_{\mathfrak{p}_2} - N(\wp_2)^{2k-1} f_{\alpha_{\mathfrak{p}_1}}(\wp_2 z)|_{2k}U_{\mathfrak{p}_2} \\ &= \lambda_{T_{\mathfrak{p}_2}}(f) f_{\alpha_{\mathfrak{p}_1}}(z)|_{2k}U_{\mathfrak{p}_2} - N(\wp_2)^{2k-1} f_{\alpha_{\mathfrak{p}_1}}(z). \end{aligned}$$

This relation rewrites as

$$f_{\alpha_{\mathfrak{p}_1}}(z)|_{2k}(U^2_{\mathfrak{p}_2} - \lambda_{T_{\mathfrak{p}_2}}(f)U_{\mathfrak{p}_2} + N(\wp_2)^{2k-1}) = 0.$$

Thus $U_{\mathfrak{p}_2}$ satisfies the quadratic polynomial $x^2 - \lambda_{T_{\mathfrak{p}_1}}(f)x + N(\wp_2)^{2k-1}$ on the two-dimensional space spanned by $\{f_{\alpha_{\mathfrak{p}_1}}(z), f_{\alpha_{\mathfrak{p}_1}}(z)|_{2k}U_{\mathfrak{p}_2}\}$ that has level at most $\mathfrak{n}\mathfrak{p}_1\mathfrak{p}_2$, see remark 6.3.3. We may factor this quadratic polynomial as $(x-\alpha_{\mathfrak{p}_2})(x-\beta_{\mathfrak{p}_2})$ where $\alpha_{\mathfrak{p}_2}$ and $\beta_{\mathfrak{p}_2}$ are algebraic integers satisfying $\alpha_{\mathfrak{p}_2}+\beta_{\mathfrak{p}_2} = \lambda_{T_{\mathfrak{p}_2}}(f)$ and $\alpha_{\mathfrak{p}_2}\beta_{\mathfrak{p}_2} = N(\wp_2)^{2k-1}$.

We now define four $U_{\mathfrak{p}_2}$ eigenforms at level $\mathfrak{n}\mathfrak{p}_1\mathfrak{p}_2 = \mathfrak{n}(p)$:

$$\begin{aligned} f_{\alpha_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z) &:= f_{\alpha_{\mathfrak{p}_1}}(z)|_{2k}(U_{\mathfrak{p}_2}-\beta_{\mathfrak{p}_2}) \quad \text{such that} \quad f_{\alpha_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z)|_{2k}U_{\mathfrak{p}_2} = \alpha_{\mathfrak{p}_2} f_{\alpha_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z); \\ f_{\alpha_{\mathfrak{p}_1},\beta_{\mathfrak{p}_2}}(z) &:= f_{\alpha_{\mathfrak{p}_1}}(z)|_{2k}(U_{\mathfrak{p}_2}-\alpha_{\mathfrak{p}_2}) \quad \text{such that} \quad f_{\alpha_{\mathfrak{p}_1},\beta_{\mathfrak{p}_2}}(z)|_{2k}U_{\mathfrak{p}_2} = \beta_{\mathfrak{p}_2} f_{\alpha_{\mathfrak{p}_1},\beta_{\mathfrak{p}_2}}(z); \\ f_{\beta_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z) &:= f_{\beta_{\mathfrak{p}_1}}(z)|_{2k}(U_{\mathfrak{p}_2}-\beta_{\mathfrak{p}_2}) \quad \text{such that} \quad f_{\beta_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z)|_{2k}U_{\mathfrak{p}_2} = \alpha_{\mathfrak{p}_2} f_{\beta_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z); \\ f_{\beta_{\mathfrak{p}_1},\beta_{\mathfrak{p}_2}}(z) &:= f_{\beta_{\mathfrak{p}_1}}(z)|_{2k}(U_{\mathfrak{p}_2}-\alpha_{\mathfrak{p}_2}) \quad \text{such that} \quad f_{\beta_{\mathfrak{p}_1},\beta_{\mathfrak{p}_2}}(z)|_{2k}U_{\mathfrak{p}_2} = \beta_{\mathfrak{p}_2} f_{\beta_{\mathfrak{p}_1},\beta_{\mathfrak{p}_2}}(z). \end{aligned} \tag{9.8}$$

Using 9.8, we can write

$$f_{\alpha_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z)|_{2k}U_{\mathfrak{p}_1} = f_{\alpha_{\mathfrak{p}_1}}(z)|_{2k}(U_{\mathfrak{p}_2}-\beta_{\mathfrak{p}_2})U_{\mathfrak{p}_1}$$

Since, $U_{\mathfrak{p}_1}$ and $U_{\mathfrak{p}_2}$ commute, it follows

$$\begin{aligned} f_{\alpha_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z)|_{2k}U_{\mathfrak{p}_1} &= f_{\alpha_{\mathfrak{p}_1}}(z)|_{2k}U_{\mathfrak{p}_1}(U_{\mathfrak{p}_2}-\beta_{\mathfrak{p}_2}) \\ &= \lambda_{T_{\mathfrak{p}_1}}(f) f_{\alpha_{\mathfrak{p}_1}}(z)|_{2k}(U_{\mathfrak{p}_2}-\beta_{\mathfrak{p}_2}) \\ &= \lambda_{T_{\mathfrak{p}_1}}(f) f_{\alpha_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z). \end{aligned}$$

This implies $f_{\alpha_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z)$ is also a $U_{\mathfrak{p}_1}$ eigenform. Similarly, $f_{\alpha_{\mathfrak{p}_1},\beta_{\mathfrak{p}_2}}(z)$, $f_{\beta_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z)$ and $f_{\beta_{\mathfrak{p}_1},\beta_{\mathfrak{p}_2}}(z)$ are also $U_{\mathfrak{p}_1}$ eigenforms.

We call $f_{\alpha_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z)$, $f_{\alpha_{\mathfrak{p}_1},\beta_{\mathfrak{p}_2}}(z)$, $f_{\beta_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z)$ and $f_{\beta_{\mathfrak{p}_1},\beta_{\mathfrak{p}_2}}(z)$, the Hilbert p-stabilised forms at level $\mathfrak{n}\mathfrak{p}_1\mathfrak{p}_2 = \mathfrak{n}(p)$ associated to the Hilbert newform $f(z)$ at level $\mathfrak{n}$.

Remark 9.4.2. Now if $f(z) \in S_{2k}^{\text{new, ord}}(\Gamma_{\mathfrak{n}}; \mathscr{O}_L)$ is p-ordinary, then its $T_{\mathfrak{p}_1}$ and $T_{\mathfrak{p}_2}$ eigenvalue must be $\mathfrak{P}_1$-adic and $\mathfrak{P}_2$-adic units, where $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are fixed prime ideals above $\mathfrak{p}_1$ and $\mathfrak{p}_2$ respectively. In other words, $|\lambda_{T_{\mathfrak{p}_1}}(f)|_{\mathfrak{P}_1} = |\lambda_{T_{\mathfrak{p}_2}}(f)|_{\mathfrak{P}_2} = 1$. We can therefore choose $\alpha_{\mathfrak{p}_1}$ and $\alpha_{\mathfrak{p}_2}$ to be a $\mathfrak{P}_1$-adic and $\mathfrak{P}_2$-adic units respectively. This then fixes a unique ordinary p-stabilised form $f_{\alpha_{\mathfrak{p}_1}, \alpha_{\mathfrak{p}_2}}(z) \in S_{2k}^{\text{ord}}(\Gamma_{\mathfrak{n}\mathfrak{p}_1\mathfrak{p}_2}; \mathscr{O}_L)$ with $U_{\mathfrak{p}_1}$ eigenvalue being $\alpha_{\mathfrak{p}_1}$, a $\mathfrak{P}_1$-adic unit and $U_{\mathfrak{p}_2}$ eigenvalue being $\alpha_{\mathfrak{p}_2}$, a $\mathfrak{P}_2$-adic unit.

9.5 Ordinary Hilbert modular forms of half-integer weight

Let $g(z) \in S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}; \mathscr{O}_L)$ be a half-integer weight Hilbert cusp form. We intend to define an idempotent element in $\widetilde{\mathbb{T}}_{k+\frac{1}{2}}(4\mathfrak{n}; \mathscr{O}_{F^{\mathfrak{P}}})$.

Definition 9.5.1 ($\mathfrak{p}$-ordinary projector). *Let p be an odd rational prime. Then for each prime ideal $\mathfrak{p} \subset \mathscr{O}_K$ lying above p, define the $\mathfrak{P}$-adic limit*

$$\widetilde{\mathfrak{e}}_{\mathfrak{p}} := \lim_{n\to\infty} T_{\mathfrak{p}^2}^{n!}.$$

The limit $\widetilde{\mathfrak{e}}_{\mathfrak{p}} \in \widetilde{\mathbb{T}}_{k+\frac{1}{2}}(4\mathfrak{n}; \mathscr{O}_{F^{\mathfrak{P}}})$ exists and $\widetilde{\mathfrak{e}}_{\mathfrak{p}}$ satisfies $\widetilde{\mathfrak{e}}_{\mathfrak{p}}^2 = \widetilde{\mathfrak{e}}_{\mathfrak{p}}$ [Hid93, Lemma 1, pg. 201].

Note 9.5.2. Note that for $\mathfrak{p} \mid \mathfrak{n}$, $T_{\mathfrak{p}^2} = U_{\mathfrak{p}^2}$ and we can alternatively write $\widetilde{\mathfrak{e}}_{\mathfrak{p}} := \lim_{n\to\infty} U_{\mathfrak{p}^2}^{n!}$.

We will now define p-ordinary Hilbert cusp forms of half-integer weight.

Definition 9.5.3 (p-ordinary Hilbert cusp form of half-integer weight). *Let $g(z) \in S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}; \mathscr{O}_L)$ be a Hilbert cusp form of half-integer weight. Then $g(z)$ is $\mathfrak{p}$-ordinary if*

$$g(z)|_{k+\frac{1}{2}}\widetilde{\mathfrak{e}}_{\mathfrak{p}} = g(z).$$

Further, we say that $g(z)$ is p-ordinary if $g(z)$ is $\mathfrak{p}$-ordinary for every prime ideal $\mathfrak{p} \subset \mathscr{O}_K$ lying above p. In other words, $g(z)$ is p-ordinary if

$$g(z)|_{k+\frac{1}{2}} \prod_{\substack{\mathfrak{p} \text{ prime} \\ \mathfrak{p}|(p)}} \widetilde{\mathfrak{e}}_{\mathfrak{p}} = g(z).$$

The image of $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}; \mathscr{O}_L)$ under the ordinary projection by $\widetilde{\mathfrak{e}}_{\mathfrak{p}}$ is called the space of ordinary Hilbert cusp forms of half-integer weight. We denote the subspace of ordinary Hilbert cusp forms of half-integer weight in $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}; \mathscr{O}_L)$ by $S_{k+\frac{1}{2}}^{ord}(\widetilde{\Gamma}_{4\mathfrak{n}}; \mathscr{O}_L)$.

We now make a few observations about p-ordinary projection of Hilbert eigenforms of half-integer weight in $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}; \mathscr{O}_L)$.

Let $g(z) \in S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}; \mathscr{O}_L)$ be a $T_{\mathfrak{p}^2}$ eigenform for each prime ideal $\mathfrak{p} \subset \mathscr{O}_K$ lying above p. Let $\lambda_{T_{\mathfrak{p}}^2}(g)$ denote the Hecke-eigenvalue of the $g(z)$ under the action of $T_{\mathfrak{p}^2}$ operator, that is,

$$g(z)|_{k+\frac{1}{2}} T_{\mathfrak{p}^2} = \lambda_{T_{\mathfrak{p}^2}}(g) g(z).$$

Then we have

$$g(z)|_{k+\frac{1}{2}}\widetilde{\mathfrak{e}}_{\mathfrak{p}} = \begin{cases} g(z) & \text{if } |\lambda_{T_{\mathfrak{p}^2}}(g)|_{\mathfrak{P}} = 1; \\ 0 & \text{if } |\lambda_{T_{\mathfrak{p}^2}}(g)|_{\mathfrak{P}} < 1. \end{cases}$$

Thus, $T_{\mathfrak{p}^2}$ eigenform $g(z)$ is said to be $\mathfrak{p}$-ordinary if its $T_{\mathfrak{p}^2}$ eigenvalue $\lambda_{T_{\mathfrak{p}^2}}(g)$ is a $\mathfrak{P}$-adic unit for the fixed prime ideal $\mathfrak{P} \subset \mathscr{O}_L$ lying above $\mathfrak{p}$.

9.6 p stabilisation of Hilbert modular forms of half-integer weight

Let K be a totally real quadratic field with narrow class number 1 and let $\mathcal{O}_K$ be its corresponding ring of integers. Let $k \in \mathbb{Z}_{>0}$ and let $\mathfrak{n}$ be a square-free integral ideal of odd norm in $\mathcal{O}_K$. Let p be an odd rational prime such that the ideal generated by p in $\mathcal{O}_K$ is co-prime to $\mathfrak{n}$. We assume that p is unramified in K.

9.6.1 p is inert in K

Let

$$g(z) = \sum_{\xi \in \mathcal{O}_K^+} b_\xi e^{2\pi i \mathrm{Tr}\left(\frac{\xi}{2} z\right)} \in S^{\mathrm{new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}; \mathcal{O}_L)$$

be a Hilbert newform of half-integer weight. Since p is inert in K, $(p) = \mathfrak{p}$ is a prime ideal in $\mathcal{O}_K$. Even though $\mathfrak{p}$ does not divide the level $4\mathfrak{n}$ of $g(z)$, we can force $\mathfrak{p}$ in the level by passing to a $\mathfrak{p}$-old Hilbert modular form of half-integer weight.

Note that $g(z)$ is a newform of half-integer weight and hence is an eigenvector for the $T_{\mathfrak{p}^2}$ operator for $\mathfrak{p} \nmid \mathfrak{n}$. Let the eigenvalue of $g(z)$ for the $T_{\mathfrak{p}^2}$ operator be denoted by $\lambda_{T_{\mathfrak{p}^2}}(g)$:

$$g(z)|_{k+\frac{1}{2}} T_{\mathfrak{p}^2} = \lambda_{T_{\mathfrak{p}^2}}(g) g(z). \tag{9.9}$$

Let $V_{(1)}$ be a map from the space $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}; \mathcal{O}_L)$ to $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}\mathfrak{p}}; \mathcal{O}_L)$, and let $V_{\mathfrak{p}^2}$, $V_{\left(\frac{\bullet}{p}\right)_2}$ be maps defined from the space $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}; \mathcal{O}_L)$ to $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}\mathfrak{p}^2}; \mathcal{O}_L)$ given by

$$|_{k+\frac{1}{2}} V_{(1)} : g(z) \mapsto g(z), \quad |_{k+\frac{1}{2}} V_{\mathfrak{p}^2} : g(z) \mapsto g(p^2 z) \quad \text{and} \quad |_{k+\frac{1}{2}} V_{\left(\frac{\bullet}{p}\right)_2} : g(z) \mapsto g_{\left(\frac{\bullet}{p}\right)_2}(z)$$

respectively.

Note 9.6.1. Here $g_{\left(\frac{\bullet}{p}\right)_2}(z) = \sum_{\xi \in \mathcal{O}_K^+} \left(\frac{\xi}{p}\right)_2 b_\xi e^{2\pi i \mathrm{Tr}\left(\frac{\xi}{2}\right) z}$ is the twist of $g(z)$ by the quadratic symbol $\left(\frac{\bullet}{p}\right)_2$. The level of $g_{\left(\frac{\bullet}{p}\right)_2}(z)$ is irrelevant to our result as we will see next that it's killed under the action of the $U_{\mathfrak{p}^2}$ operator.

From Theorem 7.2.3, $T_{\mathfrak{p}^2}$ acts on $g(z)$ as follows,

$$g(z)|_{k+\frac{1}{2}} T_{\mathfrak{p}^2} = g(z)|_{k+\frac{1}{2}} U_{\mathfrak{p}^2} + \left(\frac{-1}{p}\right)_2^k N(p)^{k-1} g_{\left(\frac{\bullet}{p}\right)_2}(z) + N(p)^{2k-1} g(p^2 z).$$

Using equation 9.9 and rearranging the terms, we can write

$$g(z)|_{k+\frac{1}{2}} U_{\mathfrak{p}^2} = \lambda_{T_{\mathfrak{p}^2}}(g) g(z) - \left(\frac{-1}{p}\right)_2^k N(p)^{k-1} g_{\left(\frac{\bullet}{p}\right)_2}(z) - N(p)^{2k-1} g(p^2 z) \tag{9.10}$$

or

$$g(z)|_{k+\frac{1}{2}} U_{\mathfrak{p}^2} = \lambda_{T_{\mathfrak{p}^2}}(g) \left(g(z)|_{k+\frac{1}{2}} V_{(1)}\right) - \left(\frac{-1}{p}\right)_2^k N(p)^{k-1} \left(g(z)|_{k+\frac{1}{2}} V_{\left(\frac{\bullet}{p}\right)_2}\right) - N(p)^{2k-1} \left(g(z)|_{k+\frac{1}{2}} V_{\mathfrak{p}^2}\right).$$

Next we act 9.10 by $U_{\mathfrak{p}^2}$ again and get

$$g(z)|_{k+\frac{1}{2}} U^2_{\mathfrak{p}^2} = \lambda_{T_{\mathfrak{p}^2}}(g) g(z)|_{k+\frac{1}{2}} U_{\mathfrak{p}^2} - \left(\frac{-1}{p}\right)_2^k N(p)^{k-1} g_{\left(\frac{\bullet}{p}\right)_2}(z)|_{k+\frac{1}{2}} U_{\mathfrak{p}^2} - N(p)^{2k-1} g(p^2 z)|_{k+\frac{1}{2}} U_{\mathfrak{p}^2}.$$

From the definition 7.1.2 of the quadratic symbol, we have $\left(\frac{p^2 \xi}{p}\right)_2 = 0$ as $p^2 \xi \in (p)$. Thus,

$$\begin{aligned} g_{\left(\frac{\bullet}{p}\right)}(z)|_{k+\frac{1}{2}} U_{\mathfrak{p}^2} &= \sum_{\xi \in \mathcal{O}_K^+} \left(\frac{p^2 \xi}{p}\right)_2 b_{p^2 \xi} e^{2\pi i \mathrm{Tr}\left(\frac{\xi}{2}\right) z} \\ &= 0. \end{aligned}$$

Hence, we conclude $g_{\left(\frac{\bullet}{p}\right)}(z)$ lies in the kernel of $U_{\mathfrak{p}^2}$. It follows,

$$g(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}^2}^2 = \lambda_{T_{\mathfrak{p}^2}}(g)g(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}^2} - N(p)^{2k-1}g(z).$$

This relation rewrites as

$$g(z)|_{k+\frac{1}{2}}\left(U_{\mathfrak{p}^2}^2 - \lambda_{T_{\mathfrak{p}^2}}(g)U_{\mathfrak{p}^2} + N(p)^{2k-1}\right) = 0.$$

Thus, $U_{\mathfrak{p}^2}$ satisfies the quadratic polynomial $x^2 - \lambda_{T_{\mathfrak{p}^2}}(g)x + N(p)^{2k-1}$ on the two dimensional space spanned by $\{g(z), g(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}^2}\}$ that has level at most $4\mathfrak{n}\mathfrak{p}$, see remark 7.3.4. We may factor this quadratic polynomial as $(x-\alpha_{\mathfrak{p}})(x-\beta_{\mathfrak{p}})$ where $\alpha_{\mathfrak{p}}$ and $\beta_{\mathfrak{p}}$ are algebraic integers satisfying $\alpha_{\mathfrak{p}}+\beta_{\mathfrak{p}} = \lambda_{T_{\mathfrak{p}}}(g)$ and $\alpha_{\mathfrak{p}}\beta_{\mathfrak{p}} = N(p)^{2k-1}$.

We now define two $U_{\mathfrak{p}^2}$ eigenforms at level $4\mathfrak{n}\mathfrak{p}$:

$$g_{\alpha_{\mathfrak{p}}}(z) := g(z)|_{k+\frac{1}{2}}(U_{\mathfrak{p}^2} - \beta_{\mathfrak{p}}) \text{ such that } g_{\alpha_{\mathfrak{p}}}(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}^2} = \alpha_{\mathfrak{p}}g_{\alpha_{\mathfrak{p}}}(z)$$

and

$$g_{\beta_{\mathfrak{p}}}(z) := g(z)|_{k+\frac{1}{2}}(U_{\mathfrak{p}^2} - \alpha_{\mathfrak{p}}) \text{ such that } g_{\beta_{\mathfrak{p}}}(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}^2} = \beta_{\mathfrak{p}}g_{\beta_{\mathfrak{p}}}(z).$$

We call $g_{\alpha_{\mathfrak{p}}}(z)$ and $g_{\beta_{\mathfrak{p}}}(z)$, the Hilbert $\mathfrak{p}$-stabilised forms at level $4\mathfrak{n}\mathfrak{p}$ associated to the half-integer weight Hilbert newform $g(z)$ at level $4\mathfrak{n}$.

Remark 9.6.2. Now if $g(z) \in S_{k+\frac{1}{2}}^{\text{new, ord}}(\widetilde{\Gamma}_{4\mathfrak{n}}; \mathcal{O}_L)$ is p-ordinary, then it's $T_{\mathfrak{p}^2}$ eigenvalue must be the fixed $\mathfrak{P}$-adic unit, where $\mathfrak{P}$ is a prime ideal above $\mathfrak{p}$. In other words, $|\lambda_{T_{\mathfrak{p}^2}}(g)|_{\mathfrak{P}} = 1$. We can therefore choose $\alpha_{\mathfrak{p}}$ to be a $\mathfrak{P}$-adic unit. This then fixes a unique oridnary $\mathfrak{p}$-stabilised form $g_{\alpha_{\mathfrak{p}}}(z) \in S_{k+\frac{1}{2}}^{\text{ord}}(\widetilde{\Gamma}_{4\mathfrak{n}\mathfrak{p}}; \mathcal{O}_L)$ with $U_{\mathfrak{p}^2}$ eigenvalue being $\alpha_{\mathfrak{p}}$, a $\mathfrak{P}$-adic unit.

9.6.2 p is split in K

Let

$$g(z) = \sum_{\xi \in \mathcal{O}_K^+} b_\xi e^{2\pi i \mathrm{Tr}\left(\frac{\xi}{2}z\right)} \in S_{k+\frac{1}{2}}^{\text{new}}(\widetilde{\Gamma}_{4\mathfrak{n}}; \mathcal{O}_L)$$

be a Hilbert newform of half-integer weight. Since p is split in K, $(p) = \mathfrak{p}_1\mathfrak{p}_2$ where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are prime ideals in $\mathcal{O}_K$. Even though (p) does not divide the level $4\mathfrak{n}$ of $g(z)$, we can force (p) in the level by passing to a (p)-old Hilbert modular form of half-integer weight.

Note that $g(z)$ is a newform of half-integer weight and hence is an eigenvector for the $T_{\mathfrak{p}_1^2}$ operator. Let the eigenvalue of $g(z)$ for the $T_{\mathfrak{p}_1^2}$ operator be denoted by $\lambda_{T_{\mathfrak{p}_1^2}}(g)$:

$$g(z)|_{k+\frac{1}{2}}T_{\mathfrak{p}_1^2} = \lambda_{T_{\mathfrak{p}_1^2}}(g)g(z). \tag{9.11}$$

Let $\wp_1 \in \mathcal{O}_K^+$ be a totally positive generator of $\mathfrak{p}_1$. Let $V_{(1)}$ be a map from the space $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}; \mathcal{O}_L)$ to $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}\mathfrak{p}_1}; \mathcal{O}_L)$, and let $V_{\mathfrak{p}_1^2}$, $V_{\left(\frac{\bullet}{\wp_1}\right)_2}$ be maps defined from the space $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}; \mathcal{O}_L)$ to $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}\mathfrak{p}_1^2}; \mathcal{O}_L)$ given by

$$|_{k+\frac{1}{2}}V_{(1)} : g(z) \mapsto g(z), \quad |_{k+\frac{1}{2}}V_{\mathfrak{p}_1^2} : g(z) \mapsto g(\wp_1^2 z) \quad \text{and} \quad |_{k+\frac{1}{2}}V_{\left(\frac{\bullet}{\wp_1}\right)_2} : g(z) \mapsto g_{\left(\frac{\bullet}{\wp_1}\right)_2}(z)$$

respectively.

From Theorem 7.2.3, $T_{\mathfrak{p}_1^2}$ acts on $g(z)$ as follows,

$$g(z)|_{k+\frac{1}{2}}T_{\mathfrak{p}_1^2} = g(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}_1^2} + \left(\frac{-1}{\wp_1}\right)_2^k N(\wp_1)^{k-1} g_{\left(\frac{\bullet}{\wp_1}\right)_2}(z) + N(\wp_1)^{2k-1} g(\wp_1^2 z).$$

Using equation 9.11 and rearranging the terms, we can write

$$g(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}_1^2} = \lambda_{T_{\mathfrak{p}_1^2}}(g)g(z) - \left(\frac{-1}{\wp_1}\right)_2^k N(\wp_1)^{k-1} g_{\left(\frac{\bullet}{\wp_1}\right)_2}(z) - N(\wp_1)^{2k-1} g(\wp_1^2 z) \tag{9.12}$$

or

$$g(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}_1^2} = \lambda_{T_{\mathfrak{p}_1^2}}(g)\left(g(z)|_{k+\frac{1}{2}}V_{(1)}\right) - \left(\frac{-1}{\wp_1}\right)_2^k N(\wp_1)^{k-1}\left(g(z)|_{k+\frac{1}{2}}V_{\left(\frac{\bullet}{\wp_1}\right)_2}\right) - N(p)^{2k-1}\left(g(z)|_{k+\frac{1}{2}}V_{\mathfrak{p}_1^2}\right).$$

Next we act 9.12 by $U_{\mathfrak{p}_1^2}$ again and get

$$g(z)|_{k+\frac{1}{2}}U^2_{\mathfrak{p}_1^2} = \lambda_{T_{\mathfrak{p}_1^2}}(g)g(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}_1^2} - \left(\frac{-1}{\wp_1}\right)_2^k N(\wp_1)^{k-1} g_{\left(\frac{\bullet}{\wp_1}\right)_2}(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}_1^2} - N(\wp_1)^{2k-1} g(\wp_1^2 z)|_{k+\frac{1}{2}}U_{\mathfrak{p}_1^2}.$$

From the definition 7.1.2 of the quadratic symbol, we have $\left(\frac{\wp_1^2\xi}{\wp_1}\right)_2 = 0$ as $\wp_1^2\xi \in (\wp_1)$. Thus,

$$\begin{aligned} g_{\left(\frac{\bullet}{\wp_1}\right)}(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}_1^2} &= \sum_{\xi\in\mathcal{O}_K^+}\left(\frac{\wp_1^2\xi}{\wp_1}\right)_2 b_{\wp_1^2\xi} e^{2\pi i \mathrm{Tr}\left(\frac{\xi}{2}\right)z} \\ &= 0. \end{aligned}$$

Hence, we conclude $g_{\left(\frac{\bullet}{\wp_1}\right)}(z)$ lies in the kernel of $U_{\mathfrak{p}_1^2}$. It follows,

$$g(z)|_{k+\frac{1}{2}}U^2_{\mathfrak{p}_1^2} = \lambda_{T_{\mathfrak{p}_1^2}}(g)g(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}_1^2} - N(\wp_1)^{2k-1}g(z).$$

This relation rewrites as

$$g(z)|_{k+\frac{1}{2}}\left(U^2_{\mathfrak{p}_1^2} - \lambda_{T_{\mathfrak{p}_1^2}}(g)U_{\mathfrak{p}_1^2} + N(\wp_1)^{2k-1}\right) = 0.$$

Thus, $U_{\mathfrak{p}_1^2}$ satisfies the quadratic polynomial $x^2 - \lambda_{T_{\mathfrak{p}_1^2}}(g)x + N(\wp_1)^{2k-1}$ on the two dimensional space spanned by $\{g(z), g(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}_1^2}\}$ that has level at most $4\mathfrak{n}\mathfrak{p}_1$, see remark 7.3.4. We may factor this quadratic polynomial as $(x-\alpha_{\mathfrak{p}_1})(x-\beta_{\mathfrak{p}_1})$ where $\alpha_{\mathfrak{p}_1}$ and $\beta_{\mathfrak{p}_1}$ are algebraic integers satisfying $\alpha_{\mathfrak{p}_1} + \beta_{\mathfrak{p}_1} = \lambda_{T_{\mathfrak{p}_1}}(g)$ and $\alpha_{\mathfrak{p}_1}\beta_{\mathfrak{p}_1} = N(\wp_1)^{2k-1}$.

We now define two $U_{\mathfrak{p}_1^2}$ eigenforms at level $4\mathfrak{n}\mathfrak{p}_1$:

$$g_{\alpha_{\mathfrak{p}_1}}(z) := g(z)|_{k+\frac{1}{2}}(U_{\mathfrak{p}_1^2} - \beta_{\mathfrak{p}_1}) \text{ such that } g_{\alpha_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}_1^2} = \alpha_{\mathfrak{p}_1}g_{\alpha_{\mathfrak{p}_1}}(z)$$

and

$$g_{\beta_{\mathfrak{p}_1}}(z) := g(z)|_{2k}(U_{\mathfrak{p}_1^2} - \alpha_{\mathfrak{p}_1}) \text{ such that } g_{\beta_{\mathfrak{p}_1}}(z)|_{2k}U_{\mathfrak{p}_1^2} = \beta_{\mathfrak{p}_1}g_{\beta_{\mathfrak{p}_1}}(z).$$

We call $g_{\alpha_{\mathfrak{p}_1}}(z)$ and $g_{\beta_{\mathfrak{p}_1}}(z)$, the Hilbert $\mathfrak{p}_1$-stabilised forms at level $4\mathfrak{n}\mathfrak{p}_1$ associated to the half-integer weight Hilbert newform $g(z)$ at level $4\mathfrak{n}$.

Again, $g(z)$ is a newform of half-integer weight and hence an eigenvector for $T_{\mathfrak{p}_2^2}$ operator for $\mathfrak{p}_2 \nmid 4\mathfrak{n}$. Let the eigenvalue of $g(z)$ for the $T_{\mathfrak{p}_2^2}$ operator be denoted by $\lambda_{T_{\mathfrak{p}_2^2}}(g)$. We now apply $T_{\mathfrak{p}_2^2}$ operator on $g_{\alpha_{\mathfrak{p}_1}}(z)$:

$$\begin{aligned} g_{\alpha_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}} T_{\mathfrak{p}_2^2} &= g(z)|_{k+\frac{1}{2}}(U_{\mathfrak{p}_1^2} - \beta_{\mathfrak{p}_1}) T_{\mathfrak{p}_2^2} \\ &= g(z)|_{k+\frac{1}{2}} U_{\mathfrak{p}_1^2} T_{\mathfrak{p}_2^2} - \beta_{\mathfrak{p}_1} g(z)|_{k+\frac{1}{2}} T_{\mathfrak{p}_2^2}. \end{aligned}$$

Since $U_{\mathfrak{p}_1^2}$ and $T_{\mathfrak{p}_2^2}$ commute, we can write

$$\begin{aligned} g_{\alpha_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}} T_{\mathfrak{p}_2^2} &= \left(g(z)|_{k+\frac{1}{2}} T_{\mathfrak{p}_2^2}\right)|_{k+\frac{1}{2}} U_{\mathfrak{p}_1^2} - \beta_{\mathfrak{p}_1}\left(g(z)|_{k+\frac{1}{2}} T_{\mathfrak{p}_2^2}\right) \\ &= \left(g(z)|_{k+\frac{1}{2}} T_{\mathfrak{p}_2^2}\right)|_{k+\frac{1}{2}}(U_{\mathfrak{p}_1^2} - \beta_{\mathfrak{p}_1}) \\ &= \lambda_{T_{\mathfrak{p}_2^2}}(g) g(z)(U_{\mathfrak{p}_1^2} - \beta_{\mathfrak{p}_1}) \\ &= \lambda_{T_{\mathfrak{p}_2^2}}(g) g_{\alpha_{\mathfrak{p}_1}}(z). \end{aligned} \tag{9.13}$$

Thus $g_{\alpha_{\mathfrak{p}_1}}(z)$ is a $T_{\mathfrak{p}_2^2}$ eigenform.

Let $\wp_2 \in \mathcal{O}_K^+$ be a totally positive generator of $\mathfrak{p}_2$. Let $V'_{(1)}$ be a map from the space $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}\mathfrak{p}_1}; \mathcal{O}_L)$ to $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}\mathfrak{p}_1\mathfrak{p}_2}; \mathcal{O}_L)$, and let $V_{\mathfrak{p}_2^2}$, $V_{\left(\frac{\bullet}{\wp_2}\right)_2}$ be maps defined from the space $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}\mathfrak{p}_1}; \mathcal{O}_L)$ to $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}\mathfrak{p}_1\mathfrak{p}_2^2}; \mathcal{O}_L)$ given by

$$|_{k+\frac{1}{2}} V'_{(1)} : g(z) \mapsto g(z), \quad |_{k+\frac{1}{2}} V_{\mathfrak{p}_2^2} : g(z) \mapsto g(\wp_2^2 z) \quad \text{and} \quad |_{k+\frac{1}{2}} V_{\left(\frac{\bullet}{\wp_2}\right)_2} : g(z) \mapsto g_{\left(\frac{\bullet}{\wp_2}\right)_2}(z)$$

respectively.

From Theorem 7.2.3, $T_{\mathfrak{p}_2^2}$ acts on $g_{\alpha_{\mathfrak{p}_1}}(z)$ as follows,

$$g_{\alpha_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}} T_{\mathfrak{p}_2^2} = g_{\alpha_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}} U_{\mathfrak{p}_2^2} + \left(\frac{-1}{\wp_2}\right)_2^k N(\wp_2)^{k-1} g_{\alpha_{\mathfrak{p}_1}, \left(\frac{\bullet}{\wp_2}\right)_2}(z) + N(\wp_2)^{2k-1} g_{\alpha_{\mathfrak{p}_1}}(\wp_2^2 z).$$

Using equation 9.13 and rearranging the terms, we can write

$$g_{\alpha_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}} U_{\mathfrak{p}_2^2} = \lambda_{T_{\mathfrak{p}_2^2}}(g) g_{\alpha_{\mathfrak{p}_1}}(z) - \left(\frac{-1}{\wp_2}\right)_2^k N(\wp_2)^{k-1} g_{\alpha_{\mathfrak{p}_1}, \left(\frac{\bullet}{\wp_2}\right)_2}(z) - N(\wp_2)^{2k-1} g_{\alpha_{\mathfrak{p}_1}}(\wp_2^2 z) \tag{9.14}$$

or

$$g_{\alpha_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}} U_{\mathfrak{p}_2^2} = \lambda_{T_{\mathfrak{p}_2^2}}(g) g_{\alpha_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}} V'_{(1)} - \left(\frac{-1}{\wp_2}\right)_2^k N(\wp_2)^{k-1} g_{\alpha_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}} V_{\left(\frac{\bullet}{\wp_2}\right)_2} - N(\wp_2)^{2k-1} g_{\alpha_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}} V_{\mathfrak{p}_2^2}.$$

Next we act 9.14 by $U_{\mathfrak{p}_2^2}$ again and get

$$g_{\alpha_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}} U_{\mathfrak{p}_2^2}^2 = \lambda_{T_{\mathfrak{p}_2^2}}(g) g_{\alpha_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}} U_{\mathfrak{p}_2^2} - \left(\frac{-1}{\wp_2}\right)_2^k N(\wp_2)^{k-1} g_{\alpha_{\mathfrak{p}_1}, \left(\frac{\bullet}{\wp_2}\right)_2}(z)|_{k+\frac{1}{2}} U_{\mathfrak{p}_2^2} - N(\wp_2)^{2k-1} g_{\alpha_{\mathfrak{p}_1}}(\wp_2^2 z)|_{k+\frac{1}{2}} U_{\mathfrak{p}_2^2}.$$

From the definition 7.1.2 of the quadratic symbol, we have $\left(\frac{\wp_2^2 \xi}{\wp_2}\right)_2 = 0$ as $\wp_2^2 \xi \in (\wp_2)$.
Hence, as before, we conclude $g_{\left(\frac{\bullet}{\wp_2}\right)}(z)$ lies in the kernel of $U_{\mathfrak{p}_2^2}$. It follows,

$$g_{\alpha_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}} U_{\mathfrak{p}_2^2}^2 = \lambda_{T_{\mathfrak{p}_2^2}}(g) g_{\alpha_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}} U_{\mathfrak{p}_2^2} - N(\wp_2)^{2k-1} g_{\alpha_{\mathfrak{p}_1}}(z).$$

This relation rewrites as

$$g_{\alpha_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}}\left(U_{\mathfrak{p}_2^2}^2 - \lambda_{T_{\mathfrak{p}_2^2}}(g) U_{\mathfrak{p}_2^2} + N(\wp_2)^{2k-1}\right) = 0.$$

Thus, $U_{\mathfrak{p}_2^2}$ satisfies the quadratic polynomial $x^2-\lambda_{T_{\mathfrak{p}_2^2}}(g)x+N(\wp_2)^{2k-1}$ on the two dimensional space spanned by $\{g_{\alpha_{\mathfrak{p}_1}}(z), g_{\alpha_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}_2^2}\}$ that has level at most $4\mathfrak{n}\mathfrak{p}_1\mathfrak{p}_2$, see remark 7.3.4. We may factor this quadratic polynomial as $(x-\alpha_{\mathfrak{p}_2})(x-\beta_{\mathfrak{p}_2})$ where $\alpha_{\mathfrak{p}_2}$ and $\beta_{\mathfrak{p}_2}$ are algebraic integers satisfying $\alpha_{\mathfrak{p}_2}+\beta_{\mathfrak{p}_2}=\lambda_{T_{\mathfrak{p}_2}}(g)$ and $\alpha_{\mathfrak{p}_2}\beta_{\mathfrak{p}_2}=N(\wp_2)^{2k-1}$.

We now define four $U_{\mathfrak{p}_2^2}$ eigenforms at level $4\mathfrak{n}\mathfrak{p}_1\mathfrak{p}_2=4\mathfrak{n}(p)$:

$$\begin{aligned}
g_{\alpha_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z) &:= g_{\alpha_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}}(U_{\mathfrak{p}_2^2}-\beta_{\mathfrak{p}_2}) \text{ such that } g_{\alpha_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}_2^2}=\alpha_{\mathfrak{p}_2}g_{\alpha_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z); \\
g_{\alpha_{\mathfrak{p}_1},\beta_{\mathfrak{p}_2}}(z) &:= g_{\alpha_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}}(U_{\mathfrak{p}_2^2}-\alpha_{\mathfrak{p}_2}) \text{ such that } g_{\alpha_{\mathfrak{p}_1},\beta_{\mathfrak{p}_2}}(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}_2^2}=\beta_{\mathfrak{p}_2}g_{\alpha_{\mathfrak{p}_1},\beta_{\mathfrak{p}_2}}(z); \\
g_{\beta_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z) &:= g_{\beta_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}}(U_{\mathfrak{p}_2^2}-\beta_{\mathfrak{p}_2}) \text{ such that } g_{\beta_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}_2^2}=\alpha_{\mathfrak{p}_2}g_{\beta_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z); \\
g_{\beta_{\mathfrak{p}_1},\beta_{\mathfrak{p}_2}}(z) &:= g_{\beta_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}}(U_{\mathfrak{p}_2^2}-\alpha_{\mathfrak{p}_2}) \text{ such that } g_{\beta_{\mathfrak{p}_1},\beta_{\mathfrak{p}_2}}(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}_2^2}=\beta_{\mathfrak{p}_2}g_{\beta_{\mathfrak{p}_1},\beta_{\mathfrak{p}_2}}(z).
\end{aligned} \tag{9.15}$$

Using 9.15, we can write

$$g_{\alpha_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}_1^2}=g_{\alpha_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}}(U_{\mathfrak{p}_2^2}-\beta_{\mathfrak{p}_2})U_{\mathfrak{p}_1^2}.$$

Since $U_{\mathfrak{p}_1^2}$ and $U_{\mathfrak{p}_2^2}$ commute, it follows

$$\begin{aligned}
g_{\alpha_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}_1^2} &= g_{\alpha_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}}U_{\mathfrak{p}_1^2}(U_{\mathfrak{p}_2^2}-\beta_{\mathfrak{p}_2}) \\
&= \lambda_{T_{\mathfrak{p}_1^2}}(g)g_{\alpha_{\mathfrak{p}_1}}(z)|_{k+\frac{1}{2}}(U_{\mathfrak{p}_2^2}-\beta_{\mathfrak{p}_2}) \\
&= \lambda_{T_{\mathfrak{p}_1^2}}(g)g_{\alpha_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z).
\end{aligned}$$

This implies $g_{\alpha_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z)$ is also a $U_{\mathfrak{p}_1^2}$ eigenform. Similarly, $g_{\alpha_{\mathfrak{p}_1},\beta_{\mathfrak{p}_2}}(z)$, $g_{\beta_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z)$ and $g_{\beta_{\mathfrak{p}_1},\beta_{\mathfrak{p}_2}}(z)$ are also $U_{\mathfrak{p}_1^2}$ eigenforms.

We call $g_{\alpha_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z)$, $g_{\alpha_{\mathfrak{p}_1},\beta_{\mathfrak{p}_2}}(z)$, $g_{\beta_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z)$ and $g_{\beta_{\mathfrak{p}_1},\beta_{\mathfrak{p}_2}}(z)$, the Hilbert p-stabilised half-integer weight forms at level $4\mathfrak{n}\mathfrak{p}_1\mathfrak{p}_2=4\mathfrak{n}(p)$ associated to the Hilbert half-integer weight newform $g(z)$ at level $4\mathfrak{n}$.

Remark 9.6.3. Now if $g(z)\in S_{k+\frac{1}{2}}^{\text{new, ord}}(\widetilde{\Gamma}_{4\mathfrak{n}};\mathcal{O}_L)$ is p-ordinary, then it's $T_{\mathfrak{p}_1^2}$ and $T_{\mathfrak{p}_2^2}$ eigenvalue must be $\mathfrak{P}_1$-adic and $\mathfrak{P}_2$-adic units, where $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are fixed prime ideals above $\mathfrak{p}_1$ and $\mathfrak{p}_2$ respectively. In other words, $|\lambda_{T_{\mathfrak{p}_1^2}}(g)|_{\mathfrak{P}_1}=|\lambda_{T_{\mathfrak{p}_2^2}}(g)|_{\mathfrak{P}_2}=1$. We can therefore choose $\alpha_{\mathfrak{p}_1}$ and $\alpha_{\mathfrak{p}_2}$ to be a $\mathfrak{P}_1$-adic and $\mathfrak{P}_2$-adic units respectively. This then fixes a unique ordinary p-stabilised form $g_{\alpha_{\mathfrak{p}_1},\alpha_{\mathfrak{p}_2}}(z)\in S_{k+\frac{1}{2}}^{\text{ord}}(\widetilde{\Gamma}_{4\mathfrak{n}\mathfrak{p}_1\mathfrak{p}_2};\mathcal{O}_L)$ with $U_{\mathfrak{p}_1^2}$ eigenvalue being $\alpha_{\mathfrak{p}_1}$, a $\mathfrak{P}_1$-adic unit and $U_{\mathfrak{p}_2^2}$ eigenvalue being $\alpha_{\mathfrak{p}_2}$, a $\mathfrak{P}_2$-adic unit.

CHAPTER 10

Congruences related to Hilbert modular forms

10.1 Background and Notation

Let K be a totally real quadratic field with narrow class number 1 and let $\mathcal{O}_K$ be its corresponding ring of integers. Let D be the discriminant of K and let χ_D be the associated quadratic character of K given by the Kronecker symbol $\left(\frac{D}{\bullet}\right)$. Let $k \in \mathbb{Z}_{>0}$ and let $\mathfrak{n}$ be a square-free integral ideal of odd norm in $\mathcal{O}_K$. Let p be an odd rational prime such that the ideal generated by p in $\mathcal{O}_K$ is co-prime to $\mathfrak{n}$. We assume that p is unramified in K, that is, it's not ramified in K. Let $\nu_\mathfrak{p}$ be the valuation determined by the prime ideal $\mathfrak{p}$ above p that extends the standard valuation ν_p on $\mathbb{Q}$. Since p does not ramify in K, the ramification index of $\mathfrak{p}$ in p is $e_\mathfrak{p} = 1$. Thus if we normalise $\nu_\mathfrak{p}$ such that for $x \in \mathcal{O}_K$, $\nu_\mathfrak{p}(x) = n/e_\mathfrak{p}$ where $x \in \mathfrak{p}^n$ and $e_\mathfrak{p} = 1$, then the image of both ν_p and $\nu_\mathfrak{p}$ lies in $\mathbb{Z}$. For simplicity, we will denote $\mathfrak{n}(p)$ by $\mathfrak{n}p$ where $(p) = p\mathcal{O}_K$.

Let $\mathfrak{m} \subset \mathcal{O}_K$ be an integral ideal such that $\mathfrak{m} \mid \mathfrak{n}$. From now on, to make notation simple, we let $q^\xi = e^{2\pi i \operatorname{Tr}\left(\frac{\xi}{2}\right)}$. Let $f(z) = \sum_{\xi \in \mathcal{O}_K^+} a_\xi q^\xi$ in $S_{2k}^{\mathrm{new}}(\Gamma_\mathfrak{m})$ be a Hilbert newform. Then by [Shi81, Section 2, pg. 650], for any normalised integer weight Hilbert eigenform, in particular $f(z)$, there exists a fixed number number field L_f with ring of integers $\mathcal{O}_f$ such that for each $\xi \in \mathcal{O}_K^+$, $a_\xi \in \mathcal{O}_f$.

Let $g(z) \in S_{k+\frac{1}{2}}^{new}(\widetilde{\Gamma}_{4\mathfrak{m}})$ be a Hilbert newform of half-integer weight associated to $f(z)$ under *Kohnen's* isomorphism for Hilbert modular forms. Then *Shimura* in [Shi87, Lemma 8.8] and [Shi87, Proposition 8.9] showed that the Fourier coefficients of any half-integer weight Hilbert eigenform are algebraic numbers in a fixed number field. In particular for $g(z)$, let this fixed number field be L_g. We now want to show that there exists a normalisation of $g(z)$ such that its Fourier coefficients are algebraic integers in $\mathcal{O}_g$-the ring of integers of L_g. We intend to prove this by generalising the result [SS77, Lemma 8, Section 5] by *Serre* and *Stark* for classical modular forms of half-integer weight to Hilbert modular forms of half-integer weight. For weight $\frac{1}{2}$, this has been shown by *S. Achimescu* and *A. Saha* in [AS08, Corollary 1.1, pg.7].

Theorem 10.1.1. *If $g(z) = \sum_{\xi \in \mathcal{O}_K^+} b_\xi e^{2\pi i \, Tr\left(\frac{\xi}{2} z\right)} \in S_{k+\frac{1}{2}}^{new}(\widetilde{\Gamma}_{4\mathfrak{m}}, \psi)$ and the Fourier coefficient's are algebraic numbers, then these have bounded denominators. In other words, there exists a non-zero algebraic integer $\mathcal{D}$ such that $\mathcal{D} b_\xi$ is an algebraic integer for all $\xi \in \mathcal{O}_K^+$.*

Proof. We closely imitate the steps in the proof of [SS77, Lemma 8, Section 5] for classical half-integer weight modular forms. We will use the familiar device of multiplying by a fixed form $g_0(z)$. Choose

$$g_0(z) = \Theta(z)^{3(2k+1)} \in M_{\frac{3(2k+1)}{2}}(\widetilde{\Gamma}_{(4)}) \tag{10.1}$$

where $\Theta(z)$ is the Hecke Theta function defined in 7.1.1.

Note 10.1.2. The reason to choose such a power of $\Theta(z)$ is to get rid of any characters involved in the space and also so that we land on to an integer weight space on multiplication by $g(z)$.

Observe that the map $g(z) \mapsto g_0(z)g(z) = \Theta^{3(2k+1)}(z)g(z)$ sends the space of half-integer weight Hilbert cuspforms $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{m}}, \psi)$ into the space of integer weight Hilbert cusp forms $S_{2(2k+1)}(\Gamma_{\mathfrak{m}}, \psi)$. Now, the Fourier coefficients of $g_0(z)$ are algebraic integers and that of $g(z)$ are algebraic numbers which implies that the Fourier coefficients of the product $g_0(z)g(z)$ are also algebraic numbers. From the remark following Theorem 1 of [Shi75, pg. 711], every integer weight Hilbert modular form whose Fourier coefficients are algebraic numbers has bounded denominators. In other words, there exists an algebraic integer $\mathscr{C}$ such that the Fourier coefficients of $\mathscr{C} g_0(z)g(z)$ are algebraic integers.

Now the main observation we make is that dividing by $g_0(z)$ does not increase denominators as Fourier coefficients of $(g_0(z))^{-1}$ are also algebraic integers. Suppose

$$g_0(z) = \sum_{\eta \in \mathcal{O}_K^+ \cup \{0\}} a_\eta q^\eta \quad \text{and} \quad (g_0(z))^{-1} = \sum_{\eta \in \mathcal{O}_K^+ \cup \{0\}} b_\eta q^\eta$$

where we already know $a_0 = 1$.

Then we have

$$\begin{aligned} g_0(z)(g_0(z))^{-1} &= \left(\sum_{\eta \in \mathcal{O}_K^+ \cup \{0\}} a_\eta q^\eta \right) \left(\sum_{\eta \in \mathcal{O}_K^+ \cup \{0\}} b_\eta q^\eta \right) \\ &= \sum_{\eta \in \mathcal{O}_K^+ \cup \{0\}} c_\eta q^\eta \end{aligned}$$

where

$$c_\eta = \sum_{\substack{\xi \in \mathcal{O}_K^+ \cup \{0\} \\ (\eta - \xi) \in \mathcal{O}_K^+ \cup \{0\}}} a_\xi b_{\eta - \xi}.$$

Since $g_0(z)(g_0(z))^{-1} = 1$, it implies

$$\begin{aligned} 1 &= \sum_{\eta \in \mathcal{O}_K^+ \cup \{0\}} \left(\sum_{\substack{\xi \in \mathcal{O}_K^+ \cup \{0\} \\ (\eta - \xi) \in \mathcal{O}_K^+ \cup \{0\}}} a_\xi b_{\eta - \xi} \right) q^\eta \\ &= \sum_{\eta \in \mathcal{O}_K^+ \cup \{0\}} \left(a_0 b_\eta + \sum_{\substack{\xi \in \mathcal{O}_K^+ \\ (\eta - \xi) \in \mathcal{O}_K^+ \cup \{0\} \\ \xi \in \mathcal{O}_K^+ \Rightarrow \mathrm{Tr}(\eta - \xi) < \mathrm{Tr}(\eta)}} a_\xi b_{\eta - \xi} \right) q^\eta \\ &= \sum_{\eta \in \mathcal{O}_K^+ \cup \{0\}} a_0 b_\eta q^\eta + \sum_{\eta \in \mathcal{O}_K^+} \left(\sum_{\substack{\xi \in \mathcal{O}_K^+ \\ (\eta - \xi) \in \mathcal{O}_K^+ \cup \{0\} \\ \xi \in \mathcal{O}_K^+ \Rightarrow \mathrm{Tr}(\eta - \xi) < \mathrm{Tr}(\eta)}} a_\xi b_{\eta - \xi} \right) q^\eta \end{aligned}$$

Note that $\xi \in \mathcal{O}_K^+$ implies $\mathrm{Tr}(\eta-\xi) < \mathrm{Tr}(\eta)$. We will now show that b_η is an algebraic integer for any $\eta \in \left(\mathcal{O}_K^+ \cup \{0\}\right)$. We will do so by using induction on $\mathrm{Tr}(\eta)$.

Let $\mathrm{Tr}(\eta)=0$. Then

$$1 = \sum_{\eta \in \mathcal{O}_K^+ \cup \{0\}} a_0 b_\eta q^\eta$$

Then,

$$a_0 b_\eta = \begin{cases} 1 & \text{for } \eta = 0; \\ 0 & \text{for } \eta \in \mathcal{O}_K^+. \end{cases}$$

Since $a_0 = 1$, this implies $b_0 = 1$ and for $\eta \in \mathcal{O}_K^+$, we have $b_\eta = 0$. Thus, in the case when $\mathrm{Tr}(\eta)=0$, b_η is an algebraic integer for all $\eta \in (\mathcal{O}_K^+ \cup \{0\})$.

Next, we assume that $b_{\eta-\xi}$ is an algebraic integer whenever $\mathrm{Tr}(\eta-\xi) < \mathrm{Tr}(\eta)$. Then, we can write

$$\sum_{\eta \in \mathcal{O}_K^+ \cup \{0\}} b_\eta q^\eta == \frac{1}{a_0}\Bigg(1 - \sum_{\eta \in \mathcal{O}_K^+}\Bigg(\sum_{\substack{\xi \in \mathcal{O}_K^+ \\ (\eta-\xi)\in \mathcal{O}_K^+ \cup \{0\} \\ \xi \in \mathcal{O}_K^+ \Rightarrow \mathrm{Tr}(\eta-\xi) < \mathrm{Tr}(\eta)}} a_\xi b_{\eta-\xi}\Bigg) q^\eta\Bigg). \tag{10.2}$$

We note that $a_0 = 1$, a_ξ is an algebraic integer for all $\xi \in \mathcal{O}_K^+$ because $g_0(z)$ is a power of $\Theta(z)$ which has integer Fourier coefficients and $b_{\eta-\xi}$ is also an algebraic integer for all $\xi \in \mathcal{O}_K^+$ by induction hypothesis. Thus, by comparing either side of equation 10.2, we conclude that b_η is an algebraic integer for all $\eta \in \left(\mathcal{O}_K^+ \cup \{0\}\right)$. This shows that the Fourier coefficients of $(g_0(z))^{-1}$ are also algebraic integers.

Then it immediately follows that the Fourier coefficients of $\mathscr{C} g(z)$ are algebraic integers. □

So, by Theorem 10.1.1, there exists a normalisation of $g(z) \in S^{new}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{m}})$ such that its Fourier coefficients are algebraic integers in $\mathcal{O}_g$, the ring of integers of its fixed number field L_g.

Let $j \in \mathbb{Z}_{>0}$ be a fixed integer and define $k' := k + \frac{p^j(p-1)}{2}$. Now, let us take a big enough number field L/K containing number fields L_f, $L_{\mathscr{F}}$, L_g and $L_{\mathscr{G}}$ for all newforms in $f(z), \mathscr{F}(z)$ in $S^{\text{new}}_{2k}(\Gamma_\mathfrak{m})$, $S^{\text{new}}_{2k'}(\Gamma_\mathfrak{m})$ and their associated Hilbert newforms $g(z), \mathscr{G}(z)$ of half-integer weight in $S^{new}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{m}})$, $S^{new}_{k'+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{m}})$ respectively over all divsiors $\mathfrak{m} \mid \mathfrak{n}p$.

Let $\mathcal{O}_L$ be the ring of integers of L. We now fix a prime ideal $\mathfrak{P} \subset \mathcal{O}_L$ above $\mathfrak{p}$ such that the valuation $v_\mathfrak{P}$ determined by $\mathfrak{P}$ is normalised as: For $x \in \mathcal{O}_L$, $v_\mathfrak{P}(x) = \frac{n}{e_\mathfrak{P}}$ where $x \in \mathfrak{P}^n$ and $e_\mathfrak{P}$ is the ramification index of $\mathfrak{P}$ in the factorisation of $\mathfrak{p}\mathcal{O}_L$. In other words, we have $v_\mathfrak{P}(\mathfrak{p}) = v_\mathfrak{p}(\mathfrak{p}) = 1$.

Denote the set of $\mathfrak{P}$-integral elements in L by $\mathcal{O}_{(\mathfrak{P})}$, that is

$$\mathcal{O}_{(\mathfrak{P})} = \left\{ \frac{x}{y} \mid x, y \in \mathcal{O}_L,\ y \notin \mathfrak{P} \right\}.$$

For a power series $f(z) = \sum_{\xi \in \mathcal{O}_K^+} a_\xi q^\xi$, we define

$$v_p(f(z)) := \inf(v_p(a_n)).$$

Let $F^\mathfrak{P}$ be a finite extension of $\mathbb{Q}_p$ containing L that extends v_p to $v_\mathfrak{P}$. Let $\mathcal{O}_{F^\mathfrak{P}}$ be its corresponding ring of integers. We can then embed $L \hookrightarrow \overline{\mathbb{Q}_p}$ or embed $L \hookrightarrow \mathbb{C}$. It therefore makes sense to view the Fourier coefficients of $f(z)$ $\mathfrak{P}$-adically embedded in $\mathcal{O}_{F^\mathfrak{P}}$.

Let $S_{2k}(\Gamma_{\mathfrak{m}};\mathcal{O}_L)$ be the $\mathcal{O}_L$-submodule of $S_{2k}(\Gamma_{\mathfrak{m}};L)$ containing Hilbert cusp forms in $S_{2k}(\Gamma_{\mathfrak{m}};L)$ that have Fourier coefficients in $\mathcal{O}_L$. Now define

$$S_{2k}(\Gamma_{\mathfrak{m}};\mathcal{O}_{F^{\mathfrak{P}}}) := S_{2k}(\Gamma_{\mathfrak{m}};\mathcal{O}_L)\otimes_{\mathcal{O}_L}\mathcal{O}_{F^{\mathfrak{P}}}.$$

Similarly, let $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{m}};\mathcal{O}_L)$ be the $\mathcal{O}_L$-submodule of $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{m}};L)$ containing half-integer weight Hilbert cusp forms in $S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{m}};L)$ that have Fourier coefficients in $\mathcal{O}_L$. Similarly, define

$$S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{m}};\mathcal{O}_{F^{\mathfrak{P}}}) := S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{m}};\mathcal{O}_L)\otimes_{\mathcal{O}_L}\mathcal{O}_{F^{\mathfrak{P}}}.$$

Let $\mathbb{T}_{2k}(\mathfrak{m};\mathcal{O}_L)$ be the commutative $\mathcal{O}_L$-subalgebra of the $\mathrm{End}(S_{2k}(\Gamma_{\mathfrak{m}};L))$ generated by $T_{\mathfrak{l}}$ where $\mathfrak{l}$ runs over all prime ideals in $\mathcal{O}_K$. Define

$$\mathbb{T}_{2k}(\mathfrak{m};\mathcal{O}_{F^{\mathfrak{P}}}) := \mathbb{T}_{2k}(\mathfrak{m};\mathcal{O}_L)\otimes_{\mathcal{O}_L}\mathcal{O}_{F^{\mathfrak{P}}}.$$

Similarly, let $\widetilde{\mathbb{T}}_{k+\frac{1}{2}}(4\mathfrak{m};\mathcal{O}_L)$ be the commutative $\mathcal{O}_L$-subalgebra of the $\mathrm{End}(S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{m}};L))$ generated by $T_{\mathfrak{l}^2}$ where $\mathfrak{l}$ runs over all prime ideals in $\mathcal{O}_K$. Define

$$\widetilde{\mathbb{T}}_{k+\frac{1}{2}}(4\mathfrak{m};\mathcal{O}_{F^{\mathfrak{P}}}) := \widetilde{\mathbb{T}}_{k+\frac{1}{2}}(4\mathfrak{m};\mathcal{O}_L)\otimes_{\mathcal{O}_L}\mathcal{O}_{F^{\mathfrak{P}}}.$$

10.2 Congruences related to Hilbert modular form-Integer weight case

In this section, we prove a series of results to establish a mod p congruence between Fourier coefficients of integer weight $\mathfrak{n}$-new Hilbert eigenforms with varying weights. Before we do that, we briefly set up the key ingredients in this section.

From equation 6.2, we have the following decomposition of complex vector spaces of Hilbert cusp forms:

$$S_{2k}(\Gamma_{\mathfrak{n}p};L) = \bigoplus_{\substack{\mathfrak{m}\mid\mathfrak{n}p\\ \mathfrak{b}\mid\mathfrak{n}p\mathfrak{m}^{-1}}} S_{2k}^{new}(\Gamma_{\mathfrak{m}};L)|_{2k}V_{\mathfrak{b}} \tag{10.3}$$

where $\mathfrak{b}$ runs over all integral ideals in $\mathcal{O}_K$ that divide $\mathfrak{n}p\mathfrak{m}^{-1}$.

Remark 10.2.1. We can replace the $V_{\mathfrak{b}}$ operator in the above definition by the $U_{\mathfrak{b}}$ operator for $\mathfrak{b}$ not dividing the level, see remark 6.3.3. Recall that $U_{\mathfrak{b}}$ is the operator which replaces every ξ^{th} Fourier coefficient for $\xi\in\mathcal{O}_K^+$ with the $(\mathscr{B}\xi)^{\text{th}}$ Fourier coefficient where $\mathscr{B}\gg 0$ is a totally positive generator of $\mathfrak{b}$.

We will now write down the space of $\mathfrak{n}$-new Hilbert cusp forms at level $\mathfrak{n}p$ explicitly.

$$S_{2k}^{\mathfrak{n}\text{-new}}(\Gamma_{\mathfrak{n}p};L) = S_{2k}^{\text{new}}(\Gamma_{\mathfrak{n}p};L)\oplus S_{2k}^{\text{new}}(\Gamma_{\mathfrak{n}};L)|_{2k}U_{(p)}\oplus S_{2k}^{\text{new}}(\Gamma_{\mathfrak{n}};L) \tag{10.4}$$

Let I denote a finite index set such that $|\,I\,| = \dim(S_{2k}^{\mathfrak{n}\text{-new}}(\Gamma_{\mathfrak{n}p};L))$. Recall from Theorem 6.4.6 that the Hilbert newspace has an orthogonal basis of newforms with Fourier coefficients in $\mathcal{O}_L$. This can be applied to each newspace in the direct sum in 10.4. Hence, we can take $\{f_i^{\mathfrak{n}}(z)\}_{i\in I}\subseteq S_{2k}^{\mathfrak{n}\text{-new}}(\Gamma_{\mathfrak{n}p};\mathcal{O}_L)$ to be a basis of the space $S_{2k}^{\mathfrak{n}\text{-new}}(\Gamma_{\mathfrak{n}p};L)$ consisting of $\mathfrak{n}$-new Hilbert Hecke-eigenforms for all Hecke operators $T_{\mathfrak{l}}$ where $\mathfrak{l}$ is a prime ideal in $\mathcal{O}_K$ (including $T_{\mathfrak{p}} = U_{\mathfrak{p}}$ for each $\mathfrak{p}\mid p$).

Note 10.2.2. We have used the superscript $\mathfrak{n}$ on top of the Hilbert cusp form $f(z)$ in order to clarify that the cusp form is new at level $\mathfrak{n}$.

Let $\mathfrak{e}_p = \prod_{\mathfrak{p}}\mathfrak{e}_{\mathfrak{p}}$ be the p-ordinary projector operator where the product runs over prime ideals $\mathfrak{p}\subset\mathcal{O}_K$ above p, see section 9.2. We apply this projection operator on each element in our basis $\{f_i^{\mathfrak{n}}(z)\}_{i\in I}$ as follows:

$$\{f_i^{\mathfrak{n}}(z)|_{2k}\mathfrak{e}_p\}_{i\in I} = \{f_i^{\mathfrak{n}}(z)\}_{i=1}^{m_{\mathfrak{n}}}\cup\{0\}.$$

Here $m_{\mathfrak{n}} = \dim(S_{2k}^{\mathfrak{n}\text{-new, ord}}(\Gamma_{\mathfrak{n}p};L))$ and the set $\{f_i^{\mathfrak{n}}(z)\}_{i=1}^{m_{\mathfrak{n}}}$ includes p-stabilised U_p Hilbert eigenforms obtained from newforms in decomposition 10.4. Hence, the set $\{f_i^{\mathfrak{n}}(z)\}_{i=1}^{m_{\mathfrak{n}}}\subset S_{2k}^{\mathfrak{n}\text{-new, ord}}(\Gamma_{\mathfrak{n}p};\mathcal{O}_L)$ consists of $m_{\mathfrak{n}}$ distinct $\mathfrak{n}$-new ordinary Hilbert eigenforms that form a basis for the ordinary space $S_{2k}^{\mathfrak{n}\text{-new,ord}}(\Gamma_{\mathfrak{n}p};L)$.

10.2.1 Main assumptions

We will now give two main assumptions that form a key ingredient in building the proof of the main result in this chapter.

Assumption 2. *Let*

$$\{f_i^{\mathfrak{n}}(z)\}_{i=1}^{m_{\mathfrak{n}}} \subseteq S_{2k}^{\mathfrak{n}\text{-new,ord}}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_L)$$

be a basis of Hilbert eigenforms for $S_{2k}^{\mathfrak{n}\text{-new,ord}}(\Gamma_{\mathfrak{n}p}; L)$ consisting of $\mathfrak{n}$-new Hilbert eigenforms with $\mathcal{O}_L$-integral Fourier coefficients and scaled in a way such that every element in the basis has at least one Fourier coefficient that is not divisible by $\mathfrak{P}$ where $\mathfrak{P} \mid \mathfrak{p}$ is fixed.

Let $\mathfrak{n}' \subset \mathcal{O}_K$ be an integral ideal such that $\mathfrak{n}' \mid \mathfrak{n}$ and let

$$f^{\mathfrak{n}'}(z) \in S_{2k}^{\mathfrak{n}'\text{-new,ord}}(\Gamma_{\mathfrak{n}'p}; \mathcal{O}_L)$$

also be a Hilbert eigenform with $\mathcal{O}_L$-integral Fourier coefficients. Suppose for all prime ideal $\mathfrak{l} \subset \mathcal{O}_K$ such that $\mathfrak{l} \nmid \mathfrak{n}p$,

$$\lambda_{T_{\mathfrak{l}}}(f_i^{\mathfrak{n}}) \equiv \lambda_{T_{\mathfrak{l}}}(f^{\mathfrak{n}'}) \bmod \mathfrak{P}$$

where $\lambda_{T_{\mathfrak{l}}}(f_i^{\mathfrak{n}})$ and $\lambda_{T_{\mathfrak{l}}}(f^{\mathfrak{n}'})$ denote the Hecke eigenvalues of $f_i^{\mathfrak{n}}(z)$ and $f^{\mathfrak{n}'}(z)$ for $T_{\mathfrak{l}}$ operator respectively. Then

$$f^{\mathfrak{n}'}(z) = \alpha f_i^{\mathfrak{n}}(z) \text{ for some } \alpha \in \mathcal{O}_L.$$

Remark 10.2.3. See remark 5.2.3 about the seriousness of assumption 2.

Assumption 3. *Let D be the discriminant of K and let χ_D be the associated quadratic character of K given by the Kronecker symbol $\left(\frac{D}{\bullet}\right)$. Let p be an odd rational prime and $\chi_p = \left(\frac{\bullet}{p}\right)$ be the Kronecker symbol. Then we will assume that $p \nmid B_{p-1,\chi_D}$ and $p \nmid B_{\frac{p-1}{2},\chi_p\chi_D}$ where B_{p-1,χ_D}, $B_{\frac{p-1}{2},\chi_p\chi_D}$ are generalised Bernoulli numbers.*

Remark 10.2.4. It would be certainly better if assumption 3 could be actually proved as it always turns out to be true when we look at various examples. However, in the example over $\mathbb{Q}(\sqrt{5})$ in Chapter 11, we have explicitly tested this assumption and shown that it can be easily verified in examples using SAGE.

10.2.2 Main Theorem

We will now state our main result. Again, note that by Hilbert Hecke eigenforms in Theorem 10.2.5, we mean Hilbert eigenforms for all Hecke operators $T_{\mathfrak{l}}$ for all prime ideals $\mathfrak{l}$ in $\mathcal{O}_K$.

Theorem 10.2.5 (Main Theorem). *Let $m_{\mathfrak{n}} = \dim(S_{2k}^{\mathfrak{n}\text{-new, ord}}(\Gamma_{\mathfrak{n}p}; L))$ and let*

$$\left\{f_i^{\mathfrak{n}}(z)\right\}_{i=1}^{m_{\mathfrak{n}}} \subset S_{2k}^{\mathfrak{n}\text{-new,ord}}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_L)$$

be a basis for $S_{2k}^{\mathfrak{n}\text{-new,ord}}(\Gamma_{\mathfrak{n}p}; L)$ consisting of $\mathfrak{n}$-new ordinary Hilbert eigenforms with $\mathcal{O}_L$-integral Fourier coefficients scaled in a way such that every element in the basis has at least one Fourier coefficient that is not divisible by $\mathfrak{P}$. Suppose assumption 2 holds for the basis $\{f_i^{\mathfrak{n}}(z)\}_{i=1}^{m_{\mathfrak{n}}}$ and also assume that $p \nmid B_{p-1,\chi_D}$ where χ_D is given by the Kronecker symbol $\left(\frac{D}{\bullet}\right)$ (Assumption 2 and 3 hold).
Let $j \in \mathbb{Z}_{\geq 0}$ be a fixed integer and define $k' := k + \frac{p^j(p-1)}{2}$. Then for each integer i such that $1 \leq i \leq m_{\mathfrak{n}}$, there exists an $\mathfrak{n}$-new ordinary Hilbert eigenform

$$\mathfrak{F}_i^{\mathfrak{n}}(z) \in S_{2k'}^{\mathfrak{n}\text{-new,ord}}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_{(\mathfrak{P})}),$$

unique up to scalar multiplication in $\mathcal{O}_{(\mathfrak{P})}$ such that we have the following congruence of Fourier coefficients:

$$f_i^{\mathfrak{n}}(z) \equiv \mathfrak{F}_i^{\mathfrak{n}}(z) \bmod p^{j+1}.$$

Figure 10.2.6.

$$f_i^{\mathfrak{n}} \in S_{2k}^{\mathfrak{n}\text{-new, ord}}(\Gamma_{\mathfrak{n}p};\mathcal{O}_L) \xrightarrow{\equiv \bmod p^{j+1} \text{ (Fourier coefficients)}} \mathscr{F}_i^{\mathfrak{n}} \in S_{2k'}^{\mathfrak{n}\text{-new, ord}}(\Gamma_{\mathfrak{n}p};\mathcal{O}_{(\mathfrak{P})})$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\boxed{\mathfrak{n}\text{-new ord. Hilbert eigenform of wt. } 2k} \qquad \boxed{\mathfrak{n}\text{-new ord. Hilbert eigenform of wt. } 2k'}$$

Remark 10.2.7. We note that Theorem 10.2.5 also holds true for all weights $k' = k + t\frac{(p-1)}{2}$ where t is a positive integer.

We will now prove a series of propositions that will be required to prove Theorem 10.2.5. We start by proving a mod p congruence between Fourier coefficients of integer weight Hilbert cusp forms of varying weights in proposition 10.2.8.

Proposition 10.2.8. *Let*

$$f(z) \in S_{2k}(\Gamma_{\mathfrak{n}p};\mathcal{O}_L)$$

be a Hilbert cusp form with $\mathcal{O}_L$-integral Fourier coefficients.
Let $j \in \mathbb{Z}_{\geq 0}$ be a fixed integer and define $k' := k + \frac{p^j(p-1)}{2}$. Assume that $p \nmid B_{p-1,\chi_D}$ where χ_D is given by the Kronecker symbol $\left(\frac{D}{\bullet}\right)$ (Assumption 3 holds). Then there exists a Hilbert cusp form

$$F(z) \in S_{2k'}(\Gamma_{\mathfrak{n}p};\mathcal{O}_{(\mathfrak{P})})$$

such that we have the following congruence of Fourier coefficients:

$$f(z) \equiv F(z) \text{ mod } p^{j+1}.$$

Figure 10.2.9.

$$f \in S_{2k}(\Gamma_{\mathfrak{n}p};\mathcal{O}_L) \xrightarrow{\equiv \bmod p^{j+1} \text{ (Fourier coefficients)}} \mathscr{F} \in S_{2k'}(\Gamma_{\mathfrak{n}p};\mathcal{O}_{(\mathfrak{P})})$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

Hilbert cusp form of weight $2k$ Hilbert cusp form of weight $2k'$

Proof. Define

$$F(z) := f(z)\mathscr{E}_p(z)^{p^j}$$

where $\mathscr{E}_p(z) \in M_{p-1}(\Gamma_p)$ is defined in terms of Hilbert Eisenstein series $E_{p-1}(z)$ as

$$\mathscr{E}_p(z) := E_{p-1}(z) - N(p\mathcal{O}_K)^{\frac{p-1}{2}} E_{p-1}(pz).$$

From equation 8.12, the Fourier expansion of $E_{p-1}(z)$ is

$$E_{p-1}(z) = 1 + 4(p-1)^2 B_{p-1}^{-1} \frac{1}{B_{p-1,\chi_D}} \sum_{\xi \in \mathcal{O}_K^+} \sigma_{p-2}(\xi) e^{2\pi i \operatorname{Tr}\left(\frac{\xi z}{2}\right)}.$$

Then using Theorem 3.2.2 and the fact that $p \nmid B_{p-1,\chi_D}$, we get that the Fourier coefficients of $E_{p-1}(z)$ lie in $\mathbb{Z}_{(p)}$. Then it's clear from the definition of $\mathscr{E}_p(z)$ that its Fourier coefficients lie in $\mathbb{Z}_{(p)}$. Also, note that it's given that Fourier coefficients of $f(z)$ lie in $\mathscr{O}_L$. This implies that the Fourier coefficients of $F(z)$ lie in $\mathscr{O}_{(\mathfrak{P})}$.

Next we can apply lemma 8.3.1 and get that $\mathscr{E}_p(z) \equiv 1 \bmod p$. Then working in the same way as in the proof of corollary 3.2.4, we have

$$\mathscr{E}_p(z)^{p^j} \equiv 1 \bmod p^{j+1}.$$

Hence, we conclude,

$$f(z) \equiv F(z) \bmod p^{j+1}.$$

□

We next try to show that the mod p action of Hecke operators $|_{2k}T_{\mathfrak{l}}$ on $f(z)$ is the same as the action of the the mod p action Hecke operators $|_{2k'}T_{\mathfrak{l}}$ on $F(z)$ for all prime ideals $\mathfrak{l} \subset \mathscr{O}_K$ where $f(z)$ and $F(z)$ are defined as in proposition 10.2.8.

Corollary 10.2.10. *Let $f(z) \in S_{2k}(\Gamma_{\mathfrak{n}p}; \mathscr{O}_L)$ and $F(z) \in S_{2k'}(\Gamma_{\mathfrak{n}p}; \mathscr{O}_{(\mathfrak{P})})$ be Hilbert cusp forms as defined in proposition 10.2.8 such that*

$$f(z) \equiv F(z) \ mod\ p^{j+1}$$

Then we have

$$f(z)|_{2k}T_{\mathfrak{l}} \equiv F(z)|_{2k'}T_{\mathfrak{l}} \ mod\ p^{j+1}$$

for all prime ideals $\mathfrak{l}$ in $\mathscr{O}_K$.

Proof. Let $f(z) = \sum_{\xi \in \mathscr{O}_K^+} a_\xi q^\xi$ and $F(z) = \sum_{\xi \in \mathscr{O}_K^+} A_\xi q^\xi$ be the Fourier expansions of $f(z)$ and $F(z)$ defined in proposition 10.2.8, where $q^\xi = e^{2\pi i \mathrm{Tr}\left(\frac{\xi}{2}z\right)}$. Then we have

$$A_\xi \equiv a_\xi \bmod p^{j+1}. \tag{10.5}$$

Case 1: $\mathfrak{l} \nmid \mathfrak{n}p$

From proposition 6.2.2, we can write for all prime ideals $\mathfrak{l} \subset \mathscr{O}_K, \mathfrak{l} \nmid \mathfrak{n}p$,

$$F(z)|_{2k'}T_{\mathfrak{l}} = \sum_{\xi \in \mathscr{O}_K^+} \left(A_{\mathscr{L}\xi} + N(\mathscr{L})^{2k'-1} A_{\xi/\mathscr{L}}\right) q^\xi \tag{10.6}$$

where $\mathscr{L} \gg 0$ is a totally positive generator of $\mathfrak{l}$.

From 10.5 and 10.6, we have

$$\begin{aligned} F(z)|_{2k'}T_{\mathfrak{l}} &\equiv \sum_{\xi \in \mathscr{O}_K^+} \left(a_{\mathscr{L}\xi} + N(\mathscr{L})^{2\left(k + \frac{p^j(p-1)}{2}\right)-1} a_{\xi/\mathscr{L}}\right) q^\xi \bmod p^{j+1} \\ &= \sum_{\xi \in \mathscr{O}_K^+} \left(a_{\mathscr{L}\xi} + N(\mathscr{L})^{2k-1} \left(N(\mathscr{L})^{p-1}\right)^{p^j} a_{\xi/\mathscr{L}}\right) q^\xi \bmod p^{j+1}. \end{aligned}$$

Since $N(\mathscr{L})$ is an integer, by *Fermat's Little Theorem*, we have $N(\mathscr{L})^{p-1} \equiv 1 \bmod p$. Then by a similar argument as in the proof of corollary 3.2.4, it follows $\left(N(\mathscr{L})^{p-1}\right)^{p^j} \equiv 1 \bmod p^{j+1}$.

Thus,

$$\begin{aligned} F(z)|_{2k'}T_{\mathfrak{l}} &\equiv \sum_{\xi \in \mathscr{O}_K^+} \left(a_{\mathscr{L}\xi} + N(\mathscr{L})^{2k-1} a_{\xi/\mathscr{L}}\right) e^{2\pi i \mathrm{Tr}\left(\frac{\xi}{2} z\right)} \bmod p^{j+1} \\ &= f(z)|_{2k} T_{\mathfrak{l}} \bmod p^{j+1}. \end{aligned}$$

Case 2: $\mathfrak{l} \mid \mathfrak{n}p$

Now for all prime ideals $\mathfrak{l} \subset \mathscr{O}_K, \mathfrak{l} \mid \mathfrak{n}p$, we can write

$$f(z)|_{2k'}T_{\mathfrak{l}} = \sum_{\xi \in \mathscr{O}_K^+} a_{\mathscr{L}\xi} q^{\xi} \quad \text{and} \quad F(z)|_{2k'}T_{\mathfrak{l}} = \sum_{\xi \in \mathscr{O}_K^+} A_{\mathscr{L}\xi} q^{\xi}. \tag{10.7}$$

From equation 10.5 and equation 10.7, it follows

$$A_{\mathscr{L}\xi} \equiv a_{\mathscr{L}\xi} \bmod p^{j+1}$$

or

$$F(z)|_{2k'}T_{\mathfrak{l}} \equiv f(z)|_{2k'}T_{\mathfrak{l}} \bmod p^{j+1}.$$

□

We next show that if we are given that $f(z)$ in proposition 10.2.8 is an ordinary Hilbert cusp form of weight $2k$, then $F(z)$ in proposition 10.2.8 is also an ordinary Hilbert cusp form of weight $2k'$.

Proposition 10.2.11. *Let*

$$f(z) \in S_{2k}^{ord}(\Gamma_{\mathfrak{n}p}; \mathscr{O}_L)$$

be an ordinary Hilbert cusp form with $\mathscr{O}_L$-integral Fourier coefficients.
Let $j \in \mathbb{Z}_{\geq 0}$ be a fixed integer and define $k' = k + \frac{p^j(p-1)}{2}$. Assume that $p \nmid B_{p-1,\chi_D}$ where χ_D is given by the Kronecker symbol $\left(\frac{D}{\bullet}\right)$ (Assumption 3 holds). Then there exists an ordinary Hilbert cusp form

$$F^{\circ}(z) \in S_{2k'}^{ord}(\Gamma_{\mathfrak{n}p}; \mathscr{O}_{(\mathfrak{P})})$$

such that we have the following congruence of Fourier coefficients:

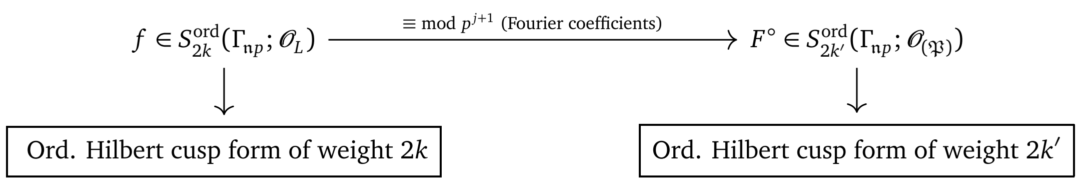
$$f(z) \equiv F^{\circ}(z) \bmod p^{j+1}.$$

Moreover,

$$f(z)|_{2k}T_{\mathfrak{l}} \equiv F^{\circ}(z)|_{2k'}T_{\mathfrak{l}} \bmod p^{j+1}$$

for all prime ideals $\mathfrak{l}$ in $\mathscr{O}_K$.

Figure 10.2.12.



Proof. Define

$$F^{\circ}(z) := F(z)|_{2k'}\mathfrak{e}_p$$

where $\mathfrak{e}_p$ is the p-ordinary projection operator defined in section 9.2 and $F(z) \in S_{2k'}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_{(\mathfrak{P})})$ is the Hilbert cusp form obtained in proposition 10.2.8.
Now $\mathfrak{e}_p$ is an idempotent operator, that is, $\mathfrak{e}_p^2 = \mathfrak{e}_p$. This implies

$$\begin{aligned} F^{\circ}(z)|_{2k'}\mathfrak{e}_p &= F(z)|_{2k'}\mathfrak{e}_p^2 \\ &= F(z)|_{2k'}\mathfrak{e}_p \\ &= F^{\circ}(z). \end{aligned}$$

Hence $F^{\circ}(z)$ is ordinary at p and lies in $S^{\mathrm{ord}}_{2k'}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_{(\mathfrak{P})})$.

Next, we use corollary 10.2.10 and write

$$\begin{aligned} F^{\circ}(z) &= F(z)|_{2k'}\mathfrak{e}_p \\ &\equiv f(z)|_{2k}\mathfrak{e}_p \bmod p^{j+1}. \end{aligned}$$

Since $f(z)$ is given to be ordinary, then $f(z)|_{2k}\mathfrak{e}_p = f(z)$. So, we can write

$$F^{\circ}(z) \equiv f(z) \bmod p^{j+1}.$$

Then the mod p^{j+1} equivalence of Hecke action follows directly from corollary 10.2.10. □

We next show that if we are given that $f(z)$ in proposition 10.2.11 is an $\mathfrak{n}$-new Hilbert eigenform of weight $2k$ for which assumption 2 holds true, then there exists $\mathscr{F}(z)$, a unique ordinary Hilbert eigenform of weight $2k'$ (unique up to scalar multiplication in $\mathcal{O}_L$) which satisfies a mod $\mathfrak{P}$ congruence of Hecke eigenvalues with $f(z)$ for all Hecke operators $T_{\mathfrak{l}}$ for all prime ideals $\mathfrak{l} \subset \mathcal{O}_K$, $\mathfrak{l} \nmid \mathfrak{n}p$.

Proposition 10.2.13. *Let* $m_{\mathfrak{n}} = dim\ (S^{\mathfrak{n}\text{-new},ord}_{2k}(\Gamma_{\mathfrak{n}p}; L))$ *and let*

$$\left\{f_i^{\mathfrak{n}}(z)\right\}_{i=1}^{m_{\mathfrak{n}}} \subset S^{\mathfrak{n}\text{-new},ord}_{2k}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_L)$$

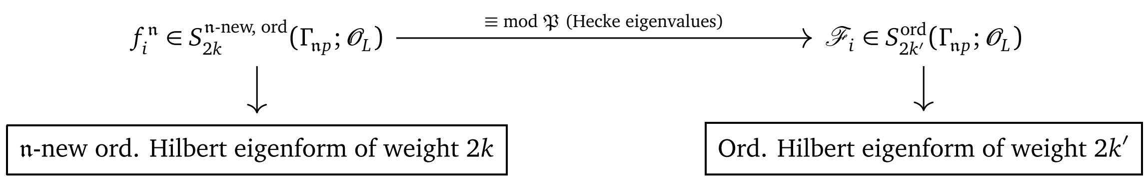
be a basis for $S^{\mathfrak{n}\text{-new},ord}_{2k}(\Gamma_{\mathfrak{n}p}; L)$ *consisting of* $\mathfrak{n}$*-new ordinary Hilbert eigenforms with* $\mathcal{O}_L$*-integral Fourier coefficients and let each* $f_i^{\mathfrak{n}}(z)$ *be scaled in a way such that it has at least one Fourier coefficient that is not divisible by* $\mathfrak{P}$*. Also, assume that assumption 2 holds true for the basis* $\left\{f_i^{\mathfrak{n}}(z)\right\}_{i=1}^{m_{\mathfrak{n}}}$*.*
Let $j \in \mathbb{Z}_{>0}$ *be a fixed integer and define* $k' := k + \frac{p^j(p-1)}{2}$*. Assume that* $p \nmid B_{p-1,\chi_D}$ *where* χ_D *is given by the Kronecker symbol* $\left(\frac{D}{\bullet}\right)$ *(Assumption 3 holds). Then for each* i *such that* $1 \leq i \leq m_{\mathfrak{n}}$*, there exists an ordinary Hilbert eigenform*

$$\mathscr{F}_i(z) \in S^{ord}_{2k'}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_L)$$

that is unique up to scalar multiplication in $\mathcal{O}_L$ *and for all prime ideals* $\mathfrak{l} \subset \mathcal{O}_K$, $\mathfrak{l} \nmid \mathfrak{n}p$ *satisfies the congruence*

$$\lambda_{T_{\mathfrak{l}}}(f_i^{\mathfrak{n}}) \equiv \lambda_{T_{\mathfrak{l}}}(\mathscr{F}_i)\ mod\ \mathfrak{P}.$$

Figure 10.2.14.



Proof. For simplicity, let us fix the integer i, say $i = 1$. Then by proposition 10.2.11, for an ordinary Hilbert cusp form $f_1^{\mathfrak{n}}(z) \in S_{2k}^{\text{n-new, ord}}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_L)$, there exists an ordinary Hilbert cusp form $F_1^{\circ}(z) \in S_{2k'}^{\text{ord}}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_{(\mathfrak{P})})$ such that

$$f_1^{\mathfrak{n}}(z) \equiv F_1^{\circ}(z) \bmod p^{j+1}.$$

Just as in equation 10.3, the space $S_{2k'}(\Gamma_{\mathfrak{n}p}; L)$ decomposes as

$$S_{2k'}(\Gamma_{\mathfrak{n}p}; L) = \bigoplus_{\substack{\mathfrak{m}|\mathfrak{n}p \\ \mathfrak{b}|\mathfrak{n}p\mathfrak{m}^{-1}}} S_{2k'}^{\text{new}}(\Gamma_{\mathfrak{m}}; L)|_{2k'} U_{\mathfrak{b}} \tag{10.8}$$

where $\mathfrak{b} \subset \mathcal{O}_K$ runs over all integral ideals that divide $\mathfrak{n}p\mathfrak{m}^{-1}$.

For each $\mathfrak{m}$, we have an orthogonal basis of Hilbert newforms for $S_{2k'}^{\text{new}}(\Gamma_{\mathfrak{m}}; L)$, see Theorem 6.4.6. These Hilbert newforms have Fourier coefficients in $\mathcal{O}_L$. Let S be a finite index set with cardinality equal to dim $(S_{2k'}(\Gamma_{\mathfrak{n}p}; L))$. Thus, we can take a basis $\{\mathscr{F}_s(z)\}_{s\in S} \subseteq S_{2k'}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_L)$ with $\mathcal{O}_L$-integral Fourier coefficients for $S_{2k'}(\Gamma_{\mathfrak{n}p}; L)$ that consists of Hilbert eigenforms for all Hecke operators $T_{\mathfrak{l}}$ for all prime ideals $\mathfrak{l} \subset \mathcal{O}_K$ (including $T_{\mathfrak{p}} = U_{\mathfrak{p}}$ for each $\mathfrak{p} \mid p$).
Then we can write $F_1^{\circ}(z) \in S_{2k'}^{\text{ord}}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_{(\mathfrak{P})})$ as a linear combination of this basis of Hilbert eigenforms:

$$F_1^{\circ}(z) = \sum_{s\in S} \alpha_s \mathscr{F}_s(z) \text{ where } \alpha_s \in L. \tag{10.9}$$

We next apply the p-ordinary projection operator $\mathfrak{e}_p$ defined in section 9.2 on either side in equation 10.9.

$$\begin{aligned} F_1^{\circ}(z)|_{2k'}\mathfrak{e}_p &= \left(\sum_{s\in S} \alpha_s \mathscr{F}_s(z)\right)|_{2k'}\mathfrak{e}_p \\ &= \sum_{s\in S} \alpha_s \left(\mathscr{F}_s(z)|_{2k'}\mathfrak{e}_p\right) \\ &= \sum_{s=1}^{t} \alpha_s \mathscr{F}_s(z) \end{aligned} \tag{10.10}$$

where $t = \dim (S_{2k'}^{\text{ord}}(\Gamma_{\mathfrak{n}p}; L))$ and the set $\{\mathscr{F}_s(z)\}_{s=1}^{t}$ includes p-stabilised $U_{\mathfrak{p}}$ Hilbert eigenforms for each prime ideal $\mathfrak{p} \mid p$ obtained from Hilbert newforms in decomposition 10.8 .

Claim: We claim that there exists an integer s such that $1 \le s \le t$, say $s = 1$, such that for all prime ideals $\mathfrak{l} \subset \mathcal{O}_K$, $\mathfrak{l} \nmid \mathfrak{n}p$, we have

$$\lambda_{T_{\mathfrak{l}}}(f_1^{\mathfrak{n}}) \equiv \lambda_{T_{\mathfrak{l}}}(\mathscr{F}_1) \bmod \mathfrak{P}.$$

We assume the contrary and try to reach a contradiction.

Suppose for every integer s such that $1 \le s \le t$, there exists some prime ideal $\mathfrak{l}_s \subset \mathcal{O}_K$ such that

$$\mathfrak{l}_s \nmid (\mathfrak{n}p) \quad \text{and} \quad \mathfrak{P} \nmid \left(\lambda_{T_{\mathfrak{l}_s}}(f_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s)\right). \tag{10.11}$$

Now let us operate equation 10.10 by the operator $|_{2k'} \prod_{s=1}^{t} \left(T_{\mathfrak{l}} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s)\right)$ on either side. We will see that this product operator kills every term in the sum on the right hand side of this equation.

$$F_1^{\circ}(z)\Big|_{2k'} \prod_{s=1}^{t}\left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s)\right) = \left(\sum_{s=1}^{t} \alpha_s \mathscr{F}_s(z)\right)\Big|_{2k'} \prod_{s=1}^{t}\left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s)\right)$$

$$
\begin{aligned}
&= \left(\sum_{s=1}^{t} \alpha_s \mathscr{F}_s(z)\right)\Bigg|_{2k'} \left(T_{\mathfrak{l}_1} - \lambda_{T_{\mathfrak{l}_1}}(\mathscr{F}_1)\right) \prod_{s=2}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s)\right) \\
&= \left(\sum_{s=1}^{t} \alpha_s (\mathscr{F}_s(z)|_{2k'} T_{\mathfrak{l}_1}) - \lambda_{T_{\mathfrak{l}_1}}(\mathscr{F}_1) \sum_{s=1}^{t} \alpha_s \mathscr{F}_s(z)\right)\Bigg|_{2k'} \prod_{s=2}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s)\right) \\
&= \left(\sum_{s=1}^{t} \alpha_s \lambda_{T_{\mathfrak{l}_1}}(\mathscr{F}_s) \mathscr{F}_s(z) - \sum_{s=1}^{t} \alpha_s \lambda_{T_{\mathfrak{l}_1}}(\mathscr{F}_1) \mathscr{F}_s(z)\right)\Bigg|_{2k'} \prod_{s=2}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s)\right) \\
&= \left(\sum_{s=1}^{t} \alpha_s \left(\lambda_{T_{\mathfrak{l}_1}}(\mathscr{F}_s) - \lambda_{T_{\mathfrak{l}_1}}(\mathscr{F}_1)\right) \mathscr{F}_s(z)\right)\Bigg|_{2k'} \prod_{s=2}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s)\right) \\
&= \sum_{s=1}^{t} \alpha_s \prod_{w=1}^{t} \left(\lambda_{T_{\mathfrak{l}_w}}(\mathscr{F}_s) - \lambda_{T_{\mathfrak{l}_w}}(\mathscr{F}_w)\right) \mathscr{F}_s(z) \\
&= 0. \qquad (10.12)
\end{aligned}
$$

Now replacing $F_1^{\circ}(z)$ by $f_1^{\mathfrak{n}}(z)$ modulo p^{j+1}, we see that the same product operator $\prod_{s=1}^{t}\left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s)\right)$ modulo p^{j+1} does not act as a zero operator on $F_1^{\circ}(z)$.

$$
\begin{aligned}
F_1^{\circ}(z)\Bigg|_{2k'} \prod_{s=1}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s)\right) &\equiv f_1^{\mathfrak{n}}(z)\Bigg|_{2k} \prod_{s=1}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s)\right) \bmod p^{j+1} \\
&= f_1^{\mathfrak{n}}(z)\Bigg|_{2k} (T_{\mathfrak{l}_2} - \lambda_{T_{\mathfrak{l}_2}}(\mathscr{F}_2)) \prod_{\substack{s=1 \\ s\neq 2}}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s)\right) \bmod p^{j+1} \\
&= \left(f_1^{\mathfrak{n}}(z)|_{2k} T_{\mathfrak{l}_2} - \lambda_{T_{\mathfrak{l}_2}}(\mathscr{F}_2) f_1^{\mathfrak{n}}(z)\right)\Bigg|_{2k} \prod_{\substack{s=1 \\ s\neq 2}}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s)\right) \bmod p^{j+1} \\
&= \left(\lambda_{T_{\ell_2}}(f_1^{\mathfrak{n}}) f_1^{\mathfrak{n}}(z) - \lambda_{T_{\mathfrak{l}_2}}(\mathscr{F}_2) f_1^{\mathfrak{n}}(z)\right)\Bigg|_{2k} \prod_{\substack{s=1 \\ s\neq 2}}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s)\right) \bmod p^{j+1} \\
&= \left(\lambda_{T_{\mathfrak{l}_2}}(f_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_2}}(\mathscr{F}_2)\right) f_1^{\mathfrak{n}}(z)\Bigg|_{2k} \prod_{\substack{s=1 \\ s\neq 2}}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s)\right) \bmod p^{j+1} \\
&= \prod_{s=1}^{t} \left(\lambda_{T_{\mathfrak{l}_s}}(f_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s)\right) f_1^{\mathfrak{n}}(z) \bmod p^{j+1}.
\end{aligned}
$$

First note that $f_1^{\mathfrak{n}}(z)$ modulo $\mathfrak{P}$ is non-zero. This is because we are given a suitable scaling under which at least one Fourier coefficient of $f_1^{\mathfrak{n}}(z)$ is not divisible by $\mathfrak{P}$. Also, assumption 10.11 implies $\mathfrak{P}$ does not divide the product $\prod_{s=1}^{t}\left(\lambda_{T_{\mathfrak{l}_s}}(f_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s)\right)$. Thus, we conclude

$$
F_1^{\circ}(z)\Bigg|_{2k'} \prod_{s=1}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s)\right) \not\equiv 0 \bmod \mathfrak{P}. \qquad (10.13)
$$

From 10.12 and 10.13, we have reached a contradiction. Hence, our assumption 10.11 is false. Therefore, there exists an integer s such that $1 \le s \le t$ (say $s = 1$) for which for all prime ideals $\mathfrak{l} \subset \mathscr{O}_K$, $\mathfrak{l} \nmid \mathfrak{n}p$,

$$
\lambda_{T_{\mathfrak{l}}}(f_1^{\mathfrak{n}}) \equiv \lambda_{T_{\mathfrak{l}}}(\mathscr{F}_s) \bmod \mathfrak{P}.
$$

We can repeat the above proof for every i such that $2 \le i \le m_{\mathfrak{n}}$. Therefore, for each $1 \le i \le m_{\mathfrak{n}}$, there exists an integer s_i where $1 \le s_i \le t$ such that for all prime ideals $\mathfrak{l} \subset \mathscr{O}_K$, $\mathfrak{l} \nmid \mathfrak{n}p$, we have

$$
\lambda_{T_{\mathfrak{l}}}(f_i^{\mathfrak{n}}) \equiv \lambda_{T_{\mathfrak{l}}}(\mathscr{F}_{s_i}) \bmod \mathfrak{P}. \qquad (10.14)
$$

Uniqueness: Let $s_2 = 2$. Then $\mathscr{F}_2(z)$ is the ordinary Hilbert eigenform that satisfies the congruence 10.14 with $f_2^{\mathfrak{n}}(z)$.

Suppose $\mathscr{F}_2(z)$ is distinct from $\mathscr{F}_1(z)$, that is, there does not exist any $\mathscr{B} \in \mathscr{O}_L$ for which $\mathscr{F}_2(z) = \mathscr{B}\mathscr{F}_1(z)$ but if possible, let $\mathscr{F}_2(z)$ also satisfy the following congruence with $f_1^{\mathfrak{n}}(z)$:

$$\lambda_{T_{\mathfrak{l}}}(f_1^{\mathfrak{n}}) \equiv \lambda_{T_{\mathfrak{l}}}(\mathscr{F}_2) \bmod \mathfrak{P}.$$

Then it follows

$$\lambda_{T_{\mathfrak{l}}}(f_1^{\mathfrak{n}}) \equiv \lambda_{T_{\mathfrak{l}}}(f_2^{\mathfrak{n}}) \bmod \mathfrak{P}.$$

Then by assumption 2, $f_2^{\mathfrak{n}}(z) = \mathscr{C} f_1^{\mathfrak{n}}(z)$ for some $\mathscr{C} \in \mathscr{O}_L$. This is not possible as the set $\{f_i^{\mathfrak{n}}(z)\}_{i=1}^{m_{\mathfrak{n}}}$ forms a basis of $\mathfrak{n}$-new ordinary Hilbert eigenforms for the space $S_{2k}^{\mathfrak{n}\text{-new,ord}}(\Gamma_{\mathfrak{n}p}; \mathscr{O}_L)$ and hence consists of $m_{\mathfrak{n}}$ distinct elements.

Hence, for each $1 \leq i \leq m_{\mathfrak{n}}$, $\mathscr{F}_{s_i}(z) \in S_{2k'}^{\text{ord}}(\Gamma_{\mathfrak{n}p}; \mathscr{O}_L)$ is the unique ordinary Hilbert eigenform up to scalar multiplication in $\mathscr{O}_L$ which satisfies congruence 10.14 with $f_i^{\mathfrak{n}}(z)$. For simplicity, we choose $s_i = i$. ☐

Proposition 10.2.13 gives us a set of ordinary Hilbert eigenforms $\{\mathscr{F}_i(z)\}_{i=1}^{m_{\mathfrak{n}}}$ in $S_{2k'}^{\text{ord}}(\Gamma_{\mathfrak{n}p}; \mathscr{O}_L)$, with each element in the set being unique up to scalar multiplication in $\mathscr{O}_L$, such that for all prime ideals $\mathfrak{l} \subset \mathscr{O}_K$ such that $\mathfrak{l} \nmid \mathfrak{n}p$, we have

$$\lambda_{T_{\mathfrak{l}}}(f_i^{\mathfrak{n}}) \equiv \lambda_{T_{\mathfrak{l}}}(\mathscr{F}_i) \bmod \mathfrak{P}.$$

We next want to show that the elements in the set $\{\mathscr{F}_i(z)\}_{i=1}^{m_{\mathfrak{n}}}$ are $\mathfrak{n}$-new and forms a basis for the space $S_{2k'}^{\mathfrak{n}\text{-new,ord}}(\Gamma_{\mathfrak{n}p}; L)$. For this, we will use induction.

Proposition 10.2.15. *Let $m_{\mathfrak{n}} = dim\ (S_{2k}^{\mathfrak{n}\text{-}new,ord}(\Gamma_{\mathfrak{n}p}; L))$ and let*

$$\left\{f_i^{\mathfrak{n}}(z)\right\}_{i=1}^{m_{\mathfrak{n}}} \subset S_{2k}^{\mathfrak{n}\text{-}new,ord}(\Gamma_{\mathfrak{n}p}; \mathscr{O}_L)$$

be a basis for $S_{2k}^{\mathfrak{n}\text{-}new,ord}(\Gamma_{\mathfrak{n}p}; L)$ consisting of $\mathfrak{n}$-new ordinary Hilbert eigenforms with $\mathscr{O}_L$-integral Fourier coefficients and let each $f_i^{\mathfrak{n}}(z)$ be scaled in a way such that it has at least one Fourier coefficient that is not divisible by $\mathfrak{P}$. Also, assume that assumption 2 holds true for the basis $\left\{f_i^{\mathfrak{n}}(z)\right\}_{i=1}^{m_{\mathfrak{n}}}$.
Let $j \in \mathbb{Z}_{>0}$ be a fixed integer and define $k' := k + \frac{p^j(p-1)}{2}$. Assume that $p \nmid B_{p-1,\chi_D}$ where χ_D is given by the Kronecker symbol $\left(\frac{D}{\bullet}\right)$ (Assumption 3 holds). Then there exists a set of $\mathfrak{n}$-new ordinary Hilbert eigenforms

$$\left\{\mathscr{F}_i^{\mathfrak{n}}(z)\right\}_{i=1}^{m_{\mathfrak{n}}} \subset S_{2k'}^{\mathfrak{n}\text{-}new,ord}(\Gamma_{\mathfrak{n}p}; \mathscr{O}_L)$$

with $\mathscr{O}_L$ integral Fourier coefficients such that it forms a basis for the space $S_{2k'}^{\mathfrak{n}\text{-}new,ord}(\Gamma_{\mathfrak{n}p}; L)$ and for all prime ideals $\mathfrak{l} \nmid \mathfrak{n}p$ satisfies the congruence

$$\lambda_{T_{\mathfrak{l}}}(f_i^{\mathfrak{n}}) \equiv \lambda_{T_{\mathfrak{l}}}(\mathscr{F}_i^{\mathfrak{n}})\ mod\ \mathfrak{P}.$$

Figure 10.2.16.

$$f_i^{\mathfrak{n}} \in S_{2k}^{\mathfrak{n}\text{-new, ord}}(\Gamma_{\mathfrak{n}p}; \mathscr{O}_L) \xrightarrow{\equiv \bmod \mathfrak{P} \text{ (Hecke eigenvalues)}} \mathscr{F}_i^{\mathfrak{n}} \in S_{2k'}^{\mathfrak{n}\text{-new, ord}}(\Gamma_{\mathfrak{n}p}; \mathscr{O}_L)$$

$\downarrow$ $\mathfrak{n}$-new ord. Hilbert eigenform of weight $2k$

$\downarrow$ $\mathfrak{n}$-new ord. Hilbert eigenform of weight $2k'$

Proof. We will use induction on the level to prove this proposition.

- *Base Step:* Let $\mathfrak{n} = (1)$, that is $\mathfrak{n} = \mathcal{O}_K$.
  Let $\{f_i^{(1)}(z)\}_{i=1}^{m_{(1)}} \subseteq S_{2k}^{(1)\text{-new,ord}}(\Gamma_p; L)$ be a basis of $S_{2k}^{(1)\text{-new,ord}}(\Gamma_p; L)$ consisting of (1)-new ordinary Hilbert eigenforms where $m_{(1)} = \dim\,(S_{2k}^{(1)\text{-new,ord}}(\Gamma_p; \mathcal{O}_L))$. Then using proposition 10.2.13, we have a set of ordinary Hilbert eigenforms $\{\mathscr{F}_i^{(1)}(z)\}_{i=1}^{m_{(1)}} \subset S_{2k'}^{\text{ord}}(\Gamma_p; \mathcal{O}_L)$ with each element unique up to scalar multiplication in $\mathcal{O}_L$ such that for all prime ideals $\mathfrak{l} \subset \mathcal{O}_K$ such that $\mathfrak{l} \nmid (p)$, we have
  $$\lambda_{T_\mathfrak{l}}(f_i^{(1)}) \equiv \lambda_{T_\mathfrak{l}}(\mathscr{F}_i^{(1)}) \bmod \mathfrak{P}.$$
  Now observe that every form in $S_{2k'}(\Gamma_p)$ is trivially (1)-new. So, $S_{2k'}(\Gamma_p) = S_{2k'}^{(1)\text{-new}}(\Gamma_p)$. In particular,
  $$\{\mathscr{F}_i^{(1)}(z)\}_{i=1}^{m_{(1)}} \subseteq S_{2k'}^{\text{ord}}(\Gamma_p; \mathcal{O}_L) = S_{2k'}^{(1)\text{-new,ord}}(\Gamma_p; \mathcal{O}_L).$$
  Next, we want to show $\dim S_{2k'}^{\text{ord}}(\Gamma_p; L) = m_{(1)}$. Since $2k' \equiv 2k \bmod p-1$. Then we can apply *Hida's Control Theorem*, see corollary 9.3.3, and get
  $$\begin{aligned}\dim(S_{2k'}^{\text{ord}}(\Gamma_p; L)) &= \dim(S_{2k}^{\text{ord}}(\Gamma_p; L)) \\ &= m_{(1)}.\end{aligned} \tag{10.15}$$
  The set $\{\mathscr{F}_i^{(1)}(z)\}_{i=1}^{m_{(1)}}$ contains $m_{(1)}$ distinct elements, the set $\{\mathscr{F}_i^{(1)}(z)\}_{i=1}^{m_{(1)}}$ forms a basis of ordinary Hilbert eigenforms for the space $S_{2k'}^{\text{ord}}(\Gamma_p; L)$.

- *Induction Hypothesis*: Let us assume that proposition 10.2.15 holds for every ideal $\mathfrak{n}' \subset \mathcal{O}_K$ such that $\mathfrak{n}' \mid \mathfrak{n}$ but $\mathfrak{n}' \neq \mathfrak{n}$.

- *Induction Step*: We are given that $\{f_i^{\mathfrak{n}}(z)\}_{i=1}^{m_\mathfrak{n}} \subset S_{2k}^{\mathfrak{n}\text{-new,ord}}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_L)$ is a basis of $\mathfrak{n}$-new ordinary Hilbert eigenforms for the space $S_{2k}^{\mathfrak{n}\text{-new,ord}}(\Gamma_{\mathfrak{n}p}; L)$ where $m_\mathfrak{n} = \dim\,(S_{2k}^{\mathfrak{n}\text{-new,ord}}(\Gamma_{\mathfrak{n}p}; L))$. Then by proposition 10.2.13, we have a set of ordinary Hilbert eigenforms $\{\mathscr{F}_i^{\mathfrak{n}}(z)\}_{i=1}^{m_\mathfrak{n}} \subset S_{2k'}^{\text{ord}}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_L)$ with each element unique up to scalar multiplication in $\mathcal{O}_L$ such that for all prime ideals $\mathfrak{l} \subset \mathcal{O}_K$, $\mathfrak{l} \nmid \mathfrak{n}p$, we have
  $$\lambda_{T_\mathfrak{l}}(f_i^{\mathfrak{n}}) \equiv \lambda_{T_\mathfrak{l}}(\mathscr{F}_i^{\mathfrak{n}}) \bmod \mathfrak{P}.$$
  Note 10.2.17. Note that we have made an abuse of notation by writing the superscript $\mathfrak{n}$ for each element in the set $\{\mathscr{F}_i^{\mathfrak{n}}(z)\}_{i=1}^{m_\mathfrak{n}}$ of ordinary Hilbert eigenforms above. However, it hasn't been shown yet that the elements are $\mathfrak{n}$-new but that is our goal. We make this abuse of notation to distinguish that this set is the nominated set that we need.

  Let $\mathfrak{n}' \subset \mathcal{O}_K$ be an integral ideal such that $\mathfrak{n}' \mid \mathfrak{n}$ but $\mathfrak{n}' \neq \mathfrak{n}$. For simplicity, let us consider $i = 1$. Suppose there exists an $\mathfrak{n}'$-new Hilbert eigenform $\mathscr{F}^{\mathfrak{n}'}(z) \in S_{2k'}^{\mathfrak{n}'\text{-new, ord}}(\Gamma_{\mathfrak{n}'p}; \mathcal{O}_L)$ for some proper divisor $\mathfrak{n}'$ of $\mathfrak{n}$ such that for all prime ideals $\mathfrak{l} \subset \mathcal{O}_K$, $\mathfrak{l} \nmid \mathfrak{n}p$, we have
  $$\lambda_{T_\mathfrak{l}}(\mathscr{F}^{\mathfrak{n}'}) = \lambda_{T_\mathfrak{l}}(\mathscr{F}_1^{\mathfrak{n}})$$
  and
  $$\lambda_{T_\mathfrak{l}}(f_1^{\mathfrak{n}}) \equiv \lambda_{T_\mathfrak{l}}(\mathscr{F}^{\mathfrak{n}'}) \bmod \mathfrak{P}. \tag{10.16}$$
  However, induction hypothesis implies that $\mathscr{F}^{\mathfrak{n}'}(z)$ is a unique form up to scalar multiplication in $\mathcal{O}_L$ that satisfies the congruence 10.16 with an $\mathfrak{n}'$-new Hilbert ordinary eigenform $f^{\mathfrak{n}'}(z) \in S_{2k}^{\mathfrak{n}'\text{-new,ord}}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_L)$. That is, for all prime ideals $\mathfrak{l} \subset \mathcal{O}_K$, $\mathfrak{l} \nmid (\mathfrak{n}'p)$, we have
  $$\lambda_{T_\mathfrak{l}}(f^{\mathfrak{n}'}) \equiv \lambda_{T_\mathfrak{l}}(\mathscr{F}^{\mathfrak{n}'}) \bmod \mathfrak{P}. \tag{10.17}$$

From congruences 10.16 and 10.17, for all prime ideals $\mathfrak{l} \subset \mathcal{O}_K$, $\mathfrak{l} \nmid (\mathfrak{n}p)$, we have

$$\lambda_{T_{\mathfrak{l}}}(f^{\mathfrak{n}'}) \equiv \lambda_{T_{\mathfrak{l}}}(f_1^{\mathfrak{n}}) \bmod \mathfrak{P}$$

which is contradiction to our assumption 2. Thus, $\mathscr{F}_1^{\mathfrak{n}}(z)$ must be $\mathfrak{n}$-new.

Now we want to show that $\dim(S_{2k'}^{\mathfrak{n}\text{-new,ord}}(\Gamma_{\mathfrak{n}p}; L)) = m_{\mathfrak{n}}$. From induction hypothesis,

$$\dim(S_{2k'}^{\mathfrak{n}'\text{-new,ord}}(\Gamma_{\mathfrak{n}'p}; L)) = \dim(S_{2k}^{\mathfrak{n}'\text{-new,ord}}(\Gamma_{\mathfrak{n}'p}; L)). \tag{10.18}$$

Since $2k' \equiv 2k \bmod p-1$, then by *Hida's Control Theorem*, see corollary 9.3.3, we have

$$\dim(S_{2k'}^{\text{ord}}(\Gamma_{\mathfrak{n}p}; L)) = \dim(S_{2k}^{\text{ord}}(\Gamma_{\mathfrak{n}p}; L)). \tag{10.19}$$

Let $\sigma_0(\mathfrak{n}) = \sum_{\substack{\mathfrak{m} \text{ integral} \\ \mathfrak{m}|\mathfrak{n}}} N(\mathfrak{m})$. Using equations 10.18 and 10.19, we get

$$\begin{aligned}
&\dim(S_{2k'}^{\mathfrak{n}\text{-new,ord}}(\Gamma_{\mathfrak{n}p}; L)) \\
&= \dim(S_{2k'}^{\text{ord}}(\Gamma_{\mathfrak{n}p}; L)) - \sum_{\substack{\mathfrak{n}'|\mathfrak{n} \\ \mathfrak{n}' \neq \mathfrak{n}}} \sigma_0(\mathfrak{n}\mathfrak{n}'^{-1}) \dim(S_{2k'}^{\mathfrak{n}'\text{-new,ord}}(\Gamma_{\mathfrak{n}'p}; L)) \\
&= \dim(S_{2k}^{\text{ord}}(\Gamma_{\mathfrak{n}p}; L)) - \sum_{\substack{\mathfrak{n}'|\mathfrak{n} \\ \mathfrak{n}' \neq \mathfrak{n}}} \sigma_0(\mathfrak{n}\mathfrak{n}'^{-1}) \dim(S_{2k}^{\mathfrak{n}'\text{-new,ord}}(\Gamma_{\mathfrak{n}'p}; L)) \\
&= \dim(S_{2k}^{\mathfrak{n}\text{-new,ord}}(\Gamma_{\mathfrak{n}p}; L)) \\
&= m_{\mathfrak{n}}.
\end{aligned}$$

Hence $\{\mathscr{F}_i^{\mathfrak{n}}(z)\}_{i=1}^{m_{\mathfrak{n}}}$ is a basis of $\mathfrak{n}$-new Hilbert ordinary eigenforms for the space $S_{2k'}^{\mathfrak{n}\text{-new,ord}}(\Gamma_{\mathfrak{n}p}; L)$.

□

10.2.3 Proof of the Main Theorem

We now give the proof of the main Theorem 10.2.5

Figure 10.2.18.

$$\begin{array}{ccc}
f_i^{\mathfrak{n}} \in S_{2k}^{\mathfrak{n}\text{-new, ord}}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_L) & \xrightarrow{\equiv \bmod p^{j+1} \text{ (Fourier coefficients)}} & \mathscr{F}_i^{\mathfrak{n}} \in S_{2k'}^{\mathfrak{n}\text{-new, ord}}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_{(\mathfrak{P})}) \\
\downarrow & & \downarrow \\
\boxed{\mathfrak{n}\text{-new ord. Hilbert eigenform of wt. } 2k} & & \boxed{\mathfrak{n}\text{-new ord. Hilbert eigenform of wt. } 2k'}
\end{array}$$

Proof. For simplicity, let us fix an integer i such that $1 \leq i \leq m_{\mathfrak{n}}$, say $i = 1$.

By proposition 10.2.13, for $f_1^{\mathfrak{n}}(z) \in S_{2k}^{\mathfrak{n}\text{-new, ord}}(\Gamma_{\mathfrak{n}p}, \mathcal{O}_L)$, there exists an ordinary Hilbert cusp form $F_1^{\circ}(z) \in S_{2k'}^{\text{ord}}(\Gamma_{\mathfrak{n}p}, \mathcal{O}_{(\mathfrak{P})})$ such that we have the following congruence of Fourier coefficients:

$$f_1^{\mathfrak{n}}(z) \equiv F_1^{\circ}(z) \bmod p^{j+1}.$$

Let $t = \dim(S_{2k'}^{\text{ord}}(\Gamma_{\mathfrak{n}p}; L))$ and $\{\mathscr{F}_s^{\mathfrak{n}}(z)\}_{s=1}^{t} \subset S_{2k'}^{\text{ord}}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_L)$ be a basis for $S_{2k'}^{\text{ord}}(\Gamma_{\mathfrak{n}p}; L)$ consisting of ordinary Hilbert eigenforms.

Note 10.2.19. We have a slight abuse of notation when we write the superscript $\mathfrak{n}$ for each ordinary Hilbert eigenform in the set $\{\mathscr{F}_s^{\mathfrak{n}}(z)\}_{s=1}^t$. These ordinary Hilbert eigenforms are not necessarily $\mathfrak{n}$-new but include the set of $\mathfrak{n}$-new ordinary Hilbert eigenforms as well. Note that $t \geq m_{\mathfrak{n}}$.

So, we can write $F_1^{\circ}(z)$ as a linear combination of elements of the basis $\{\mathscr{F}_s^{\mathfrak{n}}(z)\}_{s=1}^t$:

$$F_1^{\circ}(z) = \sum_{s=1}^{t} \alpha_s \mathscr{F}^{\mathfrak{n}}(z) \text{ where } \{\alpha_s\}_{s=1}^t \subset L.$$

By proposition 10.2.15, for $\mathfrak{n}$-new ordinary Hilbert eigenform $f_1^{\mathfrak{n}}(z) \in S_{2k}^{\mathfrak{n}\text{-new,ord}}(\Gamma_{\mathfrak{n}p}; \mathscr{O}_L)$, there exists an $\mathfrak{n}$-new ordinary Hilbert eigenform in $S_{2k'}^{\mathfrak{n}\text{-new, ord}}(\Gamma_{\mathfrak{n}p}; \mathscr{O}_L)$, say $\mathscr{F}_1^{\mathfrak{n}}(z)$, unique up to scalar multiplication in $\mathscr{O}_L$ such that for all prime ideals $\mathfrak{l} \subset \mathscr{O}_K$, $\mathfrak{l} \nmid \mathfrak{n}p$, we have

$$\lambda_{T_{\mathfrak{l}}}(f_1^{\mathfrak{n}}) \equiv \lambda_{T_{\mathfrak{l}}}(\mathscr{F}_1^{\mathfrak{n}}) \bmod \mathfrak{P}. \tag{10.20}$$

This uniqueness implies that for every integer s such that $2 \leq s \leq t$ but $s \neq 1$, there exists some prime ideal $\mathfrak{l}_s \subset \mathscr{O}_K$, $\mathfrak{l}_s \nmid \mathfrak{n}p$ such that

$$\mathfrak{P} \nmid \left(\lambda_{T_{\mathfrak{l}_s}}(f_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}})\right).$$

So, we have

$$\mathfrak{P} \nmid \prod_{s=2}^{t} \left(\lambda_{T_{\mathfrak{l}_s}}(f_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}})\right).$$

In other words,

$$\frac{1}{\prod_{s=2}^{t} \left(\lambda_{T_{\mathfrak{l}_s}}(f_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}})\right)} \in \mathscr{O}_{(\mathfrak{P})}. \tag{10.21}$$

Define

$$\mathfrak{F}_1^{\mathfrak{n}}(z) := \frac{1}{\prod_{s=2}^{t} \left(\lambda_{T_{\mathfrak{l}_s}}(f_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}})\right)} F_1^{\circ}(z)\Big|_{2k'} \prod_{s=2}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}})\right). \tag{10.22}$$

Claim: We claim that $\mathfrak{F}_1^{\mathfrak{n}}(z) = \mathscr{B}\mathscr{F}_1^{\mathfrak{n}}(z)$ for some $\mathscr{B} \in \mathscr{O}_{(\mathfrak{P})}$ and $\mathfrak{F}_1^{\mathfrak{n}}(z) \equiv f_1^{\mathfrak{n}}(z) \bmod p^{j+1}$.

We will first show that $\mathfrak{F}_1^{\mathfrak{n}}(z) = \mathscr{B}\mathscr{F}_1^{\mathfrak{n}}(z)$ for some $\mathscr{B} \in \mathscr{O}_{(\mathfrak{P})}$.

$$\begin{aligned}
&F_1^{\circ}(z)\Big|_{2k'} \prod_{s=2}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}})\right) \\
&= \left(\sum_{s=1}^{t} \alpha_s \mathscr{F}_s^{\mathfrak{n}}(z)\right)\Big|_{2k'} \prod_{s=2}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}})\right) \\
&= \left(\sum_{s=1}^{t} \alpha_s \mathscr{F}_s^{\mathfrak{n}}(z)\right)\Big|_{2k'} \left(T_{\mathfrak{l}_2} - \lambda_{T_{\mathfrak{l}_2}}(\mathscr{F}_2^{\mathfrak{n}})\right) \prod_{s=3}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}})\right) \\
&= \left(\sum_{s=1}^{t} \alpha_s (\mathscr{F}_s^{\mathfrak{n}}(z)|_{2k'} T_{\mathfrak{l}_2}) - \lambda_{T_{\mathfrak{l}_2}}(\mathscr{F}_2^{\mathfrak{n}}) \sum_{s=1}^{t} \alpha_s \mathscr{F}_s^{\mathfrak{n}}(z)\right)\Big|_{2k'} \prod_{s=3}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}})\right) \\
&= \left(\sum_{s=1}^{t} \alpha_s \lambda_{T_{\mathfrak{l}_2}}(\mathscr{F}_s^{\mathfrak{n}}) \mathscr{F}_s^{\mathfrak{n}}(z) - \sum_{s=1}^{t} \alpha_s \lambda_{T_{\mathfrak{l}_2}}(\mathscr{F}_2^{\mathfrak{n}}) \mathscr{F}_s^{\mathfrak{n}}(z)\right)\Big|_{2k'} \prod_{s=3}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}})\right) \\
&= \left(\sum_{s=1}^{t} \alpha_s \left(\lambda_{T_{\mathfrak{l}_2}}(\mathscr{F}_s^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_2}}(\mathscr{F}_2^{\mathfrak{n}})\right) \mathscr{F}_s^{\mathfrak{n}}(z)\right)\Big|_{2k'} \prod_{s=3}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}})\right).
\end{aligned}$$

$$= \sum_{s=1}^{t} \alpha_s \prod_{w=2}^{t} \left(\lambda_{T_{\mathfrak{l}_w}}(\mathscr{F}_s^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w}}(\mathscr{F}_w^{\mathfrak{n}}) \right) \mathscr{F}_s^{\mathfrak{n}}(z).$$

$$= \alpha_1 \mathscr{F}_1^{\mathfrak{n}}(z) \prod_{w=2}^{t} \left(\lambda_{T_{\mathfrak{l}_w}}(\mathscr{F}_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w}}(\mathscr{F}_w^{\mathfrak{n}}) \right)$$

or

$$\alpha_1 = \frac{F_1^{\circ}(z)|_{2k'} \prod_{s=2}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}}) \right)}{\mathscr{F}_1^{\mathfrak{n}}(z) \prod_{w=2}^{t} \left(\lambda_{T_{\mathfrak{l}_w}}(\mathscr{F}_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w}}(\mathscr{F}_w^{\mathfrak{n}}) \right)}.$$

We want to show that $\alpha_1 \in \mathscr{O}_{(\mathfrak{P})}$.

Note that $F_1^{\circ}(z)|_{2k'} \prod_{s=2}^{t} (T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}}))$ has Fourier coefficients in $\mathscr{O}_{(\mathfrak{P})}$. This is because $\lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}}) \in \mathscr{O}_L$ and the Hecke-algebra $\mathbb{T}_{2k'}(\mathfrak{n}p; \mathscr{O}_{(\mathfrak{P})}) \subseteq \mathrm{End}(S_{2k'}(\Gamma_{\mathfrak{n}p}; \mathscr{O}_{(\mathfrak{P})}))$.

Next, $\mathscr{F}_1^{\mathfrak{n}}(z)$ is unique up to scalar multiplication in $\mathscr{O}_L$ and we can scale it in a way such that at least one of its Fourier coefficient is not divisible by $\mathfrak{P}$. Thus, $\mathfrak{P} \nmid \mathscr{F}_1^{\mathfrak{n}}(z)$.
Now, suppose there exists some w such that $2 \leq w \leq t$ for which $\mathfrak{P} \mid \left(\lambda_{T_{\mathfrak{l}_w}}(\mathscr{F}_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w}}(\mathscr{F}_w^{\mathfrak{n}}) \right)$, then

$$\mathfrak{P} \mid \left(\lambda_{T_{\mathfrak{l}_w}}(\mathscr{F}_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w}}(f_1^{\mathfrak{n}}) + \lambda_{T_{\mathfrak{l}_w}}(f_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w}}(\mathscr{F}_w^{\mathfrak{n}}) \right).$$

Since $\mathfrak{P} \mid (\lambda_{T_{\mathfrak{l}_w}}(\mathscr{F}_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w}}(f_1^{\mathfrak{n}}))$ by congruence 10.20, it follows,

$$\mathfrak{P} \mid \left(\lambda_{T_{\mathfrak{l}_w}}(f_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w}}(\mathscr{F}_w^{\mathfrak{n}}) \right).$$

This is a contradiction to 10.21 . Thus,

$$\mathfrak{P} \nmid \prod_{w=2}^{t} \left(\lambda_{T_{\mathfrak{l}_w}}(\mathscr{F}_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w}}(\mathscr{F}_w^{\mathfrak{n}}) \right). \tag{10.23}$$

We can hence conclude $\alpha_1 \in \mathscr{O}_{(\mathfrak{P})}$.
We thus have

$$\begin{aligned} \mathfrak{F}_1^{\mathfrak{n}}(z) &= \frac{1}{\prod_{s=2}^{t} \left(\lambda_{T_{\mathfrak{l}_s}}(f_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}}) \right)} F_1^{\circ}(z) \bigg|_{2k'} \prod_{s=2}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}}) \right) \\ &= \alpha_1 \frac{\prod_{w=2}^{t} \left(\lambda_{T_{\mathfrak{l}_w}}(\mathscr{F}_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w}}(\mathscr{F}_w^{\mathfrak{n}}) \right)}{\prod_{s=2}^{t} \left(\lambda_{T_{\mathfrak{l}_s}}(f_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}}) \right)} \mathscr{F}_1^{\mathfrak{n}}(z) \\ &= \mathscr{B} \mathscr{F}_1(z) \end{aligned}$$

where the fact that $\alpha_1 \in \mathscr{O}_{(\mathfrak{P})}$ along with 10.21 implies

$$\mathscr{B} = \alpha_1 \frac{\prod_{w=2}^{t} \left(\lambda_{T_{\mathfrak{l}_w}}(\mathscr{F}_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w}}(\mathscr{F}_w^{\mathfrak{n}}) \right)}{\prod_{s=2}^{t} \left(\lambda_{T_{\mathfrak{l}_s}}(f_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}}) \right)} \in \mathscr{O}_{(\mathfrak{P})}.$$

We will now show $\mathfrak{F}_1^{\mathfrak{n}}(z) \equiv f_1^{\mathfrak{n}}(z) \bmod p^{j+1}$.
In order to do so, we will again look at action of $\prod_{s=2}^{t} (T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}}))$ on $F_1^{\circ}(z)$ modulo p^{j+1}.

$$F_1^{\circ}(z) \bigg|_{2k'} \prod_{s=2}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}}) \right) \equiv f_1^{\mathfrak{n}}(z) \bigg|_{2k} \prod_{s=2}^{t} \left(T_{\mathfrak{l}_s} - \lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}}) \right) \bmod p^{j+1}$$

We see how one of the terms out the product $\prod_{s=2}^{t}(T_{\mathfrak{l}_s}-\lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}}))$ acts on $f_1^{\mathfrak{n}}(z)$ to observe the pattern.

$$\begin{aligned}
F_1^{\circ}(z)\Big|_{2k'}\prod_{s=2}^{t}\left(T_{\mathfrak{l}_s}-\lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}})\right) &\equiv f_1^{\mathfrak{n}}(z)\Big|_{2k}(T_{\mathfrak{l}_2}-\lambda_{T_{\mathfrak{l}_2}}(\mathscr{F}_2^{\mathfrak{n}}))\prod_{s=3}^{t}\left(T_{\mathfrak{l}_s}-\lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}})\right) \bmod p^{j+1}\\
&= \left(f_1^{\mathfrak{n}}(z)|_{2k}T_{\mathfrak{l}_2}-\lambda_{T_{\mathfrak{l}_2}}(\mathscr{F}_2^{\mathfrak{n}})f_1^{\mathfrak{n}}(z)\right)\Big|_{2k}\prod_{s=3}^{t}\left(T_{\mathfrak{l}_s}-\lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}})\right) \bmod p^{j+1}\\
&= \left(\lambda_{T_{\mathfrak{l}_2}}(f_1^{\mathfrak{n}})f_1^{\mathfrak{n}}(z)-\lambda_{T_{\mathfrak{l}_2}}(\mathscr{F}_2^{\mathfrak{n}})f_1^{\mathfrak{n}}(z)\right)\Big|_{2k}\prod_{s=3}^{t}\left(T_{\mathfrak{l}_s}-\lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}})\right) \bmod p^{j+1}\\
&= \left(\lambda_{T_{\mathfrak{l}_2}}(f_1^{\mathfrak{n}})-\lambda_{T_{\mathfrak{l}_2}}(\mathscr{F}_2^{\mathfrak{n}})\right)f_1^{\mathfrak{n}}(z)\Big|_{2k}\prod_{s=3}^{t}\left(T_{\mathfrak{l}_s}-\lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}})\right) \bmod p^{j+1}\\
&= \prod_{s=2}^{t}\left(\lambda_{T_{\mathfrak{l}_s}}(f_1^{\mathfrak{n}})-\lambda_{T_{\mathfrak{l}_s}}(\mathscr{F}_s^{\mathfrak{n}})\right)f_1^{\mathfrak{n}}(z) \bmod p^{j+1}. \qquad (10.24)
\end{aligned}$$

Using definition 10.22 and 10.24, we get

$$\mathfrak{F}_1^{\mathfrak{n}}(z) \equiv f_1^{\mathfrak{n}}(z) \bmod p^{j+1}. \qquad (10.25)$$

Thus, we have shown that for each integer i such that $1 \le i \le m_{\mathfrak{n}}$, there exists an $\mathfrak{n}$-new ordinary Hilbert eigenform $\mathfrak{F}_i^{\mathfrak{n}}(z) \in S_{2k'}^{\text{n-new, ord}}(\Gamma_{\mathfrak{n}\mathfrak{p}};\mathscr{O}_{(\mathfrak{P})})$ with $\mathfrak{P}$-integral Fourier coefficients for fixed prime ideal $\mathfrak{P} \mid \mathfrak{p}$ for each $\mathfrak{p} \mid p$ such that we have the following congruence of Fourier coefficients:

$$\mathfrak{F}_i^{\mathfrak{n}}(z) \equiv f_i^{\mathfrak{n}}(z) \bmod p^{j+1}. \qquad (10.26)$$

Uniqueness: Let $\mathfrak{F}_2^{N}(z) \in S_{2k'}^{\text{n-new, ord}}(\Gamma_{\mathfrak{n}\mathfrak{p}};\mathscr{O}_{(\mathfrak{P})})$ be the $\mathfrak{n}$-new ordinary Hilbert eigenform that satisfies congruence 10.26 with $f_2^{\mathfrak{n}}(z)$.

Suppose there does not exist any $\mathscr{B}' \in \mathscr{O}_{(\mathfrak{P})}$ for which $\mathfrak{F}_2^{\mathfrak{n}}(z) = \mathscr{B}'\mathscr{F}_2^{\mathfrak{n}}(z)$ but if possible, let $\mathfrak{F}_2^{\mathfrak{n}}(z)$ also satisfy the congruence

$$\mathfrak{F}_2^{\mathfrak{n}}(z) \equiv f_1^{\mathfrak{n}}(z) \bmod p^{j+1}.$$

Then we can write

$$f_2^{\mathfrak{n}}(z) \equiv f_1^{\mathfrak{n}}(z) \bmod p^{j+1}$$

which implies for all prime ideals $\mathfrak{l} \subset \mathscr{O}_K$,

$$\left(\lambda_{T_{\mathfrak{l}}}(f_2^{\mathfrak{n}})-\lambda_{T_{\mathfrak{l}}}(f_2^{\mathfrak{n}})\right)f_1^{\mathfrak{n}}(z) \equiv 0 \bmod p^{j+1}.$$

Since, $\mathfrak{P} \nmid f_1^{N}(z)$ due to our choice of scaling, we have $\mathfrak{P} \mid \left(\lambda_{T_{\mathfrak{l}}}(f_2^{\mathfrak{n}})-\lambda_{T_{\mathfrak{l}}}(f_2^{\mathfrak{n}})\right)$. This is a contradiction to the assumption 2. Thus, $\mathfrak{F}_i^{\mathfrak{n}}(z)$ is the unique $\mathfrak{n}$-new ordinary eigenform up to scalar multiplication in $\mathscr{O}_{(\mathfrak{P})}$ that satisfies the congruence 5.23 with $f_i^{\mathfrak{n}}(z)$. ◻

10.3 Congruences related to Hilbert modular forms-Half-integer weight case

From formula 7.8, we have the following direct decomposition of the *Kohnen* plus space for Hilbert modular forms of half-integer weight:

$$S^{+}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p};L) = \oplus_{\mathfrak{m}|\mathfrak{n}p} \oplus_{\mathfrak{b}|\mathfrak{m}^{-1}\mathfrak{n}p} S^{\text{new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{m}};L)|_{k+\frac{1}{2}} U_{\mathfrak{b}^2}$$

where $\mathfrak{b}$ runs over all integral ideals in $\mathcal{O}_K$ that divide $\mathfrak{m}^{-1}\mathfrak{n}p$.

We now give the explicit definition of the space of $4\mathfrak{n}$-new half-integer weight Hilbert cups forms at level $4\mathfrak{n}p$.

$$S^{4\mathfrak{n}\text{-new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p};L) := S^{\text{new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p};L) \oplus \left(S^{\text{new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}};L)|_{k+\frac{1}{2}} U_{(p)^2}\right) \oplus S^{\text{new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}};L). \tag{10.27}$$

From the definition of $\mathfrak{n}$-new Hilbert modular forms of integer weight at level $\mathfrak{n}p$ in 10.4 and the definition of $4\mathfrak{n}$-new Hilbert modular forms of half-integer weight at level $4\mathfrak{n}p$ in 10.27, we obtain the following relationship.

Figure 10.3.1.

$$\begin{array}{ccccccc}
S^{\text{new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}) & \bigoplus & S^{\text{new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}) & |_{k+\frac{1}{2}} U_{(p)^2} & \bigoplus & S^{\text{new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}}) \\
\updownarrow \text{Generalised Kohnen's Iso.} & & \updownarrow \text{Generalised Kohnen's Iso.} & \updownarrow & & \updownarrow \text{Generalised Kohnen's Iso.} \\
S^{\text{new}}_{2k}(\Gamma_{\mathfrak{n}p}) & \bigoplus & S^{\text{new}}_{2k}(\Gamma_{\mathfrak{n}}) & |_{2k} U_{(p)} & \bigoplus & S^{\text{new}}_{2k}(\Gamma_{\mathfrak{n}})
\end{array}$$

Using generalised *Kohnen's* isomorphism for Hilbert modular forms in section 7.5 along with the fact that $|_{k+\frac{1}{2}} U_{(p)^2}$ operator in Hilbert half-integer weight case corresponds to $|_{2k} U_{(p)}$ operator in the Hilbert integer weight case, we conclude that the spaces $S^{4\mathfrak{n}\text{-new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p})$ and $S^{\mathfrak{n}\text{-new}}_{2k}(\Gamma_{\mathfrak{n}})$ are isomorphic and so are there respective ordinary projections.

Figure 10.3.2.

$$\begin{array}{cccc}
S^{4\mathfrak{n}\text{-new}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}) & \Big|_{k+\frac{1}{2}} \prod_{\mathfrak{p}|(p)} (\lim_{n\to\infty} U^{n!}_{\mathfrak{p}^2}) & = & S^{4\mathfrak{n}\text{-new, ord}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}) \\
\updownarrow \text{Iso.} & \updownarrow & & \updownarrow \text{Iso.} \\
S^{\mathfrak{n}\text{-new}}_{2k}(\Gamma_{\mathfrak{n}p}) & \Big|_{2k} \prod_{\mathfrak{p}|(p)} (\lim_{n\to\infty} U^{n!}_{\mathfrak{p}}) & = & S^{\mathfrak{n}\text{-new, ord}}_{2k}(\Gamma_{\mathfrak{n}p})
\end{array}$$

Let

$$\{g^{\mathfrak{n}}_i(z)\}^{m_{\mathfrak{n}}}_{i=1} \subseteq S^{4\mathfrak{n}\text{-new, ord}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p};\mathcal{O}_L)$$

be a basis for $S^{4\mathfrak{n}\text{-new, ord}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; L)$ consisting of $4\mathfrak{n}$-new ordinary Hilbert eigenforms of half-integer weight from the basis

$$\{f_i^{\mathfrak{n}}(z)\}_{i=1}^{m_{\mathfrak{n}}} \subseteq S^{\mathfrak{n}\text{-new, ord}}_{2k}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_L)$$

for $S^{\mathfrak{n}\text{-new, ord}}_{2k}(\Gamma_{\mathfrak{n}p}; L)$ consisting of $\mathfrak{n}$-new ordinary Hilbert eigenforms of integer weight with integral Fourier coefficients via the isomorphism in figure 10.3.1.

10.3.1 Main Theorem

We now state our main result about congruences between half-integer weight Hilbert eigenforms of varying weights. Note that by Hecke eigenforms in Theorem 10.3.3, we mean eigenforms for all Hecke operators $T_{\mathfrak{l}^2}$ for all prime ideals $\mathfrak{l} \subset \mathcal{O}_K$.

Theorem 10.3.3 (Main Theorem). *Let* $m_{\mathfrak{n}} = \dim(S^{4\mathfrak{n}\text{-new, ord}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; L))$ *and let*

$$\{g_i^{\mathfrak{n}}(z)\}_{i=1}^{m_{\mathfrak{n}}} \subseteq S^{4\mathfrak{n}\text{-new, ord}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_L)$$

be a basis for $S^{4\mathfrak{n}\text{-new, ord}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; L)$ *consisting of* $4\mathfrak{n}$*-new ordinary Hilbert eigenforms of half-integer weight with* $\mathcal{O}_L$*-integral Fourier coefficients that is obtained from the basis*

$$\{f_i^{\mathfrak{n}}(z)\}_{i=1}^{m_{\mathfrak{n}}} \subseteq S^{\mathfrak{n}\text{-new, ord}}_{2k}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_L)$$

for $S^{\mathfrak{n}\text{-new, ord}}_{2k}(\Gamma_{\mathfrak{n}p}; L)$ *consisting of* $\mathfrak{n}$*-new ordinary Hilbert eigenforms of integer weight with* $\mathcal{O}_L$*-integral Fourier coefficients via the isomorphism in figure 10.3.1. For each i where* $1 \leq i \leq m_{\mathfrak{n}}$*, let* $g_i^{\mathfrak{n}}(z)$ *and* $f_i^{\mathfrak{n}}(z)$ *be scaled in a way such they have at least one Fourier coefficient that is not divisible by* $\mathfrak{P}$*. Suppose assumption 2 holds for the basis* $\{f_i^{\mathfrak{n}}(z)\}_{i=1}^{m_{\mathfrak{n}}}$ *and also assume that* $p \nmid B_{\frac{p-1}{2}, \chi_p \chi_D}$ *where* χ_D *and* χ_p *are given by the Kronecker symbols* $\left(\frac{D}{\bullet}\right)$ *and* $\left(\frac{\bullet}{p}\right)$ *respectively. (Assumption 2 and 3 hold).*
Let $j \in \mathbb{Z}_{\geq 0}$ *be a fixed integer and define* $k' := k + \frac{p^j(p-1)}{2}$*. Then there for each i such that* $1 \leq i \leq m_{\mathfrak{n}}$*, there exists a* $4\mathfrak{n}$*-new ordinary Hilbert eigenform form of half-integer weight*

$$\mathfrak{G}_i^{\mathfrak{n}}(z) \in S^{4\mathfrak{n}\text{-new, ord}}_{k'+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_{(\mathfrak{P})})$$

unique up to scalar multiplication in $\mathcal{O}_{(\mathfrak{P})}$ *such that we have the following congruence of Fourier coefficients:*

$$g_i^{\mathfrak{n}}(z) \equiv \mathfrak{G}_i(z)|_{k'+\frac{1}{2}} U_{(p)} \text{ mod } p^{j+1}.$$

Figure 10.3.4.

$$g_i^{\mathfrak{n}} \in S^{4\mathfrak{n}\text{-new, ord}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_L) \xrightarrow{\equiv \text{ mod } p^{j+1} \text{ (Shifted } |U_{(p)})} \mathfrak{G}_i^{\mathfrak{n}} \in S^{4\mathfrak{n}\text{-new, ord}}_{k'+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_{(\mathfrak{P})})$$

$\downarrow$ 4𝔫-new ord. Hilbert eigenform of weight $k + \frac{1}{2}$

$\downarrow$ 4𝔫-new ord. Hilbert eigenform of weight $k' + \frac{1}{2}$

Remark 10.3.5. We note that Theorem 10.3.3 also holds true for all weights $k' = k + t\frac{(p-1)}{2}$ where t is an odd positive integer.

In order to give an organised and clear proof of Theorem 10.3.3, we will prove a series of propositions which when combined will eventually imply the result.

Proposition 10.3.6. *Let*

$$g(z) \in S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_L)$$

be a Hilbert cusp form of half-integer weight with $\mathcal{O}_L$-integral Fourier coefficients. Assume that $p \nmid B_{\frac{p-1}{2},\chi_p\chi_D}$ where χ_D and χ_p are given by the Kronecker symbols $\left(\frac{D}{\bullet}\right)$ and $\left(\frac{\bullet}{p}\right)$ respectively. (Assumption 3 holds).
Let $j \in \mathbb{Z}_{\geq 0}$ be a fixed integer and define $k' := k + \frac{p^j(p-1)}{2}$. Then there exists a Hilbert cusp form of half-integer weight

$$G(z) \in S_{k'+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_{(\mathfrak{P})})$$

such that we have the following congruence of Fourier coefficients:

$$g(z)|_{k+\frac{1}{2}} U_{(p)} \equiv G(z) \text{ mod } p^{j+1}.$$

Figure 10.3.7.

$$\begin{array}{ccc} g \in S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_L) & \xrightarrow{\text{(Shifted } |U_{(p)}) \equiv \text{mod } p^{j+1}} & G \in S_{k'+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_{(\mathfrak{P})}) \\ \downarrow & & \downarrow \\ \boxed{\text{Hilbert cusp form of weight } k+\frac{1}{2}} & & \boxed{\text{Hilbert cusp form of weight } k'+\frac{1}{2}} \end{array}$$

Before we give the proof of proposition 10.3.6, we will state and prove two prerequisite lemmas, one of which is a generalisation of lemma 5.3.8 to Hilbert modular forms while the other lemma gives us quadratic reciprocity for the quadratic symbol introduced in definition 7.1.2.

Lemma 10.3.8. *Let p be an odd rational prime. Then*

$$M_{\frac{p-1}{2}}\left(\Gamma_{4p}, \left(\frac{\bullet}{p}\right)_2\right) = M_{\frac{p-1}{2}}\left(\widetilde{\Gamma}_{4p}, \left(sgn(\bullet)\left(\frac{-1}{\bullet}\right)_2\right)^{\frac{p-1}{2}}\left(\frac{\bullet}{p}\right)_2\right).$$

Proof. Let $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \Gamma_{4p}$.
Let $f(z) \in M_{\frac{p-1}{2}}\left(\Gamma_{4p}, \left(\frac{\bullet}{p}\right)_2\right)$. Then $f(z)$ is invariant under the action of γ, that is

$$f(z)|_{\frac{p-1}{2}}\gamma = \left(\frac{a_\gamma}{p}\right)_2 f(z). \tag{10.28}$$

Now let $\widetilde{\gamma} \in \widetilde{\Gamma}_{4p}$. Then by equation 7.2, the $(\frac{p-1}{2})$-slash action of $\widetilde{\gamma}$ on $f(z)$ is

$$\begin{aligned} f(z)|_{\frac{p-1}{2}}\widetilde{\gamma} &= \Phi(z)^{-\frac{p-1}{2}} f(\gamma z). \\ &= \left(\text{sgn}(a_\gamma)\left(\frac{-1}{a_\gamma}\right)_2 \prod_{i=1,2}(c_{\gamma_i} z_i + d_{\gamma_i})\right)^{-\frac{p-1}{2}} f(\gamma z) \\ &= \left(\text{sgn}(a_\gamma)\left(\frac{-1}{a_\gamma}\right)_2\right)^{-\frac{p-1}{2}} f(z)|_{\frac{p-1}{2}}\gamma \end{aligned} \tag{10.29}$$

where

$$f(z)|_{\frac{p-1}{2}}\gamma = \prod_{i=1,2}(c_{\gamma_i}z_i + d_{\gamma_i})^{-\frac{p-1}{2}} f(\gamma z).$$

Note that $\left(\operatorname{sgn}(\bullet)\left(\frac{-1}{\bullet}\right)_2\right)^{-\frac{p-1}{2}} = \left(\operatorname{sgn}(\bullet)\left(\frac{-1}{\bullet}\right)_2\right)^{\frac{p-1}{2}}$. Then using equations 10.28 and 10.29, we get

$$f(z)|_{\frac{p-1}{2}}\widetilde{\gamma} = \left(\frac{a_\gamma}{p}\right)_2 \left(\operatorname{sgn}(a_\gamma)\left(\frac{-1}{a_\gamma}\right)_2\right)^{\frac{p-1}{2}} f(z).$$

Thus, $f(z) \in M_{\frac{p-1}{2}}\left(\widetilde{\Gamma}_{4p}, \left(\operatorname{sgn}(\bullet)\left(\frac{-1}{\bullet}\right)_2\right)^{\frac{p-1}{2}}\left(\frac{\bullet}{p}\right)_2\right)$. ☐

Lemma 10.3.9. *Let p be an odd rational prime. Then*

$$\left(\frac{p}{\bullet}\right)_2\left(\frac{\bullet}{p}\right)_2 = \left(sgn(\bullet)\left(\frac{-1}{\bullet}\right)_2\right)^{\frac{p-1}{2}}.$$

Proof. Let $\alpha \in \mathcal{O}_K$ such that $\alpha \equiv 1 \bmod 2$ and let α be co-prime to p. Then by [Lem, Theorem 12.14], for any two algebraic integers $\alpha, p \equiv 1 \bmod 2$ such that they are co-prime, we have

$$\left(\frac{\alpha}{p}\right)_2\left(\frac{p}{\alpha}\right)_2 = (-1)^{S+T} \quad \text{and} \quad \left(\frac{-1}{\alpha}\right)_2 = (-1)^{U+\operatorname{Tr}\left(\frac{\alpha-1}{2}\right)} \tag{10.30}$$

where

$$\begin{aligned}
S &= \frac{\operatorname{sgn}(\alpha_1)-1}{2}.\frac{\operatorname{sgn}(p)-1}{2} + \frac{\operatorname{sgn}(\alpha_2)-1}{2}.\frac{\operatorname{sgn}(p)-1}{2};\\
T &= \operatorname{Tr}\left(\frac{\alpha-1}{2}.\frac{p-1}{2}\right);\\
U &= \frac{\operatorname{sgn}(\alpha_1)-1}{2} + \frac{\operatorname{sgn}(\alpha_2)-1}{2}.
\end{aligned}$$

We will first use the formula in 10.30 to compute $\left(\frac{p}{\alpha}\right)_2\left(\frac{\alpha}{p}\right)_2$.
We note that $\operatorname{sgn}(p) = 1$. This implies $S = 0$. Next, $\frac{p-1}{2} \in \mathbb{Z}$ which implies $T = \frac{p-1}{2}\operatorname{Tr}\left(\frac{\alpha-1}{2}\right)$.
Thus, we conclude

$$\left(\frac{p}{\alpha}\right)_2\left(\frac{\alpha}{p}\right)_2 = (-1)^{\frac{p-1}{2}\operatorname{Tr}\left(\frac{\alpha-1}{2}\right)}.$$

Next, we use the formula in 10.30 to compute $\left(\operatorname{sgn}(\alpha)\left(\frac{-1}{\alpha}\right)_2\right)^{\frac{p-1}{2}}$.
We note that $\operatorname{sgn}(\alpha_1)$, $\operatorname{sgn}(\alpha_2)$ take values ± 1. Thus, we have the following possibilities,

$$\begin{aligned}
\left(\operatorname{sgn}(\alpha)\left(\frac{-1}{\alpha}\right)_2\right)^{\frac{p-1}{2}} &= \left(\operatorname{sgn}(\alpha)(-1)^U(-1)^{\operatorname{Tr}\left(\frac{\alpha-1}{2}\right)}\right)^{\frac{p-1}{2}}\\
&= \left(\operatorname{sgn}(\alpha_1)\operatorname{sgn}(\alpha_2)(-1)^{\frac{\operatorname{sgn}(\alpha_1)-1}{2}+\frac{\operatorname{sgn}(\alpha_2)-1}{2}}(-1)^{\operatorname{Tr}\left(\frac{\alpha-1}{2}\right)}\right)^{\frac{p-1}{2}}
\end{aligned}$$

where

$$\begin{aligned}
\operatorname{sgn}(\alpha_1)\operatorname{sgn}(\alpha_2)(-1)^{\frac{\operatorname{sgn}(\alpha_1)-1}{2}+\frac{\operatorname{sgn}(\alpha_2)-1}{2}} &= \begin{cases} 1.1.(-1)^0 & \text{if } \operatorname{sgn}(\alpha_1) = \operatorname{sgn}(\alpha_2) = 1;\\ (-1)(-1)(-1)^{-2} & \text{if } \operatorname{sgn}(\alpha_1) = \operatorname{sgn}(\alpha_2) = -1;\\ (-1)(1)(-1)^{-1} & \text{if } \operatorname{sgn}(\alpha_1) = -\operatorname{sgn}(\alpha_2) = 1 \end{cases}\\
&= 1.
\end{aligned}$$

It follows

$$\left(\operatorname{sgn}(\alpha)\left(\frac{-1}{\alpha}\right)_2\right)^{\frac{p-1}{2}} = (-1)^{\frac{p-1}{2}\operatorname{Tr}\left(\frac{\alpha-1}{2}\right)}.$$

Hence, we have shown that $\left(\frac{p}{\alpha}\right)_2\left(\frac{\alpha}{p}\right)_2 = \left(\operatorname{sgn}(\alpha)\left(\frac{-1}{\alpha}\right)_2\right)^{\frac{p-1}{2}}$ for any α in $\mathcal{O}_K$ that is odd and co-prime to p. □

Proof of proposition 10.3.6.

Define

$$G(z) := \left(g(z)|_{k+\frac{1}{2}}U_{(p)}\right)\widetilde{\mathscr{E}}_p(4z)^{p^j}$$

where $\widetilde{\mathscr{E}}_p(4z) \in M_{\frac{p-1}{2}}\left(\Gamma_{4p}, \left(\frac{\bullet}{p}\right)_2\right)$ is defined to be the generalised Hilbert Eisenstein series $E_{\frac{p-1}{2},\left(\frac{\bullet}{p}\right)_2}(4z)$, see section 8.6.

Claim 1: We claim that $G(z) \in S_{k'+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_{(\mathfrak{P})})$.

From equation 8.28, the Fourier expansion of $E_{\frac{p-1}{2},\left(\frac{\bullet}{p}\right)_2}(z)$ is

$$E_{\frac{p-1}{2},\left(\frac{\bullet}{p}\right)_2}(z) = 1 + (p-1)^2\left(B_{\frac{p-1}{2},\left(\frac{\bullet}{p}\right)}\right)^{-1}\frac{1}{B_{\frac{p-1}{2},\left(\frac{\bullet}{p}\right)\left(\frac{p}{\bullet}\right)}}\sum_{\xi\mathcal{O}_K^+}\sigma_{\frac{p-3}{2},\left(\frac{\bullet}{p}\right)_2}(\xi)e^{2\pi i\operatorname{Tr}\left(\frac{\xi}{2}z\right)}.$$

Then using Theorem 3.4.1 and the fact that $p \nmid B_{\frac{p-1}{2},\chi_p\chi_D}$, we get that the Fourier coefficients of $E_{\frac{p-1}{2},\left(\frac{\bullet}{p}\right)_2}(z)$ lie in $\mathbb{Z}_{(p)}$. Then it's clear from the definition of $\widetilde{\mathscr{E}}_p(z)$ that its Fourier coefficients lie in $\mathbb{Z}_{(p)}$. Also, note that it's given that Fourier coefficients of $g(z)$ lie in $\mathcal{O}_L$ which implies that $g(z)|_{k+\frac{1}{2}}U_{(p)}$ also has Fourier coefficients in $\mathcal{O}_L$ as action by $|_{k+\frac{1}{2}}U_{(p)}$ operator on $g(z)$ picks shifted Fourier coefficients of $g(z)$. This implies that the Fourier coefficients of $G(z)$ lie in $\mathcal{O}_{(\mathfrak{P})}$.

Next, we note that the action of $U_{(p)}$ on $g(z) \in S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_L)$ twists the character of this space by $\left(\frac{p}{\bullet}\right)_2$ (see section 7.3.3),

$$g(z)|_{k+\frac{1}{2}}U_{(p)} \in S_{k+\frac{1}{2}}\left(\widetilde{\Gamma}_{4\mathfrak{n}p}, \left(\frac{p}{\bullet}\right)_2; \mathcal{O}_L\right). \tag{10.31}$$

Also, by the lemma 10.3.8, we have

$$\widetilde{\mathscr{E}}_p(4z) \in M_{\frac{p-1}{2}}\left(\widetilde{\Gamma}_{4p}, \left(\operatorname{sgn}(\bullet)\left(\frac{-1}{\bullet}\right)_2\right)^{\frac{p-1}{2}}\left(\frac{\bullet}{p}\right)_2; \mathbb{Z}_{(p)}\right). \tag{10.32}$$

It is clear from 10.31 and 10.32 that

$$G(z) = \left(g(z)|_{k+\frac{1}{2}}U_{(p)}\right)\widetilde{\mathscr{E}}_p(4z)^{p^j} \in S_{k'+\frac{1}{2}}\left(\widetilde{\Gamma}_{4\mathfrak{n}p}, \left(\operatorname{sgn}(\bullet)\left(\frac{-1}{\bullet}\right)_2\right)^{\frac{p-1}{2}}\left(\frac{\bullet}{p}\right)_2\left(\frac{p}{\bullet}\right)_2; \mathcal{O}_{(\mathfrak{P})}\right).$$

Since lemma 10.3.9 implies that $\left(\operatorname{sgn}(\bullet)\left(\frac{-1}{\bullet}\right)_2\right)^{\frac{p-1}{2}}\left(\frac{\bullet}{p}\right)_2\left(\frac{p}{\bullet}\right)_2 = 1$, we have

$$G(z) \in S_{k'+\frac{1}{2}}\left(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_{(\mathfrak{P})}\right).$$

Claim 2: $g(z)|_{k+\frac{1}{2}}U_{(p)} \equiv G(z) \bmod p^{j+1}$.

From lemma 8.6.1, we have

$$\widetilde{\mathscr{E}}_p(4z) \equiv 1 \bmod p.$$

Then by a similar argument as in the proof of corollary 3.2.4, we get

$$\widetilde{\mathscr{E}}_p(4z)^{p^j} \equiv 1 \bmod p^{j+1}.$$

Hence, we conclude,

$$g(z)|_{k+\frac{1}{2}} U_{(p)} \equiv G(z) \bmod p^{j+1}.$$

□

We next try to show that the action of Hecke operators $|_{k+\frac{1}{2}} T_{\mathfrak{l}^2}$ for prime ideals $\mathfrak{l} \subset \mathscr{O}_K$ on $\left(g(z)|_{k+\frac{1}{2}} U_{(p)}\right)$ mod p^{j+1} is the same as the action of the Hecke operators $|_{k'+\frac{1}{2}} T_{\mathfrak{l}^2}$ on $G(z)$ mod p^{j+1} where $g(z)$ and $G(z)$ are defined as in proposition 10.3.6.

Corollary 10.3.10. *Let $g(z) \in S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathscr{O}_L)$ and $G(z) \in g(z) \in S_{k'+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathscr{O}_{(\mathfrak{P})})$ be half-integer weight Hilbert cusp forms defined in proposition 10.3.6 such that*

$$g(z)|_{k+\frac{1}{2}} U_{(p)} \equiv G(z) \text{ mod } p^{j+1}.$$

Then we have

$$\left(g(z)|_{k+\frac{1}{2}} U_{(p)}\right)|_{k+\frac{1}{2}} T_{\mathfrak{l}^2} \equiv G(z)|_{k'+\frac{1}{2}} T_{\mathfrak{l}^2} \text{ mod } p^{j+1}$$

for all prime ideals $\mathfrak{l} \subset \mathscr{O}_K$.

Proof. For simplicity, set $q^{\xi} = e^{2\pi i \operatorname{Tr}\left(\frac{\xi}{2}z\right)}$. Let $g(z) = \sum_{\xi \in \mathscr{O}_K^+} b_{\xi} q^{\xi}$ and $G(z) = \sum_{\xi \in \mathscr{O}_K^+} B_{\xi} q^{\xi}$ be the Fourier expansions of $g(z)$ and $G(z)$ are defined in proposition 10.3.6. then we have

$$b_{p\xi} \equiv B_{\xi} \bmod p^{j+1}. \tag{10.33}$$

Case 1: $\mathfrak{l} \nmid 4\mathfrak{n}p$

Recall that

$$|_{k+\frac{1}{2}} U_{(p)} : g(z) \mapsto g(z)|_{k+\frac{1}{2}} U_{(p)} \in S_{k+\frac{1}{2}}\left(\widetilde{\Gamma}_{4\mathfrak{n}p}, \left(\frac{p}{\bullet}\right)_2; \mathscr{O}_L\right).$$

From proposition 7.2.3, for all prime ideals $\mathfrak{l} \subset \mathscr{O}_K$, $\mathfrak{l} \nmid 4\mathfrak{n}p$, we have

$$G(z)|_{k'+\frac{1}{2}} T_{\mathfrak{l}^2} = \sum_{\xi \in \mathscr{O}_K^+} \left(B_{\mathscr{L}^2\xi} + N(\mathscr{L})^{k'-1} \left(\frac{-1}{\mathscr{L}}\right)_2^{k'} \left(\frac{\xi}{\mathscr{L}}\right)_2 B_{\xi} + N(\mathscr{L})^{2k'-1} B_{\xi\mathscr{L}^{-2}} \right) q^{\xi} \tag{10.34}$$

where $\mathscr{L} \gg 0$ is a totally positive generator of $\mathfrak{l}$.

From 10.34 and 10.33, we have

$$G(z)|_{k'+\frac{1}{2}} T_{\mathfrak{l}^2} \equiv \sum_{\xi \in \mathscr{O}_K^+} \left(b_{p\mathscr{L}^2\xi} + \left(\frac{-1}{\mathscr{L}}\right)_2^{k'} N(\mathscr{L})^{k'-1} \left(\frac{\xi}{\mathscr{L}}\right)_2 b_{p\xi} + N(\mathscr{L})^{2k'-1} b_{p\xi\mathscr{L}^{-2}} \right) q^{\xi} \bmod p^{j+1} \tag{10.35}$$

The term inside the summand on the right hand side of the congruence 10.35 can be rewritten as

$$b_{\mathscr{L}^2p\xi} + \left(\frac{-1}{\mathscr{L}}\right)_2^{k+\frac{p^j(p-1)}{2}} N(\mathscr{L})^{k+\frac{p^j(p-1)}{2}-1} \left(\frac{\xi}{\mathscr{L}}\right)_2 b_{p\xi} + N(\mathscr{L})^{2\left(k+\frac{p^j(p-1)}{2}\right)-1} b_{p\xi\mathscr{L}^{-2}}$$

$$=b_{\mathscr{L}^2 p\xi} + \left(\frac{-1}{\mathscr{L}}\right)_2^{k+\frac{p^j(p-1)}{2}} N(\mathscr{L})^{k-1}\left(N(\mathscr{L})^{\frac{(p-1)}{2}}\right)^{p^j}\left(\frac{p}{\mathscr{L}}\right)_2^2\left(\frac{\xi}{\mathscr{L}}\right)_2 b_{p\xi} + N(\mathscr{L})^{2k-1}\left(N(\mathscr{L})^{(p-1)}\right)^{p^j} b_{p\xi\mathscr{L}^{-2}}$$

$$=b_{\mathscr{L}^2 p\xi} + \left(\frac{-1}{\mathscr{L}}\right)_2^{k} N(\mathscr{L})^{k-1}\left(\left(\frac{-1}{\mathscr{L}}\right)_2^{\frac{(p-1)}{2}} N(\mathscr{L})^{\frac{(p-1)}{2}}\right)^{p^j}\left(\frac{p}{\mathscr{L}}\right)_2\left(\frac{p\xi}{\mathscr{L}}\right)_2 b_{p\xi} + N(\mathscr{L})^{2k-1}\left(N(\mathscr{L})^{(p-1)}\right)^{p^j} b_{p\xi\mathscr{L}^{-2}} \tag{10.36}$$

We now look at the terms $\left(N(\mathscr{L})^{(p-1)}\right)^{p^j}$ and $\left(\left(\frac{-1}{\mathscr{L}}\right)_2^{\frac{(p-1)}{2}} N(\mathscr{L})^{\frac{(p-1)}{2}}\right)^{p^j}$.

Since $N(\mathscr{L})$ is an integer, by *Fermat's Little Theorem*, we have $N(\mathscr{L})^{p-1} \equiv 1 \bmod p$. It then follows from a similar argument as in the proof of the corollary 3.2.4 that

$$\left(N(\mathscr{L})^{p-1}\right)^{p^j} \equiv 1 \bmod p^{j+1}. \tag{10.37}$$

Also, note that $N(\mathscr{L})$ is an integer. So, by *Euler's Criterion*, we have $N(\mathscr{L})^{\frac{(p-1)}{2}} = \left(\frac{N(\mathscr{L})}{p}\right) \bmod p$.
Also, by the lemma 8.5.2, we have $\left(\frac{N(\mathscr{L})}{p}\right) = \left(\frac{\mathscr{L}}{p}\right)_2$. Then it follows,

$$\begin{aligned}\left(\frac{-1}{\mathscr{L}}\right)_2^{\frac{(p-1)}{2}} N(\mathscr{L})^{\frac{(p-1)}{2}}\left(\frac{p}{\mathscr{L}}\right)_2 &\equiv \left(\frac{-1}{\mathscr{L}}\right)_2^{\frac{(p-1)}{2}}\left(\frac{\mathscr{L}}{p}\right)_2\left(\frac{p}{\mathscr{L}}\right)_2 \bmod p\\ &= 1 \bmod p.\end{aligned}$$

Hence, we have

$$\left(\left(\frac{-1}{\mathscr{L}}\right)_2^{\frac{(p-1)}{2}} N(\mathscr{L})^{\frac{(p-1)}{2}}\left(\frac{p}{\mathscr{L}}\right)_2\right)^{p^j} \equiv 1 \bmod p^{j+1}$$

or

$$\left(\frac{-1}{\mathscr{L}}\right)_2^{\frac{p^j(p-1)}{2}} N(\mathscr{L})^{\frac{p^j(p-1)}{2}}\left(\frac{p}{\mathscr{L}}\right)_2 \equiv 1 \bmod p^{j+1}. \tag{10.38}$$

Using 10.37 and 10.38 together in 10.36, we get

$$\begin{aligned}G(z)|_{k'+\frac{1}{2}}T_{\mathfrak{l}^2} &\equiv \sum_{\xi\in\mathscr{O}_K^+}\left(b_{p\mathscr{L}^2\xi} + \left(\frac{-1}{\mathscr{L}}\right)_2^{k}\left(\frac{p}{\mathscr{L}}\right)_2 N(\mathscr{L})^{k-1}\left(\frac{\xi}{\mathscr{L}}\right)_2 b_{p\xi} + N(\mathscr{L})^{2k-1} b_{p\xi\mathscr{L}^{-2}}\right)q^{\xi} \bmod p^{j+1}\\ &= \left(g(z)|_{k+\frac{1}{2}}U_{(p)}\right)|_{k+\frac{1}{2}}T_{\mathfrak{l}^2} \bmod p^{j+1}.\end{aligned}$$

Case 2: $\mathfrak{l} \mid 4\mathfrak{n}p$

Again using proposition 7.2.3, we can write for all prime ideals $\mathfrak{l} \subset \mathscr{O}_K$, $\mathfrak{l} \mid 4\mathfrak{n}p$,

$$\left(g(z)|_{k+\frac{1}{2}}U_{(p)}\right)|_{k+\frac{1}{2}}T_{\mathfrak{l}^2} = \sum_{\xi\in\mathscr{O}_K^+} b_{\mathscr{L}^2 p\xi}q^{\xi} \quad \text{and} \quad G(z)|_{k'+\frac{1}{2}}T_{\mathfrak{l}^2} = \sum_{\xi\in\mathscr{O}_K^+} B_{\mathscr{L}^2\xi}q^{\xi}. \tag{10.39}$$

From 10.39 and 10.33, it follows,

$$b_{p(\mathscr{L}^2\xi)} \equiv B_{(\mathscr{L}^2\xi)} \bmod p^{j+1}$$

or

$$\left(g(z)|_{k+\frac{1}{2}}U_{(p)}\right)|_{k+\frac{1}{2}}T_{\mathfrak{l}^2} \equiv G(z)|_{k'+\frac{1}{2}}T_{\mathfrak{l}^2} \bmod p^{j+1}.$$

□

We will next try to replace the Hilbert modular form of half-integer weight $G(z)$ defined in proposition 10.3.6 by an ordinary Hilbert modular form of the same half-integer weight $k' + \frac{1}{2}$, provided we now take $g(z)$ in proposition 10.3.6 to be ordinary.

Proposition 10.3.11. *Let*

$$g(z) \in S^{ord}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_L)$$

be an ordinary Hilbert cusp form of half-integer weight with $\mathcal{O}_L$-integral Fourier coefficients. Assume that $p \nmid B_{\frac{p-1}{2},\chi_p\chi_D}$ where χ_D and χ_p are given by the Kronecker symbols $\left(\frac{D}{\bullet}\right)$ and $\left(\frac{\bullet}{p}\right)$ respectively. (Assumption 3 holds). Let $j \in \mathbb{Z}_{\geq 0}$ be a fixed integer and define $k' := k + \frac{p^j(p-1)}{2}$. Then there exists an ordinary Hilbert cusp form of half-integer weight

$$G^\circ(z) \in S^{ord}_{k'+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_{(\mathfrak{P})})$$

such that we have the following congruence of Fourier coefficients:

$$g(z)|_{k+\frac{1}{2}}U_{(p)} \equiv G^\circ(z) \text{ mod } p^{j+1}.$$

Moreover,

$$\left(g(z)|_{k+\frac{1}{2}}U_{(p)}\right)|_{k+\frac{1}{2}}T_{\mathfrak{l}^2} \equiv G^\circ(z)|_{k'+\frac{1}{2}}T_{\mathfrak{l}^2} \text{ mod } p^{j+1}$$

for all prime ideals $\mathfrak{l} \subset \mathcal{O}_K$.

Figure 10.3.12.

$$g \in S^{\text{ord}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_L) \xrightarrow{\text{Shifted } |U_{(p)} \equiv \text{ mod } p^{j+1}} G \in S^{\text{ord}}_{k'+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_{(\mathfrak{P})})$$

$\downarrow$ Ord. Hilbert cusp form of weight $k + \frac{1}{2}$

$\downarrow$ Ord. Hilbert cusp form of weight $k' + \frac{1}{2}$

Proof. In section 9.5, we defined the p-ordinary projection operator $\widetilde{\mathfrak{e}}_p = \prod_{\mathfrak{p}} \widetilde{\mathfrak{e}}_{\mathfrak{p}}$ where $\mathfrak{p} \subset \mathcal{O}_K$ is a prime ideal above the rational prime p.

Now define

$$G^\circ(z) := G(z)|_{k'+\frac{1}{2}}\widetilde{\mathfrak{e}}_p.$$

where $G(z)$ is defined in proposition 10.3.6.

Now $\widetilde{\mathfrak{e}}_p$ is an idempotent operator, that is, $\widetilde{\mathfrak{e}}_p^2 = \widetilde{\mathfrak{e}}_p$. This implies

$$\begin{aligned} G^\circ(z)|_{k'+\frac{1}{2}}\widetilde{\mathfrak{e}}_p &= G(z)|_{k'+\frac{1}{2}}\widetilde{\mathfrak{e}}_p^2 \\ &= G(z)|_{k'+\frac{1}{2}}\widetilde{\mathfrak{e}}_p \\ &= G^\circ(z). \end{aligned}$$

Hence $G^\circ(z)$ is ordinary at p and lies in $S^{\mathrm{ord}}_{k'+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_{(\mathfrak{P})})$.

Claim: We now claim that $G^\circ(z) \equiv g(z)|_{k+\frac{1}{2}} U_{(p)} \bmod p^{j+1}$.

It is given that $g(z)$ is ordinary at p. So, $g(z)|_{k+\frac{1}{2}}\widetilde{\mathfrak{e}}_p = g(z)$. Also, note that $U_{(p)^2}$ acts in the same way on $g(z)$ as $(U_{(p)})^2$. Then by corollary 10.3.10, we can write

$$\begin{aligned}
G(z)|_{k'+\frac{1}{2}}\widetilde{\mathfrak{e}}_p &\equiv \left(g(z)|_{k+\frac{1}{2}} U_{(p)}\right)|_{k+\frac{1}{2}}\widetilde{\mathfrak{e}}_p \bmod p^{j+1} \\
&= \left(g(z)|_{k+\frac{1}{2}} U_{(p)}\right)|_{k+\frac{1}{2}} \prod_{\mathfrak{p}} \widetilde{\mathfrak{e}}_{\mathfrak{p}} \bmod p^{j+1} \\
&= \left(g(z)|_{k+\frac{1}{2}} U_{(p)}\right)|_{k+\frac{1}{2}} \prod_{\mathfrak{p}} \lim_{n\to\infty} U^{n!}_{\mathfrak{p}^2} \bmod p^{j+1} \\
&= \left(g(z)|_{k+\frac{1}{2}} U_{(p)}\right)|_{k+\frac{1}{2}} \lim_{n\to\infty} \left(\prod_{\mathfrak{p}} U^{n!}_{\mathfrak{p}^2}\right) \bmod p^{j+1} \\
&= \left(g(z)|_{k+\frac{1}{2}} U_{(p)}\right)|_{k+\frac{1}{2}} \lim_{n\to\infty} U^{n!}_{(p)^2} \bmod p^{j+1} \\
&= g(z)|_{k+\frac{1}{2}} \lim_{n\to\infty} U^{2(n!)+1}_{(p)} \bmod p^{j+1} \\
&= \left(g(z)|_{k+\frac{1}{2}} \lim_{n\to\infty} U^{2(n!)}_{(p)}\right)|_{k+\frac{1}{2}} U_{(p)} \bmod p^{j+1} \\
&= \left(g(z)|_{k+\frac{1}{2}} \lim_{n\to\infty} U^{n!}_{(p)^2}\right)|_{k+\frac{1}{2}} U_{(p)} \bmod p^{j+1} \\
&= \left(g(z)|_{k+\frac{1}{2}}\widetilde{\mathfrak{e}}_p\right)|_{k+\frac{1}{2}} U_{(p)} \bmod p^{j+1} \\
&= g(z)|_{k+\frac{1}{2}} U_{(p)} \bmod p^{j+1}.
\end{aligned}$$

Thus, we have shown that

$$G^\circ(z) \equiv g(z)|_{k+\frac{1}{2}} U_{(p)} \bmod p^{j+1}.$$

The equivalence of action of Hecke operators modulo p^{j+1} now follows in the same way as in corollary 10.3.10.
□

Our next task is to show that if we are given that the ordinary Hilbert half-integer weight cusp form $g(z) \in S^{\mathrm{ord}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_L)$ in proposition 10.3.11 is a $4\mathfrak{n}$-new eigenform that is obtained from the $\mathfrak{n}$-new ordinary eigenform of integer weight, say $f^{\mathfrak{n}}(z) \in S^{\mathfrak{n}\text{-new, ord}}_{2k}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_L)$, then there exists $\mathcal{G}(z) \in S^{4\mathfrak{n}\text{-new, ord}}_{k'+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_L)$, a unique $4\mathfrak{n}$-new ordinary Hilbert eigenform of half-integer weight $k' + \frac{1}{2}$ (unique up to scalar multiplication in $\mathcal{O}_L$) which satisfies a mod $\mathfrak{P}$ congruence of Hecke eigenvalues with $g(z)$ for all Hecke operators $T_{\mathfrak{l}^2}$ over all prime ideals $\mathfrak{l} \subset \mathcal{O}_K$.

Note that by Hecke eigenforms in proposition 10.3.13, we mean eigenforms for all Hecke operators $T_{\mathfrak{l}^2}$ for all prime ideals $\mathfrak{l} \subset \mathcal{O}_K$.

Proposition 10.3.13. *Let* $m_{\mathfrak{n}} = \mathit{dim}(S^{4\mathfrak{n}\text{-new, ord}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; L))$ *and let*

$$\{g_i^{\mathfrak{n}}(z)\}_{i=1}^{m_{\mathfrak{n}}} \subset S^{4\mathfrak{n}\text{-new, ord}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_L)$$

be a basis for $S^{4\mathfrak{n}\text{-new, ord}}_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; L)$ *consisting of* $4\mathfrak{n}$*-new ordinary Hilbert eigenforms of half-integer weight with* $\mathcal{O}_L$*-integral Fourier coefficients that is obtained from the basis*

$$\{f_i^{\mathfrak{n}}(z)\}_{i=1}^{m_{\mathfrak{n}}} \subseteq S^{\mathfrak{n}\text{-new, ord}}_{2k}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_L)$$

for $S_{2k}^{\mathfrak{n}\text{-new, ord}}(\Gamma_{\mathfrak{n}p}; L)$ *consisting of* $\mathfrak{n}$*-new ordinary Hilbert eigenforms of integer weight with* $\mathcal{O}_L$*-integral Fourier coefficients via the isomorphism in figure 10.3.1. For each* i *where* $1 \le i \le m_{\mathfrak{n}}$*, let* $g_i^{\mathfrak{n}}(z)$ *and* $f_i^{\mathfrak{n}}(z)$ *be scaled in a way such they have at least one Fourier coefficient that is not divisible by* $\mathfrak{P}$*. Suppose assumption 2 holds for the basis* $\{f_i^{\mathfrak{n}}(z)\}_{i=1}^{m_{\mathfrak{n}}}$ *and also assume that* $p \nmid B_{\frac{p-1}{2}, \chi_p\chi_D}$ *where* χ_D *and* χ_p *are given by the Kronecker symbols* $\left(\frac{D}{\bullet}\right)$ *and* $\left(\frac{\bullet}{p}\right)$ *respectively. (Assumption 2 and 3 hold).*

Let $j \in \mathbb{Z}_{\ge 0}$ *be a fixed integer and define* $k' := k + \frac{p^j(p-1)}{2}$*. Then for each* i *such that* $1 \le i \le m_{\mathfrak{n}}$*, there exists a* $4\mathfrak{n}$*-new ordinary Hilbert eigenform form of half-integer weight*

$$\mathscr{G}_i^{\mathfrak{n}}(z) \in S_{k'+\frac{1}{2}}^{4\mathfrak{n}\text{-new, ord}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_L)$$

unique up to scalar multiplication in $\mathcal{O}_L$ *such that for all prime ideals* $\mathfrak{l} \subset \mathcal{O}_K$*,* $\mathfrak{l} \nmid 4\mathfrak{n}p$*, we have*

$$\lambda_{T_{\mathfrak{l}^2}}(g_i^{\mathfrak{n}}) \equiv \lambda_{T_{\mathfrak{l}^2}}(\mathscr{G}_i^{\mathfrak{n}}) \text{ mod } \mathfrak{P}.$$

Figure 10.3.14.

$$g_i^{\mathfrak{n}} \in S_{k+\frac{1}{2}}^{4\mathfrak{n}\text{-new, ord}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_L) \xrightarrow{\text{Hecke eigenvalues} \equiv \text{mod } \mathfrak{P}} \mathscr{G}_i^{\mathfrak{n}} \in S_{k'+\frac{1}{2}}^{\text{ord}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_L)$$

$\downarrow$ 4$\mathfrak{n}$-new ord. Hilbert eigenform of weight $k + \frac{1}{2}$

$\downarrow$ 4$\mathfrak{n}$-new ord. Hilbert eigenform of weight $k' + \frac{1}{2}$

Proof. From the isomorphism given in figure 10.3.1, for all prime ideals $\mathfrak{l} \nmid 4\mathfrak{n}p$, we have

$$\lambda_{T_{\mathfrak{l}}}(f_i^{\mathfrak{n}}) = \lambda_{T_{\mathfrak{l}^2}}(g_i^{\mathfrak{n}}). \tag{10.40}$$

Then by proposition 10.2.13 and proposition 10.2.15, there exists a basis $\{\mathscr{F}_i^{\mathfrak{n}}(z)\}_{i=1}^{m_{\mathfrak{n}}} \subseteq S_{2k'}^{\mathfrak{n}\text{-new, ord}}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_L)$ of $\mathfrak{n}$-new ordinary Hilbert eigenforms for the space $S_{2k'}^{\mathfrak{n}\text{-new, ord}}(\Gamma_{\mathfrak{n}p}; L)$ such that each element in the basis is unique up to scalar multiplication in $\mathcal{O}_L$ and for all prime ideals $\mathfrak{l} \subset \mathcal{O}_K$, $\mathfrak{l} \nmid \mathfrak{n}p$ satisfies

$$\lambda_{T_{\mathfrak{l}}}(f_i^{\mathfrak{n}}) \equiv \lambda_{T_{\mathfrak{l}}}(\mathscr{F}_i^{\mathfrak{n}}) \text{ mod } \mathfrak{P} \tag{10.41}$$

In a similar way as before in figure 10.3.1, we conclude that the space $S_{k'+\frac{1}{2}}^{4\mathfrak{n}\text{-new}}(\widetilde{\Gamma}_{4\mathfrak{n}p})$ is mapped isomorphically onto $S_{2k'}^{\mathfrak{n}\text{-new}}(\Gamma_{\mathfrak{n}})$ and these spaces have isomorphic ordinary projections. We can hence take

$$\{\mathscr{G}_i^{\mathfrak{n}}(z)\}_{i=1}^{m_{\mathfrak{n}}} \subseteq S_{k'+\frac{1}{2}}^{4\mathfrak{n}\text{-new, ord}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_L)$$

to be a basis of 4$\mathfrak{n}$-new ordinary Hilbert eigenforms of half-integer weight obtained from the basis

$$\{\mathscr{F}_i^{\mathfrak{n}}(z)\}_{i=1}^{m_{\mathfrak{n}}} \subseteq S_{2k'}^{\mathfrak{n}\text{-new, ord}}(\Gamma_{\mathfrak{n}p}; \mathcal{O}_L)$$

of $\mathfrak{n}$-new ordinary eigenforms of integer weight for $S_{2k'}^{\mathfrak{n}\text{-new, ord}}(\Gamma_{\mathfrak{n}p}; L)$. Then for all prime ideals $\mathfrak{l} \subset \mathcal{O}_K$, $\mathfrak{l} \nmid 4\mathfrak{n}p$, we have

$$\lambda_{T_{\mathfrak{l}}}(\mathscr{F}_i^{\mathfrak{n}}) = \lambda_{T_{\mathfrak{l}^2}}(\mathscr{G}_i^{\mathfrak{n}}). \tag{10.42}$$

Hence, it follows from 10.40, 10.41 and 10.42 that

$$\lambda_{T_{\mathfrak{l}^2}}(g_1^{\mathfrak{n}}) \equiv \lambda_{T_{\mathfrak{l}^2}}(\mathscr{G}_1^{\mathfrak{n}}) \text{ mod } \mathfrak{P}. \tag{10.43}$$

Uniqueness: Let $\mathscr{G}_1^{\mathfrak{n}}(z)$, $\mathscr{G}_2^{\mathfrak{n}}(z) \in S_{k'+\frac{1}{2}}^{4\mathfrak{n}\text{-new, ord}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathscr{O}_L)$ be two $4\mathfrak{n}$-new ordinary Hilbert eigenforms of half-integer weight in the basis $\{\mathscr{G}_i^{\mathfrak{n}}(z)\}_{i=1}^{m_{\mathfrak{n}}}$. Then using congruence 10.43, they satisfy the respective congruences,

$$\lambda_{T_{\mathfrak{l}^2}}(g_1^N) \equiv \lambda_{T_{\mathfrak{l}^2}}(\mathscr{G}_1^N) \bmod \mathfrak{P}$$

and

$$\lambda_{T_{\mathfrak{l}^2}}(g_2^N) \equiv \lambda_{T_{\mathfrak{l}^2}}(\mathscr{G}_2^N) \bmod \mathfrak{P}$$

where $g_1^{\mathfrak{n}}(z)$, $g_2^{\mathfrak{n}}(z) \in S_{k+\frac{1}{2}}^{4\mathfrak{n}\text{-new, ord}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathscr{O}_L)$ are $4\mathfrak{n}$-new ordinary Hilbert eigenforms of half-integer weight in the basis $\{g_i^{\mathfrak{n}}(z)\}_{i=1}^{m_{\mathfrak{n}}}$.
Suppose there does not exist any $\mathscr{B} \in \mathscr{O}_L$ such that $\mathscr{G}_2^{\mathfrak{n}}(z) = \mathscr{B}\mathscr{G}_1^{\mathfrak{n}}(z)$ but for all primes $\mathfrak{l} \nmid 4\mathfrak{n}p$, $\mathscr{G}_2^{\mathfrak{n}}(z)$ also satisfies the following congruence:

$$\lambda_{T_{\mathfrak{l}^2}}(g_1^{\mathfrak{n}}) \equiv \lambda_{T_{\mathfrak{l}^2}}(\mathscr{G}_2^{\mathfrak{n}}) \bmod \mathfrak{P}.$$

Then, using equation 10.40, we have $\lambda_{T_{\mathfrak{l}}}(f_1^{\mathfrak{n}}) = \lambda_{T_{\mathfrak{l}^2}}(g_1^{\mathfrak{n}})$ where $f_1^{\mathfrak{n}}(z) \in S_{2k}^{\mathfrak{n}\text{-new, ord}}(\Gamma_{\mathfrak{n}p}; \mathscr{O}_L)$ is an $\mathfrak{n}$-new ordinary Hilbert eigenform in the basis $\{f_i^{\mathfrak{n}}(z)\}_{i=1}^{m_{\mathfrak{n}}}$. Similarly, from equation 10.42, we have $\lambda_{T_{\mathfrak{l}}}(\mathscr{F}_2^{\mathfrak{n}}) = \lambda_{T_{\mathfrak{l}^2}}(\mathscr{G}_2^{\mathfrak{n}})$. It then follows:

$$\begin{aligned}\lambda_{T_{\mathfrak{l}}}(f_1^{\mathfrak{n}}) &= \lambda_{T_{\mathfrak{l}^2}}(g_1^{\mathfrak{n}}) \\ &\equiv \lambda_{T_{\mathfrak{l}^2}}(\mathscr{G}_2^{\mathfrak{n}}) \bmod \mathfrak{P} \\ &= \lambda_{T_{\mathfrak{l}}}(\mathscr{F}_2^{\mathfrak{n}}) \bmod \mathfrak{P} \\ &\equiv \lambda_{T_{\mathfrak{l}}}(f_2^{\mathfrak{n}}) \bmod \mathfrak{P}.\end{aligned}$$

This is a contradiction to our assumption 2 as $f_1^{\mathfrak{n}}(z)$ and $f_2^{\mathfrak{n}}(z)$ are two distinct basis elements. Hence $\mathscr{G}_1^{\mathfrak{n}}(z)$ is unique up to scalar multiplication in $\mathscr{O}_L$. ☐

10.3.2 Proof of main Theorem

We are now ready to give the proof of the main theorem 10.3.3. Before we do that, we will prove the following lemma which will be required to complete it's proof.

Lemma 10.3.15. *Let*

$$g(z) \in S_{k+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathscr{O}_L)$$

be a Hilbert cusp form of half-integer weight. Then for all prime ideals $\mathfrak{l} \subset \mathscr{O}_K$, the action of $U_{(p)}$ and $T_{\mathfrak{l}^2}$ on $g(z)$ is commutative, that is,

$$\left(g(z)|_{k+\frac{1}{2}} U_{(p)}\right)_{k+\frac{1}{2}} T_{\mathfrak{l}^2} = \left(g(z)|_{k+\frac{1}{2}} T_{\mathfrak{l}^2}\right)_{k+\frac{1}{2}} U_{(p)}.$$

Note 10.3.16. For $\mathfrak{l} \mid 4\mathfrak{n}p$, lemma 10.3.15 follows trivially, hence, we will assume $\mathfrak{l} \nmid 4\mathfrak{n}p$.

Proof. For simplicity, let $e^{2\pi i \operatorname{Tr}\left(\frac{\xi}{2}z\right)} = q^{\xi}$.

Let $g(z) = \sum_{\xi \in \mathscr{O}_K^+} b_{\xi} q^{\xi}$. Then

$$g(z)|_{k+\frac{1}{2}} U_{(p)} = \sum_{\xi \in \mathscr{O}_K^+} b_{p\xi} q^{\xi} \in S_{k+\frac{1}{2}}\left(\widetilde{\Gamma}_{4\mathfrak{n}p}, \left(\frac{p}{\bullet}\right)_2; \mathscr{O}_L\right)$$

Let $\mathscr{L} \gg 0$ be a totally positive generator of $\mathfrak{l}$. Then

$$\begin{aligned}\left(g(z)|_{k+\frac{1}{2}}U_{(p)}\right)|_{k+\frac{1}{2}}T_{\mathfrak{l}^2} &= \left(\sum_{\xi\in\mathscr{O}_K^+} b_{p\xi}q^{\xi}\right)|_{k+\frac{1}{2}}T_{\mathfrak{l}^2}\\ &= \sum_{\xi\in\mathscr{O}_K^+}\left(b_{\mathscr{L}^2p\xi}+\psi^{\star}(\mathscr{L})\left(\frac{\xi}{\mathscr{L}}\right)_2 N(\mathscr{L})^{k-1}b_{\mathscr{L}p\xi}+\psi^{\star}(\mathscr{L}^2)N(\mathscr{L})^{2k-1}b_{p\xi/\mathscr{L}^2}\right)q^{\xi}\end{aligned}$$

where $\psi^{\star}(\bullet) = \left(\frac{-1}{\bullet}\right)_2^k\psi(\bullet)$ and in our case $\psi(\bullet) = \left(\frac{p}{\bullet}\right)_2$.

Thus,

$$\begin{aligned}\left(g(z)|_{k+\frac{1}{2}}U_{(p)}\right)|_{k+\frac{1}{2}}T_{\mathfrak{l}^2} &= \sum_{\xi\in\mathscr{O}_K^+}\left(b_{\mathscr{L}^2p\xi}+\left(\frac{-1}{\mathscr{L}}\right)_2^k\left(\frac{p}{\mathscr{L}}\right)_2\left(\frac{\xi}{\mathscr{L}}\right)_2 N(\mathscr{L})^{k-1}b_{\mathscr{L}p\xi}+N(\mathscr{L})^{2k-1}b_{p\xi/\mathscr{L}^2}\right)q^{\xi}\\ &= \sum_{\xi\in\mathscr{O}_K^+}\left(b_{p\mathscr{L}^2\xi}+\left(\frac{-1}{\mathscr{L}}\right)_2^k\left(\frac{p\xi}{\mathscr{L}}\right)_2 N(\mathscr{L})^{k-1}b_{p\mathscr{L}\xi}+N(\mathscr{L})^{2k-1}b_{p\xi/\mathscr{L}^2}\right)q^{\xi}\\ &= \left(\sum_{\xi\in\mathscr{O}_K^+}\left(b_{\mathscr{L}^2\xi}+\left(\frac{-1}{\mathscr{L}}\right)_2^k\left(\frac{\xi}{\mathscr{L}}\right)_2 N(\mathscr{L})^{k-1}b_{\mathscr{L}\xi}+N(\mathscr{L})^{2k-1}b_{\xi/\mathscr{L}^2}\right)q^{\xi}\right)\Bigg|_{k+\frac{1}{2}}U_{(p)}\\ &= \left(g(z)|_{k+\frac{1}{2}}T_{\mathfrak{l}^2}\right)|_{k+\frac{1}{2}}U_{(p)}.\end{aligned}$$

□

We will now give the proof the main theorem 10.3.3 of this section.

Proof of Theorem 10.3.3. For simplicity, let us fix the integer i such that $1 \le i \le m_{\mathfrak{n}}$, say $i = 1$.

By proposition 10.3.11, for a $4\mathfrak{n}$-new ordinary Hilbert eigenform $g_1^{\mathfrak{n}}(z) \in S_{k+\frac{1}{2}}^{4\mathfrak{n}\text{-new,ord}}(\widetilde{\Gamma}_{4\mathfrak{n}p};\mathscr{O}_L)$, there exists an ordinary half-integer weight Hilbert cusp form $G_1^{\circ}(z) \in S_{k'+\frac{1}{2}}^{\text{ord}}(\widetilde{\Gamma}_{4\mathfrak{n}p};\mathscr{O}_{(\mathfrak{P})})$ such that we have the following congruence of Fourier coefficients:

$$g_1^{\mathfrak{n}}(z)|_{k+\frac{1}{2}}U_{(p)} \equiv G_1^{\circ}(z) \bmod p^{j+1}.$$

Let $t = \dim(S_{k'+\frac{1}{2}}^{\text{ord}}(\widetilde{\Gamma}_{4\mathfrak{n}p};L))$ and $\{\mathscr{G}_s^{\mathfrak{n}}(z)\}_{s=1}^t \subseteq S_{k'+\frac{1}{2}}^{\text{ord}}(\widetilde{\Gamma}_{4\mathfrak{n}p};\mathscr{O}_L)$ be a basis for $S_{k'+\frac{1}{2}}^{\text{ord}}(\widetilde{\Gamma}_{4\mathfrak{n}p};L)$ consisting of ordinary half-integer weight Hilbert eigenforms.

Note 10.3.17. We have a slight abuse of notation when we wrote the superscript $\mathfrak{n}$ for each element in the set $\{\mathscr{G}_s^{\mathfrak{n}}(z)\}_{s=1}^t$. These ordinary half-integer weight Hilbert eigenforms are not necessarily $4\mathfrak{n}$-new but include the set of $4\mathfrak{n}$-new ordinary half-integer weight Hilbert eigenforms as well. Note that $t \ge m_{\mathfrak{n}}$.

So, we can write $G_1^{\circ}(z)$ as a linear combination of elements of the basis $\{\mathscr{G}_s^{\mathfrak{n}}(z)\}_{s=1}^t$.

$$G_1^{\circ}(z) = \sum_{s=1}^{t}\beta_s\mathscr{G}_s^{\mathfrak{n}}(z) \text{ where } \{\beta_s\}_{s=1}^t \subset L.$$

By proposition 10.3.13, for a $4\mathfrak{n}$-new half-integer weight Hilbert eigenform $g_1^{\mathfrak{n}}(z) \in S_{k+\frac{1}{2}}^{4\mathfrak{n}\text{-new,ord}}(\widetilde{\Gamma}_{4\mathfrak{n}p};\mathscr{O}_L)$, there exists a $4\mathfrak{n}$-new ordinary half-integer weight Hilbert eigenform in $S_{k'+\frac{1}{2}}^{4\mathfrak{n}\text{-new, ord}}(\widetilde{\Gamma}_{4\mathfrak{n}p};\mathscr{O}_L)$, say $\mathscr{G}_1^{\mathfrak{n}}(z)$, unique up to scalar multiplication by elements in $\mathscr{O}_L$ such that for all prime ideals $\mathfrak{l} \subset \mathscr{O}_K$, $\mathfrak{l} \nmid 4\mathfrak{n}p$, we have

$$\lambda_{T_{\mathfrak{l}^2}}(g_1^{\mathfrak{n}}) \equiv \lambda_{T_{\mathfrak{l}^2}}(\mathscr{G}_1^{\mathfrak{n}}) \bmod \mathfrak{P}. \tag{10.44}$$

This uniqueness implies that for every integer s such that $2 \leq s \leq t$ but $s \neq 1$, there exists some prime ideal $\mathfrak{l}_s \subset \mathcal{O}_K$, $\mathfrak{l}_s \nmid 4\mathfrak{n}p$ such that

$$\mathfrak{P} \nmid \left(\lambda_{T_{\mathfrak{l}_s^2}}(g_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right).$$

So, we have

$$\mathfrak{P} \nmid \prod_{s=2}^{t} \left(\lambda_{T_{\mathfrak{l}_s^2}}(g_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right). \tag{10.45}$$

In other words,

$$\frac{1}{\prod_{s=2}^{t} \left(\lambda_{T_{\mathfrak{l}_s^2}}(g_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right)} \in \mathcal{O}_{(\mathfrak{P})}. \tag{10.46}$$

Define

$$\mathfrak{G}_1^{\mathfrak{n}}(z) := \frac{\lambda_{U_{(p)^2}}(g_1^{\mathfrak{n}})^{-1}}{\prod_{s=2}^{t} \left(\lambda_{T_{\mathfrak{l}_s^2}}(g_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right)} G_1^{\circ}(z)\Big|_{k'+\frac{1}{2}} \prod_{s=2}^{t} \left(T_{\mathfrak{l}_s^2} - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right). \tag{10.47}$$

Claim 1: $\mathfrak{G}_1^{\mathfrak{n}}(z) = \lambda_{U_{(p)^2}}(g_1^{\mathfrak{n}})^{-1} \mathscr{C} \mathscr{G}_1^{\mathfrak{n}}(z)$ for some $\mathscr{C} \in \mathcal{O}_{(\mathfrak{P})}$.

Note 10.3.18. The eigenvalue $\lambda_{U_{(p)^2}}(g_1^{\mathfrak{n}})^{-1} \in \mathcal{O}_{(\mathfrak{P})}$. This follows from the fact that $g_1^{\mathfrak{n}}(z)$ is an ordinary Hilbert eigenform of half-integer weight and hence its $U_{(p)^2}$ eigenvalue $\lambda_{U_{(p)^2}}(g_1^{\mathfrak{n}})$ is a $\mathfrak{P}$-adic unit for the fixed prime ideal $\mathfrak{P} \subset \mathcal{O}_L$ lying above $\mathfrak{p}$, that is, $\mathfrak{P} \nmid \lambda_{U_{(p)^2}}(g_1^{\mathfrak{n}})$.

Next we want to show that $G_1^{\circ}(z)|_{k'+\frac{1}{2}} \prod_{s=2}^{t}(T_{\mathfrak{l}_s^2} - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}}))$ in definition 10.47 has Fourier coefficients in $\mathcal{O}_{(\mathfrak{P})}$.

$$\begin{aligned}
&G_1^{\circ}(z)\Big|_{k'+\frac{1}{2}} \prod_{s=2}^{t} \left(T_{\mathfrak{l}_s^2} - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right) \\
&= \left(\sum_{s=1}^{t} \beta_s \mathscr{G}_s^{\mathfrak{n}}(z)\right)\Bigg|_{k'+\frac{1}{2}} \prod_{s=2}^{t} \left(T_{\mathfrak{l}_s^2} - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right) \\
&= \left(\sum_{s=1}^{t} \beta_s \mathscr{G}_s^{\mathfrak{n}}(z)\right)\Bigg|_{k'+\frac{1}{2}} \left(T_{\mathfrak{l}_2^2} - \lambda_{T_{\mathfrak{l}_2^2}}(\mathscr{G}_2^{\mathfrak{n}})\right) \prod_{s=3}^{t} \left(T_{\mathfrak{l}_s^2} - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right) \\
&= \left(\sum_{s=1}^{t} \beta_s (\mathscr{G}_s^{\mathfrak{n}}(z)|_{k'+\frac{1}{2}} T_{\mathfrak{l}_2^2}) - \lambda_{T_{\mathfrak{l}_2^2}}(\mathscr{G}_2^{\mathfrak{n}}) \sum_{s=1}^{t} \beta_s \mathscr{G}_s^{\mathfrak{n}}(z)\right)\Bigg|_{k'+\frac{1}{2}} \prod_{s=3}^{t} \left(T_{\mathfrak{l}_s^2} - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right) \\
&= \left(\sum_{s=1}^{t} \beta_s \lambda_{T_{\mathfrak{l}_2^2}}(\mathscr{G}_s^{\mathfrak{n}}) \mathscr{G}_s^{\mathfrak{n}}(z) - \sum_{s=1}^{t} \beta_s \lambda_{T_{\mathfrak{l}_2^2}}(\mathscr{G}_2^{\mathfrak{n}}) \mathscr{G}_s^{\mathfrak{n}}(z)\right)\Bigg|_{k'+\frac{1}{2}} \prod_{s=3}^{t} \left(T_{\mathfrak{l}_s^2} - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right) \\
&= \left(\sum_{s=1}^{t} \beta_s \left(\lambda_{T_{\mathfrak{l}_2^2}}(\mathscr{G}_s^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_2^2}}(\mathscr{G}_2^{\mathfrak{n}})\right) \mathscr{G}_s^{\mathfrak{n}}(z)\right)\Bigg|_{k'+\frac{1}{2}} \prod_{s=3}^{t} \left(T_{\mathfrak{l}_s^2} - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right). \\
&= \sum_{s=1}^{t} \beta_s \prod_{w=2}^{t} \left(\lambda_{T_{\mathfrak{l}_w^2}}(\mathscr{G}_s^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w^2}}(\mathscr{G}_w^{\mathfrak{n}})\right) \mathscr{G}_s^{\mathfrak{n}}(z). \\
&= \beta_1 \mathscr{G}_1^{\mathfrak{n}}(z) \prod_{w=2}^{t} \left(\lambda_{T_{\mathfrak{l}_w^2}}(\mathscr{G}_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w^2}}(\mathscr{G}_w^{\mathfrak{n}})\right)
\end{aligned} \tag{10.48}$$

or

$$\beta_1 = \frac{G_1^{\circ}(z)|_{k'+\frac{1}{2}} \prod_{s=2}^{t} \left(T_{\mathfrak{l}_s^2} - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right)}{\mathscr{G}_1^{\mathfrak{n}}(z) \prod_{w=2}^{t} \left(\lambda_{T_{\mathfrak{l}_w^2}}(\mathscr{G}_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w^2}}(\mathscr{G}_w^{\mathfrak{n}})\right)}.$$

We want to show that $\beta_1 \in \mathcal{O}_{(\mathfrak{P})}$.

Note that $G_1^\circ(z)|_{k'+\frac{1}{2}} \prod_{s=2}^t (T_{\mathfrak{l}_s^2} - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}}))$ has Fourier coefficients in $\mathcal{O}_{(\mathfrak{P})}$. This is because $\lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}}) \in \mathcal{O}_L$ and the Hecke-algebra $\widetilde{\mathbb{T}}_{k'+\frac{1}{2}}(4\mathfrak{n}p; \mathcal{O}_{(\mathfrak{P})}) \subseteq \mathrm{End}(S_{k'+\frac{1}{2}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_{(\mathfrak{P})}))$.

Next, we are given that $\mathscr{G}_1^{\mathfrak{n}}(z) \in S_{k'+\frac{1}{2}}^{4\mathfrak{n}\text{-new, ord}}(\widetilde{\Gamma}_{4\mathfrak{n}p}; \mathcal{O}_L)$ is unique up to scalar multiplication in $\mathcal{O}_L$. We can therefore assume it is scaled in a way that it has at least one Fourier coefficient that is not divisible by $\mathfrak{P}$. Now, suppose there exists some w such that $2 \le w \le t$ for which $\mathfrak{P} \mid \left(\lambda_{T_{\mathfrak{l}_w^2}}(\mathscr{G}_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w^2}}(\mathscr{G}_w^{\mathfrak{n}})\right)$, then

$$\mathfrak{P} \mid \left(\lambda_{T_{\mathfrak{l}_w^2}}(\mathscr{G}_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w^2}}(g_1^{\mathfrak{n}}) + \lambda_{T_{\mathfrak{l}_w^2}}(g_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w^2}}(\mathscr{G}_w^{\mathfrak{n}})\right).$$

Since $\mathfrak{P} \mid (\lambda_{T_{\mathfrak{l}_w^2}}(\mathscr{G}_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w^2}}(g_1^{\mathfrak{n}}))$ by 10.44, it follows,

$$\mathfrak{P} \mid \left(\lambda_{T_{\mathfrak{l}_w^2}}(g_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w^2}}(\mathscr{G}_w^{\mathfrak{n}})\right).$$

This is a contradiction to uniqueness of $\mathscr{G}_1^{\mathfrak{n}}(z)$ in 10.44 . Thus,

$$\mathfrak{P} \nmid \prod_{w=2}^{t} \left(\lambda_{T_{\mathfrak{l}_w^2}}(\mathscr{G}_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w^2}}(\mathscr{G}_w^{\mathfrak{n}})\right). \tag{10.49}$$

We can hence conclude

$$\beta_1 \in \mathcal{O}_{(\mathfrak{P})}. \tag{10.50}$$

From equations 10.47 and 10.48, we can write

$$\begin{aligned}
\mathfrak{G}_1^{\mathfrak{n}}(z) &= \frac{\lambda_{U_{(p)^2}}(g_1^{\mathfrak{n}})^{-1}}{\prod_{s=2}^t \left(\lambda_{T_{\mathfrak{l}_s^2}}(g_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right)} G_1^\circ(z)\Big|_{k'+\frac{1}{2}} \prod_{s=2}^{t} \left(T_{\mathfrak{l}_s^2} - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right) \\
&= \lambda_{U_{(p)^2}}(g_1^{\mathfrak{n}})^{-1} \beta_1 \frac{\prod_{w=2}^t \left(\lambda_{T_{\mathfrak{l}_w^2}}(\mathscr{G}_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w^2}}(\mathscr{G}_w^{\mathfrak{n}})\right)}{\prod_{s=2}^t \left(\lambda_{T_{\mathfrak{l}_s^2}}(g_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right)} \mathscr{G}_1^{\mathfrak{n}}(z) \\
&= \lambda_{U_{(p)^2}}(g_1^{\mathfrak{n}})^{-1} \mathscr{C} \mathscr{G}_1(z)
\end{aligned}$$

where we know by 10.46 and 10.50 that

$$\mathscr{C} = \beta_1 \frac{\prod_{w=2}^t \left(\lambda_{T_{\mathfrak{l}_w^2}}(\mathscr{G}_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_w^2}}(\mathscr{G}_w^{\mathfrak{n}})\right)}{\prod_{s=2}^t \left(\lambda_{T_{\mathfrak{l}_s^2}}(g_1^{\mathfrak{n}}) - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right)} \in \mathcal{O}_{(\mathfrak{P})}.$$

Claim 2: $\mathfrak{G}_1^{\mathfrak{n}}(z)|_{k'+\frac{1}{2}} U_{(p)} \equiv g_1^{\mathfrak{n}}(z) \bmod p^{j+1}$.

We will again look at action of $\prod_{s=2}^t (T_{\mathfrak{l}_s^2} - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}}))$ on $G_1^\circ(z)$ modulo p^{j+1}.

$$G_1^\circ(z)\Big|_{k'+\frac{1}{2}} \prod_{s=2}^{t} \left(T_{\mathfrak{l}_s^2} - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right) \equiv \left(g_1^{\mathfrak{n}}(z)|_{k+\frac{1}{2}} U_{(p)}\right)\Big|_{k+\frac{1}{2}} \prod_{s=2}^{t} \left(T_{\mathfrak{l}_s^2} - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right) \bmod p^{j+1}$$

We see how one of the terms out the product $\prod_{s=2}^t (T_{\mathfrak{l}_s^2} - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}}))$ acts on $g_1^{\mathfrak{n}}(z)|_{k+\frac{1}{2}} U_{(p)}$ to observe the pattern.

$$G_1^\circ(z)\Big|_{k'+\frac{1}{2}} \prod_{s=2}^{t} \left(T_{\mathfrak{l}_s^2} - \lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right)$$

$$
\begin{aligned}
&\equiv \left(g_1^{\mathfrak{n}}(z)|_{k+\frac{1}{2}}U_{(p)}\right)\Big|_{k+\frac{1}{2}}(T_{\mathfrak{l}_2^2}-\lambda_{T_{\mathfrak{l}_2^2}}(\mathscr{G}_2^{\mathfrak{n}}))\prod_{s=3}^{t}\left(T_{\mathfrak{l}_s^2}-\lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right) \bmod p^{j+1}\\
&=\left(g_1^{\mathfrak{n}}(z)|_{k+\frac{1}{2}}U_{(p)}T_{\mathfrak{l}_2^2}-\lambda_{T_{\mathfrak{l}_2^2}}(\mathscr{G}_2^{\mathfrak{n}})\,g_1^{\mathfrak{n}}(z)|_{k+\frac{1}{2}}U_{(p)}\right)\Big|_{k+\frac{1}{2}}\prod_{s=3}^{t}\left(T_{\mathfrak{l}_s^2}-\lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right) \bmod p^{j+1}\\
&=\left(g_1^{\mathfrak{n}}(z)|_{k+\frac{1}{2}}T_{\mathfrak{l}_2^2}U_{(p)}-\lambda_{T_{\mathfrak{l}_2^2}}(\mathscr{G}_2^{\mathfrak{n}})\,g_1^{\mathfrak{n}}(z)|_{k+\frac{1}{2}}U_{(p)}\right)\Big|_{k+\frac{1}{2}}\prod_{s=3}^{t}\left(T_{\mathfrak{l}_s^2}-\lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right) \bmod p^{j+1}\\
&=\left(\lambda_{T_{\mathfrak{l}_2^2}}(g_1^{\mathfrak{n}})\,g_1^{\mathfrak{n}}(z)|_{k+\frac{1}{2}}U_{(p)}-\lambda_{T_{\mathfrak{l}_2^2}}(\mathscr{G}_2^{\mathfrak{n}})\,g_1^{\mathfrak{n}}(z)|_{k+\frac{1}{2}}U_{(p)}\right)\Big|_{k+\frac{1}{2}}\prod_{s=3}^{t}\left(T_{\mathfrak{l}_s^2}-\lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right) \bmod p^{j+1}\\
&=\left(\lambda_{T_{\mathfrak{l}_2^2}}(g_1^{\mathfrak{n}})-\lambda_{T_{\mathfrak{l}_2^2}}(\mathscr{G}_2^{\mathfrak{n}})\right)\left(g_1^{\mathfrak{n}}(z)|_{k+\frac{1}{2}}U_{(p)}\right)\Big|_{k_{\frac{1}{2}}}\prod_{s=3}^{t}\left(T_{\mathfrak{l}_s^2}-\lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right) \bmod p^{j+1}\\
&=\prod_{s=2}^{t}\left(\lambda_{T_{\mathfrak{l}_s^2}}(g_1^{\mathfrak{n}})-\lambda_{T_{\mathfrak{l}_s^2}}(\mathscr{G}_s^{\mathfrak{n}})\right)\left(g_1^{\mathfrak{n}}(z)|_{k+\frac{1}{2}}U_{(p)}\right) \bmod p^{j+1}. \qquad (10.51)
\end{aligned}
$$

Using 10.47 and 10.51, we get

$$\mathfrak{G}_1^{\mathfrak{n}}(z)\equiv\lambda_{U_{(p)^2}}(g_1^{\mathfrak{n}})^{-1}g_1^{\mathfrak{n}}(z)|_{k+\frac{1}{2}}U_{(p)} \bmod p^{j+1}. \qquad (10.52)$$

We now act on both sides in the congruence 10.52 by $U_{(p)}$ operator and get

$$
\begin{aligned}
\mathfrak{G}_1^{\mathfrak{n}}(z)|_{k'+\frac{1}{2}}U_{(p)} &\equiv \lambda_{U_{(p)^2}}(g_1^{\mathfrak{n}})^{-1}g_1^{\mathfrak{n}}(z)|_{k+\frac{1}{2}}U_{(p)}^2 \bmod p^{j+1}\\
&=\lambda_{U_{(p)^2}}(g_1^{\mathfrak{n}})^{-1}\lambda_{U_{(p)^2}}(g_1^{\mathfrak{n}})g_1^{\mathfrak{n}}(z) \bmod p^{j+1}\\
&=g_1^{\mathfrak{n}}(z) \bmod p^{j+1}. \qquad (10.53)
\end{aligned}
$$

Uniqueness: Let $\mathfrak{G}_2^{\mathfrak{n}}(z)\in S_{k'+\frac{1}{2}}^{4\mathfrak{n}\text{-new, ord}}(\widetilde{\Gamma}_{4\mathfrak{n}p};\mathcal{O}_{(\mathfrak{p})})$ be a $4\mathfrak{n}$-new ordinary half-integer weight Hilbert eigenform for all operators $T_{\mathfrak{l}^2}$ over all prime ideals $\mathfrak{l}\subset\mathcal{O}_K$ such that it satisfies congruence of the type 10.53 with $g_2^{\mathfrak{n}}(z)$:

$$\mathfrak{G}_2^{\mathfrak{n}}(z)|_{k'+\frac{1}{2}}U_{(p)}\equiv g_2^{\mathfrak{n}}(z) \bmod p^{j+1}$$

Suppose there does not exist any $\mathscr{D}\in\mathcal{O}_{(\mathfrak{P})}$ such that $\mathfrak{G}_2^{\mathfrak{n}}(z)=\mathscr{D}\mathfrak{G}_1^{\mathfrak{n}}(z)$ but $\mathfrak{G}_2^{N}(z)$ also satisfies congruence 10.53 with $g_1^{\mathfrak{n}}(z)$:

$$\mathfrak{G}_2^{\mathfrak{n}}(z)|_{k'+\frac{1}{2}}U_{(p)}\equiv g_1^{\mathfrak{n}}(z) \bmod p^{j+1}.$$

It the follows

$$g_1^{\mathfrak{n}}(z)\equiv g_2^{\mathfrak{n}}(z) \bmod p^{j+1}$$

which implies

$$g_1^{\mathfrak{n}}(z)|_{k+\frac{1}{2}}T_{\mathfrak{l}^2}\equiv g_2^{\mathfrak{n}}(z)|_{k+\frac{1}{2}}T_{\mathfrak{l}^2} \bmod p^{j+1}.$$

So, we have for all prime ideal $\mathfrak{l}\subset\mathcal{O}_K$,

$$\lambda_{T_{\mathfrak{l}^2}}(g_2^{\mathfrak{n}})g_2^{\mathfrak{n}}(z)\equiv\lambda_{T_{\mathfrak{l}^2}}(g_1^{\mathfrak{n}})g_1^{\mathfrak{n}}(z) \bmod p^{j+1}.$$

We can then write

$$\lambda_{T_{\mathfrak{l}}}(f_2^{\mathfrak{n}})g_2^{\mathfrak{n}}(z)\equiv\lambda_{T_{\mathfrak{l}}}(f_1^{\mathfrak{n}})g_1^{\mathfrak{n}}(z) \bmod p^{j+1}$$

which implies

$$\left(\lambda_{T_{\mathfrak{l}}}(f_2^{\mathfrak{n}})-\lambda_{T_{\mathfrak{l}}}(f_1^{\mathfrak{n}})\right)g_1^{\mathfrak{n}}(z)\equiv 0 \bmod p^{j+1}.$$

Now $\mathfrak{P}\nmid g_1^{\mathfrak{n}}(z)$ due to choice of scaling of $g_1^{\mathfrak{n}}(z)$. So we get for all prime ideals $\mathfrak{l}\nmid 4\mathfrak{n}p$, $\mathfrak{P}\mid(\lambda_{T_{\mathfrak{l}}}(f_2^{\mathfrak{n}})-\lambda_{T_{\mathfrak{l}}}(f_1^{\mathfrak{n}}))$. This is a contradiction to assumption 2 as $f_1^{\mathfrak{n}}(z)$ and $f_2^{\mathfrak{n}}(z)$ are distinct basis elements. Thus, $\mathfrak{G}_1^{\mathfrak{n}}(z)$ is the unique $4\mathfrak{n}$-new half-integer weight ordinary Hilbert eigenform up to scalar multiplication in $\mathcal{O}_{(\mathfrak{P})}$ that satisfies the congruence 10.53 with $g_1^{\mathfrak{n}}(z)$. ☐

CHAPTER 11

Applications

In this chapter, we will briefly recall elliptic curves over number fields and the associated L-function. We avoid going into details wherever possible and the reader might find useful to refer to [Sil09] for detailed background. Our main purpose is to work over an example of an elliptic curve E over the number field $\mathbb{Q}(\sqrt{5})$ and set-up the framework for applying our main Theorems 10.2.5 and 10.3.3 to the Hilbert newform associated to E.

11.1 Elliptic curve E over an arbitrary number field K

Every elliptic E over a number field K can be defined using a Weierstrass equation given by

$$E/K \,:\, y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

where $a_1, a_2, a_3, a_4, a_6 \in K$.

Let $E(K)$ denote the set of K-rational points. Then $E(K)$ forms an abelian group under addition law defined explicitly in [Sil09, Section III.2]. The *Mordell-Weil* Theorem tell us that $E(K)$ has the following form

$$E(K) \cong E(K)_{\text{tor}} \oplus \mathbb{Z}^r.$$

Here $E(K)_{\text{tor}}$ is the torsion subgroup of $E(K)$ and is finite while r is the rank of $E(K)$ and is a non-negative integer. The proof of *Mordell-Weil* Theorem can be found in [Sil09, Chapter VIII]. For a given elliptic curve E/K, it is relatively easy to determine the torsion subgroup while the rank turns out to be quite hard to compute.

For every elliptic curve E/K, we can define the associated L-series in terms of an Euler product given by

$$\prod_{\substack{\mathfrak{p} \text{ prime} \\ \mathfrak{p} \subset \mathcal{O}_K}} L_{\mathfrak{p}}(N(\mathfrak{p})^{-s}),$$

where $N(\mathfrak{p})$ denotes the norm of $\mathfrak{p}$.
For each $\mathfrak{p}$, the local Euler factors $L_{\mathfrak{p}}(T)$ depends on the the reduction type of E at $\mathfrak{p}$. $L_{\mathfrak{p}}(T)$ can be visualised as a power series in T and we use its linear coefficients to define the Fourier coefficients $a_{\mathfrak{p}}$ of the L-function of E/K. Then we can find the complete set of Fourier coefficients $a_{\mathfrak{n}}$ for ideals $\mathfrak{n} \subset \mathcal{O}_K$ using recursion and multiplicativity, see [Sil09, C.16, pg. 449-450].

Let $L(E/K,s)=\sum_{\mathfrak{n}\subset\mathscr{O}_K} a_{\mathfrak{n}}N(\mathfrak{n})^{-s}$ be the L-function associated to E/K. We define

$$\Lambda(E/K,s) := (\mathrm{Norm}(\mathfrak{n}_E)D_K^2)^{\frac{s}{2}}((2\pi)\Gamma(s))^{[K:\mathbb{Q}]}\, L(E,s)$$

where $\mathfrak{n}_E$ denotes the conductor of E/K and D_K denotes the discriminant of K.
The *Hasse-Weil* conjectures predicts that $\Lambda(E/K,s)$ is analytically continuous on the entire complex plane and satisfies the functional equation

$$\Lambda(E/K,s) = w_{E/K}.\Lambda(E/K,2-s), \tag{11.1}$$

where $w_{E/K}\in\{\pm 1\}$ is the global root number of E/K and determines the sign of the functional equation.
Using Modularity of real quadratic fields [FHS15, Theorem 1], this conjecture has been proved for real quadratic fields.

We now state the weak form of conjecture of *Birch* and *Swinnerton-Dyer* (BSD) for elliptic curve E/K.

Conjecture 11.1.1 (Weak form of BSD for E/K)**.** *Let E/K be an elliptic curve over number field K. Then $L(E/K,s)$ has analytic continuation to $\mathbb{C}$ and satisfies*

$$ord_{s=1}(L(E/K,s)) = rank(E/K)$$

where $ord_{s=1}(L(E/K,s))$ is the order of vanishing of the L-function $L(E/K,s)$ at $s=1$.

11.2 *Waldspurger's* Theorem - Generalisation

Let K be a totally real quadratic field of narrow class number 1 introduced in section 6.1. As usual, let $k\in\mathbb{Z}_{>0}$ and $\mathfrak{n}\subset\mathscr{O}_K$ be a square-free integral ideal of odd norm. Let $g(z)=\sum_{\xi\in\mathscr{O}_K^+} b_\xi q^\xi$ be a half-integer weight Hilbert modular form of weight $k+\frac{1}{2}$ and level $4\mathfrak{n}$ and let $f(z)=\sum_{\xi\in\mathscr{O}_K^+} a_\xi q^\xi$ be a Hilbert newform of weight $2k$, level $\mathfrak{n}$ and trivial character associated to $g(z)$ via *Shimura's* correspondence. Let $D\in K^\star$ and let $\psi_D=\left(\frac{D}{\bullet}\right)_2$ be the quadratic residue symbol defined in 7.1.2. Then the twist of $f(z)$ by ψ_D is given by $(f\otimes\psi_D)(z)=\sum_{\xi\in\mathscr{O}_K^+}\psi_D(\xi)a_\xi q^\xi$.

In 2003, the *Waldspurger*'s Theorem was generalised to Hilbert modular forms over totally real fields by *Baruch* and *Mao* [BM03] who showed that the square of b_D, the D^{th} Fourier coefficient of $g(z)$ is proportional to the central value $L(f\otimes\psi_D,k)$. In 2020, *Sirolli* and *Tornaría* [ST21] gave a more explicit formula by which $L(f\otimes\psi_D,k)$ is expressed explicitly as the product of $|b_D|^2$ with some elementary factors. This allows us to compute the central L-values $L(f\otimes\psi_D,k)$ explicitly. Before we can formally state this explicit formula as applicable to our case, we first need to introduce some notation.

We will first give the formula for the sign ϵ_f of functional equation of the $L(f\otimes\psi_D,s)$.

Let $\Sigma_{\mathfrak{n}}=\{v\ :\ v\mid\mathfrak{n}\}\cup\mathbf{a}$ is the set of all finite places dividing the level $\mathfrak{n}$ combined with the set of all infinite places $\mathbf{a}$ of K. For each place v, w_v^f denotes the *Atkin-Lehner* eigenvalue of $f(z)$ at v. These are the eigenvalues of $f(z)$ under action of *Atkin-Lehner* operators.

Definition 11.2.1 (*Atkin-Lehner* operator for Hilbert modular forms)**.**
Let $\wp,\mathscr{N}\gg 0$ be the generators of the prime ideal $\mathfrak{p}$ and the integral ideal $\mathfrak{n}$ respectively. Suppose $\mathfrak{p}\mid\mathfrak{n}$ but $\mathfrak{p}^2\nmid\mathfrak{n}$. Let $f(z)\in S_{2k}(\Gamma_{\mathfrak{n}})$. Then we define the action of Atkin-Lehner operator $W_{\mathfrak{p}}$ on $f(z)$ in terms of k-slash action as

$$f(z)|_{2k}W_{\mathfrak{p}} := f(z)\bigg|_{2k}\begin{pmatrix}\wp b & 2\delta^{-1}a\\ 2^{-1}\delta\mathscr{N} & \wp\end{pmatrix}$$

where $a,b\in\mathscr{O}_K$ such that $\wp^2 b-\mathscr{N}a=\wp$.
Note that $W_{\mathfrak{p}}$ depends only on $\wp$ and not on the choice of algebraic integers a and b.

Proposition 11.2.2. $W_{\mathfrak{p}}^2$ *acts an involution on* $S_{2k}(\Gamma_{\mathfrak{n}})$.

Proof. This easily follows from observing that $W_{\mathfrak{p}}^2$ can be rewritten as $\wp M$ where $\wp$ is equivalent to the scalar matrix $\left(\begin{smallmatrix} \wp & 0 \\ 0 & \wp \end{smallmatrix}\right)$ and $M \in \Gamma_{\mathfrak{n}}$. □

Remark 11.2.3. Thus, the *Atkin-Lehner* eigenvalues of $f(z)$ take values ± 1.

We next fix some quantities at the archimedean places as in [ST21].

***Atkin-Lehner* eigenvalue of f(z) at ∞:** $w_{\infty}^f = (-1)^k$.
Valuation of $\mathfrak{n}$ at infinity: $\mathrm{val}_{\infty}(\mathfrak{n}) = 1$.
Quadratic residue symbol ψ_D at infinity: $\psi_D(\infty) = \mathrm{sgn}(D)$.

Then the sign ϵ_f of the functional equation of $L(f \otimes \psi_D, s)$ is given by

$$\epsilon_f = \prod_{v \in \Sigma_{\mathfrak{n}}} w_v^f \psi_D(v)^{\mathrm{val}_v(\mathfrak{n})}.$$

For square-free $\mathfrak{n}$, $\mathrm{val}_v(\mathfrak{n}) = 1$. Thus,

$$\epsilon_f = \prod_{v \in \Sigma_{\mathfrak{n}}} w_v^f \psi_D(v). \tag{11.2}$$

We can now state the generalised *Waldspurger's* Theorem for Hilbert modular forms [ST21, Theorem A].

Theorem 11.2.4 (*Waldspurger's* Theorem for Hilbert modular forms). *Assume $\epsilon_f = 1$. Let $\mathfrak{n} \subset \mathcal{O}_K$ be a square-free integral ideal of odd norm. Then for every $D \in K^{\star}$ that satisfies $\psi_D(v) = w_v^f$ for every $v \mid \mathfrak{n}$, there exists a non-zero Hilbert modular form $g(z) = \sum_{\xi \in \mathcal{O}_K^+} b_{\xi} q^{\xi}$ of weight $k + \frac{1}{2}$ and level $4\mathfrak{n}$ whose Fourier coefficients are effectively computable and satisfy*

$$L(f \otimes \psi_D, k) = 2^{\omega(\mathfrak{n}, D)} c_f \langle f, f \rangle \frac{|b_D|^2}{|D|^{k-\frac{1}{2}} \langle g, g \rangle}$$

where $\omega(\mathfrak{n}, D)$ denotes the number of prime ideals dividing both $\mathfrak{n}$ and conductor of $K(\sqrt{D})/K$, c_f is the constant explicitly given in [ST21, Equation 6.10], $\langle f, f \rangle$ and $\langle g, g \rangle$ denote the Petersson inner products.

11.3 Examples over $\mathbb{Q}(\sqrt{5})$

Let us now fix $K = \mathbb{Q}(\sqrt{5})$. Let a be the generator of K with minimal polynomial $x^2 - x - 1$.

Consider an elliptic curve $E/\mathbb{Q}(\sqrt{5})$ defined by the Weierstrass equation

$$E: \quad y^2 + xy + ay = x^3 + (a+1)x^2 + ax. \tag{11.3}$$

The conductor of E is the prime ideal $\mathfrak{p}$ in $\mathcal{O}_K$ where $\mathfrak{p}$ is generated by $(5a - 2)$ and has conductor norm 31.

Remark 11.3.1. We note using SAGE [S$^+$] that the prime $p = 31$ does not divide the generalised Bernoulli numbers $B_{30,(\frac{5}{\bullet})}$ and $B_{15,(\frac{5}{\bullet})(\frac{\bullet}{31})}$ and hence the assumption 3 holds true for our example.
Let $n \in \mathbb{Z}_{\geq 0}$ and χ be a Dirichlet character of conductor N. Then we can write the generalised Bernoulli number $B_{n,\chi}$ in terms of Bernoulli numbers as

$$B_{n,\chi} = \sum_{a=1}^{N} \chi(a) \sum_{k=0}^{n} \binom{n}{k} B_n a^{n-k} N^{k-1} \tag{11.4}$$

We write the code for formula 11.4 in SAGE and compute the required generalised Bernoulli numbers $B_{30,(\frac{5}{\bullet})}$ and $B_{15,(\frac{5}{\bullet})(\frac{\bullet}{31})}$ explicitly to see if they are prime 31 divides each of them. For example, see the input and output for $B_{30,(\frac{5}{\bullet})}$ below.

INPUT

```
a = var('a')
list(a) = [binomial(30,k) * bernoulli(k) * a^(30-k)*5^(k-1) for k in range(0,30)]
innersum(a) = [sum(list(a))]
list = [kronecker(5,a) * innersum(a) for a in range(1,5)]
generalisedBernoulli=sum(list)
print(generalisedBernoulli)
```

OUTPUT

```
(1252792519982873734870069598004/5)
```

Recall that E is modular if there exists a Hilbert cuspidal eigenform $f(z)$ over K of parallel weight 2 with K-rational Hecke-eigenvalues such that Hasse-Weil L-function of E is equal to the L-function of $f(z)$. In 2014, the modularity over any real quadratic field was proved by *Freitas, Le Hung and Siksek* in [FHS15, Theorem 1].

Theorem 11.3.2 (Modularity of Elliptic curves over real quadratic fields)**.** *Let E be an elliptic curve over a real quadratic field K. Then E is modular.*

In particular, the elliptic curve E over $\mathbb{Q}(\sqrt{5})$ given in 11.3 is modular. Then we have a Hilbert newform $f(z) \in S_2(\Gamma_{\mathfrak{p}})$ of parallel weight 2 and level $\mathfrak{p} = (5a-2)$ attached to E such that

$$L(E,s) = L(f,s).$$

By using generalised *Kohnen's* isomorphism (section 7.5), there exists a half-integer weight Hilbert newform $g(z) \in S^{\text{new}}_{\frac{3}{2}}(\widetilde{\Gamma}_{4\mathfrak{p}})$ that is unique up to scalar multiplication such that

$$\lambda_{T_{\mathfrak{l}}}(f) = \lambda_{T_{\mathfrak{l}^2}}(g)$$

where $\lambda_{T_{\mathfrak{l}}}(f)$ is the Hecke-eigenvalue of $f(z)|_2 T_{\mathfrak{l}}$ and $\lambda_{T_{\mathfrak{l}^2}}(g)$ is the Hecke-eigenvalue of $g(z)|_{\frac{3}{2}} T_{\mathfrak{l}^2}$ over all prime ideals $\mathfrak{l} \subset \mathcal{O}_{\mathbb{Q}(\sqrt{5})}$.

Now the generalisation of *Waldspurger's* theorem (section 11.2) links the central critical values of twists of $f(z)$ with Fourier coefficients of $g(z)$. We want to use this link and so, we start by choosing an element $D \in K^\star$ to twist $f \mapsto f \otimes \psi_D$ where $\psi_D = \left(\frac{D}{\bullet}\right)_2$ takes values in $\{-1,0,+1\}$. Our choice of twist is not random but is rather based on making the sign ϵ_f of the functional equation of $L(f \otimes \psi_D, s)$ positive in a specific way. This would then imply that the order of vanishing of $L(f \otimes \psi_D, s)$ is even at $s = 1$. This can be observed using the functional equation for the completed L-function of $(f \otimes \psi_D)(z)$ which is:

$$\Lambda(f \otimes \psi_D, s) = (+1) \,.\, \Lambda(f \otimes \psi_D, 2-s). \tag{11.5}$$

Using Taylor series around $s = 1$, we can write,

$$\begin{aligned}\Lambda(f \otimes \psi_D, 2-s) &= \Lambda(f \otimes \psi_D, 1) + \frac{\Lambda'(f \otimes \psi_D, 1)}{1!}(1-s) + \frac{\Lambda''(f \otimes \psi_D, 1)}{2!}(1-s)^2 + \frac{\Lambda'''(f \otimes \psi_D, 1)}{3!}(1-s)^3 + \ldots \\ &= \Lambda(f \otimes \psi_D, 1) - \frac{\Lambda'(f \otimes \psi_D, 1)}{1!}(s-1) + \frac{\Lambda''(f \otimes \psi_D, 1)}{2!}(s-1)^2 - \frac{\Lambda'''(f \otimes \psi_D, 1)}{3!}(s-1)^3 + \ldots\end{aligned} \tag{11.6}$$

and

$$\Lambda(f \otimes \psi_D, s) = \Lambda(f \otimes \psi_D, 1) + \frac{\Lambda'(f \otimes \psi_D, 1)}{1!}(s-1) + \frac{\Lambda''(f \otimes \psi_D, 1)}{2!}(s-1)^2 + \frac{\Lambda'''(f \otimes \psi_D, 1)}{3!}(s-1)^3 + \ldots \tag{11.7}$$

From 11.5, 11.6 and 11.7, it follows that

$$\sum_{n=1}^{\infty} \frac{\Lambda^{2n-1}(f \otimes \psi_D, 1)}{(2n-1)!}(s-1)^{2n-1} = 0.$$

Hence, we have

$$\Lambda(f \otimes \psi_D, s) = \Lambda(f \otimes \psi_D, 1) + \sum_{n=1}^{\infty} \frac{\Lambda^{2n}(f \otimes \psi_D, 1)}{(2n)!}(s-1)^{2n}. \tag{11.8}$$

It is now easy to see from 11.8 that $\epsilon_f = +1$ implies that the order of vanishing of $L(f \otimes \psi_D, s)$ is even at $s = 1$.

We now look deeper into what criteria do we need for D so that the sign of the functional equation of $L(f \otimes \psi_D, s)$ is positive. Let $D \in K^\star$ be totally negative and co-prime to our level $\mathfrak{n}$ of $f(z)$ which in our case is the prime ideal $\mathfrak{p} = (5a - 2)$. Now from formula 11.2

$$\begin{aligned} \epsilon_f &= \prod_{v \in \Sigma_{\mathfrak{p}}} w_v^f \psi_D(v) \\ &= w_{\mathfrak{p}}^f w_\infty^f \psi_D(\mathfrak{p}) \psi_D(\infty) \end{aligned}$$

From the L-functions and modular forms database (LMFDB) [LMF22], we find $w_{\mathfrak{p}}^f = 1$ for $\mathfrak{p} = (5a - 2)$.
Also, D is totally negative implies $\operatorname{sgn}(D) = -1$ or $\psi_D(\infty) = -1$.
Finally, note that weight of $f(z)$ is 2 which implies $k = 1$ and hence $w_\infty^f = -1$.

It then follows

$$\begin{aligned} \epsilon_f &= 1.(-1).\left(\frac{D}{\mathfrak{p}}\right)_2^1.(-1) \\ &= \left(\frac{D}{\mathfrak{p}}\right)_2. \end{aligned}$$

So, in order to make the sign of the functional equation of $L(f \otimes \psi_D, s)$ positive, D must satisfy $\left(\frac{D}{\mathfrak{p}}\right)_2 = +1$. We can then apply the *Waldspurger's* theorem.

Suppose $g(z) = \sum_{\xi \in \mathcal{O}^+_{\mathbb{Q}(\sqrt{5})}} b_\xi q^\xi$. Then *Waldspurger's* theorem asserts that

$$L(f \otimes \psi_D, 1) = * \,.\, b_D^2$$

where $*$ is a well known constant term, see section 11.2. Now provided $\left(\frac{D}{\mathfrak{p}}\right)_2 = +1$ such that $\epsilon_f = +1$, then $b_D = 0$ implies that $L(f \otimes \psi_D, s)$ vanishes to order at least two at $s = 1$.

We now check some values of $D \in K^\star$ which satisfy the our criteria.

$\mathbf{D = 19a - 62,\ N(D) = 5 \times 261}$

Using `SAGE`, we first check that D is totally negative.

INPUT

```
K.<a>=NumberField(x^2 - x - 1)
D = 19 * a - 62
sign(D);
```

OUTPUT

```
-1
```

Next, we check that the quadratic residue of $\left(\frac{19a-62}{(5a-2)}\right)_2 = +1$.

INPUT

```
K.<a>=NumberField(x^2 - x - 1)
D = 19 * a - 62
N=K.ideal(5 * a - 2)
D.residue_symbol(N,2)
```

OUTPUT

```
1
```

We repeat the same steps for:

$\mathbf{D = -5a - 46,\ N(D) = 11 \times 211}$

$\mathbf{D = 16a - 67,\quad N(D) = 29 \times 109}$

$\mathbf{D = -16a - 59,\ N(D) = 11 \times 379}$

$\mathbf{D = -11a - 63,\ N(D) = 19 \times 239}$

We find that in each case, D is totally negative and satisfies the condition $\left(\frac{D}{(5a-2)}\right)_2 = +1$.

We now want to know if the Fourier coefficients b_{19a-62} b_{-5a-46}, b_{16a-67}, $b_{-16a-59}$, $b_{-11a-63}$ of the half-integer weight Hilbert newform $g(z)$ vanish or not. We can confirm these do vanish from the data available in [ST] where the Fourier coefficients of the unique half-integer weight Hilbert newform $g(z) \in S_{\frac{3}{2}}^{\text{new}}(\widetilde{\Gamma}_{4(5a-2)})$ have been computed by *Nicolás Sirolli* and *Gonzalo Tornaría*.

We now consider the corresponding twists of elliptic curve E which are given by

$$E_{19a-62}/\mathbb{Q}(\sqrt{5})\ :\ y^2 = x^3 + (-77a - 234)x^2 + (53040a - 47880)x + (523920a - 2126960);$$
$$E_{-5a-46}/\mathbb{Q}(\sqrt{5})\ :\ y^2 = x^3 + (-229a - 250)x^2 + (63024a + 11640)x + (-2748656a - 2181616);$$
$$E_{16a-67}/\mathbb{Q}(\sqrt{5})\ :\ y^2 = x^3 + (-124a - 271)x^2 + (68568a - 45312)x + (-59312a - 2814640);$$
$$E_{-16a-59}/\mathbb{Q}(\sqrt{5})\ :\ y^2 = x^3 + (-380a - 359)x^2 + (141144a + 51456)x + (-11135536a - 7606064);$$
$$E_{-11a-63}/\mathbb{Q}(\sqrt{5})\ :\ y^2 = x^3 + (-351a - 359)x^2 + (134328a + 36168)x + (-9396208a - 6892080).$$

Here our main interest lies in the order r of vanishing of the $L(E_D, s)$ at $s = 1$ which is given by

$$\lim_{s\to 1}(s-1)^r L(E_D, s)$$

we often denote r by $\mathrm{ord}_{s=1}(L(E_D,s))$.

It follows that $\mathrm{ord}_{s=1}(L(E_D,s))$ is at least 2. Now the conjecture of *Birch* and *Swinnerton-Dyer* predicts that

$$\mathrm{ord}_{s=1}(L(E_D,s)) = \mathrm{rank}(E_D).$$

We evaluate the ranks of E_{19a-62}, E_{-5a-46}, E_{16a-67}, $E_{-16a-59}$, $E_{-11a-63}$ defined over $\mathbb{Q}(\sqrt{5})$ and show its rank is at least 2 which would then provide evidence in support of the conjecture of *Birch* and *Swinnerton-Dyer*. Using SAGE, we are able to show that these rank are exactly equal to 2.

rank($\mathbf{E_{19a-62}}/\mathbb{Q}(\sqrt{5})$)

Using SAGE, we compute the lower and upper bounds for $\mathrm{rank}(E_{19a-62}/\mathbb{Q}(\sqrt{5}))$ and points of infinite order on E_{19a-62}.

INPUT

```
K.<a>=NumberField(x^2 - x - 1)
E=EllipticCurve(K,[1,a+1,a,a,0])
G=E.quadratic_twist(19*a - 62)
G.rank_bounds()
G=gens()
```

OUTPUT

```
(2,2)
[(176*a+8 : 172*a+1768 : 1)]
```

rank($\mathbf{E_{-5a-46}}/\mathbb{Q}(\sqrt{5})$)

OUTPUT

```
(2,2)
[(212*a+32 : 732*a+20 : 1),(156/5*a+1576/5 : 21084/25*a-57012/25 : 1)]
```

rank($\mathbf{E_{16a-67}}/\mathbb{Q}(\sqrt{5})$)

OUTPUT

```
(2,2)
[(176a-12 : 604a-1420 : 1)]
```

rank($\mathbf{E_{-16a-59}}/\mathbb{Q}(\sqrt{5})$)

OUTPUT

```
(2,2)
[(296a+144 : 1292a+1448 : 1)]
```

rank($\mathbf{E_{-11a-63}}/\mathbb{Q}(\sqrt{5})$)

OUTPUT

```
(2,2)
[(151a+452 : -1507a-4090 : 1)]
```

We hence conclude that the ranks of E_{19a-62}, E_{-5a-46}, E_{16a-67}, $E_{-16a-59}$, $E_{-11a-63}$ defined over $\mathbb{Q}(\sqrt{5})$ are all equal to 2 which is in line with the BSD conjecture.

11.3.1 Lower bound for D

So far, we have found five examples of totally negative $D \in K^\star$ which satisfy the condition

$$\left(\frac{D}{(5a-2)}\right)_2 = w^f_{(5a-2)} \quad \text{and} \quad b_D = 0. \tag{11.9}$$

In each of the five cases, we saw that the rank(E_D) = 2. We now want to know how many $D \in K^\star$ satisfy condition 11.9 and have even rank(E_D) ≥ 2. We will now show that based on an assumption, it is possible to show that there are infinitely many such D's.

Definition 11.3.3. *Let $\mathbf{x} = (x_1, x_2)$ denote a 2-tuple of positive real numbers. Then*

(i) We write $|\mathbf{x}|$ for the product $x_1 x_2$. This is call the size of x.

(ii) We define

$$\mathfrak{S}(\mathbf{x}) := \{ \text{ squarefree } D \in \mathcal{O}_K \ : \ |\sigma_i(D)| \leq x_i \text{ for } i = 1,2 \text{ and } rank(E_D) \geq 2, \text{ even}\}$$

where $\sigma_i(D)$ denotes the real embeddings of D in $\mathbb{R}^2$. We denote the cardinality of $\mathfrak{S}(\mathbf{x})$ by $\mathscr{S}_{E_D}(\mathbf{x})$.

We next state the *parity* conjecture which is a consequence of a quite strong assumption involving the finiteness of the *Tate-Shafarevich* group.

Conjecture 11.3.4 (Parity conjecture)**.**

$$\textit{Sign of the functional equation of } L(E_D, s) = (-1)^{rank(E_D)}.$$

We now state a result that gives a lower bound for $\mathscr{S}_{E_D}(\mathbf{x})$. This is a theorem by *F. Gouvêa* [Gou93, Theorem 1] for elliptic curves over any number field and is a generalisation of the theorem by *F. Gouvêa* and *B. Mazur* [GM91, Theorem 2] for elliptic curves over $\mathbb{Q}$. We will state it only for the case when the number field is taken to be a totally real quadratic field. However, the original theorem as stated in [Gou93] holds true for any number field.

Theorem 11.3.5. *Let $\mathbf{x} = (x_1, x_2)$ denote a 2-tuple of positive real numbers. Let E be an elliptic curve over K for which the parity conjecture 11.3.4 holds. Then, for every $\varepsilon > 0$, there exists a positive constant C such that*

$$\mathscr{S}_{E_D}(\mathbf{x}) \geq C\, |\mathbf{x}|^{\frac{1}{2}-\varepsilon}.$$

Theorem 11.3.5 implies that as the size of $\mathbf{x}$ becomes large, so will $\mathscr{S}_{E_D}(\mathbf{x})$. This means we have infinity many squarefree $D \in \mathcal{O}_K$ for which rank(E_D) ≥ 2 and even, provided the parity conjecture holds for E_D.

Remark 11.3.6. We note that $\mathscr{S}_{E_D}(\mathbf{x})$ counts the values of D and not the twists of E_D. This means we may we counting isomorphic twists more than once. However, *Goûvea* in [Gou93, pg. 111] shows that if the lower bound of $\mathscr{S}_{E_D}(\mathbf{x})$ grows large with $\mathbf{x}$, then it implies that the number of twists of E_D will also grow infinitely large.

We now make note of two points.

(1) In Theorem 11.3.5, we have assumed the parity conjecture and we are counting D's for which rank(E_D) is even. This is the equivalent to saying that the sign of the functional equation of $L(E_D, s)$ (denoted ϵ_f) must is +1. For twists E_D of elliptic curve $E/\mathbb{Q}(\sqrt{5})$ defined in 11.3 where D is totally negative, this is equivalent to saying that all D's satisfy the condition

$$\left(\frac{D}{(5a-2)}\right)_2 = +1.$$

(2) In Theorem 11.3.5, we are counting D's for which rank(E_D) ≥ 2 and even. Now if $b_D \neq 0$, then *Waldspurger's* theorem implies $L(f \otimes \psi_D, 1) \neq 0$. This means that $\text{ord}_{s=1} L(E_D, s) = 0$. In this case, we can apply theorem [PRS11, Theorem 3.3] which essentially asserts BSD holds for certain elliptic curves over a totally real number field K. This theorem is due to *Gross-Zagier* [Gro86] and *Kolyvagin* [Kol89] for elliptic curves over $\mathbb{Q}$ and is generalised to elliptic curves over totally real number fields K under certain conditions by *Zhang* [Zha01]. These conditions are satisfied if E/K is modular and also if the conductor of E/K is not a square, both of which are true in the case of the twists of elliptic curve over real quadratic field $\mathbb{Q}(\sqrt{5})$ defined in 11.3. More formally, we can state this theorem for $K = \mathbb{Q}(\sqrt{5})$ as:

Theorem 11.3.7. *Let E be an elliptic curve over K. If we assume* $\text{ord}_{s=1} L(E, s) \leq 1$, *then* $\text{rank}(E) = \text{ord}_{s=1} L(E, s)$.

This implies rank(E_D) = 0. This is a contradiction to the given condition. Hence, $b_D = 0$.

Thus, we can restate Theorem 11.3.5 for the elliptic curve over real quadratic field $\mathbb{Q}(\sqrt{5})$ defined in 11.3 as:

Theorem 11.3.8. *Let E/K be the elliptic curve over real quadratic field $\mathbb{Q}(\sqrt{5})$ defined in 11.3 for which the parity conjecture 11.3.4 holds. Then there are infinitely many twists E_D/K of E/K by square-free $D \in \mathcal{O}_K$ which satisfy each of the following conditions:*

(1) $\left(\frac{D}{(5a-2)}\right)_2 = w^f_{(5a-2)}$.
In this case, it's same as saying that the sign of the functional equation of $L(E_D, s)$ (denoted ϵ_f) is +1.

(2) *Rank*$(E_D) \geq 2$.

(3) $b_D = 0$.

Remark 11.3.9. Let E be an elliptic curve over any real quadratic field K of square-free conductor $\mathfrak{n} \subset \mathcal{O}_K$ of odd norm. Let E_D/K be a twist of E/K by a square-free $D \in \mathcal{O}_K$. Theorem 11.3.5 by *Gouvêa* holds only for those D for which the $\epsilon_f = +1$ while those square-free D may not satisfy the stronger condition analogous to (1) in Theorem 11.3.8 for every prime ideal $\mathfrak{p} \mid \mathfrak{n}$. Thus, Theorem 11.3.8 can hold true for any elliptic curve over any real quadratic field for which the parity conjecture holds as long as the conductor $\mathfrak{n} = \mathfrak{p}$. This is because $\left(\frac{D}{\mathfrak{p}}\right)_2 = w^f_{\mathfrak{p}}$ for the unique $\mathfrak{p}$ is equivalent to saying that the sign of the functional equation of $L(E_D, s)$ (denoted ϵ_f) is +1.

11.3.2 Work in progress

The final task now remains is to apply our main Theorem 10.2.5 to our example over $\mathbb{Q}(\sqrt{5})$ and the lift the congruence to half-integer weight Hilbert modular forms using Theorem 10.3.3 which shows that some Shafarevich-Tate group of the twisted Hilbert modular form in the higher weight has order divisible by 31 in the same way as discussed in the Introduction of this thesis for elliptic curves over $\mathbb{Q}$. This work is currently in progress and cannot be included in the thesis given the time constraints. This requires more work than just applying the theorem, like defining motives over real quadratic fields. Moreover, there are many ways the work in this thesis can be improved. For instance, assumption 3 about numerators of generalised Bernoulli numbers could potentially be proved and that would remove one of the assumptions in our main result. Nevertheless, I believe that the work in this thesis has achieved its goal in theory and also laid very clear direction for future research and its applications.

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