The Application of the Principles of Symmetry to the Synthesis of Multi-coloured Counterchange Patterns

By

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The candidate confirms that the work submitted is his own and that the appropriate credit has been given where reference has been made to the work of others.

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ABSTRACT

Attention is focused on the theoretical principles governing the underlying geometry of motifs, border patterns and all-over patterns. The systematic classification and construction of two-dimensional periodic patterns and tilings is introduced, with particular reference to two-colour and higher colour counterchange possibilities. An identification is made of the geometrical restraints encountered when introducing systematic interchange of colour. A wide ranging series of original patterns and tilings is constructed and fully illustrated; these designs have been printed in fabric form and are presented in the accompanying exhibition.

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1 INTRODUCTION

In general, textile designers and others concerned with two-dimensional design have been aware of the importance of geometry in the construction of repeating patterns. However, the bulk of literature on the subject has not been sufficiently accessible to practitioners, due primarily to the barrier imposed by unfamiliar symbols and distant terminology. It seems that the first serious attempt to remedy this situation was made by H.J. Woods, a physicist working in the Textile Department of the University of Leeds. In the mid-1930s Woods published a four-part paper which attempted to de-mystify the mathematical rules pertaining to the geometrical structure of patterns and other designs in two dimensions. According to Hann and Thomson [1], Woods' primary objective was to encourage an awareness, among textile designers, of the benefits to be gained from the application of the principles of geometrical symmetry to the construction of regular repeating patterns [2]. These theoretical principles had been developed by crystallographers in their attempt to understand certain three-dimensional phenomena.

In the non-mathematical context, the most influential study of pattern to be published in Europe during the nineteenth century was probably Owen Jones' 'The Grammar of Ornament' [3], which dealt with a wide range of subject matter from a number of periods and design styles. Subsequent to its publication in French and German, 'The Grammar of Ornament', as pointed out by Durant [4], acted as a stimulus for similar publications, and compendia illustrating patterns from various sources have continued to be published up to the present day. A range of such studies has been listed by Hann and Thomson [5] and a comprehensive review has been provided by Durant [6].

There have been a few occasions in the design literature where an identification of the geometrical principles governing pattern construction has been evident. Meyer, for example, in 1894, outlined his intentions when he stated that his handbook was:

> "....based on a system which is synthetic rather than analytic and intended more to construct and develop..... than to dissect and deduce."[7]

It is worth commenting that Meyer [8] grouped designs according to their spatial characteristics into "ribbon-like bands", "enclosed spaces", or "unlimited flat patterns", corresponding to border patterns, motifs and all-over patterns respectively. In addition, Meyer recognised that the foundation of every form of all-over pattern was a,

> "...certain division, a subsidiary construction or a network." [9]

He thus anticipated the use of the term "nets" (used for example by Woods [10]) to refer to the grids, or lattices underlying all-over pattern structures, a phenomenon explained later in this work. An awareness of the underlying geometrical principles fundamental to the construction of all-over patterns is also evident from other non-mathematical sources. Stephenson and Suddards [11], for example, in their appraisal of the geometry of Jacquard woven patterns, illustrated patterns with constructions based on rectangular, rhombic, hexagonal and square lattices. Likewise, Day [12], in 1903, placed much emphasis on the geometrical basis of all design and illustrated the construction of all-over patterns on square, parallelogram, rhombic and hexagonal type lattices. In 1910, Christie [13] gave numerous examples of how all-over patterns could be developed by the practitioner.

During the early twentieth century another perspective of pattern analysis and classification was evolving: the consideration of patterns by reference to their symmetries, a tradition which, as mentioned above, had its origins in the scientific study of crystals. Over the past few decades, classification systems which were developed from this same source have been used by anthropologists and design historians to analyse patterns from different cultural settings and historical periods. A review of relevant literature, dealing with both the evolution of the basic mathematical thinking on the subject as well as the application of the principles of geometrical symmetry to pattern analysis in different cultural settings, has been made by Hann and Thomson [14].

Largely absent from the literature have been attempts to present the relevant geometrical principles in a way that could prove helpful to designers in the construction of patterns. With this consideration in mind the objectives of this thesis are as follows:

(i) to review the principles of geometrical symmetry, placing an emphasis where appropriate on the construction of pattern;

(ii) to take-up the challenge proposed by Woods, and to realise the benefits to be gained from the application of the principles of geometrical symmetry to the construction of original repeating patterns (presented throughout this thesis and displayed in printed fabric form in the accompanying exhibition).

2. SYMMETRY IN PATTERN: BACKGROUND AND TERMINOLOGY

2.1 Introduction

Various scholars (such as Shubnikov and Koptsik [15], Coxeter [16], Jeger [17], Guggenheimer [18], Yale [19], Gans [20], Ewald [21], Dodge [22], Schattschneider [23], Hargittai [24] and Martin [25]) have investigated the different ways of systematically repeating a basic discrete design element (or motif) in pattern form in one direction (i.e. along a border to produce a border pattern) or in two directions (i.e. throughout the plane to produce an all-over pattern). Systematic repetition creates the whole design by using one or more of the four distinct types of geometrical actions by which a part of a design is repeated regularly without change in the shape or size of each individual part. The objective of this chapter is to further examine the geometrical operations by which motifs, border patterns and all-over patterns may be constructed.

2.2 Symmetry of the Plane: The Four Basic Symmetry Operations

The four basic geometrical operations or actions mentioned above are as follows:

(i) Translation, or repetition at regular intervals, of a motif or figure in a straight line without change of orientation.

(ii) Rotations, by which a motif or figure is rotated about a fixed point so that it undergoes repetition at regular angular intervals.

(iii) Reflection, by which a motif or figure is reflected across a straight line (or reflection axis) producing a mirror image characteristic of so-called "bilateral symmetry".

(iv) Glide-reflection, by which a motif or figure is repeated using a combination of translation and reflection.

These four geometrical actions are called symmetry operations; synonymous terms include symmetries (Grunbaum and Shephard [26]), congruence transformations (Campbell [27]), or isometries (Schattschneider [28]). Relevant schematic illustrations of these four basic geometric operations are provided in Figure 2.1. Further explanation is provided below.

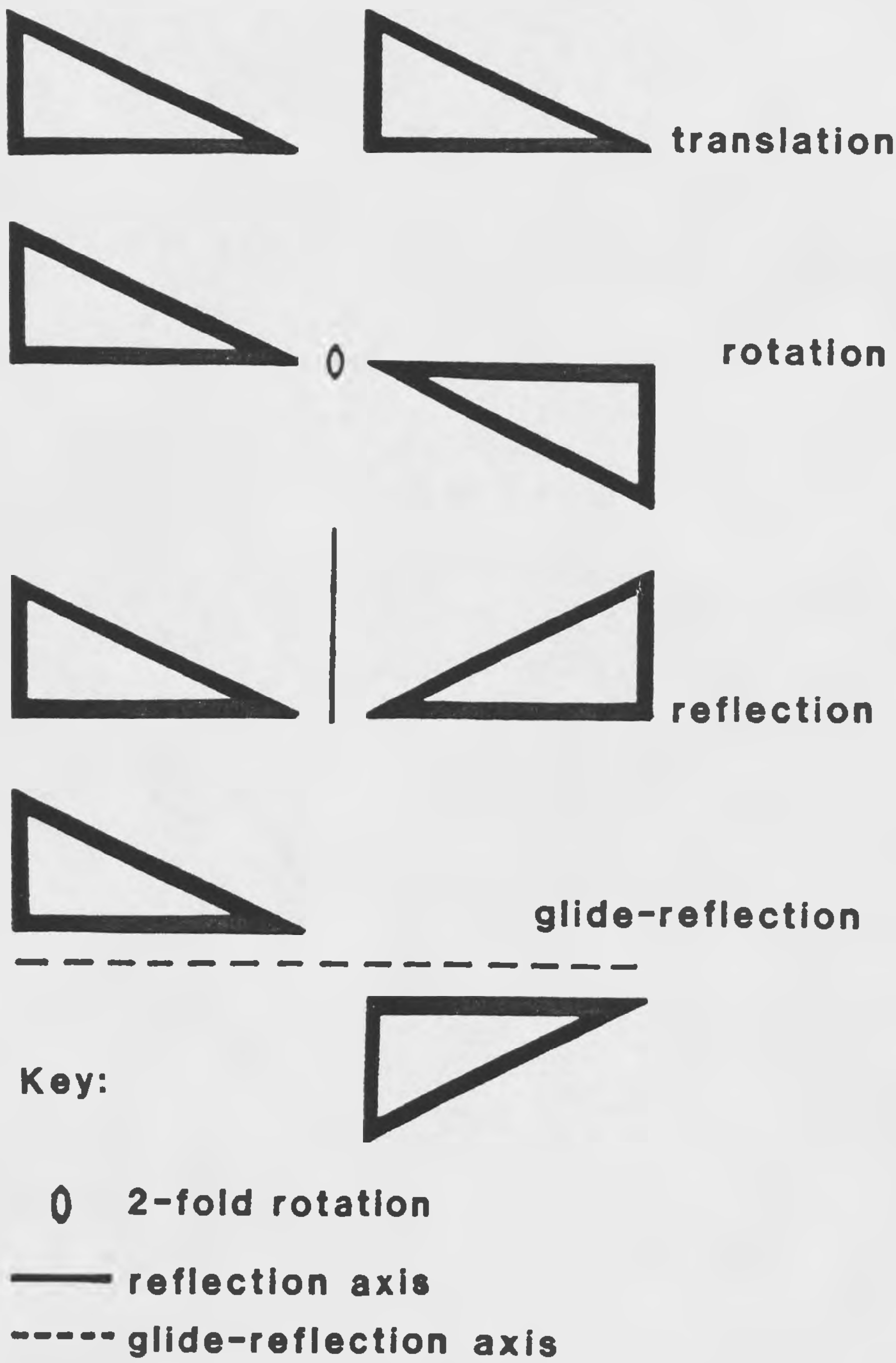
2.2.1 Translation

As stated by Schattschneider:

> " A translation of points in the plane shifts all points the same distance in the same direction" [28]

Figure 2.2a shows a series of equal asymmetrical figures. A vector (represented by an arrow) can be introduced to denote the direction of the shift. The length of the vector indicates the distance the points are shifted. If the whole row of triangular figures is moved through a distance T (the minimum translation) along the straight line L, without changing their mutual position, so that each triangle coincides with its neighbour, the whole set of triangles will assume a new position differing in no other way from the

Figure 2.1 The four basic symmetry operations (schematic illustrations).



Source: Hann, M.A. and Thomson, G.M. 'The Geometry of Regular Repeating Patterns', Textile Progress Series, vol.22, no.1, the Textile Institute, Manchester, 1992, p.5.

original. The straight line (L) is termed a "translation axis". Since displacement by a distance T does not introduce any changes, the action may be repeated as many times as desired. The displacement of figures may take place in the direction denoted by T or alternatively in the reverse direction denoted by -T with the same result (as shown in Figure 2.2b). Since it is not the actual translation axis L but its orientation in space which is of importance for the generation of a pattern, any straight line parallel to L can be taken as the translation axis. Whilst a border pattern admits translation horizontally, an all-over pattern exhibits translation not only horizontally but also vertically and diagonally (as shown in Figure 2.3).

2.2.2 Rotation

Rotation occurs through a fixed point called the centre of rotation. Repetition therefore occurs at regular angular intervals. As stated by Schattschneider:

> " A rotation of points in the plane moves points by turning the plane about a fixed point (called a centre of rotation)." [28]

A design is said to have n-fold rotational symmetry about a fixed point, when a figure (or element) in the plane is repeated by successive rotations through an angle of 360 degrees/n about a fixed point, and integral multiples of that angle; at each stage of rotation the figure (or element) will coincide with itself. The fixed point is the centre of n-fold rotation and n is an integer greater than or equal to one, which corresponds to the order of rotation.

Figure 2.2(a) A series of equal asymmetrical figures. (b) A periodic border pattern which admits translation generated by T or -T.

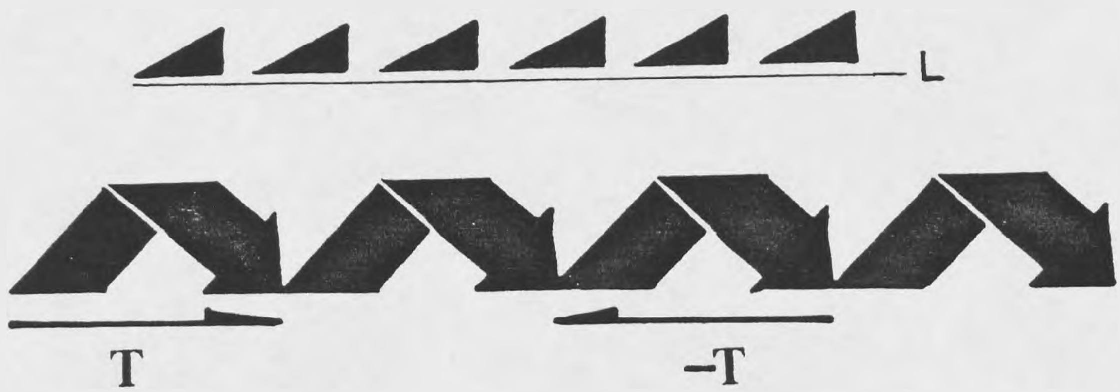
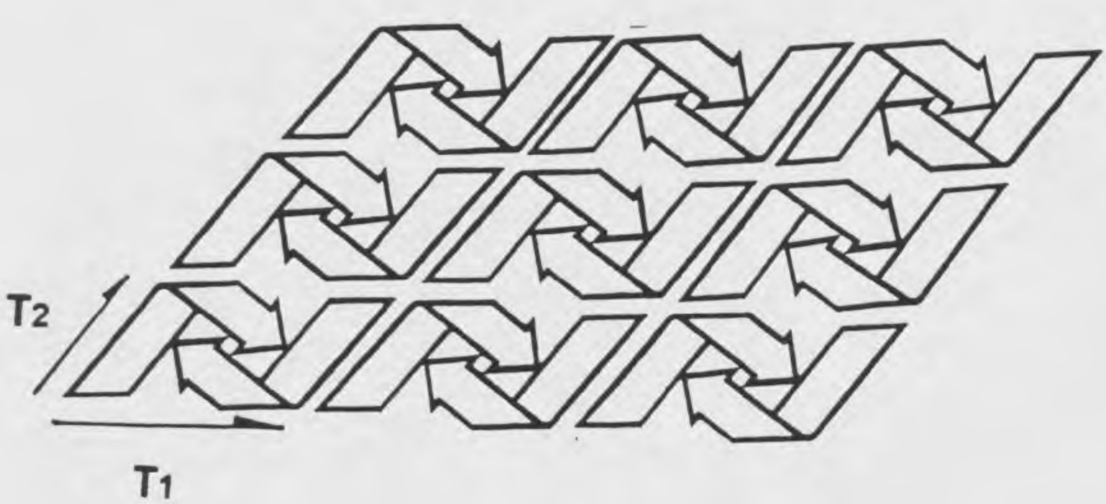


Figure 2.3 Independent translations T_1 and T_2, successively applied to a motif to generate a periodic all-over pattern.



After n successive rotations of(360/ n) degrees the figure (or element) is returned to its original position. An illustrative example of rotational transformation is provided by Figure 2.4a. In this case the figure is rotated in the plane by a given angle about an axis perpendicular to the plane; the intersection of this axis with the plane is called the centre of rotation. If a figure is repeated by successive rotation, through say 60 degrees, a design may be generated which exhibits 6-fold rotational symmetry (see, for example Figure 2.4b). Geometrically, any design in the plane having a regular circle-wise repetition is symmetric under rotation, but only by a certain minimum angle and multiples of it. The minimum rotational angle will be equal to 360/ n) degrees. It can thus be seen that:

n = 1 for rotations of 360 degrees;

n = 2 for rotations of 180 degrees (i.e. two-fold rotation);

n = 3 for rotations of 120 degrees (i.e. three-fold rotation);

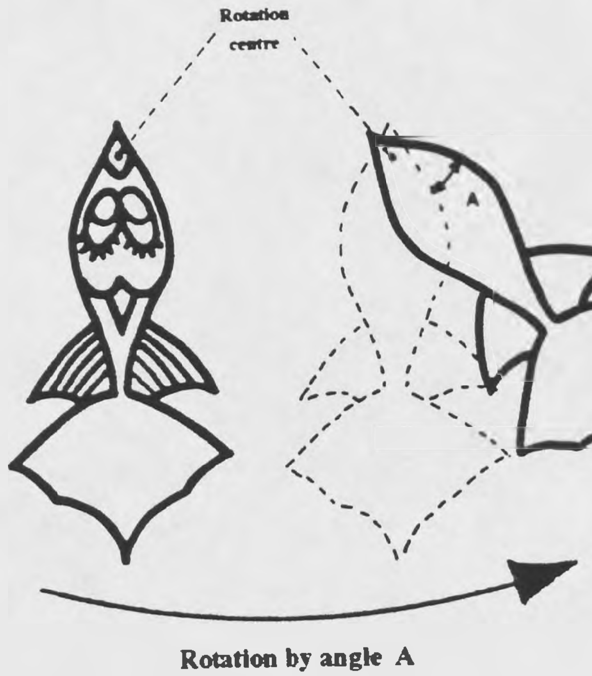
n = 4 for rotations of 90 degrees (i.e. four-fold rotation);

n = 6 for rotations of 60 degrees (i.e. six-fold rotation).

Whilst five-fold rotation (i.e. rotation through 72 degrees) may be present in individual figures or motifs, this rotational order is not possible in all-over patterns (which may only exhibit one, two, three, four and six-fold rotation). This phenomenon has become known as the "crystallographic restriction" (see for example Weyl [29], Jaswon [30], Coxeter [31] and Stevens [32]).

Figure 2.4(a) An illustration of central (or point) symmetry. (b) An example of a motif which admits transformation by rotation and exhibits six-fold rotational symmetry.

(a)



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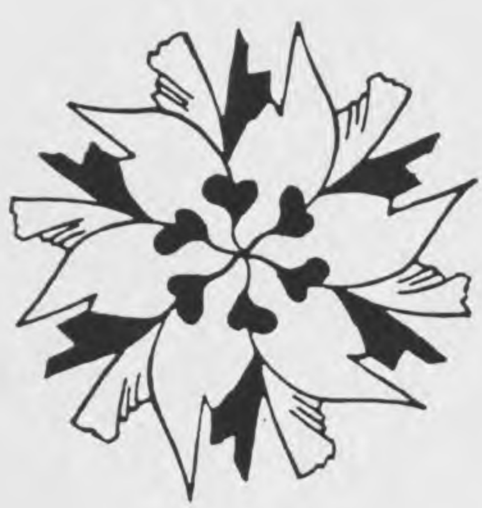


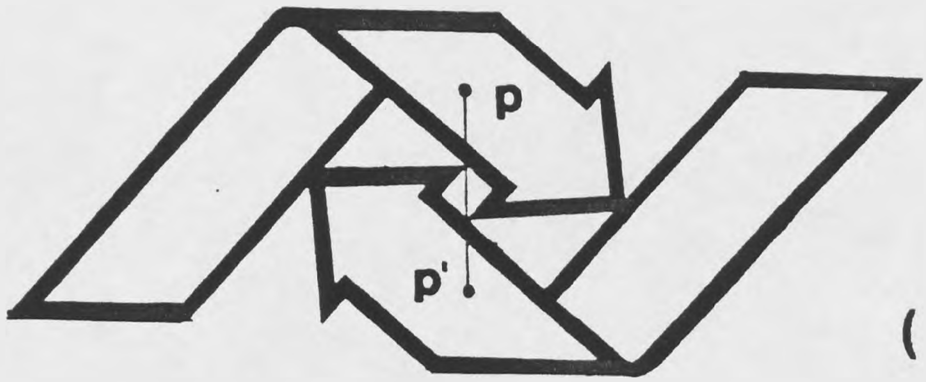
With a rotation of 360 degrees (n=1), each point in the plane is returned to its original position; this is known as "full rotational symmetry". As stated by Schattschneider:

> "A rotation of 360 degrees (n=1) sends each point in the plane to its original position. This isometry has the same effect as leaving each point fixed and is called the identity isometry" [28].

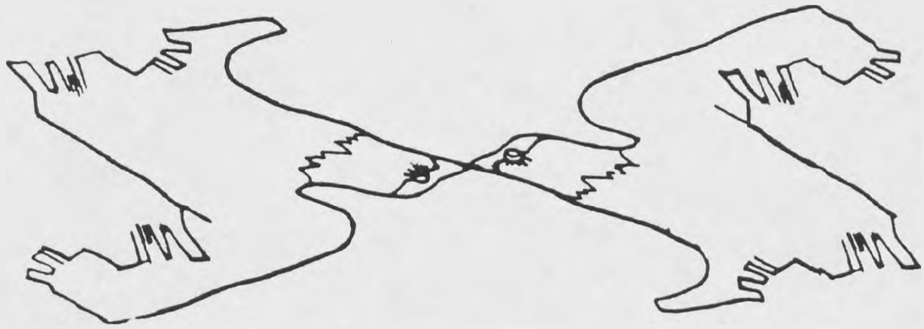
A rotation of 180 degrees (n=2) is often referred to as a "half-turn" or "central symmetry". This order of rotation is illustrated in Figure 2.5a and Figure 2.5b. As shown in Figure 2.5a, if a point p is chosen on the motif and two-fold rotation about a fixed point is introduced, a unique corresponding point p' on the transformed motif is obtained. Where p and p' interchange positions, when the motif is rotated through 180 degrees about a fixed point, such a design is said to have "central symmetry" or "point symmetry". With the repeated action of two half-turns, each point in the plane is returned to its original position.

A point in the plane might be simultaneously symmetrical under both rotations and reflections, if the two types of symmetry operations co-exist. Where this is the case, a centre of n-fold rotational symmetry will have n lines of reflection passing through it, mutually separated by half the minimum rotational angle. This phenomenon is illustrated in Figure 2.6. Where an all-over pattern does admit rotational symmetry, every centre of rotation will be repeated an infinite number of times in each direction of displacement at a

Figure 2.5(a) Two-fold rotation, with points p and p' interchanging position after rotation by 180 degrees. (b) An example of two-fold rotation (or central symmetry).



(a)



(b)

Figure 2.6 Design with three-fold rotational symmetry combined with three reflection axes which pass through the three-fold rotation centre.

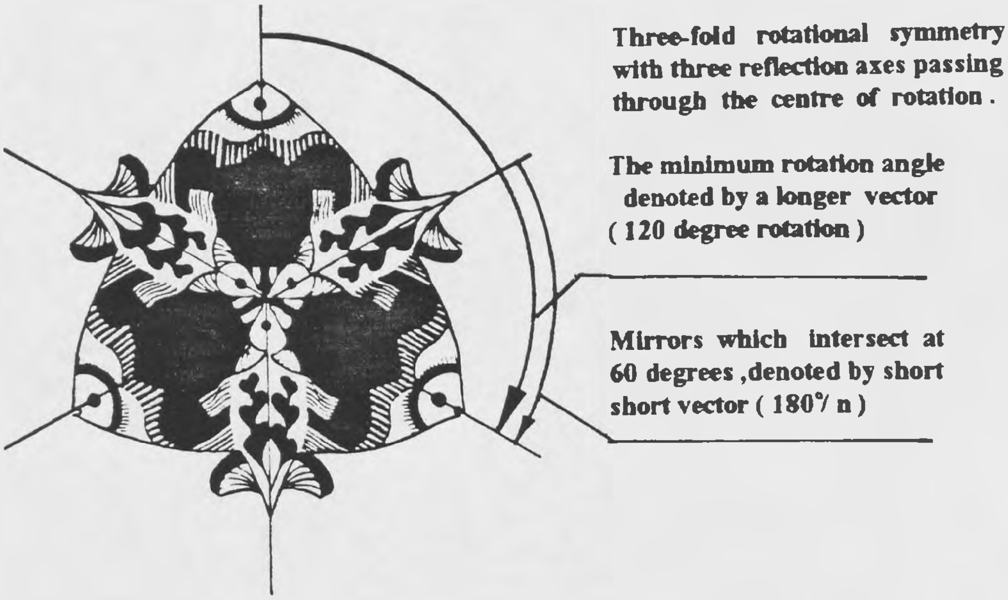
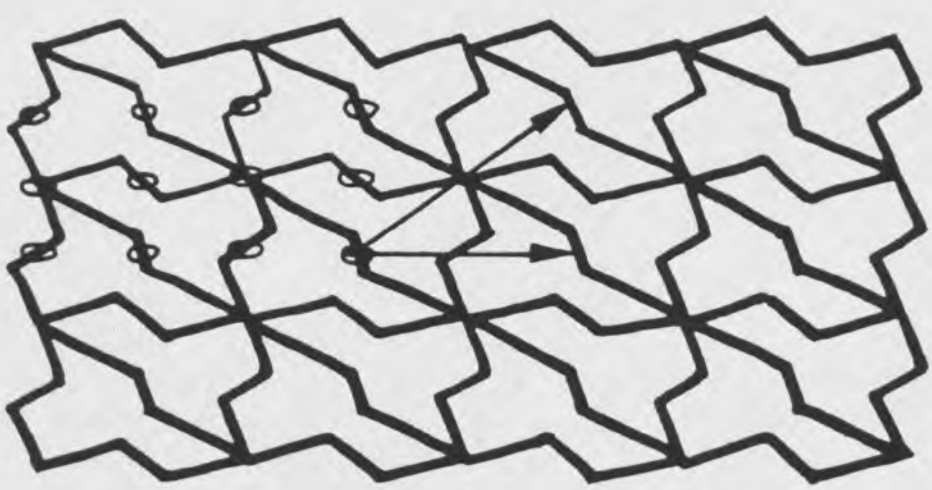


Figure 2.7 An all-over pattern admitting two-fold rotational symmetry. Translation vectors (→) indicate the direction of displacement for one of the centres of two-fold rotation.



distance equal to the relevant minimum translation distance. By way of example, Figure 2.7 illustrates an all-over pattern which admits two-fold rotational symmetry. Translation vectors indicate the direction of displacement for one of the centres of two-fold rotation.

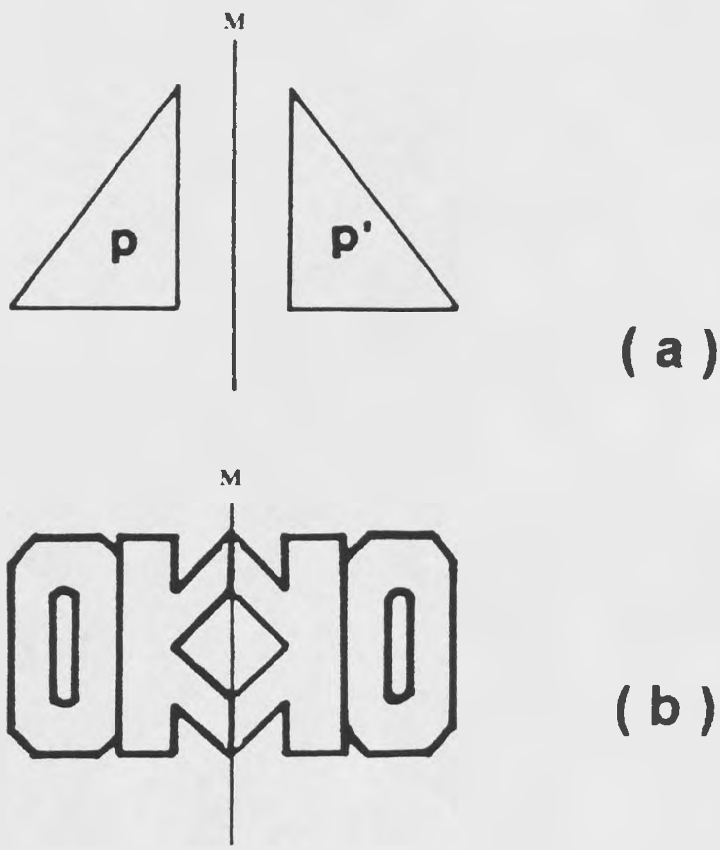
2.2.3 Reflection

Transformation through reflection can be best imagined by considering the action introduced by a mirror with its plane positioned perpendicular to the plane of the design. Schattschneider commented:

> " A reflection of points in the plane is determined by a fixed line, called a mirror line or reflection axis: every point not on the line is sent to its mirror image with respect to the line and every point on the line is left fixed" [33].

From the view point of geometry, an abstract plane has reflection symmetry through any reflection line lying in the plane. The symmetry operation which sends each point in the plane to its mirror image is known as reflection, and the mirror line M is termed the reflection axis. This concept is presented in illustrative form in Figure 2.8a and Figure 2.8b. In Figure 2.8a each point in the plane within the triangle p is reflected across the reflection axis M to an equivalent point in the transformed triangle p'. Figure 2.8b shows an intuitive example which begins with an arbitrary figure on a piece of transparent paper. On folding the paper along the reflection axis M the opposite side of the figure can be traced.

Figure 2.8(a) Point p reflected across the reflection axis M to a point p`. (b) Reflection across a central vertical axis.

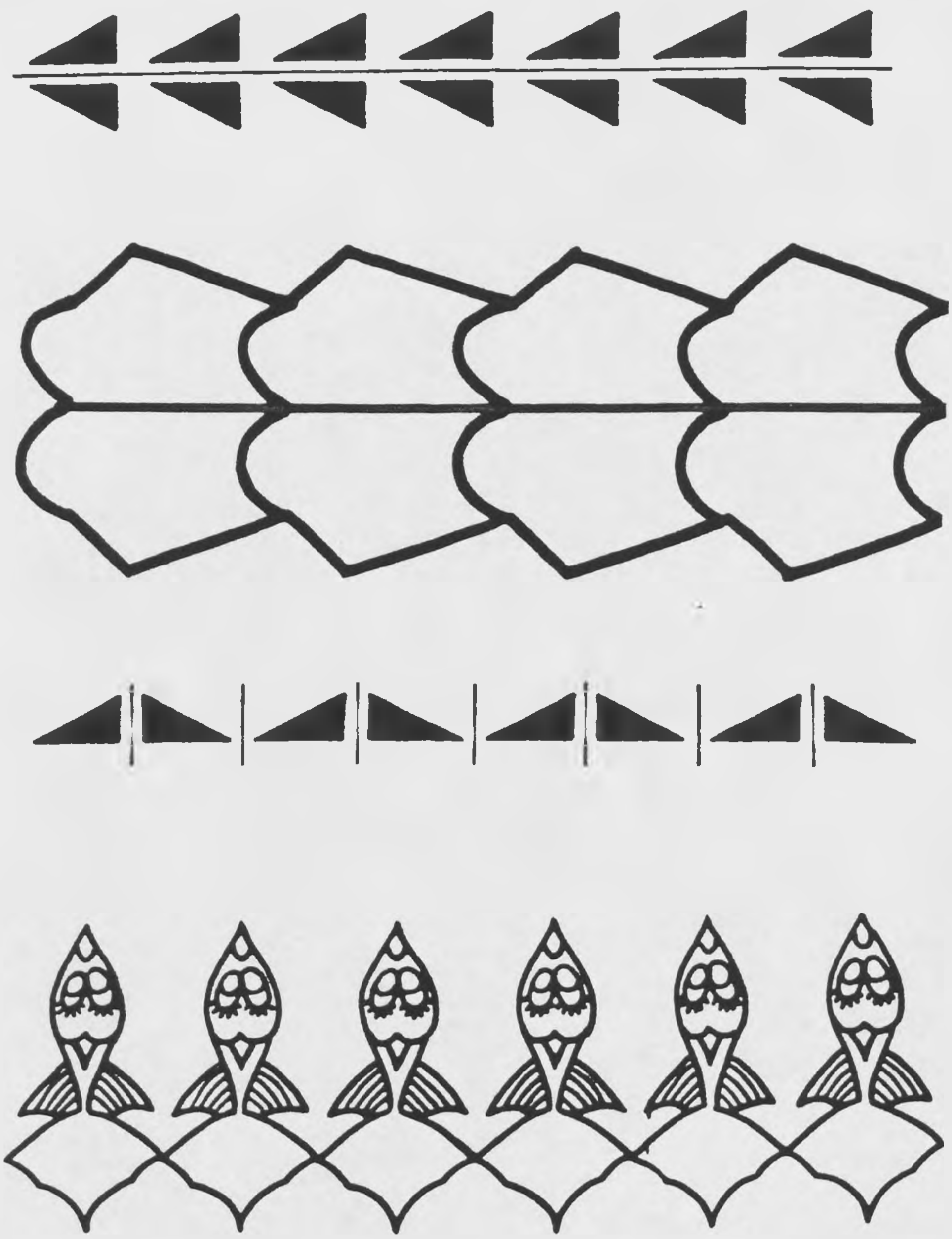


In border patterns a reflection axis can lie along a horizontal line through the middle of the border (as in Figure 2.9a and Figure 2.9b) or can occur perpendicular to the translation axis of the border (as in Figure 2.9c and Figure 2.9d). In the former case there is only one reflection axis and in the latter case there will be many reflection axes. With vertical reflection, translation will move each vertical reflection line to its equivalent (next but one) reflection line, bearing in mind that the border will have two types of alternating vertical reflection axes. This is illustrated by Figure 2.9c and Figure 2.10.

Where both horizontal and vertical reflection axes are present in a border pattern, the point of intersection between each will act as a centre of two-fold rotation. In the context of individual motifs which exhibit intersecting reflection axes, rotation will also be evident. This latter possibility is illustrated in Figure 2.11. Thus assuming n to be an integer greater than 1, n intersecting reflection axes will create n-fold rotation.

All-over patterns can also have reflectional symmetry. In this case each reflection axis is one of an infinite set of parallel lines (produced by translating the lattice unit of the design in two directions across the plane). This is illustrated in Figures 2.12a-d, where two-dimensional lattices (a and b) each admit more than one reflection axis (indicated by dashed lines). In each case rotation is generated. As stated by Washburn and Crowe:

Figure 2.9(a) & (b) Border patterns with horizontal reflection. (c) & (d) Border patterns with vertical reflection.



(a)

(b)

(c)

(d)

Figure 2.10 Successive vertical reflections of a motif in two alternating reflection axes. Two reflections have the same effect as translating the motif the distance of vector T.

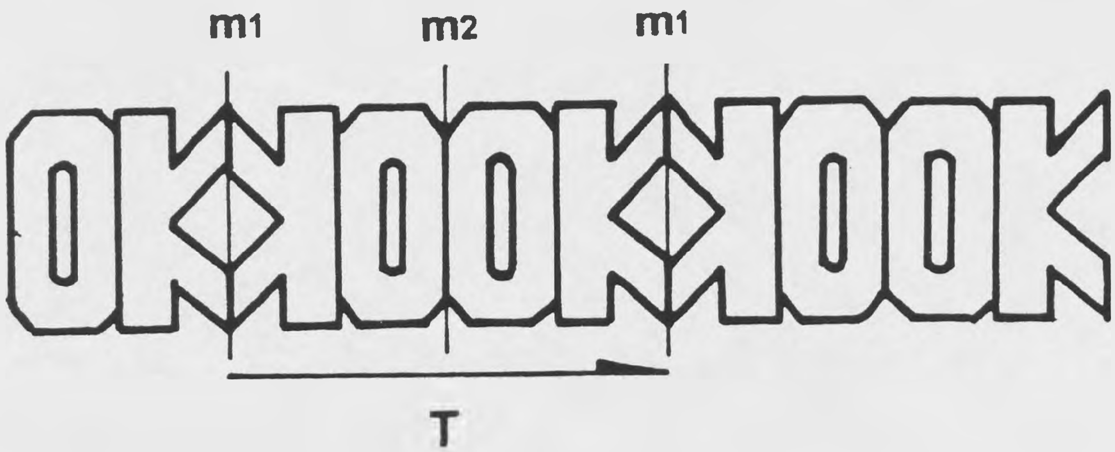
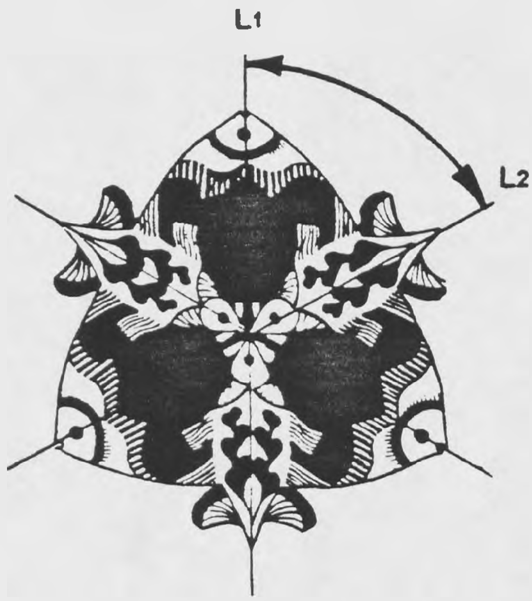


Figure 2.11(a) Successive reflections of a motif in alternating reflection axes, intersecting at 60 degrees. (b) Successive rotations of a motif through 120 degrees.



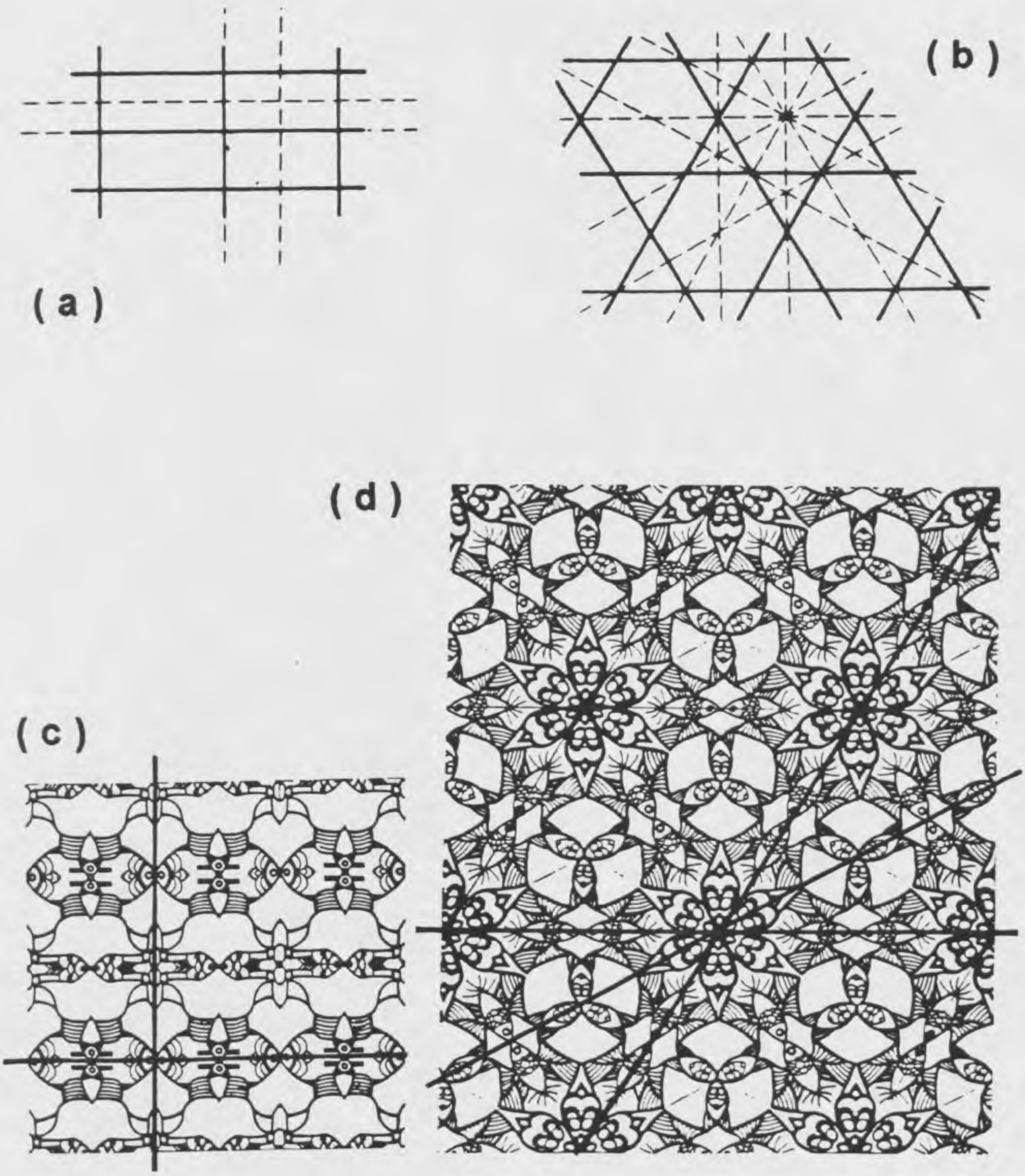
(a)

Generating region, which is bounded intersecting reflection axes denoted by L1 and L2.

(b)

Minimal repeating area (which consists of two generating regions) with minimun rotation angle denoted by the vector.

Figure 2.12(a) & (b) Two-dimensional lattices which admit reflections through centres of rotation. (c) & (d) All-over patterns with reflection axes intersecting at 90 degrees and 30 degrees respectively.



> "In two-dimensional patterns, the presence of two intersecting mirror lines implies the presence of a rotation (by an angle which is twice the angle of intersection of the two lines) about their point of intersection" [34].

Figure 2.12c shows reflection axes intersecting at right angles; two-fold rotation is thus exhibited. Figure 2.12d shows reflection axes intersecting at 30 degrees; six-fold rotation is thus exhibited.

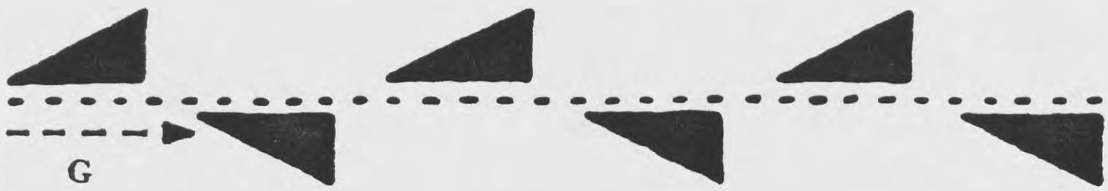
2.2.4 Glide-Reflection

Glide-reflection is best considered as a combination of translation followed by reflection or vice versa. This action is often illustrated by the continuous pattern produced by a person's footprints. Schattschneider observed:

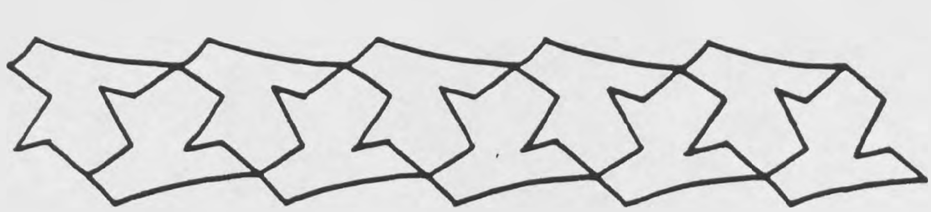
> "A glide-reflection. as its name suggests, is a transformation of points in the plane which combines a translation (glide) and a reflection. It may be obtained by a reflection followed nonstop by a translation which is parallel to the mirror line or by a translation followed by a reflection in a mirror line parallel to the translation vector" [35].

A schematic illustration of a glide-reflection and an example of a pattern which admits glide-reflection are provided by Figures 2.13a and 2.13b respectively. A single vector, called the "glide vector" or "glide-reflection axis", may be used to denote both the reflection axis and the translation vector (indicated by a dashed line and denoted by the letter G). It should be apparent that if the glide-reflection vector is repeatedly applied to a motif a continuous

Figure 2.13(a) & (b) Glide-reflection along a glide-reflection vector G.



(a)



(b)

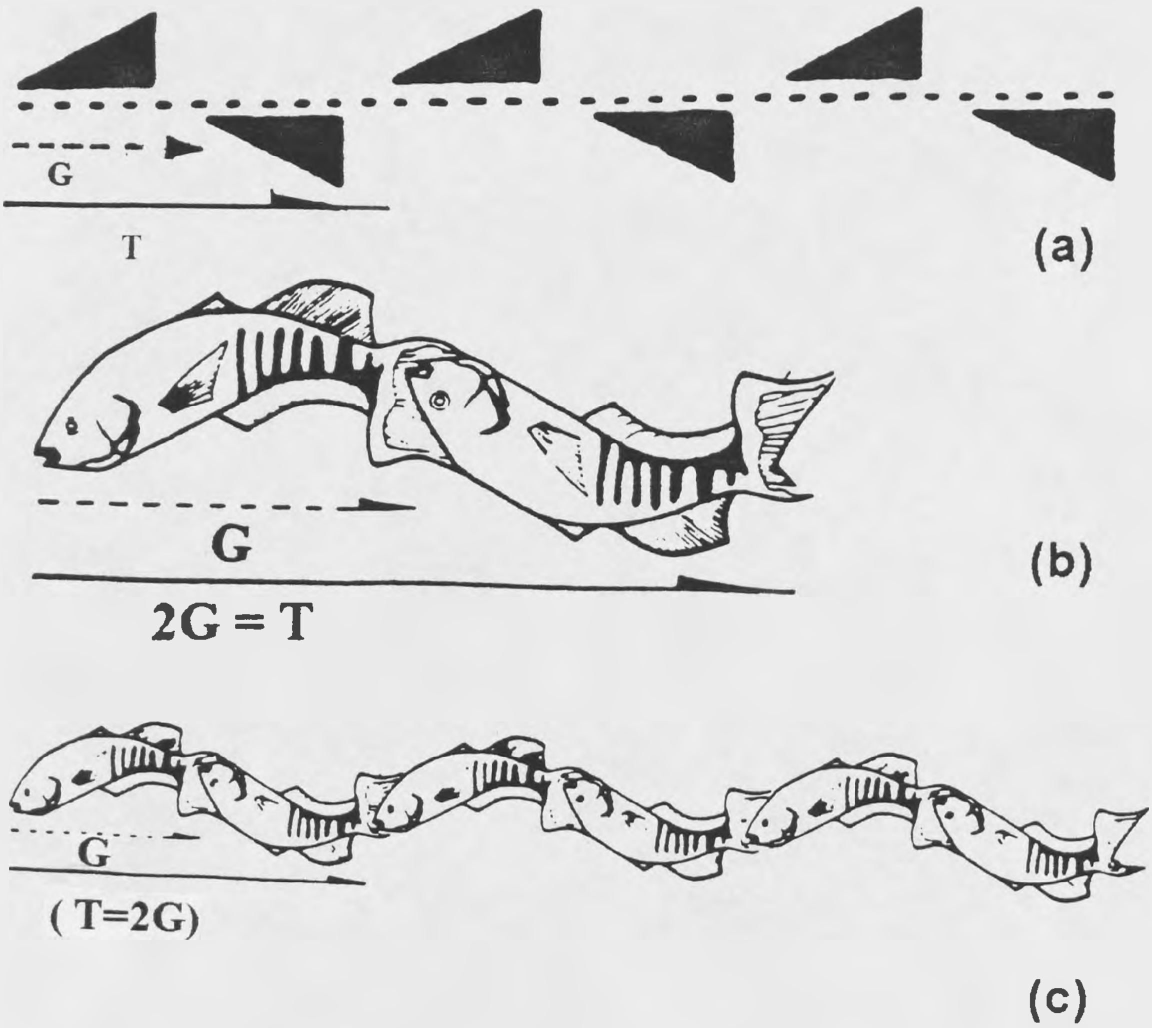
border pattern will be generated. The translation vector is parallel to the glide-reflection axis and the translation distance (T) is twice the distance of the glide (G) (see Figure 2.14).

Any figure in the plane which exhibits one or more of the four geometrical motions described above is considered to be symmetrical. Patterns are deemed to have a particular symmetry if the motion of that symmetry, when applied to the pattern as a whole, transforms each motif into another one exactly. Patterns may be classified dependent on the symmetries which they admit. The collection of symmetry operations exhibited by a pattern is referred to as its"symmetry group".

2.3 Symmetry and its Relevant Terminology: Further Considerations

Having explained briefly the four types of geometrical motions (referred to as symmetry operations) above, it is necessary to focus further attention on relevant concepts in order that a fuller understanding can be obtained of the way in which transformation geometry can impose certain restrictions in the creation of repeating patterns. Important terms, and accompanying concepts, are considered further below.

Figure 2.14(a) A schematic illustration of a periodic border pattern generated by repeated glide-reflection by vector G or by translation by distance T. (b) A unit of translation comprising a motif in its initial state and in its glide-reflected state. (c) Repeated glide-reflection (or translation after one glide-reflection) produces the border pattern.



2.3.1 Symmetry Operations and Transformations

As observed by Woods:

> "A figure is said to be symmetrical when it is possible to find two or more positions in which it is exactly super-posible on itself, and the movement necessary to bring the figure from one such equivalent position to another is said to be a symmetry operation of the figure" [36].

The resultant effect following the action of each of the four symmetry operations may be referred to as a "symmetry transformation". It is common to think of a transformation as an action that changes a system from some initial state to some other final state. What is alluded to in the geometrical context is a transformation that affects only the geometrical properties of the system. Washburn and Crowe commented that symmetry transformation is concerned with only one aspect of design: its structure [37]. From the viewpoint of geometry, any transformation that does not bring any change of shape, size or content is a symmetry transformation. Figure 2.15 illustrates a motif undergoing symmetry transformation from an initial state to what may be referred to as an "image state", the latter being indistinguishable from the former (other than its position in the plane). There is of course an inverse operation by which a figure is able to transform from the image state back to its initial state and thus its original position; the inverse of the initial transformation is also a symmetry transformation.

Figure 2.15 Symmetry transformation.

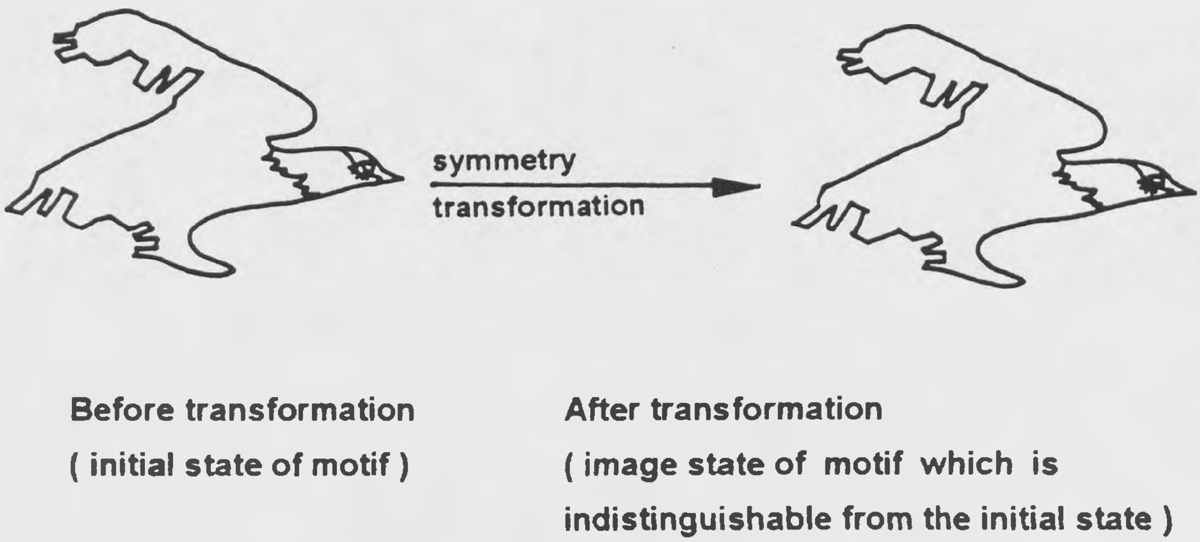
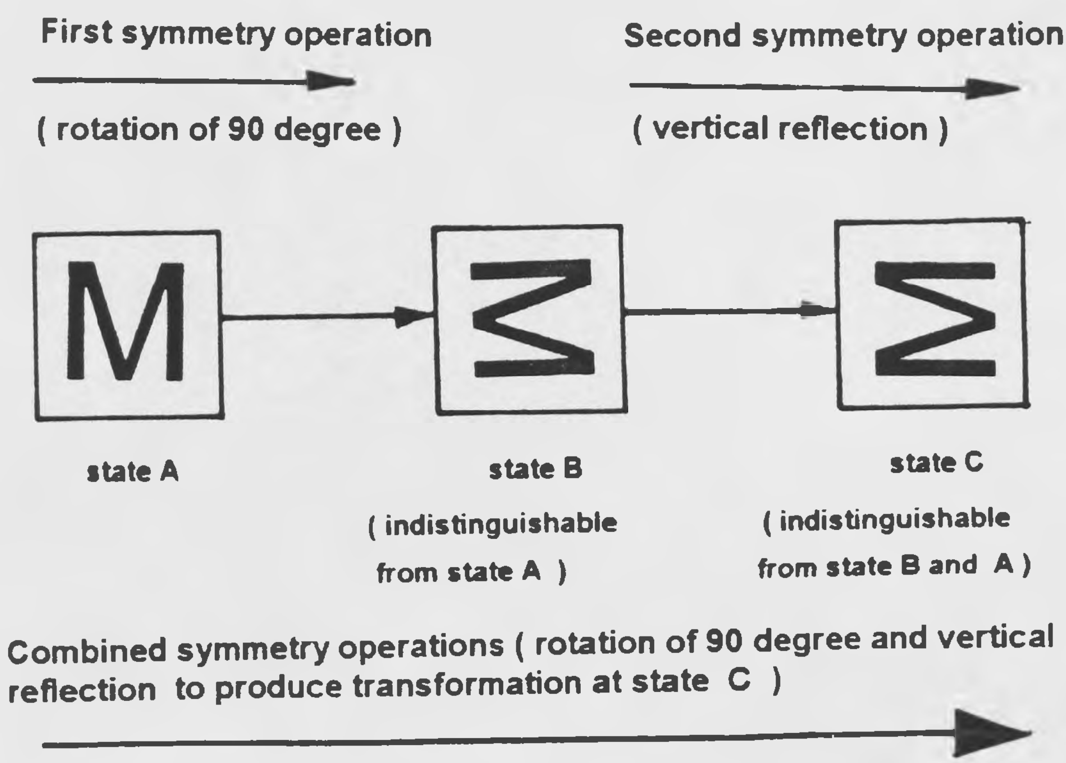


Figure 2.16 Consecutive applications of two symmetry operations.



As indicated previously, the general term "symmetry" means invariance under one or more transformations. The greater the variety of transformations that a system is invariant under, the higher the degree of symmetry. Figure 2.16 illustrates consecutive applications of two symmetry operations, where the first symmetry operation transforms the figure from its initial state A to its image state B. The second symmetry operation transforms the figure further from state B to state C. With each transformation the figure is indistinguishable from its previous state (other than a change of position in the plane). This phenomenon was recognised by Rosen, who commented:

> "When it happens that a transformation affects a system in such a way that all images are indistinguishable from their respective initial states, the system is said to be invariant or symmetric under the transformation. This transformation is then called an invariance transformation or symmetry transformation of the system" [38].

2.3.2 Symmetry Group

The collection of symmetries that a pattern possesses is called its "symmetry group". Schattschneider stated that a symmetry group is:

> "....a collection of all isometries which, when applied to the design or tiling, create an image which is superimposed exactly on the original so that, to the eye, it seems as though no transformation has taken place" [39].

From the viewpoint of geometry, the set of all symmetry transformations of a system comprises the symmetry group of the system. The term "group" is used in the mathematical sense to imply that this set has certain very definite properties. As pointed out by Stevens:

> "A symmetry group is a collection of symmetry operations that together share three characteristics: (1) each operation can be followed by a second operation to produce a third operation, that itself is a member of the group; (2) each operation can be undone by another operation, that is to say, for each operation there exists an inverse operation and (3) the position of the pattern after an operation can be the same as before the operation, that is, there exists an identical operation which leaves the figure unchanged" [40].

As indicated previously, patterns may be classified by their symmetry group; if two patterns have the same symmetry group they are thus of the same symmetry class. Finite designs (i.e. individual figures or motifs) may exhibit rotational and/or reflectional symmetry characteristics. One-dimensional designs (more commonly referred to as border patterns, strip patterns or frieze patterns) may exhibit combinations of all four symmetry operations; only seven distinct classes are possible. Two dimensional patterns (more commonly referred to as all-over patterns) admit translations in the horizontal, the vertical and the diagonal across the plane; only seventeen classes are possible.

Each of the above design types will be introduced and discussed further in subsequent chapters. Proofs for the existence of only seven classes of border patterns were provided by Grunbaum and Shephard [41]. Proofs for the existence of only seventeen classes of all-over patterns were provided by Martin [42]. A full discussion of notation, classification and pattern recognition was provided by Schattschneider [43].

2.3.3 Figures, Motifs, Patterns and Tilings

Occasionally the term "figure" is used to refer to an element of a painting, sculpture or diagram, but this usage is limited in the geometric context. More formally, a figure may be defined as a "superficial space enclosed by lines", an "image", an "illustrative drawing" or more precisely, in the context of plane patterns, as a motif or fundamental design element used in the construction of a pattern which has some form of symmetry.

The term motif is best used to refer to individual designs which allow no translations or glide-reflections and may only possess rotations about a single point and/or reflections across one or more intersecting reflection axes. As recognised by Washburn and Crowe [44] such designs are classified either as from the cyclic group, denoted by cn, with n-fold rotational symmetry about a fixed point, or from the dihedral group, denoted by dn, with n-fold rotational symmetry about a fixed point combined with reflectional symmetry about n distinct reflectional axes. Motifs may of course be used as individual components of repeating patterns.

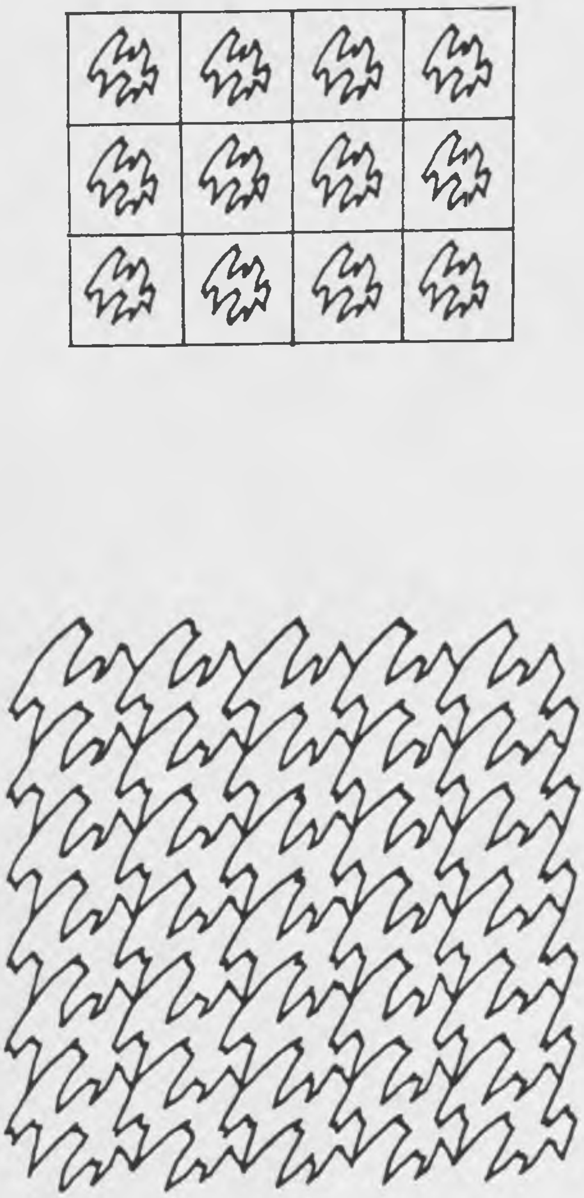
Patterns are those designs which admit translations in one or more than one direction. Washburn and Crowe described patterns as those designs which have translation symmetry and stated that,

> "... a pattern must conceptually extend to infinity; otherwise it cannot have translational symmetry" [45].

A pattern is therefore a specific type of design generated by the repetition of a motif or motifs in the plane; in addition to translation, patterns may also admit one or more of the other three symmetry operations.

A tiling may be thought of as a mosaic, a tessellated pavement or a jigsaw puzzle which when assembled will fit together without gap or overlap. A periodic plane tiling may be produced by successive translations in two directions of one or more different shaped tiles. From a geometrical viewpoint, the fundamental symmetry characteristics of periodic plane tilings and periodic all-over patterns are similar. A generating region of a tiling is the smallest element which can tile the plane without gap or overlap. In the case of a pattern, motifs may be assembled across the plane, using the same symmetry operations as a tiling, but may include a background component as a constituent part of the design. See for example Figure 2.17a and Figure 2.17b.

Figure 2.17(a) Schematic illustration of a pattern (with background).
(b) Schematic illustration of a tiling (with fitting of individual elements without gaps or overlaps).



(a)

(b)

Patterns or tilings may, as stated above, be categorised by reference to the number of directions by which they admit translations. Border patterns admit translations in one direction (and its reverse direction) whereas all-over patterns admit translations in two directions (as well as their reverse directions) so that the motif repeats horizontally, vertically and diagonally across the plane. Further classification is possible in each case by reference to the inherent symmetry characteristics. An example of a border pattern is provided by Figure 2.18 and an example of an all-over pattern by Figure 2.19.

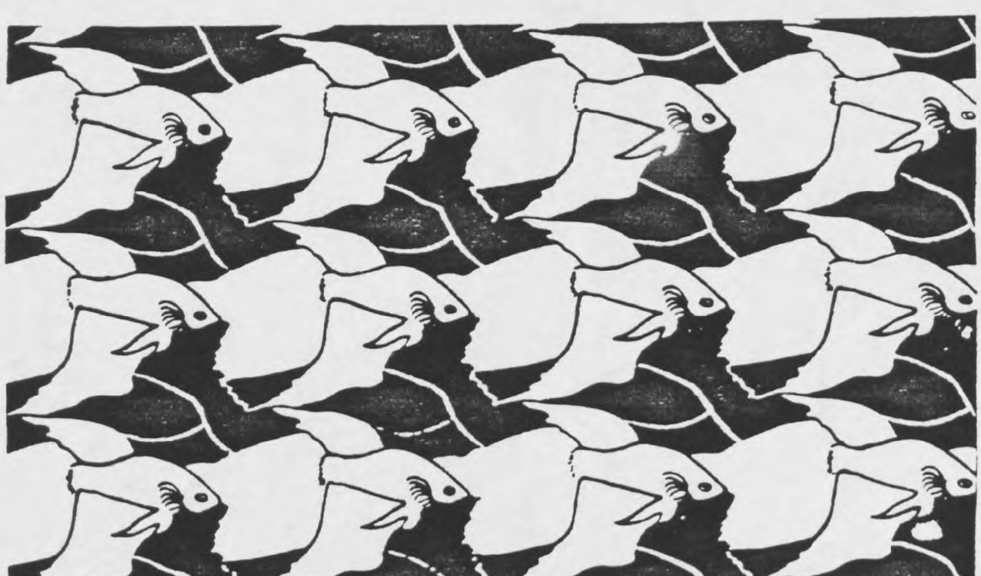
2.4 Summary

Geometric symmetry is the study of the four symmetry operations (translation, rotation, reflection and glide-reflection) and their consequential transformations. A geometric transformation is a transformation with respect to position in two-dimensional space, and does not introduce changes in size, shape or content. Motifs, when symmetrical, may exhibit only rotational and/or reflectional characteristics; border patterns, of which there are only seven distinct classes, admit translations in one direction (and its reverse direction) and possibly one or more of the other three symmetry operations. All-over patterns, of which there are only seventeen distinct classes, admit translation in two directions (and their reverse directions) and possible combinations of one or more of the other symmetry operations. The classification of all three categories of designs is based on the consideration of their symmetry characteristics only, and is not concerned directly with the abstract shape or content of units themselves. As stated by Washburn and Crowe:

> "The symmetry motions describe the specific configuration of parts for each design. Symmetry does not describe the parts, but how they are combined and arranged to make a pattern. It is concerned with only one aspect of design - its structure" [46].

Figure 2.18 One-dimensional pattern (border pattern).

Figure 2.19 Two-dimensional pattern (all-over pattern).



3. THE CLASSIFICATION AND CONSTRUCTION OF PRIMARY MOTIFS

3.1 Introduction

As indicated previously, motifs are those designs which, unlike border patterns and all-over patterns, do not admit translation or glide-reflection. They are more limited in terms of their symmetry characteristics and may exhibit rotational and/or reflectional symmetry only. The objective of this chapter is to further examine the symmetry characteristics of motifs and to give also suggestions on their construction.

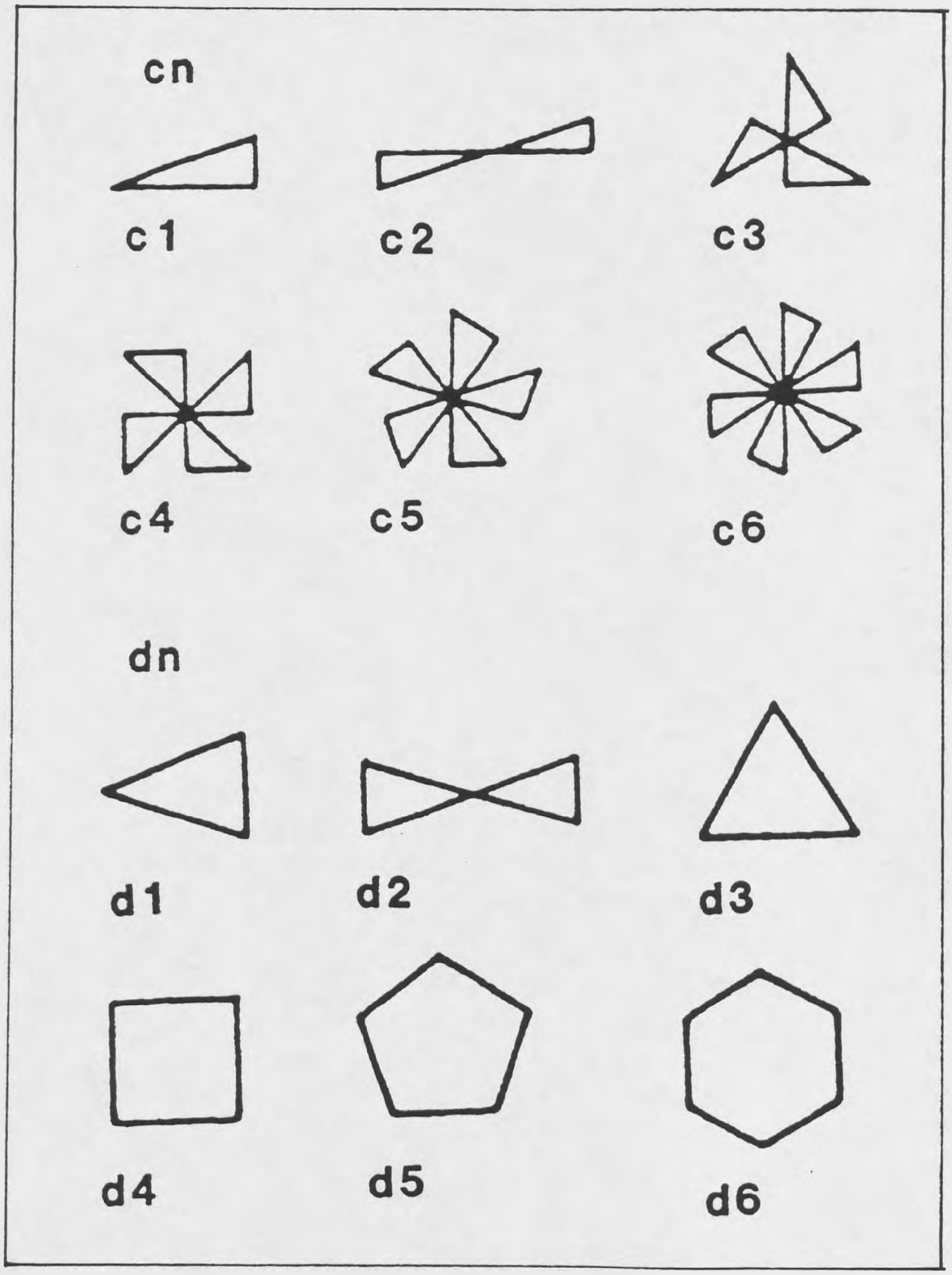
3.2 An Explanation of the Relevant Notation

In geometrical terms, motifs are referred to as point groups and are categorised into two broad classes: cn and dn, where n is some integer. Motifs from class cn exhibit n-fold rotational symmetry but no other symmetry. Dependent upon the symmetry operations present, symmetrical motifs are classified as follows:

d1, c2, d2, c3, d3, c4, d4, c5, d5, c6, d6.........dn, cn+1.

Asymmetrical motifs, as the term implies, have no symmetrical characteristics and must therefore be rotated through 360 degrees for all constituent parts to coincide with themselves; such motifs are classified as c1 motifs. Motifs from class dn exhibit n-fold rotational symmetry plus n distinct reflection axes. All class d1 motifs admit bilateral symmetry only and do not have rotational properties. Figure 3.1 illustrates schematically motifs from class cn and dn.

Figure 3.1 Schematic illustrations of class cn and class dn motifs.



A further explanation of the geometrical principles governing the structure of motifs is provided below.

3.3 Symmetries in Primary Motifs

The term symmetry applied to a motif is commonly interpreted to mean one of two characteristics:

(i) Central (or rotational) symmetry;

(ii) Axial (or reflectional) symmetry.

Where a motif has central symmetry, a fundamental design element of the motif may be rotated through a given angle about a fixed point (called a centre of rotation) and thus come into coincidence with itself. As stated by Woods:

> "We say that a figure has symmetry about a point, or central symmetry, when it is such that a rotation in its own plane about that point through a certain angle leaves the figure exactly superposible on its original position" [47].

A class cn motif is said to have n-fold rotational symmetry about a fixed point: that is to say, an element of the motif is repeated by successive rotations through an angle of (360/ n) degrees about a fixed point and integral multiples of that angle. As a result the figure is returned to its original position after n such turns about the fixed point (i.e. centre of rotation). Geometrically, any motif having a regular circlewise repetition is symmetric under rotation, but only by a certain minimum angle and multiples of it. This minimum angle of rotation will be(360/ n) degrees where n is an integer, greater than or equal to one, and corresponding with the order of rotation. The corresponding rotation centre is called a centre of n-fold rotational symmetry. A circle is an example of a system that is symmetric under rotation by any angle about its centre; such a rotational centre is called a "centre of full rotational symmetry" (Figure 3.2.a).

The other symmetry operation of relevance in the discussion of symmetry in motifs is, as mentioned above, axial (or reflectional) symmetry, where a reflection axis or series of reflection axes pass through the central point of the motif. These conditions are illustrated in Figures 3.2.b and c respectively. In the latter case the motif is simultaneously symmetrical under both rotations and reflections; it can thus be seen in this case that a centre of n-fold rotational symmetry will have n-reflection axes passing through it, mutually separated by half the minimum rotation angle.

From the viewpoint of geometry, all regular polygons exhibit rotational symmetry. The equilateral triangle, the square, the pentagram and the hexagram have respectively three-fold, four-fold, five-fold and six-fold rotational symmetry (Figures 3.3a, b, c and d). Thus, a regular n-sided polygon (called a regular n-gon) has n-fold rotational symmetry about its centre. In each case the reflection axes pass through the centre of the figure and are separated by half the minimum rotation angle. The number of reflection axes will be equal to the number of sides. As stated by Woods:

Figure 3.2(a) Full rotational symmetry. (b) Axial (or reflectional) symmetry with one reflection axis. (c) Axial (or reflectional) symmetry with a series of reflection axes.

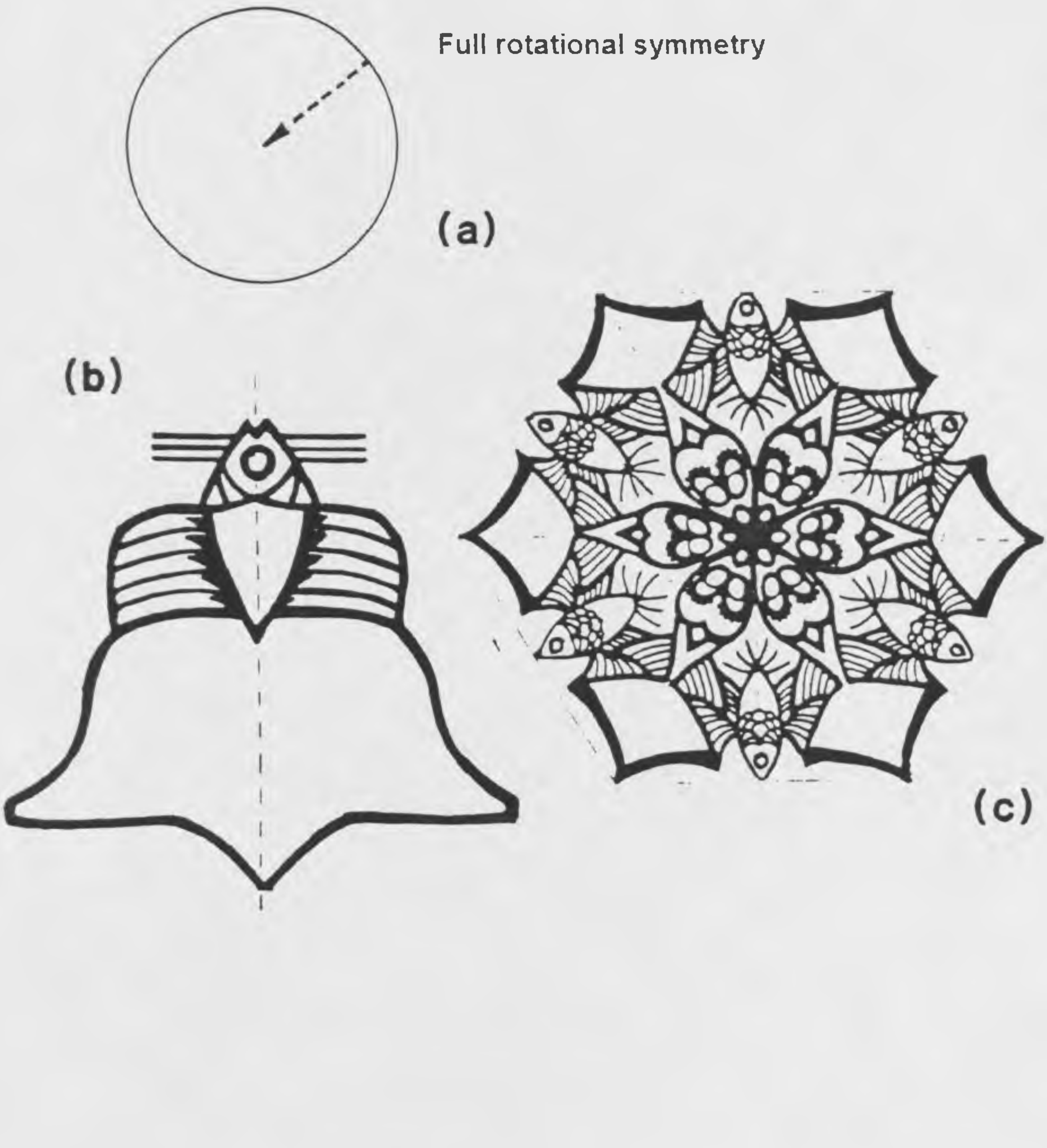
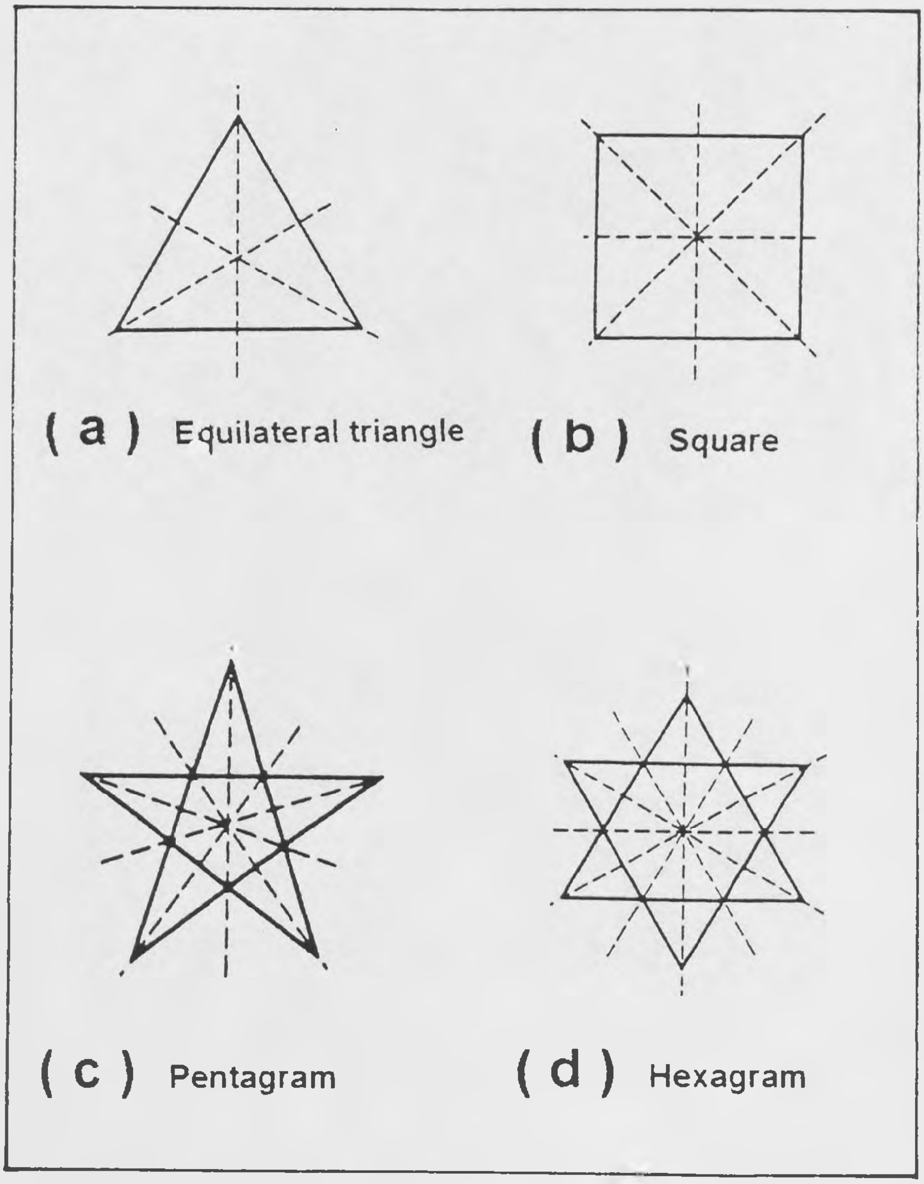


Figure 3.3(a) Equilateral triangle with three reflection axes. (b) Square with four reflection axes. (c) Pentagram with five reflection axes. (d) Hexagram with six reflection axes.



> "In general, it is easy to see that a regular polygon has the same number of symmetry axes as it has sides; if this is even there is an axis through each pair of opposite vertices and one bisecting each pair of opposite sides; if odd, an axis will pass through each vertex and bisect the opposite side perpendicularly" [48].

It should be noted that in order to refer unambiguously to the areas occupied by motifs, each motif is considered inscribed within a circle, with the centre of the circle indicated by O and where the radius can be as large as desired. Recognition charts to aid the identification of class cn and class dn motifs are provided in a subsequent section. Each motif class is further discussed and illustrated below.

3.3.1 Symmetry Operations in Class cn Motifs

As mentioned above, any motif exhibiting regular circlewise repetition is symmetrical under rotation, but only by a certain minimum angle of rotation and multiples of that minimum angle. The smallest element of such a motif which can be rotated by an angle of(360/ n) degrees is known as the "fundamental unit" of that motif. The number of times that the fundamental unit (or generating unit) comes into coincidence with itself corresponds to the older of rotation. The area of two-dimensional space occupied by the fundamental unit is known as the "fundamental region" (or generating region) of the motif. As mentioned previously, asymmetrical motifs exhibit neither rotational nor reflectional symmetry, and constituent elements of the motif can only come into coincidence following a full rotation of 360 degrees; such a motif is classed as c1 (examples are shown in Figure 3.4).

Figure 3.4 Asymmetrical motifs (c1)

Class c2 motifs possess only two-fold rotational symmetry and are comprised of two fundamental units. Through rotation of 180 degrees, each fundamental unit will come into coincidence with its neighbour. With a further 180 degree rotation each unit will be placed in its original position. As a result each point on each fundamental unit will have its equivalent point in its neighbouring unit. Typical examples of motifs exhibiting two-fold rotational symmetry are often those held within a parallelogram system, since each vertex of the figure will have a geometrically opposite equivalent vertex. Examples are shown in Figures 3.5a, b and c. In each case the generating area is half the area of the motif. In Figure 3.5c, one of the two birds has been shaded: two-colour counterchange has thus been introduced. Such motifs and patterns will be discussed fully in a later chapter.

Class c3 motifs exhibit three-fold rotational symmetry, with rotations of 120 degrees, 240 degrees and 360 degrees bringing the fundamental units of the motif into coincidence. Examples are shown in Figure 3.6.

Figure 3.7 shows examples of motifs from class c4, each of which has a minimum angle of rotation of 90 degrees. These motifs are therefore characterised by the presence of rotations through 90 degrees, 180 degrees, 270 degrees and 360 degrees.

Figure 3.5 Class c2 motifs

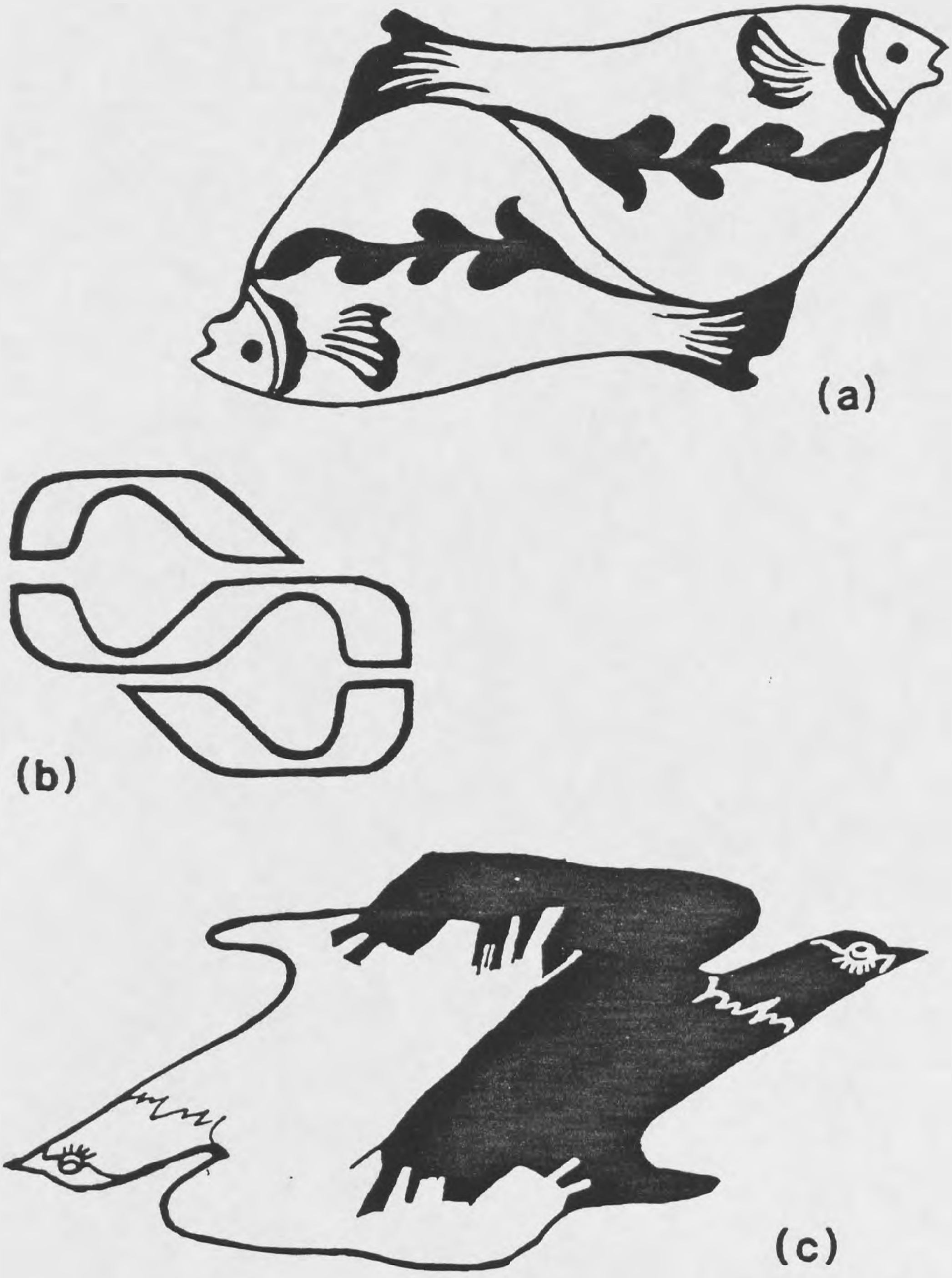


Figure 3.6 Class c3 motifs

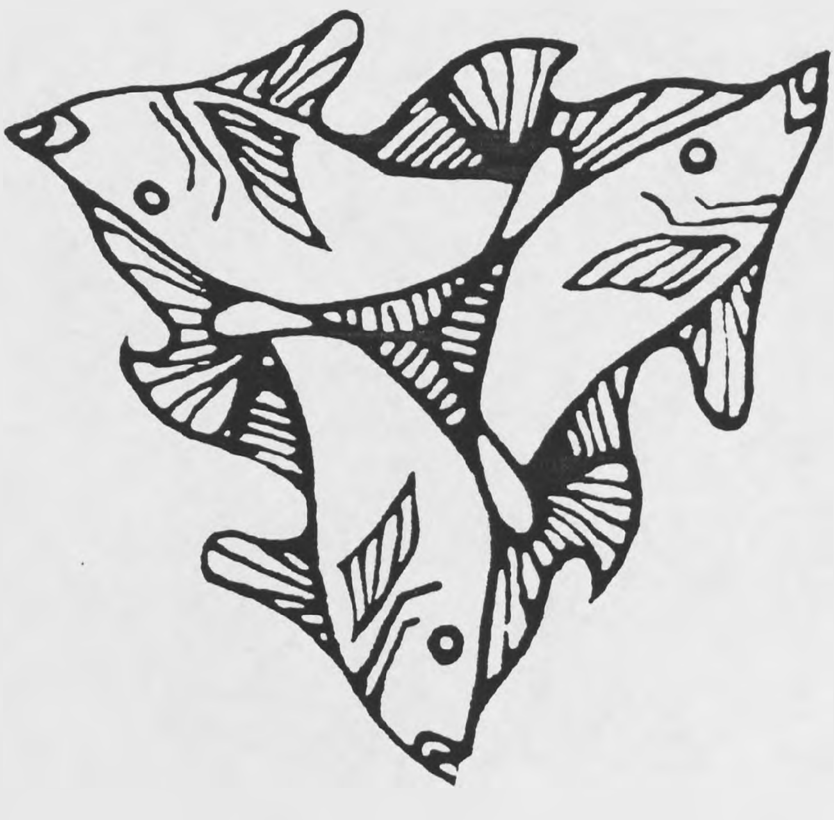


Figure 3.7 Class c4 motifs

Class c5 motifs have a minimum rotational angle of 72 degrees, and when rotated progressively by this amount fundamental elements will come into coincidence with themselves five times. Two examples are shown in Figure 3.8.

Six-fold rotation characterises c6 motifs, examples of which are shown in Figure 3.9. During a complete rotation of 360 degrees the fundamental element of the motif will coincide with itself six times (since the minimum angle of rotation is 60 degrees). The fundamental region (indicated in Figure 3.9c) is one-sixth of the motif's total area.

By way of further illustration, Figure 3.10 is presented as an aid to recognition of motifs from class c1 to class c6. Class cn motifs are infinite in number and extend to the limiting case of a circle (which, theoretically, has a rotational centre of infinite order).

Figure 3.8 Class c5 motifs

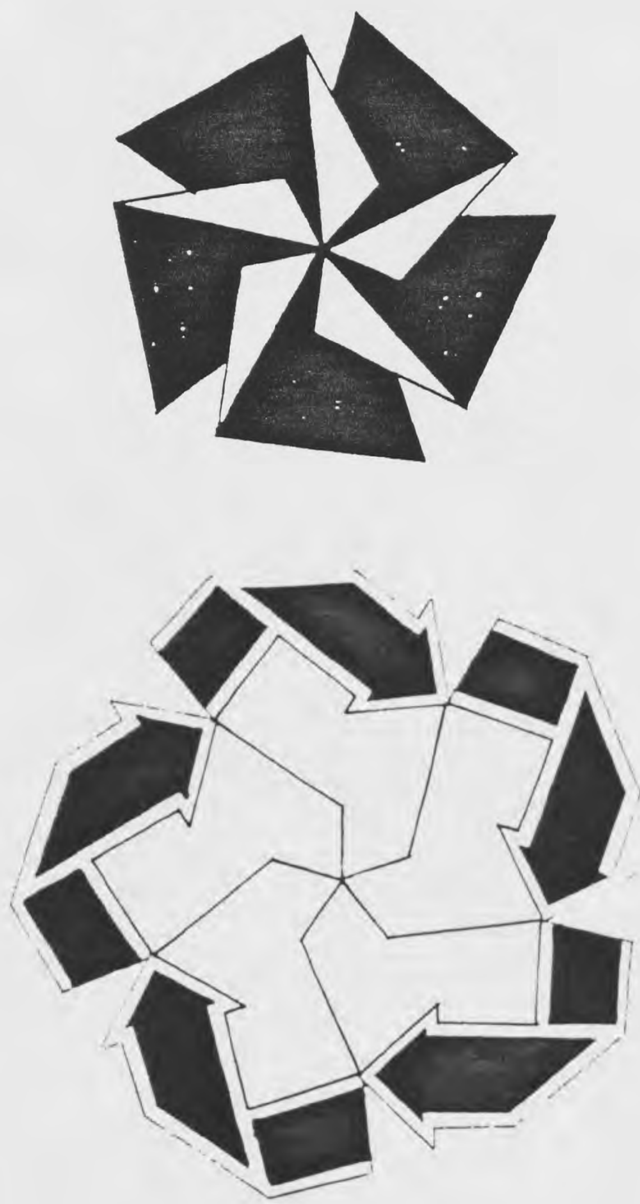
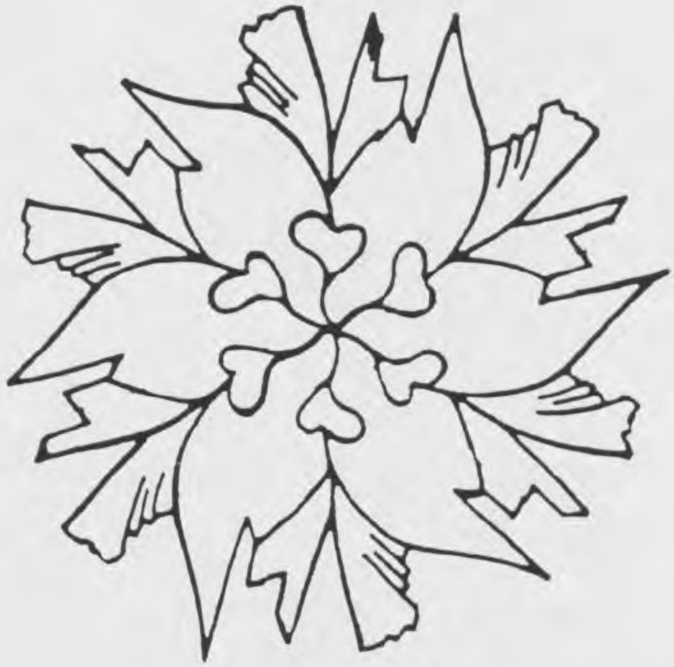
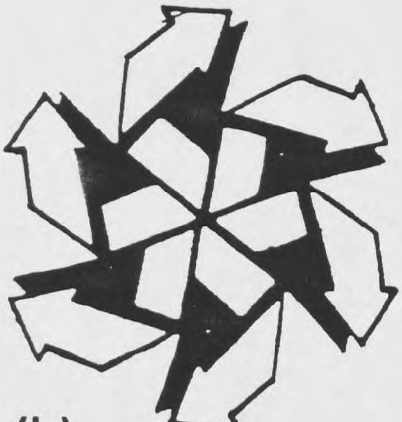


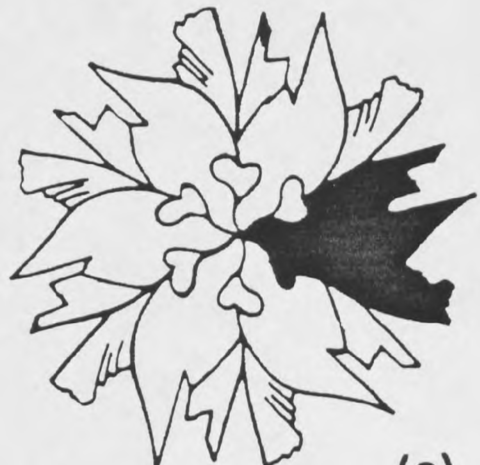
Figure 3.9 Class c6 motifs



(a)

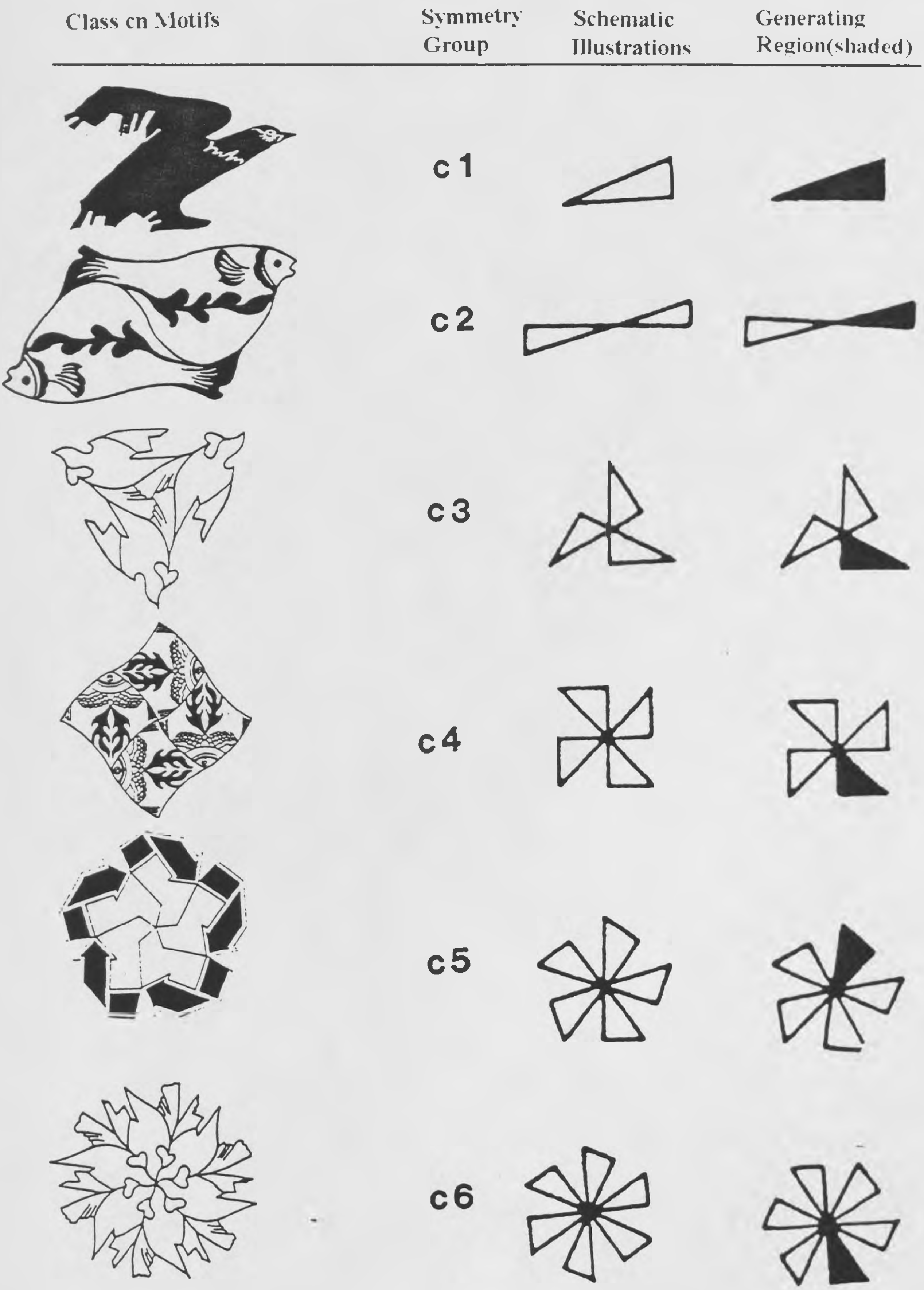


(b)



(c)

Figure 3.10 Recognition chart for class cn motifs



3.3.2 Symmetry Operations in Class dn Motifs.

Certain motifs may be simultaneously symmetrical under both rotational and reflectional symmetry operations. Where this is the case, then n-fold rotational symmetry is combined with n-reflection axes (passing through the centre of the motif) mutually separated by half the minimum angle of rotation. Such motifs are classified as class dn motifs. The first motif of the class (i.e. d1), however, has no rotational properties (other than through 360 degrees), but instead has a single reflection axis passing through its centre; this symmetrical characteristic is often referred to as "one directional bilateral symmetry". Each motif of this type has two generating units, each a mirror image of the other. Examples are shown in Figure 3.11.

Class d2 motifs, examples of which are shown in Figure 3.12, have bilateral symmetry around both their horizontal and their vertical axes. Each motif has two reflection axes, intersecting at 90 degrees. The fundamental region is one quarter of the total area. The motif may also be produced by rotating a bilaterally symmetrical unit (i.e. one half of the motif) through 180 degrees.

Class d3 motifs, examples of which are shown in Figure 3.13 are characterised by the presence of three intersecting reflection axes which produce bilaterally symmetrical units spaced at 120 degrees (Figure 3.13c). The fundamental region is, as a result, one-sixth of a circle. This class of motifs may simultaneously be produced by rotations of a bilaterally symmetrical unit through 120 degrees, 240 degrees and 360 degrees.

Figure 3.11 Class d1 motifs

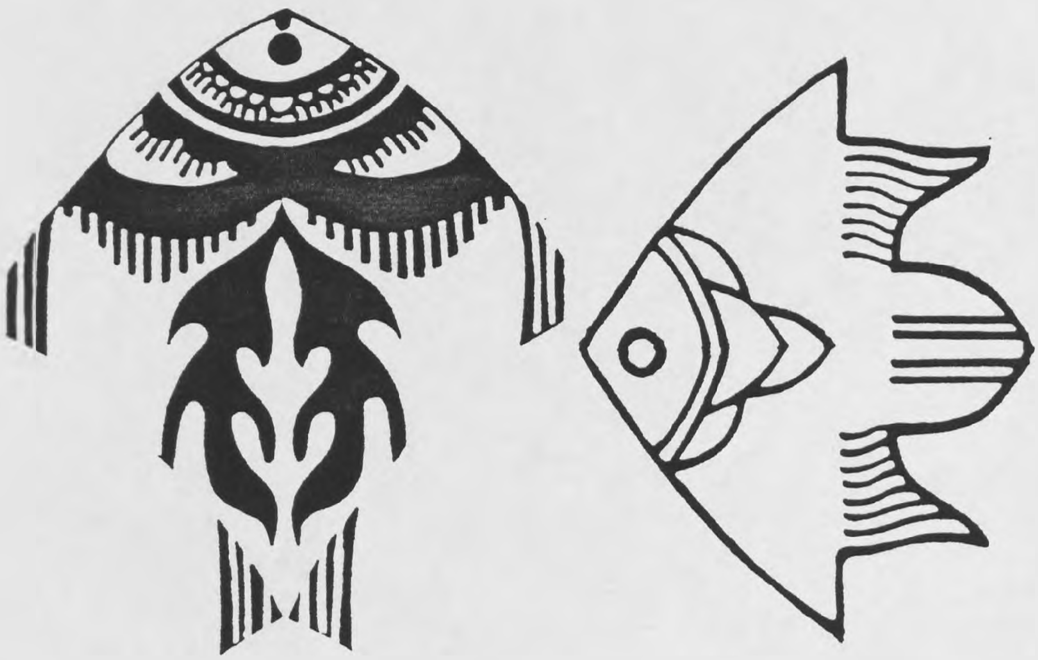


Figure 3.12 Class d2 motifs

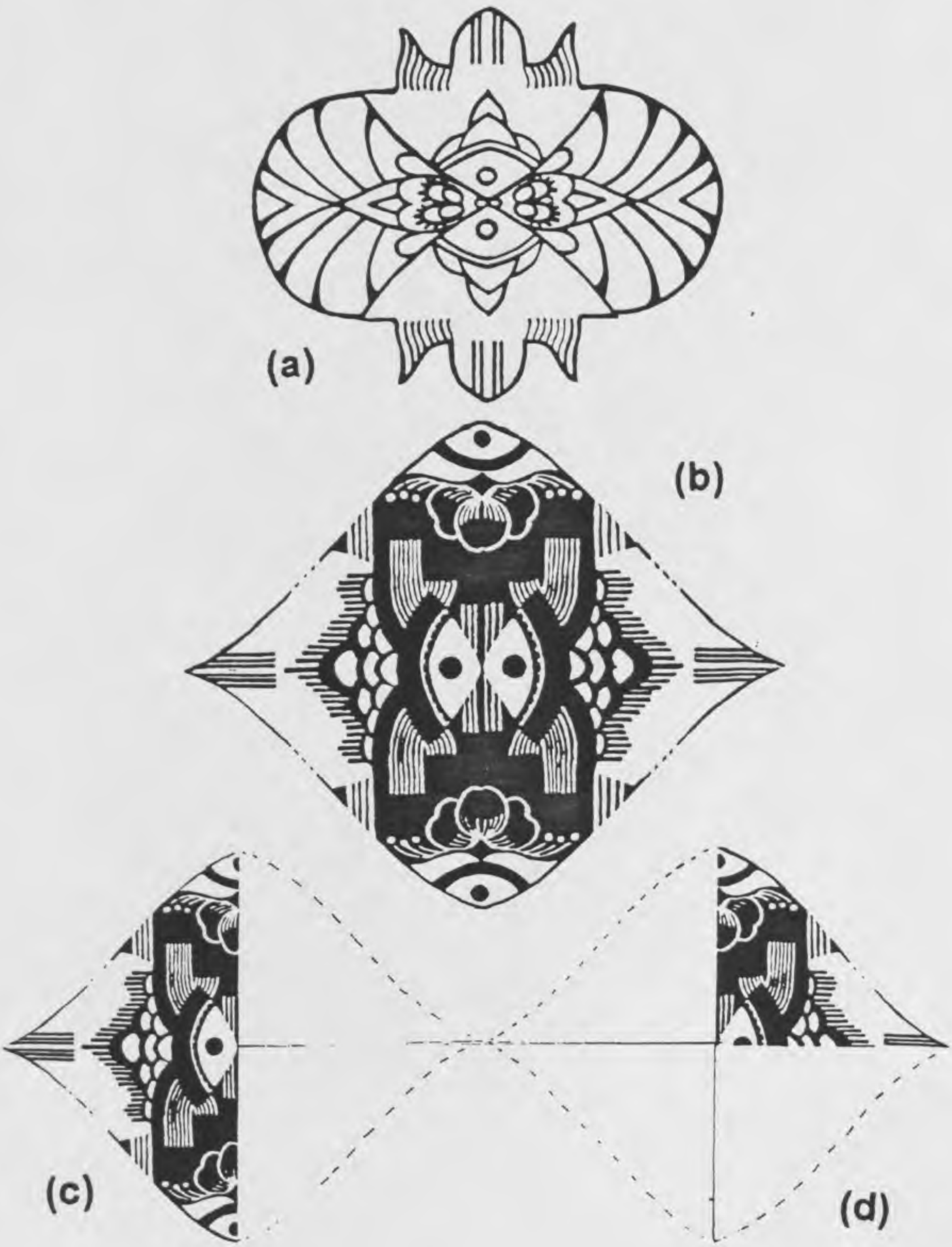
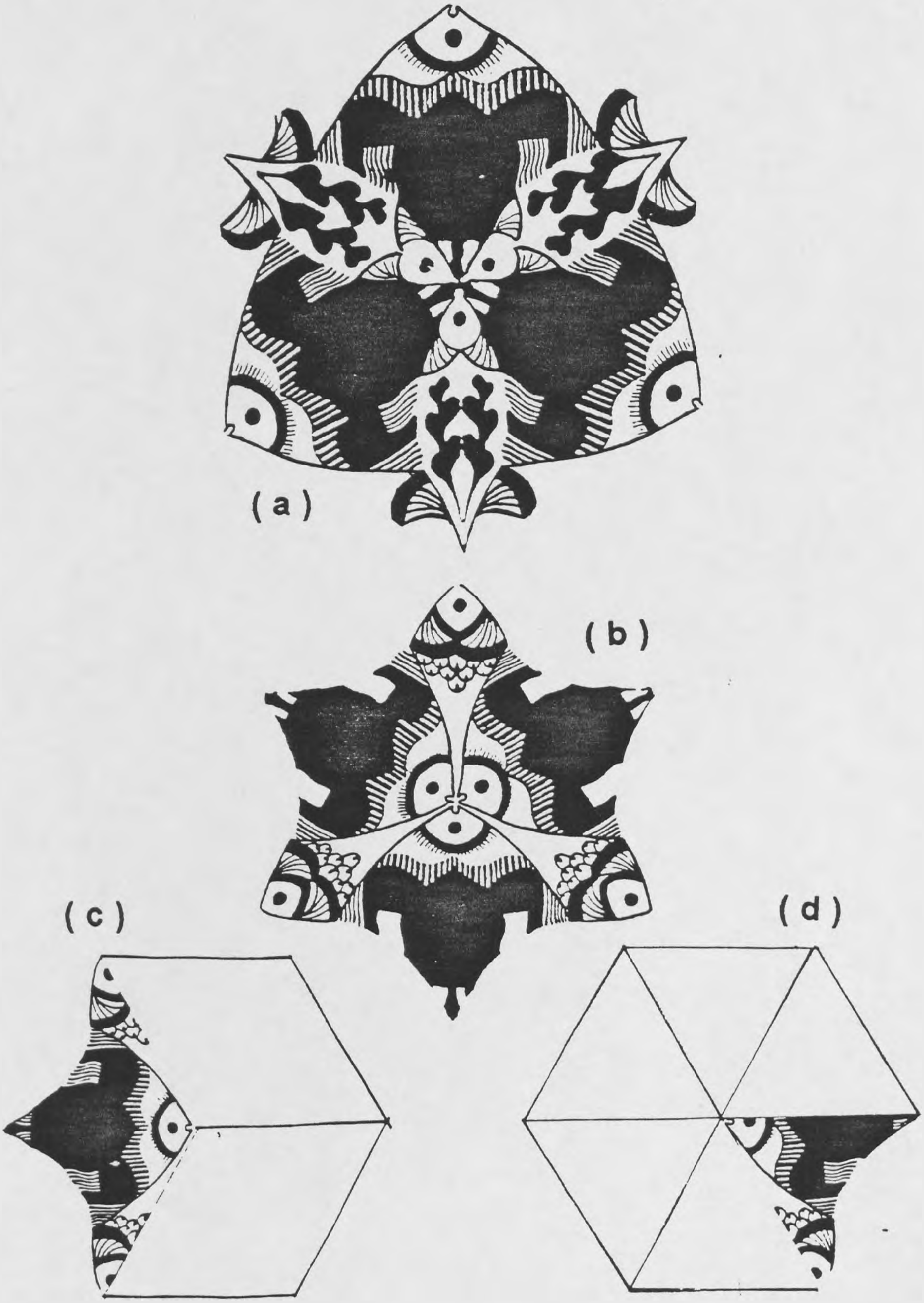


Figure 3.13 Class d3 motifs



Class d4 motifs, illustrated in Figure 3.14 are characterised by the presence of four-fold rotational symmetry together with four intersecting reflection axes (which pass through the centre of four-fold rotation and are intersected at an angle of 45 degrees). The fundamental region is one-eighth of the total motif area (see Figure 3.14c), and, when reflected, will produce a bilaterally symmetrical unit which may be rotated in 90 degree stages to produce the whole motif.

Class d5 motifs, illustrated by Figure 3.15, are characterised by the presence of five reflection axes and five-fold rotational symmetry. The fundamental region, which is one-tenth of the total motif area (see Figure 3.15c), may be reflected to produce a bilaterally symmetrical unit (Figure 3.15b) which when rotated five times produces the full motif.

Six intersecting reflection axes, combined with six-fold rotational symmetry, characterises d6 motifs. An example is shown in Figure 3.16. The fundamental region is one twelfth of the motif's area (Figure 3.16c). Motifs from this class may also be produced by rotations of a bilaterally symmetrical unit (e.g. Figure 3.15b) through 60 degrees, 120 degrees, 180 degrees, 240 degrees, 300 degrees and 360 degrees.

By way of further illustration, Figure 3.17 is provided as an aid to recognition of motifs from classes d1 to d6.

Figure 3.14(a) Class d4 motif. (b) Bilaterally symmetrical unit. (c) Fundamental unit.

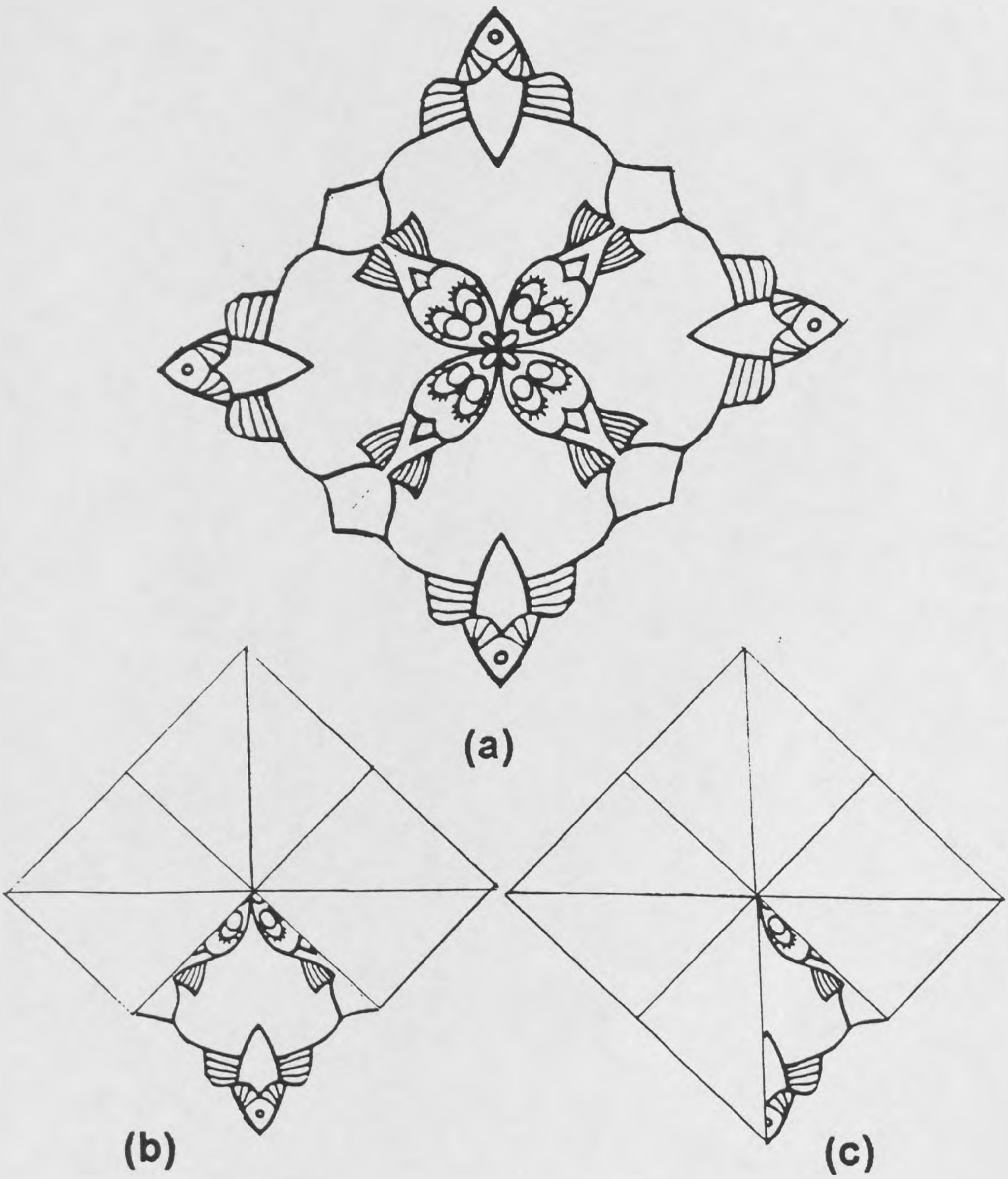


Figure 3.15(a) Class d5 motif. (b) Bilaterally symmetrical unit. (c) Fundamental unit.

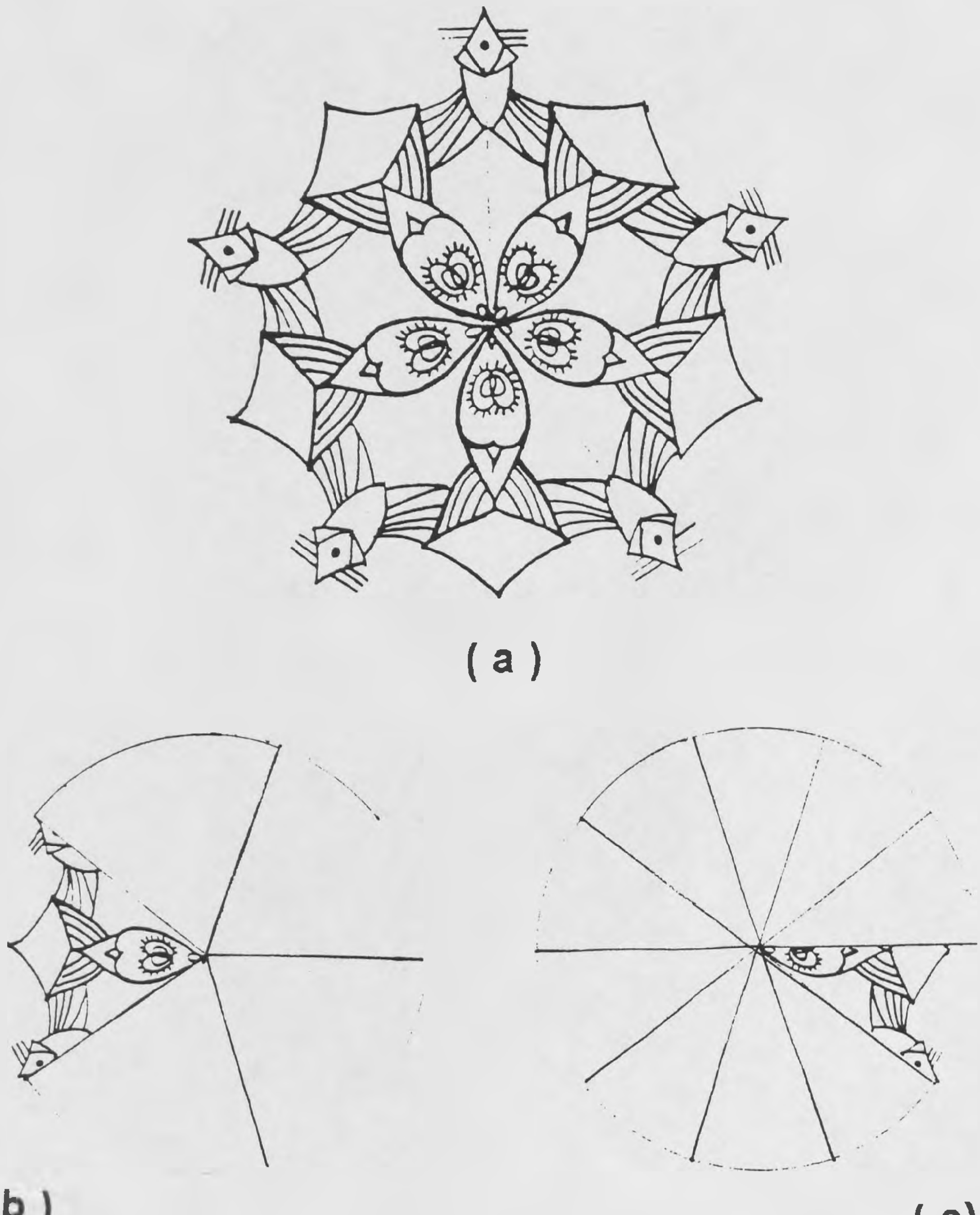


Figure 3.16(a) Class d6 motif. (b) Bilaterally symmetrical unit. (c) Fundamental unit.

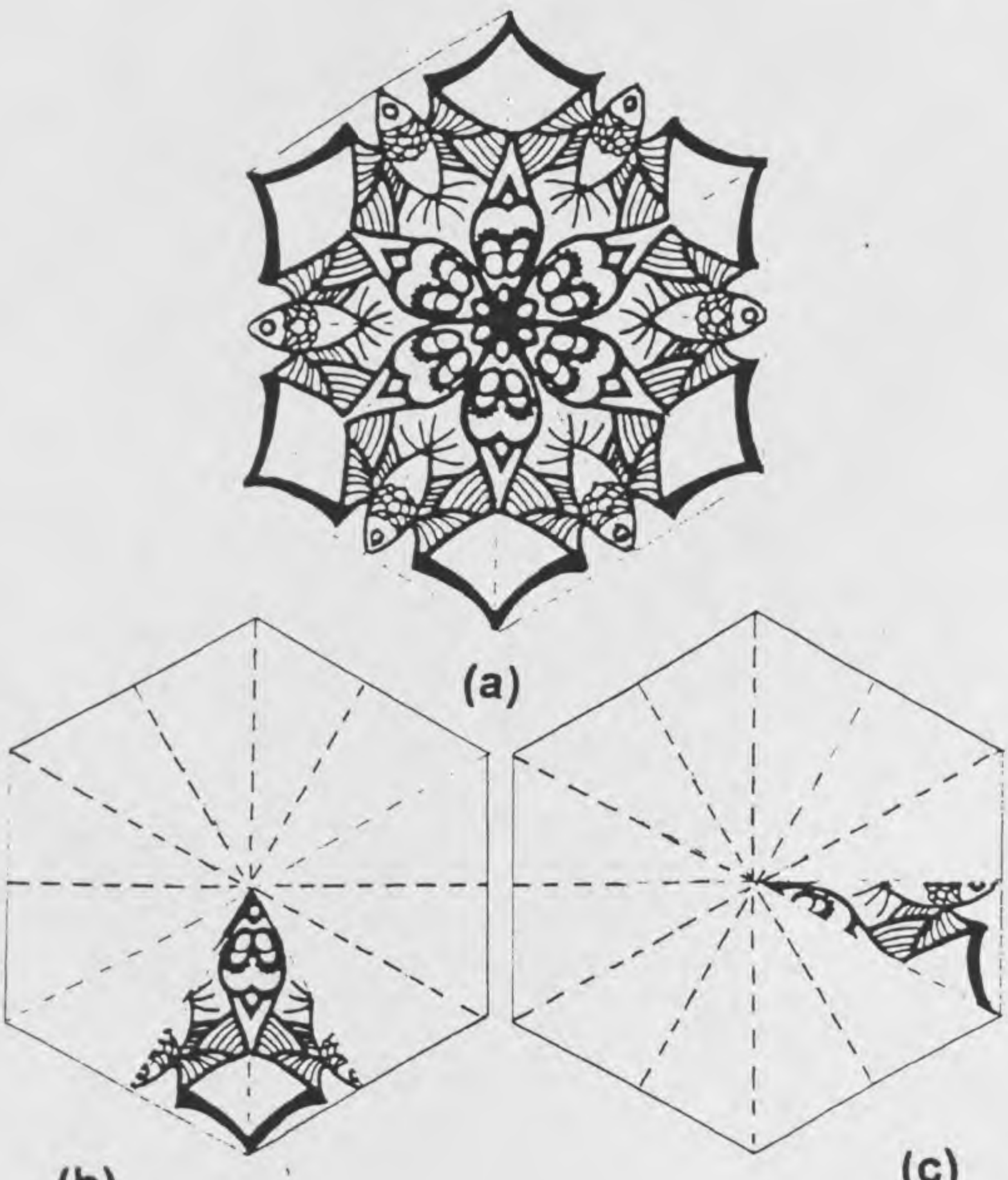
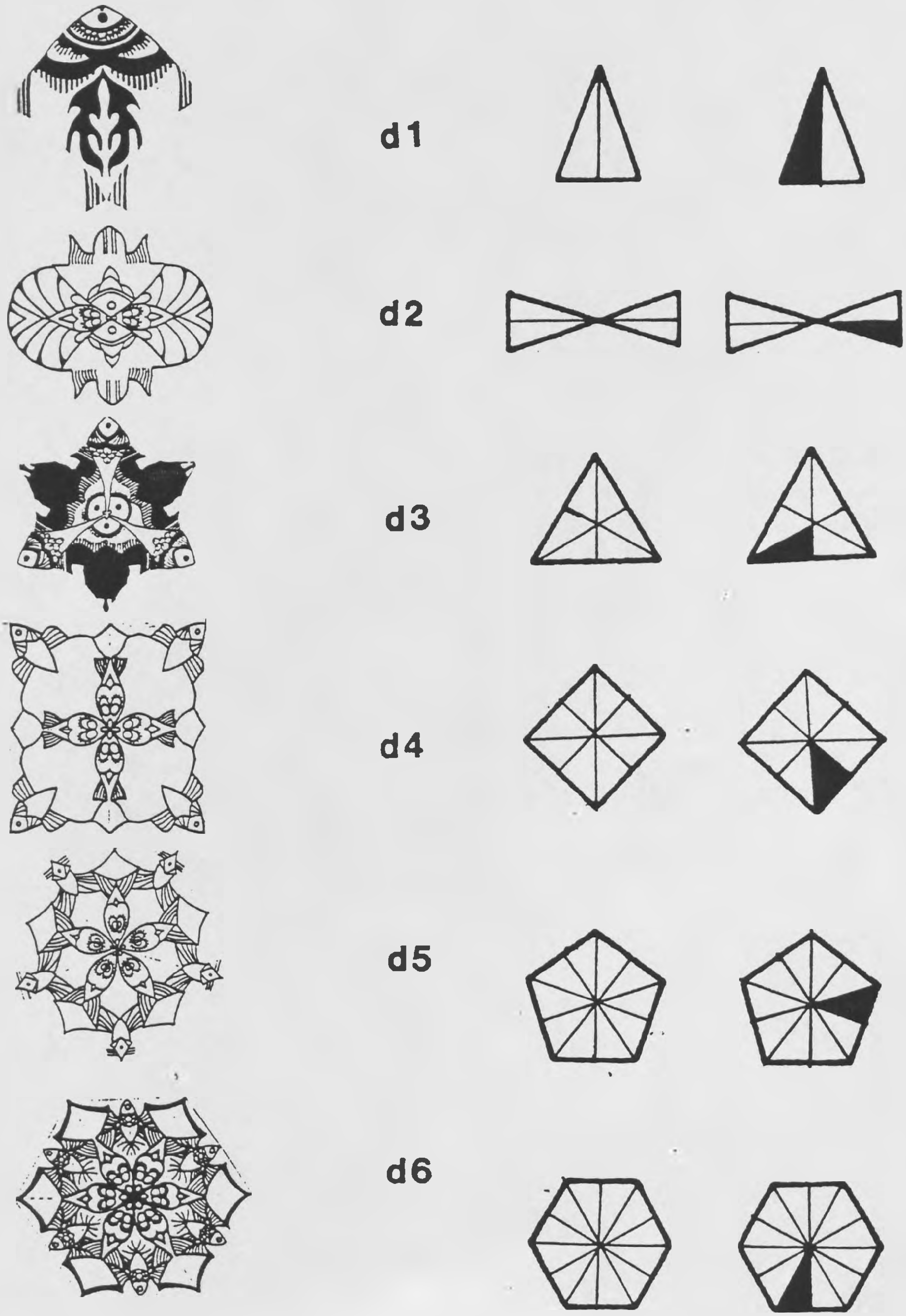


Figure 3.17 Recognition chart for class dn motifs

Class dn Motifs | Symmetry Group | Schematic Illustrations | Generating Region (shaded)



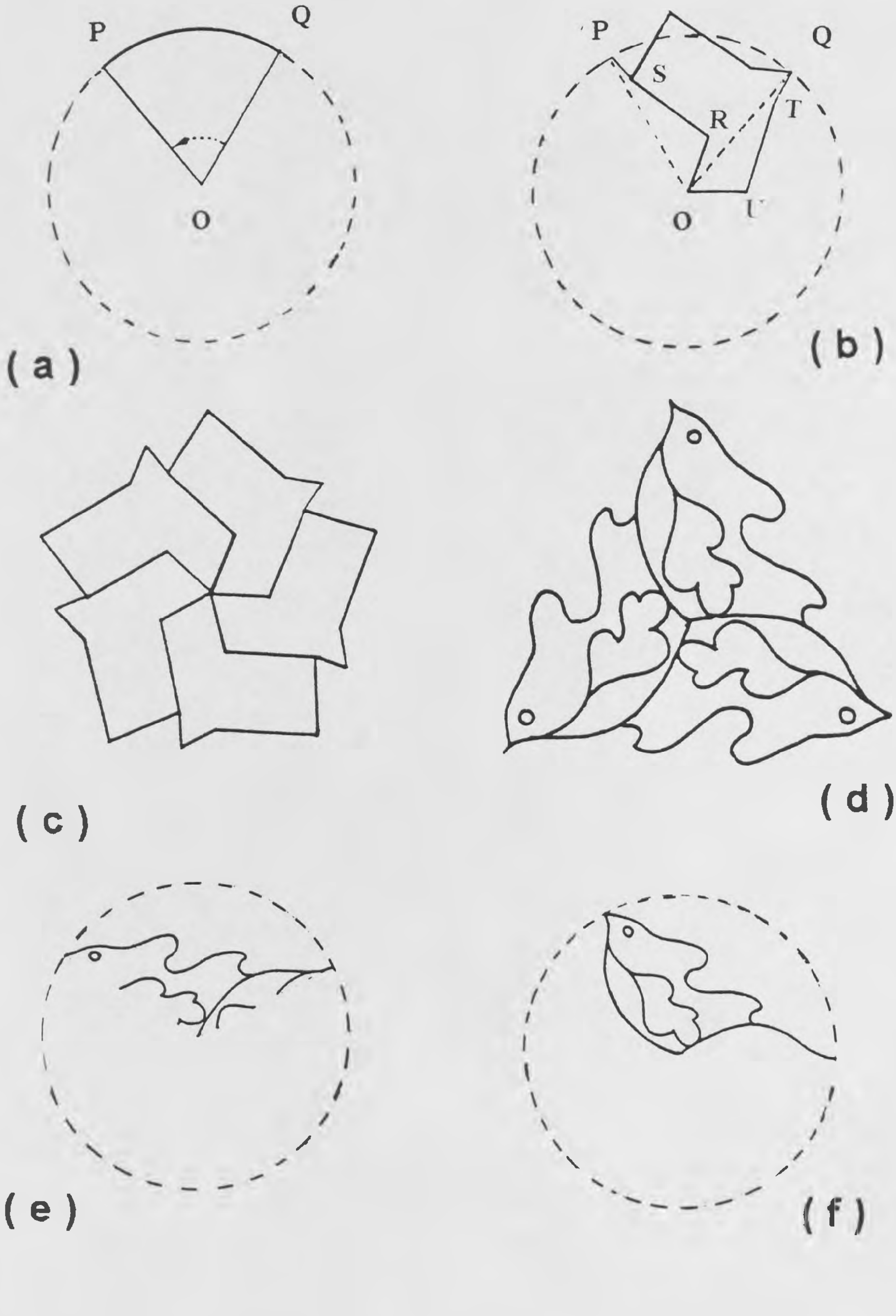
3.3.3 The Construction of Class cn and Class dn Motifs

As stated above, a motif from class cn may be produced by circlewise repetition of a fundamental unit. The area occupied by the fundamental unit is known as the "fundamental region". In the construction of a class c5 motif, for example, it is convenient to start with a circular sector or wedge denoted by OPQ with an angle of 72 degrees (Figure 3.18a). Under five-fold rotational symmetry this wedge will produce the whole circle. If the centre point O is joined to P using an arbitrary line or series of lines, the image line can be constructed joining O to Q (Figure 3.18b). The area OPSR will be equal to the area OQTU. Under five-fold rotation of the fundamental unit, a class c5 motif will be created (Figure 3.18c).

An equivalent procedure may also be used in the production of other motifs from class cn. By way of further example Figure 3.18d shows a class c3 motif with alternative fundamental units shown in Figures 3.18e and f. Although different in structure, the two units have the same area and under rotations through 120 degrees, 240 degrees and 360 degrees the full motif will be produced in each case.

As explained previously, certain motifs might be simultaneously symmetric under both rotational and reflectional symmetry; such motifs are classed as dn motifs. Motifs from this class may be produced by rotating a bilaterally symmetrical unit (which consists of two generating units) about a centre of n-fold rotation. Alternatively the motif may be created by repeated reflections

Figure 3.18(a) A circular wedge OPQ. (b) Construction of areas OPSR and OQTU. (c) Five-fold rotation of fundamental unit of c5 motif. (d) Class c3 motif. (e) Fundamental unit. (f) An alternative fundamental unit.



of the generating unit in reflection axes which intersect at 180 degrees/n. In the construction of a motif from class d3, for example, we can consider a circular wedge denoted by OPQ, with lines OP and OQ forming an angle of 60 degrees (Figure 3.19a). A fundamental unit can be constructed by joining P and Q using an arbitrary line (Figure 3.19b). The fundamental unit can then be reflected and rotated three times or alternatively undergo successive reflections using the intersecting reflection axes (Figure 3.19c).

3.4 Summary

It has been shown that motifs may be classified into one of two general classes dependent upon the symmetry operations used in their construction. Class cn motifs, of an order higher than c1, exhibit rotational characteristics only, and class dn motifs, of an order higher than d1, exhibit combinations of rotation and reflection. By way of illustrative summary, class cn and class dn motifs are further illustrated in Figure 3.20.

Figure 3.19(a) A wedge shape OPQ. (b) P and Q joined by an arbitrary line. (c) Class d3 motif.

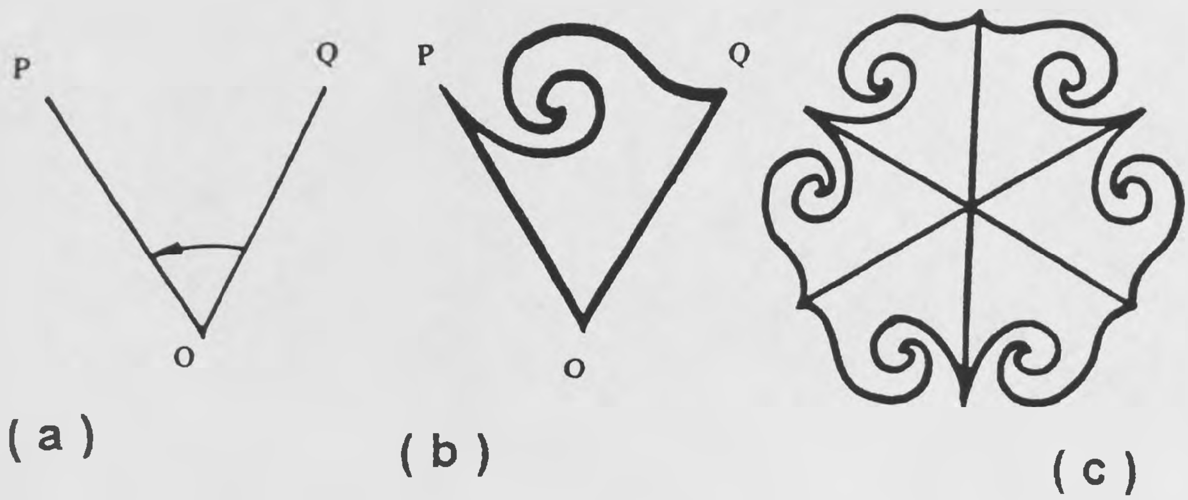
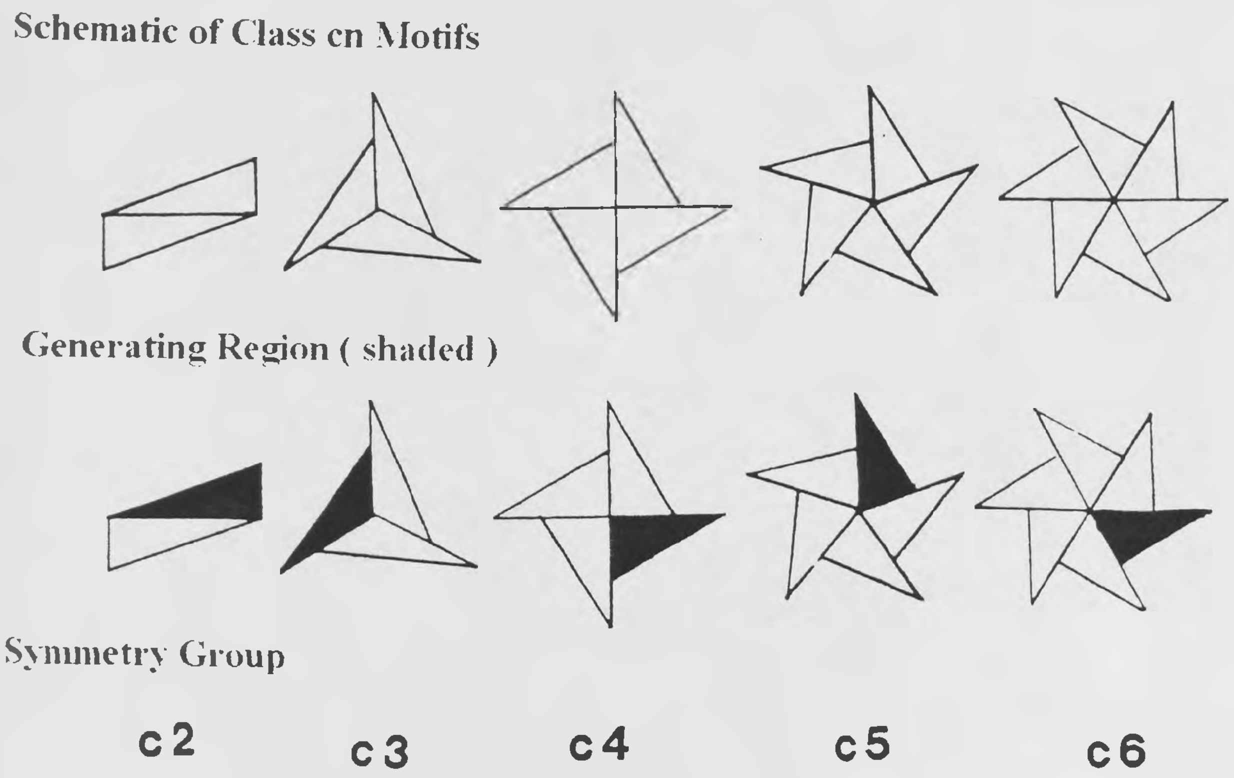
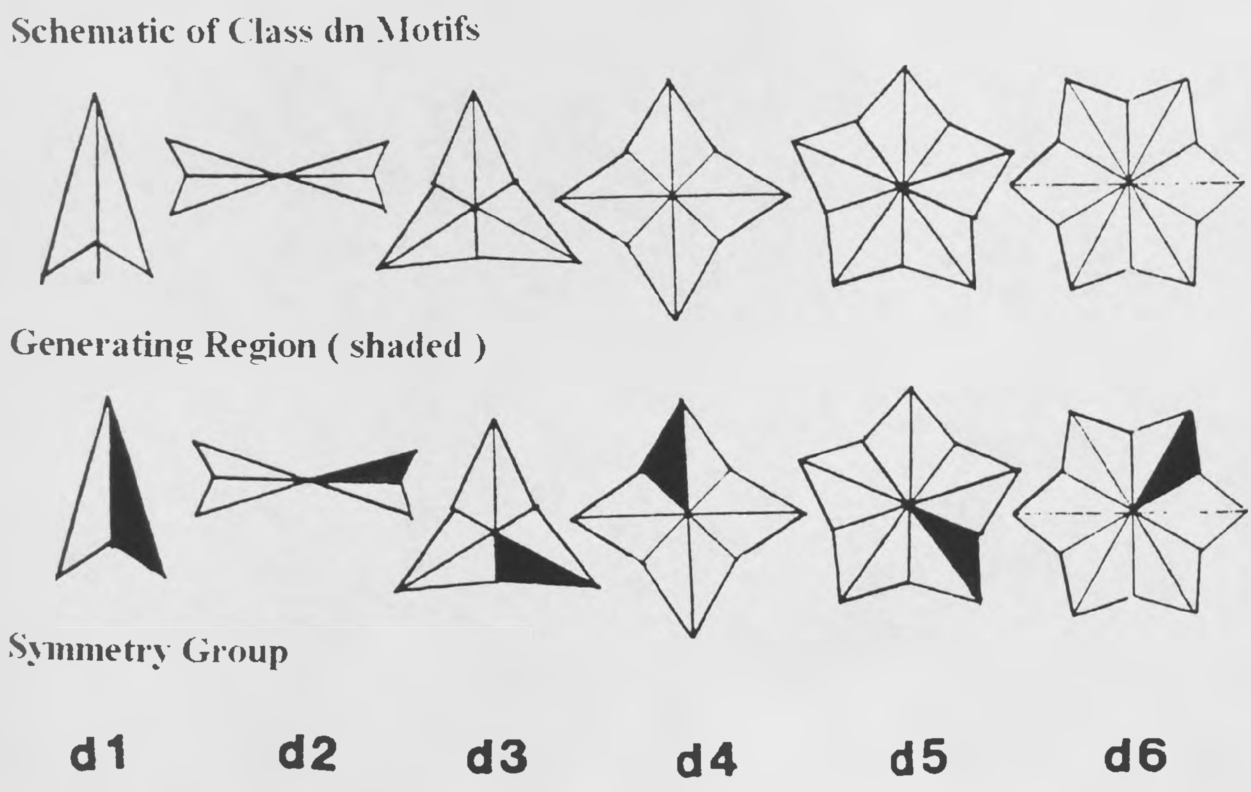


Figure 3.20 Class cn and class dn motifs.





4 THE CLASSIFICATION AND CONSTRUCTION OF PRIMARY PERIODIC BORDER PATTERNS AND TILINGS

4.1 Introduction

In this chapter attention is focused on the symmetry characteristics of border patterns. These were defined by Schattschneider as designs,

> "...... enclosed between two parallel lines (the edges of the border), that is, enclosed in a strip of finite width and infinite length, and having centreline L which is equidistant from the edges." [49]

Using combinations of the four symmetry operations, and ignoring interchange of colour, only seven distinct classes of border patterns or tilings are possible. Subsequent to directing attention to various terminology and accompanying notation, each of the seven classes are described and illustrated below. In addition, an explanation is presented of basic construction techniques for tilings from each of the seven classes.

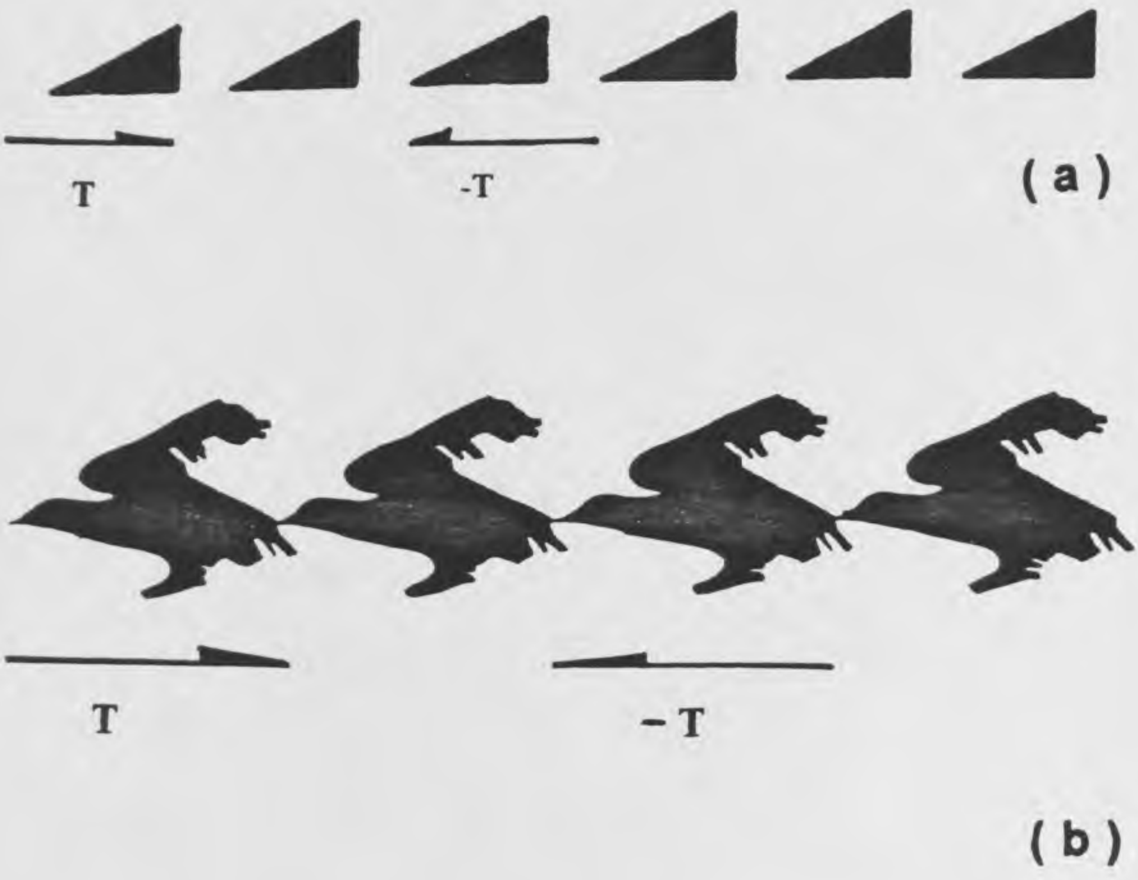
4.2 Terminology and Notation for Primary Periodic Border Patterns

As indicated previously, periodic border patterns are those patterns which admit translations along a single axis which is parallel to the sides of the border; conceptually the pattern is considered to extend to infinity. Geometrically, therefore, any system that has infinite regular repetition in one direction is symmetric under translations in that direction, but only under translation by a certain minimum displacement interval and its multiples. Figure 4.1 shows examples of border patterns which are generated by translation (T) in one direction . The reverse operation (i.e. -T) will, of course, have the same result. Theoretically, both patterns extend to infinity. It should be obvious, however, that no real system is infinite. No real border pattern can thus be extended to infinity under translation. If however a finite border pattern can be imagined as being part of an infinite system, and if the minimum distance of translation is much smaller than the border's total length, then the implications of the termination of the border at either end can be ignored.

The minimum unit that is translated may be called the "translation unit". This is the minimum area of the pattern which when successively repeated by translations produces the border pattern or tiling.

The "fundamental region", variously referred to as the "fundamental domain", the "generating region" or "asymmetric region", is the smallest region of the pattern which when acted upon by any relevant symmetry operations may then be translated to produce the border pattern. As observed by Schattschneider, a generating region of a periodic pattern is the smallest region which when acted upon by the relevant set of symmetry operations will produce the whole pattern [50]. From the viewpoint of geometric symmetry an infinite variety of outlines of generating regions may be possible but the area of each will be the same.

Figure 4.1 (a) and (b) Examples of border patterns which are generated by Translation T in one direction or by translation -T in the opposite direction.



From the viewpoint of symmetry (and ignoring interchange of colour) there exists only seven distinct classes of border patterns. Such patterns may be referred to as "primary border patterns" indicating that interchange of colour is either absent or ignored. The term "counterchange pattern" is used to refer to patterns in which colour interchange does occur. The relevant mathematical proof which determines the existence of only seven classes of primary border patterns can be found in Grunbaum and Shephard [51].

A range of different notation has been used by various authors, but the more commonly accepted notation (which is also used in this study) has been that of the International Crystallographic Union [52]. This notation, which is in the four-symbol form of pxyz, concisely summarises the relevant symmetry operations present in all types of border patterns. The seven border types are as follows:

p111, p1a1, pm11, p1m1, p112, pma2, pmm2.

Washburn and Crowe [53] summarised the use of the notation as follows. The first symbol (p) prefaces all seven classes of border patterns and tilings. Symbols in the second, third and fourth positions indicate the presence of vertical reflection, horizontal reflection or glide-reflection, and half turns respectively. More precisely, x is assigned a symbol which specifies a geometrical characteristic in the direction which is perpendicular to the longitudinal axis of the border; y is assigned a symbol which specifies a geometrical characteristic through the central axis, parallel to the sides of the border; z corresponds to the presence or absence of two-fold rotation. Two-fold rotation is the only order of rotation which is applicable in the context of border patterns, only this rotational order ensures that the sides of the border remain correctly orientated after transformation. The notation system may therefore be explained as follows:

x = m	if there exists a vertical reflection axis (perpendicular to the longitudinal axis of the border);
1	otherwise (no vertical reflection axis present);
y = m	if there exists a longitudinal reflection axis (parallel to the sides of the border);
a	if there exists a longitudinal glide-reflection axis (parallel to the sides of the border);
1	otherwise (no reflection or glide-reflection along the longitudinal axis of the border);
z = 2	if there exists a half-turn (two-fold rotation centres which lie on the longitudinal axis of the border);
1	otherwise.

4.3 Symmetry Characteristics of Primary Periodic Border Patterns

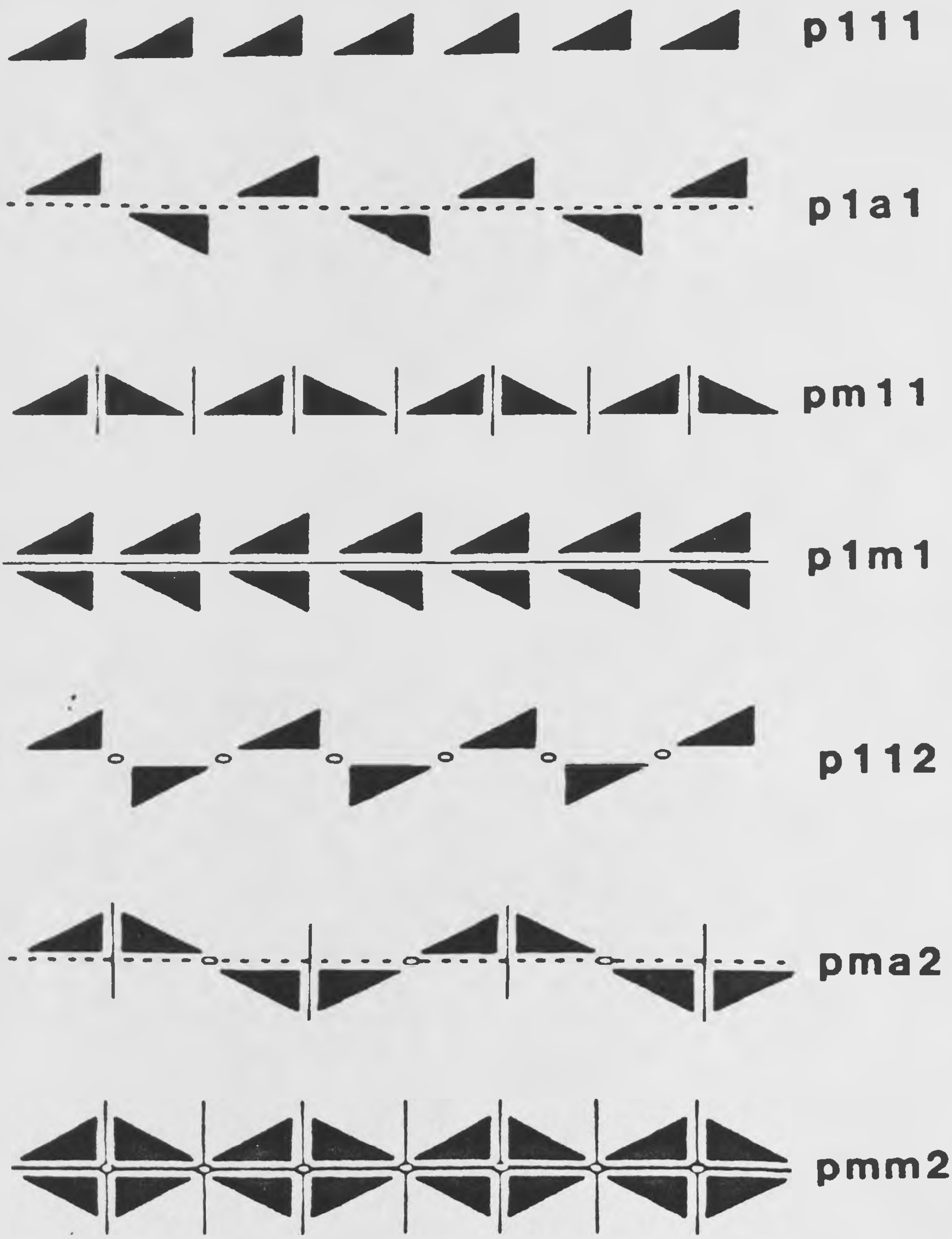
Schematic illustrations for each of the seven primary border pattern classes are provided in Figure 4.2. The symmetry characteristics of each are further described below.

4.3.1 Translational Symmetry in Class p111 Border Patterns and Tilings

From the viewpoint of symmetry, this is the most elementary of the border patterns, since the only operation that the pattern possesses is translation. Figure 4.3a shows an asymmetrical motif (of class c1) repeated at regular intervals along the longitudinal axis by translation distance T. Translation in the reverse direction does, of course, have the same result. The translation unit and the fundamental region (i.e. the generating region) in this border pattern class only, have the same area. Examples of this border pattern class are provided in Figures 4.3b, c and d.

It was explained previously that patterns and tilings are subject to the same geometrical operations. As a general rule, border patterns will contain a repeated motif together with accompanying background space, whereas border tilings will show a division of the plane (within the confines of the edges of the border) without gap or overlap.

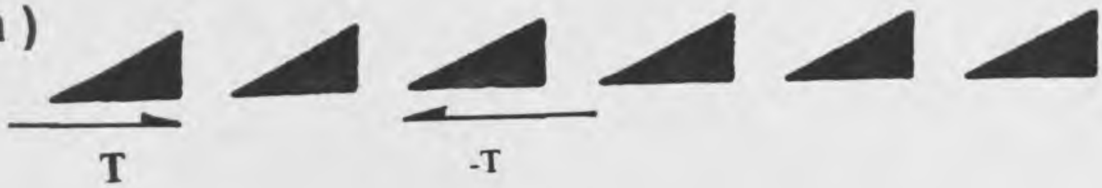
Figure 4.2. Schematic illustration of the seven primary classes of border patterns



Key: ○ 2-fold rotation
--- glide-reflection axis
—— reflection axis

Figure 4.3 (a) Schematic illustration of class p111 border pattern showing translation by T or -T. (b), (c) and (d) Further examples of border patterns from class p111.

(a)



(b)

(c)

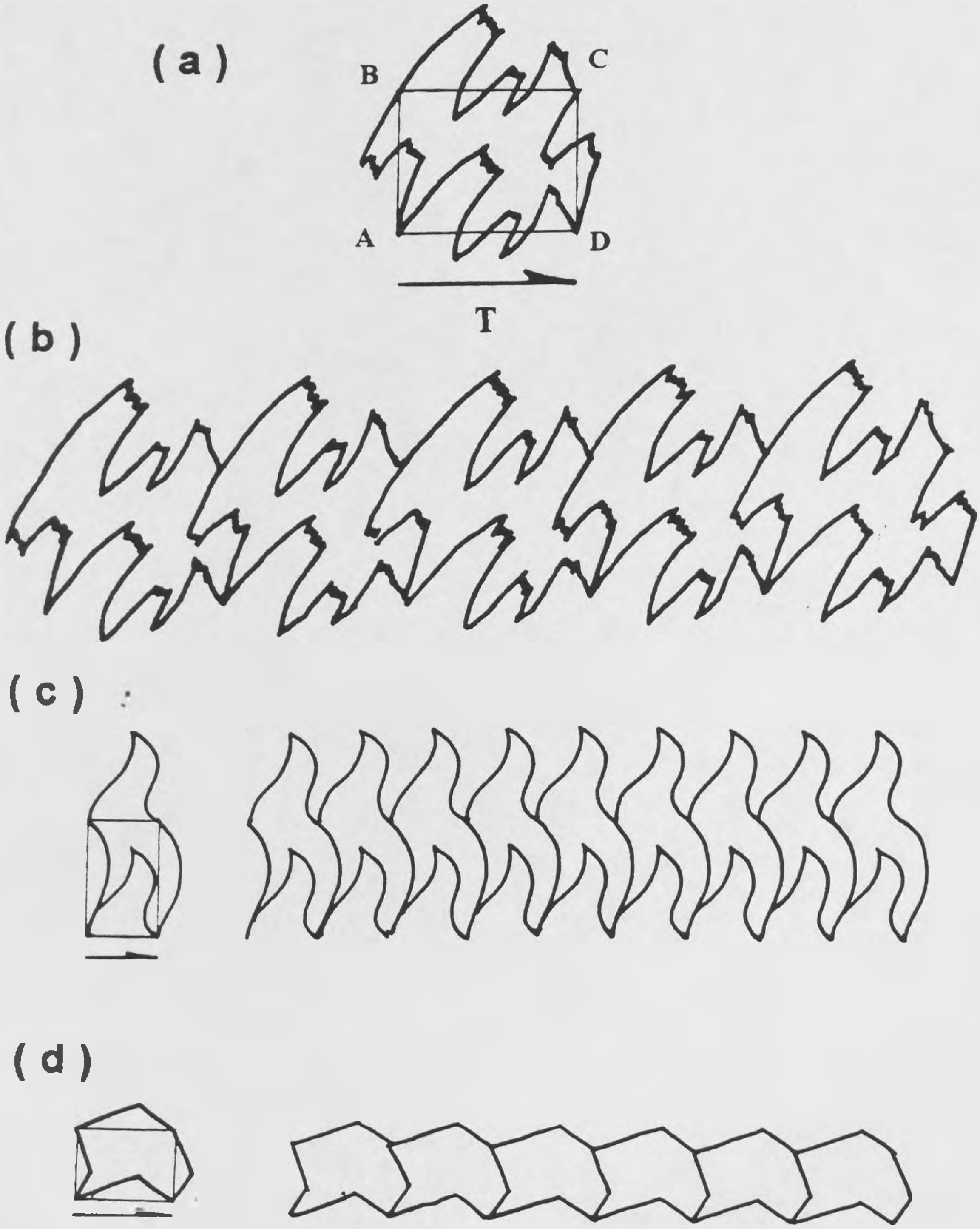
(d)

The construction of a border tiling of class p111 is by the repetition of a single tile under translation along a central axis. The generating region is a tile of minimal area which when repeated successively by translation fills out the whole border. In the construction of the single tile for such a border it is necessary to ensure that the left side of each tile has the same boundary shape as its right side in order to ensure that translation can occur without gaps or overlaps. Figure 4.4a illustrates the first stage of construction where any rectangle or parallelogram (in this case denoted by ABCD) may be taken. Point A is joined to B by a line or continuous series of lines. The translation unit of the tile is then produced by joining D to C with the same line or continuous series of lines as that which joins A to B. The line joining point A to point B will thus meet with the line joining D to C without gaps or overlaps. The tile is completed by joining A to D and B to C with any simple line or series of lines held within the imaginary border edges. Repeated translation will thus create the border tiling shown in Figure 4.4b. Two further examples are shown in Figures 4.4c and d; each may be produced following the procedure above. It should be noted that in each case the tilings illustrated are such that the boundary lines joining A to D and B to C allow further translation in a second direction; this however is not an essential feature of border patterns.

4.3.2 Glide-reflectional Symmetry in Class p1a1 Border Patterns and Tilings

Glide-reflection, as explained previously, is a combination of translation followed by a reflection in a line parallel to the translation axis of the border.

Figure 4.4 (a) First stage of construction of a border tiling unit. (b) Repeated translation to produce border tiling p111. (c) and (d) Further examples of border tilings from class p111.



In the illustrations which follow, glide-reflection is indicated by a dashed line accompanied by the letter G. A schematic illustration of a p1a1 border pattern is shown in Figure 4.5a. Figures 4.5b, c and d show further examples. In each case class c1 motifs come into coincidence through glide-reflection along the length of the border. As indicated in Figures 4.5b and c the glide distance is half the translation distance. The generating region, which is the smallest fundamental element of the pattern, is half the area of the unit of translation.

A border tiling of class p1a1 exhibits the same symmetry characteristics as a border pattern, but is produced as a division of the plane (between the two edges of the border) without gap or overlap. In this case also it is necessary during construction to ensure that the left side of the translation unit has the same boundary shape as its right side. The stages of construction are illustrated in Figures 4.6a, b and c and Figures 4.6d, e and f. Beginning with a rectangle ABCD, with a glide reflection axis passing through the centre from side AB to side CD, join point A to point B with any non-intersecting line or series of lines. The boundary line A to B is then translated the distance of G and turned over (or reflected) so that the orientation of boundary line CD is the reverse (in the vertical sense) of boundary line AB. B is then joined to C and A to D using an arbitrary line or series of lines. The generating tile has thus been produced. The portion of the tile above the glide-reflection axis can be reflected below and translated along the glide-

Figure 4.5(a) Schematic illustration of a p1a1 border pattern. (b) and (c) Border pattern showing translation distance twice glide distance. (d) Further example of class p1a1 border pattern.

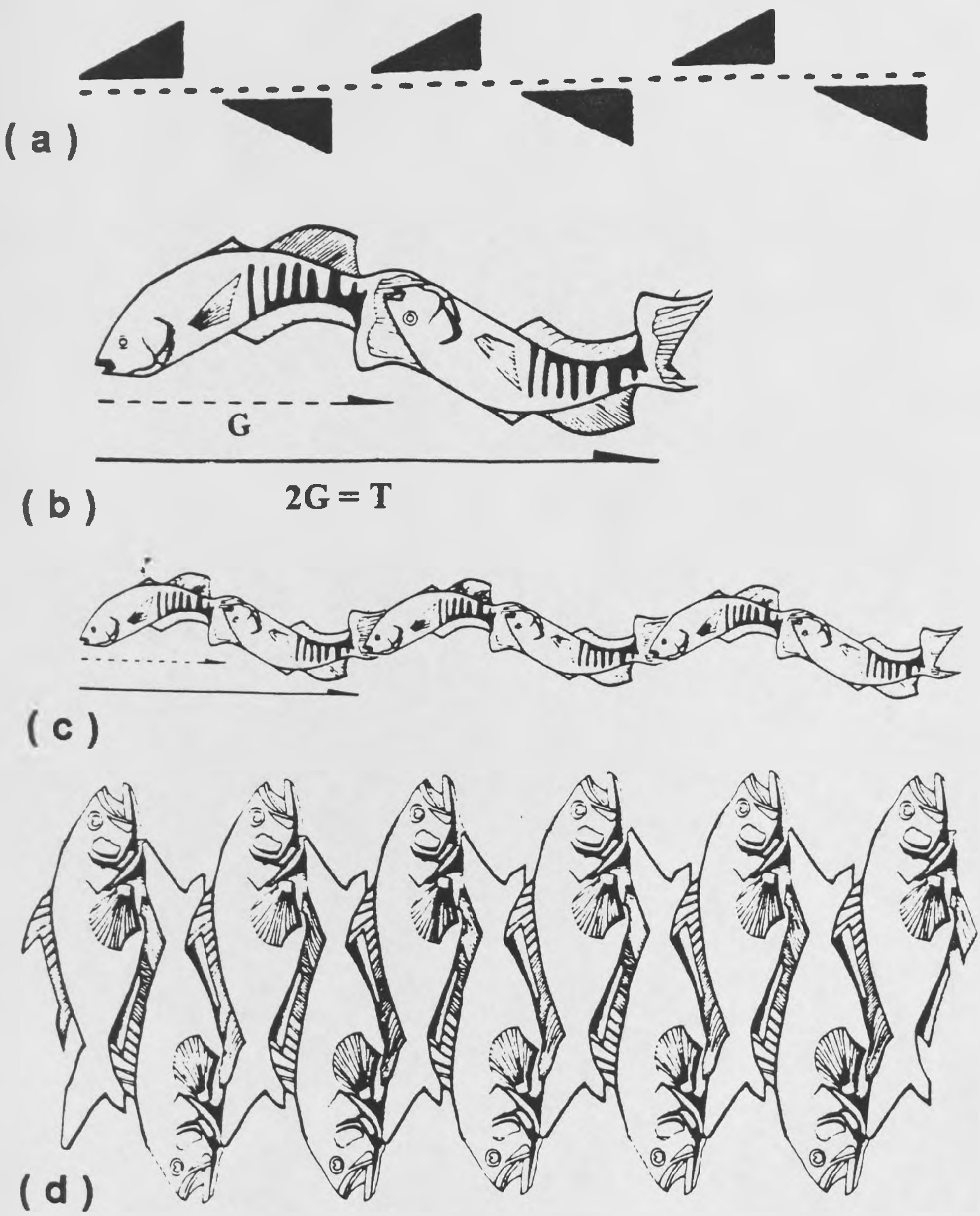
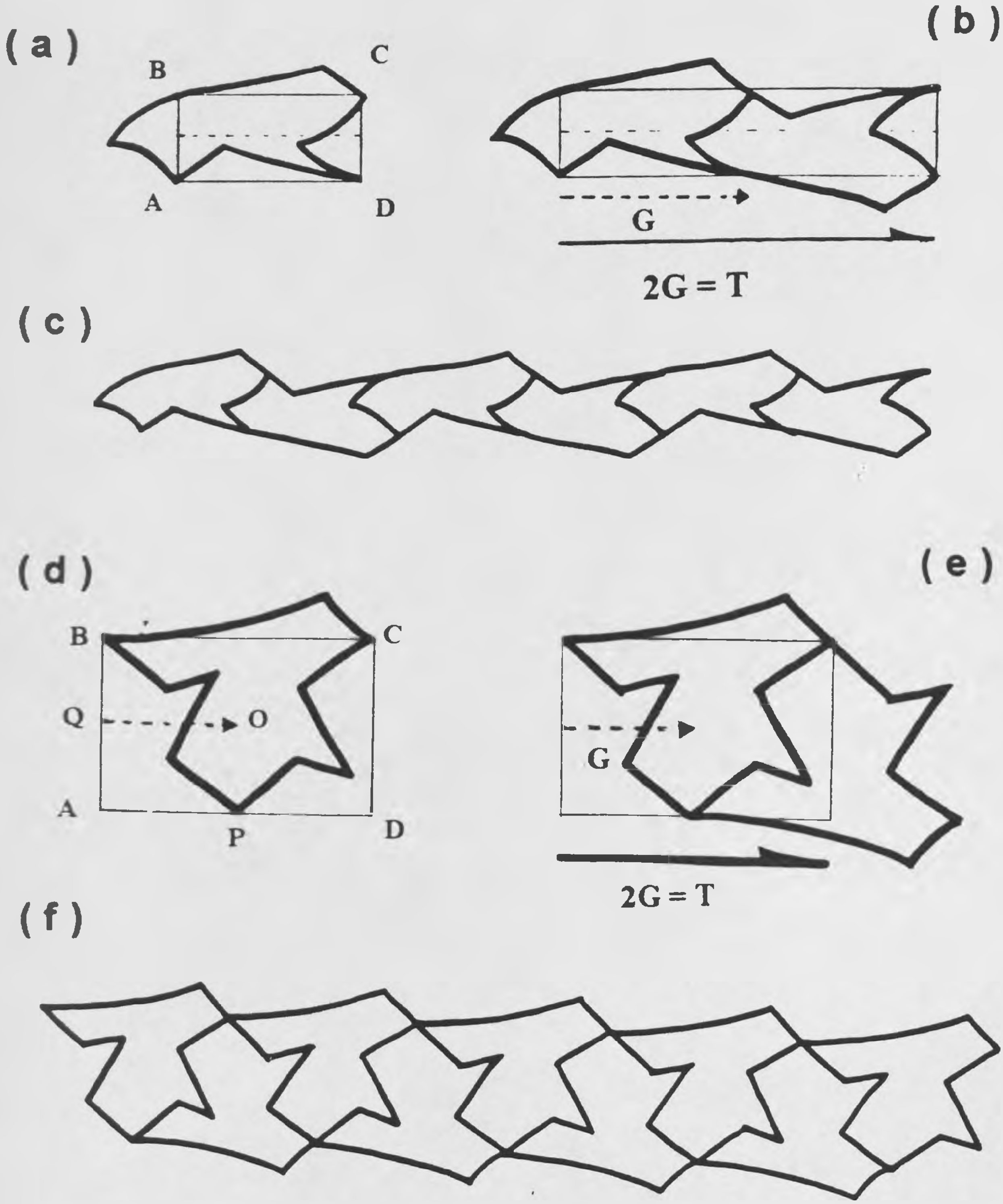


Figure 4.6(a), (b) and (c) Stages in the construction of a p1a1 border tiling. (d), (e) and (f) A further example of construction of a p1a1 border tiling.



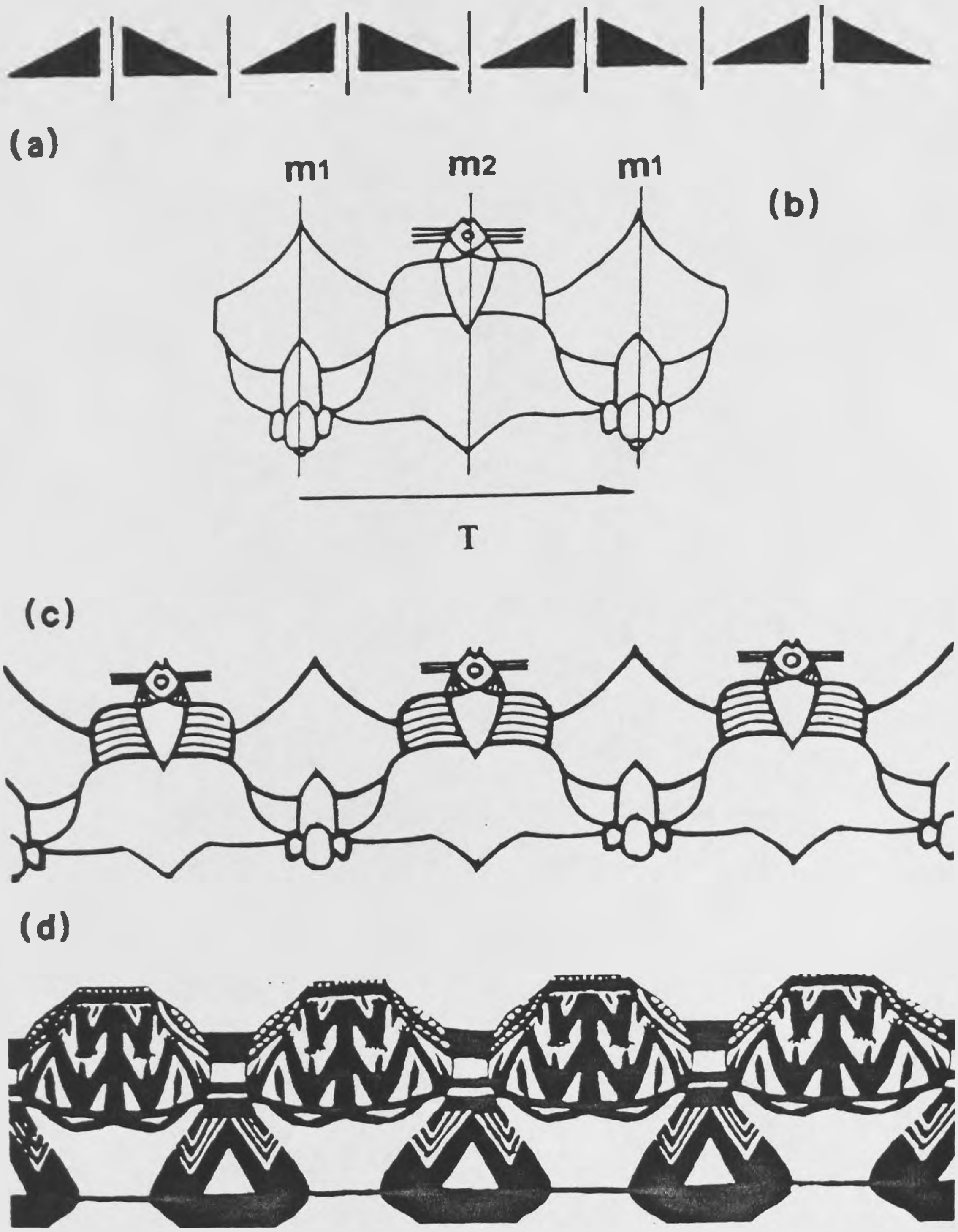
reflection axis, and the portion of the tile below the axis can be reflected upwards and translated along the axis. A full translation unit has thus been produced.

An alternative method of construction is to take rectangle ABCD (Figure 4.6d) having centre point O and points P and Q as midpoints of sides AD and AB respectively. A glide vector can be added from point Q to O. Point P can be joined to point B and glide-reflected the same distance as OQ to produce the line connecting C to P. Points B and C are then joined by an arbitrary line or series of lines. The generating tile has thus been produced, and relevant sections can be glide-reflected above and below the axis to produce the full translation unit shown in Figure 4.6e. Several translations of the unit are shown in Figure 4.6f.

4.3.3 Axial Symmetry of Class pm11 Border Patterns and Tilings

Class pm11 periodic border patterns, which are shown schematically in Figure 4.7a, are characterised by the presence of translation combined with reflections across two alternating vertical reflection axes (shown in Figure 4.7b) each perpendicular to the central axis of the border. The generating unit is half the area of the translation unit. Further examples are shown in Figures 4.7c and d.

Figure 4.7(a) Schematic illustration of a pm11 border pattern. (b) Two alternating reflection axes m1 and m2. (c) and (d) Examples of pm11 border patterns.



In the construction of a pm11 border tiling, take rectangle ABCD (Figure 4.8a). Since the generating region is bounded by two alternating reflection axes (perpendicular to the central axis of the border) the left and right hand sides of each tile must necessarily consist of straight lines. That is, AB and CD are lines positioned on the reflection axes. A tile is completed by joining lines B to C and A to D with a non-intersecting line or series of lines (Figure 4.8a). The translation unit is twice the area of the generating tile (shown in Figure 4.8b). Relevant examples are shown in Figures 4.8c and d.

4.3.4 Axial Symmetry of Class p1m1 Border Patterns and Tilings

Class p1m1 border patterns have reflection in a longitudinal axis through the centre line of the border (shown schematically in Figure 4.9a). The generating unit is half the area of the translation unit (Figure 4.9b).

In the construction of a class p1m1 border tiling, take rectangle ABCD and, as shown in Figure 4.10a, draw a central reflection line (EF). Lines joining A to B and C to D must have the same boundary line. In addition lines A to E and E to B must be a reflection of each other as must lines C to F and F to D. The tile is completed by joining points B to C with a non-intersecting line or series of lines; the resultant shape must then be reflected in the line joining A to D. The total unit can then be translated as shown in Figure 4.10b. A further example of a p1m1 border tiling is given in Figure 4.10c.

Figure 4.8(a), (b) and (c) Stages in the construction of a class pm11 border tiling. (d) A further example of a pm11 border tiling.

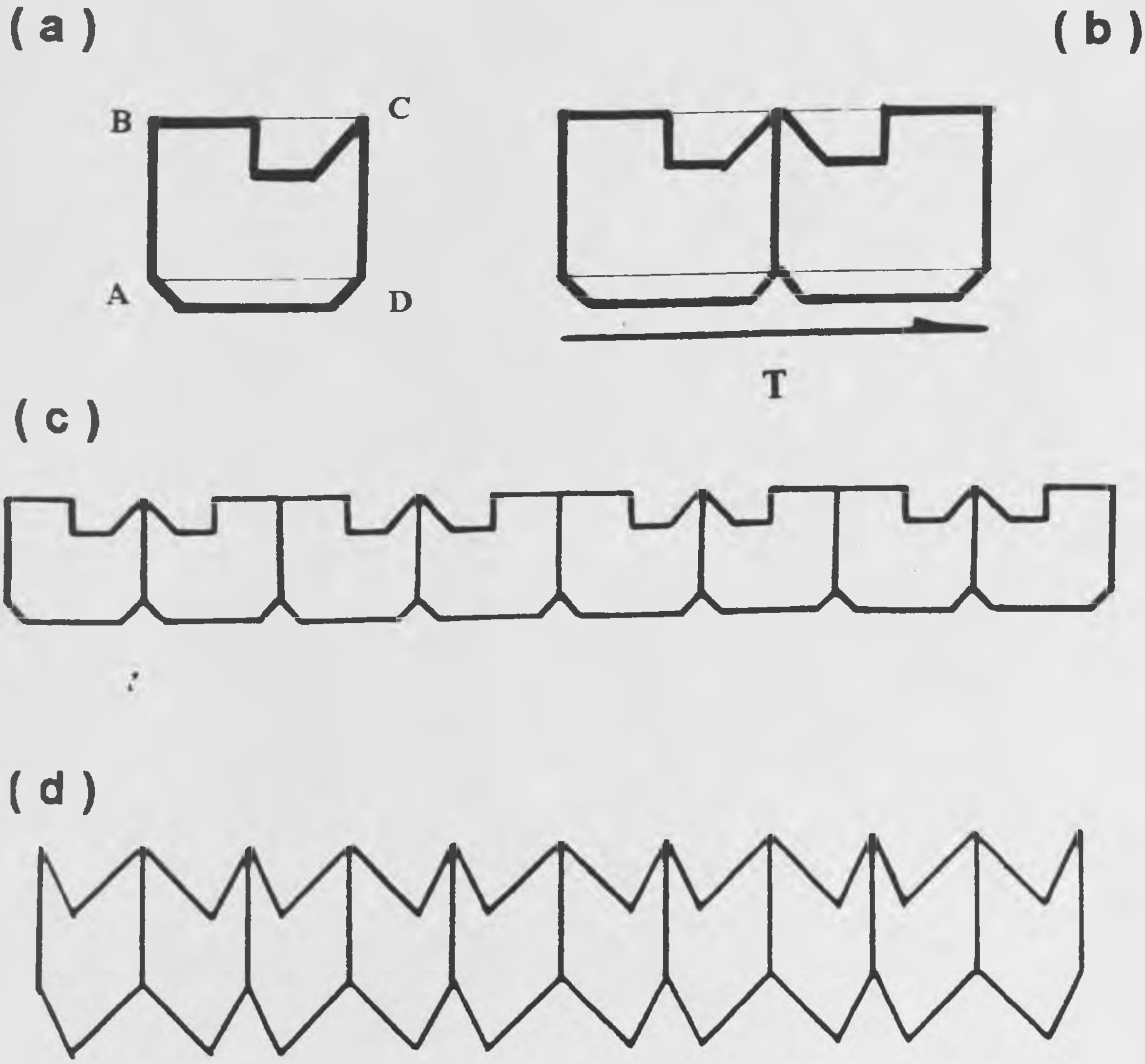


Figure 4.9(a) Schematic illustration of class p1m1 border pattern. (b) Class p1m1 border pattern with generating region shaded. (c) and (d) Further examples of class p1m1 border patterns.

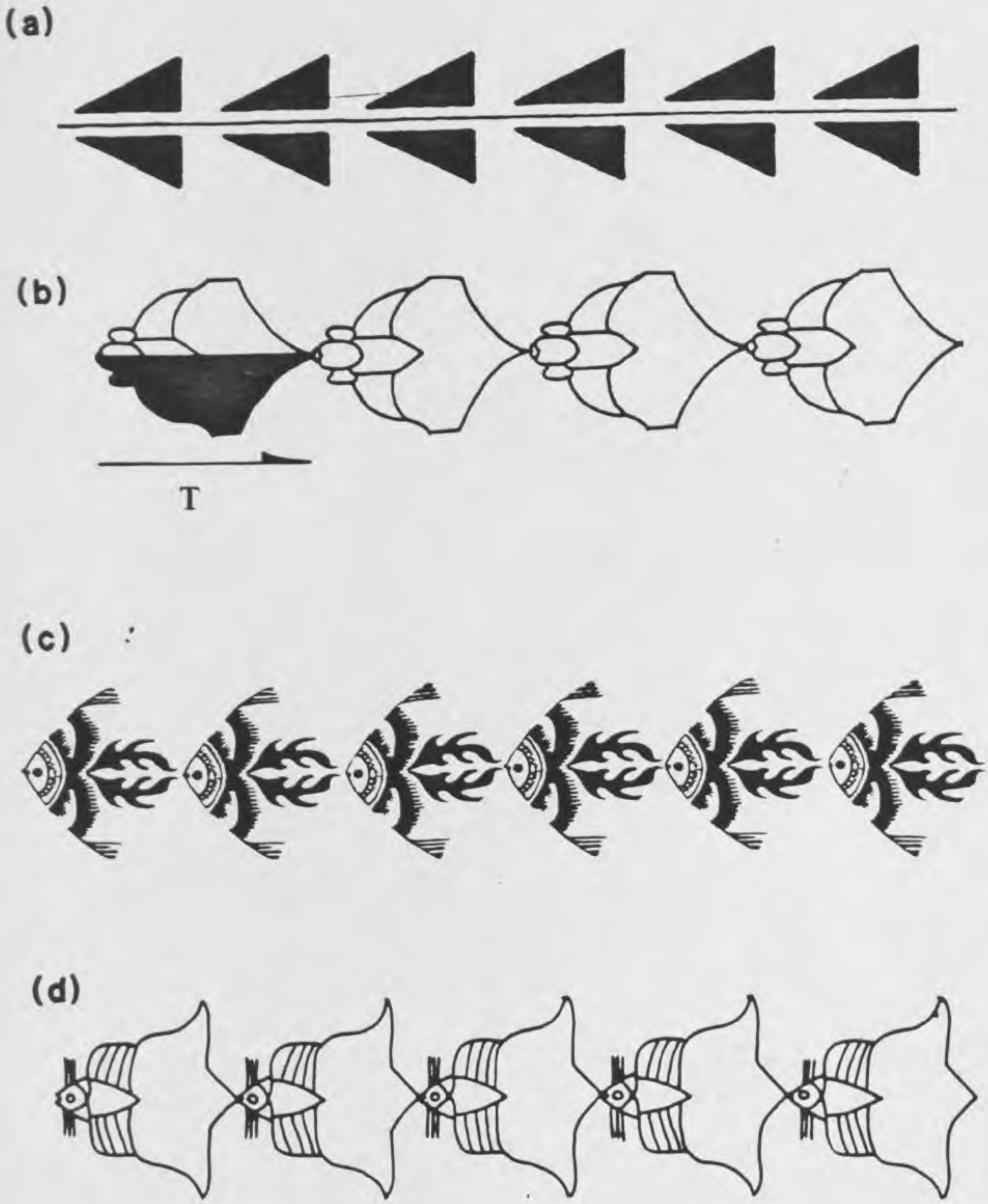
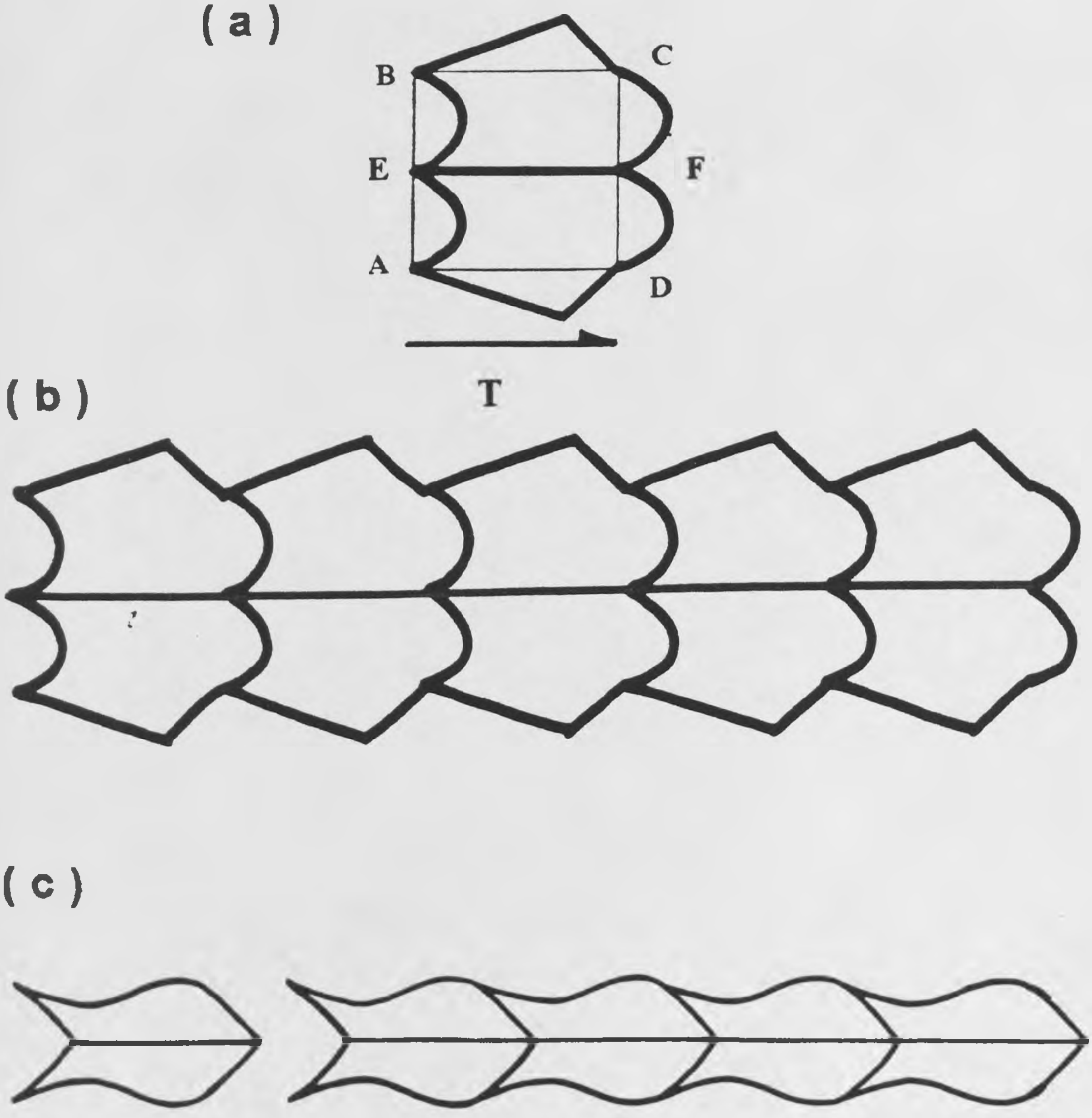


Figure 4.10(a) and (b) Stages in the construction of a class p1m1 border tiling. (c) Further examples of class p1m1 border tilings.



4.3.5 Two-fold Rotational Symmetry of Class p112 Border Patterns and Tilings

Class p112 periodic border patterns exhibit two-fold rotational symmetry. As shown schematically in Figure 4.11a, these patterns are characterised by successive translations of motifs with two-fold centres of rotation (c2 motifs). A second two-fold rotation centre is thus generated which alternates with the first two-fold rotation centre. The generating region is half the area of the translation unit. Figure 4.11b shows the translation unit of the border pattern in Figure 4.11c. A further example is shown in Figure 4.11d.

In the construction of a border tiling from this class, take rectangle ABCD (Figure 4.12a) with mid-point P of side AB and centrepoint O of the rectangle itself. Point B is joined to D by a line or series of lines passing through point O. Point A is joined to B with a line or series of non-intersecting lines passing through point P. The tile is completed by joining A to D with a line or series of lines. Following rotation through 180 degrees, the completed translation unit is shown in Figure 4.12b. The full tiling (Figure 4.12c) can either be completed through successive translation of this unit or by successive two-fold rotation. A further example is shown in Figure 4.12d.

Figure 4.11(a) Schematic illustration of a class p112 border pattern. (b) A translation unit showing two-fold rotation. (c) and (d) Examples of class p112 border patterns.

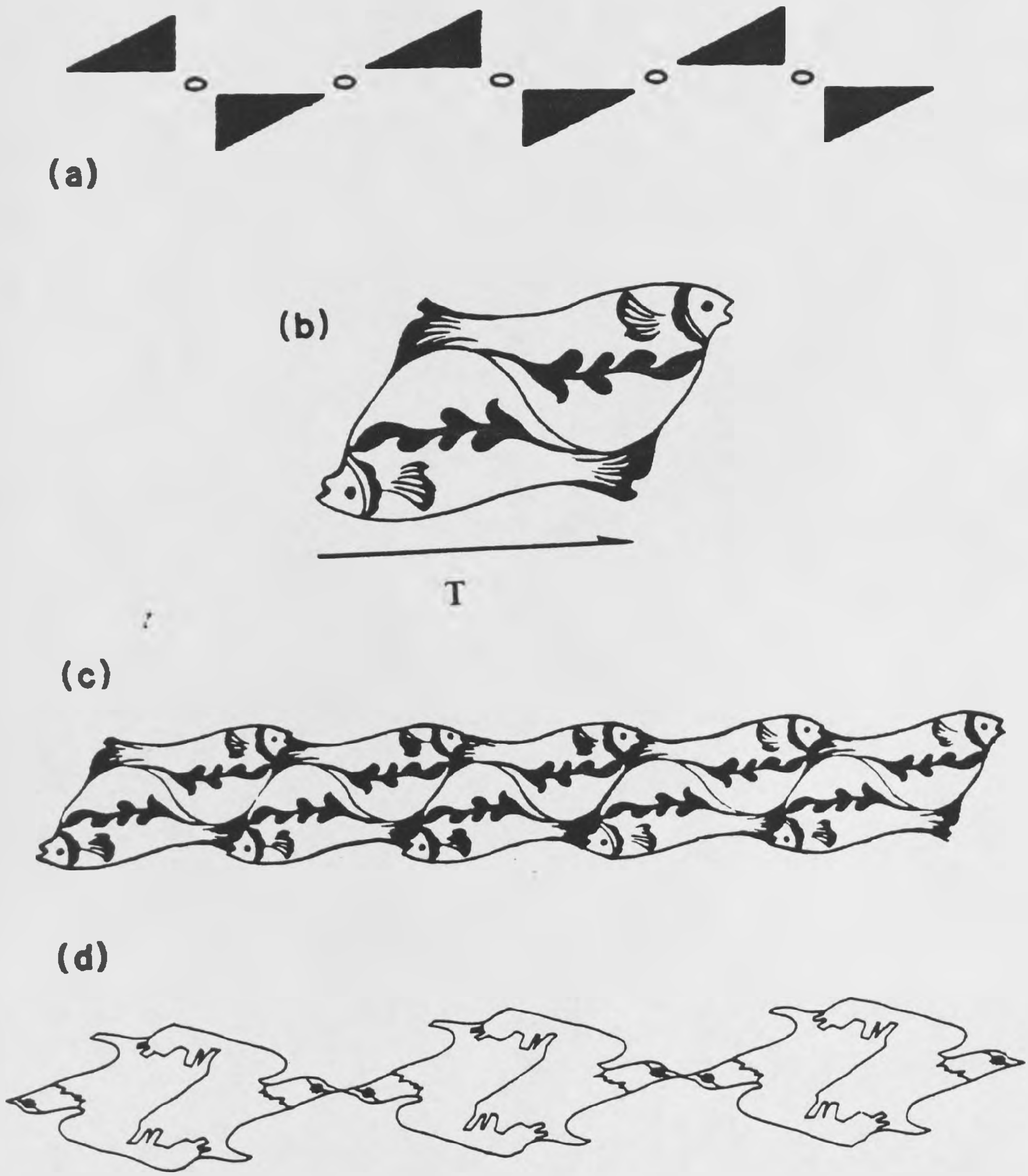
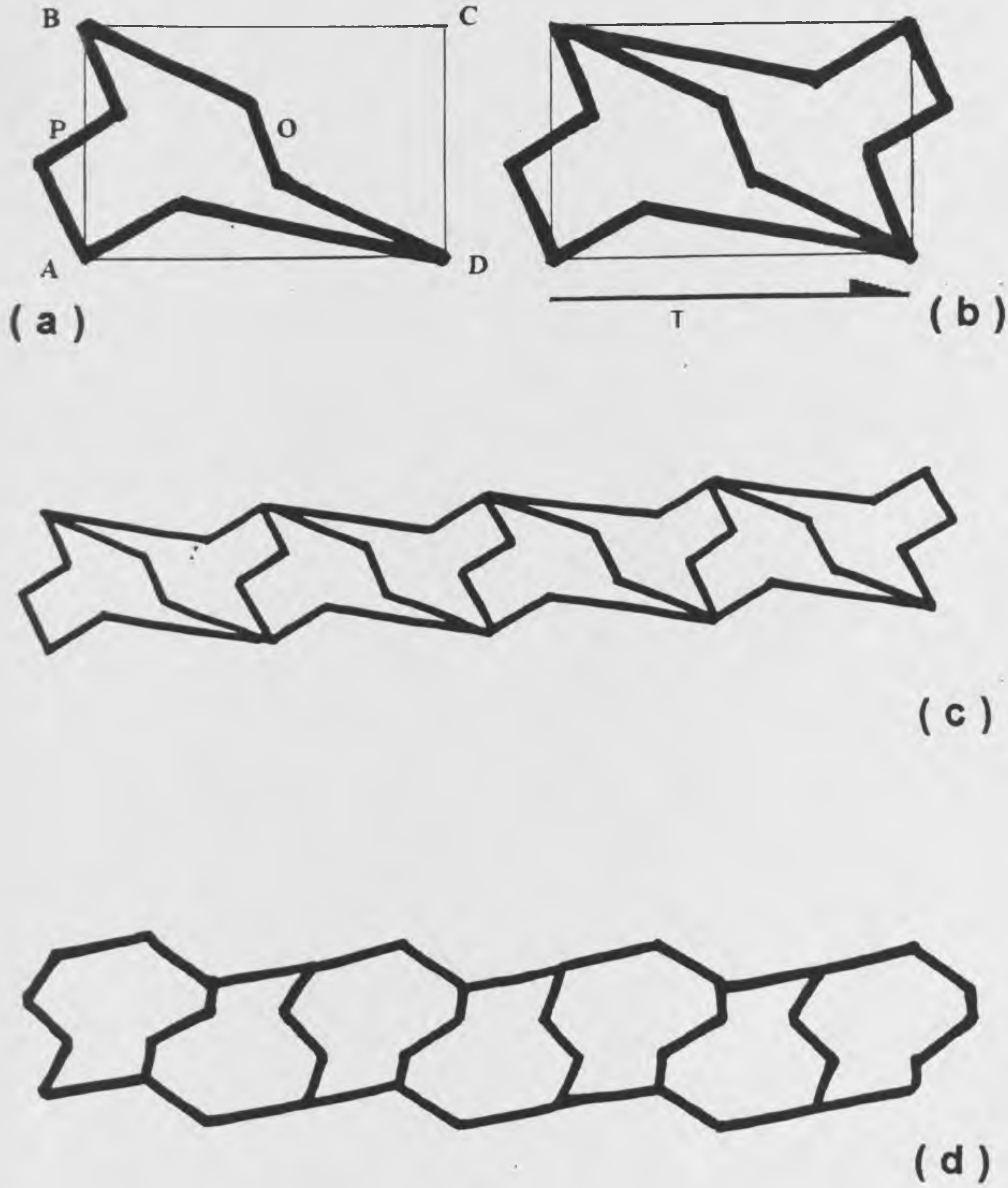


Figure 4.12(a), (b) and (c) Stages in the construction of a class p112 border tiling. (d) Further example of a class p112 border tiling.



4.3.6 Reflected Two-fold Rotational Symmetry of Class pma2 Border Patterns and Tilings

Class pma2 border patterns, which are shown schematically in Figure 4.13a, contain all four symmetry operations. Stevens [54] observed that border patterns from this class may be generated using one of four procedures: by successive reflection of a class c2 motif; by successive translation of two alternate class c2 motifs; by successive two-fold rotation of a class d1 motif; by successive glide-reflection of a class d1 motif. The translation unit consists of a generating unit which is reflected, rotated and reflected again. The generating unit is one quarter the area of the translation unit (indicated in Figure 4.13b). Further examples are shown in Figures 4.13c and d.

In the construction of a border tiling from this class, begin with a rectangle ABCD (Figure 4.14a) having points EFG and H as midpoints of sides AB, BC, CD and DA respectively. The line connecting points H and F is a reflection axis. Point A is joined to point B by a non-intersecting line or series of lines passing through the two-fold rotational centre at midpoint E. The line joining H to F acts as a reflection axis. Join B to F by a line or series of lines. A generating tile has thus been created. Through reflection and two-fold rotation, the translation unit can be produced (Figure 4.14b). The completed tiling is shown in Figure 4.14c and a further example is shown in Figure 4.14d.

Figure 4.13(a) Schematic illustration of a class pma2 border pattern. (b) Illustration showing that the generating region is one quarter of the area of translation. (c) and (d) Examples of class pma2 border patterns.

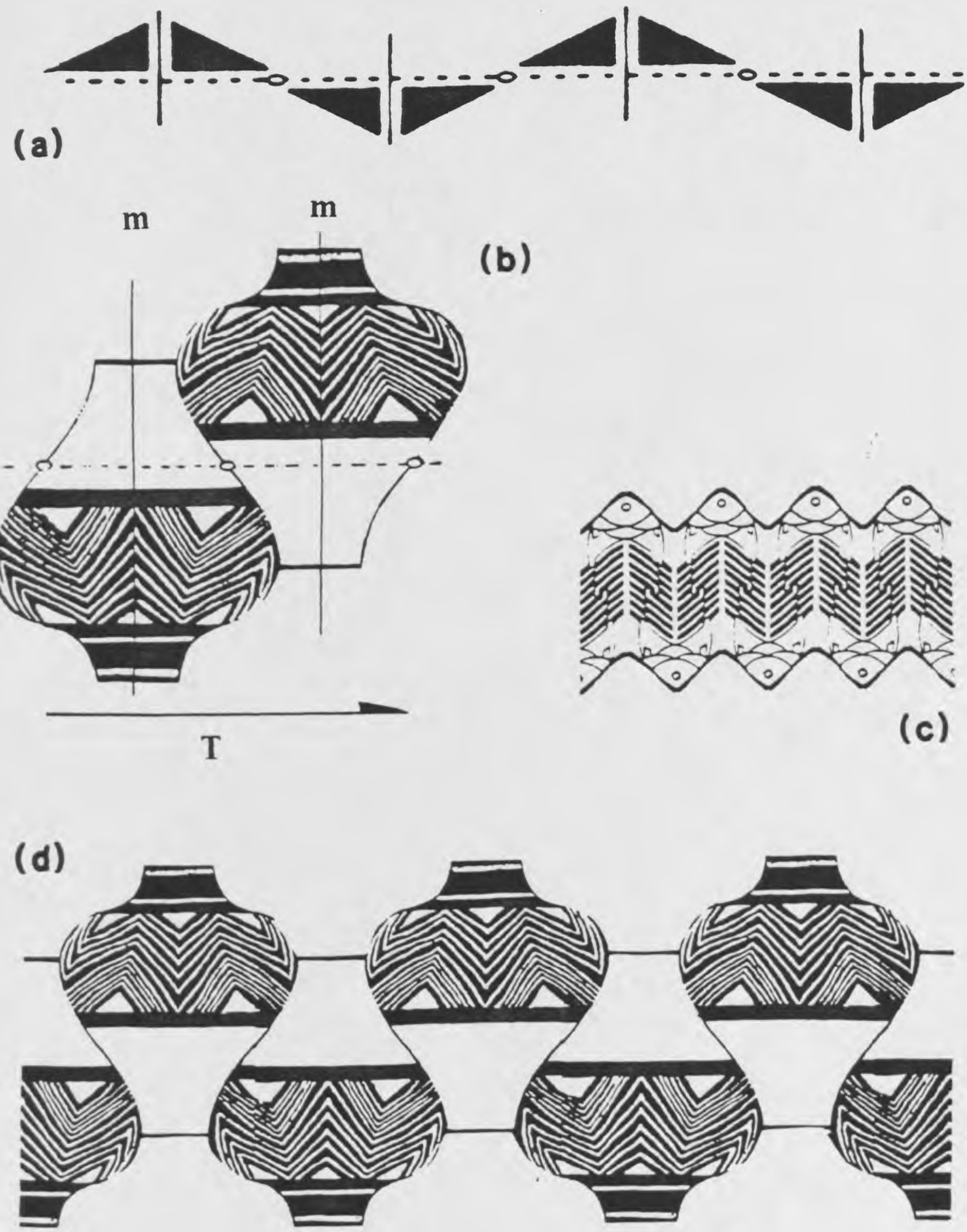
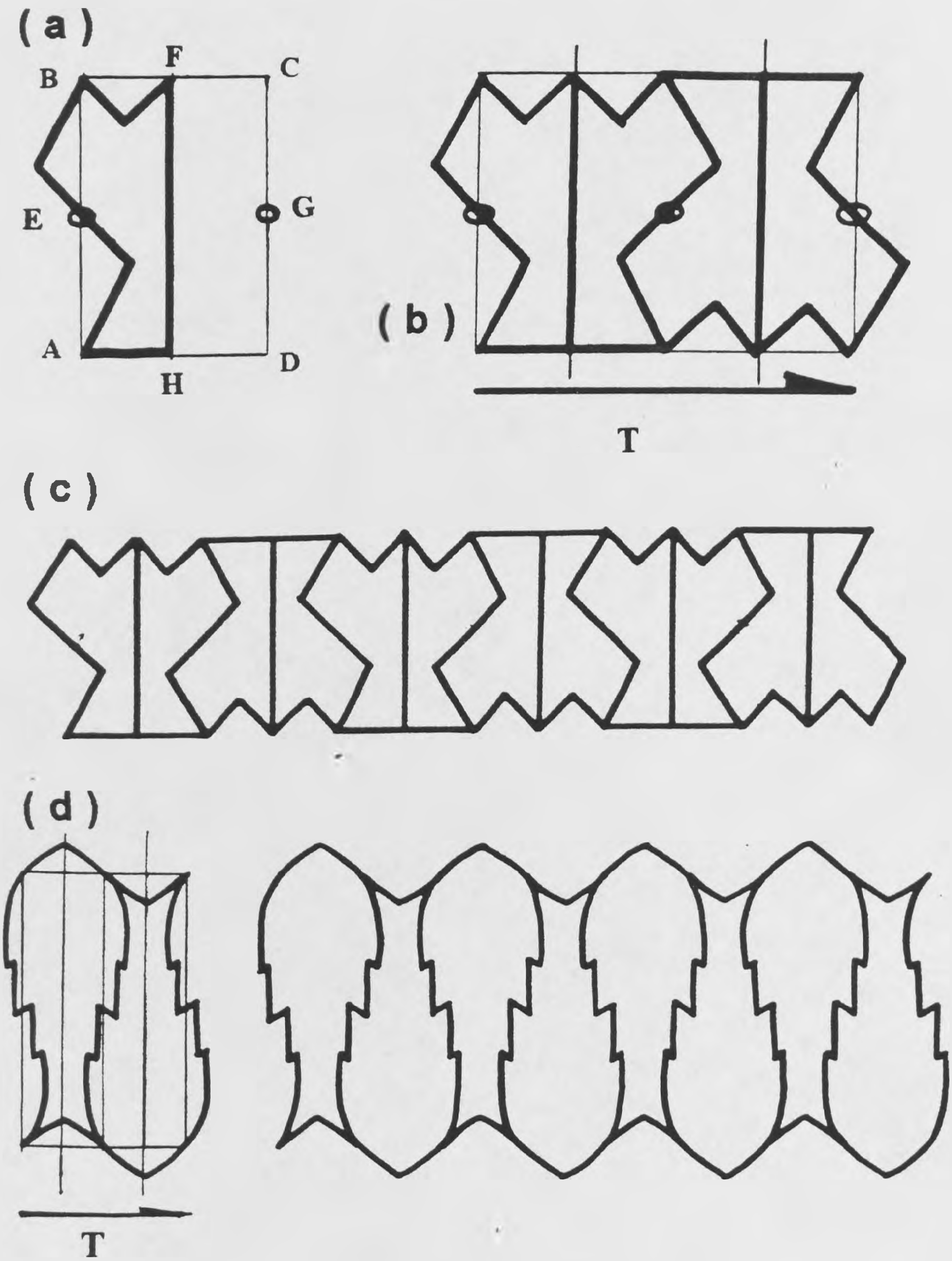


Figure 4.14(a), (b) and (c) Stages in the construction of a class pma2 border tiling. (d) Further example of a class pma2 border tiling.



4.3.7 Central-axial Symmetry of Class pmm2 Border Patterns and Tilings

Class pmm2 border patterns (Figure 4.15a) have a horizontal reflection axis which is intersected at regular points by two alternating perpendicular reflection axes. Two-fold rotation centres are thus established in the intersection points of the axes. The generating unit is one quarter the area of the translation unit. Relevant examples are shown in Figures 4.15b, c and d.

In the construction of a border tiling from this class, take rectangle ABCD (Figure 4.16a). Since the generating unit is bounded by three reflection axes (two perpendicular and one horizontal) the left and right sides of the tile and the bottom edge must be a wedge with a straight line (i.e. sides AB, CD and DA). The tile is completed by joining point B to point C by a line or series of lines. Under horizontal and vertical reflections the translation unit may be created (Figure 4.16b). The completed tiling is shown by Figure 4.16c. A further example is provided by Figure 4.16d.

4.4 Summary

The symmetry operations of translation, rotation, reflection and glide-reflection may be combined to produce a total of seven (and only seven) primary classes of periodic border patterns or tilings. Border tilings from each class may be produced by first constructing the generating unit, as described under each of the relevant sections above. Further descriptions of the seven classes of border patterns were provided by Woods [55], Crowe

[56], Budden [57] Cadwell [58], Coxeter [59], Stevens [60], and Hann and Thomson [61]. By way of summary, recognition charts for each class of periodic border pattern and tiling are provided by Figures 4.17 and 4.18 respectively. The flow-diagram in Figure 4.19, which was reproduced from Hann and Thomson [62], is a convenient aid to the identification of a border pattern or tiling's specific symmetry class.

Figure 4.15(a) Schematic illustration of a class pmm2 border pattern. (b), (c) and (d) Examples of class pmm2 border patterns.

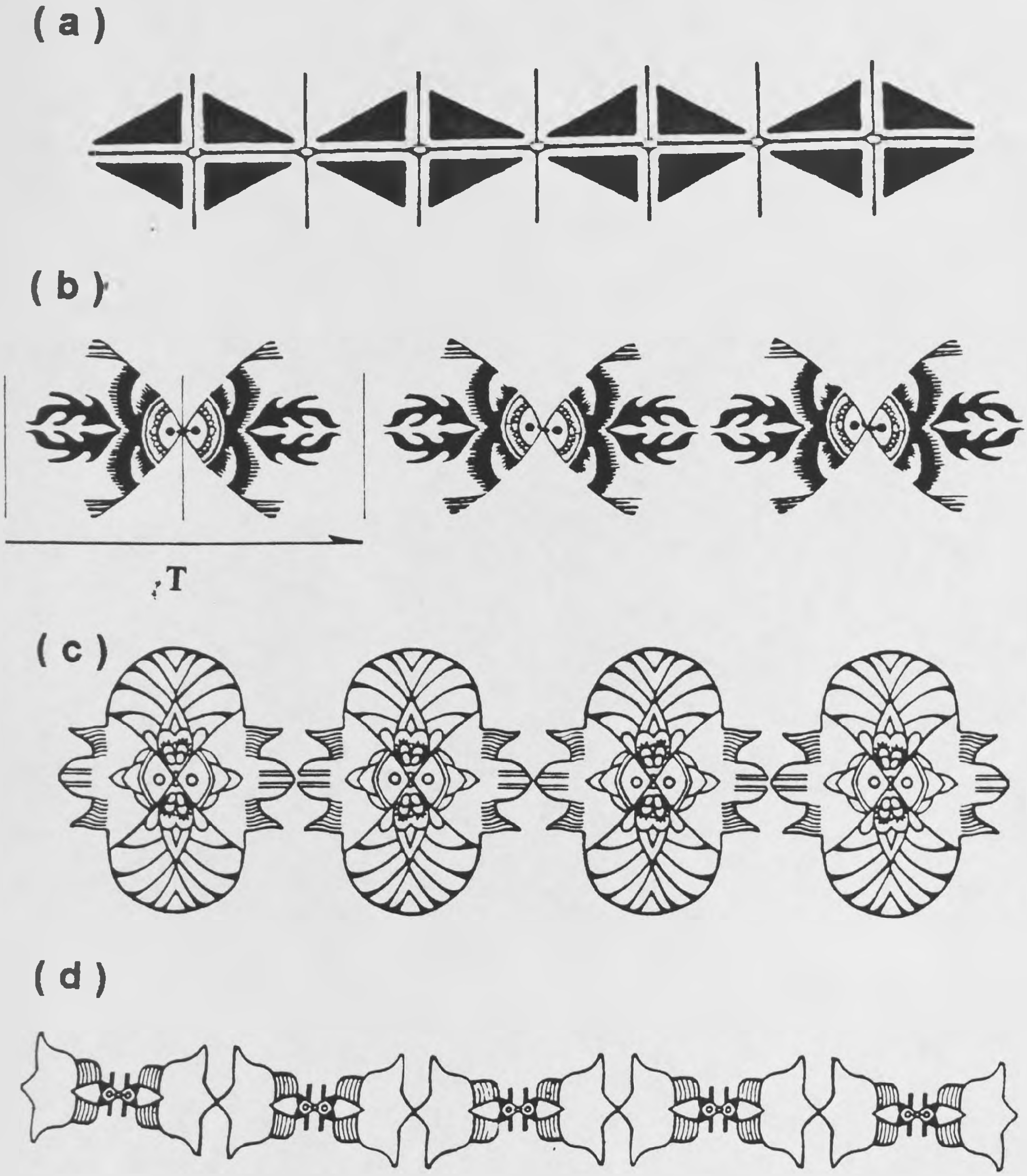


Figure 4.16(a), (b) and (c) Stages in the construction of a class pmm2 border tiling. (d) A further example of a class pmm2 border tiling.

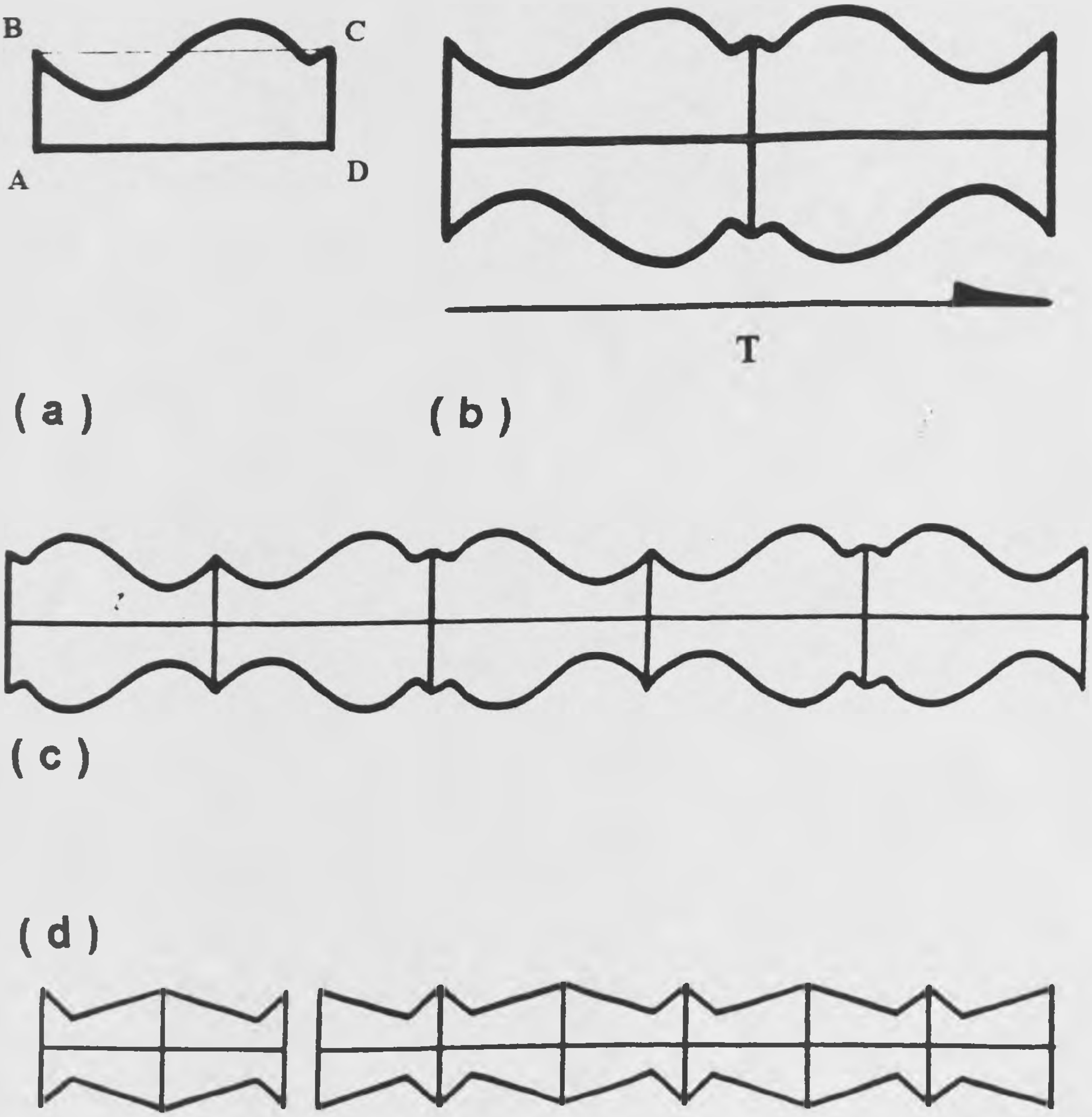


Figure 4.17 Recognition chart for the seven classes of border pattern.

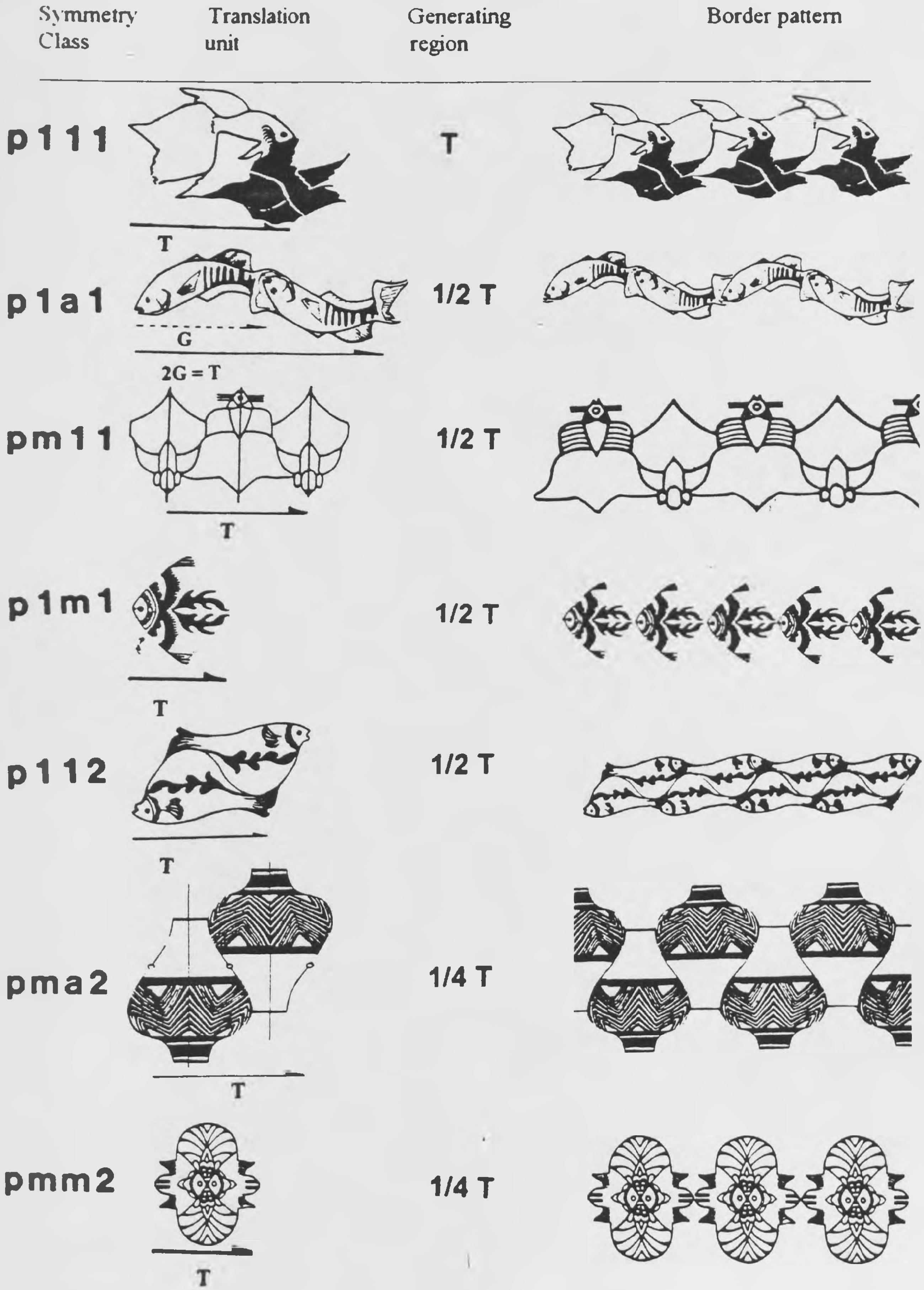


Figure 4.18 Recognition chart for the seven classes of border tilings.

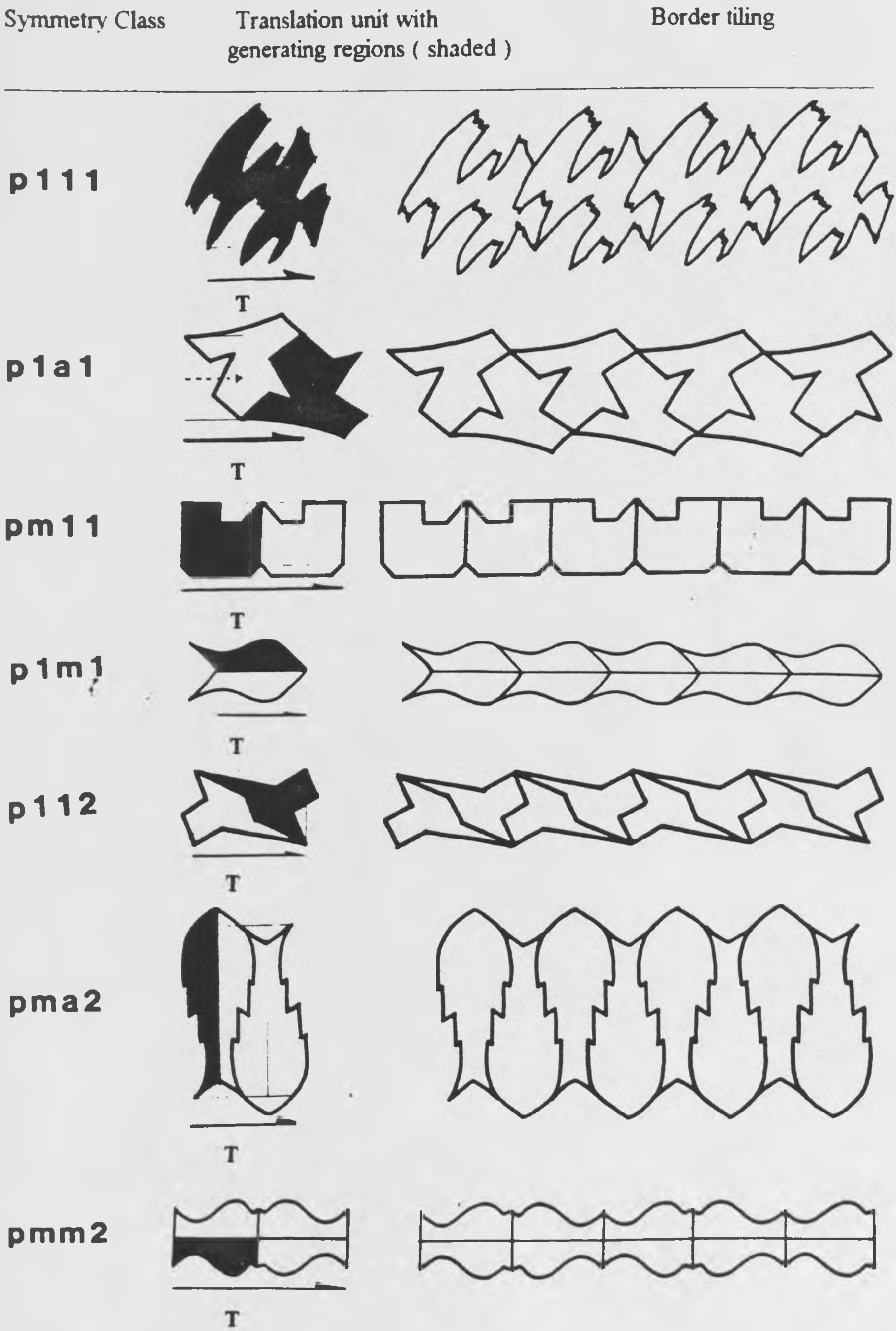
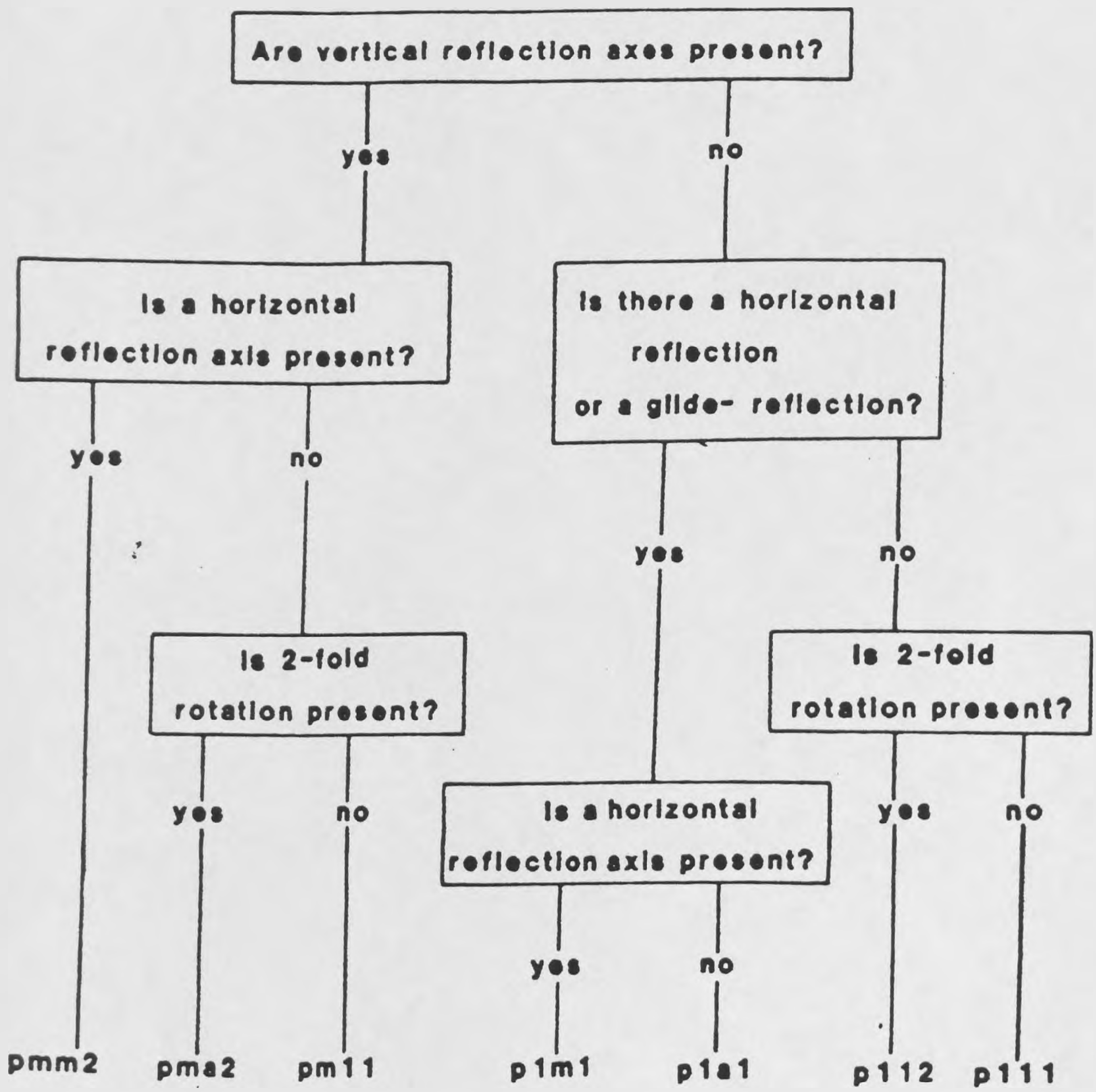


Figure 4.19 Flow-diagram to aid the identification of a border pattern or tiling's symmetry class.



5 THE CLASSIFICATION AND CONSTRUCTION OF PRIMARY ALL-OVER PATTERNS AND PERIODIC PLANE TILINGS

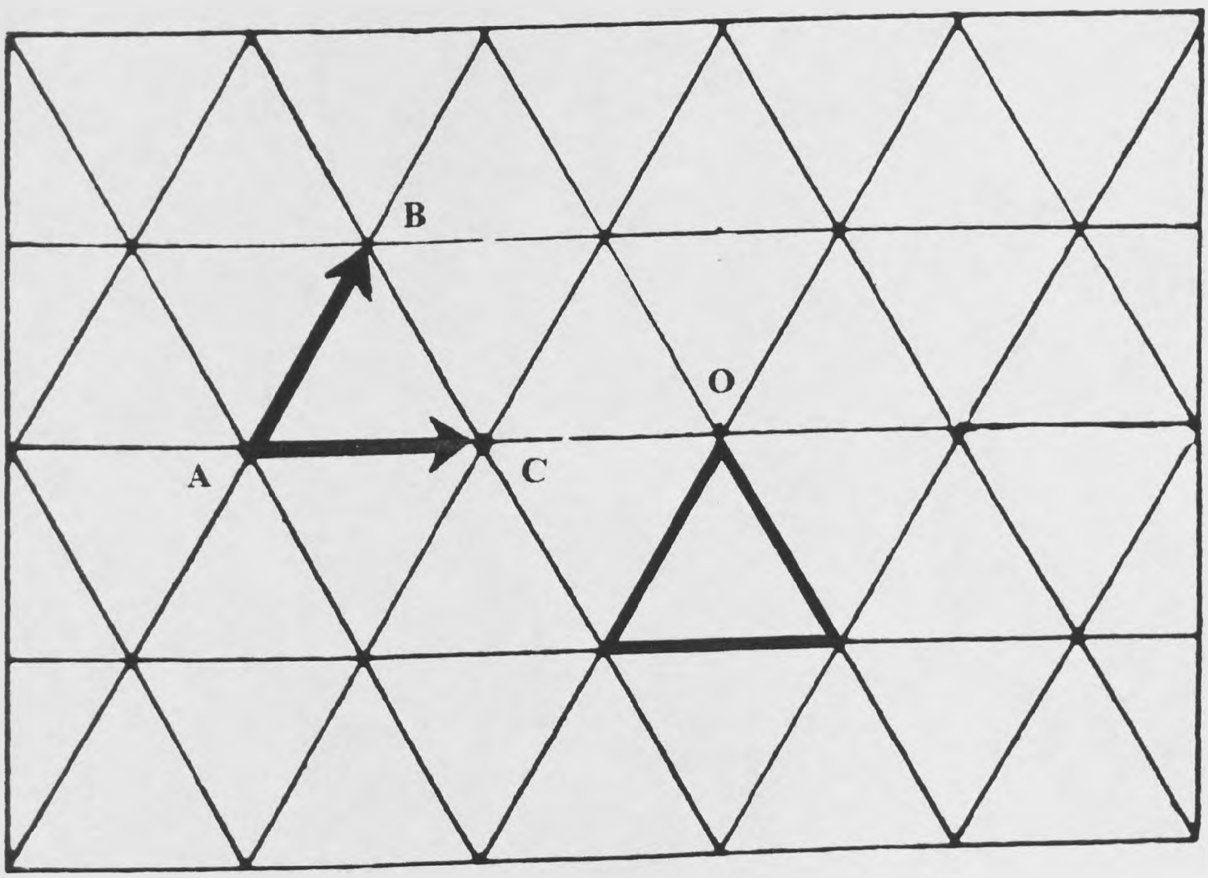
5.1 Introduction

As mentioned previously patterns may be grouped according to the number of directions in which they admit translations. It should be apparent from the previous chapter that translational symmetry acts on border patterns or tilings in one direction only. With all-over patterns translation is not confined to a single direction but rather it extends to two independent non-parallel directions. Schattschneider commented:

> "... given a periodic border design generated by a translation T1, if we take a second translation T2 whose vector is not parallel to T1 and repeatedly apply T2 and T1 to the border, then a 'wallpaper' pattern is created in which the motif repeats regularly in two directions, and the design extends throughout the plane. Such a design is called a two-dimensional (or planar) periodic design, generated by T1 and -T1 and T2 ..."[63]

In addition to exhibiting translation in two independent directions, one or more than one of the other three symmetry operations may be used in the construction of all-over patterns. Figure 5.1 shows a two dimensional surface filled with a series of equilateral triangles. There are several ways by which the equilateral triangle can come into coincidence with itself. If moved by a translation distance denoted by vector AB or vector AC, the triangle will come

Figure 5.1 Possible symmetry operations in a two-dimensional surface.



into coincidence with itself. If the tiling is rotated through 60 degrees about point O, it is mapped onto itself by six-fold rotation. Likewise, if reflected across reflection axes AB or BC or CA the design can also be generated.

All-over patterns may thus be considered as those patterns in which a motif (or motifs) is translated in two independent non-parallel directions across the plane. When combined with one or more of the other symmetry operations a total of seventeen (and only seventeen) classes of all-over pattern classes are possible. Proof for the existence of only seventeen distinct all-over pattern classes was provided by Schwarzenberger [64], Martin [65], Weyl [66], Coxeter [67] and Jaswon [68].

With the above considerations in mind the objective of this chapter is to present an explanation of the geometrical principles governing the classification and construction of all-over patterns and tilings.

5.2 An Explanation of the Relevant Notation

In addition to combinations of the four symmetry operations, a further structural element is present in all-over patterns and tilings: a framework of corresponding points which forms a regular lattice. Choosing any arbitrary point in a motif, an infinite set of images of that point will be obtained following translation. These corresponding points will be located at identical positions on motifs and will thus form a regular lattice of points. The construction of a plane lattice (or plane net) was described by Woods as follows:

> "Start with a chain of points with interval a in some straight line, and ... make each of these points a point of another chain, of interval b, making an angle Θ, say, with the first chain, we thus obtain an array of points which is such that any translation equal to a multiple of a in the direction of the first chain, or to a multiple of b in the direction of others, moves the figure into an equivalent position. Such an array is called a net of points." [69]

An illustrative example is provided by Figure 5.2. When translated in direction T1 arbitrary point P1 will thus produce a linear row of equidistant points $P1_{(1)}$, $P2_{(1)}$, $P3_{(1)}$ etc. If each point in this row is translated in the direction of T2 a two-dimensional plane lattice of points can thus be produced.

The lattice points of a given pattern may be connected to produce lattice units (generally referred to as unit cells) of the same shape, size and content. As indicated by Figure 5.3, various unit cells may be produced from a given lattice structure dependent on how the points in the lattice are joined. In each of the illustrations shown in Figure 5.3, translation of the unit cell in two independent non-parallel directions will produce the full lattice structure.

Figure 5.2 Construction of a two-dimensional lattice.

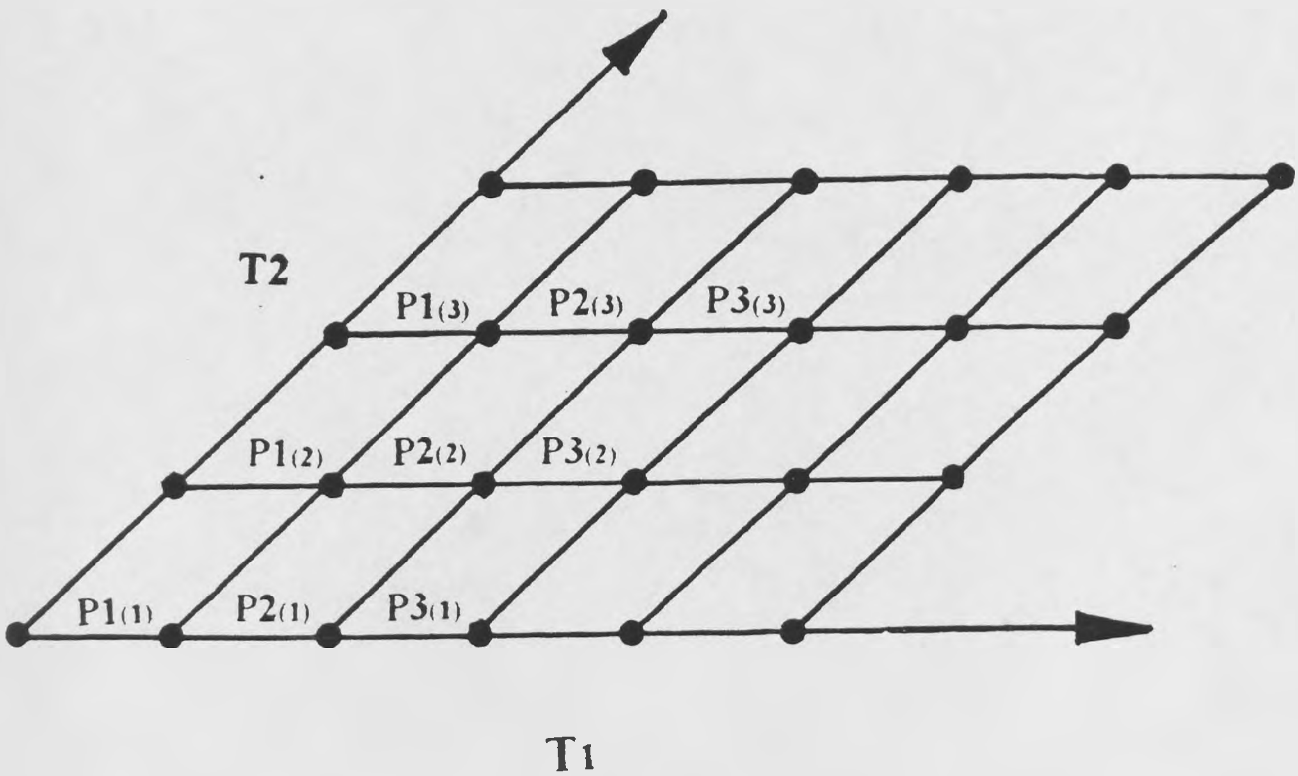
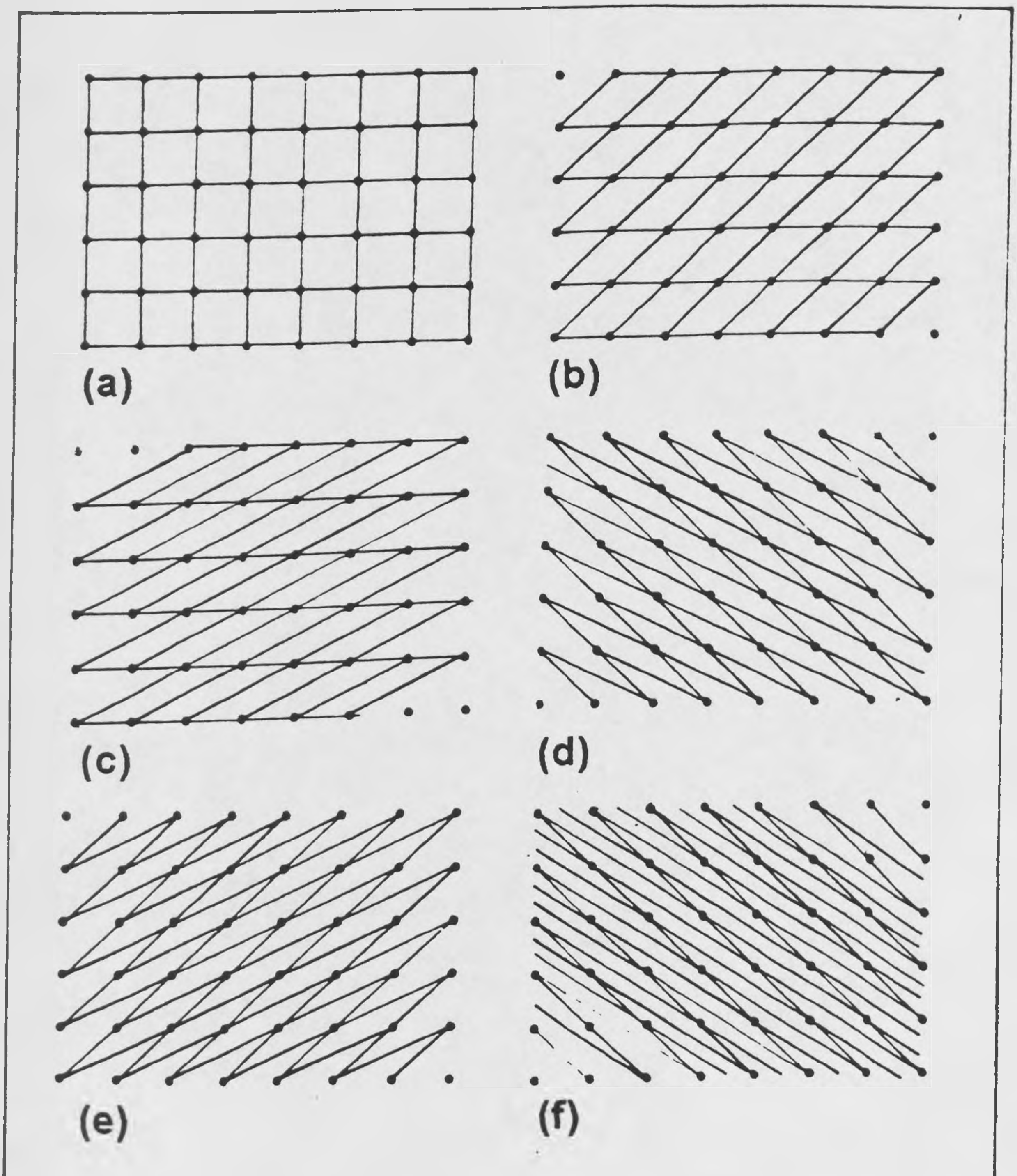


Figure 5.3 Different types of unit cell from the same lattice types.

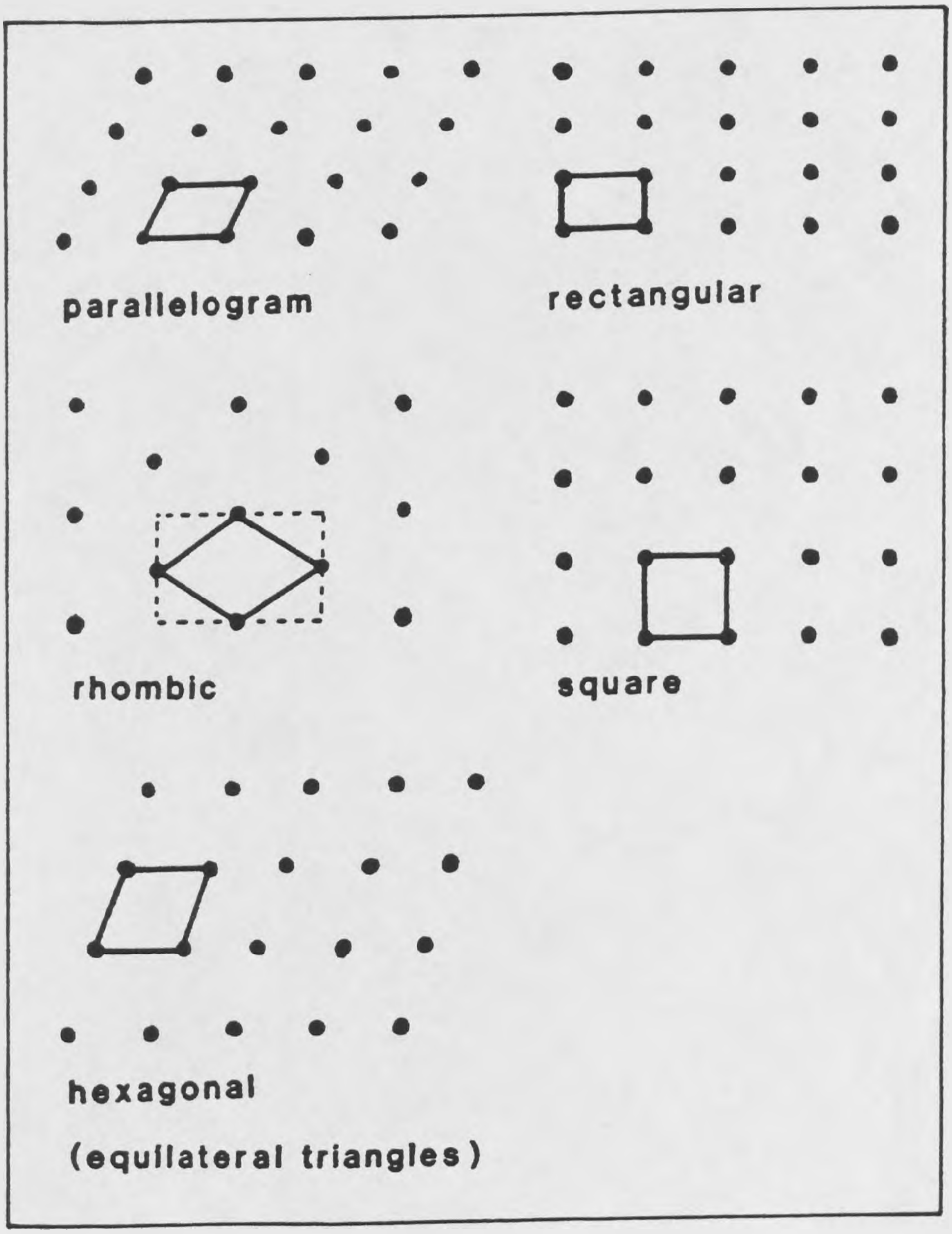


There are five distinct lattice types: parallelogram, rectangular, rhombic, square and hexagonal (the unit cell associated with this latter lattice type is a rhombus consisting of two equilateral triangles). The five lattice types (known as Bravais lattices) together with their corresponding unit cells are shown in Figure 5.4. It can be seen that the rhombic lattice unit, unlike the other lattice units, is centred and has a diamond shaped cell held within a rectangle (denoted by dashed lines) so that reflection axes can be positioned at right angles to the sides of the enlarged cell, which holds one full repeating unit (within the diamond shaped cell) and a quarter of a repeating unit at each of the enlarged cell corners.

Figure 5.5 shows schematic illustrations of unit cells for each of the seventeen all-over pattern classes, together with an indication of the relevant symmetry operations characteristic of each. Associated with each class is the widely accepted notation of the International Crystallographic Union. A shortened form of the notation (in brackets) is also provided in Figure 5.5. Table 5.1 lists the unit cell type conventionally associated with each of the seventeen pattern classes.

The notation for each of the seventeen all-over pattern classes consists of four symbols pxyz or cxyz, which identify the conventionally chosen unit cell, the highest order of rotation and other fundamental symmetries. The first symbol of the four symbol notation, either a letter p or a letter c, indicates whether the lattice cell is primitive or centred. Primitive cells which are

Figure 5.4 The five lattice types.



Source: Derived from D. Schattschneider, 'The Plane Symmetry Groups: Their Recognition and Notation', American Mathematical Monthly, vol.85, 1978, pp437-450.

Figure 5.5 Schematic illustrations of unit cells for each of the seventeen classes of all-over patterns.

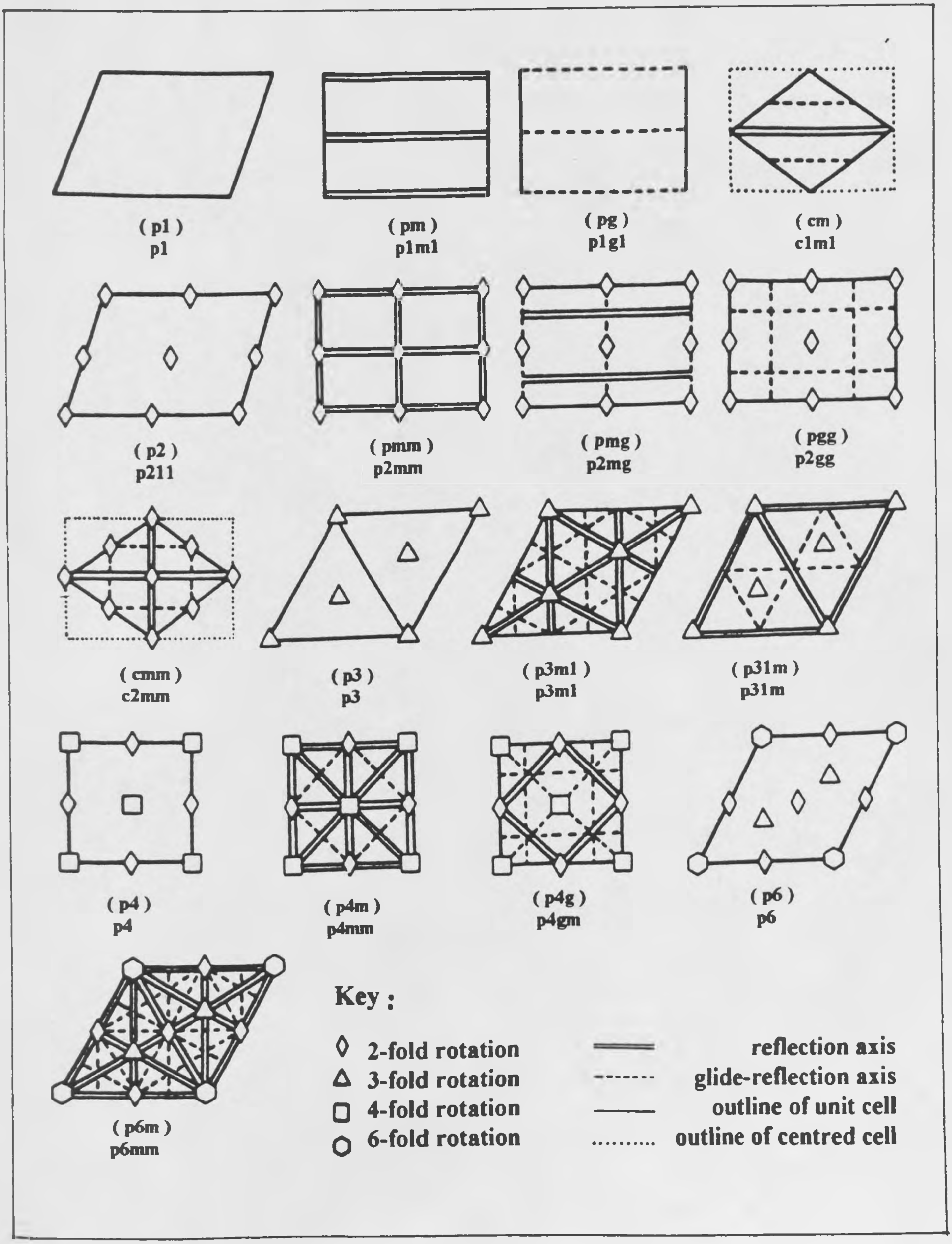


Table 5.1 Lattice Types for the Seventeen Classes of All-over Patterns.

Full International Notation	Lattice Type
p1	parallelogram
p1m1	rectangular
p1g1	rectangular
c1m1	rhombic
p211	parallelogram
p2mm	rectangular
p2mg	rectangular
p2gg	rectangular
c2mm	rhombic
p3	hexagonal
p3m1	hexagonal
p31m	hexagonal
p4	square
p4mm	square
p4gm	square
p6	hexagonal
p6mm	hexagonal

present in fifteen of the seventeen all-over pattern classes, contain the minimum area of the pattern which may be used to generate the full pattern using translational symmetry only. In the remaining two all-over pattern classes, the lattice cell is of the rhombic variety and is centred; each of these two all-over pattern classes is therefore prefaced with the letter c.

The second symbol x of the four symbol notation denotes the highest order of rotation in the pattern. As pointed out by Schattschneider [70] only rotations of orders two (180 degrees), three (120 degrees), four (90 degrees), or six (60 degrees) may generate all-over patterns. Centres of five-fold, seven-fold or higher orders of rotation are geometrically not admissible in the context of all-over patterns. This restriction, which is often referred to as the "crystallographic restriction", is discussed further by Stevens [71]. Where no rotation is present in an all-over pattern, x equals 1.

The third symbol y denotes a symmetry axis normal to the x-axis of the cell (i.e. at right angles to the left side of the cell). The letter m (for mirror) indicates an axis of reflection. The letter g (for glide) indicates the presence of a glide-reflection axis, and 1 indicates that no reflections or glide-reflections are normal to the x-axis.

The fourth symbol z indicates the presence of a symmetry axis at angle α to the x axis, with α dependent on x, the highest order of rotation. The angle α equals 60 degrees for x equals 3 or 6; α equals 45 degrees for x equals 4;

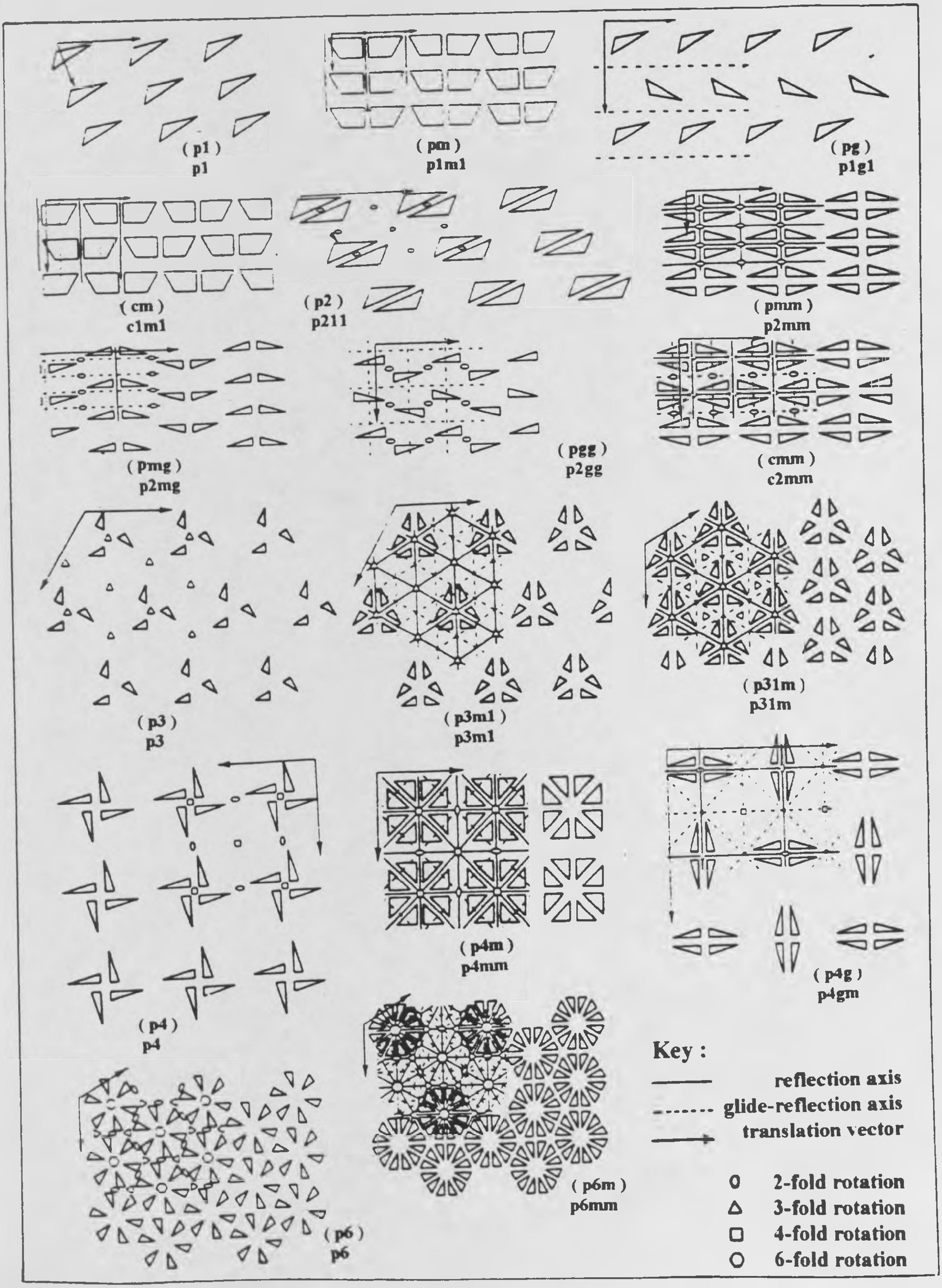
α equals 180 degrees for x equals 1 or 2. The letter m indicates that the relevant symmetry axis is an axis of reflection. The absence of a symbol in the third and fourth positions indicates that the pattern admits no reflections or glide-reflections. Schematic illustrations for each of the seventeen classes of primary all-over patterns, together with the relevant notation are provided in Figure 5.6. Each pattern class is further described and illustrated below, under headings relating to orders of rotation. As mentioned previously, all-over tilings (sometimes referred to as periodic plane tilings) are governed by the same symmetry rules as patterns. The construction of a tiling in each class is explained and illustrated.

5.3 The Classification and Construction of All-over Patterns and Tilings Without Rotational Symmetry

5.3.1 Class p1 patterns and tilings

From the viewpoint of geometrical symmetry, class p1 all-over patterns are the most straightforward in terms of analysis, classification and construction. The unit cell conventionally chosen is of the parallelogram lattice type, and the pattern does not exhibit reflections or glide reflections. In view of the fact that the highest order of rotation is one (i.e. the pattern must be rotated through 360 degrees for individual elements to coincide with themselves) the pattern is considered to have no rotational properties. The unit cell and the fundamental region are of equivalent area and the pattern is generally generated by translations of a c1 motif in two independent non-parallel

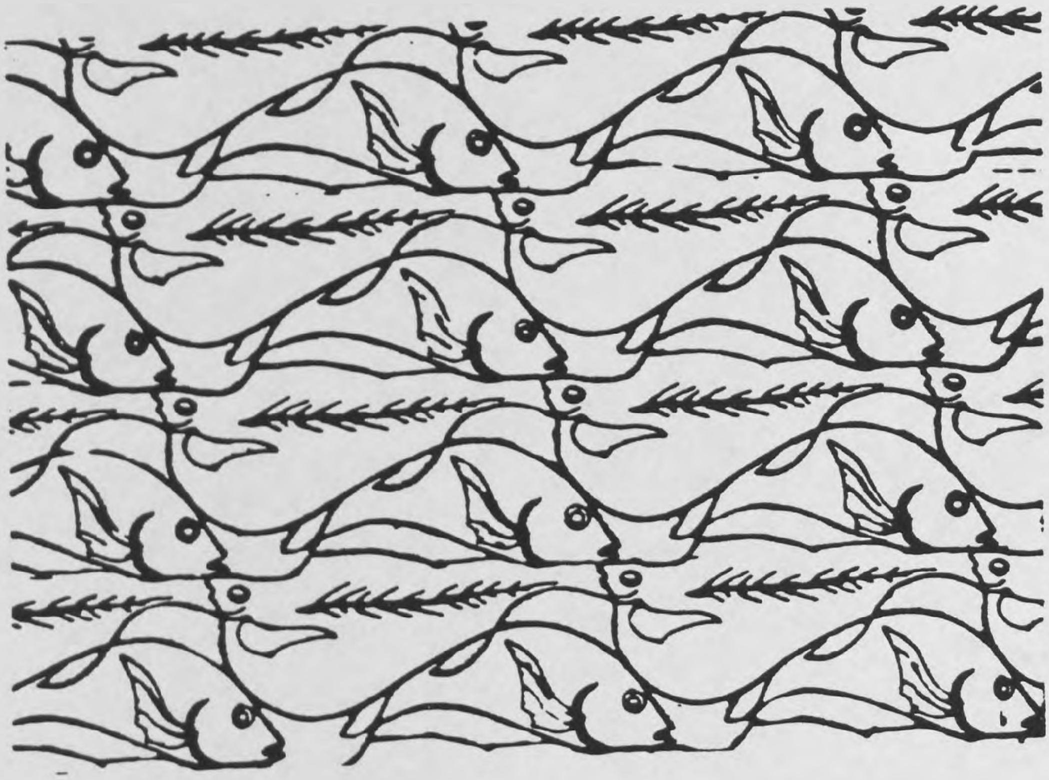
Figure 5.6 Schematic illustrations for each of the seventeen classes of all-over patterns.



directions. Relevant illustrations are shown in Figure 5.7. Figure 5.8a shows the conventionally chosen unit cell (a parallelogram), Figure 5.8b shows a schematic illustration of the pattern class, Figure 5.8c shows a notional translational unit and Figure 5.8e shows a p1 all-over pattern comprised of a fish shape and a bird shape. Unit cells are constructed by choosing four corresponding points in the pattern; two different unit cells are indicated in Figure 5.8e each containing the necessary elements to generate the whole pattern through translation in two independent non-parallel directions. Figure 5.8d shows four different generating regions for the pattern, each of the same area.

In the construction of a p1 tiling take a parallelogram or rectangular lattice type unit cell such as ABCD (Figure 5.9a). Join point A to point B with a non-intersecting line or series of lines as shown. Connect point A to point D, again using a non-intersecting line or series of lines as shown. Translate the line or lines joining A and B along the vector, denoted by T1, to position DC, and translate the line or lines joining A to D in the direction of the vector denoted by T2 to position BC. The generating unit is thus complete and can be translated in two independent directions to complete the pattern (Figure 5.9b).

Figure 5.7 Class p1 all-over patterns.



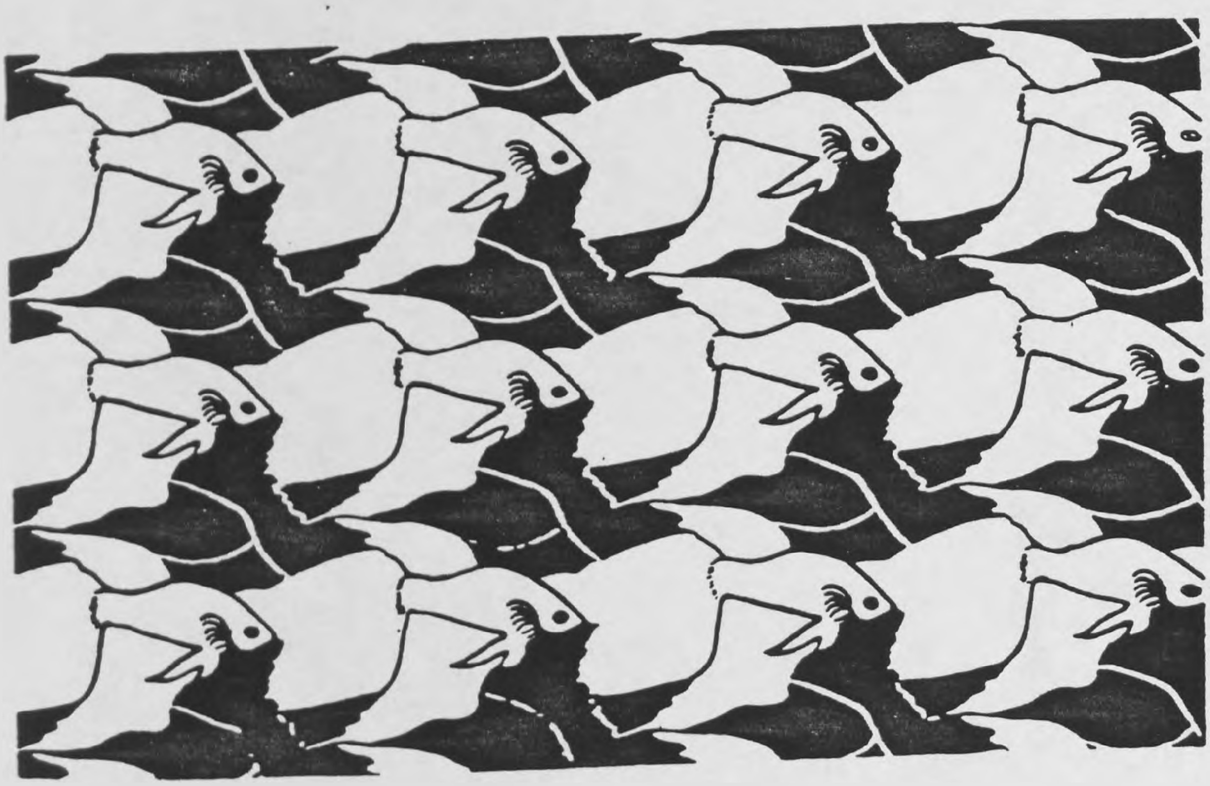


Figure 5.8 Lattice unit, schematic illustration and translation units for class p1 all-over patterns.

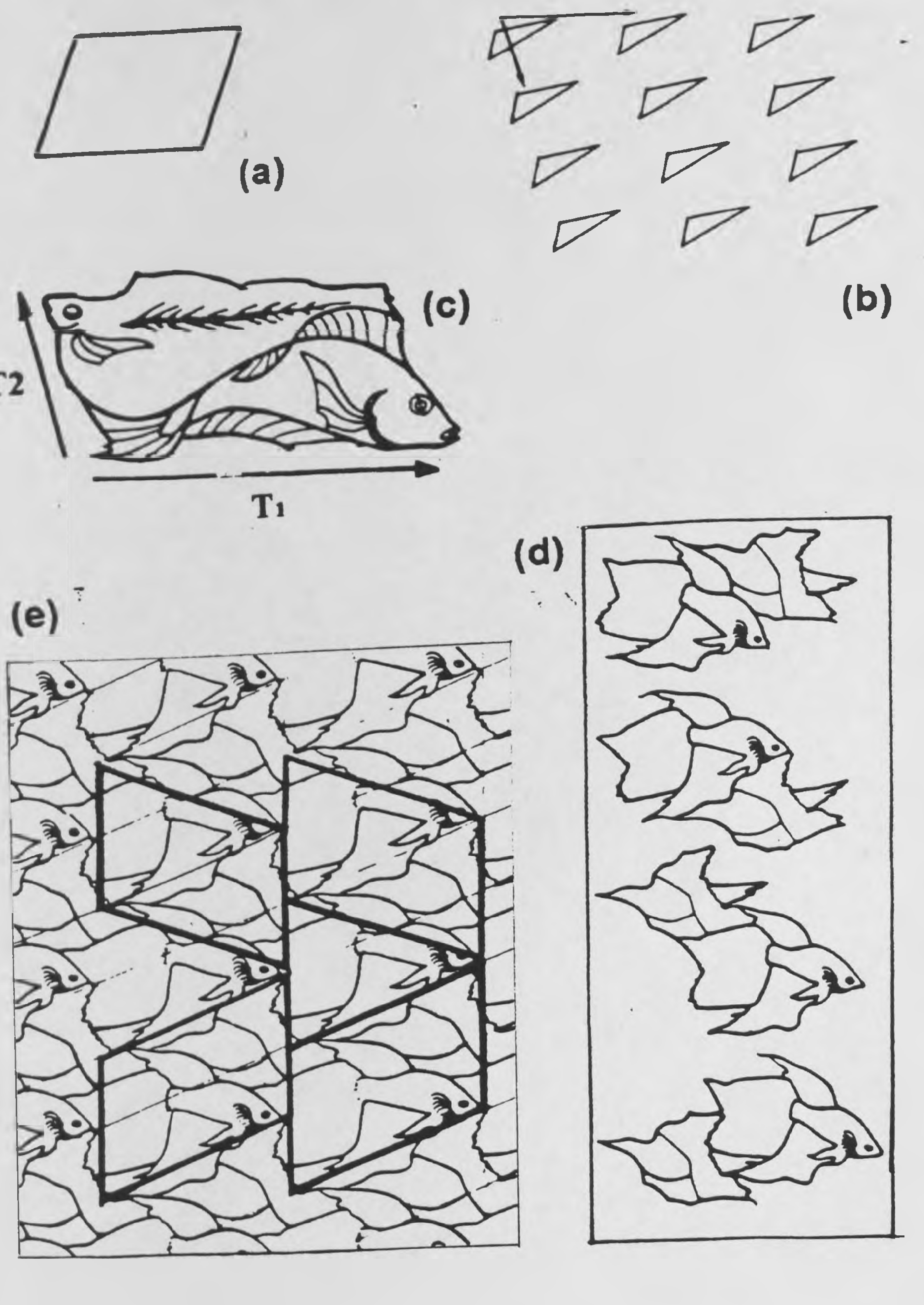
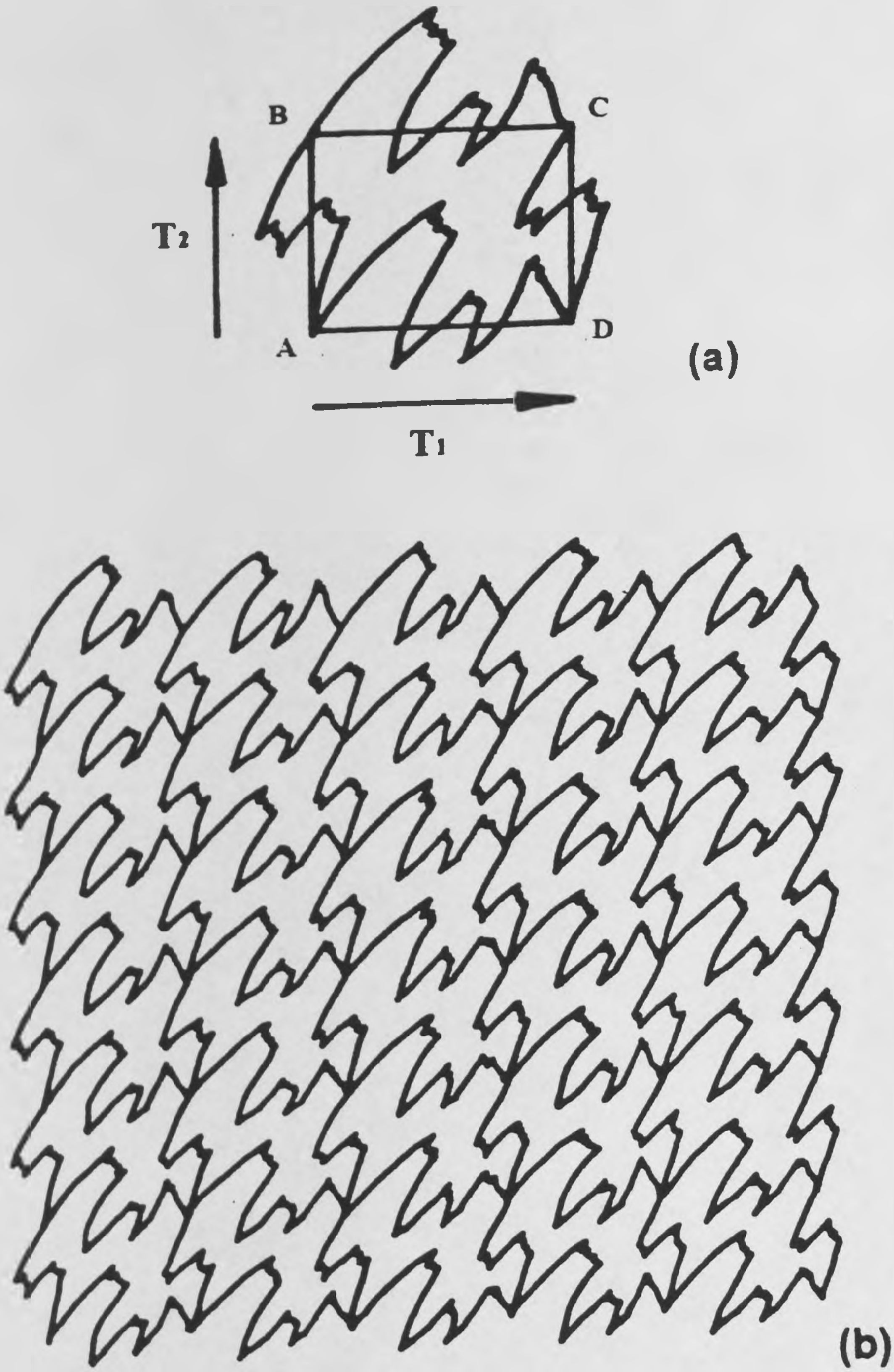


Figure 5.9a and b The construction of a p1 periodic plane tiling.



5.3.2 Class p1g1 (pg) patterns and tilings

Class p1g1 all-over patterns are generally generated by two parallel glide-reflections of a c1 motif. The corners of the unit cell (which is a rectangular lattice type cell) fall on glide-reflection axes. The highest rotational order is one and the fundamental region is half the area of the unit cell. Examples of class p1g1 all-over patterns are shown in Figure 5.10. Schematic illustrations of the lattice type and the pattern class are shown in Figures 5.11a and b respectively. Figures 5.11c and d show translation units; in each case the translation vector is twice the length of the glide vector.

In the construction of a p1g1 tiling, take two adjoining rectangles ABCD and CDEF, having M and N as mid-points of sides AB and CD respectively (Figure 5.12a). Join point A to B and point B to C with non-intersecting lines as shown. Join AD with a boundary line identical to the line joining B to C. The constructed unit can then be translated by distance G and reflected across the two-way (glide) reflection line MNO, with all elements above reflecting downward and all lower elements reflecting upward. The point A will thus have its image at C, the point B its image at D, the point C its image at F and the point D its image at E. The translation unit (which is twice the area of the generating unit) is thus complete and when translated in two independent non-parallel directions will produce the tiling shown (in diminished form) in Figure 5.12c. A further example of p1g1 tiling construction is shown in Figure 5.12b where boundary lines AE and EF are constructed. The line joining A to E is translated by distance T1 to produce

Figure 5.10 Class p1g1 primary all-over patterns.

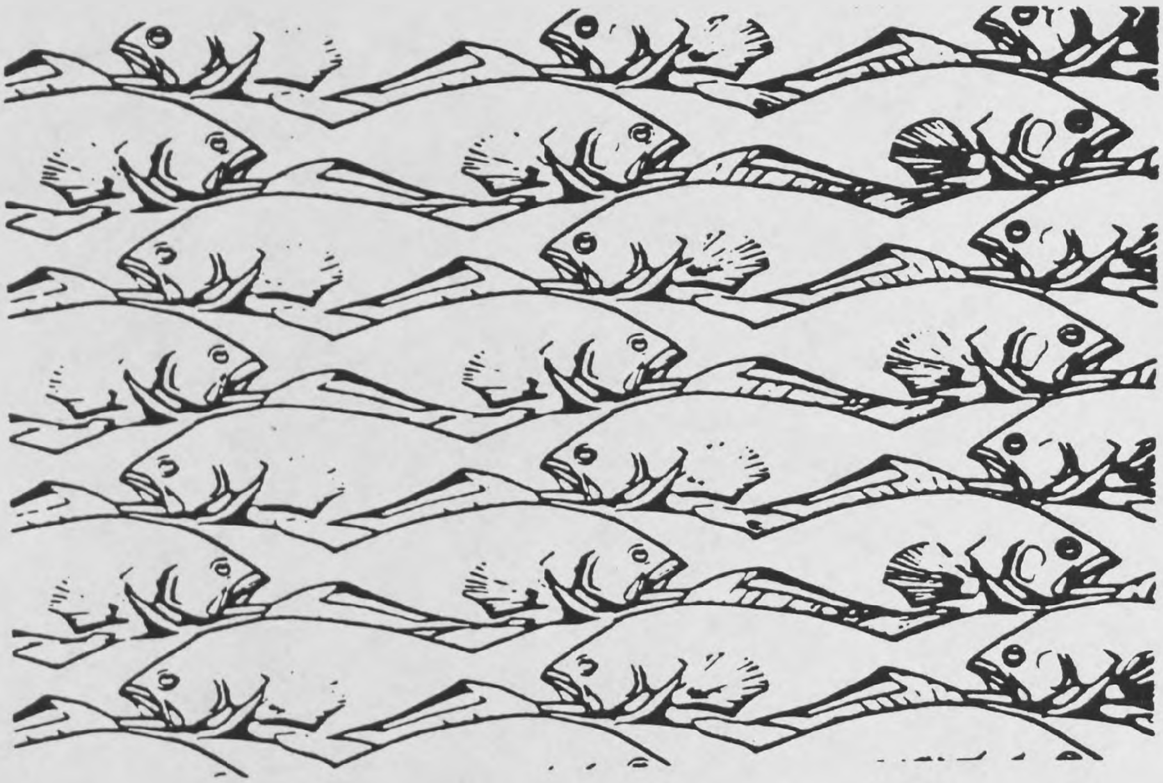


Figure 5.11 Lattice unit, schematic illustration and translation units for p1g1 all-over patterns.

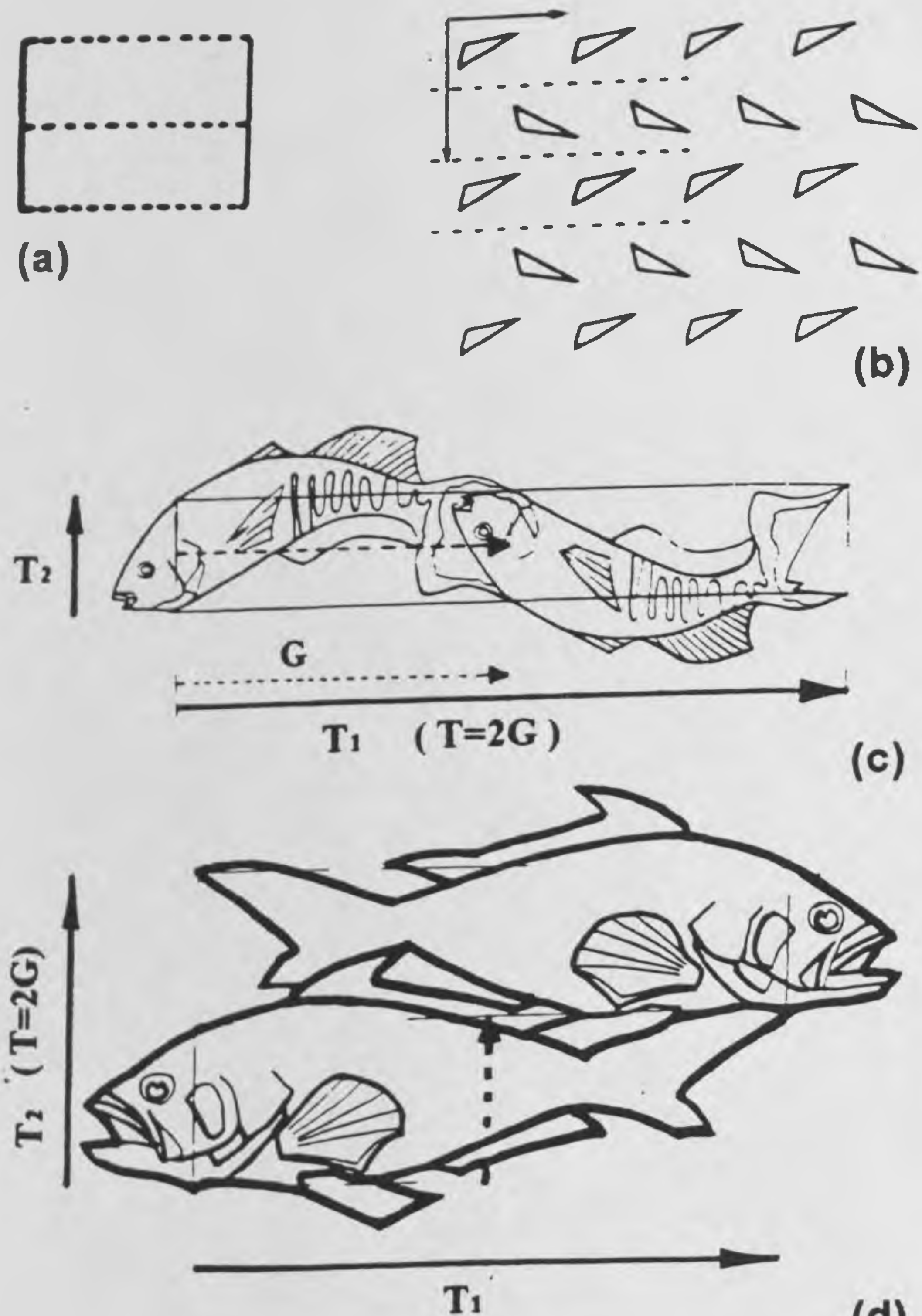
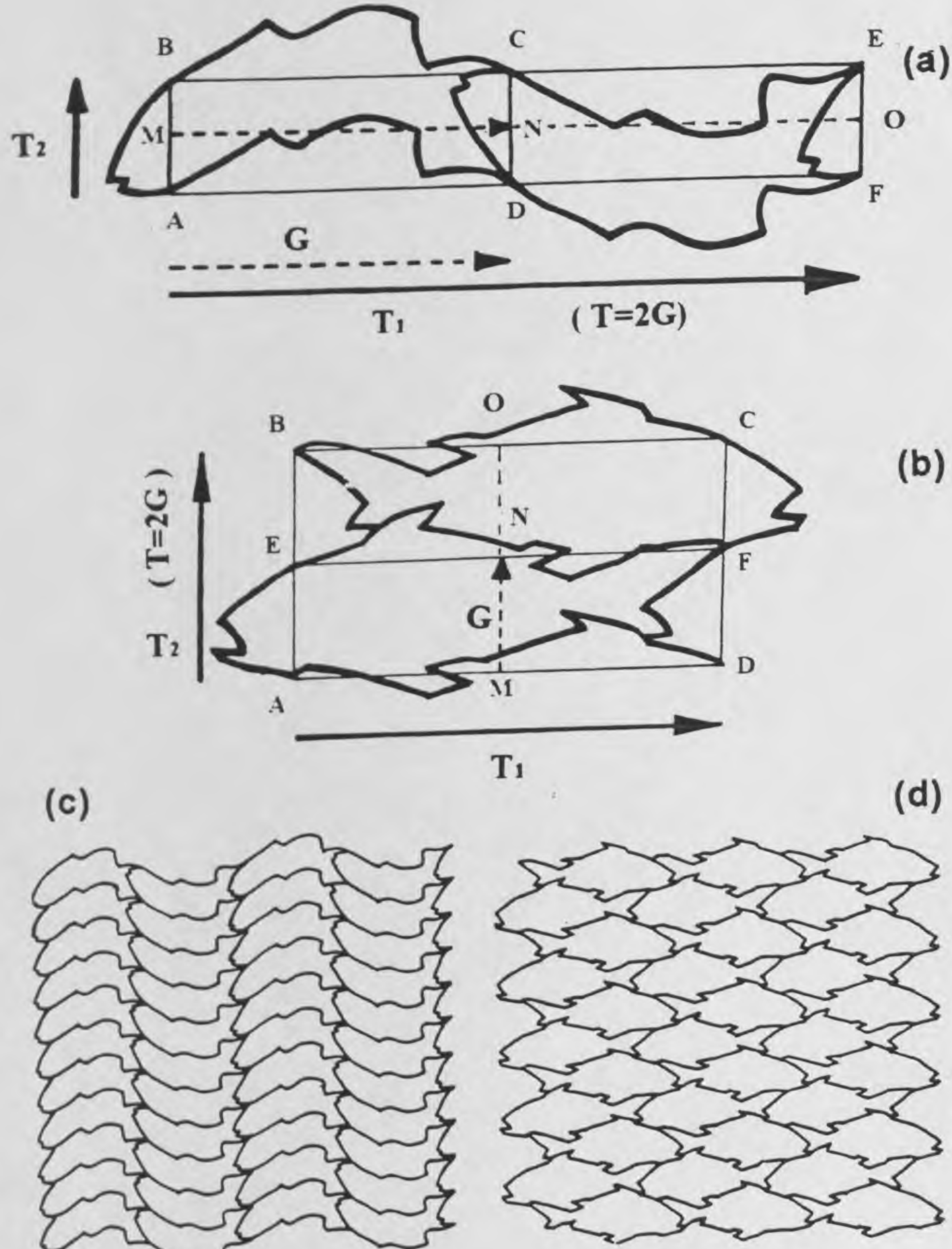


Figure 5.12 The construction of p1g1 periodic plane tilings.



a boundary line connecting D to F. The unit so far constructed is translated by distance G (between points M and N) and then two-way (glide) reflected in line NO thus producing a unit which when translated in two independent directions will produce the tiling shown in Figure 5.12d.

5.3.3 Class p1m1 (pm) patterns and tilings

Class p1m1 all-over patterns are characterised by two alternating and parallel reflection axes and have rectangular lattice units. The unit cell corners fall on reflection axes and the pattern is thus generated by two parallel reflections and subsequent translation in two independent non-parallel directions. In the case of this pattern class the fundamental region is half the area of the unit cell and is bounded on opposite sides by reflection axes. An example from this pattern class is shown in Figure 5.13. The unit cell (with double lines indicating the position of reflection axes) and a schematic illustration of the pattern class are shown in Figures 5.14a and b respectively. Figure 5.14c shows a translation unit with reflection axes m1 and m2, and Figure 5.14d shows the unit translated to produce the pattern.

To construct a p1m1 tiling, take rectangle ABCD with mid-points E and F to sides BC and AD respectively (Figure 5.15a). Join B to E with a non-intersecting line as shown, and reflect this line to produce boundary line EC. Join A to F and F to D in imitation of boundary lines BE and EC respectively.

Figure 5.13 A class p1m1 primary all-over pattern.

Figure 5.14 Lattice unit, schematic illustration and translation unit for a p1m1 all-over pattern.

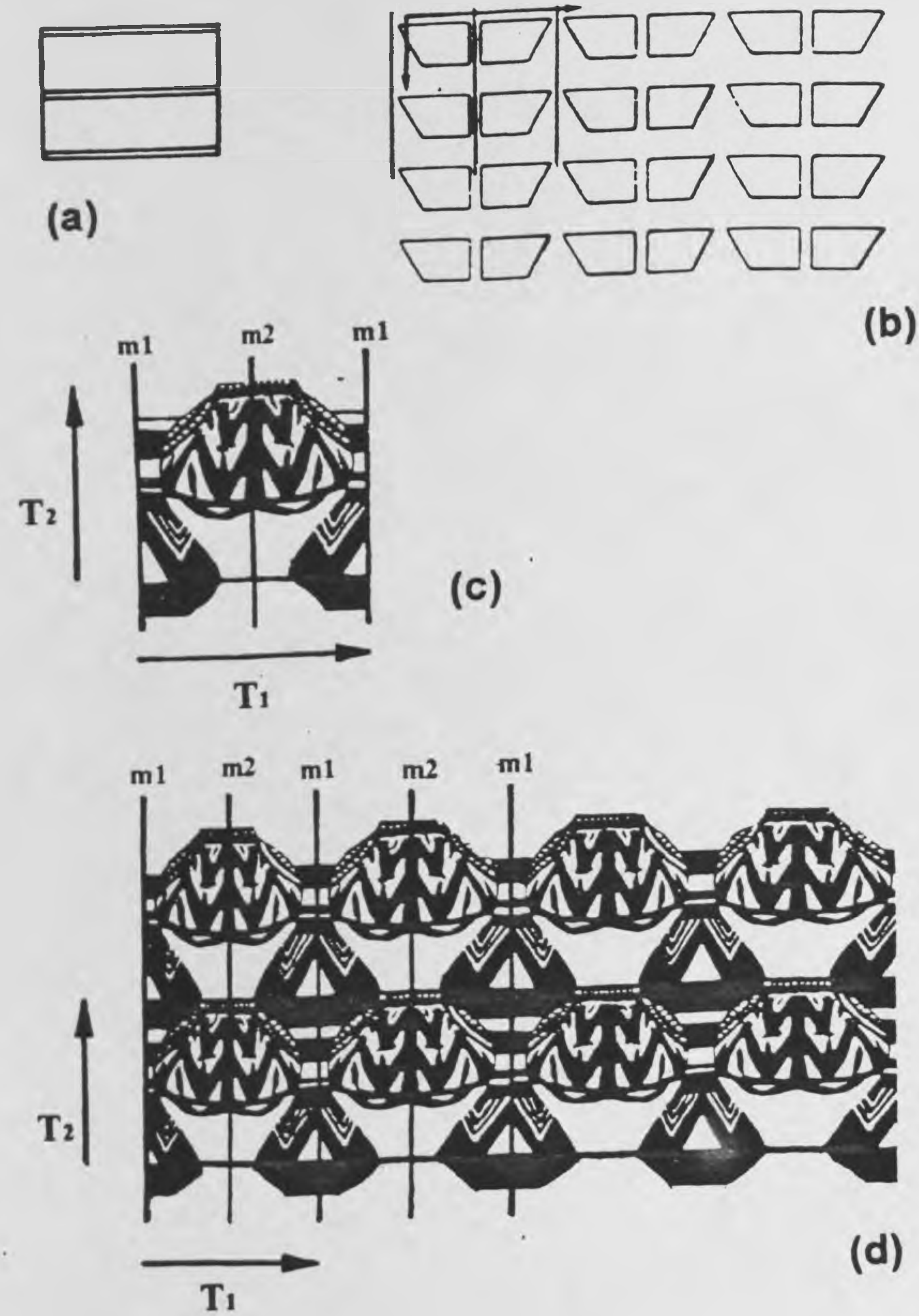
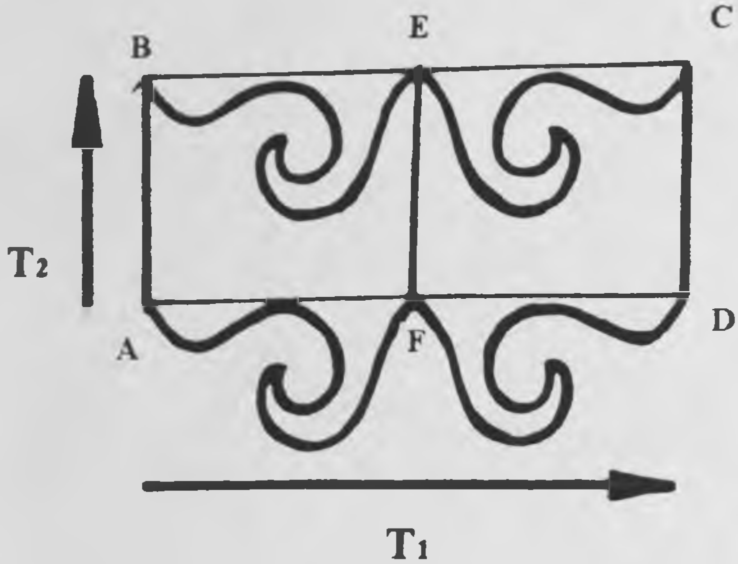
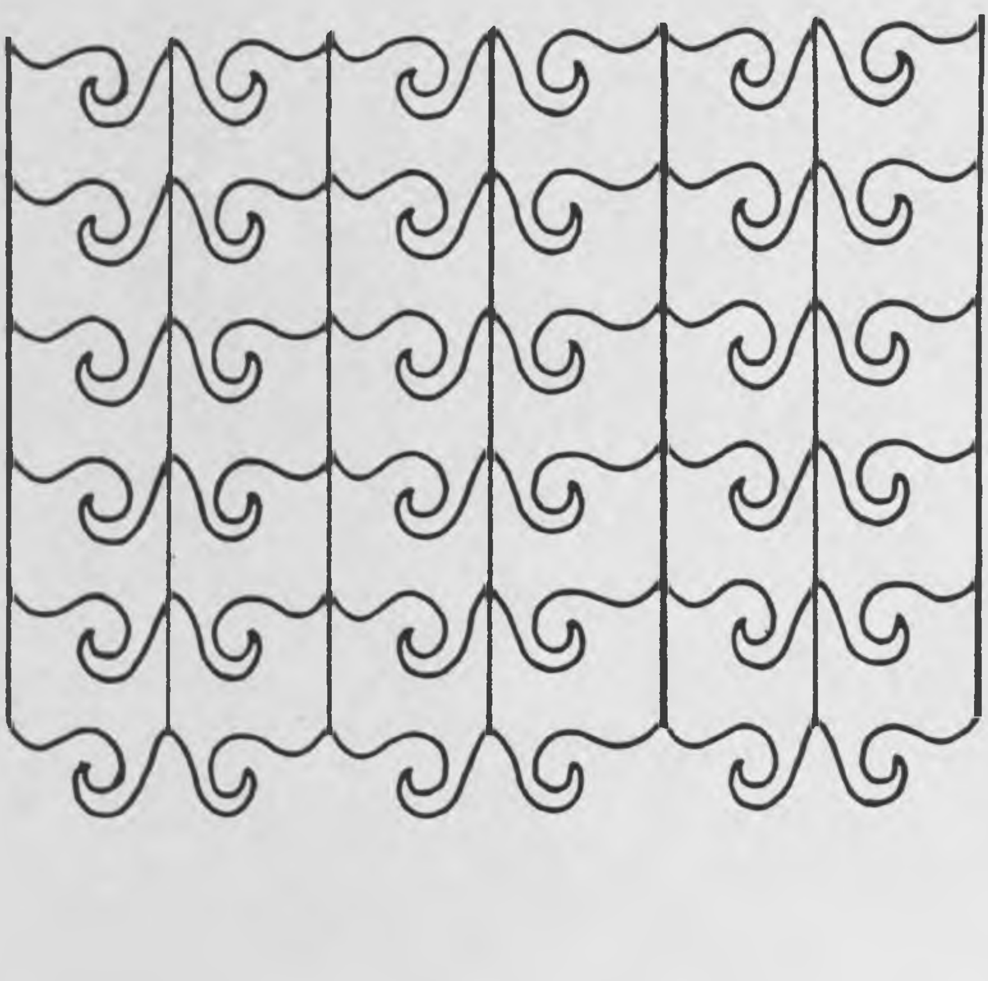


Figure 5.15 The construction of a p1m1 periodic plane tiling.



(a)



(b)

The translation unit is thus complete and can be used to produce the pattern shown in Figure 5.15b.

5.3.4 Class c1m1 (cm) patterns and tilings

Class c1m1 all-over patterns are characterised by a unit cell of the rhombic lattice type which, as mentioned previously, contains a diamond-shaped cell held within a larger rectangle. The pattern is generated by a reflection, at right angles to the enlarged cell, and by a parallel glide-reflection. Reflection axes therefore alternate with glide-reflection axes. The enlarged cell contains two repeating units, comprised of quarter units at each of the enlarged cell corners and one full repeating unit within the diamond shape. An example from this pattern class is provided by Figure 5.16.

Figures 5.17a and b show the rhombic lattice unit and a schematic illustration of the pattern class. Figure 5.17c shows the enlarged unit cell of a pattern with two repeating units: one full repeating unit within the diamond shape and quarter units at each of the enlarged cell corners. Alternating reflection axes (denoted by the letter m) and glide-reflection axes (denoted by the letter g) are also shown. Figure 5.17d shows a section of the pattern.

In the construction of a c1m1 tiling, take a diamond shaped cell ABCD held within a larger rectangle (Figure 5.18a). Join points A and B with a non-intersecting line as shown. This boundary shape is then reflected to join B to

Figure 5.16 A class c1m1 primary all-over pattern.

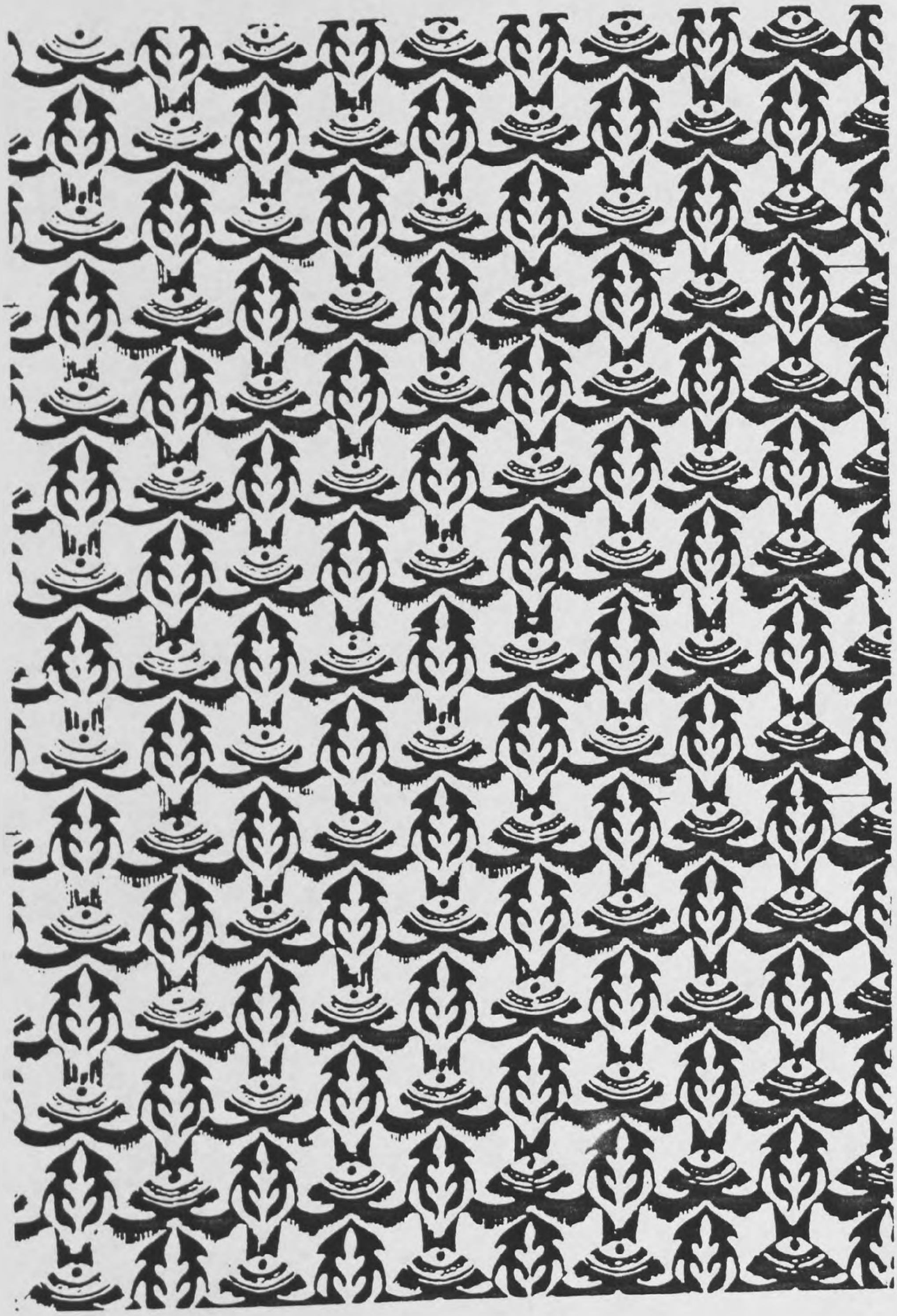


Figure 5.17 Lattice unit, schematic illustration and translation unit for a c1m1 all-over pattern.

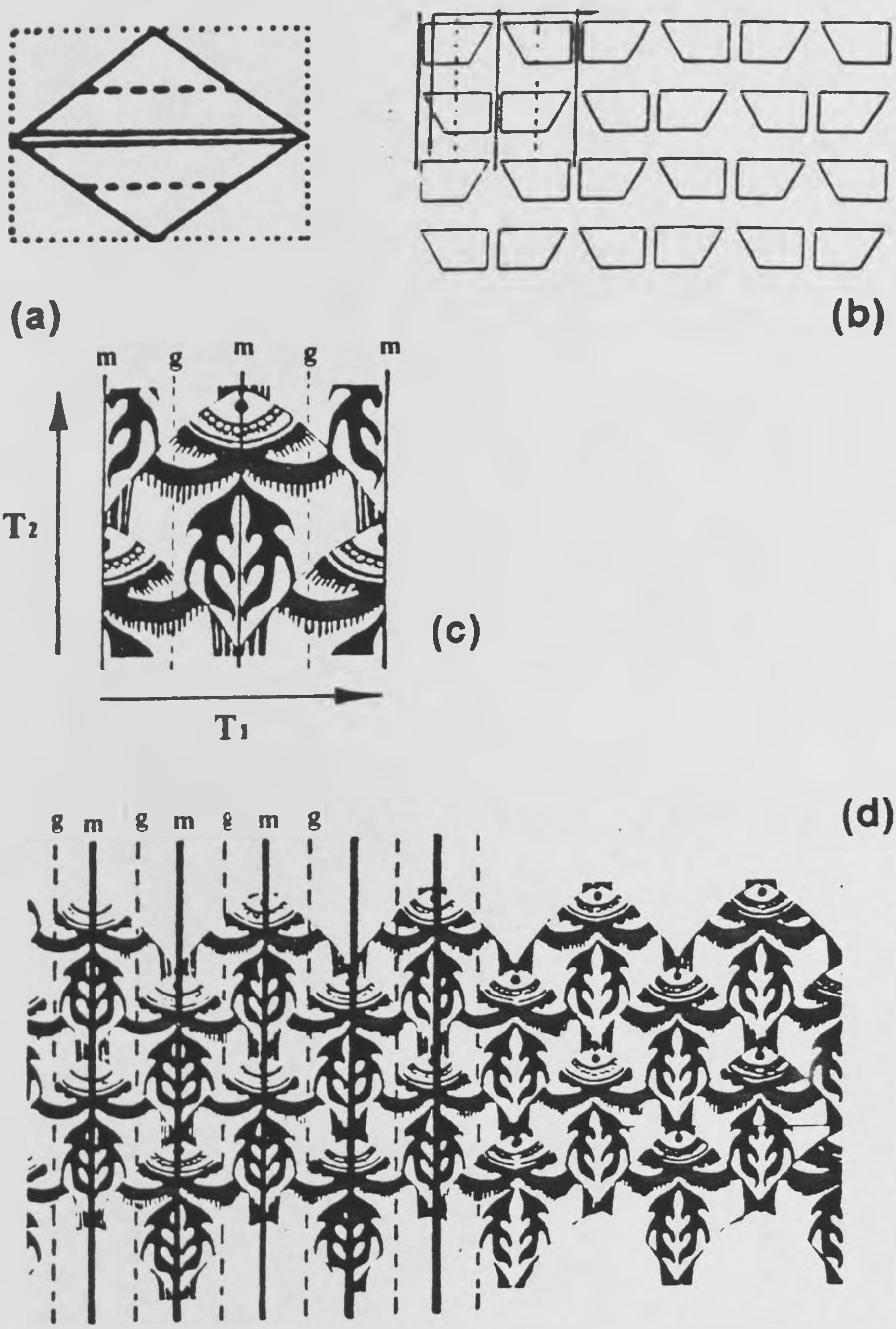
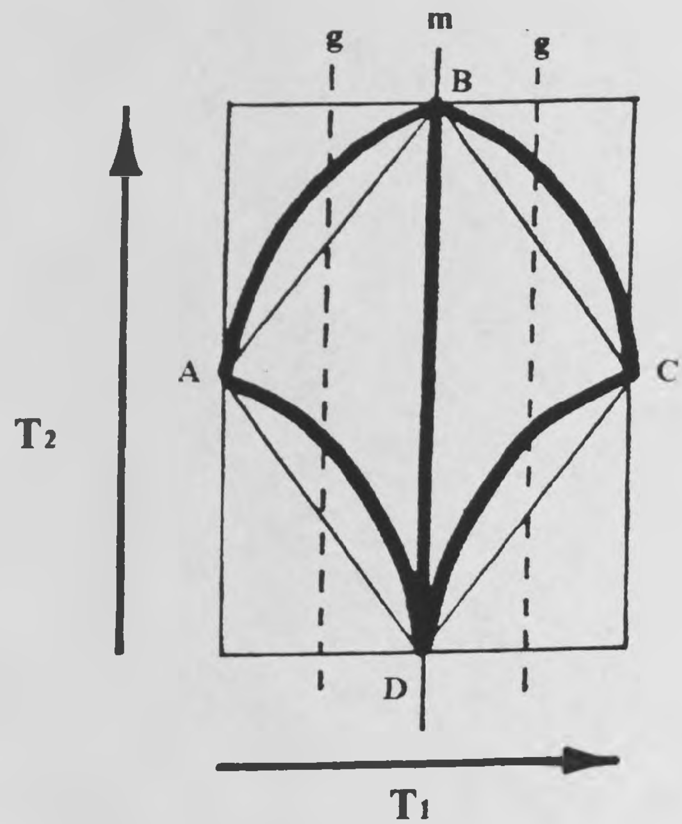
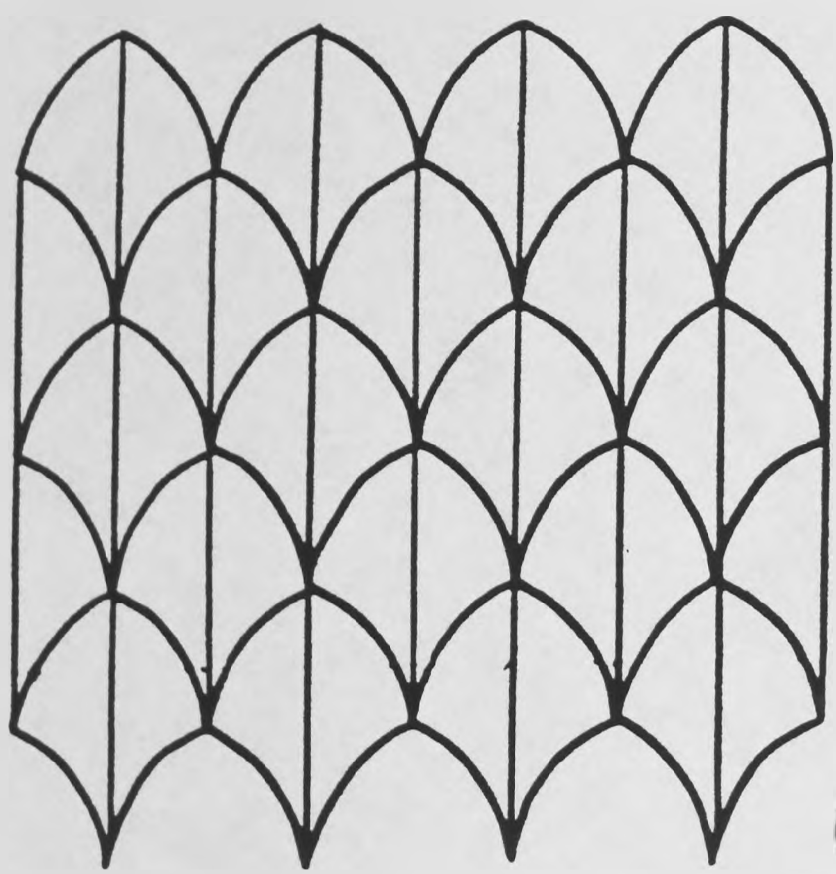


Figure 5.18 The construction of a c1m1 periodic plane tiling.



(a)



(b)

C. Using identical curves A is joined to D and D to C. The enlarged unit cell may be translated in two non-parallel independent directions to produce the pattern shown in Figure 5.18b.

5.4 The Classification and Construction of All-over Patterns and Tilings With Two-fold Rotational Symmetry

Two-fold rotational symmetry combined with other symmetry operations is evident in a total of five all-over pattern classes: p211, p2gg, p2mg, p2mm and c2mm. Each class is examined briefly below.

5.4.1 Class p211 (p2) patterns and tilings

Class p211 all-over patterns contain repetitions of four different two-fold rotation centres, with each similar rotation centre having the same orientation. A parallelogram lattice type forms the unit cell which has corners on similar two-fold rotational centres and is twice the area of the generating unit (or fundamental region). Further two-fold rotational centres are located in the centre of the unit cell and on each of its sides. An example of a pattern from this pattern class is provided in Figure 5.19. The unit cell and a schematic illustration of the pattern class are shown in Figures 5.20a and b respectively. Figure 5.20c shows the primitive cell of a p211 pattern together with constituent centres of two-fold rotation and directions of translation.

Figure 5.19 A class p211 primary all-over pattern.

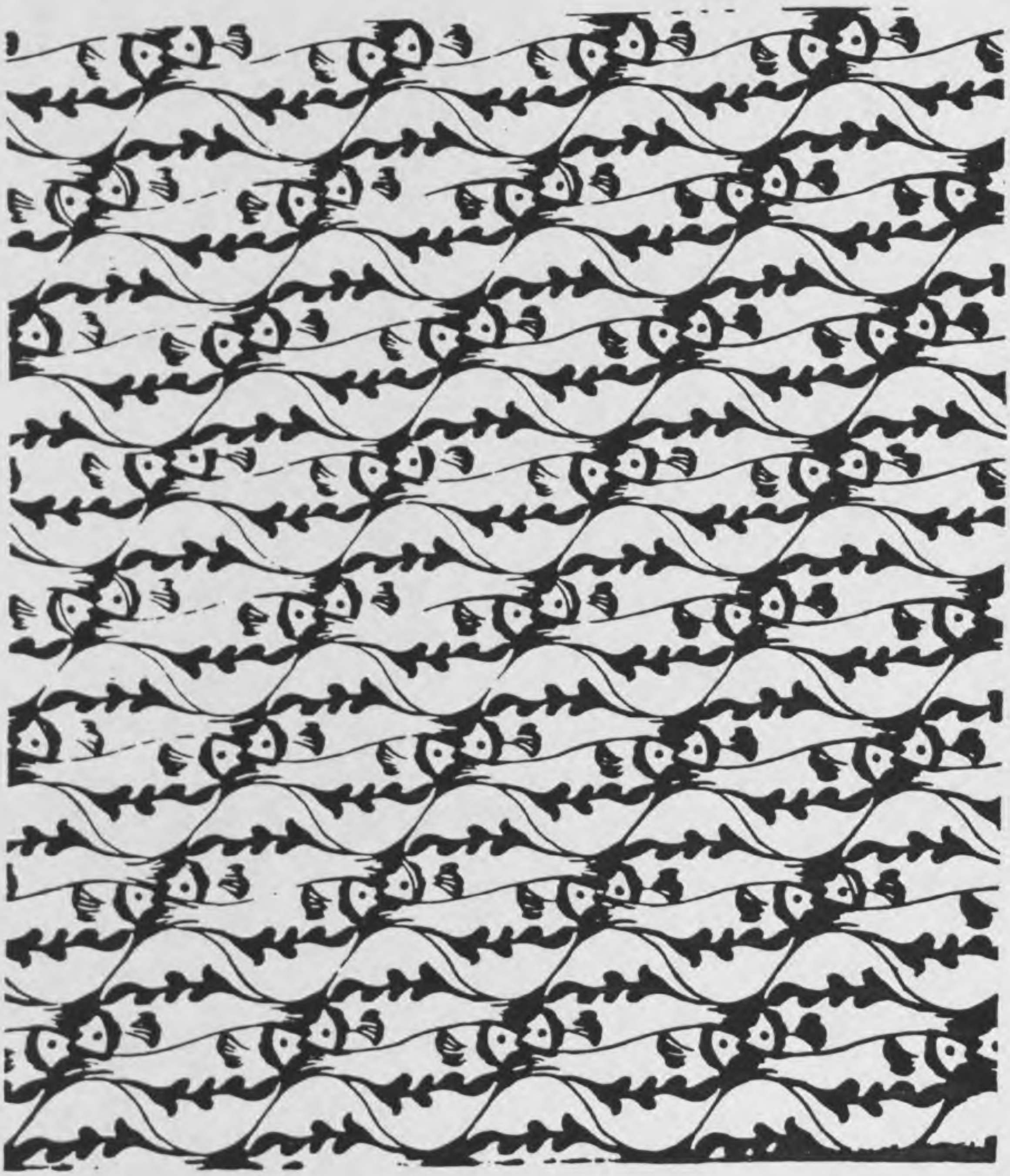
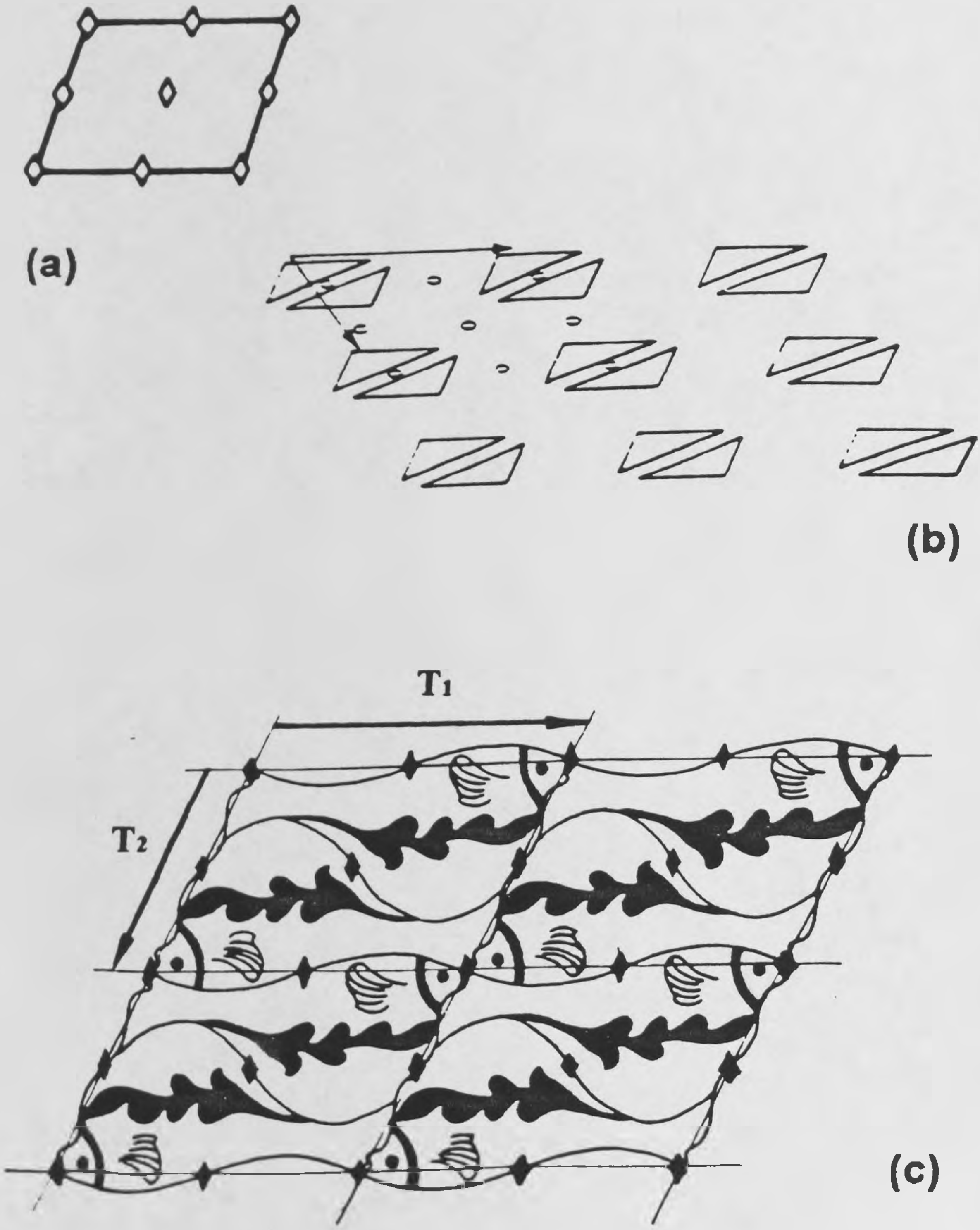


Figure 5.20 Lattice unit, schematic illustration and translation unit for a p211 all-over pattern.



In the construction of a p211 tiling, take a parallelogram (or rectangle) lattice unit with two-fold rotational points shown in either Figure 5.21a or Figure 5.21c. In each case join B to C with a line or series of lines which have two-fold rotational symmetry around the two-fold rotational centre midway between B and C. An identical boundary line joins A to D in each illustration. Subsequently join point A to the two-fold rotational centre located at mid-point along side AB. This line then undergoes two-fold rotation thus connecting to point B. An identical boundary line joins D to C in each case. Each unit cell can then be translated in two independent non-parallel directions to produce the tilings shown in Figures 5.21b and d.

5.4.2 Class p2gg (pgg) patterns and tilings

Class p2gg patterns contain glide-reflection axes which intersect at right angles within a rectangular lattice cell. The fundamental region is one quarter of the area of the unit cell, and the highest order of rotation is two. An example from this pattern class is shown in Figure 5.22. The appropriate lattice cell unit and a schematic illustration of the pattern are shown in Figures 5.23a and b respectively. Figure 5.23c indicates the relative position of glide-reflection axes and points of two-fold rotation in a section of the pattern shown in (diminished) form in Figure 5.23d.

Figure 5.21 The construction of a p211 periodic plane tiling.

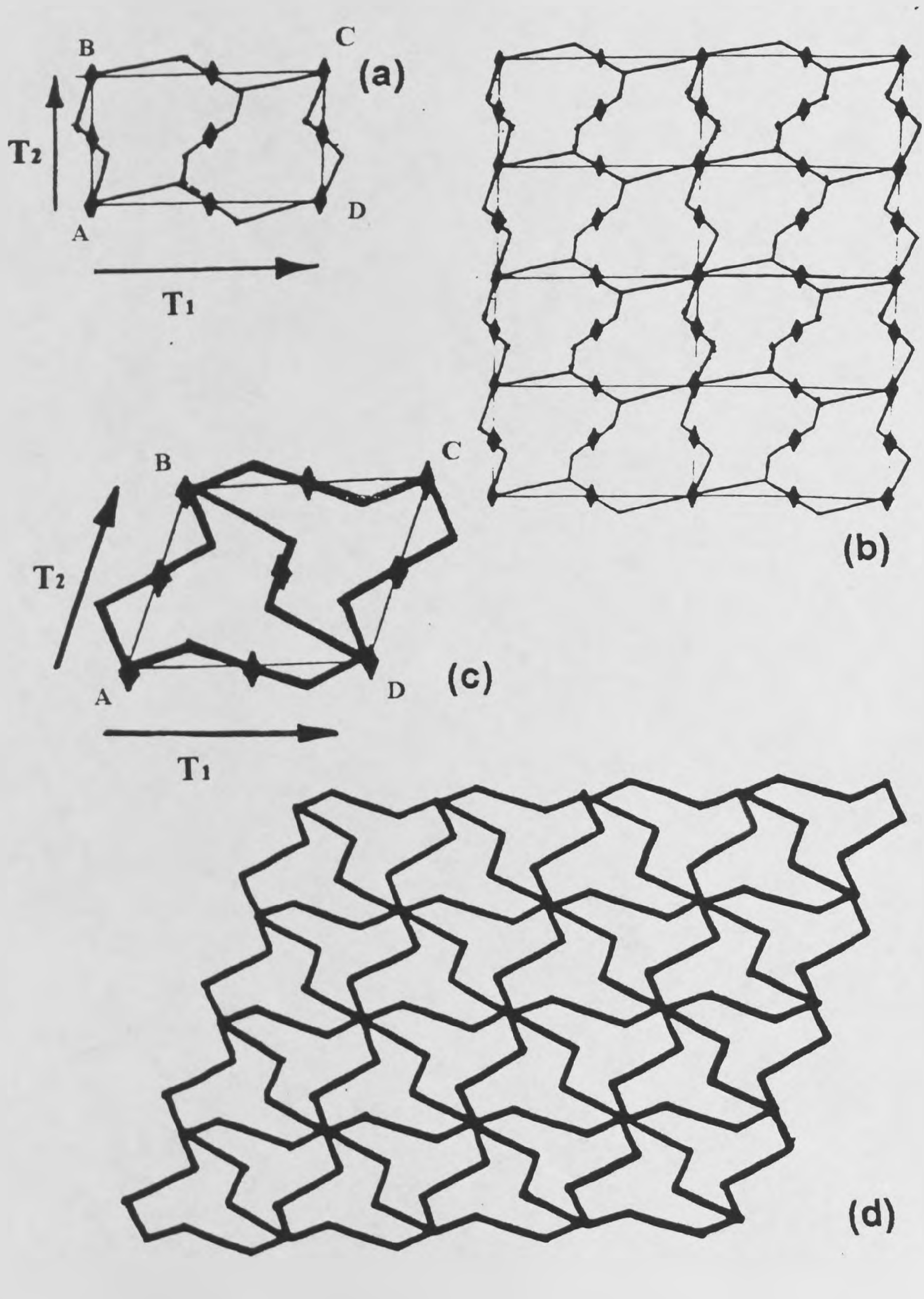


Figure 5.22 A class p2gg primary all-over pattern.

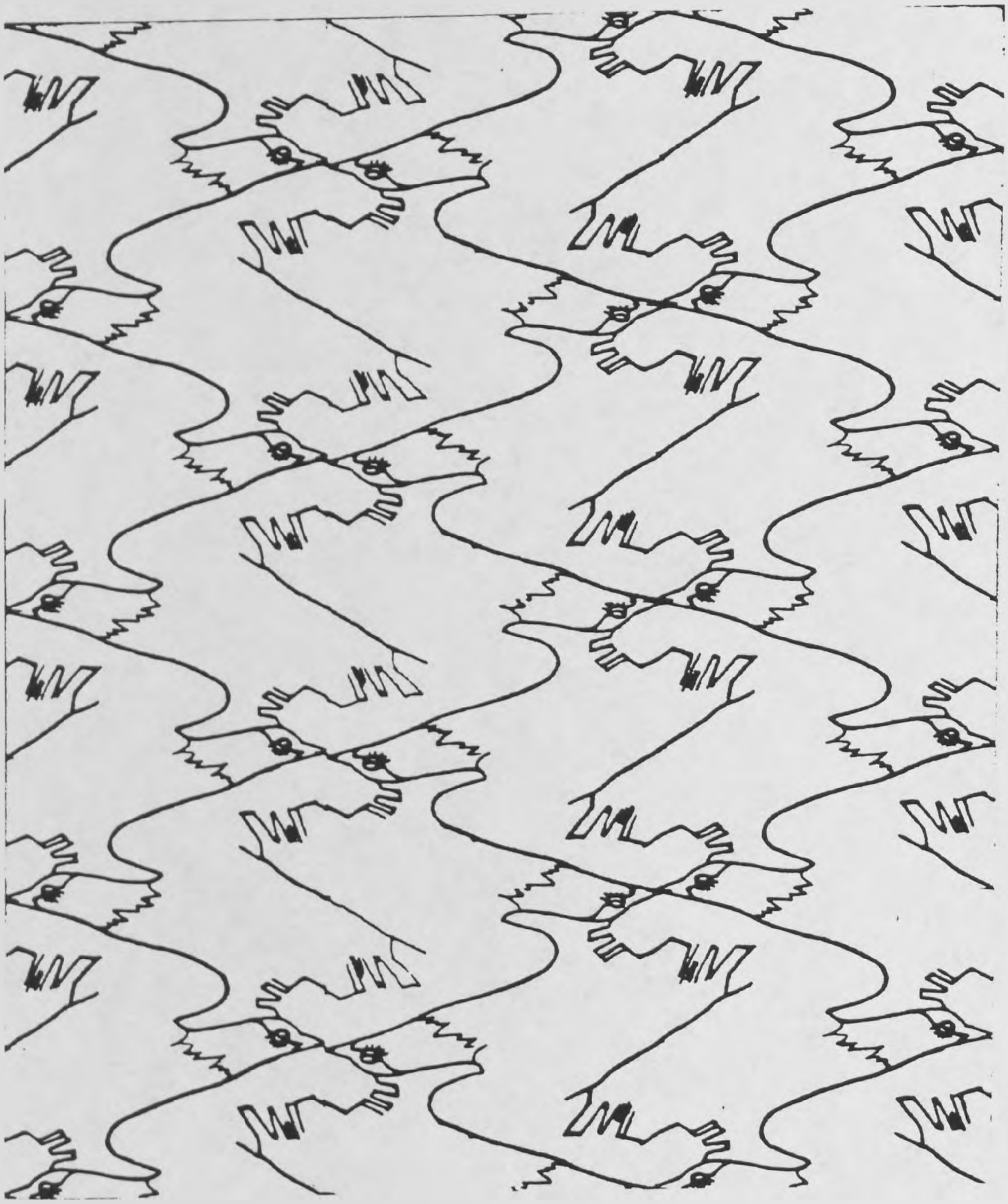
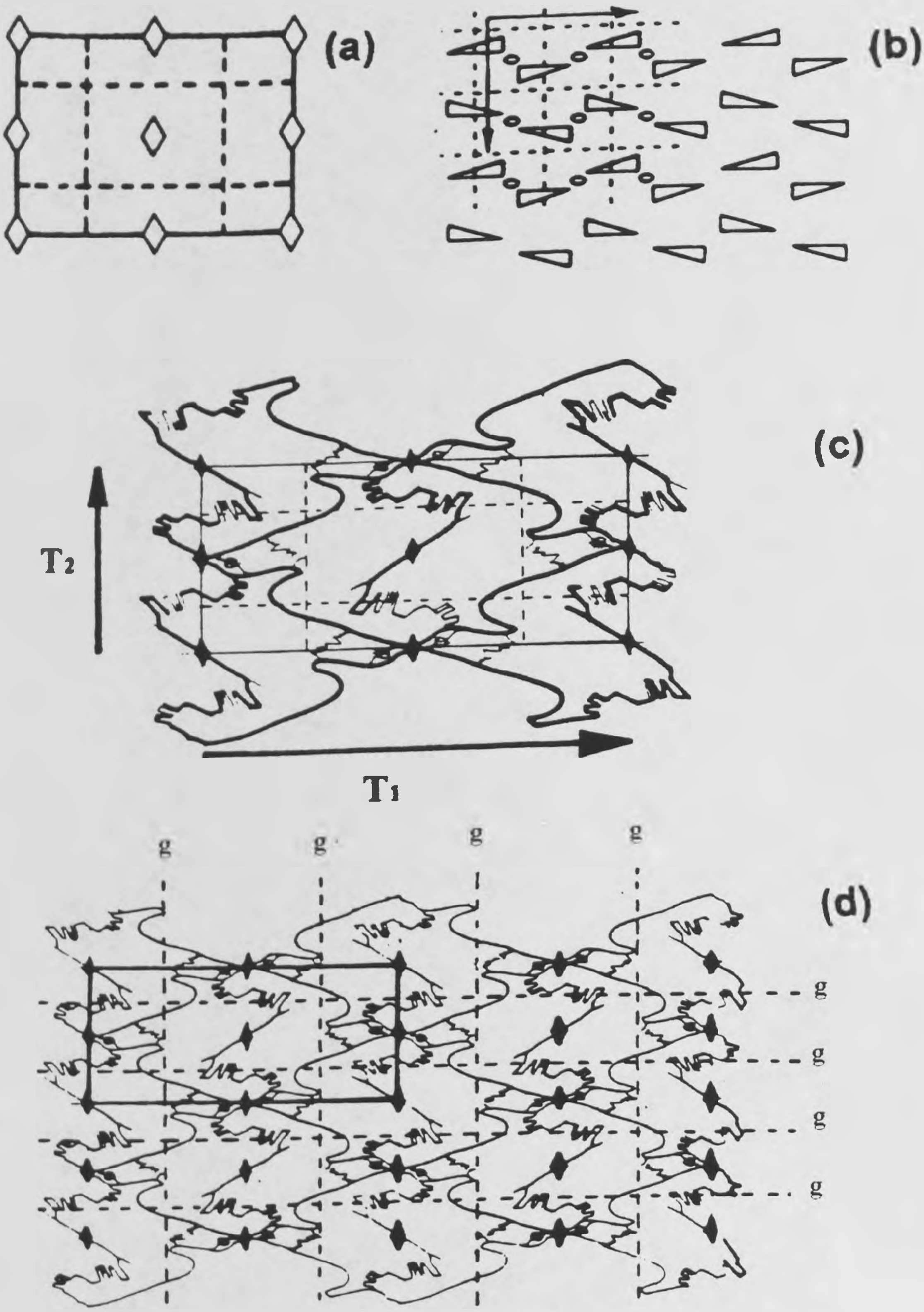


Figure 5.23 Lattice unit, schematic illustration and translation unit for a p2gg all-over pattern.

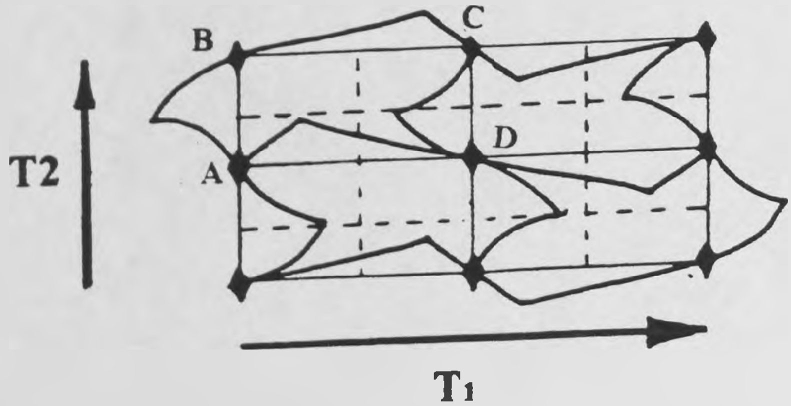


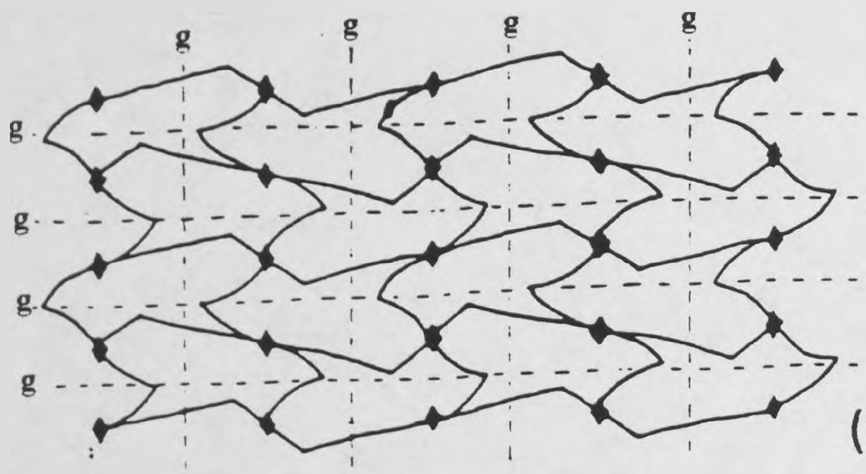
In the construction of a p2gg tiling take the unit cell shown in Figure 5.24a. Divide the cell in four by joining the mid-points on opposite sides thus producing four generating regions. Join A to B and B to C with arbitrary lines as shown. These constructions are repeated by means of existing symmetries so that boundary line CD is a glide-reflection of boundary line AB, and AD is a glide-reflection of BC. The generating tile is thus produced, and the unit cell can be readily constructed using further existing symmetries. The unit cell is subsequently translated in two independent non-parallel directions to produce the full tiling shown in Figure 5.24b. A variant of the p2gg pattern shown in Figure 5.23d is shown in Figure 5.24c.

5.4.3 Class p2mg (pmg) patterns and tilings

Class p2mg all-over patterns have two parallel reflection axes which alternate with each other and intersect at right angles with parallel glide-reflection axes. The highest order of rotation is two-fold and all centres of rotation are on glide-reflection axes. Reflection axes pass between centres of rotation. The area of the generating unit is one quarter of the unit cell area. Examples from this pattern class are provided in Figure 5.25. The unit cell and the pattern are shown schematically in Figures 5.26a and b respectively. Translation units are shown in Figures 5.26c and d. Figure 5.26e shows a section of a pattern (generated from the unit cell illustrated in Figure 5.26d) with relevant symbols to indicate reflection, glide reflection and two-fold rotation.

Figure 5.24 The construction of a p2gg periodic plane tiling.





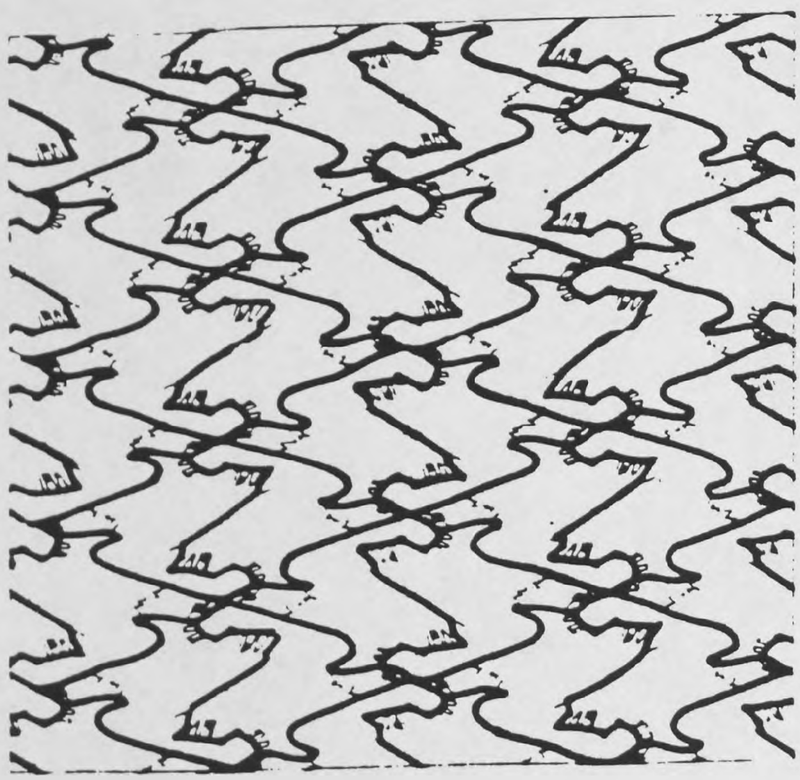
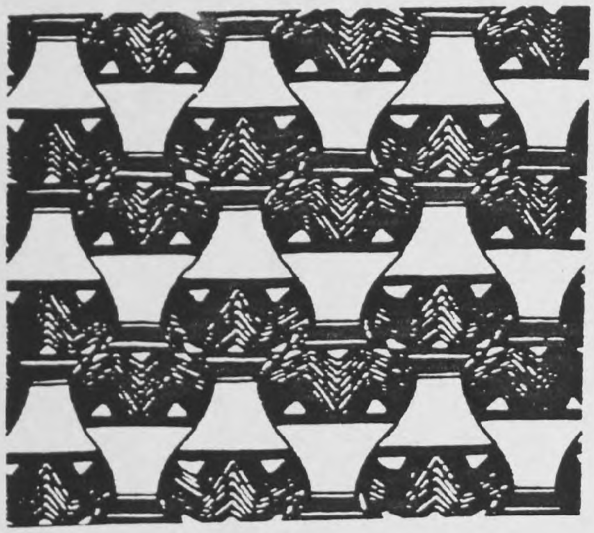


Figure 5.25 Class p2mg primary all-over patterns.



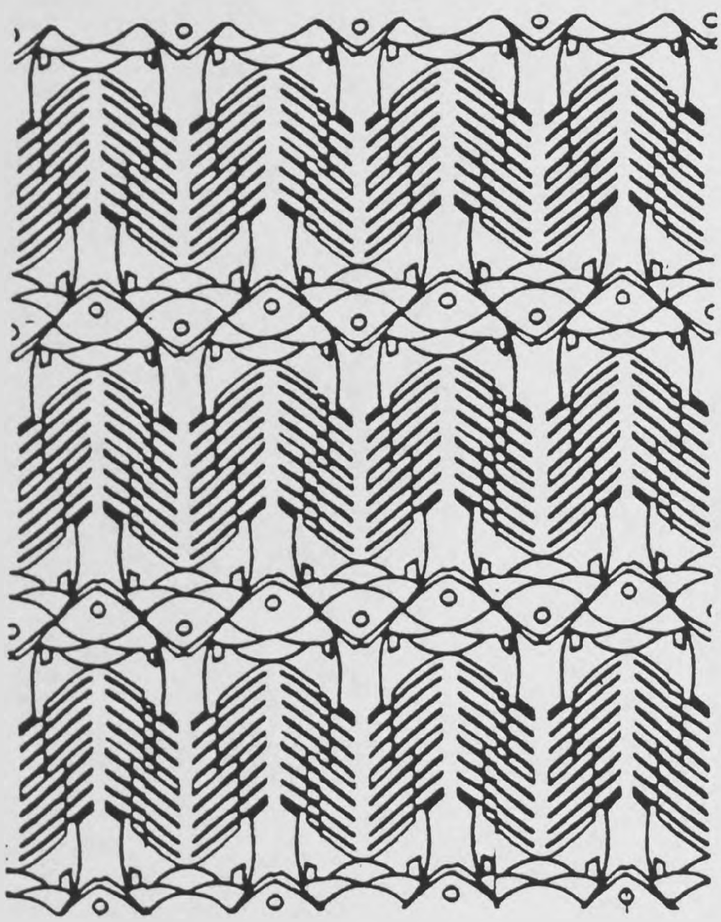
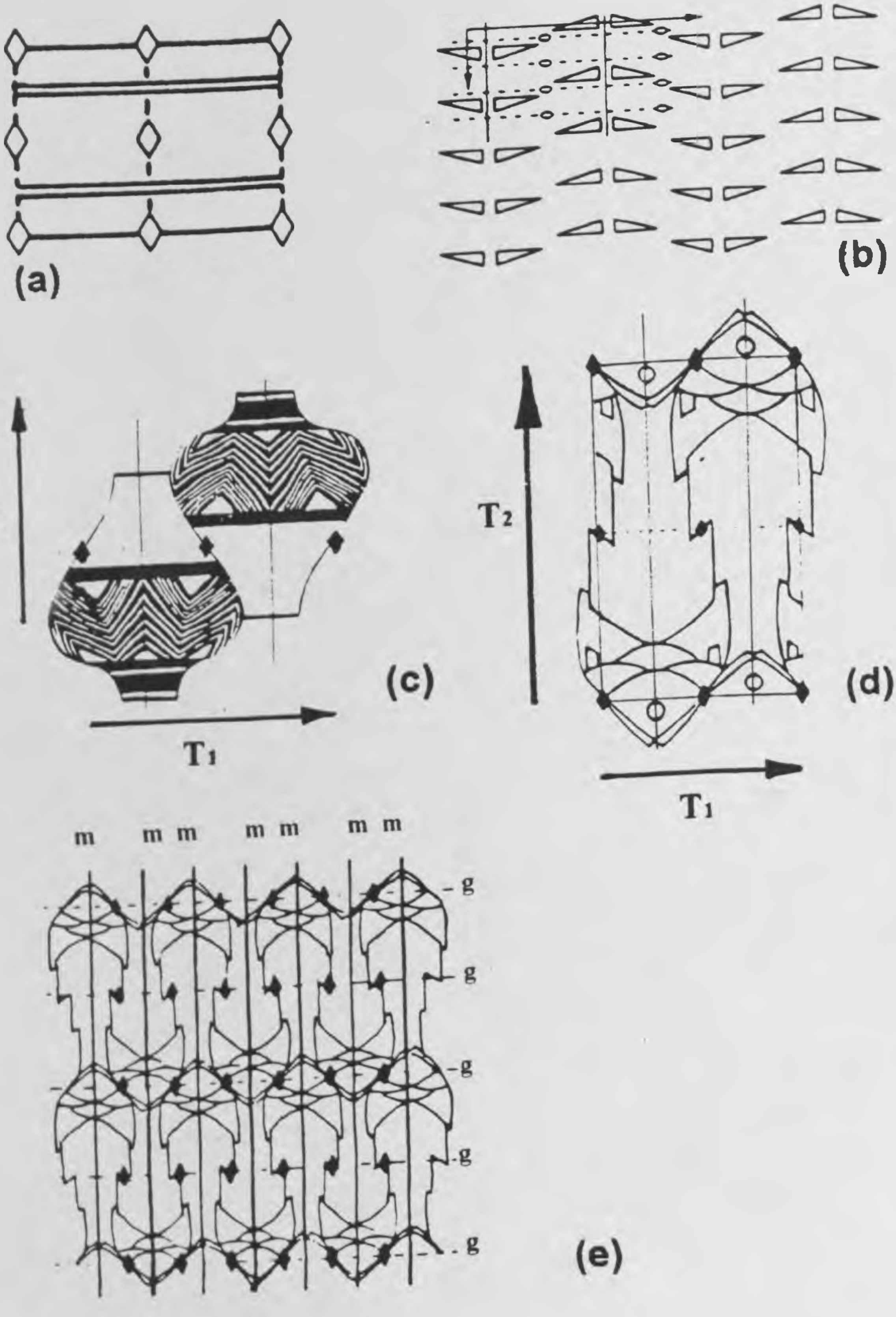


Figure 5.26 Lattice unit, schematic illustration and translation units for p2mg all-over patterns.



In the construction of a p2mg tiling, take lattice unit ABCD (Figure 5.27a) with other points marked E to K. Join point H to points E, F and I as indicated. Using existing symmetries point J can be connected to points F, G and K as shown, and ultimately the full lattice unit can be constructed using existing symmetries. Figure 5.27b shows the fully translated tiling (in diminished form). Figure 5.27c illustrates a p2mg tiling derived from a pattern shown previously.

5.4.4 Class p2mm (pmm) patterns and tilings

Class p2mm all-over patterns have rectangular lattice type unit cells. The generating region (or fundamental unit) is one quarter of the unit cell area. The highest order of rotation is two and the pattern is generated by reflection in four sides of a rectangle. Two types of horizontal reflection axes alternate with each other, as do two types of vertical reflection axes. Two-fold rotational points are present at each of the reflection axes intersections. The unit cell is constructed by joining four rotational centres of the same orientation. Examples of p2mm patterns are shown in Figure 5.28. The relevant lattice unit and a schematic illustration of the pattern are shown in Figures 5.29a and b respectively. Figures 5.29c and d show translation units of two p2mm patterns. When translated in two independent non-parallel directions the pattern is generated.

Figure 5.27 The construction of a p2mg periodic plane tiling.

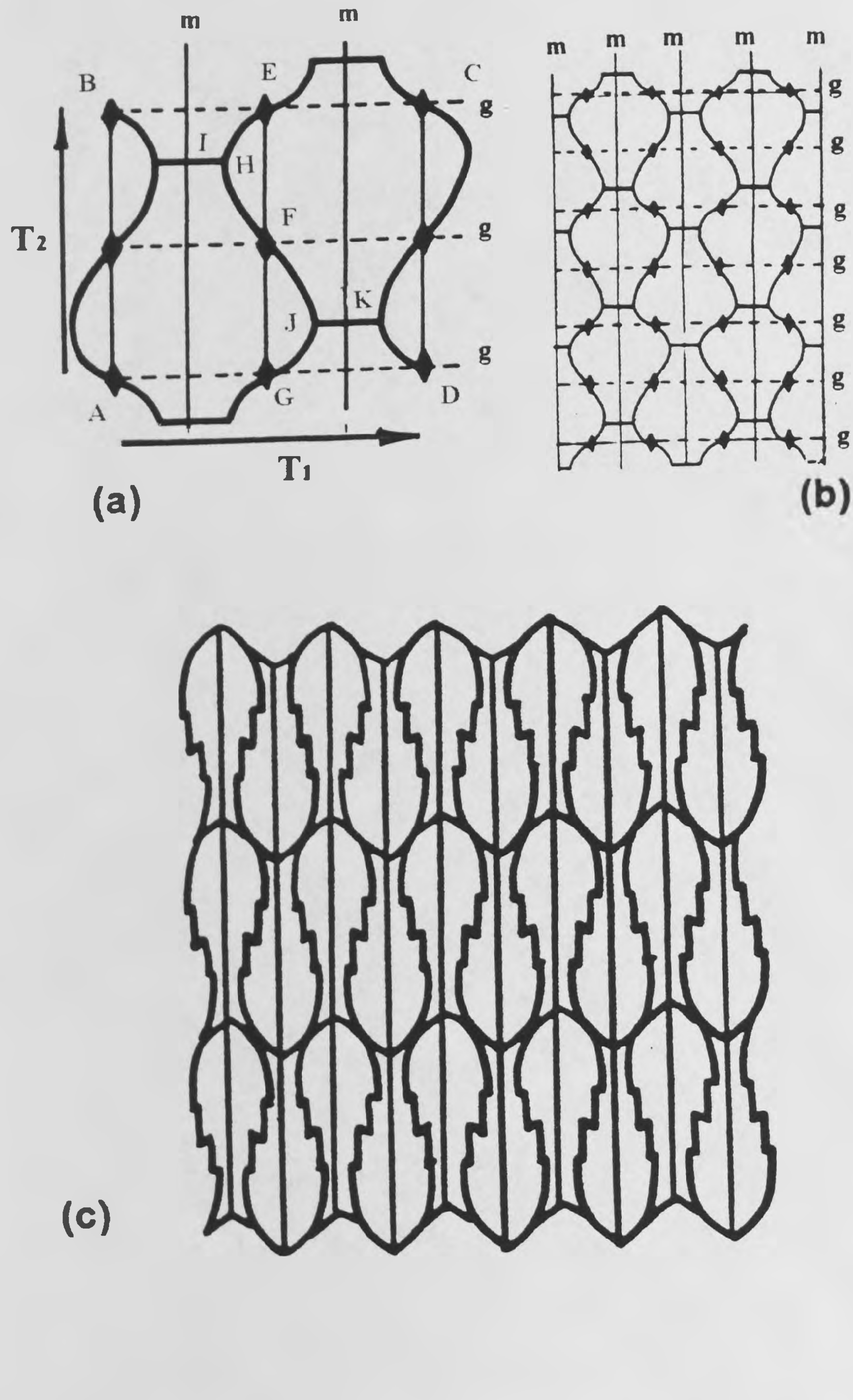
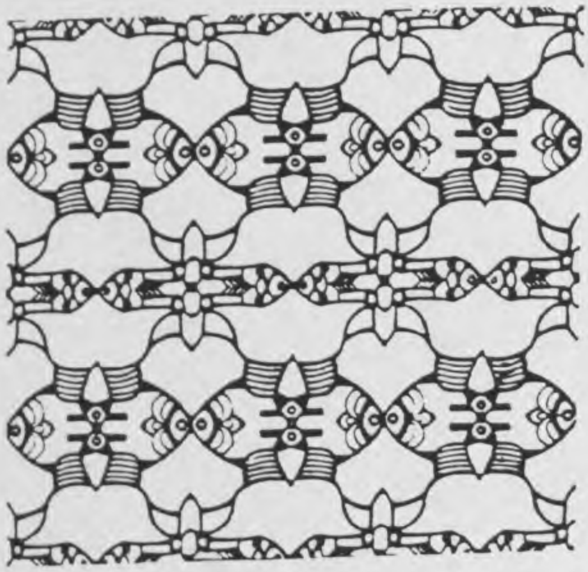


Figure 5.28 Class p2mm primary all-over patterns.



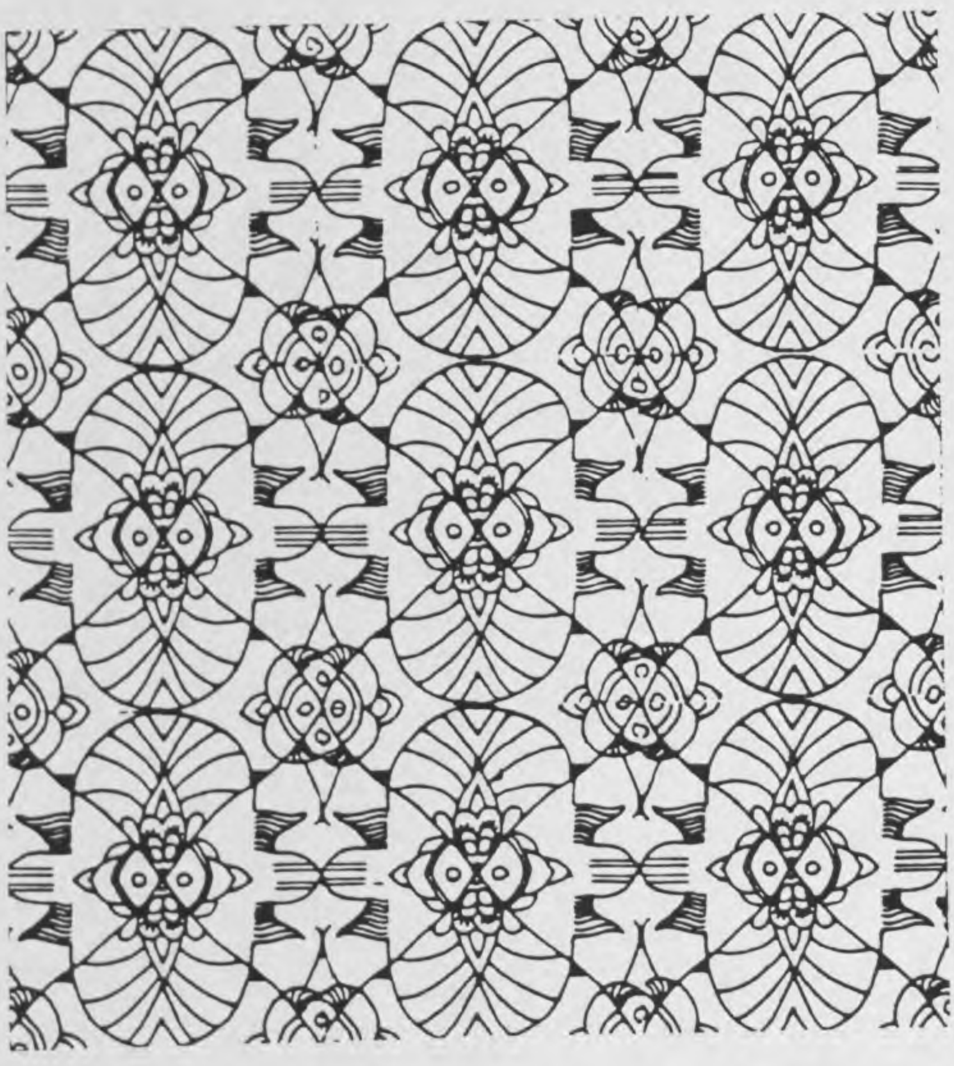
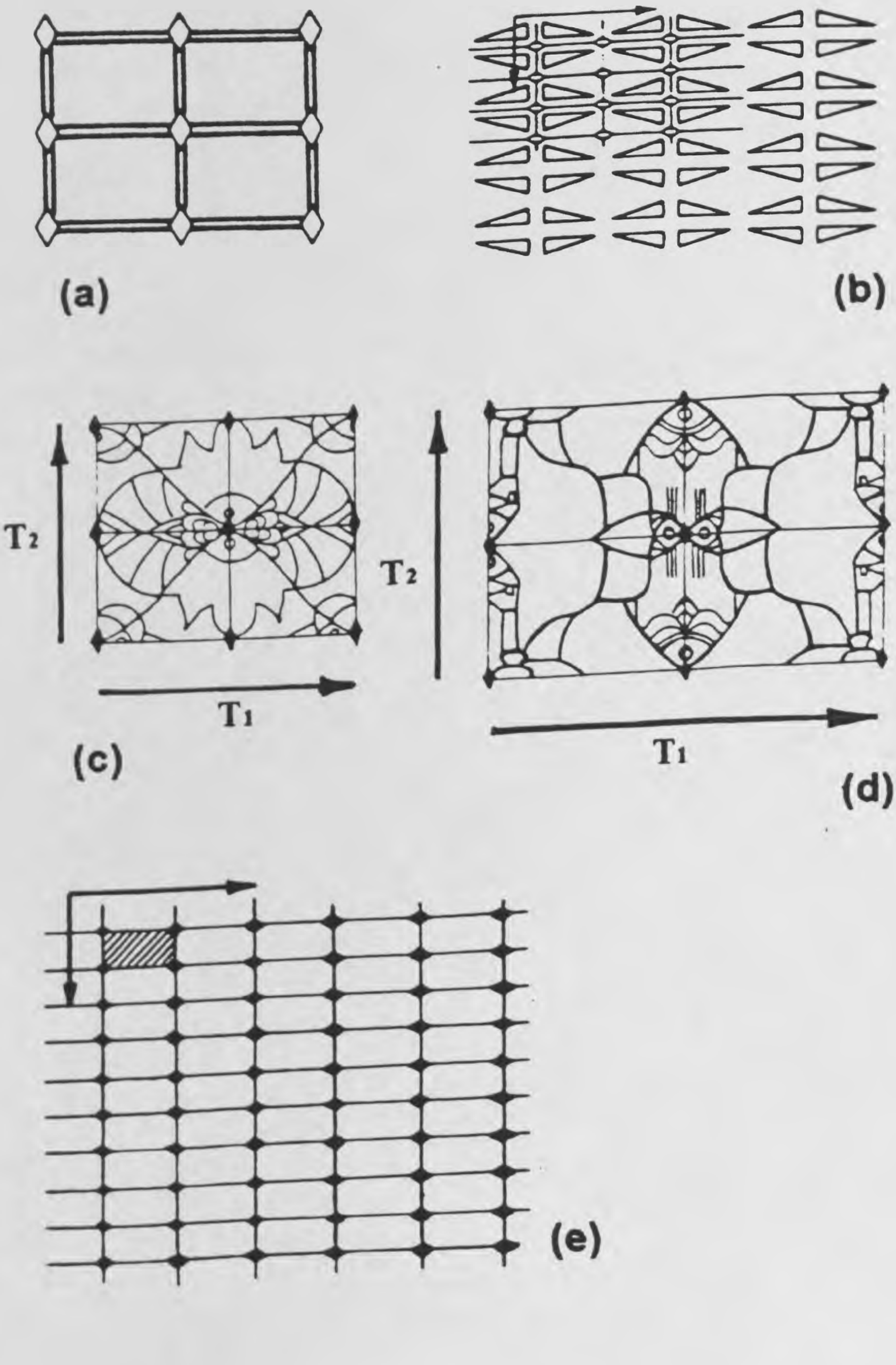


Figure 5.29 Lattice unit, schematic illustration and translation units for p2mm all-over patterns. A generating region used in the construction of a p2mm pattern or tiling.



The construction of a p2mm tiling is straightforward and requires the construction of a generating region which is one quarter of the unit cell area and bounded by reflection axes with two-fold rotation centres at each corner. The tiling is generated by simple translation in two independent non-parallel directions (Figure 5.29e).

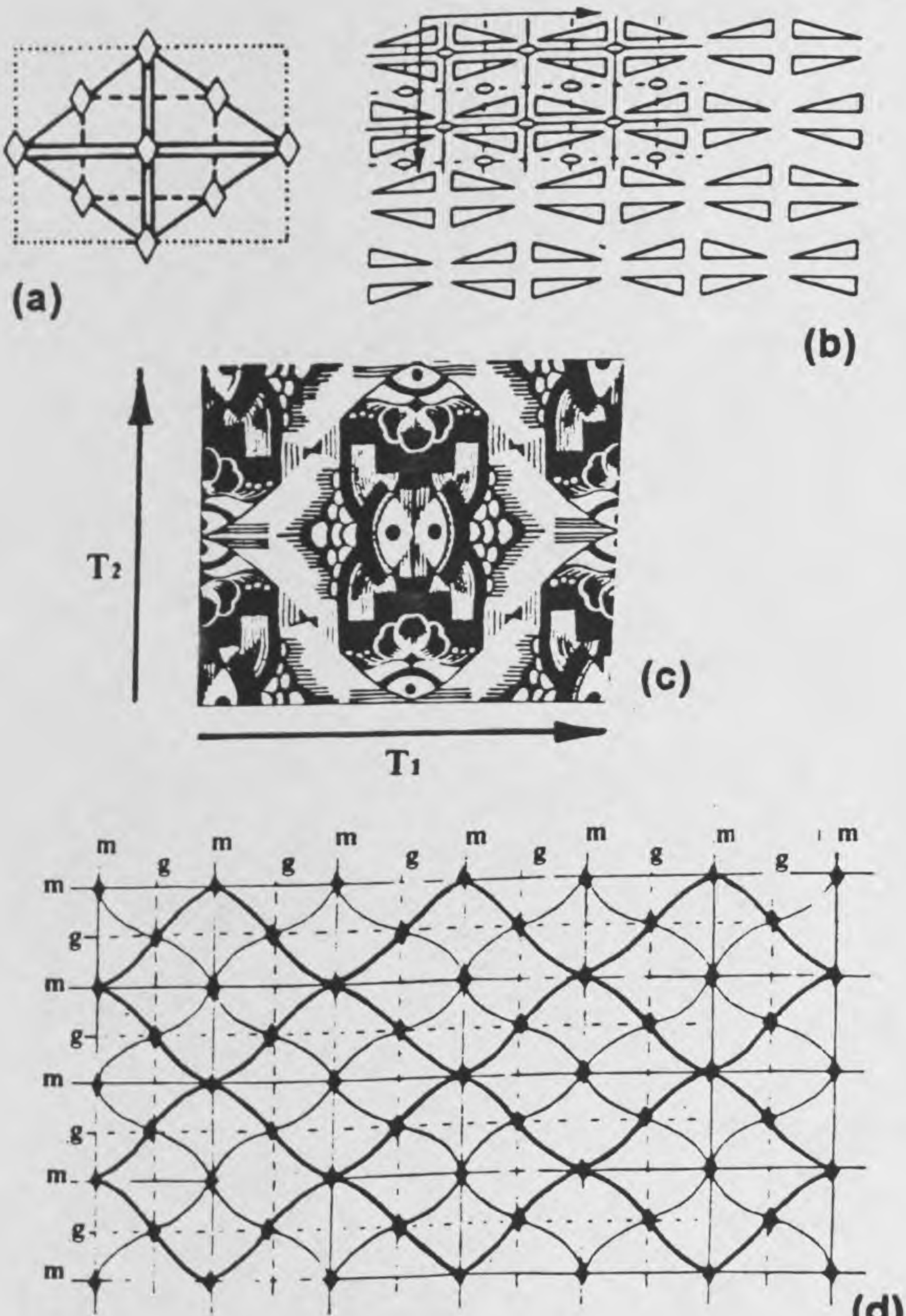
5.4.5 Class c2mm (cmm) patterns and tilings

Several symmetry characteristics are exhibited by class c2mm all-over patterns. A centred cell, whose corners and centre fall on two-fold rotational centres, is used to generate the pattern. Parallel reflection and glide-reflection axes alternate with each other in both vertical and horizontal directions. Two-fold rotation centres are present at both glide-reflection axes intersections and reflection axes intersections. An example from this pattern class is shown in Figure 5.30.

Although the diamond shaped primitive cell can be used in the generation of a c2mm pattern, convention dictates that the chosen minimal repeating area of the pattern is of the enlarged rectangular lattice type as shown in Figure 5.31a. This enlarged cell, as explained previously, holds one full repeating unit (within the diamond shaped cell) and a quarter of a repeating unit at each of the enlarged cell corners. A schematic illustration of the pattern is shown in Figure 5.31b. Figure 5.31c shows a translation unit of a class c2mm all-over pattern. Figure 5.31d shows a class c2mm tiling with the relative positions of symmetry elements indicated.

Figure 5.30 A class c2mm primary all-over pattern.

Figure 5.31 Lattice unit, schematic illustration and a translation unit for a c2mm all-over pattern.



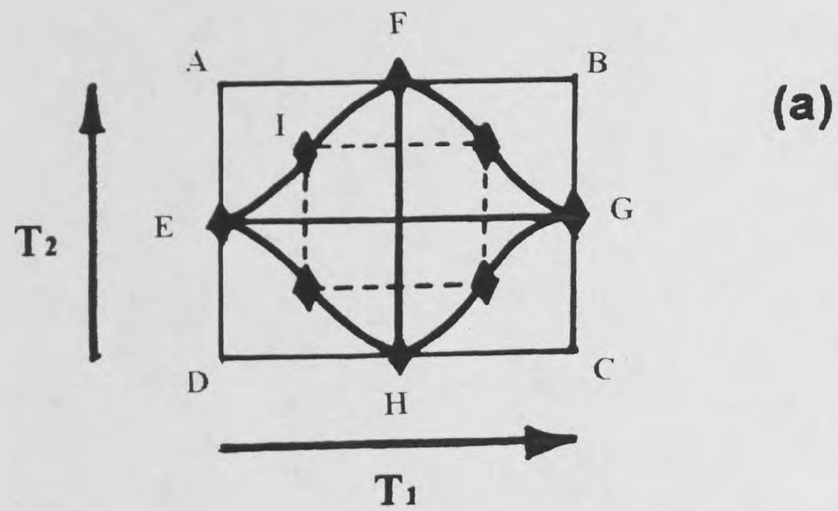
In the construction of a class c2mm periodic tiling, take rectangle ABCD with mid-points EFGH and join point E to point F allowing for two-fold rotation at point I. Join points A to B, B to C, C to D, D to A, F to H and E to G as shown. The inner sections of the tiling are complete by reflection of the boundary line EF in existing axes. The completed tiling is shown in Figure 5.32b.

5.5 The Classification and Construction of All-over Patterns and Tilings With Three-fold Rotational Symmetry

5.5.1 Class p3 patterns and tilings

Class p3 all-over patterns have an hexagonal lattice type unit cell and a highest order of rotation of 3. Three distinct three-fold rotational centres are present. The area of the fundamental region is one third of the unit cell area. An example of a pattern from this class is shown in Figure 5.33. The lattice unit and a schematic illustration of this pattern class are shown in Figures 5.34a and b respectively. The translation unit of the pattern (with example shown in Figure 5.34c) can be translated in the direction of vectors T1 and T2 to produce the pattern shown in Figure 5.34d. Alternatively the pattern may be generated under translation using the hexagonal lattice unit. Figure 5.34e shows the alternative translation unit which is three times the area of the primitive unit cell.

Figure 5.32 The construction of a c2mm periodic plane tiling.



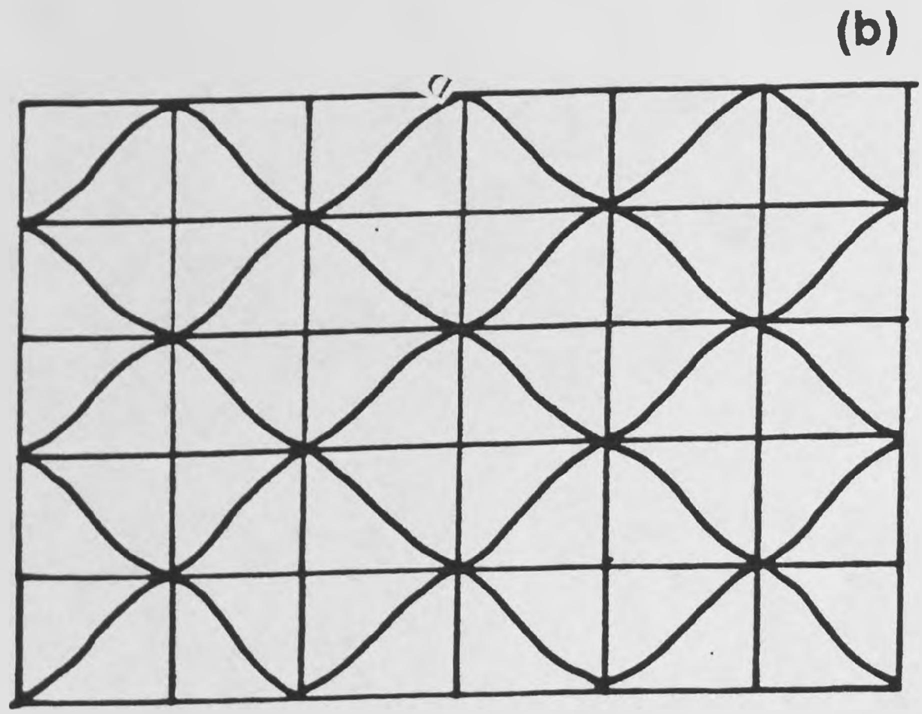


Figure 5.33 A class p3 primary all-over pattern.

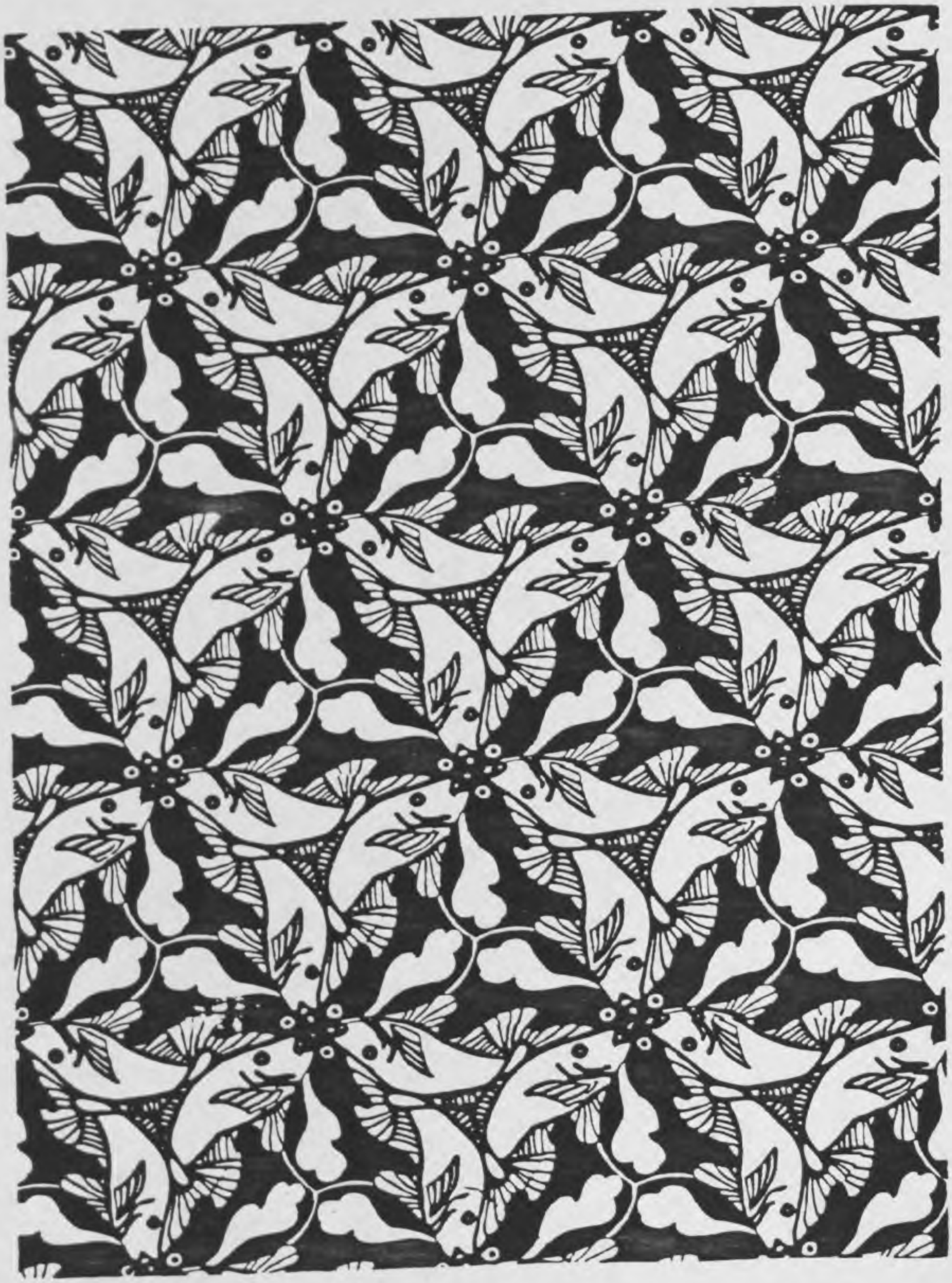
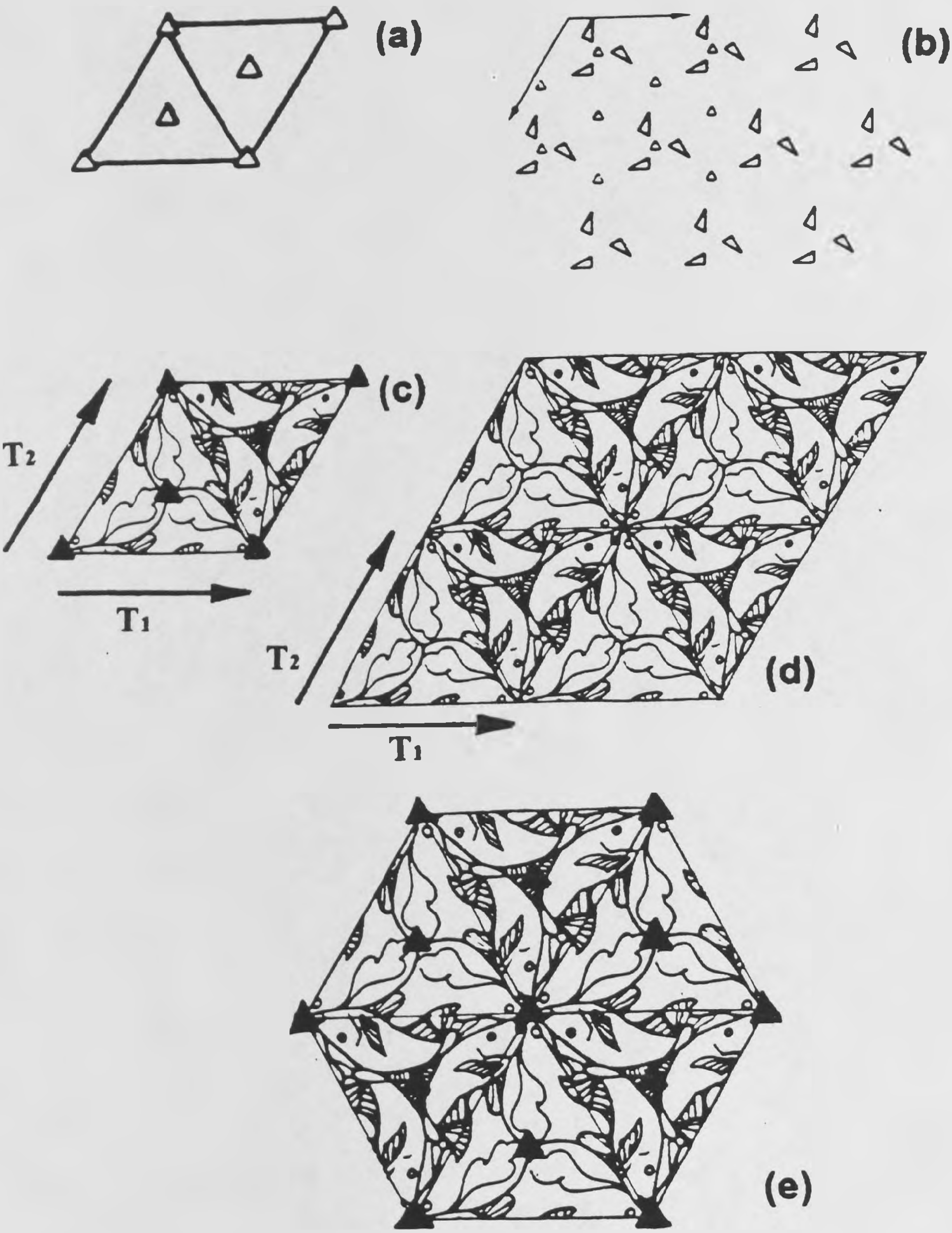


Figure 5.34 Lattice unit, schematic illustration and translation units for a p3 all-over pattern.



In the construction of a class p3 periodic tiling using the primitive unit cell, join centres of three-fold rotation as shown in Figure 5.35a. Boundary lines AB, BC, CD and DA are identical, as are boundary lines AE, BE and ED, and BF, CF and DF. A generating region is thus created within boundary lines BE, ED, DF and FB (shown in Figure 5.35b). The fully translated pattern is shown in Figure 5.35c (in diminished form).

5.5.2 Class p3m1 patterns and tilings

Class p3m1 all-over patterns have an hexagonal lattice type unit cell and a highest order of rotation of 3. The pattern class combines three-fold rotational centres with reflection axes. Each three-fold rotational centre is positioned at the intersection of reflection axes. A reflection axis is positioned along the longest diagonal of the unit cell. The generating region (or fundamental unit) is one-sixth the area of the unit cell. An example is shown in Figure 5.36.

The lattice unit and a schematic illustration of the pattern class are shown in Figures 5.37a and b respectively. Translation units, which may be either the shape of the primitive cell or hexagonal shape (shown to slightly different scale in Figures 5.37c and d respectively). Each type of translation unit will contain a total of six generating units (Figure 5.37e).

Figure 5.35 The construction of a p3 periodic plane tiling.

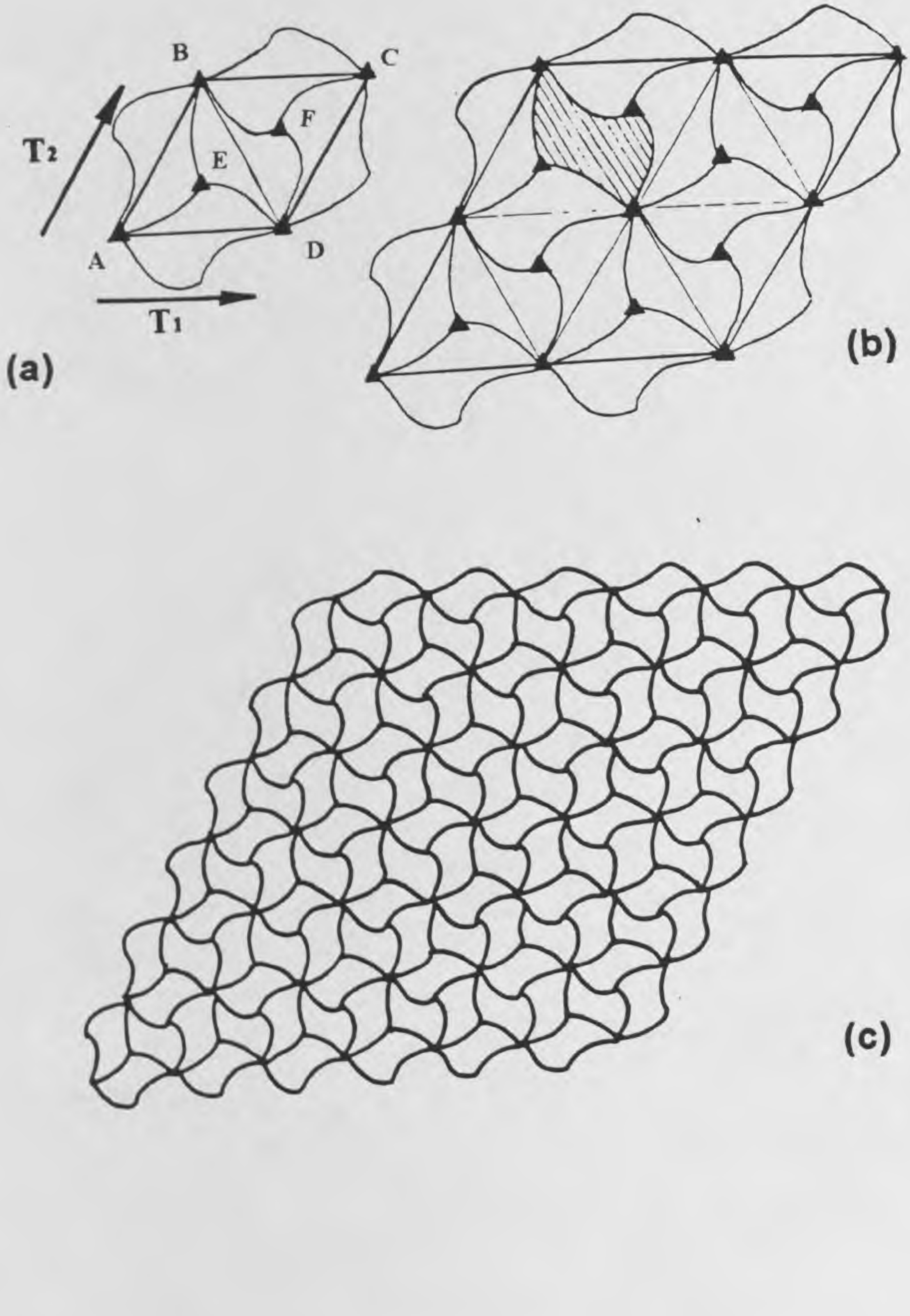


Figure 5.36 A class p3m1 primary all-over pattern.

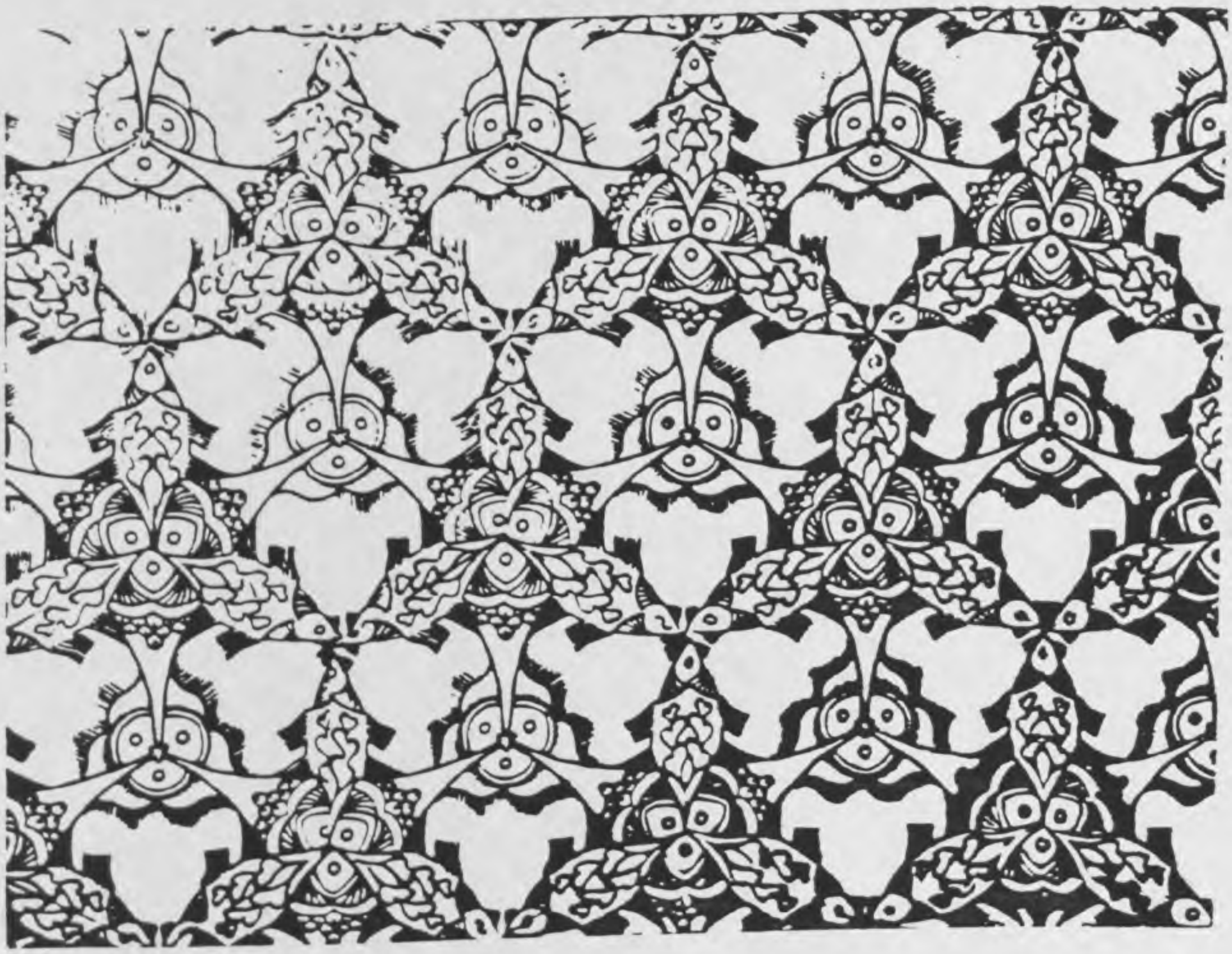
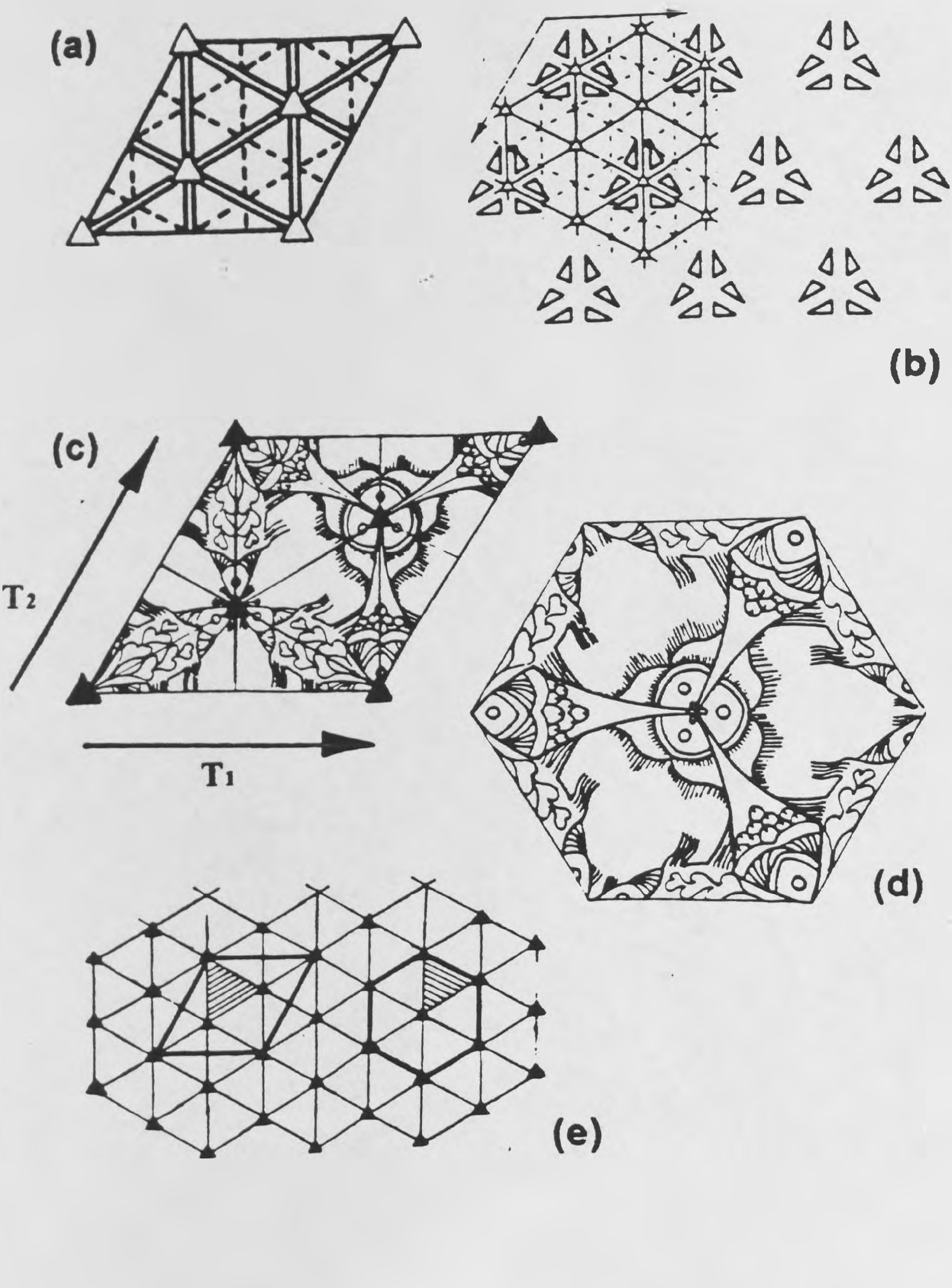


Figure 5.37 Lattice unit, schematic illustration and translation units for a p3m1 all-over pattern.



A class p3m1 periodic plane tiling of the simplest form can be produced by joining all points of three-fold rotation in the unit cell as shown in Figure 5.38a. The generating region is one-sixth the total unit cell area (shown in Figure 5.38a). When the translation unit (Figure 5.38b) is translated in two independent directions the simple tiling shown in Figure 5.38c is produced.

5.5.3 Class p31m patterns and tilings

Class p31m all-over patterns have an hexagonal lattice type unit cell and a highest order of rotation of three. A reflection axis is positioned along the shortest diagonal of the unit cell and on each side of the unit cell. Not all three-fold rotational centres are on reflection axes. The fundamental region is one-sixth of the unit cell area. An example of a pattern from this class is shown in Figure 5.39. The unit cell and a schematic illustration of the pattern is shown in Figures 5.40a and b respectively. A unit cell of a p31m pattern is shown in Figure 5.40c. When translated in two independent non-parallel directions the pattern is produced. The generating region (shaded area in Figure 5.40d) is one-sixth of the area of the unit cell. The pattern may also be obtained by translation of an enlarged hexagonal shaped translation unit (which consists of three primitive cells). This is indicated in Figure 5.40e.

In the construction of a p31m periodic plane tiling take an hexagonal (equilateral triangles) lattice unit (Figure 5.41a). Three-fold rotational points are positioned at the angles and in the middle of each equilateral triangle. A reflection axis is positioned along the shortest diagonal of the unit cell as well

Figure 5.38 The construction of a p3m1 periodic plane tiling.

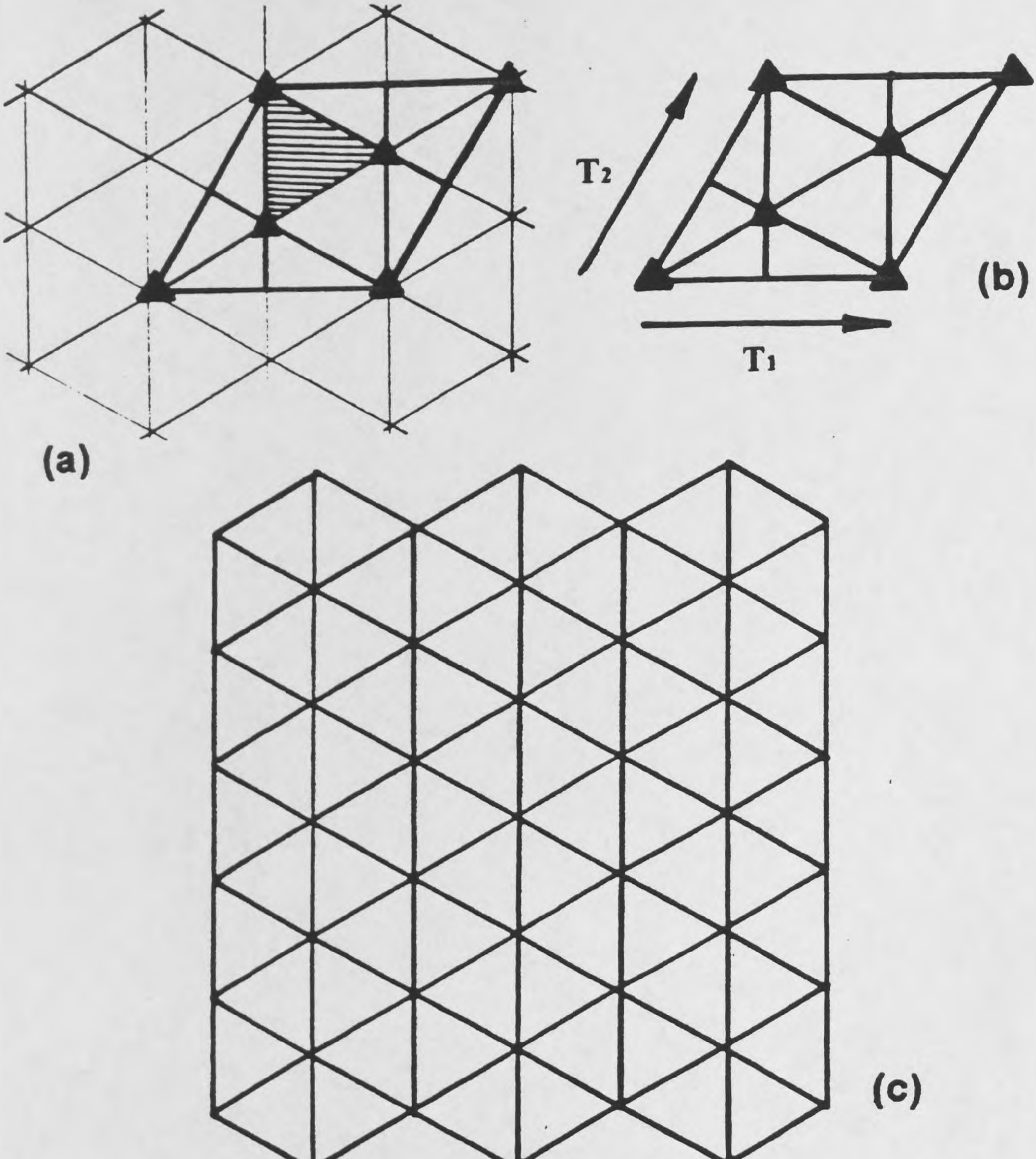


Figure 5.39 A class p31m primary all-over pattern.

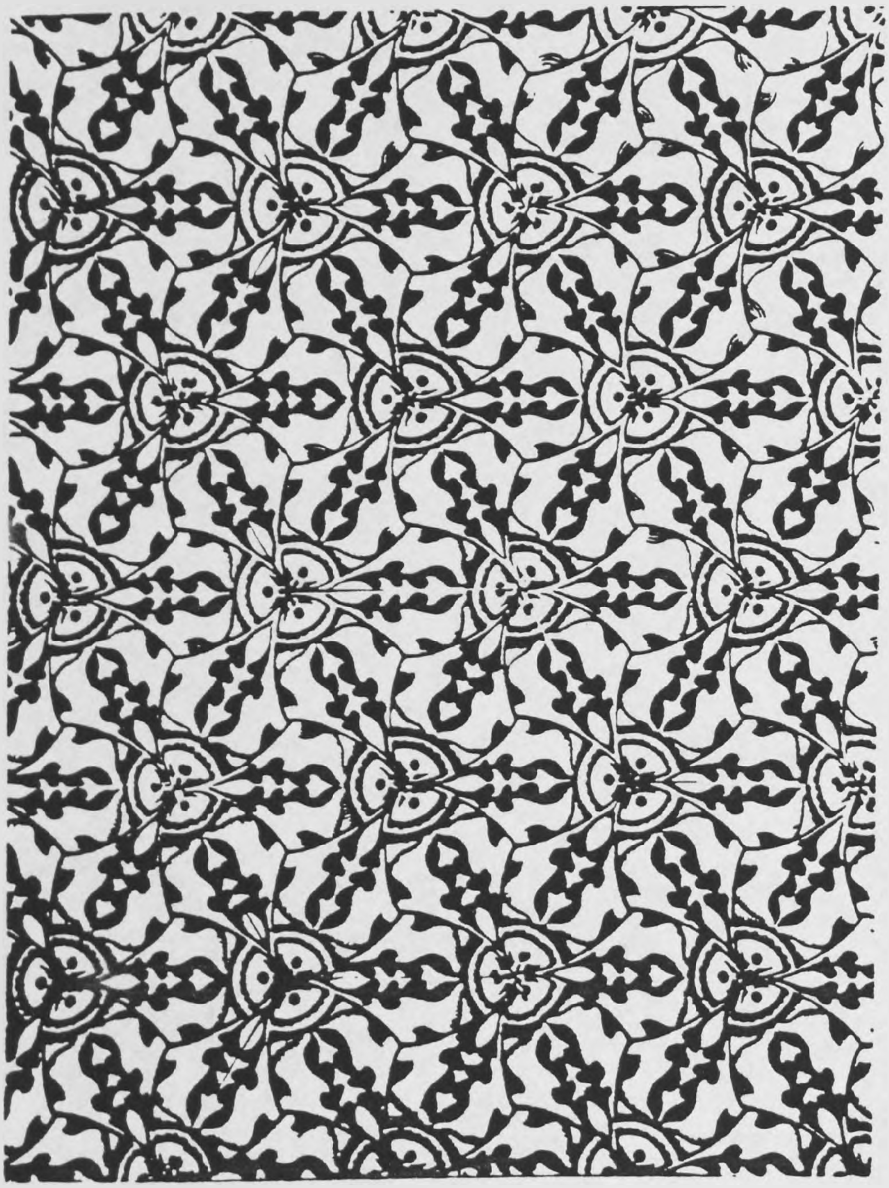


Figure 5.40 Lattice unit, schematic illustration and translation units for a p31m all-over pattern.

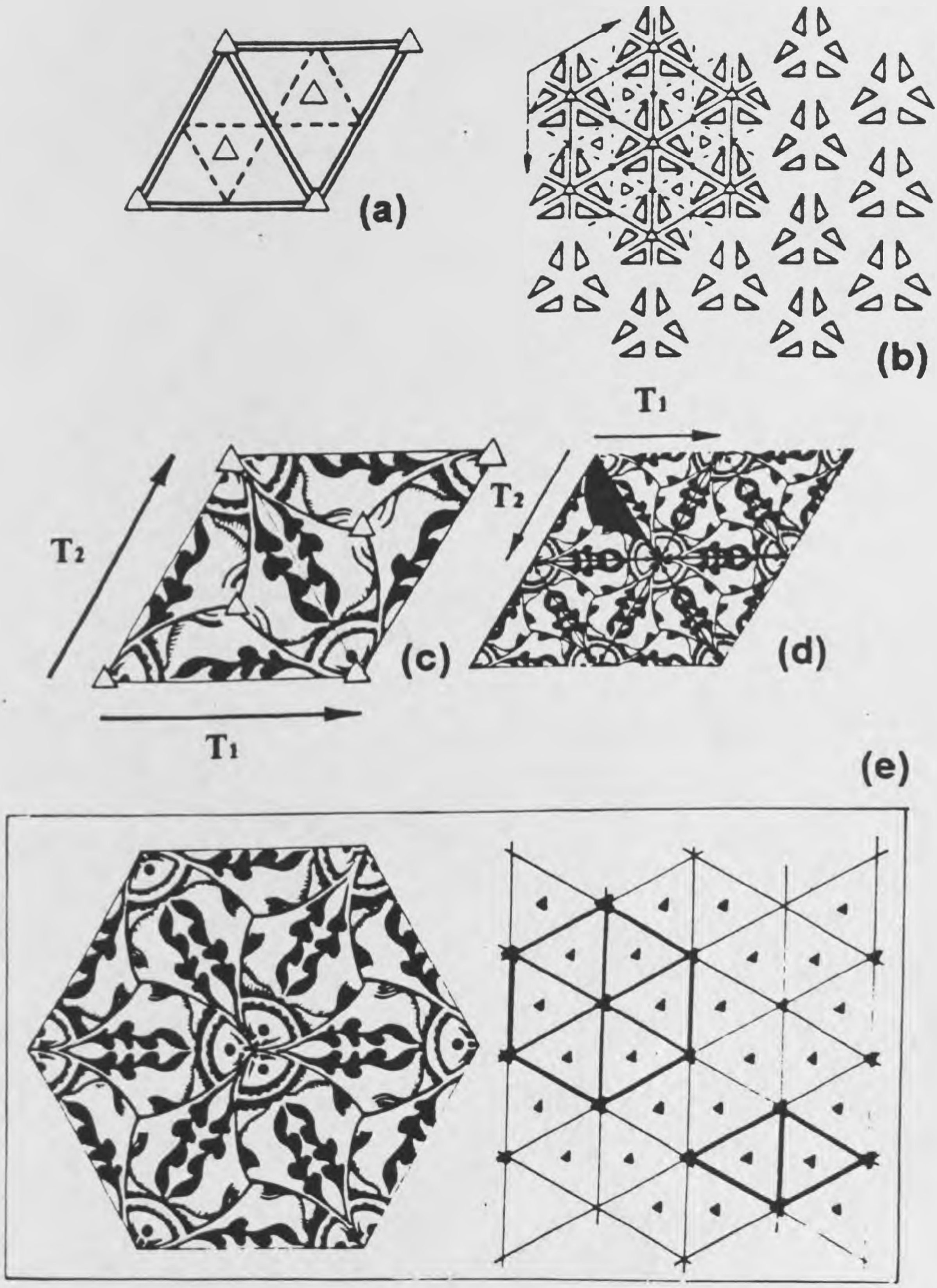
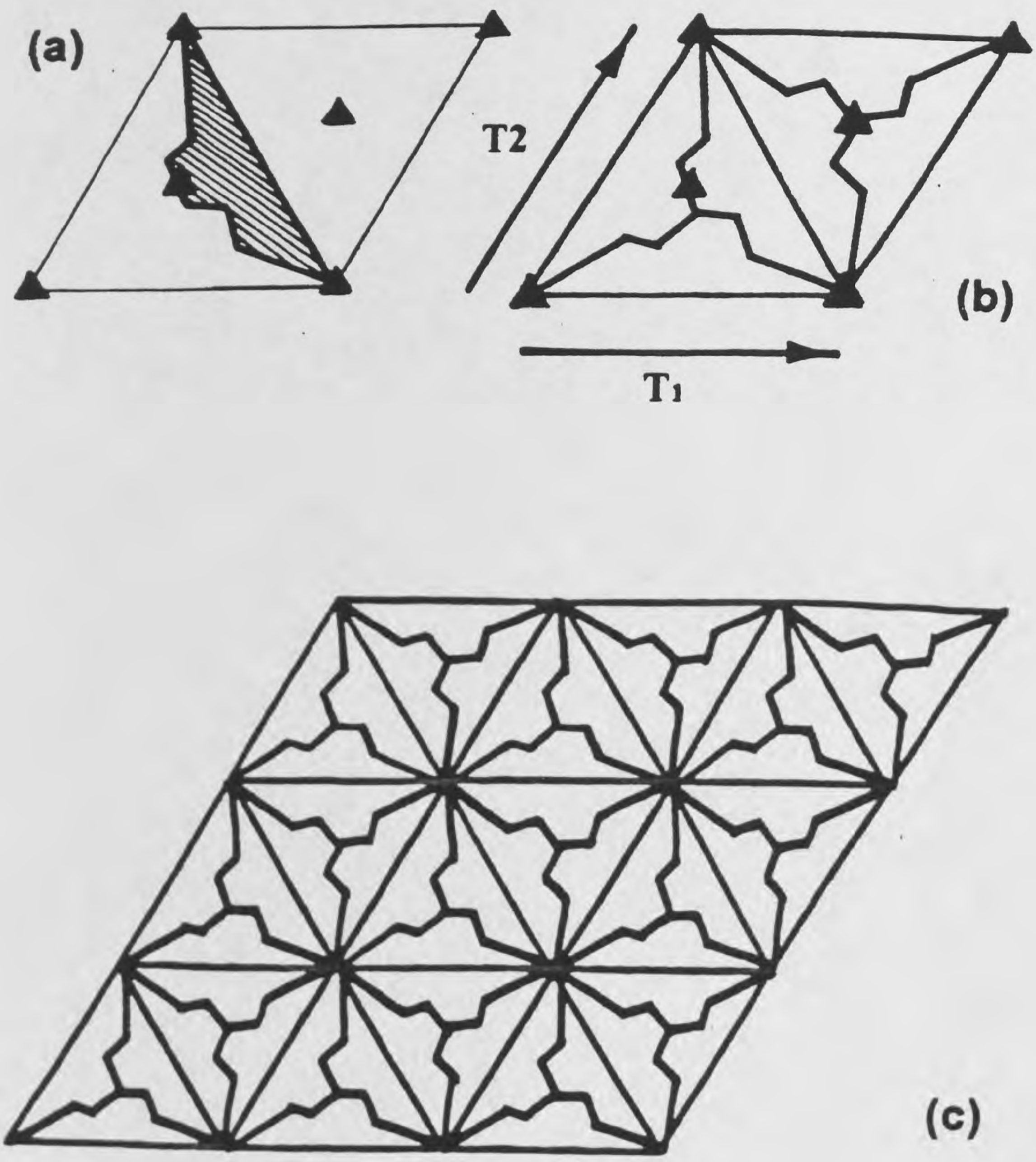


Figure 5.41 The construction of a p31m periodic plane tiling.



as on each side of the unit cell. Join the centre of the triangle to one of its three vertices with an arbitrary line or series of lines as shown. Using three-fold rotation and reflection the unit cell can be readily generated (Figure 5.41b). By translation in two independent non-parallel directions the periodic tiling may be generated (Figure 5.41c).

5.6 The Classification and Construction of All-over Patterns and Tilings With Four-fold Rotational Symmetry

5.6.1 Class p4 patterns and tilings

Class p4 all-over patterns have a square lattice type unit cell, no reflection axes and a highest order of rotation of four. Two-fold and four-fold rotational centres alternate in both horizontal and vertical directions. Centres of four-fold rotation are positioned at the centre and at each corner of the unit cell. A centre of two-fold rotation is positioned on each side of the unit cell. An example of a p4 pattern is provided by Figure 5.42. The unit cell of the pattern is based on a square shaped lattice as shown in Figure 5.43a. A schematic illustration of the pattern class is given in Figure 5.43b. A unit cell of the pattern shown in Figure 5.42 is given in Figure 5.43c.

In the construction of a p4 periodic plane tiling take a square unit cell, as shown in Figure 5.44a, and join any centre of four-fold rotation to a centre of two-fold rotation using an arbitrary line or series of lines. Using existing symmetries the unit cell can be readily generated.

Figure 5.42 A class p4 primary all-over pattern.

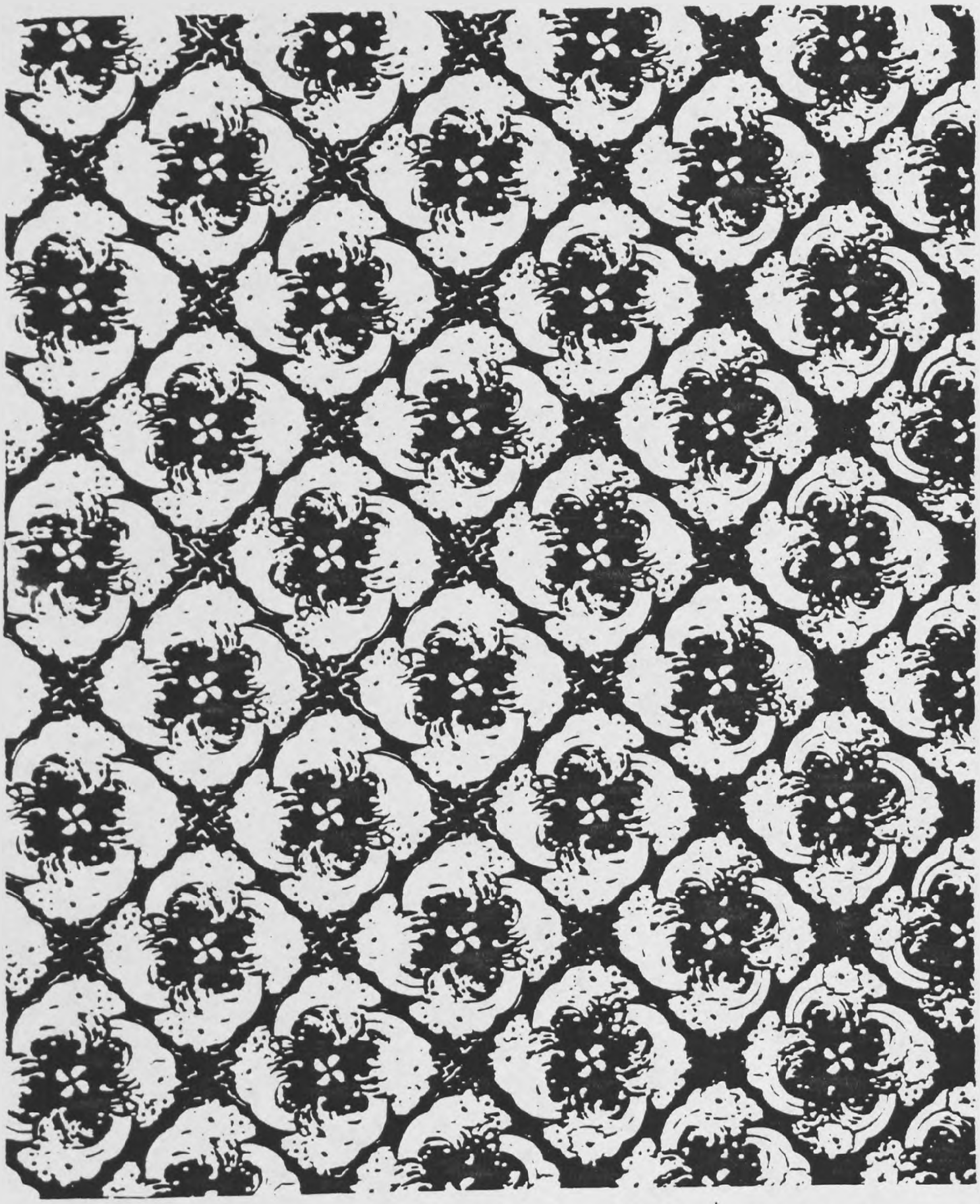


Figure 5.43 Lattice unit, schematic illustration and translation unit for a p4 all-over pattern.

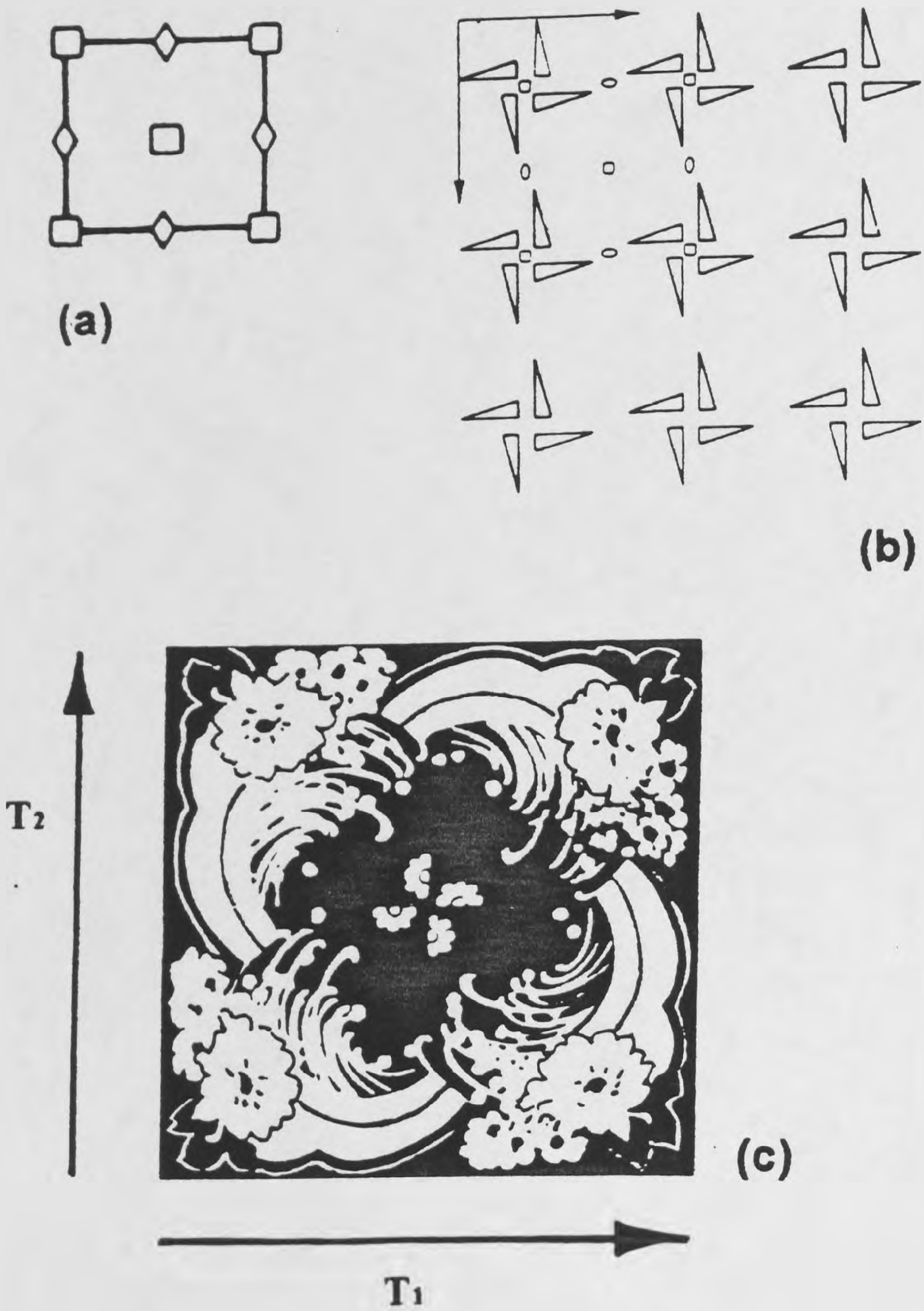
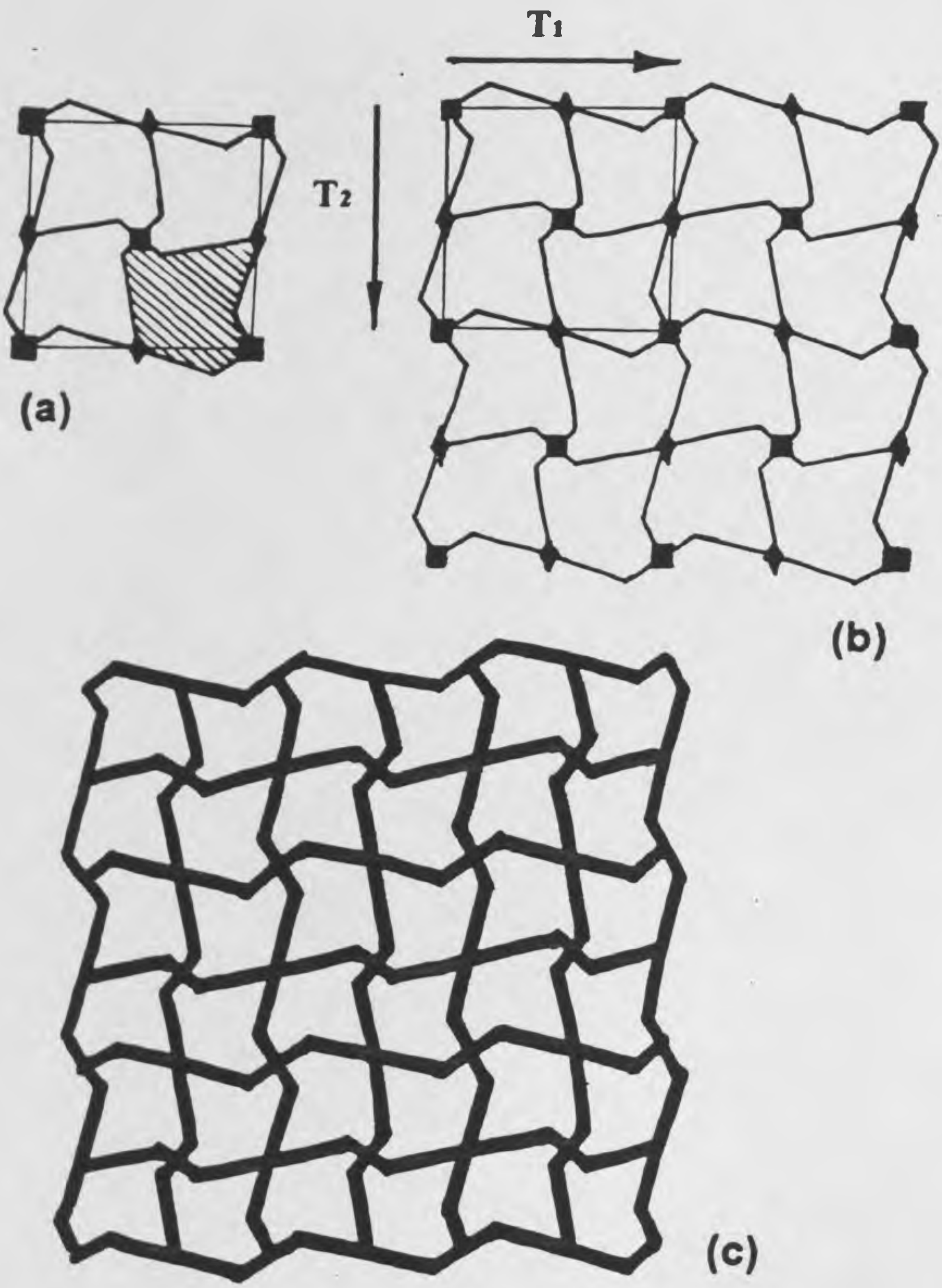


Figure 5.44 The construction of a p4 periodic plane tiling.



The generating region in the example given has been shaded (Figure 5.44a). Following translation of the unit cell the tiling can be generated (examples given in Figures 5.44b and c).

5.6.2 Class p4gm (p4g) patterns and tilings

Class p4gm patterns have a square lattice type unit cell and a highest order of rotation of four. Each corner of the unit cell is on a four-fold centre of rotation. Reflection axes intersect at right angles on two-fold centres of rotation positioned at the centre of each side of the unit cell. In this case the fundamental region is one-eighth the area of the unit cell. An example from this pattern class is shown in Figure 5.45. The unit cell of the pattern is shown in Figure 5.46a and a schematic illustration of the pattern is shown in Figure 5.46b. The generating region of the pattern is one-eighth the area of the unit cell (Figure 5.46c). A translation unit of a pattern derived from that shown in Figure 5.45 is shown in Figure 5.46d.

In the construction of a periodic plane tiling from class p4gm take a square shaped lattice unit and join a four-fold rotational centre (as shown) to a two-fold rotational centre with an arbitrary line or series of lines. Joint two centres of two-fold rotation as shown. The generating region (Figure 5.47b), from which the full tiling may be produced by translation (Figure 5.47c), is thus complete.

Figure 5.45 A class p4gm primary all-over pattern.

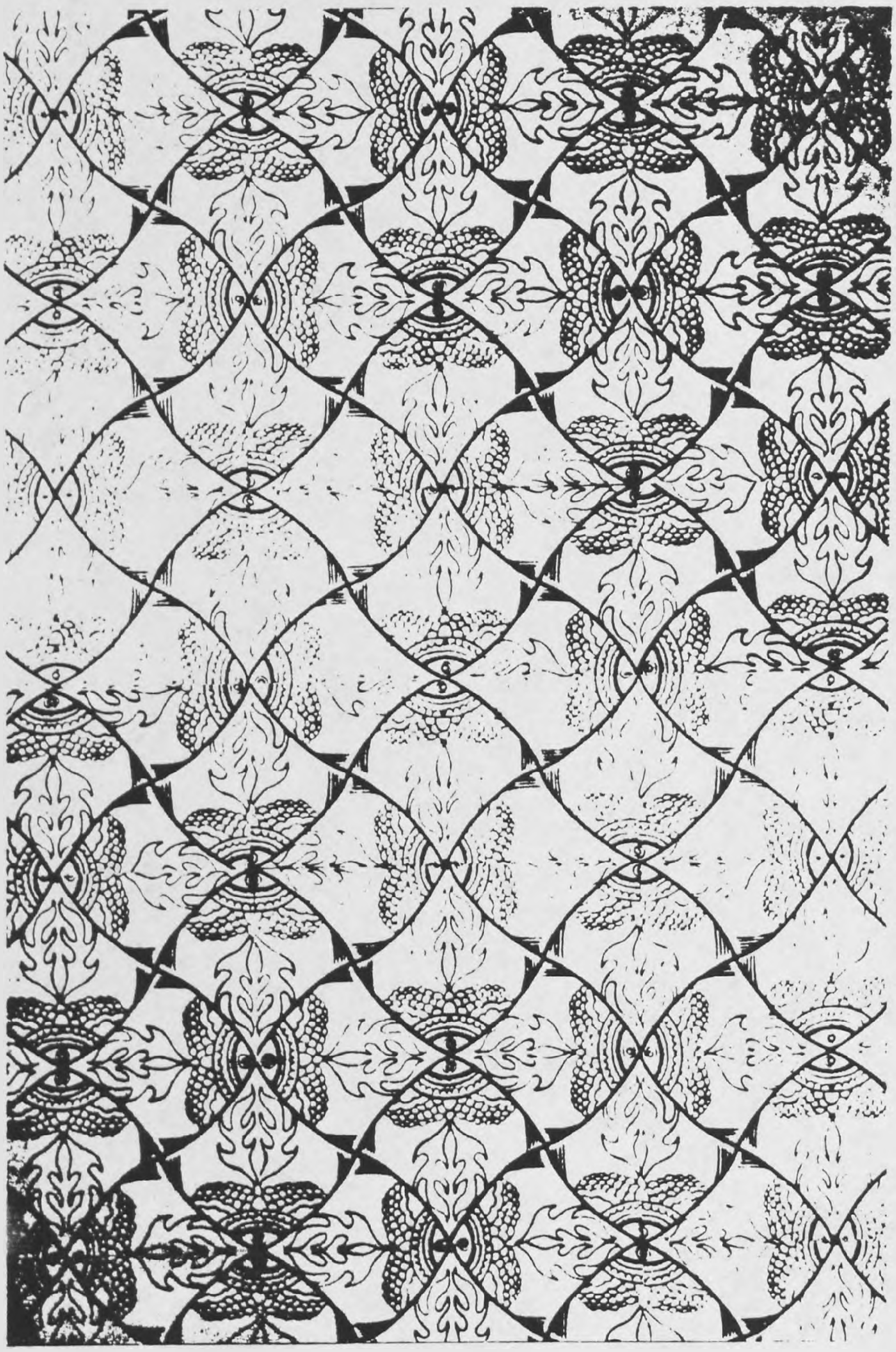


Figure 5.46 Lattice unit, schematic illustration and translation unit for a p4gm all-over pattern.

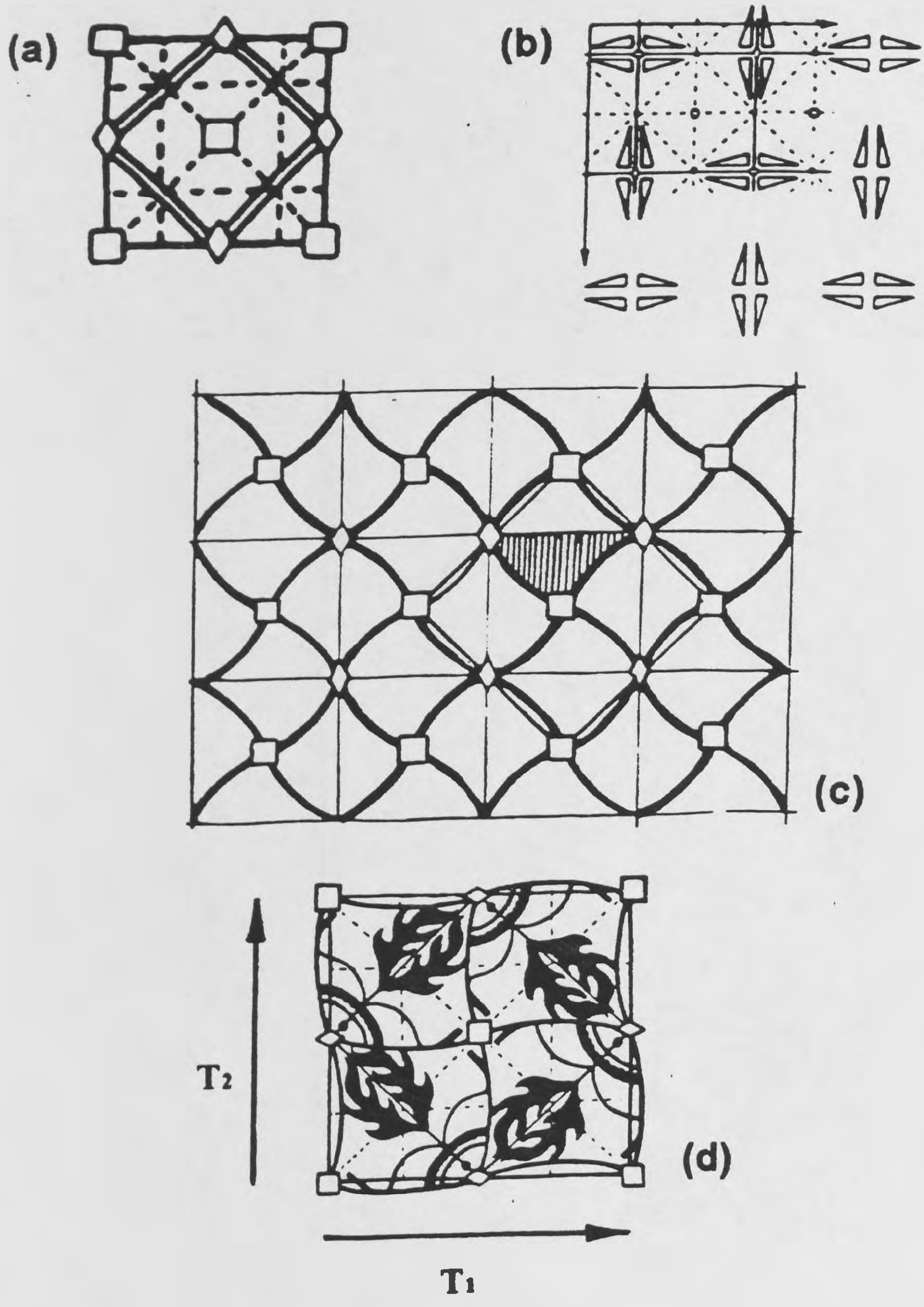
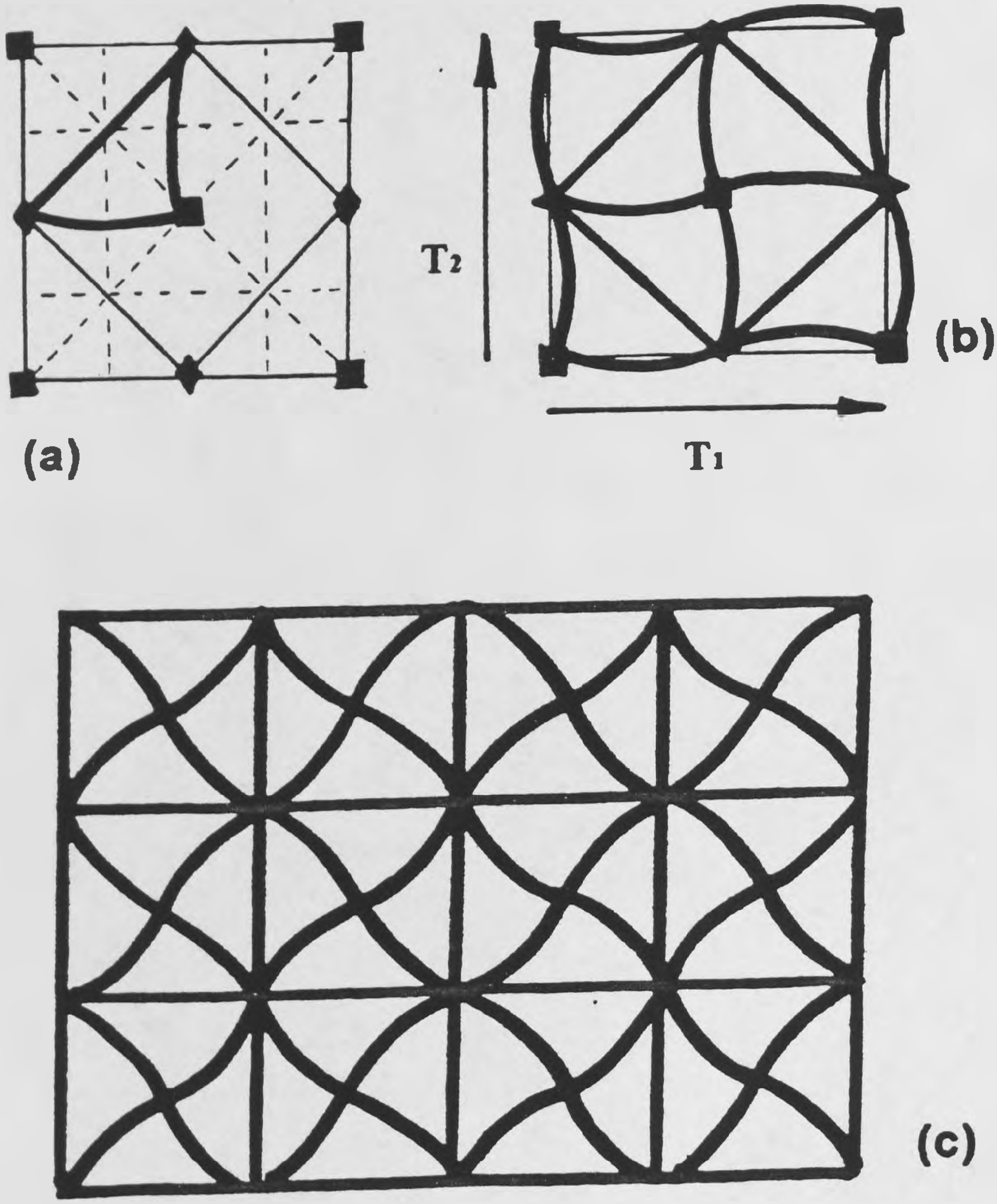


Figure 5.47 The construction of a p4gm periodic plane tiling.



5.6.3 Class p4mm (p4m) patterns and tilings

Class p4mm all-over patterns have a square lattice type unit cell, are generated by reflection in the sides of an isosceles triangle and the highest order of rotation is four. An example of a pattern from this class is provided by Figure 5.48. The conventionally chosen unit cell is illustrated in Figure 5.49a. Four-fold centres of rotation are positioned at each corner and in the centre of the unit cell. Two-fold centres are present at the mid-point of each of the four sides. Each side acts as a reflection axis, and two further reflection axes are diagonally placed within the cell itself. A schematic illustration of a p4mm pattern is shown in Figure 5.49b. The generating region is one-eighth of the unit cell area (Figure 5.49c). A unit cell of the pattern shown in Figure 5.48 is provided by Figure 5.49d.

In the construction of a simplistic p4mm periodic plane tiling, the rotation points can be connected as shown in Figure 5.50a (with a generating region shaded) to produced a translation unit shown in Figure 5.50b. The simple tiling shown in Figure 5.50c is resultant from translation of the unit in two independent non-parallel directions.

Figure 5.48 A class p4mm primary all-over pattern.

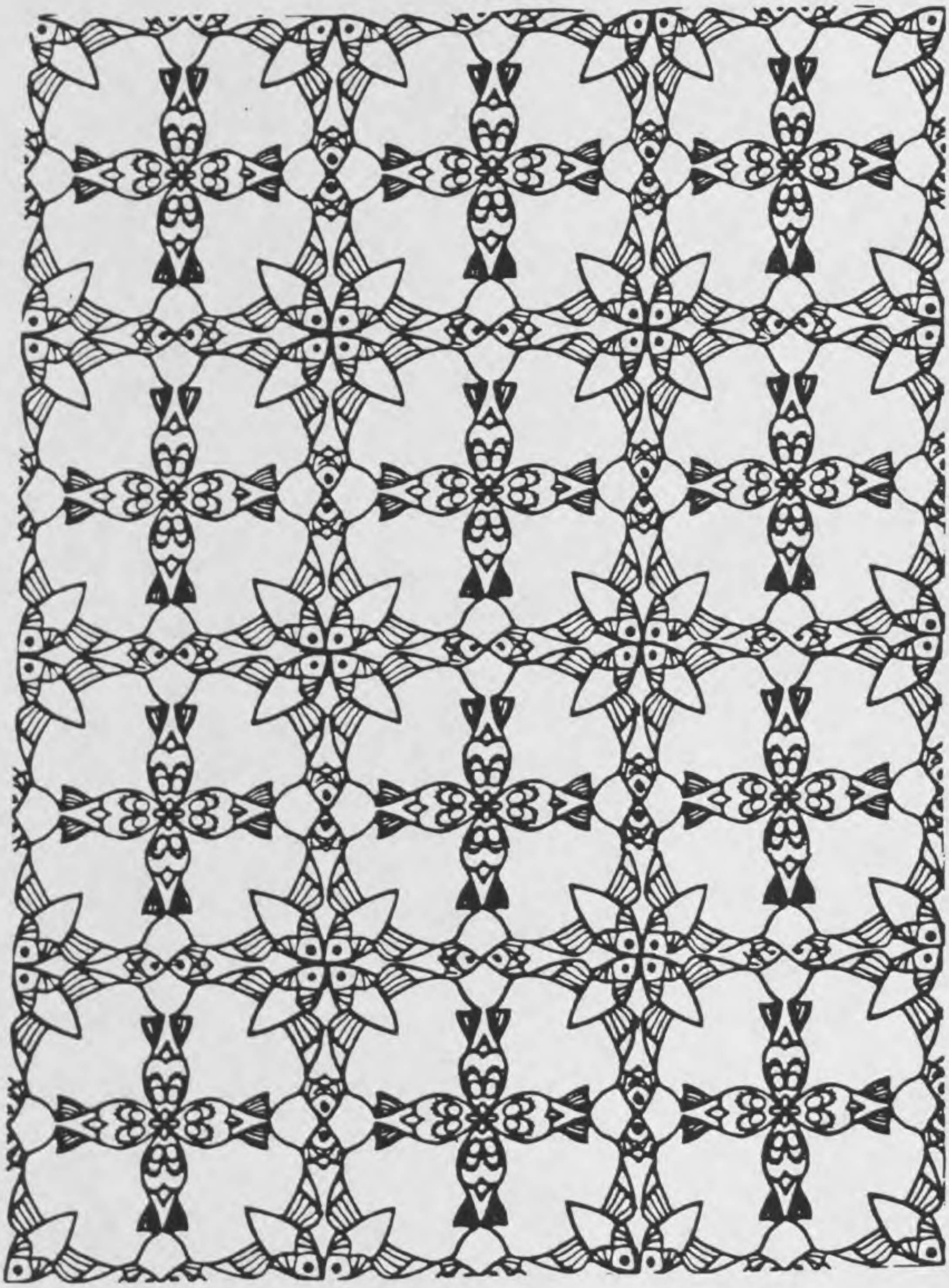


Figure 5.49 Lattice unit, schematic illustration, generating unit and translation for a p4mm all-over pattern.

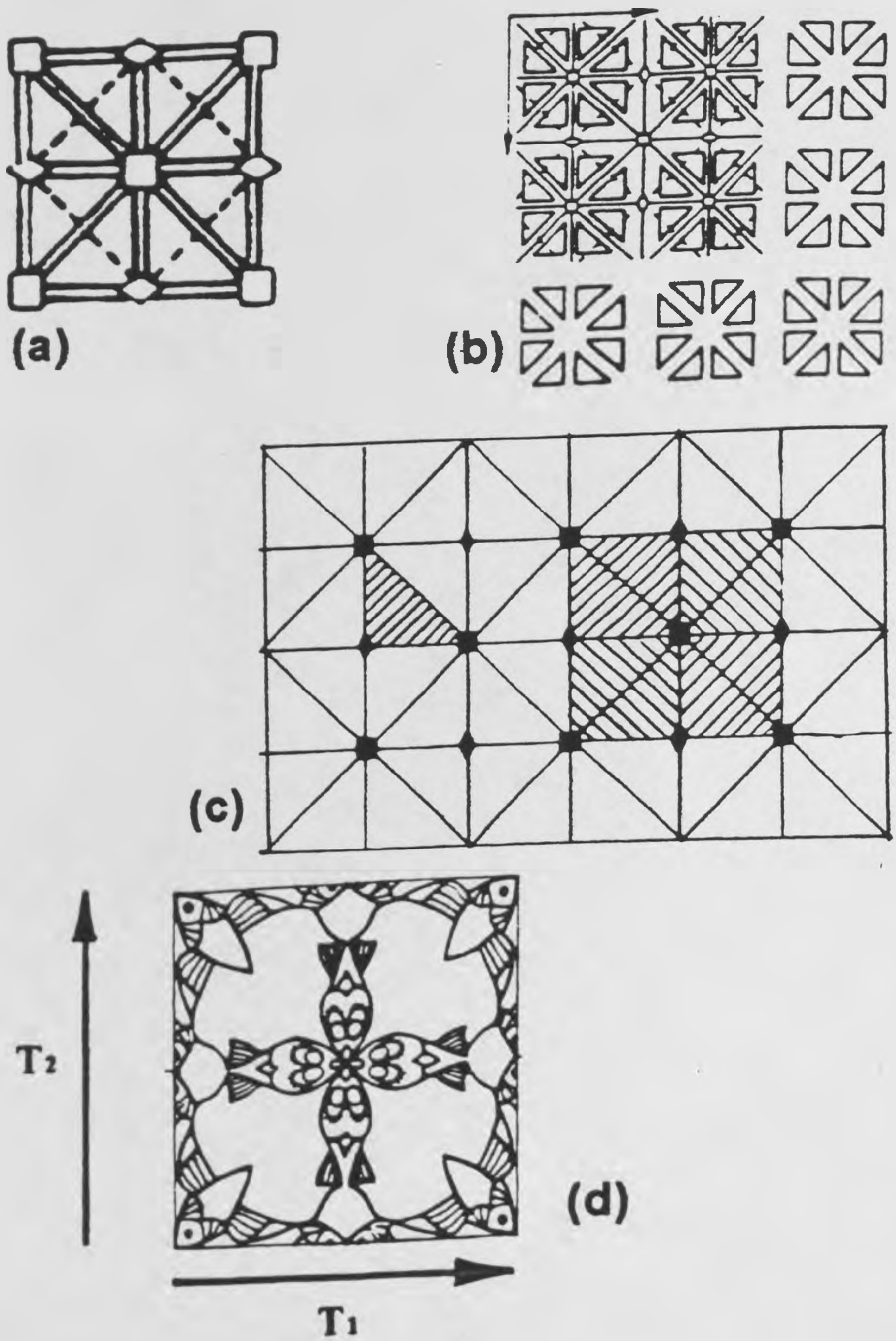
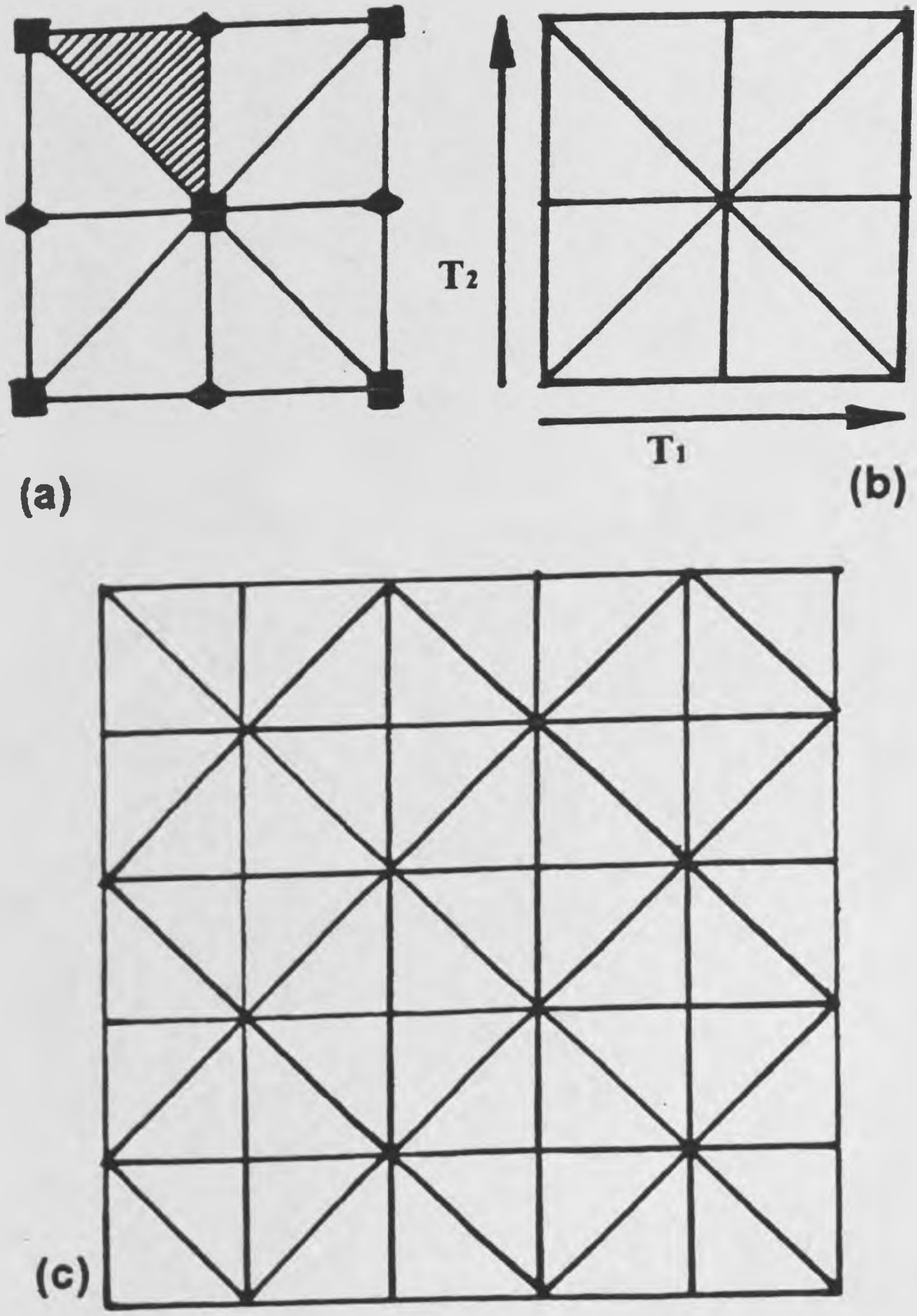


Figure 5.50 The construction of a p4mm periodic plane tiling



5.7 The Classification and Construction of All-over Patterns and Tilings With Six-fold Rotational Symmetry

5.7.1 Class p6 patterns and tilings

Class p6 all-over patterns have a hexagonal lattice type unit cell, with corners falling on six-fold rotational centres. Three-fold rotational centres and two-fold rotational centres are also present. All the six-fold rotational points have the same orientation; the three-fold rotational points have two different orientations; the two-fold rotational points have three different orientations. The fundamental region is one-sixth of the unit cell area. An example from this pattern class is given in Figure 5.51. The hexagonal lattice type unit cell is shown in Figure 5.52a and a schematic illustration of the pattern class is given in Figure 5.52b. The generating region for the pattern shown previously is illustrated by Figure 5.52c and the translation unit is illustrated in Figure 5.52d. The pattern may also be produced by repeated translations of an hexagonal shaped unit (shown in Figure 5.52e).

In the construction of a periodic plane tiling from this class, take the unit cell shape (shown in Figure 5.53a) and join one six-fold rotational centre to a two-fold rotational centre with an arbitrary line. From this two-fold rotational centre place a similar line connecting with another six-fold rotational centre. One side of one of the equilateral triangles is thus complete. Using existing symmetries repeat this line on each of the other two sides. From the three-fold rotational centre in the middle of the same equilateral triangle construct

Figure 5.51 A class p6 primary all-over pattern.

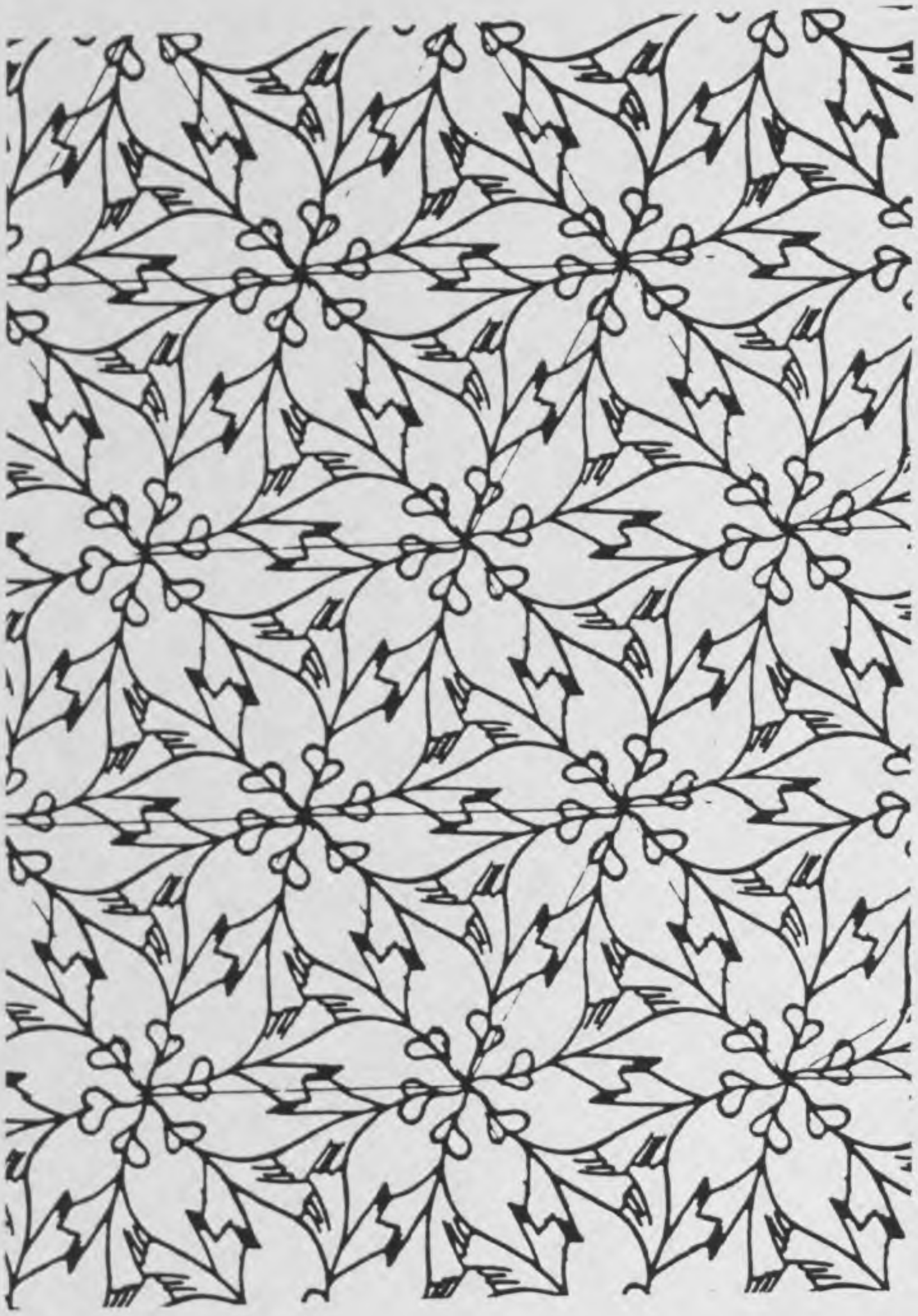


Figure 5.52 Lattice unit, schematic illustration, generating region and translation units for a p6 all-over pattern.

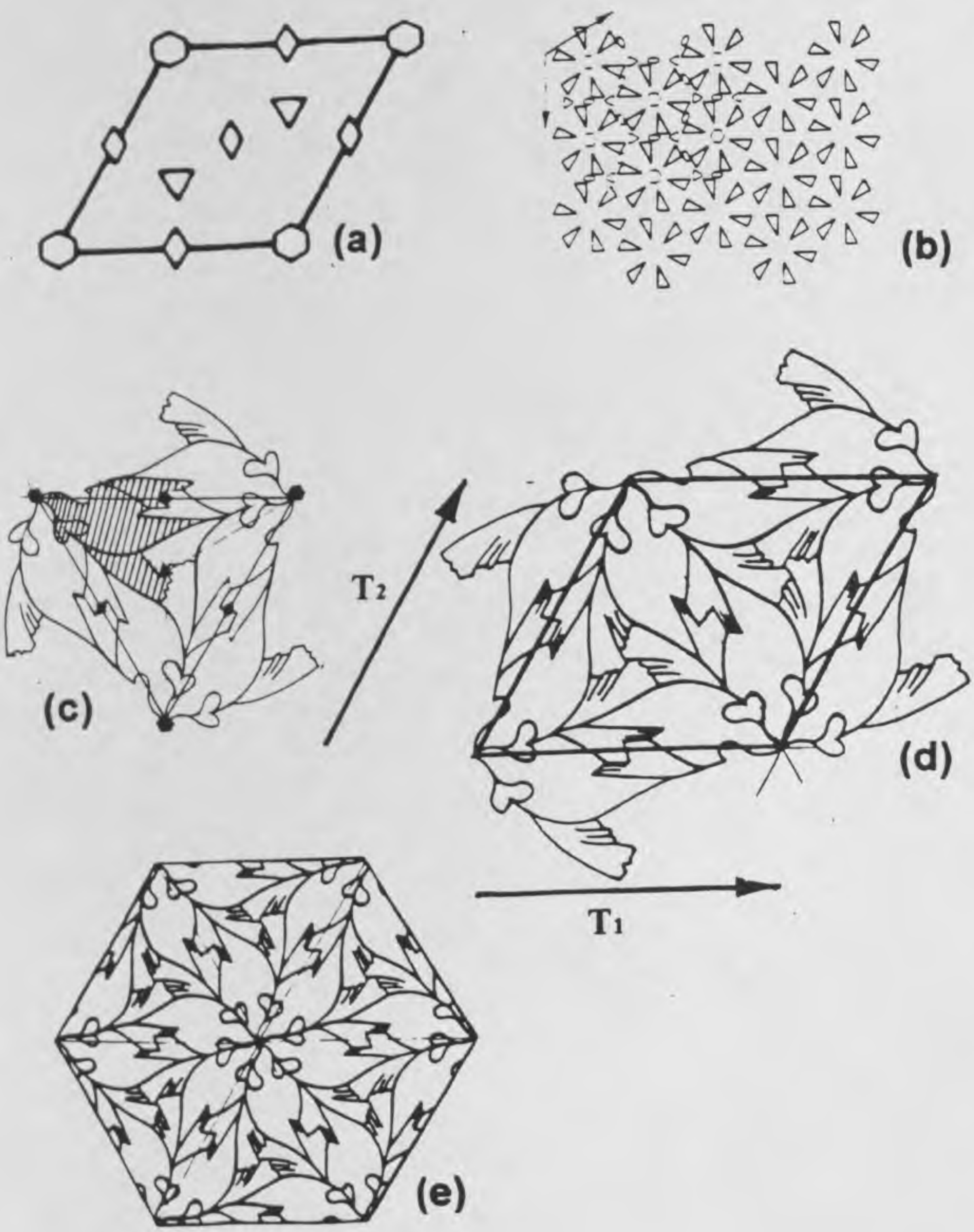
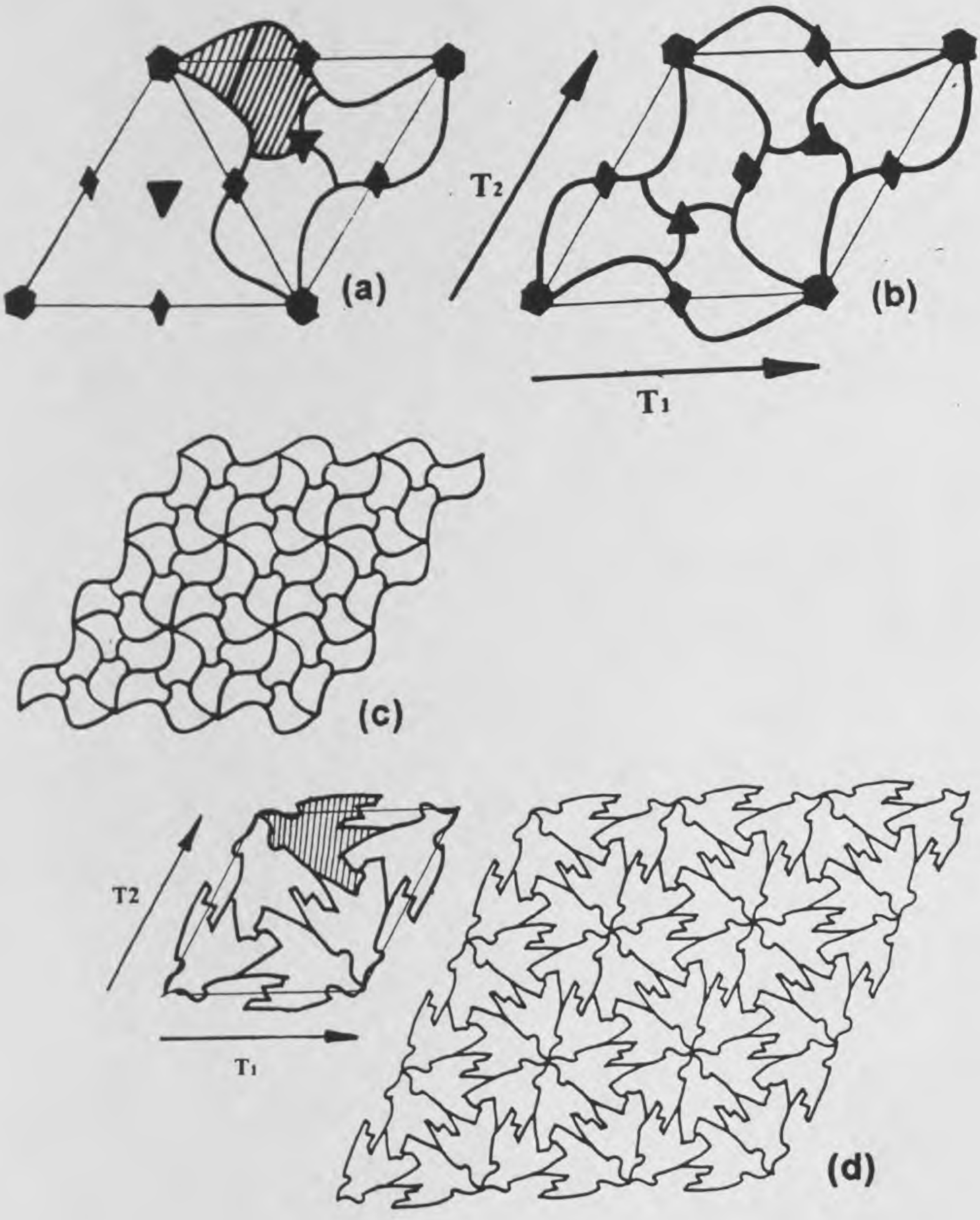


Figure 5.53 The construction of a p6 periodic plane tiling.



three similar lines to connect with each of the boundary lines associated with each side of the chosen equilateral triangle. The resultant generating region is shown shaded in Figure 5.53a and the translation unit is given in Figure 5.53b. The fully translated tiling is shown in Figure 5.53c. A further translation unit and translated tiling are given in Figure 5.53d.

5.7.2 Class p6mm (p6m) all-over patterns and tilings

Class p6mm all-over patterns are generated by a combination of reflections and rotations. Six-fold centres of rotation are present at each corner of the unit cell, which is of the hexagonal lattice type. Reflection axes connect each corner with the other three corners. In addition each side is bisected by a reflection axis. Three-fold and two-fold rotational centres are also present and located on intersections of reflection axes. The generating region is one twelfth the area of the unit cell. An example of a pattern from this class is given in Figure 5.54. An example of the conventionally chosen unit cell is given in Figure 5.55a and a schematic illustration of the pattern is given in Figure 5.55b. The symmetry operations present in the pattern, together with an indication of the relative position of the generating region is given in Figure 5.55c. Alternative translation units for the pattern shown previously are given in Figures 5.55d and e.

Figure 5.54 A class p6mm primary all-over pattern

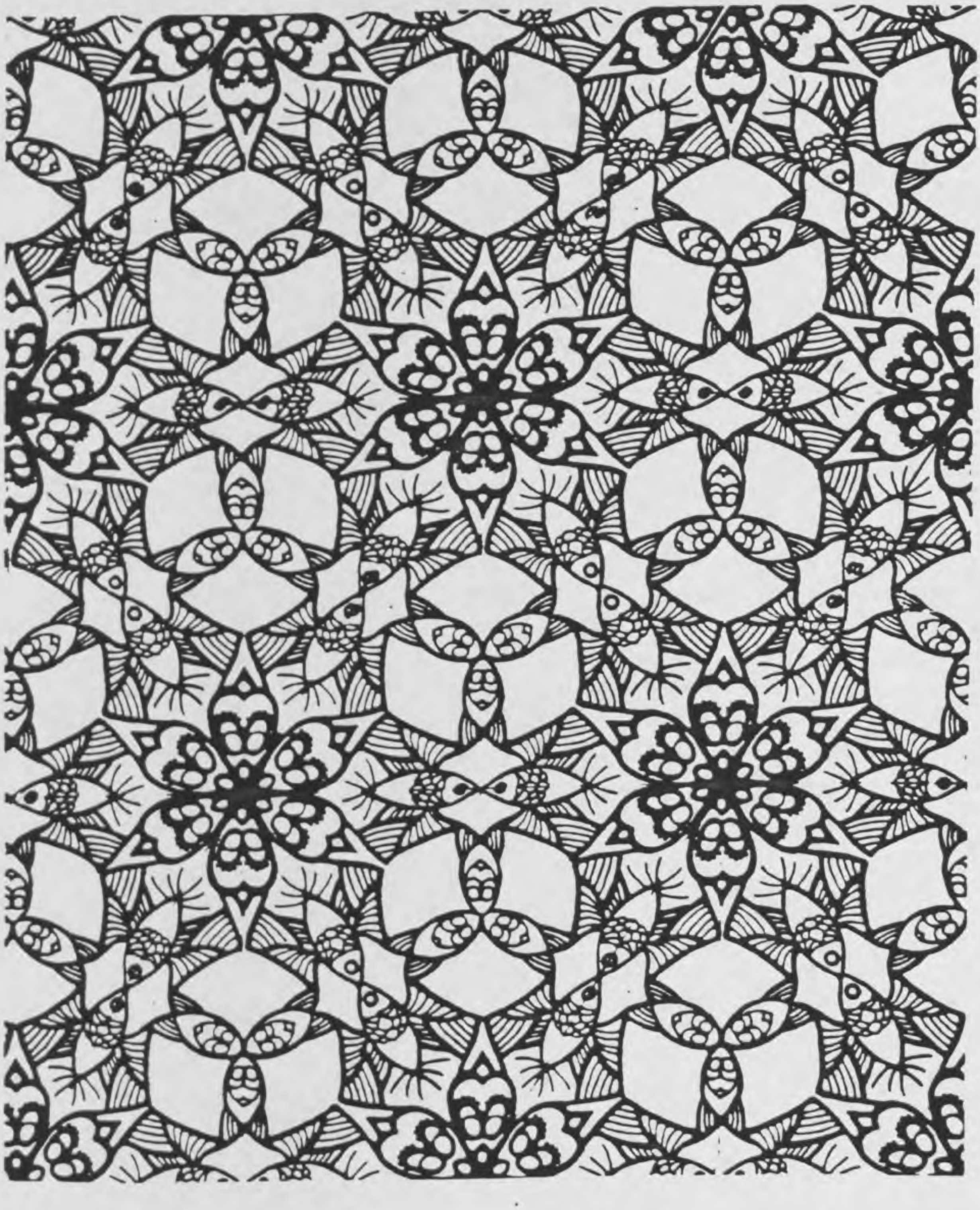
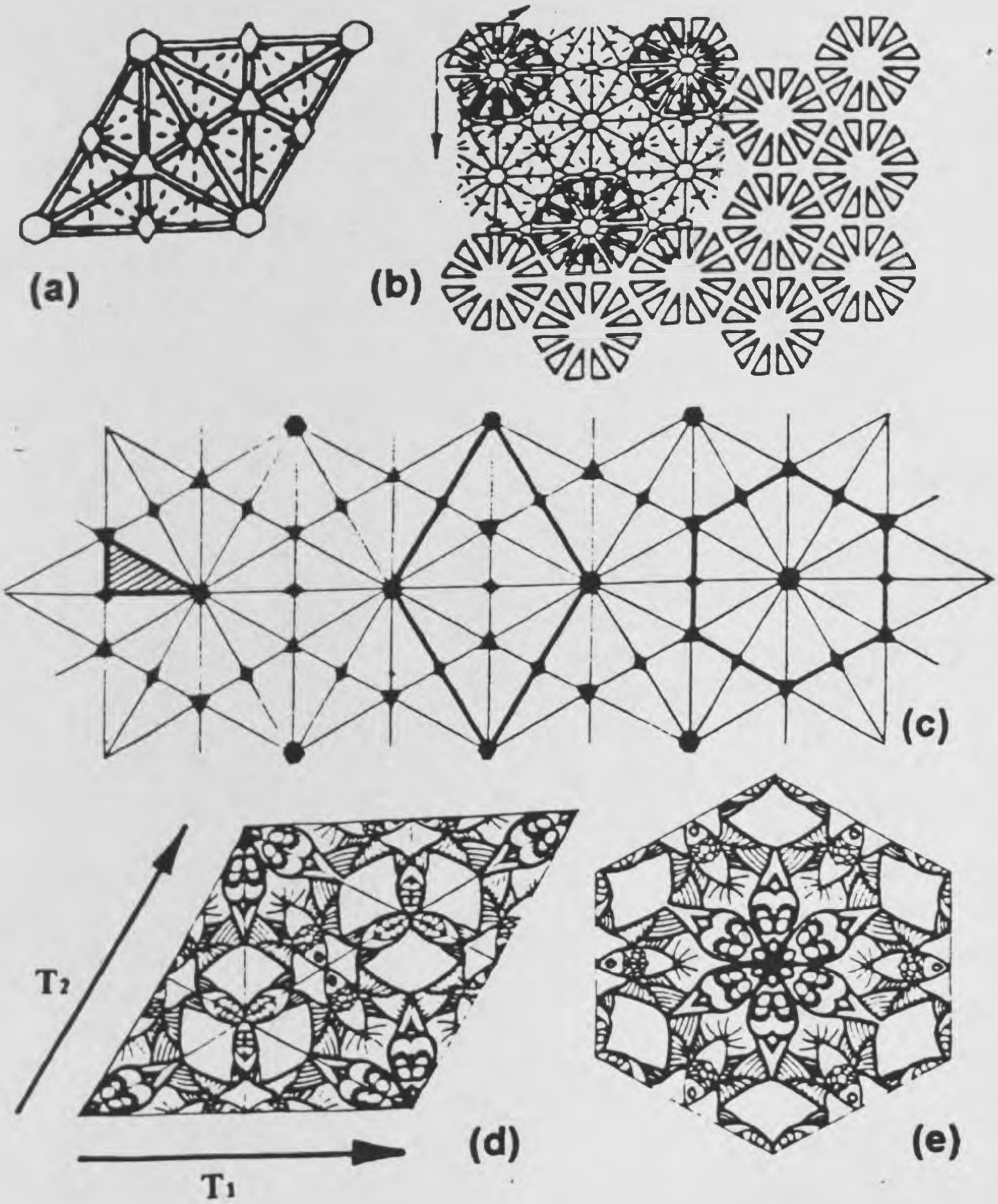


Figure 5.55 Lattice unit, schematic illustration, generating region and translation units for a p6mm all-over pattern.



In the construction of a periodic plane tiling of class p6mm, take an hexagonal type unit cell (as shown in Figures 5.56a and b) and connect all angles to opposite angles and to opposite sides as shown. When translated in two independent non-parallel directions the tiling shown in Figure 5.56c can be produced.

5.8 Summary

Primary all-over patterns and tilings may be classified into seventeen classes dependent upon the symmetry operations used in their construction. By way of illustrative summary, Figure 5.57 provides a recognition chart for each of the seventeen all-over pattern classes. Periodic plane tilings exhibit the same symmetry characteristics as all-over patterns and are classified accordingly. A recognition chart for each of these seventeen classes of tilings is provided in Figure 5.58. A flow-diagram to aid the identification of a pattern's symmetry class is provided in Figure 5.59.

Figure 5.56 The construction of a p6mm periodic plane tiling.

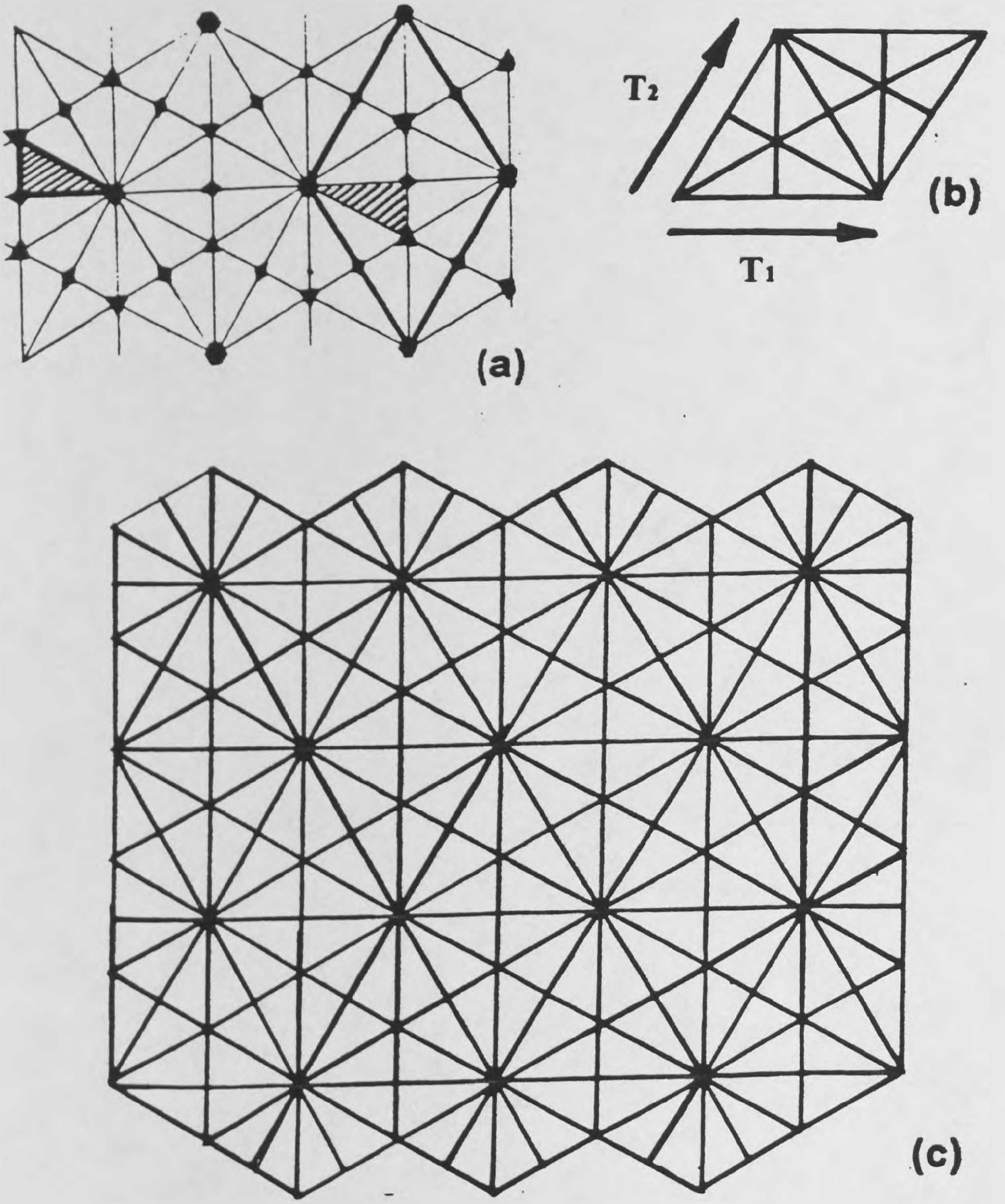


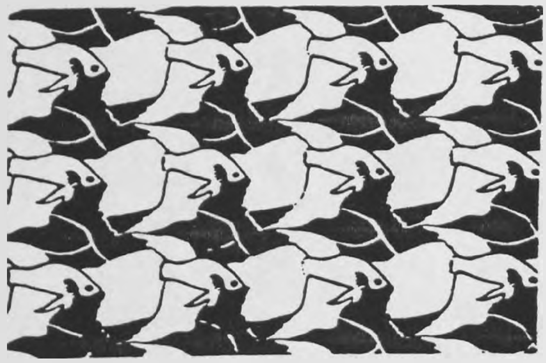
Figure 5.57 The Recognition Chart for The Seventeen Classes of Primary All-Over Patterns.

Symmetry Class | **Translation unit** | **Generating region** | **All-over pattern**

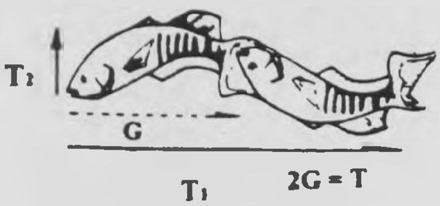
---

p1

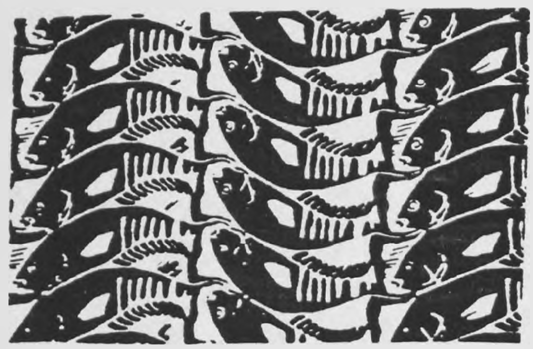
T



p1g1

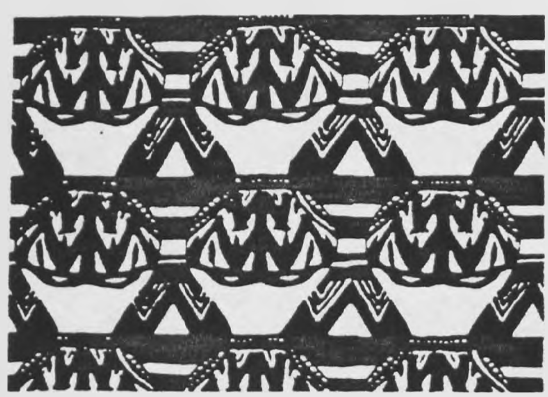


1/2T



p1m1

1/2T



c1m1

1/2T

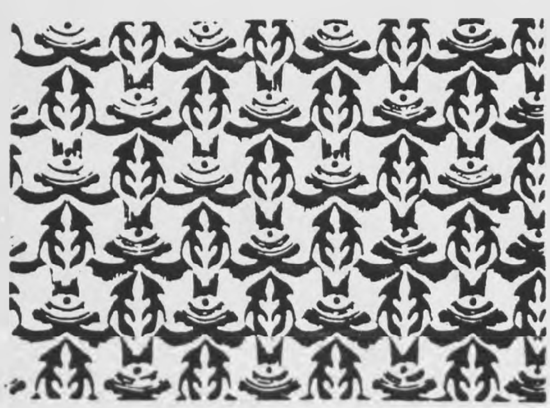
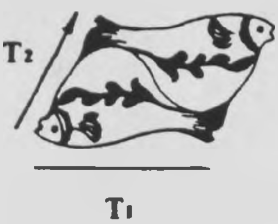
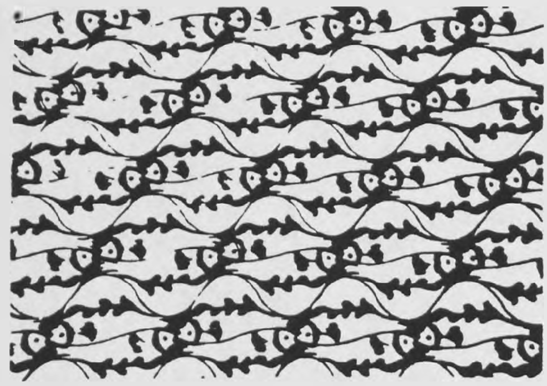


Figure 5.57 (cont.) The Recognition Chart for The Seventeen Classes of Primary All-Over Patterns.

p211

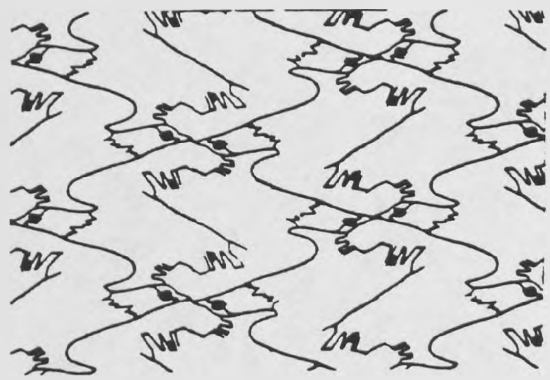


1/2T

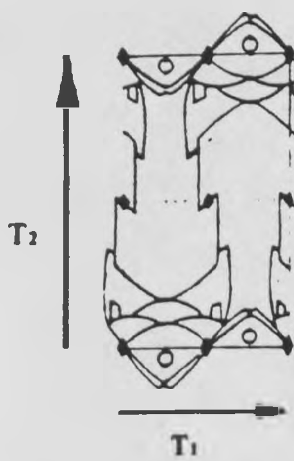


p2gg

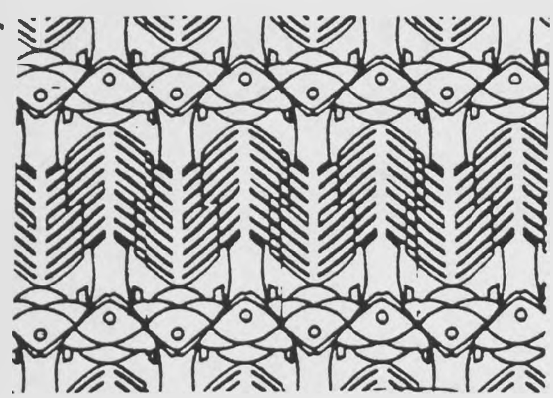
1/4T



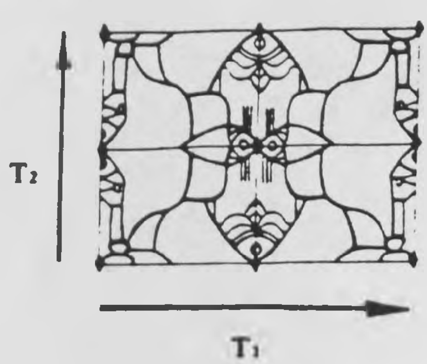
p2mg



1/4T



p2mm



1/4T

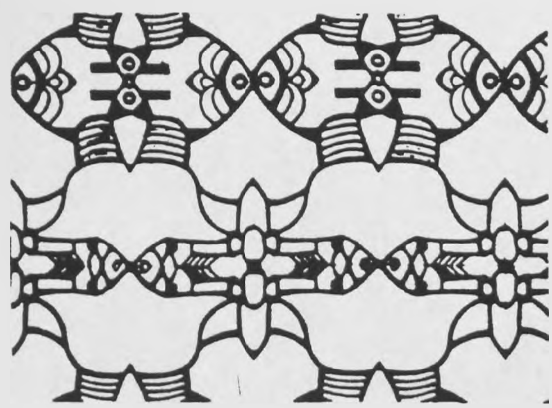
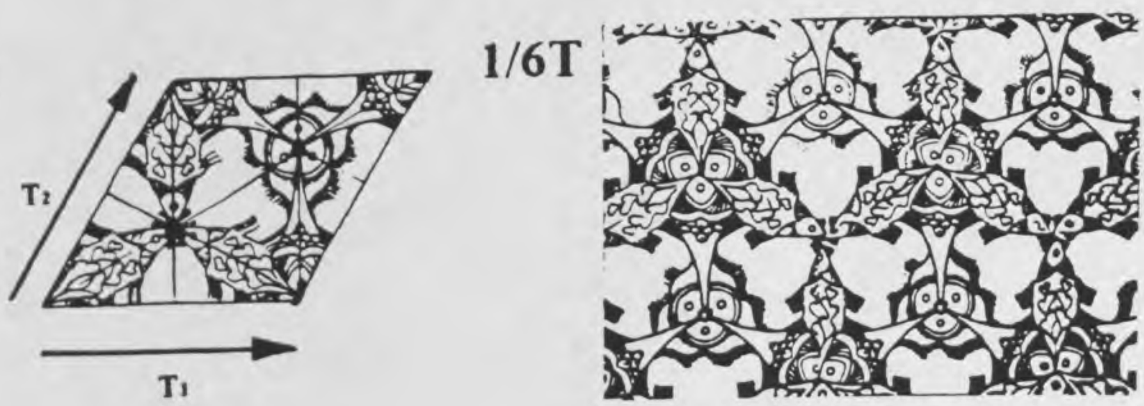


Figure 5.57 (cont.) The Recognition Chart for The Seventeen Classes of Primary All-Over Patterns.

c2mm

p3

p3m1



p31m

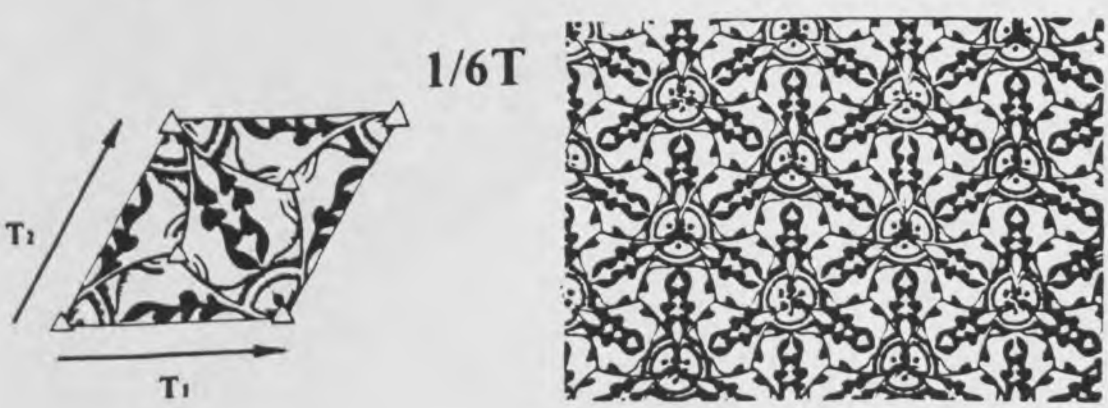


Figure 5.57 (cont.) The Recognition Chart for The Seventeen Classes of Primary All-Over Patterns.

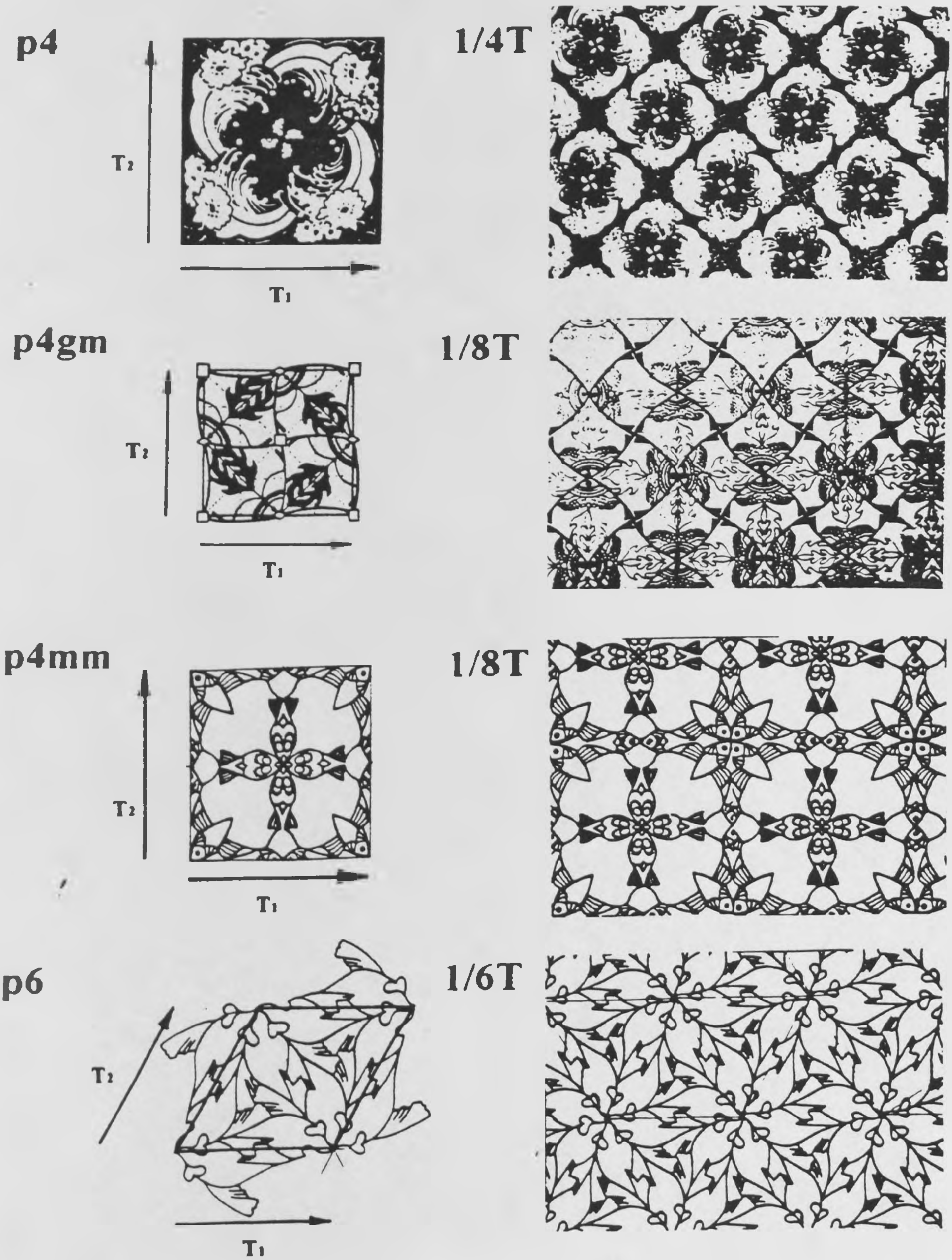
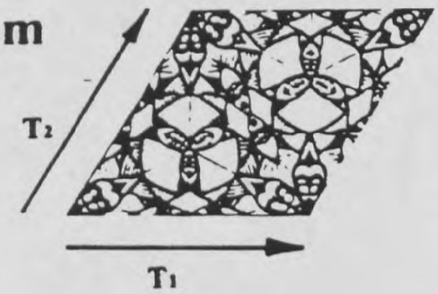


Figure 5.57 (cont.) The Recognition Chart for The Seventeen Classes of Primary All-Over Patterns.

p6mm



1/12T

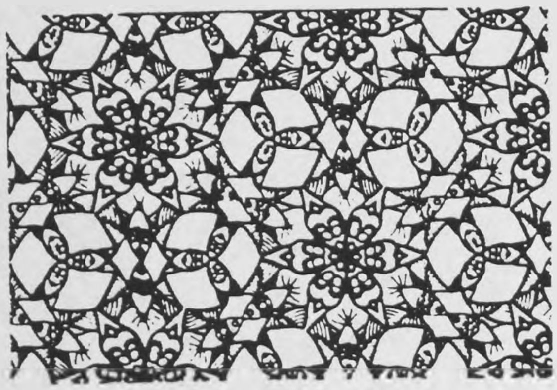
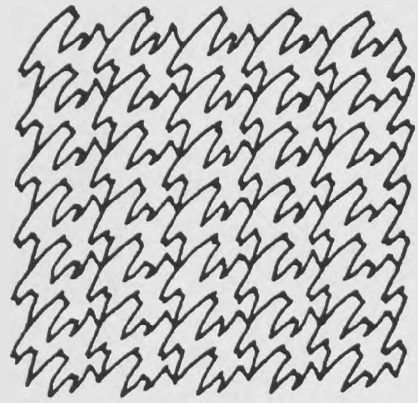


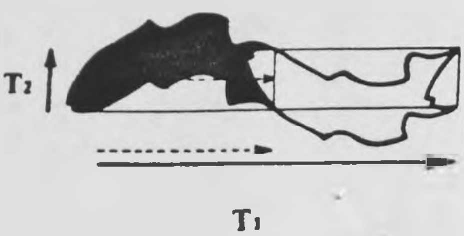
Figure 5.58 The Recognition Chart for The Seventeen Classes of Primary Periodic Plane Tilings

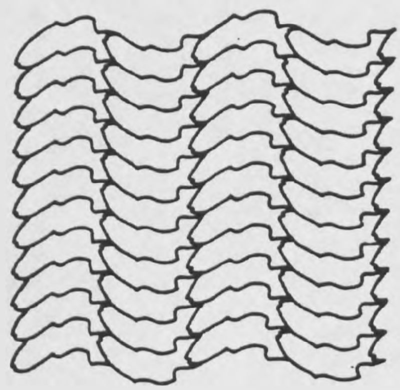
Symmetry class	Translation unit with generating regions (shaded)	Periodic plane tiling

p1

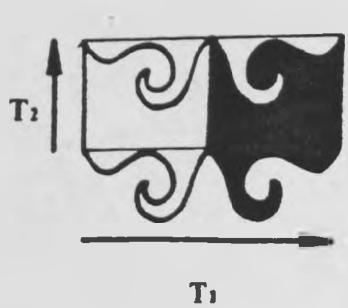


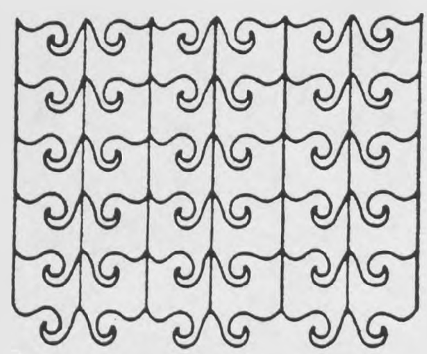
p1g1



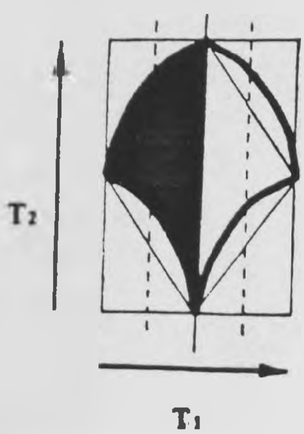


p1m1





c1m1



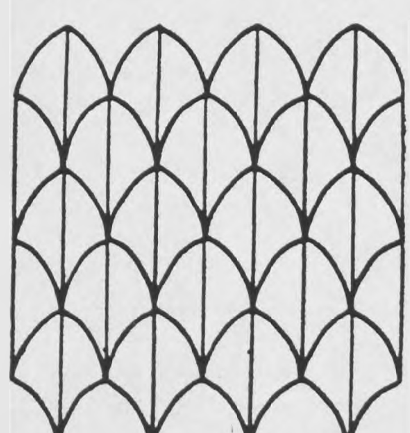
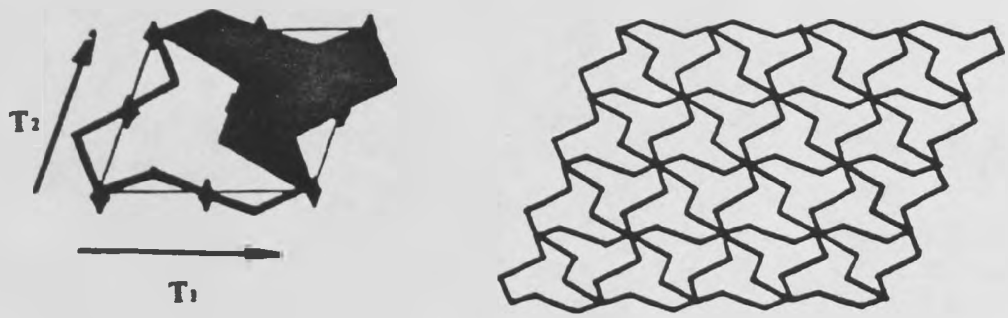
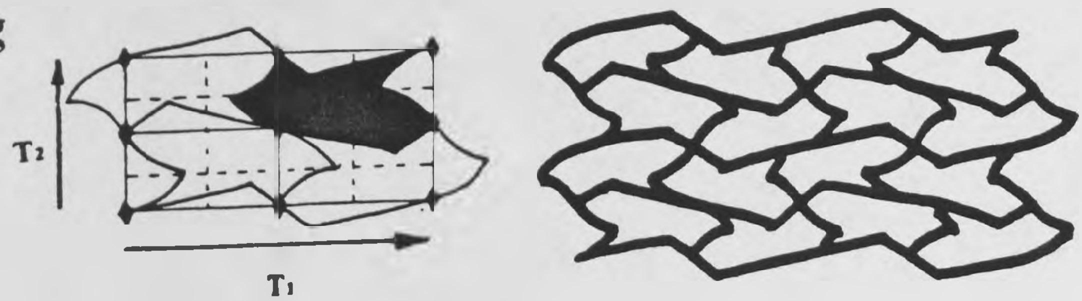


Figure 5.58 (cont) The Recognition Chart for The Seventeen Classes of Primary Periodic Plane Tilings

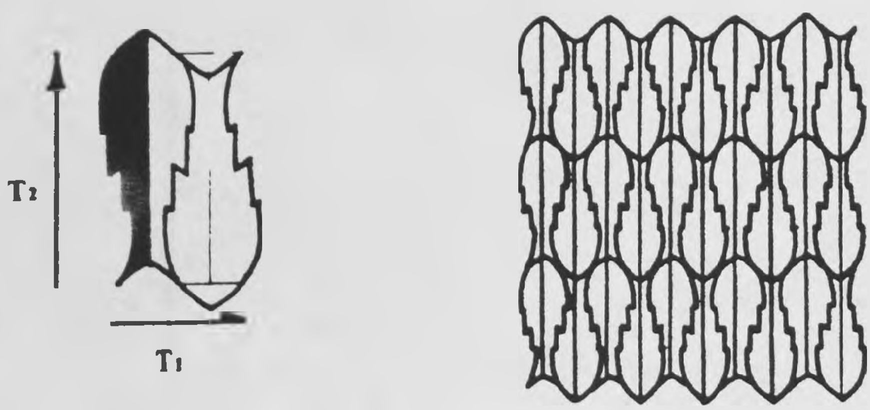
p211



p2gg



p2mg



p2mm

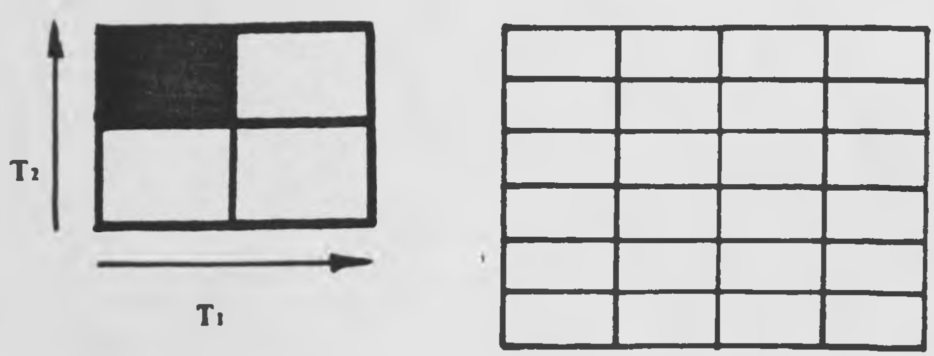
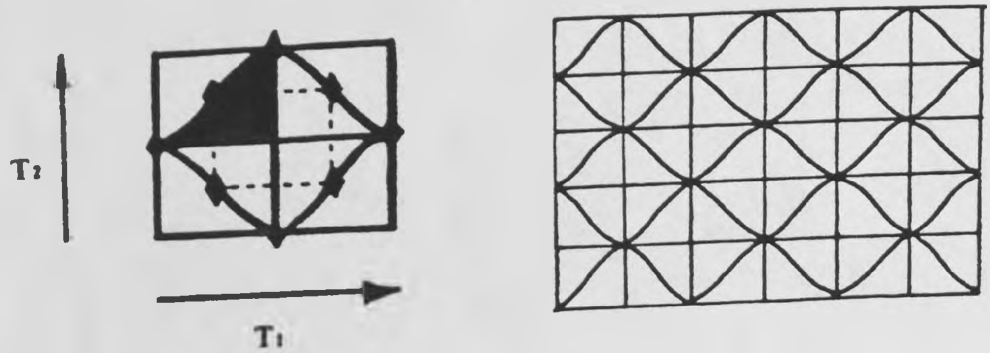
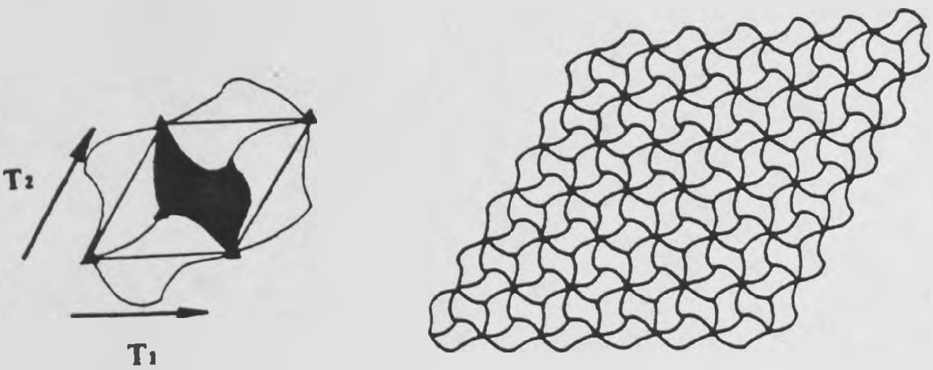


Figure 5.58 (cont) The Recognition Chart for The Seventeen Classes of Primary Periodic Plane Tilings

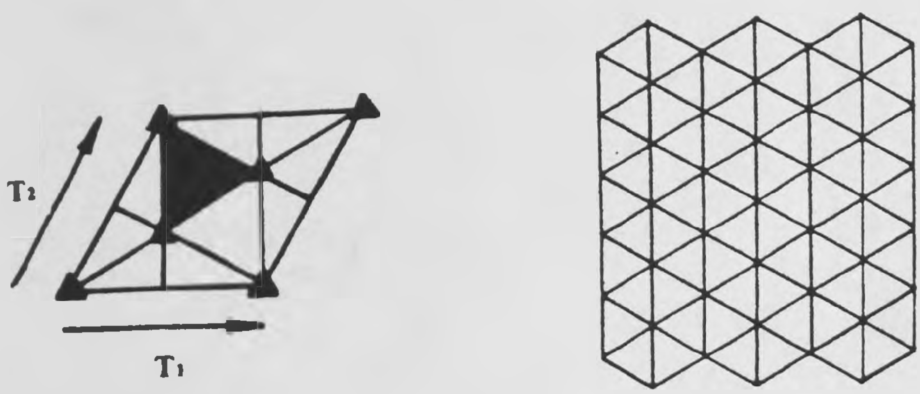
c2mm



p3



p3m1



p31m

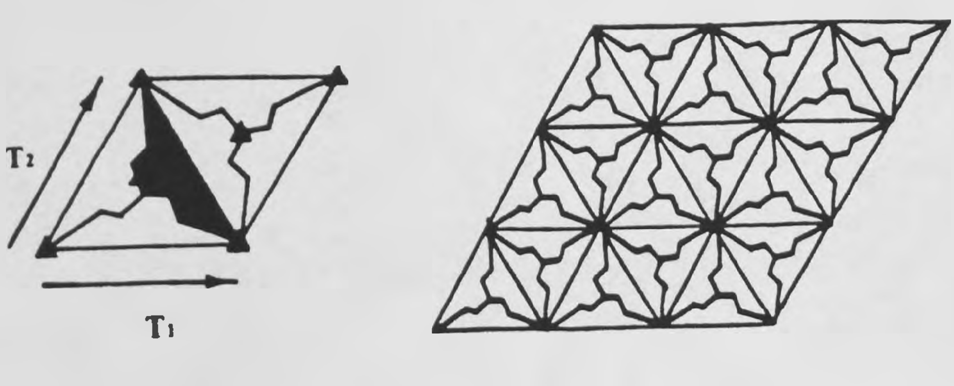
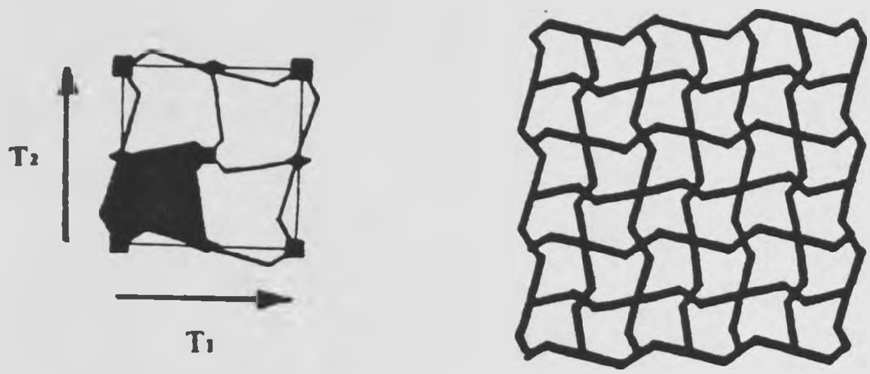
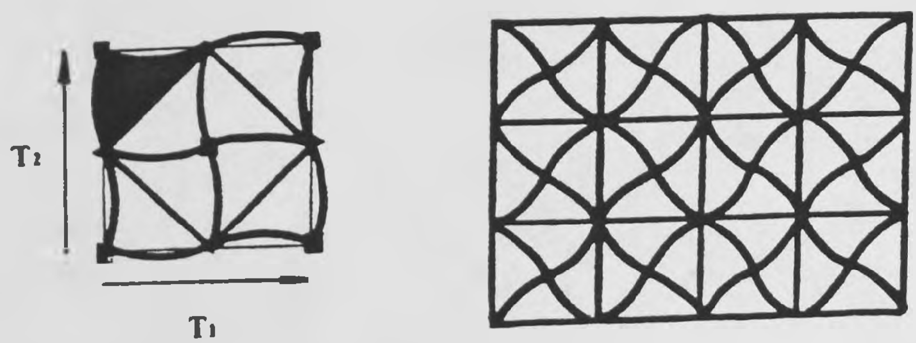


Figure 5.58 (cont) The Recognition Chart for The Seventeen Classes of Primary Periodic Plane Tilings

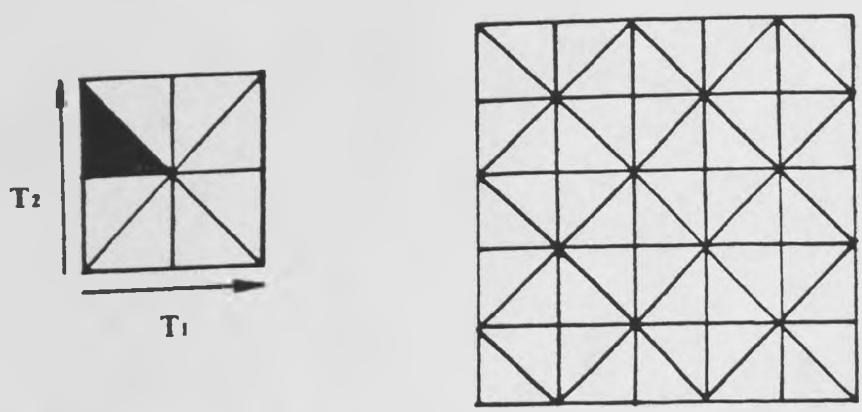
p4



p4gm



p4mm



p6

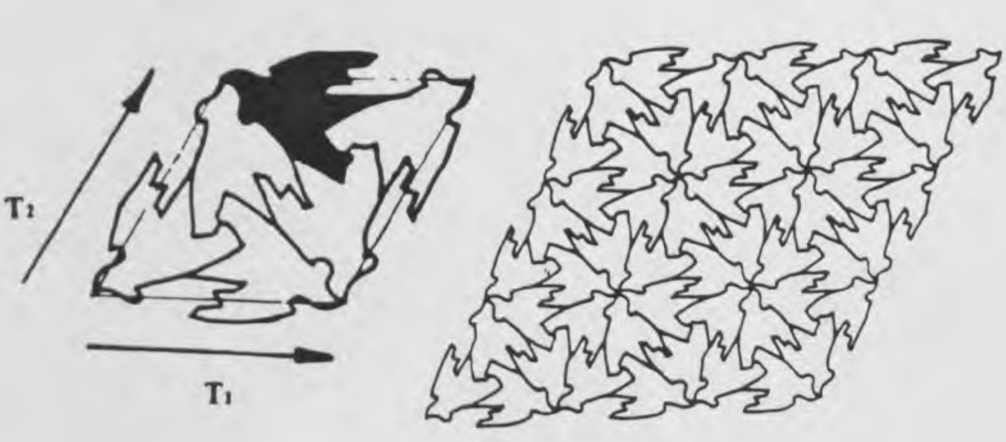
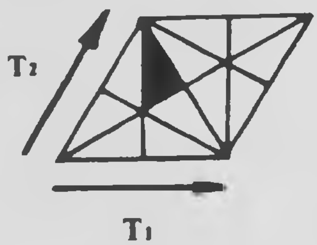


Figure 5.58 (cont) The Recognition Chart for The Seventeen Classes of Primary Periodic Plane Tilings

p6mm



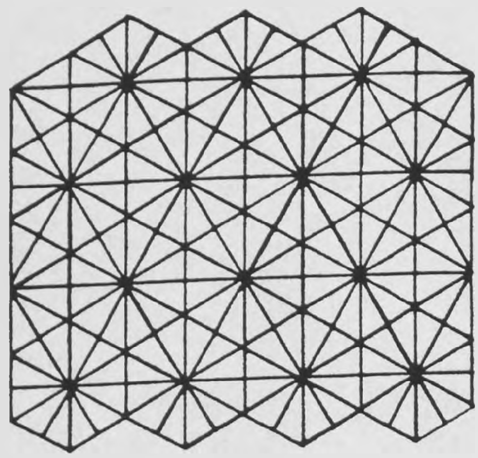
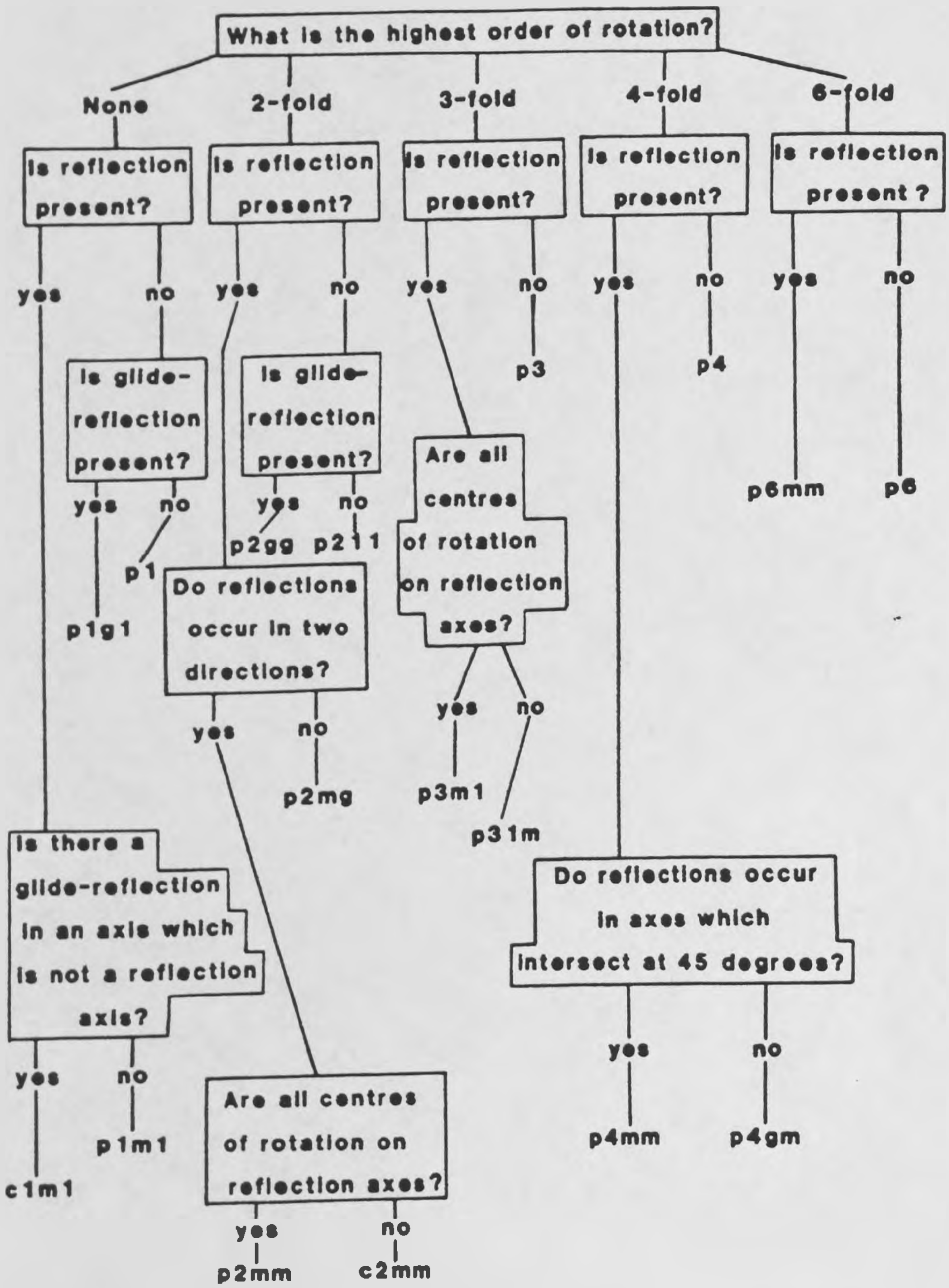


Figure 5.59 A flow diagram to aid the identification of an all-over pattern's symmetry class.



Source: Hann, M.A. and Thomson, G., 'The Geometry of Regular Repeating Patterns' Textile Progress Series, vol.22, no.1, The Textile Institute, Manchester, 1992, p.49.

6 COLOUR COUNTERCHANGE DESIGNS: CLASSIFICATION AND CONSTRUCTION

6.1 Introduction

The emphasis in explanations and discussion, so far, has been on symmetry operations which do not involve colour change, i.e. colour has been preserved following each symmetry operation. It may however by the case that, following certain symmetry operations, colours are systematically changed in a continuous way. Such designs are termed "counterchange designs" (Christie [72], Woods [73] and Gombrich [74]). Examples of two-colour counterchange patterns are given in Figures 6.1 and 6.2. In each case colour is systematically changed across all reflection axes (ABC and D in Figure 6.1 and A, B and C in Figure 6.2). In each of these counterchange examples, reflections thus reverse (or interchange) colours, and rotations preserve colours.

An array of mathematical literature is available on the subject of colour symmetry; Schwarzenberger [75], for example, lists over 100 research papers or other works related to the subject. Historically, the most important works on colour symmetry in patterns have been produced by Woods [76], Senechal [77], Loeb [78], Lockwood and Macmillan [79], Grunbaum and Shephard [80], Schattschneider [81], Washburn and Crowe [82] and Wieting [83].

Figure 6.1 An example of a two-colour counterchange all-over pattern.

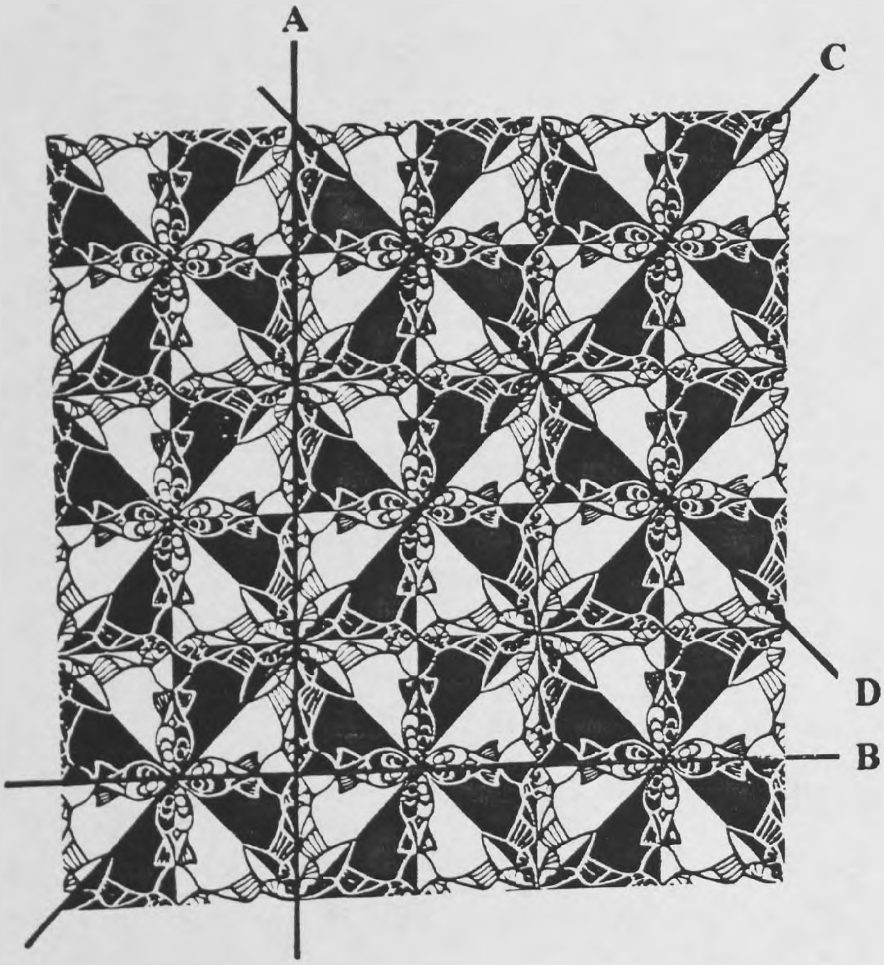
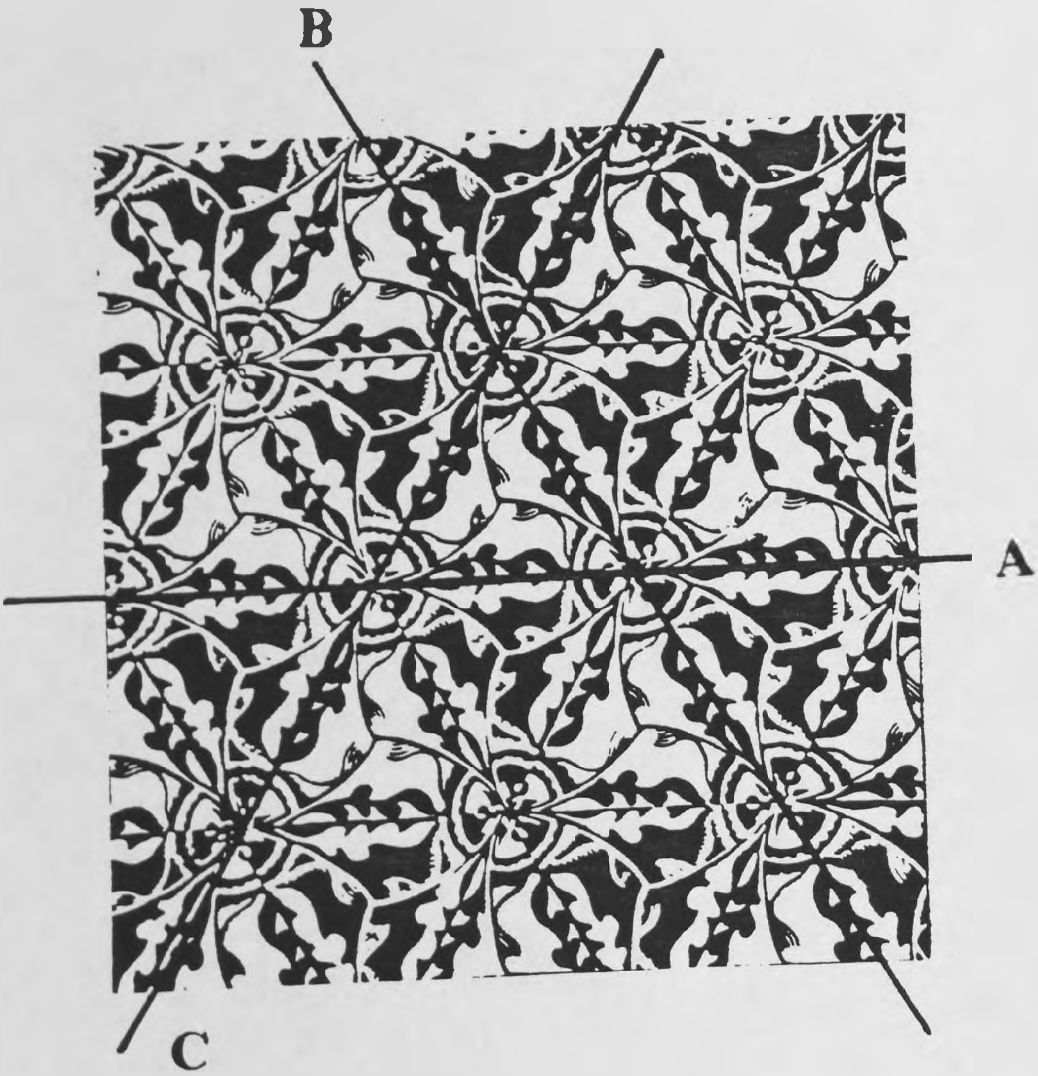


Figure 6.2 An example of a two-colour counterchange all-over pattern.



The objective of this chapter is to describe and illustrate the principles of colour counterchange applied to motifs, border patterns and all-over patterns (as well as tilings). The notation used for two-colour counterchange designs is derived from Washburn and Crowe [84] and Schattschneider [85]. Where reference is made to three-colour counterchange designs the straightforward notation of Grunbaum and Shephard [86] is used.

6.2 Colour Counterchange Motifs

As pointed out by Washburn and Crowe [87], there is only one way to colour systematically a cn motif with two colours, and that is to alternate the colours around the design. Such a colouring is only possible where n (the order of rotation) is an even number. A prime (') can be introduced into the standard notation to denote colour change when the corresponding symmetry operation is performed. Schematic illustrations of counterchange cn motifs are shown in Figure 6.3. Examples of c2', c4' and c6' motifs are shown in Figures 6.4a, b and c respectively. It can therefore be seen that class cn motifs may be perfectly coloured with two colours where n is a multiple of 2 (i.e. n equals 2 or 4 or 6 or 8 etc). Where n is an odd number it is therefore not possible to systematically introduce two-colour interchange. Where more than two colours are available further possibilities may arise. Where the number of colours is 3 (i.e. K=3) and the order of rotation is a multiple of 3 the corresponding motifs may be perfectly coloured. Where K equals 4, perfect colouring can be introduced into c4, c8, c12 (etc) motifs. Where K equals 5 perfect colouring can be introduced into c5, c10, c15 (etc) motifs.

Figure 6.3 Counterchange cn' motifs.

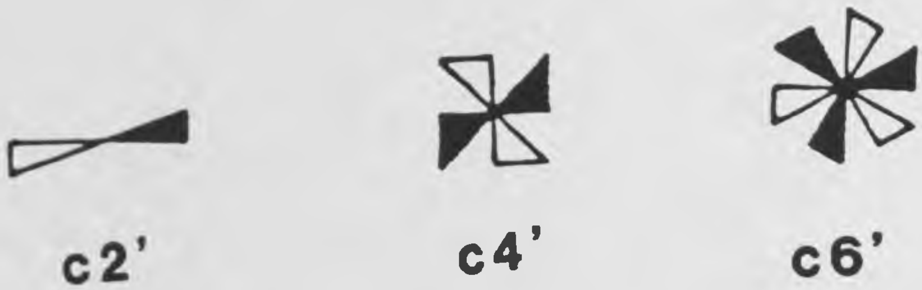
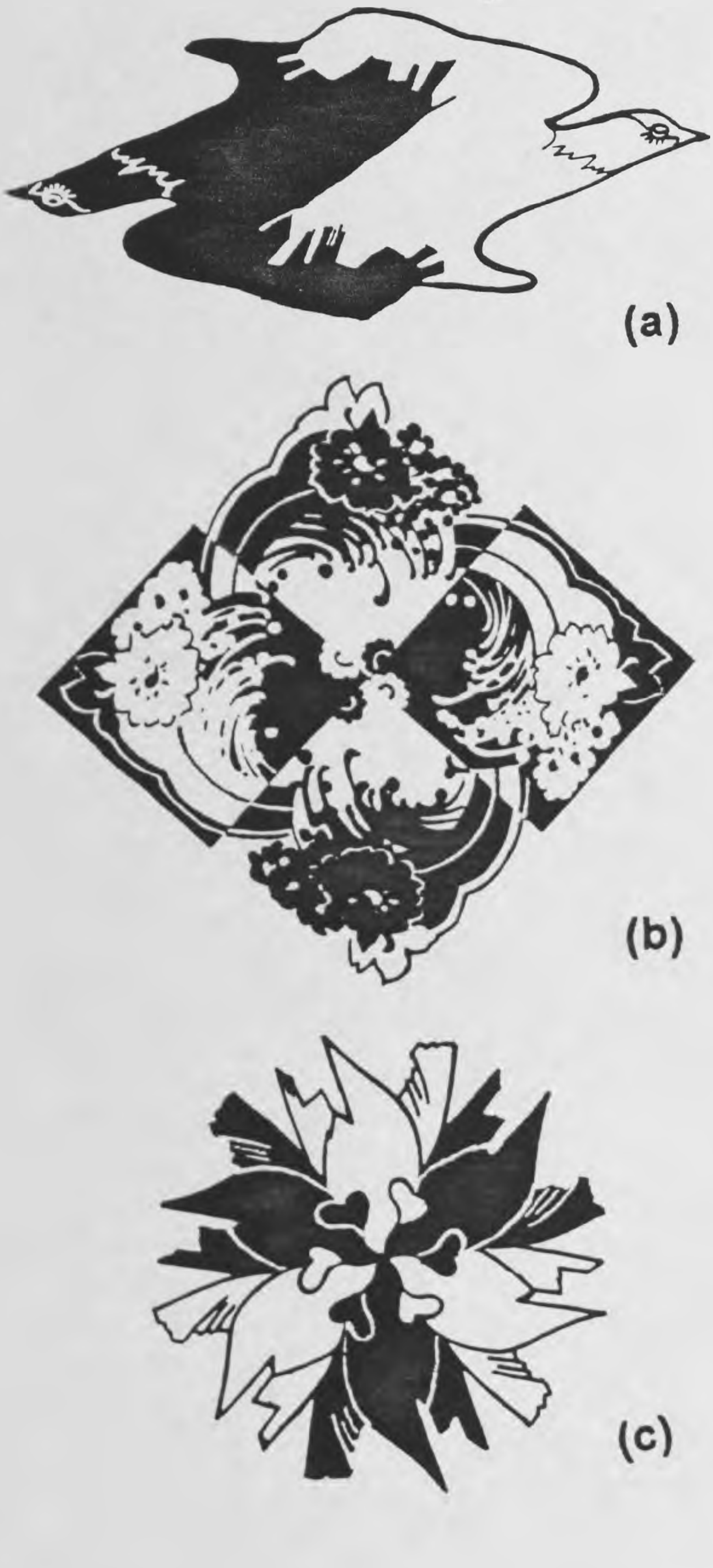


Figure 6.4 Examples of c2', c4' and c6' motifs.



Where k equals 6, perfect colouring can be introduced into c6, c12, c18 (etc) motifs. Figure 6.5 provides schematic illustrations of perfectly coloured cn motifs, from class c2 to class c6.

In the case of dn motifs (which admit n-fold rotational symmetry combined with n reflection axes) there are, as recognised by Washburn and Crowe [88], two possibilities to systematically introduced two-colour interchange. Where n is an odd number, only one type of colouring is possible; in this case all reflection operations reverse the colours and all rotation operations preserve the colours. Relevant schematic illustrations are provided by Figure 6.6. Where n is an even number , half of the reflections reverse colours and half preserve colours, and rotations by one nth of 360 degrees reverse colours. Relevant schematic illustrations are provided in Figure 6.7.

To systematically colour a dn motif with three or more colours, the number of colours involved (denoted by K) must be a factor of the number of fundamental units within the motif. Where n equals 3, 6, 9 etc (in d6, d12, d18 etc motifs) it is possible to perfectly colour with three colours. Where n equals 4, 8, 12 etc (in d8, d16, d24 etc motifs) it is possible to perfectly colour with four colours. These conditions are shown schematically in Figure 6.8 and illustrated further in Figure 6.9.

Figure 6.5 Illustrations of perfectly coloured class cn motifs (from class c2 to class c6).

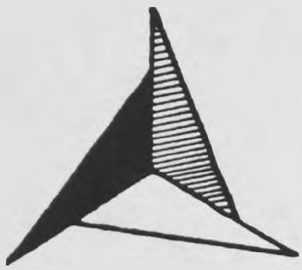
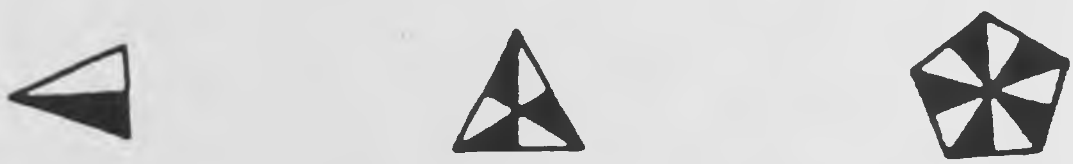
Symmetry class	**Number of colours involved in colouring (denoted by K)**	**Examples of perfectly coloured class cn motifs (From class c2 to class c6)**
C2	**K=2**	
C3	**K=3**	
C4	**K=2or 4**	 
C5	**K=5**	
C6	**K=2or3or6**	  

Figure 6.6 Counterchange d'n motifs (n = odd numbers).



d'1 d'3 d'5

Figure 6.7 Counterchange d'n and dn' motifs (n = even numbers).

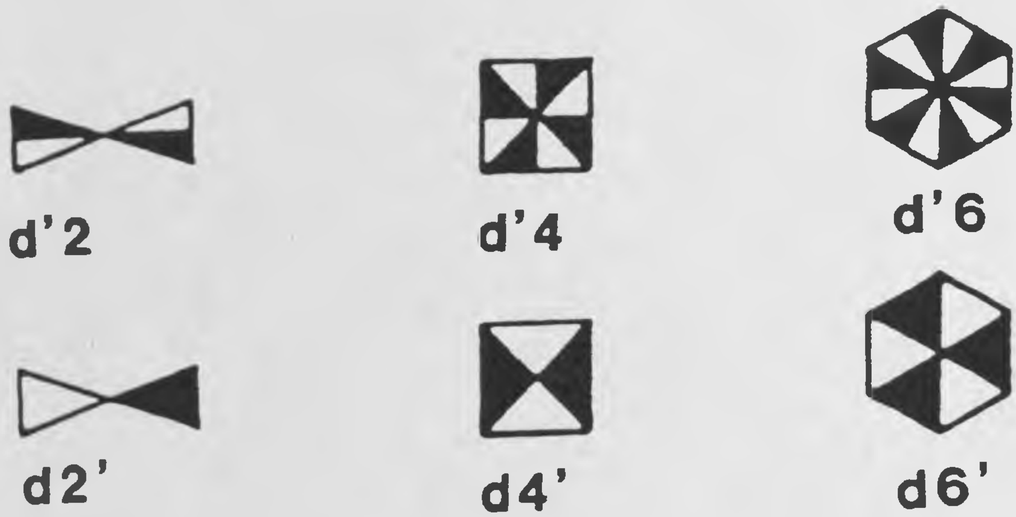


Figure 6.8 Illustrations of perfectly coloured class dn motifs.

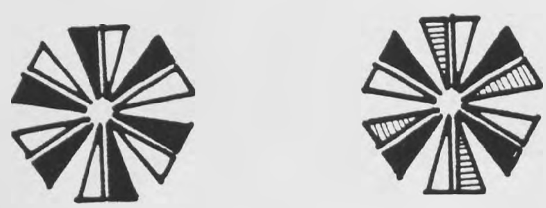
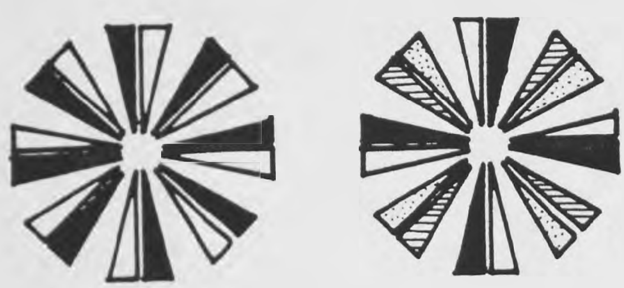
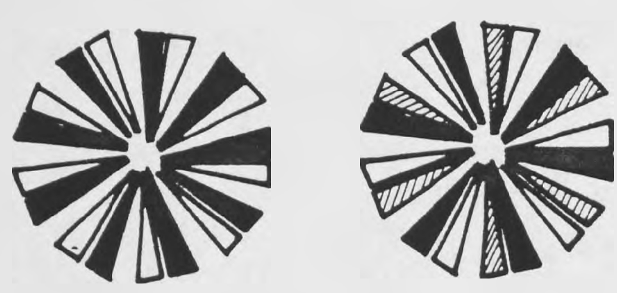
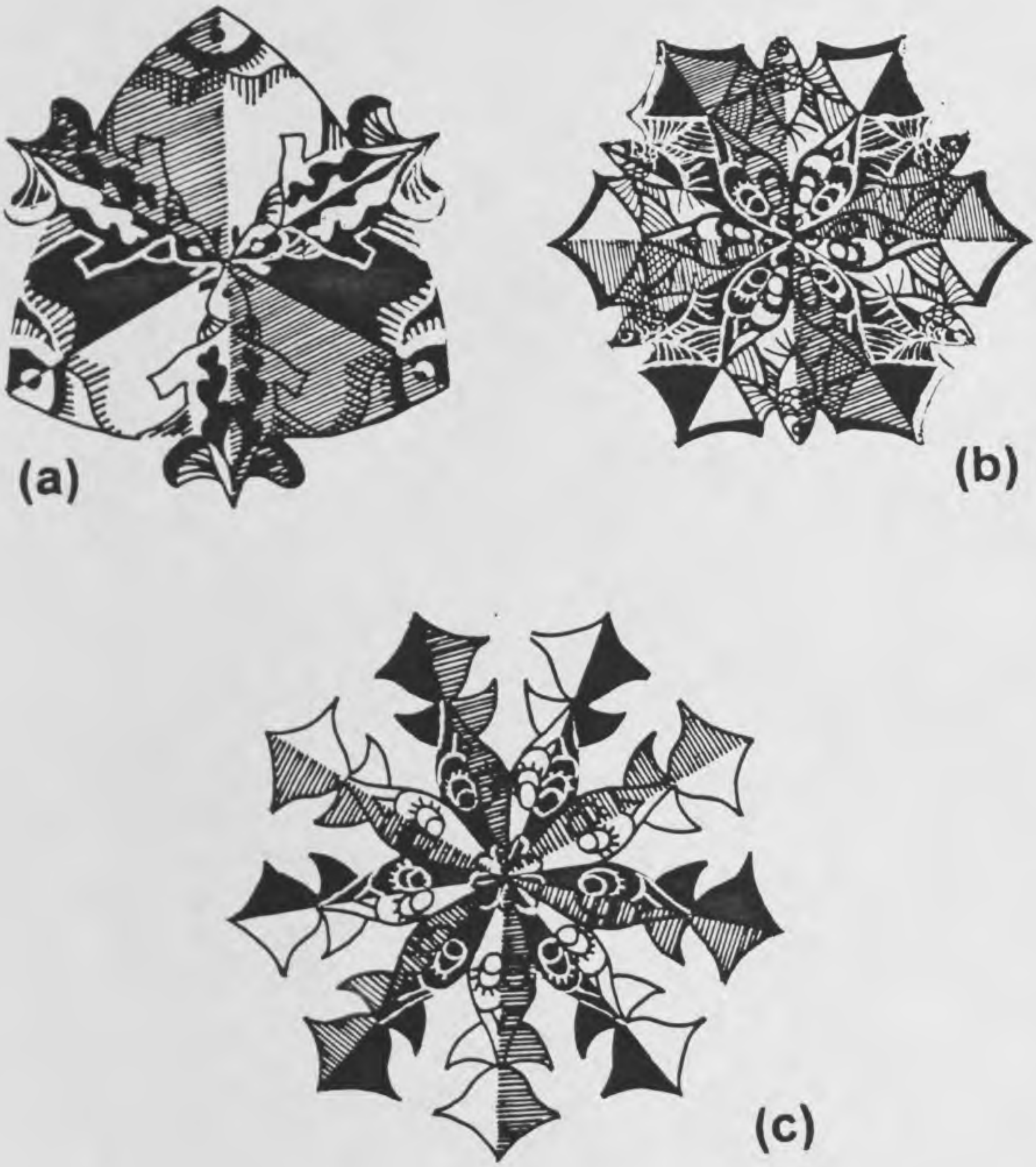
Symmetry class	Number of colours involved in colouring (denoted by K)	Examples of perfectly coloured class dn motifs
d3	**K=2 or 3**	
d4	**K=2 or 4**	
d6	**K=2 or 3**	
d8	**K=2 or 4**	
d9	**K=2 or 3**	

Figure 6.9 The systematic colouring of class dn motifs with three colours (n = three, n = six and n = nine)



6.3 Colour Counterchange Border Patterns

As stated previously, using combinations of the four symmetry operations and ignoring interchange of colour only seven distinct classes of border patterns are possible. For the sake of clarity, these seven classes may be referred to as the seven primary border pattern classes. By introducing colour interchange on these primary structures, a total of seventeen classes of two-colour counterchange border patterns are possible. The notation used in the classification of these designs is a modification of the pxyz notation used in the classification of the seven primary border pattern classes. This is the internationally accepted notation proposed by Belov [89]. A prime (') is generally associated with one of the symbols, if the corresponding symmetry operation interchanges colours. Washburn and Crowe explained the determination of the notation as follows:

> "The first symbol is p if no translation reverses the two colours; it is p' if some translation does reverse the colours. The second symbol, x, is 1 if there is no vertical reflection consistent with colour [symmetry operations consistent with colour are those which preserve colour]; m if there is a vertical reflection which preserves colour; m' otherwise (i.e. if all vertical reflections reverse the colours). The third symbol, y, is 1 if there is no horizontal reflection; m if there is a horizontal reflection which reverses colours (except in two cases beginning with p', in which two cases y is a); a' if there is no horizontal reflection, but the shortest glide-reflection reverses colours; and is a otherwise. The fourth symbol, z, is 1 if there is no half-turn consistent with colour; 2 if there are half-turns which preserve colour; 2' otherwise (i.e. if all half-turns reverse colours)." [90]

Schematic illustrations of the seventeen counterchange border possibilities are shown in Figure 6.10 in association with the seven primary classes and the appropriate notation for each category. Figure 6.11 is further illustrative and shows examples from each primary and each counterchange class.

When three colours are involved in the colouring a total of seven possibilities are evident. Although there is no universally accepted notation for three-colour border patterns, the notation used by Grunbaum and Shephard [91] would appear to be the most appropriate; in each of the seven classes a number 3 is placed in square brackets after the primary class notation. The seven possibilities are illustrated in Figure 6.12.

6.4 Two-colour Counterchange All-over Patterns

As stated previously, only seventeen distinct classes of all-over patterns may be produced using combinations of the four symmetry operations. For the sake of clarity these seventeen classes may be referred to as the seventeen primary all-over pattern classes. However, by introducing colour interchange on these primary structures, a total of forty-six classes of two-colour counterchange all-over patterns are possible. As pointed out by Washburn and Crowe [92], although there is no universally accepted international notation for the forty-six patterns, the notation proposed by Belov and Tarkhova [93]

Figure 6.10 Schematic illustrations of the seventeen two-colour counterchange border patterns.

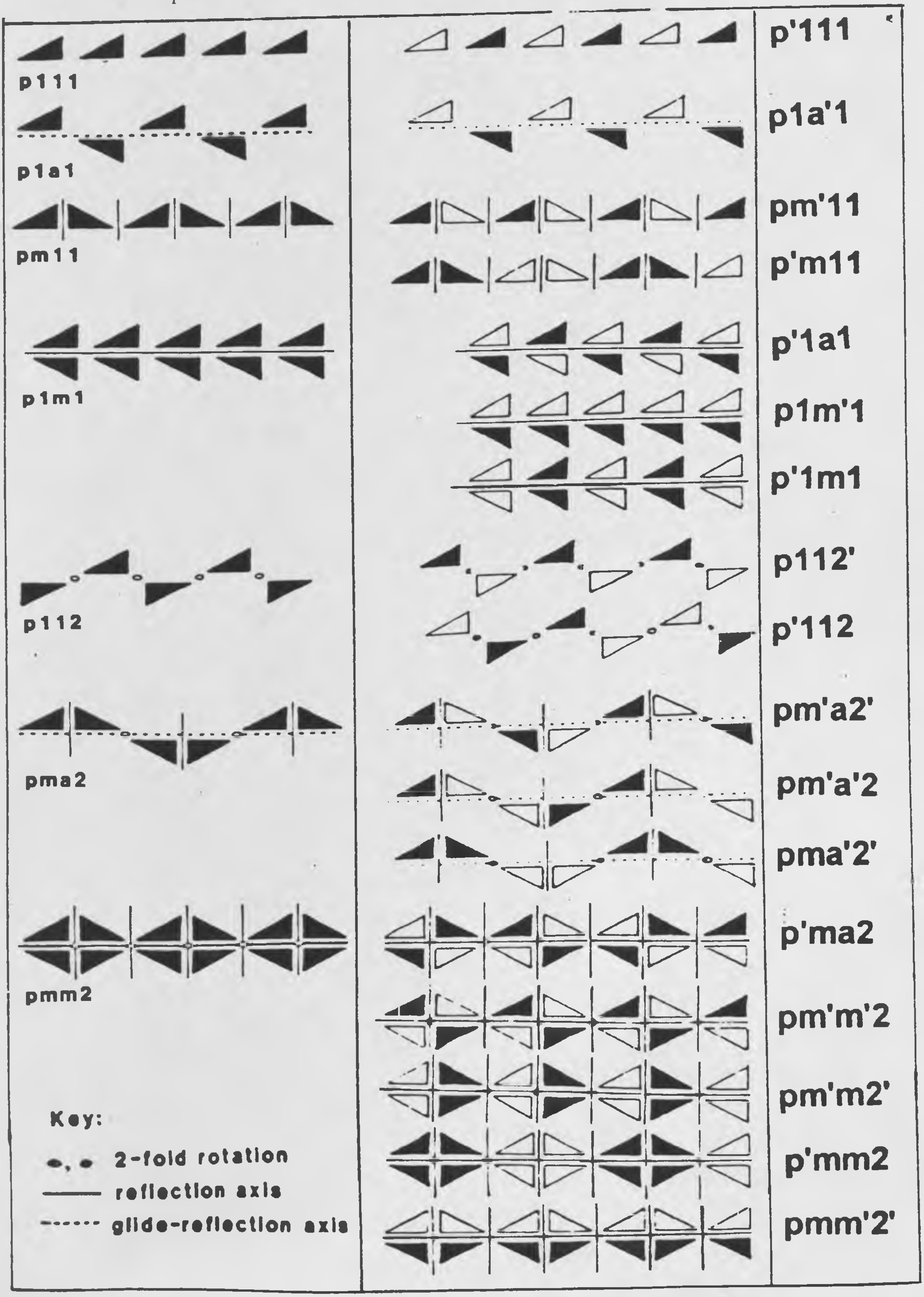


Figure 6.11 The seven primary and seventeen two-colour counterchange border patterns.

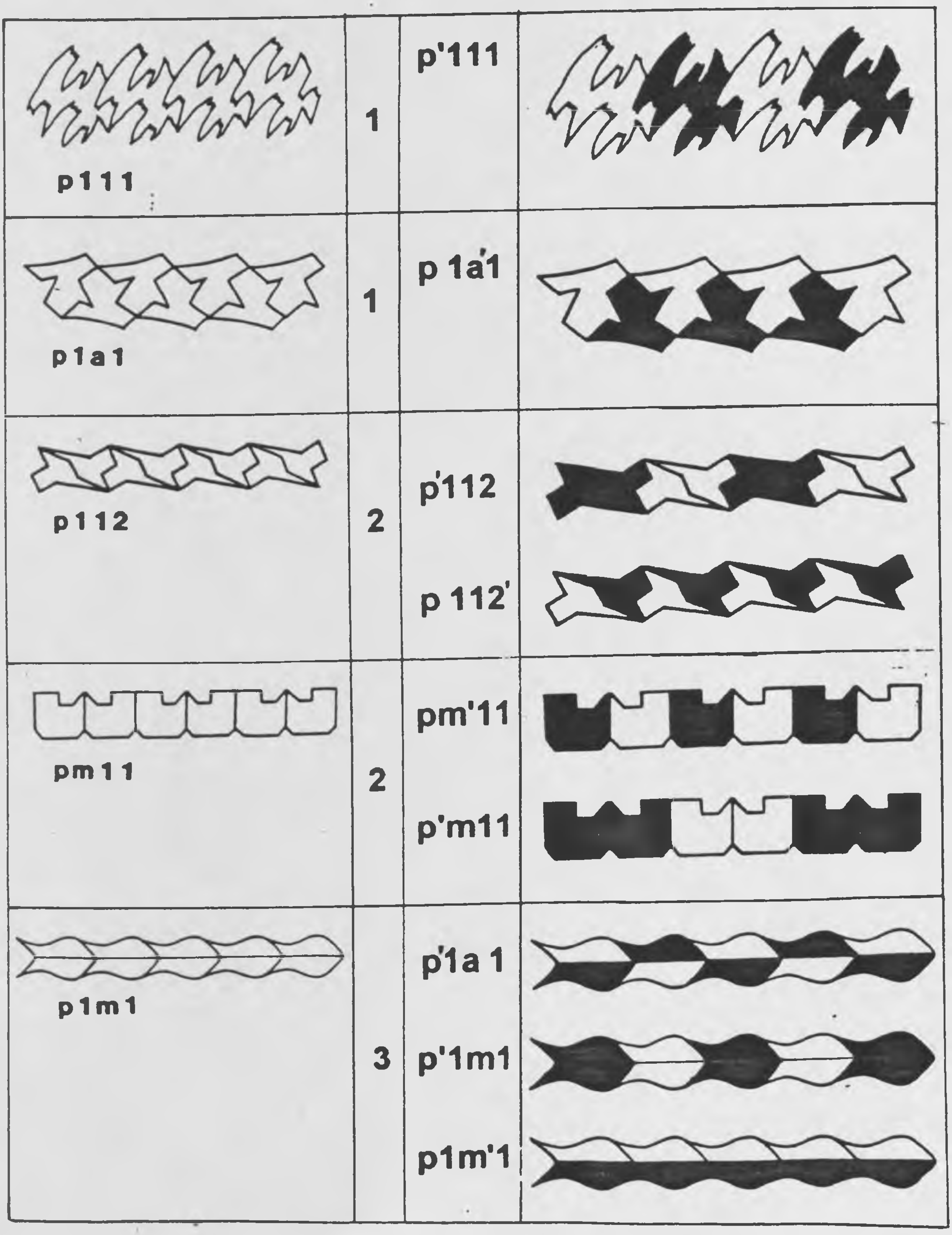


Figure 6.11 The seven primary and seventeen two-colour counterchange border patterns (continued).

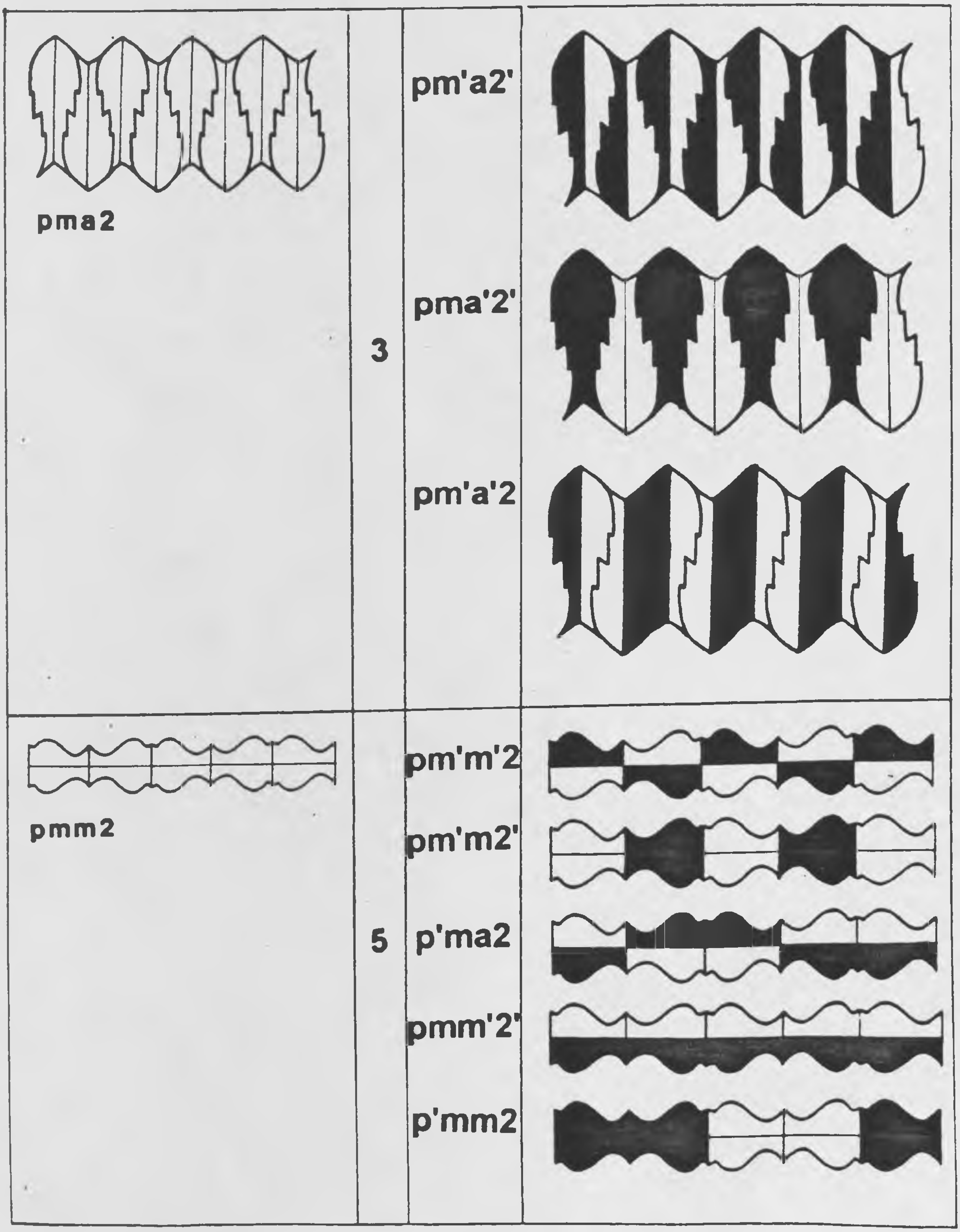
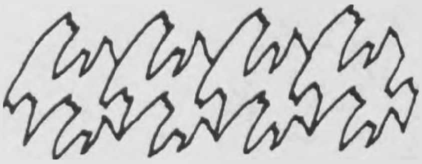


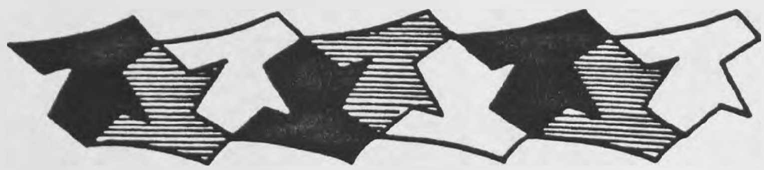
Figure 6.12 Examples of the seven three-colour counterchange patterns.



p111

p111[3]

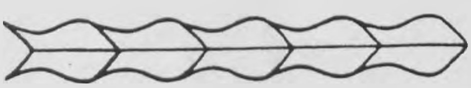
p1a1



p1a1[3]

pm11

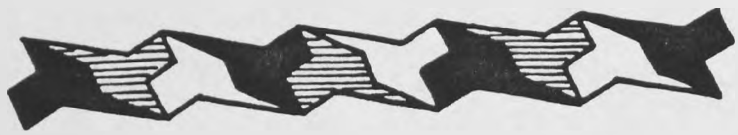
pm11[3]



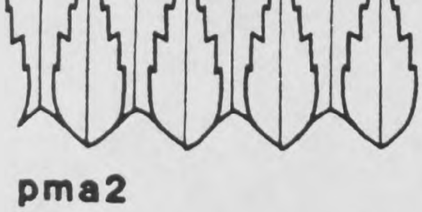
p1m1

p1m1[3]

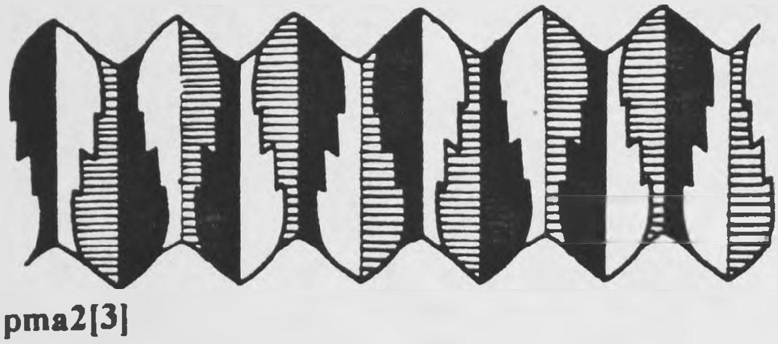
p112



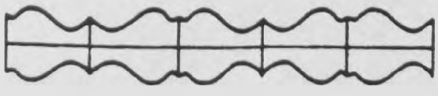
p112[3]



pma2



pma2[3]



pmm2

pmm2[3]

appears to be the most widely used. The notation is an adaptation of that used to classify primary all-over patterns, but is slightly more complex. Washburn and Crowe explained the situation as follows:

> "As a general rule (not without several exceptions!) a prime (') attached to a symbol indicates a colour change when the corresponding operation is performed. If a translation makes the colour change, the p of the symbol is changed to p_b' when the translation is along the edge of the primitive cell or to p_c' when the translation is along a diagonal of the primitive cell. (However, when p is changed to p_b' or p_c' in this way, no other symbol has a prime attached.) When all the mirror reflections in one direction reverse the colours then the corresponding m becomes m'; when all the glide-reflections in one direction reverse the colours then the corresponding g becomes g'." [94]

Schematic illustrations of all forty-six two-colour counterchange all-over patterns are provided in Figure 6.13. Each of the forty-six classes are described and illustrated further below.

6.4.1 Two-colour counterchange all-over patterns with no rotational characteristics

As stated previously, a total of four primary all-over pattern classes lack rotational characteristics. These are classes p1, p1g1, p1m1 and c1m1. By introducing colour interchange on these four primary structures, a total of eleven classes of two-colour counterchange all-over patterns are obtained. Each is further described and illustrated below.

Figure 6.13 Schematic illustrations of the forty-six two-colour counterchange all-over patterns.

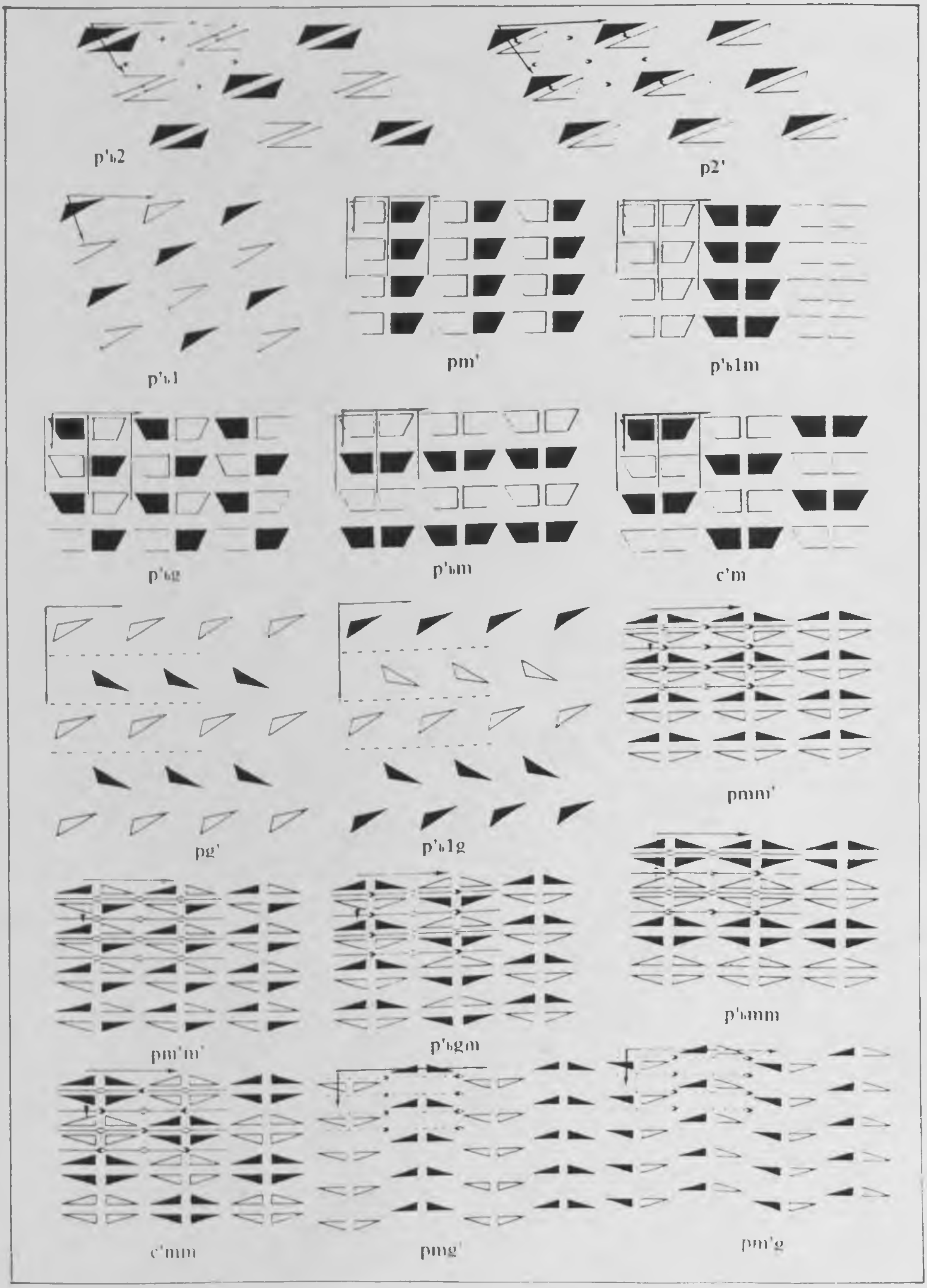


Figure 6.13 Schematic illustrations of the forty-six two-colour counterchange all-over patterns (continued).

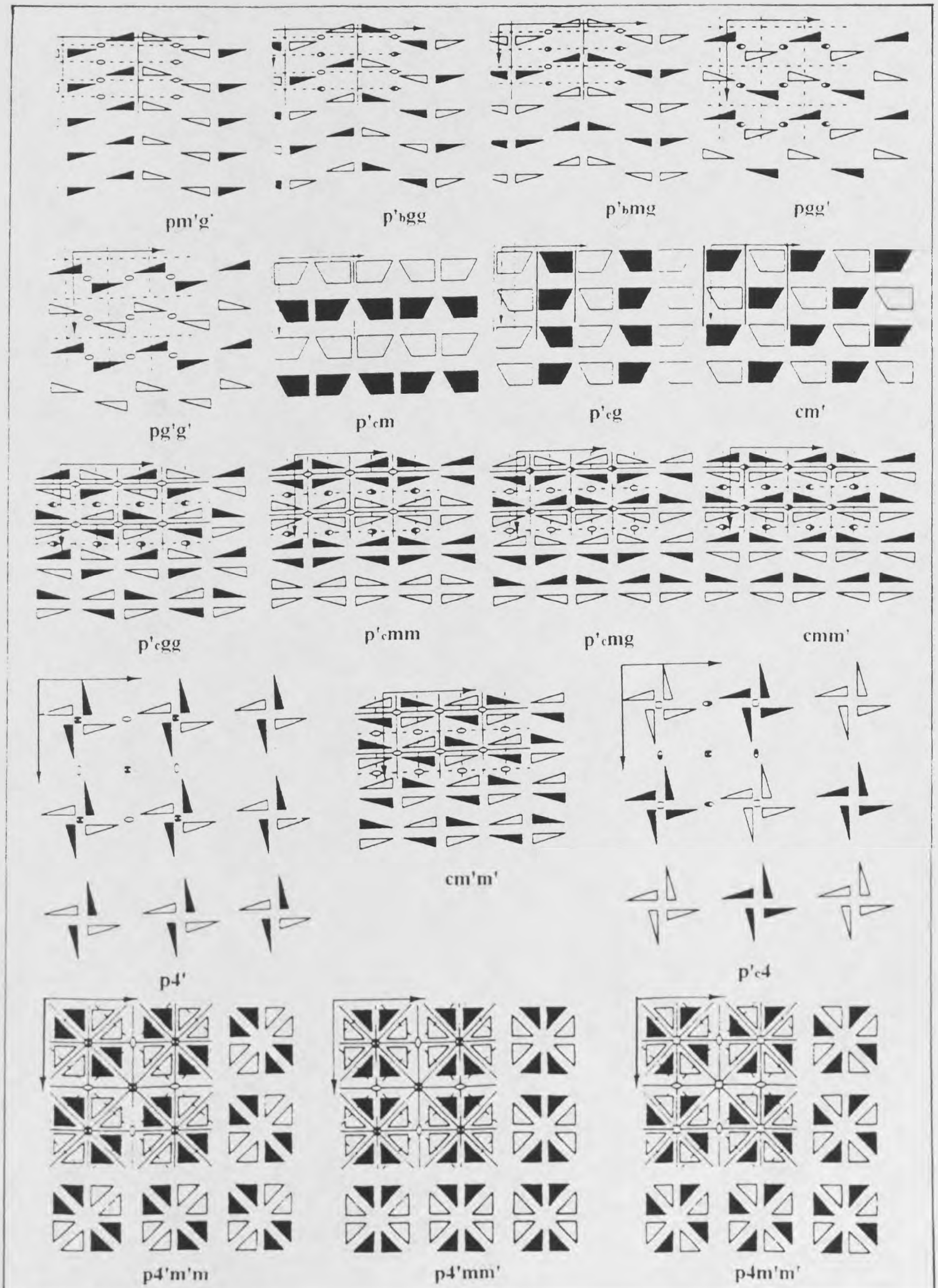
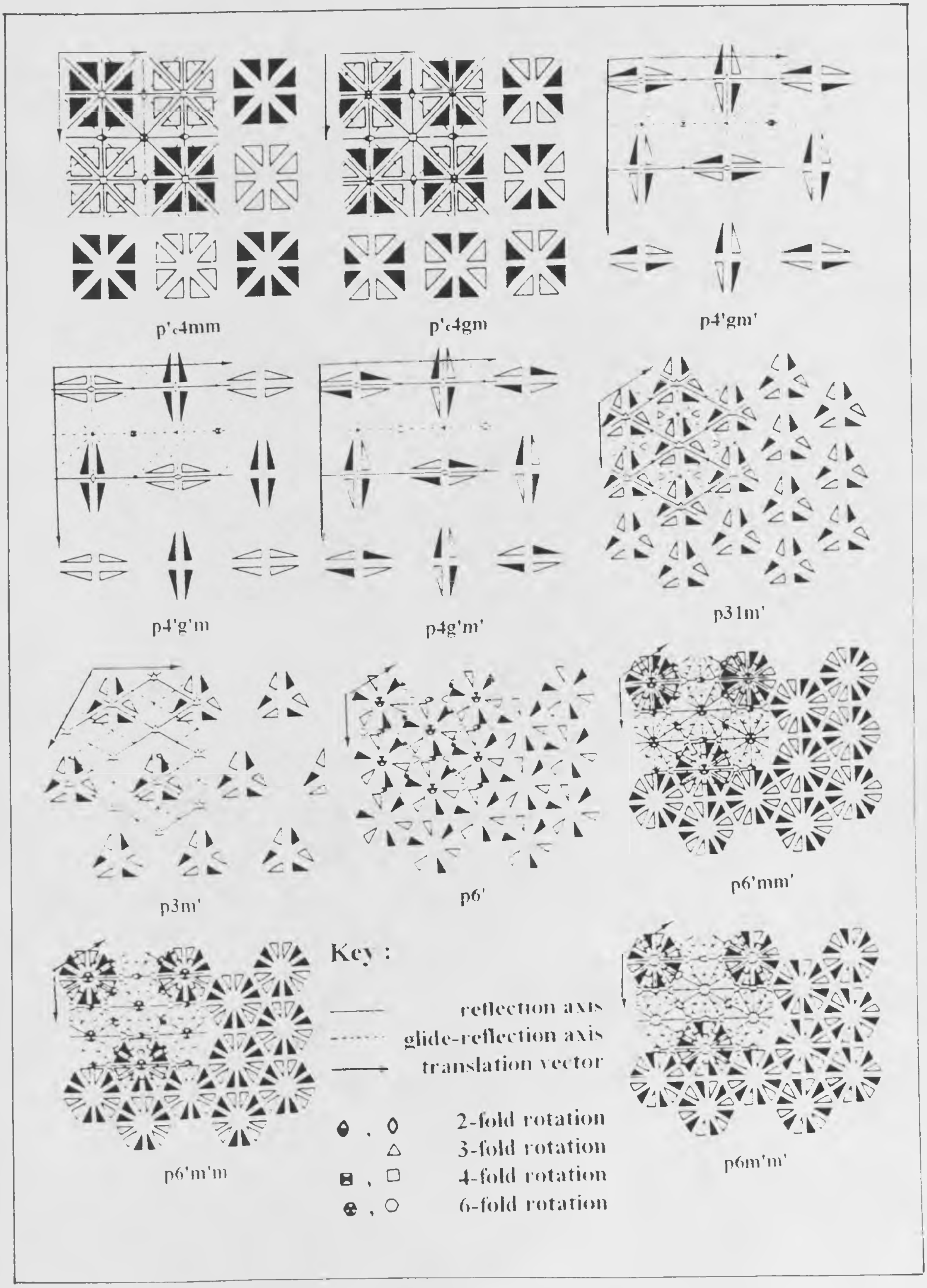


Figure 6.13 Schematic illustrations of the forty-six two-colour counterchange all-over patterns (continued).



In the case of a p1 pattern, there is only one way to introduce systematic colouring. In this case reversal of colour occurs as the pattern undergoes translation horizontally, vertically or diagonally depending on the orientation of the pattern. The pattern is thus designated the notation p_b'1. An example is provided in Figure 6.14.

There are two distinct ways of systematically colouring a class p1g1 pattern. These two counterchange classes are designated the notation pg' and p_b'1g. Examples of each are shown in Figures 6.15 and 6.16 respectively. In the former case horizontal rows of fish alternate direction, with fish in each horizontal row being the same colour. Colour is therefore reversed along vertical glide-reflection axes. In the latter case (Figure 6.16) colour reversals occur along alternating glide-reflection axes.

In the case of class p1m1 patterns, there are five distinct ways of introducing systematic colouring. The notations of these five types are: p_b'm, p_b'1m, pm', p_b'g and c'm. Relevant illustrations are shown in Figures 6.17, 6.18, 6.19, 6.20 and 6.21 respectively. In class p_b'm patterns (Figure 6.17) the units in each horizontal row are the same colour. Colour reversal therefore occurs through vertical translation. In class p_b'1m patterns (Figure 6.18) colours are preserved through vertical translation but reversed across alternating reflection axes. In class pm' patterns (Figure 6.19) colour is preserved through translation but reversed across all reflection axes.

Figure 6.14 A class p_b'1 two-colour counterchange all-over pattern.

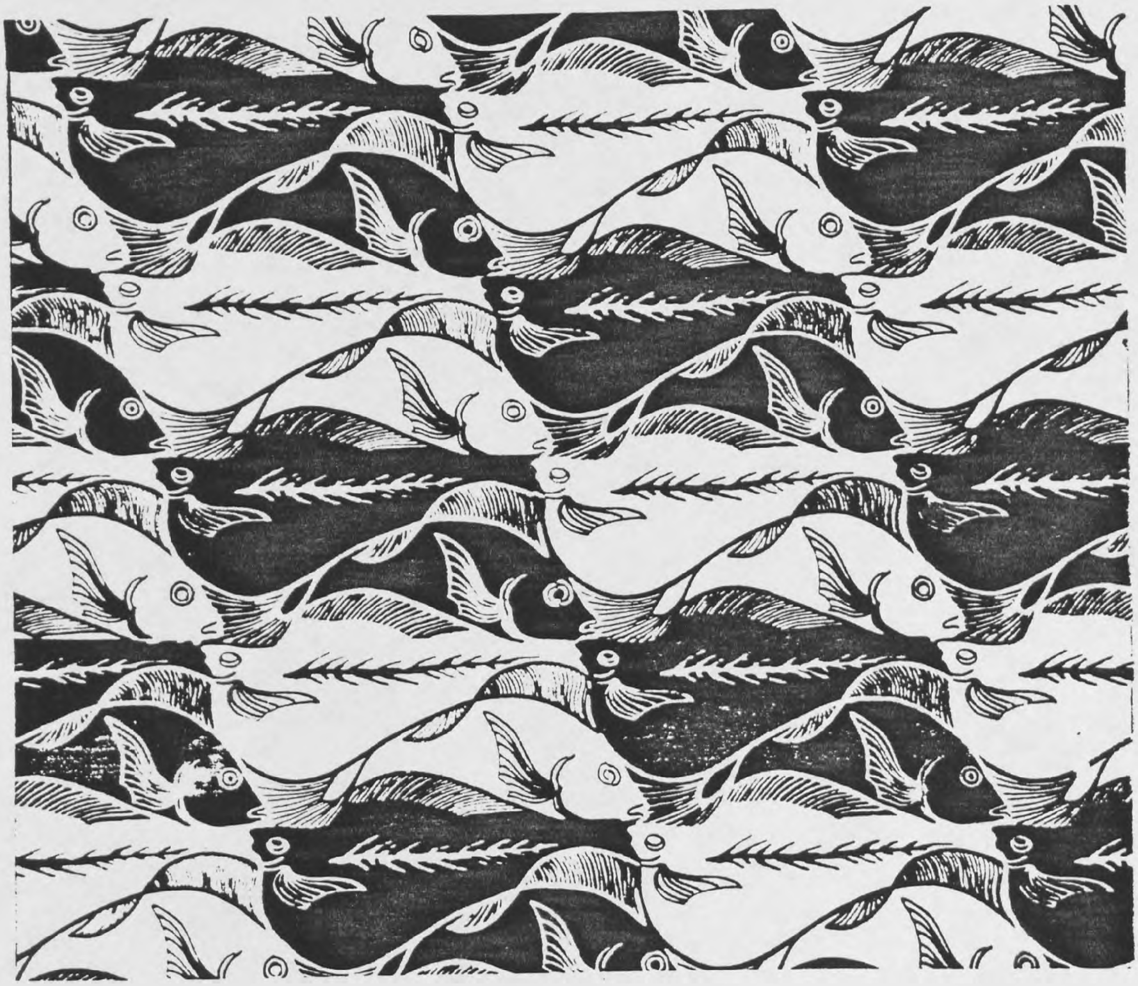


Figure 6.15 A class pg' two-colour counterchange all-over pattern.

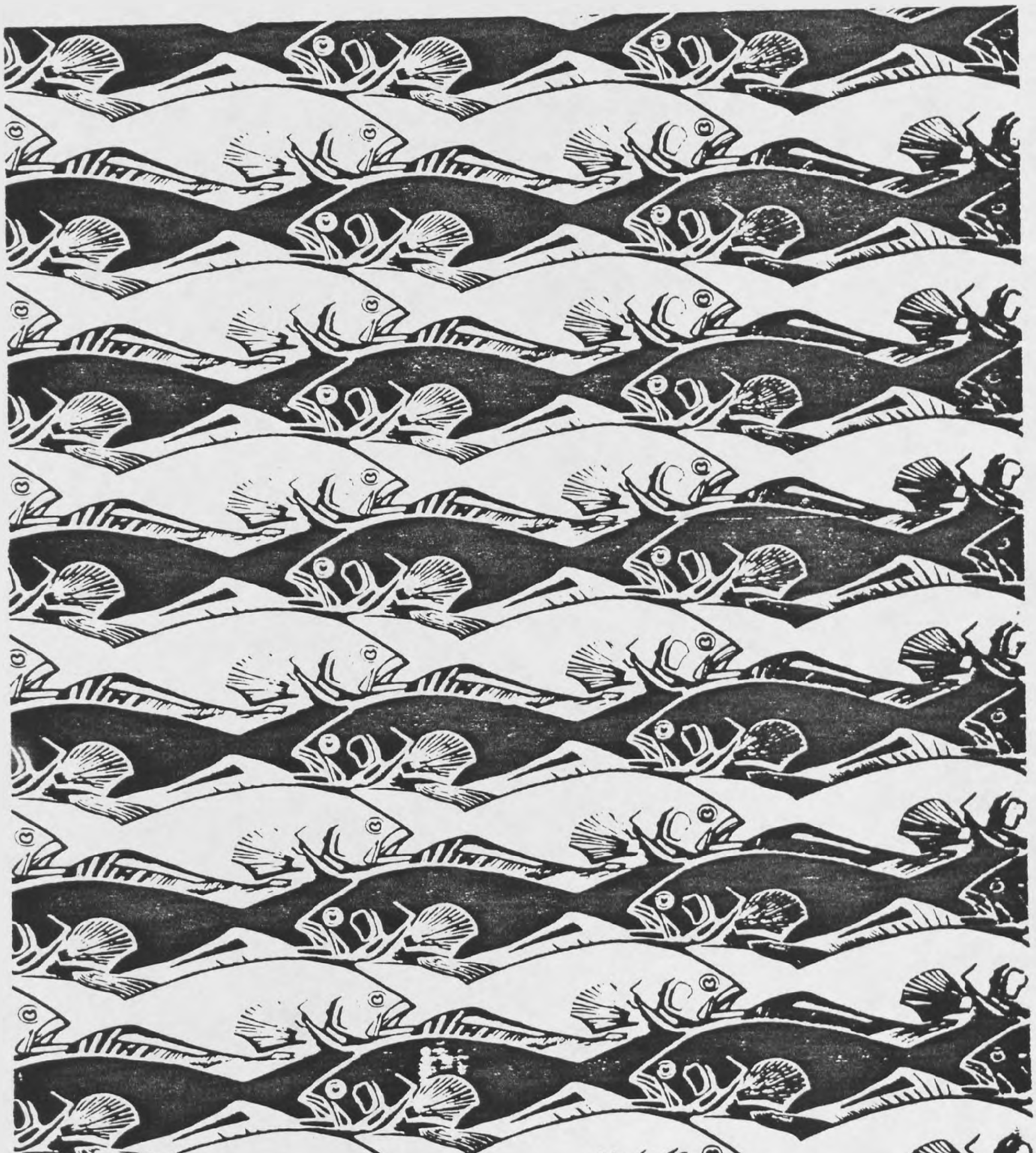


Figure 6.16 A class p_b'1g two-colour counterchange all-over pattern.

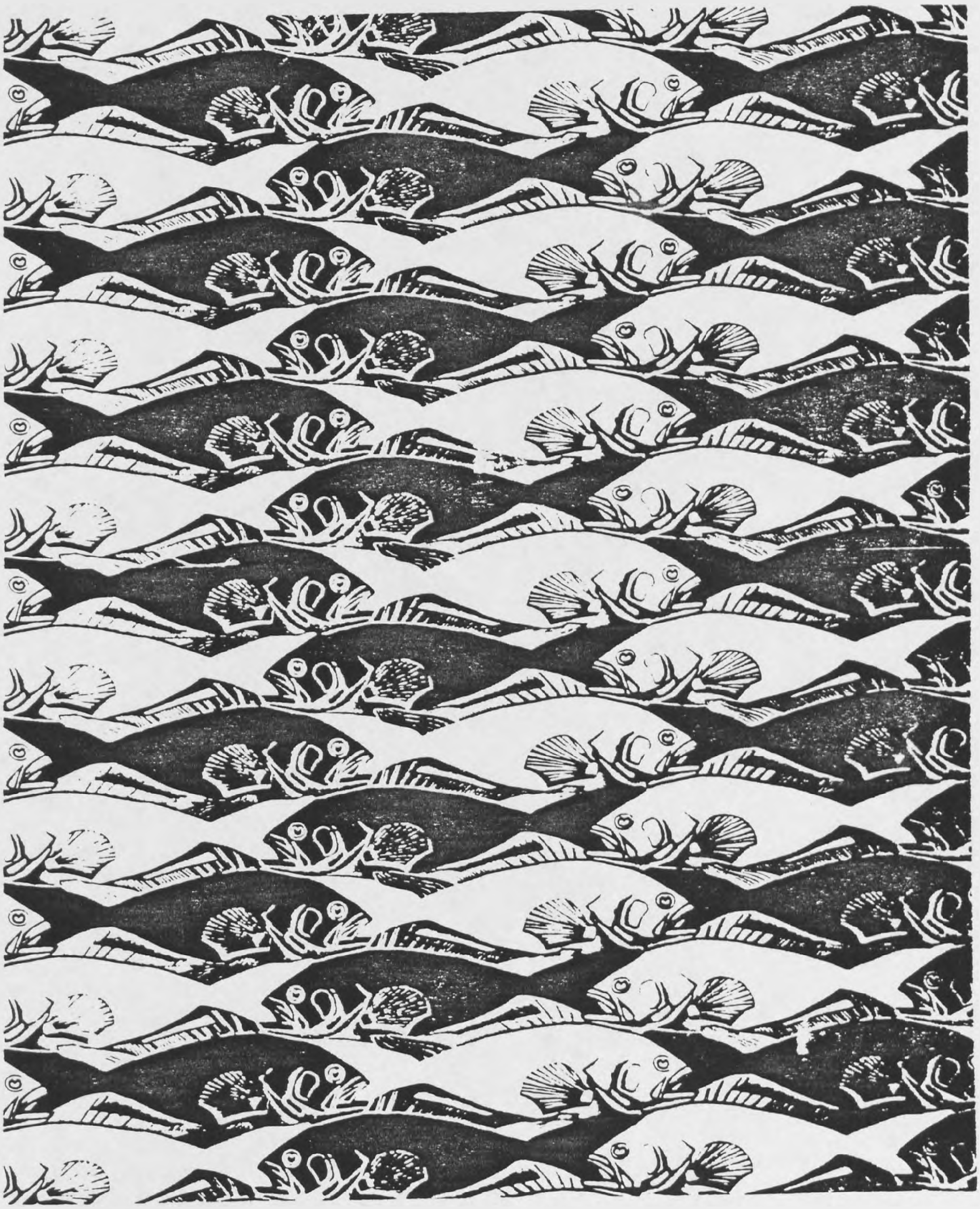


Figure 6.17 A class p_b'm two-colour counterchange all-over pattern.

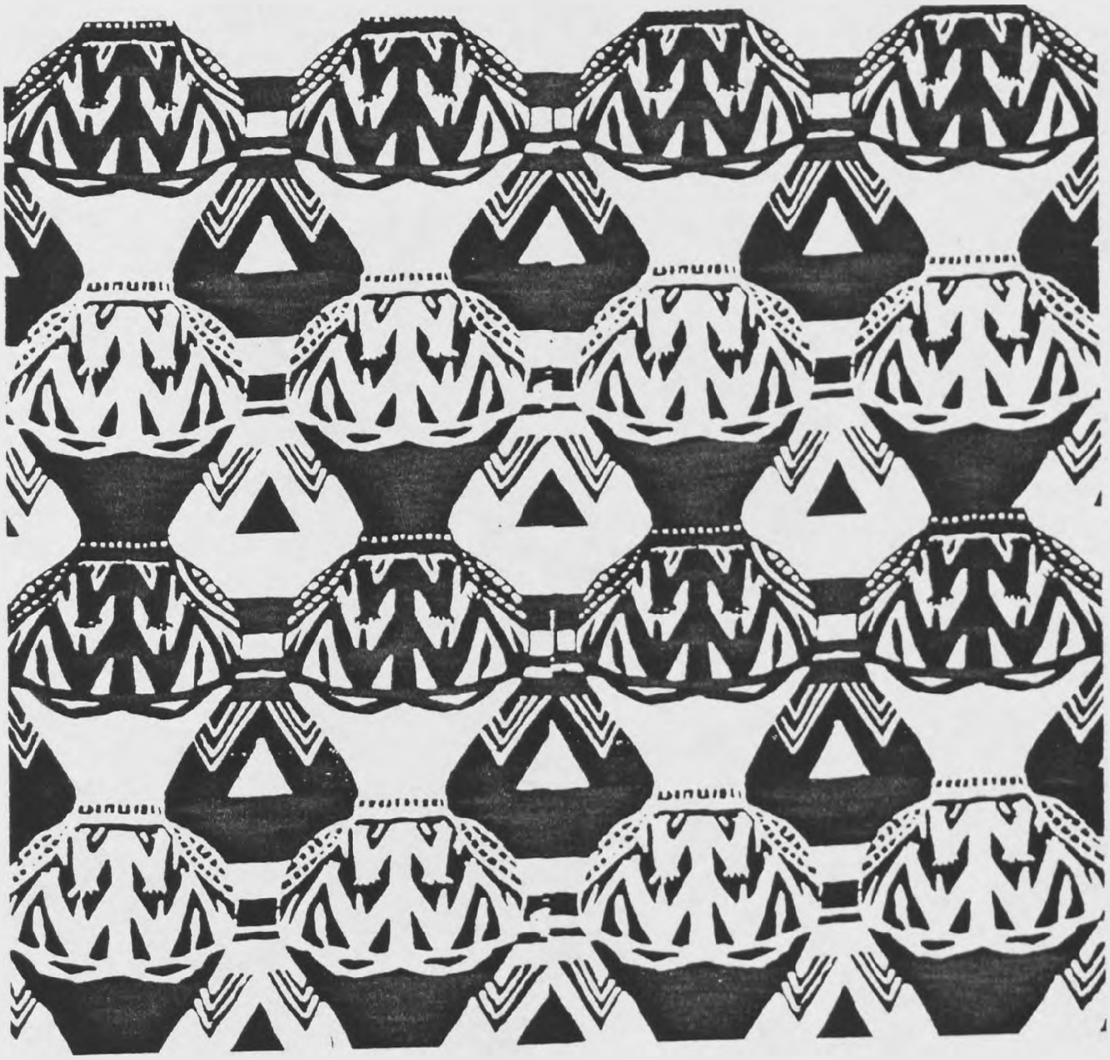


Figure 6.18 A class p_b'1m two-colour counterchange all-over pattern.

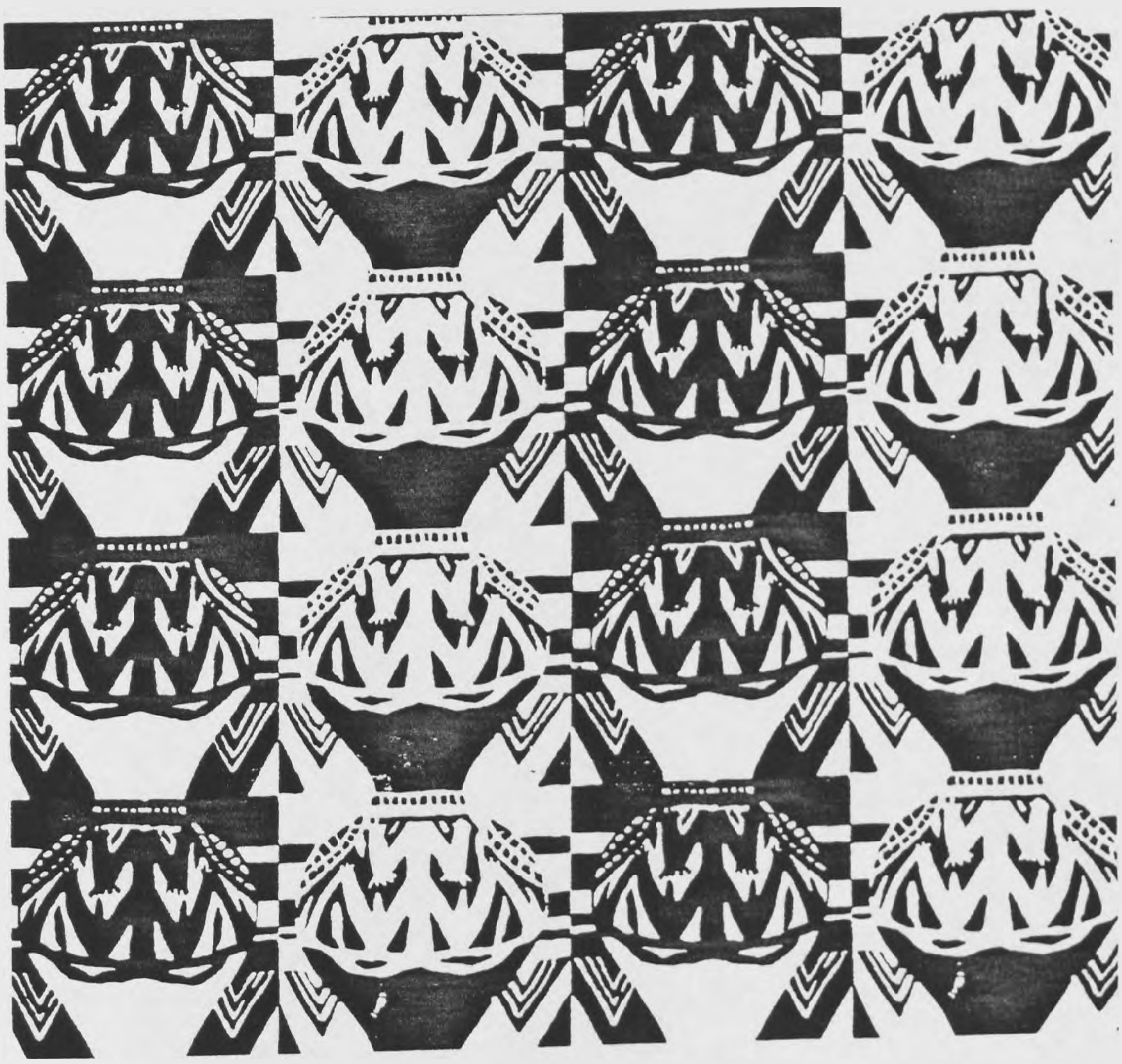


Figure 6.19 A class pm' two-colour counterchange all-over pattern.

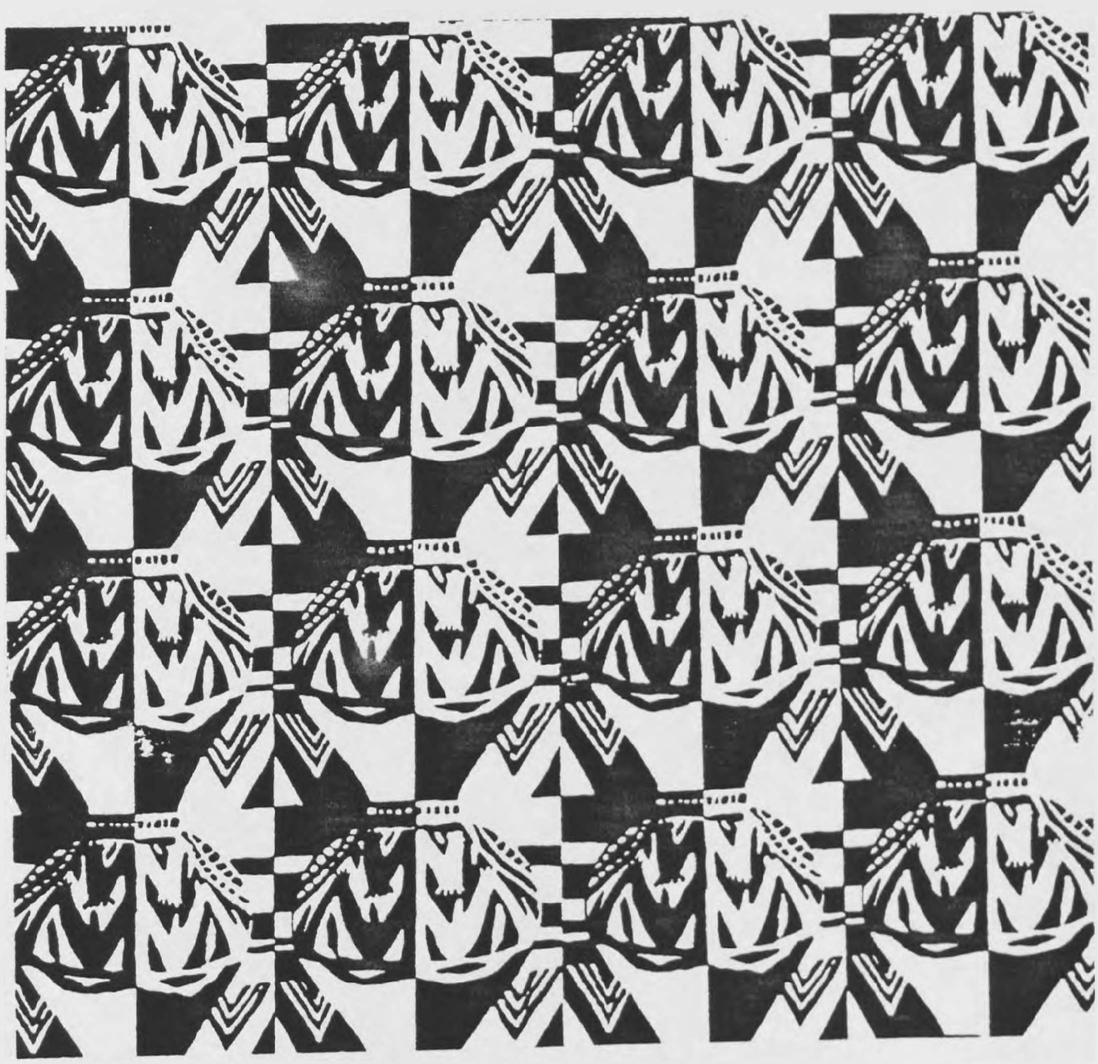


Figure 6.20 A class p_b'g two-colour counterchange all-over pattern.

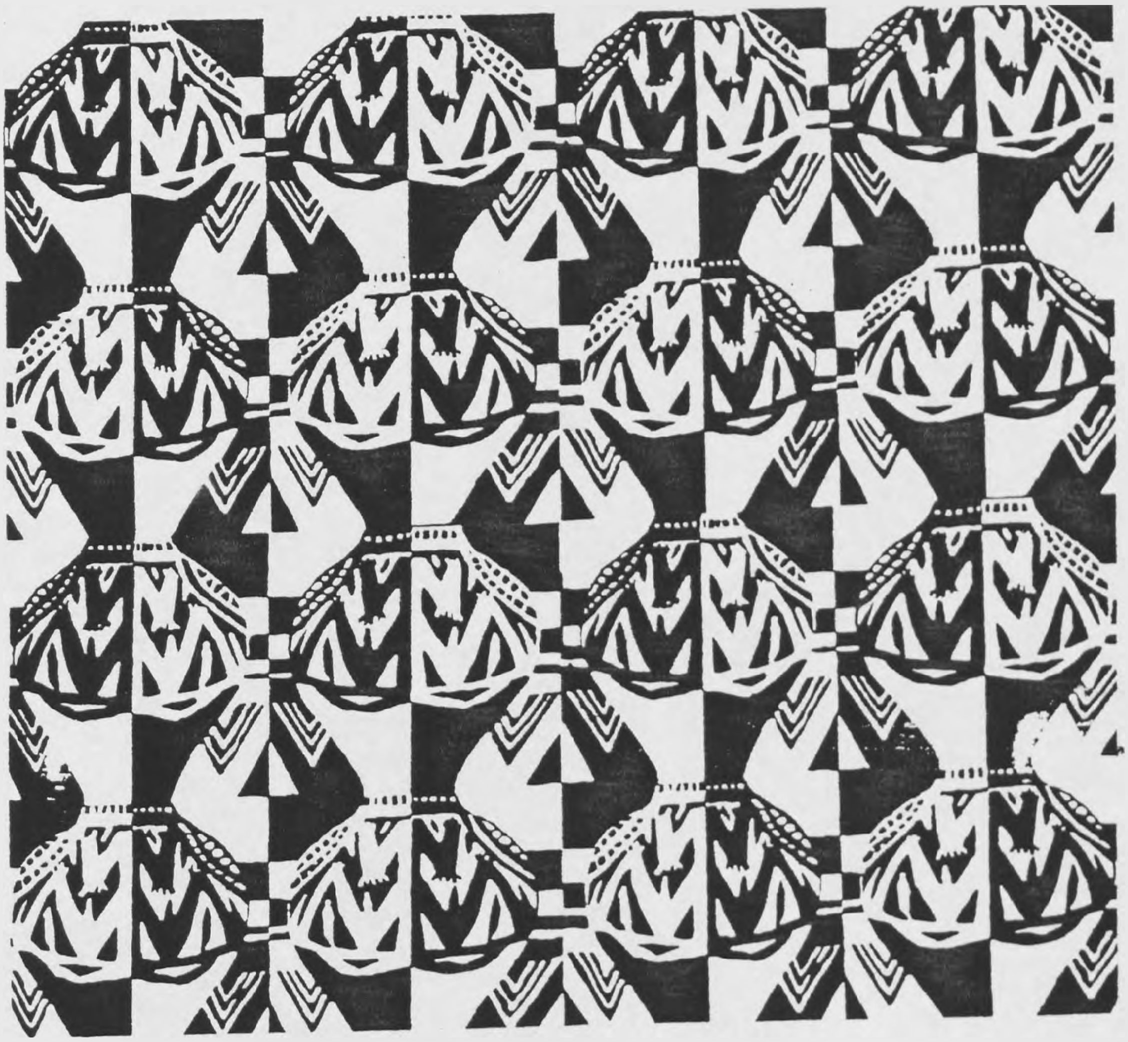
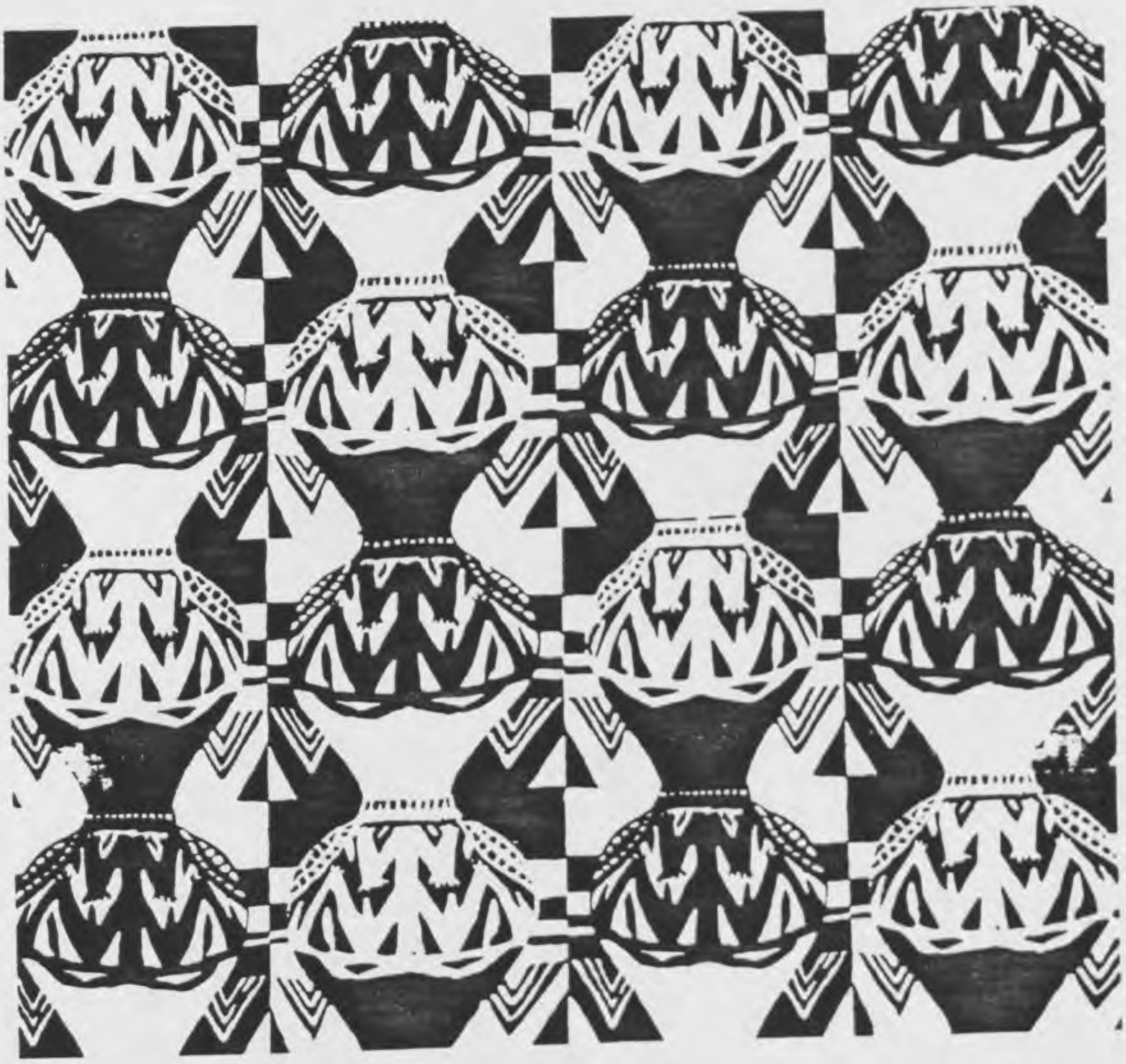


Figure 6.21 A class c'm two-colour counterchange all-over pattern.



As shown in Figure 6.20, in class p_b'g the colours reverse across all reflection axes as well as by translation. In the case of class c'm patterns, colours are reversed across alternate reflection axes and by translation in both the horizontal and the vertical (Figure 6.21).

On the primary structure of a class c1m1 pattern a total of three colouring possibilities are evident. These are designated notations of p_c'g, p_c'm and cm'. Examples of each class are provided in Figures 6.22, 6.23 and 6.24 respectively. In class p_c'g patterns (Figure 6.22) reversal of colour occurs across reflection axes but not along the glide-reflection axes. In class p_c'm patterns (Figure 6.23) colour is preserved across reflection axes but reversed along glide-reflection axes. In class cm' patterns (Figure 6.24) colour reversals occur across reflection axes as well as along glide-reflection axes.

6.4.2 Two-colour counterchange all-over patterns with two-fold rotation

Primary all-over pattern classes with two-fold rotation include classes p211, p2gg, p2mg, p2mm and c2mm. By introducing colour interchange on these primary structures a total of nineteen two-colour counterchange pattern classes are possible.

There are two distinct ways of colouring a class p211 pattern. These are designated the notations p2' and p_b'2 and are shown in Figures 6.25 and 6.26 respectively. With class p2' patterns (Figure 6.25) colour reversal occurs around all centres of two-fold rotation.

Figure 6.22 A class p_c'g two-colour counterchange all-over pattern.

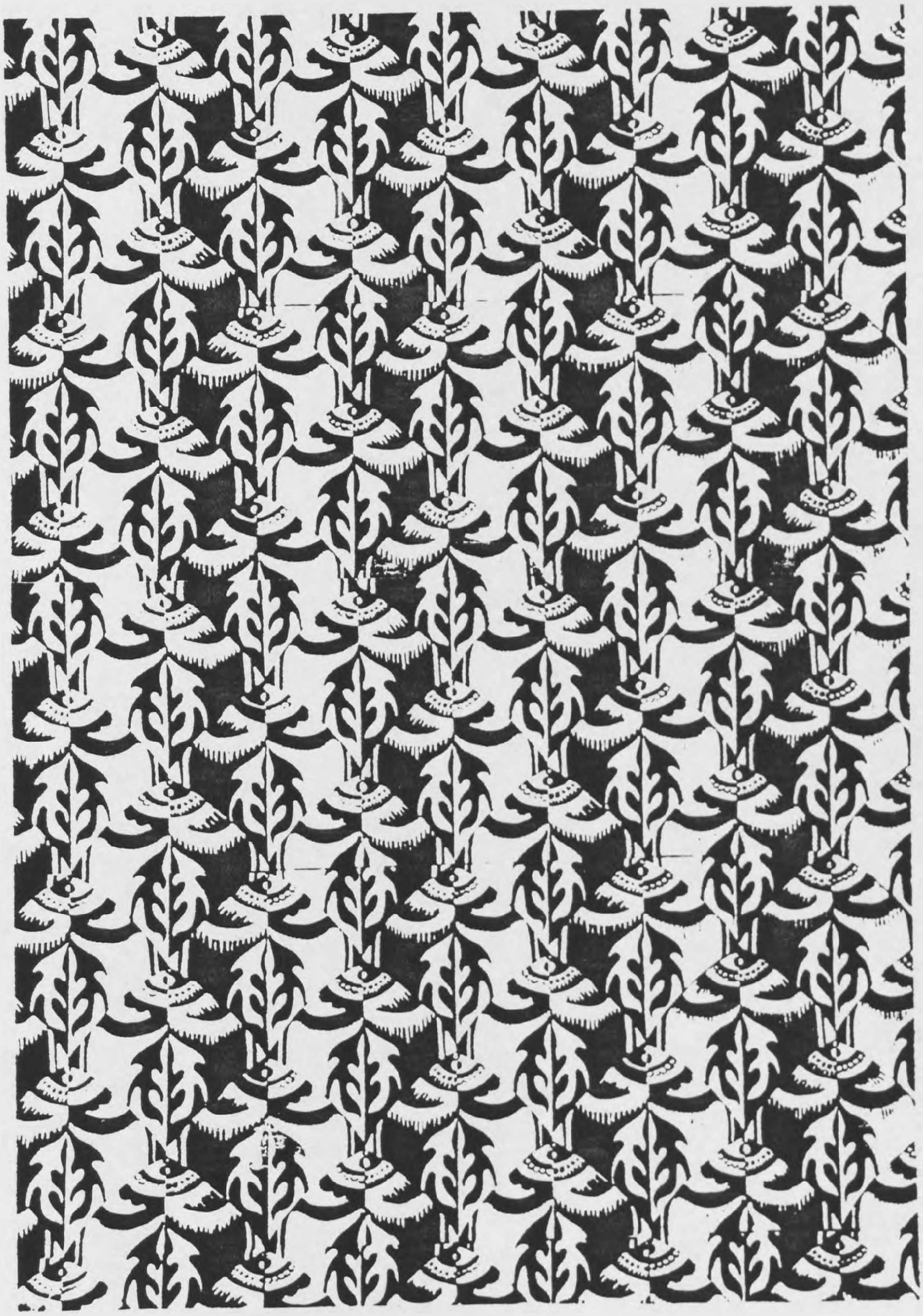


Figure 6.23 A class p_c'm two-colour counterchange all-over pattern.

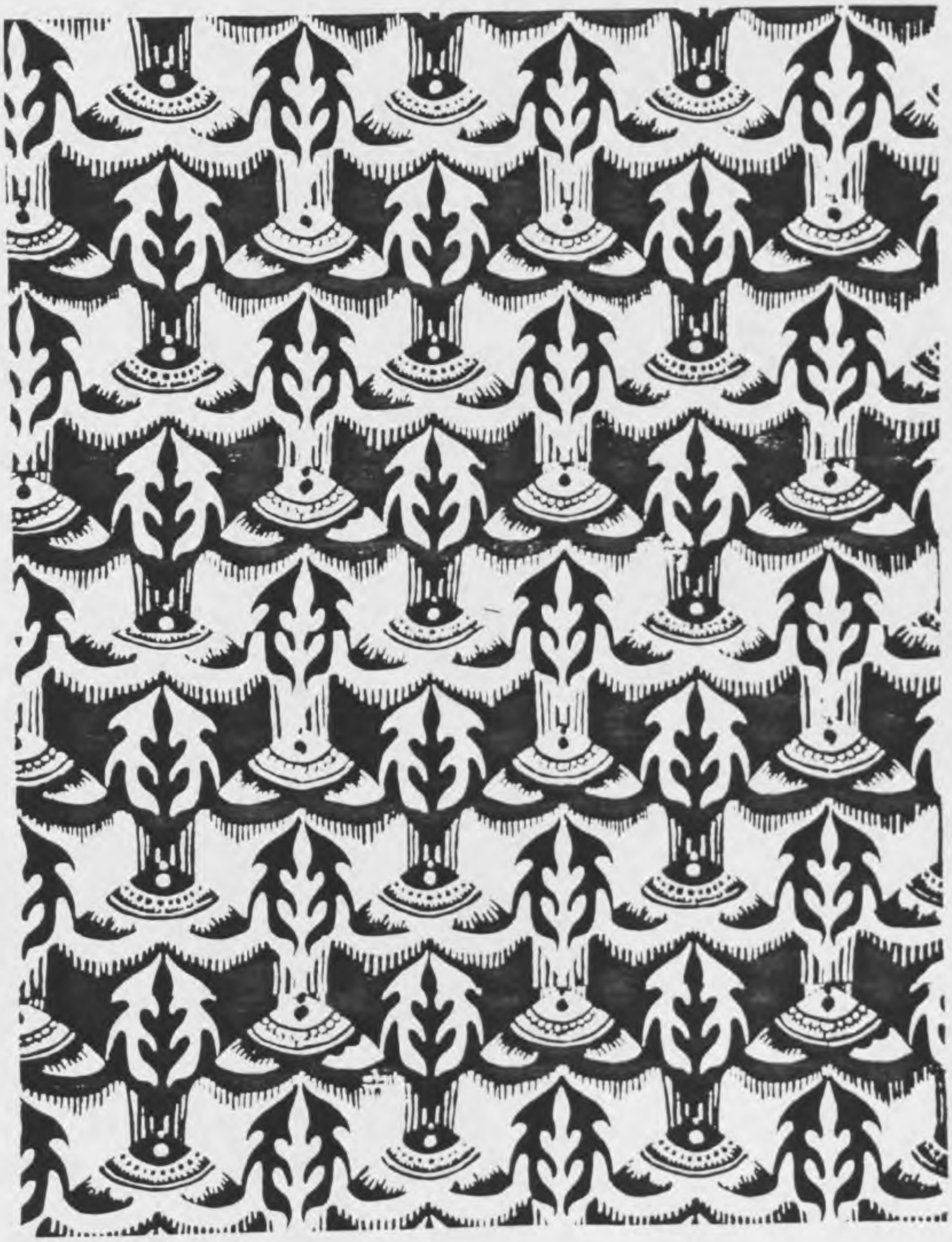


Figure 6.24 A class cm' two-colour counterchange all-over pattern.

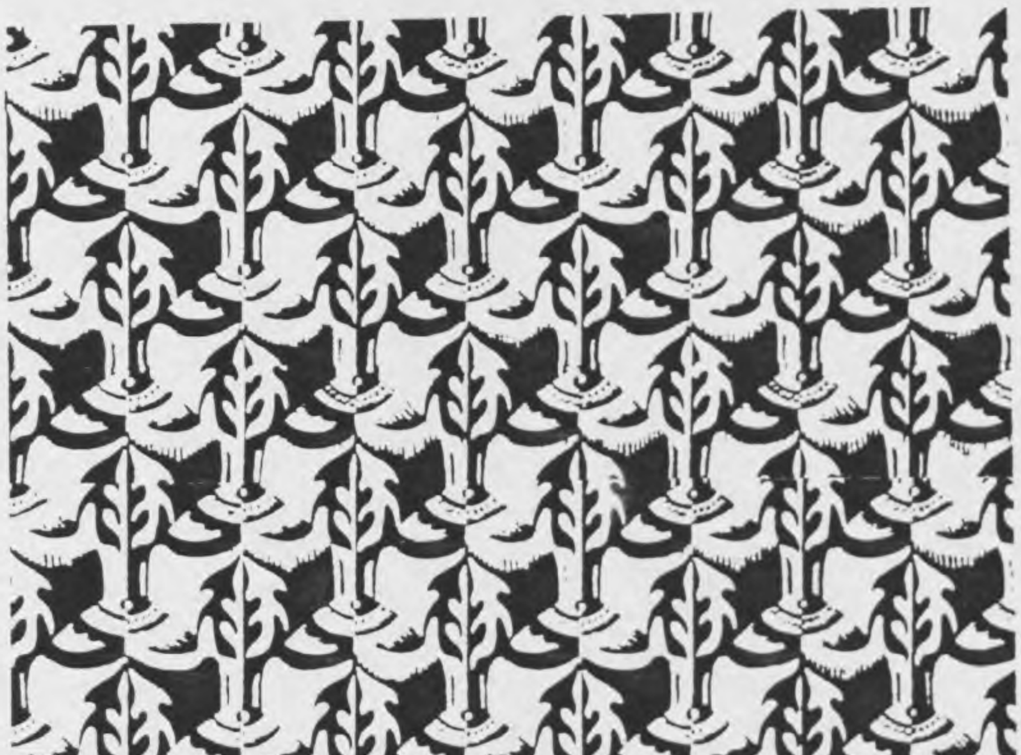
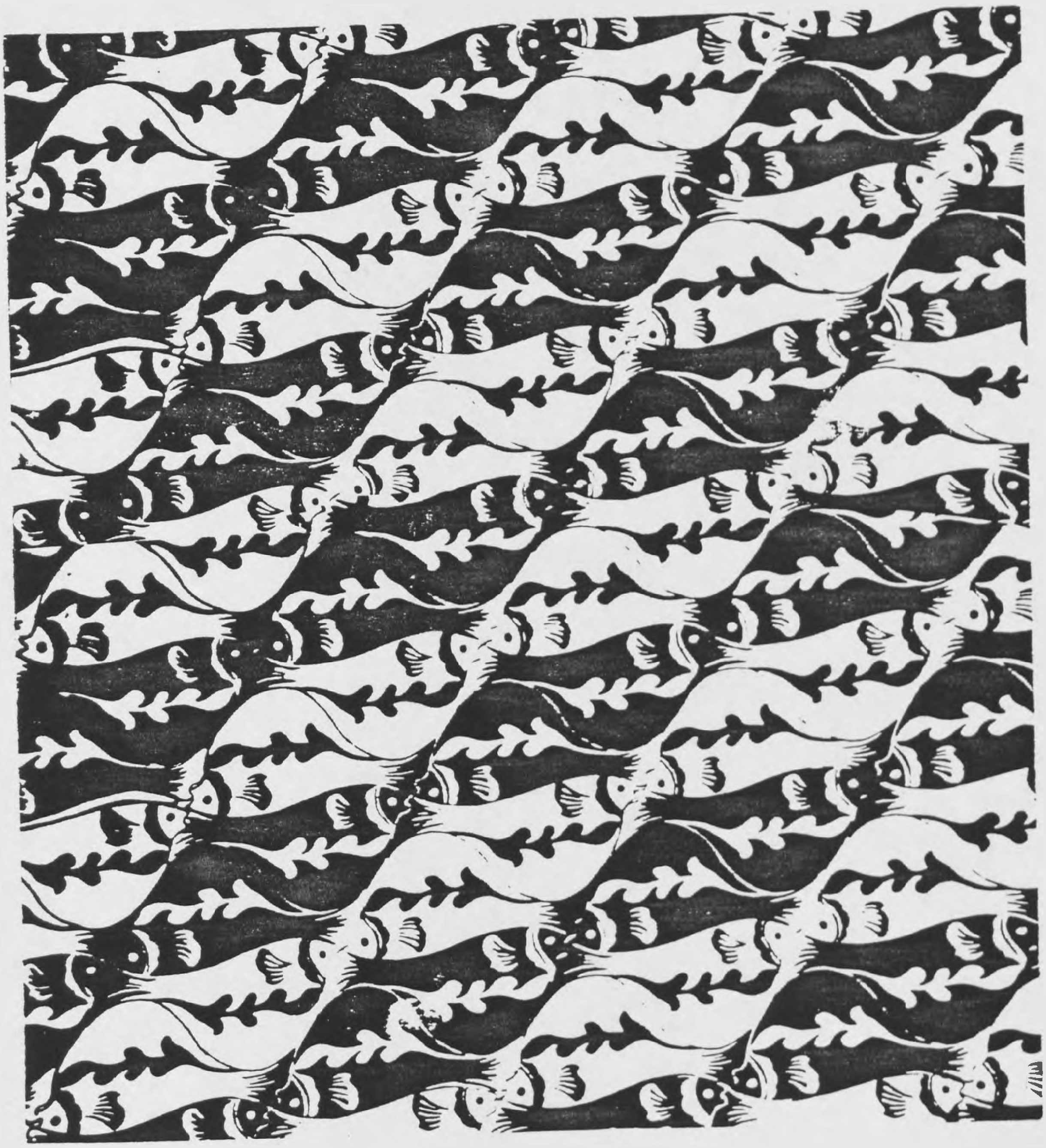


Figure 6.25 A class p2' two-colour counterchange all-over pattern.

Figure 6.26 A class p_b'2 two-colour counterchange all-over pattern.



With class p_b'2 patterns colour is reversed around alternate centres of two-fold rotation (Figure 6.26).

Two distinct counterchange possibilities are evident on systematically colouring a class p2gg primary all-over pattern; these are designated the notations pg'g' and pgg' (Figures 6.27 and 6.28 respectively). In the former case reversal of colour occurs along the glide-reflection axes running both horizontally and vertically. In the latter case (Figure 6.28) colour reversal occurs along glide-reflection axes running in one direction (vertical in the example shown). Colour is preserved along the other glide-reflection axes (horizontal in the example shown). Reversal of colour also occurs around all centres of two-fold rotation.

Five distinct counterchange classes may be derived from class p2mg primary structures. These are designated the notations p_b'gg, pmg', pm'g', p_b'mg and pm'g. Class p_b'gg counterchange patterns reverse colours across all reflection axes, around alternate centres of rotation and along alternating glide-reflection axes (Figure 6.29). In class pmg' counterchange patterns, as shown in Figure 6.30, colour is reversed around all centres of two-fold rotation and along all glide-reflection axes. Class pm'g' patterns (Figure 6.31) exhibit reversal of colour across all reflection axes and along glide-reflection axes. Colour is preserved around all centres of two-fold rotation. As shown in Figure 6.32, class p_b'mg counterchange patterns show alternating rows of each colour.

Figure 6.27 A class pg'g' two-colour counterchange all-over pattern.

Figure 6.28 A class pgg' two-colour counterchange all-over pattern.

Figure 6.29 A class p_b'gg two-colour counterchange all-over pattern.

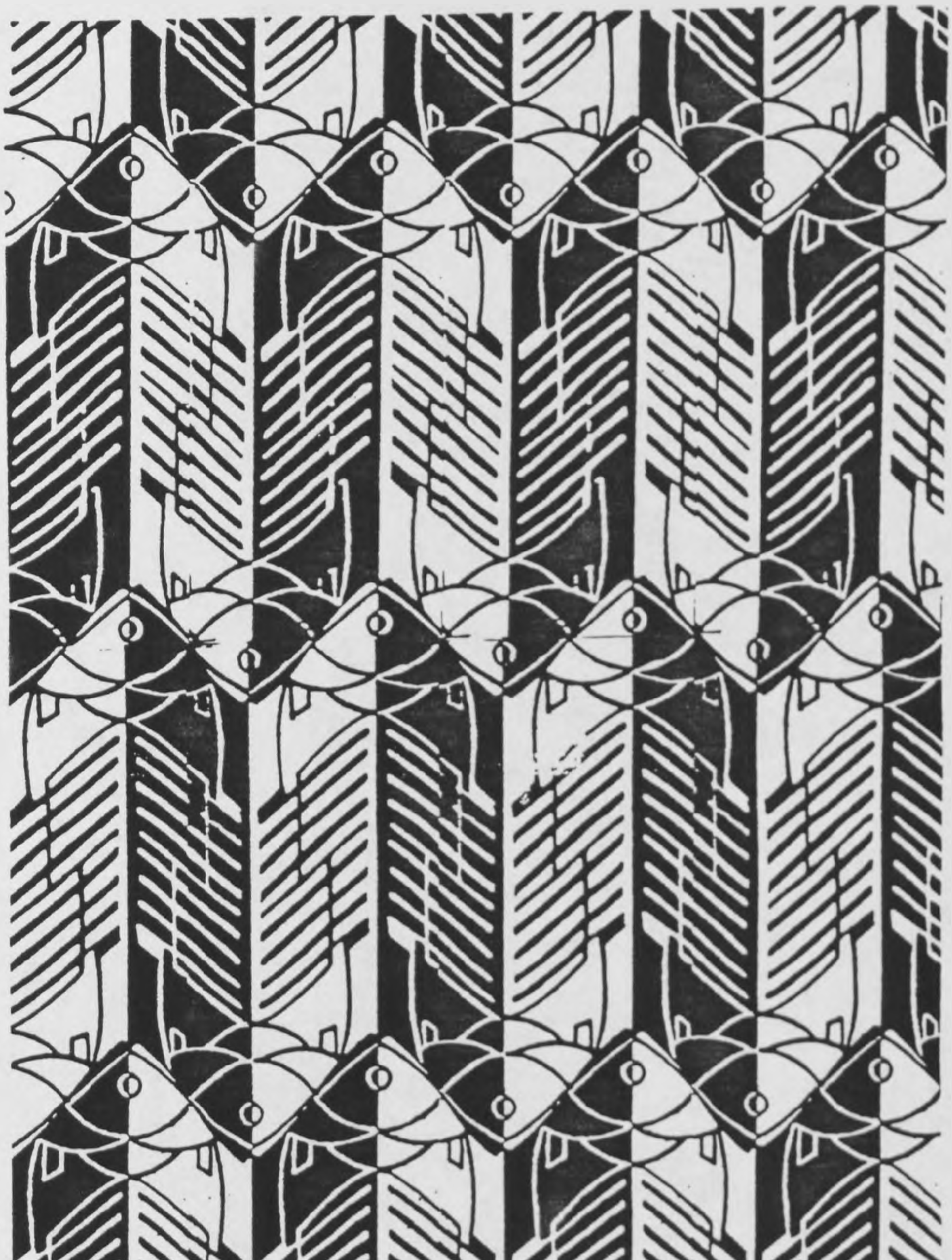


Figure 6.30 A class pmg' two-colour counterchange all-over pattern.

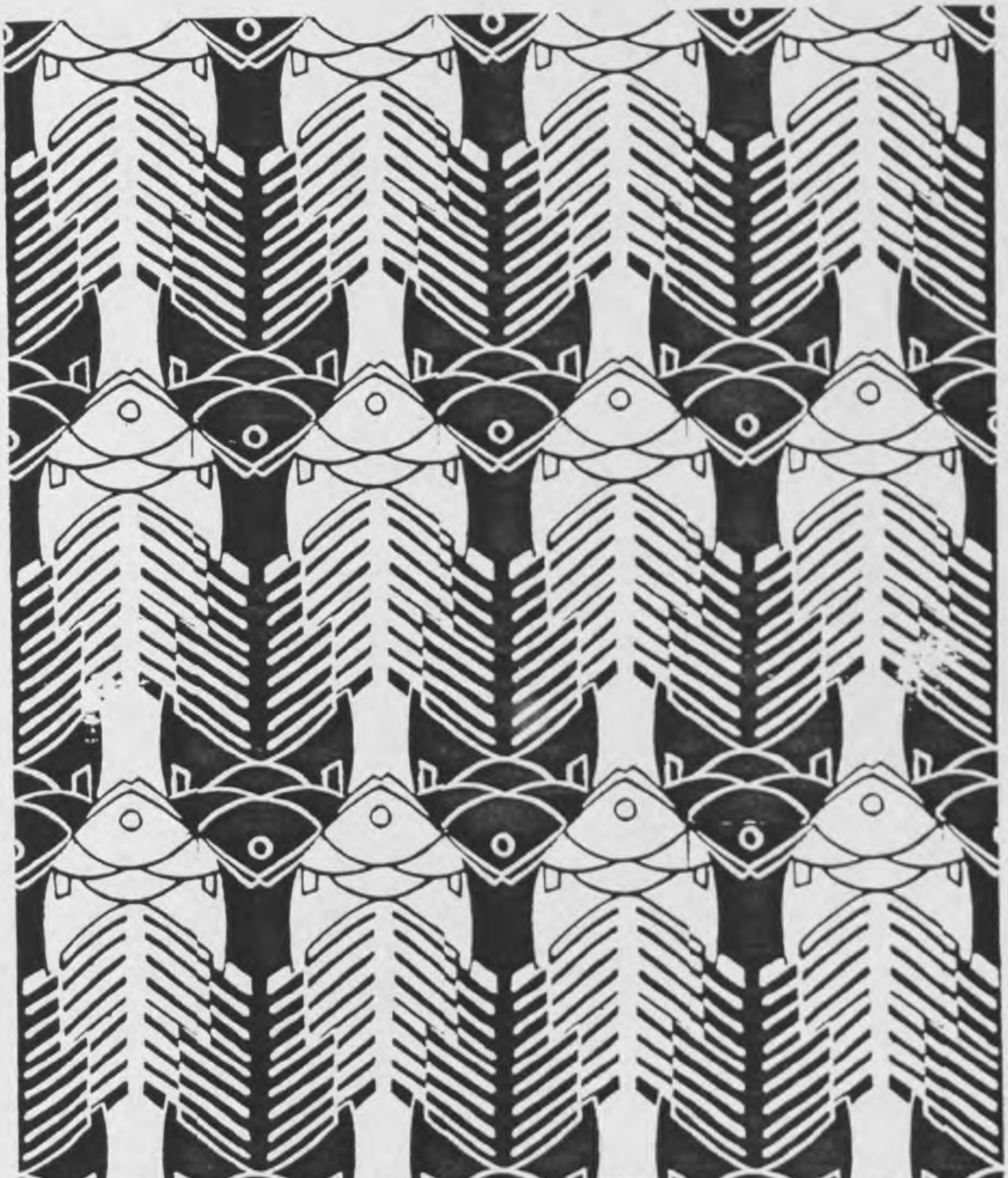


Figure 6.31 A class pm'g' two-colour counterchange all-over pattern.

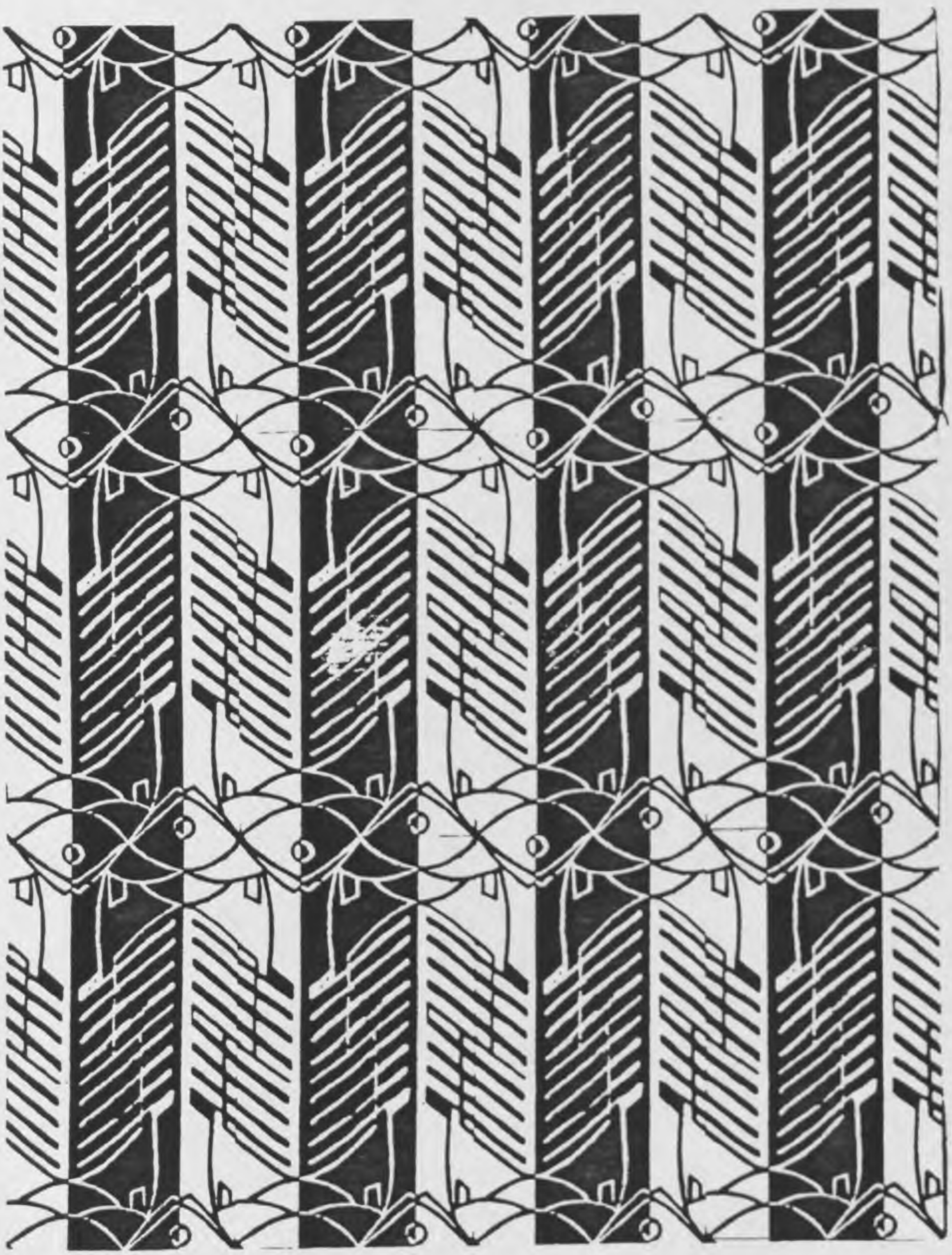


Figure 6.32 A class p_b'mg two-colour counterchange all-over pattern.

Horizontal translations (in the example shown) preserve colours, as do vertical reflections. Colour reversal occurs through vertical translation and in alternating horizontal rows of centres of two-fold rotation. Class pm'g counterchange patterns (Figure 6.33) exhibit reversal of colour across all reflection axes and around all centres of two-fold rotation. Colour is preserved along the glide-reflection axes.

Five distinct classes of counterchange patterns are evident on systematically colouring a p2mm primary structure. These five classes are designated the notations: c'mm, p_b'mm, pmm', p_b'gm and pm'm' (illustrated in Figures 6.34, 6.35, 6.36, 6.37 and 6.38 respectively). In class c'mm counterchange all-over patterns (Figure 6.34) colours alternate as on a chess board. Reversal of colour therefore occurs across alternating reflection axes and around alternating centres of two-fold rotation. As illustrated in Figure 6.35, class p_b'mm counterchange all-over patterns reverse colour across alternating reflection axes in one direction (horizontal in the example shown). Reversal of colour, therefore, only occurs around centres of two-fold rotation, positioned on these alternating axes. Class pmm' two-colour counterchange all-over patterns (Figure 6.36) have colour reversing across only one of the two sets of perpendicular reflection axes (vertical in the example given). Colour reversal also occurs around all centres of two-fold rotation. Class p_b'gm counterchange all-over patterns show reversal of colour across all reflection axes in one direction (vertical in the example shown in Figure 6.37)

Figure 6.33 A class pm'g two-colour counterchange all-over pattern.

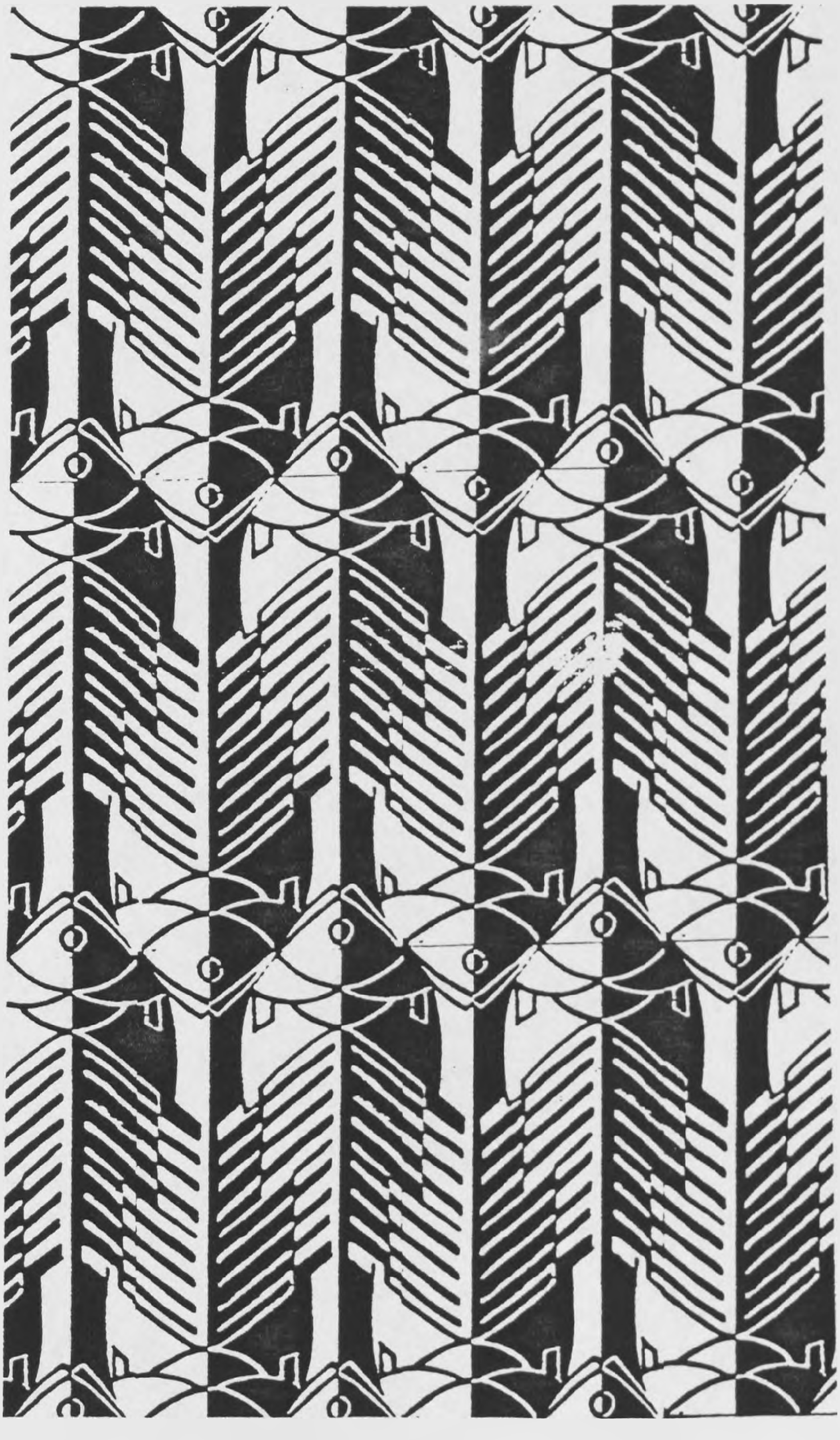


Figure 6.34 A class c'mm two-colour counterchange all-over pattern.

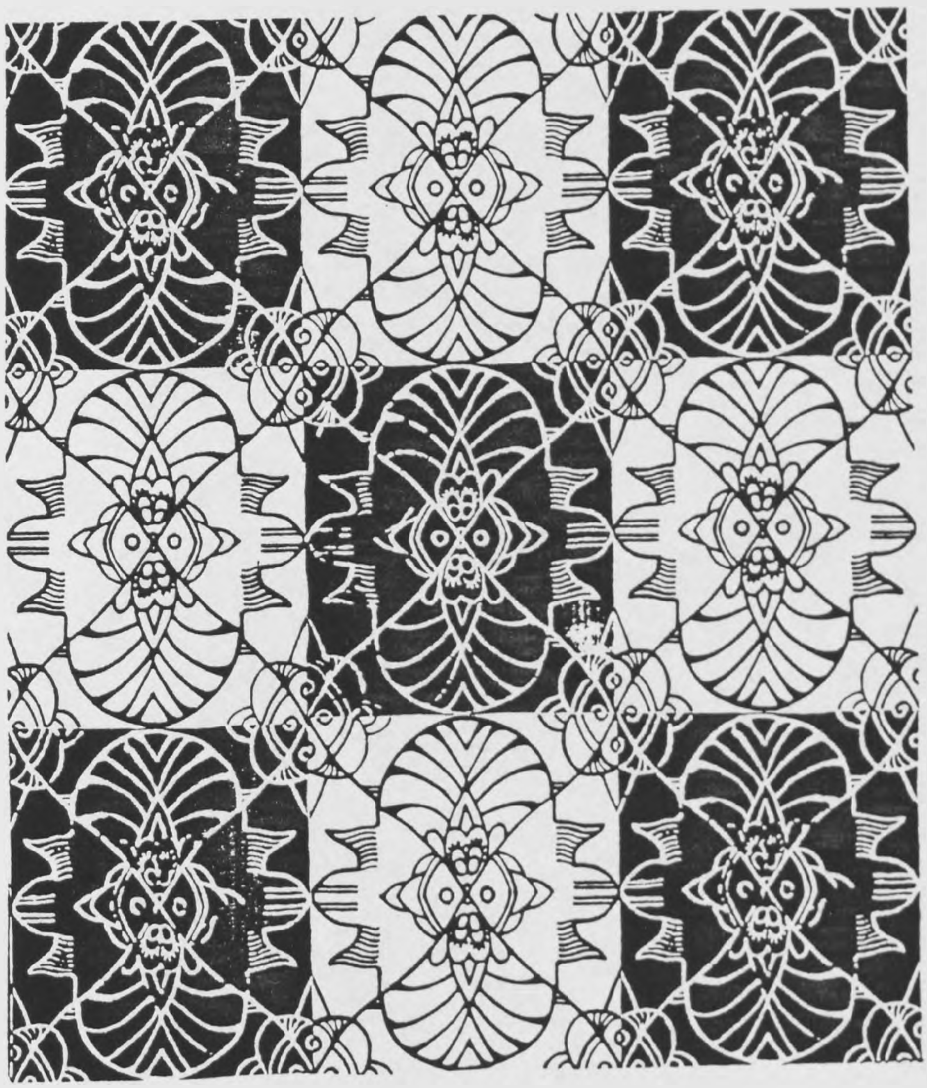


Figure 6.35 A class p_b'mm two-colour counterchange all-over pattern.

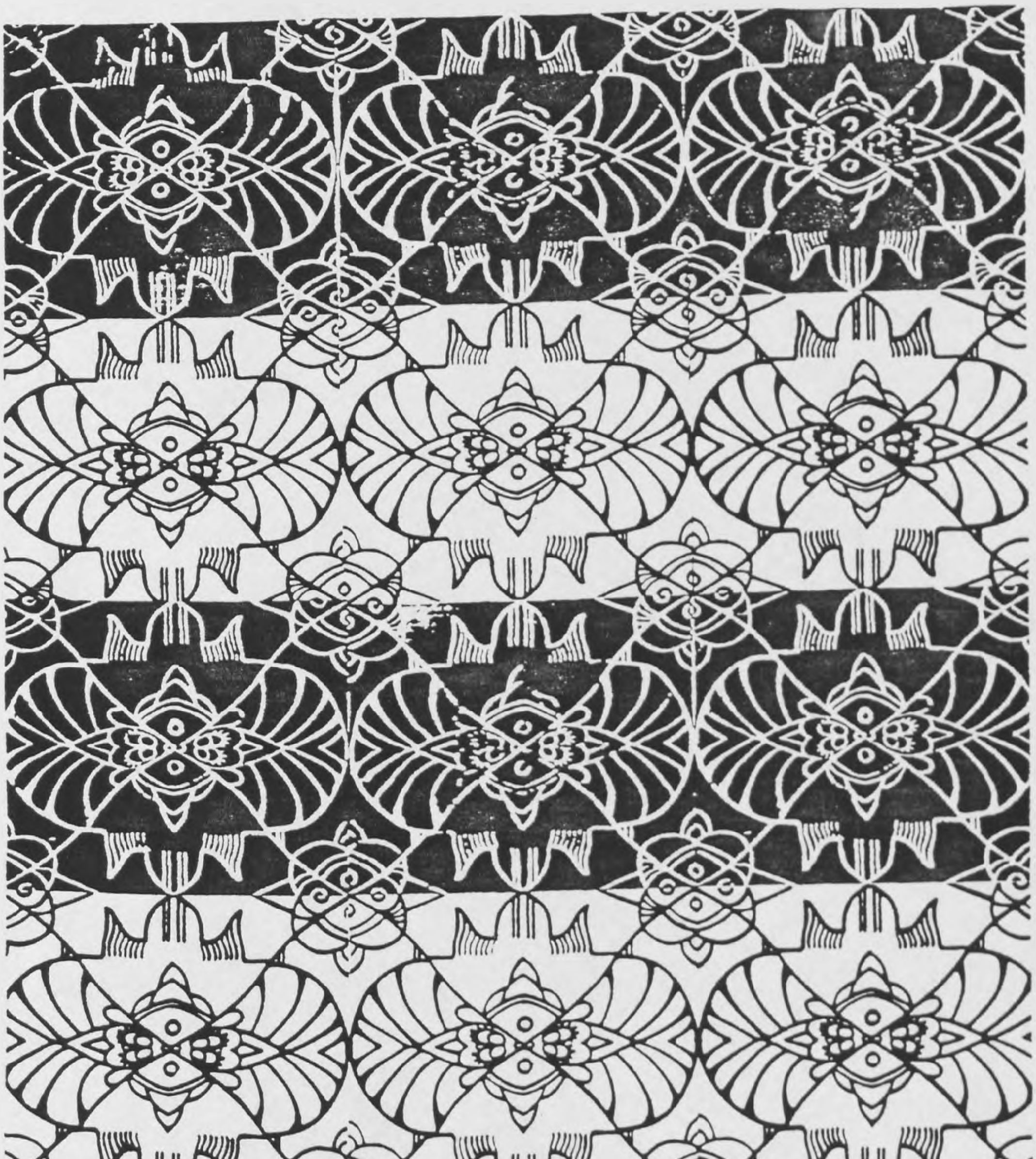


Figure 6.36 A class pmm' two-colour counterchange all-over pattern.

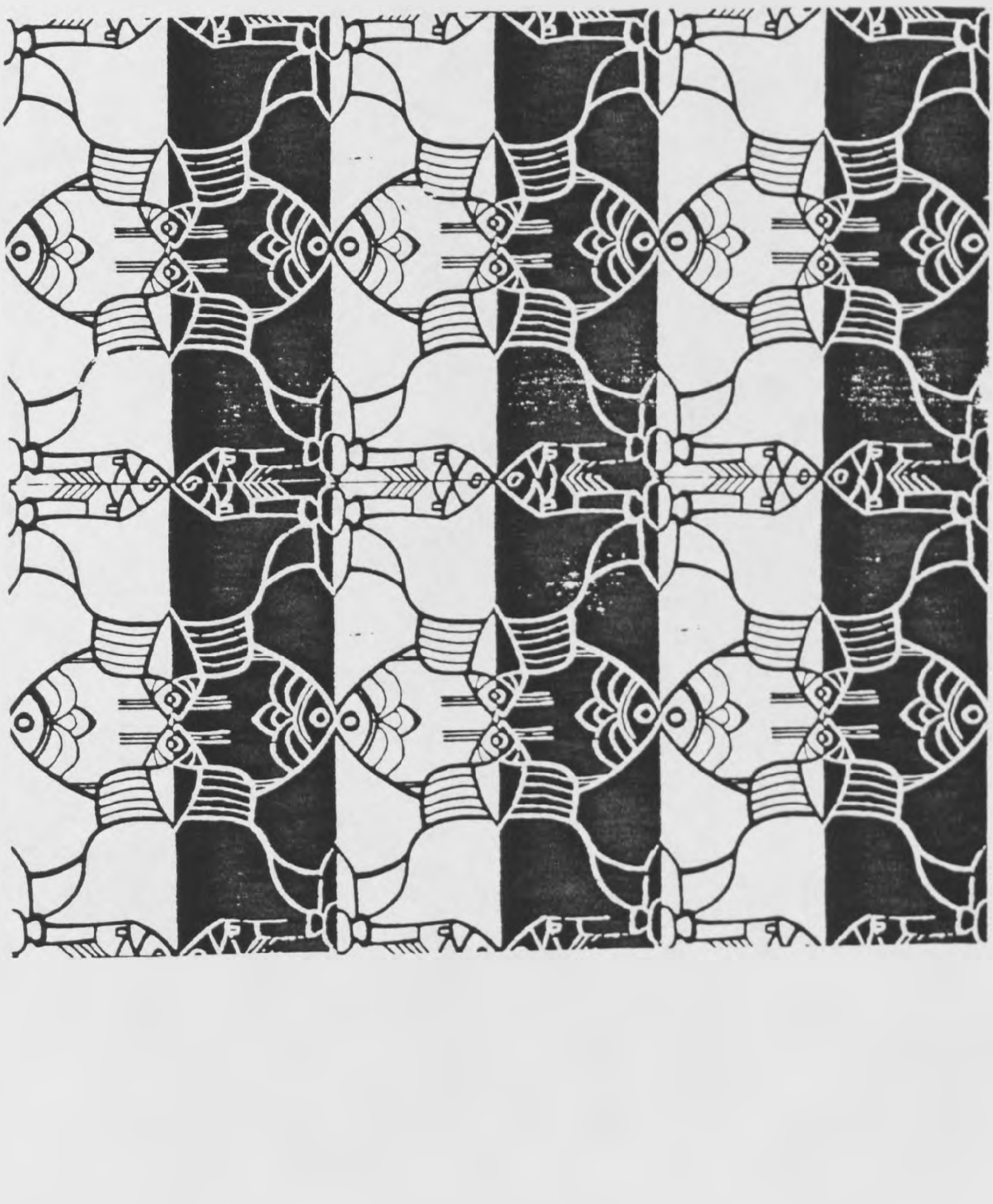


Figure 6.37 A class p_b'gm two-colour counterchange all-over pattern.

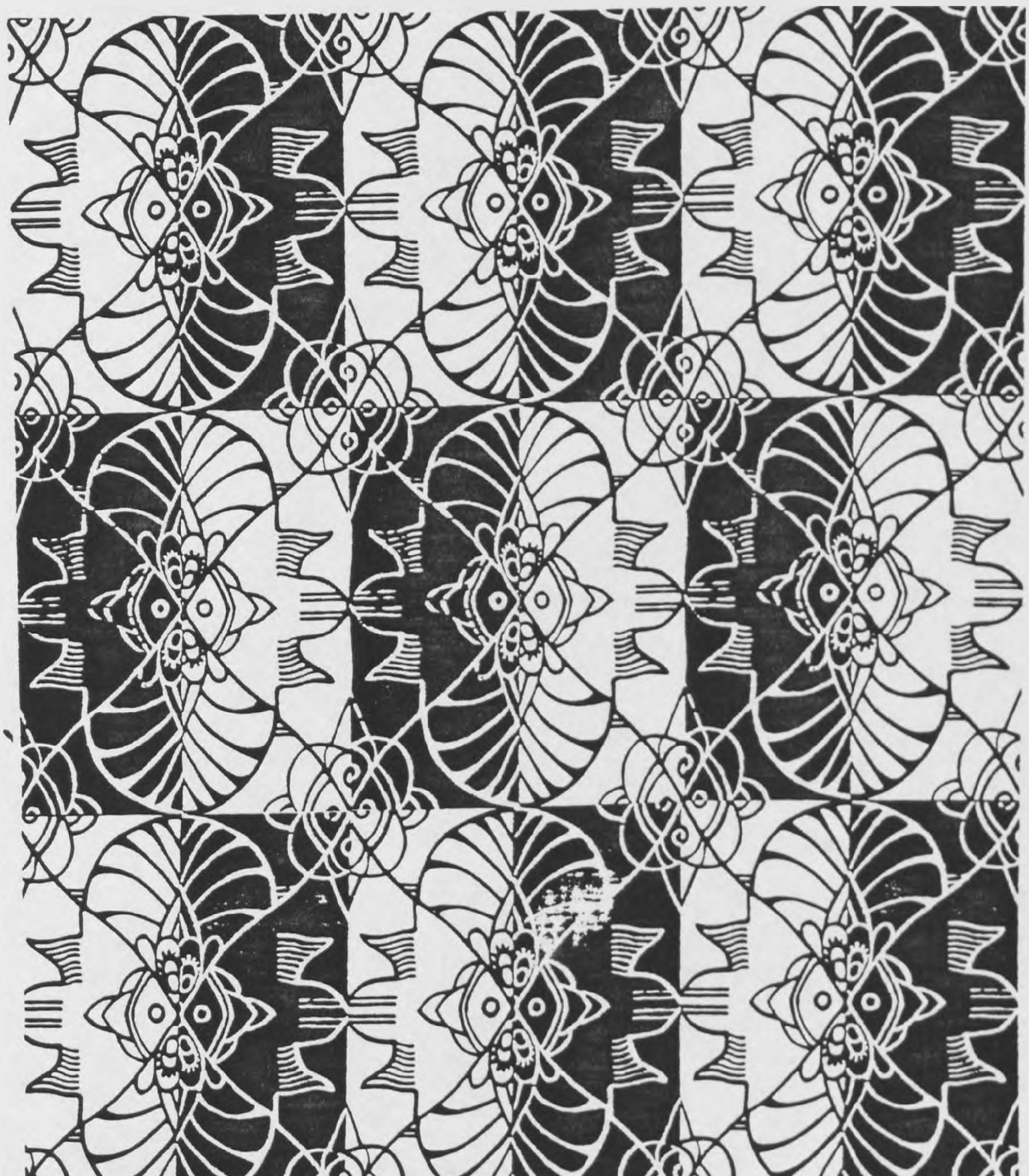
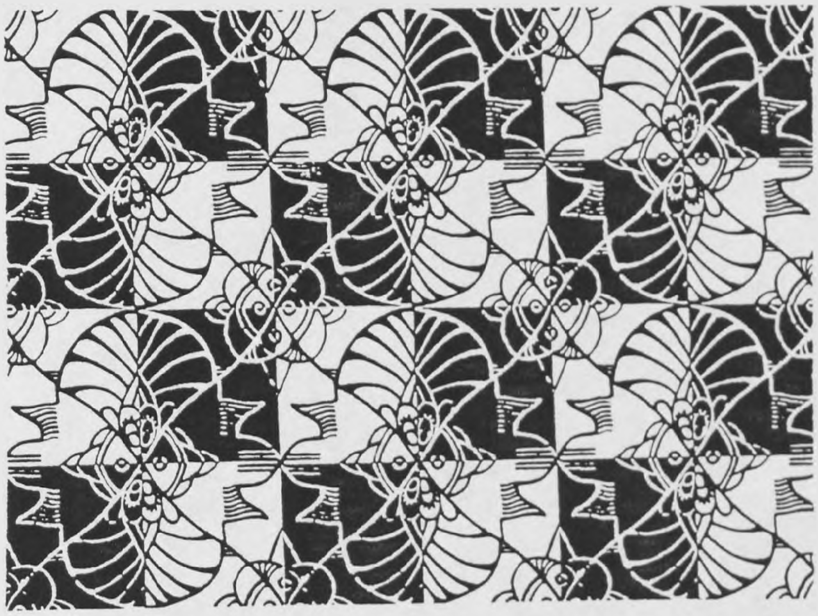
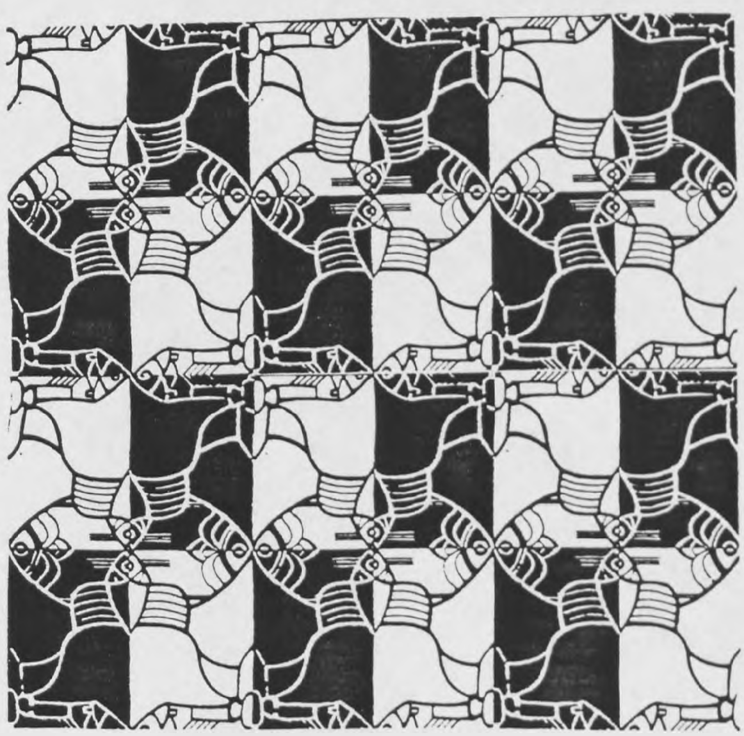


Figure 6.38 A class pm'm' two-colour counterchange all-over pattern.





and across alternating reflection axes in the other direction (horizontal in the example shown). Colours are preserved around centres of two-fold rotation lying on these latter axes, and are reversed around the others. In pm'm' patterns colours are reversed across all reflection axes and are preserved around all centres of two-fold rotation (examples are shown in Figure 6.38).

There are five distinct classes of two-colour counterchange patterns which may be derived from a class cmm primary structure. These are designated the notations p_c'mm, p_c'mg, p_c'gg, cm'm' and cmm' (Figures 6.39, 6.40, 6.41, 6.42 and 6.43 respectively). In p_c'mm two-colour counterchange all-over patterns offset rows alternate colour so that no reversal of colours occurs across reflection axes. Rather, reversal of colour occurs along glide-reflection axes and at the centres of two-fold rotation positioned at the intersection of glide-reflection axes (Figure 6.39). With class p_c'mg counterchange all-over patterns reversal of colour occurs across only one set of the perpendicular reflection axes (horizontal in the example shown in Figure 6.40) and across the glide-reflection axes perpendicular to that set. Reversal of colour also occurs around centres of two-fold rotation positioned at the intersection of reflection axes. Class p_c'gg counterchange all-over patterns exhibit reversal of colour across reflection axes and about centres of two-fold rotation positioned at the intersections of glide-reflection axes (Figure 6.41). As shown in Figure 6.42 reversal of colour in class cm'm' counterchange all-over patterns occurs across all reflection axes and along all glide-reflection axes. Colour is preserved around all centres of two-fold rotation.

Figure 6.39 A class p_c'mm two-colour counterchange all-over pattern.

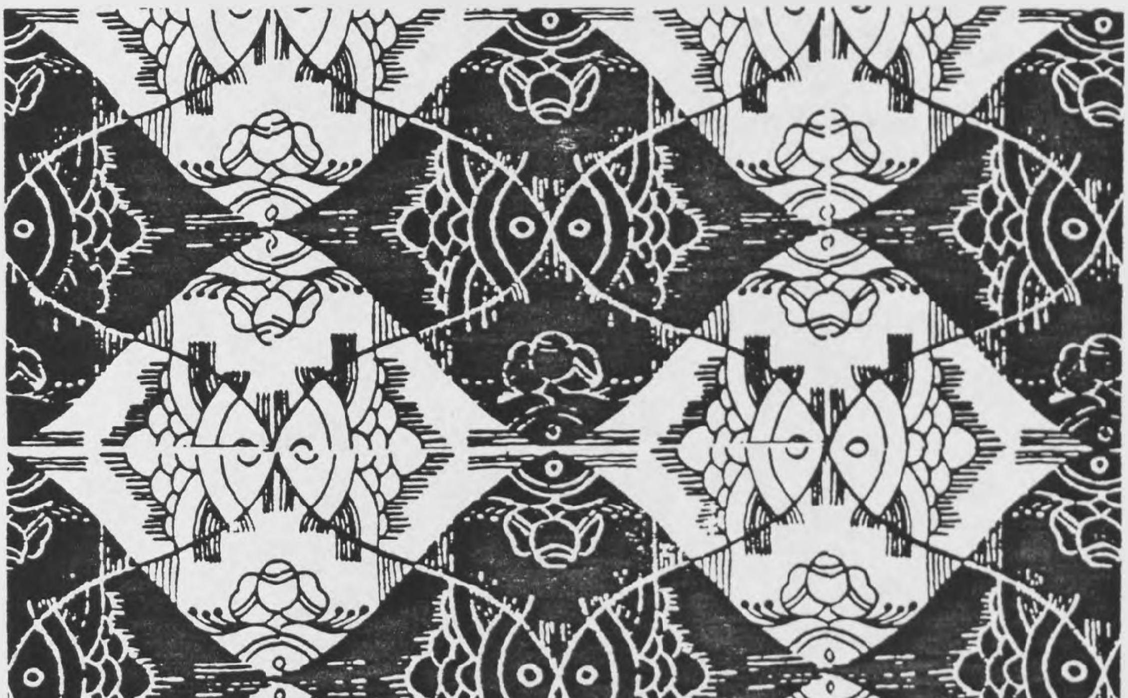


Figure 6.40 A class p_c'mg two-colour counterchange all-over pattern.

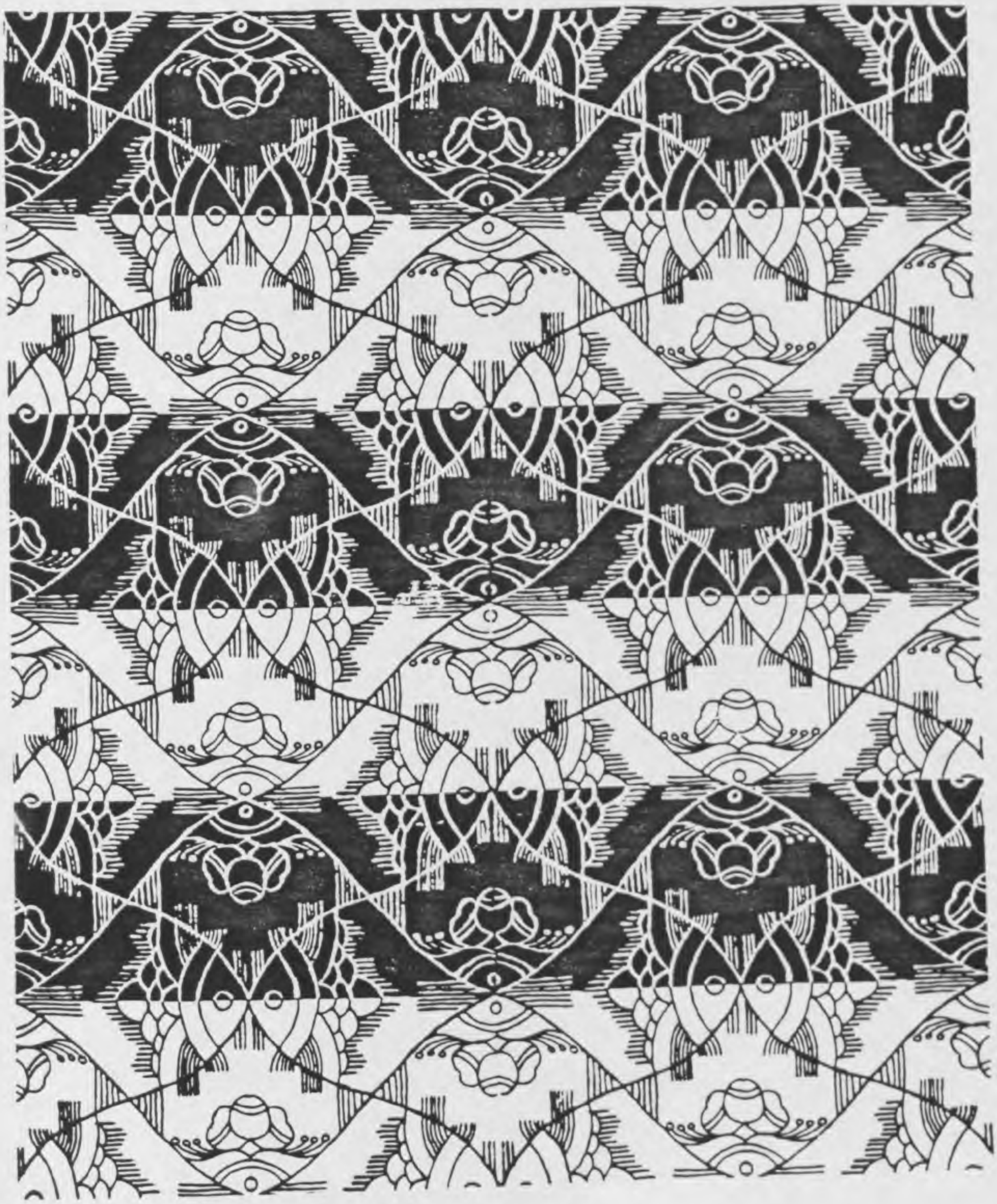


Figure 6.41 A class p_c'gg two-colour counterchange all-over pattern.

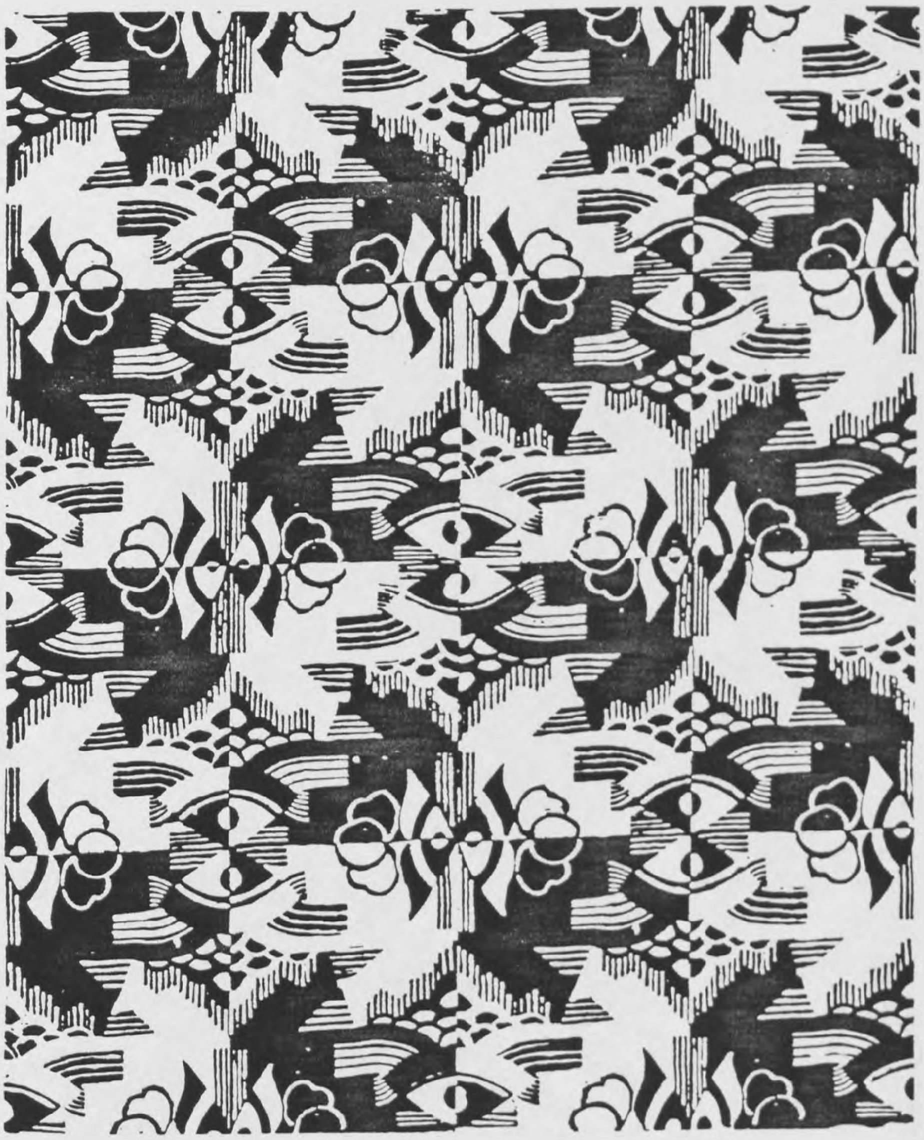
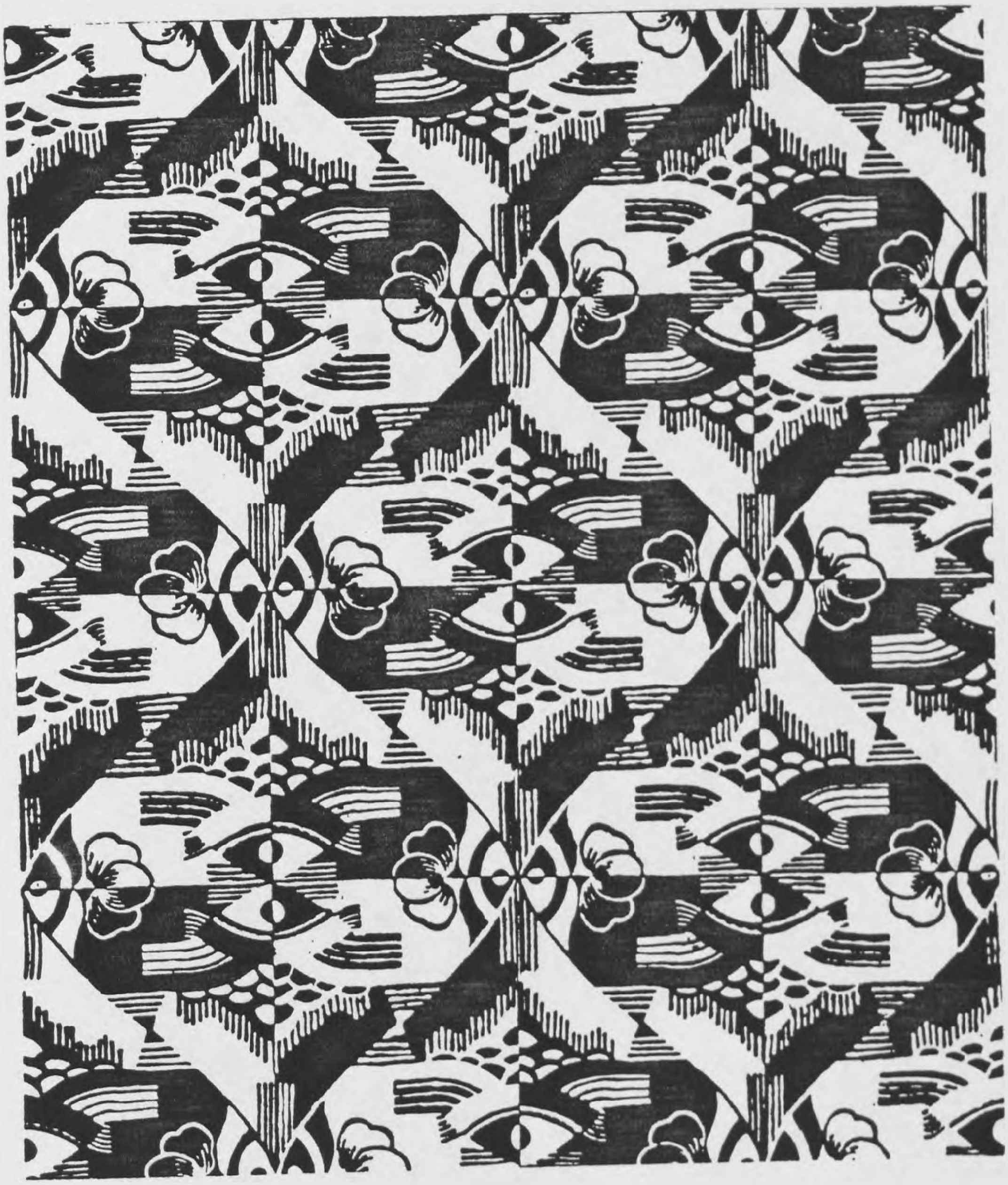


Figure 6.42 A class cm'm' two-colour counterchange all-over pattern.



In class cmm' counterchange all-over patterns (Figure 6.43) there are reversals of colour across one of the two sets of perpendicular reflection axes (horizontal in the example given) and along the glide-reflection axes with the same direction.

6.4.3 Two-colour counterchange all-over patterns with three-fold rotation

As stated previously, primary all-over patterns with a highest order of rotation of three include classes p3, p3m1 and p31m. Class p3 all-over patterns cannot be systematically coloured to produce a two-colour counterchange pattern. With classes p3m1 and p31m, systematic colouring involving reflection is however possible; relevant notations are p3m' and p31m' respectively.

With a class p3m' two-colour counterchange all-over pattern, colour reversal occurs across all reflection axes and across all glide-reflection axes (Figure 6.44). Likewise in the case of class p31m' patterns (Figure 6.45) reversal of colour occurs across all reflection axes and across all glide-reflection axes.

6.4.4 Two-colour counterchange all-over patterns with four-fold rotation

As stated previously classes p4, p4gm and p4mm each exhibit four-fold rotation. By introducing colour interchange on these primary structures a total of ten two-colour counterchange all-over pattern classes are possible. Each are described and illustrated below.

Figure 6.43 A class cmm' two-colour counterchange all-over pattern.

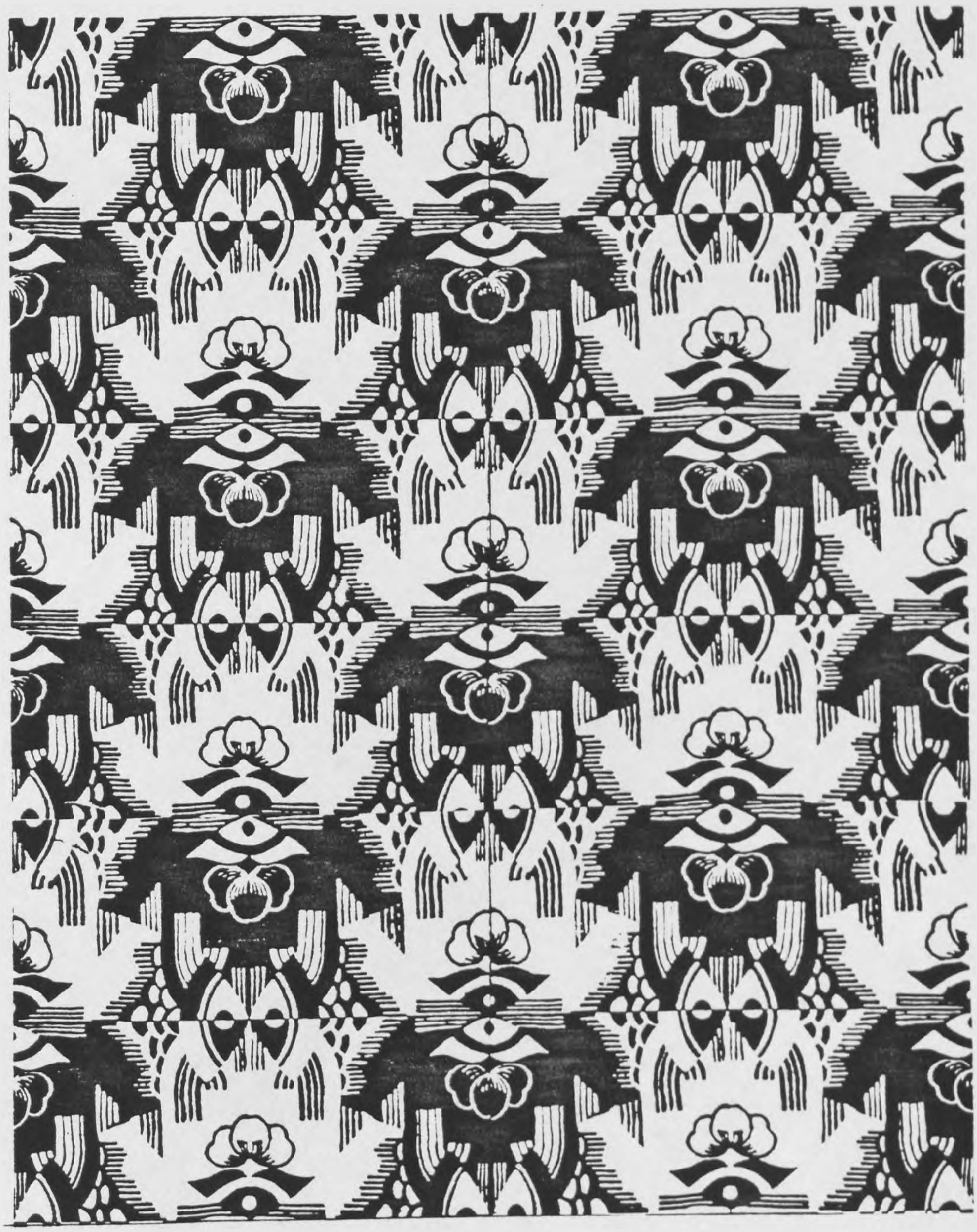


Figure 6.44 A class p3m' two-colour counterchange all-over pattern.

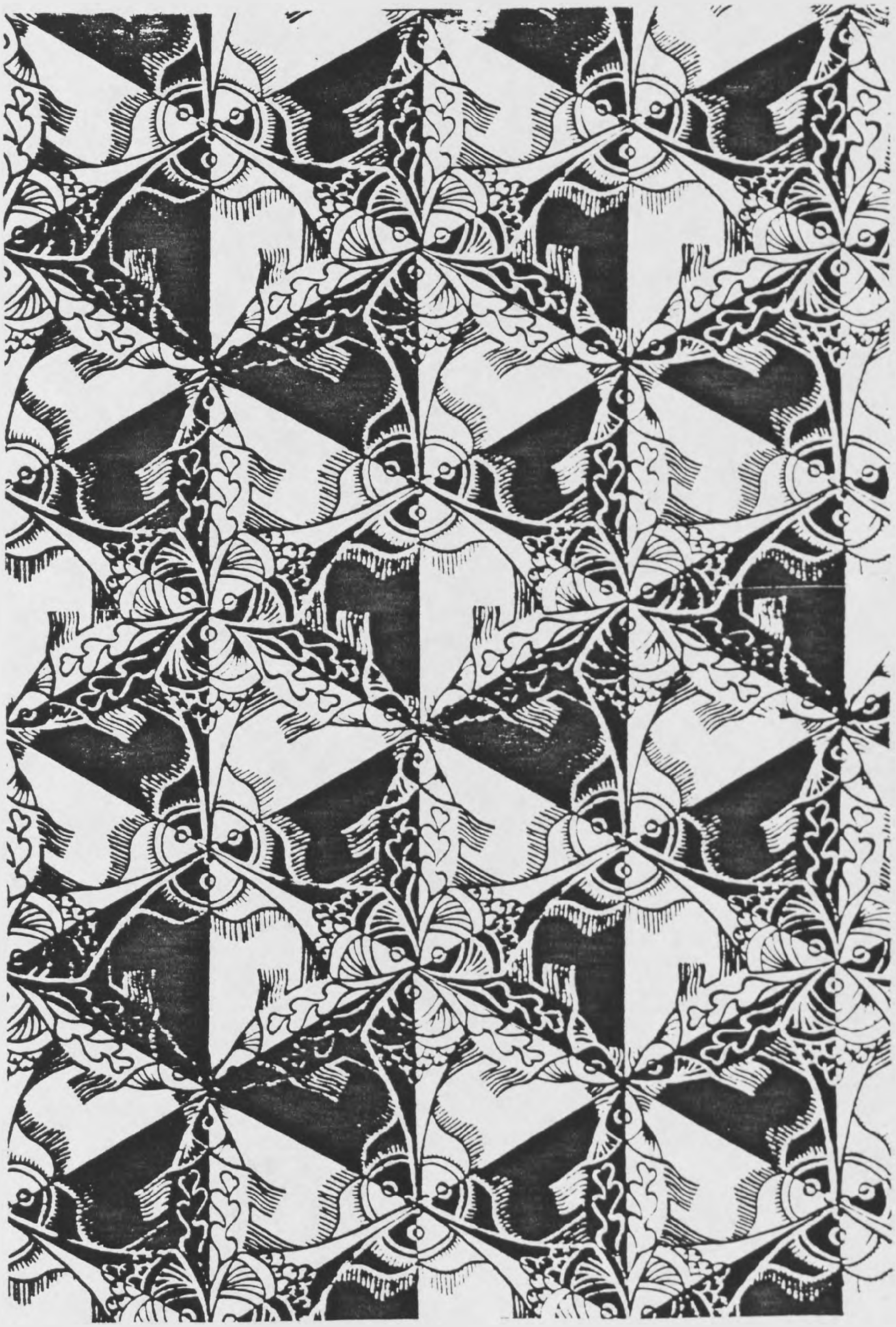
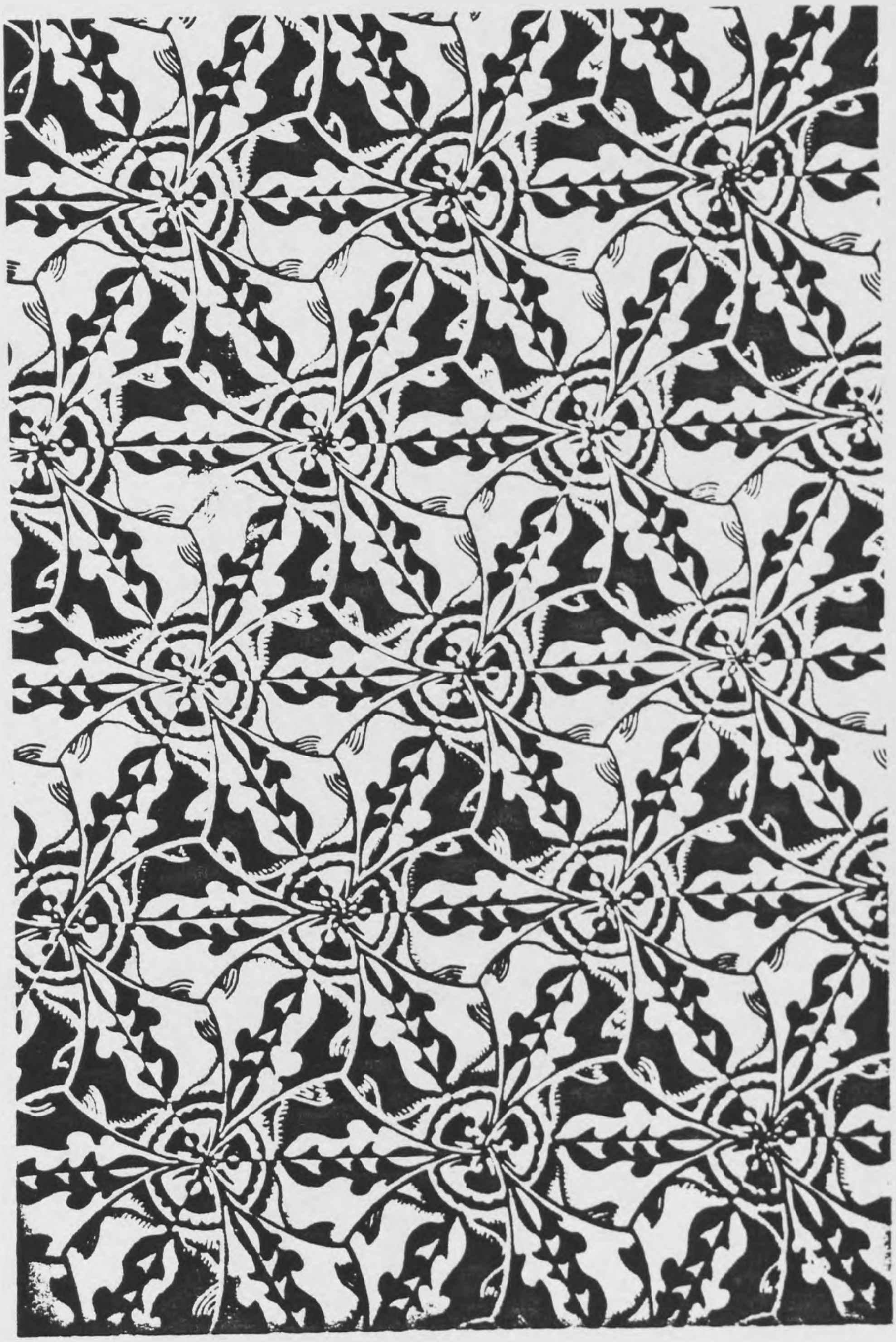


Figure 6.45 A class p31m' two-colour counterchange all-over pattern.



There are two distinct ways of systematically colouring a class p4 all-over pattern; relevant notations are p_c'4 and p4' (Figures 6.46 and 6.47 respectively). In class p_c'4 patterns reversal of colour is evident around each two-fold centre of rotation and around alternating rows of four-fold rotation (Figure 6.46). With a p4' two-colour counterchange all-over pattern reversal of colour is evident on each 90 degree rotation around all centres of four-fold rotation. Colour is preserved around centres of two-fold rotation (Figure 6.47).

Three distinct two-colour counterchange classes may be derived from p4gm primary all-over patterns. Relevant notations are p4g'm', p4'g'm and p4'gm' (Figures 6.48, 6.49 and 6.50 respectively). With class p4g'm' patterns colour reversal is evident along all glide-reflection axes and across all reflection axes. Colour is preserved around all centres of two-fold and four-fold rotation (Figure 6.48). Class p4'g'm two-colour counterchange patterns exhibit reversal of colour around centres of four-fold rotation but not centres of two-fold rotation. Reversal of colour also occurs along glide-reflection axes which pass between the centres of four-fold rotation (at a diagonal to the reflection axes) but not along the glide-reflection axes which pass through the centres of four-fold rotation (Figure 6.49). Class p4'gm' patterns exhibit reversal of colour around centres of four-fold rotation but not around centres of two-fold rotation. Reversals of colour are also evident along the glide-reflection axes passing through centres of four-fold rotation, and across reflection axes (Figure 6.50).

Figure 6.46 A class p_c'4 two-colour counterchange all-over pattern.

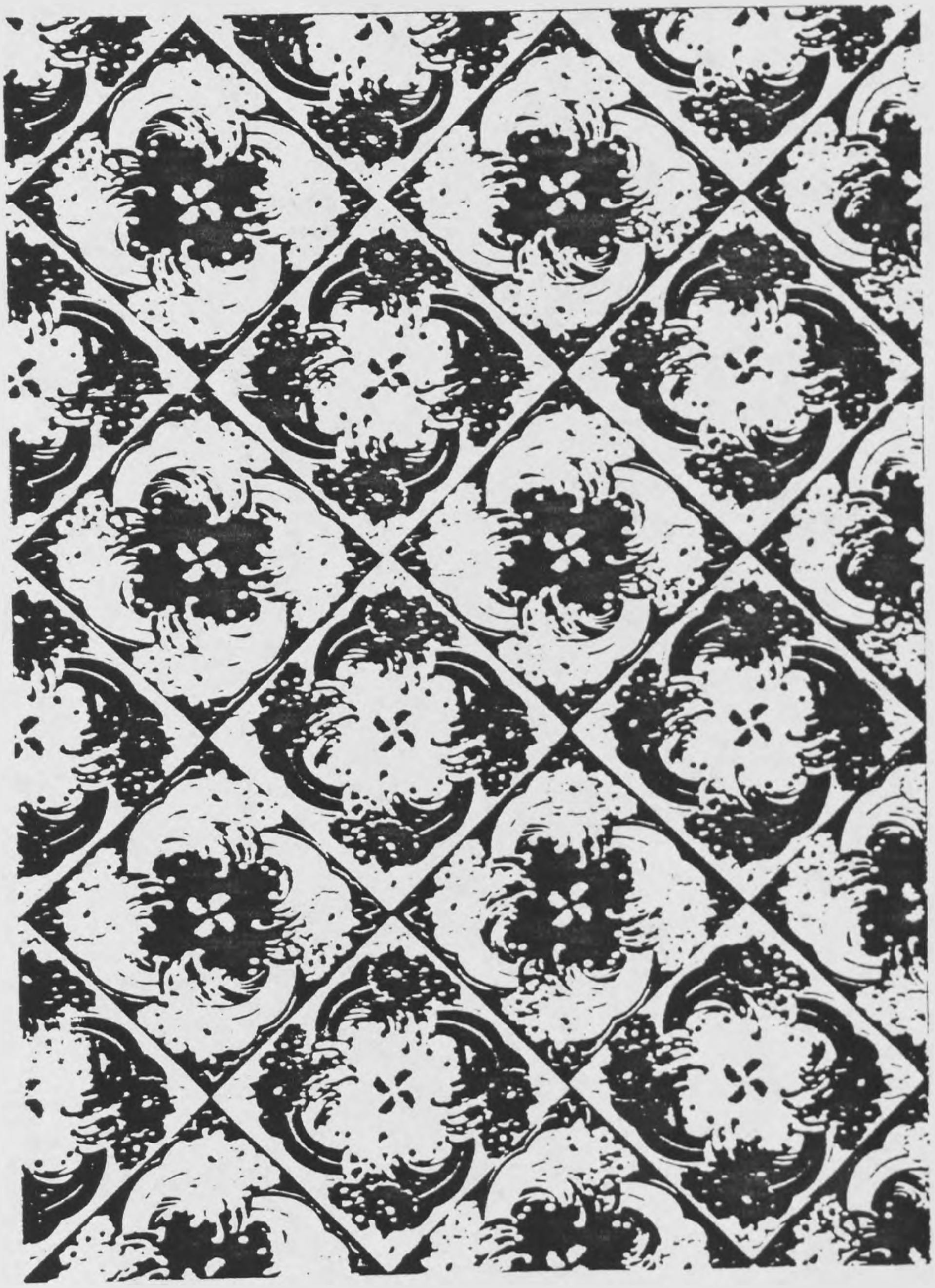


Figure 6.47 A class p4' two-colour counterchange all-over pattern.

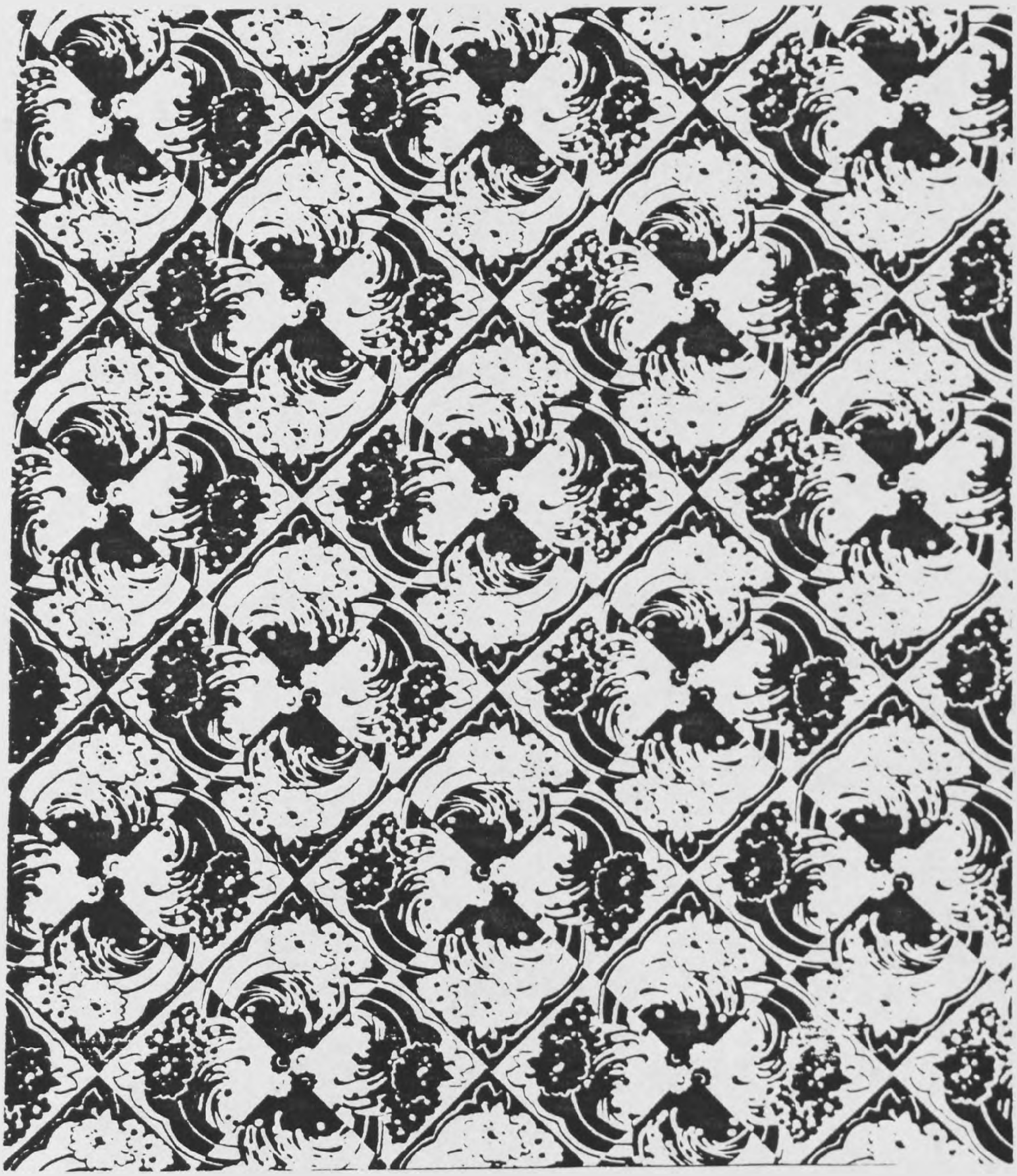


Figure 6.48 A class p4g'm' two-colour counterchange all-over pattern.

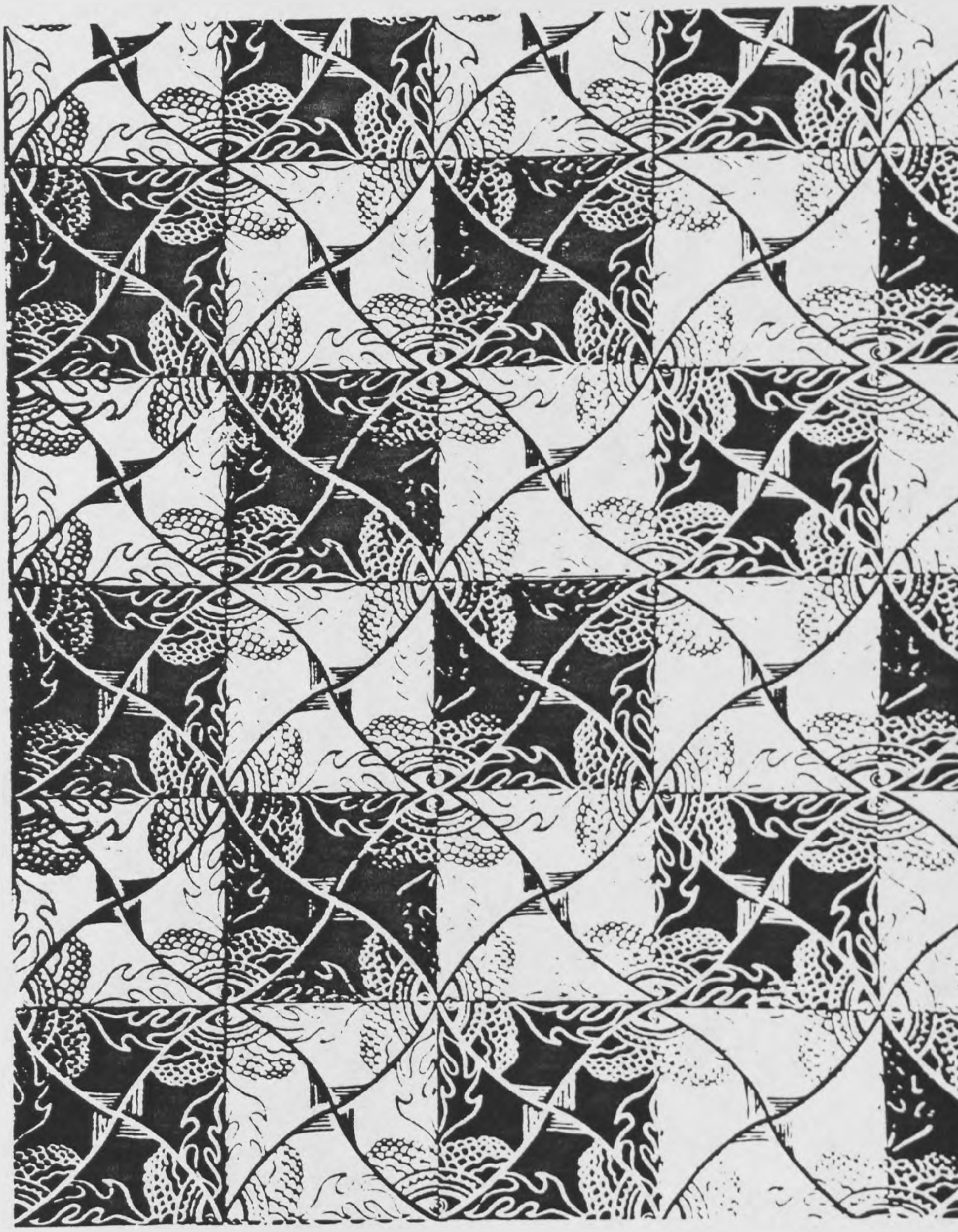


Figure 6.49 A class p4'g'm two-colour counterchange all-over pattern.

Figure 6.50 A class p4'gm' two-colour counterchange all-over pattern.

There are five distinct ways of systematically colouring a class p4mm primary all-over pattern; the corresponding notations are p4'mm', p_c'4gm, p4'm'm, p_c'4mm and p4m'm' (Figures 6.51, 6.52, 6.53, 6.54 and 6.55 respectively). Class p4'mm' patterns exhibit reversal of colour along all glide-reflection axes, across reflection axes parallel to glide-reflection axes and around all centres of four-fold rotation (Figure 6.51). Class p_c'4gm counterchange all-over patterns exhibit reversal of colour across reflection axes parallel to glide-reflection axes and across half the reflection axes in the other two directions. Colour is also reversed around all centres of two-fold rotation and around half the centres of four-fold rotation (Figure 6.52). In class p4'm'm, colour reversal is evident in reflection axes in two of the four directions and around all centres of four-fold rotation. Colour is preserved across the glide-reflection axes and around centres of two-fold rotation (Figure 6.53). In p_c'4mm two-colour counterchange all-over patterns, colour reversal occurs across reflection axes running vertically and horizontally between the square unit cells, but not across reflection axes running through the squares. Reversal of colour also occurs along the diagonal glide-reflection axes, around alternating centres of four-fold rotation and around all centres of two-fold rotation (Figure 6.54). Class p4m'm' two-colour counterchange all-over patterns exhibit colour reversal across reflection axes in all four directions and along all glide-reflection axes. Colour is preserved in all two-fold and four-fold centres of rotation (Figure 6.55).

Figure 6.51 A class p4'mm' two-colour counterchange all-over pattern.

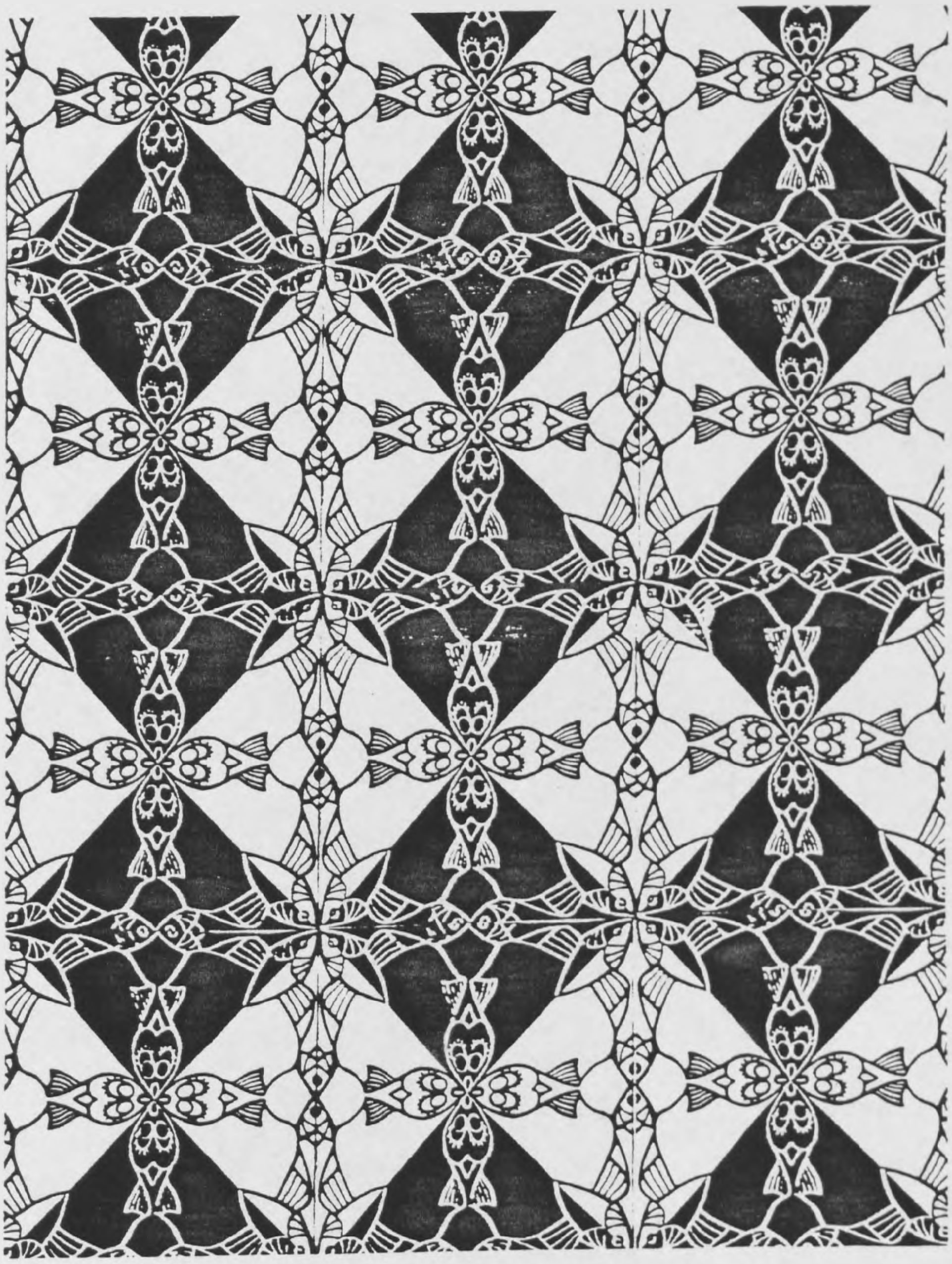
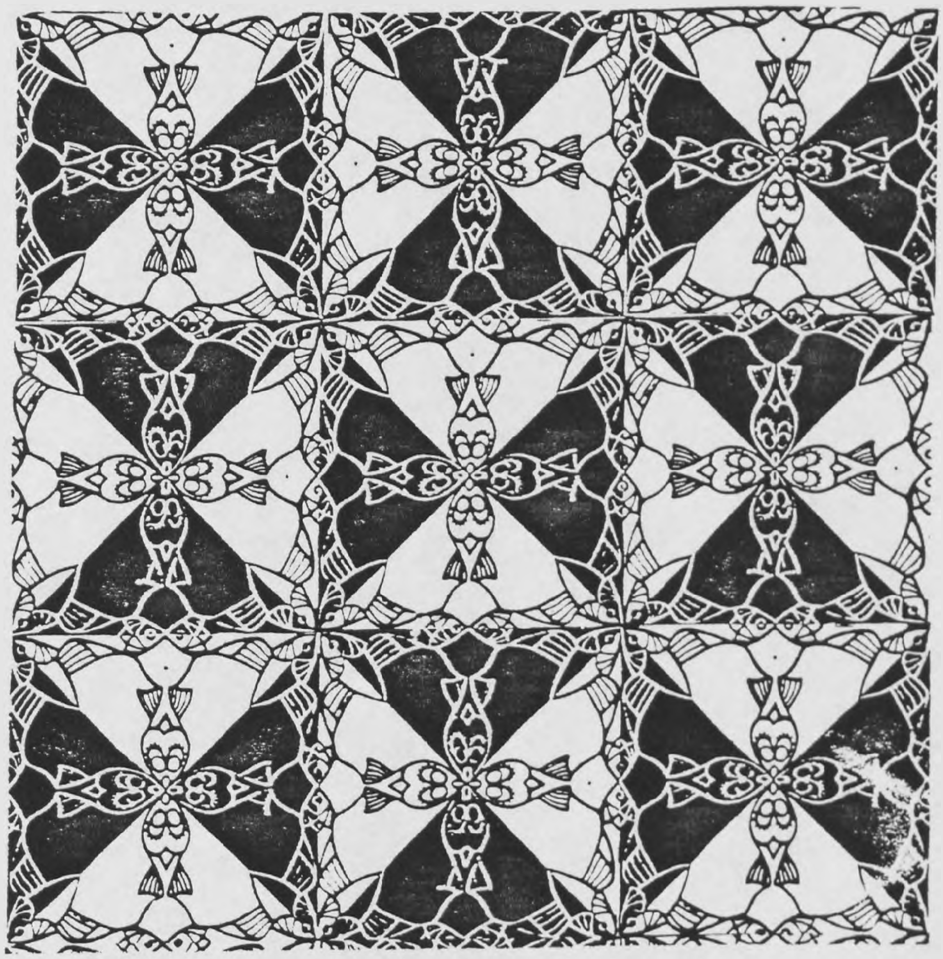


Figure 6.52 A class p_c'4gm two-colour counterchange all-over pattern.



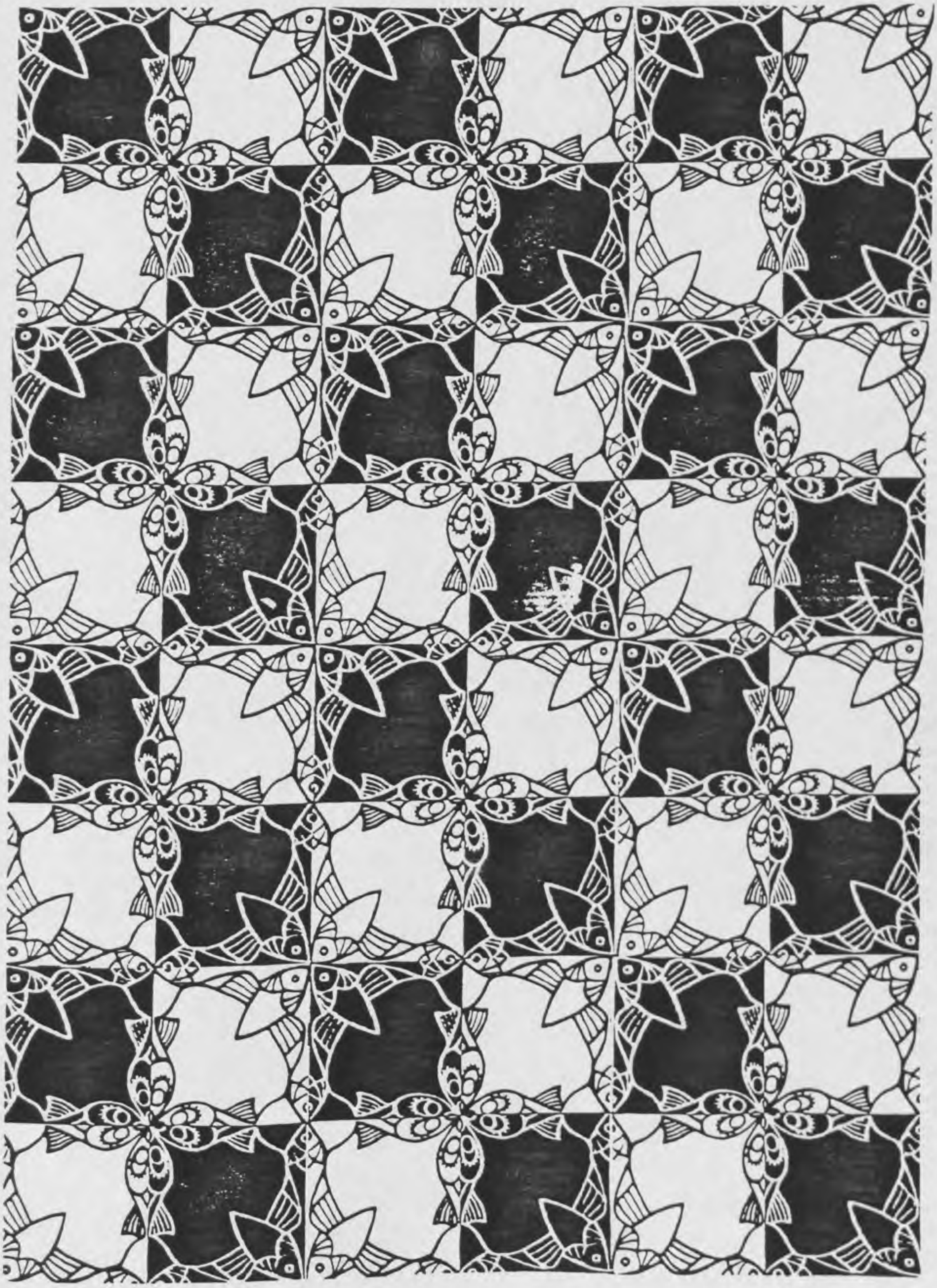


Figure 6.53 A class p4'm'm two-colour counterchange all-over pattern.

Figure 6.54 A class p_c'4mm two-colour counterchange all-over pattern.

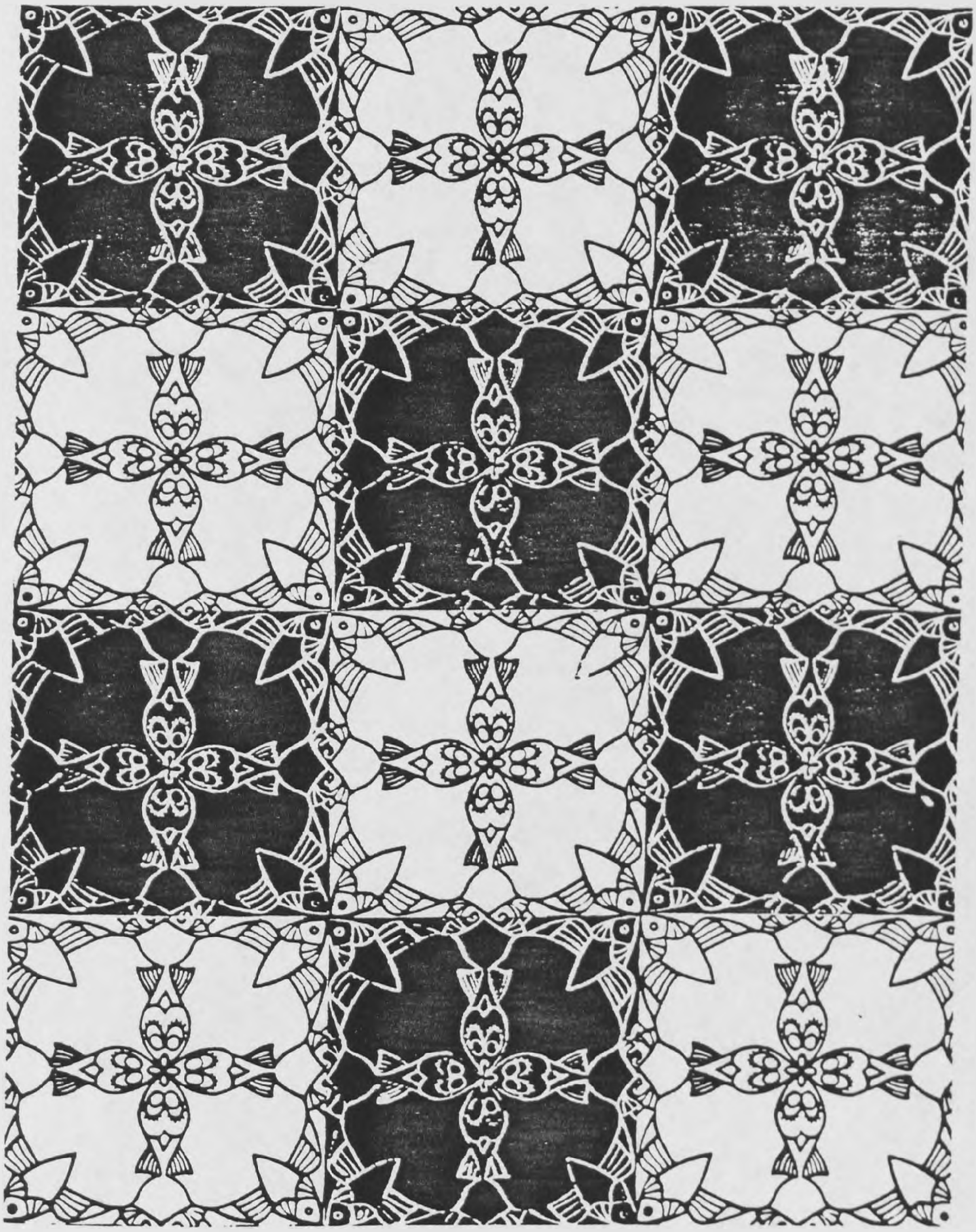
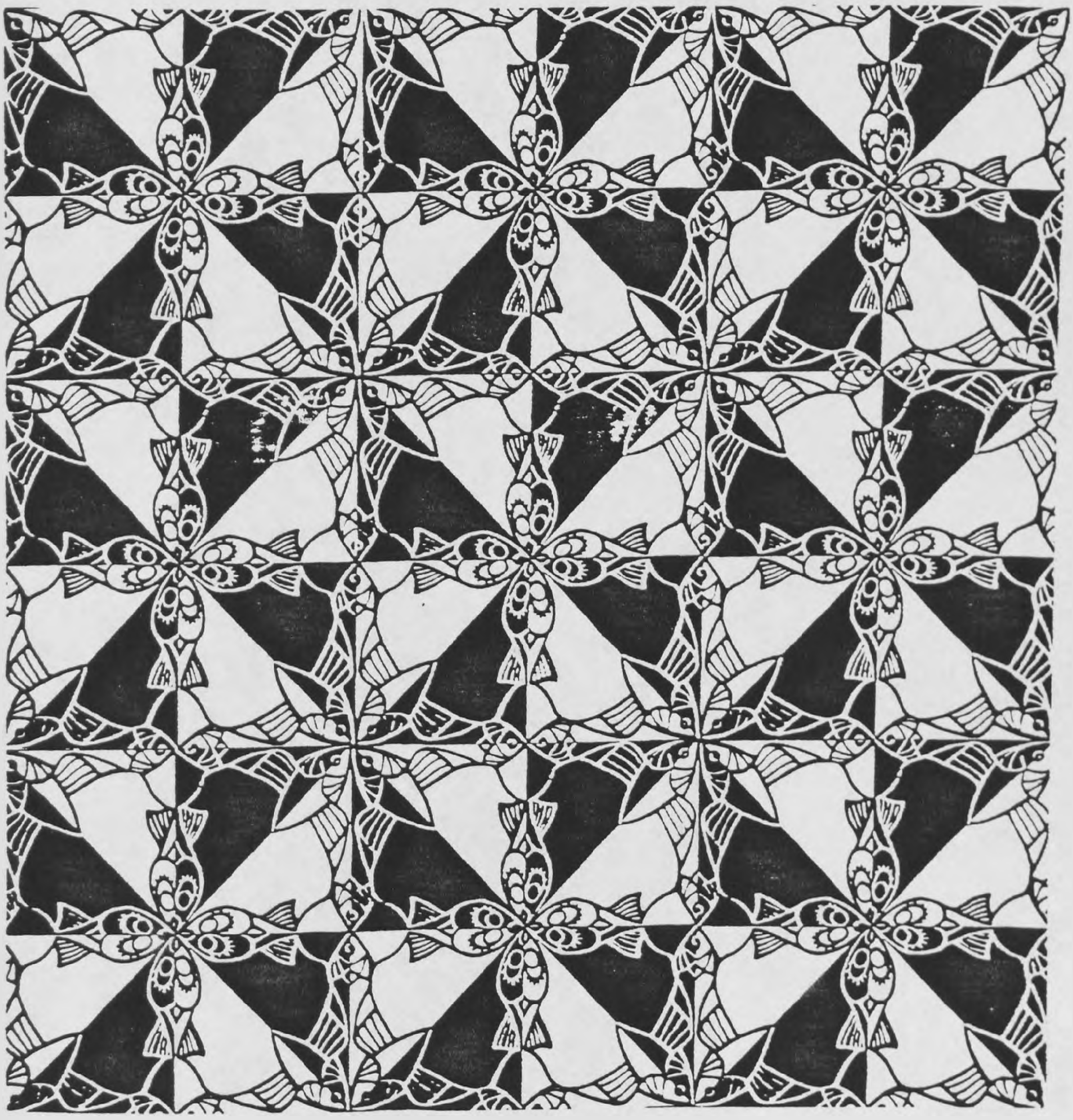


Figure 6.55 A class p4m'm' two-colour counterchange all-over pattern.



6.4.5 Two-colour counterchange all-over patterns with six-fold rotation

There is only one way to systematically colour a p6 primary all-over pattern and that is to alternate the colour around the centres of six-fold rotation. An example of this pattern type, with notation $p6'$, is given in Figure 6.56.

Class p6mm primary all-over patterns may be systematically coloured in three distinct ways. Relevant notations are $p6'mm'$, $p6'm'm$ and $p6m'm'$ (Figures 6.57, 6.58 and 6.59 respectively). In class $p6'mm'$ two-colour counterchange all-over patterns, colour is reversed around all centres of two-fold rotation and in reflection axes passing through opposite sides of the hexagonal shaped unit cell (Figure 6.57). Class $p6'm'm$ patterns (Figure 6.58) show colour reversal around all centres of two-fold rotation and in reflection axes passing through opposite angles of the hexagonal shaped unit cell. Class $p6m'm'$ patterns (Figure 6.59) exhibit colour reversal across all reflection axes. Colour is preserved in centres of rotation.

6.5 Multi-Coloured Counterchange Patterns

Having introduced the principles of two-colour counterchange patterns and tilings above, the intention below is to focus further discussion and description on patterns coloured in a systematic way with three or more colours.

When three colours are involved in the systematic colouring of the seventeen primary all-over pattern (or tiling) classes, a total of twenty-three three-colour classes are possible.

Figure 6.56 A class p6' two-colour counterchange all-over pattern.

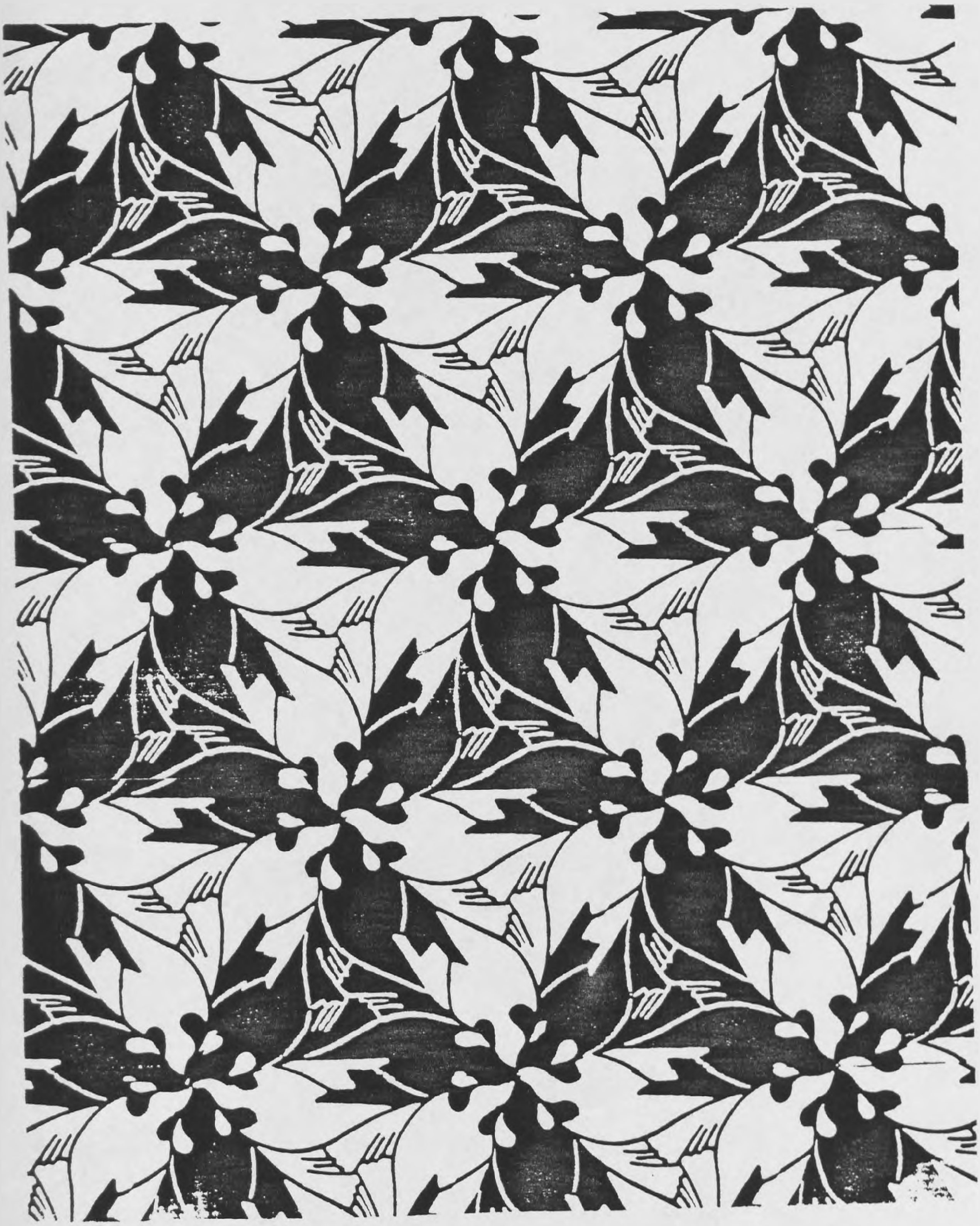


Figure 6.57 A class p6'mm' two-colour counterchange all-over pattern.

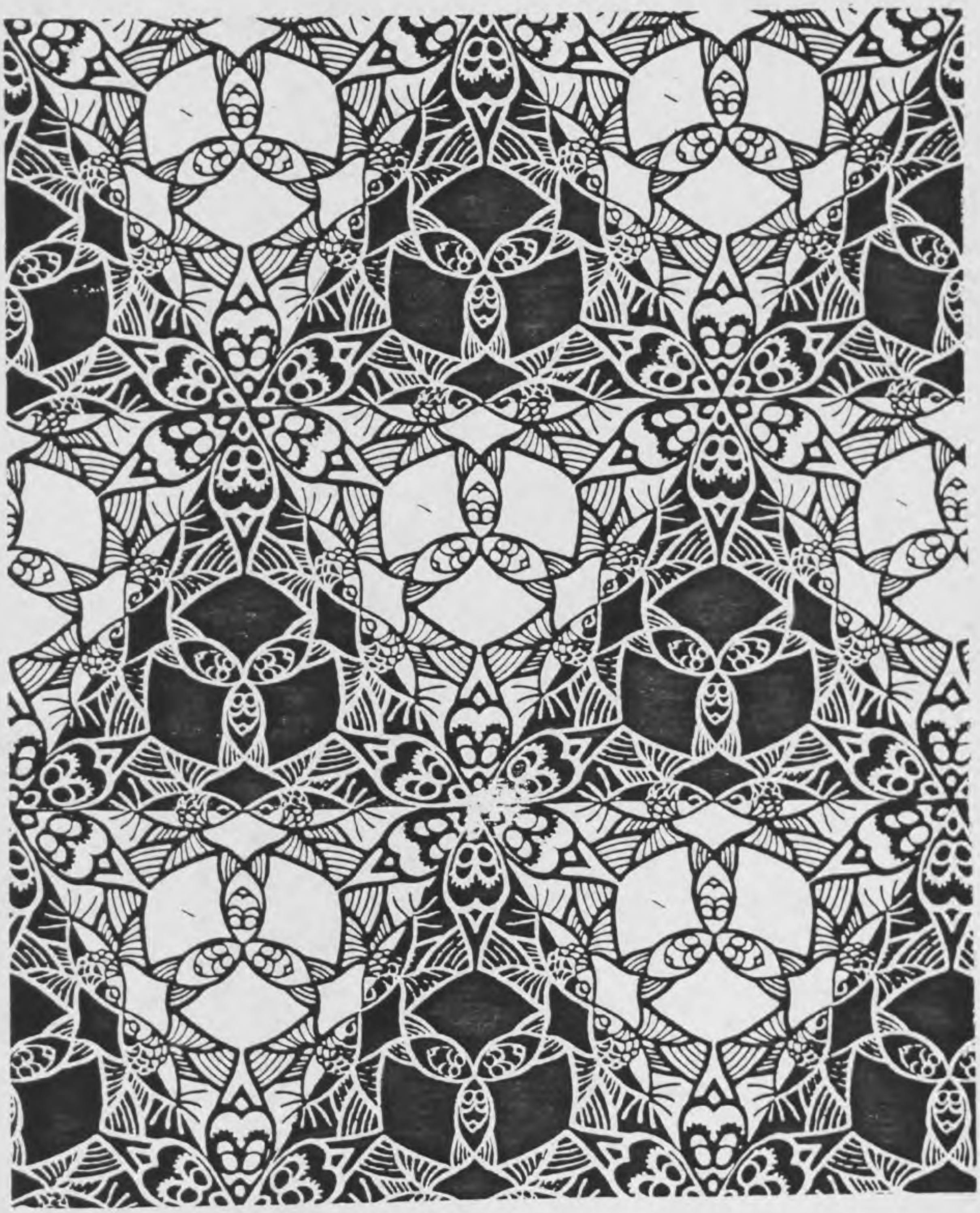


Figure 6.58 A class p6'm'm two-colour counterchange all-over pattern.

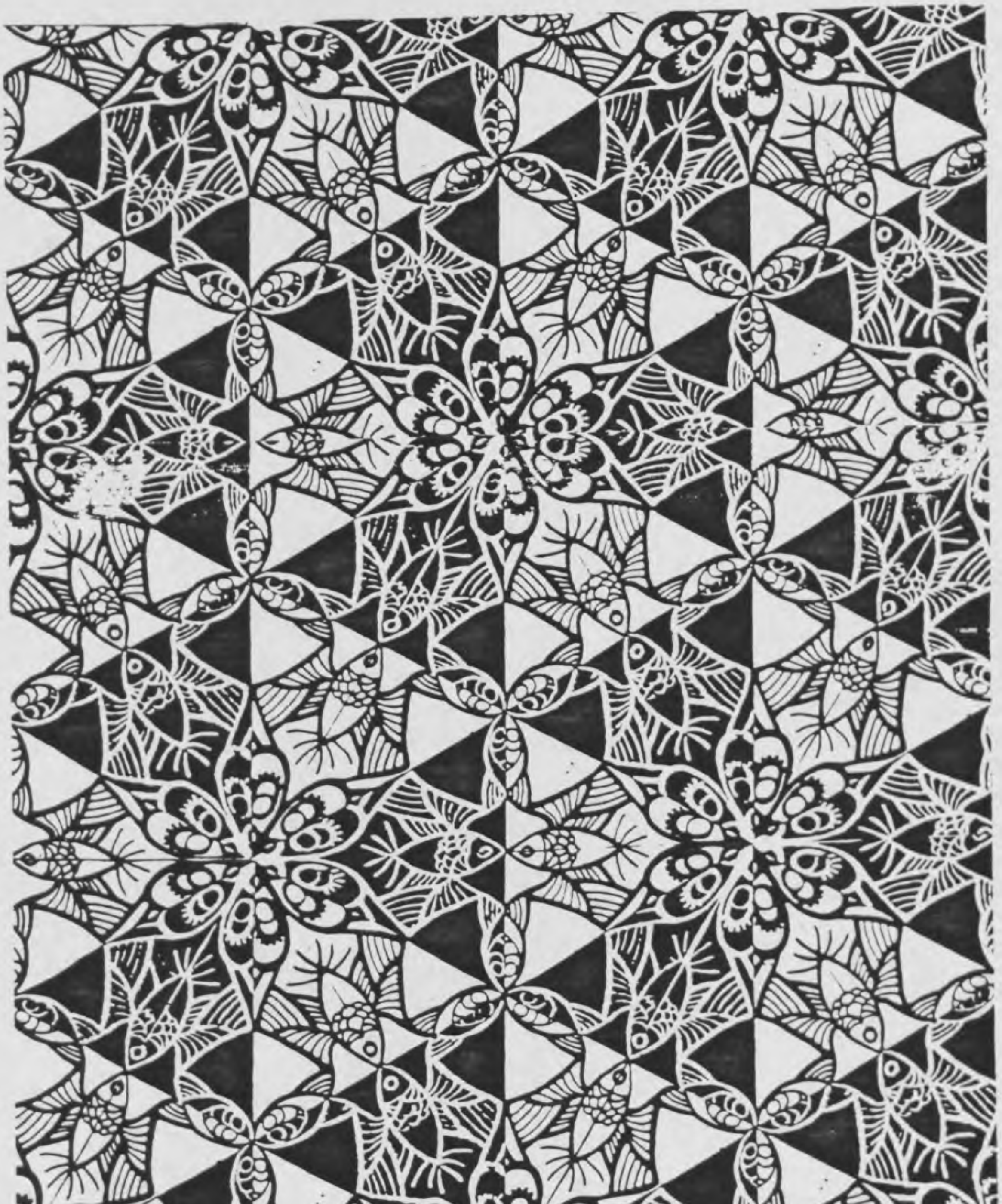
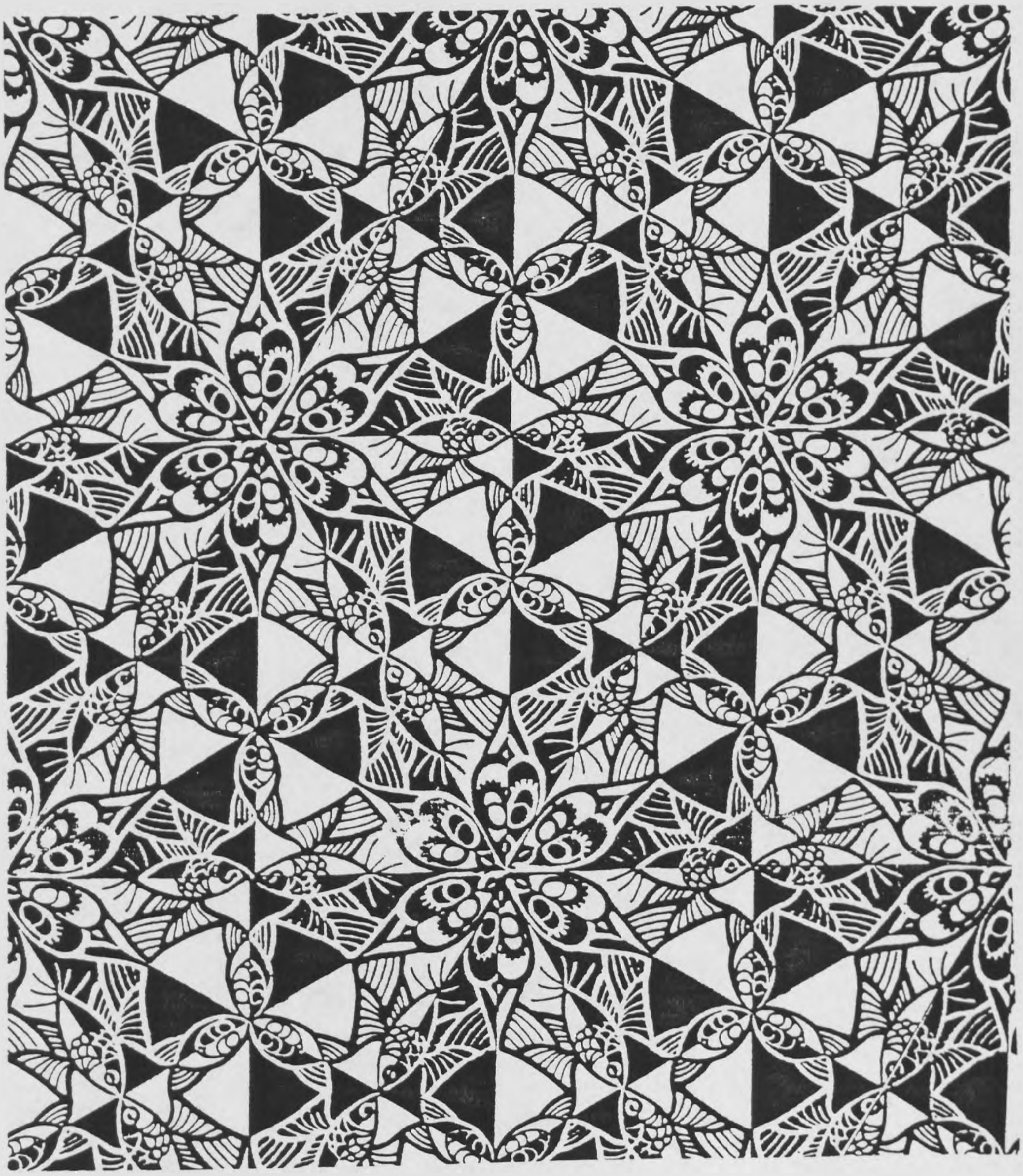


Figure 6.59 A class p6m'm' two-colour counterchange all-over pattern.



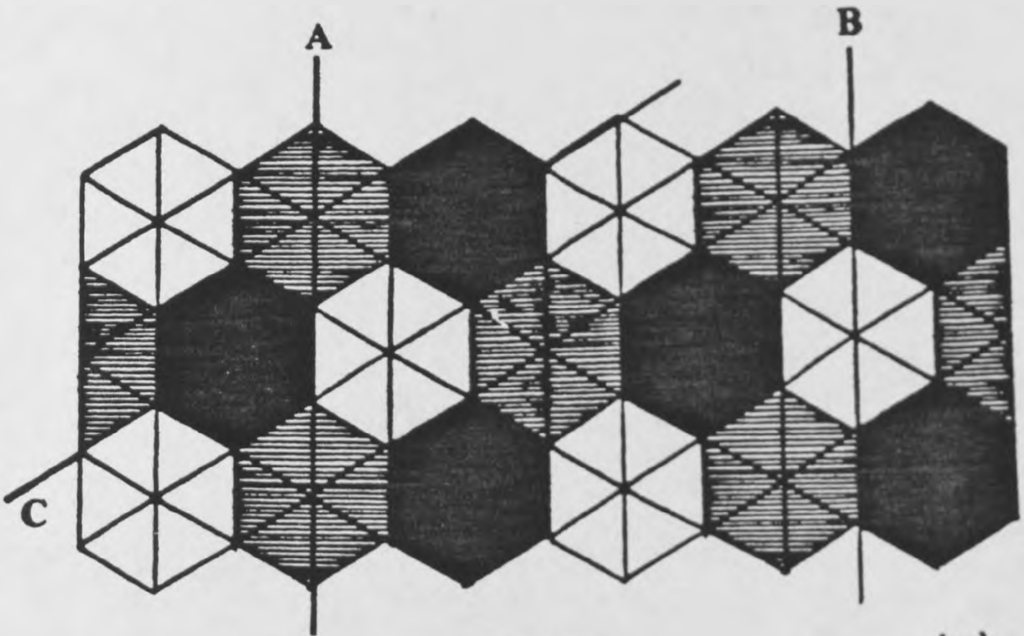
The general principles outlined above in the systematic colouring of two-colour counterchange patterns (or tilings) are applicable with three colour and higher colour counterchange patterns. That is, the colours must be systematically changed in a continuous way with respect to the relevant symmetry operations, and each symmetry operation must induce a permutation of the colours in the coloured pattern or tiling.

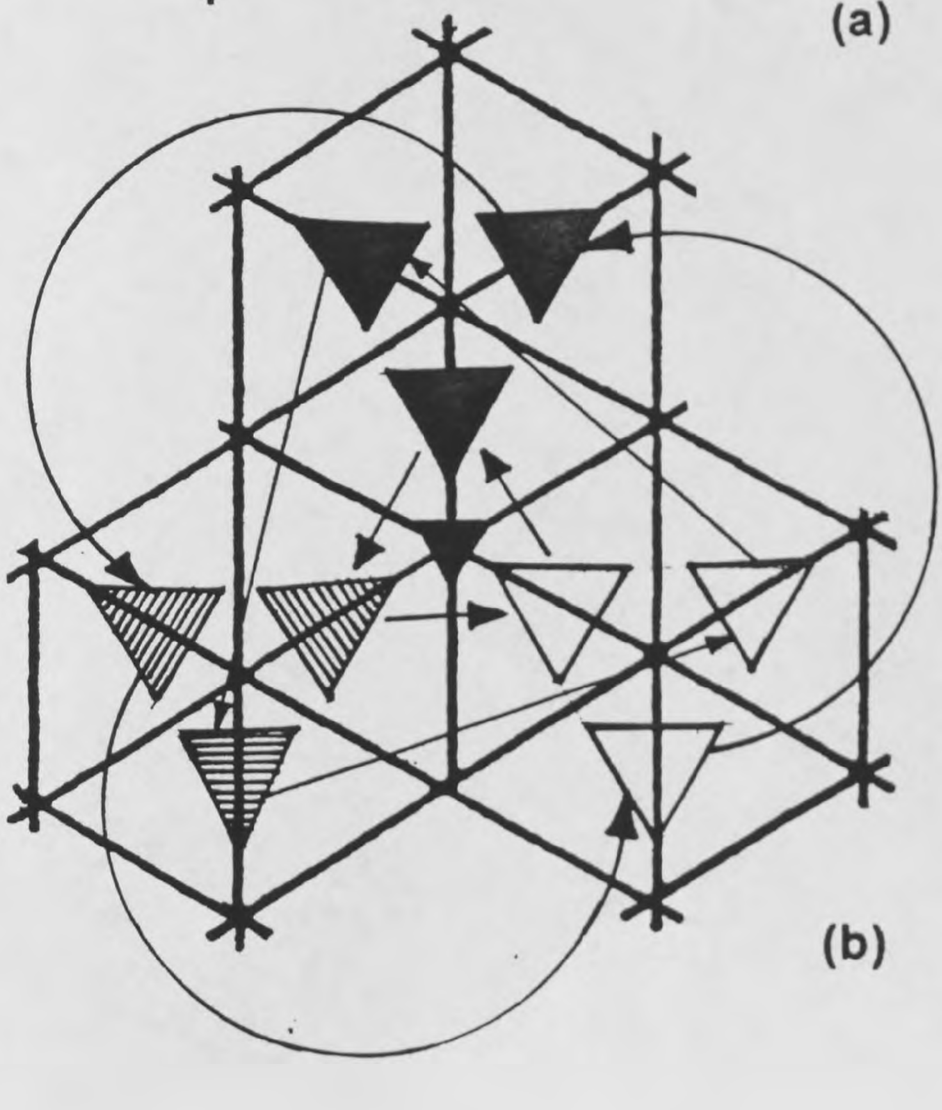
When three or more colours are involved in the colouring, the motifs (in patterns) or tiles (in tilings) are each coloured with a single colour; the letter K is used to refer to the number of colours available. Such a K-colouring is called perfect (or compatible) if each symmetry operation of the uncoloured (primary) pattern or tiling structure induces a permutation of the K colours. Schattschneider explained the characteristics of a perfectly coloured tiling with more than two colours as follows:

> "...... all tiles having the same colour (for instance, red) are transformed by the symmetry of the tiling to tiles having the same colour (for instance, blue). Although a different symmetry may send the red tiles to tiles of a different colour, say green, no symmetry of the uncoloured tiling may 'mix' colours, sending some red tiles to blue and other red tiles to green." [95]

Figure 6.60a shows a three-colour periodic tiling in which colours have been systematically interchanged in a continuous way following the corresponding symmetry operations.

Figure 6.60(a) A three-colour periodic tiling in which colours have been systematically interchanged. (b) An illustration of three-colour counterchange around a point of three-fold rotation.





No single reflection operation transforms one colour to both of the others in any portion of the tiling. That is, each reflection operation leaves one colour invariant (or unchanged) and interchanges the other two. Along vertical reflection axis A, the reflection operation changes black tiles to white tiles (or white tiles to black tiles) whilst leaving the grey unchanged. Along vertical reflection axis B, the reflection operation changes grey tiles to black tiles (or black tiles to grey tiles) and leaves white tiles unchanged. Along diagonal reflection axis C, the reflection operation changes white tiles to grey tiles (or grey tiles to white tiles) whilst leaving the black tiles unchanged. Thus with respect to the reflectional symmetry of this tiling, the colouring is compatible with the symmetry operation. That is, in such a tiling, every reflection axis either moves one colour onto the same colour (i.e. the colour is preserved) or is interchanged with one of the other colours (i.e. the colour is reversed) systematically in a continuous way.

With respect to the rotational symmetry of the tiling shown in Figure 6.60a, it can be seen that colours are also interchanged. By way of further illustration, Figure 6.60b shows that colours are interchanged around a centre of three-fold rotation in such a way that rotational operations transform one colour to each of the others systematically. In the diagram (Figure 6.60b) three-fold rotation (denoted by the small central triangle) transforms all white tiles to black tiles, all black tiles to grey tiles and all grey tiles to white tiles systematically and in a continuous way.

Figures 6.61b, c and d illustrate four-colour counterchange on the primary border tiling of class pma2 (shown in Figure 6.61a, with vertical reflection denoted by lines R and two-fold or half-turn rotation denoted by points H). The tiling is coloured in such a way that colours are systematically changed or unchanged following the corresponding symmetry operation. A prime (') is introduced to denote colour change with reflection (i.e. R') or with half-turn rotation (i.e. H'). Thus in Figure 6.61b all the reflections interchange colours and all the half-turns leave colours unchanged (i.e. R'H). In Figure 6.61c all the half-turns interchange colours and all the reflections leave colours unchanged (i.e. RH'). In Figure 6.61d all the colours are systematically interchanged in a continuous way, following all the reflections as well as all the half-turns (i.e. R'H').

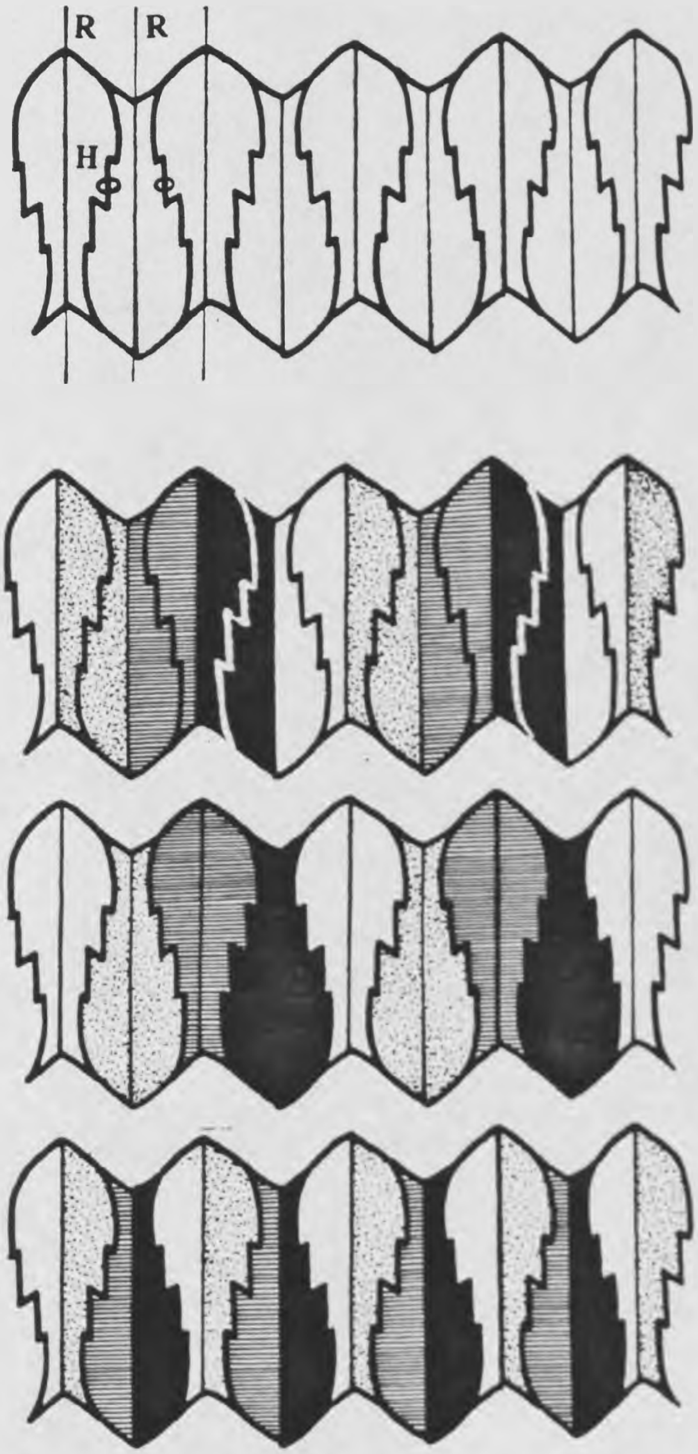
Figure 6.62 illustrates another multi-coloured counterchange possibility (where K=5). A total of three ways of colouring the tiling are possible (RH', R'H and R'H' in Figures 6.62a, b and c respectively).

In the context of all-over tilings, Figure 6.63 shows a four-colour counterchange tiling on a p1 primary structure.

6.6 Summary

By introducing interchange of colour into the seven primary border pattern classes a total of seventeen two-colour, seven three-colour, nineteen four-colour, seven five-colour, seventeen six-colour, seven seven-colour and

Figure 6.61 Four-colour counterchange on a primary border tiling of class pma2. (b), (c) and (d) show colouring possibilities of $R^{+}H$, RH^{+} and $R^{+}H^{+}$ respectively.



(a)

(b)

(c)

(d)

Figure 6.62 Five-colour counterchange on a primary border tiling of class pma2. (a), (b) and (c) show colouring possibilities of RH', R'H and R'H' respectively.

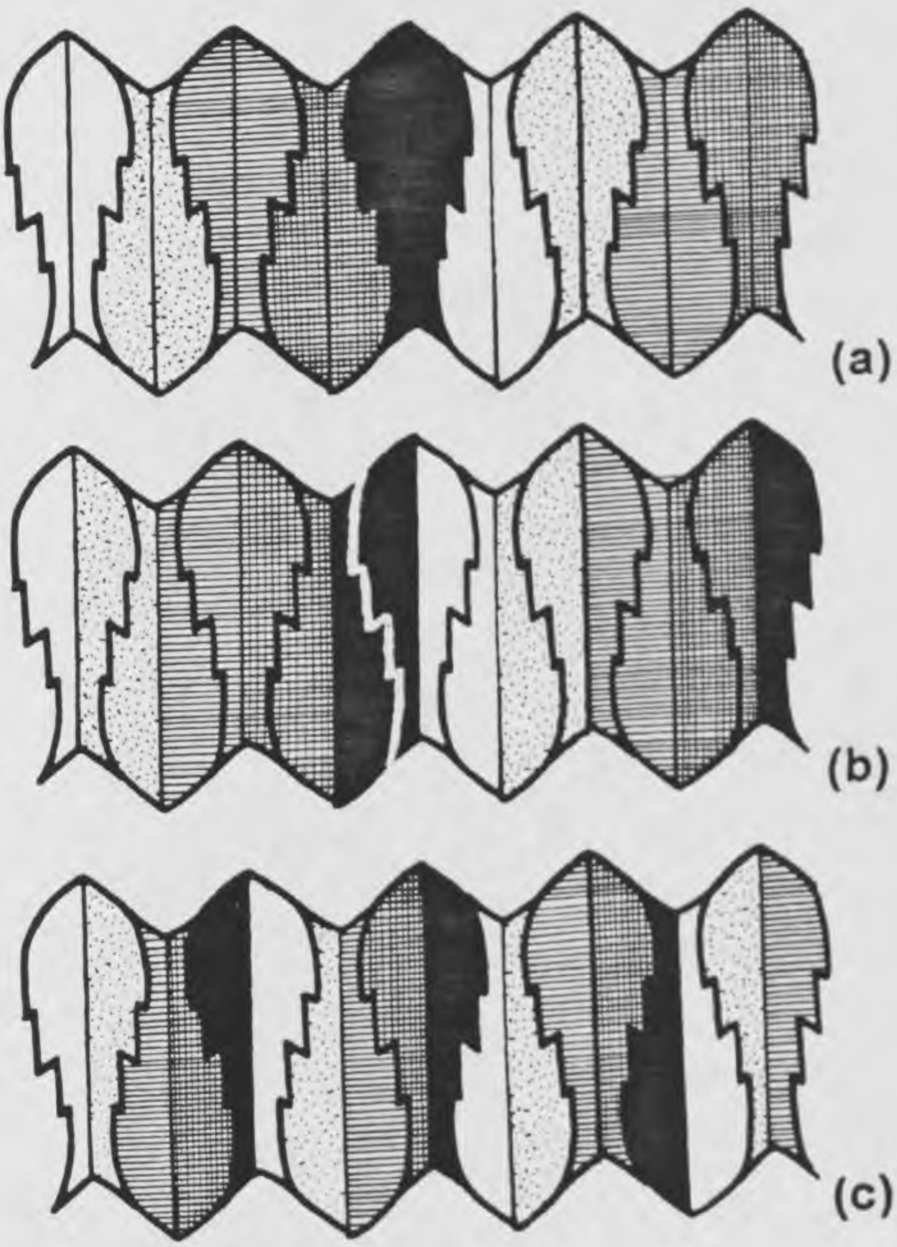
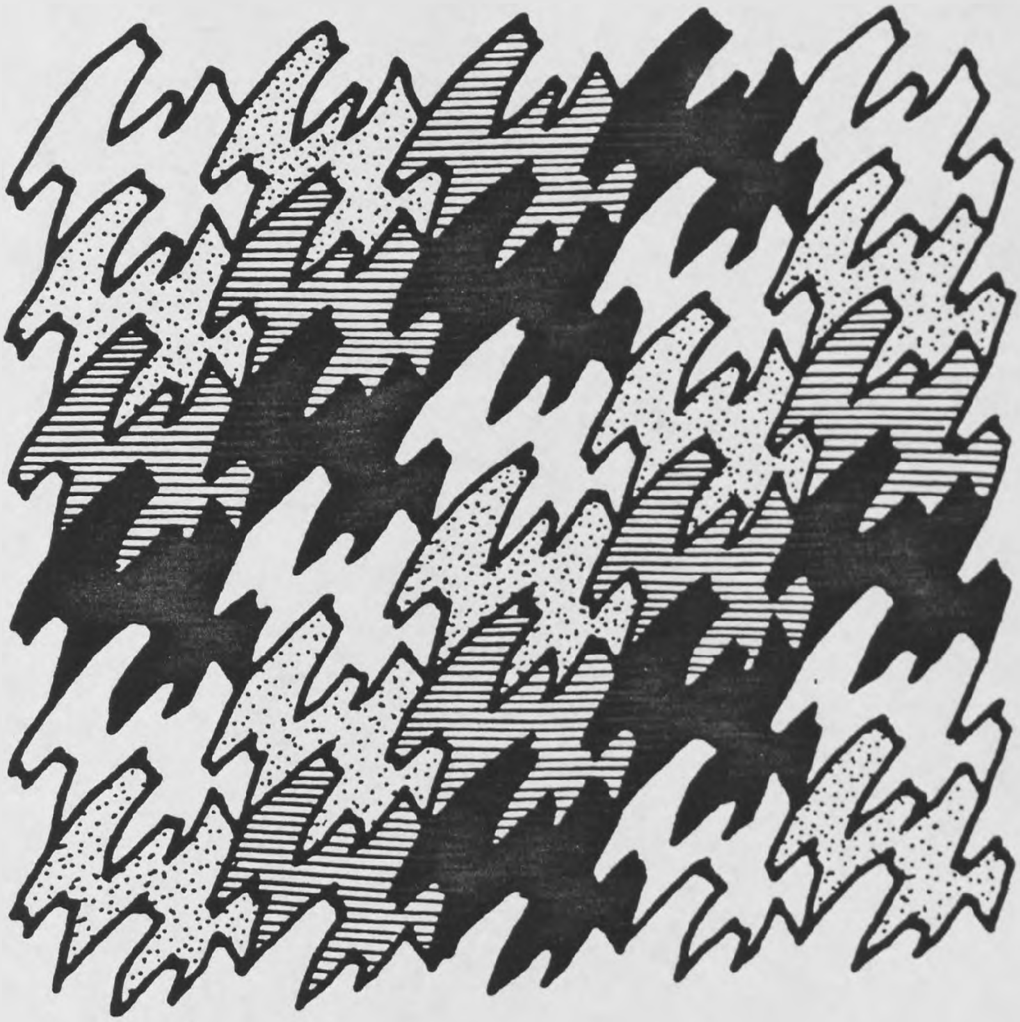


Figure 6.63 An example of a four-colour counterchange tiling on a p1 primary structure.



nineteen eight-colour counterchange possibilities unfold. A total of forty-six classes of two-colour counterchange, twenty-three classes of three-colour counterchange and ninety-six classes of four-colour counterchange all-over patterns or tilings are possible. By way of summary, Tables 6.1 and 6.2 enumerate some of colour possibilities for border patterns and all-over patterns. Examples of all forty-six classes of two-colour counterchange all-over patterns are provided in Figure 6.64. Examples for each of the twenty-three three-colour counterchange all-over patterns are provided in Figure 6.65.

Table 6.1 The enumeration of colour symmetry classes for border patterns and tilings (for K = 1 to 8).

Number of colours (denoted by K)	K=1	K=2	K=3	K=4	K=5	K=6	K=7	K=8
Class p111	1	1	1	1	1	1	1	1
Class p1a1	1	1	1	1	1	1	1	1
Class pm11	1	2	1	2	1	2	1	2
Class p1m1	1	3	1	3	1	3	1	3
Class p112	1	2	1	2	1	2	1	2
Class pma2	1	3	1	3	1	3	1	3
Class pmm2	1	5	1	7	1	5	1	7
Total number of colour classes	7	17	7	19	7	17	7	19

Table 6.2 The enumeration of colour symmetry classes for all-over patterns (for K = 1 to 4).

Number of colour (denoted by K)		K=1	K=2	K=3	K=4
Class p1	all-over pattern	1	1	1	2
Class p1g1	all-over pattern	1	2	2	4
Class p1m1	all-over pattern	1	5	2	10
Class c1m1	all-over pattern	1	3	2	7
Class p211	all-over pattern	1	2	1	3
Class pgg	all-over pattern	1	2	1	4
Class p2mg	all-over pattern	1	5	2	11
Class p2mm	all-over pattern	1	5	1	13
Class c2mm	all-over pattern	1	5	1	11
Class p3	all-over pattern	1	0	2	1
Class p3m1	all-over pattern	1	1	2	1
Class p31m	all-over pattern	1	1	2	1
Class p4	all-over pattern	1	2	0	5
Class p4gm	all-over pattern	1	3	0	7
Class p4mm	all-over pattern	1	5	0	13
Class p6	all-over pattern	1	1	2	1
Class p6mm	all-over pattern	1	3	2	2
Total number of colour classess		17	46	23	96

Figure 6.64 Examples of all forty-six classes of two-colour counterchange all-over patterns.

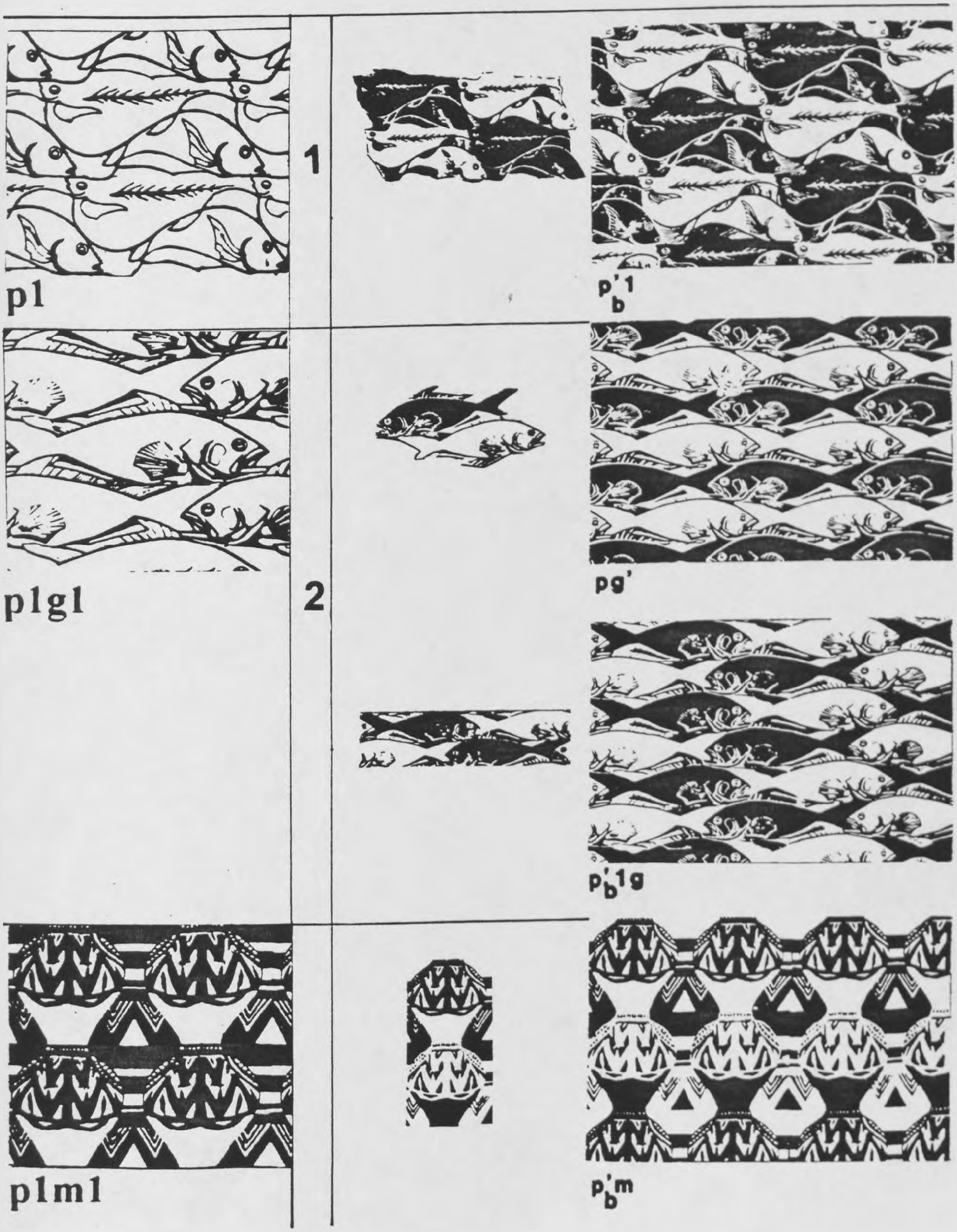
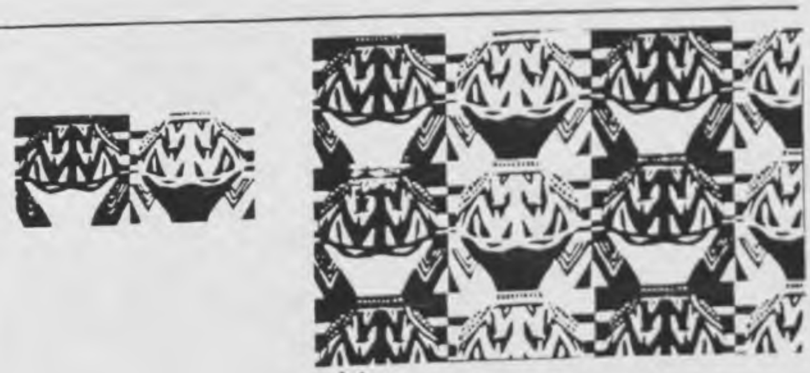


Figure 6.64 Examples of all forty-six classes of two-colour counterchange all-over patterns (continued).

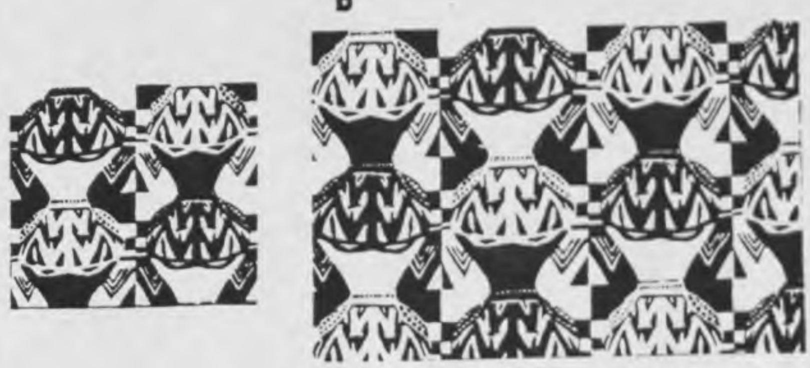
5



p'_b1m

pm'

p'_bg



$c'm$

Figure 6.64 Examples of all forty-six classes of two-colour counterchange all-over patterns (continued).

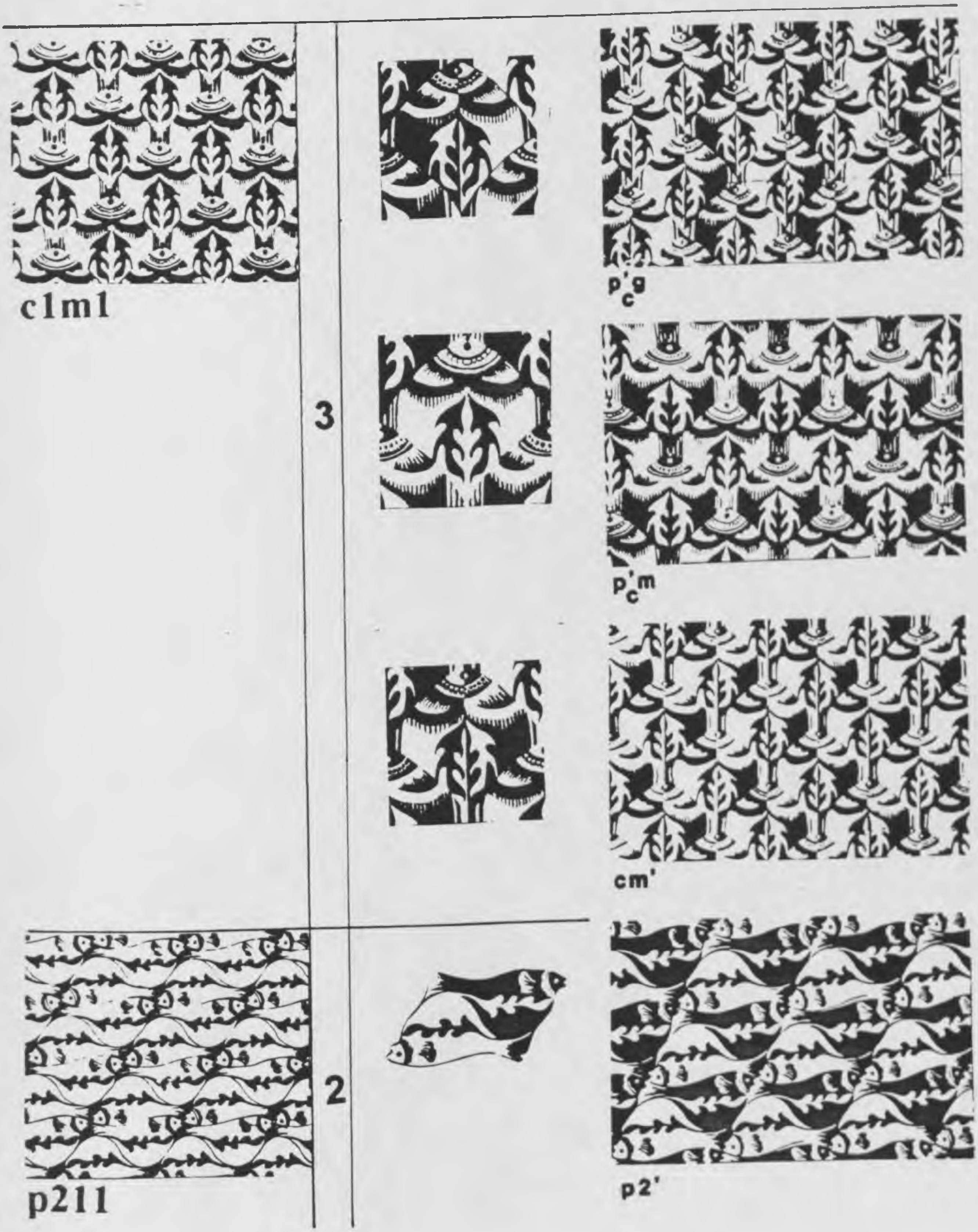


Figure 6.64 Examples of all forty-six classes of two-colour counterchange all-over patterns (continued).

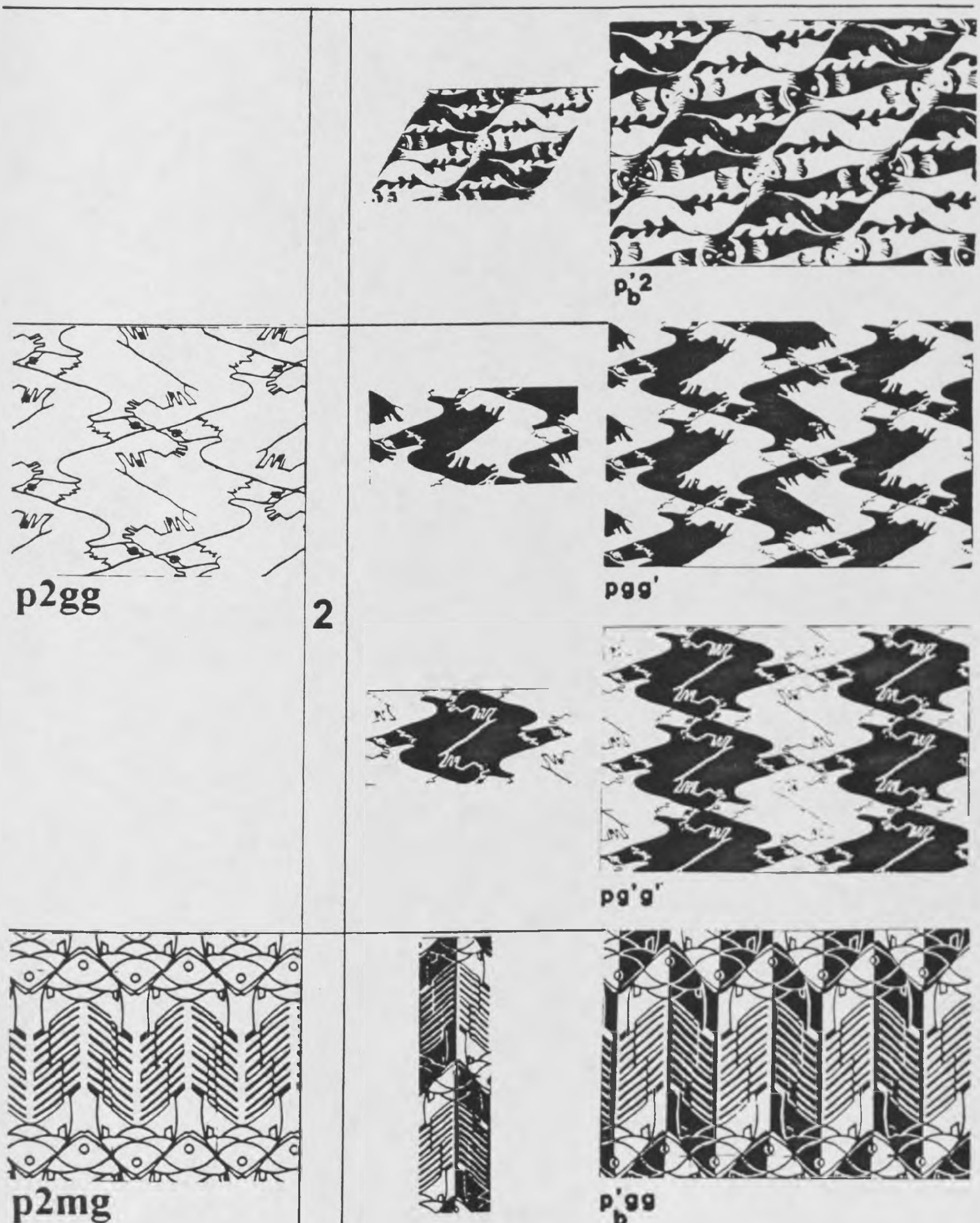


Figure 6.64 Examples of all forty-six classes of two-colour counterchange all-over patterns (continued).

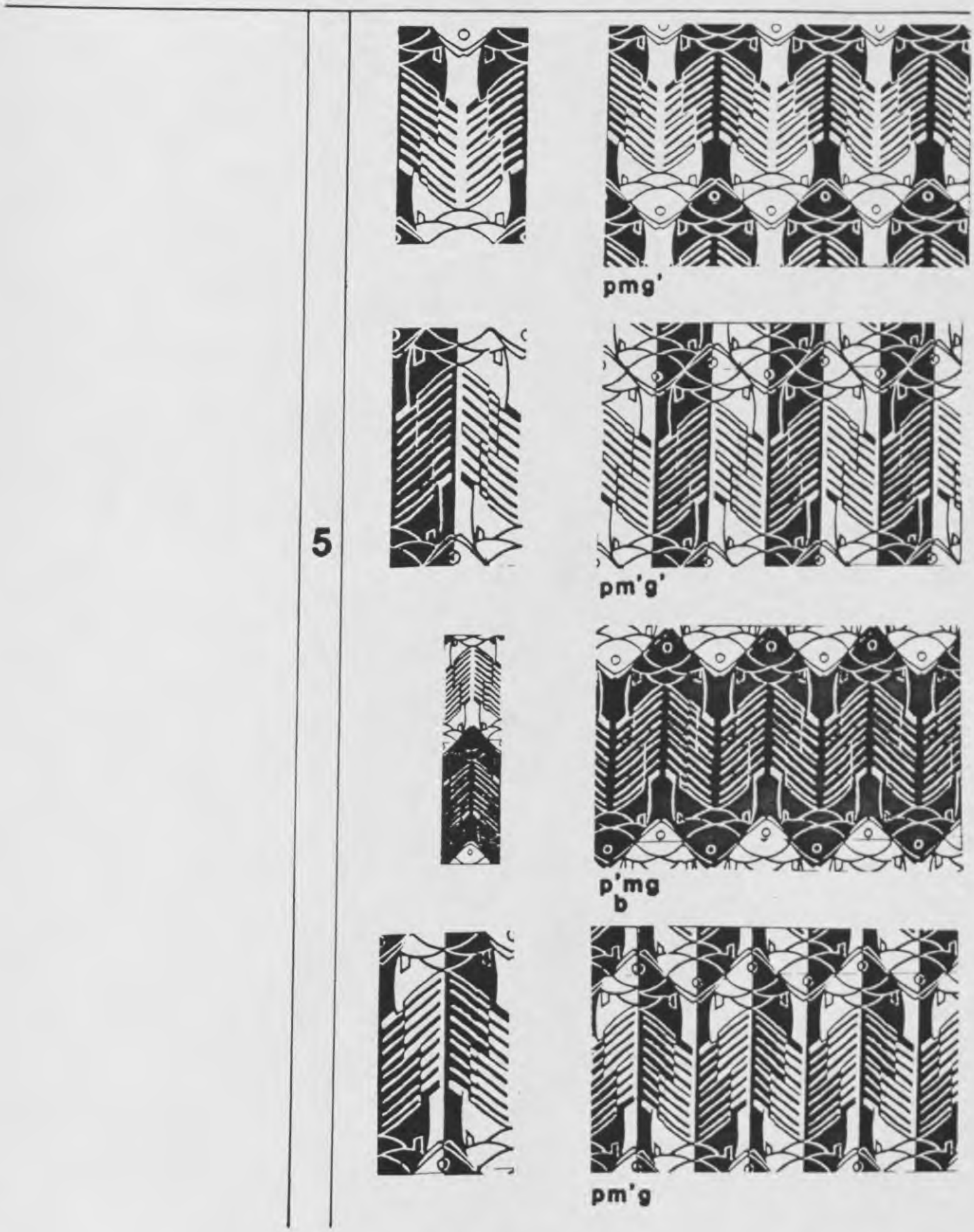
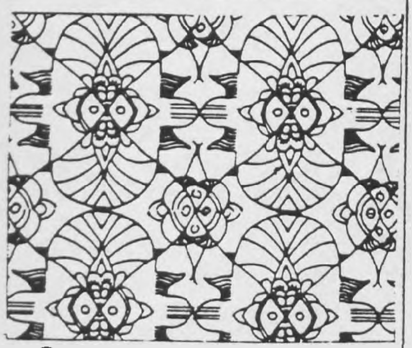
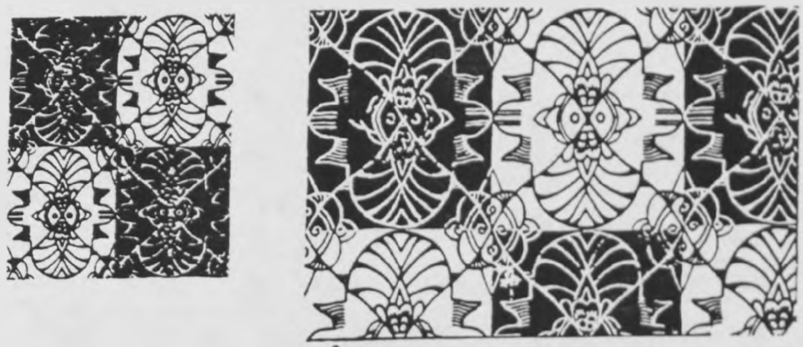


Figure 6.64 Examples of all forty-six classes of two-colour counterchange all-over patterns (continued).

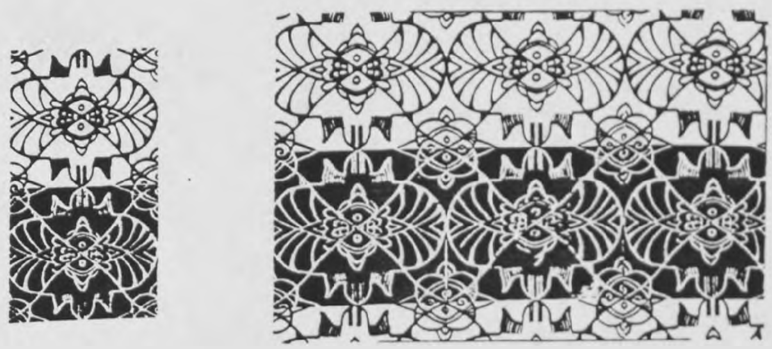


p2mm

5



c'mm

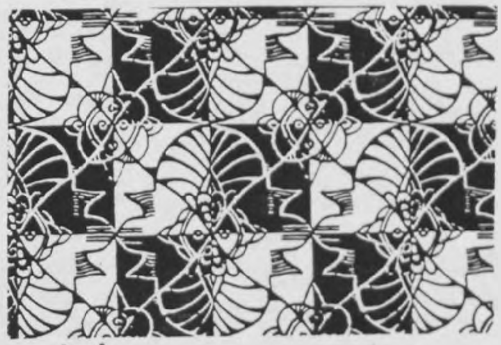


$p'_b mm$

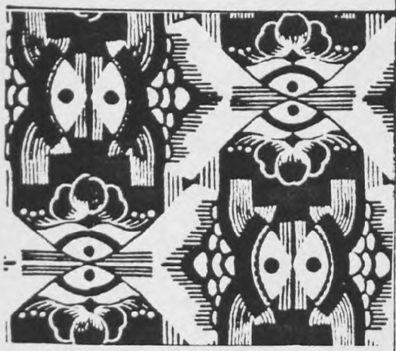
pmm'

$p'_b gm$

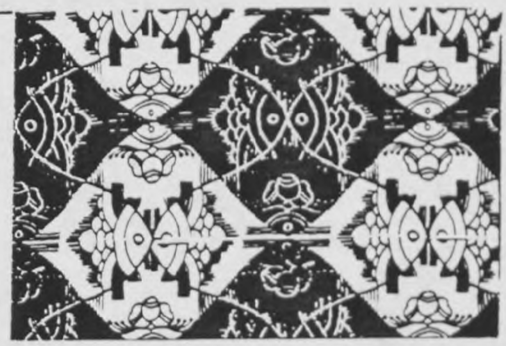
Figure 6.64 Examples of all forty-six classes of two-colour counterchange all-over patterns (continued).



pm'm'

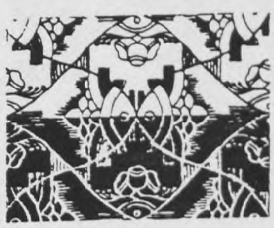


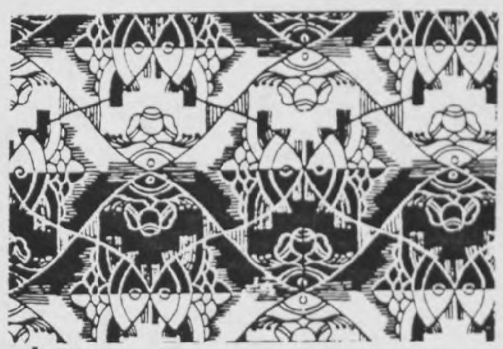
c2mm



$p'_c mm$

5

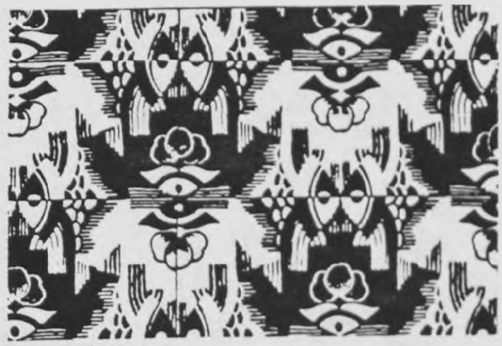




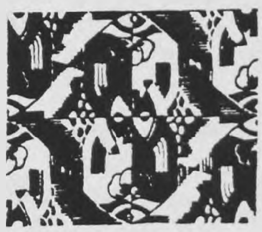
$p'_c mg$

$p'_c gg$

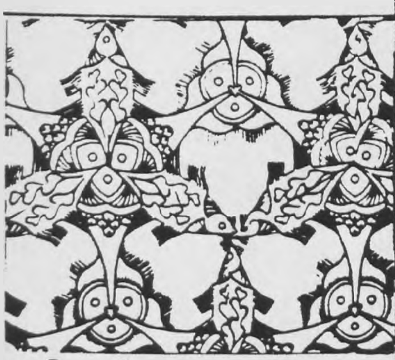
Figure 6.64 Examples of all forty-six classes of two-colour counterchange all-over patterns (continued).



cmm'

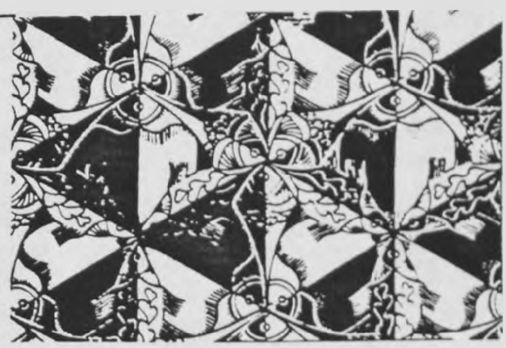


cm'm'



p3m1

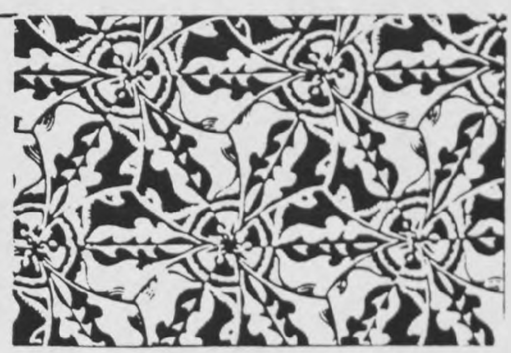
1



p3m'

p31m

1



p31m'

Figure 6.64 Examples of all forty-six classes of two-colour counterchange all-over patterns (continued).

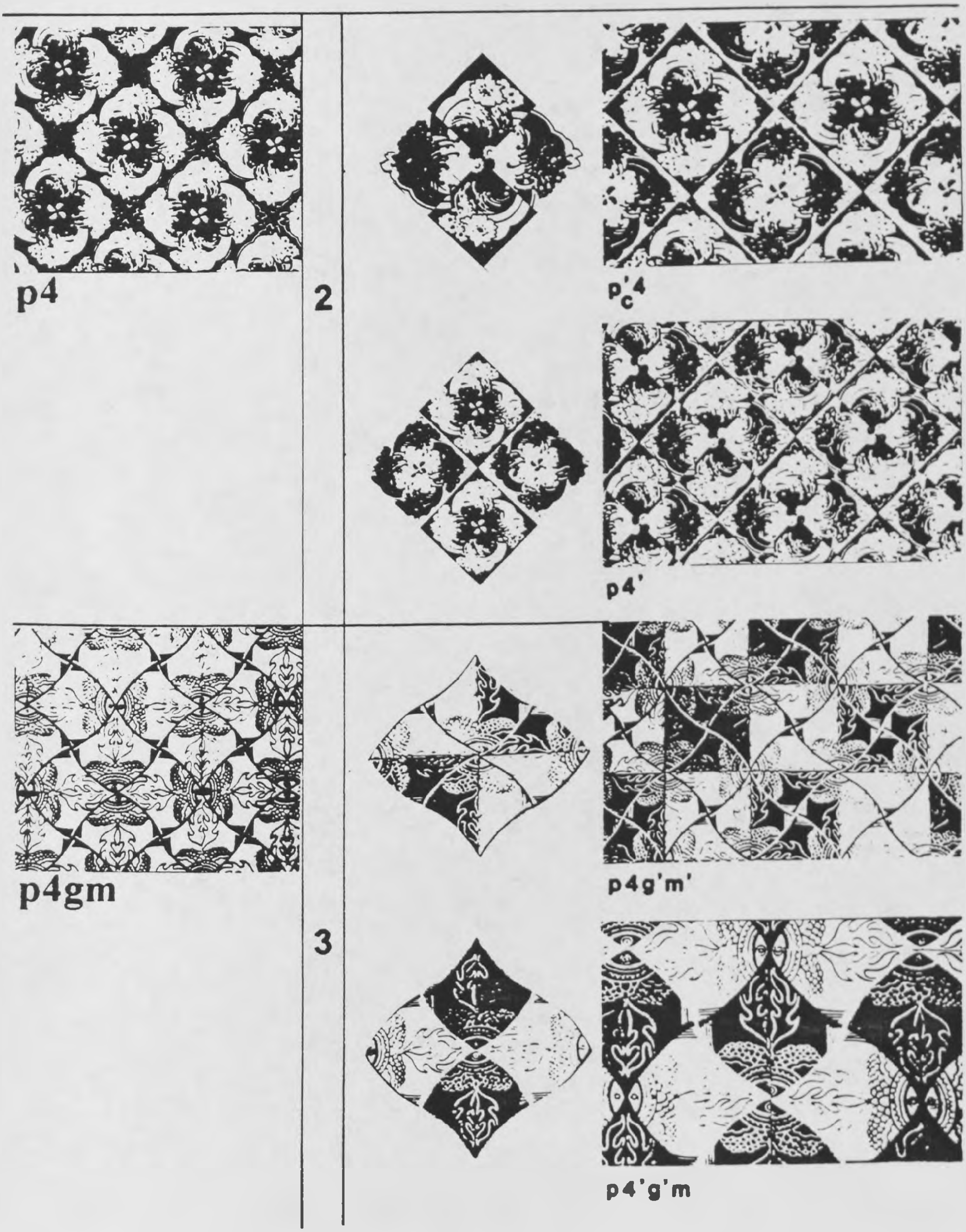


Figure 6.64 Examples of all forty-six classes of two-colour counterchange all-over patterns (continued).

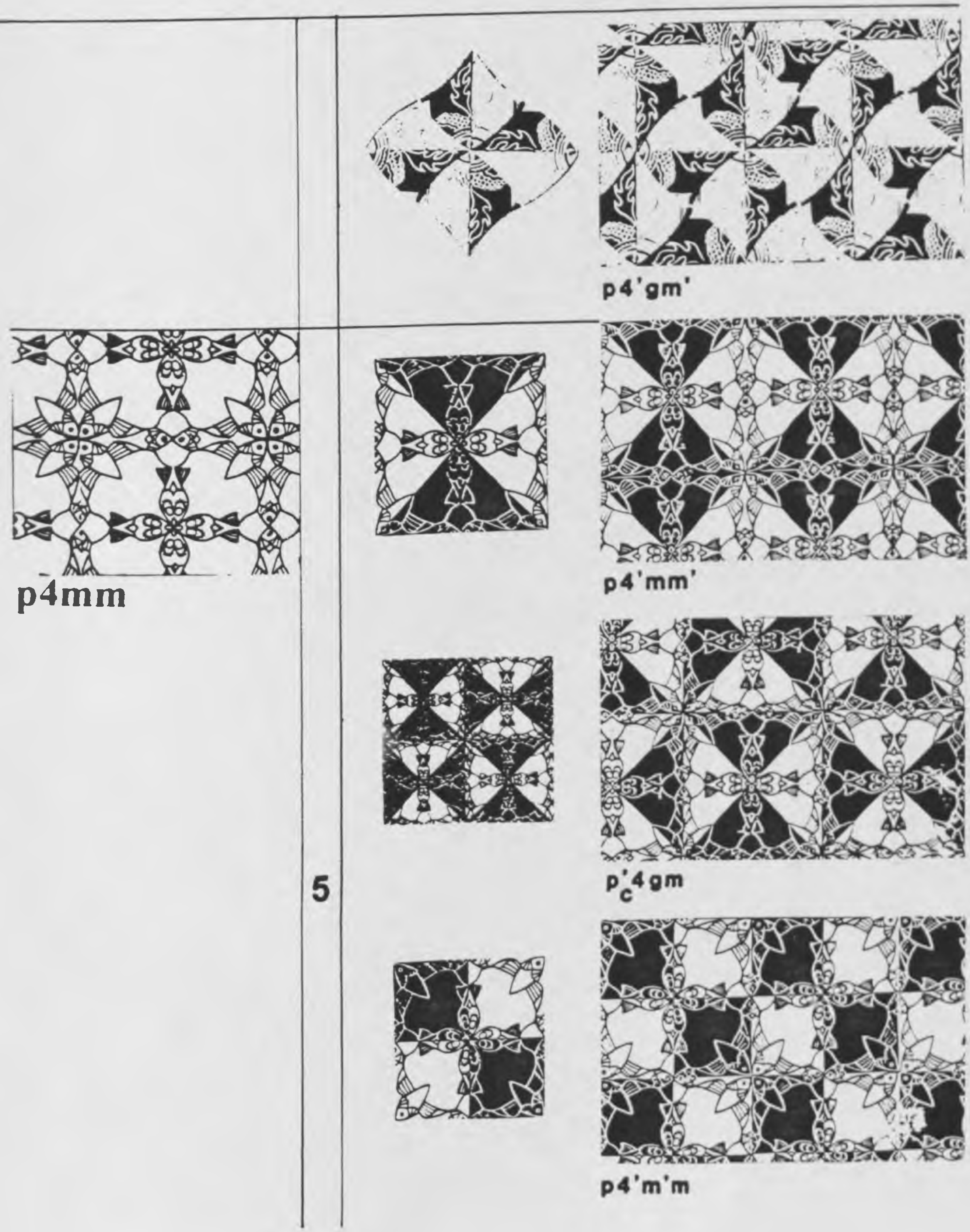


Figure 6.64 Examples of all forty-six classes of two-colour counterchange all-over patterns (continued).

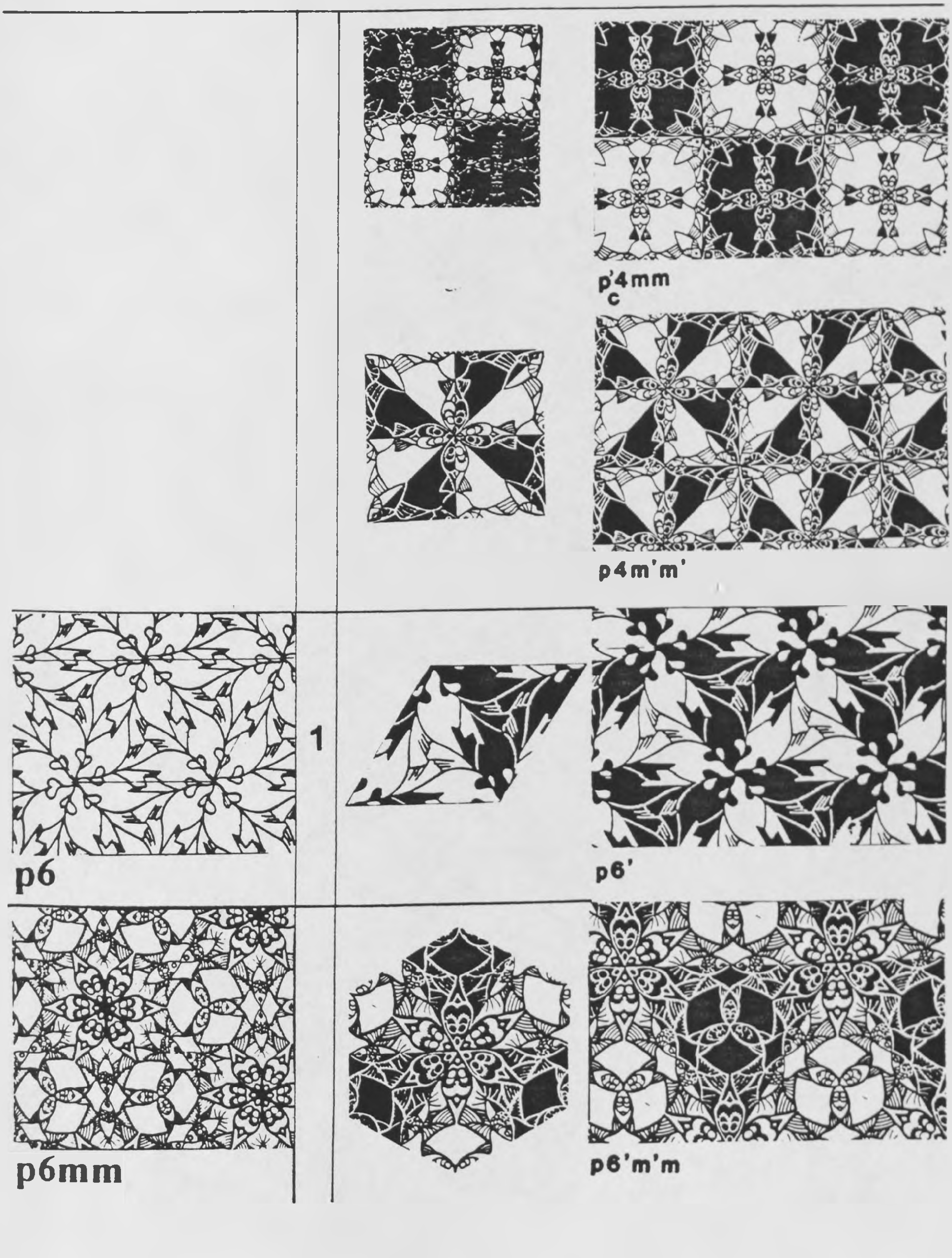


Figure 6.64 Examples of all forty-six classes of two-colour counterchange all-over patterns (continued).

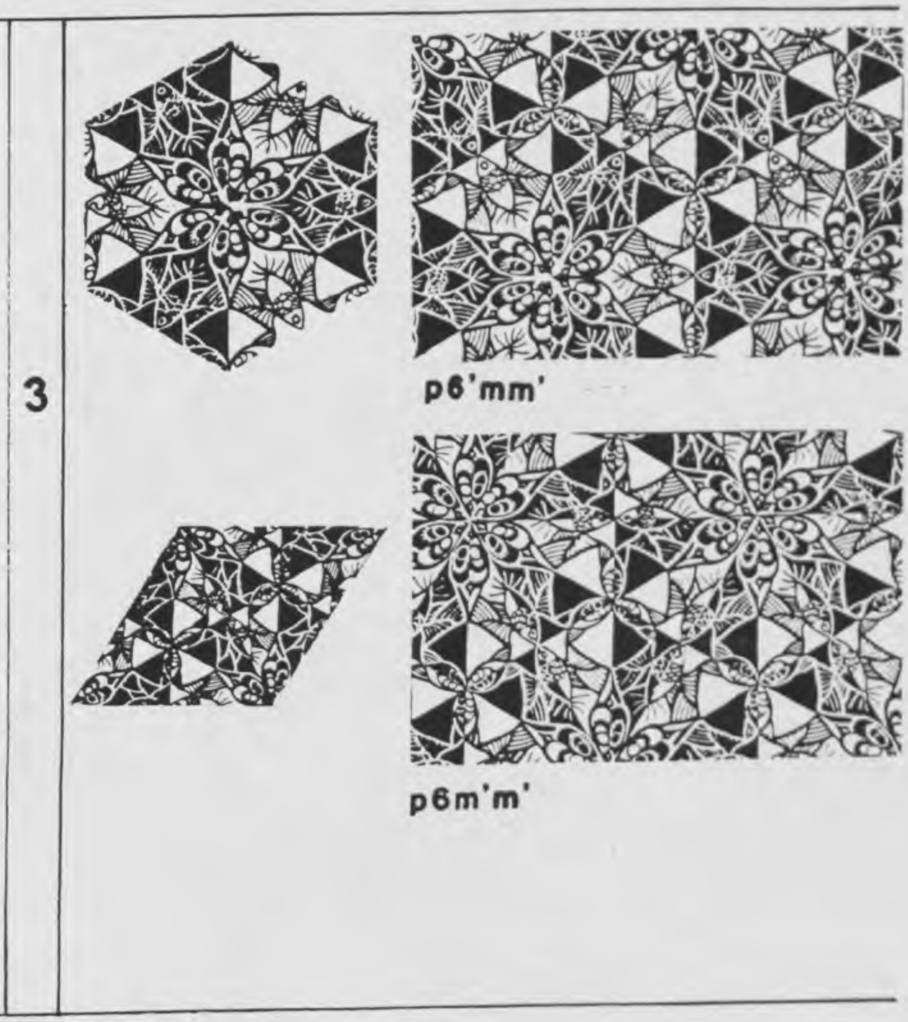


Figure 6.65 Examples of all twenty-three classes of three-colour counterchange all-over patterns.

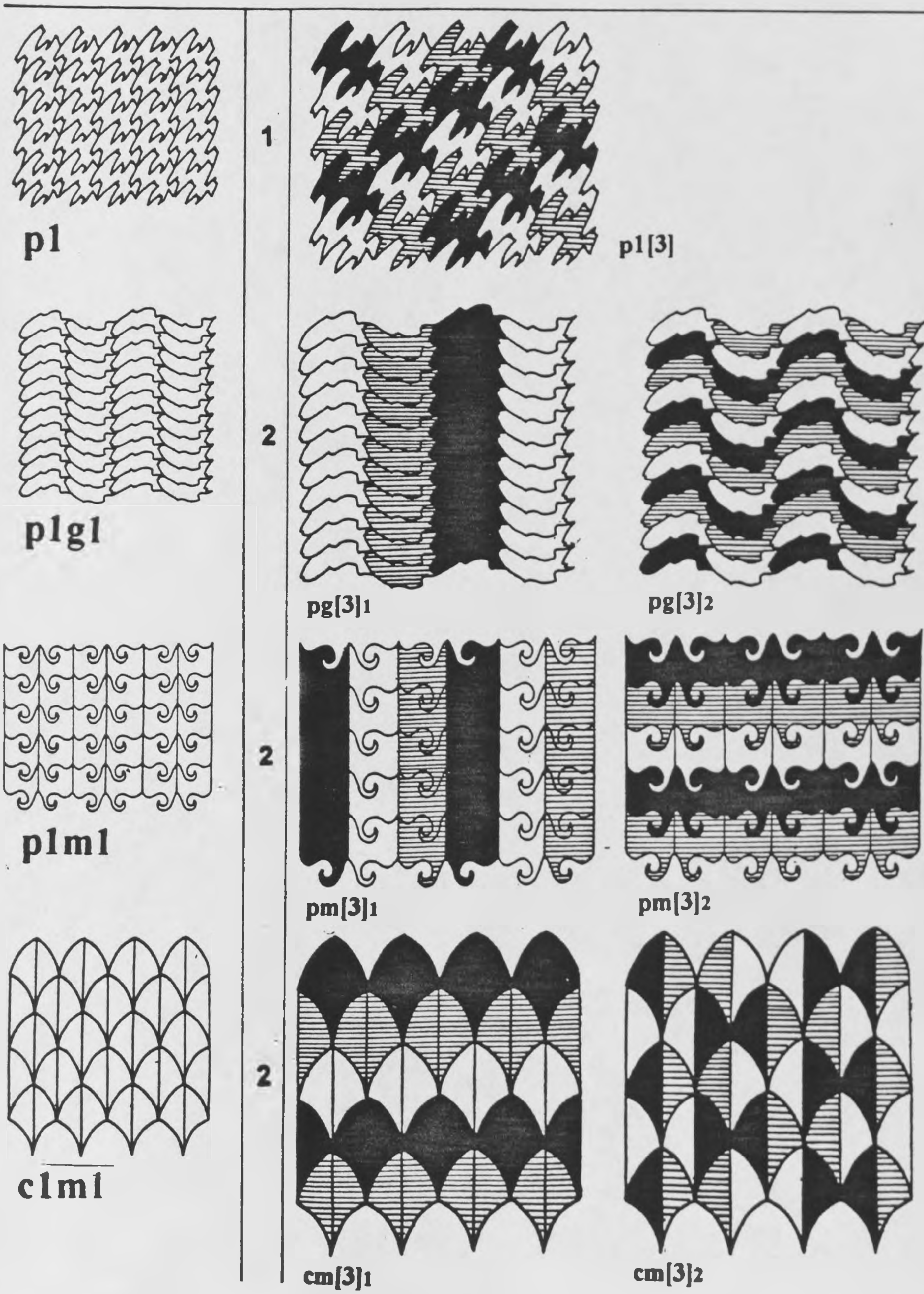


Figure 6.65 Examples of all twenty-three classes of three-colour counterchange all-over patterns (continued).

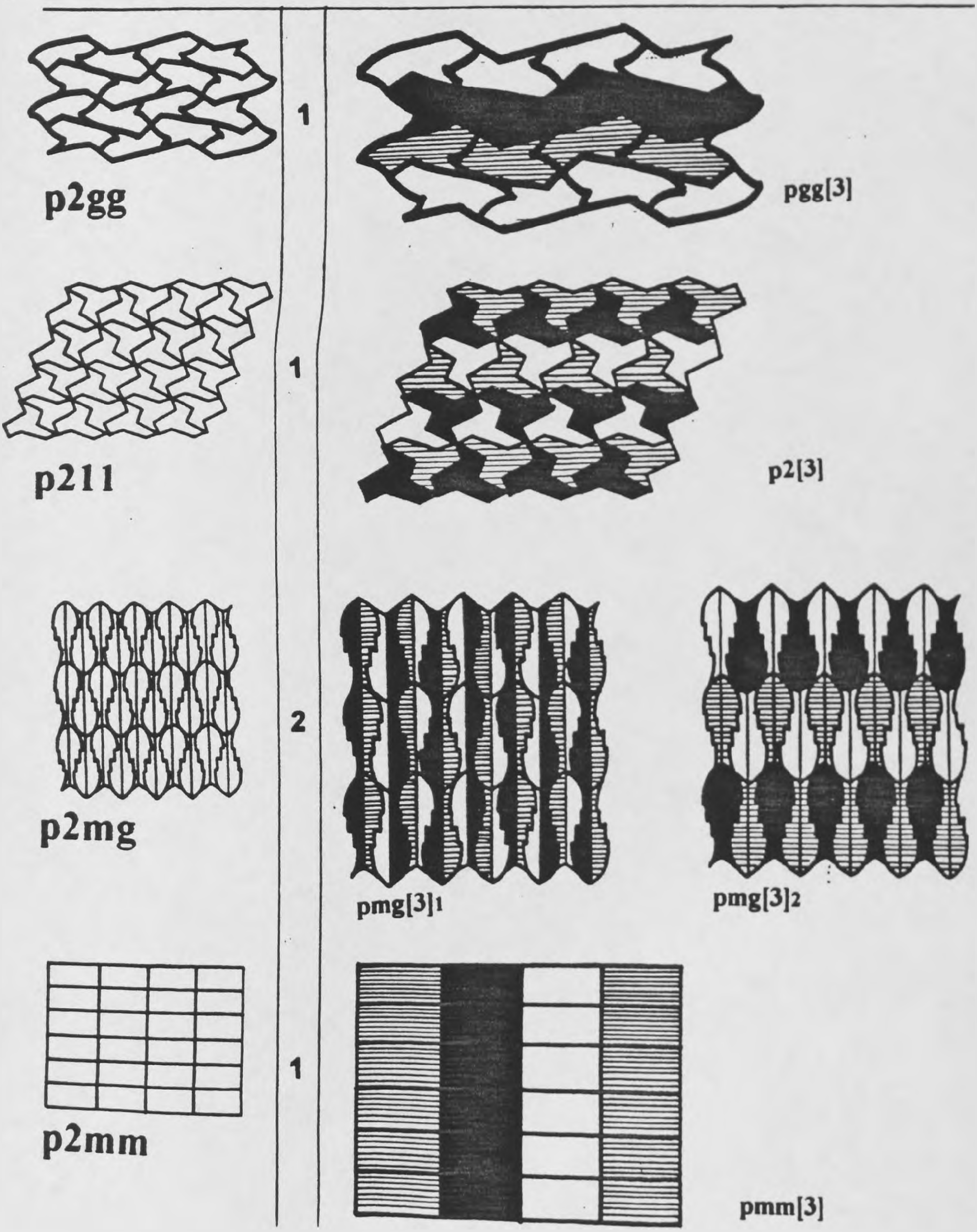


Figure 6.65 Examples of all twenty-three classes of three-colour counterchange all-over patterns (continued).

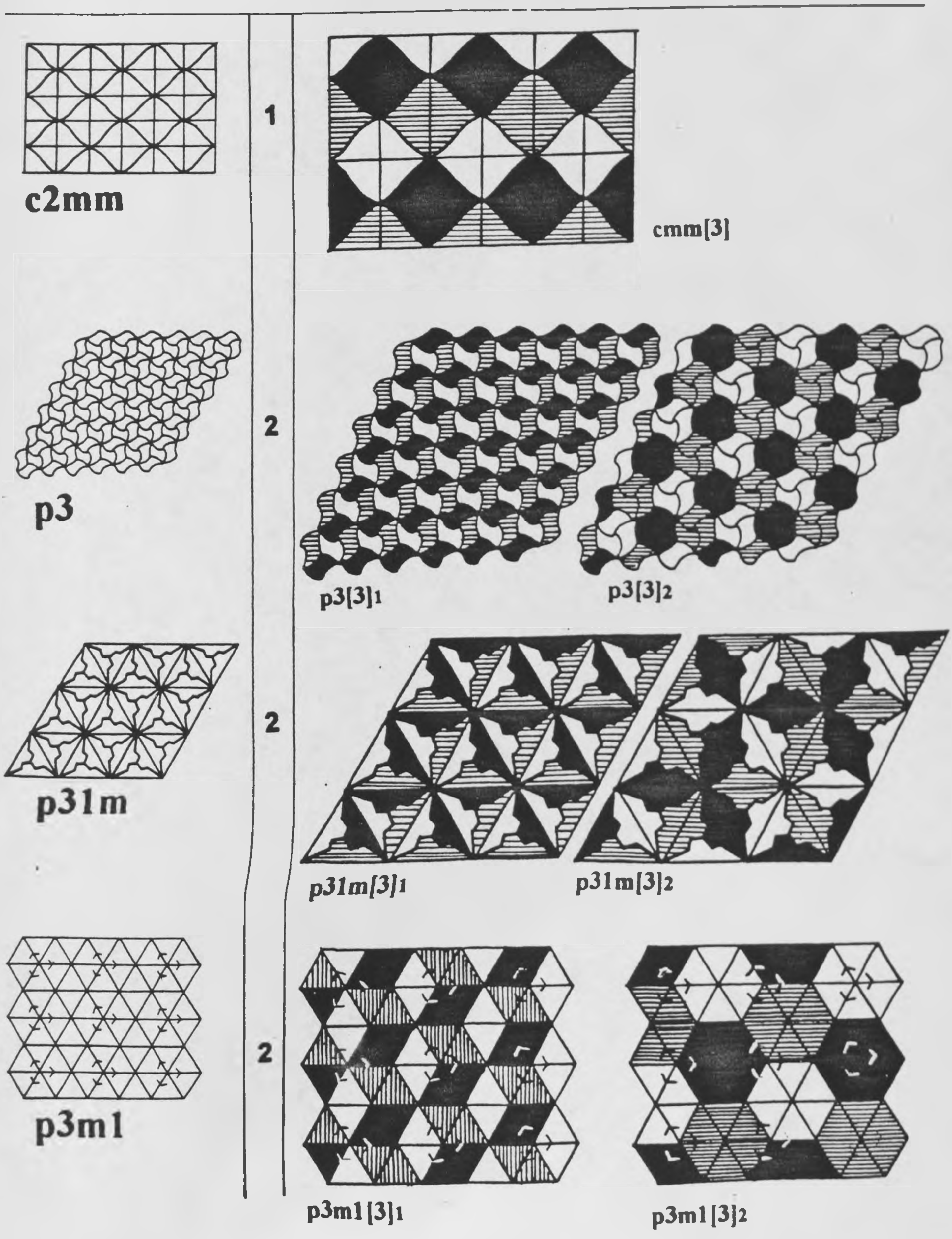
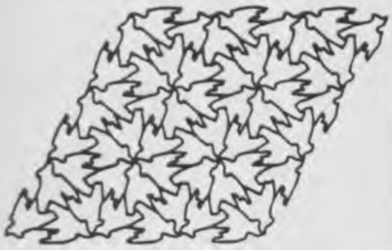
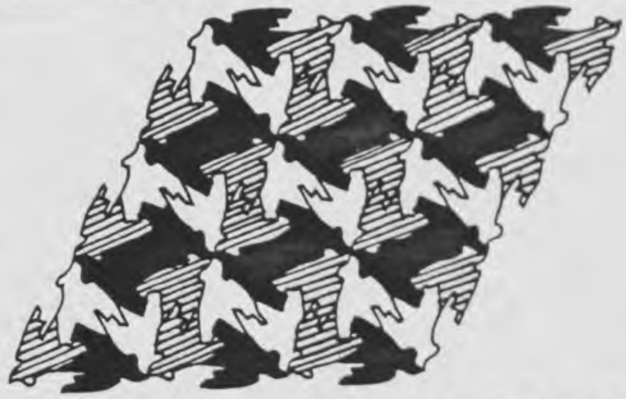


Figure 6.65 Examples of all twenty-three classes of three-colour counterchange all-over patterns (continued).

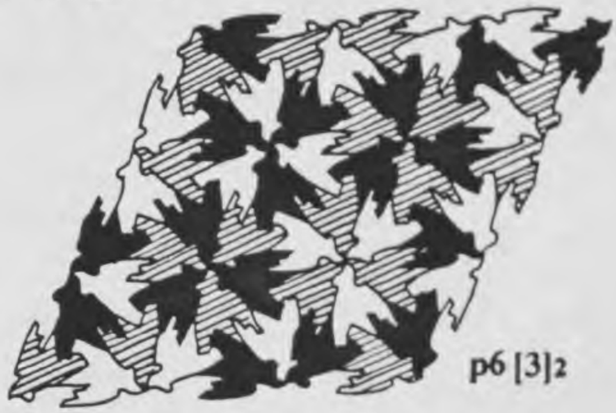


p6

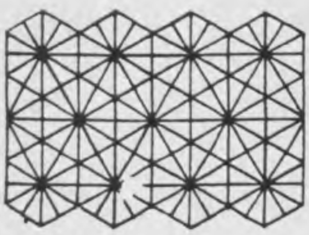
2



p6 [3]1

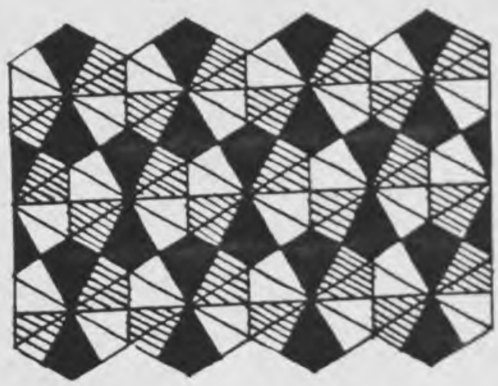


p6 [3]2

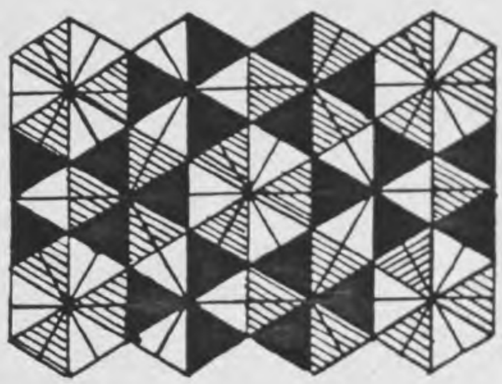


p6mm

2



p6m[3]1



p6m[3]2

7 IN CONCLUSION

This thesis has focused primarily on the presentation of a mathematically based classification system for regular repeating patterns and periodic tilings. By way of original contribution, an emphasis has been placed, where appropriate, on pattern/tiling construction. Principles relating to colour counterchange in patterns and tilings have been introduced and applied to the classification and construction of an extensive range of original designs (a selection of which is presented in printed fabric form in the accompanying exhibition). Explanations of concepts and procedures of analysis have played a major part in the thesis. The thesis has emphasised the benefit firstly, of understanding in geometrical terms how a pattern "works" and secondly, how to create original designs using the power of mathematics (a strong non-mathematical motive for acquiring an understanding of the basic theory).

Although this thesis does not present illustrative examples for each of the four-colour and higher-colour counterchange patterns/tilings, the description of the technique to create a perfect two-colour or three-colour pattern/tiling should make this a relatively easy exercise.

This thesis, as it stands, would appear to be suitable for use as a framework for the teaching of the principles of pattern construction to undergraduate design students. There appears to be a lack of awareness of the potential benefits of an understanding of these principles within the context of computer aided design; this is an obvious area for further development and application.

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