Poisson polynomial algebras and their spectra

Maram Alossaimi

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School of Mathematics and Statistics

Supervisor: Prof. Vladimir Bavula

The University of Sheffield

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Declaration

I hereby declare that the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this university or any other university, with the exception of specific references to the work of others. Except as noted in the text, everything in this thesis is my original work. It does not include any contributions from others.

Maram Alossaimi
September 2022

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Abstract

This thesis is devoted to studying Poisson algebras, their properties and classifications of Poisson prime ideals for a certain class of Poisson polynomial algebras A in three variables. The study of such algebras was first introduced by Oh in 2006, [Oh3].

This thesis consists of two parts; In the first part, general properties of Poisson algebras and their Poisson ideals are considered, and a review of known results and techniques is presented. In the second part, classifications of Poisson prime ideals, minimal Poisson ideals and maximal Poisson ideals are given.

The algebras $A = K[t][x, y]$ are the Poisson polynomial algebras in one variable t, with trivial Poisson bracket, over an algebraically closed field K of characteristic zero, that are extended into two variables x and y, under certain conditions, such that if u is a fixed polynomial in $K[t]$[1], f is an arbitrary polynomial in $K[t]$, λ is a unit element in $K^{\times}$, c is an arbitrary element in K, the partial derivations $\partial_t = \frac{d}{dt}$ and $\partial_y = \frac{d}{dy}$. The Poisson algebras A can be denoted either by

$$\big(K[t]; f\partial_t, \lambda^{-1} f\partial_t, c, u\big)^{2} \quad \text{or} \quad K\left[t\right]\left[y; f\partial_t\right]_p\left[x; \lambda^{-1} f\partial_t, u\partial_y\right]_p{}^{3}$$

with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = \lambda^{-1} fx \;\text{ and }\; \{y, x\} = cyx + u.$$

[1]The polynomial ring

[2]This notation is from [Oh3], in particular $(D; \alpha, \beta, c, u)$, see Lemma II.2.19

[3]This notation indicates the extension of Poisson algebras with some functions multiplied by partial derivatives, and p means a Poisson algebra

The class of Poisson algebras A splits into three classes: **I**, **II** and **III**. Each of them splits further into subclasses, see diagram .0.1 for detail. The classifications of Poisson prime ideals[4], minimal Poisson ideals[5] and maximal Poisson ideals[6] for the seventeen classes of Poisson algebras, that are in blue, are obtained. In addition, the classifications of special cases for the five classes of Poisson algebras, that are in green, are obtained. However, for Poisson algebras that are in red, their Poisson prime ideals cannot be classified.

[4]Poisson ideals and prime ideals of the Poisson algebra
[5]Minimal elements with respect to the inclusions of Poisson prime ideals
[6]Maximal elements with respect to the inclusions of Poisson prime ideals

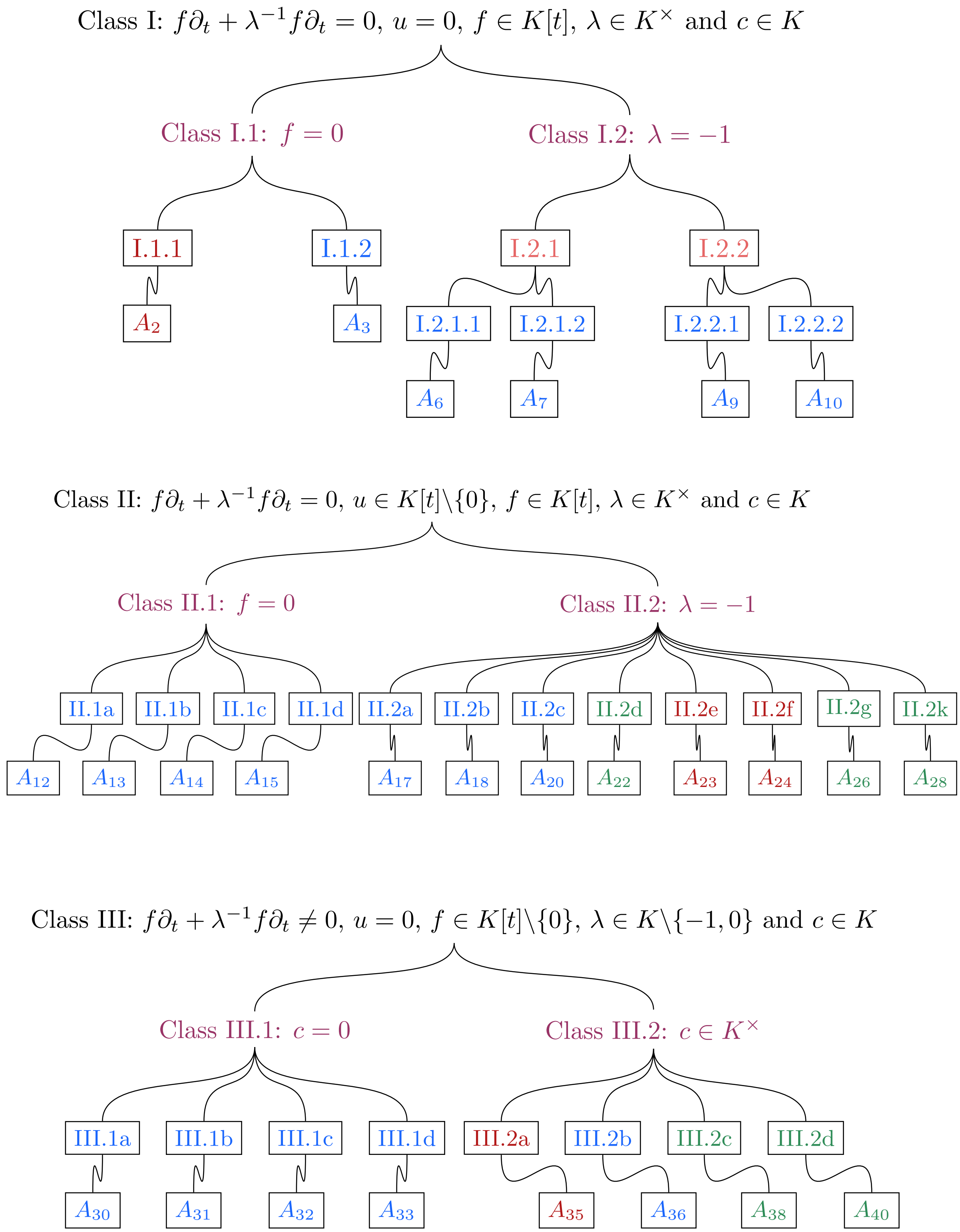


Diagram .0.1: Structure of classes of Poisson algebras A

Contents

List of diagrams

§ I Introduction

Poisson structures have their roots in nineteenth-century research by Poisson, Hamilton, Jacobi and Lie. With the groundbreaking studies of Lichnerowicz and Weinstein, it began to exist as a separate field in the 1980s. Since then, Poisson structures have been involved in a wide range of fields, including abstract algebra, representation theory, algebraic geometry, differential geometry, string theory, classical/quantum physics, and differential geometry. In each of these subjects, it turns out that the Poisson structure is a crucial component that naturally arises with the problem under investigation, and its delicate qualities are always essential for finding the solution. Recently, extensive work has been done, in particular, in abstract algebra, please refer to [Oh3], [Oh4], [Goo2], [Jor], [JoOh], [MyOh], [Bav3], [Bav4] and [Bav5].

Recall that a *Poisson algebra* D is a (commutative) K-algebra over a field K with a K-bilinear product $\{\cdot,\cdot\}$ on D, which is called a *Poisson bracket*, such that $(D, \{\cdot,\cdot\})$ is a Lie algebra and satisfies the *Leibniz's rule*

$$\{a \cdot b, c\} = a \cdot \{b, c\} + \{a, c\} \cdot b \quad \text{for all} \;\; a, b, c \in D.$$

Recall that an ideal $\mathfrak{p}$ of D is *prime* if

$$IJ \subseteq \mathfrak{p} \quad \Longrightarrow \quad I \subseteq \mathfrak{p} \quad \text{or} \quad J \subseteq \mathfrak{p},$$

where I and J are ideals of D. The *Krull dimension* of D is the supremum of the heights of all prime ideals of D.

Recall that an ideal I of a Poisson algebra D is called a *Poisson ideal* if $\{D, I\} \subseteq I$. The

Poisson ideal of D *generated by* a is denoted by (a). A Poisson ideal I is called a *Poisson prime ideal* if I is a Poisson ideal and a prime ideal of D. The set of all Poisson prime ideals of D is called the *Poisson spectrum* and is denoted by $\mathrm{PSpec}(D)$.

Classification of Poisson prime ideals is a very difficult problem which is done only for a few classes of Poisson algebras. Over a field of characteristic zero, any algebra of Krull dimension one has trivial Poisson bracket, that is

$$\{a, b\} = 0 \quad \text{for all elements } \ a, b.$$

There are plenty of Poisson algebras of Krull dimension two, and the classification of Poisson algebras of Krull dimension two up to isomorphism is a wide-open problem. Even more difficult is the classification of Poisson algebras of Krull dimension three. Moreover, Poisson algebras of Krull dimension three is a very large and complex class of Poisson algebras. It is still open problem even for the polynomial algebra in three variables. There is an excellent paper by Polishchuk [Pol], a substantial part of it is the review of results about Poisson algebras of dimension three, see Subsection II.2.6.

There is a great variety of Poisson algebras in three variables, and this thesis demonstrates this phenomenon. The main object of our study is a class of Poisson polynomial algebras in three variables over an algebraically closed field K of characteristic zero that consists of Poisson algebras

$$(K[t]; \alpha, \beta, c, u)^{1},$$

where α and β are arbitrary K-derivations of the Poisson polynomial algebra $K[t]$, that is

$$\alpha = f\partial_t, \ \ \beta = g\partial_t, \quad \text{where } \ f, g \in K[t], \ \partial_t = \frac{d}{dt}, \ c \in K \ \text{ and } \ u \in K[t].$$

Then it follows from Lemma IV.0.1 and Lemma IV.0.2 that $g = \frac{1}{\lambda}f$, where $\lambda \in K^{\times}$. There-

[1]This notation is from [Oh3], see Lemma IV.0.1

fore, the Poisson algebras A are denoted either by

$$\big(K[t]; f\partial_t, \lambda^{-1}f\partial_t, c, u\big) \quad \text{or} \quad K\big[t\big]\big[y; f\partial_t\big]_p\big[x; \lambda^{-1}f\partial_t, u\partial_y\big]_p$$ [2]

with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = \lambda^{-1}fx \quad \text{and} \quad \{y, x\} = cyx + u,$$

see Chapter IV for detail. Surprisingly, this class of Poisson algebras contains a lot of subclasses of Poisson algebras that require different techniques for their study and different types of results are obtained for each subclass.

The class of Poisson algebras A splits into three classes: **I**, **II** and **III**. Let us give more detail.

- The first class **I** consists of two subclasses: **I.1** and **I.2**.
- The second class **II** consists of two subclasses: **II.1** and **II.2**.
- The third class **III** consists of two subclasses: **III.1** and **III.2**.

Each of them splits further into subclasses. For more information please refer to diagram IV.1.1, diagram IV.2.1 and diagram IV.3.1, respectively.

Originally, the goal was to classify Poisson prime ideals for all Poisson algebras A. Now, it is obvious that it is a large task and some of the subclasses are very difficult to study. Classifications of Poisson prime ideals are obtained approximately for the half of classes of Poisson algebras A. Namely, for the Poisson algebras that belong to the following classes:

I, **II.1**, **III.1**, some classes in **II.2** and **III.2**.

The classifications of Poisson prime ideals for Poisson algebras that belong to all subclasses of the first class **I**, the first part of the second class **II.1**, and the first part of the third

[2]This notation indicates the extension of Poisson algebras with some functions multiplied by partial derivatives, and p means a Poisson algebra

class **III.1** are obtained. Additionally, in these subclasses, the minimal Poisson ideals[3] and maximal Poisson ideals[4] are classified, and the inclusions of Poisson prime ideals are given. In addition, each subclass is treated independently because techniques that are used to classify the Poisson prime ideals are different. In particular, some of them are similar to the techniques in recent papers as [Oh3], [Oh4], [JoOh] and [Bav4].

However, to classify Poisson prime ideals is a very difficult problem. There are many reasons for that and the main reason is that a description of the set of common (generalized) eigenfunctions for the hamiltonian vector fields $\{x, \cdot\}, \{y, \cdot\}$ and $\{t, \cdot\}$ is a very difficult problem, in general as we have to solve a system of partial differential equations. In the second part of the second class **II.2** and the second part of the third class **III.2**, the classifications of Poisson prime ideals for Poisson algebras that belong to some subclasses, and special subclasses with some restrictions, are given. For more information see Chapter IV.

The thesis is organised as follows: Some preliminary standard material in algebras and related subjects are given in Chapter II. In Chapter III, there are two classes of Poisson algebras of dimension two and their Poisson prime ideals′ classification, which can be considered typical examples for any other Poisson algebras of dimension two in this thesis. In Chapter IV, the class of Poisson algebras A of dimension three is introduced, and then its structure is given. In addition, the three main classes are introduced and their structure diagrams. Following that, the classifications of Poisson prime ideals, minimal Poisson ideals, and maximal Poisson ideals for Poisson algebras that belong to some subclasses are given.

Let us present some typical results on the classification of Poisson prime ideals, minimal Poisson ideals and maximal Poisson ideals for some subclasses of Poisson algebras A.

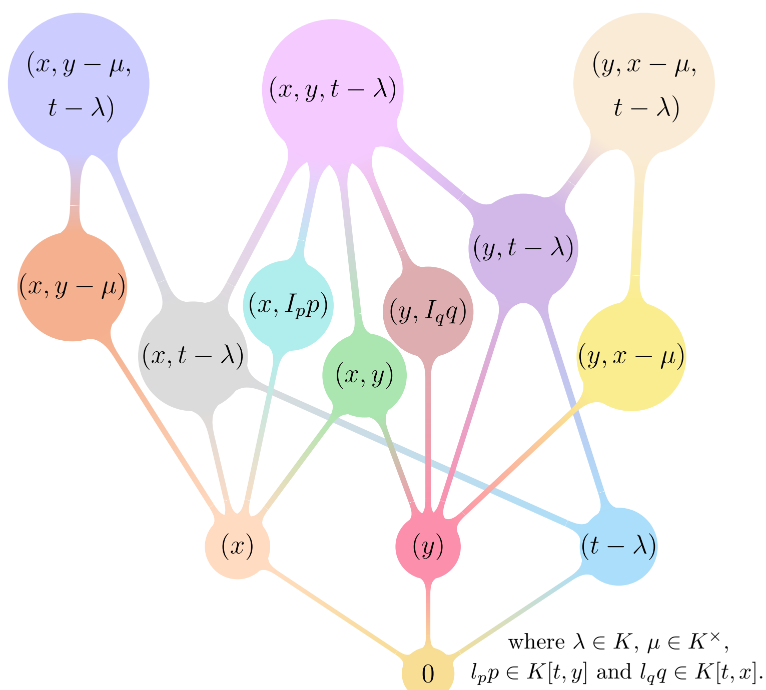
[3] Minimal elements with respect to the inclusions of Poisson prime ideals
[4] Maximal elements with respect to the inclusions of Poisson prime ideals

In the theorem below the classification of Poisson prime ideals is given for the Poisson algebra A_3 that belongs to the subclass I.1.2.

Theorem I.0.1. *[Theorem IV.1.5] Let* $A_3 = (K[t]; 0, 0, c, 0)$ *be the Poisson algebra with Poisson bracket*

$$\{t, y\} = 0, \quad \{t, x\} = 0 \quad \text{and} \quad \{y, x\} = cyx, \quad \text{where } c \in K^{\times}.$$

Then the Poisson spectrum of A_3 *is*
$\{0, (x), (y), (t-\lambda), (x, y), (x, y-\mu), (y, x-\mu), (x, l_p p), (y, l_q q), (x, t-\lambda), (y, t-\lambda), (x, y, t-\lambda), (x, y-\mu, t-\lambda), (y, x-\mu, t-\lambda) \mid \lambda \in K,\ \mu \in K^{\times}, p \in \mathrm{Irr}_m K(t)[y]$[5] *and* $q \in \mathrm{Irr}_m K(t)[x]$[6]$\}$, l_p *and* l_q *are unique monic polynomials in* $K[t]$ *of the least degree in* t *such that* $l_p p \in K[t, y]$[7] *and* $l_q q \in K[t, x]$[8]*, respectively, the inclusions of Poisson prime ideals of* A_3 *are described in the below diagram.*



[5]The set of monic irreducible polynomials of the polynomial algebra $K(t)[y]$ over the field of rational functions in the variable t

[6]The set of monic irreducible polynomials of the polynomial algebra $K(t)[x]$ over the field of rational functions in the variable t

[7]The polynomial ring in two variables t and y

[8]The polynomial ring in two variables t and x

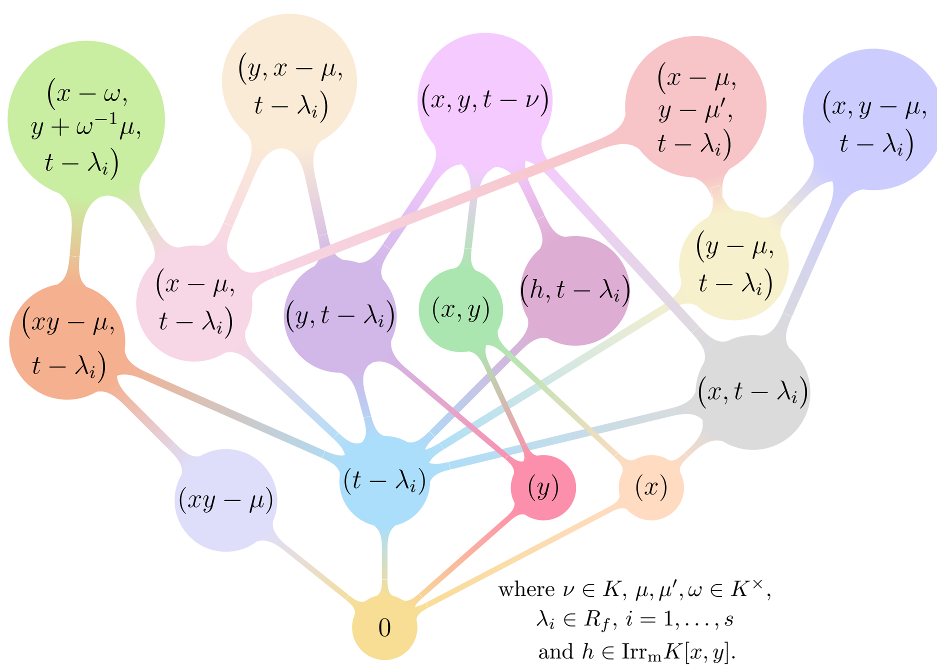
In the theorem below the classification of Poisson prime ideals is given for the Poisson algebra A_6 that belongs to the subclass I.2.1.1.

Theorem I.0.2. *[Theorem IV.1.8] Let* $A_6 = (K[t]; f\partial_t, -f\partial_t, 0, 0)$ *be the Poisson algebra with Poisson bracket*

$$\{t, y\} = fy, \quad \{t, x\} = -fx \quad \textit{and} \quad \{y, x\} = 0, \quad \textit{where } f \in K[t]\backslash K$$

and $R_f = \{\lambda_1, \dots, \lambda_s\}$ *be the set of distinct roots of the polynomial* f*. Then the Poisson spectrum of* A_6 *is*

$\{0, (x), (y), (t-\lambda_i), (x, t-\lambda_i), (y, t-\lambda_i), (xy-\mu), (x, y), (h, t-\lambda_i), (xy-\mu, t-\lambda_i), (x, y, t-\nu), (y, x-\mu, t-\lambda_i), (x, y-\mu, t-\lambda_i), (x-\mu, y-\omega, t-\lambda_i), (x-\omega, y+\omega^{-1}\mu, t-\lambda_i) \mid \nu \in K, \mu, \omega \in K^\times, \lambda_i \in R_f,\ i = 1, \dots, s \text{ and } h \in \mathrm{Irr}_m K[x, y]\}$*, the inclusions of Poisson prime ideals of* A_6 *are described in the below diagram.*



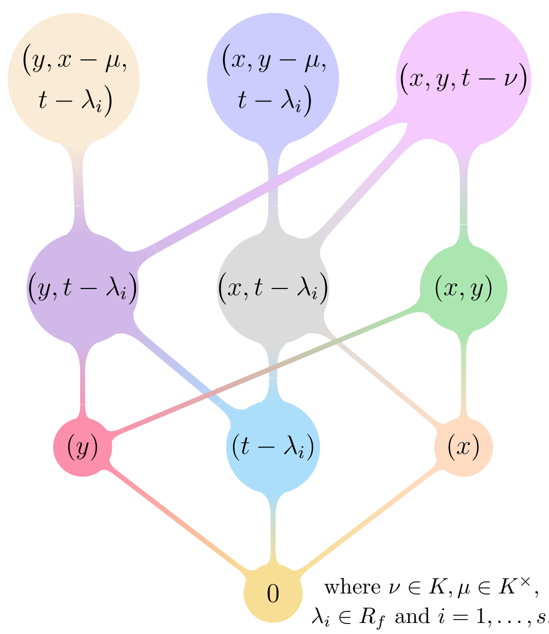
In the following theorem, the classification of Poisson prime ideals is given for the Poisson algebra A_9 that belongs to the subclass I.2.2.1.

Theorem I.0.3. *[Theorem IV.1.10] Let* $A_9 = (K[t]; f\partial_t, -f\partial_t, c, 0)$ *be the Poisson algebra with Poisson bracket*

$$\{t, y\} = fy, \quad \{t, x\} = -fx \quad \textit{and} \quad \{y, x\} = cyx, \quad \textit{where } f \in K[t]\backslash K,\ c \in K^{\times}$$

and $R_f = \{\lambda_1, \ldots, \lambda_s\}$ *be the set of distinct roots of the polynomial* f*. Then the Poisson spectrum of* A_9 *is*

$\big\{0, (x), (y), (x, y), (t-\lambda_i), (y, t-\lambda_i), (x, t-\lambda_i), (x, y, t-\nu), (x, y-\mu, t-\lambda_i), (y, x-\mu, t-\lambda_i) \mid \nu \in K, \mu \in K^{\times}, \lambda_i \in R_f$ *and* $i = 1, \ldots, s\big\}$*, the inclusions of Poisson prime ideals of* A_9 *are described in the below diagram.*



§ II Background

This chapter contains some basic facts in algebras that are significant and useful to understand throughout this thesis. This chapter divides into three sections: General knowledge in algebras II.1, Poisson algebras and related subject II.2, and Poisson enveloping algebras II.3. Readers with a basic understanding of algebras are encouraged to move on to Chapter III.

§II.1 General definitions on algebras

The aim of the section is to recall some basic definitions and properties in ring theory and K-algebra to be used throughout this thesis. The left case will only be considered and the right case has the same situation but requires a different multiplication structure.

II.1.1 Rings and modules

The following are some well-known definitions and properties of rings and modules, and the main sources for these are [GoWa] and [Sta].

Definition II.1.1. An *unital ring* R is a set on which two binary operations are defined addition $(+)$ and multiplication $(\cdot)$ such that $(R, +)$ is an abelian group, $(R, \cdot)$ is a monoid set, and the distributive laws hold:

$$a \cdot (b + c) = a \cdot b + a \cdot c \text{ and } (a + b) \cdot c = a \cdot c + b \cdot c \text{ for all } a, b, c \in R.$$

In addition, R is a *commutative ring* if $a \cdot b = b \cdot a$ for all $a, b \in R$.

Definition II.1.2. Let R be a ring. A subset S of R is a *subring* if S is a ring under the operations inherited from R.

Definition II.1.3. Let R be a ring. A subset I of R is a *left ideal* if I is an abelian subgroup of R under addition and for all $r \in R, a \in I$ implies $ar \in I$. In addition, I is an *ideal*[1] of R if I is a left and right ideal of R. A ring R is a *simple ring* if R and 0 are the only ideals of R.

Definition II.1.4. Let R be a ring. R is called an *integral domain* if $ab = 0$ then either $a = 0$ or $b = 0$ for all $a, b \in R$. In particular, R has no non-zero element as a zero divisor.

Definition II.1.5. Let K be a commutative ring. If for any non-zero $a \in K$ there exists $b \in K$ such that $ab = ba = 1$ then K is called a *field*.

Definition II.1.6. Let R and S be rings. A map $\varphi : R \to S$ is called a *ring homomorphism* if

$$\begin{aligned} \varphi(a+b) &= \varphi(a) + \varphi(b), \\ \varphi(ab) &= \varphi(a)\varphi(b), \\ \varphi(1) &= 1 \end{aligned}$$

for all $a, b \in R$. If φ is injective, surjective and a bijection then φ is called a *monomorphism*, *epimorphism* and *isomorphism*, respectively.

Definition II.1.7. Let R be a ring. An ideal $\mathfrak{p}$ of R is *prime* if

$$IJ \subseteq \mathfrak{p} \quad \Longrightarrow \quad I \subseteq \mathfrak{p} \quad \text{or} \quad J \subseteq \mathfrak{p},$$

where I and J are ideals of R. In addition, the set of all prime ideals of R is called the *spectrum* of R and is denoted by $\mathrm{Spec}(R)$.

Definition II.1.8. Let R be a ring. The *height* of a prime ideal $\mathfrak{p}$ of R is the supremum of all integers n such that there exists a chain

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$$

[1] *two-sided* ideal

of distinct prime ideals of R, and is denoted by $\text{ht}(\mathfrak{p})$. In addition, the *Krull dimension* of R is the supremum of the heights of all prime ideals of R.

Definition II.1.9. Let R be a ring. An ideal $\mathfrak{m}$ of R is a *maximal* if $\mathfrak{m} \neq R$ and the only ideal strictly containing $\mathfrak{m}$ is R. The set of all maximal ideals of R is denoted by $\text{Max}(R)$.

Definition II.1.10. Let R be a ring. The set

$$Z(R) = \{z \in R \mid zr = rz \text{ for all } r \in R\}$$

is called the *centre* of R. In addition, the centre of R is a commutative subring of R.

Definition II.1.11. Let R be a ring over a field K. If K is a subset of $Z(R)$ then R is called a *K-algebra.*

Definition II.1.12. Let R be a ring. If there is a family $\{R_i\}_{i\geq 0}$ of additive subgroups of R such that

1. $1 \in R_0$,
2. $R_i \subseteq R_j$ for all $i \leq j$,
3. $R_iR_j \subseteq R_{i+j}$ for all $i, j \geq 0$, and
4. $R = \bigcup_{i\geq 0} R_i$.

then R is called a *filtered ring* and the family $\{R_i\}$ is called a *filtration* of R.

Definition II.1.13. Let R be a ring and $\{R_i\}_{i\in\mathbb{Z}}$ be a family of additive subgroups of R then R is called a $\mathbb{Z}$-*graded* ring[2] if

1. $R = \bigoplus_{i\in\mathbb{Z}} R_i$, and
2. $R_iR_j \subseteq R_{i+j}$ for all $i, j \geq 0$.

The family $\{R_i\}$ is called a *grading* of R and a non-zero element of R_i is called an *element of homogeneous* degree i.

[2] or *a graded ring*

Definition II.1.14. Let R be a filtered ring and $\{R_i\}_{i\geq 0}$ be a filtration of R. The *associated graded ring* is

$$\mathrm{gr}(R) = \bigoplus_{i\geq 0} (R_i/R_{i-1})^{3}$$

equipped with

$$(r + R_{i-1})(s + R_{j-1}) = rs + R_{i+j-1} \quad \text{for all} \quad r \in R_i \quad \text{and} \quad s \in R_j.$$

For all $r \in R_i$ we have $\bar{r} = r + R_{i-1} \in R_i/R_{i-1}$.

Definition II.1.15. Let R be a ring. An *unital left R-module M* is an additive abelian group with a map $R \times M \to M$ that is defined by $(r, m) \mapsto rm$ such that

1. $(r_1 + r_2)m = r_1m + r_2m$ for all $r_1, r_2 \in R, m \in M$,
2. $r(m_1 + m_2) = rm_1 + rm_2$ for all $r \in R, m_1, m_2 \in M$,
3. $(r_1r_2)m = r_1(r_2m)$ for all $r_1, r_2 \in R, m \in M$, and
4. $1m = m$ for all $m \in M$.

Definition II.1.16. Let R be a ring and M be a left R-module. A subset N of M is called a *left submodule* of M if N is a left R-module under the operations inherited from M. In addition, M is a *simple* R-module if M and 0 are the only submodules of M.

Definition II.1.17. Let R be a ring and M be a left R-module. If M can be expressed as $M = \sum_{i=0}^{n} Rm_i$ for some elements $m_i \in M$ then M is called a *finitely generated R-module.* In addition, if M is generated by an element $m \in M$ such that $M = Rm = \{rm \mid r \in R\}$ is called a *cyclic* R-module.

The following is the definition of annihilator of modules.

Definition II.1.18. Let R be a ring, M be a left R-module and X be a subset of M then the *annihilator* of X is

$$\mathrm{ann}_R(X) = \{r \in R \mid rx = 0 \ \text{ for all } \ x \in X\}.$$

[3] as additive groups and $R_{-1} := 0$

Definition II.1.19. Let R be a ring. An ideal P of R is *left primitive* if $P = \operatorname{ann}_R(M)$ for some simple left R-modules M. In addition, a ring R is called a *left primitive* ring if 0 is a left primitive ideal of R.

Definition II.1.20. Let R be a ring. A left R-module M of R is a *left Noetherian* if every ascending chain

$$N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots$$

of left submodules of M is eventually stationary. Thus, there exists some integer n_0 such that $N_n = N_{n+1}$ for all $n \geq n_0$. A module M of R is a *Noetherian* if M is a left and right Noetherian.

Definition II.1.21. Let R be a ring. Then R is *left Noetherian* if and only if R is a left Noetherian R-module. In addition, R is a *Noetherian ring* if R is a left and right Noetherian.

Definition II.1.22. Let K be a field and R be a K-algebra. If R is finitely generated with finite-dimensional generating K-subspace V containing 1[4] then the real number, $\limsup \big(\log_n(\dim_K V^n)\big)$, is independent of the generating subspace V of R, this number is called the *Gelfand-Kirillov dimension* of R

$$\operatorname{GK}\dim(R) = \limsup \big(\log_n(\dim_K V^n)\big).$$

Definition II.1.23. Let R be a ring and α be an automorphism of R. A *skew Laurent ring* $T = R[x^{\pm 1}; \alpha]$ over R[5] is a ring such that

1. T is a ring and $R \subseteq T$,
2. an element x is invertible in T,
3. T is a free left R-module with basis $\{1, x, x^{-1}, x^2, x^{-2}, \ldots\}$, and
4. $xr = \alpha(r)x$ for all $r \in R$.

[4]that is, $R = \cup_{n=1}^{\infty} V^n$

[5]or *skew Laurent extension* of R

Definition II.1.24. Let R be a ring and α be an endomorphism of R. A additive map $\delta : R \to R$ is called a *left α-derivation* on R if

$$\delta(ab) = \alpha(a)\delta(b) + \delta(a)b \quad \text{for all} \quad a, b \in R.$$

Definition II.1.25. Let R be a ring, α be a ring endomorphism of R and δ be a left α-derivation on R. Then $S = R[x; \alpha, \delta]$ is called a *skew polynomial ring* over R[6] if

1. $R \subseteq S$ and S is a ring,
2. an element x is in S,
3. S is a free left R-module with basis $\{1, x, x^2, \ldots\}$, and
4. $xr = \alpha(r)x + \delta(r)$ for all $r \in R$.

II.1.2 Lie algebras

The following are some facts in Lie algebras, and the main source for these definitions is [Pre].

Definition II.1.26. A vector space L over a field K is called a *Lie algebra* if there exists a bilinear product $[\cdot, \cdot]$ on L, called a *Lie bracket*, which is *anti-commutative* and satisfies the *Jacobi identity*:

$$[a, a] = 0 \quad \text{and} \quad [a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0 \quad \text{for all} \quad a, b, c \in L.$$

Definition II.1.27. Let L be a Lie algebra. A K-subspace B of L is a *Lie subalgebra* if $[b_1, b_2] \in B$ for all b_1, $b_2 \in B$.

Definition II.1.28. Let L be a Lie algebra. A K-subspace I of L is a *Lie ideal* if $[a, b] \in I$ for all $a \in L$ and $b \in I$. In addition, L is called a *simple Lie algebra* if L is not abelian and if L and 0 are the only Lie ideals of L.

[6] or an *Ore extension* of R

Definition II.1.29. Let L_1 and L_2 be Lie algebras over a field K. A K-linear map

$$\varphi : L_1 \to L_2$$

is called a *Lie algebra homomorphism* if $\varphi([a,b]) = [\varphi(a), \varphi(b)]$ for all $a, b \in L_1$.

Definition II.1.30. Let L be a Lie algebra over a field K. A K-vector space M is an *L-module* if there is a bilinear product $[\cdot,\cdot]_M : L \times M \to M$ such that

$$[[a,b],m]_M = [a,[b,m]_M]_M - [b,[a,m]_M]_M \quad \text{for all } \; m \in M \; \text{ and } \; a, b \in L.$$

In addition, a K-subspace N of M is called an *L-submodule* of M if $[a,n]_M \in N$ for all $a \in L$ and $n \in N$. In addition, M is a *simple*[7] L-module if M and 0 are the only submodules of M.

Definition II.1.31. Let L be an algebra over a field K[8] with a binary operation $(a,b) \mapsto a \cdot b$. A K-endomorphism α of L is called *K-derivation* on L if

$$\alpha(a \cdot b) = \alpha(a) \cdot b + a \cdot \alpha(b) \quad \text{for all } \; a, b \in L.$$

The set of *K-derivations* on L is denoted by $\mathrm{Der}_K(L)$. In addition, the set of *inner derivations* on L is

$$\mathrm{IDer}_K(L) := \{\mathrm{ad}_a \mid a \in L\}, \quad \text{where } \; \mathrm{ad}_a(b) := [a,b] := ab - ba.$$

Definition II.1.32. Let L be a Lie algebra over a field K. If U is an unital associative algebra and $\varphi : L \to U$ is a Lie algebra homomorphism then the pair (φ, U) is called an *enveloping algebra* of L.

Definition II.1.33. The *universal enveloping algebra* of Lie algebra L is an enveloping algebra $\big(\psi, U(L)\big)$ which has the following universal mapping property: for any enveloping algebra (φ, U) of L there exists a unique associative algebra homomorphism $f : U(L) \to U$ such that $\varphi = f \circ \psi$.

[7] *irreducible*
[8] not necessarily associative or Lie algebra

II.1.3 The Weyl algebras and the Generalized Weyl algebras

Definition II.1.34. Let A_n be an associative K-algebra generated by $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ as determined by the defining relations

$$[Y_i, X_j] = \delta_{ij} \quad \text{and} \quad [X_i, X_j] = [Y_i, Y_j] = 0 \quad \text{for all} \;\; i, j,$$

where δ_{ij} is the Kronecker delta function then A_n is called the n'th *Weyl algebra*.

The following definition is the generalized Weyl algebra that was introduced by V. V. Bavula. Further details concerning the generalized Weyl algebra can be found in [Bav2].

Definition II.1.35. [Bav2, page 72] Let D be a ring, $\sigma = (\sigma_1, \ldots, \sigma_n)$ be an n-tuple of commuting automorphisms of D, $a = (a_1, \ldots, a_n)$ where $a_i \in Z(D)$ such that $\sigma_i(a_j) = a_j$ for all $i \neq j$. The *generalized Weyl algebra*[9] $A = D[X, Y; \sigma, a]$ of rank n is a ring generated by D and $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ as determined by the defining relations

$$Y_i X_i = a_i, \;\; X_i Y_i = \sigma_i(a_i),$$

$$X_i d = \sigma_i(d) X_i, \; Y_i d = \sigma_i^{-1}(d) Y_i, \;\; \text{for all } d \in D,$$

$$[X_i, X_j] = [X_i, Y_j] = [Y_i, Y_j] = 0 \;\; \text{for all } i \neq j,$$

where $[x, y] = xy - yx$. Notice that a and σ are called *sets of defining elements* and *automorphisms* of the generalized Weyl algebra A, respectively.

[9]in short GWA

§II.2 Poisson algebras: Review of results

The goal of this section is to recall some essential definitions and properties of Poisson algebras and related topics to be used throughout this thesis. In particular, the constructions of Poisson brackets and some criteria to classify Poisson prime ideals are discussed. In addition, our motivations for Poisson algebras are demonstrated by reviewing some studies on Poisson algebras and related subjects, see Subsections II.2.5 and II.2.6.

II.2.1 Poisson algebras

The following are some basic definitions and properties of Poisson algebras. The main sources for these are [Oh1], [Oh3], [Oh4], [JoOh] and [Bav3].

Definition II.2.1. A (commutative) K-algebra D is called a *Poisson algebra* if there exists a bilinear product $\{\cdot,\cdot\}$ on D, called a *Poisson bracket*, such that $(D,\{\cdot,\cdot\})$ is a Lie algebra and satisfies the *Leibniz's rule*

$$\{a\cdot b, c\} = a\cdot\{b,c\} + \{a,c\}\cdot b \quad \text{for all} \;\; a,b,c\in D.$$

Definition II.2.2. A K-subspace B of a Poisson algebra D is a *Poisson subalgebra* if $\{b_1,b_2\}\in B$ for all $b_1,\, b_2\in B$.

Definition II.2.3. Let D be a Poisson algebra. A Poisson bracket is called a *trivial Poisson bracket* if $\{a,b\}=0$ for all $a,b\in D$.

Definition II.2.4. Let D be a Poisson algebra. Then

1. $Z(D):=\{a\in D \mid a\cdot b = b\cdot a \;\text{ for all } b\in D\}$ is called the *centre* of D.

2. $\mathrm{PZ}(D){:=}\{a\in D \mid \{a,b\}=0 \;\text{ for all }\; b\in D\}$ is called the *Poisson centre* of D.

3. $\mathcal{Z}(D):=\{a\in D \mid a\cdot b = b\cdot a, \text{ and } \{a,b\}=0 \;\text{ for all } b\in D\}$ is called the *absolute centre* of D.

If D is a Poisson algebra with trivial Poisson bracket then the above three sets are equal.

Definition II.2.5. Let D_1 and D_2 be Poisson algebras over a field K, and define K-algebra homomorphism $\varphi : D_1 \to D_2$, which is called a *Poisson homomorphism* if

$$\varphi(\{a,b\}) = \{\varphi(a), \varphi(b)\} \quad \text{for all} \quad a, b \in D_1.$$

Definition II.2.6. Let D be a Poisson algebra. An ideal I of the algebra D is called a *Poisson ideal* of D if $\{a,b\} \in I$ for all $a \in D$ and $b \in I$. The Poisson ideal of D generated by a is denoted by (a). In addition, the algebra D is a *simple Poisson algebra*[10] if D and 0 are the only Poisson ideals of D.

Definition II.2.7. Let D be a Poisson associative algebra over a field K, and $\mathrm{Der}_K(D)$ be the set of K-derivations on D. Then

$$\mathrm{PDer}_K(D) := \big\{\delta \in \mathrm{Der}_K(D) \mid \delta(\{a,b\}) = \{\delta(a), b\} + \{a, \delta(b)\} \text{ for all } a, b \in D\big\}$$

is called the set of *Poisson derivations* on D. In addition, the set of *inner derivations* on D is

$$\mathrm{PIDer}_K(D) := \big\{\mathrm{pad}_a \mid a \in D\big\}, \quad \text{where} \;\; \mathrm{pad}_a(b) := \{a,b\}, \quad b \in D.$$

Notice that, for all $a \in D$ the derivation $\mathrm{pad}_a = \mathrm{ham}(a) := \{a, \cdot\} \in \mathrm{Der}_K(D)$ is called also the *hamiltonian vector field*[11] associated with the element a.

Definition II.2.8. Let D be a Poisson algebra. A Poisson ideal $\mathfrak{p}$ of D is called a *prime Poisson ideal* if $\mathfrak{p}$ is a prime ideal of D. In addition, a Poisson ideal $\mathfrak{q}$ of D is called a *Poisson prime ideal* of D if

$$IJ \subseteq \mathfrak{q} \implies I \subseteq \mathfrak{q} \quad \text{or} \quad J \subseteq \mathfrak{q},$$

where I and J are Poisson ideals of D.

Definition II.2.9. Let D be a Poisson algebra. A set of all Poisson prime ideals of D is called the *Poisson spectrum* of D and is denoted by $\mathrm{PSpec}(D)$.

Definition II.2.10. Let D be a Poisson algebra. A Poisson ideal I of D is called a *maximal*

[10] or *Poisson simple algebra*
[11] or a *hamiltonian derivation*

Poisson ideal of D if I is a maximal element with respect to $(\subseteq, \text{PSpec}(D))$.

$$\text{PMax}(D) = \{I \in \text{PSpec}(D) \mid I \text{ is a maximal element}\}.$$

A Poisson ideal $\mathfrak{m}$ of D is a *Poisson maximal ideal* if $\mathfrak{m}$ is a maximal ideal that is also Poisson. In addition, a Poisson ideal I of D is called a *minimal Poisson ideal* of D if I is a minimal element with respect to $(\subseteq, \text{PSpec}(D))$.

Definition II.2.11. Let D be a Poisson algebra. A Poisson ideal P of D is *Poisson primitive*[12] if there exists a maximal ideal M of D such that P is the largest Poisson ideal contained in M.

Definition II.2.12. Let D be a Poisson algebra. A Poisson ideal $\mathfrak{p}$ of D is a *Poisson height n prime ideal* if $\mathfrak{p}$ is a Poisson prime ideal of D and has height n as a prime ideal.

Definition II.2.13. Let D be a Poisson algebra, and Δ be a set of linear maps from D into itself. A Poisson ideal I of D is called Δ-*Poisson ideal*[13] if $\delta(I) \subseteq I$ for all $\delta \in \Delta$.

Definition II.2.14. Let D be a Poisson algebra and δ be a Poisson derivation on D. If D and 0 are the only δ-Poisson ideals of D then D is called an δ-*Poisson simple*. In addition, a δ-ideal $\mathfrak{p}$ of D is called a δ-*prime Poisson ideal* if $\mathfrak{p}$ is a prime Poisson ideal.

Definition II.2.15. Let G be a monoid and the Poisson algebra D be G-graded algebra such that

$$\{D_i, D_j\} \subseteq D_{i+j} \quad \text{for all } i, j \in G$$

then D is called a *G-graded Poisson algebra*.

The following remark defines the Poisson structures on factor Poisson algebras, localized Poisson algebras and the tensor product of Poisson algebras.

Remark II.2.16. Let D be a Poisson algebra.

[12] symplectic

[13] or Δ-*invariant* or Δ-*stable*

1. Let I be a Poisson ideal of D. Then the *factor algebra* D/I is a Poisson algebra with Poisson bracket defined by the rule

$$\{a+I, b+I\} = \{a,b\} + I \quad \text{for all} \quad a,b \in D.$$

2. If S is a multiplicative subset of D then the *localization algebra* $S^{-1}D$ of the algebra D is a Poisson algebra with Poisson bracket defined by the rule

$$\{as^{-1}, bt^{-1}\} = s^{-2}t^{-2}(st\{a,b\} - sb\{a,t\} - at\{s,b\} + ab\{s,t\}) \text{ for all } a,b \in D,\ s,t \in S.$$

3. Let D_1 and D_2 be Poisson algebras. Their *tensor product* $D_1 \otimes D_2$ is a Poisson algebra with Poisson bracket defined by the rule

$$\{a_1 \otimes b_1, a_2 \otimes b_2\} = \{a_1, a_2\} \otimes b_1 b_2 + a_1 a_2 \otimes \{b_1, b_2\},$$

where $a_1, a_2 \in D_1, b_1, b_2 \in D_2$ and $\{D_1, D_2\} = 0$.

II.2.2 Construction of Poisson polynomial algebras

The theorem below gives us the first extension of Poisson polynomial algebras into one variable, and the Poisson structure is defined similarly to the multiplication in skew polynomial algebras.

Theorem II.2.17. *[Oh3, Theorem 1.1] Let D be a Poisson algebra over a field K and α, δ be K-linear maps on D. Then the polynomial ring $D[x]$ becomes a Poisson algebra with Poisson bracket*

$$\{a, x\} = \alpha(a)x + \delta(a) \quad \textit{for all} \quad a \in D \tag{II.2.1}$$

if and only if α is a Poisson derivation on D and δ is a derivation on D such that

$$\delta(\{a,b\}) - \{\delta(a), b\} - \{a, \delta(b)\} = \delta(a)\alpha(b) - \alpha(a)\delta(b) \quad \textit{for all} \quad a,b \in D. \tag{II.2.2}$$

The Poisson algebra $D[x]$ *is denoted by* $D[x;\alpha,\delta]_p$ *and if* δ *is zero then it is denoted by* $D[x;\alpha]_p$.

The next remark shows that the distinction between forms of the Poisson polynomial algebras and skew polynomial algebras.

Remark II.2.18. [MyOh] The Poisson polynomial algebra $D[x]$ in Theorem II.2.17 is denoted by $D[x;\alpha,\delta]_p$ instead of $D[x;\alpha,\delta]$ to distinguish it from skew polynomial algebras.

The following lemma gives us the second extension of Poisson polynomial algebras into two variables. This is the critical method used to construct the class of Poisson algebras $\mathcal{A}$, see Chapter IV.

Lemma II.2.19. *[Oh3, Lemma 1.3] Let* D *be a Poisson algebra over a field* K*,* $c \in K$*,* $u \in D$ *and* α*,* β *be Poisson derivations on* D *such that*

$$\alpha\beta = \beta\alpha \quad \text{and} \quad \{a,u\} = (\alpha+\beta)(a)u \quad \text{for all} \quad a \in D. \tag{II.2.3}$$

Then the polynomial ring $D[x,y]$ *becomes a Poisson algebra with Poisson bracket*

$$\{a,y\} = \alpha(a)y, \quad \{a,x\} = \beta(a)x \quad \text{and} \quad \{y,x\} = cyx + u \ \text{ for all } \ a \in D. \tag{II.2.4}$$

The Poisson algebra $D[x,y]$ *with Poisson bracket* (II.2.4) *is denoted by* $(D;\alpha,\beta,c,u)$ *or* $D[y;\alpha,0]_p[x;\beta,\delta' := u\partial_y]_p$.

II.2.3 Poisson modules

The following are some basic definitions and properties of Poisson modules, and the main source is [Jor].

Notice that, the Poisson modules have been defined in several ways across the literature. The following definition was introduced by D. R. Farkas, [Far].

Definition II.2.20. Let D be a commutative Poisson algebra with Poisson bracket $\{\cdot,\cdot\}$ and M be a D-module then M is called a *Poisson D-module* if there is a bilinear product $\{\cdot,\cdot\}_M : D \times M \to M$ such that the following hold:

1. $\{\{a,b\},m\}_M = \{a,\{b,m\}_M\}_M - \{b,\{a,m\}_M\}_M$ for all $a,b \in D$, $m \in M$,

2. $\{a,bm\}_M = \{a,b\}m + b\{a,m\}_M$ for all $a,b \in D$, $m \in M$, and

3. $\{ab,m\}_M = a\{b,m\}_M + b\{a,m\}_M$ for all $a,b \in D$, $m \in M$.

Definition II.2.21. Let D be a Poisson algebra and M be a Poisson D-module. A submodule N of M is a *Poisson submodule* if $\{a,n\}_M \in N$ for all $a \in D$ and $n \in N$. In addition, M is a *simple Poisson D-module* if M and 0 are the only submodules of M.

Definition II.2.22. Let D be a Poisson algebra and M be a Poisson D-module. In addition, M is a *semisimple* Poisson module if M is a direct sum of simple Poisson D-modules.

Definition II.2.23. Let D be a Poisson algebra, and M_1 and M_2 be Poisson D-modules. *A Poisson module homomorphism* $\varphi : M_1 \to M_2$ is a D-module homomorphism such that

$$\varphi(\{a,m\}_{M_1}) = \{a,\varphi(m)\}_{M_2} \text{ for all } a \in D \text{ and } m \in M_1.$$

In addition, if φ is bijective then $\varphi^{-1} : M_2 \to M_1$ is also a Poisson module homomorphism and φ is a *Poisson module isomorphism.*

The following remarks give us some Poisson module structures expressed by factoring Poisson algebras by their Poisson ideals.

Remark II.2.24. [Jor, Remark 4] Let D be a Poisson algebra. There is a natural method for Poisson D-modules to appear in which I and J are Poisson ideals of D with $I \subseteq J$ then the factor J/I is a Poisson D-module with $\{a, j+I\}_{J/I} = \{a,j\} + I$, where $a \in D$ and $j \in J$. Every Poisson submodule of J/I is a Lie ideal, hence, if J/I is simple as a Lie algebra then it is simple as a Poisson module.

Remark II.2.25. Let D be a Poisson algebra and I, J be Poisson ideals of D then I/IJ and J/IJ are Poisson D-modules.

The following is the definition of the annihilator of Poisson modules.

Definition II.2.26. Let D be a Poisson algebra and M be a Poisson D-module. The *annihilator* of M is

$$\text{ann}_D(M) = \{a \in D \mid am = 0 \text{ for all } m \in M\}.$$

The following is the definition of the Poisson annihilator of Poisson modules.

Definition II.2.27. Let D be a Poisson algebra, M be Poisson D-module and $S \subseteq M$ then we have

$$\text{Pann}_D(S) = \{a \in D \mid \{a, m\}_M = 0 \text{ for all } m \in S\}.$$

II.2.4 Poisson algebras in algebraic geometry

The aim of this subsection is to recall some basic terminology and notations on algebraic geometry to be used throughout this thesis. The main source in this subsection is [Har].

Definition II.2.28. An irreducible closed subset of $\mathbb{A}^n_K$, with the induced topology, is called an *affine algebraic variety*[14], and an open subset of an affine variety is called a *quasi-affine variety*.

Definition II.2.29. Let Y be a quasi-affine variety in $\mathbb{A}^n_K$ and $f : Y \to K$ be a function on Y. If there is an open neighbourhood U with $P \in U \subseteq Y$, and $g, h \in A = K[x_1, \ldots, x_n]$ such that h is not zero on U and $f = g/h$ on U then f is called *regular at a point* $P \in Y$. In addition, if f is regular at every point of Y then is called *regular* on Y.

Definition II.2.30. Let X, Y be varieties and $\varphi : X \to Y$ be a continuous map such that for every open set $U \subseteq Y$, and every regular function $f : U \to K$ the function $f \circ \varphi : \varphi^{-1}(U) \to K$ is regular then φ is called a *morphism*.

Definition II.2.31. Let X be a topological space. A *presheaf* $\mathscr{F}$ of abelian groups on X consists of the data

1. for every open subset $U \subseteq X$, an abelian group $\mathscr{F}(U)$, and

[14] or simply *affine variety*

2. for every inclusion $V \subseteq U$ of open subsets of X, a morphism of abelian groups $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$

subject to the conditions

(a) $\mathscr{F}(\emptyset) = 0$, where $\emptyset$ is the empty set,

(b) $\rho_{UU} : \mathscr{F}(U) \to \mathscr{F}(U)$ is the identity map, and

(c) if $W \subseteq V \subseteq U$ are three open subsets then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

Definition II.2.32. A presheaf $\mathscr{F}$ on a topological space X is a *sheaf* if it satisfies the following supplementary conditions:

(d) If U is an open set, $\{V_i\}$ is an open covering of U and $s \in \mathscr{F}(U)$ is an element such that $s|_{V_i} = 0$ for all i then $s = 0$[15], and

(f) If U is an open set, $\{V_i\}$ is an open covering of U and $s_i \in \mathscr{F}(V_i)$ for each i with the property that for each i, j, $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ then there is an elements $s \in \mathscr{F}(U)$ such that $s|_{V_i}, = s_i$ for each i.

Definition II.2.33. Let $\mathscr{F}$ be a presheaf on X. The elements of $\mathscr{F}(U)$ are called *sections* of $\mathscr{F}$ over the open set U.

Definition II.2.34. Let X be a topological space and $\mathcal{O}_X$ be a sheaf of rings on X then a pair $(X, \mathcal{O}_X)$ is called a *ringed space*.

Definition II.2.35. A *sheaf of ideals*[16] on X is a sheaf of modules $\mathscr{F}$ which is a subsheaf of $\mathcal{O}_X$. In other words, for every open set U, $\mathscr{F}(U)$ is an ideal in $\mathcal{O}_X(U)$.

Definition II.2.36. Let $(X, \mathcal{O}_X)$ be a locally ringed space such that $(X, \mathcal{O}_X) \simeq \operatorname{Spec}(R)$ for some ring R then X is called an *affine scheme*.

Definition II.2.37. Let $(X, \mathcal{O}_X)$ be a locally ringed space in which every point has an open neighbourhood U such that the topological space U, together with the restricted sheaf $\mathcal{O}_X|_U$ is an affine scheme then X is called a *scheme*.

[15] Note condition (d) implies that s is unique

[16] or an *ideal sheaf*

Definition II.2.38. Let X be a non-singular variety X of dimension n. The sheaf Ω_X^1 *of differential 1-forms* on X is a vector bundle of rank n. Its determinant $\bigwedge^n \Omega_X^1 = \Omega_X^n$ is the *canonical bundle* of X, denoted ω_X.

Definition II.2.39. Let X be an algebraic variety and $\mathcal{O}_X$ be a sheaf of holomorphic functions on X. A *(holomorphic) Poisson structure* on X is a $\mathbb{C}$-bilinear operation

$$\{\cdot,\cdot\} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$$

which satisfies *skew-symmetry*, *Leibniz's rule* and *Jacobi identity* for all elements in $\mathcal{O}_X$.

Definition II.2.40. Let Y be an affine variety and suppose that $A(Y)$ is equipped with a Lie bracket $\{\cdot,\cdot\} : A(Y) \times A(Y) \to A(Y)$, which makes $\big(A(Y), \{\cdot,\cdot\}\big)$ into a Poisson algebra. Then $\big(Y, \{\cdot,\cdot\}\big)$ is called an *affine Poisson variety*[17].

II.2.5 Review on Poisson polynomial algebras

The first review is on the paper [Oh3]. This study gives us a significant structure of Poisson polynomial algebras, which is used to consider the class of Poisson algebras $\mathcal{A}$, see Chapter IV for detail. Following that, some valuable properties of localized and factor Poisson polynomial algebras are given in Lemma II.2.43. Additionally, some simplicity criteria for skew Poisson polynomial algebras are considered in Proposition II.2.45. After that, there are several techniques to classify Poisson prime ideals of Poisson polynomial algebras as Example II.2.50. In this example, there is an interesting approach for classifying Poisson prime ideals of a Poisson algebra that has dimension four.

Definition II.2.41. Let D be a Poisson algebra.

1. A Poisson derivation α on D is called an *inner map* if there exists an invertible element $a \in D$ such that $\alpha(b) = a^{-1}\{b, a\}$ for all $b \in D$.

2. An element a of D is called *Poisson normal* if $\{a, D\} \subseteq aD$.

The next lemma shows that any two derivations are equal, commute or skew/Poisson derivations, if this is true on a set of generators.

[17] or simple *Poisson variety*

Lemma II.2.42. *[Oh3, Lemma 1.2] Let D be a Poisson algebra over a field K and α, δ be derivations on D. If D is generated by a set X as an algebra, then*

1. *if $\alpha(a) = \delta(a)$ for all $a \in X$ then α is equal δ.*

2. *if $\alpha\delta(a) = \delta\alpha(a)$ for all $a \in X$ then α and δ commute.*

3. *if α satisfies $\alpha(\{a, b\}) = \{\alpha(a), b\} + \{a, \alpha(b)\}$ for all $a, b \in X$ then $\alpha \in \mathrm{PDer}(D)$.*

4. *if α and δ satisfy (II.2.2) for all elements in X then α and δ satisfy (II.2.2) for all elements in D.*

The next lemma gives us the structure of the localization and factorization of Poisson algebra $A = D[x; \alpha, \delta]_p$. Also, the variables can be swapped if their images are linear in one term.

Lemma II.2.43. *[Oh3, Lemma 3.2] Let D be a Poisson algebra over a field K, (α, δ) be a skew Poisson derivation on D and $A = D[x; \alpha, \delta]_p$.*

1. *If S is a multiplicative subset of D, then any derivation on D is uniquely extended to $S^{-1}D$ and*
$$(S^{-1}D)[x; \alpha', \delta']_p \cong S^{-1}A,$$
where α' and δ' are the extensions of α and δ on $S^{-1}D$, respectively.

2. *If I is an (α, δ)-Poisson ideal of D, then IA is a Poisson ideal of A and*
$$A/IA \cong (D/I)[x; \bar{\alpha}, \bar{\delta}]_p,$$
where $\bar{\alpha}$ and $\bar{\delta}$ are the maps induced on D/I by α and δ, respectively.

3. *A Poisson algebra $D[x; \alpha]_p[y; \beta]_p$ such that $\beta(D) \subseteq D$ and $\beta(x) = bx$ for some element $b \in D$ is equal to $D[y; \beta']_p[x; \alpha']_p$, where $\beta' = \beta|_D$ and $\alpha' : D[y, \beta']_p \to D[y, \beta']_p$ is defined by*
$$\alpha'(a) = \alpha(a) \quad \text{and} \quad \alpha'(y) = -by \quad \text{for all} \quad a \in D.$$

The next lemma gives us simplicity criterion for the Poisson algebra $A = D[x^{\pm 1}; \alpha]_p$.

Lemma II.2.44. *[Oh3, Lemma 3.3] Let $A = D[x^{\pm 1}; \alpha]_p$. Then A is a Poisson simple if and only if*

1. *D is α-Poisson simple, and*
2. *$n\alpha$ is not an inner map for all $n \in \mathbb{Z}^+$.*

The following proposition gives us a criterion of the simplicity of prime Poisson ideals of $A = D[x; \alpha]_p$.

Proposition II.2.45. *[Oh3, Proposition 3.4] Let $A = D[x; \alpha]_p$.*

1. *If J is a prime Poisson ideals of A such that $x \in J$ then J has the form $I + xA$, where I is a prime Poisson ideal of D.*
2. *If I is an α-prime Poisson ideal of D then IA is a prime Poisson ideal of A.*
3. *If P is a prime Poisson ideal of A such that $x \notin P$ then $P \cap D$ is an α-prime Poisson ideal of D.*
4. *If $n\alpha$ is an inner map, where $n \in \mathbb{Z}^+$ then there is $y \in \mathrm{PZ}(A)$ such that y is transcendental over D, and A is a finitely generated $D[y]$-module.*
5. *If D is α-Poisson simple and there is no positive integer n such that $n\alpha$ is an inner map then x is inside every prime Poisson ideal of A.*

The following examples give us some different structures of Poisson brackets on commutative polynomial algebras.

Example II.2.46. *The Poisson algebra $K[y, x]$.* Let $K[y]$ be a Poisson polynomial algebra with trivial Poisson bracket. Set

$$\alpha = f\partial_y \quad \text{and} \quad \delta = g\partial_y, \quad \text{where} \;\; f, g \in K[y].$$

Then α is a Poisson derivation, δ is a derivation and $(\alpha,\ \delta)$ satisfies (II.2.2). Hence, by Theorem II.2.17, $K[y, x] = K[y][x; \alpha, \delta]_p$ is a Poisson algebra with Poisson bracket

$$\{y, x\} = fx + g, \quad \text{where} \;\; f, g \in K[y].$$

The next example gives us the expression of a generalization of the Poisson algebra $D[x;\alpha,0]$ that is in Theorem II.2.17, when $\delta=0$.

Example II.2.47. *The Poisson n-space.* Let $\Lambda=(\lambda_{ij})$ be a skew-symmetric $n\times n$ matrix and $K[x_1]$ be a Poisson algebra with trivial Poisson bracket. Suppose that

$$\alpha_i=\lambda_{1i}x_1\partial_{x_1}+\cdots+\lambda_{i-1,i}x_{i-1}\partial_{x_{i-1}},\quad \text{where}\quad i=2,\dots,n$$

is a derivation on $K[x_1,\dots,x_{i-1}]$. Now, since α_2 is a Poisson derivation on $K[x_1]$ which implies that $K[x_1][x_2;\alpha_2]_p$ is a Poisson algebra. Notice that, if

$$B=K[x_1][x_2;\alpha_2]_p\dots[x_{n-1};\alpha_{n-1}]_p$$

is a Poisson algebra and by using Lemma II.2.42 we have that α_n is a Poisson derivation on B. Therefore, it follows from Theorem II.2.17 that $B[x_n;\alpha_n]_p=K[x_1,\dots,x_n]$ is a Poisson algebra. Hence, by induction on n, we have the coordinate ring $\mathcal{O}(K^n)=K[x_1,\dots,x_n]$ of the affine n-space K^n is a Poisson algebra with Poisson bracket defined by the rule

$$\{x_i,x_j\}=\lambda_{ij}x_ix_j\quad\text{for all}\quad i,j=1,\dots,n,\quad\text{see [Oh1]}.$$

The most familiar example of Poisson brackets is given in the following example.

Example II.2.48. The polynomial ring $A=K[y_1,x_1,\dots,y_n,x_n]$ is the simple Poisson $2n$-*space* with Poisson bracket defined by the rule

$$\{f,g\}=\sum\Big(\frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}-\frac{\partial g}{\partial y_i}\frac{\partial f}{\partial x_i}\Big)\quad\text{for all}\quad f,g\in A,$$

which is given in [ChPr]. Equivalently,

$$\{y_i,x_j\}=\delta_{ij},\quad\{x_i,x_j\}=0\quad\text{and}\quad\{y_i,y_j\}=0\quad\text{for all}\quad i,j,$$

where δ_{ij} is the Kronecker delta function. Let A_i be a Poisson subalgebra of A generated by $y_1,x_1,\dots,y_i,x_i$. The algebra A_i can be written as the Poisson algebra that is in Lemma

II.2.19 by

$$A_i = A_{i-1}[y_i; 0]_p[x_i; 0, \delta_i]_p = (A_{i-1}; 0, 0, 0, 1),$$

where δ_i is defined by

$$\delta_i(y_j) = 0, \quad \delta_i(x_j) = 0 \quad \text{and} \quad \delta_i(y_i) = 1 \quad \text{for all} \quad j = 1, \ldots, i-1.$$

The following is a typical example of Poisson algebra $(D; \alpha, \beta, c, u)$ that is in Lemma II.2.19, and is similar to the class of Poisson algebras A, whereas this has dimension four.

Example II.2.49. *The Poisson algebra $M_2(K)$.* Let $K[x,y]$ be a Poisson polynomial algebra with trivial Poisson bracket and $\alpha = -2x\,\partial_x - 2y\,\partial_y$ be a derivation on $K[x,y]$. Then α is a Poisson derivation and by using Lemma II.2.19 there exists

$$\big(K[x,y]; \alpha, -\alpha, 0, 4xy\big),$$

which is the Poisson algebra $\mathcal{O}(M_2(K)) = K[x,y][s,t]$ with Poisson bracket

$$\begin{aligned} \{x,y\} &= 0, & \{x,t\} &= 2xt, & \{x,s\} &= -2xs, \\ \{y,t\} &= 2yt, & \{y,s\} &= -2ys, & \{s,t\} &= 4xy, \end{aligned}$$

given in [Oh1, 2.9] and [Van, 3.13].

The next example describes a significant approach for classifying prime Poisson ideals of $A_2 = K[y_1, x_1, y_2, x_2]$. This is breaking down the Poisson algebra into factor algebras by some of their principal Poisson ideals to classify the prime Poisson ideals of A_2 that contain these ideals, and then classify the ideals of A_2 that do not contain any of these four variables.

Example II.2.50. [Oh3, Example 3.6] The coordinate ring A_2 of Poisson symplectic 4-space is the Poisson algebra $K[y_1, x_1, y_2, x_2]$ with Poisson bracket

$$\begin{aligned} \{y_1, x_1\} &= 2y_1x_1, & \{y_1, y_2\} &= y_1y_2, & \{x_1, y_2\} &= -x_1y_2, \\ \{y_1, x_2\} &= y_1x_2, & \{x_1, x_2\} &= -x_1x_2, & \{x_2, y_2\} &= 2y_2x_2 + 2y_1x_1. \end{aligned}$$

(i) Let us classify the prime Poisson ideals of A_2 that contains x_1. Set

$$A_2/x_1A_2 \cong K[y_1][y_2;\alpha]_p[x_2;\beta]_p, \tag{II.2.5}$$

where $\alpha = y_1\partial_{y_1}$ and $\beta = y_1\partial_{y_1} + 2y_2\partial_{y_2}$. Set

$$A = K[y_1][y_2;\alpha]_p, \quad B = K[y_1][y_2;\alpha]_p[x_2;\beta]_p = A[x_2;\beta]_p.$$

Notice that, by using Proposition II.2.45.(1) the prime Poisson ideals of A that contains y_2 have the form $I + y_2A$, where I is a prime ideal of $K[y_1]$. It follows from Lemma II.2.43.(3) that $A = K[y_2][y_1;\alpha']$, where $\alpha' = -y_2\partial_{y_2}$ and by using Proposition II.2.45.(1) the prime Poisson ideals of A that contains y_1 have the form $J + y_1A$, where J is a prime ideal of $K[y_2]$. Now, since every prime Poisson ideal of B not containing y_1, y_2, x_2 is (α,β,γ)-stable, and each monomial $y_1^r y_2^s x_2^t \in B$ is a common eigenvector of α,β,γ with eigenvalue $r-2t, r+2s, -s-t$, respectively, every non-zero prime Poisson ideal of B contains one of y_1, y_2, x_2. Therefore, all prime Poisson ideals of A_2 that contains x_1 are

$$\begin{aligned} &x_1A_2, && x_1A_2 + x_2A_2,\\ &x_1A_2 + y_2A_2, && y_1A_2 + x_1A_2,\\ &JA_2 + y_1A_2 + x_1A_2 + x_2A_2, && IA_2 + x_1A_2 + y_2A_2 + x_2A_2,\\ &KA_2 + y_1A_2 + x_1A_2 + y_2A_2, && \end{aligned}$$

where I, J and K are prime ideals of $K[y_1], K[y_2]$ and $K[x_2]$, respectively.

(ii) Let us classify the prime Poisson ideals of A_2 that contains y_1. Set

$$A_2/y_1A_2 \cong K[x_1][y_2;\alpha]_p[x_2;\beta]_p,$$

where $\alpha = -x_1\partial_{x_1}$ and $\beta = -x_1\partial_{x_1} + 2y_2\partial_{y_2}$. Replacing x_1 in $K[x_1][y_2;\alpha]_p[x_2;\beta]_p$ by y_1 in the Poisson algebra given in the right hand of (II.2.5), all prime Poisson ideals of

A_2 that contains y_1 are

$$\begin{array}{ll} y_1A_2, & y_1A_2 + x_1A_2, \\ y_1A_2 + y_2A_2, & y_1A_2 + x_2A_2, \\ JA_2 + y_1A_2 + x_1A_2 + x_2A_2, & LA_2 + y_1A_2 + y_2A_2 + x_2A_2, \\ KA_2 + y_1A_2 + x_1A_2 + y_2A_2, & \end{array}$$

where L, J and K are prime ideals of $K[x_1], K[y_2]$ and $K[x_2]$, respectively.

(iii) Let us classify the prime Poisson ideals that contain neither y_1 nor x_1: If P is a prime Poisson ideal that does not contain either y_2 or x_2 then $y_1 \in P$ or $x_1 \in P$ since $\{y_2, x_2\} = 2y_2x_2+2y_1x_1$. Hence, let us assume that $y_2 \notin P$, $x_2 \notin P$ and $z = y_2x_2+y_1x_1$. Then z is a Poisson normal element of A_2 and

$$A_2[y_1^{-1}, x_1^{-1}, y_2^{-1}] = K[y_1^{\pm 1}][x_1^{\pm 1}; \alpha]_p[y_2^{\pm 1}; \beta]_p[z; \gamma]_p,$$

where

$$\begin{aligned} \alpha &= 2y_1\partial_{y_1}, \\ \beta &= y_1\partial_{y_1} - x_1\partial_{x_1}, \\ \gamma &= 2y_1\partial_{y_1} - 2x_1\partial_{x_1} + 2y_2\partial_{y_2}. \end{aligned}$$

Set $A = K[y_1^{\pm 1}][x_1^{\pm 1}; \alpha]_p[y_2^{\pm 1}; \beta]_p$ and $B = A[z; \gamma]_p$. It follows from (i) that A has no non-trivial prime Poisson ideal. Suppose that there is a non-trivial Poisson ideal I of A and let P be a prime ideal minimal over I. Then the largest $\mathscr{H}$-stable ideal $(P : \mathscr{H})$ contained in P is a prime Poisson ideal containing I by [Dix], where $\mathscr{H}$ is the set of all hamiltonians in A. Hence, A is a γ-Poisson simple. Hence, it follows from (i), (ii) and

(iii) that all prime Poisson ideals of A_2 are

$$\begin{array}{ll}
0, & zA_2, \\
x_1A_2, & x_1A_2 + x_2A_2, \\
x_1A_2 + y_2A_2, & y_1A_2 + x_1A_2, \\
JA_2 + y_1A_2 + x_1A_2 + x_2A_2, & IA_2 + x_1A_2 + y_2A_2 + x_2A_2, \\
KA_2 + y_1A_2 + x_1A_2 + y_2A_2, & \\
y_1A_2, & y_1A_2 + x_2A_2, \\
y_1A_2 + y_2A_2, & y_1A_2 + x_1A_2, \\
JA_2 + y_1A_2 + x_1A_2 + x_2A_2, & LA_2 + y_1A_2 + y_2A_2 + x_2A_2, \\
KA_2 + y_1A_2 + x_1A_2 + y_2A_2, &
\end{array}$$

where $z = y_2x_2 + y_1x_1$ and I, L, J, K are prime ideals of $K[y_1], K[x_1], K[y_2], K[x_2]$, respectively.

The next review is on the paper [Oh4]. This study is an extension of the material of [Oh3]. In addition, some interesting results on finitely generated Poisson algebras over a field of characteristic zero are considered as Proposition II.2.53 and Theorem II.2.56. Following that some techniques to classify Poisson prime ideals of finitely generated Poisson algebras are given in Example II.2.60.

Definition II.2.51. Let α be a Poisson derivation of a Poisson algebra D. A K-linear map δ of D is called an *inner α-derivation* if there exists an element $a \in D$ such that

$$\delta(b) = a\alpha(b) + \{a, b\} \quad \text{for all} \quad b \in D.$$

The results of the paper [Oh4] can be considered in the following:

1. If δ is an inner α-derivation on D and then (α, δ) is a skew Poisson derivation on D then it follows from Theorem II.2.17 that there exists a Poisson polynomial algebra $D[x; \alpha, \delta]_p$.

2. Every prime Poisson ideal of D is a Poisson prime ideal of D, but the converse is not always true, see Example II.2.59.

Lemma II.2.52. *[Oh4, Lemma 1.4] Let K be a field of characteristic zero, D be a Poisson algebra over K and Λ be a set of derivations on D. Then every prime ideal minimal over a Λ-Poisson ideal is a Λ-Poisson ideal. In particular, every prime ideal minimal over a Poisson ideal is a Poisson ideal.*

The next proposition shows that any Poisson prime ideal is a prime Poisson ideal in Noetherian algebras over a field of characteristic zero.

Proposition II.2.53. *[Oh4, Proposition 1.5] Let K be a field of characteristic zero and D be a Poisson algebra over K which is finitely generated as an algebra. Then every Λ-Poisson prime ideal of D is a prime ideal, where Λ is a set of derivations on D. In particular, every Poisson prime ideal of D is a prime ideal.*

The following lemma shows that the Poisson algebra $D[x;\alpha,\delta]_p$ can be written by using the determining element of an inner map.

Lemma II.2.54. *[Oh4, Lemma 2.1] Let D be a Poisson algebra over a field of characteristic zero and (α,δ) be a skew Poisson derivation on the Poisson algebra D. If δ is an inner map α-derivation determined by $a \in D$ then*

$$D[x;\alpha,\delta]_p = D[x+a;\alpha]_p.$$

The next corollary identifies Poisson prime ideals of $D[x;\alpha,\delta]_p$ by Poisson prime ideals of D.

Corollary II.2.55. *[Oh4, Corollary 2.2] Let D be a Poisson algebra over a field of characteristic zero and Q be an α-Poisson prime ideal of D which is δ-stable. Then $QD[x;\alpha,\delta]_p$ is a Poisson prime ideal.*

The next theorem gives us the simplicity criterion for the Poisson algebra $D[x;\alpha,\delta]_p$.

Theorem II.2.56. *[Oh4, Theorem 2.3] Let D be a Poisson algebra*[18] *over a field of characteristic zero which is a field and (α, δ) be a skew Poisson derivation on D. If δ is not an inner map α-derivation then the Poisson algebra $D[x; \alpha, \delta]_p$ is a Poisson simple.*

The following lemma describes the Poisson prime ideal of $D[x; \alpha, \delta]_p$ that does not contain x.

Lemma II.2.57. *[Oh4, Lemma 2.4] Let D be a Poisson algebra over a field of characteristic zero and P be a Poisson prime ideal of $A = D[x; \alpha, \delta]_p$. Then $P \cap D$ is a prime Poisson ideal of D. In addition, if P does not contain x then the following are equivalent:*

1. *$P \cap D$ is α-stable.*
2. *$P \cap D$ is δ-stable.*
3. *$P \cap D$ is (α, δ)-stable.*

The next theorem describes the Poisson prime ideal of $D[x; \alpha, \delta]_p$ that is not α-stable.

Theorem II.2.58. *[Oh4, Theorem 2.6] Let D be a Poisson algebra over a field of characteristic zero, P be a Poisson prime ideal of $A = D[x; \alpha, \delta]_p$, and $Q = P \cap D$ be not α-stable.*

1. *The ideal P contains an element of the form $ax + b$, where $a, b \notin Q$.*
2. *Set $S = D \backslash Q$ and let α', δ' be the extensions of α, δ to $S^{-1}D$, respectively. Then the derivation δ' is an inner α'-derivation modulo $S^{-1}Q$.*

The next example shows that the Poisson prime ideals are not always prime Poisson ideals in general.

Example II.2.59. Let K be a field of characteristic $q > 0$ and $A = K[x][y; 0, \partial_x]_p$. Notice that, the ideal $I = (x^q, y^q)$ of A is a Poisson ideal hence, A/I is a Poisson algebra. Now, the factor Poisson algebra A/I is Poisson simple, i.e. A/I has no non-trivial Poisson ideals. Hence, I is a Poisson prime ideal of A, but I is not a prime ideal of the algebra A since $x \notin I$.

The next example gives us the description for classifying Poisson prime ideals of $A = D[x; \alpha, \delta]_p$ which are in three steps: Firstly, classifying the Poisson prime ideals that contain

[18] possibly infinitely generated

x. After that, classifying Poisson prime ideals that do not contain x and are (δ, α)-stable. Finally, classifying Poisson prime ideals that neither contain x nor are (δ, α)-stable. This method will be used to classify Poisson prime ideals for some Poisson algebras in Section IV.1, but it is not in a direct way and assuming $\delta = 0$.

Example II.2.60. [Oh4, Example 2.7] Let D be a Poisson algebra and (α, δ) be a skew Poisson derivation on D. Then $A = D[x; \alpha, \delta]_p$ is a Poisson algebra with Poisson bracket

$$\{a, x\} = \alpha(a)x + \delta(a), \quad \text{where } a \in D.$$

Let us classify all Poisson prime ideals of A. Notice that, any Poisson prime ideal of A that contains x is induced by a Poisson prime ideal of D.

Now, let P be a Poisson prime ideal of A such that $x \notin P$ and $Q = P \cap D$ is (α, δ)-stable. It follows from Proposition II.2.53 that the Poisson algebra D/Q is an integral domain and the ideal P/Q of $(D/Q)[x; \bar{\alpha}, \bar{\delta}]_p$ is a Poisson prime ideal that does not contain x. In addition, $(P/Q) \cap (D/Q) = 0$, where $\bar{\alpha}$ and $\bar{\delta}$ are the derivations on D/Q induced by α and δ, respectively. Hence, let us assume that D is an integral domain and $P \cap D = 0$. By localizing at the multiplicative set $D \backslash \{0\}$ and by using Lemma II.2.43.(1), one can assume that D is a field. Therefore, it follows from Theorem II.2.56 and Lemma II.2.54 that any Poisson prime ideal P is induced from Poisson prime ideals of $D[y; \bar{\alpha}]_p$. Now, one can apply Proposition II.2.45 to classify all Poisson prime ideals of $D[y; \bar{\alpha}]_p$.

Let P be a Poisson prime ideal of A such that $x \notin P$ and $Q = P \cap D$ is not(α, δ)-stable. Set $S = D \backslash Q$. Let us find the Poisson prime ideal $S^{-1}P$ of $(S^{-1}D)[y; \alpha']_p$ that contains $S^{-1}Q$ by Theorem II.2.58.(2) and Lemma II.2.54, where α' is the extension of α to $S^{-1}D$. Notice that, $S^{-1}D$ is a local ring with the maximal ideal $S^{-1}Q$. If $y \in S^{-1}P$ then we have

$$S^{-1}P = S^{-1}Q + y(S^{-1}D)[y; \alpha']_p$$

by Proposition II.2.45.(1). Suppose that $y \notin S^{-1}P$. Since $S^{-1}P \cap S^{-1}D = S^{-1}Q$, $S^{-1}Q$ is an α'-Poisson prime ideal of $S^{-1}D$ by Proposition II.2.45.(3). Hence, α' induces a Poisson derivation $\bar{\alpha}'$ on $S^{-1}D/S^{-1}Q$, which is a field. Now, one can apply Proposition II.2.45.(4)

and (5) to the Poisson algebra $(S^{-1}D/S^{-1}Q)[y;\bar{\alpha}']_p$ to classify the Poisson prime ideals of $S^{-1}P/S^{-1}Q$.

The next review is on the paper [JoOh]. This study defines a new structure of the Poisson bracket, which is in a ring in three variables over the complex field, see Definition II.2.63. This structure is determined by a fixed polynomial and its derivations. Additionally, there are various techniques for classifying Poisson prime ideals, Poisson maximal ideals and Poisson primitive ideals. In particular, there are valuable methods to classify Poisson prime ideals as Theorem II.2.72, Corollary II.2.73, Example II.2.76 and Example II.2.77, which will be used to classify Poisson prime ideals for some Poisson algebras in Chapter IV.

The next definition gives us the Zariski topology structure on the Poisson spectrum.

Definition II.2.61. [JoOh, Definition 1.6] Let D be a Poisson algebra. The Poisson spectrum of D is the subspace of $\mathrm{Spec}(D)$ consisting of the Poisson prime ideals with the induced Zariski topology. Thus, a closed set in $\mathrm{PSpec}(D)$ has the form $\mathcal{V}(I) := \{P \in \mathrm{PSpec}(D) \mid I \subseteq P\}$ for some ideal I of D. As is observed in [Goo2], replacing I by the Poisson ideal it generates, I can be assumed to be a Poisson ideal.

Definition II.2.62. Let D be a Poisson algebra and I be a Poisson ideal of D then I is called a *residually null* if the induced Poisson bracket on D/I is zero.

The following definition is the exact bracket. This is a Poisson structure, which is defined by using a fixed polynomial and its derivations.

Definition II.2.63. [JoOh, page 1] Let $D = \mathbb{C}[x, y, z]$ be a Poisson algebra and $a \in D$. Then a *Jacobian* or *exact* bracket determined by a is given by

$$\{x, y\}_a = \frac{\partial a}{\partial z}, \quad \{y, z\}_a = \frac{\partial a}{\partial x} \quad \text{and} \quad \{z, x\}_a = \frac{\partial a}{\partial y}. \tag{II.2.6}$$

The abbreviation of partial derivatives is given in the following notation.

Notation II.2.64. Let $D = \mathbb{C}[x, y, z]$ be a Poisson algebra, $Q(D) = \mathbb{C}(x, y, z)$ be the quotient field of D. Now, let us assume that $w = x, y$ or z. The derivation $\frac{\partial}{\partial w}$ of D is denoted by ∂_w and if $a \in D$ the partial derivative $\partial_w(a)$ is denoted by a_w.

In the following notation, a Poisson bracket is identified by the determinant of the Jacobian matrix.

Notation II.2.65. Let $F = (f, g, h) \in D^3$. There is a bilinear anti-symmetric product $\{\cdot,\cdot\}^F : D \times D \to D$ such that

$$\{b, c\}^F = \begin{vmatrix} f & g & h \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \quad \text{for all } \ b, c \in D.$$

Thus, $\{b, \cdot\}^F$ is the derivation

$$(gb_z - hb_y)\partial_x + (hb_x - fb_z)\partial_y + (fb_y - gb_x)\partial_z \tag{II.2.7}$$

on D. Notice that,

$$\{b, x\}^F = gb_z - hb_y, \ \{b, y\}^F = hb_x - fb_z, \ \{b, z\}^F = fb_y - gb_x \tag{II.2.8}$$

and

$$\{y, z\}^F = f, \quad \{z, x\}^F = g, \quad \{x, y\}^F = h.$$

The bracket $\{\cdot,\cdot\}^F$ is a unique bilinear anti-symmetric product $\{\cdot,\cdot\} : D \times D \to D$ such that

$$\{y, z\} = f, \quad \{z, x\} = g, \quad \{x, y\} = h$$

and $\{b, \cdot\}$ is a $\mathbb{C}$-derivation for all $b \in D$. The same conclusion holds for $Q(D)$. Let

$$J_F(a, b, c) = \{a, \{b, c\}^F\}^F + \{b, \{c, a\}^F\}^F + \{c, \{a, b\}^F\}^F \ \text{ for all } a, b, c \in D.$$

Thus, a, b and c satisfy the Jacobi identity for $\{\cdot,\cdot\}^F$ if and only if $J_F(a, b, c) = 0$.

The following proposition gives us the relation between the Poisson bracket above and the Jacobi identity of the generators.

Proposition II.2.66. *[JoOh, Proposition 1.14] Let $F \in D^3$. Then D is a Poisson algebra under $\{\cdot,\cdot\}^F$ if and only if $J_F(x, y, z) = 0$.*

Definition II.2.67. Let $F = (f, g, h) \in D^3$ and $\{\cdot, \cdot\}^F : D \to D$ be the bilinear anti-symmetric product determined by F in Notation II.2.65. Then F is called a *Poisson triple* on D if and only if there is a Poisson bracket on D such that

$$\{y, z\} = f, \quad \{z, x\} = g \quad \text{and} \quad \{x, y\} = h.$$

The following notation gives us the definitions of new functions on D^3.

Notation II.2.68. Let $F = (f, g, h) \in D^3$. Then we define the functions grad: $D \to D^3$ and curl: $D^3 \to D^3$ such that

$$\mathrm{grad}(f) = (f_x, f_y, f_z) \in D^3 \ \text{ and } \ \mathrm{curl}\, F = (h_y - g_z, f_z - h_x, g_x - f_y) \in D^3.$$

Definition II.2.69. Let $F = (f, g, h)$ be a Poisson triple on D. If F has the form:

1. $\mathrm{grad}\, a = (a_x, a_y, a_z)$ for some $a \in D$ then F is *exact* on D.
2. $b\,\mathrm{grad}\, a = (ba_x, ba_y, ba_z)$ for some $a, b \in D$ then F is a *m-exact* on D.
3. $b\,\mathrm{grad}\, a = (ba_x, ba_y, ba_z)$ for some $a, b \in Q(D)$ such that $ba_x, ba_y, ba_z \in D$ then F is *qm-exact* on D.

The following proposition describes some properties of the Poisson triple on D^3.

Proposition II.2.70. *[JoOh, Proposition 1.17] Let $f, g, h, a, b \in D$ and $F = (f, g, h) \in D^3$.*

1. *F is a Poisson triple if and only if $F \cdot (h_y - g_z, f_z - h_x, g_x - f_y) = 0$.*
2. *$\mathrm{grad}\, a = (a_x, a_y, a_z)$ is a Poisson triple on D.*
3. *If F is a Poisson triple on D then $bF := (bf, bg, bh)$ is a Poisson triple on D.*
4. *$b\,\mathrm{grad}\, a$ is a Poisson triple.*

Let $s, t \in \mathbb{C}[x, y, z]\backslash\{0\}$ be coprime and $a = st^{-1} \in \mathbb{C}(x, y, z)$. The exact bracket (II.2.6) multiplied by $b = t^2$ is the m-exact Poisson bracket $t^2\{\cdot, \cdot\}_a$ on $\mathbb{C}(x, y, z)$ restricts to a qm-exact Poisson bracket on $\mathbb{C}[x, y, z]$. Let us fix s, t and write this bracket as $\{\cdot, \cdot\}$. Then

$$\{x, y\} = ts_z - st_z, \quad \{y, z\} = ts_x - st_x \quad \text{and} \quad \{z, x\} = ts_y - st_y. \tag{II.2.9}$$

Lemma II.2.71. *[JoOh, Lemma 3.4] Let* $(\lambda, \mu) \in \mathbb{CP}^1$, $f_{\lambda,\mu} = \lambda s - \mu t$ *be a non-zero non-unit polynomial and* u *be an irreducible factor polynomial in* D *of* $f_{\lambda,\mu}$. *The ideal* $(f_{\lambda,\mu})$ *is Poisson and* (u) *is a Poisson prime ideal of* D *under the Poisson bracket* (II.2.9).

The following theorem describes the Poisson prime ideals of D under the qm-exact bracket that is determined by the polynomial a.

Theorem II.2.72. *[JoOh, Theorem 3.8] Let* $s, t \in D \backslash \{0\}$ *be coprime and* $a = st^{-1} \in Q(D)$. *The Poisson prime ideals of* D *under the qm-exact bracket determined by* a *are*

(i) 0,

(ii) the residually null Poisson prime ideals, and

(iii) the height one prime ideals (u), *where* u *is an irreducible factor of* $f_{\lambda,\mu}$ *for some* $(\lambda, \mu) \in \mathbb{CP}^1$ *such that* $f_{\lambda,\mu}$ *is a non-zero non-unit polynomial.*

The next corollary describes the Poisson prime ideals of D under the exact bracket that is determined by the non-constant polynomial a.

Corollary II.2.73. *[JoOh, Corollary 3.9] Let* $a \in D \backslash \mathbb{C}$. *The Poisson prime ideals of* D *under* $\{\cdot, \cdot\}_a$ *are*

(i) 0,

(ii) the residually null Poisson prime ideals, and

(iii) the height one prime ideals $(\mathfrak{p})$, *where* $\mathfrak{p}$ *is an irreducible factor in* D *of* $a - \mu$ *for some* $\mu \in \mathbb{C}$.

The Poisson structures in the next examples are similar to some Poisson structures that will be appeared in Chapter IV.

Example II.2.74. Let $D = \mathbb{C}[x, y, z]$ and $a = x^2/2$ then we have the Poisson bracket

$$\{y, z\} = x \quad \text{and} \quad \{x, y\} = \{z, x\} = 0.$$

Notice that, the residually null Poisson prime ideals are the prime ideals that contain x. The prime ideals generated by the irreducible factors of $x^2 - \mu$, as μ varies, are the ideals $(x - \lambda)$, where $\lambda \in \mathbb{C}$. These are all Poisson ideals and only (x) is a residually null ideal. By using Corollary II.2.73, we have

$$\mathrm{PSpec}(D) = \big\{0, (x-\lambda), (x, f), (x, y-\beta, z-\gamma) \mid \lambda, \beta, \gamma \in \mathbb{C} \text{ and } f \in \mathrm{Irr}_m \mathbb{C}[y, z]\big\}.$$

Example II.2.75. Let $D = \mathbb{C}[x, y, z]$ and $a = \alpha x + \beta y + \gamma z$, where $\alpha, \beta, \gamma \in \mathbb{C}$ are not all zero. Then we have the Poisson bracket

$$\{x, y\} = \gamma, \quad \{y, z\} = \alpha \quad \text{and} \quad \{z, x\} = \beta.$$

Notice that, by using Corollary II.2.73, we have $\mathrm{PSpec}(D) = \big\{0,\ (a-\mu) \mid \mu \in \mathbb{C}\big\}$.

The next example gives us the description for classifying the Poisson prime ideals and Poisson maximal ideals for Poisson algebras of dimension three. This can be considered as another approach for classifying the Poisson prime ideals. The idea came from fixing a polynomial a, and defining the exact bracket in $D = \mathbb{C}[x, y, z]$. There are two cases for a, and the classifications of Poisson prime ideals for both cases are given. This is a significant approach and will be used to classify Poisson prime ideals for some Poisson algebras in Section IV.3.

Example II.2.76. [JoOh, Example 4.9] Let $D = \mathbb{C}[x, y, z]$ and $a \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ be a monomial. By the symmetry between a and a^{-1}, it is enough to consider the following two cases. Let j, k and l be non-negative integers, not all 0,

Case i: $a = s = x^j y^k z^l$,

Case ii: $s = x^j z^l, t = y^k, a = x^j y^{-k} z^l$.

In Case i, we have the Poisson bracket

$$\begin{aligned} \{x, y\} = &\quad l x^j y^k z^{l-1} = (x^{j-1} y^{k-1} z^{l-1}) l x y, \\ \{y, z\} = &\quad j x^{j-1} y^k z^l = (x^{j-1} y^{k-1} z^{l-1}) j y z, \ \text{ and} \\ \{z, x\} = &\quad k x^j y^{k-1} z^l = (x^{j-1} y^{k-1} z^{l-1}) k z x. \end{aligned} \tag{II.2.10}$$

In Case ii, the Poisson bracket for $\{x,y\}$ and $\{y,z\}$ hold but

$$\{z,x\} = -kx^j y^{k-1} z^l = -(x^{j-1}y^{k-1}z^{l-1})kzx.$$

Then the Poisson principal prime ideals are

$$\begin{aligned} (x) &\quad \text{if} \quad j \neq 0, \\ (y) &\quad \text{if} \quad k \neq 0, \\ (z) &\quad \text{if} \quad l \neq 0, \\ (x^j y^k z^l - \mu), &\quad \text{where} \quad \mu \in \mathbb{C}^\times, \ \text{(in Case i), and} \\ (\lambda x^j z^l - \mu y^k), &\quad \text{where} \quad \lambda, \mu \in \mathbb{C}^\times, \ \text{(in Case ii).} \end{aligned}$$

The height two Poisson prime ideals are

- (x,y) if $j+k>1$,
- (x,z) if $j+l>1$,
- (y,z) if $k+l>1$,
- any height two Poisson prime ideal containing (x) if $j \geq 2$,
- any height two Poisson prime ideal containing (y) if $k \geq 2$, and
- any height two Poisson prime ideal containing (z) if $l \geq 2$.

Notice that, any Poisson height two prime ideal must be a residually null ideal.

In addition to 0 and the maximal ideals containing any of the residually null Poisson prime ideals listed, this specifies PSpec(D). Notice that, the Poisson principal prime ideals are Poisson primitive ideals except $(x),(y)$ and (z), if $j \geq 2$, $k \geq 2$ and $l \geq 2$, respectively.

The next example describes a certain case of Poisson algebras of dimension three in which some constants are rational. In this example, there is a critical method to classify Poisson prime ideals, which is writing the Poisson bracket as D-multiple of others. This will be used to classify Poisson prime ideals for some Poisson algebras in Sections IV.2 and IV.3.

Example II.2.77. [JoOh, Example 4.10] Let $D = \mathbb{C}[x, y, z]$ be a Poisson algebra with Poisson bracket defined by

$$\{x, y\} = \tau xy, \quad \{y, z\} = \rho yz \quad \text{and} \quad \{z, x\} = \sigma zx, \quad \text{where } \tau, \rho, \sigma \in \mathbb{C}.$$

We consider the Poisson spectrum for this bracket in $\dim_{\mathbb{Q}}(\tau\mathbb{Q}+\rho\mathbb{Q}+\sigma\mathbb{Q}) = 1$. Let us assume that $\tau = l, \rho = j$ and $\sigma = \pm k$, where j, k and l are coprime non-negative integers, and $l > 0$. Thus

$$\mathcal{B}_1 := \{x, y\} = lxy, \quad \{y, z\} = jyz \quad \text{and} \quad \{z, x\} = \pm kzx. \tag{II.2.11}$$

Then $\mathcal{B}_2 := x^{j-1}y^{k-1}z^{l-1}\mathcal{B}_1$, where $\mathcal{B}_2$ is one of the brackets considered in Example II.2.76. The Poisson spectrum of D under $\mathcal{B}_1$ can be computed from that of $\mathcal{B}_2$.

If j, k and l are non-negative integers then any Poisson prime ideal of D under $\mathcal{B}_1$ is a Poisson prime ideal of D under $\mathcal{B}_2$, but if Q is a Poisson prime ideal of D under $\mathcal{B}_2$ then Q is a Poisson prime ideal of D under $\mathcal{B}_1$ or Q contains at least one of x, y and z. In this case, it follows from Example II.2.76, that the Poisson spectrum of D under $\mathcal{B}_1$ is

$$\begin{gathered}
0, \quad (y), \quad (x), \quad (z), \\
(x^j y^k z^l - \mu), \quad \mu \in \mathbb{C}^\times, \quad \text{if} \quad \{z, x\} = kzx \quad \text{in} \quad (\text{II.2.11}), \\
(x^j z^l - \mu y^k), \quad \mu \in \mathbb{C}^\times, \quad \text{if} \quad \{z, x\} = -kzx \quad \text{in} \quad (\text{II.2.11}), \\
(x, y), \quad (y, z), \quad (x, z),
\end{gathered}$$

and the maximal ideals that contain them.

In [JoOh, Example 1.19.(2)], $D = \mathbb{C}[x, y, z]$ is a Poisson algebra with Poisson bracket

$$\{x, y\} = 0, \quad \{y, z\} = y \quad \text{and} \quad \{z, x\} = -\alpha x,$$

where $\alpha = m/n$ is rational, with m and n coprime, and $n > 0$. The Poisson spectrum of D is unchanged by multiplication by n giving

$$\mathcal{B}_1 := \{x, y\} = 0, \quad \{y, z\} = ny \quad \text{and} \quad \{z, x\} = -mx.$$

Then this is $z^{-1}\mathcal{B}_1 := x^{1-j}y^{1-k}\mathcal{B}_2$, where $j = n$, $l = 0$ and $k = |m|$. Here $\mathcal{B}_2$ is the same as, Case ii, in Example II.2.76 if $m > 0$. It is the same as, Case i, if $m < 0$. Therefore, the Poisson spectrum of D is

$$\begin{array}{llll} 0, \quad (y), & (x), & (x,y), & \\ (x^n - \mu y^m), & \mu \in \mathbb{C}^\times, & \text{if} & m > 0, \\ (x^n y^{-m} - \mu), & \mu \in \mathbb{C}^\times, & \text{if} & m < 0, \\ (x, y, z - \nu), & \nu \in \mathbb{C}. & & \end{array}$$

The next remark describes a special case of Poisson algebras of dimension three in which some constants are irrational.

Remark II.2.78. [JoOh, Remark 4.11] Let $D = \mathbb{C}[x, y, z]$ and $\rho, \sigma, \tau \in \mathbb{C}$ be such that $\dim_{\mathbb{Q}}(\rho\mathbb{Q} + \sigma\mathbb{Q} + \tau\mathbb{Q}) > 1$ the quadratic Poisson bracket in Example II.2.77. From the results in [Goo2, 9.6.(b)] the Poisson spectrum, in this case, can be obtained, including the Poisson simplicity of $B := \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$.

1. If ρ, σ and τ are all non-zero the Poisson prime ideals of D are 0, (x), (y), (z), the height two prime ideals (x, y), (y, z), (z, x), and the maximal ideals containing any one of the Poisson height two primes.

2. If $\tau = 0$ and $\dim_{\mathbb{Q}}(\rho\mathbb{Q} + \sigma\mathbb{Q}) > 1$ then the Poisson prime ideals of D are $0, (x), (y)$ and all prime ideals containing (z) or (x, y). In particular, taking $\tau = 0$, $\rho = 1$ and $\sigma = -\alpha \in \mathbb{C} \backslash \mathbb{Q}$, we get the multiple, by z, of the Poisson bracket from [JoOh, Example 1.19.(2)], where
$$\{x, y\} = 0, \quad \{y, z\} = y \quad \text{and} \quad \{z, x\} = -\alpha x.$$
The Poisson prime ideals of D are 0, $(x), (y)$ and all prime ideals containing (x, y). Following [Goo2, Example 6.4], all except (x, y) are Poisson primitive ideals.

II.2.6 Review on Poisson algebras from algebraic geometry

The following is a review on the paper [Pol]. The interest in this study for us is that the Poisson algebras of dimension three. Some properties of this algebras are given. In addition, this can be thought of as one of the Poisson structure applications and can give us a better understanding of Poisson structures.

Throughout this review, we suppose that X is a scheme of finite type over $\mathbb{C}$.

Definition II.2.79. Let X be a Poisson scheme $(X, \mathcal{O}_X, \{\cdot,\cdot\})$. Then there is a *canonical Poisson subscheme* $X_0 \subset X$ such that the induced Poisson structure on X_0 is zero, and X_0 is maximal with this property, i.e. the corresponding *Poisson ideal sheaf* is

$$\mathcal{O}_X\{\mathcal{O}_X, \mathcal{O}_X\} \subset \mathcal{O}_X.$$

Remark II.2.80. Let X be a scheme. For any Poisson structure on X there exists $\mathcal{O}_X$-linear homomorphism $H : \Omega_X \to T_X = \mathrm{Der}(\mathcal{O}_X, \mathcal{O}_X)$ such that

$$H(df)(g) = \{f, g\}.$$

In addition, if X is smooth, we denote by G the section of $\bigwedge^2 T_X$ such that

$$i(\omega)G = H(\omega) \quad \text{for all } \omega \in \Omega^1_X, \tag{II.2.12}$$

where $i(\omega)$ is the operator of contraction of ω.

Definition II.2.81. Let X be irreducible then a Poisson structure H on X is called *non-degenerate* if H has maximal rank at the general point.

Definition II.2.82. Let X be smooth with even dimension. Then the *divisor of degeneration* $Z \subset X$ of a non-degenerate Poisson structure on X can be defined as the zero locus of the Pfaffian of H, which is a section of $\det T_X \simeq \omega_X^{-1}$.

Remark II.2.83. A Poisson structure is constant along any hamiltonian flow ϕ_t, so the condition of degeneracy is preserved under ϕ_t. It follows that any hamiltonian flow moves any irreducible component of Z into itself. If f is a local equation of an irreducible component of X then for all hamiltonian vector fields H_g, the function

$$H_g(f) = \{g, f\}$$

is zero along this component.

Definition II.2.84. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. A *Poisson connection* on $\mathcal{F}$ is a $\mathbb{C}$-linear bracket $\{\cdot,\cdot\} : \mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$ which is a derivation, and satisfies the Leibniz's rule

$$\{f, gs\} = \{f, g\}s + g \cdot \{f, s\}, \quad \text{where } \ f, g \in \mathcal{O}_X \tag{II.2.13}$$

and s is a local section of $\mathcal{F}$. Equivalently, a Poisson connection is given by a homomorphism $v : \mathcal{F} \to \mathcal{H}om(\Omega_X, \mathcal{F}) = \mathrm{Der}(\mathcal{O}_X, \mathcal{F})$ which satisfies the relation

$$v(fs) = -H(df) \otimes s + f \cdot v(s), \quad \text{where } \ f \in \mathcal{O}_X. \tag{II.2.14}$$

Namely, $v(s) \in \mathrm{Der}(\mathcal{O}_X, \mathcal{F})$ is defined by the relation

$$v(s)(f) = \{f, s\}. \tag{II.2.15}$$

Definition II.2.85. Let X be a smooth Poisson variety and $\mathcal{D}$ be the sheaf of differential operators on X. A $\mathcal{D}$-Poisson module is an $\mathcal{O}_X$-module M with a Lie action of $\mathcal{O}_X$[19] given by

$$\{\cdot,\cdot\} : \mathcal{O}_X \times M \to M$$

which is a differential operator and satisfies the relation

$$\{f, gm\} = \{f, g\}m + g\{f, m\}, \ \text{ where } f, g \in \mathcal{O}_X \ \text{ and } \ m \in M.$$

In other words, this structure corresponds to some map $v : M \to \mathcal{D} \otimes_{\mathcal{O}_X} M$, where the

[19] where the Lie bracket on $\mathcal{O}_X$ is the Poisson bracket

$\mathcal{O}_X$-module left structure on $\mathcal{D}$ is

$$v(fm) = -H(df) \otimes m + fv(m), \quad \text{where } f \in \mathcal{O}_X \text{ and } m \in M.$$

Definition II.2.86. Let X be a Poisson scheme. A Poisson ideal sheaf $J \subset \mathcal{O}_X$ is called *degenerate* if

$$\{x, y\}z + \{y, z\}x + \{z, x\}y \in J^3 \quad \text{for all } x, y, z \in J.$$

Definition II.2.87. A smooth projective variety X with ample anti-canonical class is called *Fano variety.*

Let X be a smooth variety of odd dimension $n = 2k + 1$. A Poisson structure on X is non-degenerate if the corresponding morphism $H : \Omega_X \to T_X$ has rank $2k$ at the general point. In other words, if $G \in \bigwedge^2 T_X$ is the structural tensor of the Poisson structure, then the product $g = \underbrace{G \wedge G \wedge \cdots \wedge G}_{k} \in \bigwedge^{2k} T_X \simeq \Omega^1_X \otimes \omega^{-1}_X$ is non-zero, hence g induces an embedding $i : \omega_X \to \Omega^1_X$. At the general point the image $\mathrm{im}(i) \subset \Omega^1_X$ coincides with the annihilator of the Lie subsheaf $\mathrm{im}(H) \subset T_X$, hence, it defines a corank-1 foliation on X, which means that for any local section $\nu \in \omega_X$ the 1-form $\omega = i(v)$ satisfies the Pfaff equation

$$\omega \wedge d\omega = 0.$$

Theorem II.2.88. *[Pol, Theorem 9.1] Let $i : L \to \Omega^1_X$ be an embedding of a line bundle defining a corank-1 foliation on a smooth variety X. Let $c_1(L) \in H^2(X, C)$ be a first Chern class of L. Assume that either $c_1(L)^2 \neq 0$ or $c_1(n) \neq 0$ and $H^1(X, L) = 0$. Then the vanishing locus of i has a component of codimension ≤ 2.*

Corollary II.2.89. *[Pol, Corollary 9.2] The rank of a non-degenerate Poisson structure on a Fano variety of odd dimensions drops along the subset of codimension ≤ 2.*

Let X be a smooth variety of dimension three. A [20]Poisson structure on X is the same as an embedding $i : \omega_X \to \Omega^1_X$ defining a corank-1 foliation on X. Notice that, it follows from

[20]non-zero

Theorem II.2.88 and Corollary II.2.89 that if X is a Fano 3-fold then the vanishing locus Z of i has a component of dimension ≥ 1.

Theorem II.2.90. *[Pol, Theorem 13.1] Let $C \subset X$ be a smooth curve which is an irreducible component of the vanishing locus of a Poisson structure*[21] *on smooth variety X of dimension three. Then the conormal Lie sheaf of C is abelian, i.e. $\{J_C, J_C\} \subset J_C^2$, where $J_C \subset \mathcal{O}_X$ is an ideal sheaf of C.*

Let X and Y be smooth of dimensions three and one, respectively. Let $f : X \to Y$ be a morphism, and F_i be the multiple fibers of f, and m_i be their multiplicities then there is a pull-back morphism on 1-forms

$$i_f : f^*\omega_Y\Big(\sum_i (m_i - 1)F_i\Big) \to \Omega_X^1,$$

which defines generically an integrable subbundle. Let D be a divisor in the linear system

$$\Big| f^*\omega_Y\Big(\sum_i (m_i - 1)F_i\Big) \otimes \omega_X^{-1} \Big|$$

then there is a Poisson structure that is defined by the rule

$$i_{f,D} : \omega_X \simeq f^*\omega_Y(-D) \hookrightarrow f^*\omega_Y \to \Omega_X^1.$$

Lemma II.2.91. *[Pol, Lemma 13.3] Let $i : \omega_X \to \Omega_X^1$ be a Poisson structure which is defined by*

$$i(\eta) \wedge df \wedge dg = \{f, g\}\eta, \quad \text{where } \eta \in \omega_X \text{ and } f, g \in \mathcal{O}_X.$$

Then a smooth divisor $D \in X$ is Poisson with respect to i if and only if the composition

$$\omega_X|_D \xrightarrow{i|_D} (\Omega_X^1)|_D \to \Omega_D^1 \text{ is zero.}$$

Theorem II.2.92. *[Pol, Theorem 13.5] Let $f : X \to Y$ be a morphism and $i : \omega_X \to \Omega_X^1$ be a Poisson structure on X, where X has dimension three and Y has dimension one, such that*

[21] equipped with the reduced scheme structure

a general fiber of f is a Poisson divisor with respect to i. Then $i = i_{f,D}$ for some divisor

$$D \in H^0\Big(f^*\omega_Y\Big(\sum_i (m_i - 1)D_i\Big) \otimes \omega_X^{-1}\Big).$$

§II.3 Poisson enveloping algebras: Review of results

The aim of this section is to review some key definitions and properties of Poisson enveloping algebras to be used throughout the thesis. In particular, the construction of Poisson action on modules, and simplicity criterion for Poisson enveloping algebras. Additionally, our motivations for Poisson enveloping algebras are demonstrated by evaluating a paper on a related subject, see Subsection II.3.3.

II.3.1 The Generalized Weyl Poisson algebras

The following definition is the generalized Weyl Poisson algebra that was introduced by V. V. Bavula, followed by some of its Poisson simplicity criterion. The main source for these is [Bav3].

Definition II.3.1. [Bav3, page 106] Let D be a Poisson algebra, $\partial = (\partial_1, \dots, \partial_n) \in \mathrm{PDer}_K(D)^n$ be an n-tuple of commuting derivations of the Poisson algebra D, $a = (a_1, \dots, a_n)$, where $a_i \in \mathrm{PZ}(D)$ such that $\partial_i(a_j) = 0$ for all $i \neq j$. The generalized Weyl algebra

$$A = D\big[X, Y; (\mathrm{id}_D, \dots, \mathrm{id}_D), a\big] = D[X_1, \dots, X_n, Y_1, \dots, Y_n]/(X_1Y_1 - a_1, \dots, X_nY_n - a_n)$$

admits a Poisson structure which is an extension of the Poisson structure on D and is given by the rule: For all $i, j = 1, \dots, n$ and $d \in D$,

$$\{Y_i, d\} = \partial_i(d)Y_i, \ \ \{X_i, d\} = -\partial_i(d)X_i \ \text{ and } \ \{Y_i, X_i\} = \partial_i(a_i), \tag{II.3.1}$$

$$\{X_i, X_j\} = \{X_i, Y_j\} = \{Y_i, Y_j\} = 0 \ \text{ for all } \ i \neq j. \tag{II.3.2}$$

The Poisson algebra is denoted by $A = D[X, Y; a, \partial\}$ and is called the *generalized Weyl Poisson algebra* of rank n[22], where $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$.

The next lemma gives us the existence of generalized Weyl Poisson algebras of rank n.

Lemma II.3.2. *[Bav3, Lemma 2.1] Let $A = D[X, Y; a, \partial\}$ be a generalized Weyl Poisson algebra of rank n. Let $\mathcal{A} = D\big[X, Y; \partial, \partial(a)\big]$, where $\partial(a) = \big(\partial_1(a_1), \ldots, \partial_n(a_n)\big)$. Then $X_1Y_1 - a_1, \ldots, X_nY_n - a_n \in \mathrm{PZ}(\mathcal{A})$ and the generalized Weyl Poisson algebra $A = D[X, Y; a, \partial\}$ is a factor algebra of the Poisson algebra $\mathcal{A}$,*

$$A \cong \mathcal{A}/\big(X_1Y_1 - a_1, \ldots, X_nY_n - a_n\big).$$

Definition II.3.3. [Bav3, page 107] Let $A = D[X, Y; a, \partial\}$ be a generalized Weyl Poisson algebra of rank n.

1. The set $D^\partial := \big\{d \in D \,|\, \partial_1(d) = 0, \ldots, \partial_n(d) = 0\big\}$ is called the ∂-*constants* ring of D.

2. $D_\alpha = \big\{\lambda \in D^\partial \,|\, \mathrm{pad}_\lambda := \{\lambda, \cdot\} = \lambda \sum_{i=1}^n \alpha_i\partial_i,\ \lambda\alpha_i\partial_i(a_i) = 0 \text{ for all } i = 1, \ldots, n\big\}$.

The following theorem describes the simplicity criterion for generalized Weyl Poisson algebras.

Theorem II.3.4. *[Bav3, Theorem 1.1] Let D be a K-algebra and $A = D[X, Y; a, \partial\}$ be a generalized Weyl Poisson algebra of rank n. Then the Poisson algebra A is a simple Poisson algebra if and only if*

1. *the Poisson algebra D has no proper ∂-invariant Poisson ideals,*

2. *$Da_i + D\partial_i(a_i) = D$ for all $i = 1, \ldots, n$, and*

3. *the algebra $\mathrm{PZ}(A)$ is a field, i.e. $\mathrm{char}(K) = 0$, $\mathrm{PZ}(D)^\partial$ is a field and $D_\alpha = 0$ for all $\alpha \in \mathbb{Z}^n \backslash \{0\}$.*

The following proposition describes the criterion for the Poisson centre to be a field.

[22]in short GWPA

Proposition II.3.5. *[Bav3, Proposition 1.2] Let $A = D[X, Y; a, \partial\}$ be a generalized Weyl Poisson algebra of rank n. Then $\mathrm{PZ}(A)$ is a field if and only if $\mathrm{char}(K) = 0$, $\mathrm{PZ}(D)^{\partial}$ is a field and $D_\alpha = 0$ for all $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n \backslash \{0\}$.*

The construction of $\mathbb{Z}^n$-graded Poisson on generalized Weyl Poisson algebras is given in the following remark.

Remark II.3.6. [Bav3, page 110] Let $A = D[X, Y; a, \partial\}$ be a generalized Weyl Poisson algebra of rank n then

$$A := D[X, Y; a, \partial\} = \bigoplus_{\alpha \in \mathbb{Z}^n} A_\alpha \tag{II.3.3}$$

is a $\mathbb{Z}^n$-graded Poisson algebra, where $A_\alpha = Dv_\alpha, v_\alpha = \prod_{i=1}^n v_{\alpha_i}(i)$ and

$$v_j(i) = \begin{cases} X_i^j & \text{if } j > 0, \\ 1 & \text{if } j = 0, \\ Y_i^{|j|} & \text{if } j < 0. \end{cases}$$

Hence, $A_\alpha A_\beta \subseteq A_{\alpha+\beta}$ and $\{A_\alpha, A_\beta\} \subseteq A_{\alpha+\beta}$, where $\alpha, \beta \in \mathbb{Z}^n$.

The following example is a classical example of generalized Weyl Poisson algebras.

Example II.3.7. [Bav3, page 111] The *classical Poisson polynomial algebra* $P_{2n} = K[X_1, \dots, X_n, Y_1, \dots, Y_n]$ with Poisson bracket

$$\{Y_i, X_j\} = \delta_{ij} \text{ and } \{X_i, X_j\} = \{X_i, Y_j\} = \{Y_i, Y_j\} = 0 \quad \text{for all } i \neq j$$

is a GWPA

$$P_{2n} = K[H_1, \dots, H_n][X, Y; a, \partial\}, \tag{II.3.4}$$

where $K[H_1, \dots, H_n]$ is a Poisson polynomial algebra with trivial Poisson bracket, $a = (H_1, \dots, H_n)$, $\partial = (\partial_1, \dots, \partial_n)$ and $\partial_i = \frac{\partial}{\partial_{H_i}}$, via the isomorphism of Poisson algebras

$$P_{2n} \to K[H_1, \dots, H_n][X, Y; a, \partial\}, \quad X_i \mapsto X_i, \ Y_i \mapsto Y_i.$$

II.3.2 Poisson enveloping algebras

The following definition is the Poisson enveloping algebra that is recalled from [Oh2].

Definition II.3.8. Let D be a Poisson algebra over a field K, $\mathcal{U}$ be a K-algebra with an algebra homomorphism $\alpha : D \to \mathcal{U}$ and Lie homomorphism $\beta : D \to \mathcal{U}_L$[23] such that

$$\alpha(\{a,b\}) = \beta(a)\alpha(b) - \alpha(b)\beta(a) \quad \text{and} \quad \beta(ab) = \alpha(a)\beta(b) + \alpha(b)\beta(a)$$

for all $a, b \in D$. The triple $(\mathcal{U}(D), \alpha, \beta)$ is called a *Poisson enveloping algebra* of D, if the following holds: If V is K-algebra, $\gamma : D \to V$ is an algebra homomorphism, and $\delta : D \to V_L$[24] is Lie homomorphism such that

$$\gamma(\{a,b\}) = \delta(a)\gamma(b) - \gamma(b)\delta(a) \quad \text{and} \quad \delta(ab) = \gamma(a)\delta(b) + \gamma(b)\delta(a)$$

for all $a, b \in D$. There exists a unique algebra homomorphism $h : \mathcal{U} \to V$ such that $h\alpha = \gamma$ and $h\beta = \delta$, see diagram II.3.1.

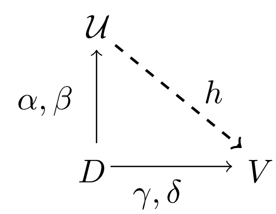


Diagram II.3.1: The unique algebra homomorphism $h : \mathcal{U} \to V$

The following theorem gives us the existence and uniqueness of Poisson enveloping algebras.

Theorem II.3.9. *[Oh2, Theorem 5] Let D be a Poisson algebra.*

1. *There exists a Poisson enveloping algebra $(\mathcal{U}(D), \alpha, \beta)$ of D.*
2. *If $(\mathcal{U}_1(D), \alpha_1, \beta_1)$ and $(\mathcal{U}_2(D), \alpha_2, \beta_2)$ are Poisson enveloping algebras of D then there exists a unique algebra isomorphism $h : \mathcal{U}_1(D) \to \mathcal{U}_2(D)$ such that $h\alpha_1 = \alpha_2$ and $h\beta_1 = \beta_2$.*

[23] $\mathcal{U}_L$ is a Lie algebra $\mathcal{U}$ with Lie bracket $[a,b] = ab - ba$
[24] V_L is a Lie algebra V with Lie bracket $[a,b] = ab - ba$

The following corollary describes the relation between Poisson modules and the modules over Poisson enveloping algebra.

Corollary II.3.10. *[Oh2, Corollary 6] Let D be a Poisson algebra over a field K and $(\mathcal{U}(D), \alpha, \beta)$ be a Poisson enveloping algebra of D. A vector space M over a field K is a Poisson D-module if and only if M is a left $\mathcal{U}(D)$-module.*

The following corollary gives us a simplicity criterion for the annihilator of Poisson modules.

Corollary II.3.11. *[Oh2, Corollary 8] Let D be a Poisson algebra. Every annihilator of a simple Poisson D-module is a symplectic ideal of D.*

The following proposition gives us a criterion for Poisson enveloping algebra to be a Noetherian ring.

Proposition II.3.12. *[Oh2, Proposition 9] Let D be a Poisson algebra over a field K and $(\mathcal{U}(D), \alpha, \beta)$ be a Poisson enveloping algebra of D. If D is finitely generated as a K-algebra then $\mathcal{U}(D)$ is a left and right Noetherian ring.*

II.3.3 Review on Poisson enveloping algebras

The following review is on the paper [Bav4]. In this study, another definition of Poisson modules is provided in Remark II.3.22. Then the relations between the universal enveloping algebra, the Poisson enveloping algebras, and the algebra of Poisson differential operators are discussed in Theorem II.3.25. Following that, the Gelfand-Kirillov dimensions of $\mathcal{U}(D)$ and $\mathrm{gr}\big(\mathcal{U}(D)\big)$ are calculated in Theorem II.3.24.

Definition II.3.13. A localization of an affine commutative algebra is called an *algebra of essentially finite type*.

Definition II.3.14. Let D be an associative algebra. Then its *dual*[25] algebra D^{op} coincides with D as a vector space, but the multiplication is given by $a * b := ba$. Similarly, given a

[25] *associative*

Poisson algebra $(D, \{\cdot,\cdot\})$, its dual associative algebra D^{op} is a Poisson algebra $(D^{op}, \{\cdot,\cdot\}^{op})$, which is called the *dual Poisson algebra* of D, where

$$\{a,b\}^{op} := -\{a,b\} \quad \text{for all} \quad a,b \in D^{op}.$$

Remark II.3.15. Let D be an associative algebra and its dual[26] D^{op}. Then every left D-module is a right D^{op}-module, and vice versa.

Definition II.3.16. Let D be a commutative algebra over a field K. The ring of [27] *differential operators* $\mathcal{D}(D)$ on D is $\mathcal{D}(D) = \bigcup_{i=0}^{\infty} \mathcal{D}_i(D)$, where $\mathcal{D}_0(D) = \mathrm{End}_K(D) \simeq D$, $(x \mapsto ax) \leftrightarrow a$,

$$\mathcal{D}_i(D) = \big\{u \in \mathrm{End}_K(D) \mid [a,u] := au - ua \in \mathcal{D}_{i-1}(D) \text{ for all } a \in D\big\}.$$

The set of D-modules $\big\{\mathcal{D}_i(D)\big\}$ is the *order filtration* for the algebra $\mathcal{D}(D)$:

$$\mathcal{D}_0(D) \subseteq \mathcal{D}_1(D) \subseteq \cdots \subseteq \mathcal{D}_i(D) \subseteq \cdots \quad \text{and} \quad \mathcal{D}_i(D)\mathcal{D}_j(D) \subseteq \mathcal{D}_{i+j}(D) \quad \text{for all} \quad i,j \geq 0.$$

Definition II.3.17. Let D be a Poisson algebra and $\{x_i\}_{i\in I}$ be a set of generators of D. The Poisson structure on an associative algebra D is uniquely determined by the Poisson structure constants $c_{ij} := \{x_i, x_j\}$, where $i,j \in I$. Let $n = \mathrm{card}(I)$ be the cardinality of the set I[28]. The $n \times n$ matrix

$$C_D := (c_{ij}) \tag{II.3.5}$$

is called the *Poisson structure constants matrix* of D and the ideal $\mathfrak{c}_D$ of D, which is generated by all the structure constants c_{ij} is called the *Poisson structure constants ideal* of D.

Remark II.3.18. [Bav4, pages 6, 7] Let D be a Poisson algebra and the set of inner derivations $\mathcal{H}_D := \mathrm{PIDer}_K(D)$. Then

1. $\mathrm{PIDer}_K(D)$ is an ideal of the Lie algebra $\mathrm{PDer}_K(D)$[29].

[26] associative
[27] K-linear
[28] the case $n = \infty$ is possible
[29] since $[\delta, \mathrm{pad}_a] = \mathrm{pad}_{\delta(a)}$ for all $\delta \in \mathrm{PDer}_K(D)$ and $a \in D$

2. the Poisson algebra D is a Lie algebra with respect to the bracket $\{\cdot,\cdot\}$. The map

$$D \to \mathrm{PIDer}_K(D), \quad a \mapsto \mathrm{pad}_a$$

is an epimorphism of Lie algebras with kernel $\mathrm{PZ}(D)$.

3. the Poisson centre $\mathrm{PZ}(D)$ is invariant under the action of $\mathrm{PDer}_K(D)$. Let $z \in \mathrm{PZ}(D), d \in D$ and $\partial \in \mathrm{PDer}_K(D)$ then applying the derivation ∂ to the equality $\{z,d\} = 0$ we obtain the equality $\{\partial(z), d\} = 0$, i.e. $\partial(z) \in \mathrm{PZ}(D)$.

Definition II.3.19. [Bav4, page 9] Let D be a Poisson algebra. The subalgebra $P\mathcal{D}(D)$ of the K-endomorphism algebra $\mathrm{End}_K(D)$, which is generated by D and $\mathcal{H}_D$ is called the *algebra of Poisson differential operators* of D.

Let us assume that K is an arbitrary field. A simplicity criterion for the algebra $P\mathcal{D}(D)$ of Poisson differential operators is given in the following theorem.

Theorem II.3.20. *[Bav4, Theorem 1.1] Let D be a Poisson algebra over K. Then the following are equivalent:*

1. *The differential operator algebra $P\mathcal{D}(D)$ is a simple algebra.*

2. *The Poisson algebra D is a simple Poisson algebra.*

A simplicity criterion for the Poisson enveloping algebra $\mathcal{U}(D)$, and the relation between $\mathcal{U}(D)$ and $P\mathcal{D}(D)$ are described in the following theorem.

Theorem II.3.21. *[Bav4, Theorem 1.2] Let D be a Poisson algebra over K. Then the following are equivalent:*

1. *The Poisson enveloping algebra $\mathcal{U}(D)$ is a simple algebra.*

2. *The differential operator algebra $P\mathcal{D}(D)$ is a simple algebra and $\mathcal{U}(D) \simeq P\mathcal{D}(D)$.*

3. *The Poisson algebra D is a simple Poisson algebra, and D is a faithful left $\mathcal{U}(D)$-module.*

If one of the above conditions holds then $\mathcal{U}(D) \simeq P\mathcal{D}(D)$.

The next remark gives us a description for Poisson action on Poisson modules.

Remark II.3.22. [Bav4, pages 7, 8] Let the commutative associative algebra D be a Poisson algebra, and M be a left D-module $\big(D \times M \to M,\ (a,m) \mapsto am\big)$. The left D-module M over D is called a *Poisson left D-module*[30] if there is a bilinear map

$$D \times M \to M, \quad (a,m) \mapsto \delta_a m$$

which is called a *Poisson action* of D on M such that for all $a,b \in D$ and $m \in M$,

(PM1) $\delta_{\{a,b\}} = [\delta_a, \delta_b]$,

(PM2) $[\delta_a, b] = \{a,b\}$, and

(PM3) $\delta_{ab} = a\delta_b + b\delta_a$.

Every left Poisson D-module M determines the homomorphism of associative algebras,

$$D \to \mathrm{End}_K(M),\ \ a \mapsto a_M : M \to M,\ \ m \mapsto am \tag{II.3.6}$$

and the homomorphism of Lie algebras,

$$D \to \mathrm{End}_K(M),\ \ a \mapsto \delta_a : M \to M,\ \ m \mapsto \delta_a m \tag{II.3.7}$$

such that

$$[\delta_a, b_M] = \{a,b\}_M \quad \text{for all } a,b \in D, \tag{II.3.8}$$

$$\delta_{ab} = a_M\delta_b + b_M\delta_a \quad \text{for all } a,b \in D, \tag{II.3.9}$$

and vice versa.

Semidirect products of algebras are described in the following remark.

[30]or *left module over the Poisson algebra*

Remark II.3.23. [Bav4, page 9] Let D be a K-algebra, $\mathcal{G}$ be a Lie algebra, $U(\mathcal{G})$ be the enveloping algebra of $\mathcal{G}$, and

$$\delta : \mathcal{G} \to \mathrm{Der}_K(D), \quad a \mapsto \delta_a$$

be a Lie algebra homomorphism[31]. Let $D \rtimes_\delta U(\mathcal{G})$ be the semidirect product of D and $U(\mathcal{G})$. It is an associative algebra that is generated by the algebras D and $U(\mathcal{G})$ subject to the defined relation

$$gd = dg + \delta_g(d) \ \text{ for all } \ d \in D \ \text{ and } \ g \in \mathcal{G}.$$

Let $\{x_i\}_{i\in I}$ be a K-basis of $\mathcal{G}$. Then

$$D \rtimes_\delta U(\mathcal{G}) = \bigoplus_{\alpha \in \mathbb{N}^{(I)}} Dx^\alpha = \bigoplus_{\alpha \in \mathbb{N}^{(I)}} x^\alpha D$$

is a free left and right D-module, where $\mathbb{N}^{(I)}$ is a direct sum of I copies of the set $\mathbb{N}$, $x^\alpha = \prod_{i\in I} x_i^{\alpha_i}$.

Let us assume that D is a domain of essentially finite type over a perfect field K and S be a multiplicative subset of P_n. The next theorem shows that the Gelfand-Kirillov dimensions of $\mathcal{U}(D)$ and $\mathrm{gr}\big(\mathcal{U}(D)\big)$ are finite.

Theorem II.3.24. *[Bav4, Theorem 1.4] Let $D = S^{-1}\big(P_n/I\big)$ a Poisson algebra over a field K, where $I = (f_1, \ldots, f_m)$ is a prime ideal of P_n and $r = r\Big(\frac{\partial f_i}{\partial x_j}\Big)$ is the rank of the Jacobian matrix $\Big(\frac{\partial f_i}{\partial x_j}\Big)$ over the field of fractions of the domain P_n/I. Then the algebra $\mathcal{U}(D)$ is a Noetherian algebra with*

$$\mathrm{GK}\,\mathcal{U}(D) = \mathrm{GK}\,\mathrm{gr}\big(\mathcal{U}(D)\big) = 2\mathrm{GK}(D) = 2(n-r).$$

The sets of generators and defining relations for the Poisson enveloping algebra are provided in the following theorem.

[31] $\delta_{[a,b]} = [\delta_a, \delta_b]$ for all $a, b \in \mathcal{G}$

Theorem II.3.25. *[Bav4, Theorem 2.2] Let D be a Poisson algebra, $U(D)$ be its universal enveloping algebra*[32] *and $\mathcal{U}(D)$ be its Poisson enveloping algebra. Then*

1. *$\mathcal{U}(D) \simeq D \rtimes_{\text{pad}} U(D)/\mathcal{I}(D)$, where $\mathcal{I}(D) = (\delta_{ab} - a\delta_b - b\delta_a)_{a,b\in D}$ is the ideal of the algebra $D \rtimes_{\text{pad}} U(D)$ generated by the set $\{\delta_{ab} - a\delta_b - b\delta_a \,|\, a, b \in D\}$.*

2. *if $D = S^{-1}K[x_i]_{i\in\Lambda}/(f_s)_{s\in\Gamma}$, where S is a multiplicative subset of the polynomial algebra $K[x_i]_{i\in\Lambda}$, and Λ, Γ are index sets. Then the algebra $\mathcal{U}(D)$ is generated by the algebra D and the elements $\{\delta_i := \delta_{x_i} \,|\, i \in \Lambda\}$ as determined by the defining relations*

   (a) $[\delta_i, \delta_j] = \sum_{k\in\Lambda} \frac{\partial\{x_i,x_j\}}{\partial_k}\delta_k$,

   (b) $[\delta_i, x_j] = \{x_i, x_j\}$, *and*

   (c) $\sum_{i\in\Lambda} \frac{\partial f_s}{\partial x_i}\delta_i = 0$,

   where $i, j \in \Lambda$ such that $i \neq j$ and $s \in \Gamma$. So, the algebra $\mathcal{U}(D)$ is generated by D and the set $\delta_D = \{\delta_a \,|\, a \in D\}$ as determined by the defining relations

   (a) $[\delta_a, \delta_b] = \delta_{\{a,b\}}$,

   (b) $[\delta_a, b] = \{a, b\}$,

   (c) $\delta_{ab} = a\delta_b + b\delta_a$,

   (d) $\delta_{\lambda a+\mu b} = \lambda\delta_a + \mu\delta_b$ *and* $\delta_1 = 0$,

   where $a, b \in D$ and $\lambda, \mu \in K$.

3. *the map $\pi_D : \mathcal{U}(D) \to \mathcal{D}(D), a \mapsto a, \delta_b \mapsto \text{pad}_b = \{b, \cdot\}$ is an algebra homomorphism, where $a, b \in D$ and its image is the algebra $P\mathcal{D}(D)$.*

The following corollary shows that a homomorphism between Poisson enveloping algebras can be defined using the homomorphism between their Poisson algebras.

Corollary II.3.26. *[Bav4, Corollary 2.3] Let D_1 and D_2 be Poisson algebras. Then every homomorphism of Poisson algebras $f : D_1 \to D_2$ can be extended to a homomorphism of their Poisson enveloping algebras $f : \mathcal{U}(D_1) \to \mathcal{U}(D_2)$ that is defined by $f(\delta_a) = \delta_{f(a)}$.*

[32] as a Lie algebra

The following corollary gives us the required condition for Poisson enveloping algebra to be commutative.

Corollary II.3.27. *[Bav4, Corollary 2.5] Let D be a Poisson algebra. Then $\mathcal{U}(D)$ is a commutative algebra if and only if D has a trivial Poisson structure.*

The following proposition describes that any endomorphism/automorphism of Poisson enveloping algebras can be obtained from an endomorphism/automorphism of their Poisson algebras.

Proposition II.3.28. *[Bav4, Proposition 2.9] Let D be a Poisson algebra. Then*

1. *the map $\mathrm{End}_{\mathrm{Pois}}(D) \to \mathrm{End}_K(\mathcal{U}(D))$, $\sigma \mapsto \sigma : a \mapsto \sigma(a)$, $\delta_a \mapsto \delta_{\sigma(a)},$ where $a \in D$, is a monoid monomorphism.*

2. *the map $\mathrm{Aut}_{\mathrm{Pois}}(D) \to \mathrm{Aut}_K(\mathcal{U}(D))$, $\sigma \mapsto \sigma : a \mapsto \sigma(a)$, $\delta_a \mapsto \delta_{\sigma(a)},$ where $a \in D$, is a group monomorphism.*

The following example gives us a construction of left Poisson modules.

Example II.3.29. Every Poisson algebra D is a left Poisson D-module since for all $a \in D$, $a_D : D \to D,\ b \mapsto ab$ and $\delta_a = \{a, \cdot\} : D \to D,\ b \mapsto \{a, b\}$.

§ III Classes of Poisson algebras of dimension two

In this chapter, two new classes of Poisson algebras are introduced and their Poisson prime ideals, minimal Poisson ideals and maximal Poisson ideals are classified. One of them is the Poisson algebra $\mathcal{P} = K[t,x]$, see Section III.1, and the other is the Poisson algebra $\mathcal{P}_2(f) = K[x,y]$, see Section III.2.

Notations. We assume that K is an algebraically closed field of characteristic zero, $K[t]$ is the Poisson polynomial algebra with trivial Poisson bracket, and $K(t)$ is the field of rational functions in the variable t.

§III.1 The Poisson algebra $\mathcal{P} = K[t,x]$

The aim of this section is to classify all Poisson prime ideals, minimal Poisson ideals and maximal Poisson ideals of the Poisson algebra $\mathcal{P}$, which are indicated in Theorem III.1.2. Following that, the containment information between Poisson prime ideals of the algebra $\mathcal{P}$ is given in diagram III.1.1. This is an interesting class of Poisson algebras of dimension two and is a significant method to classify Poisson prime ideals for some Poisson algebras in Chapter IV.

Let α be a Poisson derivation on $K[t]$. By using Theorem II.2.17, the algebra $\mathcal{P}= K[t,x] = K[t][x;\alpha]_p$ is a Poisson algebra with Poisson bracket defined by the rule

$$\{t,x\} = \alpha(t)x, \quad \text{where } \alpha = f\partial_t, \quad \text{where } f \in K[t]\backslash\{0\}. \qquad \text{(III.1.1)}$$

There are two cases to consider: $f \in K^{\times}$ and $f \in K[t]\backslash K$.

Case i: If $f \in K^{\times}$ then the Poisson algebra $K[t][x;\alpha]_p$ is isomorphic to the 2-space Poisson algebra with Poisson bracket defined by the rule

$$\{t, x\} = fx \tag{III.1.2}$$

and is denoted by $\mathcal{P}(f)$. It follows from this equality that the prime ideals of $\mathcal{P}(f)$, 0, (x) and $(x, t-\nu)$ are also Poisson ideals, $\mathcal{P}(f)/(x) \cong K[t]$, and $\mathcal{P}(f)/(x, t-\nu) \cong K$. Hence,

$$\mathrm{PSpec}\big(\mathcal{P}(f)\big) = \big\{0, (x), (x, t-\nu) \mid \nu \in K\big\}.$$

Case ii: If $f \in K[t]\backslash K$ then we have the Poisson algebra $\mathcal{P}= K[t, x]$ with Poisson bracket defined by the rule

$$\{t, x\} = fx. \tag{III.1.3}$$

The following notation gives us the explicit formula of the roots of the polynomial f and the localization of the Poisson algebra $\mathcal{P}$.

Notation III.1.1. Let $R_f = \{\lambda_1, \dots, \lambda_s\}$ be the set of distinct roots of the polynomial

$$f(t) = \prod_{i=1}^{s} \lambda_f (t-\lambda_i)^{m_i}, \tag{III.1.4}$$

where λ_f is the leading coefficient of f and $m_1, \dots, m_s \geq 1$ are the multiplicities of the roots $\lambda_1, \dots, \lambda_s$, respectively. The localization of the algebra $\mathcal{P}$ at the powers of the element fx is $\mathcal{P}_{fx}= K[t, x^{\pm 1}, f^{-1}]$, i.e. $\mathcal{P}_{fx}= S^{-1}\mathcal{P} = \big\{(fx)^{-i}p \mid i \geq 0,\ p \in \mathcal{P}\big\}$, where $S = \big\{(fx)^i \mid i \geq 0\big\}$.

The following theorem classifies the Poisson prime ideals of the Poisson algebra $\mathcal{P}$.

Theorem III.1.2. *Let $\mathcal{P}= K[t, x]$ be the Poisson algebra as above and $f \in K[t]\backslash K$, i.e. $R_f \neq \emptyset$. Then*

1. *$\mathrm{PSpec}(\mathcal{P})=\big\{0, (x), (t-\lambda_i), (x, t-\nu), (x-\mu, t-\lambda_i) \mid \nu \in K, \mu \in K^{\times}, \lambda_i \in R_f$ and $i = 1, \dots, s\big\}$, and $\mathcal{P}/(x) \cong K[t]$, $\mathcal{P}/(t-\lambda_i) \cong K[x]$, $\mathcal{P}/(x, t-\nu) \cong \mathcal{P}/(x-\mu, t-\lambda_i) \cong K$*

for all $\nu \in K$, $\mu \in K^{\times}, \lambda_i \in R_f$ *and* $i = 1, \ldots, s$, *the containment information between Poisson prime ideals of* $\mathcal{P}$ *is given in diagram III.1.1.*

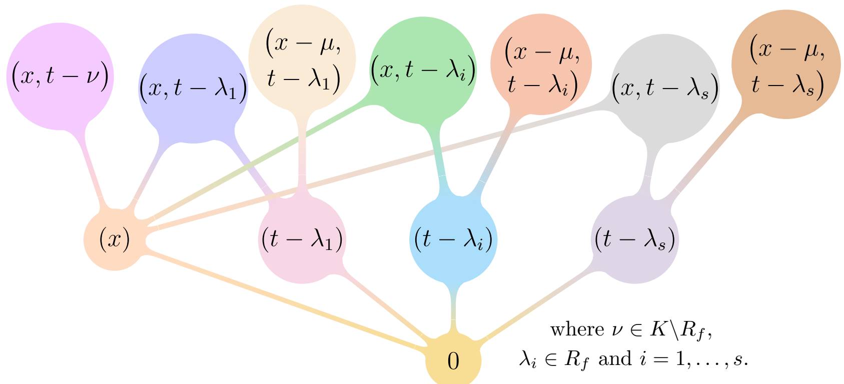


Diagram III.1.1: The containment information between Poisson prime ideals of $\mathcal{P}$

2. $\mathrm{PMax}(\mathcal{P})=\{(x, t-\nu), (x-\mu, t-\lambda_i) \mid \nu \in K, \mu \in K^{\times}, \lambda_i \in R_f$ *and* $i = 1, \ldots, s\}$

3. *the localization* $\mathcal{P}_{fx}$ *of the algebra* $\mathcal{P}$ *is a simple Poisson algebra.*

4. $\mathrm{PZ}(\mathcal{P}) = \mathrm{PZ}(\mathcal{P}_{fx}) = K$.

Proof. 3. It follows from the equality (III.1.3) that

$$\delta_t := \mathrm{pad}_t = fx\partial_x \text{ and } \delta_x := \mathrm{pad}_x = -fx\partial_t, \qquad \text{(III.1.5)}$$

where $\partial_t = \frac{d}{dt}$ and $\partial_x = \frac{d}{dx}$ are the partial derivatives of the algebras $\mathcal{P}$, and $\mathcal{P}_{fx} = K[t, x, (fx)^{-1}]$. Then

$$\partial_x = (fx)^{-1}\delta_t \text{ and } \partial_t = -(fx)^{-1}\delta_x. \qquad \text{(III.1.6)}$$

It follows from these equalities that the Poisson algebra $\mathcal{P}_{fx}$ is simple. Indeed, let I be a non-zero Poisson ideal of $\mathcal{P}_{fx}$ since $\mathcal{P} \subseteq \mathcal{P}_{fx}$, the ideal $I' := \mathcal{P} \cap I$ is a non-zero Poisson ideal of $\mathcal{P}$. Let $n := \min\{\deg_x(a) \mid a \in I' \backslash \{0\}\}$, where $\deg_x(a)$ is the x-degree of the polynomial $a = a_0 + a_1(t)x + a_2(t)x^2 + \cdots + a_d(t)x^d$, i.e. $\deg_x(a) = d$ provided $a_d(t) \neq 0$, where $a_0(t), a_1(t), \ldots, a_d(t) \in K[t]$.

(i) Suppose that $n = 0$. Then $0 \neq I' \cap K[t] = K[t]p$ for non-zero polynomial $p \in K[t]$

if $\deg_t(p) = 0$, i.e. $p \in K^\times$ then $1 \in I' \subseteq I$, hence, $I = \mathcal{P}_{fx}$. If $\deg_t(p) \geq 1$ then the polynomial $\partial_t(p) = \frac{dp}{dt} \in I' \cap K[t]$ has degree $\deg_t(p) - 1$[1] which is not possible since $0 \neq \partial_t(p) \in K[t]p$. This means that the case $\deg_t(p) \geq 1$ is impossible.

(ii) Suppose that $n \geq 1$. Fix a polynomial, say $a = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, with $\deg_x(a) = n$, where $a_0, a_1, \ldots, a_n \in K[t]$ then the polynomial $\partial_x(a) = \frac{\partial a}{\partial x} \in I' \backslash \{0\}$ and $\deg_x\big(\partial_x(a)\big) = n - 1$, a contradiction. Therefore, the case $n \geq 1$ is impossible.

1. It follows from the equality (III.1.3) that the prime ideals of $\mathcal{P}$, 0, (x), $(t - \lambda_i)$, $(x, t - \nu)$ and $(x - \mu, t - \lambda_i)$ are also Poisson ideals, $\mathcal{P}/(x) \cong K[t]$, $\mathcal{P}/(t - \lambda_i) \cong K[x]$, and $\mathcal{P}/(x, t - \nu) \cong \mathcal{P}/(x - \mu, t - \lambda_i) \cong K$, where $\nu \in K, \mu \in K^\times, \lambda_i \in R_f$ and $i = 1, \ldots, s$. Hence, all the ideals in statement 1 are Poisson prime ideals of the Poisson algebra $\mathcal{P}$. Clearly that diagram III.1.1 reflects all possible containment of these Poisson prime ideals. To finish the proof of statement 1 we have to show that there are no additional Poisson ideals of $\mathcal{P}$, but this follows from statement 3.

2. Statement 2 follows from diagram III.1.1.

4. Since $\mathrm{PZ}(\mathcal{P}) \subseteq \mathrm{PZ}(\mathcal{P}_{fx})$, it suffices to show that $\mathcal{Z} := \mathrm{PZ}(\mathcal{P}_{fx}) = K$. Given $a \in \mathcal{Z} \backslash \{0\}$. We have to show that $a \in K$. Since $a \in \mathcal{Z} \backslash \{0\}$, the ideal $(a) := a\mathcal{P}_{fx}$ is a non-zero Poisson ideal of the Poisson algebra $\mathcal{P}_{fx}$. The Poisson algebra $\mathcal{P}_{fx}$ is simple by statement 3, hence, a is a unit in the algebra $\mathcal{P}_{fx}$, i.e. $a = bf^i x^j$ for some $b \in K^\times$ and $i, j \geq 0$. It follows from the equality (III.1.6) that

$$\begin{aligned} 0 = \ (fx)^{-1}\{t, a\} = \ & \frac{\partial a}{\partial x} = jbf^i x^{j-1}, \\ 0 = \ (fx)^{-1}\{x, a\} = \ & \frac{\partial a}{\partial t} = ibf^{i-1} x^j, \end{aligned} \tag{III.1.7}$$

that $i = j = 0$, i.e. $a = b \in K^\times$, as required.

□

[1]since $\mathrm{char}(K) = 0$

§III.2 The Poisson algebra $\mathcal{P}_2(f) = K[x, y]$

The aim of this section is to classify all Poisson prime ideals, minimal Poisson ideals and maximal Poisson ideals of the Poisson algebra $\mathcal{P}_2(f)$, which are described in Proposition III.2.1. Following that, the relation between the Poisson differential operator algebras of $\mathcal{P}_2(f)$, and the Weyl algebras is given in Proposition III.2.1.(4).

It is easy to check the polynomial algebra $K[x, y]$ over an arbitrary field K admits a Poisson structure that is given by the rule $\{y, x\} = f$ for an arbitrary polynomial $f \in K[x, y]$. This Poisson algebra is denoted by $\mathcal{P}_2(f)$.

Proposition III.2.1. *Let $\mathcal{P}_2(f) = K[x, y]$ be the Poisson algebra with Poisson bracket $\{y, x\} = f$, where $f \in K[x, y]$. Then*

1. $\mathrm{PSpec}(\mathcal{P}_2(f)) = \{0, \mathfrak{p} \mid \mathfrak{p} \in \mathrm{Spec}(K[x, y]), f \in \mathfrak{p}\}$.
2. $\mathrm{PMax}(\mathcal{P}_2(f)) = \{\mathfrak{m} \mid \mathfrak{m} \in \mathrm{Max}(K[x, y]), f \in \mathfrak{m}\}$.
3. *if $f \neq 0$ then $\mathcal{P}_2(f)_f$ is the localization of $\mathcal{P}_2(f)$ at the powers of the element f is simple.*
4. *if $f \neq 0$ then the algebra $P\mathcal{D}(\mathcal{P}_2(f)_f) \simeq A_{2,f}$ is isomorphic to the localization of the Weyl algebra $A_2 = K\langle x, y, \partial_x, \partial_y\rangle$ at the powers of the element f. In particular, the algebra $P\mathcal{D}(\mathcal{P}_2(f)_f)$ is a simple Noetherian algebra.*

Proof. 4. The algebra $P\mathcal{D}(\mathcal{P}_2(f))$ is a subalgebra of the Weyl algebra $A_2 = \mathcal{D}(K[x, y])$ which is generated by the polynomial algebra $K[x, y]$, the derivations $\mathrm{pad}_x = -f\partial_y$ and $\mathrm{pad}_y = f\partial_x$, and statement 4 follows.

3. Statement 3 follows from statement 4 and Theorem II.3.20.

1. Statement 1 follows from statement 3.

2. Statement 2 follows from statement 1.

□

§ IV The Poisson algebras A of dimension three

The main purpose of this chapter is to classify Poisson prime ideals, minimal Poisson ideals and maximal Poisson ideals for a certain class of Poisson polynomial algebras in three variables.

Recall that the Poisson polynomial algebra $(D; \alpha, \beta, c, u)$ was introduced by Oh in 2006, in the following Lemma.

Lemma IV.0.1. *[Oh3, Lemma 1.3] Let D be a Poisson algebra over a field K, $c \in K$, $u \in D$ and α, β be Poisson derivations on D such that*

$$\alpha\beta = \beta\alpha \quad \textit{and} \quad \{a, u\} = (\alpha + \beta)(a)u \quad \textit{for all} \quad a \in D. \tag{IV.0.1}$$

Then the polynomial ring $D[x, y]$ becomes a Poisson algebra with Poisson bracket

$$\{a, y\} = \alpha(a)y, \quad \{a, x\} = \beta(a)x \quad \textit{and} \quad \{y, x\} = cyx + u \;\; \textit{for all} \;\; a \in D. \tag{IV.0.2}$$

The Poisson algebra $D[x, y]$ with Poisson bracket (IV.0.2) is denoted by $(D; \alpha, \beta, c, u)$ or $D[y; \alpha, 0]_p[x; \beta, \delta' := u\partial_y]_p$.

Now, let us consider an arbitrary Poisson algebra $(D; \alpha, \beta, c, u)$, where the Poisson algebra D is the Poisson polynomial algebra $K[t]$ over a field K of characteristic zero with trivial

Poisson bracket, and α, β are arbitrary K-derivations of $K[t]$, i.e.

$$\alpha = f\partial_t, \ \ \beta = g\partial_t \ \ \text{and} \ \ f, g \in K[t], \text{ where } \partial_t = \frac{d}{dt},$$

$c \in K$ and $u \in K[t]$. It follows from (IV.0.1) that

$$0 = \{d, u\} = (\alpha + \beta)(d)u \ \text{ for all } \ d \in K[t], \tag{IV.0.3}$$

hence, u is an element in the Poisson centre of $K[t]$, and

$$(\alpha + \beta)\, u = 0. \tag{IV.0.4}$$

The following lemma describes all commuting pairs $(\alpha,\, \beta)$.

Lemma IV.0.2. *Let $K[t]$ be the Poisson polynomial algebra and α, β be Poisson derivations on $K[t]$. If $\alpha = f\partial_t$ and $\beta = g\partial_t$, where $f, g \in K[t]\backslash\{0\}$ then*

$$\alpha\beta = \beta\alpha \ \textit{ if and only if } \ g = \frac{1}{\lambda} f, \ \ \textit{where } \lambda \in K^{\times}. \tag{IV.0.5}$$

Proof. Notice that,

$$[\alpha, \beta] = \alpha\beta - \beta\alpha = (fg' - gf')\partial_t = g^2 \left(\frac{f}{g}\right)' \partial_t,$$

where $(-)' = \frac{d(-)}{dt}$. Therefore, $[\alpha, \beta] = 0$ if and only if $\left(\frac{f}{g}\right)' = 0$ if and only if $\frac{f}{g} \in \ker_{K(t)}(\partial_t) = K$ if and only if $\frac{f}{g} = \lambda$ for some $\lambda \in K^{\times}$ if and only if $g = \frac{1}{\lambda} f$. □

The Poisson algebras $A = (K[t]; f\partial_t, \lambda^{-1} f\partial_t, c, u)$, where $f, u \in K[t]$, $\lambda \in K^{\times}$ and $c \in K$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = \lambda^{-1} fx \ \ \text{and} \ \ \{y, x\} = cyx + u.$$

It follows from Lemma IV.0.2 and the equality (IV.0.4) that there are three main classes:

Class I: $\alpha + \beta = f\partial_t + \lambda^{-1} f\partial_t = 0$ and $u = 0$.

Class II: $\alpha + \beta = f\partial_t + \lambda^{-1} f\partial_t = 0$ and $u \neq 0$.

Class III: $\alpha + \beta = f\partial_t + \lambda^{-1} f\partial_t \neq 0$ and $u = 0$.

In order to study Poisson algebras in each of the three classes, we need to subdivide them into subclasses.

- The first class **I** has two subclasses: **I.1** and **I.2**.
- The second class **II** has two subclasses: **II.1** and **II.2**.
- The third class **III** has two subclasses: **III.1** and **III.2**.

Each subclass has further subclasses. For more information see diagram IV.1.1, diagram IV.2.1 and diagram IV.3.1, respectively.

§IV.1 The first class

The aim of this section is to classify all Poisson prime ideals, minimal Poisson ideals and maximal Poisson ideals of Poisson algebras that belong to the first class I. This class has two subclasses: **I.1** and **I.2**, and each subclass has several subclasses. Let us give further detail, the first subclass I.1 has two subclasses: I.1.1 and I.1.2. In particular, the Poisson algebra, that belongs to the subclass I.1.1, is the polynomial ring in three variables with a trivial Poisson structure. In addition, the classification of Poisson prime ideals for the Poisson algebra, that belongs to the subclass I.1.2, is obtained in Theorem IV.1.5. The inclusions of Poisson prime ideals for this Poisson algebra are described in diagram IV.1.3.

The second subclass I.2 has two subclasses: I.2.1 and I.2.2, and each subclass has two further subclasses:

I.2.1.1, I.2.1.2, I.2.2.1 and I.2.2.2, respectively.

The classifications of Poisson prime ideals for Poisson algebras that belong to these four subclasses are obtained in Theorem IV.1.8, Theorem IV.1.9, Theorem IV.1.10 and Theorem IV.1.11, respectively. In particular, each subclass is treated individually and different techniques are used. The main methods are the localization and factorization of Poisson algebras, and Theorem III.1.2. These techniques are similar to some of the approaches in recent papers [Oh3] and [Oh4], see reviews in Subsection II.2.5 for detail. In addition, the containments information between Poisson prime ideals for these algebras are given in diagram IV.1.4, diagram IV.1.5, diagram IV.1.6 and diagram IV.1.7, respectively.

Class I:

If $\alpha+\beta = f\partial_t + \frac{1}{\lambda}f\partial_t = 0$, $u = 0$ and $c \in K$, where $\lambda \in K^\times$, $f \in K[t]$. Notice that, $f\partial_t + \frac{1}{\lambda}f\partial_t = 0$ implies that there are two subclasses:

Class I.1: If $f = 0$.

Class I.2: If $\lambda = -1$.

Structure of the first class of Poisson algebras A is given in diagram IV.1.1.

The following is the first subclass of class I and consists of two subclasses which are dependent on c.

Class I.1

If $f = 0$, i.e. $\alpha = \beta = 0$, $u = 0$ and $c \in K$ then we have the Poisson algebra $A_1 = (K[t]; 0, 0, c, 0)$ with Poisson bracket defined by the rule

$$\{t, y\} = 0, \quad \{t, x\} = 0 \quad \text{and} \quad \{y, x\} = cyx. \tag{IV.1.1}$$

There are two subclasses which are dependent on c:

Class I.1.1: If $c = 0$.

Class I.1.2: If $c \in K^{\times}$.

Remark IV.1.1. It follows from the equality (IV.1.1) that $t \in \mathrm{PZ}(A_1)$. Furthermore, the Poisson algebra $A_1 = K[t] \otimes \mathcal{P}$ is a tensor product of two Poisson algebras: $(K[t], \{\cdot, \cdot\} = 0)$ and $\mathcal{P} = K[x, y]$ with the Poisson bracket $\{y, x\} = cyx$, where $c \in K$.

The next is the Poisson algebra A_2 that belongs to the subclass I.1.1 and has a trivial Poisson structure.

Class I.1.1:

If $c = 0 = \alpha = \beta = u$ then the Poisson algebra $A_2 = (K[t]; 0, 0, 0, 0)$ has a trivial Poisson structure and the Poisson spectrum of A_2 is the spectrum of the polynomial ring in three variables, i.e.

$$\mathrm{PSpec}(A_2) = \mathrm{Spec}(A_2) = \mathrm{Spec}(K[t, x, y])$$

which is unknown yet.

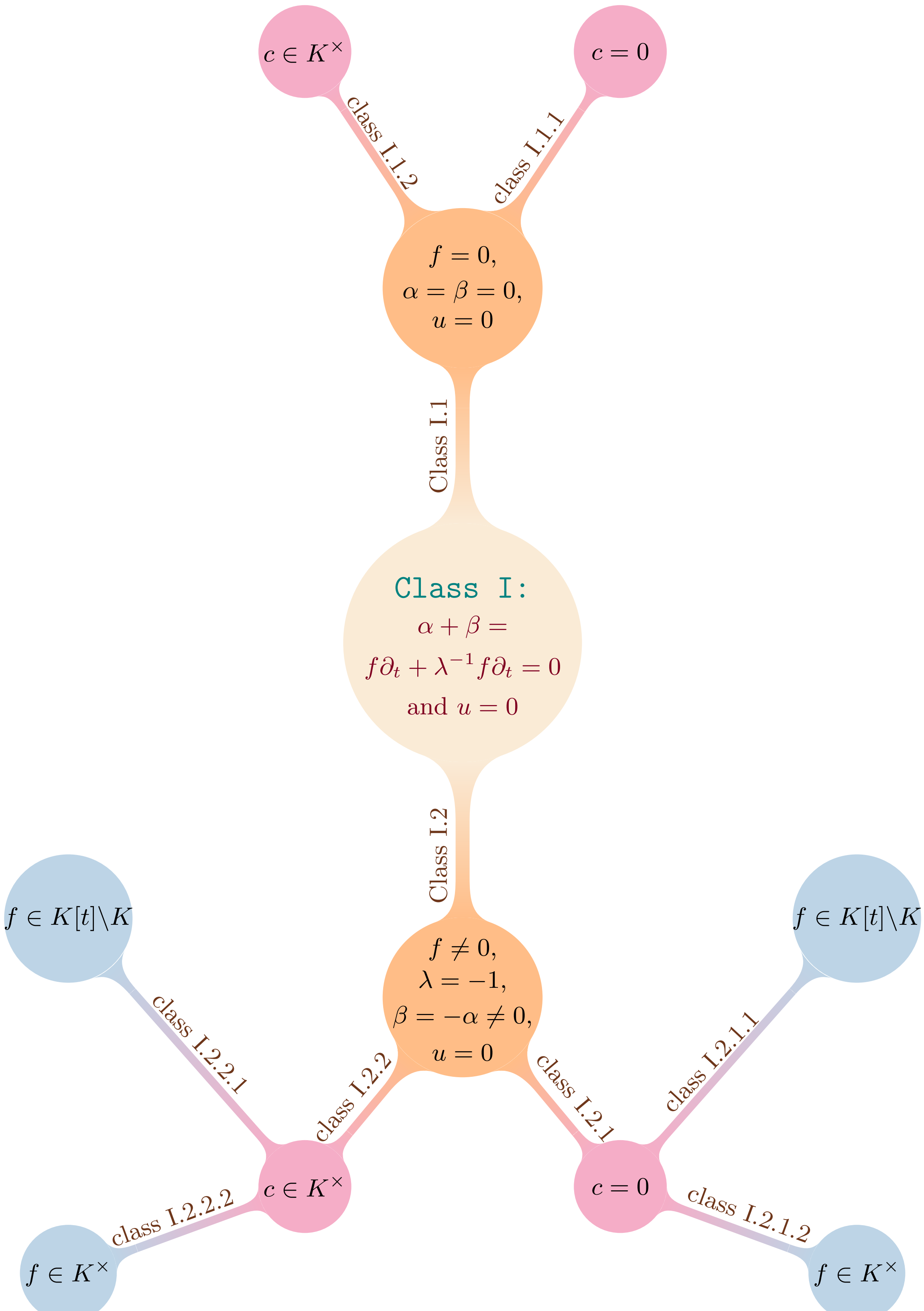


Diagram IV.1.1: Structure of the first class of Poisson algebras A

The following is the Poisson algebra A_3 that belongs to the subclass I.1.2. Notice that, c is a non-zero element in K. The techniques to classify Poisson prime ideals of A_3 have some properties of localization.

Class I.1.2:

If $c \in K^\times$ and $\alpha = \beta = u = 0$ then we have the Poisson algebra $A_3 = (K[t]; 0, 0, c, 0)$ with Poisson bracket defined by the rule

$$\{t, y\} = 0, \quad \{t, x\} = 0 \quad \text{and} \quad \{y, x\} = cyx. \tag{IV.1.2}$$

The following notation describes the localization algebra of the Poisson algebra A_3.

Notation IV.1.2. Let $S = K[t]\backslash\{0\}$. The localization of the Poisson algebra A_3 is $B = S^{-1}A_3$, i.e. $B = K(t)[x, y]$, where $K(t) = S^{-1}K[t]$ is the field of rational functions in the variable t. The algebra B is a Poisson algebra with Poisson bracket defined by the rule

$$\{y, x\} = cyx. \tag{IV.1.3}$$

The next lemma describes the Poisson prime ideals of the Poisson algebra B.

Lemma IV.1.3. *Let $A_3 = (K[t]; 0, 0, c, 0)$ be the Poisson algebra as above and B be the Poisson algebra in Notation IV.1.2, where $c \in K^\times$. Then*

1. $\mathrm{PSpec}(B)=\big\{0, (x), (y), (x, p), (y, q) \mid p \in \mathrm{Irr}_m K(t)[y]$ *and* $q \in \mathrm{Irr}_m K(t)[x]\big\}$, *the containment information between Poisson prime ideals of B is given in diagram IV.1.2.*

2. $\mathrm{PMax}(B)=\big\{(x, p), (y, q) \mid p \in \mathrm{Irr}_m K(t)[y]$ *and* $q \in \mathrm{Irr}_m K(t)[x]\big\}$.

3. *the localization B_{xy} of the algebra B at the powers of the element xy is a simple Poisson algebra.*

4. $\mathrm{PZ}(B) = \mathrm{PZ}(B_{xy})=K(t)$.

Proof. 3. It follows from the equality (IV.1.3) that

$$\delta_y := \mathrm{pad}_y = cyx\partial_x \quad \text{and} \quad \delta_x := \mathrm{pad}_x = -cyx\partial_y,$$

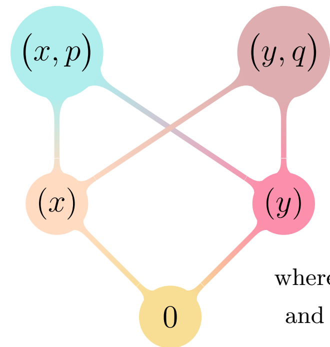


Diagram IV.1.2: The containment information between Poisson prime ideals of B

where $\partial_y = \frac{\partial}{\partial y}$ and $\partial_x = \frac{\partial}{\partial x}$ are the partial derivatives of the algebras $B = K(t)[x,y]$, and $B_{xy} = K(t)[x,y,(xy)^{-1}]$. Then

$$\partial_x = (cyx)^{-1}\delta_y \text{ and } \partial_y = -(cyx)^{-1}\delta_x.$$

It follows from these equalities that the Poisson algebra B_{xy} is simple. Indeed, let I be a non-zero Poisson ideal of B_{xy} since $B \subseteq B_{xy}$, the ideal $I' := B \cap I$ is a non-zero Poisson ideal of B. Let $n := \min\{\deg_x(a) \,|\, a \in I'\backslash\{0\}\}$, where $\deg_x(a)$ is the x-degree of the polynomial $a = a_0 + a_1(y)x + a_2(y)x^2 + \cdots + a_d(y)x^d$, i.e. $\deg_x(a) = d$ provided $a_d(y) \neq 0$, where $a_0(y), a_1(y), \ldots, a_d(y) \in K(t)[y]$.

(i) Suppose that $n = 0$. Then $0 \neq I' \cap K(t)[y] = K(t)[y]p$ for non-zero polynomial $p \in K(t)[y]$ if $\deg_y(p) = 0$, i.e. $p \in K(t)$ then p is a unit element this implies that $1 \in I' \subseteq I$ hence, $I = B_{xy}$. If $\deg_y(p) \geq 1$ then the polynomial $\partial_y(p) = \frac{\partial p}{\partial y} \in I' \cap K(t)[y]$ has degree $\deg_y(p) - 1$[1] which is not possible since $0 \neq \partial_y(p) \in K(t)[y]p$. This means that the case $\deg_y(p) \geq 1$ is impossible.

(ii) Suppose that $n \geq 1$. Fix a polynomial, say $a = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, with $\deg_x(a) = n$, where $a_0, a_1, \ldots, a_n \in K(t)[y]$ then the polynomial $\partial_x(a) = \frac{\partial a}{\partial x} \in I'\backslash\{0\}$ and $\deg_x\big(\partial_x(a)\big) = n - 1$, a contradiction. Therefore, the case $n \geq 1$ is impossible.

1. It follows from the equality (IV.1.3) that the prime ideals of B, 0, (x), (y), (x,p)

[1] since $\mathrm{char}(K) = 0$

and (y, q) are also Poisson ideals. Hence, all the ideals in statement 1 are Poisson prime ideals of the Poisson algebra B. Clearly that diagram IV.1.2 reflects all possible containment of these Poisson prime ideals. To finish the proof of statement 1 we have to show that there are no additional Poisson ideals of B, but this follows from statement 3.

2. Statement 2 follows from diagram IV.1.2.

4. Since $\mathrm{PZ}(B) \subseteq \mathrm{PZ}(B_{xy})$, it suffices to show that $\mathcal{Z} :=\mathrm{PZ}(B_{xy})=K(t)$. Given $a \in \mathcal{Z}\backslash\{0\}$. We have to show that $a \in K(t)$. Since $a \in \mathcal{Z}\backslash\{0\}$, the ideal $(a) := aB_{xy}$ is a non-zero Poisson ideal of the Poisson algebra B_{xy}. The Poisson algebra B_{xy} is simple by statement 3, hence a is a unit of the algebra B_{xy}, i.e. $a = f(t)$ for some $f \in K(t)$ as required.

□

The next remark gives us an important equality to classify the Poisson spectrum, which will be used very often throughout this thesis.

Remark IV.1.4. If $S = K[t]\backslash\{0\}$ and the localization of the Poisson algebra A is $B = S^{-1}A$ then

$$\mathrm{PSpec}(A) = \mathrm{PSpec}(B)^r \coprod \mathrm{PSpec}(A, K[t]), \tag{IV.1.4}$$

where $\mathrm{PSpec}(B)^r = \{A \cap P \mid P \in \mathrm{PSpec}(B)\}$ and

$$\mathrm{PSpec}(A, K[t]) = \{\mathfrak{p} \in \mathrm{PSpec}(A) \mid \mathfrak{p} \cap K[t] \neq 0\}. \tag{IV.1.5}$$

Notice that, the map

$$\mathrm{PSpec}(B) \to \mathrm{PSpec}(B)^r, \quad P \mapsto A \cap P \tag{IV.1.6}$$

is a bijection with the inverse $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$.

The next theorem classifies the Poisson prime ideals of the Poisson algebra A_3.

Theorem IV.1.5. *Let $A_3 = (K[t]; 0, 0, c, 0)$ be the Poisson algebra as above and B be the Poisson algebra in Notation IV.1.2, where $c \in K^{\times}$. Then*

1. $\mathrm{PSpec}(A_3) = \mathrm{PSpec}(B)^r \coprod \mathrm{PSpec}(A_3, S)$, *where* $\mathrm{PSpec}(B)^r = \{0, (x), (y), (x, l_p p), (y, l_q q) \mid p \in \mathrm{Irr}_m K(t)[y]$ *and* $q \in \mathrm{Irr}_m K(t)[x]\}$, l_p[2] *is a unique monic polynomial in* $K[t]$ *of the least degree in* t *such that* $l_p p \in K[t, y]$[3]*;* $\mathrm{PSpec}(A_3, S) = \{(t-\lambda), (x, t-\lambda), (y, t-\lambda), (x, y, t-\lambda), (x, y-\mu, t-\lambda), (y, x-\mu, t-\lambda) \mid \lambda \in K$ *and* $\mu \in K^\times\}$, *the containment information between Poisson prime ideals of* A_3 *is given in diagram IV.1.3.*

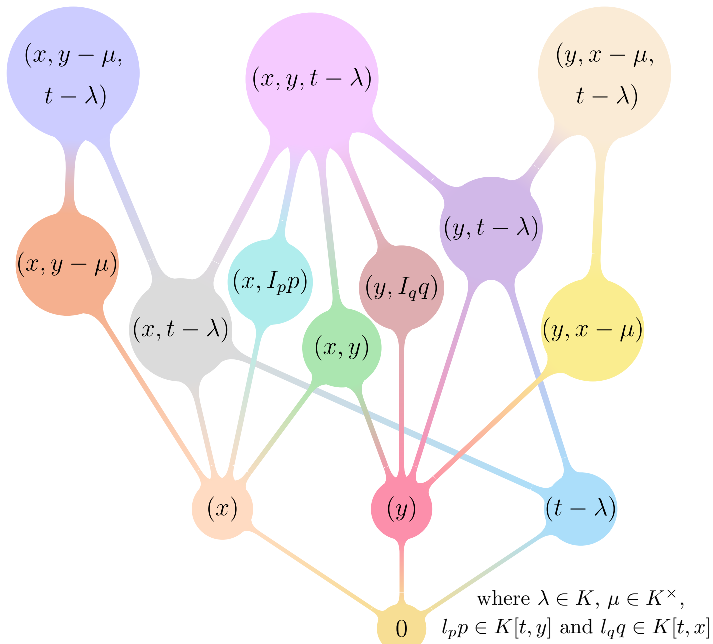


Diagram IV.1.3: The containment information between Poisson prime ideals of A_3

2. *for all* $\mathfrak{p} \in \mathrm{PSpec}(B)^r$ *and* $\mathfrak{q} \in \mathrm{PSpec}(A_3, S)$, $\mathfrak{q} \not\subseteq \mathfrak{p}$.

Proof. 1. The first equality of statement 1 follows from (IV.1.4). Then the second equality of statement 1 follows from the explicit description of the set $\mathrm{PSpec}(B)$, see Lemma IV.1.3.(1). Let $\mathfrak{p} \in \mathrm{PSpec}(A_3, S)$. Then $\mathfrak{p}' := K[t] \cap \mathfrak{p}$ is a non-zero prime ideal of the polynomial algebra $K[t]$, i.e. $\mathfrak{p}' = (t-\lambda)$ for some element $\lambda \in K$[4]. The Poisson algebra

$$A_3/(t-\lambda) \cong K[t]/(t-\lambda) \otimes K[x, y] \cong K[x, y] \qquad \text{(IV.1.7)}$$

[2]respectively, l_q
[3]respectively, $l_q q \in K[t, x]$
[4]since the field K is an algebraically closed field

is a polynomial Poisson algebra $\mathcal{P}_2 = K[x, y]$ with Poisson bracket $\{y, x\} = cyx$, where $c \in K^\times$. The Poisson prime spectrum of such algebras is described in Theorem III.1.2. So,

$$\text{PSpec}(\mathcal{P}_2) = \{0, (x), (y), (x, y), (y, x - \mu), (x, y - \mu) \mid \mu \in K^\times\}.$$

Therefore,

$\text{PSpec}(A_3, S) = \{(t-\lambda), (x, t-\lambda), (y, t-\lambda), (x, y, t-\lambda), (x, y-\mu, t-\lambda), (y, x-\mu, t-\lambda) \mid \lambda \in K \text{ and } \mu \in K^\times\}$.

2. Suppose that $\mathfrak{q} \subseteq \mathfrak{p}$ for some ideals $\mathfrak{p} \in \text{PSpec}(B)^r$ and $\mathfrak{q} \in \text{PSpec}(A_3, S)$, we seek a contradiction. Then $B = S^{-1}\mathfrak{q} \subseteq S^{-1}\mathfrak{p} \subsetneq B$, a contradiction.

□

The next is the second subclass of class I and has two subclasses which are dependent on c.

Class I.2

If $\lambda = -1$, i.e. $\beta = -\alpha = -f\partial_t$, $u = 0$ and $c \in K$, where $f \in K[t]\backslash\{0\}$ then we have the Poisson algebra $A_4 = (K[t]; f\partial_t, -f\partial_t, c, 0)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = -fx \quad \text{and} \quad \{y, x\} = cyx. \qquad \text{(IV.1.8)}$$

There are two subclasses which are dependent on c:

Class I.2.1: If $c = 0$.

Class I.2.2: If $c \in K^\times$.

Remark IV.1.6. The Poisson ideals (x) and (y) are Poisson prime ideals of $A_4 = K[t][x, y]$, and

$$A_4/(x) \cong K[t, y] \qquad \text{(IV.1.9)}$$

is a polynomial Poisson algebra $\mathcal{P}_1 = K[t, y]$ with Poisson bracket $\{t, y\} = fy$, where $f \in$

$K[t]\backslash K$. The Poisson prime spectrum of such algebras is described in Theorem III.1.2. So,

$$\mathrm{PSpec}(\mathcal{P}_1) = \big\{0, (y), (t-\lambda_i), (y, t-\nu), (y-\mu, t-\lambda_i) \,|\, \nu \in K, \mu \in K^\times, \lambda_i \in R_f \text{ and } i = 1, \dots, s\big\}.$$

In addition,

$$A_4/(y) \cong K[t, x] \tag{IV.1.10}$$

is a polynomial Poisson algebra $\mathcal{P} = K[t, x]$ with Poisson bracket $\{t, x\} = -fx$, where $f \in K[t]\backslash K$. The Poisson spectrum of such algebras is described in Theorem III.1.2. So,

$$\mathrm{PSpec}(\mathcal{P}) = \big\{0, (x), (t-\lambda_i), (x, t-\nu), (x-\mu, t-\lambda_i) \,|\, \nu \in K, \mu \in K^\times, \lambda_i \in R_f \text{ and } i = 1, \dots, s\big\}.$$

Definition IV.1.7. Let A be a Poisson algebra and I be a subset of A then

$$\mathcal{V}(I) = \big\{\mathfrak{p} \in \mathrm{PSpec}(A) \mid I \subseteq \mathfrak{p}\big\}.$$

The following is the Poisson algebra A_5 that belongs to the subclass I.2.1. This subclass has two subclasses which are dependent on the roots of f.

Class I.2.1:

If $\alpha = f\partial_t$, $\beta = -f\partial_t$ and $u = c = 0$, where $f \in K[t]\backslash\{0\}$ then we have the Poisson algebra $A_5 = (K[t]; f\partial_t, -f\partial_t, 0, 0)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = -fx \quad \text{and} \quad \{y, x\} = 0. \tag{IV.1.11}$$

There are two subclasses which are dependent on f:

Class I.2.1.1: If $f \in K[t]\backslash K$.

Class I.2.1.2: If $f \in K^\times$.

Class I.2.1.1: If $f \in K[t]\backslash K$ and $u = c = 0$ then we have the Poisson algebra $A_6 = (K[t]; f\partial_t, -f\partial_t, 0, 0)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = -fx \quad \text{and} \quad \{y, x\} = 0. \tag{IV.1.12}$$

The next theorem identifies the Poisson prime ideals of the Poisson algebra A_6.

Theorem IV.1.8. *Let $A_6 = (K[t]; f\partial_t, -f\partial_t, 0, 0)$ be the Poisson algebra as above, where $f \in K[t]\backslash K$, i.e. $R_f \neq \emptyset$. Then*

1. $\mathrm{PSpec}(A_6) = \{0\} \cup \mathcal{V}(x) \cup \mathcal{V}(y) \cup \mathcal{V}(f) \cup \bigcup_{\mu \in K^\times} \mathcal{V}(xy - \mu)$.

2. $\mathrm{PSpec}(A_6) = \big\{0, (x), (y), (t-\lambda_i), (x, t-\lambda_i), (y, t-\lambda_i), (xy-\mu), (x, y), (h, t-\lambda_i), (xy-\mu, t-\lambda_i), (x, y, t-\nu), (y, x-\mu, t-\lambda_i), (x, y-\mu, t-\lambda_i), (x-\mu, y-\mu', t-\lambda_i), (x-\omega, y+\omega^{-1}\mu, t-\lambda_i) \mid \nu \in K, \mu, \mu', \omega \in K^\times, \lambda_i \in R_f, i = 1, \ldots, s$ *and* $h \in \mathrm{Irr}_m K[x, y]\big\}$, *the containment information between Poisson prime ideals of A_6 is given in diagram IV.1.4.*

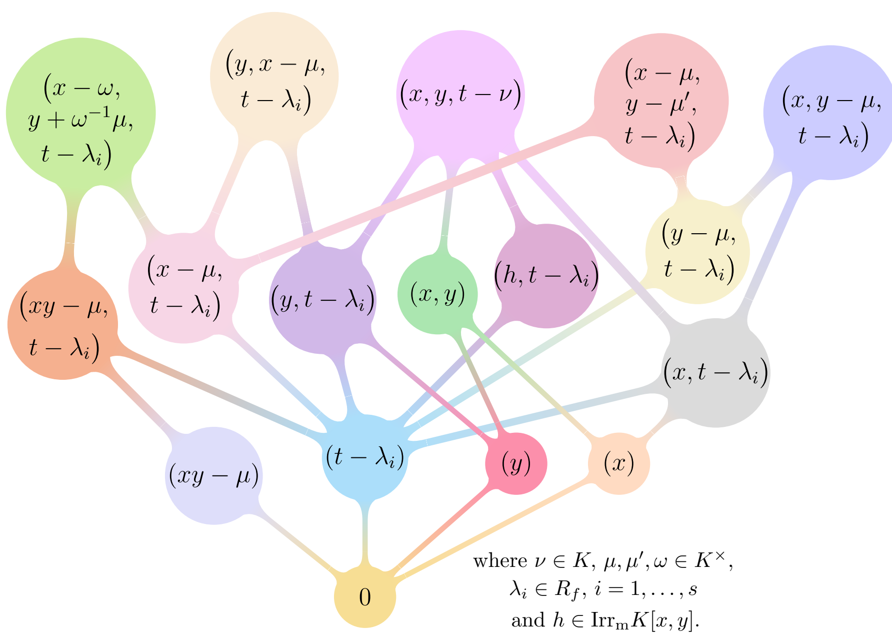


Diagram IV.1.4: The containment information between Poisson prime ideals of A_6

3. $\mathrm{PMax}(A_6) = \big\{(x, y, t-\nu), (y, x-\mu, t-\lambda_i), (x, y-\mu, t-\lambda_i), (x-\mu, y-\mu', t-\lambda_i), (x-\omega, y+\omega^{-1}\mu, t-\lambda_i) \mid \nu \in K, \mu, \mu', \omega \in K^\times, \lambda_i \in R_f$ *and* $i = 1, \dots, s\big\}$.

4. $\mathrm{PZ}(A_{6fxy}) = K[xy, (xy)^{-1}]$ *and* $I \cap \mathrm{PZ}(A_{6fxy}) \neq 0$ *for all non-zero Poisson ideals* I *of* A_{6fxy}, *where* A_{6fxy} *is the localization of the algebra* A_6 *at the powers of the element* fxy.

5. $\mathrm{PZ}(A_6) = K[xy]$.

Proof. 4. Notice that,

$$\begin{aligned} \delta_x &:= \mathrm{pad}_x = fx\partial_t, \\ \delta_y &:= \mathrm{pad}_y = -fy\partial_t, \\ \delta_t &:= \mathrm{pad}_t = f(-x\partial_x + y\partial_y). \end{aligned}$$

Therefore,

$$\partial_t = (fx)^{-1}\delta_x \text{ and } \Delta := x\partial_x - y\partial_y = -f^{-1}\delta_t.$$

Suppose that I is a non-zero Poisson ideal of the Poisson algebra A_{6fxy}. We have to show that $I \cap \mathrm{PZ}(A_{6fxy}) \neq 0$. Since $\partial_t = (fx)^{-1}\delta_x$ the ideal $I_2 := I \cap K[x, y]$ is a non-zero Poisson ideal of the Poisson algebra $K[x, y]$, where $\{y, x\} = 0$ which is Δ-invariant. Since $\ker_{K[x,y]}(\Delta) = K[xy]$ and the polynomial algebra $K[x, y] = \bigoplus_{i\in\mathbb{Z}} K[x, y]\nu_i$, where

$$\nu_i = \begin{cases} x^i & \text{if } i > 0, \\ 1 & \text{if } i = 0, \\ y^{|i|} & \text{if } i < 0 \end{cases}$$

is a direct sum of Δ-eigenspace with different eigenvectors[5]:

$$\Delta\nu_i = i\nu_i \quad \text{for all } i \in \mathbb{Z},$$

the ideal I_2 contains an element $p\nu_i$ for some polynomial $p \in K[xy]\backslash\{0\}$ and $i \in \mathbb{Z}$. The element ν_i is a unit in A_{6fxy}. Hence, $p \in I$ and so $0 \neq p \in I \cap \mathrm{PZ}(A_{6fxy})$, by the

[5] since char(K)=0

statement (ii) below.

(i) $A_{6\,fxy}\delta_x + A_{6\,fxy}\delta_y + A_{6\,fxy}\delta_t = A_{6\,fxy}\partial_t \oplus A_{6\,fxy}\Delta$ straightforward.

(ii) $\mathrm{PZ}(A_{6\,fxy}) = K[xy,(xy)^{-1}]$: By the statement (i),

$\mathrm{PZ}(A_{6\,fxy}) = \ker_{A_{6\,fxy}}(\partial_t) \cap \ker_{A_{6\,fxy}}(\Delta) = K[x^{\pm 1}, y^{\pm 1}] \cap \ker_{A_{6\,fxy}}(\Delta) = \ker_{K[x^{\pm 1},y^{\pm 1}]}(\Delta) = K[xy,(xy)^{-1}]$.

5. $\mathrm{PZ}(A_6) = A_6 \cap \mathrm{PZ}(A_{6\,fxy}) = A_6 \cap K[xy,(xy)^{-1}] = K[xy]$, by statement 4.

1. If $A_{6\,fxy} = S^{-1}A_6$, where $S = \{(fxy)^i \mid i \geq 1\}$ then any non-zero Poisson prime ideal of A_6 intersect with S contains an element $(fxy)^i$ for some $i \geq 1$. It follows from (IV.1.4) that $\mathrm{PSpec}(A_6) = \mathrm{PSpec}(A_{6\,fxy})^r \coprod \mathrm{PSpec}(A_6, S)$. Therefore,

$$\mathrm{PSpec}(A_6, S) = \{0\} \cup \mathcal{V}(fxy) = \{0\} \cup \mathcal{V}(x) \cup \mathcal{V}(y) \cup \mathcal{V}(f)$$

and by statement 4,

$$\mathrm{PSpec}(A_{6\,fxy})^r = \{0\} \cup \bigcup_{\mu \in K^\times} \mathcal{V}(xy - \mu).$$

2. It follows from statement 1,

$$\mathrm{PSpec}(A_6) = \{0\} \cup \mathcal{V}(x) \cup \mathcal{V}(y) \cup \bigcup_{i=1}^{s} (t - \lambda_i) \cup \bigcup_{\mu \in K^\times} (xy - \mu).$$

Notice that, the following Poisson factor algebras are examples of the ones appeared in Theorem III.1.2

$$\begin{aligned}
A_6/(x) &\cong \mathcal{P}_1, & & \\
A_6/(y) &\cong \mathcal{P}, & & \{t,x\} = -fx, \\
A_6/(t-\lambda_i) &\cong \mathcal{P}_0 = K[x,y], & & \{y,x\} = 0, \\
A_6/(xy-\mu) &\cong \mathcal{P}_3 = K[t,x^{\pm 1}], & & \{t,x\} = -fx.
\end{aligned}$$

Now, statement 2 follows from Theorem III.1.2.

3. It follows from diagram IV.1.4.

□

Class I.2.1.2: If $f \in K^{\times}$ then we have the Poisson algebra $A_7 = (K[t]; f\partial_t, -f\partial_t, 0, 0)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = -fx \quad \text{and} \quad \{y, x\} = 0. \tag{IV.1.13}$$

The next theorem gives us the classification of Poisson prime ideals of the Poisson algebra A_7.

Theorem IV.1.9. *Let $A_7 = (K[t]; f\partial_t, -f\partial_t, 0, 0)$ be the Poisson algebra as above, where $f \in K^{\times}$, i.e. $R_f = \emptyset$. Then*

1. $\mathrm{PSpec}(A_7) = \{0, (x), (y), (xy - \mu), (x, y), (xy - \mu, t - \nu), (x, y, t - \nu), (x - \lambda, y + \lambda^{-1}\mu, t - \nu) \mid \nu \in K$ *and* $\mu, \lambda \in K^{\times}\}$, *the containment information between Poisson prime ideals of A_7 is given in diagram IV.1.5.*

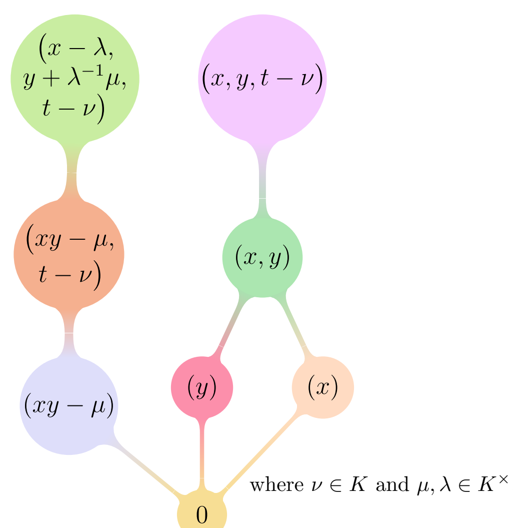


Diagram IV.1.5: The containment information between Poisson prime ideals of A_7

2. $\mathrm{PMax}(A_7) = \{(x, y, t - \nu), (x - \lambda, y + \lambda^{-1}\mu, t - \nu) \mid \nu \in K$ *and* $\mu, \lambda \in K^{\times}\}$.

Proof. 1. Since $f \in K^\times$, by Theorem IV.1.8.(1),

$$\mathrm{PSpec}(A_7) = \{0\} \cup \mathcal{V}(x) \cup \mathcal{V}(y) \cup \bigcup_{\mu \in K^\times} \mathcal{V}(xy - \mu).$$

Now, the theorem follows from Theorem III.1.2 since

$$\begin{aligned} A_7/(y) &\cong \quad \mathcal{P}(f), & \{t, x\} &= -fx, \\ A_7/(x) &\cong \quad \mathcal{P}'_1 = K[t, y], & \{t, y\} &= fy, \\ A_7/(xy - \mu) &\cong \quad \mathcal{P}'_3 = K[t, x^{\pm 1}], & \{t, x\} &= -fx. \end{aligned}$$

2. It follows from diagram IV.1.5.

□

The next is the Poisson algebra A_8 that belongs to the subclass I.2.2. This subclass has two subclasses which are dependent on the roots of f.

Class I.2.2:

If $c \in K^\times$, $\alpha = f\partial_t, \beta = -f\partial_t$ and $u = 0$, where $f \in K[t]\backslash\{0\}$ then we have the Poisson algebra $A_8 = (K[t]; f\partial_t, -f\partial_t, c, 0)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = -fx \quad \text{and} \quad \{y, x\} = cyx. \tag{IV.1.14}$$

There are two subclasses which are dependent on f:

Class I.2.2.1: If $f \in K[t]\backslash K$.

Class I.2.2.2: If $f \in K^\times$.

Class I.2.2.1: If $f \in K[t]\backslash K$ and $c \in K^\times$ then we have the Poisson algebra $A_9 = (K[t]; f\partial_t, -f\partial_t, c, 0)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = -fx \quad \text{and} \quad \{y, x\} = cyx. \tag{IV.1.15}$$

The next theorem classifies the Poisson prime ideals of the Poisson algebra A_9.

Theorem IV.1.10. *Let $A_9 = (K[t]; f\partial_t, -f\partial_t, c, 0)$ be the Poisson algebra as above, where $c \in K^\times$ and $f \in K[t]\backslash K$, i.e. $R_f \neq \emptyset$. Then*

1. $\mathrm{PSpec}(A_9) = \{0\} \cup \mathcal{V}(x) \cup \mathcal{V}(y) \cup \mathcal{V}(f)$.
2. $\mathrm{PSpec}(A_9) = \{0, (x), (y), (x, y), (t - \lambda_i), (y, t - \lambda_i), (x, t - \lambda_i), (x, y, t - \nu), (x, y - \mu, t - \lambda_i), (y, x - \mu, t - \lambda_i) \mid \nu \in K, \mu \in K^\times, \lambda_i \in R_f$ *and* $i = 1, \dots, s\}$, *the containment information between Poisson prime ideals of A_9 is given in diagram IV.1.6.*

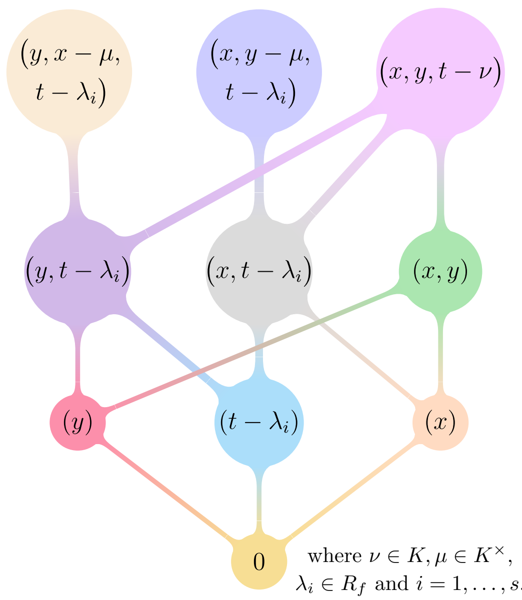


Diagram IV.1.6: The containment information between Poisson prime ideals of A_9

3. $\mathrm{PMax}(A_9) = \{(x, y, t - \nu), (y, x - \mu, t - \lambda_i), (x, y - \mu, t - \lambda_i) \mid \nu \in K, \mu \in K^\times, \lambda_i \in R_f$ *and* $i = 1, \dots, s\}$.
4. *the localization A_{9fxy} of the algebra A_9 at the powers of the element fxy is a simple Poisson algebra.*

Proof. 4. Suppose that I is a non-zero Poisson ideal of A_{9fxy}. We have to show that $I = A_{9fxy}$. Notice that, an ideal $I' := I \cap A_9$ is a non-zero Poisson ideal of the Poisson algebra A_9. Let $I_1 := I \cap K[t] = K[t]q$ be a non-zero Poisson ideal of the Poisson algebra

$K[t]$, where $q \in K[t]\backslash\{0\}$ which has no common root with f if $q \in K^\times$ then $q \in I$, and so $I = A_{9fxy}$, as required. Suppose that $q \in K[t]\backslash K$, we seek a contradiction. Notice that,

$$\delta_x := \text{pad}_x = fx\partial_t - cyx\partial_y,$$

$$\delta_y := \text{pad}_y = -fy\partial_t + cyx\partial_x,$$

$$\delta_t := \text{pad}_t = f(-x\partial_x + y\partial_y).$$

Therefore,

$$cf^{-1}\delta_t = -x^{-1}\delta_x - y^{-1}\delta_y$$

$$2f\partial_t = x^{-1}\delta_x - y^{-1}\delta_y + cy\partial_y + cx\partial_x$$

which implies that the ideal I and the algebra $K[t]$ are $f\partial_t$-invariant. Hence, I_1 and every minimal prime ideal of the ideal I_1 of $K[t]$, i.e. ideals $(t-\nu)$ are $f\partial_t$-invariant, where ν is a root of q. So, $f\partial_t(t-\nu) = f \in (t-\nu)$. Therefore, ν is a root of the polynomial f, a contradiction. Now, let $I_2 := I \cap \mathcal{P}_2$ be a non-zero Poisson ideal of the Poisson algebra $\mathcal{P}_2$, where $\{y, x\} = cyx$. In particular,

$$\partial_x = (cxy)^{-1}\delta_y,$$

$$\partial_y = -(cxy)^{-1}\delta_x.$$

Let $n := \min\{\deg_x(a) \mid a \in I_2\backslash\{0\}\}$, where $\deg_x(a)$ is the x-degree of the polynomial $a = a_0 + a_1(y)x + a_2(y)x^2 + \cdots + a_d(y)x^d$, i.e. $\deg_x(a) = d$ provided $a_d(y) \neq 0$, where $a_0(y), a_1(y), \ldots, a_d(y) \in K[y]$.

(i) Suppose that $n = 0$. Then $0 \neq I_2 \cap K[y] = K[y]p$ for non-zero polynomial $p \in K[y]$ if $\deg_y(p) = 0$, i.e. $p \in K^\times$, then p is a unit element this implies that $1 \in I_2 \subseteq I$ and so $I = A_{9fxy}$. If $\deg_y(p) \geq 1$ then the polynomial $\partial_y(p) = \frac{\partial p}{\partial y} \in I_2 \cap K[y]$ has degree $\deg_y(p) - 1$[6] which is not possible since $0 \neq \partial_y(p) \in K[y]p$. This means that the case $\deg_y(p) \geq 1$ is impossible.

(ii) Suppose that $n \geq 1$. Fix a polynomial, say $a = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, with

[6] since $\text{char}(K) = 0$

$\deg_x(a) = n$, where $a_0, a_1, \dots, a_n \in K[y]$ then the polynomial $\partial_x(a) = \frac{\partial a}{\partial x} \in I_2 \backslash \{0\}$ and $\deg_x\big(\partial_x(a)\big) = n-1$, a contradiction. Therefore, the case $n \geq 1$ is impossible.

1. By statement 4, the Poisson algebra $A_{9\,fxy}$ is simple. So, every non-zero Poisson ideal of A_9 contains an element $(fxy)^i$ for some $i \geq 1$. Therefore,

$$\mathrm{PSpec}(A_9) = \{0\} \cup \mathcal{V}(fxy) = \{0\} \cup \mathcal{V}(x) \cup \mathcal{V}(y) \cup \mathcal{V}(f). \tag{IV.1.16}$$

2. By statement 1,

$$\mathrm{PSpec}(A_9) = \{0\} \cup \mathcal{V}(x) \cup \mathcal{V}(y) \cup \bigcup_{i=1}^{s} (t - \lambda_i). \tag{IV.1.17}$$

Notice that, the following Poisson factor algebras are examples of the ones appeared in Theorem III.1.2

$$\begin{aligned} A_9/(x) &\cong \mathcal{P}_1, \\ A_9/(y) &\cong \mathcal{P}, \quad \{t, x\} = -fx, \\ A_9/(t-\lambda_i) &\cong \mathcal{P}_2. \end{aligned}$$

Now, statement 2 follows from Theorem III.1.2.

3. It follows from diagram IV.1.6.

□

Class I.2.2.2: If $f, c \in K^\times$ then we have the Poisson algebra $A_{10} = (K[t]; f\partial_t, -f\partial_t, c, 0)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = -fx \quad \text{and} \quad \{y, x\} = cyx. \tag{IV.1.18}$$

The following theorem describes the classification of Poisson prime ideals of the Poisson algebra A_{10}.

Theorem IV.1.11. *Let $A_{10} = (K[t]; f\partial_t, -f\partial_t, c, 0)$ be the Poisson algebra as above, where $c, f \in K^\times$ and $R_f = \emptyset$. Then*

1. $\text{PSpec}(A_{10}) = \{0, (x), (y), (x,y), (x,y,t-\nu) \mid \nu \in K\}$, *the containment information between Poisson prime ideals of* A_{10} *is given in diagram IV.1.7.*

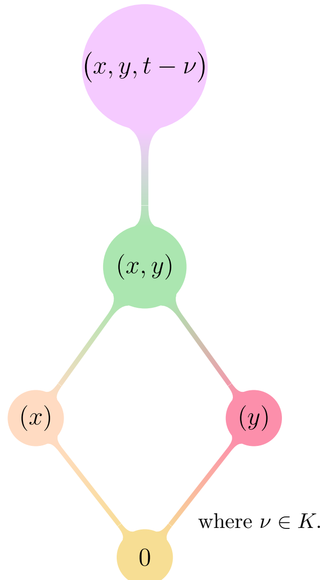


Diagram IV.1.7: The containment information between Poisson prime ideals of A_{10}

2. $\text{PMax}(A_{10}) = \{(x,y,t-\nu) \mid \nu \in K\}$.

Proof. 1. Since $f \in K^{\times}$, by Theorem IV.1.10.(1),

$$\text{PSpec}(A_{10}) = \{0\} \cup \mathcal{V}(x) \cup \mathcal{V}(y).$$

Now, statement 1 follows from Theorem III.1.2 since

$$\begin{aligned} A_{10}/(x) &\cong \mathcal{P}'_1, \\ A_{10}/(y) &\cong \mathcal{P}(f), \quad \{t,x\} = -fx. \end{aligned}$$

2. It follows from diagram IV.1.7.

□

Notes: From now, the numerical ordering for the third digit is changed to a letter instead of a number to simplify the notation.

§IV.2 The second class

The class II comprises two main subclasses: **II.1** and **II.2**, and each subclass has further subclasses. The aim of this section is to classify all Poisson prime ideals, minimal Poisson ideals and maximal Poisson ideals for Poisson algebras that belong to the class II.1 and certain subclasses of II.2.

Let us give more detail, the first subclass II.1 has four subclasses:

II.1a, II.1b, II.1c and II.1d.

The classifications of Poisson prime ideals for Poisson algebras that belong to these four subclasses are obtained in Theorem IV.2.3, Theorem IV.2.5, Theorem IV.2.6 and Theorem IV.2.7, respectively. In particular, each of these is treated individually and different techniques are involved. The main ideas to classify Poisson prime ideals for these algebras are the localization and factorization of Poisson algebras, and Lemma IV.2.2. These methods are similar to some techniques in the recent paper [Bav4], see the review in Subsection II.3.3 for detail. In addition, the inclusions of Poisson prime ideals for Poisson algebras that belong to the subclasses, II.1b, II.1c and II.1d, are given in diagram IV.2.2, diagram IV.2.3 and diagram IV.2.4, respectively.

The second subclass II.2 has eight subclasses:

II.2a, II.2b, II.2c, II.2d, II.2e, II.2f, II.2g and II.2k.

In addition, the Poisson prime ideals for Poisson algebras that belong to the subclasses, II.2a, II.2b and II.2c, are classified in Corollary IV.2.8, Corollary IV.2.9 and Theorem IV.2.11, respectively. Also, the classifications of Poisson prime ideals for Poisson algebras belong to special cases of the subclasses, II.2d, II.2g and II.2k, are obtained in Corollary Corollary IV.2.12, Corollary IV.2.13 and Corollary IV.2.14, respectively. Following that, the inclusions of Poisson prime ideals for Poisson algebras that belong to the subclasses, II.2d$'$ and II.2k$'$, are given in diagram IV.2.5 and diagram IV.2.6, respectively. However, it is difficult to classify Poisson prime ideals for the Poisson algebras that belong to the two subclasses, II.2e and II.2f.

Class II:

If $\alpha+\beta = f\partial_t + \lambda^{-1} f\partial_t = 0$, $u \in K[t]\backslash\{0\}$ and $c \in K$, where $\lambda \in K^\times$, $f \in K[t]$. Notice that, $f\partial_t + \lambda^{-1} f\partial_t = 0$ implies that there are two subclasses:

Class II.1: If $f = 0$.

Class II.2: If $\lambda = -1$.

Structure of the second class of Poisson algebras A is given in diagram IV.2.1.

The following is the first subclass of class II and consists of four subclasses. The classifications of Poisson prime ideals for Poisson algebras A_{12}, A_{13}, A_{14} and A_{15} that belong to these subclasses, II.1a, II.1b, II.1c and II.1d, respectively, are obtained. In addition, the primary key to classifying is Lemma IV.2.2, which can be considered the algebra A_{11} as a generalization of these algebras.

Class II.1

If $f = 0$, i.e. $\alpha = \beta = 0$, $u \in K[t]\backslash\{0\}$ and $c \in K$ then we have the Poisson algebra $A_{11} = (K[t]; 0, 0, c, u)$ with Poisson bracket defined by the rule

$$\{t, y\} = 0, \;\; \{t, x\} = 0 \;\text{ and }\; \{y, x\} = cyx + u. \qquad \text{(IV.2.1)}$$

There are four subclasses to consider:

Class II.1a: If $c = 0$ and $u \in K^\times$.

Class II.1b: If $c = 0$ and $u \in K[t]\backslash K$.

Class II.1c: If c and u are in $K^\times$.

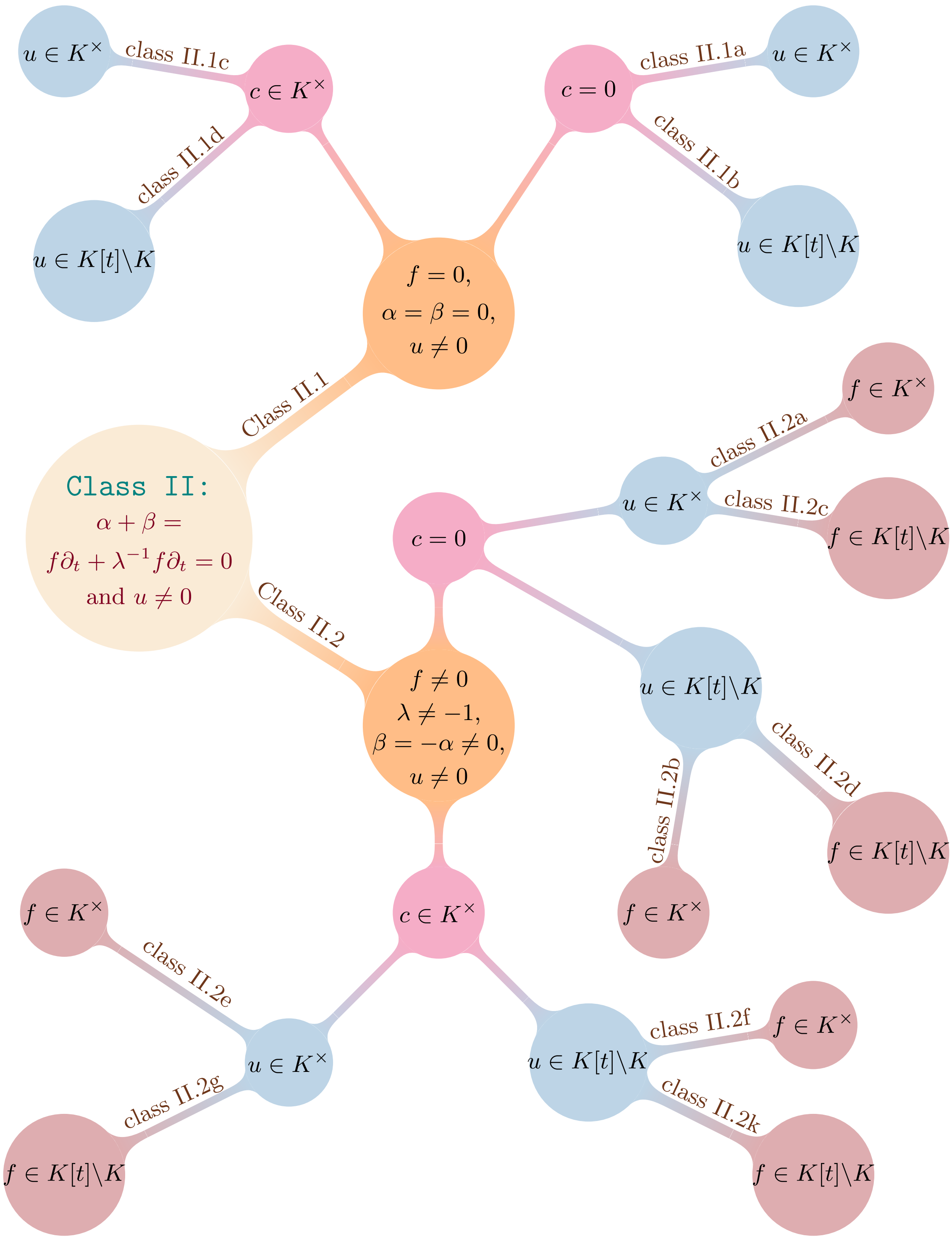


Diagram IV.2.1: Structure of the second class of Poisson algebras A

Class II.1d: If $c = 0$ and $u \in K[t]\backslash K$.

Remark IV.2.1. Clearly, $\rho := cyx + u \neq 0$, and so there is a chain of Poisson algebras

$$A_{11} \subset \mathcal{A} := K(t) \otimes K[x, y] \subset \mathcal{A}_\rho. \tag{IV.2.2}$$

The next lemma gives us the classification of Poisson prime ideals of the Poisson algebra A_{11}.

Lemma IV.2.2. *Let $A_{11} = (K[t]; 0, 0, c, u)$ be the Poisson algebra as above, where $u \in K[t]\backslash\{0\}$ and $c \in K$. Then*

1. *the Poisson algebra $\mathcal{A}_\rho$ is a simple Poisson algebra with $\mathrm{PZ}(\mathcal{A}_\rho) = K(t)$.*
2. *the algebra $P\mathcal{D}(\mathcal{A}_\rho)$ of Poisson differential operators is isomorphic to the localization of the Weyl algebra $K(t) \otimes A_2$ over the field $K(t)$ at the powers of the element ρ, $S_\rho^{-1}K(t) \otimes A_2$, where $S_\rho := \{\rho^i \mid i \geq 0\}$ and $A_2 = K\langle x, y, \partial_x, \partial_y\rangle \subseteq \mathrm{End}_K(\mathcal{A}_\rho)$. In particular, the algebra $P\mathcal{D}(\mathcal{A}_\rho)$ is a simple Noetherian domain of Gelfand-Kirillov dimension five.*
3. *if $\mathfrak{p} \in \mathrm{PSpec}(A_{11})\backslash\{0\}$ then either $t - \lambda \in \mathfrak{p}$, where $\lambda \in K$ or $\rho \in \mathfrak{p}$.*

Proof. 2. The algebra $P\mathcal{D}(\mathcal{A}_\rho)$ is a subalgebra of the algebra $\mathrm{End}_K(\mathcal{A}_\rho)$ that is generated by the algebra $\mathcal{A}_\rho$ and the derivations

$$\delta_x = \mathrm{pad}_x = -\rho\partial_y, \quad \delta_y = \mathrm{pad}_y = \rho\partial_x \quad \text{and} \quad \delta_t = \mathrm{pad}_t = 0.$$

Hence,

$$P\mathcal{D}(\mathcal{A}_\rho) \simeq S_\rho^{-1}K(t) \otimes A_2 = A_2(K(t))_\rho.$$

The Weyl algebra $A_2(K(t))$ over the field $K(t)$ is a simple Noetherian domain of Gelfand-Kirillov dimension five over K, hence so is its localization $P\mathcal{D}(\mathcal{A}_\rho)$.

1. By Theorem II.3.20, statement 1 follows from statement 2.

3. Statement 3 follows from statement 2.

□

The following is the Poisson algebra A_{12} that belongs to the subclass II.1a. Notice that, everything is zero, except u is a unit element in K. The technique to classify Poisson prime ideals of A_{12} is localizing A_{12} at u and writing A_{12} as a tensor product.

Class II.1a:

If $c = 0$ and $u \in K^{\times}$ then we have the Poisson algebra $A_{12} = (K[t]; 0, 0, 0, u)$ with Poisson bracket defined by the rule

$$\{t, y\} = 0, \quad \{t, x\} = 0 \quad \text{and} \quad \{y, x\} = u. \tag{IV.2.3}$$

The following theorem classifies the Poisson prime ideals of the Poisson algebra A_{12}.

Theorem IV.2.3. *Let $A_{12} = (K[t]; 0, 0, 0, u)$ be the Poisson algebra as above, where $u \in K^{\times}$. Then replacing the element y by $u^{-1}y$ we may assume that $u = 1$ and in this case the Poisson algebra $A_{12} = K[t] \otimes K[x, y]$ is a tensor product of the trivial Poisson algebra $K[t]$ and the simple Poisson algebra $K[x, y]$, where $\{y, x\} = 1$. Then*

1. $\mathrm{PSpec}(A_{12}) = \{\mathfrak{p} \otimes K[x, y] \mid \mathfrak{p} \in \mathrm{Spec}(K[t])\}$.
2. $\mathrm{PMax}(A_{12}) = \{\mathfrak{m} \otimes K[x, y] \mid \mathfrak{m} \in \mathrm{Max}(K[t])\}$.
3. $\mathrm{PZ}(A_{12}) = K[t]$.

Proof. We may assume that $u = 1$. Then the Poisson algebra $A_{12} = K[t] \otimes K[x, y]$ is a tensor product of the trivial Poisson algebra $K[t]$ and the simple Poisson algebra $K[x, y]$, where $\{y, x\} = 1$. Hence, $\mathrm{PZ}(A_{12}) = K[t]$ and

$$\mathrm{PSpec}(A_{12}) = \{\mathfrak{p} \otimes K[x, y] \mid \mathfrak{p} \in \mathrm{Spec}(K[t])\},$$

by Lemma IV.2.2[7]. Clearly,

$$\mathrm{PMax}(A_{12}) = \{\mathfrak{m} \otimes K[x, y] \mid \mathfrak{m} \in \mathrm{Max}(K[t])\}.$$

□

[7] since $\rho = u \in K^{\times}$

The following notation gives us the explicit formula of the roots of the polynomial u.

Notation IV.2.4. Let $R_u = \{\lambda_1, \dots, \lambda_n\}$ be the set of distinct roots of the polynomial

$$u(t) = \prod_{i=1}^{n} \lambda_u (t - \lambda_i)^{r_i}, \tag{IV.2.4}$$

where λ_u is the leading coefficient of u and $r_1, \dots, r_n \geq 1$ are the multiplicities of the roots $\lambda_1, \dots, \lambda_n$, respectively.

The following is the Poisson algebra A_{13} that belongs to the subclass II.1b. Notice that, c is zero and $u \in K[t]\backslash K$. The technique to classify Poisson prime ideals of A_{13} is similar to the techniques used in the algebra A_{12}, whereas u is a polynomial in t.

Class II.1b:

If $c = 0$ and $u \in K[t]\backslash K$ then we have the Poisson algebra $A_{13} = (K[t]; 0, 0, 0, u)$ with Poisson bracket defined by the rule

$$\{t, y\} = 0, \quad \{t, x\} = 0 \quad \text{and} \quad \{y, x\} = u. \tag{IV.2.5}$$

Let A_{13u} be the localization of the algebra A_{13} at the powers of the element u.

The next theorem describes the Poisson prime ideals of the Poisson algebra A_{13}.

Theorem IV.2.5. *Let $A_{13} = (K[t]; 0, 0, 0, u)$ be the Poisson algebra and A_{13u} be as above, where $u \in K[t]\backslash K$, i.e. $R_u \neq \emptyset$. Then*

1. *by replacing the element y by $u^{-1}y$, we may assume that $u = 1$ in the Poisson algebra A_{13u} and in this case the Poisson algebra $A_{13u} = K[t]_u \otimes K[x, y]$ is a tensor product of the trivial Poisson algebra $K[t]_u$ and the simple Poisson algebra $K[x, y]$, where $\{y, x\} = 1$,*

   (i) $\mathrm{PZ}(A_{13u}) = K[t]_u$.

   (ii) $\mathrm{PSpec}(A_{13u}) = \{\mathfrak{p} \otimes K[x, y] \mid \mathfrak{p} \in \mathrm{Spec}(K[t]_u)\}$.

2. $\mathrm{PSpec}(A_{13}) = \{0, (t-\nu), (\mathfrak{p}, t-\lambda_i) \mid \nu \in K, \lambda_i \in R_u, i = 1, \ldots, n$ *and* $\mathfrak{p} \in \mathrm{Spec}(K[x,y]) \backslash \{0\}\}$, *the containment information between Poisson prime ideals of* A_{13} *is given in diagram IV.2.2.*

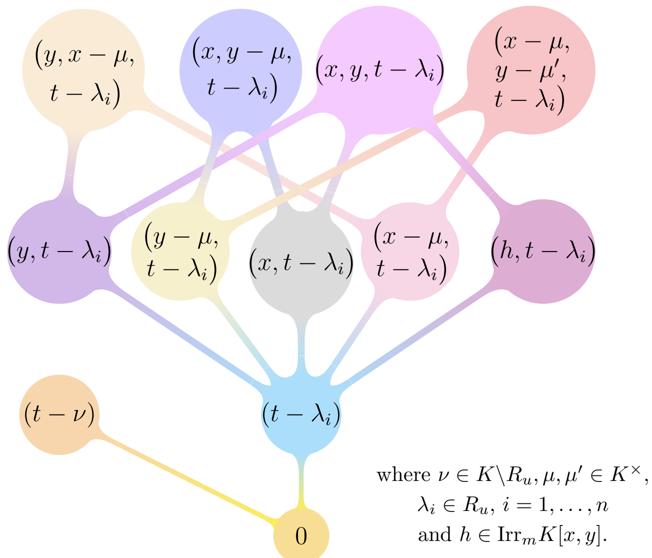


Diagram IV.2.2: The containment information between Poisson prime ideals of A_{13}

3. $\mathrm{PMax}(A_{13}) = \{(t-\nu), (\mathfrak{m}, t-\lambda_i) \mid \nu \in K\backslash R_u, \lambda_i \in R_u, i = 1, \ldots, n$ *and* $\mathfrak{m} \in \mathrm{Max}(K[x,y]) \backslash \{0\}\}$.

Proof. 1. We may assume that $u = 1$ for the algebra A_{13u}. Then the Poisson algebra $A_{13u} = K[t]_u \otimes K[x,y]$ is a tensor product of the trivial Poisson algebra $K[t]_u$ and the simple Poisson algebra $K[x,y]$, where $\{y,x\} = 1$. Hence, $\mathrm{PZ}(A_{13u}) = K[t]_u$ and

$$\mathrm{PSpec}(A_{13u}) = \{\mathfrak{p} \otimes K[x,y] \mid \mathfrak{p} \in \mathrm{Spec}(K[t]_u)\},$$

by Lemma IV.2.2.(3), since $\rho = u \in K[t]_u$.

2. For all elements $\nu \in K, (t-\nu) \in \mathrm{PSpec}(A_{13})$. For every $i = 1, \ldots, n$, the Poisson algebra $A_{13}/(t-\lambda_i) \simeq \mathcal{P}_0$ has trivial Poisson bracket, and statement 2 follows from Lemma IV.2.2.(3).

3. It follows from diagram IV.2.2.

□

The following is the Poisson algebra A_{14} that belongs to the subclass II.1c. Notice that, c and u are non-zero elements in K. The technique to classify Poisson prime ideals of A_{14} is factoring A_{14} by the irreducible polynomial ρ, and writing A_{14} as a tensor product.

Class II.1c:

If c and u are in $K^\times$ then we have the Poisson algebra $A_{14} = (K[t]; 0, 0, c, u)$ with Poisson bracket defined by the rule

$$\{t, y\} = 0, \quad \{t, x\} = 0 \quad \text{and} \quad \{y, x\} = cyx + u. \tag{IV.2.6}$$

It follows that the element $\rho = cyx + u$ is an irreducible polynomial in A_{14}.

The next theorem gives us the classification of Poisson prime ideals of the Poisson algebra A_{14}.

Theorem IV.2.6. *Let $A_{14} = (K[t]; 0, 0, c, u)$ be the Poisson algebra as above, where u and c are in $K^\times$. Then*

1. *the Poisson algebra $A_{14} = K[t] \otimes K[x, y]$ is a tensor product of the trivial Poisson algebra $K[t]$ and the Poisson algebra $K[x, y]$ with $\{y, x\} = \rho$.*

2. $\mathrm{PSpec}(A_{14}) = \big\{0, (t-\nu), (\rho), (\rho, t-\nu), \big(t-\nu, x-\mu, y-(c\mu)^{-1}u\big) \mid \nu \in K \text{ and } \mu \in K^\times\big\}$, *the containment information between Poisson prime ideals of A_{14} is given in diagram IV.2.3.*

3. $\mathrm{PMax}(A_{14}) = \big\{\big(t-\nu, x-\mu, y-(c\mu)^{-1}u\big) \mid \nu \in K \text{ and } \mu \in K^\times\big\}$.

Proof. 1. Clearly, the Poisson algebra $A_{14} = K[t] \otimes K[x, y]$ is a tensor product of the trivial Poisson algebra $K[t]$ and the Poisson algebra $\mathcal{P}_4 = K[x, y]$ with $\{y, x\} = cyx + u = \rho$, where u and c are in $K^\times$. Notice that, the ideal (ρ) is a prime Poisson ideal of the Poisson algebra A_{14} and the factor algebra

$$A_{14}/(\rho) \simeq K[t] \otimes \mathcal{P}_4/(\rho)$$

is a tensor product of trivial Poisson algebras $K[t]$ and $\mathcal{P}_4/(\rho)$.

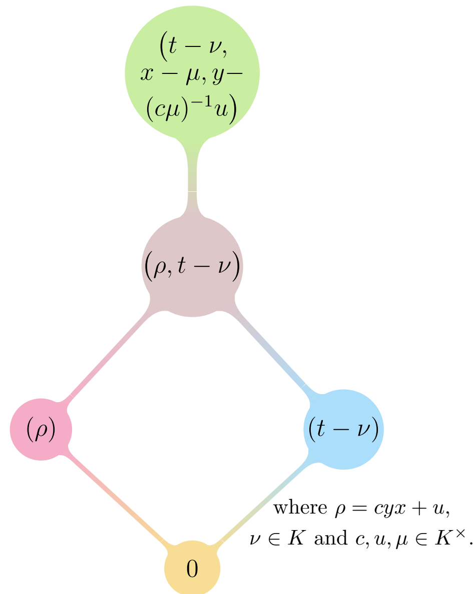


Diagram IV.2.3: The containment information between Poisson prime ideals of A_{14}

2. Now, statement 2 follows from Lemma IV.2.2.(3).

3. It follows from diagram IV.2.3.

□

The following is the Poisson algebra A_{15} that belongs to the subclass II.1d. Notice that, c is a non-zero element in K and $u \in K[t]\backslash K$. The technique to classify Poisson prime ideals of A_{15} is similar to the techniques used in the algebra A_{14}, whereas u is a polynomial in t.

Class II.1d:

If $c \in K^{\times}$ and $u \in K[t]\backslash K$ then we have the Poisson algebra $A_{15} = (K[t]; 0, 0, c, u)$ with Poisson bracket defined by the rule

$$\{t, y\} = 0, \quad \{t, x\} = 0 \quad \text{and} \quad \{y, x\} = cyx + u. \tag{IV.2.7}$$

It follows that the element $\rho = cyx + u$ is an irreducible polynomial in A_{15}.

The following theorem shows that the classification of Poisson prime ideals of the Poisson algebra A_{15}.

Theorem IV.2.7. *Let* $A_{15} = (K[t]; 0, 0, c, u)$ *be the Poisson algebra as above, where* $c \in K^\times$ *and* $u \in K[t]\backslash K$, *i.e.* $R_u \neq \emptyset$. *Then*

1. $\mathrm{PSpec}(A_{15}) = \big\{0, (t-\nu), (\rho), (\rho, t-\omega), (x, t-\lambda_i), (y, t-\lambda_i), \mathfrak{p}, (x, y, t-\lambda_i), (x, y-\mu, t-\lambda_i), (y, x-\mu, t-\lambda_i), \big(t-\omega, x-\mu, y-(c\mu)^{-1}u(\omega)\big) \mid \nu \in K, \mu \in K^\times, \lambda_i \in R_u, \omega \in K\backslash R_u, i = 1, \dots, n, \mathfrak{p} \in \mathrm{Spec}(A_{15}), \rho \in \mathfrak{p}$ *and* $\mathrm{ht}(\mathfrak{p}) = 2\big\}$, *the containment information between Poisson prime ideals of* A_{15} *is given in diagram IV.2.4.*

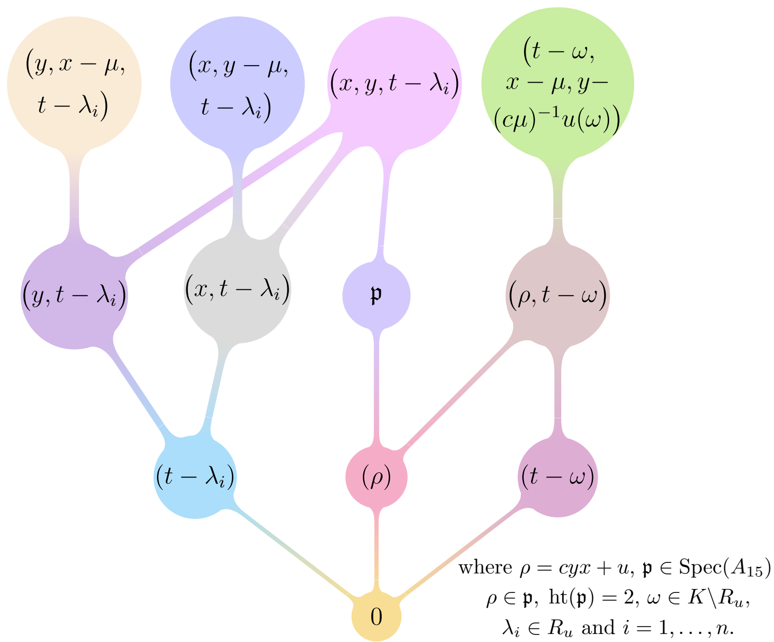


Diagram IV.2.4: The containment information between Poisson prime ideals of A_{15}

2. $\mathrm{PMax}(A_{15}) = \big\{(x, y, t-\lambda_i), (x, y-\mu, t-\lambda_i), (y, x-\mu, t-\lambda_i), \big(t-\omega, x-\mu, y-(c\mu)^{-1}u(\omega)\big) \mid \mu \in K^\times, \lambda_i \in R_u, \omega \in K\backslash R_u$ *and* $i = 1, \dots, n\big\}$.

Proof. 1. Notice that, $(t-\nu)$, where $\nu \in K$ and (ρ) are Poisson prime ideals of the Poisson algebra A_{15}. By Lemma IV.2.2.(3), each non-zero Poisson prime ideal $\mathfrak{q}$ of A_{15} contains one of the above ideals.

(i) Suppose that $t-\omega \in \mathfrak{q}$ and $(t-\omega) \neq \mathfrak{q}$, where $\omega \in K\backslash R_u$. Then the Poisson algebra

$$A_{15}/(t-\omega) \cong \mathcal{P}_4' = K[x, y]$$

admits the Poisson bracket $\{y,x\} = cyx + u(\omega)$, where $u(\omega), c \in K^{\times}$. Then by Theorem IV.2.6.(1),

$$\mathrm{PSpec}(A_{15}) \supseteq \big\{(\rho, t-\omega), \big(t-\omega, x-\mu, y-(c\mu)^{-1}u(\omega)\big) \mid \omega \in K\backslash R_u,\ \mu \in K^{\times}\big\}.$$

(ii) Suppose that $t-\lambda_i \in \mathfrak{q}$ and $(t-\lambda_i) \neq \mathfrak{q}$ for some $\lambda_i \in R_u$ and $i = 1, \ldots, n$. Then the Poisson algebra

$$A_{15}/(t-\lambda_i) \cong \mathcal{P}_2.$$

Then, $\mathrm{PSpec}(A_{15}) \supseteq \big\{(x, t-\lambda_i), (y, t-\lambda_i), (x, y, t-\lambda_i), (x, t-\lambda_i, y-\mu), (y, t-\lambda_i, x-\mu) \mid \mu \in K^{\times},\ \lambda_i \in R_u \text{ and } i = 1, \ldots, n\big\}$.

(iii) Suppose that $t-\nu \notin \mathfrak{q}$ for all $\nu \in K$. Then necessarily $\rho \in \mathfrak{q}$, by Lemma IV.2.2.(3). The Poisson algebra $A_{15}/(\rho)$ has trivial Poisson bracket, and so

$$\mathrm{PSpec}\big(A_{15}/(\rho)\big) = \mathrm{Spec}\big(A_{15}/(\rho)\big).$$

Hence, the set $\big\{\mathfrak{p} \in \mathrm{Spec}(A_{15}) \mid \rho \in \mathfrak{p} \text{ and } \mathrm{ht}(\mathfrak{p}) = 2\big\}$ contains precisely all the Poisson prime ideals of the Poisson algebra A_{15} that properly contain the ideal (ρ) and do not meet the algebra $K[t]$.

1. Now, statement 1 follows from statements (i)–(iii).

2. It follows from diagram IV.2.4.

□

The following is the second subclass of class II and contains eight subclasses. The classifications of Poisson prime ideals for the Poisson algebras A_{17}, A_{18} and A_{20} that belong to the subclasses, II.2a, II.2b and II.2c, respectively, are obtained. In addition, the classifications of Poisson prime ideals for the Poisson algebras A_{22}, A_{26} and A_{28} that belong to special cases of the subclasses, II.2d, II.2g and II.2k, respectively, are obtained. However, the Poisson prime ideals of Poisson algebras that belong to the two subclasses, II.2e and II.2f, cannot be classified.

Class II.2

If $\lambda = -1$, i.e. $\beta = -\alpha = -f\partial_t$ and $f, u \in K[t]\backslash\{0\}$ then we have the Poisson algebra $A_{16} = (K[t]; f\partial_t, -f\partial_t, c, u)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = -fx \text{ and } \{y, x\} = cyx + u. \tag{IV.2.8}$$

There are eight subclasses to consider:

Class II.2a: If $c = 0$, f and u are in $K^\times$.

Class II.2b: If $c = 0$, $f \in K^\times$ and $u \in K[t]\backslash K$.

Class II.2c: If $c = 0$, $f \in K[t]\backslash K$ and $u \in K^\times$.

Class II.2d: If $c = 0$, f and u are in $K[t]\backslash K$.

Class II.2e: If c, f and u are in $K^\times$.

Class II.2f: If $c, f \in K^\times$ and $u \in K[t]\backslash K$.

Class II.2g: If $c \in K^\times$, $f \in K[t]\backslash K$ and $u \in K^\times$.

Class II.2k: If $c \in K^\times$, f and u are in $K[t]\backslash K$.

The following is the Poisson algebra A_{17} that belongs to the subclass II.2a. Notice that, c is zero, u and f are unit elements in K. The technique to classify Poisson prime ideals of A_{17} is similar to the technique in the paper [JoOh], which identifies the non-constant polynomial a in the algebra A_{17} to turn the Poisson bracket into exact. From their study, in particular, Corollary II.2.73 the classification of Poisson prime ideals for A_{17} follows.

Class II.2a:

If $\alpha = f\partial_t, \beta = -f\partial_t$, $c = 0$ and $f, u \in K^\times$ then we have the Poisson algebra $A_{17} = (K[t]; f\partial_t, -f\partial_t, 0, u)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{x, t\} = fx \quad \text{and} \quad \{y, x\} = u. \tag{IV.2.9}$$

The following corollary shows that the classification of Poisson prime ideals for the Poisson algebra A_{17}.

Corollary IV.2.8. *Let* $A_{17} = (K[t]; f\partial_t, -f\partial_t, 0, u)$ *be the Poisson algebra as above and* $a = fxy + ut$*, where* $f, u \in K^{\times}$*. Then* $\text{PSpec}(A_{17}) = \{0, (a-\mu) \mid \mu \in K\}$.

Proof. Suppose that $a = fxy + ut$, where $f, u \in K^{\times}$ then the Poisson bracket in (IV.2.9) is exact, and $a - \mu$ is irreducible for all $\mu \in K$. It follows from Corollary II.2.73 that the Poisson spectrum of A_{17} consists of 0 and $(a - \mu)$, where $\mu \in K$. □

The following is the Poisson algebra A_{18} that belongs to the subclass II.2b. Notice that, c is zero, f is a unit element in K, and u is a polynomial in t. The technique to classify Poisson prime ideals of A_{18} is similar to the technique used in the algebra A_{17}, which identifies the non-constant polynomial a in the algebra A_{18} to turn the Poisson bracket into exact. From the study [JoOh], in particular, Corollary II.2.73 the classification of Poisson prime ideals for A_{18} follows.

Class II.2b:

If $\alpha = f\partial_t, \beta = -f\partial_t, c = 0$ and $u \in K[t] \backslash K$, where $f \in K^{\times}$ then we have the Poisson algebra $A_{18} = (K[t]; f\partial_t, -f\partial_t, 0, u)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = -fx \quad \text{and} \quad \{y, x\} = u. \tag{IV.2.10}$$

The following corollary shows that the classification of Poisson prime ideals of the Poisson algebra A_{18}.

Corollary IV.2.9. *Let* $A_{18} = (K[t]; f\partial_t, -f\partial_t, 0, u)$ *be the Poisson algebra as above and* $a = fxy + \prod_{i=1}^{n} \frac{\lambda_u}{(r_i+1)}(t-\lambda_i)^{r_i+1}$*, where* $f \in K^{\times}$*. Then* $\text{PSpec}(A_{18}) = \{0, (a-\mu), (x, y, t-\lambda_i) \mid \mu \in K, \lambda_i \in R_u \text{ and } i = 1, \dots, n\}$.

Proof. Let us assume that u is (IV.2.4) and $a = fxy + \prod_{i=1}^{n} \frac{\lambda_u}{(r_i+1)}(t-\lambda_i)^{r_i+1}$ then the Poisson bracket in (IV.2.10) is exact, and $a - \mu$ is irreducible for all $\mu \in K$. It follows from Corollary II.2.73 that the Poisson spectrum of A_{18} consists of 0, $(a - \mu)$ and $(x, y, t - \lambda_i)$, where $\mu \in K$, $\lambda_i \in R_u$ and $i = 1, \dots, n$. □

The following is a special case of the subclass II.2b.

Example IV.2.10. Let $\alpha = f\partial_t, \beta = -f\partial_t$, $c = 0$ and $u = vt$, where $f, v \in K^\times$ then we have the Poisson algebra $A_{19} = (K[t]; f\partial_t, -f\partial_t, 0, vt)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{x, t\} = fx \quad \text{and} \quad \{y, x\} = vt. \tag{IV.2.11}$$

Suppose that $a = fxy + \frac{1}{2}vt^2$, where $f, v \in K^\times$ then the Poisson bracket in (IV.2.11) is exact, and $a - \mu$ is irreducible for all $\mu \in K$. It follows from Corollary II.2.73 that the Poisson spectrum of A_{19} consists of 0, $(a - \mu)$ and (x, y, t), where $\mu \in K$.

The following is the Poisson algebra A_{20} that belongs to the subclass II.2c. Notice that, c is zero, u is a unit element in K, and f is a polynomial in t. The techniques to classify Poisson prime ideals of A_{20} have some properties of factor Poisson algebras.

Class II.2c:

If $\alpha = f\partial_t, \beta = -f\partial_t$, $c = 0$ and $u \in K^\times$, where $f \in K[t]\backslash K$ then we have the Poisson algebra $A_{20} = (K[t]; f\partial_t, -f\partial_t, 0, u)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = -fx \quad \text{and} \quad \{y, x\} = u. \tag{IV.2.12}$$

The following theorem shows that the classification of Poisson prime ideals for the Poisson algebra A_{20}.

Theorem IV.2.11. *Let $A_{20} = (K[t]; f\partial_t, -f\partial_t, 0, u)$ be the Poisson algebra as above, where $u \in K^\times$ and $f \in K[t]\backslash K$, i.e. $R_f \neq \emptyset$. Then $\mathrm{PSpec}(A_{20}) = \{0, (t - \lambda_i) \mid \lambda_i \in R_f$ and $i = 1, \ldots, s\}$.*

Proof. For some element $\lambda_i \in R_f$, where $i = 1, \ldots, s$ we have

$$A_{20}/(t - \lambda_i) \cong K[t]/(t - \lambda_i) \otimes K[x, y]$$

is a tensor product of $K[t]/(t - \lambda_i)$ and the simple Poisson algebra $\mathcal{P}(u) = K[x, y]$ with $\{y, x\} = u$, where $u \in K^\times$. Notice that, the localization A_{20_f} of algebra A_{20} is a simple

Poisson algebra. Hence, the Poisson spectrum of A_{20} consists of 0 and $(t-\lambda_i)$, where $\lambda_i \in R_f$ and $i = 1, \dots, s$. □

Class II.2d:

If $\alpha = f\partial_t, \beta = -f\partial_t$ and $c = 0$, where $f, u \in K[t]\backslash K$ then we have the Poisson algebra $A_{21} = (K[t]; f\partial_t, -f\partial_t, 0, u)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = -fx \quad \text{and} \quad \{y, x\} = u. \tag{IV.2.13}$$

The following is a special case of the subclass II.2d. In this Poisson algebra A_{22} that belongs to the subclass II.2d′, we suppose that the non-constant polynomial f divides u, and c is zero. Thus, the technique to classify Poisson prime ideals of A_{22} is similar to the technique in the paper [JoOh], which is writing the bracket as A_{22}-multiple of other. From their study, in particular, Example II.2.76 and Example II.2.77 the classification of Poisson prime ideals of A_{22} follows.

Class II.2d′:

If $u = fg$, $\alpha = f\partial_t, \beta = -f\partial_t$ and $c = 0$, where $f, g \in K[t]\backslash K$ then we have the Poisson algebra $A_{22} = (K[t]; f\partial_t, -f\partial_t, 0, fg)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = -fx \quad \text{and} \quad \{y, x\} = fg. \tag{IV.2.14}$$

The following corollary gives us the classification of Poisson prime ideals of the Poisson algebra A_{22}.

Corollary IV.2.12. *Let $A_{22} = (K[t]; f\partial_t, -f\partial_t, 0, fg)$ be the Poisson algebra as above and $a = xy + \int g$, where $f, g \in K[t]\backslash K$, i.e. $R_f \neq \emptyset$ and $R_g \neq \emptyset$. Then*

1. $\text{PSpec}(A_{22}) = \big\{0, (t-\lambda_i), (a), (a-\mu), (x, t-\lambda_i), (y, t-\lambda_i), (x-\mu', t-\lambda_i), (y-\mu', t-\lambda_i), (xy, t-\omega), (xy-\mu', t-\omega), (h, t-\lambda_i), (x, y, t-\lambda_i), (x, y-\mu', t-\lambda_i), (y, x-\mu', t-\lambda_i), (x-\mu', y-\nu, t-\lambda_i), (x, y, t-\omega) \mid \nu, \mu, \mu' \in K^{\times}, \lambda_i \in R_f, \omega \in R_g$ *and* $h \in \text{Irr}_m K[x, y]\big\}$,

the containment information between Poisson prime ideals of A_{22} is given in diagram IV.2.5.

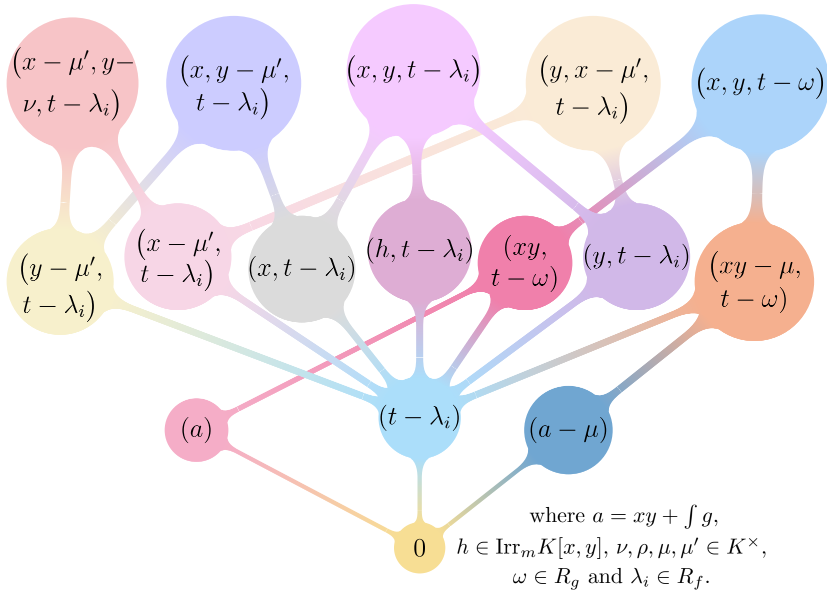


Diagram IV.2.5: The containment information between Poisson prime ideals of A_{22}

2. $\mathrm{PMax}(A_{22}) = \{(x, y, t-\lambda_i), (x, y-\mu', t-\lambda_i), (y, x-\mu', t-\lambda_i), (x-\mu', y-\nu, t-\lambda_i), (x, y, t-\omega) \mid \nu, \mu, \mu' \in K^{\times},\ \lambda_i \in R_f$ *and* $\omega \in R_g\}$.

Proof. 1. Suppose that

$$\mathcal{B}_1 := \{t, y\} = y, \quad \{x, t\} = x \quad \text{and} \quad \{y, x\} = g$$

and (IV.2.14) by $\mathcal{B}_2$ then we have $\mathcal{B}_2 = f\mathcal{B}_1$. Notice that $\mathcal{B}_1$ is exact, where $a = xy + \int g$ and it follows from Corollary IV.2.1 that the Poisson spectrum of A_{22} under $\mathcal{B}_1$ is 0, $(a-\mu)$ and $(x, y, t-\omega)$, where $\mu \in K$, $\omega \in R_g$. Hence, by using Example II.2.77 we have that any Poisson prime ideal of A_{22} under $\mathcal{B}_1$ is a Poisson prime ideal of A_{22} under $\mathcal{B}_2$, but if Q is a Poisson prime ideal of A_{22} under $\mathcal{B}_2$ then Q is a Poisson prime ideal of A_{22} under $\mathcal{B}_1$ or Q contains f. Therefore, the Poisson spectrum of A_{22} consists of $0, (t-\lambda_i), (a), (a-\mu), (x, t-\lambda_i), (y, t-\lambda_i), (x-\mu', t-\lambda_i), (y-\mu', t-\lambda_i), (xy, t-\omega)$, $(xy-\mu', t-\omega), (h, t-\lambda_i), (x, y, t-\lambda_i), (x, y-\mu', t-\lambda_i), (y, x-\mu', t-\lambda_i), (x-\mu', y-\nu, t-\lambda_i)$ and $(x, y, t-\omega)$, where $\nu, \mu, \mu' \in K^{\times}$, $\lambda_i \in R_f$, $\omega \in R_g$ and $h \in \mathrm{Irr}_m K[x, y]$.

2. It follows from diagram IV.2.5.

□

The following is the Poisson algebra A_{23} that belongs to the subclass II.2e. Notice that, c, u and f are unit elements in K. The main issue is that there is no known approach or technique to classify Poisson prime ideals of A_{23}, and it is difficult to find the common eigenvectors for δ_t, δ_x and δ_y on A_{23}.

Class II.2e:

If $\alpha = f\partial_t$, $\beta = -f\partial_t$, where $f, u, c \in K^\times$ then we have the Poisson algebra $A_{23} = (K[t]; f\partial_t, -f\partial_t, c, u)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = -fx \quad \text{and} \quad \{y, x\} = cyx + u. \tag{IV.2.15}$$

The following is the Poisson algebra A_{24} that belongs to the subclass II.2f. Notice that, c and f are unit elements in K, and u is a non-constant polynomial in t. This seems more complex than the algebra A_{23}. Also, the main issue is that there is no known approach or technique to classify Poisson prime ideals of A_{24}, and it is difficult to find the common eigenvectors for δ_t, δ_x and δ_y on A_{24}.

Class II.2f:

If $\alpha = f\partial_t$, $\beta = -f\partial_t$ and $u \in K[t]\backslash K$, where $f, c \in K^\times$ then we have the Poisson algebra $A_{24} = (K[t]; f\partial_t, -f\partial_t, c, u)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = -fx \quad \text{and} \quad \{y, x\} = cyx + u. \tag{IV.2.16}$$

Class II.2g:

If $\alpha = f\partial_t$, $\beta = -f\partial_t$ and $u, c \in K^\times$, where $f \in K[t]\backslash K$ then we have the Poisson algebra $A_{25} = (K[t]; f\partial_t, -f\partial_t, c, u)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = -fx \quad \text{and} \quad \{y, x\} = cyx + u. \tag{IV.2.17}$$

The following is a special case of the subclass II.2g. In this Poisson algebra A_{26} that belongs to the subclass II.2g$'$, we assume that f is a monomial of degree one, c and u are unit elements in K. Therefore, the technique to classify Poisson prime ideals of A_{26} is similar to the technique used in algebra A_{17}, which identifies the non-constant polynomial a in the algebra A_{26} to turn the Poisson bracket into exact. From the study [JoOh], in particular, Corollary II.2.73 the classification of Poisson prime ideals of A_{26} follows.

Class II.2g$'$:

If $f = \rho t$, i.e. $\alpha = \rho t\partial_t$, $\beta = -\rho t\partial_t$ and $\rho = c$, where $u, c \in K^\times$ then we have the Poisson algebra $A_{26} = (K[t]; ct\partial_t, -ct\partial_t, c, u)$ with Poisson bracket defined by the rule

$$\{t, y\} = cty, \quad \{x, t\} = ctx \quad \text{and} \quad \{y, x\} = cyx + u. \tag{IV.2.18}$$

The following corollary shows that the classification of Poisson prime ideals of the Poisson algebra A_{26}.

Corollary IV.2.13. *Let $A_{26} = (K[t]; ct\partial_t, -ct\partial_t, c, u)$ be the Poisson algebra as above and $a = (cyx+u)t$, where $u, c \in K^\times$. Then $\mathrm{PSpec}(A_{26}) = \big\{0, (a-\mu), \big(t, x-\lambda, y-(c\lambda)^{-1}u\big) \mid \mu \in K$ and $\lambda \in K^\times\big\}$.*

Proof. Suppose that $a = (cyx + u)t$, where $u, c \in K^\times$ then the Poisson bracket (IV.2.18) is exact, and $a - \mu$ is irreducible for all $\mu \in K^\times$. It follows from Corollary II.2.73 that the Poisson spectrum of A_{26} consists of 0, $(a - \mu)$ and $\big(t, x - \lambda, y - (c\lambda)^{-1}u\big)$, where $\mu \in K$ and $\lambda \in K^\times$. □

Class II.2k:

If $\alpha = f\partial_t, \beta = -f\partial_t$, where $f, u \in K[t]\backslash K$ and $c \in K^\times$ then we have the Poisson algebra $A_{27} = (K[t]; f\partial_t, -f\partial_t, c, u)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = -fx \quad \text{and} \quad \{y, x\} = cyx + u. \tag{IV.2.19}$$

The following is a special case of the subclass II.2k. In this Poisson algebra A_{28} that belongs to the subclass II.2k$'$, we assume that f and u are monomials of degree one, and c is a unit element in K. Therefore, the technique to classify Poisson prime ideals of A_{28} is similar to the technique used in some previous algebras, which identifies the non-constant polynomial a in the algebra A_{28} to turn the Poisson bracket into exact. From the study [JoOh], in particular, Corollary II.2.73 the classification of Poisson prime ideals for A_{28} follows.

Class II.2k$'$:

If $f = \rho t$, i.e. $\alpha = \rho t\partial_t, \beta = -\rho t\partial_t$, $u = vt$ and $\rho = c$, where $c, v \in K^\times$ then we have the Poisson algebra $A_{28} = (K[t]; ct\partial_t, -ct\partial_t, c, vt)$ with Poisson bracket defined by the rule

$$\{t, y\} = cty, \quad \{x, t\} = ctx \quad \text{and} \quad \{y, x\} = cyx + vt. \tag{IV.2.20}$$

The following corollary gives us the classification of Poisson prime ideals of the Poisson algebra A_{28}.

Corollary IV.2.14. *Let $A_{28} = (K[t]; ct\partial_t, -ct\partial_t, c, vt)$ be the Poisson algebra as above and $a = (cyx + \frac{1}{2}vt)t$, where $c, v \in K^\times$. Then*

1. $\mathrm{PSpec}(A_{28}) = \{0, (a), (t), (a-\mu), (y,t), (x,t), (t, xy-\omega), (t, xy), (x,y,t), (x,t,y-\mu'), (y,t,x-\mu'), (t, x-\lambda, y+\lambda^{-1}\omega) \mid \lambda, \omega, \mu, \mu' \in K^\times\}$, *the containment information between Poisson prime ideals of A_{28} is given in diagram IV.2.6.*

2. $\mathrm{PMax}(A_{28}) = \{(x,y,t), (x,t,y-\mu'), (y,t,x-\mu'), (t, x-\lambda, y+\lambda^{-1}\omega) \mid \lambda, \omega, \mu' \in K^\times\}$.

Proof.

1. Suppose that $a = (cyx + \frac{1}{2}vt)t$, where $c, v \in K^\times$ then the Poisson bracket (IV.2.20) is exact, and $a - \mu$ is irreducible for all $\mu \in K^\times$. It follows from Corollary II.2.73 that the Poisson spectrum of A_{28} consists of
$0, (a), (t), (a-\mu), (y,t), (x,t), (t, xy-\omega), (t, xy), (x,y,t), (x,t,y-\mu'), (y,t,x-\mu')$ and $(t, x-\lambda, y+\lambda^{-1}\omega)$, where $\lambda, \omega, \mu, \mu' \in K^\times$.

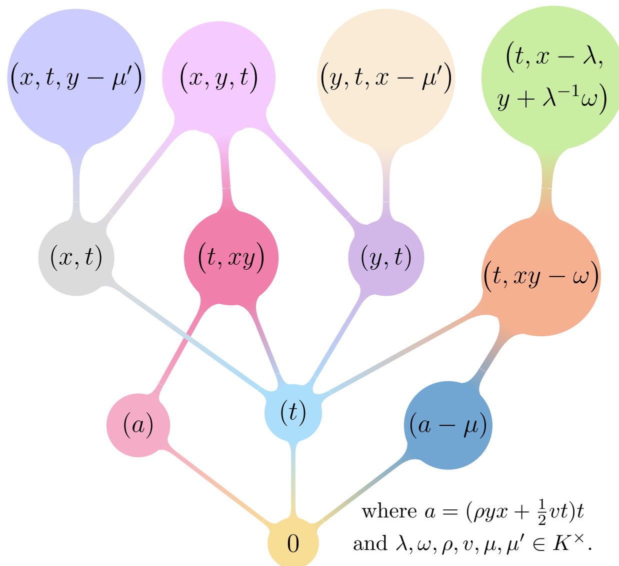


Diagram IV.2.6: The containment information between Poisson prime ideals of A_{28}

2. It follows from diagram IV.2.6. □

§IV.3 The third class

The aim of this section is to classify all Poisson prime ideals, minimal Poisson ideals and maximal Poisson ideals of Poisson algebras that belong to certain subclasses of the third class III. This class has two subclasses: **III.1** and **III.2**, and each subclass consists of several subclasses.

Let us give more detail, the first subclass III.1 has four subclasses:

$$\text{III.1a},\ \text{III.1b},\ \text{III.1c}\ \text{ and }\ \text{III.1d}.$$

The classifications of Poisson prime ideals for Poisson algebras that belong to these four subclasses are obtained in Theorem IV.3.1, Theorem IV.3.2, Corollary IV.3.3 and Corollary IV.3.4, respectively. In particular, each of these is treated individually and different techniques are involved. The main ideas to classify the Poisson prime ideals of these algebras are Example II.2.76, Example II.2.77 and Remark II.2.78. These techniques are in the recent paper [JoOh], see the review in Subsection II.2.5 for detail. In addition, the inclusions of Poisson prime ideals for these algebras are given in diagram IV.3.2, diagram IV.3.3, diagram IV.3.4 and diagram IV.3.5, respectively.

The second subclass III.2 has four subclasses:

$$\text{III.2a},\ \text{III.2b},\ \text{III.2c}\ \text{ and }\ \text{III.2d}.$$

The Poisson prime ideals for the Poisson algebra that belongs to the subclass III.2b, are classified in Corollary IV.3.5. After that, the inclusions of Poisson prime ideals for this algebra are given in diagram IV.3.6. In addition, the classifications of Poisson prime ideals for the Poisson algebras that belong to special cases of the subclasses III.2c and III.2d, are obtained in Corollary IV.3.6 and Corollary IV.3.7. Following that, the inclusions of Poisson prime ideals for these algebras are given in diagram IV.3.7 and diagram IV.3.8. However, it is difficult to classify Poisson prime ideals for the Poisson algebra that belongs to the subclass III.2a.

Class III:

If $\alpha + \beta = f\partial_t + \lambda^{-1} f\partial_t \neq 0$, $u = 0$ and $c \in K$, where $f \in K[t]\backslash\{0\}$, $\lambda \in K\backslash\{-1, 0\}$. There are two subclasses:

Class III.1: If $c = 0$.

Class III.2: If $c \in K^{\times}$.

Structure of the third class of Poisson algebras A is given in diagram IV.3.1.

Notes: We will assume that $\lambda \in \mathbb{C}\backslash\{-1, 0\}$ instead of $\lambda \in K\backslash\{-1, 0\}$.

The following is the first subclass of class III and contains four subclasses. In addition, the classifications of Poisson prime ideals for Poisson algebras A_{30}, A_{31}, A_{32} and A_{33} that belong to the subclasses III.1a, III.1b, III.1c and III.1d, respectively, are obtained. The critical key to classifying is Example II.2.76, Example II.2.77 and Remark II.2.78, which are from the study [JoOh].

Class III.1

If $c = 0$, $\alpha = f\partial_t$, $\beta = \lambda^{-1} f\partial_t$ and $u = 0$, where $f \in K[t]\backslash\{0\}$, $\lambda \in \mathbb{C}\backslash\{-1, 0\}$ then we have the Poisson algebra $A_{29} = (K[t]; f\partial_t, \lambda^{-1} f\partial_t, 0, 0)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = \lambda^{-1} fx \quad \text{and} \quad \{y, x\} = 0. \tag{IV.3.1}$$

There are four subclasses:

Class III.1a: If $f \in K^{\times}$ and $\lambda \in \mathbb{Q}\backslash\{-1, 0\}$.

Class III.1b: If $f \in K^{\times}$ and $\lambda \in \mathbb{C}\backslash\mathbb{Q}$.

Class III.1c: If $f \in K[t]\backslash K$ and $\lambda \in \mathbb{Q}\backslash\{-1, 0\}$.

Class III.1d: If $f \in K[t]\backslash K$ and $\lambda \in \mathbb{C}\backslash\mathbb{Q}$.

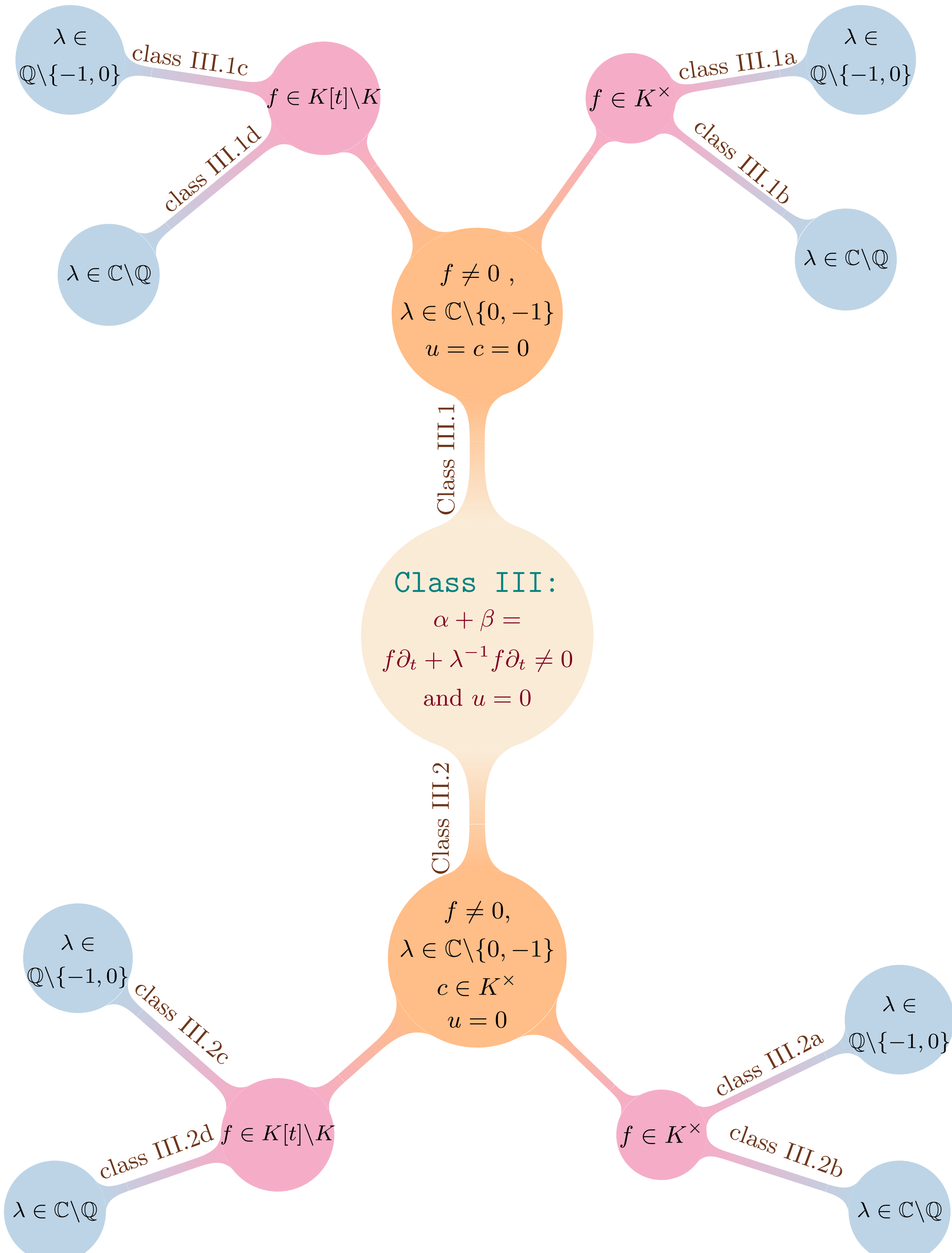


Diagram IV.3.1: Structure of the third class of Poisson algebras A

The following is the Poisson algebra A_{30} that belongs to the subclass III.1a. Notice that, f is a non-zero element in K, and $\lambda \in \mathbb{Q}\backslash\{-1, 0\}$. The technique to classify Poisson prime ideals of A_{30} is similar to the technique in the study [JoOh], which is writing the bracket as A_{30}-multiple of other. Thus, from their study, in particular, Example II.2.76 and Example II.2.77 the classification of Poisson prime ideals for A_{30} follows.

Class III.1a:

If $f \in K^{\times}$, i.e. $\alpha = f\partial_t$, $\beta = \lambda^{-1} f\partial_t$, $u = c = 0$, and n, m are non-zero integers such that $\lambda = \frac{m}{n} \neq -1$ then we have the Poisson algebra $A_{30} = (K[t]; f\partial_t, \frac{n}{m} f\partial_t, 0, 0)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = \frac{n}{m} fx \quad \text{and} \quad \{y, x\} = 0. \tag{IV.3.2}$$

The following theorem classifies the Poisson prime ideals of the Poisson algebra A_{30}.

Theorem IV.3.1. *Let $A_{30} = (K[t]; f\partial_t, \frac{n}{m} f\partial_t, 0, 0)$ be the Poisson algebra as above, where $f \in K^{\times}$, $m > 0$, $n \neq 0$ and m, n are coprime integers.*

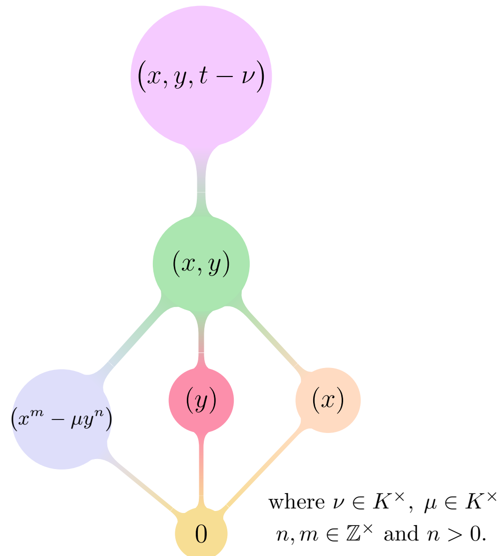
1. *If $n > 0$ then* $\text{PSpec}(A_{30}) = \{0, (x), (y), (x, y), (x^m - \mu y^n), (x, y, t - \nu) \mid \nu \in K\}$.
2. *If $n < 0$ then* $\text{PSpec}(A_{30}) = \{0, (x), (y), (x, y), (x^m y^{-n} - \mu), (x, y, t - \nu) \mid \nu \in K\}$, *the containment information between Poisson prime ideals of A_{30} is given in diagram IV.3.2.*
3. $\text{PMax}(A_{30}) = \{(x, y, t - \nu) \mid \nu \in K\}$.

Proof. 1. Let us multiply the bracket (IV.3.2) by a non-zero scalar f^{-1} then we have

$$\{t, y\} = y, \quad \{t, x\} = \frac{n}{m} x \quad \text{and} \quad \{y, x\} = 0. \tag{IV.3.3}$$

Now, multiplying by m and rearranging the bracket which implies that

$$\mathcal{B}_1 := \{y, t\} = my, \quad \{t, x\} = -nx \quad \text{and} \quad \{x, y\} = 0. \tag{IV.3.4}$$



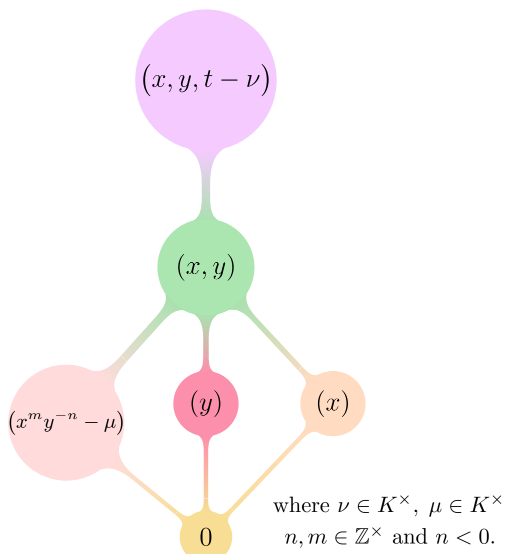


Diagram IV.3.2: The containment information between Poisson prime ideals of A_{30}

Let us call the bracket (II.2.11) in Example II.2.76 by $\mathcal{B}_2$ then $t^{-1}\mathcal{B}_1 := x^{1-j}y^{1-k}\mathcal{B}_2$ with $j = m$, $l = 0$ and $k = |n|$. It follows from Example II.2.77 that the Poisson spectrum of A_{30} consists of

$$0, \quad (y), \quad (x), \quad (x, y), \quad (x, y, t-\nu), \quad \nu \in K$$

$$(x^m - \mu y^n), \quad \mu \in K^\times, \quad \text{if} \quad n > 0,$$

$$(x^m y^{-n} - \mu), \quad \mu \in K^\times, \quad \text{if} \quad n < 0.$$

2. It follows from diagram IV.3.2.

□

The following is the Poisson algebra A_{31} that belongs to the subclass III.1b. Notice that, f is a non-zero element in K, and λ is irrational. The technique to classify Poisson prime ideals of A_{31} is similar to the technique in Remark II.2.78.

Class III.1b:

If $f \in K^{\times}$, i.e. $\alpha = f\partial_t$, $\beta = \lambda^{-1} f\partial_t$ and $u = c = 0$, where $\lambda \in \mathbb{C}\backslash\mathbb{Q}$ then we have the Poisson algebra $A_{31} = (K[t]; f\partial_t, \lambda^{-1} f\partial_t, 0, 0)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = \lambda^{-1} fx \quad \text{and} \quad \{y, x\} = 0. \tag{IV.3.5}$$

The next theorem gives us the classification of Poisson prime ideals of the Poisson algebra A_{31}.

Theorem IV.3.2. *Let $A_{31} = (K[t]; f\partial_t, \lambda^{-1} f\partial_t, 0, 0)$ be the Poisson algebra as above, where $f \in K^{\times}$ and $\lambda \in \mathbb{C}\backslash\mathbb{Q}$. Then*

1. $\text{PSpec}(A_{31}) = \{0, (y), (x), (x, y), (x, y, t - \nu) \mid \nu \in K\}$, *the containment information between Poisson prime ideals of A_{31} is given in diagram IV.3.3.*

2. $\text{PMax}(A_{31}) = \{(x, y, t - \nu) \mid \nu \in K\}$.

Proof. 1. Let us multiply the bracket (IV.3.5) by a non-zero scalar $f^{-1}\lambda$ and rearrange it then we have

$$\{y, t\} = -\lambda y, \quad \{t, x\} = x \quad \text{and} \quad \{x, y\} = 0. \tag{IV.3.6}$$

It follows from Remark II.2.78 that the Poisson spectrum of A_{31} consists of $0, (y), (x), (x, y)$ and $(x, y, t - \nu)$, where $\nu \in K$.

5. It follows from diagram IV.3.3.

□

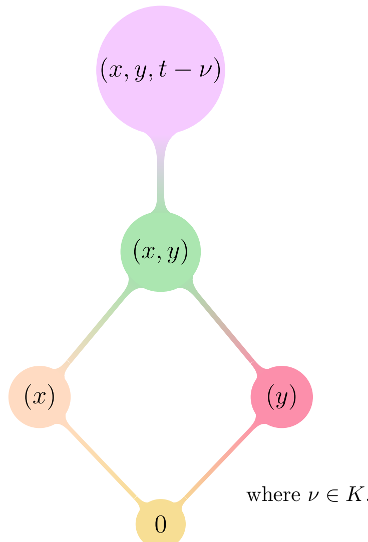


Diagram IV.3.3: The containment information between Poisson prime ideals of A_{31}

The following is the Poisson algebra A_{32} that belongs to the subclass III.1a. Notice that, f is a polynomial in t, and $\lambda \in \mathbb{Q}\setminus\{-1, 0\}$. The technique to classify Poisson prime ideals of A_{32} is similar to the technique in the study [JoOh], which is writing the bracket as A_{32}-multiple of other. Thus, from their study, in particular, Example II.2.76 and Example II.2.77 the classification of Poisson prime ideals for A_{32} follows.

Class III.1c:

If $f \in K[t]\setminus K$, i.e. $\alpha = f\partial_t$, $\beta = \lambda^{-1} f\partial_t$, $u = c = 0$, and n, m are non-zero integers such that $\lambda = \frac{m}{n} \neq -1$ then we have the Poisson algebra $A_{32} = (K[t]; f\partial_t, \frac{n}{m} f\partial_t, 0, 0)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = \frac{n}{m} fx \quad \text{and} \quad \{y, x\} = 0. \tag{IV.3.7}$$

The following corollary gives us the classification of Poisson prime ideals of the Poisson algebra A_{32}.

Corollary IV.3.3. *Let $A_{32} = (K[t]; f\partial_t, \frac{n}{m} f\partial_t, 0, 0)$ be the Poisson algebra as above, where $f \in K[t]\setminus K$, i.e. $R_f \neq \emptyset$, $m > 0$, $n \neq 0$ and m, n are coprime integers.*

1. *If* $n>0$ *then* $\mathrm{PSpec}(A_{32})=\big\{0,(x),(y),(t-\lambda_i),(x^m-\omega y^n),(x,y),(y,t-\lambda_i),(x,t-\lambda_i),(x-\mu,t-\lambda_i),(y-\mu,t-\lambda_i),(h,t-\lambda_i),(x,y,t-\nu),(x,y-\mu,t-\lambda_i),(y,x-\mu,t-\lambda_i),(x-\mu,y-\mu',t-\lambda_i)\mid \nu\in K,\ \omega,\mu,\mu'\in K^\times$ *and* $h\in \mathrm{Irr}_m K[x,y]\big\}$.

2. *If* $n<0$ *then* $\mathrm{PSpec}(A_{32})=\big\{0,(x),(y),(t-\lambda_i),(x^m y^{-n}-\omega),(x,y),(y,t-\lambda_i),(x,t-\lambda_i),(x-\mu,t-\lambda_i),(y-\mu,t-\lambda_i),(h,t-\lambda_i),(x,y,t-\nu),(x,y-\mu,t-\lambda_i),(y,x-\mu,t-\lambda_i),(x-\mu,y-\mu',t-\lambda_i)\mid \nu\in K,\ \omega,\mu,\mu'\in K^\times$ *and* $h\in \mathrm{Irr}_m K[x,y]\big\}$, *the containment information between Poisson prime ideals of* A_{32} *is given in diagram IV.3.4.*

3. $\mathrm{PMax}(A_{32})=\big\{(x,y,t-\nu),(x,y-\mu,t-\lambda_i),(y,x-\mu,t-\lambda_i),(x-\mu,y-\mu',t-\lambda_i)\mid \nu\in K$ *and* $\mu,\mu'\in K^\times\big\}$.

Proof. 1. Suppose that

$$\mathcal{B}_2:=\{y,t\}=mfy,\quad \{t,x\}=-nfx\quad\text{and}\quad \{x,y\}=0. \tag{IV.3.8}$$

If $\mathcal{B}_2:=f\mathcal{B}_1$, where the bracket $\mathcal{B}_1$ is (IV.3.4). It follows from Example II.2.77 that any Poisson prime ideal of A_{32} under $\mathcal{B}_1$ is a Poisson prime ideal of A_{32} under $\mathcal{B}_2$ and if Q is a Poisson prime ideal of A_{32} under $\mathcal{B}_2$, but not a Poisson prime ideal of A_{32} under $\mathcal{B}_1$ then $(f)\subset Q$.

The Poisson spectrum of A_{32} under $\mathcal{B}_1$ is

$$\begin{gathered}0,\quad (y),\quad (x),\quad (x,y),\quad (x,y,t-\nu),\ \ \nu\in K\\ (x^m-\mu y^n),\quad \mu\in K^\times,\quad \text{if}\quad n>0,\\ (x^m y^{-n}-\mu),\quad \mu\in K^\times,\quad \text{if}\quad n<0.\end{gathered}$$

Therefore, the Poisson spectrum of A_{32} under the bracket $\mathcal{B}_2$ is

$$\begin{array}{ccc}0,\quad (y), & (x), & (t-\lambda_i),\\ (x,y) & (x,t-\lambda_i) & (y,t-\lambda_i),\\ (x-\mu,t-\lambda_i), & (y-\mu,t-\lambda_i), & (h,t-\lambda_i),\end{array}$$

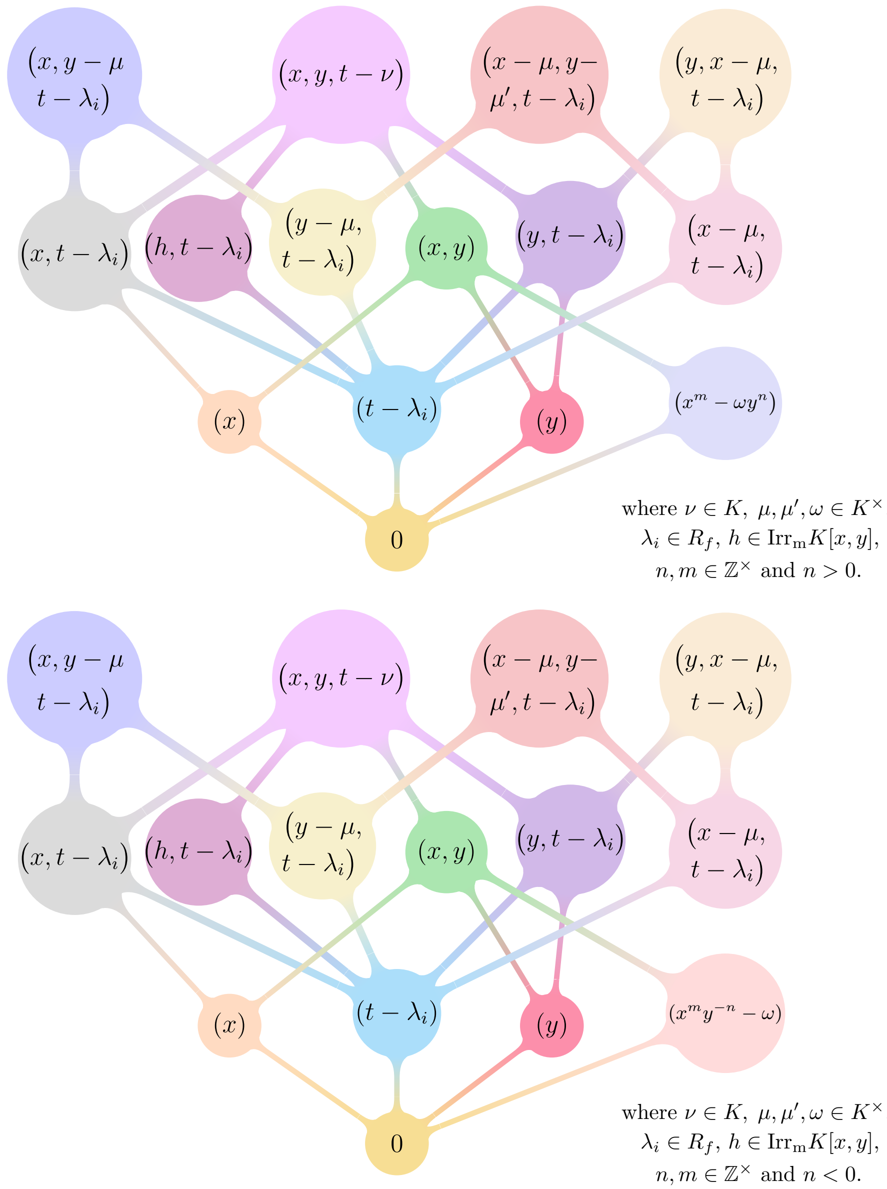


Diagram IV.3.4: The containment information between Poisson prime ideals of A_{32}

$$
\begin{aligned}
&(x, y, t-\nu), \quad (x, y-\mu, t-\lambda_i),\\
&(x-\mu, y-\mu', t-\lambda_i), \quad (y, x-\mu, t-\lambda_i),\\
&(x^m-\omega y^n), \qquad \omega \in K^\times, \qquad \text{if} \quad n > 0,\\
&(x^m y^{-n}-\omega), \qquad \omega \in K^\times, \qquad \text{if} \quad n < 0,
\end{aligned}
$$

where $\nu \in K$, $\mu, \mu' \in K^\times$, $\lambda_i \in R_f$, $i = 1, \ldots, s$ and $h \in \mathrm{Irr}_m K[x, y]$.

2. It follows from diagram IV.3.4.

□

The following is the Poisson algebra A_{33} that belongs to the subclass III.1d. Notice that, f is a polynomial in t, and λ is irrational. The technique to classify Poisson prime ideals of A_{33} is similar to the technique in the study [JoOh], which is writing the bracket as A_{33}-multiple of other. Thus, from their study, in particular, Example II.2.76 and Example II.2.77 the classification of Poisson prime ideals for A_{33} follows.

Class III.1d:

If $f \in K[t]\backslash K$, i.e. $\alpha = f\partial_t$, $\beta = \lambda^{-1} f\partial_t$ and $u = c = 0$, where $\lambda \in \mathbb{C}\backslash\mathbb{Q}$ then we have the Poisson algebra $A_{33} = (K[t]; f\partial_t, \lambda^{-1} f\partial_t, 0, 0)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = \lambda^{-1} fx \quad \text{and} \quad \{y, x\} = 0. \tag{IV.3.9}$$

The following corollary shows that the classification of Poisson prime ideals of the Poisson algebra A_{33}.

Corollary IV.3.4. *Let* $A_{33} = (K[t]; f\partial_t, \lambda^{-1} f\partial_t, 0, 0)$ *be the Poisson algebra as above, where* $f \in K[t]\backslash K$, *i.e.* $R_f \neq \emptyset$ *and* $\lambda \in \mathbb{C}\backslash\mathbb{Q}$. *Then*

1. $\mathrm{PSpec}(A_{33}) = \big\{0, (x), (y), (t-\lambda_i), (x, y), (y, t-\lambda_i), (x, t-\lambda_i), (y-\mu, t-\lambda_i), (h, t-\lambda_i), (x-\mu, t-\lambda_i), (x, y, t-\nu), (y, x-\mu, t-\lambda_i), (x, y-\mu, t-\lambda_i), (x-\mu, y-\mu', t-\lambda_i) \mid \nu \in K, \mu, \mu' \in K^{\times}, \lambda_i \in R_f,\ i = 1, \dots, s$ *and* $h \in \mathrm{Irr}_m K[x, y]\big\}$, *the containment information between Poisson prime ideals of* A_{33} *is given in diagram IV.3.5.*

2. $\mathrm{PMax}(A_{33}) = \big\{(x, y, t-\nu), (y, x-\mu, t-\lambda_i), (x, y-\mu, t-\lambda_i), (x-\mu, y-\mu', t-\lambda_i) \mid \nu \in K, \mu, \mu' \in K^{\times},\ \lambda_i \in R_f$ *and* $i = 1, \dots, s\big\}$.

Proof. 1. Suppose that

$$\mathcal{B}_2 := \{y, t\} = -\lambda fy, \quad \{t, x\} = fx \quad \text{and} \quad \{x, y\} = 0. \tag{IV.3.10}$$

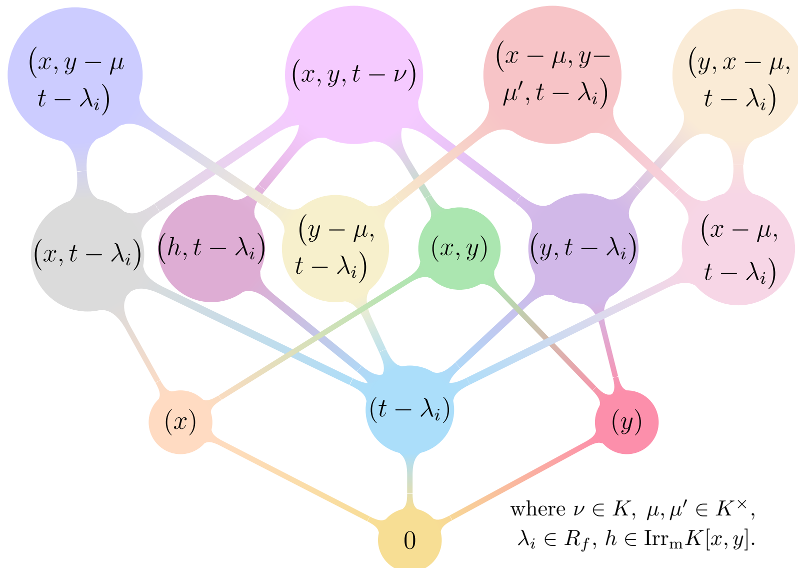


Diagram IV.3.5: The containment information between Poisson prime ideals of A_{33}

If $\mathcal{B}_2 := f\mathcal{B}_1$, where the bracket $\mathcal{B}_1$ is (IV.3.6). It follows from Example II.2.77 that any Poisson prime ideal of A_{33} under $\mathcal{B}_1$ is a Poisson prime ideal of A_{33} under $\mathcal{B}_2$ and if Q is a Poisson prime ideal of A_{33} under $\mathcal{B}_2$, but not a Poisson prime ideal of A_{33} under $\mathcal{B}_1$ then $(f) \subset Q$.

The Poisson spectrum of A_{33} under $\mathcal{B}_1$ is $0, (y), (x), (x, y)$ and $(x, y, t-\nu)$, where $\nu \in K$. Therefore, the Poisson spectrum of A_{33} under the bracket $\mathcal{B}_2$ is

$$\begin{array}{rcc} 0, \quad (y), & (x), & (t-\lambda_i), \\ (x,y) & (x,t-\lambda_i) & (y,t-\lambda_i), \\ (x-\mu,t-\lambda_i), & (y-\mu,t-\lambda_i), & (h,t-\lambda_i), \\ (x,y,t-\nu), & (x,y-\mu,t-\lambda_i), & \\ (x-\mu,y-\mu',t-\lambda_i), & (y,x-\mu,t-\lambda_i), & \end{array}$$

where $\nu \in K$, $\mu, \mu' \in K^\times$, $\lambda_i \in R_f$, $i = 1, \ldots, s$ and $h \in \mathrm{Irr}_m K[x, y]$.

2. It follows from diagram IV.3.5.

□

The following is the second subclass of class III and consists of four subclasses. In addition, the classification of Poisson prime ideals for the Poisson algebra A_{36} that belongs to the subclass III.2b, is obtained. The classifications of Poisson prime ideals for the Poisson algebras A_{38} and A_{40} that belong to special cases of the subclasses III.2c and III.2d, are obtained. However, the Poisson prime ideals of Poisson algebra that belongs to the subclass III.2a, cannot be classified.

Class III.2

If $c \in K^{\times}$, $\alpha = f\partial_t$, $\beta = \lambda^{-1} f\partial_t$ and $u = 0$, where $f \in K[t]\backslash\{0\}$, $\lambda \in \mathbb{C}\backslash\{-1, 0\}$ then we have the Poisson algebra $A_{34} = (K[t]; f\partial_t, \lambda^{-1} f\partial_t, c, 0)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = \lambda^{-1} fx \quad \text{and} \quad \{y, x\} = cyx. \tag{IV.3.11}$$

There are four subclasses:

Class III.2a: If $f \in K^{\times}$ and $\lambda \in \mathbb{Q}\backslash\{-1, 0\}$.

Class III.2b: If $f \in K^{\times}$ and $\lambda \in \mathbb{C}\backslash\mathbb{Q}$.

Class III.2c: If $f \in K[t]\backslash K$ and $\lambda \in \mathbb{Q}\backslash\{-1, 0\}$.

Class III.2d: If $f \in K[t]\backslash K$ and $\lambda \in \mathbb{C}\backslash\mathbb{Q}$.

The following is Poisson algebra A_{35} that belongs to the subclass III.2a. Notice that, f and c are unit elements in K, and λ is rational. It might be the classification of Poisson prime ideals for A_{35} is similar to the classification of Poisson prime ideals for A_{36}, but the main issue is that there is no known approach or technique to classify Poisson prime ideals of A_{35}, and it is difficult to find the common eigenvectors for δ_t, δ_x and δ_y on A_{35}.

Class III.2a:

If $\alpha = f\partial_t$, $\beta = \lambda^{-1} f\partial_t$, $u = 0$, where $f, c \in K^{\times}$ and n, m are non-zero integers such that $\lambda = \frac{m}{n} \neq -1$ then we have the Poisson algebra $A_{35} = (K[t]; f\partial_t, \frac{n}{m} f\partial_t, c, 0)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = \frac{n}{m} fx \quad \text{and} \quad \{y, x\} = cyx. \tag{IV.3.12}$$

The following is Poisson algebra A_{36} that belongs to the subclass III.2b. Notice that, f and c unit elements in K, and $\lambda \in \mathbb{C}\backslash\mathbb{Q}$. The technique to classify Poisson prime ideals of A_{36} is similar to the technique used in the algebra A_3.

Class III.2b:

If $\alpha = f\partial_t$, $\beta = \lambda^{-1} f\partial_t$ and $u = 0$, where $f, c \in K^{\times}$, $\lambda \in \mathbb{C}\backslash\mathbb{Q}$ then we have the Poisson algebra $A_{36} = (K[t]; f\partial_t, \lambda^{-1} f\partial_t, c, 0)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = \lambda^{-1} fx \quad \text{and} \quad \{y, x\} = cyx. \tag{IV.3.13}$$

Let $S = K[t]\backslash\{0\}$. The localization of the Poisson algebra A_{36} is $B_1 = S^{-1}A_{36}$, i.e. $B_1 = K(t)[x, y]$, where $K(t) = S^{-1}K[t]$ is the field of rational functions in the variable t. The algebra B_1 is a Poisson algebra with Poisson bracket defined by the rule

$$\{y, x\} = cyx. \tag{IV.3.14}$$

The following corollary shows the classification of Poisson prime ideals of the Poisson algebra A_{36}.

Corollary IV.3.5. *Let $A_{36} = (K[t]; f\partial_t, \lambda^{-1} f\partial_t, c, 0)$ be the Poisson algebra as above, where $f, c \in K^{\times}$ and $\lambda \in \mathbb{C}\backslash\mathbb{Q}$. Then*

1. $\mathrm{PSpec}(B_1) = \big\{0, (x), (y), (x, p), (y, q) \mid p \in \mathrm{Irr}_m K(t)[y]$ *and* $q \in \mathrm{Irr}_m K(t)[x]\big\}$.
2. *the localization $B_{1_{xy}}$ of the algebra B_1 at the powers of the element xy is a simple Poisson algebra.*
3. $\mathrm{PSpec}(A_{36}) = \big\{0, (x), (y), (x, y), (x, l_p p), (y, l_q q), (x, y, t - \nu) \mid \nu \in K, p \in \mathrm{Irr}_m K(t)[y]$ *and* $q \in \mathrm{Irr}_m K(t)[x]\big\}$, l_p[8] *is a unique monic polynomial in $K[t]$ of the least degree in t*

[8] respectively, l_q

such that $l_p p \in K[t,y]$[9]*, the containment information between Poisson prime ideals of* A_{36} *is given in diagram IV.3.6.*

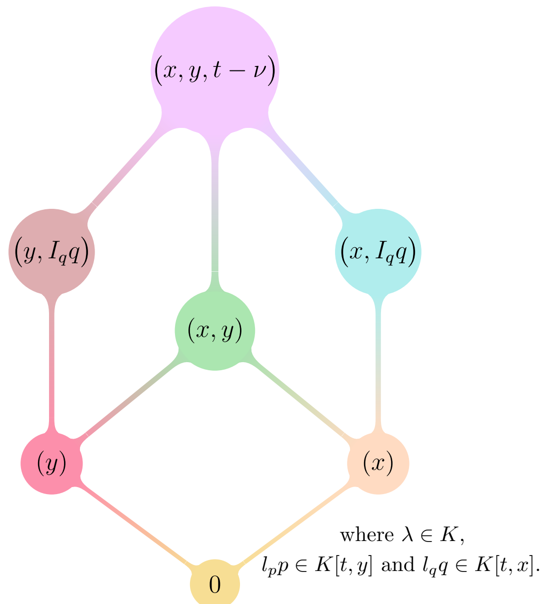


Diagram IV.3.6: The containment information between Poisson prime ideals of A_{36}

4. $\text{PMax}(A_{36}) = \{(x,y,t-\nu) \mid \nu \in K\}$.

Proof. 1. Notice that, the Poisson spectrum of B_1 can be obtained from Lemma IV.1.3.(1) and it follows from Lemma IV.1.3.(3) that the localization B_{1xy} of the algebra B_1 is a simple Poisson algebra. Let us multiple and rearrange the bracket (IV.3.13) then we have

$$\{t,y\} = \lambda f y, \quad \{t,x\} = fx \quad \text{and} \quad \{y,x\} = c\lambda yx, \tag{IV.3.15}$$

Now, let I be a non-zero Poisson prime ideal of A_{36} such that $I \in \text{PSpec}(A_{36}, S)$ then there is $I_1 := I \cap K[t] = K[t]q$ for some $q \in K[t]$. If $q \in K^{\times}$ then $1 \in I_1 \subseteq I$, so $I = A_{36}$. Therefore, let $\deg_t(q) \geq 1$. It follows from (IV.3.15) that

$$\delta_x := -fx\partial_t - c\lambda yx\partial_y,$$

$$\delta_y := -\lambda fy\partial_t + c\lambda yx\partial_x.$$

[9]respectively, $l_q q \in K[t,x]$

So, $\partial_t(q) \notin I_1$, hence $\partial_t(q) \notin I$, hence $y \in I$ and $x \in I$ this implies that $(x, y) \subseteq I$, thus, $I = (x, y, t-\nu)$ for some $\nu \in K$. The Poisson spectrum of A_{36} follows from the equality (IV.1.4);

$$\mathrm{PSpec}(A_{36}) = \mathrm{PSpec}(B_1)^r \coprod \mathrm{PSpec}(A_{36}, S).$$

2. It follows from diagram IV.3.6.

□

Class III.2c:

If $\alpha = f\partial_t$, $\beta = \lambda^{-1} f\partial_t$, $u = 0$, $c \in K^\times$, where $f \in K[t]\backslash K$, and n, m are non-zero integers such that $\lambda = \frac{m}{n} \neq -1$ then we have the Poisson algebra $A_{37} = (K[t]; f\partial_t, \frac{n}{m} f\partial_t, c, 0)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = \frac{n}{m} fx \quad \text{and} \quad \{y, x\} = cyx. \tag{IV.3.16}$$

The following is a special case of the subclass III.2c. In this Poisson algebra A_{38} that belongs to the subclass III.2c′, we assume that f is a monomial of degree one in t, $c = \frac{r}{m}$ and $\lambda = \frac{m}{n}$, where n, r, m are non-zero coprime integers. The technique to classify Poisson prime ideals of A_{38} is similar to the technique in the study [JoOh], which is writing the bracket as A_{38}-multiple of other. Thus, from their study, in particular, Example II.2.76 and Example II.2.77 the classification of Poisson prime ideals for A_{38} follows.

Class III.2c′:

If $f = t$, i.e. $\alpha = t\partial_t$, $\beta = \lambda^{-1} t\partial_t$, $u = 0$ and $c = \frac{r}{m}$, where r, n, m are non-zero coprime integers such that $\lambda = \frac{m}{n} \neq -1$ then we have the Poisson algebra $A_{38} = (K[t]; t\partial_t, \frac{n}{m} t\partial_t, \frac{r}{m}, 0)$ with Poisson bracket defined by the rule

$$\{t, y\} = ty, \quad \{t, x\} = \frac{n}{m} tx \quad \text{and} \quad \{y, x\} = \frac{r}{m} yx. \tag{IV.3.17}$$

The following corollary gives us the classification of Poisson prime ideals of the Poisson algebra A_{38}.

Corollary IV.3.6. *Let* $A_{38} = (K[t]; t\partial_t, \frac{n}{m}t\partial_t, \frac{r}{m}, 0)$ *be the Poisson algebra as above, and* m, r *and* n *are non-zero coprime integers. Then*

1. $\mathrm{PSpec}(A_{38}) = \{0, (x), (y), (t), (x^m t^r - \omega y^n), (x,t), (x,y), (y,t), (x,y,t), (x,y,t-\mu), (y,t,x-\mu), (x,t,y-\mu) \mid \mu, \omega \in K^\times\}$, *the containment information between Poisson prime ideals of* A_{38} *is given in diagram IV.3.7.*

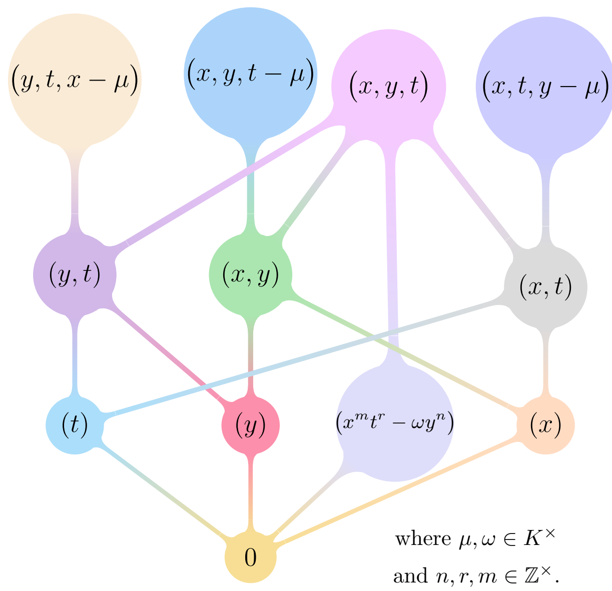


Diagram IV.3.7: The containment information between Poisson prime ideals of A_{38}

2. $\mathrm{PMax}(A_{38}) = \{(x,y,t), (x,y,t-\mu), (y,t,x-\mu), (x,t,y-\mu) \mid \mu \in K^\times\}$.

Proof. 1. Let us multiply by a non-zero scalar m and rearrange the bracket (IV.3.17) then we have

$$\mathcal{B}_1 := \{x,y\} = ryx, \quad \{t,x\} = -ntx \quad \text{and} \quad \{y,t\} = mty. \tag{IV.3.18}$$

Now, let us call the bracket (II.2.11) in Example II.2.76 by $\mathcal{B}_2$ with $l = r, j = m$ and $k = |n|$ then $\mathcal{B}_2 := x^{k-1}y^{j-1}t^{l-1}\mathcal{B}_1$. It follows from Example II.2.77 that the Poisson spectrum of A_{38} consists of

$0,\ (x), (y),\ (t), (x,t),\ (x,y), (y,t), (x,y,t),\ (x^m t^r - \omega y^n), (x,y,t-\mu),\ (y,t,x-\mu)$ and $(x,t,y-\mu)$, where $\mu, \omega \in K^\times$.

2. It follows from diagram IV.3.7.

□

Class III.2d:

If $\alpha = f\partial_t$, $\beta = \lambda^{-1} f\partial_t$, $u = 0$ and $c \in K^\times$, where $f \in K[t]\backslash K$, $\lambda \in \mathbb{C}\backslash\mathbb{Q}$ then we have the Poisson algebra $A_{39} = (K[t]; f\partial_t, \lambda^{-1} f\partial_t, c, 0)$ with Poisson bracket defined by the rule

$$\{t, y\} = fy, \quad \{t, x\} = \lambda^{-1} fx \quad \text{and} \quad \{y, x\} = cyx. \tag{IV.3.19}$$

The following is a special case of the subclass III.2d. In this Poisson algebra A_{40} that belongs to the subclass III.2d′, we assume that f is a monomial of degree one in t, and λ is irrational. The classification of Poisson prime ideals of A_{40} follows from Remark II.2.78.

Class III.2d′:

If $f = t$, i.e. $\alpha = t\partial_t$, $\beta = \lambda^{-1} t\partial_t$ and $u = 0$, where $c \in K^\times$, $\lambda \in \mathbb{C}\backslash\mathbb{Q}$ then we have the Poisson algebra $A_{40} = (K[t]; t\partial_t, \lambda^{-1} t\partial_t, c, 0)$ with Poisson bracket defined by the rule

$$\{t, y\} = ty, \quad \{t, x\} = \lambda^{-1} tx \quad \text{and} \quad \{y, x\} = cyx. \tag{IV.3.20}$$

The following corollary shows that the classification of Poisson prime ideals of the Poisson algebra A_{40}.

Corollary IV.3.7. *Let $A_{40} = (K[t]; t\partial_t, \lambda^{-1} t\partial_t, c, 0)$ be the Poisson algebra as above, $c \in K^\times$ and $\lambda \in \mathbb{C}\backslash\mathbb{Q}$. Then*

1. $\mathrm{PSpec}(A_{40}) = \{0, (x), (y), (t), (x, t), (x, y), (y, t), (x, y, t), (x, y, t - \mu), (y, t, x - \mu), (x, t, y - \mu) \mid \mu \in K^\times\}$, *the containment information between Poisson prime ideals of A_{40} is given in diagram IV.3.8.*

2. $\mathrm{PMax}(A_{40}) = \{(x, y, t), (x, y, t - \mu), (y, t, x - \mu), (x, t, y - \mu) \mid \mu \in K^\times\}$.

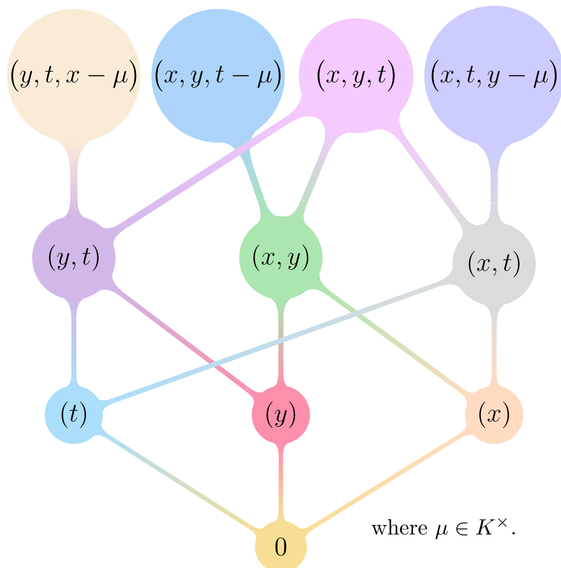


Diagram IV.3.8: The containment information between Poisson prime ideals of A_{40}

Proof. It follows from Remark II.2.78. □

Conclusion

Throughout this work, typical classes of Poisson algebras of dimension two were considered and their Poisson prime ideals were classified–see Chapter III. This thesis concentrated on a specific class of Poisson algebras of dimension three

$$A = \big(K[t]; f\partial_t, \lambda^{-1} f\partial_t, c, u\big),$$

where $f, u \in K[t]$, $\lambda \in K^{\times}$ and $c \in K$–see Chapter IV. The class of Poisson algebras A splits into three classes: **I**, **II** and **III**. Each of them splits further into subclasses, see diagram .0.1 for detail.

The classifications of Poisson prime ideals, minimal Poisson ideals and maximal Poisson ideals for the seventeen classes of Poisson algebras, which are in blue, were obtained. In addition, the classifications of special cases for the five classes of Poisson algebras, which are in green, were obtained, and some properties of these algebras were considered. Additionally, the inclusions of Poisson prime ideals for these algebras were presented in diagrams. However, for the Poisson algebras A_2, A_{23}, A_{24} and A_{36}, their Poisson prime ideals could not be classified.

Further research

This thesis has generated some ideas for future research:

The first idea is to revisit the Poisson algebras that have not been done yet and see whether their Poisson prime ideals can be classified and if there are any new methods or techniques to use.

The second idea is to classify simple finite-dimensional Poisson modules over Poisson algebras that belong to some subclasses of Poisson algebras A. This might be done by using these classifications of Poisson prime ideals. There are two studies, [Bav4] and [Jor], in Poisson modules which might be the main sources for this research. Significantly, there will be some obvious modules of some Poisson algebras that belong to some subclasses, but not all will be easy to classify, as we saw in this class of Poisson algebras.

The third idea is to classify Poisson prime ideals for a similar class of Poisson algebras, in particular, $(K[t,s];\alpha,\beta,c,u)$ that has dimension four, i.e. $K[t,s][y;\alpha][x;\beta,\delta]$. Also, it might be possible to see whether it can be generalized to arbitrary n or find a general form for this typical class of Poisson algebras. This might be done by using similar techniques and ideas from this thesis. After that, it might be possible to classify some simple finite-dimensional Poisson modules over this class of Poisson algebras.

The fourth suggestion is to classify some generalized Weyl Poisson algebras for Poisson algebras that belong to some subclasses of Poisson algebras A and study their properties. This might be done by using similar techniques in the paper [Bav3].

Bibliography

[Bav1] V. V. Bavula, The simple $D[X, Y; \sigma, a]$-modules, *Ukrainian Math. J.*, **44** (1992), no. 12, 1500–1511.

[Bav2] V. V. Bavula, Generalized Weyl algebras and their representations. *St. Petersburg Math. J*, **4** (1993), no. 1, 71–92.

[Bav3] V. V. Bavula, The generalized Weyl Poisson algebras and their Poisson simplicity criterion, *Letters in Math. Physics*, **110** (2020), no.1, 105–119.

[Bav4] V. V. Bavula, The PBW Theorem and simplicity criteria for the Poisson enveloping algebra and the algebra of Poisson differential operators, submitted, arxiv.2107.00321, 2021.

[Bav5] V. V. Bavula, The Poisson enveloping algebra and the algebra of Poisson differential operators of a generalized Weyl Poisson algebra, submitted, arXiv:2107.00944v1, 2021.

[ChPr] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Providence, RI: Cambridge University Press, 1994.

[ChOh] E.-H. Cho and S.-Q. Oh, Semiclassical limits of Ore extensions and a Poisson generalized Weyl algebra, *Lett. Math. Phys.,* **106** (2016) 997–1009.

[Dix] J. Dixmier, *Enveloping Algebras*, Graduate Studies in Mathematics, **11**. Providence, RI: Amer. Math. Soc, 1996.

[DoMa] J. Donin and L. Makar-Limanov, Quantization of quadratic Poisson brackets on a polynomial algebra of three variables, *Journal of Pure and Applied Algebra*, **129** (1998) 247-261.

[EtSt] P. Etingof and D. Stryker, Short star-products for filtered quantizations, I, *SIGMA,* **16** (2020) 14–28, arXiv:1909.13588v5.

[Far] D. R. Farkas, Modules for Poisson algebras, *Communications in Algebra,* **28** (2000) 3293–3306.

[Goo1] K. R. Goodearl, A Dixmier-Moeglin equivalence for Poisson algebras with torus actions, *Amer. Math. Soc., Providence, Contemp. Math.*, **419** (2006) 131–154.

[Goo2] K. R. Goodearl, *Semiclassical limits of quantized coordinate rings*, in Advances in Ring Theory (D. V. Huynh and S. R. Lopez-Permouth, Eds.) Basel Birkhäuser (2009) 165–204.

[GoLe] K. R. Goodearl and E. S. Letzter, Semiclassical limits of quantum affine spaces, *Proceedings of the Edinburgh Mathematical Society* (2), **52** (2009) 387–407.

[GoWa] K. R. Goodearl and R. B. Warfield, *An introduction to noncommutative Noetherian rings*. 2nd ed. New York: Cambridge University Press, (2004), pages 1–85, 105–122 and 166–186.

[Har] R. Hartshorne, *Algebraic Geometry*, Springer New York, NY, 1977.

[Jor] D. A. Jordan, Finite-dimensional simple Poisson modules, *Alg. Repr. Theory*, **13** (2010) 79–101.

[JoOh] D. A. Jordan and S.-Q. Oh, Poisson brackets and Poisson spectra in polynomial algebras. New Trends in Noncommutative Algebra, *Contemp. Math.*, **562** (2012) 169–187.

[LPV] C. Laurent-Gengoux, A. Pichereau and P. Vanhaecke, *Poisson structures*, Grundlehren der Mathematischen Wissenschaften, 347, Springer, Heidelberg, 2013.

[McRo] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*. With the cooperation of L. W. Small. Revised edition. Graduate Studies in Mathematics, 30. American Mathematical Society, Providence, RI, 2001.

[MyOh] N.-H. Myung and S.-Q. Oh, A construction of an Iterated Ore extension, arXiv:1707.05160v2, 2018.

[Oh1] S.-Q. Oh, Symplectic ideals of Poisson algebras and the Poisson structure associated to Quantum matrices, *Communications in Algebra*, **27** (1999) 2163–2180.

[Oh2] S.-Q. Oh, Poisson enveloping algebras, *Communications in Algebra*, **27** (1999), no. 5, 2181–2186.

[Oh3] S.-Q. Oh, Poisson polynomial rings, *Communications in Algebra*, **34** (2006) 1265–1277.

[Oh4] S.-Q. Oh, Poisson prime ideals of Poisson polynomial rings, *Communications in Algebra*, **35** (2007) 3007–3012.

[Per] D. Perrin, *Algebraic Geometry: An Introduction.* Springer, 2008.

[Pol] A. Polishchuk, Algebraic Geometry of Poisson Brackets, *Journal of Mathematical Sciences.* **84**, no. 5, (1997) 1413–1444.

[Pre] A. Premet, *Lie algebra*, Lecture Notes in MATH 62112 at the University of Manchester, Manchester in Spring (2017) 1–41.

[Sta] T. Stafford, *Noncommutative algebra*, Lecture Notes in MATH 62041 at the University of Manchester, Manchester in Autumn (2016) 1–87.

[Van] M. Vancliff, Primitive and Poisson spectra of twists of polynomial rings, *Algebras and Representation Theory,* **2** (1999) 269–285.

List of notations

A The Poisson algebra $(K[t]; f\partial_t, \lambda^{-1} f\partial_t, c, u)$ with Poisson bracket $\{t, y\} = fy, \{t, x\} = \lambda^{-1} fx, \{y, x\} = cyx + u$, where $f, u \in K[t]$, $c \in K$, $\lambda \in K^\times$ iv, v, 2, 3, 4, 20, 24, 28, 64, 66, 85, 105, 122, 123

A_3 The Poisson algebra $(K[t]; 0, 0, c, 0)$ with Poisson bracket $\{y, x\} = cyx$, where $c \in K^\times$ 5, 67, 69, 71, 72, 73, 116

A_6 The Poisson algebra $(K[t]; f\partial_t, -f\partial_t, 0, 0)$ with Poisson bracket $\{t, y\} = fy,\ \{t, x\} = -fx$, where $f \in K[t]\backslash K$ 6, 74, 75, 76, 77

A_9 The Poisson algebra $(K[t]; f\partial_t, -f\partial_t, c, 0)$ with Poisson bracket $\{t, y\} = fy,\ \{t, x\} = -fx,\ \{y, x\} = cyx$, where $f \in K[t]\backslash K$, $c \in K^\times$ 7, 79, 80, 82

K A field 9

Spec(R) The set of all prime ideals of R 9

Max(R) The set of all maximal ideals of R 10

$\mathbb{Z}$ The ring of integers 10

$\mathrm{Der}_K(L)$	The set of derivations on L 14
$\mathrm{IDer}_K(L)$	The set of inner derivations on L 14
δ_{ij}	The Kronecker delta function 15
$\mathrm{PDer}_K(D)$	The set of Poisson derivations on D 17
$\mathrm{PIDer}_K(D)$	The set of Poisson inner derivations on D 17
$\mathrm{PSpec}(D)$	The set of all Poisson prime ideals of D 17
$\mathrm{PMax}(D)$	The set of all maximal Poisson ideals of D 17
$\mathbb{A}_K^n$	The affine n-space over a field K 22
$\mathcal{O}_X$	The structure sheaf on a scheme X 23
$(X, \mathcal{O}_X)$	A scheme 23
Ω_X^1	The differential 1-form on X 23
$\mathbb{C}$	The field of complex numbers 24
$\mathbb{Z}^+$	The set of non-negative integers 26
$\mathbb{CP}^1$	The complex projective line 38
Irr_m	The monic irreducible polynomials 38

$\mathcal{P}_{fx}$ The localization algebra of $\mathcal{P}$ at the powers of the element fx, i.e. $\{(fx)^{-i}p \mid i \geq 0,\ p \in \mathcal{P}\}$ 59, 60, 61

$\mathcal{P}_2(f)_f$ The localization of $\mathcal{P}_2(f)$ at the powers of the non-zero element f, i.e. $\{(f)^{-i}p \mid i \geq 0,\ p \in \mathcal{P}_2(f)\}$ 62

A_1 The Poisson algebra $(K[t]; 0, 0, c, 0)$ with Poisson bracket $\{y, x\} = cyx$, where $c \in K$ 67

A_2 The Poisson algebra $(K[t]; 0, 0, 0, 0)$ with trivial Poisson structure 67, 122

B The Poisson algebra $S^{-1}A_3 = K(t)[x, y]$ with Poisson bracket $\{y, x\} = cyx$, where $c \in K^{\times}$ 69, 70, 71, 72, 73

B_{xy} The localization algebra of B at the powers of the element xy, i.e. $\{(xy)^{-i}p \mid i \geq 0,\ p \in B\}$ 69, 70, 71

$\mathcal{P}_2$ The Poisson algebra $K[x, y]$ with Poisson bracket $\{y, x\} = cyx$, where $c \in K^{\times}$ 72, 73, 81, 82, 94

A_4 The Poisson algebra $(K[t]; f\partial_t, -f\partial_t, c, 0)$ with Poisson bracket $\{t, y\} = fy,\ \{t, x\} = -fx,\ \{y, x\} = cyx$, where $f \in K[t]\backslash\{0\}, c \in K$ 73, 74

$\mathcal{P}_1$ The Poisson algebra $K[t, y]$ with Poisson bracket $\{t, y\} = fy$, where $f \in K[t]\backslash K$ 73, 77, 82

A_5 The Poisson algebra $(K[t]; f\partial_t, -f\partial_t, 0, 0)$ with Poisson bracket $\{t, y\} = fy,\ \{t, x\} = -fx$, where $f \in K[t]\backslash\{0\}$ 74

A_{13} The Poisson algebra $(K[t];0,0,0,u)$ with Poisson bracket $\{y,x\}=u$, where $u \in K[t]\backslash K$ 85, 89, 90

A_{14} The Poisson algebra $(K[t];0,0,c,u)$ with Poisson bracket $\{y,x\} = cyx+u$, where $u,c \in K^{\times}$ 85, 90, 91, 92

A_{15} The Poisson algebra $(K[t];0,0,c,u)$ with Poisson bracket $\{y,x\} = cyx+u$, where $u \in K[t]\backslash K,\ c \in K^{\times}$ 85, 92, 93, 94

A_{11} The Poisson algebra $(K[t];0,0,c,u)$ with Poisson bracket $\{y,x\} = cyx+u$, where $u \in K[t]\backslash\{0\}$, $c \in K$ 85, 87

R_u The set of distinct roots of the polynomial u 89

A_{13u} The localization algebra of A_{13} at the powers of the non-zero element u, i.e. $\{(u)^{-i}p \mid i \geq 0,\ p \in A_{13}\}$ 89, 90

$\mathcal{P}_4$ The Poisson algebra $K[x,y]$ with Poisson bracket $\{y,x\} = cyx+u$, where $u,c \in K^{\times}$ 91

$\mathcal{P}'_4$ The Poisson algebra $K[x,y]$ with Poisson bracket $\{y,x\} = cyx+u(\omega)$, where $u(\omega), c \in K^{\times}$ 93

A_{17} The Poisson algebra $(K[t]; f\partial_t, -f\partial_t, 0, u)$ with Poisson bracket $\{t,y\} = fy, \{x,t\} = fx, \{y,x\} = u$, where $f,u \in K^{\times}$ 94, 95, 96, 100

A_{18} The Poisson algebra $(K[t]; f\partial_t, -f\partial_t, 0, u)$ with Poisson bracket $\{t,y\} = fy, \{t,x\} = -fx, \{y,x\} = u$, where $u \in K[t]\backslash K$, $f \in K^\times$ 94, 96

A_{20} The Poisson algebra $(K[t]; f\partial_t, -f\partial_t, 0, u)$ with Poisson bracket $\{t,y\} = fy, \{t,x\} = -fx, \{y,x\} = u$, where $f \in K[t]\backslash K, u \in K^\times$ 94, 97

A_{22} The Poisson algebra $(K[t]; f\partial_t, -f\partial_t, 0, fg)$ with Poisson bracket $\{t,y\} = fy, \{t,x\} = -fx, \{y,x\} = fg$, where $f, g \in K[t]\backslash K$ 94, 98, 99

A_{26} The Poisson algebra $(K[t]; ct\partial_t, -ct\partial_t, c, u)$ with Poisson bracket $\{t,y\} = cty, \{x,t\} = ctx, \{y,x\} = cyx + u$, where $u, c \in K^\times$ 94, 100, 101

A_{28} The Poisson algebra $(K[t]; ct\partial_t, -ct\partial_t, c, vt)$ with Poisson bracket $\{t,y\} = cty, \{x,t\} = ctx, \{y,x\} = cyx + vt$, where $c, v \in K^\times$ 94, 101, 102

A_{16} The Poisson algebra $(K[t]; f\partial_t, -f\partial_t, c, u)$ with Poisson bracket $\{t,y\} = fy, \{t,x\} = -fx, \{y,x\} = cyx + u$, where $f, u \in K[t]\backslash\{0\}$, $c \in K$ 94

A_{19} The Poisson algebra $(K[t]; f\partial_t, -f\partial_t, 0, vt)$ with Poisson bracket $\{t,y\} = fy, \{x,t\} = fx, \{y,x\} = vt$, where $f, v \in K^\times$ 97

$\mathcal{P}(u)$ The Poisson algebra $K[x,y]$ with Poisson bracket $\{y,x\} = u$, where $u \in K^\times$ 97

A_{20_f} The localization algebra of A_{20} at the powers of the non-zero element f, i.e. $\{(f)^{-i}p \mid i \geq 0,\ p \in A_{20}\}$ 97

A_{21} The Poisson algebra $(K[t]; f\partial_t, -f\partial_t, 0, u)$ with Poisson bracket $\{t, y\} = fy, \{t, x\} = -fx, \{y, x\} = u$, where $f, u \in K[t]\backslash K$ 98

A_{23} The Poisson algebra $(K[t]; f\partial_t, -f\partial_t, c, u)$ with Poisson bracket $\{t, y\} = fy, \{t, x\} = -fx, \{y, x\} = cyx + u$, where $f, u, c \in K^{\times}$ 100, 122

A_{24} The Poisson algebra $(K[t]; f\partial_t, -f\partial_t, c, u)$ with Poisson bracket $\{t, y\} = fy, \{t, x\} = -fx, \{y, x\} = cyx + u$, where $u \in K[t]\backslash K$, $f, c \in K^{\times}$ 100, 122

A_{25} The Poisson algebra $(K[t]; f\partial_t, -f\partial_t, c, u)$ with Poisson bracket $\{t, y\} = fy, \{t, x\} = -fx, \{y, x\} = cyx + u$, where $f \in K[t]\backslash K$, $u, c \in K^{\times}$ 100

A_{27} The Poisson algebra $(K[t]; f\partial_t, -f\partial_t, c, u)$ with Poisson bracket $\{t, y\} = fy, \{t, x\} = -fx, \{y, x\} = cyx + u$, where $f, u \in K[t]\backslash K$, $c \in K^{\times}$ 101

A_{30} The Poisson algebra $(K[t]; f\partial_t, \frac{n}{m}f\partial_t, 0, 0)$ with Poisson bracket $\{t, y\} = fy$, $\{t, x\} = \frac{n}{m}fx$, where $f \in K^{\times}$, $\lambda = \frac{m}{n} \neq -1$, $m, n \in \mathbb{Z}^{\times}$ 105, 107

A_{31} The Poisson algebra $(K[t]; f\partial_t, \lambda^{-1}f\partial_t, 0, 0)$ with Poisson bracket $\{t, y\} = fy$, $\{t, x\} = \lambda^{-1}fx$, where $f \in K^{\times}$, $\lambda \in \mathbb{C}\backslash\mathbb{Q}$ 105, 109

A_{35} The Poisson algebra $(K[t]; f\partial_t, \frac{n}{m} f\partial_t, c, 0)$ with Poisson bracket $\{t, y\} = fy$, $\{t, x\} = \frac{n}{m} fx$, $\{y, x\} = cyx$, where $f, c \in K^\times$, $\lambda = \frac{m}{n} \neq -1$, $m, n \in \mathbb{Z}^\times$ 115

B_1 The Poisson algebra $S^{-1}A_{36} = K(t)[x, y]$ with Poisson bracket $\{y, x\} = cyx$, where $c \in K^\times$ 116, 117, 118

B_{1xy} The localization algebra of B_1 at the powers of the non-zero element xy, i.e. $\{(xy)^{-i}p \mid i \geq 0,\ p \in B_1\}$ 116, 117

A_{37} The Poisson algebra $(K[t]; f\partial_t, \frac{n}{m} f\partial_t, c, 0)$ with Poisson bracket $\{t, y\} = fy$, $\{t, x\} = \frac{n}{m} fx$, $\{y, x\} = cyx$, where $f \in K[t]\backslash K$, $\lambda = \frac{m}{n} \neq -1$, $m, n \in \mathbb{Z}^\times$, $c \in K^\times$ 118

A_{39} The Poisson algebra $(K[t]; f\partial_t, \lambda^{-1} f\partial_t, c, 0)$ with Poisson bracket $\{t, y\} = fy$, $\{t, x\} = \lambda^{-1} fx$, $\{y, x\} = cyx$, where $f \in K[t]\backslash K$, $\lambda \in \mathbb{C}\backslash\mathbb{Q}$, $c \in K^\times$ 120