Connections between discriminants of complex reflection groups and their representation theory.

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Submitted in accordance with the requirements for the degree of Doctor of Philosophy

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November 21, 2022

Acknowledgements

I wish to thank everyone who has made this project possible. My supervisor, Eleonore Faber for continuous help and patience, directing me both mathematically and professionally. Bethany Marsh who has provided constant support and encouragement during my time at the University of Leeds. Colin Ingalls and Marco Talarico who have given invaluable advice and collaboration. All peers who have aided mathematically and friends who have aided socially. My parents who have given me unwavering support. My partner for always being there for me when I needed it the most. Lastly the EPSRC for funding the project via the Doctoral Training Partnership (reference EP/R513258/1).

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Chapter 4 contains material from a collaboration between Eleonore Faber, Colin Ingalls and Marco Talarico. This work is still in preparation and will be submitted to a journal at a later date. The working title is "MATRIX FACTORIZATIONS OF THE DISCRIMINANT OF S_n". My contribution to the work involved writing the proofs of the statements and acting as a main author. Marco Talarico's contribution was writing code for the examples and contributing towards the proofs. Eleonore Faber and Colin Ingalls contributed towards exposition and editing of the work.

Abstract

The following thesis explores an extension to the *classical McKay correspondence*, a theorem that touches on several areas of mathematics.

Our extension comes by considering pseudo reflection groups which were not included in the original correspondence. The *discriminant* of a pseudo reflection group is a singular hypersurface expressed by a polynomial Δ in the invariant ring of the group action. A main object of study is the *matrix factorization* (z,j) of Δ, and the corresponding *Cohen-Macaulay module*, arising from the arrangement of the hyperplanes fixed by the reflections. A key idea that we frequently use is that the matrix factorization (z,j) can be decomposed using the irreducible representations of the group.

A McKay correspondence of refection groups generated by reflections of order 2 has been presented by Buchweitz–Faber–Ingalls, [BFI20]. In Chapter 3 we follow the methods from Loc. cit and consider the complex reflection groups $G(m,p,2)$, which appear in the Shephard–Todd classification and show that similar results hold. The matrix factorization is fully decomposed and the corresponding decomposition of the Cohen-Macaulay module is given.

A collaboration with Eleonore Faber, Colin Ingalls and Marco Talarico has resulted in a description of the decomposition of the matrix factorization (z,j) of Δ for the symmetric group S_n on n letters. In Chapter 4 a modification of *higher Specht polynomials* is used to present a computational way to explicitly calculate the decomposition of the corresponding Cohen-Macaulay module of (z,j).

In Chapter 5, the Lusztig algebra for the pseudo reflection group $G(m,1,2)$ which is Morita equivalent algebra to the skew group ring is calculated. This is achieved by using the McKay quivers for $G(m,1,2)$ and calculating the required relations in terms of 2-paths. The Lusztig algebra can give us insights into how the McKay correspondence can generalise to pseudo reflection groups.

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Introduction

Background

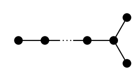
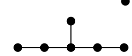
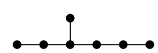
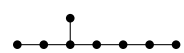
The work carried out in this thesis explores a generalised *classical McKay correspondence* which lies in the intersection between multiple areas of mathematics. On one side we have singularity theory, which aims to understand points of geometric objects where they fail to be manifolds, and on the other side representation theory of finite groups, which describes groups in terms of their actions on a vector space. A wonderful account of the history of the McKay correspondence is given by Buchweitz in [Buc12].

For a primary motivation of the McKay correspondence, we look to the work of Felix Klein who introduced *Kleinian* (sometimes referred to as *Du Val*) singularities as quotient varieties constructed from subgroups of $\mathrm{SL}(2,\mathbb{C})$ which act on the complex plane [Kle93]. Interest in this topic goes back to the Platonic solids embedded in $\mathbb{R}^3$ and their rotational symmetry groups. Finite symmetry groups of figures on a plane, $\mathbb{R}^2$ can only amount to cyclic or dihedral groups, which can then be realised as rotations in $\mathbb{R}^3$. The remaining rotation groups in $\mathbb{R}^3$ are obtained from the rotations of the Platonic solids, that is A_4 coming from the tetrahedron, S_4 from the cube (the hexahedron and the octahedron) and A_5 from the icosahedron. To study these groups, Klein lifts them to $\mathrm{SL}(2,\mathbb{C})$ with constructions using quaternions, see [MBD61] for an account.

Any finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$ that acts on $\mathbb{C}^2$, after choosing a basis, $\{x,y\}$, also naturally acts on $S := \mathbb{C}[x,y]$, see Section 1.4 for details. From such a group G one can construct the *quotient variety* $\mathbb{C}^2/G$, which is isomorphic to a complex surface in $\mathbb{C}^3$

where the ring of regular functions on $\mathbb{C}^2/G$ is given by the invariant ring $R := \mathbb{C}[x,y]^G$. It was shown by Klein that the surface $\mathbb{C}^2/G$ has an isolated singularity at the origin. *Minimal resolutions*, that is a corresponding smooth surface for these surfaces were presented by Patrick Du Val in 1934 [DV34a, DV34b, DV34c], who described their *dual resolution graphs*, which holds data about *exceptional* curves and their intersections on the minimal resolution, see Definition 1.5.9 for more detail. The dual resolution graphs correspond to the Dynkin diagrams of type A_n, D_n, E_6, E_7 and E_8, see Table 1.

Table 1: Dynkin diagram type and their corresponding diagrams

Dynkin diagram type	Diagram
A_n for $n \geq 1$	
D_n for $n > 3$	
E_6	
E_7	
E_8	

The correspondence between subgroups of SL(2, $\mathbb{C}$) and Dynkin digrams is given Table 2.

Table 2: Dynkin diagrams and their corresponding subgroups of SL(2, $\mathbb{C}$).

Dynkin diagram type	Group name	Group symbol
A_n	Cyclic group of order $n+1$	$\mathbb{Z}_{n+1}$
D_n	Binary dihedral group of order $4(n-2)$	$\mathrm{Dic}_{4(n-2)}$
E_6	Binary tetrahedral group	$\mathrm{SL}(2, \mathbb{F}_3)$
E_7	Binary octahedral group	$2S_4$
E_8	Binary icosahedral group	$\mathrm{SL}(2, \mathbb{F}_5)$

Slodowy presented arguments on how the Lie algebras of type A_n, D_n, E_6, E_7, E_8 fit into the correspondence between the singularities and their Dynkin diagrams [Slo83]. It was John McKay in 1979 who described the correspondence directly through representation theory of the groups, which is commonly known as the *classical McKay correspondence* [McK80]. The realisation of a subgroup $G \subset \mathrm{SL}(2,\mathbb{C})$ defines a representation called the *standard representation*. McKay constructed the *McKay quiver* using only the data of the embedding of the group and it's irreducible representations.

Definition. Let G be a finite group acting on V a $\mathbb{C}$ vector space of dimension n. Let $\{V_1,\ldots,V_r\}$ be a list of all non-isomorphic irreducible representations of G. The *McKay quiver* $\Xi(G)$ of G is defined as follows: the vertices of $\Xi(G)$ are the irreducible representations V_i of G and there are m_{ij} arrows from V_i to V_j if V_j appears with multiplicity m_{ij} in $V_i \otimes V$.

McKay observed that, for a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$ if one deletes the vertex corresponding to the trivial representation, or in fact any 1 dimensional irreducible representation, and collapses all 2 cycles to edges, the resulting undirected graph is the corresponding Dynkin diagram.

An algebraic approach can be used to describe the McKay correspondence in higher dimensions. Consider a finite subgroup $G \subset \mathrm{GL}(n,\mathbb{C})$, for any $n \geq 2$, acting on $S := \mathbb{C}[x_1,\ldots x_n]$ with invariant ring $R = S^G$. An algebraic extension of the McKay correspondence was explained by Auslander using the *skew group ring* (or twisted group ring) $S * G$ [Aus86], see Section 1.8 for details. One can view S as a (left) R module, with the action of left multiplication, which has the valuable *Cohen Macaulay* (*CM*) property, see Definition 1.1.1 for details. The algebraic McKay correspondence relates R direct summands of S with projective $S * G$ modules and irreducible representations of G.

A finite subgroup of $\mathrm{GL}(n,\mathbb{C})$ is called *small* if it contains no reflections, that is a diagonalisable linear isomorphism of the space that fixes a hyperplane pointwise. Auslander's theorem shows that, for a small subgroup of $\mathrm{GL}(n,\mathbb{C})$, one has $S * G \cong$

$\text{End}_R(S)$, see Theorem 1.8.2 for details. Through this, it is also shown for a subgroup of SL(2, $\mathbb{C}$) that $S * G$ is Morita equivalent to the *preprojective algebra* of the associated Dynkin diagram, see [CBH98] for the geometric context. The correspondence of CM R modules and irreducible representations of G predates the McKay correspondence by work of Jürgen Herzog [Her78]. This correspondence has since been generalised for any finite subgroup of GL(2, $\mathbb{C}$) by Wemyss using *Reconstruction algebras* [Wem11], with a geometric argument by results from Wunram [Wun88].

An interesting consequence of Auslander's theorem is that, for any n and a small subgroup G of GL(n, $\mathbb{C}$) the ring $\text{End}_R(S)$ is of finite global dimension, which leads to a result concerning a more algebraic approach to resolutions of singularities. That is, the following definition:

Definition. [DFI15, Definition 2.5] Let A be a commutative noetherian ring. Let M be a finitely generated A module such that supp$M$ =supp$A$, then $\text{End}_A(M)$ is called a *non-commutative resolution* (NCR) of A if the global dimension (gldim) of $\text{End}_A(M)$ is finite.

Using the above definition Auslander's theorem be translated into the statement that $\text{End}_R(S)$ is a NCR for R.

Extending the condition that G is a small group is the main direction in which we study. One immediate problem is that when G is generated by reflections, the invariant ring R is, by a result of Chevalley–Shephard–Todd, isomorphic to a polynomial ring and so the quotient variety $\mathbb{C}^n/G$ has no singularities [Che55]. For an example, consider the symmetric group S_n acting on $\mathbb{C}^n$ and $S = \mathbb{C}[x_1, \ldots x_n]$ which is generated by reflections given by the transpositions (i, j) which fix the hyperplanes defined by $x_i - x_j$. The invariant ring of this action is $R = S^G = \mathbb{C}[\sigma_1, \ldots \sigma_n]$ where σ_i is the i^{th} elementary symmetric polynomial.

In some sense we could be satisfied that there are no singularities, but the aim is to formulate a similar result to the McKay correspondence for these groups. We define a singularity inside of R which will mimic the role of R in the classical McKay correspondence.

Let G be a finite group generated by reflections, the hyperplane arrangement $A(G)$ in $\mathbb{C}^n$ is the union of all hyperplanes fixed by reflections of G and is defined by a product of linear forms $z \in S$. The *discriminant* $V(\Delta)$ is the image of $A(G)$ in $\mathbb{C}^n/G$ under the natural projection $\mathbb{C}^n \to \mathbb{C}^n/G$. In particular, the discriminant is a singular hypersurface in $\mathbb{C}^n/G$ and is defined by a reduced polynomial $\Delta \in R$, with coordinate ring $R/(\Delta)$. Discriminants have been studied in the context of *free divisor*s in [Sai93, OT92]. A McKay correspondence for reflection groups generated by order 2 reflections was presented in [BFI20], where $\mathrm{End}_{R/(\Delta)}(S/(z))$ is shown to be isomorphic to a quotient of the skew group ring $S * G$ by an idempotent corresponding to a 1 dimensional irreducible representation. A consequence of this is similar to Auslander's theorem, $\mathrm{End}_{R/(\Delta)}(S/(z))$ is of finite global dimension and thus is a NCR for $R/(\Delta)$. This leads to our main questions of study:

Question 1. Let G be a reflection group generated by reflections of order greater than 2. Is $\mathrm{End}_{R/(\Delta)}(S/(z))$ still a NCR for $R/(\Delta)$?

When $n = 2$ the ring $R/(\Delta)$ is of finite CM type, that is there is finitely many non-isomorphic CM modules. In this case, if each non-isomorphic CM $R/(\Delta)$ module appears as a $R/(\Delta)$ direct summand of $S/(z)$ then $\mathrm{End}_{R/(\Delta)}(S/(z))$ is of finite global dimension and is a NCR for $R/(\Delta)$. When $n = 3$ the ring $R/(\Delta)$ fails to be of finite CM type while $S/(z)$ will have finitely many direct summands. The following question arises for any reflection group.

Question 2. Let $n > 2$ and G be any reflection group in $\mathrm{GL}(n,\mathbb{C})$. Which CM $R/(\Delta)$ modules appear in the decomposition of $S/(z)$?

For reflection groups G generated by reflections of order greater than 2, we can also consider the skew group rings $S * G$ and relate it to an endomorphism ring. The result of Buchweitz–Faber–Ingalls shows that it is enough to quotient out by an idempotent corresponding to a 1 dimensional representation when considering reflection groups of order 2. One question we can ask is

Question 3. let G be a reflection group generated by reflections of order greater than 2. Can we describe the skew group ring $S * G$ and compare it to the skew group ring of a reflection group generated by reflections of order 2?

Results

For the results of this thesis, we set up a more general context. Let $G \subseteq GL(n,k)$ be a finite group acting on k^n, where k is a field whose characteristic does not divide the order of G. The group G also acts on $S := \mathrm{Sym}_k(k^n) \cong k[x_1, \dots x_n]$ where $x_1, \dots x_n$ are a basis of k^n. By the Chevalley–Shephard–Todd theorem, the invariant ring $R = S^G$ is isomorphic to a polynomial ring if and only if G is a reflection group. Thus if G is a reflection group, the quotient variety k^n/G, which has coordinate ring $k[k^n/G] \cong R$ is non-singular. Let G be a pseudo reflection group then the *discriminant* $V(\Delta)$ is the image of the hyperplane arrangement $A(G)$ in k^n/G under the natural projection $k^n \to k^n/G$. In particular the discriminant is a singular hypersurface in k^n/G and is given by a polynomial $\Delta \in R$, with coordinate ring $R/(\Delta)$. The hyperplane arrangement $A(G)$ is defined by a product of linear forms and hence a polynomial $z \in S$.

Shephard and Todd classified all complex reflection groups (when $k = \mathbb{C}$) into two cases, an infinite family, $G(m,p,n)$ with $1 \le m$, $p|m$, $1 \le n$, and 34 exceptional groups. In the case where G is of rank 2, i.e $n = 2$, Bannai calculated the discriminants of the complex reflection groups and showed that they are singular curves of type ADE [Ban76]. Notably, the coordinate rings $R/(\Delta)$ of these curve singularities all have a finitely many (non-isomorphic) CM modules, a list can be found in [Yos90].

New research is mostly contained in Chapters 3,4 and 5, with Chapters 3 and 4 mainly focusing on decomposing $S/(z)$ as a CM $R/(\Delta)$ module. The main theorem of Chapter 3 is:

Theorem A. (Theorem 3.4.22, 3.7.3, 3.7.7 and 3.7.12) Let $G = G(m,p,2)$, then all non-isomorphic CM modules of $R/(\Delta)$ appear at least once in the decomposition of $S/(z)$ as CM modules over $R/(\Delta)$.

Furthermore a precise decomposition of $S/(z)$ into CM modules over $R/(\Delta)$ is also given. Theorem A is achieved by considering the *matrix factorization*, See Definition 1.2.1 (z, j) of Δ given by the hyperplane arrangement $A(G)$. In Chapter 2, we detail how the irreducible representations of G can be used to decompose the matrix factorization (z, j) using *isotypical components*. After this, we investigate the simpler components corresponding to the 1 dimensional representations. The 1 dimensional representations of a pseudo reflection group give matrix factorizations corresponding to the reduced components of the discriminant by considering certain orbits of hyperplanes. For the higher dimensional irreducible representations the calculations require basis elements of the isotypical components for the coinvariant algebra, which in the case of the groups S_n and $G(m, p, 2)$ are given by *higher Specht polynomials* described in [ATY97].

Theorem A is a partial answer to Question 1 because an immediate corollary is that for the groups $G(m, p, 2)$, $\mathrm{End}_{R/(\Delta)}(S/(z))$ is a NCR for $R/(\Delta)$. The module $S/(z)$ admitting a NCR is an application of a result by Auslander, which was presented in the context of artin algebras and later extended to CM local rings by Leuschke and Iyama, see [Leu12, Section P] for a useful survey on NCRs. A CM local ring A is said to be of finite CM type if it has a finite number of non-isomorphic CM modules.

Theorem. (Theorem 1.3.27) [Aus71] [Leu07] [Iya07] Let A be a CM local ring which is of finite CM type then a NCR arises as follows: let P be a *representation generator* (see Definition 1.1.10) for A then $\Lambda = \mathrm{End}_A(P)$ has finite global dimension and is a NCR for A.

In Chapter 4, $S/(z)$ is decomposed into CM $R/(\Delta)$ modules for the symmetric group S_n for $n \geq 2$. The main difference between S_n and $G(m, p, 2)$ is that the discriminant $R/(\Delta)$ for S_n, $n \geq 3$ is not of finite CM type and so not all non-isomorphic CM $R/(\Delta)$ modules appear in the decomposition of $S/(z)$. The irreducible representations of S_n correspond to partitions of n, or *Young diagrams* $\lambda \vdash n$ of size n, [FH91]. Using these diagrams we can define the set of *standard Young tableau* $\mathrm{ST}(\lambda)$ on λ, see Section 3.3

for details. The main theorem of Chapter 4 is:

Theorem B. (Theorem 4.1.21.) For the discriminant Δ of S_n, the matrix factorization defined by the reduced hyperplane arrangement (z,z) can be decomposed in the following way:

$$(z,z) = \bigoplus_{\lambda \vdash n} \bigoplus_{T \in \mathrm{ST}(\lambda)} (z|_{M_T}, z|_{N_{T'}}).$$

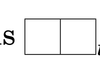
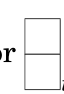
Here, $(z|_{M_T}, z|_{N_{T'}})$ are the matrix factorizations

$$M_T \xrightarrow{z|_{M_T}} N_{T'} \xrightarrow{z|_{N_{T'}}} M_T$$

where M_T is the R module generated by $\langle H_T^P | P \in \mathrm{ST}(\lambda)\rangle$ and N_T is the R module generated by $\langle F_T^P | P \in \mathrm{ST}(\lambda)\rangle$ (see Definition 4.1.9 for the definitions of the polynomials H_T^P and F_T^P).

Theorem B is achieved by considering a basis of the coinvariant algebra constructed by higher Specht polynomials and *modified higher Specht* polynomials. This is a partial answer to Question 2, since for each irreducible representation we can explicitly calculate each corresponding CM module.

Chapter 5 will focus on comparing skew group rings of complex reflection groups that have isomorphic coordinate rings of discriminants. We investigate the module categories of the skew group rings for complex reflection groups generated by reflections of order greater than 2. In particular, the groups $G(m,1,2)$ for $m > 2$, treated in Chapter 3, all have isomorphic coordinate rings of discriminants and so we compare their skew group rings. For a finite group G acting on a vector space V, with $S = \mathrm{Sym}_k(V)$, we will construct the *Lusztig algebra* of G which will be *Morita equivalent* to the skew group ring $S * G$, see [BFIL21]. The irreducible representations of $G(m,1,2)$ are described by two cases, see Lemma 3.4.13 for details:

- The 1 dimensional representations correspond to the diagrams  or , for $0 \leq i \leq m-1$.

- The 2 dimensional representations correspond to the diagrams $\square_i \square_j$, where $0 \le i < j \le m-1$.

The main theorem of Chapter 5 is:

Theorem C. (Theorem 5.3.2.) Let $m > 2$ and V, U be irreducible representations of $G(m,1,2)$, and $A^{U,V}$ be the arrow from V to U in the McKay quiver of $G(m,1,2)$. Then, the Lusztig algebra for $G(m,1,2)$ is $\tilde{A}(G(m,1,2)) \cong \mathbb{C}Q/\langle I \rangle$ where $\langle I \rangle$ is the ideal generated by:

$$A^{\square_i \square_{i+1}, \substack{\square\\\square}_{i+1}} A^{\square\square_i, \square_i \square_{i+1}}, \qquad A^{\square_i \square_{i+1}, \square\square_{i+1}} A^{\substack{\square\\\square}_i, \square_i \square_{i+1}},$$

for $0 \le i \le m-1$ and $i+1$ is taken mod m;

$$A^{\square_i \square_{j-1}, \square_i \square_j} A^{\square_{i-1} \square_{j-1}, \square_i \square_{j-1}} + A^{\square_{i-1} \square_j}, \qquad \square_i \square_j A^{\square_{i-1} \square_{j-1}, \square_{i-1} \square_j},$$

for $0 \le i,j \le m = 1$, $i+1 \ne j$ and $i-1$ is taken mod m and

$$A^{\substack{\square\\\square}_i, \square_i \square_{i+1}} A^{\square_{i-1} \square_i, \substack{\square\\\square}_i} + A^{\square\square_i, \square_i \square_{i+1}} A^{\square_{i-1} \square_i, \square\square_i} - 2A^{\square_{i-1} \square_{i+1}, \square_i \square_{i+1}} A^{\square_{i-1} \square_i, \square_{i-1} \square_{i+1}}.$$

Theorem C shows that the Lusztig algebras, and hence the skew group rings of $G(m,1,2)$ and $G(m',1,2)$ for $m \ne m'$, are not Morita equivalent. There is also another method of calculating the Morita equivalent path algebra of the skew group ring of G using superpotentials, see [BSW10].

Open problems

There are still many open problems and avenues of research in this area, for example the equivalent of Theorem A for the exceptional groups of rank 2. That is:

Open problem 1. Let G be an exceptional group of rank 2 in the Shephard–Todd classification of complex reflection group. What is the decomposition of $S/(z)$ into CM $R/(\Delta)$ modules?

One method for this would be the same as the $G(m,p,2)$ case presented here, if basis elements for the coinvariant algebra were calculated. Furthermore, for G of higher than rank 2, the modules that will appear in the decomposition of $S/(z)$ into CM modules are unclear, since $R/(\Delta)$ is of not of finite CM type. It would be interesting to explore which CM modules appear and potentially be able to describe why they appear in a geometric sense.

Open problem 2. Let G be a reflection group of rank > 2. Which CM modules appear in the decomposition of $S/(z)$?

Whilst we explore the relationship of skew group rings of groups that have isomorphic discriminants, a result similar to that [BFIL21] and Auslander can potentially be achieved. This would be done by locating a natural description of which irreducible representations to quotient out of the skew group rings.

Open problem 3. Let G be any reflection group. Is $\mathrm{End}_{R/(\Delta)}(S/(z))$ Morita equivalent to a quotient of the skew group ring $S * G$?

Structure

The structure of the thesis will be as follows, Chapter 1 contains the wider preliminaries needed in the thesis, with a goal to explain the (classical) McKay correspondence that motivates the work. Basic concepts will not be defined, while definitions for less well known objects will given. Chapter 2 contains the definitions needed to study the discriminant with some discussion and new research concerning 1 dimensional irreducible representations. Chapter 3 presents the work done on the groups $G(m,p,2)$ found in the preprint [May21], with more detailed discussion. Chapter 4 is about the decomposition of $S/(z)$ for the group S_n, with $n > 2$. These results were achieved from a collaboration with colleagues Eleonore Faber[1], Colin Ingalls[2] and Marco Talarico [3].

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Chapter 5 focuses on Lusztig algebras and aims to compare groups that have isomorphic discriminants.

Chapter 1

Preliminaries

In order to follow the content in later chapters, the preliminaries will contain the more widely known concepts, trying to be as general as it makes sense to be whilst remaining relevant to the thesis. The work contained in this thesis brings together multiple different disciplines in mathematics, mainly through the use of representation theoretic means to study certain singularities arising from pseudo reflection groups. One goal of this chapter is to describe the classical McKay correspondence, remarking on how our work will extend it.

1.1 Cohen-Macaulay modules

A central topic of this thesis will be the relationships between representations of finite groups and Cohen-Macaulay modules. One way of calculating Cohen-Macaulay modules over hypersurface rings is to use Eisenbud's matrix factorization theorem. The original can be in found [Eis80], a good summary for the context of this work can be found in [Yos90].

There are different ways of defining Cohen-Macaulay modules, depending on the context. Here is a geometric focused definition, and the one we will use throughout:

Definition 1.1.1. Let A be a noetherian commutative ring, a finitely generated A

module M is called a *Cohen-Macaulay* (CM) module if the depth of $M_{\mathfrak{m}}$ is equal to the Krull dimension of $A_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of A. A ring A is said to be a CM ring if it is a CM module over itself. Let $CM(A)$ be the category of CM modules over A.

Remark 1.1.2. If A is a local Noetherian commutative ring, then a finitely generated A module M is CM if and only if the depth(M) =dim$A$.

Example 1.1.3. A polynomial ring $S := k[x_1,\dots,x_n]$, where k is a field, is a CM ring, and the CM S modules are the free S modules. See [Yos90, Proposition 1.5].

Remark 1.1.4. The zero module is a CM module.

Definition 1.1.5. Let A be a commutative Noetherian ring which is $\mathbb{N}$-graded and graded local with a unique maximal graded proper ideal $\mathfrak{m}$. A finitely generated A module M is graded CM if $X_{\mathfrak{m}}$ is a CM $A_{\mathfrak{m}}$ module.

Definition 1.1.6. Let A be a graded local commutative Noetherian ring and M a finite generated graded A module. Consider the exact sequence of graded A modules, where F_i, for $0 \le i \le n-1$ are graded free A modules.

$$0 \longrightarrow N \longrightarrow F_{n-1} \longrightarrow F_{n-2} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0.$$

Then N is called an n^{th}*syzygy* of M. The graded A-module N can have graded free A modules as summands and so we define the *reduced* n^{th} syzygy $\mathrm{syz}^n_A(M)$ of M to be N without the graded free Amodule summands. That is , $\mathrm{syz}^n_A(M) \oplus F \cong N$ for some graded free A-module F.

Remark 1.1.7. We can split off direct summands by appealing to the following result of [IT10, Proposition 1.4]. If A_0 is Artinian then the category of graded CM modules is Krull-Schmidt.

Lemma 1.1.8. [Yos90, Proposition 1.16] Let A be a CM local ring of Krull dimension d. For any A module M, given any n greater or equal to d, $\mathrm{syz}^n_A(M)$ is either a CM module or the zero module.

Definition 1.1.9. A graded CM ring A is called *finite CM-type* if it has a finite number of non-isomorphic CM modules up to degree shift.

Definition 1.1.10. Let A be a graded CM ring of finite CM type, a CM A module is called a *representation generator* for $CM(A)$ if it contains a CM module from every isomorphism class of CM modules as a direct summand.

Definition 1.1.9 and 1.1.10 will be used to appeal to Theorem 1.3.27 for finite type CM rings.

1.2 Matrix factorizations

The context for this section is as follows: Let A be a ring of the form $A = B/I$ in which B is a either a regular local ring or a polynomial ring and I is a principal ideal generated by an element $f \neq 0$. We call A a *hypersurface* ring. Let $CM(A)$ be the category of Cohen-Macaulay modules over the ring A. Here we follow the exposition in [Yos90] and [LW12].

Let F and G be a free B module of rank n and ϕ, ψ be homomorphisms $\phi : F \to G$ and $\psi : G \to F$ such that;

$$\psi\phi = f \cdot 1_F \qquad \text{and} \qquad \phi\psi = f \cdot 1_G$$

By choosing a basis of F and G over B, the morphisms ϕ, ψ are equivalently $n \times n$ matrices with entries in B such that,

$$\psi \cdot \phi = f \cdot 1_{B^n} \qquad \text{and} \qquad \phi \cdot \psi = f \cdot 1_{B^n}$$

Definition 1.2.1. Let n be a positive integer, a pair of matrices (ϕ, ψ) with entries in B, which satisfy the above properties is called a *matrix factorization* of f.

Example 1.2.2. Let $B/(f)$ be a hypersurface ring, then trivially we have the matrix factorizations $(1, f)$ and $(f, 1)$ given by the sequences;

$$B \xrightarrow{f} B \xrightarrow{1} B \qquad B \xrightarrow{1} B \xrightarrow{f} B \ .$$

To build the category of matrix factorizations of a hypersurface, we require the definition of morphisms between matrix factorizations.

Definition 1.2.3. Let (ϕ_1,ψ_1) and (ϕ_2,ψ_2) be matrix factorizations of f. Then a morphism between them is a pair of matrices (α,β) such that the following diagram commutes:

$$\begin{array}{ccccc} B^{n_1} & \xrightarrow{\psi_1} & B^{n_1} & \xrightarrow{\phi_1} & B^{n_1} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} \\ B^{n_2} & \xrightarrow{\psi_2} & B^{n_2} & \xrightarrow{\phi_2} & B^{n_2} \end{array}$$

Remark 1.2.4. A morphism (α,β) between the matrix factorizations (ϕ_1,ψ_1) and (ϕ_2,ψ_2) induces a morphism between the R-modules Coker(ϕ_1) and Coker(ϕ_2) via

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B^{n_1} & \xrightarrow{\phi_1} & B^{n_1} & \longrightarrow & \text{Coker}(\phi_1) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \text{Coker}(\alpha)} & & \\ 0 & \longrightarrow & B^{n_2} & \xrightarrow{\phi_2} & B^{n_2} & \longrightarrow & \text{Coker}(\phi_1) & \longrightarrow & 0 \end{array}$$

Definition 1.2.5. Let (ϕ_1,ψ_1) and (ϕ_2,ψ_2) be matrix factorizations in $MF_B(f)$ then:

$$(\phi_1,\psi_1)\oplus(\phi_2,\psi_2) = \left(\begin{bmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{bmatrix}, \begin{bmatrix} \psi_1 & 0 \\ 0 & \psi_2 \end{bmatrix} \right)$$

Remark 1.2.6. [Yos90] With the morphisms and direct sum above, the matrix factorizations of f form an additive category $MF_B(f)$.

Definition 1.2.7. Two matrix factorizations are *equivalent* if there is a morphism (α,β) in which α,β are isomorphisms.

It is straightforward to see that (ϕ,ψ) is not necessarily equivalent as a matrix factorization to (ψ,ϕ), for example if f is not a unit in B, then $(1,f) \not\cong (f,1)$.

Definition 1.2.8. An object (ϕ,ψ) of $MF_B(f)$ is said to be *reduced* if ϕ and ψ, contains no units as entries.

Example 1.2.9. For any $f \neq 0$ in B, the matrix factorizations $(1,f)$ and $(f,1)$ are not reduced.

To state the equivalence between matrix factorization and CM modules we need to consider a quotient category of the category of matrix factorizations.

Definition 1.2.10. Let $\mathscr{C}$ be a pre-additive category, i.e a category where Hom-sets are abelian groups and compositions of morphisms are bilinear. Let $\mathscr{D}$ be a set of objects in $\mathscr{C}$. We define $\mathscr{C}/\mathscr{D}$ to be the category which has the same objects as $\mathscr{C}$ but the Hom-sets are:

$$\mathrm{Hom}_{\mathscr{C}/\mathscr{D}}(A,B) := \mathrm{Hom}_{\mathscr{C}}(A,B)/\mathscr{D}(A,B)$$

Where $\mathscr{D}(A,B)$ is a subgroup of $\mathrm{Hom}_{\mathscr{C}}(A,B)$ generated by all morphisms from A to B which pass through the direct sums of objects in $\mathscr{D}$.

Definition 1.2.11. With the former construction we can define the following category:

$$\underline{MF}_B(f) := MF_B(f)/\{(1,f)\}.$$

Theorem 1.2.12. (Eisenbud's matrix factorization theorem, [Eis80])
Let B be a local ring or a polynomial ring and let f be a non-zero element of B. If $A = B/(f)$ is a hypersurface then the functor $\mathrm{Coker}(\phi,\psi) := \mathrm{Coker}(\phi)$ induces an equivalence of categories:

$$\underline{MF}_B(f) \simeq CM(A).$$

Example 1.2.13. Consider the hypersurface ring $A = \mathbb{C}[x_1,x_2]/(\Delta)$ where $\Delta = x_1(x_2^2 - 4x_1^2)$. Then $(x_1, x_2^2 - 4x_1^2)$ is a matrix factorization of Δ and is equivalent to the CM module $\mathbb{C}[x_1,x_2]/(x_1)$. Another matrix factorization is given by the 2×2 matrices:

$$(\phi,\psi) = \left(\begin{bmatrix} 2x_1 & x_2x_1 \\ x_2 & 2x_1 \end{bmatrix}, \begin{bmatrix} -2x_1 & x_2x_1 \\ x_2 & -2x_1 \end{bmatrix} \right).$$

Remark 1.2.14. Eisenbud's matrix factorization theorem also induces an equivalence between the *reduced matrix factorizations* $\underline{RMf}_B(f) := MF_B(f)/\{(1,f),(f,1)\}$ and the *stable* category of Cohen Macaulay modules $\underline{CM}(A) := CM(A)/\{A\}$.

To see how the functor Coker behaves when taking syzygies we have the following Lemma:

Lemma 1.2.15. Let (ϕ,ψ) be a matrix factorization of f with $\mathrm{Coker}(\phi,\psi) = M$ where M is indecomposable and non-free, then $\mathrm{syz}^1_A(M) \cong \mathrm{Coker}(\psi,\phi)$.

Remark 1.2.16. Generally matrix factorizations are discussed in the context of regular local rings. Instead we also included polynomial rings in our definition. In [Yos90, Chapter 15] it was shown that the construction and results hold in the graded case.

1.3 Singular varieties

We will be studying finite groups G which act on a finite dimensional k-vector space V, where k is a field that is algebraically closed and of characteristic 0, and the geometric objects that arise from considering orbits of G, that is the quotient variety V/G. In most cases V/G will not be a manifold, in particular V/G will mostly have *singular* points. This section aims to introduce the notation of a non-singular variety. We will use the notation of a variety but note that this can be expanded into the language of schemes. A natural way to define a non-singular variety is to mimic the notion of a manifold in differential geometry. We follow constructions given in [Har77] but note that our varieties are not necessarily irreducible.

We start this section by fixing k as a algebraically closed field of characteristic 0.

Definition 1.3.1. Affine n-space $\mathbb{A}^n$ is the set of all n-tuples of elements of k, an element $p \in \mathbb{A}^n$ is called a *point*.

Definition 1.3.2. Let $A = k[x_1,...,x_n]$ and let T be a subset of A, then the *set of zeros* of T is

$$\mathbf{V}(T) = \{p \in \mathbb{A}^n \mid f(p) = 0 \text{ for all } f \in T\}.$$

Definition 1.3.3. A subset $X \subseteq \mathbb{A}^n$ is called a *affine variety* if there is a subset $T \subseteq k[x_1,\ldots,x_n]$ such that $X = \mathbf{V}(T)$.

The *Zarski topology* can be defined on affine space $\mathbb{A}^n$ as the topology where the open sets are the complements of affine varieties.

Definition 1.3.4. An *irreducible affine variety* is a irreducible closed subset of $\mathbb{A}^n$.

Lemma 1.3.5. [Har77, Chapter 1, Corollary 1.6] Every affine variety X is a union of finitely many irreducible affine varieties $X = \bigcup_i X_i$.

Definition 1.3.6. Let $X \subseteq \mathbb{A}^n$ and $A = k[x_1, \ldots, x_n]$, the *ideal of* X in A is

$$I(X) = \{f \in A \mid f(p) = 0 \text{ for all } p \in X\}.$$

Definition 1.3.7. Let $A = k[x_1, \ldots, x_n]$. For an affine variety $X \subseteq \mathbb{A}^n$ the *coordinate ring* $k[X]$ of X is $A/I(X)$.

Definition 1.3.8. Let X be an Noetherian topological space then the Krull dimension $\dim(X)$ of X is the supremem of all integers n such that there is a chain

$$Z_0 \subset Z_1 \subset \cdots \subset Z_n$$

of distinct irreducible subsets of X.

Definition 1.3.9. Let X be an irreducible affine variety then the Krull dimension of X is $\dim(X)$.

Definition 1.3.10. Let X be an affine variety with a decomposition $\bigcup_i V_i$ where V_i are irreducible affine varieties. Then the *dimension* $\dim_x(X)$ at a point x of X, is the

$$\max\{\dim V_i \mid x \in V_i\}.$$

Definition 1.3.11. Let X be an affine variety with coordinate ring $k[X] = k[x_1, \ldots, x_n]/I(X)$ and let $f_1, \ldots, f_t \in k[x_1, \ldots x_n]$ be a set of generators for the ideal $I(X)$. The variety X is *non-singular* at a point p of X if the rank of the Jacobian matrix,

$$\left(\frac{\partial f_i}{\partial x_j}\right)_{1 \le i \le t, 1 \le j \le n}$$

is $n - r$ where $r = \dim_p(X)$. The affine variety X is said to *non-singular* if X is non-singular at every point p of X.

Example 1.3.12. Let X be the affine variety over $\mathbb{C}^n$ with coordinate ring $\mathbb{C}[x,y]/\langle x^3 - y^2\rangle$ then the Jacobian matrix is:

$$\begin{bmatrix} 3x^2 & -2y \end{bmatrix}.$$

It has rank $2-1=1$ at every point of X except for $(0,0)$ where the curve is singular.

This definition depends on the embedding we choose for our affine variety X. To avoid this problem, Zariski instead showed that a variety being non-singular can be expressed using local rings.

Definition 1.3.13. Let A be a local ring, with maximal ideal $\mathfrak{m}$ and residue field k then A is a *regular local ring* if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$. Where dim A is the Krull dimension of A as a ring.

Definition 1.3.14. Let X be an affine variety and p a point on X, then the *local ring* of p on X denoted $\mathcal{O}_{p,X}$ is the ring of germs of regular functions on X near p.

Theorem 1.3.15. [Har77, Chapter 1, Theorem 5.1] Let X be an affine variety and p a point on X then X is non-singular at p if and only if the local ring $\mathcal{O}_{p,X}$ is a regular local ring.

The definition of non-singular can then be extended to any variety.

Definition 1.3.16. Let X be a variety, then X is non-singular at a point p if the local ring $\mathcal{O}_{p,X}$ is a regular local ring, and X is a non-singular variety if all points in X are non-singular.

Definition 1.3.17. Let X be a variety, then the *singular locus* $\mathrm{Sing}(X)$ is set of singular points of X.

Theorem 1.3.18. [Har77, Chapter 1, Theorem 5.3] Let X be a variety then $\mathrm{Sing}(X)$ is a proper closed subset of X.

The notion of a resolution of singularities is widely studied, where one relates a non-singular variety $\tilde{X}$ to the singular variety X. One geometric definition is as follows:

Definition 1.3.19. Let X be a singular variety over the field k. A non-singular variety $\tilde{X}$ together with a proper birational morphism $\pi : \tilde{X} \to X$, such that π is an isomorphism on all non-singular points, is a *resolution of singularities* of X, denoted $(\tilde{X}, \pi)$.

Remark 1.3.20. Hironaka proved that over a field of characteristic 0, there always exists a resolution of singularities for a variety X in [Hir64].

Definition 1.3.21. Let X be a singular variety with a resolution of singularities $(\tilde{X}, \pi_X)$. If every resolution of singularities factors through $(\tilde{X}, \pi_X)$ then we call $(\tilde{X}, \pi_X)$ the *minimal resolution of singularities*.

For a singular variety, the minimal resolution of singularities is unique.

Remark 1.3.22. Singular varieties in dimension 1 and 2 always have a minimal resolution, but for higher dimension there are examples of singular varieties that do not have a minimal resolution. For instance, a well known example in dimension 3 is the Atiyah flop, [Ati58], which has no minimal resolution.

In general it is hard and computationally expensive to resolve singular varieties, so we endeavour to have a more algebraic approach to the definition of a resolution of singularities.

Definition 1.3.23. Let R be a ring, if there is an integer n such that all A modules have a resolution of projective R modules of length at most n then R has *finite global dimension* and the minimal such n is the *global dimension* of R, denoted $\text{gldim} R$.

Definition 1.3.24. [DFI15, Definition 2.5] Let A be a commutative noetherian ring. Let M be a finitely generated A module such that $\text{supp} M = \text{supp} A$, then $\text{End}_A(M)$ is called a *non-commutative resolution* (NCR) of A if $\text{gldim}\, \text{End}_A(M) < \infty$.

There has been a few different definitions of NCRs, Definition 1.3.24 is a generalised version of the definition given in [DITV15, Definition 1.6]. The notion of a NCR is a weaker version of non-commutative *crepant* resolution which was first defined by Van den Bergh [VdB04, Definition 4.1] over normal Gorenstein domains.

Definition 1.3.25. [IW14, Definition 1.6] Let R be a CM ring and let A be an R-algebra, which is finitely generated as a R module, we say that A is an *non-singular order* if A is CM over R and gldim($A_{\mathfrak{p}}$) =dim($R_{\mathfrak{p}}$) for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

Definition 1.3.26. Let A be a commutative noetherian ring. Let M be a finitely generated torsion free A module such that supp$M$ =supp$A$, then $\mathrm{End}_A(M)$ is called a *non-commutative crepant resolution* (NCCR) of A if $\mathrm{End}_A(M)$ is a non-singular order.

The objects that are constructed in this thesis we will be NCRs and not NCCRs, since they fail to be non-singular orders. See [Leu12] for a thorough survey on NCCRs. The following theorem gives us a NCR for a finite CM ring

Theorem 1.3.27. [Aus71] [Leu07] [Iya07] Let A be a commutative CM graded local ring which is of finite CM type then a NCR arises as follows: Let P be a *representation generator* (see Definition 1.1.10) for A then $\Lambda = \mathrm{End}_A(P)$ has finite global dimension and in particular is a NCR for A.

1.4 Invariant theory

We will give a quick introduction to invariant theory by defining the invariant ring of a group action and giving a Cohen-Macaulay property of invariant rings. The following section will mainly be based on [Kan01, Section 16] since we will also be considering pseudo reflection groups in subsequent chapters.

Let k be any field and let V be a finite dimensional k-vector space of dimension n. Let G be a finite subgroup of GL(V) and $T(V)$ be the tensor algebra

$$T(V) = k \oplus V \oplus (V \otimes V) \oplus \cdots = \bigoplus_{i=0}^{\infty} V^{\otimes i}$$

together with the tensor product, given $x = x_1 \otimes \cdots \otimes x_i$ and $y = y_1 \otimes \cdots \otimes y_j$ then

$$x \otimes y = x_1 \otimes \cdots \otimes x_i \otimes y_1 \otimes \cdots \otimes y_j \in V^{\otimes(i+j)}$$

The k-algebra $T(V)$ is graded, where $V^{\otimes j}$ are of homogenous degree j.

Definition 1.4.1. Let k be any field, V be a finite k-vector space of dimension n and $T(V)$ be the tensor algebra. The *symmetric algebra* $\mathrm{Sym}_k(V)$ is the quotient;

$$\mathrm{Sym}_k(V) = T(V)/I$$

where I is the ideal generated by $\{x \otimes y - y \otimes x \mid x, y \in V\}$. The algebra $\mathrm{Sym}_k(V)$ inherits the grading from $T(V)$ and so $\mathrm{Sym}_k(V)$ is a commutative associative graded k-algebra with decomposition

$$\mathrm{Sym}_k(V) = \bigoplus_{i=0}^{\infty} S_i(V).$$

If we choose a basis $x_1, \ldots x_n$ for V then $\mathrm{Sym}_k(V) \cong k[x_1, \ldots, x_n]$. From now on if V is obvious we refer to $\mathrm{Sym}_k(V)$ as S.

Definition 1.4.2. Let k be any field, V a finite k-vector space of dimension n and let $\{x_1, \ldots, x_n\}$ a basis for V. Let $G \subset \mathrm{GL}(V)$ act on V, we can extend the action to $\mathrm{Sym}_k(V) \cong k[x_1, \ldots x_n]$ by using

$$g \cdot (x_1 \cdots x_n) = (g \cdot x_1)(\cdots)(g \cdot x_n).$$

and extending linearly.

Definition 1.4.3. Let k be any field, V a finite k-vector space of dimension n and $G \subset \mathrm{GL}(V)$, then the *invariant ring* of the action of G on $S = \mathrm{Sym}_k(V)$ is;

$$R = S^G = \{f \in \mathrm{Sym}(V) \mid g \cdot f = f \text{ for all } g \in G\}.$$

Example 1.4.4. [Kan01, Section 16-2] Consider S_n acting on $V = \mathbb{C}^n$ with basis $\{x_1, \ldots, x_n\}$. The transpositions (i,j) for $0 \leq i \leq j \leq n$ act on V by: $(i,j)(x_i) = x_j$, $(i,j)(x_j) = x_i$ and $(i,j)(x_k) = x_k$ for $k \neq i,j$. These transpositions are generators for S_n. The k^{th} elementary symmetric polynomial σ_k is defined by:

$$\sigma_k(x_1, \ldots, x_n) = \sum_{1 \leq j_1 < \cdots < j_k \leq n} (x_{j_1} \ldots x_{j_k})$$

and the invariant ring is:

$$\mathbb{C}[x_1, \ldots, x_n]^{S_n} = \mathbb{C}[\sigma_1, \ldots \sigma_n].$$

One important result on the structure of S^G as a ring, is the following theorem.

Theorem 1.4.5. (Hochster-Roberts) [HR74] Let k be a field, V a finite k-vector space of dimension n and G a finite subgroup of GL(V). If Char(k) does not divide $|G|$, where $|G|$ is the order of the group, then S^G is a CM ring.

Remark 1.4.6. The reason for the condition "If Char(k) does not divide $|G|$" is that the proof requires the use of the averaging operator:

$$\mathrm{Av} : S \to S$$
$$\mathrm{Av} : x \mapsto \frac{1}{|G|} \sum_{g \in G} g(x)$$

so $|G|$ must be invertible in k.

Remark 1.4.7. Some authors define $S = \mathrm{Sym}_k(V^*)$ which can be identified with the polynomial functions on V and therefore have to use the group action $g(f(x)) = f(g^{-1}x)$ for $x \in V$ and $g \in G$.

Definition 1.4.8. Let k and V a finite k-vector space of dimension n. Let G be a finite subgroup of GL(V), the $G-$orbit of a point $x \in V$ is the set

$$G(x) = \{g(x) \mid \text{ for all } g \in G\}.$$

The *orbit space* V/G is the set of all G-orbits of V.

Theorem 1.4.9. [Cox15, Section 7.4, Theorem 10] Let k be an algebraically closed field and V a finite k-vector space of dimension n. The orbit space V/G has the structure of an affine variety, with coordinate ring $k[V/G] \cong R$.

Remark 1.4.10. We will use the term *quotient variety* for the orbit space V/G with the structure of an affine variety.

Example 1.4.11. Consider the group $\mathbb{Z}_2$ acting on $V = \mathbb{C}^2$ with basis $\{x, y\}$ by the action defined by the matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \text{that is } \mathbb{Z}_2 = \left\langle \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle.$$

Note that with this representation $\mathbb{Z}_2 \subset \mathrm{SL}(V)$. The invariant ring $R = S^G$ is

$$R = \mathbb{C}[x^2, y^2, xy] \cong \mathbb{C}[u, v, w]/(w^2 - uv).$$

1.5 The classical McKay correspondence

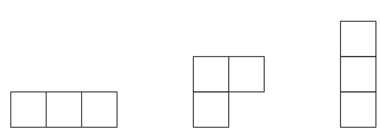
This section is dedicated to stating and discussing the classical McKay correspondence. First we need to define some objects which hold certain information about groups and their invariant rings. For more exposition about the classical McKay correspondence see [Buc12], [Rei02], [GSV83], [LW12].

Definition 1.5.1. A quiver Q is a 4-tuple (Q_0, Q_1, s, t), where Q_0 is the set of vertices, Q_1 is the set of arrows and s and t are functions $s, t : Q_1 \to Q_0$ where $s(x)$ is the source of the arrow x and $t(x)$ is the arrows target.

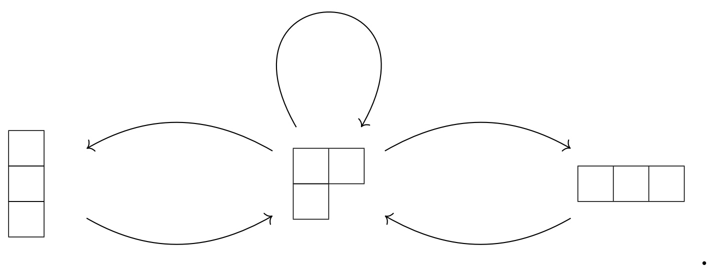
Definition 1.5.2. Let G be a finite group acting on V, where V a $\mathbb{C}$ vector space of dimension n. Let $\{V_1, \ldots, V_r\}$ be a list of all non-isomorphic irreducible representations of G. The *McKay quiver* $\Xi(G)$ of G is defined as follows: the vertices of $\Xi(G)$ are the irreducible representations V_i of G and there are m_{ij} arrows from V_i to V_j if V_j appears with multiplicity m_{ij} in $V_i \otimes V$.

Remark 1.5.3. Some authors use the opposite convention for the direction of the arrows in the McKay quiver.

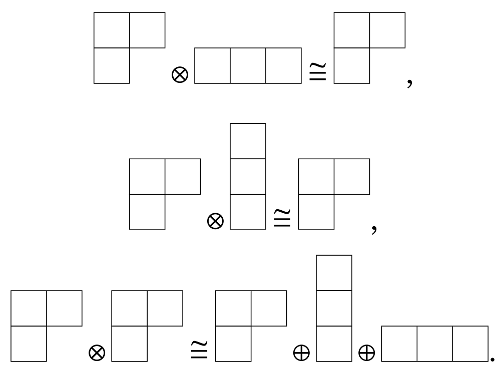
Example 1.5.4. The group S_3 has 3 irreducible representations; triv, the standard representation and det which correspond to the Young diagrams (see Section 3.3 for definitions):



respectively. The McKay quiver for S_3, when considering S_3 as a complex reflection group acting on $\mathbb{C}^2$ via the standard representation is:



This is obtained from the calculations



Using Herzog's theorem [Her78], see [LW12, Theorem 6.3] for more detail, one obtains the following theorem:

Theorem 1.5.5. Let G be a finite group acting on V where $\dim V = 2$ and assume $G \subset SL(V)$. Recall that $S = \mathrm{Sym}_k(V)$ and $R = S^G$. Then S decomposes into CM R modules in the following way

$$S \cong \bigoplus_{M \in CM(R)} M^{\alpha_M}$$

where α_M is a non-zero positive integer.

Theorem 1.5.6 (Klein, Du Val [DV34a, DV34b, DV34c])**.** Let G be a finite group that acts on V, where $\dim V = 2$, $G \leq SL(V)$, $R = S^G$, $X = V/G = \mathrm{Spec}(R)$, then X is a Du Val (or Kleinian) singularity. That is $R \cong k[x,y,w]/(f)$ where f is one of following:

- $A_n : w^2 + x^2 + y^{n+1}$,

- $D_n : w^2 + y(x^2 + y^{n-2})$ for $n \geq 4$,
- $E_6 : w^2 + x^3 + y^4$,
- $E_7 : w^2 + x(x^2 + y^3)$,
- $E_8 : w^2 + x^3 + y^5$.

Example 1.5.7. Recall for the group in Example 1.4.11 the invariant ring was

$$R = \mathbb{C}[x^2, y^2, xy] \cong \mathbb{C}[u, v, w]/(w^2 - uv)$$

and so Spec(R) is a A_1 Du Val singularity.

Definition 1.5.8. (See [Har77] for more detail) Let X be a variety and $(\tilde{X}, \pi)$ be a resolution of singularities of X, then the *exceptional divisor* $E = \bigcup_i E_i$ is the preimage $\pi^{-1}(\mathrm{Sing}(X))$ and E_i are the irreducible components of E.

Definition 1.5.9. Let X be singular surface, with a resolution of singularities $(\tilde{X}, \pi)$ then let $E = \bigcup_{i=1}^m E_i$ be the exceptional divisor on $\tilde{X}$. The *dual resolution graph* has vertices E_i and an edge from E_i to E_j if and only if $E_i \cap E_j \neq \varnothing$.

Remark 1.5.10. The dual resolution graph is an undirected graph.

In 1980 McKay observed the following in [McK80] and the geometric version was proven in [GSV83]. We present the theorem in the classical way, where instead of polynomials, the group acts on power series.

Theorem 1.5.11. (Classical McKay correspondence) Let G be a finite group that acts on $V \cong \mathbb{C}^2$, where $\dim V = 2$, $G \leq \mathrm{SL}(V)$, $S = \mathbb{C}[|x_1, x_2|]$, $R = S^G$ and $X = V/G$. The following are in a $1-1$ correspondence.

1) Irreducible components E_i of the exceptional divisor of the minimal resolution $\tilde{X}$ of X (Definition 1.5.9),

2) isomorphism classes of irreducible representations of G, excluding the trivial representation,

3) isomorphism classes of indecomposable CM modules on R, excluding the module R (Definition 1.1.1),

4) indecomposable R summands of S, excluding R (Lemma 1.5.5).

Also, if one deletes the vertex corresponding to the trivial representation in the McKay quiver of G and collapses all 2-cycles of the McKay quiver into edges one obtains the dual resolution graph of X. The correspondence of 3)–4) is given by Herzog's theorem (see Theorem 1.5.5) and 2)–4) is given by Auslander's theorem (see Theorem 1.8.2).

In this thesis we are mainly interested in parts 2–4 of Theorem 1.5.11. Part 1 is included to give a wider context.

Remark 1.5.12. The McKay quiver is also equivalent to the *Auslander-Reiten* (see [AR75]) (AR) quiver of $CM(R)$.

1.6 Reflection groups

This section will introduce the main groups that will be considered in the rest of the thesis. Let k be any field.

Definition 1.6.1. Let V be a k-vector space, then a *pseudo reflection* is a (diagonalisable) linear isomorphism $s : V \to V$, which is not the identity such that it fixes a hyperplane pointwise. A finite subgroup of V generated by pseudo reflections is called a *pseudo reflection group*. A pseudo reflection group which is generated by reflections of order 2 is called a *true* reflection group. A group that contains no pseudo reflections is called a *small* group.

Example 1.6.2. [OT92] Let S_n act on $\mathbb{C}^n$ with basis $x_1, \ldots x_n$ then the transposition (i, j) is a reflection an fixes the hyperplane $H_{i,j}$ given by the linear form $x_i - x_j$.

Remark 1.6.3. The classical McKay correspondence, Theorem 1.5.11, does not include pseudo reflection groups, since by definition a reflection has determinant not equal to 1 and thus is not in $\mathrm{SL}(V)$.

The following theorem describes the structure of the invariant ring under the action of a pseudo reflection group.

Theorem 1.6.4. (Chevalley-Shephard-Todd). [Che55]
Let V be a k-vector space, let G be a finite subgroup of $GL(V)$ and assume that the characteristic of k does not divide the order of G, then $S^G = R \cong \mathbb{C}[f_1, \ldots, f_n]$, where f_i are algebraically independent homogeneous polynomials of positive degree if and only if G is a finite pseudo refection group.

A set of polynomials $\{f_1, ..., f_n\}$ which satisfy the above are called a set of *basic invariants* for G. Basic invariants are not unique but the degrees of the basic invariant are unique [Che55].

Remark 1.6.5. In particular Theorem 1.6.4 tells us that the invariant ring of the action of a complex reflection group is non-singular.

Definition 1.6.6. A pseudo reflection group G acting on V is called reducible if $V = V_1 \oplus V_2$ where V_1 and V_2 are stable under G. The restriction of G_i of G onto V_i is a reflection group acting on V_i. If G is not reducible it is called irreducible.

Definition 1.6.7. If $k = \mathbb{C}$ then pseudo reflections are called *complex reflections* and pseudo reflection groups are called *complex reflection groups*.

A classification of finite irreducible complex reflection groups is given by Shephard-Todd:

Theorem 1.6.8 (Shephard-Todd)**.** [ST54]
All irreducible complex reflection groups fall into one of the following families:

- The infinite family $G(m,p,n)$, where $m,p,n \in \mathbb{N} \setminus \{0\}$, $p|m$ and $(m,p,n) \neq (2,2,2)$,
- The exceptional complex reflection groups $G_4, ..., G_{37}$.

Example 1.6.9. $S_n = G(1,1,n)$ is a true reflection group that acts on $\mathbb{C}^n$. The transpositions (i,j) for $0 \leq i \leq j \leq n$ are reflections of order 2 which generate S_n. Recalling

Example 1.4.4, a basic set of invariants of this action is are given by the elementary symmetric polynomials and the invariant ring is

$$\mathbb{C}[x_1,\dots,x_n]^{S_n} = \mathbb{C}[\sigma_1,\dots\sigma_n].$$

Example 1.6.10. Let $n \geq 1$, the group $G(n,1,1)$ is the cyclic group $C_n = \langle \xi_n \rangle$ where ξ_n is the primitive n^{th} root of unity. The group $G(n,1,1)$ acts on $\mathbb{C}[x]$ by $x \mapsto \xi_n x$. The invariant ring is then $S^{G(n,1,1)} \cong \mathbb{C}[x^n]$.

1.7 Path algebras

Path algebras will be used in Chapter 5 to investigate the differences between the groups $G(m,1,2)$ for $m \geq 2$, since coordinate rings of their discriminants are isomorphic to each other, see Theorem 5.1.4.We give here conventions and notations for more detail on path algebras see [Sch14].

Definition 1.7.1. Let Q be a quiver. A list of edges $(x_1,...,x_n)$ is a *path* if $t(x_i) = s(x_{i+1})$ for all $1 \leq i \leq n-1$, and is called a *cycle* if $s(x_1) = t(x_n)$.

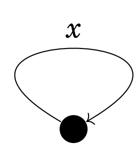
Definition 1.7.2. Let $x = (x_1,...,x_n)$ and $y = (y_1,\dots,y_m)$ be two paths, then if $s(x_n) = t(y_1)$, the *concatenation* of x and y is defined by $x \circ y = (y_1,\dots y_n, x_1 \dots x_m)$.

Definition 1.7.3. Let Q be a quiver, let kQ be the k-vector space whose basis is all the paths in Q and given paths $x = (x_1,...,x_n)$ and $y = (y_1,\dots,y_m)$ multiplication xy is given by: if $s(x_n) = t(y_1)$ then $xy = x \circ y$ and 0 otherwise.

The vector space kQ, together with the multiplication from Definition 1.7.3 forms an associative k-algebra.

Example 1.7.4. Given the quiver:

$Q =$



then $kQ \cong k[x]$.

Definition 1.7.5. Let $x \in Q_0$ be a vertex in Q, then ε_x is the empty path starting and ending at x.

The elements ε_x for all $x \in Q_0$ are pairwise orthogonal idempotent. If Q_0 is finite then the element $\varepsilon = \sum_{x \in Q_0} \varepsilon_x$ is the unit of kQ.

Example 1.7.6. Given the quiver:

$$Q = \quad x_1 \xrightarrow{\quad \alpha \quad} x_2$$

Then we have the following:

$$\varepsilon_{x_1} \alpha = \alpha \qquad \alpha \varepsilon_{x_2} = \alpha$$

and kQ is the vector space with basis $\alpha, \varepsilon_{x_1}$ and ε_{x_2}.

1.8 Skew-group rings

Let V be a vector space of dimension n over the field k and G be a finite subgroup of $\mathrm{GL}(V)$. Throughout this section we assume that the characteristic of k does not divide $|G|$, where $|G|$ is the order of the group G. The group ring is denoted by kG and by S the k-algebra $Sym_k(V) \cong k[x_1, ..., x_n]$ with G acting on S as before. We will use skew group rings in Chapter 5, relating them to path algebras. A good resource for skew group rings in this context, with more discussion is [LW12, Chapter 5].

Definition 1.8.1. Let G be a finite subgroup of $\mathrm{GL}(V)$ and $S = l[x_1, \ldots, x_n]$, the *skew group ring* of G and S is denoted $A = S * G$. As a S modules, A is free on the elements of G so $A = \bigoplus_{g \in G} Sg$. The multiplication of $S * G$ is twisted and is given by:

$$(s_1 g_1)(s_2 g_2) = s_2 g(s_1) g_1 g_2$$

which extends linearly to arbitrary elements of $S * G$.

Theorem 1.8.2. (Auslander's Theorem) [Aus86, p.515] Let V be a k-vector space of dimension n and let G be a small (recall Definition 1.6.1) subgroup of GL(V). Let S be as before, $R = S^G$ and $A = S * G$. Then there is an isomorphism of k-algebras:

$$A \cong \mathrm{End}_R(S),$$

given by

$$sg \mapsto (f \mapsto sg(f)).$$

In particular, $\mathrm{End}_R(S)$ is a CM module over R, has global dimension n.

Remark 1.8.3. The map $f \mapsto sg(f)$ can be seen to be R-linear from the following calculations. Let $r_1, r_2 \in R$, $s, f_1, f_2 \in S$ and $g \in G$.

$$\begin{aligned} r_1 f_1 + r_2 f_2 \mapsto sg(r_1 f_1 + r_2 f_2) &= sg(r_1 f_1) + sg(r_2 f_2) \text{ Since the group action is linear.} \\ &= r_1 sg(f_1) + r_2 sg(f_2) \text{ Since } r_1 \text{ and } r_2 \text{ are invariant.} \end{aligned}$$

Auslander's theorem can be translated into a statement about NCCRs:

Corollary 1.8.4. Let V be a k-vector space of dimension n and let G be a small subgroup of GL(V). Let S be as before, $R = S^G$ and $A = S * G$, then A is an NCCR for R.

Chapter 2

The Discriminant

Recall that the classical McKay correspondence (Theorem 1.5.11) only holds when the group G is a finite subgroup of $\mathrm{SL}(V)$, with $V \cong \mathbb{C}^2$, and thus G is not generated by pseudo reflections. Part of this thesis is to expand the McKay correspondence to a result that includes groups in the Shephard–Todd classification (Theorem 1.6.8). The immediate problem when G is generated by pseudo reflections is that $R = S^G$ is isomorphic to a polynomial ring and therefore not singular. To formulate a McKay correspondence for pseudo reflection groups we introduce a singularity in R called the discriminant Δ of a group action and a module $S/(z)$ over $R/(\Delta)$. The module $S/(z)$ will mimic the role that S played in part 4 of the classical McKay correspondence, Theorem 1.5.11. One problem is when the rank of G is greater than 2, $R/(\Delta)$ is not of finite CM type and so we can not appeal to Theorem 1.3.27. Most of this chapter is the context of [BFI20], with section 2.3 presenting new work in which we consider 1 dimensional representations of G and calculate which CM $R/(\Delta)$ modules they correspond to.

2.1 Discriminant of the group action

For the definition of the discriminant we will follow [OT92, Section 6]. Recall the context: Let G be a finite pseudo reflection group which acts on V, where V is a finite dimensional k-vector space of dimension n, where k is a field, $S = \mathrm{Sym}_k(V)$ and $R = S^G$.

Definition 2.1.1. Let Ref(G) be the set of reflections of G and $A(G)$ be the set of hyperplanes fixed by the reflections.

Definition 2.1.2. Let H be a hyperplane in $A(G)$, the *fixer group* $G(H)$ is the subgroup of G of all elements that fix H pointwise.

Remark 2.1.3. [OT92, Section 6] For a hyperplane H, the group $G(H)$ is cyclic.

Definition 2.1.4. Let $H \in A(G)$, the order of $G(H)$ is denoted e_H and α_H is the defining linear equation for H in S.

Example 2.1.5. Let $G = S_n$ acting on k^n. Then the transpositions (i,j) for $0 < i < j \le n$ are the generating reflections for G and (i,j) fixes the hyperplane $H_{(i,j)}$ given by $\alpha_{H_{(i,j)}} = (x_i - x_j)$.

Definition 2.1.6. Let χ be a linear character of G, i.e a one dimensional representation of G, and let

$$S_\chi^G = \{f \in S \mid gf = \chi(g)f \text{ for all } g \in G\}.$$

The elements of S_χ^G are called the relative invariants of χ.

The following theorem was first shown by Stanley in [Sta77, Theorem 3.1] for $k = \mathbb{C}$, but is given in [OT92, Theorem 6.37] for any field k, where Char(k) does not divide $|G|$.

Theorem 2.1.7. Let χ be a linear character of G. If Char(k) does not divide $|G|$ then $S_\chi^G = S^G d_\chi$ for some $d_\chi \in S$. and d_χ is called a generator for the relative invariants of χ

Definition 2.1.8. Let χ be a linear character of G. We call d_χ a generator for the relative invariants of χ if $S_\chi^G = S^G d_\chi$.

From now on Char(k) will always not divide $|G|$. We define two important polynomials for our study, see [OT92, Section 6] for a more detailed account of the following definitions and lemma.

Definition 2.1.9. Let G be a pseudo reflection group with the setup discussed at the start of the section. Then define the polynomials

$$z := \prod_{H \in A(G)} \alpha_H \in S$$

and

$$j := \prod_{H \in A(G)} \alpha_H^{e_H - 1} \in S.$$

The degree, d, of z is the number of hyperplanes of G and the degree, e, of j is the number of pseudo reflections in G, see [Bou02, Ch.5, Section 5, Prop. 6]. Note that $\mathbf{V}(z) = A(G)$, where $V(z)$ is the set of zeros of z.

Remark 2.1.10. The polynomials z and j can be defined up to scalar multiplication and are generators for the relative invariants of the linear characters det and $\det^{-1}$ respectively, i.e $S^G_{\det} \cong zR$ and $S^G_{\det^{-1}} \cong jR$.

Definition 2.1.11. Let G be a pseudo reflection group acting on V with basis $\{x_1, ..., x_n\}$ and $S = \mathrm{Sym}_k(V) = k[x_1, \ldots, x_n]$, then the *discriminant* δ of the group action is given by:

$$\delta(x_1, ..., x_n) = jz = \prod_{H \in A(G)} \alpha_H^{e_H} \in S.$$

Example 2.1.12. Let S_3 act on $\mathbb{C}^3$ with basis $\{x_1, x_2, x_3\}$ where the action permutes the basis elements, note that this is not the action of the standard representations as used in Example 1.5.4. Let $\sigma_1 = x_1 + x_2 + x_3, \sigma_2 = x_1x_2 + x_1x_3 + x_2x_3$ and $\sigma_3 = x_1x_2x_3$, then the invariant ring under this action, see Example 1.6.9, is

$$\mathbb{C}[x_1, x_2, x_3]^{S_3} = \mathbb{C}[\sigma_1, \sigma_2, \sigma_3]$$

and

$$z = j = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3),$$

$$\delta = (x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2.$$

Lemma 2.1.13. [OT92, Lemma 6.44] The discriminant of the group action is an invariant under the action of G, i.e $\delta \in R = S^G$.

Remark 2.1.14. Since the discriminant is an invariant it can be expressed as a polynomial in a set of basic invariants. Recall from Theorem 1.6.4 that basic invariants are not unique and so this expression will also not be unique. Fixing $\mathscr{F} = \{f_1,...,f_n\}$ a set of basic invariants, then δ is also a polynomial in $\mathbb{C}[\mathscr{F}]$. We define a polynomial also called the *discriminant* $\Delta_{\mathscr{F}}(X_1,...,X_n,)$ in indeterminants X_i depending on $\mathscr{F}$ such that $\Delta_{\mathscr{F}}(f_1,...,f_n,) = \delta(x_1,...,x_n)$. When the basis set of invariants is obvious, we shall just write Δ.

Example 2.1.12 (cont.)**.** The coordinate ring of the discriminant Δ in $R = S^G$ with a basic set of invariants $\sigma_1,\sigma_2,\sigma_3$ is:

$$R/(\Delta) = \mathbb{C}[\sigma_1,\sigma_2,\sigma_3]/\langle\sigma_1^2\sigma_2^2 - 4\sigma_1^3\sigma_3 - 4\sigma_2^3 + 18\sigma_1\sigma_2\sigma_3 - 27\sigma_3^2\rangle.$$

Remark 2.1.15. In the case of S_n we can set $x_1 + \cdots + x_n = 0$ and S_n still acts faithfully on $\mathbb{C}^{n-1}$. Note that the action is faithful and so S_n is still a complex reflection group. The discriminant in the S_3 case is then is a cusp singularity:

$$R/(\Delta) = \mathbb{C}[\sigma_2,\sigma_3]/\langle 4\sigma_2^3 + 27\sigma_3^2\rangle.$$

For a more interesting example, S_4 we refer to the beautiful survey [ROB], which also ties in solutions to polynomial equations.

2.2 Isotypical components

We want to have an analogue of Theorem 1.5.5 for $R/(\Delta)$ instead of R, the module $S/(z)$ will mimic S and we shall see that $S/(z)$ is a CM $R/(\Delta)$ module.

Let G be a finite pseudo reflection group acting on V a k-vector space, where Char(k) does not divide $|G|$ and let $f_1,...,f_n$ be a set of basic invariants under the action of G on $S = \mathrm{Sym}_k(V) \cong k[x_1,\ldots,x_n]$. The invariant ring of the action of G on S is $R = S^G = k[f_1,\ldots,f_n]$. We denote by irrep($G$) the set of irreducible representations of G and recall that irrep(G) is finite.

Definition 2.2.1. Define the homogenous ideal $(R_+) = (f_1,\dots,f_n)$ in S, then $S/(R_+) = S/(f_1,\dots,f_n)$ is the *coinvariant algebra*.

The structure of S as a graded free R module is given by Chevalley's Theorem [Che55]. Chevalley assumes that the field k is of characteristic 0 but the result holds more generally for Char(k) not dividing $|G|$, see [Bou81, Chapter 5, Section 2, Theorem 2]. As a graded R module S can be decomposed as

$$S \cong S/(R_+) \otimes_k R,$$

and as kG modules

$$S \cong kG \otimes_k R.$$

In particular, S is a free R module.

Example 2.2.2. Let $G = \mathbb{Z}_2$ be acting on $\mathbb{C}$ by $x \mapsto -x$, then the invariant ring is $R = \mathbb{C}[x^2]$ and $S = \mathbb{C}[x]$ decomposes as an R-module as

$$S \cong R \oplus xR$$

and $S/(R_+) = \text{span}\{[1],[x]\}$ where [1], $[x]$ are the equivalence classes of 1 and x.

Since (R_+) is a homogenous ideal of S, the coinvariant algebra $S/(R_+)$ can be viewed as a $\mathbb{Z}$ graded vector space over k with the $\mathbb{Z}$ integer grading inherited from S. We forgo writing the equivalence class notation from now on and pick a representative of lowest degree. We will be decomposing $S/(R_+)$ via isotypical (or canonical) decomposition via the action of G. The following can be found in more detail in [Ser77, Section 2.6]. Stanley also studied the isotypical decomposition of S in [Sta79], with more of a view towards the structure of the components as modules over R.

Since G is a finite group, it has finitely many irreducible representations (up to isomorphism) and finitely many distinct characters determining these irreducible representations. Let $\chi_1,\dots,\chi_r$ be the list of characters that determine the irreducible representations $V_1,\dots,V_r$ respectively. Consider a decomposition of $S/(R_+) = \bigoplus_{i=0}^m T_i$ into irreducible representations T_i. Let W_i be the direct sum of those T_i which are isomorphic to V_i. Then

$$S/(R_+) = W_1 \oplus \cdots \oplus W_r$$

is the canonical decomposition of $S/(R_+)$, and is a unique decomposition [Ser77, Theorem 8]. We call the $W_i's$ the *isotypical* components of $S/(R_+)$ and the isotypical components $S^G_{V_i}$ of S are $W_i \otimes R$, so by Chevalley's Theorem

$$S \cong \bigoplus_{V_i \in \mathrm{irrep}(G)} S^G_{V_i}.$$

Remark 2.2.3. If Char(k) does not divide $|G|$ the isotypical component $S^G_{V_i}$ of irreducible representation is normally written S^G_χ where χ is the character of V_i.

Lemma 2.2.4. The coinvariant algebra $S/(R_+)$ is a representation of G and is isomorphic to the regular representation. That is, let $\{V_1, \dots V_r\}$ be a list of all irreducible representations of G, then

$$S/(R_+) \cong \bigoplus_{0 \le i \le r} V_i^{\oplus \dim V_i}.$$

Example 2.2.5. Consider S_3 acting as a complex reflection group on $\mathbb{C}^2$ via the standard representation. The group S_3 has 3 irreducible representations, V_1, V_2 which are 1 dimensional and V_3 which is 2 dimensional. The coinvariant algebra decomposes as

$$S/(R_+) \cong V_1 \oplus V_2 \oplus V_3 \oplus V_3$$

Example 2.2.6. [OT92, Example 6.38] Let triv denote the trivial character of G, then $S^G_{\mathrm{triv}} = S^G$.

Since the coinvariant algebra $S/(R_+)$ inherits the grading from $S = k[x_1, \dots x_n]$ where $\deg(x_i) = 1$ we can calculate the degrees in which the irreducible representations appear. These can be computed by *fake degree* polynomials. An informative resource for fake degree polynomials is the website [Bel] authored by Gwyn Bellamy where the fake degrees can be found for all the exceptional groups from Theorem 1.6.8 and selected ones from the family $G(m, p, n)$.

Example 2.2.7. The isotypical component $S^G_{\det}$ (resp. $S^G_{\det^{-1}}$) of $S/(R_+)$ of det (resp. $\det^{-1}$) is in degree given by the degree of z (resp. j), which recalling Definition 2.1.9 is d, the number of hyperplanes (resp e, the number of pseudo reflections).

Example 2.2.8. Consider the group S_3 acting on $S = \mathbb{C}[x_1, x_2, x_3]$. There are 3 irreducible representations of S_3: triv, det and std, see Example 1.5.4. Note that in this example, since S_3 is a true reflection group, $z = j$ and $\det = \det^{-1}$. Then we have the decomposition:

$$S/(R_+) \cong \mathrm{triv}(0) \oplus \det(-3) \oplus (\mathrm{std}(-1) \oplus \mathrm{std}(-2)),$$

$$S \cong (\mathrm{triv}(0) \otimes R) \oplus (\det(-3) \otimes R) \oplus (\mathrm{std}(-1) \otimes R) \oplus (\mathrm{std}(-2) \otimes R).$$

Where $(-)$ denotes in which degree each module appears.

The idea is to create a matrix factorization for the discriminant using the polynomials z and j (see Definition 2.1.9). Recall that the discriminant is $\Delta = zj \in S$ and let d (resp e) be the degree of z (resp j). Since S is a free R module of rank $|G|$ then, after choosing a basis for S as a free R module, the maps (z, j) give an object in $MF_R(\Delta)$ given by;

$$\begin{array}{ccccc} S(-d-e) & \xrightarrow{z} & S(-e) & \xrightarrow{j} & S \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ R^{|G|} & \xrightarrow{z} & R^{|G|} & \xrightarrow{j} & R^{|G|} \end{array}$$

Remark 2.2.9. The $(-)$ denotes the degree shift to ensure that z and j are graded morphisms, that is they are concentrated in degree 0.

Lemma 2.2.10. The R modules $S/(z)$ and $S/(j)$ are CM $R/(\Delta)$ modules.

Proof. This follows from Eisenbud's matrix factorization Theorem 1.2.12. □

The modules $S/(z)$ and $S/(j)$ will not be indecomposable and so we will use the isotypical decomposition of $S/(R_+)$ to study $S/(z)$ and $S/(j)$. Consider the short exact sequence for the multiplication of z on S (as a free R module).

$$0 \longrightarrow S(-d) \xrightarrow{z} S \longrightarrow S/(z) \longrightarrow 0.$$

Then using the decomposition into isotypical components:

$$0 \longrightarrow \bigoplus_{\chi\in\mathrm{irrep}(G)} S_\chi^G(-d) \xrightarrow{z} \bigoplus_{\chi\in\mathrm{irrep}} S_\chi^G \longrightarrow (\bigoplus_{\chi\in\mathrm{irrep}(G)} S_\chi^G)/(z) \longrightarrow 0$$

and since z is a relative invariant for the representation det for each $\chi \in \mathrm{irrep}(G)$ we get:

$$0 \longrightarrow S_\chi^G(-d) \xrightarrow{z|_\chi} S_{\chi\otimes\det}^G \longrightarrow (S_{\chi\otimes\det}^G)/(z) \longrightarrow 0$$

where $z|_\chi$ is the restriction of left multiplication of z to S_χ^G.

Then we consider in which of the different components the restrictions map of z are defined. This gives us, for every $\chi \in \mathrm{irrep}(G)$ after choosing a basis for S_χ^G and $S_{\chi\otimes\det}^G$, the following matrix factorization of Δ:

$$S_{\chi\otimes\det}^G(-d-e) \xrightarrow{j|_{\chi\otimes\det}} S_\chi^G(-d) \xrightarrow{z|_\chi} S_{\chi\otimes\det}^G$$

Denote by e_χ the dimension of χ, and note that $e_{\chi\otimes\det} = e_\chi$. Since by Lemma 2.2.4, $S/(R_+)$ is isomorphic to the regular representation this implies that $S_\chi^G \cong R^{e_\chi^2}$.

$$R^{e_\chi^2} \xrightarrow{j|_{\chi\otimes\det}} R^{e_\chi^2} \xrightarrow{z|_\chi} R^{e_\chi^2}.$$

Lemma 2.2.11. After choosing a basis for S as a free R module, the matrix factorization

$$\begin{array}{ccccc} S(-d-e) & \xrightarrow{z} & S(-e) & \xrightarrow{j} & S \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ R^{|G|} & \xrightarrow{z} & R^{|G|} & \xrightarrow{j} & R^{|G|} \end{array}$$

of Δ is not a reduced matrix factorization.

Proof. Consider the trivial representation, then using Theorem 2.1.7 the matrix factorization of Δ,

$$S_{\det}^G(-d-e) \xrightarrow{j|_{\det}} S_{\mathrm{triv}}^G(-d) \xrightarrow{z|_{\mathrm{triv}}} S_{\det}^G$$

can be written as

$$zR \xrightarrow{j|_{\det}} R \xrightarrow{z|_{\mathrm{triv}}} zR$$

since z is a generator for the relative invariants of det. The matrix factorization of Δ above is given by the matrices $([\Delta],[1])$. This shows that $(z|_{\mathrm{triv}}, j|_{\det})$ is not reduced since 1 is a unit of R. □

Definition 2.2.12. Let χ be an irreducible representation then define the modules

$$M_\chi := \mathrm{Coker}(z|_\chi) = S^G_{\chi\otimes\det}/(z) \qquad \text{and} \qquad N_\chi := \mathrm{Coker}(j|_\chi) = S^G_{\chi\otimes\det^{-1}}/(j).$$

Lemma 2.2.13. For an irreducible representation $\chi \in \mathrm{irrep}(G)$ the module M_χ is CM over $R/(\Delta)$ and as a CM module $S/(z)$ decomposes as

$$S/(z) \cong \bigoplus_{\chi\in\mathrm{irrep}(G)} M_\chi$$

Proof. From the above discussion, M_χ are Cohen-Macaulay over $R/(\Delta)$ by the Coker functor in Eisenbud's matrix factorization theorem, Theorem 1.2.12. □

Since $(z|_\chi, j|_{\chi\otimes\det})$ are matrix factorizations for all $\chi \in \mathrm{irrep}(G)$, we get the exact sequence:

$$0 \longrightarrow \mathrm{Coker}(j|_{\chi\otimes\det}, z|_\chi) \longrightarrow R^{e_\chi^2} \longrightarrow \mathrm{Coker}(z|_\chi, j|_{\chi\otimes\det}) \longrightarrow 0.$$

Thus $\mathrm{syz}^1_R M_\chi \oplus F \cong N_{\chi\otimes\det}$, where F is a free module of $R/(\Delta)$.

Remark 2.2.14. By Eisenbud's matrix factorization theorem, the above discussion in terms of matrix factorizations is, the matrix factorization (z,j) decomposes as

$$(z,j) \cong \bigoplus_{\chi\in\mathrm{irrep}(G)} (z|_\chi, j|_\chi).$$

Example 2.2.15. Let G be S_3 then we get the following matrix factorizations:

$$\mathrm{triv}(-6)\otimes R \xrightarrow{\quad z|_{\mathrm{triv}} \quad} \det(-3)\otimes R \xrightarrow{\quad j|_{\det} \quad} \mathrm{triv}(0)\otimes R,$$

$$\det(-6)\otimes R \xrightarrow{\quad z|_{\det} \quad} \mathrm{triv}(-3)\otimes R \xrightarrow{\quad j|_{\mathrm{triv}} \quad} \det(0)\otimes R,$$

$$(\mathrm{std}(-7)\otimes R)\oplus(\mathrm{std}(-8)\otimes R) \xrightarrow{z|_{\mathrm{std}}} (\mathrm{std}(-4)\otimes R)\oplus(\mathrm{std}(-5)\otimes R) \xrightarrow{j_{\mathrm{std}}} (\mathrm{std}(-1)\otimes R)\oplus(\mathrm{std}(-2)\otimes R).$$

So

$$S/(z) \cong \mathrm{Coker}(z|_{\mathrm{triv}}, j|_{\mathrm{det}}) \oplus \mathrm{Coker}(z|_{\mathrm{det}}, j|_{\mathrm{triv}}) \oplus \mathrm{Coker}(z|_{\mathrm{std}}, j|_{\mathrm{std}}).$$

Since S_3 is a true reflection group we have $S/(z) = S/(j)$.

Lemma 2.2.16. Let $R/(\Delta)$ be of finite CM-type. If $S/(z)$ is a representation generator (recall Definition 1.1.10) for $R/(\Delta)$, then $S/(j)$ is also a representation generator.

Proof. Let M be an indecomposable CM module over $R/(\Delta)$, it corresponds to a reduced matrix factorization of Δ. Now $S/(z)$ is a representation generator for $CM(R/(\Delta))$ and $\mathrm{syz}^1_R M$ is a CM $R/(\Delta)$ module it is contained in $\mathrm{add}_{R/(\Delta)} S/(z)$. In particular there exists $\chi \in \mathrm{irrep}(G)$ such that $\mathrm{syz}^1_R M \in \mathrm{add}(M_\chi)$. Thus $\mathrm{syz}^1_R \mathrm{syz}^1_R M \cong M \in \mathrm{add}(N_{\chi\otimes\mathrm{det}})$ see Lemma 1.2.15. Hence $M \in \mathrm{add}(S/(j))$. □

2.3 Linear Characters of reflection groups

We continue with the same setup. Let G be a finite pseudo reflection group acting on V a k-vector space, where $\mathrm{Char}(k)$ does not divide $|G|$, $S = \mathrm{Sym}_k(V) \cong k[x_1, \dots, x_n]$, $R = S^G$ and Δ the discriminant of G. The linear characters are the most straightforward to investigate due to the following lemma:

Lemma 2.3.1. For χ a linear character, the module M_χ (recall Definition 2.2.12) can only be the zero module, $R/(\Delta)$ or a reduced component of $R/(\Delta)$.

Proof. This is immediate from considering the dimension of S^G_χ of S of type χ. Using Theorem 2.1.7 the isotypical component S^G_χ is of rank one as a free R module and thus the matrix factorization $(z|_\chi, j|_{\chi\otimes\mathrm{det}})$ is given by:

$$R \xrightarrow{c} R \xrightarrow{d} R$$

where c and d are invariants such that $cd = \Delta$. Since Δ is reduced, both c, d will either be reduced or the identity in R. When d is the identity then M_χ is the zero module and when c is the identity $M_\chi = R/(\Delta)$. When neither d or c is the identity, d is reduced and $M_\chi = R/(d)$ is a coordinate ring of a reduced component of $R/(\Delta)$. □

We want to describe how to construct the linear characters for reflection groups. See [Bro10] for a more in depth review of the following discussion. Recall from Definition 2.1.1 that $\mathrm{Ref}(G)$ denotes the set of reflections of G and $\alpha_H \in S$ was the defining equation of a reflecting hyperplane H, that is for $H \in A(G)$, $H = \ker(\alpha_H)$. Let $G(H)$ be the subgroup of G which fixes H and $e_H := |G(H)|$. Denote by $A(G)/G$ the set of orbits of hyperplanes under G, let $\mathfrak{O}$ be an element of $A(G)/G$.

Definition 2.3.2. With the set up above define

$$j_{\mathfrak{O}} := \prod_{H \in \mathfrak{O}} \alpha_H$$

and a linear character $\theta_{\mathfrak{O}} \in \mathrm{Hom}(G, k^{\times})$ such that

$$g(j_{\mathfrak{O}}) = \theta_{\mathfrak{O}}(g)(j_{\mathfrak{O}}) \text{ for all g} \in G.$$

The character $\theta_{\mathfrak{O}}$ then has the following property. Let $s \in \mathrm{Ref}(G)$ then

$$\theta_{\mathfrak{O}}(s) = \begin{cases} \det(s) & \text{if s} \in G(H) \text{ for some } H \in \mathfrak{O}, \\ 1 & \text{otherwise} \end{cases}$$

Note, $\det(s)$ is the determinant of the action of s on V. Thus, since G is generated by reflections, if we take $z := \prod_{H \in A(G)} \alpha_H$ we get $g(z) = \det(g)(z)$ for all $g \in G$. To describe linear characters of G it is enough to restrict to the fixer groups $G(H)$ of hyperplanes H of G.

Definition 2.3.3. Let G be a group, $K \trianglelefteq G$ and a representation W of G. The restriction of W to K, denoted Res^G_K is the representation given by;

$$W|_K = W(h), \text{where } h \in K.$$

Theorem 2.3.4. [Bro10, Theorem 4.18] Let $\theta \in \mathrm{Hom}(G, k^{\times})$ be a linear character and for a hyperplane H, denote by $\mathfrak{O}$ the orbit of H. Then there is a unique integer $m_{\mathfrak{O}}(\theta)$ such that:

$$\mathrm{Res}^G_{G(H)} \theta = \det{}^{m_{\mathfrak{O}}(\theta)} \text{ with the condition } 0 \le m_{\mathfrak{O}}(\theta) < |G(H)|.$$

That is for all $g \in G(H)$, $\mathrm{Res}^G_{G(H)} \theta(g) = \det{}^{m_{\mathfrak{O}}(\theta)}(g)$.

Proof. The proof of this theorem is essentially the fact that det generates $\mathrm{Hom}(G(H), k^{\times})$, since $G(H)$ is a cyclic group. □

Remark 2.3.5. [OT92, Lemma 6.34] The integer $m_{\mathfrak{O}}(\theta)$ is the same for all hyperplanes in an orbit $\mathfrak{O}$.

Definition 2.3.6. Let θ be a linear character of G. We define the polynomial

$$j_\theta := \prod_{\mathfrak{O} \in A(G)/G} j_{\mathfrak{O}}^{m_{\mathfrak{O}}(\theta)} \in S$$

with $m_{\mathfrak{O}}(\theta)$ satisfying Theorem 2.3.4 for each orbit $\mathfrak{O} \in A(G)/G$. Recall, for a linear character θ, we call j_θ the generator for the relative invariants of θ.

Corollary 2.3.7. [OT92, Lemma 6.37]. Let θ be a linear character of G then the isotypical component of θ is $S_\theta^G = R j_\theta$.

Definition 2.3.8. Let $\mathfrak{O}$ be an orbit of hyperplanes of G then the $\mathfrak{O}$-component of Δ denoted $\Delta_{\mathfrak{O}}$ is defined by

$$\Delta_{\mathfrak{O}} := j_{\mathfrak{O}}^{|G(H)|} = \prod_{H \in \mathfrak{O}} \alpha_H^{|G(H)|}.$$

Lemma 2.3.9. Let $\mathfrak{O}$ be an orbit of hyperplanes of G then $\Delta_{\mathfrak{O}}$ is invariant under the action of G.

Proof. Let $\mathfrak{O}$ be an orbit of hyperplanes of G, and from Definition 2.3.2, $g(j_{\mathfrak{O}}) = \theta_{\mathfrak{O}}(g)(j_{\mathfrak{O}})$ for all g $\in G$. Now $g(j_{\mathfrak{O}}^{|G(H)|}) = g(j_{\mathfrak{O}})^{|G(H)|} = \theta_{\mathfrak{O}}(g)^{|G(H)|}(j_{\mathfrak{O}}) = (j_{\mathfrak{O}}^{|G(H)}|)$ since for $s \in \mathrm{Ref}(G(H))$, $\det(s)^{|G(H)|} = 1$. □

Remark 2.3.10. If we consider all the orbits of hyperplanes we obtain

$$\prod_{\mathfrak{O} \in A(G)/G} \Delta_{\mathfrak{O}} = \Delta.$$

The following investigates which isotypical components give $\Delta_{\mathfrak{O}}$, for each orbit $\mathfrak{O}$, in the decomposition of $S/(z)$.

Lemma 2.3.11. Let $\theta \in \mathrm{Hom}(G, k^{\times})$ then $m_{\mathfrak{O}}(\theta \otimes \det) = m_{\mathfrak{O}}(\theta) + 1 \bmod e_H$ for $H \in \mathfrak{O}$.

Proof. Note that the action $-\otimes\det$ on the irreducible representations of G sends linear characters to linear characters. The lemma then follows from the observation that $m_{\mathfrak{O}}(\det) = 1$ for all orbits $\mathfrak{O}$. □

We are interested in viewing (z, j) as a matrix factorization of the discriminant, and we can use Lemma 2.3.11 to find the parts of the matrix factorization on the isotypical components for the linear characters. Recall that $z|_\theta$ denotes the map restricted on the isotypical component of type θ of S.

$$S^G_\theta \xrightarrow{z|_\theta} S^G_{\theta\otimes\det}.$$

Which, using Theorem 2.3.4 yields part of the matrix factorization

$$Rj_\theta \xrightarrow{z|_\theta} Rj_{\theta\otimes\det}.$$

Lemma 2.3.12. Let $\mathfrak{O}$ be an orbit of hyperplanes of G an H a hyperplane in $\mathfrak{O}$ then the cokernel of the map $z|_{\theta_{\mathfrak{O}}^{e_H-1}}$ is $R/\Delta_{\mathfrak{O}}$.

Proof. The relative invariants of $\theta_{\mathfrak{O}}^{e_H-1}$ and $\theta_{\mathfrak{O}}^{e_H-1}\otimes\det$ are given by:

$$j_{\theta_{\mathfrak{O}}^{e_H-1}} = j_{\mathfrak{O}}^{e_H-1} = \prod_{H\in\mathfrak{O}} \alpha_H^{e_H-1} \qquad \text{and} \qquad j_{\theta_{\mathfrak{O}}^{e_H-1}\otimes\det} = \prod_{\mathfrak{q}\in(A(G)/G)\setminus\{\mathfrak{O}\}} j_{\mathfrak{q}}$$

this means that $j_{\theta_{\mathfrak{O}}^{e_H-1}\otimes\det}$ contains all the hyperplanes that are not in the orbit $\mathfrak{O}$. Thus the following holds,

$$zj_{\theta_{\mathfrak{O}}^{e_H-1}} = j_{\mathfrak{O}}^{e_H} \prod_{\mathfrak{q}\in(A(G))/G)\setminus\mathfrak{O}} \alpha_{\mathfrak{q}} = \Delta_{\mathfrak{O}} j_{\theta_{\mathfrak{O}}^{e_H-1}\otimes\det}.$$

□

Theorem 2.3.13. For all orbits $\mathfrak{O}$, $R/(\Delta_{\mathfrak{O}})$ is a direct summand of $S/(z)$ as a $R/(\Delta)$ module.

Proof. Since $S \cong \bigoplus_{\chi\in\mathrm{irrep}(G)} S^G_\chi$ and (z, j) is a matrix factorization for Δ, the discussion above shows the map z on the isotypical component $S^G_{\theta_{\mathfrak{O}}^{e_H-1}}$ gives the component $\Delta_{\mathfrak{O}}$. □

Theorem 2.3.14. Let $\{\mathfrak{O}_1,\ldots,\mathfrak{O}_d\}$ be a set of orbits of hyperplanes of G and H_i an element of $\mathfrak{O}_i$, then the cokernel of the map $z|_{\theta_{\mathfrak{O}_1}^{e_{H_1}-1}\otimes\cdots\otimes\theta_{\mathfrak{O}_d}^{e_{H_d}-1}}$ is $R/(\Delta_{\mathfrak{O}_1}\cdots\Delta_{\mathfrak{O}_d})$.

Proof. The relative invariants of $\theta_{\mathfrak{O}_1}^{e_{H_1}-1}\otimes\cdots\otimes\theta_{\mathfrak{O}_d}^{e_{H_d}-1}$ and $\theta_{\mathfrak{O}_1}^{e_{H_1}-1}\otimes\cdots\otimes\theta_{\mathfrak{O}_d}^{e_{H_d}-1}\otimes\det$ are given by:

$$j_{\theta_{\mathfrak{O}_1}^{e_{H_1}-1}\otimes\cdots\otimes\theta_{\mathfrak{O}_d}^{e_{H_d}-1}} = j_{\mathfrak{O}_1}^{e_{H_1}-1}\cdots j_{\mathfrak{O}_d}^{e_{H_d}-1} = \prod_{i=1}^{d}\prod_{H\in\mathfrak{O}_i}\alpha_H^{e_H-1}$$

and

$$j_{\theta_{\mathfrak{O}_1}^{e_{H_1}-1}\otimes\cdots\otimes\theta_{\mathfrak{O}_d}^{e_{H_d}-1}\otimes\det} = \prod_{\mathfrak{q}\in(A(G)/G)\setminus\{\mathfrak{O}_1\cdots\mathfrak{O}_d\}} j_{\mathfrak{q}}.$$

This means that $j_{\theta_{\mathfrak{O}_1}^{e_{H_1}-1}\otimes\cdots\otimes\theta_{\mathfrak{O}_d}^{e_{H_d}-1}\otimes\det}$ contains all the hyperplanes that are not in any of the orbits $\{\mathfrak{O}_1\ldots\mathfrak{O}_d\}$. Thus the following holds,

$$\begin{aligned} z j_{\theta_{\mathfrak{O}_1}^{e_{H_1}-1}\otimes\cdots\otimes\theta_{\mathfrak{O}_d}^{e_{H_d}-1}} &= j_{\mathfrak{O}_1}^{e_{H_1}-1}\cdots j_{\mathfrak{O}_d}^{e_{H_d}-1}\prod_{\mathfrak{q}\in(A(G)/G)\setminus\{\mathfrak{O}_1\cdots\mathfrak{O}_d\}} j_{\mathfrak{q}} \\ &= \Delta_{\mathfrak{O}_1}\cdots\Delta_{\mathfrak{O}_d} j_{\theta_{\mathfrak{O}_1}^{e_{H_1}-1}\otimes\cdots\otimes\theta_{\mathfrak{O}_d}^{e_{H_d}-1}\otimes\det}. \end{aligned} \tag{2.1}$$

□

Chapter 3

Decomposition of $S/(z)$ for $G(m,p,2)$

This chapter will follow and showcase the results shown in [May21] with more discussion and detail.

We show that for the family of complex reflection groups $G = G(m,p,2)$, see Definition 3.1.1, in the context $k = \mathbb{C}$ appearing in the Shephard–Todd classification, $S/(z)$ contains a at least one copy of each isomorphism class of Cohen-Macaulay modules over the discriminant as a summand. A corollary of this is that $\mathrm{End}_{R/(\Delta)}(S/(z))$ is a non-commutative resolution (recall Definition 1.3.24) for the curve singularity $R/(\Delta)$. This furthers the work of Buchweitz, Faber and Ingalls who showed that this result holds for any true reflection group. In particular, we give a full decomposition of the matrix factorization (z, j) constructed in Section 2.2 of the discriminant Δ of G including for each irreducible representation of G a corresponding CM $R/(\Delta)$ module. The main theorem of this chapter is the following:

Theorem 3.0.1. (Thm 3.4.22, Thm 3.7.3, Thm 3.7.7, Thm 3.7.12) Let $G = G(m,p,2)$, then all non-isomorphic CM modules of R/Δ appear at least once in the decomposition of $S/(z)$ as CM modules over $R/(\Delta)$.

Furthermore, we also determine a precise decomposition of $S/(z)$ into CM modules over $R/(\Delta)$.

We first introduce the groups $G(m,p,2)$ and how their irreducible representations are formed. To calculate the map of z on the isotypical components we need to describe a basis for each isotypical component of $S/(R_+)$. Then evaluating the map on this basis will give us for each irreducible representation a corresponding CM module.

3.1 The groups $G(m,p,n)$

The representation theory of the group $G(m,p,n)$ is an extension of the representation theory of the symmetric group. In particular we can calculate a basis for the isotypical components of the coinvariant algebra, which in turn allows calculation of the modules M_χ for $\chi \in \mathrm{irrep}(G(m,p,2))$ from Section 2.2.

Definition 3.1.1. Let m,p,n be positive integers such that p divides m and let ξ_m be a primitive m^{th} root of unity. Then the complex reflection group $G(m,p,n)$ acting on a vector space $V \cong \mathbb{C}^n$ is the group of matrices of the form PD where P is a permutation matrix and D is a diagonal matrix whose entries are powers of ξ_m and $\det(D)^{\frac{m}{p}} = 1$.

For the group $G(m,p,n)$, where m,p,n are positive integers such that p divides m, we also set $q := \frac{m}{p}$. Generators and relations are given in [AK94]. The group $G(m,1,n)$ is generated by the matrices:

$$s_1 = \mathrm{diag}\{\xi_m, 1, \dots, 1\}$$

and s_i for $2 \le i \le n$, where s_i is the matrix that permutes the basis vectors in position $i-1$ and i.

Remark 3.1.2. As an abstract group $G(m,1,n)$ is identified with the wreath product $(\mathbb{Z}/m\mathbb{Z}) \wr S_n$ and $G(m,p,n)$ are normal subgroups of $G(m,1,n)$, where p must divide m.

Example 3.1.3. The group $G(1,1,n)$ is isomorphic to the symmetric group S_n and $G(m,1,1)$ is isomorphic to the cyclic group $\mathbb{Z}_m$.

Example 3.1.4. Let $m \ge 2$, the generators of the group $G(m,1,2)$ acting on $\mathbb{C}[x,y]$ are given by the matrices

$$s_1 = \begin{bmatrix} \xi_m & 0 \\ 0 & 1 \end{bmatrix} \text{ and } s_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

A basic set of invariants of this action is calculated to be: $\sigma_1 = (xy)^m$ and $\sigma_2 = x^m + y^m$, see [Ban76]. The reflections

$$s_1 = \begin{bmatrix} \xi_m & 0 \\ 0 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 & 0 \\ 0 & \xi_m \end{bmatrix}$$

of $G(m,1,2)$ fix the hyperplanes defined by $x = 0$ and $y = 0$ respectively. The reflection

$$s_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

of $G(m,1,2)$ fixes the hyperplane defined by $x = y$. The defining polynomials of the all hyperplanes fixed by reflections of $G(m,1,2)$ are, see [OT92, Section 6.4]:

$$x, y, (x - \xi_m^i y) \qquad \text{for } i \in \{0, 1, ..., m-1\} .$$

Thus the polynomial defining the hyperplane arrangement, see Definition 2.1.9, for the group $G(m,1,2)$ is:

$$z = xy(x^m - y^m).$$

The hyperplanes $H_x := \ker(x)$ and $H_y := \ker(y)$ are in the same G-orbit, seen by applying s_2 to H_x. The other hyperplanes are in a distinct orbit $\mathfrak{q}$. The Jacobian and discriminant are given as:

$$j = (xy)^{m-1}(x^m - y^m),$$

$$\Delta = (xy)^m (x^m - y^m)^2 = \sigma_1(\sigma_2^2 - 4\sigma_1).$$

Example 3.1.5. [Ban76] [OT92, Section 6.4]. For the subgroups $G(m,p,2)$, where $p \neq m$ and $q = \frac{m}{p}$ the invariants are $\sigma_1 = (xy)^q$ and $\sigma_2 = x^m - y^m$ with:

$$z = xy(x^m - y^m),$$

$$j = (xy)^{q-1}(x^m - y^m),$$

$$\Delta = (xy)^q (x^m - y^m)^2 = \sigma_1(\sigma_2^2 - 4\sigma_1^p).$$

Example 3.1.6. [Ban76] [OT92, Section 6.4]. When $p = m$ the invariants are $\sigma_1 = xy$ and $\sigma_2 = x^m + y^m$ with:

$$z = (x^m - y^m),$$

$$j = (x^m - y^m),$$

$$\Delta = (x^m - y^m)^2 = \sigma_2^2 - 4\sigma_1^m.$$

3.2 The linear characters for $G(m,p,2)$

For the examples 3.1.4, 3.1.5 and 3.1.6 the invariant ring is $R = \mathbb{C}[\sigma_1,\sigma_2]$ for their respective basic invariants, σ_1 and σ_2. Using Section 2.3 we can calculate the modules corresponding to the isotypical components of the linear characters. We consider the groups case by case. The group $G(m,1,2)$ $m \geq 2$ has two orbits of hyperplanes. The hyperplanes given by x, y are in the same orbit $\mathfrak{O}$. The rest of the hyperplanes are in the other, $\mathfrak{q}$. Then in the notation of Section 2.3:

$$\Delta_{\mathfrak{O}} = (xy)^m \qquad \text{and} \qquad \Delta_{\mathfrak{q}} = (x^m - y^m)^2.$$

Since these are invariant under the action of G, they can be expressed in terms of the basic invariants σ_1, σ_2 , so we obtain the reduced components of $\Delta \in R$

$$\Delta_{\mathfrak{O}} = \sigma_1 \qquad \text{and} \qquad \Delta_{\mathfrak{q}} = \sigma_2^2 - 4\sigma_1.$$

Lemma 3.2.1. Consider $G(m,1,2)$ with $m \geq 2$, then the isotypical components of $S/(z)$ corresponding to the linear characters $\theta_{\mathfrak{O}}^{\otimes(e_{H_x}-1)}$, $\theta_{\mathfrak{q}}^{\otimes(e_{H_{x-y}}-1)}$ are

$$M_{\theta_{\mathfrak{O}}^{\otimes(e_{H_x}-1)}} = M_{\theta_{\mathfrak{O}}^{\otimes(m-1)}} = R/(\sigma_1) \qquad \text{and} \qquad M_{\theta_{\mathfrak{q}}^{\otimes(e_{H_{x-y}}-1)}} = M_{\theta_{\mathfrak{q}}} = R/(\sigma_2^2 - 4\sigma_1).$$

Proof. This follows from Lemma 2.3.12, Theorem 2.3.13 and Theorem 2.3.14. □

Next we consider the case when $p \neq 1$: the subgroups $G(m,p,2) \trianglelefteq G(m,1,2)$ fall into 2 cases. When p is odd, it is similar to the above, there are 2 orbits $\mathfrak{O}$ and $\mathfrak{q}$ where

$\Delta_{\mathfrak{O}} = (xy)^q = \sigma_1$ and $\Delta_{\mathfrak{q}} = (x^m - y^m)^2 = x^{2m} + y^{2m} - 2x^m y^m = \sigma_2^2 - 4\sigma_1^p$. Thus we obtain a similar lemma:

Lemma 3.2.2. Consider $G(m,p,2)$, with $p \neq 1$ and p odd. Then the isotypical components of $S/(z)$ corresponding to the linear characters $\theta_{\mathfrak{O}}^{\otimes(e_{H_x}-1)}$, $\theta_{\mathfrak{q}}^{\otimes(e_{H_{x-y}}-1)}$ are

$$M_{\theta_{\mathfrak{O}}^{\otimes(e_{H_x}-1)}} = M_{\theta_{\mathfrak{O}}^{\otimes(q-1)}} = R/(\sigma_1) \qquad \text{and} \qquad M_{\theta_{\mathfrak{q}}^{\otimes(e_{H_{x-y}}-1)}} = M_{\theta_{\mathfrak{q}}} = R/(\sigma_2^2 - 4\sigma_1^p).$$

Proof. This follows from Lemma 2.3.12 and Theorem 2.3.13 and Theorem 2.3.14. □

When p is even there are 3 orbits $\mathfrak{O}_1, \mathfrak{O}_2$ and $\mathfrak{O}_3$, where $\Delta_{\mathfrak{O}_1} = \sigma_1$, $\Delta_{\mathfrak{O}_2} = \sigma_2 - 2\sigma_1^{\frac{p}{2}}$, $\Delta_{\mathfrak{O}_3} = \sigma_2 + 2\sigma_1^{\frac{p}{2}}$. With the notation from Section 2.3 we can combine the different orbits and obtain the following lemma.

Lemma 3.2.3. Consider $G(m,p,2)$, with $p \neq 1$ and p even. Then the isotypical components of $S/(z)$ corresponding to the linear characters $\theta_{\mathfrak{O}_i}^{\otimes(e_{H_{\mathfrak{O}_i}}-1)} \otimes \theta_{\mathfrak{O}_j}^{\otimes(e_{H_{\mathfrak{O}_j}}-1)}$, for $1 \leq i \neq j \leq 3$, are

$$M_{\theta_{\mathfrak{O}_i}^{\otimes(e_{H_{\mathfrak{O}_i}}-1)} \otimes \theta_{\mathfrak{O}_j}^{\otimes(e_{H_{\mathfrak{O}_j}}-1)}} = R/(\Delta_{\mathfrak{O}_i}\Delta_{\mathfrak{O}_j}).$$

Where $H_{\mathfrak{O}_i}$ is a hyperplane in the orbit $\mathfrak{O}_i$. The orders $e_{H_{\mathfrak{O}_i}}$ of the cyclic groups $G(H_{\mathfrak{O}_i})$ are as follows: $e_{H_{\mathfrak{O}_1}} = q$, $e_{H_{\mathfrak{O}_2}} = e_{H_{\mathfrak{O}_3}} = 1$.

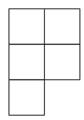
Proof. This follows from Lemma 2.3.12 and Theorem 2.3.13 and Theorem 2.3.14. □

3.3 Representation Theory of $G(m,1,n)$

Since the groups $G(m,1,n)$ are generalisations of the symmetric group, one would expect their representation theory can be described in a similar way. We briefly introduce Young Tableau and describe how they can be used to build the representation theory of $G(m,1,n)$. A standard text for Young Tableau is [Ful96, Chapter 7].

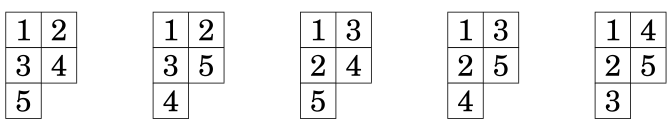
Consider $n \in \mathbb{N} \setminus \{0\}$. Let λ be a partition of n, i.e $\lambda = (\lambda_1, ..., \lambda_k)$, such that $k \leq n$, $0 < \lambda_{i+1} \leq \lambda_i$ and $\sum_i n_i = n$. A partition can also be represented as a *Young diagram*, which is constructed in the following way: given a partition λ of n, the Young diagram associated to λ is a collection of left justified rows of squares called cells. Enumerate the rows from 0 to $k-1$, top to bottom, the number of cells in row i is λ_i. The partitions uniquely determine the Young diagram so we use the same notation λ for the partition and the Young diagram. We call a Young diagram associated to a partition of n, a Young diagram of size n

Example 3.3.1. Let $n = 5$ and $\lambda = (2,2,1)$ then the corresponding Young diagram is



Definition 3.3.2. A *Young tableau* T is a Young diagram λ of size n, where each cell contains a number from 1 to n such that each number 1 to n appears only once, we call T a Young tableau of *shape* λ . A Young tableau is called *standard* when the sequence of entries in the rows and columns are strictly increasing.

Example 3.3.3. Let $n = 5$ and consider the Young diagram λ from the previous example. Then the following are all the standard Young tableaux on λ



It is known that Young diagrams of size n are in bijection with the irreducible representations, up to isomorphism, of the symmetric group on n letters, S_n, see [Ful96, Chapter 7, Proposition 1]. The representation theory of the symmetric group can be extended to the representation theory of $G(m,1,n)$ by instead considering m-tuples of Young diagrams.

Definition 3.3.4. Let $m,n \in \mathbb{N}\setminus\{0\}$.

i) Let $\mathscr{P}_{m,n}$ the set of all m-tuples of Young diagrams $\lambda = (\lambda^{(0)},\dots,\lambda^{(m-1)})$, where $\lambda^{(i)}$ is a partition of n_i for $0 \le n_i \le n$ for $0 \le i \le m-1$ and such that $\sum_{0\le i\le m-1} n_i = n$.

Let $\lambda \in \mathscr{P}_{m,n}$.

ii) An *m-tuple of Young tableaux of shape λ* is an m-tuple of Young diagrams with the numbers 1 to n enumerating the cells. The set of all m-tuples of Young tableaux of shape λ is denoted Tab(λ).

iii) An m-tuple of Young tableaux of shape λ is called *standard* if the sequence of entries in the rows and columns of each $\lambda^{(i)}$ are strictly increasing. The set of all standard m-tuples of Young tableau of shape λ is denoted ST(λ).

Example 3.3.5. Let $m = 3$ and $n = 5$ then $\lambda = (\ydiagram{1,1}, -, \ydiagram{2,1})$ is an element of $\mathscr{P}_{3,5}$ and $\lambda_1 = (\begin{ytableau} 2 \\ 5 \end{ytableau}, -, \begin{ytableau} 1 & 4 \\ 3 \end{ytableau})$ is a standard m-tuple of Young tableau of shape λ.

Lemma 3.3.6. [AK94] Every irreducible representation of $G(m,1,n)$ corresponds to a element λ of $\mathscr{P}_{m,n}$.

The following describes the action of the generators of $G(m,1,n)$ on the set of m-tuples of standard Young tableaux of shape λ and is determined in [AK94]. Take a m-tuple of Young diagrams λ and let V_λ be the vector space spanned of all linear combination of all possible m-tuples of standard Young tableaux of shape λ.

Example 3.3.7. Consider $G = G(4,1,2)$, and let λ_1 be the m-tuple of Young diagrams $(\ydiagram{2}, -, -, -)$, then there is only one standard Young tableau of shape λ_1, that is $Q = (\begin{ytableau} 1 & 2 \end{ytableau}, -, -, -)$ and V_{λ_1} is the 1 dimensional k-vector space spanned by Q. Similarly for $\lambda_2 = (\ydiagram{1}, \ydiagram{1}, -, -)$, V_{λ_2} is a 2 dimensional k-vector space since there are two standard Young tableau of shape λ_2: $(\begin{ytableau} 1 \end{ytableau}, \begin{ytableau} 2 \end{ytableau}, -, -)$ and $(\begin{ytableau} 2 \end{ytableau}, \begin{ytableau} 1 \end{ytableau}, -, -)$.

The generators of $G(m,1,n)$ act on the m-tuples of Young tableaux as follows. First let ξ be a primitive m^{th} root of unity. Let t_p be a m-tuple of standard Young tableaux, that is, a basis vector for the representation V_λ corresponding to λ. Let $2 \le k \le n-1$, assume that $k, k-1$ can be swapped and still create a m-tuple of standard Young tableaux. Let t_q be the m-tuple of tableau such that $k, k-1$ are switched in t_p. Recall from Definition 3.1.1 that the generators of $G(m,1,n)$ are denoted $s_1,\dots,s_n$, we define an action on V_λ:

$$s_1(t_p) = \xi^i t_p$$

where 1 appears in the i^{th} position of the m-tuple of tableau t_p. The other generators act as follows

$$s_k(t_p) = \begin{cases} t_p & \text{if } k-1 \text{ and } k \text{ are in the same row,} \\ -t_p & \text{if } k-1 \text{ and } k \text{ are in the same column,} \\ t_q & \text{otherwise.} \end{cases}$$

Since we are focusing on the groups $G(m,1,2)$ with $m \geq 2$, we will describe their representation theory. Recall that the generators for $G(m,1,2)$ are;

$$s_1 = \begin{bmatrix} \xi_m & 0 \\ 0 & 1 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Example 3.3.8. Continuing from Example 3.3.7, let $Q = (\boxed{1\,|\,2}, -, -, -)$ then

$$s_1(Q) = Q \qquad \text{and} \qquad s_2(Q) = Q.$$

Let $P = (\boxed{1}, \boxed{2}, -, -)$ and $T = (\boxed{2}, \boxed{1}, -, -)$ be the standard 4-tableau. The action of the generators s_1 and s_2 of $G(4,1,2)$ is as follows:

$$s_1(P) = P, \qquad s_2(P) = T,$$
$$s_1(T) = \xi_4 T, \qquad s_2(T) = P.$$

Lemma 3.3.9. Let $G = G(m,1,2)$ with $m \geq 2$, then $V_{\det}$ corresponds to the m-tuple of Young tableaux:

$$\alpha = \left(-, \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}, -, \dots, - \right)$$

Proof. We first note the representation given by α is 1 dimensional since the only possible Young tableau is: $\alpha_1 = \left(-, \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}, -, \ldots, -\right)$. Recall that s_1,s_2 are the generators of the group, and denote by ξ the m^{th} root of unity. We then have the following actions on the Young tableau α_1:

$$s_1\alpha_1 = \xi\alpha_1$$

$$s_2\alpha_1 = -\alpha_1$$

Thus this is the representation $V_{\det}$, since $s_1 \mapsto \xi = \det(s_1)$, $s_2 \mapsto -1 = \det(s_2)$. □

Remark 3.3.10. When $m = 1$ and $n = 2$, then $G = G(1,1,2) = S_2$, the symmetric group on 2 letters. In this case $V_{\det}$ corresponds to the Young tableau

$$\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}.$$

3.4 Higher Specht polynomials

Let $G = G(m,1,n)$, $S = \mathbb{C}[x_1,...,x_n]$ and R_+ be the ideal in S generated by the basic invariants $f_1,...,f_n$ of G. This section will deal with calculating bases for the isotypical components of the coinvariant algebra $S/(R_+)$ under the action of G. In [ATY97] a basis was calculated via defining higher Specht polynomials for the groups $G(m,1,n)$ and then later was generalised using Clifford theory [Cli37] for the $G(m,p,n)$ in [MY98]. We first define some more compact notation to describe representations of $G(m,1,n)$.

Definition 3.4.1. Let $G = G(m,1,n)$ and let α be a Young diagram, then let $\alpha_i = (\alpha_i^{(0)},...,\alpha_i^{(m-1)})$ be the m-tuple of Young diagrams (tableau) such that

$$\alpha_i^{(j)} = \begin{cases} -, & \text{if } i \neq j \\ \alpha. & \text{if } i = j \end{cases}$$

That is, α_i is the m-tuple of Young diagrams (resp. tableau) with α in the i^{th} position and blank in every other position. We extend this notation to $\alpha_i\beta_j....$ to mean the m-tuple of Young diagrams (resp. tableau) with α in position i, β in position j and so on.

Example 3.4.2. Let $G = G(4,1,2)$ then $V_{\det} = {\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}}_1$, and $(\square, -, \square, -) = \square_0 \square_2$.

Example 3.4.3. Let $G = G(m,1,2)$ and $\lambda = \square_a \square_b$, $Q = \boxed{1}_a \boxed{2}_b$ and $T = \boxed{2}_a \boxed{1}_b$, then $Q, T \in \mathrm{ST}(\lambda)$.

Definition 3.4.4. Let $\lambda \in \mathscr{P}_{m,n}$ and $Q \in \mathrm{ST}(\lambda)$, we create a word $w(Q)$ by first reading columns from bottom to top starting with the left most column of $Q^{(0)}$, we then move onto $Q^{(1)}$ and so on, until we have read all the components. For a word $w(Q)$ we then define the *index* $i(w(Q))$ inductively as follows: the number 1 has index $i(1) = 0$, let the number p have index $i(p) = k$, if $p+1$ is to the left of p then $i(p+1) = k+1$, if $p+1$ is to the right of p then $i(p+1) = k$. We then assign to $i(w(Q))$ a tableau $i(Q)$ of shape λ with the entries of the cells corresponding to their index. Further define $\hat{\imath}(T)$ as $i(T)$ written in non-decreasing order and $|i(T)|$ as the sum of the indexes.

Example 3.4.5. Following from Example 3.4.3, the words for the tableau are $w(Q) = 12$ and $w(T) = 21$. The indexes for the tableau are $i(w(Q)) = 00$, $i(w(T)) = 10$ and so $i(Q) = \boxed{0}_a \boxed{0}_b$ and $i(T) = \boxed{1}_a \boxed{0}_b$.

Definition 3.4.6. Given a Young tableaux T of shape λ of size n, we define two subgroups of S_n, the *row stabilizer* $R(T)$ which are all elements of S_n that permute elements within the same row, and similarly the *column stabilizer* $C(T)$ which permutes elements within the same columns of T.

Definition 3.4.7. Let T be an m-tuple of tableaux with shape λ, for each component $T^{(a)}$, the *Young Symmetrizer*, $e_{T^{(a)}}$, for $0 \le a \le m-1$ is an element of the group ring, defined by:

$$e_{T^{(a)}} := \frac{1}{\alpha_{T^{(a)}}} \sum_{\sigma \in R(T^{(a)}) \tau \in C(T^{(a)})} \mathrm{sgn}(\tau)\tau\sigma,$$

where $\alpha_{T^{(a)}}$ is the product of hook lengths, see [FH91, 4.12] of the shape $\lambda^{(a)}$ and $R(T^{(a)}), C(T^{(a)})$ are the row and column stabilizers of $T^{(a)}$ respectively.

Example 3.4.8. Continuing from Example 3.4.3, $e_{Q^{(a)}} = e_{T^{(a)}} =$id since the components of Q and T have at most one cell and thus the row and column stabilizers are the identities.

Definition 3.4.9. Let $\lambda = (\lambda^{(0)},...,\lambda^{(m-1)})$ be a m-tuple of Young diagrams $Q \in \mathrm{ST}(\lambda)$, $T \in \mathrm{Tab}(\lambda)$, $x = (x_1,...,x_n)$. Define the higher Specht polynomial as:

$$\Delta_{Q,T}(x) = \prod_{a=0}^{m-1}\left((e_{T^{(a)}} \cdot (x_{T^{(a)}}^{mi(Q)^{(a)}}))(\prod_{k\in T^{(a)}} x_k^a)\right),$$

where

$$x_{T^{(a)}}^{mi(Q)^{(a)}} = \prod_{C\in\lambda^{(a)}} x_{T^{(a)}(C)}^{mi(Q)^{(a)}(C)}.$$

and for a tableau T and a cell C of T, $T(C)$ is the number contained in the cell C.

Example 3.4.10. We again use Q,T from Example 3.4.3 then we have:

$$x_{T^{(a)}}^{mi(Q)^{(a)}} = x_2^0, \quad x_{T^{(b)}}^{mi(Q)^{(b)}} = x_1^0$$

and

$$x_{Q^{(a)}}^{mi(Q)^{(a)}} = x_1^0, \quad x_{Q^{(b)}}^{mi(Q)^{(b)}} = x_2^0.$$

The higher Specht polynomials are

$$\Delta_{Q,T} = e_{T^{(a)}}(x_2^0)(x_2^a)e_{T^{(b)}}(x_1^0)(x_1^b) = x_2^a x_1^b$$

and

$$\Delta_{Q,Q} = e_{Q^{(a)}}(x_{Q^a}^{mi(T)^a})e_{Q^{(b)}}(x_{Q^{(b)}}^{mi(T)^{(b)}}) = x_1^a x_2^b.$$

We also have

$$x_{T^{(a)}}^{mi(T)^{(a)}} = x_2^m, \quad x_{T^{(b)}}^{mi(T)^{(b)}} = x_1^0$$

and

$$x_{Q^{(a)}}^{mi(T)^{(a)}} = x_1^m, \quad x_{Q^{(b)}}^{mi(T)^{(b)}} = x_2^0.$$

The higher Specht polynomials are

$$\Delta_{T,T} = e_{T^{(a)}}(x_{T^{(a)}}^{mi(T)^{(a)}})(x_2^a)e_{T^{(b)}}(x_{T^{(b)}}^{mi(T)^{(b)}})(x_1^b) = x_2^{a+m}x_1^b$$

and

$$\Delta_{T,Q} = e_{Q^{(a)}}(x_{Q^{(a)}}^{mi(T)^{(a)}})(x_1^a)e_{Q^{(b)}}(x_{Q^{(b)}}^{mi(T)^{(b)}})(x_2^b) = x_1^{a+m}x_2^b.$$

The following theorem tells us how to decompose $S/(R_+)$ into isotypical components using the higher Specht polynomials.

Theorem 3.4.11. [ATY97, Theorem 1] Let $\lambda \in \mathscr{P}_{m,n}$ and $Q \in \mathrm{ST}(\lambda)$ then:

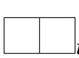
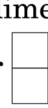
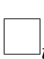
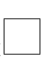
- The subspace $V_{Q,\lambda} = \sum_{T\in\mathrm{ST}(\lambda)} \mathbb{C}\Delta_{Q,T}$ of $S/(R_+)$ is isomorphic to the irreducible representation V_λ of $G(m,1,n)$.
- The set $\{\Delta_{Q,T} : T \in \mathrm{ST}(\lambda)\}$ is a basis for $V_{Q,\lambda}$.
- The coinvariant algebra $S/(R_+)$ has an irreducible decomposition:

$$S/(R_+) = \bigoplus_{\lambda\in\mathscr{P}_{m,n}} \bigoplus_{Q\in\mathrm{ST}(\lambda)} (V_{Q,\lambda} \bmod R_+).$$

Example 3.4.12. Continuing from Example 3.4.10, the collection $\{\Delta_{Q,T}, \Delta_{Q,Q}\}$ of higher Specht polynomials is a basis for $V_{Q,\lambda}$ and the collection $\{\Delta_{T,T}, \Delta_{T,Q}\}$ is a basis for $V_{T,\lambda}$ and $V_{Q,\lambda} \cong V_{T,\lambda}$.

It is straightforward to see the following:

Lemma 3.4.13. The representations of $G(m,1,2)$ are of the following form:

- The 1 dimensional representations corresponds to the m-tuple of diagrams; $_i$ or $_i$ for $0 \le i \le m-1$.
- The 2 dimensional representations corresponds to the m-tuple of diagrams $_i$$_j$, where $0 \le i < j \le m-1$.

Proof. Building representations of $G(m,1,2)$ is done by placing two cells in an m-tuple. There are two cases; the cells are in different positions or they are in the same position. In the first case these correspond to the two-dimensional representation. In the second case, there is the choice of the two cells being in the same column or in the same row - these correspond to the 1 dimensional representations. □

Lemma 3.4.14. Let α be an m-tuple of Young diagrams corresponding to the representation W_α of G. The m-tuple of Young diagrams β representing the representation $W_\alpha \otimes W_{\det}$ is obtained by the following:

- If W_α is 1 dimensional then α is of the form; $\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}_i$ or $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}_i$ for some $0 \le i < m$. Then β is; $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}_{i+1}$ or $\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}_{i+1}$ respectively where $i+1$ is taken modulo m.
- If W_α is a 2 dimensional representation, then α is of the form $\begin{array}{|c|}\hline \ \\ \hline\end{array}_i\begin{array}{|c|}\hline \ \\ \hline\end{array}_j$ where $0 \le i < j < m$, then β is of the form $\begin{array}{|c|}\hline \ \\ \hline\end{array}_{i+1}\begin{array}{|c|}\hline \ \\ \hline\end{array}_{j+1}$ where $i+1, j+1$ are taken modulo m.

Proof. Let α be of the form $\begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array}_i$ then $s_1(\alpha) = \xi_m^i \alpha$, where ξ_m is the primitive m^{th} root of unity. W_α is a one-dimensional vector space as it is spanned by the tableau $t_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}_i$, and so by Lemma 3.3.9, $W_\alpha \otimes \det$ is generated by $t_1 \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}_1$.

$$s_1(t_1 \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}_1) = s_1(t_1) \otimes s_1(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}_1) = \xi^i t_1 \otimes \xi \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}_1 = \xi^{i+1}(t_1 \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}_1)$$

$$\begin{aligned} s_2(t_1 \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}_1) &= s_2(t_1) \otimes s_2(\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}_1) = \begin{cases} t_1 \otimes - \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}_1 & \text{if 1 and 2 are in the same row in } t_1 \\ -t_1 \otimes - \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}_1 & \text{if 1 and 2 are in the same column in } t_1 \\ t_1 \otimes - \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}_1 & \text{otherwise} \end{cases} \\ &= -t_1 \otimes \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}_1 \end{aligned}$$

Thus the young tableau associated with $\alpha \otimes \det$ is $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}_{i+1}$ where $i+1$ is taken mod m. The case $\alpha = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}_i$ is similar.

Let α be of the form $\boxed{}_i\boxed{}_j$ where $0 \le i < j \le m-1$, α is then a two-dimensional representation spanned by the tableau $t_1 = \boxed{1}_i\boxed{2}_j, -t_2 = \boxed{2}_i\boxed{1}_j$. The negative sign on the basis element $-t_2$ is to make the calculations cleaner. Then

$$s_1\Big(t_1 \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}_1\Big) = s_1(t_1) \otimes s_1\Big(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}_1\Big) = \xi^i t_1 \otimes \xi \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}_1 = \xi^{i+1}\Big(t_1 \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}_1\Big),$$
$$s_1\Big(-t_2 \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}_1\Big) = s_1(-t_2) \otimes s_2\Big(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}_1\Big) = -\xi^j t_2 \otimes \xi \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}_1 = \xi^{j+1}\Big(-t_2 \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}_1\Big),$$

$$s_2\Big(t_1 \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}_1\Big) = s_2(t_1) \otimes s_2\Big(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}_1\Big) = -t_2 \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}_1,$$

$$s_2\Big(-t_2 \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}_1\Big) = s_2(t_2) \otimes s_2\Big(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}_1\Big) = t_1 \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}_1.$$

Thus $\boxed{}_i\boxed{}_j \otimes \det$ is isomorphic to the representation corresponding to the tableau $\boxed{}_{i+1}\boxed{}_{j+1}$ where $i+1, j+1$ are taken modulo m. □

Lemma 3.4.15. Let $G = G(m,1,2)$ and $\lambda = \boxed{}_i\boxed{}_j$. Then, using higher Specht polynomials, a basis for the isotypical component of $S/(R_+)$ isomorphic to W_λ is $\{x^i y^j, x^j y^i, x^{i+m} y^j, x^j y^{i+m}\}$.

Proof. Follows from calculations done in Example 3.4.10 and using Theorem 3.4.11 and Example 3.4.12. □

From now we abuse some notation and if λ is a m-tuple of Young diagrams, we will also denote the representation V_λ corresponding to it as λ.

Lemma 3.4.16. Let $G = G(m,1,2)$ and let $\lambda = \boxed{}_i\boxed{}_j$ for $0 \le i < j < m-1$ then the map of $z|_\lambda$ on the isotypical component of type λ gives rise to a matrix factorization of Δ over R equivalent to:

$$S^G_{\lambda\otimes\det} \xrightarrow{\;j|_{\lambda\otimes\det}\;} S^G_\lambda \xrightarrow{\begin{bmatrix} 0 & (4\sigma_1 - \sigma_2^2) \\ 1 & 0 \end{bmatrix} \otimes I_2} S^G_{\lambda\otimes\det}$$

Proof. The fact that $(z|_\lambda, j|_{\lambda\otimes det})$ is a matrix factorization of Δ comes from the discussion in Section 2.2. From Lemma 3.4.15, a basis for the isotypical component of S/R_+ corresponding to λ is:

$$\{x^i y^j, x^j y^i, x^{i+m} y^j, x^j y^{i+m}\}.$$

Further, by Lemma 3.4.14 and Lemma 3.4.15 , a basis for the isotypical component corresponding to $\lambda \otimes \det$ is:

$$\{x^{i+1} y^{j+1}, x^{j+1} y^{i+1}, x^{i+m+1} y^{j+1}, x^{j+1} y^{i+m+1}\}.$$

Then, multiplication of z on the basis elements can be calculated as follows:

$$\begin{aligned} z(x^i y^j) &= x^{i+1} y^{j+1}(x^m - y^m) \\ &= 2x^{m+i+1} y^{j+1} - \sigma_2(x^{i+1} y^{j+1}) \end{aligned}$$

and

$$\begin{aligned} z(x^{i+m} y^j)) &= x^{i+m+1} y^{j+1}(x^m - y^m) \\ &= \sigma_2(x^{m+i+1} y^{j+1}) - 2\sigma_1(x^{i+1} y^{j+1}). \end{aligned}$$

We obtain the matrix factorization (A,B) of Δ, where

$$A = \begin{bmatrix} -\sigma_2 & 0 & -2\sigma_1 & 0 \\ 0 & -\sigma_2 & 0 & -2\sigma_1 \\ 2 & 0 & \sigma_2 & 0 \\ 0 & 2 & 0 & \sigma_2 \end{bmatrix}$$

Which is equivalent as a matrix factorization (A,B) of Δ, where

$$A = \begin{bmatrix} 0 & (4\sigma_1 - \sigma_2^2) \\ 1 & 0 \end{bmatrix} \otimes I_2$$

□

Lemma 3.4.17. Let $G = G(m,1,2)$ and let $\lambda = \square_{m-2}\square_{m-1}$ then the map of $z|_\lambda$ on the isotypical component of type λ gives rise to a matrix factorization of Δ over R equivalent to:

$$S^G_{\lambda\otimes\det} \xrightarrow{j|_{\lambda\otimes\det}} S^G_{\lambda} \xrightarrow{\begin{bmatrix} -2\sigma_1 & \sigma_2\sigma_1 \\ \sigma_2 & -2\sigma_1 \end{bmatrix}\otimes I_2} S^G_{\lambda\otimes\det}$$

Proof. From Lemma 3.4.15, a basis for the isotypical component of S/R_+ corresponding to λ is:

$$\{x^{m-2}y^{m-1}, x^{m-1}y^{m-2}, x^{2m-2}y^{m-1}, x^{m-1}y^{2m-2}\}.$$

Further, by Lemma 3.4.14 and Lemma 3.4.15 , a basis for the isotypical component corresponding to $\lambda\otimes\det$ is:

$$\{x^{m-1}, y^{m-1}, x^{m-1}y^m, x^m y^{m-1}\}.$$

Then, multiplication of z on the basis elements can be calculated as follows:

$$\begin{aligned} z(x^{m-2}y^{m-1}) &= x^{m-1}y^m(x^m - y^m) \\ &= x^{2m-1}y^m - x^{m-1}y^{2m} \\ &= -\sigma_2(x^{m-1}y^m) + 2\sigma_1 x^{m-1} \\ z(x^{m-1}y^{m-2}) &= \sigma_2(x^m y^{m-1}) - 2\sigma_1 y^{m-1}. \end{aligned}$$

and

$$\begin{aligned} z(x^{2m-2}y^{m-1}) &= x^{2m-1}y^m(x^m - y^m) \\ &= \sigma_2\sigma_1(x^{m-1}) - 2\sigma_1(x^{m-1}y^m). \end{aligned}$$

We obtain the matrix:

$$\begin{bmatrix} -2\sigma_1 & 0 & \sigma_2\sigma_1 & 0 \\ 0 & -2\sigma_1 & 0 & \sigma_2\sigma_1 \\ \sigma_2 & 0 & -2\sigma_1 & 0 \\ 0 & \sigma_2 & 0 & -2\sigma_1 \end{bmatrix} = \begin{bmatrix} -2\sigma_1 & \sigma_2\sigma_1 \\ \sigma_2 & -2\sigma_1 \end{bmatrix} \otimes I_2$$

□

Matrix factorizations were used in [Yos90, Chapter 9] to calculate the CM modules over 1 dimensional ADE singularities. Note that Yoshino calculated them for the completed case, but from Remark 1.2.16 we can instead consider them as graded matrix factorizations over the graded local ring S.

Definition 3.4.18. We call the singularity $\operatorname{Spec}(\mathbb{C}[x,y]/(f))$, where $f = x^2 + y^4$ a A_3 singularity.

Remark 3.4.19. Let $G = G(m,1,2)$ and recall from Example 3.1.4 that $\Delta = \sigma_1(\sigma_2^2 - 4\sigma_1)$. Then $\operatorname{Spec}(R/(\Delta))$ is an A_3 singularity. This is seen by the graded coordinate change $x = \sigma_1 - \frac{\sigma_2^2}{8}$ and $y = \frac{\sigma_2}{2\sqrt{2}}$.

Theorem 3.4.20. [Yos90, Section 9.9] The non-trivial indecomposable CM modules over $\mathbb{C}[\sigma_1,\sigma_2]/(\Delta)$, where $\Delta = \sigma_1(\sigma_2^2 - 4\sigma_1)$ are given as cokernels of the matrix factorizations: $(\alpha,\beta) = (\sigma_1, \sigma_2^2 - 4\sigma_1)$, $(\beta,\alpha) = (\sigma_2^2 - 4\sigma_1, \sigma_1)$ and the matrix:

$$(\phi,\psi) = \left(\begin{bmatrix} 2\sigma_1 & \sigma_2\sigma_1 \\ \sigma_2 & 2\sigma_1 \end{bmatrix}, \begin{bmatrix} -2\sigma_1 & \sigma_2\sigma_1 \\ \sigma_2 & -2\sigma_1 \end{bmatrix} \right).$$

Denote the modules: $A = \operatorname{Coker}(\alpha,\beta)$, $B = \operatorname{Coker}(\beta,\alpha)$, $X = \operatorname{Coker}(\phi,\psi)$. Note $\operatorname{Coker}(\phi,\psi) = \operatorname{Coker}(\psi,\phi)$. These are the indecomposable modules in $CM(\mathbb{C}[\sigma_1,\sigma_2]/(\Delta))$.

Remark 3.4.21. Thus using the notation from Theorem 3.4.20, the matrix factorization of Δ from Lemma 3.4.17 corresponding to the irreducible representation $\square_{m-2}\square_{m-1}$ is equivalent (as matrix factorizations) to $\phi \oplus \phi$ (and also $\psi \oplus \psi$), and so $M_{\square_{m-2}\square_{m-1}} \cong X \oplus X$.

To show that $S/(z)$ is a representation generator of $CM(R/(\Delta))$, we need to check that each non-isomorphic indecomposable CM $R/(\Delta)$ module appears at least once in the decomposition of $S/(z)$.

Theorem 3.4.22. Let $G = G(m,1,2)$ then the following is a decomposition of $S/(z)$ as CM $R/(\Delta)$ modules:

$$S/(z) \cong R/(\sigma_1) \oplus R/(\Delta) \oplus (R/(\sigma_2^2 - 4\sigma_1))^{2\binom{m-1}{2}+m-1} \oplus X^{2m-2}$$
$$= A \oplus B^{2\binom{m-1}{2}+m-1} \oplus X^{2m-2} \oplus R/(\Delta).$$

In particular $S/(z)$ is a representation generator for $CM(R/(\Delta))$.

Proof. From Section 2.3, the linear characters give the modules $M_{\theta_{\mathfrak{D}}^{m-1}} = R/(\Delta)_{\mathfrak{D}} = R/(\sigma_1)$ and $M_{\theta_{\mathrm{q}}} = R/(\Delta)_{\mathrm{q}} = R/(\sigma_2^2 - 4\sigma_1)$. Moreover the modules $M_{\theta_{\mathfrak{D}}^{i}\otimes\theta_{\mathrm{q}}} \cong R/(\sigma_2^2 - 4\sigma_1)$ for $1 \le i \le m-2$. There are $\binom{m-1}{2}$ different irreducible representations of the form $\square_i\square_j$ for $0 \le i < j < m-1$ and for each we have $M_{\square_i\square_j} = (R/(4\sigma_1 - \sigma_2^2))^2$. Then $M_{\square_i\square_{m-1}} \cong X^2$ for $0 \le i < m-1$. □

3.5 Irreducible representations of $G(m,p,n)$

The irreducible representations of $G(m,p,n)$ were first described by Stembridge in [Ste89]. Let $G = G(m,1,n)$ and $H = G(m,p,n)$ and let $q = \frac{m}{p}$. We set a linear character of G by:

$$\delta_i = (-,\dots,\square\cdots\square,\dots,-) = \square\cdots\square_i$$

where $0 \le i \le m-1$.

Remark 3.5.1. We abuse some notation and will denote by $-\otimes\delta_i$, the operator on m-tuples of Young diagrams such that for two m-tuple of Young diagrams λ_1, λ_2, $\lambda_1 \otimes \delta_i = \lambda_2$ when $V_{\lambda_1} \otimes V_{\delta_i} \cong V_{\lambda_2}$.

It is straightforward to check that the action $-\otimes\delta_i$ "shifts" the diagrams in a m-tuple of young diagrams i places to the right. That is, if $\lambda = (\lambda^{(0)},\dots,\lambda^{(m-1)})$ then $\lambda\otimes\delta_i = (\lambda^{(0-i)},\dots,\lambda^{(m-1-i)})$ where $j-i$ is considered mod m.

The quotient group G/H is isomorphic to the cyclic group $C = \langle\delta_1^q\rangle$, where $q = \frac{m}{p}$. Through this G/H acts on the irreducible representations of G. Denote by $[\lambda]$ the G/H-orbit of the irreducible G-representation λ. Define an equivalence relation $\sim_H$ on irrep(G) such that $\lambda \sim_H \mu$ if and only if $\lambda = \mu\otimes\delta$ for some δ in G/H.

We define some numerology, let $b(\lambda)$ be the cardinality of the G/H orbit of λ and $u(\lambda) := \frac{p}{b(\lambda)}$. Define the stabilizer of λ as the subgroup of G/H:

$$(G/H)_\lambda := \{\delta \in G/H : \lambda \otimes \delta = \lambda\}$$

The subgroup $(G/H)_\lambda$ of G/H is a cyclic group of order $u(\lambda)$, generated by $\delta_1^{b(\lambda)\cdot q}$.

Example 3.5.2. Let $G = G(4,1,2)$ and $H = G(4,2,2)$. Let $\lambda_1 = (\square, \square, -, -)$, $\lambda_2 = (\square, -, \square, -)$ $\lambda_3 = (-, \square, -, \square)$ and $\lambda_4 = (-, -, \square, \square)$. The quotient group G/H is generated by δ_1^2, and for example $\lambda_1 \otimes \delta_1^2 = \lambda_4$. Then $(G/H)_{\lambda_1} = \{1\}$, $(G/H)_{\lambda_2} = \{1, \delta_1^2\}$. $[\lambda_1] = \{\lambda_1, \lambda_4\}$, $[\lambda_2] = [\lambda_2]$ and $[\lambda_3] = [\lambda_3]$

Theorem 3.5.3. (Stembridge [Ste89]) There is a one-one correspondence between the irreducible representations of $H = G(m,p,n)$ and ordered pairs $([\lambda], \delta)$ where $[\lambda]$ is the orbit of an irreducible representation λ of G and $\delta \in (G/H)_\lambda$. Furthermore, the following hold:

a) $\mathrm{Res}_H^G(\lambda) = \mathrm{Res}_H^G(\mu)$ for $\lambda \sim_H \mu$.

b) $\mathrm{Res}_H^G(\lambda) = \bigoplus_{\delta \in (G/H)_\lambda} ([\lambda], \delta)$.

Example 3.5.4. Following from Example 3.5.2, we calculate the restriction of each representation, $\mathrm{Res}_H^G \lambda_1 = ([\lambda_1], 1) = \mathrm{Res}_H^G \lambda_4$ and $\mathrm{Res}_H^G \lambda_2 = ([\lambda_2], 1) \oplus ([\lambda_2], \delta_1^2)$.

Recall from Lemma 3.4.14 that the representations of $G(m,1,2)$ are of the following form:

- The 1 dimensional representations corresponds to the m-tuple of diagrams; $\square\square_i$ or $\begin{smallmatrix}\square\\ \square\end{smallmatrix}_i$ for $0 \leq i \leq m-1$.
- The 2 dimensional representations corresponds to the m-tuple of diagrams; $\square_i \square_j$, where $0 \leq i < j \leq m-1$.

We can fully describe the irreducible representations of $G(m,p,2)$, for $p \geq 2$ in the following way.

Theorem 3.5.5. Let $G = G(m,1,2)$ and $H = G(m,p,2)$ where $p|m$ and $q = \frac{m}{p}$. The irreducible representations are given by

- $\mathrm{Res}^G_H \square\square_i$ or $\mathrm{Res}^G_H \begin{smallmatrix}\square\\ \square\end{smallmatrix}_i$ for $0 \leq i \leq m-1$.
- $\mathrm{Res}^G_H \square_i \square_j$ for $j \neq \frac{m}{2} + i$
- $(\square_i \square_{\frac{m}{2}+i}, 1)$ and $(\square_i \square_{\frac{m}{2}+i}, \delta_1^p)$ for $0 \leq i < \frac{m}{2}$.

Proof. The one-dimensional representations will again stay irreducible under restriction. By Theorem 3.5.3b), the restriction of a 2 dimensional representation λ is reducible if $(G/H)_\lambda$ is non-trivial, that means that there exists a $\delta \in G/H$ where‘ $\delta = \delta_1^{q\cdot c}$ for some $c \in \{0,..,p-1\}$, such that $\lambda \otimes \delta = \lambda$. Let $\lambda = \square_i \square_j$ and so $\lambda \otimes \delta_1^{q\cdot c} = \square_{cq+i} \square_{cq+j}$, for $\mathrm{Res}^G_H \lambda$ to be reducible $cq + i = j$ and $cq + j = m + i$. Thus $2cq = m$ and $cq = \frac{m}{2}$. Then since $q = \frac{m}{p}$ we have $c\frac{m}{p} = \frac{m}{2}$ and so $2c = p$, thus $2|p$.

From this, if m is odd, or even with p odd, all of the 2 dimensional representation stay irreducible. If m is even and p is even the only two-dimensional representations that will not be irreducible are $\square_i \square_{\frac{m}{2}+i}$ for $0 \leq i < \frac{m}{2}$. □

3.6 Higher Specht polynomials for $G(m,p,2)$

Let $G = G(m,1,n)$ and $H = G(m,p,n)$ with both acting on $V \cong \mathbb{C}^n$, $S = \mathrm{Sym}_k(V)$ and $q = \frac{m}{p}$. Let S_H be the coinvariant algebra for H. Let the operation $- \otimes \delta_1$ on the m-tuples of tableau as defined in Section 3.5, be denoted by sh, for convenience denote by Sh the operation sh^q. Note that $b(\lambda)$ is the smallest integer j such that $Sh^j(\lambda) = \lambda$. For $h = 1, ..., m$ define

$$\mathrm{ST}(\lambda)_h = \{T = (T^{(0)}, ..., T^{(m-1)}) \in \mathrm{ST}(\lambda) : 1 \in T^{(v)} \text{ for some } 0 \leq v < h\}.$$

Remark 3.6.1. If $T \in \mathrm{ST}(\lambda)_q$ then $T, Sh^{b(\lambda)}(T), ..., Sh^{(u(\lambda)-1)b(\lambda)}(T)$ are all distinct.

Lemma 3.6.2. [MY98, Lemma 2] Let Q, T be standard m-tableau of shape λ. Then the polynomial $\Delta_{Q,T}(x)$ is non-zero in S_H if and only if $Q \in \mathrm{ST}(\lambda)_q$.

Definition 3.6.3. Let $\lambda = (\lambda^{(0)}, \dots, \lambda^{(m-1)}) \in \mathscr{P}_{n,m}$, let $Q, T \in \mathrm{ST}(\lambda)$ and for $0 \le l \le u(\lambda)-1$ we define the l^{th} *higher Specht polynomial* as

$$\Delta^{(l)}_{Q,T} := \sum_{a=0}^{u(\lambda)-1} \xi_m^{laqb(\lambda)} \Delta_{Q,Sh^{mb(\lambda)}(T)}.$$

Example 3.6.4. Let $G = G(m,1,n)$, $H = G(m,p,n)$ and let $\lambda = \square_i \square_j$, where $0 \le i < j < q$ and let $Q = \boxed{1}_i \boxed{2}_j, T = \boxed{2}_i \boxed{1}_j$. Then $u(\lambda) = 1$ since the only case with $u(\lambda) \neq 1$ is when m is even and $j = \frac{m}{2} + i$ and so $j > q$. Moreover this means

$$\Delta^{(0)}_{Q,T} = \Delta_{Q,T} = x_2^i x_1^j, \qquad \Delta^{(0)}_{Q,Q} = \Delta_{Q,Q} = x_1^i x_2^j$$

and

$$\Delta^{(0)}_{T,Q} = \Delta_{T,Q} = x_1^{i+m} x_2^j \qquad \Delta^{(0)}_{T,T} = \Delta_{T,T} = x_2^{i+m} x_1^j.$$

Then since both Q, T are in $\mathrm{ST}(\lambda)_q$, all these higher Specht polynomials are non-zero in S_H. Then

$$V_Q(x) = \mathbb{C}\Delta_{Q,T} \oplus \mathbb{C}\Delta_{Q,Q} \qquad \text{and} \qquad V_T(x) = \mathbb{C}\Delta_{T,Q} \oplus \mathbb{C}\Delta_{T,T}.$$

Example 3.6.5. Let $G = G(m,1,n)$, $H = G(m,p,n)$. Let $\lambda = \square_i \square_j$, where $0 \le i < q \le j \le m-1$ with $\mathrm{Res}^G_H \lambda$ irreducible, that is $j \neq i + \frac{m}{2}$ and let $Q = \boxed{1}_i \boxed{2}_j, T = \boxed{2}_i \boxed{1}_j$, then $u(\lambda) = 1$. Moreover the higher Specht polynomials are

$$\Delta^{(0)}_{Q,T} = \Delta_{Q,T} = x_2^i x_1^j, \qquad \Delta^{(0)}_{Q,Q} = \Delta_{Q,Q} = x_1^i x_2^j$$

and

$$\Delta^{(0)}_{T,Q} = \Delta_{T,Q} = x_1^{i+m} x_2^j, \qquad \Delta^{(0)}_{T,T} = \Delta_{T,T} = x_2^{i+m} x_1^j.$$

Then since Q is in $\mathrm{ST}(\lambda)_q$, by Lemma 3.6.2 the higher Specht polynomials $\Delta_{Q,T}$ and $\Delta_{Q,Q}$ are non-zero in $S/(R_+)$. However, T is not in $\mathrm{ST}(\lambda)_q$ and thus $\Delta_{T,T}$ and $\Delta_{T,Q}$ are zero in $S/(R_+)$. Let $j = cq + r$ for some $0 \le c < p$ and $0 \le r < q$ then consider $\mu = \square_r \square_{m+i-cq}$. Let $P = \boxed{1}_r \boxed{2}_{m+i-cq}$ and $U = \boxed{2}_r \boxed{1}_{m+i-cq}$. Now $P \in \mathrm{ST}(\mu)_q$ and $U \notin \mathrm{ST}(\mu)_q$ thus the higher Specht polynomials $\Delta_{P,P}, \Delta_{P,U}$ are non-zero while $\Delta_{U,P}, \Delta_{U,U}$ are zero in $S/(R_+)$.

Theorem 3.6.6 (Morita, Yamada)**.** [MY98, Theorem 3] Let $\lambda = (\lambda^{(0)},\ldots,\lambda^{(m-1)}) \in \mathscr{P}_{n,m}$. For each $S \in \mathrm{ST}(\lambda)$ and $0 \le l \le u(\lambda)-1$, put $V_S^{(l)}(x) = \bigoplus_{T\in \mathrm{ST}(\lambda)} \mathbb{C}\Delta_{S,T}^{(l)}(x)$. Then:

- The space $V_S(x)$ decomposes as $V_S(x) = \bigoplus_{l=0}^{u(\lambda)-1} V_S^{(l)}(x)$.
- The space $V_S^{(l)}(\lambda)$ is isomorphic to an irreducible representation of $G(m,p,n)$.
- The coinvariant algebra S_H has an irreducible decomposition:

$$S_H = \bigoplus_{\lambda\vdash n}\bigoplus_{S\in\mathrm{ST}(\lambda)_q}\bigoplus_{l=0}^{u(\lambda)-1} V_S^{(l)}$$

Example 3.6.7. Continuing Example 3.6.5, the vector space $V_Q(x)$ then stays irreducible and is isomorphic to $V_T(x)$ since $\lambda \sim_H \mu$. Thus a basis for the isotypical component S_λ^H of S_H of type λ is

$$\{\Delta_{Q,T}^{(0)}, \Delta_{Q,Q}^{(0)}, \Delta_{P,P}^{(0)}, \Delta_{P,U}^{(0)}\}.$$

3.7 Decomposition of $S/(z)$ for $G(m,p,2)$

As before, let $G = G(m,1,2)$, $H = G(m,p,2)$ with $q := \frac{m}{p}$, $S = \mathbb{C}[x,y]$ and from Example 3.1.5 the invariant ring is $R = S^H \cong \mathbb{C}[\sigma_1,\sigma_2]$ where $\sigma_1 = x^m + y^m$, $\sigma_2 = (xy)^q$. For the case $p \neq m$, recall that $z = xy(x^m - y^m)$ and $\Delta = \sigma_2(\sigma_1^2 - 4\sigma_1^p)$. We first describe the action of $-\otimes \det_H$ where $\det_H$ is the determinantal representation of H.

Lemma 3.7.1. The determinantal representation of H is given by $\det_H = \mathrm{Res}_H^G \det_G$.

Proof. This is straightforward since det is 1 dimensional and for $h \in H$, $\mathrm{Res}_H^G \det(h) = \det(h)$. □

Lemma 3.7.2. Let $0 \le i < j < m$ be such that $\mathrm{Res}_H^G \square_i \square_j$ is irreducible then $\mathrm{Res}_H^G \square_i \square_j \otimes \det_H = \mathrm{Res}_H^G \square_{i+1}\square_{j+1}$ where $i+1, j+1$ are taken mod m and is irreducible.

Proof. We calculate this directly by $\mathrm{Res}_H^G \square_i \square_j \otimes \det_H = \mathrm{Res}_H^G \square_i \square_j \otimes \mathrm{Res}_H^G \det_G = \mathrm{Res}_H^G (\square_i \square_j \otimes \det_G) = \mathrm{Res}_H^G \square_{i+1}\square_{j+1}$. □

Bases for the Isotypical components of $S/(R_+)$ of type $\mathrm{Res}_H^G \square_i \square_j$ can again be calculated as higher Specht polynomials from Theorem 3.6.6. The goal of this section is to show that $S/(z)$ is a representation generator, recall Definition 1.1.10, for $CM(R/(\Delta))$. A full list of non-isomorphic indecomposable CM modules over $R/(\Delta)$ in each case can be found in [Yos90]. The remainder of this section is dedicated to giving a complete decomposition for $S/(z)$ for $G(m,p,2)$ and hence showing $S/(z)$ contains at least one copy of each indecomposable CM module.

We will breakdown the general case of $G(m,p,2)$ where m,p are positive integers such that $p|m$ into 3 distinct cases:

(1) $m \neq p$ and p odd,

(2) $m \neq p$ and p even,

(3) $m = p$.

(1) $m \neq p$ and p odd.

Recall for the groups $G(m,p,2)$ with $q = \frac{m}{p}$ the basic invariants are $\sigma_1 = (xy)^q$ and $\sigma_2 = x^m - y^m$. The discriminant $\Delta = \sigma_1(\sigma_2^2 - 4\sigma_1^p)$ of the action of H defines a D_{p+2} singularity. Here we use the list of CM modules found in [Yos90, 9.11 p.77]. When p is odd there are 2 matrix factorizations of the form $(\sigma_2^2 - 4\sigma_1^p, \sigma_1)$ and $(\sigma_1, \sigma_2^2 - 4\sigma_1^p)$, thus there are two CM modules $A = \mathrm{Coker}(\sigma_1, \sigma_2^2 - 4\sigma_1^p)$ and $B = \mathrm{Coker}(\sigma_2^2 - 4\sigma_1^p, \sigma_1)$. Using the results from Section 2.3, we can construct 1 dimensional representations λ_1, λ_2 of $G(m,p,2)$ such that $(z|_{\lambda_1}, j|_{\lambda_1 \otimes \det})$ is equivalent to $(\sigma_1, \sigma_2^2 - 4\sigma_1^p)$ and $(z|_{\lambda_2}, j|_{\lambda_2 \otimes \det})$ is equivalent to $(\sigma_2^2 - 4\sigma_1^p, \sigma_1)$.

For $R/(\Delta)$ there are four more families of indecomposable CM modules, X_j, Y_j, K_j, N_j, where $0 \le j \le p-1$. Here we note that in [Yos90] the notation for K_j is M_j, the change is due to clashing notation. For $0 \le j \le p-1$ we define the matrices

$$\phi_j = \begin{pmatrix} \sigma_2 & 2\sigma_1^j \\ 2\sigma_1^{p-j} & \sigma_2 \end{pmatrix} \qquad \text{and} \qquad \psi_j = \begin{pmatrix} \sigma_2\sigma_1 & -2\sigma_1^{j+1} \\ -2\sigma_1^{p+1-j} & \sigma_2\sigma_1 \end{pmatrix}$$

and set $K_j = \mathrm{Coker}(\phi_j, \psi_j)$ and $N_j = \mathrm{Coker}(\psi_j, \phi_j)$. For $0 \le j \le p$ define

$$\xi_j = \begin{pmatrix} \sigma_2 & 2\sigma_1^j \\ 2\sigma_1^{p+1-j} & \sigma_2\sigma_1 \end{pmatrix}, \quad \eta_j = \begin{pmatrix} \sigma_2\sigma_1 & -2\sigma_1^j \\ -2\sigma_1^{p+1-j} & \sigma_2 \end{pmatrix}$$

and set $X_j = \mathrm{Coker}(\xi_j, \eta_j)$, $Y_j = \mathrm{Coker}(\eta_j, \xi_j)$. Then one can quickly show the following:

$$X_0 \cong R/(\Delta) \cong Y_0, \qquad K_0 \cong B, \qquad N_0 \cong A \oplus R/(\Delta).$$

For $1 \le j \le p-1$ the following hold

$$N_j \cong N_{p-j}, \qquad K_j \cong K_{p-j}$$

and for $1 \le j \le p$ the following hold

$$X_j \cong Y_{p+1-j}, \qquad Y_j \cong X_{p+1-j}, \qquad X_{\frac{p+1}{2}} \cong Y_{\frac{p+1}{2}}, \qquad .$$

Using this, a complete list of non-isomorphic CM modules over $R/(\Delta)$ is given by:

i) X_j for $1 \le j \le p$,
ii) K_j for $1 \le j \le \frac{p-1}{2}$,
iii) N_j for $1 \le j \le \frac{p-1}{2}$,
iv) A,
v) B,
vi) $R/(\Delta)$.

Theorem 3.7.3. Let $H = G(m,p,2)$, with p odd, $m \neq p$ and $q = \frac{m}{p}$. The 2 dimensional representations of H correspond to the following CM modules in the decomposition of $S/(z)$:

1) $M_{\mathrm{Res}^G_H \square_i \square_{jq-1}} \cong X^2_{p+1-j}$ for $0 \le i < q-1$, $1 \le j \le p$.

2) $M_{\mathrm{Res}^G_H \square_i \square_{jq+r}} \cong K^2_{p-j} \cong K^2_j$ for $0 \le j \le p-1$, $0 \le i < q-1$ and $0 \le r < q-1$.

3) $M_{\mathrm{Res}^G_H \square_{q-1} \square_{(j+1)q-1}} \cong N^2_{p-j} \cong N^2_j$ for $1 \le j \le p-1$.

Proof. This is proved by calculations of the following form:

1) The module $M_{\mathrm{Res}^G_H \square_i \square_{jq-1}}$ is X^2_{p+1-j} for $1 \le j \le p$.

Using higher Specht polynomials, a basis for $S^G_{\mathrm{Res}^G_H \square_i \square_{jq-1}}$ where $0 \le i < q-1$ and $1 \le j \le p$ is:

$$\{x^{jq-1}y^i, x^i y^{jq-1}, x^{q-1}y^{m-(j-1)q+i}, x^{m-(j-1)q+i}y^{q-1}\}$$

and a basis for $S^G_{\mathrm{Res}\square_{i+1}\square_{jq}}$ is:

$$\{x^{jq}y^{i+1}, x^{i+1}y^{jq}, x^{jq}y, y^{m+i+1-jq}, x^{m+i+1-jq}\}$$

by calculation we can express multiplication by $z = xy(x^m - y^m)$ as the matrix:

$$\begin{aligned}
z(x^{jq-1}y^i) &= x^{jq}y^{i+1}(x^m - y^m)\\
&= \sigma_2(x^{jq}y^{i+1}) - 2\sigma_2^j(y^{m+i+1-jq})\\
z(x^i y^{jq-1}) &= x^{i+1}y^{jq}(x^m - y^m)\\
&= -\sigma_2(x^{i+1}y^{jq}) + 2\sigma_2^j(x^{m+i+1-jq})\\
z(x^{q-1}y^{m-(j-1)q+i}) &= x^q y^{m-(j-1)q+i+1}(x^m - y^m)\\
&= 2\sigma_1^{p-(j-1)}(x^{jq}y^{i+1}) - \sigma_1\sigma_2(y^{m-j+i+1})\\
z(x^{m-(j-1)q+i}y^{q-1}) &= x^{m-(j-1)q+i+1}y^q(x^m - y^m)\\
&= -2\sigma_1^{p-(j-1)}(x^{i+1}y^{jq}) - \sigma_1\sigma_2(x^{m-j+i+1})
\end{aligned}$$

which yields the matrix:

$$\begin{pmatrix}
\sigma_2 & 0 & 2\sigma_1^{p-(j-1)} & 0\\
0 & -\sigma_2 & 0 & -2\sigma_1^{p-(j-1)}\\
-2\sigma_1^j & 0 & -\sigma_1\sigma_2 & 0\\
0 & 2\sigma_1^j & 0 & \sigma_1\sigma_2
\end{pmatrix}.$$

This is then equivalent as matrix factorizations of Δ to:

$$S^G_{\mathrm{Res}^G_H \square_i \square_{jq-1}} \xrightarrow{\ j\ } S^G_{\mathrm{Res}^G_H \square_{i+1} \square_{jq}} \xrightarrow{\begin{pmatrix} \sigma_2 & 2\sigma_1^{(p+1-j)} \\ 2\sigma_1^j & \sigma_2\sigma_1 \end{pmatrix} \otimes I_2} S^G_{\mathrm{Res}^G_H \square_i \square_{jq-1}}.$$

Where j is a 4×4 matrix over R. Thus the module $M_{\mathrm{Res}_H^G\square_i\square_{jq-1}} \cong X^2_{p+1-j}$ for $1 \le j \le p$ and $0 \le i < q-1$.

2) The modules $M_{\mathrm{Res}_H^G\square_i\square_{jq+r}} \cong K^2_j$ for $1 \le j \le p-1$, $0 \le i < q-1$ and $0 \le r < 0$.

A basis for $S^G_{\mathrm{Res}_H^G\square_i\square_{jq}}$ is:

$$\{x^{jq+r}y^i, x^i y^{jq+r}, x^r y^{m-jq+i}, x^{m-jq+i}y^r\}.$$

and a basis for $S^G_{\mathrm{Res}\square_{i+1}\square_{jq+1}}$ is:

$$\{x^{jq+r+1}y^{i+1}, x^{i+1}y^{jq+r+1}, x^{r+1}y^{m-jq+i+1}, x^{m-jq+i+1}y^{r+1}\}.$$

and so by calculation we can express multiplication by z as the matrix:

$$\begin{pmatrix} \sigma_2 & 0 & 2\sigma_1^{p-j} & 0 \\ 0 & -\sigma_2 & 0 & -2\sigma_1^{p-j} \\ -2\sigma_1^j & 0 & -\sigma_2 & 0 \\ 0 & 2\sigma_1^j & 0 & \sigma_2 \end{pmatrix}$$

This is then equivalent to the matrix factorization of Δ:

$$S^G_{\mathrm{Res}_H^G\square_i\square_{jq}} \xrightarrow{\;j\;} S^G_{\mathrm{Res}_H^G\square_{i+1}\square_{jq+1}} \xrightarrow{\begin{pmatrix} \sigma_2 & 2\sigma_1^{(p-j)} \\ 2\sigma_1^j & \sigma_2 \end{pmatrix}\otimes I_2} S^G_{\mathrm{Res}_H^G\square_i\square_{jq}}$$

Where j is a 4×4 matrix over R. Thus for $1 \le j \le p-1$, $M_{\mathrm{Res}_H^G\square_0\square_{jq}} \cong K^2_{p-j}$.

3) $M_{\mathrm{Res}_H^G\square_{q-1}\square_{(j+1)q-1}} \cong N^2_{p-j}$ for $1 \le j \le p-1$.

A basis for $S^G_{\mathrm{Res}_H^G\square_{q-1}\square_{(j+1)q-1}}$ is:

$$\{x^{q-1}y^{(j+1)q-1}, x^{(j+1)q-1}y^{q-1}x^{m+q-1-jq}y^{q-1}, x^{q-1}y^{m+q-1-jq}\}$$

and a basis for $S^G_{\mathrm{Res}\square_0\square_{jq}}$ is:

$$\{x^{jq}, x^{jq}, y^{m-jq}, y^{m-jq}\}$$

and so by calculation we can express multiplication by z as the matrix:

$$\begin{pmatrix} \sigma_1\sigma_2 & 0 & 2\sigma_1^{p-(j-1)} & 0 \\ 0 & -\sigma_1\sigma_2 & 0 & -2\sigma_1^{p-(j-1)} \\ -2\sigma_1^{j+1} & 0 & -\sigma_1\sigma_2 & 0 \\ 0 & 2\sigma_1^{j+1} & 0 & \sigma_1\sigma_2 \end{pmatrix}$$

This is then equivalent to the matrix factorization:

$$S^G_{\mathrm{Res}^G_H\square_{q-1}\square_{(j+1)q-1}} \xrightarrow{\quad j \quad} S^G_{\mathrm{Res}^G_H\square_0\square_{jq}} \xrightarrow{\begin{pmatrix} \sigma_2\sigma_1 & 2\sigma_1^{p-(j-1)} \\ 2\sigma_1^{j+1} & \sigma_2\sigma_1 \end{pmatrix}\otimes I_2} S^G_{\mathrm{Res}^G_H\square_{q-1}\square_{(j+1)q-1}}$$

Where j is a 4×4 matrix over R. Thus for $1 \le j \le p-1$, $M_{\mathrm{Res}^G_H\square_{q-1}\square_{(j+1)q-1}} \cong N^2_{p-j}$. □

Lemma 3.7.4. A 2 dimensional irreducible representation of $G(m,p,2)$ is isomorphic to one of the following:

1) $\mathrm{Res}^G_H\square_i\square_{jq-1}$ for $0 \le i < q-1$, $1 \le j \le p$.

2) $\mathrm{Res}^G_H\square_i\square_{jq+r}$ for $0 \le j \le p-1$, $0 \le i < q-1$ and $0 \le r < q-1$.

3) $\mathrm{Res}^G_H\square_{q-1}\square_{(j+1)q-1}$ for $1 \le j \le p-1$.

Proof. Let λ the m-tuple of Young diagrams $\square_a\square_b$ where $0 \le a < b \le m-1$. Let $0 \le r \le q-1$ be such that $a = qd + r$ then

$$\square_a\square_b \cong \square_r\square_{b-qd}.$$

Therefore an irreducible representation is isomorphic to one of the form $\mathrm{Res}^G_H\square_i\square_j$ where $0 \le i \le q-1$ and $0 \le i \le j \le m-1$.

□

Remark 3.7.5. The irreducible representations described in Lemma 3.7.4 are not pairwise distinct, for example $\square_{q-1}\square_{(j+1)q-1} \cong \square_{q-1}\square_{m-(j-1)q-1}$ and so

$$\mathrm{Res}_H^G \square_{q-1}\square_{(j+1)q-1} \cong \mathrm{Res}_H^G \square_{q-1}\square_{m-(j-1)q-1}.$$

Theorem 3.7.6. Let $H = G(m,p,2)$, with p odd, $m \neq p$ and $q = \frac{m}{p}$ Then:

$$\begin{aligned} S/(z) &\cong \bigoplus_{j=1}^{p} X_j^{2(q-1)} \oplus \bigoplus_{j=1}^{\frac{p-1}{2}} N_j^2 \oplus \bigoplus_{j=1}^{\frac{p-1}{2}} K_j^{2(q-1)^2} \oplus K_0^{2\binom{q-1}{2}} \oplus R/(\sigma_1) \oplus (R/(\sigma_2^2 - 4\sigma_1^p))^{q-1} \oplus R/(\Delta) \\ &\cong \bigoplus_{j=1}^{p} X_j^{2(q-1)} \oplus \bigoplus_{j=1}^{\frac{p-1}{2}} N_j^2 \oplus \bigoplus_{j=1}^{\frac{p-1}{2}} K_j^{2(q-1)^2} \oplus R/(\sigma_1) \oplus (R/(\sigma_2^2 - 4\sigma_1^p))^{q-1+2\binom{q-1}{2}} \oplus R/(\Delta) \end{aligned}$$

Proof. Let $\mathfrak{O}$ be the orbit of hyperplanes that contain $\ker(x)$ and $\ker(y)$ and $\mathfrak{q}$ the other orbit. The linear characters are given by $\theta_{\mathfrak{O}}^i \otimes \theta_{\mathfrak{q}}^j$ for $0 \le i \le q-1$ and $0 \le j \le 1$. The modules $M_{\theta_{\mathfrak{O}}^i}$ for $0 \le i < q-1$ give the zero module. The module $M_{\theta_{\mathfrak{O}}^{q-1}}$ is $R/(\sigma_1)$, $M_{\theta_{\mathfrak{O}}^i \otimes \theta_{\mathfrak{q}}^1} \cong R/(\sigma_2^2 - 4\sigma_1^p)$ for $0 \le i < q-1$ and $M_{\theta_{\mathfrak{O}}^{q-1} \otimes \theta_{\mathfrak{q}}^1} \cong R/(\Delta)$. These are all of the modules that correspond to 1-dimensional representations.

From Theorem 3.7.3 and Lemma 3.7.4 we cover the 2 dimensional irreducible representations cases, but we need to count how many irreducible representations there are in each case.

The representations $\mathrm{Res}_H^G \square_i \square_{jq-1}$, for $0 \le i < q-1$, are all distinct since $i \neq q-1$. Then for each j there are $q-1$ representations of this form, giving rise to $q-1$ copies of X_{p-j}^2 in the decomposition of $S/(z)$.

Each representation of the form $\mathrm{Res}_H^G \square_{q-1}\square_{(j+1)q-1}$, for $1 \le j \le p-1$, is isomorphic to another, see Remark 3.7.5. There are $p-1$ m-tuples of Young diagrams of the form $\square_{q-1}\square_{(j+1)q-1}$ and each one is equivalent, under shifting, to exactly 1 distinct other of the same form. One can see that all representations $\mathrm{Res}_H^G \square_{q-1}\square_{(j+1)q-1}$ for $1 \le j \le \frac{p-1}{2}$ are distinct and are all of the representations appearing from m-tuples of Young diagrams of the form $\square_{q-1}\square_{(j+1)q-1}$. Thus one copy of N_j^2 for $1 \le j \le \frac{p-1}{2}$ appears in the decomposition.

The representations $\mathrm{Res}_H^G \square_i \square_{jq+r}$ for $0 \le j \le \frac{p-1}{2}$, $0 \le i < q-1$ and $0 \le r < q-1$ are all distinct and for $\frac{p-1}{2} < k < p-1$, $\square_i \square_{kq+r} \cong \square_r \square_{(p-k)q+i}$ where $0 \le p-k \le \frac{p-1}{2}$.

Each $\mathrm{Res}^G_H \square_i \square_{jq+r}$ for $0 \le j \le \frac{p-1}{2}$, $0 \le i < q-1$ and $0 \le r < q-1$, gives a copy of K^2_j in the decomposition. For $j = 0$, since $i \neq r$ in this case the choice of i and r gives $\binom{q-1}{2}$ distinct representations - all giving K^2_0. For all $1 \le j \le \frac{p-1}{2}$ we have $(q-1)^2$ distinct representations coming from the choice of i and r, these then each give a copy of K^2_j.

□

(2) $m \neq p$ and p even.

For the case when p is even, $\mathrm{Spec}(R/(\Delta))$ is also a D_{p+2} singularity. Here we use the list of CM modules found in [Yos90, 9.12 p.78]. The modules above are still indecomposable CM modules, although, since $p+2$ is even, there are a few more indecomposable CM modules. Additionally we have

- $C_+ = \mathrm{Coker}(\sigma_1(\sigma_2 + 2\sigma_1^{\frac{p}{2}}), \sigma_2 - 2\sigma_1^{\frac{p}{2}})$,
- $D_+ = \mathrm{Coker}(\sigma_2 - 2\sigma_1^{\frac{p}{2}}, \sigma_1(\sigma_2 + 2\sigma_1^{\frac{p}{2}}))$,
- $C_- = \mathrm{Coker}(\sigma_1(\sigma_2 - 2\sigma_1^{\frac{p}{2}}), \sigma_2 + 2\sigma_1^{\frac{p}{2}})$,
- $D_- = \mathrm{Coker}(\sigma_2 + 2\sigma_1^{\frac{p}{2}}, \sigma_1(\sigma_2 - 2\sigma_1^{\frac{p}{2}}))$.

We also have the isomorphisms $K_{\frac{p}{2}} \cong C_+ \oplus C_-$ and $N_{\frac{p}{2}} \cong D_+ \oplus D_-$.

Theorem 3.7.7. Let $H = G(m,p,2)$, with $m \neq p$ and p even. The discriminant defines a D_{p+2} singularity and the 2 dimensional representations correspond to the following CM modules in the decomposition of $S/(z)$:

- $M_{\mathrm{Res}^G_H \square_i \square_{jq-1}} \cong X^2_{p+1-j}$ for $1 \le j \le p-1$ and $0 \le i < q-1$.
- $M_{\mathrm{Res}^G_H \square_i \square_{jq}} \cong K^2_j$ for $1 \le j \le p-1$, $0 \le i < q-1$, $0 \le r < q-1$ and $j \neq \frac{p}{2}$.
- $M_{\mathrm{Res}^G_H \square_{q-1} \square_{(j+1)q-1}} \cong N^2_j$ for $1 \le j \le p-1$ and $j \neq \frac{p}{2}$.

Proof. The calculations are the same as in the odd case, see Theorem 3.7.3. □

Lemma 3.7.8. Let $G = G(m,p,2)$, with $m \neq p$ and p even. A 2 dimensional irreducible representation of $G(m,p,2)$ is isomorphic to one of the following:

1) $\mathrm{Res}^G_H \Box_i \Box_{jq-1}$ for $0 \le i < q-1$, $1 \le j \le p$.

2) $\mathrm{Res}^G_H \Box_i \Box_{jq+r}$ for $0 \le j \le p-1$, $0 \le i < q-1$, $0 \le r < q-1$ and $j \neq \frac{p}{2}$.

3) $\mathrm{Res}^G_H \Box_{q-1} \Box_{(j+1)q-1}$ for $1 \le j \le p-1$ and $j \neq \frac{p}{2}$.

Proof. This is the similar to the odd case, with one exception. Consider the m-tuple of Young diagrams $\Box_i \Box_{i+\frac{m}{2}}$, then $\mathrm{Res}^G_H \Box_i \Box_{i+\frac{m}{2}}$ splits into two 1 dimensional irreducible representations. The cases where this happens are excluded in the above list. □

Theorem 3.7.9. Let $H = G(m,p,2)$, with $m \neq p$ and p even, $m \neq p$ and $q = \frac{m}{p}$ then

$$\begin{aligned} S/(z) &\cong \bigoplus_{j=1}^{p} X_j^{2(q-1)} \oplus \bigoplus_{j=1}^{\frac{p-2}{2}} N_j^2 \oplus \bigoplus_{j=1}^{\frac{p-2}{2}} K_j^{2(q-1)^2} \oplus K_{\frac{p}{2}}^{2\binom{q-1}{2}} \oplus K_0^{2\binom{q-1}{2}} \oplus R/(\sigma_1) \oplus (R/\left(\sigma_2^2 - 4\sigma_1^p\right))^{q-1} \\ &\quad \oplus C_+ \oplus D_+ \oplus C_- \oplus D_- \oplus R/(\Delta) \\ &\cong \bigoplus_{j=1}^{p} X_j^{2(q-1)} \oplus \bigoplus_{j=1}^{\frac{p-2}{2}} N_j^2 \oplus \bigoplus_{j=1}^{\frac{p-2}{2}} K_j^{2(q-1)^2} \oplus R/(\sigma_1) \oplus (R/\left(\sigma_2^2 - 4\sigma_1^p\right))^{q-1+2\binom{q-1}{2}} \\ &\quad \oplus C_+^{1+2\binom{q-1}{2}} \oplus D_+ \oplus C_-^{1+2\binom{q-1}{2}} \oplus D_- \oplus R/(\Delta). \end{aligned}$$

Proof. Let $\mathfrak{O}$ be the orbit of hyperplanes such that $j_{\mathfrak{O}} := \prod_{H \in \mathfrak{O}} \alpha_H = \sigma_1$ and let $\mathfrak{q}_+, \mathfrak{q}_-$ be the orbit such that $j_{\mathfrak{q}_+} = \sigma_2 + 2\sigma_1^{\frac{p}{2}}, j_{\mathfrak{q}_-} = \sigma_2 - 2\sigma_1^{\frac{p}{2}}$. The linear characters are given by $\theta^i_{\mathfrak{O}} \otimes \theta^j_{\mathfrak{q}_+} \otimes \theta^k_{\mathfrak{q}_-}$ for $0 \le i \le q-1$, $0 \le j \le 1$ and $0 \le k \le 1$. The modules $M_{\theta^i_{\mathfrak{O}}}$ for $0 \le i < q-1$ give the zero module. The module $M_{\theta^{q-1}_{\mathfrak{O}}}$ is $R/(\sigma_1)$, $M_{\theta^i_{\mathfrak{O}} \otimes \theta^1_{\mathfrak{q}_+} \otimes \theta^1_{\mathfrak{q}_-}} \cong R/\sigma_2^2 - 4\sigma_1^p$ for $0 \le i < q-1$ and $M_{\theta^{q-1}_{\mathfrak{O}} \otimes \theta^1_{\mathfrak{q}_+} \otimes \theta^1_{\mathfrak{q}_-}} \cong R/(\Delta)$. Now we can find the isomorphisms $M_{\theta^{q-1}_{\mathfrak{O}} \otimes \theta^1_{\mathfrak{q}_+}} \cong C_+$ and $M_{\theta^i_{\mathfrak{O}} \otimes \theta^1_{\mathfrak{q}_-}} \cong D_+$ for $0 \le i < q-1$. Also $M_{\theta^{q-1}_{\mathfrak{O}} \otimes \theta^1_{\mathfrak{q}_-}} \cong C_-$ and $M_{\theta^i_{\mathfrak{O}} \otimes \theta^1_{\mathfrak{q}_+}} \cong D_-$ for $0 \le i < q-1$.

As with the odd case, we use Theorem 3.7.7 and Lemma 3.7.8, but we need to count how many irreducible representations there are in each case.

The representations $\mathrm{Res}^G_H \Box_i \Box_{jq-1}$, for $0 \le i < q-1$ are all distinct since $i \neq q-1$. Then for each j there are $q-1$ representations of this form, giving rise to $q-1$ copies of X_j in the decomposition of $S/(z)$.

Each representation in the $\mathrm{Res}_H^G \square_{q-1}\square_{(j+1)q-1}$, for $1 \le j \le p-1$ and $j \ne \frac{p}{2}$, case is isomorphic to another one. When $j = \frac{p}{2}$, $\mathrm{Res}_H^G \square_{q-1}\square_{(j+1)q-1}$ is not an irreducible 2-dimensional representation, since it is the direct sum of two 1-dimensional ones. One can see that all representations $\mathrm{Res}_H^G \square_{q-1}\square_{(j+1)q-1}$ for $1 \le j \le \frac{p-2}{2}$ are distinct and are all of the representations appearing from m-tuples of Young diagrams of the form $\square_{q-1}\square_{(j+1)q-1}$. Thus one copy of N_j^2 for $1 \le j \le \frac{p-2}{2}$ appears in the decomposition.

For the case of the representation $\mathrm{Res}_H^G \square_i\square_{jq+r}$ for $0 \le j \le \frac{p-2}{2}$, $0 \le i < q-1$ and $0 \le r < q-1$ are all distinct and for $\frac{p+2}{2} \le k < p-1$, $\square_i\square_{kq+r} \cong \square_r\square_{(p-k)q+i}$ where $0 \le p-k \le \frac{p-2}{2}$. Each $\mathrm{Res}_H^G \square_i\square_{jq+r}$ for $0 \le j \le \frac{p-2}{2}$, $0 \le i < q-1$ and $0 \le r < q-1$ gives a copy of K_j^2 in the decomposition. For $j = 0$, since $i \ne r$ in this case the choice of i and r gives $\binom{q-1}{2}$ distinct representations, which all give K_0^2. For all $1 \le j \le \frac{p-2}{2}$ we have $(q-1)^2$ distinct representations coming from the choice of i and r - each giving a copy of K_j^2. Since we are in the even case if $i = r$ and $j = \frac{p}{2}$ then the representation $\mathrm{Res}_H^G \square_i\square_{jq+r}$ splits into two irreducible 1-dimensional representations, but if $i \ne r$ and $j = \frac{p}{2}$ then $\mathrm{Res}_H^G \square_i\square_{jq+r}$ is an irreducible 2-dimensional representation and gives a copy of $K_{\frac{p}{2}}^2$. Note that for $i \ne r$ and $j = \frac{p}{2}$, $\square_i\square_{jq+r} \cong \square_r\square_{jq+i}$ and so the number of distinct representations of this form are counted by $\binom{q-1}{2}$.

□

(3) $m = p$

The group $G(m,m,2)$ is a true reflection group and thus we already know from [BFI20, Theorem 4.17] that $S/(z)$ is a NCR for $R/(\Delta)$. In loc. cit. it was shown that for a rank 2 true reflection group there is a one to one correspondence between the isomorphism classes of indecomposable CM modules over the discriminant and the isotypical components. We use the same techniques as before to calculate the matrix factorization for the discriminant, which in this case is an A_{m-1} singularity. Let $G = G(m,1,2)$ with $m > 2$ and $H = G(m,m,2)$, the representations $\mathrm{Res}_H^G \square_i\square_j \cong \mathrm{Res}_H^G \square_{i+1}\square_{j+1}$ where $0 \le i < j < m$ and $i+1$ and $j+1$ are taken mod m. From this we see that the action of

$-\otimes\det$ is the identity on all irreducible two-dimensional representations of H.

Remark 3.7.10. We avoid the case $m=2$ since $G(2,2,2)$ is a reducible reflection group and the case $m=1$ is S_2.

The 2 dimensional irreducible representations of $G(m,m,2)$ are then distinguished by the distance between the two boxes in the tuple. In particular the representations $\mathrm{Res}_H^G \square_0 \square_i$ for $1 \le i < \frac{m}{2}$ are all distinct and any other irreducible two-dimensional representation is isomorphic to one of these, as can be seen by shifting the m-tuple of diagrams. Note that if m is even the representation $\mathrm{Res}_H^G \square_0 \square_{\frac{m}{2}}$ is not irreducible.

Recalling that the discriminant is $\Delta = (x^m - y^m)^2 = \sigma_2^2 - 4\sigma_1^m$, then $\mathrm{Spec}(R/(\Delta))$ is an A_{m-1} singularity. Here we use the list of CM modules found in [Yos90, 9.9 p.75] for the m is odd case and [Yos90, 5.11 p.41] in the m is even case. In the m is odd case the CM modules come from the matrix factorizations (ϕ_j, ψ_j) where:

$$\phi_j = \begin{bmatrix} \sigma_2 & 2\sigma_1^{m-j} \\ 2\sigma_1^j & \sigma_2 \end{bmatrix} \qquad \text{and} \qquad \psi_j = \begin{bmatrix} \sigma_2 & -2\sigma_1^{m-j} \\ -2\sigma_1^j & \sigma_2 \end{bmatrix}$$

for $0 \le j \le p$. Let $X_j = \mathrm{Coker}(\phi_j, \psi_j)$, then $X_0 \cong R$, $X_j \cong X_{p-j}$. When m is odd, these are all the irreducible CM modules over $R/(\Delta)$. When m is even, then $(\sigma_2 + 2\sigma_1^{\frac{m}{2}}, \sigma_2 - 2\sigma_1^{\frac{m}{2}})$ and $(\sigma_2 - 2\sigma_1^{\frac{m}{2}}, \sigma_2 + 2\sigma_1^{\frac{m}{2}})$ are matrix factorizations. Let N_+ and N_- be the CM modules given by $(\sigma_2 + 2\sigma_1^{\frac{m}{2}}, \sigma_2 - 2\sigma_1^{\frac{m}{2}})$ and $(\sigma_2 - 2\sigma_1^{\frac{m}{2}}, \sigma_2 + 2\sigma_1^{\frac{m}{2}})$ respectively, Then $X_{\frac{m}{2}} \cong N_+ \oplus N_-$.

Lemma 3.7.11. Let $G = G(m,1,2)$ and $H = G(m,m,2)$ then the modules $M_{\mathrm{Res}_H^G \square_0 \square_i} \cong X_i^2$ for $0 \le i < \frac{m}{2}$.

Proof. A basis for the isotypical components of the coinvariant algebra S_H of H, of type $\mathrm{Res}_H^G \square_0 \square_i$ for $1 \le i < \frac{m}{2}$ is $\{x^i, y^i, x^{m-i}, y^{m-i}\}$. Recalling that $z = x^m - y^m$, calculating on this basis, multiplication by z can be expressed as the matrix:

$$\begin{pmatrix} \sigma_2 & 0 & 0 & 2\sigma_1^{m-i} \\ 0 & -\sigma_2 & -2\sigma_1^{m-i} & 0 \\ 0 & 2\sigma_1^i & \sigma_2 & 0 \\ -2\sigma_1^i & 0 & 0 & -\sigma_2 y^{m-i} \end{pmatrix}.$$

This is then equivalent as matrix factorizations to:

$$S^G_{\mathrm{Res}^G_H \square_0 \square_i} \xrightarrow{\quad j \quad} S^G_{\mathrm{Res}^G_H \square_1 \square_{i+1}} \xrightarrow{\begin{pmatrix} \sigma_2 & 2\sigma_1^{m-i} \\ 2\sigma_1^i & \sigma_2 \end{pmatrix} \otimes I_2} S^G_{\mathrm{Res}^G_H \square_0 \square_i}$$

Where j is a 4×4 matrix. □

So in the odd case we have found all the indecomposable CM modules over $R/(\Delta)$. In the even case since $\mathrm{Res}^G_H \square_0 \square_{\frac{m}{2}}$ splits into 2 one-dimensional representations, from Section 2.3 we obtain the modules corresponding to the components of the discriminant.

Theorem 3.7.12. Let $H = G(m,m,2)$, with m even, then

$$S/(z) \cong \bigoplus_{j=1}^{\frac{m-2}{2}} X_j^2 \oplus N_+ \oplus N_- \oplus R/(\Delta).$$

Let $H = G(m,m,2)$, with m odd, then

$$S/(z) \cong \bigoplus_{j=1}^{\frac{m-1}{2}} X_j^2 \oplus R/(\Delta).$$

Proof. The representations $\square_0 \square_j$ are distinct for $1 \le j \le \frac{m-1}{2}$ and $\square_0 \square_{\frac{m-1}{2}+i} \cong \square_0 \square_{\frac{m-1}{2}-i}$ for $1 \le i \le \frac{m-1}{2}$ so using Lemma 3.7.11 we get one copy of X^2 for each j. One copy of N_+, N_- and $R/(\Delta)$ come from the 1 dimensional representations. □

Bringing all of the sections together we get the following corollary of Theorem 3.4.22,3.7.6, 3.7.9 and 3.7.12.

Corollary 3.7.13. Let $G = G(m,p,2)$ where $m \ge 2$ and p divides m, then $S/(z)$ is a representation generator for $CM(R/(\Delta))$ and in particular $\mathrm{End}_{R/(\Delta)}(S/(z))$ is a NCR for $R/(\Delta)$.

3.8 Crossover with true reflection groups

The case of $G(2p,p,2)$ is similar to that of $G(m,m,2)$ for $m \geq 2$. The groups $G(2p,p,2)$ are also true reflection groups and so by [BFI20, Theorem 4.17] there is a one to one correspondence between the irreducible representations and isomorphism classes of indecomposable CM modules over $R/(\Delta)$. This section details the one to one correspondence using the notation from the previous section. We only present the p odd case, since the p even case is similar.

Lemma 3.8.1. The irreducible two-dimensional representations of $H = G(2p,p,2)$ where p is odd are given by:

1) $\mathrm{Res}^G_H \square_0 \square_{2i-1}$ for $1 \leq i \leq p$,

2) $\mathrm{Res}^G_H \square_0 \square_{2i}$ for $1 \leq i \leq \frac{p-1}{2}$,

3) $\mathrm{Res}^G_H \square_1 \square_{2i+1}$ for $1 \leq i \leq \frac{p-1}{2}$.

Proof. Since for a given $2p$-tuple of young diagrams we can always shift until there is a box in the first 2 positions, it is enough to consider the $2p$-tuples such that there is a box in position 0 or position 1. This gives us two cases:

1) $\mathrm{Res}^G_H \square_0 \square_i$ for $1 \leq i \leq 2p$.

2) $\mathrm{Res}^G_H \square_1 \square_i$ for $1 \leq i \leq 2p$.

Then noting that $\square_0 \square_{2i-1} \cong \square_1 \square_{2p-2(i-1)}$ gives us the list above. $\square$

Recall that for $G(2p,p,2)$, the discriminant $\mathrm{Spec}(R/(\Delta))$ is a D_{p+2} singularity. We use the same notation for the modules as in Theorem 3.7.3

Let p be odd, for the subgroup $H = G(2p,p,2)$ of $G = (2p,1,2)$ the following holds:

1) $M_{\mathrm{Res}^G_H \square_0 \square_{2i-1}} \cong X^2_i$for $1 \leq i \leq p$.

2) $M_{\mathrm{Res}^G_H \square_0 \square_{2i}} \cong K^2_i$ for $1 \leq i \leq \frac{p-1}{2}$.

3) $M_{\mathrm{Res}_H^G \square_1 \square_{2i+1}} \cong N_i^2$ for $1 \le i \le \frac{p-1}{2}$.

From this we can see the one to one correspondence on the 2-dimensional representations, since for the cases above there is only one representation of each form.

Example 3.8.2. Consider $H = G(6,3,2)$ and $G = G(6,1,2)$ then there are the following equivalence classes of 6-tuples of young diagrams:

$$\{\square_0\square_1, \square_2\square_3, \square_4\square_5\}, \qquad \{\square_0\square_2, \square_2\square_4, \square_0\square_4\},$$

$$\{\square_0\square_3, \square_2\square_5, \square_1\square_4\}, \qquad \{\square_1\square_2, \square_3\square_4, \square_0\square_5\},$$

$$\{\square_1\square_3, \square_3\square_5, \square_1\square_5\}.$$

Then:

$$M_{\mathrm{Res}_H^G \square_0 \square_1} = M_{\mathrm{Res}_H^G \square_2 \square_3} \cong X_1^2, \qquad M_{\mathrm{Res}_H^G \square_0 \square_2} \cong K_1^2,$$

$$M_{\mathrm{Res}_H^G \square_0 \square_3} \cong X_2^2, \qquad M_{\mathrm{Res}_H^G \square_1 \square_3} \cong N_1^2$$

and

$$M_{\mathrm{Res}_H^G \square_1 \square_2} \cong Y_1^2 \cong M_{\mathrm{Res}_H^G \square_0 \square_5} \cong X_3^2.$$

The linear characters are of the form $\mathrm{Res}_H^G \begin{array}{|c|}\hline \, \\ \hline \, \\ \hline\end{array}_0$, $\mathrm{Res}_H^G \begin{array}{|c|}\hline \, \\ \hline \, \\ \hline\end{array}_1$, $\mathrm{Res}_H^G \begin{array}{|c|c|}\hline \, & \, \\ \hline\end{array}_0$ and $\mathrm{Res}_H^G \begin{array}{|c|c|}\hline \, & \, \\ \hline\end{array}_1$. The modules corresponding to these representations are

$$M_{\mathrm{Res}_H^G M \begin{array}{|c|c|}\hline \, & \, \\ \hline\end{array}_0} \text{ is the trivial module,} \qquad M_{\mathrm{Res}_H^G M \begin{array}{|c|}\hline \, \\ \hline \, \\ \hline\end{array}_1} \cong R/(\Delta),$$

and $M_{\mathrm{Res}_H^G M \begin{array}{|c|}\hline \, \\ \hline \, \\ \hline\end{array}_0}$, $M_{\mathrm{Res}_H^G M \begin{array}{|c|c|}\hline \, & \, \\ \hline\end{array}_1}$ are the two irreducible components of Δ. This shows the one to one correspondence between irreducible representations of G and irreducible CM modules over the discriminant.

Chapter 4

Matrix Factorizations of the discriminant of S_n

The set up is the same as the previous chapters. That is, S_n naturally acts on a finite dimensional vector space V of dimension n over a field k where Char(k) does not divide the order of the group $|S_n| = n!$. By fixing a basis $\{x_1, \dots, x_n\}$ of V, we form the symmetric algebra $\mathrm{Sym}_k(V) \cong k[x_1, \dots, x_n]$ of V. The action of S_n on V can be naturally extended to $S = k[x_1, \dots, x_n]$ via $\pi \cdot f(x) = f(\pi(x))$ for $\pi \in S_n$. We denote by R the invariant ring $R = S^{S_n}$ and Δ is the discriminant of S_n.

Recall from the theorem of Chevalley–Shepard–Todd (see Theorem 1.6.4) and Example 1.4.4 and 1.6.9 that the invariant ring $R \cong k[\sigma_1, \dots, \sigma_n]$, where σ_i are the elementary symmetric polynomials in $x_1, \dots, x_n$. Note that R is a graded polynomial ring with $\deg \sigma_i = i$.

Similar methods to that of Chapter 3 can be used to study the discriminant, Δ of S_n for $n \geq 3$. There are a few key differences between the S_n and the $G(m,p,2)$ case, one immediate problem is that, for $n > 2$, $R/(\Delta)$ is not of finite CM type and not all isomorphism classes of CM $R/(\Delta)$ modules will appear in the decomposition of $S/(z)$ as a $R/(\Delta)$ module. Matrix factorizations for $n = 4$ were studied by Hovinen in his thesis and gives a classification of rank 1 homogenous modules of the discriminant of S_4 [Hov09].

The group S_n is a true reflection group and so the result that $\text{End}_{R/(\Delta)}(S/(z))$ is an NCR for $R/(\Delta)$ from [BFI20] can be used. We wish to investigate exactly which CM modules appear in the decomposition of $S/(z)$. This is achieved by explicitly calculating the CM modules which correspond to each irreducible representation using higher Specht polynomials and modifications of them.

Recall, from Lemma 2.2.4 the coinvariant $S/(R_+)$ algebra for S_n is equipped with the regular representation of S_n, so the matrix factorization given by the reduced hyperplane arrangement z is given by:

$$\begin{array}{ccccc} S(-2^{\frac{n(n-1)}{2}+1}) & \xrightarrow{z} & S(-2^{\frac{n(n-1)}{2}}) & \xrightarrow{j} & S \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ R^{n!} & \xrightarrow{\quad z \quad} & R^{n!} & \xrightarrow{\quad j \quad} & R^{n!} \end{array}$$

Where the map z is given by a $n! \times n!$ matrix over R and $(-)$ ensure that the maps z and j are graded morphisms. The degree of the polynomial z is $2^{\frac{n(n-1)}{2}}$ which, recalling Definition 2.1.9, is the number of reflections of S_n.

Remark 4.0.1. The symmetric group S_n is a true reflection group where the transposition (i,j) fixes the hyperplane $x_i - x_j$. For all the hyperplanes H in $A(S_n)$ the order of the fixer group of G is $e_H = 2$. This means that, from Definition 2.1.9, $z = j$ and $z^2 = \Delta$.

After using the techniques in Chapter 2 and decomposing the matrix factorization into the isotypical components of $S/(R_+)$, the calculations are still large. For example, the representation given by the partition $(k,1,1\cdots 1)$ where $0 < k \leq n-1$ is of dimension $\binom{n-1}{k}$ and the isotypical component of $S/(R_+)$ is of dimension $\binom{n-1}{k}^2$, see Section 2.2. The goal of this chapter is to find a basis in which we can decompose each isotypical component further.

We will first modify the higher Specht polynomials for the coinvariant algebra of S_n, which were first announced in [TY93] and constructed in more detail in [ATY97]. We then show that these modified higher Specht polynomials give a basis in which, for the isotypical component $S_\lambda^{S_n}$, the matrix factorization splits into $\dim(V_\lambda)$ many copies of equivalent matrix factorizations.

This work was achieved through a collaboration with colleagues: Eleonore Faber[1], Colin Ingalls[2] and Marco Talarico [3]. Examples were generated by code written by Marco Talarico in the language Macaulay 2 [GS].

4.1 Decomposition (z,z) for S_n

During this section fix n. We consider the decomposition of the coinvariant algebra $S/(R_+)$ and the multiplication of z restricted to each isotypical component. The group S_n is $G(1,1,n)$ from Chapter 3 and so the irreducible representations of the S_n are given by Young tableau, see [Ful96]. We have already covered Young tableau in Section 3.3, but we will recall some of the definitions in this context. Basis elements for the isotypical components are then given by higher Specht polynomials [ATY97], we follow the definitions as in loc. cit.

Definition 4.1.1. (See Definition 2.1.9) Let $A(S_n)$ be the set of reflecting hyperplanes of the action of S_n. Let $H \in A(S_n)$ be such a hyperplane, and let α_H be the linear form defining H in S. Then

$$z = \prod_{H \in A} \alpha_H = \prod_{1 \leq i < j \leq n} (x_i - x_j)$$

is a polynomial in S defining the reflection arrangement of S_n, that is $V(z) = A(S_n)$.

Definition 4.1.2. (See Definition 2.1.11) The discriminant of the S_n action on k^n is defined by:

$$\Delta = z^2 = \prod_{H \in A} \alpha_H^2 = \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \ .$$

Remark 4.1.3. Recall from Lemma 2.1.13 that Δ is also an element of $R = S^G$.

Denote the set of irreducible representations V_λ of S_n by irrep(S_n). Recall from Lemma 2.2.4, the R module $S/(R_+)$ is isomorphic as a G representation to the regular representation, in particular

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$$S/(R_+) \cong \bigoplus_{V_i \in \text{irrep}(S_n)} V_i^{\dim V_i}$$

From Definition 2.1.9 the polynomial z is the relative invariant for the determinantal representation, see [OT92, Theorem 6.37] (This was proved in [Sta77, Theorem 3.1] for characteristic 0). That is, z generates the direct summand of $S/(R_+)$ corresponding to $V_{\text{det}} = V_\lambda$ where λ is the Young diagram

$$\lambda = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array} \\ \vdots \\ \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \end{array}$$

We recall definitions from Section 3.4 in this context

Definition 4.1.4. Let $\lambda \vdash n$ and T be a standard tableau of shape λ. We define the *word* $w(T)$ to be the sequence obtained by reading each column from bottom to top starting from the left. The *index* $i(T) = i(w(T))$ is inductively defined as; $i(1) = 0$, and let $i(k) = p$. If $k+1$ is to the right of k in $w(T)$ then $i(k+1) = p$, if $k+1$ is to the left of k in $w(T)$ then $i(k+1) = p+1$. We then assign to $i(w(T))$ a tableau of shape λ with the entries of the cells corresponding to their indexes. Further we define $\hat{i}(T)$ to be $i(T)$ written in non decreasing order and $|i(T)|$ to be the sum of the indexes.

Example 4.1.5. Let $n = 5$ and consider the tableaux

$$T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}.$$

The word $w(T_1) = (3,1,5,2,4)$ and the index $i(T_1) = (1,0,2,0,1)$. The tableau assigned to $i(T_1)$ is

$$i(T_1) = \begin{array}{|c|c|c|}\hline 0 & 0 & 1 \\ \hline 1 & 2 \\ \cline{1-2} \end{array}.$$

Furthermore, $\hat{i}(T_1) = (0,0,1,1,2)$ and $|i(T_1)| = 4$.

Definition 4.1.6. Let $\lambda \vdash n$ and $T_1, T_2 \in \mathrm{ST}(\lambda)$. We define the *Last Letter Ordering (LL)* in the following way. Let $1 \le k \le n$ be the largest integer that is written in a different position for both tableaux T_1 and T_2. If the row in which k appears in T_2 is above the row it appears in T_1, then we say $T_1 < T_2$.

Example 4.1.7. Let $n = 5$, Consider the following two tableaux T_1 and T_2 on the partition $(3,2)$.

$$T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array} < \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array} = T_2$$

The set of elements that are in different positions is $\{2,3\}$ thus the maximal element which has a different position is 3. Note that in T_1 the element 3 is written in the second row, which is below the first row where 3 is written in T_2.

Definition 4.1.8. Given a Young Tableaux T of shape λ, we define two subgroups of S_n, the *Row Stabilizer* $R(T)$ which are all elements of S_n that permute elements within the same row, and similarly the *Column Stabilizer* $C(T)$ which permutes elements within the same columns of T. With these subgroups we define the following

$$r_T = \sum_{r \in R(T)} r \qquad \text{and} \qquad c_T = \sum_{c \in C(T)} \mathrm{sgn}(c)c.$$

Lastly we define the *Young Symmetrizers* as;

$$\varepsilon_T = \frac{f^\lambda}{n!} c_T r_T \qquad \text{and} \qquad \sigma_T = \frac{f^\lambda}{n!} r_T c_T$$

where f^λ is the number of standard tableau of shape λ. These are both idempotents in kS_n, see [JK81, Theorem 3.10].

In order to construct our matrix factorizations with respect to the isotypical components, we will require a basis of $S/(R_+)$ over R, this will be constructed using higher Specht polynomials from [ATY97] and also a modification of these polynomials:

Definition 4.1.9. Let T, P be two standard Young tableaux of shape λ. The *higher Specht polynomials* are defined as

$$F_T^P = \varepsilon_T . x_T^P \ ,$$

where $x_T^P = x_{w(T)}^{i(P)} = x_{w(T)^0}^{i(P)^0} \cdots x_{w(T)^{n-1}}^{i(P)^{n-1}}$ in multiindex notation. We further define the *modified higher Specht polynomials* as

$$H_T^P = \sigma_T . x_T^P \ .$$

Definition 4.1.10. For a tableau $P \in \mathrm{ST}(\lambda)$ let M^P the submodule of S generated by $\{H_T^P \mid T \in \mathrm{ST}(\lambda)\}$ and N^P the submodule of S generate by $\{F_T^P \mid T \in \mathrm{ST}(\lambda)\}$.

Theorem 4.1.11. [ATY97, Theorem 1, (2)] Let $P \in \mathrm{ST}(\lambda)$, then M^P is a S_n-subrepresentation of S isomorphic to the irreducible representation corresponding to P.

Lemma 4.1.12. [FH91, Example 4.4] Let $P \in \mathrm{ST}(\lambda)$ then N^P is a representation of S_n and is isomorphic as a representation to M^P.

Definition 4.1.13. For a tableau $T \in \mathrm{ST}(\lambda)$ let M_T be the R-submodule of $S/(R_+)$ generated by $\{H_T^P \mid P \in \mathrm{ST}(\lambda)\}$ and N_T the R-submodule generated by $\{F_T^P \mid P \in \mathrm{ST}(\lambda)\}$.

Remark 4.1.14. The modules M_T and N_T are not irreducible representations of S_n, they are free R-submodules of $S/(R_+)$.

Theorem 4.1.15. [TY93, Theorem 1] The collection:

$$\bigcup_{\lambda \vdash n} \{F_T^S \mid T \in \mathrm{ST}(\lambda), S \in \mathrm{ST}(\lambda)\}$$

form a k basis for $S/(R_+)$.

Example 4.1.16. Let T be the following standard tableau:

$$T = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline \end{array} \\ \vdots \\ \begin{array}{|c|} \hline n-1 \\ \hline n \\ \hline \end{array}$$

Note that T gives rise to the determinantal representation $V_{\det}$ of S_n. The Young Symmetrizer ε_T is given by:

$$\varepsilon_T = \frac{1}{n!}c_T r_T = \frac{1}{n!}\left(\sum_{\pi\in C(T)} \operatorname{sgn}(\pi)\pi\right) id = \frac{1}{n!}\sum_{\pi\in S_n} \operatorname{sgn}(\pi)\pi.$$

The index $i(T) = (n-1, n-2, \ldots, 1, 0)$ and so $F_T^T = \varepsilon_T(x_1^0 x_2^1 \cdots x_n^{n-1})$. The higher Specht polynomial F_T^T is the polynomial z. Moreover, in this case we also have

$$\sigma_T = \frac{1}{n!}r_T c_T = \frac{1}{n!}id\left(\sum_{\pi\in C(T)} \operatorname{sgn}(\pi)\pi\right) = \varepsilon_T.$$

and so $H_T^T = F_T^T = \frac{1}{n!}z$.

Lemma 4.1.17. If $T_1 < T_2$ then $\varepsilon_{T_1}\varepsilon_{T_2} = \sigma_{T_2}\sigma_{T_1} = 0$.

Proof. The proof for $\varepsilon_{T_1}\varepsilon_{T_2} = 0$ is widely known, see [ATY97, Lemma 4] or [Ste11, Proposition 1]. The equality $\sigma_{T_2}\sigma_{T_1} = 0$ can be seen by using a similar proof. □

Definition 4.1.18. For a Young tableau $T \in \mathrm{ST}(\lambda)$ we write T' for its *conjugate tableau*. Note that $T' \in \mathrm{ST}(\lambda')$, where λ' is the conjugate partition.

Example 4.1.19. Let T be the tableau

$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2}\end{array}$$

then T' is the partition

$$T' = \begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 & 7 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

Lemma 4.1.20. Let T be a Young tableau of shape λ then

$$\varepsilon_T(zf) = z\sigma_{T'}(f)$$

for any polynomial f in S.

Proof. We first observe that for a Young tableau T of shape λ, $R(T) = C(T')$, $C(T) = R(T')$ and so

$$\varepsilon_T = \frac{1}{n!}\sum_{r\in R(T),c\in C(T)} \mathrm{sgn}(c)rc = \sum_{c\in C(T'),r\in R(T')} \mathrm{sgn}(r)cr.$$

We also have that for any $\pi \in S_n$, $\pi(z) = \mathrm{sgn}(\pi)z$, and so for any polynomial f;

$$\begin{aligned}\varepsilon_T(zf) &= \frac{1}{n!}\sum_{r\in R(T),c\in C(T)} \mathrm{sgn}(c)rc(zf) = \frac{1}{n!}\sum_{c\in C(T'),r\in R(T')} \mathrm{sgn}(r)cr(zf)\\ &= z(\frac{1}{n!}\sum_{c\in C(T'),r\in R(T')} \mathrm{sgn}(c)cr(f)) = z(\sigma_{T'}(f)).\end{aligned}$$

□

Theorem 4.1.21. For the discriminant Δ of S_n, the matrix factorization defined by the reduced hyperplane arrangement, (z,z), can be decomposed in the following way:

$$(z,z) = \bigoplus_{\lambda\vdash n}\bigoplus_{T\in\mathrm{ST}(\lambda)} (z|_{M_T}, z|_{N_{T'}}).$$

Here $(z|_{M_T}, z|_{N_{T'}})$ are the matrix factorizations:

$$M_T \xrightarrow{z|_{M_T}} N_{T'} \xrightarrow{z|_{N_{T'}}} M_T$$

where $M_T = \langle F_T^P \,|\, P \in \mathrm{ST}(\lambda)\rangle$ and $N_T = \langle F_T^P \,|\, P \in \mathrm{ST}(\lambda)\rangle$.

Proof. For an irreducible representation λ of G, recall that the map z takes elements from the isotypical component S_λ of S to the isotypical component $S_{\lambda'}$ of S for $\lambda' = \lambda \otimes \det$. Thus the matrix factorization decomposes straight away as:

$$(z,z) = \bigoplus_{\lambda\vdash n}(z|_{S_\lambda}, z|_{S_{\lambda'}}).$$

Let $T,P \in \mathrm{ST}(\lambda)$, and consider $H_T^P \in S_\lambda$ from above, hence $zH_T^P \in S_{\lambda'}$. We can write zH_T^P as the following;

$$zH_T^P = \sum_{U,W\in\mathrm{ST}(\lambda')} g_{U,T}^{P,W} F_U^W\ , \tag{4.1}$$

where $g_{U,T}^{P,W}$ are elements of R since, by Theorem 4.1.15 the F_U^W form an R-basis of $S_{\lambda'}$. Recall that $F_T^P = \varepsilon_T.x_T^P$, and $H_T^P = \sigma_T.x_T^P$. If $T_1 < T_2$ then by Lemma 4.1 we have that $\varepsilon_{T_1}\varepsilon_{T_2} = \sigma_{T_2}\sigma_{T_1} = 0$, hence $\varepsilon_{T_1}F^P_{T_2} = \varepsilon_{T_1}\varepsilon_{T_2}.x^P_{T_2} = 0$ and $\sigma_{T_2}H^P_{T_1} = \sigma_{T_2}\sigma_{T_1}.x^P_{T_1} = 0$. Order $\mathrm{ST}(\lambda') = (T'_1, \ldots, T'_k)$ in such a way that if $i < j$ then $T'_i < T'_j$. We want to calculate $zH^P_{T_1}$. We apply $\varepsilon_{T'_1}$ to the left hand side of equation (4.1) and obtain

$$\begin{aligned} \varepsilon_{T'_1} \sum_{U,W \in \mathrm{ST}(\lambda')} g^{P,W}_{U,T_1} F^W_U &= \sum_{U,W \in \mathrm{ST}(\lambda')} g^{P,W}_{U,T_1} (\varepsilon_{T'_1} F^W_U) \\ &= \sum_{W \in \mathrm{ST}(\lambda')} g^{P,W}_{T'_1,T_1} F^W_{T'_1}. \end{aligned} \tag{4.2}$$

Since T'_1 is the least element in $\mathrm{ST}(\lambda')$. Now we apply $\varepsilon_{T'_1}$ to the right hand side of equation (4.1) and obtain

$$\begin{aligned} \varepsilon_{T'_1}(zH^P_{T_1}) &= z(\sigma_{T_1} H^P_{T_1}) \\ &= zH^P_{T_1}. \end{aligned} \tag{4.3}$$

Using Equation (4.2) and (4.3) we obtain

$$zH^P_{T_1} = \sum_{W \in \mathrm{ST}(\lambda')} g^{P,W}_{T'_1,T_1} F^W_{T'_1}.$$

Since for any $T'_i \in \mathrm{ST}(\lambda)$ there exists a permutation π that permutes T_i and T_1 we have the following equation for any $T_i \in \mathrm{ST}(\lambda)$.

$$zH^P_{T_i} = \sum_{W \in \mathrm{ST}(\lambda)} (\mathrm{sgn}(\pi) g^{P,W}_{T'_i,T_1}) F^W_{T'_i}.$$

This shows that for any H^P_T in S_λ the polynomial zH^P_T is contained in $\langle F^P_{T'} \mid P \in \mathrm{ST}(\lambda')\rangle$. Following a similar argument to that above the restriction of z to $\langle F^P_{T'} \mid P \in \mathrm{ST}(\lambda')\rangle$ yields an element of $\langle H^P_T \mid P \in \mathrm{ST}(\lambda)\rangle$. In particular:

$$zF^P_{T'_i} = \sum_{W \in \mathrm{ST}(\lambda)} (\mathrm{sgn}(\pi) h^{P,W}_{T_i,T'_k}) H^W_{T_i}.$$

Since (z,z) is a matrix factorization of Δ for a $T \in \mathrm{ST}(\lambda)$ and using the notation above we can write the matrices of the maps as

$$\left[g^{P,W}_{T,T'}\right]\left[h^{P,W}_{T',T}\right]_{P,W' \in \mathrm{ST}(\lambda)} = \Delta \mathrm{Id}_{|\mathrm{ST}(\lambda)| \times |\mathrm{ST}(\lambda)|}.$$

□

Theorem 4.1.22. Assume the same situation as in Theorem 4.1.21 and let $T_1, T_2 \in \mathrm{ST}(\lambda)$. Then there is a matrix factorization equivalence between $(z|_{M_{T_1}}, z|_{N_{T_1'}})$ and $(z|_{M_{T_2}}, z|_{N_{T_2'}})$.

Proof. Consider $\pi \in S_n$ such that $\pi(T_1) = T_2$, thus for any $P \in \mathrm{ST}(\lambda)$ we have that

$$zH^P_{T_1} = \mathrm{sgn}(\pi) z H^P_{T_2} \text{ and } zF^{P'}_{T_1'} = \mathrm{sgn}(\pi) z F^{P'}_{T_2'}.$$

This means that $z|_{M_{T_1}} = \mathrm{sgn}(\pi) z|_{M_{T_2}}$ and $z|_{N_{T_1'}} = \mathrm{sgn}(\pi) z|_{N_{T_2'}}$. Therefore the matrices are the same up to multiplication by an invertible scalar matrix and so there is a matrix factorization equivalence between them. □

Remark 4.1.23. Let λ be a Young diagram of size n, then $|\mathrm{ST}(\lambda)| = \dim S_\lambda$, so we get $\dim S_\lambda$ copies of the CM module corresponding to the matrix factorization $(z|_{N_{T_1}}, z|_{M_{T_1'}})$ of Δ in the decomposition of $S/(z)$.

We want to find a better way to computationally compute the matrix factorizations given in Theorem 4.1.21.

Definition 4.1.24. Let T be a tableau with $w(T) = (w_0, \ldots w_{n-1})$ and index $i(T) = (i_0, \ldots i_{n-1})$. Let $w(T)' = (w_{n-1}, \ldots, w_0)$ and $i(T)' = (i_{n-1}, \ldots i_0)$ then the *coindex* $j(T)$ is $i(w(T)')'$.

Example 4.1.25. Let $n = 5$ and consider the tableau

$$T_1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \cline{1-2}\end{array}.$$

from Example 4.1.5 then the coindex is $j(T_1) = (1,0,2,1,2)$ and the corresponding tableau

$$j(T_1) = \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 1 & 2 & \\ \cline{1-2}\end{array}.$$

Definition 4.1.26. Define a bilinear form $\langle -, - \rangle : S \to S$ by

$$\langle f, g \rangle = \frac{1}{z} \sum_{\pi \in S_n} \mathrm{sgn}(\pi) \pi(fg)$$

where f, g are elements of S.

Remark 4.1.27. The element $\sum_{\pi\in S_n}\operatorname{sgn}(\pi)\pi(fg)$ of S is an alternating polynomial, and so will have a factor of z. This means that the element $\langle f,g\rangle = \frac{1}{z}\sum_{\pi\in S_n}\operatorname{sgn}(\pi)\pi(fg)$ is well defined.

Remark 4.1.28. Note that this bilinear form is similar to the one used in [ATY97], except that we do not set the variables to 0

Recall from Definition 4.1.4 that for a tableau T, $\hat{\imath}(T)$ is $i(T)$ written in non decreasing order and $|i(T)|$ is the sum of the indexes. Consider the ordering on $\mathrm{ST}(\lambda)$, where $P_1 < P_2$ if and only if either $|\hat{\imath}(P_1)| < |\hat{\imath}(P_2)|$, or if $|\hat{\imath}(P_1)| = |\hat{\imath}(P_2)|$ then $\hat{\imath}(P_1) < \hat{\imath}(P_2)$ with respect to the reverse lexicographical ordering, but if $\hat{\imath}(P_1) = \hat{\imath}(P_2)$ then $P_1 < P_2$ with respect to the last letter order.

Lemma 4.1.29. Let $P_1 < P_2$ with respect to the ordering above, then $\langle F_T^{P_1}, F_{T'}^{P_2'}\rangle = 0$

Proof. Note that this bilinear form is the one used in [ATY97], except that we do not set the variables to 0. The main idea here is that if $\deg(fg) < \deg(\delta) = \frac{n(n-1)}{2}$ then $\langle f,g\rangle$ is 0 and if $\deg(fg) = \frac{n(n-1)}{2}$ then $\langle f,g\rangle$ is a 0 or a constant. In these cases the result [ATY97, Proposition 1] for $\langle -,-\rangle$ hold, thus it is sufficient to show that if $P_1 < P_2$ then $\deg(F_T^{P_1} F_T'^{P_2'}) < \frac{n(n-1)}{2}$. We consider 3 different cases:

(1) If $|\hat{\imath}(P_1)| < |\hat{\imath}(P_2)|$ then $|\hat{\imath}(P_1)| + |\hat{\jmath}(P_2)| < \frac{n(n-1)}{2}$ by [ATY97, Lemma 1].

(2) If $|\hat{\imath}(P_1)| = |\hat{\imath}(P_2)|$ then $|\hat{\imath}(P_1)| = \frac{n(n-1)}{2} - |\hat{\jmath}(P_2)|$ and $|\hat{\imath}(P_1)| + |\hat{\jmath}(P_2)| = \frac{n(n-1)}{2}$. Thus from the proof of [ATY97, Theorem 1] the results hold if $\hat{\imath}(P_1) < \hat{\imath}(P_2)$ in the reverse lexicographical ordering hold.

(3) If $|\hat{\imath}(P_1)| = |\hat{\imath}(P_2)|$ and $\hat{\imath}(P_1) = \hat{\imath}(P_2)$ then if $P_1 < P_2$ with respect to the last letter order the results hold from the same argument for (2).

□

Theorem 4.1.30. Let λ be a Young diagram, where m is the dimension of the corresponding irreducible representation and $T \in \mathrm{ST}(\lambda)$. The matrix factorization

$(z|_{N_T}, z|_{M_{T'}})$ can be written explicitly as (A,B), where

$$A = \begin{bmatrix} g_1^1 & \cdots & g_m^1 \\ \vdots & \ddots & \vdots \\ g_1^m & \cdots & g_m^m \end{bmatrix}$$

with entries $g_j^i \in R$ and B is a $m \times m$ matrix obtained by taking the first reduced syzygy $\mathrm{syz}_R^1 A$. If we order $\mathrm{ST}(\lambda) = (P_1, \ldots, P_m)$ where if $i < j$ then $P_i < P_j$, then the entries of the matrix g_i^j are defined iteratively as:

$$g_i^j = \frac{\langle F_T^{T_j}, zH_T^{T_i} - g_i^1 F_{T'}^{T'_1} - \cdots - g_i^{j-1} F_{T'}^{T_{j-1}} \rangle}{\langle F_T^{T_j}, F_{T'}^{T'_j} \rangle}$$

for $1 \le i \le m$ and $1 \le j \le m$.

Proof. Note that $\langle F_T^P, F_{T'}^{P'} \rangle$ is a non-zero constant in $\mathbb{C}$, and a formula is given in [ATY97, Proposition 1]. Order $\mathrm{ST}(\lambda) = (P_1, \ldots, P_m)$ where if $i < j$ then $P_i < P_j$, so if $i < j$ then $\langle F_T^{P_i}, F_{T'}^{P'_j} \rangle = 0$. Consider the matrix describing $z|_{H_T} : H_T \to F_{T'}$ to have the entries $[g_i^j]_{i,j}$ where i,j indexes the rows and columns. We calculate

$$zH_T^{P_i} = g_i^1 F_{T'}^{P'_1} + \cdots + g_i^m F_{T'}^{P'_m}$$

Using the bilinear form with $F_T^{P_1}$ we have

$$\langle F_T^{P_1}, zH_T^{P_i} \rangle = g_i^1 \langle F_T^{P_1}, F_{T'}^{P'_1} \rangle + \cdots + g_i^m \langle F_T^{P_1}, F_{T'}^{P'_m} \rangle$$

for $1 < j$ the term $g_i^j \langle F_T^{P_1}, F_{T'}^{P'_j} \rangle = 0$, therefore $\langle F_T^{P_1}, zH_T^{P_i} \rangle = g_i^1 \langle F_T^{P_1}, F_{T'}^{P'_1} \rangle$. Using this we can then recursively write a formula for each entry.

$$g_i^1 = \frac{\langle F_T^{P_1}, zH_T^{P_i} \rangle}{\langle F_T^{P_1}, F_{T'}^{P'_1} \rangle}$$

$$\vdots$$

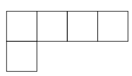
$$g_i^j = \frac{\langle F_T^{P_j}, zH_T^{P_i} - g_i^1 F_{T'}^{P'_1} - \cdots - g_i^{j-1} F_{T'}^{P_{j-1}} \rangle}{\langle F_T^{P_j}, F_{T'}^{P'_j} \rangle}.$$

These are the entries in column i, and the matrix A can be computed by considering all $i = 1, \ldots m$. □

Remark 4.1.31. This gives a quicker computational way to calculate the matrix factorization corresponding to a irreducible representation of S_n, and thus a maximal CM module over R, for a specific irreducible representation λ of S_n.

Example 4.1.32. Let S_5 act on $\mathbb{C}[x_1,x_2,x_3,x_4,x_5]$ with the basic invariants σ_1, σ_2, σ_3, σ_4, σ_5. If we quotient out by the hyperplane σ_1, and let $s = -x_2 - x_3 - x_4 - x_5$ we get a set of invariants $t_1,\dots,t_4$ of the action of S_5 on $\mathbb{C}[x_2,x_3,x_4,x_5]$, where $t_i = \sigma_{i+1}(s,x_2,x_3,x_4,x_5)$ The discriminant is given by:

$$\begin{aligned}\Delta = &-\frac{1}{3600}t_1^3t_2^2t_3^2+\frac{1}{900}t_1^4t_3^3+\frac{1}{900}t_1^3t_2^3t_4-\frac{1}{200}t_1^4t_2t_3t_4+\frac{3}{400}t_1^5t_4^2-\\ &\frac{3}{1600}t_2^4t_3^2+\frac{1}{100}t_1t_2^2t_3^3-\frac{2}{225}t_1^2t_3^4+\frac{3}{400}t_2^5t_4-\frac{7}{160}t_1t_2^3t_3t_4+\\ &\frac{7}{180}t_1^2t_2t_3^2t_4+\frac{11}{192}t_1^2t_2^2t_4^2-\frac{1}{16}t_1^3t_3t_4^2+\frac{4}{225}t_3^5-\frac{1}{9}t_2t_3^3t_4+\frac{5}{32}t_2^2t_3t_4^2+\\ &\frac{5}{36}t_1t_3^2t_4^2-\frac{25}{96}t_1t_2t_4^3+\frac{125}{576}t_4^4\end{aligned}$$

Let $\lambda =$  be a partition of n, which corresponds to the standard representation of S_n, then the matrix factorization of $S/(z)$ corresponding to λ is (A,B) where

$$A = \begin{pmatrix} t_4 & -\frac{1}{50}t_1t_3 & -\frac{1}{50}t_2t_3+\frac{1}{10}t_1t_4 & -\frac{1}{25}t_3^2+\frac{1}{10}t_2t_4 \\ -\frac{8}{5}t_3 & \frac{2}{25}t_1t_2-\frac{1}{2}t_4 & \frac{2}{25}t_2^2-\frac{2}{15}t_1t_3 & \frac{3}{50}t_2t_3-\frac{3}{10}t_1t_4 \\ \frac{6}{5}t_2 & -\frac{3}{25}t_1^2+\frac{2}{5}t_3 & -\frac{4}{75}t_1t_2+\frac{1}{3}t_4 & -\frac{1}{25}t_1t_3 \\ -\frac{4}{5}t_1 & -\frac{3}{10}t_2 & -\frac{4}{15}t_3 & -\frac{1}{2}t_4 \end{pmatrix}$$

and B is a 4×4 matrix with entries in R such that $\operatorname{Coker} B \cong \operatorname{Syz}_R^1(A)$. Note that as the dimension of the representation corresponding to λ is 4, we get 4 copies of this matrix and thus 4 copies of $\operatorname{Coker} A$ in the decomposition of $S/(z)$.

4.2 Decomposition for product submodules of S_n

In this section we generalize Theorem 4.1.21 to irreducible representations of the *Young Subgroups* of S_n. These subgroup are of the form $S_{n_1}\times\cdots\times S_{n_m} \le S_n$ for any given m-tuple $(n_1,\dots,n_m)$ with $n_i \geq 0$ and $\sum_{i=1}^{m} n_i = n$. In particular we can also decompose the matrix factorization (z,z) of the discriminant Δ of S_n using the irreducible representations of a fixed Young subgroup $S_{n_1} \times \cdots \times S_{n_m}$. The irreducible representations of the Young subgroup $S_{n_1} \times \cdots \times S_{n_m}$ will be of the form $V_{n_1} \otimes \ldots \otimes V_{n_m}$ where each V_i is a irreducible representation of S_{n_i} and we will discuss a basis for the coinvariant algebra $S/(R_+)$ indexed by these representations. While the decomposition will be coarser, the motivation for this section is that by using this construction one can describe the basis elements for the coinvariant algebra for the groups $G(m,1,n)$.

The description of the irreducible representations of $S_{n_1} \times \cdots \times S_{n_m} \leq S_n$ are similar to that of $G(m,1,n)$. We first recall some definitions from Section 3.3. Consider $m \geq 0$ and $(n_1,\dots,n_m)$ to be a list of integers such that $n_i \geq 0$ and $\sum n_i = n$. $\lambda = (\lambda_1,\dots,\lambda_m)$ is an m-tuple of Young diagrams of type $(n_1,\dots,n_m)$ if $\lambda_i \vdash n_i$ for all $1 \leq i \leq m$. An m-tableau $T = (T_1,\dots,T_m)$ is of shape λ if each T_i is of shape λ_i, and is called an *m-tableau*. An m-tableau is standard if all of its tableaux are standard, with the set of all standard m-tableau being $\mathrm{ST}(\lambda)$.

We also make some new definitions for this section.

Definition 4.2.1. Let $\mathrm{ST}(n_1,\dots,n_m)$ be the set of standard m-tableau of shape λ for all m-tuples of Young diagrams λ of type $(n_1,\dots,n_m)$.

Definition 4.2.2. Let $T = (T^1,\dots,T^m)$ be a standard m-tableau of type $(n_1,\dots,n_m)$ then the *conjugate* m-tableau is $T' = ((T^1)',\dots,(T^m)')$.

Example 4.2.3. Let $n = 7$, $m = 3$ then

$$T = \left(\begin{array}{|c|c|}\hline 1 & 7 \\ \hline 5 & \\ \cline{1-1}\end{array}, -, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 6 \\ \hline\end{array} \right)$$

is a standard m-tableau of type $(3,0,4)$. The conjugate tableau T' is given by

$$T' = \left(\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 7 & \\ \cline{1-1}\end{array}, -, \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 & 6 \\ \hline\end{array} \right)$$

Remark 4.2.4. Our definition of T' is different to that of [ATY97], where they also reverse the order of the m-tableau. This is so that in our definition given a $T \in \mathrm{ST}(\lambda)$, the m-tableau T' is in $\mathrm{ST}(\lambda') = \mathrm{ST}(\lambda \otimes \det)$. The consequence of our definition is that we will not be able to use the same bilinear form reduction to get a similar result to Theorem 4.1.30 as before.

As in the previous section, we will define an ordering on $\mathrm{ST}(n_1,\ldots,n_m)$ as an extension of the last letter ordering. First we will consider an ordering on $\mathrm{ST}(\lambda)$. Consider two standard m-tableau $T_1 = (T_1^1,\ldots,T_1^m)$ and $T_2 = (T_2^1,\ldots,T_2^m)$ of the same shape. Let $1 \le l \le n$ be the greatest number that appears in different cells in both of the tableaux. We say $T_1 < T_2$ if either l is written in T_1^i and T_2^j with $i < j$, or if it is written in a row in T_1^i below a row in T_2^i, for $1 \leqslant i \leqslant n$.

Example 4.2.5. Let $n = 4$, $m = 2$ and $\lambda = \left(\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline \\ \hline\end{array}\right)$ then, with respect to the Last Letter ordering on $\mathrm{ST}(\lambda)$:

$$\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 3 \\ \hline\end{array}\right) < \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \begin{array}{|c|}\hline 4 \\ \hline\end{array}\right)$$

Now we consider when the last number that appears in the different cells is on the same tableaux for both m-tableau. Let $n = 4, m = 2$ and $\lambda = \left(\begin{array}{|c|c|}\hline & \\ \hline & \\ \hline\end{array}, \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}\right)$ then with respect to the Last Letter ordering on $\mathrm{ST}(\lambda)$:

$$\left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 5 & 6 \\ \hline 7 \\ \cline{1-1}\end{array}\right) < \left(\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 5 & 7 \\ \hline 6 \\ \cline{1-1}\end{array}\right)$$

Theorem 4.2.6. Let λ be a Young diagram of type $(n_1,\ldots,n_m)$, then the last letter ordering is total in $\mathrm{ST}(\lambda)$.

Proof. Let $T_1 = (T_1^1,\ldots,T_1^m)$ and $T_2 = (T_2^1,\ldots,T_2^m) \in \mathrm{ST}(\lambda)$ then either $T_1 = T_2$ or there exists, at least 2 elements which appear in different boxes. Let l be the greatest number that changes. If l is written in T_1^i and T_2^j for $i \neq j$ then either $T_1 < T_2$ or $T_2 < T_1$. When

$i = j$ then l must appear in a different rows of T_1 and T_2 otherwise one of T_1, T_2 would not be standard, since l was chosen to be the greatest number that changes. Since the last number that changes appears in different rows then either $T_1 < T_2$ or $T_2 < T_1$. □

If we consider the lexicographical ordering of the partitions of type $(n_1, \dots, n_m)$ we can fully order $\mathrm{ST}(n_1, \dots, n_m)$. Let us consider a ordering on partitions of n: if $\lambda_1 = (\alpha_1, \dots, \alpha_k)$ and $\lambda_2 = (\beta_1, \dots, \beta_l)$ are two partitions of n, and let $1 \le i \le \min(k, l)$ be the first integer that $\alpha_i - \beta_i \ne 0$. Then $\alpha_i - \beta_i > 0$ if and only if $\lambda_1 < \lambda_2$. Using this ordering, it is easy to see that we can use lexicographical ordering on partition of type $(n_1, \dots, n_m)$ to have a total ordering. This way if we have two tableaux T and V, if they are in the same partition we may order them using LL-order, and if they are in different partition give their order with lexicographical order on the partitions.

Example 4.2.7. Let us consider $m = 1$ and $n = 4$ then we can order the partitions the following way;

$$(4) < (3,1) < (2,2) < (2,1,1) < (1,1,1,1)$$

If we consider $\lambda_1 = (3,1)$, $\lambda_2 = (2,2)$, $T \in \mathrm{ST}(\lambda_1)$ and $P \in \mathrm{ST}(\lambda_2)$ as below, by our ordering we have:

$$\begin{ytableau} 1 & 2 & 3 \\ 4 \end{ytableau} < \begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau}$$

Now consider $m = 3$ and $n = 6$, and define $\lambda_1 = \left(\ydiagram{2,1}, \ydiagram{1}, \ydiagram{2}\right)$ and $\lambda_2 = \left(\ydiagram{2,1}, \ydiagram{1}, \ydiagram{1,1}\right)$ then since $\ydiagram{2} < \ydiagram{1,1}$ we have the following:

$$\left(\begin{ytableau} 2 & 3 \\ 5 \end{ytableau}, \begin{ytableau} 1 \end{ytableau}, \begin{ytableau} 4 & 6 \end{ytableau}\right) < \left(\begin{ytableau} 2 & 3 \\ 5 \end{ytableau}, \begin{ytableau} 1 \end{ytableau}, \begin{ytableau} 4 \\ 6 \end{ytableau}\right)$$

Definition 4.2.8. Let λ be an m-partition and consider an m-tableau $T = (T_1, \dots, T_m)$ of shape λ. We call T *natural* if the numbers written in tableau T_i are contained in the set $\{\sum_{j=1}^{i-1} n_j + 1, \dots, \sum_{j=1}^{i} n_j\}$. We denote the set of natural standard m-tableau of shape

λ by NST(λ) and NST($n_1,\ldots,n_m$) is the set of all natural standard tableaux on all the partitions of type $(n_1,\ldots,n_m)$.

Example 4.2.9. Take $m = 3$ and $n = 5$, and consider the shape $\lambda = \left(\begin{smallmatrix}\square & \square\\ \square & \end{smallmatrix}, -, \begin{smallmatrix}\square & \square\end{smallmatrix}\right)$. To make a natural standard m-tableaux with this shape the tableau in the first position must contain $\{1,2,3\}$ and the tableau in the last position must contain $\{4,5\}$. For example, the 3-tableau

$$\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}, -, \begin{array}{|c|c|}\hline 4 & 5 \\ \hline\end{array}\right)$$

is a natural standard m-tableaux and is in NST(λ).

We can define Young symmetrizers in a similar fashion to the previous section, for a given m-tableau $T = (T_1,\ldots,T_m)$ we define $\varepsilon_T = \varepsilon_{T_1}\cdots\varepsilon_{T_m}$ and similarly $\sigma_T = \sigma_{T_1}\cdots\sigma_{T_m}$. The following theorem shows how $S/(R_+)$ decomposes into irreducible representations of a Young subgroup.

Theorem 4.2.10. [ATY97, Theorem 1] Fix n and let $(n_1,\ldots,n_m)$ be a sequence such that $\sum_{i=0}^{m} n_i = n$. Then the collection:

$$\bigcup_{\lambda\vdash(n_1,\ldots,n_m)} \{F_T^S \mid T \in \mathrm{NST}(\lambda), S \in \mathrm{ST}(\lambda)\}.$$

Form a k-basis for $S/(R_+)$. For $\lambda \vdash (n_1,\ldots,n_m)$, let $S \in \mathrm{ST}(\lambda)$. Then the collection

$$\{F_T^S \mid T \in \mathrm{NST}(\lambda)\}$$

forms a basis of the $S_{n_1} \times \cdots \times S_{n_m}$ submodule of $S/(R_+)$ which is isomorphic to irreducible representation V_λ.

Example 4.2.11. Let us consider the Young subgroup $S_1 \times S_2$ inside S_3. There are two, 2-tuples of Young diagrams that partition (1,2) namely:

$$\lambda_1 = \left(\begin{array}{|c|}\hline \\ \hline\end{array}, \begin{array}{|c|c|}\hline & \\ \hline\end{array}\right) \qquad \lambda_2 = \left(\begin{array}{|c|}\hline \\ \hline\end{array}, \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}\right)$$

$$\mathrm{NST}(\lambda_1)=\left\{\left(\begin{array}{|c|}\hline 1\\\hline\end{array},\begin{array}{|c|c|}\hline 2&3\\\hline\end{array}\right)\right\}\qquad\qquad \mathrm{NST}(\lambda_2)=\left\{\left(\begin{array}{|c|}\hline 1\\\hline\end{array},\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)\right\}$$

and

$$\mathrm{ST}(\lambda_1)=\left\{\left(\begin{array}{|c|}\hline 1\\\hline\end{array},\begin{array}{|c|c|}\hline 2&3\\\hline\end{array}\right),\left(\begin{array}{|c|}\hline 2\\\hline\end{array},\begin{array}{|c|c|}\hline 1&3\\\hline\end{array}\right),\left(\begin{array}{|c|}\hline 3\\\hline\end{array},\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}\right)\right\}$$

$$\mathrm{ST}(\lambda_2)=\left\{\left(\begin{array}{|c|}\hline 1\\\hline\end{array},\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right),\left(\begin{array}{|c|}\hline 2\\\hline\end{array},\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\right),\left(\begin{array}{|c|}\hline 3\\\hline\end{array},\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}\right)\right\}$$

Thus we get 3 copies of the irreducible representation corresponding to λ_1 and 3 copies of the irreducible representation corresponding to λ_2

Lemma 4.2.12. Let $T_1, T_2 \in \mathrm{NST}(n_1,\ldots,n_m)$. If $T_1 < T_2$, then $\varepsilon_{T_1}\varepsilon_{T_2} = 0$.

Proof. Let $T_1, T_2 \in \mathrm{NST}(n_1,\ldots,n_m)$, then we can commute terms of the Young symmetrizer such that $\varepsilon_{T_1}\varepsilon_{T_2} = \varepsilon_{T_1^1}\cdots\varepsilon_{T_1^m}\varepsilon_{T_2^1}\cdots\varepsilon_{T_2^m} = \varepsilon_{T_1^1}\varepsilon_{T_2^1}\cdots\varepsilon_{T_1^m}\varepsilon_{T_2^m}$ Let $T_1 < T_2$ and suppose that the last number that appears in different cells is contained in T_1^i. If T_1 and T_2 are of the same shape then $\varepsilon_{T_1^i}\varepsilon_{T_2^j} = \varepsilon_{T_2^j}\varepsilon_{T_1^i}$ for $i \neq j$. Now if $T_1^i < T_2^i$, then $\varepsilon_{T_1}\varepsilon_{T_2} = 0$ follows from Lemma 4.1. If T_1^i and T_2^i are of different shapes, $\varepsilon_{T_1^i}\varepsilon_{T_2^i} = 0$ and thus $\varepsilon_{T_1}\varepsilon_{T_2} = 0$. □

Lemma 4.2.13. Let T be a standard m-tableau, then for any $f \in S$

$$z\varepsilon_T(f) = \sigma_{T'}(f).$$

Proof. Let $f \in S$ then $z\varepsilon_T(f) = z\varepsilon_{T^1}\ldots\varepsilon_{T^m}(f) = \varepsilon_{T^1}\ldots\varepsilon_{T^{m-1}}z\sigma_{(T^m)'}(f) = \sigma_{(T^1)'}\ldots\sigma_{(T^m)'}(zf)$. □

Definition 4.2.14. Let λ be an m-tuple of Young diagrams of size $(n_1,\ldots,n_m)$ and let $T \in \mathrm{NST}(\lambda)$. Let $M_T = \langle H_T^S \mid S \in \mathrm{ST}(\lambda)\rangle$ and $N_T = \langle F_T^S \mid S \in \mathrm{ST}(\lambda')\rangle$.

Remark 4.2.15. These are analogous modules to the ones defined in Definition 4.1.13 and are free R-submodules of $S/(R_+)$, but are not irreducible representations of S_n .

Theorem 4.2.16. For the discriminant Δ of S_n, the matrix factorization defined by the reduced hyperplane arrangement, (z,z), can be decomposed in the following way:

$$(z,z) = \bigoplus_{\lambda \vdash (n_1,\ldots,n_m)} \bigoplus_{T \in \mathrm{NST}(\lambda)} (z|_{M_T}, z|_{N_{T'}}).$$

and $(z|_{M_T}, z|_{N_{T'}})$ are the matrix factorizations:

$$M_T \xrightarrow{z|_{M_T}} N_{T'} \xrightarrow{z|_{N_{T'}}} M_T.$$

Proof. Recall from Theorem 4.2.6 that we can order all of the natural standard tableaux with n cells, $\mathrm{NST}(n_1,\ldots,n_m)$, such that if $i<j$ then $T_i > T_j$. Thus, with this ordering if $i<j$ then, by Lemma 4.2.12 $\varepsilon_{T_j}\varepsilon_{T_i} = \sigma_{T_i}\sigma_{T_j} = 0$. Let d be the size of $\mathrm{NST}(n_1,\ldots,n_m)$, we can decompose S as $S = \oplus_{1\le i\le d} M_{T_i} = \oplus_{1\le i\le d} N_{T_i}$. It is clear that, since for a m-tableau T, ε_T and σ_T are idempotent if $i<j$ then $\varepsilon_{T_i} F_{T_j} = 0$ and $\sigma_{T_j} H_{T_i} = 0$. Therefore consider $1 \le h \le d$, let P be a standard tableaux of the same shape as T_h. Then we can split $zH^P_{T'_h}$ into the different components of S, where each $f_{T'_i} \in N_{T'_i}$, i.e

$$zH^P_{T_h} = f_{T'_1} + f_{T'_2} + \cdots + f_{T'_h} + \cdots + f_{T'_d} \; . \tag{4.4}$$

Claim: For each $1 \le j < h$, each component $f_{T'_j} = 0$.

We prove the claim by induction. Let $j = 1$, since $1 < h$ then $T_h < T_1$ hence $\sigma_{T_1} H^P_{T_h} = 0$. Then

$$\varepsilon_{T'_1} z H^P_{T_h} = \varepsilon_{T'_1}(f_{T'_1} + \cdots + f_{T'_d})$$

and from Lemma 4.2.13 we have

$$\begin{aligned} z(\sigma_{T_1} H^P_{T_h}) &= \varepsilon_{T'_1} f_{T'_1} + \cdots + \varepsilon_{T'_1} f_{T'_d} \\ 0 &= f_{T'_1} \end{aligned}$$

Assume that the claim is true for $j-1$. Since $j<h$ then $T_j > T_h$ thus $\sigma_{T_j} H^P_T = 0$.

Therefore we have the following computation;

$$\begin{aligned}
\varepsilon_{T'_j} z H^P_{T_h} &= \varepsilon_{T'_j}(f_{T'_1} + \cdots + f_{T'_j} + \cdots + f_{T'_d}) \\
z(\sigma_{T_j} H^P_{T_h}) &= \varepsilon_{T'_j} f_{T'_1} + \cdots + \varepsilon_{T'_j} f_{T'_j} + \cdots + \varepsilon_{T'_j} f_{T'_d} \\
0 &= f_{T'_j}.
\end{aligned}$$

Hence equation (4.4) reduces to

$$zH^S_{T_h} = f_{T'_h} + f_{T'_{h+1}} + \cdots + f_{T'_d}. \tag{4.5}$$

After applying $\varepsilon_{T'_h}$ to (4.5), the left hand side becomes $\varepsilon_{T'_h}(zH^P_{T_h}) = z(\sigma_{T_h} H^P_{T_h}) = zH^P_{T_h}$. If $j > h$, then $\varepsilon_{T'_h} f_{T'_j} = 0$, thus $zH^P_{T_h} = f_{T'_h}$. In other words $z|_{M_T} : M_T \to N_{T'}$. A similar argument can be made about $zF^P_{T_i}$ thus proving the statement.

The argument above shows that for a $T \in \mathrm{NST}(n_1, ..., n_m)$, $\mathrm{Im}(z|_{M_T}) = N_{T'}$, similarly one could show that $\mathrm{Im}(z|_{N_{T'}}) = M_T$. Therefore the matrix factorization splits as

$$(z, z) = \oplus_{T \in NST(n_1,\ldots,n_m)} (z|_{H_T}, z|_{F_{T'}}).$$

□

Example 4.2.17. Consider the Young subgroup $S_1 \times S_2$ inside S_3 and let $\sigma_1, \sigma_2, \sigma_3$ be the invariants of the S_3 action, then the matrix representing the multiplication by z is given by;

$$\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}$$

where B is the 3×3 matrix:

$$B = \begin{bmatrix} -\frac{1}{2}\sigma_1^2\sigma_2 + 2\sigma_2^2 - \frac{3}{2}\sigma_1\sigma_3 & -\sigma_1^2\sigma_3 + 3\sigma_2\sigma_3 & -\frac{1}{2}\sigma_1\sigma_2\sigma_3 + \frac{9}{2}\sigma_3^2 \\ 2\sigma_1^2 - 6\sigma_2 & \sigma_1\sigma_2 - 9\sigma_3 & 2\sigma_2^2 - 6\sigma_1\sigma_3 \\ -\frac{1}{2}\sigma_1\sigma_2 + \frac{9}{2}\sigma_3 & -\sigma_2^2 + 3\sigma_1\sigma_3 & -\frac{1}{2}\sigma_1\sigma_2^2 + 2\sigma_1^2\sigma_3 - \frac{3}{2}\sigma_2\sigma_3 \end{bmatrix}$$

and A is the 3×3 matrix:

$$A = \begin{bmatrix} -2\sigma_2 & -2\sigma_1\sigma_3 & -6\sigma_3 \\ 4\sigma_1 & \sigma_1\sigma_2 + 3\sigma_3 & 4\sigma_2 \\ -6 & -2\sigma_2 & -2\sigma_1 \end{bmatrix}$$

A is the matrix of $(z|_{M_T}, z|_{N_{T'}})$ where;

$$T = \left(\begin{array}{|c|}\hline 1 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array}\right) \qquad \left(\begin{array}{|c|}\hline 1 \\ \hline \end{array}, \begin{array}{|c|}\hline 2 \\ \hline 3 \\ \hline \end{array}\right)$$

of NST(1,2).

Remark 4.2.18. The matrix factorizations from Theorem 4.1.21 and 4.2.16 are equivalent as matrix factorizations, since they are matrices that describe the same map between free modules. We get the equivalence by applying a change of basis of $S/(R_+)$.

Chapter 5

Lusztig algebras of pseudo reflection groups

This chapter is based on Section 9 of my paper [May21] in which the Lusztig algebra of $G(m,1,2)$ is calculated. We have the same setup as before, with $k = \mathbb{C}$, but we need new notation to be able to differentiate between groups. Let G be a finite pseudo reflection group which acts on V_G a finite dimensional $\mathbb{C}$-vector space of dimension n_G, and $S_G = \mathrm{Sym}_{\mathbb{C}}(V_G)$. The discriminant of the group action is denoted Δ_G and is an element of the invariant ring R_G of the action of G on S_G. The Lusztig algebra is a matrix algebra that encodes the data of irreducible representations of the group $G(m,1,2)$. Let $A_G = S * G$, in [BFIL21] it was shown that the Lusztig algebra was Morita equivalent to A_G. We first define the Lusztig algebra for a group G and then discuss the case of $G = G(m,1,2)$. This work is closely related to [BSW10] in which a Morita equivalent algebra is calculated using superpotentials and we thank Michael Wemyss for pointing out that the method used to calculate the Lusztig algebra is the same as in [BS05].

5.1 Isodiscriminantal groups

We start with the motivation for this chapter, comparing groups which have similar discriminants by modules over their hyperplane arrangements. The focus is on the groups $G(m,1,2)$ which have been featured in this thesis. The algebras we consider will not necessarily be commutative, and so we remark that this chapter will consider right modules.

Definition 5.1.1. Let G and G' be two reflections groups acting on S, with Δ_G and $\Delta_{G'}$ being their discriminants respectively. We say that the two groups are *isodiscriminantal* if $R_G/(\Delta_G) \cong R_{G'}/(\Delta_{G'})$ as rings.

Theorem 5.1.2. [Bes15, Theorem 2.2] Every complex reflection group G is isodiscriminantal to a true reflection group.

Remark 5.1.3. This can also be read from the table given at the end of the paper [BMR97, Table 1,2,3,4]. The groups that are isodiscriminantal have the same base diagram.

Theorem 5.1.4. [Ban76] The groups $G(m,1,2)$ for $m \geq 2$ are isodiscriminantal to the true reflection group $G(2,1,2)$.

Proof. See Examples 3.1.4, 3.1.5, 3.1.6. □

Definition 5.1.5. Two algebras A and B are Morita equivalent if the categories of left A modules is equivalent to the categories of left B modules. If A and B are Morita equivalent and write $A \cong_M B$.

From Chapter 3 we can automatically state the following result:

Theorem 5.1.6. Let $G_m = G(m,1,2)$ for $m \geq 2$, then the endomorphism algebras $\mathrm{End}_{R_{G_m}/\Delta_m}(S_{G_m}/z_{G_m})$ are all Morita equivalent.

Proof. Theorem 3.0.1 shows that all $\mathrm{add}(S_{G_m}/(z_{G_m}))$ are equivalent, since they all contain at least one copy of every isomorphism class of indecomposable CM modules of

$R_{G_m}/(\Delta_{G_m})$, and thus of $R_{G_2}/(\Delta_{G_2})$. From a corollary of Morita's Theorem [Mor58] (see [Leu12, Corollary A.3]) $\mathrm{End}_{R_{G_m}/(\Delta_m)}(S_{G_m}/(z_{G_m})) \equiv_M \mathrm{End}_{R_{G_2}/(\Delta_2)}(S_{G_2}/(z_{G_2}))$. □

5.2 Construction of Lusztig algebra

In this section we introduce the Lusztig algebra as in [BFIL21, Section 6]. More details and in particular the construction of the group action can be found in Loc. Cit.

Definition 5.2.1. Let G be a finite group and $\{V_1,...,V_r\}$ the complete list of irreducible representations of G, the *basic representation* of G is $T = \bigoplus_{i=1}^r V_i$. If T is the basic representation of G then we define the Lusztig algebra of G as:

$$\tilde{A}(G) = (\mathrm{End}_{\mathbb{C}}(T) \otimes \mathrm{Sym}_{\mathbb{C}}(\mathbb{C}^n))^G.$$

To calculate $\tilde{A}(G)$ as a matrix algebra, let $d = \dim(T) = \sum_i \dim V_i$, and M an element of $\mathrm{End}_{\mathbb{C}}(T) \otimes \mathrm{Sym}_{\mathbb{C}}(\mathbb{C}^n)$, M is an element of $\tilde{A}(G)$ if and only if $\rho(g)(g(M))\rho(g)^{-1} = M$. In particular M can be calculated by blocks M^{ij}, which are of size $\dim(V_i) \times \dim(V_j)$ using:

$$M^{ij} = \rho_i(g) g(M^{ij}) \rho_j(g)^{-1} \tag{5.1}$$

for all $g \in G$. This method is similar to the method used in [BS05]

Theorem 5.2.2. [BFIL21, Theorem 6.14] Let G be a finite group acting on $\mathbb{C}^n$ and let $S = \mathrm{Sym}_{\mathbb{C}}(\mathbb{C}^n) \cong T_{\mathbb{C}}(\mathbb{C}^n)/I$ where I is the ideal generated by the commutativity relations $x_i x_j - x_j x_i$. Then $S * G$ (recall Definition 1.8.1) is Morita equivalent to a path algebra $\mathbb{C}Q/\langle I \rangle$ where Q is the McKay quiver (recall Definition 1.5.2) of G and $\langle I \rangle$ is the ideal of relations in $\mathbb{C}Q$ induced by I. In particular $\tilde{A}(G) \cong \mathbb{C}Q/\langle I \rangle$.

To be able to easily calculate the ideal of relations for the Lusztig alebra $\tilde{A}(G)$ from the McKay quiver, we need a result for Koszul algebra. We recall the definition of a Koszul algebra, for example see [BGS96].

Definition 5.2.3. A graded k-algebra $R = \bigoplus_{i \geq 0} R_i$ is *Koszul* if R_0 is finitely generated, semi-simple and R_0 considered as a graded module has a graded projective resolution.

The reason to consider Koszul algebras is the following result

Lemma 5.2.4. [BGS96, Corollary 2.3.3] Let R be a Koszul algebra, there is an isomorphism of Koszul algebras

$$R \cong R_0 \oplus V \oplus (V \otimes V) \oplus \cdots$$

Where V is a bi-module in degree 1 and I is generated by elements in degree 2.

We can see that $\mathbb{C}Q$ is a Koszul algebra for a quiver Q with the following set up. Let $Q = (Q_0, Q_1, s, t)$ be a quiver, then $R_0 = \prod_{x \in Q_0} \mathbb{C}$ and let V be the R_0 bimodule $V = \bigoplus \alpha \in Q_1 \mathbb{C}$. Then the path algebra $\mathbb{C}Q \cong T_{R_0} V$ and is graded by path length.

Theorem 5.2.5. [BFIL21, Theorem 6.14] Let G be a finite group acting on $\mathbb{C}^n$, then the Lusztig algebra $\tilde{A}(G) \cong \mathbb{C}Q/\langle I \rangle$ and $\tilde{A}(G)$ is Koszul.

This means that to calculate the relations for $\tilde{A}(G)$, it is enough to focus on relations in degree 2. From Equation (5.1), the block M^{ij} of M will represent the arrow $V_j \to V_i$. Since by Theorem 5.2.2 $\tilde{A}(G) \cong \mathbb{C}Q/\langle I \rangle$ we can shortcut the calculations and focus on the arrows in the McKay quiver of G.

Remark 5.2.6. In general for Definition 5.2.1 we can, instead of taking $\mathrm{Sym}_{\mathbb{C}}(\mathbb{C})$, take any (Koszul) $G - \mathbb{C}$-algebra Λ which would impose different relations on the resulting algebra $(\mathrm{End}_{\mathbb{C}}(T) \otimes \Lambda)^G$. The algebra $(\mathrm{End}_{\mathbb{C}}(T) \otimes \Lambda)^G$ would still be Koszul.

5.3 Lusztig algebra for $G(m,1,2)$

The McKay quivers of $G_m = G(m,1,2)$ can be described by the following theorem.

Theorem 5.3.1. [BFIL21, Theorem 4.10] Let $\Xi = \Xi(G_m)$ be the McKay quiver of G_m. The vertices correspond irreducible representations of G_m, Recall Lemma 3.4.13 for a description. There is an arrow from α to β if and only if the m-tuple of Young diagrams β can be obtained from α by removing a cell from position i and the adding a cell to position $i+1 \bmod m$.

We call all the 2-paths between two vertices in a quiver a mesh.

Theorem 5.3.2. Let $m > 2$, V and U be irreducible representations of $G(m,1,2)$, and $A^{U,V}$ be the arrow from V to U. The Lusztig algebra $\tilde{A}(G(m,1,2)) \cong \mathbb{C}Q/\langle I\rangle$ where $\langle I\rangle$ is the ideal generated by:

$$A^{\square_i\square_{i+1},\,\genfrac{}{}{0pt}{}{\square}{\square}_{i+1}}A^{\square\square_i,\,\square_i\square_{i+1}}, \qquad A^{\square_i\square_{i+1},\,\square\square_{i+1}}A^{\genfrac{}{}{0pt}{}{\square}{\square}_i,\,\square_i\square_{i+1}},$$

for $0 \le i \le m-1$ and $i+1$ is taken mod m;

$$A^{\square_i\square_{j-1},\,\square_i\square_j}A^{\square_{i-1}\square_{j-1},\,\square_i\square_{j-1}} + A^{\square_{i-1}\square_j,\,\square_i\square_j}A^{\square_{i-1}\square_{j-1},\,\square_{i-1}\square_j},$$

for $0 \le i,j \le m = 1$, $i+1 \ne j$ and $i-1$ is taken mod m; and

$$A^{\genfrac{}{}{0pt}{}{\square}{\square}_i,\,\square_i\square_{i+1}}A^{\square_{i-1}\square_i,\,\genfrac{}{}{0pt}{}{\square}{\square}_i} + A^{\square\square_i,\,\square_i\square_{i+1}}A^{\square_{i-1}\square_i,\,\square\square_i} - 2A^{\square_{i-1}\square_{i+1},\,\square_i\square_{i+1}}A^{\square_{i-1}\square_i,\,\square_{i-1}\square_{i+1}}.$$

Proof. We calculate the relations by looking at all the 2-paths in the McKay quiver, which fall into 3 cases. To do this we must first calculate the degree 1 part of $\tilde{A}(G)$ which encodes the arrows in the McKay quiver of G_m . Consider G_m where $m \ge 3$ and let $0 \le i < j \le m$ and $i \ne j-1$. The McKay quiver contains the mesh shown in Figure 5.1.

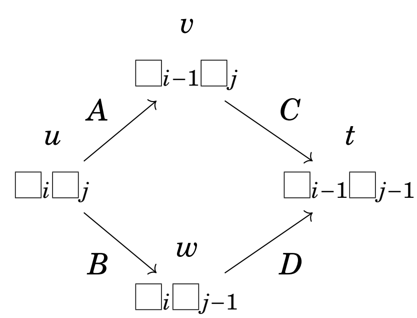


Figure 5.1: Mesh of the McKay quiver, where $i \ne j-1$.

Recall from Example 3.1.4 that the generators of G_m are:

$$s_1 := \begin{bmatrix} \xi_m & 0 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad s_2 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

We will write, see labels of vertices on Figure 5.1,

$$v = \square_{i-1}\square_j, u = \square_i\square_j, w = \square_i\square_{j-1}, t = \square_{i-1}\square_{j-1}$$

We first calculate the 2×2 matrix $M^{v,u}$ using Equation (5.1). After choosing scalars, this will correspond to the arrow labelled A. The matrix $M^{v,u}$ is of the form

$$M^{v,u} = \begin{bmatrix} a_{11}x + b_{11}y & a_{12}x + b_{12}y \\ a_{21}x + b_{21}y & a_{22}x + b_{22}y \end{bmatrix}$$

where $a_{ij}, b_{ij} \in \mathbb{C}$. We calculate the action of the generators for G_m on $M^{v,u}$:

$$s_1(M^{v,u}) = \begin{bmatrix} \xi_m a_{11}x + b_{11}y & \xi_m a_{12}x + b_{12}y \\ \xi_m a_{21}x + b_{21}y & \xi_m a_{22}x + b_{22}y \end{bmatrix},$$

and

$$s_2(M^{v,u}) = \begin{bmatrix} b_{11}x + a_{11}y & b_{12}x + a_{12}y \\ b_{21}x + a_{21}y & b_{22}x + a_{22}y \end{bmatrix}$$

Using Equation (5.1) with the generator s_1 the matrix $M^{v,u}$ has the following relations:

$$M^{v,u} = \rho_v(s_1)s_1(M^{v,u})\rho_u^{-1}(s_1) = \begin{bmatrix} \xi_m^j & 0 \\ 0 & \xi_m^{i-1} \end{bmatrix} s_1(M^{v,u}) \begin{bmatrix} \xi_m^{m-j} & 0 \\ 0 & \xi_m^{m-i} \end{bmatrix} \tag{5.2}$$

and using the generator s_2, the relation for $M^{v,u}$ are

$$M^{v,u} = \rho_v(s_2)s_2(M^{v,u})\rho_u^{-1}(s_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} s_1(M^{v,u}) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{5.3}$$

Calculating the first relation, (5.2), gives us:

$$M^{v,u} = \begin{bmatrix} \xi_m a_{11}x + b_{11}y & \xi_m^{m-i+j-1} a_{12}x + \xi_m^{m-i-j} b_{12}y \\ \xi_m^{m-j+i} a_{21}x + \xi_m^{m-j+i-1} b_{21}y & a_{22}x + \xi^{m-1} b_{22}y \end{bmatrix}$$

Since $i \neq j-1, j$ we get the following relations:

$$a_{11} = a_{12} = a_{21} = b_{12} = b_{21} = b_{22} = 0$$

Using the relations (5.3), we obtain

$$M^{vu} = \begin{bmatrix} b_{22}x + a_{22}y & b_{21}x + a_{21}y \\ b_{12}x + a_{12}y & b_{11}x + a_{11}y \end{bmatrix},$$

which in turn, gives the relations:

$$b_{22} = a_{11}, a_{22} = b_{11}, b_{21} = a_{12}, a_{21} = b_{12}$$

Using these together we see that:

$$M^{v,u} = \begin{bmatrix} ay & 0 \\ 0 & ax \end{bmatrix}$$

for a scalar $a \in \mathbb{C}$. Using similar calculations, for the other vertices we obtain:

$$M^{w,u} = \begin{bmatrix} bx & 0 \\ 0 & by \end{bmatrix}, \qquad M^{t,v} = \begin{bmatrix} cx & 0 \\ 0 & cy \end{bmatrix}, \qquad M^{t,w} = \begin{bmatrix} dy & 0 \\ 0 & dx \end{bmatrix},$$

for scalars $b, c, d \in \mathbb{C}$.

To calculate the relations that are induced by $I = \langle xy - yx \rangle$, we look at the matrix $M \cdot M'$ which encodes all paths of length 2 in the McKay quiver. We first look at the 2×2 submatrix that encodes the paths from u to t

$$(M \cdot M')^{t,u} = M^{t,-} \cdot (M^{-,u})'$$

Where $M^{t,-}$ encodes the arrows coming into t and is given by

$$M^{t,-} = \begin{bmatrix} 0 & \cdots & cx & 0 & 0 & \cdots & dy & 0 & 0 & \cdots \\ 0 & \cdots & 0 & cy & 0 & \cdots & 0 & dx & 0 & \cdots \end{bmatrix}$$

$(M^{-,u})'$ encodes the arrows originating from u and is given by

$$(M^{-,u})' = \begin{bmatrix} 0 & 0 \\ \vdots & \vdots \\ a'y & 0 \\ 0 & a'x \\ 0 & 0 \\ \vdots & \vdots \\ b'x & 0 \\ 0 & b'y \\ 0 & 0 \\ \vdots & \vdots \end{bmatrix}.$$

Multiplying we get

$$(M \cdot M')^{t,u} = \begin{bmatrix} c'axy + d'byx & 0 \\ 0 & c'ayx + d'bxy \end{bmatrix}$$

From this we get the relation of $\tilde{A}(G)$:

$$CA + DB = 0$$

Figure 5.2 shows a slightly different mesh that appears in the McKay quiver when $j = i+1$.

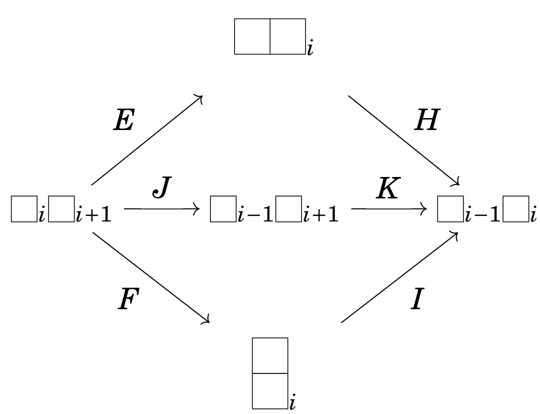


Figure 5.2: Mesh of the McKay quiver, where $i = j - 1$.

The matrices can be obtained in a similar way as before:

$$M^{\square\square_i,\square_i\square_{i+1}} = \begin{bmatrix} ex & ey \end{bmatrix}, \qquad M^{\begin{smallmatrix}\square\\\square\end{smallmatrix}_i,\square_i\square_{i+1}} = \begin{bmatrix} fx & -fy \end{bmatrix}, \qquad M^{\square_{i-1}\square_{i+1},\square_i\square_{i+1}} = \begin{bmatrix} jy & 0 \\ 0 & jx \end{bmatrix},$$

$$M^{\square_i\square_{i+1},\square\square_i} = \begin{bmatrix} hx \\ hy \end{bmatrix}, \qquad M^{\square_i\square_{i+1},\begin{smallmatrix}\square\\\square\end{smallmatrix}_i} = \begin{bmatrix} iy \\ -ix \end{bmatrix}, \qquad M^{\square_i\square_{i+1},\square_{i-1}\square_{i+1}} = \begin{bmatrix} kx & 0 \\ 0 & ky \end{bmatrix}.$$

The paths of length 2 from $\square_i\square_{i+1}$ to $\square_{i-1}\square_i$ correspond to the matrix:

$$(M\cdot M')^{\square_{i-1}\square_i,\square_i\square_{i+1}} = \begin{bmatrix} h'eyx + i'fyx + k'jxy & h'ey^2 - i'fy^2 \\ h'ex^2 - i'fx^2 & h'exy + i'fxy + k'jyx \end{bmatrix}$$

Again, using the commutativity relations we obtain the following relation in $\tilde{A}(G)$

$$HE + IF - 2KJ$$

The last mesh we have to calculate relations for is between linear characters, see Figure 5.3.

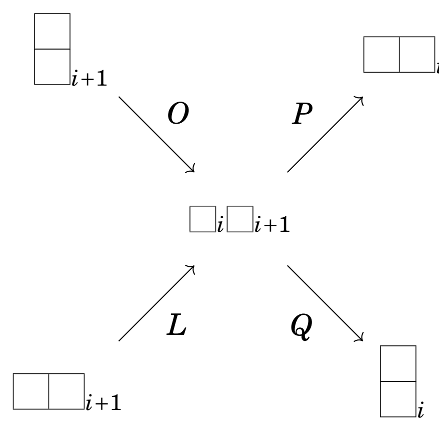


Figure 5.3: Mesh between linear characters

Using a similar calculation to the above gives us the relation of $\tilde{A}(G)$

$$PO = QL = 0.$$

These are all the relations needed in the McKay quiver for the Lusztig algebra for $G(m,1,2)$. □

The Lusztig algebra for the group G_2 can be calculated in the same way. It has the McKay quiver Q, given in Figure 5.4:

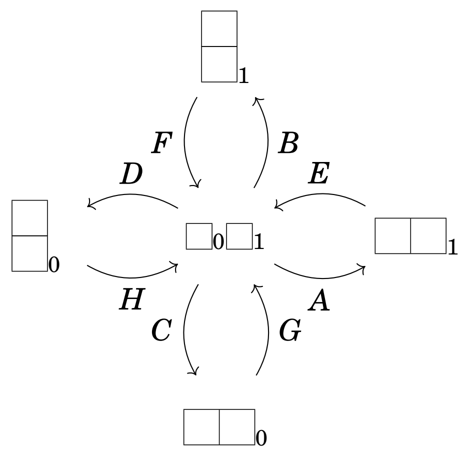


Figure 5.4: The McKay quiver for the group $G(2,1,2)$

The Lusztig algebra for $G(2,1,2)$ is:

$$\tilde{A}(\mathbb{C}^2)(G(2,1,2)) \cong \mathbb{C}Q/\langle AH, BG, CF, DE, EA+HD-BF-GC\rangle$$

These calculations can be found in [BFIL21, Example 6.15].

Remark 5.3.3. The group G_2 is a true reflection group which has the same character table as the quaternion group Q_8 which is a subgroup of $\mathrm{SL}(2,\mathbb{C})$. The Lusztig algebra for a subgroup G of $\mathrm{SL}(2,\mathbb{C})$ is the *preprojective algebra* for the McKay quiver. In the case of the quaternion group Q_8, the Lusztig algebra is isomorphic to the preprojective algebra for the extended Dynkin diagram $\tilde{D}_4$.

Remark 5.3.4. The calculations here are similar to that of [NdC12] and [WEM13] for binary dihedral groups.

5.4 Translation quivers

Reiten and Van-den-Bergh classified two-dimensional tame orders of finite representation type in [RVdB89]. Here we discuss how the Lusztig algebra of $G(m,1,2)$ fit into this classification.

Definition 5.4.1. A stable translation quiver is a triple (T, τ, σ), where T is a quiver and

$$\tau : T_0 \to T_0 \qquad \sigma : T_1 \to T_1$$

are bijections such that for any arrow $\alpha : x \to y$ in T_1 we have $\sigma\alpha : \tau y \to x$. The map τ is called the translation of T.

Theorem 5.4.2. [Aus86] Let $G \subset \mathrm{GL}(\mathbb{C}^2)$, then the McKay quiver is stable translation quiver with translation:

$$\tau = - \otimes \det .$$

Remark 5.4.3. If G does not contain pseudo reflections then this is also the AR quiver of $\mathbb{C}[|x,y|]^G$.

Definition 5.4.4. Let Q be a quiver, then $\mathbb{Z}Q$ is the quiver that has vertices (x,i), where $x \in Q$ and $i \in \mathbb{Z}$. For each arrow $\alpha : x \to y$ in Q and for every i there is an arrow $(x,i) \to (y,i)$ and an arrow $(y,i) \to (x,i+1)$.

Theorem 5.4.5. The McKay quivers for $G(m,1,2)$ for $m \geq 3$ are the quivers $\mathbb{Z}\Gamma/H$ where $\Gamma = \tilde{D}_{m+2}$ and H is an automorphism of $\mathbb{Z}\Gamma$.

Proof. We start by describing a section of the translation quivers $\mathbb{Z}\Gamma$. Let $p = m + 2 - \lfloor \frac{m+5}{2} \rfloor$

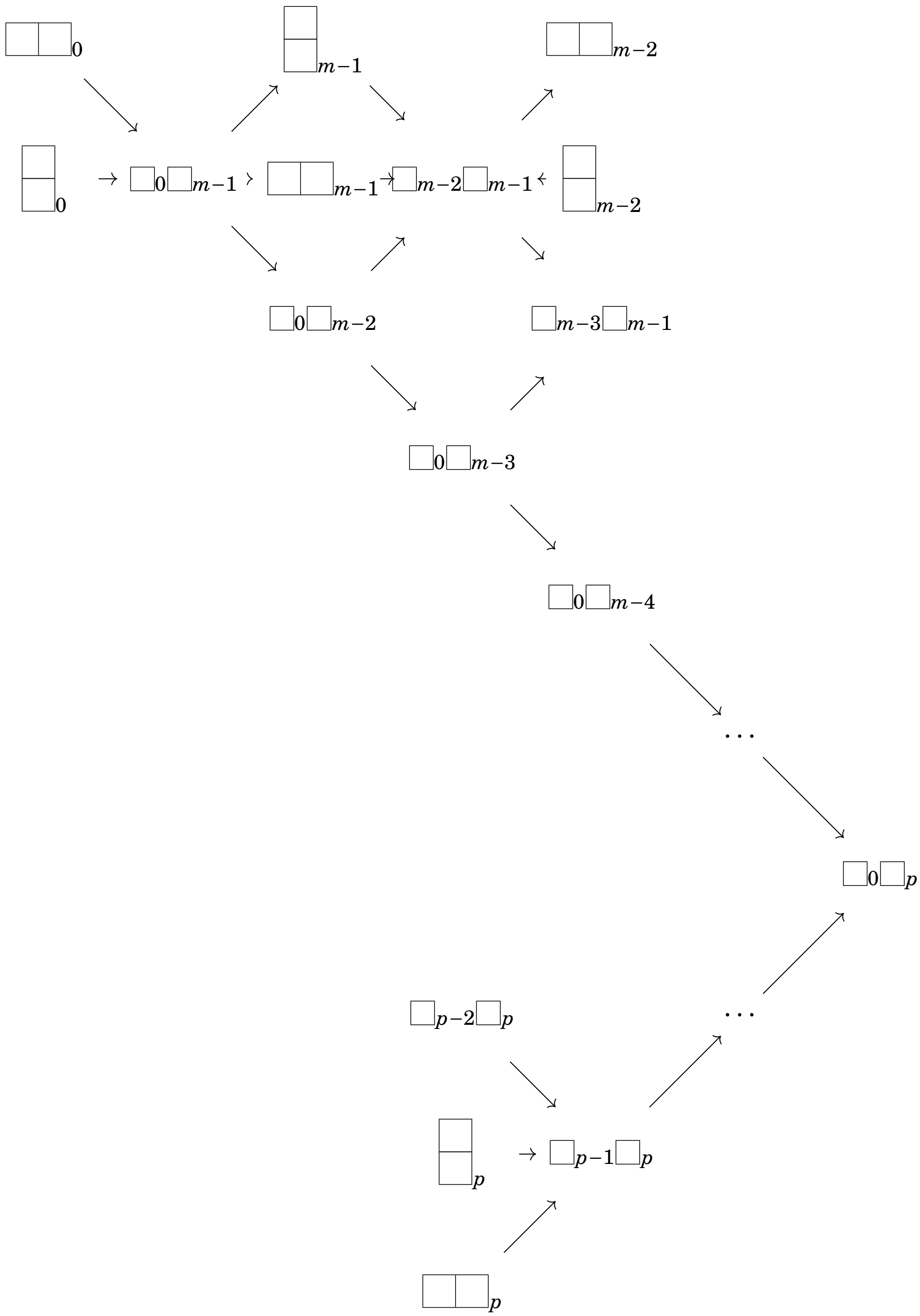
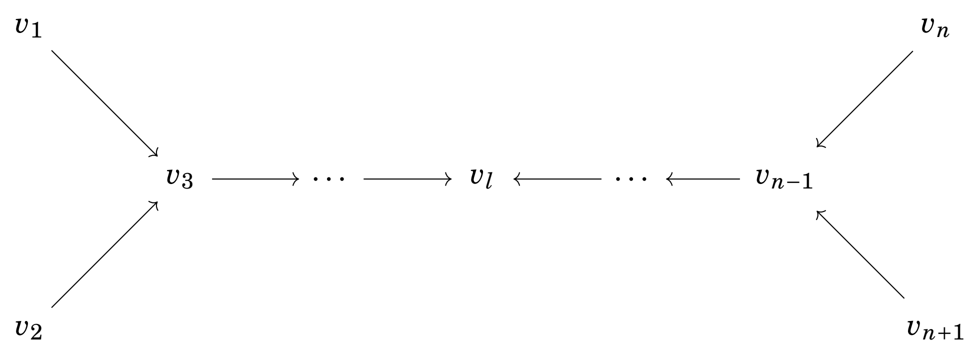


Figure 5.5: A slice of the translation quiver $\mathbb{Z}\tilde{D}_{m+2}$ decorated with irreducible representations of $G(m,1,2)$.

Here we start with a $\tilde{D}_{m+2}$ slice of the translation quiver, see Figure 5.5, where vertices are indexed by irreducible representations of $G(m,1,2)$. We can then populate $\mathbb{Z}\tilde{D}_{m+2}$ using $(v_\lambda, i+1) = (v_{\lambda\otimes\det}, i)$. The group action H is identifying vertices that are indexed by the same representation. To be explicit with the notation from [RVdB89], consider the following orientation $\overrightarrow{\tilde{D}_n}$ of $\tilde{D}_n$



where $l = \lfloor \frac{n+3}{2} \rfloor$. Define ρ as the automorphism of $\overrightarrow{\tilde{D}_n}$ given uniquely by $\rho(v_1) = v_n$ and $\rho(v_n) = v_1$.

When $m > 2$ is even, the group G which acts on the translation quiver $\mathbb{Z}\tilde{D}_{m+2}$ is generated by the automorphism

$$\phi_{\frac{m}{2}}(v,i) = (\rho(v), i - \frac{m}{2}).$$

When m is odd, the group H is generated by the element

$$\psi_m(v_j) = \begin{cases} (\rho(v_j), s - \frac{m+1}{2}) & \text{if } j \leq \frac{m+1}{2} \\ (\rho(v_j), s - \frac{m-1}{2}) & \text{if } j > \frac{m+1}{2} \end{cases}$$

□

Corollary 5.4.6. The Lusztig algebra $\tilde{A}(G(m,1,2))$ for $m > 2$ are not Morita equivalent.

Remark 5.4.7. Auslander and Reiten in [AR86] showed that for any subgroup $G \subset GL(2,\mathbb{C})$ the *separated* McKay quiver is a union of extended Dynkin quivers. Theorem 5.4.5 shows that the separated McKay quiver of $G(m,1,2)$ is $\bigsqcup_{m+2} D_{m+2}$.

Definition 5.4.8. Let T be a translation quiver and x a vertex of T, the mesh relations at x are given by

$$\sum_{\alpha: y \to x} \sigma(\alpha)\alpha$$

Remark 5.4.9. Relations for an isomorphic algebra to the Lusztig algebra can then be read from $\mathbb{Z}\Gamma$ as the mesh relations.

Let $G_m = G(m,1,2)$, since the Lusztig algebra $\bar{A}(G_m) \cong_M \mathrm{Sym}_{\mathbb{C}}(\mathbb{C}^2) * G_m$, we can relate this to the result of Buchweitz–Faber–Ingalls

Theorem 5.4.10. [BFI20, Theorem 4.17] Let $G \subset GL(V)$ be a true reflection group, Let $A_G = \mathrm{Sym}_{\mathbb{C}}(\mathbb{C}^2) * G$ and $\bar{A}_G = A_G / A_G e_\chi A_G$ where e_χ is an idempotent for a linear representation $\chi \in A_G$. Then:

$$\bar{A} \cong \mathrm{End}_{R\Delta}(S/(z)).$$

Thus for the groups $G(m,1,2)$ the following diagram shows the results of this sections:

$$\begin{array}{ccc} \mathrm{End}_{R/\Delta_2}(S_{G_2}/(z_{G_2})) & \cong_M & A_{G_2}/A_{G_2}e_\chi A_{G_2} \\ \wr\!\!\cong_M & & \not\cong_M \\ \mathrm{End}_{R/\Delta_m} G_m(S_{G_m}/(z_{G_m})) & \not\cong_M & A_{G_m}/A_{G_m}e_\chi A_{G_m} \end{array}$$

That is $\mathrm{End}_{R/\Delta_2}(S_{G_2}/(z_{G_2}))$ has equivalent module categories as $\mathrm{End}_{R/\Delta_2}(S_{G_m}/(z_{G_m}))$ and by factoring out by a 1-dimensional representation of G_2 the skew-group ring $A_{G_2}/A_{G_2}e_\chi A_{G_2}$ also has an equivalent module category. In the case of G_m, if we take $A_{G_2}/A_{G_m}e_\chi A_{G_m}$ for a 1-dimensional representation χ then this has more modules than $\mathrm{End}_{R/\Delta_2}(S_{G_m}/(z_{G_m}))$. This suggests that to expand Theorem 5.4.10 to any pseudo reflection group, we need to quotient the skew group ring with more idempotents corresponding to irreducible representations. Locating the correct irreducible representations in general would be interesting.

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